Contextual Bandits with Continuous Actions: Smoothing, Zooming, and Adapting

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Abstract

We study contextual bandit learning with an abstract policy class and continuous action space. We obtain two qualitatively different regret bounds: one competes with a smoothed version of the policy class under no continuity assumptions, while the other requires standard Lipschitz assumptions. Both bounds exhibit data-dependent “zooming” behavior and, with no tuning, yield improved guarantees for benign problems. We also study adapting to unknown smoothness parameters, establishing a price-of-adaptivity and deriving optimal adaptive algorithms that require no additional information.

1 Introduction

We consider contextual bandits: a setting in which a learner repeatedly makes an action on the basis of contextual information and observes a loss for the action, with the goal of minimizing cumulative loss over a series of rounds. Contextual bandit learning has received much attention, and has seen substantial success in practice (e.g., Auer et al., 2002; Langford and Zhang, 2007; Agarwal et al., 2014, 2017). This line of work mostly considers small, finite action sets, yet in many real-world problems actions are chosen from an interval, so the set is continuous and infinite.

How can we learn to make actions from continuous spaces based on loss-only feedback?

We could assume that nearby actions have similar losses, for example that the losses are Lipschitz continuous as a function of the action (following Agrawal, 1995, and a long line of subsequent work). Then we could discretize the action set and apply generic contextual bandit techniques (Kleinberg, 2004) or more refined “zooming” approaches (Kleinberg et al., 2019; Bubeck et al., 2011a; Slivkins, 2014) that are specialized to the Lipschitz structure.

However, this approach has several drawbacks. A global Lipschitz assumption is crude and limiting; actual problems exhibit more complex loss structures where smoothness varies with location, often with discontinuities. Second, prior works incorporating context — including the zooming approaches — employ a nonparametric benchmark set of policies, which yields a poor dependence on the context dimension and prevents application beyond low-dimensional context spaces. Finally, existing algorithms require knowledge of the Lipschitz constant, which is typically unknown.

Here we show that it is possible to avoid all of these drawbacks with a conceptually new approach, resulting in a more robust solution for managing continuous action sets. The key idea is to smooth the actions: each action $a$ is mapped to a well-behaved distribution over actions, denoted $\text{Smooth}(a)$, such as a uniform

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| Type   | Setting          | Params | Regret Bound                  | Status | Section |
|--------|------------------|--------|-------------------------------|--------|---------|
| Smooth | Worst-case       | $h \in (0, 1]$ | $\Theta \left( \sqrt{T/h} \right)$ | New    | Sec. 3.1 |
| Smooth | Data-dependent   | $h \in (0, 1]$ | $O \left( \min\{ T\epsilon + \theta_h(\epsilon) \} \right)$ | New    | Sec. 3.1 |
| Smooth | Adaptive $h \in (0, 1]$ | None | $\Theta \left( \sqrt{T/h} \right)$ | New    | Sec. 3.2 |
| Lip.   | Worst-case       | $L \geq 1$ | $\Theta \left( T^{2/3}L^{1/3} \right)$ | Generalized | Sec. 4.1 |
| Lip.   | Data-dependent   | $L \geq 1$ | $O \left( \min\{ TL\epsilon + \psi_L(\epsilon)/L \} \right)$ | Generalized | Sec. 4.1 |
| Lip.   | Adaptive $L \geq 1$ | None | $\Theta(T^{2/3}\sqrt{L})$ | New    | Sec. 4.2 |

Table 1: A summary of results for stochastic contextual bandits specialized to the interval $[0, 1]$ action space. For notation, $T$ is the number of rounds, $h$ is the smoothing bandwidth, and $\theta_h(\epsilon) \leq 1/(he)$ is the smoothing coefficient. For the Lipschitz results, $L$ is the Lipschitz constant and $\psi_L(\epsilon) \leq 1/\epsilon^2$ is the policy zooming coefficient. All algorithms take $T$ and $\Pi$ as additional inputs. Logarithmic dependence on $|\Pi|$ and $T$ is suppressed in all upper bounds.

distribution over a small interval around $a$ (when the action set is the interval $[0, 1]$). This approach leads to provable guarantees with no assumptions on the loss function, since the loss for a smoothed action is always well behaved. Essentially, we may focus on estimation considerations while ignoring approximation issues. We recover prior results that assume a small Lipschitz constant, but the guarantees are meaningful in much broader scenarios.

Our algorithms work with an abstract policy set $\Pi$ of mappings from context to actions, which we smooth as above. We measure performance by comparing the learner’s loss to the loss of the best smoothed policy, and our guarantees scale with $\log |\Pi|$, regardless of the dimensionality of the context space. This recovers results for nonparametric policy sets, but more importantly accommodates parametric policies that scale to high-dimensional context spaces. Further, in some cases we are able to exploit benign structure in the policy set and the instance to obtain faster rates.

We design algorithms that require no knowledge of problem parameters and are optimally adaptive, matching lower bounds that we prove here. For the class of problems we consider, we show how this can be done with a unified algorithmic approach.

Our contributions, specialized to the interval $[0, 1]$ action set for clarity, are:

1. We define a new notion of smoothed regret where policies map contexts to distributions over actions. These distributions are parametrized by a bandwidth $h$ governing the spread. We show that the optimal worst-case regret bound with bandwidth $h$ is $\Theta(\sqrt{T/h \log |\Pi|})$ (first row of Table 1).

2. We obtain data-dependent guarantees in terms of a smoothing coefficient, which can yield much faster rates in favorable instances (second row of Table 1).

3. We obtain an adaptive algorithm with $\sqrt{T/h}$ regret bound for all bandwidths, simultaneously. Further we show this to be optimal, demonstrating a price of adaptivity (third row of Table 1).

4. We obtain analogous results when the losses are $L$-Lipschitz (rows 3-6 of Table 1). Notably, our data-dependent result here is in terms of a policy zooming coefficient, generalizing and improving prior zooming algorithms. We also demonstrate a price of adaptivity in the Lipschitz case.

Our results hold in much more general settings, and also apply to the non-contextual case, where we obtain several new guarantees.

## 2 Smoothed regret

We work in a standard setup for stochastic contextual bandits. We have a context set $\mathcal{X}$, action set $\mathcal{A}$, policy set $\Pi : \mathcal{X} \rightarrow \mathcal{A}$, and a distribution $\mathcal{D}$ over context/loss pairs $\mathcal{X} \times \{\text{functions } \mathcal{A} \rightarrow [0, 1]\}$. The protocol
We are interested in obtaining guarantees on the expected loss that are meaningful when the loss function has discontinuities, as demonstrated by the following example. Assuming global continuity (e.g., Lipschitzness), the guarantees remain meaningful even when the expected loss function has discontinuities, as demonstrated by the following example.

Key new definitions. We depart from the standard setup by positing a smoothing operator

\[ \text{Smooth}_h : \mathcal{A} \to \Delta(\mathcal{A}), \]

where \( h \geq 0 \) is the bandwidth: a parameter that determines the spread of the distribution. Bandwidth \( h = 0 \) corresponds to the Dirac distribution. Each action \( a \) then maps to the smoothed action \( \text{Smooth}_h(a) \), and each policy \( \pi \in \Pi \) maps to a randomized smoothed policy \( \text{Smooth}_h(\pi) : x \mapsto \text{Smooth}_h(\pi(x)) \). We compete with the smoothed policy class

\[ \Pi_h \triangleq \{ \text{Smooth}_h(\pi) : \pi \in \Pi \}. \]

We then define the smoothed loss of a given policy \( \pi \in \Pi \) and the benchmark optimal loss as

\[
\lambda_h(\pi) \triangleq \mathbb{E}_{(x,t) \sim \mathcal{D}} \mathbb{E}_{a \sim \text{Smooth}_h(\pi(x))} [\ell(a)], \quad \text{and} \quad \text{Bench}(\Pi_h) \triangleq \inf_{\pi \in \Pi_h} \lambda_h(\pi). \tag{1}
\]

We are interested in smoothed regret, which compares the learner’s total loss against the benchmark:

\[
\text{Regret}(T, \Pi_h) \triangleq \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(a_t) \right] - T \cdot \text{Bench}(\Pi_h).
\]

Our regret bounds work for an arbitrary policy set \( \Pi \), leaving the choice of \( \Pi \) to the practitioner. For comparison, a standard benchmark for contextual bandits is \( \text{Bench}(\Pi) \), the best policy in the original policy class \( \Pi \), and one is interested in \( \text{Regret}(T, \Pi) \).

For the bulk of the paper, we focus on a special case when the smoothing operator is a uniform distribution over a small interval. We posit that the actions set is a unit interval: \( \mathcal{A} = [0, 1] \), endowed with a metric \( \rho(a, a') = |a - a'| \). \( \text{Smooth}_h(a) \) is defined as a uniform distribution over the closed ball \( B_h(a) \triangleq \{ a' \in \mathcal{A} : \rho(a, a') \leq h \} = [a - h, a + h] \cap [0, 1] \). Let \( \nu \) denote the Lebesgue measure, which corresponds to the uniform distribution over \([0, 1]\).

In Section 5 we discuss how the results extend to more general settings, specifically where the action set \( \mathcal{A} \) is embedded in some ambient space and the smoothing operator is given by a kernel function. The results in full generality and the proofs are deferred to the appendices.

For some intuition, the bandwidth \( h \) governs a bias-variance tradeoff inherent in the continuous-action setting: for small \( h \) the smoothed loss \( \lambda_h(\pi) \) closely approximates the true loss \( \lambda_0(\pi) \), but small \( h \) also admits worse smoothed regret guarantees. As notation, \( \text{Smooth}_{\pi, h}(a|x) \) is the probability density, w.r.t., \( \nu \), for \( \text{Smooth}_h(\pi(x)) \) at action \( a \).

Example 1. The well-studied non-contextual version of the problem fits into our framework as follows: there is only one context \( \mathcal{X} = \{ x_0 \} \) and policies are in one-to-one correspondence with actions: \( \Pi = \{ x_0 \mapsto a : a \in \mathcal{A} \} \). A problem instance is characterized by the expected loss function \( \lambda_0(a) = \mathbb{E}[\ell(a)] \) and the smoothed benchmark is simply \( \text{Bench}(\Pi_h) = \inf_{a \in \mathcal{A}} \lambda_h(a) \).

Smoothing the policy class enables meaningful guarantees in much more general settings than prior work assuming global continuity (e.g., Lipschitzness). The guarantees remain meaningful even when the expected loss function has discontinuities, as demonstrated by the following example.

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1See Appendix A.2 for how this can be relaxed.

2The term bandwidth here is in line with the literature on nonparametric statistics.
Smoothed regret guarantees

In this section we obtain smoothed-regret guarantees without imposing any continuity assumptions on the problem.

3.1 Data-dependent and worst-case guarantees

Our first result is a data-dependent smoothed regret bound for a given bandwidth $h \geq 0$.

A important part of the contribution is setting up the definitions. Recall the definition of the smoothed loss $\lambda_h(\cdot)$ from (1) and that the optimal smooth loss is $\text{Bench}(\Pi_h) \triangleq \inf_{\pi \in \Pi} \lambda_h(\pi)$. The version space of $\epsilon$-optimal policies (according to the smoothed loss) is

$$
\Pi_{h,\epsilon} \triangleq \{ \pi \in \Pi : \lambda_h(\pi) \leq \text{Bench}(\Pi_h) + \epsilon \}.
$$

For a given context $x \in \mathcal{X}$, a policy subset $\Pi' \subset \Pi$ maps to an action set $\Pi'(x) \triangleq \{ \pi(x) : \pi \in \Pi' \}$. We are interested in $\Pi_{h,\epsilon}(x)$, the subset of actions chosen by the $\epsilon$-optimal policies on context $x$, and specifically the expected packing number of this set:

$$
M_h(\epsilon, \delta) \triangleq \mathbb{E}_{x \sim D} \left[ N_\delta(\Pi_{h,\epsilon}(x)) \right],
$$

where $N_\delta(A)$ is the $\delta$-packing number of subset $A \subseteq \mathcal{A}$ in the ambient metric space $(\mathcal{A}, \rho)$.

The smoothing coefficient $\theta_h : \mathbb{R} \rightarrow \mathbb{R}$ measures how the packing numbers $M_h(12\epsilon, h)$ shrink with $\epsilon$:

$$
\theta_h(\epsilon_0) \triangleq \sup_{\epsilon \geq \epsilon_0} M_h(12\epsilon, h)/\epsilon.
$$

For the unit interval, observe that $\theta_h(\epsilon_0) \leq (he_0)^{-1}$ always, but in favorable cases we might expect $\theta_h(\epsilon_0) \leq \max\{1/h, 1/\epsilon_0\}$. Our first result is in terms of this smoothing coefficient.

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3 A subset $S$ of a metric space is a $\delta$-packing if any two points in $S$ are at distance at least $\delta$. The $\delta$-packing number of the metric space is the maximal cardinality of a $\delta$-packing. This is a standard notion in metric analysis.
Theorem 1. For any given bandwidth \( h > 0 \), SmoothPolicyElimination (Algorithm 1) with parameter \( h \) achieves

\[
\text{Regret}(T, \Pi_h) \leq O \left( \inf_{\epsilon_0 \geq 1/T} T\epsilon_0 + \theta_h(\epsilon_0) \log(|\Pi| \log_2(T)) \log(T) \right).
\]

In Appendix A.2 we present a generalized statement (Theorem 9), with a proof in Appendix D. Using the observation that \( \theta_h(\epsilon_0) \leq (h\epsilon_0)^{-1} \) we obtain a worst case guarantee as a corollary.

Corollary 2. Fix any bandwidth \( h > 0 \), SmoothPolicyElimination with parameter \( h \) achieves

\[
\text{Regret}(T, \Pi_h) \leq O \left( \sqrt{T/h \log(|\Pi| \log_2(T)) \log(T)} \right).
\]

Remarks. The worst case bound can also be achieved by a simple variation of EXP4 (Auer et al., 2002), which can operate in the adversarial version of our problem and actually achieves \( \tilde{O}(\sqrt{T/h \log |\Pi|}) \) regret, eliminating the logarithmic dependence on \( T \) (see Theorem 10 in Appendix A.2). This guarantee should be compared with the standard \( \Theta(\sqrt{T |\mathcal{A}| \log |\Pi|}) \) regret bound for finite action sets, indicating that the \( 1/h \) term can be viewed as the effective number of actions.

By making this correspondence precise, it is not hard to show a \( \Omega(\sqrt{T/h \log |\Pi|}) \) lower bound on smoothed regret. Specifically, every \( K \) arm contextual bandit instance can be reduced to a continuous action instance with bandwidth \( h = 1/(2K) \) by using piecewise constant loss functions and by mapping actions \( a \in \{1, \ldots, K\} \) to \( h \cdot (2a - 1) \). Thus, we may embed the lower bound construction for contextual bandits with finite action set into our setup to verify that Corollary 2 is optimal up to logarithmic factors (and our analysis for EXP4 is optimal up to constants).

While not technically very difficult, the worst-case bound showcases the power and generality of the new definition. In particular, we obtain meaningful guarantees for discontinuous losses as in Example 2. As we will see in the next section, under global smoothness assumptions, we can also obtain a bound on the more-standard quantity \( \text{Regret}(T, \Pi) \).

Turning to the instance-specific bound in Theorem 1, we obtain a more-refined dependence on the effective number of actions \( 1/h \). In the most favorable setting, we have \( \theta_h(\epsilon_0) = \max \{1/\epsilon_0, 1/h \epsilon_0 \} \) which yields \( \text{Regret}(T, \Pi_h) \leq O \left( \sqrt{T \log |\Pi|} + \frac{1}{h} \log |\Pi| \right) \), eliminating the dependence on \( h \) in the leading term (in fact Example 2 has this favorable behavior). Further, via the correspondence with the finite action setting, we also obtain a new data-dependent bound for standard stochastic contextual bandits, improving on prior worst case results by adapting to the effective size of the action space (Dudik et al., 2011; Agarwal et al., 2014). This result for the finite-action setting follows from our more general theorem statement, given in Appendix A.2. On the other hand, obtaining such a refined bound is more technically involved.

The algorithm. The algorithm is an adaptation of PolicyElimination from (Dudik et al., 2011), with pseudocode displayed in Algorithm 1. It is epoch based, maintaining a version space of good policies, denoted \( \Pi^{(m)} \) in the \( m \)th epoch, and pruning it over time by eliminating the provably suboptimal policies. In the \( m \)th epoch, with \( \Pi^{(m)} \) the algorithm computes a distribution \( Q_m \) over \( \Pi^{(m)} \) by solving a convex program (3). The objective function is related to the variance of the loss estimator we use, and so \( Q_m \) ensures high-quality loss estimates for all policies in \( \Pi^{(m)} \). We use \( Q_m \) to select actions at each round in the epoch by sampling \( \pi \sim Q_m \) and playing \( \text{Smooth}_h(\pi(x)) \) on context \( x \), mixing in a small amount of uniform exploration. To compute \( \Pi^{(m+1)} \) for the next epoch, we use importance weighting to form single-sample unbiased estimates for \( \lambda_h(\pi) \) in (4), and we aggregate these via a median-of-means approach. \( \Pi^{(m+1)} \) is the defined as the set of policies with low empirical regret measured via the median-of-means loss estimator.

The key changes over PolicyElimination are as follows. First, we write (3) as an optimization problem rather than a feasibility problem, which allows for data-dependent improvements in our loss estimates. Second,
Algorithm 1 SmoothPolicyElimination

Parameters: Bandwidth \( h > 0 \), policy set \( \Pi \), number of rounds \( T \).

Initialize: \( \Pi^{(1)} = \Pi \), Batches \( \delta_T = 5 \lfloor \log(\|\Pi| \log_2(T)/\delta) \rfloor \), Radii \( r_m = 2^{-m} \), \( m = 1, 2, \ldots \).

for each epoch \( m = 1, 2, \ldots \) do

// Before the epoch: compute distribution \( Q_m \) over policy set \( \Pi^{(m)} \).

Set \( V_m \leftarrow E_{x \in D} \nu \left( \bigcup_{\pi \in \Pi^{(m)}} B_h(\pi(x)) \right) \) // characteristic volume of \( \Pi^{(m)} \)

Set batch size \( \tilde{n}_m = \frac{320h}{r_m^2}, \) epoch length \( n_m = \tilde{n}_m \delta_T \), explore prob. \( \mu_m = \min \{ \frac{1}{2}, r_m \} \).

Find distribution \( Q_m \) over policy set \( \Pi^{(m)} \) which minimizes

\[
\max_{\text{policies } \pi \in \Pi^{(m)}} \mathbf{E}_{\text{context } x \sim D} \mathbf{E}_{\text{action } a \sim \text{Smooth}(\pi(x))} \left[ \frac{1}{q_m(a \mid x)} \right],
\]

(3)

for each round \( t \) in epoch \( m \) do

Observe context \( x_t \), sample action \( a_t \) from \( q_m(\cdot \mid x_t) \), observe loss \( \ell_t(a_t) \).

end for

// After the epoch: update the policy set.

for each batch \( i = 1, 2, \ldots, \delta_T \) do

Define \( S_{i,m} \) as the indices of the \((i-1)\tilde{n}_m + 1, \ldots, i\tilde{n}_m^\delta \) examples in epoch \( m \).

Estimate \( \lambda_h(\pi) \) with \( \hat{L}_{i,m}(\pi) = \frac{1}{n_m} \sum_{t \in S_{i,m}} \hat{\ell}_{t,h}(\pi) \) for each policy \( \pi \in \Pi^{(m)} \) where

\[
\hat{\ell}_{t,h}(\pi) \triangleq \frac{\text{Smooth}_{h,x}(a_t|x_t) \ell_t(a_t)}{q_m(a_t \mid x_t)}.
\]

(4)

end for

Estimate the loss \( \hat{L}_m(\pi) = \text{median} \left( \hat{L}_{1,m}^1(\pi), \hat{L}_{1,m}^2(\pi), \ldots, \hat{L}_{n_m}^\delta(\pi) \right) \).

\( \Pi^{(m+1)} = \left\{ \pi \in \Pi^{(m)} : \hat{L}_m(\pi) \leq \min_{\pi' \in \Pi^{(m)}} \hat{L}_m(\pi') + 3r_m \right\} \).

end for

Our importance weighting crucially exploits smoothing for low variance. Finally, we employ the median-of-means estimator to eliminate an unfavorable range dependence with importance weighting. Interestingly, this last step is inconsequential for Corollary 2 and for prior results with finite action sets, but it is crucial for obtaining our data-dependent bound.

For the proof, we first use convex duality to upper bound the value of (3) in terms of the characteristic volume \( V_m \), refining Dudik et al. (2011). As the objective divided by \( h \) bounds the variance of the importance weighted estimate in (4), we may use Chebyshev and Chernoff bounds to control the error of the median-of-means estimator in terms of \( V_m, h, \) and \( n_m \). Our setting of \( n_m \) then implies that \( \Pi^{(m+1)} \subset \Pi_{h,12r_{m+1}} \). Two crucial facts follow: (1) the instantaneous regret in epoch \( m + 1 \) is related to \( r_{m+1} \) and (2) \( V_{m+1} \), which determines the length of the epoch, is related to the packing number \( M_h(12r_{m+1}, h) \). Roughly speaking, this shows that the regret in epoch \( m \) is \( n_mr_m \lesssim M_h(12r_m, h)/r_m \), which we can easily relate to the smoothing coefficient.

3.2 One algorithm for all \( h \)

SmoothPolicyElimination guarantees a refined regret bound against \( \text{Bench}(\Pi_h) \) for a given \( h > 0 \). Yet choosing the bandwidth in practice seems challenging: since \( \text{Bench}(\Pi_h) \) is unknown and not monotone in general, there is no a priori way to choose \( h \) to minimize the benchmark plus the regret. As such, we seek
algorithms that can achieve a smoothed regret bound simultaneously for all bandwidths \( h \), a guarantee we call uniformly-smoothed. This is achieved by our next result.

**Theorem 3.** Fix \( \alpha \in [0, 1] \). Corral + EXP4 (with parameter \( \alpha \)) guarantees

\[
\forall h \in (0, 1] : \text{Regret}(T, \Pi_h) \leq \tilde{O} \left( T^{\frac{1}{1+\alpha}} h^{-\alpha} \right) \cdot (\log |\Pi|)^{\frac{\alpha}{1+\alpha}}.
\]

For the non-contextual setting, it achieves a uniformly-smoothed regret of \( \tilde{O} \left( T^{\frac{1}{1+\alpha}} h^{-\alpha} \right) \). Moreover, for the non-contextual setting, no algorithm achieves

\[
\forall h \in (0, 1] : \text{Regret}(T, \Pi_h) \leq \Omega \left( T^{\frac{1}{1+\alpha}} h^{-\alpha} \right).
\]

More general statements are presented in Appendix A.2 as Theorem 11 and Theorem 12 with proofs in Appendix B and Appendix C.

**Remarks.** The theorem provides a family of upper and lower bounds, one for each \( \alpha \in [0, 1] \). As two examples, taking \( \alpha = 1 \) we obtain regret rate \( \tilde{O}(\sqrt{T}/h) \) as listed in the third row of Table 1, while \( \alpha = 1/2 \) yields \( \tilde{O}(T^{2/3}/\sqrt{h}) \). These bounds are incomparable in general and so the result establishes a Pareto frontier of exponent pairs. In the non-contextual setting, all pairs are optimal, and, in particular, the \( \sqrt{T/h} \) rate from Corollary 2 is not achievable uniformly. More generally, the optimal uniformly-smoothed regret bounds are very different from those for a fixed bandwidth.

Note that while \( \alpha \) is a parameter to the algorithm, it simply governs where on the Pareto frontier the algorithm lies, and is not based on any property of the problem.

**The algorithm.** The algorithm we use here is an instantiation of Corral (Agarwal et al., 2016), which can be used to run many sub-algorithms in parallel. Corral maintains a master distribution over sub-algorithms, and in each round it samples a sub-algorithm and chooses the action the sub-algorithm recommends. Corral sends an importance weighted loss (weighted by the master distribution) to all the sub-algorithms and it updates the master distribution using online mirror descent with the log-barrier mirror map.

For the sub-algorithms we run many copies of our variant of EXP4 that is modified slightly to achieve optimal non-adaptive smoothed regret (Recall the remark after Corollary 2). Each sub-algorithm instance operates with different bandwidth scales, and if run in isolation achieves the optimal non-adaptive smoothed regret for those bandwidths. Aggregating these sub-algorithms with Corral yields the uniformly-smoothed guarantee. Note that here and elsewhere, Corral results in a worse overall regret than the best individual sub-algorithm, but in our setting it nevertheless achieves all Pareto-optimal uniformly-smoothed guarantees.

The proof for the upper bound involves a more refined analysis for EXP4 than we have previously alluded to. Specifically we discretize bandwidth to multiples of \( 1/T^2 \) and show that a single instance of EXP4 using discretized bandwidths can compete with all \( h \in [2^{-i}, 2^{-i+1}] \) simultaneously, without the Corral meta-algorithm. We also show that EXP4 is stable in the sense that, in randomized environments, the regret scales linearly with the standard deviation of the losses and that this standard deviation need not be known a priori.\(^4\) Stability is crucial for aggregating with Corral as the master’s importance weighting induces high-variance randomized losses for each sub-algorithm. We finish the proof by applying the guarantee for Corral (Agarwal et al., 2016) with \( \log(T) \) instances of EXP4 as sub-algorithms, one for each bandwidth scale \( [2^{-i}, 2^{-i+1}] \). For each \( \alpha \in [0, 1] \), we use a weakening of the EXP4 regret guarantee, essentially that

\[
\min \left\{ \sqrt{T/h} \right\} \leq T^{\frac{1}{1+\alpha}} h^{-\frac{2\alpha}{1+\alpha}} \text{ for all } \alpha \in [0, 1].
\]

\(^4\)This property was shown by Agarwal et al. (2016), but our variant of EXP4 is necessarily slightly different. Nevertheless, the proof is quite similar.
The lower bound is inspired by a construction due to Locatelli and Carpentier (2018). We show that if an algorithm, ALG, has small regret against Bench(Πh), then it must suffer large regret against Bench(Πh) for a much smaller h. The intuition is that the 1/4-smoothed regret bound prevents ALG from sufficiently exploring. Specifically, we construct one instance where small losses occur in a subinterval I0 ⊂ [0, 1] of length 1/4 and another that is identical on I0 but where even smaller losses occur in a subinterval I1 of width h ≪ 1/4. Since ALG has low 1/4-smoothed regret it cannot afford to explore to find I1, so its behavior must be similar on the instances. Thus it cannot play actions in I1 enough to compete with the h-smoothed benchmark. In comparison with Locatelli and Carpentier (2018), the details of the construction are somewhat different, since they focus on adaptivity to unknown smoothness exponent, while we are adapting to bandwidth h (and later to unknown Lipschitz constant).

4 Lipschitz regret guarantees

Our results and techniques for smoothed regret project onto the well-studied Lipschitz contextual bandits problem: each of the three results in Section 3 has a “twin” for the Lipschitz version.

In the Lipschitz version and throughout this section, we posit a Lipschitz condition on the expected loss \( \lambda(\cdot \mid x) \triangleq E[\ell(\cdot) \mid x] \):

\[
\forall x \in X, a, a' \in A : \left[ \lambda(a \mid x) - \lambda(a' \mid x) \right] \leq L \rho(a, a').
\]

The key observation enabling our results for the Lipschitz version is the following simple lemma.

**Lemma 4.** If \( f : A \to [0, 1] \) is \( L \)-Lipschitz continuous, then \( \left| E_{a' \sim \text{Smooth}_h(a)} f(a') - f(a) \right| \leq Lh. \)

In particular if \( \lambda(\cdot \mid x) \) is \( L \)-Lipschitz, we have \( \text{Bench}(\Pi_h) \leq \text{Bench}(\Pi) + Lh \), which allows us to easily obtain results for the Lipschitz version by way of smoothed regret.

4.1 Data-dependent and worst-case guarantees

In correspondence with Theorem 1, our first result here is a data-dependent regret bound. We recover the optimal worst-case regret bound for the Lipschitz setting, but we obtain an improvement when actions taken by near-optimal policies tend to lie in a relatively small region of the action space. Specializing, we recover several state-of-the-art data-dependent regret bounds from prior work.

We reuse the packing numbers \( M_h(\epsilon, \delta) \) defined in (2), but the instance-dependent complexity is slightly different. Instead of the smoothing coefficient \( \theta_h(\epsilon_0) \), we use the policy zooming coefficient:

\[
\psi_L(\epsilon_0) \triangleq \sup_{\epsilon \geq \epsilon_0} M_0(12L\epsilon, \epsilon)/\epsilon.
\]

The main differences over the smoothing coefficient are that version space of good policies is based on the unsmoothed loss \( \lambda_0(\pi) \), and we are using the \( \epsilon \)- rather than \( h \)-packing number for a fixed bandwidth \( h \). For intuition, we always have \( \psi_L(\epsilon_0) \leq O(\epsilon_0^{-2}) \) but a favorable instance might have \( \psi_L(\epsilon_0) \leq O(\epsilon_0^{-1}) \) which yields improved rates.

**Theorem 5.** Algorithm **SmoothPolicyElimination.L** with parameter \( L \) achieves regret bound

\[
\text{Regret}(T, \Pi) \leq O\left( \inf_{\epsilon_0 > 1/T} T \lambda_0 + \psi_L(\epsilon_0)/L \cdot \log(|\Pi|) \log(T) \right). \tag{5}
\]

A generalization is stated in Appendix A.3 (Theorem 13) with a proof in Appendix D. Since \( \psi_L(\epsilon_0) \leq O(\epsilon_0^{-2}) \), we obtain a the following worst-case bound, which is known to be optimal up to logarithmic factors.

**Corollary 6.** **SmoothPolicyElimination.L** with parameter \( L \) achieves

\[
\text{Regret}(T, \Pi) \leq \tilde{O}\left( T^{2/3}(L \log |\Pi|)^{1/3} \right).
\]
Remarks. The worst-case result is in correspondence with Corollary 2. It recovers the worst-case regret bound from prior work focusing on the non-contextual version (Kleinberg, 2004; Bubeck et al., 2011b). This regret bound can also be achieved by a variant of EXP4, just as with smoothed regret (see Corollary 14 in Appendix A.3).

The result can also be applied to a nonparametric policy set in the setting of Cesa-Bianchi et al. (2017). Here we assume $X$ is a $p$-dimensional metric space and the policy set is all $1$-Lipschitz mappings from $X \to A$. By a suitable discretization, Corollary 6 yields $\tilde{O} \left( T^{\frac{1+\alpha}{1+2\alpha}} L^{\frac{\alpha}{1+\alpha}} \right)$ regret, which matches their result (since the interval is a $1$-dimensional action space).

The advantage of Theorem 5 is its data-dependence. Since the packing number $M_0(\cdot, \cdot)$ is always at least $1$, the most favorable instances have $\psi_L(\epsilon_0) = O(\epsilon_0^{-1})$. In this case, Theorem 5 gives the much faster $\tilde{O}(\sqrt{T \log |\Pi|})$ regret rate.

Data-dependent bounds from prior work are often stated in terms of a packing number growth rate, called the zooming dimension. Our bound can also be stated in this way, so as to facilitate comparisons. With zooming constant $\gamma > 0$ the zooming dimension is defined as

$$ z \triangleq \inf \left\{ d > 0 : M_0(12L\epsilon, \epsilon) \leq \gamma \cdot \epsilon^{-d}, \forall \epsilon \in (0, 1) \right\}. \tag{6} $$

It is easy to see that $\psi_L(\epsilon_0) \leq \gamma \cdot \epsilon_0^{z-1}$, and so Theorem 5 may be further simplified to

$$ \text{Regret}(T, \Pi) \leq O \left( L^{\frac{1+\alpha}{1+2\alpha}} T^{\frac{1+\alpha}{1+2\alpha}} \cdot (\gamma \log |\Pi|) \right)^{\frac{1}{1+\alpha}}. \tag{7} $$

This result agrees with prior zooming results in the non-contextual setting (Kleinberg et al., 2019; Bubeck et al., 2011a). In the contextual setting, our result improves over the “contextual zooming algorithm” of Slivkins (2014) in several respects: we do not need to assume Lipschitz structure on the context space, we can handle arbitrary policy sets $\Pi$ (with regret scaling with $\log |\Pi|$), and our zooming dimension involves the “expected context” rather than the “worst context.”

On the other hand, the regret bound in Slivkins (2014) has a “zooming”-dependence on the context dimension, so the results are incomparable. See Appendix A.3 for a detailed discussion.

The algorithm. The algorithm is almost identical to SmoothPolicyElimination. The main difference is that instead of a fixed bandwidth $h$ across all epochs, we use $h_m = 2^{-m}$ in the $m$th epoch. We also set the radius parameter $r_m = L 2^{-m}$ which is slightly different from before.

At a technical level, the main difference with the Lipschitz setting is that we must carefully balance bias and variance in loss estimates. This is not an issue for smoothed regret since we have unbiased estimators for $\lambda_h(\pi)$, but not for $\lambda_0(\pi)$. We do this by decreasing the bandwidth geometrically over epochs, but the rest of the algorithm, and much of the analysis are unchanged.

4.2 Optimal Adaptivity

We now present the corresponding result to Theorem 3. We consider Lipschitz-adaptive algorithms: those that do not know any information about the problem, apart from $T$ and $\Pi$, and yet achieve regret bounds in terms of $T, L$ and $|\Pi|$ only. In particular, the algorithm does not know $L$.

Theorem 7. Fix $\alpha \in [0, 1]$. Algorithm Corral + EXP4 (with parameter $\alpha$) is Lipschitz-adaptive with

$$ \text{Regret}(T, \Pi) \leq \tilde{O} \left( T^{\frac{1+\alpha}{1+2\alpha}} L^{\frac{\alpha}{1+\alpha}} \cdot (\log |\Pi|) \right)^{\frac{1}{1+\alpha}}. $$

Formally, in the definition of the packing number (2) we take the expectation over contexts, whereas the analogous definitions in Slivkins (2014) are tailored to the worst case over contexts.
For the non-contextual version it achieves a regret $\tilde{O} \left( T^{\frac{1+\alpha}{1+2\alpha}} L^{\frac{\alpha}{1+\alpha}} \right)$ without knowing the Lipschitz constant $L$. Moreover, for the non-contextual version, no Lipschitz-adaptive algorithm achieves $\text{Regret}(T, \Pi) < \Omega \left( T^{\frac{1+\alpha}{1+2\alpha}} L^{\frac{\alpha}{1+\alpha}} \right)$.

A generalization is stated in Appendix A.3 (Theorem 15) with a proof in Appendix B and Appendix C.

**Remarks.** As in Theorem 3, we obtain a family of upper and lower bounds, one for each $\alpha \in [0, 1]$, which make up a Pareto frontier. With $\alpha = 1$ an optimal Lipschitz-adaptive rate is $T^{2/3} \sqrt{L}$ which is much worse than the $T^{2/3} L^{1/3}$ non-adaptive rate from Corollary 6. Note that it is easy to obtain the worse adaptive rate of $\tilde{O} \left( LT^{2/3} \right)$ simply by guessing that the Lipschitz constant is 1 in our variant of EXP4, or any non-adaptive algorithm.

Several prior works develop adaptive algorithms that either require knowledge of unknown problem parameters, or yield regret bounds that, in addition to $T$ and $L$, scale with such parameters (Slivkins, 2011; Bubeck et al., 2011b; Bull, 2015; Locatelli and Carpentier, 2018). These algorithms are not Lipschitz adaptive, contrasting with our algorithm that requires no additional knowledge or assumptions. However, this dependence on other parameters allows these prior results to sidestep our lower bound and achieve faster rates. See Section 6 for more discussion.

Note that Lipschitz-adaptivity is qualitatively quite different from the uniformly-smoothed adaptivity studied in Theorem 3. With Lipschitz-adaptivity there is a single fixed benchmark policy class and we simply seek a guarantee against that class, albeit in an environment with unknown smoothness parameter. However, for Theorem 3 we are effectively competing with infinitely many policy sets simultaneously ($\Pi_h$ for each $h \in (0, 1]$) and we seek a regret bound against all of them. Somewhat surprisingly, both settings demonstrate a similar price-of-adaptivity and the optimally adaptive algorithms are nearly identical.

**The algorithm.** The algorithm is again Corral with our variant of EXP4 as the sub-algorithms. The only difference is in how we set the learning rate for the master algorithm.

5 Extensions

All results presented here are special cases of the more general results that are formulated and proved in the appendix. (However, all key ideas are already present in the special case of the unit interval.)

**Higher dimensions.** All results extend to $d \geq 1$ dimensions and arbitrary convex subsets. Formally, the action set $A$ can be an arbitrary convex subset of the $d$-dimensional unit cube $[0, 1]^d$, equipped with $p$-norm $\rho(a, a') \triangleq \|a - a'\|_p$, for an arbitrary $p \geq 1$. As before, $\text{Smooth}_h(a)$ is a uniform distribution over the closed ball $B_h(a) \triangleq \{a' \in A : \rho(a, a') \leq h\}$. The data-dependent regret bounds, Theorem 1 and Theorem 5, carry over as is. Zooming dimension in (6) can take any value in $[0, d]$, depending on the problem instance. Regret bounds in the worst-case corollaries are modified so as to accommodate the dependence on $d$. In Corollary 2, the dependence on $h$ is replaced with $h^4$, and there is a matching lower bound. In Corollary 6, the dependence on $T$ becomes $\tilde{O}(T^{(d+1)/(d+2)})$, which is known to be optimal. In the “optimally adaptive” results (Theorem 3 and Theorem 7), regret bounds are modified similarly (we omit the formulas). Theorem 3 holds in a slightly weaker form: bandwidth $h$ can take only $H$ distinct values, and the regret bound scales as $\log H$. In all these results, multiplicative constants in the regret bounds scale as $2^O(d)$ and depend on the shape of $A$.
Adversarial losses. Some of our results — those based on the exponential weights technique — carry over as is to the adversarial setting, with benchmark redefined as

$$\text{Bench}(\Pi_h) \triangleq \frac{1}{T} \inf_{\pi \in \Pi_h} \mathbb{E} \left[ \sum_{t \in [T]} \ell_t(\pi(x_t)) \right].$$

This concerns Corollary 2, Corollary 6, Theorem 3, and Theorem 7.

Arbitrary action sets and smoothing operators. Both worst-case results for smoothed regret, Corollary 2 and Theorem 3, essentially admit arbitrary action sets and smoothing operators. Formally, action set $\mathcal{A}$ can be the sample space in an arbitrary probability space $(\mathcal{A}, \mathcal{F}, \nu)$, where $\mathcal{F}$ is a $\sigma$-algebra and $\nu$ is a probability measure. We call $(\mathcal{A}, \mathcal{F}, \nu)$ the ambient space, and $\nu$ the base measure. The smoothing distribution $\text{Smooth}_h(a)$ can be any distribution with a well-defined probability density, upper-bounded by $1/(2h)$. Even more generally, $\text{Smooth}_h(a)$ can be an arbitrary probability measure determined by a bounded, but otherwise arbitrary, Radon-Nikodym derivative in the ambient space. Corollary 2, using a version of EXP4 as the algorithm, carries over to this setting as is. Theorem 3 carries over with only finitely many possible values for $h$, same as above.

A notable special case is when one has a metric $\rho$ on actions (which induces the $\sigma$-algebra $\mathcal{F}$), and the smoothing distribution is supported on a ball with respect to this metric. More precisely, fix bandwidth $h > 0$ and posit that $\text{Smooth}_h(a)$ is supported on a closed ball $B_r(a)$, where radius $r$ depends on $h$, but is the same for all $a$. First, if $\text{Smooth}_h(a)$ is uniform over $B_r(a)$, like in our main presentation, then we can start with a fixed radius $r > 0$, and define $h = \inf_{a \in \mathcal{A}} \nu( B_r(a) )$. Thus, bandwidth $h$ is determined by the smallest ball volume relative to base measure $\nu$. Second, action set $\mathcal{A}$ can be an arbitrary finite subset of $[0, 1]$, with an arbitrary base measure $\nu$. One natural choice for $\nu$ would be a uniform distribution over $\mathcal{A}$. Third, $\text{Smooth}_h(a)$ need not be restricted to a uniform distribution: it can have density determined by distance to $a$, e.g., decreasing in the said distance.

6 Related Work

With small, discrete action spaces, contextual bandit learning is quite mature, with rich theoretical results and successful deployments in practice. To handle large or infinite action spaces, two high-level approaches exist. The parametric approach, including work on linear or combinatorial bandits, posits that the loss is a parametric function of the action, e.g., a linear function (c.f., Lattimore and Szepesvári (2018); Bubeck et al. (2012) for surveys). The nonparametric approach, which is closer to our results, typically makes much weaker continuity assumptions.

Bandits with Lipschitz assumptions were introduced in Agrawal (1995), and optimally solved in the worst case by Kleinberg (2004). Kleinberg et al. (2008, 2019); Bubeck et al. (2011a) achieve data-dependent regret bounds via algorithms that “zoom in” on the more promising regions of the action space. Several papers relax global smoothness assumptions with various local definitions (Auer et al., 2007; Kleinberg et al., 2008, 2019; Bubeck et al., 2011a; Slivkins, 2011; Minsker, 2013; Grill et al., 2015). While the assumptions vary, our smoothing-based approach can be used in many of these settings. More importantly, in contrast with these approaches, our guarantees remain meaningful even in pathological instances, for example when the global optimum is a discontinuity as in Example 2.

While most of this literature focuses on the non-contextual version, two papers consider contextual settings, albeit only with fixed policy sets $\Pi$. Slivkins (2014) posits that the mean loss function is Lipschitz in both context $x$ and action $a$, and the learner must compete with the best mapping from $X$ to $\mathcal{A}$. Both

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6 Previously, we considered the special case of the unit interval, with Borel $\sigma$-algebra and Lebesgue measure.

7 However, we emphasize that for smoothed regret, we make no assumptions on the loss.
algorithm and guarantees exhibit “zooming” behavior in the action space, which is qualitatively similar to ours. However, his regret bound also has a “zooming”-dependence on the context dimension, whereas our regret bound applies to arbitrary policy sets and defines packing numbers via expectation over contexts rather than supremum. We can obtain the same worst-case bound (e.g., as a corollary of (7), by discretizing the policy set). Cesa-Bianchi et al. (2017) competes with policies that are themselves Lipschitz (w.r.t. a given metric on contexts). We can recover their result via Corollary 6 and a suitable discretized policy set.

Turning to adaptivity, Bubeck et al. (2011b) develops an algorithm that adapts to the Lipschitz constant in the non-contextual setting given a bound on the second derivative. Locatelli and Carpentier (2018) obtain optimal adaptive algorithms, but require knowledge of either the value of the minimum, or a sharp bound on the achievable regret. Slivkins (2011); Bull (2015) achieve optimal regret bounds in terms of the zooming dimension, but their regret bounds depend on a certain “quality parameter.” Moreover, these results concern the stochastic setting, while our optimally adaptive guarantees carry through to the adversarial setting. Locatelli and Carpentier (2018) also obtain lower bounds against adapting to the smoothness exponent, and we build on their construction for our lower bounds.

Finally, our smoothing-based importance weighted loss estimator (4) was analyzed by Kallus and Zhou (2018) in the offline observation setting, but they do not consider the smoothed regret benchmark or the online setting, so the results are considerably different.

7 Conclusions

The main conceptual contribution is a new smoothing-based notion of regret that admits guarantees with no assumptions on the loss. Using this, we design new algorithms providing data-dependent guarantees with optimal worst-case performance and Pareto-optimal adaptivity. This also yields new performance guarantees for non-contextual and Lipschitz versions.

While our algorithms are computationally efficient in the low-dimensional non-contextual setting, they are not in general since they require enumerating the policy space. Hence, the key open question is: Are there algorithms with similar statistical performance and fast running time?

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A General development and results

A.1 Contextual bandits with continuous actions: problem setup

Let $\mathcal{X}$ be an abstract context space and let $\mathcal{A}$ be an action space endowed with a metric $\rho$ and a base probability measure $\nu$.

The contextual bandit problem involves the following $T$ round protocol: at round $t$ (1) nature chooses context $x_t \in \mathcal{X}$ and loss $\ell_t \in (\mathcal{A} \rightarrow [0, 1])$ and presents $x_t$ to the learner, (2) learner chooses action $a_t \in \mathcal{A}$, (3) learner suffers loss $\ell_t(a_t)$, which is also observed. Performance of the learner is measured relative to a class of policies $\Pi : \mathcal{X} \rightarrow \mathcal{A}$ via the notion of regret

$$\text{Regret}(T, \Pi) \triangleq \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(a_t) \right] - \min_{\pi \in \Pi} \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(\pi(x_t)) \right]$$

We consider both adversarial and stochastic settings. In the adversarial setting the contexts and losses are chosen by an adaptive adversary, meaning that $(x_t, \ell_t)$ may be a randomized function of the entire history of interaction. In the stochastic setting, we assume $(x_t, \ell_t) \sim D$ iid at each round $t$, for some unknown distribution $D$.

For notation, we use $\Delta(\mathcal{A})$ to denote the set of distributions over $\mathcal{A}$ and we typically represent them via their density with respect to $\nu$, formally their Radon-Nikodym derivative. We use $\langle \cdot, \cdot \rangle$ to denote the standard $L_2(\nu)$ inner product, $\langle f, g \rangle \triangleq \int f(a)g(a)d\nu(a)$ and at time we write $\langle a, f \rangle \triangleq f(a)$ for $a \in \mathcal{A}$. We use $B(a, r) \triangleq \{ b \in \mathcal{A} : \rho(a, b) \leq r \}$ to denote the closed ball. In the general setting, $L$-Lipschitz losses $\ell_t$ satisfy $|\ell_t(a) - \ell_t(a')| \leq L\rho(a, a')$.

We now introduce a generalization of the Smooth operator. Let $K : \Delta(\mathcal{A}) \rightarrow \Delta(\mathcal{A})$ be a smoothing kernel. For policy $\pi$, we use $K\pi : x \mapsto K(\pi(x))$ to denote the usual function composition. With $\Pi_K \triangleq \{ K\pi : \pi \in \Pi \}$ as the smoothed policy class, smoothed regret is simply given by $\text{Regret}(T, \Pi_K)$. 

B Adaptive Algorithms

B.1 EXP4 and a stability guarantee

B.2 Corral and the adaptive guarantees

B.3 Proof of the discretization lemmas

C Adaptive Lower Bounds

C.1 The constructions

C.2 Proofs of the lower bounds

D Data-dependent Regret Bounds

D.1 Proofs of the main theorems

D.2 Proofs for the lemmata
The kernels we use are always defined via linearly extending an operator from $A \to \Delta(A)$. We typically use the rectangular kernel, specified via its density w.r.t. $\nu$:

**Rectangular Kernel:** $K_h(a) : a' \mapsto \frac{1\{\rho(a, a') \leq h\}}{\nu(B(a, h))}$.

We typically use this kernel, which generalizes Smooth$_h$ from the prequel. Note, however that many of the results apply with arbitrary kernels, sometimes with support conditions.

As in the 1-dimensional interval example, our smoothed-regret bounds require no assumptions on the losses, and we use the kernel to measure the smoothness of the problem. As a convention, we write $(Ka)(a')$ is the density of the distribution $Ka$ evaluated at $a'$. Then, define

$$\kappa \triangleq \sup_{a, a'} \left| (Ka)(a') \right|,$$

which, as we will see, serves as the effective number of actions.

Translating from $\Pi_K$ to $\Pi$ with Lipschitz losses is facilitated by the following lemma, which generalizes Lemma 4.

**Lemma 8** (Smooth to Lipschitz). If $\text{supp}(Ka) \subseteq B(a, h)$ and $\ell$ is $L$-Lipschitz, then

$$|\langle Ka, \ell \rangle - \ell(a)| \leq Lh$$

**Proof.**

$$|\langle Ka, \ell \rangle - \ell(a)| = \left| \mathbb{E}_{b \sim Ka} \ell(b) - \ell(a) \right| \leq L \mathbb{E}_{b \sim Ka} \rho(a, b) \leq Lh. \qed$$

For the metric space, we make the following technical assumption.

**Assumption 1.** We assume that $\rho$ is a doubling metric with doubling dimension $d$ and that $\nu$ is the doubling measure. This means that for all $a \in A$ we have $0 < \nu(B(a, 2r)) \leq 2^d \nu(B(a, r)) < \infty$. For normalization, we assume that $A$ has diameter 1.

**Example 3** (1-dimensional interval). If $A = [0, 1)$ endowed with the metric $\rho(a, a') = |a - a'|$ and with uniform base measure, then the rectangular kernel is precisely the Smooth$_h$ operator from before. This example has doubling dimension $d = 1$, and so all of the results we presented in prequel follow from the general development.

**Example 4** (Finite metric, uniform measure). Suppose $A = \{i/M : i \in [M]\}$ is a finite set of $M$ actions with identity metric $\rho(a, a') = |a - a'|$ and that $\nu$ is the uniform distribution. Then with identity kernel $Ka \mapsto a$ we recover the classical multi-armed bandit problem. However, with a non-degenerate metric $\rho(a, a') = |a - a'|$ and the uniform kernel $K_h$, then Bench($\Pi_K$) involves taking local averages across actions. In this case, we always have $\kappa = 1/h$ and we will obtain smooth regret bounds that are independent of the number of actions $M$.

**Non-contextual setting.** Our development generalizes the non-contextual setting, but as we will give some results for this special case, we pause to clarify the notation. In the non-contextual setting there is a single context $X = \{x_0\}$ and the given policy set $\Pi : \{x_0 \mapsto a : a \in A\}$ is fully expressive. When we state results for the non-contextual version, $\Pi$ is always assumed to be this class.

Our upper bounds typically scale with $\log |\Pi|$ so they do not immediately yield meaningful guarantees when $|A| = \infty$. Nevertheless we will see how to obtain meaningful results here via discretization.
A.2 Results for smoothed regret

**Stochastic Setting.** In the stochastic setting, our main algorithm is SmoothPolicyElimination, with pseudocode in Algorithm 1. For the generalization, rather than use the Smooth operator, we use the kernel $K$ in the variance constraint, action selection scheme, and importance weighted loss. We always use the rectangular kernel $K_h$ and we also make a uniformity assumption on the metric space.

**Assumption 2** (Metric assumptions for zooming). We assume that $\sup_{a,a',h} \frac{\nu(B(a,2h))}{\nu(B(a',h))} \leq \alpha < \infty$.

If $a = a'$ in the supremum, then by the doubling property the ratio is at most $2^d$. On the other hand, for $a \neq a'$, the doubling property alone does not yield a finite bound on $\alpha$. However, in many cases, such as the unit interval or any convex subset of $\mathbb{R}^d$ and any $\ell_p$ norm, it is not hard to verify that $\alpha$ is finite. Typically $\alpha = O(2^d)$.

Let us introduce the notation for the general case, which is not substantially different from before. Recall that the smoothed loss for policy $\pi \in \Pi$ is

$$\lambda_h \triangleq \mathbb{E}_{(x,\ell)\sim D}[\langle K_h \pi(x), \ell \rangle],$$

The optimal loss is $\text{Bench}(\Pi_h) = \inf_{\pi \in \Pi} \lambda_h(\pi)$ and the $\epsilon$-optimal policies are

$$\Pi_{h,\epsilon} \triangleq \{ \pi \in \Pi : \lambda_h(\pi) \leq \text{Bench}(\Pi_h) + \epsilon \}.$$

The projection of a policy set $\Pi'$ onto context $x$ is $\Pi'(x) = \{ \pi(x) : \pi \in \Pi' \}$. The key quantity is

$$M_h(\epsilon, \delta) \triangleq \mathbb{E}_{x \sim D} [N_\delta(\Pi_{h,\epsilon}(x))],$$

where recall that $N_\delta(A)$ is the $\delta$-packing number of $A \subset A$. The smoothing coefficient is

$$\theta_h(\epsilon_0) \triangleq \sup_{\epsilon \geq \epsilon_0} M_h(12\epsilon, h)/\epsilon$$

**Theorem 9.** Let $(A, \rho, \nu)$ be a metric space satisfying Assumption 1 and Assumption 2. Then SmoothPolicyElimination (Algorithm 1) has

$$\text{Regret}(T, \Pi_h) \leq O\left( \inf_{\epsilon_0 \geq 1/T} T\epsilon_0 + \theta_h(\epsilon_0) \log(||\Pi|| \log_2(T)) \log(T) \right).$$

The proof of the theorem is deferred to Appendix D.

**Remark.** Clearly, Theorem 9 generalizes Theorem 1.

**Remark.** Actually Algorithm 1 can be analyzed under much weaker conditions. For general kernels (which may not satisfy $\text{supp}(K(a) \subset B(a, h))$) and without Assumption 2, it is not hard to extract a $\tilde{O}\left( \sqrt{T \kappa \log(||\Pi||/\delta)} \right)$ smooth regret bound from our proof. The assumptions simply enable faster rates for benign instances.

**Remark.** Further, Algorithm 1 actually achieves a high probability regret bound, which we have simplified to the expected regret bound presented in Theorem 9.

**Remark.** As we have described the algorithm, it requires knowledge of the marginal distribution over $X$, which appears in the computation of $V_m$ and in the optimization problem. Both of these can be replaced with empirical counterparts, and since the random variables are non-negative, via Bernstein’s inequality, the approximation only affects the regret bound in the constant factors. This argument has been used in several prior contextual bandit results (Dudik et al., 2011; Agarwal et al., 2014; Krishnamurthy et al., 2016), and so we omit the details here.
Moreover there exists a metric space \((\mathcal{X} \to \Delta(\mathcal{A}))\). \(W_1(\xi) \leftarrow 1\) for all \(\xi \in \mathcal{X}\).

\[\begin{align*}
\text{Algorithm 2 EXP4} \\
\text{Learning rate } \eta. \text{ Stochastic policies } \xi \in \Xi \subset (\mathcal{X} \to \Delta(\mathcal{A})). \ W_1(\xi) \leftarrow 1 \text{ for all } \xi \in \Xi.
\text{for } t = 1, \ldots, T \text{ do} \\
\quad \text{Define } P_t(\xi) \propto W_t \text{ and } p_t(\cdot | x_t) \triangleq \mathbb{E}_{\xi \sim P_t} \xi(\cdot | x_t). \ \text{Sample } a_t \sim p_t(\cdot | x_t).
\quad \text{Observe } \ell_t(a_t) \text{ and define}
\quad \hat{\ell}_t(\xi) \triangleq \frac{\xi(a_t | x_t)}{p_t(a_t | x_t)} \cdot \ell_t(a_t).
\quad \text{Update } W_{t+1}(\xi) \leftarrow W_t(\xi) \cdot \exp(-\eta \hat{\ell}_t(\xi))
\end{align*}\]

Adversarial setting. Our main algorithm for the adversarial setting is EXP4, with pseudocode displayed in Algorithm 2. As we have stated it, EXP4 takes as input a set of stochastic policies \(\Xi\) that map contexts to distributions over actions. In the general setting, we simply instantiate \(\Xi = \Pi_K\).

**Theorem 10 (EXP4 guarantee).** EXP4 with \(\Xi = \Pi_K\) admits \(\text{Regret}(T, \Pi_K) \leq O\left(\sqrt{T \kappa \log |\Pi|}\right)\).

The proof is not difficult and deferred to Appendix B.1. We now discuss some consequences, which demonstrate the flexibility and generality of our approach.

Instantiating Theorem 10 in special cases, we obtain

- \(\sqrt{KT \log |\Pi|}\) regret for the standard contextual bandits setup described in Example 4.
- \(\sqrt{T/h \cdot \log |\Pi|}\) regret against \(\Pi_h\) in the unit interval. Note that this matches the \(\Omega\left(\sqrt{T/h \cdot \log |\Pi|}\right)\) regret lower bound, as discussed in the remark following Corollary 2.

Adaptivity. For the adaptivity results, in the general setting we have a finite family of kernels \(\mathcal{K} \triangleq \{K_1, \ldots, K_M\}\) and we define \(\kappa_i \triangleq \sup_{a,a'} |(K_i)(a')| \) and \(\kappa_* = \min_{i \in [M]} \kappa_i\). Further define \(r \triangleq \frac{\max_{i \in [M]} \kappa_i}{\min_{i \in [M]} \kappa_i}\).

We consider Corral+EXP4, whose description is deferred to Appendix B. We have the following result regarding its adaptive regret against all \(\Pi_K, K \in \mathcal{K}\) simultaneously.

**Theorem 11.** Fix \(\alpha \in [0, 1]\). For any finite kernel family \(\mathcal{K} = \{K_1, \ldots, K_M\}\) Corral+EXP4 with parameter \(\alpha\) guarantees

\[\forall i \in [M] : \ \text{Regret}(T, \Pi_{K_i}) \leq O\left(T^{1/3} \left(\kappa_i \log(|\Pi|)M\right)^{1/3} \min \{M, \log r\}^{1/3} \left(\kappa_i/\kappa_*\right)^{2/3}\right)\cdot\]

Moreover there exists a metric space \((\mathcal{A}, \rho, \nu)\), a family of two kernels \(\mathcal{K} = \{K_1, K_2\}\) and functions \(f_1(T) = \Omega\left(T^{1/3} \kappa_1^{1/3}\right), f_2(T) = \Omega\left(T^{1/3} \kappa_2^{1/3}\right)\) such that no algorithm achieves

\[\text{Regret}(T, \Pi_{K_1}) < f_1(T) \quad \text{and} \quad \text{Regret}(T, \Pi_{K_2}) < f_2(T)\]

The proof of the upper bound is deferred to Appendix B and the proof of the lower bound is deferred to Appendix C.

First observe that the lower bound in Theorem 3 is essentially a consequence of the second claim here. Indeed in the proof we take \((\mathcal{A}, \rho, \nu)\) to be the \([0, 1]^d\) with \(\ell_\infty\) metric and with uniform base measure. The two
kernels are rectangular kernels with bandwidth \( h_1 = \frac{1}{4} \) and \( h_2 = h \). Since \( \kappa_1 = \Theta(1) \) and \( \kappa_2 = \Theta(h^{-d}) \), the lower bound then shows that no algorithm can achieve

\[
\forall h \in [0, 1]: \text{Regret}(T, \Pi_h) < \Omega \left( T^{\frac{1}{1+\alpha}} h^{-d\alpha} \right),
\]

in \( d \) dimensions. This yields the lower bound of Theorem 3 as a special case.

The setup above almost yields the upper bound in Theorem 3, except that we can only compete with a finite set of kernels. For example, choosing \( K_i \) as the rectangular kernel with bandwidth \( 2^{-i} \) recovers a weaker version of Theorem 3. For the stronger version that competes with all \( h \in [0, 1] \), we must exploit further structure.

**Theorem 12.** Fix \( \alpha \in [0, 1], d \in \mathbb{N} \). For \( A = [0, 1]^d \), a parametrization of Corral + EXP4 guarantees

\[
\forall h \in [0, 1]: \text{Regret}(T, \Pi_h) \leq \tilde{O} \left( T^{\frac{1}{1+\alpha}} (\log |\Pi|)^{\alpha} h^{-d\alpha} \right).
\]

In the non-contextual setting, a parametrization of Corral + EXP4 achieves a uniformly smoothed regret of \( \tilde{O} \left( T^{\frac{1}{1+\alpha}} h^{-d\alpha} \right) \). Moreover, in the non-contextual setting no algorithm achieves

\[
\forall h \in [0, 1]: \text{Regret}(T, \Pi_h) < \Omega \left( T^{\frac{1}{1+\alpha}} h^{-d\alpha} \right).
\]

The proof of the upper bounds is deferred to Appendix B and the proof of the lower bound follows from the discussions above.

Theorem 12 provides an improvement over the upper bound in Theorem 11 in that we may compete with an infinite family of kernels. The dependence on \( T, |\Pi| \) and \( \kappa \) in the regret bound is unchanged. Indeed, for the rectangular kernel \( K_h \) in \( d \) dimensions, we have \( \kappa = O(h^{-d}) \) and of course \( \kappa_* = O(1) \) here. Therefore, the main improvement is that we have eliminated the dependence on the number of kernels, \( M \).

We also provide a refinement for the non-contextual version, eliminating the dependence on \( \log |\Pi| \), which is infinite in this case. This, coupled with the lower bound that we have already discussed, demonstrates the optimal uniformly smoothed regret rate.

### A.3 Results for Lipschitz regret

**Stochastic setting.** In the stochastic setting, recall that \( \lambda_0(\pi) \) is unsmoothed expected loss for policy \( \pi \), so that \( \Pi_{0,\epsilon} \) are the \( \epsilon \)-optimal policies on the unsmoothed loss. As in the prequel, the **policy zooming coefficient** is

\[
\psi_L(\epsilon_0) \triangleq \sup_{\epsilon \geq \epsilon_0} \frac{M(12L\epsilon, \epsilon)}{\epsilon}.
\]

Recall also the **zooming dimension**, which for zooming constant \( \gamma > 0 \), is

\[
z \triangleq \inf \left\{ d > 0 : M(12L\epsilon, \epsilon) \leq \gamma \epsilon^{-d}, \forall \epsilon \in (0, 1) \right\}.
\]

**Slivkins (2014)** defines the **contextual zooming dimension**, which considers the growth of the \( \epsilon \)-covering number of the set \( \{ (x, a) : \mathbb{E}[\ell(a) | x] \leq \min_{a' \in A} \mathbb{E}[\ell(a') | x] \leq \epsilon \} \) in terms of \( \epsilon \). Although both notions measure the size of certain near-optimal sets, they are generally incomparable; we highlight several differences between the two notions:

1. **Slivkins (2014)** needs to assume a metric structure on \( A \times \mathcal{X} \), whereas we only assume a metric structure on \( A \). In addition, Slivkins (2014)'s contextual zooming dimension is at worst the covering dimension of \( A \times \mathcal{X} \), whereas our notion of zooming dimension is at worst the covering dimension of \( A \). On the other hand, our bound scales with \( \log |\Pi| \) while his does not.
2. Aside from the metric structure, Slivkins (2014)'s contextual zooming dimension is only dependent on the conditional distribution of loss given context $D(\ell | x)$. In contrast, our notion is dependent on the policy class $\Pi$, along with $D$, the joint distribution of $(x, \ell)$, which admits policy class and distribution specific upper bounds.

3. Finally, Slivkins (2014) considers a setting where contexts are adversarial chosen, and so his contextual zooming dimension considers pessimistic context arrivals. On the other hand, our definition involves an expectation over contexts, which may be more favorable.

We have the following generalization of Theorem 5.

**Theorem 13.** SmoothPolicyEliminationL with parameter $L$ has

$$
\text{Regret}(T, \Pi) \leq O \left( \inf_{\epsilon_0 > 1/T} TL\epsilon_0 + \psi_L(\epsilon_0)/L \cdot \log(|\Pi| \log_2(T)) \log(T) \right).
$$

With zooming dimension $z$ for constant $\gamma$ this is $\tilde{O}(T^{1+z/2} L^{2z/2} (\gamma \log(|\Pi| \log_2(T)/\delta))^{1/2})$.

The proof of the theorem is deferred to Appendix D. As before Theorem 5 is a special case.

**Adversarial setting.** The following is an immediate consequence of Theorem 10 and Lemma 8.

**Corollary 14.** If each $\ell_t$ is $L$-Lipschitz, $(A, \rho, \nu)$ satisfies Assumption 1, then EXP4, with rectangular kernel with $h = \Theta((\log |\Pi|/T)^{1/2} L^{1/2})$, satisfies

$$
\text{Regret}(T, \Pi) \leq O \left( L^{d/2} T^{d+1/2} (\log |\Pi|)^{1/2} \right). \tag{8}
$$

Instantiating Corollary 14 in special cases, we get:

- $T^{2/3}(L \log |\Pi|)^{1/3}$-Lipschitz regret for the unit interval metric.

- The optimal $T^{d+1/2} L^{d+1/2}$-Lipschitz regret for non-contextual problems with $d$-dimensional metric, matching prior results (Kleinberg et al., 2019; Bubeck et al., 2011b).

- $T^{d+1/2} L^{d+1/2}$ for the Lipschitz-CB setting of Cesa-Bianchi et al. (2017) with $p$-dimensional context space and $d$-dimensional action space, which matches their result.

**Adaptivity.** The next result is our general adaptive result for Lipschitz regret. It extends Theorem 7 to higher dimension.

**Theorem 15.** Fix $\alpha \in [0, 1]$ and $d \in \mathbb{N}$. For $A = [0, 1]^d$, a parametrization of Corral+EXP4 guarantees

$$
\text{Regret}(T, \Pi) \leq \tilde{O} \left( T^{1+(d+1)\alpha} (\log |\Pi|)^{1+(d+1)\alpha} L^{d+1/2} \right),
$$

when losses are $L$-Lipschitz. Crucially the algorithm does not need to know $L$. In the non-contextual setting, a parametrization achieves a regret of $\tilde{O} \left( T^{1+(d+1)\alpha} L^{d\alpha} \right)$. Moreover, there exists a function $f_L(T) = \Omega \left( T^{1+(d+1)\alpha} L^{d\alpha} \right)$ such that for any algorithm

$$
\lim \inf_{T \to \infty} \sup_{L} \sup_{\lambda \in \Lambda(L)} (f_L(T))^{-1} \text{Regret}(T, \Pi) > 1,
$$

where $\Lambda(L)$ is the set of $L$-Lipschitz instances whose loss functions take values in $[0, 1]$.

The proof of the upper bounds is deferred to Appendix B and the proof of the lower bound is deferred to Appendix C.
B Adaptive Algorithms

In this section we prove the upper bounds in Theorem 11 and Theorem 15, as well as Theorem 12. These yield the upper bounds in Theorem 3 and Theorem 7 as special cases. We start by describing the EXP4 variant that we use, and showing that it has a certain stability guarantee.

B.1 EXP4 and a stability guarantee

The main algorithm we analyze in this section is EXP4. Following the standard analysis, we start with a simple lemma.

Lemma 16. Define \( \kappa \triangleq \max_{\xi \in \Xi} \max_{x \in \mathcal{X}, a \in \mathcal{A}} \xi(a | x) \). With \( \eta = \sqrt{\frac{2 \log |\Xi|}{T \kappa}} \), Algorithm 2 satisfies

\[
\mathbb{E} \sum_{t=1}^{T} \ell_t(a_t) - \min_{\xi \in \Xi} \mathbb{E} \sum_{t=1}^{T} \ell_t \left( \hat{a}_t \right) \leq \sqrt{2TK \log |\Xi|}. \tag{9}
\]

Proof. From the standard Hedge analysis (Freund and Schapire, 1997), we have the deterministic inequality:

\[
\sum_{t=1}^{T} \mathbb{E}_{\xi \sim P_t} \hat{\ell}_t(\xi) - \min_{\xi \in \Xi} \sum_{t=1}^{T} \hat{\ell}_t(\xi) \leq \frac{\eta}{2} \sum_{t=1}^{T} \mathbb{E}_{\xi \sim P_t} \hat{\ell}_t(\xi)^2 + \frac{\log |\Xi|}{\eta}, \tag{10}
\]

where \( P_t \) is the HEDGE distribution defined in Algorithm 2.

Now, by standard importance weighting arguments we have (1) \( \mathbb{E}_{\xi \sim P_t} \hat{\ell}_t(\xi) = \ell_t(a_t) \) and (2) \( \mathbb{E}_{a_t \sim p_t} \hat{\ell}_t(\xi) = \mathbb{E}_{a_t \sim \xi(x_t)} \ell_t(a) \). For the variance term, we have

\[
\mathbb{E}_{a_t, \xi} \hat{\ell}_t(\xi)^2 \leq \kappa \mathbb{E}_{a_t} \xi(a_t | x_t)^2 \leq \kappa \int p_t(a_t | x_t) d\lambda(a) \leq \kappa \|\ell_t\|^2_{\infty}
\]

Therefore, taking expectation over both sides of (10), we have

\[
\mathbb{E} \sum_{t=1}^{T} \mathbb{E}_{\xi \sim P_t} \hat{\ell}_t(\xi) - \mathbb{E} \min_{\xi \in \Xi} \sum_{t=1}^{T} \hat{\ell}_t(\xi) \leq \mathbb{E} \sum_{t=1}^{T} \frac{\eta \kappa}{2} \|\ell_t\|^2_{\infty} + \frac{\log |\Xi|}{\eta}. \tag{11}
\]

Applying Jensen’s inequality on the left hand side, using the fact that \( \|\ell_t\|_{\infty} \leq 1 \), optimizing for \( \eta \), we obtain the guarantee. \( \square \)

Instantiating \( \Xi = \Pi_K \) proves Theorem 10.

We show a stability result of Algorithm 2 (See (Agarwal et al., 2016, Definitions 3 and 14) for formal definitions of stability and weak stability), which will be useful for our adaptive algorithms. For stability, we consider a slightly different protocol, displayed in Protocol 1. The learner is now presented with randomized loss functions \( \ell_t \) which are generated by importance weighting an original loss function \( \hat{\ell}_t \) with some probability \( p_t \) set by the adversary. Note that here, the losses presented to the learner are not guaranteed to be bounded, but we do have variance information, via \( p_t \). The original losses \( \ell_t \) are bounded in \([0, 1]\). Note further that \( p_t \) is revealed at the beginning of round \( t \).

Definition 17 (See Agarwal et al. (2016), Definitions 3 and 14). A learner with policy class \( \Xi \) is called \((\alpha, R(T))\)-stable, if in Protocol 1 it achieves

\[
\mathbb{E} \sum_{t=1}^{T} \hat{\ell}_t(a_t) - \min_{\xi \in \Xi} \mathbb{E} \sum_{t=1}^{T} \langle \xi(x_t), \hat{\ell}_t \rangle \leq \mathbb{E}[\rho]^\alpha \cdot R(T). \tag{12}
\]

where \( \rho \triangleq \max_{t \in [T]} \frac{1}{p_t} \).
Protocol 1 Learning with importance-weighted losses in Corral subalgorithms

\[
\text{for } t = 1, 2, \ldots, T: \text{ do }
\]

- Adversary generates context \( x_t \), original loss function \( \tilde{\ell}_t(\cdot) \), and a revealing probability \( p_t > 0 \).
- Adversary draws \( Q_t \sim \text{Ber}(p_t) \), and uses the revealed loss function \( \ell_t(\cdot) := \frac{Q_t}{p_t} \tilde{\ell}_t(\cdot) \).
- Learner takes action \( a_t \), and observes \( \ell_t(a_t) \).

end for

Here, the definition of \((\alpha, R(T))\)-stability is slightly different from Agarwal et al. (2016, Definition 3), in that the right hand side of the regret has term \( \mathbb{E}[\rho^\alpha] \) as opposed to \( \mathbb{E}[-\log(1 + \rho)] \). This has no bearing on the analysis of Corral, but is a requirement here as we will see. Roughly, we want to transform \( 1/2 \) stability to \( \alpha \)-stability for \( \alpha \in [0, 1/2] \), but in doing so, \( \alpha \) will appear outside the expectation.

Agarwal et al. (2016) shows that EXP4 is \((1/2, \sqrt{K T \log |\Pi|})\)-weakly stable in the discrete action setting, where \( K \) is the number of actions. Here, we give a refined characterization of the stability parameters in two aspects:

1. We replace the parameter \( K \) in the discrete action case with \( \kappa \), which acts as the “effective number of actions” for the continuous case.
2. We show that the choice of the first parameter can be extended from \( 1/2 \) to any number between 0 and \( 1/2 \), with appropriate setting of the second parameter.

For this section only, we use \( \xi(x_t) \in \Delta(A) \) to denote the distribution for expert \( \xi \) on context \( x_t \). Thus the expected loss for expert \( \xi \) on round \( t \) is \( \langle \xi(x_t), \ell_t \rangle \).

Theorem 18. A variant of EXP4 (Algorithm 2) is \( \left( \frac{\alpha}{1+\alpha}, O \left( T^{1+\alpha} \bar{\kappa} \log |\Xi| \right) \right) \) stable, for each \( \alpha \in [0, 1] \).

Proof. We first show a weaker form of stability. Suppose that \( \hat{\rho} \geq \max_{t \in [T]} \frac{1}{p_t} \) is provided to the algorithm ahead of time. Then following the proof of Lemma 16, we have

\[
\mathbb{E} \sum_{t=1}^{T} \ell_t(a_t) - \min_{\xi \in \Xi} \mathbb{E} \sum_{t=1}^{T} \langle \xi(x_t), \ell_t \rangle \leq \mathbb{E} \frac{\eta \kappa}{2} \sum_{t=1}^{T} \| \ell_t \|_\infty^2 + \frac{\log |\Xi|}{\eta}.
\]

The key observation is that in Protocol 1,

\[
\mathbb{E} \sum_{t=1}^{T} \| \ell_t \|_\infty^2 \leq \mathbb{E} \sum_{t=1}^{T} \frac{Q_t}{p_t^2} = \sum_{t=1}^{T} \mathbb{E} \frac{1}{p_t} \leq T \hat{\rho}
\]

Therefore, with the choice of \( \eta = \sqrt{\frac{2 \log |\Xi|}{T \hat{\rho}}} \), and using the fact that the conditional expectation of \( \ell_t \) is \( \tilde{\ell}_t \), we get

\[
\mathbb{E} \sum_{t=1}^{T} \tilde{\ell}_t(a_t) - \min_{\xi \in \Xi} \mathbb{E} \sum_{t=1}^{T} \langle \xi(x_t), \tilde{\ell}_t \rangle \leq \sqrt{2 \kappa T \log |\Xi| \cdot \hat{\rho}}.
\]

This proves a weaker version of stability, where a bound on \( \rho \) is specified in advance. The stronger version is based on the “doubling trick” argument in Agarwal et al. (2016, Theorem 15). We run EXP4 with a guess for \( \hat{\rho} \) and if we experience a round \( t \) where \( \frac{1}{p_t} > \hat{\rho} \), we double our guess and restart the algorithm, always with learning rate \( \eta = \sqrt{\frac{2 \log |\Xi|}{T \hat{\rho}}} \). In their Theorem 15, they prove that if an algorithm is weakly stable in the
We start with the most abstract formulation, with pseudocode presented in Algorithm 3. Given a family

\[ \text{Algorithm 3} \]

where \( B \equiv \alpha \)

\[ \text{Lemma 19.} \]

Suppose Algorithm 3 is run with learning rate \( \eta \), time horizon \( T \), kernel family \( \mathcal{K} = \{ K_1, \ldots, K_M \} \).

Define \( B = \{ \lfloor \log \kappa_K \rfloor : K \in \mathcal{K} \} \).

For \( b \in B \), define \( \mathcal{K}_b = \{ K \in \mathcal{K} : \lfloor \log \kappa_K \rfloor = b \} \), and \( \Xi_b = \{ K \pi(x) : \pi \in \Pi, K \in \mathcal{K}_b \} \).

For \( b \in B \) let \( \text{ALG}_b \) be an instance of EXP4 with restarts with policy class \( \Xi_b \), and time horizon \( T \).

Run Corral with learning rate \( \eta \), time horizon \( T \), and subalgorithms \( \{ \text{ALG}_b \}_{b \in B} \).

Define \( r \triangleq \max_{K \in \mathcal{K}} \kappa_{K} / \min_{K \in \mathcal{K}} \kappa_{K} \) and \( \kappa_* \triangleq \min_{K \in \mathcal{K}} \kappa_K \). Observe that \( B \leq \min \{ M, \log r + 1 \} \). We have the following guarantee of Algorithm 3.

**Lemma 19.** Suppose Algorithm 3 is run with learning rate \( \eta \) and horizon \( T \). Then, for all \( \alpha \in [0, 1] \), it has the following regret guarantee simultaneously for all kernels \( K \in \mathcal{K} \):

\[ \text{Regret}(T, \Pi_{K}) \leq \tilde{O} \left( \min \left\{ \frac{M \log r}{\eta} \right\} + T \eta + T \left( \eta \ln(|\Pi| M) \kappa_K \right)^\alpha \right). \]

**Proof:** This is almost a direct consequence of \textit{Agarwal et al.} (2016, Theorem 4). By the definition of \( \mathcal{K}_b \) and \( \Xi_b \), \( \kappa_b \triangleq \max_{\xi \in \Xi_b} \max_{a, x} \xi(a|x) \leq 2^b \). In addition, \( |\Xi_b| \leq |\Pi| \cdot M \). Since for all \( K \in \mathcal{K}_b \) we have \( \lfloor \log \kappa_K \rfloor = b \), therefore \( \kappa_K \in (2^{b-1}, 2^b] \). By applying Theorem 18 we see that EXP4 with restarting has the stability guarantee when measuring regret against \( \text{Bench}(\Pi_{K_i}) \) for each \( K_i \in \mathcal{K}_b \).

Now, by Theorem 4 of (Agarwal et al., 2016), Corral ensures

\[ \forall b \in [B], \forall K \in \mathcal{K}_b : \text{Regret}(T, \Pi_{K}) \leq \tilde{O} \left( \frac{B}{\eta} + T \eta - \frac{\mathbb{E}[\rho_b]}{\eta \log T} + T \frac{1}{1+\alpha} \left( \mathbb{E}[\rho_b] \kappa_K \log(|\Pi| M) \right)^\alpha \right) \]

Optimizing over \( \mathbb{E}[\rho_b] \) gives

\[ \forall K \in \mathcal{K} : \text{Regret}(T, \Pi_{K}) \leq \tilde{O} \left( \frac{B}{\eta} + T \eta + T \left( \eta \kappa_K \log(|\Pi| M) \right)^\alpha \right). \]

The result follows by observing that \( B \leq \min \{ M, \log r \} \).
Proof of upper bound in Theorem 11. We simply run Algorithm 3 with
\[ \eta = \frac{B^{1/\alpha}}{T^{1/\alpha} (\ln(|\Pi| |\mathcal{M}| \kappa_\star) + \alpha)} \]
and apply Lemma 19.

Proof of upper bounds in Theorem 12. Recall that for Theorem 12 we are in the $d$-dimensional cube with uniform base measure and with $\ell_\infty$ metric. Our goal is to obtain a uniformly-smoothed regret guarantee for all bandwidths $h \in [0, 1]$, where we are using the rectangular kernel. This requires a bit more work.

First, set $D \triangleq d^{2d + 2} T^2$ and form the discretized set:
\[ \mathcal{H} = \left\{ h \in \left\{ \frac{1}{T}, \frac{2}{T}, \ldots, 1 \right\} : 1 \leq \frac{1}{h^{\alpha}} \leq 2^{(\log_2 T) + 1} \right\}. \]
We run Corral with kernel class $\mathcal{K} = \{ K_h : h \in \mathcal{H} \}$ and use EXP4 with restarts as the sub-algorithms. As $|\mathcal{H}| \leq d^{2d + 2} T^2$, applying Theorem 11 gives
\[ \forall h \in \mathcal{H} : \text{Regret}(T, \Pi_h) \leq \tilde{O} \left( T^{1/\alpha} h^{-d\alpha} (\log |\Pi|)^{\alpha} \right). \tag{14} \]
We now must lift (14) to all $h \in [0, 1]$. We have the following lemma.

Lemma 20. For any loss $\ell : \mathcal{A} \rightarrow [0, 1]$ and bandwidth $h \geq T^{-1/d}$, there exists $\hat{h} \in \mathcal{H}$ such that $\frac{1}{h^{\alpha}} \leq \frac{2}{\hat{h}^{\alpha}}$ and $\sup_a \langle K_h(a) - K_{\hat{h}}(a), \ell \rangle \leq \frac{1}{T}$.

The proof is technical and is deferred to the end of this section. Applying this lemma allows us to obtain a smoothed regret bound for $h \notin \mathcal{H}$ by translating to $h \in \mathcal{H}$, since the former benchmark is smaller by at most $O(1)$ while the latter has $h^{-d} \leq 2(h)^{-d}$. This yields Theorem 12.

For the non-contextual upper bound, we instantiate each sub-algorithm with a policy set $\Pi' : \{ x_0 \mapsto a : a \in \mathcal{A}' \}$ where $\mathcal{A}'$ is a $\varepsilon$-covering of $\mathcal{A}$, which satisfies $|\mathcal{A}'| \leq O(e^{-d})$. The above analysis carries through, and to translate to $a \notin \mathcal{A}'$ we require a different discretization lemma.

Lemma 21. For $\rho(a, a') \leq \varepsilon$ and $\ell : \mathcal{A} \rightarrow [0, 1]$, we have $| \langle K_h(a) - K_{\hat{h}}(a'), \ell \rangle | \leq 4d \varepsilon h^{-d}$.

The proof is deferred to the end of this section.

To finish the proof set $\varepsilon = \frac{1}{4d T^{1/\alpha}}$ and note that for $h < T^{-1/d}$ the desired guarantee is trivial. Thus for all $h \geq T^{-1/d}$ the cumulative approximation error introduced by discretization is at most 1 while the policy set $\Pi'$ has $|\Pi'| \leq O(d \log d T)$.

Proof of upper bounds in Theorem 15. For a finite set of bandwidths $\mathcal{H}$ let us apply Lemma 19 with $\mathcal{K} = \{ K_h : h \in \mathcal{H} \}$ to obtain
\[ \forall h \in \mathcal{H} : \text{Regret}(T, \Pi_h) \leq \tilde{O} \left( \frac{||\mathcal{H}|}{\eta} + T\eta + T \left( \eta \log(|\Pi||\mathcal{H}|) h^{-d} \right)^\alpha \right) \]
Applying Lemma 4, we know that
\[ \min_{\pi \in \Pi} \mathbb{E} \sum_{t=1}^{T} \langle K_h \pi(x_t), \ell_t \rangle \leq \min_{\pi \in \Pi} \mathbb{E} \sum_{t=1}^{T} \ell_t(\pi(x_t)) + TLh, \]
and so we obtain
\[ \text{Regret}(T, \Pi) \leq \min_{h \in \mathcal{H}} TLh + \tilde{O} \left( \frac{||\mathcal{H}|}{\eta} + T\eta + T \left( \eta \log(|\Pi||\mathcal{H}|) h^{-d} \right)^\alpha \right). \]
Define $L = \{2^i : i \in \{1, 2, \ldots, \lceil \log_2(T) \rceil \} \}$ to be an exponentially spaced grid. If the true parameter $L \geq T$ then the bound is trivial, and otherwise $L \leq \hat{L} \leq 2L$ from some $\hat{L} \in L$. We choose $H$ of size $\lceil \log_2(T) \rceil$ to optimize the above bound for each value of $\hat{L} \in L$. Specifically, set

$$
H = \left\{ h_i = (\eta \log(|\Pi| \log_2(T)))^{\frac{\alpha}{\alpha + 1}} 2^{\frac{i}{\alpha + 1}} : i \in \lceil \log_2(T) \rceil \right\}.
$$

This yields

$$
\text{Regret}(T, \Pi) \leq \min_{h \in H} T L h + \hat{O}\left( \frac{|H|}{\eta} + T \eta + T \left( \eta \log(|\Pi| |H|) h^{-d}\right)^{\alpha}\right).
$$

$$
\leq \min_{h \in H} T \hat{L} h + \hat{O}\left( \frac{|H|}{\eta} + T \eta + T \left( \eta \log(|\Pi| |H|) h^{-d}\right)^{\alpha}\right).
$$

$$
\leq \hat{O}\left( T \hat{L} \frac{d\alpha}{\alpha + 1} (\eta \log |\Pi|)^{\frac{\alpha}{\alpha + 1}} + \frac{1}{\eta} + T \eta \right).
$$

$$
\leq \hat{O}\left( T L \frac{d\alpha}{\alpha + 1} (\eta \log |\Pi|)^{\frac{\alpha}{\alpha + 1}} + \frac{1}{\eta} + T \eta \right).
$$

We finish the proof by tuning the master learning rate $\eta$ while ignoring $L$. This gives

$$
\eta = T \frac{-(d\alpha + 1)}{1 + (d+1)\alpha} (\log |\Pi|)^{\frac{\alpha}{1 + (d+1)\alpha}},
$$

and the overall regret bound is

$$
\text{Regret}(T, \Pi) \leq \hat{O}\left( L \frac{d\alpha}{1 + (d+1)\alpha} T^{\frac{1 + d\alpha}{1 + (d+1)\alpha}} (\log |\Pi|)^{\frac{\alpha}{1 + (d+1)\alpha}} \right).
$$

As in the proof of Theorem 12, for the non-contextual case we discretize the action set to a minimal $\epsilon$ cover $\mathcal{A}'$ for $\mathcal{A}$. Choosing $\epsilon = (4dT^2)^{-1}$ as in that proof suffices here as well.

We remark that Theorem 15 is not a direct corollary of Theorem 12. Rather we must start with Lemma 19 and first tune $h$ to balance the sub-algorithm’s regret with the $TLh$ term. Then we tune the master’s learning rate. In particular for fixed exponent $\alpha$ the master learning rate for Theorem 12 and Theorem 15 are different.

### B.3 Proof of the discretization lemmas

**Proof of Lemma 20.** Recall the definition of $H$:

$$
H = \left\{ h \in \left\{ \frac{1}{D}, \frac{2}{D}, \ldots, 1 \right\} : 1 \leq \frac{1}{h^d} \leq 2^{|\log T| + 1} \right\}.
$$

We choose $h_D = \frac{|h_D|}{D}$. Note that $h_D$ is a multiple of $\frac{1}{D}$. In addition, we note that $h \geq T^{-1}$, and $h_D \geq h - \frac{1}{d2^d + 2T^2} \geq h - \frac{1}{4d^2T^2} \geq h(1 - \frac{1}{4dT})$. Therefore, by Fact 22 below, $\frac{1}{h_D^d} \leq \frac{1}{h^d}(1 - \frac{1}{4dT})^d \leq \frac{2}{h^d} \leq 2T \leq 2^{|\log T| + 1}$. Hence, $h_D$ is in $H$.

Moreover, $\nu(B(a, h)) \geq h^d$, and

$$
\nu(B(a, h) \Delta B(a, h_D)) \leq (2h)^d - (2h_D)^d
$$

$$
\leq (2h)^d(1 - (1 - \frac{1}{d2^d + 2T^2})^d)
$$

$$
\leq \frac{(2h)^d}{2^d T^d} = \frac{h^d}{2T}.
$$
Therefore, applying Fact 23, we obtain

\[ |\langle K_h(a) - K_{h_D}(a), \ell \rangle| \leq \frac{2\nu(B(a, h)\Delta B(a, h_D))}{\max \{\nu(B(a, h)), \nu(B(a, h_D)) \}} \leq \frac{1}{T}. \]

\[ \square \]

**Proof of Lemma 21.** Since we are using the \( \ell_\infty \) distance and \( \rho(a, a') \leq \varepsilon \), we have that \( \nu(B(a, h)\Delta B(a', h)) \leq 2 \|a - a'\|_1 \leq 2d\varepsilon \). Applying Fact 23 we obtain

\[ |\langle K_h(a) - K_{h}(a'), \ell \rangle| \leq \frac{2\nu(B(a, h)\Delta B(a', h))}{\max \{\nu(B(a, h)), \nu(B(a', h)) \}} \leq 4d\varepsilon h^{-d}. \]

\[ \square \]

**Fact 22.** For \( T, d \geq 1 \), \( \left( \frac{1}{4} \right)^d \leq 1 + \frac{1}{T} \).

**Proof.** We use the following simple facts: for all \( x \) in \([0, 1]\), \( e^x \leq 1 + 2x \) and \( e^{-x} \leq 1 - \frac{1}{2}x \). The proof is completed by noting that \( \frac{1}{(1 - \frac{1}{4})^d} \leq e^{\frac{1}{2}d} \leq 1 + \frac{1}{T} \).

\[ \square \]

**Fact 23.** For sets \( S_1 \) and \( S_2 \), and a loss function \( \ell : A \rightarrow [0, 1] \)

\[ \left| \frac{\int_{S_1} \ell(a) d\nu(a)}{\nu(S_1)} - \frac{\int_{S_2} \ell(a) d\nu(a)}{\nu(S_2)} \right| \leq \frac{2\nu(S_1 \Delta S_2)}{\max(\nu(S_1), \nu(S_2))} \]

**Proof.**

\[ \left| \frac{\int_{S_1} \ell(a) d\nu(a)}{\nu(S_1)} - \frac{\int_{S_2} \ell(a) d\nu(a)}{\nu(S_2)} \right| = \left| \int_{S_1} \ell(a) d\nu(a) \cdot \left( \frac{\nu(S_2)}{\nu(S_1)} - 1 \right) + \nu(S_1) \cdot \left( \int_{S_1} \ell(a) d\nu(a) - \int_{S_2} \ell(a) d\nu(a) \right) \frac{\nu(S_1)}{\nu(S_2)} \right| \]

\[ \leq \frac{\nu(S_1) \cdot \nu(S_1 \Delta S_2)}{\nu(S_1)} + \frac{\nu(S_1) \cdot \nu(S_1 \Delta S_2)}{\nu(S_2)} = \frac{2\nu(S_1 \Delta S_2)}{\nu(S_1)} \]

By symmetry, the above is also bounded by \( \frac{2\nu(S_1 \Delta S_2)}{\nu(S_1)} \). The proof is completed by taking the smaller of the two upper bounds.

\[ \square \]

### C Adaptive Lower Bounds

In this section, we prove the lower bounds in Theorem 11 and Theorem 15, showing that the exponent combinations we achieve with Corral are optimal. We start with two lemmas that describe the constructions and contain the main technical argument. In the next subsection we prove the theorems.

#### C.1 The constructions

The following two lemmas are based on a construction due to Locatelli and Carpentier (2018). Their work concerns adapting to the smoothness exponent, while ours focuses on the smoothness constant. We also use a similar construction to show lower bounds against uniformly-smoothed algorithms.

We focus on the stochastic non-contextual setting, where we consider policy class \( \Pi = \{ x_0 \mapsto a : a \in A \} \), and at each time, a dummy context \( x_0 \) is shown. We use the shorthand \( \text{Regret}(T, h) \) to denote \( \text{Regret}(T, \Pi_h) \). We define \( \Lambda \) to be the set of all functions from \( A \) to \([0, 1]\). A function \( \lambda \in \Lambda \) defines an instance where \( \ell(a) \sim \text{Ber}(\lambda(a)) \) for all \( a \in A \).
Lemma 24. Fix $h \in (0, 1/8]$. Suppose an algorithm ALG guarantees $\sup_{\lambda \in \Lambda} \text{Regret}(T, 1/4) \leq R_S(1/4, T)$ where $R_S(1/4, T) \leq \frac{\sqrt{T}}{20(8h)^3}$. Then there exists $\lambda \in \Lambda$ such that ALG has

$$\text{Regret}(T, h) \geq \min \left\{ \frac{T}{40 \cdot 2^d}, \frac{T}{400(8h)^d R_S(1/4, T)} \right\}.$$

Proof. We let $N = \lfloor 1/4h \rfloor ^d$. Note that as $h \leq 1/8$, $(1/8h)^d \leq N \leq (1/4h)^d$. We also define $\Delta = \min \left\{ \frac{N}{40(1/4,T)^d}, 1/4 \right\} \in (0, 1/4]$. By our assumption that $R_S(1/4, T) \leq \frac{\sqrt{T}}{20(8h)^3}$, we have

$$R_S(1/4, T) \leq \min \left\{ \frac{N^2 T}{200 R_S(1/4, T)}, \frac{NT}{20} \right\} = 0.2NT\Delta. \quad (15)$$

For each tuple $(s_1, \ldots, s_d) \in \lfloor [1/4h] \rfloor ^d$, we define a point $c_{s_1,\ldots,s_d} = (h(2s_1 - 1), \ldots, h(2s_d - 1))$. There are $N$ points in total, which we call $c_1, \ldots, c_N$. Define regions

$$H_i = B(c_i, h), i = 1, \ldots, N,$$

which are disjoint subsets in $[0, 1/2]^d$. Finally, define region $S = [1/2, 1]^d = B(c_0, 1/4)$, where $c_0 = (3/4, \ldots, 3/4)$. We define several plausible mean loss functions $\phi_0, \ldots, \phi_N \in \Lambda$:

$$\phi_0(a) = \begin{cases} 1/2, & a \notin S \\ 1/2 - \Delta/2, & a \in S \end{cases} \quad \text{and} \quad \phi_i(a) = \begin{cases} 1/2, & a \notin (H_i \cup S) \\ 1/2 - \Delta, & a \in H_i \\ 1/2 - \Delta/2, & a \in S \end{cases}$$

Note that $\mathbb{E}_{a \sim \text{Smooth}}(c_0) \phi_0(a) = 1/2 - \Delta/2$, and $\mathbb{E}_{a \sim \text{Smooth}}(c_i) \phi_i(a) = 1/2 - \Delta$.

The environments are parameterized by $\phi_i$ where losses are always bernoulli with mean $\phi_i$. Denote by $\mathbb{E}_i$ (resp. $\mathbb{P}_i$) the expectation (resp. probability) over the randomness of the algorithm, along with the randomness in environment $\phi_i$.

Observe that under environment $\phi_0$, for $h = 1/4$, we have $T \cdot \lambda^*_{1/4} = T \cdot (1/2 - \Delta/2)$. Since ALG guarantees that $\text{Regret}(T, 1/4) \leq R_S(1/4, T)$, we have

$$\mathbb{E}_0 \sum_{t=1}^{T} \phi_0(a_t) - T \cdot (1/2 - \Delta/2) \leq R_S(1/4, T).$$

As for all $a, \phi_0(a) - (1/2 - \Delta/2) = \Delta/2 \mathbb{1}\{a \notin S\}$, we get that

$$\sum_{t=1}^{T} \mathbb{E}_0 \mathbb{1}\{a_t \notin S\} \leq \frac{2R(1/4, T)}{\Delta}.$$

Denote by $T_i = \sum_{t=1}^{T} \mathbb{1}\{a_t \in H_i\}$ and observe that

$$\sum_{j=1}^{N} \mathbb{E}_0[T_j] \leq \mathbb{E}_0 \left[ \mathbb{1}\{a_t \in \cup_{j=1}^{N} H_j\} \right] \leq \sum_{j=1}^{T} \mathbb{E}_0 \left[ \mathbb{1}\{a_t \notin S\} \right] \leq \frac{2R(1/4, T)}{\Delta}.$$

By the pigeonhole principle, there exists at least one $i$ such that

$$\mathbb{E}_0[T_i] \leq \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}_0[T_j] \leq \frac{2R(1/4, T)}{N\Delta}. \quad (16)$$
Therefore, by Lemma 25 and the fact that $\Delta \leq \frac{1}{4}$, we have

$$KL(P_0, P_i) \leq E_0[T_i] \cdot (4\Delta^2) \leq \frac{8R(1/4, T)\Delta}{N}.$$

By the choice of $\Delta \leq \frac{N}{40R(1/4, T)}$, we have $KL(P_0, P_i) \leq 0.2$ and so Pinsker’s inequality yields $d_{TV}(P_0, P_i) \leq \sqrt{1/2KL(P_0, P_i)} \leq 0.4$. Therefore,

$$E_i[T_i] \leq E_0[T_i] + T \cdot d_{TV}(P_0, P_i) \leq \frac{2R(1/4, T)}{\sqrt{N\Delta}} + 0.4T \leq 0.8T.$$

where the first inequality is from the definition of the total variation distance and that $T_i \in [0, T]$ almost surely; the second inequality is by (16); the third inequality is by (15). Therefore, $E_i[T_i] \leq 0.8T$, which implies that on $\phi_i$

$$\text{Regret}(T, h) = E_i \sum_{t=1}^{T} \phi_i(a_t) - (1/2 - \Delta) \geq \frac{\Delta}{2} \cdot (T - E_i[T_i]) \geq \frac{\Delta}{2} \cdot 0.2T$$

$$\geq \min \left\{ \frac{T}{40 \cdot 2^d}, \frac{T}{400(8\Delta)^d R_{\lambda}(1/4, T)} \right\}.$$

Lemma 25. For $\Delta \in [0, \frac{1}{4}]$, $KL(P_0, P_i) \leq E_0[T_i] \cdot (4\Delta^2)$.

Proof. We abbreviate $l_i$ as the outcome of $l_i(a_t)$. We have the following:

$$KL(P_0, P_i) = \sum_{a_1, l_1, \ldots, a_T, l_T} P_0(a_1, l_1, \ldots, a_T, l_T) \log \frac{P_0(a_1, l_1, \ldots, a_T, l_T)}{P_i(a_1, l_1, \ldots, a_T, l_T)}$$

$$= E_0 \sum_{t=1}^{T} \log \frac{P_0(l_t | a_t)}{P_i(l_t | a_t)}$$

$$= E_0 \sum_{t=1}^{T} 1 \{ a_t \in H_i \} \cdot KL(\text{Ber}(1/2), \text{Ber}(1/2 - \Delta))$$

$$= E_0[T_i] \cdot (\frac{1}{2} \log (1 - 4\Delta^2)) \leq E_0[T_i] \cdot (4\Delta^2)$$

where the last inequality uses the fact that $\log(1 - x/2) \geq -x$ for $x \in [0, 1]$.

For the next lemma, let $\Lambda(L)$ be the set of all $L$-Lipschitz mean loss functions.

Lemma 26. Fix $L \geq 1$. Suppose an algorithm ALG guarantees $\sup_{\lambda \in \Lambda(1)} \text{Regret}(T, 0) \leq R_{Lip}(1, T)$ where $R_{Lip}(1, T) \leq \frac{T}{30} \cdot L^d$. Then there exists a loss function $\lambda \in \Lambda(L)$ such that

$$\text{Regret}(T, 0) \geq \min \left\{ \frac{T}{80}, \frac{TL^d}{3200R_{Lip}(1, T) \pi^d} \right\}.$$

Proof. We let $\Delta = \min \left\{ \left( \frac{L^d}{40R_{Lip}(1, T) \pi^d} \right)^{\frac{1}{d+1}}, 1/8 \right\} \in (0, 1/8]$, and $N = [L/4\Delta]^d$. As $L \geq 1$, $L/4\Delta \geq 2$. Therefore, $(L/8\Delta)^d \leq N \leq (L/4\Delta)^d$. Observe that by the choices of $\Delta$ and $N$:

$$\Delta \leq \frac{(\frac{L}{4\Delta})^d}{40R_{Lip}(1, T)(T)} \leq \frac{N}{40R_{Lip}(1, T)}.$$
By our assumption that $R_{\text{Lip}}(1, T) \leq \frac{T}{40} L^d$, we have that

$$R_{\text{Lip}}(1, T) \leq 0.2T \cdot \frac{L^d}{8^d (1/8)^{d-1}} \leq 0.2T \cdot \frac{L^d}{8^d \Delta^{d-1}} \leq 0.2N T \Delta,$$

where the first inequality is from that $R_{\text{Lip}}(1, T) \leq \frac{T}{40} L^d$; the second inequality is from the fact that $\Delta \leq \frac{1}{8}$; the third inequality is from the fact that $N \geq (L/8\Delta)^d$.

For each tuple $(s_1, \ldots, s_d) \in ([L/4\Delta])^d$, we define a point $c_{s_1, \ldots, s_d} = (\frac{2}{L}(2s_1 - 1), \ldots, \frac{2}{L}(2s_d - 1))$. There are $N$ points in total which we call $c_1, \ldots, c_N$. Define regions

$$H_i = B\left( c_i, \frac{\Delta}{L} \right), \ i = 1, \ldots, N,$$

which are disjoint subsets in $[0, 1/2]^d$. Finally, define region $S = [1/2, 1]^d = B(c_0, 1/4)$, where $c_0 = (3/4, \ldots, 3/4)$. We define several plausible mean loss functions $\phi_0 \in \Lambda(1), \phi_1, \ldots, \phi_N \in \Lambda(L)$:

\[
\phi_0(a) = \begin{cases} 
1/2 - (\Delta/2 - \|a - c_0\|_\infty), & a \in S \\
1/2, & \text{else}
\end{cases}
\]

and

\[
\phi_i(a) = \begin{cases} 
1/2 - (\Delta - L)\|a - c_i\|_\infty, & a \in H_i \\
1/2 - (\Delta/2 - ||a - c_0||_\infty), & a \in S \\
1/2, & \text{else}
\end{cases}
\]

Observe that $\phi_0$ is 1-Lipschitz, and each $\phi_i$ is $L$-Lipschitz for $i \geq 1$.

Each mean loss function $\phi_i$ defines an environment where realized losses are Bernoulli random variables. Denote by $E_i$ (resp. $P_i$) the expectation (resp. probability) over the randomness of the algorithm, along with the randomness in environment $\phi_i$.

For algorithm ALG, as it guarantees that $\text{Regret}(T, 0) \leq R_{\text{Lip}}(1, T)$ against all loss functions in $\Sigma(1)$, we get that

$$E_0 \sum_{t=1}^{T} \phi_0(a_t) - T \left( 1/2 - \Delta/2 \right) \leq R_{\text{Lip}}(1, T).$$

Denote by $T_i = \sum_{t=1}^{T} 1 \{ a_t \in H_i \}$. Observe that the instantaneous regret for playing in any $H_i$ is at least $\Delta/2$. Therefore, by pigeonhole principle, there exists at least one $i$ such that

$$E_0[T_i] \leq \frac{1}{N} \sum_{j=1}^{N} E_0[T_j] = \frac{1}{N} E_0 \sum_{j=1}^{N} T_j \leq \frac{2R_{\text{Lip}}(1, T)}{N \Delta}.$$ (17)

Following the exact same calculation as in the proof of Lemma 24 we get that $E_i[T_i] \leq 0.8T$, which implies that on instance $\phi_i$

$$\text{Regret}(T) \geq E_i \sum_{t=1}^{T} \phi_i(a_t) - \left( 1/2 - \Delta \right) \geq 0.2T \cdot \frac{\Delta}{2} \geq \min \left\{ \frac{T}{80}, \frac{L^{d+1}}{3200 R_{\text{Lip}}(1, T)^{\frac{d}{d+1}}} \right\}. \quad \Box$$

\subsection*{C.2 Proofs of the lower bounds}

\textbf{Proof of lower bound in Theorem 11.} The lower bound is a consequence of Lemma 24. Specifically, let $A$ be $[0, 1]^d$ equipped with $\ell_\infty$ metric and uniform base measure. Consider any $T \geq 2^{3d(1+\alpha)}$, and let $K_1$ be the rectangular kernel with bandwidth $1/4$ while $K_2$ has bandwidth $h = T^{\frac{1}{4\alpha + 11}} \in (0, \frac{1}{8})$.

Define $f_1(T) \triangleq \frac{4^d h^{1+\alpha}}{80 \cdot 2^{d(\alpha + 11)}}$ and $f_2(T) \triangleq \frac{T}{80 \cdot 2^{d(\alpha + 11)}}$. It can be easily checked that $f_1(T) = \Omega \left( T^{\frac{1}{1+\alpha}} \kappa_2^{\frac{\alpha}{\alpha + 1}} \right)$, $f_2(T) = \Omega \left( T^{\frac{1}{1+\alpha}} \kappa_2^{\frac{\alpha}{\alpha + 1}} \left( \kappa_2/\kappa_1 \right)^{\frac{\alpha^2}{\alpha + 1}} \right).$
Suppose for algorithm \( \text{ALG} \), \( \sup_{\lambda \in \Lambda} \text{Regret}(T, \Pi_{K_1}) < f_1(T) \). Now apply Lemma 24. We may take \( R_S(1/4, T) = f_1(T) \) which satisfies the precondition that \( R_S(1/4, T) \leq \frac{\sqrt{T}}{20(8h)^d} \), by our choice of \( h = T^{\frac{1}{(d+1)\alpha}} \) and \( T \geq 2^{3d(1+\alpha)} \). Provided this is satisfied, we may conclude that

\[
\sup_{\lambda \in \Lambda} \text{Regret}(T, \Pi_{K_2}) \geq \min \left\{ \frac{T}{40 \cdot 2^d}, \frac{1}{400 \cdot 8^d} T^{1+\alpha} h^{-d} \right\} = \frac{1}{5} \cdot 2^{d\alpha} T > f_2(T).
\]

This shows that for algorithm \( \text{ALG} \), \( \sup_{\lambda \in \Lambda} \text{Regret}(T, \Pi_{K_1}) < f_1(T) \) and \( \sup_{\lambda \in \Lambda} \text{Regret}(T, \Pi_{K_2}) < f_2(T) \) cannot hold simultaneously.

**Proof of lower bound in Theorem 15.** The proof is similar to above. Let \( f_L(T) \triangleq \frac{1}{600} T^{1+\frac{4d\alpha}{(d+1)\alpha}} L^{\frac{d\alpha}{1+\frac{(d+1)\alpha}{d\alpha}}} \). Assume \( \text{ALG} \) guarantees \( \sup_{\lambda \in \Lambda(1)} \text{Regret}(T, \Pi) \leq f_1(T) \), otherwise we have already proved what is required for \( \text{ALG} \). In applying Lemma 26 we may take \( L = T^{\frac{1+\frac{4d\alpha}{(d+1)\alpha}}{1-d\alpha}} \), and \( R_{\text{lip}}(1, T) = 2f_1(T) = \frac{1}{3200} T^{1+\frac{4d\alpha}{(d+1)\alpha}} \) which satisfies the precondition that \( R_{\text{lip}}(1, T) \leq \frac{1}{30} L^d T \). Provided this is satisfied, we may conclude that

\[
\sup_{\lambda \in \Lambda(L)} \text{Regret}(T, \Pi) \geq \min \left\{ \frac{T}{80}, \frac{1}{3200} T^{1-\frac{4d\alpha}{(d+1)(d+1)\alpha}} L^{\frac{d}{d+1}} \right\}
\]

\[
> \frac{1}{3200} T = 2f_L(T).
\]

This shows that for algorithm \( \text{ALG} \), \( \sup_{\lambda \in \Lambda(1)} \text{Regret}(T, \Pi) < 2f_1(T) \) and \( \sup_{\lambda \in \Lambda(L)} \text{Regret}(T, \Pi) < 2f_L(T) \) cannot hold simultaneously. Therefore, for every \( T \),

\[
\sup_{L} \sup_{\lambda \in \Lambda(L)} (f_L(T))^{-1} \text{Regret}(T, \Pi) \geq 2,
\]

which proves the lower bound.

**D Data-dependent Regret Bounds**

For the proof, we first state two lemmas, with proofs in the next subsection. Recall that we are assuming the marginal distribution over \( \mathcal{X} \) is known.

The first lemma provides a guarantee on the optimization problem (3). For a policy set \( \Pi' \subset \Pi \), bandwidth \( h \) and context \( x \), define \( A(x; \Pi', h) \triangleq \bigcup_{\pi \in \Pi'} B(\pi(x), h) = \bigcup_{\pi \in \Pi'} B(\pi, h) \) which is a subset of the action space. Similarly, let \( V(\Pi', h) = \mathbb{E}_x \nu(A(x; \Pi', h)) \). Finally, for a distribution \( Q \in \Delta(\Pi') \), bandwidth \( h \), and exploration \( \mu \), we define the action-selection distribution as

\[
q^\mu(a \mid x) \triangleq (1 - \mu) \sum_{\pi} Q(\pi)(K_h \pi(x))(a) + \mu.
\]

Note that \( q^\mu(\cdot \mid x) \) is fully supported on \( \mathcal{A} \) \( (q^\mu(a \mid x) > 0 \text{ for all } a \in \mathcal{A} ) \), so \( \mathbb{E}_{a \sim K_h \pi(x)} \frac{1}{q^\mu(a \mid x)} \) is well defined.

**Lemma 27.** For any subset \( \Pi' \subset \Pi \) with \( |\Pi'| < \infty \) and any distribution \( D \in \Delta(\mathcal{X}) \), the program (3) is convex and we have

\[
\min_{Q \in \Delta(\Pi')} \max_{\pi \in \Pi'} \mathbb{E}_{x \sim D} \mathbb{E}_{a \sim K_h \pi(x)} \left[ \frac{1}{q^\mu(a \mid x)} \right] \leq \frac{1}{1 - \mu} V(\Pi', h).
\]
Note that $V(P', h) \leq 1$, which yields a weaker, but more interpretable bound.

The following lemma gives a uniform deviation bound on $L_m(\pi)$ and $\mathbb{E}_{(x, \ell) \sim D} \langle K_{h_m}(x), \ell \rangle$ in epoch $m$.

Recall that in epoch $m$, the estimator $\hat{L}_m(\pi)$ is the median of several base estimators $\left\{ \hat{L}_m^i(\pi) \right\}_{i=1}^I$, where $I = \delta_T = 5\lceil \log(\|\Pi\| \log_2(T)/\delta) \rceil$ is the number of batches. In comparison to using the naive empirical mean estimator, this median-of-estimators has the advantage that it gets around the dependency of the range of the individual losses, therefore admitting sharper concentration.

**Lemma 28** (Concentration of median-of-means loss estimator). Fix $\Pi' \subset \Pi$, $h \in (0, 1)$, $\mu \leq 1/2$, $\delta \in (0, 1)$ and let $Q \in \Delta(\Pi')$ be the solution to (3). Let $I = 5\lceil \log(\|\Pi\| / \delta) \rceil$, $\tilde{n}$ be an integer, and $\{x_j, a_j, \ell_j(a_j)\}_{j=1}^n$ be a dataset of $n = I\tilde{n}$ samples, where $(x_j, \ell_j) \sim D$ and $a_j \sim q^\mu(\cdot | x_j)$. Define

$$\hat{L}(\pi) = \text{median}(\hat{L}^1(\pi), \ldots, \hat{L}^I(\pi)),$$

where $\hat{L}^i(\pi) = \frac{1}{\tilde{n}} \sum_{j=(i-1)\tilde{n}+1}^{i\tilde{n}} \frac{K_h(\pi(x_j))(a_j)}{q^\mu(a_j | x_j)} \ell_j(a_j)$. Then with probability at least $1 - \delta$, for all $\pi \in \Pi'$, we have

$$\left| \lambda_h(\pi) - \hat{L}(\pi) \right| \leq \sqrt{\frac{80\kappa_h V(P', h)}{n} \log(\|\Pi\| / \delta)}.$$

### D.1 Proofs of the main theorems

The proof proceeds inductively over the epochs and we will do both proofs simultaneously. In the proof of Theorem 9 we use $L(\pi) \triangleq \lambda_h(\pi)$, while for Theorem 13 we use $L(\pi) \triangleq \lambda_0(\pi) = \mathbb{E} \ell(\pi(x))$. In both cases $\pi^* \triangleq \arg\min_{\pi \in \Pi} L(\pi)$. For both proofs we use $L_m(\pi) \triangleq \lambda_{h_m}(\pi)$, noting that for Theorem 9, $L_m(\pi) = L(\pi)$. Recall the definitions of the “radii” $r_m$ which are either $2^{-m}$ or $L2^{-m}$ depending on the theorem statement. In epoch $m$ we prove two things, inductively:

1. $\pi^* \in \Pi_{m+1}$ (assuming inductively that $\pi^* \in \Pi_m$).
2. For all $\pi \in \Pi_{m+1}$ we have $L(\pi) \leq L(\pi^*) + 12r_{m+1}$.

Before proving these two claims, we first lower bound $n_m$ which provides a bound on the number of epochs. Assuming $\pi^* \in \Pi_m$, which we will soon prove, we have

$$n_m \geq \frac{\kappa_{h_m} V_m}{r_m^2} \geq \frac{\kappa_{h_m} \mathbb{E}_x V(B(\pi^*(x), h_m))}{r_m^2} \geq \frac{1}{r_m^2} = 2^{2m}.$$

The first inequality requires $\delta_T \geq 1$ (which follows since $\delta \leq 1/e$) while the third uses the fact that $\text{supp}(K_{h_m}(a)) \subset B(a, h_m)$ so that $\kappa_{h_m} \geq \sup_a \frac{1}{\nu(B(a, h_m))}$. Hence we know that there are at most $m_T \triangleq \log_2(T)$ epochs. Applying Lemma 28 to all $m_T$ epochs and taking a union bound, we have

$$\forall m \in [m_T], \forall \pi \in \Pi_m : \left| L_m(\pi) - \hat{L}_m(\pi) \right| \leq \sqrt{\frac{80\kappa_{h_m} V_m \delta_T}{n_m}}.$$

Here we are using the fact that $V_m = V(\Pi_m, h_m)$ where $V_m$ is defined in the algorithm. Plugging in the choices for $n_m \triangleq \frac{32\kappa_{h_m} V_m \delta_T}{r_m^2}$ the above inequality simplifies to

$$\forall m \in [m_T], \forall \pi \in \Pi_m : \left| L_m(\pi) - \hat{L}_m \right| \leq r_m / 2$$

We operate under the event that these inequalities hold, which occurs with probability $\geq 1 - \delta$. 29
Let us now prove the two inductive claims. For the base case, since \( \Pi_1 \leftarrow \Pi \) we clearly have \( \pi^* \in \Pi \). We also always have \( L(\pi) \leq L(\pi^*) + 2r_1 \) since the losses are bounded in \([0, 1]\). For the inductive step, first we observe that for Theorem 9, \( L(\pi) = L_m(\pi) \), and for Theorem 13, \( |L(\pi) - L_m(\pi)| \leq L_h = r_m \). In conjunction with (18), in both cases, we have

\[
\forall m \in [m_T], \forall \pi \in \Pi : |L(\pi) - \hat{L}_m| \leq 3r_m/2.
\]

By the standard analysis of empirical risk minimization, for the first claim,

\[
\hat{L}_m(\pi^*) \leq L(\pi^*) + 3r_m/2 = \min_{\pi \in \Pi_m} L(\pi) + 3r_m/2 \leq \min_{\pi \in \Pi_m} \hat{L}_m(\pi) + 3r_m.
\]

which verifies that \( \pi^* \in \Pi_{m+1} \). For the second claim, for both Theorem 9 and Theorem 13, we have for all \( \pi \) in \( \Pi_{m+1} \),

\[
L(\pi) \leq \hat{L}_m(\pi) + 3r_m/2 \leq \min_{\pi' \in \Pi_m} \hat{L}_m(\pi') + 9r_m/2 \leq L(\pi^*) + 6r_m.
\]

This proves the second claim since \( r_m = 2r_{m+1} \).

For the final regret bound, in the \( 1 - \delta \) good event, we partition into epochs:

\[
\text{Regret} \leq \sum_{m=1}^{m_T} n_m (\mu_m + 12r_m) = \sum_{m=1}^{m_T} 13n_m r_m
\]

where we have used the definition of \( \mu_m = r_m \). We optimize the bound as follows. We choose some epoch \( m^* \) and truncate the sum at \( m^* \). Using the fact that \( r_m \leq r_{m'} \) for \( m \geq m' \), we can bound the regret in the later epochs simply by \( T r_{m^*} \). For the earlier epochs we substitute the choice of \( n_m \). This gives

\[
\sum_{m=1}^{m_T} 13n_m r_m \leq 13 \min_{m^* \in [m_T]} \left( T r_{m^*} + 320 \sum_{m < m^*} \frac{\kappa_{h_m} V_m \delta T}{r_m} \right)
\]

To simplify further, by our inductive hypothesis we know that

\[
V_m \leq V(\Pi_m, h_m) = \mathbb{E}_x \nu(A(x; \Pi_m, h_m)) \leq \mathbb{E}_x N_{h_m}(\Pi_m(x)) \cdot \sup_a \nu(B(a,2h_m)).
\]

The final inequality is based on the fact that we can always cover \( A(x; \Pi_m, h_m) \) by a union of balls of radius \( 2h_m \) with centers on a \( h_m \)-covering of \( \Pi_m \), along with the fact that a maximal \( \delta \)-packing is a \( \delta \)-covering. On the other hand we have \( \kappa_{h_m} \leq \sup_a \frac{1}{\nu(B(a,h_m))} \), so that under Assumption 2 we have

\[
\kappa_{h_m} V_m \leq \alpha \cdot \mathbb{E}_x N_{h_m}(\Pi_m(x))
\]

Set \( S \triangleq \{ 2^{-i} : i \in \mathbb{N}, i \leq \log_2(T) \} \). For Theorem 9, using the definition of \( M_h(\epsilon, \delta) \), and the fact that \( \Pi_m \subseteq \Pi_{h_{12r_m}} \), we have \( \kappa_{h_m} V_m \leq \alpha \mathbb{E}_x N_{h_m}(\Pi_m(x)) \leq \alpha M_h(12r_m, r_m) \). Therefore, the bounds simplify to

\[
\text{Regret}(T, \Pi_h) \leq 13 \min_{\epsilon \in S} \left( T \epsilon^* + 320\alpha \sum_{\epsilon \in S, \epsilon \geq 2\epsilon^*} \frac{M_h(12\epsilon, \epsilon) \delta T}{\epsilon} \right) \leq 13 \min_{\epsilon_0 > 1/T} ( T \epsilon_0 + 320\alpha \theta_h(\epsilon_0) \log([\Pi] \log_2(T)/\delta) \cdot \log_2(T) )
\]

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where in the second inequality, we use the definition of $\theta_h(\epsilon)$, along with the fact that for every $\epsilon_0 > \frac{1}{T}$, there is some $\epsilon^* \in S$ such that $\epsilon^* < \epsilon_0 \leq 2\epsilon^*$.

Likewise, for Theorem 13, we have

$$\text{Regret}(T, \Pi) \leq 13 \min_{\epsilon^* \in S} \left( TL\epsilon_0 + 320\alpha \sum_{\epsilon \in S, \epsilon \geq 2\epsilon^*} \frac{M_0(12L\epsilon, \epsilon)\delta_T}{L\epsilon} \right)$$

$$\leq 13 \min_{\epsilon_0 > 1/T} \left( T\epsilon_0 + 320\alpha\psi_L(\epsilon_0)/L \cdot \log(|\Pi| \log_2(T)/\delta) \cdot \log_2(T) \right)$$

Both of these bounds are conditional on the good event, which happens with probability $1 - \delta$. In the probability $\delta$ bad event, the expected regret is at most $T$. Setting $\delta = 1/T$, the theorems follow.

### D.2 Proofs for the lemmata

**Proof of Lemma 27.** The proof follows that of Lemma 1 of Dudik et al. (2011). We introduce the following notation: for a distribution $P$ over a set of policies $\Pi'$, bandwidth $h$, denote by its induced action-selection distribution (without uniform exploration) as:

$$p(a \mid x) \triangleq \sum_{\pi \in \Pi} P(\pi)(K_h\pi(x))(a)$$

Using this notation, $p^\mu(a \mid x) \triangleq (1 - \mu)\sum_{\pi \in \Pi} P(\pi)(K_h\pi(x))(a) + \mu$ can be simplified to

$$p^\mu(a \mid x) = (1 - \mu)p(a \mid x) + \mu.$$

First, observe

$$\min_{P \in \Delta(\Pi')} \max_{\pi \in \Pi'} \mathbb{E}_{x \sim D} \mathbb{E}_{a \sim K^\pi(x)} \left[ \frac{1}{p^\mu(a \mid x)} \right] = \min_{Q \in \Delta(\Pi')} \max_{\pi \sim Q} \mathbb{E}_{x \sim D} \mathbb{E}_{a \sim K^\pi(x)} \left[ \frac{1}{p^\mu(a \mid x)} \right].$$

This latter program is linear in $Q$, convex in $P$, and defined everywhere (due to the uniform exploration), and so we may apply Sion’s minimax theorem, to obtain

$$\max_{Q \in \Delta(\Pi')} \min_{P \in \Delta(\Pi')} \mathbb{E}_{x \sim D} \mathbb{E}_{a \sim K^\pi(x)} \left[ \frac{1}{p^\mu(a \mid x)} \right] \leq \max_{Q \in \Delta(\Pi')} \mathbb{E}_{x \sim D} \mathbb{E}_{a \sim K^\pi(x)} \left[ \frac{1}{q^\mu(a \mid x)} \right] \leq \frac{1}{1 - \mu} \mathbb{E}_{x \sim D} \left[ \mathbb{1}\{q(a \mid x) > 0\} \right].$$

Applying the definition of $V(\Pi', h)$, we obtain the result.

**Proof of Lemma 28.** First, as we have seen, $\mathbb{E} \hat{\ell}_i(\pi(x)) = \lambda_h(\pi)$. Moreover,

$$\text{Var} \left( \hat{\ell}_i(\pi(x)) \right) \leq \mathbb{E} \left[ \hat{\ell}_i(\pi(x))^2 \right] = \mathbb{E}_{(x, \ell) \sim D} \left[ \int \frac{(K_h\pi(x))^2(a)\ell(a)^2}{q^\mu(a \mid x)} d\nu \right] \leq \frac{\kappa_h V(\Pi', h)}{1 - \mu} \leq 2\kappa_h V(\Pi', h).$$
where the penultimate inequality uses the fact that $Q$ is the solution to (3), so it satisfies the guarantee in Lemma 27. Note we are also using here that $\mu \leq 1/2$. Therefore, using Lemma 29 below, we have that for every $\pi \in \Pi'$, with probability at least $1 - \frac{\delta}{|\Pi|}$, the following holds:

$$\left| \hat{L}(\pi) - \bar{L}(\pi) \right| \leq \sqrt{\frac{80 \kappa h V(\Pi', h)}{n} \log(e |\Pi|/\delta)}.$$ 

The lemma is concluded by taking a union bound over all $\pi$ in $\Pi'$.

**Lemma 29.** Suppose $\delta \in (0, 1)$, $k = 5\lceil \ln \frac{1}{\delta} \rceil$, $\tilde{n}$ is an integer, and $n = k\tilde{n}$. In addition, $X_1, \ldots, X_n$ are iid random variables with mean $\mu$ and variance $\sigma^2$. Define

$$\hat{\mu} = \text{median} \left\{ \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} X_i, \frac{1}{\tilde{n}} \sum_{i=\tilde{n}+1}^{2\tilde{n}} X_i, \ldots, \frac{1}{\tilde{n}} \sum_{i=(k-1)\tilde{n}+1}^{k\tilde{n}} X_i \right\}.$$

Then with probability $1 - \delta$,

$$|\hat{\mu} - \mu| \leq \sigma \sqrt{\frac{40 \ln \frac{2}{\delta}}{n}}.$$

**Proof.** From the first part of Hsu and Sabato (2016, Proposition 5), taking $k = 5\lceil \ln \frac{1}{\delta} \rceil$, we have that with probability $1 - \delta$,

$$|\hat{\mu} - \mu| \leq \sigma \sqrt{\frac{8k}{n}}.$$

The proof is completed by noting that $k \leq 5(1 + \ln \frac{1}{\delta}) = 5 \ln \frac{e}{\delta}.$

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