A proof of the Brill–Noether method from scratch*

Elena Berardini  
CNRS; Institut de Mathématiques de Bordeaux  
Université de Bordeaux  
Talence, France, F 33405  
Department of Mathematics and Computer Science  
Eindhoven University of Technology  
Eindhoven, the Netherlands, 5612 AZ  
elena.berardini@math.u-bordeaux.fr

Alain Couvreur  
Inria & Laboratoire d’informatique de l’École polytechnique (LIX, UMR 7161)  
CNRS, École polytechnique, Institut Polytechnique de Paris  
Palaiseau, France, 91120  
alain.couvreur@inria.fr

Grégoire Lecerf  
Laboratoire d’informatique de l’École polytechnique (LIX, UMR 7161)  
CNRS, École polytechnique, Institut Polytechnique de Paris  
Palaiseau, France, 91120  
gregoire.lecerf@lix.polytechnique.fr

Abstract  
In 1874 Brill and Noether designed a seminal geometric method for computing bases of Riemann–Roch spaces. From then, their method has led to several algorithms, some of them being implemented in computer algebra systems. The usual proofs often rely on abstract concepts of algebraic geometry and commutative algebra. In this paper we present a short self-contained and elementary proof that mostly needs Newton polygons, Hensel lifting, bivariate resultants, and Chinese remaindering.

Keywords: Algebraic curves, Riemann–Roch spaces, Brill–Noether method, Hensel lemmas, Newton polygons

1 Introduction  
Riemann–Roch spaces are vector spaces of rational functions that satisfy some conditions on the localization and the multiplicity of their zeros and poles. These spaces are a cornerstone of modern applications of algebra to various areas of computer science. This paper presents the seminal method due to Brill and Noether to compute Riemann–Roch spaces. Our approach here is mostly dedicated to undergraduate students: it relies on Newton polygons, Hensel lifting, bivariate resultants, and Chinese remaindering.

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This introduction begins with elementary problems in order to motivate the study of Riemann–Roch spaces and to sketch the Brill–Noether method. Section 2 gathers well known elementary definitions and results from algebra, while Section 3 is dedicated to the Hensel lemma and its usual extensions. These tools are central in studying how valuations extend to field extensions and to introduce the notion of uniformizing parameter, that is the goal of Section 4. All these results allow us to define places and divisors on curves in Section 5. Finally Section 6 is devoted to the Brill–Noether algorithm. The proofs are self-contained. References to algorithms in the literature are gathered in our final Section 7.

Let \( \mathbb{K} \) be a field. The ring of univariate polynomials over \( \mathbb{K} \) is written \( \mathbb{K}[x] \), and we let \( \mathbb{K}[x]_{\leq d} := \{ f \in \mathbb{K}[x] : \deg f \leq d \} \). The ring of power series over \( \mathbb{K} \) is written \( \mathbb{K}[[x]] \) and its field of fractions, called the field of Laurent series, is denoted by \( \mathbb{K}((x)) \).

Let \( \mathbb{D} \) be an integral domain. A valuation \( v \) on \( \mathbb{D} \) is a map from \( \mathbb{D} \) to \( \Gamma \cup \{ \infty \} \) such that:

- \( \Gamma \) is an abelian additive group endowed with a total order written \(<\);  
- \( v(a) = \infty \) if and only if \( a = 0 \);  
- \( v(ab) = v(a) + v(b) \);  
- \( v(a + b) \geq \min(v(a), v(b)) \).

The field \( \mathbb{K}((x)) \) is endowed with the natural valuation in \( x \), written \( \text{val}_x \), normalized such that \( \text{val}_x(x) = 1 \), so \( \text{val}_x \left( \sum_{i \in \mathbb{Z}} a_i x^i \right) := \min(i \in \mathbb{Z} : a_i \neq 0) \).

1.1 Riemann–Roch spaces

Let \( \alpha_1, \ldots, \alpha_r \) be distinct values in \( \mathbb{K} \). As a basic fact of algebra, it is well known that the set of polynomials \( g \in \mathbb{K}[x] \) that vanish simultaneously at \( \alpha_1, \ldots, \alpha_r \) with respective multiplicities \( m_1, \ldots, m_r \) is the ideal of \( \mathbb{K}[x] \) generated by \( (x - \alpha_1)^{m_1} \cdots (x - \alpha_r)^{m_r} \). The polynomials in this ideal that satisfy a given degree upper bound constitute a finite dimensional \( \mathbb{K} \)-vector space, of which a basis is easily given. This setting extends to rational functions, that is, fractions of polynomials, and signed integers \( m_i \). In this context, we can pose the following problem.

**Problem A** Let \( \alpha_1, \ldots, \alpha_r \) be distinct values in \( \mathbb{K} \), let \( m_1, \ldots, m_r \) be signed integers, and set \( D_A := \{ (\alpha_1, m_1), \ldots, (\alpha_r, m_r) \} \). Compute the \( \mathbb{K} \)-vector space \( \mathcal{L}(D_A) \) of rational functions \( g/h \in \mathbb{K}(x) \) such that \( \deg g \leq \deg h \), the valuation of \( g/h \) regarded in \( \mathbb{K}((x - \alpha_i)) \) is \( \geq -m_i \) for \( i = 1, \ldots, r \), and that for any other value \( \alpha \) in \( \mathbb{K} \) the valuation of \( g/h \) regarded in \( \mathbb{K}((x - \alpha)) \) is \( \geq 0 \).

It is clear that \( \mathcal{L}(D_A) \) is a \( \mathbb{K} \)-vector space. A basis of \( \mathcal{L}(D_A) \) can be computed as follows. We let \( d := \sum_{i=1, m_i > 0}^r m_i \) and \( h(x) := \prod_{i=1, m_i > 0} (x - \alpha_i)^{m_i} \). If \( a/b \in \mathcal{L}(D_A) \) with \( a \) and \( b \) coprime, then \( b \) divides \( h \), hence there exists a unique polynomial \( g \) of degree \( \leq d \) such that \( g/h = a/b \). This shows that \( \mathcal{L}(D_A) \) has finite dimension and that \( h \) can be taken as a common denominator of a basis of \( \mathcal{L}(D_A) \). So we are led to seek for a basis \( g_1, \ldots, g_\ell \) of polynomials \( g \in \mathbb{K}[x]_{\leq d} \) such that the valuation of \( g \) regarded in \( \mathbb{K}[[x - \alpha_i]] \) is \( \geq -m_i \) for \( i = 1, \ldots, r \). This task reduces to solving a linear system in the \( d + 1 \) unknown coefficients of \( g \). If the number of equations, namely \( \sum_{i=1, m_i < 0}^r (-m_i) \), does not exceed the number of unknowns, then the set of non-zero solutions is not empty. More precisely we may take

\[
 g_1(x) := \prod_{i=1, m_i < 0} (x - \alpha_i)^{-m_i} \quad \text{and} \quad g_i := x^{i-1} g_1 \quad \text{for} \quad i = 2, \ldots, \ell,
\]

where

\[
 \ell := (d + 1) - \sum_{i=1, m_i < 0}^r (-m_i) = 1 + \sum_{i=1}^r m_i.
\]
Finally, $g_1/h, \ldots, g_t/h$ is a basis of $\mathcal{L}(D_A)$.

Problem A naturally extends to the projective setting: the affine line is replaced by the projective line $\mathbb{P}^1(\mathbb{K})$ over $\mathbb{K}$. Rational functions defined on $\mathbb{P}^1(\mathbb{K})$ are either 0 or of the form $A/B$ where $A$ and $B$ are homogeneous polynomials in $\mathbb{K}[x, z]$ of same degree. The set of such functions is written $\mathbb{K}(\mathbb{P}^1(\mathbb{K}))$. If $\zeta_i = (\alpha_i : 1)$ is a point in the affine chart $z = 1$ of $\mathbb{P}^1(\mathbb{K})$, then the valuation of $G/H$ at $\zeta_i$ is the valuation of $G(x, 1)/H(x, 1)$ regarded in $\mathbb{K}((x - \alpha_i))$. If $\zeta_i$ is the point $(1 : 0)$, called the point “at infinity”, then the valuation $G/H$ at $\zeta_i$ is the valuation of $G(1, z)/H(1, z)$ regarded in $\mathbb{K}((z))$.

**Problem B** Let $\zeta_1, \ldots, \zeta_r$ be distinct points on the projective line $\mathbb{P}^1(\mathbb{K})$, let $m_1, \ldots, m_r$ be signed integers, and set $D_B := \{(\zeta_1, m_1), \ldots, (\zeta_r, m_r)\}$. Compute the $\mathbb{K}$-vector space $\mathcal{L}(D_B)$ of rational functions $G/H \in \mathbb{K}(\mathbb{P}^1(\mathbb{K}))$ such that the valuation of $G/H$ at $\zeta_i$ is $\geq -m_i$ for $i = 1, \ldots, r$, and that for any other point $\zeta$ in $\mathbb{P}^1(\mathbb{K})$ the valuation of $G/H$ at $\zeta$ is $\geq 0$.

If all the $\zeta_i$ are in the affine chart $z = 1$, then we may write them as $\zeta_i = (\alpha_i : 1)$ and reduce Problem B to Problem A: let $g_1/h, \ldots, g_t/h$ be a basis of $\mathcal{L}(D_A)$, then with $d := \deg h$ and $H := z^d h(x/z)$, we verify routinely that $z^d g_1(x/z)/H, \ldots, z^d g_t(x/z)/H$ is a basis of $\mathcal{L}(D_B)$. If one of the $\zeta_i$ is $(1 : 0)$ and if the cardinality of $\mathbb{K}$ is sufficiently large, then we may apply a sufficiently generic linear change of variables

$$P_i = (\gamma_i : \delta_i) \mapsto (a\gamma_i + b\delta_i : c\gamma_i + d\delta_i) = \frac{(a\gamma_i + b\delta_i)}{(c\gamma_i + d\delta_i) : 1}$$

with $ad - bc \neq 0$ and such that $c\gamma_i + d\delta_i \neq 0$ for all $i = 1, \ldots, r$, in order to reduce again Problem B to Problem A.

**Remark 1** If one reformulates Problem A into the projective setting, the condition $\deg g \leq \deg h$ is equivalent to having a nonnegative valuation at infinity. Thus, Problem B can be regarded as a generalization of Problem A where any condition at infinity can be imposed.

Alternatively, we may solve Problem B directly as follows. We still set $d := \sum_{i=1}^r m_i$ but set $H(x, z) := \prod_{i=1}^r (\beta_i x - \alpha_i z)^m_i$, where $\alpha_i$ and $\beta_i$ represent coordinates of $\zeta_i = (\alpha_i : \beta_i)$. If $A/B \in \mathcal{L}(D_B)$ with $A$ and $B$ coprime, then $B$ divides $H$ and there exists a unique homogeneous polynomial $G$ of degree $d$ such that $G/H = A/B$. This shows that $\mathcal{L}(D_B)$ has finite dimension and there $H$ can be chosen as a common denominator of a basis of $\mathcal{L}(D_B)$. So it remains to seek for a basis $G_1, \ldots, G_t$ in $\mathbb{K}[x, y, z]$ of homogeneous polynomials $G$ of degree $d$ such that the valuation of $G/H$ at $P_i$ is $\geq -m_i$ for $i = 1, \ldots, r$. This task reduces to solving a linear system in the $d + 1$ unknown coefficients of $G$. Finally $G_1/H, \ldots, G_t/H$ is a basis of $\mathcal{L}(D_B)$, where $\ell$ is the same as in [1].

**Example** A particularly simple case of interest of Problem B is the following: $r = 1$, $\zeta_1 = (1 : 0)$ and $m_1 \geq 0$. Then, the above direct method shows that $x^{m_1} / z^{m_1}, x^{m_1-1} / z^{m_1-1}, \ldots, x / z, 1$ is a basis of $\mathcal{L}(D_B)$.

Problem B can be further generalized to plane algebraic curves. Let $F \in \mathbb{K}[x, y, z]$ be an irreducible homogeneous polynomial. Let $\mathbb{P}^2(\mathbb{K})$ stands for the projective plane over $\mathbb{K}$. The set of zeros of $F$ in $\mathbb{P}^2(\mathbb{K})$ is written $C$; this is an algebraic projective plane curve. A point $\zeta$ of $C$ is called singular if the partial derivatives of $F$ vanish simultaneously at $\zeta$, and non-singular otherwise. The set of singular points of a curve is called its singular locus. The set of rational functions defined on $C$ over $\mathbb{K}$, written $\mathbb{K}(C)$, is the set of functions that are either 0 or of the form $A/B$ where $A$ and $B$ are homogeneous polynomials of the same degree, with $B$ prime to $F$, and subject to the equivalence relation

$$A/B \sim A'/B' \iff AB' - A'B \in (F).$$

At a non-singular point $\zeta = (\zeta_x : \zeta_y : \zeta_z)$ of $C$, and up to an occasional linear change of variables, we may assume that $\zeta_z = 1$ and $\frac{\partial F}{\partial y}(\zeta) \neq 0$, so the implicit function theorem ensures that $C$ is locally defined by a power series

$$\varphi(x) := \zeta_y + c_1(x - \zeta_x) + c_2(x - \zeta_x)^2 + \cdots \in \mathbb{K}[[x - \zeta_x]]$$
that satisfies \( F(x, \varphi(x), 1) = 0 \); see [21] Chapter 7, Section 7.2, Corollary 7.4]. If \( G/H \in \mathbb{K}(C) \), then its valuation at \( \zeta \) is defined by

\[
\text{val}_\zeta(G/H) := \text{val}_{\zeta}(G(x, \varphi(x), 1)/H(x, \varphi(x), 1));
\]

we will see that this definition is essentially independent of the choice of coordinates. Defining valuations requires several algebraic constructions. The set \( H \) extends the above elementary ideas: one first computes a common denominator \( H \in \mathbb{K}[x, y, z] \) of \( \mathcal{L}(D) \) such that the valuation of \( G/H \) at \( \zeta_i \) is \( \geq -m_i \) for \( i = 1, \ldots, r \), and that for any other point \( \zeta \) on \( C \) the valuation of \( G/H \) at \( \zeta \) is \( \geq 0 \).

Problem C Let \( \zeta_1, \ldots, \zeta_r \) be distinct non-singular points on the projective plane curve \( C \), let \( m_1, \ldots, m_r \) be signed integers, and set \( D := \{(\zeta_1, m_1), \ldots, (\zeta_r, m_r)\} \). Compute the set \( \mathcal{L}(D) \) of rational functions \( G/H \in \mathbb{K}(C) \) such that the valuation of \( G/H \) at \( \zeta_i \) is \( \geq -m_i \) for \( i = 1, \ldots, r \), and that for any other point \( \zeta \) on \( C \) the valuation of \( G/H \) at \( \zeta \) is \( \geq 0 \). Problem B corresponds to the particular case of Problem C where \( F(x, y, z) = y \), so \( C \) is a projective line in the projective plane. However Problem C turns out to be much more difficult in general, and requires several algebraic constructions. The set \( D \) represents a divisor of \( C \); see definition in Section 5 \( \mathcal{L}(D) \) is the Riemann–Roch space of \( D \).

In their 1874 paper [9] Brill and Noether designed a method to compute a \( \mathbb{K} \)-basis of \( \mathcal{L}(D) \), that extends the above elementary ideas: one first computes a common denominator \( H \in \mathbb{K}[x, y, z] \) of \( \mathcal{L}(D) \) and then one deduces a numerator basis via linear algebra. The computation of \( H \) depends on the singular locus of \( C \). Presenting the Brill and Noether method for computing Riemann–Roch spaces is the central goal of this paper; see Section 6.

1.2 Application to error correction

For applications it is usual to take \( \mathbb{K} = \mathbb{F}_q \), the finite field with \( q \) elements. Reed–Solomon error-correcting codes are a popular technique to represent data in the form of vectors such that the data can be recovered even if some vector coordinates are corrupted during transmission or storage of the data. Let \( k < n \) be integers, let \( \alpha_1, \ldots, \alpha_n \) be distinct points in \( \mathbb{K} \). Let \( (a_1, \ldots, a_k) \in \mathbb{K}^k \) be the data to encode. We first interpolate the unique polynomial \( f \in \mathbb{K}[x]_{<k} \) such that \( f(\alpha_i) = a_i \) for \( i = 1, \ldots, k \), then we encode the data as

\[
(f(\alpha_1), \ldots, f(\alpha_n)).
\]

This encoding can be regarded as a redundant representation of \( (a_1, \ldots, a_k) \); even if a few values \( f(\alpha_i) \) are lost, one can recover \( f \) and therefore \( (a_1, \ldots, a_k) \). This recovering task is called decoding. The defining vector space of this Reed–Solomon code is the image of the injective map

\[
\mathcal{L}(\{(1 : 0), k − 1\}) \hookrightarrow \mathbb{K}^n
\]

\[
h(x, z) \mapsto (h(\alpha_1, 1), \ldots, h(\alpha_n, 1)).
\]

As seen in Example 1 functions of \( \mathcal{L}(\{(1 : 0), k − 1\}) \) are in one-to-one correspondence with \( \mathbb{K}[x]_{<k} \). Unfortunately they suffer from a limitation: the cardinality of \( \mathbb{K} \) must be \( \geq n \).

So-called Algebraic Geometry (AG) codes are a generalization of Reed–Solomon codes which enjoys the same properties with less limitation. They are based on the Riemann–Roch spaces introduced in Problem C. Let \( G_1/H, \ldots, G_k/H \) represent a basis of \( \mathcal{L}(D) \), let \( \xi_1, \ldots, \xi_n \) be distinct points on \( C \) disjoint from \( \zeta_1, \ldots, \zeta_r \), and let

\[
\mathcal{E}_n : \quad \mathcal{L}(D) \rightarrow \mathbb{K}^n
\]

\[
F/G \mapsto ((F/G)(\xi_1), \ldots, (F/G)(\xi_n)).
\]

With \( D := D \cup \{(\xi_1, −1), \ldots, (\xi_n, −1)\} \), the kernel of \( \mathcal{E}_n \) is \( \mathcal{L}(\bar{D}) \). The dimensions of \( \mathcal{L}(D) \) and \( \mathcal{L}(\bar{D}) \) can be estimated using the Riemann–Roch theorem [33] Chapter 1, Section 5. We will not enter into these.
The favorable situation for correcting codes occurs when there exists an integer \( k < n \) such that the map \( E_k \) is one-to-one. If \( K \) is a finite field \( \mathbb{F}_q \), then the maximum number \( n \) of evaluation points is the number of points of \( C \) with coordinates in \( \mathbb{F}_q \), called the rational points of the curve. An algebraic curve can have up to \( q + 1 + 2g\sqrt{q} \) rational points over the finite field \( \mathbb{F}_q \), where \( g \) is a nonnegative number called the genus of \( C \); see also Remark 3. There exist curves \( C \), said to be maximal, whose number of rational points attains this upper bound. Such curves, but also any curve with many rational points, allow to construct AG codes longer than the Reed–Solomon ones over the same finite field. AG codes became particularly famous after the work of Tsfasman, Vladuţ, and Zink [44] proving that some sequences of codes built from modular curves have better asymptotic performances than random codes.

The construction of AG codes is one of the major motivations for computing Riemann–Roch spaces. Nowadays AG codes are used in information theory protocols, such as secure multi-party computation and zero-knowledge proofs; see [7, 19], they can be used to assert resilience in distributed storage systems (e.g. clouds) [6] or to build families of quantum error-correcting codes with good parameters [37]. See [16] for a recent survey on AG codes and their applications. Of course, Riemann–Roch spaces are useful in classical contexts too. For instance, these spaces occur in arithmetic operations on Jacobian of curves [35], in number theory, and algebraic geometry. To end this section we briefly explain how Riemann–Roch spaces can be used to share secrets.

### 1.3 Application to secret sharing

A secret sharing scheme is a way for \( n \) players to carry parts of a secret message \( S \) such that the knowledge of all the parts allows to reconstruct the original message, but no proper subset of these parts leaks anything about \( S \). A secret sharing scheme with threshold \( t \) is a variation of the latter where the knowledge of \( t \) parts of the message is enough to obtain it, but any subset of \( t - 1 \) parts discloses no information at all on \( S \).

Shamir’s secret sharing scheme [41] provides an elegant construction based on Lagrange interpolation. The message \( S \) is given as an element of a finite field \( \mathbb{F}_q \), say \( a_0 \). Then, one draws a random polynomial \( f \) in \( \mathbb{F}_q[x] \) with constant coefficient \( a_0 \), and distinct points \( \alpha_1, \ldots, \alpha_n \) in \( \mathbb{F}_q \). The shared parts of the secret are \((\alpha_1, f(\alpha_1)), \ldots, (\alpha_n, f(\alpha_n))\). Hence, any subset of \( t \) players can recover the message by interpolation.

Shamir’s scheme shares multiple properties with Reed–Solomon codes, including the limitation on the number of players. Replacing polynomials with rational functions in a Riemann–Roch space, and elements over a finite field with rational points on an algebraic curve, allows to reduce drastically the size of the field with respect to the number of players \( n \). This idea was introduced in [14] and then developed in [12, 13, 19]. For instance, using maximal elliptic curves (that is when the genus of the curve is \( g = 1 \)) over a finite field \( \mathbb{F}_q \) allows to gain \( 2\sqrt{q} - 1 \) players more with respect to Shamir’s classical scheme.

**Remark 2** Actually, when replacing Reed–Solomon codes by AG ones, the secret sharing thresholds behave slightly differently. Precisely, an AG code in \( \mathbb{F}_q^{n} \) yields a secret sharing scheme with two thresholds \( t_1 \geq t_2 \) such that any coalition of \( t_1 \) players can recover the secret while any coalition of \( t_2 \) players or less cannot get any information on the secret. In the Reed–Solomon setting the two thresholds are consecutive, while they are not in the AG setting; see [15, Lemma 8] for further details.

### 2 Prerequisites

This section gathers usual definitions and constructions related to zero and one dimensional Zariski closed sets, in order to introduce plane algebraic curves and study their intersection. The proofs are not all repeated. Instead, we refer the reader to textbooks. In what follows, we let \( \mathbb{K} \) be a field and \( \overline{\mathbb{K}} \) be a fixed algebraic closure of it.
2.1 Zariski closed sets

The affine space of dimension \( n \) over \( \overline{K} \) is written \( \mathbb{A}^n \). For a set \( S \) of polynomials in \( \mathbb{K}[x_1, \ldots, x_n] \), we write \( \mathcal{V}_A(S) \) for the Zariski closed set in \( \mathbb{A}^n \) defined as the common zeros of the elements of \( S \), that is
\[
\mathcal{V}_A(S) := \{ \zeta \in \mathbb{A}^n : f(\zeta) = 0, \forall f \in S \}.
\]
If \( \mathcal{V}_A(S) \) is finite and non-empty, then it is said to have dimension zero. An ideal \( I \) is said to be zero dimensional if the dimension of \( \mathcal{V}_A(I) \) is zero.

The projective space of dimension \( n \) over \( \overline{K} \) is denoted by \( \mathbb{P}^n \). For a set \( S \) of homogeneous polynomials in \( \mathbb{K}[x_0, \ldots, x_n] \), we write \( \mathcal{V}_P(S) \) for the Zariski closed set in \( \mathbb{P}^n \) defined as the common zeros of the elements of \( S \), that is
\[
\mathcal{V}_P(S) := \{ \zeta \in \mathbb{P}^n : F(\zeta) = 0, \forall F \in S \}.
\]
If \( \mathcal{V}_P(S) \) is finite and non-empty, then it is said to have dimension zero. For a non-constant homogeneous polynomial \( F \in \mathbb{K}[x, y, z] \) the set \( \mathcal{V}_P(F) \) is called an algebraic curve. Curves are one dimensional Zariski closed sets.

If \( M := \mathbb{K}[x_1, \ldots, x_n] \) is a polynomial ring and \( \zeta \) a point in \( \mathbb{K}^n \), then \( M_\zeta \) represents the local ring of the rational functions \( A/B \) in \( \mathbb{K}(x_1, \ldots, x_n) \) such that \( B(\zeta) \neq 0 \). The following classical result will be needed for \( n = 2 \) variables.

**Proposition 1** ([22 Chapter 2, Section 9, Proposition 6], [17 Chapter 4, Section 2], or [36 Chapter 4]) Assume that \( \mathbb{K} \) is algebraically closed. Let \( I \) be a zero dimensional ideal in \( M := \mathbb{K}[x_1, \ldots, x_n] \). Then, we have
\[
M/I \cong \bigoplus_{\zeta \in \mathcal{V}_A(I)} M_\zeta/(IM_\zeta),
\]
where each summand is a local \( \mathbb{K} \)-algebra of finite dimension.

2.2 Resultant

Let \( D \) be an integral domain. The resultant of two polynomials \( f \) and \( g \) in \( D[x] \) of respective degrees \( m \) and \( n \) is defined as the determinant of the map
\[
D[x]_m \times D[x]_n \rightarrow D[x]_{m+n}, \quad (u, v) \mapsto uf + vg,
\]
and is written \( \text{Res}(f, g) \). By construction, \( \text{Res}(f, g) \) belongs to the ideal \( (f, g) \). If \( f, g \) are multivariate polynomials then their resultant with respect to an indeterminate \( x \) is denoted by \( \text{Res}_x(f, g) \). We will need the following propositions.

**Proposition 2** ([8 Chapitre 6, Corollaire 6.3] or [15 Chapter 3, Section 6, Proposition 3]) Two polynomials \( f, g \) in \( D[x] \) have no common factor of degree \( \geq 1 \) if, and only if, \( \text{Res}(f, g) \neq 0 \).

**Proposition 3** ([8 Chapitre 6, Lemme 6.9] or [17 Chapter 3, Proposition 1.5]) If \( f \) is monic, then \( \text{Res}(f, g) \) is the determinant of the multiplication endomorphism by \( g \) in \( \mathbb{K}[x]/(f(x)) \).

**Proposition 4** [8 Chapitre 6, Corollaire 6.6] For all polynomials \( f, g, \) and \( h \) in \( D[x] \) we have
\[
\text{Res}(fg, h) = \text{Res}(f, h) \text{Res}(g, h).
\]
The resultant is a long-established tool for polynomial system solving. Let \( F, G \in \mathbb{K}[x, y, z] \) be two non-constant homogeneous coprime polynomials, then the intersection of the curves defined by \( F \) and \( G \) is \( \mathcal{V}_F(F) \cap \mathcal{V}_G(G) = \mathcal{V}_{(F, G)} \). We now recall a simple method for computing curve intersection, that makes use of the resultant of \( F \) and \( G \) regarded in \( \mathbb{K}[x, z][y] \). We set
\[
R(x, z) := \text{Res}_y(F(x, y, z), G(x, y, z)),
\]
and introduce specific technical assumptions:

\begin{itemize}
  \item \( \text{A}_1 \) \( \deg_y F = \deg F \) and \( F \) is monic in \( y \);
  \item \( \text{A}_2 \) \( \deg_x R(x, z) = \deg(R(x, z)) \),
\end{itemize}

where \( \deg_y F \) and \( \deg_x R \) represent the partial degrees of \( F \) in \( y \) and of \( R \) in \( x \).

**Lemma 1** Let \( F \) and \( G \) be non-constant coprime homogeneous polynomials in \( \mathbb{K}[x, y, z] \) such that \( \text{A}_1 \) and \( \text{A}_2 \) hold. Then, the following assertions hold:

\begin{itemize}
  \item [i.] \( \mathcal{V}_F(F, G) \) is contained in the affine chart defined by \( z = 1 \);
  \item [ii.] \( \mathcal{V}_F(F(x, y, 1), G(x, y, 1)) = \bigcup_{R(\zeta_x, 1)=0} \{(\zeta_x, \zeta_y) : F(\zeta_x, \zeta_y, 1) = G(\zeta_x, \zeta_y, 1) = 0\} \);
  \item [iii.] \( \mathcal{V}_F(F(x, y, 1), G(x, y, 1)) \) is finite and non-empty.
\end{itemize}

**Proof** Let \( (\zeta_x : \zeta_y) \in \mathbb{P}^1 \), and let \( M(x, z) \) denote the matrix of the multiplication endomorphism by \( G(x, y, z) \) in \( \mathbb{K}[x, z][y]/(F(x, y, z)) \) in the basis \( 1, y, \ldots, y^{\deg_y F - 1} \). Thanks to \( \text{A}_1 \), the entries of \( M(x, z) \) belong to \( \mathbb{K}[x, z] \) and the matrix of the multiplication endomorphism by \( G(\zeta_x, y, \zeta_z) \) in \( \mathbb{K}[y]/(F(\zeta_x, y, \zeta_z)) \) in the basis \( 1, y, \ldots, y^{\deg_y F - 1} \) coincides to \( M(\zeta_x, \zeta_z) \). Proposition \( \text{A}_2 \) implies that
\[
R(\zeta_x, \zeta_z) = \text{Res}_y(F(\zeta_x, y, \zeta_z), G(\zeta_x, y, \zeta_z)).
\]

If \( (\zeta_x : \zeta_y : 0) \in \mathcal{V}_F(F, G) \) then by Proposition \( \text{A}_2 \) we have \( R(\zeta_x, 0) = 0 \), hence \( \text{A}_2 \) implies \( \zeta_x = 0 \), so \( \text{A}_1 \) yields \( \zeta_y = 0 \), whence property \( \text{[i]} \). As for \( \text{[ii]} \), by Proposition \( \text{A}_2 \) we know that \( \zeta_x \) is a root of \( R(\zeta_x, 1) = 0 \) if, and only if, \( F(\zeta_x, y, 1) \) and \( G(\zeta_x, y, 1) \) share at least one common root in \( \mathbb{K} \). Finally, thanks to \( \text{A}_1 \) the polynomial \( F(\zeta_x, y, 1) \) is non-zero and hence admits a finite number of roots in \( \mathbb{K} \), whence \( \text{[iii]} \).

Assumptions \( \text{A}_1 \) and \( \text{A}_2 \) can be recovered after suitable changes of coordinates, as stated in the following lemma.

**Lemma 2** With the notation as above, let \( (\alpha, \beta, \gamma) \in \mathbb{K}^3 \). If \( F(\alpha, 1, \beta) \neq 0 \) then the partial degree of \( F(x + \alpha y, y, z + \beta y) \) in \( y \) equals \( \deg F \). If \( R(1, \gamma) \neq 0 \) then the partial degree of \( R(x, z + \gamma x) \) in \( x \) equals \( \deg R \).

**Proof** Since \( F \) is homogeneous, the coefficient of \( y^{\deg F} \) in \( F(x + \alpha y, y, z + \beta y) \) is \( F(\alpha, 1, \beta) \). The second assertion about \( R \) is obtained in the same manner.

If the cardinality of \( \mathbb{K} \) is sufficiently large, then there exists a triple \( (\alpha, \beta, \gamma) \in \mathbb{K}^3 \) such that \( F(\alpha, 1, \beta) \neq 0 \) and \( R(1, \gamma) \neq 0 \). This simple fact will be sufficient in the sequel. Nevertheless, let us mention a stronger assertion, that is a direct consequence of the classical Schwarz–Zippel lemma [24, Lemma 6.44]: if \( S \) is a finite subset of \( \mathbb{K} \), then the probability that a triple \( (\alpha, \beta, \gamma) \) taken uniformly at random in \( S^3 \) satisfies \( F(\alpha, 1, \beta)R(1, \gamma) \neq 0 \) is \( \geq 1 - (\deg F + \deg R)/|S| \).

**Proposition 5** Let \( F \) and \( G \) be non-constant homogeneous polynomials in \( \mathbb{K}[x, y, z] \). If \( F \) and \( G \) are coprime then \( \mathcal{V}_G(F, G) \) is finite and non-empty.

**Proof** This is a consequence of Lemmas \( \text{[1]} \) and \( \text{[2]} \) used over \( \bar{\mathbb{K}} \) (that is infinite). \( \square \)
3  Hensel lemmas

The usual Hensel lemma concerns polynomials \( f \in \mathbb{K}[[x]][y] \) monic in \( y \): given \( g_0 \) and \( h_0 \) in \( \mathbb{K}[y] \) monic and coprime such that \( f = g_0 h_0 + O(x) \), then there exist unique monic polynomials \( g \) and \( h \) in \( \mathbb{K}[[x]][y] \) such that \( f = gh \), \( g = g_0 + O(x) \) and \( h = h_0 + O(x) \); see [21] Chapter 7, Theorem 7.18 or [8] Chapitre 21, Proposition 21.19 for instance. This section is devoted to usual extensions of this lemma in the context of bivariate power series.

3.1 Bivariate weighted valuations

Let \( \gamma = (\gamma_x, \gamma_y) \in \mathbb{N}^2 \) with \( \gamma_x \neq 0 \), and let \( \text{val}(x^a y^b) := \gamma_x a + \gamma_y b \) define a \textit{weighted valuation} over \( \mathbb{K}[[x, y]] \). An element in \( \mathbb{K}[[x, y]] \) is said to be \textit{quasi-homogeneous} if its non-zero terms have the same weighted valuation. It will be convenient to assume that \( \gamma_x \) and \( \gamma_y \) are coprime. This makes \( \mathbb{K}[[x, y]] \) a graded ring. The quasi-homogeneous component of \( f \) is written

\[
a = \sum_{i,j \geq 0} a_{i,j} x^i y^j \in \mathbb{K}[[x, y]]
\]

of valuation \( \gamma \) (if non-zero) is written

\[
[a]_\gamma := \sum_{\gamma_x i + \gamma_y j = \gamma} a_{i,j} x^i y^j \in \mathbb{K}[x, y].
\]

The \textit{initial form} of \( a \) is denoted by

\[
in(a) := [a]_{\text{val}(a)},
\]

and we introduce the following notation for truncations of \( a \):

\[
[a]_{\gamma + \eta} := \sum_{\gamma_x i + \gamma_y j \leq \gamma + \eta} a_{i,j} x^i y^j \in \mathbb{K}[x, y], \text{ for } \eta > 0.
\]

**Lemma 3** Let \( f, g \in \mathbb{K}[[x]][y] \). If \( g \) is monic of degree \( n \) and if \( \text{val}(g) = \text{val}(y^n) \), then the remainder \( f \bmod g \) in the division of \( f \) by \( g \) satisfies \( \text{val}(f \bmod g) \geq \text{val}(f) \).

**Proof** The result is clear whenever \( \deg f < n \). Now assume that \( m := \deg f \geq n \) and let \( f_m(x) \) denote the coefficient of \( y^m \) in \( f \). We let \( h := f - f_m(x) y^{m-n} g \). By definition we have

\[
\text{val}(f_m(x) y^m) \geq \text{val}(f).
\]

Using \( \text{val}(g) = \text{val}(y^n) \) we deduce that \( \text{val}(f_m(x) y^{m-n} g) = \text{val}(f_m(x) y^m) \), and then that

\[
\text{val}(h) \geq \min(\text{val}(f), f_m(x) y^{m-n} g) \geq \text{val}(f).
\]

Since \( \deg h < \deg f \) and \( h \bmod g = f \bmod g \), the conclusion follows from a straightforward induction on \( n \).

**Lemma 4** Let \( f \in \mathbb{K}[[x, y]] \setminus \{0\} \) be such that \( \text{in}(f) \) has bounded degree in \( y \). Then, there exist unique \( u \in \mathbb{K}[[x, y]] \) and \( g \in \mathbb{K}[[x]][y] \) such that \( f = u g \), \( g \) is monic with \( \deg_y g = \deg_y(\text{in}(f)) \), and \( \text{in}(u) \in x^m (\mathbb{K} \setminus \{0\}) \) for some \( m \in \mathbb{N} \).

**Proof** If \( u \) and \( g \) exist as specified, then we must have \( \text{in}(f) = \text{in}(u) \text{in}(g) \). Let us write

\[
\text{in}(f) = \sum_{i \geq 0} f_{m_i} x^{m_i} y^i,
\]
where $\gamma_x m_i + \gamma_y i = \text{val}(f)$ for $i = 0, \ldots, n$, and with $f_{m,n} \neq 0$. Note that $m_i \geq n_i$ for $i = 0, \ldots, n$. Consequently we set $m := m_n$, $u_1 := f_{m,n} x^m$ and $g_1 := \text{in}(f)/u_1$, so that $\text{in}(u) = u_1$ and $\text{in}(g) = g_1$ hold necessarily.

For $k \geq 1$, assume by induction that there exist $u_k \in \mathbb{K}[[x,y]]$ and $g_k \in \mathbb{K}[[x,y]]$ such that

$$[f]_{\text{val}(f):k} = [u_k g_k]_{\text{val}(f):k},$$

$g_k$ is monic, and $\text{in}(g_k) = g_1$. In order to construct $u_{k+1}$ and $g_{k+1}$, we are looking for $\tilde{u} = u_{k+1} - u_k$ and $\tilde{g} = g_{k+1} - g_k$ quasi-homogeneous of valuation $\text{val}(u) + k$ and $\text{val}(g) + k$, respectively, and such that

$$[f]_{\text{val}(f):k+1} = [(u_k + \tilde{u})(g_k + \tilde{g})]_{\text{val}(f):k+1},$$

and $\deg_y \tilde{g} < \deg_y g_1$. The latter equation rewrites into

$$[f - u_k g_k]_{\text{val}(f):k+1} = [u_k \tilde{g} + \tilde{u} g_k]_{\text{val}(f):k+1},$$

which is further equivalent to

$$[f - u_k g_k]_{\text{val}(f):k+1} = [u_1 \tilde{g} + \tilde{u} g_1]_{\text{val}(f):k+1}.$$  

Since $\deg_y \tilde{g} < \deg_y g_1$, the polynomial $u_1 \tilde{g}$ must be the remainder, written $r$, in the division of $[f - u_k g_k]_{\text{val}(f):k+1}$ by $g_1$, and $\tilde{u}$ is the corresponding quotient. This division is well defined because $g_1 = \text{in}(f)/u_1$ is monic in $y$. Lemma 3 implies $\text{val}(r) \geq \text{val}(f) + k$. Consequently, $\text{val}(r) > \text{val}(f) = \text{val}(x^m y^n)$. The inequality $\deg_y r < n$ implies that $\text{val}_x(r) > m$, so $r$ is a multiple of $u_1$, and therefore $\tilde{g}$ does exist and is given by $\tilde{g} := r/u_1$. In order to conclude the proof it suffices to take $u := \lim_{k \to \infty} u_k$ and $g := \lim_{k \to \infty} g_k$. \hfill \Box

The following lemma is sometimes called the Weierstraß normalization, or Weierstraß preparation theorem in the analytic context.

**Lemma 5** Let $f = \sum_{i=0}^{\infty} f_i(x) y^i \in \mathbb{K}[[x]][y]$ be such that there exists a positive integer $n$ satisfying $\text{val}_x(f_i(x)) = 0$ and $\text{val}_x(f_i(x)) > 0$ for $i < n$. Then, there exist $u$ invertible in $\mathbb{K}[[x,y]]$ and $g$ monic of degree $n$ in $\mathbb{K}[[x,y]]$ such that $f = ug$.

**Proof** With the notation as above, set $\gamma_x := n$ and $\gamma_y := 1$. Clearly $\text{val}(f) = n$ and $\text{in}(f)$ has degree $n$ in $y$. We have $\text{in}(g) = \text{in}(f)/f_n(0)$ and the conclusion follows from Lemma 4. \hfill \Box

### 3.2 Hensel lifting for polynomials

In the preceding lemmas we have seen how factorization in $\mathbb{K}[[x,y]]$ reduces to factorization in $\mathbb{K}[[x]][y]$. The next lemma addresses the latter case for weighted valuations.

**Lemma 6** Let $\text{val}$ denote a weighted valuation as above, and let $f \in \mathbb{K}[[x]][y]$ be monic of degree $n$ such that $\text{val}(f) = \text{val}(y^n)$. Assume that $\text{in}(f)$ factorizes into two quasi-homogeneous coprime monic polynomials $g_1$ and $h_1$. Then, there exist monic polynomials $g$ and $h$ in $\mathbb{K}[[x]][y]$ such that $f = gh$, $\text{in}(g) = g_1$, and $\text{in}(h) = h_1$.

**Proof** The Bézout relation for $g_1$ and $h_1$ regarded in $\mathbb{K}((x))[y]$ can be written

$$ug_1 + vh_1 = x^m, \tag{2}$$

where $m \in \mathbb{N}$, $u$ and $v$ belong to $\mathbb{K}[[x]][y]$ and satisfy $\deg_y u < \deg_y h_1$, $\deg_y v < \deg_y g_1$. Since $g_1$ and $h_1$ are quasi-homogeneous we may even take $u$ and $v$ quasi-homogeneous in $\mathbb{K}[[x]][y]$, so we have

$$\text{val}(u) + \text{val}(g_1) = \text{val}(v) + \text{val}(h_1) = \gamma_x m. \tag{3}$$
For \( k \geq 1 \), by induction we may assume that there exist \( g_k \) and \( h_k \) monic in \( \mathbb{K}[[x]][y] \) such that

\[
[f]_{\text{val}(f); k} = [g_k h_k]_{\text{val}(f); k},
\]

and \( \text{in}(g_k) = g_1 \). In order to construct \( g_{k+1} \) and \( h_{k+1} \), we are looking for \( \tilde{g} = g_{k+1} - g_k \) and \( \tilde{h} = h_{k+1} - h_k \) quasi-homogeneous of valuation \( \text{val}(g_1) + k \) and \( \text{val}(h_1) + k \), respectively, such that

\[
[f]_{\text{val}(f); k+1} = [(g_k + \tilde{g})(h_k + \tilde{h})]_{\text{val}(f); k+1},
\]

\( \deg_y \tilde{g} < \deg_y g_1 \), and \( \deg_y \tilde{h} < \deg_y h_1 \). The latter equation rewrites into

\[
[f - g_k h_k]_{\text{val}(f); k+1} = [g_k \tilde{h} + \tilde{g} h_k]_{\text{val}(f); k+1},
\]

which is equivalent to

\[
[f - g_k h_k]_{\text{val}(f); k+1} = g_1 \tilde{h} + \tilde{g} h_1.
\]

Thanks to the usual Chinese remaindering theorem over \( \mathbb{K}((x)) \), and using \([2]\), \( \tilde{g} \) and \( \tilde{h} \) must satisfy

\[
\tilde{g} = ((v/x^m)[f - g_k h_k]_{\text{val}(f); k+1} \text{ rem } g_1),
\]

\[
\tilde{h} = ((u/x^m)[f - g_k h_k]_{\text{val}(f); k+1} \text{ rem } h_1),
\]

where \( p \text{ rem } q \) represents the remainder in the division of \( p \) by \( q \). Lemma \([3]\) implies

\[
\text{val}(v[f - g_k h_k]_{\text{val}(f); k+1} \text{ rem } g_1) \geq \text{val}(v[f - g_k h_k]_{\text{val}(f); k+1}) > \text{val}(v) + \text{val}(f).
\]

Combined with \([3]\) we deduce that

\[
\text{val}(v[f - g_k h_k]_{\text{val}(f); k+1} \text{ rem } g_1) > \text{val}(g_1) + \gamma_x m.
\] (4)

Then, thanks to

\[
\deg_y(v[f - g_k h_k]_{\text{val}(f); k+1} \text{ rem } g_1) < \deg_y g_1,
\]
a monomial \( x^a y^b \) with non-zero coefficient in \( v[f - g_k h_k]_{\text{val}(f); k+1} \text{ rem } g_1 \) satisfies

\[
\gamma_a y^b \leq \gamma_y \deg_y g_1 = \text{val}(g_1).
\] (5)

Combining \([4]\) and \([5]\) yields

\[
\gamma_a + \text{val}(g_1) \geq \gamma_x a + \gamma_y b > \text{val}(g_1) + \gamma_x m.
\]

Since \( \gamma_x \neq 0 \), it follows that \( a \geq m \) and that the above value found for \( \tilde{g} \) actually belongs to \( \mathbb{K}[[x]][y] \). In the same way we obtain that \( \tilde{h} \in \mathbb{K}[[x]][y] \). In order to conclude the proof it suffices to take \( g := \lim_{k \to \infty} g_k \) and \( h := \lim_{k \to \infty} h_k \).

\[\square\]

### 3.3 Newton polygon and irreducibility

The Newton polygon is a tool to study the behavior of roots of polynomials. Let

\[
f = \sum_{i=0}^{n} f_i(x) y^i \in \mathbb{K}((x))[y]
\]

be of degree \( n \). The **Newton polygon** of \( f \) is the lower border of the convex hull in \( \mathbb{R}_{\geq 0} \times \mathbb{R} \) of the set of points

\[
\{(i, \text{val}_x(f_i)) : 0 \leq i \leq n, f_i \neq 0\}.
\]

Figure \([\square]\) illustrates this definition. The Newton polygon is a broken line with edges \((i_0, j_0), \ldots, (i_r, j_r)\), with \( r \geq 0 \), such that:
Figure 1: Newton polygon of \( f = x^3y + 2xy^2 - x^2y^4 + y^5 + 3xy^6 + y^7 \in \mathbb{Q}[[x]]y \).

1. \( i_0 < i_1 < \cdots < i_r \),
2. \( j_k = \text{val}(f_{i_k}) \) for \( k = 0, \ldots, r \),
3. for all \( i = 0, \ldots, n \) such that \( f_i \neq 0 \) the point \((i, \text{val}_x(f_i))\) is not strictly below the Newton polygon.

If \( f \) is zero then its Newton polygon is empty. If \( f \) has a single term then its Newton polygon reduces to a single vertex, and it is said to be degenerate. Otherwise, any \( k = 1, \ldots, r \) determines an edge \( E_k \) with vertices \((i_k - 1, j_k - 1)\) and \((i_k, j_k)\), and of slope \((j_k - j_{k-1})/(i_k - i_{k-1})\). The Newton polynomial associated to \( E_k \) is

\[
\sum_{i \in \mathbb{N}, (i,j) \in E_k} [f_i(x)]y^j x^{i - i_{k-1}} \in \mathbb{K}(x)[y],
\]

where \([f_i(x)]_j\) represents the term of degree \( j \) in \( f_i \).

**Example 2** The edge \( E_2 \) of vertices (2,1) and (5,0) in Figure 1 has associated Newton polynomial \( 2x + y^3 \).

Let \( f = \sum_{i=0}^n f_i(x)y^i \in \mathbb{K}[[x]][y] \) be monic of degree \( n \geq 1 \), and let \((i_0, j_0), \ldots, (i_r, j_r)\) represent the Newton polygon of \( f \). If this polygon is degenerate then \( f \) is reducible unless \( n = 1 \). We draw a useful consequence of Lemma 4.

**Lemma 7** Let \( f = \sum_{i=0}^n f_i(x)y^i \in \mathbb{K}((x))[y] \) be monic of degree \( n \geq 1 \), irreducible, and such that \( f_0(x) \neq 0 \) and \( \text{val}_x(f_0) \geq 0 \). Then, the Newton polygon of \( f \) has a single edge of vertices \((0, \text{val}_x(f_0))\) and \((n, 0)\).

**Proof** Let \( k \geq 0 \) be the smallest integer such that \( g := x^k f \) belongs to \( \mathbb{K}[[x]][y] \), and let \((0, \text{val}_x(x^k f_0))\) and \((l, \text{val}_x(x^k f_l))\) represent the vertices of the first edge \( E \) of the Newton polygon of \( g \). There exist coprime integers \( \gamma_x \geq 1 \), \( \gamma_y \geq 0 \) such that

\[
\frac{\gamma_y}{\gamma_x} = \frac{\text{val}_x(f_0) - \text{val}_x(f_l)}{l}.
\]

If we had \( l < n \), then the Newton polygon of \( g \) would have at least two edges, so Lemma 4 (used for \( g \)) would yield a non-trivial factor \( h \) of \( g \) of degree \( l \) in \( y \). Since \( f \) is irreducible this is not possible, whence \( l = n \), \( k = 0 \), and \( \text{in}(f) \) is the Newton polynomial of \( E \).

\(\square\)
**Proposition 6** Let \( f = \sum_{i=0}^{n} f_i(x)y^i \in \mathbb{K}[[x]][y] \) be monic of degree \( n \geq 1 \), irreducible, and such that \( f_0(x) \neq 0 \) and \( f_0(0) = 0 \). Then, the Newton polygon of \( f \) has a single edge whose Newton polynomial writes

\[
x^{\text{val}_x(f_0)}\theta \left( \frac{y^{\gamma_x}}{x^{\gamma_y}} \right)
\]

where \( \theta \in \mathbb{K}[t] \) is monic and irreducible of degree \( d = n/\gamma_y \), and \( \gamma_x \geq 1 \), \( \gamma_y \geq 1 \) are coprime integers such that

\[
\frac{\gamma_y}{\gamma_x} = \frac{\text{val}_x(f_0)}{n}.
\]

**Proof** From Lemma 7 we know that the Newton polygon of \( f \) has a single edge \( E \) and that \( \text{in}(f) \) is the Newton polynomial of \( E \). For a monomial \( x^a y^b \) of weighted-valuation

\[
\text{val}(f) = \gamma_x a + \gamma_y b = \gamma_y n = \gamma_x \text{val}_x(f_0)
\]

there exists a unique integer \( k \geq 0 \) such that \( \text{val}_x(f_0) - a = k \gamma_y \) and \( b = k \gamma_x \). We obtain:

\[
x^a y^b = x^{\text{val}_x(f_0) - k \gamma_y} y^{k \gamma_x} = x^{\text{val}_x(f_0)} \left( \frac{y^{\gamma_x}}{x^{\gamma_y}} \right)^k,
\]

hence \( \text{in}(f) \) writes

\[
\text{in}(f) = x^{\text{val}_x(f_0)}\theta \left( \frac{y^{\gamma_x}}{x^{\gamma_y}} \right),
\]

where \( \theta \) is a monic polynomial in \( \mathbb{K}[t] \). If \( \theta \) had a non-trivial irreducible factor \( \tilde{\theta} \) then

\[
x^{\gamma_y \deg \tilde{\theta}} \tilde{\theta} \left( \frac{y^{\gamma_x}}{x^{\gamma_y}} \right)
\]

would be a non-trivial factor of \( \text{in}(f) \). This is not possible because of Lemma 6 so \( \theta \) is irreducible. \( \square \)

### 4 Valuations

The notion of *valuation* has been introduced just before Section 1.1. The goal of this section is to highlight how valuations extend in algebraic extensions, and to introduce the notion of *uniformizing parameter*. If \( f \) is irreducible in \( \mathbb{K}[[x, y]] \), then the valuations on \( \mathbb{K}[[x, y]]/(f) \) are in one-to-one correspondence to those on \( \text{Frac}(\mathbb{K}[[x, y]]/(f)) \).

#### 4.1 Algebraic extensions

The valuation \( \text{val}_x \) is the unique valuation on \( \mathbb{K}((x)) \) normalized such that \( \text{val}_x(x) = 1 \). We examine how this valuation extends in algebraic extensions.

**Lemma 8** Let \( f \in \mathbb{K}[[x]][y] \) be irreducible, monic of degree \( n \) in \( y \), and such that \( f(0, 0) = 0 \). If \( v \) is a valuation on \( \mathbb{K}[[x, y]]/(f) \) that extends \( \text{val}_x \), then we have

\[
v(y \mod f) = \frac{\text{val}_x(f(x, 0))}{n},
\]

where \( y \mod f \) denotes the class of \( y \) in \( \mathbb{K}[[x, y]]/(f) \).
Proof If \( f = y \) then the statement clearly holds, so let us assume that \( f \neq y \). Let us write \( f = \sum_{i=0}^{n} f_i(x)y^i \). Since \( f \) is monic in \( y \) we have \( \text{val}_x(f_n) = 0 \). Since \( f \) is irreducible we have \( f(x, 0) = f_0(x) \neq 0 \). By Proposition 6 the Newton polygon of \( f \) has a single edge whose vertices are \( (0, \text{val}_x(f_0)) \) and \( (n, 0) \). This entails that

\[
\text{val}_x(f_i) \geq \frac{\text{val}_x(f_0)}{n}(n - i)
\]

for \( i = 0, \ldots, n \); see property (3) of the definition of the Newton polygon. If a valuation \( v \) on \( \mathbb{K}[[x, y]]/(f) \) extends \( \text{val}_x \), then we necessarily have

\[
nv(y \mod f) = v((y^n - f) \mod f) = v \left( \sum_{i=0}^{n-1} f_i(x)y^i \mod f \right) \geq \min_{i=0,\ldots,n-1} (v(f_i(x)y^i \mod f)) = \min_{i=0,\ldots,n-1} (\text{val}_x(f_i) + iv(y \mod f)) \geq \min_{i=0,\ldots,n-1} \left( \frac{\text{val}_x(f_0)}{n}(n - i) + iv(y \mod f) \right).
\]

Let \( j \) be a value of \( i \in \{0, \ldots, n - 1\} \) for which \( \frac{\text{val}_x(f_0)}{n}(n - i) + iv(y \mod f) \) is minimal, so we have

\[
nv(y \mod f) \geq \frac{\text{val}_x(f_0)}{n}(n - j) \]

that implies

\[
(n - j)v(y \mod f) \geq \frac{\text{val}_x(f_0)}{n}(n - j),
\]

whence

\[
v(y \mod f) \geq \frac{\text{val}_x(f_0)}{n}.
\] (6)

On the other hand, we verify that

\[
\text{val}_x(f_0) = v((f - f_0(x)) \mod f) = v \left( \sum_{i=1}^{n} f_i(x)y^i \mod f \right) \geq \min_{i=1,\ldots,n} (v(f_i(x)y^i \mod f)) = \min_{i=1,\ldots,n} (\text{val}_x(f_i) + iv(y \mod f)). \]

\[
\geq \min_{i=1,\ldots,n} \left( \frac{\text{val}_x(f_0)}{n}(n - i) + iv(y \mod f) \right).
\]

Again, let \( j \) be a value of \( i \in \{1, \ldots, n\} \) for which \( \frac{\text{val}_x(f_0)}{n}(n - i) + iv(y \mod f) \) is minimal. Then we obtain

\[
\text{val}_x(f_0) \geq \frac{\text{val}_x(f_0)}{n}(n - j) + jv(y \mod f),
\]

whence

\[
v(y \mod f) \leq \frac{\text{val}_x(f_0)}{n}.
\] (7)

The combination of (6) and (7) concludes the proof. \( \Box \)
Let $f \in \mathbb{K}[[x]][y]$ be a monic irreducible polynomial of degree $n$. Let $a \in \mathbb{K}((x))[y]/(f)$ be represented by $A \in \mathbb{K}((x))[y]_{<n}$. We regard $\mathbb{K}((x))[y]/(f)$ as a $\mathbb{K}((x))$-algebra of dimension $n$. The minimal polynomial of $a$ is the monic polynomial $\mu \in \mathbb{K}((x))[t]$ of smallest degree in $t$ such that $\mu(a) = 0$. We define the characteristic polynomial of $a$ to be the characteristic polynomial $\chi \in \mathbb{K}((x))[t]$ of the multiplicative endomorphism by $a$ in $\mathbb{K}((x))[y]/(f)$, so we have $\chi(a) = 0$. Since $\mathbb{K}((x))[y]/(f)$ is a field, $\mu$ is irreducible and we have
$$\chi = \mu^{n/d}, \text{ where } d := \deg \mu.$$In addition, we know from Proposition 3 that $\chi(0) = \pm \Res_y(f, A)$. For the next proposition the slight abuse of notation $\Res_y(f, a) := \Res_y(f, A)$ will be useful.

**Proposition 7** Let $f \in \mathbb{K}[[x]][y]$ be a monic irreducible polynomial of degree $n \geq 1$ such that $f(0, 0) = 0$. The map
$$v : \mathbb{K}((x))[y]/(f) \rightarrow \frac{1}{n} \mathbb{Z} \cup \{\infty\}, \quad a \mapsto \frac{\val_x(\Res_y(f, a))}{n}$$is the unique valuation on $\mathbb{K}((x))[y]/(f)$ that extends $\val_x$.

**Proof** We first prove that $v$ does define a valuation. If $v(a) = \infty$ then $a$ is zero because $f$ is irreducible. We also easily verify that $v(x) = 1$. Then, Proposition 4 implies that $v(ab) = v(a) + v(b)$ holds for all $a$ and $b$ in $\mathbb{K}((x))[y]/(f)$. Assume that $a, b \neq 0$ and that $v(a) \geq v(b)$, or equivalently that $v(a/b) \geq 0$. Let $\mu(t) = \sum_{i=0}^{d} \mu_i(t)x^i$ denote the minimal polynomial of $c := a/b$, and $\chi$ its characteristic polynomial. By Proposition 3 since $f$ is monic in $y$, we have $\chi(0) = \pm \Res_x(f, c)$. Since $\chi = \mu^{n/d}$ we deduce that $\val_x(\mu(0)) \geq 0$. Since $\mu$ is monic and irreducible in $\mathbb{K}((x))[t]$, Lemma 7 implies that the Newton polygon of $\mu$ has a single edge, hence that $\val_x(\mu_i) \geq 0$ for $i = 0, \ldots, d$. We conclude that the minimal polynomial $\mu(t - 1)$ of $a/b + 1$ has all its coefficients of nonnegative valuation. In particular, the constant coefficient of $\chi(t - 1)$ has nonnegative valuation, whence $v(a/b + 1) \geq 0$, or equivalently $v(a + b) \geq v(b)$. Finally, we have shown that $v$ is a valuation.

For the uniqueness, let now $v$ stand for a valuation on $\mathbb{K}((x))[y]/(f)$ that extends $\val_x$. Let us consider a non-zero element $c$ of $\mathbb{K}((x))[y]/(f)$ of positive valuation, and let $\mu$ and $\chi$ still denote its minimal and characteristic polynomials over $\mathbb{K}((x))$. Since $\mu$ is irreducible, $\mathbb{K}((x))[c]$ is isomorphic to $\mathbb{K}((x))[t]/(\mu(t))$ and is a subfield of $\mathbb{K}((x))[y]/(f)$, so we may endow $\mathbb{K}((x))[c]$ with $v$ and we have
$$v(c) = v(t \mod \mu(t)).$$ (8)
On the other hand, we have seen that $\mu \in \mathbb{K}[[x]][t]$ hence Lemma 8 (used with $t$ instead of $y$) yields
$$v(t \mod \mu) = \frac{\val_x(\mu(0))}{d},$$ (9)
whence
$$\frac{\val_x(\mu(0))}{d} = \frac{\val_x(\chi(0))}{n} = \frac{\val_x(\Res_y(f, c))}{n}. \quad (10)$$The combination of (8), (9), and (10) shows that $v(c)$ must equal $\val_x(\Res_y(f, c))/n$. \qed

### 4.2 Integral elements

Let $f \in \mathbb{K}[[x]][y]$ be a monic irreducible polynomial of degree $n$. An element $a \in \mathbb{K}((x))[y]/(f)$ is said to be integral if there exists a monic polynomial $p \in \mathbb{K}[[x]][t]$ such that $p(a) = 0$ holds. This is equivalent to the belonging of the minimal and characteristic polynomials of $a$ in $\mathbb{K}[[x]][t]$, thanks to the Gauss lemma 38, Chapter 4, Theorem 2.1. The following proposition gives a necessary and sufficient condition for an element to be integral, depending on its valuation. We shall use this result in the next section.
Proposition 8 Let $f \in \mathbb{K}[[x]][y]$ be a monic irreducible polynomial of degree $n$ such that $f(0,0) = 0$, and let $v$ denote the valuation of $\mathbb{K}((x))[y]/(f)$ that extends $\text{val}_x$. Then, an element $a \in \mathbb{K}((x))[y]/(f)$ is integral if, and only if, $v(a) \geq 0$.

Proof If $a$ is integral then the constant coefficient of its characteristic polynomial $\chi$ has nonnegative valuation. Proposition 7 yields $v(a) \geq 0$. Conversely, if $v(a) \geq 0$ then the constant coefficient $\chi(0)$ of the characteristic polynomial $\chi \in \mathbb{K}((x))[y]$ of $a$ has nonnegative valuation. The same holds for the minimal polynomial $\mu \in \mathbb{K}((x))[t]$. Since $\mu$ is irreducible, Lemma 7 ensures that the Newton polygon of $\mu$ has a single edge. Consequently $\mu$ must have all its coefficients of nonnegative valuation, hence $\mu \in \mathbb{K}[[x]][t]$ and by the Gauss lemma $a$ is integral.

4.3 Uniformizing parameters

In the context of Proposition 7 the valuation group

$$\Gamma := v(\mathbb{K}((x))[y]/(f) \setminus \{0\})$$

of $v$ is included in $\frac{1}{r}\mathbb{Z}$. So $\Gamma$ has the form $\frac{r}{s}\mathbb{Z}$ where $r, s \in \mathbb{N}$, $r$ divides $n$, and $s$ is prime to $r$. Since $v(x) = 1$ necessarily $\Gamma$ contains $\mathbb{Z}$. This implies that $s = 1$. The following proposition ensures that $r = n$. The integer $n$ is called the ramification index of $v$. In the present context, a uniformizing parameter of $v$ is defined as an element of minimal positive valuation.

Proposition 9 Assume that $\mathbb{K}$ is algebraically closed. Let $f \in \mathbb{K}[[x]][y]$ be a monic irreducible polynomial of degree $n$ such that $f(0,0) = 0$, let $v$ be the valuation on $\mathbb{K}((x))[y]/(f)$ that extends $\text{val}_x$, and let $\tau$ be a uniformizing parameter of $\mathbb{K}((x))[y]/(f)$. Then, $\mathbb{K}((x))[y]/(f)$ is isomorphic to $\mathbb{K}((x))[[\tau]]$ and is complete for $v$. The value group of $v$ is $\frac{n}{r}\mathbb{Z}$. In addition, there exist power series $\varphi, \psi \in \mathbb{K}[[\tau]]$ such that $x = \varphi(\tau)$ and $y = \psi(\tau)$ hold in $\mathbb{K}((x))[y]/(f)$.

Proof Let $r$ be as above, that is $v(\tau) = 1/r$, and let $\mu$ denote the minimal polynomial of $\tau$. From Proposition 8 we know that $\mu \in \mathbb{K}[[x]][t]$. Since $\mu$ is monic and irreducible, Lemma 7 implies that its Newton polygon has a single edge. From Proposition 7 we know that $\text{val}_x(\text{Res}_{y}(f, \tau)) > 0$, so Proposition 3 implies that $\mu(0,0) = 0$. The map

$$\Psi : \mathbb{K}((x))[t]/(\mu) \rightarrow \mathbb{K}((x))[y]/(f)$$

$$t \mapsto \tau$$

is injective by construction of $\mu$, so we may endow the field $\mathbb{K}((x))[t]/(\mu)$ with $v$, that is the unique valuation on $\mathbb{K}((x))[t]/(\mu)$ that extends $\text{val}_x$ according to Proposition 7. By Lemma 8 applied to $\mu$, the slope of the edge of the Newton polygon of $\mu$ is $-1/r$.

On the one hand we endow $\mathbb{K}((x))$ with the ultrametric absolute value

$$|a(x)| := \exp(-\text{val}_x(a(x)))$$

for all $a(x) \in \mathbb{K}((x))$, and the $\mathbb{K}((x))$-vector space $\mathbb{K}((x))[t]_{< \text{deg} \mu}$ with the ultrametric norm

$$\|c(t)\|_\infty := \max_{0 \leq i < \text{deg} \mu} |c_i(x)|$$

for all $c(t) := \sum_{i=0}^{\text{deg} \mu - 1} c_i(x)t^i$.

On the other hand we consider the weights $\gamma_x := r$ for $x$ and $\gamma_t := 1$ for $t$ on $\mathbb{K}((x))[t]$ and we write

$$t^m \text{rem} \mu(t) := \sum_{i=0}^{\text{deg} \mu - 1} u_{m,i}(x)t^i$$

where $u_{m,i}(x) \in \mathbb{K}[[x]]$. 

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For all $0 \leq j < \deg \mu$, we verify that

$$r \text{val}_x(u_{m,j}(x)) + j = \text{val}(u_{m,j}(x)t^j) \geq \min_{0 \leq i < \deg \mu} \text{val}(u_{m,i}(x)t^i) = \text{val}(t^m \text{rem} \mu(t)) \quad \text{(by definition of val)} \geq \text{val}(t^m) \quad \text{(by Lemma 3)} = m,$$

in order to obtain that

$$-\log(||t^m \text{rem} \mu(t)||_{\infty}) \geq \frac{m - (\deg \mu - 1)}{r}. \quad (11)$$

Conversely, a lower bound for $v$ in terms of $|| \cdot ||_{\infty}$ follows straightforwardly from the definitions: for all $c(t) \in \mathbb{K}(\!(x)\!)[t]_{< \deg \mu}$ we have

$$v(c(\tau)) \geq \min_{0 \leq i < \deg \mu} v(c_i(\tau)t^i) \geq \min_{0 \leq i < \deg \mu} \text{val}_x(c_i(x)) = -\log(||c(t)||_{\infty}). \quad (12)$$

Now let $b \in \mathbb{K}(\!(x)\!)[y]/(f)$ be non-zero and set $a := b/\tau^{rv(b)}$, so we have $v(a) = 0$, that is $\text{val}_x(\text{Res}_y(f, a)) = 0$. Let $\mu_a$ denote the minimal polynomial of $a$. From Proposition 3 we know that $\mu_a \in \mathbb{K}[[x]][t]$. Since $\mu_a$ is monic and irreducible, Lemma 7 implies that its Newton polygon has a single edge whose vertices are $(0, 0)$ and $(\deg \mu_a, 0)$. By Lemma 6, since $\mathbb{K}$ is algebraically closed the initial of $\mu_a$ cannot have $\geq 2$ distinct roots, therefore we can write

$$\mu_a(t) = (t - a_0)^{\deg \mu_a} + O(x)$$

for some $a_0 \in \mathbb{K} \setminus \{0\}$. Consequently, the minimal polynomial of $a - a_0$ writes $t^{\deg \mu_a} + O(x)$, so it has a single edge with negative slope. Propositions 3 and 7 imply $v(a - a_0) > 0$. In other words $b$ rewrites in the form $b = a_0\tau^{rv(b)} + c$ with

$$v(c) = v(\tau^{rv(b)}(a - a_0)) = v(b) + v(a - a_0) > v(b).$$

By iterating this process for $c$ we can construct unique approximate expansions of $b$ in the form

$$S_l := \tau^{rv(b)} \sum_{i=0}^{l} a_i \tau^i, \quad \text{where } a_i \in \mathbb{K}, \text{ such that } v(b - S_l) \geq v(b) + \frac{l + 1}{r}$$

holds for all $l \geq 0$. Let $\tilde{S}_l(t) \in \mathbb{K}((x))[t]_{< \deg \mu}$ represent the canonical preimage of $S_l$. The series $(\tilde{S}_l)_{l \geq 0}$ is a Cauchy sequence in $\mathbb{K}((x))[\tau]$. Inequality (11) implies that $(\tilde{S}_l(t))_{l \geq 0}$ is a Cauchy sequence for $|| \cdot ||_{\infty}$, so it converges to an element $\tilde{b}(t) \in \mathbb{K}((x))[t]_{< \deg \mu}$ because $\mathbb{K}((x))[t]_{< \deg \mu}$ is complete for $|| \cdot ||_{\infty}$, hence $||\tilde{S}_l(t) - \tilde{b}(t)||_{\infty} \to 0$. Inequality (12) applied to $c(t) := \tilde{S}_l(t) - \tilde{b}(t)$ implies that $(\tilde{S}_l)_{l \geq 0}$ converges to $\tilde{b}(\tau)$. Using

$$v(b - \tilde{b}(\tau)) \geq \min(v(b - S_l), v(S_l - \tilde{b}(\tau)))$$

for when $l$ tends to infinity, we deduce that $v(b - \tilde{b}(\tau)) = \infty$ hence that

$$b = \tilde{b}(\tau) = \tau^{rv(b)} \sum_{i \geq 0} a_i \tau^i. \quad (13)$$

In particular, this shows that $\Psi$ is surjective, whence $\deg \mu = n$. Inequality (11) and equation (13) further lead to

$$-\log(||\tilde{b}(t)||_{\infty}) \geq \min_{i \geq 0}(-\log(||t^{rv(b)} + i \text{ rem} \mu(t)||_{\infty})) \geq v(b) - \frac{\deg \mu - 1}{r}.$$
Combined with (12) the completeness of \( K((x))[t]_{\deg \mu} \) for \( \| \cdot \|_\infty \) induces the completeness of \( K((x))[\tau] \cong K((x))[y]/(f) \) for \( v \).

Expanding \( x \) and \( y \) in terms of power series as we have done it for \( b \) in (13) yields the existence and uniqueness of the requested series \( \varphi \) and \( \psi \) for \( x \) and \( y \) in terms of \( \tau \). Since \( \text{val}(\varphi) = r \), still for the weights \( \gamma_x := r \) for \( x \) and \( \gamma_t := 1 \) for \( t \), Lemma 5 ensures the existence of \( u \) invertible in \( K[[x,t]] \) and \( \tilde{\mu}(t) \in K[[x]][t] \) monic such that \( x - \varphi(t) = u(x,t)\tilde{\mu}(t) \) and \( \text{in}(\tilde{\mu}) = t^r - cx \) where \( c \in K \). We deduce that \( \mu = \tilde{\mu} \), whence \( r = n \).

\[ \square \]

**Example 3** If the characteristic \( p \) of \( K \) does not divide \( n \), then we may write

\[ x = \tau^n(c_1 + c_2\tau + c_3\tau^2 + \cdots), \]

and verify that

\[ \tilde{\tau} := \tau \left( 1 + \frac{c_2}{c_1}\tau + \frac{c_3}{c_1}\tau^2 + \cdots \right)^{1/n} \]

is a uniformizing parameter. The parametrization \( x = c_1\tilde{\tau}^n \) and \( y = \psi(\tilde{\tau}) \) is called a rational Puiseux expansion of \( f \); see details in [20].

**Example 4** Let \( K := \bar{\mathbb{F}}_2 \) and \( f := x + xy + y^2 \). Assume that there exists a uniformizing parameter \( \tau = a(x) + b(x)y \), where \( a, b \in K(x) \) such that \( x = \tau^2 \). Then we would have

\[ x = a(x)^2 + b(x)^2(x + xy), \]

whence \( b(x) = 0 \), so \( a(x) \) cannot exist. Consequently rational Puiseux expansions (defined in Example 3) do not exist in this case. In fact, \( y \) is a uniformizing parameter and we have

\[ x = \frac{y^2}{1+y} = y^2 + y^3 + y^4 + \cdots, \]

but we cannot extract the square root of \( 1 + y \).

### 5 Places and Divisors

The goal of this section is to introduce the notion of place and divisor of an algebraic plane curve. We begin by revisiting the previous results on valuations from a geometric point of view in order to achieve properties that do not depend on the ambient coordinates. *From now on \( K \) is assumed to be algebraically closed.*

#### 5.1 Integral closures and places

Recall that a *discrete valuation ring* is a principal ideal domain that admits a unique maximal ideal. We recall that \( \text{val}_x \) is the unique valuation on \( K((x)) \) such that \( \text{val}_x(x) = 1 \).

**Proposition 10** Let \( f \in K[[x]][y] \) be an irreducible polynomial such that \( f(0,0) = 0 \), and let \( \mathcal{D} := K[[x,y]]/(f) \). There exists a unique valuation \( v \) on \( \mathcal{D} \) having \( \mathbb{Z} \) for valuation group. In addition, the integral closure of \( \mathcal{D} \) in its field of fractions, written \( \bar{\mathcal{D}} \), is a discrete valuation ring whose maximal ideal is

\[ \mathfrak{P} = \{ a \in \text{Frac}(\mathcal{D}) : v(a) > 0 \}. \]
Proof If \( x \) divides \( f \) then \( f = ux \), where \( u \) is invertible in \( \mathbb{K}[[x, y]] \). In this case there clearly exists a unique valuation \( v \) on \( \mathcal{D} \) that extends \( \text{val}_y \). In addition, we have \( \mathcal{D} = \mathbb{K}[[y]] \), \( \text{Frac}(\mathcal{D}) = \mathbb{K}((y)) \), and \( \mathcal{D} = \mathcal{D} \), so the proposition clearly holds.

Now we may assume that \( f(0, y) \neq 0 \). By Lemma 5 we can write \( f = ug \) where \( u \) is invertible in \( \mathbb{K}[[x, y]] \) and \( g \in \mathbb{K}[[x]][y] \) is monic and irreducible. Proposition 7 ensures the existence and the uniqueness of a valuation \( v \) on \( \mathcal{D} \) that extends \( \text{val}_y \). In addition, we have \( \text{Frac}(\mathcal{D}) = \mathbb{K}((x))[[y]]/(g) \) and Proposition 8 asserts that an element of Frac(\( \mathcal{D} \)) has nonnegative valuation if and only if it is integral. In other words, the set of integral elements \( \mathcal{D} \) is made of the elements of Frac(\( \mathcal{D} \)) with nonnegative valuation, so \( \mathcal{D} \) is a ring. Let \( I \) be an ideal of \( \mathcal{D} \) and let \( \rho \) be an element of minimal positive valuation in \( I \). If \( a \in I \) then \( v(a/\rho) \geq 0 \), whence \( a \in (\rho) \). This proves that ideals of \( \mathcal{D} \) are principal. If \( I \) is maximal then it is necessarily generated by a uniformizing parameter (an element of minimal positive valuation). Thus \( \mathcal{D} \) has a unique maximal ideal, which concludes the proof.

Recall that since \( f \) is irreducible in \( \mathbb{K}[[x, y]] \) the valuation on \( \mathcal{D} \) is in one-to-one correspondence with that of Frac(\( \mathcal{D} \)). The ideal \( \mathfrak{P} \) of Proposition 10 does not depend on the choice of the valuation on \( \mathcal{D} \): if the coordinates are changed linearly in \( f \) then the representatives of the elements in \( \mathfrak{P} \) change accordingly. The ideal \( \mathfrak{P} \) is often called the place associated to \( f \). We denote the unique valuation on \( \mathcal{D} \) in the sense of Proposition 10 by \( w : D \to \mathbb{Z} \cup \{\infty\} \), and call it the canonical valuation; \( w^{-1}(1) \) is the set of uniformizing parameters.

### 5.2 Divisors

Let \( F \in \mathbb{K}[x, y, z] \) be a homogeneous irreducible polynomial defining a curve \( C := \mathcal{V}_F(F) \). Let \( \zeta = (\zeta_x : \zeta_y : \zeta_z) \in C \). Up to a change of coordinates we may assume in the sequel that \( \zeta = (0 : 0 : 1) \). An irreducible factor \( f \) of \( F(x, y, 1) \) in \( \mathbb{K}[[x, y]] \) defines a canonical valuation \( w \). If \( A \) and \( B \) are homogeneous polynomials in \( \mathbb{K}[x, y, z] \), then we have seen that the inequality

\[
v(A(x, y, 1) \mod f) \leq v(B(x, y, 1) \mod f)
\]

is independent of the choice of the coordinates by Proposition 10.

The local factorization of \( F(x, y, 1) \) in \( \mathbb{K}[[x, y]] \) can be uniquely written (up to a permutation of the factors)

\[
F(x, y, 1) = uf_1 \cdots f_r,
\]

where \( u \) is invertible in \( \mathbb{K}[[x, y]] \) and the \( f_i \) are irreducible in \( \mathbb{K}[[x, y]] \). The canonical valuation on \( \mathcal{D}_i := \mathbb{K}[[x, y]]/(f_i) \) is written \( w_i \). For \( A \in \mathbb{K}[x, y, z] \) homogeneous and prime to \( F \), the local divisor associated to \( A \) at \( \zeta \) is the symbolic sum

\[
\text{Div}_\zeta(A) := w_1(A)\mathfrak{P}_1 + \cdots + w_r(A)\mathfrak{P}_r,
\]

where \( \mathfrak{P}_i \) is the place associated to \( f_i \), for \( i = 1, \ldots, r \). The point \( \zeta \) is called the center of the place \( \mathfrak{P}_i \). Since \( A \) is prime to \( F \) we have \( \text{Res}_y(f_i(x, y), A(x, y, 1)) \neq 0 \), so \( w_i(A) \neq \infty \). The (global) divisor associated to \( A \) is then defined by

\[
\text{Div}(A) := \sum_{\zeta \in \mathcal{V}_F(A, F)} \text{Div}_\zeta(A).
\]

This sum is finite thanks to Proposition 5. If \( B \) is another homogeneous polynomial prime to \( F \), then we further define

\[
\text{Div}(A/B) := \text{Div}(A) - \text{Div}(B).
\]

More generally, a divisor of \( C \) is a finite \( \mathbb{Z} \)-combination of places centered at points of \( C \). The set of divisors of \( C \) is equipped with the following partial ordering:

\[
\sum_{\mathfrak{P}} c_\mathfrak{P} \mathfrak{P} \leq \sum_{\mathfrak{P}} c'_\mathfrak{P} \mathfrak{P} \iff \forall \mathfrak{P}, c_\mathfrak{P} \leq c'_\mathfrak{P}.
\]

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A divisor $D$ is said to be positive (also called effective) whenever $D \geq 0$. The degree of a divisor is defined by:

$$\deg \left( \sum_{\varphi} c_{\varphi} \varphi \right) := \sum_{\varphi} c_{\varphi}.$$

### 5.3 Bézout’s theorem

The degree of the divisor $\text{Div}(G)$ associated to a homogeneous polynomial $G \in \mathbb{K}[x, y, z]$ prime to $F$ only depends on the degrees of $G$ and $F$. This is stated in the following proposition, that can be regarded as an instance of Bézout’s well known theorem for counting the number of intersection points $V_p(G, F)$ with multiplicities.

**Proposition 11** Let $F$ be an irreducible homogeneous polynomial in $\mathbb{K}[x, y, z]$. For all homogeneous polynomial $G \in \mathbb{K}[x, y, z]$ prime to $F$ we have $\deg(\text{Div}(G)) = \deg G \deg F$.

**Proof** With the same notation as in Section 2.2 thanks to Lemma 2 we may assume that $A_1$ and $A_2$ hold, so we can use Lemma 1. Let $\zeta_x$ be a root of $R(x, 1)$ and let $\zeta_{y,1}, \ldots, \zeta_{y,s}$ be the roots of $F(\zeta_x, y)$, so that

$$F(\zeta_x, y) = \prod_{i=1}^{s} (y - \zeta_{y,i})^{m_i}$$

holds with $m_i \geq 1$ and $m_1 + \cdots + m_s = \deg F$. Thanks to the usual Hensel lemma (recalled at the beginning of Section 3) there exist unique monic polynomials $f_i \in \mathbb{K}[[x - \zeta_x]] [y]$ such that

$$F(x, y, 1) = f_1 \cdots f_s,$$

and $f_i(\zeta_x, y) = (y - \zeta_{y,i})^{m_i}$ and $\deg f_i = m_i$ for $i = 1, \ldots, s$. For all $i = 1, \ldots, s$, we write $f_{i,1}, \ldots, f_{i,r_i} \in \mathbb{K}[[x - \zeta_x]][y - \zeta_{y,i}]$ the monic irreducible factors of $f_i$. From Proposition 4 we deduce

$$\text{val}_{x - \zeta_x} (R(x, 1)) = \sum_{i=1}^{s} \sum_{j=1}^{r_i} \text{val}_{x - \zeta_x} (\text{Res}_y (G(x, y, 1), f_{i,j}(x, y))).$$

Let $w_{i,j}$ denote the canonical valuation of $\mathbb{K}[[x - \zeta_x]][y - \zeta_{y,j}]/(f_{i,j})$. By Propositions 3, 7 and 9 we have

$$w_{i,j}(G(x, y, 1) \mod f_{i,j}(x, y)) = \text{val}_{x - \zeta_x} (\text{Res}_y (G(x, y, 1), f_{i,j}(x, y))).$$

It follows that

$$\deg \left( \sum_{i=1}^{s} \text{Div}(\zeta_x; \zeta_{y,i}; 1)(G) \right) = \sum_{i=1}^{s} \sum_{j=1}^{r_i} w_{i,j}(G(x, y, 1) \mod f_{i,j}(x, y)) = \text{val}_{x - \zeta_x} (R(x, 1))$$

and that

$$\deg_x R = \sum_{R(\zeta_x, 1) = 0} \text{val}_{x - \zeta_x} (R(x, 1)) = \deg(\text{Div}(G)).$$

The conclusion follows from $A_2$ and the usual equality $\deg R = \deg G \deg F$, whose proof is elementary from the definition of the resultant; see [11, Chapter 3, Proposition 3.1.5] for instance. \qed
5.4 Functions on curves

Recall that \( \mathbb{K}(\mathcal{C}) \) denotes the set of the rational functions defined on \( \mathcal{C} \).

**Proposition 12** Let \( A/B \in \mathbb{K}(\mathcal{C}) \setminus \{0\} \). The divisor \( \text{Div}(A/B) \) is zero if, and only if, \( A/B \in \mathbb{K} \).

**Proof** If \( A/B \in \mathbb{K} \setminus \{0\} \) then \( \text{Div}(A/B) = 0 \) holds by definition. Conversely, assume that \( \text{Div}(A/B) = 0 \) and that \( A \not\in \mathbb{K} \). The assumption on \( \text{Div}(A/B) \) means that the valuations of \( A \) and \( B \) coincide for all the valuations centered at points of \( \mathcal{V}_F(A,F) \). By Proposition 5 since \( A \not\in \mathbb{K} \) there exists a point \( \zeta \) in \( \mathcal{V}_F(A,F) \). Let \( w \) be the canonical valuation centered at \( \zeta \) and let \( \tau \) be a uniformizing parameter, so \( A \) and \( B \) locally write \( A(x,y,1) = \sum_{i \geq m} a_i \tau^i \) and \( B(x,y,1) = \sum_{i \geq m} b_i \tau^i \) with \( a_m \neq 0 \) and \( b_m \neq 0 \), thanks to Proposition 6. It follows that

\[
  w \left( A - \frac{a_m}{b_m} B \right) > m = w(A).
\]

For all the other valuations \( w' \), and at other points \( \zeta \), we have \( w'(A - \frac{a_m}{b_m} B) \geq w'(A) = w'(B) \), by assumption. If \( A - \frac{a_m}{b_m} B \) were prime to \( F \) then we would have

\[
  \deg \left( \text{Div} \left( A - \frac{a_m}{b_m} B \right) \right) > \deg(\text{Div} A),
\]

which it not possible by Proposition 11. Thus \( A \in \mathbb{K} \) and consequently \( A/B \in \mathbb{K} \).

\( \square \)

5.5 Adjoint divisor

Let \( \zeta \) be a point of the curve \( \mathcal{C} = \mathcal{V}_F(F) \). Up to a change of coordinates we may assume that \( \zeta = (0 : 0 : 1) \) and write

\[
  F(x,y,1) = u(x,y)f_1(x,y) \cdots f_r(x,y)
\]

for the irreducible factorization of \( F(x,y,1) \) in \( \mathbb{K}[[x,y]] \), with \( u \) invertible. Let \( w_i, \Psi_i, \) and \( \tau_i \) represent the canonical valuation, the place, and a uniformizing parameter of \( \mathcal{D}_i := \mathbb{K}[[x,y]]/(f_i) \). Let \( \varphi_i(\tau_i) \) and \( \psi_i(\tau_i) \) denote the expansions of the images of \( x \) and \( y \) in \( \text{Frac}(\mathcal{D}_i) \). If \( F_y := \frac{\partial F}{\partial y} \neq 0 \) then the local adjoint divisor at \( \zeta \) of the curve \( \mathcal{C} \) is defined by

\[
  \mathcal{A}_\zeta := \sum_{i=1}^{r} (w_i(F_y) - w_i(\varphi'_i(\tau_i)))\Psi_i.
\]

Otherwise, when \( F_y = 0 \), we set

\[
  \mathcal{A}_\zeta := \sum_{i=1}^{r} (w_i(F_x) - w_i(\psi'_i(\tau_i)))\Psi_i, \quad \text{where} \quad F_x := \frac{\partial F}{\partial x}.
\]

Note that \( F_x \) and \( F_y \) cannot be both identically zero because \( F \) is assumed to be absolutely irreducible. In addition \( \varphi'_i \) and \( \psi'_i \) cannot vanish simultaneously: if it were so then the characteristic \( p \) of \( \mathbb{K} \) would be positive and \( \varphi_i \) and \( \psi_i \) would belong to \( \mathbb{K}[[\tau_i^p]] \), which contradicts the fact that \( \tau_i \) is a uniformizing parameter. By differentiating both sides of the equality \( F(\varphi_i(\tau_i), \psi_i(\tau_i), 1) = 0 \) in \( \tau_i \) we obtain

\[
  F_x(\varphi_i(\tau_i), \psi_i(\tau_i), 1)\varphi'_i(\tau_i) + F_y(\varphi_i(\tau_i), \psi_i(\tau_i), 1)\psi'_i(\tau_i) = 0
\]

whence

\[
  w_i(F_y) - w_i(\varphi'_i(\tau_i)) = w_i(F_x) - w_i(\psi'_i(\tau_i)) \quad \text{for} \quad i = 1, \ldots, r.
\]

Note that \( \varphi'_i \neq 0 \) implies \( F_y(\varphi_i(\tau_i), \psi_i(\tau_i), 1) \neq 0 \), and \( \psi'_i \neq 0 \) implies \( F_x(\varphi_i(\tau_i), \psi_i(\tau_i), 1) \neq 0 \).

On the other hand \( w_i(F_y) - w_i(\varphi'_i(\tau_i)) \) is left unchanged after dilatations of the coordinates or when \( y \) is replaced by \( y + \alpha x \) for all \( \alpha \in \mathbb{K} \). Consequently, \( \mathcal{A}_\zeta \) is independent of the choice of coordinates. The (global) adjoint divisor \( \mathcal{A} \) of \( \mathcal{C} \) is the sum of the \( \mathcal{A}_\zeta \) for \( \zeta \) running over the singular points of \( \mathcal{C} \).
Remark 3 If $F_y \neq 0$ then $w_i(\varphi'_i(\tau_i)) \geq 0$, so we have $\deg A^i \leq \deg(\text{Div}(F_y))$ and therefore $\deg A \leq \deg(\text{Div}(F_y))$. Proposition \cite{11} implies that $\deg(\text{Div}(F_y)) = \delta(\delta - 1)$, where $\delta := \deg F$. More precisely, it is known that $\deg A \leq (\delta - 1)(\delta - 2)$ and that $\deg A$ is even, but the proofs of these facts go beyond the scope of the present paper. The quantity $g := \frac{1}{2}(\delta - 1)(\delta - 2) - \deg A$ is a nonnegative integer called the genus of $C$.

6 The Brill–Noether method

We are now ready to present the Brill–Noether method for the computation of Riemann–Roch spaces. We still assume that $K$ is algebraically closed and $F$ still denotes an irreducible homogeneous polynomial in $K[x, y, z]$ that defines a curve written $C$. We are given a divisor $D$ of $C$ and we want to compute a $K$-basis of the Riemann–Roch space

$$\mathcal{L}(D) := \left\{ \frac{A}{B} \in K(C) \setminus \{0\} : \text{Div}(A/B) \geq -D \right\} \cup \{0\}.$$ 

As sketched in the introduction, the Brill–Noether method divides into two steps. First, we look for a homogeneous polynomial $H$ prime to $F$ that satisfies $\text{Div}(H) \geq D + A$, where $A$ is the adjoint divisor of $C$ (defined in the previous section). Second, we compute a basis $G_1, \ldots, G_\ell$ modulo $F$ of homogeneous polynomials $G$ of degree $\deg H$ such that $\text{Div}(G) \geq \text{Div}(H) - D$. Essentially, in order to prove the correctness of this method, we shall show that such a polynomial $H$ does exist and then that any $A/B \in \mathcal{L}(D)$ can be written in the form $G/H$ in $K(C)$ for some homogeneous polynomial $G$ of degree $\deg H$. The latter problem corresponds to finding $G$ homogeneous of degree $\deg H$ such that

$$AH - BG \text{ belongs to the ideal generated by } F \text{ in } K[x, y, z]. \quad (14)$$

6.1 Residue theorem

Finding a solution to \cite{14} is the purpose of the so-called residue theorem. The proof of this theorem, given in the following paragraphs, begins with local conditions for a polynomial $A$ to belong to the ideal $(F, B)$. The following central lemma sheds light on the role of the adjoint divisor in the Brill–Noether method.

Lemma 9 Let $f$ be irreducible in $K[[x, y]]$, let $D := K[[x, y]]/(f)$, let $w$ denote the canonical valuation on $\mathbb{D}$, let $\tau$ be a uniformizing parameter, and let $\varphi$ and $\psi$ be in $K[[\tau]]$ such that $x = \varphi(\tau)$ and $y = \psi(\tau)$ hold in $\text{Frac}(\mathbb{D})$. Then, any $c \in \text{Frac}(\mathbb{D})$ that satisfies

$$w(c) \geq w(f_y) - w(\varphi' (\tau)) \text{ if } \varphi' \neq 0 \quad (15)$$

or

$$w(c) \geq w(f_x) - w(\psi' (\tau)) \text{ if } \psi' \neq 0, \quad (16)$$

where $f_x := \frac{\partial f}{\partial x}$ and $f_y := \frac{\partial f}{\partial y}$, necessarily belongs to $D$.

Proof By the same reasoning as in Section 5.5 neither $\varphi' = \psi' = 0$ nor $f_x (\varphi(\tau), \psi(\tau)) = f_y (\varphi(\tau), \psi(\tau)) = 0$ can hold. Differentiating $f(\varphi(\tau), \psi(\tau)) = 0$ yields

$$f_x (\varphi(\tau), \psi(\tau)) \varphi'(\tau) + f_y (\varphi(\tau), \psi(\tau)) \psi'(\tau) = 0, \quad (17)$$

so inequalities \cite{15} and \cite{16} are equivalent whenever $\varphi' \neq 0$ and $\psi' \neq 0$. In addition, note that $\varphi' = 0$ implies $f_y (\varphi(\tau), \psi(\tau)) = 0$, and that $\psi' = 0$ implies $f_x (\varphi(\tau), \psi(\tau)) = 0$.

If $x$ divides $f$ then $D := K[[y]]$, we can choose $\tau := y$, $\varphi(\tau) := 0$, and $\psi(\tau) := \tau$ so the lemma trivially holds. The case when $y$ divides $f$ behaves similarly. Thanks to Lemma \cite{5} if neither $x$ nor $y$ divide $f,
then we may assume that $f$ belongs to $\mathbb{K}[[x]][y]$, is monic in $y$ of degree $n \geq 1$, and satisfies $f(x, 0) \neq 0$, hence $D = \mathbb{K}[[x]][y]/(f)$. From Lemma 7, it follows that the Newton polygon of $f$ admits a single edge from $(0, m)$ to $(n, 0)$ where $m := \text{val}_f(f(x, 0)) \geq 1$. Based on these preliminary remarks, the proof of the lemma is done by decreasing induction on

$$\min(w(f_x), w(f_y)) \geq 0.$$ 

The induction ends when $\min(w(f_x), w(f_y)) = 0$. In fact if $w(f_x) = 0$, equivalently if $m = 1$ holds, then we may take $\tau := y$, $\psi(\tau) := \tau$. Then, inequality (16) simplifies to $w(c) \geq 0$, hence $c$ can be expressed as a power series in $y$, whence $c \in D$. The case when $w(f_y) = 0$ is handled in a similar fashion.

From now on we assume that $m \geq 2$ and $n \geq 2$. Up to swapping the variables $x$ and $y$ we may further assume that $n \leq m$. If $n < m$ then we define

$$\tilde{f}(x, z) := \frac{f(x, xz)}{x^n} \in \mathbb{K}[[x]][z], \quad (18)$$

that is irreducible and monic of degree $n$ in $z$. By Lemma 7 its Newton polygon admits a single edge from $(0, m - n)$ to $(n, 0)$. If $n = m$ then $\frac{f(x, xz)}{x^n}$ is also irreducible but the first vertex of its Newton polygon is $(0, 0)$. By the classical Hensel lemma (recalled at the beginning of Section 3), there exists $\zeta \in \mathbb{K}$ such that

$$\frac{f(x, xz)}{x^n}(0, z) = (z - \zeta)^n.$$ 

In this case we define

$$\tilde{f}(x, z) := \frac{f(x, x(z + \zeta))}{x^n} \in \mathbb{K}[[x]][z],$$

that is irreducible and monic of degree $n$ in $z$. Its Newton polygon has a single edge from $(0, \tilde{m})$ to $(n, 0)$ with $\tilde{m} > 0$. For convenience we set $\zeta := 0$ when $n < m$.

The ring

$$\tilde{D} := \mathbb{K}[[x, z]]/(\tilde{f}) = \mathbb{K}[[x]][z]/(\tilde{f})$$

is a discrete valuation ring: $\tilde{w}(a(x, z)) := w\left(a\left(x, \frac{y}{x} - \zeta\right)\right)$ is the canonical valuation on $\tilde{D}$, $\tilde{\tau}(x, z) := \tau\left(x, \frac{y}{x} - \zeta\right)$ is a uniformizing parameter, and $x = \varphi(\tilde{\tau})$ and

$$z = \frac{\psi(\tilde{\tau})}{\varphi(\tilde{\tau})} - \zeta =: \tilde{\psi}(\tilde{\tau})$$

hold in $\text{Frac}(\tilde{D})$.

We let $\tilde{c}(x, z) := c\left(x, \frac{y}{x} - \zeta\right)$. Let us first handle the case when $\varphi' \neq 0$. We have that

$$\tilde{f}_x(x, z) = \frac{f_y(x, x(z + \zeta))}{x^{n-1}}$$

and thus inequality (15) rewrites into

$$\tilde{w}\left(\frac{\tilde{c}(\varphi(\tilde{\tau}), \tilde{\psi}(\tilde{\tau}))}{\varphi(\tilde{\tau})^{n-1}}\right) \geq \tilde{w}(\tilde{f}_x(\varphi(\tilde{\tau}), \tilde{\psi}(\tilde{\tau})) - \tilde{w}(\varphi'(\tilde{\tau})).$$

Let us now handle the case when $\varphi' = 0$. As previously noted, we necessarily have $\psi' \neq 0$, $f_y(\varphi(\tau), \psi(\tau)) = 0$, and $f_x(\varphi(\tau), \psi(\tau)) \neq 0$. From

$$\tilde{f}_x(x, z) = \frac{f_x(x, x(z + \zeta)) + (z + \zeta)f_y(x, x(z + \zeta))}{x^n} - \frac{n}{x^{n+1}},$$

we deduce that

$$\tilde{f}_x(x, z) = \frac{f_x(x, x(z + \zeta))}{x^n}$$

holds in $\text{Frac}(\tilde{D})$. 22
Inequality \(16\) rewrites into
\[
\hat{w} \left( \frac{\hat{c}(\varphi(\tilde{\tau}), \tilde{\psi}(\tilde{\tau}))}{\varphi(\tilde{\tau})^n} \right) \geq \hat{w}(\tilde{f}_x(\varphi(\tilde{\tau}), \tilde{\psi}(\tilde{\tau}))) - \hat{w}(\psi'(\tilde{\tau})),
\]
which simplifies to
\[
\hat{w} \left( \frac{\hat{c}(\varphi(\tilde{\tau}), \tilde{\psi}(\tilde{\tau}))}{\varphi(\tilde{\tau})^{n-1}} \right) \geq \hat{w}(\tilde{f}_x(\varphi(\tilde{\tau}), \tilde{\psi}(\tilde{\tau}))) - \hat{w}(\psi'(\tilde{\tau})),
\]
by using
\[
\hat{w}(\psi'(\tilde{\tau})) = \hat{w} \left( \frac{\psi'(\tilde{\tau})}{\varphi(\tilde{\tau})} \right) = \hat{w} \left( \frac{\psi'(\tilde{\tau})}{\varphi(\tilde{\tau})^2} \right) = \hat{w}(\psi'(\tilde{\tau})) - \hat{w}(\varphi(\tilde{\tau})).
\]
Since \(\min(\hat{w}(\tilde{f}_x), \hat{w}(\tilde{f}_z)) < \min(w(f_x), w(f_y))\), the induction hypothesis implies that \(\hat{c}(x, z)/x^{n-1}\) belongs to \(\hat{D}\), so it can be written
\[
\hat{c}(x, z) = \hat{c}_{n-1}(x)z^{n-1} + \cdots + \hat{c}_0(x) + q(x, z)f(x, z)
\]
with the \(\hat{c}_i(x)\) taken in \(K[[x]]\), and where \(q(x, z) \in K((x))[z]\). It follows that
\[
c(x, y) = \hat{c}_{n-1}(x)(y - \zeta x)^{n-1} + \cdots + \hat{c}_0(x)x^{n-1} + x^{n-1}q(x, z)f(x, z),
\]
hence that
\[
c(x, y) = \hat{c}_{n-1}(x)(y - \zeta x)^{n-1} + \cdots + \hat{c}_0(x)x^{n-1} + x^{n-1}q \left( \frac{y}{x} - \zeta \right) f(x, y).
\]
Consequently \(c\) belongs to \(D\). \(\square\)

**Proposition 13** Let \(\zeta\) be a point of \(C\), and consider two homogeneous polynomials \(A\) and \(B\) that are prime to \(F\). If \(\text{Div}_\zeta(A) \geq \text{Div}_\zeta(B) + A_\zeta\) then \(A\) belongs to the ideal generated by \(F\) and \(B\) in \(K[x, y, z]_\zeta\).

**Proof** Without loss of generality, up to a change of coordinates, we may assume that \(\zeta = (0 : 0 : 1)\) and that \(F_y \neq 0\). The irreducible factorization of \(F(x, y, 1)\) in \(K[[x, y]]\) can be written \(u f_1 \cdots f_r\) where \(u\) is invertible and the \(f_i\) are irreducible monic polynomials in \(K[[x]][y]\), for \(i = 1, \ldots, r\). Let \(D_i := K[[x, y]]/(f_i(x, y))\), let \(\tau_i\) be a uniformizing parameter for \(D_i\), let \(w_i\) be its canonical valuation, and let \(\varphi_i, \psi_i \in K[[\tau_i]]\) be such that \(x = \varphi_i(\tau_i)\) and \(y = \psi_i(\tau_i)\) hold in \(\text{Frac}(D_i)\). We further define \(\hat{f}_i := f_1 \cdots f_{i-1} f_{i+1} \cdots f_r\) for \(i = 1, \ldots, r\).

By definition, the hypothesis \(\text{Div}_\zeta(A) \geq \text{Div}_\zeta(B) + A_\zeta\) rephrases into
\[
w_i \left( \frac{A}{B} \right) \geq w_i(F_y) - w_i(\varphi'_i(\tau_i)) \text{ for } i = 1, \ldots, r. \tag{19}\]

Since \(u\) is invertible in \(D_i\), we have \(w_i(u) = 0\). And since \(F_y(x, y, 1) \mod f_i = u \hat{f}_i \frac{\partial f_i}{\partial y} \mod f_i\) inequality \((19)\) becomes:
\[
w_i \left( \frac{A}{f_i} \right) \geq w_i \left( \frac{\partial f_i}{\partial y} \right) - w_i(\varphi'_i(\tau_i)) \text{ for } i = 1, \ldots, r.
\]
From Lemma \(9\) we deduce that \(\frac{A(x, y, 1)}{f_i B(x, y, 1)}\) belongs to \(D_i\): we write \(c_i \in K[[x]][y]_{< \text{deg}_y f_i}\) for its canonical representative. Let \(c(x, y)\) stand for the canonical representative of \(\frac{A(x, y, 1)}{B(x, y, 1)}\) in \(K((x))[y]/(f_1 \cdots f_r)\), that is a polynomial in \(K((x))[y]_{< \text{deg}(f_1 \cdots f_r)}\). The Chinese remainder formula
\[
c(x, y) = \sum_{i=1}^r c_i \hat{f}_i
\]
shows that \(c(x, y)\) belongs to \(K(x, y) \cap K[[x, y]] = K[x, y]_{(0, 0)}\) modulo \(F\). In other words, \(A(x, y, 1)\) belongs to the ideal \((F(x, y, 1), B(x, y, 1))\) regarded in \(K[x, y]_{(0, 0)}\). \(\square\)
Lemma 10  Let $F$ and $B$ be coprime homogeneous polynomials in $\mathbb{K}[x, y, z]$ such that $F(x, y, 0)$ and $B(x, y, 0)$ have no common root in $\mathbb{P}^1$. Then, $z$ is a non-zero divisor in $\mathbb{K}[x, y, z]/(F, B)$.

Proof  Assume that there exists $E, U, V \in \mathbb{K}[x, y, z]$ homogeneous such that

$$zE(x, y, z) = U(x, y, z)F(x, y, z) + V(x, y, z)B(x, y, z). \quad (20)$$

Substituting 0 for $z$ yields

$$0 = U(x, y, 0)F(x, y, 0) + V(x, y, 0)B(x, y, 0).$$

Since $F(x, y, 0)$ and $B(x, y, 0)$ are homogeneous and have no common root in $\mathbb{P}^1$, they are coprime. Therefore there exists a homogeneous polynomial $W(x, y)$ such that

$$U(x, y, 0) = W(x, y)B(x, y, 0)$$

$$V(x, y, 0) = -W(x, y)F(x, y, 0).$$

It follows that

$$U(x, y, z) = W(x, y)B(x, y, z) + O(z)$$

$$V(x, y, z) = -W(x, y)F(x, y, z) + O(z),$$

where $O(z)$ is a shorthand for homogeneous polynomials in $z\mathbb{K}[x, y, z]$. Plugging the latter expressions into (20), we deduce that

$$zE(x, y, z) = (W(x, y)B(x, y, z) + O(z))F(x, y, z) + (-W(x, y)F(x, y, z) + O(z))B(x, y, z) = O(z)F(x, y, z) + O(z)B(x, y, z).$$

This shows that $E$ belongs to $(F, B)$ and concludes the proof. \hfill \Box

Proposition 14  Consider two homogeneous polynomials $A$ and $B$ prime to $F$. If $\text{Div}(A) \supseteq \text{Div}(B) + A$ then $A$ belongs to the ideal $(F, B)$.

Proof  By Proposition 13, the polynomial $A$ belongs to the ideal generated by $F$ and $B$ in $\mathbb{K}[x, y, z]_\zeta$ for all $\zeta \in \mathcal{C}$. Thanks to Lemma 2, the coordinates may be changed linearly in order to ensure that $A_1$ and $A_2$ hold for $F$ and $B$ instead of $\tilde{F}$ and $G$. Then Lemma 1 implies that $\mathcal{V}_\mathcal{F}(F, B)$ is in the affine chart $z = 1$. By Proposition 1 the polynomial $A(x, y, 1)$ belongs to the ideal $(F(x, y, 1), B(x, y, 1))$. Consequently, after homogenizing, we obtain that the polynomial $z^m A(x, y, z)$ lies in the ideal $(F(x, y, z), B(x, y, z))$ for some nonnegative integer $m$. Using Lemma 10, we conclude that $A(x, y, z)$ belongs to this ideal. \hfill \Box

Two divisors $D$ and $\tilde{D}$ of $\mathcal{C}$ are said to be linearly equivalent if there exists a rational function $A/B \in \mathbb{K}(\mathcal{C})$ such that $D = \tilde{D} - \text{Div}(A/B)$.

Theorem 1  (Residue theorem) Let $D$ and $\tilde{D}$ be two distinct linearly equivalent divisors of $\mathcal{C}$ and let $H$ be a homogeneous polynomial prime to $F$. If $\tilde{D} \geq 0$ and $\text{Div}(H) \supseteq D + A$, then there exists a homogeneous polynomial $G$ prime to $F$, of degree $\deg H$, and such that $\text{Div}(G/H) = \tilde{D} - D$.

Proof  Let $A/B \in \mathbb{K}(\mathcal{C}) \setminus \{0\}$ be such that $D = \tilde{D} - \text{Div}(A/B)$. We verify that

$$\text{Div}(AH) \supseteq \text{Div}(A) + D + A = \text{Div}(B) + \tilde{D} + A.$$

Since $\tilde{D}$ is positive, we obtain $\text{Div}(AH) \supseteq \text{Div}(B) + A$. By Proposition 14, the polynomial $AH$ belongs to the ideal $(F, B)$. Consequently, there exists a homogeneous polynomial $G$ such that $BG - AH \in (F)$. It follows that $G$ is prime to $F$ and that $\text{Div}(BG) = \text{Div}(AH)$, whence $\text{Div}(G/H) = \text{Div}(A/B) = \tilde{D} - D$. \hfill \Box
6.2 Main theorem

Theorem 2 Let \( \mathbb{K} \) be an algebraically closed field. Let \( \mathcal{C} \) be an irreducible plane projective curve of equation \( F = 0 \), let \( \mathcal{A} \) be its adjoint divisor, and let \( D \) be a divisor of \( \mathcal{C} \). If \( H \) is a non-zero homogeneous polynomial prime to \( F \) that satisfies \( \text{Div}(H) \geq D + \mathcal{A} \), then \( H \) is a common denominator for all the elements of \( \mathcal{L}(D) \).

Proof Let \( A/B \) be a non-zero function in \( \mathcal{L}(D) \). We will show that there exists a homogeneous polynomial \( G \) of degree \( \deg H \) such that \( A/B \) equals \( G/H \) up to a constant factor.

By definition of \( \mathcal{L}(D) \), the divisor \( \tilde{D} := D + \text{Div}(A/B) \) is positive. The use of Theorem \( 1 \) with \( D, \tilde{D} \), and \( H \) yields a homogeneous polynomial \( G \) of degree \( \deg H \) such that

\[
\text{Div}(G/H) = \tilde{D} - D = \text{Div}(A/B).
\]

It follows that \( \text{Div}((G/H)/(A/B)) \) is zero, so \( (G/H)/(A/B) \) is a constant in \( \mathbb{K}(\mathcal{C}) \) thanks to Proposition \( 12 \).

If a common denominator \( H \) is known for \( \mathcal{L}(D) \), then \( G_1/H, \ldots, G_\ell/H \) is a \( \mathbb{K} \)-basis of \( \mathcal{L}(D) \) whenever \( G_1, \ldots, G_\ell \) is a \( \mathbb{K} \)-basis of the space of homogeneous polynomials \( G \in \mathbb{K}[x, y, z]/(F) \) of degree \( \deg H \) such that \( \text{Div}(G) \geq \text{Div}(H) - D \).

6.3 Common denominator

So far we have completed the presentation of the second step of the Brill–Noether method. The first step is less intricate because it reduces to elementary linear algebra. For this purpose it is useful to be more explicit about the representation of divisors. For a place \( \mathfrak{P} \) of \( \mathcal{C} \) centered at \( (0 : 0 : 1) \) and defined by \( f(x, y) = 0 \), we assume that we are given a uniformizing parameter \( \tau \) for \( \mathfrak{D} = \mathbb{K}[x, y]/(f) \) and we let \( w \) still denote the canonical valuation. By Proposition \( 9 \) there exist \( \varphi, \psi \in \mathbb{K}[\tau] \) such that \( x = \varphi(\tau) \) and \( y = \psi(\tau) \) hold in \( \text{Frac}(\mathfrak{D}) \).

Let \( m \) be a nonnegative integer and let \( H \in \mathbb{K}[x, y, z] \) be homogeneous of degree \( d \geq 0 \). By definition, the condition \( w(H) \geq m \) rewrites into

\[
\text{val}_w(H(\varphi(\tau), \psi(\tau), 1)) \geq m,
\]

that corresponds to \( m \) linear equations in the coefficients of \( H \). More generally, if \( D \) is a positive divisor, then the condition \( \text{Div}(H) \geq D \) corresponds to \( \deg D \) linear equations in the coefficients of \( H \).

For a divisor \( D = \sum_{\mathfrak{P}} c_\mathfrak{P} \mathfrak{P} \) we write \( D_+ := \sum_{\mathfrak{P}, c_\mathfrak{P} > 0} c_\mathfrak{P} \mathfrak{P} \) for the positive part of \( D \). The condition \( \text{Div}(H) \geq D + \mathcal{A} \) is equivalent to \( \text{Div}(H) \geq [D + \mathcal{A}]_+ \) that is further equivalent to a linear system with \( \deg([D + \mathcal{A}]_+) \) equations. The number of unknowns is the number of coefficients of \( H \), that is \( \binom{d+2}{2} \). Consequently this system admits a non-zero solution as soon as

\[
\binom{d+2}{2} > \deg([D + \mathcal{A}]_+).
\]

In other words, common denominators \( H \) of \( \mathcal{L}(D) \) do exist in sufficiently large degrees.

6.4 Algorithmic aspects

The Brill–Noether method is summarized in the following algorithm for when \( \mathbb{K} \) is algebraically closed.

Algorithm 1

Input An irreducible plane projective curve \( \mathcal{C} \) defined by the equation \( F = 0 \), and a divisor \( D \) of \( \mathcal{C} \).

Output A basis of \( \mathcal{L}(D) \).
1. Compute the adjoint divisor $\mathcal{A}$ of $\mathcal{C}$.
2. Find a homogeneous polynomial $H$ prime to $F$ such that $\text{Div}(H) \geq D + \mathcal{A}$.
3. Compute $\text{Div}(H) - D$.
4. Compute a basis $G_1, \ldots, G_\ell$ in $\mathbb{K}[x, y, z]/(F(x, y, z))$ of the space of the homogeneous polynomials $G$ of degree $\deg H$ such that $\text{Div}(G) \geq \text{Div}(H) - D$.
5. Return $G_1/H, \ldots, G_\ell/H$.

The existence of $H$ in step 2 has been addressed in the preceding subsection, so the correctness of the algorithm follows from Theorem 2. A more precise algorithmic description of each step goes beyond the aim of present paper. In fact a software implementation requires suitable and efficient data structures for divisors, procedures to operate on them, and algorithms to compute $\mathcal{A}$ and $\text{Div}(H)$. References to detailed algorithms are gathered in the next section.

In practice, Riemann–Roch spaces are often needed over fields $\mathbb{K}$ that are not necessarily algebraically closed. In fact if $F \in \mathbb{K}[x, y, z]$ is homogeneous and irreducible in $\mathbb{K}[x, y, z]$, then $\mathcal{A}$ is left unchanged by the Galois group of $\mathbb{K}$ over $\mathbb{Q}$ and if $D$ is also left unchanged by this Galois group, then the linear system of step 2 can be written with coefficients in $\mathbb{K}$. Therefore $H$ can be naturally taken in $\mathbb{K}[x, y, z]$, so $\text{Div}(H)$ is also preserved by this Galois group, and the linear system in step 4 can also be written with coefficients in $\mathbb{K}$. Consequently, Algorithm [1] can compute a $\mathbb{K}$-basis of $\mathcal{L}(D)$ represented by elements in $\mathbb{K}[x, y, z]$.

**Example 5** Let us take $\mathbb{K} := \mathbb{F}_2$ and $F(x, y, z) := y^3 + x^3 + x^2z$. The singular locus of $\mathcal{C} := \mathcal{V}_F(F)$ is the singleton $\{(0 : 0 : 1)\}$. The series $f(x, y) := F(x, y, 1) = y^3 + x^3 + x^2$ in $\mathbb{F}_2[[x, y]]$ is irreducible because, for the weight 3 for $x$ and 2 for $y$, its initial form $y^3 + x^2$ is irreducible. Let $\mathfrak{P}$ and $w$ denote the corresponding place and canonical valuation, so $w(x) = 3$ and $w(y) = 2$. The adjoint divisor of $\mathcal{C}$ is $\mathcal{A} = (w(F_y) - w(x) + 1)\mathfrak{P} = (4 - 3 + 1)\mathfrak{P} = 2\mathfrak{P}$. Since $w(y^2/x) = 1$, $\tau := y^2/x$ is a uniformizing parameter for $w$. In $\mathbb{K}(x)[y]/(f(x, y))$ we have $x = \tau^3 + O(\tau^4)$ and $y = \tau^2 + O(\tau^3)$.

Now let $\mathfrak{P}_1$ represent the place of $\mathcal{C}$ centered at the smooth point $(1 : 0 : 1)$. Then $\tau_1 := y$ is a uniformizing parameter and we have $x = 1 + \tau_1 + O(\tau_1^3)$ locally. Let us take $D := \mathfrak{P}_1$. In order to obtain a basis of $\mathcal{L}(D)$ we search for a homogeneous polynomial $H$ such that $\text{Div}(H) \geq D + \mathcal{A}$. It is easy to verify that $H = y$ is suitable and that $\text{Div}(H) = D + \mathcal{A}$. The second step of the Brill–Noether method consists in computing a basis of homogenous polynomials $G(x, y, z) = ax + by + cz$ such that $\text{Div}(G) \geq \text{Div}(H) - D = \mathcal{A}$. Since $G(x, y, 1) = c + b\tau^2 + O(\tau^3)$ in $\mathbb{K}(x)[y]/(f(x, y))$, we find that $c$ must be 0 hence that $\{x, y\}$ is a basis of solutions for $G$. Note that $x$ and $y$ are linearly independent in $\mathbb{K}(x, y, z)/(F(x, y, z))$. Finally $\{1, x/y\}$ is a basis of $\mathcal{L}(D)$.

7 Notes

Good expositions of the Residue theorem stated in Theorem 1 can be found in several text books about algebraic geometry, for instance [22 [11]. Here the term “residue” comes from the notion of residual sets of points on a curve defined in [21 p. 273], and it does not correlate with the other classical Residue theorem for differential forms [43 Chapter 4, Corollary 4.3.3]. We conclude this paper with a list of articles dedicated to the Brill–Noether method and its software implementation.

**Adjoint divisor.** The notion of adjoint divisor for plane curves was introduced by Brill and Noether [9] for curves with only ordinary singularities. The extension to any curve is due to Gorenstein [25]. Then, other definitions have been studied by various authors [34, 5, 23, 17, 27]. Discarding a few subtleties, these definitions are essentially equivalent; see details in [26, 27, 23]. The proof given for Lemma 9 hides the usual desingularization method by successive blow-ups (through equation 18), used in [23] for instance. When the characteristic is zero, a simpler proof based on Lagrange interpolation is given in [1 Section 3.3]. It would be interesting to adapt it to positive characteristic.
Algorithms. The Brill–Noether method \cite{BrillNoether} to compute Riemann–Roch spaces was originally restricted to curves with only ordinary singularities. Le Brigand and Risler extended the method to arbitrary plane curves in \cite{LeBrigandRisler}. Their approach led to a software implementation by Haché \cite{Hache1,Hache2,Hache3}. Other algorithms in the vein of the Brill–Noether method have been proposed by Huang and Ierardi \cite{HuangIerardi} still for ordinary curves, and by Volcheck \cite{Volcheck} and Khuri-Makdisi \cite{Khuri-Makdisi} for computing in the Jacobian of general curves (essentially) in characteristic zero. Over fields of any characteristic, Campillo and Farrán designed a method based on Hamburger–Noether expansions \cite{CampilloFarran}. Implementations of Brill–Noether variants for general curves is available within the SINGULAR computer algebra system \cite{SingularCASA,SingularCASA2}.

More recently, faster algorithms have been designed for smooth input divisors of nodal curves \cite{AbelardBerardiniCouvreurLecerf} \cite{AbelardCouvreurLecerf}, of ordinary curves \cite{AbelardCouvreurLecerf2}, and then of any curve in characteristic zero (or positive but sufficiently large) \cite{AbelardBerardiniCouvreurLecerf2}. Up to the present time the design of a fast practical algorithm for any curve in any characteristic and for any divisor is an active research topic.

The family of algorithms derived from the Brill–Noether approach is often called “geometric”. Finally, let us mention that another family of algorithms for Riemann–Roch spaces, called “arithmetic”, exists. The current state-of-the-art algorithm of this family is due to Hess \cite{Hess}: it supports any curve and any characteristic, and relies on integral closure computations in algebraic function fields. More references in this direction can be found in \cite{AbelardCouvreurLecerf2}.

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