Clique Cover Width and Clique Sum

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Abstract

For a clique cover $C$ in the undirected graph $G$, the clique cover graph of $C$ is the graph obtained by contracting the vertices of each clique in $C$ into a single vertex. The clique cover width of $G$, denoted by $CCW(G)$, is the minimum value of the bandwidth of all clique cover graphs of $G$. When $G$ is the clique sum of $G_1$ and $G_2$, we prove that $CCW(G) \leq 3/2(CCW(G_1) + CCW(G_2))$.

1 Introduction and Summary

Throughout this paper, $G = (V(G), E(G))$ denotes a graph on $n$ vertices. $G$ is assumed to be undirected, unless stated otherwise. Let $L = \{v_0, v_1, ..., v_{n-1}\}$ be a linear ordering of vertices in $G$. The width of $L$, denoted by $w(L)$, is $\max_{v_i, v_j \in E(G)} \{|j - i|\}$. The bandwidth of $G$, denoted by $BW(G)$, is the smallest width of all linear orderings of $V(G)$ [1], [2], [3]. Unfortunately, computing the bandwidth is NP hard, even if $G$ is a tree [6]. A clique cover $C$ in $G$ is a partition of $V(G)$ into cliques. Throughout this paper, we will write $C = \{c_0, c_1, ..., c_t\}$ to indicate that $C$ is an ordered set of cliques. For a clique cover $C = \{c_0, c_1, ..., c_t\}$ in $G$, let the width of $C$, denoted by $w(C)$, denote $\max \{|j - i| \,| \, xy \in E(G), x \in c_i, y \in c_j, c_i, c_j \in C\}$. The clique cover width of $G$ denoted by $CCW(G)$, was defined by the author in [10], to be the smallest width of all ordered clique covers in $G$. Equivalently, one can define $CCW(G)$ to be the minimum value of the bandwidth overall clique cover graphs of $G$, as stated in the abstract. Combinatorial properties of $CCW(G)$ have been studied. Clearly, $CCW(G) \leq BW(G)$. In addition, it is easy to verify that $BW(G) \leq \omega(G).CCW(G)$, where $\omega(G)$ is the size of a largest clique in $G$, and that any $G$ with $CCW(G) = 1$ is an incomparability graph [10]. Furthermore, we proved in [11] that $CCW(G) \geq \lfloor s(G)/2 \rfloor - 1$ for any graph $G$, where $s(G)$ is the largest number

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of leaves in an induced star in $G$, and that for any incomparability graph $G$, $CCW(G) \leq s(G) - 1$. Additionally, in [11], we further explored the class of unit incomparability graphs, or graphs $G$ with $CCW(G) = 1$; this class is fairly large and contains the classes of the unit interval graphs and co-bipartite graphs. Furthermore, in [11] we introduced the unit incomparability dimension of $G$, or Udim$(G)$, which is a parameter similar to the cubicity of $G$ [7], [14], [15]. Specifically, Udim$(G)$ is defined as the smallest integer $d$ so that $G$ is the intersection graph of $d$ unit incomparability graphs. In [11], we also proved a decomposition theorem that establishes the inequality $Udim(G) \leq CCW(G)$, for any graph $G$. The upper bound is improved from $CCW(G)$ to $O(log(CCW(G)))$ in [13]. Finally, we have just proved that every planar $G$ is the intersection graph of a chordal graph and a graph whose clique cover width is at most seven [12]. The main application of the clique cover width is in the derivation of separation theorems in graphs; particularly the separation can be defined for more general types measures [9], instead of just the number of vertices. For instance, given a clique cover $C$ in $G$, can $G$ be separated by removing a *small* number of cliques in $C$ so that each the two remaining subgraph of $G$ can be covered by at most $k|C|$ cliques from $C$, where $k < 1$ is a constant [9, 13]? Our recent work shows a close connection between the tree width of $G$, or tw$(G)$, and $CCW(G)$. Recall that tw$(G) - 1$ is the minimum of the maximum clique sizes of all chordal graphs that are obtained by adding edges to $G$ [8],[4].

Let $G_1$ and $G_2$ be graphs so that $V(G_1) \cap V(G_2)$ is a clique in both $G_1$ and $G_2$. Then, the clique sum of $G_1$ and $G_2$, denoted by $G_1 \oplus G_2$, is a graph $G$ with $V(G) = V(G_1) \cap V(G_2)$, and $E(G) = E(G_1) \cup E(G_2)$. Clique sums are intimately related to the concept of the tree width and tree decomposition; specifically, it is known that if the tree widths of $G_1$ and $G_2$ are at most $k$, then, so is the tree width of $G_1 \oplus G_2$ [5]. Unfortunately, this is not true for the clique cover width. As seen in the following example.

**Example 1.1** Let $P_1$ and $P_2$ be paths on $2t + 1$, $t \geq 1$ vertices, then, $CCW(P_1) = CCW(P_2) = 1$. Now select the unique vertex $x$ in the middle of the two paths and take the sum of $P_1$ and $P_2$ at $x$. Then, $s(P_1 \oplus P_2) = 4$, and consequently, $CCW(P_1 \oplus P_2) \geq s(G)/2 = 2$, whereas, a simple ordering linear ordering of vertices shows that lower bound is achievable, and in fact $CCW(P_1 \oplus P_2) = 2$.

In this paper we study the clique cover width of the clique sum of two graphs and establish the inequality $CCW(G_1 \oplus G_2) \leq (3/2)(CCW(G_1) + CCW(G_2))$. In section two we define some specific technical concepts that will be used to
2 Preliminaries

Let $C = \{c_0, c_1, ..., c_l\}$ be a clique cover in $G$. Any set of consecutive cliques in $C$ is called a strip. Any strip of of cardinality $w = w(C)$ is called a block.

Let $S$ be a strip in $C$. We denote by $C^l(S)$ and $C^r(S)$, the largest strips that are entirely to the left, and to the right of $S$, respectively; thus, $C = \{C^l(S), S, C^r(S)\}$. Note that if $S$ is a block, then, the removal of cliques in $S$ disconnects the subgraphs induced on $C^l(S)$ and $C^r(S)$ in $G$. For a strip $S$ in $C$, we denote by $S^l$ and $S^r$ the largest strips of cardinality at most $w$ that immediately precede and proceed $S$, in $C$, respectively. Let $B = \{c_k, c_{k+1}, ..., c_{k+w-1}\}, k \geq 0$ be a block in a clique cover $C = \{c_0, c_1, ..., c_l\}$ for $G$. We will define a partition of $C$, denoted by $P(C, B)$, as follows. Let $C^l(B) = \{c_0, c_1, ..., c_{k-1}\}$, and $C^r(B) = \{c_{k+w}, c_{k+w+1}, ..., c_l\}$. Now, let $k = p.w + r, r \leq w - 1$ Define $S_0$ to be the set of first $r$ consecutive cliques in $C^l(B)$. For $i = 1, 2, ..., p$, let $S_i$ be the block in $C$ starting at $c_i(i1).w + r$.

It is easy to verify that $\{S_0, S_1, ..., S_p\}$ is a partition of $C^l(B)$. Similarly, one can construct a partition of $C^r(B)$ of form $\{S_{p+2}, S_{p+3}, ..., S_q\}$, for some properly defined $q$, so that $S_i$ is a block for $i = p + 2, p + 3, ..., q$, and the width of $S_p$ is at most $w(C)$. By combining these two partitions with $S_{p+1} = B$, as the middle part, we obtain the partition $P(C, B)$ of $C$ into strips. For $i, j = 0, 1, ..., q$, we define the distance of strips $S_i$ and $S_j$, in the partition $P(C, B)$, to be $|j - i|$. 

**Proposition 2.1** Let $C$ be a clique cover in $G$ and let $B$ be a block in $C$. Then, there is an ordered partition $P(C, B) = S_0, S_1, ..., S_p, B, S_p, S_{p+2}, ..., S_q$ of $C$ into strips so that $\{S_0, S_1, ..., S_p\}$ partitions $C^l(B)$, $B = S_p + 1$, and $S_p, S_{p+2}, ..., S_q$ partitions $C^r(B)$. Moreover, the first and last strips in $P(C, B)$ have widths at most $w(C)$, whereas, the remaining elements of $P(C, B)$ are blocks.

3 Main Results

Let $G = G_1 \oplus G_2$, and let $C_1$ and $C_2$ be clique covers in $G_1$ and $G_2$, respectively. Let $S_1$ and $S_2$ be strips in $C_1$ and $C_2$, respectively. The interleave ordering of $S_1$ and $S_2$, denoted by, $S_1 \oplus S_2$ is an ordered set of cliques obtained by placing the first clique in $S_1$ after the first clique in $S_2$, the second clique in $S_1$ after the second clique in $S_2$, etc., until all cliques in one $S_i, i = 1, 2$, are used, then, one places all remaining cliques in $S_{i+1}(mod 2)$ in $S_1 \oplus S_2$. Note that $S_1 \oplus \emptyset = S_1$, and $\emptyset \oplus S_2 = S_2$.

**Theorem 3.1** Let $G_1$ and $G_2$ be graphs, where $S = V(G_1) \cap V(G_2)$ induces a clique in $G_1$ and $G_2$. Then, $CCW(G_1 \oplus G_2) \leq 3/2(CCW(G_1) + CCW(G_2))$. 
Proof. To prove the claim, let $C_1 = \{c_0, c_1, ..., c_a\}$ and $C_2 = \{c'_0, c'_1, ..., c'_b\}$ be clique covers in $G_1$ and $G_2$, respectively; it suffices to construct a clique cover $C$ for $G$, of width at most $3/2(w(C_1) + w(C_2))$. Let $B_1$ and $B_2$ be the blocks in $C_1$ and $C_2$ that contain all vertices in $S$, respectively. Note that such blocks must exist, since $S$ induces a clique in $G_1$, as well as, in $G_2$. Construct the partitions $P(B_1, C_1) = \{S_0, S_1, ..., S_p, B_1, S_{p+2}, ..., S_q\}$ and $P(B_2, C_2) = \{T_0, T_1, ..., T_r, B_2, T_{r+2}, ..., T_u\}$. Now, interleave $B_1$ and $B_2$, to obtain a strip $I(B_1, B_2)$. Next, for $i = p, p1, ..., 0, j = r, r1, ..., 0$, if $pi = rj$, then, interleave $S_i$ and $T_j$, to obtain the strip $I(S_i, T_j)$. (So, we interleave those strips in $C_1$ and $C_2$ that are to left of $B_1$ and $B_2$, and are of the same distance from $B_1$ and $B_2$, respectively.) Let $L'$ denote the ordered union of all these interleaved strips. Similarly, for $i = p + 2, p + 3, ..., S_p, j = r + 2, r + 3, ..., u$, if $pi = rj$, then, interleave $S_i$ and $T_j$ to obtain $I(S_i, T_j)$, and, let $L''$ denote the ordered union of all these interleaved strips. Now, let $L = \{L', I(B_1, B_2), L''\}$. Note that $L = \{l_0, l_1, ..., l_{a+b+1}\}$ is an ordered clique cover for $G$, but the cliques in $L$ are not disjoint. It is important to observe that any clique in $L$ either belongs to $C_1$ or to $C_2$, and that the relative orderings of cliques in $C_1$ or $C_2$ remain the same in $L$.

Claim. Let $xy \in E(G_1) \cup E(G_2)$, with $x \in l_r, y \in l_i, l_r, l_i \in L$. Then, $|t - r| \leq w(C_1) + w(C_2) - 1$.

Proof. Clearly, $l_r, l_i \in C_i$, for some $i = 1, 2$. If $l_r, l_i$ are in the same strip of $L$, as prescribed in the interleave construction, then, the claim is true. So assume that $l_r$ and $l_i$ are in different strips of $L$. To verify the claim it suffices to verify that $l_r$ and $l_i$ can only be in adjacent strips. However, this is a consequence of the interleave construction.

We will now convert $L$ to a clique cover in $G$ as follows. Create a new clique in the middle of the strip $I(B_1, B_2)$, place $S$ in this clique, and remove all vertices of $S$ from any cliques in $L$. Let $C$ denote the clique cover that is constructed this way. Using the above claim, we get, $w(C) \leq w(C_1) + w(C_2) - 1 + \frac{w(C_1) + w(C_2)}{2} + 1$. Consequently, $w(C) \leq 3/2(w(C_1) + w(C_2))$, as stated.

References

[1] J. Bottcher, K. P. Pruessmann, A. Taraz, A. Wurfel, Bandwidth, treewidth, separators, expansion, and universality, European Journal of Combinatorics, 31, 2010, 1217, 1227.

[2] P.Z. Chinn, J. Chvatalov, A. K. Dewdney, N.E. Gibbs, Bandwidth problem for graphs and matrices- A Survey, Journal of Graph Theory, 6(3), 2006, 223-254.
[3] J. Diaz, J. Petit, M. Serna A survey of graph layout problems, ACM Computing Surveys (CSUR) 34(3), 2002, 313 - 356.

[4] H.L. Bodlaender, A Tourist Guide through Treewidth. Acta Cybern. 11(1-2), 1993, 1-22.

[5] H.L. Bodlaender, A partial k-arboretum of graphs with bounded treewidth. Theoretical Computer Science, 1998.

[6] M.R. Garey and D.J. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, CA, 1978

[7] F. Roberts. Recent Progresses in Combinatorics, chapter On the boxicity and cubicity of a graph, pages 301-310. Academic Press, New York, 1969.

[8] N. Robertson, P. D. Seymour, Graph minors III: Planar tree-width, Journal of Combinatorial Theory, Series B *36*, 1984, (1): 49-64.

[9] F. Shahrokhi, A New Separation Theorem with Geometric Applications, Proceedings of EuroCG2010, 2010, 253-256.

[10] F. Shahrokhi, On the clique cover width problem, Congressus Numerantium, 205 (2010), 97-103.

[11] F. Shahrokhi, Unit Incomparability Dimension and Clique Cover Width in Graphs, Congressus Numerantium, 213 (2012), 91-98.

[12] F. Shahrokhi, New representation results for planar graphs, 29th European Workshop on Computational Geom., 2013.

[13] F. Shahrokhi, in preparation.

[14] L. Sunil Chandran and Naveen Sivadasan. Boxicity and treewidth. Journal of Combinatorial Theory, Series B, 97(5):733-744, September 2007.

[15] Abhijin Adiga, L. Sunil Chandran, Rogers Mathew: Cubicity, Degeneracy, and Crossing Number. FSTTCS 2011: 176-190.