A Finitary Analogue of the Downward
 Löwenheim-Skolem Property

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Abstract
We present a model-theoretic property of finite structures, that can be seen to
be a finitary analogue of the well-studied downward Löwenheim-Skolem property
from classical model theory. We call this property as the \( \mathcal{L} \)-equivalent bounded
substructure property, denoted \( \mathcal{L} \)-EBSP, where \( \mathcal{L} \) is either FO or MSO. Intuitively
\( \mathcal{L} \)-EBSP states that a large finite structure contains a small “logically similar”
substructure, where logical similarity means indistinguishability with respect to
sentences of \( \mathcal{L} \) having a given quantifier nesting depth. It turns out that this
simply stated property is enjoyed by a variety of classes of interest in computer
science: examples include various classes of posets, such as regular languages of
words, trees (unordered, ordered or ranked) and nested words, and various classes
of graphs, such as cographs, graph classes of bounded tree-depth, those of bounded
shrub-depth and \( n \)-partite cographs. Further, \( \mathcal{L} \)-EBSP remains preserved in the
classes generated from the above by operations that are implementable using
quantifier-free translation schemes. We show that for natural tree representations
for structures that all the aforementioned classes admit, the small and logically
similar substructure of a large structure can be computed in time linear in the size of
the representation, giving linear time fixed parameter tractable (f.p.t.) algorithms
for checking \( \mathcal{L} \) definable properties of the large structure. We conclude by presenting
a strengthening of \( \mathcal{L} \)-EBSP, that asserts “logical self-similarity at all scales” for a
suitable notion of scale. We call this the logical fractal property and show that
most of the classes mentioned above are indeed, logical fractals.

1 Introduction
The downward Löwenheim-Skolem theorem is one of the earliest results of classical
model theory. This theorem, first proved by Löwenheim in 1915 [21], states that if a
first order (henceforth, FO) theory over a countable vocabulary has an infinite model,
then it has a countable model. In the mid-1920s, Skolem came up with a more general
statement: any structure \( A \) over a countable vocabulary has a countable “FO-similar”
substructure. Here, “FO-similarity” of two given structures means that the structures
agree on all properties than can be expressed in FO. This result of Skolem was further
generalized by Mal’tsev in 1936 [23], to what is considered as the modern statement of
the downward Löwenheim-Skolem theorem: for any infinite cardinal \( \kappa \), any structure \( A \)
over a countable vocabulary has an elementary substructure (an FO-similar substructure
having additional properties) that has size at most \( \kappa \). The downward Löwenheim-Skolem
theorem is one of the most important results of classical model theory, and indeed as
Lindström showed in 1969 \[20\], FO is the only logic (having certain well-defined and reasonable closure properties) that satisfies this theorem, along with the (countable) compactness theorem.

The downward Löwenheim-Skolem theorem is a statement intrinsically of infinite structures, and hence does not make sense in the finite when taken as is. While preservation and interpolation theorems from classical model theory have been actively studied over finite structures \[1\] \[2\] \[20\] \[14\] \[17\] \[25\] \[27\] \[32\] \[2\] \[6\] \[28\] \[5\] \[18\], there is very little study of the downward Löwenheim-Skolem theorem (or adaptations of it) in the finite \((\cite{15} \cite{33}) seem to be the only studies of this theorem in the contexts of finite and pseudo-finite structures respectively). In this paper, we take a step towards addressing this issue. Specifically, we formulate a finitary analogue of the model-theoretic property contained in the downward Löwenheim-Skolem theorem, and show that classes of finite structures satisfying this analogue indeed abound in computer science. We call this analogue the \(\mathcal{L}\)-equivalent bounded substructure property, denoted \(\mathcal{L}\)-EBSP(\(S\)), where \(\mathcal{L}\) is one of the logics FO or MSO, and \(S\) is a class of finite structures (Definition \[1\]).

Intuitively, this property states that over \(S\), for each \(m\), every structure \(\mathfrak{A}\) contains a small substructure \(\mathfrak{B}\) that is “\(\mathcal{L}\)-\([m]\)-similar” to \(\mathfrak{A}\), where \(\mathcal{L}\)-\([m]\) is the class of all sentences of \(\mathcal{L}\) that have quantifier nesting depth at most \(m\). In other words, \(\mathfrak{B}\) and \(\mathfrak{A}\) agree on all properties that can be described in \(\mathcal{L}\)-\([m]\). The bound on the size of \(\mathfrak{B}\) is given by a “witness” function that depends only on \(m\) (when \(\mathcal{L}\) and \(S\) are fixed). It is easily seen that \(\mathcal{L}\)-EBSP(\(S\)) has strong resemblance to the model-theoretic property contained in the downward Löwenheim-Skolem theorem, and can very well be seen as a finitary analogue of a version of the downward Löwenheim-Skolem theorem that is “intermediate” between the versions of this theorem by Skolem and Mal’tsev.

The motivation to define \(\mathcal{L}\)-EBSP(\(S\)) came from our investigations over finite structures, of a generalization of the classical Los-Tarski preservation theorem from model theory, that was proved in \[31\]. This generalization, called the generalized Los-Tarski theorem at level \(k\), denoted GLT(\(k\)), gives a semantic characterization, over arbitrary structures, of sentences in prenex normal form, whose quantifier prefixes are of the form \(\exists^k \forall^\ast\), i.e. a sequence of \(k\) existential quantifiers followed by zero or more universal quantifiers. The Los-Tarski theorem is a special case of GLT(\(k\)) when \(k\) equals 0. Unfortunately, GLT(\(k\)) fails over all finite structures for all \(k \geq 0\) (like most preservation theorems do \[26\]), and worse still, also fails for all \(k \geq 2\), over the special classes of finite structures that are acyclic, of bounded degree, or of bounded tree-width, which were identified by Atserias, Dawar and Grohe \[5\] to satisfy the Los-Tarski theorem. This motivated the search for new (and possibly abstract) structural properties of classes of finite structures, that admit GLT(\(k\)) for each \(k\). It is in this context that a version of \(\mathcal{L}\)-EBSP(\(S\)) was first studied in \[30\]. The present paper takes that study much ahead. (Most of the results of this paper are contained in the author’s Ph.D. thesis \[29\].)

The contributions of this paper are as described below.

1. A variety of classes of interest in computer science satisfy \(\mathcal{L}\)-EBSP: Our property presents a unified framework, via logic, for studying a variety of classes of finite structures that are of interest in computer science. The classes that we consider are broadly of two kinds: special kinds of labeled posets and special kinds of graphs. For the case of labeled posets, we show \(\mathcal{L}\)-EBSP holds for words, trees (of various kinds such as unordered, ordered, ranked, or “partially” ranked), and nested words over a finite alphabet, and all regular subclasses of these (Theorem \[5\]). For each of these classes, we also show that \(\mathcal{L}\)-EBSP holds with computable witness functions. While words and trees have had a long history of studies in the literature, nested words are much recent \[4\], and have attracted a lot of attention as they admit a seamless generalization of the theory of regular languages and are also closely connected with visibly pushdown languages. For the case of graphs, we show \(\mathcal{L}\)-EBSP holds for a very general, and again very recently
defined, class of graphs called \textit{n-partite cographs}, and all hereditary subclasses of this class (Theorem 5.5). This class of graphs, introduced in [13], jointly generalizes the classes of cographs (which includes several interesting graph classes such as complete \textit{r}-partite graphs, Turan graphs, cluster graphs, threshold graphs, etc.), graph classes of bounded tree-depth and those of bounded shrub-depth. Cographs have been well studied since the ’80s [9] and have been shown to admit fast algorithms for many decision and optimization problems that are hard in general. Graph classes of bounded tree-depth and bounded shrub-depth are much more recently defined [24, 13] and have become particularly prominent in the context of investigating fixed parameter tractable (f.p.t.) algorithms for MSO model checking, that have \textit{elementary dependence} on the size of the MSO sentence (which is the parameter) [12, 13]. This line of work seeks to identify classes of structures for which Courcelle-style \textit{algorithmic meta-theorems} [16] hold, but with better dependence on the parameter than in the case of Courcelle’s theorem (which is unavoidably non-elementary [11]). A different and important line of work shows that FO and MSO are equal in their expressive powers over graph classes of bounded tree-depth/shrub-depth [12, 10]. Since each of the graph classes mentioned above is a hereditary subclass of the class of \textit{n}-partite cographs for some \textit{n}, each of these satisfies \textit{L-EBSP}, further with computable witness functions, and further still, even elementary witness functions in many cases.

We give methods to construct new classes of structures satisfying \textit{L-EBSP} from classes known to satisfy \textit{L-EBSP}. Specifically, we show that \textit{L-EBSP} remains preserved under a wide range of operations on structures, that have been well-studied in the literature: unary operations like complementation, transpose and the line graph operation, binary “sum-like” operations [22] such as disjoint union and join, and binary “product-like” operations that include various kinds of products like the Cartesian, tensor, lexicographic and strong products. All of these are examples of operations that can be implemented using, what are called, \textit{quantifier-free translation schemes} [22, 16]. We show that FO-EBSP is always closed under such operations, and MSO-EBSP is closed under such operations, provided that they are unary or sum-like (Theorem 5.7). In both cases, the computability/elementariness of witness functions is preserved under the operations.

2. \textbf{Linear time f.p.t. algorithms for deciding \textit{L} properties of structures:} For each of the classes mentioned above (including those generated using the various operations) and for natural representations of structures in these classes, we give linear time f.p.t. algorithms for deciding properties of structures, that can be defined in \textit{L}. The structures in the above classes have natural \textit{tree representations} in which the leaf nodes of the tree represent simple substructures and the internal nodes represent operations that produce new structures upon being fed with input structures. Given such a tree representation for a structure, we perform appropriate “prunings” of, and “graftings” within, the tree, such that the resultant tree represents an \textit{L-[m]}-similar proper substructure of the original structure. Two key technical elements that are employed to perform these prunings and graftings are the finiteness of the index of the \textit{L-[m]}-similarity relation (which is an equivalence relation) and a Feferman-Vaught kind \textit{composition property} of the operations used in the tree representations. The latter means that the \textit{“L-[m]-similarity class”} of the structure produced by an operation is \textit{determined} by the multi-set of the \textit{L-[m]}-similarity classes of the structures that are input to the operation, and further (in the case of operations having arbitrary finite arity), determined only by a threshold number of appearances of each \textit{L-[m]}-similarity class in the multi-set, with the threshold depending solely on \textit{m}. These technical features enable \textit{generating} the “composition functions” uniformly for any operation for any given \textit{m}, and the composition functions thus generated, in turn, enable doing the compositions in time linear in the arity of the operation. Using these, we get linear time f.p.t. algorithms that when given an \textit{L} sentence of quantifier nesting depth \textit{m} (the parameter) and a tree \textit{t} as inputs, perform
the aforementioned prunings and graftings in \( t \) iteratively to produce a small subtree that represents a small \( L[m] \)-similar substructure (the “kernel”, in the f.p.t. parlance) of the structure represented by \( t \). The techniques mentioned above have been incorporated into a single abstract result concerning tree representations (Theorem 4.2). Given that this result gives unified explanations for the good computational properties of many interesting classes, we believe it might be of independent interest.

3. A strengthening of \( L \)-EBSP and connections with fractals: Fractals are classes of mathematical structures that exhibit self-similarity at all scales. That is, every structure in the class contains a similar (in some technical sense) substructure at every scale of sizes less than the size of the structure. Well-known examples of fractals in mathematics include the Mandelbrot set, the Menger Sponge, and the Koch snowflake. Remarkably, fractals are not limited to only mathematics, but in fact abound nearly everywhere in nature. Tree branching, cloud structures, galaxy clustering, fern shapes, and crystal growth patterns are some of a wide range of natural phenomena that exhibit self-similarity [7].

In the light of fractals, we observe that the \( L \)-EBSP property indeed asserts “logical self-similarity” at “small scales”. We formulate a strengthening of the \( L \)-EBSP property, that asserts logical self-similarity at all scales, for a suitable notion of scale (Definition 6.1). We call this the logical fractal property, and call a class satisfying this property as a logical fractal. Remarkably, it turns out that the aforementioned posets and graph classes, including those constructed using many of the aforementioned operations, are all logical fractals (Proposition 6.3). The classical downward Löwenheim-Skolem theorem indeed shows that the class of all infinite structures satisfies an “infinitary” variant of the logical fractal property. We believe these observations constitute the initial investigations into a potentially rich theory of logical fractals.

The paper is organized as follows. In Section 2 we introduce notation and terminology, and recall relevant notions from the literature used in the paper. In Section 3 we define the \( L \)-EBSP property and show that it holds for the class of “partially” ranked trees, which are trees in which some subset of nodes are constrained to have degrees given by a ranking function. We use this special class as a set to illustrate our techniques, that we lift to tree representations of structures in Section 4. In Section 5, we give applications of our abstract results to obtain the \( L \)-EBSP property and linear time f.p.t. algorithms for model checking \( L \) sentences, in various concrete settings, specifically those of posets and graphs mentioned earlier, and also classes that are constructed using various well-studied operations. We present the notion of logical fractals in Section 6, and conclude with open questions in Section 7.

2 Terminology and preliminaries

1. \( L \) formulae: We assume familiarity with standard notation and notions of first order logic (FO) and monadic second order logic (MSO) [19]. By \( L \), we mean either FO or MSO. We consider only finite vocabularies, represented by \( \tau \) or \( \nu \), that contain only predicate symbols (and no constant or function symbols), unless explicitly stated otherwise. All predicate symbols are assumed to have positive arity. We denote by \( L(\tau) \) the set of all \( L \) formulae over \( \tau \) (and refer to these simply as \( L \) formulae, when \( \tau \) is clear from context). A sequence \( (x_1, \ldots, x_k) \) of variables is written as \( \overline{x} \). A formula \( \varphi \) whose free variables are among \( \overline{x} \), is denoted as \( \varphi(\overline{x}) \). Free variables are always first order. A formula with no free variables is called a sentence. The rank of an \( L \) formula is the maximum number of quantifiers (first order as well as second order) that appears along any path from the root to the leaf in the parse tree of the formula. Finally, a notion or result stated for \( L \) means that the notion or result is stated for both FO and MSO.

2. Structures: Standard notions of \( \tau \)-structures (denoted \( \mathfrak{A}, \mathfrak{B} \) etc.; we refer to these
simply as structures when \( \tau \) is clear from context), substructures (denoted \( \mathfrak{A} \subseteq \mathfrak{B} \)) and extensions are used throughout the paper (see [19]). We assume all structures to be finite. As in [19], by substructures, we always mean induced substructures. Given a structure \( \mathfrak{A} \), we use \( \mathfrak{U}_{\mathfrak{A}} \) to denote the universe of \( \mathfrak{A} \), and \( |\mathfrak{A}| \) to denote its cardinality. We denote by \( \mathfrak{A} \equiv \mathfrak{B} \) that \( \mathfrak{A} \) is isomorphic to \( \mathfrak{B} \), and by \( \mathfrak{A} \rightarrow \mathfrak{B} \) that \( \mathfrak{A} \) is isomorphically embeddable in \( \mathfrak{B} \). For an \( \mathcal{L} \) sentence \( \varphi \), we denote by \( \mathfrak{A} \models \varphi \) that \( \mathfrak{A} \) is a model of \( \varphi \). We denote classes of structures by \( \mathcal{S} \) possibly with subscripts, and assume these to be closed under isomorphisms.

3. The \( \equiv_{m, \mathcal{L}} \) relation: Let \( \mathbb{N} \) and \( \mathbb{N}_+ \) denote the natural numbers including zero and excluding zero respectively. Given \( m \in \mathbb{N} \) and a \( \tau \)-structure \( \mathfrak{A} \), denote by \( \text{Th}_{m, \mathcal{L}}(\mathfrak{A}) \) the set of all \( \mathcal{L}(\tau) \) sentences of rank at most \( m \), that are true in \( \mathfrak{A} \). Given a \( \tau \)-structure \( \mathfrak{B} \), we say that \( \mathfrak{A} \) and \( \mathfrak{B} \) are \( \mathcal{L}[m] \)-equivalent, denoted \( \mathfrak{A} \equiv_{m, \mathcal{L}} \mathfrak{B} \) if \( \text{Th}_{m, \mathcal{L}}(\mathfrak{A}) = \text{Th}_{m, \mathcal{L}}(\mathfrak{B}) \). Given a class \( \mathcal{S} \) of structures and \( m \in \mathbb{N} \), we let \( \Delta_{\mathcal{S}, m} \) denote the set of all equivalence classes of the \( \equiv_{m, \mathcal{L}} \) relation over \( \mathcal{S} \). We denote by \( \Lambda_{\mathcal{S}, \mathcal{L}} : \mathbb{N} \rightarrow \mathbb{N} \) a fixed computable function with the property that \( \Lambda_{\mathcal{S}, \mathcal{L}}(m) \geq |\Delta_{\mathcal{S}, \mathcal{L}, m}| \). It is known that \( \Lambda_{\mathcal{S}, \mathcal{L}} \) always exists (see Proposition 7.5 in [19]). The notion of \( \equiv_{m, \mathcal{L}} \) has a characterization using Ehrenfeucht-Fraïssé (EF) games for \( \mathcal{L} \). We point the reader to Chapters 3 and 7 of [19] for results concerning these games.

4. Translation schemes: We recall the notion of translation schemes from the literature [22] (known in the literature by different names, like interpretations, transductions, etc). Let \( \tau \) and \( \nu \) be given vocabularies, and \( t \geq 1 \) be a natural number. Let \( x_0 \) be a fixed \( t \)-tuple of first order variables, and for each relation \( R \in \nu \) of arity \( \#R \), let \( \bar{x}_R \) be a fixed \((t \times \#R)-t\)-tuple of first order variables. A \((t, \tau, \nu, \mathcal{L})\)-translation scheme \( \Xi = (\xi, (\xi_R)_{R \in \nu}) \) is a sequence of formulas of \( \mathcal{L}(\tau) \) such that the free variables of \( \xi \) are among those in \( x_0 \), and for \( R \in \nu \), the free variables of \( \xi_R \) are among those in \( \bar{x}_R \). When \( t, \nu \) and \( \tau \) are clear from context, we call \( \Xi \) simply as a translation scheme. We call \( t \) as the dimension of \( \Xi \). One can associate with a \((t, \tau, \nu, \mathcal{L})\)-translation scheme \( \Xi \), two partial maps: (i) \( \Xi^* \) from \( \tau \)-structures to \( \nu \)-structures (ii) \( \Xi^\nu \) from \( \mathcal{L}(\nu) \) formulae to \( \mathcal{L}(\tau) \) formulae. See [22] for the definitions of these. For the ease of readability, we abuse notation slightly and use \( \Xi \) to denote both \( \Xi^* \) and \( \Xi^\nu \).

5. Fixed parameter tractability: We say that the model checking problem for \( \mathcal{L} \) over a given class \( \mathcal{S} \), denoted \( \text{MC}(\mathcal{L}, \mathcal{S}) \), is fixed parameter tractable, in short \( f.p.t. \), if there exists an algorithm \( \text{Alg} \) that when given as input an \( \mathcal{L} \) sentence \( \varphi \) of rank \( m \), and a structure \( \mathfrak{A} \in \mathcal{S} \), decides if \( \mathfrak{A} \models \varphi \), in time \( f(m) \cdot |\mathfrak{A}|^c \), where \( f : \mathbb{N} \rightarrow \mathbb{N} \) is some computable function and \( c \) is a constant. In this case, we say \( \text{Alg} \) is an \( f.p.t. \) algorithm for \( \text{MC}(\mathcal{L}, \mathcal{S}) \). We say \( \text{Alg} \) is a linear time \( f.p.t. \) algorithm for \( \text{MC}(\mathcal{L}, \mathcal{S}) \) if it is \( f.p.t. \) for \( \text{MC}(\mathcal{L}, \mathcal{S}) \) and runs in time \( f(k) \cdot |\mathfrak{A}| \) where as before, \( f \) is a computable function.

6. Miscellaneous: The \( k \)-fold exponential function \( \exp(n, k) \) is the function given inductively as: \( \exp(n, 0) = n \) and \( \exp(n, l) = 2^{\exp(n, l-1)} \) for \( 0 \leq l \leq k \). We call a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) as elementary if there exists \( k \) such that \( f(n) = \mathcal{O}(\exp(n, k)) \), and call it non-elementary if it is not elementary. Finally, we use standard abbreviations of English phrases that commonly appear in mathematical literature. Specifically, \( \text{‘w.l.o.g.’} \) stands for ‘without loss of generality’, \( \text{‘iff’} \) stands for ‘if and only if’, and \( \text{‘resp.’} \) stands for ‘respectively’.

3 The \( \mathcal{L} \)-Equivalent Bounded Substructure Property

\( \mathcal{L} \)-EBSP(\( \mathcal{S} \))

Definition 3.1 (\( \mathcal{L} \)-EBSP(\( \mathcal{S} \))). Let \( \mathcal{S} \) be a class of structures and \( \mathcal{L} \) be either FO or MSO. We say that \( \mathcal{S} \) satisfies the \( \mathcal{L} \)-equivalent bounded substructure property, abbreviated \( \mathcal{L} \)-EBSP(\( \mathcal{S} \)) is true (alternatively, \( \mathcal{L} \)-EBSP(\( \mathcal{S} \) holds), if there exists a monotonic function
We clarify that by height, we mean the maximum distance between the root and any called "root," such that for each $m \in \mathbb{N}$ and each structure $\mathfrak{A}$ of $\mathcal{S}$, there exists a structure $\mathfrak{B}$ such that (i) $\mathfrak{B} \in \mathcal{S}$, (ii) $\mathfrak{B} \subseteq \mathfrak{A}$, (iii) $|\mathfrak{B}| \leq \theta_{(S,\mathcal{L})}(m)$, and (iv) $\mathfrak{B} \equiv_{m,\mathcal{L}} \mathfrak{A}$. The conjunction of these four conditions is denoted as $\mathcal{L}$-EBSP-condition($\mathcal{S}, \mathfrak{A}, \mathfrak{B}, m, \theta_{S,\mathcal{L}}$). We call $\theta_{S,\mathcal{L}}$ a witness function of $\mathcal{L}$-EBSP($\mathcal{S}$).

We present below two simple examples of classes satisfying $\mathcal{L}$-EBSP.

1. Let $\mathcal{S}$ be the class of all $\tau$-structures, where all predicates in $\tau$ are unary. By a simple FO-EF game argument, we see that FO-EBSP($\mathcal{S}$) holds with $\theta_{S,\text{FO}}(m) = m \cdot 2^{\lceil |\tau| \rceil}$. In more detail: given $\mathfrak{A} \in \mathcal{S}$, associate exactly one of $2^{\lceil |\tau| \rceil}$ colors with each element $a$ of $\mathfrak{A}$, where the color gives the valuation of all predicates of $\tau$ for $a$ in $\mathfrak{A}$. Then consider $\mathfrak{B} \subseteq \mathfrak{A}$ such that for each colour $c$, if $A_c = \{a \mid a \in U_{\mathfrak{B}} \setminus a \text{ has colour } c \text{ in } \mathfrak{A}\}$, then $A_c \subseteq U_{\mathfrak{B}}$ if $|A_c| \leq m$, else $|A_c \cap U_{\mathfrak{B}}| = m$. It is easy to see that FO-EBSP-condition($\mathcal{S}, \mathfrak{A}, \mathfrak{B}, \theta_{S,\text{FO}})$ holds. By a similar MSO-EF game argument, one can show that MSO-EBSP($\mathcal{S}$) holds with a witness function given by $\theta_{S,\text{MSO}}(m) = m \cdot 2^{\lceil |\tau| \rceil + m}$.

2. Let $\mathcal{S}$ be the class of disjoint unions of undirected paths. It is known that for any $m$, any two paths of length $\geq p = 3^m$ are FO-equivalent. Let $\mathfrak{A} = \bigsqcup_{n \geq 0} i_n \cdot P_n$ where $P_n$ denotes the path of length $n$, $i_n \cdot P_n$ denotes the disjoint union of $i_n$ copies of $P_n$, and $\bigsqcup$ denotes disjoint union. For $n < p$, let $j_n$ be such that $j_n = i_n$ if $i_n < m$ and $j_n = m$ if $i_n \geq m$. For $n = p$, let $j_n = h = \sum_{i \geq p} i$, if $h < m$, else $j_n = m$.

One can then see using an FO-EF game argument that if $\mathfrak{B} = \bigsqcup_{n=0}^{m} j_n \cdot P_n$, then $\mathfrak{B}$ satisfies FO-EBSP-condition($\mathcal{S}, \mathfrak{A}, \mathfrak{B}, \theta_{S,\text{FO}}$) where $\theta_{S,\text{FO}}(m) = \sum_{n=0}^{m} n \cdot m$.

3.1 Partially ranked trees satisfy $\mathcal{L}$-EBSP

In this subsection, we show that the class of ordered "partially" ranked trees satisfies $\mathcal{L}$-EBSP with computable witness functions, as well as admits a linear time f.p.t. algorithm for model checking $\mathcal{L}$-sentences. This setting illustrates our reasoning and techniques that we lift in Section 4 to the more abstract setting of tree representations of structures.

An unlabeled unordered tree is a finite poset $P = (A, \preceq)$ with a unique minimal element (called "root"), such that for each $c \in A$, the set $\{b \mid b \preceq c\}$ is totally ordered by $\preceq$. Informally speaking, the Hasse diagram of $P$ is an inverted (graph-theoretic) tree. We call $A$ as the set of nodes of $P$. We use the standard notions of leaf, internal node, ancestor, descendant, parent, child, degree, height, and subtree in connection with trees. (We clarify that by height, we mean the maximum distance between the root and any leaf of the tree, as against the "number of levels" in the tree.) An unlabeled ordered tree is a pair $O = (P, \preceq)$ where $P$ is an unlabeled unordered tree and $\preceq$ is a binary relation that imposes a linear order on the children of any internal node of $P$. Unless explicitly stated otherwise, we always consider our trees to be ordered. It is clear that the above mentioned notions in connection with unordered trees can be adapted for ordered trees.

Given a countable alphabet $\Sigma$, a tree over $\Sigma$, also called a $\Sigma$-tree, or simply tree when $\Sigma$ is clear from context, is a pair $(O, \lambda)$ where $O$ is an unlabeled tree and $\lambda : A \rightarrow \Sigma$ is a labeling function, where $A$ is the set of nodes of $O$. We denote $\Sigma$-trees by $s, t, x, y, u, v$ or $z$, possibly with numbers as subscripts. Given a tree $t$, we denote the root of $t$ as root$(t)$. For a node $a$ of $t$, we denote the subtree of $t$ rooted at $a$ as $t_{\geq a}$, and the subtree of $t$ obtained by deleting $t_{\geq a}$ from $t$, as $t - t_{\geq a}$. Given a tree $s$ and a non-root node $a$ of $t$, the replacement of $t_{\geq a}$ with $s$ in $t$, denoted $t[t_{\geq a} \mapsto s]$, is a tree defined as follows. Assume w.l.o.g. that $s$ and $t$ have disjoint sets of nodes. Let $c$ be the parent of $a$ in $t$. Then $t[t_{\geq a} \mapsto s]$ is defined as the tree obtained by deleting $t_{\geq a}$ from $t$ to get a tree $t'$, and inserting $(\text{the root of } s)$ at the same position among the children of $c$ in $t'$, as the position of $a$ among the children of $c$ in $t$. For $s$ and $t$ as just mentioned, suppose the roots of both these trees have the same label. Then the merge of $s$ with $t$, denoted $t \odot s$,
is defined as the tree obtained by deleting \( \text{root}(s) \) from \( s \) and concatenating the sequence of subtrees hanging at \( \text{root}(s) \) in \( s \), to the sequence of subtrees hanging at \( \text{root}(t) \) in \( t \).

Thus the children of \( \text{root}(s) \) in \( s \) are the “new” children of \( \text{root}(t) \), and appear “after” the “old” children of \( \text{root}(t) \), and in the order they appear in \( s \).

Fix a finite alphabet \( \Sigma \), and let \( \Sigma_{\text{rank}} \subseteq \Sigma \). Let \( \rho : \Sigma_{\text{rank}} \to \mathbb{N}_+ \) be a fixed function.

We say a \( \Sigma \)-tree \( t = (\omega, \lambda) \) is partially ranked by \((\Sigma_{\text{rank}}, \rho)\) if for any node \( a \) of \( t \), if \( \lambda(a) \in \Sigma_{\text{rank}} \), then the number of children of \( a \) in \( t \) is exactly \( \rho(\lambda(a)) \). Observe that the case of \( \Sigma_{\text{rank}} = \Sigma \) corresponds to the notion of ranked trees that are well-studied in the literature [3]. Let Partially-ranked-trees\((\Sigma, \Sigma_{\text{rank}}, \rho)\) be the class of all ordered \( \Sigma \)-trees partially ranked by \((\Sigma_{\text{rank}}, \rho)\). The central result of this section is now as stated below.

**Proposition 3.2.** Given \( \Sigma \), a subset \( \Sigma_{\text{rank}} \) of \( \Sigma \) and \( \rho : \Sigma_{\text{rank}} \to \mathbb{N}_+ \), let \( S \) be the class Partially-ranked-trees\((\Sigma, \Sigma_{\text{rank}}, \rho)\). Then the following are true:

1. \( L\text{-EBSP}(S) \) holds with a computable witness function. Further, any witness function is necessarily non-elementary.
2. There is a linear time f.p.t. algorithm for \( \text{MC}(L, S) \).

We prove the two parts of the above result separately. In the remainder of this section, we fix \( L \), and also fix \( S \) to be the class Partially-ranked-trees\((\Sigma, \Sigma_{\text{rank}}, \rho)\). Given these fixings, we denote \( \Delta_{S,L,c} \) (the set of equivalence classes of the \( \equiv_{m,L} \) relation over \( S \)) simply as \( \Delta_m \) and denote \( \Lambda_{S,L}(m) \) (see point 3 in Section 2 for the definition of \( \Lambda_{S,L}(m) \)) simply as \( \Lambda(m) \). All trees will be assumed to be from \( S \).

Towards the proof of Proposition 3.2, we first present a Feferman-Vaught style \( L \)-composition lemma for ordered trees. Composition results of this kind were first studied by Feferman and Vaught, and subsequently by many others (see [22]). To state the composition lemma, we introduce some terminology. For a finite alphabet \( \Omega \), given ordered \( \Omega \)-trees \( t, s \) having disjoint sets of nodes (w.l.o.g.) and a non-root node \( a \) of \( t \), the \textit{join} of \( s \) to \( t \) to the right of \( a \), denoted \( t \uparrow_a s \), is defined as the tree obtained by making \( s \) as a new child subtree of the parent of \( a \) in \( t \), at the successor position of the position of \( a \) among the children of the parent of \( a \) in \( t \). We can similarly define the \textit{join} of \( s \) to \( t \) to the left of \( a \), denoted \( t \downarrow_a s \). Likewise, for \( t \) and \( s \) as above, if \( a \) is a leaf node of \( t \), we can define the \textit{join} of \( s \) to \( t \) below \( a \), denoted \( t \leftarrow_a s \), as the tree obtained up to isomorphism by making the root of \( s \) as a child of \( a \). The \( L \) composition lemma for ordered trees can now be stated as follows. The proof is similar to the proof of the known \( L \)-composition lemma for words. We skip presenting the proof here, but point the interested reader to Appendix A for the detailed proof.

**Lemma 3.3.** (Composition lemma for ordered trees). For a finite alphabet \( \Omega \), let \( t_1, s_1 \) be non-empty ordered \( \Omega \)-trees, and let \( a_i \) be a non-root node of \( t_i \), for each \( i \in \{1, 2\} \). Let \( m \geq 2 \) and suppose that \( (t_1, a_1) \equiv_{m,L} (t_2, a_2) \) and \( s_1 \equiv_{m,L} s_2 \). Then each of the following hold:

1. \( ((t_1 \uparrow_{a_1} s_1), a_1) \equiv_{m,L} ((t_2 \uparrow_{a_2} s_2), a_2) \)
2. \( ((t_1 \downarrow_{a_1} s_1), a_1) \equiv_{m,L} ((t_2 \downarrow_{a_2} s_2), a_2) \)
3. \( ((t_1 \leftarrow_{a_1} s_1), a_1) \equiv_{m,L} ((t_2 \leftarrow_{a_2} s_2), a_2) \) if \( a_1, a_2 \) are leaf nodes of \( t_1, t_2 \) resp.

A useful corollary of this lemma is as below.

**Corollary 3.4.** The following are true for \( m \geq 3 \):

1. Given trees \( s, t \) and a non-root node \( a \) of \( t \), let \( z = t[t_{\geq a} \mapsto s] \). If \( s \equiv_{m,L} t_{\geq a} \), then \( z \equiv_{m,L} t \).
2. Let \( s_1, s_2, t \) be given trees such that the labels of their roots are the same, and belong to \( \Sigma \setminus \Sigma_{\text{rank}} \). Suppose \( z_i = s_i \odot t \) for \( i \in \{1, 2\} \). If \( s_1 \equiv_{m,L} s_2 \), then \( z_1 \equiv_{m,L} z_2 \).
3. Let $s_1, s_2$ be given trees such that the labels of their roots are the same, and belong to $\Sigma \setminus \Sigma_{\text{rank}}$. For $i \in \{1, 2\}$, given $t_i$, let $z_i$ be the tree obtained from $s_i$ by adding $t_i$ as the (new) “last” child subtree of the root of $s_i$. If $s_1 \equiv m, \ell s_2$ and $t_1 \equiv m, \ell t_2$, then $z_1 \equiv m, \ell z_2$.

**Proof.**

1. Let $v = t - t_{\geq a}$. There are 3 possibilities:
   (i) The node $a$ has a “predecessor” sibling in $t$, call it $b$. Then $t = v \cdot b \cdot t_{\geq a}$. Then since $t_{\geq a} \equiv m, \ell s$, we have by Lemma 3.3 that $z \equiv m, \ell t$ since $z = (v \cdot b \cdot s)$.
   (ii) The node $a$ has a “successor” sibling in $t$, call it $b$. Then $t = v \cdot b^+ \cdot t_{\geq a}$. Again since $t_{\geq a} \equiv m, \ell s$, we have by Lemma 3.3 that $z \equiv m, \ell t$ since $z = (v \cdot b^+ \cdot s)$.
   (iii) The node $a$ is the sole child of its parent $b$ in $t$. Then $t = v \cdot b \cdot t_{\geq a}$. Then again by Lemma 3.3 we have that $z \equiv m, \ell t$ since $z = (v \cdot b \cdot s)$.

2. We prove this part assuming part 3. For $i \in \{1, 2\}$, let $a_i$ be the last child of the root of $s_i$ (under the linear order on the children of the root). Let $b_1, \ldots, b_n$ be the children (and in that order) of the root of $t$. Let $u_j = t_{\geq b_j}$ for $j \in \{1, \ldots, n\}$. For $i \in \{1, 2\}$, let $x^i_1 = s^i_1 \cdot b_1$ and $x^i_{j+1} = x^i_j \cdot b_{j+1}$ for $j \in \{1, \ldots, n-1\}$. Since $s_1 \equiv m, \ell s_2$, we have by part 3 of this lemma, that $x^1_1 \equiv m, \ell x^2_1$. Whereby, $x^1_j \equiv m, \ell x^2_j$ for $j \in \{1, \ldots, n\}$. Since $x^i_1 = z_i$ for $i \in \{1, 2\}$, we have $z_1 \equiv m, \ell z_2$.

3. For $i \in \{1, 2\}$, let $a_i$ be the last child of the root of $s_i$ (under the linear order on the children of the root). It is easy to verify given that $s_1 \equiv m, \ell s_2$ and $m \geq 3$, that there exists a winning strategy for the duplicator in the $m$ round $\ell$-EF game between $s_1$ and $s_2$ such that in any round, if the spoiler chooses $a_1$ from $s_1$ (resp. $a_2$ from $s_2$), then the duplicator chooses $a_2$ from $s_2$ (resp. $a_1$ from $s_1$) according to the winning strategy. Whereby, $(s_1, a_1) \equiv m, \ell (s_2, a_2)$. Then by Lemma 3.3, $z_1 = (s^1_1 \cdot a_1) \equiv m, \ell (s^2_1 \cdot a_2)$.

We use the above results to obtain a “functional” form of a composition lemma for partially ranked trees, as given by the lemma below. This lemma plays a crucial role in the proof of Proposition 3.2. Recall that $S = \text{Partially-ranked-trees}(\Sigma, \Sigma_{\text{rank}}, \rho)$.

**Lemma 3.5** (Composition lemma for partially ranked trees). For each $\sigma \in \Sigma$ and $m \geq 3$, there exists a function $f_{\sigma, m} : (\Delta_m)^m \rightarrow \Delta_m$ if $\sigma \in \Sigma_{\text{rank}}$, and functions $f_{\sigma, m, i} : (\Delta_m)^i \rightarrow \Delta_m$ for $i \in \{1, 2\}$ if $\sigma \in \Sigma \setminus \Sigma_{\text{rank}}$, with the following properties: Let $t = (\sigma, \lambda) \in S$ and $a$ be an internal node of $t$ such that $\lambda(a) = \sigma$, and the children of $a$ in $t$ are $b_1, \ldots, b_n$. Let $\delta_i$ be the $\equiv m, \ell$ class of $t_{\geq b_i}$ for $i \in \{1, \ldots, n\}$, and let $\delta$ be the $\equiv m, \ell$ class of $t_{\geq a}$.

1. If $\sigma \in \Sigma_{\text{rank}}$ (whereby $n = \rho(\sigma)$), then $\delta = f_{\sigma, m}(\delta_1, \ldots, \delta_n)$.
2. If $\sigma \in \Sigma \setminus \Sigma_{\text{rank}}$, then $\delta$ is given as follows: For $k \in \{1, \ldots, n-1\}$, let $\chi_{k+1} = f_{\sigma, m, 2}(x_k, \delta_{k+1})$ where $\chi_1 = f_{\sigma, m, 1}(\delta_1)$. Then $\delta = \chi_n$.

**Proof.** We define functions $f_{\sigma, m}$ and $f_{\sigma, m, i}$ as follows:

1. $f_{\sigma, m}$: Let $\delta_i \in \Delta_m$ for $i \in \{1, \ldots, n\}$ be given, where $n = \rho(\sigma)$ and $\sigma \in \Sigma_{\text{rank}}$. If any of the $\delta_i$’s is not realized in $S$ (i.e. there is no tree in $S$ whose $\equiv m, \ell$ class is $\delta_i$), then define $f_{\sigma, m}(\delta_1, \ldots, \delta_n) = \delta_{\text{default}}$ where $\delta_{\text{default}}$ is some fixed element of $\Delta_m$.

   Else, let $t_i \in S$ be a tree such that the $\equiv m, \ell$ class of $t_i$ is $\delta_i$ for $1 \leq i \leq n$. Let $s_{b_1, \ldots, b_n}$ be the tree obtained by making $t_1, \ldots, t_n$ as the child subtrees (and in that sequence) of a new root node labeled with $\sigma$. Let $\delta$ be the $\equiv m, \ell$ class of $s_{b_1, \ldots, b_n}$. Define $f_{\sigma, m}(\delta_1, \ldots, \delta_n) = \delta$.

2. $f_{\sigma, m, i}$: The case when $i = 1$ can be done similarly as above. We consider the case of $i = 2$. Let $\delta_1, \delta_2 \in \Delta_m$. Note that $\sigma \in \Sigma \setminus \Sigma_{\text{rank}}$. For $i \in \{1, 2\}$, find trees $t_i$ such that the $\equiv m, \ell$ class of $t_i$ is $\delta_i$ and further such that the root of $t_i$ is labeled with $\sigma$. If either $t_1$ or $t_2$ is not found, then define $f_{\sigma, m, 2}(\delta_1, \delta_2) = \delta_{\text{default}}$. Else, let $v_{\delta_1, \delta_2}$ be the tree obtained adding $t_2$ as the (new) “last” child subtree of the root of $t_1$. Let $\delta$ be the $\equiv m, \ell$ class of $v_{\delta_1, \delta_2}$. Define $f_{\sigma, m, 2}(\delta_1, \delta_2) = \delta$. 

8
We claim that $f_{\sigma,m}$ and $f_{\sigma,m,i}$ indeed satisfy the properties mentioned in the statement of this lemma. Let $t = (O, \lambda) \in S$ and $a$ be an internal node of $t$ such that $\lambda(a) = \sigma$, and the children of $a$ in $t$ are $b_1, \ldots, b_n$. Let $\delta_i$ be the $\equiv_{m,L}$ class of $t_{\geq i}$ for $i \in \{1, \ldots, n\}$, and let $\delta$ be the $\equiv_{m,L}$ class of $t_{\geq a}$.

- $f_{\sigma,m}$: Since $t_{\geq a}$ has $\equiv_{m,L}$ class $\delta_1$ for $i \in \{1, \ldots, n\}$, we see that the tree $z = s_{\delta_1, \ldots, \delta_n}$, as referred to earlier, exists. Let $d_1, \ldots, d_n$ be the children of the root of $z$; then for $i \in \{1, \ldots, n\}$, the $\equiv_{m,L}$ class of $z_{\geq d_i}$ is $\delta_i$, and hence $z_{\geq d_i} \equiv_{m,L} t_{\geq b_i}$. Since $t_{\geq a} = z \equiv t_{\geq b_1} \equiv t_{\geq b_2} \equiv t_{\geq b_n}$, we see by Corollary 3.4 that $t_{\geq a} \equiv_{m,L} z$, whereby the $\equiv_{m,L}$ class of $t_{\geq a}$ equals the $\equiv_{m,L}$ class of $z$. The latter in turn is the same as $f_{\sigma,m}(\delta_1, \ldots, \delta_n)$ by construction.

- $f_{\sigma,m,i}$: The reasoning for $i = 1$ is just as done above for $f_{\sigma,m}$. We hence consider the case of $i = 2$. We illustrate our reasoning for the example of $n = 3$. The reasoning for general $n$ can be done likewise. Let $u = t_{\geq a}$; the root of $u$ has 3 children $b_1, b_2, b_3$ such that the $\equiv_{m,L}$ class of $b_i$ is $\delta_i$ for $i \in \{1, 2, 3\}$. Consider the subtrees $x$ and $y$ of $u$ defined as $x = u - u_{\geq b_2}$ and $y = x - x_{\geq b_2}$. Let $\delta_1$ and $\delta_2$ be resp. the $\equiv_{m,L}$ classes of $x$ and $y$. Now consider the trees $u_{\delta_1, \delta_2}$ and $u_{\delta_2, \delta_3}$ which are guaranteed to exist (since $y$ is a tree whose root is labeled with $\sigma$ and $\equiv_{m,L}$ class is $\delta_1$, while $u_{\geq b_2}$ is a tree whose $\equiv_{m,L}$ class is $\delta_2$). Since $\sigma \in \Sigma \setminus \Sigma_{\text{rank}}$, we have by Corollary 3.4, that $x \equiv_{m,L} v_{\delta_1, \delta_2}$ and $u \equiv_{m,L} v_{\delta_2, \delta_3}$. By the $\equiv_{m,L}$ class of $x$ is $\delta_1 = f_{\sigma,m,2}(\delta_1, \delta_2)$ and that of $u$ is $\delta = f_{\sigma,m,2}(\delta_1, \delta_3)$. Observe that $\delta_3$ is indeed $f_{\sigma,m,1}(\delta_1)$.

Proof of part (1) of Proposition 3.2. The proof of this part has at its core, the following “reduction” lemma that shows that the degree and height of a tree can always be reduced to under a threshold, preserving $L[m]$ equivalence.

**Lemma 3.6.** There exist computable functions $\eta_1, \eta_2 : \mathbb{N} \to \mathbb{N}$ such that for each $t \in S$ and $m \in \mathbb{N}$, the following hold:

1. (Degree reduction) There exists a subtree $s_1$ of $t$ in $S$, of degree $\leq \eta_1(m)$, such that (i) the roots of $s_1$ and $t$ are the same, and (ii) $s_1 \equiv_{m,L} t$.

2. (Height reduction) There exists a subtree $s_2$ of $t$ in $S$, of height $\leq \eta_2(m)$, such that (i) the roots of $s_2$ and $t$ are the same, and (ii) $s_2 \equiv_{m,L} t$.

**Proof sketch.** For a finite subset $X$ of $\mathbb{N}$, let $\max(X)$ denote the maximal element of $X$.

1: For $n \geq 3$, define $\eta_1(n) = \max(\{p(\sigma) \mid \sigma \in \Sigma_{\text{rank}} \cup \{3\} \} \times \Lambda(n))$. For $n < 3$, define $\eta_1(n) = \eta_1(3)$. We prove this part for $m \geq 3$; then it follows that this part is also true for $m < 3$ (by taking $s_1$ for the $m = 3$ case as $s_1$ for the $m < 3$ case).

Given $m \geq 3$, let $p = \eta_1(m)$. If $t$ has degree $\leq p$, then putting $s_1 = t$, we are done. Else, some node $a$ of $t$ has degree $n > p$. Clearly then $\lambda(a) \notin \Sigma_{\text{rank}}$. Let $z = t_{\geq a}$ and let $a_1, \ldots, a_n$ be the (ascending) sequence of children of root$(z)$ in $z$. For $1 \leq j \leq n$, let $x_{1,j}, \text{resp. } y_{j+1,n}$, be the subtree of $z$ obtained from $z$ by deleting the subtrees rooted at $a_{j+1}, \ldots, a_n$, resp. deleting the subtrees rooted at $a_1, a_2, \ldots, a_{j-1}$. Then $z = x_{1,n} = x_{1,j} \cup y_{j+1,n}$ for $1 \leq j < n$. Let $g : \{1, \ldots, n\} \to \Delta_m$ be such that $g(j)$ is the $\equiv_{m,L}$ class of $x_{1,j}$. Since $n > p$, there exist $j, k \in \{1, \ldots, n\}$ such that $j < k$ and $g(j) = g(k)$, i.e. $x_{1,j} \equiv_{m,L} x_{1,k}$. If $k < n$, then let $z_1 = x_{1,j} \cup y_{k+1,n}$, else let $z_1 = x_{1,j}$. Then by Corollary 3.4, $z_1 \equiv_{m,L} z$. Let $t_1$ be the subtree of $t$ in $S$ given by $t_1 = t[z \equiv z_1]$. By Corollary 3.4 again, $t_1 \equiv_{m,L} t$. Observe that $t_1$ has strictly lesser size than $t$. Recursing on $t_1$, we are eventually done.

2: For $n \geq 3$, define $\eta_2(n) = \Lambda(n) + 1$. For $n < 3$, define $\eta_2(n) = \eta_2(3)$. As before, it suffices to prove this part for $m \geq 3$.

Given $m \geq 3$, let $p = \eta_2(m)$. If $t$ has height $\leq p$, then putting $s_2 = t$, we are done. Else, there is a path from the root of $t$ to some leaf of $t$, whose length is $> p$. Let $\Lambda$ be the set
of nodes appearing along this path. Let \( h : A \to \Delta_m \) be such that for each \( a \in A, h(a) \) is the \( \equiv_{m, \mathcal{L}} \) class of \( t_{2a} \). Since \( |A| > p \), there exist distinct nodes \( a, b \in A \) such that \( a \) is an ancestor of \( b \) in \( t \), \( a \neq \text{root}(t) \), and \( h(a) = h(b) \). Let \( t_2 = t \begin{array}{c} t_{2a} \\
arrow \end{array} t_{2b} \); then \( t_2 \) is a subtree of \( t \). Since \( h(a) = h(b) \), \( t_{2a} \equiv_{m, \mathcal{L}} t_{2b} \). By Corollary 3.4, we get \( t_2 \equiv_{m, \mathcal{L}} t \). Note that \( t_2 \) has strictly lesser size than \( t \). Recursing on \( t_2 \), we are eventually done. □

Proof of Proposition 3.2. Let \( t \in \mathcal{S} \) and \( m \in \mathbb{N} \) be given. By Lemma 3.6, there exists a subtree \( s \) of \( t \) in \( \mathcal{S} \), of degree \( \leq \eta_1(m) \) and height \( \leq \eta_2(m) \), and hence of size \( \leq \eta_1(m)(\eta_2(m)+1) \), such that \( s \equiv_{m, \mathcal{L}} t \). Then \( \mathcal{L}\text{-EBSP-condition}(\mathcal{S}, t, s, m, \theta_{\mathcal{S}, \mathcal{L}}) \) is true where \( \theta_{\mathcal{S}, \mathcal{L}}(m) = \eta_1(m)(\eta_2(m)+1) \). Since \( t \in \mathcal{S} \) and \( m \in \mathbb{N} \) are arbitrary, it follows that \( \mathcal{L}\text{-EBSP}(\mathcal{S}) \) is true.

As for the non-elementariness of witness functions for \( \mathcal{L}\text{-EBSP}(\mathcal{S}) \), observe that if there exists an elementary witness function \( \theta \) for \( \mathcal{L}\text{-EBSP}(\mathcal{S}) \), then every tree \( t \) in \( \mathcal{S} \) is \( \mathcal{L}[m] \)-equivalent to a tree \( s \) in \( \mathcal{S} \) such that \( |s| \leq \theta(m) \). Whereby the index of the \( \equiv_{m, \mathcal{L}} \) relation over \( \mathcal{S} \) is bounded by the number of trees in \( \mathcal{S} \) whose size is \( \leq \theta(m) \). Clearly then, this number, and hence the index, is bounded by an elementary function of \( m \) if \( \theta \) is elementary. However, even over words, we know that the index of the \( \equiv_{m, \mathcal{L}} \) relation is non-elementary. □

Proof of part (2) of Proposition 3.2. The following result contains the core argument for the proof of this part of Proposition 3.2. The first part of Lemma 3.7 gives an algorithm to generate the “composition” functions of Lemma 3.6 uniformly for \( m \geq 3 \). This algorithm is in turn used in the second part of Lemma 3.7 to get a “linear time” version of Lemma 3.6.

Lemma 3.7. There exist computable functions \( \eta_3, \eta_4, \eta_5 : \mathbb{N} \to \mathbb{N} \) and algorithms \( \text{Generate-functions}(m), \text{Reduce-degree}(t, m) \) and \( \text{Reduce-height}(t, m) \) such that for \( m \geq 3 \),

1. Generate-functions(\( m \)) generates in time \( \eta_3(m) \), the functions \( f_{\sigma, m} \) if \( \sigma \in \Sigma_{\text{rank}} \) and \( f_{\sigma, m, i} \) for \( i \in \{1, 2\} \) if \( \sigma \in \Sigma \setminus \Sigma_{\text{rank}} \), that satisfy the properties mentioned in Lemma 3.6.
2. For \( t \in \mathcal{S} \), \( \text{Reduce-degree}(t, m) \) computes the subtree \( s_1 \) of \( t \) as given by Lemma 3.6 in time \( \eta_4(m) \cdot |t| \). Likewise, \( \text{Reduce-height}(t, m) \) computes the subtree \( s_2 \) of \( t \) as given by Lemma 3.6 in time \( \eta_5(m) \cdot |t| \).

Using this lemma, part (2) of Proposition 3.2 can be proved as follows.

Proof of Proposition 3.2(2). We describe a simple algorithm \( \text{Evaluate}(t, \varphi) \) that when given a tree \( t \in \mathcal{S} \) and an \( \mathcal{L} \) sentence \( \varphi \) of rank \( m \), as inputs, decides, if \( t \models \varphi \) in time \( f(m) \cdot |t| \) for some computable function \( f : \mathbb{N} \to \mathbb{N} \).

\( \text{Evaluate}(t, \varphi) \):

1. Let \( m_1 = \max\{m, 3\} \).
2. Compute a subtree \( s \) of \( t \) in \( \mathcal{S} \) by invoking \( \text{Reduce-height}(\text{Reduce-degree}(t, m_1), m_1) \).
3. Evaluate \( \varphi \) on \( s \).
4. If \( s \models \varphi \), return True, else return False.

Analysis:

- Correctness: For functions \( \eta_1, \eta_2 \) as mentioned in Lemma 3.6, the subtree \( s \) in the algorithm above is such that \( |s| \leq \eta_1(m_1)(\eta_2(m_1)+1) \) and \( s \equiv_{m_1, \mathcal{L}} t \) – this follows from Lemma 3.7(2). Since \( m_1 \geq m \), we have \( s \equiv_{m, \mathcal{L}} t \) – then \( t \models \varphi \) if \( s \models \varphi \), proving that the above algorithm is indeed correct.
• Running time: By Lemma 3.7(2), the time taken for computing \( s \) is at most \( \eta_4(m_1) \cdot |t| + \eta_5(m_1) \cdot |t| \). The time taken to evaluate \( \varphi \) on \( s \) is \( \eta_6(m_1) \) for some computable function \( \eta_6 : \mathbb{N} \to \mathbb{N} \). Then the total running time of \( \text{Evaluate}(t, \varphi) \) is at most \( f(m) \cdot |t| \), where \( f(m) = \eta_4(m_1) + \eta_5(m_1) + \eta_6(m_1) \) and \( m_1 = \max\{m, 3\} \).

We now provide a proof sketch for Lemma 3.7 to complete this section.

**Proof sketch for Lemma 3.7 (Part II):** For the algorithm, we observe that the \( \mathcal{L} \)-SAT problem is decidable over \( S \) since \( \mathcal{L} \)-SAT(S) holds with a computable witness function (by Proposition 3.2(1)), if an \( \mathcal{L} \) sentence has a model in \( S \), it also has a model of size bounded by a computable function of its rank.

**Generate-functions(m):**

1. Create a list \( \mathcal{L}[m] \)-classes of the \( \equiv_{m, \mathcal{L}} \) classes over \( S \). This is done as follows:
   
   (a) Given the inductive definition of \( \mathcal{L}[m] \), there is an algorithm \( \mathcal{P}(m) \) which enumerates \( \mathcal{L}[m] \) sentences \( \varphi_1, \varphi_2, \ldots, \varphi_n \) such that every sentence \( \varphi \) captures some equivalence class of the \( \equiv_{m, \mathcal{L}} \) relation over all finite structures, and conversely, every equivalence class of the \( \equiv_{m, \mathcal{L}} \) relation over all finite structures, is captured by some \( \varphi_i \). First invoke \( \mathcal{P}(m) \) to get the \( \varphi_i \).

   (b) For each \( i \in \{1, \ldots, n\} \), if \( \varphi_i \) is satisfiable over \( S \) (whereby it represents some equivalence class of the \( \equiv_{m, \mathcal{L}} \) relation over \( S \)), then put it in \( \mathcal{L}[m] \)-classes, else discard it. (We interchangeably regard \( \mathcal{L}[m] \)-classes as a list of \( \mathcal{L}[m] \) sentences or a list of \( \equiv_{m, \mathcal{L}} \) classes.)

2. For \( \sigma \in \Sigma_{\text{rank}} \) and \( d = \rho(\sigma) \), generate \( g_{\sigma, m} : (\mathcal{L}[m] \text{-classes})^d \to (\mathcal{L}[m] \text{-classes}) \) as follows. Given \( \xi_i \in \mathcal{L}[m] \text{-classes} \) for \( i \in \{1, \ldots, d\} \), find models \( s_1 \) for \( \xi_1 \) in \( S \). Let \( s \) be the tree obtained by making \( s_1, \ldots, s_n \) as the child subtrees (and in that sequence) of a new root node labeled with \( \sigma \). Find out \( \xi \in \mathcal{L}[m] \text{-classes} \) of which \( s \) is a model. Then define \( g_{\sigma, m}(\xi_1, \ldots, \xi_d) = \xi \). Generate \( g_{\sigma, m, 1} : (\mathcal{L}[m] \text{-classes}) \to (\mathcal{L}[m] \text{-classes}) \) similarly.

3. For \( \sigma \in \Sigma \setminus \Sigma_{\text{rank}} \), generate \( g_{\sigma, m, 2} : (\mathcal{L}[m] \text{-classes})^2 \to (\mathcal{L}[m] \text{-classes}) \) as follows. For \( \xi_1, \xi_2 \in \mathcal{L}[m] \text{-classes} \), find models \( s_1 \) and \( s_2 \) resp. in \( S \) such that the root of \( s_1 \) is labeled with \( \sigma \) (this condition on the root can be captured by an FO sentence). If no \( s_1 \) is found, then define \( g_{\sigma, m, 2}(\xi_1, \xi_2) = \xi_{\text{default}} \) where the latter is some fixed element of \( \mathcal{L}[m] \text{-classes} \). Else, let \( \nu_{\xi, 1, \xi, 2} \) be the tree obtained adding \( s_2 \) as the (new) “last” child subtree of the root of \( s_1 \). Find out \( \xi \in \mathcal{L}[m] \text{-classes} \) of which \( \nu_{\xi, 1, \xi, 2} \) is a model. Define \( g_{\sigma, m, 2}(\xi_1, \xi_2) = \xi \).

It is clear that there exists a computable function \( \eta_3 : \mathbb{N} \to \mathbb{N} \) such that the running time of \( \text{Generate-functions}(m) \) is at most \( \eta_3(m) \). We now claim that \( g_{\sigma, m} \) and \( g_{\sigma, m, 1} \) generated by \( \text{Generate-functions}(m) \) indeed satisfy the composition properties of Lemma 3.5 whereby they can be indeed taken as \( f_{\sigma, m} \) and \( f_{\sigma, m, 1} \) appearing in the latter lemma. That \( g_{\sigma, m} \) and \( g_{\sigma, m, 2} \) satisfy the composition properties is easy to see using Corollary 3.4. To reason for \( g_{\sigma, m, 2} \), consider a tree \( t \) whose root is labeled with \( \sigma \), and which has say 3 children \( a_1, \ldots, a_3 \) (and in that sequence) such that the \( \equiv_{m, \mathcal{L}} \) class of \( t_{\geq a_i} \) is \( \delta_i \) for \( 1 \leq i \leq 3 \). Consider the subtrees \( x \) and \( y \) of \( t \) defined as \( x = t - t_{\geq a_3} \) and \( y = x - x_{\geq a_2} \). Let \( \delta_4 \) and \( \delta_5 \) be resp. the \( \equiv_{m, \mathcal{L}} \) classes of \( x \) and \( y \). Now consider the trees \( \nu_{\delta_4, \delta_5} \) and \( \nu_{\delta_5, \delta_3} \) which are guaranteed to be found (since indeed \( x \) and \( y \) are trees each of whose roots is labeled with \( \sigma \)). By Corollary 3.4 \( x \equiv_{m, \mathcal{L}} \nu_{\delta_4, \delta_5} \) and \( t \equiv_{m, \mathcal{L}} \nu_{\delta_5, \delta_3} \). Whereby, the \( \equiv_{m, \mathcal{L}} \) class
of $x$ is $\delta_4 = g_{\sigma,m,2}(\delta_5, \delta_2)$ and that of $t$ is $\delta = g_{\sigma,m,2}(\delta_4, \delta_3)$. Observe that $\delta_5$ is indeed $g_{\sigma,m,1}(\delta_1)$.

(Part 2: Reduce-degree($t, m$))

1. Call $\text{Generate-functions}(m)$ that returns the “composition” functions $f_{\sigma,m}$ and $f_{\sigma,m,i}$, and also gives the list $\mathcal{L}[m]$-classes as described above.

2. Using the composition functions, construct bottom-up in $t$, the function $\text{Colour} : \text{Nodes}(t) \rightarrow \mathcal{L}[m]$-classes such that for each node $a$ of $t$, $\text{Colour}(a)$ is the $\equiv_{m,C}$ class of $t_a$.

3. For $\eta_1$ as given by Lemma 3.6 if the degree of $t$ is $\leq \eta_1(m)$, then return $t$.

4. Else, let $a$ be a node of $t$ of degree $n > \eta_1(m)$. Let $x = t_{\geq a}$.

5. For each $\delta \in \mathcal{L}[m]$-classes, do the following:

   (a) Let $a_1, \ldots, a_n$ be the children of $a$ in $x$. For $k \in \{1, \ldots, n\}$, let $x_{1,k}$ be the subtree of $x$ obtained by deleting the subtrees rooted at $a_{k+1}, \ldots, a_n$. Let $g : \{1, \ldots, n\} \rightarrow \mathcal{L}[m]$-classes be such that $g(i)$ is the $\equiv_{m,C}$ class of $x_{1,k}$.

   (b) If $\delta$ appears in the range of $g$, then let $i,j$ be resp. the least and greatest indices in $\{1, \ldots, n\}$ such that $g(i) = g(j) = \delta$. Let $y$ be the subtree of $x$ obtained by deleting the subtrees rooted at $a_{i+1}, \ldots, a_j$. Set $x := y$.

6. Set $t := t[t_{\geq a} \mapsto x]$ and go to step 3.

Reasoning similarly as in the proof of Lemma 3.6[1], we can verify that Reduce-degree($t, m$) indeed returns the desired subtree $s_t$ of $t$. The time taken to compute Colour is linear in $|t|$, while that for computing $g$ is linear in the degree of $a$, whereby the time taken to reduce the degree of a node $a$ in any iteration of the loop, is $O(\Lambda(m) \cdot \text{degree}(a))$. Then, the total time taken by Reduce-degree($t, m$) is $O(\alpha(m) + \Lambda(m) \cdot |t|)$ for some computable function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$.

Reduce-height($t, m$):

1. Generate $\mathcal{L}[m]$-classes and the function Colour as in the previous part.

2. Construct bottom up in $t$, the function $\text{Lowest-subtree} : \text{Nodes}(t) \times \mathcal{L}[m]$-classes $\rightarrow \text{Nodes}(t)$ such that for any node $a$ of $t$ and $\delta \in \mathcal{L}[m]$-classes, $\text{Lowest-subtree}(a, \delta)$ gives a lowest (i.e. closest to a leaf) node $b$ in $t_{\geq a}$ such that $\text{Colour}(b) = \delta$. In other words, $b$ is the only node in $t_{\geq b}$ such that $\text{Colour}(b) = \delta$.

3. Let $a_1, \ldots, a_n$ be the children of $\text{root}(t)$. Let $x_i = \text{Rainbow-subtree}(t_{\geq a_i})$ for $i \in \{1, \ldots, n\}$, where Rainbow-subtree($x$) is described below.

4. Return $t[t_{\geq a_1} \mapsto x_1] \ldots [t_{\geq a_n} \mapsto x_n]$.

Rainbow-subtree($x$):

1. Let $a = \text{root}(x)$.

2. If $b = \text{Lowest-subtree}(a, \text{Colour}(a)) \neq a$, then return Rainbow-subtree($x_{\geq b}$).

3. Else, let $b_1, \ldots, b_n$ be the children of $\text{root}(x)$. For $i \in \{1, \ldots, n\}$, let $y_i = \text{Rainbow-subtree}(x_{\geq b_i})$.

4. Return $x[x_{\geq b_1} \mapsto y_1] \ldots [x_{\geq b_n} \mapsto y_n]$. 

12
We say \( T \) is a tree representation-feasible for \( \Sigma \rho \) function \( \rho \) if there exist alphabets \( \Sigma_{\text{int}} \) and \( \Sigma_{\text{leaf}} \), resp. \( \Sigma_{\text{int}} \), such that \( \Sigma_{\text{int}} \) is closed under (label-preserving) isomorphisms, and for all trees \( t \in T \), then the following conditions hold:

1. Labeling condition: If \( a \) is a leaf node, resp. internal node, then the label \( \lambda(a) \) belongs to \( \Sigma_{\text{leaf}} \), resp. \( \Sigma_{\text{int}} \).
2. Ranking by \( \rho \): If \( a \) is an internal node and \( \lambda(a) \) is in \( \Sigma_{\text{rank}} \), then the number of children of \( a \) in \( t \) is exactly \( \rho(\lambda(a)) \).
3. Closure under rooted subtrees: The subtree \( t_{\geq a} \) is in \( T \).
4. Closure under removal of rooted subtrees respecting \( \Sigma_{\text{rank}} \): If \( a \) is an internal node, then for every descendent \( b \) of \( a \) in \( t \), the subtree \( t_{\geq a} \rightarrow t_{\geq b} \) is in \( T \).
5. Closure under replacements with rooted subtrees: If \( a \) is an internal node, then for every descendent \( b \) of \( a \) in \( t \), the subtree \( t_{\geq a} \rightarrow t_{\geq b} \) is in \( T \).

We say \( T \) is representation-feasible if there exist alphabets \( \Sigma_{\text{leaf}} \), \( \Sigma_{\text{int}} \) and \( \Sigma_{\text{rank}} \) and function \( \rho : \Sigma_{\text{int}} \rightarrow \mathbb{N}_+ \) such that \( T \) is a class of \( (\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}}) \)-trees that is representation feasible for \( (\Sigma_{\text{rank}}, \rho) \). Given such a class \( T \) of trees and a class \( S \) of structures, let \( \text{Str} : T \rightarrow S \) be a map that associates with each tree in \( T \), a structure in \( S \). We call \( \text{Str} \) a representation map. For a tree \( t \in T \), if \( A = \text{Str}(t) \), then we say \( t \) is a tree representation of \( A \) under \( \text{Str} \). For the purposes of our result, we consider “good” maps that would allow tree reductions of the kind seen in the previous section. We formally define these below:

**Definition 4.1.** Given a class \( S \) of structures and a representation-feasible class \( T \) of trees, a representation map \( \text{Str} : T \rightarrow S \) is said to be \( L \)-good for \( S \) if it has the following properties:

- **Lifting to tree representations**

We now consider the more abstract setting of tree representations of structures, in which the internal nodes are labeled with operations coming from a finite set and the leaf nodes represent structures from a given class of structures. We show that under suitable assumptions on the tree representations (that a variety of classes of structures satisfy as seen in the forthcoming sections), we can lift the techniques seen in the previous section to show the \( L \)-EBSP property for classes of structures that admit the aforesaid representations.

Fix finite alphabets \( \Sigma_{\text{int}} \) and \( \Sigma_{\text{leaf}} \) (where the two alphabets are allowed to be overlapping). Let \( \Sigma_{\text{rank}} \subseteq \Sigma_{\text{int}} \). Let \( \rho : \Sigma_{\text{int}} \rightarrow \mathbb{N}_+ \) be a fixed function. We say a class \( T \) of \( (\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}}) \)-trees is representation-feasible for \( (\Sigma_{\text{rank}}, \rho) \) if \( T \) is closed under (label-preserving) isomorphisms, and for all trees \( t \in T \) and nodes \( a \) of \( t \), the following conditions hold:

1. Labeling condition: If \( a \) is a leaf node, resp. internal node, then the label \( \lambda(a) \) belongs to \( \Sigma_{\text{leaf}} \), resp. \( \Sigma_{\text{int}} \).
2. Ranking by \( \rho \): If \( a \) is an internal node and \( \lambda(a) \) is in \( \Sigma_{\text{rank}} \), then the number of children of \( a \) in \( t \) is exactly \( \rho(\lambda(a)) \).
3. Closure under rooted subtrees: The subtree \( t_{\geq a} \) is in \( T \).
4. Closure under removal of rooted subtrees respecting \( \Sigma_{\text{rank}} \): If \( a \) is an internal node, then for every descendent \( b \) of \( a \) in \( t \), the subtree \( t_{\geq a} \rightarrow t_{\geq b} \) is in \( T \).
5. Closure under replacements with rooted subtrees: If \( a \) is an internal node, then for
1. Isomorphism preservation: \( \text{Str} \) maps isomorphic (labeled) trees to isomorphic structures.

2. Surjectivity: Each structure in \( S \) has a unique \( \text{Str} \) representation.

3. Monotonicity: Let \( t \in T \) be a tree of size \( \geq 2 \), and \( a \) be a node of \( t \).
   (a) If \( s = t_{\geq a} \), then \( \text{Str}(s) \hookrightarrow \text{Str}(t) \).
   (b) If \( b \) is a child of \( a \) in \( t \), \( \lambda(a) \notin \Sigma_{\text{rank}} \) and \( z = (t - t_{\geq b}) \), then \( \text{Str}(z) \hookrightarrow \text{Str}(t) \).
   (c) If \( b \) is a descendent of \( a \) in \( t \) and \( z = t_{[t_{\geq a} \rightarrow t_{\geq b}]} \), then \( \text{Str}(z) \hookrightarrow \text{Str}(t) \).

4. Composition: There exists \( m_0 \in \mathbb{N} \) such that for every \( m \geq m_0 \) and for every \( \sigma \in \Sigma_{\text{int}} \), there exists a function \( f_{\sigma,m} : (\Delta_{S,E,M})^{\rho(\sigma)} \rightarrow \Delta_{S,E,M} \) if \( \sigma \in \Sigma_{\text{rank}} \), and functions \( f_{\sigma,m,i} : (\Delta_{S,E,M})^{i} \rightarrow \Delta_{S,E,M} \) for \( i \in \{1, \ldots, \rho(\sigma)\} \) if \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \), with the following properties: Let \( t = (O, \lambda) \in T \) and \( a \) be an internal node of \( t \) such that \( \lambda(a) = \sigma \) and the children of \( a \) in \( t \) are \( b_1, \ldots, b_n \). Let \( \delta_i \) be the \( \equiv_{m,\Sigma} \) class of \( \text{Str}(t_{b_i}) \) for \( i \in \{1, \ldots, n\} \), and let \( \delta \) be the \( \equiv_{m,\Sigma} \) class of \( \text{Str}(t_{a}) \).
   - If \( \sigma \in \Sigma_{\text{rank}} \) (whereby \( n = \rho(\sigma) \)), then \( \delta = f_{\sigma,m}(\delta_1, \ldots, \delta_n) \).
   - If \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \), then \( \delta \) is given as follows: Let \( d = \rho(\sigma) \) and \( n = r + q \cdot (d - 1) \) where \( 1 \leq r < d \). Let \( I = \{r + j \cdot (d - 1) | 0 \leq j \leq q\} \) and for \( k \in I, k \neq n \), let \( \chi_k = f_{\sigma,m} \circ (\chi_k, \delta_k, \delta_{k+1}, \ldots, \delta_{k+q}) \) where \( \chi_r = f_{\sigma,m} \circ (\delta_1, \ldots, \delta_r) \). Then \( \delta = \chi_n \).

We say \( S \) admits an \( L \)-good tree representation if there exists some representation map \( \text{Str} \) that is \( L \)-good for \( S \). We say an \( L \)-good tree representation \( \text{Str} : T \rightarrow S \) is effective (resp. elementary) if (i) \( T \) is recursive and (ii) there is an algorithm that, given \( t \in T \) as input, computes \( \text{Str}(t) \) (resp. computes \( \text{Str}(t) \) in time which is bounded by an elementary function of \( |t| \)).

We now present the central result of this section, which is a lifting of Proposition 3.2 to tree representations. The proof involves an abstraction of all the ideas presented in proof of Proposition 3.2.

**Theorem 4.2.** Let \( S \) be a class of structures that admits an \( L \)-good tree representation \( \text{Str} : T \rightarrow S \). Then the following are true:

1. \( L \text{-EBSP}(S) \) holds.
2. If \( \text{Str} \) is effective, then there exists a computable witness function for \( L \text{-EBSP}(S) \).
   Further, there exists a linear time f.p.t. algorithm for MC(\( L, S \)) that decides, for every \( L \) sentence \( \varphi \) (the parameter), if a given structure \( \mathfrak{A} \) in \( S \) satisfies \( \varphi \), provided that a tree representation of \( \mathfrak{A} \) under \( \text{Str} \) is given.
3. If \( \text{Str} \) is elementary, then there exists an elementary witness function for \( L \text{-EBSP}(S) \)
   iff the index of the \( \equiv_{m,\Sigma} \) relation over \( S \) has an elementary dependence on \( m \).

The rest of this section is entirely devoted to proving the above result.

We prove Theorem 4.2 analogous to Proposition 3.2. Specifically, we show the following two results which resp. are abstract versions of Lemma 3.6 and Lemma 3.7.

**Lemma 4.3.** For a class \( S \) of structures, and a representation-feasible class \( T \) of trees, let \( \text{Str} : T \rightarrow S \) be a representation map that is \( L \)-good for \( S \). Then there exist computable functions \( \eta_1, \eta_2 : \mathbb{N} \rightarrow \mathbb{N} \) such that for each \( t \in T \) and \( m \in \mathbb{N} \), we have the following:

1. (Degree reduction) There exists a subtree \( s_1 \) of \( t \) in \( T \), of degree \( \leq \eta_1(m) \), such that (i) the roots of \( s_1 \) and \( t \) are the same, (ii) \( \text{Str}(s_1) \hookrightarrow \text{Str}(t) \), and (iii) \( \text{Str}(s_1) \equiv_{m,\Sigma} \text{Str}(t) \).
2. (Height reduction) There exists a subtree \( s_2 \) of \( t \) in \( T \), of height \( \leq \eta_2(m) \), such that (i) the roots of \( s_2 \) and \( t \) are the same, (ii) \( \text{Str}(s_2) \hookrightarrow \text{Str}(t) \), and (iii) \( \text{Str}(s_2) \equiv_{m,\Sigma} \text{Str}(t) \).

Above, it additionally holds that if the index of the \( \equiv_{m,\Sigma} \) relation over \( S \) is an elementary function of \( m \), then each of \( \eta_1 \) and \( \eta_2 \) is elementary as well.
Lemma 4.4. For a class $S$ of structures, and a representation-feasible class $T$ of trees, let $\text{Str} : T \to S$ be a representation map that is $\mathcal{L}$-good for $S$ and effective. Let $m_0$ witness the composition property of $\text{Str}$, as mentioned in Definition $4.1$. There exist computable functions $\eta_1, \eta_2, \eta_3 : \mathbb{N} \to \mathbb{N}$ and algorithms $\text{Generate-functions}(m)$, $\text{Reduce-degree}(t, m)$ and $\text{Reduce-height}(t, m)$ such that for $m \geq m_0$,

1. $\text{Generate-functions}(m)$ generates in time $\eta_1(m)$, the functions $f_{\sigma, m, i}$ for $i \in \{1, \ldots, \rho(\sigma)\}$ such that for $\sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}}$, that satisfy the properties mentioned in Definition $4.1$.

2. For $t \in T$, $\text{Reduce-degree}(t, m)$ computes the subtree $s_1$ of $t$ as given by Lemma $4.3$ in time $\eta_1(m) \cdot |t|$. Likewise, $\text{Reduce-height}(t, m)$ computes the subtree $s_2$ of $t$ as given by Lemma $3.6$, in time $\eta_3(m) \cdot |t|$.

Proof of Theorem $4.2$. (1): Let $\mathfrak{A} \in S$. Let $t$ be such that $\text{Str}(t) = \mathfrak{A}$. By Lemma $4.3$ there exists a subtree $s$ of $t$, of degree $\leq \eta_2(m)$ and height $\leq \eta_2(m)$, and hence of size $\leq p = \eta_1(m)\eta_2(m) + 1$. Define $\theta_{(S, \mathcal{L})}(m) = \max\{n \in \mathbb{N} | \mathfrak{C} \in S \text{ satisfies } \mathfrak{C} = \mathfrak{A} \text{ and } |\mathfrak{C}| \leq p\}$. It is then easy to see taking $\mathfrak{A}$ to be the isomorphic copy of $\mathfrak{S}(s)$, that is a substructure of $\mathfrak{A}$, that $\mathcal{L}$-EBSP-condition($S, \mathfrak{A}, \mathfrak{B}, m, \theta_{(S, \mathcal{L})}$) holds.

(2): It is clear that if $\text{Str}$ is effective, then $\theta_{(S, \mathcal{L})}$ defined above is computable too. For the f.p.t. part, let $\mathcal{A}$ be the following algorithm. Let $\mathfrak{A} \in S$ be given as input to $\mathcal{A}$, in the form of the tree representation $t$ of $\mathfrak{A}$ under $\text{Str}$. Let $\varphi$ be an input $\mathcal{L}$ sentence. Then $\mathcal{A}$ determines the rank $m$ of $\varphi$, computes $m_1 = \max\{m, m_0\}$, and calls $\text{Reduce-degree}(t, m_1, m_1)$. By Lemma $4.1$ the aforesaid call returns, in time $(\eta_2(m_1) + \eta_2(m_1)) \cdot |t|$, a tree $s$ in $T$, of degree $\leq \eta_2(m_1)$ and height $\leq \eta_2(m_1)$, and hence of size $\leq \eta_2(m_1)\eta_2(m_1)$, such that $\text{Str}(s) \equiv_{m, \mathcal{L}} \text{Str}(t) = \mathfrak{A}$. Since $m_1 \geq m$, we have $\text{Str}(s) \equiv_{m, \mathcal{L}} \mathfrak{A}$. Checking if $\mathfrak{A} \models \varphi$ is then equivalent to checking if $\text{Str}(s) \models \varphi$, and the latter can be done in time $g(m_1)$ for some computable function $g : \mathbb{N} \to \mathbb{N}$ of $m_1$, since the size of $s$ is bounded by a computable function of $m_1$. Therefore that $\mathcal{A}$ is f.p.t. for $\mathcal{MC}(\mathcal{L}, \mathcal{S})$.

(3): It is easy to see that if there exists an elementary witness function $\theta_{(S, \mathcal{L})}$ for $\mathcal{L}$-EBSP($S$), then every structure $\mathfrak{A}$ in $S$ is $\mathcal{L}[m]$-equivalent to a structure $\mathfrak{B}$ in $S$ such that $|\mathfrak{B}| \leq \theta_{(S, \mathcal{L})}(m)$. Whereby the index of the $\equiv_{m, \mathcal{L}}$ relation over $S$ is bounded by the number of structures in $S$ whose size (of the universe) is $\leq \theta_{(S, \mathcal{L})}(m)$. Clearly then, this number, and hence the index, is bounded by an elementary function of $m$, if $\theta_{(S, \mathcal{L})}$ is elementary.

Suppose the index of the $\equiv_{m, \mathcal{L}}$ relation over $S$ is an elementary function of $m$. Then by Lemma $4.3$ $\eta_1$ and $\eta_2$ are elementary too. Wherby if $\text{Str}$ is also elementary, then $\theta_{(S, \mathcal{L})}$, as defined in part (1) above, is also elementary.

We now prove Lemma $4.3$ and Lemma $4.4$. We recall from Section $2$ that for a class $S$ of structures, $\Delta_{S, \mathcal{L}, m}$ denotes the set of all equivalence classes of the $\equiv_{m, \mathcal{L}}$ relation restricted to the structures in $S$, and $\Lambda_{S, \mathcal{L}} : \mathbb{N} \to \mathbb{N}$ is a fixed computable function with the property that $\Lambda_{S, \mathcal{L}}(m) \geq |\Delta_{S, \mathcal{L}, m}|$.

4.1 Proof of Lemma $4.3$

The following facts are easy to verify given that $\text{Str}$ satisfies the composition properties of Definition $4.1$. The proofs of these use similar ideas as in the proof of Corollary $3.4$ and are hence skipped. Below, $m_0$ witnesses the composition properties of $\text{Str}$ as given by Definition $4.1$.

Lemma 4.5. Let $s, t \in T$ and let $a$ be a node of $t$. Suppose $z = t'[t_{\geq a} \mapsto s] \in T$. Then for $m \geq m_0$, if $\text{Str}(s) \equiv_{m, \mathcal{L}} \text{Str}(t_{\geq a})$, then $\text{Str}(z) \equiv_{m, \mathcal{L}} \text{Str}(t)$. 




Lemma 4.6. Let $s_1, s_2, t \in T$ be such that the label of the root of each of these trees is $\sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}}$. Suppose $z_1 = s_1 \circ t$ is such that $z_i \in T$ for $i \in \{1, 2\}$. Suppose further that the number of children of the root of $t$ is a multiple of $(\rho(\sigma) - 1)$. Then for $m > m_0$, if $\text{Str}(s_1) \equiv_{m, \mathcal{L}} \text{Str}(s_2)$, then $\text{Str}(z_1) \equiv_{m, \mathcal{L}} \text{Str}(z_2)$.

Proof of Lemma 4.6 Part [1]: Let $m_0 \in \mathbb{N}$ be a witness to the composition property of $\text{Str}$, as mentioned in Definition 4.1. Define $\eta_1 : \mathbb{N} \rightarrow \mathbb{N}$ as follows: for $l \in \mathbb{N}$, $\eta_1(l) = \max\{\rho(\sigma) \mid \sigma \in \Sigma_{\text{int}}\} \times \Delta_{\mathcal{L}}(\max\{l, m_0\})$. Then $\eta_1$ is computable.

Given $m \in \mathbb{N}$, let $p = \eta_1(m)$. If $t$ has degree $\leq p$, then putting $s_1 = t$ we are done. Else, some node of $t$, say $a$, has degree $n > p$. Let $\sigma$ be the label of $a$; clearly $\sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}}$.

Let $z = t_{\geq a}$; then $z \in T$. Let $a_1, \ldots, a_n$ be the (ascending) sequence of children of root$(z)$ in $z$. For $d = \rho(\sigma)$, let $n = r + q \cdot (d - 1)$ for $1 \leq r < d$ and $q > 1$.

For $k \in I = \{r + l \cdot (d - 1) \mid 0 \leq l \leq q\}$, let $x_{1,k}$, resp. $y_{t+1,n}$, be the subtree of $z$ obtained from $z$ by deleting the subtrees rooted at $a_{k+1}, \ldots, a_n$, resp. deleting the subtrees rooted at $a_1, a_2, \ldots, a_k$. Then $z = x_{1,n} = x_{1,k} \odot y_{t+1,n}$ for all $k \in I$. Let $m_1 = \max\{m_0, m\}$. Define $g : I \rightarrow \Delta_{\mathcal{L}, m_1}$ such that $g(k)$ is the $\equiv_{m_1, \mathcal{L}}$ class of $\text{Str}(x_{1,k})$ for $k \in I$.

Since $n > p$, there exist $i, j \in I$ such that $i < j$ and $g(i) = g(j)$, i.e. $\text{Str}(x_{1,i}) \equiv_{m_1, \mathcal{L}} \text{Str}(x_{1,j})$. If $z_1 = x_{1,j} \odot y_{t+1,n}$, then since $T$ is closed under removal of rooted subtrees respecting $\Sigma_{\text{rank}}$, we have $z_1 \in T$. Observe that $\text{Str}(z_1) \equiv_{m_1, \mathcal{L}} \text{Str}(z)$.

Then by Lemma 4.6 and the monotonicity properties of $\text{Str}$ as mentioned in Definition 4.1, we have $\text{Str}(z_1) \preceq \text{Str}(z)$ and $\text{Str}(z_1) \equiv_{m_1, \mathcal{L}} \text{Str}(z)$. Then by Lemma 4.5 and the monotonicity properties of $\text{Str}$, we see that if $t_1 = t_{\geq t_1}$, then $t_1 \in T$ and $\text{Str}(t_1) \equiv_{m_1, \mathcal{L}} \text{Str}(t)$.

Observe that $t_1$ has strictly lesser size than $t$ (since $z_1$ has strictly lesser size than $z$), and that the roots of $t_1$ and $t$ are the same. Recursing on $t_1$, we eventually get a subtree $s_1$ of $t$ in $T$, of degree at most $p$, such that (i) the roots of $s_1$ and $t$ are the same, (ii) $\text{Str}(s_1) \preceq \text{Str}(t)$, and (iii) $\text{Str}(s_1) \equiv_{m_1, \mathcal{L}} \text{Str}(t)$. Since $m_1 = \max\{m_0, m\} \geq m$, we have $\text{Str}(s_1) \equiv_{m, \mathcal{L}} \text{Str}(t)$.

(Part 2): As in the previous part, let $m_0 \in \mathbb{N}$ be a witness to the composition property of $\text{Str}$, as mentioned in Definition 4.1. Define $\eta_2 : \mathbb{N} \rightarrow \mathbb{N}$ as follows: for $l \in \mathbb{N}$, $\eta_2(l) = 1 + \Delta_{\mathcal{L}}(\max\{l, m_0\})$. Then $\eta_2$ is computable.

Given $m \in \mathbb{N}$, let $p = \eta_2(m)$. If $t$ has height $\leq p$, then putting $s_2 = t$ we are done. Else, there is a path from the root of $t$ to some leaf of $t$, whose length is $> p$. Let $A$ be the set of nodes appearing along this path. Let $m_2 = \max\{m_0, m\}$. Consider the function $h : A \rightarrow \Delta_{\mathcal{L}, m_2}$ such that for each $a \in A$, $h(a) = \delta$ where $\delta$ is the $\equiv_{m_2, \mathcal{L}}$ class of $\text{Str}(t_{\geq a})$. Since $|A| > p$, there exist distinct nodes $a, b \in A$ such that $a$ is an ancestor of $b$ in $t$ and $h(a) = h(b)$ and $a$ is not the root of $t$. Let $t_2 = t_{\geq t_2}$. Since $T$ is closed under rooted subtrees and under replacements with rooted subtrees, we have that $t_2$ is a subtree of $t$ in $T$. By the monotonicity properties mentioned in Definition 4.1 that $\text{Str}$ satisfies, $\text{Str}(t_2) \preceq \text{Str}(t)$.

Also since $h(a) = h(b)$, we have $\text{Str}(t_{\geq a}) \equiv_{m_2, \mathcal{L}} \text{Str}(t_{\geq a})$, whereby using Lemma 4.5 we get that $\text{Str}(t_2) \equiv_{m_2, \mathcal{L}} \text{Str}(t)$. Observe that $t_2$ has strictly less size than $t$, and that the roots of $t_2$ and $t$ are the same. Recursing on $t_2$, we eventually get a subtree $s_2$ of $t$, of height at most $p$, such that (i) the roots of $s_2$ and $t$ are the same, (ii) $\text{Str}(s_2) \preceq \text{Str}(t)$, and (iii) $\text{Str}(s_2) \equiv_{m_2, \mathcal{L}} \text{Str}(t)$. Since $m_2 = \max\{m_0, m\} \geq m$, we have $\text{Str}(s_2) \equiv_{m, \mathcal{L}} \text{Str}(t)$.

It is clear from the definitions of $\eta_1$ and $\eta_2$ above, that if the index of the $\equiv_{m, \mathcal{L}}$ relation over $S$ is an elementary function of $m$, then so are $\eta_1$ and $\eta_2$.

4.2 Proof of Lemma 4.4

We now give the proofs of Lemma 4.4[1] and Lemma 4.4[2] in Section 4.2.1 and Section 4.2.2, respectively.
4.2.1 Proof of Lemma 4.4.1

Before we present the proof, we need some auxiliary lemmas that we describe below. Let $\text{All}$ denote the class of all finite structures.

**Lemma 4.7** (Enumerability of the equivalence classes of $\Delta_{\text{All}, \mathbb{L}, m}$). There exists a computable function $h : \mathbb{N} \to \mathbb{N}$ and a procedure $\mathcal{P}$ such that $\mathcal{P}$ takes as input a natural number $m$ and enumerates $\mathcal{L}[m]$ sentences $\varphi_1, \ldots, \varphi_n$ for $n = h(m)$ with the property that $\varphi_i$ captures some equivalence class $\delta$ of $\Delta_{\text{All}, \mathbb{L}, m}$ (i.e. the class of finite models of $\varphi_i$ is exactly $\delta$) for each $i \in \{1, \ldots, n\}$ and conversely, for every equivalence class $\delta$ of $\Delta_{\text{All}, \mathbb{L}, m}$, there exists some $i \in \{1, \ldots, n\}$ such that $\varphi_i$ captures $\delta$.

**Proof.** Follows from the inductive definition of $\mathcal{L}[m]$, and the proofs of Lemma 3.13 and Proposition 7.5 in [19].

Let $\mathcal{L}$-$\text{SAT}$ denote the problem of checking if a given $\mathcal{L}$ sentence is satisfiable.

**Lemma 4.8.** If $\mathcal{L}$-$\text{EBSP}(S)$ is true with a computable witness function, then $\mathcal{L}$-$\text{SAT}$ is decidable over $S$.

**Proof.** Since for any structure in $S$ and $m \in \mathbb{N}$, there is an $\mathcal{L}[m]$-equivalent substructure of size bounded by a computable function of $m$, it follows that $\mathcal{L}$ possesses the “computable” small model property over $S$. The decidability of $\mathcal{L}$-$\text{SAT}$ over $S$ then follows.

Let as usual, $m_0$ witness the composition properties of $\text{Str}$ as mentioned in Definition 4.1.

**Lemma 4.9.** There exists a computable function $\eta : \mathbb{N} \to \mathbb{N}$ with the following property: Let $t \in \mathcal{T}$ of size $\geq 2$ and $a_1, \ldots, a_n$ be the children of $\text{root}(t)$. For each $m \geq m_0$, there exists a subtree $s$ of $t$ in $\mathcal{T}$ such that

1. the roots of $s$ and $t$ are the same
2. the size of $s$ is at most $\eta(m)$
3. (a) If $\sigma \in \Sigma_{\text{rank}}$ (whereby $n = \rho(\sigma)$) or $n < \rho(\sigma)$, then the root of $s$ has exactly $n$ children $b_1, \ldots, b_n$ satisfying $\text{Str}(s_{\geq b_i}) \equiv_{m, \mathcal{L}} \text{Str}(t_{\geq a_i})$ for each $i \in \{1, \ldots, n\}$.

(b) Else, $s = x \odot y$ where:
   - $x$ is such that $\text{Str}(x) \equiv_{m, \mathcal{L}} \text{Str}(z)$ and $z$ is the tree obtained from $t$ by removing the subtrees rooted at $a_{n-d+2}, \ldots, a_n$.
   - $y$ is such that the root of $y$ has exactly $d-1$ children $b_{n-d+2}, \ldots, b_n$ for $d = \rho(\sigma)$, satisfying $\text{Str}(s_{\geq b_i}) \equiv_{m, \mathcal{L}} \text{Str}(t_{\geq a_i})$ for each $i \in \{n-d+2, \ldots, n\}$.

**Proof.** Let $k = \max\{\rho(\sigma) \mid \sigma \in \Sigma_{\text{int}}\}$. Define $\eta(m) = 1 + k \times (\eta_1(m))^{\eta_2(m)+1}$ where $\eta_1, \eta_2$ are as given by Lemma 4.3.

Consider the case when $\sigma \in \Sigma_{\text{rank}}$ (whereby $n = \rho(\sigma)$) or $n < \rho(\sigma)$. Consider the subtrees $x_i = t_{\geq a_i}$ for $i \in \{1, \ldots, n\}$; each of these belongs to $\mathcal{T}$ since $\mathcal{T}$ is representation-feasible. By parts (1) and (2) of Lemma 4.3 it follows that for each $i \in \{1, \ldots, n\}$, there exists a subtree $y_i$ of $x_i$, of degree $\leq \eta_1(m)$ and height $\leq \eta_2(m)$, and hence of size $\leq (\eta_1(m))^{\eta_2(m)+1}$, such that $\text{Str}(y_i) \equiv_{m, \mathcal{L}} \text{Str}(x_i)$. We observe from the proofs of parts (1) and (2) of Lemma 4.3 that $y_i$ is obtained from $x_i$ by removal of rooted subtrees in $x_i$ respecting $\Sigma_{\text{rank}}$, and by replacements with rooted subtrees in $x_i$. Whereby, since $\mathcal{T}$ is representation-feasible, we tree $s = t[x_1 \mapsto y_1][x_2 \mapsto y_2] \ldots [x_n \mapsto y_n]$ obtained by replacing $x_i$ in $t$ with $y_i$, is indeed a subtree of $t$ in $\mathcal{T}$, having the properties as mentioned in the statement of this lemma. Observe that the size of $s$ is at most $1 + n \times (\eta_1(m))^{\eta_2(m)+1}$.

Consider now the case when $\sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}}$ and $n \geq \rho(\sigma)$. Let $t = z \odot v$ where $z$, resp. $v$, is the subtree of $t$ obtained by deleting the subtrees rooted at $a_{n-d+2}, \ldots, a_n$, resp. $a_1, \ldots, a_{n-d+1}$. By using the reasoning above, there exists a subtree $y$ of $v$ in $\mathcal{T}$ such that (i) the roots of $y$ and $v$ are the same (and hence root($y$) is labeled with $\sigma$) (ii) the
size of $y$ is at most $1 + (d - 1) \times (\eta_1(m))^{\eta_2(m) + 1}$ and (iii) the root of $y$ has $d - 1$ children $b_{n-d+2}, \ldots, b_n$ such that $\text{Str}(s_{\geq b_i}) \equiv_{m, \mathcal{L}} \text{Str}(t_{\geq a_i})$ for each $i \in \{n - d + 2, \ldots, n\}$. Now consider $z$. Again, by Lemma 4.3 it follows that there exists a subtree $x$ of $z$ in $T$, of size $\leq (\eta_1(m))^{\eta_2(m) + 1}$, such that $\text{Str}(x) \equiv_{m, \mathcal{L}} \text{Str}(z)$ and the roots of $x$ and $z$ are the same (and hence the label of root($x$) is $\sigma$). Let $s = x \cup y$; then the size of $s$ is at most $(\eta_1(m))^{\eta_2(m) + 1} + (d - 1) \times (\eta_1(m))^{\eta_2(m) + 1}$ which in turn is at most $\eta(m)$. We check that $s$ is indeed as desired.

Proof of Lemma 4.4. The procedure Generate-functions($m$) operates in three stages that we describe below.

Stage 1: In this stage, Generate-functions($m$) creates a list $\mathcal{L}[m]$-classes of $\mathcal{L}[m]$ sentences such that every sentence of $\mathcal{L}[m]$-classes captures over $S$, some equivalence class of $\Delta S, \mathcal{L}, m$, and conversely, every equivalence classes of $\Delta S, \mathcal{L}, m$ is captured over $S$, by some sentence of $\mathcal{L}[m]$-classes. This is done as follows.

Let $\eta$ and $\mathcal{P}$ be as given by Lemma 4.7. For each $\mathcal{L}[m]$ sentence $\varphi_i$ for $i \in \{1, \ldots, \eta(m)\}$ that $\mathcal{P}$ enumerates, Generate-functions($m$) first checks if $\varphi_i$ is satisfiable over $S$ (in other words, whether $\varphi_i$ indeed represents an equivalence class of $\Delta S, \mathcal{L}, m$). This is decidable because $\mathcal{L}$-EBSP($S$) holds with a computable witness function (using part 1 and the first part of part 2 of Theorem 1.2 and the assumption that $\text{Str}$ is effective), whereby $\mathcal{L}$-SAT is decidable over $S$ by Lemma 4.8. If $\varphi_i$ is satisfiable, then $\varphi_i$ is put into $\mathcal{L}[m]$-classes, else it is discarded.

It follows that at the end of this process, $\mathcal{L}[m]$-classes gets created as desired.

(Since $\mathcal{L}[m]$-classes is a list of sentences that represent equivalence classes, we shall henceforth treat $\mathcal{L}[m]$-classes interchangeably as a list of sentences or a list of equivalence classes, depending on what is easier to understand in a given context.)

Stage II: In the following stage, the following trees from $T$ are generated by Generate-functions($m$), if they exist in $T$:

1. $s_{\sigma, \delta_1, \ldots, \delta_{\rho(\sigma)}}$ for $\sigma \in \Sigma_{\text{rank}}$ and $\delta_i \in \mathcal{L}[m]$-classes for $i \in \{1, \ldots, \rho(\sigma)\}$
2. $u_{\sigma, \delta_1, \ldots, \delta_i}$ for $\sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}}$, $\delta_j \in \mathcal{L}[m]$-classes for $j \in \{1, \ldots, i\}$ and $i \in \{1, \ldots, \rho(\sigma) - 1\}$
3. $v_{\sigma, \delta_1, \ldots, \delta_i}$ for $\sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}}$ and $\delta_i \in \mathcal{L}[m]$-classes for $i \in \{1, \ldots, \rho(\sigma)\}$

with the following properties:

1. The tree $z = s_{\sigma, \delta_1, \ldots, \delta_{\rho(\sigma)}}$ satisfies the following: (i) the label of the root of $z$ is $\sigma$, (ii) the root of $z$ has exactly $\rho(\sigma)$ children $b_1, \ldots, b_{\rho(\sigma)}$, and (iii) $\equiv_{m, \mathcal{L}}$ class of $\text{Str}(z_{\geq b_i})$ is $\delta_i$ for $i \in \{1, \ldots, \rho(\sigma)\}$.
2. The tree $z = u_{\sigma, \delta_1, \ldots, \delta_i}$ satisfies the following: (i) the label of the root of $z$ is $\sigma$, (ii) the root of $z$ has exactly $i$ children $b_1, \ldots, b_i$, and (iii) $\equiv_{m, \mathcal{L}}$ class of $\text{Str}(z_{\geq b_i})$ is $\delta_j$ for $j \in \{1, \ldots, i\}$.
3. The tree $z = v_{\sigma, \delta_1, \ldots, \delta_{\rho(\sigma)}}$ satisfies the following: (i) the label of the root of $z$ is $\sigma$, (ii) $z = x \cup y$ where the root of $y$ has exactly $d - 1$ children $b_2, \ldots, b_d$ for $d = \rho(\sigma)$, and (iii) the $\equiv_{m, \mathcal{L}}$ class of $z$ is $\delta_1$, while the $\equiv_{m, \mathcal{L}}$ class of $\text{Str}(y_{\geq b_j})$ is $\delta_j$ for $j \in \{2, \ldots, d\}$.

This is done as follows. We show this for the cases of $s_{\sigma, \delta_1, \ldots, \delta_{\rho(\sigma)}}$ and $v_{\sigma, \delta_1, \ldots, \delta_{\rho(\sigma)}}$; the case of $u_{\sigma, \delta_1, \ldots, \delta_{\rho(\sigma)}}$ can be done similarly. First, using $\eta$ as given by Lemma 4.4, Generate-functions($m$) computes $p = \eta(m)$. Since the trees in $T$ are over the finite alphabet $\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}}$ and since $T$ is recursive, Generate-functions($m$) enumerates out those trees in $T$, whose roots are labeled with $\sigma$, and whose size is $\leq p$. For a tree $t$ enumerated thus by Generate-functions($m$), let $b_1, \ldots, b_n$ be the children of the root of $t$. 

18
1. the case of \( s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \): Here \( \sigma \in \Sigma_{\text{rank}} \). Then \( \text{Generate-functions}(m) \) checks if \( n = \rho(\sigma) \). If not, then it discards \( t \). Else, \( \text{Generate-functions}(m) \) computes \( A_i = \text{Str}(t_{\geq b_i}) \) for \( i \in \{1, \ldots, \rho(\sigma)\} \). Observe that since \( \mathcal{T} \) is closed under rooted subtrees and since \( \text{Str} \) is computable, \( A_i \) can be computed too. Finally, \( \text{Generate-functions}(m) \) checks if the \( \equiv_{m,\mathcal{L}} \) class of \( A_i \) is \( \delta_i \) — this is done by checking if the formula \( \varphi \) representing \( \delta_i \) in \( \mathcal{L}[m]-\text{classes} \), is true in \( A_i \). (Checking if an \( \mathcal{L} \) sentence is true in a finite structure is decidable.) If the tree \( t \) above passes this last check, then \( \text{Generate-functions}(m) \) stores \( t \) as \( s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \). If none of the trees enumerated by \( \text{Generate-functions}(m) \) pass the last check, then \( \text{Generate-functions}(m) \) stores null for \( \sigma,\delta_1,\ldots,\delta_{\rho(\sigma)} \).

2. the case of \( v_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \): Here \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \). Let \( d = \rho(\sigma) \) and let \( t = x \circ y \) where \( x, \) resp. \( y, \) is the subtree of \( t \) obtained by deleting the subtrees rooted at \( b_{n-d+2}, \ldots, b_n \), resp. \( b_1, \ldots, b_{n-d+1} \). Then \( \text{Generate-functions}(m) \) computes \( A_1 = \text{Str}(x) \) and \( A_i = \text{Str}(t_{\geq b_{n-d+i}}) \) for \( i \in \{2, \ldots, d\} \). Observe once again that \( A_i \) can be computed for each \( i \in \{1, \ldots, d\} \). Finally, \( \text{Generate-functions}(m) \) checks if the \( \equiv_{m,\mathcal{L}} \) class of \( A_i \) is \( \delta_i \). If the tree \( t \) passes this last check, then \( \text{Generate-functions}(m) \) stores \( t \) as \( v_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \). Again if none of the trees enumerated by \( \text{Generate-functions}(m) \) pass the last check, then \( \text{Generate-functions}(m) \) stores null for \( \sigma,\delta_1,\ldots,\delta_{\rho(\sigma)} \).

In the above cases, it is clear by Lemma 4.9 that if \( \text{Generate-functions}(m) \) stores null for \( \sigma,\delta_1,\ldots,\delta_{\rho(\sigma)} \), then there is no tree in \( \mathcal{T} \) that can be taken as \( s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \), resp. \( v_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \).

**Stage III:** In this stage, the trees identified in the previous stage are used to define functions \( g_{\sigma,m} \) if \( \sigma \in \Sigma_{\text{rank}} \) and \( g_{\sigma,m,i} \) if \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \), that satisfy the composition properties mentioned in Definition 4.1, whereby these resp. can indeed be considered as the functions \( f_{\sigma,m} \) and \( f_{\sigma,m,i} \) as mentioned in Definition 4.1. We show how to define \( g_{\sigma,m} \) for \( \sigma \in \Sigma_{\text{rank}} \) using \( s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \) (if identified); analogously, for \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \), the function \( g_{\sigma,m,\rho(\sigma)} \) is defined using \( v_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \) and function \( g_{\sigma,m,i} \) is defined using \( u_{\sigma,\delta_1,\ldots,\delta_i} \) for \( i \in \{1, \ldots, \rho(\sigma) - 1\} \).

Let \( \sigma \in \Sigma_{\text{rank}} \) and \( \delta_1,\ldots,\delta_{\rho(\sigma)} \in \mathcal{L}[m]-\text{classes} \).

- If no tree \( z \) of the form \( s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \) is identified in the previous stage (i.e. \( \text{Generate-functions}(m) \) stores null for \( \sigma,\delta_1,\ldots,\delta_{\rho(\sigma)} \)), then define \( g_{\sigma,m}(\delta_1,\ldots,\delta_{\rho(\sigma)}) = \delta_{\text{default}} \) where \( \delta_{\text{default}} \) is some fixed chosen element of \( \mathcal{L}[m]-\text{classes} \).
- Else, let \( z = s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \). Identify \( \varphi \in \mathcal{L}[m]-\text{classes} \) such that \( \text{Str}(z) \models \varphi \). Let \( \delta \) be the equivalence class represented by \( \varphi \). Then define \( g_{\sigma,m}(\delta_1,\ldots,\delta_{\rho(\sigma)}) = \delta \).

Observe that since \( \text{Str} \) is assumed to be computable and since model checking an \( \mathcal{L} \) sentence on a finite structure is decidable, \( g_{\sigma,m} \) indeed gets generated after a finite amount of time. Analogously, the functions \( g_{\sigma,m,i} \) also get generated after a finite amount of time. It is easily seen from the above description of \( \text{Generate-functions}(m) \), that for some computable function \( \eta_3 : \mathbb{N} \to \mathbb{N} \), the total time taken by \( \text{Generate-functions}(m) \) is bounded by \( \eta_3(m) \).

We finally show that \( g_{\sigma,m} \) and \( g_{\sigma,m,i} \) constructed above indeed satisfy the composition properties of Definition 4.1.

Let \( t = (O, \lambda) \in \mathcal{T} \) and \( a \) be an internal node of \( t \) such that \( \lambda(a) = \sigma \) and the children of \( a \) in \( t \) are \( b_1, \ldots, b_n \). Let \( \delta_i \) be the \( \equiv_{m,\mathcal{L}} \) class of \( \text{Str}(t_{\geq b_i}) \) for \( i \in \{1, \ldots, n\} \).

1. Suppose \( \sigma \in \Sigma_{\text{rank}} \), whereby \( n = \rho(\sigma) \). Then \( t_{\geq a} \in \mathcal{T} \) since \( \mathcal{T} \) is representation-feasible. Consider the tree \( s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}} \) that then is guaranteed to be generated by \( \text{Generate-functions}(m) \) in Stage II because of Lemma 4.9. By the composition property as mentioned in Definition 4.1 it follows that \( \text{Str}(t_{\geq a}) \equiv_{m,\mathcal{L}} \text{Str}(s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}}) \),
i.e., the \( \equiv_{m,\mathcal{L}} \) classes of \( \text{Str}(t_{\geq a}) \) and \( \text{Str}(s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}}) \) are the same. Indeed, then the \( \equiv_{m,\mathcal{L}} \) class of \( \text{Str}(t_{\geq a}) \) is \( g_{\sigma,m}(\delta_1,\ldots,\delta_n) \), because the \( \equiv_{m,\mathcal{L}} \) class of \( \text{Str}(s_{\sigma,\delta_1,\ldots,\delta_{\rho(\sigma)}}) \) is \( g_{\sigma,m}(\delta_1,\ldots,\delta_n) \) by construction.

2. Suppose \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \) and \( n < \rho(\sigma) \). By similar reasoning as above, the tree \( u_{\sigma,\delta_1,\ldots,\delta_n} \) (that is guaranteed to be generated by \( \text{Generate-functions}(m) \)) is such that \( \text{Str}(t_{\geq a}) \equiv_{m,\mathcal{L}} \text{Str}(u_{\sigma,\delta_1,\ldots,\delta_n}) \). Whereby, the \( \equiv_{m,\mathcal{L}} \) class of \( \text{Str}(t_{\geq a}) \) is indeed \( g_{\sigma,m}(\delta_1,\ldots,\delta_n) \).

3. Suppose \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \) and \( n \geq \rho(\sigma) \). Let \( d = \rho(\sigma) \) and \( n = r + q \cdot (d - 1) \) where \( 1 \leq r < d \) and \( q \geq 0 \). Consider the trees \( z_{1,k} \) obtained from \( t_{\geq a} \) by deleting the subtrees of \( t_{\geq a} \) rooted at \( b_{k+1},\ldots,b_{n} \), for \( k \in I = \{ r + j \cdot (d - 1) \mid 0 \leq j \leq q \} \) (whereby, \( t_{\geq a} = z_{1,n} \)). Whereby, \( \equiv_{m,\mathcal{L}} \) class of \( \text{Str}(z_{1,k}) \). Using Lemma 4.9, it is guaranteed that in Stage II, \( \text{Generate-functions}(m) \) produces the trees \( u_{\sigma,\delta_1,\ldots,\delta_r} \) and \( v_{\sigma,\chi_k,\delta_{k+1},\ldots,\delta_{k+(d-1)}} \) for each \( k \in I \).

By the composition property of Definition 4.1 we see that for \( k \in I \setminus \{ n \} \), we have

\[
\begin{align*}
\text{Str}(z_{1,r}) &\equiv_{m,\mathcal{L}} \text{Str}(u_{\sigma,\delta_1,\ldots,\delta_r}) \\
\text{Str}(z_{1,k+(d-1)}) &\equiv_{m,\mathcal{L}} \text{Str}(v_{\sigma,\chi_k,\delta_{k+1},\ldots,\delta_{k+(d-1)}})
\end{align*}
\]

Whereby, from the very constructions of \( g_{\sigma,m,i} \) for \( i \in \{ 1,\ldots,\rho(\sigma) \} \), we get for \( k \in I \setminus \{ n \} \), that

\[
\begin{align*}
\equiv_{m,\mathcal{L}} \text{class of } \text{Str}(z_{1,r}) &= \chi_r = g_{m,\sigma,r}(\delta_1,\ldots,\delta_r) \\
\equiv_{m,\mathcal{L}} \text{class of } \text{Str}(z_{1,k+(d-1)}) &= \chi_{k+(d-1)} = g_{m,\sigma,d}(\chi_k,\delta_{k+1},\ldots,\delta_{k+(d-1)})
\end{align*}
\]

Putting \( k = r + (q - 1) \cdot (d - 1) \) above, we see that \( \chi_n \), which is the \( \equiv_{m,\mathcal{L}} \) class of \( \text{Str}(z_{1,n}) \), is indeed given by \( g_{\sigma,m,d}(g_{\sigma,m,d}(\ldots,g_{\sigma,m,d}(g_{\sigma,m,d}(\delta,\delta_{r+1},\ldots,\delta_{r+(d-1)}),\delta_{r+d},\ldots,\delta_{r+2(d-1)}),\ldots),\delta_{n-d+2},\ldots,\delta_n) \), where \( \delta = g_{\sigma,m,r}(\delta_1,\ldots,\delta_r) \).

\[\square\]

4.2.2 Proof of Lemma 4.4(2)

Proof. (1) Reduce-degree\((t,m)\):

Suppose \( t \in T \) and \( m \in \mathbb{N} \) are given as inputs. Let \( m_0 \) be a witness to the composition property of \( \text{Str} \), as mentioned in Definition 4.1 and let \( m_1 = \max\{m_0, m\} \). The algorithm \text{Reduce-degree}\((t,m)\) functions in various stages as described below.

Stage I: \text{Reduce-degree}\((t,m)\) first invokes the procedure \text{Generate-functions}\((m_1)\). The latter procedure produces the following:

1. \( \mathcal{L}[m_1]-\text{classes} \) which is a list of \( \mathcal{L}[m_1] \) sentences that represent all and exactly the equivalence classes of the \( \equiv_{m_1,\mathcal{L}} \) relation over \( S \).

2. the “composition” functions \( f_{\sigma,m_1} \) and \( f_{\sigma,m_1,i} \) which satisfy the composition properties mentioned in Definition 4.1.

The time taken to complete this step is at most \( \eta_3(m_1) \), where \( \eta_3 \) is as given by Lemma 4.4(1).

Stage II: \text{Reduce-degree}\((t,m)\) now constructs bottom-up in \( t \), the function \( \text{Colour} : t \rightarrow \Delta S_{\mathcal{L},m} \) such that \( \text{Colour}(a) \) is the \( \equiv_{m_1,\mathcal{L}} \) class of \( \text{Str}(t_{\geq a}) \). This is done inductively as follows:
• Base case: We first compute \( \text{Colour}(e) \) for each leaf node \( e \) of \( t \). This can be done in constant time as explained below.

Since \( \Sigma_{\text{leaf}} \) is finite and since \( \text{Str} \) is isomorphism preserving (see Definition 4.1), there is a finite function \( \text{leaf-structures} : \Sigma_{\text{leaf}} \to \mathcal{S} \) such that for any leaf node \( e \) of \( t \), if its label is \( \sigma \), then \( \text{Str}(t_{\geq e}) \cong \text{leaf-structures}(\sigma) \). Further, since the range of \( \text{leaf-structures} \) is finite, there exists a finite function \( \text{leaf-colour} : \text{Range(leaf-structures)} \to \Delta_{\Sigma, \mathcal{L}, m_1} \) such that for each \( \mathcal{A} \) in the range of \( \text{leaf-structures} \), we have \( \text{leaf-colour}(\mathcal{A}) \) is the \( \equiv_{m_1, \mathcal{L}} \) class of \( \mathcal{A} \). Whereby, given a leaf node \( e \), we have \( \text{Colour}(e) = \text{leaf-colour(leaf-structure}(\sigma)) \), where \( \sigma \) is the label of \( e \).

• Induction step: Assume that for an internal node \( a \), if \( b_1, \ldots, b_n \) are the children of \( a \) in \( t \), then \( \text{Colour}(b_i) \) has been computed, for \( i \in \{1, \ldots, n\} \). Let \( \sigma \) be the label of \( a \) in \( t \). We have two cases here to compute \( \text{Colour}(a) \):

- \( \sigma \in \Sigma_{\text{rank}} \): Then by the composition property of \( \text{Str} \), we have that \( \text{Colour}(a) = f_{\sigma, m_1}((\text{Colour}(b_1)), \ldots, (\text{Colour}(b_n))) \). Since \( f_{\sigma, m_1} \) is a finite function, \( \text{Colour}(a) \) can be computed in constant time.

- \( \sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}} \): Let \( n = r + q \cdot (d - 1) \) where \( d = \rho(\sigma) \) and \( 1 \leq r < d \). Let \( \xi_0 = f_{\sigma, m_1, r}((\text{Colour}(b_1)), \ldots, (\text{Colour}(b_r))) \), and \( \xi_{i+1} = f_{\sigma, m_1, d}(\xi_i, (\text{Colour}(b_{i+1:(i+1)}(d-1)))) \) for \( i \in \{0, \ldots, q - 1\} \). Then by the composition property of \( \text{Str} \), we have that \( \text{Colour}(a) = \xi_1 \) if \( n < d \), else \( \text{Colour}(a) = \xi_q \). Observe that the \( \xi_i \)s can be computed in constant time, whereby the time taken to compute \( \text{Colour}(a) \) is linear in the degree of \( a \) in \( t \).

At the end of the above process, \( \text{Colour} \) gets constructed. The time taken for this construction is linear in the sum of the degrees of the nodes of \( t \), and hence linear in \( |t| \).

Stage III: \text{Reduce-degree} \((t, m)\) finally invokes \text{Complete-degree-reduction} \((t)\) below that reduces the degrees of the nodes of \( t \) to under a threshold. The output of \text{Complete-degree-reduction} \((t)\) is the output of \text{Reduce-degree} \((t, m)\). The former in turn uses the degree reduction procedure \text{Reduce-degree-of-node} \((u, a)\) which takes in a tree \( u \) of \( \mathcal{T} \) and a node \( a \) of \( u \), and produces a subtree \( v \) of \( u \) in \( \mathcal{T} \), containing \( a \), such that (i) the degree of \( a \) in \( v \) is at most \( p \), (ii) the roots of \( v \) and \( u \) are the same, (iii) \( \text{Str}(v) \hookrightarrow \text{Str}(u) \) and (iv) \( \equiv_{m_1, \mathcal{L}} \) \( \text{Str}(u) \).

\text{Complete-degree-reduction} \((t)\):

1. Initialize \( z := t \).

2. For \( a \) ranging over the nodes of \( t \), set \( z := \text{Reduce-degree-of-node}(z, a) \).

3. Return \( z \).

It is clear that \text{Complete-degree-reduction} \((t)\), and hence \text{Reduce-degree} \((t, m)\), outputs the desired subtree \( z_1 \) as required by the statement of Lemma 4.3 (observe that \( \mathcal{L}[m_1] \) equivalence implies \( \mathcal{L}[m] \) equivalence). We now describe \text{Reduce-degree-of-node} below and show that the time taken by \text{Reduce-degree-of-node} \((u, a)\) is linear in \( \Delta_{\Sigma, \mathcal{L}, m_1}(m_1) \times |t| \). It follows then that there exists a computable function \( \eta_1 : \mathbb{N} \rightarrow \mathbb{N} \) such that the time taken by \text{Reduce-degree} \((t, m)\) to compute \( z_1 \) is indeed at most \( \eta_1(m) \times |t| \). We now complete this part of the proof by describing \text{Reduce-degree-of-node} and showing its running time to be as mentioned above.

\text{Reduce-degree-of-node} \((u, a)\):

1. Let \( \sigma \) be the label of \( a \) in \( u \). If \( a \) is a leaf node or if \( \sigma \in \Sigma_{\text{rank}} \), then return \( u \).

2. Else, for each \( \delta \in \mathcal{L}[m_1] \)-classes, do the following:
(i) Let \( x = u_{\geq a} \). Let \( a \) have \( n \) children in \( x \), call these \( a_1, \ldots, a_n \) (in ascending order). Let \( n = r+q \cdot (d-1) \) where \( 1 \leq r < d \) and \( q > 0 \). Let \( I = \{ r+l \cdot (d-1) \mid 0 \leq l \leq q \} \).

(ii) Let \( x_{1,k} \) denote the subtree of \( x \) obtained by deleting the subtrees rooted at \( a_{k+1}, \ldots, a_n \). Construct the function \( g : I \rightarrow \Delta S, L, m_1 \) such that \( g(k) \) is the \( \equiv_{m_1, L} \) class of \( \text{Str}(x_{1,k}) \) for \( k \in I \).

(iii) If \( \delta \) is in the range of \( g \), then let \( i, j \) be resp. the least and greatest indices such that \( g(i) = g(j) = \delta \). Let \( y \) be the subtree of \( x \) obtained by deleting the subtrees rooted at \( a_{i+1}, \ldots, a_j \). Set \( x := y \).

3. Let \( v = u[u_{\geq a} \mapsto x] \). Return \( v \).

Reasoning similarly as in the proof of Lemma 4.3(1), we can verify that \( \text{Reduce-degree-of-node}(u, a) \) indeed works correctly. Given that we have already computed the function \( \text{Colour} \) in Stage II, the time taken to compute \( g \) is linear in the degree of \( a \), whereby the time taken to reduce the degree of node \( a \) in any iteration of the loop, is linear in the degree of \( a \). Then, the total time taken by \( \text{Reduce-degree-of-node}(u, a) \) is then linear in \( \Delta S_L(m_1) \times (\text{the degree of } a \text{ in } u) \).

(Important note: Observe the function \( \text{Colour} \) restricted to the output \( v \) of \( \text{Reduce-degree-of-node}(u, a) \) is such that for any node \( a \) of \( v \), the \( \equiv_{m_1, L} \) class of \( \text{Str}(v_{\geq a}) \) is indeed \( \text{Colour}(a) \).

(2) \( \text{Reduce-height}(t, m) \):

Just like \( \text{Reduce-degree}(t, m) \), the algorithm \( \text{Reduce-height}(t, m) \) also functions in various stages as described below. Let as before, \( m_1 = \max(m_0, m) \).

Step I: We generate \( L[m_1] \)-classes and the function \( \text{Colour} \) as done in \( \text{Reduce-degree}(t, m) \).

Step II: We construct a 2 dimensional array \( \text{Lowest-subtree}[i][j] \) where \( i \) ranges over the nodes of \( t \) and \( j \) ranges over \( L[m_1] \)-classes, such that \( \text{Lowest-subtree}[i][j] \) stores a pointer to a lowest (i.e., closest to a leaf) node \( a \) in the subtree of \( t \) rooted at \( i \), such that the \( \equiv_{m_1, L} \) class of \( \text{Str}(t_{\geq a}) \) is \( j \). In other words, \( a \) is such that the \( \equiv_{m_1, L} \) class of \( \text{Str}(t_{\geq a}) \) is \( j \), and there is no node \( b \neq a \) in \( t_{\geq a} \) such that the \( \equiv_{m_1, L} \) class of \( \text{Str}(t_{\geq b}) \) is \( j \).

We construct \( \text{Lowest-subtree} \) bottom-up in \( t \) as described below.

- Base case: For a leaf node \( e \), since the \( \equiv_{m_1, L} \) class \( \delta_e \) of \( \text{Str}(t_{\geq e}) \) has already been computed in Step I, we simply set \( \text{Lowest-subtree}[e][\delta_e] \) to store a pointer to \( e \), and for all \( \delta \in L[m_1] \)-classes such that \( \delta \neq \delta_e \), set \( \text{Lowest-subtree}[e][\delta] = \text{null} \).

  The time taken to complete this step is linear in the number of leaf nodes of \( t \).

- Induction: Assume that for an internal node \( a \) under consideration, for all its children \( b \) in \( t \), the value of \( \text{Lowest-subtree}[b][\delta] \) has been computed for all \( \delta \in L[m_1] \)-classes. Let \( \delta_a \) be the \( \equiv_{m_1, L} \) class of \( \text{Str}(t_{\geq a}) \) (that has already been computed as \( \text{Colour}(a) \) in Step I). For \( \delta \in L[m_1] \)-classes, check if for some child \( b \) of \( a \), the value of \( \text{Lowest-subtree}[b][\delta] \) is not \text{null}. If there is such a child \( b \), set \( \text{Lowest-subtree}[a][\delta] = \text{Lowest-subtree}[b][\delta] \). If there is no such child, then if \( \delta = \delta_a \), then set \( \text{Lowest-subtree}[a][\delta] \) to store a pointer to \( a \), else set \( \text{Lowest-subtree}[a][\delta] = \text{null} \).

  Observe that \( \text{Lowest-subtree}[a][\delta] \) indeed stores a pointer to a lowest node \( c \) in \( t_{\geq a} \) such that the \( \equiv_{m_1, L} \) class of \( \text{Str}(t_{\geq c}) \) is \( \delta \). Also observe that the time taken to complete this step is linear in the degree of \( a \) in \( t \).

It is clear that eventually \( \text{Lowest-subtree} \) gets constructed in time linear in \( |t| \).

Step III: We now describe an algorithm \( \text{Rainbow-subtree}(x) \) that takes a subtree \( x \) of \( t \) in \( T \) as input and outputs a subtree \( y \) of \( x \) in \( T \) such that

1. \( \text{Str}(x) \mapsto \text{Str}(y) \)
2. \( \text{Str}(x) \equiv_{m_1, \mathcal{L}} \text{Str}(y) \)

3. in no path from the root to the leaf of \( y \) is it the case that there exist two distinct nodes \( a \) and \( b \) such that the \( \equiv_{m_1, \mathcal{L}} \) classes of \( \text{Str}(y \geq a) \) and \( \text{Str}(y \geq b) \) are the same. Whereby, the height of \( x \) is at most \( \Lambda_{S, \mathcal{L}}(m_1) \).

4. The input \( x \) and output \( y \) both satisfy the following “colour preservation” property \( \mathcal{Q}(\cdot) \): for a subtree \( s \) of \( t \), obtained from \( t \) by removal of rooted subtrees and replacements with rooted subtrees, \( \mathcal{Q}(s) \) is a predicate denoting that the function \( \text{Colour} \) computed for \( t \), when restricted to the nodes of \( s \), is such that for any node \( a \) of \( s \), \( \text{Colour}(a) \) gives the \( \equiv_{m_1, \mathcal{L}} \) class of \( s \geq a \).

Rainbow-subtree(\( x \)):
1. Let \( \delta \) be the \( \equiv_{m_1, \mathcal{L}} \) class of root(\( x \)) (by the properties mentioned above, \( \delta = \text{Colour} \text{root}(x) \)).
2. Let \( a = \text{root}(x) \).
3. If Lowest-subtree[\( a \] \( \delta \)] stores a pointer to some node \( b \) other than \( a \), then return Rainbow-subtree(\( x \geq b \)).
4. Else, let \( b_1, \ldots, b_n \) be the children of \( a \) in \( x \).
5. For \( i \in \{1, \ldots, n\} \), let \( y_i = \text{Rainbow-subtree}(x \geq b_i) \).
6. Let \( z = x[x \geq b_1 \mapsto y_1] \ldots [x \geq b_n \mapsto y_n] \) be the subtree of \( x \) obtained by replacing \( x \geq b_i \) with \( y_i \) for \( i \in \{1, \ldots, n\} \).
7. Return \( z \).

It is easy to check using the fact that \( T \) is closed under replacements with rooted subtrees and Lemma 4.5 that Rainbow-subtree(\( x \)) is indeed correct. We also see that the number of “top level” recursive calls made by Rainbow-subtree(\( x \)) is linear in the degree of root(\( x \)), whereby the total time taken by Rainbow-subtree(\( x \)) is indeed linear in |\( x \)|.

Having defined Rainbow-subtree(\( x \)), we describe Step III which consists of executing the following substeps.
1. Let \( b_1, \ldots, b_n \) be the children of root(\( t \)) in \( t \).
2. For \( i \in \{1, \ldots, n\} \), let \( u_i = \text{Rainbow-subtree}(t \geq b_i) \).
3. Let \( v = t[t \geq b_1 \mapsto u_1] \ldots [t \geq b_n \mapsto u_n] \) be the subtree of \( t \) obtained by replacing \( t \geq b_i \) with \( u_i \) for \( i \in \{1, \ldots, n\} \). Return \( v \).

Reasoning similarly as in the proof of Lemma 4.3, we observe that \( v \) above is indeed the desired subtree \( s_2 \) of \( t \). It is easy to see from the descriptions above that the time taken by Reduce-height(\( t, m \)) to compute \( s_2 \) is at most \( \eta_5(m) \times |t| \), for some computable function \( \eta_5 : \mathbb{N} \to \mathbb{N} \).

5 Applications to various concrete settings

5.1 Regular languages of words, trees (unordered, ordered, ranked or partially ranked) and nested words

Let \( \Sigma \) be a finite alphabet. The notion of unordered, ordered, ranked and partially ranked \( \Sigma \)-trees was already introduced in Section 3.1. A \( \Sigma \)-tree whose underlying poset
is a linear order is called a Σ-word. A nested word over Σ is a pair \((w, \sim)\) where \(w\) is a “usual” Σ-word and \(\sim\) is a binary relation representing a “nested matching”. Formally, if \((A, \leq)\) is the linear order underlying \(w\), then \(\sim\) satisfies the following: (i) for \(i, j \in A\), if \(i \sim j\), then \(i \leq j\) and \(i \neq j\) (ii) for \(i \in A\), there is at most one \(j \in A\) such that \(i \sim j\) and at most one \(l \in A\) such that \(l \sim i\), and (iii) for \(i_1, i_2, j_1, j_2 \in A\), if \(i_1 \sim j_1\) and \(i_2 \sim j_2\), then it is not the case that \(i_1 < i_2 \leq j_1 < j_2\). (Nested words here correspond to nested words of \([4]\), that have no pending calls or pending returns.)

For e.g., \(w = (abaabba, \{(2, 6), (4, 5)\})\) is a nested word over \(\{a, b\}\). A nested Σ-word has a natural representation using a representation-feasible tree over \(\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}}\), where \(\Sigma_{\text{leaf}} = \Sigma \cup (\Sigma \times \Sigma)\), and \(\Sigma_{\text{int}} = \Sigma_{\text{leaf}} \cup \{\epsilon\}\). The figure alongside shows the tree \(t\) for \(w\). Conversely, every representation-feasible tree over \(\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}}\) represents a nested Σ-word.

Figure 1: Nested word as a tree

The notion of regular languages of words, trees and nested words is well studied. Since this notion corresponds to MSO definability \([3] [4]\), we say a class of words, trees (of any of the aforesaid kinds) or nested words is regular if it is the class of models of an MSO sentence.

**Theorem 5.1.** Given finite alphabets \(\Sigma, \Omega\) such that \(\Omega \subseteq \Sigma\), and a function \(\rho : \Omega \to \mathbb{N}\), let \(\text{Words}(\Sigma)\), \(\text{Unordered-trees}(\Sigma)\), \(\text{Ordered-trees}(\Sigma)\), \(\text{Partially-ranked-trees}(\Sigma, \Omega, \rho)\) and \(\text{Nested-words}(\Sigma)\) denote resp. the classes of all \(\Sigma\)-words, all unordered \(\Sigma\)-trees, all ordered \(\Sigma\)-trees, all ordered \(\Sigma\)-trees partially ranked by \((\Omega, \rho)\), and all nested \(\Sigma\)-words. Let \(S\) be a regular subclass of any of these classes. Then \(\mathcal{L}\text{-EBSP}(S)\) holds with a computable witness function. Further, any witness function for \(\mathcal{L}\text{-EBSP}(S)\) is necessarily non-elementary.

To present the proof idea for the above result, we need two composition lemmas, one for unordered trees and the other for nested words, just as we needed the composition lemma for the proof of Proposition \([3][4]\).

Towards the composition lemma for unordered trees, we introduce terminology akin to that introduced for ordered trees in Section \([3][4]\). Given unordered trees \(t\) and \(s\), and a node \(a\) of \(t\), define the join of \(s\) to \(t\) at \(a\), denoted \(t\cdot a\ s\), as follows: Let \(s'\) be an isomorphic copy of \(s\) whose set of nodes is disjoint with the set of nodes of \(t\). Then \(t\cdot a\ s\) is defined up to isomorphism as the tree obtained by making \(s'\) as a new child subtree of \(a\) in \(t\). The composition lemma for unordered trees is now as stated below. The proof is similar to that of Lemma \([3][4]\) and is hence skipped.

**Lemma 5.2** (Composition lemma for unordered trees). For a finite alphabet \(\Omega\), let \(t_i, s_i\) be non-empty unordered \(\Omega\)-trees, and let \(a_i\) be a node of \(t_i\), for each \(i \in \{1, 2\}\). For \(m \in \mathbb{N}\), suppose that \((t_1, a_1) \equiv_{m, \mathcal{L}} (t_2, a_2)\) and \(s_1 \equiv_{m, \mathcal{L}} s_2\). Then \(((t_1 \cdot a_1, s_1), a_1) \equiv_{m, \mathcal{L}} ((t_2 \cdot a_2, s_2), a_2)\).

Towards the composition lemma for nested words, we first define the notion of insert of a nested word \(v\) in a nested word \(u\) at a given position \(e\) of \(u\).

**Definition 5.3** (Insert). Let \(u = (A_u, \leq_u, \lambda_u, \sim_u)\) and \(v = (A_v, \leq_v, \lambda_v, \sim_v)\) be given nested Σ-words, and let \(e\) be a position in \(u\). The insert of \(v\) in \(u\) at \(e\), denoted \(u \uparrow_e v\), is a nested Σ-word defined as below.
1. If \( u \) and \( v \) have disjoint sets of positions, then \( u \uparrow_c v = (A, \leq, \lambda, \sim) \) where
   \[ A = A_u \cup A_v \]
   \[ \leq = \leq_u \cup \leq_v \cup \{(i, j) \mid i \in A_u, j \in A_v, i \leq u \} \cup \{(j, i) \mid i \in A_u, j \in A_v, e \leq_u i, e \neq i\} \]
   \[ \lambda(a) = \lambda_u(a) \text{ if } a \in A_u, \text{ else } \lambda(a) = \lambda_v(a) \]
   \[ \sim = \sim_u \cup \sim_v \]

2. If \( u \) and \( v \) have overlapping sets of positions, then let \( v_1 \) be an isomorphic copy of \( v \) whose set of positions is disjoint with that of \( u \). Then \( u \uparrow_c v \) is defined up to
   isomorphism as \( u \uparrow_c v_1 \).

In the special case that \( e \) is the last (under \( \leq_u \)) position of \( u \), we denote \( u \uparrow_c v \) as \( u \cdot v \), and call the latter as the \textit{concatenation of \( v \) with \( u \)}.

**Lemma 5.4** (Composition lemma for nested words). For a finite alphabet \( \Sigma \), let \( u_i, v_i \in \text{Nested-words}(\Sigma), \) and let \( e_i \) be a position in \( u_i \), for \( i \in \{1, 2\} \). Then the following hold for each \( m \in \mathbb{N} \).

1. If \((u_1, e_1) \equiv_{m, L} (u_2, e_2)\) and \( v_1 \equiv_{m, L} v_2 \), then \((u_1 \uparrow_{e_1} v_1) \equiv_{m, L} (u_2 \uparrow_{e_2} v_2)\).
2. \( u_1 \equiv_{m, L} u_2 \) and \( v_1 \equiv_{m, L} v_2 \), then \( u_1 \cdot v_1 \equiv_{m, L} u_2 \cdot v_2 \).

**Proof.** We give the proof for \( \mathcal{L} = \text{MSO} \). The proof for \( \mathcal{L} = \text{FO} \) is similar.

The winning strategy \( S \) for the duplicator in the \textit{m}-round MSO-\( \text{EF} \) game between \( u_1 \uparrow_{e_1} v_1 \) and \( u_2 \uparrow_{e_2} v_2 \) is simply the composition of the winning strategies \( S_1, \) resp. \( S_2 \), of the duplicator in the \textit{m}-round MSO-\( \text{EF} \) game between \((u_1, e_1)\) and \((u_2, e_2)\), resp. \( v_1 \) and \( v_2 \). Formally, \( S \) is defined as follows.

1. Point move: If the spoiler picks an element of \( u_1 \), resp. \( v_1 \), from \( u_1 \uparrow_{e_1} v_1 \), then the duplicator picks the element of \( u_2 \), resp. \( v_2 \), from \( u_2 \uparrow_{e_2} v_2 \), that is given by the strategy \( S_1 \), resp. \( S_2 \). A similar choice of an element from \( u_1 \uparrow_{e_1} v_1 \) is made by the duplicator if the spoiler picks an element from \( u_2 \uparrow_{e_2} v_2 \).
2. Set move: If the spoiler picks a set \( Z \) from \( u_1 \uparrow_{e_1} v_1 \), then let \( Z = X \cup Y \) where \( X \) is a subset of positions of \( u_1 \) and \( Y \) is a subset of positions of \( v_1 \). Then the duplicator picks the set \( Z' \) from \( u_2 \uparrow_{e_2} v_2 \) where \( Z' = X' \cup Y' \), \( X' \) is the subset of positions of \( u_2 \) that is chosen by the duplicator in response to \( X \) according to strategy \( S_1 \), and \( Y' \) is the subset of positions of \( v_2 \) that is chosen by the duplicator in response to \( Y \) according to strategy \( S_2 \). A similar choice of a set from \( u_1 \uparrow_{e_1} v_1 \) is made by the duplicator if the spoiler picks a set from \( u_2 \uparrow_{e_2} v_2 \).

It is easy to see that \( S \) is a winning strategy in the MSO-\( \text{EF} \) game between \( u_1 \uparrow_{e_1} v_1 \) and \( u_2 \uparrow_{e_2} v_2 \). \( \square \)

**Proof idea for Theorem 5.7.** We first show MSO-\( \text{EBSP}(S) \) holds when \( S \) is exactly one of the classes mentioned in the statement of the theorem. That \( \mathcal{L} = \text{MSO-EBSP}(\cdot) \) holds for a regular subclass follows, because (i) MSO-\( \text{EBSP}(\cdot) \) is preserved under MSO definable subclasses, and (ii) MSO-\( \text{EBSP}(\cdot) \) implies FO-\( \text{EBSP}(\cdot) \).

Consider \( S = \text{Unordered-trees}(\Sigma) \) (the other cases of trees have been covered by Proposition 5.2). There is a natural map \( \text{Str}_1 : \mathcal{T}_1 \to S \), where \( \mathcal{T}_1 \) is the class of all \((\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}})\)-trees that is representation-feasible for \((\Sigma_{\text{rank}}, \rho)\), \( \Sigma_{\text{int}} = \Sigma_{\text{leaf}} = \Sigma \), \( \Sigma_{\text{rank}} = \emptyset \), \( \rho \) is the constant function of value \( 2 \), and \( \text{Str}_1 \) “forgets” the ordering among the children of any node of its input tree. The MSO composition lemma for unordered trees, given by Lemma 5.2 then shows that \( \text{Str}_1 \) is an elementary MSO-good tree representation for \( S \), whereby using Theorem 4.2 we are done.

Likewise, when \( S = \text{Nested-words}(\Sigma) \), there is a natural map \( \text{Str}_2 : \mathcal{T}_2 \to S \), where \( \mathcal{T}_2 \) is the class of all \((\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}})\)-trees that is representation-feasible for \((\Sigma_{\text{rank}}, \rho)\), \( \Sigma_{\text{int}} = \Sigma_{\text{leaf}} \cup \{\} \), \( \Sigma_{\text{leaf}} = \Sigma \cup (\Sigma \times \Sigma) \), \( \Sigma_{\text{rank}} = \emptyset \), \( \rho \) is the constant function of value \( 2 \), and \( \text{Str}_2 \) is as described in the example above. Then \( \text{Str}_2 \) is an elementary MSO-good tree representation for \( S \), due to the MSO composition lemma for nested words given by
Lemma 5.4. We are done by Theorem 4.2 again. The non-elementariness of witness functions is due to Theorem 1.2 and the non-elementariness of the index of the $\equiv_{m,c}$ relation over words [11].

5.2 n-partite cographs

The class of $n$-partite cographs, introduced in [13], can be defined up to isomorphism as the range of the map $\text{Str}$ described as follows. Let $\Sigma_{\text{leaf}} = [n] = \{1, \ldots, n\}$ and $\Sigma_{\text{int}} = \{f \mid f : [n] \times [n] \rightarrow \{0, 1\}\}$, $\Sigma_{\text{rank}} = \emptyset$ and $\rho : \Sigma_{\text{rank}} \rightarrow \mathbb{N}_+$ be the constant function 2. Let $T$ be the class of all $(\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}})$-trees that are representation-feasible for $(\Sigma_{\text{rank}}, \rho)$. Consider $\text{Str} : T \rightarrow \text{Graphs}$ be defined as follows, where $\text{Graphs}$ is the class of all undirected graphs: For $t = (O, \lambda) \in T$ where $O = ((A, \leq), \leq)$ is an ordered unlabeled tree and $\lambda$ is the labeling function, $\text{Str}(t) = G = (V, E)$ is such that (i) $V$ is exactly the set of leaf nodes of $t$ (ii) for $a, b \in V$, if $c = a \wedge b$ is the greatest common ancestor (under $\leq$) of $a$ and $b$ in $t$, then $\{a, b\} \in E$ iff $\lambda(c) (\lambda(a), \lambda(b)) = 1$. We now have the following result. Below, a $\Sigma$-labeled $n$-partite cograph is a pair $(G, \nu)$ where $G$ is an $n$-partite cograph and $\nu : V \rightarrow \Sigma$ is a labeling function. Also, “hereditary” means “closed under substructures”.

Theorem 5.5. Given $n \in \mathbb{N}$ and a finite alphabet $\Sigma$, let $\text{Labeled-n-partite-cographs}(\Sigma)$ be the class of all $\Sigma$-labeled $n$-partite cographs. Let $S$ be any hereditary subclass of this class. Then $\mathcal{L}$-EBSP($S$) holds with a computable witness function. Whereby, each of the graph classes below satisfies $\mathcal{L}$-EBSP($\cdot$) with a computable witness function. Further, the classes with bounded parameters as mentioned below have elementary functions witnessing $\mathcal{L}$-EBSP($\cdot$).
1. Any hereditary class of $n$-partite cographs, for each $n \in \mathbb{N}$.
2. Any hereditary class of graphs of bounded shrub-depth.
3. Any hereditary class of graphs of bounded SC-depth.
4. Any hereditary class of graphs of bounded tree-depth.
5. Any hereditary class of co-graphs.

The proof of Theorem 5.5 again uses a composition lemma for $n$-partite cographs. Let $T$ be the class of all $(\Sigma_{\text{int}} \cup \Sigma_{\text{leaf}})$-trees where $\Sigma_{\text{leaf}} = [n] \times \Sigma$, $\Sigma_{\text{int}} = \{f \mid f : [n] \times [n] \rightarrow \{0, 1\}\}$ and $\Sigma_{\text{rank}} = \emptyset$. Let $\rho : \Sigma_{\text{int}} \rightarrow \mathbb{N}_+$ be the constant function 2. Then $T$ is representation-feasible for $(\Sigma_{\text{rank}}, \rho)$. Further, there is a natural representation map $\text{Str} : T \rightarrow \text{Labeled-n-partite-cographs}(\Sigma)$ exactly of the kind described above for $n$-partite cographs, that maps a tree in $T$ to the labeled $n$-partite cograph that it represents. The composition lemma for $n$-partite cographs can now be stated as below.

Lemma 5.6 (Composition lemma for $n$-partite cographs). For $i \in \{1, 2\}$, let $(G_i, \nu_{i,1})$ and $(H_i, \nu_{i,2})$ be graphs in $\text{Labeled-n-partite-cographs}(\Sigma)$. Suppose $t_i$ and $s_i$ are trees of $T$ such that $\text{Str}(t_i) = (G_i, \nu_{i,1})$, $\text{Str}(s_i) = (H_i, \nu_{i,2})$, and the labels of $\text{root}(t_i)$ and $\text{root}(s_i)$ are the same. Let $z_i = t_i \odot s_i$ and $\text{Str}(z_i) = (Z_i, \nu_i)$ for $i \in \{1, 2\}$. For each $n \in \mathbb{N}$, if $(G_1, \nu_{1,1}) \equiv_{m,c} (G_2, \nu_{2,1})$ and $(H_1, \nu_{1,2}) \equiv_{m,c} (H_2, \nu_{2,2})$, then $(Z_1, \nu_1) \equiv_{m,c} (Z_2, \nu_2).

Proof. We prove the lemma for $\mathcal{L} = \text{MSO}$. A similar proof can be done for $\mathcal{L} = \text{FO}$. We can assume w.l.o.g. that $t_i$ and $s_i$ have disjoint sets of nodes for $i \in \{1, 2\}$. Let the set of vertices of $\text{Str}(t_i)$ and $\text{Str}(s_i)$ be $V\text{-Str}(t_i)$ and $V\text{-Str}(s_i)$ respectively. Then the vertex set $V\text{-Str}(z_i)$ of $\text{Str}(z_i)$ is $V\text{-Str}(t_i) \cup V\text{-Str}(s_i)$ for $i \in \{1, 2\}$.

Let $S_1$, resp. $S_2$, be the strategy of the duplicator in the $m$-round MSO-EF game between $\text{Str}(t_1)$ and $\text{Str}(t_2)$, resp. between $\text{Str}(s_1)$ and $\text{Str}(s_2)$. For the $m$-round MSO-EG game between $\text{Str}(z_1)$ and $\text{Str}(z_2)$, the duplicator follows the following strategy, call it $R$.

• Point move: If the spoiler chooses a vertex from $V\text{-Str}(t_1)$ (resp. $V\text{-Str}(t_2)$), then the duplicator chooses a vertex from $V\text{-Str}(t_2)$ (resp. $V\text{-Str}(t_1)$) according to $S_1$. 

26
We now show that

The result for the various specific classes mentioned in the statement of the theorem follows again from Theorem 4.2 and elementariness of the index of the

to the statement of Lemma 5.6, there is a natural representation map

Proof idea for Theorem 5.5. We first show the result for

Consider $a_i, a_j$ for $i \neq j$ and $i, j \in \{1, \ldots, p\}$. We show below that $a_i, a_j$ are adjacent in $\text{Str}(z_1)$ if $b_i, b_j$ are adjacent in $\text{Str}(z_2)$. This would show that $a_i \mapsto b_i$ is a partial isomorphism between $(\text{Str}(z_1), A_1, \ldots, A_r)$ and $(\text{Str}(z_2), B_1, \ldots, B_r)$ completing the proof.

We have the following three cases:

1. Each of $a_i$ and $a_j$ is from $V(\text{Str}(t_1))$: Then by the description of $R$ above, we have that (i) $b_i$ and $b_j$ are both from $V(\text{Str}(t_2))$ and (ii) $a_i, a_j$ are adjacent in $\text{Str}(t_1)$ if $b_i, b_j$ are adjacent in $\text{Str}(t_2)$.

2. Each of $a_i$ and $a_j$ is from $V(\text{Str}(s_1))$: Reasoning similarly as in the previous case, we can show that $a_i, a_j$ are adjacent in $\text{Str}(z_1)$ if $b_i, b_j$ are adjacent in $\text{Str}(z_2)$.

3. W.l.o.g. $a_i \in V(\text{Str}(t_1))$ and $a_j \in V(\text{Str}(s_1))$: Then $b_i \in V(\text{Str}(t_2))$ and $b_j \in V(\text{Str}(s_2))$. Observe now that the greatest common ancestor of $a_i$ and $a_j$ in $z_1$ is $\text{root}(z_1)$, and the greatest common ancestor of $b_i$ and $b_j$ in $z_2$ is $\text{root}(z_2)$. Since (i) the labels of $\text{root}(z_1)$ and $\text{root}(z_2)$ are the same (by assumption) and (ii) the label of $a_i$ (resp. $a_j$) in $z_1 = \text{label of } a_i$ (resp. $a_j$) in $\text{Str}(z_1) = \text{label of } b_i$ (resp. $b_j$) in $\text{Str}(z_2)$, it follows by the definition of an $n$-partite cographt that $a_i, a_j$ are adjacent in $\text{Str}(z_1)$ if $b_i, b_j$ are adjacent in $\text{Str}(z_2)$.

\[\square\]

Proof idea for Theorem 5.5. We first show the result for $S = \text{Labeled-n-partite-cographs}(\Sigma)$. The result for the various specific classes mentioned in the statement of the theorem follows from the fact that $L\text{-EBSP}(\cdot)$ is closed under hereditary subclasses, and that all of the specific classes are hereditary subclasses of $n$-partite cographs [13]. As described prior to the statement of Lemma 5.6, there is a natural representation map $\text{Str}: \mathcal{T} \to S$ that is elementary. Using the composition lemma for $n$-partite graphs given by Lemma 5.6, we can see that $\text{Str}$ is $L$-good for $S$, whereby we are done by Theorem 4.2. That the graph classes with the bounded parameters as above have elementary witness functions follows again from Theorem 4.2 and elementariness of the index of the $\equiv_{m,L}$ relation over these classes (the latter follows from Theorem 3.2 of [12]).
5.3 Classes generated using translation schemes

We look operations on classes of structures, that are “implementable” using quantifier-free translation schemes \[\Xi.\] Given a vocabulary \(\tau\), let \(\tau_{\text{disj}-\text{sum},n}\) be the vocabulary obtained by expanding \(\tau\) with \(n\) fresh unary predicates \(P_1,\ldots,P_n\). Given \(\tau\)-structures \(\mathfrak{A}_1,\ldots,\mathfrak{A}_n\) (assumed disjoint w.l.o.g.), the \(n\)-disjoint sum of \(\mathfrak{A}_1,\ldots,\mathfrak{A}_n\), denoted \(\bigoplus_{i=1}^n \mathfrak{A}_i\), is the \(\tau_{\text{disj}-\text{sum},n}\)-structure obtained up to isomorphism, by expanding the disjoint union \(\bigsqcup_{i=1}^n \mathfrak{A}_i\) with \(P_1,\ldots,P_n\) interpreted respectively as the universe of \(\mathfrak{A}_1,\ldots,\mathfrak{A}_n\). Let \(\mathcal{S}_1,\ldots,\mathcal{S}_n\) be given classes of structures. A quantifier-free \((t,\tau_{\text{disj}-\text{sum},n},\tau,\text{FO})\)-translation scheme \(\Xi\) gives rise to an \(n\)-ary operation \(O: \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \{\Xi(\bigoplus_{i=1}^n \mathfrak{A}_i) \mid \mathfrak{A}_i \in \mathcal{S}_i, 1 \leq i \leq n\}\) defined as \(O(\mathfrak{A}_1,\ldots,\mathfrak{A}_n) = \Xi(\bigoplus_{i=1}^n \mathfrak{A}_i)\). In this case, we say that \(O\) is implementable using \(\Xi\). We say an operation is quantifier-free, if it is of the kind \(O\) just described.

For a quantifier-free operation \(O\), let the dimension of \(O\) be the minimum of the dimensions of the quantifier-free translation schemes that implement \(O\). We say \(O\) is “sum-like” if its dimension is one, else we say \(O\) is “product-like”. Call \(O\) as \(\equiv_{m,\mathcal{L}}\)-preserving if whenever an input of \(O\) is replaced with an \(\mathcal{L}[m]\)-equivalent input, the output of \(O\) is replaced with an \(\mathcal{L}[m]\)-equivalent output. We say \(O\) is monotone if any input of \(O\) is embeddable in the output of \(O\). The well-studied unary graph operations like complementation, transpose, and the line-graph operation, and binary operations like disjoint union and join are all sum-like, \(\equiv_{m,\mathcal{L}}\)-preserving and monotone. Likewise, the well-studied Cartesian, tensor, lexicographic, and strong products are all product-like, \(\equiv_{m,\mathcal{L}}\)-preserving and monotone. The central result of this section is as stated below.

**Theorem 5.7.** Let \(\mathcal{S}_1,\ldots,\mathcal{S}_n,\mathcal{S}\) be classes of structures and let \(O: \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{S}\) be a surjective \(n\)-ary quantifier-free operation. Then the following are true:

1. In each of the following scenarios, it is the case that if \(\mathcal{L}\text{-EBSP}(\mathcal{S}_i)\) holds (with computable/elementary witness functions) for each \(i \in \{1,\ldots,n\}\), then \(\mathcal{L}\text{-EBSP}(\mathcal{S})\) holds as well (with computable/elementary witness functions): (i) \(O\) is sum-like (ii) \(O\) is product-like and \(\mathcal{L} = \text{FO}\).
2. Suppose \(\mathcal{S}_i\) admits an effective \(\mathcal{L}\)-good tree representation for each \(i \in \{1,\ldots,n\}\), and \(O\) is \(\equiv_{m,\mathcal{L}}\)-preserving and monotone. Then there exists an effective \(\mathcal{L}\)-good tree representation \(\text{Str}: \mathcal{T} \to \mathcal{Z}\) for the class \(\mathcal{Z} = \mathcal{S} \cup \bigsqcup_{i=1}^n \mathcal{S}_i\).
3. Let \(\text{Str}\) be as given by the previous point. Then there is a linear time f.p.t. algorithm for \(\text{MC}(\mathcal{L},\mathcal{S})\) that decides, for every \(\mathcal{L}\) sentence \(\varphi\) (the parameter), if a given structure \(\mathfrak{A}\) in \(\mathcal{S}\) satisfies \(\varphi\), provided a tree representation of \(\mathfrak{A}\) under \(\text{Str}\) is given.

Towards the proof of Theorem 5.7, we first present the following two auxiliary results.

Below, \(n\text{-disjoint-sum}(\mathcal{S}_1,\ldots,\mathcal{S}_n) = \{\bigoplus_{i=1}^n \mathfrak{A}_i \mid \mathfrak{A}_i \in \mathcal{S}_i, 1 \leq i \leq n\}\). Also, we say a quantifier-free translation scheme is scalar if its dimension is one.

**Lemma 5.8.** Let \(\mathcal{S},\mathcal{S}_1,\ldots,\mathcal{S}_n\) be classes of structures for \(n \geq 1\). If \(\mathcal{L}\text{-EBSP}(\mathcal{S}_i)\) is true for each \(i \in \{1,\ldots,n\}\), then so is \(\mathcal{L}\text{-EBSP}(n\text{-disjoint-sum}(\mathcal{S}_1,\ldots,\mathcal{S}_n))\). Further, if there is a computable/elementary witness function for \(\mathcal{L}\text{-EBSP}(\mathcal{S}_i)\) for each \(i \in \{1,\ldots,n\}\), then there is a computable/elementary witness function for \(\mathcal{L}\text{-EBSP}(n\text{-disjoint-sum}(\mathcal{S}_1,\ldots,\mathcal{S}_n))\) as well.

**Proposition 5.9.** Let \(\mathcal{S}\) be class of \(\tau\)-structures, and let \(\Xi = (\xi, (\xi_R)_{R \in \tau})\) be a quantifier-free \((t,\tau,\nu,\text{FO})\)-translation scheme. Then the following hold for each \(k \in \mathbb{N}\).

1. If \(\text{FO-EBSP}(\mathcal{S})\) is true, then so is \(\text{FO-EBSP}(\Xi(\mathcal{S}))\).
2. If \(\Xi\) is scalar and \(\text{MSO-EBSP}(\mathcal{S})\) is true, then so is \(\text{MSO-EBSP}(\Xi(\mathcal{S}))\).

In each of the implications above, a computable/elementary witness function for the antecedent implies a computable/elementary witness function for the consequent.

**Proof of Theorem 5.7.** (1): Follows easily from Lemma 5.8 and Proposition 5.9.
Let \( \mathcal{S}_i \) denote both \( \Xi \) and \( \mathcal{B} \), and \( \mathcal{B} \rangle \) is a class of trees over \((\Sigma^n_0, \Sigma^n_1)\) that is representation feasible for \((\Sigma^n_0, \rho_i)\).

Let \( O \) be a new label that is not in \((\Sigma^n_0, \Sigma^n_1)\) for any \( i \in \{1, \ldots, n\} \). Define \( \Sigma^n_0, \Sigma^n_1, \Sigma^n_0, \Sigma^n_1, \rho : \Sigma^n_0 \rightarrow \mathbb{N} \) as follows:

- \( \Sigma^n_0 = \{O\} \cup \bigcup_{i=1}^{n} \Sigma^n_i \)
- \( \Sigma^n_1 = \bigcup_{i=1}^{n} \Sigma^n_i \)
- \( \Sigma^n_0 = \{O\} \cup \bigcup_{i=1}^{n} \Sigma^n_i \)
- \( \rho = \{(O, n)\} \cup \bigcup_{i=1}^{n} \rho_i. \)

Let \( \hat{T} \) be the class of all trees over \((\Sigma^n_0, \Sigma^n_1)\) obtained by taking \( t_i \in \mathcal{T}_i \) for \( 1 \leq i \leq n \), and making \( t_1, \ldots, t_n \) as child subtrees (and in that order) of a new root node whose label is \( O \). Let \( \mathcal{T} = \hat{T} \cup \bigcup_{i=1}^{n} \mathcal{T}_i \). Verify that \( \mathcal{T} \) is indeed representation feasible for \((\Sigma^n_0, \rho)\).

Let \( \text{Str} : \mathcal{T} \rightarrow \Xi \) be such that for \( t \in \mathcal{T} \), if \( t \in \mathcal{T}_i \), then \( \text{Str}(t) = \text{Str}_i(t) \). Else, let \( a_1, \ldots, a_n \) be the children of the root of \( t \). Clearly then \( t_{\geq a} \in \mathcal{T}_i \) by construction of \( \mathcal{T} \). Then define \( \text{Str}(t) = O(\text{Str}_i(t_{\geq a_1}), \ldots, \text{Str}_i(t_{\geq a_n})) \).

Using the fact that \( O \) is \( \equiv_{m.c.} \)-preserving and monotone, and using Lemma 5.3, we conclude that \( \text{Str} \) is indeed an effective \( \mathcal{L} \)-good representation map for \( \mathcal{T} \).

(3): Since \( \text{Str} \) is effective and \( \mathcal{L} \)-good for \( \mathcal{Z} \), by Theorem 4.2 there is a linear time f.p.t. algorithm for \( MC(\mathcal{L}, \mathcal{Z}) \) that decides, for every \( \mathcal{L} \) sentence \( \varphi \), if a given structure \( \mathfrak{A} \) in \( \mathcal{Z} \) satisfies \( \varphi \), provided that a tree representation of \( \mathfrak{A} \) under \( \text{Str} \). Clearly the same algorithm is also f.p.t. for \( MC(\mathcal{L}, \mathfrak{S}) \).

The remainder of this section is devoted to proving Lemma 5.8 and Proposition 5.9. Towards the proof of Lemma 5.8, we present the following simple facts about \( n \)-disjoint sum. We skip the proof.

**Lemma 5.10.** Let \( \mathfrak{A}_i \) and \( \mathfrak{B}_i \) be \( \tau \)-structures for \( i \in \{1, \ldots, n\} \). Let \( m \in \mathbb{N} \). Then the following are true:

1. If \( \mathfrak{B}_i \vdash \mathfrak{A}_i \) for \( i \in \{1, \ldots, n\} \), then \( (\bigoplus_{i=1}^{n} \mathfrak{B}_i) \vdash (\bigoplus_{i=1}^{n} \mathfrak{A}_i) \).
2. If \( \mathfrak{B}_i \equiv_{m.c.} \mathfrak{A}_i \) for \( i \in \{1, \ldots, n\} \), then \( (\bigoplus_{i=1}^{n} \mathfrak{B}_i) \equiv_{m.c.} (\bigoplus_{i=1}^{n} \mathfrak{A}_i) \).

**Proof of Lemma 5.8** Consider a structure \( \mathfrak{A} = (\bigoplus_{i=1}^{n} \mathfrak{A}_i) \) in \( n \)-disjoint-sum(\(S_1, \ldots, S_n) \). Let \( m \in \mathbb{N} \). Since \( \mathcal{L}-\text{EBSP}(\mathfrak{A}_i) \) is true, there exists \( \mathfrak{B}_i \) such that \( \mathcal{L}-\text{EBSP-condition}(\mathfrak{A}_i, \mathfrak{B}_i, m, \theta_{m, \mathcal{L}}(m)) \) holds where \( \theta_{m, \mathcal{L}}(m) \) is a witness function for \( \mathcal{L}-\text{EBSP}(\mathfrak{S}_i) \). Then \( \mathfrak{B}_i \subseteq \mathfrak{A}_i \) and \( \mathfrak{B}_i \equiv_{m.c.} \mathfrak{A}_i \). Then by Lemma 5.10, we have that (i) \( \bigoplus_{i=1}^{n} \mathfrak{B}_i \vdash \bigoplus_{i=1}^{n} \mathfrak{A}_i \), and (ii) \( \bigoplus_{i=1}^{n} \mathfrak{B}_i \equiv_{m.c.} \bigoplus_{i=1}^{n} \mathfrak{A}_i \). Observe that \( (\bigoplus_{i=1}^{n} \mathfrak{B}_i) \) is an \( n \)-disjoint-sum(\(S_1, \ldots, S_n\)), and that \( \|\bigoplus_{i=1}^{n} \mathfrak{B}_i\| \leq \theta(m) = \sum_{i=0}^{n} \theta_{m, \mathcal{L}}(m) \). Taking \( \mathfrak{B} \) to be the substructure of \( \mathfrak{A} \) that is isomorphic to \( (\bigoplus_{i=1}^{n} \mathfrak{B}_i) \), we see that \( \mathcal{L}-\text{EBSP-condition}(n\text{-disjoint-sum}(\mathfrak{S}_1, \ldots, \mathfrak{S}_n), \mathfrak{A}, \mathfrak{B}, m, \theta) \) is true with witness function \( \theta \). Whereby \( \mathcal{L}-\text{EBSP}(n\text{-disjoint-sum}(\mathfrak{S}_1, \ldots, \mathfrak{S}_n)) \) is true. It is easy to see that if \( \theta_{m, \mathcal{L}}(m) \) is computable/elementary for each \( i \in \{1, \ldots, n\} \), then so is \( \theta \).

We now proceed proving Proposition 5.9. We use the following known facts about translation schemes [22]. To present these facts, we recall from Section 2 that one can associate with a \((t, \nu, \mathcal{L})\)-translation scheme \( \Xi \), two partial maps: (i) \( \Xi^* \) from \( \tau \)-structures to \( \nu \)-structures (ii) \( \Xi^2 \) from \( \mathcal{L}(\nu) \) formulae to \( \mathcal{L}(\tau) \) formulae. See [22] for the definitions of these. For the ease of readability, we abuse notation slightly and use \( \Xi \) to denote both \( \Xi^* \) and \( \Xi^2 \). We now have the following results from literature.
Proposition 5.11. Let \( \Xi \) be either a \( (t, \tau, \nu, FO) \)-translation scheme for \( t \geq 1 \), or a \( (t, \tau, \nu, MSO) \)-translation scheme with \( t = 1 \). Then for every \( L(\nu) \) formula \( \varphi(x_1, \ldots, x_n) \) where \( n \geq 0 \), for every \( \tau \)-structure \( \mathfrak{A} \) and for every \( n \)-tuple \( (\bar{a}_1, \ldots, \bar{a}_n) \) from \( \Xi(\mathfrak{A}) \), the following holds.

\[
(\Xi(\mathfrak{A}), \bar{a}_1, \ldots, \bar{a}_n) \models \varphi(x_1, \ldots, x_n)
\iff
(\mathfrak{A}, \bar{a}_1, \ldots, \bar{a}_n) \models \Xi(\varphi)(\bar{x}_1, \ldots, \bar{x}_n)
\]

where \( \bar{x}_i = (x_{i,1}, \ldots, x_{i,t}) \) for each \( i \in \{1, \ldots, n\} \).

Lemma 5.12. Let \( \Xi \) be a quantifier-free \( (t, \tau, \nu, FO) \)-translation scheme. Let \( m, r \in \mathbb{N} \) be such that \( r = t \cdot m \). Suppose \( \mathfrak{A} \) and \( \mathfrak{B} \) are \( \tau \)-structures.
1. If \( \mathfrak{A} \equiv_{r, FO} \mathfrak{B} \), then \( \Xi(\mathfrak{A}) \equiv_{m, FO} \Xi(\mathfrak{B}) \).
2. If \( \mathfrak{A} \equiv_{m, MSO} \mathfrak{B} \), then \( \Xi(\mathfrak{A}) \equiv_{m, MSO} \Xi(\mathfrak{B}) \), when \( \Xi \) is scalar.

Towards the proof of Proposition 5.9, we first observe the following result that shows that quantifier-free translation schemes preserve the substructure relation between any two structures of \( S \).

Lemma 5.13. Let \( S \) be a given class of finite structures. Let \( \Xi = (\xi, (\xi_R)_{R \in \nu}) \) be a quantifier-free \( (t, \tau, \nu, FO) \)-translation scheme. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be given structures from \( S \). If \( \mathfrak{A} \subseteq \mathfrak{B} \), then \( \Xi(\mathfrak{B}) \subseteq \Xi(\mathfrak{A}) \).

Proof. Consider any element of \( \Xi(\mathfrak{B}) \); it is a \( t \)-tuple \( \bar{b} \) of \( \mathfrak{B} \) such that \( (\mathfrak{B}, \bar{b}) \models \xi(\bar{x}) \). Since \( \xi(\bar{x}) \) is quantifier-free, it is preserved under extensions over \( S \). Whereby \( (\mathfrak{A}, \bar{b}) \models \xi(\bar{x}) \); then \( \bar{b} \) is an element of \( \Xi(\mathfrak{A}) \). Since \( \bar{b} \) is an arbitrary element of \( \Xi(\mathfrak{B}) \), we have \( U_{\Xi(\mathfrak{B})} \subseteq U_{\Xi(\mathfrak{A})} \).

Consider a relation symbol \( R \in \nu \) of arity say \( n \). Let \( d_1, \ldots, d_n \) be elements of \( \Xi(\mathfrak{B}) \). Then we have the following. Below \( \bar{x}_i = (x_{i,1}, \ldots, x_{i,t}) \) for each \( i \in \{1, \ldots, n\} \).

\[
(\Xi(\mathfrak{B}), d_1, \ldots, d_n) \models R(x_1, \ldots, x_n)
\iff
(\mathfrak{B}, d_1, \ldots, d_n) \models \Xi(R)(\bar{x}_1, \ldots, \bar{x}_n) \quad \text{(by Proposition 5.11)}
\iff
(\mathfrak{A}, d_1, \ldots, d_n) \models \bigwedge_{i=1}^{n} \xi(\bar{x}_i) \land \xi_R(\bar{x}_1, \ldots, \bar{x}_n) \quad \text{(by defn. of} \; \Xi(R) \text{; see [22])}
\]

Now since (i) each of \( \xi \) and \( \xi_R \) is quantifier-free, (ii) a finite conjunction of quantifier-free formulae is a quantifier-free formula, and (iii) a quantifier-free formula is preserved under substructures as well as preserved under extensions over any class, we have that

\[
(\mathfrak{A}, d_1, \ldots, d_n) \models \bigwedge_{i=1}^{n} \xi(\bar{x}_i) \land \xi_R(\bar{x}_1, \ldots, \bar{x}_n)
\iff
(\Xi(\mathfrak{A}), d_1, \ldots, d_n) \models \Xi(R)(\bar{x}_1, \ldots, \bar{x}_n) \quad \text{(by definition of} \; \Xi(R))
\iff
(\Xi(\mathfrak{B}), d_1, \ldots, d_n) \models R(x_1, \ldots, x_n) \quad \text{(by Proposition 5.11)}
\]

Since \( R \) is an arbitrary relation symbol of \( \nu \), we have that \( \Xi(\mathfrak{B}) \subseteq \Xi(\mathfrak{A}) \).

Proof of Proposition 5.9. We show the proof for part 1. The proof for part 2 is similar. Consider a structure \( \Xi(\mathfrak{A}) \in \Xi(S) \) for some structure \( \mathfrak{A} \in S \). Let \( m \in \mathbb{N} \). Since \( FO-EBSP(S) \) is true, there exists a witness function \( \theta_{(S, FO)} : N \to N \) and a structure \( \mathfrak{B} \) such that if \( r = t \cdot m \), then \( FO-EBSP-condition(S, \mathfrak{A}, \mathfrak{B}, r, \theta_{(S, FO)}) \) is true. That is, (i) \( \mathfrak{B} \in S \), (ii) \( \mathfrak{B} \subseteq \mathfrak{A} \), (iii) \( |\mathfrak{B}| \leq \theta_{(S, FO)}(r) \) and (iv) \( \mathfrak{B} \equiv_{r, FO} \mathfrak{A} \).

We now show that there exists a function \( \theta_{(\Xi(S), FO)} : N \to N \) such that \( FO-EBSP-condition(\Xi(S), \Xi(\mathfrak{A}), \Xi(\mathfrak{B}), \mathfrak{B}, m, \theta_{(\Xi(S), FO)}) \) is true. This would show \( FO-EBSP(\Xi(S)) \) is true.

1. \( \Xi(\mathfrak{B}) \in \Xi(S) \): Obvious from the definition of \( \Xi(S) \) and the fact that \( \mathfrak{B} \in S \).
2. $\Xi(\mathfrak{B}) \subseteq \Xi(\mathfrak{A})$: Follows from Lemma 5.13

3. $\Xi(\mathfrak{B}) \equiv_{m, FO} \Xi(\mathfrak{A})$: Since $\mathfrak{B} \equiv_{r, FO} \mathfrak{A}$, it follows from Lemma 5.12 that $\Xi(\mathfrak{B}) \equiv_{m, FO} \Xi(\mathfrak{A})$.

4. The existence of a function $\theta(\Xi(S), FO): \mathbb{N} \rightarrow \mathbb{N}$ such that $|\Xi(\mathfrak{B})| \leq \theta(\Xi(S), FO)(m)$:

Define $\theta(S, FO): \mathbb{N} \rightarrow \mathbb{N}$ as $\theta(S, FO)(m) = (\theta(S, FO)(t \cdot m))^t$. Since $|\mathfrak{B}| \leq \theta(S, FO)(t \cdot m)$, we have that $|\Xi(\mathfrak{B})| \leq \theta(\Xi(S), FO)(m)$.

It is clear that if $\theta(S, FO)$ is computable/elementary, then so is $\theta(\Xi(S), FO)$.

Discussion. Theorems 5.1, 5.5, 5.7 and 4.2 jointly show that the various posets and graph classes described in this section admit linear time f.p.t. algorithms for $MC(\mathcal{L}, \cdot)$, provided an $\mathcal{L}$-good tree representation of the input structure is given. In the case of words, the various kinds of trees, nested words, the class of cographs, we can indeed even construct the $\mathcal{L}$-good tree representation in quadratic time from a standard presentation of structures in these classes (this is easy to see for the first three kinds of classes; for the case of cographs, see [9]). Whereby, these classes admit quadratic time f.p.t. algorithms for $MC(\mathcal{L}, \cdot)$. The quadratic time is because we have assumed our tree-representations to be poset trees, which are required to be transitive. The graph theoretic directed trees underlying the poset trees (which are the Hasse diagrams of the poset trees) are actually constructible in linear time for each of the cases of words, the various kinds of trees, nested words and the class of cographs. One can see that the techniques that we use to get linear time f.p.t. algorithms for the aforesaid classes, given $\mathcal{L}$-good tree representations for structures in these classes, can be adapted to work even when the structures in these classes are represented using the graph theoretic (Hasse diagram) tree representations just mentioned. This indeed then enables getting linear time f.p.t. algorithms $MC(\mathcal{L}, \cdot)$ for the case of words, various kinds of trees, nested words, and cographs, thereby matching known f.p.t. results concerning these classes [11, 4, 12]. Going further, to the best of our knowledge, the f.p.t. results for $n$-partite cographs and those for classes generated using trees of quantifier-free operations, that are entailed by Theorems 5.5, 5.7 and 4.2, are new. Our proofs can then be seen as giving a different and unified technique to show existing f.p.t. results, in addition to giving new results. We mention however that if the dependence on the parameter in our f.p.t. algorithms is also considered, then our results (which give only computable parameter dependence) are weaker than those in [12] which show that for classes of bounded tree-depth/$\mathcal{SC}$-depth/shrub-depth, there are linear time f.p.t. algorithms for $\mathcal{L}$ model checking, that have elementary parameter dependence.

6 Logical fractals

We define a strengthening of the notion of $\mathcal{L}$-EBSP – instead of asserting “logical self-similarity” just at “small scales”, we assert the same “for all scales” for a suitable notion of scale. To present the formal definition, we say a function $f : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ is a scale function if it is strictly increasing. The $i^{th}$ scale, denoted $(i)_f$, is defined as the interval $[1, f(1)] = \{ j \mid 1 \leq j \leq f(i) \}$ if $i = 1$, and $[f(i - 1) + 1, f(i)] = \{ j \mid f(i - 1) + 1 \leq j \leq f(i) \}$ if $i > 1$.

Definition 6.1 (Logical fractal). Given a class $\mathcal{S}$ of structures and a logic $\mathcal{L}$ that is FO or MSO, we say $\mathcal{S}$ is an $\mathcal{L}$-fractal, if there exists a function $\theta(\mathcal{S}, \mathcal{L}) : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ such that (i) $\theta(\mathcal{S}, \mathcal{L})(m)$ is a scale function for all $m \in \mathbb{N}$, and (ii) for each structure $\mathfrak{A}$ of $\mathcal{S}$ and each $m \in \mathbb{N}$, if $f$ is the function $\theta(\mathcal{S}, \mathcal{L})(m)$ and $|\mathfrak{A}| \in (i)_f$ for some $i \in \mathbb{N}$, then for all $i < j$, there exists a substructure $\mathfrak{B}$ of $\mathfrak{A}$ in $\mathcal{S}$, such that $|\mathfrak{B}| \in (j)_f$ and $\mathfrak{B} \equiv_{m, \mathcal{L}} \mathfrak{A}$. We say $\theta(\mathcal{S}, \mathcal{L})$ is a witness to the $\mathcal{L}$-fractal property of $\mathcal{S}$.
Towards the central result of this section, we first show the following result.

**Lemma 6.2.** Let $S$ be a class of structures that admits an $L$-good tree representation $Str : T \to S$. Then there exists a strictly increasing computable function $\eta : \mathbb{N} \to \mathbb{N}$ such that for each $m \in \mathbb{N}$ and for each tree $t \in T$ of size $\geq \eta(m)$, there exists a proper subtree $s$ of $t$ in $T$ such that (i) $\text{Str}(s) \Rightarrow \text{Str}(t)$, (ii) $\text{Str}(s) \equiv_{m,\mathcal{C}} \text{Str}(t)$, and (iii) $|t| - |s| \leq \eta(m)$.

**Proof.** The proof is very much along the lines of the proof of Lemma 4.3. Let $\text{Str} : T \to S$ be an $\mathcal{L}$-good tree representation for $S$, where $T$ is a class of $(\Sigma_{\text{leaf}}\cup \Sigma_{\text{int}})$-trees that is representation feasible for $(\Sigma_{\text{rank}}, \rho)$. Let $m_0 \in \mathbb{N}$ be a witness to the composition property of $\text{Str}$, as mentioned in Definition 4.1. Let $\eta_1, \eta_2 : \mathbb{N} \to \mathbb{N}$ be defined as follows: For $l \in \mathbb{N}$, $\eta_1(l) = \max\{\rho(\sigma) : \sigma \in \Sigma_{\text{int}} \times \Delta_{\mathcal{L},\mathcal{C}}(\max\{l, m_0\})\}$ and $\eta_2(l) = 1 + \Delta_{\mathcal{L},\mathcal{C}}(\max\{l, m_0\}) \cdot \eta_1(l)^{\rho(\sigma) + 1}$. Clearly $\eta_1$ is strictly increasing and computable.

Let $b$ be a node of $t$ with the properties mentioned below.

- $b$ is a “closest-to-a-leaf” node of $t$ whose degree $\geq \eta_1(m)$. In other words, every node in $t_{\geq \eta_1(m)}$ has degree $\geq \eta_1(m)$.
- For each child $c$ of $b$ in $t$, the subtree $t_{\geq c}$ has height $\leq \eta_2(m)$.

We have the following two cases. Let $m_1 = \max\{m_0, m\}$.

1. The node $b$ exists: We then perform a “degree reduction” just as in Lemma 4.3.1. Let $\sigma$ be the label of $\text{root}(t_{\geq b})$. Since degree of $b$ is $\geq \eta_1(m)$, we have $\sigma \in \Sigma_{\text{int}} \setminus \Sigma_{\text{rank}}$. Let $z = t_{\geq b}$ and let $a_1, \ldots, a_n$ be the (ascending) sequence of children of $b$ in $t$. For $d = \rho(\sigma)$, let $n = r + q \cdot (d - 1)$ for $1 < r < d$ and $q > 1$.

For $k \in I = \{r + j \cdot (d - 1) : 0 \leq j < q\}$, let $x_{1,k}, \sigma, y_{k+1,n}$, be the subtree of $z$ obtained from $z$ by deleting the subtrees rooted at $a_{k+1}, \ldots, a_n$, resp. deleting the subtrees rooted at $a_1, a_2, \ldots, a_k$. Then $z = x_{1,n} \equiv_{m_1,\mathcal{C}} z_{1,k} \rightarrow y_{k+1,n}$ for all $k \in I$. Define $g : I \to \Delta_{\mathcal{L},\mathcal{C},m_1}$ such that $g(k)$ is the $\equiv_{m_1,\mathcal{C}}$ class of $\text{Str}(x_{1,k})$ for $k \in I$.

Since $n > \eta_1(m)$, there exist $i, j \in I$ such that $i < j$, $j - i \leq \Delta_{\mathcal{L},\mathcal{C}}(m_1)$ and $g(i) = g(j) = g(j), i.e. \text{Str}(x_{1,i}) \equiv_{m_1,\mathcal{C}} \text{Str}(x_{1,j})$. Let $z_1$ be the subtree of $z$ obtained by deleting the subtrees of $z$ that are rooted at $a_{i+1}, \ldots, a_j$. By a similar reasoning as in the proof of Lemma 4.3.1, we see that if $s = t[z \rightarrow z_1]$, then $s$ is a proper subtree of $t$ in $T$ such that $\text{Str}(s) \Rightarrow \text{Str}(t)$ and $\text{Str}(s) \equiv_{m_1,\mathcal{C}} \text{Str}(t)$, whereby $\text{Str}(s) \equiv_{m_1,\mathcal{C}} \text{Str}(t)$ (since $m_1 \geq m$). Finally, since for each $l \in \{1, 2, \ldots, j\}$, the subtree of $z$ rooted at $a_l$ has degree $\leq \eta_1(m)$ and height $\leq \eta_2(m)$, we have that $|t| - |s|$ is at most $(j - i) \cdot \eta_1(m)^{\rho(\sigma(m_1)) + 1} \leq \eta(m)$.

2. The node $b$ does not exist: Then there exists some node $c$ of $t$ such that $t_{\geq c}$ has degree $\geq \eta_1(m)$ and height $\eta_2(m) + 1$. This can be seen as follows. Either there is no node in $t$ of degree $\geq \eta_1(m)$ in which case the size of $t$ being $\geq \eta(m)$ (as assumed in the statement of the lemma) itself implies the existence of node $c$ as aforementioned. Else there is a node in $t$ of degree $\geq \eta_1(m)$, whereby there is a closest-to-a-leaf such node, call it $d$. Then some child $d_1$ of $d$ must be such that $t_{\geq d_1}$ has height $\eta_2(m)$ (for otherwise $d$ can be taken to be node $b$ which we have assumed does not exist). Then the aforementioned node $c$ can be found in $t_{\geq d_1}$.

We now perform a “height reduction” as in Lemma 4.3.2. Let $A$ be the set of nodes appearing on a path of length $\eta_2(m) + 1$ from $c$ to some leaf of $t$. Consider the function $h : A \to \Delta_{\mathcal{L},\mathcal{C},m}$ such that for each $a \in A$, $h(a) = \delta$ where $\delta$ is the $\equiv_{m_1,\mathcal{C}}$ class of $\text{Str}(t_{\geq \delta})$. Since $|A| > \eta_2(m)$, there exist distinct nodes $a, \delta \in A$ such that $a$ is an ancestor of $\epsilon$ in $t$ and $h(\epsilon) = h(\delta)$. Let $s = t[t_{\geq a} \rightarrow t_{\geq c}]$. By a similar reasoning as in the proof of Lemma 4.3.2, we can see that (i) $s$ is a subtree of $t$ in $T$, (ii) $\text{Str}(s) \Rightarrow \text{Str}(t)$ and (iii) $\text{Str}(s) \equiv_{m_1,\mathcal{C}} \text{Str}(t)$, whereby $\text{Str}(s) \equiv_{m_1,\mathcal{C}} \text{Str}(t)$ (since $m_1 \geq m$). Since $a$ is a descendant of $c$, and since $t_{\geq a}$ has height $\eta_2(m) + 1$ and degree $\leq \eta_1(m)$, the height and degree of $t_{\geq a}$ are resp. at most $\eta_2(m) + 1$ and $\eta_1(m)$, whereby the size of $t_{\geq a}$ is $\leq \eta_1(m)^{\rho(\sigma(m_1)) + 2} \leq \eta(m)$. Clearly then $|t| - |s| \leq \eta(m)$.

32
We mention below two such directions that we find challenging:

We presented a natural finitary analogue of the well-studied downward L"owenheim-Skolem

Whereby, either $|\langle \text{Str}(t) \rangle - |\text{Str}(s)\rangle| \leq \beta(n)$. In such a case, we say $S$ admits an $\mathcal{L}$-great tree representation. The central result of this section can now be stated as below.

**Proposition 6.3.** If $S$ admits an $\mathcal{L}$-great tree representation, then $S$ is an $\mathcal{L}$-fractal.

**Proof.** Let $\text{Str} : T \rightarrow S$ be an $\mathcal{L}$-great representation for $S$. Then $\text{Str}$ is $\mathcal{L}$-good, whereby by Lemma 6.2 there exists a function $\eta$ satisfying the properties mentioned in the lemma. For $m \in \mathbb{N}$, define $f(m) = \max\{|\text{Str}(t)| \mid |t| \leq \eta(m)\}$. Since $\text{Str}$ is $\mathcal{L}$-great, there exists a function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the properties mentioned in the definition of $\mathcal{L}$-greatness. Then define the function $\theta_{\langle \mathcal{L}, \mathcal{L} \rangle} : \mathbb{N}_+^2 \rightarrow \mathbb{N}_+$ as $\theta_{\langle \mathcal{L}, \mathcal{L} \rangle}(m)(n) = f(m) + (n-1) \cdot \beta(\eta(m))$. It is easily seen that $\theta_{\langle \mathcal{L}, \mathcal{L} \rangle}(m)$ is a scale function. Consider $\mathfrak{A} \in S$ and $m \in \mathbb{N}$, and suppose that $|\mathfrak{A}| \in \langle i \rangle_g$ where $g$ is the function $\theta_{\langle \mathcal{L}, \mathcal{L} \rangle}(m)$ and $i > 1$. To show that for $j < i$, there exists a substructure $\mathfrak{B}$ of $\mathfrak{A}$ in $S$ such that $|\mathfrak{B}| \in \langle j \rangle_g$ and $\mathfrak{B} \equiv_{m, \mathcal{L}} \mathfrak{A}$, we observe that it suffices to show the same simply for $j = i - 1$. Let $t \in T$ be such that $\text{Str}(t) = \mathfrak{A}$. By Lemma 6.2 there exists a subtree $s$ of $t$ in $T$ such that $|s| \subseteq \text{Str}(t)$, $|s| \equiv_{m, \mathcal{L}} \text{Str}(t)$ and $|t| - |s| \leq \eta(m)$. Since $\text{Str}$ is $\mathcal{L}$-great, it follows that $|\text{Str}(t)| - |\text{Str}(s)| \leq \beta(\eta(m))$. Whereby, either $|\text{Str}(s)| \in \langle i - 1 \rangle_g$ or $|\text{Str}(s)| \in \langle i \rangle_g$. If the former holds, then taking $\mathfrak{B} = \text{Str}(s)$, we are done. If the latter holds, then we apply Lemma 6.2 recursively to $s$ till eventually we get a subtree $x$ of $t$ in $T$ such that $|x| \subseteq \text{Str}(t)$, $|x| \equiv_{m, \mathcal{L}} \text{Str}(t)$ and $|\text{Str}(x)| \in \langle i - 1 \rangle_g$. Then taking $\mathfrak{B} = \text{Str}(x)$, we are done. 

Indeed, the diverse spectrum of posets and graphs, including those constructed using several quantifier-free operations, as seen in Section 5 admit $\mathcal{L}$-great tree representations, whereby they are all logical fractals. The logical fractal property thus appears to be a natural property that arises in a variety of interesting settings of computer science.

**7 Conclusion**

We presented a natural finitary analogue of the well-studied downward L"owenheim-Skolem property from classical model theory, denoted $\mathcal{L}$-EBSP, and showed that this property is enjoyed by various classes of interest in computer science, whereby all these classes can be seen to admit a natural finitary version of the downward L"owenheim-Skolem theorem. The aforesaid classes further admit linear time f.p.t. algorithms for FO and MSO model checking, when the structures in the classes are presented using their natural tree representations. Finally, the aforesaid classes possess a fractal like property, one based on logic. These observations open up several interesting directions for future work. We mention below two such directions that we find challenging:

1. Under what conditions on a class of structures, is it the case that the index of the $\equiv_{m, \mathcal{L}}$ relation over the class is an elementary function of $m$? Investigating this question for classes that admit elementary $\mathcal{L}$-good tree representations might yield insights for getting linear time f.p.t. algorithms for FO and MSO model checking over these classes, that have elementary parameter dependence.

2. The $\mathcal{L}$-EBSP classes (resp. logical fractals) we have identified are structurally defined. This motivates asking the converse, and hence the following: Is there a structural characterization of $\mathcal{L}$-EBSP (resp. of logical fractals)? We believe that an answer to this question, even under reasonable assumptions, would yield new classes that are well-behaved from both the logical and the algorithmic perspectives.

**Acknowledgments:** I express my deepest gratitude to Bharat Adsul for various insightful discussions and critical feedback that have helped in preparing this article.
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A Proof of composition lemma for ordered trees

We prove the composition lemma for ordered trees as given by Lemma 3.3.

Proof of Lemma 3.3. We present the result for \( L = \text{MSO} \). A similar proof can be done
for \( L = \text{FO} \). Without loss of generality, we assume \( t_1 \) and \( s_1 \) have disjoint sets of nodes
for \( i \in \{1, 2\} \). We show the result for part (1) of Lemma 3.3. The other parts can be
proved similarly. Let \( z_i = (t_i; \cap s_i) \) for \( i \in \{1, 2\} \).

Let \( \beta_1 \) be the winning strategy of the duplicator in the \( m \) round MSO-\( \text{EF} \) game
between \((t_1, a_1)\) and \((t_2, a_2)\). Let \( \beta_2 \) be the winning strategy of the duplicator in the \( m \) round
MSO-\( \text{EF} \) game between \( s_1 \) and \( s_2 \). Observe that since \( m \geq 2 \), we can assume that \( \beta_2 \) is
such that if in any round, the spoiler picks root\( (s_1) \) (resp. root\( (s_2) \)), then \( \beta_2 \) will require
the duplicator to pick root\( (s_2) \) (resp. root\( (s_1) \)). We use this observation later on. The
strategy \( \alpha \) of the duplicator in the \( m \)-round MSO-\( \text{EF} \) game between \((z_1, a_1)\) and \((z_2, a_2)\)
is defined as follows:

1. Point move: (i) If the spoiler picks an element of \( t_1 \) (resp. \( t_2 \)), the duplicator picks
the element of \( t_2 \) (resp. \( t_1 \)) given by \( \beta_1 \). (ii) If the spoiler picks an element of \( s_1 \)
(resp. \( s_2 \)), the duplicator picks the element of \( s_2 \) (resp. \( s_1 \)) given by \( \beta_2 \).
2. Set move: If the spoiler picks a set \( X \) from \( z_1 \), then let \( X = Y_1 \cup Y_2 \) where \( Y_1 \) is
a set of elements of \( t_1 \) and \( Y_2 \) is a set of elements of \( s_1 \). Let \( Y'_1 \) and \( Y'_2 \)
be the sets of elements of \( t_2 \) and \( s_2 \) respectively, chosen according to strategies \( \beta_1 \) and \( \beta_2 \).
Then in the game between \((z_1, a_1)\) and \((z_2, a_2)\), the duplicator responds with the
set \( X' = Y'_1 \cup Y'_2 \). A similar choice of set is made by the duplicator from \( z_1 \) when
the spoiler chooses a set from \( z_2 \).

We now show that the strategy \( \alpha \) is winning for the duplicator in the \( m \)-round MSO-\( \text{EF} \)
game between \((z_1, a_1)\) and \((z_2, a_2)\).

Let at the end of \( m \) rounds, the vertices and sets chosen from \( z_1 \), resp. \( z_2 \), be \( e_1, \ldots, e_p \)
and \( E_1, \ldots, E_r \), resp. \( f_1, \ldots, f_q \) and \( F_1, \ldots, F_t \), where \( p + r = m \). For \( l \in \{1, \ldots, r\} \),
let \( E_l \) resp. \( E^*_l \) be the intersection of \( E_l \) with the nodes of \( t_1 \), resp. nodes of \( s_1 \),
and likewise, let \( F_l \), resp. \( F^*_l \) be the intersection of \( F_l \) with the nodes of \( t_2 \), resp. nodes of \( s_2 \).

Firstly, it is straightforward to verify that the labels of \( e_i \) and \( f_i \) are the same for all
\( i \in \{1, \ldots, p\} \), and that for \( l \in \{1, \ldots, r\} \), \( e_i \) is in \( E^*_l \) resp. \( E^*_l \), iff \( f_i \) is in \( F^*_l \), resp.
\( F^*_l \), whereby \( e_i \in E_l \) iff \( f_i \in F_l \). For \( 1 \leq i, j \leq p \), if \( e_i \) and \( e_j \) both belong to \( t_1 \) or
both belong to \( s_1 \), then it is clear from the strategy \( \alpha \) described above, that \( f_i \) and \( f_j \)
both belong resp. to \( t_2 \) or both belong to \( s_2 \). It is easy to verify from the description
of \( \alpha \) that for every binary relation (namely, the ancestor-descendent-order \( \preceq \), and the
ordering-on-the-children-order \( \preceq_y \), the pair \((e_i, e_j)\) is in the binary relation in \( z_1 \) iff \((f_i, f_j)\)
is in that binary relation in \( z_2 \). Consider the case when without loss of generality, \( e_1 \in t_1 \)
and \( e_2 \in s_1 \). Then \( f_1 \in t_2 \) and \( f_2 \in s_2 \). We have the following cases. Assume that the
ordered tree underlying \( z_i \) is \(((A_i, \preceq_x), \preceq_y) \) for \( i \in \{1, 2\} \).

1. \( e_1 \preceq_x e_2 = \text{root}(s_1) \): Then we see that \( f_1 \preceq_x f_2 \) and \( f_2 = \text{root}(s_2) \). Observe
that \( f_2 \) must be \( \text{root}(s_2) \) by the property of \( \beta_2 \) stated at the outset. Whereby
\( e_1 \preceq_x e_2 \) and \( f_1 \preceq_x f_2 \). Likewise \( e_1 \preceq_x e_2, e_2 \preceq_x e_1 \) and \( f_2 \preceq_x f_2, f_2 \preceq_x f_1 \).
2. \( e_1 \preceq_x e_2 \neq \text{root}(s_1) \): Then we see that \( f_1 \preceq_x f_2 \) and \( f_2 \neq \text{root}(s_2) \) (again by
the property of \( \beta_2 \) stated at the outset). Whereby \( e_1 \preceq_x e_2, e_2 \preceq_x e_1 \) and \( f_1 \preceq_x f_2, f_2 \preceq_x f_1 \).
3. \( a_1 \preceq_x e_1, a_1 \neq e_1 \) and \( e_2 = \text{root}(s_1) \): Then we see that \( a_2 \preceq_x f_1, a_2 \neq f_1 \) and
\( f_2 = \text{root}(s_2) \). Observe that \( f_2 \) must be \( \text{root}(s_2) \) by the property of \( \beta_2 \) stated at the
outset. Whereby \( e_2 \preceq_x e_1 \) and \( f_2 \preceq_x f_1 \). Likewise \( e_1 \preceq_x e_2, e_2 \preceq_x e_1 \) and \( f_1 \preceq_x f_2, f_2 \preceq_x f_1 \).
4. \( a_1 \preceq_x e_1, a_1 \neq e_1 \) and \( e_2 \neq \text{root}(s_1) \): Then we see that \( a_2 \preceq_x f_1, a_2 \neq f_1 \) and
\( f_2 \neq \text{root}(s_2) \) (again by the property of \( \beta_2 \) stated at the outset). Whereby \( e_1 \preceq_x e_2, e_2 \preceq_x e_1 \) and \( f_1 \preceq_x f_2, f_2 \preceq_x f_1 \).

36
$e_2 \not\leq_1 e_1$ and $f_1 \not\leq_2 f_2$, $f_2 \not\leq_2 f_1$. Likewise, $e_1 \not\leq_1 e_2$, $e_2 \not\leq_1 e_1$ and $f_1 \not\leq_2 f_2$, $f_2 \not\leq_2 f_1$.

5. $e_1 \neq a_1, e_1 \leq_1 a_1$: Then $f_1 \neq a_1, f_1 \leq_2 a_2$. Whereby $e_1 \leq_1 e_2$ and $f_1 \leq_2 f_2$. This is because $e_1 \leq_1 c_1$ and $f_1 \leq_2 c_2$ where $c_1$ and $c_2$ are resp. the parents of $a_1$ and $a_2$ in $z_1$ and $z_2$. Also $e_1 \not\leq_1 e_2$, $e_2 \not\leq_1 e_1$ and $f_1 \not\leq_2 f_2$, $f_2 \not\leq_2 f_1$.

6. $e_1$ and $e_2$ are not related by $\leq_1$ or $\leq_1$: Then $f_1$ and $f_2$ are also not related by $\leq_2$ or $\leq_2$.

In all cases, we have that the pair $(e_i, e_j)$ is in $\leq_1$ (resp. $\leq_1$) iff $(f_i, f_j)$ is in $\leq_2$ (resp. $\leq_2$).