A Mathematical Model of Tornado

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Abstract. We propose a mathematical model of tornado in the framework of continuum mechanics. According to this model, the swirling upward air motion in a tornado can be explained even without taking into account the Coriolis force or the vertical convection of warm air. Our results show that for a tornado to appear two factors are necessary: a powerful rotational motion of air in the upper atmosphere and the growth of air pressure from the center to the periphery of an expected region of tornado. The characteristics of air flow in tornado can be found by solving a boundary value problem for a nonlinear parabolic integro-differential equation for an unknown complex-valued temperature. We formulate this problem, propose an algorithm for its numerical solution, and examine its stationary solutions by the stabilization method.

1. Introduction

The phenomenon of tornado plays an important role in the life of many regions of the Earth because of its disastrous consequences. The prediction and control of this phenomenon requires the study of the flow of air masses in a tornado. There is still no common understanding of its causes. There is a widespread opinion [1–4] that it is generated by the vertical movement of warm air, due to the local heating by the sun of parts of the earth’s or water surface and the air masses adjacent to them: ascending air flows first lead to radial, and then vortex air movements at the Earth’s surface owing to the Coriolis force. Our study suggests another possible mechanism of air motion in a tornado. It does not rely on the Coriolis force or vertical convection of warm air, but is based on small-scale phenomena in atmospheric air, in particular, its viscosity. It should be observed that the effective hydrodynamic viscosity of air in a real tornado significantly increases due to the presence in it of a large amount of water or sand dust (depending on whether it is formed above the surface of water or land). The reasons for the occurrence of a tornado, according to our research, are, firstly, the excess pressure of air in its periphery over the pressure in its central part, and secondly, the presence of a powerful vortex motion in the upper atmosphere. In this case, we show that the angular velocity of the vortex motion is directly related to the periphery-center pressure drop. Thirdly, it seems that tornado is realized as a special type of air flow, the form of which will be specified below. The combination of these three factors is uncommon, which explains the exclusivity of the tornado phenomenon.

There are dozens of works on the numerical and analytical modeling of processes in a tornado. Let us briefly dwell on some of the most important of these publications.

In [5], an analytical model is proposed for a vortex non-viscous flow with a vertical symmetry axis and a motionless core in the form of a funnel. It is assumed that the air density within the
core is smaller than that of the moving air whose rotation velocity is growing in the direction of the funnel border and is maximal near the ground. In [6], a model is proposed for the formation of atmospheric tornado-type vortices due to instability caused by the growth of the vertical component of the air velocity in the direction of the earth's surface or an increase of concentration of suspended particles. Such conditions are typical for the initial stage of the development of a tornado and are realized in thunderclouds in the atmosphere. A class of analytical solutions of the Navier-Stokes equations for viscous incompressible fluids is obtained in [7]. These solutions can be used to predict the characteristics of certain vortex flows, in particular, tornadoes. An analytical study of ascending swirling air flow is carried out in [1]. According to this investigation, the Coriolis force plays a key role in the formation of tornadoes. It should be observed that a number of assumptions made in [1] contradict the generally accepted views on the formation and stability of tornadoes.

Numerical simulation of a tornado faces the problem of setting correct initial and boundary conditions. Moreover, when using turbulent models of air flows in a tornado, the problem is the calculation of empirical turbulent transport coefficients. Nevertheless, some important results have been obtained in this direction. In [8], one considers the process of tornado formation due to convective instability near the earth's ground. In [9–12], following [8], calculations of vortices are carried out and the determining role of the swirl ratio in the formation and stability of the vortex structure of a tornado is established. The process of tornado formation due to air rotation in a thundercloud is studied in [13], with the effects of turbulence of air currents, water or sand impurities, and compressibility taken into account. In [14,15], developing the approaches of [13], one calculates three-dimensional velocity fields and air pressure in a tornado. An analysis of the dynamics of tornado-like flows on the basis of the large vortex method (LES method) was carried out in [16–20], where it was shown, in particular, that the compressibility of air has almost no effect on the vortex dynamics in a tornado. A two-fluid model of tornado is examined in [21,22], the first (main) liquid being water vapor, which condenses upon a sharp change in pressure, the second liquid consisting of solid particles involved in the air flow by the vortex tornado flow.

Here, we propose a simple mathematical model of tornado stated in terms of continuum mechanics and based on a generalization of the exact von Karman solution [23,24] that describes the motion of a viscous incompressible fluid over a uniformly rotating infinite horizontal disk. The von Karman solution shows that the disk rotation results in a “vacuum cleaner” effect: the fluid is drawn from infinity with a constant velocity in the direction of the disk and spreads over it, the fluid pressure may vary in height, but remains constant on each horizontal plane parallel to the disk. If the last condition is dropped, i.e., if the pressure on each horizontal plane is allowed to vary, being greater on the periphery than at the center, then, under certain conditions, the behavior of the flow above the disk drastically changes. The fluid moves along the disk to the vertical axis, while rotating around it, and is then ejected to infinity along that axis. A similar behavior is exhibited by the air masses in a tornado. Thus, we come to a simple mathematical model of tornado.

2. Basic Equations
Our starting point is the dynamic system of equations describing the motion of a viscous incompressible fluid in a constant field of gravity,

$$\text{div } \mathbf{U} = 0, \quad \rho = \text{const},$$

$$\frac{\partial \rho \mathbf{U}}{\partial t} + \text{Div} (\rho \mathbf{U} \mathbf{U} + p \mathbf{I}_3) = 2\mu \text{Div def} \mathbf{U} + \rho g,$$  \hspace{1cm} (1)

where $\rho$ is the density of the fluid, $p$ is pressure, $\mathbf{U}$ is its hydrodynamic velocity, $\mu$ is the dynamic viscosity coefficient, $g$ is the constant acceleration of gravity, $\mathbf{I}_3$ is the three-dimensional identity
tensor, \( \mathbf{U} \) is the strain-rate tensor, which in the Cartesian coordinates has the form

\[
\mathbf{U} = \left[ \frac{1}{2} \left( \frac{\partial U^k}{\partial x^s} + \frac{\partial U^s}{\partial x^k} \right) \right]_{1 \leq k, s \leq 3}.
\]

It is assumed that \( \mu = \text{const}, \rho = \text{const} \) are given, and we seek four unknown functions, \( p \) and three components of \( \mathbf{U} \), satisfying four scalar equations (1).

In the axisymmetric case (\( \partial / \partial \varphi = 0 \)), system (1) has a particular solution of the form

\[
U_r = r A(t, z), \quad U_\varphi = r B(t, z), \quad U_z = C(t, z), \quad p = Q(t) r^2 + R(t, z),
\]

where the axis \( z \) has the same direction as \( \mathbf{g} = (0, 0, -g) \), the function \( Q(t) \) is given and \( A, B, C, R \) are sought by substituting the expressions (2) into system (1). Taking into account the condition of axial symmetry \( \partial / \partial \varphi = 0 \) and carrying out some simple transformations, we come to the following result.

**Theorem 1.** In the axisymmetric case, (2) yields a solution of system (1) if and only if \( A, B, C \) satisfy the equations

\[
\frac{\partial A}{\partial t} + A^2 - B^2 + C \frac{\partial A}{\partial z} - \frac{\mu}{\rho} \frac{\partial^2 A}{\partial z^2} + \frac{2Q}{\rho} = 0, \\
\frac{\partial B}{\partial t} + 2AB + C \frac{\partial B}{\partial z} - \frac{\mu}{\rho} \frac{\partial^2 B}{\partial z^2} = 0, \\
\frac{\partial C}{\partial z} = -2A.
\]

If \( A, B, C \) satisfy equations (3), then the derivative \( \partial R / \partial z \) is expressed from the equation

\[
\frac{\partial C}{\partial t} + \frac{\partial}{\partial z} \left( \frac{C^2}{2} + \frac{2\mu}{\rho} A + \frac{R}{\rho} + gz \right) = 0,
\]

which yields \( R \) to within an arbitrary function of time.

Introducing a new unknown complex-valued function \( u = A + iB \) and taking into account the relations \( A = \text{Re} \, u, \, u^2 = (A^2 - B^2) + 2iAB \), we can rewrite system (3) in the form

\[
\frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - \frac{\mu}{\rho} \frac{\partial^2 u}{\partial z^2} + u^2 + \frac{2Q(t)}{\rho} = 0, \\
\frac{\partial C}{\partial z} = -2\text{Re} \, u,
\]

where \( C(t, z) \in \mathbb{R}, \ u(t, z) \in \mathbb{C} \) are unknown real- and complex-valued functions, respectively. System (5) can be written as a single integro-differential equation. Indeed, taking an arbitrary real \( C_0(t) = -C(t, 0) \), we see that system (5) is equivalent to the following integro-differential equation:

\[
\frac{\partial u}{\partial t} - \left( C_0(t) + 2 \int_0^z \text{Re} \, u \, dz \right) \frac{\partial u}{\partial z} - \frac{\mu}{\rho} \frac{\partial^2 u}{\partial z^2} + u^2 + \frac{2Q(t)}{\rho} = 0,
\]

where \( C_0(t), Q(t) \) are arbitrary given real functions.

Some stationary (\( \partial / \partial t = 0 \)) solutions of system (3) with \( Q(t) \equiv 0 \) were studied in [23] (see also [24]). In this case, (3) reduces to an autonomous nonlinear system of fifth-order ordinary differential equations. In [23,24], one considers numerical solutions of a boundary value problem for the said nonlinear fifth-order system on the semi-axis \( 0 \leq z < +\infty \) with certain (see below) boundary conditions at the points \( z = 0 \) and \( z = +\infty \).

Let us consider some initial boundary value problems for system (5).
3. Excitation of a Viscous Incompressible Fluid by a Rotating Disk

Consider the half-space \( z \geq 0 \) occupied by a viscous incompressible fluid. Its boundary \( z = 0 \) coincides with an infinite rigid disk rotating around the axis \( z \) with constant angular velocity \( \Omega \). This rotation causes, in the domain \( z > 0 \), stationary motion of the fluid described by a solution of the form (2) with \( A, B, C, Q, R \) independent of \( t \). This stationary solution can be obtained from a stabilizing solution of system (5) on the half-axis \( 0 \leq z < +\infty \) with certain boundary conditions at \( z = 0 \) and \( z = +\infty \). To obtain these boundary conditions, one should take into account the conditions of adhesion and nonpenetration on the surface of the rigid disk at each time instant. For \( z = 0 \), the adhesion condition for a solution of the form (2) is \( B(t, 0) = \Omega, A(t, 0) = 0 \), while the nonpenetration condition has the form \( C(t, 0) = 0 \). In other words, at each time instant, the identities \( u(t, 0) = i\Omega, C(t, 0) = 0 \) are required to hold. For \( z = +\infty \), no separate condition is imposed on \( C \), while the boundary conditions on \( u \) depend on our assumptions on the behavior of \( u \) and \( C \) at infinity, i.e., as \( z \to +\infty \). For instance, suppose that for \( z \to +\infty \) and any \( t \), there exist finite limits

\[
  u \to u(\infty), \quad C \frac{\partial u}{\partial z} \to 0, \quad \frac{\partial^2 u}{\partial z^2} \to 0,
\]

where \( u(\infty) \) does not depend on \( t \) and the first convergence in (7) can be differentiated in \( t \) (i.e., \( \partial u/\partial t \to \partial u(\infty)/\partial t = 0 \) as \( z \to +\infty \)). Note that the second and the third conditions in (7) follow from the first if, for each \( t \), the function \( C(t, z) \) is bounded at infinity with respect to \( z \) and the real and the imaginary parts of \( \partial u/\partial z \) and \( \partial^2 u/\partial z^2 \) are monotone in \( z \). From (5), (7), we obtain the following boundary condition for \( u(\infty) \):

\[
  u(\infty)^2 + \gamma_0 = 0 \iff u(\infty) = \sqrt{-\gamma_0}, \quad \gamma_0 = 2Q/\rho.
\]

Thus, system (5) is supplemented with the following boundary conditions for any fixed \( t \):

\[
  u|_{z=0} = i\Omega, \quad C|_{z=0} = 0, \quad u|_{z=+\infty} = \sqrt{-\gamma_0}, \quad \gamma_0 = 2Q/\rho. \tag{8}
\]

In order to find a stationary solution of the form (2) in the domain \( z > 0 \), we need to find a stabilizing solution of system (5), (8) with an arbitrary initial condition \( u|_{t=0} = u_0(z), 0 \leq z < +\infty \), whose choice should not affect the solution. The case \( \gamma_0 = 0 \) was considered in [23,24]. For \( \gamma_0 > 0 \), condition (8) ensures that at infinity, \( z = +\infty \), the incompressible fluid is rotating as a rigid body with constant angular velocity \( \sqrt{\gamma_0} \). For \( \gamma_0 < 0 \), the fluid spreads along the radii at infinity with radial velocity linearly depending on the radial coordinate, the coefficient \( \sqrt{|\gamma_0|} \) being the same for all radial directions. Keeping in mind that our theory is aimed at studying tornado, we assume that \( \gamma_0 > 0 \) in what follows.

Note that the condition that \( u(t, z) \) is bounded in \( t \) and \( z \) does not guarantee that the vertical velocity \( C(t, z) \) is bounded in \( z \). The numerical analysis described below shows that for the stationary solution of problem (5), (8), the function \( C(z) \) is not only bounded, but has a finite limit \( \lim_{z \to +\infty} C(z) \).

Let us pass to dimensionless quantities in problem (5), (8) by choosing the following characteristic scales: \( t_0 = 1/\Omega \) (time), \( U_0 = \sqrt{\mu\Omega/\rho} \) (velocity), \( L_0 = U_0 t_0 = \sqrt{\mu/(\rho\Omega)} \) (length), \( p_0 = \mu\Omega/2 \) (pressure), assuming that \( u_0 = 1/t_0 = \Omega, C_0 = U_0 \). Then system (5), (8) in dimensionless variables takes the form

\[
  \frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - \frac{\partial^2 u}{\partial z^2} + u^2 + \Gamma = 0, \quad \frac{\partial C}{\partial z} = -2\text{Re}u,
\]

\[
  u|_{z=0} = i, \quad C|_{z=0} = 0, \quad u|_{z=+\infty} = \sqrt{-\Gamma}, \tag{9}
\]

\[
  0 \leq z < +\infty, \quad t \geq 0, \quad \Gamma = 2Q/(\rho\Omega^2).
\]
Problem (9) can be reduced to the following boundary value problem for a single integro-
differential equation:
\[
\frac{\partial u}{\partial t} - 2 \int_0^z \mathrm{Re} u \, \frac{\partial u}{\partial z} \, dz - \frac{\partial^2 u}{\partial z^2} + u^2 + \Gamma = 0,
\]
\[
u|_{z=0} = i, \quad u|_{z=+\infty} = \sqrt{-\Gamma},
\]
\[0 \leq z < +\infty, \quad t \geq 0.\]  

(10)

As the initial value \(u_0(z)\) in (9), (10), one can take any smooth function on \([0, +\infty]\) satisfying the boundary conditions, for instance,
\[
u_0(z) = \frac{i + z\sqrt{-\Gamma}}{1 + z}, \quad 0 \leq z < +\infty.
\]

4. A Simple Tornado Model

The phenomenon of tornado is caused by physical processes occurring in the atmospheric air. To describe these processes we use the mathematical tools of continuum mechanics. It is necessary to take into account the rotation of the Earth with a constant angular velocity \(\Omega\) around the axis passing through the North (N) and South (S) Poles.

Consider two coordinate systems: system \(Oxyz\) is fixed, its origin \(O\) coincides with the Earth’s center and its axis \(Oz\) is directed along the rotation axis of the Earth; and system \(O'x'y'z'\) is moving, being rigidly fixed to the surface of the Earth, with its axis \(O'x'\) directed to the East along the parallel passing through \(O'\) and its axis \(O'y'\) directed to the North along the meridian passing through \(O'\) (Figure 1).

![Figure 1. Coordinate systems.](image)

We want to write the equation of atmospheric air dynamics in the moving coordinate system. Assume that the fixed coordinate system is inertial. Then, according to Newton’s second law, the equation of motion of a material point of mass \(m\) in the gravity field near the surface of the Earth and the origin \(O'\) has the form
\[
mw_a = mg, \quad g = -g_r \approx -\frac{g_0}{r_0},\]  

(11)

where \(r_0\) is the radius vector of the point \(O'\), \(r\) is the radius vector of a moving point in the fixed coordinate system, \(g = 980.665 \text{ cm/s}^2\) is the acceleration of gravity near the surface, \(w_a\) is the absolute acceleration of a point. Assuming that the speed of rotation of the Earth is constant, \(\Omega = \text{const}\), we have (see [25,26])
\[
w_a = w_r + 2[\Omega, v_r] + [\Omega, [\Omega, r']] + \frac{dv_0}{dt},\]  

(12)
where \( \mathbf{v}_r, \mathbf{w}_r \) are, respectively, the velocity and the acceleration of a moving point, \( \mathbf{r}' \) is the radius vector of a point in the moving coordinate system, \( \mathbf{r} = (0, 0, \Omega) \) in the fixed system, and \( \mathbf{r} = (0, \Omega \cos \varphi, \sin \varphi) \) in the moving coordinate system, \( \varphi \) is the latitude of \( O' \). In the moving coordinate system, \( \mathbf{g} = (0, 0, -g) \) and \( \mathbf{r}_0(t) = r_0(\cos \varphi \cos \Omega t, \cos \varphi \sin \Omega t, \sin \varphi) \). From (11), (12), we obtain the equation

\[
mw_r = mg - m \frac{d\mathbf{v}_0}{dt} - 2m[\mathbf{\Omega}, \mathbf{v}_r] - m[\mathbf{\Omega}, [\mathbf{\Omega}, \mathbf{r}']], \quad \mathbf{v}_0 = \frac{d\mathbf{r}_0}{dt}.
\]

From the explicit formula for \( \mathbf{r}_0 \), we get

\[
\frac{d\mathbf{v}_0}{dt} = \frac{d^2\mathbf{r}_0}{dt^2} = \frac{d}{dt}[\mathbf{\Omega}, \mathbf{r}_0] = 2[\mathbf{\Omega}, [\mathbf{\Omega}, \mathbf{r}_0]].
\]

Substituting this into the preceding relation and taking into account the obvious identity \( \mathbf{r} = \mathbf{r}_0 + \mathbf{r}' \), we finally obtain

\[
mw_r = \left[ mg - 2m[\mathbf{\Omega}, \mathbf{v}_r] - m[\mathbf{\Omega}, [\mathbf{\Omega}, \mathbf{r}']] \right] = \mathbf{F}_g + \mathbf{F}_{Cor} + \mathbf{F}_{cen}.
\]

Assuming air to be a continuous medium, making the replacements \( m \to \rho, \mathbf{w}_r \to d\mathbf{U}/dt, \mathbf{v}_r = \mathbf{U} \) (i.e., applying equation (13) to the unit air volume), and taking into account viscous stresses and hydrodynamic air pressure, we obtain the following system from (13):

\[
\begin{align*}
\frac{d\rho}{dt} + \rho \text{div} \mathbf{U} &= 0, \\
\frac{d\mathbf{U}}{dt} &= \mathbf{g} - 2[\mathbf{\Omega}, \mathbf{U}] - [\mathbf{\Omega}, [\mathbf{\Omega}, \mathbf{r}]] + \frac{\mu}{\rho} \Delta \mathbf{U} - \frac{\nabla \rho}{\rho}, \\
\frac{d}{dt} &= \frac{\partial}{\partial r} + \mathbf{U} \cdot \nabla, \quad \mathbf{r} = \mathbf{r}_0(t) + \mathbf{r}',
\end{align*}
\]

where all spatial derivatives are in the variables \( \mathbf{r}' \). System (14) should be supplemented by an equation for pressure \( p \), but for the sake of simplicity, we consider only incompressible air flows in tornado, and therefore, assume that \( \rho = \text{const} \), \( \text{div} \mathbf{U} = 0 \). Let us compare the orders of magnitude of the terms in the right-hand side of (14). Since \( r = |\mathbf{r}| \cong R_E = 6371 \text{ km} = 6.371 \cdot 10^8 \text{ cm}, \Omega = 7.3 \cdot 10^{-5} \text{ s}^{-1}, \mu/\rho = 0.15 \text{ cm}^2/\text{s} \) (for air temperature 20°C [24]), we have: \( |\mathbf{F}_g|/|\mathbf{F}_{cen}| \geq g/(\Omega^2 R_E) \cong 3 \cdot 10^{-2} \), and thus, the centrifugal force is much smaller than the Coriolis force. On the other hand, we have \( |\mathbf{F}_g|/|\mathbf{F}_{Cor}| \geq g/(2|\mathbf{U}|) = 671.7 \cdot 10^{-4}/|\mathbf{U}| \). Assuming that the air speed in a tornado is about \( |\mathbf{U}| \cong 100 \text{ m/s} = 10^4 \text{ cm/s} \) [3,6], we conclude that the Coriolis force is hundreds of times smaller than the gravity force. Now, let us compare the Coriolis force with the viscous stresses: \( |\mathbf{F}_{Cor}|/(\mu \rho^{-1} \Delta^2 \mathbf{U}) \cong 2|\mathbf{U}|^2/(\mu \rho^{-1}) = 1.82 \cdot 10^{-4} \cdot L_0^2 \), where \( L_0 \) is the characteristic length. It follows that when studying small-scale processes (\( L_0 \sim 1 \text{ cm} \)), we can neglect the Coriolis force, as compared with viscous stresses, while for large-scale processes (\( L_0 \sim 1 \text{ m} \)), the Coriolis force and viscous stresses are of the same order of magnitude. Below, we take into account viscous effects (which are essentially small-scale) and show that they are responsible for swirling and vertical acceleration of air masses typical for tornado. It should be emphasized (and this is one of the main results of this work) that, in our model, the said phenomena are caused exclusively by the viscous effects and not by the Coriolis force or convective processes. Therefore, in what follows, we neglect the Coriolis and the centrifugal forces and our final system of equations takes the form (1).

Note that the incompressible fluid approximation is justified if \( U^2 \ll c_s^2 \), where \( c_s \) is the speed of sound in air. According the the Fujita scale [2], the maximal value of \( U \) in a powerful tornado is about \( U \cong 100 \text{ m/s}, \) while the average speed of sound in air is \( c_s \cong 330 \text{ m/s}; \) thus, \((U/c_s)^2 \ll 0.1\). For a tornado of medium power, the Fujita scale [2] yields \((U/c_s)^2 \cong 10^{-2}\). Therefore, it can be assumed that the relation \( (U/c_s)^2 \ll 1 \) is approximately satisfied for air flows in tornado, and this justifies the above incompressible fluid approximation for this phenomenon.

Our numerical analysis of some particular solutions of system (1) in the form (2) shows that they describe the motion of air masses typical for tornadoes.
5. Numerical Solution of the Boundary Value Problem (9)
Consider an interval \([0, L]\) with a sufficiently large \(L\) (the results should not substantially change with the growth of \(L\)) and a uniform grid \(x_k = kh\) with integer \(0 \leq k \leq N, N = L/h,\) and step \(h > 0.\) We use the difference scheme
\[
\frac{u^1_k - u^0_k}{\tau} + C_k u^1_{k+1} - u^1_{k-1} - \frac{u^1_{k+1} - 2u^1_k + u^1_{k-1}}{2h} + u^0_k u^0_k + \Gamma = 0, \quad 0 < k < N,
\]
and the sweep method to find a solution \(u^1_k, 0 < k < N,\) of the following system of linear equations:
\[
u^1_{k+1} \left[ \frac{\tau}{2h} C^0_k - \frac{\tau}{h^2} \right] + u^1_k \left[ 1 + \tau u^0_k + \frac{2\tau}{h^2} \right] + u^1_{k-1} \left[ -\frac{\tau}{2h} C^0_k - \frac{\tau}{h^2} \right] = u^0_k - \tau \Gamma, \quad 0 < k < N,
\]
\[
u^1_0 = i, \quad u^1_N = \sqrt{-\Gamma}.
\]
We seek \(C\) on each upper layer as a solution of the following Cauchy problem on the interval \([0, L]\):
\[
\frac{dC^1_k}{dz} = -2\text{Re} u^1_k, \quad C^1(0) = 0,
\]
taking into account that the right-hand side of (17) is given at the nodes \(x_k.\) Since the approximation order of system (15) is \(O(\tau + h^2),\) we can use a method of a relatively small order for finding a solution of problem (17). For instance, we can use the Runge-Kutta second order method, which reduces to the quadrature trapezoid formula
\[
C^1_k = C^1_{k-1} - h \left( A^1_k + A^1_{k-1} \right), \quad k = 1, \ldots, N, \quad C^1_0 = 0,
\]
\[
A^1_k = \text{Re} u^1_k.
\]
The sweep stability condition is satisfied if the diagonal coefficients are predominant in the matrix of system (16),
\[
\left| \frac{\tau}{2h} C^0_k - \frac{\tau}{h^2} \right| + \left| \frac{\tau}{2h} C^0_k + \frac{\tau}{h^2} \right| < \left| 1 + \tau u^0_k + \frac{2\tau}{h^2} \right|, \quad 1 \leq k \leq N - 1.
\]
This condition holds a fortiori if
\[
\tau < \frac{1}{2 \max_{0 \leq k \leq N} \left| C^0_k \right|}, \quad \tau < \frac{1}{2 \max_{0 \leq k \leq N} \left| u^0_k \right|}.
\]
The inequalities (19) yield a constraint on the time step \(\tau.\)

The simplest condition for stopping the iterations for the stabilization problem has the form
\[
\max_{0 \leq k \leq N} \max \left\{ \left| u^1_k - u^0_k \right|, \left| C^1_k - C^0_k \right| \right\} < \varepsilon,
\]
where \(\varepsilon\) is a given small positive parameter, its typical value being equal to \(\varepsilon = 10^{-9}.\) As soon as the last inequality is satisfied, the calculations stop and \(u^1_k, C^1_k, 0 \leq k \leq N,\) are taken as the grid approximations of the desired stationary flow. The above algorithm for finding stationary solutions of the boundary value problem (9) is much simpler than direct methods, such as the shooting method [27] or the Bellman quasilinearization method [28], used for finding a solution of the boundary value problem for the nonlinear 5th order system of ODE’s obtained from (9)
by setting $\partial/\partial t = 0$. This can be explained by the fact that in the latter algorithms, the key role belongs to the Newton method for solving systems of nonlinear equations and the convergence of this method is determined by the initial approximation of the unknown solution, which makes calculations problematic.

Prior to describing our calculation results, we discuss the role of viscosity with regard to the existence of solutions corresponding to tornadoes.

6. The Role of Viscosity

In the case of zero viscosity, $\mu = 0$, the boundary value problem (9) does not make sense, since the boundary conditions have no physical meaning. One can try to reformulate the boundary value problem in the case of $\mu = 0$, but most likely with no result, since (as it is shown below) among the stationary solutions of system (9) with $\mu = 0$, there are no flows whose the vertical velocity grows and stabilizes as $z \to +\infty$. There is a possibility that such flows exist among nonstationary solutions of the form (2). This is an important problem, since real tornadoes are nonstationary and the funnel moves over the surface of the Earth. However, in order to analyze these processes, the original setting of the problem should take into account additional physical factors [29,30], for instance, winds over the surface. Moreover, for $\mu = 0$, the Coriolis force might take precedence [31,32], but then functions of the form (2) would not yield particular solutions of system (1) with the additional Coriolis term and the above approach to finding tornado solutions of the form (2) would become inconsistent. Note also that the role of viscosity in the generation of vortex flows is well known: the layers of a viscous fluid adhere to each other, transferring their motion to the neighboring layers, which contributes to the formation of vortices. It should also be taken into account that air in a real tornado contains a lot of water and sand dust (depending on whether it is formed above the surface of water or land), which substantially increases the hydrodynamic viscosity of the air mixture. Viscosity is also responsible for the transformation of azimuthal and radial movements of air in a tornado into a vertical movement.

Thus, consider system (9) with $\partial/\partial t = 0$:

$$C \frac{du}{dz} + u^2 + \Gamma = 0, \quad \frac{dC}{dz} = -2Re u. \tag{20}$$

It turns out that one can find all solutions of system (20) in explicit form.

**Theorem 2.** 1) For $\Gamma = 0$, the general solution of system (20) has the form $u = A + iB$ with

$$A(z) = -\frac{Rk}{2} \sin 2kz, \quad B(z) = Rk \sin^2 k z, \quad C(z) = R \sin^2 k z, \tag{21}$$

where $R, k \in \mathbb{R}$ are arbitrary real constants. Moreover, there is a particular solution of the form

$$A(z) = -Rz, \quad B(z) = 0, \quad C(z) = Rz^2, \tag{22}$$

where $R \in \mathbb{R}$ is an arbitrary constant.

2) For $\Gamma > 0$, the general solution of system (20) has the form

$$A(z) = -\sqrt{T} \frac{\sin 2pz}{\sinh 2\sqrt{T}R}, \quad B(z) = -\sqrt{T} \left( \frac{\cosh 2pz}{\sqrt{T}R} + \frac{\sinh 2pz}{\sqrt{T}R} \right), \quad C(z) = \frac{1}{2D} \left( \frac{\cos^2 pz}{\cosh 2\sqrt{T}R} + \frac{\sin^2 pz}{\sinh 2\sqrt{T}R} \right), \tag{23}$$

where $p = D\sqrt{T} \sinh 2\sqrt{T}R, R \neq 0, D \neq 0$ are arbitrary constants. Moreover, there is a particular solution of the form

$$A(z) = -\Gamma Dz, \quad B(z) = 0, \quad C(z) = \frac{1}{D} + \Gamma Dz^2, \tag{24}$$
where $D \neq 0$ is an arbitrary constant.

Theorem 2 implies that in the absence of viscosity, $\mu = 0$, the vertical speed of the stationary flow (2) is a periodic function of the height or, in exceptional cases, it has quadratic dependence on the height, which does not agree with the vertical speed variation in a tornado.

7. Some Calculation Results

Our numerical analysis of stationary solutions of the boundary value problem (9) is based on approximate solutions of this problem obtained by the stabilization method described above. These calculation results do not depend on the initial data. For instance, one can take $u(0, z) = (i + z\sqrt{-\Gamma})/(1 + z)$, $0 \leq z < +\infty$. The algorithm was tested on known solutions: for $\Gamma = 0$, we obtain the von Karman solution [23,24] and for $\Gamma = 1$, the constant solution $u(z) \equiv i$, $C(z) \equiv 0$. For $0 \leq \Gamma < 1$, the vertical speed $C(z)$ obtained in our calculations is always negative (“vacuum cleaner”) and for $\Gamma > 1$, it is positive (“tornado”). Thus, $\Gamma = 1$ is the bifurcational value of the parameter $\Gamma$ equal to the squared angular vortex velocity in upper atmosphere. Typical profiles of $A(z)$, $B(z)$, $C(z)$ for stationary flows for various values of $\Gamma$ are represented in Figure 2. Thus, the vertical velocity $C(z)$ for $\Gamma > 1$ is positive and tends to the limit value $C(\infty) > 0$ as $z \to +\infty$. The dependence of $C(\infty)$ on $\Gamma$, obtained numerically, is represented in Figure 3, which shows that with the growth of the angular vortex velocity in upper atmosphere, $C(\infty)$ grows monotonically as $\sim \sqrt{\Gamma}$ for large $\Gamma$. Another important result, which has been obtained for the left boundary condition $u(0) = 0$ and the other boundary data unchanged, shows (see Figure 4) that the presence of vortex motion near the surface due to, say, the Coriolis force, cannot be regarded as a necessary condition for the birth of tornado. Moreover, comparing the graphs of the vertical velocity in Figures 2 and 4, we see that without any vortex motion at $z = 0$, the limit vertical velocity $C(\infty)$ has increased $\approx 2.5$ times. Thus, air rotation near the surface reduces the air flow energy at infinity, while air rotation in the upper atmosphere increases that energy (see Figure 2). Another important consequence of our calculations is that if the rotation of air at infinity and that near the origin have opposite directions (say, $u(0) = i$, $u(\infty) = -i\sqrt{\Gamma}$), then no stabilization of the solution is observed, which means that in this case the problem has no stationary solutions.

![Figure 2](image-url)

**Figure 2.** Steady profiles of $C(z)$, $A(z)$, $B(z)$ for $\Gamma$: (1) $\Gamma = 0$; (2) $\Gamma = 0.5$; (3) $\Gamma = 1$; (4) $\Gamma = 4$. 
8. Conclusion
The numerical analysis described in this article can be used for some practical recommendations with regard to the tornado problem. First, it is necessary to monitor upper layers of the atmosphere for the presence of vortices and make a map of air pressure near the surface of the Earth. Secondly, tornadoes can occur only in those areas of land or water surface which are located under the corresponding vortices in upper atmosphere. A necessary condition for the emergence of a tornado is

$$\Omega^2 = \frac{1}{\rho} \left\langle \frac{\partial^2 p}{\partial r^2} \right\rangle,$$

(25)

where $\Omega$ is the angular velocity of a vortex, $\rho$ is the air density, $\left\langle \frac{\partial^2 p}{\partial r^2} \right\rangle$ is the spatial average of the second radial derivative of air pressure, the radius being measured from the center of a possible tornado, i.e., the intersection of the Earth’s surface with the straight line passing through the center of the Earth and that of a vortex in upper atmosphere. (Note that the constant $Q$ in (2) is equal to $Q = 1/2 \cdot \partial^2 p/\partial r^2$). Thirdly, to prevent a tornado, one should either create an anti-vortex, at least of low intensity, in the center of the supposed tornado near the Earth’s surface, or change the pressure on the periphery of the expected tornado in order to violate condition (25). The above considerations allow us to conclude that the role of the Coriolis force is opposite to the role usually attributed to it. The Coriolis force, which causes spinning of air currents near the surface of the Earth, does not lead to the formation of a tornado [1,2], but, on the contrary, prevents the emergence of a tornado if the direction of air rotation at the surface due to the Coriolis force is opposite to the direction of rotation of air masses in the upper layers of the atmosphere.

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