The effect of environmental coupling on tunneling of quasiparticles in Josephson junctions

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Abstract

We study quasiparticle tunneling in Josephson tunnel junctions embedded in an electromagnetic environment. We identify tunneling processes that transfer electrical charge and couple to the environment in a way similar to that of normal electrons, and processes that mix electrons and holes and are thus creating charge superpositions. The latter are sensitive to the phase difference between the superconductors and are thus limited by phase diffusion even at zero temperature. We show that the environmental coupling is suppressed in many environments, thus leading to lower quasiparticle decay rates and better superconductor qubit coherence than previously expected. Our approach is nonperturbative in the environmental coupling strength.

(Some figures may appear in colour only in the online journal)

1. Introduction

The physics of micro- and nanoscale Josephson junctions is a paradigmatic application of macroscopic dissipative quantum mechanics of open systems [1–3]. This, on the one hand, makes them ideal test-beds for that theory, with largely tunable parameters [4, 5]. On the other hand, Josephson junctions have an ample range of applications in sensing [6], amplification [7], metrology [8] and quantum information processing [9–12]. For these applications, it is imperative to thoroughly understand the dissipative quantum physics of Josephson junctions in order to achieve optimal device performance.

For a long time, work on this topic has focused on the transport properties of Josephson junctions, their current–voltage and noise characteristics [13]. The ‘\(P(E)\)'-theory of treating environmental fluctuations [3, 14] is established as a powerful tool in the case of small junctions when (Josephson or quasiparticle) tunneling can be treated perturbatively.

For applications in quantum computing, another question in this framework has emerged [15, 16]: rather than computing the current–voltage characteristic, the total quasiparticle transition rate is the quantity of interest. This is important because quasiparticle transitions in any direction are highly detrimental to qubit coherence [17]. In [17, 15, 16, 18, 19] this problem is studied in an approach that is perturbative in the coupling to the environment as measured by the environmental impedance in units of the quantum resistance \(R_Q = h^2/e^2 \approx 25.8 \, k\Omega\). In this paper we are going to generalize that study to arbitrary system–environment interaction strength. Some of the processes mix the electron-like and hole-like branches of the quasiparticle spectrum and depend on the phase difference between the superconductors. These are generally phase-dependent but highly sensitive to the
environment. These observations lead to slower relaxation than expected in previous work.

The paper is organized as follows. We introduce the mathematical formulation of the tunneling model in section 2.1. We will compute the tunneling rate in lowest order of the tunnel coupling but to all orders in the coupling of the quasiparticles to the environment in section 2.2, which can be a general linear impedance. We specialize to it being an undamped harmonic oscillator in section 3.1, which directly relates to [15, 16, 18, 19] but shows that the phase-dependent component is usually reduced due to dressing by zero-point fluctuations. We discuss an overdamped environment that may occur in other applications in section 3.2.

2. Mathematical formulation

2.1. Model

We start from the Hamiltonian

$$\hat{H} = \hat{H}_{BCS,1} + \hat{H}_{BCS,2} + \hat{H}_T + \hat{H}_{env} + \hat{H}_{env-c}. \quad (1)$$

Here, we have the BCS Hamiltonians [20, 21] in the field form describing the two electrodes

$$\hat{H}_{BCS,i} = \sum_{k, \sigma} \left( \xi_k - \mu_i \right) \hat{c}^{\dagger}_{k, \sigma} \hat{c}_{k, \sigma} + \Delta_i \sum_{k} \hat{c}_{k, \uparrow} \hat{c}_{k, \downarrow} + \Delta_i^* \sum_{k} \hat{c}^{\dagger}_{k, \downarrow} \hat{c}^{\dagger}_{k, \uparrow} \quad (2)$$

where $\xi_k = \frac{\hbar^2 k^2}{2m}$ is the kinetic energy of the electron and the $\mu_i$ are the chemical potentials in superconductors $i = 1, 2$, which are $\mu_1 = E_F$ and $\mu_2 = E_F + eV$ where $V$ is the applied voltage. We keep $\Delta_i = |\Delta| e^{i\phi/2} \text{ complex in order to allow for a phase difference.}$ The tunneling Hamiltonian is

$$\hat{H}_T = \sum_{kj} \left( T_{kj} \hat{c}^{\dagger}_{k, \sigma, 1} \hat{c}_{l, \sigma, 2} + T_{kj}^* \hat{c}^{\dagger}_{l, \sigma, 2} \hat{c}_{k, \sigma, 1} \right). \quad (3)$$

Before discussing the electromagnetic environment, we diagonalize the BCS Hamiltonians, equation (2), through the Bogoliubov transformation

$$\begin{align*}
\hat{c}_{k, \sigma} &= u_{k,i} \hat{\gamma}_{k, \sigma} + v_{k,i} \hat{\gamma}_{-k, \sigma}^\dagger, \\
\hat{c}_{k, \sigma}^{\dagger} &= -v_{k,i} \hat{\gamma}_{k, \sigma}^\dagger + u_{k,i} \hat{\gamma}_{-k, \sigma}.
\end{align*}$$

Here, the BCS coherence factors are defined

$$\begin{align*}
u_{k,i} &= \sqrt{\frac{1}{2} \left( 1 + \frac{\xi_k - \mu_i}{E_k} \right)} e^{i\phi/2}, \\
v_{k,i} &= \sqrt{\frac{1}{2} \left( 1 - \frac{\xi_k - \mu_i}{E_k} \right)} e^{-i\phi/2} \quad (4)\quad (5)
\end{align*}$$

where we introduced the quasiparticle energy $E_k = \sqrt{(\xi_k - \mu_i)^2 + |\Delta|^2}$. This allows us to rewrite the tunneling Hamiltonian, equation (3), in the following form:

$$\hat{H}_T = \sum_{k, \sigma} \left( T_{kj} |u_{k1}|u_{l2}|e^{i\phi/2} - T_{kj}^* |v_{k1}|v_{l2}|e^{-i\phi/2} \right)$$

$$\times \left( \hat{\gamma}^{\dagger}_{-k, \sigma} \hat{\gamma}_{l, \sigma}^\dagger + \hat{\gamma}^{\dagger}_{-k, \sigma} \hat{\gamma}_{l, \sigma} \right) + H_{T2} \quad (6)$$

where we introduced the phase difference $\phi = \phi_1 - \phi_2$. The term $\hat{H}_{T2}$ contains operators that change the number of quasiparticles by two, i.e., contain terms of the structure $\hat{\gamma} \hat{\gamma}^\dagger$ and $\hat{\gamma}^\dagger \hat{\gamma}^\dagger$ hence changing the total number of quasiparticles in the setup, which do not contribute to the quasiparticle rate—these terms contribute to the Josephson and Andreev processes. We now want to evaluate the Fermi golden rule rate for a transition that transfers a quasiparticle from electrode 1 to electrode 2. The relevant matrix element is

$$\langle (N_1 - 1)_{\sigma} , (N_2 + 1)_{\sigma} | \hat{H}_T | N_1_{\sigma} , N_2_{\sigma} \rangle = T_k e^{i\phi/2} |u_{k1}|u_{l2}|$$

$$- T_{kj}^* e^{-i\phi/2} |v_{k1}|v_{l2}| .$$

We now use that for a nonmagnetic barrier, $T_{kj} = T_{kj}^*$ can be chosen real. In the Fermi golden rule transition rate, we need the absolute square

$$|\langle (N_1 - 1)_{\sigma} , (N_2 + 1)_{\sigma} | \hat{H}_T | N_1_{\sigma} , N_2_{\sigma} \rangle|^2 = T_k^2 |u_{k1}|^2 + |v_{l2}|^2 - 2 |u_{k1}| |v_{l2}| \cos \phi$$

where $u(E, E') = |u_1(E)u_2(E')|$ and $v = |v_1(E)v_2(E')|$ with $u_{1/2}(E)$ and $v_{1/2}(E)$ given by the BCS coherence formulas, equations (4) and (5). Thus, the transition probability contains an interference term that is sensitive to the phase across the junction. This term describes a process that, even though it transfers a genuine quasiparticle from electrode 1 to electrode 2, actually consists of a superposition of electron and hole transfer, i.e., it is not diagonal in charge space, see the diagrammatic representation in figure 2. We will see later on how this is important in the sensitivity to the environment. This term bears analogy to the famous $\cos \phi$ term in the Josephson effect [22–25].

2.2. Tunneling rate

In order to capture the influence of the environment, we apply the ideas of $P(E)$-theory [3]. There, the environmental Hamiltonian is described by an oscillator bath $\hat{H}_{bath} = \sum_{\omega_0 \sigma} \hat{a}^{\dagger}_\omega \hat{a}_\omega$ and couples the oscillators linearly to our quasiparticle system $\hat{H}_{env-c} = \hat{N}_1 e^{i\phi} \hat{V}$, where we assume that only the first reservoir fluctuates—this corresponds to a specific choice of gauge. The total number and voltage operators are

$$\hat{N}_1 = \sum_{k, \sigma} \hat{c}^{\dagger}_{k \sigma} \hat{c}_{k \sigma}, \quad \hat{V} = \sum_{\lambda} \lambda_i \left( \hat{a}_i + \hat{a}_i^\dagger \right). \quad (7)$$

We can now work out the total tunneling rate by summing over momentum states at constant energy from the initial state $|i\rangle = |E\rangle$ to the final state $|f\rangle = |E'\rangle$ and considering the matrix element of the environmental Hamiltonian $\langle E | T_{kj} \rangle \langle \gamma^\dagger_{\sigma} | \gamma_{k \sigma} | E' \rangle$ thermal distribution of quasiparticles. The transition rate can be written formally as

$$\Gamma_1 (V) = \frac{1}{e^2 R_0} \int_{-\infty}^{\infty} dE dE' D_1 (E)D_2 (E')$$

$$\times f_1 (E) [1 - f_2 (E')] P_{\text{tot}} (E, E') \quad (8)$$

where $D_i$ is the reduced density of states and $f_i$ is the corresponding distribution function on the lead $i$. We consider
here that the nonequilibrium quasiparticle tunneling keeps the system in thermal and chemical-potential equilibrium. The influence of the external circuitry on quasiparticle tunneling is encoded in the dressed vertex $P_{\text{tot}}(e)$, the probability density for exchanging energy $\epsilon$ with the environment. From the Fermi golden rule the tunneling transition rate can be written in more detail as

$$
\Gamma_1(V) = \frac{1}{e^2 R_N} \int dE d' E' D_1(E) D_2(E') f_1(E) (1 - f_2(E')) \times \sum_{R, R'} (R' | H_T | R)^2 P_R(R) \times \delta(E + eV - E' + E_R - E_{R'}),
$$

where $P_R(R) = [R | \rho_R | R]$ for $\rho_R = e^{-\beta H_{\text{env}}} / \sum e^{-\beta H_{\text{env}}}$ and the arrow on $\Gamma$ indicates the direction of tunneling from lead 1 to 2. $R_N$ is the tunnel resistance of the junction in the normal state\(^5\). Considering that the phase has fluctuations around the classical value $\phi = \varphi + \delta \phi$ and substituting all of these one can get the transition rate

$$
\Gamma_1(V) = \frac{4}{e^2 R_N} \int dE dE' \int \frac{d\tau}{2 \pi h} e^{i E + eV - E'} \times D_1(E) D_2(E') f_1(E) (1 - f_2(E')) \times \left( (u^2 + v^2) e^{-i (\phi_0 + i \delta \phi(t)) / 2} - 2 \mu \cos \varphi e^{-i (\phi_0 + i \delta \phi(t)) / 4} \right) \times e^{-i (\delta \phi(t)/2)} \times (\delta \phi(t)/2) \times e^{-i (\phi_0 + i \delta \phi(t))/4}.
$$

(9)

where $u(E, E') = |u_1(E)u_2(E')|$ and $v = |v_1(E)v_2(E')|$ with $u_{1/2}(E)$ and $v_{1/2}(E)$ given by the BCS formulas, equations (4) and (5).

Note that the phase fluctuation defined here by the effective voltage across the junction

$$
\frac{\delta \phi(t)}{2} = \frac{e}{h} \int_0^t d\tau' \delta V(\tau')
$$

is the conjugate to the Cooper pair charge. The convention used here is consistent with assuming the resistance quantum to be $R_N / 4 = h / 4 e^2$.

The vertex probability is defined as the probability density for exchanging energy $\epsilon$ between the system and the environment and is defined formally as

$$
P_{\text{tot}}(E, E') = \int_{-\infty}^\infty \frac{d\tau}{2 \pi h} e^{i E + eV - E'} t.
$$

(11)

By comparing equations (8) and (9) one can correctly expect the definitions $e^{i(t)} = (u^2 + v^2) e^{i J_0(t) - 2 \mu \cos \varphi e^{i J_0(t)/4}}$, where $e^{i J_0(t)} = (e^{i \phi_0(t)} - e^{-i \phi_0(t)})/2$, $e^{i J_0(t)/2} = i e^{-i (\phi_0(t) + i \delta \phi(t))/4}$ and $e^{i J_0(t)/4} = i e^{-i (\phi_0(t) + i \delta \phi(t))/4}$. From these equalities one can simplify the definitions to $J_0(t) = (i (\delta \phi(t) - \phi_0(t) \delta \phi(t))/4$ and $J_0(t) = -i (\delta \phi(t) + \phi_0(t) \delta \phi(t))/4$. In this notation, one needs to pay attention to the observation that whereas the $J_{0/2}(t)$ are purely properties of the environment, the combined quantity $\exp(-J(t))$ inevitably contains coherence factors of the superconductor. Concurrently, we decompose $P_{\text{tot}}$ into two contributions

$$
P_{a/b}(E) = \int \frac{dt}{2 \pi h} e^{i E t} e^{-i J_{a/b}(t)}.
$$

(12)

Note that in the definitions we considered that the mean value of the phase is set by the external bias and the effect of the environment is summarized into the presence of phase fluctuations about this mean value. The symmetric combination $J_a(t)$ captures all-electron and all-hole processes and the corresponding vertex $P_a(E)$ coincides with the all-electron $P(E)$ known from [3].

However, $J_b(t) = (\langle \delta \phi(t) + \phi_0(t) \delta \phi(t) \rangle / 4) = J_a(t) + (\delta \phi(t)/2)$ captures processes that mix electrons and holes. Although $J_b(0) = 0$ the asymmetric correlation function does not vanish at $t = 0$ as it becomes $J_b(0) = (\langle \delta \phi^2(t) \rangle / 2)$ which captures zero-point fluctuation. $P_b(E)$ is a normalized probability as $\int P_b(E) dE = \exp(-J_b(0)) = 1$; however, $P_b(E)$ is not normalized, $\int P_b(E) dE = \exp(-J_b(0)) = \exp(\delta \phi^2(0)/2)$, as it describes electron–hole coherence for which no conservation law should be expected. We can understand this as follows. Even though electron–hole mixing processes are diagonal in the Fock space of quasiparticles, they are off-diagonal in charge space. The electromagnetic noise couples to these electrical charges. The environmental modes dress the charge in an attempt to measure the charge and localize it. This has an analogy: creation of a superposition of charge states occurs, following the charge–flux uncertainty relation, whenever the phase across a Josephson junction is becoming localized, i.e., when a Josephson junction is behaving classically. Thus, on the one hand, the scaling factor $\exp(\delta \phi^2(t)/2)$ directly measures the degree of charge localization. On the other hand, research aiming at maximum supercurrent (hence maximally localized phase and maximally extended charge) in small Josephson junctions in a dissipative environment finds $\exp(-\langle \phi(t)^2 \rangle/2)$ to be the relevant reduction factor [26, 27].

In order to compute $J_{a/b}$ we introduce an oscillator bath model and match its properties to the fluctuation-dissipation theorem as it describes Johnson-Nyquist noise. Applying the standard procedures used in $P(E)$-theory, we find

$$
S(t) = \langle \delta \phi(t) \delta \phi(0) \rangle = 2 \int_{-\infty}^\infty \frac{d\omega}{\omega} \text{Re} \text{Z}_{\text{eff}}(\omega) \frac{e^{-i \omega t}}{1 - e^{-\beta \omega h}}.
$$

(13)

Notably, this integral seems to diverge at its infrared end, where it formally appears to be $\pm \int_{-\infty}^\infty \frac{d\omega}{\omega} \text{Re} \text{Z}_{\text{eff}}(\omega)$ at $T = 0$ and $\propto T \int \frac{d\omega}{\omega} \text{Re} \text{Z}_{\text{eff}}(\omega)$ at $T > 0$. In the combination $4J_b(t) = S(t) - S(0)$ this divergence is removed, but the same argument does not apply to $4J_a(t) = S(t) + S(0)$. Thus, $\text{Z}_{\text{eff}}(\omega) \neq 0$ enforces $P_b = 0$, meaning that all electron–hole mixed processes would disappear. We will see later that this does not occur in standard physical environments.

\(^5\) By expanding the Dirac delta as a temporal integral $\delta(x) = (2\pi h)^{-1} \int_{-\infty}^\infty \frac{d\tau}{2\pi h} e^{-i \tau x}$ and writing the operator $(R' | e^{-E_{\text{env}}'} H_T e^{E_{\text{env}}'} | R) = (R' | H_T | R)$ the time evolution of the Hamiltonian makes the rate become proportional to $\langle H_T(t) H_T(0) \rangle$, where $\langle \cdot \cdot \cdot \rangle = \sum_{R,R'} (R' | \cdot \cdot \cdot | R)$. Substituting the tunneling Hamiltonian into this formula one gets $\langle H_T(t) H_T(0) \rangle = |(\alpha e^{i \phi_0(t)} - e^{-i \phi_0(t)}) (\alpha e^{-i \phi_0(t)} - e^{i \phi_0(t)})| / (\alpha e^{i \phi_0(t)} - e^{-i \phi_0(t)}) |(\alpha e^{-i \phi_0(t)} - e^{i \phi_0(t)})| = \alpha e^{i \phi_0(t)} - e^{-i \phi_0(t)} \alpha e^{-i \phi_0(t)} - e^{i \phi_0(t)} = e^{-i (\phi(t) + i \delta \phi(t))/4}$ and $e^{i (\phi(t))} e^{i (\phi(t))/2} = e^{-i (\phi(t) + i \delta \phi(t))/4}$. From these equalities one can simplify the definitions to $e^{-i (\phi(t) + i \phi(t))/4}$.
The final expression for the probability density $P(E, E')$ is thus

$$P_{\text{tot}}(E, E') = \frac{d}{2\pi} \frac{e^{-(E+E'-E')/2} e^{-i(E+E'-E')}}{Z(e^{-i(E+E'-E')})} \times \left[ (u^2 + v^2) e^{i\frac{S_0}{e}} - 2uv \cos \varphi e^{-i\frac{S_0}{e}} \right]. \tag{14}$$

### 3. Models for the environment

In order to evaluate the typical $P_{\text{s}}(E)$ and the resulting tunneling rates, we need to project likely models of the electromagnetic environment providing the impedance $Z_{\text{eff}}$. Following the resistively and capacitatively shunted junction (RCSJ) model and its microscopic analogs \cite{28, 29, 2, 1}, we consider the quasiparticle channel to be put in parallel to the supercurrent, the Josephson channel, the junction capacitance, and an external impedance $Z(\omega)$, see figure 1. We model the linear impedance of the Josephson channel by its Josephson inductance $L_J = \Phi_0/(2\pi I_c)$. The total effective impedance of this parallel setup leads to

$$\text{Re} Z_{\text{eff}} = \frac{\omega^2 \text{Re}(Z)}{C^2 |Z|^2} \left( \frac{1}{(\omega^2 - \omega_p^2)^2 + \omega_p \text{Im}(Z)^2} + \frac{\omega^2}{c^2 |Z|^2} \right) \tag{15}$$

where $\omega_p = (C L_J)^{-1/2}$ is the junction’s plasma frequency. In the case of a simple resistor or lossless transmission line, $Z(\omega) = R$ and we find

$$\text{Re}[Z_{\text{eff}}] = \frac{\omega^2}{RC^2} \left( \frac{1}{(\omega^2 - \omega_p^2)^2 + \omega^2/(RC)^2} \right). \tag{16}$$

Here, we can observe that at low frequencies, $\text{Re} Z_{\text{eff}} \propto \omega^2$. Physically, the reason for this is that the noise from the resistor gets shunted through the Josephson junction at low frequencies and does not affect the quasiparticle channel. Thus, although $P_s$ will be normalized to a value smaller than unity, the scenario of it scaling all the way to zero does not occur. This is consistent with the fact that phase qubits do show a zero-voltage current. The same general conclusion holds for other physical environments tested.

#### 3.1. Infinite-quality environmental mode

Given the high quality of qubit junctions, it is appropriate to start from the limit of an infinite-quality factor, $R \to \infty$. In that case, the effective impedance reduces to

$$Z_{\text{eff}} = \frac{\pi}{2C} \left[ \delta(\omega-\omega_p) + \delta(\omega+\omega_p) \right]. \tag{17}$$

We introduce the dimensionless parameter wave impedance $\rho_e = 4\pi Z_0/R_K$, where $Z_0 = \sqrt{L_I/C}$, which emerges from equation (17) and $\omega_p = 1/\sqrt{L_I C}$ and $L_I = \Phi_0/2\pi I_c$. The index $c$ indicates in this definition the Cooper pair quantum of resistance $R_K/4$ was considered, differently from [3]. The Cooper pair vacuum fluctuation is controlled by its corresponding wave impedance $S(0) = \rho_e$. The total environmental transition probability is defined as

$$P(E) = \sum_{k=-\infty}^{\infty} p_k(\rho_e, \omega_p, T) \delta(\omega-k\omega_p), \tag{18}$$

which has the form of a series of sidebands, corresponding to the emission/absorption of $k$ photons to/from the plasma mode. The weight in the low temperature limit of $k_B T \ll \hbar \omega_p$ is Poissonian, $p_k(\rho_e, \omega_p) = e^{-\rho_e/4}(\rho_e/4)^k/k!$. This coincides with that of [16] in the limit of $\rho_e \to 0$. The total quasiparticle tunneling rate hence reads

$$\tilde{\Gamma}_1 = \sum_{n=-\infty}^{\infty} \Gamma_{1n} = \frac{4}{R_1 e^2} \sum_{n=-\infty}^{\infty} \int_{\Delta} dE D_1(E) D_2(E + n\hbar \omega) f_1(E) \times (1 - f_2(E + n\hbar \omega)) \left[ (u^2(E, E + n\hbar \omega) + v^2(E, E + n\hbar \omega)) e^{i\frac{S_0}{e}} - 2uv(E, E + n\hbar \omega) \cos \varphi e^{-i\frac{S_0}{e}} \right] e^{-\frac{\hbar}{k} p_n(\rho_e, \omega, T)}.$$

For large capacitance junctions one can Taylor expand in small phase fluctuations $(u^2 + v^2) \exp[-S(t)/4] - 2uv \cos \varphi \exp[-S(t)/4] \sim (u - v)^2 + (u + v)^2 S(t)/4$. Considering that the phase fluctuation is small and stable, i.e. $S(t) \approx S(0) = \rho_c$, the final result is consistent with the results taken from [15, 24] except that ours shows that there is a prefactor $\exp(-(\delta \phi(0)^2)/4)$ in the rate originating from zero-point fluctuations that can reduce the quasiparticle environmentally mediated transition rate.

In order to finally compute the rate, we use equations (4) and (5). One can show that

$$u_1 u_2 v_1 v_2 = \frac{1}{2} \left( 1 + \frac{\xi_1 E_1}{E_1 E_2} \right), \tag{19}$$

$$u_1 u_2 v_1 v_2 = \frac{\Delta_1 \Delta_2}{4E_1 E_2}.$$

Substituting the density of states $D(E) = E/\xi = E/\sqrt{E^2 - \Delta^2}$ and the coherence factors in the transition rate and ignoring the pure thermal transition we arrive at the...
distribution function, the rate is

\[ \Gamma_1 = \sum_{n=-\infty}^{\infty} e^{-\frac{E_n}{T}} \frac{(\frac{e}{T})^n}{n!} \]

\[ \times \left( \frac{\Gamma_{\text{bare}}}{2} + \frac{2}{e^2 R_N} \int_{\Delta}^{\infty} dE f_1(E)(1 - f_2(E + \hbar \omega)) \right) \]

\[ \times \frac{E (E + \hbar \omega) + \Delta_1 \Delta_2}{\sqrt{(E^2 - \Delta_2^2)((E + \hbar \omega)^2 - \Delta_2^2)}} \]  

(20)

where \( \Gamma_{\text{bare}} = (4/R_N e^2) \int_{\Delta}^{\infty} dE f_1(E)(1 - f_2(E)) \) and corresponds to the normal electron tunneling rate under a bias voltage, and the second term in equation (20) is a dressed term rooted from the quasiparticle tunneling. We see several nonperturbative features in this expression. One is, of course, the occurrence of higher-order sidebands corresponding to exchange of multiple photons with the environment. The other one is that, as a consequence of normalization of the total probability, even the single-photon peak obtains a nonperturbative weight factor. In the limit of large superconducting gap \( \Delta \gg \hbar \omega \) and a large capacitance junction, one particle exchange at low-lying quasiparticle levels indicates that the total one particle rate becomes

\[ \Gamma_1 = e^{-\frac{\hbar \omega}{4}} \frac{\rho_c}{4} \Gamma_{\text{bare}} + \frac{2}{e^2 R_N} e^{-\frac{\hbar \omega}{2}} \frac{\rho_c}{4} \]

\[ \times \int_{\Delta}^{\infty} dE f_1(E)(1 - f_2(E + \hbar \omega)) \]

\[ \times \frac{E (E + \hbar \omega) + \Delta_1 \Delta_2}{\sqrt{(E^2 - \Delta_2^2)((E + \hbar \omega)^2 - \Delta_2^2)}} \],  

(21)

where the second term in the parenthesis of equation (21) is in fact the dressed tunneling rate \( \Gamma_{\text{dr.1}} \). Thus, we directly see the reduction of the phase-sensitive tunneling term by zero-point fluctuations described above made quantitative.

This rate depends on the details of the energy distribution function. One of us [30] derived the explicit temperature dependence of this rate out of equilibrium. For a junction with small macroscopic phase in thermal equilibrium at low temperatures \( T \ll \Delta \), by substituting the Fermi–Dirac distribution function, the rate is

\[ \Gamma_1 = \frac{\Delta}{2 e^2 R_N} \rho_c \exp \left( \frac{E_{\text{if}}}{2 k_B T} - \frac{\Delta}{k_B T} - \frac{\rho_c}{2} \right) K_0 \left( \frac{E_{\text{if}}}{2 k_B T} \right) \]  

(22)

where \( E_{\text{if}} \) is the parity transition energy in the qubit from odd to even states and \( K_0 \) is a Bessel function. This result is different from that of [24, see equation (35)] by the dressing factor \( \exp \left( -\sqrt{E_{\text{if}}/2E_1} \right) \). More general formulations for arbitrary phase \( \varphi \) are worked out in [30].

For phase qubits, whose impedance is engineered to be \( Z_0 \approx 50 \), this correction does not qualitatively change the physics; however, it improves the quality of fits to the data. In other types of qubits with higher impedance these corrections will be crucial. A qubit relaxation/excitation experiment involving nonequilibrium quasiparticles [15] will attempt to relax the qubit energy splitting, which is \( \xi \omega_0 \), into quasiparticles; hence, only the main band is relevant. For a quantitative estimate we can identify

\[ \rho_c = \sqrt{\frac{2E_c^2}{E_1}} \]  

(23)

where the charging energy is defined for electrons, i.e. \( E_c = e^2/2C \).

Note that the \( \rho_c^2 \) in the second term of equation (21) provides the coefficient \( 2E_c/E_1 \) for the \( \Gamma_{\text{dr.1}} \), which makes this term equivalent to equation (2) in [15]. For a transmon \( E_1/E_c \approx 30 \) and for a flux qubit \( \approx 50 \), making this correction large enough to be visible in a reduction of the quasiparticle rate. More explicitly, for a transmon we get a difference of around 7% from the perturbative approximation \( (1 - e^{-\rho_c/2}) \approx \rho_c/2 \) and for a flux qubit we get a deviation of around 5%. For a traditional charge qubit, \( \rho_c \) is large, leading to a different regime where linearization of phase fluctuations is impossible. From [31], \( E_1/E_c \approx 0.35 \), and with this caveat we can estimate a large deviation of around 40% from the perturbative approximation.

### 3.2. Overdamped environmental mode

We now study the opposite case, an environmental mode that is overdamped by an external impedance. Overdamping means that the width of the resonance in \( Z_{\text{eff}} \), equation (16), is larger than its frequency, i.e. \( \omega_0 R C \ll 1 \) or equivalently \( R \ll Z_0 \). In that case, to lowest order in the small parameter \( r = \frac{\rho_c}{Z_0} \), the poles of \( Z_{\text{eff}} \), equation (16), are at \( \omega = \pm i\gamma \) and at \( \omega = \pm i\gamma r^2 \), where \( \gamma = 1/RC \). Note that unlike other work for overdamped oscillators [32] we keep the next-to-leading order which keeps the second set of poles at zero, rendering the quasiparticle current nonzero in view of equation (14). This will have an important consequence later. We can now, in the overdamped case, rewrite equation (16) as

\[ Z_{\text{eff}} \simeq R \gamma^2 \frac{1}{1 - \rho_c^2} \left[ \frac{1}{\omega^2 + \gamma^2} - \frac{r^4}{\omega^2 + \gamma^2 + r^4} \right] \]  

(24)
This can be read as the difference between two Ohmic environments with a Drude cutoff [32]. This allows us to rely on known results [3, 14, 32, 5] for computing $J(t)$. Here, we focus on the zero-temperature regime. We find for the zero-point term $S(0) = -\frac{2R}{R_K} \log r$.

Now, interestingly, this will be a large positive term for $r \rightarrow 0$ which can be achieved by going to small $\omega_\mathrm{D}$, i.e. the $uv$-term in equation (14) will be suppressed by a factor $\rho R^2/R_K$ to a very small value.

It is known [14] that for $Z_d = \frac{R_0}{1 + a^2/\omega_0^2}$ we have

$$J_S(t) = \frac{R_0(\omega_0)}{R_K} \left[ e^{-\omega_0 \tau} E_1(-\omega_0 \tau) - e^{\omega_0 \tau} E_1(\omega_0 \tau) \right]$$

and for long times, $\omega_0 \tau \rightarrow \infty$, this reduces to

$$J_S(t) = \frac{2R_0}{R_K} \left( \log(\omega_0 \tau + \sqrt{1 + \frac{1}{\omega_0^2}}) \right),$$

where $\gamma_c \simeq 0.5772$ is the Euler–Mascheroni constant. For short times, $\omega_0 \tau \ll 1$, we find $J(t) = i\pi \alpha_0 \gamma_0 \tau t$. We can now find $P(E)$ in three regimes. For $E \ll \gamma, \gamma r^2$, both terms in equation (24) should be treated in the long-time limit. In lowest order in $r$ we find

$$J(t) = -\frac{2R}{R_K} \log r,$$

i.e., no time-dependence, in agreement with the fact that the environment is super-Ohmic at these low frequencies, see equation (24). This leads to $P_S(E) = e^{\gamma r^2} \delta(E)$. In the intermediate regime, $\gamma r^2 \ll E \ll \gamma$, we can combine the short- and long-time limits as

$$J(t) = -\frac{R_0}{R_K} \left[ \log \gamma r^2 + \pi + i \frac{\tau}{2} - i\pi \gamma r^2 t \right].$$

This leads to an Ohmic $P(E)$ that is energetically shifted by $\delta E = \frac{R_0}{R_K} \pi \gamma r^2 = \frac{K^2}{R_K}$. This results in

$$P(E) = \frac{e^{-2\gamma r^2/R_K} \pi \gamma r^2 \frac{E}{K^2}}{\Gamma(2R_K/R) E} \frac{2R_K}{R} \times \left( 1 + \frac{2R_K}{R} - 1 \right) \delta(E - E - e^2/2).$$

Finally, at large energies, and hence entirely short times and large energies, $E \gg \gamma$, we can approximate

$$J(t) \simeq \frac{i\pi}{C \tau} t,$$

hence leading to a simple capacitive contribution from the junction’s charging energy and $P(E, E') = e^{-2\gamma r^2}(u^2 + v^2) e^{v^2/2} - 2\omega e^{v^2/2} \delta(E' - E - e^2/2).$

4. Conclusion

In conclusion, we have developed the theory of quasiparticle tunneling for superconducting tunnel junctions in an arbitrary linear dissipative environment. We worked out the unperturbed tunneling rate of nonequilibrium quasiparticles in the junction. In the perturbation regime $P(E)$ governs the processes of an electron or a hole tunneling, which are represented by the diagonal transition matrix elements in charge space. We showed that processes that create charge superpositions are additionally suppressed by zero-point fluctuations of the phase. For the case of low damping, quasiparticle tunneling exchanges energy with the plasma mode in the form of a sequence of sidebands.

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