BRAID GRAPHS IN SIMPLY-LACED TRIANGLE-FREE COXETER SYSTEMS ARE CUBICAL GRAPHS

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Abstract. Any two reduced expressions for the same Coxeter group element are related by a sequence of commutation and braid moves. We say that two reduced expressions are braid equivalent if they are related via a sequence of braid moves, and the corresponding equivalence classes are called braid classes. Each braid class can be encoded in terms of a braid graph in a natural way. In this paper, we study the structure of braid graphs in simply-laced Coxeter systems. We prove that every reduced expression has a unique factorization as a product of so-called links, which in turn induces a decomposition of the braid graph into a box product of the braid graphs for each link factor. When the Coxeter graph has no three-cycles, we use the decomposition to prove that braid graphs are cubical by constructing an embedding of the braid graph into a hypercube graph whose image is an induced subgraph of the hypercube. For a special class of links, called Fibonacci links, we prove that this embedding is an isometry from the corresponding braid graph to a Fibonacci cube graph.

1. Introduction

Every element $w$ of a Coxeter group $W$ can be written as an expression in the generators, and if the number of generators in an expression (including multiplicity) is minimal, we say that the expression is reduced. According to Matsumoto’s Theorem [7, Theorem 1.2.2], every pair of reduced expressions for the same group element are related by a sequence of commutations and so-called braid moves. If all of the braid moves are of the form $sts \mapsto tst$, we say that $W$ is simply laced. In light of Matsumoto’s Theorem, we can define a connected graph on the set of reduced expressions of a given element in a Coxeter group. We define the Matsumoto graph of $w \in W$ to be the graph having vertex set equal to the set of reduced expressions of $w$, where two vertices are connected by an edge if and only if the corresponding reduced expressions are related by a single commutation or braid move. Bergeron, Ceballos, and Labbé [2] proved that for finite Coxeter groups, every cycle in a Matsumoto graph has even length. In [8], Grinberg and Postnikov extended this result to arbitrary Coxeter systems. Since every cycle in a Matsumoto graph has even length, every Matsumoto graph is always bipartite.

Two reduced expressions for the same Coxeter group element are said to be commutation equivalent if we can obtain one from the other via a sequence of commutation moves. The corresponding equivalence classes are referred to as commutation classes. Commutation classes have been studied extensively in the literature. In the case of Coxeter systems of type $A_n$, Elnitsky [4] showed that the set of commutation classes for a given permutation $w$ is in one-to-one correspondence with the set of rhombic tilings of a certain polygon determined by $w$. Meng [15] studied the number of commutation classes and their relationships via braid moves, and Bédard [1] developed recursive formulas for the number of reduced expressions in each commutation class. According to [10], every finite Coxeter group contains a unique element of maximal length, called the longest element. Determining the number of commutation classes for the longest element in Coxeter systems of type $A_n$ remains an open problem. To our knowledge, this problem was first introduced in 1992 by Knuth in Section 9 of [13] using different terminology. A more general version of the problem appears in Section 5.2 of [11]. In the paragraph following the proof of Proposition 4.4 of [17], Tenner explicitly states the open problem in terms of commutation classes. The best known bound for the number of commutation classes for the longest element in Coxeter systems of type $A_n$ appears in [5]. According to [3], the set of commutation classes for the longest element in Coxeter systems of type $A_n$ is in bijection with various interesting sets of objects, including:

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(1) Vertices of the graph obtained by contracting edges that correspond to commutations in the Matsumoto graph for the longest element in Coxeter systems of type $A_n$;

(2) Heaps for the longest element in Coxeter systems of type $A_n$;

(3) Primitive sorting networks on $n$ elements;

(4) Rhombic tilings of a regular $n$-gon;

(5) Uniform oriented matroids of rank 3 on $n$ elements;

(6) Arrangements of $n$ pseudolines.

The graph mentioned in item (1) above has been studied by several authors, often in the context of the higher Bruhat order $B(n, 2)$ [14, 19]. Even less is known about the number of commutation classes of the longest elements in other finite Coxeter groups.

Similarly, we define two reduced expressions to be braid equivalent if they are related by a sequence of braid moves, where the corresponding equivalence classes are called braid classes. Braid classes have appeared in the work of Bergeron, Ceballos, and Labbé [2] while Zollinger [20] provided formulas for the size of braid classes for permutations in the case of Coxeter systems of type $A_n$. Fishel et al. [6] provided upper and lower bounds on the number of reduced expressions for a fixed permutation in Coxeter systems of type $A_n$ by studying the commutation classes and braid classes in tandem. However, unlike commutation classes, braid classes have received very little attention. Many natural questions regarding braid classes remain open, even for type $A_n$. For example, it is unknown how many braid classes can occur for elements of a fixed length. In particular, which elements for a fixed length have maximally many braid classes? And how many braid classes does the longest element in the Coxeter system of type $A_n$ have? Answering the latter question might provide insight into the analogous question for commutation classes that was mentioned above.

The relationship among the reduced expressions in a fixed braid class can be encoded in a graph. Define the braid graph for a reduced expression to be the graph with vertex set equal to the corresponding braid class, where two vertices are connected by an edge if and only if the corresponding reduced expressions are related via a single braid move. Note that every braid graph is equal to one of the connected components of the graph obtained by deleting the edges corresponding to commutation moves in the Matsumoto graph for the corresponding group element. The overarching goal of this paper is to understand the structure of braid classes in simply-laced Coxeter systems by studying the combinatorial architecture of the corresponding braid graphs.

We begin by recalling the basic terminology of Coxeter groups and establish our notation in Section 2. In Section 3 we introduce the notion of a braid shadow in simply-laced Coxeter groups, which is simply the location where a braid move may be applied when looking across the entire braid class for a reduced expression. Loosely speaking, we say that a reduced expression is a link if it consists of a single generator or there is a sequence of overlapping braid shadows that spans the positions of the reduced expression. It turns out that every reduced expression can be written uniquely as a product of maximal links. We refer to this product as the link factorization for the reduced expression. One consequence of this factorization is that every braid graph can be decomposed as the box product of the braid graphs for the corresponding factors (Corollary 3.10). In Section 4 we completely characterize links and braid graphs in Coxeter systems of type $A_n$. Our notion of link coincides with the definition of string introduced by Zollinger [20] in the type $A_n$ case. Section 5 begins with a few technical lemmas and then concludes with one of our main results (Theorem 5.8), which states that every braid graph in a simply-laced Coxeter system having no three-cycles in its Coxeter graph is isomorphic to an induced subgraph of a hypercube. Such graphs are called cubical. In Section 6 we introduce a special class of links, called Fibonacci links, and then show that the induced embedding of the corresponding braid graph into the hypercube is an isometry to a Fibonacci cube graph (Theorem 6.9). We conclude with some open questions in Section 7.

2. Preliminaries

A Coxeter system is a pair $(W, S)$ consisting of a finite set $S$ of generating involutions and a group $W$, called a Coxeter group, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

where $e$ is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. The elements of $S$ are distinct as group elements and $m(s,t)$ is the order of $st$ [10]. Since elements of $S$ have order two, the relation
In the case that $\Gamma$ has no three-cycles, we say that $\alpha$ is of type $\Gamma$, and we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for emphasis. In the case that $\Gamma$ has no three-cycles, we say that $(W, S)$ is triangle free. If $X \subseteq S$, the induced subgraph of $\Gamma$ having vertex set equal to $X$ is denoted by $\Gamma_X$.

**Example 2.1.** The Coxeter graphs given in Figure 1 correspond to three common simply-laced Coxeter systems. We summarize the defining relations for the Coxeter systems determine by the graphs in Figures 1(a) and 1(b) below.

(a) The Coxeter system of type $A_n$ is given by the Coxeter graph in Figure 1(a). In this case, $W(A_n)$ is generated by $S(A_n) = \{s_1, s_2, \ldots, s_n\}$ and has defining relations

1. $s_i^2 = e$ for all $i$;
2. $s_is_j = s_j s_i$ when $|i - j| > 1$;
3. $s_is_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group $S_{n+1}$ under the mapping that sends $s_i$ to the adjacent transposition $(i, i + 1)$.

(b) The Coxeter system of type $D_n$ is given by the graph in Figure 1(b). The Coxeter group $W(D_n)$ has generating set $S(D_n) = \{s_1, s_2, \ldots, s_n\}$ and defining relations

1. $s_i^2 = e$ for all $i$;
2. $s_is_j = s_j s_i$ if $|i - j| > 1$ and $i, j \neq n$;
3. $s_is_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(D_n)$ is isomorphic to the index two subgroup of the group of signed permutations on $n$ letters having an even number of sign changes.

Given a Coxeter system $(W, S)$, let $\mathcal{S}^*$ denote the free monoid on the alphabet $S$. An element $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m} \in \mathcal{S}^*$ is called a word while a subword of $\alpha$ is a word of the form $s_{x_i} s_{x_{i+1}} \cdots s_{x_j}$ for $1 \leq i \leq j \leq m$, which we may denote by $\alpha_{i:j}$. If a word $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m} \in \mathcal{S}^*$ is equal to $w$ when considered as a group element of $W$, we say that $\alpha$ is an expression for $w$. If $m$ is minimal among all possible expressions for $w$, we say that $\alpha$ is a reduced expression for $w$, and we call $\ell(w) := m$ the length of $w$. Note that any subword of a reduced expression is also reduced. The set of all reduced expressions for $w \in W$ will be denoted by $\mathcal{R}(w)$. According to [10], every finite Coxeter group contains a unique element of maximal length, called the longest element, often denoted by $w_0$ if the context is clear.

![Figure 1. Examples of common simply-laced Coxeter graphs.](image-url)
For the remainder of this paper, if we are considering a particular labeling of a Coxeter graph, we will often write $i$ in place of $s_i$ for brevity.

**Example 2.2.** It is well known that the longest element in $W(A_n)$ is given in one-line notation by

$$w_0 = [n + 1, n, \ldots, 2, 1]$$

and has length $\ell(w_0) = \binom{n+1}{2}$. One possible reduced expression for $w_0 \in W(A_n)$ is given by

$$1 \mid 21 \mid 321 \mid \cdots \mid n(n−1)\cdots321,$$

where the vertical bars have been placed to aid in recognizing the given pattern. A formula for the number of reduced expressions for $w_0 \in W(A_n)$ is given in [16].

The following theorem, called Matsumoto’s Theorem [7, Theorem 1.2.2], characterizes the relationship between reduced expressions for a given group element.

**Proposition 2.3** (Matsumoto’s Theorem). In a Coxeter system $(W,S)$, any two reduced expressions for the same group element differ by a sequence of commutation and braid moves.

In light of Matsumoto’s Theorem, we can define a graph on the set of reduced expressions of a given element in a Coxeter group. For $w \in W$, define the Matsumoto graph $\mathcal{G}(w)$ to be the graph having vertex set equal to $\mathcal{R}(w)$, where two reduced expressions $\alpha$ and $\beta$ are connected by an edge if and only if $\alpha$ and $\beta$ are related via a single commutation or braid move. In order to distinguish between commutation and braid moves, an edge is colored red if it corresponds a commutation move and colored green if it corresponds to a braid move. Matsumoto’s Theorem implies that $\mathcal{G}(w)$ is connected. The graph obtained by contracting the edges corresponding to commutation moves in the Matsumoto graph for the longest element in a Coxeter systems of type $A_n$ has been studied by several authors, usually in the context of the higher Bruhat order $B(n,2)$ [14, 19] or rhombic tilings of polygons [4].

**Example 2.4.** Consider the longest element $w_0$ in the Coxeter system of type $A_3$. It turns out that $\ell(w_0) = 6$ and that there are 16 reduced expressions in $\mathcal{R}(w_0)$. The corresponding Matsumoto graph is given in Figure 2. The 16 reduced expressions are the vertices of $\mathcal{G}(w_0)$ and the edges show how pairs of reduced expressions are related via commutation or braid moves.

![Figure 2. Matsumoto graph for the longest element in $W(A_3)$](image)

We now define two different equivalence relations on the set of reduced expressions for a given element of a Coxeter group. Let $(W,S)$ be a Coxeter system of type $\Gamma$ and let $w \in W$. For $\alpha, \beta \in \mathcal{R}(w)$, $\alpha \sim_c \beta$ if we can obtain $\alpha$ from $\beta$ by applying a single commutation move of the form $st \rightarrow ts$, where $m(s,t) = 2$. We define the equivalence relation $\sim_c$ by taking the reflexive transitive closure of $\sim_c$. Each equivalence class under $\sim_c$ is called a commutation class, denoted $[\alpha]_c$ for $\alpha \in \mathcal{R}(w)$. Two reduced expressions are said to be commutation equivalent if they are in the same commutation class.

Similarly, we define $\alpha \sim_b \beta$ if we can obtain $\alpha$ from $\beta$ by applying a single braid move. The equivalence relation $\sim_b$ is defined by taking the reflexive transitive closure of $\sim_b$. Each equivalence class under $\sim_b$ is called a braid class, denoted $[\alpha]_b$ for $\alpha \in \mathcal{R}(w)$. Two reduced expressions are said to be braid equivalent if they are in the same braid class.
Example 2.5. The set of 16 reduced expressions for the longest element in the Coxeter system of type $A_3$ is partitioned into eight commutation classes and eight braid classes:

$$
\begin{align*}
[232123]_c &= \{232123\} & [123121]_b &= \{123121, 123212, 132312\} \\
[231213]_c &= \{231213, 213213, 213231, 231231\} & [312312]_b &= \{312312\} \\
[321323]_c &= \{321323, 323123\} & [312132]_b &= \{312132, 321232, 321323\} \\
[212321]_c &= \{212321\} & [123122]_b &= \{123122\} \\
[321232]_c &= \{321232\} & [123123]_b &= \{123123, 213212, 213231\} \\
[123123]_c &= \{123121, 121321\} & [213213]_b &= \{213213\} \\
[132312]_c &= \{132312, 132132, 312132, 321232\} & [231213]_b &= \{231213, 232123, 323123\} \\
[123212]_c &= \{123212\} & [231231]_b &= \{231231\}
\end{align*}
$$

In general, it is not the case that the number of commutation classes for a fixed group element is equal to the number of braid classes. Notice that the four braid classes of size 3 correspond to the vertices in the green connected components of the Matsumoto graph given in Figure 2 while the singleton braid classes correspond to the four vertices that are not incident to any green edges. A similar structure holds for the commutation classes.

The remainder of this paper will focus exclusively on braid classes in simply-laced Coxeter systems with an aim of describing their combinatorial architecture. For brevity, we will write $[\alpha]$ in place of $[\alpha]_b$.

The relationship among the reduced expressions in a fixed braid class can be encoded using the maximal green connected components of the corresponding Matsumoto graph. Let $\alpha$ be a reduced expression for $w \in W$. We define the braid graph of $\alpha$, denoted $B(\alpha)$, to be graph with vertex set equal to $[\alpha]$, where $\alpha, \beta \in [\alpha]$ are connected by an edge if and only if $\alpha$ and $\beta$ are related via a single braid move. Note that we are defining braid graphs with respect to a fixed reduced expression (or equivalence class) as opposed to the corresponding group element. The latter are the graphs that arise from contracting the edges corresponding to braid moves in the Matsumoto graph. Very little is known about the overall structure of braid graphs for reduced expressions.

Example 2.6. Below we describe three different braid classes and illustrate their corresponding braid graphs.

(a) Consider the Coxeter system of type $A_4$. The braid class for the reduced expression 1213243 consists of the following reduced expressions:

$$
\alpha_1 := 1213243, \quad \alpha_2 := 2132343, \quad \alpha_3 := 2132343, \quad \alpha_4 := 2132434.
$$

(b) In the Coxeter system of type $A_4$, the expression 1213243565 is reduced. Its braid class consists of the following reduced expressions:

$$
\begin{align*}
\beta_1 &:= 1213243565, \quad \beta_2 := 213234565, \quad \beta_3 := 213234565, \quad \beta_4 := 2132434565, \\
\beta_5 &:= 1212343656, \quad \beta_6 := 2132243656, \quad \beta_7 := 2132243656, \quad \beta_8 := 2132434656.
\end{align*}
$$

(c) Now, consider the Coxeter system of type $D_4$. The expression 2321434 is reduced and its braid class consists of the following reduced expressions:

$$
\begin{align*}
\gamma_1 &:= 2321434, \quad \gamma_2 := 3231434, \quad \gamma_3 := 3231434, \quad \gamma_4 := 2321343, \quad \gamma_5 := 3213143.
\end{align*}
$$

The braid graphs $B(\alpha_1), B(\beta_1)$, and $B(\gamma_1)$ are depicted in Figures 3(a), 3(b) and 3(c) respectively.

3. Architecture of braid classes in simply-laced Coxeter systems

In this section, we introduce the notions of braid shadow and link, which allows us to provide a factorization of reduced expressions in simply-laced Coxeter systems into products of maximal links. This in turn yields a decomposition of the braid graph for a reduced expression in simply-laced Coxeter systems into a box product of the braid graphs for the corresponding link factors. In addition, we prove several technical results concerning the structure of braid shadows.

If $i, j \in \mathbb{N}$ with $i \leq j$, then we define the interval $[i,j] := \{i,i+1,\ldots,j-1,j\}$. Note that the degenerate interval $[i,i]$ equals the singleton \{i\}. 
worth pointing out that many of the results that follow do not hold in arbitrary Coxeter systems.

Suppose $\alpha = s_{x_1}s_{x_2}\cdots s_{x_m}$ is a reduced expression for $w \in W$. We define the local support of $\alpha$ over $[i, j]$ via

$$\text{supp}_{[i, j]}(\alpha) := \{s_{x_k} \mid k \in [i, j]\}.$$ 

The local support of the braid class $[\alpha]$ over $[i, j]$ is defined by

$$\text{supp}_{[i, j]}([\alpha]) := \bigcup_{\beta \in [\alpha]} \text{supp}_{[i, j]}(\beta).$$

That is, $\text{supp}_{[i, j]}([\alpha])$ is the set consisting of the generators that appear in positions $i, i + 1, \ldots, j$ of $\alpha$ while $\text{supp}_{[i, j]}([\alpha])$ is the set of generators that appear in positions $i, i + 1, \ldots, j$ of any reduced expression braid equivalent to $\alpha$. Note that in the case of the degenerate interval $[i, i]$, we will use the notation $\text{supp}_i(\alpha)$ and $\text{supp}_i([\alpha])$ and we will simply write $\text{supp}(\alpha)$ for the set of generators that appear in $\alpha$.

Throughout the remainder of this section, we assume that $(W, S)$ is a simply-laced Coxeter system. It is worth pointing out that many of the results that follow do not hold in arbitrary Coxeter systems.

Suppose that $(W, S)$ is a simply-laced Coxeter system. If $\alpha = s_{x_1}s_{x_2}\cdots s_{x_m}$ is a reduced expression for $w \in W$, then the interval $[i, i + 2]$ is a braid shadow for $\alpha$ if and only if $s_{x_i} = s_{x_{i+2}}$ and $m(s_{x_i}, s_{x_{i+1}}) = 3$. The collection of braid shadows for $\alpha$ is denoted by $S(\alpha)$ and the set of braid shadows for the braid class $[\alpha]$ is given by

$$S([\alpha]) := \bigcup_{\beta \in [\alpha]} S(\beta).$$

The cardinality of $S([\alpha])$ is called the rank of $\alpha$, which we denote by $\text{rank}(\alpha)$.

A braid shadow is simply the location in a reduced expression where we have an opportunity to apply a braid move. A reduced expression may have many braid shadows or possibly none at all. The sets $S(\alpha)$ and $S([\alpha])$ capture the locations where braid moves can be performed in $\alpha$ and any reduced expression braid equivalent to $\alpha$, respectively. If $[i, i + 2]$ is a braid shadow for $[\alpha]$, then we may refer to position $i + 1$ in any reduced expression in $[\alpha]$ as the center of the braid shadow.

Example 3.3. Consider the reduced expressions given in Example 2.6. By inspection, we see that:

(a) $S(\alpha_1) = \{[1, 3]\}$ and $S([\alpha_1]) = \{[1, 3], [3, 5], [5, 7]\}$.
(b) $S(\beta_1) = \{[1, 3], [8, 10]\}$ and $S([\beta_1]) = \{[1, 3], [3, 5], [5, 7], [8, 10]\}$.
(c) $S(\gamma_1) = \{[1, 3], [5, 7]\}$ and $S([\gamma_1]) = \{[1, 3], [3, 5], [5, 7]\}$.

The following metric will be useful in the proof of Proposition 3.5.

Definition 3.4. Suppose $(W, S)$ is a simply-laced Coxeter system. Let $\alpha$ and $\beta$ be two braid equivalent reduced expressions for $w \in W$. The braid distance $d(\alpha, \beta)$ between $\alpha$ and $\beta$ is defined to be the minimum number of braid moves required to transform $\alpha$ into $\beta$.

Equivalently, we can interpret the number $d(\alpha, \beta)$ as the length of any minimal path from $\alpha$ to $\beta$ in the corresponding braid graph. This is a consequence of Matsumoto’s Theorem (Proposition 2.3).

If $\alpha$ is a reduced expression for $w \in W$, then a pair of braid shadows for $\alpha$ are either disjoint or overlap by a single position. This is stated explicitly for Coxeter systems of type $A_n$ in Section 2.1 of [1]. The next proposition generalizes this phenomenon and shows that braid shadows for $[\alpha]$ are either disjoint or overlap by a single position.
Proposition 3.5. Suppose \((W, S)\) is a simply-laced Coxeter system. If \(\alpha\) is a reduced expression for \(w \in W\) with \([i, i + 2] \in S([\alpha])\), then \([i + 1, i + 3] \notin S([\alpha])\).

Proof. Note that the result vacuously holds if \(\ell(w) \leq 3\). Suppose \(\ell(w) := m \geq 4\). For each \(i \in \{1, 2, \ldots, m - 3\}\), define
\[
P_i := \{(\beta, \gamma) \mid \beta, \gamma \in [\alpha], [i, i + 2] \in S(\beta) \text{ and } [i + 1, i + 3] \in S(\gamma)\}
\]
and let
\[
P := \bigcup_{i=1}^{m-3} P_i.
\]
We will prove that \(P = \emptyset\). For a contradiction, assume otherwise. Choose \((\beta, \gamma) \in P\) such that \(d(\beta, \gamma)\) is minimal among all elements of \(P\). Since \((\beta, \gamma) \in P\), there exists \(i \in \{1, \ldots, m - 3\}\) such that \((\beta, \gamma) \in P_i\). This implies that \([i, i + 2] \in S(\beta)\) while \([i + 1, i + 3] \in S(\gamma)\). Without loss of generality, assume that
\[
supp_{\{i, i+2\}}(\beta) = \{s, t\} \text{ and } supp_{\{i+1, i+3\}}(\gamma) = \{u, v\}.
\]
Suppose \(d(\beta, \gamma) = k\) and let \(\alpha_0 := \beta, \alpha_1, \ldots, \alpha_{k-1}, \alpha_k := \gamma\) be a sequence of braid equivalent reduced expressions, each one braid move apart, that transforms \(\beta\) into \(\gamma\) in \(k\) braid moves. Let \(b_j\) denote the braid move that transforms \(\alpha_{j-1}\) into \(\alpha_j\). We can represent this sequence of moves visually as follows:
\[
\begin{array}{cccccccccc}
s & \frac{t}{i+1} & \frac{\gamma}{i+2} & \frac{\zeta}{i+3} & \cdots & b_1 & \cdots & b_k & \cdots & \frac{r}{i+4} & \frac{s}{i+5} & \frac{t}{i+6} \cdots
\end{array}
\]
Suppose \(b_1\) transforms \(\alpha_0\) into \(\alpha_1\) such that \(b_1\) does not involve position \(i\) nor position \(i+3\). Then \((\alpha_1, \alpha_k) \in P_i\) while \(d(\alpha_1, \alpha_k) = k - 1\), a contradiction. So, the opening braid move \(b_1\) must involve either position \(i\) or position \(i+3\). If \(b_1\) acts on positions \([i-1, i+1]\) or \([i+1, i+3]\), then \(\alpha_0\) is not reduced. On the other hand, if \(b_1\) transforms \(\alpha_0\) into \(\alpha_1\) by applying the braid move \(sts \rightarrow tst\) in positions \([i, i+2]\), then \((\alpha_1, \alpha_k) \in P_i\) while \(d(\alpha_1, \alpha_k) = k - 1\), a contradiction. Hence \(b_1\) must act on positions \([i+2, i+4]\) or positions \([i-2, i]\).

First, assume that \(b_1\) acts on positions \([i+2, i+4]\). Hence, there exists \(x \in S\) with \(m(s, x) = 3\) such that
\[
\begin{array}{cccccccccc}
s & \frac{t}{i+1} & \frac{\gamma}{i+2} & \frac{\zeta}{i+3} & \cdots & b_1 & \cdots & b_k & \cdots & \frac{r}{i+4} & \frac{s}{i+5} & \frac{t}{i+6} \cdots
\end{array}
\]
This implies that \((\alpha_k, \alpha_1) \in P_{i+1}\) while \(d(\alpha_k, \alpha_1) = k - 1\), which is a contradiction. We can conclude that \(b_1\) acts on positions \([i-2, i]\). Now, define the subsequences
\[
L := \{b_n \mid b_n \text{ acts on } \{j, j+2\} \text{ for } j < i - 1\}, \quad R := \{b_n \mid b_n \text{ acts on } \{j, j+2\} \text{ for } j \geq i - 1\}.
\]
Note that \(b_1 \notin L\). We will show that \(R = \emptyset\). Assume otherwise and let \(b_r \in R\). If \(b_r\) acts on \([i-1, i+1]\), then \((\alpha_{r-1}, \alpha_0) \in P_{i+1}\) while \(d(\alpha_{r-1}, \alpha_0) = r - 1 < k\). Similarly, if \(b_r\) acts on \([i, i+2]\), then \((\alpha_{r-1}, \alpha_k) \in P_{i+1}\) while \(d(\alpha_{r-1}, \alpha_k) = k - (r - 1) < k\). In either case, we contradict the minimality of \(k\). This shows that the positions that the braid moves in \(L\) and \(R\) act on respectively do not overlap, and so the braid moves in \(R\) can be applied in any order relative to the braid moves in \(L\). In particular, the braid moves in \(R\) could be applied prior to any of the braid moves in \(L\). But this contradicts the fact that \(b_1 \in L\). Therefore, the sequence \(b_1, \ldots, b_k\) never acts on positions to the right of position \(i\). But this is impossible since these positions are disjoint from \([i+1, i+3]\], which we must necessarily change to arrive at \(\alpha_k\). We conclude that \(P = \emptyset\), which yields the desired result.

□

Proposition 3.5 together with its contrapositive implies that braid shadows for braid equivalent reduced expression may only overlap by a single position. This motivates the following definition.

Definition 3.6. Suppose \((W, S)\) is a simply-laced Coxeter system. Let \(\alpha = s_{x_1}s_{x_2}\cdots s_{x_m}\) be a reduced expression for \(w \in W\) with \(m \geq 1\). We say that \(\alpha\) is a link if and only if either \(m = 1\) or \(m\) is odd and \(S([\alpha]) = \{[1, 3], [3, 5], \ldots, [m-4, m-2], [m-2, m]\}\). If \(\alpha\) is a link, then the corresponding braid class \([\alpha]\) is called a braid chain.

Loosely speaking, \(\alpha\) is link if there is a sequence of overlapping braid moves that “cover” the positions \(1, 2, \ldots, m\). Note that if \(\alpha\) is a link, then the rank of \(\alpha\) is \(k\) if an only if \(\alpha\) consists of \(2k + 1\) letters. Notice that the center of every braid shadow is an even index.
Example 3.7. Consider the reduced expressions given in Example 2.6. Since \( S([\alpha_1]) = \{[1,3], [3,5], [5,7]\} \), \( \alpha_1 \) is a link and \([\alpha_1]\) is a braid chain. On the other hand, since \( S([\beta_1]) = \{[1,3], [3,5], [5,7], [8,10]\} \), it follows that \( \beta_1 \) is not a link and hence \([\beta_1]\) is not a braid chain. However, it turns out that the subwords 1213243 and 565 of \( \beta_1 \) are links in their own right. Lastly, since \( S([\gamma_1]) = \{[1,3], [3,5], [5,7]\} \), \( \gamma_1 \) is a link and \([\gamma_1]\) is a braid chain.

Definition 3.8. Assume \((W,S)\) is a simply-laced Coxeter system. Let \( \alpha \) be a reduced expression for \( w \in W \) with \( \ell(w) \geq 1 \). We say that \( \beta \) is a link factor of \( \alpha \) provided that

1. \( \beta \) is a subword of \( \alpha \),
2. \( \beta \) is a link, and
3. for every subword \( \gamma \) of \( \alpha \), if \( \beta \) is a subword of \( \gamma \) and \( \gamma \) is a link, then \( \beta = \gamma \).

It follows immediately from Definition 3.8 that every reduced expression \( \alpha \) for a nonidentity group element can be written uniquely as a product of link factors, say \( \alpha_1\alpha_2\cdots\alpha_k \), where each \( \alpha_i \) is a link factor of \( \alpha \). We refer to this product as the link factorization of \( \alpha \). For emphasis, we will often denote such a factorization via \( \alpha = \alpha_1 | \alpha_2 | \cdots | \alpha_k \). The following proposition is an immediate consequence of the definitions.

Proposition 3.9. Suppose \((W,S)\) is a simply-laced Coxeter system. If \( \alpha \) is a reduced expression for \( w \in W \) with link factorization \( \alpha_1 | \alpha_2 | \cdots | \alpha_k \), then

\[
[\alpha] = \{\beta_1 | \beta_2 | \cdots | \beta_k : \beta_i \in [\alpha_i], i = 1, 2, \ldots, k\}.
\]

Moreover, the cardinality of the braid class for \( \alpha \) is given by

\[
\text{card}(\{\alpha\}) = \prod_{i=1}^{k} \text{card}(\{\alpha_i\}),
\]

and rank of \( \alpha \) is given by

\[
\text{rank}(\alpha) = \sum_{i=1}^{k} \text{rank}(\alpha_i).
\]

Proposition 3.9 implies that the braid graph for any reduced expression for a nonidentity group element can be decomposed as the box product of the braid graphs for the corresponding factors in the link factorization. Note that the decomposition is unique if one respects the ordering of the link factors.

Corollary 3.10. Suppose \((W,S)\) is a simply-laced Coxeter system. If \( \alpha \) is a reduced expression for \( w \in W \) with link factorization \( \alpha_1 | \alpha_2 | \cdots | \alpha_k \), then

\[
B(\alpha) \cong B(\alpha_1) \square B(\alpha_2) \square \cdots \square B(\alpha_k).
\]

Proof. An isomorphism of graphs is given by \( \beta_1 | \beta_2 | \cdots | \beta_k \mapsto (\beta_1, \beta_2, \ldots, \beta_k) \), where each \( \beta_j \in [\alpha_j] \). This bijection between vertex sets respects the edges of the corresponding graphs since braid moves on different factors can be applied independently.

Example 3.11. Consider the reduced expression \( \beta_1 = 1213243565 \) defined in Example 2.6. The link factorization for \( \beta_1 \) is 1213243 | 565. The decomposition \( B(\beta_1) \cong B(1213243) \square B(565) \) guaranteed by Corollary 3.10 is shown in Figure 3. Note that we have utilized colors to help distinguish the link factors.

Example 3.12. Consider the Coxeter system of type \( D_7 \) determined by the Coxeter graph in Figure 1(b). The reduced expression \( 323134567543231343 \) has link factorization

\[
3231343 | 5 | 6 | 7 | 5 | 4 | 3231343.
\]

The braid graphs for the first and last factors are isomorphic to the braid graph in Figure 3(e). The braid graph for each singleton factor consists of a single vertex. The braid graph for the entire reduced expression and its decomposition are shown in Figure 5.

According to the next proposition, the support of a braid shadow is constant across an entire braid class in simply-laced triangle-free Coxeter systems.

Proposition 3.13. Suppose \((W,S)\) is a simply-laced triangle-free Coxeter system. If \( \alpha \) and \( \beta \) are two braid equivalent reduced expressions for \( w \in W \) with \( \ell(w) \geq 3 \), then for all \( [i, i + 2] \in S(\alpha) \cap S(\beta) \), \( \text{supp}_{[i, i+2]}(\alpha) = \text{supp}_{[i, i+2]}(\beta) \).
Proof. Let $\alpha$ and $\beta$ be two braid equivalent reduced expressions for $w \in W$ with $m \geq 3$. For each $i \in \{1, \ldots, m-2\}$, define

$$P_i := \{(\gamma, \delta) \mid \gamma, \delta \in [\alpha], [i, i+2] \in S(\gamma) \cap S(\delta), \text{ and supp}_{[i,i+2]}(\gamma) \neq \text{supp}_{[i,i+2]}(\delta)\},$$

and let

$$P := \bigcup_{i=1}^{m-2} P_i.$$ 

We will prove that $P = \emptyset$. For sake of a contradiction, suppose otherwise, and choose $(\gamma, \delta) \in P$ such that $d(\gamma, \delta)$ is minimal among all elements of $P$. Then there exists $i \in \{1, \ldots, m-2\}$ such that $(\gamma, \delta) \in P_i$, so that $[i, i+2] \in S(\gamma) \cap S(\delta)$ while

$$\text{supp}_{[i,i+2]}(\gamma) \neq \text{supp}_{[i,i+2]}(\delta).$$

Suppose $\text{supp}_{[i,i+2]}(\gamma) = \{s,t\}$ and $\text{supp}_{[i,i+2]}(\delta) = \{u,v\}$. Suppose $d(\gamma, \delta) = k$ and let

$$\alpha_0 := \gamma, \alpha_1, \ldots, \alpha_{k-1}, \alpha_k := \delta$$

be a sequence of braid equivalent reduced expressions, each one braid move apart, that transforms $\gamma$ into $\delta$ in $k$ braid moves. It is clear that $k \geq 2$. Let $b_j$ denote the braid move that transforms $\alpha_j - 1$ into $\alpha_j$. As in the proof of Proposition 3.5, we represent this sequence of moves visually as follows:

$$\cdots \xrightarrow{\frac{s}{i+1} \frac{i+2}{i}} \cdots \xrightarrow{b_1} \cdots \xrightarrow{\frac{u}{i+1} \frac{i+2}{i}} \cdots$$

Since $[i, i+2] \in S(\alpha_0)$, the intervals $[i-1, i+1]$ and $[i+1, i+3]$ are not braid shadows for every reduced expression in the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \alpha_k$ by Proposition 3.5. By minimality of $k$, the interval $[i, i+2]$ only appears as a braid shadow in $\alpha_0$ and $\alpha_k$. That is, $[i, i+2] \notin S(\alpha_l)$ for all $1 \leq l \leq k-1$. Together these facts imply that $\text{supp}_{i+1}(\alpha_l) = \{t\}$ for all $0 \leq l \leq k$. In other words, $t$ is fixed in position $i+1$ throughout the entire sequence $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \alpha_k$. This forces $v = t$, which in turn implies that $m(u, t) = 3$. Again by the minimality of $k$, it must be the case that $b_1$ acts on either $[i-2, i]$ or $[i+2, i+4]$. Without loss of generality,
assume that $b_1$ acts on $[i + 2, i + 4]$. Then there exists $x \in S$ with $m(s, x) = 3$ such that $\text{supp}_{i+3}(\alpha_0) = \{x\}$ and $\text{supp}_{i+4}(\alpha_0) = \{s\}$. In summary, we have

$$\begin{array}{cccc}
\cdots & \overset{\beta_1}{\cdots} & \overset{\beta_2}{\cdots} & \overset{\beta_3}{\cdots} \\
\beta_1 & \overset{\alpha_1}{\cdots} & \overset{\alpha_2}{\cdots} & \overset{\alpha_3}{\cdots}
\end{array}$$

Towards a contradiction, suppose $x \neq u$. In order to exchange $x$ with $u$ in position $i + 2$, there must exist a reduced expression $\alpha_\lambda$ with $2 \leq j \leq k$ such that $[i + 2, i + 4] \in S(\alpha_\lambda)$ and $\text{supp}_{i+2}[i+4](\alpha_\lambda) \neq \{x, s\}$. Yet if $\text{supp}_{i+2}[i+4](\alpha_\lambda) \neq \{x, s\}$, then $(\alpha_1, \alpha_\lambda) \in P_{i+2}$ while $d(\alpha_1, \alpha_\lambda) = j - 1 < k$, a contradiction. Hence $x = u$. This implies that $m(s, u) = 3$, and hence $m(s, u) = m(u, t) = m(t, s) = 3$. But this is contrary to the fact that $(W, S)$ is triangle free. We conclude that $P = \emptyset$, which proves the claim.

As the next example illustrates, the previous result is false without the assumption that the Coxeter graph is triangle free.

**Example 3.14.** Consider the Coxeter system of type $\widetilde{A}_2$, which is determined by the Coxeter graph in Figure 1(c). The expression $\alpha = 1213121$ is reduced, and it is easy to see that $\beta = 2123212 \in [\alpha]$. However, $\text{supp}_{[3,5]}(\alpha) = \{1, 3\}$ while $\text{supp}_{[3,5]}(\beta) = \{2, 3\}$. This shows that Proposition 3.13 is false when the Coxeter graph has a three-cycle.

If one generalizes the notions of braid shadow and link in the natural way, we conjecture that a result analogous to Proposition 5.13 holds in arbitrary Coxeter systems as long as the corresponding Coxeter graph does not contain a three-cycle with edge weights $3, 3, m$, where $m \geq 3$.

When a reduced expression has a braid shadow, the collection of generators that may appear at the center of the braid shadow in any braid equivalent reduced expression is completely determined by the support of that braid shadow. The following proposition makes this more precise.

**Proposition 3.15.** Suppose $(W, S)$ is a simply-laced triangle-free Coxeter system. If $\alpha$ is a reduced expression for $w \in W$, then $[i, i + 2] \in S(\alpha)$ if and only if $[i, i + 2] \in S([\alpha])$ and $\text{supp}_{[i, i + 2]}(\alpha) = \text{supp}_{i+1}([\alpha])$.

**Proof.** To prove the forward implication, let $[i, i + 2] \in S(\alpha)$ and assume that $\text{supp}_{[i, i + 2]}(\alpha) = \{s, t\}$. Certainly, $[i, i + 2] \in S([\alpha])$. Moreover, it is clear that $\{s, t\} \subseteq \text{supp}_{i+1}([\alpha])$. We may assume without loss of generality that $\text{supp}_{i+1}([\alpha]) = \{s\}$. Let $u \in \text{supp}_{i+1}([\alpha])$ and choose $\beta \in [\alpha]$ such that $\text{supp}_{i+1}(\beta) = \{u\}$. Then $\alpha$ and $\beta$ are related by a sequence of braid moves. If no braid move involves the $(i + 1)st$ position, then $s = u$. Otherwise, the sequence has a braid move that involves the $(i + 1)st$ position. However, by Proposition 5.13 $[i - 1, i + 1], [i + 1, i + 3] \not\in S([\alpha])$. Hence there exists $\gamma \in [\alpha]$ such that $[i, i + 2] \in S(\gamma)$ and $u \in \text{supp}_{[i, i + 2]}(\gamma)$. By Proposition 5.13, $\text{supp}_{[i, i + 2]}(\gamma) = \text{supp}_{[i, i + 2]}(\alpha)$. This shows that $u \in \{s, t\}$, and so $\text{supp}_{i+1}([\alpha]) = \{s, t\}$, as desired.

To prove the converse, assume $[i, i + 2] \in S([\alpha])$ and $\text{supp}_{[i, i + 2]}(\alpha) = \text{supp}_{i+1}([\alpha])$. Since $[i, i + 2] \in S([\alpha])$, we can choose $\beta \in [\alpha]$ such that $[i, i + 2] \in S(\beta)$. But now we can apply the forward implication to $\beta$ to conclude that $\text{supp}_{[i, i + 2]}(\beta) = \text{supp}_{i+1}([\beta])$. Hence $\text{supp}_{[i, i + 2]}(\alpha) = \text{supp}_{[i, i + 2]}(\beta)$, which implies that $[i, i + 2] \in S(\alpha)$.

Applying the previous proposition to a pair of overlapping braid shadows yields the following corollary, which says that the supports determined by the braid shadows intersect at a single element.

**Corollary 3.16.** Suppose $(W, S)$ is a simply-laced triangle-free Coxeter system. If $\alpha$ is a reduced expression for $w \in W$ such that $[i, i + 2], [i + 2, i + 4] \in S([\alpha])$, then $\text{card} \left( \text{supp}_{i+1}([\alpha]) \cap \text{supp}_{i+3}([\alpha]) \right) = 1$.

If $(W, S)$ is not triangle free, we may still conclude that $\text{card} \left( \text{supp}_{[i, i + 2]}(\beta) \cap \text{supp}_{[i + 2, i + 4]}(\gamma) \right) = 1$ whenever $\beta, \gamma \in [\alpha]$ with $[i, i + 2] \in S(\beta)$ and $[i + 2, i + 4] \in S(\gamma)$. However, in order to reach the conclusions of Proposition 3.15 and Corollary 3.16 we must require $(W, S)$ to be triangle free, as illustrated by the following example.

**Example 3.17.** Consider the reduced expression $\alpha$ from Example 3.14. We have $\text{supp}_{[3, 5]}(\alpha) = \{1, 3\}$ and $\text{supp}_{4}([\alpha]) = \{1, 2, 3\}$, contrary to Proposition 3.15. We also have $\text{supp}_{4}([\alpha]) = \{1, 2\}$ so that $\text{supp}_{4}([\alpha]) \cap \text{supp}_{0}([\alpha]) = \{1, 2\}$, which clashes with the conclusion of Corollary 3.16. Once again, this shows that the assumption that the Coxeter graph has no three-cycles cannot be discarded.
Proposition 3.15 allows us to assume that \( \text{supp}_{s+1}(\{\alpha\}) = \{s, t\} \) whenever we have \([i, i + 2] \in S(\alpha)\) with \( \text{supp}_{[i, i+2]}(\alpha) = \{s, t\} \). Moreover, if additionally we have \([i+2, i+4] \in S(\alpha)\), then we can utilize Corollary 3.10 to conclude that \( \text{supp}_{s+3}(\{\alpha\}) = \{t, u\} \). Note that in this situation \( m(s, t) = 3 \) and \( m(t, u) = 3 \). We will frequently use these facts, and we may do so without explicitly mentioning the proposition.

4. Classification of links and braid graphs in Coxeter systems of type \( A_n \)

Our notions of link and link factorization generalize Zollinger’s definitions of string and maximal string decomposition, respectively, for Coxeter systems of type \( A_n \) that appear in [20]. Let \((l, k, m, \epsilon)\) be a quadruple satisfying:

1. \( l \) is a positive integer;
2. \( k \) is a nonnegative integer less than or equal to \( l - 1 \);
3. \( m \) is a positive integer (not necessarily distinct from \( l \) or \( k \)) and;
4. \( \epsilon \) is one of \(\{+, -, 0\}\);

where \( \epsilon = 0 \) only when \( l \leq 2 \). From this quadruple, define \( \sigma_{l,k,m,\epsilon} \) with \( \epsilon \in \{+, 0\} \) to be the word in the Coxeter system of type \( A_n \), called a string, as follows. When \( l = 1, 2 \), define

\[
\sigma_{1,0,m,0} := s_m, \quad \sigma_{2,0,m,0} := s_ms_{m+1}s_m, \quad \sigma_{2,1,m,0} = s_{m+1}s_ms_{m+1}.
\]

When \( l \geq 3 \), then \( \sigma_{l,k,m,\epsilon} \) is the first \( 2l - 1 \) letters from the following product

\[
s_{m+1}s_ms_{m+2}s_{m+1}s_{m+3}^\epsilon s_{m+k}s_{m+k-1}s_{m+k}s_{m+k+1}s_{m+k+2}s_{m+k+3}s_{m+k+2}^\epsilon
\]

Note that there are two overlapping opportunities to apply a braid move, each of which has been underlined or overlined for emphasis. We define the string \( \sigma_{l,k,m,-} \) to be the reverse (i.e., inverse) of \( \sigma_{l,-l-k,m,\epsilon} \). Observe that every string consists of an odd number of letters, namely \( 2l - 1 \). According to [20], each string is a reduced expression.

Example 4.1. Below are six examples of strings in the Coxeter system of type \( A_9 \):

\[
\begin{align*}
\sigma_{6,0,4,+,+} &= 45465768798 & \sigma_{6,1,4,+,+} &= 54565768798 & \sigma_{6,2,4,+,+} &= 54656768798 \\
\sigma_{6,3,4,+,+} &= 5465767898 & \sigma_{6,4,4,+,+} &= 54657687898 & \sigma_{6,5,4,+,+} &= 54657687989
\end{align*}
\]

By applying all possible braid moves, one can verify that each expression above is a link and that these six reduced expressions comprise a single braid class.

The following result is a consequence of Lemmas 1 and 5 in [20] and completely characterizes the links in Coxeter systems of type \( A_n \).

Proposition 4.2. In the Coxeter system of type \( A_n \), a reduced expression is a link if and only if it is a string.

One consequence of the previous proposition is that every reduced expression for a nonidentity group element in Coxeter systems of type \( A_n \) can be written uniquely as a product of strings. In [20], Zollinger refers to the link factorization of a reduced expression in terms of strings as the maximal string decomposition.

Given the structure of strings, we can completely describe the corresponding braid graphs with ease. The next result is a reformulation of Lemma 1 from [20] in terms of braid graphs.

Proposition 4.3. The braid graph for the string \( \sigma_{l,k,m,\epsilon} \) is given by

\[
B(\sigma_{l,k,m,\epsilon}) = \begin{array}{cccc}
\sigma_{0,m,\epsilon} & \sigma_{1,m,\epsilon} & \cdots & \sigma_{l-1,m,\epsilon}
\end{array}
\]

Recall that Corollary 3.10 says that the braid graph for any reduced expression for a nonidentity group element can be decomposed as the box product of the braid graphs for the corresponding factors in the link factorization. In light of Proposition 4.3 we can be more explicit for Coxeter systems of type \( A_n \), as the next result indicates. This result is also a reformulation of Corollary 6 in [20] and can be thought of as a classification of braid graphs for reduced expressions in Coxeter systems of type \( A_n \).
Proposition 4.4. If $\alpha$ is a reduced expression for $w \in W(A_n)$ with link factorization $\alpha_1 | \alpha_2 | \cdots | \alpha_k$ such that each link has $2l_i - 1$ letters, then

\[
B(\alpha) \cong \begin{pmatrix}
\vdots & \vdots & \vdots \\
l_1 & \square & \vdots \\
l_2 & \cdots & \square \\
l_k & & \\
\end{pmatrix},
\]

where the $i$th factor in the decomposition is a path with $l_i$ vertices.

![Figure 6. The decomposition of the braid graph for the reduced expression in Example 4.5.](image)

Example 4.5. The word $\alpha = 12143456576$ is a reduced expression for some element in $W(A_7)$. The link factorization for $\alpha$ is $121 | 434 | 56576$. The resulting braid graph and its decomposition into a product of paths is shown in Figure 6.

5. Braid Graphs as Subgraphs of Hypercubes

For a positive integer $m$ we will denote the set of binary strings of length $m$ by $\{0,1\}^m$. That is,

\[
\{0,1\}^m := \{a_1a_2\cdots a_m \mid a_k \in \{0,1\}\}.
\]

Define the hypercube graph of dimension $m$, denoted by $Q_m$, to be the graph whose vertices are elements of $\{0,1\}^m$ with two binary strings connected by an edge exactly when they differ by a single digit.

If $G$ is a graph, let $V(G)$ denote the vertex set of $G$. An embedding of a simple graph $G$ into a simple graph $H$ is an injection $f : V(G) \to V(H)$ with the property that if $u$ and $v$ are adjacent vertices in $G$, then $f(u)$ and $f(v)$ are adjacent in $H$. If in addition, $f(u)$ and $f(v)$ adjacent in $H$ implies $u$ and $v$ are adjacent in $G$, then we say that $f$ is an induced embedding. In the graph theory literature, an embedding is often referred to as a monomorphism while an induced embedding is a faithful monomorphism. If $f$ is an induced embedding, then $G$ is isomorphic to the subgraph of $H$ induced by the image of $f$.

Assume $(W,S)$ be a simply-laced triangle-free Coxeter system and suppose $\alpha$ is a reduced expression for some $w \in W$ with link factorization $\alpha_1 | \alpha_2 | \cdots | \alpha_k$. The goal of this section is to establish an induced embedding of $B(\alpha)$ into $Q_{\text{rank}(\alpha)}$ (see Theorem 5.8). In particular, for simply-laced triangle-free Coxeter systems, we will show that the braid graph for every reduced expression is isomorphic to an induced subgraph of a hypercube graph having dimension determined by the number of braid shadows. We will utilize many of the properties of braid equivalent reduced expressions and links that were developed in Section 3.

We begin with several technical lemmas that will be useful in the the proof of Proposition 5.6.

Lemma 5.1. Suppose $(W,S)$ is a simply-laced triangle-free Coxeter system. If $\alpha$ is a reduced expression for some $w \in W$ and $[i,i+2],[i+2,i+4] \in S([\alpha])$, then $\text{supp}_{i+2}([\alpha]) = \text{supp}_{i+1}([\alpha]) \cup \text{supp}_{i+3}([\alpha])$.

Proof. According to Proposition 3.15 and Corollary 3.16, we may assume that $\text{supp}_{i+1}([\alpha]) = \{s,t\}$ and $\text{supp}_{i+3}([\alpha]) = \{t,u\}$. It is clear that $\{s,t,u\} \subseteq \text{supp}_{i+2}([\alpha])$. Now, let $v \in \text{supp}_{i+2}([\alpha])$ and consider the set

\[
X = \{\beta \in [\alpha] \mid \text{supp}_{i+2}([\beta]) = \{v\}\}.
\]

Since $\text{supp}_{i+2}([\alpha]) \neq \{v\}$, there exists $\beta \in X$ such that either $[i,i+2] \in S(\beta)$ or $[i+2,i+4] \in S(\beta)$. Then $v \in \{s,t\}$ or $v \in \{t,u\}$ according to Proposition 5.13. This proves the claim. $\square$
Lemma 5.2. Suppose \((W, S)\) is a simply-laced triangle-free Coxeter system and let \(\alpha = s_{x_1}s_{x_2} \cdots s_{x_m}\) be a link.

(a) If \(\text{supp}_2([\alpha]) = \{s, t\}\) and \(s_{x_2} = s\), then \(s_{x_1} = t\).
(b) If \(\text{supp}_{m-1}([\alpha]) = \{s, t\}\) and \(s_{x_{m-1}} = s\), then \(s_{x_m} = t\).

Proof. Suppose \(s_{x_2} = s\) and \(s_{x_1} = u\). Note that \(u \neq s\) since \(\alpha\) is reduced. Now, consider the set 
\[X = \{\beta \in [\alpha] | [1, 3] \in S(\beta) \text{ and } u \in \text{supp}_{[1, 3]}(\beta)\} .\]
Since \(\text{supp}_1([\alpha]) = \{u\}\), \(X\) is nonempty. Choose any \(\beta \in X\). Then \(\text{supp}_{[1, 3]}(\beta) = \{s, t\}\) by Proposition 5.13. Hence, \(u = t\). This proves part (a). Part (b) follows from a symmetric argument.

Lemma 5.3. Suppose \((W, S)\) is a simply-laced triangle-free Coxeter system and let \(\alpha = s_{x_1}s_{x_2} \cdots s_{x_m}\) be a link such that such that \(\text{supp}_{2k}([\alpha]) = \{s, t\}\) and \(\text{supp}_{2k+2}([\alpha]) = \{t, u\}\). If \(s_{x_{2k}} \in \{s, t\}\) and \(s_{x_{2k+2}} \in \{t, u\} \setminus \{s_{x_{2k}}\}\), then \(s_{x_{2k+1}} \in \{s, t, u\} \setminus \{s_{x_{2k}}, s_{x_{2k+2}}\}\).

Proof. Suppose \(s_{x_{2k}} \in \{s, t\}\) and \(s_{x_{2k+2}} \in \{t, u\} \setminus \{s_{x_{2k}}\}\). According to Lemma 5.1 we have \(\text{supp}_{2k+1}([\alpha]) = \{s, t, u\}\). Hence, \(s_{x_{2k+1}} \in \{s, t, u\} \setminus \{s_{x_{2k}}, s_{x_{2k+2}}\}\), otherwise \(\alpha\) would not be reduced.

Our final lemma has the important conclusion that every link is uniquely determined by the generators appearing at the even indices of the word. Moreover, every subword that begins and ends at an even index is uniquely determined by the even-index generators that are contained in the subword.

Lemma 5.4. Suppose \((W, S)\) is a simply-laced triangle-free Coxeter system and let \(\alpha = s_{x_1}s_{x_2} \cdots s_{x_m}\) and \(\beta = y_1y_2 \cdots y_m\) be two braid equivalent links. Then

(a) \(\alpha_{2k, 2k+2j} = \beta_{2k, 2k+2j}\) if and only if \(s_{x_{2k+2i}} = s_{y_{2i+2}}\) for all \(0 \leq i \leq j\),
(b) \(\alpha = \beta\) if and only if \(s_{x_{2j}} = s_{y_{2j}}\) for all \(1 \leq j \leq \frac{m-1}{2}\).

Proof. The forward implication of part (a) is immediate. Conversely, assume that \(s_{x_{2k+2i}} = s_{y_{2i+2}}\) for all \(0 \leq i \leq j\). The fact that \(s_{x_{2k+2i+1}} = s_{y_{2i+2+1}}\) for \(0 \leq i \leq j-1\) follows from Lemma 5.3 since \(\text{supp}_{2k+2}([\alpha]) = \text{supp}_{2k+2}([\beta])\). This shows that \(\alpha_{2k, 2k+2j} = \beta_{2k, 2k+2j}\). Part (b) follows by applying part (a) to the subwords \(\alpha_{2, m-1}\) and \(\beta_{2, m-1}\) and then using Lemma 5.2.

The first step to proving Theorem 5.8 is to construct an induced embedding for the braid graph for a link.

Definition 5.5. Assume that \((W, S)\) is a simply-laced triangle-free Coxeter system and let \(\alpha\) be a link of rank \(m\). For each \(k = 1, 2, \ldots, m\), assume that \(\text{supp}_{2k}([\alpha]) = \{s_{2k}, t_{2k}\}\), where \(s_{2k} := \text{supp}_{2k}([\alpha])\). Define \(\Phi_\alpha : [\alpha] \rightarrow \{0, 1\}^m \) via \(\Phi_\alpha(\beta) = a_1a_2 \cdots a_m\) when \(m \geq 1\), where 
\[a_k = \begin{cases} 0, & \text{if } \text{supp}_{2k}(\beta) = s_{2k} \\ 1, & \text{if } \text{supp}_{2k}(\beta) = t_{2k}. \end{cases} \]

In the case that \(m = 0\), the unique element (consisting of a single letter) in the braid class is mapped to the empty binary string.

In the previous definition, note that each \(s_{2k}\) and each \(t_{2k}\) are elements of \(S\). However, the index \(2k\) is specifying the generator’s position in \(\alpha\) rather than indicating a specific generator of \(S\). It is worth pointing out that the definition of the map \(\Phi_\alpha\) depends on the choice of link. Choosing a different representative of \([\alpha]\) will necessarily result in a different mapping.

We now prove the following proposition, which shows that the braid graph for every link is isomorphic to an induced subgraph of a hypercube graph. In Section 5, we will show that some braid classes contain a special choice of representative with nice recursive properties that allows us to prove a stronger version of Proposition 5.6.

Proposition 5.6. If \((W, S)\) is a simply-laced triangle-free Coxeter system and \(\alpha\) is a link of rank \(m\), then the map \(\Phi \equiv \Phi_\alpha\) given in Definition 5.5 is an induced embedding of \(B(\alpha)\) into \(Q_m\).

Proof. The fact that \(\Phi\) is injective is an immediate consequence of Lemma 5.3. Now, suppose \(\beta, \gamma \in [\alpha]\) such that \(\Phi(\beta) = a_1a_2 \cdots a_m\) and \(\Phi(\gamma) = b_1b_2 \cdots b_m\). We need to show that \(\beta, \gamma \in [\alpha]\) are adjacent in \(B(\alpha)\) if and only if \(\Phi(\beta), \Phi(\gamma) \in \Phi([\alpha])\) are adjacent in \(Q_m\).
For the forward implication, assume that $\beta, \gamma$ are adjacent. Then $\beta$ and $\gamma$ are related via a single braid move. Choose the unique $i \in \{1, m\}$ such that $\text{supp}_{2i}(\beta) \neq \text{supp}_{2i}(\gamma)$. Then $a_k = b_k$ for all $k \neq i$ and $a_i \neq b_i$. That is, $\Phi(\beta) = a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_m$ while $\Phi(\gamma) = a_1a_2\cdots a_{i-1}b_ia_{i+1}\cdots a_m$. Hence, $\Phi(\beta)$ and $\Phi(\gamma)$ are adjacent in $Q_m$.

Conversely, assume that $\Phi(\beta), \Phi(\gamma)$ are adjacent in $Q_m$. Choose the unique $i \in \{1, m\}$ such that $a_i \neq b_i$. Without loss of generality, assume that $a_i = 0$ while $b_i = 1$. By definition of $\Phi$, we have

$$\text{supp}_{2i}(\beta) = \{s_{2i}\} \text{ and } \text{supp}_{2i}(\gamma) = \{t_{2i}\}$$

while $\text{supp}_{2k}(\beta) = \text{supp}_{2k}(\gamma)$ for all $k \neq i$. According to Lemma 5.2 we can conclude that $\text{supp}_1(\beta) = \text{supp}_1(\gamma)$ and $\text{supp}_{2m+1}(\beta) = \text{supp}_{2m+1}(\gamma)$. Applying this fact in tandem with Lemma 5.4 allows us to conclude that $\beta_{1,2i-2} = \gamma_{1,2i-2}$ and $\beta_{2i+2,2m+1} = \gamma_{2i+2,2m+1}$. At this point, we only need to show that $[2i - 1, 2i + 1] \in S(\beta) \cap S(\gamma)$. We know that $\beta_{2i-2,2i+2} = x_{2i-2}u_{2i+2}x_{2i+2}$ and $\gamma_{2i-2,2i+2} = x_{2i-2}y_{2i+2}x_{2i+2}$ for some $u, v, y, z \in S$ where $x_{2i-2} \in \{s_{2i-2}, t_{2i-2}\}$ and $x_{2i+2} \in \{s_{2i+2}, t_{2i+2}\}$. We will show that $y = z = t_{2i}$ and $u = v = s_{2i}$ which will prove the claim. First of all, according to Corollary 3.11 we have $\text{card}(\{s_{2i-2}, t_{2i-2}\} \cap \{s_{2i}, t_{2i}\}) = 1$. So, we may assume without loss of generality that $t_{2i-2} = s_{2i}$. Similarly, $\text{card}(\{s_{2i-2}, t_{2i-2}\} \cap \{s_{2i+2}, t_{2i+2}\}) = 1$, and so without loss of generality, either $t_{2i+2} = s_{2i}$ or $t_{2i+2} = t_{2i}$. This leads us to consider several cases:

1. Suppose $t_{2i-2} = s_{2i}$ and $t_{2i+2} = s_{2i}$. Then $x_{2i-2} \in \{s_{2i-2}, s_{2i}\}$ and $x_{2i+2} \in \{s_{2i-2}, s_{2i}, s_{2i+2}\}$ so there are four subcases.

   a) If $x_{2i-2} = s_{2i-2}$ and $x_{2i+2} = s_{2i+2}$, then $\beta_{2i-2,2i+2} = s_{2i-2}u_{2i+2}s_{2i+2}$ and $\gamma_{2i-2,2i+2} = s_{2i-2}y_{2i+2}s_{2i+2}$.

   In this case, Lemma 5.3 implies that $u = v = t_{2i}$ and $y = z = s_{2i}$.

   b) If $x_{2i-2} = s_{2i-2}$ and $x_{2i+2} = s_{2i}$, then $\beta_{2i-2,2i+2} = s_{2i-2}u_{2i}s_{2i}$ and $\gamma_{2i-2,2i+2} = s_{2i-2}u_{2i+2}s_{2i}$. According to Proposition 5.1 $u \in \{s_{2i}, s_{2i-2}, t_{2i}\}$. If $u = s_{2i}$ or if $u = s_{2i+2}$ and $m(s_{2i}, s_{2i+2}) = 2$, then $\beta$ is not reduced. If $u = t_{2i}$ or if $u = s_{2i+2}$ and $m(s_{2i}, s_{2i+2}) = 3$, then $[2i, 2i + 2] \in S(\beta)$, contrary to Proposition 5.5. Thus, this case is impossible.

   c) If $x_{2i-2} = s_{2i}$ and $x_{2i+2} = s_{2i+2}$, then $\beta_{2i-2,2i+2} = s_{2i}u_{2i}s_{2i+2}$. This case is dismissed by considering the possibilities for $u$ and using an argument similar to Case 1(b).

   d) If $x_{2i-2} = s_{2i}$ and $x_{2i+2} = s_{2i}$, then $\beta_{2i-2,2i+2} = s_{2i}u_{2i}s_{2i+2}$. This is impossible by applying an argument as in Case 1(b) or Case 1(c).

2. Suppose $t_{2i-2} = s_{2i}$ and $t_{2i+2} = t_{2i}$. This implies that $x_{2i-2} \in \{s_{2i-2}, s_{2i}\}$ and $x_{2i+2} \in \{s_{2i+2}, t_{2i}\}$. Then there are four subcases, two of which are identical to Cases 1(a) and 1(c). The two additional subcases are as follows.

   a) If $x_{2i-2} = s_{2i-2}$ and $x_{2i+2} = t_{2i}$, then $\gamma_{2i-2,2i+2} = s_{2i-2}y_{2i+2}z_{2i}$. An argument similar to Case 1(a) shows that this case is not possible.

   b) If $x_{2i-2} = s_{2i}$ and $x_{2i+2} = t_{2i}$, then $\beta_{2i-2,2i+2} = s_{2i}u_{2i+2}v_{2i}$. Once again, use the argument from Case 1(a) to see that this is not possible.

We have shown that Case 1(a) is the only possibility and the conclusion of this case is precisely that $[2i - 1, 2i + 1] \in S(\beta) \cap S(\gamma)$. This shows that $\Phi(\beta)$ is adjacent to $\Phi(\gamma)$, which completes the proof. □

**Example 5.7.** Consider the reduced expressions $\gamma_1, \ldots, \gamma_5$ shown in Example 2.6(c). The braid graph $B(\gamma_1)$ was shown in Figure 3(c). In Example 5.7, we showed that $\gamma_1$ is a link of rank 3. Proposition 5.6 guarantees that there are at least five distinct embeddings of $B(\gamma_1)$ into $Q_3$, one for each of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and $\gamma_5$. One possible embedding, using $\gamma_4$, is shown in Figure 7.

![Figure 7. The induced embedding of $B(\gamma_1)$ into $Q_3$ as in Example 5.7.](image-url)
A graph is called cubical if it is isomorphic to an induced subgraph of a hypercube. Using this terminology, Proposition 5.6 says that the braid graph for every link in a simply-laced triangle-free Coxeter system is cubical. In light of Corollary 5.10, we can construct an induced embedding for any reduced expression in simply-laced triangle-free Coxeter systems by applying Proposition 5.6 to each link factor and then concatenating the resulting binary strings. This yields the main result of the section.

**Theorem 5.8.** If \((W, S)\) is a simply-laced triangle-free Coxeter system and \(\alpha\) is a reduced expression for some nonidentity \(w \in W\), then there is an induced embedding of \(B(\alpha)\) into \(Q_{\text{rank}(\alpha)}\). That is, \(B(\alpha)\) is cubical.

**Proof.** Suppose \(\alpha\) has link factorization \(\alpha_1 | \alpha_2 | \cdots | \alpha_k\). For each \(i\), suppose \(\text{rank}(\alpha_i) = m_i\). Note that \(\text{rank}(\alpha) = m_1 + m_2 + \cdots + m_k\) by Proposition 3.9. For each \(i\), form the induced embedding \(\Phi_{\alpha_i} : B(\alpha_i) \to Q_{m_i}\) according to Proposition 5.6. Define the induced embedding

\[
\Phi : B(\alpha_1) \sqcup B(\alpha_2) \sqcup \cdots \sqcup B(\alpha_k) \to Q_{m_1} \sqcup Q_{m_2} \sqcup \cdots \sqcup Q_{m_k}
\]

via

\[
(\beta_1, \beta_2, \ldots, \beta_k) \mapsto (\Phi_{\alpha_1}(\beta_1), \Phi_{\alpha_2}(\beta_2), \ldots, \Phi_{\alpha_k}(\beta_k)).
\]

According to Corollary 3.10, there is an isomorphism of graphs

\[
\Psi : B(\alpha) \to B(\alpha_1) \sqcup B(\alpha_2) \sqcup \cdots \sqcup B(\alpha_k).
\]

It is well known that there is an isomorphism of graphs

\[
\iota : Q_{m_1} \sqcup Q_{m_2} \sqcup \cdots \sqcup Q_{m_k} \to Q_{\text{rank}(\alpha)}.
\]

Thus, an induced embedding \(B(\alpha) \to Q_{\text{rank}(\alpha)}\) is given via the composition \(\iota \circ \Phi \circ \Psi\). \(\square\)

Since the number of vertices in \(Q_{\text{rank}(\alpha)}\) is \(2^{\text{rank}(\alpha)}\), we obtain the following corollary.

**Corollary 5.9.** If \((W, S)\) is a simply-laced triangle-free Coxeter system and \(\alpha\) is a reduced expression for some nonidentity \(w \in W\), then \(\text{card}(\{w\}) \leq 2^{\text{rank}(\alpha)}\). Moreover, if \(\alpha = \alpha_1 | \alpha_2 | \cdots | \alpha_k\), then the bound is achieved precisely when \(\text{rank}(\alpha_i) \leq 1\) for every \(i\).

If \((W, S)\) is a simply-laced Coxeter system and \(w \in W\) such that \(\ell(w) = m\), then the maximum number of braid shadows that any reduced expression for \(w\) can have is \(\lfloor \frac{m+1}{2} \rfloor\). For simply-laced Coxeter systems, it is clear that as the length increases, so too does the number of possible braid shadows. For finite Coxeter groups, the length function is bounded above by the length of the longest element. It turns out that all of the finite simply-laced Coxeter systems are triangle free [10]. Moreover, every finite simply-laced Coxeter system is isomorphic to a direct product of some combination of Coxeter systems of types \(A_n\) \((n \geq 1)\), \(D_n\) \((n \geq 4)\), \(E_6\), \(E_7\), and \(E_8\). For each of these groups, we can utilize Corollary 5.9 to obtain an upper bound on the cardinality of any braid class in that group.

Since the length of the longest element in the Coxeter system of type \(A_n\) is \(\frac{(n+1)n}{2}\), the maximum number of braid shadows that a reduced expression for an element in \(W(A_n)\) could have is \(\lfloor \frac{n^2+n-2}{4} \rfloor\). This bound is attained at least when \(n = 1, 2, 3\). Nonetheless, Corollary 5.9 implies that if \(\alpha\) is a reduced expression for some nonidentity element in \(W(A_n)\), then \(\text{card}(\{\alpha\}) \leq 2^{\lfloor \frac{n^2+n-2}{4} \rfloor}\). For example, if \(n = 3\), then the cardinality of every braid class in \(W(A_3)\) must be less than or equal to 4. However, the maximum size of a braid class in \(W(A_3)\) is 3. As \(n\) increases, our bound deteriorates. In [20], Zollinger establishes sharp upper bounds on the cardinality of a braid class for a fixed length across all Coxeter systems of type \(A_n\).

In the case of type \(D_n\), the longest element has length \(n^2 - n\), and so the maximum number of braid shadows that a reduced expression for an element in \(W(D_n)\) could have is \(\lfloor \frac{n^2+n-2}{2} \rfloor\). Thus, if \(\alpha\) is a reduced expression for some nonidentity element in \(W(D_n)\), then \(\text{card}(\{\alpha\}) \leq 2^{\lfloor \frac{n^2+n-2}{2} \rfloor}\).

**6. Fibonacci links and Fibonacci cubes**

We can view any connected graph \(G\) as a metric space by taking the standard geodesic metric. That is, the distance between \(u, v \in V(G)\) is defined via

\[
d_G(u, v) := \text{length of any minimal path between } u \text{ and } v.
\]
Note that in the case of the hypercube graph $Q_n$, the geodesic distance agrees with the Hamming distance. An isometric embedding of a graph $G$ into a graph $H$ is a function $f : V(G) \to V(H)$ with the property that $d_G(u, v) = d_H(f(u), f(v))$ for all $u, v \in V(G)$. In this case, $G$ is isometric to an induced subgraph of $H$. Since an isometry is injective and two vertices are adjacent if and only if the distance between them is one, every isometric embedding is also an induced embedding. It is worth mentioning that an induced embedding is not necessarily an isometric embedding. As an example, consider the embedding of a path with three vertices into a cycle with five vertices. This embedding is an induced embedding but is not an isometric embedding.

We conjecture that the induced embedding given in Theorem 5 is actually an isometric embedding and we will prove this claim for a special class of links that are closely related to the Fibonacci numbers. We introduce the Fibonacci links below.

**Definition 6.1.** Suppose $(W, S)$ is a simply-laced triangle-free Coxeter system. If $\varphi$ is a link with the property that $S([\varphi]) = S(\varphi)$, then we will refer to $\varphi$ as a Fibonacci link and the corresponding braid class $[\varphi]$ will be referred to as a Fibonacci chain.

The following proposition describes the connection between a Fibonacci chain and the Coxeter graph. A graph is defined to be the complete bipartite graph $K_{1,k}$.

**Proposition 6.2.** Suppose $(W, S)$ is a simply-laced triangle-free Coxeter system of type $\Gamma$. A link $\alpha$ of rank $m$ is braid equivalent to a Fibonacci link if and only if the subgraph of $\Gamma$ induced by $\text{supp}(\alpha)$ (equivalently any $\beta \in [\alpha]$) is a star graph.

**Proof.** Suppose that $\varphi \in [\alpha]$ is a Fibonacci link of rank $m$ so that $\varphi = s_{x_1}s_{x_2} \cdots s_{x_{2m+1}}$. By definition, $[2k-1, 2k+1] \in S(\varphi)$ for all $k = 1, 2, \ldots, m$ which implies that the letters in each subword $\varphi_{2k-1,2k+1} = s_{x_{2k-1}}s_{x_{2k}}s_{x_{2k+1}}$ satisfy $s_{x_{2k-1}} = s_{x_{2k+1}}$ and $m(s_{x_{2k-1}}, s_{x_{2k}}) = 3$. If we write $s = s_{x_{2k+1}}$ for all $k \in \{1, 2, \ldots, m\}$ so that $\text{supp}(\varphi) = \text{supp}(\varphi) = \{s, s_{x_2}, s_{x_4}, \ldots, s_{x_{2m}}\}$. Notice that $m(s_{x_2}, s_{x_{2i}}) = 2$ whenever $i \neq j$ since $m(s_{x_2}, s_{x_{2i}}) = 3 = m(s_{x_2}, s_{x_{2j}})$ and $\Gamma$ has no three-cycles. Thus, the subgraph of $\Gamma$ induced by $\text{supp}(\alpha)$ is a star graph.

We prove the converse by induction. If $\alpha$ is a link of rank 1, then $\alpha = stu$ for some $s, t \in S$ with $m(s, t) = 3$. In this case, $\alpha$ is a Fibonacci link. Now, assume that the statement holds for links of rank $m \geq 1$ and let $\beta$ be a link of rank $m+1$ such that the subgraph $\Gamma_{\text{supp}(\alpha)}$ of $\Gamma$ induced by the support of $\alpha$ is a star graph. Then we may assume that $\text{supp}(\alpha) = \{s, t_1, t_2, \ldots, t_k\}$ with $m(s, t_i) = 3$ for all $i$ and $m(t_i, t_j) = 2$ for all $i \neq j$. Choose a link $\beta \in [\alpha]$ with the property that $[2m-1, 2m+3] \in S(\beta)$. Then $\beta_{2m-1, 2m+3} = s_{tx_s}st_{x_s}$ for some $i \neq j$. Then the subword $\beta' := \beta_{2m-1, 2m+1}$ is a link of rank $m$ with $\text{supp}(\beta') \subseteq \text{supp}(\beta)$ so that $\Gamma_{\text{supp}(\beta')}$ is also a star graph. By the inductive hypothesis, we may choose a Fibonacci link $\hat{\varphi} \in [\beta']$. Then $\varphi := \hat{\varphi}_{tx_s}$ is a Fibonacci link of rank $m+1$ that is clearly braid equivalent to $\beta$ and thus braid equivalent to $\alpha$. □

Implicit in the proof of the preceding proposition is the fact that every Fibonacci link of rank $m$ can be written in the form

$$\varphi = s_{tx_1}st_{x_2} \cdots s_{x_{m-1}}st_{x_m}s$$

for some $s, t_{x_1}, t_{x_2}, \ldots, t_{x_m} \in S$ with $m(s, t_{x_i}) = 3$ for all $i = 1, 2, \ldots, m$ and $m(t_{x_i}, t_{x_j}) = 2$ for $i \neq j$. We will use this fact frequently in what follows.

We now construct several examples and nonexamples of Fibonacci links in some simply-laced triangle-free Coxeter systems.

**Example 6.3.** In the Coxeter system of type $A_n$, there are no Fibonacci links of rank 3 or larger. Every Fibonacci link of rank less than 3 can be described in terms of strings as follows. The Fibonacci links of rank 0 are precisely the strings $s_{\sigma_{1,0,m,0}}$ for $m \in [1, n]$. The Fibonacci links of rank 1 are the strings $s_{\sigma_{2,0,m,0}}$ and $s_{\sigma_{2,1,m,0}}$ for $m \in [1, n-1]$. Lastly, the Fibonacci links of rank 2 are the strings $s_{\sigma_{3,1,m,e}}$ for $m \in [1, n-2]$ and $e \in \{+, -\}$.

It is more difficult to write down all the Fibonacci links in the Coxeter system of type $D_n$. The following reduced expressions are Fibonacci links of ranks 1 through 5:

$$343, 34313, 3431323, 343132343, 34313234313.$$
In fact, it is not possible to find a Fibonacci link of rank greater than 5 in type $D_n$. This fact is obvious in type $D_4$ since the longest element has length 12, but remains true even for large $n$. Notice that Proposition 6.2 implies that every link in type $D_4$ is braid equivalent to a Fibonacci link.

**Example 6.4.** Not every link is a Fibonacci link. For instance, in the Coxeter system of type $A_4$, the string $s_1s_0s_1 = 1213243$ is a link, but not a Fibonacci link. In the Coxeter system of type $D_5$, the reduced expression $45343132434313$ is a link, but it is not a Fibonacci link and it is not braid equivalent to a Fibonacci link according to Proposition 6.2.

We now introduce some notation that will be used in the remainder of the section.

**Definition 6.5.** Suppose $(W, S)$ is a simply-laced triangle-free Coxeter system. Consider the Fibonacci link $\varphi = st_{x_1}st_{x_2} \cdots st_{x_{m-1}} st_{x_m}$ of rank $m \geq 2$. We define the following reduced expressions associated to $\varphi$:

(i) $\hat{\varphi} := st_{x_1}st_{x_2} \cdots st_{x_{m-1}} t_{x_m} st_{x_m}$ (obtained by applying a braid move in rightmost braid shadow of $\varphi$),

(ii) $\hat{\varphi}_0 := \varphi 1,2m-1 = st_{x_1} st_{x_2} \cdots st_{x_{m-1}}s,$

(iii) $\hat{\varphi}_1 := \psi 1,2m-3 = st_{x_1} st_{x_2} \cdots st_{x_{m-2}}s.$

We define the following maps associated with the reduced expressions $\hat{\varphi}, \hat{\psi}$:

(iv) denote by $\Omega_{\hat{\varphi}} : [\hat{\varphi}] \to S^*$ the map that adjoins to each element of $[\hat{\varphi}]$ the letters $t_{x_m}s$ on the right,

(v) denote by $\Omega_{\hat{\psi}} : [\hat{\psi}] \to S^*$ the map that adjoins to each element of $[\hat{\psi}]$ the letters $t_{x_{m-1}} t_{x_m} st_{x_m}$ on the right.

Each Fibonacci link carries with it rich recursive properties that manifest in the structure of the braid class and the corresponding braid graph. The next result summarizes these properties.

**Lemma 6.6.** Suppose $(W, S)$ is a simply-laced triangle-free Coxeter system. Define the reduced expressions $\varphi, \psi, \hat{\varphi}, \hat{\psi}$ and the maps $\Omega_{\varphi}, \Omega_{\psi}$ as in Definition 6.5. Then

(a) The reduced expressions $\hat{\varphi}$ and $\hat{\psi}$ are Fibonacci links of rank $m-1$ and $m-2$, respectively.

(b) The set $\{\text{im}(\Omega_{\varphi}), \text{im}(\Omega_{\psi})\}$ is a partition of $[\varphi]$. Then $\hat{\alpha} \in [\hat{\varphi}]$ such that $\alpha = \hat{\alpha} t_{x_m}s$. Choose a sequence of braid moves that transforms $\hat{\alpha}$ into $\hat{\psi}$. Then the same sequence transforms $\alpha = \hat{\alpha} t_{x_m}s$ so that $\alpha \in [\varphi]$. This shows that $\text{im}(\Omega_{\varphi}) \subseteq [\varphi]$. A similar argument shows that $\text{im}(\Omega_{\psi}) \subseteq [\varphi]$. Next, notice that the intersection $\text{im}(\Omega_{\varphi}) \cap \text{im}(\Omega_{\psi})$ is empty by definition of the maps $\Omega_{\varphi}$ and $\Omega_{\psi}$. Lastly, let $\alpha = s_{x_1}s_{x_2} \cdots s_{x_{2m-1}} \in \varphi$. According to Proposition 5.13 and the definition of $\varphi$ we have $s_{2m} = \{s_{t_{x_m}}\}$. If $s_{2m} = t_{x_m}$, then $s_{2m+1} = s$ by Lemma 5.2. In this case, we can transform $\alpha$ into $\varphi$ via a sequence of braid moves, each of which acts on positions within the interval $[1,2m-1]$. Hence, $\alpha_1, 2m-1 \in [\varphi]$ and $\Omega_{\varphi}(\alpha_1, 2m-1) = \alpha_1, 2m-1 t_{x_m} s = \alpha$ so that $\alpha \in \text{im}(\Omega_{\varphi})$. Otherwise $s_{2m} = s$ in which case it is easy to show that $\alpha \in \text{im}(\Omega_{\psi})$ with a similar argument. Since both sets are clearly non-empty, we have proved that $\{\text{im}(\Omega_{\varphi}), \text{im}(\Omega_{\psi})\}$ is a partition of $[\varphi]$.

To prove (b), first that $\text{im}(\Omega_{\varphi})$ and $\text{im}(\Omega_{\psi})$ are subsets of $[\varphi]$. Let $\alpha \in \text{im}(\Omega_{\varphi})$ and choose $\hat{\alpha} \in [\hat{\varphi}]$ such that $\alpha = \hat{\alpha} t_{x_m}s$. Choose a sequence of braid moves that transforms $\hat{\alpha}$ into $\hat{\psi}$. Then the same sequence transforms $\alpha = \hat{\alpha} t_{x_m}s$ so that $\alpha \in [\varphi]$. This shows that $\text{im}(\Omega_{\varphi}) \subseteq [\varphi]$. A similar argument shows that $\text{im}(\Omega_{\psi}) \subseteq [\varphi]$. Next, notice that the intersection $\text{im}(\Omega_{\varphi}) \cap \text{im}(\Omega_{\psi})$ is empty by definition of the maps $\Omega_{\varphi}$ and $\Omega_{\psi}$. Lastly, let $\alpha = s_{x_1}s_{x_2} \cdots s_{x_{2m-1}} \in \varphi$. According to Proposition 5.13 and the definition of $\varphi$ we have $s_{2m} = \{s_{t_{x_m}}\}$. If $s_{2m} = t_{x_m}$, then $s_{2m+1} = s$ by Lemma 5.2. In this case, we can transform $\alpha$ into $\varphi$ via a sequence of braid moves, each of which acts on positions within the interval $[1,2m-1]$. Hence, $\alpha_1, 2m-1 \in [\varphi]$ and $\Omega_{\varphi}(\alpha_1, 2m-1) = \alpha_1, 2m-1 t_{x_m} s = \alpha$ so that $\alpha \in \text{im}(\Omega_{\varphi})$. Otherwise $s_{2m} = s$ in which case it is easy to show that $\alpha \in \text{im}(\Omega_{\psi})$ with a similar argument. Since both sets are clearly non-empty, we have proved that $\{\text{im}(\Omega_{\varphi}), \text{im}(\Omega_{\psi})\}$ is a partition of $[\varphi]$.

To prove (c), note that both maps are injective by Lemma 5.4(b) and are surjective by definition. It is easy to see that $\hat{\alpha}, \hat{\beta} \in [\hat{\varphi}]$ are adjacent if and only if $\hat{\alpha} t_{x_m}s, \hat{\beta} t_{x_m}s$ are adjacent in $B(\varphi)$. The forward implication of this statement is immediate and converse follows from the fact that $\hat{\alpha} t_{x_m}s, \hat{\beta} t_{x_m}s \in [\varphi]$ are links. The analogous statement for $\Omega_{\psi}$ is similar to prove.

According to part (c) of Lemma 6.6, the braid graph for a Fibonacci link of rank $m$ is, in some sense, a gluing together of the braid graphs for a Fibonacci link of rank $m-1$ and a Fibonacci link of rank $m-2$. A concrete example is given below. As usual, we define the Fibonacci numbers recursively by $F_{n+1} = F_n + F_{n-1}$ with the initial condition $F_1 = F_2 = 1$.

**Example 6.7.** Let $(W(D_4), S(D_4))$ be the Coxeter system of type $D_4$. The reduced expression $\varphi = 34312343$ is a Fibonacci link of rank 4. We define $\psi = 34312344, \hat{\varphi} = 343123$, and $\hat{\psi} = 34313$ as in Definition 6.5. The partition defined in Lemma 6.6 is precisely

$$\{\{34312343, 34312344, 34312343, 341312343, 341312343\}, \{34312343, 34312344, 34312343, 341312434\}\}.$$
The first block is in bijection with \([\hat{\varphi}]\) and has \(F_5 = 5\) elements, while the second block is in bijection with \([\hat{\psi}]\) and has \(F_4 = 3\) elements. Each block of the partition induces a subgraph of \(B(\varphi)\), as depicted in Figure 8 where the dotted edges indicate the gluing of the two induced subgraphs. The subgraphs are isomorphic to \(B(\hat{\varphi})\) and \(B(\hat{\psi})\), respectively.

![Figure 8. A partition of the braid graph for a Fibonacci link of rank 4.](image)

Our next objective is to prove Proposition 6.8 which states that Fibonacci chains contain a Fibonacci number of links. The proof technique relies on the recursive structure described in Lemma 6.6.

**Proposition 6.8.** Suppose \((W,S)\) is a simply-laced triangle-free Coxeter system. If \(\varphi\) is a Fibonacci link of rank \(m \geq 0\), then \(\text{card}([\varphi]) = F_{m+2}\).

**Proof.** The proof is by induction on \(m\). The case that \(m = 0\) is trivially true. If \(m = 1\), then \(\varphi = st s\) for some \(s,t \in S\) with \(m(s,t) = 3\). In this case, \(\text{card}([\varphi]) = \text{card}([sts,tst]) = 2 = F_3\). Now, let \(m \in \mathbb{N}\) and assume that every Fibonacci link of rank \(k \leq m\) has \(F_{k+2}\) elements in its braid class. Let \(\varphi\) be a Fibonacci link of rank \(m + 1\). Define the reduced expression \(\phi, \hat{\varphi}, \hat{\psi}\) and the maps \(\Omega, \Omega_{\hat{\varphi}}, \Omega_{\hat{\psi}}\) as in Definition 6.5. Then \(\{\text{im}(\Omega_{\hat{\varphi}}), \text{im}(\Omega_{\hat{\psi}})\}\) is a partition of \([\varphi]\) according to Lemma 6.6b. Moreover, according to Lemma 6.6a and the inductive hypothesis, we obtain \(\text{card}([\text{im}(\Omega_{\hat{\varphi}})]) = F_{m+1}\) and \(\text{card}([\text{im}(\Omega_{\hat{\psi}})]) = F_m\). Then we have

\[
\text{card}([\varphi]) = \text{card}([\text{im}(\Omega_{\hat{\varphi}})]) + \text{card}([\text{im}(\Omega_{\hat{\psi}})]) = F_{m+1} + F_m = F_{m+2}.
\]

This proves the claim. \(\square\)

Next, we show that the braid graph for a Fibonacci link of rank \(m\) is isometric to a well-known induced subgraph of \(Q_m\) that is referred to as a Fibonacci cube. We define the Fibonacci cube of order \(m\) as the subgraph \(\mathcal{F}_m = Q_m[X_m]\) of \(Q_m\) induced by set of vertices

\[
X_m = \{a_1 a_2 \cdots a_m \in \{0,1\}^m : a_{i+1} = 0, 1 \leq i \leq m-1\}.
\]

That is, \(X_m\) is the collection of length \(m\) binary strings that do not contain the consecutive substring 11. Fibonacci cubes have been studied extensively in the literature. According to [12], the Fibonacci cube \(\mathcal{F}_m\) is partial cube with \(|X_m| = F_{m+2}\). Note that a graph is a partial cube if it is isometric to a subgraph of a hypercube. We will utilize the fact that \(\mathcal{F}_m\) has \(F_{m+2}\) many vertices in the proof of Theorem 6.9.

We introduce some notation that will be used in the following proof: let \(\Omega_{\hat{\varphi}}, \Omega_{\hat{\psi}}\) be the maps defined in Definition 6.5. Note that they are bijections onto their images according to Lemma 6.6c. Let \(\zeta \in [\varphi]\). If \(\zeta \in \text{im}(\Omega_{\hat{\varphi}})\), then we will use the notation \(\Omega_{\hat{\varphi}}^{-1}(\zeta)\) to denote the unique element that maps to \(\zeta\). The notation is similar when \(\zeta \in \text{im}(\Omega_{\hat{\psi}})\).

**Theorem 6.9.** Suppose \((W,S)\) is a simply-laced triangle-free Coxeter system. If \(\varphi\) is a Fibonacci link of rank \(m\), then the braid graph \(B(\varphi)\) is isomorphic to the Fibonacci cube \(\mathcal{F}_m\) of order \(m\).

**Proof.** We will use complete induction on \(m\) to prove two statements:

(a) \(\Phi_{\varphi} : [\varphi] \rightarrow \{0,1\}^m\) is an isometric embedding for all \(m \in \mathbb{N}\),

(b) \(\text{im}(\Phi_{\varphi}) \subseteq X_m\) for all \(m \in \mathbb{N}\).

Notice that (b) implies that \(\text{im}(\Phi_{\varphi}) = X_m\) since according to Theorem 6.8 we have \(\text{card}([\varphi]) = F_{m+2}\) and according to [12] we also have \(\text{card}(X_m) = F_{m+2}\). Together, (a) and (b) show that \(\Phi_{\varphi} : B(\varphi) \rightarrow \mathcal{F}_m\) is an isometry.
We now prove (a). If \( m = 0 \), the claim is trivially true. If \( m = 1 \), \( \varphi = \text{sts} \) for some \( s, t \in S \) with \( m(s, t) = 3 \). Then \( \{\varphi\} = \{\text{sts}, \text{tst}\} \) so that \( \Phi_\varphi(\text{sts}) = 0 \) and \( \Phi_\varphi(\text{tst}) = 1 \). Then \( d(\text{sts}, \text{tst}) = d_Q(0, 1) = d_Q(\{\Phi_\varphi(\text{sts}), \Phi_\varphi(\text{tst})\}) \). This proves that \( \Phi_\varphi \) is an isometric embedding. Now, let \( m \in \mathbb{N} \). Assume for all \( k \leq m \) that if \( \psi \) is a Fibonacci link of rank \( k \), then \( \Phi_\psi \) is an isometric embedding. Let \( \varphi = \text{sts}x_1\text{sts}x_2\cdots\text{sts}x_m \text{sts}x_{m+1} \) be a Fibonacci link of rank \( m + 1 \). Define the reduced expressions \( \psi, \tilde{\psi}, \tilde{\varphi} \) and the maps \( \Omega_\psi : [\tilde{\psi}] \to [\varphi] \) and \( \Omega_{\tilde{\psi}} : [\tilde{\psi}] \to [\psi] \) as in Proposition 6.5. Recall that according to Lemma 6.6(b), \( \{\text{im}(\Omega_\psi), \text{im}(\Omega_{\tilde{\psi}})\} \) is a partition of \( \{\varphi\} \). Since \( \tilde{\psi} \) is a Fibonacci link of rank \( m \), Lemma 6.6(a) together with the inductive hypothesis implies that the map \( \Phi_{\tilde{\psi}} : [\tilde{\psi}] \to \{0, 1\}^m \) is an isometric embedding of \( B(\tilde{\psi}) \) into \( Q_m \). Similarly, \( \tilde{\varphi} \) is a Fibonacci link of rank \( m - 1 \), the map \( \Phi_{\tilde{\varphi}} : [\tilde{\varphi}] \to \{0, 1\}^{m-1} \) is an isometric embedding of \( B(\tilde{\varphi}) \) into \( Q_{m-1} \).

Now, we can construct the map \( \Phi_\varphi : [\varphi] \to \{0, 1\}^{m+1} \) as follows. It is easy to see that \( \Phi_\varphi \) is given by the rule

\[
\Phi_\varphi(\zeta) = \begin{cases} 
\Phi_{\tilde{\psi}}(\Omega_{\tilde{\psi}}^{-1}(\zeta))0, & \text{if } \zeta \in \text{im}(\Omega_{\tilde{\psi}}) \\
\Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\zeta))01, & \text{if } \zeta \in \text{im}(\Omega_{\tilde{\varphi}}).
\end{cases}
\]

Now it is easy to show that \( \Phi_\varphi \) is an isometry. Let \( \alpha, \beta \in [\varphi] \). If \( \alpha, \beta \in \text{im}(\Omega_{\tilde{\varphi}}) \), then

\[
d(\Omega_{\tilde{\varphi}}^{-1}(\alpha), \Omega_{\tilde{\varphi}}^{-1}(\beta)) = d_{Q_m}(\Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\alpha)), \Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\beta)))
\]

since \( \Phi_{\tilde{\varphi}} \) is an isometric embedding. Applying this together with the definition of the map \( \Omega_{\tilde{\varphi}} \) we must have

\[
d(\alpha, \beta) = d(\Omega_{\tilde{\varphi}}(\alpha), \Omega_{\tilde{\varphi}}(\beta))
\]

\[
= d(\Omega_{\tilde{\varphi}}(\alpha)t_{x_m}s, \Omega_{\tilde{\varphi}}(\beta)t_{x_m}s)
\]

\[
= d(\Omega_{\tilde{\varphi}}(\alpha), \Omega_{\tilde{\varphi}}^{-1}(\beta))
\]

\[
= d_{Q_{m+1}}(\Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\alpha)), \Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\beta)))
\]

\[
= d_{Q_{m+1}}(\Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\alpha)), \Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\beta))).
\]

If \( \alpha, \beta \in \text{im}(\Omega_{\tilde{\varphi}}) \), then a similar argument using the fact that \( \Phi_{\tilde{\varphi}} \) is an isometry will show that

\[
d(\alpha, \beta) = d_{Q_m}(\Phi(\alpha), \Phi(\beta)).
\]

In the final case, we have \( \alpha \in \text{im}(\Omega_{\tilde{\varphi}}) \) while \( \beta \in \text{im}(\Omega_{\tilde{\varphi}}) \). Then we have

\[
d(\alpha, \beta) = d(\Omega_{\tilde{\varphi}}^{-1}(\alpha)t_{x_m}s, \Omega_{\tilde{\varphi}}^{-1}(\beta)t_{x_m}s)
\]

\[
= d(\Omega_{\tilde{\varphi}}^{-1}(\alpha)t_{x_m}s, \Omega_{\tilde{\varphi}}^{-1}(\beta)t_{x_m}s) + 1
\]

\[
= d(\Omega_{\tilde{\varphi}}^{-1}(\alpha), \Omega_{\tilde{\varphi}}^{-1}(\beta)) + 1
\]

\[
= d_{Q_m}(\Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\alpha)), \Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\beta))).
\]

The equality between (1) and (2) follows from the fact that we applied a single braid move that moved the two words closer together. The equality between (3) and (4) follows from the fact that \( \Phi_{\tilde{\varphi}} \) is an isometry. Lines (4) and (5) are equivalent since \( \tilde{\varphi} \) can be obtained from \( \varphi \) by deleting the rightmost letters \( t_{x_m}s \). Thus, we have proved that \( \Phi_\varphi \) is an isometry. This completes the proof of claim (a).

Now, we prove claim (b). We will use most of the setup developed in the preceding proof. The case that \( m = 0 \) is trivial and \( m = 1 \) holds since \( \varphi = \text{sts} \) so that \( \Phi_\varphi(\text{sts}) = 0 \) and \( \Phi_\varphi(\text{tst}) = 1 \) and \( 0, 1 \in X_1 \) trivially. Now let \( m \in \mathbb{N} \) and assume that for all \( k \leq m \) if \( \xi \) is a Fibonacci link of rank \( k \), then \( \text{im}(\Phi_\xi) \subseteq X_k \). Let \( \varphi \) be a Fibonacci link of rank \( m + 1 \) and let \( a_1a_2\cdots a_{m+1} \in \text{im}(\Phi_\varphi) \). Choose \( \zeta \in [\varphi] \) such that

\[
a_1a_2\cdots a_{m+1} = \Phi_\varphi(\zeta).
\]

If \( \zeta \in \text{im}(\Omega_{\tilde{\varphi}}) \), then \( \Phi_{\tilde{\varphi}}(\zeta) = \Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\zeta)) \) by the formulation of \( \Phi_{\tilde{\varphi}} \) from part (a) and \( a_1a_2\cdots a_{m+1} = \Phi_{\tilde{\varphi}}(\Omega_{\tilde{\varphi}}^{-1}(\zeta)) \in X_m \) by the inductive hypothesis. Thus, \( \Phi_{\varphi}(\zeta) = a_1a_2\cdots a_{m+1} \in X_{m+1} \) as well. The case that \( \zeta \in \text{im}(\Omega_{\tilde{\varphi}}) \) is similar, but will require the use of the inductive hypothesis for \( k = m - 1 \) instead. This completes the proof. \( \square \)
Example 6.10. Let \((W(D_4), S(D_4))\) be the Coxeter system of type \(D_4\). Consider the Fibonacci links from Example 6.3:

\[343, 34313, 3431323, 343132343, 34313234313.\]

According to Theorem 6.9, the corresponding braid graphs are Fibonacci cubes. Each braid graph is depicted in Figure 9. In each graph, the Fibonacci link always corresponds to the vertex of highest degree.

The Fibonacci cube is not the only interesting graph that arises as the braid graph for a reduced expression. For instance, the matchable Lucas cubes introduced [18] can be found in the Coxeter system of type \(D_5\) as seen in the following example. The matchable Lucas cubes are very similar to the Fibonacci cubes, for example, the number of vertices in the \(n\)-th matchable Lucas cube is equal to the \(n\)-th number \(L_n\) in the sequence of Lucas numbers defined by \(L_n = L_{n-1} + L_{n-2}\) with initial conditions \(L_0 = 2\) and \(L_1 = 1\).

Example 6.11. Consider the reduced expression 4534313234313 for \(w \in W(D_5)\) from Example 6.4. The braid graph, depicted in Figure 10(c), is isomorphic to the 6th matchable Lucas cube as introduced in [18]. In fact, it is not hard to see that the braid graph is a gluing together of the braid graphs for the reduced expressions 45343132343 and 453431323. The braid graph for 45343132343 is shown in Figure 10(b) and is isomorphic to the 5th matchable Lucas cube, while the braid graph for 453431323 in Figure 10(a) is isomorphic to the 4th matchable Lucas cube. The reader should compare this example with Lemma 6.6 and Example 6.7.

7. Closing

There are several open problems and potential natural generalizations that arise from this paper, which we outline below.

\[\begin{align*}
(a) & \ B(343) \cong F_3 \\
(b) & \ B(34313) \cong F_4 \\
(c) & \ B(3431323) \cong F_5 \\
(d) & \ B(343132343) \cong F_6 \\
(e) & \ B(34313234313) \cong F_7
\end{align*}\]

Figure 9. Braids graphs for several Fibonacci links in the Coxeter system \((W(D_4), S(D_4))\).

\[\begin{align*}
(a) & \ B(453431323) \\
(b) & \ B(45343132343) \\
(c) & \ B(4534313234313)
\end{align*}\]

Figure 10. Braid graphs for the reduced expressions in Example 6.11.
(1) Given a link in a simply-laced triangle-free Coxeter system, what is the image of the corresponding braid class under the map described in Definition 5.3?

(2) In Theorem 5.8 we show that every braid graph in a simply-laced triangle-free Coxeter system is cubical. Is this embedding isometric? If so, then every braid graph for a reduced expression in a simply-laced triangle-free Coxeter system is a partial cube. Moreover, can we drop the triangle-free hypothesis?

(3) Theorem 6.9 states that the braid graph for every Fibonacci link is isometric to a Fibonacci cube. In Example 6.10 we show that the Fibonacci cubes $F_1$, $F_2$, $F_3$, $F_4$, and $F_5$ are realized in the Coxeter system of type $D_4$. Is there a simply-laced triangle-free Coxeter system in which every Fibonacci cube is realized as the braid graph for a Fibonacci link?

(4) Provide a classification of braid graphs for links in other simply-laced triangle-free arbitrary Coxeter system (e.g., types $D_n$ and $D_n$).

(5) Provide a classification of braid graphs for links in arbitrary simply-laced Coxeter systems (e.g., type $A_n$).

(6) Extend our notion of braid shadow to non-simply-laced Coxeter systems and prove analogous results to those appearing in Sections 3, 5, and 10. For example, if one generalizes the notions of braid shadow and link in the natural way, we conjecture that a result analogous to Proposition 3.13 holds in arbitrary Coxeter systems as long as the corresponding Coxeter graph does not contain a triangle-cycle with edge weights $3, 3, m$, where $m \geq 3$.

(7) Say a subset $X \subseteq \{0, 1\}^m$ is admissible if there exists a Coxeter system $(W, S)$ and reduced expression $\alpha$ for $w \in W$ such that the induced subgraph $Q_n[X]$ is isomorphic to $B(\alpha)$. Are there necessary and sufficient conditions for $X$ to be admissible?

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