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Abstract. The evaluations of determinants with Legendre symbol entries have close relation with combinatorics and character sums over finite fields. Recently, Sun \cite{9} posed some conjectures on this topic. In this paper, we prove some conjectures of Sun and also study some variants. For example, we show the following result:

Let $p = a^2 + 4b^2$ be a prime with $a, b$ integers and $a \equiv 1 \pmod{4}$. Then for the determinant

$S(1, p) := \det \left[ \left( \frac{i^2 + j^2}{p} \right) \right]_{1 \leq i, j \leq p-1},$

the number $S(1, p)/a$ is an integral square, which confirms a conjecture posed by Cohen, Sun and Vsemirnov.

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1. Introduction

Given an $n \times n$ complex matrix $M = [a_{ij}]_{1 \leq i, j \leq n}$, we often use $\det M$ or $|M|$ to denote the determinant of $M$. The evaluation of determinants with Legendre symbol entries is a classical topic in number theory, combinatorics and finite fields. Krattenthaler’s survey papers \cite{7, 8} introduce many concrete examples and advanced techniques on determinant calculation.

Let $p$ be an odd prime and let $\left( \frac{\cdot}{p} \right)$ denote the Legendre symbol. Carlitz \cite{2} studied the following $(p-1) \times (p-1)$ matrix

$D_p := \left[ \left( \frac{i-j}{p} \right) \right]_{1 \leq i, j \leq p-1}.$

He obtained that the characteristic polynomial of $D_p$ is precisely

$|xI_{p-1} - D_p| = \left( x^2 - (-1)^{\frac{p+1}{2}} \right)^{\frac{p-3}{2}} \left( x^2 - (-1)^{\frac{p-1}{2}} \right),$

where $I_{p-1}$ is the $(p-1) \times (p-1)$ identity matrix.

Along this line, Chapman \cite{3} further investigated the following matrices:

$C_p(x) := \left[ x + \left( \frac{i+j-1}{p} \right) \right]_{1 \leq i, j \leq p-1}.$
and

\[ C_p^*(x) := \left[ x + \left( \frac{i + j - 1}{p} \right) \right]_{1 \leq i, j \leq \frac{p+1}{2}}, \]

where \( x \) is a variable. In the case \( p \equiv 1 \pmod{4} \), let \( \epsilon_p > 1 \) and \( h(p) \) be the fundamental unit and class number of the real quadratic field \( \mathbb{Q}(\sqrt{p}) \) respectively and let \( \epsilon_p h(p) = a_p + b_p \sqrt{p} \) with \( 2a_p, 2b_p \in \mathbb{Z} \). Chapman proved that

\[
\det C_p(x) = \begin{cases} 
(1)^{p-1/4}2^{p-1/2}(b_p - a_p x) & \text{if } p \equiv 1 \pmod{4}, \\
-2^{p-1/2}x & \text{if } p \equiv 3 \pmod{4},
\end{cases}
\]

and that

\[
\det C_p^*(x) = \begin{cases} 
(1)^{p-1/4}2^{p-1/2}(p b_p x - a_p) & \text{if } p \equiv 1 \pmod{4}, \\
-2^{p-1/2} & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

Moreover, Chapman [4] posed a conjecture concerning the determinant of the \( \frac{p+1}{2} \times \frac{p+1}{2} \) matrix

\[
C = \left( \left[ \frac{j - i}{p} \right] \right)_{1 \leq i, j \leq \frac{p+1}{2}}.
\]

Due to the difficulty of the evaluation on this determinant, he called it “evil” determinant. Finally this conjecture was confirmed completely by Vsemirnov [11, 12].

Recently Sun [9] studied various determinants of matrices involving Legendre symbol entries. Let \( p \) be a prime and \( d \) be an integer with \( p \nmid d \). Sun defined

\[ S(d, p) := \det \left( \left( \frac{j^2 + d j^2}{p} \right) \right)_{1 \leq i, j \leq \frac{p+1}{2}}. \]

In the same paper, Sun also studied some properties of the above determinant. For example, he showed that \(-S(d, p)\) is always a quadratic residue modulo \( p \) if \( \left( \frac{d}{p} \right) = 1 \) and that \( S(d, p) = 0 \) if \( \left( \frac{d}{p} \right) = -1 \). Moreover, Sun posed the following conjecture:

**Conjecture 1 (Sun).** Let \( p \equiv 3 \pmod{4} \) be a prime. Then \(-S(1, p)\) is an integral square.

This conjecture was later confirmed by Alekseyev and Krachun by using some algebraic number theory. In the case \( p \equiv 1 \pmod{4} \), Cohen, Sun and Vsemirnov also posed the following conjecture.

**Conjecture 2 (Cohen, Sun and Vsemirnov).** Let \( p = a^2 + 4b^2 \) be a prime with \( a, b \) integers and \( a \equiv 1 \pmod{4} \). Then \( S(1, p) / a \) is an integral square.

For example, if \( p = 5 = 1^2 + 4 \times 1^2 \), then \( S(1, 5) = 1 \times 1^2 \). If \( p = 13 = (-3)^2 + 4 \times 1^2 \), then \( S(1, 13) = -27 = -3 \times 3^2 \).

As the first result of this paper, by considering some character sums over finite fields, we confirm this conjecture and obtain the following result. For convenience, for each \( d \in \mathbb{Z} \) we set

\[
\epsilon(d) = \begin{cases} 
-1 & \text{if } \left( \frac{d}{p} \right) = 1 \text{ and } d \text{ is not a biquadratic residue modulo } p, \\
1 & \text{otherwise}.
\end{cases}
\]

**Theorem 3.** Let \( p = a^2 + 4b^2 \) be a prime with \( a, b \) integers and \( a \equiv 1 \pmod{4} \) and let \( d \) be an integer. Then \( \epsilon(d) S(d, p) / a \) is an integral square. In particular, when \( d = 1 \) the number \( S(1, p) / a \) is an integral square.

Sun [9] also made the following conjecture.

**Conjecture 4 (Sun).** Let \( S^*(1, p) \) denote the determinant obtained from \( S(1, p) \) via replacing the entries \( \left( \frac{j^2 + d j^2}{p} \right) (j = 1, \ldots, \frac{p+1}{2}) \) in the first row by \( \left( \frac{j}{p} \right) (j = 1, \ldots, \frac{p-1}{2}) \) respectively. Then \(-S^*(1, p)\) is an integral square if \( p \equiv 1 \pmod{4} \).
As an application of Theorem 3, we confirm this conjecture.

**Corollary 5.** Let \( p \equiv 1 \pmod{4} \) be a prime. Then \(-S^*(1, p)\) is an integral square.

For example, \( S^*(1, 5) = -1^2 \), \( S^*(1, 13) = -3^2 \), and \( S^*(1, 17) = -21^2 \).

The proofs of our main results will be given in Section 2.

## 2. Proofs of the main results

We begin with the following permutation involving quadratic residues (readers may refer to [5, 10] for details on the recent progress on permutations over finite fields). Let \( p \equiv 1 \pmod{4} \) be a prime and let \( d \in \mathbb{Z} \) with \( \left( \frac{d}{p} \right) = 1 \). If we write \( p = 2n + 1 \), then clearly the sequence

\[
d \cdot 1^2 \mod p, \ldots, d \cdot n^2 \mod p
\]

is a permutation \( \pi_p(d) \) of the sequence

\[
1^2 \mod p, \ldots, n^2 \mod p.
\]

Let \( \text{sgn}(\pi_p(d)) \) be the sign of \( \pi_p(d) \). We first have the following result:

**Lemma 6.** Let \( p \equiv 1 \pmod{4} \) be a prime, and let \( d \in \mathbb{Z} \) be a quadratic residue modulo \( p \). Then

\[
\text{sgn}(\pi_p(d)) = \begin{cases} 
1 & \text{if } d \text{ is a biquadratic residue modulo } p, \\
-1 & \text{otherwise}.
\end{cases}
\]

**Proof.** It is clear that

\[
\text{sgn}(\pi_p(d)) \equiv \prod_{1 \leq i < j \leq n} \frac{d j^2 - d i^2}{j^2 - i^2} \pmod{p}.
\]

By this we obtain

\[
\text{sgn}(\pi_p(d)) \equiv \left( d^{\frac{p-1}{2}} \right)^{n-1} \equiv d^{\frac{p-1}{4}} \pmod{p}.
\]

This implies the desired result. \( \square \)

We also need the following known result concerning eigenvalues of a matrix.

**Lemma 7.** Let \( M \) be an \( m \times m \) complex matrix. Let \( \mu_1, \ldots, \mu_m \) be complex numbers, and let \( u_1, \ldots, u_m \) be \( m \)-dimensional column vectors. Suppose that \( M u_k = \mu_k u_k \) for each \( 1 \leq k \leq m \) and that \( u_1, \ldots, u_m \) are linear independent. Then \( \mu_1, \ldots, \mu_m \) are exactly all the eigenvalues of \( M \) (counting multiplicities).

Before the proof of Theorem 3, we first introduce some notation. In the remaining part of this section, we let \( p = a^2 + 4b^2 \) be a prime with \( a, b \in \mathbb{Z} \) and \( a \equiv 1 \pmod{4} \), and let \( n = \frac{p-1}{2} \). In addition, we let \( \chi(\mathbb{Z}/p\mathbb{Z}) \) denote the group of all multiplicative characters on the finite field \( \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \), and let \( \chi_p \) be a generator of \( \chi(\mathbb{Z}/p\mathbb{Z}) \), i.e.,

\[
\chi(\mathbb{Z}/p\mathbb{Z}) = \{ \chi_p^k : k = 1, 2, \ldots, p-1 \}.
\]

Readers may refer to [6, Chapter 8] for a detailed introduction to characters on finite fields. Also, given any matrix \( M \), the symbol \( M^T \) denotes the transpose of \( M \).

Now we are in a position to prove our first theorem.

**Proof of Theorem 3.** Throughout this proof, we define

\[
M_p := \left[ \left( \frac{i^2 + j^2}{p} \right) \right]_{1 \leq i, j \leq n}.
\]
We first determine all the eigenvalues of $M_p$. For $k = 1, 2, \ldots, n$, we let
\[ \lambda_k := \sum_{1 \leq j \leq n} \left( \frac{1 + j^2}{p} \right) \chi_p^k(j^2). \]  
(1)

We claim that $\lambda_1, \ldots, \lambda_n$ are exactly all the eigenvalues of $M_p$ (counting multiplicities). In fact, for any $1 \leq i, k \leq n$ we have
\[ \sum_{1 \leq j \leq n} \left( \frac{i^2 + j^2}{p} \right) \chi_p^k(j^2) = \sum_{1 \leq j \leq n} \left( \frac{1 + j^2 i^2}{p} \right) \chi_p^k(j^2) \]
\[ = \sum_{1 \leq j \leq n} \left( \frac{1 + j^2}{p} \right) \chi_p^k(j^2) = \lambda_k \chi_p^k(i^2). \]

This implies that for each $k = 1, \ldots, n$, we have
\[ M_p v_k = \lambda_k v_k, \]
where
\[ v_k := (\chi_p^k(1^2), \chi_p^k(2^2), \ldots, \chi_p^k(n^2))^T. \]

Since
\[
\begin{pmatrix}
\chi_p^1(1^2) & \chi_p^2(1^2) & \cdots & \chi_p^n(1^2) \\
\chi_p^1(2^2) & \chi_p^2(2^2) & \cdots & \chi_p^n(2^2) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_p^1(n^2) & \chi_p^2(n^2) & \cdots & \chi_p^n(n^2)
\end{pmatrix}
= \pm \prod_{1 \leq i < j \leq n} (\chi_p(j^2) - \chi_p(i^2)) \neq 0,
\]
the vectors $v_1, \ldots, v_n$ are linear independent. By Lemma 7 our claim holds. Hence we have
\[ S(1, p) = \det M_p = \prod_{1 \leq k \leq n} \lambda_k = \prod_{1 \leq k \leq n} \left( \sum_{1 \leq j \leq n} \left( \frac{1 + j^2}{p} \right) \chi_p^k(j^2) \right). \]
(2)

Now we turn to the last product. When $k = n$, by [6, Chapter 5, Exercise 8] we have
\[ \lambda_n = \sum_{1 \leq j \leq n} \left( \frac{1 + j^2}{p} \right) \chi_p^n(j^2) = \sum_{1 \leq j \leq n} \left( \frac{1 + j^2}{p} \right) = -1. \]
(3)

When $k = n/2$, by [1, Theorem 6.2.9] we have
\[ \lambda_{n/2} = \sum_{1 \leq j \leq n} \left( \frac{1 + j^2}{p} \right) \chi^n_{p/2}(j^2) = \sum_{1 \leq j \leq n} \left( \frac{1 + j^2}{p} \right) \left( \frac{j}{p} \right) = -a. \]
(4)

As $M_p$ is a real symmetric matrix, every eigenvalue $\lambda_k$ of $M_p$ is real. Hence for any $1 \leq k \leq \frac{p-5}{4}$ we have $\lambda_k = \bar{\lambda}_{n-k}$. Let
\[ f(x) := \det(xI_n - M_p) \]
be the characteristic polynomial of $M_p$. By the above we observe that all roots of $f(x)$ apart from $\lambda_n = -1$ and $\lambda_{n/2} = -a$ are of even multiplicity. Using unique factorisation in $\mathbb{Z}[x]$, one can obtain that
\[ f(x) = (x + 1)(x + a)g(x)^2, \]
where $g(x)$ is a polynomial with integer coefficients. Therefore we obtain that $S(1, p)/a = g(0)^2$ is an integral square.

Now we consider $S(d, p)$. If $p \mid d$, then clearly $S(d, p) = 0$. If $\left( \frac{d}{p} \right) = -1$, then by [9, Theorem 1.2] we know that $S(d, p) = 0$. Suppose now that $d$ is a quadratic residue modulo $p$. Then clearly we have
\[ S(d, p) = \text{sgn}(\pi_p(d))S(1, p). \]

Now our desired result follows from Lemma 6. \hfill \square

We now prove our next result.
Proof of Corollary 5. By [1, Theorem 6.2.9] for any \(1 \leq i, j \leq n\) we have
\[
\sum_{1 \leq i \leq n} \left( \frac{i^2 + j^2}{p} \right) \left( \frac{i}{p} \right) = -a \left( \frac{j}{p} \right)
\]
and hence
\[
- \sum_{2 \leq i \leq n} \left( \frac{i^2 + j^2}{p} \right) \left( \frac{i}{p} \right) - a \left( \frac{j}{p} \right) = \left( \frac{1 + j^2}{p} \right).
\]
By this we have
\[
S^*(1, p) = \frac{-1}{a} \left| \begin{array}{cccc}
-a \left( \frac{1}{p} \right) & -a \left( \frac{2}{p} \right) & \ldots & -a \left( \frac{n}{p} \right) \\
\left( \frac{2^2 + 1^2}{p} \right) & \left( \frac{2^2 + 2^2}{p} \right) & \ldots & \left( \frac{2^2 + n^2}{p} \right) \\
\vdots & \vdots & \ddots & \vdots \\
\left( \frac{n^2 + 1^2}{p} \right) & \left( \frac{n^2 + 2^2}{p} \right) & \ldots & \left( \frac{n^2 + n^2}{p} \right)
\end{array} \right| = -S(1, p) / a.
\]
The last equality follows from (5). Now our desired result follows from Theorem 3. □

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