Generalized Local Fractional Taylor’s Formula with Local Fractional Derivative

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Abstract – In the present paper, a generalized local Taylor formula with the local fractional derivatives (LFDs) is proposed based on the local fractional calculus (LFC). From the fractal geometry point of view, the theory of local fractional integrals and derivatives has been dealt with fractal and continuously non-differentiable functions, and has been successfully applied in engineering problems. It points out the proof of the generalized local fractional Taylor formula, and is devoted to the applications of the generalized local fractional Taylor formula to the generalized local fractional series and the approximation of functions. Finally, it is shown that local fractional Taylor series of the Mittag-Leffler type function is discussed.

Keywords – Local fractional integrals; Local fractional derivatives; Fractal geometry; Mittag-Leffler function; The generalized local fractional Taylor’s formula

1. Introduction

The local fractional Taylor formula has been generalized by many authors. Kolwankar and Gangal had already written a classically formal version of the local fractional Taylor series [1, 2]

\[ f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(y)}{\Gamma(1+n)}(x-y)^n + \frac{D^\alpha f(y)}{\Gamma(1+\alpha)}(x-y)^\alpha + R_\alpha(x,y) \]  

(1.1)

where \( D^\alpha f(y) \) is the Kolwankar and Gangal local fractional derivatives, denoted by

\[ D^\alpha f(y) = \lim_{x \to y} \frac{d^\alpha}{[d(x-y)]^\alpha} \left[ f(x) - f(y) \right] \]  

(1.2)

and its reminder is

\[ R_\alpha(x,y) = \frac{1}{\Gamma(1+\alpha)} \int_{x-y}^{x-y} d^\alpha \left[ f(x) - f(y) \right] dt \]  

(1.3)

On the other hand, Adda and Cresson obtained the following relation [3]

\[ f(x) = f(y) + \frac{d^\alpha f(y)}{\Gamma(1+\alpha)}[\sigma(x-y)]^\alpha + R_\alpha(x,y) \]  

(1.4)

with

\[ R_\alpha(x,y) = \frac{1}{\Gamma(1+\alpha)} \int_{x-y}^{x-y} d^\alpha \left[ f(x) - f(y) \right] dt \]  

and

\[ \lim_{x \to y^\alpha} \frac{R_\alpha(x,y)}{(\sigma(x-y))^\alpha} = 0, \]

where

\[ F_\alpha(x,y) = D_\alpha \left[ \sigma(f(x) - f(y)) \right] \]  

and Adda and Cresson’s local fractional derivative is denoted by

\[ d^\alpha f(y) = \lim_{x \to y^\alpha} D_\alpha \left[ \sigma(f(x) - f(y)) \right] \]  

(1.5)

Recently, Yang and Gao proposed the generalized local fractional Taylor series to study the Newton iteration method and introduced the following generalized local fractional Taylor series [7]

\[ f(x) = \sum_{k=0}^{\infty} f^{(k\alpha)}(x_0) (x-x_0)^{k\alpha} \]  

(1.6)

with \( a < x_0 < \xi < x < b \), \( \forall x \in \{a,b\} \), and Gao-Yang-Kang local fractional derivative is denoted by [4-8]

\[ f^{(\alpha)}(x_0) = \lim_{x \to x_0^\alpha} \frac{\Delta^\alpha \left[ f(x) - f(x_0) \right]}{(x-x_0)^\alpha}, \]  

(1.7)

with \( \Delta^\alpha \left[ f(x) - f(x_0) \right] \equiv \Gamma(1+\alpha) \Delta^\alpha \left[ f(x) - f(x_0) \right] \).

Successively, the sequential local fractional derivatives is denoted by

\[ f^{(k\alpha)}(x) = D^\alpha \left[ D^\alpha \left[ \ldots D^\alpha \left[ f(x) \right] \ldots \right] \right]. \]  

(1.8)

If there exists the relation
\[ |f(x) - f(x_0)| < \varepsilon^\alpha \] (1.9)

with \( |x - x_0| < \delta \), for \( \varepsilon, \delta > 0 \) and \( \varepsilon, \delta \in \mathbb{R} \).

Then \( f(x) \) is called local fractional continuous on the interval \((a, b)\), denoted by

\[ f(x) \in C_\alpha (a, b). \] (1.10)

and sequential local fractional continuity is denoted by

\[ C_\alpha^k (a, b) \] (1.11)
or

\[ f(x) \in C_\alpha^k (a, b). \]

However, the proof of the generalized local fractional Taylor series is not given. As a pursuit of the work we give some results for generalized local fractional Taylor formula by using the generalized mean value theorem for local fractional integrals and prove it.

This paper is organized as follows: In section 2, a brief introduction of local fractional derivative and integral are given. The generalized local Taylor’s formula with local fractional derivative is investigated in section 3. Section 4 is devoted to the applications of the generalized local fractional Taylor formula to generalized local fractional series and approximation of functions. Conclusions are in section 5.

2. Preliminaries

Definition 1

Let \( f(x) \) is local fractional continuous on the interval \([a, b]\). Local fractional integral of \( f(x) \) of order \( \alpha \) in the interval \([a, b]\) is defined \([4, 6-7]\)

\[ \alpha I^\alpha_a f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^x f(t) (dt)^\alpha, \] (2.1)

where \( \Delta_t = t_{j+1} - t_j \), \( \Delta = \max \{\Delta_t, \Delta_{t_2}, \Delta_{t_3}, \ldots\} \), and \([t_j, t_{j+1}]\) for \( j = 0, \ldots, N - 1 \), \( t_0 = a, t_N = b \), is a partition of the interval \([a, b]\).

Here, it follows that

\[ \alpha I^\alpha_a f(x) = 0 \quad \text{if} \quad a = b; \] (2.2)

\[ \alpha I^\alpha_b f(x) = -\alpha I^\alpha_a f(x) \quad \text{if} \quad a < b; \] (2.3)

and

\[ \alpha I^\alpha_0 f(x) = f(x). \] (2.4)

Properties of the operator can be found in \([6]\). We only need here the following results:

For any \( f(x) \in C_\alpha (a, b) \), \( 0 < \alpha \leq 1 \), we have

\[ I^{(k\alpha)}_x f(x) = \frac{k \times \text{times}}{\Gamma(1+\alpha)} \int_s^x f^{(\alpha)}(t) (dt)^\alpha, \] (2.5)

\[ I^{(k\alpha)}_0 f(x) = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \int_s^x (t)^{\alpha+k} f^{(\alpha+k)}(t) (dt)^\alpha. \] (2.6)

For \( 0 < \alpha \leq 1 \), \( f^{(k\alpha)}(x) \in C_\alpha^k (a, b) \), then we have

\[ \left( \alpha I^{(k\alpha)}_x f(x) \right)^{(k\alpha)} = f(x), \] (2.7)

where

\[ \alpha I^{(k\alpha)}_x f(x) = \frac{k \times \text{times}}{\Gamma(1+\alpha)} \int_s^x f^{(\alpha)}(t) (dt)^\alpha, \]

and

\[ f^{(k\alpha)}(x) = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \int_s^x (t)^{\alpha+k} f^{(\alpha+k)}(t) (dt)^\alpha \] (2.8)

Theorem 1 (Mean value theorem for local fractional integrals)

Suppose that \( f(x) \in C_\alpha [a, b] \), we have \([6]\)

\[ \alpha I^{(\alpha)}_b f(x) = \frac{f(\xi) (b-a)^\alpha}{\Gamma(\alpha+1)}, \quad a < \xi < b. \] (2.9)

Theorem 2

Suppose that \( f^{(k\alpha)}(x), f^{(k\alpha)}(x) \in C_\alpha (a, b) \), for \( 0 < \alpha \leq 1 \), then we have

\[ I^{(k\alpha)}_x f^{(k\alpha)}(x) - f^{(k\alpha)}(x) = \frac{f^{(k\alpha)}(\xi) (x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)}, \] (2.10)

with \( a < x_0 < \xi < x < b \), where

\[ I^{(k\alpha)}_x f^{(k\alpha)}(x) = \frac{k \times \text{times}}{\Gamma(1+\alpha)} \int_s^x f^{(\alpha)}(t) (dt)^\alpha, \]

and

\[ f^{(k\alpha)}(x) = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \int_s^x (t)^{\alpha+k} f^{(\alpha+k)}(t) (dt)^\alpha \] (2.11)

Proof:

From (2.5) and (2.9), we have

\[ I^{(k\alpha)}_x f^{(k\alpha)}(x) = \frac{1}{\Gamma(1+\alpha)} \int_s^x f^{(\alpha)}(t) (dt)^\alpha \] (2.11)

and

\[ \alpha I^{(k\alpha)}_x f^{(k\alpha)}(x) = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \int_s^x (t)^{\alpha+k} f^{(\alpha+k)}(t) (dt)^\alpha. \] (2.12)
Suppose that
\[ f^{(k\alpha)}(x) = \int_{x_0}^{x} f^{(k\alpha)}(t) \, dt. \]
(2.13)
Successively, it follows from (2.13) that
\[ I_{x_0}^{(k\alpha)} f^{(k\alpha)}(x) = \frac{1}{\Gamma(1+k\alpha)} (x-x_0)^{k\alpha}. \]
(2.14)

Hence we have the result.

Remark. When \( k = 0 \), considering the formula
\[ I_{x_0}^{(0\alpha)} f^{(0\alpha)}(x) = f(x) \]
we have
\[ I_{x_0}^{(0\alpha)} f^{(0\alpha)}(x) = f(x). \]
(3.1)

Theorem 3 (Generalized mean value theorem for local fractional integrals)

Suppose that \( f(x) \in C_\alpha [a,b], f^{(\alpha)}(x) \in C(a,b) \), we have
\[ f(x) - f(x_0) = f^{(\alpha)}(\bar{x})(x-x_0)^{\alpha}, \]
(3.15)
with \( a < x_0 < \xi < x < b, \forall x \in (a, b) \), where
\[ f^{(k+1\alpha)}(x) = \frac{1}{\Gamma(k+1+\alpha)} f^{(k\alpha)}(x). \]

Proof. Form (2.10), we have
\[ x_0 I_{x_0}^{(k\alpha)} f^{(k\alpha)}(x) = x_0 f^{(k\alpha)}(x) \]
(2.16)

Successively, it follows from (2.2) that
\[ \sum_{k=0}^{n} I_{x_0}^{(n\alpha)} f^{(n\alpha)}(x) = f(x) \]
(2.27)

Applying (2.9) and (3.4), we have
\[ I_{x_0}^{(n+1\alpha)} f^{(n+1\alpha)}(x) = \frac{1}{\Gamma(1+(n+1)\alpha)} I_{x_0}^{(n\alpha)} f^{(n\alpha)}(x) \]
(3.5)

Combining the formulas (3.4) and (3.8) in (3.2), we have the result.

Theorem 5

Suppose that \( f^{(k+1\alpha)}(x) \in C_\alpha (a, b) \), for \( k = 0, 1, ..., n \) and \( 0 < \alpha \leq 1 \), then we have
\[ f(x) = \sum_{k=0}^{n} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x-x_0)^{k\alpha} + \frac{f^{(n+1\alpha)}(\bar{x})}{\Gamma(1+(n+1)\alpha)} (x-x_0)^{(n+1)\alpha}. \]
(3.9)
Proof. Applying (3.1), for \( a < x_0 < \xi < x < b \) and \( x_0 = 0 \), we have that

\[
  f(x) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(0)}{\Gamma(1+k\alpha)} x^{k\alpha}\tag{3.10}
\]

with \( 0 < \alpha < 1 \).

Hence, the proof of the theorem is completed.

4. Applications: The Generalized Local Fractional Series and Approximation of Functions

Theorem 6 (Generalized local fractional Taylor series)

Suppose that \( f^{[(k\alpha)\alpha]}(x) \in C_{\alpha}(a,b) \), for \( k = 0,1,...,n \) and \( 0 < \alpha \leq 1 \), then we have

\[
  f(x) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x-x_0)^{k\alpha}\tag{4.1}
\]

with \( a < x_0 < x < b \), \( \forall x \in (a,b) \), where

\[
  f^{[(k\alpha)\alpha]}(x) = \sum_{0}^{k+1 \text{ times}} D^{(\alpha)}_{x} \ ... \ D^{(\alpha)}_{x} f(x).
\]

Proof. From (3.1), taking the remainder

\[
  R_{n} = \frac{f^{((n+1)\alpha)}(\xi)x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}\tag{4.2}
\]

as \( n \to \infty \), we have the following relation

\[
  \lim_{n \to \infty} R_{n} = \frac{f^{((n+1)\alpha)}(\xi)x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} = 0.\tag{4.3}
\]

That is to say,

\[
  f(x) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x-x_0)^{k\alpha}.	ag{4.4}
\]

Therefore the theorem is proved.

Theorem 7 (Generalized local fractional Mc-Laurin’s series)

Suppose that \( f^{[(k\alpha)\alpha]}(x) \in C_{\alpha}(a,b) \), for \( k = 0,1,...,n \) and \( 0 < \alpha \leq 1 \), then we have

\[
  f(x) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(0)}{\Gamma(1+k\alpha)} x^k\alpha\tag{4.5}
\]

with \( a < x < b \), \( \forall x \in (a,b) \), where

\[
  f^{[(k\alpha)\alpha]}(x) = \sum_{0}^{k+1 \text{ times}} D^{(\alpha)}_{x} \ ... \ D^{(\alpha)}_{x} f(x).
\]

Proof. Taking \( x_0 = 0 \) in (4.1), we obtain the result.

Theorem 8 (Theorem for approximation of functions)

Suppose that \( f^{[(k\alpha)\alpha]}(x) \in C_{\alpha}(a,b) \), for \( k = 0,1,...,n \) and \( 0 < \alpha \leq 1 \), then we have

\[
  f(x) = \sum_{k=0}^{n-N} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x-x_0)^{k\alpha}\tag{4.6}
\]

with \( a < x_0 < x < b \), \( \forall x \in (a,b) \), where

\[
  f^{[(k\alpha)\alpha]}(x) = \sum_{0}^{k+1 \text{ times}} D^{(\alpha)}_{x} \ ... \ D^{(\alpha)}_{x} f(x).
\]

Furthermore, the error term \( R_{n} \) has the form

\[
  R_{n} = \frac{f^{((n+1)\alpha)}(\xi)x^{(n+1)\alpha}}{\Gamma(1+(N+1)\alpha)}\tag{4.7}
\]

Proof. The proof follows directly form (3.1).

Example

The Mittag-Leffler function [8] with fractal dimension \( \alpha \) is defined as

\[
  E_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{ak}}{\Gamma(1+k\alpha)}\tag{4.8}
\]

There exists a polynomial

\[
  E_{\alpha}(x^\alpha) = 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + ... + \frac{x^{N\alpha}}{\Gamma(1+N\alpha)}.
\]

5. Conclusions

This paper has pointed out the generalized local fractional Taylor formula with local fractional derivative. As well, we discussed local fractional Taylor’ series with local fractional derivative. The generalized local fractional Taylor series seems to look like fractional Taylor’s series with modified Riemann - Liouville derivative in the form [9]. However, the derivative of the former is described by local fractional derivative, the later is modified Riemann-Liouville derivative. The differences of them were discussed in [7, 9]. Hence, when we make use of the generalized local fractional Taylor formula with local fractional derivative, it is important to
defer from them. For more details of the theory and applications of local fractional calculus, see [10-17].

References

[1] K. M. Kolwankar, A. D. Gangal, Fractional differentiability of nowhere differentiable functions and dimensions, Chaos, 6 (4) (1996) 505–513.
[2] K. M. Kolwankar, A.D.Gangal, Local Fractional Fokker–Planck Equation, Phys. Rev. Lett., 80 (1998) 214–217.
[3] F. B. Adda, J. Cresson, About Non-differentiable Functions, J. Math. Anal. Appl., 263 (2001) 721–737.
[4] F. Gao, X. J. Yang, Z. X. Kang, Local Fractional Newton’s Method Derived from Modified Local Fractional Calculus. In: Proc. of the second Scientific and Engineering Computing Symposium on Computational Sciences and Optimization (CSO 2009), pp. 228–232, 2009.
[5] X. J. Yang, F. Gao. The Fundamentals of Local Fractional Derivative of the One-variable Non-differentiable Functions. World Sci-Tech R&D., 31(5) (2009) 920-921.
[6] X. J. Yang, L. Li, R. Yang, Problems of Local Fractional Definite Integral of the One-variable Non-differentiable Function, World Sci-Tech R&D., 31(4) (2009) 722-724.
[7] X. J. Yang, F. Gao. Fundamentals of Local Fractional Iteration of the Continuously Nondifferentiable Functions Derived from Local Fractional Calculus. Communications in Computer and Information Science, 153(1) (2011) 398-404.
[8] X.J. Yang, Z. X. Kang, C. H. Liu. Local Fractional Fourier’s Transform Based on the Local Fractional Calculus, In: Proc. of The International Conference on Electrical and Control Engineering (ICECE 2010), pp. 1242-1245, 2010.
[9] G. Jumarie. From Self-similarity to Fractional Derivative of Non-differentiable Functions via Mittag-Leffler Function. Appl. Math. Sci.,2 (40) (2008) 1949-1962.
[10] X. J. Yang, Research on Fractal Mathematics and Some Applications in Mechanics, M.S. thesis, China University of Mining and Technology, 2009.
[11] X. J. Yang, Local Fractional Functional Analysis and Its Applications, Asian Academic publisher Limited, Hong Kong, 2011.
[12] X. J. Yang, Local Fractional Integral Transforms, Progr. in Nonlinear Sci., 4 (2011) 1-225.
[13] W. P. Zhong, F. Gao, X.M. Shen, Applications of Yang-Fourier Transform to Local Fractional Equations with Local Fractional Derivative and Local Fractional Integral, Adv. Mat. Res., 461 (2012) 306-310.
[14] Y. Guo, Local Fractional Z Transform in Fractal Space, Advances in Digital Multimedia, 1(2) (2012) 96-102.
[15] G. S. Chen, Mean Value Theorems for Local Fractional Integrals on Fractal Space, Advances in Mechanical Engineering and its Applications, 1(1) (2012) 5-8.
[16] G. S. Chen, The Local Fractional Stieltjes Transform in Fractal Space, Advances in Intelligent Transportation Systems, 1(1) (2012) 29-31.
[17] G. S. Chen, Local Fractional Mellin Transform in Fractal Space, Advances in Electrical Engineering Systems, 1(1) (2012) 89-94.