Multispecies quantum Hurwitz numbers

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Abstract

The construction of hypergeometric 2D Toda $\tau$-functions as generating functions for quantum Hurwitz numbers is extended to multispecies families. Both the enumerative geometrical significance of multispecies quantum Hurwitz numbers as weighted enumerations of branched coverings of the Riemann sphere and their combinatorial significance in terms of weighted paths in the Cayley graph of $S_n$ are derived.

1 Introduction

In [4, 5] a general method was developed for constructing parametric families of 2D Toda $\tau$-functions [16, 18, 17] of hypergeometric type [15] that are generating functions for weighted enumeration of $n$-sheeted branched coverings of the Riemann sphere. These weighted Hurwitz numbers were shown to be interpretable equivalently as weighted enumeration of paths in the Cayley graph of the symmetric group $S_n$ generated by transpositions. In [8] this was extended, for the case of signed enumeration of paths, to multispecies enumeration of coverings, with uniform weighting.

The notion of 2D Toda $\tau$-functions of generalized hypergeometric type derives its name from the fact that, when the flow variables are restricted to the trace invariants of a pair of $N \times N$ matrices, they may be understood as hypergeometric functions of matrix arguments [3]. For the uniformly weighted case (normalized by the inverse of the order of the automorphism group), the combinatorial significance of the coefficients in their double power sum symmetric function expansion was explained in [4] in terms of path counting in the Cayley graph that are either strictly or weakly monotonically increasing sequences of transpositions having given lengths, and extended in [8] to the union of monotonic subsequences. This was shown equivalent, in geometrical terms, to the (signed) enumeration of branched covers of the Riemann sphere with multispecies branch points constrained to have fixed total ramification index within each species.

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In the following, the basic notions regarding Hurwitz numbers will be recalled, and the construction of weighted Hurwitz numbers using infinite parameter families of weight generating functions $G(z)$, as developed in [4, 5, 8] will be reviewed. Both the geometric and combinatorial significance of these weighted Hurwitz numbers are recalled and related. Restricting the general construction developed in [4, 5] to the case when the weight generating function is chosen as a variant of the quantum dilogarithm, provides the generating function for (single species) quantum Hurwitz numbers.

In Section 2 the single parameter quantum weighting will be extended multiplicatively, as in the case of signed counting [8], to the multiparameter expansions. This provides a multiparametric family of 2D Toda $\tau$-functions of hypergeometric type that are generating functions for multispecies quantum Hurwitz numbers.

1.1 Combinatorial and geometric definitions of Hurwitz numbers

For a set of $k\mathbb{N}^+$ partition $(\mu^{(1)}, \ldots, \mu^{(k)})$ of $n \in \mathbb{N}^+$, let $H(\mu^{(1)}, \ldots, \mu^{(k)})$, $|\mu^{(i)}| = n$ denote the number of $n$-sheeted branched coverings of the Riemann sphere (not necessarily connected), with $k$ branch points, whose ramification profiles are given by the partitions $(\mu^{(1)}, \ldots, \mu^{(k)})$ of $n$, weighted by the inverse of the order of the automorphism group. These are the so-called (geometrically defined) Hurwitz numbers. The genus $g$ of the covering surface is determined by the Riemann Hurwitz formula, which gives its Euler characteristic as

$$2 - 2g = 2n - \sum_{i=1}^{k} \ell^*(\mu^{(i)})$$

where

$$\ell^*(\mu) := |\mu| - \ell(\mu)$$

is the colength of the partition $\mu$; i.e., the complement of the length $\ell(\mu)$ with respect to its weight $|\mu|$, or the degree of degeneracy of the branched cover over a point with ramification profile type $\mu$.

The Frobenius-Schur formula [10, 2] expresses these in terms of sums over irreducible group characters for $S_n$

$$H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda, |\lambda| = n} h_\lambda^{k-2} \prod_{i=1}^{k} \frac{\chi_\lambda(\mu^{(i)})}{Z_{\mu^{(i)}}},$$

where $\chi_\lambda(\mu)$ is the character of the irreducible representation of symmetry type $\lambda$ evaluated on the conjugacy class $\text{cyc}(\mu)$ consisting of elements with cycle lengths equal to the parts of the partition $\mu$,

$$h_\lambda = \det \left( \frac{1}{(\lambda_i - i + j)!} \right)^{-1}$$
is the product of the hook lengths in the Young diagram of partition \( \lambda \) and 

\[
Z_\mu = \prod_{i \in \mathbb{N}_b} i^{j_i} (j_i)! ,
\]

(1.5)

where \( j_i \) the number of parts of the partition \( \mu \) equal to \( i \), is the order of the stabilizer under conjugation of any element of the conjugacy class \( \text{cyc}(\mu) \).

There is an alternative interpretation of \( H(\mu^{(1)}, \ldots, \mu^{(k)}) \) that is purely group theoretic, or combinatorial is: it equals the number of ways in which the identity element \( I \in S_n \) may be expressed as a product of \( k \) elements belonging to the conjugacy classes of cycle type \( \{ \text{cyc}(\mu^{(i)}) \} \)

\[
I = g_1 g_2 \cdots g_k , \quad \text{where } g_i \in \text{cyc}(\mu^{(i)}).
\]

(1.6)

The two are related by noting that each such factorization may be interpreted, through the monodromy map from the fundamental group of the punctured sphere with the branch points removed into \( S_n \) determined by lifting closed loops to the covering surface.

### 1.2 Weighted geometrical Hurwitz numbers

As defined in [5], given a weight generating function \( G(z) \) expressible as an infinite product 

\[
G(z) = \prod_{i=1}^{\infty} (1 + c_i z), \quad c = (c_1, c_2, \ldots)
\]

(1.7)

the weight \( W_G(\mu^{(1)}, \ldots, \mu^{(k)}) \) assigned to a configuration of \( k+2 \) branch points with ramification profiles \( \mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu, \) is solely determined by the colengths \( \{ \ell^*(\mu^{(1)}), \ldots, \ell^*(\mu^{(k)}) \} \), and is given by evaluation of the monomial sum symmetric functions at the parameter values 

\[
W_G(\mu^{(1)}, \ldots, \mu^{(k)}) := m_\lambda(c) = \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 < \cdots < i_k} \ell^*(\mu^{(1)})_{i_\sigma(1)} \cdots \ell^*(\mu^{(k)})_{i_\sigma(k)},
\]

(1.8)

Here \( \lambda \) is defined as the partition of weight \( \ell(\lambda) = k \) whose parts are \( \{ \ell^*(\mu^{(i)}) \}_{i=1}^{\ldots,k} \).

The weighted geometrically defined double Hurwitz numbers \( H^d_G(\mu, \nu) \) for \( n \) sheeted branched coverings of the sphere, give a weighted enumeration of \( n \)-sheeted branched covers of the Riemann sphere that contain a pair of fixed branch points, say at \( (0, \infty) \), with ramification profile types given by the pair of partitions \( (\mu, \nu) \) and a further set of \( k \) branch points with ramification profiles \( (\mu^{(1)}, \ldots, \mu^{(k)}) \). They are defined to be the weighted sums 

\[
H^d_G(\mu, \nu) := \sum_{k=0}^{\infty} \sum'_{\mu^{(1)}, \ldots, \mu^{(k)}} W_G(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu),
\]

(1.9)
over all $k$-tuples of nontrivial ramification profiles with weight given by $W_G(\mu^{(1)}, \ldots, \mu^{(k)})$, satisfying the condition

$$d = \sum_{i=1}^{k} \ell^*(\mu^{(i)}) = |\lambda|. \quad (1.10)$$

The Riemann-Hirwitz formula for the genus $g$ of the covering surface is then

$$2 - 2g = \ell(\mu) + \ell(\nu) - d, \quad (1.11)$$

An alternative is to use the dual weight generating function

$$\tilde{G}(z) = \prod_{i=1}^{\infty} (1 - \tilde{c}_i z)^{-1}, \quad \tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots), \quad (1.12)$$

for which the geometrical weight $W_{\tilde{G}}(\mu^{(1)}, \ldots, \mu^{(k)})$ is given by the “forgotten” symmetric function $f_{\lambda}(\tilde{c})$

$$W_{\tilde{G}}(\mu^{(1)}, \ldots, \mu^{(k)}) := f_{\lambda}(\tilde{c}) = (-1)^{\ell^*(\lambda)} [\text{aut}(\lambda)] \sum_{\sigma} \sum_{1 \leq i_1 \leq \cdots \leq i_k} c_{\ell^*(\mu^{(1)})}^{(1)} \cdots c_{\ell^*(\mu^{(k)})}^{(k)}, \quad \tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots), \quad (1.13)$$

where the partition $\lambda$ is again defined as above, with parts consisting of the colengths $\{\ell^*(\mu^{(i)})\}_{i=1,\ldots,k}$. The dually weighted geometrical Hurwitz numbers are similarly defined by the weighted sum

$$H_{G}^d(\mu, \nu) := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)}, \ldots, \mu^{(k)} \in S_k \text{ \ s.t. } \ell^*(\mu^{(i)}) = d \ \forall i}} W_{\tilde{G}}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu), \quad (1.14)$$

### 1.3 Weighted combinatorial Hurwitz numbers

Following [4, 5], we may alternatively define a combinatorial Hurwitz number $F_{G}^d(\mu, \nu)$ that gives the weighted enumeration of $d$-step paths in the Cayley graph of the symmetric group $S_n$ generated by transpositions $(a, b)$, $b > a$ starting at an element $h \in \text{cyc } \mu$ in the conjugacy class $\text{cyc}(\nu)$ consisting of elements with cycle lengths equal to the parts of $\mu$ and ending in the conjugacy class $\text{cyc}(\nu)$

$$(a_{d}b_{d}) \ldots (a_{1}b_{1})h \in \text{cyc}(\nu). \quad (1.15)$$

Every such path has a signature $\lambda$, which is defined to be the partition of weight $d$, whose parts are, in weakly decreasing order, the number of times any given second element $b_i$, $i = 1, \ldots, \ell(\lambda)$ is repeated. In the case of the weight generating function $G(z)$, we assign
to any path with signature $\lambda$ a combinatorial weight equal to the product $e_{\lambda}(c)$ of the elementary symmetric functions [11], evaluated at the parameters $(c_a, c_2, \ldots)$

$$e_{\lambda}(c) = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}(c)$$

In the case of the dual generating functions $\tilde{G}(z)$, a combinatorial weight equal to the product $h_{\lambda}(\tilde{c})$ of the complete symmetric functions [11], evaluated at the parameters $($\tilde{c}_a, \tilde{c}_2, \ldots)$

$$h_{\lambda}(\tilde{c}) = \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}(\tilde{c})$$

Let $m_{\mu \nu}^\lambda$ be the number of $d = |\lambda|$ step paths of signature $\lambda$ starting at $h \in \text{cyc}(\mu)$ and ending in the conjugacy class $\text{cyc}(\nu)$. Then the combinatorially defined weighted Hurwitz numbers $F_d^G(\mu, \nu), F_d^{\tilde{G}}(\mu, \nu)$ are defined to be the weighted sums

$$F_d^G(\mu, \nu) := \frac{1}{n!} \sum_{\lambda} e_{\lambda}(c)m_{\mu \nu}^\lambda,$$

$$F_d^{\tilde{G}}(\mu, \nu) := \frac{1}{n!} \sum_{\lambda} h_{\lambda}(\tilde{c})m_{\mu \nu}^\lambda.$$  

In [5] it is proved that these two notions of weighted Hurwitz numbers in fact coincide:

**Theorem 1.1** (5).

$$F_d^G(\mu, \nu) = H_d^G(\mu, \nu), \quad F_d^{\tilde{G}}(\mu, \nu) = H_d^{\tilde{G}}(\mu, \nu)$$

### 1.4 Hypergeometric 2D-toda $\tau$-functions as generating functions

As shown in [5], for any given generating function of type $G(z)$ or $\tilde{G}(z)$, there is a naturally associated 2D Toda $\tau$-function of hypergeometric type, expressible as a diagonal double Schur function expansion

$$\tau_G^{G}(N, t, s) := \sum_{\lambda} r_{\lambda}^G(N)s_{\lambda}(t)s_{\lambda}(s),$$

$$\tau_{\tilde{G}}^{\tilde{G}}(N, t, s) := \sum_{\lambda} r_{\lambda}^{\tilde{G}}(N)s_{\lambda}(t)s_{\lambda}(s),$$

where

$$t = (t_1, t_2, \ldots), \quad s = (s_1, s_2, \ldots)$$

are the 2D Toda flow variables, which may be identified in this notation with the power sums

$$t_i = \frac{p_i}{i}, \quad s_i = \frac{p'_i}{i}, \quad N \in \mathbb{Z}$$
in two independent sets of variables. (See [11] for notation and further definitions involving symmetric functions.) The coefficients have the standard “content product” form that characterize such 2D \( \tau \)-functions of hypergeometric kind:

\[
r^G_\lambda(N) := r^G_0(N) \prod_{(i,j) \in \lambda} G(z(N + j - i)),
\]

where

\[
r^G_0(N) := N - 1 \prod_{j=1}^{N-1} G((N-j)z), \quad r^G_0(0) := 1, \quad r^G_0(-N) := \prod_{j=1}^{N} G((j-N)z)^{-1}, \quad N \geq 1.
\]

(1.26)

and identical formulae for \( G \) replaced by \( \tilde{G} \).

The main result of [5] is that the resulting \( \tau \)-functions (1.21), (1.22), for \( N = 0 \)

\[
\tau^G(z)(t,s) := \tau^G(z)(0,t,s), \quad \tau^{\tilde{G}}(z)(t,s) := \tau^{\tilde{G}}(0,t,s),
\]

(1.27)

when expanded in the basis of tensor products of pairs of power sum symmetric functions \( \{p_\mu\} \), are generating functions for the weighted double Hurwitz numbers.

**Theorem 1.2 ([5]).**

\[
\tau^G(z)(t,s) = \sum_{d=0}^{\infty} \sum_{|\mu| = |\nu|} z^d H^d_G(\mu, \nu)p_\mu(t)p_\nu(s),
\]

(1.28)

\[
\tau^{\tilde{G}}(t,s) = \sum_{d=0}^{\infty} \sum_{|\mu| = |\nu|} z^d H^d_{\tilde{G}}(\mu, \nu)p_\mu(t)p_\nu(s).
\]

(1.29)

### 1.5 Quantum Hurwitz numbers

Amongst the examples studied in [5], four special classes of weighted Hurwitz numbers were introduced in which the generating functions \( G(z), \tilde{G}(z) \) were chosen to depend on an auxiliary quantum parameter \( q \), by setting the parameters \( c \) in their infinite product representations equal to powers of this

\[
c_i := q^i, \quad \tilde{c}_i := q^i.
\]

(1.30)

By suitable interpretation of the parameter \( q \) in terms of Planck’s constant \( \hbar \) and Boltzmann factors for a Bosonic gas with linear energy spectrum, the resulting weighted counting of branched covers was related in [5] to that for a Bosonic gas, which further justifies terming these “quantum” Hurwitz numbers.
Recalling this construction, the quantum Hurwitz numbers introduced in [5], have four variants, for which a 1-parameter family of 2D Toda $\tau$-function generating functions were constructed. In the first case, the weight generating function is:

$$E(q, z) := \prod_{i=0}^{\infty} (1 + q^i z) = 1 + \sum_{i=1}^{\infty} E_i(q) z^i,$$

and hence the parameters $c_i$ are identified as $\{c_i := q^{i-1}\}_{i \in \mathbb{N}^+}$. The second is a slight modification of this, with weight generating function

$$E'(q, z) := \prod_{i=1}^{\infty} (1 + q^i z) = 1 + \sum_{i=1}^{\infty} E'_i(q) z^i,$$

and hence the zero power $q^0$ is omitted, and $\{c_i := q^i\}_{i \in \mathbb{N}^+}$.

The third case is based on the weight generating function

$$H(q, z) := \prod_{i=1}^{\infty} (1 - q^i z)^{-1} = 1 + \sum_{i=1}^{\infty} H_i(q) z^i,$$

and hence is the generating function dual to the first case, with $\{\tilde{c}_i := q^{i-1}\}_{i \in \mathbb{N}^+}$. The final case is a hybrid, formed from the product of the first and third for two distinct quantum deformation parameters $q$ and $p$, with weight generating function

$$Q(q, p, z) := \prod_{k=0}^{\infty} (1 + q^k z)(1 - p^k z)^{-1} = \sum_{i=0}^{\infty} Q_i(q, p) z^i,$$

and hence are all expressible as exponentials of the quantum dilogarithm function

$$\text{Li}_2(q, z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1 - q^k)},$$

$$E(q, z) = e^{-\text{Li}_2(q, -z)}, \quad E'(q, z) = (1 + z)^{-1} e^{-\text{Li}_2(q, -z)}$$

$$H(q, z) = e^{\text{Li}_2(q, z)}, \quad Q(q, p, z) = e^{\text{Li}_2(p, z) - \text{Li}_2(q, -z)}.$$
The content products formulae for the first and third of these are
\[ r^E_{\lambda}(z)(N) := \prod_{(i) \in \lambda} E(q, (N + j - i)z) \]  \hspace{1cm} (1.42)
\[ r^H_{\lambda}(z)(N) := \prod_{(i) \in \lambda} H(q, (N + j - i)z). \]  \hspace{1cm} (1.43)

The associated hypergeometric 2D Toda \( \tau \)-functions have diagonal double Schur function expansions with these as coefficients:
\[ \tau^E_{\lambda}(N, t, s) = \sum_{\lambda} r^E_{\lambda}(N) S_{\lambda}(t) S_{\lambda}(s) \] \hspace{1cm} (1.44)
\[ \tau^H_{\lambda}(N, t, s) = \sum_{\lambda} r^H_{\lambda}(N) S_{\lambda}(t) S_{\lambda}(s). \] \hspace{1cm} (1.45)

Using the Frobenius character formula [11],
\[ S_{\lambda} = \sum_{\mu, |\mu|=|\lambda|} \frac{\chi_{\lambda} P_\mu}{Z_\mu} \] \hspace{1cm} (1.46)
and setting \( N = 0 \), these may be rewritten as double expansions in the power sum symmetric functions [5]:
\[ \tau^E_{\lambda}(t, s) := \tau^E_{\lambda}(0, t, s) = \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu|=|\nu|} H^d_{E(q)}(\mu, \nu) P_\mu(t) P_\nu(s), \] \hspace{1cm} (1.47)
\[ \tau^H_{\lambda}(t, s) := \tau^H_{\lambda}(0, t, s) = \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu|=|\nu|} H^d_{H(q)}(\mu, \nu) P_\mu(t) P_\nu(s). \] \hspace{1cm} (1.48)

The coefficients are the corresponding quantum Hurwitz numbers \( H^d_{E(q)}(\mu, \nu) \), \( H^d_{H(q)}(\mu, \nu) \), which count weighted \( n \)-sheeted branched coverings of the Riemann sphere, defined by
\[ H^d_{E(q)}(\mu, \nu) := \sum_{k=0}^{\infty} \sum_{(\mu^{(1)}, \ldots, \mu^{(k)})} W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu), \] \hspace{1cm} (1.49)
\[ H^d_{H(q)}(\mu, \nu) := \sum_{k=0}^{\infty} (-1)^{k+d} \sum_{(\mu^{(1)}, \ldots, \mu^{(k)})} W_{H(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu). \] \hspace{1cm} (1.50)

where the weightings for such covers with \( k \) additional branch points are [5]:
\[ W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} q^{(k-1)\ell^*(\mu^{(1)})} \ldots q^{\ell^*(\mu^{(k-1)})} (1 - q^{\ell^*(\mu^{(\sigma(1))})}) \ldots (1 - q^{\ell^*(\mu^{(\sigma(k))})}). \] \hspace{1cm} (1.51)
\[ W_{H(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} (1 - q^{\ell^*(\mu^{(\sigma(1))})}) \ldots (1 - q^{\ell^*(\mu^{(\sigma(k))})}). \] \hspace{1cm} (1.52)
The sum \( \sum_{i=1}^k c^*(\mu^{(i)}) \) is over all \( k \)-tuples of partitions having nontrivial ramification profiles that satisfy the constraint \( \sum_{i=1}^k c^*(\mu^{(i)}) = d \), and \( H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \) is the number of branched \( n \)-sheeted coverings, up to isomorphism, having \( k + 2 \) branch points with ramification profiles \( (\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \).

These thus count weighted covers with a pair of branch points, say \((0, \infty)\), having ramification profiles of type \((\mu, \nu)\) and an arbitrary number of further branch points, whose profiles \( (\mu^{(1)}, \ldots, \mu^{(k)}) \) are constrained only by the requirement that the sum of the colengths, which is related to the genus by the Riemann-Hurwitz formula

\[
\sum_{i=1}^k \ell^*(\mu^{(i)}) = 2g - 2 + \ell(\mu) + \ell(\nu) = d,
\]

be fixed to equal \( d \).

The combinatorial interpretation of the quantum Hurwitz numbers \( F_{E(q)}^d(\mu, \nu) \) and \( F_{H(q)}^d(\mu, \nu) \) appearing in (1.48) is as follows. Let \((a_1b_1) \cdots (a_db_d)\) be a product of \( d \) transpositions \((a_ib_i) \in S_n\) in the symmetric group \( S_n \) with \( a_i < b_i, \ i = 1, \ldots, d \). If \( h \in S_n \) is in the conjugacy class \( \text{cyc}(\mu) \subset S_n \), we may view the successive steps in the product

\[
(a_1b_1) \cdots (a_db_d)h
\]

as a path in the Cayley graph generated by all transpositions, whose signature is the partition \( \lambda \) of \( d \), \( |\lambda| = d \), whose parts \( \lambda_i \) consist of the number of transpositions \((a_ib_i)\) sharing the same second element. If we further require that the ones with equal second elements be grouped together into consecutive subsequences in which these second elements are constant, with the consecutive subsequences strictly increasing in their second elements, then the number \( \tilde{N}_\lambda \) of elements with signature \( \lambda \) is related to the number \( N_\lambda \) of such ordered sequences by

\[
\tilde{N}_\lambda = \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} N_\lambda
\]

(1.55)

Denote the number of such paths from the conjugacy class of cycle type \( \text{cyc}(\mu) \) to the one of type \( \text{cyc}(\nu) \) having signature \( \lambda \) as \( \tilde{m}_\mu^\lambda \), and the number of ordered sequences of type \( \lambda \) as \( m_\mu^\lambda \). Thus

\[
\tilde{m}_\mu^\lambda = \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} m_\mu^\lambda.
\]

(1.56)

For all paths of signature \( \lambda \) we assign the weights

\[
\tilde{E}_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \lambda_i! E_{\lambda_i}(q) = \prod_{i=1}^{\ell(\lambda)} \lambda_i! q^{\lambda_i(\lambda_i-1)/2} \prod_{j=1}^\lambda (1 - q^j),
\]

(1.57)

\[
\tilde{H}_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \lambda_i! H_{\lambda_i}(q) = \prod_{i=1}^{\ell(\lambda)} \lambda_i! \prod_{j=1}^\lambda (1 - q^j)
\]

(1.58)
to paths in the Cayley graph, and obtain the pair of corresponding combinatorial weighted Hurwitz numbers
\[
F_{E(q)}^d(\mu, \nu) := \frac{1}{d!} \sum_{\lambda, |\lambda|=d} \tilde{E}_\lambda(q) \tilde{m}_\mu^\lambda, \tag{1.59}
\]
\[
F_{H(q)}^d(\mu, \nu) := \frac{1}{d!} \sum_{\lambda, |\lambda|=d} \tilde{H}_\lambda(q) \tilde{m}_\mu^\lambda, \tag{1.60}
\]
that give the weighted enumeration of paths, using the weighting factors \( \tilde{E}_\lambda(q) \) and \( \tilde{H}_\lambda(q) \) respectively for all paths of signature \( \lambda \).

As shown in general in [5], the enumerative geometrical and combinatorial definitions of these quantum weighted Hurwitz numbers coincide:
\[
H_{E(q)}^d(\mu, \nu) = F_{E(q)}^d(\mu, \nu), \quad H_{H(q)}^d(\mu, \nu) = F_{H(q)}^d(\mu, \nu). \tag{1.61}
\]
A similar result holds for weights generated by the function \( E'(q, z) \), with corresponding quantum Hurwitz numbers defined by
\[
H_{E'(q)}^d(\mu, \nu) := \sum_{k=0}^\infty \sum_{\sum_{i=1}^{\ell^*(\mu)} \ell_i = d} W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu), \tag{1.62}
\]
where the weights \( W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \) are given by
\[
W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{k!} \sum_{\sigma \in S_k} \frac{q^{(k)} \ell^* (\mu^{(1)}) \cdots q^{(k)} \ell^* (\mu^{(k)})}{(1 - q^{\ell^* (\mu^{(1)})}) \cdots (1 - q^{\ell^* (\mu^{(1)})}) \cdots q^{\ell^* (\mu^{(k)})}}, \quad \tag{1.63}
\]
\[
:= \frac{1}{k!} \sum_{\sigma \in S_k} \frac{1}{(q^{\ell^* (\mu^{(1)})} - 1) \cdots (q^{\ell^* (\mu^{(1)})} - 1)} \tag{1.64}
\]
Choosing \( q \) as a positive real number, parametrizing it as
\[
q = e^{-\beta \hbar \omega}, \quad \beta = \frac{1}{kT}, \tag{1.65}
\]
and identifying the energy levels \( \epsilon_k \) as those for a Bose gas with linear spectrum in the integers, as for a 1-D harmonic oscillator
\[
\epsilon_k := \hbar \omega, \quad k \in \mathbb{N}, \tag{1.66}
\]
we see that if we assign the energy
\[
\epsilon(\mu) := \epsilon_{\ell^* (\mu)} = \ell^* (\mu) \hbar \omega \tag{1.67}
\]
to each branch point with ramification profile of type \( \mu \), it contributes a factor
\[
n(\mu) = \frac{1}{e^{\beta \epsilon(\mu)} - 1} \tag{1.68}
\]
to the weighting distributions, the same as that for a bosonic gas.
2D Toda \( \tau \)-functions as generating functions for multispecies quantum Hurwitz numbers

2.1 The multiparameter family of \( \tau \)-functions \( \tau^{Q(q; w; p, z)}(N, t, s) \)

We now consider a multiparameter family of weight generating functions \( Q(q, w; p, z) \) obtained by taking the product of any number of the generating functions \( E(q_i, z_i) \) and \( H(p_j, w_j) \) for distinct sets of generating function parameters \( w = (w_1, \ldots, w_l) \), \( z = (z_1, \ldots, z_m) \), and quantum parameters \( q = (q_1, \ldots, q_l) \), \( p = (p_1, \ldots, p_m) \)

\[
Q(q, w; p, z) := \prod_{\alpha=1}^{l} E(q_{\alpha}, w_{\alpha}) \prod_{\beta=1}^{m} H(p_{\beta}, z_{\beta}). \tag{2.1}
\]

Following the approach developed in [5], we define an associated element of the center \( Z(C[S_n]) \) of the group algebra \( C[S_n] \) by

\[
Q_n(q, w; p, z, J) := \prod_{a=1}^{n} Q(q, J_a w; p, J_a z), \tag{2.2}
\]

where \( J := (J_1, \ldots, J_n) \) are the Jucys-Murphy elements [9, 12, 1]

\[
J_1 := 0, \quad J_b := \sum_{a=1}^{b-1}(ab), \quad b = 1, \ldots, n, \tag{2.3}
\]

which generate an abelian subalgebra of \( Z(C[S_n]) \). The element \( Q_n(q, w; p, z, J) \) defines an endomorphism of \( Z(C[S_n]) \) under multiplication, which is diagonal in the basis \( \{F_\lambda\} \) of \( Z(C[S_n]) \) consisting of the orthogonal idempotents, corresponding to irreducible representations, labelled by partitions \( \lambda \) of \( n \):

\[
Q_n(q, w; p, z, J) F_\lambda = r^Q_{\lambda}(q, w; p, z) F_\lambda \tag{2.4}
\]

where

\[
r^Q_{\lambda}(q, w; p, z) = \prod_{\alpha=1}^{l} t^E_{\lambda}(q_{\alpha}, w_{\alpha}) \prod_{\beta=1}^{m} t^H_{\lambda}(p_{\beta}, z_{\beta}). \tag{2.5}
\]

More generally, defining

\[
r^Q_{\lambda}(q, w; p, z) (N) = \prod_{\alpha=1}^{l} t^E_{\lambda}(q_{\alpha}, w_{\alpha}) (N) \prod_{\beta=1}^{m} t^H_{\lambda}(p_{\beta}, z_{\beta}) (N), \tag{2.6}
\]

where

\[
r^E_{\lambda}(q_{\alpha}, w_{\alpha}) (N) := \prod_{(ij) \in \lambda} E(q_{\alpha}, (N + j - i)w_{\alpha}) \tag{2.7}
\]

\[
r^H_{\lambda}(p_{\beta}, z_{\beta}) (N) := \prod_{(ij) \in \lambda} H(p_{\beta}, (N + j - i)z_{\beta}), \tag{2.8}
\]
we have

$$\tau^{Q(q,z,p,w)}_\lambda = \tau^{Q(q,z,p,w)}_\lambda(0).$$ (2.9)

The double Schur function series

$$\tau^{Q(q,z,p,w)}(N,t,s) := \sum_\lambda \tau^{Q(q,z,p,w)}_\lambda(N) S_\lambda(t) S_\lambda(s)$$ (2.10)

then defines a family of 2D Toda $\tau$-functions of hypergeometric type.

### 2.2 Multispecies geometric quantum Hurwitz numbers

We now consider coverings in which the branch points are divided into two different classes, \(\{\mu^{(\alpha,u_\alpha)}\}_{\alpha=1,\ldots,l; u_\alpha=1,\ldots,k_\alpha}\) and \(\{\nu^{(\beta,v_\beta)}\}_{\beta=1,\ldots,m; v_\beta=1,\ldots,k_\beta}\), corresponding to weight generating functions of type \(E(q_\alpha)\) and \(H(p_\beta)\) respectively, each of which is further divided into \(l\) and \(m\) distinct species (or “colours”), of which there are \(k_\alpha\) and \(k_\beta\) branch points of types \(E\) and \(H\) and colours \(\alpha\) and \(\beta\) respectively. The weighted number of such coverings with specified total colengths \(d = (d_1, \ldots, d_l)\), \(\bar{d} = (d_1, \ldots, d_m)\), \(d_\alpha,\bar{d}_\beta \in \mathbb{N}\)

$$d_\alpha = \sum_{u_\alpha=1}^{k_\alpha} \ell^*(\mu^{(\alpha,u_\alpha)})$$

$$\bar{d}_\beta = \sum_{v_\beta=1}^{k_\beta} \ell^*(\nu^{(\beta,v_\beta)})$$ (2.11)

for each class and colour is the multispecies quantum Hurwitz number

$$H^{(d,\bar{d})}_{(q,p)}(\mu, \nu) := \sum_{\alpha=1,\ldots,l; \beta=1,\ldots,m} \sum_{\mu^{(\alpha,u_\alpha)}; \nu^{(\beta,v_\beta)}}^{\{k_\alpha\}_{\alpha=1,\ldots,l} \{k_\beta\}_{\beta=1,\ldots,m}} \prod_{\alpha=1}^l W_{E(q_\alpha)}(\mu^{(\alpha,1)}, \ldots, \mu^{(\alpha,k_\alpha)}) \prod_{\beta=1}^m W_{H(p_\beta)}(\nu^{(\beta,1)}, \ldots, \nu^{(\beta,k_\beta)})$$

$$\times \prod_{\alpha=1}^l W_{E(q_\alpha)}(\mu^{(\alpha,1)}, \ldots, \mu^{(\alpha,k_\alpha)}) \prod_{\beta=1}^m W_{H(p_\beta)}(\nu^{(\beta,1)}, \ldots, \nu^{(\beta,k_\beta)}) H(\mu^{(\alpha,u_\alpha)}; \nu^{(\beta,v_\beta)}), \mu, \nu).$$ (2.12)

Substituting the Frobenius-Schur formula (1.3) and the Frobenius character formula (1.46) into (2.10), it follows that

$$\tau^{Q(q,z,p,w)}(t,s) := \tau^{Q(q,z,p,w)}(0, t, s)$$ (2.13)

is the generating function for \(H^{(d,\bar{d})}_{(q,p)}(\mu, \nu)\). Using multi-index notion to denote

$$\prod_{\alpha=1}^l w_\alpha^{d_\alpha} \prod_{\beta=1}^m z_\beta^{\bar{d}_\beta} := w^d z^\bar{d},$$

we have:

**Theorem 2.1.**

$$\tau^{Q(q,w,p,z)}(t,s) := \sum_{d \in \mathbb{N}} W^d \sum_{d \in \mathbb{N}} z^\bar{d} \sum_{\mu, \nu} H^{(d,\bar{d})}_{(q,p)}(\mu, \nu) P_\mu(t) P_\nu(s).$$ (2.15)
2.3 Multispecies combinatorial quantum Hurwitz numbers

Another basis for $\mathbb{Z}(\mathbb{C}[S_n])$ consists of the cycle sums

$$C_\mu := \sum_{h \in \text{cyc}(\mu)} h,$$  \hspace{1cm} (2.16)

where $\text{cyc}(\mu)$ denotes the conjugacy class of elements $h \in \text{cyc}(\mu)$ with cycle lengths equal to the parts of $\mu$. The two are related by

$$F_\lambda = h_\lambda^{-1} \sum_{\mu, |\mu|=|\lambda|} \chi_\lambda(\mu) C_\mu,$$  \hspace{1cm} (2.17)

where $\chi_\lambda(\mu)$ denotes the irreducible character of the irreducible representation of type $\lambda$ evaluated on the conjugacy class $\text{cyc}(\mu)$. Under the characteristic map, this is equivalent to the Frobenius character formula (1.46).

The Macmahon generating function

$$\prod_{i=1}^{\infty} (1 - q^i)^{-1} = \sum_{n=0}^{\infty} D_n q^n,$$  \hspace{1cm} (2.18)

gives the number $D_n$ of partitions of $n$. We denote by $F_{E(q_\alpha)}^{(n,d_\alpha)}$ and $F_{H(p_\beta)}^{(n,\tilde{d}_\beta)}$ the $D_n \times D_n$ matrices, whose elements are $\left(F_{E(q_\alpha)}^{(n,d_\alpha)}(\mu, \nu)\right)_{|\mu|=|\nu|=n}$ and $\left(F_{H(p_\beta)}^{(n,\tilde{d}_\beta)}(\mu, \nu)\right)_{|\mu|=|\nu|=n}$, respectively, for $\alpha = 1, \ldots, l$, $\beta = 1, \ldots, m$. Since these represent central elements of the group algebra $\mathbb{C}[S_n]$, they commute amongst themselves. Defining the matrix product:

$$F_{(q,p)}^{(d,d)} = \prod_{\alpha=1}^{l} F_{E(q_\alpha)}^{(n,d_\alpha)} \prod_{\beta=1}^{m} F_{H(p_\beta)}^{(n,\tilde{d}_\beta)},$$  \hspace{1cm} (2.19)

its matrix elements, denoted $F_{(q,p)}^{(c,d)}(\mu, \nu)$, may be interpreted as the weighted number of step paths in the Cayley graph from the conjugacy class of cycle type $\mu$ to the one of type $\nu$, where all paths are divided into equivalence classes, according to their multisignatures $\{\lambda^\alpha, \tilde{\lambda}^\beta\}_{\alpha=1, \ldots, l, \beta=1, \ldots, m}$.

These consist of a partition of the $d$ steps into $l + m$ parts, each of which is a subsequence assigned a “colour” and a “class” with $l$ of them of the first class and $m$ of the second. The number of partitions of first class with colour $\alpha$ is $d_\alpha$ while the number of second class with colour $\beta$ is $\tilde{d}_\beta$. The partitions $\lambda^{(\alpha)}$ of weights $d_\alpha$ are defined to have parts $\{\lambda_{u_\alpha}^{(\alpha)}\}_{u_\alpha=1, \ldots, k_\alpha}$.
equal to the number of a transposition appears within that subsequence, having the same second element, and similarly for \( \tilde{\lambda}_{v_\beta} \) with

\[
k_\alpha = \ell(\lambda^{(\alpha)}), \quad \tilde{k}_\beta = \ell(\tilde{\lambda}^{(\beta)})
\]  

(2.21)

the number of such parts.

For each such subpartition, the weight assigned is the product of the weights for each subsegment:

\[
\prod_{\alpha=1}^{l} \tilde{E}_{\lambda^{(\alpha)}}(q_\alpha, w_\alpha) \prod_{\beta=1}^{m} \tilde{H}_{\tilde{\lambda}^{(\beta)}}(p_\beta, z_\beta)
\]

(2.22)

and \( F_{(c,d)}^{(q,p)}(\mu, \nu) \) is the sum of these, each multiplied by the number of elements of the equivalence class of paths, with the given multisignature. The multispecies generalization of (1.61) is equality of the geometric and combinatorial Hurwitz numbers:

**Theorem 2.2.**

\[ F_{(q,p)}^{(c,d)}(\mu, \nu) = H_{(q,p)}^{(c,d)}(\mu, \nu). \]  

(2.23)

**Proof.** Applying the central element (2.4) to the cycle sum \( C_\mu \) gives

\[
\hat{Q}(q, z; p, w, J) C_\mu = \sum_{\nu, |\nu| = |\mu|} F_{(q,p)}^{(c,d)}(\mu, \nu) C_\nu
\]

(2.24)

and the orthogonality of group characters implies that

\[
\tau_{\tilde{Q}(q,z;p,w)}(t, s) := \sum_{\mu, \nu} F_{(q,p)}^{(c,d)}(\mu, \nu) \left( P_\mu(t) P_\nu(s) \right).
\]

(2.25)

Comparing this with eq. (2.15) proves the result.

\[ \square \]

**Remark 2.1. Multispecies Bose gases**

Returning to the interpretation of the quantum Hurwitz weighting distributions in terms of Bose gases, if we choose each \( q_i \) to be a positive real number with \( q_i < 1 \), and parametrize it, as before,

\[ q_i = e^{-\beta \hbar \omega_i} \]

(2.26)

for some ground state energy \( \hbar \omega_i \), and again choose a linear energy spectrum, with energy assigned to the branchpoint \( \mu \) of type \( i \) with profile type \( \mu \) to be

\[ \epsilon^{(i)}(\mu) := \ell^*(\mu) \hbar \omega_i \]

(2.27)

we see that the resulting contributions to the weighting distributions distributions from each species of branch points of ramification type \( \mu^{(j)} \) are given by

\[
n_B^{(i)}(\mu) = \frac{1}{e^{\beta \epsilon^{(i)}(\mu)} - 1},
\]

(2.28)

which are those of a multispecies mixture of Bose gases.
2.4 General multispecies Hurwitz numbers

The concept of multispecies weighted Hurwitz numbers is of course not restricted to the particular quantum weightings considered here. For any choice of weight generating functions $G_1(z_1), G_2(z_2), \ldots$, we may form composites by similarly using the product $\prod_i G_i(z_i)$ as generating function for multiple weighting types. The resulting content product coefficients $r_{\lambda}^{\prod_i G_i(z_i)}$ are again just the product of the individual ones

$$r_{\lambda}^{\prod_i G_i(z_i)} = \prod_i r_{\lambda}^{G_i(z_i)}.$$

(2.29)

The weighted enumerative geometrical and combinatorial interpretations remain the same as in the cases studied here, but with the individual weight generators $E(q, z_i)$ or $H(p_j, w_j)$ replaced by $G_1(z_1), G_2(z_2), \ldots$. In particular, we may include factors of the type $E'(q, z)$ or $Q(q, p, z)$ to arrive at similar formulae for the weighted Hurwitz numbers, both in the weighted enumerative geometrical and combinatorial sense, differing only in the detailed form of the various component weights.

We may also include weight factors in which some or all of the parameters $z_1, z_2 \ldots$ are repeated in the product. This only affects the constraints on the sums of the colengths in the weighted multispecies Hurwitz numbers. (See, e.g. example 3.3 in [5], in which the weights are uniform, but the linear generating function that gives Hurwitz numbers for Belyi curves and strictly monotonic paths is replaced by a power of the latter, resulting in multiple branch points, with the total colength fixed, and multimononic paths.)

General multispecies Hurwitz numbers, with arbitrary multiplicative weight generating functions, are developed in [6]. A more general class of quantum deformations of classically weighted Hurwitz numbers, including a pair of deformation parameters $(q, t)$ related to Macdonald polynomials is developed in [7].

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