Weyl law for fat fractals

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Abstract. It has been conjectured that for a class of piecewise linear maps the closure of the set of images of the discontinuity has the structure of a fat fractal, that is, a fractal with positive measure. An example of such maps is the sawtooth map in the elliptic regime. In this work we analyze this problem quantum mechanically in the semiclassical regime. We find that the fraction of states localized on the unstable set satisfies a modified fractal Weyl law, where the exponent is given by the exterior dimension of the fat fractal.

PACS numbers: 05.45.Mt, 05.45.Df, 05.45.Pq
1. Introduction

The general question of how, and at which rate, quantum states explore phase spaces with a complex structure has drawn much attention recently. In the case of chaotic repellers the ordinary Weyl law of one quantum state per phase space cell \((2\pi\hbar)^d\) is substituted by a conjectured fractal Weyl law with the integer dimension \(d\) replaced by the fractal dimension of the repeller \([1]\). In this paper we address this question with respect to elliptic quantum maps with a discontinuity. These maps exhibit a complex phase space where regular open islands coexist with the set consisting of the iterations of the discontinuity. This irregular set has a very complex structure and the question is whether it supports a finite fraction of stationary states in the semiclassical limit. This has been answered rigorously by the negative in \([2]\) with the assumption that the \(d\)-Minkowski content of this set vanishes, i.e. that its closure has zero Lebesgue measure.

A paradigmatic example of such maps is the sawtooth map in the elliptic regime \([3, 4]\). In \([4]\) Ashwin shows numerical evidence indicating that for this map the irregular sea, i.e. the closure of the images of the discontinuity, has fractal structure and positive Lebesgue measure. An analogous result is presented in \([5]\) for the case of a discontinuous Hamiltonian mapping on the sphere \([6]\), which is conjectured to produce circle packing in phase space. Also in this case, the irregular sea appears to have non zero measure. These subsets of fractal structure and positive measure are loosely called ‘fat fractals’ and can be characterized by their exterior dimension and their Lebesgue measure \([7]\).

In this work we conjecture, and validated by numerical evidence, that in the case where the irregular set is a ‘fat’ fractal, for which the condition in \([2]\) is not met, a non vanishing fraction of eigenstates is localized in it and the rate at which this is achieved follows a generalized Weyl law governed by its exterior dimension. In order to obtain the results presented, an effective method to count the eigenstates living in the irregular set is needed. We introduce a procedure which consists in opening the map along the discontinuity line. The eigenstates supported by the irregular set have a large overlap with the open area and thus the corresponding eigenvalues migrate to the interior of the unit disk. By computing the fraction of these short-living states as a function of the resolution, we are able to find a scaling law, depending on the Lebesgue measure and exterior dimension of the set.

We organized our contribution as follows. Our calculations will be performed for the the elliptic sawtooth map. We will first recall some properties of the classical version of this map in section \([2]\). Then in section \([3]\) we consider its quantized version and using the proposed method we compute the fraction of eigenstates supported by the irregular subset. Finally we give some concluding remarks in section \([4]\).
2. The classical sawtooth map

The sawtooth map is an area preserving map on the torus defined by

\[
\begin{align*}
\bar{p} &= p + K \text{saw}(q) \\
\bar{q} &= q + \bar{p}
\end{align*}
\]

for \( q, p \in [0, 1)^2 \), with \( \text{saw}(q) = q \mod 1 - 1/2 \). This map is linear and continuous except on the discontinuity line \( q = 0 \). For \(-4 < K < 0\) the map is elliptic. In this regime the phase space is filled with chains of regular islands. Their boundary corresponds to the infinite iteration of the discontinuity. In the elliptic case a rotation parameter can be defined related to \( K \) as \( K = 2 \cos \theta - 2 \) with \( 0 < \theta < \pi \). When \( \theta \) is a rational fraction of \( \pi \), the islands are convex polygons whose interior is filled with periodic trajectories (see Fig. 1 (right), otherwise they are ellipses (see Fig. 1 (left) filled with tori. The phase space of the map can be decomposed into two invariant domains: the stable set \( D^c \) consisting of the open elliptic islands, and the unstable set \( D \) which is the closure of the set of all images and preimages of the discontinuity. In Fig. 1 box counting in a discretised phase space of \( 2^n \times 2^n \) is used to compute the measure of the \( \epsilon \)-neighborhood of the unstable set \( D \), where \( \epsilon = 2^{-n} \). This measure scales with \( \epsilon \) as

\[
\ell_\epsilon(D) = \ell_\infty(D) + A \epsilon^{2-d_{\text{ext}}}
\]

where \( d_{\text{ext}} \) is the exterior dimension of the fat fractal \( \mathbb{H} \) and \( \ell_\infty(D) \) is the Lebesgue measure of the closure of the discontinuity. For rational values of \( \frac{\theta}{\pi} \), \( \ell_\epsilon(D) \) converges to zero and \( d_{\text{ext}} \) is the Hausdorff dimension of the fractal. For irrational angles, the numerical data are consistent with a positive two-dimensional Lebesgue measure of \( D \). Fig. 1 gives a very clear graphical view of this fact.
3. The quantum open map

As the classical map is the product of two shears, its quantization is immediate \cite{3} and in the position representation using antisymmetric boundary conditions yields the unitary matrix:

\[ \langle q_n' | \hat{U} | q_n \rangle = \frac{1}{N} \exp[i 2\pi K (n + \frac{1}{2} - \frac{N}{2})^2] \]

\[ \sum_{k=0}^{N-1} \exp[-i 2\pi \frac{(k + \frac{1}{2})^2}{N} + (k + \frac{1}{2})(n' - n)] \]  

(3)

where the dimension of the Hilbert space \( N \) plays the role of an effective Planck’s constant \( \hbar = (2\pi N)^{-1} \), the semiclassical limit corresponding to \( N \to \infty \).

All the eigenvalues of the quantum map \( \hat{U} \) lie on the unit circle. In the semiclassical limit the eigenfunctions can be classified into two types: the regular ones, whose Husimi representation is localized on the tori resolved at each value of \( N \) and the irregular ones which spread along the irregular sea. The former have vanishingly small overlap with the neighborhood of the discontinuity while the latter have a significant overlap with it. Counting the number of states in these two sets as \( N \to \infty \) will allow us to determine the relative measure of these sets in the semiclassical limit. In order to distinguish between the two kinds of eigenfunctions we open the map along the discontinuity line. This is implemented by the nonunitary matrix \( \hat{U}_{\text{open}} = \hat{U}(I - \hat{\Pi}) \), where \( \hat{\Pi} = |0\rangle\langle 0| + |N - 1\rangle\langle N - 1| \) is a projector modeling an \( \hbar \) vicinity of the discontinuity.
discontinuity. The spectrum of the open map will show complex resonances with eigenvalues very close to the unit circle corresponding to the regular eigenfunctions while the irregular states will have faster decays and will be further away from the unit circle. This procedure is in some respects a quantum version of the box counting procedure in the classical case where a classical strip of width $\epsilon$ generates the irregular invariant set at $\epsilon$ resolution whose measure scales with $\epsilon$ as in eq.(2). In the quantum case the projector $\Pi$ (of support $2/N$) provides the analogous strip vanishing in the semiclassical limit and endowing the irregular states with a lifetime that allows their counting. It is worth stressing that the problem we are considering is different from the one leading to the formulation of the fractal Weyl law for open chaotic [1] and mixed [8] systems. In those studies the interest lies in the long lived resonances arising from an opening of classical size involving a finite fraction of the phase space.

**Figure 3.** Husimi representation of the projector $H_N(z, \bar{z})$ comprising resonances with $\gamma > \gamma_c$ for the same values of $K$ as in Fig[1] The black dot shows the area of a quantum state for this value of $N (=360)$.

Fig[2] displays the ordered moduli (starting with the short-lived states) $|\lambda_i| = \exp(-\gamma_i)$ of the first $N/3$ eigenvalues of operator $\hat{U}_{\text{open}}$ with $K = -0.5$ as a function of the index $i/N$, for values of $N$ going from $N = 512$ to $N = 8800$. We show the band $0.99 < |\lambda_i| < 1$ where most of the eigenvalues concentrate. The cut-off value $|\lambda_c| = \exp(-\gamma_c)$ that we use to distinguish between regular states (associated with an escape rate $\gamma_i < \gamma_c$) and irregular ones (with $\gamma_i > \gamma_c$) is indicated for each value of $N$. This cut-off value which should depend on the fraction of the total area in phase space occupied by the opening has been taken equal to $\frac{2}{N}$. We tested the validity of this approximation by building the Husimi representation of the projector constituted
by the eigenfunctions with escape rate faster than $\gamma_c(N)$

$$H_N(z, \bar{z}) = \sum_{\gamma_i > \gamma_c} \frac{\langle z | R_i \rangle \langle L_i | z \rangle}{\langle L_i | R_i \rangle}$$

(4)

where $| R_i \rangle (\langle L_i |)$ are the right (left) eigenstates of $\hat{U}_{\text{open}}$ with $\gamma_i > \gamma_c(N)$ and $| z \rangle$ are coherent states centered at $z = q + ip$. Fig.3 (corresponding to $N = 360$ and the values of parameter $K$ used in Fig. 1) clearly shows that the Husimi density of the fast-decaying resonances up to $\gamma_c = \frac{2}{N}$ concentrates on the irregular set, that is, on the complement of the islands that can be resolved at this value of $N$ and are classically decoupled from the opening (a cell of area $h = \frac{1}{N}$ is indicated). Similar plots were obtained for several Hilbert space dimensions, validating our choice of $\gamma_c(N)$.

Fig.4 displays the fraction of irregular states $\frac{N_{\text{irr}}}{N}$ as a function of $N$ for $K = -0.5$ (○ symbols), upper points (fitted with $\frac{N_{\text{irr}}}{N} = 0.14 + 1.74/N^{-0.57}$) and $K = 2 \cos \frac{\pi}{4} - 2$ (□ symbols) (fitted with $\frac{N_{\text{irr}}}{N} = 0.03 + 4.4/N^{-0.61}$). The inset shows the corresponding log-log plot.

Fig.4 displays the fraction $\frac{N_{\text{irr}}}{N}$ of states with $\gamma_i > \gamma_c(N)$ that we associate with the states localized on the $\epsilon$ - neighborhood of the unstable set $D$ with $\epsilon = \frac{2}{N}$, as a function of $N$ for $K = 2 \cos \frac{\pi}{4} - 2$ (□ symbols) and $K = -0.5$ (○ symbols). In both cases our results are consistent with a scaling

$$\frac{N_{\text{irr}}}{N}(N) = \frac{N_{\text{irr}}}{N}(\infty) + C\left(\frac{1}{N}\right)^\beta$$

(5)

analogous to the scaling law followed by the coarse grained measure of the unstable set (eq.(2)). In the rational case $(K = 2 \cos \frac{\pi}{4} - 2)$, where the analytical values of the
exterior dimension of the fractal set $d_{\text{ext}}$ and of its Lebesgue measure $\ell_{\infty}(D) = 0$ are known [9], our fitted values of $\beta$ and $\frac{N_{\text{irr}}}{N}$ reproduce them with a reasonable accuracy (see Table 1). In the irrational case ($K = -0.5$) the fluctuations of our data for small $N$ do not allow a precise determination of the exponent $\beta$. Therefore we fixed $\beta = 2 - d_{\text{ext}}$ to the value obtained in [4] for the exterior dimension of the fractal set and extracted $\frac{N_{\text{irr}}}{N}(\infty)$ from the fitted curve. As shown in Table 1, our value is somewhat larger than the value of $\ell_{\infty}(D)$ obtained in [4]. In spite of this overestimation of the asymptotic value, also noticeable in the rational case, our data strongly suggest that the fraction of irregular states in the semiclassical limit goes to a finite value for $K = -0.5$ (where $D$ is a fat fractal, according to [4]) while it practically vanishes for $K = 2 \cos \frac{\pi}{4} - 2$. This can also be appreciated from the log-log plot of the inset. The overestimation of $\frac{N_{\text{irr}}}{N}(\infty)$ is probably due to the the use of the classical escape rate as a cut-off criterium which in the quantum case is not fully justified. We have remarked that the opening we use is not classical - its width goes to zero in the semiclassical limit - and therefore strong diffraction effects may be expected. Thus the cut-off value corresponding to $\epsilon = 2/N$ that we have used may very well be modified by an ‘effective’ escape rate $\alpha_{\text{eff}}/N$. In fact a much better agreement can be obtained if we tailor an effective cut-off $\approx 2.5/N$ to fit the data, but we could not properly justify such an ad hoc procedure.

| $K$         | $\beta$ | $2 - d_{\text{ext}}$ | $\frac{N_{\text{irr}}}{N}(\infty)$ | $\ell_{\infty}(D)$ |
|-------------|---------|-----------------------|-------------------------------------|-----------------|
| $2 \cos \frac{\pi}{4} - 2$ | 0.61    | 0.62                  | 0.03                                | 0.              |
| -0.5        | 0.57*   | 0.57                  | 0.14                                | 0.12            |

The oscillations observed in our numerical data (especially in the $K = -0.5$ case) can be explained by the discontinuities in the filling of the elliptic structures. For values of $N$ corresponding to a resolution at which a new chain of islands becomes visible, states which spread along the irregular sea will be able to locate in the resolved regular domain, and the number of irregular states will drop. In fact, we checked that for values of $K$ for which the classical phase space has smoother distribution in the size of the islands, these oscillations are less pronounced.

4. Conclusion

We have provided a procedure that makes it possible to obtain the fraction of states with support in the irregular set for a map with a phase space divided into separate domains. We have shown that this fraction scales with $N$ following a power law analogous to the classical one for the coarse grained Lebesgue measure of the set. We presented numerical calculations for the quantum sawtooth map which exhibits two distinct behaviors depending on the parameter $K = 2 \cos \theta - 2$, for $\theta$ either a rational or irrational fraction of $\pi$. In the rational case, where the set is of zero measure, our
findings verify the rigorous results found in [2]. In the irrational case our results strongly suggest that the generalized Weyl law described in [2] — which states that for large \( N \) the fraction of states located in a given set is approximately proportional to the measure of the set — can be extended to the case where this set has the structure of a fat fractal.

The general question explored in this paper is also of interest in other situations, where the coexistence at all scales of regular and irregular regions provides a substrate to quantum mechanics. The neighborhood of a rational torus in perturbed integrable systems and the quantum mechanics of sphere or circle packing are also instances of particular interest.

Acknowledgments

This work was partially supported by CONICET and ANPCyT (PICT 25373).

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