Invertibility Preserving Linear Maps On Semi-Simple Banach Algebras

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Abstract

In this paper, we show that the essentiality of the socle of an ideal $B$ of the semi-simple Banach algebra $A$ implies that any invertibility preserving isomorphism $\phi : A \rightarrow A$ is a Jordan homomorphism. Specifically if, the unitary semi-simple Banach algebra $A$ has an essential minimal ideal then $\phi \mid_{soc(A)}$ is a Jordan homomorphism.

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1 Introduction

Linear invertibility preserving maps of algebras, were noteworthy from years ago. For example, the famous theorem of Kahan-Zelasco which asserts that any invertibility preserving isomorphism into the scalar field is homomorphism. This problem discussed on different algebras previously. Let $A$ be an unitary Banach algebra and $a \in A$ is invertible. Then the inverse of $a$ is denoted by $a^{-1}$ and the set of all invertible elements of $A$ is denoted by $Inv(A)$. Also, let $A, B$ are tow algebras and $\phi : A \rightarrow B$ is a linear map. The $\phi$ is called an invertibility preserving map if,

$$a \in Inv(A) \Rightarrow \phi(a) \in Inv(B) \text{ for all } a \in A$$

The reader is referred to [1] for undefined terms and notations.
2 Main Results

The following lemma, is useful for the proof of our next theorem.

**Lemma 2.1** [1] Let \( A, B \) are unitary semi-simple Banach algebras and \( \phi : A \rightarrow B \) is an invertibility preserving isomorphism. Then,

\[
\phi^{-1}(\phi(a^2) - \phi^2(a)).soc(A) = 0 \quad \forall a \in A
\]

Moreover, if the \( \text{soc}(A) \) is an essential ideal of \( A \), then \( \phi \) is a Jordan homomorphism (i.e. \( \phi(a^2) = \phi^2(a) \quad \forall a \in A \))

Now, let us to state of our main theorem:

**Theorem 2.2** Let \( A \) is a semi-simple Banach algebra and \( \phi : A \rightarrow A \) is an invertibility preserving isomorphism. Then \( \phi \) is a Jordan homomorphism, whenever \( A \) has an ideal \( B \) that \( \text{soc}(B) \) is an essential ideal.

**proof.** At first, we suppose that \( A \) is unitary. Since \( \text{soc}(B) \) is an essential ideal of \( A \), \( \text{soc}(A) \) is an essential ideal, too [1] and so \( \phi \) is a Jordan homomorphism, by lemma 2.1. If now, \( A \) is not unitary \( \tilde{A} = A \oplus \mathbb{C} \) is an unitary semi-simple Banach algebra with \( (a_1, \lambda_1) \cdot (a_2, \lambda_2) = (a_1a_2 + \lambda_2a_1 + \lambda_1a_2, \lambda_1\lambda_2) \).

Let \( \tilde{\phi}(a, \lambda) = (\phi(a), \lambda) \) for \( (a, \lambda) \in \tilde{A} \). The \( \tilde{\phi} : \tilde{A} \rightarrow \tilde{A} \) is a well defined invertibility preserving isomorphism. If \( \text{soc}(A) = K \) and \( (a, \lambda)\text{soc}(\tilde{A}) = 0 \), then \( (a, \lambda)k = 0 \) for all \( k \in K \) since \( \text{soc}(A, 0) \subseteq \text{soc}(A, 0) \subseteq \text{soc}(\tilde{A}) \).

So for all \( k \in K \), \( ak = -\lambda k \) and therefore \( \lambda = 0 \). Because \( \lambda \neq 0 \) implies that \(-\frac{a}{\lambda}k = k \), for all \( k \in K \). So \(-\frac{a}{\lambda} \) is a left unit of \( A \). Let \( d \) is an other left unit of \( A \). Then

\[
(-\frac{a}{\lambda} - d)A = 0 \Rightarrow (-\frac{a}{\lambda} - d)k = 0 \quad \forall k \in K \Rightarrow -\frac{a}{\lambda} = d
\]

Note that \( K \) is an essential ideal. So \( A \) is unitary which contradicts our hypothesis. Therefore \( a = 0 \) and \( \text{soc}(\tilde{A}) \) is an essential ideal. Now lemma 2.1 implies that,

\[
\tilde{\phi}(a, \lambda)^2 = \tilde{\phi}^2(a, \lambda) \quad \forall (a, \lambda) \in \tilde{A}
\]

But, for all \( (a, \lambda) \in \tilde{A} \) we have,

\[
\tilde{\phi}(a, \lambda)^2 = (\phi(a^2) + 2\lambda\phi(a), \lambda^2) \quad \text{and} \quad \tilde{\phi}^2(a, \lambda) = (\phi^2(a), \lambda)
\]

Hence, for all \( a \in A \), \( \phi^2(a) = \phi(a^2) \) and \( \phi \) is a Jordan homomorphism.
Lemma 2.3 [4] Let $\mathcal{A}$ is an unitary semi-simple Banach algebra and $a \in \mathcal{A}$. Then

(i) $a \in \text{soc}(\mathcal{A})$ if and only if $|\sigma(xa)| < \infty$ for all $x \in \mathcal{A}$

(ii) $a \in \text{soc}(\mathcal{A})$ if and only if there exists $n \in \mathbb{N}$ such that $\bigcap_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$

for all $x \in \mathcal{A}$ for which $F$ is the set of $n$-tuples of $\mathbb{C} \setminus \{0\}$.

Lemma 2.4 Let $\phi : \mathcal{A} \to \mathcal{A}$ is a spectrum preserving isomorphism on the unitary semi-simple Banach algebra $\mathcal{A}$. Then $\phi(\text{soc}(\mathcal{A})) = \text{soc}(\mathcal{A})$.

proof. Let $a \in \text{soc}(\mathcal{A})$. Since $\phi$ is spectrum preserving, we have,

$$\sigma(\phi(y + ta)) = \sigma(y + ta) \text{ for all } t \in \mathbb{C} \text{ and } y \in \mathcal{A}$$

If now, $x = \phi(y)$ by the lemma 2.3, there exists $n \in \mathbb{N}$ such that,

$$\bigcap_{t \in F} \sigma(x + t\phi(a)) = \bigcap_{t \in F} \sigma(y + ta) \subseteq \sigma(x) \text{ for all } x \in \mathcal{A}$$

where $F$ is $n$-tuples of $\mathbb{C} \setminus \{0\}$. Hence $\phi(a) \in \text{soc}(\mathcal{A})$ and so $\phi(\text{soc}(\mathcal{A})) \subseteq \text{soc}(\mathcal{A})$.

Now, we show that $\text{soc}(\mathcal{A}) \subseteq \phi(\text{soc}(\mathcal{A}))$. Let $a \in \text{soc}(\mathcal{A})$. Then there exists $b \in \mathcal{A}$ such that $\phi(b) = a$ and there exists $n \in \mathbb{N}$ such that,

$$\bigcap_{t \in F} \sigma(x + tb) = \bigcap_{t \in F} \sigma(\phi(x) + t\phi(b)) \subseteq \sigma(\phi(x)) = \sigma(x) \text{ for all } x \in \mathcal{A}$$

where $F$ is $n$-tuples of $\mathbb{C} \setminus \{0\}$. This implies that $b \in \text{soc}(\mathcal{A})$.

Let us mention that if the Banach algebra $\mathcal{A}$ has an essential minimal ideal, then $\text{soc}(\mathcal{A})$ is essential. Thus we obtained the following consequence:

Corollary 2.5 If $\phi : \mathcal{A} \to \mathcal{A}$ is an invertibility preserving isomorphism on the unitary semi-simple Banach algebra $\mathcal{A}$ with an essential minimal ideal, then $\phi \mid_{\text{soc}(\mathcal{A})}$ is a Jordan homomorphism.

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