Mild solutions to the dynamic programming equation for stochastic optimal control problems

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Abstract

We show via the nonlinear semigroup theory in $L^1(\mathbb{R})$ that the 1-D dynamic programming equation associated with a stochastic optimal control problem with multiplicative noise has a unique mild solution $\varphi \in C([0,T]; W^{1,\infty}(\mathbb{R}))$ with $\varphi_{xx} \in C([0,T]; L^1(\mathbb{R}))$. The $n$-dimensional case is also investigated.

1 Introduction

Consider the following stochastic optimal control problem

Minimize $E\left\{ \int_0^T \left( g(X(t)) + h(u(t)) \right) dt + g_0(X(T)) \right\}$, \hspace{1cm} (1)

subject to $u \in U$ and to state equation

$$\begin{cases}
    dX = f(X) \, dt + \sqrt{u} \sigma(X) \, dW, & \text{for } t \in (0,T) \\
    X(0) = X_0
\end{cases}$$ \hspace{1cm} (2)

where $U$ is the set of all $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes $u : (0,T) \to \mathbb{R}^+ = [0,\infty]$ and $W : \mathbb{R} \to \mathbb{R}$ is an 1-D Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, provided the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Here $X_0 \in \mathbb{R}$, while $X : [0,T] \to \mathbb{R}$ is the strong solution to (2).

We would like to underline that the studied optimization problem is related to the so called stochastic volatility models, used in the financial framework, whose relevance has raised exponentially during last years. In fact such models, contrarily to the constant volatility ones as, e.g., the standard Black and Scholes approach, the Vasicek interest rate model, or the Cox-Ross-Rubenstein model, allow to consider the more realistic situation of volatility levels changing in time. As an example, the latter is the case of the Heston model, see [9], where the variance is assumed to be a stochastic process following a Cox-Ingersoll-Ross (CIR) dynamic, see [10] or [4] and references therein for more recent related techniques, as well as the case of the Constant Elasticity of Variance (CEV) model, see [5], where the volatility is expressed by a power of the underlying level, which is often referred as a local stochastic volatility model. Other interesting examples, which is the object of our ongoing research particularly from the numerical point of view, include the Stochastic Alpha, Beta, Rho (SABR) model, see, e.g., [8], and models which are used to estimate the stochastic volatility by exploiting directly markets data, as happens using the GARCH approach and its variants.

Within latter frameworks and due to several macroeconomic crises that have affected different (type of) financial markets worldwide, governments decided to become active players of the game, as, e.g., in the recent case of the Volatility Control Mechanism (VCM) established for the securities, resp. for the derivatives, market established in August 2016, resp. in January 2017, within the Hong Kong Stock Exchange (HKEX) framework, see, e.g., [12, 13] and

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Hypotheses:

1. $h : \mathbb{R} \rightarrow \mathbb{R}$ is convex, continuous and $h(u) \geq \alpha_1 |u|^2 + \alpha_2$, $\forall u \in \mathbb{R}$, for some $\alpha_1 > 0, \alpha_2 \geq 0$.

2. $f \in C^2_b(\mathbb{R})$, $f'' \in L^1(\mathbb{R})$, $g, g_0 \in W^{2,\infty}(\mathbb{R})$.

3. $\sigma \in C^1_b(\mathbb{R})$, and

$$|\sigma(x)| \geq \rho > 0, \ \forall x \in \mathbb{R}. \quad (3)$$

We set

$$H(u) = h(u) + I_{[0,\infty)}(u) = \begin{cases} h(u) & \text{if } u \geq 0 \\
+\infty & \text{otherwise} \end{cases}$$

and we denote by $H^*$ the Legendre conjugate of $H$, namely,

$$H^*(p) = \sup \{ pu - H(u) : u \in \mathbb{R}, \ \forall p \in \mathbb{R} \}. \quad (4)$$

Moreover, if $\varphi$ is a smooth solution to (6) the associated feedback controller

$$u(t) = \arg \min_u \left\{ \frac{1}{2} \sigma^2 \varphi_{xx}(t, X(t)) u + H(u) \right\}, \quad (8)$$

is optimal for problem (1).

Up to our knowledge, in literature the rigorous treatment of existence theory for equation (6) has been shown, so far within the theory of viscosity solutions only. (See, e.g., [6].) Here we shall exploit a different approach, namely we use a suitable transformation aiming at reducing (6) to a one dimensional Fokker-Planck equation which is then treated as a nonlinear Cauchy problem in $L^1(\mathbb{R})$. The $n$-dimensional case is also studied in section 4. As regards the non-degenerate hypothesis (3) it will be later on dispensed by assuming more regularity on function $\sigma$. (See section 4 below.)

### 1.1 Notation and basic results

We shall use the standard notation for functional spaces on $\mathbb{R}$. In particular $C^k_b(\mathbb{R})$ is the space of functions $y : \mathbb{R} \rightarrow \mathbb{R}$, differentiable of order $k$ and with bounded derivatives until order $k$. By $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, we denote the classical space of Lebesgue measurable $p$-integrable functions on $\mathbb{R}$ with the norm $||y||_p$ and by $H^k(\mathbb{R}^n)$, $W^{k,p}(\mathbb{R}^n)$, $k=1,2$, the standard Sobolev spaces on $\mathbb{R}^n$, $n=1,2$. We set also $y_x = y' = \partial y/\partial x$, $y_t = \partial y/\partial t$, $y_{xx} = \partial^2 y/\partial x^2$, for $x \in \mathbb{R}$ and $\Delta y(x) = \sum_{i=1}^n \partial_{x_i}^2 y$, for $x \in \mathbb{R}^n$. By $D'(\mathbb{R}^n)$ we denote the space of Schwartz distributions on $\mathbb{R}^n$.

**Definition 1.1 (Accretive operator)** Given a Banach space $X$, a nonlinear operator $A$ from $X$ to itself, with domain $D(A)$, is said to be accretive if $\forall u_i \in D(A), \forall v_i \in A u_i, \ i = 1, 2$, there exists $\eta \in J(u_1 - u_2)$ such that

$$\langle x(v_1 - v_2, \eta) \rangle_{X'} \geq 0, \quad (9)$$

where $X'$ is the dual space of $X$, $\langle \cdot, \cdot \rangle_{X'}$ is the duality pairing and $J : X \rightarrow X'$ is the duality mapping of $X$. (See, e.g., [11].)

An accretive operator $A$ is said to be $m$-accretive if
\( \mathbb{R}(\lambda I + A) = X \) for all (equivalently some) \( \lambda > 0 \), while it is said to be quasi-m-accretive if there is \( \lambda_0 \in \mathbb{R} \) such that \( \lambda_0 I + A \) is m-accretive.

We refer to [11] for basic results on m-accretive operators in Banach spaces and the corresponding associated Cauchy problem.

## 2 Existence results

We set

\[
y(t, x) = -\varphi_{xx}(T - t, x), \quad \forall t \in [0, T], x \in \mathbb{R},
\]

(10)

and we rewrite eq. (7) as

\[
\begin{cases}
y_t(t, x) - \left( H^*\left( \frac{\partial^2}{\partial x^2} y(t, x) \right) \right)_{xx} + \varphi_x(T - t, x) - 2f'(x)y(t, x) - f(x)y_x(t, x) = -g'(x), \\
y(0, x) = -g''(x), \quad x \in \mathbb{R}.
\end{cases}
\]

(11)

We recall (see [3] for details), that, for \( z \in L^1(\mathbb{R}) \), the equation

\[
-\Psi'' = z, \quad \text{in} \; \mathcal{D}'(\mathbb{R}),
\]

(12)

has a unique solution \( \Psi = \Phi(z) \in W^{1,\infty}(\mathbb{R}) \) and \( \|\Psi\|_{W^{1,\infty}(\mathbb{R})} \leq C\|z\|_1 \). Then by (10) we have

\[
\varphi(t, x) = -\Phi(y(T - t, x)) \in W^{1,\infty}(\mathbb{R}), \quad \forall t \in [0, T].
\]

(13)

Setting

\[
B y = -f''(\Phi(y))' - 2f'y, \quad \forall y \in L^1(\mathbb{R}),
\]

(14)

and taking into account that \( f' \in L^\infty(\mathbb{R}) \), \( f'' \in L^1(\mathbb{R}) \), and \( \|(\Phi(y))''\|_\infty \leq \|\Phi\|_{W^{1,\infty}(\mathbb{R})} \leq C\|y\|_1 \), we obtain for operator \( B \) the estimate

\[
\|By\|_1 \leq C\|y\|_1, \quad \forall y \in L^1(\mathbb{R}).
\]

(15)

Therefore eq. (11) can be rewritten as follows

\[
\begin{cases}
y_t - \left( H^*\left( \frac{\partial^2}{\partial x^2} y \right) \right)_{xx} - f y_x + B y = g_1, \quad \text{in} \; [0, T] \times \mathbb{R} \\
y(0) = y_0 \in L^1(\mathbb{R}),
\end{cases}
\]

where \( y_0 = -g''_0 \) and \( g_1 = -g'' \) in \( \mathcal{D}'(\mathbb{R}) \).

### Definition 2.1

The function \( y: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is said to be a mild solution to equation (10) if \( y \in C([0, T]; L^1(\mathbb{R})) \) and

\[
y(t) = \lim_{\epsilon \to 0} y_\epsilon(t) \in L^1(\mathbb{R}), \quad \forall t \in [0, T],
\]

(17)

\[
y_\epsilon(t) = y_\epsilon, \quad \text{for} \; t \in [i \epsilon, (i+1) \epsilon], \; i = 0, 1, \ldots, N = \left\lceil \frac{T}{\epsilon} \right\rceil,
\]

(18)

\[
\frac{1}{\epsilon}(y_\epsilon^{i+1} - y_\epsilon^i) - \left( H^*\left( \frac{\partial^2}{\partial x^2} y_\epsilon^{i+1} \right) \right)' - f(y_\epsilon^{i+1})' + B y_\epsilon^{i+1} = g_1, \quad \text{in} \; \mathcal{D}'(\mathbb{R}),
\]

(19)

\[
y_0 = y_0, \; y_\epsilon^i \in L^1(\mathbb{R}), \; i = 0, 1, \ldots, N.
\]

We have

### Theorem 2.2

Under hypotheses (1)-(3) eq. (11) has a unique mild solution \( y \). Assume further that \( j(\frac{\partial^2}{\partial x^2} y_0) \in L^1(\mathbb{R}) \). Then \( j(\frac{\partial^2}{\partial x^2} y_\epsilon) \in L^\infty([0, T]; L^1(\mathbb{R})) \) and \( \left( H^*\left( \frac{\partial^2}{\partial x^2} y_\epsilon \right) \right) \in L^2([0, T] \times \mathbb{R}) \).

Theorem 2.2 will be proven by using the standard existence theory for the Cauchy problem in Banach spaces with nonlinear quasi-m-accretive operators. Now taking into account that for \( y \in C([0, T]; L^1(\mathbb{R})) \) equation (12) uniquely defines the function \( \varphi \in C([0, T]; W^{1,\infty}(\mathbb{R})) \), by Theorem 2.2 we obtain the following existence result for the dynamic programming equation (6).

### Theorem 2.3

Under hypothesis (1)-(3) there is a unique mild solution

\[
\varphi \in C([0, T]; W^{1,\infty}(\mathbb{R})), \; \varphi'' \in C([0, T]; L^1(\mathbb{R})),
\]

(20)

to equation (6). Moreover, if \( h(\lambda u) \leq C \lambda h(u) \) \( \forall u \in \mathbb{R}, \lambda > 0 \) and \( j(-\frac{\partial^2}{\partial x^2} g_0) \in L^1(\mathbb{R}) \), then \( H^*\left( -\frac{\partial^2}{\partial x^2} \varphi_{xx}(T - t, x) \right) \in L^2([0, T] \times \mathbb{R}) \).

According to the Definition 2.1 and (13), by mild solution \( \varphi \) to equation (6), we mean a function \( \varphi \in C([0, T]; W^{1,\infty}(\mathbb{R})) \) defined by

\[
\varphi(t) = \lim_{\epsilon \to 0} \varphi_\epsilon(t) \in W^{1,\infty}(\mathbb{R}), \quad \forall t \in [0, T],
\]

(21)
For each \( D \), the controller can be computed explicitly by the finite difference scheme (16). Therefore, the feedback controller \( \bar{\nu} \) is well defined on \([0, T]\).

Remark 2.4 The principal advantage of Theorem 2.2 compared with standard existence results expressed in terms of viscosity solutions is the regularity of \( \varphi \) and the fact that the optimal feedback controller can be computed explicitly by the finite difference scheme (21)-(22). This will be treated in a forthcoming paper.

3 Proof of Theorem 2.2

The idea is to write equation (16) as a Cauchy problem of the form

\[
\begin{align*}
\frac{dy}{dt} + Ay + By &= g_1, \quad \text{in } [0, T] \\
y(0) &= y_0
\end{align*}
\]

(23)
in the space \( L^1(\mathbb{R}) \), where \( A \) is a suitable nonlinear quasi-m-accretive operator. The operator \( A : D(A) \subset L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}) \) is defined as follows

\[
A y = -\left( H^*\left( \frac{\sigma^2}{2} y \right) \right)^\prime - f y' \quad \text{in } D'(\mathbb{R}), \forall y \in D(A),
\]

(24)

\[
D(A) = \left\{ y \in L^1(\mathbb{R}) : H^*\left( \frac{\sigma^2}{2} y \right) \in L^\infty(\mathbb{R}), \quad Ay \in L^1(\mathbb{R}) \right\},
\]

where the derivatives are taken in the \( D'(\mathbb{R}) \) sense.

Lemma 3.1 For each \( \eta \in L^1(\mathbb{R}) \) and \( \lambda \geq \lambda_0 = ||y'||_\infty \), there exists a unique solution \( y = y(\eta) \) to equation

\[
\lambda y + Ay = \eta.
\]

Moreover, it holds

\[
||y(\eta) - y(\tilde{\eta})||_1 \leq (\lambda - \lambda_0)^{-1} ||\eta - \tilde{\eta}||_1,
\]

(26)

\( \forall \eta, \tilde{\eta} \in L^1(\mathbb{R}), \lambda > \lambda_0, \) hence \( A \) turns to be quasi-m-accretive in \( L^1(\mathbb{R}) \).

Proof. [Proof of Lemma 3.1] Assume first that \( \eta \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). For each \( \nu > 0 \) consider the equation

\[
\lambda y - \nu y'' - \left( H^*\left( \frac{\sigma^2}{2} y \right) \right)^\prime + \nu H^*\left( \frac{\sigma^2}{2} y \right) - fy' = \eta,
\]

(27)
in \( D'(\mathbb{R}) \). Equivalently,

\[
(\lambda - \nu^2)\left( \nu I - \frac{d^2}{dx^2} \right)^{-1} y + H^*\left( \frac{\sigma^2}{2} y \right) + vy
\]

\[
- \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (fy') = \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} \eta,
\]

(28)

where \( z = \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} y \) is defined by equation

\[
\nu z - z'' = y, \quad \text{in } D'(\mathbb{R}).
\]

(29)

Note that by Hypothesis (2) the operator \( \Gamma y = (\lambda - \nu^2)(\nu I - \frac{d^2}{dx^2})^{-1} y -(\nu I - \frac{d^2}{dx^2})^{-1}(fy') + vy \) is linear continuous in \( L^2(\mathbb{R}) \) and by (28) we have that

\[
\langle z, (y, y) \rangle_2 = \nu ||z||^2 + ||z'||^2.
\]

\[
-\langle \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (fy'), y \rangle_2 = -\langle f y', z \rangle_2
\]

\[
= \langle y, f' z + f z' \rangle_2 \leq ||f'||_\infty ||y||_2 ||z||_2 + ||f||_\infty ||y||_2 ||z'||_2.
\]

(30)

(31)

Here \( || \cdot ||_2 \) and \( \langle \cdot , \cdot \rangle_2 \) are the norm and the scalar product in \( L^2(\mathbb{R}) \), respectively, and by \( || \cdot ||_p, 1 \leq p \leq \infty \) we denote the norm of \( L^p(\mathbb{R}) \). We note that Hypothesis (1) and (2) imply that the function \( H^* \) is continuous, monotonically non-decreasing, and

\[
C_1 \leq H^* (\nu) \leq C_2 \nu^2, \quad \forall \nu \in \mathbb{R}.
\]

(32)

Furthermore, by (29), (31), we have

\[
(\Gamma y, y)_2 = \nu ||y||^2_2 + (\lambda - \nu^2)(\nu ||y||^2_2 + ||z'||^2_2)
\]

\[
\geq \nu ||y||^2_2 - (\lambda - \nu^2)(\nu ||z||^2_2 + ||z'||^2_2)
\]

\[
- ||y||_2 (||f'||_\infty ||z||_2 + ||f||_\infty ||z'||_2)
\]

\[
\geq \nu ||y||^2_2 - (\lambda - \nu^2)(\nu ||z||^2_2 + ||z'||^2_2)
\]

\[
- C(f) ||y||_2 (||z||_2 + ||z'||_2)
\]

\[
\geq \nu ||y||^2_2 - (\lambda - \nu^2)(\nu ||z||^2_2 + ||z'||^2_2)
\]

\[
- C(f) ||y||_2 (||z||_2 + ||z'||_2)
\]
The latter yields

\[ \langle \Gamma y, y \rangle \geq \frac{\nu}{2} \| y \|_2^2, \quad \lambda \geq C \left( \frac{1}{\nu} + \nu^2 \right), \forall \nu > 0, \tag{33} \]

where \( C \) is dependent on \( \nu \). By assumption (3) we have that the operator \( y \to \mathcal{H}(y) \equiv H^*(\frac{\sigma^2}{2} y) \) is maximal monotone in \( L^2(\mathbb{R}) \), hence, by [35], \( \Gamma \) is maximal monotone and coercive, i.e. positively definite, therefore we have

\[ \mathbb{R}(\Gamma + \mathcal{H}) = L^2(\mathbb{R}), \]

for \( \lambda \geq \lambda^* = C(\frac{1}{\nu} + \nu^2) \). Consequently, for each \( \nu > 0 \) and \( \lambda \geq \lambda^* \), eq. [28] (equivalently eq. [24]) has a unique solution \( y = y_{\lambda, \nu} \in L^2(\mathbb{R}) \), with \( H^*(\frac{\sigma^2}{2} y_{\lambda, \nu}) \in L^2(\mathbb{R}) \).

We have also

\[ z_{\lambda, \nu} + z_{\nu, \lambda}^* \in L^2(\mathbb{R}), \]

so that \( z_{\lambda, \nu} \in H^2(\mathbb{R}) \).

Since by assumption (3) the operator \( z \to \nu z + H^*(\frac{\sigma^2}{2} z) \) is invertible in \( L^2(\mathbb{R}) \), and its inverse maps \( H^1(\mathbb{R}) \) into itself, we infer that \( y_{\lambda, \nu} \in H^1(\mathbb{R}) \).

It is worth to mention that by [24], we have

\[ \lambda \| y_{\lambda, \nu}(\eta) - y_{\lambda, \nu}(\eta') \|_1 \leq \| f' \|_\infty \| y_{\lambda, \nu}(\eta) - y_{\lambda, \nu}(\eta') \|_1 + \| \eta - \eta' \|_1 \]

\[ \forall \eta, \eta' \in L^1(\mathbb{R}), \] so that

\[ \| y_{\lambda, \nu}(\eta) - y_{\lambda, \nu}(\eta') \|_1 \leq \frac{1}{\lambda - \lambda_0} \| \eta - \eta' \|_1, \tag{34} \]

\[ \forall \eta, \eta' \in L^1(\mathbb{R}), \text{ for } \lambda \geq \max(\lambda_0, \lambda^*) \text{ and where } \lambda_0 = \| f' \|_\infty. \] To get [34], we simply multiply the equation

\[ \lambda(y_{\lambda, \nu}(\eta) - y_{\lambda, \nu}(\eta')) - \nu(y_{\lambda, \nu}(\eta) - y_{\lambda, \nu}(\eta'))'' + \nu \left( H^*(\frac{\sigma^2}{2} y_{\lambda, \nu}(\eta)) - H^*(\frac{\sigma^2}{2} y_{\lambda, \nu}(\eta')) \right) - \left( H^*(\frac{\sigma^2}{2} y_{\lambda, \nu}(\eta)) - H^*(\frac{\sigma^2}{2} y_{\lambda, \nu}(\eta')) \right)'' + f(y_{\lambda, \nu}(\eta) - y_{\lambda, \nu}(\eta'))' = \eta - \eta' \]

by \( \zeta \in L^\infty(\mathbb{R}) \)

\[ \zeta \in \text{sgn}(y_{\lambda, \nu}(\eta) - y_{\lambda, \nu}(\eta')) = \]

\[ = \text{sgn} \left( H^*(\frac{\sigma^2}{2} y_{\lambda, \nu}(\eta)) - H^*(\frac{\sigma^2}{2} y_{\lambda, \nu}(\eta')) \right), \]

where \( \text{sgn} r = \frac{r}{|r|} \) for \( r \neq 0 \), \( \text{sgn} 0 = [-1, 1] \) and we integrate on \( \mathbb{R} \), taking into account that

\[ - \int_\mathbb{R} y'' \text{sgn} y dx \geq 0, \quad \forall y \in H^1(\mathbb{R}), \]

\[ \int_\mathbb{R} f' \text{sgn} y dx = \int_\mathbb{R} f' \text{sgn} y dy. \]

For a rigorous proof of these relations we replace \( \text{sgn} y \) by \( X_\delta(y) \), where \( X_\delta \) is a smooth approximation of signum function, while \( \delta \to 0 \), see, e.g., [1], p. 115. If \( y \in L^1(\mathbb{R}) \) and \( \{ \eta_n \}_{n=1}^\infty \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) is strongly convergent to \( \eta \in L^1(\mathbb{R}) \), we can proceed as above to obtain for the corresponding solution \( y_n \) to [24] the estimate [34], namely,

\[ \| y_n - y_m \|_1 \leq (\lambda - \lambda_0)^{-1} \| \eta_n - \eta_m \|_1, \quad \forall \lambda > \max(\lambda^*, \lambda_0). \]

Hence there exists \( y \in L^1(\mathbb{R}) \) such that

\[ y_n \to y \quad \text{in } L^1(\mathbb{R}) \text{ as } n \to \infty. \tag{35} \]

By [28], we have

\[ (\lambda - \nu^2) \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} y_n + H^*(\frac{\sigma^2}{2} y_n) + \nu y_n \]

\[ - \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f y_n) = \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} \eta_n. \tag{36} \]

By [12] and [29], we have

\[ \| z_n \|_{W^{1, \infty}(\mathbb{R})} \leq \| \nu z_n - y_n \|_1 \leq (\nu + 1) \| y_n \|_1. \tag{37} \]

Let \( \theta_n := \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f y_n) \),

that is \( \nu \theta_n - \theta_n'' = f y_n' = (f y_n)' - f' y_n \) in \( D'(\mathbb{R}) \).

Equivalently

\[ \nu \left( \theta_n(x) + \int_0^x f y_n d\xi \right)' - \left( \theta_n(x) + \int_0^x f y_n d\xi \right)'' = \]

\[ = \nu \int_0^x f y_n d\xi - f' y_n. \tag{38} \]
This yields
\[
\left\| \nu \theta_n + \nu \int_{0}^{x} f y_n d\xi \right\|_1 \leq \nu \left\| \int_{0}^{x} f y_n d\xi \right\|_1 + \|f' y_n\|_{L^1} \leq \nu \|f\|_\infty \|y_n\|_1 + \|f'\|_\infty \|y_n\|_1
\]
and then
\[
\nu \|\theta_n\|_1 \leq ((\nu + 1)\|f\|_\infty \|y_n\|_1 + \|f'\|_\infty) \|y_n\|_1.
\]
On the other hand, by (38), we have
\[
\left\| \theta_n + \nu \int_{0}^{x} f y_n d\xi \right\|_{W^{1,\infty}(\mathbb{R})} \leq \nu \|\theta_n + \nu \int_{0}^{x} f y_n d\xi\|_1 + \nu \|f y_n - f' y_n\|_1 \leq \nu \|\theta\|_1 + (2\nu \|f\|_\infty + \|f'\|_\infty) \|y\|_1.
\]
Hence
\[
\|\theta_n\|_{W^{1,\infty}(\mathbb{R})} \leq ((3\nu + 1)\|f\|_\infty + 2\|f'\|_\infty) \|y_n\|_1.
\]
This yields
\[
\left\| \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f y_n) \right\|_\infty \leq C \|y_n\|_1 \leq \frac{C_1}{\lambda - \lambda_0} \|\eta_n\|_1
\]
and therefore, by (36), we derive the estimate
\[
\left\| H^* \left( \frac{\sigma^2}{2} y_n \right) + \nu y_n \right\|_1 \leq C \|y_n\|_1 \leq \frac{C_1}{\lambda - \lambda_0} \|\eta_n\|_1.
\]
Since, by hypothesis (1) \( H^*(\nu) \nu \geq 0, \forall \nu \in \mathbb{R} \), the latter implies that
\[
\left\| H^* \left( \frac{\sigma^2}{2} y_n \right) \right\|_\infty + \nu \|y_n\|_\infty \leq \frac{C_1}{\lambda - \lambda_0} \|\eta_n\|_1, \quad \forall n,
\]
where \( C_1 \) is still independent of \( n \) as well as on \( \nu \).
By (33) and (12), it follows that
\[
H^* \left( \frac{\sigma^2}{2} y_n \right) \overset{n \to \infty}{\longrightarrow} H^* \left( \frac{\sigma^2}{2} y \right),
\]
strongly in \( L^1(\mathbb{R}) \), and therefore \( y = y_{\lambda, \nu} \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \) solves (27). Furthermore, by (34) and (12), we have
\[
\|y_{\lambda, \nu}\|_1 + \left\| H^* \left( \frac{\sigma^2}{2} y_{\lambda, \nu} \right) \right\|_\infty + \nu \|y_{\lambda, \eta}\|_\infty \leq \frac{C_1}{\lambda - \lambda_0} \|\eta\|_1
\]
\[\forall \lambda > \max(\lambda^*, \lambda_0), \quad \text{where} \quad C_1 \quad \text{is independent of} \quad \nu. \]
We also obtain that inequality (34) holds for solution \( y_{\lambda, \nu} \) to (27), with \( \eta \in L^1(\mathbb{R}) \) only. Now we are going to extend the solution \( y_{\lambda, \nu} \) to (27) for all \( \lambda > \lambda_0 \). To this end we set \( G_\lambda = \Gamma + H \), rewriting (27) as follows
\[
G_\lambda = \eta. \quad \forall \lambda > 0, \quad \text{we can equivalently write this as}
\]
\[
y = (G_{\lambda+\delta})^{-1}(\eta) + \delta(G_{\lambda+\delta})^{-1}(\eta) \quad \text{for every} \quad \lambda > 0.
\]
By (33) we also have
\[
\left\| (G_{\lambda+\delta})^{-1} \right\|_{L(L^1(\mathbb{R}), L^1(\mathbb{R}))} \leq \frac{1}{\lambda - \lambda_0},
\]
then, by contraction principle, (33) has a unique solution \( y = y_{\lambda, \nu} \in L^1(\mathbb{R}) \), for all \( \lambda > \lambda_0 \). Estimate (36) extends for all \( \lambda > \lambda_0 \). In order to complete the proof of Lemma (31) we are going to let \( \nu \to 0 \) in equation (27), or, more precisely, in (28) which holds for all \( \lambda > \lambda_0 \). As noted before, for all \( z \in L^1(\mathbb{R}) \), we have
\[
\left\| \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} z \right\|_{W^{1,\infty}(\mathbb{R})} \leq C \|z\|_1
\]
and
\[
\lim_{\nu \to 0} \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} z = \left( \frac{d^2}{dx^2} \right)^{-1} z \text{ in } W^{1,\infty}(\mathbb{R})
\]
consequently
\[
\left\| \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f z') \right\|_\infty \leq C \|z\|_1
\]
and
\[
\lim_{\nu \to 0} \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f z') = \lim_{\nu \to 0} \left( \frac{d}{dx} \right) \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f z)
\]
\[+ \lim_{\nu \to 0} \left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f' z) \quad \text{strongly in } L^\infty(\mathbb{R}).
\]
We set \( u_\nu = (\nu I - \frac{d^2}{dx^2})^{-1} y_{\lambda, \nu} \). Then, for \( \nu \to 0 \), we have \( \nu u_\nu \to 0 \) in \( L^1(\mathbb{R}) \) and

\[-\lim u'' = \lim y_{\lambda, \nu} = y \text{ in } D'(\mathbb{R}).\]

Hence

\[
\left( \nu I - \frac{d^2}{dx^2} \right)^{-1} y_{\lambda, \nu} \to \left( -\frac{d^2}{dx^2} \right)^{-1} y
\]

strongly in \( W^{1, \infty}(\mathbb{R}) \), and

\[
\left( \nu I - \frac{d^2}{dx^2} \right)^{-1} (f y_{\lambda, \nu}) \to \left( -\frac{d^2}{dx^2} \right)^{-1} (f y')
\]

strongly in \( L^\infty(\mathbb{R}) \), where \( y \in L^1(\mathbb{R}) \), and

\[
\lambda y - H^* \left( \frac{\sigma^2}{2} y'' \right) - f y' = \eta \text{ in } D'(\mathbb{R}),
\]

for \( \lambda > \lambda_0 \). Moreover, by (24), the map \( \eta \to y \) is Lipschitz in \( L^1(\mathbb{R}) \), with Lipschitz constant \( (\lambda - \lambda_0)^{-1} \), then \( y \) solves (25), and (26) follows. This completes the proof of Lemma 3.1.

Proof. [Proof of Theorem 2.2 (continued)] Coming back to equation (24), by Lemma 3.1 and (14), it follows that the operator \( A + B \) is quasi-m-accretive in \( L^1(\mathbb{R}) \). Then by the Crandall & Liggett theorem, see [1], p. 147, the Cauchy problem (23) has a unique solution \( y \) in \( C([0, T]; L^1(\mathbb{R})) \), that is

\[
y(t) = \lim_{\epsilon \to 0} y_\epsilon(t) \text{ in } L^1(\mathbb{R}), \forall t \in [0, T],
\]

\[
y_\epsilon(t) = y_\epsilon^i \text{ for } t \in [i\epsilon, (i + 1)\epsilon], i = 0, \ldots, N = \left[ \frac{T}{\epsilon} \right],
\]

\[
\frac{1}{\epsilon}(y_{\epsilon}^{i+1} - y_\epsilon^i) + (A + B)(y_\epsilon^i) = g_1 \quad i = 0, \ldots, N,
\]

\[
y_\epsilon^0 = y_0.
\]

The function \( y \) is a mild solution to (11) in the sense of Definition 2.1.

Assume now that \( j(\lambda v) \leq C \lambda j(v) \) \( \forall v \in \mathbb{R} \) and \( \lambda > 0 \). Taking into account that \( j(v) \leq \lambda (2\nu) - \nu H^*(v), \forall v \in \mathbb{R} \), it is easily seen that this implies that

\[
H^*(v) \leq (C_2 - 1) j(v), \quad \forall v \in \mathbb{R}.
\]

Assume also that \( j(\frac{\sigma^2}{2} y_0) \in L^1(\mathbb{R}) \). Then, if we take in (19), \( z^i = \frac{\sigma^2}{2} y^i_\epsilon \) and get

\[
\frac{2}{\sigma^2} (z^{i+1} - z^i) - (H^*(z^{i+1}))'' - f(\frac{2}{\sigma^2} z^{i+1})' + B\left(\frac{2}{\sigma^2} z^{i+1}\right) = g_1.
\]

Multiplying by \( H^*(z^{i+1}) \) and integrating on \( \mathbb{R} \) we get

\[
\frac{2}{\epsilon} \int_\mathbb{R} \frac{1}{\sigma^2} (j(z^{i+1}) - j(z^i)) \, dx + \int_\mathbb{R} \left( H^*(z^{i+1}) \right)^2 \, dx
\]

\[
+ 2 \int_\mathbb{R} f \left( \frac{z^{i+1}}{\sigma^2} \right) H^*(z^{i+1}) \, dx
\]

\[
+ 2 \int_\mathbb{R} B\left(\frac{z^{i+1}}{\sigma^2}\right) H^*(z^{i+1}) \, dx = \int_\mathbb{R} g_1 H^*(z^{i+1}) \, dx.
\]

Integrating by parts in \( \int_\mathbb{R} f \left( \frac{z^{i+1}}{\sigma^2} \right) H^*(z^{i+1}) \, dy \), summing up, after some calculation involving (14) and (27), we get the estimate \( \forall k \)

\[
2 \int_\mathbb{R} \frac{1}{\sigma^2} j(z^{i+1}) \, dx + \epsilon \sum_{i=0}^k \int_\mathbb{R} \left( H^*(z^{i+1}) \right)^2 \, dx \leq C,
\]

which implies the desired conclusion

\[
\left( H^* \left( \frac{\sigma^2}{2} y \right) \right) \in L^2((0, T) \times \mathbb{R}),
\]

\[
j \left( \frac{\sigma^2}{2} y \right) \in L^\infty([0, T]; L^1(\mathbb{R})).
\]

4 A multi-dimensional case

Consider the problem (11) in \( \mathbb{R}^n \) with the drift \( f \equiv 0 \), namely

\[
\text{Minimize } \mathbb{E} \left\{ \int_0^T g(X(t)) + h(u(t)) \, dt + g_0(X(T)) \right\},
\]

subject to \( u \in \mathcal{U} \), and to stochastic differential equation

\[
\begin{cases}
    dX = \sqrt{\sigma} \sigma(X) \, dW, & \text{in } (0, T) \times \mathbb{R}^n \\
    X(0) = X_0
\end{cases}
\]
Here \( W: [0, T] \rightarrow \mathbb{R}^m \) is a Wiener process, \( h : \mathbb{R} \rightarrow \mathbb{R} \) satisfies assumption (1) and

(i) \( g, g_0 \in W^{2,\infty}([0, T]; \mathbb{R}) \)

(ii) \( \sigma(x) = \sigma_0(x)a \), where \( \sigma_0 \in C^1_b(\mathbb{R}) \) satisfies condition (3), while the matrix \( a = \|a_{ij}\|_{i,j=1}^m \) is such that \( b = aa^T \) is positive defined.

Let \( L \) be the elliptic second order operator

\[
Lz(x) = \sum_{i,j=1}^n b_{ij} \frac{\partial^2 z(x)}{\partial x_i \partial x_j}, \quad \forall x \in \mathbb{R}^n
\]

where \( b_{ij} = \sum_{k=1}^m a_{ik}a_{jk}. \) The corresponding dynamic programming equation for \( \psi \) reads as follows

\[
\begin{aligned}
&\phi_t(t, x) + \min_{u} \left\{ \frac{1}{2} \sigma_0^2(x) L \phi_t(t, x) u + H(u) \right\} \\
&\phi(t, x) = g_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\]

If

\[
y(t, x) = -L \phi(T - t, x), \quad \forall t \in [0, T], x \in \mathbb{R}^n,
\]

equation (52) reduces to

\[
\begin{aligned}
&y_t(t, x) - L \left( H^* \left( \frac{\sigma_0^2}{2} y(t, x) \right) \right) = g_1(x), \\
&y(0, x) = y_0(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

(53)

see (11), where \( y_0 = -Lg_0, g_1 = -Lg. \) By (3), for \( z \in L^1(\mathbb{R}^n) \) the elliptic equation \( -L \psi = z \) in \( D'(\mathbb{R}^n) \) has a unique solution \( \psi \) which satisfies \( \psi \in W^{1,\infty}(\mathbb{R}) \) if \( n = 1, \psi \in W^{1,1}_0(\mathbb{R}^2) \) if \( n = 2 \) and \( \psi \in L^1_{loc}(\mathbb{R}) \cap M^{\infty,2}(\mathbb{R}^n) \) if \( n = 3 \), where here \( M^{\infty,2}(\mathbb{R}^n) \) is the Marcinkiewicz space. The latter implies that any solution \( y \in C([0, T]; L^1(\mathbb{R}^n)) \) to (53) leads to a unique solution \( \phi \in C([0, T]; W^{1,\infty}(\mathbb{R})) \) for \( n = 1, \phi \in C([0, T]; W^{1,1}_0(\mathbb{R}^2)) \), for \( n = 2 \) and, respectively, \( \phi \in C([0, T]; M^{\infty,2}(\mathbb{R}^n)) \) for \( n = 3 \). Concerning the existence of a solution to eq. (53), we have a result similar to the one stated in Theorem 2.2, namely

**Theorem 4.1** Under assumption (i)-(ii)-(iii) there is a unique mild solution \( y \in C([0, T]; L^1(\mathbb{R}^n)) \), in the sense of Definition 2.1.

**Proof.** We shall proceed as in the proof of Theorem 2.2. In particular, we consider the operator \( A : D(A) \subset L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n) \)

\[
Ay = -L \left( H^* \left( \frac{\sigma_0^2}{2} y \right) \right) \forall y \in D(A),
\]

(54)

\( D(A) = \{ y \in L^1(\mathbb{R}^n), \frac{\partial}{\partial t} \left( H^* \left( \frac{\sigma_0^2}{2} y \right) \right) \in L^1(\mathbb{R}^n) \}, \)

and we write equation (53) as

\[
\begin{aligned}
&\frac{dy}{dt} + Ay = g_1, \quad \text{in } [0, T] \\
y(0) = y_0
\end{aligned}
\]

(55)

**Lemma 4.1** The operator \( A \) is m-accretive in \( L^1(\mathbb{R}^n) \).

**Proof.** Since the operator \( -\Delta \) is m-accretive in \( L^1(\mathbb{R}^n) \), see, e.g., [2, 3], then the same holds for the operator \( -L \), moreover, taking into account that \( |\sigma_0(x)| \geq \rho > 0 \), it follows the m-accretivity of the operator \( A \), as claimed. Indeed, equation

\[
\lambda y + Ay = \eta \text{ in } D'(\mathbb{R}^n)
\]

is equivalent to

\[
\lambda \beta(z) - \Delta z = \eta \text{ in } D'(\mathbb{R}^n)
\]

where \( \beta = \frac{\sigma_0}{\rho} z \) and this implies the conclusion.

Again invoking the Crandall & Ligget Theorem, we get that the eq. (55) has a unique mild solution \( y \in C([0, T]; L^1(\mathbb{R}^n)), \) which is given by

\[
y(t) = \lim_{\epsilon \to 0} y_\epsilon(t) \text{ in } L^1(\mathbb{R}^n), \forall t \in [0, T],
\]

\[
y_\epsilon(t) = y_\epsilon^i \text{ for } t \in [i\epsilon, (i + 1)\epsilon], \ i = 0, \ldots, N = \left\lceil \frac{T}{\epsilon} \right\rceil
\]

\[
\frac{1}{\epsilon} (y_{i+1}^\epsilon - y_i^\epsilon) + A(y_i^\epsilon + 1) = g_1, \quad i = 0, \ldots, N
\]

\[
y_0^\epsilon = y_0
\]

hence completing the proof of Theorem 4.1.

By Theorem 4.1, it follows the existence and uniqueness of a solution \( \phi \in C([0, T]; L^1_{loc}(\mathbb{R}^n)) \cap M^{\infty,2}(\mathbb{R}^n)). \)
Remark 4.2 In the general n-dimensional case, where \( f \in C^2_b(\mathbb{R}^n) \), the dynamic programming equation corresponding to \( 1 \) reduces to
\[
\begin{cases}
y_t - L \left( H^* \left( \frac{\sigma^2}{2} y \right) \right) - f \cdot \nabla y + By = Lg_1, \\
y(0) = y_0, \quad x \in \mathbb{R}^n,
\end{cases}
\]
where
\[
By = -2 \sum_{i,j,k=1}^n b_{ij} D_j f_k \frac{\partial^2}{\partial x_i \partial x_k} L^{-1}(y)
- \sum_{i,j,k=1}^n b_{ij} D_j f_k \frac{\partial^2}{\partial x_k} L^{-1}(y)
\]
Therefore eq. (56) can be treated analogously to what we have seen in the 1-dimensional case, at least if the operator \( B \) is continuous in \( L^1(\mathbb{R}^n) \), which happens under some additional conditions on \( f = \{ f_k \}_{k=1}^n \).

We note that, for \( L = \Delta \), the linear Fokker-Planck equation (56), has been treated in [2].

5 The degenerate 1-D case

Consider here equation (16), that is
\[
\begin{cases}
y_t - \left( H^* \left( \frac{\sigma^2}{2} y \right) \right)_{xx} - f y_x + By = g_1, \quad \text{in } [0,T] \times \mathbb{R} \\
y(0) = y_0 \in \mathbb{R}
\end{cases}
\]
where \( \sigma \) is assumed to satisfy the condition \( \sigma \in C^2_b(\mathbb{R}) \) only. Moreover, if we consider, as above, the operator \( A : D(A) \subset L^1(\mathbb{R}) \to L^1(\mathbb{R}) \), such that
\[
Ay = -\left( H^* \left( \frac{\sigma^2}{2} y \right) \right)'' - f y',
\]
\( D(A) = \{ y \in L^1(\mathbb{R}) : f y' + \left( H^* \left( \frac{\sigma^2}{2} y \right) \right)'' \in L^1(\mathbb{R}) \} \), we have the following holds

Lemma 5.1 \( A \) is quasi-\( m \)-accractive in \( L^1(\mathbb{R}) \).

Proof. For each \( \epsilon > 0 \) we consider the operator
\[
A_\epsilon y = -\left( H^* \left( \frac{\sigma^2 + \epsilon}{2} y \right) \right)'' - f y',
\]
which is quasi-\( m \)-accractive, see Lemma 5.1. Hence, for each \( \eta \in L^1(\mathbb{R}) \) and \( \lambda \geq \lambda_0 \) the equation
\[
\lambda y_t - \left( H^* \left( \frac{\sigma^2 + \epsilon}{2} y \right) \right)'' - f y' = \eta, \quad \text{in } \mathbb{R},
\]
has a unique solution \( y_\epsilon \in L^1(\mathbb{R}) \), with \( H^* \left( \frac{\sigma^2 + \epsilon}{2} y_\epsilon \right) \in L^\infty(\mathbb{R}) \).

Dynamic estimates. As in the proof of Lemma 3.1 we have
\[
\lambda \| y_\epsilon \|_1 \leq \| \eta \|_1 + \| f' \|_1 \| y_\epsilon \|_1, \quad \forall \epsilon > 0,
\]
that is for \( \lambda > \| f' \|_\infty \)
\[
\| y_\epsilon \|_1 \leq (\lambda - \| f' \|_1)^{-1} \| \eta \|_1, \quad \forall \epsilon > 0.
\]
Assume now that \( \eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), then, by (60) we see that for each \( M > 0 \)
\[
\lambda (y_\epsilon - M) - \left( H^* \left( \frac{\sigma^2 + \epsilon}{2} y_\epsilon \right) - H^* \left( \frac{\sigma^2 + \epsilon}{2} M \right) \right)''
- f (y_\epsilon - M)' = \eta - \lambda M + \left( H^* \left( \frac{\sigma^2 + \epsilon}{2} M \right) \right)'' = \tilde{\eta}.
\]
Moreover, by (56), we also have
\[
\tilde{\eta}(x) \leq \eta - M \lambda + M^2 \| (H^*)'' \|_\infty \| \sigma' \|_\infty +
+ M \| (H^*)' \|_\infty \| \sigma'' \| + (\sigma')^2 \|_\infty \leq 0
\]
for \( M \) and \( \lambda \) large enough (independently of \( \epsilon \)). This yields
\[
\lambda \| (y_\epsilon - M)^+ \|_1 \leq \| f' \|_\infty \| (y_\epsilon - M)^+ \|_1.
\]
Hence \( y_\epsilon \leq M \) in \( \mathbb{R} \) for \( \lambda > \| f' \|_\infty \). Similarly, it follows that
\[
\lambda (y_\epsilon + M) - \left( H^* \left( \frac{\sigma^2 + \epsilon}{2} y_\epsilon \right) - H^* \left( -\frac{\sigma^2 + \epsilon}{2} M \right) \right)''
+ f (y_\epsilon + M)' = \eta + \lambda M + \left( H^* \left( -\frac{\sigma^2 + \epsilon}{2} M \right) \right)''
= \eta + \lambda M \geq 0,
\]
if \( M \) is large enough, but independent of \( \epsilon \). Therefore, if multiply the equation by \((y_\epsilon + M)^-\) and integrate on
\[ \text{Theorem 5.1} \text{ There is a unique mild solution } y \in C([0,T];\mathbb{R}) \text{ to equation } (1). \]

As in previous case Theorem 5.1 implies via (13) the existence of a mild solution \( \varphi \) to equation (1) satisfying (20). We omit the details.

6 Conclusions

In this paper it is shown, via nonlinear semigroup theory in \( L^1 \), both the existence and the uniqueness of a mild solution for the dynamic programming equation for stochastic optimal control problem with control in the volatility term. Latter problem is related to the analysis of controlled stochastic volatility models, within the financial frameworks, whose related computational study is the subject of our ongoing research.

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