Limits on Threshold and "Sommerfeld" Enhancements in Dark Matter Annihilation

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We find model-independent upper limits on rates of dark matter annihilation in galactic halos. The Born approximation generally fails, while exotic threshold enhancements akin to "Sommerfeld factors" also turn out to be baseless. The most efficient annihilation mechanism involves perturbatively small decay widths that have largely been ignored. Widths that are very small compared to TeV mass scales suffice to cause large enhancements in the velocity averaged cross sections. Bound state formation in weakly coupled theories produces small effects due to wave function normalizations. Unitarity shows the Sommerfeld factor cannot produce large enhancements of cross sections, and serves to identify where those approximations break down.

I. THRESHOLD ENHANCEMENTS

There is great interest in recent data from the PAMELA [1], FERMI [2] and PPB-BETS [3] experiments. The observations suggest a significant signal in excess positron production in galactic halos, as long suggested by the HEAT [4] and ATIC [5] experiments. Possible explanations range from exotic mechanisms [6], uncertain features of pulsars [7], to dark matter decays [8] and dark matter annihilation [9].

In considering annihilation there are puzzles from comparing predictions of relic densities with rates of particle production in the current era. This has led to invoking more or less exotic threshold enhancements under the catch-phrase of "Sommerfeld factors" [10, 11].

One reason to appeal to a Sommerfeld factor is to boost cross sections of TeV-scale particles well above Born-level estimates. We find the starting point of Born-level cross sections is not a good approximation for much different reasons. Basic facts of finite width particle physics substantially revise estimates of annihilation rates in galactic halos. We find that annihilation of TeV-scale dark matter with typical electroweak couplings can actually saturate unitarity limits over the observable range. We also obtain upper limits to halo annihilation rates that do not depend on fine details of the dark matter velocity distribution.

Non-relativistic scattering amplitudes can be classified by their analytic properties in the complex momentum plane. Stable bound states are described by poles on the positive imaginary axis. It follows that stable bound states produce no remarkable enhancement of annihilation rates in the physical region of real momentum k. Metastable particles or resonances, described by poles of finite width, are in no way comparable with stable bound states, because everything observable (and potentially large) is a strong function of the width.

Unless one is considering an absolutely stable intermediate state, all intermediate states in particle physics have a finite lifetime. Pursuing the consequences of finite lifetimes with galactic halo kinematics re-directs attention from exotic mechanisms to ordinary physics. There are two salient cases. If the width of an intermediate annihilation state is limited by the initial state velocity v, then the peak of the cross section goes like 1/v^2. This case produces the largest reaction rates in halos, and most conservative bounds. If the width of the intermediate state is constant, the peak of the cross section goes like 1/v. In these and intermediate cases the peak cross section actually dominates the entire halo velocity distribution for a surprisingly broad range of dark matter parameters. As a result, our more conservative bounds merge smoothly with reasonable estimates predicting surprisingly large rates.

II. BREIT-WIGNER FORMULAS

Relic particles trapped in galactic halos will be non-relativistic, with velocities v ~ 10^-3. There are several distinctly different non-relativistic "Breit-Wigner" formulas. Most Breit-Wigner cross sections \( \sigma_{res} \) can be cast into the form

\[
\sigma_{res} = \frac{4\pi v^N}{k^2} \frac{(1/2)^2 B_i B_f}{(E - E_{res})^2 + (1/2)^2} = \frac{4\pi v^N}{k^2} BW(\Gamma, E_{res}).
\]

Here \( B_i \) and \( B_f \) are the branching fractions to the initial and final state, and \( k \) is the momentum of an initial state particle in the center of mass frame. Different values of the parameter \( N = 0, 1 \) distinguish two classic limits:

- **Phase Space Limited Case, N = 0**: It is common for 2 \( \rightarrow \) 2 non-relativistic physics to be quasi-elastic. In particular, the final state phase space may be severely limited by the initial state velocity v. Ignoring spin and matrix elements, the Lorentz-invariant phase space integral LIPS for two particles of momentum \( p_i, p_f \) and
mass $m_f$: is

$$LIPS = \int \frac{d^3p_f}{2p_f} \frac{d^3p_f}{2p_f} 5^4 (Q - p_f - p_f) \sqrt{1 - 4m_f^2/Q^2} = 2\pi \sqrt{1 - 4m_f^2/Q^2} = 2\pi v_f. \quad (2)$$

Here $v_f$ is the final state velocity of either particle in the CM frame. When initial and final state masses are comparable, and the 2-body states dominate, the total width $\Gamma \sim K v_f \sim K v$, where $K$ absorbs coupling constants and matrix elements. Incorporating the explicit velocity dependence with an $s$-channel propagator leads to Eq. 1 with $N = 0$. Note that the peak of the cross section scales like $1/(m^2 v^2)$, making this case potentially capable of saturating elastic unitarity bounds.

**Relativistic Phase Space Case, $N = 1$:** Annihilation may also proceed to final states which are ultrarelativistic. Then the square root in Eq. 2 approaches 1, and the partial width $\Gamma_f \sim constant$ in this limit. Any other kinematic situation where $Q^2/m_f^2$ goes to a finite constant as $v \to 0$ will produce the same outcome. This includes the “exoergic” resonances long known in low-energy nuclear physics, and associated with the “1/v law” of low energy cross sections. These cross sections do not increase as fast as unitarity would allow as $v \to 0$.

The difference between $1/v$ and $1/v^2$ velocity dependence is dramatic. Yet is only part of the story, because resonances may produce large cross sections either way. For example, neutron absorption cross sections on Gadolinium-157 exceeding one hundred million barns have been observed. This comes in the seemingly mild $1/v$ case not impinging on a unitarity limit. The experimental stunt simply exploits neutrons with grossly small velocities of order 3 meters per second. In much the same way, galactic halo velocities of order $10^{-3}$ are grossly small on the scale of particle physics. The combination of low speed halo kinematics and very ordinary widths produces surprisingly large enhancements.

### III. General Limits on the Velocity Averaged Breit-Wigner Cross Section

The halo annihilation rate via a single $s$-wave resonance is governed by the velocity-weighted cross section $\langle \sigma v \rangle_{res}$:

$$\langle \sigma v \rangle_{res} = \int dv \frac{4\pi v^N}{m_X^2 v^2} \times \frac{(\Gamma/2)^2 B_{k} B_f}{(m_X v^2/2 - m_X v^2_{res}/2)^2 + (\Gamma/2)^2} \Phi_{halo}(v).$$

Here $\Phi_{halo}(v) = dN/dv^3$ is the normalized dark matter relative velocity distribution, assumed from astrophysics to be a smooth function on the scale of 100-500 km/s. In an isothermal halo model the velocity distribution is in equilibrium,

$$\frac{dN}{d^3v} = \frac{constant}{E_0} e^{-E/2E_0}; \quad \frac{dN}{dv} = 4\pi v^2 \left( \frac{2\pi m_0^2}{3} \right)^{1/2} e^{-v^2/2v_0^2}. \quad (3)$$

While the actual velocity distribution is uncertain the phase space factors of $v^2$ are general. The isothermal halo will illustrate the method, but none of our upper bounds depend on it.

The rate $\langle \sigma v \rangle_{res}$ is a function of $E_0$, $E_{res}$, $\Gamma$ and $m_X$. If other scales are expressed in units of $m_X \sim TeV$ the conjunction of several rapidly varying functions making analysis troublesome, as noted by Griest and Seckel. However in the present universe the halo energy $m_X v_0^2/2 \sim 10^{-6} m_X$ is rather small on particle physics scales. It is natural to rescale variables in units of the halo characteristic energy, defining

$$\gamma_0 = \frac{\Gamma}{2E_0}; \quad \delta_0 = \frac{E_{res}}{E_0}. \quad \text{Assuming the equilibrium distribution, some algebra gives}$$

$$\langle \sigma v \rangle_{res} = \frac{2^{2-N} (2\pi)^{N+1} v_0^N}{m_X^2} I_N(\gamma_0, \delta_0), \quad (4)$$

where

$$I_N(\gamma_0, \delta_0) = \int_0^{\infty} dzz^{N} \left( \frac{\gamma_0^2 e^{-z/2}}{(z - \delta_0)^2 + \gamma_0^2} \right). \quad (5)$$

Note that $I_N(\gamma_0, \delta_0)$ is analytic for all $\gamma_0 > 0$ and $\delta_0$ regardless of the sign of $\delta_0$. It can be computed exactly in terms of Exponential Integral ($Ei$) functions. We found it more useful to observe that $I_N(\gamma_0, \delta_0)$ has certain absolute upper limits for all possible values of $\gamma_0 > 0$ and $\delta_0$.

Consider the derivative $\partial I_0(\gamma_0, \delta_0)/\partial \gamma_0$:

$$\frac{\partial I_0(\gamma_0, \delta_0)}{\partial \gamma_0} = \int dz \left( \frac{2\gamma_0 (\delta_0 - z)^2}{(\gamma_0^2 + (\delta_0 - z)^2)^2} e^{-z} \right). \quad (6)$$

Since the integrand above is positive definite, the integral achieves its maximum at $\gamma_0 \to \infty$. For $\gamma_0 >> 1$ the integration becomes trivial, yielding $I_N(\gamma_0, \delta_0) \lesssim 1$. A stronger limit notes the integrand of Eq. 5 is cut off for $z \lesssim \gamma_0$ when $\gamma_0 \lesssim 1$, $\delta_0 \lesssim 1$, implying $I_N(\gamma_0, \delta_0) \lesssim 1 - e^{-C \gamma_0}$, where $C$ is a constant. Numerical work shows that for all parameters

$$I_0(\gamma_0, \delta_0) \lesssim 1 - e^{-\frac{5}{4} \gamma_0}, \quad I_1(\gamma_0, \delta_0) \lesssim 1 - e^{-\frac{3}{4} \gamma_0}. \quad (7)$$

These are close to equality for positive $\delta_0 << 1$. Figure I shows a plot of $I_N(\gamma_0, \delta_0)$ for a wide range of $\gamma_0, \delta_0$ and how the integral approaches the upper bound.
The positivity property of Eq. (6) holds for all halo distributions. The upper limit $BW \rightarrow 1$ produces a universal inequality:

$$\langle \sigma v \rangle_{res} < \frac{4\pi}{m_X^2} \left( \frac{1}{v_0^{1-N}} \right).$$

The expected value $\langle 1/v \rangle$ is relative to the distribution $\Phi_{halo}(v)$, not $dN/dv$. If the equilibrium distribution is assumed, then

$$\langle \sigma v \rangle_{res} < \frac{2^{2-N}(2\pi)^{N+1}v_0^{-N}E_{res}}{m_X^2}(1 - e^{-\pi\gamma_0/2^{N+1}}).$$

The result is a possible significant enhancement factor (EF) (“boost factor”) for annihilation rates. The enhancement factor is defined relative to a typical Born approximation $\sigma_{Born} = 4\pi\alpha_X^2/m_X^2v^{2-N}$:

$$EF = \frac{\langle \sigma v \rangle_{res}}{\langle \sigma v \rangle_{Born}} \lesssim \frac{1}{\alpha_X^2}.$$ (10)

Note that the upper limit does not depend on the position of the resonance nor on any halo properties.

1. $N = 0$ Enhancement Factors

For $N = 0$, Eq. (10) leads to substantial enhancements approaching the unitarity bound when the fundamental width $\Gamma$ is large enough. Obtaining a “large enough” width from a weakly coupled theory might appear special. Yet remember that halo annihilations are driven by the width in units of the rather small scale $E_0 \sim 10^{-6}m_X$. For TeV-scale dark matter a width $\Gamma \gtrsim$ MeV is large enough to dominate the halo width and make $BW(\Gamma, E_{res}) \sim 1$. Recall that the $J/\psi$ has a width of order 0.1 MeV and is exceedingly “narrow”. For an elementary particle on any mass scale of GeV-TeV not to have widths exceeding $10^{-6}m_X$ requires special conspiracies or selection rules.

Figure 2 shows that even a tiny value of $\Gamma/m_X \sim 10^{-8}$ can produce rates much larger than the oft-cited value $\langle \sigma v \rangle \sim 3 \times 10^{-26}cm^3/s$. It is a new insight that merely including physics of widths tends to saturate unitarity bounds in halo annihilation.
Yet just as above, everything about any significant enhancement depends on the width, and won’t proceed without it. To estimate widths, first note that bound states are spatially large for small coupling constant $\alpha_X$. The size of a weakly coupled bound state is roughly estimated by the “Bohr radius” $a_0$, where

$$a_0 \sim 1/m_X \alpha_X.$$ 

Similarly, the binding energy is $E_{\text{res}} \sim m_X \alpha_X^2$. Next recall that the Schrödinger wave function at the origin $\psi(0)$ determines the width via $\Gamma \sim |\psi(0)|^2 \sigma_c$, where $\sigma_c$ is a continuum cross section.

The wave function at the origin is set by the inverse of the size of the bound state:

$$|\psi(0)|^2 \sim a_0^{-3} \sim \alpha_X^3.$$ 

The continuum annihilation cross section $\sigma_c \sim \alpha_X^{1+A}$ for $A > 0$ depending on the model. For reference the annihilation rates of ortho (para) positronium via three (two) photons go like $\sigma_{em}^0 (\alpha_{em})^3$. Thus bound state widths follow a general pattern

$$\Gamma \sim \alpha_X^{1+A} m_X \lesssim 10^{-8} m_X.$$ 

The right hand side is a fair upper limit for $\alpha_X \sim 10^{-2}$.

Restricted phase space factors and branching ratios can only reduce this. Comparing $E_0 \sim 10^{-6} m_X$, we find that $\gamma_0 \ll 1$ is by far the generic case for annihilation from a bound state. As a consistency check, consider the definite case of spin-1/2 dark matter interacting with vector particles. Nature has already done this calculation with the $J/\psi$ decay via gluons, which has $\Gamma_{J/\psi}/m_{J/\psi} \lesssim 10^{-4}$. The $J/\psi$ is sufficiently heavy that the perturbative phase space factors are driven by dimensional analysis, as expected for TeV-scale physics. The raw $J/\psi$ ratio needs to be re-scaled by $(\alpha_X/\alpha_s)^{1+A} \sim 10^{-4}$, which gives satisfactory agreement.

When $\gamma_0 \ll 1$ it is a good approximation to replace $BW(\Gamma, E_{\text{res}}) \sim \pi(\Gamma/2)\delta(E - E_{\text{res}})$. A short calculation then gives

$$EF(\gamma_0 \ll 1) = \frac{\pi \Gamma/2}{\alpha_X^2 m_X v_{\text{res}}^2} \frac{\Phi_{\text{halo}}(v_{\text{res}})}{(\frac{1}{m_{\text{res}}})},$$ (11)

where $E_{\text{res}} = m_X v_{\text{res}}^2 / 2$. This formula has no singularity as $v_{\text{res}} \to 0$ because $\Phi_{\text{halo}}(v_{\text{res}}) \sim v_{\text{res}}^2$ has compensating factors from phase space (Eq. 3). If a metastable bound state resonance lies above threshold in an expected electroweak range the effects are quite small. Taking $E_{\text{res}} = m_X \alpha_X^2 / 2 \sim 10^{-8} m_X$, and the equilibrium halo model with scale $v_0 = 10^{-3}$, the factor $e^{-E_{\text{res}}/E_0} \sim e^{-100}$ is too small to consider further. When the resonance is below threshold it must have width $\Gamma \gtrsim E_{\text{res}}$ to intrude into the physical region. Since $\Gamma$ is proportional to several powers of $\alpha_X$ compared to $E_{\text{res}}$ this case can also be set aside. If $E_{\text{res}} \to 0$ with $\Gamma >> E_{\text{res}}$ is contemplated, it implies the decay time scale is much less than a binding
(orbital) time scale, which is not consistent with bound states forming in the first place.

An exponentially small suppression can be avoided by adjusting the binding into the range probed by the halo velocity. For example choose \( \alpha_X \sim 10^{-3} \). This device rapidly loses consistency because the bound state criterion \( \alpha_X \gtrsim \mu/m_X \) needs couplings not too small. If a bound state is tuned to the vicinity of the peak, then the halo factors will be order unity. Meanwhile there remains in Eq. \( 11 \) an overall factor of \( \Gamma/(m_X \alpha_X^2) \) << 1.

Figure 2 compares the upper limits from annihilation of continuum processes (\( \gamma_0 = \gamma M_{res}/2E_0 \)). As long as \( \gamma_0 >> 1 \), the upper bounds on the halo annihilation cross section will be saturated. This key feature is conceptually absent if the halo annihilation cross section is simply re-scaled by factors invoked for relic evolution. Thus the reported “boost factors” of the relic calculations do not take into account the Breit-Wigner effects on halo annihilation we have found. This explains why the suggestion \( 15, 16 \) that very small \( \Gamma/M_{res} \) is necessary or tends to enhance halo rates is not general, and appears different from our conclusion. It is clearly possible to find models and parameter regions where both, or neither of the correct relic density and halo enhancement phenomena can be accommodated.

We note that relic densities are also subject to many uncertainties of galaxy formation and the other boost factors representing “clumpiness”. For purposes of confronting experimental data, it seems best to separate the problems of halo annihilation and relic evolution entirely, despite mathematical similarities in how they are calculated.

IV. SOMMERFELD FACTORS

We have shown that Breit-Wigner width effects of typical particle physics type can be surprisingly large, while bound state effects have little chance to compete. Sommerfeld factors have also been claimed as a mechanism to produce large enhancements not involving particle widths \( \gamma \).

Given an \( s \)-wave cross section \( \sigma^0 \), which has been computed in the plane wave basis, the Sommerfeld-based recipe to include Coulomb wave effects is to make a replacement

\[
\sigma^0 \rightarrow \sigma^0 S(v, \alpha);
\]

\[
S(v, \alpha) = \frac{\alpha}{v} \frac{2\pi}{1 - e^{-2\pi\alpha/v}}.
\]

Here \( \alpha \) is the fine structure constant. Since cross sections contain many other terms of different orders in \( \alpha^j/v^k \), together with logarithmic type dependence, the recipe is an approximation by re-summation of selected contributions \( 18 \).

2. Motivation for Re-summation

Non-relativistic QED has complicated logarithmic and power-behaved infrared singularities. Singular terms must be summed or controlled in some way to avoid upsetting perturbation theory. There are reasons to believe that the leading singularities appear \( 28 \) order by order as a series in \( \alpha^j/v^k \). Evidently such a series is summed in \( S(v, \alpha) = S(\alpha/v) \).
Sub-leading terms are dropped in any re-summation - for example, a term of order $\alpha^j/v^k$ with $j > k$ is sub-leading as $v \to 0$. The fact that infinitely many sub-leading terms exist comes from the fact that it is always possible to add a photon exchange loop to any diagram, and all loops have some integration region not singular as $v \to 0$.

The purpose of re-summation is to extend the reach of perturbation theory into the difficult, non-perturbative regime. Some examples illustrate typical limitations. Expand $S(v, \alpha) \sim 1 + \pi(\alpha/v) + \pi^2\alpha^2/3v^2 + \ldots$. For $\alpha = 10^{-2}$, and $v = 5 \times 10^{-2}$ the third term is $\pi^2/75 \sim 0.13$. It happens to be larger than a typical non-singular term of order $\alpha$; retaining it is well-motivated for these kinematics. The series is also stable in this regime: a high order term $(\pi\alpha/v)^{10} \sim 0.0096$ is small. Yet at the smaller velocity $v = 5/1000$, the sub-leading correction $\alpha^{11}(\pi/v)^{10} \to 9589560$ is not included, is hardly small, and the leading-order re-summation fails. It is the nature of such re-summation that self-consistency breaks down in the region $\alpha/v \gtrsim 1$, exactly where $S(v, \alpha) \gtrsim 1$.

If one believes a re-summation recipe might generate corrections of order 30-50%, say, there’s seldom any reason to invoke it for factors of “10” or more. Yet recent treatments of dark matter annihilation have imagined the Sommerfeld factor to be very general. It has been held responsible for extremely large enhancement factors of $S \gg 10$, while also coming from any generic interaction involving light Yukawa particles \[20\]. The perception comes from a practice of citing continuum Coulomb normalization factors $|\psi_C|^2(0)$. Since Coulomb normalization factors are known exactly, the procedure has been thought to be “exact.”

We have traced early literature to find several logical and historical contradictions. Guth and Mullin highlighted the approximations in 1951 \[21\]. In lowest order approximation, while using Coulomb wave functions $\psi_C$ for a basis, one encounters matrix elements $M$ of the form

$$ M = \int d^3x \psi_C^* \mathcal{V} \psi_C. $$

Insert complete sets of momentum eigenstates $|k\rangle$:

$$ M = \int d^3k d^3k' \psi_C^* (k) \mathcal{V}_{kk'} \psi_C (k'). $$

The Coulomb wave functions are sharply peaked in the vicinity of certain momenta $k_1, k_2$ identified by taking the limit $\alpha = 0$. Assume the plane-wave matrix elements $V_{k_1, k_2} = \langle k_1 | \mathcal{V} | k_2 \rangle$ are relatively smooth functions of momentum transfer. Make a rough approximation moving $V_{k_1, k_2}$ outside the integral:

$$ M \to V_{k_1, k_2} \int d^3k \psi_C^* (k) \int d^3k' \psi_C (k'), $$

$$ \to V_{k_1, k_2} \psi_C^* (x = 0) \psi_C (x = 0). \quad (12) $$

In the last line $\psi_C (x = 0)$ appears as the coordinate-space wave function at the origin, “improving” the plane wave calculation. Inserting the analytically known normalization of $|\psi_C|^2$ then produces the factor $S(v, \alpha)$ for the cross section.

The operations of separating the collision and wave function integration into products is one of leading power factorization. It is used in QCD calculations separating “hard” and “soft” regions of perturbation theory, but in that case while attempting to be systematic. Careful work with positronium annihilation \[22\] does not use the factorized approximation. Instead, reference to re-summation is made after the full calculations are carried out. An early work \[22\] on one-loop corrections to positronium decay states that “Coulomb effects are included by this (factored) method to all orders in $e^2$, though only, of course, approximately.” (Italics are ours.)

What did Sommerfeld actually do? We consulted his 1931 article, in German, to see it introduced exact Coulomb wave functions to calculate bremsstrahlung, while it never suggested factorization. It is a tour de force of early quantum theory; consulting it for a renormalization factor actually perpetuates a normalization mistake. Cross sections are defined by ratios relative to a flux computed with a given normalization. The overall normalization of physical states cancels out in total cross sections: and so Eq. \[12\] is not only approximate, it is incomplete.

Elwert’s 1939 dissertation \[23\] recognized this, as did Guth in 1941 \[24\]. These papers abandoned Sommerfeld’s calculation and used the ratio of two in- and out- Coulomb factors as an approximate factorized ansatz. “Elwert factors” are used in atomic and molecular physics to cancel spurious pre-factors going like $v$ from other approximations, but only when their effects are not too large. Experimental confrontation of the Elwert factor finds errors of relative order unity in the region the factors are of order unity \[22, 25\]. Elwert and collaborators find this kind of breakdown reasonable \[23\]. In no event are very large corrections ever credited.

3. Multiplicative Factors Must Fail

In retrospect, we find the concept behind generating singularities via multiplicative factors questionable on general grounds.

A general scattering amplitude has the partial wave expansion

$$ f(\theta, k) = \frac{1}{k} \sum_l (2l + 1) f_l(k) P_l(\cos \theta). $$

For each partial wave cross section $\sigma_l$ of angular momentum $l$, elastic unitary gives the upper limit

$$ \sigma_l = \frac{4\pi(2l + 1)|f_l(k)|^2}{k^2} \leq \frac{4\pi(2l + 1)}{k^2}. $$

This summarizes the unitarity bound of Ref. \[26\]. Since each partial wave has a finite cross section, no partial
wave can possibly have a singularity. “Improving” the s-wave cross section - or any particular partial wave cross section - by singular terms of order $\alpha_X/v$ then contradicts unitarity for $\alpha_X/v \gtrsim 1$. This is just the same region where the claimed Sommerfeld factor $S(v, \alpha) >> 1$.

Can one escape the contradiction by appealing to small $\alpha_X$? It seems not: No small value for $\alpha_X$ is used in the logic of an exact normalization citing Sommerfeld. Small coupling is also no protection from internal inconsistency. Unitarity and analyticity in perturbation theory are exact facts maintained in a systematic way, order by order, regardless of the size of the coupling constant, small or large. When violated, it shows the calculation was bad, just as indicated by sub-leading terms [29].

This problem of consistency is different from the one previously recognized. Dark matter interactions have finite range, while the infrared singularities of resummation come from infinite range. To account for this the authors of Ref. [11] argued that for a finite range potential, a Sommerfeld enhancement would saturate when the de Broglie wavelength is tiny compared to the range. This is just the upper limit $l_{\text{max}} \sim r_{\text{max}} k$, where $r_{\text{max}}$ is the range of the potential. This gives

$$\sigma \lesssim 4\pi r_{\text{max}}^2 |f_{\text{max}}|^2 \lesssim 4\pi r_{\text{max}}^2.$$

The Coulomb singularity occurs because (1) the effective range $r_{\text{max}} \to \infty$, and (2) an infinite number of partial waves actually can contribute. Closely related is Wigner’s classic theorem [27] that power-law potentials $V(r) \gtrsim 1/r^2$ are needed to develop any kind of singularity.

V. CONCLUDING REMARKS

We have explored significant effects in halo annihilation rates due to natural widths of intermediate states. The problem is intricate due to subtle interplay of energy scales. The Born approximation almost always fails, giving gross underestimates of reaction rates. Tiny values of galactic halo velocities reverse an assumption that propagator widths might be “small corrections.” That perception comes from comparing widths to particle masses, and does not capture the important features of halo annihilation.

Given that TeV-scale particles with typical electroweak type couplings may easily have $\Gamma/E_\odot >> 1$, Breit-Wigner factors of ordinary radiative corrections must generally be taken into account. Consistency of rates in particular channels, such as the apparent dominance of leptons, still needs to be considered model by model. The fact that merely including basic physics of widths may be quite significant. It revises the basic picture of annihilation scenarios confronting the PAMELA-FERMI-PPB-BETS data in a positive way that increases the possibilities to find new physics.

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[1] O. Adriani et al. [PAMELA Collaboration], Nature 458, 607 (2009) [arXiv:0810.4995 [astro-ph]].

[2] E. A. Baltz et al., JCAP 0807, 013 (2008) [arXiv:0806.2911 [astro-ph]].

[3] S. Torii et al. [PPB-BETS Collaboration], arXiv:0809.0760 [astro-ph].
