THE FIXED POINT SUBALGEBRA OF A LATTICE VERTEX OPERATOR ALGEBRA BY AN AUTOMORPHISM OF ORDER THREE

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Abstract. We study the subalgebra of the lattice vertex operator algebra $V_{\sqrt{2}A_2}$ consisting of the fixed points of an automorphism which is induced from an order 3 isometry of the root lattice $A_2$. We classify the simple modules for the subalgebra. The rationality and the $C_2$-cofiniteness are also established.

1. Introduction

The space of fixed points of an automorphism group of finite order in a vertex operator algebra is a vertex operator subalgebra. The study of such subalgebras and their modules is called the orbifold theory. It is a rich field both in the conformal field theory and in the theory of vertex operator algebras. However, the orbifold theory is difficult to study in general. One of the reason is that the subalgebra of fixed points usually has more complicated structure than the original vertex operator algebra.

The first example of the orbifold theory in vertex operator algebras is the moonshine module $V^\natural$ by Frenkel, Lepowsky, and Meurman [17]. In their book $V^\natural$ was constructed as an extension of $V^+_A$ by $V^+_\Lambda$, where $V^+_A$ is the space of fixed points of an automorphism $\theta$ of order two in the Leech lattice vertex operator algebra $V_\Lambda$. This construction is called a $2B$-orbifold construction because $\theta$ corresponds to a $2B$ involution of the monster simple group. More generally, a vertex operator algebra $V_L$ associated with an arbitrary positive definite even lattice $L$ was defined in [17]. Those lattice vertex operator algebras provide a large family of vertex operator algebras. Such a lattice vertex operator algebra admits an automorphism $\theta$ of order two, which is a lift of the isometry $\alpha \mapsto -\alpha$ of the underlying lattice $L$. The orbifold theory for the fixed point subalgebra $V^+_L$ of $\theta$ has been developed extensively. In fact, the simple $V^+_L$-modules were classified [2] and the fusion rules were determined [3]. Furthermore, the $C_2$-cofiniteness of $V^+_L$ was established [1, 35].

In this paper we study the fixed point subalgebra by an automorphism of order three for a certain lattice vertex operator algebra. Namely, let $L = \sqrt{2}A_2$ be $\sqrt{2}$ times an ordinary root lattice of type $A_2$ and $\tau$ be an isometry of the root lattice of type $A_2$ induced from an order three permutation on the set of positive roots. We classify the simple modules for the subalgebra $V^\tau_L$ of fixed points by $\tau$. Moreover, we show that $V^\tau_L$ is rational and $C_2$-cofinite.

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In previous papers \cite{8, 20} we have already discussed the vertex operator algebra $V_L^\tau$. It was shown that $V_L^\tau = M^0 \oplus W^0$ is a direct sum of a subalgebra $M^0$ and its simple highest weight module $W^0$. Actually, $M^0$ is a tensor product of a $W_3$ algebra of central charge $6/5$ and a $W_3$ algebra of central charge $4/5$. The property of a $W_3$ algebra of central charge $6/5$ as the first component of the tensor product $M^0$ was investigated in \cite{8}. It is generated by the Virasoro element $\tilde{\omega}^1$ and a weight 3 vector $J$. On the other hand, the second component of $M^0$, which is a $W_3$ algebra of central charge $4/5$, was studied in \cite{21}. It is generated by the Virasoro element $\tilde{\omega}^2$ and a weight 3 vector $K$. Each of these $W_3$ algebras possesses a symmetry of order three. The order three symmetry of the second $W_3$ algebra is related to the $\mathbb{Z}_3$ part of $L^+/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $L^+$ denotes the dual lattice of $L$. As an $M^0$-module, $W^0$ is generated by a highest weight vector $P$ of weight 2. Thus the vertex operator algebra $V_L^\tau$ is generated by the five elements $\tilde{\omega}^1$, $\tilde{\omega}^2$, $J$, $K$, and $P$.

There are 12 inequivalent simple $V_L^\tau$-modules, which correspond to the cosets of $L$ in its dual lattice $L^+$ (cf. \cite{20}). Let $(U, Y_U)$ be a simple $V_L^\tau$-module. One can define a new simple $V_L^\tau$-module $(U \circ \tau, Y_{U\circ\tau})$ by $U \circ \tau = U$ as vector spaces and $Y_{U\circ\tau}(v, z) = Y_U(\tau v, z)$ for $v \in V_L$. Then $U \mapsto U \circ \tau$ is a permutation on the set of simple $V_L^\tau$-modules. In the case where $U$ and $U \circ \tau$ are equivalent $V_L^\tau$-modules, $U$ is said to be $\tau$-stable. If $U$ is $\tau$-stable, then the eigenspace $U(\xi)$ of $\tau$ with eigenvalue $\xi^\epsilon$, where $\xi = \exp(2\pi \sqrt{-1}/3)$, $\epsilon = 0, 1, 2$, is a simple $V_L^\tau$-module, while if $U$ belongs to a $\tau$-orbit of length 3, then $U$ itself is a simple $V_L^\tau$-module and the three members in the $\tau$-orbit are equivalent each other (cf. \cite{15} Theorem 6.14). Among the 12 inequivalent simple $V_L^\tau$-modules, three of them are $\tau$-stable and the remaining nine simple $V_L^\tau$-modules are divided into three $\tau$-orbits. In this way we obtain 12 simple $V_L^\tau$-modules. It is known that there are three inequivalent simple $\tau$-twisted (resp. $\tau^2$-twisted) $V_L^\tau$-modules $V_L^{T\chi(j)}(\tau)$ (resp. $V_L^{T\chi(j)}(\tau^2)$), $j = 0, 1, 2$. The automorphism $\tau$ acts on these $\tau$-twisted or $\tau^2$-twisted $V_L^\tau$-modules and each eigenspace of $\tau$ is a simple $V_L^\tau$-module (cf. \cite{30} Theorem 2). There are 18 such simple $V_L^\tau$-modules. Furthermore, all of these simple $V_L^\tau$-modules are inequivalent. Hence there are at least 30 inequivalent simple $V_L^\tau$-modules.

The main part of our argument is to show that every simple $V_L^\tau$-module is isomorphic to one of the 30 above mentioned simple $V_L^\tau$-modules. Recall that $V_L^\tau = M^0 \oplus W^0$ and $M^0$ is a tensor product of two $W_3$ algebras. The $W_3$ algebra of central charge $6/5$ (resp. $4/5$) possesses 20 (resp. 6) inequivalent simple modules. Thus there are 120 inequivalent simple $M^0$-modules. It turns out that among these simple $M^0$-modules, 60 of them can not appear as an $M^0$-submodule in any simple $V_L^\tau$-module and that each simple $V_L^\tau$-module is a direct sum of two of the remaining 60 simple $M^0$-modules. We note that $W^0$ is not a simple current $M^0$-module, namely, $V_L^\tau$ is a nonsimple current extension of $M^0$. A discussion on simple modules for another nonsimple current extension of a certain vertex operator algebra can be found in \cite{23} Appendix C).

The organization of this paper is as follows. In Section 2 we review various notions about untwisted or twisted modules for vertex operator algebras, together with some basic tools which will be used in later sections. In Section 3 we fix notation for the vertex operator algebra $V_L^\tau$ and collect its properties. We clarify an argument on the simplicity of $M^0_\tau(\tau^i)$ and $W^0_\tau(\tau^i)$, $i = 1, 2$, in \cite{20} Proposition 6.8]. Furthermore, we correct some misprints in \cite{20} (6.46)] and in an equation of \cite{8} page 265 concerning a decomposition
of a simple \(\tau\)-twisted \(V_L\)-module \(V_L^{T_\omega}(r)\), \(j = 1, 2\) as a \(\tau\)-twisted \(\mathfrak{g}_k^0 \otimes \mathfrak{m}_l^0\)-module (see Remark 4.51). In Section 4 we discuss the structure of the 30 known simple \(V_L^r\)-modules. In particular, we calculate the action of \(o(\tilde{\omega}^1), o(\tilde{\omega}^2), o(J), o(K),\) and \(o(P)\) on the top level of these simple modules. Finally, in Section 5 we complete the classification of simple \(V_L^r\)-modules. We also show the rationality of \(V_L^r\).

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2. Preliminaries

We recall some notation for untwisted or twisted modules for a vertex operator algebra. We also review the twisted version of Zhu’s theory. A basic reference to twisted modules is [11]. For untwisted modules, see also [25]. Let \((V, Y, \omega)\) be a vertex operator algebra and \(g\) be an automorphism of \(V\) of finite order \(T\). Set \(V^r = \{v \in V \mid gv = e^{2\pi \sqrt{-1}r/T}v\}\), so that \(V = \oplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r\).

**Definition 2.1.** A weak \(g\)-twisted \(V\)-module \(M\) is a vector space equipped with a linear map

\[
Y_M(\cdot, z) : v \in V \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \in (\text{End } M)\{z\}
\]

which satisfies the following conditions.

1. \(Y_M(v, z) = \sum_{n \in \mathbb{Z}+r\mathbb{Z}} v_n z^{-n-1}\) for \(v \in V^r\).
2. \(v_n w = 0\) if \(n \gg 0\), where \(v \in V\) and \(w \in M\).
3. \(Y_M(1, z) = \text{id}_M\).
4. For \(u \in V^r\) and \(v \in V\), the \(g\)-twisted Jacobi identity holds.

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2) Y_M(u, z_1)
= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0) v, z_2).
\]

(2.1)

Compare the coefficients of \(z_0^{-l-1}z_1^{-m-1}z_2^{-n-1}\) in both sides of (2.1) for \(u \in V^r, v \in V^s, l \in \mathbb{Z}, m \in \mathbb{Z} + \mathbb{T}, n \in \mathbb{Z} + \mathbb{T}\). Then we obtain the following identity.

\[
\sum_{i=0}^{\infty} \left( \begin{array}{c} m \\ i \end{array} \right) (u_{l+i} v)_{m+n-i} = \sum_{i=0}^{\infty} (-1)^i \left( \begin{array}{c} l \\ i \end{array} \right) (u_{l+m-i} v_{n+i} - (-1)^i v_{l+n-i} u_{m+i}).
\]

(2.2)

In the case \(l = 0\), (2.2) reduces to

\[
[u_m, v_n] = \sum_{i=0}^{\infty} \left( \begin{array}{c} m \\ i \end{array} \right) (u_i v)_{m+n-i}.
\]

(2.3)
The Virasoro element $\omega$ is contained in $V^0$. Let $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$. Then

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\text{rank}V),$$

$$\frac{d}{dz} Y_M(v, z) = Y_M(L(-1)v, z)$$

for $v \in V$ \cite{11} (3.8), (3.9).

An important consequence of \cite{21} is the associativity formula \cite{11} (3.5)

$$(z_0 + z_2)^{k + t/T}Y_M(u, z_0 + z_2)Y_M(v, z_2)w = (z_2 + z_0)^{k + r/T}Y_M(Y(u, z_0)v, z_2)w,$$  \tag{2.4}

where $u \in V^r$, $v \in V$, $w \in M$, and $k$ is a nonnegative integer such that $z^{k + r/T}Y_M(u, z)w \in M[[z]]$.

Let $(M, Y_M)$ and $(N, Y_N)$ be weak $g$-twisted $V$-modules. A homomorphism of $M$ to $N$ is a linear map $f : M \to N$ such that $fY_M(v, z) = Y_N(v, z)f$ for all $v \in V$.

Let $\mathbb{N}$ be the set of nonnegative integers.

**Definition 2.2.** A $\frac{1}{T} \mathbb{N}$-graded weak $g$-twisted $V$-module $M$ is a weak $g$-twisted $V$-module with a $\frac{1}{T} \mathbb{N}$-grading $M = \bigoplus_{n \in \frac{1}{T} \mathbb{N}} M(n)$ such that

$$v_m M(n) \subset M(n + \text{wt}(v) - m - 1)$$  \tag{2.5}

for any homogeneous vectors $v \in V$.

A $\frac{1}{T} \mathbb{N}$-graded weak $g$-twisted $V$-module here is called an admissible weak $g$-twisted $V$-module in \cite{11}. Without loss we can shift the grading of a $\frac{1}{T} \mathbb{N}$-graded weak $g$-twisted $V$-module $M$ so that $M(0) \neq 0$ if $M \neq 0$. We call such an $M(0)$ the top level of $M$.

**Definition 2.3.** A $g$-twisted $V$-module $M$ is a weak $g$-twisted $V$-module with a $\mathbb{C}$-grading $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$. Moreover, each $M_\lambda$ is a finite dimensional space and for any fixed $\lambda$, $M_{\lambda + n/T} = 0$ for all sufficiently small integers $n$.

A $g$-twisted $V$-module is sometimes called an ordinary $g$-twisted $V$-module. By \cite{11} Lemma 3.4, any $g$-twisted $V$-module is a $\frac{1}{T} \mathbb{N}$-graded weak $g$-twisted $V$-module. Indeed, assume that $M$ is a $g$-twisted $V$-module. For each $\lambda \in \mathbb{C}$ with $M_\lambda \neq 0$, let $\lambda_0 = \lambda + m/T$ be such that $m \in \mathbb{Z}$ is minimal subject to $M_{\lambda_0} \neq 0$. Let $\Lambda$ be the set of all such $\lambda_0$ and let $M(n) = \bigoplus_{\lambda \in \Lambda} M_{n + \lambda}$. Then $M(n)$ satisfies the condition in Definition 2.2. Thus we have the following inclusions.

$$\{g\text{-twisted }V\text{-modules}\} \subset \{\frac{1}{T} \mathbb{N}\text{-graded weak }g\text{-twisted }V\text{-modules}\} \subset \{\text{weak }g\text{-twisted }V\text{-modules}\}$$

**Definition 2.4.** A vertex operator algebra $V$ is said to be $g$-rational if every $\frac{1}{T} \mathbb{N}$-graded weak $g$-twisted $V$-module is semisimple, that is, a direct sum of simple $\frac{1}{T} \mathbb{N}$-graded weak $g$-twisted $V$-modules.

Let $M$ be a weak $g$-twisted $V$-module. The next lemma is a twisted version of \cite{28} Lemma 3.12]. In fact, using the associativity formula \cite{21} we can prove it by essentially the same argument as in \cite{28}. 
Lemma 2.5. Let \( u \in V^r, v \in V^s, w \in M, \) and \( k \) be a nonnegative integer such that \( z^{k+r/T}Y_M(u,z)w \in M[[z]]. \) Let \( p \in \frac{r}{T} + \mathbb{Z}, q \in \frac{s}{T} + \mathbb{Z}, \) and \( N \) be a nonnegative integer such that \( z^{N+1+q}Y_M(v,z)w \in M[[z]]. \) Then

\[
 u_p v_q w = \sum_{i=0}^{N} \sum_{j=0}^{\infty} \binom{p - k - r/T}{i} \binom{k + r/T}{j} (u_{p-k-r/T} v_{q+k+r/T} z^{-i}) w.
\]  

(2.6)

Conversely, \((u_p v_q) w\) can be written as a linear combination of some vectors of the form \(u, v_j \).

Lemma 2.6. Let \( u \in V^r, v \in V^s, w \in M. \) Then for \( p \in \mathbb{Z} \) and \( q \in \frac{r+s}{T} + \mathbb{Z}, \) the vector \((u_p v_q) w\) is a linear combination of \(u_i v_j w\) with \( i \in \frac{p}{T} + \mathbb{Z}\) and \( j \in \frac{q}{T} + \mathbb{Z}.\)

Proof. Let \( X = \text{span}\{u_i v_j w \mid i \in \frac{p}{T} + \mathbb{Z}, j \in \frac{q}{T} + \mathbb{Z}\}. \) We use (2.2). Take \( m \in \frac{p}{T} + \mathbb{Z} \) such that \( u_{m+i} w = 0 \) for \( i \geq 0. \) Let \( N \in \mathbb{Z} \) be such that \( u_{N+i} w = 0 \) for \( i > 0. \) If \( p > N, \) then \( u_p v_0 w = 0 \) and the assertion is trivial. Assume that \( p \leq N. \) For \( j = 0, 1, \ldots, N - p, \) let \( l = p + j \) and \( n = q - m - j \) in (2.2). Then

\[
 \sum_{i=0}^{\infty} \binom{m}{i} (u_{p+j+i} v_{q-j-i} w) = \sum_{i=0}^{\infty} (-1)^i \binom{p+j}{i} u_{p+m+j-i} v_{q-m-j+i} w.
\]

The right hand side of this equation is contained in \( X. \) Consider the left hand side for each of \( j = N-p, N-p-1, \ldots, 1, 0. \) Then we see that \( (u_N v_q) w \in X, (u_{N-1} v_q) w \in X, \ldots, \) and \( (u_p v_q) w \in X. \) \( \square \)

For subsets \( A, B \) of \( V \) and a subset \( X \) of \( M, \) set \( A \cdot X = \text{span}\{u_n w \mid u \in A, w \in X, n \in \frac{1}{T} \mathbb{Z}\} \) and \( A \cdot B = \text{span}\{u_n v \mid u \in A, v \in B, n \in \mathbb{Z}\}. \) Then it follows from (2.6) that \( A \cdot (B \cdot X) \subset (A \cdot B) \cdot X. \) For a vector \( w \in M, \) this in particular implies that \( V \cdot w \) is a weak \( g \)-twisted \( V \)-submodule of \( M. \) If \( w \) is an eigenvector for \( L(0), \) then \( V \cdot w \) is a direct sum of eigenspaces for \( L(0). \) Each eigenspace is not necessarily of finite dimension. Thus \( V \cdot w \) is not a \( g \)-twisted module in general. This subject was discussed in [11][33]. We will review it later in this section.

In [36], Zhu introduced an associative algebra \( A(V) \) called the Zhu algebra for a vertex operator algebra \( V, \) which plays a crucial role in the study of representations for \( V. \) Later, Dong, Li and Mason [11] constructed an associative algebra \( A_g(V) \) called the \( g \)-twisted Zhu algebra in order to generalize Zhu’s theory to \( g \)-twisted representations for \( V. \) The definition of \( A_g(V) \) is similar to that of \( A(V). \) Let \( V, g, T, \) and \( V^r \) be as before. Roughly speaking, \( A_g(V) = V/O_g(V) \) is a quotient space of \( V \) with a binary operation \( *_g. \) It is in fact an associative algebra with respect to \( *_g. \) If \( r \neq 0, \) then \( V^r \subset O_g(V). \) Thus \( A_g(V) = (V^0 + O_g(V))/O_g(V). \) For the case \( g = 1, \) see (5.1) in Section 5.

A certain Lie algebra \( V[g] \) was considered in [11]. Any weak \( g \)-twisted \( V \)-module is a module for the Lie algebra \( V[g] \) (cf. [11] Lemma 5.1]). Moreover, for a \( V[g] \)-module \( M, \) the space \( \Omega(M) \) of lowest weight vectors with respect to \( V[g] \) was defined. If \( M \) is a weak \( g \)-twisted \( V \)-module, then \( \Omega(M) \) is the set of \( w \in M \) such that \( v_{wt(v) - 1 + n} w = 0 \) for all homogeneous vectors \( v \in V \) and \( 0 < n \in \frac{1}{T} \mathbb{Z}. \) The map \( v \mapsto \alpha(v) \) for homogeneous vectors \( v \in V^0 \) induces a representation of the associative algebra \( A_g(V) \) on \( \Omega(M), \) where \( \alpha(v) = v_{wt(v) - 1}. \) If \( M \) is a \( \frac{1}{T} \mathbb{N} \)-graded weak \( g \)-twisted \( V \)-module, then the top level \( M(0) \) is contained in \( \Omega(M). \) In the case where \( M \) is a simple \( \frac{1}{T} \mathbb{N} \)-graded weak \( g \)-twisted \( V \)-module, \( M(0) = \Omega(M) \) and \( M(0) \) is a simple \( A_g(V) \)-module (cf. [11] Proposition 5.4]).
For any $A_g(V)$-module $U$, a certain $\frac{1}{7}N$-graded $V[g]$-module $M(U)$ such that $M(U)_{(0)} = U$ was defined (cf. [11 (6.1)]). Let $W$ be the subspace of $M(U)$ spanned by the coefficients of

$$(z_0 + z_2)^{\text{wt}(u)-1+\delta_r+r/T}Y_M(u, z_0 + z_2)Y_M(v, z_2)w$$

for all homogeneous $u \in V^*, v \in V, w \in U$ (cf. [11 (6.3)]). Set $\bar{M}(U) = M(U)/U[V[g]]W$, which is a quotient module of $M(U)$ by the $V[g]$-submodule generated by $W$.

The following results will be necessary in Sections 3 and 4.

**Theorem 2.7.** ([11 Theorem 6.2]) $\bar{M}(U)$ is a $\frac{1}{7}N$-graded weak $g$-twisted $V$-module such that its top level $\bar{M}(U)_{(0)}$ is equal to $U$ and such that it has the following universal property: for any weak $g$-twisted $V$-module $M$ and any homomorphism $\varphi : U \to \Omega(M)$ of $A_g(V)$-modules, there is a unique homomorphism $\bar{\varphi} : \bar{M}(U) \to M$ of weak $g$-twisted $V$-modules which is an extension of $\varphi$.

Let $J$ be the sum of all $\frac{1}{7}N$-graded $V[g]$-submodules of $M(U)$ which intersect trivially with $U$. Since $M(U)_{(0)} = U$, it is a unique $\frac{1}{7}N$-graded $V[g]$-submodule of $M(U)$ being maximal subject to $J \cap U = 0$. The principal point is that $U(V[g])W \subset J$. Set $L(U) = M(U)/J$.

**Theorem 2.8.** ([11 Theorem 6.3]) $L(U)$ is a $\frac{1}{7}N$-graded weak $g$-twisted $V$-module such that $\Omega(L(U)) \cong U$ as $A_g(V)$-modules.

**Remark 2.9.** If $M$ is a $\frac{1}{7}N$-graded weak $g$-twisted $V$-module and $\varphi : U \to M_{(0)}$ is a homomorphism of $A_g(V)$-modules, then the homomorphism $\bar{\varphi} : \bar{M}(U) \to M$ of weak $g$-twisted $V$-modules in Theorem 2.7 preserves the $\frac{1}{7}N$-grading. Indeed, $\bar{M}(U) = \text{span}\{v_n U \mid v \in V, n \in \frac{1}{7}Z\}$ by (2.6), since $\bar{M}(U)$ is generated by $U$ as a $\frac{1}{7}N$-graded weak $g$-twisted $V$-module. By (2.5), $v_{\text{wt}(v) - 1 - n}\bar{M}(U)_{(0)} \subseteq \bar{M}(U)_{(n)}$ for any homogeneous $v \in V$ and $n \in \frac{1}{7}Z$. Since $\bar{M}(U)_{(0)} = U$, it follows that $\bar{M}(U)_{(n)}$ is spanned by $v_{\text{wt}(v) - 1 - n}U$ for all homogeneous $v \in V$. Now, $\bar{\varphi}(v_{\text{wt}(v) - 1 - n}U) = v_{\text{wt}(v) - 1 - n}\varphi(U)$ is contained in $v_{\text{wt}(v) - 1 - n}M_{(0)} \subset M_{(n)}$. Hence $\bar{\varphi}(\bar{M}(U)_{(n)}) \subseteq M_{(n)}$ as required. In the case where both of $\bar{M}(U)$ and $M$ are ordinary $g$-twisted $V$-modules, $\bar{\varphi}$ becomes a homomorphism of ordinary $g$-twisted $V$-modules since $\bar{\varphi}$ commutes with $L(0)$.

**Lemma 2.10.** Let $U$ be an $A_g(V)$-module. Let $S$ be a $\frac{1}{7}N$-graded weak $g$-twisted $V$-module such that it is generated by its top level $S_{(0)}$ and such that $S_{(0)}$ is isomorphic to $U$ as an $A_g(V)$-module. Then there is a surjective homomorphism $S \to L(U)$ of weak $g$-twisted $V$-modules which preserves the $\frac{1}{7}N$-grading.

**Proof.** By Theorem 2.7 and Remark 2.9, an isomorphism $\varphi : U \to S_{(0)}$ of $A_g(V)$-modules can be extended to a surjective homomorphism $\bar{\varphi} : \bar{M}(U) \to S$ of weak $g$-twisted $V$-modules which preserves the $\frac{1}{7}N$-grading. The kernel $\ker \bar{\varphi}$ of $\bar{\varphi}$ intersects trivially with $\bar{M}(U)_{(0)}$ and so it is contained in $\oplus_{0 \leq n \leq \frac{1}{7}N} \bar{M}(U)_{(n)}$. Let $I$ be a $\frac{1}{7}N$-graded $V[g]$-submodule of $M(U)$ such that $\ker \bar{\varphi} = I/U(V[g])W$. Then $I \cap U = 0$. This implies that $I \subset J$. Hence $L(U) = M(U)/J$ is a homomorphic image of $M(U)/I \cong S$. $\square
Theorem 2.11. ([11, Theorem 7.2]) \( L \) is a functor from the category of simple \( A_g(V) \)-modules to the category of simple \( \frac{1}{N} \)-graded weak \( g \)-twisted \( V \)-modules such that \( \Omega \circ L = \text{id} \) and \( L \circ \Omega = \text{id} \).

Theorem 2.12. ([11, Theorem 8.1]) If \( V \) is a \( g \)-rational vertex operator algebra, then the following assertions hold.

1. \( A_g(V) \) is a finite dimensional semisimple associative algebra.
2. \( V \) has only finitely many isomorphism classes of simple \( \frac{1}{N} \)-graded weak \( g \)-twisted \( V \)-modules.
3. Every simple \( \frac{1}{N} \)-graded weak \( g \)-twisted \( V \)-module is an ordinary \( g \)-twisted \( V \)-module.

In case of \( g = 1 \), the above argument reduces to the untwisted case. In particular, \( A_g(V) \) is identical with the original Zhu algebra \( A(V) \) if \( g = 1 \).

There is an important intrinsic property of a vertex operator algebra, namely, the \( C_2 \)-cofiniteness. Let \( C_2(V) = \text{span} \{ u_{-2}v \mid u, v \in V \} \). More generally, we set \( C_2(M) = \text{span} \{ u_{-2}w \mid u \in V, w \in M \} \) for a weak \( V \)-module \( M \). If the dimension of the quotient space \( V/C_2(V) \) is finite, \( V \) is said to be \( C_2 \)-cofinite. Similarly, a weak \( V \)-module \( M \) is said to be \( C_2 \)-cofinite if \( M/C_2(M) \) is of finite dimension. The notion of \( C_2 \)-cofiniteness of a vertex operator algebra was first introduced by Zhu [36]. The subspace \( C_2(M) \) of a weak \( V \)-module \( M \) was studied in Li [26]. We refer the reader to [31] also.

Theorem 2.13. ([12, Proposition 3.6]) If \( V \) is \( C_2 \)-cofinite, then \( A_g(V) \) is of finite dimension.

If \( V = \bigoplus_{n=0}^{\infty} V_n \) and \( V_0 = \text{C1} \), then \( V \) is said to be of CFT type. Here \( V_n \) denotes the homogeneous subspace of weight \( n \), that is, the eigenspace of \( L(0) = \omega_1 \) with eigenvalue \( n \).

Theorem 2.14. ([14, Lemma 3.3]) Suppose \( V \) is \( C_2 \)-cofinite and of CFT type. Choose a finite dimensional \( L(0) \)-invariant and \( g \)-invariant subspace \( U \) of \( V \) such that \( V = U + C_2(V) \). Let \( W \) be a weak \( g \)-twisted \( V \)-module generated by a vector \( w \). Then \( W \) is spanned by the vectors of the form \( u_{-n_1}^1 u_{-n_2}^2 \cdots u_{-n_k}^k w \) with \( n_1 > n_2 > \cdots > n_k > -N \) and \( u^i \in U \), \( i = 1, 2, \ldots, k \), where \( N \in \frac{1}{N} \mathbb{Z} \) is a constant such that \( u_m w = 0 \) for all \( u \in U \) and \( m \geq N \).

Theorem 2.15. ([31, Corollaries 3.8 and 3.9]) Suppose \( V \) is \( C_2 \)-cofinite and of CFT type. Then the following assertions hold.

1. Every weak \( g \)-twisted \( V \)-module is a \( \frac{1}{N} \)-graded weak \( g \)-twisted \( V \)-module.
2. Every simple weak \( g \)-twisted \( V \)-module is a simple ordinary \( g \)-twisted \( V \)-module.

Remark 2.16. Suppose \( V \) is \( C_2 \)-cofinite and of CFT type. Let \( M \) be a weak \( g \)-twisted \( V \)-module and \( w^1, \ldots, w^k \) be eigenvectors of \( L(0) \) in \( M \). Then the weak \( g \)-twisted \( V \)-submodule \( W \) generated by \( w^1, \ldots, w^k \) is an ordinary \( g \)-twisted \( V \)-module. Indeed, \( W \) is a direct sum of eigenspaces for \( L(0) \) and each homogeneous subspace is of finite dimension by Theorem 2.14.

For the untwisted case, that is, the case \( g = 1 \), we refer the reader to [1, 5, 10, 26, 31]. A spanning set for a vertex operator algebra was first studied in [18, Proposition 8].
3. Fixed point subalgebra \((V_{\sqrt{2}A_2})^\tau\)

In this section we fix notation. We tend to follow the notation in \[8, 19, 20\] unless otherwise specified. We also recall certain properties of the lattice vertex operator algebra \(V_{\sqrt{2}A_2}\) associated with \(\sqrt{2}\) times an ordinary root lattice of type \(A_2\) and its subalgebras (cf. \[8, 19, 20, 21\]).

Let \(\alpha_1, \alpha_3\) be the simple roots of type \(A_2\) and set \(\alpha_0 = -(\alpha_1 + \alpha_2)\). Thus \(\langle \alpha_i, \alpha_j \rangle = 2\) and \(\langle \alpha_i, \alpha_j \rangle = -1\) if \(i \neq j\). Set \(\beta_i = \sqrt{2} \alpha_i\) and let \(L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2\) be the lattice spanned by \(\beta_1\) and \(\beta_2\). We denote the cosets of \(L\) in its dual lattice \(L^+ = \{ \alpha \in \mathbb{Q} \odot \mathbb{Z} | \langle \alpha, L \rangle \subset \mathbb{Z} \}\) as follows.

\[
L^0 = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L,
\]

\[
L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,
\]

\[
L^{(i,j)} = L_i + L^j
\]

for \(i = 0, a, b, c\) and \(j = 0, 1, 2\), where \(\{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) is Klein’s four-group. Note that \(L^{(i,j)}\), \(i \in \{0, a, b, c\}, j \in \{0, 1, 2\}\) are all the cosets of \(L\) in \(L^1\) and \(L^1/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3\).

We adopt the standard notation for the vertex operator algebra \((V_L, Y(\cdot, z))\) associated with the lattice \(L\) (cf. \[17\]). In particular, \(\mathfrak{h} = \mathbb{C} \odot L\) is an abelian Lie algebra, \(\hat{\mathfrak{h}} = \mathfrak{h} \odot \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c\) is the corresponding affine Lie algebra, \(M(1) = \mathbb{C}[\alpha(n) : \alpha \in \mathfrak{h}, n < 0]\), where \(\alpha(n) = \alpha \otimes t^n\) is the unique simple \(\hat{\mathfrak{h}}\)-module such that \(\alpha(n)1 = 0\) for all \(\alpha \in \mathfrak{h}\) and \(n > 0\) and \(c = 1\). As a vector space \(V_L = M(1) \otimes \mathbb{C}[L]\) and for each \(v \in V_L\), a vertex operator \(Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in \text{End}(V_L)[[z, z^{-1}]]\) is defined. The vector \(1 = 1 \otimes 1\) is called the vacuum vector. In our case \(\langle \alpha, \beta \rangle \in 2\mathbb{Z}\) for any \(\alpha, \beta \in L\). Thus the twisted group algebra \(\mathbb{C}\{L\}\) of \(\mathbb{C}\{L^1\}\) is naturally isomorphic to the ordinary group algebra \(\mathbb{C}\{L\}\).

There are exactly 12 inequivalent simple \(V_L\)-modules, which are represented by \(V_{L^{(i,j)}}\), \(i = 0, a, b, c\) and \(j = 0, 1, 2\) (cf. \[2\]). We use the symbol \(e^\alpha, \alpha \in L^1\) to denote a basis of \(\mathbb{C}\{L^1\}\).

We consider the following three isometries of \((L, \langle \cdot, \cdot \rangle)\).

\[
\tau : \beta_1 \rightarrow \beta_2 \rightarrow \beta_0 \rightarrow \beta_1,
\]
\[
\sigma : \beta_1 \rightarrow \beta_2, \quad \beta_2 \rightarrow \beta_1,
\]
\[
\theta : \beta_i \rightarrow -\beta_i, \quad i = 1, 2.
\]

Note that \(\tau\) is fixed-point-free and of order 3. The isometries \(\tau, \sigma, \) and \(\theta\) of \(L\) can be extended linearly to isometries of \(L^1\). Moreover, the isometry \(\tau\) lifts naturally to an automorphism of \(V_L\):

\[
\alpha^1(-n_1) \cdots \alpha^k(-n_k)e^\beta \mapsto (\tau^\alpha)(-n_1) \cdots (\tau^\alpha)(-n_k)e^{\tau \beta}.
\]

By abuse of notation, we denote it by \(\tau\) also. We can consider the action of \(\tau\) on \(V_{L^{(i,j)}}\) in a similar way. We apply the same argument to \(\sigma\) and \(\theta\). Our purpose is the classification of simple modules for the fixed point subalgebra \(V_L^\tau = \{ v \in V_L | \tau v = v \}\) of \(V_L\) by the automorphism \(\tau\).

For a simple \(V_L\)-module \((U, Y_U)\), let \((U \circ \tau, Y_{U \circ \tau})\) be a new \(V_L\)-module such that \(U \circ \tau = U\) as vector spaces and \(Y_{U \circ \tau}(v, z) = Y_U(\tau v, z)\) for \(v \in V_L\) (cf. \[12\]). Then \(U \mapsto U \circ \tau\) induces a permutation on the set of simple \(V_L\)-modules. If \(U\) and \(U \circ \tau\) are equivalent \(V_L\)-modules,
$U$ is said to be $\tau$-stable. The following lemma is a straightforward consequence of the definition of $V_{L,(\omega, \beta)}$.

**Lemma 3.1.** (1) $V_{L,(\omega, \beta)}$, $j = 0, 1, 2$ are $\tau$-stable.

(2) $V_{L,(\omega, \beta)} \circ \tau = V_{L,(\omega, \beta)}$, $V_{L,(\omega, \beta)} \circ \tau = V_{L,(\omega, \beta)}$, and $V_{L,(\omega, \beta)} \circ \tau = V_{L,(\omega, \beta)}$, $j = 0, 1, 2$.

A family of simple twisted modules for lattice vertex operator algebras was constructed in [9 24]. Following [9], three inequivalent simple $\tau$-twisted $V_L$-modules $(V_L^{T_{\chi_j}}(\tau), Y^\tau(\cdot, z))$, $j = 0, 1, 2$ were studied in [8 20]. By the above lemma and [12 Theorem 10.2], we know that $(V_L^{T_{\chi_j}}(\tau), Y^\tau(\cdot, z))$, $j = 0, 1, 2$, are all the inequivalent simple $\tau$-twisted $V_L$-modules. Similarly, there are exactly three inequivalent simple $\tau^2$-twisted $V_L$-modules $(V_L^{T_{\chi_j}^2}(\tau^2), Y^{\tau^2}(\cdot, z))$, $j = 0, 1, 2$.

We use the same notation for $(V_L^{T_{\chi_j}}(\tau), Y^\tau(\cdot, z))$ and $(V_L^{T_{\chi_j}^2}(\tau^2), Y^{\tau^2}(\cdot, z))$ as in [8 Section 4]. Thus

$$V_L^{T_{\chi_j}}(\tau) = S[\tau] \otimes T_{\chi_j},$$

where $T_{\chi_j}$, $j = 0, 1, 2$ are the one-dimensional representations of a certain central extension of $L$ affording the character $\chi_j$. Let

$$h_1 = \frac{1}{3}(\beta_1 + \xi^2\beta_2 + \xi\beta_0), \quad h_2 = \frac{1}{3}(\beta_1 + \xi\beta_2 + \xi^2\beta_0).$$

Then $\tau h_1 = \xi h_1$, $\langle h_1, h_1 \rangle = \langle h_2, h_2 \rangle = 0$, and $\langle h_1, h_2 \rangle = 2$. Moreover, $\beta_i = \xi^{i-1}h_1 + \xi^{2(i-1)}h_2$, $i = 0, 1, 2$. As a vector space, $S[\tau]$ is isomorphic to a polynomial algebra with variables $h_1(1/3+n)$, $h_2(2/3+n)$, $n \in \mathbb{Z}_{<0}$. The isometry $\tau$ acts on $S[\tau]$ by $\tau h_j = \xi^2 h_j$. We define the action of $\tau$ on $T_{\chi_j}$ to be the identity. The weight in $S[\tau]$ is given by $\text{wt } h_i(i/3+n) = -i/3 - n$, $i = 1, 2$ and $\text{wt } 1 = 1/9$. The weight of any element of $T_{\chi_j}$ is defined to be 0. Note that the weight in $V_L^{T_{\chi_j}}(\tau)$ is identical with the eigenvalue for the action of the coefficient of $z^{-2}$ in the $\tau$-twisted vertex operator $Y^\tau(\omega, z)$, where $\omega$ denotes the Virasoro element of $V_L$.

The simple $\tau^2$-twisted $V_L$-modules $(V_L^{T_{\chi_j}^2}(\tau^2), Y^{\tau^2}(\cdot, z))$, $j = 0, 1, 2$ is

$$V_L^{T_{\chi_j}^2}(\tau^2) = S[\tau^2] \otimes T_{\chi_j},$$

where $T_{\chi_j}$, $j = 0, 1, 2$ are the one-dimensional representations of a certain central extension of $L$ affording the character $\chi_j$. Moreover, $S[\tau^2]$ is isomorphic to a polynomial algebra with variables $h'_1(1/3+n)$, $h'_2(2/3+n)$, $n \in \mathbb{Z}_{<0}$ as a vector space, where $h'_1 = h_2$ and $h'_2 = h_1$. Thus $\tau^2 h'_i = \xi^2 h'_i$, $i = 1, 2$. The action of $\tau$ on $S[\tau^2]$ is given by $\tau h'_i = \xi^2 h'_i$, $i = 1, 2$. The action of $\tau$ on $T_{\chi_j}$ is defined to be the identity. The weight in $S[\tau^2]$ is given by $\text{wt } h'_i(i/3+n) = -i/3 - n$, $i = 1, 2$ and $\text{wt } 1 = 1/9$. The weight of any element of $T_{\chi_j}$ is defined to be 0. The weight in $V_L^{T_{\chi_j}^2}(\tau^2)$ is identical with the eigenvalue for the action of the coefficient of $z^{-2}$ in the $\tau^2$-twisted vertex operator $Y^{\tau^2}(\omega, z)$.

By Lemma [5 14 Theorem 4.4], and [15 Theorem 6.14],

$$V_{L,(\omega, \beta)}(\varepsilon) = \{ v \in V_{L,(\omega, \beta)} \mid \tau v = \xi^\varepsilon v \}, \quad j, \varepsilon = 0, 1, 2$$

are inequivalent simple $V_L^\tau$-modules. For each of $j = 0, 1, 2$, we have that $V_{L,(\omega, \beta)}$, $i = a, b, c$ are equivalent simple $V_L^\tau$-modules. Moreover, $V_{L,(\omega, \beta)}$, $j = 0, 1, 2$ are inequivalent simple
$V_L^\tau$-modules. From [30, Theorem 2], it follows that

$$V_L^{T \omega_j}(\tau)(\varepsilon) = \{ v \in V_L^{T \omega_j}(\tau) \mid \tau v = \xi \varepsilon v \}, \quad j, \varepsilon = 0, 1, 2$$

are inequivalent simple $V_L^\tau$-modules. Similar assertions hold for simple $\tau^2$-twisted modules, namely,

$$V_L^{T \omega_j}(\tau^2)(\varepsilon) = \{ v \in V_L^{T \omega_j}(\tau^2) \mid \tau^2 v = \xi \varepsilon v \}, \quad j, \varepsilon = 0, 1, 2$$

are inequivalent simple $V_L^\tau$-modules. In this way we obtain 30 simple $V_L^\tau$-modules. These 30 simple $V_L^\tau$-modules are inequivalent by [30, Theorem 2]. We summarize the result as follows.

**Lemma 3.2.** The following 30 simple $V_L^\tau$-modules are inequivalent.

1. $V_{L(0,0)}(\varepsilon), j, \varepsilon = 0, 1, 2$,
2. $V_{L(c,0)}(\varepsilon), j = 0, 1, 2$,
3. $V_{L^{T \omega_j}}(\tau)(\varepsilon), j, \varepsilon = 0, 1, 2$,
4. $V_{L^{T \omega_j}}(\tau^2)(\varepsilon), j, \varepsilon = 0, 1, 2$.

We consider the structure of $V_L^\tau$ in detail. Let

$$x(\alpha) = e^{\sqrt{2} \alpha} + e^{-\sqrt{2} \alpha}, \quad y(\alpha) = e^{\sqrt{2} \alpha} - e^{-\sqrt{2} \alpha}, \quad w(\alpha) = \frac{1}{2} \alpha(-1)^2 - x(\alpha)$$

for $\alpha \in \{ \pm \alpha_0, \pm \alpha_1, \pm \alpha_2 \}$ and let

$$\omega = \frac{1}{6}(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2),$$

$$\tilde{\omega}^1 = \frac{1}{5}(w(\alpha_1) + w(\alpha_2) + w(\alpha_0)), \quad \tilde{\omega}^2 = \omega - \tilde{\omega}^1,$$

$$\omega^1 = \frac{1}{4} w(\alpha_1), \quad \omega^2 = \tilde{\omega}^1 - \omega^1.$$ 

Then $\omega$ is the Virasoro element of $V_L$ and $\tilde{\omega}^1$, $\tilde{\omega}^2$ are mutually orthogonal conformal vectors of central charge $6/5, 4/5$ respectively. The subalgebra $\text{Vir}(\tilde{\omega}^i)$ generated by $\tilde{\omega}^i$ is isomorphic to the Virasoro vertex operator algebra of given central charge, namely, $\text{Vir}(\omega^1) \cong L(6/5, 0)$ and $\text{Vir}(\omega^2) \cong L(4/5, 0)$. Moreover, $\tilde{\omega}^1$ is a sum of two conformal vectors $\omega^1$ and $\omega^2$ of central charge $1/2$ and $7/10$ respectively and $\omega^1$, $\omega^2$ and $\tilde{\omega}^2$ are mutually orthogonal. Note that $\tilde{\omega}^2$ was denoted by $\omega^3$ in [19, 20, 21]. Such a decomposition of the Virasoro element of a lattice vertex operator algebra into a sum of mutually orthogonal conformal vectors was first studied in [13].

Set

$$M_k^i = \{ v \in V_L \mid (\omega^2)_1 v = 0 \},$$

$$W_k^i = \{ v \in V_L \mid (\omega^2)_1 v = \frac{2}{5} v \}, \quad i = 0, a, b, c,$$

$$M_k^i = \{ v \in V_L \mid (\omega^1)_1 v = (\omega^2)_1 v = 0 \},$$

$$W_k^i = \{ v \in V_L \mid (\omega^1)_1 v = 0, (\omega^2)_1 v = \frac{3}{5} v \}, \quad j = 0, 1, 2.$$ 

Then $M_k^i$ and $W_k^i$ are simple vertex operator algebras. Moreover, $\{ M_k^i, W_k^i ; i = 0, a, b, c \}$ and $\{ M_k^j, W_k^j ; j = 0, 1, 2 \}$ are complete sets of representatives of isomorphism classes of
simple modules for $M^0_k$ and $M^0_t$, respectively (cf. [19, 21, 22]). As $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$-modules,

$$M^0_k \cong \left( L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \right) \oplus \left( L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \right),$$

$$M^0_t \cong M^0_k \cong L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{7}{10}, \frac{7}{16}\right),$$

$$M^1_k \cong \left( L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, 0\right) \right) \oplus \left( L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \right),$$

$$W^0_k \cong \left( L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \right) \oplus \left( L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{1}{10}\right) \right),$$

$$W^0_t \cong W^0_k \cong L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{7}{10}, \frac{3}{80}\right),$$

$$W^c_k \cong \left( L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \right) \oplus \left( L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, \frac{1}{10}\right) \right),$$

and as $\text{Vir}(\tilde{\omega}^2)$-modules,

$$M^0_t \cong L\left(\frac{4}{5}, 0\right) \oplus L\left(\frac{4}{5}, \frac{4}{3}\right),$$

$$M^1_t \cong M^2_t \cong L\left(\frac{4}{5}, \frac{2}{3}\right),$$

$$W^0_t \cong L\left(\frac{4}{5}, \frac{2}{5}\right) \oplus L\left(\frac{4}{5}, \frac{7}{5}\right),$$

$$W^1_t \cong W^2_t \cong L\left(\frac{4}{5}, \frac{1}{15}\right).$$

Furthermore,

$$V_{L^{(c,j)}} \cong (M^0_k \otimes M^j_t) \oplus (W^i_k \otimes W^j_t)$$

as $M^0_k \otimes M^0_t$-modules. In particular,

$$V_L \cong (M^0_k \otimes M^0_t) \oplus (W^0_k \otimes W^0_t).$$

Note that $M^i_j = \{ v \in V_L \mid (\tilde{\omega}^i_1)v = 0 \}$ and that $M^0_k$, $W^0_k$ and $M^j_t, j = 0, 1, 2$ are $\tau$-invariant. However, $W^j_t, j = 0, 1, 2$ are not $\tau$-invariant.

The fusion rules for $M^0_k$ and $M^0_t$ were determined in [22] and [29], respectively. They are as follows.

$$M^i_k \times M^j_k = M^{i+j}_k,$$

$$M^i_k \times W^j_k = W^{i+j}_k,$$

$$W^i_k \times W^j_k = M^{i+j}_k + W^{i+j}_k$$

for $i, j = 0, a, b, c$ and

$$M^i_t \times M^j_t = M^{i+j}_t,$$

$$M^i_t \times W^j_t = W^{i+j}_t,$$

$$W^i_t \times W^j_t = M^{i+j}_t + W^{i+j}_t$$

for $i, j = 0, 1, 2$.

The following two weight 3 vectors are important.

$$J = w(\alpha_1)w(\alpha_2) - w(\alpha_2)w(\alpha_1)$$

$$= -\frac{1}{6}(\beta_1(-2)(\beta_2 - \beta_0)(-1) + \beta_2(-2)(\beta_0 - \beta_1)(-1) + \beta_0(-2)(\beta_1 - \beta_2)(-1))$$

$$- (\beta_2 - \beta_0)(-1)y(\alpha_1) - (\beta_0 - \beta_1)(-1)y(\alpha_2) - (\beta_1 - \beta_2)(-1)y(\alpha_0),$$
\[
K = -\frac{1}{9}(\beta_1 - \beta_2)(-1)(\beta_2 - \beta_0)(-1)(\beta_0 - \beta_1)(-1) \\
+ (\beta_2 - \beta_0)(-1)x(\alpha_1) + (\beta_0 - \beta_1)(-1)x(\alpha_2) + (\beta_1 - \beta_2)(-1)x(\alpha_0).
\]

Let \( M(0) = (M^0_k)^\tau = \{ u \in M^0_k | \tau u = u \} \). The vertex operator algebra \( M(0) \) was studied in [8]. Among other things, the classification of simple modules, the rationality and the \( C_2 \)-cofiniteness for \( M(0) \) were established. It is known that \( M(0) \) is a \( W_3 \) algebra of central charge \( 6/5 \) with the Virasoro element \( \tilde{\omega}^1 \). In fact, \( M(0) \) is generated by \( \tilde{\omega}^1 \) and \( J \). The following equations hold \([8, (3.1)]\).

\[
\begin{align*}
J_5 J &= -84 \cdot 1, \\
J_4 J &= 0, \\
J_3 J &= -420 \tilde{\omega}^1, \\
J_2 J &= -210(\tilde{\omega}^1)_0 \tilde{\omega}^1, \\
J_1 J &= 9(\tilde{\omega}^1)_0(\tilde{\omega}^1)_0 - 240(\tilde{\omega}^1)_1 \tilde{\omega}^1, \\
J_0 J &= 22(\tilde{\omega}^1)_0(\tilde{\omega}^1)_0 - 120(\tilde{\omega}^1)_0(\tilde{\omega}^1)_1 \tilde{\omega}^1.
\end{align*}
\] (3.8)

Let \( L^1(n) = (\tilde{\omega}^1)_{n+1} \) and \( J(n) = J_{n+2} \) for \( n \in \mathbb{Z} \), so that the weight of these operators is \( \text{wt} L^1(n) = \text{wt} J(n) = -n \). Then

\[
[L^1(m), L^1(n)] = (m - n)L^1(m + n) + \frac{m^2 - m}{12} \cdot \frac{6}{5} \cdot \delta_{m+n,0}, \tag{3.9}
\]

\[
[L^1(m), J(n)] = (2m - n)J(m + n), \tag{3.10}
\]

\[
[J(m), J(n)] = (m - n)\left(22(m + n + 2)(m + n + 3) + 35(m + 2)(n + 2)\right)L^1(m + n) \\
- 120(m - n)\left(\sum_{k \leq -2} L^1(k)L^1(m + n - k) + \sum_{k \geq -1} L^1(m + n - k)L^1(k)\right) \\
- \frac{7}{10}m(m^2 - 1)(m^2 - 4)\delta_{m+n,0}. \tag{3.11}
\]

The vertex operator algebra \( M^0_l \) is known as a 3-State Potts model. It is a \( W_3 \) algebra of central charge \( 4/5 \) with the Virasoro element \( \tilde{\omega}^2 \) and is generated by \( \tilde{\omega}^2 \) and \( K \). Both of \( \tilde{\omega}^2 \) and \( K \) are fixed by \( \tau \), so that \( \tau \) is the identity on \( M^0_l \). The rationality of \( M^0_l \) was established in [21] and the \( C_2 \)-cofiniteness of \( M^0_l \) follows from [9]. By a direct calculation, we can verify that

\[
\begin{align*}
K_5 K &= 104 \cdot 1, \\
K_4 K &= 0, \\
K_3 K &= 780 \tilde{\omega}^2, \\
K_2 K &= 390(\tilde{\omega}^2)_0 \tilde{\omega}^2, \tag{3.12}
\end{align*}
\]

\[
\begin{align*}
K_1 K &= -27(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0 \tilde{\omega}^2 + 480(\tilde{\omega}^2)_{-1} \tilde{\omega}^2, \\
K_0 K &= -46(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0 \tilde{\omega}^2 + 240(\tilde{\omega}^2)_0(\tilde{\omega}^2)_{-1} \tilde{\omega}^2.
\end{align*}
\]
Let $L^2(n) = (\tilde{\omega}^2)_{n+1}$ and $K(n) = K_{n+2}$ for $n \in \mathbb{Z}$. Then

$$[L^2(m), L^2(n)] = (m - n)L^2(m + n) + \frac{m^3 - m}{12} \cdot \frac{4}{5} \cdot \delta_{m+n,0},$$  \hspace{1cm} (3.13)$$

$$[L^2(m), K(n)] = (2m - n)K(m + n),$$  \hspace{1cm} (3.14)$$

$$[K(m), K(n)] = -(m - n)\left(46(m + n + 2)(m + n + 3) + 65(2n + 1) \right)L^2(m + n)$$

$$\quad + 240(m - n)\left(\sum_{k \leq -2} L^2(k)L^2(m + n - k) + \sum_{k \geq -1} L^2(m + n - k)L^2(k)\right)$$

$$\quad + \frac{13}{15}(m^2 - 1)(m^2 - 4)\delta_{m+n,0}. \hspace{1cm} (3.15)$$

**Remark 3.3.** Let $L_n = L^1(n)$, $W_n = \sqrt{-1/210}J(n)$, and $c = 6/5$. Then the above commutation relations coincide with (2.1) and (2.2) of \cite{4}. The same commutation relations also hold if we set $L_n = L^2(n)$, $W_n = K(n)/\sqrt{390}$, and $c = 4/5$.

Let us review the 20 inequivalent simple $M(0)$-modules studied in \cite{8}. Among those simple $M(0)$-modules, eight of them appear in simple $M^0_k$-modules, namely,

$$M(\varepsilon) = \{u \in M^0_k | \tau u = \varepsilon u\}, \quad W(\varepsilon) = \{u \in W^0_k | \tau u = \varepsilon u\}$$

for $\varepsilon = 0, 1, 2$, $M^0_k$ and $W^0_k$. The remaining 12 simple $M(0)$-modules appear in simple $\tau$-twisted or $\tau^2$-twisted $V_L$-modules. Let

$$M_T(\tau)(\varepsilon) = \{u \in V_L^{\tau \varepsilon}(\tau) | (\tilde{\omega}^2)_1 u = 0, \tau u = \varepsilon u\},$$

$$W_T(\tau)(\varepsilon) = \{u \in V_L^{\tau \varepsilon}(\tau) | (\tilde{\omega}^2)_1 u = \frac{2}{5} u, \tau u = \varepsilon u\}.$$  

Then $M_T(\tau)(\varepsilon)$, $W_T(\tau)(\varepsilon)$, $\varepsilon = 0, 1, 2$ are inequivalent simple $M(0)$-modules. Similarly,

$$M_T(\tau^2)(\varepsilon) = \{u \in V_L^{\tau_2 \varepsilon}(\tau^2) | (\tilde{\omega}^2)_1 u = 0, \tau^2 u = \varepsilon u\},$$

$$W_T(\tau^2)(\varepsilon) = \{u \in V_L^{\tau_2 \varepsilon}(\tau^2) | (\tilde{\omega}^2)_1 u = \frac{2}{5} u, \tau^2 u = \varepsilon u\}$$

for $\varepsilon = 0, 1, 2$ are inequivalent simple $M(0)$-modules. In \cite{8}, it was shown that $M(\varepsilon)$, $W(\varepsilon)$, $M^0_k$, $W^0_k$, $M_T(\tau)(\varepsilon)$, $W_T(\tau)(\varepsilon)$, $M_T(\tau^2)(\varepsilon)$, and $W_T(\tau^2)(\varepsilon)$, $\varepsilon = 0, 1, 2$ form a complete set of representatives of isomorphism classes of simple $M(0)$-modules.

Let us describe the structure of the fixed point subalgebra $V^+_L$. By the definition of $M(0)$ and $M^0_0$, we see that $V^+_L \supset M(0) \otimes M^0_0$. Since both of $M(0)$ and $M^0_0$ are rational, $M(0) \otimes M^0_0$ is also rational. Thus $V^+_L(\varepsilon) = \{u \in V_L | \tau u = \varepsilon u\}$, $\varepsilon = 0, 1, 2$ can be decomposed into a direct sum of simple modules for $M(0) \otimes M^0_0$. Any simple module for $M(0) \otimes M^0_0$ is of the form $A \otimes B$, where $A$ and $B$ are simple modules for $M(0)$ and $M^0_0$, respectively. By (3.5), it follows that $B \cong M^0_0$ or $W^0_0$. Moreover, $V^+_L(\varepsilon)$ contains the simple $M(0)$-modules $M(\varepsilon)$ and $W(\varepsilon)$. The eigenvalues of $(\tilde{\omega}^1)_1$ in $M(\varepsilon)$ (resp. $W(\varepsilon)$) are integers (resp. of the form $3/5 + n$, $n \in \mathbb{Z}$), while the eigenvalues of $(\tilde{\omega}^2)_1$ in $M^0_0$
(resp. \(W_0^0\)) are integers (resp. of the form \(2/5 + n, n \in \mathbb{Z}\)). Since the eigenvalues of \(\omega_1 = (\tilde{\omega}^1)_1 + (\tilde{\omega}^2)_1\) in \(V_L\) are integers, we conclude that
\[
V_L(\varepsilon) \cong (M(\varepsilon) \otimes M_0^0) \oplus (W(\varepsilon) \otimes W_0^0)
\]
as \(M(0) \otimes M_0^0\)-modules, \(\varepsilon = 0, 1, 2\). In particular,
\[
V_L^+ \cong (M(0) \otimes M_0^0) \oplus (W(0) \otimes W_0^0).
\]
From now on we set \(M^0 = M(0) \otimes M_0^0\) and \(W^0 = W(0) \otimes W_0^0\). Thus \(V_L^+ = V_L(0) \cong M^0 \oplus W^0\). Let
\[
P = y(\alpha_1) + y(\alpha_2) + y(\alpha_0).
\]
Then we can verify that \((\tilde{\omega})_n P = (\tilde{\omega})_n P = 0\) for \(n \geq 2\), \((\tilde{\omega})_1 P = (8/5) P\), and \((\tilde{\omega})_1 P = (2/5) P\). Moreover, \(J_n P = K_n P = 0\) for \(n \geq 2\). Thus \(W^0\) is a simple \(M^0\)-module with \(P\) a highest weight vector of weight \((8/5, 2/5)\). The vertex operator algebra \(V_L^+\) is generated by \(\tilde{\omega}, \tilde{\omega}, J, K\) and \(P\).

**Theorem 3.4.** \(V_L^+\) is a simple \(C_2\)-cofinite vertex operator algebra.

**Proof.** We know that \(M(0)\) and \(M_0^0\) are \(C_2\)-cofinite. Thus \(M^0\) is also \(C_2\)-cofinite. Since \(W^0\) is generated by \(P\) as an \(M^0\)-module, it follows from \([5]\) that \(V_L^+\) is \(C_2\)-cofinite. By \([14]\), Theorem 4.4), \(V_L^+\) is simple. \(\square\)

Following the outline of the argument in \([3, 20]\), we discuss the structure of the simple \(\tau\)-twisted \(V_L\)-modules \(V_L^{T_\lambda}(\tau), j = 0, 1, 2\) as \(\tau\)-twisted \(M_0^0 \otimes M_0^0\)-modules. Furthermore, we correct an error in \([3, 20]\) concerning a decomposition of \(V_L^{T_\lambda}(\tau)\) for \(j = 1, 2\). We first consider \(V_L^{T_{\lambda 0}}(\tau)\). Let \(0 \neq v \in T_{\lambda 0}\) and \(1\) be the identity of \(S[\tau]\). Then \(1 \otimes v \in S[\tau] \otimes T_{\lambda 0} = V_L^{T_{\lambda 0}}(\tau)\). Since \(M_0^0 \subset V_L^+\), we can decompose \(V_L^{T_{\lambda 0}}(\tau)\) into a direct sum of simple \(M_0^0\)-modules. By a direct calculation, we can verify that
\[
(\tilde{\omega}^1)_1 (1 \otimes v) = 0, \quad (\tilde{\omega}^2)_1 (h_2(-\frac{1}{3}) \otimes v) = \frac{2}{5} h_2(-\frac{1}{3}) \otimes v.
\]
Thus we see that \(M_0^0\) and \(W_0^0\) appear as direct summands. Since \(V_L^{T_{\lambda 0}}(\tau)\) is simple as a \(\tau\)-twisted \(V_L\)-module, \([3, 35]\) and the fusion rule \(W_0^0 \times W_0^0 = M_0^0 + W_0^0\) (cf. \([3, 34]\)) imply that any simple \(M_0^0\)-submodule of \(V_L^{T_{\lambda 0}}(\tau)\) is isomorphic to \(M_0^0\) or \(W_0^0\). Hence
\[
V_L^{T_{\lambda 0}}(\tau) \cong (M_0^0(\tau) \otimes M_0^0) \oplus (W_0^0(\tau) \otimes W_0^0)
\]
as \(\tau\)-twisted \(M_0^0 \otimes M_0^0\)-modules, where
\[
M_0^0(\tau) = \{ u \in V_L^{T_{\lambda 0}}(\tau) \mid (\tilde{\omega}^2)_1 u = 0 \},
\]
\[
W_0^0(\tau) = \{ u \in V_L^{T_{\lambda 0}}(\tau) \mid (\tilde{\omega}^2)_1 u = \frac{2}{5} u \}.
\]
The \(\tau\)-twisted \(M_0^0\)-modules \(M_0^0(\tau)\) and \(W_0^0(\tau)\) are simple. Indeed, if \(N\) is a \(\tau\)-twisted \(M_0^0\)-submodule of \(M_0^0(\tau)\), then \(N \otimes M_0^0\) is a \(\tau\)-twisted \(M_0^0 \otimes M_0^0\)-submodule of \(M_0^0(\tau) \otimes M_0^0\). By \([2, 6]\), \(V_L \cdot (N \otimes M_0^0) = \text{span}\{ a_n (N \otimes M_0^0) \mid a \in V_L, n \in \mathbb{Q} \}\) is a \(\tau\)-twisted \(V_L\)-submodule of \(V_L^{T_{\lambda 0}}(\tau)\). The fusion rule \(W_0^0 \times M_0^0 = W_0^0\) and \([3, 35]\) imply that \(V_L \cdot (N \otimes M_0^0)\) is contained in \((N \otimes M_0^0) \oplus (W_0^0(\tau) \otimes W_0^0)\). Since \(V_L^{T_{\lambda 0}}(\tau)\) is a simple \(\tau\)-twisted \(V_L\)-module, we conclude that \(M_0^0(\tau)\) is a simple \(\tau\)-twisted \(M_0^0\)-module.
Because of the fusion rule $W^0_t \times W^0_t = M^0_t + W^0_t$, we cannot apply a similar argument to $W^0_T(\tau)$. Note that there are at most two inequivalent simple $\tau$-twisted $M^0_k$-modules by [8, Lemma 4.1] and [12, Theorem 10.2]. Note also that a weight in $M^0_T(\tau)$ or in $W^0_T(\tau)$ means an eigenvalue of $(\tilde{\omega}^i)_1$. First several terms of the characters of $M^0_T(\tau)$ and $W^0_T(\tau)$ can be calculated easily from (3.18) (cf. [8]).

\[
\begin{align*}
\text{ch } M^0_T(\tau) &= q^{1/9} + q^{1/9 + 2/3} + q^{1/9 + 1} + q^{1/9 + 4/3} + \cdots, \\
\text{ch } W^0_T(\tau) &= q^{2/45} + q^{2/45 + 1/3} + q^{2/45 + 2/3} + q^{2/45 + 1} + \cdots.
\end{align*}
\]

Suppose $W^0_T(\tau)$ is not a simple $\tau$-twisted $M^0_k$-module. Let $N$ be the $\tau$-twisted $M^0_k$-submodule of $W^0_T(\tau)$ generated by the top level of $W^0_T(\tau)$. Then the top level of $N$ is a one dimensional space of weight 2/45. If $N$ is not a simple $\tau$-twisted $M^0_k$-module, then the sum $U$ of all proper $\tau$-twisted $M^0_k$-submodules of $N$ is a unique maximal $\tau$-twisted $M^0_k$-submodule of $N$. The quotient $N/U$ is a simple $\tau$-twisted $M^0_k$-module whose top level is of weight 2/45. Denote the top level of $U$ by $U_\lambda$, where the weight $\lambda$ is 2/45 + m/n for some 1 ≤ m ∈ $\mathbb{Z}$. Consider the $\tau$-twisted Zhu algebra $A_\tau(M^0_k)$ of $M^0_k$. Since $U_\lambda$ is a finite dimensional $A_\tau(M^0_k)$-module, we can choose a simple $A_\tau(M^0_k)$-submodule $S$ of $U_\lambda$. By [11, Proposition 5.4 and Theorem 7.2], there is a simple 1/3-$\mathbb{N}$-graded weak $\tau$-twisted $M^0_k$-module $R$ with top level $R_\lambda$ being isomorphic to $S$ as an $A_\tau(M^0_k)$-module. It follows from [31, Corollary 3.8] that $R$ is in fact a simple $\tau$-twisted $M^0_k$-module. Here we note that $M^0_k$ is $C_2$-cofinite and of CFT type by its structure [32]. Since the top levels of $M^0_T(\tau)$, $N/U$, and $R$ have different weight, they are inequivalent simple $\tau$-twisted $M^0_k$-modules. If $N$ is a simple $\tau$-twisted $M^0_k$-module, then it is not equal to $W^0_T(\tau)$ by our assumption. The quotient $W^0_T(\tau)/N$ is a $\tau$-twisted $M^0_k$-module and the weight of its top level, say $\mu$, is 2/45 + m/3 for some 1 ≤ m ∈ $\mathbb{Z}$. By a similar argument as above, we see that there is a simple $\tau$-twisted $M^0_k$-module whose top level is of weight $\mu$. Hence we have three inequivalent simple $\tau$-twisted $M^0_k$-modules in both cases. This contradicts the fact that there are at most two inequivalent simple $\tau$-twisted $M^0_k$-modules. Thus $W^0_T(\tau)$ is a simple $\tau$-twisted $M^0_k$-module.

Next, let 0 ≠ $v \in T_{x_j}, j = 1, 2$. From the definition of $V^T_{Lx_j}(\tau)$ in [8, 20], we can calculate that

\[
(\tilde{\omega}^2)_1(1 \otimes v) = \frac{1}{15}(1 \otimes v), \quad (\tilde{\omega}^2)_1u^j = \frac{2}{3}u^j,
\]

where $u^j = h_1(-\frac{3}{4}) \otimes v - (-1)^j\sqrt{3}h_2(-\frac{1}{3})^2 \otimes v$. Thus $M^1_t$ or $M^2_t$ and $W^1_t$ or $W^2_t$ appear as $M^0_t$-submodules of $V^T_{Lx_j}(\tau)$. In order to distinguish $M^1_t$ and $M^2_t$ (resp. $W^1_t$ and $W^2_t$), we need to know the action of $K_2$ on these vectors (cf. [21]). By a direct calculation, we can verify that

\[
K_2(1 \otimes v) = (-1)^j\frac{2}{9}(1 \otimes v), \quad K_2u^j = (-1)^j\frac{52}{9}u^j.
\]

Hence $M^3_{-j}$ and $W^3_{-j}$ appear in $V^T_{Lx_j}(\tau)$ for $j = 1, 2$. Let

\[
\begin{align*}
M^3_t(\tau) &= \{u \in V^T_{Lx_j}(\tau) \mid (\tilde{\omega}^2)_1u = \frac{2}{3}u\}, \\
W^3_t(\tau) &= \{u \in V^T_{Lx_j}(\tau) \mid (\tilde{\omega}^2)_1u = \frac{1}{15}u\}, \quad j = 1, 2.
\end{align*}
\]
Then, $V_{L}^{T_{x_{0}}} (\tau) \cong (M_{T}^{i}(\tau) \otimes M_{T}^{i-2}) \oplus (W_{T}^{i}(\tau) \otimes W_{T}^{i-2})$ as $\tau$-twisted $M_{k}^{0} \otimes M_{l}^{0}$-modules for $j = 1, 2$. Moreover, $M_{T}^{i}(\tau)$ and $W_{T}^{i}(\tau)$, $j = 1, 2$ are simple $\tau$-twisted $M_{k}^{0}$-modules.

Recall that there are at most two inequivalent simple $\tau$-twisted $M_{k}^{0}$-modules. Looking at the smallest weight of $M_{T}^{i}(\tau)$ and $W_{T}^{i}(\tau)$, we have that $M_{T}^{i}(\tau)$, $j = 0, 1, 2$ are equivalent and $W_{T}^{i}(\tau)$, $j = 0, 1, 2$ are equivalent, but $M_{T}^{0}(\tau)$ and $W_{T}^{0}(\tau)$ are not equivalent. For simplicity, set $M_{T}(\tau) = M_{T}^{0}(\tau)$ and $W_{T}(\tau) = W_{T}^{0}(\tau)$. Then

$$V_{L}^{T_{x_{0}}} (\tau) \cong (M_{T}(\tau) \otimes M_{l}^{0}) \oplus (W_{T}(\tau) \otimes W_{l}^{0}),$$

$$V_{L}^{T_{x_{j}}} (\tau) \cong (M_{T}(\tau) \otimes M_{l}^{3-j}) \oplus (W_{T}(\tau) \otimes W_{l}^{3-j}), \quad j = 1, 2$$

(3.19)

as $\tau$-twisted $M_{k}^{0} \otimes M_{l}^{0}$-modules.

The structure of the simple $\tau^{2}$-twisted $V_{L}$-module $V_{L}^{T_{x_{j}}} (\tau^{2})$, $j = 0, 1, 2$ as a $\tau^{2}$-twisted $M_{k}^{0} \otimes M_{l}^{0}$-module is similar to that of the case for $V_{L}^{T_{x_{j}}} (\tau)$. Let $0 \neq v \in T_{x_{0}}$ and 1 be the identity of $S[\tau^{2}]$. Then

$$(\tilde{\omega}^{2})_{1}(1 \otimes v) = 0, \quad (\tilde{\omega}^{2})_{1}(h_{2}^{1}(-\frac{1}{3}) \otimes v) = \frac{2}{5} h_{2}^{1}(-\frac{1}{3}) \otimes v$$

and so

$$V_{L}^{T_{x_{0}}} (\tau^{2}) \cong (M_{T}^{0}(\tau^{2}) \otimes M_{l}^{0}) \oplus (W_{T}^{0}(\tau^{2}) \otimes W_{l}^{0})$$

as $\tau^{2}$-twisted $M_{k}^{0} \otimes M_{l}^{0}$-modules, where

$$M_{T}^{0}(\tau^{2}) = \{ u \in V_{L}^{T_{x_{0}}} (\tau^{2}) | (\tilde{\omega}^{2})_{1}u = 0 \},$$

$$W_{T}^{0}(\tau^{2}) = \{ u \in V_{L}^{T_{x_{0}}} (\tau^{2}) | (\tilde{\omega}^{2})_{1}u = \frac{2}{5}u \}.$$

By a similar argument as in the $\tau$-twisted case, we can show that $M_{T}^{0}(\tau^{2})$ and $W_{T}^{0}(\tau^{2})$ are inequivalent simple $\tau^{2}$-twisted $M_{k}^{0}$-modules.

Let $0 \neq v \in T_{x_{j}}$, $j = 1, 2$. Then

$$(\tilde{\omega}^{2})_{1}(1 \otimes v) = \frac{1}{15} (1 \otimes v), \quad (\tilde{\omega}^{2})_{1}v^{j} = \frac{2}{3} v^{j},$$

where $v^{j} = h_{1}^{1}(-\frac{2}{3}) \otimes v - (-1)^{j} \sqrt{-3} h_{2}^{1}(-\frac{1}{3})^{2} \otimes v$. Furthermore,

$$K_{2}(1 \otimes v) = (-1)^{j} \frac{2}{9} (1 \otimes v), \quad K_{2}v^{j} = -(-1)^{j} \frac{52}{9} v^{j}.$$

Hence $V_{L}^{T_{x_{j}}} (\tau^{2}) \cong (M_{T}^{j}(\tau^{2}) \otimes M_{l}^{j}) \oplus (W_{T}^{j}(\tau^{2}) \otimes W_{l}^{j})$ as $\tau^{2}$-twisted $M_{k}^{0} \otimes M_{l}^{0}$-modules for $j = 1, 2$, where

$$M_{T}^{j}(\tau^{2}) = \{ u \in V_{L}^{T_{x_{j}}} (\tau^{2}) | (\tilde{\omega}^{2})_{1}u = \frac{2}{3}u \},$$

$$W_{T}^{j}(\tau^{2}) = \{ u \in V_{L}^{T_{x_{j}}} (\tau^{2}) | (\tilde{\omega}^{2})_{1}u = \frac{1}{15}u \}, \quad j = 1, 2.$$

As in the $\tau$-twisted case, $M_{T}^{j}(\tau^{2})$, $j = 0, 1, 2$ are equivalent and $W_{T}^{j}(\tau^{2})$, $j = 0, 1, 2$ are equivalent. Set $M_{T}(\tau^{2}) = M_{T}^{0}(\tau^{2})$ and $W_{T}(\tau^{2}) = W_{T}^{0}(\tau^{2})$. Then

$$V_{L}^{T_{x_{j}}} (\tau^{2}) \cong (M_{T}(\tau^{2}) \otimes M_{l}^{j}) \oplus (W_{T}(\tau^{2}) \otimes W_{l}^{j}), \quad j = 0, 1, 2$$

(3.20)
as $\tau^2$-twisted $M^0_k \otimes M^0_l$-modules.

**Remark 3.5.** The weight 3 vector $K$ was denoted by different symbols in previous papers, namely, $v_1$, $v^3$, and $q$ were used in [8], [20], and [21], respectively. They are related as follows: $K = -2\sqrt{2}v_1 = -2\sqrt{2}v^3 = 2\sqrt{2}q$. Thus, in the proof of [20] Proposition 6.8 $(v^3)^2$ should act on the top level of $V^{TX_j}_L(\tau)$ as a scalar multiple of $(-1)^j/9\sqrt{2}$ for $j = 1, 2$. Moreover, (6.46) of [20] and the equation for $V^{TX_j}_L(\tau)$ on page 265 of [8] should be replaced with Equation (3.19). This correction does not affect the results in [8]. However, certain changes must be necessary in [20] along the correction.

Note that

$$M_T(\tau^i)(\varepsilon) = \{ u \in M_T(\tau^i) \mid \tau^iu = \xi^\varepsilon u \},$$

$$W_T(\tau^i)(\varepsilon) = \{ u \in W_T(\tau^i) \mid \tau^iu = \xi^\varepsilon u \}$$

for $i = 1, 2$, $\varepsilon = 0, 1, 2$. Another notation was used in [20], namely,

$$M_T(\tau^i)^\varepsilon = \bigoplus_{n \in 1/9+\varepsilon/3+\mathbb{Z}} (M_T(\tau^i))^n, \quad W_T(\tau^i)^\varepsilon = \bigoplus_{n \in 2/45+\varepsilon/3+\mathbb{Z}} (W_T(\tau^i))^n,$$

where $U_n$ denotes the eigenspace of $U$ with eigenvalue $n$ for $(\tilde{\omega}^1)_1$. They are related as follows.

$$M_T(\tau^i)^\varepsilon = M_T(\tau^i)(2\varepsilon), \quad W_T(\tau^i)^\varepsilon = W_T(\tau^i)(2\varepsilon - 1). \quad (3.21)$$

Likewise,

$$(V^{TX_j}_L(\tau))^\varepsilon = \bigoplus_{n \in 1/9+\varepsilon/3+\mathbb{Z}} (V^{TX_j}_L(\tau))^n_n, \quad (V^{TX_j}_L(\tau^2))^\varepsilon = \bigoplus_{n \in 1/9+\varepsilon/3+\mathbb{Z}} (V^{TX_j}_L(\tau^2))^n$$

of [20] are denoted by

$$(V^{TX_j}_L(\tau))^\varepsilon = V^{TX_j}_L(\tau)(2\varepsilon), \quad (V^{TX_j}_L(\tau^2))^\varepsilon = V^{TX_j}_L(\tau^2)(2\varepsilon) \quad (3.22)$$

in our notation for $j = 0, 1, 2$ and $\varepsilon = 0, 1, 2$, where $U_n$ is the eigenspace of $U$ with eigenvalue $n$ for $\omega_1$.

By [3.3], the minimal eigenvalues of $(\tilde{\omega}^2)_1$ on $M^0_t$ and $W^0_t$ are 0 and 2/5, respectively, while those on $M^1_t$ and $W^1_t$, $j = 1, 2$, are 2/3 and 1/15, respectively. Hence it follows from (3.19) that

$$(V^{TX_0}_L(\tau))^\varepsilon \cong (M_T(\tau)^\varepsilon \otimes M^0_t) \oplus (W_T(\tau^\varepsilon)^{\varepsilon-1} \otimes W^0_t), \quad (3.23)$$

$$(V^{TX_j}_L(\tau))^\varepsilon \cong (M_T(\tau)^{\varepsilon+1} \otimes M^3_{t-j}) \oplus (W_T(\tau)^{\varepsilon} \otimes W^3_{t-j}), \quad j = 1, 2$$

as $M^0$-modules for $\varepsilon = 0, 1, 2$, where $M^0 = M(0) \otimes M^0_t$. Similarly,

$$(V^{TX_0}_L(\tau^2))^\varepsilon \cong (M_T(\tau^2)^\varepsilon \otimes M^0_t) \oplus (W_T(\tau^2)^{\varepsilon-1} \otimes W^0_t), \quad (3.24)$$

$$(V^{TX_j}_L(\tau^2))^\varepsilon \cong (M_T(\tau^2)^{\varepsilon+1} \otimes M^3_{t-j}) \oplus (W_T(\tau^2)^{\varepsilon} \otimes W^3_{t-j}), \quad j = 1, 2$$

as $M^0$-modules for $\varepsilon = 0, 1, 2$ (cf. [20] (7.17)).
The following fusion rules of simple $M(0)$-modules will be necessary for the study of simple $V_L^\tau$-modules.

\[
\begin{align*}
W(0) \times M_\epsilon^c &= W_\epsilon^c, \\
W(0) \times W_\epsilon^c &= M_\epsilon^c + W_\epsilon^c, \\
W(0) \times M(\epsilon) &= W(\epsilon), \\
W(0) \times W(\epsilon) &= M(\epsilon) + W(\epsilon), \\
W(0) \times M_T(\tau^\epsilon)(\epsilon) &= W_T(\tau^\epsilon)(\epsilon), \\
W(0) \times W_T(\tau^\epsilon)(\epsilon) &= M_T(\tau^\epsilon)(\epsilon) + W_T(\tau^\epsilon)(\epsilon)
\end{align*}
\]  

(3.25)

for $i = 1, 2$ and $\epsilon = 0, 1, 2$. In fact, the first four fusion rules, that is, fusion rules among simple $M(0)$-modules which appear in untwisted simple $V_L$-modules can be found in [32]. The last two fusion rules involve simple $M(0)$-modules which appear in $\tau^\epsilon$-twisted simple $V_L$-modules. Their proofs are given in Appendix A.

Fusion rules possess certain symmetries. Let $M^i$, $i = 1, 2, 3$ be modules for a vertex operator algebra $V$. Then by [10] Propositions 5.4.7 and 5.5.2

\[
\dim I_V \begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix} = \dim I_V \begin{pmatrix} M^3 \\ M^2 \ M^1 \end{pmatrix} = \dim I_V \begin{pmatrix} (M^2)' \\ M^1 \ (M^3)' \end{pmatrix},
\]

where $(M^i)'$ is the contragredient module of $M^i$. Recall that the contragredient module $(U', Y_{U'})$ of a $V$-module $(U, Y_U)$ is defined as follows. As a vector space $U' = \oplus_n (U_n)^*$ is the restricted dual of $U$ and $Y_{U'}(\cdot, z)$ is determined by

\[
\langle Y_{U'}(a, z)v, u \rangle = \langle v, Y_U(e^{z L(1)}(-z^{-2})L(0)^2 a, z^{-1})u \rangle
\]

for $a \in V$, $u \in U$, and $v \in U'$.

In our case $M(0)$ is generated by the Virasoro element $\tilde{\omega}^1$ and the weight 3 vector $J$. Moreover, $\langle L(0)^2(v, u) \rangle = \langle v, L(1)^2(u) \rangle$ and $\langle J(0)(v, u) \rangle = -\langle v, J(0)(u) \rangle$. Since the 20 simple $M(0)$-modules are distinguished by the action of $L(0)^1$ and $J(0)$ on their top levels, we know from Tables 1, 3, and 4 of [8] that the contragredient modules of the simple $M(0)$-modules are as follows.

\[
M(\epsilon)' \cong M(2\epsilon), \quad W(\epsilon)' \cong W(2\epsilon), \quad \epsilon = 0, 1, 2,
\]

\[
(M_\epsilon^c)' \cong M_\epsilon^c, \quad (W_\epsilon^c)' \cong W_\epsilon^c,
\]

\[
M_T(\tau)(\epsilon)' \cong M_T(\tau^{\epsilon})(\epsilon), \quad W_T(\tau)(\epsilon)' \cong W_T(\tau^{\epsilon})(\epsilon), \quad \epsilon = 0, 1, 2
\]

(see also [14] Lemma 3.7 and [32] Section 4.2).

4. Structure of simple modules

Recall that $V_L^\tau = V_L(0) = M^0 \oplus W^0$ with $M^0 = M(0) \otimes M(0)$ and $W^0 = W(0) \otimes W(0)$. In this section we study the structure of the 30 known simple $V_L^\tau$-modules listed in Lemma [32]. We discuss decompositions of these simple modules as modules for $M^0$. Those decompositions have been obtained in [20]. We review them briefly. (Some corrections are necessary in [20], see Remark [32] for details.)

A vector in a $V_L^\tau$-module is said to be of weight $h$ if it is an eigenvector for $L(0) = \omega_0$ with eigenvalue $h$. We calculate the action of $(\tilde{\omega}^1)_1$, $(\tilde{\omega}^2)_1$, $J_2$, $K_2$, $P_1$, $(J_1 P)_2$, and $(K_1 P)_2$ on the top levels of the 30 known simple $V_L^\tau$-modules. Recall that the top level
of a module means the homogeneous subspace of the module of smallest weight. The calculation is accomplished directly from the definition of untwisted or twisted vertex operators associated with the lattice $L$ and the automorphisms $\tau$ and $\tau^2$ (cf. [9, 17, 25]). The results in this section will be used to determine the Zhu algebra $A(V_L^\tau)$ of $V_L^\tau$ in Section 3.

The vectors $J_1 P$ and $K_1 P$ are of weight 3. Their precise form in terms of the lattice vertex operator algebra $V_L$ is as follows.

$$J_1 P = 2\beta_1(-1)^3 + 3\beta_1(-1)^2\beta_2(-1) - 3\beta_1(-1)\beta_2(-1)^2 - 2\beta_2(-1)^3$$

$$-4\left((\beta_2 - \beta_0)(-1)x(\alpha_1) + (\beta_0 - \beta_1)(-1)x(\alpha_2) + (\beta_1 - \beta_2)(-1)x(\alpha_0)\right)$$

$$= \frac{13}{9}\left(2\beta_1(-1)^3 + 3\beta_1(-1)^2\beta_2(-1) - 3\beta_1(-1)\beta_2(-1)^2 - 2\beta_2(-1)^3\right) - 4K,$$

$$K_1 P = 3\left(\beta_1(-2)\beta_2(-1) - \beta_2(-2)\beta_1(-1)\right)$$

$$-\left((\beta_2 - \beta_0)(-1)y(\alpha_1) + (\beta_0 - \beta_1)(-1)y(\alpha_2) + (\beta_1 - \beta_2)(-1)y(\alpha_0)\right)$$

$$= \frac{7}{2}\left(\beta_1(-2)\beta_2(-1) - \beta_2(-2)\beta_1(-1)\right) + J.$$

4.1. The simple module $V_L(0)$. $V_L(0) = M^0 \oplus W^0$ as $M^0$-modules. The top level of $V_L(0)$ is $\mathbb{C}1$. By a property of the vacuum vector, all of $(\tilde{\omega}^1)_1$, $(\tilde{\omega}^2)_1$, $J_2$, $K_2$, $P_1$, $(J_1 P)_2$, and $(K_1 P)_2$ act as 0 on $\mathbb{C}1$.

4.2. The simple module $V_L(\varepsilon)$, $\varepsilon = 1, 2$. By (3.10), $V_L(\varepsilon) \cong (M(\varepsilon) \otimes M^0_\tau) \oplus (W(\varepsilon) \otimes W^0)$ as $M^0$-modules for $\varepsilon = 1, 2$. The top level of $V_L(\varepsilon)$ is $\mathbb{C}v^{2,\varepsilon}$, where $v^{2,\varepsilon} = \alpha_1(-1) - \xi^\varepsilon\alpha_2(-1) \in W(\varepsilon) \otimes W^0$. The following hold.

$$(\tilde{\omega}^1)_1 v^{2,\varepsilon} = \frac{3}{5} v^{2,\varepsilon}, \quad (\tilde{\omega}^2)_1 v^{2,\varepsilon} = \frac{2}{5} v^{2,\varepsilon}, \quad J_2 v^{2,\varepsilon} = -(1)^\varepsilon 2\sqrt{3} v^{2,\varepsilon},$$

$$K_2 v^{2,\varepsilon} = 0, \quad P_1 v^{2,\varepsilon} = 0, \quad (J_1 P)_2 v^{2,\varepsilon} = 0, \quad (K_1 P)_2 v^{2,\varepsilon} = (-1)^\varepsilon 12\sqrt{3} v^{2,\varepsilon}.$$

4.3. The simple module $V_{L(0,j)}(0)$, $j = 1, 2$. For $j = 1, 2$, (3.14) implies that $V_{L(0,j)}$ is a direct sum of simple $M^0$-modules of the form $A \otimes B$, where $A$ is a simple $M(0)$-module and $B$ is a simple $M^0$-module isomorphic to $M^0_i$ or $W^0_i$. For convenience, set $U^j(\varepsilon) = V_{L(0,j)}(\varepsilon)$, $j = 1, 2$, $\varepsilon = 0, 1, 2$. Let

$$v^{3,j} = e^{(-1)^j(\beta_1-\beta_2)/3} + e^{(-1)^j(\beta_2-\beta_0)/3} + e^{(-1)^j(\beta_0-\beta_1)/3}.$$

Then $v^{3,j} \in U^j(0)$. Moreover, $(\omega^1)_1 v^{3,j} = (\omega^2)_1 v^{3,j} = 0$ and $(\tilde{\omega}^2)_1 v^{3,j} = (2/3) v^{3,j}$. Hence $v^{3,j} \in M^0_i$ and $U^j(0)$ contains an $M^0$-submodule isomorphic to $M^0_i$. By the fusion rule $M^0_i \times W^0_i = W^0_i$ of $M^0$-modules and Proposition 11.9, $U^j(0)$ contains an $M^0$-submodule isomorphic to $W^0_i$ also. Thus $U^j(0)$ contains simple $M^0$-submodules of the form $A \otimes M^0_i$ and $A' \otimes W^0_i$ for some simple $M(0)$-modules $A$ and $A'$. The minimal weight of $V_{L(0,j)}$ is 2/3. Its weight subspace is of dimension 3 and spanned by $e^{(-1)^j(\beta_1-\beta_2)/3}$, $e^{(-1)^j(\beta_2-\beta_0)/3}$, and $e^{(-1)^j(\beta_0-\beta_1)/3}$. Thus the weight 2/3 subspace of $U^j(0)$ is $C v^{3,j}$. Since $(\omega^1)_1 v^{3,j} = 0$ and since only $M(0)$ is the simple $M(0)$-module whose minimal weight (= eigenvalue of $(\omega^1)_1$) is 0 by [8], we conclude that $U^j(0)$ contains a simple $M^0$-submodule isomorphic to $M(0) \otimes M^0_i$. 
The minimal eigenvalue of \((\tilde{\omega}^2)\) in \(W^j\) is 1/15. Thus the eigenvalues of \((\tilde{\omega}^1)\) on 
\(A'\) must be of the form \(3/5 + n, n \in \mathbb{Z}\). By [3.1], only \(W(0), W(1), W(2)\) are the simple \(M(0)\)-
modules whose weights are of this form. The minimal weight of these simple modules are
\(8/5, 3/5\) and 3/5, respectively. Since the weight 2/3 subspace of \(U^j(0)\) is one dimensional, we see that \(U^j(0)\) contains a simple \(M^0\)-submodule isomorphic to \(W(0) \otimes W^j\).

From the fusion rules for \(M^0\)-modules, we obtain the fusion rules

\[
(M(\varepsilon) \otimes M^0(\varepsilon)) \times (M(0) \otimes M^j) = M(\varepsilon) \otimes M^j,
\]
\[
(W(\varepsilon) \otimes W^0(\varepsilon)) \times (M(0) \otimes M^j) = W(\varepsilon) \otimes W^j
\]

for \(M^0\)-modules. Hence \(U^j(\varepsilon) \cong (M(\varepsilon) \otimes M^j) \oplus (W(\varepsilon) \otimes W^j)\) for \(j = 1, 2\) and \(\varepsilon = 0, 1, 2\) by [3.1] and [3.11]. In particular, \(V_{L(0,j)}(0) \cong (M(0) \otimes M^j) \oplus (W(0) \otimes W^j)\) as \(M^0\)-modules, \(j = 1, 2\). The top level of \(V_{L(0,j)}(0)\) is \(Cv^{3,j} \subset M(0) \otimes M^j\). The following hold.

\((\tilde{\omega}^1) v^{3,j}, 0 = 0, (\tilde{\omega}^2) v^{3,j} = 2/3 v^{3,j}, J_2 v^{3,j} = 0, K_2 v^{3,j} = -(-1)^j \frac{52}{9} v^{3,j}, P_1 v^{3,j} = 0, (J_1 P)_2 v^{3,j} = 0, (K_1 P)_2 v^{3,j} = 0.

4.4. The simple module \(V_{L(0,j)}(\varepsilon)\), \(j = 1, 2, \varepsilon = 1, 2\). We have shown above that \(V_{L(0,j)}(\varepsilon) \cong (M(\varepsilon) \otimes M^j) \oplus (W(\varepsilon) \otimes W^j)\) as \(M^0\)-modules, \(j = 1, 2, \varepsilon = 1, 2\). The top level of \(V_{L(0,j)}(\varepsilon)\) is \(Cv^{4,j,\varepsilon}\), where

\(v^{4,j,\varepsilon} = e^{(-1)^j (\beta_1 - \beta_2)/3} + \xi^{2\varepsilon} e^{(-1)^j (\beta_2 - \beta_3)/3} + \xi^{\varepsilon} e^{(-1)^j (\beta_3 - \beta_1)/3} \in W(\varepsilon) \otimes W^j\).

The following hold.

\((\tilde{\omega}^1) v^{4,j,\varepsilon} = \frac{3}{5} v^{4,j,\varepsilon}, (\tilde{\omega}^2) v^{4,j,\varepsilon} = \frac{1}{15} v^{4,j,\varepsilon}, J_2 v^{4,j,\varepsilon} = -(-1)^\varepsilon \sqrt{-3} v^{4,j,\varepsilon}, \]
\(K_2 v^{4,j,\varepsilon} = (-1)^j \frac{2}{9} v^{4,j,\varepsilon}, P_1 v^{4,j,\varepsilon} = -(-1)^{j+\varepsilon} \sqrt{3} v^{4,j,\varepsilon}, \]
\((J_1 P)_2 v^{4,j,\varepsilon} = -(-1)^j 2 v^{4,j,\varepsilon}, (K_1 P)_2 v^{4,j,\varepsilon} = -(-1)^\varepsilon 2 \sqrt{-3} v^{4,j,\varepsilon}.\)

4.5. The simple module \(V_{L(c,0)}\). By [3.1], \(V_{L(c,0)} \cong (M^c \otimes M^0) \oplus (W^c \otimes W^0)\) as \(M^0\)-modules. The top level of \(V_{L(c,0)}\) is of dimension 2 with basis \(\{v^{5,1}, v^{5,2}\}\), where \(v^{5,1} = e^{\beta_1/2} - e^{-\beta_1/2} \in M^c \otimes M^0, v^{5,2} = e^{\beta_1/2} + e^{-\beta_1/2} \in W^c \otimes W^0\). The following hold.

\((\tilde{\omega}^1) v^{5,1} = \frac{1}{2} v^{5,1}, (\tilde{\omega}^1) v^{5,2} = \frac{1}{10} v^{5,2}, (\tilde{\omega}^2) v^{5,1} = 0, (\tilde{\omega}^2) v^{5,2} = \frac{2}{5} v^{5,2}, \]
\(J_2 v^{5,j} = 0, K_2 v^{5,j} = 0, j = 1, 2, P_1 v^{5,1} = -v^{5,2}, P_1 v^{5,2} = v^{5,1}, \]
\((J_1 P)_2 v^{5,j} = 0, (K_1 P)_2 v^{5,j} = 0, j = 1, 2.\)

4.6. The simple module \(V_{L(c,j)}\), \(j = 1, 2\). By [3.1], \(V_{L(c,j)} \cong (M^c \otimes M^j) \oplus (W^c \otimes W^j)\) as \(M^0\)-modules, \(j = 1, 2\). The top level of \(V_{L(c,j)}\) is \(Cv^{6,j}\), where \(v^{6,j} = e^{(-1)^j (\beta_2 - \beta_0)/6} \in W^c \otimes W^j\). The following hold.

\((\tilde{\omega}^1) v^{6,j} = \frac{1}{10} v^{6,j}, (\tilde{\omega}^2) v^{6,j} = \frac{1}{15} v^{6,j}, J_2 v^{6,j} = 0, K_2 v^{6,j} = -(-1)^j \frac{2}{9} v^{6,j}, \]
\(P_1 v^{6,j} = 0, (J_1 P)_2 v^{6,j} = -(-1)^j 2 v^{6,j}, (K_1 P)_2 v^{6,j} = 0.\)
4.7. The simple module $V^{T_{x_0}}_L(\tau)(0)$. By \eqref{3.23}, $V^{T_{x_0}}_L(\tau)(0) \cong (M_T(\tau)(0) \otimes M_t^0) \oplus (W_T(\tau)(0) \otimes W_t^0)$ as $M^0$-modules. The top level of $V^{T_{x_0}}_L(\tau)(0)$ is $\mathbb{C}v^7$, where $v^7 = 1 \otimes v \in M_T(\tau)(0) \otimes M_t^0$ and $0 \neq v \in T_{x_0}$. The following hold.

\[
(\tilde{\omega}^1)_1v^7 = \frac{1}{9}v^7, \quad (\tilde{\omega}^2)_1v^7 = 0, \quad J_2v^7 = \frac{14}{81}\sqrt{-3}v^7, \quad K_2v^7 = 0, \\
P_1v^7 = 0, \quad (J_1P)_2v^7 = 0, \quad (K_1P)_2v^7 = 0.
\]

4.8. The simple module $V^{T_{x_0}}_L(\tau)(1)$. By \eqref{3.23}, $V^{T_{x_0}}_L(\tau)(1) \cong (M_T(\tau)(1) \otimes M_t^0) \oplus (W_T(\tau)(1) \otimes W_t^0)$ as $M^0$-modules. The top level of $V^{T_{x_0}}_L(\tau)(1)$ is of dimension 2 with basis $\{v^{8,1}, v^{8,2}\}$, where $v^{8,1} = h_2(-1/3)^2 \otimes v \in M_T(\tau)(1) \otimes M_t^0$, $v^{8,2} = h_1(-2/3) \otimes v \in W_T(\tau)(1) \otimes W_t^0$ and $0 \neq v \in T_{x_0}$. The following hold.

\[
(\tilde{\omega}^1)_1v^{8,1} = \left(\frac{1}{9} + \frac{23}{3}\right)v^{8,1}, \quad (\tilde{\omega}^2)_1v^{8,1} = 0, \quad J_2v^{8,1} = -\frac{238}{81}\sqrt{-3}v^{8,1}, \quad K_2v^{8,1} = 0, \\
P_1v^{8,1} = -\frac{4}{9}v^{8,2}, \quad (J_1P)_2v^{8,1} = \frac{104}{9}\sqrt{-3}v^{8,2}, \quad (K_1P)_2v^{8,1} = 0,
\]

\[
(\tilde{\omega}^1)_1v^{8,2} = \left(\frac{2}{45} + \frac{1}{3}\right)v^{8,2}, \quad (\tilde{\omega}^2)_1v^{8,2} = \frac{2}{5}v^{8,2}, \quad J_2v^{8,2} = -\frac{22}{81}\sqrt{-3}v^{8,2}, \quad K_2v^{8,2} = 0, \\
P_1v^{8,2} = 2v^{8,1}, \quad (J_1P)_2v^{8,2} = -\frac{52}{3}\sqrt{-3}v^{8,1}, \quad (K_1P)_2v^{8,2} = -\frac{20}{3}\sqrt{-3}v^{8,2}.
\]

4.9. The simple module $V^{T_{x_0}}_L(\tau)(2)$. By \eqref{3.23}, $V^{T_{x_0}}_L(\tau)(2) \cong (M_T(\tau)(2) \otimes M_t^0) \oplus (W_T(\tau)(2) \otimes W_t^0)$ as $M^0$-modules. The top level of $V^{T_{x_0}}_L(\tau)(2)$ is $\mathbb{C}v^9$, where $v^9 = h_2(-1/3) \otimes v \in W_T(\tau)(2) \otimes W_t^0$ and $0 \neq v \in T_{x_0}$. The following hold.

\[
(\tilde{\omega}^1)_1v^9 = \frac{2}{45}v^9, \quad (\tilde{\omega}^2)_1v^9 = \frac{2}{5}v^9, \quad J_2v^9 = -\frac{4}{81}\sqrt{-3}v^9, \quad K_2v^9 = 0, \\
P_1v^9 = 0, \quad (J_1P)_2v^9 = 0, \quad (K_1P)_2v^9 = \frac{4}{3}\sqrt{-3}v^9.
\]

4.10. The simple module $V^{T_{x_j}}_L(\tau)(0)$, $j = 1, 2$. By \eqref{3.23}, $V^{T_{x_j}}_L(\tau)(0) \cong (M_T(\tau)(2) \otimes M_t^{3-j}) \oplus (W_T(\tau)(2) \otimes W_t^{3-j})$ as $M^0$-modules for $j = 1, 2$. The top level of $V^{T_{x_j}}_L(\tau)(0)$ is $\mathbb{C}v^{10,j}$, where $v^{10,j} = 1 \otimes v \in W_T(\tau)(2) \otimes W_t^{3-j}$ and $0 \neq v \in T_{x_j}$. The following hold.

\[
(\tilde{\omega}^1)_1v^{10,j} = \frac{2}{45}v^{10,j}, \quad (\tilde{\omega}^2)_1v^{10,j} = \frac{1}{15}v^{10,j}, \quad J_2v^{10,j} = -\frac{4}{81}\sqrt{-3}v^{10,j}, \\
K_2v^{10,j} = -(1)^j\frac{2}{9}v^{10,j}, \quad P_1v^{10,j} = -(1)^j\frac{1}{9}\sqrt{-3}v^{10,j}, \\
(J_1P)_2v^{10,j} = -(1)^j\frac{8}{9}v^{10,j}, \quad (K_1P)_2v^{10,j} = -\frac{2}{9}\sqrt{-3}v^{10,j}.
\]

4.11. The simple module $V^{T_{x_j}}_L(\tau)(1)$, $j = 1, 2$. By \eqref{3.23}, $V^{T_{x_j}}_L(\tau)(1) \cong (M_T(\tau)(0) \otimes M_t^{3-j}) \oplus (W_T(\tau)(0) \otimes W_t^{3-j})$ as $M^0$-modules for $j = 1, 2$. The top level of $V^{T_{x_j}}_L(\tau)(1)$ is
of dimension 2 with basis \( \{ \mathbf{v}^{11,j,1}, \mathbf{v}^{11,j,2} \} \), where

\[
\mathbf{v}^{11,j,1} = h_1(-2/3) \otimes v - (-1)^j \sqrt{-3} h_2(-1/3)^2 \otimes v \in M_T(\tau)(0) \otimes M_i^{0-3-j},
\]
\[
\mathbf{v}^{11,j,2} = 2h_1(-2/3) \otimes v + (-1)^j \sqrt{-3} h_2(-1/3)^2 \otimes v \in W_T(\tau)(0) \otimes W_i^{0-3-j}
\]

and \( 0 \neq v \in T_{x_j} \). The following hold.

\[
(\tilde{\omega}^1)_{1} \mathbf{v}^{11,j,1} = \frac{1}{9} \mathbf{v}^{11,j,1}, \quad (\tilde{\omega}^2)_{1} \mathbf{v}^{11,j,1} = \frac{2}{3} \mathbf{v}^{11,j,1}, \quad J_2 \mathbf{v}^{11,j,1} = \frac{14}{81} \sqrt{-3} \mathbf{v}^{11,j,1},
\]
\[
K_2 \mathbf{v}^{11,j,1} = (-1)^j \frac{52}{9} \mathbf{v}^{11,j,1}, \quad P_1 \mathbf{v}^{11,j,1} = -(-1)^j \frac{4}{9} \sqrt{-3} \mathbf{v}^{11,j,2},
\]
\[
(J_1 P)_{2} \mathbf{v}^{11,j,1} = (-1)^j \frac{52}{9} \mathbf{v}^{11,j,2},
\]
\[
(K_1 P)_{2} \mathbf{v}^{11,j,1} = -\frac{28}{9} \sqrt{-3} \mathbf{v}^{11,j,2},
\]

\[
(\tilde{\omega}^1)_{1} \mathbf{v}^{11,j,2} = \left( \frac{2}{45} + \frac{2}{3} \right) \mathbf{v}^{11,j,2}, \quad (\tilde{\omega}^2)_{1} \mathbf{v}^{11,j,2} = \frac{1}{15} \mathbf{v}^{11,j,2}, \quad J_2 \mathbf{v}^{11,j,2} = \frac{176}{81} \sqrt{-3} \mathbf{v}^{11,j,2},
\]
\[
K_2 \mathbf{v}^{11,j,2} = -(-1)^j \frac{2}{9} \mathbf{v}^{11,j,2}, \quad P_1 \mathbf{v}^{11,j,2} = -(-1)^j \frac{8}{9} \sqrt{-3} \mathbf{v}^{11,j,1} + (-1)^j \frac{5}{9} \sqrt{-3} \mathbf{v}^{11,j,2},
\]
\[
(J_1 P)_{2} \mathbf{v}^{11,j,2} = (-1)^j \frac{104}{9} \mathbf{v}^{11,j,1} - (-1)^j \frac{200}{9} \mathbf{v}^{11,j,2},
\]
\[
(K_1 P)_{2} \mathbf{v}^{11,j,2} = -\frac{56}{9} \sqrt{-3} \mathbf{v}^{11,j,1} - \frac{10}{9} \sqrt{-3} \mathbf{v}^{11,j,2}.
\]

4.12. The simple module \( V_{L}^{T_{x_j}}(\tau)(2), \ j = 1, 2 \). By \([3,23]\), \( V_{L}^{T_{x_j}}(\tau)(2) \cong (M_T(\tau)(1) \otimes M_i^{0-3-j}) \oplus (W_T(\tau)(1) \otimes W_i^{0-3-j}) \) as \( M_0 \)-modules for \( j = 1, 2 \). The top level of \( V_{L}^{T_{x_j}}(\tau)(2) \) is \( \mathbb{C} \mathbf{v}^{12,j} \), where \( \mathbf{v}^{12,j} = h_2(-1/3) \otimes v \in W_T(\tau)(1) \otimes W_i^{3-3-j} \) and \( 0 \neq v \in T_{x_j} \). The following hold.

\[
(\tilde{\omega}^1)_{1} \mathbf{v}^{12,j} = \left( \frac{2}{45} + \frac{1}{3} \right) \mathbf{v}^{12,j}, \quad (\tilde{\omega}^2)_{1} \mathbf{v}^{12,j} = \frac{1}{15} \mathbf{v}^{12,j}, \quad J_2 \mathbf{v}^{12,j} = -\frac{22}{81} \sqrt{-3} \mathbf{v}^{12,j},
\]
\[
K_2 \mathbf{v}^{12,j} = -(-1)^j \frac{2}{9} \mathbf{v}^{12,j}, \quad P_1 \mathbf{v}^{12,j} = -(-1)^j \frac{5}{9} \sqrt{-3} \mathbf{v}^{12,j},
\]
\[
(J_1 P)_{2} \mathbf{v}^{12,j} = -(-1)^j \frac{8}{9} \mathbf{v}^{12,j}, \quad (K_1 P)_{2} \mathbf{v}^{12,j} = \frac{10}{9} \sqrt{-3} \mathbf{v}^{12,j}.
\]

4.13. The simple module \( V_{L}^{T_{x_0}}(\tau^2)(0) \). By \([3,23]\), \( V_{L}^{T_{x_0}}(\tau^2)(0) \cong (M_T(\tau^2)(0) \otimes M_i^0) \oplus (W_T(\tau^2)(0) \otimes W_i^0) \) as \( M_0 \)-modules. The top level of \( V_{L}^{T_{x_0}}(\tau^2)(0) \) is \( \mathbb{C} \mathbf{v}^{13} \), where \( \mathbf{v}^{13} = 1 \otimes v \in M_T(\tau^2)(0) \otimes M_i^0 \) and \( 0 \neq v \in T_{x_0} \). The following hold.

\[
(\tilde{\omega}^1)_{1} \mathbf{v}^{13} = \frac{1}{9} \mathbf{v}^{13}, \quad (\tilde{\omega}^2)_{1} \mathbf{v}^{13} = 0, \quad J_2 \mathbf{v}^{13} = -\frac{14}{81} \sqrt{-3} \mathbf{v}^{13}, \quad K_2 \mathbf{v}^{13} = 0,
\]
\[
P_1 \mathbf{v}^{13} = 0, \quad (J_1 P)_{2} \mathbf{v}^{13} = 0, \quad (K_1 P)_{2} \mathbf{v}^{13} = 0.
\]
4.14. The simple module $V_L^{T_{X_0}^0}(\tau^2)(1)$. By (3.24), $V_L^{T_{X_0}^0}(\tau^2)(1) \cong (M_T(\tau^2)(1) \otimes M_0^0) \oplus (W_T(\tau^2)(1) \otimes W_0^0)$ as $M_0^0$-modules. The top level of $V_L^{T_{X_0}^0}(\tau^2)(1)$ is of dimension 2 with basis $\{v^{14,1}, v^{14,2}\}$, where $v^{14,1} = h_2'(-1/3) \otimes v \in M_T(\tau^2)(1) \otimes M_0^0$, $v^{14,2} = h_1'(-2/3) \otimes v \in W_T(\tau^2)(1) \otimes W_0^0$ and $0 \neq v \in T_{X_0}^0$. The following hold.

$$(\tilde{\omega})_1^1 v^{14,1} = \left(\frac{1}{9} + \frac{2}{3}\right)v^{14,1}, \quad (\tilde{\omega})_1^2 v^{14,1} = 0, \quad J_2 v^{14,1} = \frac{238}{81} \sqrt{-3}v^{14,1}, \quad K_2 v^{14,1} = 0,$$

$$P_1 v^{14,1} = -\frac{4}{3} v^{14,2}, \quad (J_1 P) v^{14,1} = -\frac{104}{9} \sqrt{-3}v^{14,2}, \quad (K_1 P) v^{14,1} = 0,$$

$$(\tilde{\omega})_1^1 v^{14,2} = \left(\frac{2}{45} + \frac{1}{3}\right)v^{14,2}, \quad (\tilde{\omega})_1^2 v^{14,2} = \frac{2}{5} v^{14,2}, \quad J_2 v^{14,2} = \frac{22}{81} \sqrt{-3}v^{14,2}, \quad K_2 v^{14,2} = 0,$$

$$P_1 v^{14,2} = 2v^{14,1}, \quad (J_1 P) v^{14,2} = \frac{52}{3} \sqrt{-3}v^{14,1}, \quad (K_1 P) v^{14,2} = \frac{20}{3} \sqrt{-3}v^{14,2}.$$

4.15. The simple module $V_L^{T_{X_0}^0}(\tau^2)(2)$. By (3.24), $V_L^{T_{X_0}^0}(\tau^2)(2) \cong (M_T(\tau^2)(2) \otimes M_0^0) \oplus (W_T(\tau^2)(2) \otimes W_0^0)$ as $M_0^0$-modules. The top level of $V_L^{T_{X_0}^0}(\tau^2)(2)$ is $\mathbb{C}v^{15}$, where $v^{15} = h_2'(-1/3) \otimes v \in W_T(\tau^2)(2) \otimes W_0^0$ and $0 \neq v \in T_{X_0}^0$. The following hold.

$$(\tilde{\omega})_1^1 v^{15} = \frac{2}{45} v^{15}, \quad (\tilde{\omega})_1^2 v^{15} = \frac{2}{5} v^{15}, \quad J_2 v^{15} = \frac{4}{81} \sqrt{-3}v^{15}, \quad K_2 v^{15} = 0,$$

$$P_1 v^{15} = 0, \quad (J_1 P) v^{15} = 0, \quad (K_1 P) v^{15} = -\frac{4}{3} \sqrt{-3}v^{15}.$$

4.16. The simple module $V_L^{T_{X_0}^j}(\tau^2)(0)$, $j = 1, 2$. By (3.24), $V_L^{T_{X_0}^j}(\tau^2)(0) \cong (M_T(\tau^2)(2) \otimes M_0^0) \oplus (W_T(\tau^2)(2) \otimes W_0^0)$ as $M_0^0$-modules for $j = 1, 2$. The top level of $V_L^{T_{X_0}^j}(\tau^2)(0)$ is $\mathbb{C}v^{16,j}$, where $v^{16,j} = 1 \otimes v \in W_T(\tau^2)(2) \otimes W_0^0$ and $0 \neq v \in T_{X_0}^j$. The following hold.

$$(\tilde{\omega})_1^1 v^{16,j} = \frac{2}{45} v^{16,j}, \quad (\tilde{\omega})_1^2 v^{16,j} = \frac{1}{15} v^{16,j}, \quad J_2 v^{16,j} = \frac{4}{81} \sqrt{-3}v^{16,j},$$

$$K_2 v^{16,j} = (-1)^j \frac{2}{9} v^{16,j}, \quad P_1 v^{16,j} = (-1)^j \frac{1}{9} \sqrt{-3}v^{16,j},$$

$$(J_1 P) v^{16,j} = (-1)^j \frac{8}{9} v^{16,j}, \quad (K_1 P) v^{16,j} = \frac{2}{9} \sqrt{-3}v^{16,j}.$$

4.17. The simple module $V_L^{T_{X_0}^j}(\tau^2)(1)$, $j = 1, 2$. By (3.24), $V_L^{T_{X_0}^j}(\tau^2)(1) \cong (M_T(\tau^2)(0) \otimes M_0^0) \oplus (W_T(\tau^2)(0) \otimes W_0^0)$ as $M_0^0$-modules for $j = 1, 2$. The top level of $V_L^{T_{X_0}^j}(\tau^2)(1)$ is of dimension 2 with basis $\{v^{17,j,1}, v^{17,j,2}\}$, where

$$v^{17,j,1} = h_1'(-2/3) \otimes v - (-1)^j \sqrt{-3}h_2'(-1/3)^2 \otimes v \in M_T(\tau^2)(0) \otimes M_j^0,$$

$$v^{17,j,2} = 2h_1'(-2/3) \otimes v + (-1)^j \sqrt{-3}h_2'(-1/3)^2 \otimes v \in W_T(\tau^2)(0) \otimes W_j^0.$$
and $0 \neq v \in T_{\chi_j}$. The following hold.

$$(\tilde{\omega})_1^1 v^{17,j,1} = \frac{1}{9} v^{17,j,1}, \quad (\tilde{\omega})_2^1 v^{17,j,1} = \frac{2}{3} v^{17,j,1}, \quad \sigma K v^{17,j,1} = -\frac{14}{81} \sqrt{-3} v^{17,j,1},$$
$$K_2 v^{17,j,1} = (-1)^j \frac{52}{9} v^{17,j,1}, \quad P_1 v^{17,j,1} = (-1)^j \frac{4}{9} \sqrt{-3} v^{17,j,2},$$
$$\sqrt{(J_1 P)_2 v^{17,j,1} = (-1)^j \frac{52}{9} v^{17,j,2},}$$
$$\sqrt{(K_1 P)_2 v^{17,j,1} = \frac{28}{9} \sqrt{-3} v^{17,j,2}}.$$

4.18. The simple module $V_L^{T_{\chi_j}(\tau^2)}(2)$, $j = 1, 2$. By (3.24), $V_L^{T_{\chi_j}(\tau^2)}(2) \cong (M_T(\tau^2)(1) \otimes M^0) \oplus (W_T(\tau^2)(1) \otimes W^j)$ as $M^0$-modules for $j = 1, 2$. The top level of $V_L^{T_{\chi_j}(\tau^2)}(2)$ is $\mathbb{C} v^{18,j}$, where $v^{18,j} = h_j(-1/3) \otimes v \in W_T(\tau^2)(1) \otimes W^j$ and $0 \neq v \in T_{\chi_j}$. The following hold.

$$(\tilde{\omega})_1^1 v^{18,j} = \left(\frac{2}{45} + \frac{1}{3}\right) v^{18,j}, \quad (\tilde{\omega})_2^1 v^{18,j} = \frac{1}{15} v^{18,j}, \quad \sigma K v^{18,j} = \frac{22}{81} \sqrt{-3} v^{18,j},$$
$$K_2 v^{18,j} = (-1)^j \frac{2}{9} v^{18,j}, \quad P_1 v^{18,j} = (-1)^j \frac{5}{9} \sqrt{-3} v^{18,j},$$
$$\sqrt{(J_1 P)_2 v^{18,j} = (-1)^j \frac{8}{9} v^{18,j},} \quad (K_1 P)_2 v^{18,j} = \frac{10}{9} \sqrt{-3} v^{18,j}.$$

4.19. Symmetries by $\sigma$. Let us consider the automorphisms $\sigma$ and $\theta$ of $V_L$ which are lifts of the isometries $\sigma$ and $\theta$ of the lattice $L$ defined by (3.1). Clearly, $\sigma T \sigma = T$, $\sigma \theta \theta = \theta \sigma$, and $\tau \sigma = \sigma \tau$. Thus $\sigma$ and $\theta$ induce automorphisms of $V_L^T$ of order 2. We have $\sigma J = -J$, $\sigma K = -K$, $\sigma P = P$, $\theta J = J$, $\theta K = -K$, and $\theta P = -P$. Hence $\sigma$ and $\theta$ induce the same automorphism of $M^0$ and $\theta$ is the identity on $M(0)$. Note also that $\sigma (J_1 P) = -J_1 P$ and $\sigma (K_1 P) = -K_1 P$.

From the action of $\sigma$ on the top level of the 30 known simple $V_L^T$-modules or the action of $J_2, K_2, (J_1 P)_2$, and $(K_1 P)_2$, we know how $\sigma$ permutes those simple $V_L^T$-modules. In fact, $\sigma$ transforms $V_L^{T_{\chi_j}(\tau^2)}(2)$ into an equivalent simple $V_L^T$-module and interchanges the remaining simple $V_L^T$-modules as follows.

$$V_L(1) \leftrightarrow V_L(2), \quad V_L^{T_{\chi_j}(\tau^2)}(2) \leftrightarrow V_L^{T_{\chi_j}(\tau^2)}(2), \quad \varepsilon = 0, 1, 2,$$
$$V_L^{T_{\chi_j}(\tau^2)}(2) \leftrightarrow V_L^{T_{\chi_j}(\tau^2)}(2), \quad \varepsilon = 0, 1, 2.$$
Note that $\sigma h_i = \xi^3 i h_i', \ i = 1, 2$. The top level of $V^{T_{\xi_j}}_{L}(\tau^2)(\varepsilon)$ can be obtained by replacing $h_i(i/3 + n)$ with $h_i'(i/3 + n)$ in the top level of $V^{T_{\xi_j}}_{L}(\tau)(\varepsilon)$ for $j, \varepsilon = 0, 1, 2$. The corresponding action of $\sigma$ on the simple $M(0)$-modules was discussed in [36, Section 4.4].

5. Classification of simple modules

We keep the notation in the preceding section. Thus $V^r_L = M^0 \oplus W^0$ with $M^0 = M(0) \otimes M^0_i$ and $W^0 = W(0) \otimes W^0_i$. In this section we show that any simple $V^r_L$-module is equivalent to one of the 30 simple $V^r_L$-modules listed in Lemma 3.2. The result will be established by considering the Zhu algebra $A(V^r_L)$ of $V^r_L$.

First, we review some notation and basic formulas for the Zhu algebra $A(V)$ of a vertex operator algebra $(V, Y, 1, \omega)$. Define two binary operations

$$u \ast v = \sum_{i=0}^{\infty} \left( \text{wt} \ u \right)_i u_{i-1}v, \quad u \circ v = \sum_{i=0}^{\infty} \left( \text{wt} \ u \right)_i u_{i-2}v$$

for $u, v \in V$ with $u$ being homogeneous and extend $\ast$ and $\circ$ for arbitrary $u \in V$ by linearity. Let $O(V)$ be the subspace of $V$ spanned by all $u \circ v$ for $u, v \in V$. Set $A(V) = V/O(V)$. By [36, Theorem 2.1.1], $O(V)$ is a two-sided ideal with respect to the operation $\ast$. Thus $\ast$ induces an operation in $A(V)$. Denote by $[v]$ the image of $v \in V$ in $A(V)$. Then $[u] \ast [v] = [u \ast v]$ and $A(V)$ is an associative algebra by this operation. Moreover, $[1]$ is the identity and $[\omega]$ is in the center of $A(V)$. For $u, v \in V$, we write $u \sim v$ if $[u] = [v]$. For $\varphi, \psi \in \text{End} V$, we write $\varphi \sim \psi$ if $\varphi v \sim \psi v$ for all $v \in V$. We need some basic formulas (cf. [36]).

$$v \ast u \sim \sum_{i=0}^{\infty} \left( \text{wt} \ u \right)_i u_{i-1}v, \quad (5.2)$$

$$\sum_{i=0}^{\infty} \left( \text{wt}(u) + m \right)_i u_{i-n-2}v \in O(V), \quad n \geq m \geq 0. \quad (5.3)$$

Moreover (cf. [33]),

$$L(-n) \sim (-1)^n \left\{ (n-1)(L(-2) + L(-1)) + L(0) \right\}, \quad n \geq 1, \quad (5.4)$$

$$[\omega] \ast [u] = [(L(-2) + L(-1))u], \quad (5.5)$$

where $L(n) = \omega_{n+1}$. From (5.4) and (5.5) we have

$$[L(-n)u] = (-1)^n (n-1)[\omega] \ast [u] + (-1)^n[L(0)u], \quad n \geq 1. \quad (5.6)$$

If $u \in V$ is of weight 2, then $u(-n - 3) + 2u(-n - 2) + u(-n - 1) \sim 0$ by (5.3), where $u(n) = u_{n+1}$. Hence

$$u(-n) \sim (-1)^n ((n-1)u(-2) + (n-2)u(-1)) \quad (5.7)$$

for $n \geq 1$. Then it follows from (5.1) and (5.2) that

$$u(-n)w \sim (-1)^n \left\{ -u \ast w + nw \ast u + u(0)w \right\} \quad (5.8)$$
for \( n \geq 1, \) \( w \in V. \) Likewise, if \( u \) is of weight 3 and \( u(n) = u_{n+2} \), then

\[
u(-n) \sim (-1)^{n+1} \left( \frac{1}{2}(n-1)(n-2)u(-3) + (n-1)(n-3)u(-2) + \frac{1}{2}(n-2)(n-3)u(-1) \right), \tag{5.9}\]

\[
u(-n)w \sim (-1)^{n+1} \left( nu(-1)w + (n-1)u(0)w - (n-1)u * w + \frac{1}{2}n(n-1)w * u \right) \tag{5.10}\]

for \( n \geq 1, w \in V. \)

For a homogeneous vector \( u \in V, \) \( o(u) = u_{\text{wt}(u)-1} \) is the weight zero component operator of \( Y(u, z). \) Extend \( o(u) \) for arbitrary \( u \in V \) by linearity. Note that we call a module in the sense of \([36]\) an \( N \)-graded weak module here. If \( M = \bigoplus_{n=0}^{\infty} M(n) \) is an \( N \)-graded weak \( V \)-module with \( M(0) \neq 0 \), then \( o(u) \) acts on its top level \( M(0). \) Zhu’s theory \([36]\) says: (1) \( o(u) o(v) = o(u * v) \) as operators on the top level \( M(0) \) and \( o(u) \) acts as 0 on \( M(0) \) if \( u \in O(V). \) Thus \( M(0) \) is an \( A(V) \)-module, where \([u] \) acts on \( M(0) \) as \( o(u). \) (2) The map \( M \mapsto M(0) \) is a bijection between the set of isomorphism classes of simple \( N \)-graded weak \( V \)-modules and the set of isomorphism classes of simple \( A(V) \)-modules.

Let us return to our \( V_L^* \). As in Section \([3]\) we write \( L_i(n) = (\tilde{\omega}^i)_{n+1}, i = 1, 2, J(n) = J_{n+2}, \) and \( K(n) = K_{n+2}. \) The Zhu algebras \( A(M(0)) \) and \( A(M_0^0) \) were determined in \([8]\) and \([21]\), respectively. Since \( O(M_0^0) \subset O(V_L^*) \), the image of \( M(0) \) (resp. \( M_0^0 \)) in \( A(V_L^*) \) is a homomorphic image of \( A(M(0)) \) (resp. \( A(M_0^0) \)). It is generated by \([\tilde{\omega}^1]\) and \([J]\) (resp. \([\tilde{\omega}^2]\) and \([K]\)).

By a direct calculation, we have

\[
P_1 P = -16\tilde{\omega}^1 - 6\tilde{\omega}^2, \]

\[
P_0 P = -8(\tilde{\omega}^1)_{-2}1 - 3(\tilde{\omega}^2)_{-2}1,
\]

\[
P_{-1} P = \frac{5}{273} J_1 K_1 P - \frac{12}{7}(\tilde{\omega}^1)_{-3}1 - \frac{18}{13}(\tilde{\omega}^2)_{-3}1 - \frac{36}{7}(\tilde{\omega}^1)_{-1}(\tilde{\omega}^1)_{-1}1 - \frac{9}{13}(\tilde{\omega}^2)_{-1}(\tilde{\omega}^2)_{-1}1 - 16(\tilde{\omega}^1)_{-1}(\tilde{\omega}^2)_{-1}1, \tag{5.11}\]

\[
P_{-2} P = \frac{1}{84} J_0 K_1 P + \frac{1}{156} J_1 K_0 P - \frac{8}{7}(\tilde{\omega}^1)_{-4}1 - \frac{12}{13}(\tilde{\omega}^2)_{-4}1 - \frac{36}{7}(\tilde{\omega}^1)_{-2}(\tilde{\omega}^1)_{-1}1 - \frac{9}{13}(\tilde{\omega}^2)_{-2}(\tilde{\omega}^2)_{-1}1 - 8(\tilde{\omega}^1)_{-2}(\tilde{\omega}^2)_{-1}1 - 8(\tilde{\omega}^1)_{-1}(\tilde{\omega}^2)_{-2}1.\]

Moreover, \( J_2 P = K_2 P = 0. \) Then, using the formulas \((5.11)-(5.10), \) we obtain

\[
[P] * [P] = \frac{5}{273} [J_1 K_1 P] - \frac{36}{7} [\tilde{\omega}^1] * [\tilde{\omega}^1] - \frac{9}{13} [\tilde{\omega}^2] * [\tilde{\omega}^2] - 16[\tilde{\omega}^1] * [\tilde{\omega}^2] + \frac{4}{7}[\tilde{\omega}^1] + \frac{6}{13}[\tilde{\omega}^2], \tag{5.12}\]
\[ [P \circ P] = \frac{1}{84} [J] * [K_1 P] - \frac{1}{84} [K_1 P] * [J] + \frac{1}{156} [K] * [J_1 P] - \frac{1}{156} [J_1 P] * [K] \] (5.13)

\[ [P \circ P] = 0. \]

It turns out that \( A(V^*_L) \) is generated by \([\tilde{\omega}^1], [\tilde{\omega}^2], [J], [K], \) and \([P] \) (cf. Corollary 5.11). However, we first prove the following intermediate assertion.

**Proposition 5.1.** The Zhu algebra \( A(V^*_L) \) is generated by \([\tilde{\omega}^1], [\tilde{\omega}^2], [J], [K], [P], [J_1 P], \) and \([K_1 P] \).

**Proof.** Recall that \( L^i(n) P = 0 \) for \( i = 1, 2 \), \( n \geq 1 \), \( L^1(0) P = (8/5) P \), \( L^2(0) P = (2/5) P \), and \( J(n) P = K(n) P = 0 \) for \( n \geq 0 \). Thus from the commutation relations (3.9)–(3.11) and (3.13)–(3.15) we see that \( W^0 \) is spanned by the vectors of the form

\[ L^1(-j_1) \cdots L^1(-j_r) L^2(-k_1) \cdots L^2(-k_s) J(-m_1) \cdots J(-m_p) K(-n_1) \cdots K(-n_q) P \] (5.14)

with \( j_1 \geq \cdots \geq j_r \geq 1, k_1 \geq \cdots \geq k_s \geq 1, m_1 \geq \cdots \geq m_p \geq 1, n_1 \geq \cdots \geq n_q \geq 1 \). Let \( v \) be a vector of this form. Its weight is

\[ j_1 + \cdots + j_r + k_1 + \cdots + k_s + m_1 + \cdots + m_p + n_1 + \cdots + n_q + 2. \]

Since \( V_L^* = M^0 \oplus W^0 \) and since the image of \( M^0 \) in \( A(V_L^*) \) is generated by \([\tilde{\omega}^1], [\tilde{\omega}^2], [J], \) and \([K] \), it suffices to show that the image \([v] \) of \( v \) in \( A(V_L^*) \) is contained in the subalgebra generated by \([\tilde{\omega}^1], [\tilde{\omega}^2], [J], [K], [P], [J_1 P], \) and \([K_1 P] \). We proceed by induction on the weight of \( v \). By the formula (5.8) with \( u = \tilde{\omega}^i \), \( i = 1, 2 \) and the induction on the weight, we may assume that \( r = s = 0 \), that is,

\[ v = J(-m_1) \cdots J(-m_p) K(-n_1) \cdots K(-n_q) P. \]

Moreover, by the formula (5.10) with \( u = J \), we may assume that \( m_1 = \cdots = m_p = 1 \). Since \( J(m) \) and \( K(n) \) commute, we may also assume that \( n_1 = \cdots = n_q = 1 \) by a similar argument. Then \( v = J(-1)^p K(-1)^q P \).

Next, we reduce \( v \) to the case \( p \leq 1 \). For this purpose, we use a singular vector

\[ 5J(-1)^2 P + 2496L^1(-2) P - 195L^1(-2) P = 0. \] (5.15)

in \( W(0) \). Suppose \( p \geq 2 \). Then, since \( K(-1) \) commute with \( J(m) \) and \( L^1(n) \), (5.15) implies that \( v = J(-1)^p K(-1)^q P \) is a linear combination of \( J(-1)^{p-2} L^1(-2) K(-1)^q P \) and \( J(-1)^{p-2} L^1(-1)^2 K(-1)^q P \). By the commutation relation (3.10), these two vectors can be written in the form \( L^1(-2) H K(-1)^q P \) and \( L^1(-1)^2 H' K(-1)^q P \), where \( H \) (resp. \( H' \)) is a polynomial in \( J(-1) \) and \( J(-3) \) (resp. \( J(-1), J(-2) \), and \( J(-3) \)). Then by the formula (5.8) with \( u = \tilde{\omega}^1 \) and the induction on the weight, the assertion holds for \( v \). Hence we may assume that \( p \leq 1 \).

There is a singular vector

\[ K(-1)^2 P - 210 L^2(-2) P = 0 \] (5.16)

in \( W^0 \). Thus, by a similar argument as above, we may assume that \( q \leq 1 \). Finally, it follows from (5.12) that \( [J(-1) K(-1) P] \) can be written by \([\tilde{\omega}^1], [\tilde{\omega}^2], \) and \([P] \) in \( A(V_L^*) \). The proof is complete. 

\[ \square \]
Let us classify the simple $V^\tau_L$-modules. Our argument is based on the knowledge of simple modules for $M(0)$ and $M^0_i$ together with fusion rules (3.25) and (3.7) among them. Set

$$\mathcal{M}_1 = \{M(\varepsilon), M^\tau_{\varepsilon}, M^\tau_T(\varepsilon) | i = 1, 2, \varepsilon = 0, 1, 2\},$$

$$\mathcal{W}_1 = \{W(\varepsilon), W^\tau_{\varepsilon}, W^\tau_T(\varepsilon) | i = 1, 2, \varepsilon = 0, 1, 2\},$$

$$\mathcal{M}_2 = \{M^\tau_i | j = 0, 1, 2\}, \quad \mathcal{W}_2 = \{W^\tau_i | j = 0, 1, 2\}.$$

Then $\mathcal{M}_1 \cup \mathcal{W}_1$ (resp. $\mathcal{M}_2 \cup \mathcal{W}_2$) is a complete set of representatives of isomorphism classes of simple $M(0)$-modules (resp. simple $M^0$-modules). A main point is that the fusion rules of the following form hold.

$$W(0) \times M^1 = W^1, \quad W(0) \times W^1 = M^1 + W^1,$$

$$W^0 \times M^2 = W^2, \quad W^0 \times W^2 = M^2 + W^2,$$  \hspace{1cm} (5.17)

where $M^i \in \mathcal{M}_i$, $i = 1, 2$, and $W^i \in \mathcal{W}_i$ is determined by $M^i$ through the fusion rule $W(0) \times M^1 = W^1$ or $W^0 \times M^2 = W^2$.

Recall that $M^0$ is rational, $C_2$-cofinite, and of CFT type. Thus every $N$-graded weak $M^0$-module is a direct sum of simple $M^0$-modules. As a result, every $N$-graded weak $V^\tau_L$-module is decomposed into a direct sum of simple $M^0$-modules, and in particular $L(0) = \omega_1$ acts semisimply on it. Each weight subspace, that is, each eigenspace for $L(0)$ is not necessarily a finite dimensional space. However, any simple $V^\tau_L$-module is a simple ordinary $V^\tau_L$-module by [11, Corollary 5.8], since $V^\tau_L$ is $C_2$-cofinite and of CFT type.

We note that

$$W^0 \cdot W^0 = V^\tau_L.$$  \hspace{1cm} (5.18)

Indeed, $W^0 \cdot W^0 = \text{span}\{a_n b | a, b \in W^0, n \in \mathbb{Z}\}$ is an $M^0$-submodule of $V^\tau_L$ by (2.6). Since $P, J_1 K_1 P \in W^0$ and $\tilde{\omega}^1, \tilde{\omega}^2 \in M^0$, (5.17) implies that $W^0 \cdot W^0 = M^0 \oplus W^0$.

Each simple $M^0$-module is isomorphic to a tensor product $A \otimes B$ of a simple $M(0)$-module $A$ and a simple $M^0_i$-module $B$. We show that only restricted simple $M^0$-modules can appear in $N$-graded weak $V^\tau_L$-modules.

**Lemma 5.2.** Let $U$ be an $N$-graded weak $V^\tau_L$-module. Then any simple $M^0$- submodule of $U$ is isomorphic to $M^1 \otimes M^2$ or $W^1 \otimes W^2$ for some $M^1 \in \mathcal{M}_i$ and $W^i \in \mathcal{W}_i$, $i = 1, 2$.

**Proof.** Suppose $U$ contains a simple $M^0$-submodule $S^0 \cong M^1 \otimes W^2$ with $M^1 \in \mathcal{M}_1$ and $W^2 \in \mathcal{W}_2$. Let $S = V^\tau_L \cdot S^0 = \text{span}\{a_n w | a \in V^\tau_L, w \in S^0, n \in \mathbb{Z}\}$. Then (2.6) implies that $S$ is the $N$-graded weak $V^\tau_L$-submodule of $U$ generated by $S^0$. By the construction of $S$, the difference of any two eigenvalues of $L(0)$ in $S$ is an integer. In fact, $S$ is an ordinary $V^\tau_L$-module by Remark 2.16.

If $v$ is a nonzero vector in $V^\tau_L$, then $v_n S^0 \neq 0$ for some $n \in \mathbb{Z}$. Indeed, Lemma 2.6 implies that the set $\{v \in V^\tau_L | v_n S^0 = 0 \text{ for all } n \in \mathbb{Z}\}$ is an ideal of $V^\tau_L$. It is in fact 0, since $V^\tau_L$ is a simple vertex operator algebra and $S^0$ is a simple $M^0$-module. Then by the fusion rules (5.17), a simple $M^0$-module isomorphic to $W^1 \otimes M^2$ or $W^1 \otimes W^2$ must appear in $S$. However, the difference of the minimal eigenvalues of $L(0)$ in $M^1 \otimes W^2$ and $W^1 \otimes M^2$, or in $M^1 \otimes W^2$ and $W^1 \otimes W^2$ is not an integer. This is a contradiction. Thus $U$ does not contain a simple $M^0$-submodule isomorphic to $M^1 \otimes W^2$. By a similar argument, we can also show that there is no simple $M^0$-submodule isomorphic to $W^1 \otimes M^2$ in $U$. Hence the assertion holds.
Set $\mathcal{M} = \{M^i \otimes M^j | M^i \in \mathcal{M}_i, i = 1, 2\}$ and $W = \{W^i \otimes W^j | W^i \in \mathcal{W}_i, i = 1, 2\}$. Then each of $\mathcal{M}$ and $W$ consists of 30 inequivalent simple $M^0$-modules. The top level of every simple $M^0$-module is of dimension one.

**Lemma 5.3.** If $U$ is a simple $\mathbb{N}$-graded weak $V_\tau^+$-module whose top level is of dimension one, then $U$ is isomorphic to one of the 23 known simple $V_\tau^+$-modules with one dimensional top level, namely, $V_{L(0,\varepsilon)}(\varepsilon)$, $j = 0, 1, 2$, $\varepsilon = 0, 1, 2$, $V_{L(\varepsilon,j)}(\varepsilon)$, $j = 1, 2$, $V_{L(\varepsilon,j)}^T(\varepsilon)$, $j = 0, 1, 2$, $\varepsilon = 0, 2$, and $V_{L(\varepsilon,j)}^T(\varepsilon)$, $j = 0, 1, 2$, $\varepsilon = 0, 2$.

**Proof.** Since $U$ is a direct sum of simple $M^0$-modules and since the top level, say $U_\lambda$ of $U$ is assumed to be of dimension one, it follows from Lemma 5.2 that $U_\lambda$ is isomorphic to the top level of $M^1 \otimes M^2$ or the top level of $W^1 \otimes W^2$ as an $A(M^0)$-module for some $M^i \in \mathcal{M}_i$, $W^i \in \mathcal{W}_i$, $i = 1, 2$. The Zhu algebra $A(M^0) \cong A(M(0)) \times A(M_0^0)$ is commutative and the action of $A(M^0)$ on the top level of $M^1 \otimes M^2$ and the top level of $W^1 \otimes W^2$ are known. Indeed, we know all possible action of the elements $[\tilde{x}^1]$, $[\tilde{x}^2]$, $[J]$, and $[K]$ of $A(V_\tau^+)$ on $U_\lambda$. Let $[\tilde{x}^1]$, $[\tilde{x}^2]$, $[J]$, and $[K]$ act on $U_\lambda$ as scalars $a_1$, $a_2$, $b_1$, and $b_2$, respectively. There are 60 possible such quadruplets $(a_1, a_2, b_1, b_2)$.

Let $[P]$, $[J_1P]$, and $[K_1P]$ act on $U_\lambda$ as scalars $x_1$, $x_2$, and $x_3$, respectively. Then it follows from (5.12) that $[J_1K_1P]$ acts on $U_\lambda$ as a scalar

\[(273/5)x_1^2 + (1404/5)a_1^2 + (189/5)a_2^2 + (4368/5)a_1a_2 - (156/5)a_1 - (126/5)a_2.\] (5.19)

From Appendix B and formulas (5.4)–(5.10) we see that $[P \circ (J_1P)] = 0$ gives

\[15b_2x_1 + 5a_2x_3 - 2x_3 = 0.\] (5.20)

Likewise,

\[(15a_2 - 1)x_2 = 0,\] (5.21)

since $[P \circ (K_1P)] = 0$.

Using (5.19), we can calculate $[(J_1P) \ast (J_1P)]$, $[(K_1P) \ast (K_1P)]$, and $[(J_1P) \ast (K_1P)]$ in a similar way and verify that the following equations hold.

\[x_2^2 = \left( (229164/575)a_1 - (37856/425)a_2 + 1669382/48875 \right)x_1^2 - (56/85)b_2x_2 - (4056/115)b_1x_3 + (348994464/107525)a_1^2 + (137149584/9775)a_2^2 - (1030224/1375)a_1a_2 + (704876/9775)a_1a_2^2 - (40788488/48875)a_1a_2 + (16160456/537625)a_1 - (419184/9775)a_2^3 - (200994/48875)a_2^2 + (1065516/48875)a_2 - (3042/187)b_1^2,\] (5.22)

\[x_3^2 = \left( - (37044/575)a_1 - (5684/85)a_2 + 741713/97750 \right)x_1^2 + (28/221)b_2x_2 + (216/115)b_1x_3 - (54559344/107525)a_1^2 - (28217448/9775)a_1a_2^2 + (254982/1375)a_2^2 - (25042724/9775)a_1a_2^2 + (26308184/48875)a_1a_2 - (8127098/537625)a_1 - (4775148/25415)a_2^3 + (188338017/1270750)a_2^2 - (9722139/635375)a_2 - (180/187)b_1^2.\] (5.23)
\[ x_2x_3 = \left( -\frac{864}{5}a_1^2 + \frac{1248}{25}a_1a_2 + \frac{1152}{5}a_2^2 + \frac{5904}{125}a_1 + \frac{184176}{125}a_2 - \frac{62112}{625} \right)x_1 - 36b_1b_2. \] (5.24)

We have obtained a system of equations (5.20)–(5.24) for \(x_1, x_2, x_3\). We can solve this system of equations with respect to the 60 possible quadruplets \((a_1, a_2, b_1, b_2)\). Actually, there is no solution for 37 quadruplets of \((a_1, a_2, b_1, b_2)\). For each of the remaining 23 quadruplets \((a_1, a_2, b_1, b_2)\), the system of equations possesses a unique solution \((x_1, x_2, x_3)\).

Furthermore, the 23 sets \((a_1, a_2, b_1, b_2, x_1, x_2, x_3)\) of values determined in this way coincide with the action of \(\tilde{\omega}^1\), \(\tilde{\omega}^2\), \([J]\), \([K]\), \([P]\), \([J_1P]\), and \([K_1P]\) on the top level of the 23 known simple \(V^\tau_L\)-modules with one dimensional top level described in Section 4. Since \(A(V^\tau_L)\) is generated by these seven elements, this implies that \(U_\lambda\) is isomorphic to the top level of one of the 23 simple \(V^\tau_L\)-modules listed in the assertion as an \(A(V^\tau_L)\)-module. Thus the lemma holds by Zhu’s theorem. \(\square\)

**Remark 5.4.** We also obtain some equations for \(x_1x_2\) and \(x_1x_3\) from \([P \ast (J_1P)]\) and \([P \ast (K_1P)]\). However, they are not sufficient to determine \(x_1\), \(x_2\), and \(x_3\).

**Lemma 5.5.** Every \(\mathbb{N}\)-graded weak \(V^\tau_L\)-module contains a simple \(M^\theta\)-submodule isomorphic to a member of \(\mathcal{M}\).

**Proof.** Suppose false and let \(U\) be an \(\mathbb{N}\)-graded weak \(V^\tau_L\)-module which contains no simple \(M^\theta\)-submodule isomorphic to a member of \(\mathcal{M}\). Then by Lemma 5.2, there is a simple \(M^\theta\)-submodule \(W\) in \(U\) such that \(W \cong W^1 \otimes W^2\) for some \(W^i \in \mathcal{W}_1\), \(i = 1, 2\). The top level of \(W\), say \(W_\lambda\) for some \(\lambda \in \mathbb{Q}\), is a one dimensional space. Take \(0 \neq w \in W_\lambda\) and let \(S = V^\tau_L \cdot w = \{a_nw \mid a \in V^\tau_L, n \in \mathbb{Z}\}\), which is an ordinary \(V^\tau_L\)-module by (2.6) and Remark 2.16. Since \(V^\tau_L = M^\theta \oplus W^0\), it follows from our assumption and the fusion rules (5.17) that \(S\) is isomorphic to a direct sum of finite number of copies of \(W\) as an \(M^\theta\)-module. Thus \(\tilde{\omega}^1\), \(\tilde{\omega}^2\), \([J]\), and \([K]\) act on the top level \(S_\lambda\) of \(S\) as scalars, say \(a_1, a_2, b_1, b_2\), respectively. Then by a similar calculation as in the proof of Lemma 5.3 we see that \([P \circ (K_1P)] = 0\) implies

\[ (15a_2 - 1) o(J_1P) = 0 \] (5.25)
as an operator on the top level \(S_\lambda\). Recall that \([u] \in A(V^\tau_L)\) acts on \(S_\lambda\) as \(o(u) = u_{\text{wt}(u)} - 1\) for a homogeneous vector \(u\) of \(V^\tau_L\). Furthermore, we can calculate that

\[ o(J_1P) o(P) - o(P) o(J_1P) = 0, \]

\[ o(K_1P) o(P) - o(P) o(K_1P) = \frac{2}{13} (15a_2 - 1) o(J_1P), \] (5.26)

\[ o(J_1P) o(K_1P) - o(K_1P) o(J_1P) = \frac{96}{125} (15a_2 - 1) (65a_1 + 100a_2 + 441) o(P) \]
as operators on \(S_\lambda\).

By (5.25), \(15a_2 - 1 = 0\) or \(o(J_1P) = 0\) and so \(o(P), o(J_1P),\) and \(o(K_1P)\) commute each other. Thus the action of \(A(V^\tau_L)\) on \(S_\lambda\) is commutative. Hence we can choose a one dimensional \(A(V^\tau_L)\)-submodule \(T\) of \(S_\lambda\). Zhu’s theory tells us that there is a simple \(\mathbb{N}\)-graded weak \(V^\tau_L\)-module \(R\) whose top level \(R_\lambda\) is isomorphic to \(T\) as an \(A(V^\tau_L)\)-module. Since \(\dim R_\lambda = 1\), \(R\) is isomorphic to one of the 23 simple \(V^\tau_L\)-modules listed in Lemma
5.3. In particular, $R$ contains a simple $M^0$-submodule $M$ isomorphic to a member of $\mathcal{M}$. Now, consider the $V^T_L$-submodule $V^T_L \cdot T$ of $S$ generated by $T$. By Lemma 2.10, there is a surjective homomorphism of $V^T_L$-modules from $V^T_L \cdot T$ onto $R$. Then $V^T_L \cdot T$ must contain a simple $M^0$-submodule isomorphic to $M$. This contradicts our assumption. The proof is complete. 

Lemma 5.6. Let $U$ be an $\mathbb{N}$-graded weak $V^T_L$-module and $M$ be a simple $M^0$-submodule of $U$ such that $M \cong M^1 \otimes M^2$ as $M^0$-modules for some $M^i \in \mathcal{M}$, $i = 1, 2$. Then $V^T_L \cdot M = \text{span}\{a_n u | a \in V^T_L, u \in M, n \in \mathbb{Z}\}$ is a simple $V^T_L$-module. Moreover, $V^T_L \cdot M = M \oplus W$, where $W$ is a simple $M^0$-module isomorphic to $W^1 \otimes W^2$ and $W^i$, $i = 1, 2$ are determined from $M^i$ by the fusion rules $W(0) \times M^1 = W^1$ and $W^0 \times M^2 = W^2$ of (5.17).

Proof. By Remark 2.10, $V^T_L \cdot M$ is an ordinary $V^T_L$-module. Note that $V^T_L \cdot M = (M^0 + W^0) \cdot M = M + W^0 \cdot M$. We see that $W^0 \cdot M \neq 0$ by a similar argument as in the proof of Lemma 5.2. Actually, $W^0 \cdot (W^0 \cdot M) \supseteq (W^0 \cdot W^0) \cdot M = V^T_L \cdot M$ (cf. Lemma 2.6 and (5.18)) implies $W^0 \cdot M \neq 0$ also. Moreover, $W^0 \cdot M$ is an $M^0$-module by (2.6). Since $M^0$ is rational, $W^0 \cdot M$ is decomposed into a direct sum of simple $M^0$-modules, say $W^0 \cdot M = \oplus_{\gamma \in \Gamma} S^\gamma$. Let $W = W^1 \otimes W^2$, where $W^i \in \mathcal{W}$, $i = 1, 2$ are determined by the fusion rules $W(0) \times M^1 = W^1$ and $W^0 \times M^2 = W^2$. The space $I_{M^0}(W^0, M)$ of intertwining operators of type $(W^0, M)$ is of dimension one and each $S^\gamma$ is isomorphic to $W$.

We want to show that $|\Gamma| = 1$. Suppose $\Gamma$ contains at least two elements and take $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$. Let $\psi : S^{\gamma_2} \rightarrow S^{\gamma_1}$ be an isomorphism of $M^0$-modules and $p_\gamma : W^0 \cdot M \rightarrow S^\gamma$ be a projection. For $a \in W^0$ and $u \in M$, set

$$G_{\gamma_1}(a, z)u = p_{\gamma_1} Y_U(a, z)u, \quad G_{\gamma_2}(a, z)u = \psi p_{\gamma_2} Y_U(a, z)u,$$

where $Y_U(a, z)$ is the vertex operator of the $\mathbb{N}$-graded weak $V^T_L$-module $U$. Then $G_{\gamma_1}(\cdot, z)$, $i = 1, 2$ are nonzero members in the one dimensional space $I_{M^0}(W^0, M)$, so that $\mu G_{\gamma_1}(\cdot, z) = G_{\gamma_1}(\cdot, z)$ for some $0 \neq \mu \in \mathbb{C}$. Let $0 \neq v \in S^{\gamma_1}$. Then $v \in W^0 \cdot M$ and so $v = \sum_j (a^j)_{n_j} u^j$ for some $a^j \in W^0$, $u^j \in M$, $n_j \in \mathbb{Z}$. Take the coefficients of $z^{-n_j-1}$ in both sides of $\mu G_{\gamma_1}(a^j, z)u^j = G_{\gamma_1}(a^j, z)u^j$. Then $\mu p_{\gamma_1}(a^j)_{n_j} u^j) = \psi p_{\gamma_2}(a^j)_{n_j} u^j)$. Summing up both sides of the equation with respect to $j$, we have $\mu p_{\gamma_1} v = \psi p_{\gamma_2} v$. However, $v \in S^{\gamma_1}$ implies that $p_{\gamma_1} v = v$ and $p_{\gamma_2} v = 0$. This is a contradiction since $\mu \neq 0$ and $v \neq 0$. Thus $|\Gamma| = 1$ and $W^0 \cdot M \cong W$ as required.

If $V^T_L \cdot M$ is not a simple $V^T_L$-module, then there is a proper $V^T_L$-submodule $N$ of $V^T_L \cdot M$. Since $M$ and $W$ are simple $M^0$-modules, $N$ must be isomorphic to $M$ or $W$ as an $M^0$-module. Then the top level of $N$ is of dimension one. The simple $V^T_L$-modules with one dimensional top level are classified in Lemma 3.3. Each of them is a direct sum of two simple $M^0$-modules. However, $N$ is not of such a form. Thus $V^T_L \cdot M$ is a simple $V^T_L$-module.

Proof. Assume that $(U, Y_1)$ and $(U, Y_2)$ are simple $V^T_L$-modules such that $Y_i(a, z) = Y(a, z)$ for all $a \in M^0$, $i = 1, 2$, where $(U, Y)$ is the given $M^0$-module structure. We denote the vertex operator of $V^T_L$ by $\hat{Y}(v, z)$ for $v \in V^T_L$. Let $p_{M^0} : V^T_L \rightarrow M^0$ and $p_{W^0} : V^T_L \rightarrow W^0$ be
projections and define $\mathcal{I}(\cdot, z)$ and $\mathcal{J}(\cdot, z)$ by

$$\mathcal{I}(a, z)b = p_M\bar{Y}(a, z)b, \quad \mathcal{J}(a, z)b = p_W\bar{Y}(a, z)b$$

for $a, b \in W^0$. Then by (5.18), $\mathcal{I}(\cdot, z)$ and $\mathcal{J}(\cdot, z)$ are nonzero intertwining operators of type $(M^0_{w^0}, W^0_{w^0})$ and $(W^0_{w^0}, W^0_{w^0})$, respectively. By the fusion rules (5.17), the space $I_{M^0}(M^0_{w^0}, W^0_{w^0})$ of $M^0$-intertwining operators of type $(M^0_{w^0}, W^0_{w^0})$ is of dimension one. Likewise, dim $I_{M^0}(W^0_{w^0}, W^0_{w^0}) = 1$. Note that $W^0 \cdot M^0 \subset W^0$ and that $\mathcal{I}(a, z)b + \mathcal{J}(a, z)b = \bar{Y}(a, z)b$.

Let $p_M : U \to M$ and $p_W : U \to W$ be projections. Define $\mathcal{F}_i^M(\cdot, z)$ and $\mathcal{F}_i^W(\cdot, z)$, $i = 1, 2$ by

$$\mathcal{F}_i^M(a, z)w = p_MY_i(a, z)w, \quad \mathcal{F}_i^W(a, z)w = p_WY_i(a, z)w$$

for $a \in W^0$ and $w \in W$. Then $\mathcal{F}_i^M(\cdot, z)$ and $\mathcal{F}_i^W(\cdot, z)$ are intertwining operators of type $(M^0_{w^0}, W^0_{w^0})$ and $(W^0_{w^0}, W^0_{w^0})$, respectively. Clearly, $\mathcal{F}_i^M(a, z)w + \mathcal{F}_i^W(a, z)w = Y_i(a, z)w$. If $\mathcal{F}_i^M(\cdot, z) = 0$, then $W^0 \cdot W \subset W$ and so $V^*_w \cdot W = M^0 \cdot W + W^0 \cdot W \subset W$. This is a contradiction, since $U$ is a simple $V^*_w$-module. Hence $\mathcal{F}_i^M(\cdot, z) \neq 0$. Let

$$G_i^W(a, z)v = Y_i(a, z)v$$

for $a \in W^0, v \in M$. Then $G_i^W(\cdot, z)$ is a nonzero intertwining operator of type $(M^0_{w^0}, W^0_{w^0})$ by (5.17). The space of $M^0$-intertwining operators $I_{M^0}(M^0_{w^0}, W^0_{w^0})$ of type $(M^0_{w^0}, W^0_{w^0})$ is of dimension one by (5.17). Similarly, dim $I_{M^0}(M^0_{w^0}, W^0_{w^0}) = \dim I_{M^0}(W^0_{w^0}, W^0_{w^0}) = 1$. Therefore, $\mathcal{F}_2^M(\cdot, z) = \lambda\mathcal{F}_1^M(\cdot, z)$, $\mathcal{F}_2^W(\cdot, z) = \mu\mathcal{F}_1^W(\cdot, z)$, and $G_2^W(\cdot, z) = \gamma G_1^W(\cdot, z)$ for some $\lambda, \mu, \gamma \in \mathbb{C}$ with $\lambda \neq 0$ and $\gamma \neq 0$.

Now,

$$Y_i(a, z_1)Y_i(b, z_2)v = (\mathcal{F}_i^M(a, z_1) + \mathcal{F}_i^W(a, z_1))G_i^W(b, z_2)v,$n $$Y_i(b, z_2)Y_i(a, z_1)v = (\mathcal{F}_i^M(b, z_2) + \mathcal{F}_i^W(b, z_2))G_i^W(a, z_1)v,$n $$Y_i(\bar{Y}(a, z_0)b, z_2)v = Y_i(\mathcal{I}(a, z_0)b, z_2)v + G_i^W(\mathcal{J}(a, z_0)b, z_2)v$$

for $a, b \in W^0$ and $v \in M$. Taking the image of both sides of the Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_i(a, z_1)Y_i(b, z_2)v - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)Y_i(b, z_2)Y_i(a, z_1)v$$

(5.27)

under the projection $p_M$, we obtain

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\mathcal{F}_i^M(a, z_1)G_i^W(b, z_2)v - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)\mathcal{F}_i^M(b, z_2)G_i^W(a, z_1)v$$

(5.28)

Likewise, if we take the image of both sides of (5.27) under the projection $p_W$, then

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\mathcal{F}_i^W(a, z_1)G_i^W(b, z_2)v - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)\mathcal{F}_i^W(b, z_2)G_i^W(a, z_1)v$$

(5.29)

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_i(\mathcal{I}(a, z_0)b, z_2)v.$$
Comparing Equation (5.29) for \( i = 1 \) and \( i = 2 \), we have
\[
\gamma(\mu - 1)z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)G_1^W(\mathcal{J}(a, z_0)b, z_2)v = 0,
\]
since \( \mathcal{F}_2^M(\cdot, z) = \lambda \mathcal{F}_1^M(\cdot, z) \), \( \mathcal{F}_2^W(\cdot, z) = \mu \mathcal{F}_1^W(\cdot, z) \), and \( G_2^W(\cdot, z) = \gamma G_1^W(\cdot, z) \). Now,\( z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) = z_1^{-1}\delta\left(\frac{z_1 - z_0}{z_1}\right) \) by [17 Proposition 8.8.5] and so the above equation is equivalent to the following assertion.
\[
\gamma(\mu - 1)(z_2 + z_0)^kG_1^W(\mathcal{J}(a, z_0)b, z_2)v = 0 \quad \text{for all} \quad k \in \mathbb{Z}.
\]
This implies that
\[
\gamma(\mu - 1)G_1^W(\mathcal{J}(a, z_0)b, z_2)v = 0,
\]
since \( G_1^W(\mathcal{J}(a, z_0)b, z_2)v \in W((z_0))[[z_2, z_2^{-1}]] \). Then since \( \mathcal{J}(\cdot, z) \neq 0 \) and \( G_1^W(\cdot, z) \neq 0 \), we conclude that \( \mu = 1 \).

Next, we use Equation (5.28). Since \( \mathcal{I}(a, z_0)b \in M^0((z_0)) \), we have \( Y_1(\mathcal{I}(a, z_0)b, z_2)v = Y_2(\mathcal{I}(a, z_0)b, z_2)v \) by our assumption. Then it follows from (5.28) for \( i = 1, 2 \) that
\[
(\lambda \gamma - 1)z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_1(\mathcal{I}(a, z_0)b, z_2)v = 0.
\]
Since \( \mathcal{I}(\cdot, z) \neq 0 \) and \( M \) is a simple \((M^0, Y_1)\)-module, a similar argument as above gives that \( \lambda \gamma = 1 \).

For \( a \in M^0, b \in W^0, v \in M, \) and \( w \in W \),
\[
Y_1(a + b, z)\((v + w) = Y_1(a, z)v + Y_1(a, z)w + G_i^W(b, z)v + (\mathcal{F}_i^M(b, z) + \mathcal{F}_i^W(b, z))w.
\]
Note that \( Y_1(a, z)v, \mathcal{F}_i^M(b, z)w \in M((z)) \) and \( Y_1(a, z)w, G_i^W(b, z)v, \mathcal{F}_i^W(b, z)w \in W((z)) \). Define \( \varphi : U \to U \) by \( \varphi(u) = \lambda u \) if \( u \in M \) and \( \varphi(u) = u \) if \( u \in W \). Since \( \mu = 1 \) and \( \lambda \gamma = 1 \), we can verify that
\[
Y_2(a + b, z)\varphi(v + w) = \varphi(Y_1(a + b, z)(v + w))
\]
Thus \( \varphi \) is an isomorphism of \( V_L^\ast \)-modules from \((U, Y_1)\) onto \((U, Y_2)\). This completes the proof. \( \square \)

**Remark 5.8.** The proof of the above lemma is essentially the same as that of [23, Lemma C.3]. Consider the Jacobi identity for \( a, b \in W^0 \) and \( w \in W \) and take the images of both sides of the identity under the projections \( p_M \) and \( p_W \), respectively. Then
\[
z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\mathcal{F}_i^M(a, z_1)\mathcal{F}_i^W(b, z_2)w - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)\mathcal{F}_i^M(b, z_2)\mathcal{F}_i^W(a, z_1)w
\]
\[
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{F}_i^M(\mathcal{J}(a, z_0)b, z_2)w,
\]
\[
z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)(\mathcal{G}_i^W(a, z_1)\mathcal{F}_i^M(b, z_2) + \mathcal{F}_i^W(a, z_1)\mathcal{F}_i^W(b, z_2))w
\]
\[
- z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)(\mathcal{G}_i^W(b, z_2)\mathcal{F}_i^M(a, z_1) + \mathcal{F}_i^W(b, z_2)\mathcal{F}_i^W(a, z_1))w
\]
\[
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)(Y_i(\mathcal{I}(a, z_0)b, z_2) + \mathcal{F}_i^W(\mathcal{J}(a, z_0)b, z_2))w.
\]
Each of these two equations gives the identical equations in case of \(i = 1\) and \(i = 2\) provided that \(\mu = 1\) and \(\lambda_\gamma = 1\).

**Theorem 5.9.** There are exactly 30 inequivalent simple \(V_L^\tau\)-modules. They are represented by the 30 simple \(V_L^\tau\)-modules listed in Lemma 5.2.

**Proof.** Let \(U\) be a simple \(V_L^\tau\)-module. Then by Lemma 5.3 \(U\) contains a simple \(M^0\)-submodule \(M\) isomorphic to a member of \(\mathcal{M}\). Since \(U\) is a simple \(V_L^\tau\)-module, Lemma 5.6 implies that \(U = M \oplus W\) for some simple \(M^0\)-submodule \(W\) isomorphic to a member of \(\mathcal{W}\). In fact, the isomorphism class of \(W\) is uniquely determined by \(M\). By Lemma 5.7 \(U\) admits a unique \(V_L^\tau\)-module structure. Since \(\mathcal{M}\) consists of 30 members, it follows that there are at most 30 inequivalent simple \(V_L^\tau\)-module. Hence the assertion holds. \(\square\)

**Theorem 5.10.** \(V_L^\tau\) is a rational vertex operator algebra.

**Proof.** It is sufficient to show that every \(N\)-graded weak \(V_L^\tau\)-module \(U\) is a sum of simple \(V_L^\tau\)-modules. Since \(M^0\) is rational, \(U\) is a direct sum of simple \(M^0\)-modules. Thus by Lemma 5.2 we may assume that \(U = (\oplus_{\gamma \in \Gamma} S^\gamma) \oplus (\oplus_{\lambda \in \Lambda} S^\lambda)\), where \(S^\gamma\) is isomorphic to a member of \(\mathcal{M}\) and \(S^\lambda\) is isomorphic to a member of \(\mathcal{W}\). We know that \(V_L^\tau \cdot S^\gamma\) is a simple \(V_L^\tau\)-module by Lemma 5.6. Set \(N = \sum_{\gamma \in \Gamma} V_L^\tau \cdot S^\gamma\). Since \(U/N\) has no simple \(M^0\)-submodule isomorphic to a member of \(\mathcal{M}\), it follows from Lemma 5.5 that \(U = N\) and the proof is complete. \(\square\)

**Corollary 5.11.** The Zhu algebra \(A(V_L^\tau)\) of \(V_L^\tau\) is a 51 dimensional semisimple associative algebra isomorphic to a direct sum of 23 copies of the one dimensional algebra \(\mathbb{C}\) and 7 copies of the algebra \(\text{Mat}_2(\mathbb{C})\) of \(2 \times 2\) matrices. Moreover, \(A(V_L^\tau)\) is generated by \([\hat{\omega}^1], [\hat{\omega}^2], [\hat{J}], [\hat{K}],\) and \([\hat{P}]\).

**Proof.** Since \(V_L^\tau\) is rational, \(A(V_L^\tau)\) is a finite dimensional semisimple associative algebra (cf. [11] Theorem 8.1, [36] Theorem 2.2.3). We know all the simple \(V_L^\tau\)-modules and the action of \([\hat{\omega}^1], [\hat{\omega}^2], [\hat{J}], [\hat{K}],\) and \([\hat{P}]\) on their top levels in Section 4. Hence we can determine the structure of \(A(V_L^\tau)\) as in the assertion. \(\square\)

**Appendix A. Some fusion rules for \(M(0)\)**

We give a proof of the fusion rules

\[
W(0) \times M_T(\tau^i)(\varepsilon) = W_T(\tau^i)(\varepsilon),
\]

\[
W(0) \times W_T(\tau^i)(\varepsilon) = M_T(\tau^i)(\varepsilon) + W_T(\tau^i)(\varepsilon),
\]

\(i = 1, 2, \varepsilon = 0, 1, 2\) of simple \(M(0)\)-modules in [37, 24].

Recall that \(V_L^\tau \cong M^0 \oplus W^0\), where \(M^0 = M(0) \otimes M^0\) and \(W^0 = W(0) \otimes W^0\). Set \(\tilde{M}_T(\tau^i)(\varepsilon) = M_T(\tau^i)(\varepsilon) \otimes M^0\) and \(\tilde{W}_T(\tau^i)(\varepsilon) = W_T(\tau^i)(\varepsilon) \otimes W^0\), which are simple \(M^0\)-modules. Then

\[
V_L^{T\omega_0}(\tau)(\varepsilon) \cong \tilde{M}_T(\tau)(\varepsilon) \oplus \tilde{W}_T(\tau)(\varepsilon),
\]

\[
V_L^{T\omega_0}(\tau^2)(\varepsilon) \cong \tilde{M}_T(\tau^2)(\varepsilon) \oplus \tilde{W}_T(\tau^2)(\varepsilon)
\]
as \(M^0\)-modules by [37, 23] and [37, 24]. Denote by \(Y_1(\cdot, z)\) (resp. \(Y_2(\cdot, z)\)) the vertex operator of the simple \(V_L^\tau\)-module \(V_L^{T\omega_0}(\tau)(\varepsilon)\) (resp. \(V_L^{T\omega_0}(\tau^2)(\varepsilon)\)). Let \(p_M : V_L^{T\omega_0}(\tau)(\varepsilon) \rightarrow\)
Then $F = \hat{W}_T(\tau^i)(\varepsilon)$ be projections. We also use the same symbol $p_M$ or $p_W$ to denote a projection from $V_L^{T\tau_0}(\tau^2)(\varepsilon)$ onto $\hat{M}_T(\tau^2)(\varepsilon)$ or onto $\hat{W}_T(\tau^2)(\varepsilon)$. We fix $i = 1, 2$ and $\varepsilon = 0, 1, 2$. For simplicity of notation, set $\hat{M} = \hat{M}_T(\tau^i)(\varepsilon)$ and $\hat{W} = \hat{W}_T(\tau^i)(\varepsilon)$.

Let $\mathcal{F}_i^M(a, z)w = p_M Y_i(a, z)w$ and $\mathcal{F}_i^W(a, z)w = p_W Y_i(a, z)w$ for $a \in W^0$ and $w \in \hat{W}$. Then $\mathcal{F}_i^M(\cdot, z)$ and $\mathcal{F}_i^W(\cdot, z)$ are intertwining operators of type $(\hat{M}_\omega \hat{W})$ and $(\hat{W}_\omega \hat{W})$, respectively. Likewise, let $\mathcal{G}_i^W(a, z)v = Y_i(a, z)v$ for $a \in W^0$ and $v \in \hat{M}$. Then $\mathcal{G}_i^W(\cdot, z)$ is an intertwining operator of type $(\hat{W}_0 \hat{M})$, since the fusion rule $W^0 \times M^0 = W^0$ of $M_i^0$-modules implies that $W^0 \cdot \hat{M} = \text{span}\{a_n \hat{M} | a \in W^0, n \in \mathbb{Z}\}$ is contained in $\hat{W}$. If $\mathcal{G}_i^W(\cdot, z) = 0$, then $V^\tau_L \cdot \hat{M} = (M^0 + W^0) \cdot \hat{M} \subset \hat{M}$. This is a contradiction, since $V_L^{T\tau_0}(\tau^i)(\varepsilon)$ and $V_L^{T\tau_0}(\tau^2)(\varepsilon)$ are simple $V_L^\tau$-modules. Thus $\mathcal{G}_i^W(\cdot, z) \neq 0$. Similarly, $\mathcal{F}_i^M(\cdot, z) \neq 0$. Indeed, if $\mathcal{F}_i^M(\cdot, z) = 0$, then $V_L^\tau \cdot \hat{W} \subset \hat{W}$, which is a contradiction. Assume that $\mathcal{F}_i^W(\cdot, z) = 0$. Then $W^0 \cdot \hat{W} \subset \hat{M}$ and so $W^0 \cdot (W^0 \cdot \hat{W}) \subset \hat{W}$. However, $W^0 \cdot (W^0 \cdot \hat{W}) \supset (W^0 \cdot W^0) \cdot \hat{W} = V^\tau_L \cdot \hat{W}$ by Lemma 2.3 and (5.13). This contradiction implies that $\mathcal{F}_i^W(\cdot, z) \neq 0$.

Restricting the three nonzero intertwining operators $\mathcal{F}_i^M(\cdot, z)$, $\mathcal{F}_i^W(\cdot, z)$, and $\mathcal{G}_i^W(\cdot, z)$ to the first component of each of the tensor products $W^0 = W(0) \otimes W^0$, $\hat{M} = M_T(\tau^i)(\varepsilon) \otimes M^0_i$, and $\hat{W} = W_T(\tau^i)(\varepsilon) \otimes W^0$, we obtain nonzero intertwining operators of type $(\hat{W}^0 W_T(\tau^i)(\varepsilon))$, $(\hat{W}_0 \hat{W}_T(\tau^i)(\varepsilon))$, and $(\hat{W}^0 \hat{W}_T(\tau^i)(\varepsilon))$ for $M(0)$-modules, respectively.

Let $N^2$ be one of $M_T(\tau^i)(\varepsilon)$, $W_T(\tau^i)(\varepsilon)$, $i = 1, 2$, $\varepsilon = 0, 1, 2$ and let $N^3$ be any of the 20 simple $M(0)$-modules. Then the top level $N^i_0$ of $N^3$ is of dimension one. By [8], the Zhu algebra $A(M(0))$ of $M(0)$ is generated by $[\hat{a}^i_1]$ and $[J]$. Moreover, we know the action of $\sigma(\hat{a}^i_1)$ and $\sigma(J)$ on $N^j_0$. Thus by a similar argument as in pages 192 and 193 of [32], we can calculate that the dimension of

$$\text{Hom}_{A(M(0))}(A(W(0)) \otimes A(M(0)) N^2_0, N^3_0)$$

is at most one and it is equal to one if and only if the pair $(N^2, N^3)$ is one of

$$(M_T(\tau^i)(\varepsilon), W_T(\tau^i)(\varepsilon)), \quad (W_T(\tau^i)(\varepsilon), M_T(\tau^i)(\varepsilon)), \quad \text{or} \quad (W_T(\tau^i)(\varepsilon), W_T(\tau^i)(\varepsilon))$$

for $i = 1, 2$, $\varepsilon = 0, 1, 2$. Note that $W(0)$ was denoted by $W^0_{00}$ in [32]. Now, the desired fusion rules are obtained by [27] Proposition 2.10 and Corollary 2.13.

### Appendix B. Some vectors in $V_L^\tau$

We describe certain vectors in $V_L^\tau$ which are used for determining the Zhu algebra of $V_L^\tau$ in Section 5. The calculation was done by a computer algebra system Risa/Asir.
\( P_1(J_1 P) = -(312/7)J_{-1}1 - (80/7)K_1 P, \)
\( P_0(J_1 P) = -(104/7)J_{-2}1 - (15/7)(\omega^1)0K_1 P -(40/17)(\omega^2)0K_1 P -(40/17)K_0 P, \)
\( P_{-1}(J_1 P) = -(1404/119)(\omega^1)_{-1}1 - (312/7)(\omega^2)_{-1}J_{-1}1 - (156/119)J_{-3}1 - (208/35)(\omega^1)_{-1}K_1 P + (12/7)(\omega^1)0K_1 P -(15/34)(\omega^1)0(\omega^2)0K_1 P -(832/357)(\omega^2)_{-1}K_1 P - (15/34)(\omega^1)0K_0 P -(16/17)K_{-1} P, \)
\( P_{-2}(J_1 P) = -(156/17)(\omega^1)_{-2}1 - (156/7)(\omega^2)_{-2}J_{-1}1 - (208/119)(\omega^1)_{-1}J_{-2}1 - (104/7)(\omega^2)_{-1}J_{-2}1 + (78/119)J_{-4}1 - (52/5)(\omega^1)_{-2}K_1 P +(286/35)(\omega^1)_{-1}(\omega^1)0K_1 P - (9/7)(\omega^1)0(\omega^1)0K_1 P - (104/85)(\omega^1)_{-1}(\omega^2)0K_1 P + (6/17)(\omega^1)0(\omega^1)0(\omega^2)0K_1 P - (52/119)(\omega^1)0(\omega^2)_{-1}K_1 P + (132/259)(\omega^2)_{-2}K_1 P - (164/259)(\omega^2)_{-1}(\omega^2)0K_1 P - (104/85)(\omega^1)_{-1}K_0 P + (6/17)(\omega^1)0(\omega^1)0K_0 P - (164/259)(\omega^2)_{-1}K_0 P - (3/17)(\omega^1)0K_{-1}P -(492/259)(\omega^2)0K_{-1}P + (1152/259)K_{-2}P; \)

\( (J_1 P)_1 P = -(312/7)J_{-1}1 - (80/7)K_1 P, \)
\( (J_1 P)_0 P = -(208/7)J_{-2}1 - (65/7)(\omega^1)0K_1 P -(40/17)(\omega^2)0K_1 P -(40/17)K_0 P, \)
\( (J_1 P)_{-1} P = -(1404/119)(\omega^1)_{-1}1 - (312/7)(\omega^2)_{-1}J_{-1}1 -(1924/119)J_{-3}1 -(208/35)(\omega^1)_{-1}K_1 P -(13/7)(\omega^1)0(\omega^1)0K_1 P -(65/34)(\omega^1)0K_0 P -(16/17)K_{-1} P, \)
\( (J_1 P)_{-2} P = -(312/119)(\omega^1)_{-2}1 -(156/7)(\omega^2)_{-2}J_{-1}1 -(1196/119)(\omega^1)_{-1}J_{-2}1 -(208/7)(\omega^2)_{-1}J_{-2}1 + (78/119)11 - (156/35)(\omega^1)_{-2}K_1 P -(494/35)(\omega^1)_{-1}(\omega^1)0K_1 P +(13/6)(\omega^1)0(\omega^1)0K_1 P -(104/85)(\omega^1)_{-1}(\omega^2)0K_1 P -(13/34)(\omega^1)0(\omega^1)0(\omega^2)0K_1 P -(676/357)(\omega^1)0(\omega^2)_{-1}K_1 P +(132/259)(\omega^2)_{-2}K_1 P -(164/259)(\omega^2)_{-1}(\omega^2)0K_1 P -(104/85)(\omega^1)_{-1}K_0 P -(13/34)(\omega^1)0(\omega^1)0K_0 P -(164/259)(\omega^2)_{-1}K_0 P -(13/17)(\omega^1)0K_{-1}P -(492/259)(\omega^2)0K_{-1}P + (1152/259)K_{-2}P; \)

\( P_1(K_1 P) = -(63/13)K_{-1}1 + (20/13)J_1 P, \)
\( P_0(K_1 P) = -(21/13)K_{-2}1 + (10/23)(\omega^1)0J_1 P -(15/13)(\omega^2)0J_1 P + (10/23)J_0 P, \)
\( P_{-1}(K_1 P) = -(168/13)(\omega^1)_{-1}K_{-1}1 -(189/299)(\omega^2)_{-1}K_{-1}1 -(126/299)K_{-3}1 + (1176/1495)(\omega^1)_{-1}J_{-1}P + (42/253)(\omega^1)0(\omega^1)0J_1 P -(15/46)(\omega^1)0(\omega^2)0J_1 P -(2/13)(\omega^2)_{-1}J_{-1}P + (84/253)(\omega^1)0J_0 P -(15/46)(\omega^2)0J_0 P -(144/115)J_{-1}P, \)
\( P_{-2}(K_1 P) = -(84/13)(\omega^1)_{-2}K_{-1}1 -(9/23)(\omega^2)_{-2}K_{-1}1 -(56/13)(\omega^1)_{-1}K_{-2}1 -(48/299)(\omega^2)_{-1}K_{-2}1 -(27/299)K_{-4}1 -(24/65)(\omega^1)_{-2}J_{-1}P + (36/65)(\omega^1)_{-1}(\omega^1)0J_1 P -(347/3354)(\omega^1)0(\omega^1)0(\omega^2)0J_1 P -(882/1495)(\omega^1)_{-1}(\omega^2)0J_1 P -(63/506)(\omega^1)0(\omega^2)0J_1 P -(1/23)(\omega^1)0(\omega^2)_{-1}J_{-1}P + (18/13)(\omega^2)_{-2}J_1 P -(17/13)(\omega^2)_{-1}J_0 P + (36/65)(\omega^1)_{-1}J_0 P -(347/1118)(\omega^1)0(\omega^2)0J_0 P -(63/253)(\omega^1)0(\omega^2)0J_0 P -(1/23)(\omega^2)_{-1}J_0 P + (108/65)(\omega^1)0J_{-1}P + (108/115)(\omega^2)0J_{-1}P -(228/65)J_{-2}P; \)
\((K_1 P)_1 P = -(63/13)K_{-1}1 + (20/13)J_1 P,\)
\((K_1 P)_0 P = -(42/13)K_{-2}1 + (10/23)(\tilde{\omega}^1)_{0} J_1 P + (35/13)(\tilde{\omega}^2)_{0} J_1 P + (10/23)J_0 P,\)
\((K_1 P)_{-1} P = -(168/13)(\tilde{\omega}^1)_{-1} K_{-1}1 - (189/299)(\tilde{\omega}^2)_{-1} K_{-1}1 - (609/299)K_{-3}1
+ (1176/1495)(\tilde{\omega}^1)_{-1} J_1 P + (42/253)(\tilde{\omega}^1)_{0} J_1 P + (35/46)(\tilde{\omega}^1)_{0} (\tilde{\omega}^2)_{0} J_1 P
+ (28/13)(\tilde{\omega}^2)_{-1} J_1 P + (84/253)(\tilde{\omega}^1)_{0} J_0 P + (35/46)(\tilde{\omega}^2)_{0} J_0 P - (144/115)J_{-1} P,\)
\((K_1 P)_{-2} P = -(84/13)(\tilde{\omega}^1)_{-2} K_{-1}1 - (72/299)(\tilde{\omega}^2)_{-2} K_{-1}1 - (112/13)(\tilde{\omega}^1)_{-1} K_{-2}1
- (141/299)(\tilde{\omega}^2)_{-1} K_{-2}1 - (27/23)K_{-4}1 - (24/65)(\tilde{\omega}^1)_{-2} J_1 P + (36/65)(\tilde{\omega}^1)_{-1} (\tilde{\omega}^1)_{0} J_1 P
- (347/3354)(\tilde{\omega}^1)_{0} (\tilde{\omega}^1)_{0} J_1 P + (2058/1495)(\tilde{\omega}^1)_{-1} (\tilde{\omega}^2)_{0} J_1 P
+ (147/506)(\tilde{\omega}^1)_{0} (\tilde{\omega}^1)_{0} (\tilde{\omega}^2)_{0} J_1 P + (14/23)(\tilde{\omega}^1)_{0} (\tilde{\omega}^2)_{-1} J_1 P - (7/13)(\tilde{\omega}^2)_{-2} J_{1} P
+ (28/13)(\tilde{\omega}^2)_{-1} (\tilde{\omega}^2)_{0} J_1 P + (36/65)(\tilde{\omega}^1)_{-1} J_0 P - (347/1118)(\tilde{\omega}^1)_{0} (\tilde{\omega}^1)_{0} J_0 P
+ (147/253)(\tilde{\omega}^1)_{0} (\tilde{\omega}^2)_{0} J_0 P + (14/23)(\tilde{\omega}^2)_{-1} J_0 P + (108/65)(\tilde{\omega}^1)_{0} J_{-1} P
- (252/115) (\tilde{\omega}^2)_{0} J_{-1} P - (228/65) J_{-2} P,\)

\((J_1 P)_{2} (J_1 P) = 8112(\tilde{\omega}^1)_{-2} 1 + 1872(\tilde{\omega}^2)_{-2} 1,\)
\((J_1 P)_{1} (J_1 P) = 8112(\tilde{\omega}^1)_{-1} (\tilde{\omega}^1)_{-1} 1 + 16224(\tilde{\omega}^1)_{-1} (\tilde{\omega}^2)_{-1} 1 + 864(\tilde{\omega}^2)_{-2} 1 + 432(\tilde{\omega}^2)_{-1} (\tilde{\omega}^2)_{-1} 1 - 8 J_1 K_1 P,\)
\((J_1 P)_{0} (J_1 P) = 8112(\tilde{\omega}^1)_{-1} (\tilde{\omega}^1)_{-1} 1 + 8112(\tilde{\omega}^1)_{-2} (\tilde{\omega}^2)_{-1} 1 + 8112(\tilde{\omega}^1)_{-1} (\tilde{\omega}^2)_{-2} 1 + 576(\tilde{\omega}^2)_{-1} 1
+ 432(\tilde{\omega}^2)_{-2} (\tilde{\omega}^2)_{-1} 1 - (52/23)(\tilde{\omega}^1)_{0} J_1 K_1 P - (28/17)(\tilde{\omega}^2)_{0} J_1 K_1 P - (28/17) J_1 K_0 P
- (52/23)J_0 K_1 P,\)
\((J_1 P)_{-1} (J_1 P) = - (146016/77)(\tilde{\omega}^1)_{-3} 1 - (101400/1309)(\tilde{\omega}^1)_{-3} (\tilde{\omega}^1)_{-1} 1 + (2314962/1309)(\tilde{\omega}^1)_{-2} (\tilde{\omega}^1)_{-2} 1
+ (1565616/1309)(\tilde{\omega}^1)_{-1} (\tilde{\omega}^1)_{-1} (\tilde{\omega}^1)_{-1} 1 + 4056(\tilde{\omega}^1)_{-2} (\tilde{\omega}^2)_{-2} 1
+ 8112(\tilde{\omega}^1)_{-1} (\tilde{\omega}^1)_{-1} (\tilde{\omega}^2)_{-1} 1 + 3744(\tilde{\omega}^1)_{-1} (\tilde{\omega}^2)_{-1} 1 + 1872(\tilde{\omega}^1)_{-1} (\tilde{\omega}^2)_{-1} (\tilde{\omega}^2)_{-1} 1
+ (7776/23)(\tilde{\omega}^2)_{-3} 1 + (5472/23)(\tilde{\omega}^2)_{-3} (\tilde{\omega}^2)_{-1} 1 + (2304/23)(\tilde{\omega}^2)_{-2} (\tilde{\omega}^2)_{-2} 1
+ (432/23)(\tilde{\omega}^2)_{-2} (\tilde{\omega}^2)_{-1} (\tilde{\omega}^2)_{-1} 1 - (3042/187) J_{-1} J_{-1} 1 + (5876/805)(\tilde{\omega}^1)_{-1} J_1 K_1 P
+ (5590/1771)(\tilde{\omega}^1)_{0} (\tilde{\omega}^1)_{0} J_1 K_1 P - (182/391)(\tilde{\omega}^1)_{0} (\tilde{\omega}^2)_{0} J_1 K_1 P - (416/255)(\tilde{\omega}^2)_{-1} J_1 K_1 P
+ (11180/1771)(\tilde{\omega}^1)_{0} J_0 K_1 P - (182/391)(\tilde{\omega}^2)_{0} J_0 K_1 P - (182/391)(\tilde{\omega}^1)_{0} J_1 K_0 P
- (4056/115) J_{-1} K_1 P - (182/391) J_0 K_0 P - (56/85) J_{-1} K_{-1} P;\)
\( (K_1 P)_2 (K_1 P) = -672(\omega^2)_{-2} 1 - 882(\omega^2)_{-2} 1, \)
\( (K_1 P)_1 (K_1 P) = -144(\omega^1)_{-3} 1 - 432(\omega^1)_{-1} (\omega^1)_{-1} 1 - 4704(\omega^1)_{-1} (\omega^2)_{-1} 1 \)
\( - (2646/13)(\omega^2)_{-1} (\omega^2)_{-1} (\omega^2)_{-1} 1 - (30/13) J_1 K_1 P, \)
\( (K_1 P)_0 (K_1 P) = -96(\omega^2)_{-4} 1 - 432(\omega^2)_{-2} (\omega^2)_{-1} 1 - 2352(\omega^2)_{-2} (\omega^2)_{-1} 1 - 2352(\omega^2)_{-2} (\omega^2)_{-1} 1 \)
\( - (1764/13)(\omega^2)_{-1} (\omega^2)_{-1} (\omega^2)_{-1} 1 - (15/23) J_1 K_1 P \)
\( - (105/221)(\omega^2)_{0} J_1 K_1 P - (15/23) J_0 K_1 P - (105/221) J_0 K_0 P, \)
\( (K_1 P)_1 (K_1 P) = - (2688/11)(\omega^1)_{-5} 1 + (15792/187)(\omega^1)_{-3} (\omega^1)_{-1} 1 - (50316/187)(\omega^1)_{-2} (\omega^1)_{-2} 1 \)
\( - (32928/187)(\omega^1)_{-1} (\omega^1)_{-1} (\omega^1)_{-1} 1 - 504(\omega^1)_{-3} (\omega^2)_{-2} 1 - 1176(\omega^1)_{-2} (\omega^2)_{-2} 1 \)
\( - 1512(\omega^1)_{-1} (\omega^2)_{-1} (\omega^2)_{-1} 1 - (7056/13)(\omega^1)_{-1} (\omega^2)_{-3} 1 \)
\( - (18816/13)(\omega^2)_{-3} (\omega^2)_{-1} 1 + (49392/299)(\omega^2)_{-2} (\omega^2)_{-1} 1 \)
\( - (31752/299)(\omega^2)_{-3} (\omega^2)_{-1} 1 + (42336/299)(\omega^2)_{-1} (\omega^2)_{-1} 1 - 180/187 J_1 J_1 K_1 P \)
\( - (1764/1495)(\omega^2)_{-1} J_1 K_1 P - (63/253)(\omega^2)_{0} J_1 K_1 P - (105/782) (\omega^2)_{0} J_1 K_1 P \)
\( - (812/663)(\omega^2)_{0} J_0 K_1 P - (126/253)(\omega^2)_{0} J_0 K_0 P - (105/782) (\omega^2)_{0} J_0 K_0 P \)
\( - (105/782) (\omega^2)_{0} J_0 K_0 P + (216/115) J_{-1} K_1 P - (105/782) J_0 K_0 P + (28/221) J_1 K_{-1} P, \)
\( (J_1 P)_2 (K_1 P) = -780(\omega^1)_{0} P + 720(\omega^2)_{0} P, \)
\( (J_1 P)_1 (K_1 P) = - (2496/5)(\omega^1)_{-1} 1 - 156(\omega^1)_{0} 1 P + 585(\omega^1)_{0} (\omega^2)_{0} P + 96(\omega^2)_{-1} 1, \)
\( (J_1 P)_0 (K_1 P) = (1872/5)(\omega^1)_{-2} 1 - (5928/5)(\omega^1)_{-1} (\omega^1)_{0} P + 182(\omega^1)_{0} (\omega^1)_{0} 1 P + (1872/5)(\omega^1)_{-1} (\omega^2)_{0} P \)
\( + 117(\omega^1)_{0} (\omega^2)_{0} P + 78(\omega^1)_{0} (\omega^2)_{0} P - 864(\omega^2)_{-2} P + 816(\omega^2)_{-1} (\omega^2)_{0} P, \)
\( (J_1 P)_1 (K_1 P) = - 36 J_{-1} K_1 1 + (3936/5)(\omega^1)_{-3} 1 - (2064/5)(\omega^1)_{-2} (\omega^1)_{0} P - (864/5)(\omega^1)_{-1} (\omega^1)_{-1} 1 \)
\( - (2352/5)(\omega^1)_{-1} (\omega^2)_{0} P + 98(\omega^1)_{0} (\omega^2)_{0} P - (1404/5)(\omega^1)_{-2} (\omega^2)_{0} P \)
\( + (4446/5)(\omega^1)_{1} (\omega^2)_{0} P - (237/2)(\omega^1)_{0} (\omega^2)_{0} P \)
\( + (1248/25)(\omega^1)_{0} (\omega^2)_{0} P - (84/5)(\omega^1)_{0} (\omega^2)_{0} P - 702(\omega^1)_{0} (\omega^2)_{0} P \)
\( + 663(\omega^1)_{0} (\omega^2)_{0} P - 576(\omega^2)_{-3} P + 144(\omega^2)_{-2} (\omega^2)_{0} P + (1152/5)(\omega^2)_{-2} (\omega^2)_{0} P, \)
\( (K_1 P)_2 (J_1 P) = - 180(\omega^1)_{0} P - 1680(\omega^2)_{0} P, \)
\( (K_1 P)_1 (J_1 P) = - (2496/5)(\omega^1)_{-1} P + 144(\omega^1)_{0} (\omega^1)_{0} P - 315(\omega^1)_{0} (\omega^2)_{0} P - 1344(\omega^2)_{-1} 1, \)
\( (K_1 P)_0 (J_1 P) = - (4368/5)(\omega^1)_{-2} P + (3432/5)(\omega^1)_{-1} (\omega^1)_{0} P - 108(\omega^1)_{0} (\omega^1)_{0} (\omega^1)_{0} P \)
\( - (4368/5)(\omega^1)_{-1} (\omega^2)_{0} P + 252(\omega^1)_{0} (\omega^2)_{0} P - 252(\omega^1)_{1} (\omega^2)_{-1} P + 336(\omega^2)_{-2} P \)
\( - 1344(\omega^2)_{-2} (\omega^2)_{0} P, \)
\( (K_1 P)_1 (J_1 P) = - 36 J_{-1} K_1 1 - (2304/5)(\omega^1)_{-3} P - (504/5)(\omega^1)_{-2} (\omega^1)_{0} P - (864/5)(\omega^1)_{-1} (\omega^1)_{-1} 1 \)
\( + (2328/5)(\omega^1)_{-1} (\omega^2)_{0} P - 72(\omega^1)_{0} (\omega^2)_{0} P - (7644/5)(\omega^1)_{-2} (\omega^2)_{0} P \)
\( + (6006/5)(\omega^1)_{-1} (\omega^2)_{0} P - 189(\omega^1)_{0} (\omega^2)_{0} P - (17472/25)(\omega^1)_{-2} (\omega^2)_{-1} P \)
\( + (1008/5)(\omega^1)_{0} (\omega^2)_{0} P - 63(\omega^1)_{0} (\omega^2)_{0} P - 252(\omega^1)_{0} (\omega^2)_{-1} (\omega^2)_{-1} P \)
\( + 864(\omega^2)_{-3} P - 96(\omega^2)_{-2} (\omega^2)_{0} P - (4608/5)(\omega^2)_{-2} (\omega^2)_{-1} P. \)
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