A Novel Approach to Elastodynamics: II. The Three-Dimensional Case

A. S. Fokas\textsuperscript{a,*}, D. Yang\textsuperscript{a,b†}

\textsuperscript{a} Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK

\textsuperscript{b} Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China

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Abstract

A new approach was recently introduced by the authors for constructing analytic solutions of the linear PDEs describing elastodynamics. Here, this approach is applied to the case of a homogeneous isotropic half-space body satisfying arbitrary initial conditions and Lamb’s boundary conditions. A particular case of this problem, namely the case of homogeneous initial conditions and normal point load boundary conditions, was first solved by Lamb using the Fourier-Laplace transform. The general problem solved here can also be analysed via the Fourier transform, but in this case, the solution representation involves transforms of unknown boundary values; this necessitates the formulation and solution of a cumbersome auxiliary problem, which expresses the unknown boundary values in terms of the Laplace transform of the given boundary data. The new approach, which is applicable to arbitrary initial and boundary conditions, bypasses the above auxiliary problem and expresses the solutions directly in terms of the given initial and boundary conditions.

Keywords: elastodynamics, three dimensions, half space, initial boundary value problem, Lamb’s problem, global relation.

1 Introduction

The problem considered in this paper has a long and illustrious history, which begins with the classic works of Sir Horace Lamb in 1904 \cite{1}. In \cite{1}, Lamb treated

\footnotesize{\textsuperscript{*}T.Fokas@damtp.cam.ac.uk
\textsuperscript{†}yangd04@mails.tsinghua.edu.cn}
four basic problems, the so-called Lamb’s problems, which are formulated in either two or three dimensions.

These problems have been studied by several authors, see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

A new approach to elastodynamics was introduced in [14]. This approach is based on the unified method for solving linear and integrable nonlinear PDEs introduced by one of the authors in [15]. In [14], the new approach was applied to the Lamb’s problem in two dimensions. Here, we study three-dimensional problems. In particular, we consider arbitrary initial conditions and general stress boundary conditions, including normal point load, tangential point load, and mixed point load. We refer to the latter three stress conditions as Lamb’s boundary conditions.

Most studies of Lamb’s problems were based on the Helmholtz decomposition and on the use of the Laplace transform in time. Helmholtz decomposition has the advantage of decomposing the P-waves and S-waves. However, it has the disadvantage of introducing higher order derivatives to the boundary conditions. The use of Laplace transform, essentially restricts the problem to the case of homogeneous initial conditions.

A crucial role of the method of Fokas [15, 16] is played by the so-called global relations, which are algebraic relations coupling appropriate transforms of the unknown boundary values with transforms of the given data. For linear PDEs the method of Fokas uses three novel steps [16]–[45]: 1. Derive a representation for the solution in terms of an integral involving a contour in the complex Fourier plane. This representation is not yet effective, because in addition to transforms of the given initial and boundary data, it also contains transforms of unknown boundary values. 2. Analyse certain transformations in the complex Fourier plane which leave invariant the transforms of the unknown boundary values. 3. Eliminate the transforms of the unknown boundary values, by combining the results of the first two steps and by employing Cauchy’s theorem (or more precisely Jordan’s lemma).

This paper is organised as follows. In section 2, we recall the governing equations of elastodynamics and derive the global relations. In section 3 we implement step 1. In section 4 we implement steps 2 and 3. In section 5, by employing the general representations derived in section 4, we analyse Lamb’s problems. These results are further discussed in section 6.

For certain complicated boundary value problems it seems that it is not possible to eliminate from the integral representation of the solution the transforms of the unknown boundary values. However, for some of these problems, by using the global relations, one can derive expressions for the Laplace transforms of the unknown boundary values in terms of the given data [46]. A summary of this less effective approach is presented in the Appendix. It is interesting that for the particular case of zero initial conditions, the formulae presented in the Appendix reduce to the formulae first derived in the classic works of Lamb.
2 Governing Equations and Global Relations

The transient problem for three dimensional elastodynamics in the half space with Lamb’s boundary conditions is defined as follows: Let \( u = u(x, y, z, t) \), \( v = v(x, y, z, t) \), \( w = w(x, y, z, t) \), denote the displacements of a homogeneous isotropic half space body. The governing equations of motion without external body forces, are the Lamé-Navier equations:

\[
\begin{align*}
(\lambda + 2\mu)u_{xx} + (\lambda + \mu)(v_{xy} + w_{xz}) + \mu(u_{yy} + u_{zz}) - \rho u_{tt} &= 0, \quad (2.1a) \\
(\lambda + 2\mu)v_{yy} + (\lambda + \mu)(w_{yz} + u_{yx}) + \mu(v_{zz} + v_{xx}) - \rho v_{tt} &= 0, \quad (2.1b) \\
(\lambda + 2\mu)w_{zz} + (\lambda + \mu)(u_{zx} + v_{zy}) + \mu(w_{xx} + w_{yy}) - \rho w_{tt} &= 0, \quad (2.1c)
\end{align*}
\]

\(- \infty < x < \infty, \ - \infty < y < \infty, \ z > 0, \ t > 0, \)

where \( \lambda, \mu \) are the Lamé constants and \( \rho \) denotes the density of the material, which without loss of generality is normalized to unity, i.e., \( \rho \equiv 1 \). Let the initial conditions be denoted by

\[
\begin{align*}
u(x, y, z, 0) &= u_0(x, y, z), \quad u_t(x, y, z, 0) = u_1(x, y, z), \quad (2.2a) \\
v(x, y, z, 0) &= v_0(x, y, z), \quad v_t(x, y, z, 0) = v_1(x, y, z), \quad (2.2b) \\
w(x, y, z, 0) &= w_0(x, y, z), \quad w_t(x, y, z, 0) = w_1(x, y, z), \quad (2.2c)
\end{align*}
\]

\(- \infty < x < \infty, \ - \infty < y < \infty, \ z > 0. \)

Let the stress boundary conditions be denoted by

\[
\begin{align*}
(u_z + w_x)(x, y, 0, t) &= g_1(x, y, t), \quad (2.3a) \\
(v_z + w_y)(x, y, 0, t) &= g_2(x, y, t), \quad (2.3b) \\
\left( w_z + \frac{\lambda}{\lambda + 2\mu}(u_x + v_y) \right)(x, y, 0, t) &= g_3(x, y, t), \quad (2.3c)
\end{align*}
\]

\(- \infty < x < \infty, \ - \infty < y < \infty, \ t > 0. \)

For a tangential point load, the functions \( g_1, g_2 \) and \( g_3 \) are given by

\[
g_1(x, y, t) = \sigma_0 \delta(x, y) X(t)/\mu, \quad g_2(x, y, t) = 0, \quad g_3(x, y, t) = 0; \quad (2.4)
\]

for a normal point load,

\[
g_1(x, y, t) = 0, \quad g_2(x, t) = 0, \quad g_3(x, y, t) = \sigma_0 \delta(x, y) Y(t)/(\lambda + \mu); \quad (2.5)
\]

for a moving normal line load with a constant speed \( C \) along the \( x - axis \),

\[
g_1(x, y, t) = 0, \quad g_2(x, y, t) = 0, \quad g_3(x, y, t) = \sigma_0 \delta(x - Ct, y)/(\lambda + \mu). \quad (2.6)
\]

Here, \( \sigma_0 \) is a constant, \( \delta(x, y) \) denotes the Dirac-\( \delta \) function, \( X(t) \) and \( Y(t) \) are functions which depend only on \( t \).
Notations  
Hat, “\(^\land\)”, will denote the three dimensional Fourier transform with respect to \(x, y\) and \(z\), whereas tilde, “\(\sim\)”, will denote the two dimensional Fourier transform with respect to \(x\) and \(y\). In particular,

\[
\hat{u}(k, l, m, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \ e^{-ikx-ily-imz} u(x, y, z, t), \quad (2.7a)
\]

\[
\hat{v}(k, l, m, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \ e^{-ikx-ily-imz} v(x, y, z, t), \quad (2.7b)
\]

\[
\hat{w}(k, l, m, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \ e^{-ikx-ily-imz} w(x, y, z, t), \quad (2.7c)
\]

\[
\hat{u}_j(k, l, m, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \ e^{-ikx-ily-imz} u_j(x, y, z, t), \quad (2.7d)
\]

\[
\hat{v}_j(k, l, m, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \ e^{-ikx-ily-imz} v_j(x, y, z, t), \quad (2.7e)
\]

\[
\hat{w}_j(k, l, m, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \ e^{-ikx-ily-imz} w_j(x, y, z, t), \quad (2.7f)
\]

where \(\mathbb{C}^-\) denotes the lower half complex \(m\)-plane.

Furthermore,

\[
\hat{u}(k, l, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-ikx-ily} u(x, y, 0, t), \quad (2.8a)
\]

\[
\hat{v}(k, l, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-ikx-ily} v(x, y, 0, t), \quad (2.8b)
\]

\[
\hat{w}(k, l, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-ikx-ily} w(x, y, 0, t), \quad k, l \in \mathbb{R}, t > 0. \quad (2.8c)
\]

\[
\hat{g}_j(k, l, t) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-ikx-ily} g_j(x, y, t), \quad k, l \in \mathbb{R}, \ t > 0, \ j = 1, 2, 3. \quad (2.9)
\]

We emphasise that since \(x, y \in \mathbb{R}\), the \(x, y\)-Fourier transform is well-defined only for \(k, l \in \mathbb{R}\); on the other hand, since \(0 < z < \infty\), the \(z\)-Fourier transform is well-defined for \(m\) in the lower half complex \(m\)-plane.

By applying the three dimensional Fourier transform to equations \((2.7)\) and by using in the resulting equations the boundary conditions \((2.3)\), we obtain the following equations:

\[
-[(\lambda + 2\mu)k^2 + \mu l^2 + \mu m^2]\hat{u} - (\lambda + \mu)kl\hat{v} - (\lambda + \mu)km\hat{w} - \mu\hat{g}_1 - \mu\hat{v} - \lambda\hat{w} = \hat{u}_{tt}, \quad (2.10a)
\]

\[
-[(\lambda + 2\mu)l^2 + \mu k^2 + \mu m^2]\hat{v} - (\lambda + \mu)kl\hat{u} - (\lambda + \mu)lm\hat{w} - \mu\hat{g}_2 - \mu\hat{u} - \lambda\hat{w} = \hat{v}_{tt}, \quad (2.10b)
\]

\[
-[(\lambda + 2\mu)m^2 + \mu k^2 + \mu l^2]\hat{w} - (\lambda + \mu)km\hat{u} - (\lambda + \mu)lm\hat{v} - (\lambda + 2\mu)\hat{g}_3 - (\lambda + 2\mu)i\mu\hat{u} - \mu\hat{v} = \hat{w}_{tt}, \quad (2.10c)
\]
Introducing the notations \( \omega \) where the dispersion relations equations (2.10) become

\[
P(k, l, m, t) = k \hat{u}(k, l, m, t) + l \hat{v}(k, l, m, t) + m \hat{w}(k, l, m, t),
\]
(2.11a)

\[
Q(k, l, m, t) = m \hat{v}(k, l, m, t) - l \hat{w}(k, l, m, t),
\]
(2.11b)

\[
R(k, l, m, t) = k \hat{w}(k, l, m, t) - m \hat{u}(k, l, m, t),
\]
(2.11c)

equations (2.10) become

\[
P_{tt} + (\lambda + 2\mu)(k^2 + l^2 + m^2)P = F_P,
\]
(2.12a)

\[
Q_{tt} + \mu(k^2 + l^2 + m^2)Q = F_Q,
\]
(2.12b)

\[
R_{tt} + \mu(k^2 + l^2 + m^2)R = F_R, \quad k, l \in \mathbb{R}, \quad m \in \mathbb{C}^-,
\]
(2.12c)

where the functions \( F_P(k, l, m, t), F_Q(k, l, m, t) \) and \( F_R(k, l, m, t) \) are defined as follows:

\[
F_P = -(\mu k \tilde{g}_1 + \mu l \tilde{g}_2 + i\lambda k^2 \tilde{w} + i\lambda l^2 \tilde{v}) - m(2i\mu k \hat{u} + 2i\mu \hat{v})
+ (\lambda + 2\mu) \tilde{g}_3) - im^2(\lambda + 2\mu) \hat{v},
\]
(2.13a)

\[
F_Q = ((\lambda + 2\mu) \tilde{g}_3 + i\mu k \hat{u} + i\mu l \hat{v}) + m(-\mu \tilde{g}_2 + 2i\mu \hat{w}) - m^2 i\mu \hat{v},
\]
(2.13b)

\[
F_R = -(\lambda + 2\mu) \tilde{g}_3 + i\mu k^2 \hat{u} + i\mu l \hat{v}) + m(\mu \tilde{g}_1 - 2i\mu \hat{w}) + m^2 i\mu \hat{v}.
\]
(2.13c)

Solving equations (2.12) for \( \{P, Q, R\} \), we obtain the following expressions:

\[
P = \frac{1}{-2i\omega_1} \left( e^{-i\omega_1 t} \int_0^t e^{i\omega_1 s} F_P(k, l, m, s) ds - e^{i\omega_1 t} \int_0^t e^{-i\omega_1 s} F_P(k, l, m, s) ds \right)
+ \left( \frac{1}{2} P_0 + i \frac{P_1}{2 \omega_0} \right) e^{i\omega_1 t} + \left( \frac{1}{2} P_0 - i \frac{P_1}{2 \omega_0} \right) e^{-i\omega_1 t},
\]
(2.14a)

\[
Q = \frac{1}{-2i\omega_1} \left( e^{-i\omega_1 t} \int_0^t e^{i\omega_1 s} F_Q(k, l, m, s) ds - e^{i\omega_1 t} \int_0^t e^{-i\omega_1 s} F_Q(k, l, m, s) ds \right)
+ \left( \frac{1}{2} Q_0 + i \frac{Q_1}{2 \omega_1} \right) e^{-i\omega_1 t} + \left( \frac{1}{2} Q_0 - i \frac{Q_1}{2 \omega_1} \right) e^{i\omega_1 t},
\]
(2.14b)

\[
R = \frac{1}{-2i\omega_1} \left( e^{-i\omega_1 t} \int_0^t e^{i\omega_1 s} F_R(k, l, m, s) ds - e^{i\omega_1 t} \int_0^t e^{-i\omega_1 s} F_R(k, l, m, s) ds \right)
+ \left( \frac{1}{2} R_0 + i \frac{R_1}{2 \omega_1} \right) e^{-i\omega_1 t} + \left( \frac{1}{2} R_0 - i \frac{R_1}{2 \omega_1} \right) e^{i\omega_1 t}, \quad k, l \in \mathbb{R}, \quad m \in \mathbb{C}^-,
\]
(2.14c)

where the dispersion relations \( \omega_1 \) and \( \omega_2 \) are given by

\[
\omega_1^2 = (\lambda + 2\mu)(k^2 + l^2 + m^2), \quad \omega_2^2 = \mu(k^2 + l^2 + m^2), \quad k, l \in \mathbb{R}, \quad m \in \mathbb{C},
\]
(2.15)
and the known functions $P_j(k, l, m)$, $Q_j(k, l, m)$, $R_j(k, l, m)$, $j = 0, 1$ are given in terms of the initial conditions by

\begin{align}
P_0 &= k\hat{u}_0 + l\hat{v}_0 + m\hat{w}_0, \quad P_1 = k\hat{u}_1 + l\hat{v}_1 + m\hat{w}_1, \\
Q_0 &= m\hat{v}_0 - l\hat{w}_0, \quad Q_1 = m\hat{v}_1 - l\hat{w}_1, \\
R_0 &= k\hat{w}_0 - m\hat{v}_0, \quad R_1 = k\hat{w}_1 - m\hat{v}_1,
\end{align}

with $\hat{u}_j$, $\hat{v}_j$, $\hat{w}_j$, $j = 0, 1$, defined in equations (2.7).

In the following, we take

\[ \omega_1 = \sqrt{\lambda + 2\mu(k^2 + l^2 + m^2)\frac{1}{2}}, \quad \omega_2 = \sqrt{\mu(k^2 + l^2 + m^2)\frac{1}{2}}. \tag{2.17} \]

The function $(k^2 + l^2 + m^2)\frac{1}{2}$ has the branch points $\pm i\sqrt{k^2 + l^2}$ in the complex $m-$plane; we connect these two branch points by a branch cut and we fix a branch in the cut plane by the requirement that,

\[ (k^2 + l^2 + m^2)\frac{1}{2} \sim m + O\left(\frac{1}{m}\right), \quad \text{as } m \to \infty. \]

Let $u^{(j)\pm}(k, l, m, t)$, $v^{(j)\pm}(k, l, m, t)$, $U^{(j)}(k, l, m, t)$, $V^{(j)}(k, l, m, t)$, $W^{(j)}(k, l, m, t)$, $j = 1, 2$, denote the following unknown functions:

\begin{align}
u^{(j)\pm}(k, l, m, t) &= \int_0^t e^{\pm i\omega_j s} \tilde{u}(k, l, s) ds, \\
v^{(j)\pm}(k, l, m, t) &= \int_0^t e^{\pm i\omega_j s} \tilde{v}(k, l, s) ds, \\
w^{(j)\pm}(k, l, m, t) &= \int_0^t e^{\pm i\omega_j s} \tilde{w}(k, l, s) ds, \\
U^{(j)}(k, l, m, t) &= \frac{1}{2\omega_j} (e^{-i\omega_j t} u^{(j)\pm}(k, l, m, t) - e^{i\omega_j t} u^{(j)\mp}(k, l, m, t)), \\
V^{(j)}(k, l, m, t) &= \frac{1}{2\omega_j} (e^{-i\omega_j t} v^{(j)\pm}(k, l, m, t) - e^{i\omega_j t} v^{(j)\mp}(k, l, m, t)), \\
W^{(j)}(k, l, m, t) &= \frac{1}{2\omega_j} (e^{-i\omega_j t} w^{(j)\pm}(k, l, m, t) - e^{i\omega_j t} w^{(j)\mp}(k, l, m, t)), \quad \tag{2.18f}\end{align}

$k, l \in \mathbb{R}$, $m \in \mathbb{C}$, $t \geq 0$, $j = 1, 2$. 


Similarly, let \( g^{(j)\pm}(k, l, m, t) \), \( f^{(j)\pm}(k, l, m, t) \), \( e^{(j)\pm}(k, l, m, t) \), \( G^{(j)}, F^{(j)}, E^{(j)} \), \( j = 1, 2 \), denote the following known functions:

\[
g^{(j)\pm}(k, l, m, t) = \int_0^t e^{\pm i\omega_j t} \tilde{g}_1(k, l, s) ds, \tag{2.19a}
\]

\[
f^{(j)\pm}(k, l, m, t) = \int_0^t e^{\pm i\omega_j t} \tilde{g}_2(k, l, s) ds, \tag{2.19b}
\]

\[
e^{(j)\pm}(k, l, m, t) = \int_0^t e^{\pm i\omega_j t} \tilde{g}_3(k, l, s) ds, \tag{2.19c}
\]

\[
G^{(j)}(k, l, m, t) = \frac{1}{2\omega_j}(e^{-i\omega_j t}g^{(j)+}(k, l, t) - e^{i\omega_j t}g^{(j)-}(k, l, m, t)), \tag{2.19d}
\]

\[
F^{(j)}(k, l, m, t) = \frac{1}{2\omega_j}(e^{-i\omega_j t}f^{(j)+}(k, l, t) - e^{i\omega_j t}f^{(j)-}(k, l, m, t)), \tag{2.19e}
\]

\[
E^{(j)}(k, l, m, t) = \frac{1}{2\omega_j}(e^{-i\omega_j t}e^{(j)+}(k, l, t) - e^{i\omega_j t}e^{(j)-}(k, l, m, t)), \tag{2.19f}
\]

\[k \in \mathbb{R}, l \in \mathbb{C}, t \geq 0, j = 1, 2.\]

Using the above notations, equations (2.14) become

\[ku + lv + mw = 2\mu kmU^{(1)} + 2\mu mlV^{(1)} + (\lambda(k^2 + l^2) + m^2(\lambda + 2\mu))W^{(1)} + N_P, \tag{2.20a}\]

\[m\tilde{v} - \tilde{w} = -\mu klU^{(2)} + \mu(m^2 - l^2)V^{(2)} - 2\mu mlW^{(2)} + N_Q, \tag{2.20b}\]

\[k\tilde{w} - \tilde{u} = \mu(k^2 - m^2)U^{(2)} + \mu klV^{(2)} + 2\mu kmW^{(2)} + N_R, \tag{2.20c}\]

where the known functions \(N_P(k, l, m, t), N_Q(k, l, m, t)\) and \(N_R(k, l, m, t)\) are defined as follows:

\[
N_P(k, l, m, t) = -i\mu k G^{(1)}(k, l, m, t) - i\mu F^{(1)}(k, l, m, t) - im(\lambda + 2\mu)E^{(1)}(k, l, m, t)
+ \left(\frac{1}{2}P_0(k, l, m) + \frac{i}{2\omega_1} P_1(k, l, m)\right) e^{-i\omega_1 t} + \left(\frac{1}{2}P_0(k, l, m) - \frac{i}{2\omega_1} P_1(k, l, m)\right) e^{i\omega_1 t}, \tag{2.21a}
\]

\[
N_Q(k, l, m, t) = -i\mu m F^{(2)}(k, l, m, t) - il(\lambda + 2\mu)E^{(2)}(k, l, m, t)
+ \left(\frac{1}{2}Q_0(k, l, m) + \frac{i}{2\omega_2} Q_1(k, l, m)\right) e^{-i\omega_2 t} + \left(\frac{1}{2}Q_0(k, l, m) - \frac{i}{2\omega_2} Q_1(k, l, m)\right) e^{i\omega_2 t}. \tag{2.21b}
\]

\[
N_R(k, l, m, t) = i\mu m G^{(2)}(k, l, m, t) - i\kappa(\lambda + 2\mu)E^{(2)}(k, l, m, t)
+ \left(\frac{1}{2}R_0(k, l, m) + \frac{i}{2\omega_2} R_1(k, l, m)\right) e^{-i\omega_2 t} + \left(\frac{1}{2}R_0(k, l, m) - \frac{i}{2\omega_2} R_1(k, l, m)\right) e^{i\omega_2 t}. \tag{2.21c}
\]

We will refer to equations (2.20) as the global relations. The global relations express the three dimensional Fourier transforms of the solution \((u, v, w)\) in terms of the given initial and boundary data, as well as in terms of the transforms.
the functions $U^{(1)}, U^{(2)}, V^{(1)}, V^{(2)}, W^{(1)}, W^{(2)}$ of the unknown boundary values. The important observation is that equations (2.20) are valid for all values of $m$ in the lower half complex $m-$plane. It turns out that using this fact it will be possible to eliminate the unknown transforms.

3 An Integral Representation Involving the Unknown Transforms

We first observe that for any fixed $k, l \in \mathbb{R}$ and any fixed $t$, $0 \leq t < T$, $T > 0$, the functions $U^{(j)}, V^{(j)}, W^{(j)}, G^{(j)}, F^{(j)}, E^{(j)}, j \in \{1, 2\}$, are analytic in the complex $m-$plane.

Indeed, these functions are all single-valued, thus it only remains to establish the analyticity in the neighbourhood $m = \pm i\sqrt{k^2 + l^2}$. The definition of $W^{(1)}$, i.e. equation (2.18), implies that this function posses the following expansion for $m$ near $\pm i\sqrt{k^2 + l^2}$:

$$W^{(1)}(k, l, m, t) = i \sum_{n=1}^{\infty} \frac{\omega_1^{2n-2}}{(2n-1)!} \int_0^t (s-t)^{2n-2} \hat{w}(k, l, s) ds. \quad (3.1)$$

Similar expansions are also valid for $U^{(1)}, U^{(2)}, V^{(1)}, V^{(2)}, W^{(1)}, W^{(2)}, G^{(1)}, G^{(2)}, F^{(1)}, F^{(2)}, E^{(1)}, E^{(2)}$.

Solving equations (2.20) we find

$$\dot{u} = \frac{1}{m(k^2 + l^2 + m^2)} [2\mu k^2 m^2 U^{(1)} + 2\mu m^2 k l V^{(1)} + [\lambda k m (k^2 + l^2) + km^3(\lambda + 2\mu)]W^{(1)} - \mu m^2 (k^2 - l^2 - m^2)U^{(2)} - 2\mu m^2 k l V^{(2)} - 2\mu k m^3 W^{(2)} + km N_P - k l N_Q - (m^2 + l^2) N_R], \quad (3.2a)$$

$$\dot{v} = \frac{1}{m(k^2 + l^2 - m^2)} [2\mu k m^2 U^{(1)} + 2\mu m^2 k l V^{(1)} + [\lambda m (k^2 + l^2) + lm^3(\lambda + 2\mu)]W^{(1)} - 2\mu m^2 k l U^{(2)} + \mu m^2 (m^2 + k^2 - l^2) V^{(2)} - 2\mu lm^3 W^{(2)} + lm N_P + (k^2 + m^2) N_Q + lk N_R], \quad (3.2b)$$

$$\dot{w} = \frac{1}{k^2 + l^2 + m^2} [2\mu k m^2 U^{(1)} + 2\mu m^2 V^{(1)} + [\lambda k^2 m + \lambda l^2 m + m^3(\lambda + 2\mu)]W^{(1)} + \mu k (l^2 + k^2 - m^2) U^{(2)} + \mu l (k^2 + l^2 - m^2) V^{(2)} + 2\mu k m (l^2 + m^2) W^{(2)} + m N_P - l N_Q + k N_R]. \quad (3.2c)$$

We observe that the following important transformation is valid in the complex $m-$plane:

$$m \to -m \text{ maps } \omega_1 \text{ to } -\omega_1, \omega_2 \text{ to } -\omega_2, \text{ and leaves } U^{(j)}, V^{(j)}, W^{(j)}, j = 1, 2 \text{ invariant.} \quad (3.3)$$

Employing the transformation $m \to -m$ in equations (3.2) and then adding the
resulting equations to (3.2), we obtain the following equations:

\[
\begin{align*}
\hat{u}(k,l,m,t) + \hat{u}(k,l,-m,t) &= \frac{1}{m(k^2 + l^2 + m^2)} \left[ 2km[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W^{(1)} - 4\mu km^3W^{(2)} + \\
&\quad km[N_P(k,l,m,t) + N_P(k,l,-m,t)] - kl[N_Q(k,l,m,t) - N_Q(k,l,-m,t)] - \\
&\quad (m^2 + l^2)[N_R(k,l,m,t) - N_R(k,l,-m,t)] \right], \\
\hat{v}(k,l,m,t) + \hat{v}(k,l,-m,t) &= \frac{1}{m(k^2 + l^2 + m^2)} \left[ 2lm[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W^{(1)} - 4\mu lm^3W^{(2)} + \\
&\quad lm[N_P(k,l,m,t) + N_P(k,l,-m,t)] - kl[N_Q(k,l,m,t) - N_Q(k,l,-m,t)] - \\
&\quad (m^2 + l^2)[N_R(k,l,m,t) - N_R(k,l,-m,t)] \right], \\
\hat{w}(k,l,m,t) + \hat{w}(k,l,-m,t) &= \frac{1}{k^2 + l^2 + m^2} \left[ 4\mu km^2U^{(1)} + 4\mu lm^2V^{(1)} + 2\mu l(k^2 + l^2 - m^2)U^{(2)} + \\
&\quad 2\mu l(k^2 + l^2 - m^2)V^{(2)} + m[N_P(k,l,m,t) - N_P(k,l,-m,t)] - \\
&\quad l[N_Q(k,l,m,t) + N_Q(k,l,-m,t)] + k[N_R(k,l,m,t) + N_R(k,l,-m,t)] \right].
\end{align*}
\]
Applying the inverse Fourier transform formula to these equations, we obtain

\[
\begin{align*}
\text{u}(x, y, z, t) &= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \ e^{ikx+ily+imz} \hat{u}(k, l, m, t) \\
&= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \ e^{ikx+ily+imz} \\
&\left\{ 2k(\lambda(k^2 + l^2) + m^2(\lambda + 2\mu))W^{(1)} - 4\mu km^2W^{(2)} + \\
k[N_P(k, l, m, t) + N_P(k, l, -m, t)] - \frac{1}{m} kl[N_Q(k, l, m, t) - N_Q(k, l, -m, t)] - \\
\frac{1}{m}(m^2 + l^2)[N_R(k, l, m, t) - N_R(k, l, -m, t)] \right\}, \\
\end{align*}
\]

\[
\begin{align*}
\text{v}(x, y, z, t) &= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \ e^{ikx+ily+imz} \hat{v}(k, l, m, t) \\
&= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \ e^{ikx+ily+imz} \\
&\left\{ 2l(\lambda(k^2 + l^2) + m^2(\lambda + 2\mu))W^{(1)} - 4\mu lm^2W^{(2)} + \\
l[N_P(k, l, m, t) + N_P(k, l, -m, t)] + \frac{1}{m}(k^2 + m^2)[N_Q(k, l, m, t) - \\
N_Q(k, l, -m, t)] + \frac{1}{m} kl[N_R(k, l, m, t) - N_R(k, l, -m, t)] \right\}, \\
\end{align*}
\]

\[
\begin{align*}
\text{w}(x, y, z, t) &= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \ e^{ikx+ily+imz} \hat{w}(k, l, m, t) \\
&= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \ e^{ikx+ily+imz} \\
&\left\{ 4\mu km^2U^{(1)} + 4\mu lm^2V^{(1)} + 2\mu kl(k^2 + l^2 - m^2)U^{(2)} + \\
2\mu l(k^2 + l^2 - m^2)V^{(2)} + m[N_P(k, l, m, t) - N_P(k, l, -m, t)] - \\
l[N_Q(k, l, m, t) + N_Q(k, l, -m, t)] + k[N_R(k, l, m, t) + N_R(k, l, -m, t)] \right\}, \\
\end{align*}
\]

\[-\infty < x < \infty, \ -\infty < y < \infty, \ 0 < z < \infty, \ t > 0.\]

Denote by \(H_j(k, l, m, t), \ j = 1, 2, 3,\) the following functions appearing in
equations (3.5):

\begin{align*}
H_1(k, l, m, t) &= \frac{1}{k^2 + l^2 + m^2} \left\{ 2\lambda \left[ k(k^2 + l^2) + m^2(\lambda + 2\mu) \right] W^{(1)} - 4\mu km^2 W^{(2)} \right\}, \\
H_2(k, l, m, t) &= \frac{1}{k^2 + l^2 + m^2} \left\{ 2l \left[ k(k^2 + l^2) + m^2(\lambda + 2\mu) \right] W^{(1)} - 4\mu lm^2 W^{(2)} \right\}, \\
H_3(k, l, m, t) &= \frac{1}{k^2 + l^2 + m^2} \left\{ 4\mu km^2 U^{(1)} + 4\mu lm^2 V^{(1)} + 2\mu k(l^2 + k^2 - m^2) U^{(2)} + 2\mu l(k^2 + l^2 - m^2) V^{(2)} \right\}.
\end{align*}

(3.6a) \hspace{1cm} (3.6b) \hspace{1cm} (3.6c)

We observe that for any fixed \( k, l \in \mathbb{R} \) and fixed \( t \), \( 0 \leq t < T, T > 0 \), the above functions are analytic in the entire complex \( m \)-plane. Indeed, equation (3.1) and the analogous equation for \( W^{(2)} \), imply that in the neighbourhood of \( m = \pm i \sqrt{k^2 + l^2} \), the following expansion is valid:

\[ H_1(k, l, m, t) = 2i\lambda k \int_0^t \tilde{u}(k, l, s) ds + o(\omega^2_1). \]  

(3.7)

Similarly,

\[ H_2(k, l, m, t) = 2i\lambda l \int_0^t \tilde{v}(k, l, s) ds + o(\omega^2_1), \]  

(3.8)

\[ H_3(k, l, m, t) = 2i\mu k \int_0^t \tilde{u}(k, l, s) ds + 2i\mu l \int_0^t \tilde{v}(k, l, s) ds + o(\omega^2_1). \]  

(3.9)

The restriction \( z > 0 \), as well as the analyticity of the functions \( H_j(k, l, m, t) \) \( j = 1, 2, 3 \), allow us to deform the contour of integration from the real axis to a contour \( \gamma_{k,l} \) in the upper half \( m \)-plane (the particular choice of \( \gamma_{k,l} \) will be
determined in the next section):

\[ u(x, y, z, t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{\gamma_{k,l}} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \]

\[ \left\{ 2k[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W^{(1)} - 4\mu km^2W^{(2)} \right\} \]

\[ + \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \]

\[ \left\{ k[N_P(k, l, m, t) + N_P(k, l, -m, t)] - \frac{1}{m}kl[N_Q(k, l, m, t) - N_Q(k, l, -m, t)] - \frac{1}{m}(m^2 + l^2)[N_R(k, l, m, t) - N_R(k, l, -m, t)] \right\}, \]

\[ v(x, y, z, t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{\gamma_{k,l}} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \]

\[ \left\{ 2l[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W^{(1)} - 4\mu lm^2W^{(2)} \right\} \]

\[ + \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \]

\[ \left\{ l[N_P(k, l, m, t) + N_P(k, l, -m, t)] + \frac{1}{m}(k^2 + m^2)[N_Q(k, l, m, t) - N_Q(k, l, -m, t)] \right\} \]

\[ \left\{ \frac{1}{m}lk[N_R(k, l, m, t) - N_R(k, l, -m, t)] \right\}, \]

\[ w(x, y, z, t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{\gamma_{k,l}} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \]

\[ \left\{ 4\mu km^2U^{(1)} + 4\mu lm^2V^{(1)} + 2\mu(k^2 + l^2 - m^2)U^{(2)} + 2\mu(k^2 + l^2 - m^2)V^{(2)} \right\} \]

\[ + \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \]

\[ \left\{ m[N_P(k, l, m, t) - N_P(k, l, -m, t)] - l[N_Q(k, l, m, t) + N_Q(k, l, -m, t)] + k[N_R(k, l, m, t) + N_R(k, l, -m, t)] \right\} \]

\[ (3.10c) \]

4 The Elimination of the Transforms of the Unknown Boundary Values

Let

\[ m_{21} = -m\left(\frac{\lambda + 2\mu}{\mu} + \frac{\lambda + \mu}{\mu} \frac{k^2 + l^2}{m^2}\right)^{\frac{1}{2}} \]

and

\[ m_{12} = -m\left(\frac{\mu}{\lambda + 2\mu} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{k^2 + l^2}{m^2}\right)^{\frac{1}{2}}. \]
The function $m_{12}$ has two branch points $\pm \sqrt{k^2 + l^2} \sqrt{\frac{\lambda + \mu}{\mu}}$, which we connect by a horizontal branch cut; the function $m_{21}$ has two branch points $\pm i \sqrt{k^2 + l^2} \sqrt{\frac{\lambda + \mu}{\lambda + 2\mu}}$, which we connect by a vertical branch cut, see Fig. 1. We fix the branches by the requirements that,

$$m_{21} \sim -m \sqrt{\frac{\lambda + 2\mu}{\mu}}, \quad m_{12} \sim -m \sqrt{\frac{\mu}{\lambda + 2\mu}}, \quad \text{as } m \to \infty.$$

The following transformations are valid in the cut $m$-plane:

$m \to m_{21}$ maps $\omega_2$ to $-\omega_1$, $U^{(2)}$ to $U^{(1)}$, $V^{(2)}$ to $V^{(1)}$, $W^{(2)}$ to $W^{(1)}$;

$m \to m_{12}$ maps $\omega_1$ to $-\omega_2$, $U^{(1)}$ to $U^{(2)}$, $V^{(1)}$ to $V^{(2)}$, $W^{(1)}$ to $W^{(2)}$.

Using in equations (2.20a),(2.20b),(2.20c) the transformations $m \to m_{12}$, $m \to -m$, $m \to -m$, respectively, and then combining the three resulting equations, we obtain the following equations:

\begin{align*}
  k\hat{u}(k, l, m_{12}, t) + l\hat{v}(k, l, m_{12}, t) + m_{12}\hat{w}(k, l, m_{12}, t) &= 2\mu km_{12}U^{(2)}(k, l, m, t) + 2\mu m_{12}(V^{(2)}(k, l, m, t) + (\lambda(k^2 + l^2) + m_{12}^2(\lambda + 2\mu))W^{(2)}(k, l, m, t) + N_P(k, l, m_{12}, t) \quad (4.4a) \\
  -m\hat{v}(k, l, -m, t) - l\hat{w}(k, l, -m, t) &= -\mu klU^{(2)}(k, l, m, t) + \mu(m^2 - l^2)V^{(2)}(k, l, m, t) + 2\mu mlW^{(2)}(k, l, m, t) + N_Q(k, l, -m, t), \quad (4.4b) \\
  k\hat{w}(k, l, -m, t) + m\hat{u}(k, l, -m, t) &= \mu(k^2 - m^2)U^{(2)}(k, l, -m, t) + \mu klV^{(2)}(k, l, -m, t) - 2\mu kmW^{(2)}(k, l, -m, t) + N_R(k, l, -m, t). \quad (4.4c)
\end{align*}
Equations (4.4) imply
\[ m \to -m, \quad m \to m_{21}, \quad m \to m_{21} \]
respectively, and then combining the three resulting equations, we obtain the following equations:

\[ k\hat{u}(k, l, -m, t) + l\hat{v}(k, l, -m, t) - m\hat{w}(k, l, -m, t) = -2\mu kmU^{(1)}(k, l, -m, t) - 2\mu mlV^{(1)}(k, l, -m, t) + (\lambda(k^2 + l^2) + m^2(\lambda + 2\mu))W^{(1)}(k, l, -m, t) + N_P(k, l, -m, t), \]

\[ (4.5a) \]

\[ m_{21}\hat{v}(k, l, m_{21}, t) - l\hat{w}(k, l, m_{21}, t) = -\mu klU^{(1)}(k, l, m, t) + \mu(m^2_{21} - l^2)W^{(1)}(k, l, m, t) - 2\mu ml_{21}W^{(1)}(k, l, m, t) + N_Q(k, l, m_{21}, t), \]

\[ (4.5b) \]

\[ k\hat{w}(k, l, m_{21}, t) - m_{21}\hat{u}(k, l, m_{21}, t) = \mu(k^2 - m^2_{21})U^{(1)}(k, l, m, t) + \mu klV^{(1)}(k, l, m, t) + 2\mu km_{21}W^{(1)}(k, l, m, t) + N_R(k, l, m_{21}, t). \]

\[ (4.5c) \]

Let

\[ C = \begin{pmatrix} 2\mu km_{12} & 2\mu ml_{12} & \lambda(k^2 + l^2) + m^2_{12}(\lambda + 2\mu) \\ -\mu kl & \mu(m^2 - l^2) & 2\mu ml \\ -\mu(m^2 - k^2) & \mu kl & -2\mu km \end{pmatrix}, \]

\[ (4.6a) \]

\[ D = \begin{pmatrix} -2\mu km & -2\mu ml & \lambda(k^2 + l^2) + m^2(\lambda + 2\mu) \\ -\mu kl & \mu(m^2_{21} - l^2) & -2\mu ml_{21} \\ \mu(k^2 - m^2_{21}) & \mu kl & 2\mu km_{21} \end{pmatrix}, \]

\[ (4.6b) \]

\[ \Delta_1 = det(C), \quad \Delta_2 = det(D). \]

\[ (4.6c) \]

Simplifying the expressions for \( \Delta_1 \) and \( \Delta_2 \) we find

\[ \Delta_1 = \mu^3 m^2 [(k^2 + l^2 - m)^2 - 4(k^2 + l^2)mm_{12}], \]

\[ (4.7a) \]

\[ \Delta_2 = \mu m_{21}^2 [(\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)^2 - 4\mu^2(k^2 + l^2)mm_{21}]. \]

\[ (4.7b) \]

Equations (4.4) imply

\[ \begin{pmatrix} U^{(2)}(k, l, m, t) \\ V^{(2)}(k, l, m, t) \\ W^{(2)}(k, l, m, t) \end{pmatrix} = C^{-1} \begin{pmatrix} k\hat{u}(k, l, m_{12}, t) + l\hat{v}(k, l, m_{12}, t) + m_{12}\hat{w}(k, l, m_{12}, t) - N_P(k, l, m_{12}, t) \\ -m\hat{v}(k, l, -m, t) - l\hat{w}(k, l, -m, t) - N_Q(k, l, -m, t) \\ k\hat{w}(k, l, -m, t) + m\hat{u}(k, l, -m, t) - N_R(k, l, -m, t) \end{pmatrix}, \]

\[ (4.8) \]

Equations (4.5) imply

\[ \begin{pmatrix} U^{(1)}(k, l, m, t) \\ V^{(1)}(k, l, m, t) \\ W^{(1)}(k, l, m, t) \end{pmatrix} = D^{-1} \begin{pmatrix} k\hat{u}(k, l, -m, t) + l\hat{v}(k, l, -m, t) - m\hat{w}(k, l, -m, t) - N_P(k, l, -m, t) \\ m_{21}\hat{v}(k, l, m_{21}, t) - l\hat{w}(k, l, m_{21}, t) - N_Q(k, l, m_{21}, t) \\ k\hat{w}(k, l, m_{21}, t) - m_{21}\hat{u}(k, l, m_{21}, t) - N_R(k, l, m_{21}, t) \end{pmatrix}, \]

\[ (4.9) \]

where \( C^{-1} \) and \( D^{-1} \) are the inverse matrices of \( C \) and \( D \) respectively.

We fix the choice of the contour \( \gamma_{k,l} \) by requiring that every term in the RHS of (4.5) and (4.9) does not have a pole or a branch point above this contour, see Fig.2.
Regarding the zeros of $\Delta_j$, $j = 1, 2$, we note that they are of the form $m = \alpha \sqrt{k^2 + l^2}$, for some constant $\alpha \in \mathbb{C}$. For example, if $\lambda = 2\mu$, we find that the zeros of $\Delta_1$ are

$$m = 0, m \approx (-1.624 \pm 0.126i)\sqrt{k^2 + l^2}, m \approx \pm 0.357i\sqrt{k^2 + l^2},$$

$$m \approx \pm 1.056i\sqrt{k^2 + l^2}, m \approx (1.624 \pm 0.126i)\sqrt{k^2 + l^2};$$

whereas the zeros of $\Delta_2$ are

$$m \approx \pm 0.866i\sqrt{k^2 + l^2}, m \approx (-0.295 \pm 0.442i)\sqrt{k^2 + l^2}, m \approx \pm 0.885i\sqrt{k^2 + l^2},$$

$$m \approx \pm i\sqrt{k^2 + l^2}, m \approx (0.295 \pm 0.442i)\sqrt{k^2 + l^2}.$$

Substituting equations (4.8) and (4.9) in equations (3.10), and using Jordan’s lemma in the complex $m$–plane above the contour $\gamma_k$, it follows that $\hat{u}(k, l, -m, t), \hat{u}(k, l, m_{12}, t), \hat{v}(k, l, -m, t), \hat{v}(k, l, m_{21}, t), \hat{w}(k, l, -m, t), \hat{w}(k, l, m_{12}, t), \hat{w}(k, l, m_{21}, t)$ yield a zero contribution. Let

\[
\begin{pmatrix}
U_2(k, l, m, t) \\
V_2(k, l, m, t) \\
W_2(k, l, m, t)
\end{pmatrix}
= -C^{-1}
\begin{pmatrix}
N_P(k, l, m_{12}, t) \\
N_Q(k, l, -m, t) \\
N_R(k, l, -m, t)
\end{pmatrix},
\]

\[
\begin{pmatrix}
U_1(k, l, m, t) \\
V_1(k, l, m, t) \\
W_1(k, l, m, t)
\end{pmatrix}
= -D^{-1}
\begin{pmatrix}
N_P(k, l, -m, t) \\
N_Q(k, l, m_{21}, t) \\
N_R(k, l, m_{21}, t)
\end{pmatrix},
\]

where $N_P, N_Q, N_R$ are known functions defined in (2.21), and $(m_{12}, m_{21})$ are

![Figure 2: $\gamma_{k, l}$, the deformed path of integration.](image)
defined by equations (4.2) and (4.1) respectively. Equations (3.10) become:

\[ u(x, y, z, t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} \frac{\mathrm{e}^{ikx+ily+imz}}{k^2 + l^2 + m^2} \left\{ 2k[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W_1 - 4\mu km^2W_2 \right\} \]

+ \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx+ily+imz}}{k^2 + l^2 + m^2} \left\{ k[N_P(k, l, m, t) + N_P(k, l, -m, t)] - \frac{1}{m}kl[N_Q(k, l, m, t) - N_Q(k, l, -m, t)] - \frac{1}{m}(m^2 + l^2)[N_R(k, l, m, t) - N_R(k, l, -m, t)] \right\}, \tag{4.11a} \]

\[ v(x, y, z, t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx+ily+imz}}{k^2 + l^2 + m^2} \left\{ 2l[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W_1 - 4\mu lm^2W_2 \right\} \]

+ \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx+ily+imz}}{k^2 + l^2 + m^2} \left\{ l[N_P(k, l, m, t) + N_P(k, l, -m, t)] + \frac{1}{m}(k^2 + m^2)[N_Q(k, l, m, t) - N_Q(k, l, -m, t)] \right\}, \tag{4.11b} \]

\[ w(x, y, z, t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx+ily+imz}}{k^2 + l^2 + m^2} \left\{ 4\mu km^2U_1 + 4\mu lm^2V_1 + 2\mu(l^2 + k^2 - m^2)U_2 + 2\mu(k^2 + l^2 - m^2)V_2 \right\} \]

+ \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx+ily+imz}}{k^2 + l^2 + m^2} \left\{ m[N_P(k, l, m, t) - N_P(k, l, -m, t)] - l[N_Q(k, l, m, t) + N_Q(k, l, -m, t)] + k[N_R(k, l, m, t) + N_R(k, l, -m, t)] \right\}, \tag{4.11c} \]

where \( C, D \) are defined in (4.6), the known functions \( U_j, V_j, W_j, \ j = 1, 2 \), are defined in (4.10), and the known functions \( N_P, N_Q, N_R \) are defined in (2.21).

We summarize the above result in the following proposition:

**Proposition 4.1** Let \((u, v, w)\) satisfy the Lamé-Navier equations (2.1) in the
with the initial conditions \(2.2\) and the stress boundary conditions \(2.3\). Assume that the given functions

\[
\{u_j(x, y, z), v_j(x, y, z), w_j(x, y, z)\}_{j=0,1}, \quad \{g_j(x, y, t)\}_{j=1,2,3}
\]

have sufficient smoothness and decay. A solution of the above initial-boundary value problem, which decays for large \((x, y, z)\), is given by equations \(4.11\), where:

a) The known functions \((N_P, N_Q, N_R)\) are defined in \(2.21\) in terms of the transforms of the initial and boundary data (see equations \(2.15\)-\(2.19\)).

b) The known functions \((U_j, V_j, W_j)\) are defined in \(4.10\) in terms of \((N_P, N_Q, N_R)\) and of the matrices \(C\) and \(D\) given by \(4.6\).

c) The contours \(\gamma_{k,l}\) depicted in Fig.2 are deformations of the real axis and determined by the requirement that the zeros of \(\Delta_j, j = 1, 2\) and the branch points of \(m_{21}\) and \(m_{12}\) are all below \(\gamma_{k,l}\).

5 The Normal Point Load with Homogeneous Initial Conditions

Consider a normal point load suddenly applied to an isotropic elastic half space body. In this case,

\[
u_0 = v_1 = v_0 = w_0 = w_1 = 0; g_1 = 0, \quad g_2 = 0, \quad g_3 = \sigma_0 \delta(x, y) h(t)/(\lambda + \mu),
\]

where \(\sigma_0\) is a constant and \(h(t)\) is the Heaviside function defined by \(h(t) = 0, \quad t \leq 0; \quad h(t) = 1, \quad t > 0\).

In what follows, we compute the functions needed in equations \(4.10\) and \(4.11\). Equations \(2.9\) and \(2.19\) imply

\[
\begin{align*}
\tilde{g}_1(k, l, t) &= 0, \quad \tilde{g}_2(k, l, t) = 0, \quad \tilde{g}_3(k, l, t) = \frac{\sigma_0}{\lambda + \mu} h(t), \quad (5.2a) \\
g^{(j)}(k, l, m, t) &= 0, \quad G^{(j)}(k, l, m, t) = 0, \quad (5.2b) \\
f^{(j)}(k, l, m, t) &= 0, \quad F^{(j)}(k, l, m, t) = 0, \quad (5.2c) \\
e^{(j)}(k, l, m, t) &= \frac{\sigma_0}{\lambda + \mu} \left( e^{\pm i\omega_j t} - 1 \right), \quad (5.2d) \\
E^{(j)}(k, l, m, t) &= - \frac{i\sigma_0}{\lambda + \mu} \left( \frac{1}{\omega_j} - \frac{\cos(\omega_j t)}{\omega_j^2} \right), \quad j = 1, 2. \quad (5.2e)
\end{align*}
\]
Thus, the known functions $N_P$, $N_Q$, $N_R$, defined in (4.1), are given by

\begin{align*}
N_P(k, l, m, t) &= -m\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_1^2} \cos (\omega_1 t) - \frac{1}{\omega_2^2} \cos (\omega_2 t) \right), \\
N_Q(k, l, m, t) &= l\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_1^2} - \frac{1}{\omega_2^2} \cos (\omega_2 t) \right), \\
N_R(k, l, m, t) &= -k\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_2^2} - \frac{1}{\omega_1^2} \cos (\omega_1 t) \right).
\end{align*}

The above equations imply

\begin{align*}
N_P(k, l, m, t) + N_P(k, l, -m, t) &= 0, \\
N_Q(k, l, m, t) + N_Q(k, l, -m, t) &= 2l\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_1^2} - \frac{1}{\omega_2^2} \cos (\omega_2 t) \right), \\
N_R(k, l, m, t) + N_R(k, l, -m, t) &= -2k\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_1^2} - \frac{1}{\omega_2^2} \cos (\omega_1 t) \right), \\
N_P(k, l, m, t) - N_P(k, l, -m, t) &= -2m\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_1^2} \cos (\omega_1 t) - \frac{1}{\omega_2^2} \cos (\omega_2 t) \right),
\end{align*}

\begin{align*}
N_Q(k, l, m, t) - N_Q(k, l, -m, t) &= 0, \\
N_R(k, l, m, t) - N_R(k, l, -m, t) &= 0,
\end{align*}

\begin{align*}
N_P(k, l, m_{12}, t) &= -\frac{m_{12}}{k} N_Q(k, l, m, t) = \frac{m_{12}}{k} N_R(k, l, m, t), \\
N_Q(k, l, m_{21}, t) &= -\frac{l}{m} N_P(k, l, t), \\
N_R(k, l, m_{21}, t) &= \frac{k}{m} N_P(k, l, t)
\end{align*}

and

\begin{align*}
\begin{pmatrix}
U_2(k, l, m, t) \\
V_2(k, l, m, t) \\
W_2(k, l, m, t)
\end{pmatrix}
&= -\sigma_0 \frac{\lambda + 2\mu}{\omega_2^2} \frac{1}{\lambda + \mu} \begin{pmatrix} -m_{12} + m_{12} \cos (\omega_2 t) \\ l + l \cos (\omega_2 t) \\ -k + k \cos (\omega_2 t) \end{pmatrix}, \\
\begin{pmatrix}
U_1(k, l, m, t) \\
V_1(k, l, m, t) \\
W_1(k, l, m, t)
\end{pmatrix}
&= -\sigma_0 \frac{\lambda + 2\mu}{\omega_1^2} \frac{1}{\lambda + \mu} \begin{pmatrix} m - m \cos (\omega_1 t) \\ l + l \cos (\omega_1 t) \\ -k + k \cos (\omega_1 t) \end{pmatrix},
\end{align*}

where the matrices $C$ and $D$ are defined in (4.1). It should be noted that the definitions of $C$ and $D$ do not depend on the particular initial-boundary values, i.e., they are fundamental matrix-valued functions for elastodynamics in half-space. Another fundamental function appears in the alternative approach described in Appendix, which is related to Rayleigh’s function, see equation (6.3).

**Proposition 5.1** Let $(u, v, w)$ satisfy the Lamé-Navier equations (2.1) in the half space

$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 < z < \infty, \quad t > 0,$$

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with the homogeneous initial conditions and Lamb’s boundary conditions (5.1). A solution of this initial-boundary value problem, which decays for large \((x, y, z)\), is given by

\[
\begin{align*}
\frac{1}{8\pi^3} & \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{\gamma_{k,l}} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \\
& \left\{ 2k[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W_1 - 4\mu km^2W_2 \right\}, \\
v(x, y, z, t) &= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{\gamma_{k,l}} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \\
& \left\{ 2l[\lambda(k^2 + l^2) + m^2(\lambda + 2\mu)]W_1 - 4\mu lm^2W_2 \right\}, \\
w(x, y, z, t) &= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{\gamma_{k,l}} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \\
& \left\{ 4\mu km^2U_1 + 4\mu lm^2V_1 + 2\mu k(l^2 + k^2 - m^2)U_2 + 2\mu l(k^2 + l^2 - m^2)V_2 \right\} \\
& - \frac{\sigma_1^2}{4\pi^3} \lambda + 2\mu \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dm \frac{e^{ikx + ily + imz}}{k^2 + l^2 + m^2} \\
& \left\{ m^2 \left[ \frac{1}{\omega_1^2} - \frac{\cos(\omega_1 t)}{\omega_1^2} \right] + l^2 \left[ \frac{1}{\omega_2^2} - \frac{\cos(\omega_2 t)}{\omega_2^2} \right] + k^2 \left[ \frac{1}{\omega_3^2} - \frac{\cos(\omega_3 t)}{\omega_3^2} \right] \right\},
\end{align*}
\]

where \((\omega_1, \omega_2)\) are defined by (2.17), and the known functions \((U_j, V_j, W_j), j = 1, 2\) are computed in (5.5).

Similar expressions are valid for the other Lamb’s problems.

6 Conclusions

The main result of this paper is the derivation of equations (4.11). These equations express the displacements \((u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))\) in terms of integrals along the real line and integrals along contours \(\gamma_{k,l}\) of the complex \(m\)-plane; these integrals involve transforms of the given initial and boundary data. Indeed, equations (4.11) involve the functions \(N_P(k, l, m, t), N_Q(k, l, m, t)\) and \(N_Q(k, l, m, t)\), which are defined in equations (2.21) in terms of the Fourier transforms \(P_j(k, l, m, t), Q_j(k, l, m, t), R_j(k, l, m, t), j = 1, 2\), of the initial data (see equations (2.16)), as well as in terms of certain known transforms \(G^{(j)}(k, l, m, t), F^{(j)}(k, l, m, t), E^{(j)}(k, l, m, t), j = 1, 2\), of the boundary data (see equations (2.19)).

The starting point of the derivations of equations (4.11) is the derivation of the global relations (2.20). These equations are the direct consequence of the application of the three-dimensional Fourier transform and of the substitution
in the resulting equations of the given initial and boundary conditions. Equations (2.20) involve the known functions \( N_P, N_Q, N_R \), as well as the unknown functions \( U^{(j)}, V^{(j)}, W^{(j)}, j = 1, 2 \), (these functions involve certain transforms of the unknown boundary values \( u(x, y, 0, t), v(x, y, 0, t), w(x, y, 0, t) \) see equations (2.8) and (2.18)). The elimination of the above unknown functions is achieved by the following steps: 1. By employing the transformation \( m \to -m \), which leave the unknown functions \( (U^{(j)}, V^{(j)}, W^{(j)}), j = 1, 2 \), invariant, and by utilising the analyticity properties of these functions, we obtain the integral representations (3.10). These representations involve an integral along the real \( k, l \) axes and an integral along the contour \( \gamma_{k,l} \) of the complex \( m \)-plane. 2. Using the transformations \( m \to m_{21} \) and \( m \to m_{12} \) which map \( U^{(2)} \) to \( U^{(1)} \), \( V^{(2)} \) to \( V^{(1)} \), \( W^{(2)} \) to \( W^{(1)} \) and \( U^{(1)} \) to \( U^{(2)} \), \( V^{(1)} \) to \( V^{(2)} \), \( W^{(1)} \) to \( W^{(2)} \) respectively, we express the unknown functions \( (U^{(j)}, V^{(j)}, W^{(j)}), j = 1, 2 \), in terms of known functions as well as in terms of certain unknown functions which however are analytic in a certain domain of the complex \( m \)-plane, see equations (4.8) and (4.9). 3. Using (4.8) and (4.9) in equations (3.10) and employing Jordan’s lemma we obtain equations (4.11).

The main advantages of the new approach are the following:

1. The new method provides an analytic solution of the three dimensional Lamb’s problem with arbitrary initial and boundary conditions.

2. This solution is expressed in terms of the given initial and boundary data. The relevant representation is novel even for the particular case of homogeneous initial conditions (this case has been analysed by several authors). An alternative approach using the Laplace transform is briefly discussed in the Appendix. By comparing equations (4.11) and equation (6.1), the advantage of the new formulae becomes clear. Actually, taking into consideration that the initial-boundary value problems of the equations of elastodynamics are well posed for any finite \( t \), the use of the Laplace transform, which requires \( t \to \infty \), is clearly inappropriate.

3. The new method can be employed for the solution of several related initial-boundary value problems, including Lamb’s problem of the orthotropic half space.

**Appendix**

It is also possible to analyse Lamb’s problem by using only the transformations (3.3) instead of using the transformations (3.3) and the transformations (4.3). However, in this case one cannot eliminate directly all unknown boundary values. Instead, one can derive a complicated expression for the unknown boundary values in terms of the given initial and boundary data. (This approach is similar with the one used in [46] for solving Crighton’s problem). Indeed, let the contour \( \gamma_{k,l,2} \) be a simple curve in the lower half \( m \)-plane, determined by the requirement that it does not cross the branch cut associated with \( \omega_1 \) and \( \omega_2 \),
see Fig[3] Let $K$ be the integral operator defined by
\[
(K[f])(k, l, t) = \frac{1}{2\pi} \int_{\gamma} f(k, l, m, t) \frac{dm}{m(k^2 + l^2 + m^2)^{1/2}},
\]
for any function $f(k, l, m, t)$ with appropriate smoothness and decay. Integrating the global relations (2.20) along $\gamma_{k,l,2}$ we find that the functions
\[
\tilde{h}(k, l, t) = \begin{pmatrix} \tilde{u}(k, l, t) \\ \tilde{v}(k, l, t) \\ \tilde{w}(k, l, t) \end{pmatrix}, \quad \tilde{g}(k, l, t) = \begin{pmatrix} \tilde{g}_1(k, l, t) \\ \tilde{g}_2(k, l, t) \\ \tilde{g}_3(k, l, t) \end{pmatrix},
\]
satisfy a system of Volterra integral equations of the second kind:
\[
\begin{align*}
\tilde{h}(k, l, t) &= (K[N])(k, l, t) \ast \tilde{g}(k, l, t) + (K[M])(k, l, t) \ast \tilde{h}(k, l, t) + (K[H])(k, l, t), \\
0 \leq t < T, T > 0; \quad k, l \in \mathbb{R},
\end{align*}
\]
where $\ast$ denotes the convolution operation with respect to $t$.

\begin{align*}
N(k, l, m, t) &= \begin{pmatrix} im\sqrt{\mu}e^{i\omega_1 t} & 0 & -ik\sqrt{\frac{\lambda+2\mu}{\mu}}e^{i\omega_1 t} \\ 0 & im\sqrt{\mu}e^{i\omega_1 t} & -il\sqrt{\frac{\lambda+2\mu}{\mu}}e^{i\omega_1 t} \\ ik \sqrt{\lambda+2\mu} e^{i\omega_1 t} & il \sqrt{\lambda+2\mu} e^{i\omega_1 t} & im\sqrt{\lambda+2\mu} e^{i\omega_1 t} \end{pmatrix}, \\
M(k, l, m, t) &= \begin{pmatrix} k^2 \sqrt{\mu}e^{i\omega_1 t} & kl \sqrt{\mu}e^{i\omega_1 t} & 2km\sqrt{\mu}e^{i\omega_1 t} \\ kl \sqrt{\mu}e^{i\omega_1 t} & l^2 \sqrt{\mu}e^{i\omega_1 t} & 2lm\sqrt{\mu}e^{i\omega_1 t} \\ km \sqrt{\lambda+2\mu} e^{i\omega_1 t} & lm \sqrt{\lambda+2\mu} e^{i\omega_1 t} & (k^2 + l^2)\sqrt{\lambda+2\mu} e^{i\omega_1 t} \end{pmatrix},
\end{align*}

and the known function $H(k, l, m, t)$ is defined by
\[
H(k, l, t) = \left( \frac{i\omega_1 P_0 + P_1}{\lambda + 2\mu} e^{i\omega_1 t}, \frac{i\omega_2 Q_0 + Q_1}{\mu} e^{i\omega_2 t}, \frac{i\omega_2 R_0 + R_1}{\mu} e^{i\omega_2 t} \right).
\]
The solution of the integral equations (6.1) yields the unknown transforms appearing in (2.20).

Equations (6.1) can be solved in closed form by using the Laplace transform in \( t \). Let “ \( \circ \) ” denote the Laplace transform with respect to \( t \) and let

\[
\begin{align*}
    w_1 &= (p^2 + (\lambda + 2\mu)(k^2 + l^2))^{\frac{1}{2}},
    \quad w_2 = (p^2 + \mu(k^2 + l^2))^{\frac{1}{2}}, \\
    \Delta &= \frac{(p^2 + 2\mu(k^2 + l^2))^2}{w_1^2w_2^2} - 4\sqrt{\frac{\mu}{\lambda + 2\mu}} \frac{\mu(k^2 + l^2)}{w_1w_2}.
\end{align*}
\]

The solution of (6.1) is given by

\[
\hat{h} = \frac{1}{\Delta} (I - K[M])^{*T}(K[N]\hat{g} + K[H]),
\]

where “ \( * \) ” denotes the adjoint operation, and the functions \( K[M], K[N] \) which are independent of the initial and boundary conditions, are given by

\[
K[M] = \begin{pmatrix}
    -\frac{k^2\mu}{w_2} & \frac{-ki\mu}{w_2} & \frac{-2k\sqrt{\lambda}}{w_2} \\
    \frac{-ki\mu}{w_2} & \frac{-l^2\mu}{w_2} & \frac{-2il\sqrt{\lambda}}{w_2} \\
    \sqrt{\frac{\mu}{\lambda + 2\mu}} & \sqrt{\frac{\mu}{\lambda + 2\mu}} & \frac{-\lambda(k^2 + l^2)}{w_1}
\end{pmatrix}
\]

and

\[
K[N] = \begin{pmatrix}
    0 & \frac{i(\lambda + 2\mu)k}{w_2} & \frac{i(\lambda + 2\mu)l}{w_2} \\
    \frac{-i(\lambda + 2\mu)k}{w_2} & 0 & \frac{-i(\lambda + 2\mu)l}{w_2} \\
    \frac{-ik\mu}{w_1} & \frac{-il\mu}{w_1} & \frac{-\sqrt{\lambda + 2\mu}}{w_1}
\end{pmatrix}.
\]

The zeros of \( \Delta \) coincide with the zeros of Rayleigh’s function. When \( \mu/\lambda > 0.906 \), the known transforms \( \hat{g}(k, l, p) \), for each fixed \( k, l \), do not have poles with positive real parts.

In the particular case of the problem of homogeneous initial condition and the normal point load boundary condition, equation (6.4) reduces to the classic Lamb’s solution \[1\][2].

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