LARGE DEVIATION PRINCIPLE ASSOCIATED WITH TRANSIENT SAMPLING FORMULA OF NEUTRAL MODEL

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ABSTRACT. In this article, the weak limit of the infinitely-many-neutral-alleles model at various small time scales is considered. Correspondingly, under time scale $t(\theta)$, where $\lim_{\theta \to +\infty} \theta t(\theta) = +\infty$, the large deviation deviation for the transient sampling formula is obtained as well. Of particular interest is the phase transition of rate function. It turns out that $2 \log \theta$ is a critical time scale of the neutral model.

1. Introduction

In population genetics, we often face such a population in which the entire collection of types are unknown, but distinguishable types, if any, can be recognized. Therefore, a sample of size $n$ from this population is usually described by integer partitions. A vector $\eta = (\eta_1, \eta_2, \ldots, \eta_l)$ is called a partition of $n$ if $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_l \geq 1$ and $\sum_{i=1}^l \eta_i = n$. We denote $\sum_{i=1}^l \eta_i$ by $|\eta|$. Clearly, $\eta_i$ indicates that one type shows up exactly $\eta_i$ times in the sample. $l$ is the total number of types and is usually denoted by $l(\eta)$. Furthermore, consider 

$$\alpha_i(\eta) = \# \{ j \mid \eta_j = i, 1 \leq j \leq l \}, 1 \leq i \leq n.$$ 

Then $(\alpha_1(\eta), \alpha_2(\eta), \ldots, \alpha_n(\eta))$ is another representation of integer partitions, and $\sum_{i=1}^n i \alpha_i(\eta) = n$, $\sum_{i=1}^n \alpha_i(\eta) = l$. Define $\mathcal{M} = \{ \eta \mid |\eta| \geq 1 \}$ and $\tilde{\mathcal{M}} = \{ 1, \eta \mid \eta(l(\eta)) \geq 2 \}$. Consequently $\tilde{\mathcal{M}} \subset \mathcal{M}$. We can define an order “$>$” in both $\tilde{\mathcal{M}}$ and $\mathcal{M}$ as follows:

$$\varphi_\eta > \varphi_\xi, \text{ if } |\eta| > |\xi|;$$

when $|\eta| = |\xi|$, we use lexicographic order.

Suppose that the allele spectrum of a population has frequency $x \in \nabla_\infty$, where

$$\nabla_\infty = \{ x \in [0, 1]^{\infty} \mid x_1 \geq x_2 \geq \sum_{i=1}^\infty x_i = 1 \}.$$ 

We know the sampling probability is

$$P_n(\eta) = p_\eta(x) = \frac{n!}{\eta_1! \cdots \eta_l \alpha_1(\eta)! \cdots \alpha_n(\eta)!} \sum_{i_1, i_2, \ldots, i_l \neq j} x_1^{\eta_{i_1}} \cdots x_l^{\eta_{i_l}},$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_l)$ is one specific sample of size $n$ from the population with frequency $x$.

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If, however, \( x \in \mathbb{N}_1 \) and \( \sum_{i=1}^{\infty} x_i < 1 \), where 
\[
\mathbb{N}_1 = \{ x \in [0, 1]^\infty | x_1 \geq x_2 \geq \cdots \sum_{i=1}^{\infty} x_i \leq 1 \},
\]
then the population has a mixed allele spectrum, i.e., a discrete spectrum with frequency \( x \) and a continuous spectrum with weight \( 1 - \sum_{i=1}^{\infty} x_i \). The type distribution of this allele spectrum is 
\[
\mu(dy) = \sum_{i=1}^{\infty} x_i \delta_{C_i}(dy) + (1 - \sum_{i=1}^{\infty} x_i) \nu_c(dy),
\]
where \( \{C_i, i \geq 1\} \) is the collection of types in discrete spectrum, and \( \nu_c(dy) \) is a continuous distribution in the continuous spectrum.

The sampling from the population with spectrum distribution \( \mu(dy) \) can be characterized by Kingman’s paintbox process\(^9,10\). Since continuous spectrum can only contribute singletons, we have the sampling probability

\[ P_n(\tilde{\eta}) = p_\eta(x) = \frac{n!}{\tilde{\eta}_1! \cdots \tilde{\eta}_l! |\alpha_1(\tilde{\eta})|! \cdots |\alpha_n(\tilde{\eta})|!} \sum_{\tilde{x}_{i_1} \cdots \tilde{x}_{i_{l+\alpha_1(\tilde{\eta})}}} x_{i_1}^{\tilde{\eta}_{i_1}} \cdots x_{i_{l+\alpha_1(\tilde{\eta})}}^{\tilde{\eta}_{i_{l+\alpha_1(\tilde{\eta})}}}, \]

where \( \tilde{\eta} = (\tilde{\eta}_1, \cdots, \tilde{\eta}_l, 1 \cdots, 1) \), \( \tilde{\eta} \geq 2 \), and \( x_{i_j}, l + 1 \leq j \leq l + \alpha_1(\tilde{\eta}) \), is \( x_{i_j} + (1 - \sum_{i=1}^{\infty} x_i) \). Therefore, all the singletons in the sample could either be from discrete spectrum or continuous spectrum, but all non-singletons are only from discrete spectrum. Moreover, since

\[
\sum_{\tilde{x}_{i_1} \cdots \tilde{x}_{i_{l+\alpha_1(\tilde{\eta})}}} x_{i_1}^{\tilde{\eta}_{i_1}} \cdots x_{i_{l+\alpha_1(\tilde{\eta})}}^{\tilde{\eta}_{i_{l+\alpha_1(\tilde{\eta})}}} = \sum_{\tilde{x}_{i_1} \cdots \tilde{x}_{i_{l+\alpha_1(\tilde{\eta})}}} x_{i_1}^{\tilde{\eta}_{i_1}} \cdots x_{i_{l+\alpha_1(\tilde{\eta})}}^{\tilde{\eta}_{i_{l+\alpha_1(\tilde{\eta})}}} \left( 1 - \sum_{j=1}^{l+\alpha_1(\tilde{\eta})-1} x_{i_j} \right),
\]

we can explicitly rewrite (2) as a function of \( x \), which happens to be the continuous extension of (1). In this article, an explicit continuous extension of (1) is obtained. With abuse of notation, we still denote the continuous extension of (1) by \( p_\eta(x) \) and

\[
p_\eta(x) = \frac{n!}{\eta_1! \cdots \eta_l! |\alpha_1(\eta)|! \cdots |\alpha_n(\eta)|!} \sum_{d=1}^{l(\eta)} (-1)^{l(\eta) - d} \sum_{\beta \in \pi(l(\eta), d)} (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i},
\]

where \( \beta \) is the partition of set \( \{1, 2, \cdots, l(\eta)\} \) into \( d \) components, satisfying

\[
\min \beta_1 < \min \beta_2 < \cdots < \min \beta_d;
\]

and \( |\beta_i| \) denotes the number of elements in \( \beta_i \); \( \varphi_k(x) = \sum_{i=1}^{\infty} x_i^k \), \( k \geq 2 \).

The above continuous extension describes not only the sampling probability from a discrete spectrum but also the sampling probability from a continuous allele spectrum. Making use of this expression, Kingman’s one-to-one correspondence \(^9,8\) associated with partition structures can be refined as

\[
P_n(\eta) = \int_{\mathbb{N}_1} p_\eta(x) \mu(dx),
\]
where $\mu(dx)$ is the unique probability measure in $\nabla_\infty$, and is called representation measure. For example, $\mu(dx) = \delta_{(0,0,\ldots)}(dx)$ corresponds to the following trivial partition structure:

$$P_n(\eta) = I_{(1,1,\ldots,1)}(\eta).$$

In fact, partition structure essentially describes the sampling distributions satisfying a natural consistency condition. The famous Ewens sampling formula is one specific partition structure. Therefore, for the infinitely-many-neutral-alleles model $\mathcal{X}_t$, the transient sampling formula is

$$P_n(\eta) = \mathbb{E}_{\eta}(\mathcal{X}_t).$$

In this article, we consider the weak limit of $\mathcal{X}_{t(\theta)}$, where $t(\theta)$ is a small-time scale, and therefore the limiting sampling formula $\lim_{\theta \to \infty} \mathbb{E}_{\eta}(\mathcal{X}_{t(\theta)})$ is known. The small-time behaviour of line-of-descent process and transient sampling formula are studied in [7] and [6] respectively. However, in [7] the strength of mutation rate $\theta$ is fixed; while in this article the asymptotic behaviour at small-time scale $t(\theta)$ uncovers more details hidden at small-time region. Moreover, when $\lim_{\theta \to \infty} \theta t(\theta) = \infty$, the large deviation principle (henceforth LDP) of $P_n(\eta)$ is also obtained. Intuitively, the large deviation estimates tell us the growth of the entropy or the amount of information carried by the samples. We can also see that the growth of the information asymptotically behaves as power law. In [4], LDP for Ewens sampling formula, the stationary sampling probability, was considered; in this article, a phase transition is observed. It turns out that $\frac{2\log \theta}{\theta}$ is the critical time scale.

This article is organized as follows. In section 2, the continuous extension of (1) is obtained, and the refinement of Kingman’s one-to-one correspondence is stated. In section 3, the weak limit of $\mathcal{X}_t$ are discussed. Finally, in section 4, the LDP for the transient sampling formula is proved.

### 2. Sampling Formula

To our surprise, the sampling probability $p_{\eta}(x)$ shows up in various scenarios. In [2], S.N. Ethier obtain the explicit transition density $p(t,x,y)$ of $\mathcal{X}_t$ where

$$p(t,x,y) = 1 + \sum_{m=2}^{\infty} e^{-\lambda_m t} Q_m(x,y),$$

$$\lambda_m = \frac{m(m-1+\theta)}{2},$$

and

$$Q_m(x,y) = \frac{2m + \theta - 1}{m!} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} (n + \theta)(m-1) p_n(x,y),$$

$$p_n(x,y) = \sum_{|\eta| = n} \frac{p_\eta(x)p_\eta(y)}{\int_{\Theta} p_\eta dPD(\theta)}.$$  

(3)  

In [3], $p_\eta(x)$ is the sampling probability, which is the continuous extension of (1).
**Proposition 1.** For a given partition \( \eta \) of the positive integer \( n \) and \( x \in \mathcal{N}_\infty \), \( p_\eta(x) \) can be rewritten as

\[
p_\eta(x) = \frac{n!}{\eta_1! \cdots \eta_{l(\eta)}! \alpha_1(\eta)! \cdots \alpha_{l(\eta)}(\eta)!} \sum_{d=1}^{l(\eta)} (-1)^{l(\eta) - d} \sum_{\beta \in \pi(l(\eta),d)} \left( |\beta_1| - 1 \right) ! \cdots \left( |\beta_d| - 1 \right) ! \varphi_{\sum_{i \in \beta_1} \eta_i(x)} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i(x)},
\]

where \( \beta \) varies over partitions of the set \( \{1, 2, \ldots, l(\eta)\} \) into \( d \) subsets, \( \beta_1, \ldots, \beta_d \), satisfying

\[
\min \beta_1 < \min \beta_2 < \cdots < \min \beta_d,
\]

and \( |\beta_i| \) denotes the cardinality of \( \beta_i \). Moreover, \( \varphi_k(x) = \sum_{i=1}^{\infty} x^i, k \geq 2 \).

**Proof.** We can rewrite (1) as

\[
\sum_{i_1, \ldots, i_{l(\eta)} \neq 1} \frac{n!}{\eta_1! \cdots \eta_{l(\eta)}! \alpha_1(\eta)! \cdots \alpha_{l(\eta)}(\eta)!} x_{i_1}^{\eta_1} \cdots x_{i_{l(\eta)}}^{\eta_{l(\eta)}}.
\]

If we define \( p_\eta^0(x) \) to be \( \sum_{i_1, \ldots, i_{l(\eta)} \neq 1} x_{i_1}^{\eta_1} \cdots x_{i_{l(\eta)}}^{\eta_{l(\eta)}} \), then it only suffices to show that

\[
p_\eta^0(x) = \sum_{d=1}^{l(\eta)} (-1)^{l(\eta) - d} \sum_{\beta \in \pi(l(\eta),d)} \left( |\beta_1| - 1 \right) ! \cdots \left( |\beta_d| - 1 \right) ! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i}.
\]

We can prove this by mathematical induction on \( l(\eta) \). When \( l(\eta) = 1 \), it is trivial; when \( l(\eta) = 2 \),

\[
p_\eta^0(x) = \sum_{i \neq j} x_i^{\eta_1} x_j^{\eta_2} = \sum_{i=1}^{\infty} x_i^{\eta_1} (\varphi_{\eta_2}(x) - x_i^{\eta_2}) = \varphi_{\eta_1}(x) \varphi_{\eta_2}(x) - \varphi_{\eta_1 + \eta_2}(x).
\]

Now assuming that, for \( l(\eta) = l \),

\[
p_\eta^0 = \sum_{d=1}^{l} (-1)^{l-d} \sum_{\beta \in \pi(l,d)} \left( |\beta_1| - 1 \right) ! \cdots \left( |\beta_d| - 1 \right) ! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i},
\]

we have, for \( l(\eta) = l + 1 \),

\[
p_{\eta_1, \ldots, \eta_{l+1}}(x) = \sum_{u=1}^{l} \varphi_{\eta_{l+1}} - \sum_{u=1}^{l} p_{\eta_1, \ldots, \eta_u + \eta_{l+1}, \ldots, \eta_{l}}(x)
\]

\[
= \sum_{d=1}^{l} (-1)^{l-d} \sum_{\beta \in \pi(l,d)} \left( |\beta_1| - 1 \right) ! \cdots \left( |\beta_d| - 1 \right) ! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i} \varphi_{\eta_{l+1}}
\]

\[
- \sum_{u=1}^{l} \sum_{d=1}^{l} (-1)^{l-d} \sum_{\beta \in \pi(l,d)} \left( |\beta_1| - 1 \right) ! \cdots \left( |\beta_d| - 1 \right) ! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i} \varphi_{\eta_{u+1}},
\]

where

\[
\eta_i^u = \begin{cases} 
\eta_i & \text{if } i \neq u \\
\eta_i + \eta_{l+1} & \text{if } i = u.
\end{cases}
\]
By switching the order of summation in (7), we have

\[\sum_{u=1}^{l} \sum_{d=1}^{l} (1)^{l-d} \sum_{\beta \in \pi(l,d)} (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i} \]

\[= \sum_{d=1}^{l} (1)^{l-d} \sum_{\beta \in \pi(l,d)} (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! \sum_{u=1}^{l} \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i},\]

where, as a matter of fact,

\[\sum_{u=1}^{l} \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i} = \sum_{u=1}^{d} \sum_{v=1}^{u} \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_u} \eta_i} \varphi_{\sum_{i \in \beta_{u+1}} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i} \]

\[= \sum_{v=1}^{d} (\beta_v | \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_v} \eta_i} + \eta_{v+1} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i}. \]

Therefore, (8) becomes

\[\sum_{d=1}^{l} (1)^{l-d} \sum_{v=1}^{d} (|\beta_1| - 1)! \cdots (|\beta_{v-1}| - 1)! |\beta_{v+1}| (|\beta_{v+1}| - 1)! \cdots (|\beta_d| - 1)! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_v} \eta_i} \varphi_{\sum_{i \in \beta_{v+1}} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i}. \]

Here \(\beta_v = \beta_1 \cup \cdots \cup \beta_v,\)

\[\beta_i^v = \begin{cases} \beta_i & \text{if } i \neq v \\ \beta_i \cup \{i + 1\} & \text{if } i = v, \end{cases} \]

and \(\beta = \beta_1 \cup \cdots \cup \beta_d \in \pi(l, d).\) Thus,

\[p_{(\eta_1, \ldots, \eta_l, \eta_{l+1})}^{l+1} = \sum_{d=2}^{l+1} (1)^{l+1-d} \sum_{\beta \in \pi(l+1,d), \beta_d = \{l+1\}} (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! \varphi_{\sum_{i \in \beta_1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_d} \eta_i} \varphi_{\eta_{l+1}}, \]

(8)

\[= (\Theta) \sum_{d=1}^{l} (1)^{l+1-d} \sum_{v=1}^{d} (|\beta_v^1| - 1)! \cdots (|\beta_v^d| - 1)! \varphi_{\sum_{i \in \beta_v^1} \eta_i} \cdots \varphi_{\sum_{i \in \beta_v^d} \eta_i}. \]

Let us separate the terms associated with \(d = l + 1\) from (8) and separate the terms related to \(d = 1\) from (9); then we combine other terms in (8) and (9). Therefore,
Applying Gram-Schmidt orthogonalization procedure to
\{ \varphi_{\eta} \}_{\eta \in \mathcal{M}} associated with partition structure is represented as
of (1). We can also easily conclude that Kingman’s one-to-one correspondence
where
\[ \mu \]
Remark 1. Clearly, the right hand side of (3) is the explicit continuous extension of (1). We can also easily conclude that Kingman’s one-to-one correspondence associated with partition structure is represented as
\[ \int_{\varnothing} p_\eta(x) \mu(dx), \]
where \( \mu(dx) \) may concentrate on \( \varnothing \). Under this representation, the convergence of representation measures is characterized by point-wise convergence of partition structures.

3. Weak Limits of \( X_t \)

In this section, the weak limit of \( X_t \) is discussed. To this end, we need to apply S.N. Ethier’s representation of transition density for \( X_t \). However, we only use the fact that \( Q_\eta(x,y) \) is an orthogonal kernel defined as \( \sum_{\eta} \chi_\eta^\theta(x) \chi_\eta^\theta(y) \), where \( \{ \chi_\eta^\theta(x) \mid \eta \in \mathcal{M} \} \) is just an orthogonal basis of the Hilbert space \( L^2(PD(\theta)) \). The inner product \( \langle ., . \rangle_\theta \) is defined as \( \langle f, g \rangle_\theta = \int_{\mathcal{M}} f(x) g(x) PD(\theta)(dx) \) and the induced \( L^2 \) norm is \( \| f \|_2^{\theta} = \sqrt{\langle f, f \rangle_\theta} \). Define \( \varphi_\eta(x) = \varphi_{\eta_1}(x) \cdots \varphi_{\eta_l}(x) \) for partition \( \eta \in \mathcal{M} \). Consequently, \( \mathcal{P} = \text{span}\{ 1, \varphi_\eta(x) \mid \eta \in \mathcal{M} \} \) is a dense subspace of \( C(\varnothing_\infty) \).

Applying Gram-Schmidt orthogonalization procedure to \( \{ 1, \varphi_\eta(x) \mid \eta \in \mathcal{M} \} \), we have
\begin{align*}
\psi_1^\theta(x) &= 1, \\
\psi_2^\theta(x) &= \varphi_2(x) - \frac{1}{1 + \theta}, \\
\psi_3^\theta(x) &= \varphi_3(x) - \langle \varphi_3, \psi_2^\theta \rangle_\theta \psi_2^\theta(x) - \langle \varphi_3, 1 \rangle_\theta, \\
\psi_4^\theta(x) &= \varphi_4(x) - \langle \varphi_4, \psi_3^\theta \rangle_\theta \psi_3^\theta(x) - \langle \varphi_4, \psi_2^\theta \rangle_\theta \psi_2^\theta(x) - \langle \varphi_4, 1 \rangle_\theta, \\
\psi_{2,2}^\theta(x) &= \varphi_{2,2}(x) - \langle \varphi_{2,2}, \psi_2^\theta \rangle_\theta \psi_2^\theta(x) - \langle \varphi_{2,2}, \psi_3^\theta \rangle_\theta \psi_3^\theta(x) - \langle \varphi_{2,2}, \psi_2^\theta \rangle_\theta \psi_2^\theta(x) - \langle \varphi_{2,2}, 1 \rangle_\theta, \\
&\quad \cdots \\
\psi_\eta^\theta(x) &= \varphi_\eta - \sum_{\xi < \eta} \langle \varphi_\eta, \psi_\xi^\theta \rangle_\theta \psi_\xi^\theta(x) \quad | \eta | \geq 1,
\end{align*}

The proof is thus completed. \( \square \)
then $\chi^\theta_\eta(x) = \frac{\psi^\theta_\omega(x)}{\parallel \psi^\theta_\xi \parallel_{2,\theta}}, \eta \in \tilde{M}$.

**Theorem 3.1.** Suppose that $X_t$ starts with $x \in \bar{\nabla}_\infty$. As $\theta \rightarrow +\infty$, we have

$$X_t(\theta) \rightarrow \begin{cases} \delta_{(0,0,\ldots)}, & \text{if } \lim_{\theta \rightarrow +\infty} \theta t(\theta) = +\infty \\ \delta_{-\frac{1}{2}x}, & \text{if } \lim_{\theta \rightarrow +\infty} \theta t(\theta) = c > 0 \\ \delta_x, & \text{if } \lim_{\theta \rightarrow +\infty} \theta t(\theta) = 0. \end{cases}$$

**Remark 2.** As the time scale increases, the effect of mutations is enhanced gradually. Because the allele spectrum starts to show some properties of continuous spectra. When $t(\theta) \gg 1$, mutations completely take over. Thus the allele spectrum is a purely continuous spectrum. In fact, this theorem unveils the fine structure of the interactions between mutations and random sampling.

Now we are going to prove Theorem 3.1. To this end, we need the following lemmas.

**Lemma 1.** For any given partition $\omega \in \tilde{M}$, we have

$$|\psi^\theta_\omega| \leq M(\omega),$$

where $M(\omega)$ is a positive constant independent of $\theta$. Thus, we have

$$\lim_{\theta \rightarrow +\infty} \langle \varphi_\omega, \psi^\theta_\xi \rangle_\theta = 0, \quad \forall \xi \leq \omega,$$

and

$$\lim_{\theta \rightarrow +\infty} \psi^\theta_\omega = \varphi_\omega.$$  

**Proof.** We can show the first statement by mathematical induction. Apparently, 1 and $\psi^\theta_2$ are bounded by 1 and 2. Thus we assume that for all $\xi < \omega$, we have

$$|\psi^\theta_\xi| \leq M(\xi).$$

Then consider $\psi^\theta_\omega$. Due to (10), we have

$$|\psi^\theta_\omega| \leq \varphi_\omega + \sum_{\xi < \omega} |\langle \varphi_\omega, \psi^\theta_\xi \rangle_\theta| \cdot |\psi^\theta_\xi|$$

by the Cauchy-Schwarz inequality and $\varphi_\omega \leq 1$, we have

$$\leq 1 + \sum_{\xi < \omega} \parallel \varphi_\omega \parallel_{2,\theta} \cdot \parallel \psi^\theta_\xi \parallel_{2,\theta} \cdot |\psi^\theta_\xi|$$

$$\leq 1 + \sum_{\xi < \omega} M(\xi)^2.$$  

Therefore, each $\psi^\theta_\omega$ has a bound independent of $\theta$. Next, by the Cauchy-Schwarz inequality, we have

$$0 \leq |\langle \varphi_\omega, \psi^\theta_\xi \rangle_\theta| \leq \parallel \varphi_\omega \parallel_{2,\theta} \cdot \parallel \psi^\theta_\xi \parallel_{2,\theta} \leq M(\xi) \parallel \varphi_\omega \parallel_{2,\theta} \rightarrow 0.$$  

The limit is due to the weak convergence of PD(\theta). Then we have $\lim_{\theta \rightarrow +\infty} \langle \varphi_\omega, \psi^\theta_\xi \rangle_\theta = 0$, for $\xi \leq \omega$. Thus it is easy to see that $\lim_{\theta \rightarrow +\infty} \psi^\theta_\omega = \varphi_\omega$ because of (10).

**Lemma 2.** For any partition $\omega \in \tilde{M}$, we have the following

$$\int_{\bar{\nabla}_\infty} Q_{m}(x,y)\psi^\theta_\omega(y)PD(\theta)(dy) = \begin{cases} \psi^\theta_\omega(x), & |\omega| = m \\
0, & \text{otherwise}. \end{cases}$$
Proof. Notice that
\[ Q_m(x,y) = \sum_{|\eta|=m} \chi^\theta_{\eta}(x)\chi^\theta_{\eta}(y), \]
where \( \chi^\theta_{\eta}(x) \) can be chosen as the normalized \( \psi^\theta_{\eta} \), i.e. \( \chi^\theta_{\eta}(x) = \frac{\psi^\theta_{\eta}(x)}{\|\psi^\theta_{\eta}\|_2} \). Thus, we have
\[
\int_{\bar{\Omega}} Q_m(x,y)\psi^\theta_{\omega}(y)PD(\theta)(dy) = \|\psi^\theta_{\omega}\|_2\int_{\bar{\Omega}} Q_m(x,y)\chi^\theta_{\omega}(y)PD(\theta)(dy)
= \|\psi^\theta_{\omega}\|_2\delta_{\omega,|\omega|}\chi^\theta_{\omega}(x) = \psi^\theta_{\omega}(x)\delta_{\omega,|\omega|}. \]

Lemma 3. For any given partition \( \omega \in \hat{\Omega} \), we have
\[
\lim_{\theta \to +\infty} E_x(\varphi_{\omega}(X_t(\theta))) = \begin{cases} 
0, & \text{if } \lim_{\theta \to +\infty} \theta t(\theta) = +\infty \\
\varphi_{\omega}(e^{-\frac{t}{\theta}}x), & \text{if } \lim_{\theta \to +\infty} \theta t(\theta) = c \\
\varphi_{\omega}(x), & \text{if } \lim_{\theta \to +\infty} \theta t(\theta) = 0
\end{cases}
\]

Proof. Since the transition density function of \( X_t \) is
\[ p(t,x,y) = 1 + \sum_{m=2}^\infty e^{-\lambda_m t}Q_m(x,y), \]
we have
\[
\int_{\bar{\Omega}} \varphi_{\omega}(y)p(t(\theta),x,y)PD(\theta)(dy)
= \int_{\bar{\Omega}} \varphi_{\omega}(y)PD(\theta)(dy) + \sum_{m=2}^\infty e^{-\lambda_m t(\theta)}\int_{\bar{\Omega}} \varphi_{\omega}(y)Q_m(x,y)PD(\theta)(dy).
\]
Then
\[ \varphi_{\omega} = \psi^\theta_{\omega} + \sum_{\xi<\omega}\varphi_{\omega},\psi^\theta_{\xi}\psi^\theta_{\xi}; \]
hence, by Lemma 2, we have
\[
\int_{\bar{\Omega}} \varphi_{\omega}(y)Q_m(x,y)PD(\theta)(dy) = \psi^\theta_{\omega}(x)\delta_{m,|\omega|} + \sum_{\xi<\omega}\varphi_{\omega},\psi^\theta_{\xi}\psi^\theta_{\xi}(x)\delta_{m,|\xi|}.
\]
Therefore,
\[
\int_{\bar{\Omega}} \varphi_{\omega}p(t(\theta),x,y)PD(\theta)(dy) = \int_{\bar{\Omega}} \varphi_{\omega}(y)PD(\theta)(dy) + \sum_{m=2}^\infty e^{-\lambda_m t(\theta)}\psi^\theta_{\omega}(x)\delta_{m,|\omega|}
+ \sum_{\xi<\omega} \sum_{m=2}^\infty e^{-\lambda_m t(\theta)}\varphi_{\omega},\psi^\theta_{\xi}\psi^\theta_{\xi}(x)\delta_{m,|\xi|}
= \int_{\bar{\Omega}} \varphi_{\omega}(y)PD(\theta)(y) + e^{-\lambda_{|\omega| t(\theta)}\psi^\theta_{\omega}(x)}
+ \sum_{\xi<\omega} e^{-\lambda_{|\xi| t(\theta)}\varphi_{\omega},\psi^\theta_{\xi}\psi^\theta_{\xi}(x)}.\]
By Lemma \ref{lem:1} we have
\[
\lim_{\theta \to +\infty} \int_{\bar{\mathbb{R}}_+} \varphi_\omega(x)p(t(\theta), x, y)PD(\theta)(dy) = \begin{cases} 0, & \text{if } \lim_{\theta \to +\infty} \theta t(\theta) = +\infty \\
\varphi_\omega(e^{-\frac{\theta}{2}}x), & \text{if } \lim_{\theta \to +\infty} \theta t(\theta) = c \\
\varphi_\omega(x), & \text{if } \lim_{\theta \to +\infty} \theta t(\theta) = 0. \end{cases}
\]

Now we are ready to show Theorem \ref{thm:3.1}

**Proof.** We only need to show that \(\forall f \in C(\bar{\mathbb{R}}_+),\)
\[
\lim_{\theta \to +\infty} Ef(X_{t(\theta)}) = \begin{cases} f(0), & \lim_{\theta \to +\infty} \theta t(\theta) = +\infty \\
f(e^{-\frac{\theta}{2}}x), & \lim_{\theta \to +\infty} \theta t(\theta) = c \\
f(x), & \lim_{\theta \to +\infty} \theta t(\theta) = 0. \end{cases}
\]

Notice that \(P\) is dense in \(C(\bar{\mathbb{R}}_+).\) Therefore, \(\forall \epsilon > 0,\) there is \(p \in P,\) such that
\[||f - p||_\infty < \epsilon.\]

By Lemma \ref{lem:2} we know
\[
\lim_{\theta \to +\infty} Exp(X_{t(\theta)}) = \begin{cases} p(0), & \lim_{\theta \to +\infty} \theta t(\theta) = +\infty \\
p(e^{-\frac{\theta}{2}}x), & \lim_{\theta \to +\infty} \theta t(\theta) = c \\
p(x), & \lim_{\theta \to +\infty} \theta t(\theta) = 0. \end{cases}
\]

Moreover,
\[
|Ef(X_{t(\theta)}) - f(x)| \leq 2||f - p||_\infty + |Exp(X_{t(\theta)}) - p(x)| \leq 2\epsilon + |Exp(X_{t(\theta)}) - p(x)|.
\]

If we replace \(x\) in the above by \(0, e^{-\frac{\theta}{2}}x,\) and let \(n \to +\infty,\) and then let \(\epsilon \to 0,\) we have
\[
\lim_{\theta \to +\infty} Ef(X_{t(\theta)}) = \begin{cases} f(0), & \lim_{\theta \to +\infty} \theta t(\theta) = +\infty \\
f(e^{-\frac{\theta}{2}}x), & \lim_{\theta \to +\infty} \theta t(\theta) = c \\
f(x), & \lim_{\theta \to +\infty} \theta t(\theta) = 0. \end{cases}
\]

Theorem \ref{thm:3.1} is thus proved. \hfill \Box

4. **LDP of Transient Sampling Formula**

Because of the weak limit of \(X_{t(\theta)},\) we know the limits of the associated transient sampling formula is \(P_n(\eta) = p_n(e^{\frac{\theta}{2}-\eta}),\) \(c \in [0, \infty],\) where 0 and \(\infty\) are included. In this section, we consider the small-time LDP for the transient sampling formula of \(X_{t(\theta)}\) where \(\lim_{\theta \to +\infty} \theta t(\theta) = \infty.\) Interestingly, one can also observe some phase transitions concerned with the LDP rate functions.

**Lemma 4.** As \(\theta \to +\infty,\) for any given partition \(\eta = (\eta_1, \cdots, \eta_l)(\eta_l \geq 2),\) we have the following estimation:
\[
(11) \quad \langle \varphi_\eta, 1 \rangle_\theta \sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{1}{\theta^{\sigma_{\eta} - l}}.
\]
For $\xi = (\xi_1, \cdots, \xi_p)(\xi_p \geq 2, \xi \leq n)$, we have

$$
\langle \varphi_\eta, \psi^0_\eta \rangle_\theta \sim \left[ \sum_{i=1}^{l} \sum_{j=1}^{p} \frac{(\eta_i + \xi_j - 1)!}{(\eta_i - 1)!((\xi_j - 1)!)} \right] \frac{1}{\theta^{\eta_1 - 1}(\eta_1)} \cdot \cdots \cdot \frac{1}{\theta^{\eta_l - 1}(\eta_l)}
$$

(12)

$$(\eta_1 - 1)! \cdots (\eta_l - 1)!((\xi_1 - 1)! \cdots (\xi_p - 1)! \frac{1}{\theta^{\eta_1 - 1}(\eta_1)} \cdots \frac{1}{\theta^{\eta_l - 1}(\eta_l)}$$

Proof. Since $\varphi_\eta = \sum_{d=1}^{l} \sum_{\beta \in \pi(l,d)} \theta^d \sum_{i \in \beta_1} \eta_i \cdots \sum_{i \in \beta_d} \eta_i$, where $\pi(l, d)$ is the set of partitions $\beta$ of $\{1, \cdots, l\}$ into $d$ subsets, $\beta_1, \cdots, \beta_d$, satisfying $\min \beta_1 < \cdots < \min \beta_d$, we have

$$
\langle \varphi_\eta, 1 \rangle_\theta = \sum_{d=1}^{l} \sum_{\beta \in \pi(l,d)} \int \theta^d \sum_{i \in \beta_1} \eta_i \cdots \sum_{i \in \beta_d} \eta_i dPD(\theta).
$$

By the Ewens sampling formula, we have

$$
\langle \varphi_\eta, 1 \rangle_\theta = \sum_{d=1}^{l} \sum_{\beta \in \pi(l,d)} (\sum_{i \in \beta_1} \eta_i - 1)! \cdots (\sum_{i \in \beta_d} \eta_i - 1)! \frac{\theta^d}{\theta^{\eta_1 - 1}(\eta_1)} \cdots \frac{\theta^d}{\theta^{\eta_l - 1}(\eta_l)}
$$

(13)

therefore, the leading term in (13) is the term associated with partition $\beta = \{1\} \cup \{2\} \cup \cdots \{l\}$. Then

$$
\langle \varphi_\eta, 1 \rangle_\theta \sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{1}{\theta^{\eta_1 - 1}(\eta_1)} \cdots \frac{1}{\theta^{\eta_l - 1}(\eta_l)}
$$

Now we can use mathematical induction on partition $\xi(\xi_p \geq 2)$ to show the second statement (12). Let us first check $\langle \varphi_\eta, \psi^0_2 \rangle_\theta$. Since $\psi^0_2 = \varphi_2 - \frac{1}{1 + \theta}$, it yields that

$$
\langle \varphi_\eta, \psi^0_2 \rangle_\theta = \langle \varphi_\eta, \varphi_2 \rangle_\theta - \frac{1}{1 + \theta} \langle \varphi_\eta, 1 \rangle_\theta.
$$

(14)

Let us define $\tilde{\eta} = (\eta_1, \cdots, \eta_l, 2)$. Then $\langle \varphi_\eta, \varphi_2 \rangle_\theta = \langle \varphi_{\tilde{\eta}}, 1 \rangle_\theta$. By the Ewens sampling formula, we have

$$
\langle \varphi_\eta, \varphi_2 \rangle_\theta = \sum_{d=1}^{l+1} \sum_{\beta \in \pi(l+1,d)} (\sum_{i \in \beta_1} \tilde{\eta}_i - 1)! \cdots (\sum_{i \in \beta_d} \tilde{\eta}_i - 1)! \frac{\theta^d}{\theta^{\tilde{\eta}_1 - 2}(\tilde{\eta}_1)} \cdots \frac{\theta^d}{\theta^{\tilde{\eta}_l - 2}(\tilde{\eta}_l)}
$$

(15)

Substituting (15) and (13) into (14), we have

$$
\langle \varphi_\eta, \psi^0_2 \rangle_\theta = \sum_{d=1}^{l+1} \sum_{\beta \in \pi(l+1,d)} (\sum_{i \in \beta_1} \tilde{\eta}_i - 1)! \cdots (\sum_{i \in \beta_d} \tilde{\eta}_i - 1)! \frac{\theta^d}{\theta^{\tilde{\eta}_1 - 2}(\tilde{\eta}_1)} \cdots \frac{\theta^d}{\theta^{\tilde{\eta}_l - 2}(\tilde{\eta}_l)} - \frac{1}{1 + \theta}
$$

(16)

$$
- \sum_{d=1}^{l} \sum_{\beta \in \pi(l,d)} (\sum_{i \in \beta_1} \eta_i - 1)! \cdots (\sum_{i \in \beta_d} \eta_i - 1)! \frac{\theta^d}{\theta^{\eta_1 - 1}(\eta_1)} \cdots \frac{\theta^d}{\theta^{\eta_l - 1}(\eta_l)} \frac{1}{1 + \theta}
$$

(17)

In (16), for a given $d$, the corresponding term is of the order $\frac{1}{\theta^{\eta_1 - 1}(\eta_1)}$; in (17), for a given $d$, the associated term is of the order $\frac{1}{\theta^{\eta_1 - 1}(\eta_1)}$. Let us first check terms that are associated with $d = l$ or $l + 1$ in (16) and with $d = l$ or $d = l - 1$ in (17).
Clearly, the above summation (18) can be rewritten as

\[
(\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{\theta^{l+1}}{\theta|\eta|+2} + (\eta_1 - 1)! \cdots (\eta_l - 1)! \sum_{u=1}^{l} (\eta_u + 1)\eta_u \frac{\theta^l}{\theta|\eta|+2} \\
+ (\eta_1 - 1)! \cdots (\eta_l - 1)! \sum_{1 \leq u < v \leq l} \frac{(\eta_u + \eta_v - 1)!}{(\eta_u - 1)! (\eta_v - 1)!} \frac{\theta^l}{(\theta + |\eta|)(\theta + |\eta| + 1)} \\
- (\eta_1 - 1)! \cdots (\eta_l - 1)! \sum_{1 \leq u < v \leq l} \frac{(\eta_u + \eta_v - 1)!}{(\eta_u - 1)! (\eta_v - 1)!} \frac{\theta^{l-1}}{(\theta + |\eta|)(\theta + |\eta| + 1)} \\
- (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{\theta^l}{\theta|\eta|+1}.
\]  

(18)

Then the summation of those terms are

\[
(\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{\theta^l}{\theta|\eta|+2} \left[ \sum_{u=1}^{l} \eta_u (\eta_u + 1) + \theta - \frac{(|\eta| + \theta)(|\eta| + \theta + 1)}{\theta + 1} \right] \\
+ \sum_{1 \leq u < v \leq l} \frac{(\eta_u + \eta_v - 1)!}{(\eta_u - 1)! (\eta_v - 1)!} \left( 1 - \frac{(\theta + |\eta|)(\theta + |\eta| + 1)}{\theta + 1} \right) \\
= (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{\theta^l}{\theta|\eta|+2} \left[ \sum_{u=1}^{l} \eta_u (\eta_u + 1) - \frac{2\theta + 1}{\theta + 1} \right] \\
- \sum_{1 \leq u < v \leq l} \frac{(\eta_u + \eta_v - 1)!}{(\eta_u - 1)! (\eta_v - 1)!} \frac{|\eta|^2}{\theta + 1} \left[ |\eta| + (2\theta + 1) \right] \\
\sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \left( \sum_{u=1}^{l} \eta_u (\eta_u + 1) - 2|\eta| \right) \frac{1}{\theta |\eta|+2-1} \quad \text{as } \theta \to +\infty.
\]

All the remaining terms in (16) and (17) are at least of the order \( \frac{1}{\theta |\eta|+2-1} \). Therefore,

\[
\langle \varphi_{\eta}, \psi^0_{\delta} \rangle_{\theta} \sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \left( \sum_{u=1}^{l} \eta_u (\eta_u + 1) - 2|\eta| \right) \frac{1}{\theta |\eta|+2-1}.
\]

Now we assume that, for a given partition \( \xi(\xi_p > 2), \forall \delta < \xi(\delta_l(\delta) \geq 2) \) and \( \forall \eta \geq 2 \), we have

\[
\langle \varphi_{\eta}, \psi^0_{\bar{\delta}} \rangle_{\theta} \sim \left( \sum_{i=1}^{l(\delta)} \sum_{j=1}^{l(\bar{\delta})} \frac{(\eta_i + \delta_j - 1)!}{(\eta_i - 1)! (\delta_j - 1)!} - |\eta||\bar{\delta}| \right) \\
(\eta_1 - 1)! \cdots (\eta_l - 1)! (\delta_1 - 1)! \cdots (\delta_{l(\delta)} - 1)! \frac{1}{\theta |\eta|+l+|\bar{\delta}|+l(\delta)+1}.
\]  

(19)

Now we consider \( \langle \varphi_{\eta}, \psi^{0}_{\xi} \rangle_{\theta} \). Since

\[
\psi^{0}_{\xi} = \varphi_{\xi} - \sum_{\delta < \xi} \langle \varphi_{\xi}, \psi^{0}_{\delta} \rangle_{\theta} \psi^{0}_{\delta},
\]

\[
\langle \varphi_{\eta}, \psi^{0}_{\xi} \rangle_{\theta} = \langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} - \sum_{2 \leq \delta < \xi} \langle \varphi_{\xi}, \psi^{0}_{\delta} \rangle_{\theta} \langle \varphi_{\eta}, \psi^{0}_{\delta} \rangle_{\theta} - \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta}.
\]  

(20)
By the above assumption \([13]\), for each partition \(\delta < \xi\), the associated summands in the second part of \((20)\) are of the order \(\frac{1}{\theta_{1}^{l} \cdot \cdots \cdot \theta_{1}^{l}}\). Notice that \(2(|\delta| - l(\delta)) + 1 \geq 3\) for \(\delta \geq (2)\). So the leading term in

\[
\sum_{2 \leq \delta < \xi} \langle \varphi_{\eta}, \psi_{\eta}^{\delta} \rangle_{\theta} \langle \varphi_{\gamma}, \psi_{\gamma}^{\delta} \rangle_{\theta}
\]

is of the order \(\frac{1}{\theta_{1}^{l} \cdot \cdots \cdot \theta_{1}^{l}}\); but the leading term in \(\langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta}\) is of the order \(\frac{1}{\theta_{1}^{l} \cdot \cdots \cdot \theta_{1}^{l}}\). Therefore, we should first check \(\langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} - \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta}\). Let us define \(\omega = (\eta_{1}, \ldots, \eta_{i}, \xi_{1}, \ldots, \xi_{p})\). Then \(\langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} = \langle \varphi_{\omega}, 1 \rangle_{\theta}\). By the Ewens sampling formula, we have

\[
\langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} = \sum_{d=1}^{l+p} \sum_{\beta \in \pi(l+p,d)} \int_{q_{\omega}} p_{\omega}^{\beta} \sum_{j \in \beta_{1}} \omega_{j} \cdots \sum_{j \in \beta_{d}} \omega_{j} d\text{PD}(\theta)
\]

\[
= \sum_{d=1}^{l+p} \sum_{\beta \in \pi(l+p,d)} (\sum_{j \in \beta_{1}} \omega_{j} - 1) \cdots (\sum_{j \in \beta_{d}} \omega_{j} - 1) \frac{\theta^{d}}{\theta_{\eta}^{l} + |\xi|}.
\]

Thus, we should consider terms of the order \(\frac{1}{\theta_{1}^{l} \cdot \cdots \cdot \theta_{1}^{l}}\) and \(\frac{1}{\theta_{1}^{l} \cdot \cdots \cdot \theta_{1}^{l}}\) in

\[
\langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} - \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta};
\]

and the summation of these terms is

\[
(\eta_{1} - 1)! \cdots (\eta_{l+p} - 1) \frac{\theta^{l+p}}{\theta_{\eta}^{l} + |\xi|} \theta_{\eta}^{l+p} - \langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} + \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta};
\]

\[
(\xi_{1} - 1)! \cdots (\xi_{j} - 1)! \frac{\theta^{l+p}}{\theta_{\xi}^{l} + |\eta|} \theta_{\xi}^{l+p} - \langle \varphi_{\xi}, \varphi_{\eta} \rangle_{\theta} + \langle \varphi_{\eta}, 1 \rangle_{\theta} \langle \varphi_{\xi}, 1 \rangle_{\theta}.
\]

Recall that

\[
(\omega_{1} - 1)! \cdots (\omega_{l+p} - 1)! = (\eta_{1} - 1)! \cdots (\eta_{l} - 1)! (\xi_{1} - 1)! \cdots (\xi_{p} - 1)!
\]

Then the above summation, \((21) + (22)\), is the following:

\[
(\eta_{1} - 1)! \cdots (\eta_{l} - 1)! (\xi_{1} - 1)! \cdots (\xi_{p} - 1)! \frac{\theta^{l+p}}{\theta_{\eta}^{l} + |\xi|} \theta_{\eta}^{l+p} - \langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} + \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta} + \langle \varphi_{\eta}, \varphi_{\xi} \rangle_{\theta} - \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta};
\]

\[
= \sum_{1 \leq i < j \leq l+p} \frac{(\omega_{i} + \omega_{j} - 1)!}{(\omega_{i} - 1)! (\omega_{j} - 1)!} \frac{\theta^{l+p}}{\theta_{\eta}^{l} + |\xi|} \theta_{\eta}^{l+p} + \theta_{\xi}^{l+p} - \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta} - \langle \varphi_{\xi}, 1 \rangle_{\theta} \langle \varphi_{\eta}, 1 \rangle_{\theta}.
\]
Assume that Theorem 4.1. This lemma is thus proved! □

\[(\eta_1 - 1)! \cdots (\eta_l - 1)! (\xi_1 - 1)! \cdots (\xi_p - 1)! \frac{\theta^{l+p-1}}{\theta^{|\eta|+|\xi|}} \]

\[
\sum_{1 \leq i < j \leq l+p} (\omega_i + \omega_j - 1)! \frac{(\eta_i + \eta_j - 1)!}{(\eta_i - 1)! (\eta_j - 1)!} \frac{\theta^{(\eta_i+|\xi|)}}{\theta^{(\eta_j+|\xi|)}} + \theta \left( \prod_{i=0}^{||\eta||-1} (\theta + i) - \prod_{i=0}^{||\eta||-1} (\theta + \delta) \right)
\]

\[
- \sum_{1 \leq i < j \leq l} \frac{(\eta_i + \eta_j - 1)!}{(\eta_i - 1)! (\eta_j - 1)!} \frac{\theta^{(\eta_i+|\xi|)}}{\theta^{(\eta_j+|\xi|)}} - \sum_{1 \leq i < j \leq p} \frac{(\xi_i + \xi_j - 1)!}{(\xi_i - 1)! (\xi_j - 1)!} \frac{\theta^{(\xi_i+|\xi|)}}{\theta^{(\xi_j+|\xi|)}}
\]

\[
\sim (\eta_1 - 1)! \cdots (\eta_l - 1)! (\xi_1 - 1)! \cdots (\xi_p - 1)! \frac{1}{\theta^{|\eta|+l+|\xi|-p+1}}
\]

\[
\left[ \sum_{1 \leq i < j \leq l+p} \frac{\omega_i + \omega_j - 1)!}{(\omega_i - 1)! (\omega_j - 1)!} - |\xi| \frac{\theta^{||\eta||-1}}{\theta + \delta} \right]
\]

\[
- \sum_{1 \leq i < j \leq l} \frac{(\eta_i + \eta_j - 1)!}{(\eta_i - 1)! (\eta_j - 1)!} - \sum_{1 \leq i < j \leq p} \frac{(\xi_i + \xi_j - 1)!}{(\xi_i - 1)! (\xi_j - 1)!}
\]

\[
\sim (\eta_1 - 1)! \cdots (\eta_l - 1)! (\xi_1 - 1)! \cdots (\xi_p - 1)! \frac{1}{\theta^{|\eta|+l+|\xi|-p+1}}
\]

\[
\sum_{i=1}^{l} \sum_{j=1}^{p} \frac{(\eta_i + \xi_j - 1)!}{(\eta_i - 1)! (\xi_j - 1)!} - |\eta||\xi| \frac{1}{\theta^{|\eta|+l+|\xi|-p+1}}.
\]

Therefore, the leading term should be

\[
(\eta_1 - 1)! \cdots (\eta_l - 1)! (\xi_1 - 1)! \cdots (\xi_p - 1)! \frac{1}{\theta^{l+|\eta|+|\xi|-p+1}}.
\]

This lemma is thus proved! □

**Theorem 4.1.** Assume that \( \lim_{\theta \to +\infty} \theta t(\theta) = \infty \). As \( \theta \to +\infty \), for any given integer \( n \geq 2 \), \( P_n^\theta \) has the following LDPs:

- If \( \lim_{\theta \to +\infty} \frac{\theta(t(\theta))}{\log \theta} = k \geq 2 \), then \( P_n^\theta \) has an LDP with speed \( \log \theta \) and rate function \( I_n(\eta) = n - l(\eta) \).
Proposition 1, we know

\[
\log P_X \quad \text{and} \quad \theta \to +\infty.
\]

Remark 3. This theorem indicates that each \( k \in (0, 2) \) serves as a critical value. But since all the partition distributions together determine the distribution of \( X_{\theta(t)} \), \( k = 2 \) should be a critical point for \( X_{\theta(t)} \). Therefore, \( X_{\theta(t)} \) has a critical time scale \( 2 \log \theta \). Furthermore, \( \frac{n \alpha(\eta)}{\theta(t)} \) is the average density of a given partition excluding singletons. Similar result also holds for two-parameter infinite dimensional diffusion in [11] and [5].

Proof. Since

\[
P_{\theta}^\eta(\eta) = \frac{n!}{\eta_1! \cdots \eta_\ell(\eta) ! \alpha_1(\eta)! \cdots \alpha_\ell(\eta)!} \, E_p^\eta(X_{\theta(t)}),
\]

and \( X_{\theta(t)} \to \delta_{(0,0,\ldots)} \), one can easily have

\[
P_{\theta}^\eta(\eta) \to \delta_{(1,1,\ldots,1)}(\eta).
\]

Then \( \log P_{\theta}^\eta(1,1,\ldots,1) \to 0 \), as \( \theta \to +\infty \). So we only need to consider the case \( \eta \neq (1,1,\ldots,1) \). Now we assume \( \eta = (\eta_1, \ldots, \eta_\ell) \) thereby \( l(\eta) = l \). Thanks to Proposition [11] we know

\[
p_{\theta}^\eta = \sum_{d=1}^{l} \prod_{i \in \beta_{(d)}} (-1)^{l-d} \sum_{\beta \in \pi(l,d)} (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! \varphi \sum_{i \in \beta_1} \eta_i \cdots \varphi \sum_{i \in \beta_d} \eta_i.
\]

Let \((\sum_{i \in \beta_{(i)}} \eta_i, \ldots, \sum_{i \in \beta_{(d)}} \eta_i)\) be the decreasing rearrangement of

\[
(\sum_{i \in \beta_{(i)}} \eta_i, \ldots, \sum_{i \in \beta_{(d)}} \eta_i).
\]

We define \( \eta^\beta = (\sum_{i \in \beta_{(i)}} \eta_i, \ldots, \sum_{i \in \beta_{(d)}} \eta_i) \). If \( \sum_{i \in \beta_{(i)}} \eta_i = 1 \), then we delete it such that \( \eta^\beta \in \mathcal{M} \) for we will repeatedly apply lemma [2] but we always have \(|\eta^\beta| - l(\eta^\beta) = |\eta| - d \) and \( |\eta^\beta| \geq |\eta| - \alpha_1(\eta) \). Since

\[
E_p^\eta(X_{\theta(t)}) = \int_{\mathbb{R}^\infty} p_{\theta}^\eta(y) PD(\theta)(dy) + \sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} p_{\theta}^\eta(y) Q_m(x,y) PD(\theta)(dy)
\]

\[
= (\eta_1 - 1)! \cdots (\eta_\ell - 1)! \alpha_1(\eta)! \cdots \alpha_\ell(\eta)! + \sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \sum_{\beta \in \pi(l,d)} \prod_{i=1}^{d} (|\beta_i| - 1)!
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{i \in \beta_{(i)}} (-1)^{l-d} \sum_{\beta \in \pi(l,d)} \prod_{i=1}^{d} (|\beta_i| - 1)!
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]

\[
\sum_{m=2}^{\infty} e^{-\lambda_m t(\theta)} \int_{\mathbb{R}^\infty} \varphi_{\theta^\beta}(y) Q_m(x,y) PD(\theta)(dy),
\]
and \( \varphi_{\eta} = \psi_{\eta} + \sum_{\delta < \xi} \langle \varphi_{\eta}, \psi_{\delta} \rangle \psi_{\delta} \), one can easily have

\[
E p_{\eta}^{\theta}(X_{t(\theta)}) = (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{\theta^l}{\theta(\eta_l)} + \sum_{d=1}^{l} (-1)^{l-d} \prod_{\beta \in \pi(l,d)} (|\beta_i| - 1)! e^{-\lambda_{\eta,\beta} t(\theta)}
\]

(23)

or

\[
= \sum_{m=2}^{\infty} e^{-\lambda_{m,\eta} t(\theta)} \left( \psi_{\eta} \delta_{m,|\eta|} + \sum_{2 \leq \delta < |\eta|} \langle \varphi_{\eta}, \psi_{\delta} \rangle \psi_{\delta} \delta_{\delta m,|\delta|} \right)
\]

(25)

or

\[
= (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{\theta^l}{\theta(\eta_l)} + \sum_{d=1}^{l} (-1)^{l-d} \prod_{\beta \in \pi(l,d)} (|\beta_i| - 1)! \sum_{2 \leq \delta < |\eta|} \langle \varphi_{\eta}, \psi_{\delta} \rangle \psi_{\delta} (x) e^{-\lambda_{\eta,\beta} t(\theta)}
\]

(26)

Recall that \( |\eta| \geq n - 2 \alpha_1(\eta) \); then

\[
\sum_{d=1}^{l} (-1)^{l-d} \prod_{\beta \in \pi(l-d,\alpha_1(\eta), d)} (|\beta_i| - 1)! \langle \varphi_{\eta}, 1 \rangle \psi_{\delta} (x) e^{-\lambda_{\eta,\beta} t(\theta)}
\]

(24)

or

\[
\sim \sum_{d=1}^{l} (-1)^{l-d} \prod_{\beta \in \pi(l-d,\alpha_1(\eta), d)} (|\beta_i| - 1)! \langle \varphi_{\eta}, 1 \rangle \psi_{\delta} (x) e^{-\lambda_{\eta,\beta} t(\theta)}
\]

(25)

By Lemma 3 we know \( \forall \delta < \eta \),

\[
\langle \varphi_{\eta}, \psi_{\delta} \rangle \sim \left( \sum_{i=1}^{l} \sum_{j=1}^{l} \frac{(\eta_i^\beta + \delta_j - 1)!}{(\eta_i^\beta - 1)! (\delta_j - 1)!} - |\eta^\beta||\delta| \right) \]

\[
\sim (\eta_1^\beta - 1)! \cdots (\eta_l^\beta - 1)! (\delta_1 - 1)! \cdots (\delta_l - 1)! \left( \frac{1}{\theta^{\eta^\beta + \delta} - \theta^{\eta^\beta} + \delta} \right)
\]

Since \( |\eta^\beta| - l(\eta^\beta) = n - d \) and \( |\delta| - l(\delta) \geq 1 \), for \( \delta \geq (2) \), one can easily conclude that

\[
|\eta^\beta| - l(\eta^\beta) + |\delta| - l(\delta) + 1 = n - d + |\delta| - l(\delta) + 1 \geq n - d + 2.
\]
Moreover, $1 \leq d \leq l$; then $|\eta| - l(\eta^2) + \delta| - l(\delta) + 1 \geq n - l + 2$, where the equality holds if and only if $d = l$ and $\delta = (2)$. So if $\eta^2 = (2)$, then $(24) = 0$; if $\eta^2 > (2)$, then

$$
(25) \sim \left[ \sum_{i=1}^{l-\alpha_1(\eta)} \frac{(\eta_i + 1)!}{(\eta_i - 1)!} - 2(|\eta| - \alpha_1(\eta)) \right] \prod_{i=1}^{l}(\eta_i - 1)! \phi_2(x) \frac{1}{\theta |\eta| - l + 2} e^{-(d+1)t(\theta)}
$$

and

$$
(26) \sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{1}{\theta^n - l} e^{-\frac{|n-\alpha_1(\eta)|}{2} t(\theta)},
$$

$$
(23) \sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{1}{\theta^{n-1}}.
$$

Obviously, $(23)$ is the leading term among $(23)$, $(25)$ and $(26)$. Then

$$
(23) + (25) + (26) \sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{1}{\theta^{n-1}};
$$

and $(24) \sim p_{\eta_1, \ldots, \eta_{l-1}}(x) e^{-\frac{|n-\alpha_1(\eta)|}{2} t(\theta)}$. Hence,

$$
E p_{\eta}^\theta(X_t(\theta)) \sim (\eta_1 - 1)! \cdots (\eta_l - 1)! \frac{1}{\theta^n - l} + p_{\eta_1, \ldots, \eta_{l-1}}(x) e^{-\frac{|n-\alpha_1(\eta)|}{2} t(\theta)}.
$$

Let us consider three cases: $\lim_{\theta \to +\infty} \frac{\theta(\theta)}{\log \theta} \geq 2$, $\lim_{\theta \to +\infty} \frac{\theta(\theta)}{\log \theta} = k \in (0, 2)$ and $\lim_{\theta \to +\infty} \frac{\theta(\theta)}{\log \theta} = 0$.

**Case I:** $\lim_{\theta \to +\infty} \frac{\theta(\theta)}{\log \theta} = 0$

We know

$$
E p_{\eta}^\theta(X_t(\theta)) \sim e^{-\frac{|n-\alpha_1(\eta)|}{2} t(\theta)} \left[ p_{\eta_1, \ldots, \eta_{l-1}}(x) + \prod_{i=1}^{l} (\eta_i - 1)! e^{-\log \theta - \frac{|n-\alpha_1(\eta)|}{2} t(\theta)} \right].
$$

Then

$$
E p_{\eta}^\theta(X_t(\theta)) \sim e^{-\frac{|n-\alpha_1(\eta)|}{2} t(\theta)} p_{\eta_1, \ldots, \eta_{l-1}}(x).
$$

Therefore,

$$
\lim_{\theta \to +\infty} \frac{1}{\theta(\theta)} \log P_{\eta}^\theta(\eta) = - \frac{|\eta| - \alpha_1(\eta)}{2}.
$$

Now for the remaining two cases, we have

$$
E p_{\eta}^\theta(X_t(\theta)) \sim e^{-\frac{|n-\alpha_1(\eta)|}{2} k \log \theta} \left[ \prod_{i=1}^{l} (\eta_i - 1)! e^{-\log \theta - \frac{|n-\alpha_1(\eta)|}{2} t(\theta)} + p_{\eta_1, \ldots, \eta_{l-1}}(x) e^{-\frac{|n-\alpha_1(\eta)|}{2} t(\theta)/k \log \theta} \right].
$$

Define

$$
D(\theta) = \frac{1}{\log \theta} \log \left[ \prod_{i=1}^{l} (\eta_i - 1)! e^{-\log \theta - \frac{|n-\alpha_1(\eta)|}{2} t(\theta)} + p_{\eta_1, \ldots, \eta_{l-1}}(x) e^{-\frac{|n-\alpha_1(\eta)|}{2} t(\theta)/k \log \theta} \right].
$$
We claim that \( \lim_{\theta \to +\infty} D(\theta) = \max \left\{ 0, -\left[ (n-l) - \frac{|\eta| - \alpha_1(\eta)}{2} \right] \right\} \). Indeed, by lemma 1.2.15 in [1],

\[
\limsup_{\theta \to +\infty} \frac{1}{\log \theta} \log \left[ \prod_{i=1}^{l} (\eta_i - 1)! e^{-(n-l) - \frac{|\eta| - \alpha_1(\eta)}{2} k} \log \theta \right],
\]

\[
\liminf_{\theta \to +\infty} \frac{1}{\log \theta} \log \left[ p_{\eta_1, \ldots, \eta_l}^e (x) e^{-\frac{|\eta| - \alpha_1(\eta)}{2} \log \theta} \log \theta \right],
\]

\[
\geq \max \left\{ -\left[ (n-l) - \frac{|\eta| - \alpha_1(\eta)}{2} k \right], 0 \right\}.
\]

Moreover,

\[
\limsup_{\theta \to +\infty} \frac{1}{\log \theta} \log \left[ \prod_{i=1}^{l} (\eta_i - 1)! e^{-(n-l) - \frac{|\eta| - \alpha_1(\eta)}{2} k} \log \theta \right],
\]

\[
\liminf_{\theta \to +\infty} \frac{1}{\log \theta} \log \left[ p_{\eta_1, \ldots, \eta_l}^e (x) e^{-\frac{|\eta| - \alpha_1(\eta)}{2} \log \theta} \log \theta \right],
\]

\[
\geq \max \left\{ -\left[ (n-l) - \frac{|\eta| - \alpha_1(\eta)}{2} k \right], 0 \right\}.
\]

Therefore, the claim is true. Then we have

\[
\lim_{\theta \to +\infty} \frac{1}{\log \theta} \log P_n^\theta(\eta) = -\frac{n - \alpha_1(\eta)}{2} k + \max \left\{ 0, -\left[ (n-l) - \frac{|\eta| - \alpha_1(\eta)}{2} k \right] \right\}.
\]

**Case II:** \( \lim_{\theta \to +\infty} \frac{\partial \log P_n^\theta(\eta)}{\log \theta} = k \geq 2 \)

For fixed \( k > 0 \), if \( k \geq 2 \), then

\[
\frac{n - \alpha_1(\eta)}{2} k \geq n - \alpha_1(\eta) \geq n - l, \text{ for } \alpha_1(\eta) \leq l.
\]

Additionally \( (n-l) - \frac{n - \alpha_1(\eta)}{2} k \leq 0 \); thus,

\[
\lim_{\theta \to +\infty} \frac{1}{\log \theta} \log P_n^\theta(\eta) = -(n-l).
\]

**Case III:** \( \lim_{\theta \to +\infty} \frac{\partial \log P_n^\theta(\eta)}{\log \theta} = k \in (0, 2) \)

When \( 0 < k < 2 \), we have \( n-l = n - \alpha_1(\eta) - (l - \alpha_1(\eta)) \). Then

\[
(n-l) - \frac{n - \alpha_1(\eta)}{2} k \geq 0 \text{ if and only if } \frac{n - \alpha_1(\eta)}{l - \alpha_1(\eta)} \geq \frac{2}{2-k}
\]

and

\[
(n-l) - \frac{n - \alpha_1(\eta)}{2} k < 0 \text{ if and only if } \frac{n - \alpha_1(\eta)}{l - \alpha_1(\eta)} < \frac{2}{2-k}
\]

Thus,

\[
\lim_{\theta \to +\infty} \frac{1}{\log \theta} \log P_n^\theta(\eta) = \begin{cases} \frac{n - \alpha_1(\eta)}{2} k, & \text{if } \frac{n - \alpha_1(\eta)}{l - \alpha_1(\eta)} \geq \frac{2}{2-k} \\ -(n-l), & \text{if } \frac{n - \alpha_1(\eta)}{l - \alpha_1(\eta)} < \frac{2}{2-k}. \end{cases}
\]

\( \square \)
Finally, due to the LDP for transient sampling formula, we know
\[-\log P_n(\eta) \sim I_n(\eta) \log \theta.\]
Therefore, asymptotically $P_n(\eta)$ behaves as power law.

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