Evolution of Anisotropies in Eddington-Born-Infeld Cosmology

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Recently a Born-Infeld action for dark energy and dark matter that uses additional affine connections was proposed. At background level, it was shown that the new proposal can mimic the standard cosmological evolution. In Bianchi cosmologies, contrary to the scalar field approach (e.g., Chaplygin gas), the new approach leads to anisotropic pressure, raising the issues of stability of the isotropic solution under anisotropic perturbations and, being it stable, how the anisotropies evolve. In this work, the Eddington-Born-Infeld proposal is extended to a Bianchi type I scenario and residual post-inflationary anisotropies are shown to decay in time. Moreover, it is shown that the shears decay following a damped oscillatory pattern, instead of the standard exponential-like decay. Allowing for some fine tuning on the initial conditions, standard theoretical bounds on the shears can be avoided.

Keywords: Dark energy, dark matter, Bianchi cosmology, isotropization.

I. INTRODUCTION

One of the greatest puzzles in current fundamental physics is the nature of dark energy and dark matter\[1,2\]. Observationally their existence is well rooted in the results of many independent experiments, e.g.\[2,3\], while theoretically the understanding of these entities is far from satisfactory. The simplest way to model the dark sector, i.e. dark energy and dark matter, in good agreement with observations is through the now standard ΛCDM model\[2\]. Following this approach, at background level, dark matter enters as an additional dust contribution and dark energy as a positive cosmological constant.

Considering their apparent dissimilarities, that dark matter and dark energy might be two faces of a single mysterious entity is an appealing proposal. This direction was first explored through the employment of the Chaplygin gas and its generalized version\[5\], which use a (generalized) Born-Infeld action for scalar fields. Albeit compatible with cosmological data at background level\[6\], once density perturbations are considered the structure formation and the cosmic microwave background power spectrum\[7\].

Recently, motivated from a general relativity extension to spaces whose space-time metric can be degenerated, a new model that describes a unified dark sector was proposed\[8\]. In this Eddington-Born-Infeld model (or Einstein-Eddington-Born-Infeld), the space-time metric interacts with the Eddington action\[9\] through a Born-Infeld coupling.

The Eddington action was proposed as an alternative action for gravitation\[9\]. Its fundamental fields are the affine connections and no metric appears in the action.

It reads

\[S_{\text{Edd}}[\Gamma] = \int \sqrt{|R_{\mu\nu}(\Gamma)|} d^4x.\] \hspace{1cm} (1)

In above, \(\Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha}\) are independent affine connections and \(|R_{\mu\nu}|\) is the modulus of the determinant of the symmetric Riemann curvature tensor. The action (1) does not depend on the space-time metric but it is equivalent to the Einstein-Hilbert action in the presence of a non-null cosmological constant\[10\]. Considering that current cosmological observations favors a non-null cosmological constant, the latter point is welcome. Nevertheless, the coupling of gravity in this formulation to the rest of the universe is not a straightforward issue. The fundamental fields are the affine connections, but direct couplings of these to other structures are typically inconsistent, since the former are not tensors\[27\].

The above property turns to be a welcome feature in the Eddington-Born-Infeld model. In the latter, the additional affine connections that come from the Eddington part are interpreted as the dark sector fundamental fields, which couples to the rest of the universe only through gravity.

Besides the aforementioned motivation, the employment of additional connections in cosmology, instead of scalar fields, is a new curious possibility which deserves to be further investigated. The Eddington-Born-Infeld model can be seen as an affine connection version for the Chaplygin gas approach, where the scalar kinetic term inside the determinant is replaced by the curvature tensor for the additional “dark” connections.

The purpose of this work is to analyze the Eddington-Born-Infeld cosmology beyond the isotropic and homogeneous background. Of particular interest is post-inflationary shear perturbations, which will be studied in a Bianchi I framework. It generalizes the flat Friedmann-Lemaître-Robertson-Walker cosmology to homogeneous and anisotropic spaces without rotation. For reviews on Bianchi cosmologies see\[12\]. For some recent developments on the Bianchi I phenomenology see\[13,14\] and

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II. REVISITING THE EDDINGTON-BORN-INFELD ACTION AND ITS HOMOGENEOUS AND ISOTROPIC COSMOLOGY

This section reviews some key results of \( S \) and introduces procedures that will be useful for the following sections. The action proposed in \( S \) reads

\[
S[g, C, \Psi] = \int L_m(\Psi, g) \, d^4x + \frac{k}{2} \int \left[ \sqrt{|g_{\mu\nu}|} R + 2 \frac{\alpha}{l^2} \sqrt{|g_{\mu\nu} - l^2 K_{\mu\nu}|} \right] \, d^4x,
\]

where \( (g_{\mu\nu}) \) is the space-time metric, \( k \equiv 1/(8\pi G) \), \( \Psi \) stands for any additional matter fields, \(|X|_{\mu\nu}\) is the absolute value of the \((X)_{\mu\nu}\) determinant, \( K_{\mu\nu} \) is the symmetric Ricci tensor constructed with the symmetric connection \( C_{\alpha\beta}^{\mu\nu} \) and \( R \) is the standard curvature scalar constructed with the space-time metric and the metric connection \( \Gamma(g) \). The dimensions of the constants \( l^2 \) and \( \alpha \) read \( [l^2] = m^{-2} \), \([\alpha] = 1 \). The affine connection \( C_{\alpha\beta}^{\mu\nu} \) couples to \( g_{\mu\nu} \) through a Born-Infeld interaction, and only influence additional fields indirectly through the coupling of \( g_{\mu\nu} \) to them.

In the absence of \( L_m \), the equations of motion of the action \( S \) read

\[
G_{\mu\nu} = \frac{1}{\alpha l^2} \sqrt{|g|} g_{\mu\alpha}(g - l^2 K)^{\alpha\beta\gamma} g_{\beta\gamma},
\]

\[
\nabla^C_{\epsilon} \left( \sqrt{|g|} (g - l^2 K)^{\alpha\beta\gamma} \right) = 0,
\]

where the indices inside the determinants were omitted, \((g - l^2 K)^{\alpha\beta\gamma}\) is the inverse of the matrix \((g_{\alpha\beta} - l^2 K_{\alpha\beta})\) and \( \nabla^C \) is the covariant derivative constructed with the \( C \) connection.

The equations of motion of the pure Eddington action \( S \) are given by \( 4 \) without \( g_{\mu\nu} \). In order to express such equations in a form closer to the Einstein equations, one introduces an auxiliary non-degenerate symmetric tensor \( q^{\mu\nu} \) such that \( \nabla^C q^{\mu\nu} = 0 \), which implies that \( C_{\beta\rho}^\lambda \) can be written as functions of \( q_{\mu\nu} \) and derivatives. Letting \( q_{\mu\nu} \propto K_{\mu\nu} \), Eq. \( 4 \) (without \( g_{\mu\nu} \)) becomes an identity and the equations of motion are now simply \( q_{\mu\nu} \propto K_{\mu\nu} \). A constant \( \lambda \) is interpreted \( \propto K_{\mu\nu}(g) \), where \( \lambda \) is a constant. Interpreting \( q_{\mu\nu} \) as the space-time metric, one sees that the latter equations are just Einstein equations for a de Sitter space-time with \( \lambda \) as the cosmological constant. For other details on the correspondence of this formulation with the standard one, see \( 10 \).

A similar trick is also useful for the action \( 2 \). The main differences are that the auxiliary tensor \( q_{\mu\nu} \) is not the space-time metric, and that the value of the arbitrary constant \( \lambda \) corresponds to a rescaling of \( q_{\mu\nu} \) that has no dynamical role. The resulting new expressions for the equations of motion are

\[
G_{\mu\nu} = -\Lambda \sqrt{|g|} g_{\mu\alpha} q^{\alpha\beta\gamma} g_{\beta\gamma},
\]

\[
K_{\mu\nu}(g) = \Lambda \left[ \alpha q_{\mu\nu} + (1 - \alpha) g_{\mu\nu} \right].
\]

In above we have re-parametrized the model from \( \{l^2, \alpha\} \) to \( \{\Lambda, \alpha\} \) using the same relation of \( S \),

\[
l^2 = \frac{1}{1 - \alpha \Lambda},
\]

but a different normalization for \( q_{\mu\nu} \) was selected. To change to the original normalization of \( S \) one should replace every \( q_{\mu\nu} \) by \((1 - \alpha) q_{\mu\nu} \).

The simplest nontrivial solution of Eqs. \( 5 \) is found for \( C_{\mu\nu}^{\alpha} = \Gamma_{\mu\nu}^{\alpha} \). This implies that \( q_{\mu\nu} = \gamma g_{\mu\nu} \), for constant \( \gamma \). From \( 6 \), one concludes that this corresponds to a de Sitter solution with \( \Lambda \gamma \) as the cosmological constant; and requiring compatibility with \( 6 \) leads to \( \gamma = 1 \). The latter defines the selected \( q_{\mu\nu} \) normalization.

Eq. \( 9 \) can be set in the standard form of Einstein equations,

\[
Q_{\mu\nu} = \Lambda (1 - \alpha) \left( \frac{\alpha}{\alpha - 1} g_{\mu\nu} - \frac{1}{2} g_{\alpha\beta} q^{\alpha\beta} q_{\mu\nu} \right).
\]
where $Q_{\mu\nu} = K_{\mu\nu} - \frac{1}{2} K g^{\alpha\beta} q_{\mu\nu}$.

Now we turn to the isotropic and homogeneous cosmology spawned by the equations (8) with

$$
(g_{\mu\nu}) = \text{diag} (-N^2 \quad a^2 \quad a^2 \quad a^2),
$$

$$
(q_{\mu\nu}) = \text{diag} (-X^2 \quad Y^2 \quad Y^2 \quad Y^2).
$$

All the above variables depend on time alone. The equations of motion, in the presence of usual matter and radiation, can be stated as

$$
3H^2 = N^2 \left( \frac{\Lambda NY^3}{Xa^3} + \frac{\rho_b}{k} + \frac{\rho_r}{k} \right),
$$

$$
3H_Y^2 = \frac{\Lambda(1 - \alpha)X^2}{2} \left( \frac{2\alpha}{1 - \alpha} - \frac{N^2}{X^2} + \frac{3a^2}{Y^2} \right),
$$

$$
3a^2H \frac{XY}{N} = \frac{d}{dt} \left( \frac{NY^3}{X} \right),
$$

where $H \equiv \dot{a}/a$, $H_Y \equiv \dot{Y}/Y$, $\rho_b$ is the baryon energy density and $\rho_r$ is the radiation energy density. Once the temporal gauge is fixed, one has three equations for three unknowns. The above equations are the $G_{00}$ constraint, the $Q_{0\mu}$ constraint (12), and the energy momentum conservation for the “dark fluid” (13). The latter can be found from the Bianchi identities of either $G_{\mu\nu}$ or $Q_{\mu\nu}$ together with (8) or (9).

Eqs. (11) (12) (13) were the equations selected in [8] for evaluating the numerical solutions, which is the simplest choice in the isotropic picture. Nevertheless, for plotting anisotropic cosmologies one does not have the possibility of selecting only the first order equations. To prepare to the next section, we will use the second order equations,

$$
2\dot{H} + 3H^2 - 2HH_N = \frac{\Lambda NY}{a} - \frac{N^2\rho_r}{3k},
$$

$$
2\dot{H}_Y + 3H_Y^2 - 2H_Y H_X = \frac{\Lambda(1 - \alpha)X^2}{2} \left( \frac{2\alpha}{1 - \alpha} - \frac{1}{X^2} + \frac{a^2}{Y^2} \right),
$$

together with (13) as the equations to be solved numerically. With $H_X \equiv \dot{X}/X$ and $H_N \equiv \dot{N}/N$. Eqs. (11) (12) constrains the initial conditions.

In order to plot the numerical solutions, we fix the temporal gauge with $N = 1$ and search for a consistent set of initial conditions. We will use $X_0$, $H_0$, $Y_0$, $Y_0$ to denote the values of $X$, $H$, $Y$, $\dot{Y}$ at present time $t_0$, and we set $a(t_0) = 1$. The initial conditions will be set at $t = t_0$. Letting (11) to set the value of $X_0$ as a function of $H_0$ and $Y_0$, the $Y$ initial condition reads

$$
\dot{Y}_0 = \pm \frac{\sqrt{1 - \alpha} \Lambda^2 a \sqrt{2} Y_0^4}{\sqrt{6} \left( -3H_0^2 + \frac{\rho_r}{k} + \frac{\rho_b}{k} \right)} \left[ \frac{2\alpha}{1 - \alpha} + \frac{3}{Y_0^2} - \frac{(-3H_0^2 + \frac{\rho_r}{k} + \frac{\rho_b}{k})^{\frac{1}{2}}}{\Lambda^2 Y_0^2} \cdot \right]
$$

Besides $\rho_r$ and $\rho_b$, two parameters, $\alpha$ and $\Lambda$, and two initial conditions, $Y_0$ and $H_0$, are still free. The proper way of fixing them is to search for the best fit directly from cosmological data [16, 17, 18]. Here we follow the original proposal and assume that this best fit should lead to values of $\rho_r$, $\rho_b$, $\Lambda$ and $H_0$ close to the ones found from the $\Lambda$CDM best fit. Any choice of these parameters close to the values found from WMAP5+BAO+SN [4] is perfectly consistent with this paper proposal, and we will take precisely those values — except for radiation, for it a mean value is adopted. Regarding the values of $Y_0$ and $\alpha$, $\alpha$ is set to 0.99 (it should be close to one to allow solutions that are close to the standard cosmological model [8]), and $Y_0$ is set such that the Hubble parameter directly evolves toward $\sqrt{\Lambda}/3$, leading to $Y_0 = 1.083$ [28]. Fig. (1) shows a plot for the evolution of all the isotropic Eddington-Born-Infeld functions. And Fig. (2) shows the evolution of $H$, $H_Y$ and the pressure induced in the new fluid, it will be useful for the next section.

**III. ANISOTROPIC PERTURBATIONS**

The approach of this section is to study the behavior of the anisotropies assuming that they are fluctuations around the isotropic solution. The isotropic quantities, whose analytical solutions are unknown, are analyzed numerically, while the anisotropic quantities are analyzed analytically. In this model, as it will be shown in the following section, there are six initial conditions that...
can be expressed as \[29\]

Quantities (using these three quantities are in above considered as adimensional quantities (using \(t_0 = 1\)). Although the ratio of the pressure \(P\) by the energy density goes to zero as \(t\) approaches the big bang [3], the pressure \(P\) does not go to zero.

parametrize the evolution of the anisotropies, therefore a pure numerical analysis for some sets of initial conditions does not unveil significantly the general picture. In this section it will be shown that, independently on the anisotropic initial conditions, the isotropic solution acts as an attractor for both early and late times, and that the Eddington-Born-Infeld isotropization has a characteristic pattern.

The Bianchi I cosmology follows from general relativity with the homogeneous metric [12]

\[ (g_{\mu\nu}) = \text{diag}( -N^2, a_1^2, a_2^2, a_3^2). \] (17)

Introducing \(H_i \equiv \dot{a}_i / a_i\), its four equations of motion can be expressed as \[29\]

\[
H_1H_2 + H_2H_3 + H_1H_3 = \frac{N^2}{k} \rho,
\]

\[
\sum_{j=1}^{3} \left( \dot{H}_j + H_j^2 - H_N H_j \right) + \frac{H_1H_2H_3}{H_i} = -\frac{N^2}{k} p_i .
\] (19)

These generalize the usual Friedmann equations for a flat anisotropic universe without rotation, with energy density \(\rho\) and pressures \(p_i\).

Using \(H \equiv \frac{1}{3} \sum_j H_j\), the shears evolution is found from

\[
\frac{d}{dt} (H_i - H_k) = (H_N - 3H)(H_i - H_k) + \frac{N^2}{k} (p_i - p_k).
\] (20)

The cosmology proposed in [3] is generalized to a Bianchi type I one by using the metric [17] and the auxiliary tensor field

\[ (q_{\mu\nu}) = \text{diag}( -X^2, Y_1^2, Y_2^2, Y_3^2 ). \] (21)

The equations of motion in \[8\] can be written in the form of \[13\] \[19\] with the appropriate metric variables, energy density and pressures. The energy density and pressures induced by the \(q_{\mu\nu}\) field on the equations for the \(g_{\mu\nu}\) dynamics read, henceforth fixing \(N = 1\),

\[
\rho_q = k\Lambda \frac{X_1Y_2Y_3}{Xa_1a_2a_3},
\]

\[
p_{q_i} = -k\Lambda \frac{X_1Y_2Y_3}{a_1a_2a_3} Y_i^2.
\] (23)

Analogously, the metric induces a fluid on the \(q_{\mu\nu}\) equations whose energy density and pressures are

\[
\rho_g = \frac{k\Lambda (1 - \alpha)}{2} \left( \frac{2\alpha - 1}{X^2} + \frac{3}{1} \sum_{j=1} Y_j^2 \right),
\]

\[
p_{q_i} = -\frac{k\Lambda (1 - \alpha)}{2} \left( \frac{2\alpha + 1}{X^2} + \frac{3}{1} \sum_{j=1} Y_j^2 - 2 Y_i^2 \right).
\] (25)

In the above we could have introduced another constant different from \(k\), but that is superfluous since the additional \(\Gamma_{\alpha\beta\gamma}\) connections do not couple to anything aside from gravity. The above energies and pressures reduce to their corresponding isotropic quantities if \(a_i = a\) and \(Y_i = Y\) (see \[11\] \[12\] \[13\] \[14\] \[15\]).

With the purpose of separating the isotropic contribution from the anisotropic ones, we set

\[
p_{q_i} = \mathcal{P} + \xi_i,
\]

\[
H_i = H + \varepsilon_i
\]

\[
H_{Y_i} = H_Y + \gamma_i.
\] (28)

After the gauge is fixed \((N = 1)\) a residual symmetry is still present, namely the rescalings of the spacial coordinates by constant factors, leading to the following transformations for the scale factors: \(a_i \rightarrow c_i a_i\) and \(Y_i \rightarrow c_i Y_i\), where the \(c_i\)'s are arbitrary non-null constants. One should take care when dealing with isotropization directly from the metric components; in particular, that all the metric components approach the same value is not sufficient (nor necessary) for isotropization to happen; it can be just an artifact of the selected coordinate system. On the other hand, the quantities \(H_i\), \(H_{Y_i}\) and \(p_{q_i}\) are all invariant under the above symmetry, and the six independent quantities \(p_{q_i} - p_{q_j}\), \(H_i - H_j\), \(H_{Y_i} - H_{Y_j}\) fully characterize the anisotropies.

To first order on the anisotropic perturbations \((\varepsilon_i, \xi_i, \gamma_i)\), the equations of motion satisfied by \(\mathcal{H}, \mathcal{H}_Y\) and \(\mathcal{P}\) are the same of their corresponding isotropic quantities \((H, H_Y, \rho_g)\) if we set \(\sum_i \xi_i = \sum_i \varepsilon_i = \sum_i \gamma_i = 0\).
This result only depends on the assumption that $\varepsilon_i, \gamma_i$ are small in regard to $\mathcal{H}, \mathcal{H}_Y$ respectively. That is, $\varepsilon_i$ is assumed to be small in regard to $\mathcal{H}$, but its relation to other quantities is arbitrary.

The evolution of the pressure anisotropies (anisotropic stresses) read

$$\frac{d}{dt}(p_i - p_j) = \frac{d}{dt}(\xi_i - \xi_j) = (\mathcal{H}_Y - \mathcal{H} + H_X)(\xi_i - \xi_j) + 2p(\varepsilon_i - \gamma_i) + 2\xi_i(\varepsilon_i - \gamma_i) - 2\xi_j(\varepsilon_j - \gamma_j). \quad (29)$$

Whenever the anisotropies are small in regard to their corresponding isotropic values, the two last terms from (29) can be dropped.

All the dynamics of the anisotropies can be found from the two versions of (20) (one for $g_{\mu\nu}$ and the other for $q_{\mu\nu}$) and (20). In a standard Bianchi I cosmology there are two groups of two interacting anisotropic quantities, which are the shears ($H_i - H_j$) and the pressure anisotropies ($p_i - p_j$). In the Eddington-Born-Infeld extension to a Bianchi I scenario, the evolution of the latter depends on the physical shears ($H_i - H_j$) and the shears from $q_{\mu\nu}$ ($H_{Y_i} - H_{Y_j}$). There is not an independent second set of pressure anisotropies, see (32).

### A. Early time behavior

To some extent, how the shears in the Eddington-Born-Infeld model will behave just after inflation depends on the role of the connections $C_{\alpha\beta}^\gamma$ during inflation. However, at early times, if the anisotropic quantities are smaller than their isotropic counterparts, or the shears will decay or their derivatives will approach zero. There is no guarantee that the shears will decay from the early universe to the present time, and Figs. [6, 7] illustrate that, but the isotropic solution acts as an attractor.

For the shears modulus to increase it is necessary that $|\xi_i - \xi_j| > 3k\mathcal{H}(\varepsilon_i - \varepsilon_j)$. The assumption of small anisotropies does not rule out this possibility at early times, for the pressure exerted by the unified dark fluid starts at minus infinity. Moreover, no upper limit in $|\xi_i - \xi_j|$ can avoid that inequality, since there is no lower bound for the shears. Nevertheless, if that inequality holds at early times, then

$$\frac{d}{dt}(\xi_i - \xi_k) \sim (\mathcal{H}_Y - \mathcal{H} + H_X)(\xi_i - \xi_j) + 2p(-\gamma_i + \gamma_k). \quad (30)$$

The latter comes from

$$|(\mathcal{H}_Y - \mathcal{H} + H_X)(\xi_i - \xi_j)| > 3k\mathcal{H}|(\varepsilon_i - \varepsilon_j)| \Rightarrow 2|p(\varepsilon_i - \varepsilon_j)|,$$

since, as $t$ goes to zero, $\mathcal{H}$ increases faster than $|p|$ (see also Fig. [3]) — the first evolves proportionally to $a^{-3/2}$ (matter phase) or $a^{-2}$ (radiation phase) while the second to $a^{-1}$.

Therefore, if after inflation the pressure anisotropies have a relevant contribution to the shears evolution, the former will quickly decay if the last term of (30) can be neglected. It might happen that the pressure anisotropies will not decrease just after inflation, for this to happen it is necessary that $2|p(\gamma_i + \gamma_k)| > |(\mathcal{H}_Y - \mathcal{H} + H_X)(\xi_i - \xi_j)|$. Which implies, at early times, that $|p(\gamma_i + \gamma_k)| > |\mathcal{H} - (\gamma_i + \gamma_k)|$, and hence, from (20, 32), that $|\gamma_i + \gamma_k|$ will decay.

In conclusion, it is possible that the modulus of the physical shears increase at early times, but if this happens the pressure anisotropies will decrease, reducing the shears derivative. The isotropic solution acts as an attractor at early times.

### B. Late time behavior

For the late time behavior, another approach is capable of unveiling more details. Using the following relations that come directly from the definitions,

$$p_{qi} - p_{qj} = -\frac{X^2}{k\Lambda(1 - \alpha)}(p_{qi} - p_{qj}), \quad (32)$$

$$\mathcal{P} = (\mathcal{H}_Y - \mathcal{H} + H_X)\mathcal{P}, \quad (33)$$

$$\dot{p}_q = (3\mathcal{H}_Y - \mathcal{H})\mathcal{P}, \quad (34)$$

it is possible to write the second order expressions for the evolution of the shears and pressure anisotropies as follows,

$$\frac{d^2}{dt^2}(\varepsilon_i - \varepsilon_j) = \left(-3\mathcal{H} + \frac{2\mathcal{P}}{k}\right)(\varepsilon_i - \varepsilon_j) - 3\mathcal{H} \frac{d}{dt}(\varepsilon_i - \varepsilon_j) + \frac{1}{k}(\mathcal{H}_Y - \mathcal{H} + H_X)(\xi_i - \xi_j) - 2\frac{\mathcal{P}}{k}(\gamma_i - \gamma_j), \quad (35)$$

$$\frac{d^2}{dt^2}(\xi_i - \xi_j) = \left(\mathcal{H}_Y - \mathcal{H} + H_X + 2\frac{\mathcal{P}}{k} - 2\frac{\mathcal{P}\Lambda(1 - \alpha)}{\rho_q}\right)(\xi_i - \xi_j) + (\mathcal{H}_Y - \mathcal{H} + H_X)\frac{d}{dt}(\varepsilon_i - \varepsilon_j) + 2\mathcal{P}(\gamma_i - \gamma_j), \quad (36)$$

$$\frac{d^2}{dt^2}(\gamma_i - \gamma_j) = \left(H_X - 3\mathcal{H}_Y + \frac{2\mathcal{P}\Lambda(1 - \alpha)}{\rho_q}\right)(\gamma_i - \gamma_j) + (H_X - 3\mathcal{H}_Y)\frac{d}{dt}(\varepsilon_i - \varepsilon_j) - 2\frac{\mathcal{P}\Lambda(1 - \alpha)}{\rho_q}(\varepsilon_i - \varepsilon_j).$$
\[ + \frac{2\Lambda(1-\alpha)}{\rho_q} (\mathcal{H}_Y - \mathcal{H} - H_X)(\xi_i - \xi_j). \] (37)

Albeit less compact, the above form is useful for interpreting the evolution of anisotropies as the interaction of three parametric damped oscillators.

The differential equation \( \ddot{x} = -b\dot{x} - \omega^2 x \) describes a damped oscillator with damping factor \( b > 0 \) and resonance frequency \( \omega \). The effective frequency of oscillation is \( \frac{1}{2}\sqrt{4\omega^2 - b^2} \). If this value is complex the \( x \) evolution is given by hyperbolic trigonometric functions, otherwise oscillations appear for \( x \). We will use these observations as approximations to the late time behavior of \( x \); firstly without interaction, i.e. neglecting the two last terms in each of the above equations [30].

From the numerical solutions of the three damping coefficients, \( b_i = 3\mathcal{H}, b_2 = -(\mathcal{H} - 3\mathcal{H}_Y) \) and \( b_3 = -(\mathcal{H}_Y - \mathcal{H} + H_X) \), it is found that they are always positive, leading to a tendency towards isotropization. In Fig. 3 the values of \( b_i^2, b_2^2 \) and \( b_3^2 \) are plotted and compared to \( 4\omega^2 = 4 \left( 3\mathcal{H} - \frac{2}{t^2} \right) \), \( 4\omega_i^2 = 4 \left( -\mathcal{H}_X + 3\mathcal{H}_Y - 2\frac{\rho\Lambda(1-\alpha)}{\rho_q} \right) \) and \( 4\omega_2^2 = 4 \left( -\mathcal{H}_Y + \mathcal{H} - H_X - 2\frac{\rho\Lambda(1-\alpha)}{\rho_q} \right) \).

The effective frequency of oscillation \( \frac{1}{2}\sqrt{4\omega^2 - b^2} \) for the physical shears and the pressures anisotropies becomes real after \( t \sim 0.8 \, t_0 \) and \( t \sim 0.2 \, t_0 \) respectively. These quantities oscillate with approximate asymptotic effective (angular) frequencies of \( 2/t_0 \) and \( 3/(2t_0) \), and hence their half periods are \( 3/2 \, t_0 \) and \( 2 \, t_0 \) respectively. These late time low frequency oscillations can reduce dramatically the decrement of the physical shears, when compared to the standard Bianchi I exponential decay, but do not spoil the isotropization tendency, since the damping factors are always positive.

The shears from \( q_{\mu\nu} \) have an intrinsic oscillatory tendency at high redshift, but with low and quickly decreasing frequency. While for late times, it simply decay with an exponential-like behavior.

When the two last terms of [35 36 37] are considered, turning on the interaction among the three damped parametric oscillators, the picture at \( t \lesssim 1 \) can change considerably depending on the initial conditions (see next section). During these times, due to the interactions, some oscillations with ill defined large period (\( \gtrsim 0.5 \)) can appear. High frequency oscillations cannot appear since all of the oscillators have low frequency. For latter times, the behavior is less dependent on the initial conditions. The physical shears, in particular, oscillate with a well defined low frequency and with decreasing amplitude, interacting only with the shears from \( q_{\mu\nu} \) (since \( \mathcal{H}_Y - \mathcal{H} + H_X \to 0 \)), whose natural tendency is to decay in an exponential way.

![FIG. 3: Evolution of the damping and resonance parameters on time.](image)

**FIG. 3:** Evolution of the damping and resonance parameters on time. Generated from the same isotropic solution displayed in Fig. 1. In particular, even though at late times this model isotropic version behaves as a standard de Sitter universe, its Bianchi I version does not display the typical exponential decay for the shears, instead they decay oscillating (see Conclusions for other details).

**IV. NUMERICAL SOLUTIONS FOR THE EVOLUTION OF ANISOTROPIES**

In this section we present an approach to the numerical analysis of this problem and numerical solutions that complement Sec. III results. For late times \( (t > 1) \) the anisotropies always decay oscillating with well defined low frequency, but many possibilities are left for the details on the anisotropies evolution at both high and low redshift. This section displays some peculiar possibilities of this model, considering small anisotropies that fluctuate around the isotropic solution presented in Sec. II.

The Eddington-Born-Infeld model with a Bianchi I background is described by seven equations of motion and seven unknowns, once the temporal gauge is fixed. Analogously to the isotropic case as presented in Sec. II, the equations selected to be solved numerically are the six second order ones (i.e., 19) with the appropriate variables and energy momentum conservation,

\[
XY_1Y_2Y_3 \left( \sum_{j=1}^{3} \frac{a_j^2}{Y_j^2} H_j \right) = \frac{d}{dt} \left( \frac{Y_1Y_2Y_3}{X} \right). \tag{38}
\]

We use \( X_0, H_{0i}, Y_{0i} \) and \( Y_0 \) to denote the values of \( X, H_i, Y_i \) and \( \dot{Y} \) at the present time \( t_0 \), and set \( a_i(t_0) = 1 \). The first order equation 15 for the metric determines \( X_0 \) as a function of \( H_{0i}, Y_{0i} \), while the first order equation for \( q_{\mu\nu} \) determines one of the \( Y_{0i} \)’s a function of \( H_{0i}, Y_{0i} \) and the two others \( Y_{0j} \)’s.

In general, the value of \( X_0 \) in the anisotropic case is different from the isotropic one. It remains to be imposed that the isotropic quantities \( H_0 \) and \( Y_0 \) are the mean values of the anisotropic quantities. To this end, one sets
$H_0 = \frac{1}{3} \sum_{i=1}^{3} H_{0i}$ and $Y_0^3 = Y_{01}Y_{02}Y_{03}$. The latter, for small anisotropies, are sufficient for finding a $X_0$ that is equal to the isotropic one and $H_0 = \frac{1}{3} \sum_{i=1}^{3} H_{0i}$.

Figs. (4, 5) exemplify the evolution of anisotropies at late times for two sets of initial conditions set at $t_0$. Some details on the evolution at low redshift depend strongly on the initial conditions (e.g., the shears may or may not become null at $t \lesssim 1$), but at latter times their behavior follows the general pattern described in Sec. III.

![FIG. 4: Evolution of all anisotropic quantities at late time. All the quantities in this plot refer to the corresponding adimensional ratios, i.e. $(H_i - H_j)/H$, $(H_Y - H_{Y,j})/H_Y$, $(p_{qi} - p_{qj})/P$. Only the two shears with greater magnitude are displayed, together with the pressure anisotropies and $q_{uv}$ shears of same indices. All the anisotropies were set to be small at $t = 1$, but with a considerable magnitude difference. The physical shears were set to be $10^{-5}$, being close to the usual bounds, and the others were set to $10^{-1}$. This solution only has small anisotropies at late times.](image)

![FIG. 5: Like in Fig. (4), but all anisotropies are set to have about the same order $\sim 10^{-5}$ at $t_0$, and the sign of the pressure anisotropies at $t_0$ was inverted, preventing the shears to continue to decay before reaching zero.](image)

From the numerical solution of the isotropic case (or from the anisotropic equations solution with isotropic initial conditions), one can infer the value of all model parameters at any time, and hence generate numerical solutions from initial conditions at any time. Figs. (4, 5) show two sets of solutions for the shears evolution with nonstandard features. Both were generated from initial conditions set at matter-radiation decoupling time — which is a natural choice since the strongest shear observational constraints come from the cosmic microwave background (CMB) [12]. The dashed solution shows that, in spite of the isotropic solution being an attractor, it is possible that the shears have been increasing since the last scattering surface formation and up to the present time. This idea was explored recently in other models as a way for solving certain cosmological anomalies, e.g. [22, 24, 26]. The solid shears solution shows the possibility that the shears could have been high at at the last scattering and high at present time, but with inverted signs. This possibility can significantly reduce the standard observational constraints on the shears — at the price of adding one coincidence problem in cosmology, since the CMB will seem more isotropic at current time than at the past or the future. Note that all solutions of this model have shears that invert sign, since at late times the shears approach zero like damped oscillators. The special feature of this solution (the solid one) is when the first sign flip happens.

![FIG. 6: Evolution of two sets of shears with initial conditions set at decoupling time (about $10^{-5}t_0$ after the big bang). In above the instant $t = 0$ was shifted to be the decoupling time and the shears are measured in $H_0$ units. For the dashed solution it was used, at decoupling: $H_i - H_j \sim 10^{-9}H$, $H_{Y,i} - H_{Y,j} \sim 10^{-2}H_Y$, $p_{qi} - p_{qj} \sim 10^{-3}P$. For the solid solution, the same as above, but $H_i - H_j \sim 10^{-5}H$.](image)

V. CONCLUSIONS

The Eddington-Born-Infeld (EBI) model [8] is a new candidate for an unified dark sector and uses additional connections, instead of scalar fields, to model dark energy and dark matter. The approach with additional connections makes its extension to Bianchi cosmologies nontrivial since it induces anisotropic pressures (stresses).
In this work the Eddington-Born-Infeld cosmology was extended to a Bianchi type I cosmology and it was shown that i) the full dynamics of anisotropies depends on two noninteracting groups of three anisotropic quantities (the physical shears, the shears from $q_{\mu\nu}$, and the pressure anisotropies); ii) for both early and late times the isotropic solution acts as an attractor, implying that the shears tend to decay, and that observational constraints on the shears can be fulfilled [12, 13]; iii) the shears do not decay monotonically, but they typically oscillate with decreasing amplitude; iv) letting some fine tuning on the initial conditions that parametrize this model anisotropies, we could construct two solutions qualitatively different from usual anisotropic models Figs. (6, 7); specifically, these are capable of avoiding standard theoretical bounds on the shears.

Since here we have evaluated anisotropic fluctuations around the isotropic EBI solution found in [8], some of our results rely on the same assumption considered in [8], namely that some of the EBI isotropic parameters should have values similar to the corresponding ones from $\Lambda$CDM. We stress that further EBI developments may lead to different parameters, for the Ref. [4] results should be more than a phenomenological model for dark energy and dark matter; the role of the $C_{\beta\rho}^{q_{\mu\nu}}$ connections on the shears need not to be satisfied in this model.

Recently anisotropic forms of dark energy have been receiving increasing attention [22, [23, 24, 25, 26]. These proposals, besides constraining what dark energy can and cannot be, are generating their first observational consequences and perhaps are the explanation behind some anomalies found in the standard cosmological model. The Eddington-Born-Infeld model naturally generates an anisotropic form of dark energy that is considerably intricate. It is not the purpose of this paper to pinpoint a set of initial conditions capable of solving a particular anomaly, but we would like to stress that standard theoretical issues are being evaluated.

The present work is a step toward the proof that the EBI model is in conformity with observational data for modeling dark energy and dark matter. Of special relevance to this proof is the analysis of density fluctuations and consequences for the CMB anisotropies [16], since this step the Chaplygin gas approach could not overcome as a unified dark sector model [7]. If the EBI model can indeed be a realistic “quartessence” model it has yet to be proven, but this paper results do not depend on that.

Considering the original motivation [8], the EBI model should be more than a phenomenological model for dark energy and dark matter; the role of the $C_{\beta\rho}^{q_{\mu\nu}}$ connections during inflation have yet to be evaluated and other theoretical issues are being evaluated.

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[27] Another possibility is to use the Riemann tensor in place of the metric. Considering the coupling between gravity and electromagnetism, this nonstandard coupling form was explored in [11].

[28] In [8] the approach for fixing $Y_0$ was finding the closest scale factor evolution to the standard cosmological model, leading to $Y_0 = 1.059$. But the analytical analysis is more transparent with $Y_0 = 1.083$, in particular since $H$ goes directly to $\sqrt{\Lambda/3}$, instead of going to a lower value and then slowly increasing to $\sqrt{\Lambda/3}$ for $t > 1$. There are no changes in our conclusions if either of these values of $Y_0$ is used.

[29] We are using a notation as close as possible to the standard one in isotropic cosmologies. There is no need to proceed with the 1+3 covariant decomposition here.

[30] In this sense, without interactions is not a physical limit, it is just a first approach to the Eqs. (35, 36, 37).

[31] Shears oscillations, albeit following a different pattern, were also found in another cosmological context [10].

[32] Actually the anisotropic pressure will at first help the shears to decay even faster than the standard exponential decay.