Capacity of Quantum Channels
Using Product Measurements

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dedicated to Robert Schrader and Ruedi Seiler
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Abstract

The capacity of a quantum channel for transmission of classical information depends in principle on whether product states or entangled states are used at the input, and whether product or entangled measurements are used at the output. We show that when product measurements are used, the capacity of the channel is achieved with product input states,

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so that entangled inputs do not increase capacity. We show that this result continues to hold if sequential measurements are allowed, whereby the choice of successive measurements may depend on the results of previous measurements.

We also present a new simplified expression which gives an upper bound for the Shannon capacity of a channel, and which bears a striking resemblance to the well-known Holevo bound.

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1 Introduction

1.1 Overview

Bennett and Shor [2] note that there are, in principle, four basic types of channel capacities for “classical” communication using quantum signals, i.e., communications in which signals are sent using an “alphabet” of pure states of quantum systems and decoded using measurements on the (possibly mixed state) signals which arrive. The mixed states are the result of noise which is represented by a stochastic or completely positive, trace-preserving map $\Phi$. The four possible capacities correspond to using product or entangled states at the input, and using product or entangled measurements at the output. These are denoted as follows:
In more precise language “using product” means restricting to products and “using entangled” means using arbitrary (product or entangled) states or measurements. Hence, it is evident that \( C_{PP} \leq \{ C_{EP}, C_{PE} \} \leq C_{EE} \). The main purpose of this note is to show that \( C_{PP} = C_{EP} \), i.e., that if one is restricted to using product measurements, then using entangled inputs does not increase the capacity. Thus \( C_{PP} = C_{EP} \leq C_{PE} \leq C_{EE} \). It is known [6, 9, 10] that one can have strict inequality in \( C_{PP} < C_{PE} \) for certain non-unital channels. The question of whether or not one can have strict inequality in \( C_{PE} \leq C_{EE} \) is open, although numerical evidence [1, 23] suggests equality.

1.2 Notation and Definitions

To give precise definitions, we use relatively standard notation in which \( \mathcal{M} = \{ E_b \} \) denotes a “positive operator valued measurement” (POVM) i.e., \( E_b > 0 \) and \( \sum_b E_b = I \). Let \( \rho_j \) denote a set (or alphabet) of pure state density matrices, \( \pi_j \) a discrete probability vector, and \( \rho = \sum_j \pi_j \rho_j \). We let \( \mathcal{E} = \{ \pi_j, \rho_j \} \) denote this ensemble of input states. Both \( E_b \) and \( \rho_j \) are operators on a Hilbert space \( \mathcal{H} \), so that the stochastic map \( \Phi \) (representing the noise in the channel) acts on \( B(\mathcal{H}) \), the algebra of bounded operators on \( \mathcal{H} \). We will write \( \tilde{\mathcal{E}} = \{ \pi_j, \Phi(\rho_j) \} \) for the ensemble of output states emerging from the channel.

We write the dual of \( \Phi \) (or adjoint with respect to the Hilbert-Schmidt inner product) as \( \hat{\Phi} \) so that \( \text{Tr} [\Phi(\rho) E] = \text{Tr} [\rho \hat{\Phi}(E)] \). The adjoint of a stochastic map takes a POVM \( \mathcal{M} = \{ E_b \} \) to another POVM \( \hat{\mathcal{M}} = \{ \hat{E}_b \} \) since the trace-preserving condition on \( \Phi \) is equivalent to \( \hat{\Phi}(I) = I \).

The information content of a noiseless quantum channel with a fixed input ensemble and a fixed POVM can be described using the standard Shannon formula of classical information theory.

**Definition 1** For a fixed ensemble \( \mathcal{E} = \{ \pi_j, \rho_j \} \) and a POVM \( \mathcal{M} = \{ E_b \} \) on a Hilbert space \( \mathcal{H} \), the quantum mutual information is given by

\[
I^q(\mathcal{E}; \mathcal{M}) = S(\text{Tr} [\rho E_b]) - \sum_j \pi_j S(\text{Tr} [\rho_j E_b]),
\]

(1)
where $S(\text{Tr}[\rho E_b])$ denotes the Shannon entropy $-\sum_b p_b \log p_b$ of the probability vector with elements $p_b = \text{Tr}[\rho E_b]$ (and similarly for $S(\text{Tr}[\rho_j E_b])$).

The information content of a noisy channel defined by the stochastic map $\Phi$ is obtained from (1) by replacing $E$ by the output ensemble $\tilde{E} = \{\pi_j, \Phi(\rho_j)\}$. Alternatively, since $\text{Tr} [\Phi(\rho_j) E] = \text{Tr} [\rho_j \tilde{\Phi}(E)]$, we could instead choose to regard the “noise” as acting on the POVM, and obtain the capacity from (1) by replacing $\mathcal{M}$ by $\tilde{\mathcal{M}}$. Although this viewpoint is atypical, it can be useful, as we will see in Section 4.

**Definition 2** For a stochastic map $\Phi$, an input ensemble $E = \{\pi_j, \rho_j\}$ and a POVM $\mathcal{M} = \{E_b\}$, the quantum information content is given by

$$I_q^q(\tilde{E}; \mathcal{M}) = I_q^q(\tilde{\tilde{E}}; \tilde{\mathcal{M}}) = S(\text{Tr}[\Phi(\rho) E_b]) - \sum_j \pi_j S(\text{Tr}[$$ $\Phi(\rho_j) E_b])$$

We consider memoryless channels in which multiple uses of the channel are described by the n-fold tensor product $\Phi \otimes \Phi \ldots \otimes \Phi$ acting on the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H} \ldots \otimes \mathcal{H}$ which we denote by $\Phi^\otimes n$ and $\mathcal{H}^\otimes n$ respectively. This allows us to define the ‘ultimate’ information capacity of the channel as the asymptotic rate achievable when entangled inputs and measurements are used.

**Definition 3** The entangled signals/entangled measurements capacity of a quantum channel is defined as

$$C_{EE}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \sup_{\tilde{E}, \mathcal{M}} I_q^q(\tilde{E}; \mathcal{M})$$

where the supremum is taken over all possible (product or entangled) signals and measurements on $\mathcal{H}^\otimes n$.

To define capacity restricted to product measurements, we write $\mathcal{M}^\otimes n$ for a product POVM of the form $\{E_{b_1} \otimes E_{b_2} \ldots \otimes E_{b_n}\}$.

**Definition 4** The entangled signals/product measurements capacity of a quantum channel is defined as

$$C_{EP}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \sup_{\tilde{E}, \mathcal{M}^\otimes n} I_q^q(\tilde{E}; \mathcal{M}^\otimes n)$$
Note that the existence of the limits follows from superadditivity of the classical capacity.

The capacities $C_{PP}$ and $C_{PE}$ can be similarly defined. We write $\mathcal{E}^\otimes n$ to denote an ensemble of the form $\{\pi_{j_1, \ldots, j_n}, \rho_{j_1} \otimes \cdots \otimes \rho_{j_n}\}$, where $\{\rho_j\}$ is a fixed collection of states, and $\{\pi_{j_1, \ldots, j_n}\}$ is some joint probability distribution.

**Definition 5** The product signals/entangled measurements capacity of a quantum channel is defined as

$$C_{PE}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \sup_{\mathcal{E}^\otimes n, \mathcal{M}} I^q_{\Phi} (\mathcal{E}^\otimes n; \mathcal{M}).$$

**Definition 6** The product signals/product measurements capacity of a quantum channel is defined as

$$C_{PP}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \sup_{\mathcal{E}^\otimes n, \mathcal{M}^\otimes n} I^q_{\Phi^\otimes n} (\mathcal{E}^\otimes n; \mathcal{M}^\otimes n).$$

The additivity of classical information capacity immediately implies the following result.

**Theorem 7** The product signals/product measurements capacity of a quantum channel is given by

$$C_{PP}(\Phi) = C_{\text{Shan}}(\Phi) = \sup_{\mathcal{E}, \mathcal{M}} I^q_{\Phi} (\mathcal{E}; \mathcal{M}).$$

which we call the Shannon capacity.

A far deeper result is that $C_{PE}(\Phi)$ can be re-expressed in terms of the well-known Holevo bound [8, 9, 17]. This result was proved independently in [9] and [22], building on earlier work in [10] and [7].

**Theorem 8** (Holevo-Schumacher-Westmoreland) The product signals/entangled measurements capacity of a quantum channel is given by

$$C_{PE}(\Phi) = C_{\text{Holv}}(\Phi) = \sup_{\mathcal{E}} \left( S[\Phi(\rho)] - \sum_j \pi_j S[\Phi(\rho_j)] \right)$$

where $S(P) = -\text{Tr} (P \log P)$ denotes the von Neumann entropy of the density matrix $P$. We call this the Holevo capacity of the channel.
1.3 Summary of Results

Our main result, that using entangled inputs with product measurements does not increase the capacity of a channel, can be stated as

**Theorem 9** For any stochastic map, $C_{EP}(\Phi) = C_{Shan}(\Phi)$.

There is another implementation of product measurements which has the potential for a greater capacity. It involves a sequence of POVM’s on the product spaces $H^\otimes n$, whereby the POVM for the second measurement depends on the result of the first measurement, the POVM for the third measurement depends on the results of the first two measurements, and so on. The idea is that “Bob” can choose his successive POVM’s based on the results of previous measurements. We write $C_{EP}^{cond}(\Phi)$ for the maximum asymptotic rate achievable for such a sequence of conditional POVM’s, with entangled inputs allowed. (The precise definition of a conditional POVM is postponed to Section 4 and the capacity is given by (34).) Our next result shows that using such conditional POVM’s with entangled inputs again does not increase the channel capacity.

**Theorem 10** For any stochastic map, $C_{EP}^{cond}(\Phi) = C_{Shan}(\Phi)$.

Theorem 10 was proved independently (and simultaneously), using different methods, by P. Shor [20], and also later proved independently by A. Holevo [12]. A conditional POVM is not the most general situation involving product measurements, which would be a POVM in which each measurement can be written as a tensor product. Except for the obvious bounds, we know of no results for the capacity associated with such POVM’s.

The capacity of a classical channel can be written as the (suitably restricted) supremum of the classical mutual information. We extend this observation to the quantum case, using a tensor product formulation whereby the first two (and possibly all four) of these basic capacities are realized using mutual information in the form of the relative entropy of a density matrix and the product of its reduced density matrices. This leads to the following upper bound.

**Theorem 11** For any stochastic map,

$$C_{EP}(\Phi) \leq \sup_{M,\rho} \left[ S(\rho) - \sum_b S\left( \sqrt{\rho} \Phi(\rho) E_b \sqrt{\rho} \right) + S(\tau) \right]$$

where $\tau_b = \text{Tr} [\Phi(\rho) E_b] = \text{Tr} [\rho \Phi(E_b)]$. 

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We call the quantity on the right $U_{EP}$, and we conjecture that it is equal to $C_{EP}$, i.e. that equality holds in Theorem 11 above. We motivate and study $U_{EP}$ in Section 2.3 where we show that it can be rewritten in a form similar to the Holevo capacity. Combined with Theorem 9 above, this conjectured equality would provide a simplified expression for the Shannon capacity of any channel, whereby the sup over both input ensemble and POVM is replaced by a sup over one average input state and the POVM.

Although the proof of Theorem 10 does not depend on our tensor product reformulation, we present this material first, in the following section, because we feel it gives some useful insights. Section 2 is largely pedagogical and provides the motivation for our conjectured expression for $C_{EP}$. Section 3 is also primarily pedagogical; it introduces the reader to Holevo’s C-Q and Q-C channels [9]. This leads to a short proof of both the well-known Holevo bound and the new bound in Theorem 11. Moreover, the additivity of Q-C channels implies Theorem 9 and motivates our proof of Theorem 10. The reader primarily interested in this proof can skip directly to Section 4.

2 Capacity from Mutual Information

2.1 Classical background

The classical mutual information of two random variables $X$ and $Y$ measures how much information they have in common and is given by

$$I_c(X;Y) \equiv \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}.$$  \hspace{1cm} (9)

If $X$ and $Y$ represent the input and output distributions of a channel, then the classical Shannon capacity is the supremum of $I_c(X;Y)$ taken over all possible joint distributions allowed by the channel.

The Shannon capacity of a quantum channel can also be obtained in this way provided that the joint distribution arises from a quantum communication process $(\Phi, E, M)$ as

$$p(j, b) = \pi_j \text{Tr} [\Phi(\rho_j)E_b] = \pi_j \text{Tr} [\rho_j \hat{\Phi}(E_b)]$$  \hspace{1cm} (10)

Although the stochastic map $\Phi$ is usually regarded as noise acting on the signals $\rho_j$, it is important to recognize that it has another interpretation corresponding to the second expression for $p(j, b)$ in (10) above. In the second case, the channel transmits signals faithfully, but the “noise” distorts the measurement process by
converting the POVM \( \{ E_b \} \) to a modified POVM \( \{ \hat{E}_b = \hat{\Phi}(E_b) \} \) implemented by the action of the dual of \( \Phi \).

In order to make the transition from classical to quantum communication, it is sometimes useful to consider a classical probability vector \( p(x) \) as the diagonal of a matrix \( P \). We can then write the relative entropy

\[
H(P, Q) = \text{Tr}[P \log P - P \log Q]
\]

in a form which reduces to the usual classical expression when \( P \) and \( Q \) are diagonal, but is also valid when \( P \) and \( Q \) are density matrices representing mixed quantum states. In this notation (9) becomes

\[
I^c(X; Y) = H[P_{12}, P_1 \otimes P_2]
\]

where \( P_{12}, P_1, \) and \( P_2 \) are diagonal matrices with non-zero entries \( p(x, y), p(x) \) and \( p(y) \) respectively.

### 2.2 Tensor Product Reformulation

A reformulation and generalization of mutual information and capacity can be made using formal tensor products. It should be emphasized that this is done for convenience of notation and is distinct from the tensor products used in describing multiple uses of the channel. Let \( \mathcal{H}_{ABQR} = C^J \otimes C^M \otimes \mathcal{H} \otimes \mathcal{H} \) where \( j = 1 \ldots J, \ b = 1 \ldots M \) and \( \mathcal{H}_Q = \mathcal{H}_R = \mathcal{H} \) is the original Hilbert space on which \( \rho \) and \( E_b \) act. The partial traces then correspond to \( T_A = \sum_j, T_B = \sum_b, T_Q = \text{Tr}, \) and \( T_R = \text{Tr}. \)

Let \( P_{ABQ} \) be the block diagonal matrix with blocks \( \pi_j \sqrt{\Phi(\rho_j)} E_b \sqrt{\Phi(\rho_j)} \) and \( \hat{P}_{ABQ} \) the block diagonal matrix with blocks \( \pi_j \sqrt{\hat{\Phi}(E_b)} \sqrt{\hat{\Phi}(E_b)} \). Then \( P_{AB} \equiv T_Q P_{ABQ} = T_Q \hat{P}_{ABQ} \equiv \hat{P}_{AB} \) and

\[
P_{AB} \text{ is a diagonal matrix with (non-zero) elements } p(j, b) = \pi_j \text{Tr} [\Phi(\rho_j) E_b],
\]

\[
P_A \equiv T_{BC} P_{ABQ} = T_B P_{AB} \text{ is a diagonal matrix with elements } \delta_{ij} \pi_j,
\]

\[
P_B \equiv T_{AQ} P_{ABQ} = T_A P_{AB} \text{ is a diagonal matrix with elements } \delta_{ab} \tau_b \text{ where } \tau_b = \text{Tr} \Phi(\rho) E_b = \text{Tr} \rho \hat{\Phi}(E_b) \text{ as in Theorem 11.}
\]

It is straightforward to verify that

\[
C_{PP} \equiv C_{\text{Shan}}(\Phi) = \sup_{\varepsilon, \mathcal{M}} [S(P_B) - S(P_{AB}) + S(P_A)]
\]

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\]
\[
\begin{align*}
= & \sup_{\mathcal{E}, \mathcal{M}} H(P_{AB}, P_A \otimes P_B) = \sup_{\mathcal{E}, \mathcal{M}} I_{\Phi}^q(\mathcal{E}; \mathcal{M}) \\
= & \sup_{\mathcal{E}, \mathcal{M}} I^q(\tilde{\mathcal{E}}; \mathcal{M}) = \sup_{\mathcal{E}, \mathcal{M}} I^q(\mathcal{E}; \tilde{\mathcal{M}}).
\end{align*}
\] (14)

where the last line in (14), although redundant is included to emphasize the fact that we can suppress the explicit dependence on \(\Phi\) by using either a restricted ensemble with \(\tilde{\rho}_j = \Phi(\rho_j)\) or a restricted POVM of the form \(\tilde{\Phi}(E_b)\).

Note that all the matrices in (13) above are diagonal and could be replaced by probability vectors. The quantum character of the channel is hidden in the fact that \(P_{AB}\) must be the reduced density matrix of a \(P_{ABQ}\) of the form above with quantum blocks. Thus we might have replaced \(\sup_{\mathcal{E}, \mathcal{M}}\) above by either \(\sup_{P_{ABQ}} H(P_{AB}, P_A \otimes P_B)\) or \(\sup_{\tilde{P}_{ABQ}} H(P_{AB}, P_A \otimes P_B)\) with the understanding that the supremum was to be taken over those \(P_{ABQ}\) or \(\tilde{P}_{ABQ}\) with the block diagonal form given above.

We can find a similar expression for the Holevo capacity by noting that

\[
P_{AQ} \equiv T_B P_{ABQ}\]

is a block diagonal matrix with blocks \(\pi_j \Phi(\rho_j)\), and

\[
P_Q \equiv T_{AB} P_{ABQ} = T_A P_{AQ} = \Phi(\rho).
\]

It is again straightforward to verify that

\[
C_{PE} \equiv C_{\text{Holv}}(\Phi) = \sup_{\mathcal{E}} \left[ S(P_Q) - S(P_{AQ}) + S(P_A) \right] = \sup_{\mathcal{E}} H(P_{AQ}, P_A \otimes P_Q).
\] (15)

We can interpret this as a classical to quantum mutual information between the classical probability distribution \(\pi_j\) of the input alphabet and the average quantum distribution \(\Phi(\rho)\) which emerges from the channel.

We conclude by observing that the entanglement assisted capacity of [4] can be written in a similar way as

\[
\sup \{ H(\rho_{QR}, \rho_Q \otimes \rho_R) : \rho_{QR} = (\Phi \otimes I)(|\Psi\rangle\langle\Psi|) \} = \sup \{ H(\rho_{QR}, \rho_Q \otimes \rho_R) : \rho_{QR} = (\Phi \otimes I)(|\Psi\rangle\langle\Psi|) \} = \sup \{ H(\rho_{QR}, \rho_Q \otimes \rho_R) : \rho_{QR} = (\Phi \otimes I)(|\Psi\rangle\langle\Psi|) \}
\] (16)

with \(\Psi \in \mathbb{C}^2 \otimes \mathbb{C}^2\). This differs slightly from eq. (4) of [4]. However, because \(|\Psi\rangle\langle\Psi|\) is pure, their \(S(\rho) = S[T_2(|\Psi\rangle\langle\Psi|)] = S[T_1(|\Psi\rangle\langle\Psi|)] = S(\rho_R)\) in our notation. Thus the expression in (16) above is equivalent to eq. (4) of [4]. This is a form of quantum to quantum mutual information between the subsystems of an entangled pair, one of which is subjected to noise via transmission through the channel.
We also expect that the capacity $C_{EE}$ can be expressed as a (different) quantum to quantum mutual information. Unfortunately the precise form has eluded us. This approach does, however, lead in a natural way to a new expression related to $C_{EP}$.

2.3 Proposed expression for $C_{EP}$

To motivate our new candidate for $C_{EP}$, we let $P_{BR}$ be the block diagonal matrix with blocks $\sqrt{\rho} \hat{\Phi}(E_b) \sqrt{\rho}$. Then $P_B \equiv T_R P_{BR}$ is a diagonal matrix with elements $\tau_b$.

$P_R \equiv T_B P_{BR} = \rho$

and define

$$U_{EP}(\Phi) = \sup_{M,\rho} \left[ S(P_R) + S(P_B) - S(P_{BR}) \right]$$

$$= \sup_{M,\rho} H(P_{BR}, P_R \otimes P_B)$$

$$= \sup_{M,\rho} \left[ S(\rho) - \sum_b S\left( \sqrt{\rho} \hat{\Phi}(E_b) \sqrt{\rho} \right) + S(\tau) \right]$$

$$= \sup_{\tau_b,\gamma_b} \left[ S(\gamma) - \sum_b \tau_b S(\gamma_b) \right]$$

(17)

(18)

where $\gamma_b = \frac{1}{\tau_b} \tau_b \sqrt{\rho} \hat{\Phi}(E_b) \sqrt{\rho}$ and $\gamma = \sum_b \tau_b \gamma_b = \rho$. The last form (18), looks like the Holevo capacity with the input ensemble $\mathcal{E} = \{\pi_j, \rho_j\}$ replaced by a new “output measurement ensemble” $\{\tau_b, \gamma_b\}$. How can we characterize this ensemble? Using Kraus operators we can write $\Phi(\rho) = \sum_k A_k^\dagger \rho A_k$, where $\sum_k A_k A_k^\dagger = I$. It follows that $\gamma_b = \sum_k B_k^\dagger E_b B_k$ with $B_k = A_k^\dagger \sqrt{\rho}$. Hence $\gamma_b$ is a density matrix in the range of a completely positive map which, rather than being trace-preserving or unital, satisfies $\sum_k B_k B_k^\dagger = \Phi(\rho)$. If we define $\Gamma_\rho(P) = \sqrt{\rho} \hat{\Phi}(P) \sqrt{\rho}$ we can write

$$U_{EP}(\Phi) = \sup_{\rho,M} \left( S[\Gamma_\rho(I)] - \sum_b \tau_b S[\Gamma_\rho(\tau_b^{-1} E_b)] \right).$$

(19)

A different characterization is given in the next section as a condition on $P_{BR}$. We can interpret (17) as a quantum to classical mutual information between the average input $\rho$ and the classical probability vector $\tau_b$ associated with the correspondingly averaged output measurements $\text{Tr}[\rho \hat{\Phi}(E_b)]$. 

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We conjecture that $U_{EP} = C_{EP}$ although we can only show $U_{EP} \geq C_{EP}$, which is proved in the next section. Note if $\Phi$ is the completely noisy channel which maps every density matrix to the identity, then $P_{BR} = P_B \otimes P_R$ so that $H(P_{BR}, P_B \otimes P_R) = 0$ as expected. This also holds if $\rho$ is a one-dimensional projection.

### 2.4 Optimization constraints

We can rewrite all of these expressions for capacity as the suitably constrained supremum of an “Input-Output” mutual information, $H(\rho_{IO}, \rho_I \otimes \rho_O)$, i.e.,

$$\sup \{H(\rho_{IO}, \rho_I \otimes \rho_O) : \rho_{IO} \text{ is a density matrix in } X_{IO}\} \quad (20)$$

where the subset $X_{IO}$ lies in $A_I \otimes A_O$ and the algebra $A$ is either $\mathbb{C}^{n \times n}$ or $\mathbb{D}^n$, the algebra of diagonal $n \times n$ matrices. We will let $G = \{E : 0 \leq E \leq I\}$ denote the set of positive semi-definite operators less than the identity, $D$ the set of density matrices, and $\leq D$ the set of positive semi-definite matrices with trace $\leq 1$, i.e., the set of matrices $\lambda P$ where $P$ is a density matrix and $0 \leq \lambda \leq 1$.

- $C_{PP} : X_{IO} = \{\rho_{AB} = \text{Tr}_Q \rho_{ABQ} : \rho_{AQ}^{-1/2} \rho_{ABQ} \rho_{AQ}^{-1/2} \in \mathbb{D}^n \otimes \mathbb{D}^n \otimes \hat{\Phi}(G)\}$.

  In the case of maps on $\mathbb{C}^{2 \times 2}$ we expect this to be a subset of $\mathbb{D}^2 \otimes \mathbb{D}^2$ although, in principle, it could be a subset of $\mathbb{D}^4 \otimes \mathbb{D}^4$.

- $C_{PE} : X_{IO} = \{\rho_{AQ} \in \mathbb{D}^n \otimes \hat{\Phi}(\leq D)\}$.

- $U_{EP} : X_{IO} = \{\rho_{BR}^{-1/2} \rho_{BR} \rho_{BR}^{-1/2} \in \mathbb{D}^n \otimes \hat{\Phi}(G)\}$

- $C_{EE} : \text{ We know only that } X_{IO} \subset \mathbb{C}^{n \times n} \otimes \mathbb{C}^{n \times n}$.

In order to conclude that these expressions are equivalent to those given previously, we need to verify that when $\rho_{IO}$ is in the indicated set, one can always find a corresponding ensemble $E$ and/or POVM $M$. The block diagonal conditions implicit in the notation above and the fact that $\Phi$ and $\hat{\Phi}$ are trace-preserving and identity preserving respectively, makes this quite straightforward.

When $n = 2$, we can describe $G$ explicitly by writing $E = w_0 I + w \cdot \sigma$ where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ denotes the formal vector of Pauli matrices and $w$ in $\mathbb{R}^3$. Then $0 \leq E \leq I$ if and only if $|w| \leq \min\{w_0, 1 - w_0\}$ so that

$$G = \bigcup_{w_0 \in [0, 1]} \left\{ E = w_0 I + w \cdot \sigma : |w| \leq \min\{w_0, 1 - w_0\} \right\}.$$
3 Bounds via Q-C Channels

Holevo [11] introduced an extremely useful family of stochastic maps of the form

$$\Omega(P) = \sum_k R_k \text{Tr}(PX_k) \quad (21)$$

where $R_k$ is a family of density matrices, $X_k$ is a POVM. He also distinguished two important subclasses of these channels

$\Omega_{QC}$ Quantum-classical channels in which $R_k = |e_k\rangle\langle e_k|$ so that each density matrix is a one-dimensional projection from an orthonormal basis $\{e_k\}$.

$\Omega_{CQ}$ Classical-quantum channels in which $X_k = |e_k\rangle\langle e_k|$ so that the POVM is a partition of unity arising from an orthonormal basis $\{e_k\}$.

Holevo [9] showed that the quantum capacity of such channels is additive, i.e.,

$$C_{PE}(\Phi_{QC} \otimes \Phi_{QC} \ldots \otimes \Phi_{QC}) = C_{PE}(\Phi_{QC}^n) = n C_{PE}(\Phi_{QC})$$

and similarly for $C_{PE}(\Phi_{CQ}^n) = n C_{PE}(\Phi_{CQ})$. In the next section, we use Holevo’s strategy for proving additivity for $\Phi_{CQ}$ to prove Theorem 10.

We now show that both the celebrated “Holevo bound” $C_{PP}(\Phi) \leq C_{PE}(\Phi)$ and the new bound $C_{PP}(\Phi) \leq U_{EP}(\Phi)$ follow easily from the monotonicity of relative entropy under $\Omega_{QC}$ channels. Our strategy is similar to one used earlier by Yuen and Ozawa [25].

In the first case, we let $\Omega_{QB}$ be a Q-C map of the form (21) with $X_b = E_b$ and $R_b = |e_b\rangle\langle e_b|$. Then

$$H(P_{AB}, P_A \otimes P_B) = H[\Omega_{QB}(P_{AQ}), \Omega_{QB}(P_A \otimes P_Q)]$$

$$\leq H(P_{AQ}, P_A \otimes P_Q) \quad (22)$$

where $P_{AQ}$ and $P_{AB}$ are as in Section 2 and we have suppressed the identity in $I \otimes \Omega_{QB}$. Taking the supremum over $\mathcal{E}$ yields $C_{PP}(\Phi) \leq C_{PE}(\Phi)$.

For the new bound, let $\Omega_{RA}$ be a Q-C map of the form (21) with $X_j = \pi_j \rho^{-1/2} \rho_j \rho^{-1/2}$ and $R_j = |e_j\rangle\langle e_j|$, so that $\Omega_{RA}(P_{BR}) = P_{AB}$. Then

$$H(P_{AB}, P_A \otimes P_B) = H[\Omega_{RA}(P_{BR}), \Omega_{RA}(P_B \otimes P_R)]$$

$$\leq H(P_{BR}, P_B \otimes P_R)$$

from which it follows that $C_{PP}(\Phi) \leq U_{EP}(\Phi)$. 

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Remark: It may appear that the argument in (22) above yields a simple proof of the Holevo bound without using the strong subadditivity (SSA) of relative entropy [15] as in [21]. However, Lindblad [16] made the useful observation that any stochastic map can be represented as the partial trace after interaction with an auxiliary system, i.e., $\Phi(P) = T_B[U_{AB}P \otimes E_B U_{AB}^\dagger]$ In fact, he used this representation to obtain monotonicity as a corollary of SSA. Thus, the arguments used to obtain the Holevo bound via monotonicity (as above or in [24]) and via SSA (as in [21]) are essentially equivalent. In the latter approach, an auxiliary system is added explicitly and then discarded; in the former, this is done implicitly via Lindblad’s representation theorem. Further discussion of the history of the closely connected properties of SSA, monotonicity of relative entropy and the joint convexity of relative entropy is given in [18, 19, 24].

4 Proof of Additivity Using Q-C Channels

Theorem 9 can be obtained from Holevo’s result [11] that $C_{\text{Holv}}(\Omega_{QC})$ is additive, i.e., if $\Gamma$ is a Q-C channel of the form following (21), then $C_{\text{Holv}}(\Gamma)$ is additive. To show how this follows, we define

$$\Gamma_{\Phi,\mathcal{M}}(P) = \sum_b |e_b\rangle\langle e_b| \text{Tr} [\hat{\Phi}(E_b)].$$ (23)

Then $\Gamma_{\Phi,\mathcal{M}}(P)$ is a Q-C channel with $X_n = \hat{\Phi}(E_b)$. Moreover, $\sup_{\mathcal{E}} I_\Phi^b(\mathcal{E};\mathcal{M}) = C_{\text{Holv}}(\Gamma_{\Phi,\mathcal{M}})$, and the additivity of $C_{\text{Holv}}(\Gamma_{\Phi,\mathcal{M}})$ implies $\sup_{\mathcal{E}} I_{\Phi \otimes^n}^b(\mathcal{E};\mathcal{M} \otimes^n) = C_{\text{Holv}}(\Gamma_{\Phi,\mathcal{M}})^n = n C_{\text{Holv}}(\Gamma_{\Phi,\mathcal{M}})$. Then Theorem 9 follows from

$$C_{\text{Holv}}(\Phi) = \sup_{\mathcal{E},\mathcal{M}} I_{\Phi}^b(\mathcal{E};\mathcal{M}) = \sup_{\mathcal{M}} C_{\text{Holv}}(\Gamma_{\Phi,\mathcal{M}}).$$

In order to prove Theorem 10, we will need to extend Holevo’s result. Our extension, which we present below, follows Holevo’s strategy [11] with the identity (27) replacing subadditivity. This also provides a self-contained proof of Theorem 9, since a product measurement is a special case of a conditional measurement.

First consider a product channel with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and noise operator $\Phi_1 \otimes \Phi_2$. Let $\mathcal{E}_{12} = \{\pi_j, \rho_j\}$ be an ensemble of possibly entangled input states on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\mathcal{M}_1 = \{E_b\}$ denote the POVM on $\mathcal{H}_1$ which implements the first measurement, and for each $b$ let $\mathcal{M}_2(b) = \{E_c^{(b)}\}$ denote the POVM on $\mathcal{H}_2$ which implements the second measurement. We then define a joint POVM $\mathcal{M}_{12}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, namely $\{E_b \otimes E_c^{(b)}\}$. Note that although each element of $\mathcal{M}_{12}$ is a product, the joint measurement need not be the product of independent measurements.
\( \mathcal{M}_1 \otimes \mathcal{M}_2 \). This is the result of the fact that the second measurement may be conditioned on the results of the first. Nevertheless, it is easy to verify that \( \mathcal{M}_{12} \) is a POVM since

\[
\sum_{b,c} E_b \otimes E_c^{(b)} = \sum_b E_b \otimes \left( \sum_c E_c^{(b)} \right) = \sum_b E_b \otimes I.
\]

The information content of a channel using such conditioned measurements is

\[
I_{\Phi_1 \otimes \Phi_2}(\mathcal{E}_{12}; \mathcal{M}_{12}) = I^q(\mathcal{E}_{12}; \hat{\mathcal{M}}_{12}) = I^q(\tilde{\mathcal{E}}_{12}; \mathcal{M}_{12})
\]

(24)

where \( \hat{\mathcal{M}}_1, \hat{\mathcal{M}}_2(b) \) and \( \hat{\mathcal{M}}_{12} \) denote the POVM’s in which \( E_b \) is replaced by \( F_b = \hat{\Phi}_1(E_b) \) and \( E_c^{(b)} \) is replaced by \( F_c^{(b)} = \hat{\Phi}_2(E_c^{(b)}) \), and we have used the notation defined in (1) and (2). Because we are interested in studying the capacity for a fixed set of POVM’s, we use the form \( I^q(\mathcal{E}_{12}; \hat{\mathcal{M}}_{12}) \) and proceed as if we were considering a noiseless channel with a restricted POVM of the above form. Although this viewpoint is useful, it is not essential. The argument would work equally well if we explicitly included the stochastic maps or used the form \( I^q(\tilde{\mathcal{E}}_{12}; \mathcal{M}_{12}) \) and defined reduced density matrices using partial traces acting on, e.g., \((\Phi_1 \otimes \Phi_2)(\rho_j)\).

For any input ensemble \( \mathcal{E}_{12} \) we now define a pair of associated input ensembles on \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively. For this purpose it is useful to let \( T_j \) denote the partial trace over \( \mathcal{H}_j \). First, let \( \rho_j^{(1)} = T_2[\rho_j] \) be the indicated reduced density matrix and \( \mathcal{E}_1 = \{ \pi_j, \rho_j^{(1)} \} \). This is our ensemble on \( \mathcal{H}_1 \). Second, for each \( j \) and \( b \), define a state on \( \mathcal{H}_2 \) by

\[
\rho_{j,b}^{(2)} = p(b|j)^{-1} T_1[(\rho_j)(F_b \otimes I)],
\]

(25)

where \( p(b|j) = \text{Tr}[\rho_j(F_b \otimes I)] \). Then the corresponding input ensemble on \( \mathcal{H}_2 \) is \( \mathcal{E}_2(b) = \{ p(j|b), \rho_{j,b}^{(2)} \} \), where \( p(j|b) = p(b|j)\pi_j/p(b) \) and

\[
p(b) = \sum_j \pi_j p(b|j) = \text{Tr} \left[ \left( \sum_j \pi_j \rho_j \right) (F_b \otimes I) \right].
\]

(26)

We claim that

\[
I^q(\mathcal{E}_{12}; \hat{\mathcal{M}}_{12}) = I^q(\mathcal{E}_1; \hat{\mathcal{M}}_1) + \sum_b p(b) I^q[\mathcal{E}_2(b); \hat{\mathcal{M}}_2(b)].
\]

(27)

Since

\[
I^q[\mathcal{E}_2(b); \hat{\mathcal{M}}_2(b)] = I_{\hat{\Phi}_2}(\mathcal{E}_2(b); \mathcal{M}_2(b)) \leq C_{\text{Shan}}(\Phi_2)
\]

(28)
it follows immediately from (27) that

\[ I^q(\mathcal{E}_{12}; \widetilde{\mathcal{M}}_{12}) \leq I^q(\mathcal{E}_1; \widetilde{\mathcal{M}}_1) + \sum_b p(b) C_{\text{Shan}}(\Phi_2) \]

\[ = I^q(\mathcal{E}_1; \widetilde{\mathcal{M}}_1) + C_{\text{Shan}}(\Phi_2). \]  

(29)

Taking the supremum over channels of this type, which we now emphasize by writing \( \mathcal{M}_{12}^{\text{cond}} \), gives

\[ \sup_{\mathcal{E}_{12}, \mathcal{M}_{12}^{\text{cond}}} I^q_{\Phi_1 \otimes \Phi_2}(\mathcal{E}_{12}; \mathcal{M}_{12}^{\text{cond}}) = \sup_{\mathcal{E}_{12}, \mathcal{M}_{12}^{\text{cond}}} I^q(\mathcal{E}_{12}; \widetilde{\mathcal{M}}_{12}^{\text{cond}}) \]

\[ \leq C_{\text{Shan}}(\Phi_1) + C_{\text{Shan}}(\Phi_2). \]  

(30)

However by restricting to product ensembles and product POVM’s in the sup on the left side of (30), and using additivity of the classical capacity (7), we deduce

\[ \sup_{\mathcal{E}_{12}, \mathcal{M}_{12}^{\text{cond}}} I^q_{\Phi_1 \otimes \Phi_2}(\mathcal{E}_{12}; \mathcal{M}_{12}^{\text{cond}}) \geq C_{\text{Shan}}(\Phi_1) + C_{\text{Shan}}(\Phi_2). \]  

(31)

Hence we have equality in (30).

Now consider the \( n \)-fold product channel \( \Phi_1 \otimes \cdots \otimes \Phi_n \). Let \( \mathcal{M}_{\text{cond}} \) be a conditional POVM on \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \). By assumption, every operator in this POVM has the form \( E_b \otimes E_c^{(b)} \) where \( \{E_b\} \) is a conditional POVM \( \mathcal{N}_{\text{cond}} \) on \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{n-1} \), and for each \( b \), \( E_c^{(b)} \) constitutes a POVM on \( \mathcal{H}_n \). Also, for any input ensemble \( \mathcal{E} \) on \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \), let \( \mathcal{E}' \) be the ensemble of reduced density matrices on \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{n-1} \). Then (30) implies

\[ \sup_{\mathcal{E}, \mathcal{M}_{\text{cond}}} I^q_{\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n}(\mathcal{E}; \mathcal{M}_{\text{cond}}) \leq \sup_{\mathcal{E}', \mathcal{N}_{\text{cond}}} I^q_{\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_{n-1}}(\mathcal{E}'; \mathcal{N}_{\text{cond}}) + C_{\text{Shan}}(\Phi_n). \]  

(32)

Iterating (32) gives

\[ \sup_{\mathcal{E}, \mathcal{M}_{\text{cond}}} I^q_{\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n}(\mathcal{E}; \mathcal{M}_{\text{cond}}) \leq \sum_{k=1}^n C_{\text{Shan}}(\Phi_k). \]  

(33)

The definition of conditional capacity is

\[ C_{\text{EP}}^{\text{cond}}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \sup_{\mathcal{E}, \mathcal{M}_{\text{cond}}} I^q_{\Phi \otimes n}(\mathcal{E}, \mathcal{M}_{\text{cond}}). \]  

(34)
Hence if we let $\Phi_k = \Phi$, ($k = 1, 2 \ldots$) it follows immediately from (33) that
\[ C_{\text{EP}}^{\text{cond}}(\Phi) \leq C_{\text{Shan}}(\Phi). \] (35)

Since the capacity of the product channel is never less than the sum of the channel capacities, i.e, $C_{\text{EP}}^{\text{cond}}(\Phi) \geq C_{\text{Shan}}(\Phi)$ we must have equality in (33) which proves Theorem 1.

It is worth noting that our argument can be used to prove a somewhat stronger result, namely the additivity of $\sup_{E} I_q^\epsilon(E; \mathcal{M}^{\text{cond}})$ for any fixed conditional measurement $\mathcal{M}^{\text{cond}}$.

All that remains is to verify (27) which is, except for notation, equivalent to the following result from classical information theory: for any random variables $J, B, C$
\[ I_c(J; B, C) = I_c(J; B) + I_c(J; C|B). \] (36)

Although the derivation of (36) is quite elementary (see for example [5, 17]), for completeness we include it in Appendix A, where we also show its equivalence to (27).

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A Appendix: A Useful Information Identity.

First we relate (27) to an expression involving classical mutual information. The input alphabet of the product channel can be described by a classical discrete random variable $J$, whose distribution is given by the input ensemble $\mathcal{E}_{12}$, that is $P(J=j) = \pi_j$. The output alphabet can be described similarly by a pair of random variables $B, C$, corresponding to the joint POVM $\tilde{\mathcal{M}}_{12}$. The joint distribution of $J, B, C$ is given by application of the formula (14), namely
\[ P(J=j, B=b, C=c) = p(j, b, c) = \pi_j \text{Tr}[(\rho_j) F_b \otimes F_c^{(b)}]. \] (37)

Applying the definitions in (14), (3) and (14) gives directly
\[ I_c^\epsilon(J; B, C) = I_q^\epsilon(\mathcal{E}_{12}; \tilde{\mathcal{M}}_{12}). \] (38)
Furthermore, by summing over $c$ in (37) and conditioning on $j$, it follows that

$$p(b|j) = \text{Tr}[\rho_j F_b \otimes I] = \text{Tr}[\rho_j^{(1)} F_b].$$

(39)

Comparing with the definition of the ensemble $\mathcal{E}_1$, it follows that

$$I^c(J; B) = I^q(\mathcal{E}_1; \hat{\mathcal{M}}_1).$$

(40)

For the second term on the right side of (36), recall that by definition

$$I^q(J; C | B) = \sum_b p(b) I^q(J; C | \{B = b\}).$$

(41)

Also

$$p(c|j, b) = \frac{p(j, b, c)}{p(j, b)} = \text{Tr}[\rho_{j,b}^{(2)} F_c^{(b)}]$$

(42)

and $p(j|b) = p(j, b)/p(b) = p(b|j)\pi_j/p(b)$, so therefore

$$I^q(J; C | \{B = b\}) = I^q(\mathcal{E}_2(b); \hat{\mathcal{M}}_2(b)).$$

(43)

Hence equations (27) and (36) are identical.

As noted before, (36) is a standard result in information theory. We include its derivation for completeness. The left side can be rewritten as

$$I(J; B, C) = H(J) + H(B, C) - H(J, B, C)$$

(44)

where $H(X)$ is the classical entropy of the random variable $X$. The two terms on the right side are respectively

$$I(J; B) = H(J) + H(B) - H(J, B)$$

(45)

and

$$I(J; C | B) = H(J|B) + H(C|B) - H(J, C|B).$$

(46)

Further, for any random variables $X$ and $Y$,

$$H(X|Y) = H(X, Y) - H(Y),$$

(47)

and therefore (46) can be written as

$$I(J; C | B) = H(J, B) - H(B) + H(C, B) - H(B) - H(J, C, B) + H(B).$$

(48)

Adding (45) and (48) gives the right side of (44), which proves the result.
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