Non-linear Affine Processes with Jumps

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Abstract

We present a probabilistic construction of $\mathbb{R}^d$-valued non-linear affine processes with jumps. Given a set $\Theta$ of affine parameters, we define a family of sublinear expectations on the Skorokhod space under which the canonical process $X$ is a (sublinear) Markov process with a non-linear generator. This yields a tractable model for Knightian uncertainty for which the sublinear expectation of a Markovian functional can be calculated via a partial integro-differential equation.

Keywords: sublinear expectation, non-linear affine processes, dynamic programming, PIDE

Mathematics Subject Classification (2020): 60G65, 60G07

1 Introduction

Aim of this paper is to introduce multi-dimensional non-linear affine processes with jumps. Classical affine processes are highly relevant for applications in finance, e.g. for modelling phenomena like volatility, stock prices, credit default or interest rates and have been studied in several contributions in the classical setting, see [15], [20], [22], [8], [23], [24]. Over the last years, several different approaches have been developed in order to establish so-called robust settings, which are independent of the underlying priors, see among others [1], [4], [9], [10], [16], [30], [31] and [33].

To our knowledge, the first definition of affine processes under model uncertainty is from [14], where they consider the one-dimensional continuous case with canonical state space. Here, non-linear continuous affine processes are represented by a family of semimartingale laws, such that the differential characteristics are bounded from above and below by affine functions depending on the current state. In particular, the coefficients of the affine functions are assumed to be in a parameter set $\Theta$ which is fixed. The idea in [14] is based on the papers [29] and [17], which consider non-linear Lévy processes (with jumps) and non-linear Markov processes, respectively. In all these works, a sublinear expectation associated to the underlying sets of priors is defined and a dynamic programming principle is derived. A different approach has been studied in [11], [27], [26], where non-linear semigroup theory is used to define non-linear Markov processes.

The scope of this paper is to extend the results on non-linear affine processes in [14] to include jumps and an arbitrary closed state space $S \subseteq \mathbb{R}^d$ for a given set of parameters $\Theta$. This is achieved by combining the

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probabilistic construction from [14] and [29] with findings from [17]. In particular, the latter paper provides sufficient conditions on the uncertainty sets such that the dynamic programming principle follows.

More specifically, we consider the canonical process $X$ on the Skorokhod space $\Omega = D(\mathbb{R}_+, \mathbb{R}^d)$ equipped with a family of sublinear expectations $\{\mathcal{E}^x\}_{x \in \mathbb{R}^d}$ defined as

$$\mathcal{E}^x(f) := \sup_{\mathcal{P}_x} E_P[f],$$

where $\mathcal{P}_x$ is a set of probability measures, and $E_P$ is the usual expectation with respect to $P$. In this setting, $\mathcal{P}_x$ is the set of priors such that under every $P \in \mathcal{P}_x$, $X$ is a semimartingale starting at $x$ with differential characteristics evolving in the set $\Theta(X)$. Here, by a slight abuse of notation $\Theta(\cdot)$ denotes an affine set-valued map for a fixed set of parameters $\Theta$. In line with [29] and [14], we call this process non-linear affine process.

In this framework, it is enough to assume that the set of parameters $\Theta$ is closed to prove the dynamic programming principle and the Markov property for sublinear expectations. In contrast to the approach in [14] and [29], our argument relies only on the separability of the space of Lévy measures. Hence, it can be extended to other classes of processes, e.g., polynomial processes. Moreover, since we consider a more general parameter set $\Theta$, we obtain a broader class of non-linear affine processes than in [14], even for the continuous case with dimension $d = 1$ on the canonical state space.

Furthermore, we slightly modify the construction in [14] in order to admit an arbitrary closed state space $S \subseteq \mathbb{R}^d$. In particular, we fix the state space $S$ a priori and use it to define the set-valued map $\Theta(X)$. In this way, we can guarantee that the process $X$ is hindered from continuously exiting the state space unless it jumps outside of $S$. These properties are formalised in Lemma 2.3 and Corollary 2.7.

Finally, as in [14] and [29], we establish a link to the corresponding parabolic partial integro-differential equation (PIDE)

$$\partial_t v(t, x) - A_x v(t, x) = 0,$$

which can be understood as the analogue of the Kolmogorov equation. The non-linear generator $A_x$ is the supremum of the generators of classical affine processes with parameters in $\Theta$. More specifically, we show that if the value function $v(t, x) := \mathcal{E}^x(\varphi(X_t))$ is continuous for a non-linear affine process $X$, then $v$ is a viscosity solution of the PIDE (1.2) with initial condition $v(0, x) = \varphi(x)$. The consideration of a general parameter set $\Theta$ and of a general state space $S$ comes at the cost of many technical subtleties compared to the corresponding approaches in [14] and [28]. In particular, the PIDE (1.2) does not satisfy the degenerate ellipticity condition in general. Thus most existing comparison and uniqueness results in the literature are not applicable. Hence, we cannot prove the uniqueness of a viscosity solution of the PIDE (1.2) in full generality, as done in [29]. However, we are able to establish a uniqueness result for a special case with $S = \mathbb{R}^d$ and a modified parameter map $\hat{\Theta}$ with associated family of sublinear expectations $\{\hat{\mathcal{E}}^x\}_{x \in \mathbb{R}^d}$ and a non-linear generator $\hat{A}_x$. In this case, if the value function $\hat{v}(t, x) := \hat{\mathcal{E}}^x(\varphi(X_t))$ is continuous, then $\hat{v}$ is the unique viscosity solution to the PIDE induced by $\hat{A}_x$ with initial condition $\hat{v}(0, x) = \varphi(x)$. This result is neither covered by the comparison results used in [29] or [14], nor we assume a uniform bounded condition as in [17] or [25]. Instead, we only impose a Lipschitz condition to prove the uniqueness.

The remainder of this paper is organised as follows. In Section 2, we introduce the definition of a multidimensional non-linear affine process with jumps and state space $S \subseteq \mathbb{R}^d$. In Section 3, we prove the dynamic programming principle and the Markov property for the family of sublinear expectations in Section 4, we establish the connection with the Kolmogorov-type backward equation. In Section 5, we give a uniqueness result for a special case. Finally, we present some examples in Section 6.

## 2 Non-linear affine processes

For some $d \in \mathbb{N}$, let $\Omega = D(\mathbb{R}_+, \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued càdlàg paths equipped with the Skorokhod topology and the corresponding Borel $\sigma$-algebra $\mathcal{F}$. Moreover, let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ denote the raw filtration
generated by the canonical process $X = (X_t)_{t \geq 0}$ given by $X_t(\omega) = \omega(t)$. In the sequel the $i$-th coordinate of the vector-valued process $X$ is denoted by $X^i$. Similarly, the $i, j$-th coordinate of a matrix-valued process $M$ is denoted by $M^{i,j}$. The same notation applies to elements in $\mathbb{R}^d$.

Let $\mathfrak{P}(\Omega) := \mathfrak{P}(\Omega, \mathcal{F})$ denote the Polish space of all probability measures on $(\Omega, \mathcal{F})$. Let $P \in \mathfrak{P}(\Omega)$. The process $X$ is called a $(P, \mathcal{F})$-semimartingale, if there exists right-continuous $\mathcal{F}$-adapted processes $M$ and $B$ such that $M_0 = 0 = B_0$, $M$ is a local martingale, $B$ is of locally finite variation, and $X = X_0 + M + B$ $P$-a.s. Fix a truncation function $h$ of zero, and let $X$ be the jump measure of $X$. Then using the unique canonical semimartingale decomposition as in [18, Theorem II.2.34], we have $P$-a.s.

$$X_t = X_0 + \alpha^P_t + \int_0^t \int_{\mathbb{R}^d} h(z) (\mu - K^P)(dz) \, ds + \int_0^t \int_{\mathbb{R}^d} h(z) K^P(dz, dz) + \beta^P_t + \int_0^t \int_{\mathbb{R}^d} h(z) \mu(dz, dz),$$

where $K^P$ is the predictable compensator of the jump measure $\mu$, $\alpha^P$ and $\beta^P$ are the continuous and purely discontinuous local martingale parts respectively, $B^P := \beta^P + B^P$ is the predictable finite variation part, and $J^P$ is the unbounded jump part.

If $B^P, \alpha^P, K^P$ are absolutely continuous with respect to the Lebesgue measure $P$-a.s., there exists an $\mathcal{F}$-adapted process $(\beta^P, a^P, k^P)$ such that $P$-a.s.

$$d\beta^P_t = b^P_t \, dt \quad d\alpha^P_t = a^P_t \, dt \quad dK^P([0,t]; dz) = k^P_t(dz) \, dt.$$

Such a process $(\beta^P, a^P, k^P)$ is called $(P, \mathcal{F})$-differential semimartingale characteristics of $X$ relative to $h$.

Note that $a^P$ takes values in the cone of non-negative definite symmetric matrices $\mathbb{S}_+$, where $\mathbb{S}$ denotes the vector space of symmetric matrices. The process $k^P$ takes values in the cone of non-negative Lévy measures $\mathcal{L}_+ \subseteq \mathcal{L}$, which are defined as

$$\mathcal{L}_+ := \left\{ k \in \mathcal{M}_+(\mathbb{R}^d) : \int_{\mathbb{R}^d} \|z\|^2 \, k(dz) < \infty \right\},$$

and

$$\mathcal{L} := \left\{ k \in \mathcal{M}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \|z\|^2 \, k(dz) < \infty \right\},$$

where $\mathcal{M}_+(\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d)$ are the cone of all non-negative measures and the vector space of all signed measures on $\mathbb{R}^d$, respectively. By [6, Theorem 8.9.4], the space of all finite measures $\mathcal{M}(\mathbb{R}^d)$ endowed with the Kantorovich-Rubinstein metric $d_{\mathcal{M}}$ is a separable metric space. A Lévy measure $k \in \mathcal{L}$ can be associated with the finite measure $\hat{k} : \mathcal{A} \to \int_{\mathbb{R}^d} \|z\|^2 \, k(dz)$, $A \in \mathcal{B}(\mathbb{R}^d)$. Thus, $\mathcal{M}(\mathbb{R}^d)$ is a separable metric space with the metric $d_{\mathcal{L}}(k, m) := d_{\mathcal{M}}(\hat{k}, \hat{m})$ on $\mathcal{L}$, cf. [28, Lemma 2.3]. We endow $\mathcal{L}$ with the corresponding Borel $\sigma$-algebra. Moreover, we can define a metric $d_{\mathcal{L}}$ such that $\mathbb{R}^d \times \mathcal{L}$ is again a separable metric space (and so are subspaces and finite product spaces of it), cf. [34, Theorem 16.4c].

In classical stochastic analysis, an affine process is a Markov semimartingale with differential characteristics that are affine functions of the current state, i.e.

$$(b^P, a^P, k^P) = (\beta(X), \alpha(X), \nu(X)) \quad dt \otimes dP\text{-a.e.}$$

for some affine functions $\beta, \alpha, \nu$ on the state space of $X$. Note that the functions $\beta, \alpha, \nu$ do not depend on $P$. In the following, we define an affine process under parameter uncertainty by allowing the differential characteristics $(b^P, a^P, k^P)$ to evolve in a random set $\Theta(X)$ which depends on the value of $X$ in an affine manner. For this purpose, fix a closed, non-empty state space $S \subseteq \mathbb{R}^d$. Let $\beta = (\beta_0, \ldots, \beta_d) \in (\mathbb{R}^d)^{d+1}$, $\alpha = (\alpha_0, \ldots, \alpha_d) \in S^{d+1}$, $\nu = (\nu_0, \ldots, \nu_d) \in S^{d+1}$. Define for $x = (x^1, \ldots, x^d)^T \in \mathbb{R}^d$ the following functions

$$\beta(x) := (\beta_0 + (\beta_1, \ldots, \beta_d) x) 1_S(x) = (\beta_0 + \sum_{i=1}^d x^i \beta^i) 1_S(x) \in \mathbb{R}^d,$$
\[ \alpha(x) := (\alpha_0 + (\alpha_1, \ldots, \alpha_d)) \mathbf{1}_S(x) = (\alpha_0 + \sum_{i=1}^d x^i \alpha_i) \mathbf{1}_S(x) \in \mathbb{S}, \quad (2.7) \]

\[ \nu(x) := (\nu_0 + (\nu_1, \ldots, \nu_d)) \mathbf{1}_S(x) = (\nu_0 + \sum_{i=1}^d x^i \nu_i) \mathbf{1}_S(x) \in \mathcal{L}. \quad (2.8) \]

From now on, we refer to any \( \theta = (\beta, \alpha, \nu) \in (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathcal{L}^{d+1} \) as parameter, and identify it with the map

\[ \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{S} \times \mathcal{L}, \quad x \mapsto \theta(x) := (\beta(x), \alpha(x), \nu(x)), \quad (2.9) \]

where \( \alpha, \beta, \nu \) as in (2.6) - (2.8). Similarly for a subset \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathcal{L}^{d+1} \) and \( x \in \mathbb{R}^d \), define

\[ \Theta(x) := \{ \theta(x) : \theta \in \Theta \} \subseteq \mathbb{R}^d \times \mathbb{S} \times \mathcal{L}. \quad (2.10) \]

A set \( \Theta \) of parameters is closed, if it is closed with respect to the topology which makes \((\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathcal{L}^{d+1}\) a separable metric space.

Finally, for \( x \in \mathbb{R}^d \), define the following set of probability measures

\[ \mathcal{P}_x(\Theta) := \left\{ P \in \mathcal{P}_{sem}^{ac}(\Omega) : P(X_0 = x) = 1; (b^P, a^P, k^P) \in \Theta(X) \, dt \otimes dP \text{-}a.e. \right\}, \quad (2.11) \]

with

\[ \mathcal{P}_{sem}^{ac}(\Omega) := \left\{ P \in \mathcal{P}_{sem}(\Omega) : X \text{ admits } (P, \mathcal{F})\text{-diff. characteristics } (b^P, a^P, k^P) \right\} \quad (2.12) \]

and

\[ \mathcal{P}_{sem} := \left\{ P \in \mathcal{P}(\Omega) : X \text{ is a semimartingale on } (\Omega, \mathcal{F}, \mathcal{F}, P) \right\}. \quad (2.13) \]

Note that \( \mathcal{P}_{sem}^{ac} \) is not empty, as it always contains the Dirac measures on \( \Omega \) with point mass on a constant path. To each \( \mathcal{P}_x(\Theta) \), we can associate the sublinear expectation

\[ \mathcal{E}^x(f) := \sup_{P \in \mathcal{P}_x(\Theta)} E_P[f] \quad (2.14) \]

for any Borel-measurable function \( f : \Omega \to \mathbb{R} \) with

\[ E_P[f] := E_P[f^+] - E_P[f^-], \quad (2.15) \]

where \( f^+ \) and \( f^- \) denote the positive and negative parts of \( f \) respectively. We use the convention \( \sup \emptyset = -\infty \)

\[ \text{and } \inf \emptyset = -\infty. \]

**Definition 2.1.** Let \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathcal{L}^{d+1} \) be non-empty and closed. The tuple \((X, \{\mathcal{P}_x(\Theta)\})_{x \in S, S}\), where \( X \) is the canonical process and \( \mathcal{P}_x(\Theta) \) is defined in (2.11), is called *non-linear affine process* with parameter set \( \Theta \) and state space \( S \).

**Remark 2.2.** 1. Note that in (2.12) we consider \((P, \mathcal{F})\)-semimartingales, where \( \mathcal{F} \) is the raw filtration.

   We could also work with \( \mathbb{F}_+ = (\mathbb{F}_t^+)_{t \geq 0} \) its right-continuous version or \( \mathbb{F}_+^P = (\mathbb{F}_t^P)_{t \geq 0} \) its usual augmentation under \( P \). By [28, Proposition 2.2], the choice of filtration is not crucial. In particular, the associated semimartingale characteristics are \( P \)-a.s. the same. Hence, whenever the probability measure \( P \) is fixed, we may consider \( X \) as stochastic process on the augmented space \((\Omega, \mathcal{F}^P, \mathbb{F}_+^P, P)\) to avoid measurability issues.

2. Another approach for defining affine processes under parameter uncertainty is to consider the set of all probability measures \( P \) such that \( X \) is a \((P, \mathcal{F})\)-semimartingale with differential characteristics \( \theta(X) \) for some \( \theta \in \Theta \). That is, the family \( \{\hat{\mathcal{P}}_x(\Theta)\}_{x \in S} \) given by

\[ \hat{\mathcal{P}}_x(\Theta) = \left\{ P \in \mathcal{P}_{sem}^{ac}(\Omega) : P(X_0 = x) = 1; \exists \theta \in \Theta : \theta(X) = (b^P, a^P, k^P) \, dt \otimes dP \text{-}a.e. \right\}. \quad (2.16) \]
We now study the behaviour outside of the state space $S$ of the non-linear affine process $(X, \{\mathcal{P}_x(\Theta)\}_{x \in S}, S)$. For this purpose, we define the random times

$$\sigma := \inf \{ t \in \mathbb{R}_+ : X_t \not\in S \}, \quad \tau := \inf \{ t > \sigma : X_t \in S \}. \quad (2.17)$$

For every $x \in \mathbb{R}^d$, $P \in \mathcal{P}_x(\Theta)$, $\sigma$ and $\tau$ are $\mathbb{R}^d_+$-stopping times by the Début Theorem, cf. [18, Theorem I.1.27]. Note that, if $x \notin S$, then $\sigma = 0$ $P$-a.s. for all $P \in \mathcal{P}_x(\Theta)$.

**Lemma 2.3.** For all $x \in S$, $P \in \mathcal{P}_x(\Theta)$, the set

$$\{\sigma < \infty, X_\sigma \in S\} \quad (2.18)$$

is $P$-null.

**Proof.** Fix $x \in S$ with $\mathcal{P}_x(\Theta) \neq \emptyset$, and let $P \in \mathcal{P}_x(\Theta)$. Then, for any $\varepsilon > 0$ we have $P$-a.s.

$$\begin{align*}
(X_{\sigma + \varepsilon} - X_{\sigma}) \mathbf{1}_{\{\sigma < \infty\}} &= \left( \int_\sigma^{\sigma + \varepsilon} \tilde{b}_s^P \, ds + \int_\sigma^{\sigma + \varepsilon} \tilde{a}_s^P \, dW^P_s + \int_\sigma^{\sigma + \varepsilon} h(z) \mu(ds,dz) \right) \mathbf{1}_{\{\sigma < \infty\}} \\
&\quad + \int_\sigma^{\sigma + \varepsilon} \int_{\mathbb{R}^d} (z - h(z)) \mu(ds,dz) \mathbf{1}_{\{\sigma < \infty\}} \quad (2.19)
\end{align*}$$

for some $P$-Wiener process $W^P$ and for $(\tilde{b}^P, \tilde{a}^P, \tilde{h}^P)$ taking values in $\Theta(X)$ for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$.

Let $\omega \in \{\sigma < \infty, X_{\sigma} \in S\}$, then there exists an $\varepsilon_\ast > 0$ with $\sigma(\omega) + \varepsilon_\ast < \tau(\omega)$. Due to the right-continuity of $X(\omega)$, we can choose $\varepsilon_\ast$ small enough such that $u \mapsto X_u(\omega)$ is continuous on $[\sigma(\omega), \sigma(\omega) + \varepsilon_\ast]$. Since $\Theta(X) = \{(0,0,0)\}$ on $[\sigma, \tau]$, we have in particular $(\tilde{b}^P_s(\omega), \tilde{a}^P_s(\omega), \tilde{h}^P_s(\omega)) = (0,0,0)$ for $s \in [\sigma(\omega), \sigma(\omega) + \varepsilon_\ast]$ by our choice of $\omega$ and $\varepsilon_\ast$. Moreover, we have $\mu(\{\sigma(\omega) \in [\sigma(\omega), \sigma(\omega) + \varepsilon_\ast]\}, \mathbb{R}^d) = 0$ since the path is continuous on $[\sigma(\omega), \sigma(\omega) + \varepsilon_\ast]$. Hence, the right-hand side of (2.19) is zero for $\omega$ and $\varepsilon_\ast$. Note that the left-hand side of (2.19) is not zero since $X_{\sigma(\omega)}(\omega) \in S$ and $X_{\sigma(\omega) + \varepsilon_\ast}(\omega) \notin S$. Thus, $\omega$ is contained in the $P$-null set on which (2.19) does not hold for $\varepsilon = \varepsilon_\ast$. We conclude that

$$\{\sigma < \infty, X_\sigma \in S\} \subseteq \bigcup_{n \in \mathbb{N}} \left\{ (2.19) \text{ does not hold for } \varepsilon = \frac{1}{n} \right\}, \quad (2.20)$$

which as countable union of $P$-null sets is again $P$-null.

**Lemma 2.4.** For all $x \in \mathbb{R}^d$ and $P \in \mathcal{P}_x(\Theta)$, the set

$$\{\tau < \infty\} \quad (2.21)$$

is $P$-null.

**Proof.** Fix $x \in \mathbb{R}^d$ with $\mathcal{P}_x(\Theta) \neq \emptyset$, and let $P \in \mathcal{P}_x(\Theta)$. Note that $\sigma \leq \tau$, and $X_\tau \in S$ on $\{\tau < \infty\}$. Thus,

$$\{\tau < \infty, X_\tau = X_\sigma\} \subseteq \{\sigma < \infty, X_\sigma \in S\}. \quad (2.22)$$

If $x \in S$, the right-hand side is $P$-null by Lemma 2.3. Otherwise for $x \notin S$, we have $X_\sigma = X_0 = x \notin S$ $P$-a.s., and the right-hand side is $P$-null.

Now, consider the set $\{\tau < \infty, X_\tau \neq X_\sigma\}$. As in the proof of Lemma 2.3, fix a process $(b,a,k)$ taking values in $\Theta(X)$ such that $P$-a.s.

$$(X_\tau - X_\sigma) \mathbf{1}_{\{\tau < \infty\}} = \left( \int_\sigma^{\tau} b(s) \, ds + \int_\sigma^{\tau} a(s) \, dW^P_s \right)$$
\[ + \int_\sigma^T \int_{\mathbb{R}^d} h(z) \left( \mu(\text{d}s, \text{d}z) - k(s, \text{d}z) \text{d}s \right) + \int_\sigma^T \int_{\mathbb{R}^d} (z - h(z)) \mu(\text{d}s, \text{d}z) \right) \mathbf{1}_{\{\tau < \infty\}} \]

(2.23)

for some \( \text{P}-\text{Wiener} \) process \( W^P \). We have

\[ E_P \left[ \int_\sigma^T \int_{\mathbb{R}^d} (\|z\|^2 \wedge 1) \mu(\text{d}s, \text{d}z) \mathbf{1}_{\{\tau < \infty\}} \right] = E_P \left[ \int_\sigma^T \int_{\mathbb{R}^d} (\|z\|^2 \wedge 1) k(s, \text{d}z) \text{d}s \mathbf{1}_{\{\tau < \infty\}} \right] = 0 \]

since \( \Theta(X) = \{(0,0,0)\} \) on \( \sigma, \tau \) and the boundary \( \{\sigma, \tau\} \) has Lebesgue measure zero. Consequently, the right-hand side of (2.23) is zero \( \text{P}\)-a.s. That is, \( X_\tau = X_\sigma \) \( \text{P}\)-a.s. and \( \{\tau < \infty, X_\sigma \neq X_\tau\} \) is \( \text{P}\)-null. Hence, \( \{\tau < \infty\} = \{\tau < \infty, X_\sigma = X_\tau\} \cup \{\tau < \infty, X_\sigma \neq X_\tau\} \) is also \( \text{P}\)-null. \( \square \)

As immediate consequence we obtain the following corollaries.

**Corollary 2.5.** For all \( x \in \mathbb{R}^d \) and \( P \in \mathcal{P}_x(\Theta) \), \( X \) is constant \( \text{P}\)-a.s. on \( [\sigma, \infty] \).

**Proof.** This follows immediately from a close inspection of (2.23) and Lemma 2.4. \( \square \)

**Remark 2.6.** By Lemmas 2.3, 2.4 and Corollary 2.5 we can conclude if the process \( X \) exits the set \( S \), it will never reenter \( S \). Furthermore, the process \( X \) can only exit the set \( S \) through a jump.

**Corollary 2.7.** Let \( x \in \mathbb{R}^d \setminus S \). Then \( \mathcal{P}_x(\Theta) \neq \emptyset \), and for all \( P \in \mathcal{P}_x(\Theta) \) and \( t \geq 0 \), we have \( X_t = x \) \( \text{P}\)-a.s.

**Proof.** Fix \( x \in \mathbb{R}^d \setminus S \). Let \( \delta_x \) denote the Dirac measure on \( \Omega \) with unit mass on the constant path \( \omega \equiv x \). Then clearly \( \delta_x \in \mathcal{P}_x(\Theta) \), and hence \( \mathcal{P}_x(\Theta) \neq \emptyset \). The second statement follows from Corollary 2.5 by observing that \( \sigma = 0 \) \( \text{P}\)-a.s. for all \( P \in \mathcal{P}_x(\Theta) \). \( \square \)

Corollary 2.7 states that for all \( x \notin S \) and \( P \in \mathcal{P}_x(\Theta) \) the process \( X \) is a \( P\)-modification of the constant process with state \( x \). In particular, for \( x \notin S \), \( t \geq 0 \), and any Borel-measurable function \( f : \mathbb{R}^d \to \mathbb{R} \), we have

\[ \mathcal{E}^x(f(X_t)) = \sup_{P \in \mathcal{P}_x(\Theta)} E_P[f(X_t)] = f(x). \]

(2.24)

**Remark 2.8.** Note that \( \Theta \neq \emptyset \) does not immediately imply \( \mathcal{P}_x(\Theta) \neq \emptyset \) for all \( x \in \mathbb{R}^d \). In order to avoid difficulties in that direction, we will henceforth assume that \( \mathcal{P}_x(\Theta) \neq \emptyset \) for all \( x \in S \). This assumption is not particularly strong as it follows immediately for many common choices of \( S \) and \( \Theta \). For instance, for the canonical state space \( S = \mathbb{R}_+^m \times \mathbb{R}^{d-m} \), we have a one-to-one correspondence between admissible parameters in the sense of [12, Definition 2.6] and affine processes due to [12, Theorem 2.7]. More precisely, if \( \Theta \) contains some admissible parameters \( (\beta, \alpha, \nu) \), then there exists a family \( \mathcal{P}_x(\Theta) \) of probability measures \( \{P_x\}_{x \in S} \) such that \( X \) together with \( \{P_x\}_{x \in S} \) is a linear affine process with affine characteristics \( (\beta, \alpha, \nu) \). In particular, this implies \( \emptyset \neq \mathcal{P}_x(\Theta) \subseteq \mathcal{P}_x(\Theta) \) for all \( x \in S \). Clearly, by Corollary 2.7, \( \mathcal{P}_x(\Theta) \neq \emptyset \) for all \( x \notin S \).

### 3 Dynamic Programming Principle and Markov Property

In this section, we prove that the family of subsets \( \{\mathcal{P}_x(\Theta)\}_{x \in S} \), or rather the associated sublinear expectations \( \{\mathcal{E}^x\}_{x \in S} \), is amenable to the dynamic programming principle as formalised in Proposition 3.6. In particular, we show that, for each initial value \( x \), there exists a family of conditional sublinear operators \( \{\mathcal{E}_t^x\}_{t \geq 0} \) that satisfies the dynamic programming property. As an immediate consequence we obtain a strong Markov property in Lemma 3.7.

We build on results from [31], which have been extended in [17]. Thus, we first introduce the notation and the main results of [17, Chapter 4], which we use later.
Let $\tilde{\omega}, \omega \in \Omega$, and $\tau : \Omega \to \mathbb{R}_+$ be a finite random time. The concatenation of paths $\tilde{\omega}$ and $\omega$ at $\tau$ is defined as
\[
(\tilde{\omega} \otimes \tau \omega)(t) = \tilde{\omega}(t) 1_{[0, \tau(\tilde{\omega})]}(t) + \left(\tilde{\omega}(\tau(\tilde{\omega})) - \omega(0) + \omega(t - \tau(\tilde{\omega}))\right) 1_{[\tau(\tilde{\omega}), \infty]}(t), \quad t \geq 0.
\] (3.1)

Given a function $f$ on $\Omega$, define the function $f^{\tau,\tilde{\omega}}$ on $\Omega$ by
\[
f^{\tau,\tilde{\omega}}(\omega) := f(\tilde{\omega} \otimes \tau \omega).
\] (3.2)

Let $P \in \mathfrak{P}(\Omega)$ and $\tau$ be a finite $\mathcal{F}$-stopping time. By [5, Theorem 33.3], there exists a family of regular conditional probability measures $\{P^\tau_x\}_{x \in \Omega}$ given $\mathcal{F}$, which can be chosen such that $P^\tau_x \in \mathfrak{P}(\Omega)$ is concentrated on $\omega \otimes \tau \Omega$. More precisely, for any $A \in \mathcal{F}$, the map $\omega \mapsto P^\tau_x(A)$ is $\mathcal{F}_\tau$-measurable,
\[
E_{P^\tau_x}[f] = E_P[f_{|\mathcal{F}_\tau}](\omega) \quad \text{for } P\text{-a.e. } \omega \in \Omega
\] (3.3)
when $f$ is bounded and $\mathcal{F}$-measurable, and
\[
P^\tau_x \left( \left\{ \omega' \in \Omega : \omega = \omega' \text{ on } [0, \tau(\omega)] \right\} \right) = 1 \quad \text{for all } \omega \in \Omega.
\] (3.4)

Define the probability measure $P^{\tau, \omega} \in \mathfrak{P}(\Omega)$ by
\[
P^{\tau, \omega}(A) := P^\tau_x(\omega \otimes \tau A), \quad A \in \mathcal{F},
\] (3.5)
then
\[
E_{P^{\tau, \omega}}[f^{\tau, \omega}] = E_{P^\tau_x}[f] = E_P[f_{|\mathcal{F}_\tau}](\omega) \quad \text{for } P\text{-a.e. } \omega \in \Omega.
\] (3.6)

For $x \in \mathbb{R}^d$, let $\Omega_x \subseteq \Omega$ be the subspace of all paths starting at $x$. Moreover, let $\mathfrak{P}(\Omega_x)$ denote the space of all probability measures on $\Omega_x$. In the following, we consider a family $\{\mathcal{P}_x(t, \omega)\}_{(t, \omega) \in \mathbb{R}_+ \times \Omega_x}$ of subsets of $\mathfrak{P}(\Omega_x)$ for a fixed $x \in \mathbb{R}^d$.

**Assumption 3.1.** For all $s \geq 0$, finite $\mathcal{F}$-stopping times $s \leq \tau$, paths $\tilde{\omega} \in \Omega_x$, probability measures $P \in \mathcal{P}_x(s, \tilde{\omega})$ and $\sigma := \tau^s \tilde{\omega} - s$, the following holds.

(i) Adaptedness: If $\tilde{\omega} = \omega$ on $[0, t]$, then $\mathcal{P}_x(t, \tilde{\omega}) = \mathcal{P}_x(t, \omega)$;

(ii) Measurability: The graph
\[
\{(P', \omega') : \omega' \in \Omega_x, P' \in \mathcal{P}_x(\tau(\omega), \omega')\} \subseteq \mathfrak{P}(\Omega_x) \times \Omega_x
\] (3.7)
is analytic;

(iii) Invariance: $P^{\sigma, \omega} \in \mathcal{P}_x(\tau(\tilde{\omega} \otimes_s \omega), \tilde{\omega} \otimes_s \omega)$ for $P$-a.e. $\omega \in \Omega_x$;

(iv) Stability: If $\nu : \Omega_x \to \mathfrak{P}(\Omega_x)$ is an $\mathcal{F}_\sigma$-measurable kernel with
\[
\nu(\omega) \in \mathcal{P}_x(\tau(\tilde{\omega} \otimes_s \omega), \tilde{\omega} \otimes_s \omega) \quad \text{for } P\text{-a.e. } \omega \in \Omega_x,
\] (3.8)
then the measure $\overline{P}$ defined by
\[
\overline{P}(A) := \int_{\Omega_x} \int_{\Omega_x} (1_A)^{\sigma, \omega}(\omega') \nu(\omega; d\omega') P(d\omega) \quad \text{for all } A \in \mathcal{F}
\] (3.9)
is an element of $\mathcal{P}_x(s, \tilde{\omega})$.

If Condition 3.1 holds, we set $\mathcal{P}_x := \mathcal{P}_x(0, \omega)$ since $\mathcal{P}_x(0, \omega)$ is independent of $\omega$ due to Condition 3.1 (i).
We now state the relevant results from [17], cf. Theorem 4.29 and Lemma 4.30.

**Theorem 3.2.** Suppose that \( \{P_x(t,\omega)\}_{(t,\omega)\in\mathbb{R}_+\times\Omega_x} \) is a family which satisfies Condition 3.1. Then for any finite \( \mathbb{F} \)-stopping time \( \tau \) and upper semianalytic function \( f : \Omega_x \to \mathbb{R} \), the function

\[
E_\tau^x(f)(\omega) := \sup_{P \in \mathcal{P}_x(\tau(\omega),\omega)} E_P [f^\tau|\omega]
\]

is upper semianalytic and \( \mathcal{F}_\tau^x \)-measurable. Moreover, for finite stopping \( \mathbb{F} \)-stopping times \( \sigma < \tau \), it holds

\[
E_\sigma^x (E_\tau^x(f)) (\omega) = E_\sigma^x(f)(\omega)
\]

for all \( \omega \in \Omega_x \).

Further, if for each \( x \in \mathbb{R}^d \), the family \( \{P_x(t,\omega)\}_{(t,\omega)\in\mathbb{R}_+\times\Omega_x} \) satisfies Condition 3.1., and the graph of \( x \mapsto \mathcal{P}_x \), i.e., the set

\[
\{(x,P) \in \mathbb{R}^d \times \mathcal{B}(\Omega) : x \in \mathbb{R}^d \text{ and } P|\Omega_x \in \mathcal{P}_x\}
\]

is analytic, then \( x \mapsto \mathcal{E}_0^x(f) := \mathcal{E}_0^x(f|\Omega_x) \), where \( \mathcal{E}_0^x \) is defined as in (3.10), is upper semianalytic for every upper semianalytic function \( f \).

For \( s \geq 0 \), let \( \vartheta_s : \Omega \to \Omega \) denote the time shift operator defined by

\[
\vartheta_s(\omega)(t) := \omega(s+t), \quad t \geq 0.
\]

We now summarise Proposition 4.31 and Lemma 4.32 in [17].

**Proposition 3.3.** Suppose that the map

\[
H : \mathbb{R}_+ \times \Omega \to 2^{\mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+}
\]

is such that

\[
\{(t,\omega,h) \in [0,T] \times \Omega \times (\mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+) : h \in H(t,\omega)\} \in \mathcal{B}([0,T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+)
\]

for all \( T \geq 0 \). Then for all \( x \in \mathbb{R}^d \), the family \( \{P^H_x(s,\tilde{\omega})\}_{(s,\tilde{\omega})\in\mathbb{R}_+\times\Omega_x} \) defined by

\[
P^H_x(s,\tilde{\omega}) := \left\{ P \in \mathcal{P}^{ac}_{sem}(\Omega_x) : (b^P, a^P, k^P) \in H(s + \cdot, \tilde{\omega} \otimes \cdot, \cdot) \right\}
\]

satisfies all assumptions from Condition 3.1 and the set in (3.12) is analytic.

In particular, (3.15) holds for all \( T \geq 0 \) if \( H(t,\omega) := H(\omega(t)) \), where \( H : \mathbb{R}^d \to 2^{\mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+} \) is a map with a Borel-measurable graph, i.e., the set

\[
\{(x,h) \in \mathbb{R}^d \times (\mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+) : h \in H(x)\}
\]

is Borel. In this case, the corresponding sublinear conditional expectation from Theorem 3.2 satisfies

\[
\mathcal{E}_0^x \left( \mathcal{E}_0^{X_t}(f) \right) = \mathcal{E}_0^x(f \circ \vartheta_t)
\]

for every upper semianalytic function \( f : \Omega \to \mathbb{R} \), \( t \geq 0 \) and \( x \in \mathbb{R}^d \), where \( \mathcal{E}_0^{X_t} \) is defined as

\[
\mathcal{E}_0^{X_t}(\cdot) = \mathcal{E}_0^x(\cdot)|_{x=X_t}.
\]

The results from [17] provide a method for constructing a family of subsets that satisfies Condition 3.1. In particular, by Proposition 3.3 it suffices to prove that the graph of \( x \mapsto \Theta(x) \) is Borel-measurable.
Lemma 3.4. Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{L}^{d+1}$ be non-empty and closed. Then the set
\[
\{(x, b, a, k) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+ : (b, a, k) \in \Theta(x)\}
\tag{3.19}
\]
is Borel, where $\Theta(x)$ is defined in (2.10).

Proof. Recall that $\Theta(x) = \{\theta(x) : \theta \in \Theta\}$, and every $\theta \in \Theta$ defines a Borel-measurable map $x \mapsto \theta(x)$, cf. (2.9). For each $\theta \in \Theta$, consider the Borel-measurable map $d_\pi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+ \to \mathbb{R}_+$ given by
\[
d_\pi(x, b, a, k) := d_\pi((b, a, k), \theta(x)),
\tag{3.20}
\]
where $\mathbb{L}_+$ is defined in (2.3) and $d_\pi$ denotes a metric on $\mathbb{R}^d \times \mathbb{S} \times \mathbb{L}$. As a subspace of a separable metric space, $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{L}^{d+1}$ is separable, i.e., there exists a countable, dense subset $\Theta_c$ in $\Theta$. Fix such a subset $\Theta_c$, and consider
\[
\inf_{\theta \in \Theta_c} d_\pi((b, a, k), \theta(x)).
\tag{3.21}
\]
The pointwise infimum of countably many Borel-measurable maps is again Borel-measurable. Hence, the set
\[
G := \left\{(x, b, a, k) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+ : \inf_{\theta \in \Theta_c} d_\pi((b, a, k), \theta(x)) = 0\right\}
\tag{3.22}
\]
is Borel. Further, observe that
\[
G = \{(x, b, a, k) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_+ \times \mathbb{L}_+ : (b, a, k) \in \Theta(x)\}.
\tag{3.23}
\]
The $\supseteq$-inclusion is trivial. For the other direction, fix $(x, b, a, k) \in G$. Then there exists a sequence $(\theta_n)_{n \in \mathbb{N}}$ in $\Theta_c$ such that
\[
\lim_{n \to \infty} d_\pi((b, a, k), \theta_n(x)) = 0.
\tag{3.24}
\]
Since $\Theta_c = \Theta$, we have $\theta := \lim_{n \to \infty} \theta_n \in \Theta$. Moreover, by the continuity of $d_\pi$ we have
\[
d_\pi((b, a, k), \theta(x)) = d_\pi((b, a, k), \theta_n(x)) = \lim_{n \to \infty} d_\pi((b, a, k), \theta_n(x)) = 0.
\tag{3.25}
\]
Hence, $(b, a, k) = \theta(x) \in \Theta(x)$, and the identity holds. \hfill \Box

Remark 3.5. The proof is stated for affine parameters, but solely relies on the separability of the space $\mathbb{R}^d \times \mathbb{S} \times \mathbb{L}$. Thus, it also carries over for polynomial processes, i.e., when we consider polynomial functions in (2.6)-(2.8). In particular, this implies that non-linear polynomial processes can be defined analogously and satisfy the dynamic programming principle. We refer to e.g. [7] for an introduction to classical polynomial processes.

Proposition 3.6. Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{L}^{d+1}$ be non-empty and closed. For $x \in \mathbb{R}^d$, let the family
\[
\{\mathcal{P}_x(s, \omega)\}_{(s, \omega) \in \mathbb{R}_x \times \Omega}
\]
be defined by
\[
\mathcal{P}_x(s, \omega) := \left\{P \in \mathcal{P}^{\mathbb{R}}_{\text{sem}}(\Omega) : X_0 = x \ P\text{-a.s., } (b^P, a^P, k^P) \in \Theta(X_{s+},(\omega \otimes \cdot))\ dt \otimes dP\text{-a.e.}\right\}
\tag{3.26}
\]
for $\omega \in \Omega_x$, and $\mathcal{P}_x(s, \omega) := \emptyset$ otherwise. Then, for any upper semianalytic function $f : \Omega \to \mathbb{R}$ and finite $\mathbb{F}$-stopping times $\sigma \leq \tau$, the function
\[
\mathcal{E}_\tau^\sigma(f)(\omega) := \sup_{P \in \mathcal{P}_x(\tau(\omega), \omega)} E_P[f^{\tau^\omega}]
\tag{3.27}
\]
is upper semianalytic and $\mathcal{F}_\tau^\sigma$-measurable, and the dynamic programming principle holds, i.e.,
\[
\mathcal{E}_\sigma^\tau(\mathcal{E}_\tau^\sigma(f)) = \mathcal{E}_\sigma^\tau(f) \text{ on } \Omega.
\tag{3.28}
\]
\textbf{Proof.} The proof is a straightforward application of Theorem 3.2 and Proposition 3.3. As an immediate consequence of Lemma 3.4, the graph of the map $\overline{H} : \mathbb{R}^d \to 2^{\mathbb{R}d \times \mathbb{S}_+ \times \mathcal{L}_+}$ given by
\begin{equation}
\overline{H}(x) = \Theta(x) \cap (\mathbb{R}^d \times \mathbb{S}_+ \times \mathcal{L}_+)
\end{equation}
is Borel since $\Theta$ is non-empty and closed. Fix $x \in \mathbb{R}^d$, and consider the map $H : (t, \omega) \mapsto \overline{H}(\omega(t)) = \overline{H}(X_t(\omega))$. Let $\mathcal{P}_x^H(s, \omega)$ with $s \geq 0$ and $\omega \in \Omega_x$ be defined as in Proposition 3.3. Then
\begin{equation}
\mathcal{P}_x^H(s, \omega) = \left\{ P \in \mathcal{P}_{ac}^\Theta(\Omega_x) : (b^P, a^P, k^P) \in H(s + \cdot, \omega \otimes \cdot ) \ dt \otimes dP \text{-a.e.} \right\}
\end{equation}
where the last identity is immediate by (3.29) since the characteristics $(b^P, a^P, k^P)$ takes values in $\mathbb{R}^d \times \mathbb{S}_+ \times \mathcal{L}_+$. Observe that for all $\omega \in \Omega_x$, we have
\begin{equation}
\mathcal{P}_x^H(s, \omega) = \left\{ P\|_{\Omega_x} : P \in \mathcal{P}_x(s, \omega) \right\},
\end{equation}
and thus
\begin{equation}
\mathcal{E}_x^\tau(f)(\omega) = \sup_{P \in \mathcal{P}_x(\tau(\omega), \omega)} E_P [f^{\tau, \omega}] = \sup_{P \in \mathcal{P}_x^\Theta(\tau(\omega), \omega)} E_P [(f|_{\Omega_x})^{\tau, \omega}].
\end{equation}
Since $f|_{\Omega_x}$ is upper semianalytic for an upper semianalytic $f : \Omega \to \mathbb{R}$, Proposition 3.3 and Theorem 3.2 yields the desired properties on $\Omega_x$. For $\omega \in \Omega \setminus \Omega_x$, we have
\begin{equation}
\mathcal{E}_x^\tau(f)(\omega) = -\infty,
\end{equation}
thus the desired measurability properties and the dynamic programming property on $\Omega$ follow immediately. \hfill \Box

Note that the conditional sublinear expectations from Proposition 3.6 are compatible with the sublinear expectations $\mathcal{E}^x$ defined in (2.14) in the sense that for all $x \in \mathbb{R}^d$ and upper semianalytic $f : \Omega \to \mathbb{R}$, it holds that
\begin{equation}
\mathcal{E}^x(f) = \mathcal{E}_0^x(f) \quad \text{on } \Omega_x.
\end{equation}

In the classical setting, the Markov property is quintessential to affine processes. The dynamic programming principle of Proposition 3.6 yields an analogue for sublinear conditional expectations.

\textbf{Corollary 3.7.} Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}_+^{d+1} \times \mathcal{L}_+^{d+1}$ be non-empty and closed. Then for any $x \in \mathbb{R}$, $s, t \geq 0$ and any upper semianalytic function $f : \Omega \to \mathbb{R}$, we have
\begin{equation}
\mathcal{E}_x^s(f \circ \vartheta_s) = \mathcal{E}_x^s(f) \quad \text{on } \Omega_x,
\end{equation}
where $\vartheta_s$ is the time shift defined in (3.13). In particular,
\begin{equation}
\mathcal{E}_x^s(f(X_{t+s})) = \mathcal{E}_x^s(f(X_t)) \quad \text{on } \Omega_x.
\end{equation}
\textbf{Proof.} It follows immediately from Proposition 3.3. \hfill \Box
4 Kolmogorov PIDE under model uncertainty

In the classical setting, an affine process \( X = (X_t)_{t \geq 0} \) with characteristics \( (\beta, \alpha, \nu) \) is a Markov process with generator

\[
A_x^{(\beta, \alpha, \nu)} f(x) = D_x f(x)^T \beta(x) + \frac{1}{2} \text{tr} \left[ D_x^2 f(x) \alpha(x) \right] + \int_{\mathbb{R}^d} \left[ f(x + z) - f(x) - D_x f(x)^T h(z) \right] \nu(x; dz). \tag{4.1}
\]

The generator induces a PIDE, usually referred to as evolution or Kolmogorov equation given by

\[
\partial_t v(t, x) - A_x^{(\beta, \alpha, \nu)} v(t, x) = 0. \tag{4.2}
\]

Under suitable conditions on \( \varphi \), the value function \( v(t, x) := E^x[\varphi(X_t)] \) is a viscosity solution of (4.2) satisfying the initial condition \( v(0, x) = \varphi(x) \). The goal of this section is to establish a similar result for non-linear affine processes. More precisely, for a bounded, Lipschitz continuous function \( \varphi \in \text{Lip}_b(\mathbb{R}^d) \), a non-linear affine process \( X \) with parameter set \( \Theta \) and state space \( S \), we show that if the value function \( v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) given by

\[
v(t, x) := E^x(\varphi(X_t)) \tag{4.3}
\]

is continuous, then \( v \) is a viscosity solution of the PIDE

\[
\partial_t v(t, x) - A_x v(t, x) = 0 \quad \text{for } v(0, x) = \varphi(x), \tag{4.4}
\]

where the operator \( A_y \) is defined by

\[
A_y f(x) := \sup_{(\beta, \alpha, \nu) \in \Theta} \left\{ D_x f(x)^T \beta(y) + \frac{1}{2} \text{tr} \left[ D_x^2 f(x) \alpha(y) \right] + \int_{\mathbb{R}^d} \left[ f(x + z) - f(x) - D_x f(x)^T h(z) \right] \nu(y; dz) \right\}. \tag{4.6}
\]

To be specific, we use the following definition of a viscosity solution, as in [29].

**Definition 4.1 (Viscosity Solution).** Let \( G : \mathbb{R}^d \times \mathbb{R}^d \times S \times C^2_b(\mathbb{R}^d) \to \mathbb{R} \) be some operator. An upper semicontinuous function \( v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) is called a viscosity subsolution of

\[
\partial_t v(t, x) + G(x, D_x v(t, x), D_x^2 v(t, x), v(t, \cdot)) = 0 \quad \text{in a domain } D \subseteq \mathbb{R}_+ \times \mathbb{R}^d \text{ if for any interior point } (t, x) \in D^c \text{ and any } \psi \in C^2_b(\mathbb{R}_+ \times \mathbb{R}^d) \text{ with } \psi(t, x) = v(t, x) \text{ and } v \leq \psi \text{ on } \mathbb{R}_+ \times \mathbb{R}^d, \text{ we have }
\]

\[
\partial_t \psi(t, x) + G(x, D_x v(t, x), D_x^2 v(t, x), v(t, \cdot)) \leq 0. \tag{4.8}
\]

The definition of a viscosity supersolution of (4.7) in \( D \) is obtained by reversing the inequalities and considering a lower semicontinuous function \( v \). A function \( v \) is called viscosity solution of (4.7) in \( D \) if it is both a viscosity sub- and a supersolution.

To obtain the desired result, we need to impose some boundedness conditions on \( \Theta \). Let \( \| \cdot \| \) denote both the Euclidean norm on \( \mathbb{R}^d \) and the spectral norm on \( S \). For \( k \in \mathfrak{L} \), define

\[
\| k \| := \int_{\mathbb{R}^d} (\| z \|^2 \wedge \| z \|) |k(dz)|, \tag{4.9}
\]

which is clearly an extended norm on \( \mathfrak{L} \) since \( k(\{0\}) = 0 \) for all \( k \in \mathfrak{L} \).
Moreover, define (extended) norms for \( \beta = (\beta_0, \ldots, \beta_d) \in (\mathbb{R}^d)^{d+1}, \alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{S}^{d+1} \) and \( \nu = (\nu_0, \ldots, \nu_d) \in \mathbb{L}^{d+1} \) by

\[
\begin{align*}
\|\beta\| &:= \sup_{x \in \mathbb{R}^d} \frac{\|\beta_0 + (\beta_1, \ldots, \beta_d) x\|}{\|x\|+1}, \\
\|\alpha\| &:= \sup_{x \in \mathbb{R}^d} \frac{\|\alpha_0 + (\alpha_1, \ldots, \alpha_d) x\|}{\|x\|+1}, \\
\|\nu\| &:= \sup_{x \in \mathbb{R}^d} \frac{\|\nu_0 + (\nu_1, \ldots, \nu_d) x\|}{\|x\|+1}.
\end{align*}
\]

(4.10) (4.11) (4.12)

By the properties of \( \|\cdot\| \) it follows that each map in (4.10) - (4.12) satisfy the defining properties of an (extended) norm. Next, we state similar conditions as in [29], [14].

**Assumption 4.2.** Let \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{L}^{d+1} \) be non-empty, closed and such that the following holds. For all \( x \in \mathbb{R}^d \), we have \( \mathcal{P}_x(\Theta) \neq \emptyset \) and

\[
\lim_{\delta \downarrow 0} K_\delta(x) = 0,
\]

where for \( \delta > 0 \),

\[
K_\delta(x) := K_\delta(x; \Theta) := \sup_{(\beta, \alpha, \nu) \in \Theta} \int_{\|z\| \leq \delta} \|z\|^2 \nu(x; dz).
\]

Further,

\[
\mathcal{K} := \mathcal{K}(\Theta) := \sup_{(\beta, \alpha, \nu) \in \Theta} \left\{ \|\beta\| + \|\alpha\| + \|\nu\| \right\} < \infty.
\]

(4.13) (4.14) (4.15)

Note that (4.15) is equivalent to \( \Theta \) being bounded with respect to the operator norm induced by the Euclidean norm on \( \mathbb{R}^{d+1} \). In particular, (4.15) implies that the linear components of the parameters are uniformly bounded.

**Corollary 4.3.** Let \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{L}^{d+1} \) satisfy Assumption 4.2. Then

\[
\sup_{(\beta, \alpha, \nu) \in \Theta} \left\{ \|\beta_1, \ldots, \beta_d\|_{\text{op}} + \|\alpha_1, \ldots, \alpha_d\|_{\text{op}} + \|\nu_1, \ldots, \nu_d\|_{\text{op}} \right\} \leq 3 \mathcal{K},
\]

(4.16)

where \( \beta = (\beta_0, \beta_1, \ldots, \beta_d), \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \) and \( \nu = (\nu_0, \nu_1, \ldots, \nu_d) \), and \( \|\cdot\|_{\text{op}} \) denote the extended operator norms induced by \( \|\cdot\| \).

**Proof.** By the definition of the operator norm and the triangle inequality, we have that

\[
\|\beta_1, \ldots, \beta_d\|_{\text{op}} = \sup_{x \in \mathbb{R}^d; \|x\|=1} \|\beta_1, \ldots, \beta_d\| x
\]

\[
\leq \sup_{x \in \mathbb{R}^d; \|x\|=1} \|\beta_0 + (\beta_1, \ldots, \beta_d) x\| + \|\beta_0\|
\]

\[
= \sup_{x \in \mathbb{R}^d; \|x\|=1} 2 \frac{\|\beta_0 + (\beta_1, \ldots, \beta_d) x\|}{\|x\|+1} + \frac{\|\beta_0 + (\beta_1, \ldots, \beta_d) 0\|}{\|0\|+1}
\]

\[
\leq 3 \|\beta\|.
\]

(4.17)

Analogously, we can show (4.17) for \( \alpha \) and \( \nu \).

In order to avoid subtleties related to measurability, from now on we work with the augmentation \( \mathbb{R}^d \), as this does not affect the the semimartingale characteristics, see Remark 2.2 Point 1.

We start with some auxiliary results before we turn to the value function. We follow the line of argument in the proof of [14, Lemma 3] and adapt it to the case with jumps and consider a \( d \)-dimensional setting.
Lemma 4.4. Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{S}^{d+1}$ satisfy Assumption 4.2, and let $1 \leq p \leq 2$. Then there exists an $\varepsilon := \varepsilon(p) > 0$ such that for all $0 \leq t \leq \varepsilon$, $x \in \mathbb{R}^d$ and $P \in \mathcal{P}_x(\Theta)$, we have

$$E_P \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] \leq C_{K,p} (1 + \| x \|)^p (t^p + t^\frac{p}{p-1})$$

(4.18)

for some constant $C_{K,p}$ independent of $t$, $x$ and $P$.

Proof. For some $x \in \mathbb{R}^d$, fix a $P \in \mathcal{P}_x(\Theta)$ and let $(b^p, a^p, k^p)$ denote the associated differential characteristics. Then we have

$$E_P \left[ \sup_{0 \leq s \leq t} \| B^p_s \|^p \right] = E_P \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s b^p_u \, du \right\|^p \right]$$

$$= E_P \left\{ \left( \int_0^t \| b^p_u \|\, du \right)^p \right\}$$

$$\leq K^p E_P \left[ \left( \int_0^t (\| X_u \| + 1) \, du \right)^p \right]$$

$$\leq K^p t^p E_P \left[ \left( \sup_{0 \leq s \leq t} \| X_s \| + 1 \right)^p \right]$$

$$\leq K^p t^p E_P \left[ \left( \sup_{0 \leq s \leq t} \| X_s - X_0 \| + \| X_0 \| + 1 \right)^p \right]$$

$$\leq K^p t^p 2^{p-1} \left( E_P \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] + (\| x \| + 1)^p \right).$$

(4.19)

The last step follows by Jensen’s inequality, as $x \to x^p$ is convex on $\mathbb{R}_+$ for $p \geq 1$. Moreover, we will use repeatedly the following inequality for a different number of non-negative summands, i.e.,

$$\left( \sum_{i=1}^n y_i \right)^p \leq n^{p-1} \sum_{i=1}^n y_i^p.$$ 

(4.20)

By the Burkholder-Davis-Gundy inequality in (A.1), we have for the continuous local martingale part $^c M^p$ of $X$ that

$$E_P \left[ \sup_{0 \leq s \leq t} \| ^c M^p_s \|^p \right] \leq C_{p,d} E_P \left[ \left\| [^c M^p]_t \right\|^\frac{p}{p-1} \right]$$

$$= C_{p,d} E_P \left[ \left\| \int_0^t a^p_u \, du \right\|^\frac{p}{p-1} \right]$$

$$\leq C_{p,d} K^\frac{p}{p-1} E_P \left[ \left( \int_0^t (\| X_u \| + 1) \, du \right)^\frac{p}{p-1} \right]$$

$$\leq C_{p,d} K^\frac{p}{p-1} t^\frac{p}{p-1} E_P \left[ \left( \sup_{0 \leq s \leq t} \| X_s \| + 1 \right)^\frac{p}{p-1} \right]$$

$$\leq C_{p,d} K^\frac{p}{p-1} t^\frac{p}{p-1} 2^{p-1} \left( E_P \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] + (\| x \| + 1)^p \right).$$

(4.21)
where $C_{p,d}$ is the constant from the Burkholder-Davis-Gundy inequality. Similarly, we obtain for the purely discontinuous local martingale part $\lambda^{P}$,

$$E_P \left[ \sup_{0 \leq s \leq t} \| \lambda^{P} \|^{p} \right] = C_{p,d} E_P \left[ \left\| \lambda^{P} \right\|^{2} \right]$$

$$\leq C_{p,d} E_P \left[ \left( \int_{0}^{t} \int_{\mathbb{R}^{d}} \| h(z) \|^{2} \mu(ds,dz) \right)^{\frac{p}{2}} \right]$$

$$\leq C_{p,d} C_{h}^{p} E_P \left[ \left( \int_{0}^{t} \int_{\mathbb{R}^{d}} \left( \| z \|^{2} \wedge 1 \right) \mu(ds,dz) \right)^{\frac{p}{2}} \right]$$

$$\leq C_{p,d} C_{h}^{p} E_P \left[ \int_{0}^{t} \int_{\mathbb{R}^{d}} \left( \| z \|^{2} \wedge 1 \right) k_{u}^{p}(dz) \mu(ds) \right]$$

$$\leq C_{p,d} C_{h}^{p} E_P \left[ \int_{0}^{t} \int_{\mathbb{R}^{d}} \left( \| z \|^{2} \wedge \| z \| \right) k_{u}^{p}(dz) \mu(ds) \right]$$

$$\leq C_{p,d} C_{h}^{p} E_P \left[ \int_{0}^{t} \left( \| X_{u} \| + 1 \right) du \right]$$

$$\leq C_{p,d} C_{h}^{p} t^{\frac{p}{2}} E_P \left[ \sup_{0 \leq s \leq t} \| X_{s} \| + 1 \right]$$

$$\leq C_{p,d} C_{h}^{p} t^{\frac{p}{2}} E_P \left[ \sup_{0 \leq s \leq t} \left( \| X_{s} \| + 1 \right)^{p} \right]$$

$$\leq C_{p,d} C_{h}^{p} K_{x}^{p} t^{p} 2^{p-1} \left( E_P \left[ \sup_{0 \leq s \leq t} \| X_{s} - X_{0} \|^{p} \right] + (\| x \| + 1)^{p} \right), \quad (4.22)$$

where $C_{h}$ depends only on $h$ and is such that $\| z - h(z) \| \leq C_{h}(\| z \|^{2} \wedge \| z \|)$ and $\| h(z) \| \leq C_{h}(\| z \| \wedge 1)$ for all $z \in \mathbb{R}^{d}$, and $\| z \|, \| h(z) \| \leq C_{h}(\| z \|^{2} \wedge \| z \|)$ outside some neighbourhood of zero.\(^1\) Further, for the unbounded jump part $\lambda^{P}$,

$$E_P \left[ \sup_{0 \leq s \leq t} \| \lambda^{P} \|^{p} \right] = E_P \left[ \sup_{0 \leq s \leq t} \left\| \int_{0}^{s} \int_{\mathbb{R}^{d}} [z - h(z)] \mu(du,dz) \right\|^{p} \right]$$

$$= E_P \left[ \left( \int_{0}^{t} \int_{\mathbb{R}^{d}} \| z - h(z) \| \mu(du,dz) \right)^{p} \right]$$

$$\leq C_{h}^{p} E_P \left[ \left( \int_{0}^{t} \int_{\mathbb{R}^{d}} \left( \| z \|^{2} \wedge \| z \| \right) k_{u}^{p}(dz) \mu(du) \right)^{p} \right]$$

$$\leq C_{h}^{p} K_{x}^{p} E_P \left[ \left( \int_{0}^{t} \left( \| X_{u} \| + 1 \right) du \right)^{p} \right]$$

$$\leq C_{h}^{p} K_{x}^{p} t^{p} 2^{p-1} \left( E_P \left[ \sup_{0 \leq s \leq t} \| X_{s} - X_{0} \|^{p} \right] + (\| x \| + 1)^{p} \right). \quad (4.23)$$

\(^1\)Such a constant exists. By the definition of $h$, there exists a $\delta \in [0,1]$ such that $h(z) = z$ for all $z$ with $\| z \| \leq \delta$. Fix such a $\delta$ and set $C_{h} := \delta^{-2} \| h \|_{\infty}$. Then the desired inequalities hold. For a detailed proof, see Corollary A.2.
We can choose $C_{p,d}, C_h \geq 1$. Then by the canonical semimartingale decomposition in (2.1), the triangle inequality, and the inequalities (4.19) - (4.23), we have for all $\varepsilon \geq t$,

$$
E_p \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] \leq 4^{p-1} E_p \left[ \sup_{0 \leq s \leq t} \| B^s_p \|^p + \sup_{0 \leq s \leq t} \| M^s_p \|^p + \sup_{0 \leq s \leq t} \| J^s_p \|^p \right]
$$

$$
\leq 4^{p-1} 2^p C^p_h C_{p,d} \left( K^p + C^p t^p \right) \left( E_p \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] + (\| x \|+1)^p \right)
$$

$$
\leq 2^{3p-2} C^p_h C_{p,d} (K^p + 1) (t^p + \tilde{\varepsilon}^p) \left( E_p \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] + (\| x \|+1)^p \right)
$$

$$
\leq \tilde{C} \left( \varepsilon^p + \tilde{\varepsilon}^p \right) E_p \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] + \tilde{C} (t^p + \tilde{\varepsilon}^p)(\| x \|+1)^p. \quad (4.24)
$$

Fix a sufficiently small $\varepsilon := \varepsilon(p) > 0$ such that $1 - \tilde{C} \left( \varepsilon^p + \tilde{\varepsilon}^p \right) > 0$. Then we obtain

$$
E_p \left[ \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right] \leq \tilde{C} \frac{1}{1 - C \left( \varepsilon^p + \tilde{\varepsilon}^p \right)} (\| x \|+1)^p (t^p + \tilde{\varepsilon}^p), \quad (4.25)
$$

which proves Lemma 4.4.

**Corollary 4.5.** Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{S}^{d+1}$ satisfy Assumption 4.2, and let $1 \leq p \leq 2$. Then there exists an $\varepsilon := \varepsilon(p) > 0$ such that for all $0 \leq t \leq \varepsilon$ and $x \in \mathbb{R}^d$,

$$
\mathcal{E}^x \left( \sup_{0 \leq s \leq t} \| X_s - X_0 \|^p \right) \leq C_{K,p} \left( 1 + \| x \| \right)^p (t^p + \tilde{\varepsilon}^p) \quad (4.26)
$$

for some constant $C_{K,p}$ independent of $t$ and $x$.

**Proof.** Follows immediately from Lemma 4.4.

**Lemma 4.6.** Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{S}^{d+1}$ satisfy Assumption 4.2. Then for all $x \in \mathbb{R}^d$ and $t \geq 0$,

$$
\mathcal{E}^x (\| X_t \|) < \infty.
$$

**Proof.** Let $\varepsilon := \varepsilon(1) > 0$ be the constant from Corollary 4.5, and let $x \in \mathbb{R}^d$. For $t \leq \varepsilon$, the statement follows directly from Corollary 4.5. Suppose that $t > \varepsilon$. The tower property in (3.11) and the Markov property in (3.36) give

$$
\mathcal{E}^x (\| X_t \|) = \mathcal{E}^x (\mathcal{E}^x_{\varepsilon}(\| X_t \|))
$$

$$
= \mathcal{E}^x (\mathcal{E}^{X_{t-\varepsilon}}(\| X_t \|))
$$

$$
\leq \mathcal{E}^x (\mathcal{E}^{X_{t-\varepsilon}}(\| X_{t-\varepsilon} \|)) + \mathcal{E}^x (\mathcal{E}^{X_{t-\varepsilon}}(\| X_0 \|))
$$

$$
= \mathcal{E}^x (\mathcal{E}^{X_{t-\varepsilon}}(\| X_{t-\varepsilon} \|)) + \mathcal{E}^x (\mathcal{E}^y (\| X_0 \|) y = X_{t-\varepsilon})
$$

$$
= \mathcal{E}^x (\mathcal{E}^{X_{t-\varepsilon}}(\| X_{t-\varepsilon} \|)) + \mathcal{E}^x (\| y \|) y = X_{t-\varepsilon}
$$

$$
\leq \mathcal{E}^x \left( \mathcal{E}^{X_{t-\varepsilon}} \left( \sup_{0 \leq u \leq \varepsilon} \| X_u - X_0 \| \right) \right) + \mathcal{E}^x (\| X_{t-\varepsilon} \|)
$$

$$
\leq C_{K,1} (\varepsilon + \tilde{\varepsilon}^p) (\mathcal{E}^x (\| X_{t-\varepsilon} \|) + 1) + \mathcal{E}^x (\| X_{t-\varepsilon} \|) \quad (4.27)
$$

$$
\leq \left( C_{K,1} (\varepsilon + \tilde{\varepsilon}^p) + 1 \right) (\mathcal{E}^x (\| X_{t-\varepsilon} \|) + 1), \quad (4.28)
$$

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where we use Corollary 4.5 in (4.27). For every $t > \varepsilon$, there exists an $N \in \mathbb{N}$ such that $N\varepsilon < t \leq (N + 1)\varepsilon$. Hence, the statement follows by repeating the procedure $N$ times and applying Corollary 4.5 to $E_x(\|X_{t-N\varepsilon}\|)$.

Now we can turn to the value function $v$. We start by proving its continuity and then show that it is a viscosity solution of (4.4).

**Lemma 4.7.** Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{S}^{d+1}$ satisfy Assumption 4.2 and $\varphi \in \text{Lip}_b(\mathbb{R}^d)$. The value function $v$ from (4.3) satisfies for all $t, s \geq 0$ and $x \in \mathbb{R}^d$,

$$v(t + s, x) = E^x(v(t, X_s)).$$  

(4.29)

**Proof.** The proof is essentially the same as in [29, Lemma 5.1]. Let $s, t \geq 0$ and $x \in \mathbb{R}^d$. By the definition of the value function and the Markov property from (3.36),

$$v(t, X_s) = E^{X_s}(\varphi(X_t)) = \mathcal{E}_s^x(\varphi(X_{t+s})).$$  

(4.30)

Applying $\mathcal{E}^x(\cdot)$ on both sides, the dynamic programming property in (3.11) yields

$$\mathcal{E}^x(v(t, X_s)) = \mathcal{E}^x(\mathcal{E}_s^x(\varphi(X_{t+s}))) = \mathcal{E}^x(\varphi(X_{t+s})) = v(t + s, x).$$  

(4.31)

**Lemma 4.8.** Let $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{S}^{d+1}$ satisfy Assumption 4.2 and $\varphi \in \text{Lip}_b(\mathbb{R}^d)$ with Lipschitz constant $L_\varphi$. Let $v$ be the value function from (4.3). Then for fixed $x \in \mathbb{R}^d$, the map $t \mapsto v(t, x)$ is locally $\frac{1}{2}$-Hölder continuous. Moreover, if $\cup_{\varepsilon \in K} P_\varepsilon(\Theta)$ is sequentially compact in itself (with respect to weak convergence of measures)\footnote{i.e. every sequence of measures in $\cup_{\varepsilon \in K} P_\varepsilon(\Theta)$ has a convergent subsequence converging with respect to the weak convergence of measures to another measure in $\cup_{\varepsilon \in K} P_\varepsilon(\Theta)$.} for compact $K \subset \mathbb{R}^d$, then for fixed $t \geq 0$, $x \mapsto v(t, x)$ is upper semicontinuous.

**Proof.** For the local $\frac{1}{2}$-Hölder right-continuity, fix an $s \geq 0$ and let $t \geq 0$. By the Markov property in (3.36), we have for all $x \in \mathbb{R}^d$ that

$$|v(s + t, x) - v(s, x)| = |\mathcal{E}^x(\varphi(X_{s+t})) - \mathcal{E}^x(\varphi(X_s))|$$

$$\leq \mathcal{E}^x\left(\|\varphi(X_{s+t}) - \varphi(X_s)\|\right)$$

$$= \mathcal{E}^x(\mathcal{E}^{X_s}(\|\varphi(X_t) - \varphi(X_0)\|))$$

$$\leq L_\varphi \mathcal{E}^x(\mathcal{E}^{X_s}(\|X_t - X_0\|))$$

$$\leq L_\varphi \mathcal{E}^x\left(\sup_{0 \leq u \leq t} \|X_u - X_0\|\right).$$  

(4.32)

Let $\varepsilon := \varepsilon(1) > 0$ be the constant from Corollary 4.5. Then (4.32) yields for all $0 \leq t \leq \varepsilon$,

$$|v(s + t, x) - v(s, x)| \leq L_\varphi C_{K,1} \left( (t + \frac{t^2}{2}) \left( \mathcal{E}^x(\|X_s\|) + 1 \right), \right.$$  

(4.33)

and $\mathcal{E}^x(\|X_s\|) < \infty$ by Lemma 4.6. Analogously, for the local $\frac{1}{2}$-Hölder left-continuity, we have that

$$|v(s - t, x) - v(s, x)| \leq L_\varphi C_{K,1} \left( (t + \frac{t^2}{2}) \left( \mathcal{E}^x(\|X_{s-t}\|) + 1 \right) \right.$$  

(4.34)

for all $0 \leq t \leq \varepsilon$. Combining both (4.33) and (4.34) yields for all $-\varepsilon \leq t \leq \varepsilon$,

$$|v(s + t, x) - v(s, x)| \leq |v(s + t, x) - v(s, x)| + |v(s - t, x) - v(s, x)|$$

$$+ |v(s - t, x) - v(s, x)| \leq L_\varphi C_{K,1} \left( (t + \frac{t^2}{2}) \left( \mathcal{E}^x(\|X_s\|) + 1 \right) \right.$$  

(4.35)
\[ \leq L_\varphi C_{K,1} (|t|+|t|^{1/2}) \left( \mathcal{E}^x (\|X_s\|) + \mathcal{E}^x (\|X_{s-t}\|) + 2 \right) \]
\[ \leq L_\varphi C_{K,1} (|t|+|t|^{1/2}) \left[ \left( C_{K,1} (\varepsilon + \varepsilon \frac{1}{2}) + 1 \right) \left( \mathcal{E}^x (\|X_{s-t}\|) + 1 \right) + \left( C_{K,1} ((\varepsilon - t) + (\varepsilon - t) \frac{1}{2}) + 1 \right) \left( \mathcal{E}^x (\|X_{s-t}\|) + 1 \right) + 2 \right] \]
\[ = H_{s,x} \]

where we apply twice (4.28) in (4.35).

The upper semicontinuity of \( v \) follows directly by the assumptions on \( \cup_{x \in K} \mathcal{P}_x(\Theta) \) and by Lemma 4.42 in [17]. \( \square \)

**Remark 4.9.**

1. Note that the value function \( v \) in (4.3) is not any longer continuous, as it is for example the case in [29], where the sublinear operator and the value function to the set \( \mathcal{P}_0(\Theta) \) in (2.16) is considered. However, in our setting the equation in (4.37) is not any longer correct, which would allow to conclude the continuity of \( v \). For \( t \geq 0 \) and \( x,y \in \mathbb{R}^d \) it is not true that

\[ |\mathcal{E}^x (\varphi(X_t)) - \mathcal{E}^y (\varphi(X_t))| = |\mathcal{E}^x (\varphi(X_t)) - \mathcal{E}^x (\varphi(X_t - x + y))|. \] (4.37)

2. The continuity of the value function \( v \) has also been discussed in [17]. In particular, in Lemma 4.42 in [17] provides a criteria which guarantees the upper semicontinuity of \( v \) and which we use in Lemma 4.8. However, as pointed out in [17] such a criteria does not exist for the lower semicontinuity of \( v \).

**Proposition 4.10.** Let \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathbb{S}^{d+1} \) satisfy Assumption 4.2 and \( \varphi \in \text{Lip}(\mathbb{R}^d) \). If \( \cup_{x \in K} \mathcal{P}_x(\Theta) \) is sequentially compact in itself (with respect to weak convergence of measures) for compact \( K \subset \mathbb{R}^d \), then the value function \( v \) from (4.3) is a viscosity subsolution of (4.4) in \( \mathbb{R}_+ \times S \), and satisfies the initial condition (4.5) on \( \mathbb{R}^d \). If in addition the value function \( v \) is also lower semicontinuous, then \( v \) is a viscosity solution of (4.4) in \( \mathbb{R}_+ \times S \), and satisfies the initial condition (4.5) on \( \mathbb{R}^d \).

**Proof.** We largely follow the line of arguments presented in the proofs of [14, Theorem 1], and [29, Proposition 5.4] for the jump part. The initial condition (4.5) follows from the definition of the value function in (4.3). Further, the condition for the upper semicontinuity of the value function \( v \) follows directly by Lemma 4.8. We prove that \( v \) is a viscosity subsolution of the PIDE (4.4). In case \( v \) is also lower semicontinuous, the supersolution property follows analogously.

Fix an interior point \((t,x) \in \mathbb{R}_+ \times S\), and let \( \psi \in C^{2,3}_0(\mathbb{R}_+ \times \mathbb{R}^d) \) with \( \psi(t,x) = v(t,x) \) and \( \psi \geq v \) on \( \mathbb{R}_+ \times \mathbb{R}^d \). By Lemma 4.7, for all \( 0 \leq s \leq t \), we have

\[ 0 = \mathcal{E}^x (v(t-s,X_s) - v(t,x)) \leq \mathcal{E}^x (\psi(t-s,X_s) - \psi(t,x)). \] (4.38)

In the following, we study the sublinear expectation on the right-hand side by considering the associated linear expectations. Therefore, fix a \( P \in \mathcal{P}_x(\Theta) \), and let \((b^P, a^P, k^P)\) denote the differential characteristics of \( X \) under \( P \). For \( 0 \leq s \leq t \), Itô’s formula yields \( P \)-a.s. that

\[
\begin{align*}
\psi(t-s,X_s) - \psi(t,x) &= \int_0^s -\partial_t \psi(t-u,X_{u-}) \, du + \int_0^s D_x \psi(t-u,X_{u-})^T b_u^P \, du \\
&\quad + \int_0^s D_x \psi(t-u,X_{u-})^T dM_u^P + \frac{1}{2} \int_0^s \text{tr} \left[ D_x^2 \psi(t-u,X_{u-}) a_u^P \right] \, du \\
&\quad + \int_0^s \int_{\mathbb{R}^d} \left[ \psi(t-u,X_{u-} + z) - \psi(t-u,X_{u-}) - D_x \psi(t-u,X_{u-})^T h(z) \right] \mu(du,dz),
\end{align*}
\] (4.39)
Similarly for the second term on the right-hand side of (4.40), we have
\[ E_P \left[ \int_0^s D_x \psi(t-u, X_{u-})^T dM^P_u \right] s \]
\[ = E_P \left[ \int_0^s D_x \psi(t-u, X_{u-})^T d[M^P]_u D_x \psi(t-u, X_{u-}) \right] \]
\[ = E_P \left[ \int_0^s D_x \psi(t-u, X_{u-})^T a_u D_x \psi(t-u, X_{u-}) du \right] \]
\[ + E_P \left[ \int_0^s D_x \psi(t-u, X_{u-})^T \int_{\mathbb{R}^d} h(z) h(z)^T d\mu(du, dz) D_x \psi(t-u, X_{u-}) \right] \]
\[ \leq \|D_x \psi\|^2 \left( E_P \left[ \int_0^s \|a_u\|^2 du \right] + E_P \left[ \int_0^s \|h(z)\|^2 \mu(du, dz) \right] \right) < \infty, \] (4.41)
due to the inequalities (4.21) and (4.22). Hence, the integral with respect to \( M^P \) in (4.39) is a true martingale on \([0, \varepsilon]\), cf. [32, Corollary II.6.3], and
\[ E_P \left[ \int_0^s D_x \psi(t-u, X_{u-})^T dM^P_u \right] = 0. \] (4.42)

To estimate the expectation of the other terms, note that \( \psi, \partial_t \psi, D_x \psi \) and \( D_x^2 \psi \) are Lipschitz continuous in \( t, x \) since \( \psi \in C^{2,3}_b(\mathbb{R}_+ \times \mathbb{R}^d) \). From now on we use the notation \( L := \max(L_t, L_x, L_{xx}) < \infty \). Observe that for the first term on the right-hand side of (4.40),
\[ \int_0^s -\partial_t \psi(t-u, X_{u-}) du = \int_0^s \left( \partial_t \psi(t, x) - \partial_t \psi(t-u, X_{u-}) \right) du - \int_0^s \partial_t \psi(t, x) du. \] (4.43)

For sufficiently small \( s \), Lemma 4.4 gives the estimation
\[ \left| E_P \left[ \int_0^s \left( \partial_t \psi(t, x) - \partial_t \psi(t-u, X_{u-}) \right) du \right] \right| \leq L E_P \left[ \int_0^s (u + \|X_{u-} - x\|) du \right] \]
\[ \leq L s E_P \left[ \frac{s}{2} + \sup_{0 \leq u \leq s} \|X_u - x\| \right] \]
\[ \leq L s \left( \frac{s}{2} + C_{K,1} (\|x\|+1)(s + s^2) \right) \]
\[ = L \left( \frac{1}{2} + C_{K,1} (\|x\|+1) \right) s^2 + 1 s^2. \] (4.44)

Similarly for the second term on the right-hand side of (4.40), we have
\[ \int_0^s D_x \psi(t-u, X_{u-})^T b_u^P du = \int_0^s \left( D_x \psi(t-u, X_{u-}) - D_x \psi(t, x) \right)^T b_u^P du \]
\[ + \int_0^s D_x \psi(t, x)^T b_u^P du. \] (4.45)
Applying Lemma 4.4, this yields for sufficiently small $0 \leq s$ that

$$
\left| E P \left[ \int_0^s \left( D_x \psi(t-u,X_{u-}) - D_x \psi(t,x) \right)^T b_u^P \, du \right] \right|
$$

$$
\leq L E P \left[ \int_0^s (u + \|X_{u-} - x\|) \|b_u^P\| \, du \right]
$$

$$
\leq L KE P \left[ \int_0^s (u + \|X_{u} - x\|) (\|X_u\|+1) \, du \right]
$$

$$
\leq L K s E P \left[ \left( \frac{s}{2} + \sup_{0 \leq u \leq s} \| X_u - x \| \right) \left( \sup_{0 \leq u \leq s} \| X_u - x \| + \| x \| + 1 \right) \right]
$$

$$
= L K s \left( \left( \frac{s}{2} + \| x \| + 1 \right) E P \left[ \sup_{0 \leq u \leq s} \| X_u - x \| \right] + \frac{s}{2} (\| x \| + 1) + E P \left[ \sup_{0 \leq u \leq s} \| X_u - x \|^2 \right] \right)
$$

$$
\leq L K s \left( \left( \frac{s}{2} + \| x \| + 1 \right) C_{K,1} (s + s^2) (\| x \| + 1) + \frac{s}{2} (\| x \| + 1) + C_{K,2} (s^2 + s) (\| x \| + 1)^2 \right)
$$

$$
\leq L K (\| x \| + 1)^2 \left( C_{K,1} + 1 + C_{K,2} \right) (s^2 + s + s^2 + 1) s^2. \tag{4.46}
$$

For the forth term on the right-hand side of (4.40),

$$
\int_0^s \text{tr} \left[ D_x^2 \psi(t-u,X_{u-}) a_u^P \right] \, du = \int_0^s \text{tr} \left[ \left( D_x^2 \psi(t-u,X_{u-}) - D_x^2 \psi(t,x) \right) a_u^P \right] \, du
$$

$$
+ \int_0^s \text{tr} \left[ D_x^2 \psi(t,x) a_u^P \right] \, du. \tag{4.47}
$$

Since the trace is an inner product on $S$, we get by the Cauchy-Schwarz inequality, we obtain that

$$
E P \left[ \int_0^s \text{tr} \left[ \left( D_x^2 \psi(t-u,X_{u-}) - D_x^2 \psi(t,x) \right) a_u^P \right] \, du \right]
$$

$$
\leq E P \left[ \int_0^s \sqrt{\text{tr} \left[ \left( D_x^2 \psi(t-u,X_{u-}) - D_x^2 \psi(t,x) \right)^2 \right]} \, \text{tr} \left[ (a_u^P)^2 \right] \, du \right]
$$

$$
\leq E P \left[ \int_0^s \sqrt{d^2} \| D_x^2 \psi(t-u,X_{u-}) - D_x^2 \psi(t,x) \|^2 \| a_u^P \|^2 \, du \right]
$$

$$
= E P \left[ \int_0^s d \| D_x^2 \psi(t-u,X_{u-}) - D_x^2 \psi(t,x) \| \| a_u^P \| \, du \right]
$$

$$
\leq d L E P \left[ \int_0^s (u + \| X_{u-} - x \|) \| a_u^P \| \, du \right]
$$

$$
\leq d L K (\| x \| + 1)^2 \left( C_{K,1} + 1 + C_{K,2} \right) (s^2 + s + s^2 + 1) s^2, \tag{4.48}
$$

where the last steps follows as in (4.46).

For the last term in (4.40), i.e., the integral with respect to the jump measure $\mu$, we consider large and small jumps separately. Therefore, take some $0 < \delta \leq 1$ such that $h(z) = z$ for all $z$ with $\| z \| \leq \delta$. By a Taylor expansion and the Mean-Value Theorem, there exists for each $z \in \mathbb{R}^d$ such that $P$-a.s.,

$$
\int_0^s \int_{\mathbb{R}^d} \left[ \psi(t-u,X_{u-} + z) - \psi(t-u,X_{u-}) - D_x \psi(t-u,X_{u-})^T h(z) \right] \mu(du,dz)
$$

$$
= \int_0^s \int_{\| z \| \geq \delta} \left[ \psi(t-u,X_{u-} + z) - \psi(t-u,X_{u-}) - D_x \psi(t-u,X_{u-})^T h(z) \right] \mu(du,dz)
$$

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For the expectation of the small jumps in (4.49), we have:

\[
\begin{align*}
&+ \int_0^s \int_{||z||<\delta} \left[ \psi(t, x + z) - \psi(t, x) - D_x \psi(t, x)^T z \right] \mu(du, dz) \\
&= \int_0^s \int_{||z|| \geq \delta} \left[ \psi(t - u, X_{u-} + z) - \psi(t - u, X_{u-}) - D_x \psi(t - u, X_{u-})^T h(z) \right] \mu(du, dz) \\
&+ \frac{1}{2} \int_0^s \int_{||z|| \leq \delta} \text{tr} \left[ D_x^2 \psi(t - u, X_{u-} + \zeta z) z z^T \right] \mu(du, dz). 
\end{align*}
\]

(4.49)

Both terms are \( P \)-integrable due to Assumption 4.2. Moreover for the large jumps,

\[
\begin{align*}
&\int_0^s \int_{||z|| \geq \delta} \left[ \psi(t - u, X_{u-} + z) - \psi(t - u, X_{u-}) - D_x \psi(t - u, X_{u-})^T h(z) \right] \mu(du, dz) \\
&= \int_0^s \int_{||z|| \geq \delta} \left[ \psi(t - u, X_{u-} + z) - \psi(t, x + z) + \psi(t, x) - \psi(t - u, X_{u-}) \\
&\quad + \left( D_x \psi(t, x) - D_x \psi(t - u, X_{u-}) \right)^T h(z) \right] \mu(du, dz) \\
&+ \int_0^s \int_{||z|| \geq \delta} \left[ \psi(t, x + z) - \psi(t, x) - D_x \psi(t, x)^T h(z) \right] \mu(du, dz). 
\end{align*}
\]

(4.50)

And by the same arguments as above, we obtain for sufficiently small \( s \) that

\[
\left| E_P \left[ \int_0^s \int_{||z|| \geq \delta} \left[ \psi(t - u, X_{u-} + z) - \psi(t, x + z) + \psi(t, x) - \psi(t - u, X_{u-}) \\
&\quad + \left( D_x \psi(t, x) - D_x \psi(t - u, X_{u-}) \right)^T h(z) \right] \mu(du, dz) \right| \\
\leq E_P \left[ \int_0^s \int_{||z|| \geq \delta} \left[ 2 L (u + ||X_{u-} - x||) + L ||h(z)|| (u + ||X_{u-} - x||) \right] \mu(du, dz) \right] \\
\leq L (2 + C_h) \mathcal{E}_P \left[ \int_0^s (u + ||X_{u-} - x||) \int_{||z|| \geq \delta} ||z||^2 \mu(du, dz) \right] \\
\leq L (2 + C_h) \mathcal{K} (||x||+1)^2 (C_{K,1} + 1 + C_{K,2}) (s^2 + s + s^2 + 1) s^2. 
\]

(4.51)

For the expectation of the small jumps in (4.49), we have:

\[
\begin{align*}
&\left| E_P \left[ \int_0^s \int_{||z|| < \delta} \left[ \psi(t, x + z) - \psi(t, x) - D_x \psi(t, x)^T z \right] \mu(du, dz) \right] \right| \\
&= \frac{1}{2} \left| E_P \left[ \int_0^s \int_{||z|| < \delta} \text{tr} \left[ D_x^2 \psi(t - u, X_{u-} + \zeta z) z z^T \right] \mu(du, dz) \right] \right| \\
&= \frac{1}{2} E_P \left[ \int_0^s \int_{||z|| < \delta} d \left| D_x^2 \psi(t - u, X_{u-} + \zeta z) \right| ||z||^2 \mu(du, dz) \right] \\
&\leq \frac{d}{2} ||D_x^2 \psi||_{\infty} E_P \left[ \int_0^s \int_{||z|| < \delta} ||z||^2 k_u^p(dz) \right] \\
&\leq \frac{d}{2} ||D_x^2 \psi||_{\infty} E_P \left[ \int_0^s \sup_{(\beta, \alpha, \rho) \in \Theta} \int_{||z|| < \delta} ||z||^2 \nu(X_u; dz) \right] 
\end{align*}
\]

(4.52)
\[
\begin{align*}
&\left(\frac{d}{2}\right)\|D^2_x\psi\|_\infty E_P \left[ \int_0^s K_\delta(X_u) \, du \right] \\
&\leq \left(\frac{d}{2}\right)\|D^2_x\psi\|_\infty E_P \left[ \sup_{0 \leq u \leq s} K_\delta(X_u) \right] s,
\end{align*}
\]

where such a \( \zeta \in \mathbb{R}^d \) in (4.52) exists by using a Taylor expansion and \( K_\delta(x) \) is as defined in Assumption 4.2.

Choose \( \bar{C} > 0 \) such that \( \bar{C}\bar{s}^2 \) is equal or greater than the sum of the right-hand side of equations (4.44), (4.46), (4.48) and (4.51). For instance, consider

\[
\bar{C} := L \left( 4 + d + C_h \right) (K + 1) (|x| + 1)^2 (C_{K,1} + 1 + C_{K,2}) (s^2 + s + s^2 + 1)
\]

Then, combining (4.40) with (4.43) - (4.53), yields

\[
E_P [\psi(t - s, X_s) - \psi(t, x)] \\
\leq \bar{C} s^2 + C_{\delta,s} s - \int_0^s \partial_t \psi(t, x) \, du \\
+ E_P \left[ \int_0^s D_x\psi(t, x)^T b_u^P + \int_0^s \partial_x\psi(t, x) a_u^P \right] \, du \\
+ \int_0^s \int_{\|z\| \geq \delta} \left[ \psi(t, x + z) - \psi(t, x) - D_x\psi(t, x)^T h(z) \right] \mu(du, dz) \\
= \bar{C} s^2 + (C_{\delta,s} - \bar{C}\psi(t, x)) s \\
+ E_P \left[ \int_0^s \left\{ D_x\psi(t, x)^T b_u^P + \partial_x\psi(t, x) a_u^P \right\} \right. \\
\left. + \int_{\mathbb{R}^d} \left[ \psi(t, x + z) - \psi(t, x) - D_x\psi(t, x)^T h(z) \right] k_u^P (dz) \right] \, du \\
- E_P \left[ \int_0^s \int_{\|z\| \leq \delta} \left[ \psi(t, x + z) - \psi(t, x) - D_x\psi(t, x)^T h(z) \right] \mu(du, dz) \right] \\
\leq \bar{C} s^2 + (2C_{\delta,s} - \bar{C}\psi(t, x)) s \\
+ E_P \left[ \sup_{(\beta, \alpha, \nu) \in \Theta} \left\{ D_x\psi(t, x)^T \beta(X_u) + \partial_x\psi(t, x) \alpha(X_u) \right\} \\
\left. + \int_{\mathbb{R}^d} \left[ \psi(t, x + z) - \psi(t, x) - D_x\psi(t, x)^T h(z) \right] \nu(X_u; dz) \right\} \, du \right] \\
= \bar{C} s^2 + (2C_{\delta,s} - \bar{C}\psi(t, x)) s + E_P \left[ \int_0^s A_{X_u}\psi(t, x) \, du \right] \\
\leq \bar{C} s^2 + (2C_{\delta,s} - \bar{C}\psi(t, x)) s + s E_P \left[ \sup_{0 \leq u \leq s} A_{X_u}\psi(t, x) \right],
\]

where \( A_p \) is introduced in (4.6). Since \( P \in \mathcal{P}_x(\Theta) \) was arbitrary, we conclude from (4.55) and (4.38), divided by \( s > 0 \), that

\[
0 \leq \bar{C} s^2 + 2C_{\delta,s} - \bar{C}\psi(t, x) + \mathcal{E}^x \left( \sup_{0 \leq u \leq s} A_{X_u}\psi(t, x) \right).
\]
We proceed by studying the convergence of the last term for $s \downarrow 0$. Therefore, note that

$$\left| \mathcal{E}^x \left( \sup_{0 \leq u \leq s} A_{X_u} \psi(t, x) - A_x \psi(t, x) \right) \right| \leq \mathcal{E}^x \left( \left| \sup_{0 \leq u \leq s} A_{X_u} \psi(t, x) - A_x \psi(t, x) \right| \right) \leq \mathcal{E}^x \left( \sup_{0 \leq u \leq s} |A_{X_u} \psi(t, x) - A_x \psi(t, x)| \right).$$  (4.57)

Further, observe that we have $A_{X_u} \psi(t, x) = 0$ on $\{X_u \notin S\}$. On $\{X_u \in S\}$, Corollary 4.3 together with the definition of $A_y$ yields

$$|A_{X_u} \psi(t, x) - A_x \psi(t, x)| \leq \sup_{(\beta, \alpha, \nu) \in \Theta} \left\{ |D_x \psi(t, x)^T (\beta_1, \ldots, \beta_d)(X_u - x)| + \frac{1}{2} \left| \text{tr} \left[ D_x^2 \psi(t, x) (\alpha_1, \ldots, \alpha_d) (X_u - x) \right] \right| + \int_{\|z\| \geq \delta} \left| (\psi(t, x + z) - \psi(t, x) - D_x \psi(t, x)^T h(z) ) \right| (\nu_1(dz), \ldots, \nu_d(dz)) (X_u - x) \right\}$$

$$\leq 3 \mathcal{K} \|D_x \psi\|_\infty \|X_u - x\| + 3 \mathcal{K} \frac{d}{2} \|D_x^2 \psi\|_\infty \|X_u - x\| + \int_{\|z\| \geq \delta} \left[ L \|z\| + \|D_x \psi\|_\infty \|h(z)\| \right] (\nu_1(dz), \ldots, \nu_d(dz)) (X_u - x) \right] + \frac{1}{2} \int_{\|z\| < \delta} \left\{ \frac{d}{2} \|D_x^2 \psi(t, x + \zeta_z z T) \right| (\nu_1(dz), \ldots, \nu_d(dz))(X_u - x)|$$

$$\leq 3 \mathcal{K} \left( \frac{d}{2} \|D_x^2 \psi\|_\infty \|X_u - x\| + \int_{\|z\| \geq \delta} \left[ L \|z\| + \|D_x \psi\|_\infty \|h(z)\| \right] (\nu_1(dz), \ldots, \nu_d(dz)) (X_u - x) \right] + \frac{1}{2} \int_{\|z\| < \delta} \left( \frac{d}{2} \|D_x^2 \psi(t, x + \zeta_z z T) \right| (\nu_1(dz), \ldots, \nu_d(dz))(X_u - x)|$$

$$\leq 3 \mathcal{K} \left( \frac{d}{2} \|D_x^2 \psi\|_\infty \|X_u - x\| + L + \|D_x \psi\|_\infty \right) C_h \|X_u - x\| + 3 \mathcal{K} \frac{d}{2} \|D_x^2 \psi\|_\infty \|X_u - x\|$$

$$\leq \left( L + 2 \|D_x \psi\|_\infty \right) C_h + d \|D_x^2 \psi\|_\infty \mathcal{K} \|X_u - x\|.$$  (4.59)

Thus, we obtain

$$\mathcal{E}^x \left( \sup_{0 \leq u \leq s} |A_{X_u} \psi(t, x) - A_x \psi(t, x)| \right) \leq \mathcal{E}^x \left( \sup_{0 \leq u \leq s} |A_{X_u} \psi(t, x) - A_x \psi(t, x)| \cdot 1_{\{X_u \notin S\}} \right) + \mathcal{E}^x \left( \sup_{0 \leq u \leq s} |A_x \psi(t, x)| \cdot 1_{\{X_u \notin S\}} \right) \leq \mathcal{E}^x \left( \sup_{0 \leq u \leq s} \|X_u - x\| \cdot 1_{\{X_u \notin S\}} \right) + |A_x \psi(t, x)| \mathcal{E}^x \left( \sup_{0 \leq u \leq s} 1_{\{X_u \notin S\}} \right).$$  (4.60)
For estimating the latter expectation, recall that \( x \in \mathcal{S} \) and \( x \notin \partial \mathcal{S} \), i.e., \( \rho := \text{dist}(x, \mathbb{R}^d \setminus \mathcal{S}) > 0 \), and

\[
\rho \mathcal{E}^x \left( \sup_{0 \leq u \leq s} 1_{\{X_u \notin \mathcal{S}\}} \right) \leq \mathcal{E}^x \left( \sup_{0 \leq u \leq s} \|X_u - x\| 1_{\{X_u \notin \mathcal{S}\}} \right).
\]

Combining (4.57) - (4.61), yields

\[
\left| \mathcal{E}^x \left( \sup_{0 \leq u \leq s} \mathcal{A}_{X_u} \psi(t, x) \right) - \mathcal{A}_x \psi(t, x) \right|
\leq C \mathcal{E}^x \left( \sup_{0 \leq u \leq s} \|X_u - x\| 1_{\{X_u \in \mathcal{S}\}} \right) + |\mathcal{A}_x \psi(t, x)| \rho^{-1} \mathcal{E}^x \left( \sup_{0 \leq u \leq s} \|X_u - x\| 1_{\{X_u \notin \mathcal{S}\}} \right)
\leq (C + |\mathcal{A}_x \psi(t, x)| \rho^{-1}) C_{K, 1} (\|x\| + 1) (s + s^{\frac{1}{2}})
\]

for small enough \( s \geq 0 \) due to Corollary 4.5. This establishes the point-wise limit

\[
\lim_{s \downarrow 0} \mathcal{E}^x \left( \sup_{0 \leq u \leq s} \mathcal{A}_{X_u} \psi(t, x) \right) = \mathcal{A}_x \psi(t, x).
\]

Finally, since \( C_{\delta, s} \) is continuous in \( s \), cf. (4.53), first letting \( s \downarrow 0 \) and then \( \delta \downarrow 0 \) in (4.55), we obtain

\[
0 \leq \hat{C} s^{\frac{1}{2}} + 2 C_{\delta, s} - \partial_t \psi(t, x) + \mathcal{E}^x \left( \sup_{0 \leq u \leq s} \mathcal{A}_{X_u} \psi(t, x) \right)
\leq 2 C_{\delta, 0} - \partial_t \psi(t, x) + \mathcal{A}_x \psi(t, x)
\leq -\partial_t \psi(t, x) + \mathcal{A}_x \psi(t, x),
\]

where we used that \( C_{\delta, 0} \) is a multiple of \( K_\delta(x) \), and \( \lim_{\delta \downarrow 0} K_\delta(x) = 0 \) for all \( x \in \mathbb{R}^d \) due to Assumption 4.2. Thus, the value function \( v \) satisfies (4.8).

5 Uniqueness of the Viscosity Solution

We now study uniqueness of viscosity solutions of the PIDE (4.4) - (4.5). Unfortunately, comparison and uniqueness results for viscosity solutions tend to become more involved when considering an arbitrary domain. Even for the choice \( \mathcal{S} = \mathbb{R}^d \), the associated PIDE (4.4) fails to satisfy the necessary assumptions, which allow to apply comparison and uniqueness results, cf. eg. [3], [29, pp. 30 - 31], [17, Chapter 2], [19]. This is due to our definition in (2.6) - (2.8). More precisely, since we do not have that \( \Theta(x) \subseteq \mathbb{R}^d \times S_+ \times \mathcal{L}_+ \) for all \( x \in \mathcal{S} \), the degenerate ellipticity condition is not satisfied. One possible solution would be to restrict the supremum in the definition of \( \mathcal{A}_x \) in (4.6) to parameters \( (\beta, \alpha, \nu) \in \Theta \) such that \( \alpha(y) \in S_+ \) and \( \nu(y) \in \mathcal{L}_+ \). This is in line with the probabilistic reasoning behind the construction since the differential characteristics \( (\beta, \nu, k) \) takes values in \( \mathbb{R}^d \times S_+ \times \mathcal{L}_+ \), but it would cause problems in the first estimation in (4.59). In order to circumvent these issues, we consider the canonical state space \( \mathcal{S} = \mathbb{R}^d \) and an alternative parameter function by following the approach of [14].

For \( (\beta, \alpha, \nu) \in (\mathbb{R}^d)^{d+1} \times S^{d+1} \times \mathcal{L}^{d+1} \), define

\[
\hat{\beta}(x) := \beta_0 + (\beta_1, \ldots, \beta_d) x, \quad (\hat{\beta}(x), \hat{\alpha}(x), \hat{\nu}(x)) := (\beta, \alpha, \nu) \in \Theta \}
\]

for all \( x \in \mathbb{R}^d \), where \( x^+ := x \vee 0 \in \mathbb{R}^d \) coordinate-wise. Analogously, for any subset \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times S^{d+1} \times \mathcal{L}^{d+1} \) and \( x \in \mathbb{R}^d \), define

\[
\hat{\Theta}(x) := \left\{ (\hat{\beta}(x), \hat{\alpha}(x), \hat{\nu}(x)) : (\beta, \alpha, \nu) \in \Theta \right\}.
\]
As in (2.11) we define the family \( \{ \mathcal{P}_x(\hat{\Theta}) \}_{x \in \mathbb{R}} \) by

\[
\mathcal{P}_x(\hat{\Theta}) := \left\{ P \in \mathcal{F}_{ac}\left( \Omega \right) : P(X_0 = x) = 1; (b^p, a^p, k^p) \in \hat{\Theta}(X) \, dt \otimes dP \text{-a.e.} \right\},
\]

and let \( \{ \hat{\mathcal{E}}^x \}_{x \in \mathbb{R}^d} \) denote the associated sublinear expectations.

**Remark 5.1.** Note that the function \( \hat{\nu}(x) \) is constant. This is necessary in order to prove that the corresponding value function is a unique viscosity solution of the associated PIDE.

**Theorem 5.2.** Let \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times S_{d+1} \times S_{d+1} \) satisfy Condition 4.2. Assume that there exists a constant \( \hat{L} > 0 \) such that for all \( x, y \in \mathbb{R}^d \)

\[
\sup_{(\alpha,\beta,\nu) \in \Theta} \left\| \sqrt{\alpha(x)} - \sqrt{\alpha(y)} \right\| \leq \hat{L} \|x - y\|,
\]

where \( \sqrt{\alpha(\cdot)} \) denotes the unique square root of the matrix \( \alpha(\cdot) \). Let \( \phi \in \text{Lip}_b(\mathbb{R}^d) \). Define the value function

\[
\hat{\nu}(t,x) := \hat{\mathcal{E}}^x(\phi(X_t)).
\]

If \( \hat{\nu} \) is continuous, then \( \hat{\nu} \) is the unique viscosity solution of the PIDE

\[
\partial_t v(t,x) - \hat{A}_x v(t,x) = 0
\]

in \( \mathbb{R}_+ \times \mathbb{R}^d \) such that the initial condition (4.5) holds on \( \mathbb{R}^d \), where the operator \( \hat{A}_y \) is given by

\[
\hat{A}_y f(x) := \sup_{(\beta,a,\nu) \in \Theta} \left\{ D_x f(x)^T \hat{\beta}(y) + \frac{1}{2} \text{tr} \left[ D_x^2 f(x) \hat{\alpha}(y) \right] + \int_{\mathbb{R}^d} \left[ f(x + z) - f(x) - D_x f(x)^T h(z) \right] \hat{\nu}(y;dz) \right\},
\]

with \( h(z) := z 1_{\{\|z\| \leq 1\}}(z) \) for all \( z \in \mathbb{R}^d \).

As the proofs in Sections 3 - 4 in general do not depend specifically on the form of the functions \( \beta, \alpha, \nu \), the main results of these two sections carry over to this setting. Thus, the following results hold.

**Lemma 5.3.** Let \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times S_{d+1} \times S_{d+1} \) satisfy Assumption 4.2. Then the family \( \{ \mathcal{P}_x(\hat{\Theta}) \}_{x \in \mathbb{R}^d} \) is amenable to dynamic programming in the sense of Proposition 3.6, and the associated family of sublinear expectations \( \{ \hat{\mathcal{E}}^x \}_{x \in \mathbb{R}^d} \) satisfies the Markov property from Lemma 3.7.

**Proof.** By Condition 4.2, \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times S_{d+1} \times S_{d+1} \) is non-empty and closed. Hence, the set

\[
\left\{ (x,b,a,k) \in \mathbb{R}^d \times \mathbb{R}^d \times S \times \mathcal{L} : (b,a,k) \in \hat{\Theta}(x) \right\}
\]

is Borel. This follows by the same arguments as in the proof of Lemma 3.4 since they solely rely on the separability of the space \( (\mathbb{R}^d)^{d+1} \times S_{d+1} \times S_{d+1} \), cf. Remark 3.5. The dynamic programming principle and the Markov property follow as in the proofs of Proposition 3.6 and Lemma 3.7. \(\square\)

**Lemma 5.4.** Let \( \Theta \subseteq (\mathbb{R}^d)^{d+1} \times S_{d+1} \times S_{d+1} \) satisfy Assumption 4.2 and \( \phi \in \text{Lip}_b(\mathbb{R}^d) \). Assume that the value function \( \hat{\nu} \) from (5.7) is continuous. Then \( \hat{\nu} \) is a viscosity solution of (5.8) in \( \mathbb{R}_+ \times \mathbb{R}^d \), and satisfies the initial condition (4.5) on \( \mathbb{R}^d \).
Proof. Note that in the proofs of Corollaries 4.3, 4.5 and Lemmas 4.4, 4.6 - 4.8, the specific form of the functions $\beta, \alpha, \nu$ is not used. Instead, these results follow from the dynamic programming principle, the sublinear Markov property, and the inequalities in Assumption 4.2.

Since $\Theta \subseteq (\mathbb{R}^d)^{d+1} \times \mathcal{S}^{d+1} \times \mathcal{L}^{d+1}$ satisfies Condition 4.2, we have

$$\lim_{\delta \downarrow 0} \sup_{(\beta, \alpha, \nu) \in \Theta} \int_{\|z\| < \delta} \|z\|^2 \hat{v}(x; dz) \equiv \lim_{\delta \downarrow 0} \sup_{(\beta, \alpha, \nu) \in \Theta} \int_{\|z\| < \delta} \|z\|^2 \nu_0(dz)$$ (5.11)

Moreover,

$$\sup_{x \in \mathbb{R}^d} \|\hat{\beta}(x)\| = \|\beta\|, \quad \sup_{x \in \mathbb{R}^d} \|\hat{\alpha}(x)\| \leq \|\alpha\|, \quad \sup_{x \in \mathbb{R}^d} \|\hat{\nu}(x)\| \leq \|\nu\|$$ (5.13)

for all $(\beta, \alpha, \nu) \in \Theta$. Hence, Corollaries 4.3, 4.5 and Lemmas 4.4, 4.6 - 4.8 are also valid for the family $\{\hat{\xi}_x\}_{x \in \mathbb{R}^d}$ of sublinear expectations and the value function $\hat{v}$.

By careful examining the arguments in the proof of Proposition 4.10 we can see that the specific form of the Hamilton-Jacobi-Bellman equations from [17], see the Appendix.

$$\hat{A}_x \hat{\xi}_x(t, x) - \hat{A}_x \hat{\xi}_x(t, x) = \sum_{(\beta, \alpha, \nu) \in \Theta} \left\{ D_x \hat{\psi}(t, x)^T (\beta_1, \ldots, \beta_d) (X_u - x) + \frac{1}{2} \text{tr} \left[ D_x^2 \hat{\psi}(t, x) (\alpha_1, \ldots, \alpha_d) (X_u^+ - x^+) \right] \right\}$$

$$\leq 3 \mathcal{K} \left( \|D_x \hat{\psi}\|_\infty + \frac{d}{2} \|D_x^2 \hat{\psi}\|_\infty \right) \|X_u - x\|,$$ (5.14)

where we used Assumption (4.2).

Proof of Theorem 5.2. Due to Lemmas 5.3 and 5.4, we are only left with proving the uniqueness of the viscosity solution $\hat{v}$. This in turn follows from Assumption A.3 together with the comparison result for Hamilton-Jacobi-Bellman equations from [17], see the Appendix.

The PIDE (5.8) clearly has the form as in (A.12) since $\hat{\alpha}(x) \in \mathcal{S}_+$ and its unique square root exists. For the boundedness condition on the local part, observe that for every $x \in \mathbb{R}^d$,

$$\sup_{(\beta, \alpha, \nu) \in \Theta} \left\{ \|\hat{\beta}(x)\| + \|\sqrt{\hat{\alpha}(x)}\| \right\} \leq \sup_{(\beta, \alpha, \nu) \in \Theta} \left\{ \|\hat{\beta}(x)\| + \|\hat{\alpha}(x)\| + 1 \right\}$$

$$\leq \mathcal{K} (\|x\| + 1) + 1 < \infty$$ (5.15)

due to (5.13) and Assumption 4.2. Analogously, the non-local part is bounded

$$\sup_{(\beta, \alpha, \nu) \in \Theta} \int_{\mathbb{R}^d} \|z\|^2 \wedge \|\hat{v}(x; dz)\| \equiv \sup_{(\beta, \alpha, \nu) \in \Theta} \int_{\mathbb{R}^d} \|z\|^2 \wedge \|\nu(0, dz) \leq \mathcal{K} < \infty$$ (5.16)

by (5.3). The tightness condition follows from (5.11). By Corollary 4.3 and (5.6),

$$\sup_{(\alpha, \beta, \nu) \in \Theta} \left\{ \|\hat{\beta}(x) - \hat{\beta}(y)\| + \|\sqrt{\hat{\alpha}(x)} - \sqrt{\hat{\alpha}(y)}\| \right\} \leq (3 \mathcal{K} + \hat{L}) \|x - y\|$$ (5.17)

for all $x, y \in \mathbb{R}^d$, which establishes the continuity condition.

Applying Corollary A.4 immediately yields that $\hat{v}$ is the unique viscosity solution of (5.8) in $\mathbb{R}_+ \times \mathbb{R}^d$ such that the initial condition (4.5) holds on $\mathbb{R}^d$. \qed
Remark 5.5. Note that although the results in Sections 3 and 4 do not depend on the specific form of the affine functions $\beta, \alpha, \nu$, the results in Lemmas 2.3, 2.4 and Corollaries 2.5, 2.7 in Section 2 do. By choosing the state space $S = \mathbb{R}^d$ for the non-linear affine process with uncertainty subsets $\{\mathcal{P}_x(\Theta)\}_{x \in S}$, we avoid further technicalities.

6 Examples

In the following, we give an example of a non-linear affine process. For the sake of simplicity, we consider one-dimensional processes with canonical state space $S = \mathbb{R}$ or $S = \mathbb{R}_+$. We focus here on a class of pure jump processes with admissible parameters. For affine diffusion processes we refer to [14] and the examples therein. For the sake of convenience, we state here the definition of an admissible parameter for $d = 1$, cf. [12, Definition 2.6], [21, pp. 19 - 21].

Definition 6.1. A parameter $(\beta_0, \beta_1, \alpha_0, \alpha_1, \nu_0, \nu_1) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{L}^2$ is called admissible for the state space $S = \mathbb{R}$ if

$$\beta_0 \in \mathbb{R}_+, \beta_1 \in \mathbb{R}, \alpha_0 = 0, \alpha_1 \in \mathbb{R}_+, \nu_0, \nu_1 \in \mathbb{L}_+$$

$$\text{supp}(\nu_0), \text{supp}(\nu_1) \subseteq \mathbb{R}_+, \int_0^\infty z \wedge 1 (\nu_0(dz) + \nu_1(dz)) < \infty;$$

and for the state space $S = \mathbb{R}_+$ if

$$\beta_0 \in \mathbb{R}, \beta_1 \in \mathbb{R}, \alpha_0 \in \mathbb{R}_+, \alpha_1 = 0, \nu_0 \in \mathbb{L}_+, \nu_1 = 0.$$ (6.2)

Consider a compound Poisson process given by

$$X_t := \sum_{n=1}^{N_t} Y_n,$$ (6.3)

where $(Y_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. random variables with law $\nu$, and $N = (N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda > 0$ independent of $(Y_n)_{n \in \mathbb{N}}$, cf. [2, p. 449]. For simplicity, assume that $\nu \in \mathbb{L}_+$, i.e., $\nu(\{0\}) = 0$ and $\int(z^2 \wedge 1) d\nu(dz) < \infty$. Fix a truncation function $h : \mathbb{R} \to \mathbb{R}$. Then $X$ admits the canonical decomposition

$$X_t = \lambda t \underbrace{\int_\mathbb{R} h(z) \nu(dz)}_{= \beta_t} + \underbrace{\sum_{n=1}^{N_t} h(Y_n) - \lambda t \int_\mathbb{R} h(z) \nu(dz) + \sum_{n=1}^{N_t} (Y_n - h(Y_n))}_{= \beta_t}.$$ (6.4)

That is, $X$ is a Lévy process and thus an affine process. All parameters are zero except for $\beta_0 = \lambda \int \mathbb{R} h(z) \nu(dz)$ and $\nu_0 = \lambda \nu$. If supp($\nu$) $\subseteq \mathbb{R}_+$ and $\int_0^\infty (z^2 \wedge 1) \nu(dz) < \infty$, the state space of $X$ is $S = \mathbb{R}_+$, or $S = \mathbb{R}$ otherwise. Now, fix some constants $0 \leq \lambda \leq \overline{\lambda}$ and a convex, compact set $\emptyset \neq L \subseteq \mathbb{L}_+$. Set

$$\Theta := \left\{ \left( \lambda \int_\mathbb{R} h(z) \nu(dz), 0, 0, 0, \lambda \nu, 0 \right) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{L}^2 : \lambda \in [\underline{\lambda}, \overline{\lambda}], \nu \in L \right\}$$

$$= \left[ \lambda \min_{\nu \in L} \int_\mathbb{R} h(z) \nu(dz), \lambda \max_{\nu \in L} \int_\mathbb{R} h(z) \nu(dz) \right] \times \{0\} \times \{0\} \times \{0\} \times \lambda^L \times \{0\},$$ (6.5)

where $L^\lambda := \{ \lambda \nu : \lambda \in [\underline{\lambda}, \overline{\lambda}], \nu \in L \}$, which is convex and compact. Since $L$ is compact, the maximum and minimum in (6.5) are attained, and all intermediate values can be obtained choosing suitable $\lambda \nu \in L^\lambda$ due to its convexity. As in the classical setting, we choose $S = \mathbb{R}_+$ if supp($\nu$) $\subseteq \mathbb{R}_+$ and $\int_0^\infty (z^2 \wedge 1) \nu(dz) < \infty$ for all $\nu \in L$, or $S = \mathbb{R}$ otherwise. We call $X$ with associated uncertainty sets $\{\mathcal{P}_x(\Theta)\}_{x \in S}$ a non-linear compound Poisson process with state space $S$. 

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Note that the linear parameters of this non-linear affine processes are zero, i.e., it is a non-linear Lévy process as in [29]. In the classical setting, we can generalise the compound Poisson process to an affine pure jump process by replacing the constant intensity \( \lambda \) by an affine intensity \( \lambda(x) := \lambda_0 + \lambda_1 x > 0 \), cf. [13, p. 1349]. That is, for the state space \( S = \mathbb{R}_+ \), we can consider

\[
\Theta := \left\{ \left( \lambda_0 t \int_R h(z) \nu(dz), \lambda_1 t \int_R h(z) \nu(dz), 0, 0, \lambda_0, \lambda_1 \right) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{L}^2 : \right.

\lambda_0 \in [\lambda_0, \lambda_0], \lambda_1 \in [\lambda_1, \lambda_1], \nu \in L, \left. \right\}, \tag{6.6}
\]

where \( 0 \leq \lambda_0 \leq \lambda_0, 0 \leq \lambda_1 \leq \lambda_1 \) are some constants, and \( \emptyset \neq L \subseteq \mathcal{L}_+ \) is convex and compact with \( \text{supp}(\nu) \subseteq \mathbb{R}_+ \) and \( \int_0^\infty (z \wedge 1) \nu(dz) < \infty \) for all \( \nu \in L \). We call \( X \) with associated uncertainty sets \( \{ \mathcal{P}_x(\Theta) \}_{x \in S} \) a non-linear generalised compound Poisson process with state space \( S = \mathbb{R}_+ \).

Clearly, \( \Theta \) in (6.5), respectively in (6.6), is closed and non-empty, such that the dynamic programing principle holds. Further, note that for \( \Theta \) in (6.5), respectively (6.6), the uncertainty sets are non-empty, i.e., \( \mathcal{P}_x(\Theta) \neq \emptyset \) for all \( x \in \mathbb{R} \) due to the admissibility of the parameters, cf. Remark 2.8. Moreover, it is obvious that for the non-linear compound Poisson process the definition of \( \Theta(x) \) in (2.10) and the one of \( \Theta(x) \) in (5.4) coincide. Thus, by Theorem 5.2 the value function \( v(t, x) := \mathcal{E}^x(\varphi(X_1)) \) is the unique viscosity solution of the corresponding PIDE (4.4)-(4.5). Unfortunately, for the non-linear generalised compound Poisson process we only get an existence result for the solution of the associated PIDE.

**Remark 6.2.** Note that in Sections 2 - 4 we have not required \( \Theta \) to be a set of admissible parameters but allowed it to be an arbitrary non-empty, closed subset of \((\mathbb{R}^d)^{d+1} \times \mathbb{S}^{d+1} \times \mathcal{L}^{d+1}\). In fact, it is possible to choose \( S \) and \( \Theta \) such that no \( \theta \in \Theta \) is admissible for the state space \( S \) but \( \mathcal{P}_x(\Theta) \) is non-empty for all \( x \in \mathbb{R}^d \). This is due to our definition of \( \mathcal{P}_x(\Theta) \) in (2.11). More precisely, allow the differential characteristics to evolve in \( \Theta(X) \), i.e., we ask for

\[
(b^P, a^P, k^P) \in \Theta(X) \quad dt \otimes dP\text{-a.e.} \tag{6.7}
\]

rather than to require that

\[
\exists \theta \in \Theta : \quad (b^P, a^P, k^P) = \theta(X) \quad dt \otimes dP\text{-a.e.} \tag{6.8}
\]

Assumption (6.8) corresponds to parameter uncertainty in the narrow sense and to the set \( \hat{\mathcal{P}}_x(\Theta) \) as defined in (2.16). The set \( \hat{\mathcal{P}}_x(\Theta) \) being non-empty is equivalent to \( \Theta \) containing some admissible parameters. More precisely, [12, Theorem 2.7] yields that for every \( P \in \hat{\mathcal{P}}_x(\Theta) \) the process \( X \) is affine with admissible parameters. On the contrary, if \( \theta \in \Theta \) is admissible, then there exists a \( P \in \mathcal{P}_x(\Theta) \) such that \( X \) is affine with parameters \( \theta \). This approach is especially suitable to take into account statistical uncertainty for parameter estimation of a stochastic phenomenon (e.g. short rates, volatility or credit default) by considering confidence intervals instead of one single estimator. On the other hand, our construction is more general and allows for more flexibility.

### Appendix

In this Appendix we summarize some important results we used in the previous sections. First, we state the Burkholder-Davis-Gundy inequality for \( \mathbb{R}^d \)-valued càdlàg local martingales. Second, we summarize some properties of truncation functions and conclude with a comparison result for PIDEs.

**Lemma A.1.** Let \( p \geq 1 \). There exists a constant \( 0 < C_{p,d} < \infty \) such that for all \( \mathbb{R}^d \)-valued càdlàg local martingales \( M \) and \( t \geq 0 \),

\[
E \left[ \sup_{0 \leq s \leq t} \|M_s - M_0\|^p \right] \leq C_{p,d} E \left[ \|M_t\|^p \right]. \tag{A.1}
\]
Proof. This follows from the Burkholder-Davis-Gundy inequality for real-valued local martingales, cf. e.g. [32, Theorem IV.48]. Let $C_p$ denote the constant for the one-dimensional case, and $M = (M^1, \ldots, M^d)^T$ be an $\mathbb{R}^d$-valued càdlàg local martingale. By the convexity of $x \mapsto x^p$ on $\mathbb{R}_+$ and the triangle inequality, we obtain that

$$E \left[ \sup_{0 \leq s \leq t} \|M_s\|^p \right] \leq E \left[ \left( \sum_{i=1}^d \sup_{0 \leq s \leq t} |M^i_s| \right)^p \right]$$

$$\leq d^{p-1} \sum_{i=1}^d C_p E \left[ (|M^i_t|^2)^{\frac{p}{2}} \right]$$

$$\leq d^p C_p \max_{1 \leq i \leq d} E \left[ (|M^i_t|^4)^{\frac{p}{2}} \right]$$

$$= d^p C_p \max_{1 \leq i \leq d} E \left[ (e_i^T [M]_t e_i)^2 \right]$$

$$\leq d^p C_p E \left[ \| [M]_t \|^2 \right], \quad (A.2)$$

where $e_i$ denotes the $i$-th canonical basis vector. \hfill \square

Corollary A.2. Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a truncation function. Then there exists a constant $C_h > 0$ such that,

$$\|z - h(z)\| \leq C_h (\|z\|^2 \wedge \|z\|) \quad \text{and} \quad \|h(z)\| \leq C_h (\|z\| 1) \quad \text{for all} \ z \in \mathbb{R}^d; \quad (A.3)$$

$$\|z\| \leq C_h (\|z\|^2 \wedge \|z\|) \quad \text{outside some neighbourhood of zero}. \quad (A.4)$$

Proof. By the definition of $h$, there exists a $0 < \delta \leq 1$ such that $h(z) = z$ for all $z$ with $\|z\| \leq \delta$. Fix $\delta > 0$. Then clearly $\delta \leq \|h\|_{\infty}$, and

$$\|z - h(z)\| = \|z - h(z)\| 1_{\{\|z\| \geq \delta\}}(z) \leq \|z\| 1_{\{\|z\| \geq \delta\}}(z) + \|h\|_{\infty} 1_{\{\|z\| \geq \delta\}}(z),$$

$$\|h(z)\| \leq \|z\| 1_{\{\|z\| < \delta\}}(z) + \|h\|_{\infty} 1_{\{\|z\| \geq \delta\}}(z). \quad (A.5)$$

Further, observe that

$$\|z\| 1_{\{\|z\| \leq \delta\}}(z) \leq (\|z\| 1) \frac{\|h\|_{\infty}}{\delta} \frac{1}{\delta} 1_{\{\|z\| \leq \delta\}}(z) = (\|z\| 1) \delta^{-2} \|h\|_{\infty} 1_{\{\|z\| \leq \delta\}}(z), \quad (A.7)$$

$$\|z\| 1_{\{\|z\| \geq \delta\}}(z) \leq \|z\| \frac{\|z\|^2}{\delta} 1_{\{\|z\| \geq \delta\}}(z) = (\|z\|^2 \wedge \|z\|) \delta^{-2} \|h\|_{\infty} 1_{\{\|z\| > \delta\}}(z), \quad (A.8)$$

$$\|h\|_{\infty} 1_{\{\|z\| \geq \delta\}}(z) \leq \|z\| \frac{\|z\|^2}{\delta} \|h\|_{\infty} 1_{\{\|z\| \geq \delta\}}(z) \leq \left( \frac{\|z\|^2 \wedge \|z\|}{\delta} \right) \delta^{-2} \|h\|_{\infty} 1_{\{\|z\| > \delta\}}(z), \quad (A.9)$$

where we used $\frac{\|z\|}{\delta}, \frac{1}{\delta} \geq 1$ in the last step. Hence, $C_h = \delta^{-2} \|h\|_{\infty}$ is a possible choice. \hfill \square

We now state a comparison principle for Hamilton-Jacobi-Bellman equations from [17] in a simplified way which is suitable for our setting. Let $G : \mathbb{R}^d \times \mathbb{R}^d \times S \times C_b^2(\mathbb{R}^d) \to \mathbb{R}$ be some operator. Set $h(z) :=$
$z 1_{\{\|z\| \leq 1\}}(z)$ for all $z \in \mathbb{R}^d$. Suppose that there exist an index set $\Theta$ and a family $\{G_\theta\}_{\theta \in \Theta}$ of linear operators $G_\theta : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times C_b^2(\mathbb{R}^d) \to \mathbb{R}$ given by

$$G_\theta(x, p, X, f) = -p^T b_\theta(x) - \text{tr} \left[ X a_\theta(x) \right] - \int_{\mathbb{R}^d} \left[ f(x + z) - f(x) - D_x f(x)^T h(z) \right] k_\theta(dz) \quad (A.10)$$

for some Borel-measurable functions $b_\theta, a_\theta, k_\theta : \mathbb{R}^d \to \mathbb{R}^d, \mathbb{S}, \mathcal{L}_+$ satisfying Condition A.3, such that

$$G = \inf_{\theta \in \Theta} G_\theta \quad (A.11)$$
on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times C_b^2(\mathbb{R}^d)$. We need a preliminary condition.

**Assumption A.3.** Let $\{G_\theta\}_{\theta \in \Theta}$ be a family of operators as defined in (A.10) with coefficient functions $b_\theta, a_\theta, k_\theta$ on $\mathbb{R}^d$. Set $\sigma_\theta(x) := \sqrt{a_\theta(x)}$ for all $x \in \mathbb{R}^d$.

(i) **Boundedness:** The coefficients of the local part are bounded, i.e.,

$$\sup_{\theta \in \Theta} \left\{ \|b_\theta(x)\| + \|\sigma_\theta(x)\| \right\} < \infty$$
in every $x \in \mathbb{R}^d$, and the family of measures of the nonlocal part satisfy

$$\sup_{\theta \in \Theta} \int_{\mathbb{R}^d} \left( \|z\|^2 1_{\{\|z\| \leq 1\}}(z) + 1_{\{\|z\| > 1\}}(z) \right) k_\theta(dz) < \infty$$
a uniform integrability condition at zero and infinity.

(ii) ** Tightness:** The family of measures of the nonlocal part satisfies

$$\lim_{\delta \to 0} \sup_{\theta \in \Theta} \int_{\|z\| \leq \delta} \|z\|^2 k_\theta(dz) = 0 = \lim_{R \to \infty} \sup_{\theta \in \Theta} \int_{R < \|z\|} 1 k_\theta(dz)$$
a uniform tightness condition at the origin and infinity.

(iii) **Continuity:** There exists some constant $C > 0$ such that

$$\sup_{\theta \in \Theta} \left\{ \|b_\theta(x) - b_\theta(y)\| + \|\sigma_\theta(x) - \sigma_\theta(y)\| \right\} \leq C \|x - y\|$$
for all $x, y \in \mathbb{R}^d$.

**Proposition A.4.** Let $G$ be an operator satisfying (A.11) for linear operators $\{G_\theta\}_{\theta \in \Theta}$ given in (A.10). Let $v_1, v_2 : \mathbb{R}_+ \times \mathbb{R}^d$ be bounded viscosity sub- and supersolutions respectively of

$$\partial_t v(t, x) + G(x, D_x v(t, x), D_x^2 v(t, x), v(t, \cdot)) = 0 \quad (A.12)$$
in $\mathbb{R}_+ \times \mathbb{R}^d$. If the initial values $v_1(0, \cdot), v_2(0, \cdot)$ are continuous and

$$v_1(0, x) \leq v_2(0, x)$$

for all $x \in \mathbb{R}^d$, then $v_1(t, x) \leq v_2(t, x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

**Proof of Proposition A.4.** This follows immediately from [17, Corollary 2.34].

\[\square\]

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