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Poincaré inequalities and compact embeddings from Sobolev type spaces into weighted $L^q$ spaces on metric spaces

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**A B S T R A C T**

We study compactness and boundedness of embeddings from Sobolev type spaces on metric spaces into $L^q$ spaces with respect to another measure. The considered Sobolev spaces can be of fractional order and some statements allow also nondoubling measures. Our results are formulated in a general form, using sequences of covering families and local Poincaré type inequalities. We show how to construct such suitable coverings and Poincaré inequalities. For locally doubling measures, we prove a self-improvement property for two-weighted Poincaré inequalities, which applies also to lower-dimensional measures.

We simultaneously treat various Sobolev spaces, such as the Newtonian, fractional Hajlasz and Poincaré type spaces, for rather general measures and sets, including fractals and domains with fractal boundaries. By considering lower-dimensional measures on the boundaries of such domains, we obtain trace embeddings for the above spaces. In the case of Newtonian spaces we exactly characterize when embeddings into $L^q$ spaces with respect to another measure are compact.

Our tools are illustrated by concrete examples. For measures satisfying suitable dimension conditions, we recover several...
classical embedding theorems on domains and fractal sets in $\mathbb{R}^n$.

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1. Introduction

The classical Rellich–Kondrachov compactness theorem says that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary and $1 \leq p < n$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $1 \leq q < np/(n-p)$, see e.g. Ziemer [59, Theorem 2.5.1 and Exercise 2.3]. For $p \geq n$ this holds for all $q \geq 1$.

In Hajłasz–Koskela [24, Section 8], similar compactness results were proved in the setting of metric measure spaces which in most typical situations support global Poincaré inequalities and are either geometrically doubling or equipped with doubling measures.

In this paper we study compactness and boundedness of embeddings from subsets $\mathcal{Y}$ of Sobolev type spaces, defined on a metric measure space $X = (X,d,\mu)$, into $L^q$ spaces on a measurable totally bounded set $E \subset X$ and with respect to possibly another measure $\nu$. Our goal is to formulate and prove the results under least possible assumptions, which include various earlier results as special cases. We are able to simultaneously treat various spaces of Sobolev type, including the Newtonian $N^{1,p}$, fractional Hajlasz $M^{\alpha,p}$ and Poincaré $P_{\tau}^{\alpha,p}$ spaces, for rather general measures and in metric spaces, including fractals. Our results can be extended to Besov spaces, but we omit such treatment here.

Our main idea for compactness, formulated in an abstract form in Assumptions (C) and Theorem 2.3, is to construct countably many local Poincaré inequalities involving the measures $\mu$ and $\nu$, and satisfied on certain covering sets $E_i = E_i(m)$ of $E$ and with suitable $E_i' = E_i'(m) \subset X$, for each $m = 1, 2, \ldots$:

$$\left( \int_{E_i} |u - a_{E_i}(u)|^q \, d\nu \right)^{1/q} \leq C_m \left( \int_{E_i'} g^p \, d\mu \right)^{1/p}, \quad (1.1)$$

whenever $u : X \to [-\infty, \infty]$ belongs to the appropriate function space. As a byproduct, we also obtain a sufficient condition for the boundedness of the above embeddings, in which case it suffices to have one covering net, see Assumptions (B) and Theorem 2.6. We consider mainly the case when $1 \leq p \leq q < \infty$.

The generalized “gradient” $g$ in (1.1) serves as a substitute for $|\nabla u|$ in the classical Poincaré inequalities in the Euclidean spaces and on manifolds. For a fixed $m$, the covering sets $E_i$ and $E_i'$ form a finite covering family of the sets for which the embedding is considered and, intuitively, shrink as $m \to \infty$. They will often be constructed from balls with radii $r_m \to 0$, but other choices are possible.
The most important property of such a covering is that we can control simultaneously the constants \( C_m \) in the Poincaré inequalities \((1.1)\), as well as the overlap \( N_m \) of the covering family \( \{E'_i\} \) associated with each \( m \). This is enforced by the requirement that \( C_m N_m^{1/p} \to 0 \) as \( m \to \infty \). A concrete application of this approach is in Theorem 3.1, where we use integrability assumptions on the weights \( w \) and \( v \) in \( \mathbb{R}^n \) to obtain an explicit sufficient condition for the compactness of the embedding

\[
W^{1,p}(\mathbb{R}^n, w \, dx) \hookrightarrow L^q(E, v \, dx)
\]

for bounded measurable sets \( E \subset \mathbb{R}^n \). Its sharpness for the limiting exponent is illustrated in Example 3.3, where the embedding \( W^{1,2}(\mathbb{R}^2, w \, dx) \hookrightarrow L^2(B(0,1), v \, dx) \) is compact, while there is no bounded embedding into \( L^q(B(0,1), v \, dx) \) for any \( q > 2 \).

Poincaré inequalities with suitable exponents \( p \) and \( q \) are the main ingredient in our method. In Theorem 4.1 we therefore prove the two-weighted Poincaré type inequality \((1.1)\) in spaces with a good domain measure \( \mu \) and a rather general target measure \( \nu \). An explicit special case is the following result, proved in Section 4. It applies for example in the setting of weighted \( \mathbb{R}^n \), as in Heinonen–Kilpeläinen–Martio \([30]\), as well as on many metric measure spaces, together with a possibly “lower-dimensional” measure \( \nu \) defined on good subsets as in \((1.14)\), including fractals. This interpretation seems to be new even in unweighted \( \mathbb{R}^n \) and is one of our main contributions.

**Theorem 1.1 (Self-improvement of Poincaré inequalities on subsets).** Assume that \( 1 < p < q < \infty \), that \( \mu \) and \( \nu \) satisfy the doubling condition \((1.5)\) for all balls \( B \subset X \) centred in a uniformly perfect set \( E \subset X \), and that \( \mu \) supports the Poincaré inequality

\[
\int_B |u - u_{B,\mu}| \, d\mu \leq C[\text{diam}(B)]^\alpha \left( \int_{\lambda B} g^p \, d\mu \right)^{1/p},
\]

\((1.2)\)

with \( \alpha > 0 \) and dilation \( \lambda \geq 1 \), for all such balls and a pair of functions \((u, g)\), where \( u \) is assumed to have \( \mu \)-Lebesgue points \( \nu \)-a.e. in \( E \).

Then the Poincaré type inequality \((1.1)\) holds for the pair \((u, g)\) and all balls \( B = B(x, r) \subset X \) centred in \( E \), with \( E_i = E \cap B, E'_i = 2\lambda B, a_{E_i}(u) = \mu(B)^{-1} \int_B u \, d\mu \), the exponent \( q \) in the left-hand side of \((1.1)\) replaced by any \( q' < q \) and the Poincaré constant \( C_m \) replaced by

\[
C(r) = C' \nu(E \cap B)^{1/q'} \sup_{0 < \rho \leq r} \sup_{x \in E \cap B} \frac{\rho^{\alpha} \nu(B(x, \rho))^{1/q}}{\mu(B(x, \lambda \rho))^{1/p}}.
\]

\((1.3)\)

The constant \( C' \) in \((1.3)\) depends only on \( C \) in \((1.2)\) and on other fixed parameters (including \( q' \)), but not on \( B, u \) and \( g \). If the pair \((u, g)\) satisfies the truncation property, then also \( q' = q \) is allowed.
When $\alpha = 1$ and $\mu = \nu$ satisfies the lower bound $(1.7)$, we partially recover Theorem 5.1 in Hajłasz–Koskela [24] and Theorem 6 in Alvarado–Górka–Hajłasz [3]. Self-improvements of Poincaré inequalities with two measures were earlier obtained in e.g. Chanillo–Wheeden [15] and Björn [11] in the setting of $A_p$ weights in $\mathbb{R}^n$.

The “gradient” $g$ is allowed to be both nonlinear and nonlocal, for example:

- The Hajłasz gradient coming from the pointwise inequality
  \[ |u(x) - u(y)| \leq d(x, y)^\alpha (g(x) + g(y)) \text{ for } \mu\text{-a.e. } x, y \in X, \]
  and defining the fractional Hajłasz Sobolev space $M^{\alpha,p}(X, \mu)$ in Section 3.1.
- The upper gradient on metric spaces, defined by curve integrals
  \[ |u(x) - u(y)| \leq \int_{\gamma} g \, ds \quad \text{for all } x, y \in X \text{ and all curves } \gamma \text{ connecting them}, \]
  and defining the Newtonian Sobolev space $N^{1,p}(X, \mu)$ in Section 6.

Our next result holds for fractional Hajłasz spaces with very mild assumptions on the measure $\mu$ and generalizes Theorem 2 in Kalamajska [36], which dealt with $\alpha = 1$. It is a direct consequence of Proposition 3.9 with $\theta = 0$, where the doubling assumption on $X$ is partially relaxed. Note that $\mu$ need not be doubling.

**Proposition 1.2** (Compactness of $M^{\alpha,p}$ in $L^p$ with general $\mu$). Assume that $X$ is bounded and doubling as in Definition 2.11, and that $0 < \mu(B) < \infty$ for all balls $B$ in $X$. Then the embedding $M^{\alpha,p}(X, \mu) \hookrightarrow L^p(X, \mu)$ is compact for all $p \geq 1$ and $\alpha > 0$. In particular, if $\mu$ is not a finite sum of atoms, then $M^{\alpha,p}(X, \mu) \neq L^p(X, \mu)$.

Two-weighted compactness results for $M^{\alpha,p}$ and the more general $E$-restricted Poincaré Sobolev spaces $P_{\tau,E}^{\alpha,p}$, introduced in Definition 3.5, are provided in Theorem 5.1 and Corollary 5.2. The following local doubling and dimension conditions for $\mu$ and $\nu$ on $E \subset X$ play a major role in these results.

**D** For all $x \in E$ and all $0 < r' < r \leq r_0$, with $C, C', C'', \delta, s, \sigma > 0$ independent of $x$, $r'$ and $r$:

\[
\begin{align*}
0 < \mu(B(x, 2r)) &\leq C \mu(B(x, r)) < \infty, \\
0 < \nu(B(x, 2r)) &\leq C \nu(B(x, r)) < \infty, \\
\frac{\nu(B(x, r'))}{\nu(B(x, r))} &\leq C \left( \frac{r'}{r} \right)^\delta, \\
\mu(B(x, r)) &\geq C'r^s, \\
\nu(B(x, r)) &\leq C''r^\sigma.
\end{align*}
\]
In particular, Corollary 5.2 (i) with $E = X$ recovers the well-known compactness of the embedding $M^{\alpha,p}(X,\mu) \hookrightarrow L^q(X,\mu)$ for totally bounded $X$ and $q(s - \alpha p) < sp$, and generalizes it to two measures, including lower-dimensional target measures $\nu$. When discussing embeddings into $L^q(E,\nu)$, we mean that $\bar{u} \in L^q(E,\nu)$, where

$$\bar{u}(x) := \limsup_{r \to 0} \int_{B(x,r)} u \, d\mu, \quad x \in E$$

and the integral average $\bar{f}$ is as in (2.1).

For the Newtonian spaces $N^{1,p}(X,\mu)$, defined by upper gradients, a standard assumption is the following $p$-Poincaré inequality for all (or some) balls $B$ and all $u \in L_{loc}^1(X,\mu)$ with a minimal $p$-weak upper gradient $g_u \in L^p(X,\mu)$,

$$\int_B |u - u_{B,\mu}| \, d\mu \leq C \, \text{diam}(B) \left( \int_{\lambda B} g_u^p \, d\mu \right)^{1/p},$$

where $C > 0$ and $\lambda \geq 1$ are independent of $u$ and $B$, and we implicitly assume that $0 < \mu(B) < \infty$ for all such balls $B$. We prove the following characterization of compact embeddings for $N^{1,p}$, which seems new in the context of metric spaces. It is a direct consequence of Theorem 5.1 (ii), together with Propositions 7.1 and 7.3, and generalizes a similar condition from $\mathbb{R}^n$ due to Maz’ya [50, Theorem 8.8.3]. For the definition of $N^{1,p}$ and $D^p$, see Section 6.

**Theorem 1.3 (Characterization of embeddings for $N^{1,p}$).** Assume that $E \subset X$ is totally bounded and that for all balls centred in $E$ and of radius at most $r_0$:

(a) The doubling conditions (1.5) and (1.6) in Assumptions (D) hold for $\mu$ and $\nu$.

(b) The domain measure $\mu$ supports the $p$-Poincaré inequality (1.10) with $g_u$.

Let $\mathcal{Y} = N^{1,p}(X,\mu)$ or $\mathcal{Y} = D^p(X,\mu) \cap L^1(E,\nu)$, equipped with the norms

$$\|u\|_{N^{1,p}(X)} \quad \text{or} \quad \|g_u\|_{L^p(X,\mu)} + \|u\|_{L^1(E,\nu)},$$

respectively.

Then the embedding $\mathcal{Y} \hookrightarrow L^q(E,\nu)$ for $q > p$ is bounded whenever

$$\sup_{0 < r < r_0} \sup_{x \in E} \frac{r \nu(B(x,r))^{1/q}}{\mu(B(x,r))^{1/p}} < \infty,$$

and compact whenever

$$\sup_{x \in E} \frac{r \nu(B(x,r))^{1/q}}{\mu(B(x,r))^{1/p}} \to 0, \quad \text{as} \ r \to 0.$$
If ν also satisfies the measure density condition
\[ \nu(B(x, r) \cap E) \geq c \nu(B(x, r)) \quad \text{with some } c > 0, \] (1.13)
for all \( x \in E \) and \( 0 < r \leq r_0 \) (in particular, if \( \nu(X \setminus E) = 0 \)), then (1.11) and (1.12) are also necessary for the boundedness/compactness of \( Y \hookrightarrow L^q(E, \nu) \), respectively.

When \( \mu \) and \( \nu \) satisfy also the dimension conditions (1.7) and (1.8), Theorem 1.3 yields the sufficient conditions \( q(s - p) \leq \sigma p \) and \( q(s - p) < \sigma p \) for boundedness/compactness of the embedding \( N^{1,p}(X, \mu) \hookrightarrow L^q(E, \nu) \).

Two particular settings for our results are when the target measure \( \nu \) is a lower-dimensional or codimensional measure, restricted to suitable subsets of well-behaved metric measure spaces (Example 8.1), and when the domain measure \( \mu \) is restricted to sufficiently regular domains in such spaces (Theorem 8.2). By combining these two approaches, we obtain the following trace embedding result, proved at the end of Section 8. These trace embeddings are similar to those in Malý [46, Proposition 4.16], proved for \( M^{1,p} \) and \( N^{1,p} \) by means of Besov spaces. However, the assumptions on the target measure \( \nu \) in [46] are different from ours, while our assumptions on the domain measure \( \mu \) are somewhat weaker. See Remark 8.5 for typical situations when \( \mu|\Omega \) satisfies these assumptions. The obtained exponents for embeddings into \( L^q \) are sharp for the dimensions associated with \( \mu \) and \( \nu \). The proof shows that \( M^{\alpha,p}(\Omega, \mu) \) can be replaced by the space \( P_{\tau,E}^{\alpha,p}(\Omega, \mu) \) from Definition 3.5.

**Proposition 1.4 (Trace embeddings).** Let \( \Omega \subset X \) and \( E \subset F \subset \overline{\Omega} \) be such that the \( d \)-dimensional Hausdorff measure \( \Lambda_d \) satisfies the dimension condition
\[ Cr^d \leq \Lambda_d(F \cap B(x, r)) \leq C'r^d \quad \text{for all } x \in E \text{ and } 0 < r \leq r_0. \] (1.14)
Assume that \( E \) is totally bounded and \( \Lambda_d \)-measurable and that for all balls centred in \( E \) and of radius at most \( r_0 \), the restriction \( \mu|\Omega \) satisfies the doubling and dimension conditions (1.5) and (1.7) in Assumptions (D) with exponent \( s \).

If \( \alpha > 0 \) and \( d > s - \alpha p \), then functions in \( M^{\alpha,p}(\Omega, \mu) \) have traces on \( E \), defined by the integral average
\[ u(x) := \lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} u \, d\mu \quad \text{for } \Lambda_d\text{-a.e. } x \in E, \] (1.15)
and the trace mapping \( M^{\alpha,p}(\Omega, \mu) \hookrightarrow L^q(E, \Lambda_d) \) is compact whenever \( q(s - \alpha p) < dp \).

If moreover \( \Omega \) is open, \( d > s - p \) and the restriction \( \mu|\Omega \) supports the \( p \)-Poincaré inequality (1.10) for \( g_u \) on \( \Omega \), then the trace mapping \( N^{1,p}(\Omega, \mu) \hookrightarrow L^q(E, \Lambda_d) \), defined by (1.15), is bounded whenever \( q(s - p) \leq dp \), and compact if \( q(s - p) < dp \).
The tight connection between Poincaré inequalities and compact embeddings was in the setting of metric spaces exploited in Hajłasz–Koskela [24, Section 8]. Another approach, based on the pointwise inequality (1.4) and suitable coverings by balls with controlled overlap was used in Kalamajska [36]. See also Ivanishko–Krotov [34], Krotov [41] and Romanovskiǐ [54] for compact embeddings of other abstractly defined Sobolev spaces on metric spaces. The paper Górka–Kostrzewa [20] deals with compact embeddings for Sobolev spaces defined on metrizable groups. Hajłasz–Koskela [23] and Hajłasz–Liu [26] related compactness of Sobolev embeddings to the existence of embeddings into better spaces of Orlicz type.

The method proposed in this paper combines the Poincaré inequalities from [24] and the well-controlled coverings from [36]. We formulate our results in the setting of metric measure spaces but some more general formulations are possible as well. Theorems 2.3 and 2.6 could equally well be formulated in measure spaces, without an underlying metric, or in quasimetric spaces, but we omit such generalizations.

We have recently learned about the paper Chua–Rodney–Wheeden [16], where a similar abstract treatment, also based on covering families and a good control of certain generalized gradients, was used to derive compact embeddings for degenerate Sobolev spaces associated with a nonnegative quadratic form in $\mathbb{R}^n$. Despite some similarities, our assumptions on the involved sets, measures, exponents and inequalities are different from [16], see Remark 2.5.

Our main contributions to the current literature are the following:

- We allow two different measures, both for the embeddings and the self-improvement of Poincaré inequalities. The two-weighted Theorem 1.1 has recently found applications in [10] and Butler [14] to Lebesgue points for Besov spaces on metric spaces.

- By using a lower-dimensional target measure $\nu$, we can include also trace embeddings (Proposition 1.4 and Example 8.1). Our approach is direct, without the use of Besov spaces on the boundary. The conditions imposed on the measures $\mu$ and $\nu$ are rather flexible and natural.

- Some of the measures are not required to be doubling but can still support Poincaré type inequalities as in (1.1) with good enough $C_m$, see Remark 3.4. In Theorem 3.1, we consider nondoubling $B_p$ weights on $\mathbb{R}^n$ and prove compactness up to, and including, a limiting exponent $q$. Proposition 1.2 for $M^{\alpha,p}$ and a similar weaker embedding for the Poincaré Sobolev space $P^{\alpha,p}_{\tau,E}$ hold for very general measures, since Poincaré type inequalities come for free in these cases.

- Even when the doubling condition is used, it is required only for small balls centred in $E$. This is not the same as assuming that the restriction of the measure to $E$ is doubling and cannot be treated by considering $E$ as a metric space in its own right, see Example 5.4. In particular for the trace results, this assumption is much weaker than a global doubling condition, see Example 5.3. Also the Poincaré inequality is only required for balls centred in $E$, which is captured in Definition 3.5 of the $E$-restricted Poincaré Sobolev space $P^{\alpha,p}_{\tau,E}$. 
• All smoothness exponents $\alpha > 0$ are allowed for the fractional Sobolev spaces $M^{\alpha,p}$ and $P^{\alpha,p}_{r,E}$, as well as for the Poincaré inequality (1.2). Most earlier results deal with $\alpha = 1$, while $\alpha < 1$ can be obtained from $\alpha = 1$ by snowflaking. The case of $\alpha > 1$ is less studied and applies in particular to fractals, see Remark 3.7.

• We formulate our sufficient conditions for compactness/boundedness using (1.11) and (1.12), rather than in terms of dimensions, see Theorem 5.1. This gives sharper results in general. For Newtonian spaces $N^{1,p}$ with a good domain measure $\mu$, it leads to the exact characterization of compact embeddings in Theorem 1.3.

The paper is organized as follows. In Section 2 we give the necessary definitions and prove our abstract compactness and boundedness results, Theorems 2.3 and 2.6. In Section 2.2 we provide a more thorough discussion of the general assumptions in these theorems, such as the choice of covering families and the functionals $a_{E_i}(u)$ in (1.1). Section 3 contains compactness results with nondoubling measures.

In Section 4 we construct the local $(q,p)$-Poincaré inequalities (1.1) for locally doubling measures, starting from the weaker Poincaré inequality (1.2). Theorem 1.1 and its general version, Theorem 4.1, are proved there. The self-improving argument is based on maximal functions as in Hajłasz–Koskela [24, Theorem 5.3] and Heinonen–Koskela [31, Lemma 5.15], together with Maz’ya’s truncation technique from [49].

The Poincaré inequalities derived in Section 4 are further used in Section 5 to prove the general compactness Theorem 5.1 and Corollary 5.2, assuming the weaker inequality (1.2), together with more precise information about the measures $\mu$ and $\nu$. The spaces $P^{\alpha,p}_{r,E}$ considered in these statements, with Poincaré inequalities required only for balls centred in $E$, could be of independent interest. The spaces $M^{\alpha,p}$ and $P^{\alpha,p}_{r,E}$ are included as special cases. Section 6 deals with Newtonian spaces $N^{1,p}$ and the sufficiency part of Theorem 1.3 is proved there, while various necessary conditions for compact/bounded embeddings are obtained in Section 7.

Section 8 is devoted to concrete examples and to embeddings on uniform domains with distance weights. Example 8.1 deals with lower-dimensional measures and Proposition 1.4 is proved in this section. Examples 8.6 and 8.7 recover trace embeddings for Lipschitz domains in $\mathbb{R}^n$ and for the von Koch snowflake domain.

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2. Compactness and boundedness of embeddings

In this section we prove a general result about compact embeddings from certain function spaces into $L^q$ spaces, possibly with respect to a different measure. Unless said otherwise, we assume the following setting throughout the paper.
(A) General assumptions:

- $1 \leq p, q < \infty$,
- $X$ is a metric space equipped with a metric $d$ and positive complete Borel measures $\mu$ and $\nu$, called the domain measure and the target measure,
- $E \subset X$ is $\nu$-measurable and $\nu(E) > 0$.

Note that $X$ need not be complete. For example, $X$ can be an open set in $\mathbb{R}^n$, as in Examples 2.2, 2.4 and 2.7. We wish to cover the following possible situations:

- $E$ is a domain (i.e. an open connected set) in $X$ and $\nu$ is a restriction to $E$ of some Borel measure defined on $X$ (such as $\mu$),
- $E$ is the boundary of a domain in $X$ and $\nu$ is a lower dimensional Hausdorff measure defined on such a boundary.

If $\nu$ is a positive complete Borel measure defined only on $E$, then we extend it to $X$ by setting $\nu(A) := \nu(A \cap E)$, whenever $A \cap E$ is $\nu$-measurable; we will not distinguish between $\nu$ and its extension. We use the following notation: For a ball

$$B = B(x, r) := \{y \in X : d(x, y) < r\},$$

let $\lambda B := B(x, \lambda r)$. In metric spaces, it can happen that balls with different centres and/or radii denote the same set. We therefore adopt the convention that a ball is open and comes with a predetermined centre and radius. For $A \subset X$ with $0 < \mu(A) < \infty$, let

$$u_{A, \mu} := \int_A u \, d\mu := \frac{1}{\mu(A)} \int_A u \, d\mu,$$  \hspace{1cm} (2.1)

and similarly for any other measure. Lebesgue spaces are denoted by $L^p(X, \mu)$, where $\mu$ is often omitted from the notation if it is meant to be the Lebesgue measure on an Euclidean space. The same simplification will be used for Sobolev type spaces. As in [22], by $u \in L^p_{\text{loc}}(X, \mu)$ we mean that $u \in L^p(B, \mu)$ for all balls $B$ in $X$.

The letter $C$ denotes various positive constants whose exact values depend only on unimportant parameters and may vary even within the same line.

2.1. General compactness and boundedness results

Let $\mathcal{X}$ be a normed space and $\mathcal{Y} \subset \mathcal{X}$. Recall that a mapping from $\mathcal{Y}$ into a Banach space $\mathcal{Z}$ is compact if the image of every bounded sequence from $\mathcal{Y}$ has a convergent subsequence with respect to the norm on $\mathcal{Z}$. Our primary interest is in situations when $\mathcal{Y}$ is a subset of some space $\mathcal{X}$ of functions defined on a metric space $X$. For example, $\mathcal{X}$ can be a Sobolev space, which has a natural norm associated with it. Since $\mathcal{Y}$ need
not be a linear space, the restriction of such a norm need not be a norm on \( \mathcal{Y} \), only a metric. We will, however, often use the term norm for it as well, in order to distinguish it from the metric \( d \) defined on the underlying metric space \( X \), which carries the functions under consideration.

One reason why we formulate our result in terms of the subset \( \mathcal{Y} \), rather than the normed space \( X \) itself, is that in applications, Assumptions (B) and (C) below might only be satisfied for a certain class of functions, coming with suitable a priori estimates. Such a class need not be a linear space. We use the notation \( \mathcal{Y} \hookrightarrow \mathcal{Z} \) for the identity mapping, whenever \( \mathcal{Y} \subset \mathcal{Z} \), but also when suitable restrictions or extensions of functions from \( \mathcal{Y} \) belong to \( \mathcal{Z} \), as in the trace results in Proposition 1.4.

Given the exponents \( 1 \leq p, q < \infty \) and a set \( \mathcal{Y} \) of \( \nu \)-measurable functions defined on the metric space \( X \), we consider the following assumptions, where (C) stands for compactness and (B) for boundedness:

\( \text{(C) } \) Existence of a countable sequence of finite covering families with a controlled overlap and a subordinate Poincaré type inequality valid for all functions in \( \mathcal{Y} \):

- For every \( m \in \mathbb{N} \), there is a finite covering family \( \{E_i, E'_i\}_{i=1}^{K_m} \), such that
  (i) \( E_i = E_i(m) \subset X \) are \( \nu \)-measurable, \( \nu(\bigcup_{i=1}^{K_m} E_i) = 0 \) and
  \[
  0 < \nu(E_i) < \infty, \quad \text{for all } i = 1, 2, \ldots, K_m, \tag{2.2}
  \]
  (ii) \( E'_i = E'_i(m) \subset X \) are \( \mu \)-measurable.
- The following Poincaré type inequality holds for each fixed \( m \) and the corresponding sets \( E_i \) and \( E'_i \), \( i = 1, \ldots, K_m \): There exist mappings \( a_{E_i} : \mathcal{Y} \to \mathbb{R} \) and \( G_m : \mathcal{Y} \to L^p(X, \mu) \) such that \( G_m(u) \geq 0 \) \( \mu \)-a.e. and
  \[
  \left( \int_{E_i} |u - a_{E_i}(u)|^q \, d\nu \right)^{1/q} \leq C_m \left( \int_{E'_i} G_m(u)^p \, d\mu \right)^{1/p}, \tag{2.3}
  \]
  whenever \( u \in \mathcal{Y} \). The constant \( C_m \) can depend on \( m, p, q \) and on the family \( \{E_i, E'_i\}_{i=1}^{K_m} \), but not on \( u \) and \( i \).
- For such a covering family \( \{E_i, E'_i\}_{i=1}^{K_m} \) we define the overlap
  \[
  N_m := \text{ess sup}_{x \in X} \sum_{i=1}^{K_m} \chi_{E'_i}(x), \tag{2.4}
  \]
  where \( \chi_A \) is the characteristic function of a set \( A \) and the ess sup is taken with respect to \( \mu \).

The requirement that \( \nu(E_i) > 0 \) in (2.2) can clearly be fulfilled by discarding some of the \( E_i \)'s. The sets \( E_i \) and \( E'_i \), as well as the mappings \( u \mapsto a_{E_i}(u) \) and \( u \mapsto G_m(u) \),
may depend on \( m \). Also the overlap \( N_m \) is allowed to depend on \( m \), as long as it is compensated by the Poincaré constant \( C_m \) in (2.3), see Theorem 2.3.

**Remark 2.1 (Typical situations).** Concrete examples are when \( E = X \) and \( \nu = \mu \), or when \( X = \overline{\Omega} \) and \( E = \partial \Omega \), where \( \Omega \) is a bounded open set, equipped with suitable measures. With \( \mathcal{Y} \) equal to an appropriate Sobolev or fractional space, our results then imply sharp compact Sobolev and trace type embeddings for such spaces.

Note that \( E_i \) need not be a subset of \( E'_i \). For example, in trace theorems it may be convenient to have \( E_i \subset \partial \Omega \) and \( E'_i \subset \Omega \) for some open set \( \Omega \) and \( X = \overline{\Omega} \).

Typical examples of \( G_m(u) \) are \(|\nabla u|\) or its metric space analogue \( g_u \), but other choices are possible. Neither linearity nor locality is required for \( G_m(u) \). The possible dependence of \( G_m(u) \) on \( m \) allows gradients adapted to various scales, such as the ones introduced for fractional Hajłasz, Besov and Triebel–Lizorkin spaces on metric spaces in Koskela–Yang–Zhou [39]. We omit such generalizations here.

**Example 2.2 (Choice of covering sets).** The simplest choice of a covering family is to let \( E_i \) and \( E'_i \) be balls with fixed radius \( r_m \), such that \( r_m \to 0 \) as \( m \to \infty \). The Poincaré type inequality (2.3) (as well as (2.5) below) then often follows from the structure of the set \( \mathcal{Y} \) or from the usual \( p \)-Poincaré inequality on \( X \). However, this choice would exclude e.g. the following natural situation:

Let \( X = E \) be the slit disc (i.e. a disc with a radius removed) in the plane. Covering \( E \) by balls defined in \( X \) by the Euclidean metric inevitably leads to disconnected balls divided by the slit. Such balls do not support any Poincaré inequality for functions in \( W^{1,p}(X) \) with \( G_m(u) = |\nabla u| \), while their connected components can be considered as two sets \( E_i \) in our general assumptions.

For other types of gradients or other classes \( \mathcal{Y} \) of functions it might still be possible that some kind of Poincaré inequality holds on such disconnected balls. Alternatively, connected balls with respect to the inner metric in \( X \) can be used.

The number of covering families in (C) is countable but each family is finite. Typically, the cardinality \( K_m \) will tend to \( \infty \), as \( m \to \infty \). The following assumptions for boundedness require only one covering family, which can be infinite.

**B** Existence of at most countable covering family with a controlled overlap and a subordinate Poincaré type inequality valid for all functions in \( \mathcal{Y} ":

- There exists a covering family \( \{E_i, E'_i\}_{i \in I} \) with \( I \subset \mathbb{N} \), such that
  (i) \( E_i \subset X \) are \( \nu \)-measurable, \( \nu(E \setminus \bigcup_{i \in I} E_i) = 0 \),
  \[ \nu(E_i) < \infty \quad \text{for every } i \in I, \quad \text{and} \quad \inf_{i \in I} \nu(E_i \cap E) > 0, \]
  (ii) \( E'_i \subset X \) are \( \mu \)-measurable.
• The following Poincaré type inequality holds for all $E_i$ and $E'_i$, $i \in \mathcal{I}$: There exist mappings $a_{E_i} : \mathcal{Y} \to \mathbb{R}$ and $G : \mathcal{Y} \to L^p(X, \mu)$ such that $G(u) \geq 0 \, \mu$-a.e. and
\[
\left( \int_{E_i} |u - a_{E_i}(u)|^q \, d\nu \right)^{1/q} \leq C_E \left( \int_{E'_i} G(u)^p \, d\mu \right)^{1/p},
\]
whenever $u \in \mathcal{Y}$. The constant $C_E$ can depend on $p$, $q$ and on the family $\{E_i, E'_i\}_{i \in \mathcal{I}}$, but not on $u$ and $i$.

• The sets $E_i$ and $E'_i$, $i \in \mathcal{I}$, have a bounded overlap
\[
N := \text{ess sup}_{x \in X} \sum_{i \in \mathcal{I}} \chi_{E_i}(x) + \text{ess sup}_{x \in X} \sum_{i \in \mathcal{I}} \chi_{E'_i}(x) < \infty,
\]
where the essential suprema are taken with respect to $\nu$ and $\mu$, respectively.

Our first result reads as follows. We postpone the proof to Section 2.3.

**Theorem 2.3** (General sequential compactness). Let $E$ and $\mathcal{Y}$ be such that Assumptions (C) are satisfied with $C_m$, $N_m$, $K_m$, $G_m(\cdot)$ and $a_{E_i}(\cdot)$. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{Y}$, such that the sequence $\{a_{E_i}(u_n)\}_{n=1}^{\infty}$ is bounded for every fixed $m$ and $i = 1, \ldots, K_m$.

Assume that one of the following conditions holds:

(i) $1 \leq p \leq q < \infty$ and
\[
\lim_{m \to \infty} C_m N_m^{1/p} \sup_n \|G_m(u_n)\|_{L^p(X, \mu)} = 0.
\]

(ii) $1 \leq q < p < \infty$ and
\[
\lim_{m \to \infty} C_m N_m^{1/p} K_m^{1/q-1/p} \sup_n \|G_m(u_n)\|_{L^p(X, \mu)} = 0.
\]

Then $\{u_n\}_{n=1}^{\infty}$ has a subsequence converging in $L^q(E, \nu)$.

**Example 2.4** (Rellich–Kondrachov theorem). If $X = \mathbb{R}^n$ with the Euclidean metric and $\mu = \nu = dx$ is the Lebesgue measure, then the Sobolev–Poincaré inequality
\[
\left( \int_B |u - u_B, dx|^q \, dx \right)^{1/q} \leq Cr \left( \int_B |\nabla u|^p \, dx \right)^{1/p}
\]
holds for all $q \leq p^* := np/(n - p)$, every ball $B = B(x_0, r)$ and every $u \in W^{1,p}(\mathbb{R}^n)$, where $p < n$. This implies that the Poincaré type inequality (2.3) is satisfied with $E_i = E'_i = B(x_{i,m}, r_m)$ for some choice of $x_{i,m} \in E \subset \mathbb{R}^n$ and with
\[ C_m = Cr_m^{1+n/q-n/p} \to 0 \quad \text{as } r_m \to 0 \text{ when } q < p^*. \]

Clearly, \( E \) can be covered by such balls with a bounded overlap only depending on \( n \). For each fixed \( E_i \) and for \( a_{E_i}(u) = u_{E_i, dx} \) we have the uniform bound

\[ |a_{E_i}(u)| \leq \mu(E_i)^{-1/p} \|u\|_{L^p(\mathbb{R}^n)}. \]

Theorem 2.3 thus implies the classical Rellich–Kondrachov result about compactness of the embedding \( W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(E) \) for every bounded measurable set \( E \) when \( q < p^* \). For bounded domains \( \Omega \subset \mathbb{R}^n \), compactness of the embedding

\[ W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \]

can be obtained in a similar way if the intersections \( \Omega \cap B(x, r_m), x \in \Omega, \) support (2.3). This is possible e.g. for uniform domains, and thus for bounded Lipschitz domains, see Remark 8.5, Aikawa [1, p. 120] and Maz’ya [51, Sections 1.1.8–1.1.11].

**Remark 2.5** (Discussion of sharpness). For certain normed spaces, compactness of an embedding into \( L^q(X, \nu) \) is equivalent to the boundedness of an embedding into a “better” Orlicz space \( L^\Phi(X, \nu) \), provided that \( \nu(X) < \infty \), see Hajłasz–Liu [26]. On the other hand, in Remark 3.2 and Example 3.3 we present a compact embedding with a target space that is optimal among \( L^q \) spaces.

Theorem 1.3 shows that a sufficient condition on measures, deduced from Theorem 2.3, is also essentially necessary for compactness. A measure which exactly satisfies this condition for the optimal exponent is presented in Example 3.3.

In Theorem 2.3 we prove compactness in \( L^q \) for the same exponent \( q \) as in the left-hand side of (2.3), provided that \( C_m \) and \( N_m \) are well-controlled as \( m \to \infty \). This is in contrast to Chua–Rodney–Wheeden [16], which uses uniformly bounded overlap (as in Euclidean spaces) and obtains compactness in \( L^q \) only for \( q' < q \), assuming initial boundedness in \( L^q \). For \( q \geq p \) we need stronger Poincaré inequalities than [16], but we can also reach the limiting exponent \( q \) if \( C_m \) is good enough. This is exhibited e.g. in Theorem 3.1, Example 3.3 and Proposition 3.9.

Our next result deals with the boundedness of embeddings. Note that finiteness is not required for \( \mu \) and \( \nu \), not even locally. For instance, in Example 2.7, finiteness of the measure fails for balls containing the origin.

**Theorem 2.6** (General boundedness). Let \( E \) and \( \mathcal{Y} \) be such that the Assumptions (B) are satisfied. Assume that one of the following conditions holds:

(i) The index set \( \mathcal{I} \) is finite, \( 1 \leq q < \infty \) and \( t \geq 1 \).
(ii) \( \mathcal{I} \) is countably infinite, \( 1 \leq p \leq q < \infty \) and \( 1 \leq t < q \).
Then there is \( C > 0 \) such that for all \( u \in \mathcal{Y} \cap L^1(E, \nu) \),

\[
\|u\|_{L^q(E, \nu)} \leq C \left( \|G(u)\|_{L^p(X, \mu)} + \|u\|_{L^1(E, \nu)} \right).
\]

Also the proof of Theorem 2.6 is postponed to Section 2.3.

**Example 2.7 (Bounded embeddings on bad domains).** Consider the cusp

\[
\Omega := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < x_n^\gamma < 1 \}, \quad \gamma > 1,
\]

and let \( X = E = \Omega \) with the Euclidean metric. For each \( k = 1, 2, \ldots \), let \( j_k \) be the smallest integer such that \( j_k \geq 2^{k(\gamma - 1)} \) and cover \( \Omega \) up to a set of zero measure by countably many “chunks”

\[
\Omega_{k,j} := \{ x \in \Omega : 2^{-k} + (j - 1)2^{-k\gamma} < x_n < 2^{-k} + j2^{-k\gamma} \}, \quad j = 1, 2, \ldots, j_k,
\]

of length \( 2^{-k\gamma} \), \( k = 1, 2, \ldots \). Each \( \Omega_{k,j} \) is biLipschitz equivalent to the unit cylinder

\[
T_k(\Omega_{k,j}) = B(0, 1) \times (j - 1, j) \subset \mathbb{R}^n, \quad j = 1, 2, \ldots, j_k,
\]

by means of the mapping \( T_k : (x', x_n) \mapsto (x'/x_n^\gamma, 2^{k\gamma}(x_n - 2^{-k})) \). The triangle inequality implies that

\[
|T_k(x) - T_k(y)| \leq \frac{1}{x_n^\gamma} \left( |x' - y'| + \frac{|y'|}{x_n^\gamma} |x_n^\gamma - y_n^\gamma| \right) + 2^{k\gamma}|x_n - y_n| \leq C2^{k\gamma}|x - y|.
\]

Similar estimates hold also for \( T_k^{-1} \) and we conclude that \( |T_k(x) - T_k(y)| \) is comparable to \( 2^{k\gamma}|x - y| \). Consider the measures

\[
d\mu(x) = x_n^\alpha \, dx \quad \text{and} \quad d\nu = x_n^\beta \, dx,
\]

which on each \( \Omega_{k,j} \) are comparable to \( 2^{-k\alpha} \, dx \) and \( 2^{-k\beta} \, dx \), respectively. Starting from the classical Sobolev–Poincaré inequality on the unit cylinder \( T_k(\Omega_{k,j}) \), we arrive after a suitable change of variables at the Sobolev–Poincaré type inequality

\[
\left( \int_{\Omega_{k,j}} |u - a_{k,j}(u)|^q \, d\nu \right)^{1/q} \leq C2^{-k\theta} \left( \int_{\Omega_{k,j}} |\nabla u|^p \, d\mu \right)^{1/p},
\]

where \( C \) does not depend on \( k \) and \( j \),

\[
a_{k,j}(u) = \int_{T_k(\Omega_{k,j})} u \circ T_k^{-1}(y) \, dy, \quad \text{(2.9)}
\]
\[ \theta = \gamma + \frac{\beta + n\gamma}{q} - \frac{\alpha + n\gamma}{p}, \quad q \leq p^* = \frac{np}{n-p}, \quad p < n. \]

For simplicity, choose \( \alpha = \beta = -n\gamma \) which gives \( \theta \geq 0 \) and \( 1/C \leq \nu(\Omega_{k,j}) \leq C \) for all \( k,j \). Theorem 2.6, with \( E_i \) and \( E_i' \) replaced by \( \Omega_{k,j} \), now guarantees boundedness of the embedding

\[ W^{1,p}(\Omega, \mu) \hookrightarrow L^q(\Omega, \nu), \quad p \leq q \leq p^*. \quad (2.10) \]

Since \( \nu(\Omega) = \infty \), we cannot conclude embedding for \( q < p \). In fact, for

\[ q < \frac{p(\gamma - 1)}{p + \gamma - 1}, \]

the function \( u(x) = x_n^{(\gamma - 1)/q} \) shows that the embedding \( W^{1,p}(\Omega, \mu) \hookrightarrow L^q(\Omega, \nu) \) fails. As \( \gamma \to \infty \), this includes all \( q < p \) and hence the above range \( q \geq p \) is at least asymptotically sharp. Since \( \mu \) and \( \nu \) are comparable to a multiple of the Lebesgue measure on each \( \Omega_{k,j} \), the upper end point \( p^* \) in (2.10) is sharp in the same way as for unweighted Sobolev spaces.

We leave it to the interested reader to make the necessary modifications for other weights and domains. A similar argument with \( \gamma = 1 \) can be applied to the punctured ball \( B(0,1) \setminus \{0\} \subset \mathbb{R}^n \), equipped with the measure \( d\mu(x) = |x|^{-n} dx \), where the role of the sets \( E_i \) is played by Whitney cubes near the origin. Whitney type decompositions of other domains can also be considered.

2.2. Discussion of Assumptions (B) and (C)

Our aim is to obtain compactness and boundedness under least possible assumptions on the covering family and the involved measures. Several remarks are therefore in order to clarify these assumptions.

**Remark 2.8 (Choice of \( a_{E_i}(u) \)).** A standard choice in (2.3) and (2.5) is \( a_{E_i}(u) = u_{E_i,\nu} \) but other integral averages, medians, traces or even nonlinear functionals can also be used, see e.g. (2.9) in Example 2.7 and the proof of Theorem 3.1.

An application of the triangle inequality shows that if (2.3) and (2.5) hold with some \( a_{E_i}(u) \) then they also hold with \( a_{E_i}(u) = u_{E_i,\nu} \), whenever defined, at the cost of enlarging the constant on the right-hand side.

**Remark 2.9 (Covering by balls).** Since \( E \subset \bigcup_{x \in E} B(x,r) \), the Hausdorff maximality principle (see e.g. Ziemer [59, p. 7]) provides us for every \( r > 0 \) with a maximal pairwise disjoint family of balls \( \frac{1}{2}B_\alpha \) of radius \( \frac{1}{2}r \) and centres in \( E \). The maximality of the family implies that the balls \( B_\alpha \) cover \( E \). In general, this construction does not guarantee any bounds on the overlap of the balls \( B_\alpha \), as shown by the following example.
Example 2.10 (Unbounded overlap). Let $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n = [0, 2^{-n}]^n \subset \mathbb{R}^n$, $n = 1, 2, \ldots$, and the origins in all $X_n$ are identified as one point 0 in $X$. The metric on $X_n$ is defined as the $n$-dimensional Euclidean metric $d_n$, while for $x \in X_n$ and $y \in X_m$ with $n \neq m$ we let $d(x, y) = d_n(x, 0) + d_m(y, 0)$. Note that $X$ is compact. We shall now see how $E := X$ can be covered by open balls with radius $r = 2^{-m}$, which will give us the covering family $\{E_i, E'_i\}_{i=1}^{K_n}$.

The ball $B(0, r)$ contains all $X_n$ with $2^{-n} \sqrt{n} \leq r$, while the remaining $X_n$ have to be covered by additional balls of radius $r$. For such $n$ and sufficiently large $m$, the required overlap of Euclidean balls is determined by the Lebesgue covering dimension of $\mathbb{R}^n$, and equals $n + 1$, see Munkres [52, p. 305]. Since the minimal overlap $N_m$ on $X$ majorizes the minimal overlap for all such $X_n$ with $2^{-n} \sqrt{n} > r$, we conclude that $N_m \to \infty$ as $m \to \infty$.

We shall now see that a doubling property guarantees a bounded overlap.

Definition 2.11 (Doubling spaces and measures). A metric space $X$ is doubling if every ball of radius $r$ can be covered by at most $M$ balls of radius $r/2$, where $M$ is independent of $r$. A measure $\mu$ is doubling if there exists a constant $C_\mu \geq 1$ such that for all balls $B$ in $X$,

$$0 < \mu(2B) \leq C_\mu \mu(B) < \infty.$$ 

Every space carrying a doubling measure is doubling. The following lemma, which may be of independent interest, implies that if $X$ or $\mu$ is doubling and the covering family is obtained from balls with the same radius then such a family can always be chosen with a bounded overlap independent of the radius.

The doubling property also implies that such a family is finite whenever it is confined to a bounded set. In particular, every bounded set is totally bounded. This need not be true if the doubling property only holds for balls up to a certain radius. Thus, total boundedness of $E$ will be explicitly assumed in those cases.

Lemma 2.12 (Doubling and controlled overlap). Assume that $\lambda \geq \frac{1}{2}$ and that for a function $M_\lambda : (0, \infty) \to (0, \infty)$, one of the following conditions holds for all $r > 0$:

(a) Every ball of radius $\lambda r$ can be covered by $M_\lambda(r)$ balls of radius $\frac{1}{2} r$.
(b) For all balls $B$ of radius $\frac{1}{2} r$,

$$0 < \mu((4\lambda + 1)B) \leq M_\lambda(r)\mu(B) < \infty.$$ 

Let $\{B_\alpha = B(x_\alpha, r)\}_{\alpha \in A}$ be a collection of balls such that the balls $\{\frac{1}{2} B_\alpha\}_{\alpha \in A}$ are pairwise disjoint. Then the balls $\{\lambda B_\alpha\}_{\alpha \in A}$ have overlap at most $M_\lambda(r)$.

In particular, for doubling spaces and measures, the overlap is independent of $r$. 


Proof. Let \( x \in X \) be arbitrary and let \( A_x = \{ \alpha \in A : x \in \lambda B_{\alpha} \} \). Then
\[
x_{\alpha} \in B(x, \lambda r) \quad \text{and} \quad \frac{1}{2} B_{\alpha} \subset B(x, (\lambda + \frac{1}{2})r) \quad \text{for all} \quad \alpha \in A_x.
\]
(2.11)
Suppose that (a) holds. Then the ball \( B(x, \lambda r) \) can be covered by \( M_{\lambda}(r) \) balls \( B'_{\alpha} \) of radius \( \frac{1}{2} r \). Hence, every \( x_{\alpha} \) with \( \alpha \in A_x \) belongs to some \( B'_{\alpha} \), whose centre in turn belongs to \( \frac{1}{2} B_{\alpha} \). Since the balls \( \{ \frac{1}{2} B_{\alpha} \}_{\alpha \in A} \) are pairwise disjoint, it follows that \( x_{\alpha} \) and \( x_{\alpha'} \), with \( \alpha \neq \alpha' \), cannot belong to the same \( B'_{\alpha} \) and consequently, there are at most \( M_{\lambda}(r) \) indices \( \alpha \) such that \( x \in \lambda B_{\alpha} \). Similarly, if (b) holds then for every \( \alpha \in A_x \),
\[
\mu(B(x, (\lambda + \frac{1}{2})r)) \leq \mu(B(x_{\alpha}, (2\lambda + \frac{1}{2})r)) \leq M_{\lambda}(r)\mu(\frac{1}{2} B_{\alpha}).
\]
Hence, by (2.11) and the pairwise disjointness of the balls \( \frac{1}{2} B_{\alpha} \), \( \alpha \in A \),
\[
\mu(B(x, (\lambda + \frac{1}{2})r)) \geq \sum_{\alpha \in A_x} \mu(\frac{1}{2} B_{\alpha}) \geq \frac{\mu(B(x, (\lambda + \frac{1}{2})r))}{M_{\lambda}(r)} \sum_{\alpha \in A_x} 1,
\]
from which it follows that the index set \( A_x \) can have at most \( M_{\lambda}(r) \) elements. \( \square \)

2.3. Proofs of Theorems 2.3 and 2.6

Proof of Theorem 2.3. We start by considering the case (i). For every fixed \( m = 1, 2, \ldots \), consider the covering family \( \{ E_1, E'_1 \}_{i = 1}^{K_m} \). Let \( u \in Y \) be arbitrary, with \( G_m(u) \) as in (2.3). Then, by the local Poincaré type inequality (2.3),
\[
\sum_{i=1}^{K_m} \int_{E_i} |u - a_{E_i}(u)|^q \, d\nu \leq C_m^q \sum_{i=1}^{K_m} \left( \int_{E'_i} G_m(u)^p \, d\mu \right)^{q/p}.
\]
(2.12)
Since \( q \geq p \), the elementary inequality \( \sum_i x_i^{q/p} \leq (\sum_i x_i)^{q/p} \) yields
\[
\sum_{i=1}^{K_m} \left( \int_{E'_i} G_m(u)^p \, d\mu \right)^{q/p} \leq \left( \sum_{i=1}^{K_m} \int_{E'_i} G_m(u)^p \, d\mu \right)^{q/p}.
\]
Inserting this into (2.12), together with (2.4), implies
\[
\sum_{i=1}^{K_m} \int_{E_i} |u - a_{E_i}(u)|^q \, d\nu \leq C_m^q \left( N_m \int_{X} G_m(u)^p \, d\mu \right)^{q/p}.
\]
(2.13)
Now, consider a sequence \( \{ u_n \}_{n=1}^{\infty} \subset Y \), such that \( \{ G_m(u_n) \}_{n=1}^{\infty} \) satisfies (2.6) and \( \{ a_{E_i}(u_n) \}_{n=1}^{\infty} \) is bounded for every fixed \( E_i \) in the covering family. We shall now show that \( \{ u_n \}_{n=1}^{\infty} \) has a Cauchy subsequence in \( L^q(E, \nu) \). We have for all \( n, k \geq 1 \),
\[
\int_{E} |u_n - u_k|^q \, d\nu \leq 3^{q-1} \sum_{i=1}^{K_m} \left( \int_{E_i} |u_n - a_{E_i}(u_n)|^q \, d\nu \right) + \int_{E_i} |u_k - a_{E_i}(u_k)|^q \, d\nu + \int_{E_i} |a_{E_i}(u_n) - a_{E_i}(u_k)|^q \, d\nu \right). 
\]

The sums of the first two integrals on the right-hand side can be estimated using (2.13) with \( u \) replaced by \( u_n \) and \( u_k \), respectively. Hence, because of (2.6), we can for every \( \varepsilon > 0 \) choose a sufficiently large \( m \) in (2.14) such that for all \( n, k \geq 1 \),

\[
\int_{E} |u_n - u_k|^q \, d\nu \leq \varepsilon + 3^{q-1} \sum_{i=1}^{K_m} |a_{E_i}(u_n) - a_{E_i}(u_k)|^q \nu(E_i), 
\]

where \( \{E_i\}_{i=1}^{K_m} \) is the covering family corresponding to \( m \). With this family fixed, the sequence \( \{a_{E_i}(u_n)\}_{n=1}^{\infty} \) is bounded in \( R \) for every \( i = 1, \ldots, K_m \). Hence, applying the Bolzano–Weierstrass theorem, we can for \( \varepsilon_1 = \frac{1}{2} \) and a suitable family \( E_i = E_i(m_1) \), corresponding to \( m_1 \), find a subsequence \( \{u_n^{(1)}\}_{n=1}^{\infty} \) of \( \{u_n\}_{n=1}^{\infty} \) such that the sequence \( \{a_{E_i}(u_n^{(1)})\}_{n=1}^{\infty} \) is convergent for every \( i = 1, \ldots, K_{m_1} \) and (2.15) becomes

\[
\int_{E} |u_n^{(1)} - u_k^{(1)}|^q \, d\nu \leq \frac{1}{2} + 3^{q-1} \sum_{i=1}^{K_{m_1}} |a_{E_i}(u_n^{(1)}) - a_{E_i}(u_k^{(1)})|^q \nu(E_i) < 1
\]

for all \( n, k \geq 1 \). Similarly, we can find another family corresponding to \( \varepsilon_2 = \frac{1}{4} \) and \( m_2 \), and a subsequence \( \{u_n^{(2)}\}_{n=1}^{\infty} \) of \( \{u_n^{(1)}\}_{n=1}^{\infty} \) such that

\[
\int_{E} |u_n^{(2)} - u_k^{(2)}|^q \, d\nu < \frac{1}{2} \text{ when } n, k \geq 1.
\]

Continuing in this way and choosing the diagonal sequence \( \{u_n^{(n)}\}_{n=1}^{\infty} \) we construct a Cauchy sequence in \( L^q(E, \nu) \). Since \( L^q(E, \nu) \) is complete, we are done with the case (i). In the case (ii), the only change needed in the above proof is to use Hölder’s inequality instead of the elementary inequality just after (2.12). \( \square \)

**Proof of Theorem 2.6.** By integrating over smaller sets in the left-hand side of (2.5), if needed, we can assume that \( E = \bigcup_{i \in I} E_i \). According to Remark 2.8 we can assume that \( a_{E_i}(u) = u_{E_i, \nu} \) in (2.5). Then

\[
\int_{E} |u|^q \, d\nu \leq 2^{q-1} \sum_{i \in I} \left( \int_{E_i} |u - a_{E_i, \nu}(u)|^q \, d\nu + \int_{E_i} |u_{E_i, \nu}|^q \, d\nu \right)
\]

\[
\leq 2^{q-1} \sum_{i \in I} \left( C_{E_i}^q \left( \int_{E_i} G(u)^p \, d\mu \right)^{q/p} + \nu(E_i) \left( \int_{E_i} |u|^q \, d\nu \right)^{q/t} \right), \quad (2.16)
\]
where in the last term we used Hölder’s inequality. When \( \mathcal{I} \) is finite this immediately yields that

\[
\int_E |u|^q \, d\nu \leq C \left( \|G(u)\|_{L^p(X,\mu)}^q + \|u\|_{L^q(E,\nu)}^q \right).
\]

For infinite \( \mathcal{I} \) and \( q \geq p \), the elementary inequality \( \sum_i x_i^{q/p} \leq \left( \sum_i x_i \right)^{q/p} \) yields

\[
\sum_{i \in \mathcal{I}} \left( \int_{E_i} G(u)^p \, d\mu \right)^{q/p} \leq \left( \sum_{i \in \mathcal{I}} \int_{E_i} G(u)^p \, d\mu \right)^{q/p} \leq N^{q/p} \|G(u)\|_{L^p(X,\mu)}^q,
\]

because of the bounded overlap of the sets \( E_i \). Since \( q/t > 1 \), the lower bound on \( \nu(E_i) \), the above elementary inequality and the bounded overlap of \( E_i \) imply that

\[
\sum_{i \in \mathcal{I}} \nu(E_i) \left( \int_{E_i} |u|^t \, d\nu \right)^{q/t} \leq \left( \inf_{i \in \mathcal{I}} \nu(E_i) \right)^{1-q/t} N^{q/t} \|u\|_{L^q(E,\nu)}^q.
\]

Inserting the last two estimates into (2.16) concludes the proof also for infinite \( \mathcal{I} \). \( \square \)

3. Embeddings with nondoubling measures

We now provide some concrete results about compact embeddings based on conditions (2.6) and (2.7). Our first result deals with the weighted spaces

\[
W^{1,p}(\mathbb{R}^n, \mu) = \{ u \in L^p(\mathbb{R}^n, \mu) : \nabla u \in L^p(\mathbb{R}^n, \mu) \},
\]

\[
D^p(\mathbb{R}^n, \mu) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^n, \mu) : \nabla u \in L^p(\mathbb{R}^n, \mu) \},
\]

where \( \nabla u \) is the distributional gradient of \( u \) and \( d\mu = w \, dx \) with a \( B_p \) weight \( w \), i.e. \( w^{1/(1-p)} \in L^1_{\text{loc}}(\mathbb{R}^n) \). Such weights were introduced in Kufner–Opic [43] and are suitable for Sobolev spaces based on the distributional gradient \( \nabla u \). See Ambrosio–Pinamonti–Speight [5] and Zhikov [58] for a discussion of various types of weighted Sobolev spaces in \( \mathbb{R}^n \) and on metric spaces.

**Theorem 3.1** (Precise embeddings based on integrability conditions). Let \( w, v \) be weights such that \( w, w^{-\alpha}, w^\beta \in L^1_{\text{loc}}(\mathbb{R}^n) \) with some \( \alpha > 0 \) and \( \beta > 1 \). Assume that

\[
\max \left\{ n \left( \frac{1}{\alpha} + \frac{1}{\beta} \right), 1 + \frac{1}{\alpha} \right\} \leq p < n \left( 1 + \frac{1}{\alpha} \right),
\]

and let \( d\mu = w \, dx \) and \( d\nu = v \, dx \). Then the embeddings

\[
W^{1,p}(\mathbb{R}^n, \mu) \hookrightarrow L^q(E, \nu) \quad \text{and} \quad D^p(\mathbb{R}^n, \mu) \cap L^1(E, \nu) \hookrightarrow L^2(E, \nu)
\]
are compact for every bounded Lebesgue measurable set $E \subset \mathbb{R}^n$ and all exponents

$$1 \leq q \leq \frac{np(1 - 1/\beta)}{(n - p) + n/\alpha}. \quad (3.1)$$

A similar result can be proved for weighted Sobolev spaces in metric spaces with a well-behaved underlying measure. We shall not dwell upon such generalizations. The assumptions on $\alpha$ and $p$ imply that $w$ is a $B_p$ weight.

Example 3.3 below demonstrates the sharpness of the limiting exponent in (3.1). Note that the exponent $q$ has the correct asymptotics $p^* = np/(n - p)$, as $\alpha, \beta \to \infty$. Moreover, the proof shows that if $w \geq C > 0$, then the statement of Theorem 3.1 holds for $1 \leq q \leq p^*(1 - 1/\beta)$ and similarly, it holds for $1 \leq q \leq np/((n - p) + n/\alpha)$ when $v \leq C'$. This means that in (3.1), we may also allow the limiting cases $\alpha = \infty$ or $\beta = \infty$, but not both at the same time. This case is excluded because then both integrals in (3.5) would be replaced by $L^\infty$-norms of $1/w$ and $v$, and thus, $\tilde{C}(B)$ would not be small for small balls $B$.

**Proof.** The result will be deduced from Theorem 2.3 with $X = \mathbb{R}^n$. We start by a verification of Assumptions (C). It clearly suffices to consider

$$q = \frac{np(1 - 1/\beta)}{(n - p) + n/\alpha}.$$ 

The covering family will consist of suitably chosen balls $E_i = E'_i = B$, whose precise construction we postpone until the end of the proof. To obtain the Poincaré type inequality (2.3) for such balls, let $u \in D^p(\mathbb{R}^n, \mu)$ be arbitrary. The Hölder inequality implies that for $t = \alpha p/(\alpha + 1)$ and every ball $B \subset \mathbb{R}^n$,

$$\left( \int_B |\nabla u|^t \, dx \right)^{1/t} \leq \left( \int_B |\nabla u|^p \, dx \right)^{1/p} \left( \int_B w^{-\alpha} \, dx \right)^{1/\alpha p} \quad (3.2)$$

and hence $u \in D^t(B, dx)$. Note that $1 \leq t < n$ and $q = t^*(1 - 1/\beta) < t^*$, where

$$t^* = \frac{nt}{n - t} = \frac{np}{(n - p) + n/\alpha}$$

is the Sobolev exponent associated with $t$. Another use of the Hölder inequality, together with the usual $(t^*, t)$-Sobolev–Poincaré inequality for the Lebesgue measure in $\mathbb{R}^n$ and (3.2), then yields for all balls $B \subset \mathbb{R}^n$ that

$$\left( \int_B |u - u_B, dx|^q \, d\nu \right)^{1/q} \leq \left( \int_B |u - u_B, dx|^{t^*} \, dx \right)^{1/t^*} \left( \int_B v^{\beta} \, dx \right)^{1/\beta q} \quad (3.3)$$
\[
\leq C(n,t)|B|^{1/t^*} \operatorname{diam}(B) \left( \frac{1}{|B|} \int_B |\nabla u|^t \, dx \right)^{1/t} \left( \int_B |v|^q \, dx \right)^{1/b^q}
\]

\[
\leq C^*(n,t) \left( \int_B |\nabla u|^p \, dx \right)^{1/p} \left( \int_B w^{-\alpha} \, dx \right)^{1/\alpha p} \left( \int_B |v|^\beta \, dx \right)^{1/b^q},
\]

where \( C(n,t) \) comes from the \((t^*,t)\)-Sobolev–Poincaré inequality in \( \mathbb{R}^n \). The triangle inequality allows us to replace \( u_{B,\nu} \) at the cost of an additional factor 2 on the right-hand side. We have thus shown that for every ball \( B \subset \mathbb{R}^n \),

\[
\left( \int_B |u - a_B(u)|^q \, d\nu \right)^{1/q} \leq \tilde{C}(B) \left( \int_B |\nabla u|^p \, dx \right)^{1/p},
\]

where both possibilities \( a_B(u) = u_{B,\nu} \) and \( a_B(u) = u_{B,\nu} \) are allowed and

\[
\tilde{C}(B) := 2C^*(n,t) \left( \int_B w^{-\alpha} \, dx \right)^{1/\alpha p} \left( \int_B |v|^\beta \, dx \right)^{1/b^q - 1/t^*}. \tag{3.5}
\]

Let \( E \) be as in the statement of the theorem. For each \( m = 1, 2, \ldots \), we will now find a suitable covering family satisfying (2.6). Since \( w^{-\alpha}, \theta^\beta \in L^1_{\text{loc}}(\mathbb{R}^n) \), we can for every \( x \in \mathbb{R}^n \) find a ball \( B_x \ni x \) such that \( \tilde{C}(2B_x) < 1/m \). By compactness, the closure \( \overline{E} \) can be covered by finitely many such balls \( B_{x_j} \) with radii \( \rho_j \).

Choose \( r_m = \min\{1/m, \min_j \rho_j\} \). Use the Hausdorff maximality principle as in Remark 2.9 and Lemma 2.12 to cover \( E \) by balls \( B_i \) with radius \( r_m \), centres in \( E \) and a bounded overlap independent of \( m \). Since \( r_m \leq \rho_j \), we see that each \( B_i \) is contained in some \( 2B_{x_j} \) and hence \( \tilde{C}(B_i) \leq \tilde{C}(2B_{x_j}) < 1/m \). These balls form a covering family for \( E \) and (3.4) shows that the Assumptions (C) are satisfied with \( E_i = B_i \cap E, E_i' = B_i \),

\[
C_m = \max_i \tilde{C}(B_i) < 1/m, \quad a_{E_i}(u) = u_{B_i,\nu} \quad \text{and} \quad a_{E_i}(u) = u_{B_i,\nu}.
\]

The condition \( q \geq p \) is guaranteed by \( p \geq n(1/\alpha + 1/\beta) \). Finally, as in (3.2),

\[
|u_{B_i,\nu}| \leq \left( \int_{B_i} |u|^t \, dx \right)^{1/t} \leq \left( \int_{B_i} |u|^p \, dx \right)^{1/p} \left( \int_{B_i} w^{-\alpha} \, dx \right)^{1/\alpha p},
\]

and thus the sequence of integral averages \( a_{E_i}(u_n) := (u_n)_{B_i,\nu} \) is bounded whenever \( \{u_n\}_{n=1}^\infty \) is bounded in \( W^{1,p}(B,\mu) \). If instead \( \{u_n\}_{n=1}^\infty \) is bounded in \( D^p(B,\mu) \cap L^1(B,\nu) \), we use \( a_{E_i}(u) = u_{B_i,\nu} \). Since the overlap is independent of \( m \), we see that (2.6) holds and Theorem 2.3 concludes the proof. \( \square \)

**Remark 3.2.** Let the notation be as in the proof of Theorem 3.1. The Hölder inequality shows, as in (3.2), that the following embeddings are bounded:
\[ W^{1,p}(\mathbb{R}^n, \mu) \hookrightarrow W^{1,t}(B, dx) \quad \text{and} \quad L^\tau(E) \hookrightarrow L^{\tau(1-1/\beta)}(E, \nu) \]

for every ball \( B \subset \mathbb{R}^n \) and every \( \tau \geq 1 \). Hence, by directly using the classical compact embedding \( W^{1,t}(B, dx) \hookrightarrow L^\tau(B) \), we obtain compactness of the embedding

\[ W^{1,p}(\mathbb{R}^n, \mu) \hookrightarrow L^{\tau(1-1/\beta)}(E, d\nu) \quad \text{whenever} \quad \tau < t^* = \frac{n\alpha p}{(n-p)\alpha + n}, \]

i.e. for \( q < t^*(1-1/\beta) \). On the other hand, Theorem 3.1 makes it possible to reach also the limiting exponent \( q = t^*(1-1/\beta) \). The following example shows that it is optimal among all \( q \)'s for which one has a compact embedding into \( L^q(E, \nu) \).

Other results concerning compact embeddings for limiting exponents were under certain assumptions recently obtained in Gaczkowski–Górka–Pons [18].

**Example 3.3 (Optimal compactness).** Let \( n = p = q = \alpha = \beta = 2 \) in Theorem 3.1 and set for \( x \in \mathbb{R}^2 \),

\[
w(x) = \begin{cases} |x| \log(1/|x|), & \text{if } 0 < |x| < \frac{1}{2}, \\ \frac{1}{2} \log 2, & \text{otherwise,} \end{cases} \quad \text{and} \quad v(x) = \frac{1}{w(x)}.
\]

Elementary calculations show that the assumptions in Theorem 3.1 are satisfied and that for sufficiently small \( r > 0 \),

\[
\mu(B(0, r)) \simeq r^3 \log \frac{1}{r} \quad \text{and} \quad \nu(B(0, r)) \simeq \frac{r}{\log(1/r)}.
\]

From Theorem 3.1 we deduce that the embedding \( W^{1,2}(\mathbb{R}^n, \mu) \hookrightarrow L^2(B(0,1), \nu) \) is compact. At the same time, for \( q > 2 \) and sufficiently small \( r > 0 \),

\[
\frac{r\nu(B(0,r))^{1/q}}{\mu(B(0,r))^{1/2}} \simeq \frac{r^{1/q-1/2}}{(\log(1/r))^{1/q+1/2}} \rightarrow \infty, \quad \text{as } r \rightarrow 0.
\]

The proof of Proposition 7.1 below then shows that there is no compact embedding \( W^{1,2}(\mathbb{R}^n, \mu) \hookrightarrow L^q(B(0,1), \nu) \) for any \( q > 2 \). In fact, \( L^2(B(0,1), \nu) \) is optimal among \( L^q \) spaces both for bounded and for compact embeddings. Note that the exponent 2 in the conditions \( w^{-2} \in L^1_{\text{loc}}(\mathbb{R}^2) \) and \( v^2 \in L^1_{\text{loc}}(\mathbb{R}^2) \) cannot be replaced by any larger exponent.

Obvious modifications can be done for other exponents and dimensions. It is also possible to construct weights that are singular in a similar way at a dense set.

**Remark 3.4.** Examples of nondoubling measures in \( \mathbb{R} \) and \( \mathbb{R}^2 \), with singular parts and good enough \( C_m \) for Theorem 2.3 can be found in Björn–Björn [7, pp. 206–207 and Proposition 10.6]. See also Alvarado–Hajłasz [4, Example 4] for a nondoubling weighted measure on \([0, \infty)\) that supports a \((q,p)\)-Poincaré inequality (as in (2.8)) precisely when
q ≤ p. At the same time, [4, Theorem 1] shows in the generality of metric spaces that measures supporting (q,p)-Poincaré inequalities with q > p must be doubling.

3.1. Fractional Sobolev spaces with nonlocal gradients

Let α > 0, r₀ > 0 and τ ≥ 1 be fixed in this section.

**Definition 3.5 (Spaces based on Poincaré inequalities).** The E-restricted Poincaré Sobolev space \( P_{τ,E}^{α,p}(X,μ) \) consists of all \( u \in L^1_{loc}(X,μ) \) which satisfy the p-Poincaré inequality

\[
\int_B |u - u_{B,μ}| \, dμ \leq r^α \left( \int_{τB} |g|^p \, dμ \right)^{1/p}
\]

(3.6)

for some \( g \in L^p(X,μ) \) and all balls \( B = B(x,r) \subset X \) centred in \( E \) and of radius at most \( r₀ \), where we implicitly assume that \( 0 < μ(B) < ∞ \). The space \( P_{τ,E}^{α,p}(X,μ) \) is equipped with the seminorm \( \inf_g \|g\|_{L^p(X,μ)} \), where the infimum is taken over all \( g \) satisfying (3.6). When \( E = X \), we omit the subscript \( E \) and write \( P_{τ}^{α,p} \).

The space \( P_{τ}^{α,p}(X,μ) \) was for \( α = 1 \) (with \( E = X \) and all \( 0 < r < ∞ \) in (3.6)) introduced in Koskela–MacManus [37], while the general case \( α > 0 \) was studied in Heikkinen–Koskela–Tuominen [28, Section 3]. The spaces \( P_{τ,E}^{α,p}(X,μ) \) depend also on \( r₀ \), but we suppress this dependence in the notation. Note that

\[
P_{τ}^{α,p}(X,μ) \subset P_{τ,E}^{α,p}(X,μ) \quad \text{for all } E \subset X
\]

(3.7)

and that \( P_{τ,E}^{α,p}(X,μ) \) is in general not the same as \( P_{τ}^{α,p}(E,μ) \). If \( μ \) satisfies the doubling condition (1.5) in Assumptions (D) for balls centred in \( E \) and of radius at most \( r₀ \), then also (upon replacing \( r₀ \) in \( P_{τ,r₀}^{α,p} \) with \( τ r₀/τ₂ \))

\[
P_{τ₁,E}^{α,p}(X,μ) \subset P_{τ₂,E}^{α,p}(X,μ) \quad \text{whenever } τ₁ ≤ τ₂.
\]

(3.8)

Next, we recall the definition of Hajłasz spaces, which were for \( α = 1 \) introduced by Hajłasz [21]. Similar spaces with \( α ≠ 1 \) were considered e.g. in [28], Koskela–Yang–Zhou [38] and implicitly already in Hajłasz–Martio [27].

**Definition 3.6.** A nonnegative Borel function \( g \) is a *Hajłasz α-gradient* of a function \( u : X → \mathbb{R} \) if for \( μ \text{-a.e. } x, y ∈ X \),

\[
|u(x) - u(y)| \leq d(x,y)^α (g(x) + g(y)).
\]

(3.9)

The *Hajłasz space* \( M^{α,p}(X,μ) \) consists of all \( u ∈ L^p(X,μ) \) for which there exists \( g ∈ L^p(X,μ) \) satisfying (3.9). It is equipped with the norm

\[
\|u\|_{M^{α,p}(X,μ)} := \|u\|_{L^p(X,μ)} + \inf\{\|g\|_{L^p(X,μ)} : g \text{ is as in (3.9)}\}.
\]
By Hajłasz [22, Theorems 2.1 and 2.2], the spaces \( P_1^{1,p}(\mathbb{R}^n, dx) \) and \( M^{1,p}(\mathbb{R}^n, dx) \) coincide with the usual Sobolev space \( W^{1,p}(\mathbb{R}^n) \), \( p > 1 \). In this case, \( g \) in (3.9) can be the Hardy–Littlewood maximal function of \(|\nabla u|\), while \(|\nabla u|\) itself will do as \( g \) in (3.6). The equality between the three types of spaces is true also for sufficiently smooth bounded Euclidean domains, cf. [22, p. 196], but in general, the Hajłasz space \( M^{1,p}(\Omega, dx) \) can be substantially smaller than \( W^{1,p}(\Omega) \), e.g. for the slit disc in the plane. In fact, by Proposition 1 in Hajłasz–Koskela–Tuominen [25], for any Euclidean domain \( \Omega \) satisfying a measure density condition, the equality \( M^{1,p}(\Omega, dx) = W^{1,p}(\Omega) \) is equivalent to \( \Omega \) being a \( W^{1,p} \)-extension domain.

A repeated integration of (3.9) over a ball \( B \) shows that for all \( u \in M^{\alpha,p}(X, \mu) \),

\[
\int_B |u(x) - u_{B,\mu}|^p d\mu(x) \leq \int_B \int_B |u(x) - u(y)|^p d\mu(y) d\mu(x) \leq C r^{\alpha p} \int_B g^p d\mu, \tag{3.10}
\]

where \( r \) is the radius of \( B \). In particular,

\[
M^{\alpha,p}(X, \mu) \subset P_1^{\alpha,p}(X, \mu) \cap L^p(X, \mu). \tag{3.11}
\]

A detailed analysis of the spaces \( M^{1,p} \) and \( P_1^{1,p} \), as well as comparisons with other types of Sobolev spaces can be found in [22].

**Remark 3.7** (Nontriviality of \( M^{\alpha,p} \) for \( \alpha > 1 \)). If \( \Omega \subset \mathbb{R}^n \) is a domain and \( \alpha > 1 \), then Proposition 2 in Brezis [13] implies that every \( u \in M^{\alpha,p}(\Omega, dx) \) is constant. On the other hand, for more general sets with few rectifiable curves, \( M^{\alpha,p}(X, \mu) \) can be nontrivial even for \( \alpha > 1 \), see [28, Example 6.3] and Hu [33]. In fact, it is easily verified that if \( \mathcal{I} \) is the unit interval \((0, 1)\) equipped with the snowflaked metric \( \hat{d}(x, y) := |x - y|^{1/2} \), then \( M^{2,p}(\mathcal{I}, dx) = M^{1,p}((0, 1), dx) = W^{1,p}((0, 1)) \) is the usual Sobolev space when \( p > 1 \).

Similarly, the fractional spaces \( M^{\alpha,p}(X, \mu) \) and \( P_1^{\alpha,p}(X, \mu) \) for \( 0 < \alpha < 1 \) coincide with the spaces \( M^{1,p}(X_\alpha, \mu) \) and \( P_1^{1,p}(X_\alpha, \mu) \), respectively, where \( X_\alpha \) denotes the space \( X \) equipped with the snowflaked metric \( d_\alpha(x, y) := d(x, y)^\alpha \).

Theorem 2.3 implies the following simple compactness results for \( P_1^{\alpha,p} \) and \( M^{\alpha,p}(X, \mu) \) with very general measures. Note that \( \mu \) need not be doubling.

**Proposition 3.8** (Embeddings for \( P_1^{\alpha,p} \) with the same measure). Let \( E \subset X \) be bounded Lebesgue measurable subsets of \( \mathbb{R}^n \), equipped with the measure \( d\mu(x) = w(x) \, dx \), where \( 0 < w \in L^t(X) \) for some \( t \geq 1 \). Then the embedding

\[
P_1^{\alpha,p}(X, \mu) \cap L^1(X, \mu) \hookrightarrow L^1(E, \mu)
\]

is compact for all \( \alpha \geq n(1 - 1/p)/t \). In particular, for the classical exponents \( p = 2 \) and \( \alpha = 1 \), this holds for any weight \( 0 < w \in L^1_{\text{loc}}(\mathbb{R}^2) \).
A similar statement, with \( n \) replaced by \( d \), holds if \( X \) is (a measurable subset of) a bounded self-similar Cantor set in \( \mathbb{R}^n \) of Hausdorff dimension \( 0 < d < n \), equipped with \( d\mu = w \, d\Lambda_d \), where \( \Lambda_d \) is the \( d \)-dimensional Hausdorff measure.

**Proof.** For any \( r_m > 0, m = 1, 2, \ldots \), a simple geometrical argument shows that \( X \) can be covered by \( K_m \leq Cr_m^{-n} \) many balls \( B(x_i, r_m) \) with \( x_i \in E \) and bounded overlap \( N_m = N \) depending only on the dimension \( n \). Definition 3.5 implies that functions in \( P_{r,E}^{\alpha,E}(X, \mu) \) satisfy the Poincaré type inequality (2.3) with \( q = 1, \nu = \mu \), \( E_i = B(x_i, r_m) \cap X \), \( E_i' = B(x_i, \tau r_m) \cap X \), integral averages \( a_{E_i}(u) = u_{E_i, \mu} \) and

\[
C_m = r_m^\alpha \sup_{z \in E} \mu(B(z, r_m) \cap X)^{1-1/p}.
\]

Hölder’s inequality implies that

\[
\mu(B(z, r_m) \cap X) \leq Cr_m^{n(1-1/t)} \left( \int_{B(z, r_m) \cap X} w(x)^t \, dx \right)^{1/t}
\]

and hence

\[
C_m N_m^{1/p} K_m^{1-1/p} \leq C r_m^{\alpha-n(1-1/p)/t} \sup_{z \in E} \left( \int_{B(z, r_m) \cap X} w(x)^t \, dx \right)^{(1-1/p)/t}.
\]

Absolute continuity of the integral shows that the last supremum tends to zero as \( r_m \to 0 \). Since \( \alpha - n(1-1/p)/t \geq 0 \) and the integral averages \( u_{E_i, \mu} \) are bounded for each fixed \( E_i \), Theorem 2.3 applied to \( E = X \) concludes the proof. \( \square \)

Because of the inclusions (3.7) and (3.11), Proposition 3.8 applies also to the Hajlasz space \( M^{\alpha,p}(X, \mu) \). The next result gives a compact embedding also into \( L^p \). It partially generalizes (to \( \alpha \neq 1 \)) Theorem 2 in Kalamajska [36].

**Proposition 3.9** (Embeddings into \( L^p \) with the same measure). Assume that \( E \) is totally bounded and that one of the conditions (a) and (b) in Lemma 2.12 holds with \( \lambda = 1 \), \( M_1(r) := Cr^{-\theta} \) and \( \theta \geq 0 \), for all balls centred in \( E \) and of radius at most \( r_0 > 0 \). Assume also that \( 0 < \mu(B) < \infty \) for all such balls. Then the embedding

\[
M^{\alpha,p}(X, \mu) \hookrightarrow L^p(E, \mu)
\]

is compact when \( \alpha > \theta/p \).

Moreover, if there exists a totally bounded set \( \emptyset \neq E_0 \subset X \) such that \( \mu|_{E_0} \) is not a finite sum of atoms, then \( M^{\alpha,p}(X, \mu) \neq L^p(X, \mu) \) in this case.

**Proof.** Remark 2.9 and Lemma 2.12 with \( \lambda = 1 \), provide us for any \( r_m > 0, m = 1, 2, \ldots \), with a covering family \( \{E_i\}_{i=1}^{K_m} \), consisting of finitely many balls of radius \( r_m \leq r_0 \) and with overlap \( N_m \leq Cr_m^{-\theta} \). The statement about compactness now follows from (3.10) and
Theorem 2.3 with \( q = p, C_m = C_{r_m}^\alpha, G_r(u) = g \) and the integral averages \( a_{E_i}(u) = u_{E_i,\mu} \), whose boundedness is easily justified.

Finally, if \( M^{\alpha,p}(X,\mu) = L^p(X,\mu) \) and \( E_0 \subset X \) is as in the statement of the proposition, then \( L^p(E_0,\mu) \) is continuously embedded in \( L^p(X,\mu) \), by means of the zero extension of functions, and hence compactly embedded in itself, because of the first part of the proposition. This implies that \( L^p(E_0,\mu) \) must be finite-dimensional, by the Riesz lemma [40, Theorem 2.5-5], i.e. \( \mu|_{E_0} \) is a finite sum of atoms. \( \Box \)

**Proof of Proposition 1.2.** Since \( X \) is bounded and doubling, it is totally bounded and we can apply Proposition 3.9 with \( E = E_0 = X \). Note that condition (a) in Lemma 2.12 is satisfied with constant \( M_1(r) \), i.e. \( \theta = 0 \). \( \Box \)

4. Derivation of Poincaré type inequalities

Next, we prove the Poincaré type inequalities (2.3) and (2.5) for balls in spaces with a good domain measure \( \mu \). This generalizes Theorem 7 in Björn [11] to measures restricted to certain (possibly lower-dimensional) subsets. The proof has been inspired by Hajłasz–Koskela [24, Theorem 5.3] and Heinonen–Koskela [31, Lemma 5.15].

**Theorem 4.1 (Self-improvement of Poincaré inequalities).** Let \( 1 \leq p < q < \infty, r_0 > 0, \alpha > 0 \) and \( \lambda \geq 1 \). Assume that the measures \( \mu \) and \( \nu \) satisfy the doubling conditions (1.5) and (1.6) in Assumptions (D) for all balls centred in \( E \) and with radius at most \( r_0 \). Let \( u \in L^1_{\text{loc}}(X,\mu) \) be such that

\[
u \cdot \frac{u - u_{B(x, \lambda r)}}{\mu(B(x, \lambda r))^{1/p}} \leq C_{q, \lambda} \Theta_{q, \lambda}(r),
\]

and, for some function \( g \in L^p_{\text{loc}}(X,\mu) \), the Poincaré inequality (1.2) holds with dilation \( \lambda \geq 1 \) on all balls \( B \) centred in \( E \) and with radius at most \( r_0 \).

If the local Poincaré constant on \( E \),

\[
\Theta_{q, \lambda}(r) := \sup_{0 < \rho \leq r} \sup_{x \in E} \frac{\rho^\alpha \nu(B(x, \rho))^{1/q}}{\mu(B(x, \lambda \rho))^{1/p}},
\]

satisfies \( \Theta_{q, \lambda}(r_0) < \infty \), then the following are true with \( C_{q, \lambda} > 0 \) depending only on \( p, q, \mu, \nu, \alpha, \lambda \) and \( \delta \), but not on \( x, r, u \) and \( g \):

(i) For all \( 1 \leq q' < q \), the following two-weighted Poincaré type inequality

\[
\left( \int_{B \cap E} |u - u_{B,\mu}|^{q'} \, d\nu \right)^{1/q'} \leq C_{q, \lambda} \Theta_{q, \lambda}(r) \frac{\nu(B \cap E)^{1/q - 1/q}}{q - q'} \left( \int_{2\lambda B} g^p \, d\mu \right)^{1/p}
\]

holds for all balls \( B = B(x, r) \) with \( x \in E \) and \( 10\lambda r \leq r_0 \).
(ii) If the pair $(u,g)$ also satisfies the truncation property, i.e. for all $l < k$, the Poincaré inequality $(1.2)$ holds with $u$ and $g$ replaced by

$$u_{l,k} := \max\{l, \min\{u,k\}\} \quad \text{and} \quad g_{l,k} := g\chi_{\{l<u<k\}},$$

(4.4)

respectively, then $(4.3)$ holds also for $q' = q$.

A straightforward application of the triangle inequality shows that the integral average $u_{B,\mu}$ in the left-hand side of $(4.3)$ can always be replaced by $u_{B\cap E, \nu}$ at the cost of an additional factor 2 on the right-hand side.

The truncation property in Theorem 4.1 (ii) is satisfied if $g = |\nabla u|$ in $\mathbb{R}^n$ or if $g = g_u$ is the minimal $p$-weak upper gradient of $u$, but not for the nonlocal Hajlasz gradients, both in $\mathbb{R}^n$ and in metric spaces. Before proving Theorem 4.1, we formulate some remarks about its assumptions.

Remark 4.2 (Choice of $\alpha$ and $q$). In $(1.2)$ and $(4.2)$ we allow also $\alpha > 1$. In the classical $p$-Poincaré inequality in Euclidean spaces, such a choice would force $u$ to be constant, by Heikkinen–Koskela–Tuominen [28, Corollary 1.2] or Brezis [13]. On the other hand, for functions defined on fractal sets, even $\alpha > 1$ can give nontrivial results, cf. Remark 3.7.

The proof of Theorem 4.1 requires $q > p$, but once $(4.3)$ has been proved for some $q > p$, a similar inequality holds also for smaller exponents, by Hölder’s inequality. However, even in that case, $(4.2)$ has to be assumed with some initial $q > p$ for the proof to apply. It would be interesting to see which Poincaré inequalities can be obtained from $(1.2)$ when $\Theta_{q,\lambda}(r_0) < \infty$ only for some $q \leq p$.

Remark 4.3 (Role of $\delta$ and uniform perfectness). The exact value of $\delta$ in $(1.6)$ of Assumptions (D) is unimportant and only has effect on the constant $C_{q,\lambda}$ in the Poincaré inequality $(4.3)$. It can be proved as in [6, Corollary 3.8] that $(1.6)$ follows for some $\delta > 0$ from the local doubling condition $(1.5)$ if $E$ is connected or, more generally, locally uniformly perfect, i.e. there is $0 < a < 1$ such that the set $(E \cap B(x,r)) \setminus B(x,ar)$ is nonempty whenever $x \in E$ and $0 < r \leq r_0$. Note that many fractal sets, which are natural candidates for $E$, are uniformly perfect but not connected. If $\nu$ is supported on $E$ and satisfies $(1.5)$, then $(1.6)$ with some $\delta > 0$ is equivalent to the local uniform perfectness of $E$, see Martín–Ortiz [47, Lemma 7].

Remark 4.4 ($\mu$-Lebesgue points). Assumption $(4.1)$ in Theorem 4.1 is not too restrictive for our applications. Since $u \in L^1_{\text{loc}}(X,\mu)$ and $\mu$ satisfies the local doubling condition $(1.5)$ on $E$, it can be shown as in Heinonen [29, Theorems 1.6 and 1.8], using only balls centred in $E$, that $\mu$-a.e. $x \in E$ satisfies the equality in $(4.1)$.

Lemma 4.5 below then shows that an $L^1_{\text{loc}}(X,\mu)$-representative of $u$ in Theorem 4.1 always has $\mu$-Lebesgue points $\nu$-a.e. in $E$. 
Lemma 4.5 (Non-Lebesgue points). Assume that the measures $\mu$ and $\nu$ satisfy (1.5) and (1.6) in Assumptions (D). Let $u \in L^1_{\text{loc}}(X, \mu)$ be such that the Poincaré type inequality (1.2) with parameters $\alpha > 0$ and $\lambda \geq 1$ holds for some function $g \in L^p_{\text{loc}}(X, \mu)$ and all balls centred in $E$ and of radius at most $r_0 > 0$. Assume that

$$\sup_{0 < r \leq r_0} \frac{r^\alpha \nu(B(x, r))^{1/q}}{\mu(B(x, \lambda r))^{1/p}} < \infty \quad \text{for every } x \in E,$$

where $q > p$. Then $\nu(E_0) = 0$, where

$$E_0 := \{x \in E : \text{the limit in (4.1) does not exist or is not finite}\}.$$

In particular, a representative $\bar{u}$ of $u$ in $L^1_{\text{loc}}(X, \mu)$ has $\mu$-Lebesgue points $\nu$-a.e. in $E$ in the sense of (4.1).

Proof. By splitting $E$ into subsets and using the countable subadditivity of $\nu$, we can assume that $E$ is contained in a ball $B_0$ of radius $r_0$ and that for some $M$,

$$\sup_{0 < r \leq r_0} \frac{r^\alpha \nu(B(x, r))^{1/q}}{\mu(B(x, \lambda r))^{1/p}} \leq M \quad \text{for all } x \in E. \quad (4.5)$$

The Poincaré type inequality (1.2) and the local doubling property (1.5) of $\mu$ imply as in [6, (5.1) and the proof of Theorem 5.1] that

$$\lim_{r \to 0} \int_{B(x, r)} u \, d\mu \text{ exists and is finite,} \quad (4.6)$$

whenever $x \in E$ is such that the fractional maximal function

$$\sup_{0 < r \leq r_0/5\lambda} r^\alpha' \left( \int_{B(x, r)} g^p \, d\mu \right)^{1/p}$$

is finite for some $0 < \alpha' < \alpha$. It therefore suffices to estimate $\nu(E_t)$ for $t > 0$, where

$$E_t = \left\{ x \in E : \sup_{0 < r \leq r_0/5\lambda} r^\alpha' \int_{B(x, r)} g^p \, d\mu > t \right\}.$$

Let $\alpha' = \alpha - \delta(1/p - 1/q)$, where $\delta$ is as in (1.6). For each $x \in E_t$, find $0 < r_x \leq r_0/5\lambda$ such that

$$\mu(B(x, r_x)) < \frac{r_x^\alpha' p}{t} \int_{B(x, r_x)} g^p \, d\mu.$$
Together with the local doubling property (1.5) of $\mu$ and (4.5), this shows that

$$\nu(B(x, r_x))^{p/q} \leq \frac{M \mu(B(x, \lambda r_x))}{r_x^{\alpha p}} \leq \frac{C M r_x^{(\alpha' - \alpha)p}}{t} \nu(B(x, r_x))^{q/p} \int_{B(x, r_x)} g^p \, d\mu.$$ 

Moreover, (1.6) yields

$$\nu(B(x, r_x))^{1-p/q} \leq C \nu(B(x, r_0))^{1-p/q} \left(\frac{r_x}{r_0}\right)^{\delta(1-p/q)}.$$

Combining the last two estimates with the local doubling property (1.5) for $\nu$ and the fact that $(\alpha - \alpha')p = \delta(1 - p/q)$, we get

$$\nu(E_t) \leq \sum_j \nu(5B_j) \leq \frac{C}{t} \sum_j \int_{B_j} g^p \, d\mu \leq \frac{C}{t} \int_{\bigcup_j B_j} g^p \, d\mu.$$

Since $g \in L^p_{\text{loc}}(X, \mu)$ and $E_0 \subset E_t$ for all $t > 0$, letting $t \to \infty$ shows that $\nu(E_0) = 0$. Redefining $u$ on $E \setminus E_0$ by (4.6) then provides the desired $L^1_{\text{loc}}(X, \mu)$-representative, see Remark 4.4.  

4.1. Proof of Theorem 4.1

**Proof.** Let $x_0 \in E$ and $B = B(x_0, r)$ with $10\lambda r \leq r_0$ be fixed. Let $u$ and $g$ be as in the statement of the theorem. Assume that $x \in B \cap E$ is a Lebesgue point of $u$ with respect to $\mu$ and set

$$B_0 = 2B \quad \text{and} \quad B_j = B(x, 2^{1-j}r), \quad j = 1, 2, \ldots.$$

The Poincaré inequality (1.2), a telescoping argument and (1.5) for $\mu$ imply that

$$|u(x) - u_{B_0, \mu}| = \lim_{j \to \infty} |u_{B_j, \mu} - u_{B_0, \mu}| \leq \sum_{j=0}^{\infty} \int_{B_{j+1}} |u - u_{B_j, \mu}| \, d\mu.$$
\[ \leq C \sum_{j=0}^{\infty} \int_{B_j} |u - u_{B,j,\mu}| \, d\mu \leq C \sum_{j=0}^{\infty} (2^{-j} r)^\alpha \left( \int_{\lambda B_j} g^p \, d\mu \right)^{1/p} \]

and

\[ |u_{B,\mu} - u_{B_0,\mu}| \leq C r^\alpha \left( \int_{\lambda B_0} g^p \, d\mu \right)^{1/p}. \]

Applying (4.2) to the balls \( B_j \) and \( B_0 \) then yields

\[ |u(x) - u_{B,\mu}| \leq C \Theta_{q,\lambda(r)} \sum_{j=0}^{\infty} \frac{1}{\nu(B_j)^{1/q}} \left( \int_{\lambda B_j} g^p \, d\mu \right)^{1/p}. \] (4.7)

Write the above sum as \( \Sigma' + \Sigma'' \), where the summation in \( \Sigma' \) and \( \Sigma'' \) is over \( j < j_0 \) and \( j \geq j_0 \), respectively, and \( j_0 \) will be chosen later. By the local dimension condition (1.6) for \( \nu \) we have

\[ \nu(B_j) \geq C 2^{(j_0-j)\delta} \nu(B_{j_0}) \quad \text{for } j < j_0, \]
\[ \nu(B_j) \leq C 2^{(j_0-j)\delta} \nu(B_{j_0}) \quad \text{for } j \geq j_0, \] (4.8)

and hence,

\[ \Sigma' = \sum_{j=0}^{j_0-1} \nu(B_j)^{-1/q} \left( \int_{\lambda B_j} g^p \, d\mu \right)^{1/p} \leq C \sum_{j=0}^{j_0-1} 2^{(j-j_0)\delta q} \nu(B_{j_0})^{-1/q} \left( \int_{\lambda B_j} g^p \, d\mu \right)^{1/p} \]

At the same time, because of (4.8) and since \( 1/p - 1/q > 0 \), we have

\[ \Sigma'' = \sum_{j=j_0}^{\infty} \nu(B_j)^{1/p-1/q} \left( \frac{1}{\nu(B_j)} \int_{\lambda B_j} g^p \, d\mu \right)^{1/p} \leq C \nu(B_{j_0})^{1/p-1/q} M(x)^{1/p}, \]

where

\[ M(x) := \sup_{B' \subset B} \frac{1}{\nu(B')} \int_{\lambda B'} g^p \, d\mu \] (4.9)

is a generalized maximal function, with the supremum taken over all balls \( B' \), containing \( x \) and of radius \( r' \leq 2r \). Inserting this into (4.7) yields
\[ |u(x) - u_{B,\mu}| \leq C\Theta_{q,\lambda}(r)(\Sigma' + \Sigma'') \leq C \frac{\Theta_{q,\lambda}(r)}{\nu(B_{j_0})^{1/q}} \left( \left( \int_{\lambda B_0} g^p \, d\mu \right)^{1/p} + (\nu(B_{j_0})M(x))^{1/p} \right). \]

To minimize the right-hand side of (4.10), we will choose \( j_0 \) so that \( \Sigma' \) and \( \Sigma'' \) are comparable. We can assume that \( M(x) > 0 \), as otherwise (4.9), (1.2) and (4.1) imply that \( g \equiv 0 \) \( \mu \text{-a.e. in } \lambda B \) and \( u \equiv u_{B,\mu} \) both \( \mu \text{-a.e. and } \nu \text{-a.e. in } B \), so that the statement of the theorem holds trivially.

A standard argument using the Vitali type 5-covering lemma [31, Lemma 5.5], together with the doubling property (1.5) of \( \nu \) on \( E \) and the fact that \( 10\lambda r \leq r_0 \), shows for all \( \tau > 0 \) the weak type inequality

\[ \nu(\{ x \in B \cap E : M(x) \geq \tau \}) \leq \frac{C}{\tau} \int_{2\lambda B} g^p \, d\mu, \]

cf. Chapter 1 in Stein [57]. Hence \( M(x) < \infty \) for \( \nu \text{-a.e. } x \in B \cap E \). We consider only such \( x \) in the rest of the proof. Since \( B' = B_0 \) is allowed in (4.9) and \( \nu(B_j) \to 0 \), as \( j \to \infty \), we can use the doubling property of \( \nu \) to find \( j_0 = j_0(x, r) \geq 0 \) such that

\[ \frac{1}{\nu(B_{j_0})} \int_{\lambda B_0} g^p \, d\mu \leq M(x) \leq \frac{1}{\nu(B_{j_0}+1)} \int_{\lambda B_0} g^p \, d\mu \leq \frac{C}{\nu(B_{j_0})} \int_{\lambda B_0} g^p \, d\mu. \]

Then \( \nu(B_{j_0}) \) is comparable to \( M(x)^{-1} \int_{\lambda B_0} g^p \, d\mu \) and inserting this into (4.10) yields

\[ |u(x) - u_{B,\mu}| \leq C\Theta_{q,\lambda}(r) \left( \int_{2\lambda B} g^p \, d\mu \right)^{1/p-1/q} M(x)^{1/q}, \]

since \( \lambda B_0 = 2\lambda B \). Using (4.11) we therefore conclude that

\[ t^q \nu(\{ x \in B \cap E : |u(x) - u_{B,\mu}| \geq t \}) \leq C[\Theta_{q,\lambda}(r)]^q \left( \int_{2\lambda B} g^p \, d\mu \right)^{q/p}. \]

(4.12)

Now, for \( 1 \leq q' < q \), we have by the Cavalieri principle that

\[ \int_{B \cap E} |u - u_{B,\mu}|^{q'} \, d\nu = q' \int_0^\infty t^{q'-1} \nu(\{ x \in B \cap E : |u(x) - u_{B,\mu}| \geq t \}) \, dt. \]

Splitting the integral as \( \int_0^\infty = \int_0^{t_0} + \int_{t_0}^\infty \) and estimating the integrands by \( \nu(B \cap E) \) and (4.12), respectively, yields
\[
\int_{B \cap E} |u - u_{B,\mu}|^{q'} \, d\nu \leq t_0^{q'} \nu(B \cap E) + \frac{C q'^{q'-q} \Theta_{q,\lambda}(r)^q}{q - q'} \left( \int_{2\lambda B} g^p \, d\mu \right)^{q/p}.
\]

Choosing
\[
t_0 = \frac{\Theta_{q,\lambda}(r)}{\nu(B \cap E)^{1/q}} \left( \int_{2\lambda B} g^p \, d\mu \right)^{1/p}
\]
then gives (4.3) for \( q' < q \).

To obtain (4.3) also for \( q' = q \), we proceed by Maz'ya’s truncation method [49]. We shall estimate the integral
\[
\int_{B \cap E} v^q \, d\nu, \quad \text{where } v := \max\{u - u_{B,\mu}, 0\}.
\]

We can clearly assume that \( v \neq 0 \). Let \( k_0 \) be the largest integer such that
\[
2^{k_0-1} \leq \int_{B} v \, d\mu,
\]
and for \( k > k_0 \) let \( u_k := u_{l, l+2^k} \) be the truncations of \( u \) at levels \( l := u_{B,\mu} + 2^k \) and \( l + 2^k = u_{B,\mu} + 2^{k+1} \), defined as in (4.4). Note that \( u_k = (u - u_{B,\mu})_{2^k, 2^{k+1}} + u_{B,\mu} \leq v + l \).

Since (4.13) fails for \( k > k_0 \), we have for such \( k \),
\[
(u_k)_{B,\mu} \leq l + \int_{B} v \, d\mu < l + 2^{k-1}.
\]
Moreover, \( v(x) \geq 2^{k+1} \) if and only if \( u_k(x) \geq l + 2^k \). Hence the estimate (4.12), applied with \( t = 2^{k-1} \) to the truncations \( u_k \), yields that for \( k > k_0 \),
\[
2^{(k-1)q} \nu(\{x \in B \cap E : v(x) \geq 2^{k+1}\})
\leq 2^{(k-1)q} \nu(\{x \in B \cap E : |u_k(x) - (u_k)_{B,\mu}| \geq 2^{k-1}\})
\leq C \Theta_{q,\lambda}(r)^q \left( \int_{2\lambda B} g_k^p \, d\mu \right)^{q/p},
\]
where \( g_k := g\chi(2^k < v < 2^{k+1}) \). Summing over \( k > k_0 \) we get that
\[
\int_{\{x \in B \cap E : v(x) > 2^{k_0 + 2}\}} v^q \, d\nu \leq \sum_{k=k_0+1}^{\infty} 2^{(k+2)q} \nu(\{x \in B \cap E : 2^{k+2} \geq v(x) \geq 2^{k+1}\})
\]
\[ \leq C[\Theta_{q,\lambda}(r)]^q \sum_{k=k_0+1}^{\infty} \left( \int_{2\lambda B} g_k^p \, d\mu \right)^{q/p} \]

\[ \leq C[\Theta_{q,\lambda}(r)]^q \left( \int_{2\lambda B} g^p \, d\mu \right)^{q/p}, \quad (4.14) \]

where in the last step we used the elementary inequality \( \sum_{j=1}^{\infty} a_j^{q/p} \leq \left( \sum_{j=1}^{\infty} a_j \right)^{q/p} \), valid since \( q > p \). At the same time, (4.13) together with the fact that \( 0 \leq v \leq |u - u_{B,\mu}| \) and the assumed Poincaré type inequality (1.2) implies that

\[ \hat{\{ x \in B \cap E : v(x) \leq 2(k_0+2) \}} \]

\[ \leq 2^{(k_0+2)} \nu(B \cap E) \leq C \nu(B \cap E) \left( \int_B v \, d\nu \right)^q \]

\[ \leq C \frac{\nu(\lambda B)^q}{\mu(\lambda B)^{q/p}} \left( \int_{\lambda B} g^p \, d\mu \right)^{q/p}. \]

Adding this to (4.14) we conclude, using the definition of \( \Theta_{q,\lambda}(r) \), that

\[ \int_{B \cap E} \max\{ u - u_{B,\mu}, 0 \}^q \, d\nu \leq C[\Theta_{q,\lambda}(r)]^q \left( \int_{2\lambda B} g^p \, d\mu \right)^{q/p}. \]

The integral \( \int_{B \cap E} \max\{ u_{B,\mu} - u, 0 \}^q \, d\nu \) can be estimated similarly, since (1.2) and the truncation property remain invariant when replacing the pair \((u, g)\) by \((-u, g)\). Hence, we can conclude that (4.3) holds and \( u \in L^q(B \cap E, \nu) \).

**Proof of Theorem 1.1.** Since \( E \) is uniformly perfect, Remark 4.3 implies that the dimension condition (1.6) holds for \( \nu \). For a fixed ball \( B \subset X \) centred in \( E \), the statement now follows by applying Theorem 4.1 with \( E \) replaced by \( E \cap B \). Note that the assumptions in Theorem 4.1 are satisfied for any \( 0 < r_0 < \infty \).

5. **Embeddings with locally doubling measures**

Theorems 2.3 and 4.1 imply the following practical conditions for compact and bounded embeddings. Recall from the introduction that an embedding into \( L^q(E, \nu) \) means that \( \bar{u} \in L^q(E, \nu) \), where \( \bar{u} \) is given by (1.9). Condition (1.5) in Assumptions (D) implies that \( \bar{u} = u \) \( \mu \)-a.e. in \( E \), see Remark 4.4. Recall Definition 3.5 of the \( E \)-restricted Poincaré Sobolev space \( P_{r,E}^{\alpha,p}(X, \mu) \).

**Theorem 5.1** (Embeddings with locally doubling measures). Assume that \( E \) is totally bounded and that the measures \( \mu \) and \( \nu \) satisfy the doubling conditions (1.5) and (1.6)
in Assumptions (D) for all balls \( B \) centred in \( E \) and with radius at most \( r_0 \). Let \( Y \subset P_{\lambda,E}(X,\mu) \), \( \lambda \geq 1 \), be equipped with one of the norms

\[
\|u\|_{L^1(E,\nu)} + \|G(u)\|_{L^p(X,\mu)} \quad \text{or} \quad \|u\|_{L^1(X,\mu)} + \|G(u)\|_{L^p(X,\mu)},
\]

(5.1)

where \( G(u) \) is such that (3.6) in Definition 3.5 holds with \( g = G(u) \).

Then the following hold for \( q > p \):

(i) If \( \Theta_{q,\lambda}(r_0) < \infty \) then the embedding \( Y \hookrightarrow L^q(E,\nu) \) is compact for all \( q' < q \).

(ii) If functions in \( Y \) satisfy the truncation property in Theorem 4.1 (ii), then the embedding \( Y \hookrightarrow L^q(E,\nu) \) is bounded when \( \Theta_{q,\lambda}(r_0) < \infty \) and compact when \( \Theta_{q,\lambda}(r) \to 0 \), as \( r \to 0 \).

Since \( E \) is totally bounded, we have \( \nu(E) < \infty \), and so embeddings into \( L^q \) for \( q \leq p \) follow from those with \( q > p \). Propositions 7.1 and 7.3 below show that statement (ii) admits a converse, at least for some choices of \( Y \). See also Example 3.3 for a situation where (i) is not strong enough to include the optimal exponent \( q = 2 \) for a compact embedding. Total boundedness of \( E \) is natural since e.g. the Sobolev space \( W^{1,p}(\mathbb{R}^n) \) does not embed compactly into any \( L^q(\mathbb{R}^n) \). By Proposition 7.4, it is also necessary at least for some \( Y \) when \( q = p \), \( E = X \) and \( \nu = \mu \) is doubling.

To estimate \( \Theta_{q,\lambda}(r) \), we can use the dimension conditions (1.7) and (1.8) from Assumptions (D). Note that (1.8) with \( \sigma = \delta \) follows from (1.6) if \( E \) can be covered by finitely many balls \( \{B_j^0\}_j \) of radius \( r_0 \) and \( \nu(2B_j^0) < \infty \). Indeed, for each \( x \in E \) there is \( B_j^0 \) such that \( x \in B_j^0 \) and hence for \( r \leq r_0 \),

\[
\nu(B(x,r)) \leq C \nu(B(x,r_0)) \left( \frac{r}{r_0} \right)^\delta \leq \frac{C \max_j \nu(2B_j^0)}{r_0^\delta} r^\delta. \tag{5.2}
\]

Similarly, an iteration of (1.5) implies (1.7) with some \( s > 0 \), see [6, Lemma 3.3].

**Proof of Theorem 5.1.** Choose a sequence \( r_m \to 0 \) so that \( 0 < r_m \leq r_0/10\lambda \). As in Remark 2.9, and using the total boundedness of \( E \), the set \( E \) can be for each fixed \( m = 1,2,... \), covered by finitely many balls \( B_i \) with centres in \( E \) and radius \( r_m \) so that the balls \( \frac{1}{2}B_i \) are pairwise disjoint. This gives us a covering family with \( E_i = B_i \cap E \) and \( E_i' = 2\lambda B_i \), Lemma 2.12 with \( X \) replaced by \( E \), together with the local doubling property on \( E \), shows that the dilated balls \( 2\lambda B_i \) have a bounded overlap \( N \) independent of \( m \).

Note that \( \nu(E_i') \leq C r_i^\delta_m \) with \( C \) independent of \( m \) and \( i \), by (5.2). Since \( \Theta_{q,\lambda}(r_0) < \infty \) and (1.6) in Assumptions (D) holds, functions in \( Y \) (modified by (1.9)) satisfy (4.1), by Remark 4.4 and Lemma 4.5.

To prove (i), consider a sequence in \( Y \), which is bounded with respect to one of the norms in (5.1). Theorem 4.1 implies that (4.3) holds with \( G_m(u) = G(u) \) for all \( 1 \leq q' < q \). Thus, Assumptions (C) are satisfied for all such \( q' \) with
\[ C_m := C \Theta_{q,\lambda}(r_m) \max_i \nu(E_i)^{1/q'-1/q} \leq C \Theta_{q,\lambda}(r_0)^{\nu(r_0)^{1/q'-1/q}\delta} \]  

(5.3)

and with both integral averages \( a_{E_i}(u) = u_{B_i,\mu} \) and \( a_{E_i}(u) = u_{E_i,\nu} \). These averages are clearly bounded for each fixed \( E_i \) and each of the norms in (5.1). Since \( \Theta_{q,\lambda}(r_0) < \infty \), we have \( C_m \to 0 \) and an application of Theorem 2.3 proves (i).

If the truncation property holds in (ii), then (4.3) and (5.3) hold also for \( q' = q \) with \( C_m := C \Theta_{q,\lambda}(r_m) \). Boundedness then follows from Theorem 2.6 since \( \Theta_{q,\lambda}(r_0) < \infty \) implies that Assumptions (B) are satisfied with the above integral averages \( a_{E_i}(u) \). Similarly, \( \Theta_{q,\lambda}(r_m) \to 0 \) in (ii) implies that Assumptions (C) hold with \( C_m \to 0 \), which gives a compact embedding also into \( L^q(E,\nu) \). \( \square \)

**Corollary 5.2** (Embeddings with dimension conditions). Assume that \( E \) is totally bounded and that \( \mu \) and \( \nu \) satisfy the doubling and dimension conditions (1.5)–(1.8) in Assumptions (D) with exponents \( s \) and \( \sigma > s - \alpha p \), for all balls \( B \) centred in \( E \) and with radius at most \( r_0 \). Let \( Y \subset P_{\alpha,p}^\nu(X,\mu), \lambda \geq 1 \), be equipped with one of the norms (5.1). Then the following hold for \( q > p \):

(i) The embedding \( Y \hookrightarrow L^q(E,\nu) \) is compact whenever \( q(s - \alpha p) < \sigma p \). When \( \nu = \mu \), condition (1.8) is not needed and the embedding is compact whenever \( q(s - \alpha p) < sp \).

(ii) If functions in \( Y \) satisfy the truncation property in Theorem 4.1 (ii), then the embedding \( Y \hookrightarrow L^q(E,\nu) \) is bounded whenever \( q(s - \alpha p) \leq \sigma p \). When \( \nu = \mu \), condition (1.8) is not needed and the embedding is bounded whenever \( q(s - \alpha p) \leq sp \).

**Proof.** For (i), choose \( \tilde{q} > \max\{q,p\} \) such that \( \tilde{q}(s - \alpha p) < \sigma p \). Consider the local Poincaré constant \( \Theta_{\tilde{q},\nu}(r) \), defined as in (4.2) with \( q \) replaced by \( \tilde{q} \). Using (1.7) and (1.8) we see that \( \Theta_{\tilde{q},\nu}(r_0) < \infty \). Theorem 5.1 (i), with \( q \) and \( q' \) replaced by \( \tilde{q} \) and \( q \), then concludes the proof in the general case \( \nu \neq \mu \). The special case \( \nu = \mu \) is similar and uses only (1.7).

Part (ii) follows directly from Theorem 5.1 (ii) and the fact that the dimension conditions (1.7) and (1.8) in Assumptions (D) imply

\[ \Theta_{q,\lambda}(r) \leq C_0 \sup_{0 < \rho \leq r} \rho^{\alpha + \sigma/q - s/p}. \]  

\( \square \)

Theorem 5.1 and Corollary 5.2 apply in particular to \( Y = M^{\alpha,p}(X,\mu) \) and extend earlier results from Hajłasz [22, Theorem 8.7], dealing with \( \nu = \mu \), \( E = X \) and \( \alpha \leq 1 \) (the case \( \alpha < 1 \) follows from [22] by using the snowflaked metric \( d(x,y)^\alpha \)). Note that the dimension condition (1.7) in Assumptions (D) is necessary for such Sobolev embeddings when \( X \) is bounded and uniformly perfect, by Alvarado–Górka–Hajłasz [3, Corollary 24]. In Proposition 7.3, we extend their result to the two-weighted situation.

Since certain types of Besov spaces embed into \( M^{\alpha,p} \), by e.g. Gogatishvili–Koskela–Shanmugalingam [19, Lemma 6.1] and Malý [46, Lemma 3.16], similar embeddings for Besov spaces also follow from Theorem 5.1 and Corollary 5.2.
The following examples illustrate why we only impose the doubling and dimension conditions in Assumptions (D) on balls centred in $E$. The local assumption about the Poincaré inequalities (1.2) and (1.10) can be justified similarly.

**Example 5.3.** Consider $X = \mathbb{R}^n$ with the weight

$$w(x_1,\ldots,x_n) = |x_1|^{-\theta}, \quad x_1 \neq 0 \text{ and } 0 < \theta < 1.$$ 

Since $w$ is easily verified to be an $A_1$ weight (see [30, p.10]), the measure $d\mu = w \, dx$ is doubling and supports a 1-Poincaré inequality (i.e. (1.2) with $p = \alpha = \lambda = 1$) for all balls $B \subset \mathbb{R}^n$. It is easily verified that for $x = (x_1, x_2, \ldots, x_n)$ and $r > 0$, the measure $\mu(B(x,r))$ is comparable to $r^{n-\theta}$ if $r \geq \frac{1}{2}|x_1|$ and to $|x_1|^{-\theta}r^n$ otherwise.

This implies that if $E$ is contained in the hyperplane \( \{x \in \mathbb{R}^n : x_1 = 0\} \) then the dimension condition (1.7) for $\mu$ and balls centred on $E$ holds with $s = n - \theta < n$, while for general balls centred in $\mathbb{R}^n$ we must have $s \geq n$. Since the critical exponent \( q = \sigma p/(s - \alpha p) \) in Corollary 5.2 increases as $s$ decreases, allowing for the smaller local “dimension” $s = n - \theta$ gives better embedding results than $s = n$. A similar argument applies e.g. when considering trace embeddings with respect to the weight $w(x) = \text{dist}(x, \partial \Omega)^{-\alpha}$ for sufficiently regular domains $\Omega \subset \mathbb{R}^n$, see Theorem 8.2 and Example 8.6 below.

**Example 5.4.** Let $E \subset \mathbb{R}^n$ be compact and consider a weight $0 < w \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfying $w \equiv 1$ on $E$ and \( \int_Q w \, dx = 1 \) for every cube $Q$ in the Whitney decomposition of $\mathbb{R}^n \setminus E$. The weighted measure $d\mu(x) = w(x) \, dx$ then satisfies, for all balls centred in $E$, the same doubling and dimension conditions in Assumptions (D) as the Lebesgue measure. It can therefore be used in Theorem 5.1 and Corollary 5.2.

However, $\mu$ can fail any doubling conditions for balls outside $E$ and the restriction $\mu|_E$ need not be doubling either. Thus, Assumptions (D) are substantially weaker than similar global conditions or assumptions for the restricted measure.

6. Newtonian spaces and Theorem 1.3

Another generalization of Sobolev spaces to metric spaces is the Newtonian spaces $N^{1,p}$. Their definition is based on the following notion of upper gradients, introduced in Heinonen–Koskela [31]: A Borel function $g : X \to [0,\infty]$ is an upper gradient of $u : X \to [-\infty,\infty]$ if for all rectifiable curves $\gamma : [0,l_\gamma] \to X$,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \quad (6.1)$$

where $ds$ is the arc length parametrization of $\gamma$ and the left-hand side is interpreted as $\infty$ if at least one of the terms therein is infinite.
If $u$ has an upper gradient in $L^p_{\text{loc}}(X, \mu)$, then it has a $\mu$-a.e. unique minimal ($p$-weak) upper gradient $g_{u} \in L^p_{\text{loc}}(X, \mu)$, see Shanmugalingam [56, Corollary 3.7].

**Definition 6.1.** The **Newtonian space** on $X$ is

$$N^{1,p}(X, \mu) = \left\{ u : \|u\|_{N^{1,p}(X, \mu)} := \left( \int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p} < \infty \right\},$$

where the infimum is taken over all ($p$-weak) upper gradients $g$ of $u$. The **Dirichlet space** $D^p(X, \mu)$ consists of all $u \in L^1_{\text{loc}}(X, \mu)$ having an upper gradient in $L^p(X, \mu)$, and is equipped with the seminorm $\|g_{u}\|_{L^p(X, \mu)}$.

Newtonian spaces were defined by Shanmugalingam [55] and are in general larger than both $M^{1,p}$ and $P^{1,p}_{\tau}$, see Hajłasz [22, Corollary 10.5]. For example, if $X$ has no rectifiable curves then $N^{1,p}(X, \mu) = L^p(X, \mu)$, while $M^{1,p}(X, \mu)$ is in general different from $L^p(X, \mu)$, see Proposition 3.9. The truncation property assumed in Theorems 4.1(ii) and 5.1(ii) and Corollary 5.2(ii) is valid for $N^{1,p}$ and the minimal $p$-weak upper gradients and makes it possible to treat also the limiting exponent $q' = q$ and to obtain the optimal sufficient and necessary conditions in Theorem 1.3.

For various properties of Newtonian functions we refer to Shanmugalingam [55, 56], Björn–Björn [6] and Heinonen–Koskela–Shanmugalingam–Tyson [32]. For (6.1) to make sense, functions in $N^{1,p}(X, \mu)$ and $D^p(X, \mu)$ are pointwise defined. Their equivalence classes are up to sets of zero $p$-capacity, where the **(Sobolev) $p$-capacity** of a set $A \subset X$ is

$$C^p_p(A) = \inf \|u\|^p_{N^{1,p}(X, \mu)},$$

with the infimum taken over all $u \in N^{1,p}(X, \mu)$ such that $u \geq 1$ on $A$.

**Remark 6.2 (N^{1,p} for domains in $\mathbb{R}^n$).** By [55], the space $N^{1,p}(\Omega, dx)$ coincides for $p > 1$ and open $\Omega \subset \mathbb{R}^n$ with the usual Sobolev space $W^{1,p}(\Omega)$, and

$$g_{u} = \left| \nabla u \right| \text{ a.e.,} \quad (6.2)$$

where $\nabla u$ is the distributional gradient. However, since the equivalence classes in $N^{1,p}(\Omega, dx)$ are taken with respect to sets of $p$-capacity zero, $N^{1,p}(\Omega, dx)$ consists only of the quasicontinuous representatives from $W^{1,p}(\Omega)$, considered e.g. in Evans–Gariepy [17, Chapter 4].

Similarly, [6, Appendix A.2] shows that if $0 < w \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a $p$-admissible weight in the sense of Heinonen–Kilpeläinen–Martio [30], then $N^{1,p}(\Omega, \mu)$ corresponds to the weighted (refined) Sobolev space studied in [30, Chapter 4]. In this case, (6.2) holds
with \( \nabla u \) defined through the completion of \( C^\infty \) as in [30, Section 1.9] (and equal to the distributional gradient when \( w^{1/(1-p)} \in L^1_{\text{loc}}(\mathbb{R}^n) \)).

For general weights, the relationship between \( N^{1,p} \) and other definitions of weighted Sobolev spaces is rather subtle, see Ambrosio–Pinamonti–Speight [5] and Zhikov [58] for counterexamples and sufficient integrability conditions on \( w \).

When \( \mu \) is doubling and supports the \( p \)-Poincaré inequality (1.10) with \( g_u \) on \( X \), we have for \( p > 1 \) and up to suitable equivalence classes and equivalent norms,

\[
N^{1,p}(X, \mu) = M^{1,p}(X, \mu) = P^{1,p}(X, \mu) \cap L^p(X, \mu),
\]

see [22, Theorem 11.3] (together with [8, Theorem 5.1] when \( X \) is not complete).

**Remark 6.3 (Embeddings and Lebesgue points for \( N^{1,p} \)).** The question of \( \bar{u} \) in (1.9) and \( \mu \)-Lebesgue points in Lemma 4.5 is more delicate for \( N^{1,p} \) than for \( P^{\alpha,p} \) and \( M^{\alpha,p} \), since the equivalence classes in \( N^{1,p} \) are up to sets of zero \( p \)-capacity. More precisely, \( \bar{u} \) might differ from \( u \) more than on a set of zero \( p \)-capacity, in which case it is not a representative of \( u \) in \( N^{1,p}(X, \mu) \), only a \( \mu \)-a.e.-representative. It is this function \( \bar{u} \) that realizes the embeddings into \( L^q(E, \nu) \) in Theorem 1.3.

A sufficient condition for \( \bar{u} \in N^{1,p}(X, \mu) \) is that the doubling property for \( \mu \) and the \( p \)-Poincaré inequality (1.10) with \( g_u \) hold for all balls contained in some open neighbourhood of \( E \). Indeed, Proposition 4.8 in [8] (for \( p > 1 \)) and Proposition 3.7 in [9] (for \( p = 1 \)) guarantee that \( \bar{u} = u \) on \( E \) outside a set of zero \( p \)-capacity.

**Proof of Theorem 1.3.** For the sufficiency part, note that the minimal \( p \)-weak upper gradients \( g_u \) satisfy the truncation property. More precisely, for all \( l < k \), the function \( g_u \chi_{\{l \leq u < k\}} \) is a \( p \)-weak upper gradient of the truncation \( \max\{l, \min\{u, k\}\} \) and hence the \( p \)-Poincaré inequality (1.10) (and thus (1.2) with \( \alpha = 1 \)) holds for such truncations as well. Theorem 5.1 (ii) thus applies with \( G(u) = g_u \). Since \( \alpha = 1 \), the conditions \( \Theta_{q,\lambda}(r) < \infty \) and \( \Theta_{q,\lambda}(r) \to 0 \) become (1.11) and (1.12).

The necessity part follows from Propositions 7.1 and 7.3, since their conclusions imply (1.11) and (1.12), by the measure density condition (1.13).

**7. Necessary conditions for compact embeddings**

Condition (1.12) can be compared to Theorem 8.8.3 in Maz’ya [50], where the embedding \( W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n, \nu) \), was shown to be compact if and only if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{r \nu(B(x, r))^{1/q}}{|B(x, r)|^{1/p}} = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \sup_{0 < r \leq 1} \frac{r \nu(B(x, r))^{1/q}}{|B(x, r)|^{1/p}} = 0,
\]

(7.1)

where \( q > p \) and \( |B(x, r)| \) is the Lebesgue measure of \( B(x, r) \). Of course, for embeddings into \( L^q(E, \nu) \) with bounded \( E \), the latter condition in (7.1) is superfluous.
We shall now show that similar conditions are necessary for compactness and boundedness also in metric spaces, under very mild assumptions on the measures. Throughout this section, \( \mathcal{Y} \) stands for

\[
N^{1,p}(X,\mu), \quad D^p(X,\mu), \quad M^{1,p}(X,\mu), \quad P^{1,p}_1(X,\mu) \quad \text{or} \quad P^{1,p}_{1,E}(X,\mu), \quad (7.2)
\]

with the usual (semi)norms.

**Proposition 7.1 (Necessity of Maz’ya’s condition).** Assume that \( 0 < \mu(B) < \infty \) for all balls centred in \( E \) and that \( \nu \) is nonatomic on \( E \), i.e. \( \nu(\{x\}) = 0 \) for all \( x \in E \). If the embedding \( \mathcal{Y} \hookrightarrow L^q(E,\nu) \) is compact for \( \mathcal{Y} \) as in \( (7.2) \), then for all \( \lambda > 1 \),

\[
\sup_{x \in E} \frac{r \nu(B(x,r) \cap E)^{1/q}}{\mu(B(x,\lambda r))^{1/p}} \to 0, \quad \text{as } r \to 0. \quad (7.3)
\]

**Proof.** Let \( \{r_j\}_{j=1}^\infty \) be an arbitrary sequence converging to zero as \( j \to \infty \). For each \( j = 1, 2, \ldots, \) find a ball \( B_j = B(x_j, r_j) \) such that \( x_j \in E \) and

\[
\frac{r_j \nu(B_j \cap E)^{1/q}}{\mu(\lambda B_j)^{1/p}} \geq \min \left\{ 1, \frac{S(r_j)}{2} \right\},
\]

where \( S(r) \) denotes the supremum in \( (7.3) \). Consider the functions

\[
u_j(x) = a_j \left( 1 - \frac{\text{dist}(x, B_j)}{(\lambda - 1)r_j} \right)_+ \quad \text{with constants } a_j = \frac{(\lambda - 1)r_j}{\mu(\lambda B_j)^{1/p}}. \quad (7.4)
\]

It is easily verified that the distance function \( x \mapsto \text{dist}(x, B_j) \) is 1-Lipschitz, and hence \( g_j := \chi_{\lambda B_j}/\mu(\lambda B_j)^{1/p} \) is a \( p \)-weak upper gradient of \( u_j \) and can be used in the definition of \( M^{1,p}(X,\mu) \) and the other spaces as well, cf. (3.7) and (3.11). As both \( u_j \) and \( g_j \) vanish outside \( \lambda B_j \), we have

\[
\int_X u_j^p \, d\mu \leq a_j^p \mu(\lambda B_j) \leq ((\lambda - 1)r_j)^p \quad \text{and} \quad \int_X g_j^p \, d\mu \leq 1,
\]

i.e. \( \{u_j\}_{j=1}^\infty \) is bounded in \( \mathcal{Y} \). By compactness of the embedding \( \mathcal{Y} \hookrightarrow L^q(E,\nu) \), there exists a subsequence (also denoted \( \{u_j\}_{j=1}^\infty \) which converges in \( L^q(E,\nu) \) to some \( u \in L^q(E,\nu) \). Since \( 0 < \mu(B) < \infty \) for all balls centred in \( E \), [6, Proposition 1.6] implies that \( E \) is separable. We therefore conclude from Lemma 7.2 below that \( u = 0 \ \nu\text{-a.e. in } E \). Hence, as \( u_j = a_j \) in \( B_j \), it follows from the choice of \( B_j \) that

\[
\min \left\{ 1, \frac{S(r_j)}{2} \right\} \leq \frac{r_j \nu(B_j \cap E)^{1/q}}{\mu(\lambda B_j)^{1/p}} \leq \frac{1}{\lambda - 1} \|u_j\|_{L^q(E,\nu)} \to 0 \quad \text{as } j \to \infty,
\]

which proves \( (7.3) \). \( \Box \)
Lemma 7.2. Let $Y$ be a separable metric space equipped with a nonatomic Borel measure $\nu$. Assume that $u_j \in L^q(Y, \nu)$ are such that $\text{supp } u_j \subset B(y_j, r_j) =: B_j$ for some $y_j \in Y$ and $r_j \to 0$. If $u_j \to u$ in $L^q(Y, \nu)$ then $u = 0$ $\nu$-a.e. in $Y$.

Proof. By passing to a subsequence, we can assume that $u_j \to u$ $\nu$-a.e. in $Y$. Let $B = B(x, r)$ be an arbitrary ball. Then there are two possibilities:

(a) There exists a subsequence $(B_{j_k})_{k=1}^\infty$ of $(B_j)_{j=1}^\infty$ such that $B_{j_k} \cap B = \emptyset$ for all $k = 1, 2, \ldots$. Then $u = 0$ $\nu$-a.e. in $B$ because of the $\nu$-a.e. convergence of $u_j$.

(b) There exists $k$ such that $B_j \cap B \neq \emptyset$ for all $j \geq k$, and hence $\text{supp } u_j$ lies within an $r_k$-neighbourhood of $B$. Letting $k \to \infty$ and hence $r_k \to 0$, the $\nu$-a.e. convergence of $u_j$ then implies that $u = 0$ $\nu$-a.e. in $Y \setminus \overline{B}$.

Now, we distinguish two cases:

If there is $x \in Y$ such that (b) holds for all $r > 0$ then $u = 0$ $\nu$-a.e. in $Y \setminus \{x\}$ and hence also $\nu$-a.e. in $Y$ by the assumption that $\nu$ is nonatomic.

Assume therefore that for every $x \in Y$, there exists $r_x > 0$ such that (b) fails for $r_x$, in which case (a) must hold for $B_x = B(x, r_x)$. Since $Y$ is separable, and thus Lindelöf (by e.g. [6, Proposition 1.5]), we can choose among the balls $B_x, x \in Y$, countably many balls $B_{x_j}, j = 1, 2, \ldots$, so that $Y \subset \bigcup_{j=1}^\infty B_{x_j}$. As $u = 0$ $\nu$-a.e. in each $B_{x_j}$, the claim follows by the subadditivity of $\nu$. \qed

Proposition 7.3. If the embedding $\mathcal{Y} \hookrightarrow L^q(E, \nu)$ is bounded for $\mathcal{Y}$ as in (7.2), then for all $\lambda > 1$ and $r_0 > 0$,

$$\sup_0 < r \leq r_0 \quad \sup_{x \in X} \frac{r\nu(B(x, r) \cap E)^{1/q}}{\mu(B(x, \lambda r))^{1/p}} < \infty,$$

where the suprema are taken only over balls such that $0 < \mu(B(x, \lambda r)) < \infty$.

Proof. For $z \in X$ and $0 < r \leq r_0$, let $B = B(z, r)$ with $0 < \mu(\lambda B) < \infty$ and

$$u(x) = a \left(1 - \frac{\text{dist}(x, B)}{(\lambda - 1)r}\right)_+ \quad \text{with } a = \frac{(\lambda - 1)r}{\mu(\lambda B)^{1/p}}.$$

As in the proof of Proposition 7.1, $g = \chi_{\lambda B}/\mu(\lambda B)$ serves as a gradient in the above choices of $\mathcal{Y}$. The boundedness of $\mathcal{Y} \hookrightarrow L^q(E, \nu)$ yields

$$\frac{r\nu(B(z, r) \cap E)^{1/q}}{\mu(B(z, \lambda r))^{1/p}} \leq \frac{1}{\lambda - 1} \|u\|_{L^q(E, \mu)} \leq \frac{C}{\lambda - 1} \|g\|_{L^p(X, \mu)} \leq \frac{C}{\lambda - 1} < \infty.$$

Taking supremum over all $z \in X$ and $0 < r \leq r_0$ concludes the proof. \qed
Proposition 7.4. Assume that \( \mu \) satisfies both the doubling condition \( (1.5) \) in Assumptions (D) and a measure density condition as in \( (1.13) \), for all balls centred in \( E \) and of radius at most \( r_0 > 0 \). (For example, let \( \mu \) be doubling on \( E = X \).) If the embedding \( \mathcal{Y} \hookrightarrow L^p(E, \mu) \) is compact for \( \mathcal{Y} \) as in \( (7.2) \), then \( E \) is totally bounded.

Proof. Assume that \( E \) is not totally bounded. Then there exists \( r \leq r_0 \) and infinitely many balls \( B_j = B(x_j, r) \subset X, j = 1, 2, ... \), such that \( x_j \in E \) and \( 2B_j \cap 2B_k = \emptyset \) whenever \( j \neq k \). As in the proof of Proposition 7.1, the sequence \( \{u_j\}_{j=1}^\infty \), defined by \( (7.4) \) with \( \lambda = 2 \) and \( r_j = r \), is bounded in \( \mathcal{Y} \).

By assumption, \( \{u_j\}_{j=1}^\infty \) contains a subsequence (also denoted \( \{u_j\}_{j=1}^\infty \)), which converges both in \( L^p(E, \mu) \) and \( \mu \)-a.e. in \( E \) to some \( u \in L^p(E, \mu) \). Since for every \( x \in X \) at most one of \( u_j(x) \) is nonzero by the choice of \( B_j \), we conclude that \( u = 0 \) \( \mu \)-a.e. in \( E \) and hence \( \|u_j\|_{L^p(E, \mu)} \to 0 \). This contradicts the doubling property \( (1.5) \) and the measure density \( (1.13) \) of \( \mu \), since

\[
\|u_j\|_{L^p(E, \mu)} \geq a_j \mu(B_j \cap E)^{1/p} = \frac{r\mu(B_j \cap E)^{1/p}}{\mu(2B_j)^{1/p}} \geq Cr > 0.
\]

\[ \square \]

8. Examples

We now give concrete examples when our results can be applied. Our aim is mainly to demonstrate their flexibility.

Example 8.1 (Lower-dimensional target measures). A suitable candidate for \( \nu \) is the \( d \)-dimensional Hausdorff measure \( \Lambda_d \) on a lower-dimensional Ahlfors regular \( d \)-set, as in Jonsson–Wallin [35, Sections 2.1.1 and 2.1.2]. More precisely, assume that the dimension condition \( (1.14) \) holds for a totally bounded set \( E \subset F \subset X \). Assumptions (D) then hold for \( \nu = \Lambda_d|_F \) with \( \delta = \sigma = d \) and Corollary 5.2 yields compactness of the embedding \( \mathcal{Y} \hookrightarrow L^q(E, \Lambda_d) \) when \( \mathcal{Y} \subset P^p_{\tau, E}(X, \mu), \tau \geq 1, q(s - \alpha p) < dp \), and \( \mu \) satisfies the local doubling and dimension conditions \( (1.5) \) and \( (1.7) \).

A traditional approach to such embeddings is to first use trace theorems into Besov spaces on \( F \) and then embeddings of these Besov spaces into \( L^q \) spaces, followed by the compactness results from Hajłasz–Koskela [23, Theorem 5] or Hajłasz–Liu [26]. See [35, Theorem VII.1 and Proposition VIII.6] together with Peetre [53, Théorème 8.1] for such results in \( \mathbb{R}^n \), and L. Malý [46, Corollary 3.18 and Proposition 4.16] on metric spaces.

Our approach is direct and avoids the use of Besov spaces. We impose assumptions only on small balls centred in \( E \), which can give better exponents, as seen in Examples 5.3 and 5.4. Our condition \( (1.7) \) for the exponent \( s \) is in general less restrictive than the one in [46]. We use the Hausdorff measure \( \Lambda_d \) satisfying \( (1.14) \), rather than the codimensional bounds in [46, Proposition 4.16]. However, a codimensional measure, introduced by J. Malý [45], could also be used instead of \( \Lambda_d \).
Another application of our results is to replace $X$ and $E$ by a sufficiently regular domain, seen as a metric space in its own right, and equipped with weights given by the distance to the boundary.

An open set $\Omega \subset X$ is a uniform domain, if there is a constant $A \geq 1$ such that for every pair $x, y \in \Omega$ there is a curve $\gamma$ in $\Omega$ connecting $x$ and $y$, so that its length is at most $Ad(x, y)$ and for all $z \in \gamma$,

$$\text{dist}(z, X \setminus \Omega) \geq A^{-1} \min\{\ell_{xz}, \ell_{yz}\},$$

where $\ell_{xz}$ and $\ell_{yz}$ are the lengths of the subcurves of $\gamma$ connecting $z$ to $x$ and $y$, respectively. Uniform domains were introduced by Martio–Sarvas [48]. Typical examples include convex sets and bounded Lipschitz domains in $\mathbb{R}^n$, see Aikawa [1, p. 120] and Maz’ya [51, Sections 1.1.8–1.1.11]. There are also many examples of fractal nature, such as the interior of the von Koch snowflake. By considering uniform domains as metric spaces in their own right, we obtain the following embedding result. The assumptions on $\mu$ and $\nu$ guarantee that results from [12] can be used to obtain suitable estimates and Poincaré inequalities for the measures in (8.1).

**Theorem 8.2** (Uniform domains with distance weights). Assume the following for a bounded uniform domain $\Omega \subset X$:

(a) The doubling and dimension conditions (1.5)–(1.8) in Assumptions (D) hold for $\mu$ and $\nu$ with exponents $s$ and $\sigma > s - p$, for all balls centred in $\Omega$ and with radii $0 < r' < r < \infty$.

(b) The domain measure $\mu$ supports the $p$-Poincaré inequality (1.10) with $g_u$ and $\lambda \geq 1$ for all balls $B \subset \Omega$.

Consider the measures

$$d\mu_\alpha := d(\cdot)^\alpha d\mu|_\Omega \quad \text{and} \quad d\nu_\beta := d(\cdot)^\beta d\nu|_\Omega \quad \text{on} \ \Omega, \quad (8.1)$$

where $d(x) := \text{dist}(x, X \setminus \Omega)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist constants $\alpha_0, \beta_0 < 0$ such that if $\alpha > \alpha_0$ and $\beta > \beta_0$, then the embeddings

$$D^p(\Omega, \mu_\alpha) \cap L^1(\Omega, \nu_\alpha) \hookrightarrow L^q(\Omega, \nu_\alpha) \quad \text{and} \quad N^{1,p}(\Omega, \mu_\alpha) \hookrightarrow L^q(\Omega, \nu_\alpha)$$

are compact whenever one of the following holds:

(i) $q > p$ and

$$q(s - p) < \sigma p + \min\{\beta p - \alpha q, 0\}, \quad (8.2)$$

(ii) $q \leq p$ and $s - p < \sigma + \beta - \alpha$. 

If \( q > p \) and the inequality in (8.2) is nonstrict, then the above embeddings are bounded. Moreover, if \( \nu = \mu \), then assumption (1.8) in (a) is not needed and the embedding \( N^{1,p}(\Omega, \mu_\alpha) \hookrightarrow L^q(\Omega, \mu_\beta) \) is compact (resp. bounded) whenever the above assumptions (resp. a nonstrict analogue of (8.2)) hold with \( \sigma \) replaced by \( s \).

**Remark 8.3.** Compactness of the embedding \( M^{\theta,p}(\Omega, \mu_\alpha) \hookrightarrow L^q(\Omega, \mu_\beta) \) for \( \theta > 0 \) can be shown similarly when \( \Omega \), rather than being uniform, satisfies the corkscrew condition (i.e. there is \( c > 0 \) such that for all \( x \in \Omega \) and \( 0 < r \leq \operatorname{diam} \Omega \), the set \( B(x, r) \cap \Omega \) contains a ball of radius \( cr \)) and \( s - p \) in assumptions (a), (8.2) and (ii) of Theorem 8.2 is replaced by \( s - \theta p \). Indeed, the \( p \)-Poincaré inequality for \( g_\alpha \) in (b) can be replaced by (3.10).

**Remark 8.4.** The proof presented below shows that \( \mu_\alpha \) and \( \nu_\beta \) satisfy (1.7) and (1.8) with \( s \) and \( \sigma \) replaced by

\[
\sigma_\alpha = \max\{s, s + \alpha\} \quad \text{and} \quad \sigma_\beta = \min\{\sigma, \max\{\sigma + \beta, 0\}\},
\]

respectively. However, a direct use of Corollary 5.2 with these exponents would instead of \( \sigma > s - p \) and the assumptions (i) and (ii) in Theorem 8.2 require that

\[
\sigma_\beta > s_\alpha - p \quad \text{and} \quad q(s_\alpha - p) < \sigma_\beta p.
\]

This would give a less general result than our direct proof (e.g. when \( \alpha, \beta > 0 \)).

**Proof of Theorem 8.2.** We shall use Theorem 5.1 with \( X \) and \( E \) replaced by \( \Omega \). The doubling property of \( \mu \) (and thus of \( X \)) implies that the bounded set \( \Omega \) is totally bounded, see Heinonen [29, Section 10.13]. Theorem 4.4 in [12] and its proof ensure that for some \( \alpha_0 < 0 \) and all \( \alpha > \alpha_0 \), the measure \( \mu_\alpha \) is doubling and supports the \( p \)-Poincaré inequality (1.10) for \( g_\alpha \) on \( \Omega \) as the underlying space, with dilation \( \lambda = 3A \), where \( A \) is the uniformity constant for \( \Omega \). Hence, \( D^p(\Omega, \mu_\alpha) \subset P^{1,p}_{3A,\Omega}(\Omega, \mu_\alpha) \).

Since \( \Omega \) satisfies the corkscrew condition, by [12, Lemma 4.2 and Definition 2.4], we conclude from [12, Theorem 2.8] that \( \nu_\beta \) is doubling on \( \Omega \) for all \( \beta > \beta_0 \) and some \( \beta_0 < 0 \). In particular, (1.5) is satisfied for \( \mu_\alpha \) and \( \nu_\beta \). As \( \Omega \) is connected, also (1.6) holds for \( \nu_\beta \), see Remark 4.3. To be able to control the local Poincaré constant

\[
\Theta_{\alpha,\lambda}(r) = \sup_{0 < \rho \leq r} \sup_{x \in \Omega} \frac{\rho \nu_\beta(B(x, \rho))^{1/q}}{\mu_\alpha(B(x, \lambda \rho))^{1/p}},
\]

we need to estimate \( \mu_\alpha(B(x, \rho)) \) and \( \nu_\beta(B(x, \rho)) \) for \( x \in \Omega \) and \( 0 < \rho \leq r_0 \).

First, [12, Theorem 2.8 (with \( \nu \) replaced by \( \mu_\alpha \)) and Lemma 4.2] show that

\[
\mu_\alpha(B(x, \rho)) \geq C \rho^\alpha \mu(B(x, \rho) \cap \Omega) \geq C \rho^\alpha \mu(B(x, \rho)) \quad \text{for} \quad \rho \geq \frac{1}{2}d(x).
\]

Similarly, when \( \beta \leq 0 \), [12, Theorem 2.8] with \( \mu \) replaced by \( \nu_\beta \) implies that
\[ \nu_\beta(B(x, \rho)) \leq C \rho^\beta \nu(B(x, \rho)) \quad \text{for } \rho \geq \frac{1}{2} d(x), \]

while the inequality is immediate when \( \beta > 0 \). If \( 0 < \rho \leq \frac{1}{2} d(x) \) then \( d(y) \) is comparable to \( d(x) \) on \( B(x, \rho) \), which immediately yields that \( \mu_\alpha(B(x, \rho)) \) and \( \nu_\beta(B(x, \rho)) \) are comparable to \( d(x)\alpha \mu(B(x, \rho)) \) and \( d(x)\beta \nu(B(x, \rho)) \), respectively.

Using this, we now have that

\[
\frac{\nu_\beta(B(x, \rho))^{1/q}}{\mu_\alpha(B(x, \rho))^{1/p}} \leq \begin{cases} 
\frac{C \nu(B(x, \rho))^{1/q}}{\mu(B(x, \rho))^{1/p}} d(x)^{\beta/q-\alpha/p}, & \text{if } 0 < \rho \leq \frac{1}{2} d(x), \\
\frac{C \nu(B(x, \rho))^{1/q}}{\mu(B(x, \rho))^{1/p}} \rho^{\beta/q-\alpha/p}, & \text{if } \rho \geq \frac{1}{2} d(x).
\end{cases}
\]

Since

\[ d(x)^{\beta/q-\alpha/p} \leq C \rho^{\min\{\beta/q-\alpha/p, 0\}}, \quad \text{when } 0 < \rho \leq \frac{1}{2} d(x) \leq \frac{1}{2} \text{ diam } \Omega < \infty, \]

we therefore conclude, using (1.7) and (1.8), that in both cases,

\[ \Theta_{q, \lambda}(r) \leq \sup_{0 < \rho \leq r} \sup_{x \in \Omega} \rho^{\nu_\beta(B(x, \rho))^{1/q}} \leq C_0 \sup_{0 < \rho \leq r} \rho^{1+\sigma/q-s/p+\min\{\beta/q-\alpha/p, 0\}}, \]

where \( C_0 \) depends on \( r_0 \) and other fixed parameters, but not on \( r \). It follows that \( \Theta_{q, \lambda}(r) \to 0 \) whenever the exponent

\[ 1 + \frac{\sigma}{q} - \frac{s}{p} + \min\left\{ \frac{\beta}{q} - \frac{\alpha}{p}, 0 \right\} > 0, \]

and that \( \Theta_{q, \lambda}(r) < \infty \) if the inequality in (8.4) is nonstrict. It is easily verified that (8.4) follows from (8.2), and hence an application of Theorem 5.1 concludes the proof of the general case \( \nu \neq \mu \) when \( q > p \).

When \( q \leq p \), the assumptions \( \sigma > s - p \) and \( s - p < \sigma + \beta - \alpha \) imply that there is always \( \bar{q} > p \) satisfying (8.2) (instead of \( q \)). Thus, an application of the already proved part, with \( q > p \) replaced by \( \bar{q} > p \), provides compact embeddings into \( L^\bar{q} \) and thus also into \( L^q \).

If \( \nu = \mu \), then replacing \( \nu \) by \( \mu \) in (8.3) gives that

\[ \Theta_{q, \lambda}(r) \leq C \sup_{0 < \rho \leq r} \sup_{x \in \Omega} \mu(B(x, \rho))^{1/q-1/p} \rho^{1+\min\{\beta/q-\alpha/p, 0\}}. \]

The requirement (8.4) is therefore replaced by

\[ 1 + \frac{s}{q} - \frac{s}{p} + \min\left\{ \frac{\beta}{q} - \frac{\alpha}{p}, 0 \right\} > 0, \]

which holds when \( \sigma \) in (8.2) is replaced by \( s \). \( \square \)
Proof of Proposition 1.4. Note that $M^{\alpha,p}(\Omega, \mu) = M^{\alpha,p}(\overline{\Omega}, \mu|_{\Omega})$, with $\mu|_{\Omega}$ extended by zero to $\partial \Omega$. Remark 4.4 and Lemma 4.5 then imply that the limit in (1.15) exists $\Lambda_d$-a.e. in $E$. Corollary 5.2 (i), applied to $\mathcal{Y} = M^{\alpha,p}(\overline{\Omega}, \mu|_{\Omega})$ and $\nu = \Lambda_d|_F$, proves the statement for $M^{\alpha,p}$.

For $N^{1,p}$, it follows from Aikawa–Shanmugalingam [2, Proposition 7.1] that the measure $\mu|_{\Omega}$, extended by zero to $\partial \Omega$, is doubling and supports the $p$-Poincaré inequality (1.10) for $g_u$ on $\overline{\Omega}$ as the underlying metric space. Moreover, $N^{1,p}(\Omega, \mu) = N^{1,p}(\overline{\Omega}, \mu|_{\Omega})$ with extensions given by (1.15) and with the same norm, see Heinonen–Koskela–Shanmugalingam–Tyson [32, Lemma 8.2.3] together with Theorem 4.1 and Remark 4.2 in Björn–Björn [8] (when $X$ is not complete). Applying (i) and (ii) in Corollary 5.2 to $\mathcal{Y} = N^{1,p}(\overline{\Omega}, \mu|_{\Omega})$ concludes the proof. \qed

Remark 8.5. Theorems 2.8 and 4.4 in Björn–Shanmugalingam [12] imply that the assumptions on $\mu|\Omega$ in Proposition 1.4 are satisfied e.g. if

- $\Omega$ satisfies the corkscrew condition and $\mu$ is doubling on $X$ (for $M^{\alpha,p}$).
- $\Omega$ is a uniform domain and $\mu$ is doubling and supports the $p$-Poincaré inequality (1.10) for $g_u$ on $X$ (for embeddings from $N^{1,p}$).

Example 8.6 (Trace embeddings for Lipschitz domains). Since bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$ are uniform, Proposition 1.4 and Remark 8.5 imply that the trace embedding

$$W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega, \Lambda_{n-1})$$

is bounded whenever $q(n-p) \leq p(n-1)$ and compact if $q(n-p) < p(n-1)$. This recovers a classical result, see Section 6.10.5 in Kufner–John–Fučík [42].

The following example seems new in the weighted setting. A similar case with $\alpha < 0$ was recently considered in Lindquist–Shanmugalingam [44].

Example 8.7 (von Koch snowflake with a weight). Let $\Omega$ be the bounded domain in $\mathbb{R}^2$ whose boundary is the von Koch snowflake curve of Hausdorff dimension $d = \log 4/\log 3$. Then $\Omega$ is a uniform domain. Consider the weighted measure on $\Omega$,

$$d\mu_\alpha(x) = \text{dist}(x, \partial \Omega)^\alpha \, dx, \quad 0 \leq \alpha < d + p - 2.$$ 

Theorems 2.8 and 4.4 in [12] and Proposition 1.4 then show that the trace embedding

$$W^{1,p}(\Omega, \mu_\alpha) \hookrightarrow L^q(\partial \Omega, \Lambda_d)$$

is bounded whenever $q(2+\alpha-p) \leq dp$, and compact if $q(2+\alpha-p) < dp$. Other domains with fractal boundaries can be treated similarly.
Declaration of competing interest

None.

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