Nonlinear Sequence Transformations: Computational Tools for the Acceleration of Convergence and the Summation of Divergent Series

Ernst Joachim Weniger
Institut für Physikalische und Theoretische Chemie
Universität Regensburg, D-93040 Regensburg, Germany
joachim.weniger@chemie.uni-regensburg.de

Submitted to the topical issue
Time Series and Time Evolution of Generic Systems. Spectral Analysis beyond Fourier Transform, Auto- and Cross-Correlation Functions, Green’s Resolvent Spectra. Nonlinear Convergence Accelerators of Series and Sequences

of
Journal of Computational Methods in Sciences and Engineering (JCMSE)
Corrected Version 11 June 2001

Abstract

Convergence problems occur abundantly in all branches of mathematics or in the mathematical treatment of the sciences. Sequence transformations are principal tools to overcome convergence problems of the kind. They accomplish this by converting a slowly converging or diverging input sequence \( \{s_n\}_{n=0}^{\infty} \) into another sequence \( \{s'_n\}_{n=0}^{\infty} \) with hopefully better numerical properties. Padé approximants, which convert the partial sums of a power series to a doubly indexed sequence of rational functions, are the best known sequence transformations, but the emphasis of the review will be on alternative sequence transformations which for some problems provide better results than Padé approximants.

1 Introduction

Many numerical techniques as for example iterative schemes, discretization methods, perturbation techniques, or series expansions produce results which are actually sequences. Obviously, a numerical technique of that kind is practically useful only if the resulting sequence converges sufficiently fast. Unfortunately, it frequently happens that the resulting sequence either converges too slowly to be practically useful, or it may even diverge.

Problems with slow convergence or divergence were of course already encountered in the early days of calculus. Accordingly, numerical techniques for the acceleration of convergence or the summation of divergent series are almost as old as calculus itself. According to Knopp [1, p. 249], the first series transformation was published by Stirling [2] already in 1730, and in 1755 Euler [3] published the series transformation which now bears his name. In rudimentary form, convergence acceleration methods are even older. In a book by Brezinski [4, pp. 90 - 91] it is mentioned that convergence acceleration methods were already used in 1654 by Huygens and in 1674 by Seki Kowa, the probably most famous
Japanese mathematician of his time. Both Huygens and Seki Kowa tried to obtain better approximations to $\pi$. Huygens used a linear extrapolation scheme which is a special case of what we now call Richardson extrapolation [5], and Seki Kowa used the so-called $\Delta^2$ process, which is usually attributed to Aitken [6]. Then, in a book by Liem, Lü, and Shih [7, p. ix] it is mentioned that extrapolation methods were already used by the Chinese mathematicians Liu Hui (A.D. 263) and Zhu Chongzhi (429 - 500) for obtaining better approximations to $\pi$, but no further details are given.

Sequence transformations are principal tools to overcome convergence problems of the kind mentioned above. In this approach, a slowly convergent or divergent sequence $\{s_n\}_{n=0}^{\infty}$, whose elements may for instance be the partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

of an infinite series, is converted into a new sequence $\{s'_n\}_{n=0}^{\infty}$ with hopefully better numerical properties. The history of sequence transformations and related topics starting from the 17th century until today is discussed by Brezinski in a monograph [4] or in two articles [8,9].

Before the invention of electronic computers, mainly linear sequence transformations were used, which compute the elements of the transformed sequence $\{s'_n\}_{n=0}^{\infty}$ as weighted averages of the elements of the input sequence $\{s_n\}_{n=0}^{\infty}$ according to

$$s'_n = \sum_{k=0}^{n} \mu_{nk} s_k .$$

(1.2)

The theoretical properties of these matrix transformations are now very well understood [4,14,15]. Their main appeal lies in the fact that for the weights $\mu_{nk}$ in (1.2) some necessary and sufficient conditions could be formulated which guarantee that the application of such a matrix transformation to a convergent sequence $\{s_n\}_{n=0}^{\infty}$ yields a transformed sequence $\{s'_n\}_{n=0}^{\infty}$ converging to the same limit $s = s_\infty$. Theoretically, this regularity is extremely desirable, but from a practical point of view, it is a disadvantage. This sounds paradoxical. However, Wimp remarks in the preface of his book [15, p. X] that the size of the domain of regularity of a transformation and its efficiency seem to be inversely related. Accordingly, regular linear transformations are in general at most moderately powerful, and the popularity of most linear transformations has declined considerably in recent years.

Modern nonlinear sequence transformations as for instance Wynn’s epsilon [16] and rho [17] algorithm or Brezinski’s theta algorithm [18] have largely complementary properties: They are nonregular, which means that the convergence of the transformed sequence to the correct limit is not guaranteed. In addition, their theoretical properties are far from being completely understood. Nevertheless, they often accomplish spectacular results. Consequently, nonlinear transformations now dominate both mathematical research as well as practical applications, as documented by the large number of recent books [7,15,19–26] and review articles [27–29] on this topic.

There is considerable evidence that the culprit for the frequently unsatisfactory performance of regular matrix transformation is not their linearity, but their regularity. In [30] it was shown that suitably chosen linear but nonregular transformations can at least for special problems be as efficient as the most powerful nonlinear transformations.

The best known class of sequence transformations are Padé approximants which convert the partial sums

$$f_n(z) = \sum_{k=0}^{n} \gamma_k z^k$$

(1.3)
Nonlinear Sequence Transformations

of a (formal) power series for some function \( f \) into a doubly indexed sequence of rational functions

\[
[l/m]f(z) = P_l(z)/Q_m(z), \quad l, m \in \mathbb{N}_0. 
\]  

(1.4)

Here, \( P_l(z) = p_0 + p_1 z + \ldots + p_l z^l \) and \( Q_m(z) = 1 + q_1 z + \ldots + q_m z^m \) are polynomials in \( z \) of degrees \( l \) and \( m \), respectively. The \( l + m + 1 \) polynomial coefficients \( p_0, p_1, \ldots, p_l \) and \( q_1, q_2, \ldots, q_m \) are chosen in such a way that the Taylor expansion of the ratio \( P_l(z)/Q_m(z) \) at \( z = 0 \) agrees with the power series for \( f \) as far as possible:

\[
f(z) - P_l(z)/Q_m(z) = O(z^{l+m+1}), \quad z \to 0.
\]  

(1.5)

This asymptotic error estimate leads to a system of \( l + m + 1 \) linear equations for the coefficients of the polynomials \( P_l(z) \) and \( Q_m(z) \) \cite{33}. Moreover, several recursive algorithms are known. The merits and weaknesses of the various computational schemes for Padé approximants are discussed in \cite[Section II.3]{23}.

In applied mathematics and in theoretical physics, Padé approximants have become the standard tool to overcome convergence problems with power series. Accordingly, there is an extensive literature, and any attempt to provide a complete bibliography would be beyond the scope of this article (see for example the extensive bibliography compiled by Brezinski \cite{33}). The popularity of Padé approximants in theoretical physics can be traced back to an article by Baker \cite{34}, who also wrote the first modern monograph \cite{31}. The currently most complete treatment of the theory of Padé approximants is the 2nd edition of the monograph by Baker and Graves-Morris \cite{32}. More condensed treatments can be found in books on continued fractions \cite{33,35}, in a book by Bender and Orszag on mathematical physics \cite[Section 8]{35}, or in a book by Baker on critical phenomena \cite[Part III]{34}. Then, there is a book by Pozzi on the use of Padé approximants in fluid dynamics \cite{41} as well as several review articles \cite{42,43}. The generalization of Padé approximants to operators and multivariate power series is discussed in a book by Cuyt \cite{46} and in articles by Cuyt \cite{47} and by Guillaume and A. Huard \cite{48}. Another generalization of Padé approximants − Padé-type approximants − are described in a book by Brezinski \cite{21}.

The emphasis of this article is not on Padé approximants, which are well known as well as extensively documented in the literature, but on alternative sequence transformations, which are not so well known yet. This is quite undeserved. In some cases, sequence transformations outperform Padé approximants. For example, the present author has applied sequence transformations successfully in such diverse fields as the evaluation of special functions \cite{28,30,49}, the evaluation of molecular multicenter integrals of exponentially decaying functions \cite{54,58}, the summation of strongly divergent quantum mechanical perturbation expansions \cite{19,23,49}, and the extrapolation of crystal orbital and cluster calculations for oligomers to their infinite chain limits of stereoregular quasi-onedimensional organic polymers \cite{71,73}. In the majority of these applications, it was either not possible to use Padé approximants, or sequence transformations did a better job.

It is of course impossible to present something as complex and diverse as sequence transformations in a single and comparatively short article. Because of obvious space limitations, this article can at best provide some basic facts about some of the most useful sequence transformations and review the recent literature. However, it cannot be a substitute for more detailed treatments as for example an older review of the author \cite{28} or the book by Brezinski and Redivo Zaglia \cite{25}.

2 On the Construction of Sequence Transformations

The basic step for the construction of a sequence transformation is the assumption that the elements of a convergent or divergent sequence \( \{s_n\}_{n=0}^{\infty} \) can be partitioned into a limit or
antilimit $s$ and a remainder $r_n$ according to

$$s_n = s + r_n, \quad n \in \mathbb{N}_0.$$  \hspace{1cm} (2.1)

If a sequence $\{s_n\}_{n=0}^\infty$ converges, the remainders $r_n$ in (2.1) can be made negligible by increasing $n$ as much as necessary. However, many sequences converge so slowly that this is not feasible. Moreover, increasing $n$ does not help in the case of divergence.

Alternatively, one can try to improve convergence by computing approximations to the remainders which are then eliminated from the sequence elements. Of course, this is more easily said than done. The remainders $\{r_n\}_{n=0}^\infty$ of a sequence $\{s_n\}_{n=0}^\infty$ are in general unknown, and their determination is normally not easier than the determination of the (generalized) limit. For example, if the input data $\{s_n\}_{n=0}^\infty$ are the partial sums (1.1) of an infinite series, the remainders satisfy

$$r_n = - \sum_{k=n+1}^\infty a_k.$$  \hspace{1cm} (2.2)

Thus, the straightforward elimination of exact remainders is normally not possible. However, the remainders of some infinite series can be approximated with the help of the Euler-Maclaurin formula (see for example [74, pp. 279 - 295]). Let us consider a convergent infinite series $\sum_{\nu=0}^\infty g(\nu)$, and let us assume that its terms $g(\nu)$ are smooth and slowly varying functions of the index $\nu$. Then, the integral

$$\int_M^N g(x) \, dx$$

with $M, N \in \mathbb{Z}$ provides a good approximation to the sum

$$\frac{1}{2}g(M) + g(M + 1) + \ldots + g(N - 1) + \frac{1}{2}g(N)$$

and vice versa. In the years between 1730 and 1740, Euler and Maclaurin derived independently correction terms, which ultimately yielded what we now call the Euler-Maclaurin formula:

$$\sum_{\nu=M}^N g(\nu) = \int_M^N g(x) \, dx + \frac{1}{2} [g(M) + g(N)]
+ \sum_{j=1}^k \frac{B_{2j}}{(2j)!} \left[ g^{(2j-1)}(N) - g^{(2j-1)}(M) \right] + R_k(g),$$

$$R_k(g) = - \frac{1}{(2k)!} \int_M^N B_{2k} \left( x - \lfloor x \rfloor \right) g^{(2k)}(x) \, dx.$$  \hspace{1cm} (2.5a)

Here, $\lfloor x \rfloor$ is the integral part of $x$, $B_m(x)$ is a Bernoulli polynomial, and $B_m = B_m(0)$ is a Bernoulli number.

If we set $M = n + 1$ and $N = \infty$, the leading terms of the Euler-Maclaurin formula, which is actually an asymptotic series, yield for sufficiently large $n$ rapidly convergent approximations to the truncation error $\sum_{\nu=n+1}^\infty g(\nu)$.

An example, which demonstrates the usefulness of the Euler-Maclaurin formula, is the Dirichlet series for the Riemann zeta function:

$$\zeta(z) = \sum_{\nu=0}^\infty (\nu + 1)^{-z}.$$  \hspace{1cm} (2.6)
Nonlinear Sequence Transformations

This series converges for \( \Re(z) > 1 \). However, it is notorious for extremely slow convergence if \( \Re(z) \) is only slightly larger than one.

The terms \((\nu+1)^{-z}\) of the Dirichlet series (2.6) are obviously smooth and slowly varying functions of the index \( \nu \) and they can be differentiated and integrated easily. Thus, we can apply the Euler-Maclaurin formula (2.5) with \( M = n + 1 \) and \( N = \infty \) to the truncation error of the Dirichlet series:

\[
\sum_{\nu=n+1}^{\infty} (\nu+1)^{-z} = \frac{(n+2)^{1-z}}{z-1} + \frac{1}{2} (n+2)^{-z} + \sum_{j=1}^{k} \frac{(z)_{2j-1} B_{2j} z}{(2j)!} (n+2)^{-z-2j+1} + R_k(n, z),
\]

\( R_k(n, z) = - (z)_{2k} B_{2k} \int_{n+1}^{\infty} \frac{x-[x]}{(1+x)^{z+2k}} \, dx . \) (2.7a)

Here, \((z)_m = z(z+1)\cdots(z+m-1) = \Gamma(z+m)/\Gamma(z)\) is a Pochhammer symbol.

In [39, Tables 8.7 and 8.8, p. 380] or in [73, Section 2] it was shown that a few terms of the Euler-Maclaurin approximation (2.7) suffice for a convenient and reliable computation of \( \zeta(z) \) even if \( z \) is so close to one. The arguments \( z = 1.1 \) and \( z = 1.01 \) considered in [39, 73] lead to such a slow convergence of the Dirichlet series (2.6) that its evaluation via a straightforward addition of its terms is practically impossible.

Thus, it looks like an obvious idea to use the Euler-Maclaurin formula routinely in the case of slowly convergent series. Unfortunately, this is not possible. The Euler-Maclaurin formula requires that the terms of the series can be differentiated and integrated with respect to the index. This excludes many series of interest. Moreover, the Euler-Maclaurin formula cannot be applied in the case of alternating or divergent series since their terms are neither smooth nor slowly varying. However, the probably worst drawback of the Euler-Maclaurin formula is that it is an analytic convergence acceleration method. This means that it cannot be applied if only the numerical values of the terms of a series are known.

Sequence transformations also try to compute approximations to the remainders and to eliminate them from the sequence elements. However, they require only relatively little knowledge about the \( n \)-dependence of the remainders of the sequence to be transformed. Consequently, they can be applied in situations in which apart from the numerical values of a finite string of sequence elements virtually nothing else is known.

Since it would be futile to try to eliminate a remainder \( r_n \) with a completely unknown and arbitrary \( n \)-dependence, a sequence transformation has to make some assumptions, either implicitly or explicitly. Therefore, a sequence transformation will only work well if the actual behavior of the remainders is in sufficient agreement with the assumptions made. Of course, this also implies that a sequence transformation, which makes certain assumptions, may fail to accomplish something if it is applied to a sequence with remainders of a sufficiently different behavior. Thus, the efficiency of a sequence transformation for certain sequences and its inefficiency or even nonregularity for other sequences are intimately related.

Sequence transformations normally eliminate only approximations to the remainders. In such a case, the elements of the transformed sequence \( \{s'_n\}_{n=0}^{\infty} \) will also be of the type of (2.1), which means that \( s'_n \) can also be partitioned into the (generalized) limit \( s \) and a transformed remainder \( r'_n \) according to

\[
s'_n = s + r'_n , \quad n \in \mathbb{N}_0 .
\] (2.8)

The transformed remainders \( \{r'_n\}_{n=0}^{\infty} \) are normally different from zero for all finite values of \( n \). However, convergence is accelerated if the transformed remainders \( \{r'_n\}_{n=0}^{\infty} \) vanish more
rapidly than the original remainders \( \{r_n\}_{n=0}^\infty \),

\[
\lim_{n \to \infty} \frac{s'_n - \bar{s}}{s_n - \bar{s}} = \lim_{n \to \infty} \frac{r'_n}{r_n} = 0 ,
\]

and a divergent sequence is summed if the transformed remainders \( r'_n \) vanish as \( n \to \infty \).

Assumptions about the \( n \)-dependence of the truncation errors can be incorporated into the transformation process via \textit{model sequences}. In this approach, a sequence transformation \( E_k^{(n)} \) is constructed in such a way that it produces the (generalized) limit \( \bar{s} \) of a model sequence

\[
\bar{s}_n = \bar{s} + \bar{r}_n = \bar{s} + \sum_{j=0}^{k-1} \tilde{c}_j \phi_j(n) ,
\]

if it is applied to a set of \( k + 1 \) consecutive elements of this model sequence:

\[
E_k^{(n)} = E_k^{(n)}(\bar{s}_n, \bar{s}_{n+1}, \ldots, \bar{s}_{n+k}) = \bar{s} .
\]

The \( \phi_j(n) \) are assumed to be known functions of \( n \), and the \( \tilde{c}_j \) are unspecified coefficients.

The elements of this model sequence contain \( k + 1 \) unknowns, the limit or antilimit \( \bar{s} \) and the \( k \) coefficients \( \tilde{c}_0, \tilde{c}_1, \ldots, \tilde{c}_{k-1} \). Since all unknowns occur linearly, Cramer’s rule implies that a sequence transformation, which is able to determine the limit or antilimit \( \bar{s} \) in (2.10) from the numerical values of \( k + 1 \) sequence elements \( \bar{s}_n, \bar{s}_{n+1}, \ldots, \bar{s}_{n+k} \), is given as the ratio of two determinants \( [25, p. 56] \). Since determinantal representations are computationally unattractive, alternative computational schemes for \( E_k^{(n)} \) are highly desirable. A recursive scheme for \( E_k^{(n)} \) was derived independently by Schneider \([75, 76, 77]\), Brezinski, and Hävie \([78]\). A more economical implementation was later obtained by Ford and Sidi \([78]\).

The sequence transformation \( E_k^{(n)} \) contains the unspecified quantities \( \phi_j(n) \). The majority of all currently known sequence transformations can be obtained by specializing the \( \phi_j(n) \) (see for example \([25, pp. 57 - 58]\)). So, from a purely formal point of view either the determinantal representation for \( E_k^{(n)} \) or the recursive schemes provide a complete solution to the majority of all convergence acceleration and summation problems. However, even the recursive scheme of Ford and Sidi \([78]\) is considerably more complicated than the recursive schemes for other transformations that can be obtained by specializing the \( \phi_j(n) \) in (2.10).

So, it is usually simpler to use instead of \( E_k^{(n)} \) one of its special cases. Important is also the following aspect: It is certainly helpful to know that for arbitrary functions \( \phi_j(n) \) the sequence transformation \( E_k^{(n)} \) can be computed, but it is more important to find out which set \( \{\phi_j(n)\}_{j=0}^\infty \) produces the best results for a given sequence \( \{s_n\}_{n=0}^\infty \).

If the remainders of the model sequence (2.11) are capable of producing sufficiently accurate approximations to the remainders of a sequence \( \{s_n\}_{n=0}^\infty \), then the application of the sequence transformation \( E_k^{(n)} \) to \( k + 1 \) sequence elements \( s_n, s_{n+1}, \ldots, s_{n+k} \) should produce a sufficiently accurate approximation to the (generalized) limit \( s \) of the input sequence. This is usually the case if the truncation errors can be expressed as \textit{infinite} series in terms of the \( \phi_j(n) \). Further details as well as many examples can be found in \([23, 28]\).

Simple asymptotic conditions are used in the literature to classify the type of convergence of a sequence. For example, many practically relevant sequences \( \{s_n\}_{n=0}^\infty \) converging to some limit \( s \) can be characterized by the asymptotic condition

\[
\lim_{n \to \infty} \frac{s_{n+1} - s}{s_n - s} = \rho ,
\]

which closely resembles the ratio test in the theory of infinite series. If \( |\rho| < 1 \), the sequence is called \textit{linearly} convergent, and if \( \rho = 1 \), it is called \textit{logarithmically} convergent.
Typical examples of linearly convergent sequences are the partial sums of a power series with a nonzero, but finite radius of convergence. In contrast, the partial sums $\sum_{\nu=0}^{n}(\nu+1)^{-z}$ of the Dirichlet series (2.6) for $\zeta(z)$ converge logarithmically.

3 The Aitken Formula, Wynn’s Epsilon Algorithm, and Related Transformations

It will now be shown how Aitken’s $\Delta^2$ formula can be constructed by assuming that the truncation error consists of a single exponential term according to

$$s_n = s + c \lambda^n, \quad c \neq 0, \quad |\lambda| \neq 1, \quad n \in \mathbb{N}_0.$$  \hfill (3.1)

For $0 < |\lambda| < 1$, this sequence converges to its limit $s$, and for $|\lambda| > 1$, it diverges away from its generalized limit or antilimit $s$. Thus, if this sequence converges, it converges linearly according to (2.12).

By considering $s, c,$ and $\lambda$ in (3.1) as unknowns of the linear system $s_{n+j} = s + c \lambda^{n+j}$ with $j = 0, 1, 2$, a sequence transformation can be constructed which is able to determine the (generalized) limit $s$ of the model sequence (3.1) from the numerical values of three consecutive sequence elements $s_n, s_{n+1}$ and $s_{n+2}$. A short calculation shows that

$$A_1^{(n)} = s_n - \frac{[\Delta s_n]^2}{\Delta^2 s_n}, \quad n \in \mathbb{N}_0,$$  \hfill (3.2)

produces the (generalized) limit $s$ of the model sequence (3.1) according to $A_1^{(n)} = s$. Alternative expressions for $A_1^{(n)}$ are discussed in [28, Section 5.1]. The forward difference operator $\Delta$ in (3.2) is defined according to $\Delta G(n) = G(n+1) - G(n)$.

The power and practical usefulness of Aitken’s $\Delta^2$ formula is limited since it is designed to eliminate only a single exponential term. However, the output data $A_1^{(n)}$ can be used as input data in the $\Delta^2$ formula (3.2). Hence, the $\Delta^2$ process can be iterated, which leads to the following nonlinear recursive scheme [28, Eq. (5.1-15)]:

$$A_0^{(n)} = s_n, \quad n \in \mathbb{N}_0,$$  \hfill (3.3a)

$$A_{k+1}^{(n)} = A_k^{(n)} - \frac{[\Delta A_k^{(n)}]^2}{\Delta^2 A_k^{(n)}}, \quad k, n \in \mathbb{N}_0.$$  \hfill (3.3b)

In the case of doubly indexed quantities like $A_k^{(n)}$, it will always be assumed that $\Delta$ only acts on the superscript $n$ but not on the subscript $k$ according to $\Delta A_k^{(n)} = A_{k+1}^{(n)} - A_k^{(n)}$.

There is an extensive literature on Aitken’s $\Delta^2$ process and its iteration. Its numerical performance was studied in [28, 53, 74]. It was also discussed in articles by Lubkin [80], Shanks [81], Tucker [82, 83], Clark, Gray, and Adams [84], Cordellier [85], Hillion [86], Jurkat [87], Bell and Phillips [88], and Weniger [28, 53, 89], or in books by Baker and Graves-Morris [32], Brezinski [19, 20], Brezinski and Redivo-Zaglia [25], Delahaye [24], Walz [26], and Wimp [15]. A multidimensional generalization of Aitken’s transformation to vector sequences was discussed by MacLeod [90]. Modifications and generalizations of Aitken’s $\Delta^2$ process were proposed by Drummond [91], Jamieson and O’Beirne [92], Bjørstad, Dahlquist, and Grosse [93], and Sablonniere [94]. The iteration of other sequence transformations is discussed in [95].
An obvious generalization of the model sequence (3.1) would be the following model sequence which contains \( k \) exponential terms:

\[
s_n = s + \sum_{j=0}^{k-1} c_j \lambda_j^n, \quad |\lambda_0| > |\lambda_1| > \ldots > |\lambda_{k-1}|.
\]  

(3.4)

Although the \( \Delta^2 \) process (3.3) is by construction exact for the model sequence (3.1), its iteration (3.3) is not exact for the model sequence (3.4). Instead, this is – as first shown by Wynn [16], and later extended by Sidi [17] – true for Wynn’s epsilon algorithm (3.5):

\[
\epsilon_{n-1}^{(n)} = 0, \quad \epsilon_{0}^{(n)} = s_n, \quad n \in \mathbb{N}_0, \\
(3.5a)
\]

\[
\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + 1/|\epsilon_{k-1}^{(n+1)} - \epsilon_{k}^{(n)}|, \quad k, n \in \mathbb{N}_0.
\]  

(3.5b)

Only the elements \( \epsilon_{2k}^{(n)} \) with even subscripts provide approximations to the limit \( s \) of the sequence \( \{s_n\}_{n=0}^{\infty} \) to be transformed. Efficient recursive schemes based on the moving lozenge technique introduced by Wynn [16] are discussed in [28, Section 4.3].

A short calculation shows \( A_1^{((n)} = e_2^{(n)} \). For \( k > 1 \), \( A_k^{(n)} \) and \( e_{2k}^{(n)} \) are in general different, but they have similar properties in convergence acceleration and summation processes.

If the input data \( s_n \) for Wynn’s epsilon algorithm are the partial sums (1.3) of the (formal) power series for some function \( f(z) \) according to \( s_n = f_n(z) \), then it produces Padé approximants according to [16]

\[
\epsilon_{2k}^{(n)} = [n + k/k] f(z), \quad k, n \in \mathbb{N}_0.
\]  

(3.6)

Since the epsilon algorithm can be used for the computation of Padé approximants, it is discussed in books on Padé approximants such as [12], but there is also an extensive literature dealing directly with it. In Wimp’s book [14, p. 120], it is mentioned that over 50 articles on the epsilon algorithm were published by Wynn alone, and at least 30 articles by Brezinski. As a fairly complete source of references Wimp recommends Brezinski’s first book [19]. However, this book was published in 1977, and since then many more articles dealing with the epsilon algorithm have been published. Thus, a detailed bibliography would be beyond the scope of this article. Moreover, the epsilon algorithm is not restricted to scalar sequences but can be generalized to cover for example vector sequences. A very recent review can be found in [15].

Aitken’s iterated \( \Delta^2 \) process (3.3) as well as Wynn’s epsilon algorithm (3.5) are powerful accelerators for sequences which according to (2.12) converge linearly, and they are also able to sum many alternating divergent series. However, they fail completely in the case of logarithmic convergence (compare for example [11, Theorem 12]). Moreover, in the case of divergent power series whose series coefficients grow more strongly than factorially, Padé approximants or equivalently Wynn’s epsilon algorithm either converge too slowly to be numerically useful [100, 101] or are not at all able to accomplish a summation to a unique finite generalized limit [102].

Brezinski showed that the inability of Wynn’s epsilon algorithm of accelerating logarithmic convergence can be overcome by a suitable modification of the recursive scheme (3.3). This leads to the so-called theta algorithm [18]:

\[
\vartheta_{-1}^{(n)} = 0, \quad \vartheta_{0}^{(n)} = s_n, \quad n \in \mathbb{N}_0, \\
(3.7a)
\]

\[
\vartheta_{2k+1}^{(n)} = \vartheta_{2k-1}^{(n+1)} + 1/|\vartheta_{2k-1}^{(n+1)} - \vartheta_{2k}^{(n)}|, \quad k, n \in \mathbb{N}_0, \\
(3.7b)
\]

\[
\vartheta_{2k+2}^{(n)} = \vartheta_{2k}^{(n+1)} + \frac{|\vartheta_{2k}^{(n+1)}|}{\Delta^2 \vartheta_{2k+1}^{(n+1)}}, \quad k, n \in \mathbb{N}_0.
\]  

(3.7c)
As in the case of Wynn’s epsilon algorithm (3.5), only the elements $\vartheta_{2k}^{(n)}$ with even subscripts provide approximations to the (generalized) limit of the sequence to be transformed.

The theta algorithm was derived with the intention of overcoming the inability of the epsilon algorithm to accelerate logarithmic convergence. In that respect, the theta algorithm was a great success. Extensive numerical studies of Smith and Ford [79, 103] showed that the theta algorithm is not only very powerful, but also much more versatile than the epsilon algorithm. Like the epsilon algorithm, it is an efficient accelerator for linear convergence and it is also able to sum many divergent series. However, it is also able to accelerate the convergence of many logarithmically convergent sequences and series. Further details as well as additional references can be found in [25, Section 2.9] or in [28, Sections 10 and 11].

As for example discussed in [95], new sequence transformations can be constructed by iterating explicit expressions for sequence transformations with low transformation orders. This approach is also possible in the case of the theta algorithm. A suitable closed-form expression, which may be iterated, is [28, Eq. (10.3-1)]

$$\vartheta_{2k}^{(n)} = s_{n+1} - \frac{[\Delta s_n][\Delta s_{n+1}][\Delta^2 s_{n+1}]}{[\Delta s_{n+2}][\Delta^2 s_n] - [\Delta s_n][\Delta^2 s_{n+1}]}, \quad n \in \mathbb{N}_0.$$  \hfill (3.8)

Its iteration yields the following nonlinear recursive scheme [28, Eq. (10.3-6)]:

$$\mathcal{J}_0^{(n)} = s_n, \quad n \in \mathbb{N}_0,$$ \hfill (3.9a)

$$\mathcal{J}_{k+1}^{(n)} = \mathcal{J}_k^{(n+1)} - \frac{[\Delta \mathcal{J}_k^{(n)}][\Delta \mathcal{J}_k^{(n+1)}][\Delta^2 \mathcal{J}_k^{(n+1)}]}{[\Delta \mathcal{J}_k^{(n+2)}][\Delta^2 \mathcal{J}_k^{(n)}] - [\Delta \mathcal{J}_k^{(n)}][\Delta^2 \mathcal{J}_k^{(n+1)}]}, \quad k, n \in \mathbb{N}_0. \hfill (3.9b)$$

The iterated transformation $\mathcal{J}_k^{(n)}$ has similar properties as the theta algorithm from which it was derived: Both are very powerful as well as very versatile. $\mathcal{J}_k^{(n)}$ is not only an effective accelerator for linear convergence as well as able to sum many divergent series, but it is also an effective accelerator for logarithmic convergence [28, 94, 95, 104–107].

4 Richardson Extrapolation, Wynn’s Rho Algorithm, and Related Transformations

As long as $\rho$ in (2.12) is not too close to one, the acceleration of linear convergence is comparatively simple. With the help of Germain-Bonne’s formal theory of convergence acceleration [108] and its extension [28, Section 12], it can be decided whether a sequence transformation is capable of accelerating linear convergence or not. Moreover, several sequence transformations are known that accelerate linear convergence effectively.

Logarithmic convergence leads to more challenging computational problems than linear convergence. An example is the Dirichlet series (2.6) for the Riemann zeta function. As already discussed in Section 2, its convergence of can become so slow that the evaluation of $\zeta(z)$ by successively adding up the terms $(\nu + 1)^{-z}$ is practically impossible.

There are also principal theoretical problems. Delahaye and Germain-Bonne [109, 110] showed that no sequence transformation can exist which is able to accelerate the convergence of all logarithmically convergent sequences. Consequently, an analogue of Germain-Bonne’s beautiful formal theory of the acceleration of linear convergence [108] and its extension [28, Section 12] cannot exist, and the success of a convergence acceleration process cannot be guaranteed unless additional information is available.

Nevertheless, many sequence transformations are known which work at least for suitably restricted subsets of the class of logarithmically convergent sequences. Examples are Richardson extrapolation [1], Wynn’s rho algorithm [17] and its iteration [28, Section 6],
Osada’s modification of the rho algorithm \[11\], and the modification of the $\Delta^2$ process by Bjørstad, Dahlquist, and Grosse \[93\]. However, there is considerable evidence that sequence transformations speed up logarithmic convergence less efficiently than linear convergence (see, for example, the discussion in \[52, Appendix A\]).

Another disadvantage of logarithmic convergence is that serious numerical instabilities are much more likely. A sequence transformation accelerates convergence by extracting and utilizing information on the index-dependence of the truncation errors from a finite set of input data. This is normally accomplished by forming higher weighted differences. If the input data are the partial sums of a strictly alternating series, the formation of higher weighted differences is a remarkably stable process, but if the input data all have the same sign, numerical instabilities are quite likely. Thus, if the sequence to be transformed converges logarithmically, numerical instabilities are to be expected, and it is usually not possible to obtain results that are close to machine accuracy.

In some cases, these instability problems can be overcome with the help of a condensation transformation due to Van Wijngaarden, which converts input data having the same sign to the partial sums of an alternating series, whose convergence can be accelerated more effectively. The condensation transformation was first mentioned in \[112, pp. 126 - 127\] and only later published by Van Wijngaarden \[113\]. It was used by Daniel \[114\] in combination with the Euler transformation \[3\], and recently, it was rederived by Pelzl and King \[115\]. Since the transformation of a strictly alternating series by means of nonlinear sequence transformations is a remarkably stable process, it was in this way possible to evaluate special functions, that are defined by extremely slowly convergent monotone series, not only relatively efficiently but also close to machine accuracy \[12\], or to perform extensive quantum electrodynamical calculations \[116\]. Unfortunately, the use of this combined nonlinear-condensation transformation is not always possible: The conversion of a monotone to an alternating series requires that terms of the input series with large indices can be computed.

For the construction of sequence transformations, which are able to accelerate logarithmic convergence, the standard interpolation and extrapolation methods of numerical mathematics \[117, 118\] are quite helpful. For that purpose, let us postulate the existence of a function $S$ of a continuous variable which coincides on a set of discrete arguments $\{x_n\}_{n=0}^\infty$ with the elements of the sequence $\{s_n\}_{n=0}^\infty$ to be transformed:

$$S(x_n) = s_n, \quad n \in \mathbb{N}_0.$$  \hspace{1cm} (4.1)

This ansatz reduces the convergence acceleration problem to an extrapolation problem. If a finite string $s_0, s_1, \ldots, s_{n+k}$ of $k+1$ sequence elements is known, one can construct an approximation $S_k(x)$ to $S(x)$ which satisfies the $k+1$ interpolation conditions

$$S_k(x_{n+j}) = s_{n+j}, \quad n \in \mathbb{N}_0, \quad 0 \leq j \leq k.$$  \hspace{1cm} (4.2)

In the next step, the value of $S_k(x)$ has to be determined for $x \to x_\infty$. If this can be done, we can expect that $S_k(x_\infty)$ will provide a better approximation to the limit $s = s_\infty$ of the sequence $\{s_n\}_{n=0}^\infty$ than the last sequence element $s_{n+k}$ used for the construction of $S_k(x)$.

The most common interpolating functions are either polynomials or rational functions. In the case of polynomial interpolation, it is implicitly assumed that the $k$-th order approximant $S_k(x)$ is a polynomial of degree $k$ in $x$:

$$S_k(x) = c_0 + c_1 x + \cdots + c_k x^k.$$  \hspace{1cm} (4.3)

For polynomials, the most natural extrapolation point is $x = 0$. Accordingly, the interpolation points $x_n$ have to satisfy the conditions

$$x_0 > x_1 > \cdots > x_m > x_{m+1} > \cdots > 0,$$

$$\lim_{n \to \infty} x_n = 0.$$  \hspace{1cm} (4.4a)
The choice \( x_\infty = 0 \) implies that the approximation to the limit is to be identified with the constant term \( c_0 \) of the polynomial \((4.3)\).

Several different methods for the construction of interpolating polynomials \( S_k(x) \) are known. Since only the constant term \( c_0 \) of a polynomial \( S_k(x) \) has to be computed and since in most applications it is desirable to compute simultaneously a whole string of approximants \( S_0(0), S_1(0), S_2(0), \ldots \), the most economical choice is Neville’s scheme \([11]\) for the recursive computation of interpolating polynomials. If we set \( x = 0 \), Neville’s algorithm reduces to the following 2-dimensional linear recursive scheme \([20\text{, p. 6}]\):

\[
N_0^{(n)} = s_n, \quad n \in \mathbb{N}_0, \quad (4.5a)
\]

\[
N_{k+1}^{(n)} = \frac{x_m N_k^{(n+1)} - x_{n+k+1} N_k^{(n)}}{x_n - x_{n+k+1}}, \quad k, n \in \mathbb{N}_0, \quad (4.5b)
\]

In the literature, this variant of Neville’s scheme is called Richardson extrapolation \([1]\).

There are functions that can be approximated more effectively by rational functions than by polynomials. Consequently, at least for some sequences \( \{s_n\}_{n=0}^\infty \) rational extrapolation should give better results than polynomial extrapolation. Let us therefore assume that \( S_k(x) \) can be expressed as the ratio of two polynomials of degrees \( l \) and \( m \), respectively:

\[
S_k(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_l x^l}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m}, \quad k, l, m \in \mathbb{N}_0. \quad (4.6)
\]

This rational function contains \( l + m + 2 \) coefficients \( a_0, \ldots, a_l \) and \( b_0, \ldots, b_m \). However, only \( l + m + 1 \) coefficients are independent since they are determined only up to a common nonvanishing factor. Usually, one requires either \( b_0 = 1 \) or \( b_m = 1 \). Consequently, the \( k + 1 \) interpolation conditions \((4.3)\) will determine the coefficients \( a_0, \ldots, a_l \) and \( b_0, \ldots, b_m \) provided that \( k = l + m \) holds.

The extrapolation point \( x_\infty = 0 \) is again an obvious choice. In this case, the interpolation points \( \{x_n\}_{n=0}^\infty \) have to satisfy \((4.4)\), and the approximation to the limit is to be identified with the ratio \( a_0/b_0 \) of the constant terms in \((4.6)\).

If \( l = m \) holds in \((4.6)\), extrapolation to infinity is also possible. In that case the interpolation points \( \{x_n\}_{n=0}^\infty \) have to satisfy

\[
0 < x_0 < x_1 < \cdots < x_{m} < x_{m+1} < \cdots, \quad (4.7a)
\]

\[
\lim_{n \to \infty} x_n = \infty, \quad (4.7b)
\]

and the approximation to the limit is to be identified with the ratio \( a_l/b_l \) in \((4.6)\).

As in the case of polynomial interpolation, several different algorithms for the computation of rational interpolants are known. A discussion of the relative merits of these algorithms as well as a survey of the relevant literature can be found in \([23\text{, Chapter III}]\).

The most frequently used rational extrapolation technique is probably Wynn’s rho algorithm \([17]\):

\[
\rho_{-1}^{(n)} = 0, \quad \rho_0^{(n)} = s_n, \quad n \in \mathbb{N}_0, \quad (4.8a)
\]

\[
\rho_{k+1}^{(n)} = \rho_{k-1}^{(n+1)} + \frac{x_{n+k+1} - x_n}{\rho_k^{(n+1)} - \rho_k^{(n)}}, \quad k, n \in \mathbb{N}_0. \quad (4.8b)
\]

Formally, the only difference between Wynn’s epsilon algorithm \([3,7]\) and Wynn’s rho algorithm is the sequence \( \{x_n\}_{n=0}^\infty \) of interpolation points which have to satisfy \((4.7)\). As in the case of the epsilon algorithm, only the elements \( \rho_{2k}^{(n)} \) with even subscripts provide approximations to the limit.
In spite of their formal similarity, the epsilon and the rho algorithm have complementary features. The epsilon algorithm is a powerful accelerator for linear convergence and is also able to sum many divergent alternating series, whereas the rho algorithm fails to accelerate linear convergence and is not able to sum divergent series. However, it is a very powerful accelerator for many logarithmically convergent sequences.

The properties of Wynn’s rho algorithm are discussed in books by Brezinski [13, 20] and Wimp [13]. Then, there is an article by Osada [120] discussing its convergence properties, but otherwise, relatively little seems to be known about its theoretical properties.

As in the case of Aitken’s \( \Delta^2 \) formula [12], iterated transformations can be constructed. For \( k = 1 \), we obtain from (4.8):

\[
\rho_2^{(n)} = s_{n+1} + \frac{(x_{n+2} - x_n)[\Delta s_{n+1}][\Delta s_n]}{[\Delta x_n][\Delta s_n] - [\Delta x_n][\Delta s_{n+1}]}, \quad n \in \mathbb{N}_0.
\]

This expression can be iterated yielding [28, Eq. (6.3-3)]

\[
\begin{align*}
\mathcal{W}_0^{(n)} &= s_n, & n \in \mathbb{N}_0, \\
\mathcal{W}_k^{(n)} &= \mathcal{W}_k^{(n+1)} + \frac{(x_{n+2k+2} - x_n)[\Delta W_k^{(n+1)}][\Delta W_k^{(n)}]}{(x_{n+2k+2} - x_n)[\Delta W_k^{(n)}] - (x_{n+2k+1} - x_n)[\Delta W_k^{(n+1)}]}, & k, n \in \mathbb{N}_0.
\end{align*}
\]

This is not the only possibility of iterating \( \rho_2^{(n)} \). However, the iterations derived by Bhowmick, Bhattacharya, and Roy [21] are significantly less efficient than \( \mathcal{W}_k^{(n)} \), which has similar properties as Wynn’s rho algorithm [28, 32].

The main practical problem with sequence transformations based upon interpolation theory is that for a given sequence \( \{s_n\}_{n=0}^\infty \) one has to find suitable interpolation points \( \{x_n\}_{n=0}^\infty \) that produces good results. For example, the Richardson extrapolation scheme (4.3) is normally used in combination with the interpolation points \( x_n = 1/(n + \beta) \) with \( \beta > 0 \). Then, \( N_k^{(n)} \) possesses a closed form expression (see, for example, [22, Lemma 2.1, p. 313] or [28, Eq. (7.3-20)]),

\[
N_k^{(n)} = \Lambda_k^{(n)}(\beta, s_n) = (-1)^k \sum_{j=0}^k (-1)^j \frac{(\beta + n + j)^k}{j!(k-j)!} s_{n+j}, \quad k, n \in \mathbb{N}_0,
\]

and the recursive scheme (4.3) assumes the following form [28, Eq. (7.3-21)]:

\[
\begin{align*}
\Lambda_0^{(n)}(\beta, s_n) &= s_n, & n \in \mathbb{N}_0, \\
\Lambda_{k+1}^{(n)}(\beta, s_n) &= \Lambda_k^{(n+1)}(\beta, s_{n+1}) + \frac{\beta + n}{k + 1} \Delta \Lambda_k^{(n)}(\beta, s_n), & k, n \in \mathbb{N}_0.
\end{align*}
\]

Similarly, Wynn’s rho algorithm (4.8) and its iteration (4.10) are normally used in combination with the interpolation points \( x_n = n + 1 \), yielding the standard forms (see for example [28, Eq. (6.2-4)])

\[
\begin{align*}
\rho_0^{(n)} &= 0, & \rho_1^{(n)} = s_n, & n \in \mathbb{N}_0, \\
\rho_k^{(n)} &= \rho_{k-1}^{(n+1)} + \frac{k + 1}{\rho_k^{(n+1)} - \rho_k^{(n)}}, & k, n \in \mathbb{N}_0,
\end{align*}
\]

and [28, Section 6.3]

\[
\begin{align*}
\mathcal{W}_0^{(n)} &= s_n, & n \in \mathbb{N}_0, \\
\mathcal{W}_k^{(n)} &= \mathcal{W}_k^{(n+1)} - \frac{(2k + 2)[\Delta \mathcal{W}_k^{(n+1)}][\Delta \mathcal{W}_k^{(n)}]}{(2k + 1)\Delta^2 \mathcal{W}_k^{(n)}}, & k, n \in \mathbb{N}_0.
\end{align*}
\]
Nonlinear Sequence Transformations

Many practically relevant logarithmically convergent sequences \( \{s_n\}_{n=0}^{\infty} \) can be represented by series expansions of the following kind:

\[
s_n = s + (n + \beta)^{-\alpha} \sum_{j=0}^{\infty} c_j/(n + \beta)^j, \quad n \in \mathbb{N}_0. \tag{4.15}
\]

Here, \( \alpha \) is a positive decay parameter and \( \beta \) is a positive shift parameter. In [28, Theorem 14-4], it was shown that the standard form (4.12) of Richardson extrapolation accelerates the convergence of sequences of the type of (4.15) if \( \alpha \) is a positive integer, but fails if \( \alpha \) is nonintegral. This is also true for the standard form (4.13) of the rho algorithm [111, Theorem 3.2]. In the case of the iteration of Wynn’s rho algorithm, no rigorous theoretical result seems to be known but there is considerable empirical evidence that it only works if \( \alpha \) is a positive integer [28, Section 14.4].

If the decay parameter \( \alpha \) of a sequence of the type of (4.15) is known, then Osada’s variant of Wynn’s rho algorithm can be used [111, Eq. (3.1)]:

\[
\bar{\rho}_2^{(n)} = s_{n+1} - \frac{(\alpha + 1)}{\alpha} \frac{[\Delta s_n][\Delta s_{n+1}]}{[\Delta^2 s_n]}, \quad n \in \mathbb{N}_0. \tag{4.16a}
\]

Osada also demonstrated that his variant accelerates the convergence of sequences of the type of (4.15) for arbitrary \( \alpha > 0 \), and that the transformation error satisfies the following asymptotic estimate [111, Theorem 4]:

\[
\bar{\rho}_2^{(n)} - s = O(n^{-\alpha-2k}), \quad n \to \infty. \tag{4.17}
\]

Osada’s variant of the rho algorithm can be iterated. From (4.16) we obtain the following expression for \( \bar{\rho}_2^{(n)} \) in terms of \( s_n, s_{n+1}, \) and \( s_{n+2} \):

\[
\bar{\rho}_2^{(n)} = s_{n+1} - \frac{(\alpha + 1)}{\alpha} \frac{[\Delta s_n][\Delta s_{n+1}]}{[\Delta^2 s_n]}, \quad n \in \mathbb{N}_0. \tag{4.18}
\]

If the iteration is done in such a way that \( \alpha \) is increased by two with every recursive step, we obtain the following recursive scheme [95, Eq. (2.29)] which was originally derived by Bjørstad, Dahlquist, and Grosse [93, Eq. (2.4)]:

\[
\begin{align*}
\bar{W}_0^{(n)} &= s_n, \quad n \in \mathbb{N}_0, \tag{4.19a} \\
\bar{W}_k^{(n+1)} &= \frac{\bar{W}_k^{(n+1)} - (2k + \alpha + 1)}{(2k + \alpha)} \frac{[\Delta \bar{W}_k^{(n)}][\Delta \bar{W}_k^{(n)}]}{[\Delta^2 \bar{W}_k^{(n)}]}, \quad k, n \in \mathbb{N}_0. \tag{4.19b}
\end{align*}
\]

Bjørstad, Dahlquist, and Grosse also showed that \( \bar{W}_k^{(n)} \) accelerates the convergence of sequences of the type of (4.17), and that the transformation error satisfies the following asymptotic estimate [28, Eq. (3.1)]

\[
\bar{W}_k^{(n)} - s = O(n^{-\alpha-2k}), \quad n \to \infty. \tag{4.20}
\]

The explicit knowledge of the decay parameter \( \alpha \) is crucial for an application of the transformations (4.11) and (4.13) to a sequence of the type of (4.15). An approximation to \( \alpha \) can be obtained with the help of the following nonlinear transformation, which was first derived in a somewhat disguised form by Drummond [31] and later rederived by Bjørstad, Dahlquist, and Grosse [33]:

\[
T_n = \frac{[\Delta^2 s_n][\Delta^2 s_{n+1}]}{[\Delta s_{n+1}][\Delta^2 s_{n+1}] - [\Delta s_{n+2}][\Delta^2 s_n]} - 1, \quad n \in \mathbb{N}_0. \tag{4.21}
\]
$T_n$ is essentially a weighted $\Delta^3$ method, which implies that it is potentially very unstable. Thus, stability problems are likely to occur if the relative accuracy of the input data is low. Bjørstad, Dahlquist, and Grosse [93, Eq. (4.1)] also showed that

$$\alpha = T_n + O(1/n^2), \quad n \to \infty,$$

if the elements of a sequence of the type of (4.13) are used as input data.

## 5 Transformations with Explicit Remainder Estimates

As discussed before, the action of a sequence transformation corresponds at least conceptually to the construction an approximations to the truncation error $r_n$, whose elimination from $s_n$ leads to an acceleration of convergence or a summation. However, the remainders $r_n$ may depend on $n$ in a very complicated way, and the construction of approximations to the $r_n$ and their subsequent elimination can be very difficult. In some cases, however, structural information on the $n$-dependence of the $r_n$ is available. For example, the truncation error of a convergent series with strictly alternating and monotonously decreasing terms is bounded in magnitude by the first term not included in the partial sum and has the same sign as this term [1, p. 132]. The first term neglected is also the best simple estimate for the truncation error of a strictly alternating nonterminating hypergeometric series $2F_0(\alpha, \beta; -x)$ with $\alpha, \beta, x > 0$ [124, Theorem 5.12-5], which diverges for every nonzero argument $x$. Such an information should be extremely helpful. Unfortunately, the sequence transformations considered so far cannot benefit from it.

Structural information of that kind can be incorporated into the transformation process via explicit remainder estimates $\{\omega_n\}_{n=0}^\infty$, as it was first done by Levin [123]. For that purpose, let us assume that the remainders $r_n$ of a sequence $\{s_n\}_{n=0}^\infty$ can be partitioned into a remainder estimate $\omega_n$ multiplied by a correction term $z_n$ according to $r_n = \omega_n z_n$. The remainder estimates are chosen according to some rule and may depend on $n$ in a very complicated way. If the remainder estimates $\{\omega_n\}_{n=0}^\infty$ correctly describe the essential features of the remainders $\{r_n\}_{n=0}^\infty$, then the $\{z_n\}_{n=0}^\infty$ should depend on $n$ in a relatively smooth way. Of course, we tacitly assume here that the products $\omega_n z_n$ are in principle capable of producing sufficiently accurate approximations to the remainders.

Thus, we have to find a sequence transformation that is exact for the model sequence

$$\tilde{s}_n = \tilde{s} + \omega_n z_n, \quad n \in \mathbb{N}_0,$$

where the remainder estimates $\{\omega_n\}_{n=0}^\infty$ are assumed to be known. The principal advantage of this approach is that only approximations to the correction terms $\{z_n\}_{n=0}^\infty$ have to be determined. If good remainder estimates can be found, the determination of $z_n$ and the subsequent elimination of $\omega_n z_n$ from $s_n$ often leads to clearly better results than the construction and elimination of other approximations to $r_n$. The explicit utilization of information contained in remainder estimates is the major difference between the sequence transformations discussed in this Section and the other transformations of this article.

The model sequence (5.1) has another indisputable advantage: There is a systematic way of constructing a sequence transformation which is exact for this model sequence. Let us assume that a linear operator $\hat{T}$ can be found which annihilates the correction term $z_n$ according to $\hat{T}(z_n) = 0$. Then, a sequence transformation, which is exact for the model sequence (5.1), can be obtained by applying $\hat{T}$ to $[\tilde{s}_n - \tilde{s}]/\omega_n = z_n$. Since $\hat{T}$ annihilates $z_n$ and is by assumption linear, the following sequence transformation $\mathcal{T}$ is exact for the model sequence (5.1) [28, Eq. (3.2-11)]:

$$\mathcal{T}(\tilde{s}_n, \omega_n) = \frac{\hat{T}(\tilde{s}_n/\omega_n)}{\hat{T}(1/\omega_n)} = \tilde{s}.$$

(5.2)
Nonlinear Sequence Transformations

Originally, the construction of sequence transformations via annihilation operators was introduced in [28, Section 3.2] in connection with a rederivation of Levin’s transformation [123] and the construction of some other, closely related sequence transformations [28, Sections 7 - 9]. Later, this operator approach was also used and discussed by Brezinski [125], Brezinski and Redivo Zaglia [25, 125, 126], Brezinski and Salam [127], Brezinski and Matos [128], Matos [129], and Homeier [29].

If the annihilation operator $\hat{T}$ in (5.2) is based upon the finite difference operator $\Delta$, simple and yet very powerful sequence transformations are obtained [28, Sections 7 - 9]. As is well known, $\Delta^k$ annihilates a polynomial $P_{k-1}(n)$ of degree $k - 1$ in $n$. Thus, the correction terms should be chosen in such a way that multiplication of $z_n$ by some $w_k(n)$ yields a polynomial $P_{k-1}(n)$ of degree $k - 1$ in $n$.

Since $\Delta^k w_k(n) z_n = \Delta^k P_{k-1}(n) = 0$, the weighted difference operator $\tilde{T} = \Delta^k w_k(n)$ annihilates $z_n$, and the corresponding sequence transformation (5.2) is given by the ratio

$$T_k^{(n)}(w_k(n)|s_n, \omega_n) = \frac{\Delta^k\{w_k(n)s_n/\omega_n\}}{\Delta^k\{w_k(n)/\omega_n\}}. \tag{5.3}$$

Several different sequence transformations are obtained by specializing $w_k(n)$. For instance, $w_k(n) = (n + \zeta)^{k-1}$ with $\zeta > 0$ yields Levin’s sequence transformation [123]:

$$L_k^{(n)}(\zeta, s_n, \omega_n) = \frac{\Delta^k\{(n + \zeta)^{k-1} s_n/\omega_n\}}{\Delta^k\{(n + \zeta)^{k-1}/\omega_n\}} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\zeta + n + j)^{k-1} s_{n+j}}{(\zeta + n + k)^{k-1} \omega_{n+j}}. \tag{5.4}$$

The shift parameter $\zeta$ has to be positive to allow $n = 0$ in (5.4). The most obvious choice is $\zeta = 1$. According to Smith and Ford [79, 103], Levin’s transformation is among the most powerful and most versatile sequence transformations that are currently known.

The unspecified weights $w_k(n)$ in (5.3) can also be chosen to be Pochhammer symbols according to $w_k(n) = (n + \zeta)^{k-1}$ with $\zeta > 0$, yielding [28, Eq. (8.2-7)]

$$S_k^{(n)}(\zeta, s_n, \omega_n) = \frac{\Delta^k\{(n + \zeta)^{k-1} s_n/\omega_n\}}{\Delta^k\{(n + \zeta)^{k-1}/\omega_n\}} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\zeta + n + j)^{k-1} s_{n+j}}{(\zeta + n + k)^{k-1} \omega_{n+j}}. \tag{5.5}$$

As shown in several articles, this transformation is very effective if strongly divergent alternating series are to be summed [28, 54, 49, 52, 57, 55, 53, 59, 60, 130, 132]. Again, the most obvious choice for the shift parameter is $\zeta = 1$.

Levin’s transformation $L_k^{(n)}(\zeta, s_n, \omega_n)$ is by construction exact for the model sequence $s_n = s + \omega_n \sum_{j=0}^{k-1} c_j/(n + \zeta)^j$. Consequently, Levin’s transformation should work well if the ratio $|s_n - s|/\omega_n$ can be expressed as a power series in $1/(n + \zeta)$. Similarly, $S_k^{(n)}(\zeta, s_n, \omega_n)$ is by construction exact for the model sequence $s_n = s + \omega_n \sum_{j=0}^{k-1} c_j/(n + \zeta)^j$, which is a truncated factorial series [28, Eq. (8.2-1)]. Accordingly, $S_k^{(n)}(\zeta, s_n, \omega_n)$ should give good results if the ratio $|s_n - s|/\omega_n$ can be expressed as a factorial series.
Given a suitable sequence \( \{\omega_n\}_{n=0}^\infty \) of remainder estimates, the sequence transformations \( L_k^{(n)} \) and \( S_k^{(n)} \) can be computed via their explicit expressions \([\ref{5.4}]\) and \([\ref{5.5}]\). However, it is usually more effective to compute the numerator and denominator sums in \([\ref{5.4}]\) and \([\ref{5.5}]\) with the help of three-term recursions \([28]\) Eqs. (7.2-8) and (8.3-7).

The explicit incorporation of the information contained in the remainder estimates makes the transformations \([\ref{5.4}]\) and \([\ref{5.5}]\) potentially very powerful. However, this is also their major potential weakness. If remainder estimates can be found such that the products \(\omega_n z_n\) provide good approximations to the remainders, Levin-type sequence transformations should work very well. If, however, good remainder estimates cannot be found, sequence transformations of that kind perform poorly. Consequently, a fortunate choice of the remainder estimates in \([\ref{5.4}]\) and \([\ref{5.5}]\) is of utmost importance since it ultimately determines success or failure.

The difference operator \(\Delta\) is linear. Consequently, the effect of the general sequence transformation \([\ref{5.3}]\) on an arbitrary sequence \(\{s_n\}_{n=0}^\infty\) with (generalized) limit \(s\) can be expressed as follows:

\[
T_k^{(n)}(w_k(n) | s_n, \omega_n) = s + \frac{\Delta^k \{w_k(n)(s_n - s)/\omega_n\}}{\Delta^k \{w_k(n)/\omega_n\}}.
\]

Obviously, \(T_k^{(n)}(w_k(n) | s_n, \omega_n)\) converges to \(s\) if the ratio on the right-hand side can be made arbitrarily small. This is the case if \(\Delta^k w_k(n)\) annihilates \([s_n - s]/\omega_n\) more effectively than \(1/\omega_n\). Thus, one should try to find remainder estimates such that the ratios \([s_n - s]/\omega_n\) depend on \(n\) only weakly:

\[
[s_n - s]/\omega_n = c + O(1/n), \quad c \neq 0, \quad n \to \infty.
\]

This asymptotic condition does not determine the remainder estimates uniquely. Therefore, it is at least in principle possible to find for a given sequence an unlimited variety of different remainder estimates, which all satisfy this asymptotic condition.

On the basis of heuristic and asymptotic arguments, Levin \([123]\) suggested the following simple remainder estimates, which nevertheless often work remarkably well:

\[
\omega_n = (\zeta + n) \Delta s_{n-1}, \quad \zeta > 0,
\]

\[
\omega_n = \Delta s_{n-1},
\]

\[
\omega_n = \frac{\Delta s_{n-1} \Delta s_n}{\Delta s_{n-1} - \Delta s_n}.
\]

The use of these remainder estimates in \([\ref{5.3}]\) yields Levin’s \(u\), \(t\), and \(v\) transformation, respectively \([28]\) Eqs. (7.3-5), (7.3-7), and (7.3-11):

\[
u_k^{(n)}(\zeta, s_n) = L_k^{(n)}(\zeta, s_n, (\zeta + n) \Delta s_{n-1})\]

\[
u_k^{(n)}(\zeta, s_n) = L_k^{(n)}(\zeta, s_n, \Delta s_{n-1})\]

\[
u_k^{(n)}(\zeta, s_n) = L_k^{(n)}(\zeta, s_n, (\Delta s_{n-1} \Delta s_n)/(\Delta s_{n-1} - \Delta s_n))\].

Later, Smith and Ford \([103]\) suggested the remainder estimate

\[
\omega_n = \Delta s_n,
\]

which yields Levin’s \(d\) transformation \([28]\) Eq. (7.3-9):

\[
d_k^{(n)}(\zeta, s_n) = L_k^{(n)}(\zeta, s_n, \Delta s_n).
\]
Nonlinear Sequence Transformations

Levin’s $t$ and $d$ transformations are capable of accelerating linear convergence and they are particularly efficient in the case of alternating series, but fail to accelerate logarithmic convergence. Levin’s $u$ and $v$ transformation are more versatile since they not only accelerate linear convergence but also many logarithmically convergence sequences and series. A more detailed discussion of the properties of these remainder estimates, some generalizations, additional heuristic motivation, and a description of the types of sequences, for which these estimates should be effective, can be found in [28, Sections 7 and 12 - 14]. The main advantage of the simple remainder estimates (5.8) - (5.10) and (5.14) is that they can be used in situations in which only the numerical values of a few elements of a slowly convergent or divergent sequence are known.

The remainder estimates (5.8) - (5.10) and (5.14) can also be used in the case of the sequence transformation (5.5), yielding [28, Eqs. (8.4-2), (8.4-3), (8.4-4), and (8.4-5)]

$$y_k^{(n)}(\zeta, s_n) = S_k^{(n)}(\zeta, s_n, (\zeta + n)\Delta s_{n-1})$$

$$t_k^{(n)}(\zeta, s_n) = S_k^{(n)}(\zeta, s_n, \Delta s_{n-1})$$

$$\phi_k^{(n)}(\zeta, s_n) = S_k^{(n)}(\zeta, s_n, (\Delta s_{n-1}\Delta s_n)/(\Delta s_{n-1} - \Delta s_n))$$

$$\delta_k^{(n)}(\zeta, s_n) = S_k^{(n)}(\zeta, s_n, \Delta s_n)$$

Alternative remainder estimates for the sequence transformations (5.4) and (5.5) are discussed in [28, 58]. Convergence properties of the sequence transformations (5.4) and (5.5) and their variants were analyzed in articles by Sidi [133–135], in [28, Sections 12 - 14], in [62, Section 4], and also in [29].

Other sequence transformations, which are also special cases of the general sequence transformation (5.2), can be found in [28, Sections 7 - 9], in the book by Brezinski and Redivo Zaglia [25, Section 2.7], or in a recent review by Homeier [29], which is the currently most complete source of information on Levin-type transformations. A sequence transformation, which interpolates between the transformations (5.4) and (5.5), was described in [61].

6 Numerical Aspects

As discussed before, sequence transformations try to accomplish an acceleration of convergence or a summation by detecting and utilizing regularities in the behavior of the elements of the sequence to be transformed. For sufficiently large indices $n$, one can expect that certain asymptotic regularities do exist. However, sequence transformations are normally used with the intention of avoiding the asymptotic domain, i.e., one tries to construct the transforms from the leading elements of the input sequence. Unfortunately, sequence elements $s_n$ with small indices $n$ often behave irregularly. Consequently, it can happen that a straightforward application of a sequence transformation is ineffective and even leads to completely nonsensical results. In fact, one should not be too surprised that a strategy, which tries to avoid the asymptotic domain by extracting asymptotic information from the leading elements of an input sequence, occasionally runs into trouble.

As a possible remedy, one should analyze the behavior of the input data as a function of the index and exclude highly irregular sequence elements – usually the leading ones – from the transformation process. This gives a much better chance of obtaining good and reliable transformation results.

In view of the fact that irregular input data can never be excluded, one might expect that there are many references dealing with this topic. However, apart from a recent reference of my own [53], I am only aware of an article by Gander, Golub, and Grunts [136] where it is shown that the convergence of extrapolations of iteration sequences can be improved by excluding the leading elements of the input sequence from the extrapolation process.
Otherwise, a discussion of the impact of irregular input data on convergence acceleration and summation processes seems to be part of the oral tradition only.

A numerical process can only involve a finite number of arithmetic operations. Thus, a sequence transformation $\mathcal{T}$ can only use finite subsets of the type $\{s_n, s_{n+1}, \ldots, s_{n+l}\}$ for the computation of a new sequence element $s'_{n}$. Moreover, all the commonly used sequence transformations $\mathcal{T}$ can be represented by infinite sets of doubly indexed quantities $T_{k}^{(n)}$ with $k, n \in \mathbb{N}_0$ that can be displayed in a two-dimensional array called the *table* of $\mathcal{T}$. The superscript $n$ always indicates the minimal index occurring in the finite subset of input data used for the computation of $T_{k}^{(n)}$. The subscript $k$ – usually called the *order* of the transformation – is a measure for the complexity of the transformation process which yields $T_{k}^{(n)}$. The elements $T_{k}^{(n)}$ are gauged in such a way that $T_{0}^{(n)}$ corresponds to an untransformed sequence element according to $T_{0}^{(n)} = s_n$. An increasing value of $k$ implies that the complexity of the transformation process increases. Moreover, $l = l(k)$ in the substring $\{s_n, s_{n+1}, \ldots, s_{n+l}\}$ also increases. This means that for every $k, n \in \mathbb{N}_0$ the sequence transformation $\mathcal{T}$ produces a new transform according to

$$T_{k}^{(n)} = \mathcal{T}(s_n, s_{n+1}, \ldots, s_{n+l(k)}).$$

(6.1)

The relationship, which connects $k$ and $l$, is specific for a given sequence transformation $\mathcal{T}$.

Let us assume that a sequence transformation $\mathcal{T}$ is to be used to speed up the convergence of a sequence $\{s_n\}_{n=0}^{\infty}$ to its limit $s = s_\infty$. One can try to obtain a better approximation to $s$ by proceeding on an unlimited variety of different *paths* in the table of $\mathcal{T}$. Two extreme types of paths – and also the ones which are predominantly used in practical applications – are *order-constant* paths $\{T_{k}^{(n+v)}\}_{v=0}^{\infty}$ with fixed transformation order $k$ and increasing superscript, and *index-constant* paths $\{T_{k+n}^{(n)}\}_{n=0}^{\infty}$ with fixed minimal index $n$ and increasing subscript.

These two types of paths differ significantly. In the case of an order-constant path, a *fixed* number of $l + 1$ sequence elements $\{s_n, s_{n+1}, \ldots, s_{n+l}\}$ is used for the computation of $T_{k}^{(n)}$, and the starting index $n$ of this string of fixed length is increased successively until either convergence is achieved or the available elements of the input sequence are used up. In the case of an index-constant path, the starting index $n$ is kept fixed at a low value (usually $n = 0$ or $n = 1$) and the transformation order $k$ is increased and with it the number of elements contained in the subset $\{s_n, s_{n+1}, \ldots, s_{n+l(k)}\}$. Thus, on an index-constant path $T_{k}^{(n)}$ is always computed with the highest possible transformation order $k$ from a given set of input data.

In order to clarify the differences between order-constant and index-constant paths, let us consider the computation of Padé approximants with the help of Wynn's epsilon algorithm (3.7). It follows from (3.0) that the epsilon algorithm (3.7) effects the following transformation of the partial sums (1.3) of the power series for some function $f$ to Padé approximants:

$$\{f_n(z), f_{n+1}(z), \ldots, f_{n+2k}(z)\} \rightarrow [n + k/k].$$

(6.2)

Thus, if we use a window consisting of $2k + 1$ partial sums $f_{n+j}(z)$ with $0 \leq j \leq 2k$ on an order-constant path and increase the minimal index $n$ successively, we obtain the following sequence of Padé approximants:

$$[n + k/k], [n + k + 1/k], \ldots, [n + k + m/k], \ldots$$

(6.3)

Only $2k + 1$ partial sums are used for the computation of the Padé approximants, although many more may be known. Obviously, the available information is not completely utilized on such an order-constant path.
Nonlinear Sequence Transformations

Then, the degree of the numerator polynomial of \([n + k + m/k]\) increases with increasing \(m \in \mathbb{N}_0\), whereas the degree of the denominator polynomial remains fixed. Thus, these Padé approximants look unbalanced. Instead, it seems to be much more natural to use diagonal Padé approximants, i.e., Padé approximants with numerator and denominator polynomials of equal degree, or – if this is not possible – to use Padé approximants with degrees of the numerator and denominator polynomials that differ as little as possible.

This approach has many theoretical as well as practical advantages. Wynn could show that if the partial sums \(f_0(z), f_1(z), \ldots, f_{2n}(z)\) of a Stieltjes series are used for the computation of Padé approximants, then the diagonal approximant \([n/n]\) provides the most accurate approximation to the corresponding Stieltjes function \(f(z)\), and if the partial sums \(f_0(z), f_1(z), \ldots, f_{2n+1}(z)\) are used for the computation of Padé approximants, then for \(z > 0\) either \([n + 1/n]\) or \([n/n + 1]\) provides the most accurate approximation \([37]\). A detailed discussion of Stieltjes series and their special role in the theory of Padé approximants can be found in \([2\,Section\,5]\).

Thus, it is apparently an obvious idea to try to use either diagonal Padé approximants or their closest neighbors whenever possible. If the partial sums \(f_0(z), f_1(z), \ldots, f_{m}(z), \ldots\) are computed successively and used as input data in the epsilon algorithm \([33]\), then we obtain the following staircase sequence in the Padé table \([28, \text{Eq. (4.3-7)}]\), which exploits the available information optimally:

\[
[0/0], [1/0], [1/1], \ldots [\nu/\nu], [\nu + 1/\nu], [\nu + 1/\nu + 1], \ldots
\]  

This example indicates that index-constant paths are in principle computationally more efficient than order-constant paths since they exploit the available information optimally. This is also true for all other sequence transformations considered in this article.

Another serious disadvantage of order-constant paths is that they cannot be used for the summation of divergent sequences and series since increasing \(n\) in the set \(\{s_n, s_{n+1}, \ldots, s_{n+l}\}\) of input data normally only increases divergence. Thus, it is apparently an obvious idea to use exclusively index-constant paths, and preferably those which start at a very low index \(n\), for instance at \(n = 0\) or \(n = 1\). This is certainly a good idea if all elements of the input sequence contain roughly the same amount of useful information. If, however, the leading terms of the sequence to be transformed behave irregularly, they cannot contribute useful information, or – to make things worse – they contribute wrong information. In such a case it is usually necessary to exclude the leading elements of the input sequence from the transformation process. Then, one should use either an order-constant path or an index-constant path with a sufficiently large starting index \(n\). The use of an order-constant path has the additional advantage that the diminishing influence of irregular input data with small indices \(n\) should become obvious from the transformation results as \(n\) increases.

In \([23,22,25,37]\) I did extensive quantum mechanical calculations for anharmonic oscillators by summing strongly divergent perturbation expansions. The coefficients of these perturbation expansions are all rational numbers. Thus, they can be computed free of errors with the help of the exact rational arithmetics of a computer algebra system like Maple. In such a case, it is not only an obvious idea but in fact necessary to do the summation calculations on an index-constant path. The maximum transformation order in these summation calculations was only limited by the number of perturbative coefficients that could be computed before the available memory of the computer was exhausted.

The situation was completely different when I was involved in the extrapolation of quantum chemical oligomer calculations to the infinite chain limit \([71,72]\). As discussed before, sequence transformations have to access the information stored in the later digits of the input data. However, the input data are produced by molecular ab initio programs which are huge packages of FORTRAN code (more than 100 000 lines of code). Normally, these programs operate in DOUBLE PRECISION which corresponds to an accuracy of 14 - 16
decimal digits (depending on the compiler). Unfortunately, this does not mean that their results have this accuracy. Some of the leading digits may have converged, and some of the trailing digits are corrupted because of internal approximations used in the molecular program or because of inevitable rounding errors. Thus, in most cases there is only a relatively narrow window of digits that can provide useful information for the transformation process. Moreover, the complexity of the calculations done in such a molecular program makes it impossible to obtain realistic estimates of the errors by the standard approaches of numerical mathematics. In such a case, it is recommendable to do the extrapolations on order-constant paths with low transformation orders, because high transformation orders can easily lead to meaningless extrapolation results. Further details can be found in [73].

It may look like an obvious idea to use the results of a transformation process as input data for another sequence transformation. At least from a conceptual point of view, this is essentially the same as iterating a low order transformation like Aitken’s $\Delta^2$ formula (3.2). If, however, the input data for the second transformation were obtained on an index-constant path by the first transformation, then I see basically two problems. The law, which governs the convergence of the first transformation, may be unknown and/or hard to find. Thus, it may be difficult to find a suitable second transformation. The second and – as I feel – more serious restriction is numerical in nature. The trailing digits of the input data for the second transformation may be largely corrupted by the first transformation. Thus, they cannot provide the information needed for a further improvement of convergence. There are articles which describe successful attempts of combining different sequence transformations [70,138]. However, I suspect that many more attempts were not reported in the literature because they failed to accomplish something substantial.

7 Outlook

If a review article is to written and if there is only a limited amount of space available, there is always the danger that the author emphasizes those aspects he knows particularly well, whereas other aspects are not treated as thoroughly as they probably should. This is of course also true for this review. So, I will now try to give a more balanced view by mentioning some additional applications of sequence transformations.

In this review, exclusively the transformation of sequences of real or complex numbers was treated. However, sequence transformations can also be formulated for vector or matrix problems. Sequence transformations for vector sequences are for instance treated in the books by Brezinski and Redivo Zaglia [25, Section 4] and Brezinski [124], or in articles by MacLeod [10], Graves-Morris and Saff [139,140], Sidi [141], Smith, Ford, and Sidi [142,143], Sidi and Bridger [144], Jbilou and Sadok [145,146], Osada [147,149], Brezinski and Sadok [150], Matos [151], Graves-Morris [152,153], Graves-Morris and Roberts [154,155], Brezinski and Salam [127], Homeier, Rast, and Krienke [154], Salam [157,159], Graves-Morris and Van Iseghem [160,161], Roberts [161,162], and Graves-Morris, Roberts, and Salam [99].

In the practice of the author, convergence acceleration and summation of infinite series has dominated. However, sequence transformations can also be very useful in the context of numerical quadrature, in particular if an oscillatory function is to be integrated over a semiinfinite interval. The use of sequence transformations in connection with numerical quadratures is for instance discussed in a book by Evans [163], or in articles by Sidi [164,171], Levin and Sidi [172], Levin [173], Greif and Levin [174], Safouhi and Hoggan [175,178], Safouhi, Pinchon, and Hoggan [179], and Safouhi [180,181].

So far, sequence transformations were typically used to obtain better approximations to the limits of sequences, series, or numerical quadrature schemes. However, it is possible to use Padé approximants or other sequence transformations to make predictions for unknown perturbation series coefficients. For example, in some subfields of theoretical physics it is
Nonlinear Sequence Transformations

extremely difficult to compute more than just a few perturbative coefficients, and even these few coefficients are usually affected by large relative errors. Thus, sequence transformations – mainly Padé approximants – were used to make predictions for unknown coefficients or to check the accuracy of already computed coefficients. Further details as well as many references can be found in [69,70,89,182]. In the majority of these predictions calculations, rational expressions in an unspecified symbolic variable were constructed, which were then expanded in a Taylor polynomial of suitable length with the help of a computer algebra program like Maple or Mathematica. However, in the case of Aitken’s iterated $\Delta^2$ process (3.3), of Wynn’s epsilon algorithm (3.5), and of the iteration (3.9) of Brezinski’s theta algorithm explicit recursive schemes for predictions were recently derived [89]. With the help of these recursive schemes, it is possible to make predictions for power series coefficients with very large indices. In this way, it was possible to gain some insight into the mathematical properties of a perturbation expansion for a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian [83].

I hope that this review could show that sequence transformations are extremely useful numerical tools, and that there is a lot of active research going on, both on the mathematical properties of sequence transformations as well as on their application in applied mathematics and in the mathematical treatment of the sciences. Moreover, I am optimistic and expect further progress in the near future.

Acknowledgments

I would like to thank Professor Dzevad Belkić for his invitation to contribute to this topical issue of Journal of Computational Methods in Sciences and Engineering. Financial support by the Fonds der Chemischen Industrie is gratefully acknowledged.

References

[1] K. Knopp, Theorie und Anwendung der unendlichen Reihen, Springer-Verlag, Berlin, 1964.
[2] J. Stirling, Methodus differentialis sive tractatus de summatione et interpolacione serium infinitarum, London, 1730. English translation by F. Holliday, The differential method, or, a treatise concerning the summation and interpolation of infinite series, London, 1749.
[3] L. Euler, Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina seriem. Part II.1. De transformatione serium, Academia Imperialis Scientiarum Petropolitana, 1755. This book was reprinted as Vol. X of Leonardi Euleri Opera Omnia, Seria Prima, Teubner, Leipzig and Berlin, 1913.
[4] C. Brezinski, History of continued fractions and Padé approximants, Springer-Verlag, Berlin, 1991.
[5] L.F. Richardson, The deferred approach to the limit. I. Single lattice, Phil. Trans. Roy. Soc. London A, 226, pp. 229 - 349 (1927).
[6] A.C. Aitken, On Bernoulli’s numerical solution of algebraic equations, Proc. Roy. Soc. Edinburgh, 46, pp. 289 - 305 (1926).
[7] C.B. Liem, T. Lü, and T.M. Shih, The splitting extrapolation method, World Scientific, Singapore, 1995.
[8] C. Brezinski, Extrapolation algorithms and Padé approximations: a historical survey, Appl. Numer. Math., 20, pp. 299 - 318 (1996).
[9] C. Brezinski, Convergence acceleration during the 20th century, J. Comput. Appl. Math., 122, pp. 1 - 21 (2000). Reprinted in C. Brezinski (Editor), Numerical analysis 2000, Vol. 2: Interpolation and extrapolation, Elsevier, Amsterdam, 2000, pp. 1 - 21.
[10] G.H. Hardy, Divergent series, Clarendon Press, Oxford, 1949.
[11] G.M. Petersen, Regular matrix transformations, McGraw-Hill, London, 1966.
[12] A. Peyerimhoff, Lectures on summability, Springer-Verlag, Berlin, 1969.
[13] K. Zeller and W. Beekmann, Theorie der Limitierungsverfahren, Springer-Verlag, Berlin, 1970.
[14] R.E. Powell and S.M. Shah, Summability theory and its applications, Prentice-Hall of India, New Delhi, 1988.
[15] J. Wimp, Sequence transformations and their applications, Academic Press, New York, 1981.
[16] P. Wynn, On a device for computing the $e_m(S_n)$ transformation, Math. Tables Aids Comput., 10, pp. 91 - 96 (1956).
[17] P. Wynn, On a Procrustean technique for the numerical transformation of slowly convergent sequences and series, Proc. Camb. Phil. Soc., 52, pp. 663 - 671 (1956).
[18] C. Brezinski, Accélération de suites à convergence logarithmique, C. R. Acad. Sc. Paris, 273, pp. 727 - 730 (1971).
[19] C. Brezinski, Accélération de la convergence en analyse numérique, Springer-Verlag, Berlin, 1977.
[20] C. Brezinski, Algorithmes d’accélération de la convergence – étude numérique, Editions Technip, Paris, 1978.
[21] C. Brezinski, Padé-type approximation and general orthogonal polynomials, Birkhäuser, Basel, 1980.
[22] G.I. Marchuk and V.V. Shaidurov, Difference methods and their extrapolations, Springer-Verlag, New York, 1983.
[23] A. Cuyt and L. Wuytack, Nonlinear methods in numerical analysis, North-Holland, Amsterdam, 1987.
[24] J.-P. Delahaye, Sequence transformations, Springer-Verlag, Berlin, 1988.
[25] C. Brezinski and M. Redivo Zaglia, Extrapolation methods, North-Holland, Amsterdam, 1991.
[26] G. Walz, Asymptotics and extrapolation, Akademie Verlag, Berlin, 1996.
[27] A.J. Guttmann, Asymptotic analysis of power series expansions, in C. Domb and J.L. Lebowitz (Editors), Phase transitions and critical phenomena Vol. 13, Academic Press, London, 1989, pp. 3 - 234.
[28] E.J. Weniger, Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series, Comput. Phys. Rep., 10, pp. 189 - 371 (1989).
[29] H.H.H. Homeier, Scalar Levin-type sequence transformations, J. Comput. Appl. Math., 122, pp. 81 - 147 (2000). Reprinted in C. Brezinski (Editor), Numerical analysis 2000, Vol. 2: Interpolation and extrapolation, Elsevier, Amsterdam, 2000, pp. 81 - 147.
[30] E.J. Weniger, On the efficiency of linear but nonregular sequence transformations, in A. Cuyt (Editor), Nonlinear numerical methods and rational approximation Vol. II, Kluwer, Dordrecht, 1994, pp. 269 - 282.
[31] G.A. Baker, Jr., Essentials of Padé approximants, Academic Press, New York, 1975.
[32] G.A. Baker, Jr., and P. Graves-Morris, Padé approximants, 2nd edition, Cambridge U.P., Cambridge, 1996.
[33] C. Brezinski, A bibliography on continued fractions, Padé approximation, extrapolation and related subjects, Prensas Universitarias de Zaragoza, Zaragoza, 1991.
[34] G.A. Baker, Jr., The theory and application of the Padé approximant method, Adv. Theor. Phys., 1, pp. 1 - 58 (1965).
[35] H.S. Wall, Analytic Theory of continued fractions, Chelsea, New York, 1973.
[36] W.B. Jones and W.T. Thron, Continued fractions, Addison Wesley, Reading, Mass., 1980.
[37] K.O. Bowman and L.R. Shenton, Continued fractions in statistical applications, Marcel Dekker, New York, 1989.
[38] L. Lorentzen and H. Waadeland, Continued fractions with applications, North-Holland, Amsterdam, 1992.
[39] C.M. Bender and S.A. Orszag, Advanced mathematical methods for scientists and engineers, McGraw-Hill, New York, 1978.
[40] G.A. Baker, Jr., Quantitative theory of critical phenomena, Academic Press, San Diego, 1990.
[41] A. Pozzi, Applications of Padé approximation theory in fluid dynamics, World Scientific, Singapore, 1994.
[42] J. Zinn-Justin, Strong interaction dynamics with Padé approximants, Phys. Rep., 1, pp. 55 - 102 (1971).
[43] J.L. Basdevant, The Padé approximation and its physical applications, Fortschr. Physik; 20, pp. 283 - 331 (1972).
[44] C. Brezinski and J. Van Iseghem, Padé approximations, in P.G. Ciarlet and J.L. Lions (Editors), Handbook of numerical analysis Vol. III, North-Holland, Amsterdam, 1994, pp. 47 - 222.
[45] C. Brezinski and J. Van Iseghem, A taste of Padé approximation, in A. Iserles (Editor), Acta Numerica, Cambridge U.P., Cambridge, 1995, pp. 53 - 103.
[46] A. Cuyt, Padé approximants for operators: Theory and applications, Springer-Verlag, Berlin, 1984.
[47] A. Cuyt, How well can the concept of Padé approximant be generalized to the multivariate case?, J. Comput. Appl. Math., 105, pp. 25 - 50 (1999).
[48] P. Guillaume and A. Huard, Multivariate Padé approximation, J. Comput. Appl. Math., 121, pp. 197 - 219 (2000). Reprinted in L. Wuytack and J. Wimp (Editors), Numerical analysis 2000, Vol. 1: Approximation theory, Elsevier, Amsterdam, 2000, pp. 197 - 219.
[49] E.J. Weniger, On the summation of some divergent hypergeometric series and related perturbation expansions, J. Comput. Appl. Math., 32, pp. 291 - 300 (1990).
[50] E.J. Weniger and J. Čížek, Rational approximations for the modified Bessel function of the second kind, Comput. Phys. Commun., 59, pp. 471 - 493 (1990).
[51] E.J. Weniger, Computation of the Whittaker function of the second kind by summing its divergent asymptotic series with the help of nonlinear sequence transformations, Comput. Phys., 10, pp. 496 - 503 (1996).
[52] U.D. Jentschura, P.J. Mohr, G. Soff, and E.J. Weniger, Convergence acceleration via combined nonlinear-condensation transformations, Comput. Phys. Commun., 116, pp. 28 - 54 (1999).
[53] E.J. Weniger, Irregular input data in convergence acceleration and summation processes: General considerations and some special Gaussian hypergeometric series as model problems, Comput. Phys. Commun., 133, pp. 202 - 228 (2001).
[54] E.J. Weniger, J. Grotendorst, and E.O. Steinborn, Some applications of nonlinear convergence accelerators, Int. J. Quantum Chem. Symp., 19, pp. 181 - 191 (1986).
[55] J. Grotendorst, E.J. Weniger, and E.O. Steinborn, Efficient evaluation of infinite-series representations for overlap, two-center nuclear attraction, and Coulomb integrals using nonlinear convergence accelerators, Phys. Rev. A, 33, pp. 3706 - 3726 (1986).
[56] E.J. Weniger and E.O. Steinborn, Nonlinear sequence transformations for the efficient evaluation of auxiliary functions for GTO molecular integrals, in M. Defranceschi and J. Delhalle (Editors), Numerical Determination of the Electronic Structure of Atoms, Diatomic and Polyatomic Molecules, Kluwer, Dordrecht, 1989, pp. 341 - 346.
[57] E.O. Steinborn and E.J. Weniger, Sequence transformations for the efficient evaluation of infinite series representations of some molecular integrals with exponentially decaying basis functions, J. Mol. Struct. (Theochem), 210, pp. 71 - 78 (1990).
H.H.H. Homeier and E.J. Weniger, On remainder estimates for Levin-type sequence transformations, Comput. Phys. Commun., 92, pp. 1 - 10 (1995).

E.J. Weniger, J. Čížek, and F. Vinette, Very accurate summation for the infinite coupling limit of the perturbation series expansions of anharmonic oscillators, Phys. Lett. A, 156, pp. 169 - 174 (1991).

J. Čížek, F. Vinette, and E.J. Weniger, Examples on the use of symbolic computation in physics and chemistry: Applications of the inner projection technique and of a new summation method for divergent series, Int. J. Quantum Chem. Symp., 25, pp. 209 - 223 (1991).

E.J. Weniger, Interpolation between sequence transformations, Numer. Algor., 3, pp. 477 - 486 (1992).

E.J. Weniger, J. Čížek, and F. Vinette, The summation of the ordinary and renormalized perturbation series for the ground state energy of the quartic, sextic, and octic anharmonic oscillators using nonlinear sequence transformations, J. Math. Phys., 34, 571 - 609 (1993).

J. Čížek, F. Vinette, and E.J. Weniger, On the use of the symbolic language Maple in physics and chemistry: Several examples, in R.A. de Groot and J. Nadrchal (Editors), Proceedings of the Fourth International Conference on Computational Physics PHYSICS COMPUTING ’92, World Scientific, Singapore, 1993, pp. 31 - 44.

E.J. Weniger, Nonlinear sequence transformations: A computational tool for quantum mechanical and quantum chemical calculations, Int. J. Quantum Chem., 57, pp. 265 - 280 (1996); Erratum, Int. J. Quantum Chem., 58, pp. 319 - 321 (1996).

E.J. Weniger, A convergent renormalized strong coupling expansion for the ground state energy of the quartic, sextic, and octic anharmonic oscillator, Ann. Phys. (NY), 246, pp. 133 - 165 (1996).

J. Čížek, E.J. Weniger, P. Bracken, and V. Špirko, Effective characteristic polynomials and two-point Padé approximants as summation techniques for the strongly divergent perturbation expansions of the ground state energies of anharmonic oscillators, Phys. Rev. E, 53, pp. 2925 - 2939 (1996).

E.J. Weniger, Construction of the strong coupling expansion for the ground state energy of the quartic, sextic and octic anharmonic oscillator via a renormalized strong coupling expansion, Phys. Rev. Lett., 77, pp. 2859 - 2862 (1996).

E.J. Weniger, Performance of superconvergent perturbation theory, Phys. Rev. A, 56, pp. 5165 - 5168 (1997).

U. Jentschura, J. Becher, E.J. Weniger, and G. Soff, Resummation of QED perturbation series by sequence transformations and prediction of perturbative coefficients, Phys. Rev. Lett., 85, pp. 2446 - 2449 (2000).

U. Jentschura, E.J. Weniger, and G. Soff, Asymptotic improvement of resummations and perturbative predictions in quantum field theory, J. Phys. G: Nucl. Part. Phys., 26, pp. 1545 - 1568 (2000).

E.J. Weniger and C.-M. Liegener, Extrapolation of finite cluster and crystal-orbital calculations on trans-polyacetylene, Int. J. Quantum Chem., 38, pp. 55 - 74 (1990).

J. Cioslowski and E.J. Weniger, Bulk properties from finite cluster calculations. VIII. Benchmark calculations on the efficiency of extrapolation methods for the HF and MP2 energies of polyacenes, J. Comput. Chem., 14, pp. 1468 - 1481 (1993).

E.J. Weniger and B. Kirtman, Extrapolation methods for improving the convergence of oligomer calculations to the infinite chain limit of quasi-onedimensional stereoregular polymers, Comput. Math. Applic., in press. Los Alamos preprint math.NA/0004115 (http://arXiv.org).

F.W.J. Olver, Asymptotics and special functions, Academic Press, New York, 1974.

C. Schneider, Vereinfachte Rekursionen zur Richardson-Extrapolation in Spezialfällen, Numer. Math., 24, pp. 177 - 184 (1975).

C. Brezinski, A general extrapolation algorithm, Numer. Math., 35, pp. 175 - 180 (1980).
Nonlinear Sequence Transformations

[77] T. Håvie, Generalized Neville type extrapolation schemes, BIT, 19, pp. 204 - 213 (1979).

[78] W.F. Ford and A. Sidi, An algorithm for a generalization of the Richardson extrapolation process, SIAM J. Numer. Anal., 24, pp. 1212 - 1232 (1987).

[79] D.A. Smith and W.F. Ford, Numerical comparisons of nonlinear convergence accelerators, Math. Comput., 38, pp. 481 - 499 (1982).

[80] S. Lubkin, A method of summing infinite series, J. Res. Natl. Bur. Stand., 48, pp. 228 - 254 (1952).

[81] D. Shanks, Non-linear transformations of divergent and slowly convergent sequences, J. Math. and Phys. (Cambridge, Mass.), 34, pp. 1 - 42 (1955).

[82] R.R. Tucker, The $\delta^2$ process and related topics, Pacif. J. Math., 22, pp. 349 - 359 (1967).

[83] R.R. Tucker, The $\delta^2$ process and related topics II, Pacif. J. Math., 28, pp. 455 - 463 (1969).

[84] W.D. Clark, H.L. Gray, and J.E. Adams, A note on the T-transformation of Lubkin, J. Res. Natl. Bur. Stand. B, 73, pp. 25 - 29 (1969).

[85] F. Cordellier, Sur la régularité des procédés $\delta^2$ d’Aitken et W de Lubkin, in L. Wuytack (Editor), Padé Approximation and Its Applications, Springer-Verlag, Berlin, 1979, pp. 20 - 35.

[86] P. Hillion, Méthode d’Aitken itérée pour les suites oscillantes d’approximations, C. R. Acad. Sc. Paris A, 280, pp. 1701 - 1704, (1975).

[87] M.P. Jurkat, Error analysis of Aitken’s $\Delta^2$ process, Comp. Math. Appl., 9, pp. 317 - 322 (1983).

[88] G.E. Bell and G.M. Phillips, Aitken acceleration of some alternating series, BIT, 24, pp. 70 - 77 (1984).

[89] E.J. Weniger, Prediction properties of Aitken’s iterated $\Delta^2$ process, of Wynn’s epsilon algorithm, and of Brezinski’s iterated theta algorithm, J. Comput. Appl. Math., 122, pp. 329 - 356 (2000). Reprinted in C. Brezinski (Editor), Numerical analysis 2000, Vol. 2: Interpolation and Extrapolation, Elsevier, Amsterdam, 2000, pp. 329 - 356.

[90] A.J. MacLeod, Acceleration of vector sequences by multi-dimensional $\Delta^2$-methods, Commun. Appl. Numer. Meth., 2, pp. 385 - 392 (1986).

[91] J.E. Drummond, Summing a common type of slowly convergent series of positive terms, J. Austral. Math. Soc. B, 19, 416 - 421 (1976).

[92] M.J. Jamieson and T.H. O’Beirne, A note on a generalization of Aitken’s $\delta^2$ transformation, J. Phys. B, 11, pp. L31 - L35 (1978).

[93] P. Bjorstad, G. Dahlquist, and E. Grosse, Extrapolations of asymptotic expansions by a modified Aitken $\delta^2$-formula, BIT, 21, pp. 56 - 65 (1981).

[94] P. Sablonniere, Asymptotic behaviour of iterated modified $\Delta^2$ and $\theta_2$ transforms on some slowly convergent sequences, Numer. Algor., 3, pp. 401 - 409 (1992).

[95] E.J. Weniger, On the derivation of iterated sequence transformations for the acceleration of convergence and the summation of divergent series, Comput. Phys. Commun., 64, pp. 19 - 45, (1991).

[96] P. Wynn, On the convergence and the stability of the epsilon algorithm, SIAM J. Numer. Anal., 3, pp. 91 - 122 (1966).

[97] A. Sidi, Extension and completion of Wynn’s theory on convergence of columns of the epsilon table, J. Approx. Theor., 86, pp. 21 - 40 (1996).

[98] P. Wynn, A note on programming repeated applications of the $\epsilon$-algorithm, R.F.T.I. - Chiffres, 8, pp. 23 - 62 (1965).

[99] P.R. Graves-Morris, D.E. Roberts, and A. Salam, The epsilon algorithm and related topics, J. Comput. Appl. Math., 122, pp. 51 - 80 (2000). Reprinted in C. Brezinski (Editor), Numerical analysis 2000, Vol. 2: Interpolation and extrapolation, Elsevier, Amsterdam, 2000, pp. 51 - 80.
[100] J. Čížek and E.R. Vrscay, Large order perturbation theory in the context of atomic and molecular physics – Interdisciplinary aspects, Int. J. Quantum Chem., 21, pp. 27 - 68 (1982).
[101] B. Simon, Large orders and summability of eigenvalue perturbation theory: A mathematical overview, Int. J. Quantum Chem., 21, pp. 3 - 25 (1982).
[102] S. Graffi and V. Grecchi, Borel summability and indeterminacy of the Stieltjes moment problem: Application to the anharmonic oscillators, J. Math. Phys., 19, pp. 1002 - 1006 (1978).
[103] D.A. Smith and W.F. Ford, Acceleration of linear and logarithmic convergence, SIAM J. Numer. Anal., 16, pp. 223 - 240 (1979).
[104] S. Bhownick, R. Bhattacharya, and D. Roy, Iterations of convergence accelerating nonlinear transforms, Comput. Phys. Commun., 54, pp. 31 - 36 (1989).
[105] P. Sablonniere, Convergence acceleration of logarithmic fixed point sequences, J. Comput. Appl. Math., 19, pp. 55 - 60 (1987).
[106] P. Sablonniere, Comparison of four algorithms accelerating the convergence of a subset of logarithmic fixed point sequences, Numer. Algor., 1, pp. 177 - 197 (1991).
[107] P. Sablonniere, Comparison of four nonlinear transforms on some classes of logarithmic fixed point sequences, J. Comput. Appl. Math., 62, pp. 103 - 128 (1995).
[108] B. Germain-Bonne, Transformations de suites, Rev. Française Automat. Informat. Rech. Operat., 7 (R-1), pp. 84 - 90 (1973).
[109] J.P. Delahaye and B. Germain-Bonne, Résultats négatifs en accélération de la convergence, Numer. Math., 35, pp. 443 - 457 (1980).
[110] J.P. Delahaye and B. Germain-Bonne, The set of logarithmically convergent sequences cannot be accelerated, SIAM J. Numer. Anal., 19, pp. 840 - 844, (1982).
[111] N. Osada, A convergence acceleration method for some logarithmically convergent sequences, SIAM J. Numer. Anal., 27, pp. 178 - 189, (1990).
[112] C.W. Clenshaw, E.T. Goodwin, D.W. Martin, G.F. Miller, F.J.W. Olver, and J.H. Wilkinson (Editors), Modern computing methods, 2nd edition, Notes on applied science Vol. 16, H. M. Stationary Office, London, 1961.
[113] A. Van Wijngaarden, Cursus: Wetenschappelijk Rekenen B, Process Analyse, Stichting Mathematisch Centrum, Amsterdam, 1965, pp. 51 - 60.
[114] J.W. Daniel, Summation of series of positive terms by condensation transformations, Math. Comput., 23, pp. 91 - 96 (1969).
[115] P.J. Pelzl and F.W. King, Convergence accelerator approach for the high-precision evaluation of three-electron correlated integrals, Phys. Rev. E, 57, 7268 - 7273 (1998).
[116] U.D. Jentschura, P.J. Mohr, and G. Soff, Calculation of the electron self-energy for low nuclear charge, Phys. Rev. Lett., 82, pp. 53 - 56 (1999).
[117] P.J. Davis, Interpolation and approximation, Dover, New York, 1975.
[118] D.C. Joyce, Survey of extrapolation processes in numerical analysis, SIAM Rev., 13, pp. 435 - 490 (1971).
[119] E.H. Neville, Iterative interpolation, J. Indian Math. Soc., 20, pp. 87 - 120 (1934).
[120] N. Osada, An acceleration theorem for the ρ algorithm, Numer. Math., 73, pp. 521 - 531 (1996).
[121] S. Bhownick, R. Bhattacharya, and D. Roy, Iterations of convergence accelerating nonlinear transforms, Comput. Phys. Commun., 54, pp. 31 - 36 (1989).
[122] B.C. Carlson, Special Functions of Applied Mathematics, Academic Press, New York, 1977.
[123] D. Levin, Development of non-linear transformations for improving convergence of sequences, Int. J. Comput. Math. B, 3, pp. 371 - 388, (1973).
[124] C. Brezinski, Projection methods for systems of equations, Elsevier, Amsterdam, 1997.
Nonlinear Sequence Transformations

[125] C. Brezinski and M. Redivo Zaglia, A general extrapolation procedure revisited, Advances Comput. Math., 2, pp. 461 - 477 (1994).

[126] C. Brezinski and M. Redivo Zaglia, On the kernel of sequence transformations, Appl. Numer. Math., 16, pp. 239 - 244 (1994).

[127] C. Brezinski and A. Salam, Matrix and vector sequence transformations revisited, Proc. Edinb. Math. Soc., 38, pp. 495 - 510 (1995).

[128] C. Brezinski and A.C. Matos, A derivation of extrapolation algorithms based on error estimates, J. Comput. Appl. Math., 66, pp. 5 - 26 (1996).

[129] A.C. Matos, Linear difference operators and acceleration methods, IMA J. Numer. Anal., 20, pp. 359 - 388 (2000).

[130] D. Roy, R. Bhattacharya, and S. Bhowmick, Rational approximants using Levin-Weniger transforms, Comput. Phys. Commun., 93, pp. 159 - 178 (1996).

[131] R. Bhattacharya, D. Roy, and S. Bhowmick, Rational interpolation using Levin-Weniger transforms, Comput. Phys. Commun., 101, pp. 213 - 222 (1997).

[132] D. Roy, R. Bhattacharya, and S. Bhowmick, Multipoint Levin-Weniger approximants and their application to the ground state energies of quantum anharmonic oscillators, Comput. Phys. Commun., 113, 131 - 144 (1998).

[133] A. Sidi, Convergence properties of some nonlinear sequence transformations, Math. Comput., 33, pp. 315 - 326 (1979).

[134] A. Sidi, Analysis of convergence of the $T$-transformation for power series, Math. Comput., 35, pp. 833 - 850 (1980).

[135] A. Sidi, Borel summability and converging factors for some everywhere divergent series, SIAM J. Math. Anal., 17, pp. 1222 - 1231 (1986).

[136] W. Gander, G.H. Golub, and D. Gruntz, Solving linear equations by extrapolation, in J.S. Kowalik (Editor), Supercomputing, NATO ASI Series Vol. 62, Springer-Verlag, Berlin, 1990, pp. 279 - 293.

[137] P. Wynn, Upon the Padé table derived from a Stieltjes series, SIAM J. Numer. Anal., 4, pp. 805 - 834 (1968).

[138] D. Belkić, New hybrid non-linear transformations of divergent perturbation series for quadratic Zeeman effects, J. Phys. A, 22, pp. 3003 - 3010 (1989).

[139] P.R. Graves-Morris and E.B. Saff, Row convergence theorems for generalised vector-valued Padé approximants, J. Comput. Appl. Math., 23, pp. 63 - 85 (1988).

[140] P.R. Graves-Morris and E.B. Saff, An extension of a row convergence theorem for vector Padé approximants, J. Comput. Appl. Math., 34, pp. 315 - 324 (1991).

[141] A. Sidi, Application of vector extrapolation methods to consistent singular linear systems, Appl. Numer. Math., 6, pp. 487 - 500 (1989/90).

[142] D.A. Smith, W.F. Ford, and A. Sidi, Acceleration of convergence of vector sequences, SIAM J. Numer. Anal., 23, pp. 178 - 196 (1986).

[143] D.A. Smith, W.F. Ford, and A. Sidi, Correction to “Extrapolation methods for vector sequences”, SIAM Rev., 30, pp. 623 - 624 (1988).

[144] A. Sidi and J. Bridger, Convergence and stability analyses for some vector extrapolation methods in the presence of defective iteration matrices, J. Comput. Appl. Math., 22, pp. 35 - 61 (1988).

[145] K. Jbilou and H. Sadok, Some results about vector extrapolation methods and related fixed-point iterations, J. Comput. Appl. Math., 36, pp. 385 - 398 (1991).

[146] K. Jbilou and H. Sadok, Analysis of some vector extrapolation methods for solving systems of linear equations, Numer. Math., 70, pp. 73 - 89 (1995).

[147] N. Osada, Acceleration methods for vector sequences, J. Comput. Appl. Math., 38, pp. 361 - 371 (1991).
[148] N. Osada, Extensions of Levin’s transformation to vector sequences, Numer. Algor., 2, pp. 121 - 132 (1992).
[149] N. Osada, Vector sequence transformations for the acceleration of logarithmic convergence, J. Comput. Appl. Math., 66, 391 - 400 (1996).
[150] C. Brezinski and H. Sadok, Some vector sequence transformations with applications to systems of equations, Numer. Algor., 3, pp. 75 - 80 (1992).
[151] A. Matos, Convergence and acceleration properties for the vector ε-algorithm, Numer. Algor., 3, pp. 313 - 320 (1992).
[152] P.R. Graves-Morris, Extrapolation methods for vector sequences, Numer. Math., 61, pp. 475 - 487 (1992).
[153] P. Graves-Morris, A new approach to acceleration of convergence of a sequence of vectors, Numer. Algor., 11, pp. 189 - 201 (1996).
[154] P.R. Graves-Morris and D.E. Roberts, From matrix to vector Padé approximants, J. Comput. Appl. Math., 51, pp. 205 - 236 (1994).
[155] P.R. Graves-Morris and D.E. Roberts, Problems and progress in vector Padé approximation, J. Comput. Appl. Math., 77, pp. 173 - 200 (1997).
[156] H.H.H. Homeier, S. Rast, and H. Krienke, Iterative solution of the Ornstein-Zernike equation with various closures using vector extrapolation, Comput. Phys. Commun., 92, pp. 188 - 202 (1995).
[157] A. Salam, An algebraic approach to the vector ε-algorithm, Numer. Algor., 11, pp. 327 - 337 (1996).
[158] A. Salam, What is a vector Hankel determinant, Lin. Alg. Applic., 278, pp. 147 - 161 (1998).
[159] A. Salam, Vector Padé-type approximants and vector Padé approximants, J. Approx. Theor., pp. 97, 92 - 112 (1999).
[160] P.R. Graves-Morris and J. Van Iseghem, Row convergence theorems for vector-valued Padé approximants, J. Approx. Theor., 90, pp. 153 - 173 (1997).
[161] D.E. Roberts, On a vector q-d algorithm, Adv. Comput. Math., 8, pp. 193 - 219 (1998).
[162] D.E. Roberts, A vector Chebyshev algorithm, Numer. Algor., 17, pp. 33 - 50 (1998).
[163] G. Evans, Practical numerical integration, Wiley, Chichester, 1993.
[164] A. Sidi, Numerical quadrature and nonlinear sequence transformations; Unified rules for efficient computation of integrals with algebraic and logarithmic endpoint singularities, Math. Comput., 35, pp. 851 - 874 (1980).
[165] A. Sidi, Extrapolation methods for oscillatory infinite integrals, J. Inst. Math. Appl., 26, 1 - 26 (1980).
[166] A. Sidi, Numerical quadrature rules for some infinite range integrals, Math. Comput., 38, pp. 127 - 142 (1982).
[167] A. Sidi, The numerical evaluation of very oscillatory infinite integrals by extrapolation, Math. Comput., 38, pp. 517 - 529 (1982).
[168] A. Sidi, Extrapolation methods for divergent oscillatory infinite integrals that are defined in the sense of summability, J. Comput. Appl. Math., 17, pp. 105 - 114 (1987).
[169] A. Sidi, A user-friendly extrapolation method for oscillatory infinite integrals, Math. Comput., 51, pp. 249 - 266 (1988).
[170] A. Sidi, On rates of acceleration of extrapolation methods for oscillatory infinite integrals, BIT, 30, pp. 347 - 357 (1990).
[171] A. Sidi, Further convergence and stability results for the generalized Richardson extrapolation process GREP(1) with an application to the D(1)-transformation for infinite integrals, J. Comput. Appl. Math., 112, pp. 269 - 290 (1999).
[172] D. Levin and A. Sidi, Two new classes of nonlinear transformations for accelerating the convergence of infinite integrals and series, Appl. Math. Comput., 9, pp. 175 - 215 (1981).
Nonlinear Sequence Transformations

[173] D. Levin, Procedures for computing one- and two-dimensional integrals of functions with rapid irregular oscillations, Math. Comput., 38, 531 - 538 (1982).

[174] C. Greif and D. Levin, The $d_2$-transformation for infinite double series and the $D_2$-transformation for infinite double integrals, Math. Comput., 67, pp. 695 - 714 (1998).

[175] H. Safouhi and P.E. Hoggan, Efficient evaluation of Coulomb integrals: The nonlinear $D$- and $D$-transformations, J. Phys. A, 31, pp. 8941 - 8951 (1998).

[176] H. Safouhi and P.E. Hoggan, Three-centre two-electron Coulomb and hybrid integrals evaluated using nonlinear $D_-$ and $D$-transformations, J. Phys. A, 32, pp. 6203 - 6217 (1999).

[177] H. Safouhi and P.E. Hoggan, Efficient and rapid evaluation of three-center two electron Coulomb and hybrid integrals using nonlinear transformations, J. Comput. Phys., 155, pp. 331 - 347 (1999).

[178] H. Safouhi and P.E. Hoggan, Non-linear transformations for rapid and efficient evaluation of multicenter bielectronic integrals over $B$ functions, J. Math. Chem., 25, pp. 259 - 280 (1999).

[179] H. Safouhi, D. Pinchon, and P.E. Hoggan, Efficient evaluation of integrals for density functional theory: Nonlinear $D$ transformation to evaluate three-center nuclear attraction integrals over $B$ functions, Int. J. Quantum Chem., 70, pp. 181 - 188 (1998).

[180] H. Safouhi, The $HD$ and $H^\phi D$ methods for accelerating the convergence of three-center nuclear attraction and four-center two-electron Coulomb integrals over $B$ functions and their convergence properties, J. Comput. Phys., 165, pp. 473 - 495 (2000).

[181] H. Safouhi, An extremely efficient approach for accurate and rapid evaluation of three-centre two-electron Coulomb and hybrid integrals over $B$ functions, J. Phys. A, 34, pp. 881 - 902 (2001).

[182] F. Chishstie, V. Elias, V.A. Miransky, and T.G. Steele, Padé-summation approach to QCD $\beta$-function infrared properties, Prog. Theor. Phys., 104, pp. 603 - 631 (2000).

[183] C.M. Bender and E.J. Weniger, Numerical evidence that the perturbation expansion for a non-Hermitian $PT$-symmetric Hamiltonian is Stieltjes, J. Math. Phys., 42, pp. 2167 - 2183 (2001).