ON APPELL SEQUENCES OF POLYNOMIALS OF BERNOULLI
AND EULER TYPE

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ABSTRACT. A construction of new sequences of generalized Bernoulli polynomials of first and second kind is proposed. These sequences share with the classical Bernoulli polynomials many algebraic and number-theoretical properties. A new class of Euler-type polynomials is also presented.

Contents

1. Introduction 1
2. Finite operator calculus 4
3. Difference delta operators 7
4. A generalization of the Stirling numbers 8
5. Bernoulli–type polynomials and numbers of the first kind 10
6. New Bernoulli–type polynomials of the second kind 14
7. New Euler–type polynomials 16
8. References 18

1. Introduction

In this article several new classes of polynomials of Appell type, which generalize the classical Bernoulli and Euler polynomials, will be introduced and discussed.

We recall that the Bernoulli polynomials [6] are defined by the generating function

\[
\frac{t}{e^t - 1} e^{xt} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k.
\]

For \(x = 0\), formula (1.1) reduces to the generating function of the Bernoulli numbers:

\[
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k.
\]

The Bernoulli polynomials are also determined by the two properties

\[
DB_n(x) = nB_{n-1}(x),
\]

\[
\Delta B_n(x) = nx^{n-1},
\]
with the condition \( B_0(x) = 1 \). Here \( D \) is the continuous derivative, \( \Delta := (T - 1) \) is a discrete derivative, and \( T \) is the shift operator, defined by \( Tf(x) = f(x + 1) \). Standard references are [14] and [37] and the books [17] and [23]. By virtue of relation (1.3), the Bernoulli polynomials belong to the class of Appell polynomials.

The Bernoulli polynomials play an important rôle in several branches of mathematical analysis, such as the theory of distributions in \( p \)-adic analysis [25], the theory of modular forms [26], the study of polynomial expansions of analytic functions [7], etc. Formula (1.4) is the basis for the application of these polynomials to interpolation theory [32]. Several generalizations of Bernoulli polynomials have also been proposed. Particularly important is that of Leopold and Iwasawa, motivated by a connection with the theory of \( p \)-adic \( L \)-functions [24]. Another example is provided by the work of Carlitz [9]. Recently, new applications of the Bernoulli polynomials have also been found in mathematical physics, in connection with the theory of the Korteweg–de Vries equation [18] and Lamé equation [20], and in the study of vertex algebras [13].

The Bernoulli numbers are relevant in several branches of number theory, in particular in the computation of special values of zeta functions (see, e.g., [16], [44]), in the theory of cyclotomic fields [42] and, since Kummer’s work, in connection with Fermat’s last theorem [23]. They are also useful in singularity theory [2] and in the study of Coxeter groups [3]. Standard applications to algebraic topology are found in the computation of Todd characteristic classes and in the Hirzebruch signature theorem, as well as, more recently, in complex homology theory [5], [33]. In the last years, Bernoulli number identities have found applications in quantum field theory [15] and in the computation of Gromov–Witten invariants [19].

In the papers [39], [40] the following generalization of the Bernoulli polynomials was introduced.

**Definition 1.** Let us consider the polynomial ring \( \mathbb{Q}[c_1, c_2, \ldots] \) and the formal power series

\[
F(s) = s + c_1 s^2 / 2 + c_2 s^3 / 3 + \ldots
\]

Let \( G(t) \) be the compositional inverse series:

\[
G(t) = t - c_1 t^2 / 2 + (3c_1^2 - 2c_2) t^3 / 6 + \ldots
\]

so that \( F(G(t)) = t \). The universal higher–order Bernoulli polynomials

\[
B_{k,a}^G(x, c_1, \ldots, c_n) \equiv B_{k,a}^G(x)
\]

are defined by

\[
\left( \frac{t}{G(t)} \right)^a e^{xt} = \sum_{k \geq 0} B_{k,a}^G(x) \frac{t^k}{k!},
\]

where \( a \neq 0 \).

This definition is clearly motivated by the works by Clarke [12], Ray [33] and Adelberg [1] on universal Bernoulli numbers. We observe that many known polynomial sequences can be obtained by suitable choices of the coefficients appearing in the generating function (1.7). For instance, the case of the standard Bernoulli polynomials corresponds to the choice \( a = 1 \), \( c_i = (-1)^i \), since now \( F(s) = \log(1 + s) \) and \( G(t) = e^t - 1 \). When \( a \) is a rational integer, \( c_i = (-1)^i \) we obtain the higher–order Bernoulli polynomials, which have also been extensively studied.
ON APPELL SEQUENCES OF POLYNOMIALS OF BERNOULLI AND EULER TYPE 3

(e.g. in [10], [11], [22], [35]). When \( x = 0, a \in \mathbb{R} \), formula (1.7) can be reduced to the generating function of the Nörlund polynomials and their generalizations.

By construction, the polynomials \( B_{k,1}^U (0) \equiv \tilde{B}_k \in \mathbb{Q} [c_1, \ldots, c_n] \), where \( \tilde{B}_k \) are the universal Bernoulli numbers introduced by Clarke in [12]. The connection with algebraic topology relies on the fact that \( G (F (s_1) + F (s_2)) \) represents the universal formal group [21]; the series \( F \) is called the formal group logarithm and \( G \) the formal group exponential. The universal formal group is defined over the Lazard ring \( L \), which is the subring of \( \mathbb{Q} [c_1, \ldots, c_n] \) generated by the coefficients of the power series \( G (F (s_1) + F (s_2)) \). In algebraic topology, in particular in complex cobordism theory, the coefficients \( c_n \) are identified with the cobordism classes of \( \mathbb{C}P^n \) (see [4], [31] and [33]). Generalizations of famous congruences valid for the classical Bernoulli numbers, like the celebrated Kummer and Clausen–von Staudt congruences [1], [23], are satisfied by these universal numbers.

The universal polynomials (1.7) possess as well several interesting general properties studied in [39], [40].

The first aim of this paper is to provide nontrivial realizations of polynomial sequences of the type (1.7). In particular, we will focus on a specific class of Bernoulli–type polynomials, constructed using the finite operator calculus, as formulated by G. C. Rota and S. Roman [35], [36]. For a recent review of the vast literature existing on this approach (also known, in its earlier formulations, as the Umbral Calculus) see, e.g., [8]. In other words, the Bernoulli–type polynomials of order \( p \) considered here correspond to a class of formal power series \( G (t) \) representing suitable difference delta operators of order \( p \) (denoted by \( \Delta_p \)). These delta operators have been introduced in [28], [29], where a version of Rota’s operator approach, based on the theory of representations of the Heisenberg–Weyl algebra, has been outlined. In this paper we wish to further illustrate the connection between Rota’s approach and the theory of Appell and Sheffer polynomials. The basic sequences associated with the delta operators \( \Delta_p \) are here studied in detail and some of their combinatorial properties derived. In addition, their connection constants with the basic sequence \( \{ x^n \} \) (and the associated inverse relations) define generalized Stirling numbers of the first and second kind. The problem of classifying all Sheffer sequences associated with the delta operators considered in this paper is essentially open.

By analogy with the classical case, the Bernoulli–type polynomials, parametrized by a real variable \( a \) introduced in this work are determined by the relations

\[
(1.8) \quad D B_{n,a}^p (x) = n B_{n-1,a}^p (x),
\]

\[
(1.9) \quad \Delta_a^p B_{n,a}^p (x) = n x^{n-1},
\]

and by the condition \( B_{0,a}^p (x) = 1 \); here \( \Delta_a^p \) is a suitable operator (defined in Sec. 5.2). For \( a = 1, p = 1 \), the usual characterization of the classical Bernoulli polynomials is recovered.

The use of the Roman–Rota formalism has the advantage of providing us with a natural language which allows to unify the mathematical treatment of many polynomial sequences.

The second aim of this paper is to introduce the notions of Bernoulli polynomials of the second kind and of Euler–type polynomials.
All these polynomials can also play a role in discrete mathematics. Indeed, they satisfy certain interesting linear difference equations of order \( p \) in one variable, defined in a two-dimensional space of parameters.

The future research plans on generalized Bernoulli polynomials include possible applications in analytic number theory, in particular to the theory of Dirichlet \( L \)-series and Riemann–Hurwitz zeta function. Results in this direction have been obtained in [39] and [40], where a new construction relating generalized Bernoulli polynomials, formal groups and \( L \)-series is proposed.

The paper is organized as follows. In Section 2, a brief introduction to finite operator calculus is presented, with a discussion in Section 3 of difference delta operators. In Section 4, generalized Stirling numbers are considered. In Section 5, new Bernoulli–type polynomials of the first kind are introduced. The Bernoulli–type polynomials of the second kind are introduced and discussed in Section 6. A generalization of Euler polynomials is discussed in the final Section 7.

2. Finite operator calculus

In this section, some basic results concerning the theory of finite difference operators and their relation with polynomial sequences of Appell and Sheffer type are reviewed. For further details and proofs, see the monographs [35], [36], where an extensive and modern treatment of this topic is proposed.

Let \( \mathcal{F} \) denote the algebra of formal power series in one variable \( t \), endowed with the operations of sum and multiplication of series. Let \( \mathcal{P} \) be the algebra of polynomials in one variable \( x \) and \( \mathcal{P}^* \) the vector space of all linear functionals on \( \mathcal{P} \). If \( L \in \mathcal{P}^* \), following Dirac we will denote the action of \( L \) on \( p(x) \in \mathcal{P} \) by \( \langle L \mid p(x) \rangle \).

A remarkable fact of the finite operator calculus is that any element of \( \mathcal{F} \) can play a threefold rôle: It can be regarded as a formal power series, as a linear functional on \( \mathcal{P} \) and also as a linear operator on \( \mathcal{P} \). To prove this, let us first notice that, given a formal power series

\[
f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k,
\]

we can associate it with a linear functional via the correspondence

\[
\langle f(t) \mid x^n \rangle = a_n.
\]

Therefore

\[
f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} t^k.
\]

In particular, since

\[
\langle t^k \mid x^n \rangle = n! \delta_{n,k},
\]

it follows that, for any polynomial \( p(x) \),

\[
\langle t^k \mid p(x) \rangle = p^{(k)}(0),
\]

where \( p^{(k)}(0) \) denotes the \( k \)-derivative of \( p(x) \) evaluated at \( x = 0 \). It is easily shown that any linear functional \( L \in \mathcal{P}^* \) is of the form (2.3).
If we interpret $t^k$ as the $k$-th order derivative operator on $P$, given a polynomial $p(x)$ we have that $t^k p(x) = p^{(k)}(x)$. Hence the formal series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!}$$

is also regarded as a linear operator acting on $P$. Depending on the context, $t$ will play the rôle of a formal variable or that of a derivative operator.

Now, some basic definitions and theorems of finite operator theory, necessary in the subsequent considerations, are in order.

**Definition 2.** An operator $S$ commuting with the shift-operator $T$, i.e. $[S,T] = 0$, is said to be shift-invariant.

Relevant examples of operators belonging to this class are provided by the delta operators.

**Definition 3.** A delta operator $Q$ is a shift-invariant operator such that $Qx = \text{const} \neq 0$.

As has been proved in [36], there is an isomorphism between the ring of formal power series in a variable $t$ and the ring of shift-invariant operators, carrying

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \text{ into } \sum_{k=0}^{\infty} \frac{a_k Q^k}{k!}.$$  

In the following let us denote by $p_n(x)$, $n = 0, 1, 2, \ldots$, a sequence of polynomials in $x$, where $p_i(x)$ is of order $i$ for all $i$.

**Definition 4.** A polynomial sequence $p_n(x)$ is called a sequence of basic polynomials for a delta operator $Q$ if it satisfies the following conditions:

1) $p_0(x) = 1$;
2) $p_n(0) = 0$ for all $n > 0$;
3) $Qp_n(x) = np_{n-1}(x)$.

For each delta operator there exists a unique sequence of associated basic polynomials.

**Definition 5.** A polynomial sequence $s_n(x)$ is called a set of Sheffer polynomials for the delta operator $Q$ if

1) $s_0(x) = c \neq 0$;
2) $Qs_n(x) = ns_{n-1}(x)$.

**Definition 6.** An Appell sequence of polynomials is a Sheffer sequence for the delta operator $D$.

Any shift invariant operator $S$ can be expanded into a formal power series in terms of a delta operator $Q$:

$$S = \sum_{k=0}^{\infty} \frac{a_k}{k!} Q^k,$$

with $a_k = [Sp_k(x)]_{x=0}$, where $p_k$ is the basic polynomial of order $k$ associated with $Q$. By using the isomorphism (2.7), a formal power series $s(t)$ is defined, which is called the indicator of $S$. 

Remark 1. A shift invariant operator is a delta operator if and only if it corresponds, under the isomorphism (2.7), to a formal power series \( G(t) \) such that \( G(0) = 0 \) and \( G'(0) \neq 0 \). This series admits a unique compositional inverse.

The umbral formalism also allows us to characterize the generating functions of polynomial sequences of Sheffer and Appell type.

If \( s_n(x) \) is a Sheffer sequence for the operator \( Q \), then there exists an invertible shift–invariant operator \( S \) such that

\[
S \ s_n(x) = q_n(x),
\]

where \( q_n(x) \) is a basic set for \( Q \).

Let us denote by \( s(t) \) and \( q(t) \) the indicators of the operators \( S \) and \( Q \). We will say that \( s_n(x) \) is the Sheffer sequence associated with \((s(t), q(t))\). The following result holds:

\[
(2.8) \quad \frac{1}{s(q^{-1}(t))} e^{xq^{-1}(t)} = \sum_{n \geq 0} \frac{s_n(x)}{n!} t^n,
\]

where \( q^{-1}(t) \) denotes the compositional inverse of \( q(t) \).

Another characterization of a Sheffer sequence is provided by the identity

\[
(2.11) \quad s_n(x + y) = \sum_{k \geq 0} \binom{n}{k} q_k(y) s_{n-k}(x).
\]

In the specific case of the Appell sequences, which obey the equation

\[
(2.9) \quad D \ s_n(x) = n s_{n-1}(x),
\]

there exists an invertible operator \( g(t) \) such that

\[
(2.10) \quad g(t) \ s_n(x) = x^n.
\]

Therefore, we will say that \( s_n(x) \) is the Appell sequence associated with \( g(t) \). Its generating function is defined by

\[
(2.11) \quad \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}.
\]

The Appell identity is

\[
(2.12) \quad s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} q_k(y) x^{n-k}.
\]

We also recall the polynomial expansion theorem. If \( s_n(x) \) is a Sheffer set for the operators \((s(t), q(t))\), then for any polynomial \( p(x) \) we have:

\[
(2.13) \quad p(x) = \sum_{k \geq 0} \frac{\langle s(t)q(t)^k \mid p(x) \rangle}{k!} s_k(x).
\]

In particular, if \( q_n(x) \) is a basic sequence for \( q(t) \), this formula reduces to

\[
(2.14) \quad p(x) = \sum_{k \geq 0} \frac{\langle q(t)^k \mid p(x) \rangle}{k!} q_k(x).
\]
3. Difference delta operators

In this paper, we will consider a specific class of difference operators, introduced in [28], having the general form

\[ Q \equiv \Delta_p = \frac{1}{\sigma} \sum_{k=l}^{m} a_k T^k, \quad l, m \in \mathbb{Z}, \quad l < m, \quad m - l = p, \]

where \( a_k \) and \( \sigma \) are constants. In order to satisfy the definition of delta operator, we must assume that

\[ \sum_{k=l}^{m} a_k = 0, \quad \sum_{k=l}^{m} k a_k = c. \]

We also require that in the continuous limit analogue, \( \Delta_p \) reproduces the standard derivative \( D \); this implies \( c = 1 \), i.e.

\[ \sum_{k=l}^{m} k a_k = 1. \]

**Definition 7.** A difference operator of the form (3.1) which satisfies Eqs. (3.2) and (3.3) is called a delta operator of order \( p = m - l \).

We observe that Eq. (3.1) involves \( m - l + 1 \) constants \( a_k \), subject to conditions (3.2) and (3.3). To fix all constants \( a_k \) we have to impose \( m - l - 1 \) further conditions. A possible choice is, for instance,

\[ \gamma_p := \sum_{k=l}^{m} k^p a_k = 0, \quad p = 2, 3, ..., m - l. \]

When conditions (3.2), (3.3) and (3.4) are satisfied, the operator (3.1) provides an approximation of order \( p \) of the continuous derivative \( D \), since

\[ \Delta_p f \sim f'(x) + \frac{\sigma^{m-l}}{(m-l+1)!} f^{(m-l-1)}(x) \sum_{k=l}^{m} a_k k^{m-l-1}. \]

From now on, we will put \( \sigma = 1 \). The Pincherle derivative of a delta operator is defined by the relation

\[ Q' = [Q, x]. \]

Now, let us introduce a shift–invariant operator \( \beta \) such that

\[ [\Delta_n, x\beta] = 1. \]

It follows that \( \beta = (\Delta_n')^{-1} \) (see [27]–[29]). Such an operator is invertible [36] and finite. Indeed, using the identity \( \beta \beta^{-1} = 1 \), the action of \( \beta \) on a monomial of order \( n \) is easily seen to be

\[ \beta x^n = x^n - \sum_{j=0}^{n-1} \alpha_j^n x^{n-1-j}, \]

where \( \alpha_j^n \) are defined via the recursion relation

\[ \alpha_j^n = \binom{n}{j+1} \gamma_{j+2} - \sum_{l=0}^{j-1} \binom{n}{l+1} \gamma_{l+2} \alpha_{j-l-1}^{n-l-1}, \]
with $\gamma_j = \sum_{k=1}^{m} k^j a_k$. We see that $\beta$ preserves polynomial structures. When $\Delta = D$, $\beta = 1$. Other specific cases are listed below: for
\begin{equation}
\Delta = \Delta^+ = T - 1, \quad \beta = T^{-1};
\end{equation}
for
\begin{equation}
\Delta = \Delta^- = 1 - T^{-1}, \quad \beta = T.
\end{equation}
When
\begin{equation}
\Delta_2 = \Delta^s = \frac{T - T^{-1}}{2}, \quad \beta = \left(\frac{T + T^{-1}}{2}\right)^{-1}.
\end{equation}
Other nontrivial examples of higher–order operators are provided by
\begin{equation}
\Delta_3 = -(T^2 - 2T + T^{-1}),
\end{equation}
\begin{equation}
\Delta_4 = T^2 - \frac{3}{2}T + \frac{3}{2}T^{-1} - T^{-2},
\end{equation}
\begin{equation}
\Delta_5 = T^3 - 2T^2 + 2T - 2T^{-1} + T^{-2},
\end{equation}
\begin{equation}
\Delta_7 = T^4 - T^3 + T^2 - 2T + T^{-1} - T^{-2} + T^{-3}.
\end{equation}

Finally, we recall that $Q$ is a delta operator if and only if is of the form $Q = DP$, where $P$ is an invertible, shift–invariant operator [36]. It follows that the sequence of basic polynomials for $Q$ is expressible in the form
\begin{equation}
p_n(x) = xP^{-n}x^{n-1}.
\end{equation}

4. A generalization of the Stirling numbers

According to Rota’s approach, given any delta operator of the form (3.1)–(3.3) it is possible to associate it with a sequence of basic polynomials. In perfect analogy to the classical theory of Stirling numbers, this allows us to study generalizations of Stirling numbers. In this section, several concrete examples of basic polynomials as well as generalized Stirling numbers are obtained. This construction represents a natural realization of the very general scheme proposed by Ray in [33], who first studied the basic sequences and defined the universal Stirling numbers for any delta operator constructed via the complex homology theory. Here we will follow a different but equivalent philosophy: the basic sequences for the operators (3.1)–(3.3) are simply derived from the “discrete” representations of the Heisenberg–Weyl algebra; the associated Stirling numbers furnish the connection coefficients between the basic sequences and the standard power sequence $\{x^n\}_{n \in \mathbb{N}}$.

We recall that the classical Stirling numbers of the first and second kind $s(n, k)$ and $S(n, k)$ are defined by the relations (see, e.g., [34])
\begin{equation}
(s)_{n} = \sum_{k=0}^{n} s(n, k) x^k
\end{equation}
and
\begin{equation}
x^n = \sum_{k=0}^{n} S(n, k) (x)_k,
\end{equation}
with $\gamma_j = \sum_{k=1}^{m} k^j a_k$. We see that $\beta$ preserves polynomial structures. When $\Delta = D$, $\beta = 1$. Other specific cases are listed below: for
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\end{equation}
and
\begin{equation}
x^n = \sum_{k=0}^{n} S(n, k) (x)_k,
\end{equation}
respectively. Here \((x)_n = x(x-1) \ldots (x-n+1)\) denotes the lower factorial polynomial of order \(n\). The numbers \(S(n,k)\) also admit a representation in terms of a generating function:

\[
\sum_{n=k}^{\infty} S(n,k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}.
\]

As is well known, the lower factorial polynomials are the basic sequence associated with the forward derivative: \(\Delta (x)_n = n (x)_{n-1}\). More generally, the basic sequence associated with the operator \(\Delta_p\), denoted by \((x)_n^p\), is \([28]\)

\[
(x)_n^p = (x_\beta)^n 1. \tag{4.4}
\]

Therefore,

\[
\Delta_p(x)_n^p = n(x)_n^p - 1. \tag{4.5}
\]

Explicitly, the first basic polynomials are expressed by

\[
(x)_n^0 = (x_\beta)^0 1 = 1, \\
(x)_n^1 = (x_\beta)^1 1 = x, \\
(x)_n^2 = (x_\beta)^2 1 = x^2 - \gamma_2 x, \\
(x)_n^3 = (x_\beta)^3 1 = x^3 - 3 \gamma_2 x^2 - (\gamma_3 - 3 \gamma_2^2) x,
\]

and so on, with \(\gamma_j\) defined by

\[
\gamma_j = \sum_{k=1}^{m} a_k j^j, \quad \gamma_0 = 0, \quad \gamma_1 = 1, \quad j = 0, 1, 2, \ldots \tag{4.6}
\]

**Definition 8.** The generalized Stirling numbers of the first kind and order \(p\), associated with the operators \((3.1)-(3.3)\), denoted by \(s^p(n,k)\), are defined by

\[
(x)_n^p = \sum_{k=0}^{n} s^p(n,k) x^k. \tag{4.7}
\]

**Remark 2.** In the subsequent considerations, with an abuse of notation, we will use Roman numerals for denoting the values of \(p\), when \(p\) appears as an index in sequences of polynomials or numbers.

Observe that \((x)_n^I = (x)_n\) and \(s^I(n,k) = s(n,k)\). Since for any polynomial \(p(x)\) the expansion

\[
p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k | p(x) \rangle}{k!} x^k, \tag{4.8}
\]

holds, it follows that

\[
s^p(n,k) = \frac{1}{k!} \langle t^k | (x)_n^p \rangle. \tag{4.9}
\]

**Definition 9.** The generalized Stirling numbers of the second kind and order \(p\), denoted by \(S^p(n,k)\), are defined by

\[
x^n = \sum_{k=0}^{n} S^p(n,k) (x)_n^p. \tag{4.10}
\]
They admit the generating function

\[ \sum_{n=k}^{\infty} \frac{S_p(n,k)}{n!} t^n = \frac{[\Delta_p(t)]^k}{k!}, \]

where \( \Delta_p(t) = \sum_{j} a_j e^{jt} \) is the indicator of the operator \( \Delta_p \). As a consequence of formula (2.14), one immediately obtains that

\[ S_p(n,k) = \frac{1}{k!} \left( \sum_{j=l}^{m} a_j e^{jt} \right)^k |x^n\rangle, \quad m - l = p. \]

The polynomials \((x)^n_p\) satisfy the binomial identity

\[ (x + y)^n_p = \sum_{k=0}^{n} \binom{n}{k} (x)^k_p (y)^{n-k}_p. \]

A relevant feature of relations (4.7) and (4.10) is that they are inverse to each other. This immediately implies that

\[ \delta_{mn} = \sum_{k} S_p(n,k) S_p(k,m) = \sum_{k} S_p(n,k) S_p(k,m). \]

**Remark 3.** Due to formulae (3.5) and (4.4), it is natural to interpret the operators \( \Delta \) and \( x \beta \) as quantum mechanical annihilation and creation operators \( a \) and \( a^\dagger \) acting on a finitely generated space, as noticed in [38], [41]. For instance,

\[ (x \beta)^n \cdot 1 = (a^\dagger)^n |0\rangle = |n\rangle. \]

\[ \Delta_p(x \beta)^n \cdot 1 = n |n - 1\rangle. \]

**5. Bernoulli–type polynomials and numbers of the first kind**

In this section, some realizations of the polynomial class (1.7) are presented. We shall restrict to sequences of polynomials obtainable in the umbral context described above. We propose the following definition.

**Definition 10.** The higher-order Bernoulli–type polynomials of the first kind are the polynomials generated by means of the relation

\[ \sum_{k=0}^{\infty} B_{k,a}^n(x) \frac{t^k}{k!} = (J_p(t))^a e^{xt}, \]

with \( a \neq 0 \) and

\[ J_p(t) = \frac{t}{\sum_{j=1}^{m} a_j e^{tj}}, \]

where \( \sum_{j=1}^{m} a_j e^{tj} \) is the indicator of a delta operator of order \( p \).

**Remark 4.** Usually, we will assume that \( a_j \in \mathbb{Q} \), as turns out to be the case in many applications. However, more general operator structures with real coefficients \( a_j \) could be considered within the same framework.

In the following, when \( a = 1 \), the index \( a \) will be omitted for simplicity.

**Remark 5.** When \( p = a = 1 \), \( J_1(t) = \frac{t}{e^t - 1} \), and in this case we obtain the classical Bernoulli polynomials. In the case \( a \neq 1, p = 1 \), the polynomials (5.2) reduce to the standard higher–order Bernoulli polynomials.
Definition 11. The Bernoulli–type numbers \( B_{p,k,a} \) are defined by the relation
\[
\sum_{k=0}^{\infty} \frac{B_{p,k,a}}{k!} t^k = J_p(t)^a, \quad a \neq 0.
\]

Remark 6. The operators \( J_p(t) \) and the corresponding numbers for \( a = 1 \) have been studied in [33] in the context of complex oriented homology theory. Here again the construction of these operators is realized using a difference operator approach.

Remark 7. Let us introduce the operator \( \Delta_p(x) \), which is defined to be the operator whose indicator is
\[\Delta_p(x) := (J_p(t))^{-a}.\]
In the special case \( a = 1 \), we simply get the difference operator \( \Delta_1 := \Delta_p \).

We can now prove a result stated in the Introduction 2

Lemma 1. The polynomials \( B_{p,m,a}(x) \) are determined by the two properties
\[
DB_{p,m,a}(x) = mB_{p,m-1,a}(x),
\]
\[
\Delta_p^a B_{p,m,a}(x) = mx^{m-1},
\]
with the condition \( B_{p,0,a}(x) = 1 \).

Proof. The argument we propose is very simple and closer to the spirit of the finite operator theory [35]. Identity (5.4) follows immediately from (5.1). Observe that, being an Appell sequence, the higher-order Bernoulli-type polynomials satisfy the relation
\[
B_{p,m,a}(x) = (J_p(t))^a x^m.
\]
The result follows by applying to both sides the operator \( \Delta_p^a(t) \), taking into account the initial condition. \( \square \)

Remark 8. For \( a = 1 \), we obtain a new family of linear difference equations involving the Bernoulli-type polynomials. Also, for \( a = 1, p = 1, \Delta_1 \equiv \Delta \) so that we obtain again the conditions characterizing the classical Bernoulli polynomials.

Examples. The case \( p = 2 \). We get a sequence of polynomials which we will call the central Bernoulli polynomials (of the first kind). The generating function reads
\[
\sum_{k=0}^{\infty} \frac{B_{k,a}^{II}(x)}{k!} t^k = \left( \frac{t \sinh t}{e^t} \right)^a e^{xt}.
\]

For \( x = 0 \) and \( a = 1 \), one obtains the numbers \( B_{n}^{II} \). From the generating function it emerges that \( B_{k}^{II} = 0 \) for \( k \) odd. The first central Bernoulli numbers \( B_{n}^{II} \) are:
\[1, -\frac{1}{3}, \frac{7}{15}, -\frac{31}{21}, \frac{127}{15}, -\frac{2555}{33}, \text{ etc.}\]
The polynomials (5.7) can be directly expressed in terms of the classical Bernoulli polynomials:
\[
B_n^{II}(x) = 2^n B_n^I \left( \frac{x+1}{2} \right), \quad B_n^I(x) = \frac{B_n^{II}(2x-1)}{2^n}.
\]
Other properties are now briefly discussed. From the polynomial expansion theorem we get

\[ B_{II}^n(x) = \sum_{k=0}^{n} \binom{e^t - e^{-t}}{2}^k \left( \frac{1}{k!} \right) B_{n,k}^{II} (x) \]  

(5.8)

Since

\[ \left\langle \binom{e^t - e^{-t}}{2}^k \left| B_{II}^n (x) \right. \right\rangle = \left\langle \binom{e^t - e^{-t}}{2}^{k-1} \left| n x^{n-1} \right. \right\rangle = n (k-1)! S_{II}^n (n-1, k-1), \]

(5.9)

we have:

\[ B_{II}^n(x) = B_{n}^{II} (0) + \sum_{k=1}^{n} \frac{n}{k} S_{II}^n (n-1, k-1) (x)^k. \]

(5.10)

For \( p = 3 \), the corresponding delta operator is given by formula (3.10), with the generating function

\[ \sum_{k=0}^{\infty} \frac{B_{III}^n (x)}{k!} t^k = -\frac{t e^{xt}}{e^{2t} - 2e^t + e^{-t}}. \]

(5.11)

For the other cases, we get similar results. For instance,

\[ \sum_{k=0}^{\infty} \frac{B_{IV}^n (x)}{k!} t^k = \frac{t e^{xt}}{e^{3t} - 2e^{2t} + 2e^t - 2e^{-t} + e^{-2t}}, \]

(5.12)

\[ \sum_{k=0}^{\infty} \frac{B_{VII}^n (x)}{k!} t^k = -\frac{t e^{xt}}{e^{4t} - e^{3t} + 2e^{2t} - 2e^t - e^{-t} - e^{-2t} + e^{-3t}}, \]

(5.13)

and so on. As an example, the first polynomials \( B_{III}^n (x) \) are given by

\[ B_{III}^0 (x) = 1, \quad B_{III}^1 (x) = x + \frac{3}{2}, \quad B_{III}^2 (x) = x^2 + 3x + \frac{37}{6}, \]

\[ B_{III}^3 (x) = x^3 + \frac{9}{2} x^2 + \frac{37}{2} x + 39, \]

\[ B_{III}^4 (x) = x^4 + 6x^3 + 37x^2 + 156x + \frac{9719}{30}, \]

\[ B_{III}^5 (x) = x^5 + \frac{15}{2} x^4 + \frac{185}{3} x^3 + 390x^2 + \frac{9719}{6} x + 3365, \]

\[ B_{III}^6 (x) = x^6 + 9x^5 + \frac{185}{2} x^4 + \frac{185}{3} x^3 + 780x^2 + \frac{9719}{2} x^2 + 20190x + \frac{1762237}{42}, \ldots \]

The relation (5.10) among higher order Stirling–type numbers, Bernoulli–type numbers and Bernoulli–type polynomials can be generalized as follows:

\[ B_{p}^{n} (x) = B_{p}^{n} (0) + \sum_{k=1}^{n} \frac{n}{k} S_{p}^{n} (n-1, k-1) (x)^k. \]

(5.14)
The recurrence relation for the polynomials \( B_{n,a}^p(x) \) is formally derived from the defining relation (5.1):

\[
\sum_{j=1}^{m} a_j e^{jt} B_{n,a}^p(x) = \left( \frac{t}{\sum_{j=1}^{m} a_j e^{jt}} \right)^{a-1} tx^n = n B_{n-1,a-1}^p(x).
\]

Therefore, the difference equation solved by the Bernoulli–type polynomials is, for every \( p \),

\[
\sum_{j=1}^{m} a_j B_{n,a}^p(x + j) = n B_{n-1,a-1}^p(x),
\]

where the \( a_k \) satisfy the constraints (3.2), (3.3). When \( p = 2 \), we obtain

\[
B_{n,a}^{II}(x + 1) - B_{n,a}^{II}(x - 1) - 2n B_{n-1,a-1}^{II}(x) = 0.
\]

Also we have the recurrence

\[
B_{n+1,a}^p(x) = \left( x - \frac{g'(t)}{g(t)} \right) B_{n,a}^p(x),
\]

where \( g(t) \) is the operator

\[
g(t) = \left( \sum_{k=1}^{m} a_k e^{kt} \right)^a.
\]

Using the relations \([t, x] = 1\) and \( B_{n,a}^p(x) = g(t)^{-1} x^n\), which is a consequence of Eq. (2.10), we get

\[
(n + 1) B_{n,a}^p(x) = \left( xt + 1 - a \sum_{k=1}^{m} a_k e^{kt} (tk - 1) \right) B_{n,a}^p(x).
\]

From the previous equation we derive the formula expressing the Bernoulli–type polynomials of order \( a + 1 \) in terms of those of order \( a \):

\[
\sum_{k} k a_k B_{n,a+1}^p(x + k) = \left( 1 - \frac{n}{a} \right) B_{n,a}^p(x) + \frac{n x}{a} B_{n-1,a}^p(x).
\]

Observe that, in accordance with Eq. (3.14), the polynomials

\[
p_n(x) = x \left( \frac{t}{\sum_{k=1}^{m} a_k e^{kt}} \right)^{na} x^{n-1} = x B_{n-1,n,a}^p(x)
\]

represent the basic sequence associated with the operator \( Q = D = D \left( \frac{\Delta_a}{t} \right)^a \equiv \Delta_a^p \). Consequently, for \( a = 1 \), we deduce the relation expressing the basic sequence \( (x)_n^p \) in terms of Bernoulli–type polynomials:

\[
(x)_n^p = x B_{n-1,n,1}^p(x).
\]

An interesting connection between the Bernoulli numbers and the Stirling numbers of the same order holds. Indeed, using Eq. (5.22) and its differential consequences, after some algebraic manipulations we get

\[
s^p(n,r) = \frac{1}{r!} \langle t^r \mid (x)_n^p \rangle = \frac{1}{r!} \langle t^0 \mid t^r (x)_n^p \rangle = \left( \frac{n-1}{r-1} \right) B_{n-r,n}^p(0).
\]
Therefore,

\[(x)_n^p = \sum_{r=0}^{n} \binom{n-1}{r-1} B_{n-r,n}^p (0)x^r.\]

Analogously,

\[S_p(n, r) = \frac{1}{r!} \left( \sum_{k=0}^{m} a_ke^{kt} \right) t^r x^n = \binom{n}{r} B_{n-r,-r}^p (0),\]

and

\[x^n = \sum_{r=0}^{n} \binom{n}{r} B_{n-r,-r}^p (0) x^r.\]

6. New Bernoulli–Type Polynomials of the Second Kind

By analogy with the standard Bernoulli polynomials of the second kind (see, e.g., [35]), we introduce the following sequences of polynomials.

**Definition 12.** The higher–order Bernoulli–type polynomials of the second kind are the polynomials defined by

\[(6.1) \quad b_n^p(x) = J_p(t)^{-1}(x)_n^p, \quad n \in \mathbb{N},\]

where \(J_p(t)\) is the operator (5.2).

These polynomials represent a Sheffer sequence for the operator \(\Delta_p\) of Eqs. (3.1), (3.2):

\[(6.2) \quad \Delta_p b_n^p(x) = nb_n^{p-1}(x),\]

and therefore they satisfy the identity

\[(6.3) \quad b_n^p(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k^p(y)(x)_{n-k}^p,\]

which relates them with the higher factorial polynomials. In particular, for \(y = 0\), we get

\[(6.4) \quad b_n^p(x) = \sum_{k=0}^{n} \binom{n}{k} b_k^p(0)(x)_{n-k}^p.\]

From Eq. (6.2) we deduce the difference equation satisfied by the polynomials (6.1):

\[(6.5) \quad \sum_{k} a_k b_n^p(x+k) = nb_n^{p-1}(x).\]

The case \(J_1(t)^{-1} = \frac{e^t-1}{t}\) just reproduces the standard Bernoulli polynomials of the second kind. The generating function of the sequences (6.1) can be obtained in some specific cases. Let us briefly discuss the second-order case. Since the operator \(\mathcal{J}^I := \frac{e^t-1}{2t}\) acts as follows:

\[\mathcal{J}^I p(x) = \frac{1}{2} \int_{x-1}^{x+1} p(u) du,\]
we deduce an explicit expression for the central Bernoulli polynomials of the second kind:

\begin{equation}
(6.6) \quad b_{II}^n (x) = 3^{II} (x)_n^{II} = \frac{1}{2} \int_{x-1}^{x+1} (u)_n^{II} \, du.
\end{equation}

The generating function is

\begin{equation}
(6.7) \quad \sum_{k=0}^{\infty} \frac{b_{II}^k (x)}{k!} t^k = \frac{t}{\log (t + \sqrt{1 + t^2})} \left( t + \sqrt{1 + t^2} \right)^x.
\end{equation}

For \( x = 0 \) we get

\begin{equation}
(6.8) \quad \sum_{k=0}^{\infty} \frac{b_{II}^k (0)}{k!} t^k = \frac{t}{\log (t + \sqrt{1 + t^2})}.
\end{equation}

Let us derive the recurrence relation satisfied by the polynomials \( b_{II}^n (x) \). From Eq. (6.6) we get

\begin{equation}
(6.9) \quad t b_{II}^n (x) = \frac{e^t - e^{-t}}{2} (x)_n^{II} = n (x)_n^{II}.
\end{equation}

Integrating we obtain

\begin{equation}
(6.10) \quad b_{II}^n (x) - b_{II}^n (0) = n \int_0^x (u)_n^{II} \, du.
\end{equation}

The difference equation (6.5) reduces to

\begin{equation}
(6.11) \quad b_{II}^n (x + 1) = b_{II}^n (x - 1) + 2 n b_{II}^{n-1} (x).
\end{equation}

Since

\begin{equation}
(6.12) \quad b_{II}^n (1) = \frac{1}{2} \int_0^2 (u)_n^{II} \, du,
\end{equation}

another representation of \( b_{II}^n (0) \) is

\begin{equation}
(6.13) \quad b_{II}^n (0) = \frac{1}{2} \int_0^2 (u)_n^{II} \, du - n \int_0^1 (u)_n^{II} \, du.
\end{equation}

Notice the analogy with the classical Bernoulli numbers of the second kind, defined by (see [35])

\begin{equation}
(6.14) \quad b_n (0) = \left\langle \frac{e^t - 1}{t} \mid (x)_n \right\rangle = \int_0^1 (u)_n \, du.
\end{equation}

There is a connection between Bernoulli-type polynomials of the second kind and generalized Stirling numbers of the first kind. Since

\[ b_n^p (x) = \sum_{k=0}^{n} \frac{1}{k!} \left\langle t^k \mid b_n^p (x) \right\rangle x^k, \]

we get

\begin{equation}
(6.15) \quad b_n^p (x) = b_n^p (0) + \sum_{k=1}^{n} \frac{n}{k} s^p (n - 1, k - 1) x^k.
\end{equation}
Many other properties and identities, which we will not discuss here for the sake of brevity, can be derived using operator techniques.

7. **New Euler–Type Polynomials**

In this Section, a new class of polynomial sequence of Appell type is discussed. For many aspects, it can be considered to be a natural generalization of the Euler polynomials. The definition proposed in this paper is different from that in [43].

Euler polynomials and numbers (introduced by Euler in 1740) also possess an extensive literature and several interesting applications in Number Theory (see, for instance, [14], [35], [43]). In many respects, they are closely related to the theory of Bernoulli polynomials and numbers.

**Definition 13.** The Euler–type polynomials are the Appell sequence generated by

\[
\sum_{k=0}^{\infty} \frac{E_{k}^{p,a,\omega}(x)}{k!} t^{k} = \left(1 + \frac{\Delta_{p}(t)}{\omega}\right)^{-a} e^{xt},
\]

where \(a, \omega \neq 0\) and \(\Delta_{p}(t)\) is the indicator of the operator \(\Delta_{p}\).

For \(a = 1\), \(\Delta_{p}(t) = e^{t} - 1\), \(\omega = 2\), we obtain the classical Euler polynomials. It will be shown that these new polynomials possess many of the properties of their classical analogues.

We recall that the Euler polynomials of order \(a\), which will be denoted by the symbol \(E_{n}^{a}(x)\), are the Appell sequence associated with the operator

\[ g(t) = \left(1 + \frac{e^{t} - 1}{2}\right)^{a}, \quad a \neq 0. \]

Consequently

\[
E_{n}^{a}(x) = \left(1 + \frac{e^{t} - 1}{2}\right)^{-a} x^{n}. \tag{7.2}
\]

By analogy, we have the following

**Lemma 2.** The Euler–type polynomial sequences are given by the relation

\[
E_{n}^{p,a,\omega}(x) = \left(1 + \frac{\Delta_{p}(t)}{\omega}\right)^{-a} x^{n}, \tag{7.3}
\]

where \(a, \omega \in \mathbb{R}\), \(a, \omega \neq 0\).

For the sake of clarity, we will omit the superscript \(\omega\). The Euler–type polynomials satisfy the Appell property

\[ DE_{n}^{p,a}(x) = n E_{n-1}^{p,a}(x) \]

and the Appell binomial identity

\[
E_{n}^{p,a}(x + y) = \sum_{k=0}^{n} \binom{n}{k} E_{k}^{p,a}(y) x^{n-k}. \tag{7.4}
\]

For every \(p\), the polynomials (7.1) are an Appell cross–sequence:

\[
E_{n}^{p,a+b}(x + y) = \sum_{k=0}^{n} \binom{n}{k} E_{k}^{p,a}(y) E_{n-k}^{p,b}(x). \tag{7.5}
\]
We also have a recurrence formula:

\[(7.6)\]

\[E_{n+1}^{p,a}(x) = \left( x - \frac{g_p'(t)}{g_p(t)} \right) E_n^{p,a}(x),\]

where \(g_p(t) = \left( 1 + \frac{\Delta_p(t)}{\omega} \right)^a\).

There exists an interesting analogue of the Boole summation formula. Since for any function \(h(t)\),

\[h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \rangle}{k!} 1 + \frac{\Delta_p(t)}{\omega} \]

then, choosing \(h(t) = \exp(yt)\), we get

\[(7.7)\]

\[\exp(yt) = \sum_{k=0}^{\infty} \frac{E_k^{p,a}(y)}{k!} 1 + \frac{\Delta_p(t)}{\omega} \]

Once applied to a polynomial \(p(x)\), we obtain for any \(p\) the expansion

\[p(x + y) = \sum_{k=0}^{\infty} \frac{E_k^{p,a}(y)}{k!} 1 + \frac{\Delta_p(t)}{\omega} \]

Let us consider in more detail the operator (for \(p = 2\))

\[(7.8)\]

\[g^{II}(t) = \left( 1 + \frac{e^t - e^{-t}}{\omega} \right)^a.\]

Correspondingly,

\[(7.9)\]

\[E_n^{II,a}(x) = \left( 1 + \frac{e^t - e^{-t}}{\omega} \right)^{-a} x^n.\]

Since

\[(7.10)\]

\[g^{II}(t)^{-1} = \left( \frac{1}{1 + \frac{e^t - e^{-t}}{\omega}} \right)^a = \sum_{j=0}^{\infty} \frac{(-a)}{j!} \left( \frac{e^t - e^{-t}}{\omega} \right)^j,\]

the polynomials \((7.9)\) are of the form

\[(7.11)\]

\[E_n^{II,a}(x) = \sum_{j=0}^{\infty} \frac{(-a)}{j!} \left( \frac{e^t - e^{-t}}{\omega} \right)^j x^n = \sum_{k=0}^{n} \sum_{j=0}^{n} \frac{(-a)}{k!} \frac{j!}{\omega^j} S^{II}(k,j) x^{n-k}.\]

From this equation and from the definition \((7.9)\) we derive the difference equation

\[(7.12)\]

\[E_n^{II,a}(x + 1) - E_n^{II,a}(x - 1) = \omega(E_n^{II,a-1}(x) - E_n^{II,a}(x)).\]

The analogue of the Newton expansion is given by the formula

\[(7.13)\]

\[E_n^{II,a}(x) = \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{(-a)}{k!} \frac{j^2}{\omega^k} S^{II}(n,j) (x)^{II}_{j-k},\]

which is easily derived from eq. \((7.11)\) and from the formula

\[(7.14)\]

\[\left( \frac{e^t - e^{-t}}{2} \right)^k x^n = \sum_{j=k}^{n} S^{II}(n,k) (x)^{II}_{j-k}.\]
Finally, from the recurrence formula (7.6) (for $p = 2$) we have:

$$E_{n+1}^{II,a}(x) = \left( x - a \frac{e^t + e^{-t}}{e^t - e^{-t}} \right) E_n^{II,a}(x),$$

providing another difference equation satisfied by $E_n^{II,a}(x)$:

$$(7.15) \quad E_{n+1}^{II,a}(x) = x E_n^{II,a}(x) - \frac{a}{\omega} \left( E_n^{II,a+1}(x + 1) + E_n^{II,a+1}(x - 1) \right).$$

Other cases can be studied similarly.

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**Errata corrig**

The following main changes have been done and typos corrected w.r.t. the published version:

1. A new, more general statement and proof of Lemma 1 has been provided. In particular, in Eq. (5.5) of Lemma 1, $\Delta_p \rightarrow \Delta_p^a$.
2. The Remarks 5, 7 and 8 have been added.
3. The factor 2 in the r.h.s. of Eq. (5.17) has been restored. The factor $a$ missing in the r.h.s. of Eq. (7.15) has been restored. The r.h.s. of Eq. (5.24) has been corrected.

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