Quantum Logic Maps and Triangular Norms on D-posets

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Abstract. In this paper, we propose a type of generalization of triangular norms and quantum logic functions on d-poset algebra. We compare the modified constructions, and properties to the classical properties of the generalized concepts. We show several structures with proof of each type of the proposed generalization. We show several relationships that connect the triangular norms to the quantum logic maps, and also show their relationships to the classical probability space. We provide some explanatory examples that show each structure on d-poset with extra properties that depends on a definition of state.

Keywords: Probability Space, Quantum Logic Functions, T-norms, D-poset, Orthoalgebra.

1. Introduction

There are some modern trends that aim to connect various types of measurements to the algebraic concepts. One of these trends related to the concepts of triangular norms and their concepts with respect to some algebras and their properties. On the other hand, the studies of quantum logic maps show that the solutions via such maps are much effective and smoother than the classical solutions that only depend on Boolean algebra concepts. The benefit of such solutions is to find some techniques that can be used to solve the problems which are related to the quantum mechanics. One of the these solutions is related to the problem of compatibility, non-compatibility, and the properties of the correlated components under some algebraic systems. These types of studies leads to find and construct many new structures and properties, for example, constructing quantum logic maps. In this work, we propose a generalization of some quantum logic maps and triangular norms on d-poset. We also discuss some properties that associate the modified triangular norms with each other. An interpretation to the problems of compatible and non-compatible elements is presented by means of examples that show how the quantum logic maps is working. In this note we point out an explicit relationship of difference posets with structures already established in the foundation of quantum mechanics with the orthoalgebras see [8], [6], [10]. By means of state a very important notion was mention in [5] that refer to the set of automaton state. It has discussed the automaton state as portion with respect to identifiability in input/output experiments see [14, 15, 5]. (In 2003), Na'na'siova' has presented a notion of conditional state. Also, notions of quantum logic functions were defined and shown their structures. It has shown that probability space is not enough to describe causality, see [13].
Another essential notion that we need to mention is the notion of triangular norms. This notion was firstly occurred in the fuzzy set theory [9], [2], etc. Afterwords, many other researchers presented various $T$-operators in fuzzy set theory, see [3, 16, 17]. In particular, zadeh has presented $T$-operator like, min and max, as special cases of fuzzy set operators, see [18].

This work is organized by the following way:

In the next section, we review the basic notions, and concepts that are related to the dposet, orthomodular lattice, othoalgebras, state quantum logic maps, and $T$-norms. While, the third section deals with the main ideas of this study. Finally, conclusions and remarks are presented in section four which are related to the results that we have obtained from section three.

2. Foundations

In this part, we review some basic concepts related to a d-poset, probability space, quantum logic functions, $T$-norms, and present examples that explain their characteristics and how they are implemented.

**Definition 2.1** [4] A D-poset, or a difference poset, is a partially ordered set $L$ with a partial ordering $\leq$, greatest element $1$, and partial binary operation $\ominus: L \times L \rightarrow L$, called a difference, such that, for $u, v, w \in L$, $v \ominus u$ is defined if and only if $u \leq v$. Then the following axioms hold for all $u, v, w \in L$:

1. $v \ominus u \leq v$.
2. $u \leq v \leq w \Rightarrow v \ominus u \leq w \ominus u$.

Another definition that need to be recalled is the definition of orthomodular lattice (OML). The importance of OML follows from the fact that all the quantum logic are illustrated on it.

**Definition 2.2** [4] Let $A$ be a lattice with the greatest element $I$, the smallest element $O$, respectively and partial ordering $\leq$, endowed with a unary operation $\perp: A \rightarrow A$, such that the following hold:

i. $(u)^{\perp} = u$;

ii. $u \leq v$ implies $v^{\perp} \leq u^{\perp}$;

iii. $u \lor u^{\perp} = I$;

iv. $u \leq v$ implies $u \lor (u^{\perp} \land v)$.

Then the system $L = (A, O, I, V, \perp, \lor)$ is said to be an orthomodular lattice (OML). The importance of OML follows from the fact that all the quantum logic are illustrated on it.

**Definition 2.3** [4, 5] An othoalgebra is a set $L$ with two particular elements $0, 1$, and with a partial binary operation $\oplus: L \times L \rightarrow L$ such that for all $u, v, w \in L$ we have:

i. If $u \oplus v \in L$, then $v \oplus u \in L$ and $u \oplus v = v \oplus u$ (commutativity).

ii. If $v \oplus w \in L$ and $u \oplus (v \oplus w) \in L$, then $u \oplus v \in L$ and $(u \oplus v) \oplus w \in L$ and, $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ (associativity).

iii. For any $u \in L$ there is a unique $v \in L$ such that $u \oplus v$ is defined, and $u \oplus v = 1$ (orthocomplementation).

iv. If $u \oplus u$ is defined, then $u = 0$ (consistency).

According to the property number (iv), it shows that the following statements are true.

a. Let $u, v \in L$, then $u$ is orthogonal to $v$, written $u \perp v$, if and only if, $u \oplus v$ is define in $L$;

b. $u$ is less or equal to $v$ and written $u \leq v$ if and only if there exists an element $w \in L$ such that $u \perp w$ and $u \oplus w = v$ (in this case we also write $v \geq u$);

c. $v$ is the orthocomplement of $u$, if and only if, $v$ is a (unique) element in $L$ such that $v \perp u$ and $u \oplus v = I$, and it is written as $u^{*}$.
If \( u \perp v \), for the element \( w \) in (b) with \( u \ominus w = v \) we write \( w = v \ominus u \), and \( w \) is called the deference of \( u \) in \( v \). It is evident that \( v \ominus u = (u \ominus v)^\perp \).

**Definition 2.4** [10] Let \( \delta \) be a d-poset. A map \( m : L \rightarrow [0,1] \) is called state if the following conditions hold:
1. \( m(O) = 0 \);
2. If \( u \perp v \) then \( m(u \ominus v) = m(u) + m(v) \). Where \( u \neq v \).

It is well-known that \( m(l) = 1 \).

Furthermore, let us turn to review the definitions of the quantum logic maps on OML.

**Definition 2.5** [1] let \( \mathcal{L} \) be an OML. The map \( p : A^2 \rightarrow [0,1] \) is called an s-map if the following conditions are hold:
1. \( p(l,l) = 1 \);
2. If \( u \perp v \) then \( p(u,v) = 0 \);
3. If \( u \perp v \) then for any \( w \in A \),
   \[
   p(u \vee v, w) = p(u, w) + p(v, w);
   \]
   \[
   p(w, u \vee v) = p(w, u) + p(w, v).
   \]

We notice that s-map has the following property, see [1].
\[
 p(u, l) = p(u, v \uparrow) = p(u, u) + p(u, v \uparrow) = p(l, u).
\]

Furthermore, let us recall a definition of a join map which represents a dual function of s-map on OML.

**Definition 2.6** [1,11] let \( \mathcal{L} \) be an OML. A map \( d : A^2 \rightarrow [0,1] \) is called a join map (briefly, j-map) if the following conditions hold:
1. \( d(O,O) = 0 \) and \( d(l,l) = 1 \);
2. If \( u, v \in A \) and \( u \perp v \) then \( d(u,v) = d(u,u) + d(v,v) \);
3. If \( u, v \in A \) and \( u \perp v \) then for each \( w \in A \),
   \[
   q(u \vee v, w) = q(u, w) + q(v, w) - q(w, w);
   \]
   \[
   q(w, u \vee v) = q(w, u) + q(w, v) - q(w, w).
   \]

A third important map that we need to review is known by a difference map, briefly d-map. That is

**Definition 2.7** [1,11] let \( \mathcal{L} \) be an OML. The map \( d : A^2 \rightarrow [0,1] \) is called a difference map (d-map) if the following conditions are hold:
1. \( d(O,O) = 0 \) and \( d(l,l) = 1 \); for \( \forall u \in A \), \( d(u,u) = 0 \);
2. \( d(u,v) = d(u,O) + d(O,v) \) for \( u \perp v \);
3. If \( u \perp v \), \( w \in A \),
   \[
   d(u \vee v, w) = d(u, w) + d(v, w) - d(O, w);
   \]
   \[
   d(w, u \vee v) = d(w, u) + d(w, v) - d(O, w).
   \]

Finally, we recall the classical definitions of triangular norms on the unit interval \([0,1]\).

**Definition 2.8** [7] A function \( T : [0,1]^2 \rightarrow [0,1] \) is called T-norm, if and only if, for all \( u,v \in [0,1] \):
1. \( T(u,v) = T(v,u) \) (commutativity),
2. \( T(u,v) \leq T(u,w), \) if \( v \leq w \) (monotonicity),
3. \( T(u,T(v,w)) = T(T(u,v),w) \) (associativity),
4. \( T(u,1) = u \).

**Definition 2.9** [7] A function \( T' : [0,1]^2 \rightarrow [0,1] \) is called a T-conorm, if \( T' \) holds the properties of T-norm from (1)-(3), and the extra condition, that is \( T'(u,0) = T'(0,u) = u \).

Notice that \( T' \) is also known as dual function to \( T \). Indeed, there are several families that hold the conditions of \( T \), and also there are others that hold \( T'^\perp \).

3. Generalization of T-norms and Quantum Logic Maps on D-poset
In advance, a generalization of each chosen structure of the classical t-norms, or even of the quantum logic maps needs the following necessary notations and properties that we can summarize them in the remark below.
Remark 3.1 On d-poset, we need to recall the following notions in terms of homomorphism property because they are essential to illustrate proper structures of T-norms, and other maps on d-poset, see [4].

1. \((\ominus, 0, 1, \leq, \ast)\) homomorphism with to \((\oplus, 0, 1, \leq, \ast)\);
2. A d-poset is homomorphism to the orthalgebra system. This means that for all \(u, v \in D\), there exist \(u, v \in L\) then \(u \lor v = u \oplus v\);
3. For all \(u, v \in D\), when \(u \leq v\) implies that \(v \ominus u \in D\), and this can be written as \(u \perp v\).

Remark 3.2 There is no loss in the expression, if we write the symbol \(\delta\) as the whole system of d-poset \((\ominus, 0, 1, \leq, \ast)\).

Upon the notation of Remark 3.1, we can now illustrate our proposed structures (T-norms, and Quantum logic maps) on the d-poset. We begin by constructing a definition of s-map on d-poset.

Definition 3.1 Let \(\delta\) be a d-poset. A map \(p_{\delta}: D^2 \rightarrow [0, 1]\) is called an s-map on d-poset, if it holds the following conditions.

1. \(p_{\delta}(1, 1) = 1\);
2. For all \(u, v \in D\), such that \(v \ominus u \in D\), and \(u \perp v\), then \(p_{\delta}(u, v) = 0\);
3. For all \(u, v \in D\), such that \(u \perp v\), then for any \(w \in D\),
   \[p_{\delta}(u \oplus v, w) = p_{\delta}(u, w) + p_{\delta}(v, w);\]
   \[p_{\delta}(w, u \oplus v) = p_{\delta}(w, u) + p_{\delta}(w, v).\]

Now, let's try to clarify the properties or conditions of the definition of s-map and explaining how they work in case that the elements are compatible or non-compatible. We have to mention that these properties for the elements in Boolean algebra are not exist because the elements are always compatible.

Example 3.1 Let \(p\) be an s-map of compatible elements on the d-poset \(\delta\). Then according to [12] we have the following table of s-map.

| \(p_{\delta}(\cdot, \cdot)\) | \(O\) | \(u\) | \(u'\) | \(v\) | \(v'\) | \(l\) |
|--------------------------|-----|------|------|-----|-----|-----|
| \(O\)                     | 0   | 0    | 0    | 0   | 0   | 0   |
| \(u\)                     | 0   | 0.3  | 0    | 0.2 | 0.1 | 0.3 |
| \(u'\)                    | 0   | 0    | 0.6  | 0.4 | 0.3 | 0.7 |
| \(v\)                     | 0   | 0.2  | 0.4  | 0.6 | 0   | 0.6 |
| \(v'\)                    | 0   | 0.1  | 0.3  | 0   | 0.4 | 0.4 |
| \(l\)                     | 0   | 0.3  | 0.7  | 0.6 | 0.4 | 1   |

We can, see that the values of \(p_{\delta}\) in Table 1 are compatible, for example \(p_{\delta}(u, v) = p_{\delta}(v, u)\), and etc. While the following example can be illustrated with an s-map that has non-compatible elements.

Example 3.2 Let \(p\) be an s-map of Non-compatible elements on the d-poset \(\delta\). Then according to [12] we have the following table.

| \(p_{\delta}(\cdot, \cdot)\) | \(O\) | \(u\) | \(u'\) | \(v\) | \(v'\) | \(l\) |
|--------------------------|-----|------|------|-----|-----|-----|
| \(O\)                     | 0   | 0    | 0    | 0   | 0   | 0   |
| \(u\)                     | 0   | 0.3  | 0    | 0.1 | 0.2 | 0.3 |
| \(u'\)                    | 0   | 0    | 0.7  | 0.5 | 0.2 | 0.7 |
| \(v\)                     | 0   | 0.2  | 0.4  | 0.6 | 0   | 0.6 |
In particular, we can see that in Table 2, \( p_{dp}(u, v) = 0.1 \), while \( p_{dp}(v, u) = 0.2 \). It is clear that this situation leads to non-compatible elements of s-map. Therefore, \( p_{dp}(u, v) \neq p_{dp}(v, u) \), and this is true for all comparable pairs in Table 2.

Next, we illustrate a dual function to s-map on d-poset \( \delta \), which its basic form was presented in the previous part.

**Definition 3.2** Let \( \delta \) be a d-poset. A map \( q_{dp}: D^2 \rightarrow [0, 1] \) is called a j-map on d-poset, if it holds the following conditions.

1. \( q_{dp}(O, O) = 0, q_{dp}(1, 1) = 1; \)
2. For all \( u, v \in D \), such that \( v \ominus u \in D \), and \( u \perp v \), then \( q_{dp}(u, v) = q_{dp}(u, u) + q_{dp}(v, v); \)
3. For all \( u, v \in D \), such that \( v \ominus u \in D \), and \( u \perp v \), then for any \( w \in D, q_{dp}(u, v) = q_{dp}(u, u) + q_{dp}(w, w); \)
4. \( q_{dp}(w, u \oplus v) = q_{dp}(w, u) + q_{dp}(v, v) - q_{dp}(w, w). \)

Again, we explain the properties or conditions of j-map and how it acts when the elements they are compatible or non-compatible by the following examples.

**Example 3.3** Let \( q \) be an j-map of compatible elements on the d-poset \( \delta \). Then according to [12], we have the following table.

| \( q_{dp}(\ldots) \) | \( O \) | \( u \) | \( u' \) | \( v \) | \( v' \) | \( I \) |
|---------------------|-----|-----|-----|-----|-----|-----|
| \( O \)             | 0   | 0.2 | 0.8 | 0.1 | 0.9 | 1   |
| \( u \)             | 0.2 | 1   | 0.8 | 1   | 1   | 1   |
| \( v \)             | 0.9 | 0.1 | 0.8 | 0.9 | 1   | 1   |
| \( v' \)            | 1   | 1   | 1   | 1   | 1   | 1   |

While a non-compatible case can be shown in the following example.

**Example 3.4** Let \( q \) be an j-map of non-compatible elements on the d-poset \( \delta \). Then according to [12], we have the following table.

| \( q_{dp}(\ldots) \) | \( O \) | \( u \) | \( u' \) | \( v \) | \( v' \) | \( I \) |
|---------------------|-----|-----|-----|-----|-----|-----|
| \( O \)             | 0   | 0.7 | 0.3 | 0.6 | 0.34| 1   |
| \( u \)             | 0.7 | 1   | 0.3 | 0.7 | 0.8 | 1   |
| \( v \)             | 0.3 | 0.7 | 0.7 | 0.66| 1   | 1   |
| \( v' \)            | 0.34| 0.82| 0.52| 1   | 0.34| 1   |
| \( I \)             | 1   | 1   | 1   | 1   | 1   | 1   |

In a similar way to s-map and j-map on d-poset \( \delta \), we can illustrate a definition of d-map. This map corresponds to the notion of symmetric difference.

**Definition 3.3** Let \( \delta \) be a d-poset. A map \( d_{dp}: D^2 \rightarrow [0, 1] \) is called a difference map on d-poset, if it holds the following conditions.

1. \( d_{dp}(O, O) = d_{dp}(1, 1) = 1 \); 
2. For all \( u, v \in D \), such that \( v \ominus u \in D \), and \( u \perp v \), then \( d_{dp}(u, v) = d_{dp}(u, u) + d_{dp}(v, v); \)
3. For all \( u, v \in D \), such that \( v \ominus u \in D \), and \( u \perp v \), then for any \( w \in D, d_{dp}(u, v) = d_{dp}(u, u) + d_{dp}(w, w); \)
4. \( d_{dp}(w, u \oplus v) = d_{dp}(w, u) + d_{dp}(v, v) - d_{dp}(w, w). \)
\[ d_{dp}(u \oplus v, w) = d_{dp}(u, w) + d_{dp}(v, w) - d_{dp}(0, w); \]
\[ d_{dp}(w, u \oplus v) = d_{dp}(w, u) + d_{dp}(w, v) - d_{dp}(w, 0). \]

Once again, we can also present examples of d-map of elements whether they are compatible or non-compatible.

**Example 3.5** Let \( d \) be an d-map of compatible elements on the d-poset \( \delta \). Then according to [12], we have the following table.

**Table 5.** Compatible elements of d-map.

| \( d_{dp}(., .) \) | \( O \) | \( u \) | \( u' \) | \( v \) | \( v' \) | \( l \) |
|-----------------|-----|-----|-----|-----|-----|-----|
| \( O \)        | 0   | 0.2 | 0.8 | 0.1 | 0.9 | 1   |
| \( u \)        | 0.2 | 0   | 1   | 0.3 | 0.7 | 0.8 |
| \( u' \)       | 0.8 | 1   | 0   | 0.7 | 0.3 | 0.2 |
| \( v \)        | 0.1 | 0.3 | 0.7 | 0   | 1   | 0.9 |
| \( v' \)       | 0.9 | 0.7 | 0.3 | 1   | 0   | 0.1 |
| \( l \)        | 1   | 0.8 | 0.2 | 0.9 | 0.1 | 0   |

While the following example explains the case of non-compatible elements with dp-map.

**Example 3.6** Let \( d \) be an d-map of non-compatible elements on the d-poset \( \delta \). Then according to [12], we have the following table.

**Table 6.** Non-compatible elements of d-map.

| \( d_{dp}(., .) \) | \( O \) | \( u \) | \( u' \) | \( v \) | \( v' \) | \( l \) |
|-----------------|-----|-----|-----|-----|-----|-----|
| \( O \)        | 0   | 0.2 | 0.8 | 0.1 | 0.9 | 1   |
| \( u \)        | 0.2 | 0   | 1   | 0.4 | 0.6 | 0.8 |
| \( u' \)       | 0.8 | 1   | 0   | 0.6 | 0.4 | 0.2 |
| \( v \)        | 0.1 | 0.3 | 0.7 | 0   | 1   | 0.9 |
| \( v' \)       | 0.9 | 0.7 | 0.3 | 1   | 0   | 0.1 |
| \( l \)        | 1   | 0.8 | 0.2 | 0.9 | 0.1 | 0   |

Furthermore, we illustrate definitions of t-norms on d-poset. We begin by a definition of T-norm on \( \delta \).

**Definition 3.4** let \( \delta \) be a d-poset. A function \( T_{dp} : D^2 \to [0, 1] \) is called an dp-T-norm on \( \delta \), if it holds the following conditions.

1. \( T_{dp}(u, 0) = T_{dp}(0, u) = 0 \; \forall \; u \in D \) (grounded)
2. \( T_{dp}(u, v) = T_{dp}(v, u) \; \forall \; u, v \in D \) (commutative)
3. \( T_{dp}(u, v) \leq T_{dp}(u, w), \text{ for } u, v, w \in D, \text{ such that } w \ominus v \in D \) (monotonicity)
4. \( T_{dp}(l, .), T_{dp}(., l) \) are states.

In the following example, we show how the properties of **Definition 3.4** are satisfied. The main property that we focus on is the property of state and how it works.

**Example 3.7** Let \( \delta \) be a d-poset. Let the dp-T-norm is defined by the following relation

\[ T_{dp}(u, v) = \min(s(u), s(v)) \] (3.1)

\( \forall \; u, v \in D \). Then, we can see that the dp-T-norm in (3.1) fulfills the properties of **Definition 3.4**.

In this example, it is sufficient to prove a state property because the other properties are already satisfied and trivial. So to prove \( T_{dp}(., l) \) is state, we have.

1. \( T_{dp}(., l)(l) = T_{dp}(l, l) = \min(s(l), s(l)) = \min(1, 1) = 1. \)
2. Let \( u, v \in D \), such that \( u \ominus v \in D \), such that \( u \perp v \). We have to prove that

\[ T_{dp}(., l)(u \oplus v) = T_{dp}(., l)(u) + T_{dp}(., l)(v) \]

Then
\[ T_{dp}(., I)(u \oplus v) = T_{dp}((u \oplus v), I) = T_{dp}(u, I) + T_{dp}(v, I) = \min(s(u), s(I)) + \min(s(v), s(I)) = s(u) + s(v) \]

Conversely,
\[
T_{dp}(., I)(u) = T_{dp}(u, I) = \min(s(u), s(I)), \text{ and } T_{dp}(., I)(v) = T_{dp}(v, I) = \min(s(v), s(I)).
\]

Thus
\[
T_{dp}(., I)(u) + T_{dp}(., I)(v) = T_{dp}(u, I) + T_{dp}(v, I) = \min(s(u), s(I)) + \min(s(v), s(I)) = s(u) + s(v)
\]

Hence,
\[
T_{dp}(., I)(u \oplus v) = T_{dp}(., I)(u) + T_{dp}(., I)(v)
\]

Similarly, we obtain that \( T_{dp}(I, .) \) is state. Therefore, \( T_{dp} \) is a dp-T-norm on \( \delta \).

**Definition 3.5** Let \( \delta \) be a d-poset. A function \( T_{dp}^*: D^2 \rightarrow [0, 1] \) is called an dp-T-conorm on \( D \), if it holds following conditions.
1. \( T_{dp}^*(u, I) = T_{dp}^*(I, u) = 1 \)
2. \( T_{dp}^* \) is commutative
3. \( T_{dp}^* \) is monotonic
4. \( T_{dp}^*(., 0), T_{dp}^*(0, .) \) are states.

We notice that, property number four is not exist in the classical definition of T-conorm, in order to have full properties of classical T-norms (T-norm, and T-conorm) we propose the following notion that hold the associative property which is not exist in **Definition 3.4**, and **Definition 3.5**.

**Definition 3.6** Let \( \delta \) be a poset. A map \( \odot : D \rightarrow D \) is called state and has the following properties.
1. \( u \odot v = v \odot u \) (commutative)
2. \( u \odot 1 = u \) (unary)
3. \( u \odot v \leq u \odot w \) when \( w \ominus v \in D \) (monotonic)
4. \( (u \odot v) \odot w = u \odot (v \odot w) \) (associative)

**Example 3.8** Let \( \delta \) be a d-poset, let \( \odot \) be the multiplication binary operation that satisfies the following relation
\[
v \odot u = v, u \] (3.2)

We can see \( \odot \) that satisfies the following properties
- \( v \odot u = v, u \) and \( u \odot v = u \odot v \) (commutative).
- \( u \odot 1 = u, 1 = u \)
- Let \( w \odot v \in D, u \odot v \leq u \odot w \)
- We know that \( u \odot v = u, v \) but since \( v \leq w \) and \( v \leq u \), \( v \leq u, w \)
- \( u \odot (v \odot w) = (u \odot v) \odot w \)
- We know \( u \odot (v \odot w) = u, (v, w) \) but the multiplication is associative Thus
- \( u \odot (v \odot w) = (u \odot v) \odot w \)

In association with definitions of dp-T-norm and dp-T-conorm, we can reformulate their definitions with respect to \( \odot \). This enable us to add the associative property to the definitions of dp-T-norms, and have a full properties of classical T-norms.

**Definition 3.7** Let \( \delta \) be a d-poset, and \( T_{dp}^*, T_{dp}^* \) be a dp-T-norm, dp-T-conorm, respectively.
Then \( T_{dp}^*, T_{dp}^* \) are called complete dp-T-norm, and complete dp-T-conorm, respectively, if they hold all the properties of **Definition 3.4**, **Definition 3.5**, and they hold the relation,
\[
T_{dp}(u \odot v, w) = T_{dp}(u, v \odot w)
\]

where \( T_{dp} \) has the associative property.

One of the interesting properties that we can associate it to \( T_{dp} \), and \( T_{dp}^* \) is well-known by the convex property. This can be shown by the following way
**Proposition 3.1** let $\mathfrak{d}$ be a $d$-poset, let $T'_{dp}$, and $T''_{dp}$ be complete dp-T-norms, respectively. Then for $k \in [0, 1]$

$$T_{dp}(u,v) = kT'_{dp}(u,v) + (1-k)T''_{dp}(u,v)$$

is complete dp-T-norm.

**Proof:**

1. $\forall u \in D, T_{dp}(u,0) = kT'_{dp}(u,0) + (1-k)T''_{dp}(u,0)$

$$= k(0) + (1-k)(0) = 0$$

$\therefore T_{dp}(u,0) = 0$

2. $T_{dp}(1,0), T_{dp}(0,1)$ are states

   a. $T_{dp}(1,1) = 1 \Rightarrow T_{dp}(1,1) = kT'_{dp}(1,1) + (1-k)T''_{dp}(1,1) = k + 1 - k = 1$

   b. $T_{dp}(w,1) = kT'_{dp}(w,1) + (1-k)T''_{dp}(w,1)$

$$= kT'_{dp}(w,1) + (1-k)T''_{dp}(w,1)$$

$\therefore T_{dp}(w,1) = T_{dp}(1,1) + T_{dp}(0,0)$

3. $\forall u,v \in D, T_{dp}(u,v) = kT'_{dp}(u,v) + (1-k)T''_{dp}(u,v)$

$$= kT'_{dp}(u,v) + (1-k)T''_{dp}(u,v)$$

$$= T_{dp}(u,v)$$

4. Let $u,w \in D$, such that $u \leq v$, implies that $T_{dp}(u,w) \leq T_{dp}(v,w)$

   We know that

$$T_{dp}(u,w) = kT'_{dp}(u,w) + (1-k)T''_{dp}(u,w) \quad (3.3)$$

$$T_{dp}(v,w) = kT'_{dp}(v,w) + (1-k)T''_{dp}(v,w) \quad (3.4)$$

But $T'_{dp}(u,w) \leq T'_{dp}(v,w)$, and $T''_{dp}(u,w) \leq T''_{dp}(v,w)$

Hence, $T_{dp}(u,w) \leq T_{dp}(v,w)$.

Therefore, $T_{dp}$ has convex property.

5. Let $u,v,w \in D, T_{dp}(u,v \otimes w) = kT''_{dp}(u,v \otimes w) + (1-k)T''_{dp}(u,v \otimes w)$

But

$$T''_{dp}(u,v \otimes w) = T''_{dp}(u \otimes v, w)$$

And

$$T''_{dp}(u \otimes v, w) = T''_{dp}(u \otimes v, w)$$

Thus

$T_{dp}(u \otimes v, w) = kT''_{dp}(u \otimes v, w) + (1-k)T''_{dp}(u \otimes v, w) = T_{dp}(u \otimes v, w)$

Therefore, $T_{dp}$ is complete dp-T-norm with convex property.

In the same manner, we can present a convex property with respect of dp-T-conorm.

**Proposition 3.2** Let $\mathfrak{d}$ be a $d$-poset and $T'_{dp}, T''_{dp}$ be two complete dp-T-conorms. For any constant $k \in [0, 1]$, and $T''_{dp}(u,v) = kT'_{dp}(u,v) + (1-k)T''_{dp}(u,v)$, is complete dp-T-conorm.

**Proof:**

1. $\forall u \in D, T''_{dp}(u,0) = kT''_{dp}(u,0) + (1-k)T''_{dp}(u,0)$

$$= k(u) + (1-k)(u) = uk + u - uk = u$$

Therefore $T''_{dp}(u,0) = u$

2. $T''_{dp}(0,0), T''_{dp}(0,0)$ are states

   a. $T''_{dp}(0,0) = kT''_{dp}(0,0) + (1-k)T''_{dp}(0,0)$
\( = k(0) + (1 - k)(0) = 0 \)

b. \( T^*_{d_p}(u \oplus v, 0) = kT^*_{d_p}(u \oplus v, 0) \)
\( = k[T^*_{d_p}(u, 0) + T^*_{d_p}(v, 0)] + (1 - k)[T^*_{d_p}(u, 0) + T^*_{d_p}(v, 0)] \)
\( = kT^*_{d_p}(u, 0) + kT^*_{d_p}(v, 0) + (1 - k)T^*_{d_p}(u, 0) + (1 - k)T^*_{d_p}(v, 0) \)
\( = kT^*_{d_p}(u, 0) + (1 - k)T^*_{d_p}(v, 0) + kT^*_{d_p}(u, 0) + (1 - k)T^*_{d_p}(v, 0) \)
\( = T^*_{d_p}(u, 0) + T^*_{d_p}(v, 0) \)
\( \therefore T^*_{d_p}(u \oplus v, 0) = T^*_{d_p}(u, 0) + T^*_{d_p}(v, 0) \)

Similarly
\( \forall u, v \in D, T^*_{d_p}(u, v) = kT^*_{d_p}(u, v) + (1 - k)T^*_{d_p}(v, u) \)

3. Let \( u, v \in D \), such that \( u \leq v \), implies that \( T^*_{d_p}(u, w) \leq T^*_{d_p}(v, w) \).
We know that
\( T^*_{d_p}(u, w) = kT^*_{d_p}(u, w) + (1 - k)T^*_{d_p}(v, w) \)
\( T^*_{d_p}(v, w) = kT^*_{d_p}(v, w) + (1 - k)T^*_{d_p}(v, w) \)

But \( T^*_{d_p}(u, w) \leq T^*_{d_p}(v, w) \), and \( T^*_{d_p}(u, w) \leq T^*_{d_p}(v, w) \)

Hence, \( T^*_{d_p}(u, w) \leq T^*_{d_p}(v, w) \)

Therefore, \( T^*_{d_p} \) has convex property.

4. Let \( u, v, w \in D, T^*_{d_p}(u \oplus v \ominus w) = kT^*_{d_p}(u \oplus v \ominus w) + (1 - k)T^*_{d_p}(u \oplus v \ominus w) \)
But \( T^*_{d_p}(u \oplus v \ominus w) = T^*_{d_p}(u \ominus v, w) \), and \( T^*_{d_p}(u \ominus v, w) = T^*_{d_p}(u \ominus v, w) \)

Thus \( T^*_{d_p}(u \ominus v, w) = kT^*_{d_p}(u \ominus v, w) + (1 - k)T^*_{d_p}(u \ominus v, w) = T^*_{d_p}(u \ominus v, w) \)

Therefore, \( T^*_{d_p} \) is complete dp-T-conorm with convex property.

In the following proposition, we clarify the relationship between dp-T-norm and dp-T-conorm by taking elements of which dp-T-conorm is a function of orthogonality to achieve the conditions of dp-T-norm.

**Proposition 3.3** let \( \mathcal{D} \) be d-poset. If \( T^*_{d_p} \) is complete dp-T-conorm, then for all \( u, v \in D, T^*_{d_p}(u, v) = 1 - T^*_{d_p}(u', v') \) is complete dp-T-norm.

**Proof:**

1. We have to prove that \( T^*_{d_p}(u, 0) = 0 \)
   Thus \( T^*_{d_p}(u, 0) = 1 - T^*_{d_p}(u', 0') \)
   \( = 1 - T^*_{d_p}(u', 1) \)
   \( = 1 - 0 = 0 \)

2. \( T^*_{d_p}(u, v) = 1 - T^*_{d_p}(u', v') \)
   \( = 1 - T^*_{d_p}(v', u') \)

3. To prove that \( T^*_{d_p}(u \oplus v) = T^*_{d_p}(u \oplus v) + T^*_{d_p}(u \oplus v) \)?? We have
   \( T^*_{d_p}(u \oplus v) = T^*_{d_p}(u \oplus v) \), \( I = 1 - T^*_{d_p}(u \oplus v), I' = 1 - T^*_{d_p}(u \oplus v), 0 \)
   But \( T^*_{d_p}(u \oplus v, 0) = 1 - T^*_{d_p}(u \oplus v, 0) \)
   Therefore, \( T^*_{d_p}(u \oplus v, 0) = 1 - T^*_{d_p}(u \oplus v, 0) = T^*_{d_p}(u \oplus v, 0) \)
   \( \therefore T^*_{d_p}(u \oplus v, 0) = T^*_{d_p}(u, 0) + T^*_{d_p}(v, 0) \)

\( \Leftrightarrow \) we know that

\( T^*_{d_p}(u, v) = T^*_{d_p}(u, v) = 1 - T^*_{d_p}(u', 0') \)
But \( T^*_{d_p}(u', 0') = 1 - T^*_{d_p}(u', 0') \)
Thus, $T_{dp}(u, I) = 1 - \left(1 - T^*_{dp}(u, 0)\right) = T^*_{dp}(u, 0)$

Similarly, we can show that $T_{dp}(v, I) = T^*_{dp}(v, 0)$

Thus, $T_{dp}(u, I) + T_{dp}(v, I) = T^*_{dp}(u, 0) + T^*_{dp}(v, 0)$

Hence, $T_{dp}\left((u \oplus v), I\right) = T_{dp}(u, I) + T_{dp}(v, I)$

Therefore, $T_{dp}(., I)$ is state.

4. Let $u, v, w \in D$, such that $u \leq v$, we have to prove that $T_{dp}(u, w) \leq T_{dp}(v, w)$.

We know that

$$T_{dp}(u, w) = 1 - T^*_{dp}(u', w') \quad (3.7)$$

$$T_{dp}(v, w) = 1 - T^*_{dp}(v', w') \quad (3.8)$$

It is given $u \leq v$ implies that $v' \leq u'$

Hence, $T^*_{dp}(v', w') \leq T^*_{dp}(u', w')$, implies that $1 - T^*_{dp}(v', w') \leq 1 - T^*_{dp}(u', w')$

Therefore, $T_{dp}(u, w) \leq T_{dp}(v, w)$

5. To prove the associative property, let $u, v, w \in D$, $T_{dp}(u, v \odot w) = 1 - T^*_{dp}(u', v' \odot w')$

But $T^*_{dp}$ is complete dp-T-conorm

Thus, $T^*_{dp}(u', v' \odot w') = T^*_{dp}(u' \odot v', w')$

Hence, $1 - T^*_{dp}(u' \odot v', w') = T_{dp}(u \odot v, w)$

Therefore, $T_{dp}(u, v \odot w) = T_{dp}(u \odot v, w)$

Therefore, $T_{dp}$ is complete dp-T-norm with convex property. $\blacksquare$

**Corollary 3.1**

Let $\mathfrak{d}$ be d-poset, if $T_{dp}$ is complete dp-T-norm then for all $u, v \in D, T^*_{dp}(u, v) = 1 - T_{dp}(u', v')$ is complete dp-T-conorm.

The proof is clear and similar to the proof of Proposition 3.3.

4 Conclusion

A generalization of quantum logic maps, and T-norms on d-poset has been performed via a notion of state. The notions of compatible and non-compatible elements with respect to the quantum logic maps have been presented and shown in several examples. Relationships between dp-T-norm and dp-T-conorm and prove that each one of them can be written with respect to the complement property. A convex property was also shown in terms of dp-T-norm, and dp-T-conorm. A modification of associativity property was proposed in order to have the full properties of classical T-norms. This process leads us to have a complete dp-T-norms.

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