On Quantum - Classical Correspondence for Baker’s Map

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Abstract

Quantum baker’s map is a model of chaotic system. We study quantum dynamics for the quantum baker’s map. We use the Schack and Caves symbolic description of the quantum baker’s map. We find an exact expression for the expectation value of the time dependent position operator. A relation between quantum and classical trajectories is investigated. Breakdown of the quantum-classical correspondence at the logarithmic timescale is rigorously established.
1 Introduction

The quantum-classical correspondence for dynamical systems has been studied for many years, see for example [1, 2] and reference therein. A significant progress in understanding of this correspondence has been achieved in a mathematical approach when one considers the Planck constant $h$ as a small variable parameter. It is well known that in the limit $h \to 0$ quantum theory is reduced to the classical one [3, 4].

However in physics the Planck constant is a fixed constant although it is very small. Therefore it is important to study the relation between classical and quantum evolutions when the Planck constant is fixed. There is a conjecture [5, 6, 7] that a characteristic timescale $t_h$ appears in the quantum evolution of chaotic dynamical systems. For time less then $t_h$ there is a correspondence between quantum and classical expectation values, while for times greater that $t_h$ the predictions of the classical and quantum dynamics no longer coincide.

An important problem is to estimate the dependence $t_h$ on the Planck constant $h$. Probably a universal formula expressing $t_h$ in terms of $h$ does not exist and every model should be studied case by case. It is expected that certain quantum and classical expectation values diverge on a timescale inversely proportional to some power of $h$. Other authors suggest that for chaotic systems a breakdown may be anticipated on a much smaller logarithmic timescale (see [1, 8] for a discussion). Numerous works are devoted to the analytical and numerical study of classical and quantum chaotic systems [9] - [33].

Most results concerning various timescales are obtained numerically. In this paper we will present some exact results on a quantum chaos model. We compute explicitly an expectation value for the quantum baker’s map and prove rigorously the appearance of the logarithmic timescale.

The quantum baker’s map is a model invented to study the chaotic behavior [15]. The model has been studied in [16] - [24].

In this paper quantum dynamics of the position operator for the quantum baker’s map is considered. We use a simple symbolic description of the quantum baker’s map proposed by Schack and Caves [22]. We find an exact expression for the expectation value of the time dependent position operator. In this sense the quantum baker’s map is an exactly solvable model though stochastic one. A relation between quantum and classical trajectories is investigated. For some matrix elements the breakdown of the quantum-
classical correspondence at the logarithmic timescale is established.

Here we would like to note that in fact the notion of the timescale is not a uniquely defined notion. Actually we will obtain the formula

$$\langle \hat{q}_m \rangle - q_m = \hbar 2^{m-1}$$

where $\hat{q}_m$ and $q_m$ are quantum and classical positions respectively at time $m$. This formula will be interpreted as the derivation of the logarithmic timescale (see discussion in Sect.5). The main result of the paper is presented in Theorem 1 in Sect. 4.

In another paper [33], semiclassical properties and chaos degree for the quantum baker’s map are considered.

## 2 Classical Baker’s Transformation

The classical baker’s transformation maps the unit square $0 \leq q, p \leq 1$ onto itself according to

$$(q, p) \rightarrow \begin{cases} 
(2q, p/2), & \text{if } 0 \leq q \leq 1/2 \\
(2q - 1, (p + 1)/2), & \text{if } 1/2 < q \leq 1
\end{cases}$$

This corresponds to compressing the unit square in the $p$ direction and stretching it in the $q$ direction, while preserving the area, then cutting it vertically and stacking the right part on top of the left part.

The classical baker’s map has a simple description in terms of its symbolic dynamics [11]. Each point $(q, p)$ is represented by a symbolic string with a dot

$$\xi = \cdots \xi_2 \xi_1 \xi_0 \cdot \xi_1 \xi_2 \cdots ,$$

where $\xi_k \in \{0, 1\}$, and

$$q = \sum_{k=1}^{\infty} \xi_k 2^{-k}, \quad p = \sum_{k=0}^{\infty} \xi_{-k} 2^{-k-1}$$

The action of the baker’s map on a symbolic string $\xi$ is given by the shift map (Bernoulli shift) $U$ defined by $U\xi = \xi'$, where $\xi'_m = \xi_{m+1}$. This means
that, at each time step, the dot is shifted one place to the right while entire string remains fixed. After \( m \) steps the \( q \) coordinate becomes

\[
q_m = \sum_{k=1}^{\infty} \xi_{m+k}2^{-k}
\]  

(2)

This relation defines the classical trajectory with the initial data

\[
q = q_0 = \sum_{k=1}^{\infty} \xi_k2^{-k}
\]  

(3)

3 Quantum Baker’s Map

Quantum baker’s maps are defined on the \( D \)-dimensional Hilbert space of the quantized unit square. To quantize the unite square one defines the Weyl unitary displacement operators \( \hat{U} \) and \( \hat{V} \) in \( D \)-dimensional Hilbert space, which produces displacements in the momentum and position directions, respectively, and the following commutation relation is obeyed

\[
\hat{U}\hat{V} = \epsilon\hat{V}\hat{U},
\]

where \( \epsilon = \exp(2\pi i/D) \). We choose \( D = 2^N \), so that our Hilbert space will be the \( N \) qubit space \( \mathbb{C}^{\otimes N} \). The constant \( h = 1/D = 2^{-N} \) can be regarded as the Planck constant. The space \( \mathbb{C}^2 \) has a basis

\[
|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

The basis in \( \mathbb{C}^{\otimes N} \) is

\[
|\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_N\rangle, \quad \xi_k = 0, 1
\]

We write

\[
\xi = \sum_{k=1}^{N} \xi_k2^{N-k}
\]

then \( \xi = 0, 1, \ldots, 2^N - 1 \) and denote

\[
|\xi\rangle = |\xi_1\xi_2\cdots\xi_N\rangle = |\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_N\rangle
\]
We will use for this basis also notations \( \{|\eta\rangle = |\eta_1\eta_2 \cdots \eta_N\rangle, \ \eta_k = 0, 1\} \) and 
\( \{|j\rangle = |j_1j_2 \cdots j_N\rangle, \ j_k = 0, 1\} \).

The operators \( \hat{U} \) and \( \hat{V} \) can be written as
\[
\hat{U} = e^{2\pi i \hat{q}}, \  \hat{V} = e^{2\pi i \hat{p}}
\]
where the position and momentum operators \( \hat{q} \) and \( \hat{p} \) are operators in \( \mathbb{C}^{\otimes N} \) which are defined as follows. The position operator is
\[
\hat{q} = \sum_{j=0}^{2^N-1} q_j |j\rangle \langle j| = \sum_{j_1 \cdots j_N} q_{j_N \cdots j_1} |j_N \cdots j_1\rangle \langle j_1 \cdots j_N|
\]
where
\[|j\rangle = |j_1j_2 \cdots j_N\rangle, \ j_k = 0, 1\]
is the basis in \( \mathbb{C}^{\otimes N} \),
\[j = \sum_{k=1}^{N} j_k 2^{N-k}\]
and
\[q_j = j + 1/2, \ j = 0, 1, \ldots, 2^N - 1\]
The momentum operator is defined as
\[
\hat{p} = F_N \hat{q} F_N^* \]
where \( F_N \) is the quantum Fourier transform acting to the basis vectors as
\[
F_N |j\rangle = \frac{1}{\sqrt{D}} \sum_{\xi=0}^{D-1} e^{2\pi i \xi j/D} |\xi\rangle,
\]
here \( D = 2^N \).

A quantum baker's map is the unitary operator \( T \) in \( \mathbb{C}^{\otimes N} \) with the following matrix elements
\[
\langle \xi | T | \eta \rangle = \frac{1 - i}{2} \exp \left( \frac{\pi i}{2} |\xi_1 - \eta_N| \right) \prod_{k=2}^{N} \delta (\xi_k - \eta_{k-1}), \tag{4}
\]
where \( |\xi\rangle = |\xi_1\xi_2 \cdots \xi_N\rangle, \ |\eta\rangle = |\eta_1\eta_2 \cdots \eta_N\rangle \) and \( \delta(x) \) is the Kronecker symbol, \( \delta(0) = 1; \ \delta(x) = 0, x \neq 0 \). This transformation belongs to a family of quantizations of baker’s map introduced by Schack and Caves \cite{22} and studied in \cite{23, 24}. 

5
4 Expectation Value

We consider the following mean value of the position operator \( \hat{q} \) for time \( m = 0, 1, \ldots \) with respect to a vector \( |\xi\rangle \):

\[
r_m^{(N)} = \langle \xi | T^m \hat{q} T^{-m} | \xi \rangle,
\]

where \( |\xi\rangle = |\xi_1 \xi_2 \cdots \xi_N\rangle \). First we show that there is an explicit formula for the expectation value \( r_m^{(N)} \). In this sense the quantum baker's map is an explicitly solvable model. Then we compare the dynamics of the mean value \( r_m^{(N)} \) of position operator \( \hat{q} \) with that of the classical value \( q_m \), Eq. (2). We will establish a logarithmic timescale for the breakdown of the quantum-classical correspondence for the quantum baker's map.

From Eq. (4) one gets for \( m = 0 \), \( 1 \), \( \ldots , N-1 \)

\[
\langle \xi | T^m | \eta \rangle = \left( \frac{1-i}{2} \right)^m \left( \prod_{k=1}^{N-m} \delta (\xi_{m+k} - \eta_k) \right) \left( \prod_{l=1}^{m} \exp \left( \frac{\pi}{2} i |\xi_l - \eta_{N-m+l}| \right) \right),
\]

and for \( m = N \)

\[
\langle \xi | T^N | \eta \rangle = \left( \frac{1-i}{2} \right)^N \left( \prod_{l=1}^{N} \exp \left( \frac{\pi}{2} i |\xi_l - \eta_l| \right) \right)
\]

Using this formula we will prove the following

**Theorem 1.** One has the following expression for the expectation value \( r_m^{(N)} \) of the position operator

\[
r_m^{(N)} = \langle \xi | T^m \hat{q} T^{-m} | \xi \rangle = \sum_{k=1}^{N-m} \frac{\xi_{m+k}}{2^k} + \frac{1}{2^{N-m+1}}
\]

for \( 0 \leq m < N \). For \( m = N \) we have

\[
r_N^{(N)} = \frac{1}{2}
\]

**Proof.** By a direct calculation, we obtain

\[
r_m^{(N)} = \langle \xi | T^m \hat{q} T^{-m} | \xi \rangle
\]

\[
= \langle \xi | T^m \left( \sum_{j=0}^{2N-1} \frac{j + 1/2}{2^N} |j\rangle \langle j| \right) T^{-m} | \xi \rangle
\]
\[
\sum_{j=0}^{2N-1} j + 1/2 \left\langle \xi \left| T^m \right| j \right\rangle \left\langle j \left| T^m \right| \xi \right\rangle
= \sum_{j=0}^{2N-1} j + 1/2 \left| \left\langle \xi \left| T^m \right| j \right\rangle \right|^2.
\]

Using (6) we write
\[
\begin{align*}
\sum_{j=0}^{2N-1} &\left( j + 1/2 \right) \left| \left\langle \xi \left| T^m \right| j \right\rangle \right|^2 \\
&= \sum_{j=0}^{2N-1} \left( j + 1/2 \right) \left| \left\langle \xi \left| T^m \right| j \right\rangle \right|^2 \\
&= \sum_{j=0}^{2N-1} \left( j + 1/2 \right) \left( \prod_{k=1}^{N-m} \delta (\xi_{m+k} - j_k) \right) \\
&= \sum_{j=0}^{2N-1} \left( j + 1/2 \right) \left( \prod_{k=1}^{N-m} \delta (\xi_{m+k} - j_k) \right) \\
&= \sum_{j=0}^{2N-1} \left( j + 1/2 \right) \left( \prod_{k=1}^{N-m} \delta (\xi_{m+k} - j_k) \right)
\end{align*}
\]

Using the Kronecker symbols one gets
\[
\begin{align*}
\sum_{j=0}^{2N-1} &\left( j + 1/2 \right) \left( \prod_{k=1}^{N-m} \delta (\xi_{m+k} - j_k) \right)
\end{align*}
\]

We can write it as
\[
\begin{align*}
\sum_{j=0}^{2N-1} &\left( j + 1/2 \right) \left( \prod_{k=1}^{N-m} \delta (\xi_{m+k} - j_k) \right)
\end{align*}
\]
Finally we obtain (8) for $0 \leq m < N$

$$r_m^{(N)} = \frac{2m}{2^N + m} \left( \sum_{l=1}^{N-m} \xi_{m+l} 2^{N-l} \right) + \frac{1}{2^N + m} \sum_{j_{N-m+1} \cdots j_N} \left( \sum_{l=1}^{m} j_{N-m+l} 2^{m-l} \right) + \frac{1}{2^{N+1}}$$

In the case $m = N$ we have

$$r_N^{(N)} = \frac{2^{N-1}}{2^N} \sum_{j=0}^{2^N-1} \left| \langle \xi | T^N | j \rangle \right|^2$$

$$= \frac{2^{N-1}}{2^N} \sum_{j=0}^{2^N-1} \left| \left( \frac{1-i}{2} \right)^N \right|^2 = \frac{1}{2^{2N}} \sum_{j=0}^{2^N-1} (j + 1/2) = \frac{1}{2}.$$ 

The theorem is proved.

5 Time Scales

We consider here the quantum-classical correspondence for the quantum baker's map. First let us mention that $2^N = 1/h$ and the limit $h \to 0$ corresponds to the limit $N \to \infty$. Therefore from Theorem 1 and Eq. (2) one has the mathematical correspondence between quantum and classical trajectories as $h \to 0$:

$$\lim_{N \to \infty} r_m^{(N)} = q_m, \quad m = 0, 1, \ldots$$

Now let us fix the Planck constant $h = 2^{-N}$ and investigate on which time scale the quantum and classical expectation values start to differ from each other. From Theorem 1 and Eq. (2) we obtain the following
Proposition 1. Let $r_m^{(N)}$ be the mean value of position operator $\hat{q}$ at the time $m$ and $q_m$ is the classical trajectory Eq. (2). Then we have

$$q_m - r_m^{(N)} = \sum_{j=N-m+1}^{\infty} \xi_{m+j} 2^{-j} - \frac{1}{2^{N-m+1}}$$

(10)

for any $0 \leq m \leq N$.

Let us estimate the difference between the quantum and classical trajectories.

Proposition 2. Let $q_m$ and $r_m^{(N)}$ be the same as in the Proposition 1. Then we have

$$|r_m^{(N)} - q_m| \leq \frac{1}{2^{N-m+1}}$$

(11)

for any string $\xi = \xi_1\xi_2...$ and any time $0 \leq m \leq N$.

Proof. Note that

$$0 \leq \sum_{j=N-m+1}^{\infty} \xi_{m+j} 2^{-j} \leq \frac{1}{2^{N-m+1}} \left( 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots \right) = \frac{1}{2^{N-m}}.

Using the above inequality, one gets from Eq. (10)

$$-\frac{1}{2^{N-m+1}} \leq q_m - r_m^{(N)} \leq \frac{1}{2^{N-m}} - \frac{1}{2^{N-m+1}} = \frac{1}{2^{N-m+1}}

This means that we have

$$|r_m^{(N)} - q_m| \leq \frac{1}{2^{N-m+1}}$$

for any $0 \leq m \leq N$.

Proposition 2 shows an exact correspondence between quantum and classical expectation value for baker's map. We can write the relation (11) in the form

$$|r_m^{(N)} - q_m| \leq \frac{h}{2^m}$$

(12)

since the Planck constant $h = 2^{-N}$. In particular for $m = 0$ we have

$$|r_0^{(N)} - q_0| \leq \frac{h}{2}$$

(13)
for any $\xi = \xi_1 \xi_2 \ldots$.

Now let us estimate at what time $m = t_h$ there appears an essential
difference between classical trajectory and quantum expectation value. From
Eq. (12) we can expect that the time $m = t_h$ corresponds to the maximum
of the function $2^m / 2^{N-1}$ for $0 \leq m \leq N$, i.e.

$$t_h = N = \log_2 \frac{1}{\hbar}$$

(14)

For time $0 \leq m < t_h$ the difference between classical and quantum trajecto-
ries in (12) is bounded by $1/4$ since

$$h2^{m-1} = \frac{1}{2^{N-m+1}} \leq \frac{1}{4}$$

One can see that the bound is saturated. Indeed let us take a string $\xi$ with
arbitrary $\xi_1, \ldots, \xi_N$ but with $\xi_{N+1} = 0, \xi_{N+2} = 0, \ldots$. Then one has

$$r_m^{(N)} - q_m = h2^{m-1}, \quad m = 0, 1, \ldots, N$$

Therefore we have established the logarithmic dependence of the timescale
on the Planck constant $\hbar$.

6 Conclusions

In this paper we have computed the expectation values for the position op-
erator in the quantum baker’s map. Breakdown of the quantum-classical
 correspondence at the logarithmic timescale is rigorously estab-
lished. For better understanding of the quantum-classical correspondence and the de-
coherence process it is important to perform similar computations for more
general matrix elements which include also the momentum operators and
coherent vectors.

Only the simplest quantization of the baker’s map was considered in the
paper. It would be interesting to extend the computations to the whole
family of quantizations of quantum baker’s map proposed in [22]. Some of
these questions will be investigated in another paper [33].

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