Finding $3 \times 3$ Hermitian Matrices over the Octonions with Imaginary Eigenvalues

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Abstract

We show that any 3-component octonionic vector which is purely imaginary, but not quaternionic, is an eigenvector of a 6-parameter family of Hermitian octonionic matrices, with imaginary eigenvalue equal to the associator of its elements.

1 Introduction

The eigenvalue problem for $3 \times 3$ Hermitian octonionic matrices, henceforth referred to as \textit{Jordan matrices}, contains some surprises. Notable among these is that, whereas each Jordan matrix satisfies its characteristic equation, its real eigenvalues do not. As shown in \cite{1, 2}, each Jordan matrix admits six real eigenvalues, rather than three. However, the eigenvalues divide naturally into two families of three, and the corresponding families of eigenvectors do have the expected properties, such as orthonormality, provided that these properties are properly formulated.

Due to the nonassociativity of the octonions, most Jordan matrices also appear to admit eigenvalues which are not real; several examples were discussed in \cite{3, 4}. However, to our knowledge there is no known algorithm for finding the non-real eigenvalues of such matrices, nor is it clear how many there are.
In this paper, which is an extension of [5], we take a different approach. Rather than attempt to find the non-real eigenvalues and corresponding eigenvectors of a given Jordan matrix, we instead find the Jordan matrices which admit a given eigenvector/eigenvalue pair. Specifically, for any vector \( v \in \mathbb{O}^3 \) which is not quaternionic, we use the associator of the elements of \( v \) as the eigenvalue, and find all Jordan matrices for which \( v \) is an eigenvector with that eigenvalue, which is nonzero by assumption. We show below that a necessary condition for such matrices to exist is that \( \text{Re}(v) = 0 \), and that if this condition is satisfied there is a 6-parameter family of such matrices.

We begin in Section 2 by reviewing the octonions and their properties, and then briefly summarize some known examples [4] of \( 3 \times 3 \) Hermitian octonionic matrices with imaginary eigenvalues in Section 3. In Section 4 we present our new results, which we then summarize in Section 5, where we also propose some further conjectures.

## 2 Octonions

We use the standard basis \( \{1, i, j, k\} \) for the quaternions \( \mathbb{H} \), and we construct the octonions \( \mathbb{O} \) via the Cayley-Dickson process as \( \mathbb{H} \oplus \mathbb{H} \ell \). The resulting multiplication table is neatly summarized by the oriented Fano geometry shown in Figure 1. As is well-known, the octonions are neither commutative nor associative.
Writing the components of an octonion \( w \) as
\[
w = w_1 + w_2 i + w_3 j + w_4 k + w_5 k\ell + w_6 j\ell + w_7 i\ell + w_8 \ell
\] (1)
we have
\[
\text{Re}(w) = w_1 \quad (2)
\]
\[
\text{Im}(w) = w - \text{Re}(w) \quad (3)
\]
and
\[
\overline{w} = 2\text{Re}(w) - w \quad (4)
\]
\[
|w|^2 = w\overline{w} \quad (5)
\]
Any three octonions \( x, y, z \in \mathbb{O} \) can be assumed without loss of generality to take the form
\[
x = x_1 + x_2 i \quad (6)
\]
\[
y = y_1 + y_2 i + y_3 j \quad (7)
\]
\[
z = z_1 + z_2 i + z_3 j + z_4 k + z_8 \ell \quad (8)
\]
by suitable choice of basis; we refer to \( x, y, z \) as \textit{generic octonions}. Choosing
\[
v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{O}^3 \quad (9)
\]
the associator of \( v \) is by definition the associator of its elements, that is
\[
[v] = [x, y, z] = (xy)z - x(yz) = 2x_2y_3z_8 k\ell \quad (10)
\]
with the last equality holding for generic octonions.

We seek solutions of the (right) eigenvalue problem
\[
Av = v\lambda \quad (11)
\]
where \( \lambda \not\in \mathbb{R} \); in what follows we will consider only the case \( \lambda = [v] \neq 0 \), noting that \([v]\) is pure imaginary. Any Jordan matrix can be written in the form
\[
A = \begin{pmatrix} p & a & \overline{c} \\ \overline{a} & m & b \\ c & \overline{b} & n \end{pmatrix} \quad (12)
\]
with \( p, m, n \in \mathbb{R} \) and \( a, b, c \in \mathbb{O} \). Then (11) takes the form
\[
px + ay + \overline{c}z = x\lambda \quad (13)
\]
\[
\overline{a}x + my + bz = y\lambda \quad (14)
\]
\[
\overline{c}x + \overline{b}y + nz = z\lambda \quad (15)
\]
As shown below, (11) admits solutions only if \( \text{Re}(v) = 0 \), in which case there is a 6-parameter family of Jordan matrices \( A \) which satisfy it.
3 Examples

Little is known about solutions of (11) with non-real eigenvalues. Although the problem of finding real eigenvalues for $3 \times 3$ Hermitian octonionic matrices has been completely solved [1], no such solution exists for finding imaginary eigenvalues [3, 4]. In fact, we know of only a handful of explicit examples of families of $3 \times 3$ Hermitian octonionic matrices admitting imaginary eigenvalues, such as those given in [4], which are reproduced below. Note in each case that $p$ can always be chosen so that the eigenvalue has no real part. Furthermore, in the first example, the eigenvectors have no real part, the associator of each eigenvector is a (possibly zero) multiple of $k\ell$, the imaginary direction of the eigenvalue, and this last property also holds for the elements of $A_1$. However, the first two properties fail to hold in the second example, while the last property fails in the third.

Example 1  The matrix

$$A_1 = \begin{pmatrix} p & iq & kqs \\ -iq & p & jq \\ -kqs & -jq & p \end{pmatrix}$$

(16)

with $p, q \in \mathbb{R}$ and

$$s = \cos \theta + k\ell \sin \theta$$

(17)

has, among others, the eigenvalues and eigenvectors,

$$\lambda_u = p \pm q\bar{s} : \quad u_{\pm} = \begin{pmatrix} i \\ 0 \\ j \end{pmatrix} S_{\pm}$$

(18a)

$$\lambda_v = p \pm q\bar{s} : \quad v_{\pm} = \begin{pmatrix} j \\ 2ks \\ i \end{pmatrix} S_{\pm}$$

(18b)

$$\lambda_w = p \mp 2q\bar{s} : \quad w_{\pm} = \begin{pmatrix} j \\ -ks \\ i \end{pmatrix} S_{\pm}$$

(18c)

where

$$S_{\pm} = \begin{cases} -k\ell \\ 1 \end{cases}$$

(19)

Example 2  The matrix

$$A_2 = \begin{pmatrix} p & qi & \frac{q}{2}(\sqrt{5}k + 2\ell) \\ -qi & p & \frac{q}{2}j \\ -\frac{q}{6}(\sqrt{5}k + 2\ell) & -\frac{q}{2}j & p \end{pmatrix}$$

(20)
has, among others, the eigenvectors and eigenvalues,

\begin{align}
\lambda_{u_1} &= (p + \frac{\sqrt{5}}{2} q) - \frac{q}{2} k \ell : \quad u_1 = \begin{pmatrix} 3k \\ \sqrt{5} j - 2 i \ell \\ 1 + \sqrt{5} k \ell \end{pmatrix} \quad (21a) \\
\lambda_{u_2} &= (p + \frac{\sqrt{5}}{2} q) + \frac{q}{2} k \ell : \quad u_2 = \begin{pmatrix} 3j \\ \sqrt{5} k + 2 \ell \\ \sqrt{5} - k \ell \end{pmatrix} \quad (21b) \\
\lambda_{v_1} &= (p - \frac{\sqrt{5}}{3} q) + \frac{2q}{3} k \ell : \quad v_1 = \begin{pmatrix} 3k \\ \sqrt{5} j - 2 i \ell \\ 0 \end{pmatrix} \quad (21c) \\
\lambda_{v_2} &= (p - \frac{\sqrt{5}}{3} q) - \frac{2q}{3} k \ell : \quad v_2 = \begin{pmatrix} 3j \\ \sqrt{5} k + 2 \ell \\ 0 \end{pmatrix} \quad (21d) \\
\lambda_{w_1} &= (p - \frac{\sqrt{5}}{6} q) - \frac{q}{6} k \ell : \quad w_1 = \begin{pmatrix} 3k \\ \sqrt{5} j - 2 i \ell \\ -7 - \sqrt{5} k \ell \end{pmatrix} \quad (21e) \\
\lambda_{w_2} &= (p - \frac{\sqrt{5}}{6} q) + \frac{q}{6} k \ell : \quad w_2 = \begin{pmatrix} 3j \\ \sqrt{5} k + 2 \ell \\ -3 \sqrt{5} - 3 k \ell \end{pmatrix} \quad (21f)
\end{align}

**Example 3**  The matrix

\begin{equation}
A_3 = \begin{pmatrix}
p & qi & -q(j - i \ell - j \ell) \\
\pmi & p & q(1 + k + \ell) \\
q(j - i \ell - j \ell) & q(1 - k - \ell) & p
\end{pmatrix}
\end{equation}

admits the eigenvector

\begin{equation}
v = \begin{pmatrix} j \\ i \\ 0 \end{pmatrix}
\end{equation}

with eigenvalue

\begin{equation}
\lambda_v = p - q k \ell
\end{equation}
4 Results

As already noted, for each example in the previous section, \( p \) can be chosen so that a given eigenvalue is purely imaginary. More generally, the real part of an eigenvalue can be changed by adding a suitable multiple of the identity matrix to the original matrix. More formally, we have the following result:

**Lemma 1.** The Hermitian matrices having a given eigenvector can be divided into families which differ only by (real) multiples of the identity matrix. The imaginary part of the corresponding eigenvalue is the same for each member of such a family, and each family contains a unique member such that the real part of the corresponding eigenvalue vanishes.

**Proof.** If \( v \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), then \( v \) is also an eigenvector of \( A + pI \) for any \( p \in \mathbb{R} \), with eigenvalue \( \lambda + p \). In particular, \( v \) is an eigenvector of \( A - \text{Re}(\lambda)I \), with eigenvalue \( \text{Im}(\lambda) \).

Note that this technique can in general be used to eliminate the real part of only one eigenvalue at a time. Nonetheless, any eigenvector with a non-real eigenvalue is also an eigenvector of a closely related matrix with a purely imaginary eigenvalue.

This suggests the following strategy for trying to find eigenvectors with non-real eigenvalues: Rather than trying to find eigenvector/eigenvalue pairs \( v \) and \( \lambda \) satisfying (11) for given \( A \), with \( \lambda \) non-real, we will instead seek to categorize the matrices which admit such eigenvalues. Lemma 1 now tells us that we can assume \( \text{Re}(\lambda)=0 \) without loss of generality, at least so long as we consider only a single eigenvector. We will therefore attempt to find the matrices \( A \) which admit a given vector \( v \) as an eigenvector, with given eigenvalue \( \lambda \) satisfying \( \text{Re}(\lambda)=0 \). Motivated by the first example, we will further assume that \( \lambda \) is a real multiple of \([v]\), and we will consider only the case where \([v] \neq 0 \). Finally, by rescaling \( A \), the constant of proportionality can be assumed to be 1. Thus, we assume that

\[
\lambda = [v] \neq 0
\]  

(25)

Since \([v] \neq 0 \) by assumption, none of \( x, y, z \) can be zero. In particular, \( x \neq 0 \), and it is straightforward to solve (11) for \( \bar{\alpha} \) and (15) for \( c \), yielding

\[
\bar{\alpha} = \left( y(\lambda - m) - bz \right) \frac{x}{|x|^2}
\]  

(26)

\[
c = \left( z(\lambda - n) - \bar{b}y \right) \frac{\bar{\alpha}}{|x|^2}
\]  

(27)

Inserting these expressions into (13) reduces (11) to the form

\[
\left( x (\lambda y - \bar{z}b) \right) y + \left( x (\lambda z - \bar{y}b) \right) z - x|\lambda|^2 = x (m|y|^2 + n|z|^2 - p|x|^2)
\]  

(28)
Lemma 2. If $|v| \neq 0$, then $b \perp \lambda$, that is, $b_5 = 0$.

Proof. Multiply both sides of (23) on the left by $\overline{\pi}$. The LHS of the resulting expression must be real, since the RHS is, but direct computation shows that the coefficient of $i$ on the left is $2|x|^2y_3z_8b_5$. Since each factor except for $b_5$ is nonzero by assumption, $b_5$ must be zero.

Theorem 1. If $|v| \neq 0$, then there are no solutions to (11) unless $\text{Re}(v) = 0$.

Proof. Direct computation, as follows. Inserting $b_5 = 0$ into (23), the $j$-component yields $4x_2^2y_3^2z_8z_1 = 0$. Since each factor except for $z_1$ is nonzero by assumption, $\text{Re}(z) = 0$. In a separate computation, the $k$-component of (23) can be solved for $b_8$, yielding

$$b_8 = \frac{2y_3z_2z_8x_2^2 + b_6y_1x_2 - b_6x_1y_2 + b_7x_1y_3}{x_2y_3}$$

Inserting the result into the $\ell$-component of (23), along with $b_5 = 0 = z_1$, results in $-4x_2^2y_3^2z_8y_1 = 0$, which forces $\text{Re}(y) = 0$. Finally, the $i\ell$- and $j\ell$-components of (23) can be solved for $b_2$ and $b_3$, yielding

$$b_2 = \frac{1}{x_2^2y_3^2z_8}(y_3^2z_8x_2^5 - y_3^2z_8x_2^3 + y_3^2z_8x_2^3 + x_1^2y_3^2z_8x_2^3 + y_2^2y_3^2z_8x_2^3 + y_3^2z_8x_2^3 - y_3^2z_8x_2^3 - y_3^2z_8x_2^3 + b_7y_3^2z_4z_8x_2^2 + b_7x_1y_3^2z_2x_2 - b_7x_1y_2y_3z_3x_2 + b_4x_1y_3z_3x_2 + b_4x_1y_3z_3x_2 + b_6x_1^2y_2^2z_4 - b_7x_1^2y_3z_4)$$

$$b_3 = \frac{1}{x_2^2y_3^2z_8}(2y_2y_3^2z_8x_2^3 + 2y_3^2z_3z_8x_2^3 - b_7y_3^2z_4x_2^2 - 2x_1y_3z_2z_4z_8x_2^2 + b_6x_1y_3z_2x_2 - b_6x_1y_2z_3x_2 + b_4x_1y_3z_3x_2 + b_6x_1^2y_2^2z_4 - b_7x_1^2y_3z_4)$$

and the result inserted into the $k$-component (along with $b_5 = 0 = z_1 = y_1$ and the above expression for $b_8$), resulting in $-2x_1|x|^2\lambda = 0$, which forces $\text{Re}(x) = 0$. Thus, $\text{Re}(v) = 0$.

Theorem 2. If $|v| \neq 0$ and $\text{Re}(v) = 0$, then there is a 6-parameter family of solutions to (11).

Proof. Inserting the above expressions for $b_2, b_3$ and $b_8$, as well as the condition $\text{Re}(v) = 0$, into (23) results in a single nonzero component, which can be solved for $b_6$, yielding

$$b_6 = \left( -2y_3z_4z_8x_2^3 - p_2z_8x_2^2 + 2y_3z_4z_3z_2^2x_2 + 2y_3z_4^3z_8x_2 + 2y_3^3z_4z_8x_2 - 2y_3z_2^2z_4z_8x_2 + 2y_3z_2^2z_8x_2 + 2y_2y_3^2z_3z_4x_2 + 4y_2z_2z_3z_4z_8x_2 + nz_3^2 - 2b_7y_2z_4^2 - 2b_7y_2z_8^2 + my_2^2z_8 + my_3^2z_8 + nz_2^2z_8 + nz_2^2z_8 + nz_4^2z_8 + 2b_1y_2z_2z_8 + 2b_4y_3z_3z_8 - 2b_4y_3z_3z_8 + 2b_1y_3z_3z_8) / (2y_3(z_4^2 + z_8^2)) \right)$$

As with the equations solved above for $b_2, b_3$ and $b_8$, the relevant coefficients are nonzero under the stated assumptions, so that the given solutions always exist. We have thus constructed $A$ explicitly, with $b_1, b_4, b_7, p, m, n$ and $n$ as free parameters.
5 Conclusion

We have created a method for finding a Hermitian matrix $A \in \mathbb{O}^{3 \times 3}$ which has an eigenvalue relationship with an imaginary vector $v \in \mathbb{O}^3$, with the associator of $v$, assumed to be nonzero, playing the role of the eigenvalue $\lambda$. For our method to be successful, rather than fixing the matrix, we must begin by fixing the vector $v$, thus fixing $\lambda$ as well.

The question we must ask is if the resulting eigenvalue/eigenvector system from our construction method represents a variation of one of the existing three family examples presented by Dray, Janesky and Manogue [4].

The last two eigenvectors in Example 1 satisfy the conditions of Theorem 2 (vanishing real part and non-vanishing associator), and $p$ and $q$ can be chosen so that the eigenvalue is precisely the vector associator, as required by our hypotheses. It is straightforward to verify that the corresponding matrix $A_1$ is indeed contained in the 6-parameter family constructed in Theorem 2. Lemma 1 can now be used, along with some obvious renormalization, to construct the matrices $A_1$ for any values of $p$ and $q$. In this sense, Example 1 is contained within our solution method, although our method generates many more solutions — but can not yet handle the first eigenvector shown, whose associator vanishes.

Each of the eigenvectors in Examples 2 and 3, however, either has a non-zero real part or a vanishing associator, so that our results do not apply to these cases. Note that in Example 2, although the imaginary part of the eigenvalues is indeed in the direction of $[A_2]$ (the associator of the elements of $A_2$), namely $k\ell$, none of the given eigenvectors has an associator in this direction. Furthermore, in Example 3, the imaginary part of the eigenvalue no longer points in the direction of the matrix associator $[A_3]$. These examples therefore make clear that the results in this paper represent only the tip of the iceberg; our assumptions are too restrictive.

By understanding more about the problems encountered in trying to find a characteristic eigenvalue equation for $3 \times 3$ Hermitian matrices over the octonions, we hope our work will aid in the discovery of a method for finding the imaginary eigenvalues (if any), and their corresponding eigenvectors, for any given $3 \times 3$ Hermitian octonionic matrix.
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