Computer geometric modeling of quasi-rotation surfaces

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Abstract. The paper presents an analytic description of quasi-rotation correspondence. The formulas for calculation of values characterizing geometric objects are provided on the basis of constructive description of the quasi-rotation correspondence. The values characterizing circles generated through quasi-rotation of the initial point are calculated. A system of parametric equations describing a generic quasi-rotation surface is provided. Every type of curvilinear second-degree axis – circular, elliptic, parabolic, and hyperbolic – is considered. An algorithm for surface modeling through computer algebra systems has been developed on the basis of the analytic description and tested by means of Maple software. The algorithm allows one to acquire plots of separate sheets of the surface as well as all four sheets simultaneously. The capability to acquire 3D models of quasi-rotation surfaces given the desired shape and mutual location of the axis and the generatrix is demonstrated. The developed algorithm is applicable to any flat generating curve belonging to the axis plane and defined by a single-parameter system of equations. The resultant 3D model polygon count is user-defined and only limited by computer’s processing capability assuring high surface quality. The images of several four-sheet quasi-rotation surfaces are provided. The developed algorithm allows one to design surfaces of pre-defined properties.

1. Introduction
Quasi-rotation as geometric correspondence has been previously considered in paper [1]. This correspondence is constructively based on the assumption that spherical, conical and cylindrical surfaces as well as Dupin cyclides are generated through a common method. Papers [2,3] provide basic background for definition of the quasi-rotation correspondence and demonstrate its applicability to surface formation. According to the classification known from source [4], quasi-rotation surfaces are classified as channel cyclic surfaces and include a certain number of Joachimsthal surfaces, e.g. surfaces shaped through quasi-rotation around a circle. Geometric properties of such surfaces are considered in study [5]. It is worth noting that the curvature line theorem proven by Joachimsthal in 1846 is essential in modern studies of surface properties [6]. There are other examples of quasi-rotation surfaces being also Joachimsthal surfaces. For instance, the epi-hypocycloidal channel surfaces considered in paper [7] can be shaped through quasi-rotation. However, in the general case, circles generated upon quasi-rotation around an ellipse, a hyperbola or a parabola do not belong to the planes of a single bundle and therefore cannot belong to a Joachimsthal surface [8]. All of the mentioned known cyclic surfaces are applied in mechanical engineering and architecture [9,10]. Development of a universal and simple analytic description for the quasi-rotation correspondence is an urgent problem, since it reveals a wide variety of cyclic surfaces with unstudied properties.
2. Problem definition
It is required to develop an algorithm of analytic formation of three-dimensional plots of quasi-rotation surfaces on the basis of the existing constructive definition of the quasi-rotation correspondence. The calculation has to be performed for a pair of curves – a curvilinear axis (second-degree curve) and a generatrix – assigned on a plane arbitrarily. The generatrix has to be described with a system of parametric equations, while the curvilinear axis has to be defined by major semiaxis value and focal distance. The algorithm has to be realized through the use of Maple computer algebra system.

3. Theory
The mathematical description of the quasi-rotation surfaces is based on the fact that these surfaces are cyclic, i.e. generated through a family of circles. A circle in three-dimensional space is defined by its radius, position of its center, and position of the plane it belongs to. Let us place a curvilinear axis of quasi-rotation (further referred to as “axis”) into plane $XY$. Then, as it follows from the constructive definition of the studied correspondence [1,2,3], the planes of circles comprising the quasi-rotation surface are perpendicular to plane $XY$, while their centers belong to plane $XY$. Let us express a system of equations describing such circles.

$$\begin{align*}
x &= r \cos(\beta + 1) \cos \theta + x_l \\
y &= r \cos(\beta + 1) \sin \theta + y_l \\
z &= r \sin \beta
\end{align*}$$

(1)

In the system of equations (1) the value of angle $\beta$ is the parameter defining a point of a circle. The values of radius $r$ and angle $\theta$ depend on position of the generating point $L(x_l, y_l)$ as well as shape and position of the quasi-rotation axis. The correlation between the coordinates $X, Y$ and angles $\beta, \theta$ of the system of equations (1) is presented on figure 1.

![Figure 1.](image)

The applicate $Z$ value does not depend on angle $\theta$ value since the plane of the described circle is parallel to $OZ$.

In order for the system (1) to describe a family of curves rather than a single curve, the values of radius $r$ and angle $\theta$ have to be described with expressions, where axis parameters are constant, while axis location $L(x_l, y_l)$ is variable and corresponds to the second parameter of the sought system of equations. For example, if we appoint a circle as a generating curve, the values $x_l$ and $y_l$ satisfy the following system:

$$\begin{align*}
x_l &= x_0 + R \cos \tau \\
y_l &= y_0 + R \sin \tau
\end{align*}$$

(2)

Let us consider quasi-rotation of a circle belonging to plane $XY$ and described with the system of equations (2) around a hyperbolic axis. The hyperbola is described with the value of the major semiaxis $a$ and the distance $c$ between the focus and the coordinate center; its asymptotes intersect at point $O(0,0,0)$ and are symmetric with respect to axis $OX$. Figure 2 depicts plane $XY$ containing a
hyperbola $g$, a point $L$, circle projections $k2'$ and $k2''$ constituting the result of quasi-rotation of the point $L$ around the hyperbolic axis $g$ with respect to focus $F2$. The angle $\theta$ between the quasi-rotation plane and axis $OX$ as well as the radius $LC2'$ of quasi-rotation depend on the position of the point $L(x_l, y_l)$.

![Figure 2](image2.png)

**Figure 2.** Quasi-rotation of a point $L$ around a hyperbolic axis $g$ with respect to focus $F2$.

The algorithm of construction of quasi-rotation center $S2'$ and $S2''$ as well as projection of circles $k2'$ and $k2''$ applied here is identical to the one in the case of elliptic axis [3]. Further only quasi-rotation around the closer center of quasi-rotation $S2'$ with generation of circles $k2'$ is considered. Similar reasoning is applicable to quasi-rotation around centers $S2''$, $S1'$, and $S1''$.

![Figure 3](image3.png)

**Figure 3.** Constructive correlation between the values $r$, $\theta$ and the location of points $S2'$, $L$, $F1$, and $F2$.

In order for the system of equations (1) to describe a circle $k2'$, the values of $r$ and $\theta$ have to be calculated with respect to the constructive correlation depicted on figure 3. Then the expression for radius $r$ calculation takes the following form:

$$r = (v-u)\sin\left(\frac{\varphi-\delta}{2}\right),$$  \hspace{1cm} (3)

where $v$ represents segment $F2\ S2'$ length, $u$ represents segment $F2\ L$ length. Angles $\varphi$ and $\delta$ are designated on figure 3. The distance $v$ between the focus and a point of hyperbola $g$ is calculated through the following formula:

$$v = \frac{p}{1-\varepsilon \cos \varphi},$$  \hspace{1cm} (4)

where focal parameter $p$ and excentricity $\varepsilon$ can be expressed through values $a$ and $c$:uxtaposition.
The value of angle \( \phi \) depends only on location of the point \( L \) and is calculated through the following formula:

\[
\varphi = \arccot \left( \frac{c - x_i}{y_i \text{signum}(y_i)} \right) \text{signum}(y_i)
\]  

(6)

The function \( \text{signum}(y_i) \) of the formula (6) is a built-in Maple command that returns the sign of a real or complex number. It provides negative values of angle \( \varphi \) at negative values of coordinate \( y_i \). The segment length \( u \) is calculated through the formula:

\[
u = \left( y_i^2 + (c - x_i)^2 \right)^{\frac{1}{2}}
\]  

(7)

The function \( \text{signum}(y_i) \) of the formula (8) changes sign of the angle \( \delta \) according to the sign of angle \( \varphi \). The angle between the circle plane and axis \( OX \) depends on angles \( \varphi \) and \( \delta \) according to the following formula:

\[
\theta = \frac{\pi - \delta - \varphi}{2}
\]  

(9)

Through the formulas above (1) – (9) one can express a system of equations that defines a single sheet of a surface acquired through quasi-rotation of a circle around a hyperbolic axis.

The algorithms of construction of quasi-rotation surfaces with elliptic and parabolic rotation axis are developed with similar reasoning. Figure 4 represents the constructive correspondence between the location of point \( L \), incline angle \( \theta \) of quasi-rotation plane to axis \( OX \) and quasi-rotation radius for the cases of parabolic (a) and elliptic (b) axes.

4. Results of experiments

The conducted research has resulted in development of the algorithm for construction of 3D plots of surfaces generated through the quasi-rotation correspondence. The images of these 3D plots provided...
in the present paper include projections of a surface on three coordinate planes as well as plane of
general position displayed on the bottom right. For example, figure 5 (a) depicts a single sheet of a
quasi-rotation surface for the case of generating circle $l$ of radius $R = 1$ centered at point $F_2$ quasi-
rotated around a hyperbolic axis $g$ with parameters $a = 2$, $c = 4$. The depicted sheet is generated
through quasi-rotation with respect to focus $F_2$ and the closer center of rotation $S_2'$. The algorithm for
construction of this sheet is implemented through formulas (1) – (9).

![Figure 5. 3D plots of a quasi-rotation surface with generating circle centered at the focus of
hyperbolic axis: a single sheet of a surface (a) and the whole four-sheet surface (b).]

The algorithms for construction of the other sheets of the quasi-rotation surface are implemented in a
similar fashion and return the results depicted on figure 5 (b). Figure 6 (a) represents a four-sheet
surface for the case of quasi-rotation of a circle $l$ around a parabolic axis $p$. Radius of the generating
circle $R = 0.5$, focal parameter $f = 3$, the circle is centered at the focus of the parabolic axis $p$. Figure 6
(b) represents a four-sheet surface for the case of quasi-rotation of a circle $l$ around an elliptic axis $e$.
Here, the radius $R$ of the generating circle equals 0.5, the major $a$ and the minor $b$ semiaxes of the
ellipse equal 5 and 3 respectively, the circle is centered at the elliptic axis focus.

![Figure 6. Four-sheet quasi-rotation surfaces with generating circles centered at the focus of
parabolic (a) and elliptic (b) axes.]
Quasi-rotation around a circular axis is a special case of quasi-rotation around an elliptic axis at $a = b$. This case results in generation of a two-sheet surface.

5. Consideration of the results

Figures 5 and 6 represent the results of quasi-rotation of circles centered at the axis focus. One can observe that in these three cases two out of four sheets of the generated surfaces constitute Dupin cyclides. The sheet depicted on figure 5 (a) can be acquired through quasi-rotation of a circle around an elliptic axis. A similar sheet is generated in the case depicted in yellow on figure 6 (b). Only one of the four sheets of the surface depicted on figure 6 (a) constitutes a closed surface and is shown in full. This sheet is depicted in purple. The other three sheets are shown in fragments. For example, the sheets depicted in green and black are in fact the results of quasi-rotation of an arc of the generating circle. The sheet depicted in yellow constitutes a cylindrical surface, since it is generated with circles of infinite radius ($L S_2'' = \infty$). A similar circle is labeled $k_2''$ on figure 4 (a). Figure 6 (a) depicts a fragment of such cylindrical surface.

The developed algorithm is applicable to any generating curves located in the axis plane. Figure 7 (a) represents a four-sheet surface for the case of quasi-rotation of a circle centered at the center of an elliptic axis. Study [11] considers the physical properties of bodies bounded by one of the sheets of such surface. The accuracy of similar studies can be significantly improved through the application of 3D models calculated with the tools presented in this paper and based on the analytic description. Figure 7 (b) depicts a four-sheet surface for the case of quasi-rotation of a circle with center shifted from the center of the elliptic axis on the value of its radius along both $X$ and $Y$ axis. Obviously, the geometric objects depicted on figure 7 are neither bodies of rotation, nor Dupin cyclides, however, they are generated through the same constructive scheme.

![Figure 7. Four-sheet surfaces for the case of quasi-rotation around an elliptic axis featuring generating circle centered at the elliptic axis center (a) and shifted from the elliptic axis center (b).](image)

The proposed algorithm allows one to acquire 3D plots of surfaces generated through quasi-rotation of any flat curve belonging to the axis plane and defined with a system of two equations with one parameter. For example, in order to generate a model of a surface acquired through quasi-rotation of a straight line $l$ around an elliptic axis $e$, the system of equations (2) has to be replaced with following system of equations:

\[
\begin{align*}
  x_t &= x_0 + t \cot \tau \\
  y_t &= t
\end{align*}
\]

where $t$ is the parameter, $x_0$ represents the abscissa of intersection between the line $l$ and axis $OX$. Figure 8 (a) depicts a surface of quasi-rotation of a straight line passing through the center of coordinates at angle 135° to axis $OX$ around an elliptic axis with semiaxis values of 3 and 5.
Figure 8. Four-sheet surfaces of quasi-rotation around an elliptic axis: the generating line passing through the coordinate center at angle 135° to axis $OX$ (a), the generating line parallel to axis $OX$ (b).

In case the line is parallel to axis $OX$, the system of equations describing it takes the following form:

$$
\begin{align*}
&x_i = t \\
y_i = \text{const}
\end{align*}
$$

The quasi-rotation surface generated with such line is depicted on figure 8 (b). The provided examples demonstrate the capability of construction of surface models generated through quasi-rotation of any curve by application of the respective parametric equations.

Generation of surface models of a particular type given pre-determined parameters is also possible through the equations (1) – (9). For example, it is required to construct the surface sheet depicted in black on figure 7 (a) given the quasi-rotation axis and the pre-determined minimal opening radius $r$. In this case, it is possible to calculate the generating circle radius corresponding to the given minimal opening radius $r$ and then construct a model with pre-determined minimal opening radius.

6. Conclusions

In the present study an algorithm for construction of 3D plots of surfaces generated through the quasi-rotation correspondence has been developed. The capability of creation of models given generatrix parameters and quasi-rotation axis parameters is demonstrated. Any second-degree curve can act as a quasi-rotation axis. Any curve defined by a system of parametric equations can act as a generatrix. The described algorithm can be realized in any available computer geometric modeling system. In the present paper the calculations were performed through the Maple computer algebra system that allows one to export both 2D images and high-quality 3D models of constructed surfaces. The resultant 3D model polygon count is user-defined and only limited by computer’s processing capability assuring high surface quality. These models can be applied in solutions to the problems of research and design. Application of quasi-rotation surfaces as medial surfaces of shells requires one to determine their differential properties that can later be applied in calculations of various physical properties of real-world objects [12]. Analytic description of geometric objects allows one to apply various instruments of research available in the Maple software. For example, construction of a plane tangent to a quasi-rotation surface demonstrated in paper [5] can be performed through a special function of Maple software.

The analytic description of the quasi-rotation correspondence and its application results show that this correspondence allows one to consider quasi-rotation surfaces and Dupin cyclides as a uniform class of surfaces. This study confirms the idea of existence of a wide variety of quasi-rotation surfaces stated in papers [1,2,3]. The properties of these surfaces are of high theoretical and practical interest.
The results of the present study are going to be utilized in further studies of properties of quasi-rotation correspondence and quasi-rotation surfaces.

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