Markovian Memory Embedded in Two-State Natural Processes

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Markovian type of memory is considered as an inseparable ingredient in a variety of natural two-state processes within a vast range of interdisciplinary fields. The Markovian memory embedded in a binary system is shaping its evolution on the basis of its current state. In doing so, this type of memory introduces either clustering or dispersion of binary states. The consequence is directly observed in the lengthening or shortening of the runs of the same binary state and also in the way the proportion of a state within a sequence of state measurements scatters about its true average. In the presence of clustering, this scatter will become broader. Conversely, it will become narrower when dispersion of states is present. Both trends are directly quantifiable through the Markovian self-transition probabilities. It is shown that the Markovian memory can even imitate the evolution of a random process, regarding the long-term behavior of the frequencies of its binary states. This situation occurs when the associated binary state self-transition probabilities are balanced. To exemplify the behavior of Markovian memory, two natural processes are selected from diverse scientific disciplines, belonging to a wide range of systems classified as two-state systems. The first example is studying the preferences of nonhuman troglodytes regarding handedness. The Markovian model in this case assesses the extent of influence two contiguous individuals may have on each other. The other example studies the hindering of the quantum state transitions that rapid state measurements introduce, known as the Quantum Zeno effect (QZE). Based on the current mathematical methodology, simulations of the experimentally observed clustering of states allowed for the estimation of the two self-transition probabilities in this quantum system. Through these, one can appreciate how the particular hindering of the evolution of a quantum state may have originated. Namely, through a quantifiable degree of preference for the same binary state combined with a quantifiable degree of avoidance of the rival state. The aim of this work is to illustrate as merits of the current mathematical approach, its wide range applicability and its potential to provide a variety of information regarding the dynamics of the studied process.

INTRODUCTION

Markov sources provide phenomenological representations of a wide range of natural processes. In a Markov chain involving state measurements, the system under observation experiences state transitions according to specified probabilities. A specific class of Markov sources are the two-state systems, frequently employed in physics on a multitude of occasions such as to represent spins in the Ising model, or the outcome of particle collisions in the Galton board binomial experiment. Binary systems can also represent sequences of quantum measurements, as in the case of the quantum Zeno effect (QZE), where the natural evolution of the two-level atomic system is hindered by rapid observations. Moreover, binary systems are often encountered in electronic solid-state diffusion, turbulent flow dynamics, bistable liquid crystal displays, Josephson junctions, or flip-flop electrical circuits in computing machines, which fit well as types of two-state systems. Binary Markov processes can, therefore, describe adequately a whole host of physical micro- as well as macro-systems.

Yet, two-state systems are not only pertinent in physics and computing, but also highly relevant to probabilistic systems commonly studied within the medical and social sciences. It is customary, for instance, in medicine, psychiatry, or anthropology, to test a hypothesis against two alternatives. In medicine, the effect of a drug treatment is decided from the proportion of successes (against failures) across many independent studies. While the degree of reliability of the results depends on the trustworthiness of the database, specific tests are designed to determine whether the database is free from biases. One such mainly visual test is the construction of a scatter plot of independent results that constitute the database. If the distribution of a large number of data on the scatter plot appears asymmetric, it implies that the database is biased by, as an example, publication biases and thus rendered inappropriate source for reliable statistical inferences regarding the studied effect. Less emphasis has been sited so far on the scatter plot’s breadth and the information that it can provide concerning the dynamics of the underlying mechanism. The relationship between them, however, can be exemplified on the basis of two-state Markov processes, as will be shown in this work.
Markovian memory is introduced by the requirement that each new state depends on previous states some $n$ steps back, so that past probabilities determine the future ones \cite{18,19}. As it is discussed here, the dynamics of two-state systems fall under two specific categories of Markovian memory, which either introduce a clustering or a dispersion of binary states.

The purpose for the application of the Markov memory approach is to study the first-order interaction dynamics in a variety of two-state systems. In doing so, it can estimate the conditional likelihood for the system to change state or remain at the same state. It also estimates the consequent lengthening or shortening of runs of the same state in a sequence of state measurements.

A brief description of the two-state Markov process is offered in section 2. The statistics of Markov chains in relation to the clustering of states (lengthening of runs) and dispersion of states (shortening of runs) is discussed in section 3. Two applications of the mathematical approach implicating Markovian memory, in physics (QZE) and anthropology (Handedness) are presented in section 4. A more detailed analysis of the present mathematical treatment is provided in the appendix.

**FIRST-ORDER, TWO-STATE MARKOV PROCESS**

A first-order, two-state Markov process is driven by four transition probabilities, $p_{ij}$, $i, j = 1, 2$. The sequence of measurements of the binary state, $x_n$, represents the occurrence of an event (state A), otherwise exemplified by $x_n = 1$, or the failure of its occurrence (state B) corresponding to $x_n = 0$. The initial absolute probabilities of finding the system at either state A or B are $p_1$ and $1 - p_1$, respectively. Once the system is at state A, the conditional probability that it remains at the same state after a single measurement is $p_{11} = p$, whereas the conditional probability, $p_{21}$, to make a transition to B will be $1 - p$. In a similar fashion, probabilities $p_{22} = q$ and $p_{12} = 1 - q$ are assigned to transitions from state B. In both cases, the self-transition probabilities satisfy the inequality $0 < p, q < 1$. Whereas $p, q = 0.5$ underlines random variability of state measurements, the range of probabilities $0 < p, q < 1$ and $0 < p, q < 0.5$ introduce persistence of the same state and anti-persistence, respectively. The latter case implies an increased probability to avoid transitions to the same state over a sequence of measurements. We shall next discuss how the expected frequency of the one state of the Markov process (A), in a sequence of $n$ consecutive measurements, depends on the self-transition probabilities $p$ and $q$.

The average frequency, $\bar{p}_n$, of occurrence of state A in a Markov chain of $n$ steps is \cite{20}

$$\bar{p}_n = \varphi + \frac{p_1 - \varphi}{n} \cdot \frac{1 - a^n}{1 - a} \quad (1)$$

The parameter $a \neq 1$ is

$$a = p + q - 1 \quad (2)$$

After a large enough number of state measurements, $n$, (Markov transitions) the frequencies of observed states A and B become $\varphi$ and $1 - \varphi$, respectively, at any value of parameter $a$, $a \neq 1$.

$$\varphi = \lim_{n \to \infty} p_n = \frac{1 - q}{2 - (p + q)}$$

$$1 - \varphi = \frac{1 - p}{2 - (p + q)} \quad (3)$$

If $p = q$, the frequencies of observed states A and B become $\varphi = 0.5$ and $1 - \varphi = 0.5$. This result holds true not only in the absence of memory when $p = q = 0.5$, but most importantly, when $p = q \neq 0.5$. This is a curious condition which turns a Markov process with memory into a random process, as far as the long-term state frequencies $\varphi$ and $1 - \varphi$ are concerned.

Even in such an odd situation, the non-randomness of the Markov process is directly observed through the variance of the binary state in the sequences. The standard deviation of the expected proportion of binary state A, $\bar{p}_n$, is estimated to be \cite{20}. 

\[\]
\[
\sigma = \sqrt{\frac{\varphi(1-\varphi)}{n}} \cdot \sqrt{\frac{p+q}{2-(p+q)}}
\] (4)

Relation (4) is also written \( \sigma = \sigma_0 \cdot \nu \), where \( \sigma_0 = \sqrt{\varphi(1-\varphi)/n} \), is the standard deviation of outcomes of a memory-free Markov process and it indicates that the variance of \( \bar{p}_n \) can be modulated by a factor \( \nu^2 \neq 1 \) introduced by the Markov self-transition probabilities \( p \) and \( q \). Assuming \( p \) and \( q \neq 0 \) the factor which modulates the variance is

\[
\nu^2 = \frac{p+q}{2-(p+q)}
\] (5)

A large variety of natural processes can be represented as Markov processes. In such cases the characteristic parameters \( \varphi \) and \( \nu^2 \) can be assigned accordingly. These provide insights into the process dynamics on first neighbor level. Often a process is studied through its statistical behavior with the help of meta-analyses. In such approaches, the two characteristic parameters, \( \varphi \) and \( \nu^2 \), can be easily estimated through the so-called scatter plots. The application of the Markov process on such statistical ensembles can provide useful information on the investigated process as will be shown next.

**SCATTER PLOT OF MARKOVIAN BINARY STATES**

The combination of results from independent studies of a phenomenon constitutes the so-called meta-analysis. The accuracy of the result of each individual study depends proportionally on the size, \( n \), of the study, since the associated error is inversely proportional to the square root of the standard deviation.

The scatter plot, \( n = f(\bar{p}_n) \), where the size of studies, \( n \), is plotted against the associated proportion of binary state, \( \bar{p}_n \), will be shaped like an inverted funnel \[21\] centered at \( \varphi \), the single true average, or in other words the value to which the averages \( \bar{p}_n \) converge. This is due to the fact, as mentioned above, that the estimate of the underlying effect becomes more accurate as the sample size of component studies increases. Scatter plots can thus provide useful information not only on the magnitude, \( \varphi \), of an investigated effect, but also about the dynamics of the mechanism involved \[22\].

The frequency of state A, \( \bar{p}_n \), in a sequence of \( n \) individual measurements of a Markov process will range within a confidence interval. The 95\% of them on the scatter plot are expected to be enclosed by the confidence interval, represented by the two funnel-shaped red curves in Fig. 1, \( \bar{p}_n = f(n) \) or \( n = f(\bar{p}_n) \)

\[
\bar{p}_n = \varphi \pm 1.96 \cdot \sqrt{\frac{\varphi(1-\varphi)}{n}} \cdot \nu
\] (6)

and

\[
n = 3.84 \cdot \frac{\varphi(1-\varphi) \cdot \nu^2}{(\bar{p}_n - \varphi)^2}
\] (7)

Computer-simulated Markovian data sequences of size \( n \) were generated and plotted against the frequency of binary state A, \( p_n \), in each sequence. Fig. 1 illustrates two such examples; (a) one of a symmetric \( (p = q) \) Markov process exhibiting anti-persistence, \( p, q < 0.5 \), and (b) of an asymmetric \( (p \neq q) \) Markovian process with \( p = 0.88, q = 0.5 \) exhibiting one-sided persistence. In the symmetrical case, \( p = q \), the correlation factor \( C_m \) between the two states in the Markov chain

\[
C_m = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} s_n s_{n+m}
\] (8)

becomes the assemble expectation value of the correlation between first-order neighbors \[23\].
FIG. 1: Scatter plot of Markovian data. Dots: computer simulated data, Lines: 95% confidence interval. (a) Solid lines: Symmetric Markovian process, \( p = q = 0.12 \), \( \nu = 0.37 \) and \( \varphi = 0.5 \). Probabilities \( p, q < 0.5 \) introduce anti-persistence and \( \nu < 1 \) introduces dispersion of states and narrowing of variance. Dotted lines: memory-free binary process, \( \nu = 1 \). (b) Solid lines: Asymmetric Markovian process, \( p = 0.88 \) and \( q = 0.5 \), \( \nu = 1.49 \), \( \varphi = 0.81 \). The persistence introduced by the self-transition probability, \( p \), of state-A, introduces broadening of the variance (\( \nu > 1 \)), according to relation (4) and as compared to a Markov process with \( \varphi = 0.81 \) and \( \nu = 1 \).

\[ C_1 = s_n s_{n+1} = 2p - 1 \]  

When \( p = q = 0.88 \) the strong, positive correlation of neighbors \( (C1 = +76\%) \) introduces persistence and therefore clustering of states, while the condition \( p = q = 0.12 \) introduces strong negative correlation, anti-persistence and dispersion of states \( (C1 = -76\%) \). The graphical representation of Fig. 1 confirms that the simulated Markovian
data obey well the statistical estimations: the 95% of them were found enclosed under the two 95% confidence interval curves, Eq. (7), suggesting that this statistical tool applied on scatter plots of meta-analyses is relatively reliable.

The modulated variance, as described above, indicates absence of random variability in independent measurements of an effect [24]. This type of observed irregularity within a meta-analysis has been referred to by the term "statistical heterogeneity" [25]. There are numerous reasons possible behind statistical heterogeneity in a database. We consider here the two types that generate either dispersion or clustering of Markovian states.

Looking closer at the shape of the plot we notice that, according to Eqs. (4) and (5), when $\nu > 1$ its scatter becomes broader at all sample sizes $n$, as compared that of a memory-free process ($\nu = 1$). Similarly, the variance will be narrowed in the case where $p + q < 1$, (i.e. $\nu < 1$). The broadening of a scatter plot is the direct consequence of the persistence of a binary state and the occurrence of longer-than-usual runs in the data sequences due to state clustering, as Fig. 2 clearly illustrates.

![Normalized number of runs](image)

**FIG. 2:** Normalized length of run (averaged over 10 sequences) in a 10,000-unit long computer generated Markovian sequence. (a) Circles, $p = q = 0.50$, memory-free Markovian process; (b) triangles, $p = q = 0.88$, clustering of states; (c) squares, $p = q = 0.12$, dispersion of states; (d) dotted line: theoretical expectation for memory-free processes, Eq. (10).

It is expected that, when $p = q = 0.5$, the number of runs having length $m$, $a_m$, i.e. sequences of the same binary state occurring at frequency $p_m$ in a sequence of total length $n$, will be

$$a_m = (n - m - 1) \cdot [p_m^2 (1 - p_m)^m + (1 - p_m)^2 p_m^m]$$

The variance $\sigma$ of $a_m$ for long enough sequences is: $\sigma = a_m$. In Fig. 2 the normalized average number of runs (over 10 computer-simulated symmetric Markov sequences, each of length $n = 10,000$), is plotted against the length of run, $m$. When the Markovian memory ($p = q = 0.88$) introduces persistence and clustering of the same binary state, the average length of the computer-simulated runs is considerably longer than that of the memory-free case ($p = q = 0.5$). On the contrary, in the anti-persistent case ($p = q = 0.12$), the average length of runs is shorter than that of the memory-free sequences. This result supports that, in the latter condition, the states tend to disperse. The simulated Markovian memory-free sequences, on the other hand, behave according to Eq. (10), as expected by theory. Estimating the self-transition probabilities ($p, q$) and the variance factor $\nu^2$ provides a quantifiable description of the Markovian dynamics in natural processes within a diversity of scientific disciplines.

In the first example that follows, the above analysis will be directly applied on scatter plots referring to anthropological data. In the second example the analysis is applied on data recording two-state quantum transitions (QZE), to directly estimate the lengthening and shortening of runs of the same binary state. Then through simulations, based on the mathematical formulation developed here, it goes on to estimate the Markovian self-transition probabilities. From these, information regarding the transitions dynamics will be drawn.
EXAMPLES OF THE MARKOV MEMORY MODEL IN NATURAL PROCESSES

Handedness as a Markov process

Often in scientific disciplines such as medicine, anthropology, sociology, or psychology the evaluation of evidence regarding an effect makes use of scatter plots. On one such occasion within anthropology, a meta-analysis, that merged the results of 32 studies, was performed to investigate the strength and consistency of right-handedness in nonhuman primates [10]. Fig. 3. This application refers to an even blend of primates raised in the wild as well as in captivity. The constructed scatter plot in this study was intended to shed light on questions regarding the consistency of the data variability with normal sampling variation and the nature of biases in the reports of statistically significant handedness. One could, however, get additional information from these scatter plots, with regard to the internal dynamics of the processes involved that lead to handedness preference.

Anthropologists discovered that chimpanzees develop cultures in the same way as humans do. According to Whiten et al. [23], handedness in nonhuman primates tends to be lateraled. Yet, unlike humans where there is only 10% use of left hand, this proportion in nonhuman primates in the wild reaches 50%. Chimps in captivity, however, exhibit a weak preference for the use of right hand, which is believed to result from the human influence. These observations are in agreement with results of the present Markov analysis.

We assume that the interactions among the nonhuman primates that influence handedness are driven by a Markovian binary process, where preference for the right and left hand is represented by the self-transition probabilities, \( p_{11} = p \) and \( p_{22} = q \), respectively. Surely there exist multiple interactions among the individuals in a group. Yet, as it is a common practice in many scientific disciplines, interactions are often reduced to first neighbors, as long as this simplification does not distort their true representation. In Fig. 3 the number, \( n \), of significantly-handed individuals (pan troglodytes) was plotted against the proportion, \( \bar{p}_n \), of them who were right-handed. The confidence interval curves according to Eq. (7) that best envelope the 95% of the experimental data are drawn by adjusting the Markov memory parameters and are given by

\[
n = \frac{1.24}{(\bar{p}_n - 0.58)^2}
\]  

(11)

The confidence interval representing data of random variability (\( \nu = 1 \)), which implies an equal preference for right and left hand use, is also marked on the graph for comparison. It is clearly not fitting adequately the totality of these data, apart from these referring to individuals living in the wild.

The Markov analysis rendered the parameters, \( p = 0.64, q = 0.50, \nu = 1.15, \varphi = 58\% \) for the individuals in captivity and \( p = q = 0.50, \nu = 1, \varphi = 50\% \) for those in the wild (dark circles). The accuracy of the three parameters above is limited to 1% and so the parameters are rounded up to the second decimal digit [23]. It is a known fact that the members of a group positively influence each other with regard to handedness. Therefore, the estimated self-transition probabilities for either the use of the right or the left hand should be \( \geq 0.5 \). That fact together with the condition that the 95% of the data points should be enveloped by the confidence interval curves enables a relatively accurate adjustment of the two parameters, \( p \) and \( q \), within the limitations of the size of this database.

The current results can be understood as follows. They first imply that the studied individuals have a tendency to use both left and right hands equally, unless they are in captivity (open circles). In that latter case, the human influence modifies their handedness habits: Therefore, the initial 50% preference for the use of the right hand rises to a 58% (\( \varphi = 0.58 \)). This conclusion represents only part of the information available by the Markov memory analysis. As it confirms the independent observations in the reference study, one can be relatively confident about the effectiveness, reliability and applicability of the Markov approach used.

Apart from the static average value above, the current Markov analysis provides extra information towards the understanding of the chimpanzees interactions. The collective evaluation of independent studies in a meta-analysis presupposes that they are all measures of one the same phenomenon within samples of different size. We can, therefore, assume that the size of a group of the individuals in captivity can increase for a number of reasons (that do not concern the scope of this paper). Every newly added member would take after the handedness habit of the one close to them, whether this is a mother, a partner etc, in the following likely scheme. In this representation of handedness habits of individuals in captivity by the first-order Markov memory model, there will be a 64% probability that an added member of the group will prefer to use their right hand, as the individual close to them does. Also that 36% of them will prefer to use their left hand instead, if they are close to a right-handed individual. The use of the left hand appears not predisposed by human presence, however. There will be a balanced 50% probability to use either their left or right hand if the individual close to them is a left-handed individual.
The merit of the current Markov analysis, when applied on scatter plots, lies on the fact that it provides in one graph more information than just the static parameters directly available from other qualified statistical methods. Additionally, it offers a quantitative description of the interactions between the studied system units. In that sense, the current analysis sheds additional light on the dynamics of the system under study which otherwise constitute complicated processes. We showed here that the Markov analysis widens the scope of applicability of scatter plots, from mere graphic examinations of the presence of biases in data bases to effective tools for the understanding of the interactions among the group units.

The quantum Zeno effect

The second example treats a quantum two-state system exhibiting the so-called quantum Zeno effect (QZE) \[2, 3\], that has taken its name from the famous Zeno paradox. In the QZE the natural state transitions can be impeded by fast repetitive state-monitoring measurements. Therefore, a series of state measurements will also observe the persistence of this preferable state from which transitions have been hindered. Longer runs of that state will be recorded in a series of measurements, as Fig. 2 illustrates. Therefore, the persisting state will appear to cluster while the state that is avoided will appear to disperse in the sequence of state measurements. The clustering is marked by an increased frequency of runs of the same state (triangles in Fig. 2) as compared to the memory-free unhindered situation (circles and dotted line in Fig. 2). As the current Markov memory model mathematically handles this clustering and dispersing behavior of the two-state system, the QZE represents an attractive candidate for its application.

Specific details about the experimental settings and the generation of quantum states by Balzer et al that can be found directly in the source paper are not pertinent to the current analysis. It suffices to mention that in this quantum system the transitions between the two-states are driven by the application of an appropriate field. As the system evolves in time driven between the two states, it is possible to observe at which state the system is at a certain time by the application of a second field, the probe field. The probe field monitors the state of the system. Emission of scattered light indicates that the system is at the lower state-1, called 'on' state. Absence of scattered light indicates that the system is at the upper state-2 called 'off' state. It was observed that under certain experimental settings introducing fast repeated state measurements, the evolution of the quantum system was hindered. The suppression of transitions is observed by a clustering of the state that persists, or by the dispersion of the second state, in a long...
sequence of state measurements. It was documented in the source paper by a graph of the normalized frequency of "uninterrupted sequences" of state 'on' and state 'off', against the length of the corresponding sequences. The degree to which the probing process interacted with the drive field to impede the system’s quantum evolution was thus established.

The Markov memory approach developed in this paper goes a step further to estimate the conditional probabilities that determine the readiness of the two-state system to make a transition, or remain at the same state with each probing measurement. The frequency of uninterrupted sequences (runs) of states 'on' and 'off' were simulated, by the selection of appropriate self-transition probabilities, and fitted on the experimental data. The associated first-order Markov self-transition probabilities, \( p_{11} \) and \( p_{22} \) thus estimated at each of the three experimental settings employed, provide an insight into the state transition dynamics.

![Graph showing the normalized number of runs, \( N_{m}/N_{(m-1)} \), against the length of run, \( m \), for different settings of the system's state transitions.](image)

**FIG. 4:** Clustering and dispersion of Markovian states in the quantum Zeno effect. Experimental data for three different detuning settings. Open squares: 'on' events, state-1. Dark squares: 'off' events, state-2. Lines: computer simulated runs of a two-state Markov memory process, modulated by the self-transition probabilities, \( p_{11} \) and \( p_{22} \). See text for details.

In the first experimental setting, the system starts from state-2 at conditions favoring its persistence. The estimated self-transition Markov probabilities were \( p_{22} = 0.65 \) and \( p_{11} = 0.25 \), or equivalently \( p_{21} = 0.75 \). The latter probability implies that once the system is at state-1 it is very eager to make a transition back to state-2. On the other hand, once the system is at state-2 it prefers to remain at it by a 65% probability with every consecutive measurement. The system therefore, was lead to persistence of the 'off' state-2 due to a combination of two factors: its preference to remain at this state combined with the most effective 75% preference to avoid the competing 'on' state-1. This analysis shows therefore that a quantum transition is impeded not only because the experimental conditions favor that state, but mainly because they obstruct the rival state. Longer runs of state-2 will be observed in that case, Fig. 4a.
There were further two more experimental settings employed in this experiment. The first one, fig. 4c, favors the persistence of state-1. The application of the Markov memory analysis on these data has turned the values $p_{11} = 0.80$ and $p_{22} = 0.55$. These imply that once the system is at state-1 transitions from it are hindered, because it exhibits a strong 80% preference to remain at this state combined with a small 45% tendency to avoid state-2. Unlike in the previous experimental setting, here it is mainly the strong preference for state-1 that hinders transitions from it.

Finally, the third experimental setting brings the quantum system somewhere in between the previous two behaviors, fig. 4b. The Markov memory analysis has rendered the values $p_{11} = 0.60$ and $p_{22} = 0.65$. In this case there is an equal, yet relatively weak, 60 – 65% preference for the system to remain at either state. The evolution of the system from both states is moderately hindered, but the experimental conditions level the competition between the two rival states. Thus, the system shows evidence of similarly administered preference, where state-2 is temporarily and mildly winning.

This example shows how the QZE can alternatively be accounted for as a process driven by a Markovian type of memory. In that procedure, an insight regarding the inner dynamics of the observed effect is obtained. The agreement between simulated and experimentally observed QZE data is very good, especially for frequencies up to about $10^{-4}$.

**SUMMARY**

In this work it is shown how the natural evolution of two-state processes is shaped by the presence of Markovian memory in them. The presented mathematical formulation provides an alternative, quantitative as well as qualitative, description of a wide interdisciplinary range of processes, classified as two-state systems. The presence of Markov memory will enhance either the clustering or the dispersion of these binary states. When independent studies of a phenomenon are combined and graphically presented in scatter plots, for instance, the clustering of states is recognized by a broadening of the associated scatter plot. Conversely, the dispersion of states is marked by a narrowing of the scatter plot’s breadth. These changes occur for the reason that the Markovian memory has affected the length of runs of the same state in the two-state system. Two examples are invoked to exemplify the applicability of the Markov memory approach and the kind of prospective information that can be gained from it. The first one is a two-state system taken from anthropology, while the second one is taken from quantum theory.

In the first example, independent studies testing handedness in a mixture of cultures of chimpanzees were shown to exhibit Markov memory with regard to how strongly-related individuals may influence each other. Individuals raised in the wild exhibit equal preference for the use of both hands, regardless of what is the preference of their closest relatives. The presence of humans influences this attitude as far as the use of the right hand is concerned. There was, therefore, a mildly increased preference for the use of the right hand assessed to a 64% probability, that the closely related individuals exhibit the same preference. There was no similar effect observed in the use of the left hand, however. It appears that a new member of the group, closely associating with a left-handed individual will exhibit an independent handedness preference. Overall, a proportion of 58% of individuals across the mixture of studied groups, both in captivity and in the wild, prefer to use their right hand. These estimated tendencies in handedness, influenced by presence of humans, agree with independent anthropological studies, conveying a degree of confidence to them.

The current Markovian mathematical formulation was also applied on the experimental observation of the quantum Zeno effect (QZE) in a two-state system. The experimental evidence in this example exhibits a combination of characteristic clustering and dispersion of the binary states, across the three experimental conditions employed by the source paper of Balzer et al. Through simulations of the length of runs in each of the three experimental conditions, it was possible to observe that a synchronous clustering of the one state and dispersion of the other can be responsible for the characteristic hindering of state transitions. The probabilities that determine the likelihood for the system to make one transition to the other state, or remain at the same binary state, were thus estimated. These probabilities should not be confused with those estimating the frequency of each state in a long sequence of measurements. It was, thus, concluded that the hindering of a binary state in the quantum Zeno effect may be effected not so much by the 65% preference for state-2 through two consecutive measurements, as through the synchronous avoidance of the rival binary state-1, by a 75% probability that the rival state will precede the preferred state. Similarily, the hindering of the evolution of state-1 can be effected by a strong 80% preference for that state, winning over a weak preference 55% for the rival state.

The Markov memory analysis of two-state processes presented here treats complicated processes in terms of their first-order, first-neighbor interactions. This simplification is an initial step towards the understanding of rather complex processes that can describe the system and where comparisons are possible, are in agreement with experimental evidence. Our understanding of nature builds in steps of gradual complication. Occasionally, simplified versions of reality, rather than the more complicated or unattainable ones are tried, as long as their exercise does not conflict.
with established experimental evidence.

APPENDIX A

Following the treatment of von Mises [20] we assume that \(|p+q-1| \neq 1\) excluding the cases \(p = q = 1\) and \(p = q = 0\), as they present no interest but \(0 < p, q < 1\). The following recursion formula holds for the state probabilities

\[
p_i^{(n)} = \sum_{j=1}^{2} p_{ij} p_j^{(n-1)}, \quad i = 1, 2; \quad n = 1, 2, \ldots \tag{12}
\]

Equations \(12\) are equivalent to the iteration set up of the following homogeneous equations

\[-x_i + \sum_{j=1}^{2} p_{ij} x_j = 0, \quad i = 1, 2 \tag{13}\]

The sum of transition probabilities of each column is equal to 1. Also the absolute probabilities at every step of the system’s evolution sum up to unity

\[
\sum_{j=1}^{2} p_{ij} = 1 \tag{14}
\]

\[
\sum_{j=1}^{2} p_i^{(n)} = \sum_{j=1}^{2} p_i^{(0)} = 1 \tag{15}
\]

Since the 2x2 transition matrix having elements \(p_{ij}, i, j = 1, 2\) is regular, \(p_i^{(n)}\) tends to a unique fixed probability vector \(\vec{p}_i^{(\infty)}\) that can be estimated by solving the system of Eqs. \(13\). These lead to two non-zero solutions \(u\) and \(1 - u\), to which the probabilities \(p_n\) and \(1 - p_n\) converge after a large number of trials, \(n\). The two roots of system \(17\) are the values \(\lambda\) for which its determinant \(|P(\lambda)|\) vanishes

\[
|P(\lambda)| = \begin{vmatrix} p - \lambda & 1 - q \\ 1 - p & q - \lambda \end{vmatrix} = 0 \tag{16}
\]

where \(\lambda\) cannot be greater than 1 in absolute value. In this case the two roots are

\[
\lambda_1 = p + q - 1
\]

\[
\lambda_2 = 1 \tag{17}
\]

The equations \(12\), and \(17\), can then be written as

\[
u_1 = p \cdot u_1 + (1 - q) \cdot u_2
\]

\[
u_2 = (1 - p) \cdot u_1 + q \cdot u_2 \tag{18}\]

yielding the two not uniquely determined solutions \(u\) and \(1 - u\)

\[
u \equiv \varphi = \lim_{n \to \infty} p_n = \frac{1 - q}{2 - (p + q)}
\]
1. $1-u \equiv 1-\wp = \frac{1-p}{2-(p+q)}$ (19)

Equations (19) have an important consequence. The probabilities of finding the system at either binary state after $n$ trials converge to $\wp = 50\%$, provided the two self-transition probabilities are equal, $p = q$, and regardless if they are different from $50\%$. In other words, in the long run binary states (A and B) will occur in the Markov chain at the same frequency as if no memory was involved. From the point of view of long-run state probabilities the Markov process will resemble a memoryless Bernoulli case. We shall next estimate the frequency of state A.

As stated in the text, the number $x_\nu = 1$ is associated with the occurrence of state A after a measurement and $x_\nu = 0$ is associated with the occurrence of state B. The expectation value of $x_\wp$, i.e., $p_\wp$, will be the probability that the outcome of the $\nu$th measurement will be the number 1. The expectation value of all the '1' states present in a Markov sequence of $n$ trials will be then

$$E[x_1 + x_2 + ... + x_n] = p_1 + p_2 + ... + p_n$$ (20)

and the expected value of proportion of ones in the chain in will be

$$E\left[\frac{x_1}{n}\right] = \frac{p_1 + p_2 + ... + p_n}{n} = \bar{p}_n$$ (21)

The recursion formula (16) can be written

$$p_n = 2 \sum_{j=1}^{2} p_{1j} p_j^{(n-1)} = p_{11} p_1^{(n-1)} + p_{12} p_2^{(n-1)} = a \cdot p_{n+1} + b$$ (22)

where $a = p + q - 1 \neq 1$, and $b = \frac{1-p}{2-(p+q)}$ since $0 < p < 1$ and $0 < q < 1$. Given that the initial value of $p_n$ is $p_1^{(0)} = p_1$, the recursion formula (26) yields

$$p_n = a^{(n-1)} p_1 + a^{n-2} + a^{n-3} + ... + a + 1 \cdot b = a^{n-1} p_1 + \frac{1-a^{n-1}}{1-a} \cdot b$$ (23)

or

$$p_n = a^{n-1} [p_1 - \frac{1-q}{2-(p+q)} + \frac{1-q}{2-(p+q)}] = a^{n-1} [p_1 - \wp] + \wp$$ (24)

where $\wp$ is defined in (23). The average expected proportion of state A, $p_n$, in the chain of $n$ trials would be written as

$$\bar{p}_n = \frac{1}{n} \sum_{i=1}^{n} p_i = \frac{1}{n} (p_1 + p_2 + .. + p_n) = \wp + \frac{p_1 - \wp}{n} \cdot \frac{1-a^n}{1-a}$$ (25)

and

$$\bar{p}_n \rightarrow_{n \rightarrow \infty} \wp$$ (26)

The absolute probability of finding the system at state A after $n$ measurements of its state is written $p_1^{(n)} = p_n$ and the probability of finding it at state B is $p_2^{(n)} = 1 - p_n$. This is equivalent to saying that the $n$-th measurement of the system's state has a probability $p_n$ of finding it at state A. Since $p^{(n)} = p_{11}^{(n)}$ and $q^{(n)} = p_{12}^{(n)}$ the recursion formula (16) applies to the transition probabilities between states too.
The recursion formula then yields, since \( p^{(0)}_{11} = 1 \) and \( p^{(0)}_{12} = 0 \)

\[
p^{(n)} = a^n p^{(0)} + \left[ \frac{a^{n-1} + a^{n-2} + \ldots + a + 1}{n \text{ terms}} \right] \cdot b = a^n + \frac{1 - a^n}{1 - a} \cdot b = a^n \cdot [1 - \varphi] + \varphi
\]

and

\[
q^{(n)} = a^n q^{(0)} + \left[ \frac{a^{n-1} + a^{n-2} + \ldots + a + 1}{n \text{ terms}} \right] \cdot b = a^n \cdot [1 - \varphi]
\]

If the system is at state A the probability that after \( n \) steps, where \( n \) is very large, the measurement will yield again state A is \( \varphi \): \( p^{(n)} \to \varphi \) and to yield state B is zero \( q^{(n)} \to 0 \).

Von Mises has estimated the standard deviation of the mean value \( \varphi \), in other words, the asymptotic value of the standard deviation of the proportion of state A, \( \bar{p}_n \), in the sequence of \( n \) trials as

\[
\sigma = \sqrt{\frac{\varphi(1 - \varphi)}{n}} \cdot \sqrt{\frac{1 + a}{1 - a}} = \sigma_o \cdot \nu
\]

The proportion of state A in sequences of trials of length \( n \) of a Markov process with memory scatters about the asymptotic value \( \varphi \), while their scatter has been modulated with respect to the memory-less case

\[
\sigma_o = \sqrt{\frac{\varphi(1 - \varphi)}{n}}
\]

The modulating variance factor in (34)

\[
\nu = \sqrt{\frac{1 + a}{1 - a}}
\]

takes values either above or below 1 depending on the self transition probabilities according to

\[
\nu > 1 \Rightarrow p + q > 0.5
\]

\[
\nu < 1 \Rightarrow p + q < 0.5
\]
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[33] The sum of the converging geometric series has now $n$ terms.