BLOW-UP CRITERION FOR AN INCOMPRESSIBLE NAVIER-STOKES/ALLEN-CAHN SYSTEM WITH DIFFERENT DENSITIES

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Dedicated to Professor Lishang Jiang on the occasion of his 80th birthday

Abstract. This paper is concerned with a coupled Navier-Stokes/Allen-Cahn system describing a diffuse interface model for two-phase flow of viscous incompressible fluids with different densities in a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$). We establish a criterion for possible break down of such solutions at finite time in terms of the temporal integral of both the maximum norm of the deformation tensor of velocity gradient and the square of maximum norm of gradient of phase field variable in 2D. In 3D, the temporal integral of the square of maximum norm of velocity is also needed. Here, we suppose the initial density function $\rho_0$ has a positive lower bound.

1. Introduction. In this paper, we investigate a diffusive interface model, which describes the motion of a mixture of two viscous incompressible fluids. Especially, the fluids have not “matched densities” but “different densities”. In this model the sharp interfaces are replaced by narrow transition layers. The latter feature has the advantage to deal with interfaces that merge, reconnect and hit conditions. A phase field variable $\chi$ is introduced and a mixing energy is defined in terms of $\chi$ and its spatial gradient. The model consists of Navier-Stokes equations governing the fluid velocity coupled with a convective Allen-Cahn equation for the change of the concentration caused by diffusion. The effects of phase transitions can also be described by different modified convective Cahn-Hilliard or other types of dynamics, see [3, 9, 15].

In this paper, we are interested in the following coupled Navier-Stokes/Allen-Cahn system for viscous incompressible fluids with different densities

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p = \text{div}(2\eta(\chi) Du) - \delta \text{div}(\nabla \chi \otimes \nabla \chi), \\
\text{div} u = 0, \\
(\rho \chi)_t + \text{div}(\rho u \chi) = -\mu, \\
\rho u = -\delta \Delta \chi + \rho \frac{\partial f}{\partial \chi}
\end{cases}$$

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for \((x, t) \in \Omega \times (0, +\infty)\), where \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N = 2, 3)\) with smooth boundary \(\partial \Omega\), \(\rho \geq 0\) is the total density, \(u\) denotes the mean velocity of the fluid mixture, \(Du = \frac{1}{2}(\nabla u + \nabla u^T)\), \(p\) is the pressure, \(\chi\) represents the concentration difference of the two fluids, \(\mu\) is the chemical potential, \(\eta(\chi) > 0\) is the viscosity of the mixture, the free energy density satisfies double-well structure \(f(\chi) = \frac{1}{\delta}(\chi^4 - \chi_0^2)\), positive constant \(\delta\) denotes the width of the interface. The usual Kronecker product is denoted by \(\otimes\), i.e. \((a \otimes b)_{ij} = a_i b_j\) for \(a, b \in \mathbb{R}^N\).

The equations (1)\(_{1-3}\) are nonhomogeneous incompressible Navier-Stokes equations, which have an extra term \(\nabla \chi \otimes \nabla \chi\) describing capillary effect related to the free energy

\[
F(\rho, \chi) = \int_\Omega \left( pf(\chi) + \frac{\delta}{2} |\nabla \chi|^2 \right) \, dz.
\]

The equations (1)\(_{4-5}\) are Allen-Cahn equations. Obviously, the system (1) is a highly nonlinear system coupling hyperbolic equations with parabolic equations.

The diffuse interface models for two-phase flow of incompressible viscous fluids with “matched densities” have been extensively studied. We refer the readers to [1, 4, 12, 19, 20, 21] for details. It is evident that, the densities in two fluids are often quite different. Within our knowledge, there are only a few theoretical results available to compressible models. For compressible Navier-Stokes/Allen-Cahn system, Feireisl et al. [10] proved the existence of weak solutions in 3D. In [8], we obtained the global well-posedness in 1D with constant mobility. We prove the existence of the initial boundary value problem in various regularity classes, as well as uniqueness for strong solutions. For compressible Navier-Stokes/Cahn-Hilliard system, Abels and Feireisl [2] derived the existence of weak solutions.

In this paper, we investigate the Navier-Stokes/Allen-Cahn system for two fluids with non-matched densities, but the velocity \(u\) satisfies the divergence-free condition \(\text{div} u = 0\), i.e. the fluids are incompressible and with different densities. Following our works in [14], where the existence of unique local strong solution has been obtained, we deal with the main mechanism for possible breakdown of such a local strong solution.

We supplement the system (1) with the following initial conditions

\[
(\rho, u, \chi) \bigg|_{t=0} = (\rho_0, u_0, \chi_0), \quad x \in \Omega,
\]

the usual no-slip boundary condition on the velocity and Neumann boundary condition on the phase field variable

\[
\left( u, \frac{\partial \chi}{\partial n} \right) \bigg|_{\partial \Omega} = (0, 0), \quad t \geq 0,
\]

where \(n\) is the unit outward normal vector of \(\partial \Omega\).

**Notations.** For \(p \geq 1\), denote \(L^p = L^p(\Omega)\) as the \(L^p\) space with the norm \(\| \cdot \|_{L^p}\). For \(k \geq 1\) and \(p \geq 1\), denote \(W^{k,p} = W^{k,p}(\Omega)\) for a Sobolev space, whose norm is denoted by \(\| \cdot \|_{W^{k,p}}\) and specially \(H^k = W^{k,2}(\Omega)\).

**Definition 1.1.** For \(T > 0\), \((\rho, u, p, \chi, \mu)\) is called a strong solution of the coupled Navier-Stokes/Allen-Cahn system (1) in \(\Omega \times (0, T]\), if

\[
\rho \in L^\infty(0, T; W^{2,6}), \quad \rho_t \in L^\infty(0, T; W^{1,6}), \quad 0 < c^{-1} \leq \rho \leq c,
\]

\[
u \in L^\infty\left(0, T; H^2 \cap H^1_0\right) \cap L^2(0, T; W^{2,6}), \quad u_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1_0),
\]
\[ p \in L^\infty(0, T; H^1) \cap L^2(0, T; W^{1,6}), \]
\[ \chi \in L^\infty(0, T; H^3) \cap L^2(0, T; H^4), \quad \chi_t \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \]
\[ \mu \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \quad \mu_t \in L^2(0, T; L^2), \]
and \( (\rho, u, p, \chi, \mu) \) satisfies (1) a.e. in \( \Omega \times (0, T] \).

The existence and uniqueness of local strong solutions have been proved in [14].

**Theorem 1.2.** [14] Assume that \( \rho_0 \in W^{2,6}(\Omega) \) satisfies \( 0 < c_0^{-1} \leq \rho_0 \leq c_0 \) for some constant \( c_0, u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \chi_0 \in H^3(\Omega) \) and \( u_0|_{\partial \Omega} = 0, \text{div} u_0 = 0 \) for \( x \in \Omega \). Then there exist a time \( T_* > 0 \), a constant \( c = c(c_0, T_*) \) and a unique strong solution \( (\rho, u, p, \chi, \mu) \) of the problem (1)–(3) in \( \Omega \times (0, T_*] \).

The existence of global solutions is closely related to the estimate for \( \| \nabla \rho \|_{L^\infty(Q_T)} \), which is the main difficulty we can not handle. Besides the global existence, another interesting question is the main mechanism of possible breakdown of local strong solutions. For such question, the pioneering work is obtained by Beale, Kato and Majda [6], they proved that the maximum norm of the vorticity \( \nabla \times u \) controls the breakdown of smooth solutions of the 3D Euler equations. Later, Ponce [16] derived the same blow-up criterion when the vorticity was substituted by the deformation tensor \( Du \), that is, a solution remains smooth if
\[
\int_0^T \| Du \|_{L^\infty} \, dt
\]
remains bounded. The works on blow-up criterion for incompressible Navier-Stokes equation, we refer the readers to Serrin [17] and Struwe [18] for example. Recently, for nonhomogeneous incompressible Navier-Stokes equations, i.e. the velocity is divergence-free, but the density is not assumed to be a constant, Kim [13] established a weak Serrin class blow-up criterion.

Motivated by these works, we will establish in this paper the blow-up criterion of breakdown of local strong solutions in finite time. Our main result is as follows.

**Theorem 1.3.** Let \( (\rho, u, p, \chi, \mu) \) be a strong solution of the initial boundary value problem (1)–(3). If \( 0 < T_* < +\infty \) is the maximum time of existence, then
\[
\int_0^{T_*} \left( \| Du \|_{L^\infty} + \| \nabla \chi \|_{L^2}^2 \right) \, dt = +\infty, \quad \text{if } N = 2, \tag{4}
\]
\[
\int_0^{T_*} \left( \| Du \|_{L^\infty} + \| \chi \|_{L^2}^2 + \| \nabla \chi \|_{L^\infty}^2 \right) \, dt = +\infty, \quad \text{if } N = 3. \tag{5}
\]

**Remark.** We should point out that, because of the appearance of the density \( \rho \) in Allen-Cahn equation (1)\textsubscript{4,5}, we can not handle the vacuum state. Moreover, our proof strongly depends on the divergence-free condition, which ensure the solutions being away from vacuum. So the conclusions in this paper can not be extended to corresponding compressible system directly.

Since the constant \( \delta \) play no role in the analysis, we assume henceforth that \( \delta = 1 \). Throughout this paper, we assume that \( \eta(s) \in C^1(\mathbb{R}) \) and there exist positive constants \( \underline{\eta}, \bar{\eta}, \tilde{\eta} \) such that
\[
0 < \underline{\eta} \leq \eta(s) \leq \bar{\eta}, \quad |\eta'(s)| \leq \tilde{\eta}. \tag{6}
\]
Moreover, we denote by \( A \lesssim B \) if there exists a positive constant \( C \) such that \( A \leq CB \).
2. Proof of our main result. Let \( 0 < T_* < +\infty \) be the maximum time for the existence of strong solution \((\rho, u, p, \chi, \mu)\) to the problem (1)-(3). In other words, \((\rho, u, p, \chi, \mu)\) is a strong solution of (1)-(3) in \( \Omega \times (0, T) \) for any \( 0 < T < T_* \), but not a strong solution in \( \Omega \times (0, T_* \). We prove (4) and (5) by contradiction: if not, i.e.

\[
\int_0^{T_*} (\|Du\|_{L^\infty} + \|\nabla \chi\|_{L^\infty}^2) \, dt \leq M_0 < +\infty, \quad \text{if } N = 2, \tag{7}
\]

\[
\int_0^{T_*} (\|Du\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\nabla \chi\|_{L^\infty}^2) \, dt \leq M_0 < +\infty, \quad \text{if } N = 3 \tag{8}
\]

holds for some constant \( M_0 > 0 \), then there exists a positive constant \( C \) depending only on \( \rho_0, u_0, \chi_0, T_* \) and \( M_0 \) such that

\[
\sup_{0 \leq t < T_*} \left( \|\rho\|_{W^{2,s}}^s + \|\mu\|_{W^{1,s}}^2 + \|u\|_{H^2}^2 + \|u_t\|_{L^2}^2 \right.
\]

\[
+ \|p\|_{H^1}^2 + \|\chi\|_{H^3}^2 + \|\chi_t\|_{H^1}^2 + \|\mu\|_{H^1}^2 \bigg) \tag{9}
\]

\[
+ \int_0^{T_*} \left( \|u\|_{W^{2,s}}^s + \|u_t\|_{H^1}^2 + \|p\|_{W^{1,1}}^2 + \|\chi\|_{H^2}^2 \right.
\]

\[
+ \|\chi_t\|_{H^2}^2 + \|\mu\|_{H^2}^2 + \|\mu_t\|_{L^2}^2 \bigg) \, dt \leq C.
\]

In terms of the a priori estimates (9), we can prove that \( T_* \) is not the maximum time, which is the desired contradiction.

Firstly, we deal with the bounds of \( \rho \) and \( \nabla \rho \), which is very important in the proof of the following estimates.

**Lemma 2.1.** Let \( 0 < T_* < +\infty \) be the maximum time of a strong solution \((\rho, u, p, \chi, \mu)\) to the problem (1)-(3). If (7) and (8) hold, then

\[
0 < C^{-1} \leq \rho(x, t) \leq C, \quad (x, t) \in Q_{T_*}, \tag{10}
\]

\[
\sup_{0 \leq t < T_*} \|\nabla \rho\|_{L^\infty} \leq C, \tag{11}
\]

where \( Q_{T_*} = \Omega \times (0, T_\ast) \).

**Proof.** For any \( 1 \leq r < +\infty \), multiplying (1) by \( r \rho^{r-1} \), integrating the result with respect to \( x \) over \( \Omega \) and by using (1)\(_3\), we get

\[
\frac{d}{dt} \int_{\Omega} \rho^r \, dx = - \int_{\Omega} u \cdot \nabla (\rho^r) \, dx = \int_{\Omega} \rho^r \, \text{div} \, u \, dx = 0.
\]

From which we have

\[
\|\rho(\cdot, t)\|_{L^r} = \|\rho_0\|_{L^r}, \quad 0 \leq t < T_*\]

Sending \( r \to +\infty \) and recalling \( 0 < c_0^{-1} \leq \rho_0 \leq c_0 \), we obtain (10).

Differentiating (1)\(_1\) with respect to \( x \), multiplying the result by \( r|\nabla \rho|^{r-2} \nabla \rho \), integrating the result with respect to \( x \) over \( \Omega \) and recalling (1)\(_3\), we have

\[
\frac{d}{dt} \int_{\Omega} |\nabla \rho|^r \, dx = - \int_{\Omega} (u \cdot \nabla) (|\nabla \rho|^r) \, dx - r \int_{\Omega} |\nabla \rho|^{r-2} \nabla \rho \cdot \nabla (u \cdot \nabla) \rho \, dx
\]

\[
= \int_{\Omega} (|\nabla \rho|^r) \, \text{div} \, u \, dx - r \int_{\Omega} |\nabla \rho|^{r-2} \nabla \rho \cdot D(u \cdot \nabla) \rho \, dx
\]

\[
\leq r \|Du\|_{L^\infty} \int_{\Omega} |\nabla \rho|^r \, dx,
\]
which implies that
\[
\frac{d}{dt} \| \nabla \rho \|_{L^r} \leq \| Du \|_{L^\infty} \| \nabla \rho \|_{L^r}.
\]
Then Gronwall’s inequality implies
\[
\sup_{0 \leq t < T} \| \nabla \rho \|_{L^r} \leq \| \nabla \rho_0 \|_{L^r} \exp \left\{ \int_0^T \| Du \|_{L^\infty} \, dt \right\} \leq C.
\]
Hence by sending \( r \to +\infty \) and recalling \( \rho_0 \in W^{2,6} \), we arrive at (11). Then lemma 2.1 follows.

Next lemma is concerned with basic energy estimate.

**Lemma 2.2.** Under the same assumptions in Lemma 2.1, we have
\[
\sup_{0 \leq t < T} \int_\Omega \left( \frac{|u|^2}{2} + \frac{|\nabla \chi|^2}{2} + \frac{\rho(\chi^4 - 2\chi^2)}{4} \right) \, dx
\]
\[
+ \int_0^T \int_\Omega (2\eta(\chi)|Du|^2 + \mu^2) \, dx \, dt = E_0,
\]
where
\[
E_0 := \int_\Omega \left( \frac{\rho_0|u_0|^2}{2} + \frac{\nabla \chi_0|^2}{2} + \frac{\rho_0(\chi_0^4 - 2\chi_0^2)}{4} \right) \, dx.
\]
Furthermore,
\[
\sup_{0 \leq t < T} \int_\Omega \chi^4 \, dx + \int_0^T \int_\Omega (|\nabla u|^2 + |\nabla^2 \chi|^2 + \chi_t^2) \, dx \, dt \leq C.
\]

**Proof.** Multiplying (1)₂ by \( u \), integrating the result over \( \Omega \) and recalling (1)₁,₃, we have
\[
\frac{d}{dt} \int_\Omega \frac{\rho|u|^2}{2} \, dx + \int_\Omega 2\eta(\chi)|Du|^2 \, dx = -\int_\Omega u \cdot \nabla \chi \Delta \chi \, dx.
\]
Multiplying (1)₄ by \( \mu \) and integrating over \( \Omega \), by using (1)₁,₅ we get
\[
\frac{d}{dt} \int_\Omega \left( \frac{|
abla \chi|^2}{2} + \frac{\rho(\chi^4 - 2\chi^2)}{4} \right) \, dx + \int_\Omega \mu^2 \, dx = \int_\Omega u \cdot \nabla \chi \Delta \chi \, dx.
\]
Adding (14) to (15), then integrating the resulting equation with respect to \( t \) from 0 to \( T_* \), we obtain (12).

From (12), by using Cauchy inequality and (10) we have
\[
\int_\Omega \rho \chi^4 \, dx \leq \int_\Omega \rho \chi^2 \, dx + 1 \lesssim \frac{1}{2} \int_\Omega \rho \chi^4 \, dx + 1.
\]
It follows that
\[
\int_\Omega \rho \chi^4 \, dx \leq C.
\]
Applying the standard \( H^2 \)-estimate to the equation (1)₅ yields
\[
\| \nabla^2 \chi \|_{L^2}^2 \lesssim \| \Delta \chi \|_{L^2}^2 + \| \nabla \chi \|_{L^2}^2 \lesssim \| \mu \|_{L^2}^2 + \| \rho(\chi^3 - \chi) \|_{L^2} + \| \nabla \chi \|_{L^2}^2
\]
\[
\lesssim \| \mu \|_{L^2}^2 + \| \chi \|_{L^6}^6 + \| \chi \|_{L^2}^2 + \| \nabla \chi \|_{L^2}^2
\]
\[
\lesssim \| \mu \|_{L^2}^2 + \| \chi \|_{H^1}^6 + \| \chi \|_{L^2}^2 + \| \nabla \chi \|_{L^2}^2
\]
\[
\lesssim \| \mu \|_{L^2}^2 + 1,
\]
(17)
where we have used (10), (12) and (16). On the other hand, from the equation (14) we get
\[ \|\rho\chi_t\|^2_{L^2} \lesssim \|\rho u \cdot \nabla \chi\|^2_{L^2} + \|\mu\|^2_{H^2} \lesssim \|\nabla \chi\|^2_{L^\infty} \|u\|^2_{L^2} + \|\mu\|^2_{L^2}. \]
(18)

The well-known Korn’s inequality [5, 7] implies that, for bounded connected open domain Ω ⊂ R^d (N = 2, 3), there exists a (generic) positive constant C_Ω such that
\[ \|\nabla v\|_{L^2(Ω)} \leq C_Ω(\|Dv\|_{L^2(Ω)} + \|v\|_{L^2(Ω)}), \quad \forall v \in (H^1(Ω))^N. \]
(19)

Integrating (17) and (18) with respect to t over (0, T^*), by using (7), (8), (12) and Korn’s inequality (19), we derive (13). Then Lemma 2.2 follows.

The following estimate is crucial in the proof of the forthcoming lemmas.

**Lemma 2.3.** We assume that the hypotheses in Lemma 2.1 hold. Then for any 2 ≤ r ≤ 6, there holds
\[ \sup_{0 \leq t < T^*} \int_Ω \rho|\nabla \chi|^r dx + \int_0^{T^*} \int_Ω \frac{1}{r} |\nabla \chi|^{r-2} |\nabla^2 \chi|^2 dx dt \leq C. \]
(20)

**Proof.** From (1)4,5 we have
\[ \rho \chi_t + \rho u \cdot \nabla \chi = \frac{1}{\rho} \Delta \chi - (\chi^3 - \chi). \]
(21)

Differentiating the above equation with respect to x yields
\[ \rho \nabla \chi_t + \chi_t \nabla \rho + \rho (u \cdot \nabla) \nabla \chi + \nabla \rho (u \cdot \nabla) \chi + \rho \nabla (u \cdot \nabla) \chi = \nabla \left( \frac{1}{\rho} \Delta \chi \right) - (3\chi^2 - 1) \nabla \chi. \]
(22)

Multiplying (22) by \( r |\nabla \chi|^{r-2} \nabla \chi \) (2 ≤ r ≤ 6) and integrating the result over Ω, we get
\[ \int_Ω \rho(|\nabla \chi|^r) dx + r \int_Ω |\nabla \chi|^{r-2} \chi_t \nabla \chi \cdot \nabla \rho dx + \int_Ω \rho (u \cdot \nabla) (|\nabla \chi|^r) dx + r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla (u \cdot \nabla) \chi dx \]
\[ = r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla \left( \frac{1}{\rho} \Delta \chi \right) dx - r \int_Ω (3\chi^2 - 1) |\nabla \chi|^r dx. \]
(23)

We calculate the first term on the right hand side
\[ r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla \left( \frac{1}{\rho} \Delta \chi \right) dx \]
\[ = r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \cdot \text{div} \left( \frac{1}{\rho} \nabla^2 \chi \right) dx + r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla \left( \frac{1}{\rho} \right) \Delta \chi dx \]
\[ - r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \nabla \left( \frac{1}{\rho} \right) : \nabla^2 \chi dx \]
\[ = - r \int_Ω \frac{1}{\rho} |\nabla \chi|^{r-2} |\nabla^2 \chi|^2 dx - r (r-2) \int_Ω \frac{1}{\rho} |\nabla \chi|^{r-2} \nabla |\nabla \chi|^2 dx \]
\[ + r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla \left( \frac{1}{\rho} \right) \Delta \chi dx - r \int_Ω |\nabla \chi|^{r-2} \nabla \chi \nabla \left( \frac{1}{\rho} \right) : \nabla^2 \chi dx. \]
(24)
Putting (24) into (23), integrating by parts and using (1), we have
\[
\frac{d}{dt} \int_{\Omega} \rho |\nabla \chi|^r \, dx + r(r-1) \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^{r-2} |\nabla^2 \chi|^2 \, dx \\
= -r \int_{\Omega} |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla \rho \, dx - r \int_{\Omega} |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla (u \cdot \nabla) \chi \, dx \\
- r \int_{\Omega} \rho |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla (u \cdot \nabla) \chi \, dx + r \int_{\Omega} |\nabla \chi|^{r-2} \nabla \chi \cdot \nabla \left( \frac{1}{\rho} \right) \Delta \chi \, dx \\
- r \int_{\Omega} |\nabla \chi|^{r-2} \nabla \chi \otimes \nabla \left( \frac{1}{\rho} \right) : \nabla^2 \chi \, dx - r \int_{\Omega} (3\chi^2 - 1)|\nabla \chi|^r \, dx := \sum_{i=1}^{6} I_i. \quad (25)
\]
In what follows, we estimate \( I_i \) \((i = 1, 2, 3, 4, 5, 6)\) one by one in dimension three for example.
\[
I_1 \lesssim \|\chi\|_{L^2} \left( \int_{\Omega} (|\nabla \chi|^{r-1})^2 \, dx \right)^{1/2} \\
\lesssim \|\chi\|_{L^2} \left( \int_{\Omega} \left( |\nabla(|\nabla \chi|^{r-1})|^{5/2} + |\nabla \chi|^{5/2(r-1)} \right) \, dx \right)^{5/6} \\
\lesssim \|\chi\|_{L^2} \left( \int_{\Omega} |\nabla \chi|^{5/2(r-2)} |\nabla^2 \chi|^{5/2} \, dx \right)^{5/6} + \|\chi\|_{L^2} \left( \int_{\Omega} \rho |\nabla \chi|^r \, dx + 1 \right) \\
\lesssim \|\chi\|_{L^2} \left( \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^{r-2} |\nabla^2 \chi|^2 \, dx \right)^{1/2} \left( \int_{\Omega} \rho |\nabla \chi|^{3/2(r-2)} \, dx \right)^{1/3} \\
+ \|\chi\|_{L^2}^2 \left( \int_{\Omega} \rho |\nabla \chi|^r \, dx + 1 \right) \\
\lesssim \varepsilon \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^{r-2} |\nabla^2 \chi|^2 \, dx + \|\chi\|_{L^2} \left( \int_{\Omega} \rho |\nabla \chi|^{3/2(r-2)} \, dx \right)^{2/3} \\
+ \|\chi\|_{L^2} \left( \int_{\Omega} \rho |\nabla \chi|^r \, dx + 1 \right) \\
\lesssim \varepsilon \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^{r-2} |\nabla^2 \chi|^2 \, dx + \|\chi\|_{L^2} \left( \int_{\Omega} \rho |\nabla \chi|^r \, dx + 1 \right) \\
\int_{\Omega} |\nabla \chi|^r \, dx + \|\chi\|_{L^2} \left( \int_{\Omega} \rho |\nabla \chi|^r \, dx + 1 \right)
\]
where we have used Sobolev embedding theorem, Poincaré’s inequality, Hölder’s inequality, Cauchy inequality and \( \varepsilon \) is a sufficiently small constant to be determined later. Observing the second term on the right hand side, it should be satisfied that
\[
0 \leq \frac{5}{2}(r-1) \leq r, \quad 0 \leq \frac{3}{2}(r-2) \leq r, \quad \text{i.e.} \quad 2 \leq r \leq 6 \text{ should be satisfied. Similarly,}
\]
\[
I_2 \lesssim \|u\|_{L^6} \left( \int_{\Omega} |\nabla \chi|^\frac{6}{r} \, dx \right)^{5/6} \lesssim \|u\|_{L^2} \left( \int_{\Omega} |\nabla \chi|^r \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla \chi|^\frac{6}{r} \, dx \right)^{1/3} \\
\lesssim \|\nabla u\|_{L^2}^2 \int_{\Omega} |\nabla \chi|^r \, dx + \left( \int_{\Omega} (|\nabla \chi|^r)^{5/2} \, dx \right)^{2/3} \\
\lesssim (\|\nabla u\|_{L^2}^2 + 1) \int_{\Omega} |\nabla \chi|^r \, dx + \int_{\Omega} |\nabla(|\nabla \chi|^r)| \, dx \\
\lesssim (\|\nabla u\|_{L^2}^2 + 1) \int_{\Omega} |\nabla \chi|^r \, dx + \int_{\Omega} |\nabla \chi|^{r-1} |\nabla^2 \chi| \, dx \\
\lesssim \varepsilon \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^{r-2} |\nabla^2 \chi|^2 \, dx + (\|\nabla u\|_{L^2}^2 + 1) \int_{\Omega} \rho |\nabla \chi|^r \, dx.
\]
Moreover, by using Cauchy inequality, we have

\[ I_3 = -r \int_{\Omega} \rho |\nabla \chi|^{-2} \nabla \chi \cdot D(u \cdot \nabla \chi) \, dx \lesssim \|Du\|_{L^\infty} \int_{\Omega} |\nabla \chi|^r \, dx, \]

\[ I_4 + I_5 \lesssim \int_{\Omega} |\nabla \chi|^{-1} |\nabla^2 \chi| \, dx \leq \varepsilon \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^{-2} |\nabla^2 \chi|^2 \, dx + \int_{\Omega} \rho |\nabla \chi|^r \, dx, \]

\[ I_6 \lesssim (\|\chi\|_{L^\infty}^2 + 1) \int_{\Omega} |\nabla \chi|^r \, dx \lesssim (\|\nabla^2 \psi\|_{L^2}^2 + 1) \int_{\Omega} |\nabla \chi|^r \, dx. \]

Putting all these estimates into (25) and choosing \( \varepsilon > 0 \) small enough, we obtain

\[ \frac{d}{dt} \int_{\Omega} \rho |\nabla \chi|^r \, dx + \int_{\Omega} \rho |\nabla \chi|^{-2} |\nabla^2 \chi|^2 \, dx \]

\[ \lesssim (\|\chi\|_{L^\infty}^2 + \|\nabla u\|_{L^2}^2 + \|Du\|_{L^\infty} + \|\nabla^2 \chi\|_{L^2}^2 + 1) \int_{\Omega} \rho |\nabla \chi|^r \, dx + \|\chi\|_{L^2}^2. \]

Applying Gronwall’s inequality and using (7), (8), (12), (13), we derive (20). The case of dimension two is similar. Therefore, the proof of this lemma is complete. \( \Box \)

Then we continue to do some estimates for \( \chi \) and \( u \).

**Lemma 2.4.** Suppose that the assumptions in Lemma 2.1 are satisfied, we have

\[ \sup_{0 \leq t < T} \int_{\Omega} (|\nabla u|^2 + |\chi_t|^2 + |\Delta \chi|^2 + |\mu|^2) \, dx \]

\[ + \int_0^T \int_{\Omega} (|u|^2 + |\nabla \chi|^2 + |\nabla^2 u|^2 + |\nabla^3 \chi|^2) \, dx \, dt \leq C. \]  

(26)

**Proof.** Multiplying (1.2) by \( u_t \) and integrating over \( \Omega \) yield

\[ \frac{d}{dt} \int_{\Omega} \eta(\chi)|Du|^2 \, dx + \int_{\Omega} \rho |u_t|^2 \, dx \]

\[ = \int_{\Omega} \eta' \chi_t |Du|^2 \, dx - \int_{\Omega} \rho (u \cdot \nabla) u \cdot u_t \, dx - \int_{\Omega} \text{div}(\nabla \chi \otimes \nabla \chi) \cdot u_t \, dx. \]  

(27)

If \( N = 2 \), by Nirenberg’s interpolation inequality, we have

\[ \frac{d}{dt} \int_{\Omega} \eta(\chi)|Du|^2 \, dx + \int_{\Omega} \rho |u_t|^2 \, dx \]

\[ \lesssim \|\chi_t\|_{L^2} \|Du\|_{L^\infty} \|Du\|_{L^2} + \|u\|_{L^1} \|\nabla u\|_{L^1} \|u_t\|_{L^2} + \|\nabla \chi\|_{L^2} \|\nabla^2 \chi\|_{L^2} \|u_t\|_{L^2} \]

\[ \lesssim \|Du\|_{L^\infty} \int_{\Omega} \eta(\chi)|Du|^2 \, dx + \|Du\|_{L^\infty} \|\chi_t\|_{L^2}^2 + \frac{1}{2} \int_{\Omega} \rho |u_t|^2 \, dx \]

\[ + \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) + \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^2 \|\nabla^2 \chi\|^2 \, dx. \]

From which we get

\[ \frac{d}{dt} \int_{\Omega} \eta(\chi)|Du|^2 \, dx + \int_{\Omega} \rho |u_t|^2 \, dx \]

\[ \lesssim \|Du\|_{L^\infty} \int_{\Omega} \eta(\chi)|Du|^2 \, dx + \|Du\|_{L^\infty} \|\chi_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^3 \]

\[ + \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} + \int_{\Omega} \frac{1}{\rho} |\nabla \chi|^2 \|\nabla^2 \chi\|^2 \, dx. \]  

(28)

It follows from \([11]\) that

\[ \|\nabla^2 u\|_{L^2} \lesssim \|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\text{div}(\nabla \chi \otimes \nabla \chi)\|_{L^2} \]
Moreover, from the equations (1)–(2) and the Sobolev embedding theorem, the estimate (20) for \( r = 6 \) and the fact \( \| \chi \|_L^\infty \leq C \| \chi \|_{W^{1,\infty}} \leq C \), we see that
\[
\| \chi \|_L^2 \lesssim \| u \|_2^2 \| \nabla \chi \|_L^2 + \| \Delta \chi \|_L^2 + \| \chi^3 - \chi \|_L^2 \lesssim \| \nabla u \|_L^2 + \| \Delta \chi \|_L^2 + 1 \tag{30}
\]
Witch together with (28), (29), and by using Korn’s inequality yield
\[
\frac{d}{dt} \int_\Omega \eta(\chi) |Du|^2 \, dx + \int_\Omega \rho |u_t|^2 \, dx + \int_\Omega \| \nabla u \|^2 \, dx \\
\lesssim (\| Du \|_L^\infty + \| u \|_L^2) \int_\Omega \eta(\chi) |Du|^2 \, dx + \| Du \|_L^\infty \| \Delta \chi \|_L^2 \\
+ \int_\Omega \frac{1}{\rho} \| \nabla \chi \|^2 \| \nabla u \|^2 \, dx + \| Du \|_L^\infty + \| u \|_L^2. \tag{31}
\]
If \( N = 3 \), from (27) and by using (30), we get
\[
\frac{d}{dt} \int_\Omega \eta(\chi) |Du|^2 \, dx + \int_\Omega \rho |u_t|^2 \, dx \\
\lesssim \| \chi_t \|_L^2 \| Du \|_L^\infty \| Du \|_L^2 + \| u_t \|_L^\infty \| \nabla u \|_L^2 \| u_t \|_L^2 + \| \nabla \chi \|_L^2 \| \nabla u \|_L^2 \| u_t \|_L^2 \\
\lesssim (\| Du \|_L^\infty + \| u \|_L^2) \int_\Omega \eta(\chi) |Du|^2 \, dx + \| Du \|_L^\infty \| \Delta \chi \|_L^2 + \frac{1}{2} \int_\Omega \rho |u_t|^2 \, dx \\
+ \| u \|_L^2 \| u \|_L^2 + \int_\Omega \frac{1}{\rho} \| \nabla \chi \|^2 \| \nabla u \|^2 \, dx + \| Du \|_L^\infty + 1,
\]
where we have used Korn’s inequality (19) in the last step. Then it follows that
\[
\frac{d}{dt} \int_\Omega \eta(\chi) |Du|^2 \, dx + \int_\Omega \rho |u_t|^2 \, dx \\
\lesssim (\| Du \|_L^\infty + \| u \|_L^2) \int_\Omega \eta(\chi) |Du|^2 \, dx + \| Du \|_L^\infty \| \Delta \chi \|_L^2 \\
\quad + \int_\Omega \frac{1}{\rho} \| \nabla \chi \|^2 \| \nabla u \|^2 \, dx + \| u \|_L^2 + \| Du \|_L^\infty + 1. \tag{32}
\]
Next, need to estimate the term \( \| \Delta \chi \|_L^2 \). We deal with the case \( N = 3 \), the case of \( N = 2 \) is similar. Multiplying (22) by \( \nabla \chi_t \) and integrating the result over \( \Omega \) yield
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{1}{\rho} \| \Delta \chi \|^2 \, dx + \int_\Omega \rho |\nabla \chi_t|^2 \, dx \\
= \int_\Omega \left( \frac{1}{\rho} \right)_t |\Delta \chi|^2 \, dx - \int_\Omega \chi_t \nabla \rho \cdot \nabla \chi_t \, dx - \int_\Omega \rho (u \cdot \nabla) \nabla \chi \cdot \nabla \chi_t \, dx \\
- \int_\Omega (u \cdot \nabla) \nabla \rho \cdot \nabla \chi_t \, dx - \int_\Omega \rho \nabla \chi_t \cdot (u \cdot \nabla) \chi \, dx - \int_\Omega (3 \chi^2 - 1) \nabla \chi \cdot \nabla \chi_t \, dx \\
\lesssim \| u \|_L^2 \| \Delta \chi \|_L^2 + \| \chi_t \|_L^2 \| \nabla \chi_t \|_L^2 + \| u \|_L^2 \| \nabla \chi \|_L^2 + \| \nabla \chi \|_L^2 \| \nabla u \|_L^2 \\
+ \| u \|_L^2 \| \nabla \chi \|_L^\infty \| \nabla \chi_t \|_L^2 + \| \nabla \chi_t \|_L^2 \| \nabla u \|_L^2 \| \nabla \chi \|_L^\infty \\
+ \| 3 \chi^2 - 1 \|_L^\infty \| \nabla \chi \|_L^2 \| \nabla \chi_t \|_L^2 \| \nabla \chi \|_L^\infty \\
+ \| 3 \chi^2 - 1 \|_L^\infty \| \nabla \chi \|_L^2 \| \nabla \chi_t \|_L^2 \| \nabla \chi \|_L^\infty.
\[
\lesssim \|\Delta \chi\|^2_{L^2} + e\|\nabla \chi\|^2_{L^2} + \int_{\Omega} |u|^2 |\nabla^2 \chi|^2 \, dx + \|\nabla u\|^2_{L^2} \\
+ \|\nabla \chi\|^2_{L^\infty} \|\nabla u\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} + 1,
\]
(33)
where we have used (12), (13), (30) and (19) in the last step. In what follows, we estimate the first term on the right hand side. By Nirenberg's interpolation inequality
\[
\|\nabla^2 \chi\|_{L^3} \lesssim \|\nabla \chi\|_{L^6}^{1/2} \|\nabla^3 \chi\|_{L^2}^{1/2},
\]
and (20) (with \( r = 6 \)), we see that
\[
\int_{\Omega} u^2 |\nabla^2 \chi|^2 \, dx \lesssim \|u\|^2_{L^3} \|\nabla^2 \chi\|^2_{L^2} \lesssim \|\nabla u\|^2_{L^2} \|\nabla \chi\|_{L^6} \|\nabla^3 \chi\|_{L^2} \\
\lesssim e \|\nabla^3 \chi\|^2_{L^2} + \|\nabla u\|^4_{L^2}.
\]
(34)
It remains for us to estimate \( \|\nabla^3 \chi\|^2_{L^2} \). Applying the standard \( H^3 \)-estimate to the Neumann boundary value problem of the equation
\[
\frac{1}{\rho} \nabla \Delta \chi = \rho \nabla \chi_t + \chi_t \nabla \rho + \rho (u \cdot \nabla) \nabla \chi + \nabla (u \cdot \nabla) \chi \\
+ \rho \nabla (u \cdot \nabla) \chi - \nabla \left( \frac{1}{\rho} \right) \Delta \chi + (3\chi^2 - 1) \nabla \chi,
\]
by using (12), (13), (30) and (34), we have
\[
\|\nabla^3 \chi\|^2_{L^2} \lesssim \|\nabla \Delta \chi\|^2_{L^2} + \|\nabla \chi\|^4_{H^1} \\
\lesssim \|\nabla \chi_t\|^2_{L^2} + \|\chi_t\|^2_{L^2} + \|\nabla (u \cdot \nabla) \chi\|^2_{L^2} + \|\nabla^2 \chi\|^2_{L^2} + \|\Delta \chi\|^2_{L^2} \\
+ \|\nabla \chi\|^2_{L^2} + \|\nabla \rho\|^2_{L^\infty} \\
\lesssim \|\nabla \chi_t\|^2_{L^2} + \|\chi_t\|^2_{L^2} + \int_{\Omega} |u|^2 |\nabla^2 \chi|^2 \, dx + \|u\|^2_{L^2} \|\nabla \chi\|^2_{L^6} \\
+ \|\nabla u\|^2_{L^2} \|\nabla \chi\|^2_{L^\infty} + \|\Delta \chi\|^2_{L^2} + \|\chi\|^2_{L^6} \|\nabla \chi\|^2_{L^\infty} + \|\nabla^2 \chi\|^2_{L^2} + \|\nabla \chi\|^2_{L^2} \\
\lesssim \|\nabla \chi_t\|^2_{L^2} + e \|\nabla^3 \chi\|^2_{L^2} + \|\nabla \chi\|^2_{L^6} \|\nabla \chi\|^2_{L^\infty} \|\nabla u\|^2_{L^2} \\
+ \|\Delta \chi\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} + 1.
\]
It follows that
\[
\|\nabla^3 \chi\|^2_{L^2} \lesssim \|\nabla \chi_t\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} \|\nabla u\|^2_{L^2} \\
+ \|\Delta \chi\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} + 1.
\]
(35)
Putting (35) into (34) yields
\[
\int_{\Omega} u^2 |\nabla^2 \chi|^2 \, dx \lesssim e \|\nabla \chi_t\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} \|\nabla u\|^2_{L^2} \\
+ \|\Delta \chi\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} + 1.
\]
Substituting the above inequality into (33), we get
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{\rho} |\Delta \chi|^2 \, dx + \int_{\Omega} \rho |\nabla \chi_t|^2 \, dx \\
\lesssim \int_{\Omega} \frac{1}{\rho} |\Delta \chi|^2 \, dx + \|\nabla u\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} \int_{\Omega} \eta(\chi) |Du|^2 \, dx + \|\nabla u\|^2_{L^2} + \|\nabla \chi\|^2_{L^\infty} + 1.
\]
(36)
Putting (31), (32) and (36) together gives
\[
\frac{d}{dt} \int_\Omega \left( \eta(\chi)|Du|^2 + \frac{1}{\rho} |\Delta \chi|^2 \right) dx + \int_\Omega \left( \rho |u_t|^2 + |\nabla^2 u|^2 + \rho |\nabla \chi_t|^2 \right) dx \\
\leq \left( \|Du\|_{L^\infty} + \|\nabla u\|_{L^2}^2 + \|\nabla \chi\|_{L^\infty}^2 \right) \left( \int_\Omega \left( \eta(\chi)|Du|^2 + \frac{1}{\rho} |\Delta \chi|^2 \right) dx + \int_\Omega \frac{1}{\rho} |\nabla \chi|^2 |\nabla^2 \chi|^2 dx + 1 \right) \quad \text{if } N = 2
\]
and if \( N = 3 \),
\[
\frac{d}{dt} \int_\Omega \left( \eta(\chi)|Du|^2 + \frac{1}{\rho} |\Delta \chi|^2 \right) dx + \int_\Omega \left( \rho |u_t|^2 + \rho |\nabla \chi_t|^2 \right) dx \\
\leq \left( \|Du\|_{L^\infty} + \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \chi\|_{L^\infty}^2 \right) \left( \int_\Omega \left( \eta(\chi)|Du|^2 + \frac{1}{\rho} |\Delta \chi|^2 \right) dx + \int_\Omega \frac{1}{\rho} |\nabla \chi|^2 |\nabla^2 \chi|^2 dx + 1 \right)
\]
Hence, applying Gronwall’s inequality and using (7), (8), (12), (20) for \( r = 4 \), we obtain
\[
\sup_{0 \leq t < T} \int_\Omega \left( \eta(\chi)|Du|^2 + \frac{1}{\rho} |\Delta \chi|^2 \right) dx + \int_0^T \int_\Omega \left( \rho |u_t|^2 + \rho |\nabla \chi_t|^2 \right) dx dt \leq C. \quad (37)
\]
Moreover, from Korn’s inequality (19), (29), (30), (35) and equation (1)_4, we can easily see that (26) holds. Then Lemma 2.4 is obtained. \( \square \)

**Lemma 2.5.** Under the assumptions in Lemma 2.1, we have the inequality
\[
\sup_{0 \leq t < T} \int_\Omega \left( \rho |u_t|^2 + \frac{1}{\rho} |\nabla \chi_t|^2 \right) dx + \int_0^T \int_\Omega (|\nabla u|^2 + \rho |\chi_t|^2) dx dt \leq C. \quad (38)
\]

**Proof.** Differentiating (1)_2 with respect to \( t \) yields
\[
\rho u_{tt} + \rho_t u_t + \rho u \cdot \nabla u_t + \rho u_t \cdot \nabla u + \rho_x u \cdot \nabla p_t = \text{div} (2\eta(\chi)Du) - \text{div} (\nabla \chi_t \otimes \nabla \chi + \nabla \chi \otimes \nabla \chi_t).
\]
Multiplying the above equation by \( u_t \), integrating the result over \( \Omega \), and recalling (1)_{1,3}, we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u_t|^2 dx + \int_\Omega 2\eta(\chi)|Du_t|^2 dx \\
= - \int_\Omega 2\eta \chi_t Du : Du_t dx - \frac{1}{2} \int_\Omega \rho |u_t|^2 dx - \int_\Omega \rho(u \cdot \nabla)u_t \cdot u_t dx \\
- \int_\Omega \rho (u_t \cdot \nabla)u \cdot u_t dx - \int_\Omega \rho (u_t \cdot \nabla)u \cdot u_t dx \\
+ \int_\Omega \nabla \chi_t \otimes \nabla \chi : \nabla u_t dx + \int_\Omega \nabla \chi \otimes \nabla \chi_t : \nabla u_t dx \\
\leq \| \chi_t \|_{L^2} \| Du \|_{L^2} \| Du_t \|_{L^2} + \| u_t \|_{L^\infty} \| u_t \|_{L^2} + \| u_t \|_{L^2} \| Du \|_{L^2} \\
+ \| Du_t \|_{L^\infty} \| u_t \|_{L^2} + \| u_t \|_{L^2} \| \nabla u \|_{L^2} \| u_t \|_{L^2} + \| \nabla \chi_t \|_{L^2} \| Du_t \|_{L^2} \\
\leq \int_\Omega \eta(\chi)|Du_t|^2 dx + \left( \| u_t \|_{H^1}^2 + \| Du \|_{L^\infty} + 1 \right) \int_\Omega \rho |u_t|^2 dx \\
+ \left( \| u_t \|_{H^1}^2 + \| \nabla \chi_t \|_{L^2} \right) \| \nabla \chi_t \|_{L^2} \| u_t \|_{H^1} + 1, \]
where we have used (12), (26) and Korn’s inequality (19) in the last step. From which we have

\[
\frac{d}{dt} \int \rho |u_t|^2 \, dx + \int \eta(\chi) |Du_t|^2 \, dx
\]

\[
\lesssim (\|u\|^2_{H^2} + \|Du\|_{L^\infty} + 1) \int \rho |u_t|^2 \, dx + \left( \|u\|^2_{H^2} + \|\nabla \chi \|^2_{L^2} \right) \|\nabla \chi_t\|^2_{L^2} + \|u\|^2_{H^2} + 1.
\]

(39)

In the following, we deal with the term \(\|\nabla \chi_t\|^2_{L^2}\). Differentiating (21) with respect to \(t\) gives

\[
\rho \chi_{tt} + \rho_t \chi_t + \rho u \cdot \nabla \chi_t + \rho \chi_t \cdot \nabla \chi + \rho u \cdot \nabla \chi = \frac{1}{\rho} \Delta \chi_t + \left( \frac{1}{\rho} \right)_t \Delta \chi - (3 \chi^2 - 1) \chi_t.
\]

(40)

Multiplying (40) by \(\chi_t\), integrating the result over \(\Omega\) and noticing

\[
\int \frac{1}{\rho} \Delta \chi_t \chi_t \, dx = - \int \frac{1}{\rho} \nabla \chi_t \cdot \nabla \chi_t \, dx - \int \nabla \left( \frac{1}{\rho} \right)_t \chi_t \, dx
\]

\[
= - \frac{1}{2} \frac{d}{dt} \int \rho |\nabla \chi_t|^2 \, dx + \frac{1}{2} \int \left( \frac{1}{\rho} \right)_t |\nabla \chi_t|^2 \, dx
\]

\[
- \int \nabla \left( \frac{1}{\rho} \right)_t \cdot \nabla \chi_t \, dx
\]

we have

\[
\frac{1}{2} \frac{d}{dt} \int \rho |\nabla \chi_t|^2 \, dx + \int \rho |\chi_t|^2 \, dx
\]

\[
\lesssim \int |u| |\chi_t||\chi_t| \, dx + \int |u||\nabla \chi_t||\chi_t| \, dx + \int |u_t||\nabla \chi_t| \, dx
\]

\[
+ \int |u|^2 |\nabla \chi_t| \, dx + \int |u||\nabla \chi_t|^2 \, dx + \int |\nabla \chi_t| |\chi_t| \, dx
\]

\[
+ \int |u| |\Delta \chi_t| |\chi_t| \, dx + \int \left( (|\chi|^2 + 1) |\chi_t| \right) \, dx
\]

\[
\lesssim \|u\|_{L^\infty} \|\chi_t\|_{L^2} \|\chi_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla \chi_t\|_{L^2} \|\chi_t\|_{L^2} + \|u_t\|_{L^6} \|\nabla \chi_t\|_{L^2} \chi_t \|_{L^2}
\]

\[
+ \|u\|_{L^\infty} \|\Delta \chi_t\|_{L^2} \|\chi_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla \chi_t\|_{L^2} \|\chi_t\|_{L^2} + \|u_t\|_{L^6} \|\nabla \chi_t\|_{L^2} \|\chi_t\|_{L^2}
\]

\[
\lesssim \|u\|_{H^2}^2 + \|u_t\|_{H^2}^2 + \|\nabla \chi_t\|_{L^2}^2 + \|u_t\|_{H^2}^2 \|\chi_t\|_{L^2}^2 + \|\nabla \chi_t\|_{L^2}^2 \|\chi_t\|_{L^2} + 1,
\]

where we have used (10), (11), (12) and (26). It follows that

\[
\frac{d}{dt} \int \rho |\nabla \chi_t|^2 \, dx + \int \rho |\chi_t|^2 \, dx
\]

\[
\leq C(\|u\|^2_{H^2} + 1) \int \rho |\nabla \chi_t|^2 \, dx + C\|u\|^2_{H^2} + C\|Du_t\|^2_{L^2} + C\|u_t\|^2_{L^2} + C.
\]

(41)
Multiplying (41) by \(\eta/C\), adding the resulting inequality to (39) and by using (6) yield
\[
\frac{d}{dt} \int_\Omega \left( \rho |u_t|^2 + \frac{1}{\rho} |\nabla \chi_t|^2 \right) dx + \int_\Omega (|Du_t|^2 + \rho |\chi_t|^2) dx \lesssim (||u||_{L^2}^2 + ||Du||_{L^\infty} + ||\nabla \chi||_{L^\infty}^2 + 1) \int_\Omega \left( \rho |u_t|^2 + \frac{1}{\rho} |\nabla \chi_t|^2 \right) dx + ||u||_{L^2}^2 + 1.
\]
From the equations (12.4) and the assumptions on the initial data \(\rho_0, u_0, \chi_0\), applying Gronwall’s inequality, by (7) and (26), we obtain (38). Thus we complete the proof of this lemma.

In terms of the results for the stationary Stokes equation, we derive the higher order estimates for \(u\) and \(p\).

**Lemma 2.6.** If the assumptions in Lemma 2.1 are valid, there holds
\[
\sup_{0 \leq t < T} \left( ||u||_{H^2}^2 + ||p||_{H^1}^2 + ||\nabla^3 \chi||_{L^2}^2 + ||\nabla \mu||_{L^2}^2 \right) + \int_0^T \left( ||u||_{W^{2,6}}^2 + ||p||_{W^{1,6}}^2 + ||\mu_t||_{L^2}^2 \right) dt \leq C. 
\] (42)

**Proof.** From (26) and Nirenberg’s interpolation inequality
\[
||\chi||_{L^\infty} \lesssim ||\chi||_{L^2}^{1/4} ||\nabla^2 \chi||_{L^2}^{3/4} + ||\chi||_{L^2},
\]
we have
\[
||\nabla^3 \chi||_{L^2} \lesssim ||\nabla \chi_t||_{L^2} + ||\chi_t||_{L^2} + ||\nabla^2 u||_{L^2} + ||\nabla \chi_t||_{L^2}^{1/4} ||\nabla^3 \chi||_{L^2}^{3/4} + ||\Delta \chi||_{L^2} + 1
\lesssim \frac{1}{2} ||\nabla^3 \chi||_{L^2} + ||\nabla \chi_t||_{L^2} + ||\chi_t||_{L^2} + ||\nabla^2 u||_{L^2} + ||\nabla \chi||_{L^2} + ||\Delta \chi||_{L^2} + 1.
\]
By (12), (26), (38), we get
\[
||\nabla^3 \chi||_{L^2} \lesssim ||\nabla^2 u||_{L^2} + 1. 
\] (43)
On the other hand, (12) can be rewritten as
\[
-\text{div}(2\eta(\chi) Du) + \nabla p = -\rho u \cdot \nabla u - \text{div}(\nabla \chi \otimes \nabla \chi) - \rho \mu_t. 
\] (44)
By the estimates for the stationary Stokes equations (see [11]), we have
\[
||u||_{H^2} + ||p||_{H^1} \lesssim ||u \cdot \nabla u||_{L^2} + ||\text{div}(\nabla \chi \otimes \nabla \chi)||_{L^2} + ||u_t||_{L^2}
\lesssim ||u||_{L^6} ||\nabla u||_{L^3} + ||\nabla \chi||_{L^\infty} ||\nabla^2 \chi||_{L^2} + ||u_t||_{L^2}
\lesssim ||\nabla u||_{L^2}^{3/2} ||\nabla^2 u||_{L^2}^{1/2} + ||\nabla \chi||_{L^2}^{1/4} ||\nabla^3 \chi||_{L^2}^{3/4} ||\nabla^2 \chi||_{L^2} + ||u_t||_{L^2}
\lesssim ||\nabla^2 u||_{L^2}^{1/2} + ||\nabla^3 \chi||_{L^2}^{3/4} + 1 \leq ||\nabla^2 u||_{L^2}^{1/2} + ||\nabla^2 u||_{L^2}^{3/4} + 1
\lesssim \frac{1}{2} ||\nabla^2 u||_{L^2} + 1,
\] (45)
where we have used Nirenberg’s interpolation inequality, (10), (12), (26),and (38). From (43) and (45) we obtain
\[
\sup_{0 \leq t < T} \left( ||u||_{H^2}^2 + ||p||_{H^1}^2 + ||\nabla^3 \chi||_{L^2}^2 \right) \leq C. 
\] (46)
The estimates for the stationary Stokes equation (44) (see [11]) and (46) also imply
\[
||u||_{W^{2,6}} + ||p||_{W^{1,6}} \lesssim ||u \cdot \nabla u||_{L^6} + ||\text{div}(\nabla \chi \otimes \nabla \chi)||_{L^6} + ||u_t||_{L^6}
\lesssim ||u||_{L^\infty} ||\nabla u||_{L^6} + ||\nabla \chi||_{L^\infty} ||\nabla^2 \chi||_{L^6} + ||\nabla u_t||_{L^2}
\]
where we have used Hölder’s inequality for $2 \leq r \leq 6$. From the above inequality and (38), we have

$$\int_0^{T_\star} (\|u\|^2_{W^{2,6}} + \|p\|^2_{W^{1,6}})dt \lesssim \int_0^{T_\star} (\|\nabla u_t\|^2_{L^2} + 1)dt \leq C.$$ 

Recalling (1) and using (10), (11), (26), (38), (46), we get

$$\|\nabla \mu\|_{L^2} \lesssim \|\nabla^3 \chi\|_{L^2} + \|\Delta \chi\|_{L^2} + (\|\chi\|^2_{L^\infty} + 1)\|\nabla \chi\|_{L^2} \leq C,$$

and

$$\|\mu_t\|^2_{L^2} \lesssim \|\chi_{tt}\|^2_{L^2} + \|\chi_t(u \cdot \nabla)\|^2_{L^2} + \|\nabla \chi_t\|^2_{L^2} + \|\nabla \chi(u \cdot \nabla)\|^2_{L^2} \lesssim \|\chi_{tt}\|^2_{L^2} + \|\chi_{t\chi}\|^2_{L^2} + \|\chi_{t\nabla}^2\|^2_{L^2} + \|\nabla \chi\|^2_{L^2} \lesssim \|\chi_{tt}\|^2_{L^2} + \|\nabla \chi_t\|^2_{L^2} + 1.$$ 

Integrating the above inequality over $(0, T_\star)$ and by using (38) again, we arrive at (42). The proof of this lemma is complete.

At last, we deal with the higher order estimates for $\rho$, $\chi$ and $\mu$.

**Lemma 2.7.** Assume that the hypotheses in Lemma 2.1 hold, then we have

$$\sup_{0 \leq t < T_\star} (\|\rho\|^2_{W^{2,6}} + \|\rho_t\|^2_{W^{1,6}}) + \int_0^{T_\star} (\|\nabla^4 \chi\|^2_{L^2} + \|\nabla^2 \chi_t\|^2_{L^2} + \|\nabla^2 \mu\|^2_{L^2})dt \leq C.$$ 

(47)

**Proof.** Differentiating (1) with respect to $x$ twice and using (1), we have

$$\nabla^2 \rho_t + (u \cdot \nabla)\nabla^2 \rho + \nabla^2(u \cdot \nabla)\rho + 2\nabla(u \cdot \nabla)\nabla \rho = 0.$$ 

Multiplying the above equation by $r|\nabla^2 \rho|^r - 2|\nabla^2 \rho|^2$, integrating the result over $\Omega$ and noticing that

$$\int_\Omega (u \cdot \nabla)\nabla^2 \rho : (r|\nabla^2 \rho|^r - 2|\nabla^2 \rho|^2)dx = \int_\Omega (u \cdot \nabla)(|\nabla^2 \rho|^r)dx = -\int_\Omega |\nabla^2 \rho|^r \text{div} u dx = 0,$$

we have

$$\frac{1}{2} d\int_\Omega |\nabla^2 \rho|^rdx \lesssim \int_\Omega |\nabla^2 u||\nabla \rho|||\nabla^2 \rho|^r - 1dx + \int_\Omega |\nabla u|||\nabla^2 \rho|^r - 1dx$$

$$\lesssim \|\nabla^2 u\|_{L^\infty} \|\rho\|_{L^\infty} \left(\int_\Omega |\nabla^2 \rho|^\frac{r}{2}dx\right)^5 \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^r}^r$$

$$\lesssim \|\nabla^2 u\|_{L^\infty} \left(\int_\Omega |\nabla^2 \rho|^r dx\right)^{(r-1)/r} \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^r}^r$$

$$\lesssim (\|u\|^2_{W^{2,6}} + \|\nabla u\|_{L^\infty} + 1)\|\nabla^2 \rho\|_{L^r} + \|u\|^2_{W^{2,6}} + 1,$$

where we have used Hölder’s inequality for $2 \leq r \leq 6$ in the third step. Then by Gronwall’s inequality and (42), we obtain

$$\sup_{0 \leq t < T_\star} \|\nabla^2 \rho\|_{L^r} \leq C.$$ 

(48)
Moreover, differentiating (1) with respect to \(x\), we can derive that
\[
\|\nabla \rho_t\|_{L^r} \lesssim \|\nabla u\|_{L^r} \|\nabla \rho\|_{L^\infty} + \|u\|_{L^\infty} \|\nabla^2 \rho\|_{L^r} \leq C.
\]

Applying the standard \(H^2\)-estimate to the equation (40), we get
\[
\|\nabla^2 \chi_t\|_{L^2} \lesssim \|\Delta \chi_t\|_{L^2} + \|\nabla \chi_t\|_{L^2}
\leq \|\chi_t\|_{L^2} + \|u \chi_t\|_{L^2} + \|u \cdot \nabla \chi_t\|_{L^2} + \|u_t \cdot \nabla \chi_t\|_{L^2}
+ \|\nabla u\|_{L^2} \|\chi_t\|_{L^2} + \|u\|_{L^2} \|\nabla \chi_t\|_{L^2} + \|u_t\|_{L^2} \|\nabla \chi_t\|_{L^2}
+ \|\nabla \chi_t\|_{L^2} \|\nabla \chi_t\|_{L^2} + \|u_t\|_{L^2} \|\Delta \chi_t\|_{L^2}
+ (\|\chi_t\|_{L^6}^2 + 1)\|\nabla \chi_t\|_{L^2} + \|\nabla \chi_t\|_{L^2}
\leq \|\chi_t\|_{L^2} + \|\nabla u\|_{L^2} \|\chi_t\|_{H^1} + \|\nabla u\|_{L^2} \|\nabla \chi_t\|_{L^2} \|\nabla^2 \chi_t\|_{L^2}^{1/2}
+ \|\nabla u_t\|_{L^2} \|\nabla \chi_t\|_{H^1} + \|\nabla u\|_{L^2} \|\nabla \chi_t\|_{L^2} + \|\nabla \chi_t\|_{L^2}
+ (\|\chi_t\|_{L^6}^2 + 1)\|\nabla \chi_t\|_{L^2} + \|\nabla \chi_t\|_{L^2}
\leq \frac{1}{2} \|\nabla^2 \chi_t\|_{L^2} + \|\chi_t\|_{L^2} + \|\nabla u_t\|_{L^2} + 1,
\]
where we have use Nirenberg’s interpolation inequality, (10), (11), (38) and (42) in the last step. From the above inequality and (38) we have
\[
\int_0^T \|\nabla^2 \chi_t\|^2_{L^2} \, dt \leq \int_0^T (\|\chi_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2} + 1) \, dt \leq C.
\] (49)

Applying the standard \(L^2\)-estimate to the equation (21) and by using (10) and (11),
\[
\|u\|_{L^\infty} \lesssim \|u\|_{H^2} \leq C, \quad \|\chi\|_{L^\infty} \lesssim \|\chi\|_{H^2} \leq C, \quad \|\nabla \chi\|_{L^\infty} \lesssim \|\nabla \chi\|_{H^2} \leq C,
\]
we have
\[
\|\nabla^4 \chi\|_{L^2} \lesssim \|\nabla^2 (\rho \chi)\|_{L^2} + \|\nabla^2 (\rho^2 u \cdot \nabla \chi)\|_{L^2} + \|\nabla^2 (\rho(\chi^3 - \chi))\|_{L^2} + \|\nabla^3 \chi\|_{L^2}
\lesssim \|\nabla^4 \chi_t\|_{L^2} + \|\nabla \chi_t\|_{L^2} + \|\nabla \chi\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} + \|\nabla^3 \chi\|_{L^2}
+ \|\nabla^2 \chi\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^2 \chi\|_{L^3} + \|\nabla^2 u_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} + 1
\lesssim \|\nabla^3 \chi_t\|_{L^2} + 1,
\] (50)
where we have used (38), (26) and the estimates for \(\rho\) in the last step. Furthermore, from (1) we can deduce that
\[
\|\nabla^2 \mu\|_{L^2} \lesssim \|\nabla^4 \chi\|_{L^2} + \|\nabla^3 \chi\|_{L^2} + \|\nabla^2 \rho\|_{L^2} \|\Delta \chi\|_{L^\infty} + \|\nabla^2 \chi\|_{L^2} + 1
\lesssim \|\nabla^4 \chi\|_{L^2} + 1.
\] (51)

From (49), (50) and (51), we obtain (47). Therefore, Lemma 2.7 is established. □

From Lemma 2.1–Lemma 2.7 we obtain (9). Hence
\[
(r, u, \chi)(x, T_\ast) = \lim_{t \to T_\ast} (r, u, \chi)(x, t)
\]
exists. Moreover, \(u(x, T_\ast)|_{\partial \Omega} = 0\) and \(\text{div} \, u(x, T_\ast) = 0\) for \(x \in \Omega\). Thus, we can extend the strong solution to \(T_\ast + \delta\) with some constant \(\delta > 0\), which contradicts the definition of \(T_\ast\). Therefore, (7) and (8) are false. The proof of Theorem 1.3 is now complete. □
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