Research Article

Differences of Positive Linear Operators on Simplices

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The aim of the paper is twofold: we introduce new positive linear operators acting on continuous functions defined on a simplex and then estimate differences involving them and/or other known operators. The estimates are given in terms of moduli of smoothness and K-functionals. Several applications and examples illustrate the general results.

1. Introduction

Differences of positive linear operators were intensively investigated in the last years; see [1–14] and the references therein. The operators involved in these studies act usually on continuous functions defined on real intervals, and the differences are estimated in terms of moduli of smoothness and K-functionals. In some papers, operators having equal central moments up to a certain order are considered. Other articles deal with operators constructed with the same fundamental functions and different functionals in front of them.

The study of differences of positive linear operators is important from a theoretical point of view, but also from a practical one. Let \((U_n)\) and \((V_n)\) be certain positive linear operators. If we know that \(|U_n(f) - V_n(f)|\) is small, we can choose \((U_n)\) or \((V_n)\) taking into account other qualities of them like shape-preserving properties and smoothness/Lipschitz preserving properties.

This paper is concerned with differences of positive linear operators acting on continuous functions defined on simplices. For the sake of simplicity, we consider only the case of the canonical simplex in \(\mathbb{R}^2\), where the notation is simpler, but the results can be easily translated to an arbitrary simplex in \(\mathbb{R}^n\).

We consider the bivariate versions of some classical operators like Bernstein, Durrmeyer, Kantorovich, and genuine Bernstein-Durrmeyer operators. These bivariate versions were already studied in literature from other points of view. We introduce the bivariate versions of other operators: \(U_n^\rho\) (see [15, 16]) and the operators defined in [17]. All these operators are constructed with the fundamental Bernstein polynomials on the two-dimensional simplex. A different kind of operator is the bivariate version of the univariate Beta operator of Mühlbach and Lupas (see [18–20]); we introduce it and use it in composition with the Bernstein operator to get a useful representation of \(U_n^\rho\).

We get estimates of differences of the abovementioned operators, in terms of suitable moduli of smoothness and K-functionals.

To resume, the aim of our paper is twofold: we introduce new operators on a simplex and then estimate differences involving them and other known operators.

The list of applications and examples can be enlarged. In particular, we will be interested for a future work in studying differences of bivariate versions of operators, which preserve exponential functions (see [21–23]). We also intend to deepen the study of the newly introduced Beta operators on the simplex and to consider the composition of it with other operators, leading to new applications and—why not—new theoretical aspects/problems. Given a Markov operator (i.e., a positive linear operator which preserves the constant functions), the study of its iterate is important not only in Approximation Theory but also in Ergodic Theory and other areas of research. We intend to investigate from this point of
view the newly introduced operators, which are in fact Markov operators.

We end this Introduction by presenting some notation and a fundamental inequality expressed in Lemma 1. Section 2 contains the main theoretical results, while Section 3 is devoted to applications and examples.

Let \( S = \{(x, y) \in \mathbb{R}^2 | x \geq 0, x + y \leq 1\} \) be the canonical simplex in \( \mathbb{R}^2 \) and \( E(S) \) denote a space of real-valued continuous functions of two variables defined on \( S \), containing the polynomials. Throughout the paper, we will denote by \( \mathbf{1} \) the constant function, namely,

\[
\mathbf{1} : S \longrightarrow \mathbb{R}, \mathbf{1}(x, y) = 1, (x, y) \in S,
\]

and \( \text{pr}_i : S \longrightarrow \mathbb{R}, i = 1, 2, \) will denote the \( i \)th coordinate functions restricted on \( S \), which are given by

\[
\text{pr}_1(x, y) = x \text{ and } \text{pr}_2(x, y) = y, (x, y) \in S.
\]

Let \( F : E(S) \longrightarrow \mathbb{R} \) be a positive linear functional such that \( F(\mathbf{1}) = 1 \). Set

\[
b_i^F = F(\text{pr}_i), \quad b_i^F = F(\text{pr}_i),
\]

\[
\mu_{ij}^F = F\left((\text{pr}_1 - b_i^F \mathbf{1})(\text{pr}_2 - b_j^F \mathbf{1})\right), \quad i, j \in \mathbb{N}.
\]

Then, one has

\[
\mu_{0,0}^F = 0, \mu_{2,0}^F = F(\text{pr}_2) - (b_2^F)^2 \geq 0,
\]

\[
\mu_{0,2}^F = 0, \mu_{0,2}^F = F(\text{pr}_2) - (b_2^F)^2 \geq 0.
\]

Let \( C^2(S) \) be the space of all real-valued (continuous) functions, differentiable on \( int(S) \) and whose partial derivatives of order \( \leq 2 \) can be continuously extended to \( S \), having

\[
\|f\| = \sup \{|f(x, y)| : (x, y) \in S\} < \infty.
\]

Lemma 1. If \( f \in C^2(S) \), then

\[
|f - F(f)(b_1^F, b_2^F)| \leq M_f(\mu_{0,0}^F + \mu_{0,2}^F),
\]

where \( M_f = \max\{\|f_{xx}\|, \|f_{xy}\|, \|f_{yy}\|\} \).

\[
\text{Proof.} \text{ Consider the line segment connecting } (b_1^F, b_2^F) \text{ with } (t_1, t_2) \in S. \text{ From Taylor’s formula (see [24], p.245), there is a point } (c_1, c_2) \text{ on this line segment, different from } (b_1^F, b_2^F)
\]

and \((t_1, t_2)\), such that

\[
f(t_1, t_2) = f(b_1^F, b_2^F) + f_x(b_1^F, b_2^F)(t_1 - b_1^F) + f_y(b_1^F, b_2^F)(t_2 - b_2^F) + \frac{1}{2}f_{xx}(c_1, c_2)(t_1 - b_1^F)^2 + 2f_{xy}(c_1, c_2)(t_1 - b_1^F)(t_2 - b_2^F) + f_{yy}(c_1, c_2)(t_2 - b_2^F)^2.
\]

Therefore, we can write

\[
|f - F(f)(b_1^F, b_2^F)| \leq M_f(\mu_{0,0}^F + \mu_{0,2}^F).
\]

2. Difference of Bivariate Positive Linear Operators

Denote by \( C(S) \) the space of real-valued continuous functions on \( S \) with the norm \( \|f\| = \max_{(x,y) \in S} |f(x,y)| \), \( f \in C(S) \). Let \( K \) be a set of nonnegative integers and for \( k, l \in K \) let \( p_{kl} \in C(S), p_{kl} \geq 0 \), satisfy \( \sum_{k,l \in K} p_{kl} = 1 \). Let \( F_{kl} : E(S) \longrightarrow \mathbb{R} \) and \( G_{kl} : E(S) \longrightarrow \mathbb{R}, k, l \in K \), be positive linear functionals such that \( F_{kl}(\mathbf{1}) = 1 \) and \( G_{kl}(\mathbf{1}) = 1 \). Moreover, let \( D(S) \) be the set of all \( f \in E(S) \) for which

\[
\sum_{k,l \in K} F_{kl}(f)p_{kl} \in C(S),
\]

\[
\sum_{k,l \in K} G_{kl}(f)p_{kl} \in C(S).
\]

Now, consider the bivariate positive linear operators \( V \)
and $W$ acting from $D(S)$ into $C(S)$ defined, for $f \in D(S)$, by
\begin{equation}
V(f)(x, y) = \sum_{k,l \in K} F_{k,l}(f)p_{k,l}(x, y),
\end{equation}
\begin{equation}
W(f)(x, y) = \sum_{k,l \in K} G_{k,l}(f)p_{k,l}(x, y),
\end{equation}
respectively. For future correspondences, we denote
\begin{equation}
\sigma(x, y) = \sum_{k,l \in K} \left( \mu_{2,0}^l + \mu_{2,0}^l + \mu_{0,2}^l + \mu_{0,2}^l \right) p_{k,l}(x, y),
\end{equation}
\begin{equation}
\delta = \sup_{k,l \in K} \left| \left( b_1^{G_l} - b_2^{G_l} \right) - \left( b_1^{G_l} - b_2^{G_l} \right) \right|,
\end{equation}
where $|\cdot|$ is the $l_1$-norm in $\mathbb{R}^2$.

In the following, we adopt the definitions of $K$-functional and modulus of smoothness from [25, 26]. Let
\begin{equation}
S(h) = \{ x \in [x + th \in S \text{ for } 0 \leq t \leq 1] \}, \ h \in \mathbb{R}^2.
\end{equation}

For $r \in \mathbb{N}$, $r$th order differences on the subset $S(rh)$ are defined as
\begin{equation}
\Delta_r f(x) = \sum_{k=0}^{r} (-1)^{k} \binom{r}{k} f(x + kh).
\end{equation}

The $r$th order modulus of smoothness of $f$ is a function $\omega_r : C(S) \times (0, \infty) \to [0, \infty)$ given by
\begin{equation}
\omega_r(f, \alpha) = \sup_{0 < |h| < \alpha} \| \Delta_r f \|, \ \alpha > 0.
\end{equation}

Let $C^r(S)$ be the space of all real-valued (continuous) functions, differentiable on int ($S$) and whose partial derivatives of order $\leq r$ can be continuously extended to $S$, with the seminorm
\begin{equation}
\| g \|_{C^r(S)} = \sum_{\gamma_1, \gamma_2 = r} \| D^{\gamma_1 + \gamma_2} g \| < \infty, \gamma_1 \geq 0, i = 1, 2, \gamma_1 + \gamma_2 = r.
\end{equation}

For $f \in C(S)$, we shall use the following $K$-functional:
\begin{equation}
K_r(f, t) = \inf \left\{ \| f - g \| + t\| g \|_{C^r(S)} : g \in C(S) \right\}.
\end{equation}

Then, there exist $c_1, c_2 > 0$ such that for any $t > 0$ (see [25, 26])
\begin{equation}
c_1 K_r(f, t) \leq \omega_r(f, t) \leq c_2 K_r(f, t).
\end{equation}

Here, $c_2$ depends only on $r$ (for the general definition on the $L_p$, $1 \leq p \leq \infty$, spaces of functions on bounded domains, see [25] or, on unbounded domains see [27], p.341.

**Theorem 2.** If $f \in D(S) \cap C^2(S)$, then
\begin{equation}
\|(V - W)(f)(x, y)\| \leq M_f \sigma(x, y) + \omega_1(f, \delta),
\end{equation}
where $M_f$ is defined in Lemma 1.

**Proof.** Let $(x, y) \in S$. From Lemma 1, we get
\begin{equation}
\|(V - W)(f)(x, y)\| \leq \sum_{k,l \in K} \| F_{k,l}(f) - G_{k,l}(f) \| p_{k,l}(x, y)
\leq \sum_{k,l \in K} p_{k,l}(x, y) \left( \| F_{k,l}(f) - f(b_1^{G_l}, b_2^{G_l}) \| + \| G_{k,l}(f) - f(b_1^{G_l}, b_2^{G_l}) \| \right)
\leq \sum_{k,l \in K} \left( \| F_{k,l}(f) - f(b_1^{G_l}, b_2^{G_l}) \| + \| G_{k,l}(f) - f(b_1^{G_l}, b_2^{G_l}) \| \right)
\leq M_f \sigma(x, y) + \omega_1(f, \delta).
\end{equation}

**Theorem 3.** If $f \in C(S)$, then
\begin{equation}
\|(V - W)(f)(x, y)\| \leq \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2\left( f, \sqrt{\sigma(x, y)} \right),
\end{equation}
where $\eta_1, \eta_2 > 0$, and $\delta = \sup_{k,l \in K} \left( \| b_1^{G_l} - b_2^{G_l} \| + \| b_2^{G_l} - b_2^{G_l} \| \right)$.

**Proof.** Let $g \in C^2(S)$. From Theorem 2, we get
\begin{equation}
\|(V - W)(f)(x, y)\| \leq \| V(f - g)(x, y) \| + \| W(g - f)(x, y) \|
\leq 2 \| f - g \| + M_f \| g \| + \| W(g)(x, y) \| + \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2\left( f, \sqrt{\sigma(x, y)} \right)
\leq M_g \| f - g \| + \| W(g)(x, y) \| + \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2\left( f, \sqrt{\sigma(x, y)} \right)
\leq M_g \| f - g \| + \| W(g)(x, y) \| + \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2\left( f, \sqrt{\sigma(x, y)} \right)
\leq M_g \| f - g \| + \| W(g)(x, y) \| + \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2\left( f, \sqrt{\sigma(x, y)} \right)
\end{equation}
where $M_g$ is the same notation as in Lemma 1 for $g$. Since partial derivatives of $g$ exist and are continuous everywhere in $S$, it follows that $g$ is differentiable at every point of the line segment connecting the points $(b_1^{G_l}, b_2^{G_l})$ and $(b_1^{G_l}, b_2^{G_l})$ in $S$, $k, l \in K$. By the mean value theorem (see, e.g., [24], p. 239), there is a point $(a_1, a_2)$ on this line segment such that
\begin{equation}
g(b_1^{G_l}, b_2^{G_l}) = g_x(a_1, a_2) \left( b_1^{G_l} - b_1^{G_l} \right)
+ g_y(a_1, a_2) \left( b_2^{G_l} - b_2^{G_l} \right).
\end{equation}

From (16), we get
\begin{equation}
\| g(b_1^{G_l}, b_2^{G_l}) - g(b_1^{G_l}, b_2^{G_l}) \| \leq \| g_x \| \| b_1^{G_l} - b_1^{G_l} \|
+ \| g_y \| \| b_2^{G_l} - b_2^{G_l} \|
\leq \| g \|_{C^r(S)} \sup_{k,l \in K} \left( \| b_1^{G_l} - b_1^{G_l} \| + \| b_2^{G_l} - b_2^{G_l} \| \right).
\end{equation}
Moreover, since \( M_g \leq |g|_{C^1(S)} \), (22) gives that

\[
(V - W)(f)(x, y) \leq 2|f - g| + \delta|g|_{C^1(S)} + \sigma(x, y)|g|_{C^1(S)} \leq K_1(f, \delta) + K_2(f, \sigma(x, y)).
\]

Finally, from (18), we obtain

\[
|(V - W)(f)(x, y)| \leq \eta_1 \omega_1(f, \delta) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x, y)}\right).
\]

(26)

3. Applications

3.1. Difference of Bivariate Bernstein Operators and Their Durrmeyer Variants. For every \( n \geq 1, f \in C(S), \) and \( (x, y) \in S \) , the \( n \)th bivariate Bernstein operator \( B_n : C(S) \longrightarrow C(S) \) is defined by

\[
B_n(f)(x, y) = \sum_{k,l=0}^{n} f\left(\frac{k}{n}, \frac{l}{n}\right) p_{n,k,l}(x, y),
\]

(27)

where

\[
p_{n,k,l}(x, y) = \frac{n!}{k!(n-k-l)!} x^k y^l (1-x)(1-y)^{n-k-l},
\]

(28)

with \( k, l = 0, \ldots, n, k + l \leq n, (x, y) \in S \), (see, e.g., [28], p. 115).

For \( f \in L^1(S) \), the bivariate Durrmeyer operators \( M_n : L^1(S) \longrightarrow C(S) \) are defined by

\[
M_n(f)(x, y) = \sum_{k,l=0}^{n} \left( (n+1)(n+2) \right)^{\frac{1}{2}} p_{n,k,l}(s, t) \cdot f(s, t) dt ds p_{n,k,l}(x, y),
\]

(29)

see, e.g., [29].

Now, denoting

\[
F_{n,k,l}(f) = f\left(\frac{k}{n}, \frac{l}{n}\right), 0 \leq k + l \leq n,
\]

\[
G_{n,k,l}(f) = (n+1)(n+2) \int_0^{1-s} p_{n,k,l}(s, t) f(s, t) dt ds,
\]

(30)

the bivariate Bernstein operators and bivariate Durrmeyer operators can be written as

\[
B_n(f)(x, y) = \sum_{k,l=0}^{n} F_{n,k,l}(f)p_{n,k,l}(x, y),
\]

(31)

\[
M_n(f)(x, y) = \sum_{k,l=0}^{n} G_{n,k,l}(f)p_{n,k,l}(x, y),
\]

respectively.

Proposition 4. For bivariate Bernstein operators and their Durrmeyer variants, the following properties hold:

(i) If \( f \in C^2(S) \), then

\[
|(B_n - M_n)(f)(x, y)| \leq M_j \sigma(x, y) + \omega_1\left(f, \frac{3}{n+3}\right),
\]

(32)

where \( M_j \) is the same as in Lemma 1 and

\[
\sigma(x, y) = \frac{(-x^2 - y^2 + x + y)n^2 + (x^2 + y^2 + 2)n + 4}{(n+3)^2(n+4)} \leq \frac{1}{n+4}.
\]

(33)

(ii) If \( f \in C(S) \), then

\[
|(B_n - M_n)(f)(x, y)| \leq \eta_1 \omega_1\left(f, \frac{3}{n+3}\right) + \eta_2 \omega_2\left(f, \sqrt{\sigma(x, y)}\right).
\]

(34)

Proof. We need to evaluate the terms in (11). So, we get the following results:

\[
b_1^{F_{ab}} = \frac{k}{n}, b_2^{F_{ab}} = \frac{l}{n} - 1,
\]

(35)

\[
b_1^{G_{ab}} = \frac{k+1}{n+3}, b_2^{G_{ab}} = \frac{l+1}{n+3},
\]

\[
0 \leq k + l \leq n. \text{ Therefore, we easily obtain that}
\]

\[
\frac{F_{ab}}{\mu_{2,0}} = 0, \frac{G_{ab}}{\mu_{2,0}} = \frac{(k+1)(n+2-k)}{(n+3)^2(n+4)},
\]

(36)

\[
\frac{F_{ab}}{\mu_{0,2}} = 0, \frac{G_{ab}}{\mu_{0,2}} = \frac{(l+1)(n+2-l)}{(n+3)^2(n+4)}.
\]
Using Maple, one obtains

\[
\sigma(x, y) = \sum_{k,l=0,\ldots,n} \left[ \frac{(k+1)(n+2-k)}{(n+3)^2(n+4)} + \frac{(l+1)(n+2-l)}{(n+3)^2(n+4)} \right] \\
\rho_{n,k,l}(x, y) = \frac{-x^2 - y^2 + x + y}{n+3} + \frac{x^2 + y^2 + 2}{n+4}.
\]

(37)

It is easy to verify that \(\sigma(x, y) \leq 1/(n+4)\). Now, for \(\delta\), we obtain

\[
\delta = \max_{0 \leq k, l \leq n} \left\{ \left| b_1^{F_{k,l}} - b_1^{G_{k,l}} \right| + \left| b_2^{F_{k,l}} - b_2^{G_{k,l}} \right| \right\} \\
= \max_{0 \leq k, l \leq n} \left\{ \frac{n-3k}{n(n+3)} + \frac{n-3l}{n(n+3)} \right\} = \frac{3}{n+3}.
\]

(38)

The rest of the proof follows from Theorems 2 and 3.

3.2. Difference of Bivariate Bernstein Operators and the Bivariate Operators \(A_n\). Let \(\Pi_n\) be the space of polynomials over \([0, 1]\) of degree at most \(n\). In [17], Aldaz et al. introduced a Bernstein operator \(A_n : C[0, 1] \rightarrow \Pi_n\) that fixes 1 and \(x^2\). The operators \(A_n\) are given by

\[
A_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k(k-1)}{n(n-1)}\right)^{1/2} \left(\frac{n}{k}\right) x^{n-k}. 
\]

(39)

Here, for \(f \in C(S)\) and \((x, y) \in S\), we introduce the bivariate form of the operators \(A_n\) as follows

\[
A_n(f)(x, y) = \sum_{k,l=0,\ldots,n} f\left(\frac{k(k-1)}{n(n-1)}\right) \left(\frac{l(l-1)}{n(n-1)}\right) \rho_{n,k,l}(x, y).
\]

(40)

Denoting

\[
F_{n,k,l}(f) = f\left(\frac{k}{n}, \frac{l}{n}\right), G_{n,k,l}(f)
\]

\[
= f\left(\frac{k(k-1)}{n(n-1)}, \frac{l(l-1)}{n(n-1)}\right),
\]

(41)

for \(k, l = 0, \ldots, n, k + l \leq n\), we get

\[
b_1^{F_{k,l}} = \frac{k}{n}, b_1^{G_{k,l}} = \frac{l}{n},
\]

\[
b_1^{F_{k,l}} = \sqrt{\frac{k(k-1)}{n(n-1)}}, b_1^{G_{k,l}} = \sqrt{\frac{l(l-1)}{n(n-1)}},
\]

\[
\mu_{k,0} = \mu_{l,0} = \mu_{0,0} = \mu_{0,2} = 0,
\]

\[
\delta = \max_{0 \leq k, l \leq n} \left\{ \left| b_1^{F_{k,l}} - b_1^{G_{k,l}} \right| + \left| b_2^{F_{k,l}} - b_2^{G_{k,l}} \right| \right\} = \frac{2}{n}.
\]

(42)

**Proposition 5.** For bivariate Bernstein operators and bivariate operators \(A_n\), the following properties hold:

(i) If \(f \in C^2(S)\), then

\[
|B_n - A_n| f \leq \omega_1 \left( f, \frac{2}{n}\right).
\]

(43)

(ii) If \(f \in C(S)\), then

\[
|B_n - A_n| f \leq \eta_1 \omega_1 \left( f, \frac{2}{n}\right).
\]

(44)

3.3. Difference of Bivariate Bernstein Operators and Bivariate Genuine Bernstein-Durrmeyer Operators. In 1987, Chen [30] and Goodman and Sharma [31] constructed the following positive linear operators

\[
U_{n,1}(f)(x) = f(0) p_{n,0}(x) + f(1) p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) (n-1) \\
\cdot \int_0^1 p_{n-2,k-1}(t) f(t) dt,
\]

(45)

where \(n \in \mathbb{N}, f \in C[0, 1]\), and

\[
p_{n,k}(x) = \left(\frac{n}{k}\right) x^k (1-x)^{n-k}, x \in [0, 1], 0 \leq k \leq n.
\]

(46)

For the historical background of these operators, we refer to [32]. In 1991, Goodman and Sharma [33] constructed and studied the multivariate form of the operators \(U_{n,1}\) on a simplex. In [34], Sauer deeply studied the multivariate genuine Bernstein-Durrmeyer operators. Here, for \(f \in L^1(S)\), we
consider the bivariate form given by
\[
U_n(f)(x, y) = f(0, 0)(1 - x - y)^n + f(1, 0)x^n + f(0, 1)y^n \\
+ \sum_{i=1}^{n-1} p_{n,i}(x, y)(n-1) \int_0^1 p_{n-2,i-1}(t)f(0, t)dt \\
+ \sum_{k=1}^{n-1} p_{n,k}(x, y)(n-1) \int_0^1 p_{n-2,k-1}(s)f(s, 0)ds \\
+ \sum_{k=1}^{n} p_{n,k,n-k}(x, y)(n-1)
\]
\[
\cdot \int_0^1 p_{n-2,k-1}(t)f(t, 1-t)dt \\
+ \sum_{k+l \leq n-1 \atop k \geq 1, l \geq 1} \sum_{j=1}^{l} p_{n,k,j}(x, y)(n-1)(n-2) \\
\cdot \int_0^1 p_{n-3,k-1,j-1}(s, t)f(s, t)ds dt
\]
(47)
with the bivariate Bernstein’s fundamental functions given by (28) (see [33], Formula 1.7). These operators satisfy \( U_n(f)(x, y) = f(x, y) \) at the vertices of \( S \).

**Proposition 6.** For bivariate Bernstein operators and bivariate genuine Bernstein-Durrmeyer operators, the following properties hold:

(i) If \( f \in C^2(S) \), then
\[
|\langle B_n - U_n \rangle(f)(x, y) | \leq M_f \sigma(x, y),
\]
where \( M_f \) is the same as in Lemma 1 and
\[
\sigma(x, y) = \frac{(x + y - x^2 - y^2)(n-1)}{n(n+1)} \leq \frac{1}{2(n+1)}.
\]
(ii) If \( f \in C(S) \), then
\[
|\langle B_n - U_n \rangle(f)(x, y) | \leq \eta_2 \omega_2 \left( f, \sqrt{\sigma(x, y)} \right).
\]

**Proof.** If we denote
\[
F_{n,k}(f) := f \left( \frac{k}{n}, \frac{1}{n} \right), \quad 0 \leq k + l \leq n,
\]
\[
G_{n,k,0}(f) = f(0, 0), \quad k = l = 0,
\]
\[
G_{n,n,0}(f) = f(1, 0), \quad k = n, l = 0,
\]
\[
G_{n,0,n}(f) = f(0, 1), \quad k = 0, l = n,
\]
\[
G_{n,k,0}(f) = (n-1) \int_0^1 p_{n-2,k-1}(s)f(s, 0)ds, \quad 1 \leq k \leq n-1, l = 0,
\]
\[
G_{n,0,l}(f) = (n-1) \int_0^1 p_{n-2,l-1}(t)f(0, t)dt, \quad k = 0, 1 \leq l \leq n-1,
\]
\[
G_{n,k,n-k}(f) = (n-1) \int_0^1 p_{n-2,k-1}(t)f(t, 1-t) \\
\cdot dt, \quad 1 \leq k \leq n-1, l = n - k,
\]
\[
G_{n,k,l}(f) = (n-1)(n-2) \int_0^1 \int_0^1 p_{n-3,k-1,j-1}(s, t)f(s, t) \\
\cdot ds dt, \quad 1 \leq k + l \leq n-1,
\]
(51)
then for the bivariate Bernstein operators, we have
\[
B_n(f)(x, y) = \sum_{k+l \leq n \atop k \geq 1} F_{n,k}(f)P_{n,k}(x, y).
\]
(52)
The bivariate genuine Bernstein-Durrmeyer operators are given by
\[
U_n(f)(x, y) = \sum_{k+l \leq n \atop k \geq 1} G_{n,k}(f)P_{n,k}(x, y).
\]
(53)
Now, for \( 0 \leq k + l \leq n \), we get
\[
b_1^{F_{n,k}} = \frac{k}{n}, \quad b_2^{F_{n,k}} = \frac{1}{n}, \quad b_1^{G_{n,k}} = \frac{k}{n}, \quad b_2^{G_{n,k}} = \frac{l}{n}.
\]
(54)
Hence, we obtain
\[
\mu_{2,0}^{F_{n,k,l}} = 0, \quad \mu_{2,0}^{G_{n,k,l}} = \frac{k(n-k)}{n^2(n+1)},
\]
\[
\mu_{2,2}^{F_{n,k,l}} = 0, \quad \mu_{2,2}^{G_{n,k,l}} = \frac{l(n-l)}{n^2(n+1)}.
\]
(55)
Therefore, \( \delta = 0 \) and
\[
\sigma(x, y) = \sum_{k+l \leq n \atop k \geq 1} \left[ \frac{k(n-k)}{n^2(n+1)} + \frac{l(n-l)}{n^2(n+1)} \right] P_{n,k}(x, y)
\]
\[
= \frac{(x + y - x^2 - y^2)(n-1)}{n(n+1)}, \quad (x, y) \in S.
\]
(56)
The proof is concluded by using Theorems 2 and 3.
4. The Difference $U^n_\rho - U^n_\eta$

Let $\rho > 0$ and $n \in \mathbb{N}$. The operators $U^n_\rho : C[0,1] \rightarrow \prod_n$ are introduced by Paltanea in [35] (see also [15, 16]). These operators are defined by

$$U^n_\rho(f;x) := \sum_{k=1}^{n+1} \left( \int_0^1 \frac{k^\rho - (1-t)^{(n-k)\rho - 1}}{\beta(k\rho, (n-k)\rho)} f(t) \, dt \right) \cdot p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n,$$

where $f \in C[0,1]$, $x \in [0,1]$, and $\beta(\cdot, \cdot)$ are Euler’s Beta function.

Here, for $f \in L^1(S)$, we consider the bivariate form of these operators, given by

$$U^n_{\rho\eta}(f, y) = f(0, 0)(1-x-y)^n + f(1, 0)x^n + f(0, 1)y^n + \sum_{l=1}^{n-1} \sum_{k=1}^{n-l} F^n_{\rho,0,l}(f) p_{n,0,l}(x, y) + \sum_{l=1}^{n-1} \sum_{k=1}^{n-l} F^n_{\rho,1,l}(f) p_{n,1,l}(x, y) + \sum_{l=1}^{n-1} \sum_{k=1}^{n-l} F^n_{\rho,0,l-k}(f) p_{n,0,l-k}(x, y) + \sum_{l=1}^{n-1} \sum_{k=1}^{n-l} F^n_{\rho,1,l-k}(f) p_{n,1,l-k}(x, y),$$

where

$$F^n_{\rho,0,l}(f) = \int_0^1 \frac{t^{l\rho - (n-l)\rho - 1}}{B(l\rho, (n-l)\rho)} f(0, t) \, dt,$$

$$F^n_{\rho,1,l}(f) = \int_0^1 \frac{(1-t)^{(n-l)\rho - 1}}{B((n-l)\rho)} f(t, 0) \, dt,$$

$$F^n_{\rho,0,l-k}(f) = \int_0^1 \frac{t^{l\rho - (n-l-k)\rho - 1}}{B(l\rho, (n-l-k)\rho)} f(0, t) \, dt,$$

$$F^n_{\rho,1,l-k}(f) = \int_0^1 \frac{(1-t)^{(n-l-k)\rho - 1}}{B((n-l-k)\rho)} f(t, 0) \, dt.$$

(59)

It can be easily seen that, for $\rho = 1$, we obtain the genuine Bernstein-Durrmeyer operators $U^n_\eta$. On the other hand, these operators have the following limiting behavior.

**Theorem 7.** For any $f \in C(S)$, one has $\lim_{\rho \to \infty} U^n_\rho(f) = B_n(f)$, uniformly.

**Proof.** Let $f = pr^j_1$, $j = 0, 1, \ldots$. Then,

$$F^n_{\rho,0,l}(pr^j_1) = \frac{(kp+j-1)\cdots(kp)}{(np+j-1)\cdots(np)}, \quad \text{for} \quad k, l \geq 1,$$

$$\lim_{\rho \to \infty} F^n_{\rho,0,l}(pr^j_1) = \left(\frac{k}{n}\right)^j = pr^j_1\left(\frac{k}{n}\right).$$

(60)

Since

$$F^n_{0,0,l}(pr^j_1) = \begin{cases} 1, j = 0, \\ 0, j > 0, \end{cases} F^n_{\rho,0,0}(pr^j_1) = F^n_{\rho,0,0-k}(pr^j_1) = \frac{(kp+j-1)\cdots(kp)}{(np+j-1)\cdots(np)},$$

we get

$$\lim_{\rho \to \infty} F^n_{\rho,0,0}(pr^j_1) = pr^j_1(0, \frac{1}{n}), \quad \lim_{\rho \to \infty} F^n_{\rho,0,0}(pr^j_1) = pr^j_1(\frac{k}{n}, 0),$$

$$\lim_{\rho \to \infty} F^n_{\rho,0,0}(pr^j_1) = pr^j_1(\frac{k}{n}, \frac{n-k}{n}).$$

(62)

Similar results can be obtained for $pr^j_2, j = 0, 1, \ldots$.

Using Korovkin’s theorem (see [36], p. 534, C.4.3.3), it follows $\lim_{\rho \to \infty} F^n_{\rho,0,0}(f) = f(\mu/n, \nu/n)$. Therefore,

$$\lim_{\rho \to \infty} U^n_\rho(f) = B_n(f), f \in C(S).$$

(63)

**Proposition 8.** For the bivariate operators $U^n_\rho$, the following properties hold:

(i) If $f \in C^2(S)$, then

$$|U^n_\rho - U^n_\eta(f)(x, y)| \leq M_f \sigma(x, y),$$

where $M_f$ is the same as in Lemma 1 and

$$\sigma(x, y) = \frac{(nr + np + 2)(x + y - x^2 - y^2)(n-1)}{n(np + 1)(nr + 1)}.$$

(65)

(ii) If $f \in C(S)$, then

$$|U^n_\rho - U^n_\eta(f)(x, y)| \leq \eta_2 \omega_2(f; \sqrt{\sigma(x, y)}).$$

(66)

**Proof.** Since $b_{1,0}^{\rho,0} = k/n, b_{1,0}^{\rho,0} = l/n$, we get

$$\frac{F^n_{\rho,0,0}}{n^2(np + 1)} = \frac{k(n-k)}{n^2(np + 1)}, \frac{F^n_{\rho,0,0}}{n^2(np + 1)} = \frac{l(n-l)}{n^2(np + 1)}.$$

(67)
Therefore, \( \delta = 0 \) and
\[
\sigma(x, y) = \frac{(nr + np + 2)(x + y - x^2 - y^2)(n - 1)}{n(n \rho + 1)(n + 1)}.
\] (68)

5. Difference of Bivariate Bernstein Operators and Their Kantorovich Variants

In 2017, F. Altomare et al. [37] introduced Kantorovich operators on \( S \) as follows
\[
C_n(f)(x, y) = \sum_{k,l=0 \cdots n} P_{n,k,l}(x,y) \int f (\frac{k + as}{n + a}, \frac{l + at}{n + a}) dsdt,
\] (69)
where \( P_{n,k,l}(x,y) \) is given by (28). It can be easily seen that, for \( a = 1 \), we obtain Kantorovich operators \( K_n \) introduced in [38].

If we denote
\[
G_{k,l}(f) = 2 \int f (\frac{k + as}{n + a}, \frac{l + at}{n + a}) dsdt,
\] (70)
the bivariate Kantorovich operators can be written as
\[
C_n(f)(x,y) = \sum_{k,l=0 \cdots n} G_{k,l}(f) P_{n,k,l}(x,y).
\] (71)

**Proposition 9.** For bivariate Bernstein operators and bivariate Bernstein-Kantorovich operators, the following properties hold:

(i) If \( f \in C^2(S) \), then
\[
|B_n - C_n(f)(x,y)| \leq M_f \sigma(x,y) + \omega_1 \left( f, \frac{4a}{3(n + a)} \right),
\] (72)
where \( M_f \) is the same as in Lemma 1 and
\[
\sigma(x, y) = \frac{a^2}{9(n + a)^2}.
\] (73)

(ii) If \( f \in C(S) \), then
\[
|B_n - C_n(f)(x,y)| \leq \eta_1 \omega_1 \left( f, \frac{4a}{3(n + a)} \right) + \eta_2 \omega_2 \left( f, \sqrt{\sigma(x,y)} \right).
\] (74)

**Proof.** As in the previous examples, taking Bernstein operators as
\[
B_n(f)(x,y) = \sum_{k,l=0 \cdots n} F_{k,l}(f) P_{n,k,l}(x,y),
\] (75)
we get
\[
\begin{align*}
F_{k,l}^{1_1} = \frac{k}{n}, & \quad F_{k,l}^{2_1} = \frac{1}{n}, \\
F_{k,l}^{1_2} = \frac{3k + a}{3(n + a)}, & \quad F_{k,l}^{2_2} = \frac{3l + a}{3(n + a)}.
\end{align*}
\] (76)

Therefore, we easily obtain that
\[
\begin{align*}
\mu_{2,1} = 0, & \quad \mu_{2,2} = \frac{a^2}{18(n + a)^2}; \\
\mu_{0,1} = 0, & \quad \mu_{0,2} = \frac{a^2}{18(n + a)^2}.
\end{align*}
\] (77)

Then
\[
\sigma(x, y) = \sum_{k,l=0 \cdots n} \frac{a^2}{9(n + a)^2} P_{n,k,l}(x,y) = \frac{a^2}{9(n + a)^2}.
\] (78)

Moreover, we have
\[
\begin{align*}
\delta = \max_{0 \leq k,l \leq n} \left\{ |b_1^{1_1} - b_1^{2_1}| + |b_2^{1_2} - b_2^{2_2}| \right\} \\
= \max_{0 \leq k,l \leq n} \left\{ \frac{|a|3k - \eta_1}{3n(n + a)} + \frac{|a|3l - \eta_2}{3n(n + a)} \right\} \leq \frac{4a}{3(n + a)}.
\end{align*}
\] (79)

Then, the proof follows from Theorems 2 and 3.

6. A Beta Operator on \( C(S) \)

For \( \rho \in (0, \infty) \), \( f \in C(S) \), and \( (x, y) \in S \), let us define
\[
\mathbb{B}_\rho(f)(x,y) = \begin{cases} 
\frac{f(x,y)}{B(\rho x, \rho(1-x))}, & x \in (0, 1), \ y = 0, \\
\frac{f(x,y)}{B(\rho y, \rho(1-y))}, & x = 0, \ y \in (0, 1), \\
\frac{f(x,y)}{B(\rho x, \rho(1-x))}, & \rho x = 1 - x, \ x \in (0, 1), \\
\frac{f(x,y)}{B(\rho y, \rho(1-y))}, & y = 1 - x, \ x \in (0, 1), \\
\int \frac{f(x,y)}{B(\rho x, \rho(1-x))} dsdt, & x \in (0, 1), \\
\int \frac{f(x,y)}{B(\rho y, \rho(1-y))} dsdt, & y \in (0, 1), \\
\end{cases}
\] (80)

For \( \rho = n \in \mathbb{N} \), this is the bivariate version of the operator \( \mathbb{B}_n \); see [39] and the references therein.
**Theorem 10.** \( \mathcal{B}_\rho \) is a positive linear operator acting between \( C(S) \) and \( C(S) \). Moreover,

\[
\mathcal{B}_\rho(1) = 1, \tag{81}
\]

and if \( \varphi_{ij}(x,y) = pr_i pr_j(x,y), \ (x,y) \in S, \ i \geq 0, \ j \geq 0, \ integers, \) then

\[
\mathcal{B}_\rho \left( \varphi_{ij} \right)(x,y) = \frac{px(px+1) \cdots (px+i-1)}{\rho (\rho+1) \cdots (\rho+i-1)} \cdot \frac{\rho_y (\rho_y + 1) \cdots (\rho_y + j-1)}{\rho (\rho+1) \cdots (\rho+j-1)}. \tag{82}
\]

**Proof.** It is easy to prove (81) and (82) by direct calculation. It remains to prove that if \( f \in C(S) \), then, \( \mathcal{B}_\rho(f) \in C(S) \). To do this, it suffices to verify that \( \mathcal{B}_\rho(f) \) is continuous at each point of the boundary of \( S \). Let us prove that if \( 0 < a < 1 \)

\[
\lim_{(x,y) \to (a,0)} \mathcal{B}_\rho(f)(x,y) = \mathcal{B}_\rho(f)(a,0). \tag{83}
\]

Let \( V_{(a,0)} : C(S) \to \mathbb{R}, \ V_{(a,0)}(g) = \mathcal{B}_\rho(g)(a,0) \). For \( (x,y) \in \text{int}(S) \) define \( U_{(x,y)} : C(S) \to \mathbb{R}, \ U_{(x,y)}(g) = \mathcal{B}_\rho(g)(x,y) \). Then, \( U_{(x,y)} \) and \( V_{(a,0)} \) are positive linear functionals of norm 1.

Let \( \varepsilon > 0 \). Then, there exists a polynomial function \( p \) on \( S \) such that \( \| f - p \|_\infty \leq \varepsilon /4 \). Using (82), it is easy to verify that

\[
\lim_{(x,y) \to (a,0)} U_{(x,y)}(p) = V_{(a,0)}(p). \tag{84}
\]

Consequently, there exists \( \delta > 0 \) with

\[
\left| U_{(x,y)}(p) - V_{(a,0)}(p) \right| \leq \frac{\varepsilon}{2} \| (x,y) - (a,0) \|_1 \leq \delta. \tag{85}
\]

So, if \( \| (x,y) - (a,0) \|_1 \leq \delta \), we have

\[
\left| U_{(x,y)}(f) - V_{(a,0)}(f) \right| \leq \left| U_{(x,y)}(f) - U_{(x,y)}(p) \right| + \left| U_{(x,y)}(p) - V_{(a,0)}(p) \right| + \left| V_{(a,0)}(p) - V_{(a,0)}(f) \right| \tag{86}
\]

\[\leq \| f - p \|_\infty + \frac{\varepsilon}{2} + \| p - f \|_\infty \leq \varepsilon.\]

This shows that

\[
\lim_{(x,y) \to (a,0)} U_{(x,y)}(f) = V_{(a,0)}(f), \ f \in C(S), \tag{87}
\]

and then (83) is proved.

The continuity of \( \mathcal{B}_\rho(f) \) at the other boundary points can be proved similarly.

**Proposition 11.** For each \( f \in C(S) \), one has

\[
\lim_{\rho \to \infty} \mathcal{B}_\rho(f) = f. \tag{88}
\]

**Proof.** Using Theorem 10, it is easy to verify that (88) is valid for the functions \( 1, pr_1, pr_2, pr_1^2 + pr_2^2 \). But these functions form a Korovkin test system (see [36], p. 534, C.4.3.3), so that (88) holds for each \( f \in C(S) \).

In what follows, we formulate a

**Conjecture 12.** If \( f \in C(S) \) is convex and \( (x,y) \in S \), then, the function \( \rho \to \mathcal{B}_\rho(f)(x,y) \) is decreasing on \( (0, \infty) \).

It is supported by the following facts.

(i) The unidimensional version of the conjecture is valid: see [14, 40].

(ii) \( \mathcal{B}_\rho \) is a positive linear operator preserving the affine functions; this implies

\[
\mathcal{B}_\rho(f) \geq f, \ f \in C(S) \text{convex.} \tag{89}
\]

Now, (88) combined with (89) support the conjecture.

(iii) The conjecture is valid for the functions \( pr_1^k, pr_2^k, \ (1 - pr_1 - pr_2)^k, \ k \in \mathbb{N} \).

In the sequel, we present two results under the hypothesis that the conjecture is true. To this end, let us introduce some notation.

Let \( f \in C^2(S) \) and

\[
m_1(f) := \min \left\{ f_{xx}(x,y) - f_{xy}(x,y) : (x,y) \in S \right\},
\]

\[
m_2(f) := \min \left\{ f_{yy}(x,y) - f_{xy}(x,y) : (x,y) \in S \right\},
\]

\[
M_1(f) := \max \left\{ f_{xx}(x,y) + f_{xy}(x,y) : (x,y) \in S \right\},
\]

\[
M_2(f) := \max \left\{ f_{yy}(x,y) + f_{xy}(x,y) : (x,y) \in S \right\}. \tag{90}
\]

Then, the functions \( \varphi(x,y) := f(x,y) - 1/2m_1(f)x^2 - 1/2m_2(f)y^2 \) and \( \psi(x,y) := 1/2M_1(f)x^2 + 1/2M_2(f)y^2 - f(x,y) \) are convex on \( S \); indeed, for each of them, the Hessian matrix is positive semidefinite.

**Theorem 13.** If \( f \in C^2(S) \) and \( \sigma > \rho > 0 \), then

\[
\frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} \left[ m_1(f)x(1-x) + M_2(f)y(1-y) \right] \leq \mathcal{B}_\rho(f)(x,y) - \mathcal{B}_\sigma(f)(x,y) \leq \frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} \left[ M_1(f)x(1-x) + M_2(f)y(1-y) \right], \ (x,y) \in S. \tag{91}
\]
Proof. Let $\sigma > \rho > 0$. If the Conjecture is true, we have $B_p(\sigma) \geq B_p(\phi)$ and $B_p(\psi) \geq B_p(\psi)$. Thus

$$0 \leq B_p(\sigma)(x, y) - B_p(\psi)(x, y) = B_p(f)(x, y) - B_p(\phi)(x, y) + \frac{\rho - \sigma}{2(\sigma + 1)(\rho + 1)} \left[ 2M_1(f)(x - 1) + 2M_2(f)(1 - y) \right].$$

(92)

Consequently,

$$B_p(f)(x, y) - B_p(\phi)(x, y) \geq \frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} [M_1(f)x(1 - x) + M_2(f)y(1 - y)].$$

(93)

Moreover,

$$0 \leq B_p(\sigma)(x, y) - B_p(\psi)(x, y) = \frac{x}{2} M_1(f)(\sigma - \rho)(1 - x) + \frac{\rho - \sigma}{2(\sigma + 1)(\rho + 1)} B_p(f)(x, y) + B_p(\phi)(x, y).$$

(94)

Now,

$$B_p(f)(x, y) - B_p(\phi)(x, y) \leq \frac{\sigma - \rho}{2(\sigma + 1)(\rho + 1)} [M_1(f)x(1 - x)]$$

$$+ M_2(f)y(1 - y)].$$

(95)

So, combining (93) and (95), we have proved the theorem.

Theorem 14. If $f \in C^2(S)$ and $\sigma > \rho > 0$, then

$$0 \leq U^n_p(\sigma)(x, y) - U^n_p(\phi)(x, y) = U^n_p(f)(x, y) - U^n_p(\phi)(x, y) + \frac{(n - 1)(\sigma - \rho)}{2(n\sigma + 1)(n\rho + 1)} [M_1(f)x(1 - x)]$$

$$+ M_2(f)y(1 - y)],$$

(96)

Thus,

$$U^n_p(f)(x, y) - U^n_p(\phi)(x, y) \leq \frac{(n - 1)(\sigma - \rho)}{2(n\sigma + 1)(n\rho + 1)} [M_1(f)x(1 - x)]$$

$$+ M_2(f)y(1 - y)].$$

(100)

From $U^n_p(\psi) \geq U^n_p(\psi)$, we get a similar upper bound for $U^n_p(f)(x, y) - U^n_p(f)(x, y)$, which concludes the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no competing financial interests.

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