Equivalence of relativistic three-particle quantization conditions

Based on work with Tyler Blanton:
arXiv:2007.16188 and
arXiv:2007.16190
Scope & Notation

- Identical spinless particles of mass $m$ (e.g. $3\pi^+$)
- $\mathbb{Z}_2$ symmetry — no $2 \rightarrow 3$ transitions
- All quantities in QC3 are infinite-dimensional matrices with indices $\{\vec{k}, \ell, m\}$ describing 3 on-shell particles with total energy-momentum $(E, \vec{P})$
Relativistic QC3 landscape 2019

RFT = generic relativistic EFT

\[ \mathcal{H}_{\text{df,3}} \rightarrow \mathcal{M}_3 \]

Integral eqs.

Hansen & SS, 15

\[ L \rightarrow \infty \]

QC3 (all \( \ell \))

\[ \det \left[ F_3^{-1} + \mathcal{H}_{\text{df,3}} \right] = 0 \]

Hansen & SS, 14

\[ F_3 = \tilde{F} \left[ \frac{1}{3} - \frac{1}{\tilde{H}} \right] \]

\[ \tilde{H} = \tilde{F} + \tilde{G} + \frac{1}{(2\omega L^3 \mathcal{H}_2)} \]

FVU = finite-volume unitarity

Equivalent representations of \( \mathcal{M}_3 \) (infinite volume)

Jackura, Dawid, Fernández-Ramírez, Mathieu, Mikhasenko, Pilloni, SS, Szczepaniak, 19

Unitary representation of \( \mathcal{M}_3 \) in terms of \( \mathcal{R}^{(u,u)} \)

Mai, Hu, Döring, Pilloni, Szczepaniak, 17; Jackura, Fernández-Ramírez, Mathieu, Mikhasenko, Nys, Pilloni, Saldaña, Sherril & Szepaniak, 18

QC3 (\( \ell = 0 \))

\[ \det \left[ \tilde{H}_s + \frac{1}{2\omega L^3} \tilde{C}_s^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \]

Mai & Döring, 17

Form given in Hansen & SS, Review 19

Can show that \( \tilde{C}_s^{(u,u)} = \mathcal{R}_s^{(u,u)} \)

(Quantities are defined & full references are given in backup slides)

S. Sharpe, “Equivalence of relativistic three-particle quantization conditions,” APLAT 2020, 8/3/2020
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\]

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Equivalent representations of \( M_3 \) (infinite volume)

Unitary representation of \( M_3 \) in terms of \( R^{(u,u)} \)

Mai et al., 17; Jackura et al., 18

EQ3 (\( \ell = 0 \))

\[
\det \left[ H_s + \frac{1}{2\omega L^3} C_s^{(u,u)} \frac{1}{2\omega L^3} \right] = 0
\]

Mai & Döring, 17

\( FVU = \) finite-volume unitarity

Alternate form of QC3

\[
\det \left[ 1 + (2\omega L^3 H_2 + H_{df,3}^{(u,u)})(F + G) \right] = 0
\]

Blanton & SS, 20

New derivation of QC3

\[
\det \left[ 1 + (2\omega L^3 H_2 + H_{df,3}^{(u,u)})(F + G) \right] = 0
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Hansen & SS, 15

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Hansen & SS, 14

L → ∞

QC3 (all ℓ)

\[ \det \left[ 1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}^{(u,u)}_{df,3})(\widetilde{F} + \widetilde{G}) \right] = 0 \]

Blanton & SS, 20

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\[ \det \left[ \widetilde{H}_s + \frac{1}{2\omega L^3} \mathcal{C}_s^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \]

Mai & Döring, 17

Discussed in Tyler Blanton’s talk

Alternate form of QC3

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\[ \text{QC3 (} \ell = 0 \text{)} \]

\[ \det \left[ H_s + \frac{1}{2\omega L^3} \tilde{C}^{(u,u)}_s \frac{1}{2\omega L^3} \right] = 0 \]

Mai & Döring, 17


equivalent representations of \( \mathcal{M}_3 \) (in infinite volume)

Jackura et al., 19

Relativistic QC3 landscape 2020

Alternate form of QC3

\[ \det \left[ 1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{df,3}^{(u,u)}(\tilde{F} + \tilde{G})) \right] = 0 \]

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This talk: Establishing the RFT to FVU connection for QC3s

S. Sharpe, ``Equivalence of relativistic three-particle quantization conditions;'' APLAT 2020, 8/3/2020
Definitions of asymmetric kernels

- In original RFT approach (using Feynman diagrams & Bethe-Salpeter kernels)

\[
\left[ \mathcal{M}_3^{(u,u)} \right]_{k;pr} = \begin{array}{c}
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\end{array} + \begin{array}{c}
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\end{array} + \begin{array}{c}
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\end{array} + \cdots
\]

\( k \text{ & } p \text{ assigned to spectators} \)

- In our approach (using time-ordered perturbation theory)

\[
\left[ \widetilde{\mathcal{M}}_3^{(u,u)} \right]_{k;pr} = \begin{array}{c}
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_2 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\end{array} + \begin{array}{c}
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\end{array} + \begin{array}{c}
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\text{B}_3 \quad \text{p} \\
\text{B}_2 \quad \text{r} \\
\end{array} + \cdots
\]

\( k \text{ & } p \text{ assigned to spectators} \)

Cuts in time-ordered PT

TOPT kernels (no 3-particle cuts)
Asymmetric kernels differ!

- Consider a particular Feynman diagram

\[
\begin{align*}
\text{a} & \quad \text{p} \quad \in \\
\text{k} & \quad \text{r}
\end{align*}
\]

Asymmetric

- In TOPT the two time orderings are put into different terms—one being symmetrized

\[
\begin{align*}
\text{a} & \quad \text{p} \quad \in \\
\text{k} & \quad \text{r}
\end{align*}
\]

Asymmetric

\[
\begin{align*}
\text{a} & \quad \text{p} \quad \in \\
\text{k} & \quad \text{r}
\end{align*}
\]

Symmetrized

- Thus \( M_3^{(u,u)} \neq \tilde{M}_3^{(u,u)} \), although both symmetrize to \( M_3 \)
Asymmetric kernels ⇒ redundancy

- E.g., asymmetric form of QC3 holds with (at least) two different kernels
  \[
  \det \left[ 1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{df,3}^{(u,u)})(\mathcal{F} + \mathcal{G}) \right] = 0 \\
  \det \left[ 1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{df,3}^{(u,u)})(\mathcal{F} + \mathcal{G}) \right] = 0
  \]

- R matrix representation of \( M_3^{(u,u)} \) holds for all choices of asymmetry

\[ M_3^{R,(u,u)} = \]

\[ \begin{align*}
  & = \begin{array}{c}
    \includegraphics[width=0.3\textwidth]{fig1}
  \end{array} + \begin{array}{c}
    \includegraphics[width=0.3\textwidth]{fig2}
  \end{array}
  \end{align*} \]

We call it \( R^{(u,u)} \) to emphasize its asymmetry

Can set this equal to either \( M_3^{(u,u)} \) or \( \bar{M}_3^{(u,u)} \): leads to different, equally valid, \( R^{(u,u)} \)
Key steps in derivation

1. Rewrite asymmetric QC3

\[
\det \left[ 1 + (2\omega L^3) \mathcal{K}_2 + \mathcal{H}_\text{df,3}^{(u,u)} (\mathcal{F} + \mathcal{G}) \right] = 0 \quad \Rightarrow \quad \det \left[ \mathcal{H} - X^{(u,u)} \right] = 0
\]

\[
\mathcal{H} = \mathcal{F} + \mathcal{G} + \mathcal{H}^{-1}_{2,L} \quad \quad \quad \mathcal{H}_{2,L} = (2\omega L^3) \mathcal{K}_2
\]

\[
X^{(u,u)} = \mathcal{H}^{-1}_{2,L} \mathcal{H}_\text{df,3}^{(u,u)} \mathcal{H}^{-1}_{2,L} \frac{1}{1 + \mathcal{H}_\text{df,3}^{(u,u)} \mathcal{H}^{-1}_{2,L}}
\]

2. Equate \( M_3^{\mathcal{R},(u,u)} \) to \( M_3^{(u,u)} \) to determine the relation between \( \mathcal{R}^{(u,u)} \) and \( \mathcal{H}_\text{df,3}^{(u,u)} \), leading to

\[
\left[ (2\omega L^3) X^{(u,u)} (2\omega L^3) \right]_{klm;pl'm'} = \left[ \mathcal{R}^{(u,u)} \right]_{klm;pl'm'} + \mathcal{O}(e^{-mL})
\]

3. Combine these results

\[
\det \left[ \mathcal{H} - (2\omega L^3)^{-1} \mathcal{R}^{(u,u)} (2\omega L^3)^{-1} \right] = 0
\]
Conclusions & Outlook

• RFT quantization conditions can be rewritten in terms of R matrix

\[
\det \left[ \tilde{F} + \tilde{G} + \frac{1}{2\omega L^3 K_2} - \frac{1}{2\omega L^3} \mathcal{R}^{(u,u)} \frac{1}{2\omega L^3} \right] = 0
\]

• Holds for both choices of \( \mathcal{R}^{(u,u)} \)—we conjecture it holds for family of redundant choices

• Provides generalization of FVU quantization condition

\[
\det \left[ \tilde{H}_s + \frac{1}{2\omega L^3} \tilde{C}_s^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \text{ to all } \ell
\]

• Derivation requires use of smooth cutoff function, and barrier factors in \( \tilde{G} \)

• We expect that by taking the NR limit we would obtain the generalization of the NREFT form of QC3 to all \( \ell \)
Conclusions & Outlook

- Which form of QC3 is the most useful in practical applications?

\[
\det \left[ F_3^{-1} + \mathcal{H}_{df,3} \right] = 0
\]

- Symmetric three-particle $K$ matrix
  - Threshold expansion requires fewer parameters
  - Little intuition in presence of three-particle resonances
  - $\mathcal{H}_{df,3}$ depends on PV pole prescription

\[
\det \left[ \tilde{F} + \tilde{G} + \frac{1}{2\omega L^3 K_2} - \frac{1}{2\omega L^3} R^{(u,u)} \frac{1}{2\omega L^3} \right] = 0
\]

- Asymmetric three-particle $R$ matrix
  - Threshold expansion requires more (redundant) parameters
  - Considerable experience and intuition from JPAC studies of fitting amplitudes to experimental data
  - $R^{(u,u)}$ independent of pole prescription
Conclusions & Outlook

• Which form of QC3 is the most useful in practical applications?

\[ \det \left( F_3^{-1} + \mathcal{K}_{df,3} \right) = 0 \]

- Symmetric three-particle K matrix
  - Threshold expansion requires fewer parameters
  - Little intuition in presence of three-particle resonances
  - \( \mathcal{K}_{df,3} \) depends on PV pole prescription

\[ \det \left( \mathcal{F} + \mathcal{G} + \frac{1}{2\omega L^3} \mathcal{K}_2 - \frac{1}{2\omega L^3} \mathcal{R}^{(u,u)} \frac{1}{2\omega L^3} \right) = 0 \]

- Asymmetric three-particle R matrix
  - Threshold expansion requires more (redundant) parameters
  - Considerable experience and intuition from JPAC studies of fitting amplitudes to experimental data
  - \( \mathcal{R}^{(u,u)} \) independent of pole prescription

Thank you. Questions?
Backup slides
F_3 collects 2-particle interactions

\[ F_3 = \left[ \frac{\tilde{F}}{3} - \tilde{F} \frac{1}{(2\omega L^3 \mathcal{K}_2)^{-1} + \tilde{F} + \tilde{G}} \tilde{F} \right] \]
\[ F_3 = \left[ \frac{\widetilde{F}}{3} - \frac{1}{(2\omega L^3 \mathcal{K}_2)^{-1}} + \frac{\widetilde{F} + \widetilde{G}}{F} \right] \]

- F & G are known geometrical functions, containing cutoff function H

\[ \widetilde{F}_{p\ell^\prime m';k\ell m} = \frac{1}{2\omega_k L^3} \delta_{pk} H(\vec{k}) F_{PV,\ell^\prime m';\ell m}(E - \omega_k, \vec{P} - \vec{k}, L) \]

\[ \widetilde{G}_{p\ell^\prime m';k\ell m} = \frac{1}{2\omega_p L^3} \left( \frac{k^*}{q_p^*} \right)^\ell \left( \frac{4\pi Y_{\ell^\prime m'}(k^*) H(\vec{p}^*) H(\vec{k}) Y_{\ell m}^*(\vec{p}^*)}{(P - k - p)^2 - m^2} \right) \left( \frac{p^*}{q_k^*} \right)^\ell \frac{1}{2\omega_k L^3} \]
3-particle papers: RFT approach

Max Hansen & SRS:
“Relativistic, model-independent, three-particle quantization condition,”
arXiv:1408.5933 (PRD) [HS14]
“Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,”
arXiv:1504.04028 (PRD) [HS15]
“Perturbative results for 2- & 3-particle threshold energies in finite volume,”
arXiv:1509.07929 (PRD) [HSPT15]
“Threshold expansion of the 3-particle quantization condition,”
arXiv:1602.00324 (PRD) [HSTH15]
“Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,”
arXiv: 1609.04317 (PRD) [HSBS16]

“Lattice QCD and three-particle decays of Resonances,”
arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]
Raúl Briceño, Max Hansen & SRS:
“Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles,” arXiv:1701.07465 (PRD) [BHS17]

“Numerical study of the relativistic three-body quantization condition in the isotropic approximation,” arXiv:1803.04169 (PRD) [BHS18]

“Three-particle systems with resonant sub-processes in a finite volume,” arXiv:1810.01429 (PRD 19) [BHS19]

SRS
“Testing the threshold expansion for three-particle energies at fourth order in $\phi^4$ theory,” arXiv:1707.04279 (PRD) [SPT17]

Tyler Blanton, Fernando Romero-López & SRS:
“Implementing the three-particle quantization condition including higher partial waves,” arXiv:1901.07095 (JHEP) [BRS19]

“I=3 three-pion scattering amplitude from lattice QCD,” arXiv:1909.02973 (PRL)
Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

“Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states,” arXiv:1908.02411 (JHEP)

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

“Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism,” arXiv:1905.11188 (PRD)

Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

“On the Equivalence of Three-Particle Scattering Formalisms,” arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

“Generalizing the relativistic quantization condition to include all three-pion isospin channels”, arXiv:2003.10974 (JHEP)
Alternate 3-particle approaches

★ NREFT approach

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]

★ Finite-volume unitarity (FVU) approach

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of $M_3$ involving $R$ matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of $R$ matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [$3$ pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]

★ HALQCD approach

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [$3$ nucleon potentials in NR regime]