On the Morita invariance of Categorical Enumerative Invariants

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Abstract

Categorical Enumerative Invariants (CEI) are invariants associated with unital cyclic $A_{\infty}$-categories that are smooth, proper and satisfy the Hodge-to-de-Rham degeneration property. In this paper, we formulate and prove their Morita invariance. In particular, when applied to derived categories of coherent sheaves, this yields new birational invariants of smooth and proper Calabi-Yau 3-folds.

1 Introduction

1.1 Main results.

Let $\mathbb{K}$ be a field of characteristic zero. Let $A$ be a $\mathbb{Z}/2\mathbb{Z}$-graded cyclic $A_{\infty}$-algebra over $\mathbb{K}$. Assume that $A$ is

(†) smooth, finite dimensional, unital, and satisfies the Hodge-to-de-Rham degeneration property. (The Hodge-de Rham degeneration property is automatic if $A$ is $\mathbb{Z}$-graded [22].) Assume that cyclic pairing is of parity $d \in \mathbb{Z}/2\mathbb{Z}$. Let $s$ be a choice of splitting of the non-commutative Hodge filtration, see Definition 3.14. Associated to the pair $(A, s)$, we obtain from [15, 7] a set of invariants that resembles the Gromov-Witten invariants in symplectic geometry:

$$\langle \alpha_1 u^{k_1}, \ldots, \alpha_n u^{k_n} \rangle_{g,n}^{A,s} \in \mathbb{K},$$

$$\alpha_1, \ldots, \alpha_n \in HH_*(A)[d], \ k_1, \ldots, k_n \in \mathbb{N}.$$ 

These invariants are called categorical enumerative invariants (CEI for short) in [7]. In this paper we consider these invariants for $A_{\infty}$-categories and then formulate and prove their Morita invariance.
The difficulty lies in the very formulation of the Morita invariance, since cyclic structures don’t even pull-back via $A_\infty$ quasi-isomorphisms. For this reason, we need to replace cyclic structures by something more flexible, namely Calabi-Yau structures. There are two a priori different definitions in the literature: proper Calabi-Yau structures [26] and smooth Calabi-Yau structures [27]. In this paper, we shall not distinguish between them since by a result of Shklyarov [38] these two definitions are equivalent for smooth and proper categories.

Let $\mathcal{C}$ be a small $\mathbb{Z}/2\mathbb{Z}$-graded $A_\infty$-category over $K$. Assume that $\mathcal{C}$ is

$$(\dagger\dagger)$$ smooth, proper, unital, and satisfies the Hodge-to-de-Rham degeneration property.

Notice that since we no longer assume cyclicity, we have replaced the finite dimensional condition in $(\dagger)$ by the properness condition.

An element $\omega \in HH_\bullet(\mathcal{C})$ is called a weak Calabi-Yau structure if its induced pairing on $\mathcal{C}$ defined by $a \otimes b \mapsto \langle m_2(a, b), \omega \rangle_{\text{Muk}}$ is homologically non-degenerate, see Definition 2.12.

Fix a parity $d \in \mathbb{Z}/2\mathbb{Z}$. Define a set $\mathcal{M}_C^d$ consisting of pairs $(\omega, s)$ with

- $\omega$ a weak Calabi-Yau structure of $\mathcal{C}$ of parity $d$,
- $s$ a splitting of the non-commutative Hodge filtration of $\mathcal{C}$.

We shall refer to a pair $(\omega, s)$ as an extended Calabi-Yau structure. Next, we formulate a unitality condition for the pair $(\omega, s)$. We shall denote by $\text{tw}^\pi \mathcal{C}$ the triangulated split-closure of $\mathcal{C}$. It is well-known that the embedding $\iota : \mathcal{C} \hookrightarrow \text{tw}^\pi \mathcal{C}$ induces isomorphisms on Hochschild and cyclic homologies, and hence we can identify extended Calabi-Yau structures $\mathcal{C}$ with those in $\text{tw}^\pi \mathcal{C}$. In the following, we shall make use of this identification freely.

An extended Calabi-Yau structure $(\omega, s)$ is called unital if there exists a split-generating subcategory $\mathcal{A} \subset \text{tw}^\pi \mathcal{C}$ such that

\[
\langle 1_X, s(\omega) \rangle_{\text{hres}} \in K, \quad \forall X \in \mathcal{A},
\]

where $1_X$ in the identity morphism of an object $X$ in the category $\mathcal{A}$, and $\langle -, - \rangle_{\text{hres}}$ denotes the higher residue pairing (see Subsection 3.3).

Then we show that given an unital pair $(\omega, s) \in \mathcal{M}_C^d$, the split-generating subcategory $\mathcal{A} \subset \text{tw}^\pi \mathcal{C}$ admits a cyclic, unital model denoted by $\mathcal{A}'$ to which we can apply the CEI construction to define a function

\[
F_{g,n}^C : \mathcal{M}_C^d \times HH_\bullet(\mathcal{C})[d][[u]]^\otimes_n \rightarrow K
\]

for each pair of integers $(g, n)$ such that $2 - 2g - n < 0$. 

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Our first main result is that this construction is independent of the choice of $\mathcal{A}$ and its cyclic unital model $\mathcal{A}'$, see Section 5.5 for details.

In order to formulate the Morita invariance of this construction first recall [37] that two $A_\infty$-categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if and only if $\text{tw}^\pi \mathcal{C}$ and $\text{tw}^\pi \mathcal{D}$ are quasi-equivalent.

**Theorem 1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathbb{Z}/2\mathbb{Z}$-graded $A_\infty$-category over a field $K$ of characteristic zero. Assume that they are both small categories and satisfy Condition (††). Assume $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent through a quasi-equivalence $f : \text{tw}^\pi \mathcal{C} \rightarrow \text{tw}^\pi \mathcal{D}$. Then there exists a naturally defined push-forward map

$$(f)_* : \mathcal{M}^d_\mathcal{C} \times HH_\bullet(\mathcal{C})[[u]]^{\otimes n} \rightarrow \mathcal{M}^d_\mathcal{D} \times HH_\bullet(\mathcal{D})[[u]]^{\otimes n}, \forall d \in \mathbb{Z}/2\mathbb{Z},$$

such that

$$F^c_{g,n} = F^D_{g,n} \circ (f)_*, \forall (g, n), 2 - 2g - n < 0.$$
when $|\omega|$ is even, not all strong Calabi-Yau structures determine unital, cyclic models, which is why we need to restrict to unital splittings to ensure that $s(\omega)$ determines an unital cyclic model $\mathcal{A}'$. This model is then unique up to unital cyclic $A_\infty$-morphisms, see Proposition 3.13. On the contrary, in the odd case, we always have existence, but the uniqueness of $\mathcal{A}'$ fails to hold - see Example 3.11. Then we use the CEI of $\mathcal{A}'$ to define the function $F_{g,n}^C$ applied to the pair $(\omega, s) \in \mathcal{M}_d$.

It is rather non-trivial to prove that the definition of $F_{g,n}^C$ is independent of the choice of the cyclic model $\mathcal{A}'$. We first show that CEI are invariant under cyclic $A_\infty$-isomorphism. In order to do this we prove that a cyclic $A_\infty$-isomorphism is equivalent to a cyclic pseudo-isotopy - a notion introduced by Fukaya [17]. And then show that CEI can be defined in families which allows us to prove that they are invariant under pseudo-isotopies. This is accomplished in Theorem 5.5. In the even case, this is enough to show that $F_{g,n}^C$ are independent of the model, since $\mathcal{A}'$ is unique up to unital, cyclic isomorphism. In the odd case the cyclic model is not unique. To get around this problem, in Appendix B we prove that CEI is invariant under tensoring with the Clifford algebra $\text{Cl} = \mathbb{K}[\epsilon]$. Thus, by applying this tensor trick, we can reduce the odd parity case to the even case. Once we prove that $F_{g,n}^C$ is well-defined, its Morita invariance follows relatively easily, see Subsection 5.5.

1.3 Organization of the paper

Section 2 is mainly devoted to prove that a Calabi-Yau category (i.e. an $A_\infty$-category endowed with a strong Calabi-Yau structure) admits a cyclic model which is unique up to $A_\infty$ quasi-isomorphisms. Section 3 is the unital version of Section 2. Adding the unitality condition requires to introduce a new version of cyclic homology.

After briefly reviewing the definition of CEI in Section 4, we proceed to formulate and prove their Morita invariance in Section 5. In Section 6, we present some basic examples and applications, as well as some conjectures in the case of Fukaya categories and derived categories of coherent sheaves.

Finally, Appendix A deals with sign diagrams that appear in different places of the paper. In Appendix B, we prove that CEI is invariant under tensoring with the Clifford algebra $\text{Cl} = \mathbb{K}[\epsilon]$. This tensor trick is used to formulate the Morita invariance of CEI of odd Calabi-Yau categories, thus bypassing the non-uniqueness of unital cyclic models in this case.

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1.5 Conventions and Notations.

We shall always work over a field $\mathbb{K}$ of characteristic zero. Given a $\mathbb{Z}/2$-graded vector space $V$ and an element $v \in V$, we denote by $|v|$ the degree of $v$ and by $|v|'$ its shifted degree. That is $|v|' = |v| - 1$.

For the sign convention of $A_\infty$ multiplications or $A_\infty$ homomorphisms, unless otherwise stated, we shall always use the shifted sign convention.

For an $A_\infty$-category $C$ over $\mathbb{K}$, we use the notation $\text{hom}_C(-,-)$ for the morphism space of $C$, while the notation $\text{Hom}(-,-)$ for the space of $\mathbb{K}$-linear maps.

We use bold letters to denote elements $x = x_1 \otimes \ldots \otimes x_n$. Furthermore, we use Sweedler notation for the coproduct on a tensor coalgebra, namely:

$$\sum x^{(1)} \otimes x^{(2)} = \sum_{i=0}^{n} (x_1 \otimes \ldots \otimes x_i) \otimes (x_{i+1} \otimes \ldots \otimes x_n).$$

2 Calabi–Yau $A_\infty$-categories

2.1 $A_\infty$-bimodules and Hochschild invariants

In this subsection we recall some basic definitions of $A_\infty$-bimodules, Hochschild (co)homology and related constructions.

Let $C$ be an $A_\infty$-category. We use the following notation

$$C(X_0, X_1, \ldots, X_n) := \text{hom}_C(X_0, X_1)[1] \otimes \cdots \otimes \text{hom}_C(X_{n-1}, X_n)[1],$$

with the convention that $C(X_0, X_1, \ldots, X_n) = \mathbb{K}$, when $n = 0$.

**Definition 2.1.** Hochschild cochains of length $n$ are defined as:

$$CC^\bullet(C)^n = \prod_{X_0, \ldots, X_n} \text{Hom}^\bullet(C(X_0, X_1, \ldots, X_n), C(X_0, X_n))[-1].$$

The Hochschild cochain complex is defined as $CC^\bullet(C) = \prod_{n \geq 0} CC^\bullet(C)^n$. A Hochschild cochain $\varphi = \prod_n \varphi_n$ is said to be of order $n$ if $\varphi_0 = \ldots = \varphi_{n-1} = 0$.

There is a normalized (or reduced) version of the Hochschild complex $CC_{\text{red}}^\bullet(C)$. It is defined as the subspace of Hochschild cochains $\varphi$ satisfying $\varphi_n(\ldots, 1_{X_i}, \ldots) = 0$ for all $X_i$ and $n \geq 1$. On the cohomology level, this brings nothing new as the inclusion map $CC_{\text{red}}^\bullet(C) \to CC^\bullet(C)$ is a quasi-isomorphism, see [28], [13].

On the Hochschild cochain complex one defines the Gerstenhaber product:

$$\varphi \circ \psi(x) = \sum (-1)^{\delta(\varphi)} \varphi(x^{(1)}, \psi(x^{(2)})) \psi(x^{(3)}).$$
and the corresponding bracket $[\varphi, \psi] := \varphi \circ \psi - (-1)^{[\varphi][\psi]} \psi \circ \varphi$. The differential on $CC^\bullet(C)$ is defined as $\delta(-) := [m, -]$, where $m = \prod_n m_n$ are the $A_\infty$-operations. The corresponding cohomology is called the Hochschild cohomology of $C$ and is denoted by $HH^\bullet(C)$.

It is elementary to check that $CC^\bullet_{red}(C)$ is closed under the Gerstenhaber product and a subcomplex of $CC^\bullet(C)$, assuming $C$ is strictly unital.

One can reinterpret Hochschild cochains as vector fields in the following way: first define

$$BC := \bigoplus_{n \geq 0} C(X_0, \ldots, X_n).$$

As explained in [26, Example 2.1.6], $BC$ is naturally a coalgebra. A Hochschild cochain $\varphi$ can be extended uniquely to a coderivation of this coalgebra by the formula

$$\hat{\varphi}(x) := \sum_n \sum \left((-1)^{\ast \varphi'} x \otimes \varphi_n(x^{(2)}) \otimes x^{(3)}\right).$$

An easy computation shows that $[\varphi, \psi] = [\hat{\varphi}, \hat{\psi}]$.

**Remark 2.2.** In fact, one can check that any coderivation of $BC$ with the following support condition

$$\hat{\varphi}(C(X_0, \ldots, X_n)) \subset \bigoplus_{0 \leq i_1 \leq \ldots \leq i_k \leq n} C(X_0, X_{i_1}) \otimes C(X_{i_1}, X_{i_2}) \otimes \cdots \otimes C(X_{i_k}, X_n)$$

is uniquely determined by a Hochschild cochain by the formula above.

**Definition 2.3.** The Hochschild complex of $C$ is defined as the vector space

$$CC^\bullet(C) = \bigoplus_{X_0, \ldots, X_n} C(X_n, X_0) \otimes C(X_0, X_1, \ldots, X_n)[-1],$$

with differential

$$b(x_0 \otimes x) = \sum (-1)^{\ast x_0} x_0 \otimes x^{(1)} \otimes m(x^{(2)}) \otimes x^{(3)} + \sum (-1)^{\ast x_0} m(x^{(3)}, x_0, x^{(1)}) \otimes x^{(2)}$$

where $\ast = |x_0|' + |x^{(1)}|'$ and $@ = (|x_0|' + |x^{(1)}|' + |x^{(2)}|')|x^{(3)}|'$.

As for cochains, there is a reduced Hochschild chain complex $CC^\bullet_{red}(C)$. It is defined as the quotient of $CC^\bullet(C)$ by the subcomplex spanned by chains of the form $x_0 \otimes x_1 \cdots \otimes x_i \otimes 1_{X_i} \cdots \otimes x_n$. Once again the natural map $CC^\bullet(C) \to CC^\bullet_{red}(C)$ is a quasi-isomorphism [28, 13].

We also recall the cyclic complex of $C$. We introduce two additional operators $b', t : CC^\bullet(C) \to CC^\bullet(C)$. Let $t$ be the map (of degree $0$) defined as

$$t(x_0 \otimes \cdots \otimes x_n) = (-1)^{\ast x_n} x_n \otimes x_0 \otimes \cdots \otimes x_{n-1}.$$
The map $b'$, of degree 1, is defined as
\[ b'(x_0 \otimes x) = \sum m(x_0 \otimes x^{(1)}) \otimes x^{(2)} + \sum (1)^* x_0 \otimes x^{(1)} m(x^{(2)}) \otimes x^{(3)}. \]

An important fact about this map is that it is a differential, that is $(b')^2 = 0$.

**Definition 2.4.** Let $C^{\text{bar}}$ be the complex whose underlying vector space is $CC_\bullet(C)$ equipped with the differential $b'$. This is called the bar complex of $C$.

The two maps above satisfy the following relation
\[ (\text{id} - t)b' = b(\text{id} - t). \]
This is a straightforward check. For the case of algebras see [28].

**Definition 2.5.** Let $C^{\lambda}_\bullet(C) = CC_\bullet(C)/\text{Im}(\text{id} - t)$. The relation in Equation (4) implies that the differential $b$ induces a differential on $C^\lambda_\bullet(C)$. We will refer to the resulting complex as the (positive) cyclic complex of $C$, it is also sometimes called the Connes’ complex. Its homology is called the cyclic homology of $C$, $HC^\lambda_\bullet(C)$.

**Remark 2.6.** It is often convenient to use different chain models for the cyclic homology of $C$, sometimes also referred to as positive cyclic homology. We will make use of the $u$-model, $(CC^{\text{red}}_\bullet(C)[u^{-1}], b + uB)$ where $B$ is the Connes’ differential (see [2] for example). It is proved in [28] (for algebras) and [13] (in the $A_\infty$ case) that the homology $H_\bullet(CC^{\text{red}}_\bullet(C)[u^{-1}], b + uB)$, usually denoted by $HC^+_\bullet(C)$, is isomorphic to $HC^\lambda_\bullet(C)$.

Let $C$ and $D$ be $A_\infty$-categories. An $A_\infty$-pre-functor $F : C \to D$ consists of a map on objects and a sequence of maps
\[ F_n : C(X_0, \ldots, X_n) \to D(FX_0, FX_n). \]
We say $F$ is unital if $F_1(1_{X_1}) = 1_{FX_1}$ and $F_n(\ldots, 1_{X_1}, \ldots) = 0$ for all $n \geq 2$.

These can be extended to a coalgebra homomorphism $\hat{F} : BC \to BD$, by the formula
\[ \hat{F}(x) = \sum F(x^{(1)}) \otimes F(x^{(2)}) \otimes \cdots \otimes F(x^{(k)}). \]

$F$ is called an $A_\infty$-functor if $\hat{F} \circ \hat{m}_C = \hat{m}_D \circ \hat{F}$. $A_\infty$-functors can be composed ([32, Section 1]) and in the case when a functor $F$ is a bijection on objects and all the $F_1$ maps are linear isomorphisms it admits a strict inverse, which we denote by $F^{-1}$.

An $A_\infty$-functor $F$ induces a chain map $F_* : CC_\bullet(C) \to CC_\bullet(D)$ defined as follows
\[ F_*(x_0 \otimes x) = \sum (-1)^k F(x^{(k+1)}, x_0, x^{(1)}) \otimes F(x^{(2)}) \otimes \cdots \otimes F(x^{(k)}). \]
We remark that when \( F \) is unital, this formula determines a chain map between the reduced complexes as well. There is a similar induced chain map between the bar complexes, \( F_*: C^{\text{bar}} \to D^{\text{bar}} \) defined as

\[
F'_*(x_0 \otimes x) = F(x_0, x^{(1)}) \otimes F(x^{(2)}) \otimes \cdots \otimes F(x^{(k)})
\]

The two chain maps are related by the equation

\[
(id - t)F'_* = F_*(id - t).
\]

In particular, this relation implies that \( F_* \) induces a chain map between the cyclic complexes \( C^\bullet_\Lambda(\mathcal{C}) \) and \( C^\bullet_\Lambda(\mathcal{D}) \).

We now recall some facts about \( A_\infty \)-bimodules following [32, 33].

**Definition 2.7.** An \( A_\infty \)-bimodule over \( \mathcal{C} \) (or \( \mathcal{C} - \mathcal{C} \) bimodule) consists of a graded vector space \( \mathcal{P}(X, Y) \) for each pair of objects \( X, Y \) in \( \mathcal{C} \) and a family of maps

\[
n_{r,s} : \mathcal{C}(X_0, \ldots, X_r) \otimes \mathcal{P}(X_r, Y_0) \otimes \mathcal{C}(Y_0, \ldots, Y_s) \to \mathcal{P}(X_0, Y_s)[1],
\]

for objects \( X_0, \ldots, X_r, Y_0, \ldots, Y_s \), satisfying

\[
\begin{align*}
\sum (-1)^s n_{r,s}(x^{(1)}, m_i(x^{(2)}), x^{(3)}, p, y) \\
+ \sum (-1)^s n_{r,s}(x, p, y^{(1)}, m_i(y^{(2)}), y^{(3)}) \\
+ \sum (-1)^s n_{r,s}(x^{(1)}, n_{r,s}(x^{(2)}, p, y^{(1)}), y^{(2)}) = 0,
\end{align*}
\]

where \( * \) is the sum of the degrees to the left of the second multilinear map. For example, in the second line above, \( * = |x_1| + \ldots + |x_r| + |y_1| + \ldots + |y_j| + |p| \).

\( A_\infty \)-bimodules form a dg-category \( [\mathcal{C}, \mathcal{C}] \) (see [32]). Given \( A_\infty \)-bimodules \( \mathcal{P} \) and \( \mathcal{Q} \) an element \( \rho \in hom^k_{[\mathcal{C}, \mathcal{C}]}(\mathcal{P}, \mathcal{Q}) \) of degree \( k \), called a pre-homomorphism, consists of maps

\[
\rho_{r,s} : \mathcal{C}(X_0, \ldots, X_r) \otimes \mathcal{P}(X_r, Y_0) \otimes \mathcal{C}(Y_0, \ldots, Y_s) \to \mathcal{Q}(X_0, Y_s)[k].
\]

The differential in \( hom^*_{[\mathcal{C}, \mathcal{C}]}(\mathcal{P}, \mathcal{Q}) \) is defined as

\[
(\partial \rho)_{r,s}(x, p, y) = \sum (-1)^{\rho} n_{r_1, s_1}(x^{(1)}, \rho_{r_2, s_2}(x^{(2)}, p, y^{(1)}), y^{(2)})
\]

\[
+ \sum (-1)^{1+\rho+\rho_{r_1, s_1}} n_{r_2, s_2}(x^{(1)}, \rho_{r_1, s_1}(x^{(2)}, p, y^{(1)}), y^{(2)})
\]

\[
+ \sum (-1)^{1+\rho+\rho_{r_1, s_1}} n_{r_3, s_3}(x^{(1)}, m_i(x^{(2)}), p, y)
\]

\[
+ \sum (-1)^{1+\rho+\rho_{r_1, s_1}} n_{r_4, s_4}(x^{(1)}, m_i(x^{(2)}), y^{(1)}, y^{(3)}).
\]

**Definition 2.8.** Given \( A_\infty \)-bimodules \( \mathcal{P} \) and \( \mathcal{Q} \), a pre-homomorphism \( \rho \in hom^*_{[\mathcal{C}, \mathcal{C}]}(\mathcal{P}, \mathcal{Q}) \) with \( \partial \rho = 0 \) is called a homomorphism. A homomorphism is called a quasi-isomorphism if \( \rho_{0,0} : \mathcal{P}(X, Y) \to \mathcal{Q}(X, Y) \) induces an isomorphism on cohomology for all \( X, Y \).
A standard fact of the $A_\infty$ world is that any quasi-isomorphism has a homotopy inverse, see for example [32]. Now we define the two bimodules we are mostly interested in.

**Definition 2.9.** Let $C_{sd}$ be the bimodule defined as $C_{sd}(X, Y) := \text{hom}_C(X, Y)[1]$, with operations

$$n_{r,s}(x, p, y) := m_{r+s+1}(x, p, y).$$

We will refer to $C_{sd}$ as the (shifted) diagonal bimodule.

**Definition 2.10.** Let $C_{sd'}$ be the bimodule defined as $C_{sd'}(X, Y) := \text{hom}_C(Y, X)^\lor[1]$, where $\lor$ stands for the $\mathbb{K}$-linear dual. We define the operations

$$n_{r,s}(x, \pi, y)(p) := (-1)^{|x|' + |y|' + |p|'} \pi(m_{r+s+1}(y, p, x)).$$

We will refer to $C_{sd'}$ as the (shifted) dual diagonal bimodule.

In order to define Calabi–Yau structure we need the following lemma. Denote by $CC_\bullet(C)^\lor$ the linear dual to $CC_\bullet(C)$ equipped with the dual differential $b^\lor(\varphi)(x) = (-1)^{|\varphi|'} \varphi(b(x))$.

**Lemma 2.11.** There is a quasi-isomorphism of chain complexes

$$\Psi : CC_\bullet(C)^\lor \to \text{hom}_{[C, C]}(C_{sd}, C_{sd'}),$$

given explicitly as

$$\Psi(\varphi)(x, y)(w) = \sum (-1)^{|\varphi|} \varphi(m(y^{(3)}, w, x, y^{(1)}), y^{(2)})$$

**Proof.** It follows from [33, Formula (2.27)] that $\text{hom}_{[C, C]}(C_{sd}, C_{sd'})$ is canonically isomorphic to the dual of $CC_\bullet(C, C_d \otimes_C C_d)$, the Hochschild complex of the bimodule $C_d \otimes_C C_d$, where $C_d$ is the unshifted diagonal bimodule and $\otimes_C$ and is the tensor product of bimodules as defined in [32]. The bimodule $C_d \otimes_C C_d$ is naturally quasi-isomorphic to $C_d$, see [32, Formula (2.21)]. This induces a quasi-isomorphism on Hochschild complexes $CC_\bullet(C, C_d \otimes_C C_d) \simeq CC_\bullet(C, C_d)$, where $CC_\bullet(C, C_d)$ is by definition the complex $CC_\bullet(C)$ from Definition 2.3. Tracing through these quasi-isomorphisms one obtains the map $\Psi$ in the statement. \hfill \Box

**Definition 2.12.** An element $\phi \in HH_\bullet(C)^\lor$ is called non-degenerate if $\Psi(\phi)$ is a quasi-isomorphism.

A **weak Calabi-Yau** structure on $C$ is a non-degenerate element $\phi \in HH_\bullet(C)^\lor$.

A **strong Calabi-Yau** structure on $C$ is an element $\tilde{\phi} \in HC_\bullet(C)^\lor$, such that $\tilde{\phi} \circ \pi_* \in HH_\bullet(C)^\lor$ is non-degenerate, where $\pi_*$ is induced by the natural projection $\pi : CC_\bullet(C) \to C^\lambda(C)$. We call the pair $(C, \tilde{\phi})$ a Calabi-Yau $A_\infty$-category.
2.2 From Calabi–Yau to homotopy cyclic

In this subsection we give an alternative description of strong Calabi-Yau structures. This is essentially a reinterpretation of a result from [26] for $A_\infty$-algebras. The main difference is that instead of using the language of non-commutative geometry we work with the more algebraic notion of a strong homotopy inner product [12].

Lemma 2.13. There is a degree one chain map $S : CC^\bullet(C) \to \text{hom}^\bullet_{[C,C]}(C_{sd},C_{sd}^\vee)$ defined by the formula

$$(S\varphi)(x, v, y)(w) = (-1)^\theta \varphi(v, y, w, x) - (-1)^\theta \varphi(w, x, v, y),$$

where the symbol $\@$ stands for the sign obtained, following the Koszul convention for the shifted degrees, from rotating the inputs of the expression from their original order.

Proof. This a long but straightforward calculation that we omit.

Remark 2.14. The proof of Lemma 2.13 does not use the $A_\infty$ relations. Therefore, one can prove that the map $S$ commutes with Lie derivatives $L_{\vee}Z$ and $L_Z$, introduced in Definitions 2.28 and 2.27, for any Hochschild cochain $Z$. When $Z = m$, this is exactly the content of Lemma 2.13.

Definition 2.15. Given $\rho \in \text{hom}^\bullet_{[C,C]}(C_{sd},C_{sd}^\vee)$, we say

- $\rho$ is anti-symmetric if

$$\rho(x, v, y)(w) = -(-1)^\@ \rho(y, w, x)(v).$$

- $\rho$ is closed if for any $x_0 \otimes \cdots \otimes x_n \in C(X_n, X_0, X_1, \ldots, X_n)$ and $0 \leq i < j < k \leq n$ we have

$$(-1)^{\epsilon_i} \rho(\ldots, x_{i}, \ldots) (x_j) + (-1)^{\epsilon_k} \rho(\ldots, x_j, \ldots)(x_k) + (-1)^{\epsilon_i} \rho(\ldots, x_k, \ldots)(x_i) = 0,$$

where $\epsilon_l = (|x_0|' + \ldots |x_l|')(|x_{l+1}|' + \ldots + |x_n|').$

Denote by $\Omega^{2,cl}(C)$ the subspace of $\text{hom}^\bullet_{[C,C]}(C_{sd},C_{sd}^\vee)$ consisting of closed, anti-symmetric pre-homomorphisms.

Lemma 2.16. $\Omega^{2,cl}(C)$ is a subcomplex of $\text{hom}^\bullet_{[C,C]}(C_{sd},C_{sd}^\vee)$. Moreover $\Omega^{2,cl}(C)$ coincides with the image of the map $S$.

Proof. We first show that the image of $S$ equals $\Omega^{2,cl}(C)$. Since $S$ is a chain map, this implies $\Omega^{2,cl}(C)$ is a subcomplex.
A direct calculation shows that the image of $S$ is contained in $\Omega^{2,cl}(\mathcal{C})$. Next given $\rho \in \text{hom}^\bullet_{[\mathcal{C},\mathcal{C}]}(\mathcal{C}_{sd}, \mathcal{C}_{sd}^\vee)$ define $h(\rho) \in CC^\bullet(\mathcal{C})^\vee$ as follows

\begin{equation}
(11)
 h(\rho)(x_0 \otimes x_1 \cdots \otimes x_n) = \sum_{i=1}^{n} \frac{(-1)^i}{n+1} \rho(\ldots, x_0, \ldots)(x_i),
\end{equation}

for $n \geq 1$ and $h(\rho)(x_0) = 0$. We claim that if $\rho \in \Omega^{2,cl}(\mathcal{C})$ we have $S(h(\rho)) = \rho$, which gives the desired result. For this we compute

\[
S(h(\rho))(x, v, y)(w) = \frac{1}{r + s + 2} \left( \rho(x, v, y)(w) + \sum (-1)^i \rho(x^{(3)}, v, y, w, x^{(1)})(x_i) + \right.
\]
\[
+ \sum (-1)^i \rho(y^{(3)}, w, x, v, y^{(1)})(y_j) - (-1)^i \rho(y, w, x)(v)
\]
\[
- \sum (-1)^i \rho(y^{(3)}, w, x, v, y^{(1)})(y_j) - (-1)^i \rho(x^{(3)}, v, y, w, x^{(1)})(x_i) \right)
\]
\[
= \frac{1}{r + s + 2} \left( \rho(x, v, y)(w) - \sum (-1)^i \rho(y, w, x^{(1)}, x_i, x^{(3)})(v)
\]
\[
+ \sum (-1)^i \sum \rho(y^{(3)}, w, x, v, y^{(1)})(y_j) + \rho(x, v, y)(w)
\]
\[
+ \sum (-1)^i \rho(x, v, y^{(1)}, y_j, y^{(3)})(w) - \sum (-1)^i \rho(x^{(3)}, v, y, w, x^{(1)})(x_i) \right)
\]
\[
= \frac{1}{r + s + 2} \left( 2\rho(x, v, y)(w) - \sum (-1)^i \rho(y^{(1)}, y_j, y^{(3)}, w, x)(v)
\]
\[
+ \sum \rho(x^{(1)}, x_i, x^{(3)}, v, y)(w) \right)
\]
\[
= \frac{1}{r + s + 2} \left( 2\rho(x, v, y)(w) + s\rho(x, v, y)(w) + r\rho(x, v, y)(w) \right)
\]
\[
= \rho(x, v, y)(w).
\]

Here, on the second and fourth equality we used anti-symmetry of $\rho$. On the third equality, we used closedness of $\rho$ to combine the third and fifth terms, and the second and sixth.

We now identify the kernel of the map $S$. Recall from its definition, there is a natural map of complexes

$$
\pi : CC^\bullet(\mathcal{C}) \to C^\lambda_{\bullet}(\mathcal{C}).
$$

We denote by $\mathfrak{K}_C$ the kernel of $\pi$ equipped with the induced differential.

**Proposition 2.17.** We have the following short exact sequence of complexes

$$
0 \to C^\lambda_{\bullet}(\mathcal{C})^\vee \xrightarrow{\pi^\vee} CC^\bullet(\mathcal{C})^\vee \xrightarrow{S} \Omega^{2,cl}(\mathcal{C})[1] \to 0.
$$
Proof. Lemmas 2.13 and 2.16 imply the map $S$ is a surjective chain map. From its definition, it is clear that the kernel of $S$ consists of $\varphi \in CC_\bullet(C)^\vee$ satisfying

$$\varphi(x_0 \otimes \cdots \otimes x_n) = (-1)^n \varphi(x_n \otimes x_0 \otimes \cdots \otimes x_{n-1}).$$

But this is exactly the image of $\pi^\vee$.

This short exact sequence is functorial in the following sense.

Proposition 2.18. Let $F : C \to D$ be an $A_\infty$-functor. We have the following commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & C^\bullet(C)^\vee & \xrightarrow{\pi^\vee} & CC_\bullet(C)^\vee & \xrightarrow{S} & \Omega^{cl}(C)[1] & \longrightarrow & 0 \\
& & \downarrow{F_*} & & \downarrow{F_*} & & \downarrow{F_*} & & \\
0 & \longrightarrow & C^\bullet(D)^\vee & \xrightarrow{\pi^\vee} & CC_\bullet(D)^\vee & \xrightarrow{S} & \Omega^{cl}(D)[1] & \longrightarrow & 0
\end{array}
$$

Here $F^*$ in the first two vertical arrows is just the dual to $F_*$ in (6). The third map $F^*$ is a chain map defined as follows:

$$F^*\rho(x, v, y)(w) = \sum (-1)^{\hat{\alpha}} \rho(\hat{F}(\hat{x}^{(2)}), \hat{F}(\hat{x}^{(3)}, v, \hat{y}^{(1)}), \hat{F}(\hat{y}^{(2)})) F(\hat{y}^{(3)}, w, \hat{x}^{(1)})$$

Proof. Commutativity of the left square is obvious, as the left most vertical arrow is induced by the middle one (Equation 7).

It is straightforward to check that the square on the right commutes once we observe that (by definition) $|F_k(x_1, \ldots, x_k)|' = |x_1'| + \ldots + |x_k'|$. Finally one can check that $F^*$ on the right is a chain map, directly using the $A_\infty$ functor equation for $F$. Alternatively, one can use the fact that $S$ is surjective and that all the other sides of the commutative square are chain maps.

Next we describe $\mathfrak{K}_C$.

Lemma 2.19. Let $p : \mathcal{C}^{\text{bar}} \to \mathfrak{K}_C$ be the map $p = \text{id} - \alpha$ and let $N : C^\bullet(C) \to \mathcal{C}^\text{bar}$ be the map $N(x_0 \otimes \cdots \otimes x_n) = \sum_{k=0}^{n} t^k(x_0 \otimes \cdots \otimes x_n)$.

The maps $p$ and $N$ are well defined chain maps.

Proof. By definition, $\mathfrak{K}_C = \ker(\pi)$ is the image of $\text{id} - \alpha$, so the map $p$ is well defined. The fact that $p$ is a chain map is exactly Equation (4).

Well-definedness of $N$ follows from the simple observation $N \circ \alpha = N$. The condition $N \circ b = b' \circ N$ can be checked directly, see [28] for the case of associative algebras.

Proposition 2.20. We have the following short exact sequence of complexes

$$0 \to C^\bullet(C) \xrightarrow{N} \mathcal{C}^{\text{bar}} \xrightarrow{p} \mathfrak{K}_C \to 0$$
Proof. As observed before $p \circ N = 0$. Also, by definition of $\mathfrak{R}_C$, $p$ is surjective. Given $x \in \ker(p)$, we have $t(x) = x$. Therefore $x = N((\frac{1}{n+1}[x])$.

All that is left to show is that $N$ is injective. If $N([x]) = 0$, then $\sum_{k=1}^{n} t^k(x) = -x$. We define $\gamma := -\frac{1}{n+1} \sum_{k=1}^{n} k t^k(x)$, then

$$(\text{id} - t)(\gamma) = -\frac{1}{n+1} \left( \sum_{k=1}^{n} t^k(x) - nx \right) = -\frac{1}{n+1} \left( -(n + 1)x \right) = x.$$ 

Hence $[x] = 0 \in C^\wedge(C)$.

This short exact sequence is also functorial.

**Proposition 2.21.** Let $F : C \to D$ be an $A_{\infty}$-functor. We have the following commutative diagram

$$
\begin{array}{ccll}
0 & \longrightarrow & C^\wedge(C) & \longrightarrow & \bar{C}^* \longrightarrow & \mathfrak{R}_C & \longrightarrow & 0 \\
0 & \longrightarrow & C^\wedge(D) & \longrightarrow & \bar{D}^* \longrightarrow & \mathfrak{R}_D & \longrightarrow & 0
\end{array}
$$

**Proof.** Commutativity of the square on left is easy to check once we note $F_k$ has degree $1-k$. The square on the right is exactly Equation (7).

**Lemma 2.22.** Assume $C$ is unital. Then $\bar{C}^*$ is an acyclic complex.

**Proof.** There is an explicit contracting homotopy $h : \bar{C}^* \to \bar{C}^*$. Given $x_0 \otimes \cdots x_n \in C(X_n, X_0, \ldots, X_n)$ one defines $h(x_0 \otimes \cdots \otimes x_n) = 1_{X_n} \otimes x_0 \otimes \cdots \otimes x_n$. One easily checks that $hb + b'h = \text{id}_{\bar{C}^*}$.

Before we state the main result of this subsection, we introduce some terminology.

**Definition 2.23.** A strong homotopy inner product on $C$, is a quasi-isomorphism $\rho \in \Omega^{2,cl}(C) \subseteq \text{hom}_{[C,C]}^*(C_{sd}, C_{sd}^\vee)$.

When $C$ is minimal and $\rho$ is a constant strong homotopy inner product, meaning $\rho_{r,s} = 0$ for $r, s > 0$, then

$$\rho_{0,0} := \langle -, - \rangle : \text{hom}_C(X,Y) \otimes \text{hom}_C(Y,X) \to \mathbb{K}$$

defines a symmetric, non-degenerate pairing. Moreover, $\partial \rho = 0$ is equivalent to the following equation

$$\langle x_0, m_n(x_1, \ldots, x_n) \rangle = (-1)^{(n-1)} \langle x_1, m_n(x_2, \ldots, x_n, x_0) \rangle.$$ 

In this case, $C$ together with the pairing $\langle -, - \rangle$ is called a cyclic $A_{\infty}$-category.

Given two cyclic $A_{\infty}$-categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is called cyclic if $F^* \langle -, - \rangle_{\mathcal{C}} = \langle -, - \rangle_{\mathcal{D}}$. This condition is equivalent to the following equations

$$\langle F_1(x), F_1(y) \rangle \in (x, y)_{\mathcal{C}}; \quad \sum_{i=1}^{n} \langle F_i(x_1, \ldots, x_i), F_{n-i}(x_{i+1}, \ldots, x_n) \rangle = 0, n \geq 3.$$ 

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Corollary 2.24. The complexes $\Omega^{2,cl}(C)$ and $C^\lambda_\bullet(C)^\vee$ are quasi-isomorphic. This quasi-isomorphism is functorial with respect to $A_\infty$-functors. Moreover, it identifies strong Calabi–Yau structures with strong homotopy inner products.

Proof. Proposition 2.17 gives the isomorphism

$$\Omega^{2,cl}(C)[1] \cong CC_\bullet(C)^\vee / C^\lambda_\bullet(C)^\vee \cong \mathcal{K}_C^\vee.$$ 

Proposition 2.20, together with $C^\text{bar}$ being acyclic, implies that $\mathcal{K}_C$ and $C^\lambda_\bullet(C)[1]$ are quasi-isomorphic. Combining these we obtain the first statement, and therefore, on cohomology level we obtain an isomorphism

$$HC^\lambda_\bullet(C)^\vee \cong H^\bullet(\Omega^{2,cl}(C), \partial).$$

Let $\phi \in HC^\lambda_\bullet(C)^\vee$ and $\rho \in H^\bullet(\Omega^{2,cl}(C), \partial)$ be related by the above isomorphism. We claim that $\phi$ is a Calabi-Yau structure if and only if $\rho$ is a quasi-isomorphism of bimodules. We assume, without loss of generality that $C$ is a minimal $A_\infty$-category. Recall $\phi$ is Calabi-Yau when $\Psi(\phi \circ \pi_\ast)$ is a quasi-isomorphism, which is equivalent to the pairing

$$\Psi(\phi \circ \pi_\ast)(v) = (-1)^{\phi_0(m_2(w, v))},$$

being non-degenerate as a map $\text{hom}_C(X, Y) \otimes \text{hom}_C(Y, X) \to \mathbb{K}$, for all objects $X, Y$. Similarly, $\rho$ is a quasi-isomorphism if the pairing $\rho_{0,0}(v)(w)$ is non-degenerate. Take $\theta \in CC_\bullet(C)^\vee$ with $S(\theta) = \rho$. Then $\rho_{0,0}(v)(w) = \theta_1(v \otimes w - (-1)^{\theta_0} w \otimes v)$. Diagram chasing one sees that $v \otimes w - (-1)^{\theta_0} w \otimes v \in \mathcal{K}_C$ is sent to $(-1)^{\theta_0} m_2(w, v) \in C^\lambda_\bullet(C)[1]$. Therefore under the above isomorphism

$$\rho_{0,0}(v)(w) = \theta_1(v \otimes w - (-1)^{\theta_0} w \otimes v) = (-1)^{\theta_0} \phi_0(m_2(w, v)).$$

Hence $\rho_{0,0}(v)(w) = \Psi(\phi \circ \pi_\ast)(0, 0)(v)(w)$, which proves the claim.

Finally, the statement on functoriality follows directly from Propositions 2.18 and 2.21. 

Example 2.25. Let $C$ be a cyclic $A_\infty$-category. We claim that the the cyclic structure $\rho := \langle -, - \rangle \in \Omega^{2,cl}(C)$ corresponds, by Corollary 2.24, to the strong Calabi–Yau structure $\Phi \in HC^\lambda_\bullet(C)^\vee$ defined as

$$\Phi([x_0 \otimes \cdots \otimes x_n]) = \begin{cases} 0 & n > 0 \\ \langle 1_{x_0}, x_0 \rangle & n = 0. \end{cases}$$

In order to see this first note that if we define $\theta \in CC_\bullet(C)^\vee$ as $\theta_1(x_0 \otimes x_1) := \frac{1}{2} \langle x_0, x_1 \rangle$, and all other entries zero, then $\theta$ determines a class on the cohomology on $\mathcal{K}_C^\vee$ and $\Psi(\theta) = \rho$. Diagram chasing one sees that the corresponding Calabi–Yau structure is given by

$$\Phi([x_0 \otimes \cdots \otimes x_n]) = \theta(\rho(1_{x_0} \otimes N(x_0 \otimes \cdots \otimes x_n))),$$

which by definition of $\theta$ is no-zero only when $n = 0$ in which case it equals $\theta(1_{x_0} \otimes x_0 - (-1)^{\phi_0} x_0 \otimes 1_{x_0}) = \langle 1_{x_0}, x_0 \rangle$, by anti-symmetry of the inner product.
2.3 From homotopy cyclic to cyclic

In this subsection we prove an analogue of the Darboux theorem in symplectic geometry for strong homotopy inner products. Namely we show that up to an $A_\infty$-isomorphism any strong homotopy inner product can be made constant. Therefore any $A_\infty$-category with a strong homotopy inner product has a cyclic model. We also study the uniqueness of this cyclic model.

From now on we assume that $\mathcal{C}$ is minimal. Recall, the homological perturbation lemma guarantees one can always achieve this. Together with our starting assumption that $\mathcal{C}$ is proper, this implies that all hom spaces in $\mathcal{C}$ are finite dimensional.

**Definition 2.26.** Let $\rho \in \text{hom}_{[\mathcal{C},\mathcal{C}]}(\mathcal{C}_{sd}, \mathcal{C}_{sd}^\vee)$ be a bimodule pre-homomorphism and $Z \in CC^\bullet(\mathcal{C})$ a Hochschild cochain. We define the contraction $\iota_Z \rho \in CC^\bullet(\mathcal{C})^\vee$ as follows

$$\iota_Z \rho(x_0 \otimes x_1 \cdots \otimes x_n) = \sum (-1)^{\tilde{\tau} + \tilde{\upsilon}} \rho(x_1, \ldots, Z(x_i, \ldots, x_j) \ldots x_n)(x_0).$$

where $\tilde{\tau} := |Z|(|\rho| + |x_1| + \ldots + |x_{i-1}|)$.

**Definition 2.27.** Let $Z \in CC^\bullet(\mathcal{C})$ a Hochschild cochain. We define the Lie derivative

$$\mathcal{L}_Z : \text{hom}_{[\mathcal{C},\mathcal{C}]}(\mathcal{C}_{sd}, \mathcal{C}_{sd}^\vee) \to \text{hom}_{[\mathcal{C},\mathcal{C}]}(\mathcal{C}_{sd}, \mathcal{C}_{sd}^\vee)$$

by the formula

$$(\mathcal{L}_Z \rho)_{r,s}(x, u, y)(w) = \sum (-1)^{\tilde{\tau} + \tilde{\upsilon} + |Z|(|\rho| + |y_1| + |v| + |x_2|)} \rho(x^{(2)}, u, y^{(1)})(Z(y^{(2)}, w, x^{(1)}))$$

$$+ \sum (-1)^{\tilde{\tau} + \tilde{\upsilon} + |Z|(|\rho| + |v| + |y_1| + |x_2|)} \rho_{r_1,s_1}(x^{(1)}, Z(x^{(2)}, v, y^{(1)}), y^{(2)})(w)$$

$$+ \sum (-1)^{\tilde{\tau} + \tilde{\upsilon} + |Z|(|\rho| + |v| + |x_1| + |x_3|)} \rho_{r_1,s_1}(x^{(1)}, Z(x^{(2)}, x^{(3)}, v, y), y^{(2)})(w)$$

$$+ \sum (-1)^{\tilde{\tau} + \tilde{\upsilon} + |Z|(|\rho| + |v| + |x_1| + |x_3|)} \rho_{r_1,s_1}(x, \overline{v}, y^{(1)}, Z(y^{(2)}, y^{(3)})(w))$$

Note that when $Z = \mathfrak{m}$ we have $\mathcal{L}_Z \rho = \partial \rho$.

**Definition 2.28.** Let $Z \in CC^\bullet(\mathcal{C})$ a Hochschild cochain. We define a Lie derivative map

$$\mathcal{L}_Z : CC^\bullet(\mathcal{C}) \to CC^\bullet(\mathcal{C})$$

by the formula

$$\mathcal{L}_Z(x_0 \otimes x) = \sum (-1)^{|x|} x_0 \otimes x^{(1)} \otimes Z(x^{(2)}) \otimes x^{(3)} + \sum (-1)^{|Z|} Z(x^{(3)}, x_0, x^{(1)} \otimes x^{(2)}$$

Please note that when $Z = \mathfrak{m}$, we have $\mathcal{L}_\mathfrak{m} = b$, the Hochschild differential.

**Lemma 2.29.** Let $\rho \in \Omega^{2,cl}(\mathcal{C})$ be a bimodule pre-homomorphism.
1. If \( \rho \) is a quasi-isomorphism then the map \( \iota_\rho : CC_\bullet(C) \to CC_\bullet(C)^{\vee} \) is an isomorphism.

2. For Hochschild cochains \( Y, Z \), we have the identity

\[
\iota[Z,Y]\rho = \iota_Z(\mathcal{L}_Y \rho) + (-1)^{|Z||Y|} \mathcal{L}_Y^{\vee}(\iota_Z \rho),
\]

where \( \mathcal{L}_Y^{\vee}(\psi)(x) = (-1)^{|\psi||Y|} \psi(\mathcal{L}_Y(x)) \). In particular when \( Y = m \) we have \( \iota_{(Z)\rho} = (-1)^{|Z|} \iota_Z(\partial \rho) - b^\vee(\iota_Z \rho) \).

3. For a Hochschild cochain \( Z \) we have \( \mathcal{L}_Z \rho = S(\iota_Z \rho) \). In particular \( \mathcal{L}_Z \rho \in \Omega^{2,cl}(C) \).

4. Let \( F : C \to C \) be an \( A_\infty \)-pre-functor such that each \( F_1 \) is an isomorphism and let \( F^{-1} \) be its inverse. Given a Hochschild cochain \( Z \) we define \( F^*Z := F^{-1} \circ \bar{Z} \circ \bar{F} \). We have \( F^*(\mathcal{L}_Z \rho) = \mathcal{L}_{F^*Z}(F^* \rho) \).

**Proof.** We have assumed that \( C \) is minimal, therefore \( \rho \) is a quasi-isomorphism if and only if

\[
\rho_{0,0} : \text{hom}_C(X,Y) \otimes \text{hom}_C(Y,X) \to \mathbb{K}
\]

is a non-degenerate pairing for all \( X, Y \). Take \( \varphi \in CC_\bullet(C)^{\vee} \), we will show by induction on length that there exists a unique \( Z \in CC_\bullet(C) \) satisfying \( \iota_Z \rho = \varphi \). Assume we have found \( Z_k \) for \( k < n \), in order to define \( Z_n \) one must solve

\[
\varphi(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = (-1)^{\alpha + \frac{n}{2}} \rho_{0,0}(Z_n(x_1, \ldots, x_n))(x_0)
\]

\[
+ \sum_{0 \leq k < n} (-1)^{\alpha + \frac{n}{2}} \rho(x_1, \ldots, Z_k(x_{i+1}, \ldots, x_{i+k}), \ldots x_n)(x_0).
\]

Since \( \rho_{0,0} \) is non-degenerate there is a unique \( Z_n(x_1, \ldots, x_n) \) solving this equation.

The second item in this lemma follows from a direct computation that we leave to
We refer to the interested reader. For the third item we compute

\[
S(t_z \rho)(x, y, z)(w) = \\
\sum (-1)^{\alpha + \iota} (y^1, Z(y^2, w, x^1), x^2)(v) - \sum (-1)^{\alpha + \iota} \rho(x^{(1)}, Z(x^2, v, y^1), y^{(2)}(w)) \\
+ \sum (-1)^{\alpha + \iota} \rho(y^1, Z(y^2), y^3, w, x)(v) - \sum (-1)^{\alpha + \iota} \rho(x^1, Z(x^2), x^3, v, y)(w) \\
+ \sum (-1)^{\alpha + \iota} \rho(y, w, x^1, Z(x^2), x^3)(v) - \sum (-1)^{\alpha + \iota} \rho(x, v, y^1, Z(y^2), y^3)(w) \\
= - \sum (-1)^{\alpha + \iota} \rho(x^2, v, y^1)(Z(y^2, w, x^1)) - \sum (-1)^{\alpha + \iota} \rho(x^{(1)}, Z(x^2, v, y^1), y^{(2)}(w)) \\
- \sum (-1)^{\alpha + \iota} \rho(y^3, w, x, v, y^1)(Z(y^2)) + \sum (-1)^{\alpha + \iota} \rho(x^3, v, y, w, x^1)(Z(x^2)) \\
+ \sum (-1)^{\alpha + \iota} \rho(y, w, x^1, Z(x^2), x^3)(v) - \sum (-1)^{\alpha + \iota} \rho(x, v, y^1, Z(y^2), y^3)(w) \\
= - \sum (-1)^{\alpha + \iota} \rho(x^2, v, y^1)(Z(y^2, w, x^1)) - \sum (-1)^{\alpha + \iota} \rho(x^{(1)}, Z(x^2, v, y^1), y^{(2)}(w)) \\
+ \sum (-1)^{\alpha + \iota} \rho(y^1, Z(y^2), y^3, w, x)(v) - \sum (-1)^{\alpha + \iota} \rho(x^1, Z(x^2), x^3, v, y)(w) \\
= \sum (-1)^{\alpha + \iota} \rho(x^2, v, y^1)(Z(y^2, w, x^1)) + \sum (-1)^{\alpha + \iota} \rho(x^{(1)}, Z(x^2, v, y^1), y^{(2)}(w)) \\
+ \sum (-1)^{\alpha + \iota} \rho(x, v, y^1, Z(y^2), y^3)(w) + \sum (-1)^{\alpha + \iota} \rho(x^1, Z(x^2), x^3, v, y)(w) \\
= \mathcal{L}_Z \rho.
\]

Here the first and last equality follow from the definitions, the second and fourth from anti-symmetry of \( \rho \) and the third equality follows from closedness of \( \rho \).

The final item follows from a direct calculation, simply using the definitions and the fact that \( F \circ \hat{F} \circ Z = Z \circ \hat{F} \).

\[ \square \]

Next we define (one-parameter) families of Hochschild cochains as follows

\[
CC^\bullet(C)\{t\} = \prod_{n \geq 0} \prod_{X_0, \ldots, X_n} \text{Hom}^\bullet(C(X_0, X_1, \ldots, X_n), C(X_0, X_n)) \otimes \mathbb{K}[t][-1].
\]

Similarly we can define families of \( \mathbb{A}_\infty \)-pre-functors, which are constant on objects.

**Lemma 2.30.** Let \( Z^t \in CC^1(C)\{t\} \) be a family of degree one Hochschild cochains of order two, that is \( Z_0^t = Z_1^t = 0 \). Then there exists an unique family of \( \mathbb{A}_\infty \) pre-functors \( F^t : C \to C \), which is the identity on objects and satisfies

\[
F^0 = \text{id}, \quad \frac{d}{dt} F^t = Z^t \circ \hat{F}^t.
\]

We refer to \( F^t \) as the flow of \( Z^t \). In addition, if \( Z^t \) is a reduced cochain, then \( F^t \) is unital. Moreover, if \( \delta(Z^t) = [m, Z^t] = 0 \) then \( F^t \) is an \( \mathbb{A}_\infty \)-functor.
Proof. We will construct the $A_\infty$ pre-functor maps $F^t_n$ by induction on $n$. For $n = 1$, Equation (16) immediately gives $F^t_1|_{\mathcal{C}(X_0,X_1)} = \text{id}_{\mathcal{C}(X_0,X_1)}$, since $Z^t$ has order two. For the induction step we have to solve the equation
\[
\frac{d}{dt} F^t_n(x_1,\ldots,x_n) = \sum_{k \geq 2} \sum_{i_1+\ldots+i_k = n} Z^t_k(F^t_{i_1}(x_1,\ldots),\ldots,F^t_{i_k}(\ldots,x_n)).
\]
Note that every term on the right-hand side of the equation was constructed in the previous induction steps, therefore we can find $F^t_n(x_1,\ldots,x_n)$, by integrating the right-hand side, with the initial condition $F^t_0 = 0$. Note that, since $Z^t$ has degree 1, $F^t_n$ has (shifted) degree 0 as required.

By construction $F^t_1(1_{X_1}) = 1_{X_1}$. For $n \geq 2$, if $Z^t$ is reduced it is clear that the right-hand side of the above equation vanishes whenever one of the $x_j = 1_{X_j}$. Therefore $F^t_n(\ldots,1_{X_j},\ldots) = 0$, hence $F^t$ is unital.

We now assume that $[m,Z^t] = 0$ and compute
\[
\frac{d}{dt} (\hat{F}^t \circ \hat{m} - \hat{m} \circ \hat{F}^t) = \hat{Z}^t \circ \hat{F}^t \circ \hat{m} - \hat{m} \circ \hat{Z}^t \circ \hat{F}^t
\]
\[
= \hat{Z}^t \circ (\hat{F}^t \circ \hat{m} - \hat{m} \circ \hat{F}^t).
\]
Here we used the fact that $\frac{d}{dt} \hat{F}^t = \hat{Z}^t \circ \hat{F}^t$, which is equivalent to Equation (16). Therefore, as before we conclude, by uniqueness of the solution to this equation, that $\hat{F}^t \circ \hat{m} - \hat{m} \circ \hat{F}^t = 0$ which is equivalent to the $A_\infty$-functor equation for $F^t$. \qed

Lemma 2.31. Let $Z^t$ and $F^t$ be as in the previous lemma.

1. Given $\rho \in \text{hom}_{[\mathcal{C},\mathcal{C}]}(\mathcal{C}_{sd},\mathcal{C}^\vee_{sd})$, we have $\frac{d}{dt} ((F^t)^* \rho) = -(F^t)^* (\mathcal{L}_{Z^t} \rho)$.

2. Let $x \in CC_*(\mathcal{C})$, we have $\frac{d}{dt} ((F^t)^* (x)) = \mathcal{L}_{Z^t}(F^t_*(x))$.

3. Given $\theta \in CC_*(\mathcal{C})^\vee$, we have $\frac{d}{dt} ((F^t)^* \theta) = (F^t)^* (\mathcal{L}_{Z^t}^\vee \theta)$.

Proof. We observe that the third statement is simply the dual to the second one. The first two claims have identical proofs, we prove the second one:
\[
\frac{d}{dt} ((F^t)^*(x_0 \otimes x)) = \sum (-1)^{\alpha} \frac{d}{dt} F^t(x^{(3)}, x_0, x^{(1)}) \otimes \hat{F}^t(x^{(2)}) + \sum (-1)^{\alpha} F^t(x^{(3)}, x_0, x^{(1)}) \otimes \frac{d}{dt} \hat{F}^t(x^{(2)}) = \sum (-1)^{\alpha} Z^t \circ \hat{F}^t(x^{(3)}, x_0, x^{(1)}) \otimes \hat{F}^t(x^{(2)}) + \sum (-1)^{\alpha} F^t(x^{(3)}, x_0, x^{(1)}) \otimes \hat{Z}^t \circ \hat{F}^t(x^{(2)}),
\]
where we used Equation (16) in the second equality. The expression above then equals

\[ \mathcal{L}_{Z}^{-1}(\sum (-1)^{i} F^t(x^{(i)}, x_0, x^{(1)}) \otimes \hat{F}^t(x^{(2)})) = \mathcal{L}_{Z}^{-1}((F^t)_*(x_0 \otimes x)). \]

We are now ready to prove the analogue of the Darboux theorem in our setting.

**Proposition 2.32.** Let \( \rho \in \Omega^{2,cl}(\mathcal{C}) \) be a strong homotopy inner product. There exists a cyclic \( A_\infty \)-category \( \mathcal{C}' \), with the same objects and morphism spaces as \( \mathcal{C} \), and an \( A_\infty \)-isomorphism \( F : \mathcal{C}' \rightarrow \mathcal{C} \). Moreover, if we denote by \( \langle - , - \rangle \) the cyclic pairing (or constant strong homotopy inner product on \( \mathcal{C}' \)), we have \( F^* \rho = \langle - , - \rangle \).

**Proof.** Define \( \rho^t = (1 - t)\rho_{0,0} + t\rho \in \text{hom}^*_{[\mathcal{C},\mathcal{C}]}(\mathcal{C}_{sd}, \mathcal{C}_{sd})\{t\} \). It is easy to see that \( \rho^t \) is anti-symmetric, closed and non-degenerate since \( \rho^t_{0,0} = \rho_{0,0} \). By Proposition 2.17, there exists \( \theta \in CC^*(\mathcal{C})^v \) with \( S(\theta) = \rho - \rho^0 \). In fact, since \( \langle \rho - \rho^0, 0 \rangle = 0 \), we can take \( \theta \) satisfying \( \theta_0 = \theta_1 = 0 \). Lemma 2.29 then implies the existence of a Hochschild cochain \( Z^t \) satisfying \( \iota_{Z^t} \rho^t = \theta \). Moreover from the construction we see that we can choose \( Z^t \) of order two, that is \( Z^t_0 = Z^t_1 = 0 \). Also note that, since \( S \) has degree 1, \( |\theta| = |\rho| + 1 \) which implies \( Z \) has degree one.

Then applying Lemma 2.30 to \( Z^t \) we obtain a family of \( A_\infty \)-pre-functors \( F^t \) and we compute

\[
\frac{d}{dt}(F^t)^* \rho^t = (F^t)^*(\frac{d\rho^t}{dt} - \mathcal{L}_{Z^t} \rho^t)
= (F^t)^*(\rho - \rho^0 - S(\iota_{Z^t} \rho^t))
= (F^t)^*(S(\theta) - S(\theta)) = 0.
\]

Here we used Lemma 2.31(1) in the first equality and Lemma 2.29(3) in the second. Therefore \( (F^t)^* \rho = (F^t)^* \rho^t = (F^t)^* \rho^0 = \rho_{0,0} \) and we define \( F := F^1 \). We then define a new \( A_\infty \)-category \( \mathcal{C}' \), with the same objects and hom spaces as \( \mathcal{C} \) and operations given by the formula

\[ \hat{m} : = \hat{F}^{-1} \circ \hat{m} \circ \hat{F}. \]

By construction \( F : \mathcal{C}' \rightarrow \mathcal{C} \) defines an \( A_\infty \)-isomorphism with \( F^* \rho = \rho_{0,0} \). Finally we have

\[ \partial \rho_{0,0} = \mathcal{L}_{\omega} \rho_{0,0} = \mathcal{L}_{F^* m} (F^* \rho) = F^* (\mathcal{L}_m \rho) = F^* (\partial \rho) = 0, \]

where the third equality follows from Lemma 2.29(4). This implies \( \rho_{0,0} \) is a constant strong homotopy inner product for \( \mathcal{C}' \), hence \( \mathcal{C}' \) is cyclic.

The previous proposition guarantees a (minimal) \( A_\infty \)-category equipped with a strong homotopy inner product is isomorphic to a cyclic one. We want to examine to what extent this is unique. For this purpose we need the following lemma.
Lemma 2.33. Let $\rho^0$ and $\rho^1$ be two strong homotopy inner products with $[\rho^0] = [\rho^1] \in H^\bullet(\Omega^{2,cl}(C), \partial)$. Then there exists an $A_\infty$-isomorphism $F : C \to C$ with $F^* \rho^1 = \rho^0$.

Proof. By assumption $\rho^1 - \rho^0 = \partial \beta$ for some $\beta \in \Omega^{2,cl}(C)$. This, in particular, implies $(\rho^1)_{0,0} = (\rho^0)_{0,0}$, since $C$ is minimal. Therefore $\rho^t := (1-t)\rho^0 + t\rho^1$ is a quasi-isomorphism for all $t$. Moreover, by Lemma 2.16, $\beta = S(\theta)$ for some $\theta \in CC_\bullet(C)\vee$, with $\theta_0 = 0$.

Lemma 2.29(1) guarantees the existence of $Z^t$ (of degree 1) satisfying $\iota_{Z^t}\rho^t = b^\vee \theta$. Since $\theta_0 = 0$ and $C$ is minimal, $(b^\vee \theta)_0 = (b^\vee \theta)_1 = 0$ and therefore $Z^t$ has order 2. Hence we can apply Lemma 2.30 to $Z^t$ and obtain a family of $A_\infty$-pre-functors $F^t$. As in the previous proof, we have

\[ \frac{d}{dt}(F^t)^* \rho^t = (F^t)^* \left( \frac{d\rho^t}{dt} - L_{Z^t} \rho^t \right) = (F^t)^* (\rho^1 - \rho^0 - S(\iota_{Z^t} \rho^t)) = (F^t)^* (\partial S(\theta) - S(b^\vee (\theta))) = 0, \]

since $S$ is a chain map. Hence $(F^t)^* \rho^t = \rho^0$. Additionally, by Lemma 2.29(2), one has

\[ \iota_{[Z^t,m]}^t \rho^t = \iota_{Z^t} (L_m \rho^t) - L^\vee_m (\iota_{Z^t} \rho^t) = -b^\vee (\iota_{Z^t} \rho^t) = b^\vee (b^\vee \theta) = 0. \]

Therefore, since $\rho^t$ is non-degenerate we conclude $[m, Z^t] = 0$ and therefore $\tilde{F}^t$ are $A_\infty$-functors by Lemma 2.30. Hence $F^1$ is the required $A_\infty$-isomorphism.

Theorem 2.34. Let $(C, \phi_C)$ and $(D, \phi_D)$ be minimal Calabi–Yau $A_\infty$-categories and let $F : C \to D$ be an $A_\infty$-functor with $F^* \phi_D = \phi_C$. Then there exist cyclic $A_\infty$-categories $C'$ and $D'$, $A_\infty$-isomorphisms $G_C : C' \to C$ and $G_D : D' \to D$. Moreover there is a cyclic $A_\infty$-functor $F' : C' \to D'$ making the following diagram commute.

\[
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
G_C \uparrow & & \uparrow G_D \\
C' & \xrightarrow{F'} & D'
\end{array}
\]

Proof. The Calabi-Yau structures determine strong homotopy inner products $\rho_C$ and $\rho_D$ by Corollary 2.24. Moreover $[F^* \rho_D] = [\rho_C] \in H^\bullet(\Omega^{2,cl}(C), \partial)$. Applying Proposition 2.32 to $\rho_C$ and $\rho_D$ one obtains cyclic categories $C'$ and $D'$ and $A_\infty$-isomorphisms $G_C : C' \to C$ and $G_D : D' \to D$.

Let $\tilde{F} := G_D^{-1} \circ F \circ G_C$, by assumption $[\tilde{F}^* \langle -,- \rangle_{D'}] = [\langle -,- \rangle_C]$. By the previous lemma there exists an $A_\infty$-isomorphism $E : C' \to C'$ satisfying $E^* (\tilde{F}^* \langle -,- \rangle_{D'}) = \langle -,- \rangle_C$. Now replace $G_C$ by $G_C \circ E$ and take $F' := \tilde{F} \circ E$. With these choices we have the required commutative diagram and the condition $(F')^* \langle -,- \rangle_{D'} = \langle -,- \rangle_C$, since $(\tilde{F} \circ E)^* \rho = E^* \tilde{F}^* \rho$. \qed
3 Unital Calabi–Yau structures

We would like to have unital versions (meaning all the $A_\infty$-categories and functors are unital) of Proposition 2.32 and Theorem 2.34. As we will see, this is not true in the later case. For this purpose we need to introduce an unital version of cyclic homology. Unlike the case of Hochschild (co)homology this is not isomorphic to the normalized or reduced versions [28, Section 2.2.12].

Throughout this chapter we assume all the $A_\infty$-categories are minimal.

3.1 Unital cyclic homology

Definition 3.1. Denote by $Q$ the subspace of $C^\lambda(C)$ spanned by chains of the form $[1_{X_0} \otimes x_1 \cdots \otimes x_n]$, for $n \geq 1$. This is a subcomplex and we define $C^{\lambda,un}(C)$ to be quotient complex $C^\lambda(C)/Q$. We will refer to the homology of this complex as unital cyclic homology and denote it by $HC^{\lambda,un}(C)$.

Remark 3.2. Loday [28, Section 2.2.12] introduces the notion of reduced cyclic homology for unital, associative algebras. The notion we just introduced, in the case of associative algebras, is slightly different from the one in Loday.

In order to compare $HC^{\lambda,un}(C)$ and $HC^\lambda(C)$ we will use the $u$-model for cyclic homology: $CC^\lambda_+(C) := (CC^{\text{red}}_+(C)[u^{-1}], b + uB)$. We define $CC^{\lambda,un}_+(C)$ as the cokernel of the inclusion

\[
\bigoplus_{X} u^{-1}K[u^{-1}] \to CC^\lambda_+(C)
\]

\[ (u^{-k})_X \to u^{-k}1_X. \]

Note that this is a chain map (from a complex with trivial differential) since $1_X$ is $(b + uB)$-closed on $CC^{\text{red}}_+(C)[u^{-1}]$.

Proposition 3.3. The map $\Pi : CC^{\lambda,un}_+(C) \to CC^{\lambda,un}_+(C)$ defined as

\[
\Pi(u^{-k}x_0 \otimes \cdots \otimes x_n) = \begin{cases} 
0 & k > 0 \\
[x_0 \otimes \cdots \otimes x_n] & k = 0
\end{cases}
\]

is a quasi-isomorphism.

Proof. We first observe that, by definition of $Q$, $\Pi$ is a well-defined map on $CC^{\text{red}}_+(C)$. Moreover $\Pi$ is a chain map since $\Pi(\text{Im}(B)) \subseteq Q$. Now consider the (non-negative) increasing filtrations

\[
\mathcal{F}^pC^{\lambda,un}_+(C) := \text{span} \{ [x_0 \otimes \cdots \otimes x_n] | n \leq p \}
\]

\[
\tilde{\mathcal{F}}^pC^{\lambda,un}_+(C) := \text{span} \{ u^{-k}x_0 \otimes \cdots \otimes x_n | n \leq p \} \oplus \text{span} \{ u^{-k}1_{X_n} \otimes x_0 \otimes \cdots \otimes x_n | n = p \}
\]
Again by definition of $Q$ we see that $\Pi$ preserves the above filtrations. We claim $\Pi$ induces quasi-isomorphisms on the associated graded pieces, which implies the statement. For $p = 0$ we have $\mathcal{F}^0 = \bigoplus_{X} \mathcal{C}(X, X)$ and $\mathcal{F}^0$ is given by the complex
\[
\bigoplus_{X} \mathcal{C}(X, X) \xrightarrow{b} 1_{X} \otimes \overline{\mathcal{C}(X, X)} \xleftarrow{uB} u^{-1}\mathcal{C}(X, X) \xrightarrow{b} u^{-1}1_{X} \otimes \overline{\mathcal{C}(X, X)} \xleftarrow{uB} u^{-2}\mathcal{C}(X, X) \cdots
\]
where $\overline{\mathcal{C}(X, X)} = \mathcal{C}(X, X)/\overline{\text{span}\{1_{X}\}}$. The above complex is isomorphic to
\[
\bigoplus_{X} \mathcal{C}(X, X) \xrightarrow{0} \overline{\mathcal{C}(X, X)} \xleftarrow{id} \mathcal{C}(X, X) \xrightarrow{0} \overline{\mathcal{C}(X, X)} \xleftarrow{id} \mathcal{C}(X, X) \cdots.
\]
Hence it is quasi-isomorphic to $\mathcal{F}^0$ via $\Pi$. For $p > 0$ we have
\[
\mathcal{F}^p / \mathcal{F}^{p-1} \simeq \bigoplus_{X_0, \ldots, X_p} \overline{\mathcal{C}(X_p, X_0, \ldots, X_p)} / \text{Im}(\text{id} - t),
\]
where, analogously, $\overline{\mathcal{C}(X_p, X_0, \ldots, X_p)}$ is the quotient of $\mathcal{C}(X_p, X_0, \ldots, X_p)$ by the span of tensors with a unit. Similarly to the $p = 0$ case we have
\[
\overline{\mathcal{F}^p} / \overline{\mathcal{F}^{p-1}} \simeq \bigoplus_{X_0, \ldots, X_p} \overline{\mathcal{C}(X_p, X_0, \ldots, X_p)} \xrightarrow{b} 1_{X_p} \otimes \overline{\mathcal{C}(X_p, X_0, \ldots, X_p)} \xleftarrow{uB}
\]
\[
\xleftarrow{u^{-1}\mathcal{C}(X_p, X_0, \ldots, X_p)} \xrightarrow{b} u^{-1}1_{X_p} \otimes \overline{\mathcal{C}(X_p, X_0, \ldots, X_p)} \xleftarrow{uB} \cdots
\]
\[
\xleftarrow{\mathcal{C}(X_p, X_0, \ldots, X_p)} \xleftarrow{\text{id} - t} \overline{\mathcal{C}(X_p, X_0, \ldots, X_p)} \xleftarrow{\text{id}^{-1}} \overline{\mathcal{C}(X_p, X_0, \ldots, X_p)} \xleftarrow{\text{id}^{-1}} \cdots
\]
since $b(1_{X_n} \otimes x_0 \otimes \cdots \otimes x_p) = (\text{id} - t)(x_0 \otimes \cdots \otimes x_p) + \overline{\mathcal{F}^{p-1}}$. Therefore, as follows from Proposition 2.20, this is a resolution of $\bigoplus_{X_0, \ldots, X_p} \overline{\mathcal{C}(X_p, X_0, \ldots, X_p)} / \text{Im}(\text{id} - t)$ which proves the claim.

It now follows we have two different chain models for computing the unital cyclic homology. Using $CC_{+}^{*,\text{un}}(\mathcal{C})$ we immediately obtain the following result.

**Corollary 3.4.** We have the following exact sequence
\[
0 \rightarrow HC_{1}^{+}(\mathcal{C}) \rightarrow HC_{1}^{\text{un}}(\mathcal{C}) \rightarrow \bigoplus_{X} u^{-1}\mathbb{K}[u^{-1}] \rightarrow HC_{0}^{+}(\mathcal{C}) \rightarrow HC_{0}^{\text{un}}(\mathcal{C}) \rightarrow 0.
\]

Using $CC_{+}^{*,\text{un}}(\mathcal{C})$ we can relate $HC_{0}^{\text{un}}(\mathcal{C})$ to an unital version of $\Omega^{2,cl}(\mathcal{C})$.

**Definition 3.5.** An element $\rho \in \Omega^{2,cl}(\mathcal{C})$ is called unital if
\[
\rho(x^{(1)}, 1_{X_i}, x^{(2)}, v, y)(w) = 0,
\]
for all $x^{(1)}, x^{(2)}, v, y, w$. By symmetry, this also implies $\rho(x, v, y^{(1)}, 1_{Y_j}, y^{(2)})(w) = 0$. 

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We denote by $\Omega_{un}^{2,cl}(C)$ the subset of unital pre-homomorphisms. It is easy to check that $\Omega_{un}^{2,cl}(C)$ is a subcomplex of $\Omega^{2,cl}(C)$. We need to modify the domain of the map $S$ (Lemma 2.16) for $\Omega_{un}^{2,cl}(C)$. We introduce the sub-complexes

$I := \text{span}\{x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n \mid n \geq 2, 1 \leq i \leq n - 1\} \subseteq CC_*(C)$

and $U := (\text{id} - t)(I)$. With these notions one can prove the following analogues of Propositions 2.17 and 2.20, with only minor modifications.

**Lemma 3.6.** There are short exact sequences of complexes

$$0 \to C^\lambda_*(C)^\vee \overset{\pi^\vee}{\longrightarrow} \left(CC_*(C)/U\right)^\vee \overset{S}{\longrightarrow} \Omega_{un}^{2,cl}(C)[1] \to 0,$$

$$0 \to C^\lambda_*(C)/N^{-1}(I) \overset{N}{\longrightarrow} C^{\text{bar}}/I \overset{p}{\longrightarrow} \mathcal{S}_C/U \to 0.$$

We now arrive at the unital version of Corollary 2.24.

**Proposition 3.7.**

$$HC^\text{un}_*(C)^\vee \cong H^*(\Omega_{un}^{2,cl}(C), \partial).$$

**Proof.** The homotopy $h$ used in Lemma 2.22 to prove $C^{\text{bar}}$ is acyclic preserves $I$ therefore we can use it to show acyclicity of $C^{\text{bar}}/I$. Then combining this with the two short exact sequences in Lemma 3.6 we conclude that

$$H^*(\Omega_{un}^{2,cl}(C), \partial) \cong H_*(C^\lambda_*(C)/N^{-1}(I))^\vee.$$

We compute the later cohomology by considering the projection

$$q : C^\lambda_*(C)/N^{-1}(I) \to C^\lambda_*(C)/Q = C^\lambda_{\text{un}}(C).$$

Since $N^{-1}(I) \subseteq Q$, $q$ is surjective and has kernel $\ker(q) \cong Q/N^{-1}(I)$. We claim that $\ker(q)$ is acyclic, which shows that $q$ is a quasi-isomorphism and in turn proves the statement.

On $\ker(q)$ we consider the filtration

$$\mathcal{F}^p \ker(q) := \text{span} \{ [x_1 \otimes \cdots \otimes x_n] \in Q \mid n \leq p \} \oplus \text{span} \{ [1_{X_1} \otimes 1_{X_1} \otimes x_1 \otimes \cdots \otimes x_{p-1}] \in Q \}$$

By definition of $I$ we see that $N^{-1}(I)$ is spanned by tensors with at least three units, or two non-consecutive units. Therefore we have

$$\mathcal{F}^p/\mathcal{F}^{p-1} \cong \text{span} \{ [1_{X_1} \otimes x_1 \otimes \cdots \otimes x_{p-1}], [1_{X_1} \otimes 1_{X_1} \otimes x_1 \otimes \cdots \otimes x_{p-1}] \}$$

A simple computation gives

$$b([1_{X_1} \otimes 1_{X_1} \otimes x_1 \otimes \cdots \otimes x_{p-1}]) = -[1_{X_1} \otimes x_1 \otimes \cdots \otimes x_{p-1}] + [1_{X_1} \otimes 1_{X_1} \otimes b'(x_1 \otimes \cdots \otimes x_{p-1})].$$

By definition the second term is in $\mathcal{F}^{p-1}$, thus we conclude $\mathcal{F}^p/\mathcal{F}^{p-1}$ is acyclic. Hence $\ker(q)$ is acyclic. \[\Box\]
3.2 Unital Darboux theorem

Lemma 3.8. Let $\rho \in \Omega^2_{\text{un}}(\mathcal{C})$ be an unital pre-homomorphism.

1. If $\rho$ is a quasi-isomorphism then the map $\iota_\rho : CC^\bullet_{\text{red}}(\mathcal{C}) \to (CC^\bullet_{\text{red}}(\mathcal{C}))^\vee$ is an isomorphism.

2. Assume $\rho_{0,0}(1_X)(1_X) = 0$ for all $X$, then there is $\theta \in (CC^\bullet_{\text{red}}(\mathcal{C}))^\vee$ satisfying $S(\theta) = \rho$.

Proof. From Definition 2.26 it is clear that if $Z$ is reduced and $\rho$ is unital then $\iota_Z\rho$ is reduced. Moreover, close inspection shows that the proof of Lemma 2.29(1) still applies in the reduced setting.

For the second claim, we first observe that the condition on $\rho_{0,0}$ plus the fact that $\rho$ is closed and unital implies that any expression of the form

$$\rho(x, 1_X, y)(1_Y)$$

vanishes. Next, one picks basis for the hom spaces, which in the case of self-homs extend the units. Then one defines $\theta(x_0, x_1, \ldots, x_n)$ on the basis elements as follows: when none of the $x_i$ are units one uses $h(\rho)$ as in Equation (11); when one or more of the $x_i$ (for $i > 0$) is a unit then $\theta$ must be zero, for it to be reduced; and when $x_0 = 1_X$ one takes

$$\theta(1_X, x_1, \ldots, x_n) := \rho(1_X, x_1, \ldots, x_{n-1})(x_n).$$

Equation (18) guarantees these conditions are compatible. We claim that $S(\theta) = \rho$. This can be proved in essentially the same way as in Lemma 2.16.

Proposition 3.9. Let $\rho \in \Omega^2_{\text{un}}(\mathcal{C})$ be an unital strong homotopy inner product. There exists a cyclic, unital $A_\infty$-category $C'$, with the same objects, morphism spaces and units as $C$, and an unital $A_\infty$-isomorphism $F : C' \to C$. Moreover, if we denote by $\langle -, - \rangle$ the cyclic pairing on $C'$, we have $F^*\rho = \langle -, - \rangle$.

Proof. The proof follows the same argument as Proposition 2.32, just taking the unital version of every ingredient. Since $\rho$ is unital so is $\rho'$ and $\rho - \rho^0$. Since $(\rho - \rho^0)_{0,0} = 0$, we can apply the previous lemma to obtain a reduced $\beta$ with $S(\beta) = \rho - \rho^0$. Moreover, we can take $\theta$ satisfying $\theta_0 = \theta_1 = 0$. By Lemma 3.8(1) there is a reduced family of cochains $Z^t$ satisfying $\iota_{Z^t}\rho^t = \theta$. For reduced $Z^t$, the family $F^t$ provided by Lemma 2.30 are also unital. Then it is easy to check that the new $A_\infty$ structure $m'$ is unital (for the same units $1_X$) and $F^1$ is an unital $A_\infty$-isomorphism.

Lemma 3.10. If $\rho^0, \rho^1$ are unital cohomologous strong homotopy inner products in $C$ of even degree, then there is an unital $A_\infty$-isomorphism $F : C \to C$ with $F^*\rho^1 = \rho^0$.

Proof. By assumption $\rho^1 - \rho^0 = \partial \beta$ for some $\beta \in \Omega^2_{\text{un}}(\mathcal{C})$ of odd degree. Since $\beta$ has odd degree, it satisfies the condition in Lemma 3.8(2), which then gives $\theta \in (CC^\bullet_{\text{red}}(\mathcal{C}))^\vee$ satisfying $S(\theta) = \beta$. The remainder of the proof is identical to Lemma 2.33.
The following example shows how the above lemma fails in the odd case.

**Example 3.11.** Let \( Cl \) be the (one-dimensional) Clifford algebra, this is an \( A_\infty \)-algebra with underlying vector space \( \mathbb{K}[\epsilon] \), where \( \epsilon \) is odd. It is strictly unital, the only non-trivial \( A_\infty \) operation is \( m_2(\epsilon, \epsilon) = \frac{1}{2} \mathbb{I} \) and it has a cyclic odd inner product \( \langle \mathbb{1}, \epsilon \rangle = 1 \).

Let \( A \) denote the cyclic, unital \( A_\infty \)-algebra with the same vector space and inner product as \( Cl \), but with \( A_\infty \) operations \( m_k(\epsilon, \ldots, \epsilon) = \frac{1}{2} C_{k-1} \mathbb{I} \), for \( k \geq 2 \), where \( C_n \) is the \( n \)-th Catalan number.

There is an unital \( A_\infty \)-isomorphism \( G : A \to Cl \), defined by the formula \( G_k(\epsilon, \ldots, \epsilon) = C_{k-1} \epsilon \), \( k \geq 1 \). One can check that \( G^* \langle -, - \rangle = \langle -, - \rangle - \partial G^* \beta \), where \( \beta \in \Omega^2_{un}(Cl) \) is defined by \( \beta(1)(1) = -2 \). By Lemma 2.33 there is an \( A_\infty \)-isomorphism \( F : A \to Cl \) satisfying \( F^* \langle -, - \rangle = \langle -, - \rangle \), in other words \( F \) is cyclic. However we claim there is no cyclic, unital \( A_\infty \)-isomorphism \( F : A \to Cl \). Any such unital \( A_\infty \)-isomorphism is determined by \( F_k(\epsilon, \ldots, \epsilon) \), but it easy to check that cyclicity implies \( F_k(\epsilon, \ldots, \epsilon) = 0 \) for \( k \geq 2 \), which does not satisfy the \( A_\infty \) equations.

We are now ready to state the unital versions of Theorem 2.34. The situation is different in the odd and even cases.

**Proposition 3.12.** Let \( C \) be an \( A_\infty \)-category with an odd strong Calabi–Yau structure \( \phi \). Then there exists a cyclic, unital \( A_\infty \)-category \( C' \) and an unital \( A_\infty \)-isomorphism \( G_C : C' \to C \).

**Proof.** It follows from (the dual of) Corollary 3.4 that the odd Calabi–Yau structure \( \phi \in HC^1(C) \) lifts to an unital Calabi–Yau structure, meaning it is in the image of the natural map \( HC^1_{un}(C) \to HC^1(C) \). Note the lift will in general not be unique. This lift determines an unital strong homotopy inner product by Proposition 3.7. Applying Proposition 3.9 to this unital strong homotopy inner product gives the desired result. \( \square \)

**Proposition 3.13.** Let \( C \) and \( D \) be \( A_\infty \)-categories with even Calabi–Yau structures \( \phi_C \) and \( \phi_D \), respectively, which have unital lifts. And let \( F : C \to D \) be an unital \( A_\infty \)-functor with \( F^* \phi_D = \phi_C \). Then there exist cyclic, unital \( A_\infty \)-categories \( C' \) and \( D' \), unital \( A_\infty \)-isomorphisms \( G_C : C' \to C \) and \( G_D : D' \to D \). Moreover there is a cyclic, unital \( A_\infty \)-functor \( F' : C' \to D' \) making the following diagram commute

![Diagram](https://via.placeholder.com/150)

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Proof. Let $\phi_C^{un}$ and $\phi_D^{un}$ be the unital lifts of $\phi_C$ and $\phi_D$, respectively. Corollary 3.4 implies these lifts are unique (since they are of even degree) which in turn implies $[\mathcal{F}^*(\phi_D^{un})] = [\phi_C^{un}]$. These lifts determine unital strong homotopy inner products $\rho_C$ and $\rho_D$, by Proposition 3.7. One then applies Proposition 3.9, to obtain unital $C', D', G_C$ and $G_D$. Next one observes that unitality implies that $\tilde{\mathcal{F}}^*(\langle - , - \rangle^D)$ is an unital strong homotopy inner product and $[\tilde{\mathcal{F}}^*(\langle - , - \rangle^D)] = [\langle - , - \rangle^C]$ (in the notation of Theorem 2.34).

Now, using Lemma 3.10 we can proceed as in the proof of Theorem 2.34. 

3.3 Trivializations and CY structures

In order to define the CEI we need to choose a splitting of the Hodge filtration. In this subsection we recall how a splitting together with a weak Calabi–Yau structure determine a strong Calabi–Yau structure. We first review some preliminary notions, following [2].

The chain level Mukai pairing on $CC_{red}^\bullet(C)$ descends to a pairing which we still denote by $\langle - , - \rangle_{Muk} : HH_\bullet(L) \otimes HH_\bullet(L) \to \mathbb{K}$.

It is non-degenerate when $C$ is smooth and proper [38]. One can also extend $\langle - , - \rangle_{Muk}$ sesquilinearly to the negative cyclic complex $CC_{red}^\bullet(C)[[u]]$ to obtain the so-called higher residue pairing

$$\langle \alpha u^i, \beta u^j \rangle_{hres} := (-1)^j \langle \alpha, \beta \rangle_{Muk} \cdot u^{i+j}.$$  

Taking the $(b + uB)$-homology yields the higher residue pairing in homology

$$\langle - , - \rangle_{hres} : HC_{red}^\bullet(C) \otimes HC_{red}^\bullet(C) \to \mathbb{K}[[u]].$$

The following definition is taken from [2, Definition 3.7], except the last condition which is new.

**Definition 3.14.** A $\mathbb{K}$-linear map $s : HH_\bullet(C) \to HC_{red}^\bullet(C)$ is called a splitting of the Hodge filtration of $C$ if it satisfies

S1. (Splitting condition) $s$ splits the canonical map $\pi : HC_\bullet^-(C) \to HH_\bullet(C)$, defined as $\pi(\sum_{n \geq 0} \alpha_n u^n) = \alpha_0$.

S2. (Lagrangian condition) $\langle s(\alpha), s(\beta) \rangle_{hres} = \langle \alpha, \beta \rangle_{Muk}$, $\forall \alpha, \beta \in HH_\bullet(C)$.

A splitting $s$ is called **good** if it satisfies

S3. (Homogeneity) The subspace $\bigoplus_{l \in \mathbb{N}} u^{-1} \cdot \operatorname{Im}(s)$ is stable under the $u$-connection $\nabla_{u \frac{d}{du}}$.

This is equivalent to requiring $\nabla_{u \frac{d}{du}} s(\alpha) \in u^{-1} \operatorname{Im}(s) + \operatorname{Im}(s)$, $\forall \alpha \in HH_\bullet(C)$.

Let $\omega \in HH_\bullet(C)$ be an element of the Hochschild homology.
S4. (ω-Compatibility) A splitting $s$ is called $ω$-compatible if
\[ \nabla_{u^{-1}} s(ω) \in r \cdot s(ω) + u^{-1} \cdot \text{Im}(s) \] for some $r \in \mathbb{K}$.

S5. (Unitality) For $X \in \text{tw}^\pi \mathcal{C}$, let $[1_X]$ be the (potentially zero) class in $HC^{-}_\bullet(\text{tw}^\pi \mathcal{C})$ determined by the unit. A splitting $s$ is called unital if there exists a split-generating subcategory $\mathcal{A} \subset \text{tw}^\pi \mathcal{C}$ such that we have
\[ \langle [1_X], s(ω) \rangle_{\text{hres}} \in \mathbb{K}, \]
i.e. the higher residue pairing is a constant, for any object $X$ in $\mathcal{A}$.

Remark 3.15. The unitality condition is empty when $ω$ is an odd class, since the higher residue pairing is even. Furthermore, the unitality condition is for a split-generating subcategory, instead of $\mathcal{C}$ itself, so that this condition is stable under Morita equivalences.

As mentioned above, Shklyarov [38] proved that the Mukai pairing is non-degenerate, which gives a natural isomorphism
\[ HH_\bullet(\mathcal{C})^\vee \cong HH_\bullet(\mathcal{C}). \]
Moreover it gives an isomorphism $HC^+_\bullet(\mathcal{C})^\vee \cong HC^-_\bullet(\mathcal{C})$ via the pairing
\[ \langle \alpha u^i, \beta u^j \rangle := \delta_{i+j=0} \cdot \langle \alpha, \beta \rangle_{\text{Muk}} \]
Using the above isomorphisms, we can identify elements of $HH_\bullet(\mathcal{C})$ (respectively $HC^-_\bullet(\mathcal{C})$) satisfying the same non-degeneracy condition with weak (respectively strong) Calabi–Yau structures.

Let $ω \in HH_\bullet(\mathcal{C})$ be a weak Calabi–Yau structure and $s$ a splitting. By construction, the element $s(ω) \in HC^-_\bullet(\mathcal{C})$ (via the identification above) is non-degenerate and therefore defines a strong Calabi–Yau structure. We have the following

Lemma 3.16. Let $ω \in HH_\bullet(\mathcal{C})$ be a weak Calabi–Yau structure and $s$ an unital splitting with respect to $\mathcal{A} \subset \text{tw}^\pi \mathcal{C}$. Then the strong Calabi–Yau structure $s(ω)$, viewed as an element of $HC^-_\bullet(\mathcal{A})$, admits a lift to an unital Calabi–Yau structure of $\mathcal{A}$.

Proof. From the definition of the unital cyclic complex $CC^+_{\bullet,\text{un}}(\mathcal{C})$, an element $ϕ \in HC^+_{\bullet}(\mathcal{C})^\vee$ lifts to $HC^\text{un}_{\bullet}(\mathcal{C})^\vee$ if and only if $ϕ(u^{-k}[1_X]) = 0$ for any object $X$ and $k > 0$. When $ϕ = \langle -, s(ω) \rangle$, this exactly the Unitality condition for $s$. \(\square\)

Since the Unitality condition is new, we end this section by giving two settings where it is implied by the more conventional condition of $ω$-compatibility.

Proposition 3.17. Let $s$ be a $ω$-compatible splitting for some $ω \in HH_\bullet(\mathcal{C})$, with $r \notin \mathbb{Z}_{\geq 0}$. Assume that $\mathcal{C}$ satisfies one of the following conditions:
1. \( C \) is a \( \mathbb{Z} \)-graded \( A_\infty \)-category;

2. \( C \) has semi-simple Hochschild cohomology and \( s \) is the semi-simple splitting, as defined in [2, Corollary 3.8].

Then \( s \) is unital with respect to \( C \) itself.

**Proof.** When \( C \) is \( \mathbb{Z} \)-graded, the \( u \)-connection has a simple pole at \( u = 0 \), instead of a pole of order 2, as proved in [9, Lemma 3.2]. Therefore \( \omega \)-compatibility takes the form \( \nabla_{u \frac{d}{du}} s(\omega) = r \cdot s(\omega) \). We compute

\[
\frac{d}{du} \langle 1_X, s(\omega) \rangle_{\text{hres}} = \langle \nabla_{u \frac{d}{du}} 1_X, s(\omega) \rangle_{\text{hres}} + \langle 1_X, \nabla_{u \frac{d}{du}} s(\omega) \rangle_{\text{hres}} = r(1_X, s(\omega))_{\text{hres}}.
\]

It is elementary that, since \( r \notin \mathbb{Z}_{>0} \), the only possible solutions of the differential equation \( u \frac{d}{du} f(u) = r f(u) \) are constants. Therefore we conclude that \( \langle 1_X, s(\omega) \rangle_{\text{hres}} \) is constant.

For the second condition, we freely use the notation from [2]. Recall \( C \) has the decomposition \( C = \prod_i C_{\lambda_i} \), and consider \( X \in C_{\lambda_i} \). A simple computation shows \( \nabla_{u \frac{d}{du}} 1_X = u^{-1}\lambda_i 1_X \). Under the decomposition above we write \( \omega = \sum_i \omega_i \in \bigoplus_i HH_\bullet(C_{\lambda_i}) \). Then \( \omega \)-compatibility gives \( \nabla_{u \frac{d}{du}} s(\omega) = r \cdot s(\omega) + u^{-1}s(\xi(\omega)) \), where \( \xi(\omega) = \sum_i \lambda_i \omega_i \). As above we compute

\[
\frac{d}{du} \langle 1_X, s(\omega) \rangle_{\text{hres}} = u^{-1}\langle 1_X, s(\omega) \rangle_{\text{hres}} + \langle 1_X, rs(\omega) \rangle_{\text{hres}} + u^{-1}s(\sum_i \lambda_i \omega_i)_{\text{hres}} = u^{-1}\lambda_i \langle 1_X, s(\omega) \rangle_{\text{hres}} + r(1_X, s(\omega))_{\text{hres}} - u^{-1}\langle 1_X, \sum_i \lambda_i s(\omega_i) \rangle_{\text{hres}} = u^{-1}\lambda_i \langle 1_X, s(\omega) \rangle_{\text{hres}} + r(1_X, s(\omega))_{\text{hres}} - u^{-1}\lambda_i \langle 1_X, s(\omega) \rangle_{\text{hres}} = r(1_X, s(\omega))_{\text{hres}}.
\]

Here in the third and fourth equality rely on the fact that \( s \) is the semi-simple splitting and therefore respects the orthogonal decomposition \( C = \prod_i C_{\lambda_i} \). As before we can now conclude that \( \langle 1_X, s(\omega) \rangle_{\text{hres}} \) is constant. \( \square \)

4 Definition of categorical enumerative invariants

We briefly recall the construction of categorical enumerative invariants following [15, 7]. In [7], the construction is done for a finite dimensional cyclic \( A_\infty \)-algebra. As in [15], we shall work with cyclic \( A_\infty \)-categories. Observe that by requiring morphisms be composable in the definition of the Hochschild chain complex of \( C \) (see Section 2), the constructions in [7] generalize to the categorical setting.

Throughout the section, let \( C \) be a \( \mathbb{Z}/2\mathbb{Z} \)-graded cyclic \( A_\infty \)-category over a field \( K \) of characteristic zero. Note that in particular, the non-degeneracy condition of the cyclic
pairing implies that the Hom spaces of $C$ are finite dimensional. We also assume that $C$ is strictly unital and its cyclic pairing is of parity $d \in \mathbb{Z}/2\mathbb{Z}$. For a $\mathbb{Z}$-graded complex, we use homological grading conventions: for a graded vector space $V = \oplus_n V_n$, its shift $V[k]$ is the graded vector space whose $n$-th graded piece is $V_{n-k}$.

### 4.1 TCFT’s and DGLA’s

Let $M_{g,k,l}^{fr}$ be the moduli space of smooth Riemann surfaces of genus $g$, with $k$ incoming framed marked points, $l$ outgoing framed marked points. Here a framing of a marked point means a choice of an embedded disk around the marked point. We also assume the framings along all $k + l$ marked points be disjoint from each other, i.e. the closures of all the embedded disks be disjoint. Denote by $C_{\text{comb}}^{\bullet}(M_{g,k,l}^{fr})$ the combinatorial model of these moduli spaces with coefficients in $\mathbb{K}$, as described in Costello [14], Kontsevich-Soibelman [26], and Wahl-Westerland [42].

**Theorem 4.1.** Let $CC_{\text{red}}^{\bullet}(C)[d]$ denote the $d$-shifted reduced Hochschild chain complex of the $A_{\infty}$-category $C$. Then it carries a 2-dimensional topological conformal field theory structure, i.e. there are action maps compatible with sewing of Riemann surfaces

$$\rho_{g,k,l}^C : C_{\text{comb}}^{\bullet}(M_{g,k,l}^{fr}) \to \text{Hom}(CC_{\text{red}}^{\bullet}(C)[d] \otimes^k, CC_{\text{red}}^{\bullet}(C)[d] \otimes^l)$$

with $g \geq 0$, $k \geq 1$, $l \geq 0$, and $2 - 2g - k - l < 0$.

To obtain invariants from $\rho_{g,k,l}^C$’s one needs to overcome the difficulty that the moduli spaces $M_{g,k,l}^{fr}$ are non-compact since we are only dealing with smooth surfaces. One natural approach to this problem is to extend this action to the Deligne-Mumford compactification, assuming furthermore that $C$ is smooth (and hence it also satisfies the Hodge-to-de-Rham degeneration property by Kaledin [22]). This was conjectured by Kontsevich-Soibelman [26].

Following Costello, in [7] we take a different (but still closely related) route which completely bypasses the Deligne-Mumford compactification. One of the key idea is due to Sen-Zwiebach [34] who introduced a differential graded Lie algebra (DGLA) structure on normalized singular chains of moduli spaces of smooth curves. This idea is further implemented in the combinatorial setup in [6].

Indeed, for an integer $k \geq 1$, denote by $\text{sgn}_k[-k]$ the rank one local system over $M_{g,k,l}^{fr}$ whose fiber over a Riemann surface $(\Sigma, p_1, \ldots, p_k, q_1, \ldots, q_l, \phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_l)$ is the sign representation of the symmetric group $S_k$ on the set $\{p_1, \ldots, p_k\}$, shifted by $[-k]$. For simplicity, we often write this local system as $\text{sgn}$ when the integer $k$ is clear from the context. The combinatorial version of Sen-Zwiebach’s DGLA is given by

$$\widehat{\mathfrak{g}} := \bigoplus_{g \geq 0, k \geq 1, l \geq 0 \atop 2g - 2 + k + l > 0} C_{\text{comb}}^{\bullet}(M_{g,k,l}^{fr}, \text{sgn})_{ns}[2][[\hbar, \lambda]],$$
where the subscript $hS$ denotes the homotopy quotient by the symmetric group $Σ_k × Σ_l$ and the $k + l$ circle actions that rotate the framings; and the formal variables $ℏ$ and $λ$ are both of homological degree $−2$. The additional shift by $[2]$ is explained in the remark following Theorem 4.2 below.

In the construction of Sen-Zwiebach’s DGLA structure, we need to perform the “twisted sewing operation” where the moduli space of annuli plays an important role. Indeed, let us consider the Mukai ribbon graph

\begin{equation}
M := \begin{array}{cc}
& \\
\end{array}
\end{equation}

which is a zero chain inside $C^\text{comb}_*(M^{fr}_{0,2,0})$. Here, the two crosses in $M$ correspond to its 2 inputs (cycles of the ribbon graph $M$). Using $M$ we may define a map

\[ ι : C^\text{comb}_*(M^{fr}_{g,k,l}; \text{sgn})_{hS} \to C^\text{comb}_*(M^{fr}_{g,k,l+1,l-1}; \text{sgn})_{hS}, \]

obtained by sewing with one input of $M$ with an output of a chain in $C^\text{comb}_*(M^{fr}_{g,k,l}; hS)$, hence this operation reduces the number of outputs by one while increases the number of inputs also by one. There is a thickened version of $M$ which we shall denote by

\begin{equation}
\bar{M} := \begin{array}{cc}
& \\
\end{array}
\end{equation}

It is obtained from $M$ by a circle action from one of its two inputs. This is a degree one chain inside $C^\text{comb}_*(M^{fr}_{0,0,2})$. Sewing with $\bar{M}$ at outputs yields a map of homological degree one

\[ Δ : C^\text{comb}_*(M^{fr}_{g,k,l}; \text{sgn})_{hS} \to C^\text{comb}_*(M^{fr}_{g,k,l-2}; \text{sgn})_{hS}, \]

called the self twisted sewing map. In a similar way, twisted sewing between $r$ ($r ≥ 1$) outputs and $r$ inputs yields a map

\[ \{−, −\}_r : C^\text{comb}_*(M^{fr}_{g',k',l'}; \text{sgn})_{hS} \otimes C^\text{comb}_*(M^{fr}_{g'',k'',l''}; \text{sgn})_{hS} \to C^\text{comb}_*(M^{fr}_{g,k,l}; \text{sgn})_{hS}, \]

called the $r$-th twisted sewing map, where $g = g' + g'' + r - 1$, $k = k' + k'' - r$, and $l = l' + l'' - r$. We refer to [6] for more details (such as signs) of the these constructions.

**Theorem 4.2.** [Caldararu-Costello-T. [6]] There exists a $\mathbb{Z}$-graded DGLA structure on $\hat{g}$ whose differential and Lie bracket are of the form

\begin{itemize}
  \item $d := \bar{δ} + ι + ℏΔ$ with $\bar{δ}$ the boundary map of $C^\text{comb}_*(M^{fr}_{g,k,l}; hS)$, the and $Δ$ is the self twisted sewing operator.
  \item $\{−, −\}_h := \sum_{r≥1} \frac{1}{r!} · \{−, −\}_r h^{r-1}$ with $\{−, −\}_r$ the $r$-th twisted sewing map.
\end{itemize}
Furthermore, the DGLA \( \hat{\mathfrak{g}} \) has a unique Maurer-Cartan element \( \hat{V} \) (up to gauge equivalence) of homological degree \(-1\), and of the form

\[
\hat{V} = \sum_{g,k,l} \hat{V}_{g,k,l} \hbar^g \lambda^{2g-2+k+l}
\]

\[
\hat{V}_{0,1,2} = \frac{1}{2}
\]

We shall refer to \( \hat{V}_{g,k,l} \)'s as combinatorial string vertices.

**Remark 4.3.** The \( \mathbb{Z} \)-grading of \( \hat{\mathfrak{g}} \) is designed so that the boundary map \( \partial \) has homological degree \(-1\), the Lie bracket \( \{-,-\}_h \) has degree 0, and that the equivariant chain \( \hat{V}_{g,k,l} \) is of homological degree \( 6g - 5 + 2(k + l) \) in the shifted chain complex \( C^\text{comb}_\bullet(M^\text{fr}_{g,k,l}, \text{sgn})_{\text{hs}}[2] \). This explains the extra shift \([2]\) in the definition of \( \hat{\mathfrak{g}} \).

On the algebraic side, there is also a DGLA associated with the \( \mathcal{A}_\infty \)-category \( \mathcal{C} \). Following the notations in [7], we have the following chain complexes

\[
\begin{align*}
L & := (CC^\text{red}_\bullet(\mathcal{C})[d], b) \\
L_+ & := (CC^\text{red}_\bullet(\mathcal{C})[d][u], b + uB) \\
L^\text{Tate} & := (CC^\text{red}_\bullet(\mathcal{C})[d]((u)), b + uB) \\
L_- & := L^\text{Tate}/u \cdot L_+ = (CC^\text{red}_\bullet(\mathcal{C})[d][u^{-1}], b + uB)
\end{align*}
\]

With these notations, define a DGLA

\[
\hat{h}_C := \bigoplus_{k \geq 1, l \geq 0} \text{Hom}^c(S^k(L_+[1]), S^l L_-)[2][[\hbar, \lambda]],
\]

where \( \text{Hom}^c \) stands for \( u \)-adic continuous \( \mathbb{K} \)-linear maps. The construction of \( \hat{h}_C \) is in complete parallel to that of the DGLA \( \hat{\mathfrak{g}} \). For example, the Mukai graph \( M \) under the TCFT action \( \rho^c \) yields the chain level Mukai pairing \( \langle -, - \rangle_{\text{Muk}} \equiv \rho^c(M) : L \otimes L \to \mathbb{K} \), which we use to define a map

\[
\iota : \text{Hom}^c(S^k(L_+[1]), S^l L_-) \to \text{Hom}^c(S^{k+1}(L_+[1]), S^{l-1}L_-).
\]

Similarly, we use \( \langle B-, - \rangle_{\text{Muk}} = \rho^c(M) : L \otimes L \to \mathbb{K} \) to define the self twisted sewing map

\[
\Delta : \text{Hom}^c(S^k(L_+[1]), S^l L_-) \to \text{Hom}^c(S^k(L_+[1]), S^{l-2}L_-).
\]

As a conclusion, we obtain a DGLA structure on \( \hat{h}_C \) of the form:

- Its differential is of the form \( b + uB + \iota + \hbar \Delta \) with \( \Delta \) the algebraic analogue of the self twisted sewing operator.
• Its Lie bracket is also of the form \( \{-, -\}_\hbar := \sum_{r \geq 1} \frac{1}{r!} \cdot \{-, -\}_r \hbar^r \) with \( \{-, -\}_r \) the algebraic version of the \( r \)-th twisted sewing operator.

By the very constructions, the action map \( \rho^C \) induces a morphism of \( \mathbb{Z}/2\mathbb{Z} \)-graded DGLA’s which we still denote by

\[
\rho^C : \hat{g} \to \hat{h}_C.
\]

The push-forward of the combinatorial string vertex \( \hat{V} \) yields a Maurer-Cartan element of the form

\[
\hat{\beta}^C := \rho^C \hat{V} = \sum_{g,k,l} \rho^C(\hat{V}_{g,k,l}) \hbar^g \lambda^{2g-2+k+l}.
\]

Sometimes we also denote the components of \( \hat{\beta}^C \) by \( \hat{\beta}_{g,k,l}^C := \rho^C(\hat{V}_{g,k,l}) \).

### 4.2 Weyl algebras and Fock spaces

We may use the Mukai pairing to define a symplectic pairing on the Tate complex \( L^{\text{Tate}} \) explicitly defined by

\[
\langle \alpha u^i, \beta u^j \rangle_{\text{res}} := (-1)^i \delta_{i+j=1} \cdot \langle \alpha, \beta \rangle_{\text{Muk}}
\]

Associated with the symplectic vector space \( (L^{\text{Tate}}, \langle -,-\rangle_{\text{res}}) \) is the Weyl algebra (with two formal variables \( \hbar \) and \( \lambda \) of even parity), defined as

\[
W(L^{\text{Tate}}) := T(L^{\text{Tate}})[[\hbar, \lambda]]/(\alpha \otimes \beta - (-1)^{[\alpha][\beta]} \beta \otimes \alpha = \hbar \langle \alpha, \beta \rangle_{\text{res}}).
\]

We shall also need a localized (inverting of \( \hbar \)) and completed version of the Weyl algebra \( W(L^{\text{Tate}}, \langle -,-\rangle_{\text{res}}) \). We define the localized and completed Weyl algebra denoted by

\[
\hat{W}_h(L^{\text{Tate}}) := \lim_{\leftarrow n} W(L^{\text{Tate}})[\hbar^{-1}]/(\lambda^n)
\]

to be the original Weyl algebra \( W(L^{\text{Tate}}) \) localized at \( \hbar \), and then completed in the \( \lambda \)-adic topology. For instance, infinite power series of the form

\[
\sum_{k \geq 0} \alpha_k \lambda^k \hbar^{-k}
\]

would exist in \( \hat{W}_h(L^{\text{Tate}}) \), but not in \( W(L^{\text{Tate}}) \).

The positive subspace \( u \cdot L_+ \subset L^{\text{Tate}} \) is a sub-complex. Hence the left ideal generated by this subspace \( (u \cdot L_+) \) is a dg-ideal. Its quotient, known as the Fock space

\[
F := W(L^{\text{Tate}})/(u \cdot L_+)
\]

is a left dg-module of the Weyl algebra. On the other hand, the linear subspace \( L_- \subset L^{\text{Tate}} \) is not a sub-complex. However, it is still isotropic with respect to the residue
pairing, which yields an embedding of algebras $SL_-[[h, \lambda]] \hookrightarrow W(L^{\text{Tate}})$. Post-composing with the canonical projection to the Fock space yields an isomorphism (as graded vector spaces):

$$SL_-[[h, \lambda]] \cong F.$$ 

Similar to the previous paragraph, we denote by $\hat{S}_h L_-[[h, \lambda]]$ and $\hat{F}_h$ the localized and completed versions of the symmetric algebra and the Fock space.

A key observation due to Costello [15] is that under the above isomorphism, the differential on the Fock space pulls back to the differential $b + uB + h\Delta$ on the symmetric algebra $SL_-[[h, \lambda]]$. The additional operator $\Delta$ defines a differential graded Batalin–Vilkovisky algebra structure, which in particular yields a DGLA structure on $SL_-[[h, \lambda]][1]$ with its Lie bracket defined by $\{x, y\} := (-1)^{|x|}(\Delta(x \cdot y) - \Delta x \cdot y - (-1)^{|x|}x \cdot \Delta y)$. Inside the DGLA $SL_-[[h, \lambda]][1]$ lies the scalars $\mathbb{K}[[h, \lambda]][1]$ which is clearly central.

Let us denote the DGLA’s by

$$h_C := SL_-[[h, \lambda]][1],$$
$$h_C^+ := SL_-[[h, \lambda]][1]/\mathbb{K}[[h, \lambda]][1] = S^\geq 1 L_-[[h, \lambda]][1]$$

where the space $S^\geq 1 L_-[[h, \lambda]]$ is endowed with the quotient DGLA structure. From this we obtain a short exact sequence of DGLA’s:

$$0 \rightarrow \mathbb{K}[[h, \lambda]][1] \rightarrow h_C \rightarrow h_C^+ \rightarrow 0,$$

**Lemma 4.4.** Assume the $A_\infty$-category $\mathcal{C}$ (which is already proper by our assumption) is smooth and satisfies the Hodge-to-de-Rham degeneration property. Then there is a quasi-isomorphism of DGLA’s

$$\iota : h_C^+ \rightarrow \hat{h}_C.$$ 

Using the above lemma, we obtain a unique (up to gauge equivalence) Maurer-Cartan element $\beta_C \in SL_-[[h, \lambda]][1]$ such that $\iota \beta_C$ is gauge equivalent to $\hat{\beta}_C \in \hat{h}_C$. The previous discussions may be summarized into the following diagram of DGLA’s:

$$\begin{array}{ccc}
\hat{\mathfrak{g}} & \rightarrow & \hat{h}_C \\
\downarrow_{\rho_C} & & \\
\hat{h}_C & \rightarrow & \hat{h}_C \\
\hat{\beta}_C & \rightarrow & \beta_C & \rightarrow & \hat{\beta}_C = \rho_C^* \hat{\mathfrak{g}}
\end{array}$$

In [7], it was proved that the Maurer-Cartan element $\beta_C$ can be lifted to a Maurer-Cartan element $\hat{\beta}_C \in h_C$. The set of liftings is a torsor over the additive group underlying the scalars $\mathbb{K}[[h, \lambda]]$. A preferred lifting may be chosen using the Dilaton equation. Nevertheless, the particular choice of lifting will not be important in this paper.
For a differential graded Batalin–Vilkovisky algebra, it is well-known that the Maurer-Cartan equation satisfied by $\tilde{\beta} C$ is equivalent to the equation
\[(b + uB + h\Delta) \exp\left(\frac{\tilde{\beta} C}{h}\right) = 0,\]
where the left hand side is computed in the localized and completed Fock space $\hat{S}_h L_+[[h, \lambda]]$. From this, we obtain a $(b + uB + h\Delta)$-homology class
\[[\exp\left(\frac{\tilde{\beta} C}{h}\right)] \in H_\bullet(\hat{S}_h L_-[[h, \lambda]])].
This was called the abstract total descendant potential of $C$. Note that the abstract total descendant potential of $C$ is independent of the choice of string vertices by the homotopy uniqueness.

4.3 Trivializations and CEI

To obtain a more familiar looking invariants, we need to “trivialize” the BV operator $\Delta$. This shall involve a choice of splittings of the non-commutative Hodge filtration, whose definition we recalled in Definition 3.14.

Let $s : H_\bullet(L) \to H_\bullet(L_+)$ be a splitting of the Hodge filtration of $C$. Extending it $u$-linearly yields an isomorphism denoted by $\tilde{s} : H_\bullet(L)[[u]] \to H_\bullet(L_+)$. Further localizing at $u$ gives an isomorphism still denoted by
\[\tilde{s} : H_\bullet(L)((u)) \to H_\bullet(L_{\text{Tate}}).\]

We may also define a symplectic structure on the left hand side by endowing it with the pairing
\[\langle \alpha u^i, \beta u^j \rangle_{\text{res}} := (-1)^i \delta_{i+j=1} \cdot \langle \alpha, \beta \rangle_{\text{Muk}}\]

Conditions $(S1.)$ and $(S2.)$ above are equivalent to requiring that

- $\tilde{s}$ is an isomorphism of symplectic vector spaces.
- $\tilde{s}$ is compatible with the action by $u$, takes $H_\bullet(L_+)$ to $H_\bullet(L_+)$, and is the identity map on the associated graded spaces of the $u$-filtration.

Indeed, these conditions are taken as the definition of splittings in [15, Lemma 11.0.12]. Let us proceed to consider the following diagram:
\[
\begin{array}{ccc}
\hat{W}_h(H_\bullet(L)_{\text{Tate}}) & \xrightarrow{\Phi_s} & \hat{W}_h(H_\bullet(L_{\text{Tate}})) \\
\uparrow \psi^s & & \downarrow p \\
\hat{S}_h H_\bullet(L_-[[h, \lambda]]) & \xrightarrow{\Psi_s} & H_\bullet(\hat{F}_h) \cong H_\bullet(\hat{S}_h L_-[[h, \lambda]])
\end{array}
\]
Here the map \( \Phi^s \) is induced by the symplectic isomorphism \( \tilde{s} \). The left vertical map is the inclusion of the symmetric algebra generated by the Lagrangian subspace \( H_*(L)_- \) into the Wyel algebra, while the right vertical map is the canonical quotient map from the Wyel algebra to the Fock space. And we define \( \Psi^s = p\Phi^si \) to be the composition of these three maps so that the diagram above is commutative.

**Definition 4.5.** The total descendant potential of the pair \((C, s)\) is defined by the preimage of the abstract total descendant potential \( \exp(\frac{\beta^C}{\hbar}) \), i.e. we set

\[
\mathcal{D}_{C, s} := (\Psi^s)^{-1}\exp(\frac{\beta^C}{\hbar}),
\]

The \( n \)-point function of genus \( g \) denoted by \( F^{C, s}_{g, n} \in S^g H_*(L)_- \) is defined by the identity

\[
\sum_{g, n} F^{C, s}_{g, n} \cdot \hbar^g \lambda^{2g-2+n} := \hbar \cdot \ln \mathcal{D}_{C, s}.
\]

From the definition above, it is not immediately clear how to compute these invariants. In [7], an explicit formula is obtained expressing \( F^{C, s}_{g, n} \)’s using \( \beta^C_{g, n} \)’s and the splitting data \( s \). The formula is obtained by constructing a \( L_\infty \) quasi-isomorphism (which depends on the splitting data \( s \)):

\[
(24) \quad K^s : \mathfrak{h}_C \to \mathfrak{h}_C^{\text{triv}}
\]

where \( \mathfrak{h}_C^{\text{triv}} \) has the same underlying graded vector space as \( \mathfrak{h}_C \) but is endowed with the differential \( b + uB \) and zero Lie bracket. Namely, the morphism \( K^s \) kills the extra differential \( \hbar \Delta \) and the Lie bracket \( \{-, -\} \) of \( \mathfrak{h}_C \). The explicit formula of \( F^{C, s}_{g, n} \) is then obtained by pushing forward the Maurer-Cartan element \( \tilde{\beta}^C \) via the morphism \( K^s \), i.e. we have

\[
(25) \quad \sum_{g, n} F^{C, s}_{g, n} \hbar^g \lambda^{2g-2+n} = K^s \tilde{\beta}^C.
\]

The right hand side can be expanded using the definition of \( K^s \) as a stable graph sum.

**Lemma 4.6.** Let \( C \) be a minimal, unital, cyclic \( A_\infty \)-category. Let \( I : \mathcal{A} \hookrightarrow C \) be an inclusion from a fully faithful \( A_\infty \)-subcategory. Assume that \( I \) induces an isomorphism \( I_* : HH_*(\mathcal{A}) \to HH_*(C) \). Let \( s \) be a splitting of the Hodge filtration of \( \mathcal{A} \). Then, for any \((g, n)\) such that \( 2g - 2 + n > 0 \), we have

\[
I_* F^{\mathcal{A}, s}_{g, n} = F^{C, I_* s I_*^{-1}}_{g, n}.
\]
Proof. As $I$ is a linear $A_{\infty}$-embedding we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{h}_{\mathcal{C}} & \xrightarrow{I_*} & \mathfrak{h}_{\mathcal{C}} \\
\downarrow & & \downarrow^R \\
\mathfrak{h}_{\mathcal{A}} & \xrightarrow{I_*} & \mathfrak{h}_{\mathcal{A}}
\end{array}
\]

From this, we claim that $I_*\beta^\mathcal{A}$ is gauge equivalent to $\beta^\mathcal{C}$. Indeed, since the morphisms in the square are all quasi-isomorphisms of DGLA’s, we have

\[
I_*\beta^\mathcal{A} \cong \beta^\mathcal{C} \\
\Leftrightarrow R_I I_*\beta^\mathcal{A} \cong R_I \beta^\mathcal{C} \\
\Leftrightarrow R_I I_*\beta^\mathcal{A} \cong R \beta^\mathcal{C} \\
\Leftrightarrow R_I I_*\beta^\mathcal{A} \cong \rho^\mathcal{C} \hat{\nu} \\
\Leftrightarrow R_I I_*\beta^\mathcal{A} \cong \rho^\mathcal{A} \hat{\nu} \\
\Leftrightarrow I_*\beta^\mathcal{A} \cong \hat{\beta}^\mathcal{A},
\]

proving the claim by definition of $\beta^\mathcal{A}$. Since $\mathcal{A} \subset \mathcal{C}$ is a subcategory, we have a commutative diagram of the trivialization map:

\[
\begin{array}{ccc}
\mathfrak{h}_{\mathcal{A}} & \xrightarrow{K^*} & \mathfrak{h}_{\mathcal{A}}^{\text{triv}} \\
\downarrow & & \downarrow \\
\mathfrak{h}_{\mathcal{C}} & \xrightarrow{K^* I_* I_*^{-1}} & \mathfrak{h}_{\mathcal{C}}^{\text{triv}}
\end{array}
\]

The Lemma then follows from Equation (25):

\[
I_* \sum_{g,n} F_{g,n}^A h^g \chi^{2g-2+n} = I_* K_* \beta^\mathcal{A} = K_* I_* I_*^{-1} I_* \beta^\mathcal{A} = K_* I_* I_*^{-1} \beta^\mathcal{C} = \sum_{g,n} F_{g,n}^C I_* I_*^{-1} h^g \chi^{2g-2+n}
\]

\[\square\]

5 Morita invariance of CEI

5.1 Cyclic Pseudo-isotopies

The notion of cyclic pseudo-isotopy was introduced by Fukaya [17]. Let $\mathcal{C}$ be an $A_{\infty}$-category over a field $\mathbb{K}$ of characteristic zero. Denote its $A_{\infty}$-composition maps by $m_k$ ($k \geq 1$). Let $\mathcal{C}'$ be an $A_{\infty}$-category which has the same underlying objects and Hom-space as $\mathcal{C}$, but is endowed with $A_{\infty}$-maps $m'_k$ ($k \geq 1$). Without loss of generality, we assume both $\mathcal{C}$ and $\mathcal{C}'$ be minimal, i.e. $m_1 = m'_1 = 0$. 

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Lemma 5.2. Let $F$ be a cyclic functor, then $F$ is called unital if $F_{0}(t)$ is unital, meaning $\sum_{r,i+j=k+1}(-1)^{r+i+j}|x_{i}|^{r+i+j}|x_{j}|^{r+j}l_{i}(t)(x_{1},...,x_{r},m_{j}(t)(x_{r+1},...,x_{j+r}),...,x_{k})$

Finally, we require the boundary conditions $m_{k}(0) = m_{k}$, while $m_{k}(1) = m'_{k}$.

Assume furthermore that the $A_{\infty}$-category $C$ is endowed with a cyclic structure $\langle -, - \rangle$, and $C'$ is also endowed with the same cyclic pairing.

- A minimal pseudo-isotopy $m(t) + l(t)dt$ between $C$ and $C'$ is called cyclic if the structure maps $m_{k}(t)$'s and $l_{k}(t)$'s are all cyclic with respect to $\langle -, - \rangle$, meaning they satisfy Equation (12).

- It is called unital if $m_{k}(t)$’s form a strictly unital $A_{\infty}$ structure, and the $l_{k}(t)$ are reduced cochains.

Lemma 5.2. Let $(C, \langle -, -, \rangle, \{m_{k}\})_{k=2}^{\infty}$ and $(C', \langle -, -, \rangle, \{m'_{k}\})_{k=2}^{\infty}$ be two minimal $A_{\infty}$ categories with the same underlying set of objects and $\text{Hom}$-spaces. Let $F : C \rightarrow C'$ be an $A_{\infty}$-functor with $F_{1} = \text{id}$. Then there is a degree one Hochschild cochain $Z \in CC^{1}(C)$ of order two, such that its flow, that is the family of $A_{\infty}$-pre-functors determined by Lemma 2.30, satisfies $F^{1} = F$.

If $F$ is unital, then $Z$ is reduced and therefore the $F^{t}$ are unital. If $C$ and $C'$ are cyclic and $F$ is a cyclic functor, then $Z$ and $F^{t}$ are cyclic.

Proof. Let $\widehat{F}$ be the colagebra extension of $F$, as before, and denote by $G = \widehat{F} - \text{id}$. Since $F_{1} = \text{id}$, the map $G$ decreases the length of elements in $BC$. Therefore the following map is convergent:

$\hat{Z} := \log(\widehat{F}) = \log(\text{id} + G) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{G^{(n)}}{n}$

This is a coderivation associated to a degree one Hochschild cochain $Z$ of order 2. Since $Z$ is independent of $t$, the differential equation in Lemma 2.30 can be easily solved:

$\widehat{F}^{t} = \exp(t\hat{Z}) = \text{id} + \sum_{n=1}^{\infty}\frac{t^{n}\hat{Z}^{(n)}}{n!}$.  

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Then we have, $\hat{F}^t = \exp(\log(\hat{F})) = \hat{F}$.

When $F$ is unital, it is clear from the construction that $Z$ is reduced. Finally when $F$ is cyclic, then $Z$ is cyclic, see [21, Proposition 4.14]. This is equivalent to $S(\iota_Z\rho) = 0$, where $\rho$ is the cyclic structure. Then using Lemmas 2.31 and 2.29 we calculate

$$\frac{d}{dt}(F^t)^*\rho = (F^t)^*(L_Z\rho) = (F^t)^*(S(\iota_Z\rho)) = 0.$$ 

Hence $(F^t)^*\rho = (F^0)^*\rho = \rho$. $\square$

**Proposition 5.3.** Let $(\mathcal{C}, \langle -, - \rangle, \{\mathfrak{m}_k\}_{k=2}^\infty)$ and $(\mathcal{C}', \langle -, - \rangle, \{\mathfrak{m}'_k\}_{k=2}^\infty)$ be two minimal cyclic $A_\infty$-categories with the same underlying set of objects, $Hom$-spaces and cyclic structure. Then there exists an unital, cyclic $A_\infty$ functor 

$$f = (f_1, f_2, \ldots) : \mathcal{C} \to \mathcal{C}'$$

with $f_1 = id$ if and only if there exists an unital, cyclic, minimal pseudo-isotopy $\mathfrak{m}(t) + l(t)dt$.

**Proof.** Assume that $f = (id, f_2, f_3, \ldots)$ is such a cyclic isomorphism. Since $f_1 = id$, $f$ has a strict inverse $f^{-1}$ and by the previous lemma there is a cyclic, reduced $\xi$, whose flow $f^{-1}$ satisfies $f^{-1} = F^1$.

Pulling-back the $A_\infty$-structure $\mathfrak{m}$ via $F^t$ yields a family of cyclic $A_\infty$ structures $\mathfrak{m}(t) := (F^t)^*\mathfrak{m} = F^{-t} \circ \hat{\mathfrak{m}} \circ \hat{F}^t$ such that

$$\mathfrak{m}(0) = \mathfrak{m} \text{ and } \mathfrak{m}(1) = \mathfrak{m}'.$$ 

Next, we verify that $\mathfrak{m}(t) + \xi dt$ forms a cyclic pseudo-isotopy. Indeed, we have

$$\frac{d}{dt} \mathfrak{m}(t) = \frac{d}{dt}(F^{-t} \circ \hat{\mathfrak{m}} \circ \hat{F}^t)$$

$$= -\xi \circ \hat{F}^{-t} \circ \hat{\mathfrak{m}} \circ \hat{F}^t + F^{-t} \circ \hat{\mathfrak{m}} \circ \hat{\xi} \circ \hat{F}^t$$

$$= -\xi \circ \hat{F}^{-t} \circ \hat{\mathfrak{m}} \circ \hat{F}^t + F^{-t} \circ \hat{\mathfrak{m}} \circ \hat{F}^t \circ \hat{\xi}$$

$$= \mathfrak{m}(t) \circ \hat{\xi} - \xi \circ \hat{\mathfrak{m}}(t) = [\mathfrak{m}(t), \xi].$$

In the second equality we used the differential equation defining $F^t$ and on the third we used that $\hat{\xi}$ and $\hat{F}^t$ commute, as can be seen from the explicit description of $F^t$ in the previous lemma.

The other direction in the statement is proved by Fukaya [17]. $\square$
5.2 Homotopic TCFT’s

Let \( m(t) + l(t) dt \) be a minimal, unital and cyclic pseudo-isotopy on an \( A_\infty \)-category category \( C \). We shall use the notation \( C \otimes \Omega^* \) to denote this one parameter family of \( A_\infty \)-structures. Our next goal is to extend the construction of TCFT’s in Theorem 4.1 to pseudo-isotopic families. In order to do this, it is necessary to deal with the signs. The sign was discussed in [42, Appendix], and will be treated with greater detail in [10].

For exposition purpose, we shall only describe the construction of the chain map \( \rho^C \) in Theorem 4.1 with that the Calabi-Yau dimension \( d \equiv 0 \pmod{2} \) and with the number of output \( l = 0 \). The discussion in general case is essentially similar, but involves more work, which we refer to [10].

Keeping the above assumptions in mind, let us consider a running example of ribbon graph \( G \in C_1^{\text{comb}}(M_{1,1,0}^\text{fr}) \) depicted as

In the first step we redraw the graph \( G \) in the manner as illustrated in the black part of following picture.
In doing so, we need to choose
(1.) an ordering of vertices,
(2.) an ordering of half-edges of $G$ (including leaves),
(3.) an ordering of edges.

Also by definition an orientation of a ribbon graph is given by a choice of (1.) and (2.). We choose (1.) and (2.) so that it is compatible with the orientation of $G$. Since the Calabi-Yau pairing is of parity $d$ which we assumed to be even, the choice of (3.) is not relevant. In the blue part of the picture, there is also a Hochschild chain $x_0|x_1|x_2$ which we insert into the unique input cycle of the ribbon graph $G$. The red lines are attached to odd maps: the top red line is due to the shift map $s : x_0|x_1|x_2 \mapsto sx_0|x_1|x_2$; while the bottom red lines are due to the odd map

$$\tilde{m}_v := \langle m_v | -1(-,-,-), - \rangle$$

associated with black vertices of $G$.

In the evaluation $\rho^C(G)$, we simply put the Koszul sign associated with the permutation $\sigma_G$ as shown in the dashed box. Observe that this is indeed well-defined, i.e. we need to check that $\rho^C(G)$ flips its sign under (1.) and (2.), and is unchanged under (3.). At a black vertex $v$, its associated map is odd map, which shows that switching $v_1$ with $v_2$ indeed yields a sign $-1$. At an edge $e$, we put the inverse tensor of the pairing $\langle -, - \rangle^{-1}$ which is even, and hence the ordering of edges doesn’t matter in this case. However, the orientation of each edge does flip the sign since the pairing $\langle -, - \rangle$ is anti-symmetric in the shifted degree.

It is useful to write down the evaluation $\rho^C(G)$ (as explained above graphically) by the following equation:

$$\rho^C(G) := \left( \prod_{v \in V_G^{\text{black}}} \tilde{m}_v \right) \circ \sigma_G \circ \left( \underbrace{s \otimes \cdots \otimes s}_{k \text{ copies}} \otimes \langle -,- \rangle^{-1} \otimes \cdots \otimes \langle -,- \rangle^{-1} \right)$$

We choose the orderings involved in the above formula so that

$$v_1 \wedge \cdots \wedge v_n \wedge l_1 \wedge \cdots \wedge l_k \wedge e_1^+ \wedge e_1^- \wedge \cdots \wedge e_m^+ \wedge e_m^-$$

agrees with the orientation of $G$, with $n = |V_G^{\text{black}}|$ and $m = |E_G|$.

With the above construction of $\rho^C$, one can prove that it is a chain map, i.e.

$$\rho^C(\partial G) = [L_m, \rho(G)].$$

See the Appendix A for a proof of this equation.

Let $\Omega^\bullet := \mathbb{K}[t, dt]$ be the space of polynomial differential forms on $\mathbb{A}^1$ endowed with the de Rham differential.
Theorem 5.4. There exist chain maps
\[ \rho^C_{g,k,l} : C_{\text{comb}}(M_{g,k,l}) \rightarrow \text{Hom}(CC_*(C)[d] \otimes^k_\mathbb{K} \Omega^*) \]
with \( g \geq 0, k \geq 1, l \geq 0, \) and \( 2 - 2g - k - l < 0. \) The differential on the right hand side is \( L_{\text{m}(t)} + L_{\text{m}(t)} \, dt + d_{\text{DR}}. \) Furthermore, these chain maps satisfy

(1.) The restriction \( \rho^C_{g,k,l} |_{t=t_0} = (\rho^C_{g,k,l}(t_0)) \) for any fixed \( t_0. \)

(2.) They are compatible with the composition maps on both sides.

Proof. The action map \( \rho^C_{g,k,l} \) is defined in the same way as in Theorem 4.1, except at one of the black vertices we put the operator \( l \, dt. \) Again, we shall only describe the construction in the case \( l = 0 \) and \( d \equiv 0 \) \((\text{mod } 2).\) In view of Equation (26), we set

(28) \[ \rho^C_{g,k,l} \left( G \right) := \left( \prod_{v \in V_G^{\text{black}}} (\tilde{m}_v + \tilde{v}_v \, dt) \right) \circ \sigma_G \circ \left( s \otimes \cdots \otimes s \otimes \langle -, - \rangle^{-1} \otimes \cdots \otimes \langle -, - \rangle^{-1} \right) \]

Since \( dt^2 = 0, \) we have

\[ \rho^C_{g,k,l} \left( G \right) = \rho^C_{[0]}_{g,k,l} \left( G \right) \]

(29) \[ \rho^C_{[0]} \left( G \right) := \left( \prod_{v \in V_G^{\text{black}}} (\tilde{m}_v) \right) \circ \sigma_G \circ \left( s \otimes \cdots \otimes s \otimes \langle -, - \rangle^{-1} \otimes \cdots \otimes \langle -, - \rangle^{-1} \right) \]

To verify that \( \rho^C \) defined as above is indeed a chain map, we need to prove

\[ \rho^C \left( \partial G \right) = [L_m + L_0 \, dt + d_{DR}, \rho^C \left( G \right)] \]

In the case \( l = 0, \) writing the above equation in its components yields

(29) \[ \left( \begin{array}{c}
\rho^C_{[0]}(\partial G) + (-1)^{|G|} \rho^C_{[0]}(G)L_m = 0 \\
\rho^C_{[1]}(\partial G) + (-1)^{|G|} \rho^C_{[0]}(G)L_0 \, dt + (-1)^{|G|} \rho^C_{[1]}(G)L_m = [d_{DR}, \rho^C_{[0]}(G)]
\end{array} \right) \]

The top equation is the same as in Equation (27). For the second equation, since

\[ [d_{DR}, \rho^C_{[0]}(G)] = \]

\[ \sum_{j=1}^n (-1)^j (\tilde{m}_v \otimes \cdots \otimes [\tilde{m}, \tilde{v}_j \, dt] \otimes \cdots \otimes \tilde{m}_{v_n}) \circ \sigma_G \circ \left( s \otimes \cdots \otimes s \otimes \langle -, - \rangle^{-1} \otimes \cdots \otimes \langle -, - \rangle^{-1} \right) \]

\[ k \text{ copies} \]

\[ |E_G| \text{ copies} \]
In this equation, there are three types of configurations in the above composition, depicted in the following picture.

The leftmost case is when the inputs of the upper vertex involves an edge in $G$, this term corresponds to $(i)$ in Equation (23) above. The middle and the rightmost cases correspond to $(ii)$ and $(iii)$ respectively. The signs involved here are explained in the Appendix A.

\[ \begin{align*}
\text{[L_m, L_idt]} & , \\
L_m \circ L_idt & , \\
L_idt \circ L_m & .
\end{align*} \]

5.3 Homotopic DGLA’s

One of the main difficulties in proving that CEI is invariant under unital cyclic $A_\infty$-isomorphisms is that the construction of the DGLA $\hat{h}_C$ in Equation (21) associated with $\mathcal{C}$ is not functorial. To see this, it suffices to observe that the Mukai pairing is already not functorial on the chain-level. To get around this issue, in this subsection we extend the construction of $\hat{h}_C$ associated with an $A_\infty$ category $\mathcal{C}$ to the family version. Again, we shall work with $\mathfrak{m}(t) + \mathfrak{l}(t)dt$ that is a minimal, cyclic and unital pseudo-isotopy on an $A_\infty$-category $\mathcal{C}$.

The family version of the DGLA construction is defined as follows. As a graded vector space, it is given by

\[ \hat{h}_{C \otimes \Omega^\bullet} := \bigoplus_{k \geq 1, l \geq 0} \text{Hom}^c(S^k(L_+[1]), S^l(L_-)[2][[h, \lambda]] \otimes_k \Omega^\bullet). \]

Its DGLA structure is analogously defined as in Subsection 4.1:

- Its differential is of the form $b + uB + d_{DR} + \iota + h\Delta$ with $b = \mathcal{L}_{\mathfrak{m}(t)}$ the Hochschild differential, $\Delta$ is the self twisted sewing operator, i.e. sewing with $\rho^c \otimes \Omega^\bullet(M)$, and the map $\iota$ is sewing with $\rho_c \otimes \Omega^\bullet(M)$.

- Its Lie bracket is of the form $\{-, -\}_r := \sum_{r \geq 1} \frac{1}{r!} \cdot \{-, -\}_r h^{r-1}$ with $\{-, -\}_r$ the $r$-th twisted sewing operator. This is the same Lie bracket (extended $\Omega^\bullet$ linearly) as in the DGLA $\hat{h}_C$. 

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By definition, the isotopic family of TCFT’s $\rho^{\mathcal{C} \otimes \Omega^\bullet}$ then gives us a morphism of DGLA’s:

$$\rho^{\mathcal{C} \otimes \Omega^\bullet} : \hat{g} \to \hat{h}_{\mathcal{C} \otimes \Omega^\bullet},$$

which specializes to $\rho^{(\mathcal{C}, m(t_0))}$ in Equation (22) for any fixed $t_0$.

### 5.4 Cyclic invariance

Let $(\mathcal{C}, \langle -, - \rangle, \{m_k\})$ and $(\mathcal{C}', \langle -, - \rangle, \{m'_k\})$ be two minimal, unital and cyclic $A_\infty$-categories with the same underlying set of objects, $\text{Hom}$-spaces, and the cyclic pairing. Let $f$ be an unital and cyclic $A_\infty$ functor $f = (f_1, f_2, \ldots) : \mathcal{C} \to \mathcal{C}'$ with $f_1 = \text{id}$.

For a splitting map $s : H_* (L_{\mathcal{C}})^{\text{Tate}} \to H_* (L_{\mathcal{C}'})^{\text{Tate}}$ of $\mathcal{C}$, there is a naturally associated splitting of $\mathcal{C}'$ defined by the composition

$$f_* s f_*^{-1} : H_* (L_{\mathcal{C}'})^{\text{Tate}} \to H_* (L_{\mathcal{C}})^{\text{Tate}} \to H_* (L_{\mathcal{C}'}) \to H_* (L_{\mathcal{C}'})^{\text{Tate}}.$$

**Theorem 5.5.** Let $f$ be an unital and cyclic $A_\infty$-isomorphism

$$f = (f_1, f_2, \ldots) : (\mathcal{C}, \langle -, - \rangle, \{m_k\}) \to (\mathcal{C}', \langle -, - \rangle, \{m'_k\})$$

with $f_1 = \text{id}$. Then we have

$$f_* F^{\mathcal{C}, s} = F^{\mathcal{C}'}, f_* s f_*^{-1}.$$

**Proof.** We first use Proposition 5.3 to construct a pseudo-isotopic family of $A_\infty$ categories given by

$$m(t) + l(t) dt := \exp(t \xi)^* m + \xi dt$$

with $f^{-1} = \exp(\xi)$. Using the associated isotopic family of TCFT’s we obtain a morphism $\rho^{\mathcal{C} \otimes \Omega^\bullet} : \hat{g} \to \hat{h}_{\mathcal{C} \otimes \Omega^\bullet}$ of DGLA’s. Use it to push-forward the string vertex $\hat{\mathcal{V}}$ we obtain a Maurer-Cartan element $\hat{\beta}^{\mathcal{C} \otimes \Omega^\bullet} = \rho^{\mathcal{C} \otimes \Omega^\bullet}_* (\hat{\mathcal{V}})$. Consider the following

$$\begin{array}{cccc}
\hat{g} & \downarrow \rho^{\mathcal{C} \otimes \Omega^\bullet} \\
\hat{h}_{\mathcal{C} \otimes \Omega^\bullet} & \longrightarrow & \hat{h}_{\mathcal{C} \otimes \Omega^\bullet}^+ & \longrightarrow & \hat{h}_{\mathcal{C} \otimes \Omega^\bullet}^\epsilon & \longrightarrow & \hat{h}_{\mathcal{C} \otimes \Omega^\bullet}.
\end{array}$$

Since $\epsilon$ is a quasi-isomorphism of DGLA’s, denote by $\beta^{\mathcal{C} \otimes \Omega^\bullet}$ the unique (up to gauge equivalence) Maurer-Cartan element of $\hat{h}_{\mathcal{C} \otimes \Omega^\bullet}^+$, such that $\epsilon (\beta^{\mathcal{C} \otimes \Omega^\bullet})$ is gauge equivalent to $\hat{\beta}^{\mathcal{C} \otimes \Omega^\bullet}$. As in Subsection 4.2, we may consider $\beta^{\mathcal{C} \otimes \Omega^\bullet}$ as a Maurer-Cartan element of $\hat{h}_{\mathcal{C} \otimes \Omega^\bullet}^+$ via the inclusion $\hat{h}_{\mathcal{C} \otimes \Omega^\bullet}^+ \subset \hat{h}_{\mathcal{C} \otimes \Omega^\bullet}$.

Next, we proceed to construct the family version of the trivialization map used in Equation (24) to compute CEI. Recall from *Loc. Cit.* this is an $L_\infty$ quasi-isomorphism

$$\mathcal{K} : h_{\mathcal{C} \otimes \Omega^\bullet} \to h_{\mathcal{C} \otimes \Omega^\bullet}^{\text{triv}}.$$
The construction of $K$ is similar to that of $K$, and relies on a homotopy trivialization of the twisted sewing map map $\rho^{C\otimes\Omega^*}(M) = \rho^{C\otimes\Omega^*}(\bullet \to \cdot)$. That is, we need to construct an operator
\[
\mathcal{H} : L_- \otimes L_- \to \mathbb{K}[t, dt]
\]
such that
\[
[b(t) + uB + d_{DR} + \mathcal{L}_\xi dt, \mathcal{H}](\alpha u^{-i}, \beta u^{-j}) = \begin{cases} 
\rho^{C\otimes\Omega^*}(M)(\alpha, \beta) & \text{if } i = j = 0; \\
0 & \text{otherwise.}
\end{cases}
\]
Indeed, let us choose a chain level lift of the splitting map $s$ of the form
\[
R = \text{id} + R_1u + R_2u^2 + \cdots \in \text{End}(L_C)[[u]].
\]
From this, we form the operator
\[
R(t) := \exp(-t\xi)_* \circ R \circ \exp(t\xi)_* \in \text{End}(L_C)[[u]]
\]
which is a chain level lift of the splitting map $\exp(-t\xi)_* \circ s \circ \exp(t\xi)_*$. Denote by $T(t)$ the inverse operator of $R(t)$. As in [7, Proposition 4.5], we set
\[
\mathcal{H}(\alpha u^{-i}, \beta u^{-j}) := \rho^{C\otimes\Omega^*}(M)\left((-1)^j \sum_{i=0}^{j} R_iT_{i+j+1-i}\alpha, \beta\right)
\]
To verify the commutator identity in Equation (30), in the case of $i = j = 0$, we compute as
\[
[L_m(t) + d_{DR} + \mathcal{L}_\xi dt, \mathcal{H}](\alpha, \beta)
= -\rho^{C\otimes\Omega^*}(M)(T_1L_m(t) + d_{DR} + \mathcal{L}_\xi dt)(\alpha, \beta) - (-1)^{\lvert\alpha\rvert}\rho^{C\otimes\Omega^*}(M)(T_1\alpha, (L_m(t) + d_{DR} + \mathcal{L}_\xi dt)\beta)
=\rho^{C\otimes\Omega^*}(M)((L_m(t) + d_{DR} + \mathcal{L}_\xi dt)T_1\alpha, \beta) - \rho^{C\otimes\Omega^*}(M)(T_1(L_m(t) + d_{DR} + \mathcal{L}_\xi dt)\alpha, \beta)
=\rho^{C\otimes\Omega^*}(M)([L_m(t), T_1]\alpha, \beta) + \rho^{C\otimes\Omega^*}(M)((d_{DR}T_1)\alpha, \beta) + \rho^{C\otimes\Omega^*}(M)([\mathcal{L}_\xi dt, T_1]\alpha, \beta)
\]
To this end, observe that we have $[L_m(t), T_1] = B$ since $R$ is a chain map. For the term $d_{DR}T_1$ we have
\[
d_{DR}T_1 = d_{DR}(\exp(-t\xi)_* T_1 \exp(t\xi)_*) = -[\mathcal{L}_\xi dt, T(t)_1]
\]
Putting together we obtain
\[
[L_m(t) + d_{DR} + \mathcal{L}_\xi dt, \mathcal{H}](\alpha, \beta) = \rho^{C\otimes\Omega^*}(M)(B\alpha, \beta) = \rho^{C\otimes\Omega^*}(M)(\alpha, \beta),
\]
as we have desired. The other cases with $i \neq 0$ or $j \neq 0$ can be computed similarly using the commutator relations proved in [7, Proposition 4.5].
Using the homotopy operator $\mathcal{H}$, we obtain an $L_\infty$ map

$$\mathcal{K} : h_{\mathfrak{C} \otimes \Omega^\bullet} \to h_{\mathfrak{C} \otimes \Omega^\bullet}^{\text{triv}}.$$  

defined using a stable graph sum formula [7, Section 4.8].

Since by construction $\mathcal{K}$ specializes to $K(0)$ and $K(1)$ at $t = 0$ and $t = 1$ respectively, the push-forward $\mathcal{K}_* \mathfrak{C} \otimes \Omega^\bullet$ specializes to $\sum_{g,n} F_{g,n}^C \alpha_{g} \lambda^{2g-2+n}$ and $\sum_{g,n} F_{g,n}^{C'} \alpha_{g} \lambda^{2g-2+n}$ at $t = 0$ and $t = 1$. The theorem now follows from the following lemma.

**Lemma 5.6.** Let $\gamma(t)$ be a Maurer-Cartan element of the trivialized DGLA $h_{\mathfrak{C} \otimes \Omega^\bullet}^{\text{triv}}$, i.e. it satisfies the equation

$$(b(t) + uB + d_{\mathcal{D}R} + L_\xi dt)\gamma(t) = 0.$$  

Then we have $[\exp(-\xi)_* \gamma(0)] = [\gamma(1)]$ in the $(b + uB)$-homology $H_* (h_{\mathfrak{C} \otimes \Omega^\bullet}^{\text{triv}})$.

**Proof.** Writing $\gamma(t)$ as $\theta(t) + \eta(t)dt$ and rewrite the Maurer-Cartan equation yields

$$(b(t) + uB)\theta(t) = 0$$

$$\frac{d}{dt} \theta(t) + (b(t) + uB)\eta(t) + L_\xi \theta(t) = 0$$

We shall prove the difference $D(t) := \exp(-t\xi)_* \theta(0) - \theta(t) \equiv 0$ in the $(b(t) + uB)$-homology of all fixed $t \in [0, 1]$. For this we differentiate, using Lemma 2.31,

$$\frac{d}{dt} D(t) = -L_\xi (\exp(-t\xi)_* \theta(0)) - \frac{d}{dt} \theta(t)$$

$$= -L_\xi D(t) + (b(t) + uB)\eta(t)$$

Taking $(b(t) + uB)$-homology both sides yields the equation $\frac{d}{dt} D(t) = -L_\xi D(t)$. We also have the initial condition that $D(0) = 0$. Hence the result follows from uniqueness of solutions of ODE’s. \qed

**Corollary 5.7.** Let $\mathcal{C}$ and $\mathcal{C}'$ be minimal, unital, cyclic $A_\infty$-categories and let $f$ be an unital and cyclic $A_\infty$-isomorphism

$$f = (f_1, f_2, \ldots) : (\mathcal{C}, \langle -, - \rangle, \{m_k\}) \to (\mathcal{C}', \langle -, - \rangle', \{m'_k\})$$

not necessarily with $f_1 = \text{id}$. Then we have

$$f_* F_{g,n}^{\mathcal{C},s} = F_{g,n}^{\mathcal{C}',s} f_* s^{-1}.$$
Proof. We may factor the $A_\infty$-equivalence $f$ as a composition

$$\mathcal{C} \xrightarrow{h} \tilde{\mathcal{C}} \xrightarrow{g} \mathcal{C},$$

where $g$ is a linear $A_\infty$-equivalence and $h$ has $h_1 = \text{id}$. We define $\tilde{\mathcal{C}}$ to have the same objects, morphism spaces and pairing as $\mathcal{C}$ and define the $A_\infty$-operations as $\tilde{m}_k = f_1^{-1} \circ m'_k \circ f^{\otimes k}$. This is possible since by the assumptions, $f_1$ are linear isomorphisms. We then define $h_k := f_1^{-1} \circ f_k$, for $k \geq 1$ and $g_1 = f_1, g_k = 0$ for $k \geq 2$. It is easy to check these define cyclic $A_\infty$-functors.

The functor $h$ satisfies the conditions in Theorem 5.5 and hence preserves the CEI. Since the $A_\infty$-isomorphism $g$ is linear, it induces isomorphisms on $\mathfrak{h}_\mathcal{C}$ and $\hat{\mathfrak{h}}_\mathcal{C}$, hence the invariance of the CEI is obvious in this case. 

5.5 Morita invariance of CEI

In this subsection, we formulate and prove the Morita invariance of CEI. Let $\mathcal{C}$ be an $A_\infty$-category (not necessarily cyclic) that is proper and smooth. In the $\mathbb{Z}/2\mathbb{Z}$-graded case, we also assume that it satisfies the Hodge-to-de-Rham degeneration property. Note that in the $\mathbb{Z}$-graded case, this condition is automatic by Kaledin’s theorem [22].

As in Subsection 3.3 we shall refer to elements of $HH_\bullet(\mathcal{C}) (HC^\lambda_\bullet(\mathcal{C}))$ satisfying the same non-degeneracy condition as weak (strong) Calabi-Yau structures, via the duality isomorphisms:

$$HH_\bullet(\mathcal{C})^\vee \cong HH_\bullet(\mathcal{C})$$
$$HC^\lambda_\bullet(\mathcal{C})^\vee \cong HC^\lambda_\bullet(\mathcal{C})$$

Fix a parity $d \in \mathbb{Z}/2\mathbb{Z}$. Associated with the $A_\infty$-category $\mathcal{C}$ we define a set $\mathcal{M}^d_\mathcal{C}$ consisting of pairs $(\omega, s)$ such that

- $\omega \in HH_\bullet(\mathcal{C})$ is a weak Calabi-Yau structure of parity $d$,
- $s$ is an unital splitting with respect some split-generating subcategory $\mathcal{A} \subset \text{tw}^\pi \mathcal{C}$ (see Definition 3.14).

We refer to a pair $(\omega, s)$ as an extended Calabi–Yau structure on $\mathcal{C}$.

One may also define variants of $\mathcal{M}^d_\mathcal{C}$ by requiring $s$ be good, or $\omega$-compatible, or both. Since the $u$-connection is Morita invariant [37], all these variants are also preserved under Morita equivalences.

Fix a pair of natural numbers $(g,n)$ such that $2g - 2 + n > 0$. We proceed to define a function

$$F_{g,n}^\mathcal{C} : \mathcal{M}^d_\mathcal{C} \to \text{Hom}^c (HH_\bullet(\mathcal{C})[d],[[u]]^\otimes n, \mathbb{K}) .$$

Indeed, given a pair $(\omega, s) \in \mathcal{M}^d_\mathcal{C}$, since $s$ is unital, we may choose a split-generating subcategory $\mathcal{A} \subset \text{tw}^\pi \mathcal{C}$ such that $s(\omega)$ admits a lift to an unital Calabi–Yau structure of
Thus, we may apply Proposition 3.13 (if \( \omega \) is even) or Proposition 3.12 (if \( \omega \) is odd) to obtain an unital and cyclic \( A_\infty \)-model:

\[
G_A : \mathcal{A}' \to \mathcal{A}.
\]

Denote by \((G_C)_*\) the induced map on the Hochschild invariants associated with the following zig-zag compositions:

\[
G_C : \mathcal{A}' \to \mathcal{A} \hookrightarrow \mathcal{C} \twoheadleftarrow \mathcal{C}.
\]

Define \( F_{g,n}^C(\omega, s) \) to be the linear functional \( F_{g,n}^C,\omega,s \) explicitly given by

\[
F_{g,n}^C,\omega,s(\alpha_1, \ldots, \alpha_n) := \langle (G_C)_s^{-1} \alpha_1, \ldots, (G_C)_s^{-1} \alpha_n \rangle_{g,n} \mathcal{A}'(G_C)_s^{-1} \mathcal{A}'(G_C)_s.
\]

**Lemma 5.8.** *The definition of \( F_{g,n}^C,\omega,s \) is independent of the choice of the category \( \mathcal{A} \) as well as its unital cyclic \( A_\infty \)-model \( \mathcal{A}' \).*

**Proof.** When \( \omega \) is even, Proposition 3.13 guarantees any two cyclic, unital models are related by an unital, cyclic \( A_\infty \)-isomorphism. Applying Corollary 5.7 to this \( A_\infty \)-isomorphism implies the independence of the choice of \( \mathcal{A}' \). To argue the independence of the choice of \( \mathcal{A} \), if \( s(\omega) \) is unital with respect to \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), both split-generate \( \mathcal{C} \), then we may consider a third subcategory \( \mathcal{A} \subset \mathcal{C} \) defined by the union of objects in \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). Clearly, \( \mathcal{A} \) contains both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). Furthermore, \( s(\omega) \) is also unital with respect to \( \mathcal{A} \). Thus, we may choose an unital cyclic model \( \mathcal{A}' \) of \( \mathcal{A} \), which induces cyclic models \( \mathcal{A}'_1, \mathcal{A}'_2 \) of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) respectively. At this point, it suffices to apply Lemma 4.6 to conclude the invariants of both \( \mathcal{A}'_1 \) and \( \mathcal{A}'_2 \) match with the invariants of \( \mathcal{A}' \).

In the odd case, the cyclic unital model \( \mathcal{A}' \) may not be unique up to cyclic unital \( A_\infty \)-isomorphisms, as shown in Example 3.11. However, we may use Theorem B.6 proved in the Appendix B to argue the desired independence. Indeed, assume that we have two unital cyclic models \( G_1 : \mathcal{A}' \to \mathcal{A} \) and \( G_2 : \mathcal{A}' \to \mathcal{A} \). Tensoring with the Clifford algebra \( \text{Cl} \) yields two unital cyclic models of \( \mathcal{A} \otimes \text{Cl} \). But now we are back to the even case, unital cyclic model is unique up to unital cyclic \( A_\infty \)-isomorphisms. Together with Theorem B.6, it follows that \( F_{g,n}^C \) is independent of the choice of the unital cyclic \( A_\infty \)-model \( \mathcal{A}' \). The independence of the choice of \( \mathcal{A} \) is the same as the even case.

Now we formulate and prove Morita invariance of the \( F_{g,n}^C,\omega,s \). As proved in [37, Theorem A.3] two \( A_\infty \)-categories are Morita equivalent if and only if \( \mathcal{C} \) and \( \mathcal{D} \) are quasi-equivalent, where \( \mathcal{C} \) is the triangulated split-closure of \( \mathcal{C} \). Inspired by [20] we give the definition.

**Definition 5.9.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( A_\infty \)-categories with extended Calabi–Yau structures \((\omega_C, s_C)\) and \((\omega_D, s_D)\). We say \((\mathcal{C}, \omega_C, s_C)\) and \((\mathcal{D}, \omega_D, s_D)\) are Morita equivalent if there is a quasi-equivalence \( f : \mathcal{C} \to \mathcal{D} \) with \([f_* \omega_C] = [\omega_D]\) and \( f_* \circ s_C = s_D \circ f_* \).
Theorem 5.10. Let \((\mathcal{C}, \omega_{\mathcal{C}}, s_{\mathcal{C}})\) and \((\mathcal{D}, \omega_{\mathcal{D}}, s_{\mathcal{D}})\) be \(A_{\infty}\)-categories with extended Calabi–Yau structures and let \(f: \tw^s \mathcal{C} \to \tw^s \mathcal{D}\) be a Morita equivalence between these extended Calabi–Yau categories. Then the CEI of \((\mathcal{C}, \omega_{\mathcal{C}}, s_{\mathcal{C}})\) and \((\mathcal{D}, \omega_{\mathcal{D}}, s_{\mathcal{D}})\) agree. More precisely, for any \((g, n)\) such that \(2g - 2 + n > 0\), we have
\[
F_{g,n}^{\mathcal{C}, \omega_{\mathcal{C}}, s_{\mathcal{C}}}(\alpha_1, \ldots, \alpha_n) = F_{g,n}^{\mathcal{D}, \omega_{\mathcal{D}}, s_{\mathcal{D}}}(f_*(\alpha_1), \ldots, f_*(\alpha_n)).
\]

Proof. The left hand side is defined using the CEI of an unital cyclic model \(\mathcal{A}'\) of \(\mathcal{A} \subset \tw^s \mathcal{C}\). Since \(\mathcal{C}\) is a small category, by applying Lemma 4.6 we may replace \(\mathcal{A}\) by its skeleton. Equivalently, we may assume that no two objects in \(\mathcal{A}\) are isomorphic. Now, since \(f\) is an equivalence, it preserves the Hochschild invariants as well as the higher residue pairing. Thus, the image full subcategory \(f(\mathcal{A}) \subset \tw^s \mathcal{D}\) is split-generating, and with respect to which the pair \((\omega_{\mathcal{D}}, s_{\mathcal{D}})\) is unital. Thus, the unital cyclic model \(\mathcal{A}'\), via the composition 
\[f \circ G_A: \mathcal{A}' \to \mathcal{A} \to f(\mathcal{A}),\]

is an unital cyclic model of \(f(\mathcal{A})\) whose CEI is by definition the right hand side.

6 Examples and Applications

6.1 CEI of Frobenius associative algebras

Throughout this subsection, let \(A\) be a finite-dimensional, \(\mathbb{Z}/2\mathbb{Z}\)-graded, cyclic \(A_{\infty}\)-algebra such that its higher products \(m_3, m_4, \ldots\) all vanish. As before, we continue to assume \(A\) is unital, smooth, and satisfies the Hodge-to-de-Rham degeneration property. The cyclic pairing is of parity \(d \in \mathbb{Z}/2\mathbb{Z}\). Recall that the notation \(L = \text{CC}^\text{red}_*(A)[d]\) is the shifted reduced Hochschild chain complex of \(A\).

The simplest example is when \(A\) is the ground field.

Example 6.1. Consider the ground field \(\mathbb{K}\) as an \(A_{\infty}\)-category with one object. Its Hochschild invariants are \(HH_* (\mathbb{K}) \cong \mathbb{K}\) and \(HC_*^+ (\mathbb{K}) \cong \mathbb{K}[\![u]\!]\). In this case, the parity \(d\) necessarily equals \(0 \in \mathbb{Z}/2\mathbb{Z}\). Furthermore, it has an unique unital splitting \(s^\text{can}(1) = 1\). Thus, the set \(\mathcal{M}_\mathbb{K} \cong \mathbb{K}^*\) with the correspondence given by
\[
\lambda \in \mathbb{K}^* \mapsto (\lambda, s^\text{can}) \in \mathcal{M}_\mathbb{K}.
\]

It was shown in [41] that we have
\[
F_{g,n}^{\mathbb{K}, \lambda, s^\text{can}}(u^{k_1}, \ldots, u^{k_n}) = \lambda^{2-2g-n} \cdot \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}
\]
where \(\psi_j\)'s are the \(\psi\)-classes on the Deligne-Mumford moduli space \(\overline{M}_{g,n}\).

For more complicated \(A\) we need a few more lemmas.
Lemma 6.2. The chain level Mukai pairing \( \langle - , - \rangle_{\text{Muk}} : L \otimes L \to \mathbb{K} \) satisfies that
\[
\langle a_0|a_1| ⋯ |a_k, b_0|b_1| ⋯ |b_l \rangle_{\text{Muk}} = 0, \quad \text{if } k \geq 1 \text{ or } l \geq 1.
\]

Proof. This follows from the definition of the Mukai pairing, since if \( k \geq 1 \) or \( l \geq 1 \), then one needs higher products in order to be non-zero. \( \Box \)

Lemma 6.3. The canonical inclusion map \( A[d] \to L \) as length zero Hochschild chains induces a surjective map \( H_\ast(A[d]) \to H_\ast(L) \) in homology.

Proof. Since \( A \) only has \( \mathfrak{m}_2 \), the homology \( H_\ast(L) \) is also graded by the length of Hochschild chains. Let \( \alpha = [a_0|a_1| ⋯ |a_k] \in H_\ast(L) \) be a Hochschild homology class with \( k \geq 1 \). Then by the previous lemma we have
\[
\langle \alpha, \beta \rangle_{\text{Muk}} = 0, \quad \forall \beta \in H_\ast(L).
\]
But \( A \) is smooth and proper, by Shklyarov’s non-degeneracy result we must have \( \alpha = 0 \). Thus any Hochschild homology class must be represented by a length zero chain. \( \Box \)

Lemma 6.4. There exists a unique splitting denoted by \( s^{\text{can}} : H_\ast(L) \to H_\ast(L^+) \) of the Hodge filtration of \( A \) characterized by the property that \( \nabla u \frac{\partial}{\partial n} s^{\text{can}}(\alpha) = 0, \quad \forall \alpha \in H_\ast(L) \).

Proof. Since there is only \( \mathfrak{m}_2 \), the \( u \)-connection is given by the operator
\[
\nabla u \frac{\partial}{\partial n} (a_0|a_1| ⋯ |a_k \cdot u^n) = (n - \frac{k}{2}) \cdot a_0|a_1| ⋯ |a_k \cdot u^n.
\]
Let \( \alpha = [a] \in H_\ast(L) \) be a Hochschild class represented by a length zero chain \( a \in A[d] \). Since both the Hochschild differential \( b \) and the Connes operator \( B \) are homogeneous respect to the length grading, i.e. \( b \) is degree \(-1\) while \( B \) is degree \( 1 \), we may choose \( s(\alpha) \) to be of the form
\[
s^{\text{can}}(\alpha) = [a + \alpha_1 u + \alpha_2 u^2 + ⋯], \quad \alpha_k \in A \otimes A^{\otimes 2k}[d], \quad \forall k \geq 1.
\]
Then one easily checks that we have \( \nabla u \frac{\partial}{\partial n} s(\alpha) = 0 \). The uniqueness follows from the non-degeneracy of the higher residue pairing and Lemma 6.2. \( \Box \)

Recall from [2, Section 2.3] the cyclic pairing on \( A \) induces a duality isomorphism
\[
D : HH^\ast(A) \cong H_\ast(L)
\]
Let us denote by \( \phi := D(1_A) \). By construction \( \phi \) is a weak Calabi-Yau structure.

Lemma 6.5. Under the correspondence in Corollary 2.24, the cyclic structure on \( A \) corresponds to the strong Calabi-Yau structure \( s^{\text{can}}(\phi) \in H_\ast(L^+) = HC_\ast^{-}(A)[d] \).
Proof. We denote \( s^{\text{can}}(\phi) = \sum_{k \geq 0} \phi_k u^k \) and compute the action of this Calabi–Yau structure on \( HC^+ (A) \):

\[
\left\langle \sum_{k \geq 0} u^{-k} \alpha_k, s^{\text{can}}(\phi) \right\rangle = \sum_{k \geq 0} \langle \alpha_k, \phi_k \rangle_{\text{Muk}} = \langle \alpha_0, D(1_A) \rangle_{\text{Muk}}.
\]

In the second equality we have used the fact that all the \( \phi_k, k > 0 \) can be represented by chains of positive length (see proof of Lemma 6.4) and therefore the corresponding pairings vanish by Lemma 6.2. Again by Lemma 6.2, the remaining term also vanishes unless \( \alpha_0 \) has length zero, in which case \( \langle \alpha_0, D(1_A) \rangle_{\text{Muk}} = \langle 1_A, \alpha_0 \rangle \) by definition of \( D \).

As computed in Example 2.25, the cyclic pairing on \( A \) corresponds to the Calabi–Yau structure \( \Phi \) defined in (14). The comparison map between the \( u \)-model and cyclic complexes for cyclic homology (see [28]) sends \( \sum_{k \geq 0} \phi_k u^{-k} \) to \([\phi_0]\). Therefore we conclude that \( \Phi \) agrees with the map described above. □

Observe that \( s^{\text{can}} \) is \( \phi \)-compatible (with the scalar \( r = 0 \) in Condition \((S4.)\) Definition 3.14) and therefore unital, by Proposition 3.17, since \( A \) is \( \mathbb{Z} \)-graded. Thus we may use the Frobenius algebra \( A \) as a cyclic model to compute the CEI of the triple \((A, \phi, s^{\text{can}})\).

Using the duality map \( D \) above, we can transfer the cup product structure on \( HH^\bullet (A) \) to \( H^\bullet (L) \). Together with the Mukai pairing we obtain a unital commutative Frobenius algebra \((H^\bullet (L), \phi, \cup, \langle -,- \rangle_{\text{Muk}})\). We refer to [2, Theorem 2.4] for a proof. It is well-known that a commutative Frobenius algebra is equivalent to the structure of a 2-dimensional topological field theory. In particular, this yields linear functionals

\[
(31) \quad \omega^A_{g,n} : H^\bullet (L)^{\otimes n} \to \mathbb{K}
\]

**Theorem 6.6.** The CEI of the triple \((A, \phi, s^{\text{can}})\) is given by

\[
F^{A,\phi,s^{\text{can}}}_{g,n}(\alpha_1 \cdot u^{k_1}, \ldots, \alpha_n \cdot u^{k_n}) = \omega^A_{g,n}(\alpha_1, \ldots, \alpha_n) \cdot \int_{[\Sigma_{g,n}]} \psi_{k_1}^{k_1} \cdots \psi_{k_n}^{k_n}
\]

where \( \alpha_1, \ldots, \alpha_n \in H^\bullet (L) \), and \( k_1, \ldots, k_n \geq 0 \).

**Proof.** As in the proof of Lemma 6.4, since the operators \( b \) and \( B \) are both homogeneous with respect to the length of Hochschild chains, we may choose a chain level lift \( R : L \to L_+ \) of the splitting operator \( s^{\text{can}} \) such that it is of the form

\[
R(\alpha) = \alpha + u R_1 \alpha + u^2 R_2 \alpha + \cdots,
\]

with \( R_n \alpha \in A \otimes \overline{A}^{2n+k}, \forall n \geq 0 \) if \( \alpha \in A \otimes \overline{A}^k \). Using Lemma 6.2, we see that \( R \) trivially satisfies the symplectic condition, i.e.

\[
\langle \alpha, \beta \rangle_{\text{Muk}} = \langle R(\alpha), R(\beta) \rangle_{\text{hres}}, \forall \alpha, \beta \in L.
\]
To this end, we may use of the following explicit formula from [7] expressing $F_{g,n}$’s as a type of graph sum:

$$F_{g,n}^{A,\phi,s_{\text{can}}} (\alpha_1 \cdot u^{k_1}, \ldots, \alpha_n \cdot u^{k_n}) = \sum_{j=1}^{n} \sum_{G} \text{wt}(G) \cdot \langle \rho(G) (\alpha_j u^{k_j}), \alpha_1 u^{k_1} \cdots \alpha_j u^{k_j} \cdots \alpha_n u^{k_n} \rangle_{\text{Muk}}$$

By Lemma 6.3, we may assume that $\alpha_i = a_i \in A[d]$. Again by the vanishing in Lemma 6.2, in the above summation only the star graph $\ast_{g,1,n-1}$ with one input and $n-1$ outputs can contribute. This reduces the above summation into

$$F_{g,n}^{A,\phi,s_{\text{can}}} (\alpha_1 \cdot u^{k_1}, \ldots, \alpha_n \cdot u^{k_n}) = \sum_{j=1}^{n} \frac{1}{(n-1)!} \langle \rho(\mathcal{V}_{g,1,n-1}) (\alpha_j u^{k_j}), \alpha_1 u^{k_1} \cdots \alpha_j u^{k_j} \cdots \alpha_n u^{k_n} \rangle_{\text{Muk}}$$

But since $A$ only has $m_2$, only degree zero ribbon graphs in $\mathcal{V}_{g,1,n-1}$ can contribute. Thus the invariant $F_{g,n}^{A,\phi,s_{\text{can}}} (\alpha_1 \cdot u^{k_1}, \ldots, \alpha_n \cdot u^{k_n})$ is the product of two parts: the first part is by action of degree zero ribbon graphs which give exactly the topological part $\omega^A_{g,1}(\alpha_1, \ldots, \alpha_n)$; the second part is the $\psi$-class contribution which is computed in [41] and is given by $\int_{[\overline{M}_{g,n}]} \psi_1^{k_1} \cdots \psi_n^{k_n}$. □

**Corollary 6.7.** Assume that $H_\bullet(L) \cong \mathbb{K}$ as a Frobenius algebra. Then we have

$$F_{g,n}^{A,\phi,s_{\text{can}}} (\phi \cdot u^{k_1}, \ldots, \phi \cdot u^{k_n}) = \int_{[\overline{M}_{g,n}]} \psi_1^{k_1} \cdots \psi_n^{k_n}.$$

**Example 6.8.** Consider the Clifford algebra $\text{Cl}_1 = \mathbb{K} \oplus \mathbb{K} \cdot \epsilon$ generated by an odd element $\epsilon$, with $m_2(\epsilon, \epsilon) = \frac{1}{2}$. Its cyclic pairing is given by $\langle 1, \epsilon \rangle = 1$. Then it was computed in [9] that $H_\bullet(L) \cong \mathbb{K}$ as a Frobenius algebra. The weak Calabi-Yau element is $[\epsilon] \in H_\bullet(L)$. Thus, we have

$$F_{g,n}^{\text{Cl}_1,[\epsilon],s_{\text{can}}} ([\epsilon] \cdot u^{k_1}, \ldots, [\epsilon] \cdot u^{k_n}) = \int_{[\overline{M}_{g,n}]} \psi_1^{k_1} \cdots \psi_n^{k_n}.$$

The same holds for the Clifford algebra $\text{Cl}_d = \mathbb{K}[\epsilon_1, \ldots, \epsilon_d]$ generated by $d$ odd elements, with $m_2(\epsilon_i, \epsilon_j) = \frac{1}{2} \cdot \delta_{ij}$, and with cyclic pairing is given by $\langle 1, \epsilon_i \rangle = 1$. The corresponding weak Calabi-Yau element is $[\epsilon_1 \cdots \epsilon_d] \in H_\bullet(L)$.

**Remark 6.9.** There are many splittings of the Hodge filtration for $\text{Cl}$, other than $s_{\text{can}}$. These other splitting are unital (since we are in the odd case) but not homogeneous. We expect the CEI for these other splittings will give different (including non homogeneous) Cohomological Field Theories. In particular, we expect that for an appropriately chosen splitting for $\text{Cl}$, one recovers the Cohomological Field Theory given by the total Chern class of the Hodge bundle on $\overline{M}_{g,n}$ [31, Section 1]. We will further study this problem in a future work.
The next corollary serves to illustrate how one can use our Morita invariance result to compute the CEI of $A_\infty$-categories relevant to Mirror Symmetry.

**Corollary 6.10.** Let $W \in \mathbb{K}[[x_1, \ldots, x_n]]$ be a potential with an isolated, Morse (non-degenerate) singularity at the origin. The category of matrix factorizations $\text{MF}(W)$ is smooth, proper, has the Hodge-to-de-Rham degeneration property and $HH_\bullet(\text{MF}(W)) \cong \mathbb{K} \cdot \phi$. Moreover there is an unique $\phi$-compatible (and unital) splitting $s_{\text{can}}$ and the corresponding CEI are equal to

\[ F_{g,n}^{\text{MF}(W),\phi,s_{\text{can}}} (\phi \cdot u^{k_1}, \ldots, \phi \cdot u^{k_n}) = \int_{[\overline{\mathcal{M}}_{g,n}]} \psi_1^{k_1} \cdots \psi_n^{k_n}. \]

**Proof.** The facts that $\text{MF}(W)$ is smooth, proper and has the Hodge-to-de-Rham degeneration property are proved in [16], and in fact hold without the Morse assumption. It is also proved in [16] that $\text{MF}(W)$ has a compact generator $K^{\text{stab}}$, and the cohomology of the endomorphism dg-algebra $\text{end}(K^{\text{stab}})$ is isomorphic to a Clifford algebra corresponding to the Hessian of $W$. When $W$ has a Morse singularity the Clifford algebra is intrinsically formal [36, Section 6.1] and therefore $\text{end}(K^{\text{stab}})$ is quasi-isomorphic to the Clifford algebra $\text{Cl}_n$. Hence $\text{MF}(W)$ is Morita equivalent to $\text{Cl}_n$. The result now follows from Theorem 5.10 and the previous example. \(\square\)

### 6.2 A-model comparison with Gromov-Witten theory

Let $X$ be a compact symplectic manifold of real dimension $2d$ and let $\mathcal{C} = \text{Fuk}(X)$ be the Fukaya category of $X$. This is an $A_\infty$-category linear over $\mathbb{K} = \Lambda$ the Novikov field (with complex coefficients). It is natural to expect that for the “correct” choice of Calabi–Yau structure and splitting, the induced CEI should coincide with the geometrically defined Gromov–Witten invariants of $X$.

We expect the correct splitting to be determined by the geometrically defined *open-closed map* $\text{OC} : HH_\bullet(\text{Fuk}(X)) \to H^\bullet(X, \Lambda)[d]$, and its cyclic enhancement

\[ \text{OC}_{\text{cyc}} : HC^\bullet_\bullet(\text{Fuk}(X)) \to H^\bullet(X, \Lambda)[d][[u]]. \]

The open-closed map $\text{OC}$ has been constructed in several settings (starting with [18]) and is expected to be an isomorphism for a wide class of symplectic manifolds. The cyclic version $\text{OC}_{\text{cyc}}$ was more recently constructed by Ganatra [19]. Although the construction in [19] is carried out in a somewhat limited technical setup, the arguments should generalize to any other setting where one constructs the Fukaya category.

From now on we make the assumption on $X$ that the open-closed map $\text{OC}$ is an isomorphism. Then, as explained in [19] for example, one can define a weak Calabi–Yau
structure on \( \text{Fuk}(X) \) by taking the composition

\[
\omega^{OC} : HH_\bullet(\text{Fuk}(X)) \xrightarrow{\omega^{OC}} H^\bullet(X,\Lambda)[d] \xrightarrow{f} \Lambda[-d],
\]

where \( f \) is the integration map (or Poincare pairing) on \( X \).

Assuming \( OC \) is an isomorphism also implies that \( OC^{\text{cyc}} \) is an isomorphism. Therefore it determines a splitting \( s^{OC} \), by requiring the following diagram be commutative

\[
\begin{array}{ccc}
HH_\bullet(\text{Fuk}(X)) & \xrightarrow{s^{OC}} & HC^\bullet(\text{Fuk}(X))[d] \\
\downarrow{\omega^{OC}} & & \downarrow{OC^{\text{cyc}}} \\
H^\bullet(X,\Lambda)[d] & \xrightarrow{i} & H^\bullet(X,\Lambda)[d][[u]],
\end{array}
\]

where \( i \) is the inclusion by constant map. It should follow from general properties of open-closed maps that \( s^{OC} \) is a good, \( \omega^{OC} \)-compatible and unital splitting. Therefore, we can use these to construct the CEI and obtain maps

\[
F_{g,n}^{\text{Fuk}(X)}(\omega^{OC}, s^{OC} ; -) : HH_\bullet(\text{Fuk}(X))[u][[u]] \to \Lambda
\]

for each pair of integers \( (g,n) \) such that \( 2 - 2g - n < 0 \). We make the following conjecture.

**Conjecture 6.11.** Assume the open-closed map \( OC : HH_\bullet(\text{Fuk}(X)) \to H^\bullet(X,\Lambda)[d] \) is an isomorphism and let \( (\omega^{OC}, s^{OC}) \) be the extended Calabi–Yau structure defined above. Then, for any \( \alpha_1, \ldots, \alpha_n \in HH_\bullet(\text{Fuk}(X)) \) we have

\[
F_{g,n}^{\text{Fuk}(X)}(\omega^{OC}, s^{OC}, \alpha_1 u^{k_1}, \ldots, \alpha_n u^{k_n}) = \langle OC(\alpha_1) \psi^{k_1}, \ldots, OC(\alpha_n) \psi^{k_n} \rangle^X_g,
\]

where the right-hand side denotes the descendant Gromov–Witten invariants of \( X \).

In [2] the authors use the closed-open map (dual to the open-closed map above) to construct a splitting \( s^\mu \) for \( \text{Fuk}(X) \), under the extra assumption that \( HH^\bullet(\text{Fuk}(X)) \) is a semi-simple ring. One can show that \( s^\mu \) agrees with the splitting \( s^{OC} \) in the conjecture. Furthermore, it is proved in [2] that for the splitting \( s^\mu \) the genus zero categorical Gromov–Witten invariants, computed using a categorical analogue of Saito’s primitive forms, agree with the Gromov–Witten invariants of \( X \). Even though it is not yet known these invariants agree with the CEI, this provides some evidence for the above conjecture.

### 6.3 B-model: derived invariants of smooth projective Calabi-Yau’s.

Consider the case when \( \mathcal{C} = D^b_{dg}(\text{Coh}(X)) \) is the dg enhancement of the derived category of coherent sheaves on a smooth projective Calabi-Yau variety \( X \) over \( \mathbb{C} \). Blanc [4] developed a theory of complex topological \( K \)-theory for \( \mathbb{C} \)-linear differential graded categories,
based on Toën’s proposal [40]. A immediate consequence of this construction is that for the category $D^b_{dg}(\text{Coh}(X))$ there exists an intrinsic splitting of the non-commutative Hodge filtration which we denote by $s^{BT}$ (for “Blanc-Toën”). Under the comparison result obtained by Blanc [4, Theorem 1.1 (b), (d)], this splitting corresponds to the complex conjugate splitting of the classical Hodge filtration of $H^*(X, \mathbb{C})$, through a comparison map (see [4, Section 4.6]) $HP_\bullet(C) \cong H^*(X, \mathbb{C})$.

Assume the complex dimension of $X$ is $d$. Using the splitting $s^{BT}$, the CEI gives a map

$$F^{C,s^{BT}}_{g,n} : HH_d(C)^* \times HH_\bullet(C)[d][u]^\otimes n \to \mathbb{C}.$$ 

Note that here the space of weak (=strong) Calabi-Yau structures $HH_d(C)^*$ can be identified with the space of nowhere vanishing holomorphic volume forms on $X$ which is a $\mathbb{C}^*$-torsor since $X$ is a smooth and projective Calabi-Yau. Since the splitting map $s^{BT}$ is intrinsic to the dg-category $C$, our main theorem above implies that $F^{C,s^{BT}}_{g,n}$ only depends on the dg category $C = D^b_{dg}(\text{Coh}(X))$. In fact, by Lunts-Orlov’s uniqueness of dg-enhancements [29], we can conclude that for each pair $(g,n)$, the map $F^{C,s^{BT}}_{g,n}$ is an invariant of the derived category $D^b(\text{Coh}(X))$.

In the case $d = 0$ when $X = \text{Spec} \mathbb{C}$, and we set the weak Calabi-Yau structure $\phi = 1 \in HH_0(\mathbb{C})^* = \mathbb{C}^*$, the CEI agrees with Gromov-Witten invariants of a point, i.e.

$$F^{C,s^{BT}}_{g,n}(u^{k_1}, \ldots, u^{k_n}) = \int_{[M_{g,n}]} \psi_1^{k_1} \cdots \psi_\tau^{k_n},$$

as explained in Example 6.1.

In the case $d = 1$ when $X = \mathbb{C}/(1,\tau)$ is an elliptic curve. The invariant $F^{C,s^{BT}}_{1,1}$ is computed in [8]. It turns out the only non-trivial invariant is computed as

$$F^{C,s^{BT}}_{1,1}(2\pi i [dz]; \frac{1}{\tau - \bar{\tau}}[\bar{dz}]) = -\frac{1}{24} E_2^*(\tau).$$

In this formula, the weak Calabi-Yau structure is $\phi = 2\pi i [dz] \in HH_1(C) \cong H^{1,0}(X,\tau)$, and the insertion $\alpha = \frac{1}{\tau - \bar{\tau}}[\bar{dz}] \in HH_{-1}(C) \cong H^{0,1}(X,\tau)$. The function $E_2^*(\tau)$ is the non-holomorphic, but modular Eisenstein series $E_2^*(\tau) = 1 - 24 \sum_{k=1}^{\infty} \frac{k \tau^k}{1 - q^k} - \frac{6i}{(\tau - \bar{\tau})\pi}$ with $q = \exp(2\pi i \tau)$.

In the case $d = 2$ when $X$ is a K3 surface. In view of mirror symmetry that the Gromov-Witten invariants are essentially trivial on a K3, we make the following

**Conjecture 6.12.** Let $C = D^b(\text{Coh}(X))$ with $X$ a smooth projective K3 surface. Then

- if $g = 0$, $k_1 + \cdots + k_n = n - 3$, $|\alpha_1| + \cdots + |\alpha_n| = 2n - 4$, we have
  $$F^{C,s^{BT}}_{0,n}(\phi; \alpha_1 \cdot u^{k_1}, \ldots, \alpha_n \cdot u^{k_n}) = \omega^{C,\phi}_{0,n}(\alpha_1, \ldots, \alpha_n) \cdot \frac{(n - 3)!}{k_1! \cdots k_n!}.$$
• if \( g = 1, \ k_1 + \cdots + k_n = n, \ |\alpha_1| = \cdots = |\alpha_n| = 2, \) we have
\[
F_{1,n}^{C,s_{BT}}(\phi; \alpha_1 \cdot u^{k_1}, \ldots, \alpha_n \cdot u^{k_n}) = \omega_{1,n}^{C,\phi}(\alpha_1, \ldots, \alpha_n) \cdot \int_{[\overline{M}_{1,n}]} \psi^{k_1} \cdots \psi^{k_n}
\]

• if \( g \geq 2, \) we have \( F_{g,n}^{C,s_{BT}} = 0. \)

Here the contribution \( \omega_{0,n}^{C,\phi}(\alpha_1, \ldots, \alpha_n) \) and \( \omega_{1,n}^{C,\phi}(\alpha_1, \ldots, \alpha_n) \) are the “topological invariants” obtained from the unital Frobenius algebra \((HH_\bullet(C), \phi, \cup, \langle - , - \rangle_Muk), \) see Equation (31).

The case \( d = 3 \) is arguably the most interesting case due to its physics interpretation as the partition functions in topological string theory, which was already pointed out by Costello [15]. Combined with a famous theorem [5] of Bridgeland that in complex dimension 3 birational equivalence implies derived equivalence, our theorem implies that the CEI \( F_{g,n}^{C,s_{BT}} \)'s are also birational invariants. This echoes well with related works of Hu-Li-Ruan [24] in symplectic geometry and Maillot-Rössler [30] in genus one.

In general the CEI \( F_{g,n}^{C,s_{BT}} \) are maps, with Hochschild classes as inputs. A remarkable property of dimension \( d = 3 \) is that we may obtain actual numbers in \( \mathbb{C} \) as invariants. Indeed, we may define a complex number
\[
F_g^{C,s_{BT}} := \begin{cases} 
\frac{1}{2g-2} F_{g,1}^{C,s_{BT}} ([\omega] \cdot u), & \text{if } g \geq 1, \\
F_{g,1}^{C,s_{BT}} ([\omega] \cdot u), & \text{if } g = 1.
\end{cases}
\]

The factor \( \frac{1}{2g-2} \) is from the Dilaton equation in Gromov-Witten theory. It is easy to show that if we scale \( \omega \) to \( \lambda \cdot \omega, \) the invariants \( F_{g,n}^{C,s_{BT}} \) scale by \( \lambda^{2g-2-n}. \) Thus using the above definition we see that \( F_{g}^{C,s_{BT}} \) scales as \( \lambda^{2g-2-1} \cdot \lambda = \lambda^{2-2g}. \) In particular, \( F_1^{C,s_{BT}} \) is independent of the choice of \( \omega, \) and is a birational invariant of \( X. \) More generally, one may consider the ring
\[
R := \mathbb{C}[F_1, F_2, F_3, \ldots],
\]
and assign weights \( \text{wt}(F_g) = 2g-2. \) Then the weight zero elements in the field of fractions \( K(R)_0 \) are all birational invariants of \( X \) whenever the denominator is not zero, such as \( F_2^2, F_3^2, F_3^2+F_4^2, \) and so on. We refer to the upcoming work [35] for some recent progresses on CEI of Calabi-Yau 3-folds.

A Sign diagrams in TCFT’s
A.1 Signs in proving the Equation (27)

We shall deal with the case \( l = 0 \), i.e. the number of outputs is zero. In this case, the right hand side of Equation (27) is \((-1)^{|G|} \cdot \rho^C(G) \mathcal{L}_m\). The composition \( \rho^C(G) \mathcal{L}_m \) is illustrated in the following figure:

Pulling the top operation \( \mathcal{L}_m \) down yields the following diagram:

The extra intersections of the red lines in these two diagrams is exactly equal to \( k \), the number of inputs. Algebraically, this is because

\[
\underbrace{(s \otimes \cdots \otimes s)}_{k \text{ copies}} \circ \mathcal{L}_m = (-1)^k \mathcal{L}_m' \circ \underbrace{(s \otimes \cdots \otimes s)}_{k \text{ copies}},
\]

where \( \mathcal{L}_m \) is the Hochschild differential of \( CC_\bullet(C)^{\otimes k} \), while \( \mathcal{L}_m' \) is the Hochschild differential of \( CC_\bullet(C)[1]^{\otimes k} \). Similarly, the left hand side of Equation (27) \( \rho^C(\partial G) \) is illustrated.
by the figure:

We may move the first black vertex (from left to right) up to obtain the following figure:

The number of crossings of the red lines in the bottom is exactly equal to $|V_G^{\text{black}}|$, the number of black vertices. To this point, we observe that the degree of a ribbon graph $G$ (in the case $l = 0$) is equal to

$$|G| \equiv \sum_{v \in V_G^{\text{black}}} (\text{val}(v) - 3) \equiv \sum_{v \in V_G^{\text{black}}} \text{val}(v) + |V_G^{\text{black}}| \equiv k + |V_G^{\text{black}}| \pmod{2}.
$$

Putting the above together, and using the $A_\infty$ relation yields the desired identity

$$\rho^c(\partial G) + (-1)^{|G|} \rho^c(G) \mathcal{L}_m = 0$$
A.2 Signs in Equation (29)

We first compare the signs of the term (\(i\)) with the first type term in \([d_{DR}, \rho_{[0]}^{C\otimes\Omega^*}(G)]\) of Equation (29). This is illustrated in the following figure:

\[
(-1)^j [L_m, L_t dt] = (-1)^{j-1} \rho_{[ii]}^{C\otimes\Omega^*}(\partial G)
\]

Here the sign \((-1)^{j-1}\) appears from the definition of \(\partial G\), while the extra \((-1)\) sign comes from the intersection of the two red lines.

For the \((ii)\) term, the corresponding term on the right hand side of Equation (29) is given by a composition of the form:

\[
(-1)^j (\tilde{m}_{v_1} \otimes \cdots \otimes \tilde{m} \circ \tilde{dt} \otimes \cdots \otimes \tilde{m}_{v_n}) \circ \sigma_G \circ \left( s \otimes \cdots \otimes s \otimes \langle -, - \rangle^{-1} \otimes \cdots \otimes \langle -, - \rangle^{-1} \right)
\]

To match with the \((ii)\) term on the left hand side, we need to move the operator \(\tilde{dt}\) to the rightmost part. This move yields a sign \((-1)^{n-j} \cdot (-1)^k\). Together with the sign \((-1)^j\), we obtain \((-1)^{n+k}\) which is equal to \((-1)^{|G|}\) by Equation (32). The other term \((iii)\) is similar.

B Boson-Fermion correspondence

In this section, we prove a type of Boson-Fermion correspondence for CEI. Denote by \(\text{Cl}\) the Clifford algebra \(\mathbb{K}[\epsilon]\) in one odd generator \(\epsilon\). This is a 2-dimensional vector space spanned by a strict unit \(1\) and \(\epsilon\) over the base field \(\mathbb{K}\). Its \(A_\infty\) product is given by \(m_2(\epsilon, \epsilon) = \frac{1}{2} I\). We shall also use its usual associative product which is simply denoted by \(\epsilon^2 = -\frac{1}{2} I, \epsilon^3 = -\frac{1}{2} \epsilon, \ldots\). Furthermore, \(\text{Cl}\) is endowed with an odd inner product by setting \(\langle 1, \epsilon \rangle = 1\). It forms a \(\mathbb{Z}/2\mathbb{Z}\)-graded, cyclic, and unital \(A_\infty\)-algebra over \(\mathbb{K}\). It is known that \(\text{Cl}\) is smooth and satisfies the Hodge-to-de-Rham degeneration property. Indeed, recall from \([9, \text{Section 2}]\) that its Hochschild homology is 1-dimensional generated by \([\epsilon]\). Furthermore, it has an unique homogeneous splitting of its non-commutative Hodge filtration explicitly given by

\[
\epsilon \mapsto \sum_{j \geq 0} (-1)^j (2j - 1)!! \cdot \epsilon [\epsilon] \cdots [\epsilon] u^j.
\]
Let $\mathcal{C}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded, cyclic, and unital $A_\infty$-category over $\mathbb{K}$. Assume also that $\mathcal{C}$ is smooth, and satisfies the Hodge-to-de-Rham degeneration property. We may form another such $A_\infty$-category $\mathcal{C} \otimes \mathcal{Cl}$. It has the same underlying objects. Between two objects $X$ and $X'$ the Hom space is given by

$$\mathcal{C} \otimes \mathcal{Cl}(X, X') := \mathcal{C}(X, X') \otimes \mathcal{Cl}.$$  

The $A_\infty$ composition in the tensor product category is given by

$$\mu_n(x_1 \otimes \epsilon^{k_1}, \ldots, x_n \otimes \epsilon^{k_n}) = (-1)^\bullet \mu_n(x_1, \ldots, x_n) \otimes \epsilon^{k_1+\cdots+k_n},$$  

where \(\bullet = \sum_{j=2}^n (k_1 + \cdots + k_{j-1}) |x_j|'.\)  

The sign $(-1)^\bullet$ is given by the Koszul sign of moving the $\epsilon$’s to the right side of the $x$’s. This is a special case of the tensor product defined in [1].

We endow $\mathcal{C} \otimes \mathcal{Cl}$ with the tensor product inner product. An important observation is that the parity of the tensor product inner product is different from that of $\mathcal{C}$. Moreover this construction is functorial [1], meaning if $f : \mathcal{C} \to \mathcal{C}'$ is a (cyclic/unital) $A_\infty$-functor there is an induced (cyclic/unital) $A_\infty$-functor $F \otimes \text{id} : \mathcal{C} \otimes \mathcal{Cl} \to \mathcal{C}' \otimes \mathcal{Cl}$.

The goal of this appendix is to prove that the CEI of $\mathcal{C}$ and $\mathcal{C} \otimes \mathcal{Cl}$ are the same. This is formulated more precisely in Theorem B.6 below.

### B.1 Hochschild invariants

There is a shuffle product map (see [28, Section 4.2]) $\text{sh} : \mathcal{L}\mathcal{C} \otimes \mathcal{LCl} \to \mathcal{L}\mathcal{C} \otimes \mathcal{Cl}$. Since $\epsilon \in \mathcal{LCl}$ is an even closed element, it determines a chain map denote by

$$i_0 : \mathcal{L}\mathcal{C} \to \mathcal{L}\mathcal{C} \otimes \mathcal{Cl}, \quad i_0 := \text{sh}(-, \epsilon).$$

Explicitly, this map is given by

$$i_0(x_0|x_1| \cdots |x_n) := (-1)^\bullet (x_0 \otimes \epsilon)|x_1| \cdots |x_n,'$$

with $(-1)^\bullet$ the Koszul sign $(-1)^{|x_1|'+\cdots+|x_n|'}$. In the above equation, we have also abused the notation $x_j$ for $x_j \otimes \mathbf{1}$ in the tensor product.

**Lemma B.1.** The map $i_0 : \mathcal{L}\mathcal{C} \to \mathcal{L}\mathcal{C} \otimes \mathcal{Cl}$ is a quasi-isomorphism.

**Proof.** Consider the length filtration of Hochschild chains on both sides. Obviously the map $i_0$ respects the filtration. Hence it induces maps on the associated spectral sequences. In the first page, we obtain the homology of $\mathcal{m}_1$. To this point, it suffices to observe that after taking $\mathcal{m}_1$, the product $\mathcal{m}_2$ becomes associative, which implies that the induced map of $i_0$ on the first page is already an isomorphism by the K"unneth formula of Hochschild homology in the case of associative algebras [28, Theorem 4.2.5]. This finishes the proof. \[\Box\]
It is easy to see that the map $i_0$ is not compatible with the circle operators. However, there is a cyclic extension of $\text{sh}$ using cyclic shuffle product map (see [28, Section 4.3] and [39, Section 2]) of the form

$$\text{sh} + u\text{Sh} : L^C((u)) \otimes L^{Cl}((u)) \to L^{C \otimes Cl}((u)).$$

Then the element $\tilde{\epsilon} = \sum_{j \geq 0} (-1)^j (2j - 1)!! \cdot \epsilon | \cdots | \epsilon$ induces a cyclic extension of $i_0$ which we denote by

$$i : L^C((u)) \to L^{C \otimes Cl}((u)), \quad i := \text{sh}(-, \tilde{\epsilon}) + u\text{Sh}(-, \tilde{\epsilon}).$$

Since $\tilde{\epsilon}$ has no negative powers in $u$, the map $i$ preserves the positive sub-complexes, and hence it also induces a map on the quotient complexes. We shall still denote these induced maps by

$$i : L^+_C \to L^+_C \otimes Cl,$$

$$i : L^-_C \to L^-_C \otimes Cl.$$  

**Lemma B.2.** The map $i : L^C((u)) \to L^{C \otimes Cl}((u))$ is a quasi-isomorphism. The same holds for its induced maps on the positive and negative subspaces.

**Proof.** We may consider the $u$-filtration on both sides. The map $i$ preserves this filtration, and induces an isomorphism on the first page of the associated spectral sequences by the previous lemma. 

We shall also need a backward chain map $p : L^{C \otimes Cl} \to L^C$ defined explicitly by

$$p(x_0 \otimes \epsilon^{k_0} | x_1 \otimes \epsilon^{k_1} | \cdots | x_n \otimes \epsilon^{k_n}) := (-1)^{\epsilon} \pi(\epsilon^{k_0 + \cdots + k_n - 1}) \cdot x_0 | x_1 | \cdots | x_n,$$

where the sign $(-1)^{\epsilon}$ is as in Equation (33) and $\pi : Cl \to \mathbb{K}$ is the projection to scalar map.

**Lemma B.3.** The map $p$ defined above is a chain map, i.e. $pb = bp$. Furthermore, it is also compatible with the circle actions, i.e. $pB = Bp$. Thus, it induces a chain map still denoted by

$$p : L^{C \otimes Cl}((u)) \to L^C((u)).$$

Then, we also have $p \circ i = \text{id}$. In particular, since $i$ is a quasi-isomorphism, so is $p$.

**Proof.** Checking the two equations $pb = bp$ and $pB = Bp$ is a straight-forward calculation. For the identity $p \circ i = \text{id}$, we first observe that $p \circ i_0 = \text{id}$. Thus it suffices to prove that $p \circ (i - i_0) = 0$. Indeed, using the explicit formula of the shuffle product and cyclic shuffle product, the terms appearing in $(i - i_0)(x_0 | x_1 | \cdots | x_n)$ are all of the form:

$$x_0 \otimes \epsilon | \cdots | 1_{X_j} \otimes \epsilon | \cdots \quad \text{or} \quad 1_{X_0} \otimes 1 | \cdots | 1_{X_j} \otimes \epsilon | \cdots.$$

In either case, applying the map $p$ yields zero since we are using the reduced Hochschild chain complex of $C$. 

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Since \( p \circ i = \text{id} \), we obtain a direct sum decomposition

\[
L^C \otimes \text{Cl} \cong L^C_+ \oplus \ker(p)_+,
\]

where \( \ker(p) \subset L^C \otimes \text{Cl} \) denotes the kernel of the map \( p \). Note that since \( pB = Bp \), the subspace \( \ker(p) \) is \( B \)-invariant. The equivariant chain complex \( \ker(p)_+ \) is formed using the subspace circle action.

Let \( s : H_\bullet(L^C) \to H_\bullet(L^C_+) \) be a splitting of the non-commutative Hodge filtration. Let \( S : L^C \to L^C_+ \) be a chain-level splitting which lifts the map \( s \). Since \( \ker(p) \) is \( B \)-invariant and acyclic, we may choose a chain-level splitting of it, say

\[
T : \ker(p) \to \ker(p)_+.
\]

Define a chain-level splitting of \( C \otimes \text{Cl} \) by setting

\[
S^\otimes := \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}
\]

Its induced splitting in homology is denoted by \( s^\otimes : H_\bullet(L^C \otimes \text{Cl}) \to H_\bullet(L^C_+ \otimes \text{Cl}) \). By construction, we have

\[
p \circ S^\otimes = S \circ p.
\]

### B.2 CEI of \( A \) and \( A \otimes \text{Cl} \)

Recall from Subsection 4.1 the TCFT action of ribbon graphs associated with the cyclic \( A_\infty \)-category \( C \) is given by a multi-linear map

\[
\rho^C_{g,k,l}(G) : (L^C)^{\otimes k} \to (L^C)^{\otimes l},
\]

for each ribbon graph \( G \) of genus \( g \) and with \( k \) inputs and \( l \) outputs.

**Lemma B.4.** We have

\[
p^{\otimes l} \circ \rho^C_{g,k,l}(G) = \rho^C_{g,k,l}(G) \circ p^{\otimes k},
\]

as maps \( (L^C \otimes \text{Cl})^{\otimes k} \to (L^C)^{\otimes l} \).

**Proof.** We shall only sketch the proof since this is largely by inspection. Indeed, when we evaluate \( \rho^C_{g,k,l}(G) \) the result splits into two parts: one from \( C \) in which case the evaluation gives \( \rho^C_{g,k,l}(G) \); the other one from evaluating a trivalent graph obtained by replacing all black vertices of \( G \) by a binary tree which yields 1, see Subsection 6.1. \( \square \)
To compare the CEI of $C$ and $C \otimes Cl$, we shall make use of an explicit formula of these invariants obtained in [7]. It is of the following form:

\begin{equation}
\iota (F_{g,n}^{C}) = \sum_{m \geq 1} \sum_{G \in \Gamma((g,1,n-1))_{m}} (-1)^{m-1} \frac{\text{wt}(G)}{\text{Aut}(G)} \prod_{v} \text{Cont}(v) \prod_{e} \text{Cont}(e) \prod_{l} \text{Cont}(l)
\end{equation}

Here $\iota : \text{sym}^{n} L_{-}^{C} \rightarrow \text{Hom}(L_{+}^{C}[1], \text{sym}^{n-1} L_{-}^{C})$ is a kind of Koszul type differential turning an output to an input. The point is that $\iota$ is injective in homology, and hence it suffices to consider the right hand side. The second summation in the above equation is over isomorphism classes of partially directed graphs with 1 input and $n - 1$ outputs. The fraction $\frac{\text{wt}(G)}{\text{Aut}(G)}$ is a rational weight associated to such a graph. Finally, the contributions $\prod_{v} \text{Cont}(v) \prod_{e} \text{Cont}(e) \prod_{l} \text{Cont}(l)$ from vertices, edges and legs are explicitly given by:

- At a vertex $v$, we assign the multi-linear maps given by the image of combinatorial string vertices $\rho_{g,k,l}^{\text{comb}}$.
- At an incoming leg, we put the operator $S : L_{+}^{C} \rightarrow L_{+}^{C}$ which is a chain-level splitting lifting the given splitting map $s$ in homology.
- At an outgoing leg, we put the inverse operator of $S$, denoted by $R = S^{-1}$.
- For a directed edge in the spanning tree $K$ of $G$, assign an homotopy operator $F : L_{-}^{C} \rightarrow L_{+}^{C}[1]$, while at other directed edges we put the operator $\Theta : L_{-}^{C} \rightarrow L_{+}^{C}[1]$.
- For an un-directed edge in the spanning tree $K$, assign the homotopy operator $\delta/2 : L_{-}^{C} \otimes L_{-}^{C} \rightarrow \mathbb{K}$, while at other un-directed edges we put the homotopy operator $H^{\text{sym}} : L_{-}^{C} \otimes L_{-}^{C} \rightarrow \mathbb{K}$.

The operators $H^{\text{sym}}$, $\delta$, $F$, and $\Theta$ will be recalled in Lemma B.5. Composing along $G$ using the above operators yields the desired map in $\text{Hom}(L_{+}^{C}[1], \text{sym}^{n-1} L_{-}^{C})$.

**Lemma B.5.** The following identities hold:

(a.) $H^{\text{sym}} \circ (p \otimes p) = (H^{\otimes})^{\text{sym}}$.

(b.) $p \circ F^{\otimes} = F \circ p$.

(c.) $\delta/2 \circ (p \otimes p) = \delta^{\otimes}/2$.

(d.) $p \circ \Theta^{\otimes} = \Theta \circ p$.

**Proof.** We begin with part (a.). Recall from [7] we have an operator $H : L_{-}^{C} \otimes L_{-}^{C} \rightarrow \mathbb{K}$ defined using $S$ and its inverse map $R = S^{-1}$ explicitly by

$$H(\alpha, \beta) := - \langle S_{\geq 1} R(\alpha), \beta \rangle_{\text{res}}.$$
where \( \tau_{\geq 1} : L^C((u)) \to u \cdot L^C_+ \) is the projection map onto strictly positive powers of \( u \). Then, the operator \( H_{\text{sym}} \) is the symmetrization of \( H \). Thus, to prove \((a.)\), it suffices to prove that \( H \circ (p \otimes p) = H^\otimes \). For this, we compute

\[
H^\otimes(\alpha, \beta) = -\langle S^\otimes \tau_{\geq 1} R^\otimes(\alpha), \beta \rangle_{\text{res}}
= -\langle p S^\otimes \tau_{\geq 1} R^\otimes(\alpha), p \beta \rangle_{\text{res}}
= -\langle S \tau_{\geq 1} R(\alpha), p \beta \rangle_{\text{res}}
= H(\alpha, p \beta)
\]

For part \((b.)\), recall \([7]\) that the operator \( F : L^C_- \to L^C_+ [1] \) is defined by

\[
F(\beta) := -u^{-1} \cdot S \tau_{\geq 1} R(\beta).
\]

From this, one similarly deduce that \( p \circ F^\otimes = F \circ p \) as in part \((a.)\).

For part \((c.)\), the operator \( \delta : L^C_- \otimes L^C_- \to \mathbb{K} \) is defined to satisfy the following equation:

\[
[b + uB, \frac{\delta}{2}] = H - H_{\text{sym}},
\]
i.e. it bounds the failure of \( H \) being a symmetric operator. From \((a.)\) we see that \( H^\otimes(\alpha, \beta) = 0 \) if \( \alpha \in \ker(p) \) or \( \beta \in \ker(p) \). Hence we may simply take

\[
\delta^\otimes = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix},
\]

which implies \((c.)\) immediately. Part \((d.)\) is a straightforward calculation. Indeed, by definition \( \Theta : L^C_- \to L^C_+ [1] \) is given by

\[
\Theta(\alpha_0 + \alpha_1 u^{-1} + \cdots) = B \alpha_0.
\]

Hence we have

\[
p(\Theta^\otimes(\alpha_0 + \alpha_1 u^{-1} + \cdots)) = p B \alpha_0 = B p \alpha_0 = \Theta(p(\alpha)).
\]

Finally, we may deduce our desired comparison result between the CEI of \( C \) and \( C \otimes \text{Cl} \).

**Theorem B.6.** Let \( s \) be a splitting of \( C \). Denote by \( s^\otimes \) the induced splitting of \( C \otimes \text{Cl} \) (see Subsection B.1). Then we have

\[
\langle \alpha_1 u^{k_1}, \ldots, \alpha_n u^{k_n} \rangle_{\mathcal{C}^s} = \langle \text{sh}(\alpha_1, \epsilon) u^{k_1}, \ldots, \text{sh}(\alpha_n, \epsilon) u^{k_n} \rangle_{\mathcal{C}^\otimes \text{Cl}, s^\otimes}.
\]

Here \( \text{sh} \) is the K"unneth map given by shuffle product.
Proof. Since the map $\iota$ is an embedding in homology, the homology class $[\iota(F_{g,n}^{A,s})]$ determines the CEI of $(A,s)$ completely. Thus, to prove this theorem, it suffices to show that we have

$$p^{\otimes n-1} \circ \iota(F_{g,n}^{\mathcal{C}\mathcal{I},s}) \circ i = \iota(F_{g,n}^{\mathcal{C},s}),$$

as both $p$ and $i$ are quasi-isomorphisms. Using Equation (34), both sides are expressed as a sum over partially directed graphs. Since the rational number $\frac{\wrt(G)}{\aut(G)}$ is only depends on $G$, it suffices to prove that the contributions at vertices, edges and legs are equal. For this, the idea is to make use of Lemma B.4 to move the projection maps $p^{\otimes n-1}$ from the output legs all the way “up” to the input leg in a given partially directed graph $G$. In the following comparison we shall refer to $p^{\otimes n-1} \circ \iota(F_{g,n}^{\mathcal{C}\mathcal{I},s}) \circ i$ as LHS, and $\iota(F_{g,n}^{\mathcal{C},s})$ as RHS.

There is a partial ordering $>$ defined on the set of vertices $V_G$ of $G$: we have $w > v$ if there is a directed path from $w$ to $v$. Let $v$ be a minimal element in this partial order. Then an outgoing half-edge at the vertex $v$ can be exactly one of the following three cases:

1. an outgoing leg,
2. part of an un-directed edge not in the spanning tree $K$,
3. part of an un-directed edge in the spanning tree $K$.

In the case (1), at an output leg of $G$, by construction in $\iota(F_{g,n}^{\mathcal{C}\mathcal{I},s})$ it is given by the operator $R^{\otimes}$. By construction, we have

$$p \circ R^{\otimes} = R \circ p.$$ 

This shows that indeed, after moving $p$, the output leg contribution becomes $R$ which matches with the RHS $\iota(F_{g,n}^{\mathcal{C},s})$. In the case (2), the un-directed edge is decorated by $(H^{\otimes})^{\text{sym}}$. We use the identity

$$H^{\text{sym}} \circ p^{\otimes 2} = (H^{\otimes})^{\text{sym}}.$$ 

While in the case (3), the un-directed edge is by $\delta^{\otimes}/2$ for which we use the identity

$$\delta/2 \circ p^{\otimes 2} = \delta^{\otimes}/2.$$ 

After these replacements, every outgoing half-edge of $v$ has a projection operator $p$ adjacent to it. Thus, we can apply Lemma B.4 to move the $p$’s to the input half-edges of $v$.

To continue moving the $p$’s “up”, let $w$ be a second minimal element of $V_G$ in the partial order $>$. At such a vertex, two more cases can appear for an outgoing half-edge:

4. part of a directed edge not in the spanning tree $K$,
(5) part of a directed edge in the spanning tree $K$.

In the case (4), we move the operator $p$ using the identity
\[ p \circ \Theta^\otimes = \Theta \circ p. \]
While in the case (5), we use the identity
\[ p \circ F^\otimes = F \circ p \]
After these replacements, every outgoing half-edge of $w$ has a projection operator $p$ adjacent to it. Using Lemma B.4 we may move the $p$’s to the input half-edges of $w$ as well.

Continuing to perform the above replacements at a third minimal element of $V_G$, and so on so forth, we arrive at the desired equality that
\[ p^{\otimes n-1} \circ \iota(F_{Cg,s}^{\otimes} \otimes g,n) \circ i = \iota(F_{g,n}^{C,s}). \]

The proof is complete.

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\Box
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