Optimal design and quantum benchmarks for coherent state amplifiers

Giulio Chiribella\textsuperscript{1,*} and Jinyu Xie\textsuperscript{1,‡}
\textsuperscript{1}Center for Quantum Information, Institute for Interdisciplinary
Information Sciences, Tsinghua University, Beijing 100084, China

We establish the ultimate quantum limits to the amplification of an unknown coherent state, both in the deterministic and probabilistic case, investigating the realistic scenario where the expected photon number is finite. In addition, we provide the benchmark that experimental realizations have to surpass in order to beat all classical amplification strategies and to demonstrate genuine quantum amplification. Our result guarantees that a successful demonstration is in principle possible for every finite value of the expected photon number.

Continuous-variable quantum systems, such as coherent light pulses, are promising information carriers for the new quantum technology\textsuperscript{1,2}. One of the cornerstones of continuous-variable quantum information is the amplification of signals encoded into quantum states of the radiation field\textsuperscript{3,4}. Unlike classical amplifiers, quantum amplifiers are subject to fundamental limits, typically expressed as a reduction of the signal-to-noise ratio (SNR) as a function of the amplification parameter\textsuperscript{5–7}. Despite these limits, quantum amplifiers are an essential piece of technology\textsuperscript{8}, for they enable the detection of ultra-weak signals—such as gravitational waves—that would not trigger the detectors otherwise.

Determining the ultimate quantum limits to amplification is both a topic of immediate technological import and a fundamental chapter of quantum theory, deeply connected with the no-cloning theorem, the uncertainty principle, and the quantum-classical transition in the limit of large amplification. Up to now, however, the performances of quantum amplifiers have been discussed mostly in classical terms (SNR), which are well suited for tasks such as signal detection, but less suited for applications in quantum information processing. For example, the role of the amplifier could be to coherently copy quantum data\textsuperscript{9,10} and to broadcast them to the users of a quantum internet\textsuperscript{11}. For quantum tasks, the most natural figure of merit is the fidelity between the desired output states and the states effectively produced by the amplifier, which can be interpreted operationally as the probability that the output state passes a test set up by a verifier who knows the input state.

In the fidelity setting, the works on optimal cloning of coherent states\textsuperscript{12–15} give a first insight in the problem of optimal amplification, suggesting that two-mode squeezing should be the best deterministic process allowed by quantum mechanics. If confirmed in a realistic scenario, this conclusion would be of high practical importance, as it would allow one to construct the best possible amplifiers using an optical element that is already in the toolbox of most laboratories. However, the optimality of two-mode squeezing, long conjectured, has never been proved without invoking strong simplifying assumptions, either on the nature of the amplifier—typically assumed to be Gaussian—or on the probability distribution of the states to be amplified—typically assumed to be uniform over all coherent states. Both assumptions are far from trivial: On the one hand, it is well known that non-Gaussian operations often outperform Gaussian ones, even for the manipulation of coherent states\textsuperscript{10}. Hence, there is no\textit{ a priori} reason to expect that the best amplifier of coherent states is Gaussian. From a fundamental point of view, any restriction on the allowed operations can hardly be satisfactory: if one wants to discover the ultimate quantum limits, one should not restrict the search to a subset, such as the subset of Gaussian operations, which has measure zero in the set of all possible operations. On the other hand, assuming a uniform distribution over coherent states means assuming that the expected photon number is infinite, or equivalently, that there is no bound on the energy of the source producing the coherent pulses—a quite unphysical assumption. In a realistic setting one can only have a large photon number, and in order to know how large this number should be to be effectively treated as infinite, one needs to gain first a full grasp of the finite photon number scenario.

Further motivation to go beyond the assumption of uniform distribution comes from the recent proposals of noiseless probabilistic amplifiers\textsuperscript{17–22}, whose performances are almost ideal for low photon numbers but decay exponentially as the photon number increases. In this case, it is most natural to test the performances of the amplifier on input states with low photon number, because these are the states where the amplifier is expected to work. Furthermore, in order to claim the demonstration of a genuine quantum amplifier, a real experiment should surpass the classical fidelity threshold (CFT)\textsuperscript{23,24}, i.e., the maximum fidelity achieved by “classical” amplifiers that produce an estimate of the input state and, conditional to the estimate, reprepare amplified states. In the case of probabilistic amplifiers, where the photon number is necessarily finite, it would be unfair to compare the experimental fidelity with a lower CFT computed for the uniform distribution. However, despite the urge to have suitable criteria to assess the new experimental breakthroughs on probabilistic amplification\textsuperscript{19–22}, the correct value of the CFT for proba-
bilistic quantum amplifiers has never been derived up to now.

In this Letter we establish the ultimate limits on the fidelity of quantum and classical amplifiers, treating both the deterministic and probabilistic case without making any assumption on the type of amplifying process, and without making the assumption of infinite expected photon number. We focus on the realistic scenario where the coherent states are distributed according to a Gaussian prior, which is the most studied case for applications in coherent-state quantum cryptography [27–31], cloning [14], and teleportation or storage [23]. In the deterministic case, we show that the maximum quantum fidelity can be achieved through a two-mode squeezing process with the amount of squeezing depending critically on the variance of the prior. In the probabilistic case, the critical behavior persists, with a dramatic effect: for variances below the critical value the optimal amplifier becomes non-Gaussian and its fidelity can be arbitrarily close to 1.

We then provide the value of the classical fidelity threshold (CFT) that must be experimentally surpassed in order to demonstrate the implementation of a genuine quantum amplifier. The value of the CFT is the same for both deterministic and probabilistic protocols and, luckily, it guarantees that a successful demonstration is possible for every finite value of the expected photon number. For example, for a gain $g = 2$ and variance $1/3$, the value of the CFT is 50%, while the fidelity achieved by the optimal deterministic amplifier is 85%. The general techniques developed in this work are not limited to quantum amplification, but apply more broadly to the optimization of quantum devices for any desired quantum task, including e.g. cloning, time reversal, and purification. At this level, they establish a tight relation between the demonstration of genuine quantum processing and the advantage of entanglement in the maximization of a suitable Bell-type correlation.

Let us start the derivation of our results. We begin from a general problem: finding the best physical process that approximates a desired transformation $\rho_x \mapsto \psi_x$, where $\{\rho_x\}_{x \in X}$ is a set of (possibly mixed) input states, given with prior probabilities $\{p_x\}_{x \in X}$, and $\{\psi_x = |\psi_x\rangle\langle\psi_x|\}_{x \in X}$ is a set of pure target states. Finding the best coherent-state amplifiers is a special case of this problem, corresponding to the input $\rho_\alpha = |\alpha\rangle\langle\alpha|$ and the output $|\psi_\alpha\rangle = |g\alpha\rangle$, where $g > 1$ is the desired gain. To approximate the transformation $\rho_x \mapsto \psi_x$, we will consider the most general deterministic process, described by a quantum channel (completely positive trace-preserving map) $C$. The performances of the channel will be ranked by the average fidelity $F = \sum_{x \in X} p_x \langle \psi_x | C(\rho_x) | \psi_x \rangle$. In addition to the deterministic processes we will also consider probabilistic ones, described by quantum operations (completely positive trace non-increasing maps). The average fidelity of a quantum operation $Q$, conditional on its occurrence, is given by $F' = \sum_{x \in X} p_x \langle \psi_x | Q(\rho_x) | \psi_x \rangle / (\sum_{x \in X} p_x \text{Tr}[Q(\rho_x)])$. The optimal fidelity, defined as the supremum of the fidelity over all possible deterministic (probabilistic) processes, will be denoted by $F^{\text{det}}$ ($F^{\text{prob}}$).

**Theorem 1** [32] [34] For deterministic processes, the optimal fidelity for the transformation $\rho_x \mapsto \psi_x$ is given by

$$F^{\text{det}} = \inf_{\sigma > 0, |\langle x | \sigma | x \rangle| = 1} \| A_\sigma \|_\infty$$

where $\| A_\sigma \|_\infty$ denotes the operator norm $\| A_\sigma \|_\infty := \sup_{\|\psi\| = 1}\langle \psi | A_\sigma | \psi \rangle$, and $T$ denotes the transpose.

For probabilistic processes, the optimal fidelity is given by

$$F^{\text{prob}} = \| A_\tau \|_\infty$$

where $\tau := \sum_{x \in X} p_x \rho_x$.

Theorem 1 is a powerful tool for the optimization of quantum devices: since every quantum state $\sigma > 0$ gives an upper bound on the fidelity, finding a channel that achieves any of these bounds means finding an optimal channel.

In addition to the performances of the best quantum processes, it is important to know the CFT for the transformation $\rho_x \mapsto \psi_x$. The CFT is the maximum fidelity that can be achieved with a classical, measure-and-prepare protocol, where the input state is measured with a positive operator-valued measure (POVM) $\{P_y\}_{y \in Y}$ and, conditionally on outcome $y$, a state $\rho'_y$ is prepared. In the deterministic case, the fidelity of the protocol is the fidelity of the measure-and-prepare channel $C(\rho) = \sum_{y \in Y} \text{Tr}[P_y \rho] \rho'_y$. In the probabilistic case, the POVM $\{P_y\}_{y \in Y}$ includes an outcome $y = \tau$, conditionally to which no output state is produced. The fidelity is then the fidelity of the measure-and-prepare quantum operation $\tilde{Q}(\rho) = \sum_{y \in Y, y \neq \tau} \text{Tr}[P_y \rho] \rho'_y$. In the following, the CFT will be denoted by $\tilde{F}^{\text{det}}$ ($\tilde{F}^{\text{prob}}$) in the deterministic (probabilistic) case.

**Theorem 2** [33] For deterministic protocols, the CFT for the transformation $\rho_x \mapsto \psi_x$ is given by

$$\tilde{F}^{\text{det}} = \inf_{\sigma > 0, |\langle x | \sigma | x \rangle| = 1} \| A_\sigma \|_\times$$

where $\| A_\sigma \|_\times$ denotes the injective cross norm $\| A_\sigma \|_\times := \sup_{\|\psi\| = 1} \langle \psi | A_\sigma | \psi \rangle$, and $|\langle x | \sigma | x \rangle| = 1$.

For probabilistic protocols, the CFT is given by

$$\tilde{F}^{\text{prob}} = \| A_\tau \|_\times$$

Remark: quantum-classical gap and Bell-type correlations. Note that the trace of the separable operator $A_\sigma$ with a quantum state is a Bell-type correlation. Remarkably, Eqs. (2) and (3) state that for probabilistic
processes the gap between the quantum fidelity and the CFT is equal to the gap between the maximum Bell correlation achievable with entangled states and the maximum Bell correlation achievable with separable states. This relation establishes a tight connection between the demonstration of genuine quantum processing and the violation of suitable Bell-type inequalities.

We are now ready to tackle the optimal design of quantum amplifiers and to find the corresponding CFT. To account for the prior information about the input, we introduce a probability distribution \( p(\alpha) \), normalized as \( \int d^2 \alpha p(\alpha) = 1 \). The most popular choice for \( p(\alpha) \), typically considered in the literature \[23\] [27] [31], is a Gaussian distribution with mean \( \alpha_0 \) and variance \( V = 1/\lambda \). The idealized “uniform prior” can be retrieved here in the special case of 1-to-2 cloning, this fact was noted by Cochrane, Ralph, and Dolińska \[13\], who assumed from the start cloning processes based on two-mode squeezing. Armed with Theorem \[4\], we are now in position to prove that no deterministic process can beat two-mode squeezing:

**Theorem 1** (Deterministic amplification) Two-mode squeezing is the best deterministic process for the amplification of Gaussian-distributed coherent states.

For probabilistic amplifiers, however, the situation is very different. Evaluating Eq. (2) we get \[33\]

\[
F^{\text{prob}}_{g,\lambda} = \begin{cases} 
\frac{\lambda + 1}{g^2}, & \lambda \leq g^2 - 1 \\
1, & \lambda > g^2 - 1.
\end{cases}
\]  

(7)

The difference with the deterministic case is dramatic: above the critical value \( \lambda^{c,\text{prob}} = g^2 - 1 \) probabilistic processes allow for noiseless amplification. Fidelity arbitrarily close to \( F^{\text{prob}}_{g,\lambda} \) can be reached as follows:

**Theorem 2** (Optimal design of probabilistic amplifiers \[33\]) The best probabilistic amplifier for Gaussian-distributed coherent states is

1. for \( \lambda \leq \lambda^{c,\text{det}} \), the two-mode squeezer \[5\] with squeezing parameter \( r = \cosh^{-1}[g/(\lambda + 1)] \)

2. for \( \lambda^{c,\text{det}} < \lambda \leq \lambda^{c,\text{prob}} \), a quantum operation \( Q_N(\rho) = Q_N \rho Q_N \) with \( Q_N \propto \sum_{n=0}^{N} [g^n|n\rangle\langle n| \), achieving fidelity \( F^{\text{prob}}_{g,\lambda} = (1 + \lambda)/g^2 \) exponentially fast in the limit \( N \to \infty \)

3. for \( \lambda > \lambda^{c,\text{prob}} \), a quantum operation \( Q_N(\rho) = Q_N \rho Q_N \) with \( Q_N \propto \sum_{n=0}^{N} g^n|n\rangle\langle n| \), achieving the fidelity \( F^{\text{prob}}_{g,\lambda} = 1 \) exponentially fast in the limit \( N \to \infty \).

Note that for \( \lambda > g - 1 \) the optimal quantum operations are non-Gaussian, whereas for \( \lambda = 0 \) (“uniform prior”) the optimal deterministic and probabilistic amplifiers coincide and are Gaussian. Noiseless amplification is only possible when the expected photon number is finite.

Suppose now that an experiment aims at demonstrating quantum amplification—or equivalently, cloning—of a coherent state. Thanks to Theorem \[2\], we can easily find the analytical expression of the CFT, also specifying the best measure-and-prepare channel. The result applies to both deterministic and probabilistic protocols, and, as an extra bonus, provides a concise derivation of the quantum benchmark for teleportation and storage of coherent states found by Hammerer, Wolf, Polzik, and Cirac \[23\], which is retrieved here in the special case of no amplification \( g = 1 \).
Theorem 5 (Benchmark for quantum amplifiers) [33]

The CFT for the amplification of Gaussian-distributed coherent states is given by

$$\tilde{F}_{g,\lambda} = \frac{1 + \lambda}{1 + \lambda + g^2}$$  \hspace{1cm} (8)

both for deterministic and probabilistic protocols. The above value is achieved by a heterodyne measurement $P(\hat{a}) \frac{d^2\hat{a}}{\pi} = |\hat{a}\rangle \langle \hat{a}| \frac{d^2\hat{a}}{\pi}$ followed by the conditional preparation of the coherent state $|\alpha_{g,\lambda}\rangle$.

Eqs. [6][25] and (8) represent good news for experimental demonstrations: they prove that genuine quantum amplification can be demonstrated for every finite value of the expected photon number. As an illustration, consider the demonstration of probabilistic amplification provided by Zavatta, Fiurášek and Bellini in Ref. [22]. In this case, the amplifier is designed to achieve gain $g = 2$. By Eq. (25), noiseless amplification requires at least $\lambda \geq 3$, which is actually a reasonable value in the experiment (choosing $\lambda = 3$ puts the maximum amplitude tested in the experiments $|\alpha_{\text{max}}|^2 \approx 1.0$, at three standard deviations from the mean photon number $\langle n \rangle = 1/3$, effectively cutting off the values $|\alpha| > 1$). For $\lambda = 3$, Eqs. [6] and (25) give $F_{g=2,\lambda=3}^{\text{squeez}} = 85\%$ and $\tilde{F}_{g=2,\lambda=3} = 50\%$ for the fidelity of the best deterministic amplifier and for the CFT, respectively [35]. The average of the experimental fidelities $F_{\text{exp}} \approx 0.99/0.91/0.67$, corresponding to the amplitudes $|\alpha| \approx 0.4/0.7/1.0$, gives a value that is well above the benchmark for genuine quantum processing, but also very close to the value that can be achieved by deterministic amplifiers. One should observe, however, that the small number of values of $|\alpha|$ probed in the experiment precludes an accurate data analysis, as the average over few values of $\alpha$ is very sensitive to statistical fluctuations. Our analysis suggest that, although the available data show a neat quantum advantage over measure-and-prepare strategies, further experimental investigations would be desirable to establish a statistically significant analysis of the advantage of probabilistic amplifiers. To guarantee a fair sampling, the ideal setup would be to test the amplifier on Gaussian-distributed coherent states generated randomly by a heterodyne measurement on one side of a two-mode squeezed state.

The classical limit of quantum amplifiers. For $\lambda \leq g - 1$, the gain between the quantum fidelity and the CFT is equal to the gap between entangled and separable states in the Bell correlation ($A_\eta$). The gap vanishes in the limit $g \to \infty$, and the fundamental reason is that an amplifier with infinite gain is classical, like a cloning device producing infinite clones [37][39]. This point is made very clear by our results: denoting by $C_{g,\lambda}$ and by $\tilde{C}_{g,\lambda}$ the optimal quantum amplifier and the optimal measure-and-prepare amplifier, for $\lambda \leq g - 1$ we have the remarkable relation [33]

$$\tilde{C}_{g,\lambda} = A_\eta \frac{g}{\sqrt{g^2 + (\lambda + 1)^2}} C \sqrt{\frac{g^2 + (\lambda + 1)^2}{\.lambda}}$$

where $A_\eta$ is the attenuation channel transforming the coherent state $|\alpha\rangle$ into $|\eta\rangle$, $\eta \leq 1$. In words, the best measure-and-prepare strategy with gain $g$ is equivalent to the best quantum strategy with gain $g' = \sqrt{g^2 + (\lambda + 1)^2}$, followed by an attenuation of $\eta = g/\sqrt{g^2 + (\lambda + 1)^2}$ that reduces the gain from $g'$ to $g$. When the desired gain is large compared to the prior information available ($g \gg \lambda$) we have $g' \approx g$ and $\eta \approx 1$, which imply $C_{g,\lambda} \approx \tilde{C}_{g,\lambda}$.

In conclusion, we established the ultimate quantum limits to the deterministic and probabilistic amplification of Gaussian-distributed coherent states, without making any assumption on the nature of the amplifier and without making the unrealistic assumption of uniform distribution over coherent states. For probabilistic amplifiers, we discovered the presence of a critical value of the expected photon number, below which noiseless amplification becomes possible. Furthermore, we provided the quantum benchmark that has to be surpassed in order to establish the successful experimental demonstration of a genuine quantum amplifier. Our results show an intriguing link between genuine quantum amplification and the maximization of a suitable Bell-type correlation, and, in addition, they guarantee that a successful demonstration is possible for any finite value of the expected photon number.

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* Electronic address: gchiribella@mail.tsinghua.edu.cn
\footnote{Electronic address: xiejiv09@mails.tsinghua.edu.cn}
\footnote{URL: \url{http://iitis.tsinghua.edu.cn}}

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Proof of Theorem 1: General expression for the optimal quantum fidelity

1. Deterministic case. For a generic quantum channel $\mathcal{C}$ and an arbitrary quantum state $\sigma > 0$, it is easy to prove the upper bound

$$F \leq \|A_\sigma\|_\infty$$

where

$$A_\sigma := \sum_{x \in X} p_x |\psi_x\rangle \langle \psi_x| \otimes \left(\sigma^{-\frac{1}{2}} \rho_x \sigma^{-\frac{1}{2}}\right)^T.$$

The proof runs as follows:

$$F = \sum_{x \in X} p_x \langle \psi_x| \mathcal{C} \left(\sigma^{\frac{1}{2}} \left(\sigma^{-\frac{1}{2}} \rho_x \sigma^{-\frac{1}{2}}\right) \left(\sigma^{\frac{1}{2}}\right)\right) |\psi_x\rangle$$

$$= \sum_{x \in X} p_x \text{Tr} \left( |\psi_x\rangle \langle \psi_x| \otimes \left(\sigma^{-\frac{1}{2}} \rho_x \sigma^{-\frac{1}{2}}\right)^T \Phi_{\sigma,\mathcal{C}} \right)$$

$$= \text{Tr}[A_\sigma \Sigma_\mathcal{C}],$$

where $\Phi_{\sigma,\mathcal{C}}$ is the quantum state defined by

$$\Phi_{\sigma,\mathcal{C}} := (\mathcal{C} \otimes I)(|\sigma^{\frac{1}{2}}\rangle \langle |\sigma^{\frac{1}{2}}|) |m\rangle |n\rangle := \sum_{m,n} \langle m| \sigma^{\frac{1}{2}} |n\rangle |m\rangle |n\rangle.$$
The bound of Eq. (9) then follows from the inequality $\|\text{Tr}[A_\sigma \Phi_\sigma, C]\| \leq \|A_\sigma\|_\infty$, valid for every quantum state $\Phi_\sigma, C$. Hence, we conclude that the maximum of the fidelity over all quantum channels, denoted by $F^{\text{det}}$ satisfies

$$F^{\text{det}} \leq \inf_{\sigma > 0, \text{Tr}[\sigma] = 1} \|A_\sigma\|_\infty.$$  \hfill (10)

On the other hand, using the duality of semidefinite programming, one can show that the bound can be achieved. In the case where the input and output states live in finite-dimensional Hilbert spaces, the proof was given by König, Renner, and Schaffner in Ref. [32]. For completeness, we present it here in the language of our paper. Without loss of generality, we assume that the average input state $\tau := \sum_{x \in X} p_x \rho_x$ is strictly positive [the latter condition can be imposed by restricting the action of the channel $C$ to the support of $\tau$]. In this case, the fidelity can be written as

$$F = \text{Tr}[AC], \quad A := \sum_{x \in X} p_x |\psi_x\rangle\langle \psi_x| \otimes \rho_x^T, \quad C := (C \otimes I)(|I_{in}\rangle\langle I_{in}|), \quad |I_{in}\rangle := \sum_{n=1}^{\dim(\mathcal{H}_{in})} |n\rangle|n\rangle.$$  \hfill (11)

where $C$, the Choi operator of the channel $C$, satisfies the normalization condition

$$\text{Tr}_{\text{out}}[C] = I_{in},$$  \hfill (12)

$\text{Tr}_{\text{out}}$ and $I_{in}$ denoting the partial trace on the output Hilbert space and the identity on the input Hilbert space, respectively. Since every positive operator $C \geq 0$ satisfying Eq. (23) is the Choi operator of some channel, the maximum fidelity can be computed by the semidefinite program

$$F^{\text{det}} = \max_{C \geq 0, \text{Tr}[C] = I_{in}} \text{Tr}[CA],$$

whose value, by strong duality, is equal to

$$F^{\text{det}} = \min_{\Lambda \geq 0, I_{out} \otimes \Lambda \geq A} \text{Tr}[\Lambda].$$

Note that, actually, $\Lambda$ must be strictly positive, because the average state $\tau$ is strictly positive. Defining $\Lambda = t\sigma^T$, where $t = \text{Tr}[\Lambda]$ and $\sigma > 0$ is a density matrix, we then have

$$F^{\text{det}} = \min_{t \geq 0} \left\{ t \geq 0 \mid \exists \sigma > 0, \text{Tr}[\sigma] = 1, t(I_{out} \otimes I_{in}) \geq \left(I_{out} \otimes \sigma^{-\frac{1}{2}}\right)^T A \left(I_{out} \otimes \sigma^{-\frac{1}{2}}\right)^T \right\}$$

$$= \min_{\sigma > 0, \text{Tr}[\sigma] = 1} \left\| \left(I_{out} \otimes \sigma^{-\frac{1}{2}}\right)^T A \left(I_{out} \otimes \sigma^{-\frac{1}{2}}\right)^T \right\|_\infty$$

$$= \min_{\sigma > 0, \text{Tr}[\sigma] = 1} \|A_\sigma\|_\infty.$$  \hfill (13)

Hence, in finite dimensions we have a guarantee that the bound of Eq. (10) can be achieved by a quantum channel. When the input and output Hilbert spaces are infinite dimensional, we show that, in most relevant cases, one can reduce the problem to the finite dimensional case by truncating the dimension. The technical details of the truncation are provided in the last section of this supplemental material.

2. **Probabilistic case** [32]. For an arbitrary quantum operation $Q$, the fidelity is given by

$$F = \frac{\sum_{x \in X} p_x \langle \psi_x|Q(\rho_x)|\psi_x\rangle}{\text{Tr}[Q(\tau)]}.$$  

Following the same proof as in the deterministic case, we get

$$F = \frac{\text{Tr}[\Phi_\tau Q A_\tau]}{\text{Tr}[Q(\tau)]} = \frac{\text{Tr}[\Phi_\tau Q A_\tau]}{\text{Tr}[\Phi_\tau Q]} = \text{Tr}[\Sigma_\tau Q A_\tau] \quad \Sigma_{\tau, Q} := \Phi_\tau Q / \text{Tr}[\Phi_\tau Q].$$  \hfill (14)
Hence, we have the upper bound $F \leq \|A_{\tau}\|_{\infty}$. In finite dimensions, the bound can be achieved by taking the eigenvector of $A_{\tau}$ with maximum eigenvalue, denoted by $|\Psi\rangle$, and using the state

$$|	ilde{\Psi}\rangle := \frac{(I_{\text{out}} \otimes \tau^{-\frac{1}{2}})^T |\Psi\rangle}{\| (I_{\text{out}} \otimes \tau^{-\frac{1}{2}})^T |\Psi\rangle \|}$$

as the resource state in a probabilistic teleportation protocol. In infinite dimensions, the optimal fidelity is achieved in the limit, using approximate teleportation. Note that $A_{\tau}$ itself may have only approximate eigenvectors.

Proof of theorem 2: general expression for the CFT

1. Deterministic case. For a generic measure-and-prepare channel $\tilde{C}$ and for every state $\sigma > 1$, it is easy to prove the upper bound

$$\tilde{F} \leq \|A_{\sigma}\|_\times$$

$$A_{\sigma} := \sum_{x \in X} p_x |\psi_x\rangle \langle \psi_x| \otimes \left( \sigma^{-\frac{1}{2}} \rho_x \sigma^{-\frac{1}{2}} \right)^T . \quad (15)$$

The proof is the same as the proof of Lemma 1, with the only difference that now the state $\Phi_{\sigma, \tilde{C}} = (\tilde{C} \otimes I)(|\sigma\rangle \langle \sigma|)$ is separable. By definition of the injective cross norm, we have $\text{Tr}[A_{\sigma} \Phi_{\sigma, \tilde{C}}] \leq \|A_{\sigma}\|_\times$, for every separable quantum state $\Phi_{\sigma, \tilde{C}}$. Hence, the CFT will be bounded as

$$\tilde{F}_{\text{det}}^\ast \leq \inf_{\sigma > 0, \text{Tr}[\sigma] = 1} \|A_{\sigma}\|_\times . \quad (16)$$

Like in the proof of Lemma 1, we can use the duality of semidefinite programming to show that the upper bound is actually an equality. Again, we first consider first the case where the input and output Hilbert spaces are finite-dimensional Hilbert spaces, and the average input state

$$\tau := \sum_{x \in X} p_x \rho_x$$

is strictly positive. In this case, the fidelity can be written as

$$F = \text{Tr}[A \tilde{C}] \quad A := \sum_{x \in X} p_x |\psi_x\rangle \langle \psi_x| \otimes \rho_x^T, \quad \tilde{C} := (\tilde{C} \otimes I)(|I_{\text{in}}\rangle \langle I_{\text{in}}|), \quad (17)$$

where the Choi operator $\tilde{C}$ is now separable. To turn the separability condition into a semidefinite program, we now use the $n$-extendability condition of Ref. [40], stating that $\tilde{C}$ is separable if and only if, for every $n \in \mathbb{N}$ there exists an $n$-symmetric extension, that is, an operator $\tilde{C}_n$ on $\mathcal{H}_{\text{out}}^\otimes n \otimes \mathcal{H}_{\text{in}}$ such that

(a) $\tilde{C}_n$ extends $\tilde{C}$, i.e.

$$\text{Tr}_{n-1}[\tilde{C}_n] = \tilde{C}$$

($\text{Tr}_k$ denoting the partial trace over the first $k$ output spaces), and

(b) $\tilde{C}_n$ is invariant under permutation of the outputs.

Permutation invariance can be expressed as $(\Pi_n \otimes I_{\text{in}})(\tilde{C}_n) = C_n$, where $\Pi_n$ is the permutation-twirling

$$\Pi_n(\rho) = \frac{1}{n!} \sum_{\pi \in S_n} U_{\pi} \rho U_{\pi}^\dagger,$$
$U_r$ being the unitary operator that implements the permutation $\pi \in S_n$ of the $n$ output spaces. We can then express the maximum fidelity over all measure-and-prepare channels as

$$
\tilde{F}^{\text{det}} = \inf_{n \in \mathbb{N}} \max \{ \tilde{C}_n \geq 0, (\Pi_n \otimes \mathbb{I}_m)(\tilde{C}_n) = \tilde{C}_n, \text{Tr}_n[\tilde{C}_n] = I_n \} \\text{Tr}\{\tilde{C}_n(I_{n-1} \otimes A)\}
$$

$$
= \inf_{n \in \mathbb{N}} \max \{ C_n \geq 0, \text{Tr}_n[C_n] = I_n \} \\text{Tr}\{C_n(\Pi_n \otimes \mathbb{I}_m)(I_{n-1} \otimes A)\}
$$

Using strong duality for the maximization over $C_n$, we obtain

$$
\tilde{F}^{\text{det}} = \inf_{n \in \mathbb{N}} \min \{ \Lambda_n \geq 0, I_n \otimes \Lambda_n \geq (\Pi_n \otimes \mathbb{I}_m)(I_{n-1} \otimes A) \} \\text{Tr}[\Lambda_n].
$$

Now, since $\tau$ is strictly positive, also $\Lambda_n$ must be strictly positive. Writing $\Lambda_n = t_n \sigma$, with $t_n = \text{Tr}[\Lambda_n]$ and $\sigma > 0$, we have

$$
\tilde{F}^{\text{det}} = \inf_{n \in \mathbb{N}} \min\{ t_n | \exists \sigma > 0, \text{Tr}[\sigma] = 1, t(I_n \otimes I_n) \geq (\Pi_n \otimes \mathbb{I}_m)(I_{n-1} \otimes A) \}
$$

$$
= \inf_{n \in \mathbb{N}} \min\{ \sigma > 0, \text{Tr}[\sigma] = 1 \} \| (\Pi_n \otimes \mathbb{I}_m)(I_{n-1} \otimes A) \| \infty
$$

$$
= \inf_{n \in \mathbb{N}} \min\{ \rho_n > 0, \text{Tr}[\rho_n] = 1 \} \max\{ \textrm{Tr}\{\rho_n(\Pi_n \otimes \mathbb{I}_m)(I_{n-1} \otimes A)\}\} = \inf_{n \in \mathbb{N}} \rho_n > 0, \text{Tr}[\rho_n] = 1, (\Pi_n \otimes \mathbb{I}_m)(\rho_n) = \rho_n
$$

$$
= \min\{ \sigma > 0, \text{Tr}[\sigma] = 1 \} \max\{ \textrm{Tr}[\rho A]\} = \equiv \| A_x \|_\infty.
$$

The validity of the formula $\tilde{F}^{\text{det}} = \inf_{\sigma > 0, \text{Tr}[\sigma] = 1} \| A_x \|_\infty$ in the case where the input and output Hilbert spaces are infinite dimensional can be proved using the truncation argument provided in the end of this supplementary material.

2. Probabilistic case. Inserting a measure-and-prepare quantum operation $\bar{Q}$ into Eq. (14), we get $\tilde{F} = \text{Tr}[A_x \Sigma_{r, \bar{Q}}]$, where $\Sigma_{r, \bar{Q}}$ is a separable state. This implies the bound $\tilde{F}^{\text{prob}} \leq \| A_x \|_\infty$. In finite dimensions, a quantum operation that achieves the bound can be obtained by taking two unit vectors $|\psi\rangle \in \mathcal{H}_{\text{out}}$ and $|\varphi\rangle \in \mathcal{H}_{\text{in}}$ such that $\| A_x \|_\infty = \langle \psi | \langle \varphi | A_x | \psi \rangle | \varphi \rangle$, and by defining $\bar{Q}(\rho) \propto |\psi\rangle \langle \psi | \langle \varphi | | \varphi \rangle^{-\frac{1}{2}} \rho | \varphi \rangle^{-\frac{1}{2}} | \varphi \rangle$. In infinite dimensions, one may have to truncate the vector $\tau^{-\frac{1}{2}} | \varphi \rangle$ to a finite dimensional subspace to make it normalizable. Letting the dimension of the subspace grow, one obtains a sequence of quantum operations with fidelity converging to $\tilde{F}^{\text{prob}}$.

---

**Proof of Eq. (6): The performances of two-mode squeezing**

A parametric amplifier $C_r(\rho) := \text{Tr}_B[\rho e^{(a^b \dagger - ab)}(\rho \otimes |0\rangle \langle 0|) e^{-(a^b \dagger - ab)}]$ satisfies the covariance property

$$
C_r(D(\alpha)\rho D(\alpha)^\dagger) = D(\alpha \cosh r)C_r(\rho)D(\alpha \cosh r) \quad \forall \alpha \in \mathbb{C},
$$

for every trace-class operator $\rho \in \mathcal{T}(\mathcal{H})$. Moreover, we have

$$
C_r(|0\rangle \langle 0|) = (1 - x) \sum_{n=0}^{\infty} x^n |n\rangle \langle n| := \rho_x \quad x = \tanh^2 r. \quad (18)
$$

Combining these two facts, the amplification fidelity of the channel $C_r$ is given by

$$
F_{g,\lambda}^r = \int_{\alpha \in \mathbb{C}} d^2\alpha \frac{\lambda e^{-\lambda |\alpha|^2}}{\cosh^2 r} \langle (g - \cosh r)\alpha | \rho_x | (g - \cosh r)\alpha \rangle
$$

$$
= \int_{\alpha \in \mathbb{C}} d^2\alpha \frac{\lambda e^{-\lambda |\alpha|^2}}{\cosh^2 r} \frac{e^{-(g - \cosh r)^2 |\alpha|^2}}{\cosh^2 r}
$$

$$
= \frac{\lambda \cosh^2 r + (g - \cosh r)^2}{\lambda \cosh^2 r + (g - \cosh r)^2}.
$$
The maximum of the function $F_{g,\lambda}^r$ is achieved by $\cosh r = g/(\lambda + 1)$ when $g \geq \lambda + 1$ and by $\cosh r = 1$ otherwise, thus giving

\[ F_{g,\lambda}^{opt} = \begin{cases} 
\frac{\lambda + 1}{g^2}, & \lambda \leq g - 1 \\
\frac{\lambda}{\lambda + (g - 1)^2}, & \lambda > g - 1.
\end{cases} \quad (19) \]

which is what we wanted to prove.

**Proof of Theorem 3: Optimal design of deterministic amplifiers**

**Proof.** We show that the performances of two-mode squeezing, given by Eq. (19), are the best among the performances of all quantum channels. To this purpose, our strategy is to find a state $\sigma$ such that the upper bound provided by Eq. (9), matches the lower bound of Eq. (19).

As an ansatz, we assume $\sigma$ to be a thermal state, of the form

\[ \sigma_x := (1 - x) \sum_{n=0}^{\infty} x^n |n\rangle\langle n|, \quad (20) \]

so that the operator $A_{g,\lambda,\sigma}$ becomes

\[ A_{g,\lambda,x} := \frac{\lambda}{1 - x} \int \frac{d^2\alpha}{\pi} e^{-(\lambda + 1 - \frac{1}{x})|\alpha|^2} \langle g\alpha | g\alpha \rangle \otimes \left| \frac{\bar{\alpha}}{\sqrt{x}} \right\rangle \langle \frac{\bar{\alpha}}{\sqrt{x}} \right|, \quad (21) \]

The operator norm of $A_{g,\lambda,x}$ can be computed using the relation $||A_{g,\lambda,x}||_{\infty} = \lim_{p \to \infty} (\text{Tr}|A_{g,\lambda,x}|^p)^{\frac{1}{p}}$. For each fixed $p$, the calculation consists only of Gaussian integrals: By definition, we have

\[ \text{Tr}[A_{g,\lambda,x}^p] = \left( \frac{\lambda}{1 - x} \right)^p \int \frac{d^2\alpha}{\pi^p} \prod_{j=1}^{p} \left( e^{-(\lambda + 1 - \frac{1}{x})|\alpha_j|^2} \langle g\alpha_j | g\alpha_{(j+1) \mod p} \rangle \left( \frac{\bar{\alpha_j}}{\sqrt{x}} \right) \langle \frac{\bar{\alpha_j}}{\sqrt{x}} \right) \right), \]

where $\bar{\alpha}$ is the complex vector $\bar{\alpha} := (\alpha_1, \ldots, \alpha_p)^T \in \mathbb{C}^p$ and $d^2\alpha := \prod_{j=1}^{p} d^2\alpha_j$. Now, using the relation $\langle \alpha | \beta \rangle = e^{-||\bar{\alpha}||^2 - ||\bar{\beta}||^2 + 2\bar{\alpha}S\bar{\beta}}$ we obtain

\[ \text{Tr}[A_{g,\lambda,x}^p] = \left( \frac{\lambda}{1 - x} \right)^p \int \frac{d^2\bar{\alpha}}{\pi^p} e^{-(\lambda + 1 - \frac{1}{x})||\bar{\alpha}||^2} e^{g^2(-||\bar{\alpha}||^2 + \bar{\alpha}S\bar{\alpha})} e^{\frac{1}{x}(-||\bar{\alpha}||^2 + \bar{\alpha}S\bar{\alpha})}, \]

where $S$ is the shift matrix defined by $S_{jk} := \delta_{k,(j+1) \mod p}$. Elementary algebra then gives

\[ \text{Tr}[A_{g,\lambda,x}^p] = \left( \frac{\lambda}{1 - x} \right)^p \int \frac{d^2\bar{\alpha}}{\pi^p} e^{-(\lambda + 1 + g^2)||\bar{\alpha}||^2} e^{g^2\bar{\alpha}S\bar{\alpha}} e^{\frac{1}{x}\bar{\alpha}S\bar{\alpha}}, \]

where

\[ \Gamma_p = \begin{pmatrix}
\lambda + 1 + g^2 & -g^2 & 0 & \cdots & 0 & -\frac{1}{x} \\
-\frac{1}{x} & \lambda + 1 + g^2 & -g^2 & \cdots & 0 & 0 \\
0 & -\frac{1}{x} & \lambda + 1 + g^2 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & \lambda + 1 + g^2 & -g^2 \\
-g^2 & 0 & 0 & \cdots & -\frac{1}{x} & \lambda + 1 + g^2
\end{pmatrix} \]

Now, $\Gamma_p$ is a circulant matrix, and, therefore, can be unitarily diagonalized using the discrete Fourier transform. Hence, the Gaussian integral in Eq. (22) can be computed with a simple change of variables, giving

\[ \text{Tr}[A_{g,\lambda,x}^p] = \frac{\lambda^p}{(1 - x)^p \det \Gamma_p}. \quad (23) \]
Taking the $p$-th root and the limit $p \to \infty$ we finally obtain

$$\| A_{g, \lambda, x} \|_\infty = \frac{\lambda}{(1-x) \lim_{p \to \infty} (\det \Gamma_p)^{\frac{1}{p}}}.$$ 

Now, the eigenvalues a circulant matrix are easily found by Fourier transforming its entries [41]. In our specific case, the eigenvalues of $\Gamma_p$ are $\gamma_{p, n} = a - b \omega_p^n - c \omega_p^{-n}$, with $\omega_p := \exp(2\pi i / p)$ and $n = 0, \ldots, p - 1$. Hence, we have

$$\lim_{p \to \infty} \ln (\det \Gamma_p)^{\frac{1}{p}} = \lim_{p \to \infty} -\frac{1}{p} \sum_{n=0}^{p-1} \ln(a - b \omega_p^n - c \omega_p^{-n})$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} \ln(a - be^{i\theta} - ce^{-i\theta}).$$

For $x \geq 1/(\lambda + 1)$ we can decompose $a - be^{i\theta} - ce^{-i\theta} = b(e^{i\theta} - y_+)(y_+ - e^{-i\theta} - 1)$ with $y_+ = \frac{\lambda + g^2 + 1 + \sqrt{(\lambda + g^2 + 1)^2 - 4g^2/x}}{2g^2}$, we finally obtain

$$\lim_{p \to \infty} \ln (\det A_{g, \lambda, x}^p)^{\frac{1}{p}} = \int_0^{2\pi} \frac{d\theta}{2\pi} \ln(b(y_+ - e^{i\theta})) + \ln[1 - y_+ e^{-i\theta}]$$

$$= \ln(b y_+),$$

which, inserted in Eq. (23) gives

$$\| A_{g, \lambda, x} \|_\infty = \frac{2\lambda}{(1-x)(\lambda + g^2 + 1 + \sqrt{(\lambda + g^2 + 1)^2 - 4g^2/x}).}$$

Finally, we separate the two cases $\lambda > g - 1$ and $\lambda \leq g - 1$. For $\lambda > g - 1$, we choose $x = \frac{g}{\lambda + g^2 + 1}$ and obtain $\| A_{g, \lambda, x} \|_\infty = \frac{\lambda}{\lambda + g^2 + 1}$, matching the lower bound provided by (trivial) two-mode squeezing [Eq. (19)]. For $\lambda \leq g - 1$, we choose $x = 1/(\lambda + 1)$ and obtain $\| A_{g, \lambda, x} \|_\infty = (\lambda + 1)/g^2$, again, matching the lower bound provided by two-mode squeezing [Eq. (19)].

**Proof of Eq. (7): fidelity of optimal probabilistic amplifiers**

In the special case of Gaussian prior $p_\alpha(\alpha) = \lambda e^{-\lambda |\alpha|^2}$ and for coherent input states $\rho_\alpha = |\alpha\rangle \langle \alpha|$, the average state $\tau$ is the thermal state $\sigma_x = (1-x) \sum_{n=0}^{\infty} x^n |n\rangle \langle n|$ for $x = 1/(\lambda + 1)$. Then, using Eq. (24) for $x = 1/(\lambda + 1)$ we get

$$F_{g, \lambda}^{\text{prob}} \leq \frac{2(\lambda + 1)}{\lambda + g^2 + 1 + |\lambda + 1 - g^2|},$$

giving the bound $F_{g, \lambda}^{\text{prob}} \leq (\lambda + 1)/g^2$ for $\lambda \leq g^2 - 1$ and $F_{g, \lambda}^{\text{prob}} \leq 1$ for $\lambda > g^2 - 1$.

**Proof of Theorem 4: Optimal Design of Probabilistic Amplifiers**

To prove the theorem, we exhibit suitable quantum operations that reach the fidelity in Eq. (25).

1) Case $\lambda > g^2 - 1$. For the quantum operation $Q_N(\rho) = Q_N \rho Q_N^\dagger$ with $Q_N \propto \sum_{n=0}^{N-1} g^n |n\rangle \langle n|$, the fidelity is given by

$$F_{g, \lambda, N} = \frac{\int \frac{d^2\alpha}{\pi} p_\alpha(\alpha) |\langle g\alpha|Q_N|\alpha\rangle|^2}{\int \frac{d^2\beta}{\pi} p_\beta(\beta) |\langle g\beta|Q_N|\beta\rangle|^2} = \frac{\int \frac{d^2\alpha}{\pi} e^{-(\lambda + 1 - g^2)|\alpha|^2} |\langle g\alpha|P_N|g\alpha\rangle|^2}{\int \frac{d^2\beta}{\pi} e^{-(\lambda + 1 - g^2)|\beta|^2} |\langle g\beta|P_N|g\beta\rangle|^2}$$

$$= \frac{\int \frac{d^2\alpha}{\pi} (\lambda + 1 - g^2) e^{-(\lambda + 1 - g^2)|\alpha|^2} [1 - 2g\alpha(I - P_N)g\alpha]}{P_N := \sum_{n=0}^{N} |n\rangle \langle n|}$$

$$= 1 - 2 \left( \frac{g^2}{\lambda + 1} \right)^{N+1},$$

and

where $\mathcal{H}$ is the Hilbert space, $x$ is the thermal state $\tau$,

$$\langle g\alpha|Q_N|\alpha\rangle = \sum_{n=0}^{N-1} g^n \langle \alpha|n\rangle \langle n|\alpha\rangle = \sum_{n=0}^{N-1} g^n \langle \alpha|\alpha\rangle$$

$$= \sum_{n=0}^{N-1} g^n \langle \alpha|\alpha\rangle = \sum_{n=0}^{N-1} g^n = \frac{g^N}{1 - g},$$

and

$$P_N := \sum_{n=0}^{N} |n\rangle \langle n| = \mathbf{1} - P_N = \mathbf{1} - \sum_{n=0}^{N} |n\rangle \langle n|.$$
which converges to 1 exponentially fast as $N$ increases.

2) Case $g - 1 < \lambda \leq g^2 - 1$. For the quantum operation $Q_N(\rho) = Q_N \rho Q_N^\dagger$ with $Q_N \propto \sum_{n=0}^{N} x^n |n\rangle \langle n|$, the fidelity is given by

$$F_{g,\lambda,N} = \frac{\int \frac{d^2 \alpha}{\pi} p_{\lambda}(\alpha) |\langle ga|Q_N|\alpha\rangle|^2}{\int \frac{d^2 \beta}{\pi} p_{\lambda}(\beta) |\langle\beta|Q_N^\dagger|\beta\rangle|^2}$$

$$= \frac{\int \frac{d^2 \alpha}{\pi} e^{-(\lambda+1-x^2)|\alpha|^2} |\langle ga|P_N|x\alpha\rangle|^2}{\int \frac{d^2 \beta}{\pi} e^{-(\lambda+1-x^2)|\beta|^2} |\langle x\beta|P_N|x\beta\rangle|^2}$$

$$\geq \frac{1}{\lambda + 1 - x^2} \int d^2 \alpha \left( \frac{\lambda + 1 - x^2}{\lambda + 1 - x^2} - 2 \sqrt{\mathbb{E}(|\langle ga|(I-P_N)|\alpha\rangle|^2)} \right)$$

$$\geq \frac{\lambda + 1 - x^2}{\lambda + 1 - x^2 + (g-x)^2} - 2 \sqrt{\mathbb{E}(|\langle ga|(I-P_N)|\alpha\rangle|^2)}$$

where $\mathbb{E}(f_\alpha)$ denotes the expectation value of $f_\alpha$ over the Gaussian distribution $p_{\lambda+1-x^2}(\alpha) = (\lambda+1-x^2)e^{-(\lambda+1-x^2)|\alpha|^2}$.

Now, it is easy to obtain

$$\mathbb{E}(|\langle ga|(I-P_N)|\alpha\rangle|^2) = \left( \frac{g^2}{g^2 + 1 - x^2} \right)^{N+1}$$

$$\mathbb{E}(|\langle x\alpha|(I-P_N)|\alpha\rangle|^2) = \left( \frac{x^2}{\lambda + 1} \right)^{N+1}$$

Hence, for $x^2 < \lambda + 1$ the fidelity converges to $\frac{\lambda+1-x^2}{\lambda+1-x^2+g}$, exponentially fast as $N$ increases. If $\lambda < g^2 - 1$, the condition $x^2 < \lambda + 1$ is satisfied by choosing $x = (\lambda + 1)/g$, which gives fidelity $(\lambda + 1)/g^2$ in the limit $N \to \infty$. If $\lambda = g^2 - 1$, the condition $x^2 < \lambda + 1$ is satisfied by choosing $x = g - \epsilon$, which gives fidelity $1 - O(\epsilon^2)$ in the limit $N \to \infty$.

Case 3) $\lambda \leq g - 1$. Already treated in the deterministic case: a two-mode squeezer is optimal here.

**Proof of theorem 5: Benchmark for quantum amplifiers**

**Proof.** It is immediate to check that a heterodyne measurement followed by re-preparation of the state $|\frac{g\hat{a}}{1+\lambda}\rangle$, corresponding to the measure-and-prepare channel

$$\tilde{C}(\rho) = \int \frac{d^2 \alpha}{\pi} \langle \hat{a}|\rho|\hat{a}\rangle \left| \frac{g\hat{a}}{1+\lambda} \right\rangle \left\langle \frac{g\hat{a}}{1+\lambda} \right|$$

achieves the fidelity $F_{g,\lambda} = \frac{\lambda+1-x^2}{\lambda+1-x^2+g}$.

We now prove that no measure-and-prepare channel can do better, both in the deterministic and in the nondeterministic case. Let us start from the deterministic case. Here we use Eq. (15) and the fact that $\|A_{g,\nu,\sigma}\|_x = \|A_{g,\nu,\sigma}\|_x$, $T_2$ denoting the transposition on the second Hilbert space. For the Gaussian distribution $p_\lambda(\alpha) = \lambda e^{-\lambda|\alpha|^2}$, we choose $\sigma$ equal to $\tau$, the average state of the source, given by $\tau = (1-x) \sum_{n=0}^{\infty} x^n |n\rangle \langle n|$, $x = 1/(1+\lambda)$. Denoting the corresponding operator by $A_{g,\lambda,\tau}$, we have

$$A_{g,\lambda,\tau} = \left( \frac{\lambda}{1-x} \right) \int \frac{d^2 \alpha}{\pi} e^{-(\lambda+1-x^2)|\alpha|^2} |\alpha\rangle \langle \alpha| \otimes \left| \frac{\alpha}{\sqrt{\tau}} \right\rangle \left\langle \frac{\alpha}{\sqrt{\tau}} \right|$$

$$= \left( \frac{\lambda}{1-x} \right) \int \frac{d^2 \alpha}{\pi} e^{-(\lambda+1-x^2)|\alpha|^2} V_\theta^\dagger \left( |\sqrt{g^2 + x^2-1}\rangle \langle \sqrt{g^2 + x^2-1}| \otimes |0\rangle \langle 0| \right) V_\theta,$$

where $V_\theta = e^{\theta(ab^\dagger - a^\dagger b)}$ is a beamsplitter operator with $\theta = \tan^{-1}(g\sqrt{x})$. The calculation is particularly easy for $x = 1/(1+\lambda)$, where we have

$$A_{g,\lambda,\tau} = \left( \frac{\lambda}{1-x} \right) \left( \frac{\lambda + 1}{g^2 + \lambda + 1} \right) \int \frac{d^2 \alpha}{\pi} V_\theta^\dagger (|\alpha\rangle \langle \alpha| \otimes |0\rangle \langle 0|) V_\theta$$

$$= \left( \frac{\lambda + 1}{g^2 + \lambda + 1} \right) V_\theta^\dagger (I \otimes |0\rangle \langle 0|) V_\theta.$$
Now, we have \( \|A_{g,\lambda,\sigma}^n\|_\infty = \frac{\lambda+1}{g^2+\lambda+1} = (0|0)(A_{g,\lambda,\sigma}^T|0) \). By definition of the cross norm and of the operator norm, this implies

\[
\|A_{g,\lambda,\sigma}\|_x = \|A_{g,\lambda,\sigma}^T\|_x = \|A_{g,\lambda,\sigma}\|_\infty = \frac{\lambda+1}{g^2+\lambda+1}.
\]

Using Eq. (15) we conclude that every measure-and-prepare channel \( \tilde{C} \) has fidelity \( \tilde{F}_{g,\lambda} \leq (\lambda + 1)/(g^2 + \lambda + 1) \). This proves that the heterodyne measure-and-prepare channel of Eq. (26) is optimal among all measure-and-prepare channels.

It remains to prove that the heterodyne channel is optimal also among the probabilistic measure-and-prepare protocols, described by quantum operations of the form \( \tilde{Q}(\rho) = \sum_{j \in Y} \text{Tr}[P_j \rho] \), with \( P_j \geq 0 \)\( \forall j \in Y \) and \( \sum_{j \in Y} P_j \leq 1 \). In this case, the fidelity is given by

\[
F_{\text{prob}}^{g,p} = \frac{\int d^2 \alpha p(\alpha) \langle g\alpha|Q(\rho_\alpha)|g\alpha\rangle}{\text{Tr}[\tilde{Q}(\tau)]}
\]

where \( \tau \) is the average state of the source. Using Eq. (14) we then get \( F_{\text{prob}}^{g,p} \leq \text{Tr}[A_{g,p,\tau} \Sigma] \), where \( \Sigma \) is the separable quantum state \( \Sigma := \Phi_g \tilde{Q} / \tilde{\Phi}_g \tilde{Q} \). Hence, we obtain the bound \( F_{\text{prob}}^{g,p} \leq \|A_{g,p,\tau}\|_\infty \). Now, in the case of a Gaussian distribution we already showed in the first part of the proof that \( \|A_{g,p,\tau}\|_\infty = (\lambda + 1)/(g^2 + \lambda + 1) \). This proves that nondeterministic measure-and-prepare protocols have to satisfy the bound \( F_{\text{prob}}^{g,p} \leq (\lambda + 1)/(g^2 + \lambda + 1) \). Hence, the heterodyne measure-and-prepare channel of Eq. (26) is optimal also among non-deterministic protocols. ■

Relation between the optimal quantum channel and the optimal measure-and-prepare channel for \( \lambda \leq g - 1 \)

Consider the measure-and-prepare channel \( \tilde{C}_r \) defined as \( \tilde{C}_r(\rho) := \int_{\alpha \in \mathbb{C}} \frac{d^2 \alpha}{\pi} |(\alpha|0)^2 \rho |(\alpha \cosh\tau)^r (\alpha \cosh\tau)\rangle \langle \alpha | \rangle \). Like the channel \( C_r \), the measure-and-prepare channel \( \tilde{C}_r \) satisfies the covariance property

\[
C_r(D(\alpha) \rho D(\alpha)^\dagger) = D(\alpha \cosh\tau) C_r(\rho) D(\alpha \cosh\tau)^\dagger \quad \forall \alpha \in \mathbb{C},
\]

for every trace-class operator \( \rho \in T(H) \). Moreover, we have \( \tilde{C}_r(\langle 0|0\rangle) = (1 - y) \sum_{n=0}^\infty y^n |n\rangle \langle n| \), with \( y = \cosh^2 r / \cosh^2 r + 1 \).

Recalling Eq. (18) we then have \( \tilde{C}_r(\langle 0|0\rangle) = A_{g,\lambda,\sigma}^{\cosh r / \cosh r + 1} C_r(\langle 0|0\rangle) \), where \( A_{g,\lambda,\sigma}^{\cosh r / \cosh r + 1} \) is an attenuation channel with attenuation parameter \( \cosh r / \sqrt{\cosh r^2 + 1} \), and \( r := \text{tanh}^{-1} \sqrt{\cosh^2 r / \cosh^2 r + 1} \). Using the covariance properties of \( \tilde{C}_r, C_r \) and \( A_{g,\lambda,\sigma}^{\cosh r / \cosh r + 1} \) we then obtain

\[
\tilde{C}_r(\alpha \langle \alpha |) = A_g C_r(\langle \alpha | \rangle \alpha \rangle) \quad \forall \alpha \in \mathbb{C},
\]

which in turn implies

\[
\tilde{C}_r = A_{g,\lambda}^{\cosh r / \cosh r + 1} C_r.
\]

Since the optimal quantum and classical channels are given by \( C_{g,\lambda} = C_{g,\lambda}^{-1}[g/(\lambda + 1)] \) and \( \tilde{C}_{g,\lambda} = \tilde{C}_{g,\lambda}^{-1}[g/(\lambda + 1)] \) we have proven the relation

\[
\tilde{C}_{g,\lambda} = A_g^{g / \sqrt{g^2 + (\lambda + 1)^2}} C_{g,\lambda}^{\sqrt{g^2 + (\lambda + 1)^2},\lambda}.
\]

Coping with infinite dimensions: the truncation argument

Here we show how to extend the validity of theorems 1 and 2 to the case when the input and/or output Hilbert spaces are infinite dimensional, by showing that the expression for the fidelity given therein can be achieved by a suitable sequence of quantum channels. The proof is based on a truncation argument, that works when the average input state \( \tau = \sum_{x \in \mathcal{X}} \rho_x p_x \)—diagonalized as \( \tau = \sum_{n=1}^\infty p_n |n\rangle \langle n| \)—has eigenvalues that decay sufficiently fast, in the sense that \( \sum_{n=1}^\infty p_n E_n = E < \infty \) for some increasing sequence \( (E_{n+1} \geq E_n \geq 0, \forall n \in \mathbb{N}) \) such that \( \lim_{n \to \infty} E_n = \infty \).
This is the case in all relevant examples: for example, a thermal state \( \tau = (1 - x) \sum_{n=0}^{\infty} x^n |n\rangle \langle n| \) satisfies the required condition with \( E_n = n \).

We illustrate the truncation argument for general quantum channels, as the use of the argument for measure-and-prepare channels is exactly the same.

Let us show that there exists a sequence of quantum channels reaching the value \( F^{det} = \inf_{\sigma > 0, \text{Tr}[\sigma] = 1} \| A_\sigma \|_\infty \). To prove the achievability of the value \( F^{det} \), for every finite \( N \) we define the value

\[
F^{det}_N = \inf_{\rho_N \geq 0, \text{Supp}(\rho_N) = \text{Supp}(P_N), \text{Tr}[\rho_N] = 1} \| A_{\rho_N} \|_\infty \quad A_{\rho_N} := (I_{out} \otimes \rho_N^{-\frac{1}{2}})A(I_{out} \otimes \rho_N^{-\frac{1}{2}})
\]

where \( P_N = \sum_{n=1}^{N} |n\rangle \langle n| \) and \( \rho_N^{-1/2} \) is the inverse of \( \rho_N^{1/2} \) on its support. It is easy to see that the value \( F^{det}_N \) is a lower bound to the fidelity that can be achieved by quantum channels of the form \( C_N(\rho) = C_N(P_N \rho P_N) + \text{Tr}[(I_n - P_n) \rho] \rho_0 \), where \( \rho_0 \) is a fixed state. Hence,

\[
\lim_{N \to \infty} F^{det}_N \leq F^{det}.
\]

We now show that, in fact, \( \lim_{N \to \infty} F^{det}_N = F^{det} \). Let \( \rho_N \) be a state such that \( \text{Supp}(\rho_N) = \text{Supp}(P_N) \), and let \( \sigma_N \) be the state

\[
\sigma_N := \rho_N \rho_N + \chi_N^T, \quad p_N := \sum_{n=1}^{N} \frac{p_n E_n}{E}, \quad \chi_N := \sum_{n=N+1}^{\infty} \frac{p_n E_n}{E} |n\rangle \langle n|
\]

With this definition, we have

\[
\| A_{\sigma_N} \|_\infty = \| A_{\frac{1}{p_N}}(I_{out} \otimes \sigma_N^{-1}) A_{\frac{1}{p_N}} \|_\infty \\
\leq \frac{1}{p_N} \| A_{\frac{1}{p_N}}(I_{out} \otimes \rho_N^{-1}) A_{\frac{1}{p_N}} \|_\infty + \| A_{\frac{1}{p_N}}(I_{out} \otimes \chi_N^{-1}) A_{\frac{1}{p_N}} \|_\infty \\
= \frac{1}{p_N} \| A_{\rho_N} \|_\infty + \left\| \left(I_{out} \otimes \chi_N^{-\frac{1}{2}}\right)^T A \left(I_{out} \otimes \chi_N^{-\frac{1}{2}}\right)^T \right\|_\infty.
\]

Observe that, by construction, the second term vanishes in the limit \( N \to \infty \). Indeed, we have

\[
\left\| \left(I_{out} \otimes \chi_N^{-\frac{1}{2}}\right)^T A \left(I_{out} \otimes \chi_N^{-\frac{1}{2}}\right)^T \right\|_\infty = \left\| \left(I_{out} \otimes \chi_N^{-\frac{1}{2}}\right)^T \left(\sum_{x \in X} p_x |\psi_x\rangle \langle \psi_x| \otimes \rho_x^T\right) \left(I_{out} \otimes \chi_N^{-\frac{1}{2}}\right)^T \right\|_\infty \\
\leq \left\| I_{out} \otimes \left(\chi_N^{-\frac{1}{2}} \chi_N^{-\frac{1}{2}}\right) \right\|_\infty \\
= \left\| \sum_{n=N+1}^{\infty} \frac{E}{E_n} |n\rangle \langle n| \right\|_\infty \\
= \frac{E}{E_{N+1}}.
\]

Hence, for sufficiently large \( N \) we have

\[
\| A_{\sigma_N} \|_\infty - \| A_{\rho_N} \|_\infty < \epsilon. \quad (27)
\]

In particular, choosing the state \( \rho_N \) to satisfy

\[
\| A_{\rho_N} \|_\infty < F^{det}_N + \epsilon
\]

we obtain

\[
F^{det} \leq \| A_{\sigma_N} \|_\infty < F^{det}_N + 2\epsilon,
\]

and, therefore \( F^{det} \leq \lim_{N \to \infty} F^{det}_N \). Since by definition \( F^{det}_N \leq F^{det}_N \), for every \( N \), this implies \( F^{det} = \lim_{N \to \infty} F^{det}_N \). Hence, the value \( F^{det} \) can be achieved by a suitable sequence of quantum channels.