Quasirationality and prounipotent crossed modules

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Abstract
We study quasirational presentations (QR-presentations) of (pro-p) groups, which contain aspherical presentations and their subpresentations, and also still mysterious pro-p-groups with a single defining relation. Using schematization of QR-presentations and embedding of the rationalized module of relations into a diagram related to certain prounipotent crossed module, we study cohomological properties of pro-p-groups with a single defining relation (a question of J.-P. Serre).

Keywords: quasirationality, prounipotent crossed modules.

1 Introduction
In the paper [20] we introduced the notion of quasirational (pro-p) presentation, which in the discrete case contains aspherical presentations and their subpresentations, and also pro-p-groups with a single defining relation in the pro-p-case. Quasirational presentations naturally generalize discrete combinatorially aspherical presentations introduced by J. Huebschmann, and aspherical (pro-p)presentations studied by O. V. Melnikov in [16]. In the paper [17] it has been shown that the notion of existence of quasirational presentation of (pro-p)group $G$ is equivalent to the statement that the homology groups $H_2(G, \mathbb{Z})$ (respectively $H_2(G, \mathbb{Z}_p)$ in the pro-p-case) have no torsion, and therefore quasirationality is independent on the choice of presentation, and is the property of the (pro-p)group itself. Earlier in [20] we have proved a simple criterion of quasirationality stating that quasirationality is equivalent to absence of torsion in the (trivial) coinvariant module of the relations module by the action of the group. In the discrete case, as shown in [17], quasirationality clarifies the difference of properties of aspherical presentation and its subpresentation, which is curious from the viewpoint of Whitehead’s celebrated “asphericity” conjecture. Regarding quasirational pro-p-presentations, in [17] we proved O. V. Melnikov’s Conjecture “on existence of envelope” of the class of aspherical pro-p-presentations, pointing out that quasirational pro-p-presentations correspond to all requirements of the Conjecture stated in 1997.

In this paper we shall give the description, announced in [21], of modules of relations of quasirational pro-p-presentations by means of affine group schemes technique. For these purposes, after recalling necessary constructions in Section 2, in Section 3 we construct a prounipotent presentation [21] from a finite presentation of a pro-p-group [21] by means of $\mathbb{Q}_p$-prounipotent completion (Definition [3]) of finitely generated free pro-p-groups (“schematization”). Using the analogy with the discrete and profinite...
cases, we study the prounipotent analogs of 2-reduced free simplicial groups, crossed and pre-crossed modules. Proposition 1 shows that the prounipotent crossed module constructed from a prounipotent presentation is free (Definition 9).

In discrete and profinite algebraic homotopy theories \[2\leq2\leq4\] the main benefits from systematical development of the theory of crossed modules are obtained after their abelianizations. Such approach is effective also for prounipotent crossed modules. Proposition 2, Lemma 2, Lemma 3 show that the introduced abelianizations have structures of topological modules (Definition 10), and in Theorem 11 and Corollary 2 we include such objects into a commutative diagram, which seems an analog of the Gaschutz theory \[3\leq3\leq1\leq7\leq6\] for quasirational presentations of pro-p-groups. Since in Lemma 2 the rationalized relation module \(\mathcal{R}\otimes\mathbb{Q}_p = \lim_{\varphi} R/[R,RM]\) of a presentation \(1\) is identified with Abelianization of a continuous prounipotent completion of \(R\), then \(\mathcal{R}\otimes\mathbb{Q}_p\) is included into a commutative diagram \(6\). Theorem 11 and Corollary 2 should be considered also as variations of ideas of comparison of homotopy types \[2\leq2\leq5\] in dimension 2. We apply the obtained results to the following question of J.-P. Serre from a remark to the \[3\leq2\leq1\leq3\].

Let \(G_r = F/(r)_{\mathbb{F}}\), where \((r)_{\mathbb{F}}\) is the normal closure of \(r \in F_p[F,F]\) in a free pro-p-group of finite rank \(F\), then J.-P. Serre asks the following: “Can it be true that \(cd(G_r) = 2\), only if \(G_r\) is torsion free (and \(r \neq 1\))?”

Let us first note that pro-p-groups are \(\mathbb{F}_p\)-points of prounipotent affine group schemes defined over \(\mathbb{F}_p\) - the prime field of characteristics \(p \geq 2\). Actually, consider the complete group algebra \(\mathbb{F}_pG = \lim_{\varphi} \mathbb{F}_p[G_n]\) of a pro-p-group \(G = \lim_{\varphi} G_n\), where \(G_n\) are finite pro-p-groups. Each group algebra \(\mathbb{F}_p[G_n]\) is obviously a cocommutative Hopf algebra over the field \(\mathbb{F}_p\). Then the dual Hopf algebra \(\mathbb{F}_p[G_n]^*\) {for details see the beginning of the third part of the paper} is a finitely generated commutative Hopf algebra, and hence it determines certain affine algebraic group scheme. Let \(G\) be the functor of group like elements of a Hopf algebra. Note that \(G_\alpha = G_{\mathbb{F}_p}[G_\alpha] \cong Hom_{\mathcal{A}_{\mathcal{G}_p}}(\mathbb{F}_p[G_\alpha]^*, \mathbb{F}_p)\).

Dimension shifting enables one to compute cohomology of a \(\mathcal{L}_{p}\)-group \(H\) as invariants of certain modules. From this viewpoint, in order to look like (cohomologically) a group with elements of finite order, it suffices for the elements of a group \(H\) to act as if they were of finite order, although they can actually be not so. In \[2\leq2\leq4\], using multiplication of defining relations \(r = y^p\) in a free pro-p-group by elements \(\zeta^p\) of special kind, we managed to obtain elements of a free pro-p-group \(F\) of the form \(y^p\cdot \zeta^p\) which act on finite dimensional modules of arbitrarily high dimension exactly as the initial relation, but are not \(p\)-th powers themselves. A more detailed description of cohomology of \(\mathbb{F}_p\)-affine group schemes (based not only on presence or absence of elements of finite order at \(\mathbb{F}_p\)-points) should be expected from the study of the Frobenius homomorphism on the algebra of regular functions.

If a finitely generated discrete nilpotent group has no torsion, then by the known Malcev theorem it is embedded \[3\leq4\] into its own rational prounipotent completion. For \(\mathbb{F}_p\)-prounipotent groups, for which pro-p-groups are their \(\mathbb{F}_p\)-points, it would be too optimistic to hope for a similar statement, but we shall show that existence of an embedding into a type of completion implies that cohomological dimension
of such a group is less or equal to 2. Using a description of the relations module of a prounipotent group with one defining relation [21, Corollary 12] we point out (Proposition 3 and Corollary 3), in terms of continuous prounipotent completion, the condition under which a finitely generated pro-p-group with one defining relation has cohomological dimension 2. The proof is based on Theorem 1 and Corollary 2. From our viewpoint this condition contains a variation of J.-P. Serre’s question with a positive answer.

2 Schematization and quasirationality

By definition, a finite type presentation of a discrete group $G$ is an exact sequence

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

(1)

in which $F = F(X)$ is the free group with a finite set $X$ of generators, and $R$ is a normal subgroup in $F$ generated by a finite number of defining relations $r \in R$. By a pro-p-group one calls a group isomorphic to projective limit of finite $p$-groups. This is a topological group (with the topology of direct product) which is a compact totally disconnected group. For such groups one has a presentation theory which is in many aspects similar to the combinatorial theory of discrete groups [12], [28]. By analogy with finite type presentation of a discrete group, we shall say that a pro-p-group $G$ is given by a finite type pro-p- presentation if $G$ is included into an exact sequence (1) in which $F$ is a free pro-p-group with finite number of generators, and $R$ is a closed normal subgroup topologically generated by a finite number of elements in $F$, contained in the Frattini subgroup of the group $F$ [12, 28].

Let $R = \varprojlim R_\alpha$ be a profinite ring ($R_\alpha$ are finite rings), then denote by $RG$ the completed group algebra of a pro-p-group $G$. By a completed group algebra we understand the topological algebra $RG = \varprojlim RG_\alpha$. [23, 5.3], where $G = \varprojlim G_\alpha$ is a decomposition of the pro-p-group $G$ into the projective limit of finite $p$-groups $G_\mu$.

For discrete groups, $p$ will run over all primes, while for pro-p-groups $p$ is fixed. Let $G$ be a (pro-p)group with a finite type (pro-p)presentation (1), and $\mathcal{R} = R/[R, R]$ be the corresponding $G$-module of relations, where $[R, R]$ is the commutant, and the action of $G$ is induced by conjugation by $F$ on $R$. For each prime number $p \geq 2$ denote by $\Delta_p$ the augmentation ideal of the ring $\mathbb{F}_p G$. In the pro-p-case by $\Delta^n$ we understand the closure of the module generated by the $n$-th powers of elements from $\Delta = \Delta_p$, and in the discrete case this is the $n$-th power of the ideal $\Delta_p$. The properties of this filtration in the pro-p-case are exposed in [12, 7.4], and in the discrete case the properties of the Zassenhaus filtration are similar [23, Chap.11], the difference is in the use of the usual group ring instead of the completed one.

Denote by $\mathcal{M}_n, n \in \mathbb{N}$ its Zassenhaus filtration in $F$ with coefficients in the field $\mathbb{F}_p$, which is defined by the rule $\mathcal{M}_{n,p} = \{f \in F \mid f - 1 \in \Delta_p^n\}$. We shall denote these filtrations simply by $\mathcal{M}_n$, omitting the prime $p$, since its choice will be always clear from the context. Introduce the notation $\mathbb{Z}_{(p)}$ for $\mathbb{Z}$ in the case of discrete groups and $\mathbb{Z}_p$ in the case of pro-p-groups.

Definition 1: We shall call the (pro-p)presentation (1) quasirational ($QR$-(pro-p) presentation) if one of the following three equivalent conditions is satisfied:

(i) for each $n > 0$ and for each prime $p \geq 2$, the $F/\mathcal{M}_n$-module $R/[R, R]\mathcal{M}_n$ has no $p$-torsion ($p$ is fixed for pro-p-groups and runs over all primes $p \geq 2$ and the corresponding Zassenhaus $p$-filtrations in the discrete case);
(ii) the quotient module of coinvariants $\overline{R}_G = \overline{R}_F = R/[R,F]$ has no torsion;
(iii) $H_2(G, Z_{(p)})$ has no torsion.

The proof of equivalence of conditions (i) – (iii) is contained in [20 Proposition 4] and [19 Proposition 1]. $QR$-presentations are curious in particular due to the fact that they contain aspherical presentations of discrete groups and their subpresentations, and also pro-$p$-presentations of pro-$p$-groups with one defining relation [20].

By an affine group scheme over a field $k$ one calls a representable $k$-functor $G$ from the category $\text{Alg}_k$ of commutative $k$-algebras with unit to the category of groups. If $G$ is representable by an algebra $\mathcal{O}(G)$, then for any commutative $k$-algebra $A$ the functor $G$ is given by the formula

$$G(A) = \text{Hom}_{\text{Alg}_k}(\mathcal{O}(G), A).$$

Of course, we assume that the considered homomorphisms $\text{Hom}_{\text{Alg}_k}$ map the unit of the algebra $\mathcal{O}(G)$ to the unit of $A$. The algebra $\mathcal{O}(G)$ representing the functor $G$ is usually called the algebra of regular functions of $G$. The Yoneda lemma implies anti-equivalence of the categories of affine group schemes and commutative Hopf algebras (with unit) [22, 1.3]. Let us say that an affine group scheme $G$ is algebraic if its coordinate Hopf algebra $\mathcal{O}(G)$ is a finitely generated algebra.

Let $A$ be a coalgebra over a field $k$ with the coproduct $\Delta : A \to A \otimes A$ and the counit $\varepsilon : A \to k$. Let us say that $A$ is conilpotent (or “connected” in the terminology of [27, B.3]) if there exists an element $1_A \in A$ such that $\varepsilon(1_A) = 1_k$, where $1_k$ is the unit of the field $k$, $\Delta(1_A) = 1_A \otimes 1_A$, and there exists $n \in \mathbb{N}$ such that $A = F_n A$, where $F_rA$ is the filtration in $A$ defined recursively by the following formulas: $F_0A = k \cdot 1_A, F_rA = \{x \in A | \Delta x = x \otimes 1_A - 1_A \otimes x \in F_{r-1} \otimes F_{r-1}\}$.

Definition 2: By a unipotent group one calls an affine algebraic group scheme $G$ whose Hopf algebra of regular functions $\mathcal{O}(G)$ is conilpotent (see [32] 8 and [31, Proposition 16], where equivalent definitions are given). An affine group scheme $G = \varinjlim G_\alpha$, where $G_\alpha$ are affine algebraic group schemes over a field $k$, is called a prounipotent group if each $G_\alpha$ is a unipotent group.

Let $G$ be a prounipotent group with the algebra of regular functions $\mathcal{O}(G)$, then the unipotency condition $G_\alpha$ in Definition 2 is equivalent to the condition that the so-called conilpotent filtration $0 \subset C_0 = I^1 \subset C_1 = (I^2)^1 \subset \ldots \subset C_k = (I^{k+1})^1 \subset \ldots$ is exhausting (such coalgebras are usually called locally conilpotent), i.e. $\mathcal{O}(G) = \text{lim} \mathcal{C}_i$, where, as usual, $I$ is the augmentation ideal (the kernel of the counit) in the Hopf algebra $\mathcal{O}(G)^*$, and $(I^r)^1 = \{r \in \mathcal{O}(G) : r(\phi) = 0, \forall \phi \in I^1\}$ (see Section 3 regarding the definition of $\mathcal{O}(G)^*$ and the duality questions).

Let $A$ be a Hopf algebra over a field $k$ of characteristic 0, in which: 1) the product is commutative; 2) the coproduct is conilpotent. Then, as an algebra, $A$ is isomorphic to a free commutative algebra [4, Theorem 3.9.1]. Therefore, each unipotent group $G_\alpha$ is isomorphic as an algebraic variety to certain $n_\alpha$-dimensional affine space $k^n_{(\alpha)}$ [32, Theorem 4.4] and hence it is an affine algebraic group, and consequently [32, Corollary 4.4] a linear algebraic group. Thus, we can use results from the theory of linear algebraic groups in characteristic 0.

Also one has a well known correspondence between unipotent groups over $k$ and nilpotent Lie algebras over $k$, which assigns to a unipotent group its Lie algebra. This correspondence is easily extended to a correspondence between prounipotent groups over $k$ and pronilpotent Lie algebras over $k$ [27, Appendix 1].
of this correspondence enables one to interpret, when it is convenient, problems on
prounipotent groups in the language of Lie algebras. For example, the image of a
closed subgroup under a homomorphism of prounipotent groups will be always a closed
subgroup.

The group of \( Q_p \)-points of any affine group scheme \( G \) over \( Q_p \) has the \( p \)-adic
topology. Actually, [5 Part 2] shows that \( G \) can be represented as a projective limit
\( G = \lim Q \alpha \) of a surjective projective system of linear algebraic groups. Each \( G_\alpha(Q_p) \)
has a canonical \( p \)-adic topology induced by an embedding \( G_\alpha \hookrightarrow GL_\alpha \). Let us define
a topology on \( G(Q_p) \) as the topology of the projective limit \( G(Q_p) = \lim Q \alpha \).

**Definition 3:** [11 A.2], [20] Let us fix a group \( G \) (with \( p \)-adic topology). Define the
(continuous) prounipotent completion of \( G \) as the following universal diagram, in which \( \rho \)
is a (continuous) Zariski dense homomorphism from \( G \) to the group of \( Q_p \)-points of a
prounipotent affine group \( G^\wedge \):

\[
\begin{array}{ccc}
G & \xrightarrow{\rho} & G^\wedge(Q_p) \\
\downarrow{\alpha} & & \downarrow{\tau} \\
H(Q_p) & \xleftarrow{\uparrow{\gamma}} & \end{array}
\]

We require that for each continuous and Zariski dense homomorphism \( \chi \) there exists
a unique homomorphism \( \tau \) of prounipotent groups making the diagram commutative.

Such an object always exists. Indeed, let \( G = \lim Q \alpha \). By the Zariski closure of a
subgroup \( K \) in \( G(k) \) we understand the least proalgebraic \( k \)-subgroup \( H \leq G \) such
that \( H(k) \geq K \). It is evident that \( H = \lim Q H_\alpha \), where \( H_\alpha \) is the Zariski closure of \( K_\alpha \)
in \( G_\alpha(k) \). Consider the set of pairs \( (\phi_\alpha, U_\alpha) \), where \( \phi_\alpha : G \to U_\alpha(Q_p) \) is a continuous
homomorphism. This is a projective system, since the pair

\[
(\phi_\alpha \times \phi_\beta, \phi_\alpha(G) \times \phi_\beta(G)) = (\phi_{\alpha\beta}, U_{\alpha\beta}) \geq (\phi_\alpha, U_\alpha), (\phi_\beta, U_\beta).
\]

Here for any \( \alpha, \beta \) we have denoted by \( \phi_\alpha(G) \times \phi_\beta(G) \) the Zariski closure of the direct
product \( \phi_\alpha(G) \times \phi_\beta(G) \) in \( U_\alpha(Q_p) \times U_\beta(Q_p) \). The partial ordering is given by the
rule \( (\phi_\alpha, U_\alpha) \geq (\phi_\beta, U_\beta) \) if there exists \( U_\alpha \to U_\beta \) such that \( U_\alpha(Q_p) \to U_\beta(Q_p) \) is an
epimorphism compatible on the images of \( G \). Now put \( (\phi, G^\wedge) = \lim (\phi_\alpha, U_\alpha) \). It is
not difficult to see that the prounipotent completion of a free group \( F(X) \) satisfies
the universal properties inherent to a free object, and we shall, by analogy with the
discrete or \( p \)-adic cases, call such a group free and denote it by \( F^\wedge(X) \).

### 3 Prounipotent crossed modules

One has the following standard chain of equivalences: the category of simplicial proalgebraic
affine groups over a field \( k \) is equivalent to the category of affine simplicial
group schemes [6 2.1], which are dual to cosimplicial commutative Hopf algebras [22
1.6], and the latter are dual to simplicial cocommutative linearly compact \( (\pro-finite \)
dimensional) Hopf algebras. Let us describe the latter duality in more detail.

For an arbitrary discrete Hopf \( k \)-algebra \( A \), the dual Hopf algebra \( A^* = Hom_k(A, k) \)
is a linearly compact Hopf algebra, i.e. it is representable as the inverse limit
$A^* \cong \varprojlim A^*/U^\bot$, where $U$ are finite dimensional subspaces in $A$. The basis of the system of neighborhoods of zero in $A^*$ is formed by the $k$-subspaces

$$U^\bot = \{ \phi \in A^* \mid \phi(U) = 0, \text{where } U \text{ is a finite dimensional } k\text{-subspace of } A \}. $$

For any linearly compact space $V$ it makes sense to speak about the dual space $V^\vee$ of continuous linear functions, and the evaluation map defines the Pontryagin duality [7, 1.2] (here $V$ is a discrete or linearly compact space)

$$e : V \to V^\vee, \ v \mapsto (\phi \mapsto \phi(v)).$$

Unifying what was said above, we obtain the following Corollary [7, §2,14]:

**Corollary 1:** Assigning $V \mapsto V^*$ yields a one-to-one correspondence between the structures of (commutative) Hopf algebras on a vector $k$-space $V$ with discrete topology and the structures of (cocommutative) linearly compact Hopf algebras on $V^*$.

From this moment and below we shall consider only either discrete commutative or linearly compact cocommutative Hopf algebras.

First, following [27, A.2], let us state the notion of a complete augmented algebra (CAA) over a field $k$. By a filtration of an algebra $A$ we shall understand a decreasing sequence of subspaces $A = F_0A \supset F_1A \supset \ldots$ such that $1 \in F_0$ and $F_n \cdot F_m \subset F_{n+m}$. In this case each space of the filtration is a two-sided ideal in $A$, and $grA = \oplus_{i=0}^\infty gr_iA = \oplus_{i=0}^\infty F_iA/F_{i+1}A$ has the natural structure of a graded algebra.

An augmented algebra $A$ with the augmentation ideal $I$ and a filtration $F_nA$ is called a complete augmented algebra if the following conditions are satisfied:

1) $\hat{A} = \varprojlim A/I^n$ (i.e. $A$ is complete in the $I$-adic topology);
2) the algebra $grA$ is generated by $gr_1A$;
3) $F_1A = I$.

Quillen notes that condition 2), taking into account conditions 1) and 3) from the definition of CAA, is equivalent to the requirement that $F_nA$ coincides with the closure of $I^n$ in the $I$-adic topology.

Recall that the completed tensor product $E \hat{\otimes}_k F$ of topological $k$-vector spaces $E$ and $F$ is the completion of $E \otimes_k F$ with respect to the topology (called the topology of tensor product) given by the fundamental system of neighborhoods of 0 consisting of the sets $V \otimes_k F + E \otimes_k W$, where $V$ (respectively $W$) is an arbitrary element of the fundamental system of neighborhoods of 0 consisting of vector subspaces of $E$ (respectively $F$) [7 1.2.4].

**Definition 4:** Let us say that a complete augmented linearly compact algebra $A$ is a complete Hopf algebra (below CHA for short) if $A$ is endowed with a map $\triangle : A \to A \hat{\otimes} A$ of complete linearly compact algebras, which is included into the cocommutativity and coassociativity diagrams, and has the augmentation $A \to k$ as a counit.

Our definition is somewhat different from the definition of CHA in Quillen’s paper cited above, since we additionally require linear compactness, having in mind that $CHA$ arises as a dual of Hopf algebra.

It is convenient to control linear compactness in the definition of CHA, following Quillen [27, A.3], using the augmentation ideal $I$, assuming $\dim kI/I^2 < \infty$. In this situation one can construct the prounipotent completion of a finitely generated discrete group explicitly. Let $kG$ be the group ring of a finitely generated discrete group $G$. 


over certain field $k$, and let $I$ be the kernel of augmentation (the fact that $G$ is finitely generated yields the fact that the quotients $kG/I^n$ have finite dimension). Taking into account continuity of the coproduct $\Delta : kG \to kG \otimes kG$ in the Hopf algebra $kG$ with respect to the $I$-adic topology (the topology on $\otimes$ is defined using the filtration $F_n(\hat{k}G) = \sum_{i+j=n} I^i \otimes I^j \subset kG \otimes kG$), using the $I$-adic completion we obtain the continuous coproduct $\hat{\Delta} : \hat{k}G \to \hat{k}G \hat{\otimes} \hat{k}G \cong kG \hat{\otimes} kG$, determining the structure of a complete linearly compact cocommutative Hopf algebra on $\hat{k}G$. A direct check shows that the prounipotent group with the algebra of regular functions $\hat{k}G^\circ$ is the prounipotent completion of $G$ in the sense of Definition 3.

In simplicial group theory, by analogy with glueing two-dimensional cells, it is convenient to identify presentation (1) with the second step of construction of free prounipotent completion of $G$.

Let us assign to the simplicial finite type presentation (2) a presentation in the category of complete Hopf algebras.

\[
\begin{array}{c}
F(X \cup Y) \xrightarrow{s_0} F(X) \xrightarrow{d_1} G, \\
\end{array}
\]

here $d_0, d_1, s_0$ for $x \in X, y \in Y, r_\gamma \in R$ are defined by the identities $d_0(x) = x, d_0(y) = 1, d_1(x) = x, d_1(y) = r_\gamma, s_0(x) = x$.

Recall [12] that (2) is a free simplicial (pro-$p$) group of finite type, degenerate in dimensions greater than two. If a pro-$p$-presentation (1) is minimal then $|Y| = dim_p H^2(G, F_p), |X| = dim_p H^1(G, F_p)$.

Let us assign to the simplicial finite type presentation (2) a presentation in the category of complete Hopf algebras. First, let us consider the corresponding diagram of group rings, $kF(X \cup Y) \xrightarrow{s_0} kF(X)$. Then we obtain from (2) using the $I$-adic completion, taking into account finite generation of groups, the following diagram of free prounipotent groups:

\[
\begin{array}{c}
\hat{k}F(X \cup Y) \xrightarrow{s_0} \hat{k}F(X) \\
\end{array}
\]

where $\hat{k}F(X) = \lim_{\rightarrow} kF(X)/I^n$, and $I$ is the augmentation ideal in $kF(X)$. Applying the Pontryagin duality and the anti-equivalence of the categories of commutative Hopf algebras and affine group schemes, we obtain a diagram of free prounipotent groups

\[
\begin{array}{c}
F_u(X \cup Y) \xrightarrow{s_0} F_u(X) \\
\end{array}
\]

The fact that simplicial identities hold in (2) implies similar identities for the obtained diagram of prounipotent groups, which is implicitly used in all the constructions of the paper. For $k = F_p$ the constructions yield pro-$p$-completions and the homotopy theory developed in [13], and also to the concepts of $p$-adic homotopy theory.

Definition 5: Let us say that we are given a finite type presentation of a prounipotent group $G_u$ if there exist finite sets $X$ and $Y$ such that $G_u$ is included into the following diagram of free prounipotent groups:

\[
\begin{array}{c}
F_u(X \cup Y) \xrightarrow{s_0} F_u(X) \xrightarrow{G_u} G_u \\
\end{array}
\]
in which the identities similar to (2) and

\[ G_u \cong F_u(X)/d_1(Kerd_0) \]

hold. Denote \( R_u = d_1(Kerd_0); \) this is a normal subgroup in \( F_u(X), \) and hence we obtain an analog of the notion of presentation \( \mathbb{I} \) for a prounipotent group \( G, \) to which we shall refer also as to a presentation of type \( \mathbb{I}. \)

By the schematic homotopy type of a pair

\( (X, k \mid X \text{ is a connected simplicial set}, k \text{ is a field}) \)

one calls a simplicial \( k \)-proalgebraic group \( GX_{\text{alg}} \) \([11, \text{p.}655], \ [25]\), where \( G \) is the Kan functor \([\text{[18, page 13–14]}], \) and \( GX_{\text{alg}} \) is the componentwise \( k \)-proalgebraic complement of the free simplicial group \( GX \) \([25, \text{Def.}1.6]\). The constructions above are particular cases of the notion of scheme homotopy type, expressed here in a concrete form of prounipotent completions of finite type presentations of \( (\text{pro-p}) \) groups.

Below we shall work in the category of prounipotent groups over the field \( k = \mathbb{Q}_p. \)

By an action from the left of an affine group scheme \( G_1 \) on an affine group scheme \( G_2 \) one understands a natural transformation of functors \( G_1 \times G_2 \to G_2 \) included into the standard action diagrams \([22, 6n]\).

**Definition 6:** By a prounipotent pre-crossed module one calls a triple \((G_2, G_1, \partial)\), where \( G_1, G_2 \) are prounipotent groups, \( \partial : G_2 \to G_1 \) is a homomorphism of prounipotent groups, \( G_1 \) acts on \( G_2 \) from the left, satisfying, for any \( \mathbb{Q}_p \)-algebra \( A \), the identity

\[ \partial(g_1 g_2) = g_1 \partial(g_2) g_1^{-1}, \]

where the action is written in the form \((g_1, g_2) \to g_1 g_2, g_2 \in G_2(A), g_1 \in G_1(A)\).

**Definition 7:** A prounipotent pre-crossed module is called crossed if for any \( \mathbb{Q}_p \)-algebra \( A \) the following additional identity holds:

\[ \partial(g_2) g'_2 = g_2 g'_2 g_2^{-1}. \]

The identity \( \text{CM 2)} \) is called the Peifer identity. Such prounipotent crossed module will be denoted by \((G_2, G_1, \partial)\).

**Definition 8:** By a morphism of prounipotent crossed modules

\((G_2, G_1, \partial) \to (G'_2, G'_1, \partial')\)

one calls a pair of homomorphisms \( \varphi : G_2 \to G'_2 \) and \( \psi : G_1 \to G'_1 \) of prounipotent groups such that \( \varphi(g_2) = \psi(g_1) g_2 \) and \( \psi'(g_2) = \psi \partial(g_2) \).

For us the main role will be played by prounipotent crossed modules arising from finite presentations of \( (\text{pro-p}) \) groups. Recall the known construction of crossed modules.

Assume we are given a \( (\text{pro-p}) \)presentation \( \mathbb{I} \) in the simplicial form \( \mathbb{I}. \) The group \( F \) acts on \( R \) by conjugation \( g \to f^{-1} g f \to f^{-1}, g \in R, f \in F. \) Let us construct a pre-crossed module of this presentation. To this end, let us first pass to the beginning of the Moore complex of the simplicial group \( \mathbb{I}: \)

\[ Kerd_0 = NF_1 \overset{d_1}{\to} NF_0, \]
where $NF_0 = F(X), NF_1 = Ker d_0$. Note that both in the discrete case [2] Proposition 1 and in the pro-$p$-case [28] 8.1.3 one has $NF_1 = Ker d_0 \cong F(Y \times F(X))$. Now we can construct the free pre-crossed module of the (pro-$p$) presentation [2] on the set $Y$,

$$F(Y \times F(X)) \xrightarrow{d_1} F(X),$$

where the action of $F(X)$ is given by $\nu a = s_0(f)as_0(f)^{-1}$, where $f \in F(X), a \in F(Y \times F(X))$. A detailed study of this construction in the pro-$p$ and discrete cases can be found in [15] Proposition 12, [2] 3].

Below we shall define the notion of free (pre-)crossed module for prounipotent presentations of the form (8).

**Definition 9:** A prounipotent (pre-)crossed module $(G_2, G_1, d)$ is called a free prounipotent pre-crossed module with the base $Y \in G_2(Q_p)$ if $Y$ generates a Zariski dense $G_1(Q_p)$- subgroup in $G_2(Q_p)$ with the following universal property: for any prounipotent (pre-) crossed module $(G'_2, G'_1, d')$ and a subset $\nu(Y) \in G'_2(Q_p)$, for a function $\nu : Y \to G'_2(Q_p)$, with a Zariski dense $G'_1(Q_p)$- group closure, and for any epimorphism of prounipotent groups $f : G_1 \to G'_1$ such that $fd(Q_p)(Y) = d'(Q_p)\nu(Y)$, there exists a unique homomorphism of prounipotent groups $h : G_2 \to G'_2$ such that $h(Q_p)(Y) = \nu(Y)$ and the pair $(h, f)$ is a homomorphism of (pre-)crossed modules.

**Proposition 1:** Assume we are given a prounipotent finite type presentation [4] then $Kerd_0 \xrightarrow{d_1} F_u(X)$ is a free prounipotent pre-crossed module on the set $Y$.

**Proof:** For each function $\nu : Y \to A(Q_p)$, where $A \xrightarrow{d} G$ is a prounipotent pre-crossed module, let us construct the $\psi$ - homomorphism in the following diagram, with $\overline{\partial}(A) = \partial(A), \partial(G) = id_G$, $\partial(a, b) = \partial(a) \cdot b$ for $a \in A(Q_p), b \in G(Q_p)$.

$$\xymatrix{ Kerd_0 \ar[d]_{\overline{\partial}} \ar[r]^{d_1} & F_u(X \cup Y) \ar[r]^\nu \ar[d]_{\psi} & F_u(X) \ar[d]_f \ar[r]^{\partial} & \ar[r] & G }$$

The semidirect product $A \ltimes G$ [22 7] is a prounipotent group (prounipotence follows from the fact that a semidirect product of nilpotent Lie algebras is a nilpotent Lie algebra), since it is constructed from a prounipotent action, which is given in the pre-crossed module $(A, G, \partial)$. The fact that $\partial$ is a homomorphism follows by direct computation using the fact that for $\partial$ the property 1 from Definition 6 holds.

Set $\psi(Q_p)$ on $X$ as the composite $f(Q_p)d_1(Q_p)$, and set $\psi(Q_p)$ on $Y$ equal to $\nu(Y)$. The universal property of $F_u(X \cup Y)$ yields a homomorphism of prounipotent groups $\psi : F_u(X \cup Y) \to A \ltimes G$. Put $\varphi : Kerd_0 \to A \ltimes G$ equal to $\varphi = \psi |_{Kerd_0}$, i. e. restriction of $\psi$ to $Kerd_0$.

Constructing the prounipotent crossed module of the presentation

$$(C_u, F_u(X), d_1)$$

from the prounipotent presentation is standard and made by compilation of the constructions from [2] [15]:

$$\xymatrix{ Kerd_0 \ar[r]^{|P_u} & F_u(X) }$$
here $P_a = \langle d_1(a) = bab^{-1} \rangle$ is the normal Zariski closure of the subgroup of Peiffer commutators. By the homotopy groups of a simplicial group $F_\bullet$ one calls the homology groups of its Moore complex $(NF_\bullet = \cap_{n=0}^{\infty} ker d_n, d_n[K_{F_n}])$. In [8 5.7] (24 4.3.8), [18 page 23]) it is proved that in any simplicial group degenerate in dimension two and in particular in the simplicial group $F_\bullet$, arising from the presentation $\{1\}$, one has the coincidence of subgroups $Imd_2 = P_a$, where $d_2 : NF_2 \rightarrow NF_1$, and hence $\pi_1(F_\bullet) \cong ker d_1$ (recall that in the discrete case $\pi_1(F_\bullet)$ coincides with $\pi_2$ of the standard 2-complex of the presentation $\{1\}$ of the group).

Lemma 1: Assume we are given a prounipotent finite type presentation $\{1\}$, then one has the isomorphism of prounipotent crossed modules arising from coincidence of subgroups $P_u$ and $[Kerd_0, Kerd_1]$ in $Kerd_0$,

$$(C_u, F_u(X), d_1) = (Kerd_0/P_u, F_u(X), d_1) \cong (Kerd_0/[Kerd_0, Kerd_1], F_u(X), d_1).$$

Proof: The proof consists of two steps:

1) using elementary substitutions we see that Peiffer commutators are representable in the form of commutators $[x^{-1}sod_1(x), y]$, where $x, y \in Ker d_0$;

2) elements of $Ker d_1$ are exactly the elements $x^{-1}sod_1(x)$, where $x \in Ker d_0$.

For details, see [18 p.23].

Below, by the crossed module of a prounipotent presentation $\{1\}$ we shall understand the crossed module from Lemma 1. Recall the notation

$R_u = im(d_1)$.

Definition 10: Let $A$ be a complete linearly compact Hopf algebra over a field $k$ or a topological group (the field is considered with discrete topology). By a left (or right) complete topological $A$-module we shall call a linearly compact topological $k$-vector space $M$ with a structure of $A$-module such that the corresponding $k$-linear action $A \widehat{\otimes} M \rightarrow M$ is continuous. We assume that the topology on $M$ is given by a fundamental system of neighborhoods of zero $M = M^0 \supseteq M^1 \supseteq M^2 \supseteq \ldots$, where $M^j$ are topological $A$-submodules in $M$ of finite codimension (finiteness of type) and $M \cong \lim_{\leftarrow} M/M^j$. By a homomorphism of topological $A$-modules one calls a continuous $A$-module homomorphism.

If the filtration $M^j$ admits a compression such that for each $j$ the compression quotients $M^j/M^{j+1}$ are trivial $A$-modules (i. e. the action of $A$ is trivial), then we shall call such topological $A$-module prounipotent.

In contrast to [9], in the definition of a topological module we always require “finiteness of type”. If $A$ is a complete linearly compact Hopf algebra, then the corresponding category of topological $A$-modules is Abelian [9 Theorem 3.4].

Let $G$ be a finitely generated group and $k$ a field $char(k) = 0$, and let $G_a$ be its prounipotent completion. The lower central series $C_n(G_a)$ of the prounipotent group $G_a = \lim_{\leftarrow} G_n (G_a$ are unipotent groups) is defined by the rule $C_n(G_a) = \lim_{\leftarrow} C_n G_n(G_a)$. Since $G_\alpha$ are linear algebraic groups, there is an opportunity to define $G_\alpha(G_n(G_a))$ as subgroups of the lower central series of the linear algebraic group (details see in [10]). Denote the $n$-th element of the lower central series of a group $G$ by $L_n(G)$.

The prounipotent completion of any group is the inverse limit of unipotent completions of its nilpotent torsion free quotients

$G_n^\wedge = \lim_{\leftarrow} G_n(G/D_n)^\wedge \cong \lim_{\leftarrow} G_n(G_n(G_a)),$
where \([23, \text{Chapter 11, Theorem 1.10}]\) \(D_n = \sqrt{L_n(G)} = \{x \in G | x^n \in L_n(G), n \geq 1\}\).

By analogy with discrete and pro-\(p\) cases, on the rational points of the relations module \(\overline{R_u}(Q_p) = R_u/[R_u, R_u](Q_p) \cong R_u(Q_p)/[R_u(Q_p), R_u(Q_p)]\) of a prounipotent presentation \([4]\) \(Q\) there is a structure of topological \(\mathcal{O}(F_u)^*\) module with the filtration

\[
\overline{R_u}(Q_p)^j = \frac{[R_u(Q_p), R_u(Q_p)C_j(F_u(X))(Q_p)]}{[R_u(Q_p), R_u(Q_p)]}.
\]

Similarly, for \(\overline{C_u}(Q_p)\) the filtration is given using the rule

\[
\overline{C_u}(Q_p)^j = \frac{(\ker_d(\overline{Q_p}), \ker_d(Q_p)C_j(F_u(X))(Q_p))}{(\ker_d(\overline{Q_p}), \ker_d(Q_p)\ker_d(Q_p))}.
\]

**Proposition 2:** Assume we are given a prounipotent finite type presentation \([3]\), then \(\overline{C_u}(Q_p)\) and \(\overline{R_u}(Q_p)\) are prounipotent topological \(\mathcal{O}(G_u)^*\) modules with the filtrations \(C_u(Q_p)^j\) and \(R_u(Q_p)^j\).

**Proof:** Let us give the proof for \(\overline{C_u}(Q_p)\), for \(\overline{R_u}(Q_p)\) the argument is completely similar.

The fact that \(\cap C_j(F_u(X \cup Y))(Q_p) = 1\) implies \(\cap \overline{C_u}(Q_p)^j = 0\). CM 2) implies that \(R_u\) acts on \(\overline{C_u}(Q_p)\) trivially, and hence \(\overline{C_u}(Q_p)\) is a \(\mathcal{O}(G_u)^*\) module (for details see [2, 2.4] or [21, 3.2.5]). Finite dimension of quotients \(\overline{C_u}(Q_p)^j/\overline{C_u}(Q_p)^{j+1}\) follows from finiteness of the presentation. Indeed, since \(\overline{C_u}(Q_p)\), as an Abelian prounipotent group, is generated by the elements \(\cup_{\bar{y} \in Y} G_u(Q_p) \cdot \bar{y}\) (the finite set \(Y\) corresponds to the relations of the presentation \([3]\)), one has an epimorphism of \(\mathcal{O}(G_u)^*\) modules \(\mathcal{O}(G_u)^*|_{Y^1} \rightarrow \overline{C_u}(Q_p)\). The action by conjugation is prounipotent since \(F_u(X \cup Y)\) is a prounipotent group.

In [10, A.2] it is proved that the continuous prounipotent completion of a finitely generated free pro-\(p\)-group with a basis \(X\) and the prounipotent completion of the discrete free group contained in it with the same basis (and the induced pro-\(p\)-topology) are naturally isomorphic. This follows from the fact that any homomorphism from a finitely generated pro-\(p\)-group into the group of \(Q_p\)-points of a unipotent group is always continuous [10, Lemma A.7]. Hence for any finite \(X\) we have \(F(X)\) \(\cong F_u(X)\), where \(F_u(X)\) is the free prounipotent group [11]. Since \(d_0, d_1\) are epimorphisms, \(d_1(\ker_d)\) is a normal subgroup in \(F_u\) the quotient map \(F_u \rightarrow G_u = F_u/d_1(\ker_d)\) is a coequalizer [15, 3.3] of the diagram \(d_0, d_1\). The functor of continuous prounipotent completion has a right adjoint functor (\(Q_p\)-points of a prounipotent group) and hence preserves coequalizers [13, 5.5], and hence for finite type presentations one has an isomorphism (similar idea can be found in [27, p.284])

\[
G_u^w \cong G_u,
\]

where \(G_u^w\) is the continuous prounipotent completion of the pro-\(p\)-group from presentation [2], here \(G_u\) is the prounipotent group from [3] and the presentation [3] is obtained from [2] by means of prounipotent completion of pro-\(p\)-groups \(F(X)\) and \(F(X \cup Y)\).

The following results describe the structure of Abelianization of continuous prounipotent completions of the crossed modules of \(QR\) pro-\(p\)-presentations announced in [20].

**Lemma 2:** Let [11] be a finite \(QR\) pro-\(p\)-presentation, then using the isomorphism \(\overline{R_u}(Q_p) \cong \lim R/\langle R, RM_u \rangle \otimes Q_p\) of Abelian prounipotent groups, \(\overline{R_u}(Q_p)\) is endowed with a structure of topological \(G\)-module.
Proof: The idea of construction is contained in [20, Theorem 1], let us expose the construction completely. Let us check that the topological vector space $\mathbb{R} \otimes \mathbb{Q}_p$ considered as the group of $\mathbb{Q}_p$-points of an Abelian pronipotent group, has the universal property inherent to the group of $\mathbb{Q}_p$-points of $\mathbb{G}_m$, with respect to continuous (in the pro-$p$-topology induced from $F(X)$) Zariski-dense homomorphisms of $\mathbb{R}$ into Abelian pronipotent groups. Since each pronipotent group is the projective limit of a surjective projective system of unipotent groups, it suffices to check the universal property for homomorphisms into Abelian unipotent groups.

Let $\phi: R \rightarrow U(\mathbb{Q}_p) \cong \mathbb{Q}_p^d$ be a continuous in pro-$p$-topology and Zariski dense homomorphism. Since $\phi(R)$ is dense in $\mathbb{Q}_p^d$ in the Zariski topology and $\mathbb{Z}_p$ is a principal ideal domain, then, by compactness of $R$, we obtain $\phi(R) \cong \mathbb{Z}_p^d$. Let $\gamma_n : \mathbb{Z}_p^d \rightarrow \mathbb{Z}_p^d/p^n\mathbb{Z}_p^d = Z_1^d/p^nZ_1^d$, $W = \ker(\gamma_n \circ \phi)$.

By construction of the topology on $R$, we can find $k \in \mathbb{N}$ such that $R \cap \mathcal{M}_k \subseteq W$ [12], but $[R, \mathcal{M}_k] \subseteq R \cap \mathcal{M}_k$. $R/[R, \mathcal{M}_k]$ is a free Abelian pro-$p$-group (due to quasirationality), hence we can define a homomorphism of free Abelian pro-$p$-groups $\nu_k : R/[R, \mathcal{M}_k] \rightarrow \phi(R) \cong \mathbb{Z}_p^d$ by the formula $\nu_k(b) = \phi(a_i)$ on a free basis $b \in R/[R, \mathcal{M}_k], i \in I$. Let us describe the construction of this basis in more detail.

1) in $\mathbb{R}$ there exists a convergent basis $\{a_i\}, j \in J$, 2) in $[R, \mathcal{M}_k] \subseteq J \cap \mathcal{M}_k$, $R/[R, \mathcal{M}_k]$ is a free Abelian pro-$p$-group, $\Phi(\phi(R))$ is the Frattini subgroup in $\phi(R)$, and hence in $R/[R, \mathcal{M}_k]$;

3) now $\kappa_i$ is defined in the unique way putting $\kappa_k(b_i) = \bar{a}_i, i \in I$, and $\kappa_k(b_J) = 0$ on the rest of free generators of $R/[R, \mathcal{M}_k]$.

We construct $\psi_k : \mathbb{R} \otimes \mathbb{Q}_p \rightarrow U(\mathbb{Q}_p) = \mathbb{Q}_p^d$, extending $\phi$, by the rule $\psi_k = (\kappa_k \otimes id) \circ pr_k$, where $pr_k : R \otimes \mathbb{Q}_p \rightarrow [R, \mathcal{M}_k] \otimes \mathbb{Q}_p$ is the projection. Let $ker(\gamma_n) = ker(R \otimes \mathbb{R} \xrightarrow{\gamma_n} R \otimes [R, \mathcal{M}_k])$.

$kern(\gamma_n \otimes \mathbb{Q}_p) = ker(R \otimes \mathbb{R} \otimes \mathbb{Q}_p \xrightarrow{\gamma_n \otimes id} R \otimes [R, \mathcal{M}_k] \otimes \mathbb{Q}_p)$.

Now check that $ker(\gamma_n \otimes \mathbb{Q}_p)$ provide a $G$-filtration, which will be of finite type, due to finiteness of considered presentations. First of all note that $ker(\gamma_n \otimes \mathbb{Q}_p) \cong \lim_{\leftarrow k \geq n} [R, \mathcal{M}_k]/[R, \mathcal{M}_k] \otimes \mathbb{Q}_p$. Indeed, for each $k \geq n$ we have a short exact sequence

$$0 \rightarrow [R, \mathcal{M}_k] \rightarrow R \rightarrow R \rightarrow 0$$

of finitely generated free Abelian pro-$p$-groups (due to quasirationality), hence the following sequence of finite dimensional vector spaces will be also exact:

$$0 \rightarrow R \otimes \mathbb{Q}_p \rightarrow R \otimes \mathbb{Q}_p \rightarrow R \otimes \mathbb{Q}_p \rightarrow 0.$$
but since on the right we have a homomorphism of $G$-modules, on the left we have also a $G$-module. Now, since $\lim_{\gamma}^1 \frac{[R, R\mathcal{M}_n]}{[R, R\mathcal{M}_k]} \otimes \mathbb{Q}_p = 0$, we have an exact sequence

$$0 \rightarrow \lim_{n \geq 1} \frac{[R, R\mathcal{M}_n]}{[R, R\mathcal{M}_k]} \otimes \mathbb{Q}_p \rightarrow R \otimes \mathbb{Q}_p \rightarrow R \otimes \frac{[R, R\mathcal{M}_n]}{[R, R\mathcal{M}_k]} \otimes \mathbb{Q}_p \rightarrow 0,$$

which implies the required statement.

**Lemma 3:** Assume we are given a finite type pro-$p$-presentation (1), then $\mathbb{Q}_p$-points of the Abelianization $C_w^G$ of the prounipotent group

$$C_w^G := \left( \frac{\ker d}{[\ker d, \ker d]} \right)^\wedge_w$$

have a structure of topological finite type $G$-module, given by the decomposition $C_w^G(\mathbb{Q}_p) \cong \lim_{\gamma} \frac{\ker d}{[\ker d, \ker d]} \otimes \mathbb{Q}_p \cong \lim_{\gamma} (\mathbb{Q}_p(G/\mathcal{M}_k))^\wedge_\gamma$.

**Proof:** As noted above, the topology on $\ker d_0$ is induced by the topology on $F(Y \cup X)$. Then the argument of Lemma 2 (with small deviations) yields a presentation $(R\ker d_0)_w = \lim_{\gamma} \left( F(Y \times X)/\mathcal{M}_k \right)_w$, and hence, identifying $C_w^G$ with the projective limit of linear algebraic groups, the group of $\mathbb{Q}_p$-points of $C_w^G(\mathbb{Q}_p)$, we obtain

$$C_w^G(\mathbb{Q}_p) \cong \lim_{\gamma} \left( \frac{\ker d}{[\ker d, \ker d]} \right)_0^\wedge_w (\mathbb{Q}_p) \cong \lim_{\gamma} (\mathbb{Q}_p(G/\mathcal{M}_k))^\wedge_\gamma.$$

The latter isomorphism is a consequence of the general theory of crossed modules of pro-$p$-presentations. Indeed, [20] Theorem 1.13, Proposition 14 gives an isomorphism of $G$-modules $\ker d_0/\ker d_0 \cong \mathbb{Z}_p[G]^\wedge_\gamma$. It remains to define, similarly to Lemma 2 a finite type $G$-filtration by the rule

$$\lim_{n \geq 1} \frac{[\ker d_0, \ker d_1 \ker d_0, \mathcal{M}_k]}{[\ker d_0, \ker d_1 \ker d_0, \mathcal{M}_k]} \otimes \mathbb{Q}_p.$$

Consider the following diagram of Abelian prounipotent groups:

$$\begin{array}{ccc}
C_w^G & \xrightarrow{\gamma} & R_w^G \\
\downarrow{\kappa} & & \downarrow{\tau} \\
C_u & \xrightarrow{\mu} & R_u
\end{array}$$

We have denoted by $C_w^G, C_u, R_w^G, R_u$ the Abelianizations of the corresponding prounipotent groups. The homomorphisms $\kappa$ and $\tau$ arise from the universal properties of $C_w^G$ and $R_w^G$ with respect to the homomorphisms induced by a continuous embedding of a pro-$p$-presentation (2) into the rational points of the corresponding prounipotent presentation (3) ($\ker d_0 \hookrightarrow \ker d_0(\mathbb{Q}_p)$ and $R \hookrightarrow R_u(\mathbb{Q}_p)$). The homomorphism $\gamma$ is induced, in the notations (2), by the homomorphism of pro-$p$-groups $\nu : \ker d_0 \rightarrow R$.

**Theorem 1:** Let (1) be a finite $QR$ pro-$p$-presentation of a pro-$p$-group $G$, then one has a commutative diagram (5) of Abelian prounipotent groups in which on $\mathbb{Q}_p$-points $\gamma(\mathbb{Q}_p)$ is a homomorphism of topological $G$-modules, $\mu(\mathbb{Q}_p)$ is a homomorphism of $\mathcal{O}(G_u)^*$-modules, and $\kappa$ and $\tau$ are induced by embedding of a pro-$p$-presentation (2) into the corresponding prounipotent presentation (3) and are $G$-homomorphisms on Zariski dense subgroups $R$ and $\frac{[\ker d_0, \ker d_1 \ker d_0]}{[\ker d_0, \ker d_1 \ker d_0]}$ in $R_w^G(\mathbb{Q}_p)$ and $C_w^G(\mathbb{Q}_p)$, respectively.
Proof: Commutativity of the diagram follows from the fact that the morphisms in (3) are defined using the morphisms in (2). Since the action both in the upper and the lower rows of the commutative diagram is defined by conjugation, these are $G$-homomorphisms on dense subgroups $\ker_{d_0}(\ker_{d_1}(\ker_{d_0}))$ and $\overline{\mathcal{R}_u}$, where $G$ acts on $\mathcal{C}_u(Q_p)$ and $\overline{\mathcal{R}_u}(Q_p)$ through the homomorphism into its prounipotent completion. Continuity and the module homomorphism property of $\gamma$ are obvious from the construction in Lemma 2 and Lemma 3. The properties of $\mu$ follow from Proposition 2 and by general constructions from the beginning of this part of the paper. Recall that since $\mathcal{R}_u$ acts on $\overline{\mathcal{R}_u}$ trivially, $\overline{\mathcal{R}_u}$ is an $\mathcal{O}(G_u)^*$-module.

It is clear that $\kappa$ and $\tau$ determine continuous maps of topological vector spaces in the introduced in Lemma 2 and Lemma 3 filtrations on $\overline{\mathcal{R}_u}(Q_p)$ and $\mathcal{C}_u(Q_p)$, and continuity is actually shown there.

We have already noted (before Lemma 1) that in any simplicial group degenerate in dimensions two and greater, and in particular in the simplicial group (3) one has an isomorphism $\pi_1(F_\bullet) \cong \ker d_1$. And hence for the prounipotent presentation (3) in the notations of Lemma 1 it is natural to introduce the second homotopy group of the presentation as

$$u_2(Q_p) = \ker \{ \mathcal{C}_u(Q_p) \to \mathcal{R}_u(Q_p) \},$$

and also for the $QR$ pro-$p$-presentation (2) the continuous prounipotent completion of its second homotopy group

$$\pi_2 \otimes Q_p = \lim_{\leftarrow} \pi_2 \cap \ker (\ker d_0(\ker d_1(\ker d_0))) \otimes Q_p.$$

The next Corollary yields a relation between the second homotopy group of the $QR$-pro-$p$-presentation $\pi_2 = \ker (d_1 : C \to R)$ and the diagram from Theorem 1.

Corollary 2: In the notations of the previous Theorem, one has the diagram of topological vector spaces

$$\begin{array}{ccc}
\pi_2 \otimes Q_p & \xrightarrow{\gamma(Q_p)} & \overline{\mathcal{R}_u}(Q_p) \\
\downarrow \kappa(Q_p) & & \downarrow \tau(Q_p) \\
\overline{\gamma(Q_p)} & \xrightarrow{\mu(Q_p)} & \mathcal{R}_u(Q_p)
\end{array}$$

with exact rows, and an isomorphism of $\mathcal{O}(G_u)^*$-modules $\overline{\mathcal{C}_u}(Q_p) \cong \mathcal{O}(G_u)^{[1]}$.

Proof: Actually, generalizing [14], one can see that $\mathcal{R}_u$ is a free prounipotent group, and hence the canonical surjection $\mathcal{C}_u \to \mathcal{R}_u$ splits, and we can use the same argument as [2] Proposition 4, taking into account Theorem on commutant closure [22 4.3]. In particular, compiling [13] Proposition 14 using the prounipotent analog of the Brown–Loday Lemma (3 5.7, [24 4.3.8]), we see that $u_2(Q_p)$ is the prounipotent analog of the second homotopy group of the presentation. Coincidence of $\ker_{\mathcal{R}_u}(Q_p)$ with $\pi_2 \otimes Q_p$ follows from quasirationality and the fact that $\lim_{\leftarrow} \pi_1 = 0$ for a surjective projective system of finite dimensional vector spaces.

The standard isomorphism $\overline{\mathcal{C}_u}(Q_p) \cong \mathcal{O}(G_u)^{[1]}$ [24 Proposition 29] follows from Proposition 2 since $\overline{\mathcal{C}_u}(Q_p)$ possesses the universal property of a direct sum of algebras with respect to homomorphisms into prounipotent $\mathcal{O}(G_u)^*$-modules. Details can be completely reconstructed from the arguments in [24 Proposition 29] using Proposition 4.
Proposition 3: Let $G$ be a finitely generated pro-$p$-group given by a presentation (1) with one relation $r \neq 1$, and assume that the natural homomorphism from $G$ to the $\mathbb{Q}_p$-points $G_\mathbb{Q}_p$ of its prounipotent completion is an embedding, then $cd(G) = 2$.

Proof: First, note that the condition $G \hookrightarrow G_\mathbb{Q}_p$ implies that the homomorphism $\kappa(Q_p)$ from the commutative diagram (Corollary 2)

\[
\begin{array}{ccc}
\pi_2 \otimes \mathbb{Q}_p & \longrightarrow & C_2^G(\mathbb{Q}_p) \\
\gamma(Q_p) & \longrightarrow & \mathcal{R}_2^G(\mathbb{Q}_p) \\
\kappa(Q_p) & \longrightarrow & \tau(Q_p) \\
\eta_2(\mathbb{Q}_p) & \longrightarrow & C_2^G(\mathbb{Q}_p) \\
\mu(Q_p) & \longrightarrow & \mathcal{R}_2(\mathbb{Q}_p)
\end{array}
\]

is an isomorphism. Actually, since the image $\kappa(Q_p)$ is dense, $\kappa(Q_p)$ is an epimorphism. On the other hand, Lemma 3 implies that the elements $G \cdot y$, where $y \in Y$ (in the notation of Definition 2) form a pseudo-basis of $C_2^G(\mathbb{Q}_p)$ as a topological vector $\mathbb{Q}_p$-space. But Corollary 2 implies that $C_u(\mathbb{Q}_p) \cong \mathcal{O}(G_u)^*|Y]$ and therefore, as a consequence, $\kappa(Q_p)$ is injective.

Assume that $\mathcal{R}_u(\mathbb{Q}_p) \cong \mathcal{O}(G_u)^*$ [21 Corollary 12], then our diagram (only as a diagram of topological vector spaces) takes the form

\[
\begin{array}{ccc}
\mathcal{O}(G_u)^* & \longrightarrow & \mathcal{R}_u(\mathbb{Q}_p) \\
\kappa(Q_p) & \longrightarrow & \tau(Q_p) \\
\mathcal{O}(G_u)^* & \longrightarrow & \mathcal{O}(G_u)^*.
\end{array}
\]

Due to quasirationality of presentations of pro-$p$-groups with a single defining relation [21 Proposition 1], Lemma 2 gives an isomorphism

$\mathcal{R}_u(\mathbb{Q}_p) \cong \mathcal{R}_\otimes \mathbb{Q}_p = \lim_{\leftarrow} R/[R, R\mathcal{M}_u] \otimes \mathbb{Q}_p$.

Then left exactness of the functor $\lim_{\leftarrow}$ yields injectivity of $\mathcal{R} \hookrightarrow \mathcal{R}_\otimes \mathbb{Q}_p$, which implies $G$-embedding $\mathcal{R} \hookrightarrow \mathcal{R}_u(\mathbb{Q}_p)$. Therefore, $\mathcal{R} \hookrightarrow \mathcal{O}(G_u)^*$, and hence the elements of the form $G \cdot y$ form a permutational basis in $\mathcal{R}$. But then $G$ is an aspherical pro-$p$-group [15 §2], and since it is torsion free, one has $cd(G) = 2$ [15 (3.2)]. Let us note that $G_\mathbb{Q}_p \cong G_u(\mathbb{Q}_p)$ [3], and hence the statement of Proposition 3 one could equivalently assume $G \hookrightarrow G_u(\mathbb{Q}_p)$.

The following Corollary generalizes and explains the known group theory results (see, for example, [13 and 29], where absence of divisors of 0 in the graded algebra of a filtration is used). We shall say that a pro-$p$-group $G$ is $p$-regular if for any finite quotient $G/\mathcal{M}_u(G)$ there exists a torsion free nilpotent quotient $G/V$ and $V \subset \mathcal{M}_u(G)$. We shall also say that a pro-$p$-group $G$ has a $p$-regular filtration $V = (V_n, n \in \mathbb{N})$ if for any finite quotient $G/\mathcal{M}_k(G)$ there exists $m(k) \in \mathbb{N}$ such that $V_{m(k)} \subset \mathcal{M}_k(G)$ and the quotients of this filtration are torsion free.

Corollary 3: Assume that a pro-$p$-group $G$ with one relation is $p$-regular, then $cd(G) = 2$. In particular, if $G$ has a $p$-regular filtration $V$, then $cd(G) = 2$. 
Proof: If $G_λ$ is a finitely generated nilpotent pro-$p$-group without torsion, then, due to [10] Corollary A.4, $G_λ$ is included into the group of $Q_p$-points of a unipotent group $U$ over $Q_p$, and hence $G_λ$ is embedded into the group of $Q_p$-points of its continuous prounipotent completion $(G_λ)^w_w$.

Since $G$ is $p$-regular, one has $G \cong \varprojlim G_λ$, where $G_λ$ are finitely generated nilpotent torsion-free pro-$p$-groups. For each $λ$ we have an embedding $G_λ \hookrightarrow (G_λ)^w_w(Q_p)$, and due to left exactness of the projective limit functor, we have also an embedding $G \hookrightarrow \varprojlim(G_λ)^w_w(Q_p)$. Obviously, there exists an epimorphism $ζ : G^w_w(Q_p) \twoheadrightarrow \varprojlim(G_λ)^w_w(Q_p)$, and the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{β} & G^w_w(Q_p) \\
\downarrow{γ} & & \downarrow{ζ(Q_p)} \\
\varprojlim(G_λ)^w_w(Q_p) & & \\
\end{array}
\]

is commutative, but since $γ$ is injective, $β$ is also an embedding. And we can apply Proposition 3. Clearly, the absence of torsion on the graded quotients of a $p$-regular filtration $V$ is sufficient for $p$-regularity of $G$.

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