Abstract This paper is concerned with an optimal reinsurance and investment problem for an insurance firm under the criterion of mean-variance. The driving Brownian motion and the rate in return of the risky asset price dynamic equation cannot be directly observed. And the short-selling of stocks is prohibited. The problem is formulated as a stochastic linear-quadratic control problem where the control variables are constrained. Based on the separation principle and stochastic filtering theory, the partial information problem is solved. Efficient strategies and efficient frontier are presented in closed forms via solutions to two extended stochastic Riccati equations. As a comparison, the efficient strategies and efficient frontier are given by the viscosity solution to the HJB equation in the full information case. Some numerical illustrations are also provided.

Keywords Mean-variance, optimal reinsurance and investment, partial information, stochastic filtering, viscosity solution.

1 Introduction

In recent years, there has been an increasing research interest in applying stochastic control theory to optimal reinsurance and optimal investment problems for various models. As is well known, reinsurance is an effective method to reduce insurance risk, while investment is also a very important element in the insurance business. Maximizing the utility and minimizing the probability of ruin are the two main optimization criteria in the literature. A partial list of recent work in such field includes: Browne[1], Yang and Zhang[2], Promislow and Young[3], Gu and Li[4], Li, et al.[5], Bai and Guo[6], Liang, et al.[7], Xu, et al.[8], etc. It is worth mentioning that Bai and Guo[6] explicitly derived the optimal value functions and optimal strategies by solving the corresponding Hamilton-Jacobi-Bellman (HJB, for short) equations. They also showed that
in some special cases, the optimal strategies for maximizing the expected exponential utility and minimizing the probability of ruin are equivalent. Liang, et al.\textsuperscript{[7]} studied the optimal investment and reinsurance strategy with the instantaneous rate of investment return follows an Ornstein-Uhlenbeck process. Xu, et al.\textsuperscript{[8]} considered that the financial market was driven by a drifted Brownian motion with coefficients modulated by an external Markov process. They derived the explicit optimal investment and reinsurance policy with the expected terminal utility.

However, all these works are predominantly done within the expected utility framework. It should be noted that mean-variance analysis and expected utility formulation are two important models in the financial market. The reader is referred to Bielecki, et al.\textsuperscript{[9]}, Steinbach\textsuperscript{[10]} and MacLean, et al.\textsuperscript{[11]} for discussion on crucial differences between the expected utilities and mean-variance models. The mean-variance criterion is firstly proposed in portfolio selection by Markowitz\textsuperscript{[12]} considering the expected return as well as the variance of the investment in a single period. Li and Ng\textsuperscript{[13]} extended Markowitz’s mean-variance model to the multi-period setting by using an idea of embedding the problem into a tractable auxiliary one. In the paper by Zhou and Li\textsuperscript{[14]}, the continuous-time mean-variance problem is studied by using stochastic linear-quadratic (LQ, for short) control theory. Considering the constraint that short-selling of stocks is prohibited, the corresponding HJB equation inherently has no smooth solutions. To tackle this difficulty, Li, et al.\textsuperscript{[15]} constructed a continuous function via two Riccati equations and showed that this function is a viscosity solution to the HJB equation. Hu and Zhou\textsuperscript{[16]} studied a stochastic LQ control problem where the control variable was constrained in a cone and all the coefficients of the problem were random processes. By Tanaka’s formula, they explicitly obtained optimal control and optimal cost via solutions to two extended stochastic Riccati equations.

Recently, more researching attentions are drawn to adopt the mean-variance criterion in insurance modeling. For example, Bai and Zhang\textsuperscript{[17]} derived the optimal proportional reinsurance and investment strategy in both classical model and its diffusion approximation under the mean-variance criterion. Bi, et al.\textsuperscript{[18]} considered the optimal investment and optimal reinsurance problems for an insurer under the criterion of mean-variance with bankruptcy prohibition. Zhang, et al.\textsuperscript{[19]} considered the mean-variance criterion to proportional reinsurance and investment problem of an insurer whose risk process is driven by the diffusion approximation of a controlled compound Poisson process.

However, in all these works it is assumed that the driving Brownian motions are completely observable by an investor, which in reality is more an exception than a rule. Practically, the investor can observe only the stock prices on which he will base his decisions. In fact, optimal portfolio problems with partial information in financial markets under various setups have been studied extensively in the financial economic literature. Di Nunno and Øksendal\textsuperscript{[20]} considered an optimal portfolio problem for a dealer who has access to some information that in general is smaller than the one generated by market events. Peng and Hu\textsuperscript{[21]} studied the optimal proportional reinsurance and investment strategy for an insurer that only has partial information at its disposal. Malliavin calculus for Lévy processes are applied in their analysis. Wang and Wu\textsuperscript{[22]} obtained some general maximum principles for the partially observed risk-sensitive stochastic.
control problems. Huang, et al.\cite{23} studied the optimal premium policy of an insurance firm in two situations: Full information and partial information. In both situations, they characterized the optimal premium policy with the associated optimal cost functionals. Very recently, Al-Hussein and Gherbal\cite{24} obtained necessary and sufficient optimality conditions for relaxed and strict control of forward-backward doubly SDEs with jumps under full and partial information.

Different from previous expected utility criteria with partial information, Pham\cite{25} considered a mean-variance hedging problem for a general semimartingale model and proved a separation principle for a diffusion model by the martingale method. Xiong and Zhou\cite{26} proved a separation principle in a continuous-time mean-variance portfolio selection problem. Pang, et al.\cite{27} studied a continuous-time mean-variance portfolio selection problem under a stochastic environment. A partial information stochastic control problem with random coefficients was formulated. They showed that the optimal portfolio strategy was constructed by solving a deterministic Riccati-type ordinary differential equation (ODE, for short) and two deterministic backward ODEs. Liang and Song\cite{28} studied an optimal investment and reinsurance problem under partial information for insurer with mean-variance utility, with unobservable Markov-modulated regime switching drift process. Time-consistent equilibrium strategy is obtained within a game theoretic framework. Cao, et al.\cite{29} considered a problem of the optimal time-consistent investment and proportional reinsurance strategy under the mean-variance criterion, in which the insurer has some inside information at her disposal concerning the future realizations of her claims process. A verification theorem on the extended HJB equations is provided, and the optimal strategy was obtained. Recently, Li, et al.\cite{30} solved a mean-variance portfolio selection problem, by a stochastic maximum principle of mean-field models governed by Lévy processes involving continuous and impulse control.

In the present paper, we shall consider a new partial information problem of an insurance firm towards optimal reinsurance and investment. We assume the insurance firm is allowed to take reinsurance and invest its wealth in a Black-Scholes market. However, we cannot directly observe the Brownian motion and the rate of return in the risky asset price dynamic equation. In fact, only partial information concerning the past risky asset prices and the randomness from the insurance claims are available to the policymaker. We overcome the difficulties encountered by stochastic filtering technique. Different from the criterion considered in Xu, et al.\cite{8} and Liang, et al.\cite{7}, we apply the mean-variance criteria in this paper. The stochastic LQ control approach, stochastic Riccati equations and viscosity solution theory are applied to obtain the efficient strategies and efficient frontier.

The rest of this paper is organized as follows. In Section 2, the optimal reinsurance and investment problems with partial information are formulated. Section 3 focuses on the filtering problem and the mean-variance criteria. Efficient strategies and efficient frontier are presented in closed forms via solutions to two extended stochastic Riccati equations. Section 4 presents the efficient strategies and efficient frontier by the viscosity solution to the HJB equation in the full information case. Some numerical illustrations are provided here. Section 5 concludes this paper.
2 Problem Formulation

We fix a finite time horizon $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which three 1-dimensional standard Brownian motions $W^0(\cdot)$, $W^1(\cdot)$, and $W^2(\cdot)$ are defined. We assume that they are mutually independent. For notational clarity, we denote $\{\mathcal{F}_t^{W^0}\}_{0 \leq t \leq T}$ and $\{\mathcal{F}_t^{W^1}\}_{0 \leq t \leq T}$ to be the filtrations generated by $W^0(\cdot)$, $W^1(\cdot)$ and $W^2(\cdot)$, respectively, and denote $\mathcal{F}_t = \mathcal{F}_t^{W^0} \otimes \mathcal{F}_t^{W^1} \otimes \mathcal{F}_t^{W^2}$ for $0 \leq t \leq T$. Let $\mathcal{F} = \mathcal{F}_T$ and $\mathbb{E}[\cdot]$ be the expectation with respect to $\mathbb{P}$.

Now we consider an insurance firm whose claim process is denoted by $C(\cdot)$. Following the framework of Promislow and Young\cite{31}, we model the claim process $C(\cdot)$ according to a Brownian motion with drift as follows:

$$dC(t) = adt - bdW^0(t), \quad t \in [0, T],$$

where $a$ and $b$ are positive constants. $W^0(t)$ represents the randomness from the insurance claim. Assume that the premium is paid continuously at the constant rate $c$, which is calculated by the expected value principle, i.e., $c = (1 + \theta)a$, where $\theta > 0$ is the relative safety loading of the insurer.

Suppose that the insurer is allowed to invest its surplus in a financial market consisting of a risk-free asset and a risky asset, whose price dynamics are described by the following:

$$\begin{align*}
    dB(t) &= r(t)B(t)dt, \quad t \in [0, T], \\
    dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)dW^1(t), \quad t \in [0, T],
\end{align*}$$

respectively. Here the interest rate $r(\cdot) > 0$ is a deterministic, uniformly bounded function, the rate of return $\mu(\cdot)$ is an $\mathcal{F}_t$-adapted process which satisfies

$$d\mu(t) = h(t)\mu(t)dt + l(t)dW^1(t) + z(t)dW^2(t), \quad t \in [0, T],$$

where $h(\cdot), l(\cdot)$, and $z(\cdot)$ are deterministic functions. The volatility rate $\sigma(\cdot)$ a deterministic, uniformly bounded function together with $\sigma(\cdot)^{-1}$ is also bounded.

**Remark 1** The above assumption (2.3) about the rate of return $\mu(\cdot)$ can refer to Gennotte\cite{31}. In fact, there are two different sources of uncertainty in financial market. The first one, $W^1(\cdot)$, affects stock prices and the rate of return. We can think of it as economic cycle factors. The second one, $W^2(\cdot)$, affects the rate of return. We can put it as size of firms that issues shares or book-to-market values according to economical model.

In this paper, we shall assume that $\mathcal{F}_t^{W^0}$ and $\mathcal{F}_t^S := \sigma(S_u, 0 \leq u \leq t)$ are independent, and denote $\mathcal{G}_t := \mathcal{F}_t^{W^0} \otimes \mathcal{F}_t^S$ which is the only information available to the insurance firm at time $t$. That is to say, we only know the randomness from the insurance claim and the price process of the risky asset.

In addition to investment, we assume that the insurer can purchase proportional reinsurance to reduce the underlying insurance risk. The reinsurance level is associated with the value $1 - q(t)$ at time $t$ with $q(t) \geq 0$ for all $t$. 

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A strategy $u(t) := (\pi(t), q(t))'$, where $\pi(t)$ represents the amount invested in the risky asset at time $t$. Here, $\pi(\cdot) \in [0, \infty)$ is in the case when short-selling is not allowed. On the other hand, $q(\cdot) \in [0, 1]$ corresponds to a proportional reinsurance and $q(\cdot) > 1$ corresponds to acquiring new reinsurance business.

Under the above assumptions, the surplus/wealth process $X(\cdot)$ of the insurance firm satisfies:

$$
\begin{cases}
    dX(t) = cdt - q(t)dC(t) - (1 + \eta)a(1 - q(t))dt \\
    \quad + (X(t) - \pi(t))r(t)dt + \pi(t)\frac{dS(t)}{S(t)} \\
    \quad + bq(t)dW^0(t) + \pi(t)\sigma(t)dW^1(t), \quad t \in [0, T],
\end{cases}
$$

where $x_0 > 0$ is the initial wealth and $\eta$ represents the safety loading of reinsurance. In the case of proportional reinsurance, the insurance company pays to the reinsurance company a proportional part of the premium, the reinsurance company covers some part of the claim. There are two types of reinsurance contracts: Cheap-reinsurance and noncheap-reinsurance. Cheap-reinsurance is a contract in which the safety loading of the reinsurer and the cedent are the same, that is, the fraction of the premiums diverted to the reinsurer is the same as the fraction of each claim covered by the reinsurer, i.e., $\eta = \theta$. When the safety loading of the reinsurer is higher than that of the cedent, i.e., $\eta > \theta$, such a contract is called a noncheap-reinsurance. See Azcue and Muler[32] and Luo, et al.[33].

**Definition 2.1** A strategy $u(\cdot) := (\pi(\cdot), q(\cdot))'$ is said to be admissible if $\pi(\cdot) \in [0, \infty)$ and $q(\cdot) \in [0, \infty)$ are $\mathcal{G}_t$-progressively measurable, satisfying $\mathbb{E} \int_0^T \pi^2(t)dt < \infty$ and $\mathbb{E} \int_0^T q^2(t)dt < \infty$. Denote the set of all admissible strategies by $\mathcal{U}_{ad}[0, T]$.

The mean-variance problem refers to finding admissible strategies such that the expected terminal wealth satisfies $\mathbb{E}X(T) = d > 0$, while the risk measured by the variance of the terminal wealth

$$
\text{Var}[X(T)] = \mathbb{E}[X(T) - \mathbb{E}X(T)]^2 = \mathbb{E}[X(T) - d]^2
$$

is minimized.

It is reasonable to impose $d \geq d_0$, where

$$
d_0 := e^{\int_0^T r(s)ds} \left\{ x_0 + a(\theta - \eta) \left( \int_0^T e^{-\int_0^s r(u)du} dt - 1 \right) \right\}
$$

is the terminal wealth at time $T$, if the insurance firm invests all of its wealth at hand into the risk-free asset and transfers all forthcoming risks to the reinsurer.

**Definition 2.2** The mean-variance problem is formulated as the following optimization
problem with partial information:

\[
\min J_{MV}(x_0, u(\cdot)) := \text{Var}[X(T)] \equiv \mathbb{E}[X(T) - \mathbb{E}X(T)]^2,
\]

subject to

\[
\begin{cases}
\mathbb{E}X(T) = d, \\
u(\cdot) \in \mathcal{U}_{ad}[0,T], \\
(X(\cdot), u(\cdot)) \text{ satisfy Equation (4)}. 
\end{cases}
\]

Morever, the optimal control \( u^*(\cdot) \) satisfying (6) is called an efficient strategy, and \( (\text{Var}[X^*(T)], d) \) is called an efficient point. The set of all efficient points, when the parameter \( d \) runs over \([d_0, +\infty)\), is called the efficient frontier.

3 Efficient Strategies and Efficient Frontier with Partial Information

A notorious difficulty in tackling general stochastic optimization problems with partial information is that one usually cannot separate the filtering and optimization, except for some very rare situations. However, the separation principle in Xiong and Zhou\[26\] shows that for some specific mean-variance problems, the separation principle happens to hold: The partial information stochastic control problem separates into a filtering problem and an optimal control problem. For simplicity, we consider the cheap-reinsurance in this partial information section, i.e., \( \eta = \theta \). At this time, \( d_0 \) is simplified to \( d_0 = x_0e^{\int_0^T r(s)ds} \). There are also some papers that focus on the cheap reinsurance contract, for example, \[32\] and \[33\].

3.1 Separate Principle and Stochastic Filtering

In this subsection, we first consider the filtering problem associated with our model (6) and establish a separation principle. Specifically, we introduce the innovation process and reduce the partial information problem to a full information one. Actually, the innovation process contains the same information as the observation process \( S(\cdot) \). Then, at this time, the state equation \( X(\cdot) \) is \( \mathcal{G}_t \)-adapted, and the control process \( u(\cdot) \) is also \( \mathcal{G}_t \)-adapted. It means one can choose the control variable based on “full information” \( \mathcal{G}_t \), but this “full information” \( \mathcal{G}_t \) here is some subset of \( \mathcal{F}_t \).

Firstly, we introduce the innovation process and established the separate principle. We are just here to draw a conclusion, details can be found in Xiong and Zhou\[26\].

Lemma 3.1 For any admissible control \( u(\cdot) \in \mathcal{U}_{ad}[0,T] \), the corresponding wealth process \( X(\cdot) \) satisfies the following SDE:

\[
\begin{cases}
dX(t) = [a\eta\xi(t) + r(t)X(t) + (m(t) - r(t))\pi(t)]dt \\
+ b\eta(t)dW^0(t) + \pi(t)\sigma(t)d\overline{W}(t), \quad t \in [0, T], \\
X(0) = x_0,
\end{cases}
\]

where \( m(t) \equiv \mathbb{E}[\mu(t)|\mathcal{G}_t] \) is the optimal filter of \( \mu(t) \), and the innovation process \( \overline{W}(\cdot) \) given by

\[
d\overline{W}(t) := \frac{1}{\sigma(t)} \left[ \frac{dS(t)}{S(t)} - m(t)dt \right]
\]

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is a Brownian motion with respect to \( P \) and \( \{ \mathcal{G}_t \}_{0 \leq t \leq T} \).

Next, we study the filtering problem and obtain the best estimate \( m(t) \) of \( \mu(t) \) based on the observation information \( \{ \mathcal{G}_t \} \). According to Theorem 11.1 in Liptser and Shiryaev\(^{34} \), we obtain the following lemma.

**Lemma 3.2** We denote \( n(t) \equiv \mathbb{E}[(\mu(t) - m(t))^2|\mathcal{G}_t] \). Let the conditional distribution \( F_{\mathcal{G}_t}(x) = \mathbb{P}(\mu(0) \leq x|\mathcal{G}_0) \) be Gaussian, \( N(m(0), n(0)) \), with \( 0 \leq n(0) < \infty \). Then the conditional distributions \( F_{\mathcal{G}_t}(x) = \mathbb{P}(\mu(t) \leq x|\mathcal{G}_t) \) are Gaussian, \( N(m(t), n(t)) \), for all \( t \).

Thus, \( m(\cdot) \) is the optimal estimate after obtaining the information \( \{ \mathcal{G}_t \}_{0 \leq t \leq T} \). Since in reality, it is difficult to observe the information of the rate \( \mu(\cdot) \) in return of risky asset, but one can usually know the information of the risky asset prices \( S(\cdot) \), therefore, we could estimate \( \mu(\cdot) \) based on the risky asset price \( S(\cdot) \), and \( m(t) = \mathbb{E}[\mu(t)|\mathcal{G}_t] \) is denoted as its best estimate. According to Theorem 12.1 in Liptser and Shiryaev\(^{34} \), optimal estimates \( m(\cdot) \) and \( n(\cdot) \) can be obtained in the following lemma.

**Lemma 3.3** Suppose stochastic processes \( \mu(\cdot) \) and \( S(\cdot) \) satisfy

\[
\begin{align*}
\frac{d\mu(t)}{dt} &= h(t)\mu(t)dt + l(t) dW^1(t) + z(t) dW^2(t), \quad t \in [0, T], \\
\frac{dS(t)}{dt} &= \mu(t)S(t)dt + \sigma(t) S(t) dW^1(t), \quad t \in [0, T],
\end{align*}
\]

respectively. Then the optimal estimates \( m(\cdot) \) and \( n(\cdot) \) satisfy

\[
\begin{align*}
\frac{dm(t)}{dt} &= h(t)m(t)dt + \left[ l(t) + \frac{n(t)}{\sigma(t)} \right] \frac{1}{\sigma(t)} \left[ \frac{dS(t)}{S(t)} - m(t) dt \right], \quad t \in [0, T], \\
\frac{dn(t)}{dt} &= 2h(t)n(t) + l(t)^2 + z(t)^2 - \left[ l(t) + \frac{n(t)}{\sigma(t)} \right]^2, \quad t \in [0, T].
\end{align*}
\]

From the above lemma, we can also obtain

\[
\begin{align*}
\frac{dS(t)}{dt} &= m(t)S(t)dt + \sigma(t) S(t) d\overline{W}(t), \quad t \in [0, T], \\
\frac{dm(t)}{dt} &= h(t)m(t)dt + \left[ l(t) + \frac{n(t)}{\sigma(t)} \right] d\overline{W}(t), \quad t \in [0, T].
\end{align*}
\]

**3.2 Solution to the Stochastic LQ Control Problem**

In this subsection, we derive the efficient strategies and efficient frontier via solutions to two extended stochastic Riccati equations. According to Lemma 3.1, we can solve the resulting problem as in the full information case.

First, Equation (7) can be rewritten as the following linear SDE:

\[
\begin{align*}
\frac{dX(t)}{dt} &= [r(t)X(t) + B(t)u(t)]dt + u(t) D(t) dW(t), \quad t \in [0, T], \\
X(0) &= x_0,
\end{align*}
\]

where \( u(\cdot) \equiv (q(\cdot), \pi(\cdot))' \in \mathcal{U}_a[0, T] \), \( W(\cdot) \equiv (W^0(\cdot), \overline{W}(\cdot))' \) and \( B(t) \equiv (an, m(t) - r(t)) \), \( D(t) \equiv (D^1(t), D^2(t))' \), \( D^1(t) \equiv (b, 0) \), \( D^2(t) \equiv (0, \sigma(t)) \).
This problem is exactly a stochastic LQ model with random coefficients and the control variable is constrained. Similar to the results by Hu and Zhou\[16\], efficient strategies and efficient frontier could be presented in closed form via solutions to two extended stochastic Riccati equations.

Now we introduce the following two nonlinear backward stochastic differential equations (BSDEs, for short):

\[
\begin{align*}
\frac{dP_+(t)}{dt} &= -[2r(t)P_+(t) + H^*_+(t, P_+(t), A_+(t))] dt + A_+(t)'dW(t), \quad t \in [0, T], \\
P_+(T) &= 1, \\
P_+(t) &> 0,
\end{align*}
\]

(13)

\[
\begin{align*}
\frac{dP_-(t)}{dt} &= -[2r(t)P_-(t) + H^*_-(t, P_-(t), A_-(t))] dt + A_-(t)'dW(t), \quad t \in [0, T], \\
P_-(T) &= 1, \\
P_-(t) &> 0,
\end{align*}
\]

(14)

where

\[
\begin{align*}
H^*_+(t, P, A) &:= \min_{u \in \mathcal{U}_u_P(t, T]} \left\{ u'PD(t)D(t)'u + 2u' [B(t)'P + D(t)A] \right\}, \\
H^*_-(t, P, A) &:= \min_{u \in \mathcal{U}_u_P(t, T]} \left\{ u'PD(t)D(t)'u - 2u' [B(t)'P + D(t)A] \right\}.
\end{align*}
\]

(15)

Also, define

\[
\begin{align*}
\xi_+(t, P, A) &:= \arg\min_{u \in \mathcal{U}_u_P(t, T]} \left\{ u'PD(t)D(t)'u + 2u' [B(t)'P + D(t)A] \right\}, \\
\xi_-(t, P, A) &:= \arg\min_{u \in \mathcal{U}_u_P(t, T]} \left\{ u'PD(t)D(t)'u - 2u' [B(t)'P + D(t)A] \right\},
\end{align*}
\]

(16)

\[ (t, P, A) \in [0, T] \times \mathbb{R} \times \mathbb{R}. \]

Similar to Theorem 5.2 in [16], we see that (13) and (14) admit unique bounded, uniformly positive solutions \( P_+(\cdot) \) and \( P_-(\cdot) \), respectively.

Since (6) is a convex optimization problem, the equality constraint \( \mathbb{E}[X(T)] = d \) can be dealt with by introducing a Lagrange multiplier \( \gamma \in \mathbb{R} \). In this way the problem (6) can be solved via the following stochastic control problem (for every fixed \( \gamma \)). Define

\[
J(x_0, u(\cdot), \gamma) := \mathbb{E}\left\{ X(T)^2 - d^2 - 2\gamma[X(T) - d] \right\} = \mathbb{E} \left[ (X(T) - \gamma)^2 \right] - (\gamma - d)^2, \quad \gamma \in \mathbb{R}.
\]

(17)

Based on Lagrange duality theorem (see Luenberger\[35\]), we may first solve the following unconstrained problem parameterized by the Lagrange multiplier \( \gamma \in \mathbb{R} \):

\[
\begin{align*}
\text{minimize } J(x_0, u(\cdot), \gamma) &:= \mathbb{E} \left[ (X(T) - \gamma)^2 \right] - (\gamma - d)^2, \\
\text{subject to } (X(\cdot), u(\cdot)) \text{ is admissible for (12)}.
\end{align*}
\]

(18)

We now consider the state feedback control for the problem (18). For any real number \( x \) we define \( x^+ := \max\{x, 0\} \) and \( x^- := \max\{-x, 0\} \).
Theorem 3.4 Let \((P_+ (\cdot), A_+)\) and \((P_- (\cdot), A_-)\) be the unique bounded, uniformly positive solutions to the BSDEs (13) and (14), respectively. Then the state feedback control

\[
u^\ast (t) = \xi_+ (t, P_+(t), A_+(t)) \left( X(t) - \gamma e^{-\int_0^t r(s) ds} \right)^+ \\
+ \xi_- (t, P_-(t), A_- (t)) \left( X(t) - \gamma e^{-\int_0^t r(s) ds} \right)^-
\]

is optimal for the problem (18). Moreover, in this case the optimal cost is

\[
J^\ast (x_0, \gamma) := \inf_{u(\cdot) \in \mathcal{U}_{x_0}^\ast} J(x_0, u(\cdot), \gamma)
\]

\[
= \begin{cases} 
P_+(0)e^{-\int_0^T r(s) ds} \gamma^2 - 2 \left[ x_0P_+(0)e^{-\int_0^T r(s) ds} - d \right] \gamma + P_+(0)x_0^2 - d^2, & \text{if } x_0 > \gamma e^{-\int_0^T r(s) ds}, \\
P_-(0)e^{-\int_0^T r(s) ds} \gamma^2 - 2 \left[ x_0P_-(0)e^{-\int_0^T r(s) ds} - d \right] \gamma + P_-(0)x_0^2 - d^2, & \text{if } x_0 \leq \gamma e^{-\int_0^T r(s) ds}. 
\end{cases}
\]

\[\text{(20)}\]

Proof Set

\[y(t) := X(t) - \gamma e^{-\int_0^t r(s) ds}.\]

It turns out the wealth equation (12) in terms of \(y(\cdot)\) has exactly the following same form except for the initial condition:

\[
\begin{cases} 
dy(t) = [r(t)y(t) + B(t)u(t)]dt + u(t)'D(t)dW(t), & t \in [0, T], \\
y(0) = x_0 - \gamma e^{-\int_0^T r(s) ds}, 
\end{cases}
\]

\[\text{(21)}\]

whereas the cost functional (17) can be rewritten as

\[
J(y_0, u(\cdot), \gamma) = \mathbb{E}y(T)^2 - (\gamma - d)^2.
\]

\[\text{(22)}\]

The above problem (21)–(22) is exactly a stochastic LQ control problem with random coefficients and the control variable is constrained. Hence the optimal feedback control (19) follows from Theorem 5.1 in [16]. Finally, the optimal cost is

\[
J^\ast (x_0, \lambda) = P_+(0) \left[ \left( x_0 - \gamma e^{-\int_0^T r(s) ds} \right)^+ \right]^2 + P_-(0) \left[ \left( x_0 - \gamma e^{-\int_0^T r(s) ds} \right)^- \right]^2 - (\gamma - d)^2,
\]

which equals the right-hand side of (20) after some simple manipulations.

\[\Box\]

Theorem 3.5 The efficient strategy corresponding to \(d \geq d_0\), where \(d_0 := x_0e^{\int_0^T r(s) ds}\), as a feedback of the wealth process, is

\[
u^\ast (t) = \xi_+ (t, P_+(t), A_+(t)) \left( X^\ast (t) - \gamma e^{-\int_0^T r(s) ds} \right)^+ \\
+ \xi_- (t, P_-(t), A_- (t)) \left( X^\ast (t) - \gamma e^{-\int_0^T r(s) ds} \right)^-,
\]

\[\text{(23)}\]
where
\[ \gamma^* := \frac{d - x_0 P_-(0)e^{-\int_0^T r(s)ds}}{1 - P_-(0)e^{-2\int_0^T r(s)ds}}. \]

Moreover, the efficient frontier is
\[ \text{Var}[x^*(T)] = \frac{P_-(0)e^{-2\int_0^T r(s)ds}}{1 - P_-(0)e^{-2\int_0^T r(s)ds}} \left[ \mathbb{E}X^*(T) - x_0 e^\int_0^T r(s)ds \right]^2, \quad \mathbb{E}X^*(T) \geq d_0. \quad (25) \]

**Proof** First, if \( d = x_0 e^\int_0^T r(s)ds \), then it is straightforward that the corresponding efficient strategy is \( u^*(t) \equiv (0,0)' \). The resulting wealth process is \( X^*(t) = x_0 e^\int_0^T r(s)ds \). On the other hand, in this case the associated \( \gamma^* = x_0 e^\int_0^T r(s)ds \). This implies that (25) is indeed the efficient frontier when \( d = x_0 e^\int_0^T r(s)ds \).

So we need only to prove the theorem for any fixed \( d > x_0 e^\int_0^T r(s)ds \). Applying the Lagrange duality theorem again, we have
\[ J_{\text{MV}}^*(x_0) := \inf_{u(\cdot) \in \mathcal{U}_2} \sup_{\gamma \in \mathbb{R}} J_{\text{MV}}(x_0, u(\cdot), \gamma) := \sup_{\gamma \in \mathbb{R}} \inf_{u(\cdot) \in \mathcal{U}_2} J(x_0, u(\cdot), \gamma) > -\infty, \quad (26) \]
and the optimal feedback control for (6) is (19), with \( \gamma \) replaced by \( \gamma^* \) in (24) which maximizes \( J^*(x_0, \gamma) \) over \( \gamma \in \mathbb{R} \), due to Theorem 3.4.

If \( \gamma < x_0 e^\int_0^T r(s)ds \), then the expression (20) and the fact that \( d \geq x_0 e^\int_0^T r(s)ds \), give
\[
\begin{align*}
\frac{\partial}{\partial \gamma} J^*(x_0, \gamma) &= 2\left[ P_+(0)e^{-2\int_0^T r(s)ds} - 1 \right] \gamma - 2\left[ x_0 P_+(0)e^{-\int_0^T r(s)ds} - d \right] \geq 2\left[ P_+(0)e^{-2\int_0^T r(s)ds} - 1 \right] x_0 e^\int_0^T r(s)ds - 2\left[ x_0 P_+(0)e^{-\int_0^T r(s)ds} - x_0 e^\int_0^T r(s)ds \right] \\
&= 0.
\end{align*}
\]

In the above calculation, we have used the result \( P_+(0)e^{-\int_0^T r(s)ds} - 1 \leq 0 \) in Lemma 6.1 of [16]. Hence,
\[
\sup_{\gamma \in \mathbb{R}} J^*(x_0, \gamma) = \sup_{\gamma \in \left(-\infty, x_0 e^\int_0^T r(s)ds \right)} J^*(x_0, \gamma).
\]

But for \( \gamma \geq x_0 e^\int_0^T r(s)ds \), it follows from (20) that \( J^*(x_0, \gamma) \) is a quadratic function in \( \gamma \) whose maximizer is given by (24), whereas
\[
J_{\text{MV}}^*(x_0) = \sup_{\gamma \in \mathbb{R}} \inf_{u(\cdot) \in \mathcal{U}_2} J^*(x_0, \gamma)
\]
\[
= \sup_{\gamma \in \mathbb{R}} \left\{ \left[ P_+(0)e^{-\int_0^T r(s)ds} - 1 \right] \gamma - 2\left[ x_0 P_+(0)e^{-\int_0^T r(s)ds} - d \right] \right. \gamma + P_-(0)x_0^2 - d^2 \right\}
\]
\[
= \frac{P_+(0)e^{-2\int_0^T r(s)ds}}{1 - P_-(0)e^{-2\int_0^T r(s)ds}} \left[ d - x_0 e^\int_0^T r(s)ds \right]^2, \quad d \geq x_0 e^\int_0^T r(s)ds.
\]
This proves (25), noting that \( \mathbb{E}X^*(T) = d \). The proof is completed.
It is interesting to note that, after using the stochastic filtering, the wealth process (7) contains random coefficient \( m(\cdot) \). Accordingly, the problem becomes more complex. Specifically, the conventional stochastic Riccati equations turn to two BSDEs (13) and (14). Usually, this kind of nonlinear BSDEs have no analytical solutions. However, if all the market coefficients are deterministic, then \( \Lambda \equiv 0 \) and the equations (13) and (14) turn to ODEs. We can see this in detail in the next section.

4 Mean-Variance Problem with Full Information

In this section, we derive the efficient frontier of the full information mean-variance problem. Specifically, the insurer is allowed to invest its surplus in a financial market and purchase proportional reinsurance. We consider the noncheap-reinsurance in this section, i.e., \( \eta > \theta \).

From (4), the wealth process \( X(\cdot) \) satisfies:

\[
\begin{align*}
\frac{dX(t)}{dt} &= \left[ a_\theta - a_\eta + a_\eta q(t) + (\mu - r)\pi(t) \right] + \sigma \pi(t) dW^1(t), \quad t \in [0, T], \\
X(0) &= x_0,
\end{align*}
\]

where we have let \( \mu(t) \equiv \mu, r(t) \equiv r \) and \( \sigma(t) \equiv \sigma \) for all \( t \).

**Definition 4.1** A strategy \( u(\cdot) := (\pi(\cdot), q(\cdot))' \) is said to be admissible if \( \pi(\cdot) \in [0, \infty) \) and \( q(\cdot) \in [0, \infty) \) are \( \mathcal{F}_t \)-progressively measurable, satisfying \( \mathbb{E} \int_0^T \pi^2(t) dt < \infty \) and \( \mathbb{E} \int_0^T q^2(t) dt < \infty \). Denote the set of all admissible strategies by \( \mathcal{U}_{ad}[0, T] \).

It is reasonable to impose \( d \geq d_1 \), where \( d_1 := x_0 e^{Tr} + \frac{\theta - \alpha}{r} (e^{Tr} - 1) \) is the terminal wealth at time \( T \), if insurance company invests all of its wealth at hand into the risk-free asset and transfers all forthcoming risks to the reinsurer.

The mean-variance problem is formulated as the following optimization problem with full information:

\[
\begin{align*}
\text{minimize} & \quad \text{Var}[X(T)] = \mathbb{E} \left[ (X(T) - \mathbb{E}X(T))^2 \right] \\
\text{subject to} & \quad \mathbb{E}X(T) = d, \\
& \quad u(\cdot) \in \mathcal{U}_{ad}[0, T], \\
& \quad (X(\cdot), u(\cdot)) \text{ satisfy Equation (27).}
\end{align*}
\]

4.1 Value Function for Auxiliary Problem

Similarly, the problem (28) can be solved via the following stochastic LQ control problem (for every fixed \( \gamma \))

\[
\begin{align*}
\text{minimize} & \quad \mathbb{E} \left[ (X(T) - d)^2 + 2\gamma(\mathbb{E}X(T) - d) \right] \\
\text{subject to} & \quad u(\cdot) \in \mathcal{U}_{ad}[0, T], \\
& \quad (X(\cdot), u(\cdot)) \text{ satisfy Equation (27),}
\end{align*}
\]
where the factor 2 in front of the multiplier $\gamma$ is introduced in the objective function just for convenience. Clearly, this problem is equivalent to the following auxiliary problem

$$\begin{align*}
\text{minimize} & \quad \mathbb{E}\left\{ \frac{1}{2}[X(T) - (d - \gamma)]^2 \right\} \\
\text{subject to} & \quad u(\cdot) \in \mathcal{U}_{ad}[0, T], \\
& \quad (X(\cdot), u(\cdot)) \text{ satisfy Equation (27)}. 
\end{align*}$$

(30)

Set

$$x(t) := X(t) - (d - \gamma),$$

(31)

then (27) is equivalent to the following controlled linear SDE:

$$\begin{align*}
\frac{dx(t)}{dt} &= [rx(t) + Bu(t) + f]dt + D^1u(t)dW^0(t) + D^2u(t)dW^1(t), \quad t \in [0, T], \\
x(0) &= x_0 - (d - \gamma),
\end{align*}$$

(32)

where $u(\cdot) \equiv (q(\cdot), \pi(\cdot))' \in \mathcal{U}_{ad}[0, T]$ and

$$B \equiv (a\eta, \mu - r), \quad f \equiv a\theta - a\eta + (d - \gamma)r, \quad D^1 \equiv (b, 0), \quad D^2 \equiv (0, \sigma).$$

Our objective is to find an optimal control $u^*(\cdot)$ that minimizes the quadratic cost functional

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}x(T)^2.$$ 

(33)

The problem is an indefinite stochastic LQ control problem. An important feature in this problem is that the control variable is constrained. In this subsection, we use HJB equation and viscosity solution theory to solve it.

The value function associated with the problem (32)–(33) is defined by

$$V(s, y) = \inf_{u(\cdot) \in \mathcal{U}_{ad}[s, T]} J(s, y; u(\cdot)),$$

(34)

where $x(s) = y \in \mathbb{R}, s \in [0, T]$.

From standard arguments (for example, Yong and Zhou [36]), we see that if $V(\cdot, \cdot) \in C^{1,2}[0, T] \times \mathbb{R}$, then it satisfies the following HJB equation:

$$\begin{align*}
\frac{\partial v(t, x)}{\partial t} + \inf_{q \geq 0, \pi \geq 0} \left\{ \frac{\partial v(t, x)}{\partial x} [rx + a\eta q(t) + (\mu - r)\pi(t) + a\theta - a\eta + (d - \gamma)r] + \frac{1}{2}[b^2q^2(t) + \sigma^2\pi^2(t)] \frac{\partial^2 v(t, x)}{\partial x^2} \right\} &= 0, \quad t \in [0, T], \\
v(T, x) &= \frac{1}{2}x^2.
\end{align*}$$

(35)

Owing to the nonnegativity constraint of the control, the HJB equation (35) does not have a smooth solution. Hence, the idea here is to construct a function, to show that it is a viscosity solution to it, and then employ the verification theorem to construct the optimal control. We will do this in the next subsection.
4.2 Optimal Control and Viscosity Solution

This subsection is devoted to verify the following result.

**Theorem 4.2** Define

\[
\begin{align*}
  g_1(t) &:= \frac{a\theta - a\eta + (d - \gamma)r}{r} [e^{r(T-t)} - 1], \\
  A_1 &:= -\frac{(\mu - r)^2}{2\sigma^2} - \frac{u^2\eta^2}{2b^2}.
\end{align*}
\]  

Then the value function

\[
V(t, x) = \begin{cases} 
  \frac{1}{2} \left[ e^{r(T-t)}x + g_1(t) \right]^2, & \text{if } x + g_1(t)e^{-r(T-t)} \geq 0, \\
  \frac{1}{2} \left[ e^{(A_1+r)(T-t)}x + e^{r(T-t)}A_1g_1(t) \right]^2, & \text{if } x + g_1(t)e^{-r(T-t)} < 0
\end{cases}
\]

(37)

is a continuous viscosity solution to the HJB equation (35), and

\[
u^*(t, x) = \begin{cases} 
  -\frac{\mu - r}{\sigma^2} \left[ x + g_1(t)e^{-r(T-t)} \right], & \text{if } x + g_1(t)e^{-r(T-t)} \geq 0, \\
  \frac{a\eta}{b^2} \left[ x + g_1(t)e^{-r(T-t)} \right], & \text{if } x + g_1(t)e^{-r(T-t)} < 0, \\
  (0,0), & \text{if } x + g_1(t)e^{-r(T-t)} = 0.
\end{cases}
\]

(38)

is the associated optimal feedback control.

**Proof** First we show that \( V(\cdot,\cdot) \) constructed in (37) is a viscosity solution to (35).

Suppose that it has a solution \( v(\cdot,\cdot) \in C^{1,2}[0,T] \times \mathbb{R} \) satisfying \( \frac{\partial v}{\partial x} > 0 \). Then, if \( \frac{\partial v}{\partial x} > 0 \), the minimum of the left hand side of (35) is attained at \( u^*(\cdot) = (q^*(\cdot),\pi^*(\cdot))' = (0,0)' \). Assuming that \( v(t,x) \) has the following form:

\[
v(t,x) = \frac{1}{2} P(t)x^2 + Q(t)x + R(t),
\]

(39)

where \( P(\cdot),Q(\cdot),R(\cdot) \) are differentiable functions to be determined. Inserting (39) and \( u^*(\cdot) = (q^*(\cdot),\pi^*(\cdot))' = (0,0)' \) into (35), we have

\[
\begin{align*}
  \dot{P}(t) + 2rP(t) &= 0, \quad P(T) = 1, \\
  \dot{Q}(t) + rQ(t) + [a\theta - a\eta + (d - \gamma)r]P(t) &= 0, \quad Q(T) = 0, \\
  \dot{R}(t) + [a\theta - a\eta + (d - \gamma)r]Q(t) &= 0, \quad R(T) = 0.
\end{align*}
\]  

(40)

Solving them, we obtain

\[
P_1(t) = e^{2r(T-t)}, \quad Q_1(t) = g_1(t)e^{r(T-t)}, \quad R_1(t) = \frac{g_1^2(t)}{2},
\]

(41)

with \( g_1(t) \) being defined in (36). Considering the assumption \( \frac{\partial v}{\partial x} > 0 \), we have

\[
v(t,x) = \frac{1}{2} [e^{r(T-t)}x + g_1(t)]^2
\]
in the region
\[ \mathcal{A}_1 = \{(t, x) \in [0, T] \times \mathbb{R} : x + g_1(t)e^{-r(T-t)} > 0 \}, \]
and the minimum is attained at \((\pi^*(\cdot), q^*\cdot(\cdot)) = (0, 0)\).

For \((t, x) \in \{(t, x) \in [0, T] \times \mathbb{R} : x + g_1(t)e^{-r(T-t)} < 0\}, we have \(\frac{\partial v}{\partial x} < 0\). Assume that the minimum of (35) is attained in the interior of the control region. Then
\[
\pi^*(t, x) = -\frac{\mu - r}{\sigma^2} \frac{\partial v(t, x)}{\partial x}, \quad q^*(t, x) = -\frac{\alpha}{\beta^2} \frac{\partial^2 v(t, x)}{\partial x^2}.
\]
Inserting this into (35), the HJB equation becomes
\[
\begin{align*}
\frac{\partial v(t, x)}{\partial t} + \left[ r x + a \theta - a \eta + (d - \gamma) r \right] \frac{\partial v(t, x)}{\partial x} &+ \frac{(\mu - r)^2}{2 \sigma^2} \left( \frac{\partial v(t, x)}{\partial x} \right)^2 - \frac{\alpha^2 \eta^2}{2 \beta^2} \left( \frac{\partial^2 v(t, x)}{\partial x^2} \right)^2 = 0, \quad t \in [0, T], \\
v(T, x) &= \frac{1}{2} x^2.
\end{align*}
\]
Inserting (39) and (42) into (43), we obtain
\[
\begin{align*}
\dot{P}(t) + [2r + 2A_1]P(t) &= 0, \quad P(T) = 1, \\
\dot{Q}(t) + [r + 2A_1]Q(t) + [a \theta - a \eta + (d - \gamma) r]P(t) &= 0, \quad Q(T) = 0, \\
\dot{R}(t) + A_1 \frac{Q^2(t)}{P(t)} + [a \theta - a \eta + (d - \gamma) r]Q(t) &= 0, \quad R(T) = 0,
\end{align*}
\]
where \(A_1\) is defined in (36). Solving them, we have
\[
P_2(t) = e^{(2A_1+2r)(T-t)}, \quad Q_2(t) = g_1(t)e^{(2A_1+r)(T-t)}, \quad R_2(t) = \frac{1}{2} e^{2(T-t)A_1} g_1^2(t).
\]
Since \(\frac{\partial v}{\partial x} < 0\), we have
\[
v(t, x) = \frac{1}{2} \left[ e^{(A_1+r)(T-t)}x + e^{(T-t)A_1}g_1(t) \right]^2.
\]
In the region
\[ \mathcal{A}_2 = \{(t, x) \in [0, T] \times \mathbb{R} : x + g_1(t)e^{-r(T-t)} < 0 \}, \]
the minimum is attained at
\[
(\pi^*(t), q^*(t)) = \left( -\frac{\mu - r}{\sigma^2} [x + g_1(t)e^{-r(T-t)}], -\frac{\alpha}{\beta^2} [x + g_1(t)e^{-r(T-t)}] \right).
\]
In the inner regions \(\mathcal{A}_i (i = 1, 2), v(\cdot, \cdot) \in C^{1,2}[0, T] \times \mathbb{R}, \) thus it is a classical solution inside these regions. However, the switching curve \(\mathcal{A}_3\) defined by
\[ \mathcal{A}_3 = \{(t, x) \in [0, T] \times \mathbb{R} : x + g_1(t)e^{-r(T-t)} = 0 \}
\]is where the non-smoothness of \(V(\cdot, \cdot)\) happens.
Firstly, a direct calculation shows that
\[ V(t, x) = \frac{1}{2} [e^{(T-t)x} + g_1(t)]^2 = \frac{1}{2} [e^{(A_1+r)(T-t)x} + e^{(T-t)A_1}g_1(t)]^2 = 0 \]
on \mathcal{A}_3. Therefore, \( V(\cdot, \cdot) \) is continuous at points on \( \mathcal{A}_3 \). In addition, we also easily obtain
\[
\begin{align*}
\frac{\partial V(t,x)}{\partial t} &= \frac{1}{2} \dot{P}_1(t)x^2 + \dot{Q}_1(t)x + \dot{R}_1(t) = \frac{1}{2} \dot{P}_2(t)x^2 + \dot{Q}_2(t)x + \dot{R}_2(t) = 0, \\
\frac{\partial V(t,x)}{\partial x} &= P_1(t)x + Q_1(t) = P_2(t)x + Q_2(t) = 0.
\end{align*}
\]
(46)

That is, \( V(\cdot, \cdot) \) is also continuously differentiable at points on \( \mathcal{A}_3 \). However, \( \frac{\partial^2 V}{\partial x^2} \) does not exist on \( \mathcal{A}_3 \), since \( P_1(t) \neq P_2(t) \). This means that \( V \) does not have the smoothness property to qualify as a classical solution to the HJB equation (35). For this reason, we need to work within the framework of viscosity solutions (please refer to Yong and Zhou[36], Li, et al.[15] for some basic terminologies of viscosity solutions).

It can be shown that for any \((t, x) \in \mathcal{A}_3, \)
\[
\begin{align*}
D_{t,x}^{1,2,+} V(t, x) &= \{0\} \times \{0\} \times [P_1(t), +\infty), \\
D_{t,x}^{1,2,-} V(t, x) &= \{0\} \times \{0\} \times (-\infty, P_2(t)].
\end{align*}
\]
(47)

For the HJB equation (35), we define
\[
G(t, x, u, p, P) := p [rx + a\eta q(t) + (\mu - r)\pi(t) + a\theta - a\eta + (d - \gamma) x] \\
+ \frac{1}{2} P \left[ b^2 q^2(t) + \sigma^2 \pi^2(t) \right].
\]
(48)

For any \((q, p, P) \in D_{t,x}^{1,2,+} V(t, x), \) when \((t, x) \in \mathcal{A}_3, \) we have
\[
q + \inf_{u \geq 0} G(t, x, u, p, P) = \inf_{u \geq 0} \left\{ \frac{1}{2} P \left[ b^2 q^2(t) + \sigma^2 \pi^2(t) \right] \right\} \\
\geq \inf_{u \geq 0} \left\{ \frac{1}{2} P_1(t) \left[ b^2 q^2(t) + \sigma^2 \pi^2(t) \right] \right\} \\
= 0.
\]
(49)

Therefore, \( V(\cdot, \cdot) \) is a viscosity sub-solution to (35). On the other hand, for \((q, p, P) \in D_{t,x}^{1,2,-} V(t, x), \) when \((t, x) \in \mathcal{A}_3, \) we have
\[
q + \inf_{u \geq 0} G(t, x, u, p, P) = \inf_{u \geq 0} \left\{ \frac{1}{2} P \left[ b^2 q^2(t) + \sigma^2 \pi^2(t) \right] u \right\} \\
\leq \inf_{u \geq 0} \left\{ \frac{1}{2} P_2(t) \left[ b^2 q^2(t) + \sigma^2 \pi^2(t) \right] \right\} \\
= 0.
\]
(50)

Therefore, \( V(\cdot, \cdot) \) is also a viscosity super-solution to (35).

Finally, it is easy to see that the terminal condition \( V(T, x) = \frac{1}{2} x^2 \) is satisfied. Hence, it follows that \( V(\cdot, \cdot) \) is a viscosity solution to the HJB equation (35).
Moreover, for any \((t, x) \in \mathcal{A}_3\), take \((q^*(t, x), p^*(t, x), P^*(t, x), u^*(t, x)) = (0, 0, P_1(t), 0) \in D^2_{t,x} \times \mathcal{U}_{ad}[s, T]\), then
\[
q^*(t, x) + G(t, x, u^*(t, x), p^*(t, x), P^*(t, x)) = 0.
\]
(51)

It then follows from the verification theorem (Zhou, et al.\[37\]) that \(u^*(t, x)\) defined by (38) is the optimal feedback control. The proof is completed.

4.3 Efficient Strategies and Efficient Frontier

In this subsection, we give the efficient frontier for the problem (28), i.e., we derive the connection between the expected value and the variance of the terminal wealth for each efficient strategy. First of all, noting (31) and (33), we have
\[
\mathbb{E}\left\{\frac{1}{2}x(T)^2\right\} = \mathbb{E}\left\{\frac{1}{2}(X(T) - (d - \gamma))^2\right\} = \mathbb{E}\left\{\frac{1}{2}(X(T) - d)^2\right\} + \gamma[{\mathbb{E}X(T) - d}] + \frac{1}{2}\gamma^2.
\]
Hence, for every fixed \(\gamma\), we have
\[
\min_{u(\cdot) \in \mathcal{U}_{ad}[0, T]} \mathbb{E}\left\{\frac{1}{2}(X(T) - d)^2 + \gamma[{\mathbb{E}X(T) - d}]\right\} = \min_{u(\cdot) \in \mathcal{U}_{ad}[0, T]} \mathbb{E}\left\{\frac{1}{2}x(T)^2\right\} - \frac{1}{2}\gamma^2
\]
\[
= V(0, x) - \frac{1}{2}\gamma^2
\]
\[
= \frac{1}{2}P(0)x^2 + Q(0)x + R(0) - \frac{1}{2}\gamma^2
\]
\[
= \frac{1}{2}P(0) [x_0 - (d - \gamma)]^2 + Q(0) [x_0 - (d - \gamma)] + R(0) - \frac{1}{2}\gamma^2,
\]
(52)

where \(P(\cdot), Q(\cdot)\) and \(R(\cdot)\) are specified in (39). If \(x + g_1(t)e^{-rT} < 0\), we have a concave quadratic function in \(\gamma\):
\[
\min_{u(\cdot) \in \mathcal{U}_{ad}[0, T]} \mathbb{E}\left\{\frac{1}{2}(X(T) - d)^2 + \gamma[{\mathbb{E}X(T) - d}]\right\} = \frac{1}{2}P_2(0) [x_0 - (d - \gamma)]^2 + Q_2(0) [x_0 - (d - \gamma)] + R_2(0) - \frac{1}{2}\gamma^2
\]
\[
= \frac{1}{2}e^{2A_T} \left[ x_0e^{Tr} + \frac{a\theta - an\eta}{r} - (d - \gamma) \right]^2 - \frac{1}{2}\gamma^2.
\]
If \( x + g_1(t)e^{-rT} \geq 0 \), we have a linear function in \( \gamma \):

\[
\min_{u(\cdot) \in U^*_{ad}[0,T]} \mathbb{E} \left\{ \frac{1}{2}[X(T) - d]^2 + \gamma[\mathbb{E} X(T) - d] \right\} \\
= \frac{1}{2} P(0) [x_0 - (d - \gamma)]^2 + Q(0) [x_0 - (d - \gamma)] + R(0) - \frac{1}{2} \gamma^2
\]

\[
= \frac{1}{2} \left[ x_0e^{Tr} + \frac{a\theta - a\eta}{r} - (d - \gamma) \right]^2 - \frac{1}{2} \gamma^2
\]

\[
= \frac{1}{2} \left[ x_0e^{Tr} - d + \frac{a\theta - a\eta}{r} \right]^2 + \left[ x_0e^{Tr} - d + \frac{a\theta - a\eta}{r} \right] \gamma.
\]

Therefore, we conclude that under the optimal strategy (42), the optimal cost for the problem (28) is

\[
\min_{u(\cdot) \in U^*_{ad}[0,T]} \mathbb{E} \left\{ [X(T) - d]^2 + 2\gamma[\mathbb{E} X(T) - d] \right\}
\]

\[
= \begin{cases} \\
e^{2A_1T} \left[ x_0e^{Tr} + \frac{a\theta - a\eta}{r} - (d - \gamma) \right]^2 - \gamma^2, & \text{if } x_0 - (d - \gamma) + g_1(t)e^{-rT} < 0, \\
\left[ x_0e^{Tr} - d + \frac{a\theta - a\eta}{r} \right]^2 + 2 \left[ x_0e^{Tr} - d + \frac{a\theta - a\eta}{r} \right] \gamma, & \text{if } x_0 - (d - \gamma) + g_1(t)e^{-rT} \geq 0.
\end{cases}
\]

(53)

Note that the above still relies on the Lagrange multiplier \( \gamma \). Similarly, according to the Lagrange duality theorem, one needs to maximize the value in (53) over \( \gamma \in \mathbb{R} \). A simple calculation shows that (53) achieves its maximum value

\[
\frac{(d - x_0e^{Tr} - \frac{a\theta - a\eta}{r})^2}{e^{-2A_1T} - 1} \quad \text{at} \quad \gamma^* = \frac{x_0e^{Tr} + \frac{a\theta - a\eta}{r} - d}{e^{-2A_1T} - 1}.
\]

The above derivation leads to the following result.

**Theorem 4.3** The efficient strategy of the problem (28) corresponding to the expected terminal wealth \( \mathbb{E} X(T) = d \), as a function of time \( t \) and wealth \( X \), is

\[
u^*(t, X) \equiv (\pi^*(t, X), q^*(t, X))
\]

\[
\begin{cases} \\
= \begin{cases} \\
\left( -\frac{\mu - r}{\sigma^2} [X + g_1(t)e^{-r(T-t)}] - \frac{a\eta}{b^2} [X + g_1(t)e^{-r(T-t)}] \right)', & \text{if } x_0 - (d - \gamma^*) + g_1(t)e^{-r(T-t)} < 0, \\
(0, 0)', & \text{if } x_0 - (d - \gamma^*) + g_1(t)e^{-r(T-t)} \geq 0,
\end{cases}
\end{cases}
\]

(54)

where \( \gamma^* = \frac{x_0e^{Tr} + \frac{a\theta - a\eta}{r} - d}{e^{-2A_1T} - 1} \). Moreover, the efficient frontier is

\[
\text{Var}[X(T)] = \frac{(x_0e^{Tr} + \frac{a\theta - a\eta}{r} - \mathbb{E} X(T))^2}{e^{-2A_1T} - 1}.
\]

(55)
Remark 2 We will compare with some classic results in [6] here. Different from the objective function (maximizing the expected terminal wealth and minimizing the probability of ruin) of the problem in [6], we study the optimal investment and reinsurance problem under the mean-variance criterion. Although we both use the HJB equation method, the value function we get is not necessarily smooth, so we deal with it under the framework of the viscosity solution.

4.4 Numerical Examples

In this subsection, we give some numerical examples to illustrate the theoretical results obtained in this paper.

Firstly, we consider the filtering of $\mu(t)$ in (9). Set $l(t) \equiv 3, z(t) \equiv 2, \sigma(t) \equiv 1$. In detail, from (10), the filtering of the appreciation rate process $m(t)$ increases as the value of $h$ increases. It is shown in Figure 1. In fact, since $\mu(t)$ is increasing when $h(t)$ increases, then our best estimate $m(t)$ should also increase as $h(t)$ increases. Figure 2 shows that the variance $n(t)$ of the conditional distribution $F_{G_t}(x) = \mathbb{P}(\mu(t) \leq x|G_t)$ goes up when $h$ rises. On the other hand, we can also see that the filtering error $E_n(t) = \mathbb{E}(\mu(t) - m(t))^2$ tends to be stable with the increase of $t$.

![Figure 1](image1.png)  ![Figure 2](image2.png)

**Figure 1** Comparison of $m(t)$  **Figure 2** Comparison of $n(t)$

Next, we consider the following examples for the full information case. Let the initial wealth $x_0 = 50$, time duration $T = 100$, safety loading of the insurer $\theta = 0.2$, safety loading of the reinsurer $\eta = 0.3$, the first-order and second-order moment of claim sizes $a = b = 1$, expected return rate of the risky asset $\mu = 0.06$, volatility of the risky asset $\sigma = 1$ and the interest rate $r = 0.04$. 
Figure 3 is a description of the value function region. We can find that the value function $v(t, x)$ is $C^{1,2}[0, T] \times \mathbb{R}$ in the interior of the control region $A_1$ and $A_2$ in Figure 3 and $v(t, x)$ is non-smoothness in the switching curve $A_3$. Figure 4 is the image of the value function (53) and $\gamma$. From Figure 4, we can see that the value function reaches the maximum value when $\gamma = -0.9$, which is the same as the calculated results in formula $\gamma^* = x_0 e^{Tr} + \frac{d}{e^{Tr}} - d - \frac{a}{e^{Tr}} + \frac{a - d}{e^{Tr}}$.

From Figure 5, we notice that efficient frontier is a quadratic curve. Obviously, the higher expected return $d$, the higher the risk. In addition, if $\text{Var}[X(T)] = 0$, we can see that expected return $E[X(T)] = d = 2727.4$. In fact, in this case, the insurance firm invests all of its wealth at hand into the risk-free asset and transfers all forthcoming risks to the reinsurer. Thus, there is no risk for insurance firm here.
5 Concluding Remarks

A new partial information problem of an insurance firm towards optimal reinsurance and investment under the criterion of mean-variance, has been studied in this paper. We assume that we cannot directly observe the Brownian motion and the rate of return in the risky asset price dynamic equation. In fact, only partial information is available to the policymaker. This is more realistic. Based on separation principle and stochastic filtering theory, we can simply replace the rate of return with its filter in the wealth equation and then solve the partial information problem as in the full information case. Efficient strategies and efficient frontier are presented in closed forms via solutions to two extended stochastic Riccati equations. As a comparison, we also obtain the efficient strategies and efficient frontier by the viscosity solution to the HJB equation in the full information case.

It is worth noting that the mean-variance problem is a time-inconsistent problem owing to the term \( \mathbb{E}[X(T)]^2 \) in the cost functional. In this paper, we fix one initial point and then try to find the admissible control \( u^*(\cdot) \) which maximizes the cost functional. We then simply disregard the fact that at a later points in time the control \( u^*(\cdot) \) will not be optimal for the functional. In the economics literature, this is known as pre-commitment.

Possible extension to the mean-variance problem is in another different way. Inspired by Björk, et al.[38], Björk and Murgoci[39], we could take the time inconsistency seriously and formulate the problem in game theoretic terms. We will study this topic in our forthcoming papers.

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References

[1] Browne S, Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research*, 1995, 20(4): 937–958.

[2] Yang H L and Zhang L H, Optimal investment for insurer with jump-diffusion risk process, *Insurance: Mathematics and Economics*, 2005, 37(3): 615–634.

[3] Promislow S D and Young V R, Minimizing the probability of ruin when claims follow Brownian motion with drift, *North American Actuarial Journal*, 2005, 9(3): 110–128.

[4] Gu A and Li Z, Optimal reinsurance and investment strategies for insurers with regime-switching and state-dependent utility function, *Journal of Systems Science and Complexity*, 2016, 29(6): 1658–1682.

[5] Li D, Rong X, and Zhao H, Optimal investment problem for an insurer and a reinsurer, *Journal of Systems Science and Complexity*, 2015, 28(6): 1326–1343.

[6] Bai L H and Guo J Y, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, *Insurance: Mathematics and Economics*, 2008, 42(3): 968–975.
[7] Liang Z B, Yuen K C, and Guo J Y, Optimal proportional reinsurance and investment in a stock market with Ornstein-Uhlenbeck process, *Insurance: Mathematics and Economics*, 2011, 49(2): 207–215.

[8] Xu L, Zhang L M, and Yao D J, Optimal investment and reinsurance for an insurer under Markov-modulated financial market, *Insurance: Mathematics and Economics*, 2017, 74: 7–19.

[9] Bielecki T R, Jin H Q, Pliska S R, et al., Continuous-time mean-variance portfolio selection with bankruptcy prohibition, *Mathematical Finance*, 2005, 15(2): 213–244.

[10] Steinbach M C, Markowitz revisited: Mean-variance models in financial portfolio analysis, *SIAM review*, 2001, 43(1): 31–85.

[11] MacLean L C, Zhao Y G, and Ziemba W T, Mean-variance versus expected utility in dynamic investment analysis, *Computational Management Science*, 2011, 8(1–2): 3–22.

[12] Markowitz H, Portfolio selection, *The Journal of Finance*, 1952, 7(1): 77–91.

[13] Li D and Ng W L, Optimal dynamic portfolio selection: Multiperiod mean-variance formulation, *Mathematical Finance*, 2000, 10(3): 387–406.

[14] Zhou X Y and Li D, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization*, 2000, 42(1): 19–33.

[15] Li X, Zhou X Y, and Lim A E, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal on Control and Optimization*, 2002, 40(5): 1540–1555.

[16] Hu Y and Zhou X Y, Constrained stochastic LQ control with random coefficients, and application to portfolio selection, *SIAM Journal on Control and Optimization*, 2015, 44(2): 444–466.

[17] Bai L H and Zhang H Y, Dynamic mean-variance problem with constrained risk control for the insurers, *Mathematical Methods of Operations Research*, 2008, 68(1): 181–205.

[18] Bi J N, Meng Q B, and Zhang Y J, Dynamic mean-variance and optimal reinsurance problems under the no-bankruptcy constraint for an insurer, *Annals of Operations Research*, 2014, 212(1): 43–59.

[19] Zhang N, Chen P, Jin Z, et al., Markowitz's mean-variance optimization with investment and constrained reinsurance, *Journal of Industrial and Management Optimization*, 2017, 13(1): 375–397.

[20] Di Nunno G and Øksendal B, Optimal portfolio, partial information and Malliavin calculus, *Stochastics: A Probability and Stochastic Processes*, 2009, 81(3–4): 303–322.

[21] Peng X C and Hu Y J, Optimal proportional reinsurance and investment under partial information, *Insurance: Mathematics and Economics*, 2013, 53(2): 416–428.

[22] Wang G C and Wu Z, General maximum principles for partially observed risk-sensitive optimal control problems and applications to finance, *Journal of Optimization Theory and Applications*, 2009, 141(3): 677–700.

[23] Huang J H, Wang G C, and Wu Z, Optimal premium policy of an insurance firm: Full and partial information, *Insurance: Mathematics and Economics*, 2010, 47(2): 208–215.

[24] Al-Hussein A and Gherbal B, Necessary and sufficient optimality conditions for relaxed and strict control of forward-backward doubly SDEs with jumps under full and partial information, *Journal of Systems Science and Complexity*, 2020, 33(6): 1804–1846.

[25] Pham H, Mean-variance hedging for partially observed drift processes, *International Journal of Theoretical and Applied Finance*, 2001, 4(2): 263–284.

[26] Xiong J and Zhou X Y, Mean-variance portfolio selection under partial information, *SIAM Jou-
nal on Control and Optimization, 2007, 46(1): 156–175.

[27] Pang W K, Ni Y H, Li X, et al., Continuous-time mean-variance portfolio selection with partial information, Journal of Mathematical Finance, 2014, 4: 353–365.

[28] Liang Z X and Song M, Time-consistent reinsurance and investment strategies for mean-variance insurer under partial information, Insurance: Mathematics and Economics, 2015, 65: 66–76.

[29] Cao J, Peng X C, and Hu Y J, Optimal time-consistent investment and reinsurance strategy for mean-variance insurers under the inside information, Acta Mathematicae Applicatae Sinica, English Series, 2016, 32(4): 1087–1100.

[30] Li C L, Liu Z M, Wu J B, et al., The stochastic maximum principle for a jump-diffusion mean-field model involving impulse controls and applications in finance, Journal of Systems Science and Complexity, 2020, 33(1): 26–42.

[31] Gennote G, Optimal portfolio choice under incomplete information, The Journal of Finance, 1986, 41(3): 733–746.

[32] Azcue P and Muler N, Optimal reinsurance and dividend distribution policies in the Cramer-Lundberg model, Mathematical Finance, 2005, 15(2): 261–308.

[33] Luo S, Wang M, and Zeng X, Optimal reinsurance: Minimize the expected time to reach a goal, Scandinavian Actuarial Journal, 2016, 8: 741–762.

[34] Liptser R S and Shiryaev A N, Statistics of Random Processes: II. Applications, 2nd Edition, Springer-Verlag, New York, 1977.

[35] Luenberger D G, Optimization by Vector Space Methods, John Wiley and Sons, New York, 1969.

[36] Yong J M and Zhou X Y, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, 1999.

[37] Zhou X Y, Yong J M, and Li X J, Stochastic verification theorems within the framework of viscosity solutions, SIAM Journal on Control and Optimization, 1997, 35(1): 243–253.

[38] Björk T, Khapko M, and Murgoci A, On time-inconsistent stochastic control in continuous time, Finance & Stochastics, 2017, 21: 331–360.

[39] Björk T and Murgoci A, A general theory of Markovian time inconsistent stochastic control problems, Preprint, 2010, Electronic copy available at: http://ssrn.com/abstract=1694759.