SYMMETRY ANALYSIS OF A LANE-EMDEN-KLEIN-GORDON-FOCK SYSTEM WITH CENTRAL SYMMETRY

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ABSTRACT. In this paper we introduce a sort of Lane-Emden system derived from the Klein-Gordon-Fock equation with central symmetry. Point symmetries are obtained and, since the system can be derived as the Euler-Lagrange equation of a certain functional, a Noether symmetry classification is also considered and conservation laws are derived from the point symmetries.

1. Introduction. The Klein-Gordon-Fock equation with central symmetry, namely,

\[ u_{tt} - u_{rr} - \frac{2}{r} u_r + \frac{b}{r^2} u = 0, \quad (1) \]

has been studied in various fields of mathematical physics for different values of the parameter \( b \). For example Eq. (1) occurs very often in phenomena in which central symmetry plays a key role, such as in certain quantum physical problems.

Very recently, Kochetov [11] studied the Klein-Gordon-Fock equation with central symmetry using the classical Lie symmetry method. Later on, in [1] conservation laws for the same equation were established.

Inspired by the works of [11, 1] we study the following natural two-component extension of equation (1),

\[
\begin{align*}
  u_{tt} - u_{rr} - \frac{2}{r} u_r + \frac{u^p}{r^2} &= 0, \\
  v_{tt} - v_{rr} - \frac{2}{r} v_r + \frac{u^q}{r^2} &= 0,
\end{align*}
\]

where \( p, q \) are non-zero constants.

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System (2) is quite similar to the so-called Lane-Emden system. Actually, the Lane-Emden system is an elliptic one in $\mathbb{R}^n$, $n > 2$, see, for instance, [3] and references therein, with nonlinearities appearing in the form of powers such as in (2), but usually with $pq > 1$, $p \neq q$ and $p, q \neq 1$. Such system, on the other hand, is a natural generalisation of the Lane-Emden equation, which models stars.

Along time several extensions and generalisations of the Lane-Emden systems have been considered: for example, eliminating the restrictions mentioned above on the parameters $p$ and $q$, see [4, 5, 13, 8], or relaxing conditions on the dimension of the space, see [4, 5]. More recently, hyperbolic versions of such a system have also been studied, see [12]. Then, in line with these generalisations of the Lane-Emden systems, we shall refer to system (2) as a Lane-Emden-Klein-Gordon-Fock system with central symmetry.

The organisation of this paper is as follows. In Section 2, we find the Lie symmetries of systems (2) with the following restriction: $p, q \notin \{0, 1\}$. This technical condition is made to bring a truly nonlinearity to system (2). Otherwise, we could either eliminate one of the equations or then consider two decoupled systems of linear equations, which are out of our interest. Next, we determine Noether symmetries together with the associated conservation laws of system (2) in Section 3. In Section 4 we use some point generators obtained in Section 2 to transform system (2) in a system of ordinary differential equations.

### 2. Lie group classification.
Let $x = (x^1, \ldots, x^n)$ be $n$ independent variables and $u = (u^1, \ldots, u^m)$ $m$ dependent variables. An operator (the sum over repeated indices is presupposed)

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

is called Lie point symmetry generator of the system of differential equations

$$F_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, s,$$

if the following condition, called invariance condition, holds

$$X^{[k]}(F_\alpha) \big|_S = 0,$$

where in (4) $u_{(\ell)}$ means the set of $\ell$-th derivatives of $u$, $S$ means the manifold defined by the set of equations given in (4) and

$$X^{[k]} = X + \sum_{s=1}^{k} \sum_{i_1, \ldots, i_s = 1} D_{i_1} \cdots D_{i_s} (W^\alpha) \frac{\partial}{\partial u^{\alpha}_{i_1, \ldots, i_s}}.$$

The invariance condition lead us to an overdetermined, linear system of partial differential equations involving the coefficients of (4), which can easily be solved by using convenient softwares. In our case, we made use of the packages developed in [6, 7], from which we can prove the following result.

**Theorem 2.1.** The Lie point symmetry generators for system (2) are spanned by the operators

$$X_1 = \frac{\partial}{\partial t},$$

and

$$X_2 = r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t}.$$
For some specific values of $p$ and $q$ it is possible to enlarge the basis. These values are:

1. If $pq = 1$, $p \neq \pm 1$, then we have
   \[ X^q = -(q + 1) \left( r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t} \right) + u \frac{\partial}{\partial u} + qv \frac{\partial}{\partial v}, \]
   (8)

2. If $p = q = -1$, we have
   \[ X_3 = \frac{\partial}{\partial r} - \frac{u}{r} \frac{\partial}{\partial u} - \frac{v}{r} \frac{\partial}{\partial v}, \]
   (9)

   and
   \[ X_4 = t \frac{\partial}{\partial r} + r \frac{\partial}{\partial t} - \frac{tu}{r} \frac{\partial}{\partial u} - \frac{tv}{r} \frac{\partial}{\partial v}. \]
   (10)

3. Noether symmetries and conservation laws. The operator given by the formal sum
   \[ \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{j_1 \cdots j_s} \frac{\partial}{\partial u_{j_1 \cdots j_s}^\alpha}, \]
   (11)

   is called Euler-Lagrange operator.

   From the differential operator (3) we can construct the so-called Noether operators
   \[ N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha} + \sum_{s=1}^{\infty} D_{j_1 \cdots j_s} (W^\alpha) \frac{\delta}{\delta u_{j_1 \cdots j_s}^\alpha}, \]
   (12)

   where $W^\alpha := \eta^\alpha - \xi^i u_i$, see Ibragimov [10], page 202.

   The following (formal) identity is known as Noether identity (see, for instance, [10], page 203, for further details):
   \[ X[\infty](\cdot) + D_i \xi^i(\cdot) = W^\alpha \frac{\delta}{\delta u^\alpha}(\cdot) + D_i \left( N^i(\cdot) \right), \]
   (13)

   where
   \[ X[\infty] = X + \sum_{s=1}^{\infty} \sum_{i_1 \cdots i_s=1} D_{i_1} \cdots D_{i_s} (W^\alpha) \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}. \]

   If the system (4) is a critical point of the functional
   \[ J[u] = \int_{\mathbb{R}} \mathcal{L}(x, u, \cdots, u_\ell) dx, \]
   (14)

   that is,
   \[ F_\alpha = \frac{\delta \mathcal{L}}{\delta u^\alpha} \]

   which means that (4) is a Euler-Lagrange system of equations, and if $X[\infty] \mathcal{L} + D_i (\xi^i) \mathcal{L} = D_i B^i$, for a certain vector field $B$ of components $B^i$, then the differential operator (3) is called Noether symmetry generator and, in particular, equation (13) reads
   \[ D_i (N^i(\mathcal{L}) - B^i) = 0. \]
   (15)

   If we define $C^i := N^i(\mathcal{L}) - B^i$, we conclude that $D_i C^i \equiv 0$ on the solutions of (4).

   Therefore, the vector field $C = (C^i)$ is said to be a conserved vector and (15) is a conservation law of equation (4). In particular, we have just made a demonstration of the celebrated Noether theorem.
Noticing that system (2) is the Euler-Lagrange equations of the functional

\[ J[u, v] = \int_0^\infty \int_0^\infty \mathcal{L}(t, r, u_t, u_r, v_t, v_r) dt dr, \]

where the equivalent function of Lagrange is defined by

\[ \mathcal{L} = \frac{1}{2} \left( r^2 u_t v_t - r^2 u_r v_r - K(u) - M(v) \right) \]

where

\[ K(u) = \begin{cases} \frac{u^{q+1}}{q+1}, & \text{if } q \neq -1, \\ \ln |u|, & \text{if } q = -1, \end{cases} \quad M(v) = \begin{cases} \frac{v^{p+1}}{p+1}, & \text{if } p \neq -1, \\ \ln |v|, & \text{if } p = -1. \end{cases} \]  

The foregoing machinery applied to the Lane-Emden-Klein-Gordon-Fock system (2) gives the following Noether symmetries and their corresponding conserved vector, summarised in the next table.

| Generator | \( p \) and \( q \) | Components \( C^0 \) |
|-----------|-----------------|------------------|
| \( X_1 \) | \( \forall \) | \( C^0 = \frac{r^2}{2} (u_t v_t + u_r v_r) + \frac{1}{2} (K(u) + M(v)) \), \( C^1 = -\frac{r^2}{2} (u_t v_r + u_r v_t) \). |
| \( X^q \) | \( pq = 1 \), \( p \neq \pm 1 \) | \( C^0 = -\frac{r^2}{2} u_t v_t - \frac{r^2}{2} u_r v_r - \frac{r^3}{2} u_t v_t - \frac{r^3}{2} u_r v_r - \frac{r^3}{2} u_t v_t - \frac{r^3}{2} u_r v_r \), \( C^1 = \frac{r^3}{2} u_t v_t + \frac{r^3}{2} u_r v_r + \frac{r^3}{2} u_t v_t + \frac{r^3}{2} u_r v_r + \frac{r^3}{2} u_t v_t + \frac{r^3}{2} u_r v_r \), \( \frac{r^3}{2} u_t v_t + \frac{r^3}{2} u_r v_r + \frac{r^3}{2} u_t v_t + \frac{r^3}{2} u_r v_r + \frac{r^3}{2} u_t v_t + \frac{r^3}{2} u_r v_r \). |
| \( X_3 \) | \( -1 \) | \( C^0 = -\frac{1}{2} r^2 u_t v_t - \frac{1}{2} r^2 u_r v_r - \frac{1}{2} r^2 u_t v_t - \frac{1}{2} r^2 u_r v_r \), \( C^1 = \frac{1}{2} r^2 u_t v_t + \frac{1}{2} r^2 u_r v_r + \frac{1}{2} r^2 u_t v_t + \frac{1}{2} r^2 u_r v_r + \frac{1}{2} r^2 u_t v_t + \frac{1}{2} r^2 u_r v_r \). |
| \( X_4 \) | \( p = q = -1 \) | \( C^0 = \frac{1}{2} r^3 u_t v_t - \frac{1}{2} r^3 u_r v_r - \frac{1}{2} r^3 u_t v_t - \frac{1}{2} r^3 u_r v_r \), \( C^1 = \frac{1}{2} r^3 u_t v_t + \frac{1}{2} r^3 u_r v_r + \frac{1}{2} r^3 u_t v_t + \frac{1}{2} r^3 u_r v_r + \frac{1}{2} r^3 u_t v_t + \frac{1}{2} r^3 u_r v_r \). |
4. Reductions and integrability of system (2). In this section we use some
generators obtained in Theorem 2.1 in order to transform system (2) into a system
of ordinary differential equations through the invariant surface condition, see, for
instance, [2], page 169.

We do it very shortly, since the calculations are quite straightforward.

- We begin with generator (6). It leads to the following general group invariant
  solutions: \( u(r,t) = y(r), v(r,t) = z(r) \), where \( y(r) \) and \( z(r) \) are any solutions
  of the second order ODE system

\[
\begin{align*}
  y''(r) + \frac{2}{r} y'(r) - \frac{z(r)^p}{r^2} &= 0, \\
  z''(r) + \frac{2}{r} z'(r) - \frac{y(r)^q}{r^2} &= 0.
\end{align*}
\]

(18)

Thus, we conclude that \( u(r,t) = \psi(\xi), v(r,t) = \zeta(\xi) \) is a general group invari-
ant solution of system (2), with \( \psi = \psi(\xi) \) and \( \zeta = \zeta(\xi) \) satisfying the second
order system (19).

- Generator (7) enables us to construct the following three invariants: \( \xi = t/r \),
  \( u = \psi \) and \( v = \zeta \). Making use of these invariants, system (2) becomes

\[
\begin{align*}
  (1 - \xi^2) \psi''(\xi) + \xi \psi'(\xi) - \frac{1}{\psi(\xi)} &= 0, \\
  (1 - \xi^2) \zeta''(\xi) + \xi \zeta'(\xi) - \frac{1}{\zeta(\xi)} &= 0.
\end{align*}
\]

(19)

Consequently, \( u(r,t) = \psi(\xi), v(r,t) = \zeta(\xi) \) is a general group invariant solution of
system (2), where \( \psi(\xi) \) and \( \zeta(\xi) \) are any solutions of the second order ODE system
(20).

- Generator (10) leads to the following invariants: \( \xi = r^2 - t^2, u = \psi, v = \zeta \) and
  then making use of them, system (2) can then be rewritten as

\[
\begin{align*}
  A''(t) + \frac{1}{B(t)} &= 0, \\
  B''(t) + \frac{1}{A(t)} &= 0.
\end{align*}
\]

(20)

(21)

It is quite interesting to observe that for this exceptional case, system (21) is a
Lagrangian formulation system and it possess two Noether point symmetries
and consequently two Noetherian integrals. Making use of the Noetherian
integrals we obtain the exact solution

\[
u(r,t) = \frac{1}{\sqrt{b_1 r}} \exp \left[ -\text{erf}^{-1}\left( \sqrt{\frac{2}{\pi}}(t + ib_2) \right)^2 \right].
\]

(22)
\[ v(r, t) = \frac{1}{\sqrt{b_1 r}} \exp \left[ -\text{erf}^{-1} \left( \sqrt{\frac{2}{\pi}} (t + ib_2) \right)^2 \right] \]  

(23)

for system (2), where \(b_1\) and \(b_2\) are arbitrary constants and \(\text{erf}^{-1}\) is the inverse error function [14, 9].

5. Discussion and conclusion. In this paper we considered system (2) from the point of view of classical Lie symmetry analysis. We found point symmetries and conservation laws and also transformed the original system into ODE’s systems and some general group invariant solutions for system (2) were derived. In addition, an exact solution for an exceptional case for system (2) was obtained. Further explicit solutions for system (18) can be found by using the ansatz \(y = Ar^e\) and \(z = Br^s\). We, however, omit them since they are straightforward.

We would like to call attention to the fact that the system of hyperbolic equations (2) can be transformed into a system of elliptic equations under the complex change \(t \mapsto iy\). Then, renaming variables, we get the new system

\[
\begin{align*}
    u_{xx} + u_{yy} + \frac{2}{r} u_x + \frac{v^p}{x^2} &= 0, \\
v_{xx} + v_{yy} + \frac{2}{r} v_x + \frac{u^q}{x^2} &= 0.
\end{align*}
\]

(24)

The symmetries of system (24) can be effortlessly obtained from Theorem 2.1 by using the change \((r, t) \mapsto (x, iy)\). Apart from the generator (10), the symmetry generators will be preserved. Concerning (10), its projection on the \((x, y)\) plane will correspond to a rotation, instead of a hyperbolic rotation generated by the action of the group generated by (10) in the \((r, t)\) plane. Additionally, the same complex transformation enables us to classify the Noether symmetries of (24) from those obtained to (10) and the conservation laws as well.

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