The effect of inter-edge Coulomb interactions on the transport between quantum Hall edge states

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In a recent experiment, Milliken et al. demonstrated possible evidence for a Luttinger liquid through measurements of the tunneling conductance between edge states in the $\nu = 1/3$ quantum Hall plateau. However, at low temperatures, a discrepancy exists between the theoretical predictions based on Luttinger liquid theory and experiment. We consider the possibility that this is due to long-range Coulomb interactions which become dominant at low temperatures. Using renormalization group methods, we calculate the cross-over behaviour from Luttinger liquid to the Coulomb interaction dominated regime. The cross-over behaviour thus obtained seems to resolve one of the discrepancies, yielding good agreement with experiment.

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Recent progress in semiconductor nanostructure fabrication technology has ushered us into a world of reduced dimensionality which includes one-dimensional quantum wires, two-dimensional electron systems like GaAs/AlGaAs heterostructures and so on. In addition, the discovery of high-$T_c$ superconductors, in which we believe the physics of correlated electrons is at the origin of the many interesting and novel properties such as non-Fermi liquid like normal phase behaviour, has increased our interest in strongly correlated electron systems. Accordingly, the properties of one-dimensional interacting electron systems have been studied in great detail. In one-dimension, inclusion of short-range electron-electron interactions leads to the strongly correlated electron systems known as Luttinger liquids. Recently, Kane and Fisher have studied the transport properties of a Luttinger liquid in the presence of a local impurity. The physical realization of this system in terms of tunneling between quantum Hall edge states is a particularly clean example of ‘boundary critical phenomena.’

K. Moon et al. have investigated the resonant tunneling between quantum Hall edge states and numerically computed the universal resonance line shape, obtaining results close to the exact result subsequently discovered by Fendley et al. These results show reasonable agreement with the experiment done by Milliken et al. provided that the data is normalized to achieve the correct peak value ($1/3)(g^2/h)$. However, at very low temperatures, there is an additional discrepancy in that the tunneling conductance falls below the theoretical predictions.

Recently, Oreg and Finkel’stein have considered the effect of the inter-edge Coulomb interaction on the propagation of the edge modes in samples with constrictions.

In this paper, we study the effects of residual long-range Coulomb interactions on the transport properties of a Luttinger liquid using a renormalization group analysis and show that these residual interactions may explain the discrepancy between theory and experiment at low temperatures.

In the following, we assume that the bulk system exhibits the fractional quantum Hall effect with filling factor $\nu = 1/n$, where $n$ is an odd integer. Since there is a charge excitation gap in the bulk, the low temperature physics is controlled by the gapless edge excitations. Consider a quantum Hall bar with width $w$ and length $L(>w)$ as illustrated in Fig.1. In the quantum Hall bar geometry, electrons at the two edges are moving in opposite directions. The Lagrangian for the two corresponding edge excitations including the long-range Coulomb interactions is given by two-coupled chiral Luttinger liquids

$$\mathcal{L} = \frac{1}{4\pi\nu} \int dx \left\{ \partial_x \phi_R (i\partial_t \phi_R + v\partial_x \phi_R - \partial_x \phi_L (i\partial_t \phi_L - v\partial_x \phi_L) \right\} + \frac{1}{8\pi^2} \int dxdx' \sum_{i=R,L} V_a(x-x') \partial_x \phi_i \partial_x \phi_i + \frac{1}{4\pi^2} \int dxdx' V_m(x-x') \partial_x \phi_R (-\partial_x \phi_L), \quad (1)$$

where $v$ is the bare sound velocity of the edge wave and $R(L)$ specifies right(left)-moving electrons. The first term corresponds to the Lagrangian for pure chiral Luttinger liquids for the right and left-movers. The second and third terms describe the intra- and inter-edge Coulomb interactions, respectively. The operators $\phi_R$ and $\phi_L$ satisfy the following commutation relations

$$[\phi_R(x),\phi_R(x')] = -[\phi_L(x),\phi_L(x')] = -i\pi\nu \epsilon(x-x'), \quad (2)$$

where $\epsilon(x) = 1(-1)$ for $x > 0(<0)$ and the intra- and inter-edge Coulomb interactions are given by
\[ V_o(x) = \frac{e^2}{\epsilon \sqrt{\alpha^2 + a^2}}, \quad V_w(x) = \frac{e^2}{\epsilon \sqrt{\alpha^2 + w^2}} \]  

(3)

where \( a \) is an ultraviolet cutoff on the scale of the mean particle spacing and \( \epsilon \) is the dielectric constant. We define symmetric and antisymmetric modes

\[ \theta = \frac{1}{2\sqrt{\pi}}(\phi_R - \phi_L), \quad \phi = \frac{1}{2\sqrt{\pi}}(\phi_R + \phi_L) \]  

(4)

where the \( \theta(x) \)- and \( \phi(x) \)-fields satisfy the following algebra

\[ [\theta(x), \phi(x')] = -i \frac{\nu}{2}(x - x'). \]  

(5)

In the experiment, current is injected from source to drain as shown in Fig. 1 and the voltage drop \( V_{SD} \) is measured, from which longitudinal conductance can be obtained. The net current \( J(= J_R - J_L) \) and charge operators \( \rho(= \rho_R + \rho_L) \) are related to the \( \phi_R \)- and \( \phi_L \)-fields as follows

\[ \rho = \frac{1}{2\pi}(\partial_x \phi_R - \partial_x \phi_L) = \frac{1}{\sqrt{\pi}} \partial_x \theta, \]  

(6)

\[ J = \frac{1}{2\pi}(\partial_x \phi_R + \partial_x \phi_L) = \frac{1}{\sqrt{\pi}} \partial_x \phi. \]  

(7)

Following the above transformation and subtracting and adding \( V_w(x) \), we obtain the following Lagrangian

\[ \mathcal{L} = \frac{v}{2\pi} \int dx \left( (\partial_x \theta)^2 + (\partial_x \phi)^2 \right) 
+ \frac{1}{2\pi} \int dx dx' V_w \partial_x \partial_x \theta 
+ \frac{1}{4\pi} \int dx dx' (V_n - V_w)(\partial_x \theta \partial_x \phi + \partial_x \phi \partial_x \phi). \]  

(8)

Upon Fourier transformation and taking the long wavelength limit \( (k < w^{-1}) \), one obtains the following Lagrangian

\[ \mathcal{L} \approx \frac{v_R}{2\pi} \int dk \left[ \lambda(k)|\tilde{\theta}|^2 + |\tilde{\phi}|^2 \right] \]  

(9)

where \( v_R \) is a renormalized sound velocity given by

\[ v_R = v \left[ 1 + \frac{\nu \alpha}{\pi \epsilon v} \ln \left( \frac{w}{\alpha} \right) \right], \]  

(10)

\( \alpha \) is the fine structure constant and \( \lambda(k) \) is given by

\[ \lambda(k) = 1 + 2\chi \ln \left( \frac{2}{kw} \right), \]  

(11)

where we have defined the following dimensionless parameter \( \chi \) which is a measure of the strength of inter-edge Coulomb interactions

\[ \chi \equiv \frac{\nu \alpha}{\pi \epsilon} \left( \frac{t}{v_R} \right). \]  

(12)

Hence we obtain the generic Luttinger liquid Lagrangian with a renormalized sound velocity \( v_R \) plus the inter-edge Coulomb interaction term, which gives \( \lambda(k) \) a logarithmic divergence at small wavevectors.

Since \( \partial_x \theta/\nu \) is canonically conjugate to the \( \phi \)-field from Eq. (5), it is convenient to integrate out the \( \theta(x) \)-fields and obtain the following Euclidean action for the \( \phi(x) \)-fields

\[ S_E \approx \frac{v_R}{2\nu} \sum_{\omega_n} \int dk \left\{ k^2 + \frac{1}{\lambda(k)} \left( \frac{\omega_n}{v_R} \right)^2 \right\} |\tilde{\phi}|^2 \]  

(13)

where \( \omega_n = 2n\pi/\beta \) is a Boson Matsubara frequency. One can now directly read off the dispersion relation for the edge magnetoplasmon

\[ \hbar \omega = v_R k \sqrt{1 + 2\chi \ln \left( \frac{2}{kw} \right)} \]  

(14)

which agrees with the one obtained by M. Wassermieier et al.\[13\]. For small \( k \), the linear dispersion relation, which is a characteristic of a Luttinger liquid, is modified to \( k \sqrt{\ln k} \).

Now we are ready to study the transport through a narrow constriction defined electrostatically by gates on both sides near the middle of a quantum Hall bar as shown in Fig. 1. Since the right- and left-movers are spatially close together near the constriction, tunneling from one edge to the other can occur. Using a renormalization group analysis, Kane and Fisher [9] showed that for weak back scattering, the effective barrier strength is renormalized to be

\[ v_{\text{eff}} \approx \frac{v_0}{T_1 - \nu}. \]  

(15)

Hence, as the temperature is lowered, the effective barrier strength grows and the perturbative RG analysis based on the weak barrier limit breaks down. Furthermore the effects of Coulomb interactions become important in this regime. In this strong barrier limit, the main contributions to the transport come from weak electron transmission as opposed to the quasiparticle tunneling between edges which occurs in the weak barrier limit \[11\]. Since the electron creation operators \( \psi_R^\dagger(x) \) and \( \psi_L^\dagger(x) \) are respectively given by \( e^{i\phi_R(x)/\nu} \) and \( e^{-i\phi_L(x)/\nu} \), the electron transmission operator can be written \( e^{i(\phi_L(x)+\phi_R(x))/\nu} \approx e^{i2\sqrt{2\pi}/\nu} \). Hence we have the following tunneling action \( H_t \) (taking the tunneling to occur only at \( x = 0 \))

\[ H_t = -t \int \cos(2\sqrt{\pi}/\nu). \]  

(16)

Using momentum-shell RG analysis, we obtain the following RG flow

\[ \frac{d}{d\nu} \lambda(k) = -\frac{1}{\nu} \frac{d\lambda(k)}{d\nu}. \]  

(17)

where \( \lambda(k) \) is the linear conductance which increases logarithmically with decreasing wavevector.

\[ \chi \equiv \frac{\nu \alpha}{\pi \epsilon} \left( \frac{t}{v_R} \right). \]  

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\[ \frac{d}{d\nu} \lambda(k) = -\frac{1}{\nu} \frac{d\lambda(k)}{d\nu}. \]  

(17)
\[
\frac{dt}{dt} = \left[ 1 - \frac{2}{\nu \pi} \int_0^\infty dk \frac{\Lambda}{\Lambda^2/\lambda(k) + k^2} \right] t
\]  
(17)

with \( \Lambda \cong e^{-\ell} \). At finite temperature, the renormalization group flow is cut-off at \( \Lambda \cong T/\alpha T \) where \( \alpha T \) is a dimensionless number of order unity. After integrating the above RG equation up to \( \Lambda \cong T/\alpha T \), we obtain the following effective (renormalized) tunneling amplitude \( t_{\text{eff}} \)

\[
t_{\text{eff}} \cong \begin{cases} 
 t_0 \frac{T_0}{T} \exp \left\{ -\frac{2}{3 \nu \chi} \left[ \left( 1 + 2 \chi \frac{\ln T_0}{T} \right)^{3/2} - 1 \right] \right\} 
 \text{for } T \leq T_0, \\
 t_0 \left( \frac{T_0}{T} \right)^{1-1/\nu} \text{for } T > T_0.
\end{cases}
\]  
(18)

where \( T_0 \equiv 2 \alpha T \hbar v_R/w \) is the cross-over temperature scale from Luttinger liquid to the inter-edge interaction dominated regime, and the upper expression is valid for \( T \leq T_0 \) while the lower applies for \( T > T_0 \). At low temperatures in the tail of the resonance, the conductance \( G_0 \), can be obtained from the single particle hopping (or one-loop level) and is proportional to \( t_{\text{eff}}^2 \)

\[
G_C^0 \cong t_0^2 \left( \frac{T_0}{T} \right)^2 \exp \left\{ -\frac{2}{3 \nu \chi} \left[ \left( 1 + 2 \chi \frac{\ln T_0}{T} \right)^{3/2} - 1 \right] \right\}.
\]  
(19)

In the absence of inter-edge Coulomb interactions (i.e. \( \chi \to 0 \)), we recover the familiar Luttinger liquid behaviour [3]

\[
G_L^0 \equiv G_C^0(\chi \to 0) \cong t_0^2 \left( \frac{T_0}{T} \right)^{2(1-1/\nu)}.
\]  
(20)

In the extreme Coulomb interaction dominated regime (\( \chi \gg 1 \)), Eq. (14) reduces to the following result

\[
G_C^0(\chi \gg 1) \equiv t_0^2 \left( \frac{T_0}{T} \right)^2 \exp \left\{ -\frac{2}{3 \nu \chi} \sqrt{2} \chi \left( \ln \frac{T_0}{T} \right)^{3/2} \right\}
\]  
(21)

which I.I. Glazman et al. [14] obtained by studying the transport properties of a charge density wave through a single barrier. The underlying physics can be understood as follows. We know that the Luttinger liquid with \( \nu < 1 \) has a charge density wave (CDW) correlation function which decays algebraically with distance. In the presence of the Coulomb interaction, the CDW correlation function decays slower than any power law [15]. Since a CDW is easily pinned by even a single impurity, one can expect the conductance to be further reduced in the presence of Coulomb interactions.

The universal resonance line shape \( G_L \) for the Luttinger liquid with short range interactions at \( \nu = 1/3 \) was obtained previously by K. Moon et al. [3] using Monte carlo methods and subsequently the exact solution was obtained by Fendley et al. [7]. Using Eq. (20) and Eq. (19) one can calculate the tunneling conductance in the presence of Coulomb interactions by taking into account the leading corrections in the following approximate way

\[
G_\chi(T) \cong \begin{cases} 
 G_L \left( \frac{G_0}{G_L} \right) \text{ for } T \leq T_0, \\
 G_L \text{ for } T \geq T_0.
\end{cases}
\]  
(22)

The conductance \( G_\chi(T) \) thus obtained has no explicit dependence on the bare tunneling amplitude \( t_0 \) and satisfies the following properties. When the tunneling amplitude is very small, \( G_0/G_L \) approaches unity and the conductance is given by \( G_L \), as expected. At relatively high temperatures, since the thermal correlation length \( (\xi_\delta \equiv v_R/2\pi T \nu) \) is rather short and the long-range Coulomb interaction is less effective, \( G_0^0/G_L^0 \) becomes 1 and the conductance is given by the Luttinger liquid value \( G_L \). We want to emphasize that there exist two crossover scales [16]. In addition to the scale \( T_0 \), describing the onset of Coulomb interaction, there is also the scale \( T_1 \) (\( \propto \nu_{C0}^{1/(1-\nu)} \)), which governs the crossover from the weak to the strong barrier regime. Following Eq. (22), for \( T \leq T_0 \leq T_1 \), the conductance can be written

\[
\tilde{G}_\chi \left( \frac{T_1}{T}, \frac{T_0}{T} \right) = G_L \left( \frac{T_1}{T} \right) F_\chi \left( \frac{T_0}{T} \right)
\]  
(23)

where \( G_L \) is the Luttinger liquid contribution and the function \( F_\chi \) is given by

\[
F_\chi(x) = x^{2/\nu} \exp \left\{ -\frac{2}{3 \nu \chi} \left[ (1 + 2 \chi \ln x)^{3/2} - 1 \right] \right\}.
\]  
(24)

One can clearly see in Eq. (23) the breakdown of single-variable scaling due to the existence of the additional Coulomb scale \( T_0 \).

Now we want to compare our results with the recent experiment by Milliken et al. [8]. The experiment is performed on a GaAs/AlGaAs heterostructure. A point contact gate voltage applied in the middle of the sample pinches off a narrow channel. The bulk system is in the \( \nu = 1/3 \) fractional quantum Hall regime. Fine tuning of the gate voltage produced several resonance peaks and they measured the temperature dependence of the tunneling conductance. The nominal size of the quantum Hall bar is roughly \( 60 \times 60 \mu m^2 \) and dielectric constant \( \epsilon \cong 13 \).

The mean electron density \( n \) is about \( 1.0 \times 10^{11} cm^{-2} \). Hence the mean particle spacing \( a \) is approximately 300\( \AA \) and the edge wave velocity \( v_R \) is about \( 10^5 m/sec \) [17].

Using the parameters above, fixes \( \chi \sim 0.18 \). In Fig.2, the experimental data obtained by Milliken et al. [8] are compared to our results showing nice agreement with a best fit value for the cross-over temperature \( T_0 \) of about 60 mK which gives \( a_T \approx 2.5 \) close to unity as expected and \( T_1 \) is chosen to be 60 mK. The somewhat poorer data collapse in the tail of the scaling curve at low temperatures might be a manifestation of the breakdown of scaling in terms of a single variable due to Coulomb interactions.

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Unfortunately, we can not know for certain at this point how significant this good fit (using essentially one additional parameter) really is. While the nominal sample geometry is a 60µ square, the actual edge geometry under the near pinch-off conditions of the experiment is unknown. At the cross over temperature, the thermal correlation length is only about 15µ so that the asymptotic long-distance approximation (k < w⁻¹) is not likely to be valid. In addition, the sample has a back gate, which presumably screens out the long-range piece of Coulomb interaction leading to the dipole-like interaction (∝ 1/r³).

Nevertheless, the fact that a good fit is obtained with physically reasonable values of the parameters lends credence to the idea that long range Coulomb forces are playing a significant role at the lowest temperatures. Clearly however, more experimental studies with well-defined geometries are needed to address these issues.

In conclusion, we have approximately computed the cross-over from the Luttinger liquid to the inter-edge Coulomb interaction dominated regime of a quantum Hall bar with back scattering at a narrow constriction. It is clear that further experiments on narrower samples with well-characterized geometries would be very useful in probing this interesting phenomenon.

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FIG. 1. Schematic diagram for a quantum Hall bar of width w and length L. The gates G are placed in the middle of the Hall bar to make a narrow constriction. Charges can tunnel through the barrier or be reflected. The source S and drain D are located at each end of the bar.

FIG. 2. The (+) symbols correspond to the experimental data obtained by Milliken et al. and the open circles are from the Monte Carlo data by K. Moon et al. based on the Luttinger liquid theory. The solid curves are obtained by taking χ ≈ 0.18 and T₁ (∼ T₀) is chosen to be about 60 mK.
K. Moon et al. Fig. 1
$X = (T_{1}/T)^{2/3}$

$T_{0} \sim 60 \text{ mK}$

$\chi \sim 0.18$

K. Moon et. al. Fig.2