Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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Abstract

The new manifestation of conformal invariance for a massless scalar particle in a non-Minkowskian spacetime is found. Conformal transformations conserve the Hamiltonian and wave function in the Foldy-Wouthuysen representation. Similarity of conformal transformations for massless scalar and Dirac particles is proved. Exact FW Hamiltonians are derived for both massive and massless scalar particles in general static inertial and gravitational fields and for particles in a frame rotating in the field of a rotating source. The latter case covers an observer on the ground of the Earth or on a satellite and takes into account not only the rotation but also the Lense-Thirring effect. High-precision formulas are obtained for an arbitrary spacetime metric. General quantum-mechanical equations of motion are derived. Their classical limit coincides with corresponding classical equations.

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INTRODUCTION

Penrose \[1\] has discovered fifty years ago that the covariant Klein-Fock-Gordon (KFG) equation \[2\] for a massless scalar particle in a non-Minkowskian spacetime added by an appropriate term describing a nonminimal coupling to the scalar curvature is conformally invariant. Chernikov and Tagirov \[3\] have given clear explanations of this wonderful result. Their study involved the case of a nonzero mass and \(n\)-dimensional Riemannian spacetime. The inclusion of the Penrose-Chernikov-Tagirov (PCT) term has been argued for both massive and massless particles \[3\]. Next step in investigation of the problem of conformal invariance of the KFG equation has been made by Accioly and Blas \[4\]. They have performed the exact Foldy-Wouthuysen (FW) transformation for a massive spin-0 particle in static spacetimes and have found new telling arguments in favour of the PCT coupling to the scalar curvature. A derivation of the relativistic FW Hamiltonian is important for a comparison of gravitational (and inertial) effects for scalar and Dirac particles. However, the transformation method used in Ref. \[4\] is inapplicable to massless particles. In addition, it cannot be applied for nonstatic spacetimes. This does not allow to obtain an information about a manifestation of the conformal invariance in the FW representation.

In the present work, we consider a scalar particle in arbitrary spacetimes. To obtain a Hamiltonian form of the initial covariant KFG equation not only for massive particles but also for massless ones, we use the generalization of the Feshbach-Villars transformation \[5\] proposed in Ref. \[6\]. Then we fulfil the FW transformation and prove the conformal invariance of the relativistic FW Hamiltonian for a wide class of inertial and gravitational fields. We derive general quantum-mechanical equations of motion and obtain their classical limit.

We denote world and spatial indexes by Greek and Latin letters \(\alpha, \mu, \nu, \ldots = 0, 1, 2, 3, \ i, j, k = 1, 2, 3\), respectively. Tetrad indexes are denoted by Latin letters from the beginning of the alphabet, \(a, b, c, \ldots = 0, 1, 2, 3\). Separate tetrad indexes are distinguished by hats. The signature is \((+ - - -)\), the Ricci scalar curvature is defined by \(R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\alpha_{\mu\nu\alpha}\), where \(R^\alpha_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \ldots\) is the Riemann curvature tensor. We use the system of units \(\hbar = 1, \ c = 1\) except for some specific expressions.
IMPORTANT OF THE PENROSE-CHERNIKOV-TAGIROV TERM

The covariant KFG equation which describes a scalar particle in a non-Minkowskian spacetime and is added by the PCT term is given by

\[
\left(\Box + m^2 - \lambda R\right)\psi = 0, \quad \Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu.
\] (1)

Minimal (zero) coupling corresponds to \(\lambda = 0\), while the PCT coupling is defined by \(\lambda = 1/6\). For noninertial frames, the spacetime is flat and \(R = 0\).

For massless particles, the conformal transformation

\[
\tilde{g}_{\mu\nu} = O^{-2} g_{\mu\nu}
\] (2)

conserves the form of Eq. (1) but changes the operators and the wave function \([1, 3]:\)

\[
(\tilde{\Box} - \frac{1}{6} \tilde{R})\tilde{\psi} = 0, \quad \tilde{\psi} = O\psi.
\] (3)

In Ref. [3], higher dimensionality was also considered.

The corresponding classical equation

\[
g^{\mu\nu} p_\mu p_\nu - m^2 = 0
\]

is also conformal for a massless particle. It does not contain any nonminimal coupling to the scalar curvature. Therefore, the square of the classical momentum corresponds to the operator \(-\hbar^2 (\Box - R/6)\) \([3]\).

Chernikov and Tagirov \([3]\) have shown the importance of the additional term for massive particles. They have proved that the requirement for motion to be quasiclassical for a large momentum is satisfied for massive and massless particles only when \(\lambda = 1/6\). This choice of \(\lambda\) has been additionally grounded in Refs. \([8, 9]\).

Important development of problem of the PCT coupling for massive particles has been made by Accioly and Blas \([4]\). They considered the diagonal static metric

\[
ds^2 = V(r)^2 (dx^0)^2 - W(r)^2 (dr)^2
\] (4)

with arbitrary \(V(r), W(r)\). After the Feshbach-Villars transformation (unappropriate for massless particles), initial Eq. (1) takes the Hamiltonian form. Next nonunitary and FW transformations result in the FW Hamiltonian \([4]:\)

\[
\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + \mathcal{F} \mathcal{F}^2 - \frac{1}{4} \nabla \mathcal{F} \cdot \nabla \mathcal{F} + \mathcal{D}_\lambda (V, W)},
\] (5)
where \( p = -i \nabla \) is the momentum operator and \( \rho_i (i = 1, 2, 3) \) are the Pauli matrices. Only for \( \lambda = 1/6 \), the Darwin term \( D_\lambda(V,W) \) has the simple form and is equal to \( F_\Delta F/6 \) \([4]\).

However, the important result obtained by Accioly and Blas demonstrates only a shadow of the conformal invariance, because it does not cover the case of \( m = 0 \). We perform general examination of the problem.

**GENERALIZED FESHBACH-VILLARS TRANSFORMATION**

General form of the covariant KFG equation reads

\[
\left( \partial_0^2 + \frac{1}{g^{00} \sqrt{-g}} \{ \partial_i, \sqrt{-g} g^{0i} \} \partial_0 + \frac{1}{g^{00} \sqrt{-g}} \partial_i \sqrt{-g} g^{ij} \partial_j + \frac{m^2 - \lambda R}{g^{00}} \right) \psi = 0.
\]

(6)

There is an ambiguity \([10]\) in the definition of parameter of the Feshbach-Villars transformation. We use the generalized Feshbach-Villars (GFV) transformation proposed in Ref. \([6]\) and based on this ambiguity. In the considered case, the transformation consists in the following definition of components of the wave function:

\[
\psi = \phi + \chi, \quad i (\partial_0 + \Upsilon) \psi = N(\phi - \chi),
\]

\[
\Upsilon = \frac{1}{2g^{00} \sqrt{-g}} \{ \partial_i, \sqrt{-g} g^{0i} \},
\]

(7)

where \( N \) is an arbitrary nonzero real parameter. For the Feshbach-Villars transformation, it is definite and equal to the particle mass \( m \). This generalization allows us to represent Eq. (6) in the Hamiltonian form describing both massive and massless particles:

\[
i \frac{\partial \Psi}{\partial t} = \mathcal{H} \Psi, \quad \mathcal{H} = \rho_3 N^2 + T + i \rho_2 \frac{N^2 + T}{2N} - i \Upsilon,
\]

\[
T = \frac{1}{g^{00} \sqrt{-g}} \partial_i \sqrt{-g} g^{ij} \partial_j + \frac{m^2 - \lambda R}{g^{00}} - \Upsilon^2.
\]

(8)

Similarly to Ref. \([4]\), then we perform the nonunitary transformation \( \Psi' = f \Psi \) to obtain a pseudo-Hermitian (more exactly, \( \rho_3 \)-pseudo-Hermitian) Hamiltonian: \( \mathcal{H}' = f \mathcal{H} f^{-1} \), \( \mathcal{H}' = \rho_3 \mathcal{H}' \dagger \rho_3 \). In the case under consideration,

\[
f = \sqrt{g^{00} \sqrt{-g}}, \quad \Upsilon' = \frac{1}{2f} \{ \partial_i, \sqrt{-g} g^{0i} \}, \quad \frac{1}{f},
\]

\[
T' = \frac{1}{f} \partial_i \sqrt{-g} g^{ij} \partial_j \frac{1}{f} + \frac{m^2 - \lambda R}{g^{00}} - (\Upsilon^2)'.
\]

(9)
Transformed operators are denoted by primes and \((\Upsilon^2)'=(\Upsilon')^2\). Tedious but simple calculations result in

\[
H' = \rho_3 \frac{N^2 + T'}{2N} + i\rho_2 \frac{-N^2 + T'}{2N} - iT',
\]

\[
T' = \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \frac{m^2 - \lambda R}{g^{00}} + \frac{1}{f} \nabla_i \left( \sqrt{-g} G^{ij} \right) \nabla_j \left( \frac{1}{f} \right)
\]

\[
+ \sqrt{\frac{-g}{g^{00}}} \nabla_i \nabla_j \left( \frac{1}{f} \right) + \frac{1}{4f^4} \left[ \nabla_i (\Gamma^i) \right]^2
\]

\[
- \frac{1}{2f^2} \nabla_i \left( \frac{g^{0i}}{g^{00}} \right) \nabla_j (\Gamma^j) - \frac{g^{0i}}{2g^{00}f^2} \nabla_i \nabla_j (\Gamma^j),
\]

\[
\Upsilon' = \frac{1}{2} \left\{ \partial_i, \frac{g^{0i}}{g^{00}} \right\}, \quad G^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}, \quad \Gamma^i = \sqrt{-gg^{0i}},
\]

where the nabla operators act only on the operators in brackets and the primes denote transformed operators. Equation (10) is exact and covers any inertial and gravitational fields.

**FOLDY-WOUTHUYSEN TRANSFORMATION**

General methods of the FW transformation for relativistic particles have been developed in Refs. [11, 12]. They belong to step-by-step methods performing the transformation as a result of subsequent iterations. We use the version [6] adapted to scalar particles. In this case, the relativistic FW transformation is carried out with the \(\rho_3\)-pseudounitary operator

\[
(U^\dagger = \rho_3 U^{-1} \rho_3)
\]

\[
U = \frac{\epsilon + N + \rho_1 (\epsilon - N)}{2\sqrt{\epsilon N}}, \quad \epsilon = \sqrt{T'}, \quad (11)
\]

It is important that the Hamiltonian obtained as a result of the transformation does not depend on \(N\). This shows a self-consistency of the used transformation method. Next transformation [6] eliminates residual odd terms and leads to the final form of the approximate relativistic FW Hamiltonian:

\[
\mathcal{H}_{FW} = \rho_3 \epsilon - i\Upsilon' - \frac{1}{2\sqrt{\epsilon}} \left[ \sqrt{\epsilon}, \left[ \sqrt{\epsilon}, (i\partial_0 + i\Upsilon') \right] \right] \frac{1}{\sqrt{\epsilon}}, \quad (12)
\]

**EXACT FOLDY-WOUTHUYSEN TRANSFORMATION AND CONFORMAL INVARIANCE**

The used method ensures the exact FW transformation for a wide class of spacetime metrics. The manifestation of conformal invariance can also be investigated in detail.
The sufficient condition of the exact FW transformation [6, 11, 12] applied to scalar particles is given by \( \partial_0 T' - [T', \Upsilon'] = 0 \). When it is satisfied, the exact FW Hamiltonian reads
\[
\mathcal{H}_{FW} = \rho_3 \sqrt{T'} - i\Upsilon'.
\] (13)
Equation (13) covers all static spacetimes (\( \Upsilon' = 0 \)) and some important cases of stationary ones.

Since general expressions for the scalar Ricci curvature are very cumbersome, we restrict ourselves to an analysis of several special cases. For the metric defined by Eq. (4), the result of our calculations formally coincides with Eq. (5). However, the case of \( m = 0 \) can now be considered. Explicit expression for \( D_\lambda(V, W) \) shows the presence of conformal invariance for massless particles if and only if \( \lambda = \frac{1}{6} \). In this case, conformal transformation (2) does not change the FW Hamiltonian and the FW wave function \( \Psi_{FW} \). These manifestations of conformal invariance radically differ from those for the covariant KFG equation and the corresponding wave function.

The validity of the found properties can be checked for the scalar particle in nonstatic spacetimes. The metric of the rotating Kerr source has been reduced to the Arnowitt-Deser-Misner form [13] by Hergt and Schäfer [14]. Its form can be additionally simplified due to an introduction of spatially isotropic coordinates and dropping terms violating the isotropy [15]:
\[
ds^2 = V^2(dx^0)^2 - W^2 \delta_{ij}(dx^i - K^i dx^0)(dx^j - K^j dx^0), \quad K = \omega \times r.
\] (14)
The exact FW transformation can be fulfilled when \( V, W, \) and \( \omega \) depend only on the isotropic radial coordinate \( r \). In this approximation, the metric is defined by
\[
V(r) = \frac{\kappa_-}{\kappa_+} + \mathcal{O}\left(\frac{\mu a^2}{r^3}\right), \quad W(r) = \kappa_+^2 + \mathcal{O}\left(\frac{\mu a^2}{r^3}\right),
\]
\[
\omega(r) = \frac{2\mu c}{r^3}a \left[ 1 - \frac{3\mu}{r} + \frac{21\mu^2}{4r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right) \right].
\] (15)
Here \( \kappa_\pm = 1 \pm \mu/(2r) \), \( a = J/(Mc), \mu = GM/c^2 \); the total mass \( M \) and the total angular momentum \( J \) (directed along the z axis) define the Kerr source uniquely. The leading term in the expression for \( \omega(r) = \omega(r)e_z \) corresponds to the Lense-Thirring approximation.

We can pass on to a frame rotating in the Kerr field with the angular velocity \( \omega \) after the transformation \( dx^i \rightarrow dX^i = dx^i + (\omega \times r)dx^0 \). The stationary metric of this frame can
be obtained from Eqs. (14), (15) with the replacement $\omega \rightarrow \Omega = \omega - o$. It covers an observer on the ground of a rotating source like the Earth or on a satellite. In this case, $o = J/I$, where $I$ is the moment of inertia. The exact FW Hamiltonian is given by Eq. (13) where

$$T' = m^2V^2 + \mathcal{F}p^2\mathcal{F} - \frac{1}{4}\nabla \mathcal{F} \cdot \nabla \mathcal{F} + \mathcal{D}_\lambda (V, W)$$

$$+ \frac{\lambda}{2} (x^2 + y^2)(\Omega'_r)^2, \quad \mathcal{D}_\lambda (V, W) = \lambda \mathcal{F} \Delta \mathcal{F}$$

$$+ (1 - 6\lambda) \frac{V}{2W^2} \left[ \mathcal{F} \left( \frac{2W'_r}{r} + W''_{rr} \right) + \frac{2V'_r + V''_{rr}}{r} \right],$$

and derivatives with respect to $r$ are denoted by indexes. In particular, for the Lense-Thirring metric $\Omega(r) = 2GJ/r^3$, $V(r) = 1 - GM/r$, $W(r) = 1 + GM/r$.

If and only if $\lambda = 1/6$, conformal transformation (2) changes neither $T'$ nor $H_{FW}$, $\Psi_{FW}$. This property is the same as for the static metric.

**GENERAL EQUATIONS OF MOTION**

Equations for the FW Hamiltonian allow us to derive general quantum-mechanical equations of motion and then obtain their classical limit ($\hbar \rightarrow 0$). The quantum-mechanical equations of motion defining the force, velocity, and acceleration read ($p_0 \equiv H_{FW}$)

$$F^i \equiv \frac{dp^i}{dt} = \frac{1}{2} \frac{\partial}{\partial t} \left\{ g^{ij}, p_j \right\} + \frac{i}{2\hbar} \left[ H_{FW}, \left\{ g^{ij}, p_j \right\} \right],$$

$$V^i \equiv \frac{dx^i}{dt} = \frac{i}{\hbar} \left[ H_{FW}, x^i \right], \quad W^i = \frac{\partial V^i}{\partial t} + \frac{i}{\hbar} \left[ H_{FW}, V^i \right].$$

Any commutation adds the factor $\hbar$ as compared with the product of operators.

It has been proved in Ref. [17] that satisfying the condition of the Wentzel-Kramers-Brillouin approximation allows to use this approximation in the relativistic case and to obtain a classical limit of the relativistic quantum mechanics. Determination of the classical limit reduces to the replacement of operators in the FW Hamiltonian and quantum-mechanical equations of motion in the FW representation by the respective classical quantities. The classical limit of the general FW Hamiltonian is given by

$$H = \left( \frac{m^2 - G^{ij}p_ip_j}{g^{00}} \right)^{1/2} - \frac{g^{0i}p_i}{g^{00}}.$$

It coincides with the classical Hamiltonian derived in Ref. [18]. The classical limit of Eq.
\[ V^i = \frac{G^{ij} p_j}{\sqrt{g^{00}(m^2 - G^{ij} p_i p_j)}} + \frac{g^{0i}}{g^{00}}, \]
\[ F^i = p_\mu \dot{g}^{i\mu} + g^{0j} \frac{\partial H}{\partial t} + g^{ij} \partial_j H + p_\mu V^j g^{i\mu}. \]

(19)

It coincides with the corresponding classical equations which follow from Hamiltonian (18) and the Hamilton equations. Thus, the quantum-mechanical and classical equations are in the best compliance.

For example, the exact metric of a general noninertial frame characterized by the acceleration \( \mathbf{a} \) and the rotation \( \omega \) of an observer is defined by

\[ V = 1 + \mathbf{a} \cdot \mathbf{r}, \quad W = 1, \quad \omega = -\mathbf{o}. \]

In this case, the classical limit of the equations of motion is given by (\( p \equiv (-p_1, -p_2, -p_3) \))

\[ V = (1 + \mathbf{a} \cdot \mathbf{r}) \frac{p}{\sqrt{m^2 + p^2}} - \mathbf{o} \times \mathbf{r}, \]
\[ W = -a(1 + a \cdot r) - 2o \cdot V - o \times (o \times r) \]
\[ + \frac{2a \cdot V + a \cdot (o \times r)}{1 + a \cdot r} (V + o \times r). \]

(20)

Equation (20) agrees with the classical results [20].

CONFORMAL INVARIANCE FOR DIRAC AND CLASSICAL PARTICLES. CONCLUSIONS

It is important to compare the conformal transformations for massless scalar, Dirac, and classical particles. Our analysis shows that the general Hermitian Dirac Hamiltonian for a massless particle in an arbitrary metric in the presence of an electromagnetic field [15] is not changed by the transformation (2). The FW transformation operator for particles in strong external fields obtained in Ref. [12] is also conformally invariant in the case of \( m = 0 \). As a result, the Dirac and FW wave functions, \( \psi \) and \( \psi_{FW} \), remain unchanged. These properties are the same for Dirac and scalar particles. The Hamiltonian of massless classical particles is conformally invariant even if its spin-dependent part defined by Eqs. (3.18) and (4.12) in Ref. [15] is taken into account.

In the general case, the transformation of the initial covariant Dirac equation to the Hermitian Hamiltonian form is performed by the nonunitary operator \( f_D = (\sqrt{-\hat{g}e_0^0})^{1/2} \) [15]. Since the transformation (2) leads to \( \tilde{f}_D = O^{-3/2} f_D \), the conformally transformed wave
function of the initial covariant Dirac equation, \( \tilde{\Psi} \), reads

\[
\tilde{\Psi} = f_D^{-1} \tilde{\psi} = O^{3/2} f_D^{-1} \psi = O^{3/2} \Psi.
\]  

(21)

While its transformation is similar to that for the scalar particles, the powers of \( O \) in Eqs. (3) and (21) differ.

The second-order wave equation for the Dirac particles in general electromagnetic and gravitational fields derived in Ref. [15] includes the term describing a nonminimal coupling to the scalar curvature \( R \). As the definitions of \( R \) in Ref. [15] and the present work differ in sign, this term corresponds to \( \lambda = 1/4 \).

The use of the GFV and relativistic FW transformations allows to describe the both massive and massless scalar particles in general noninertial frames and gravitational fields. The present work demonstrates the new manifestation of the conformal invariance for massless particles. The conformal transformation conserves the FW Hamiltonian and the FW wave function but changes the wave function of the initial KFG equation. The similar conclusion is valid for the Dirac particles. The nonminimal coupling to the scalar curvature is not a unique property of scalar particles.

Contemporary methods of (pseudo)unitary and nonunitary transformations make it possible to derive exact FW Hamiltonians for i) both massive and massless scalar particles in the general static inertial and gravitational fields and ii) particles in the frame rotating in the field of a rotating source. The latter result covers an observer on the ground of the Earth or on a satellite and takes into account not only the rotation but also the Lense-Thirring effect. For an arbitrary metric, high-precision formula (12) is obtained. The classical limit of the derived general quantum-mechanical equations of motion coincides with corresponding classical equations.

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