A simpler sublinear algorithm for approximating the triangle count

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Abstract
A recent result of Eden, Levi, and Ron (ECCC 2015) provides a sublinear time algorithm to estimate the number of triangles in a graph. Given an undirected graph $G$, one can query the degree of a vertex, the existence of an edge between vertices, and the $i$th neighbor of a vertex. Suppose the graph has $n$ vertices, $m$ edges, and $t$ triangles. In this model, Eden et al provided a $O(poly(\varepsilon^{-1} \log n)(n/t^{1/3} + m^{3/2}/t))$ time algorithm to get a $(1+\varepsilon)$-multiplicative approximation for $t$, the triangle count. This paper provides a simpler algorithm with the same running time (up to differences in the $poly(\varepsilon^{-1} \log n)$ factor) that has a substantially simpler analysis.

1 Introduction

Counting the number of triangles in a graph is a fundamental algorithmic problem. It has been studied widely in theory and well as practice. With the recent study of complex networks and massive real-world graphs, triangle counting is a key operation in graph analysis for bioinformatics, social networks, community analysis, and graph modeling [HL70, Col88, EM02, Por98, MSI+02, Bur04, BBCG08, WDC10, BHL11, SKP12].

There has been much study on triangle counting by the theoretical computer science community, where the primary hammer is fast matrix multiplication [IR78, AYZ97, BPWZ14]. On the more applied side, there is a plethora of provable and practical algorithms that employ clever sampling methods for approximate triangle counting [CN85, SW05b, SW05a, Tso08, TKMF09, KMPT10, Avr10, CC11, SV11, TKM11, AKM12, SPK13, TPT13]. Triangle counting has also been a popular problem for the streaming setting [BYKS02, JG05, BFL+06, AGM12, KMSS12, JSP13, PTW13, TPT13, ADNK14].

Yet all these algorithms always read the entire graph. A recent result of Eden, Levi, and Ron [ELR15] provided the first truly sublinear algorithm. For a graph with $n$ vertices, $m$ edges, and $t$ triangles, their running time is $O(poly(\varepsilon^{-1} \log n)(n/t^{1/3} + m^{3/2}/t))$. Observe that when $t = \Omega(m^{1/2+\delta})$ (for constant $\delta > 0$), this algorithm does not even read the entire graph. They also prove a lower bound, tight up to the $poly(\varepsilon^{-1} \log n)$ factor. The algorithm and analysis are quite intricate, and require a number of ideas. This paper provides a simpler algorithm with a simpler analysis.

Let us explain the formal result. The input is an undirected graph $G = (V, E)$, stored as an adjacency list. We set $V = [n]$, where $n$ is known to the algorithm. The following queries are allowed:

- Degree queries: for any $v \in V$, we can get $d_v$, the degree of $v$.
- Neighbor queries: for any $v \in V$ and $i \leq d_v$, we can get the $i$th neighbor of $v$.
- Edge queries: for any $u, v \in V$, we can check if $(u, v) \in E$.
We denote the number of edges by $m$ and the number of triangles by $t$. Our main result is the following (identical to that of [ELR15]).

**Theorem 1.** There exists an algorithm with the following guarantee. Given query access to graph $G$ and an approximation parameter $\varepsilon > 0$, the algorithm outputs an estimate for $t$ in the range $[(1 - \varepsilon)t, (1 + \varepsilon)t]$ with probability at least $\frac{2}{3}$. The expected running time of the algorithm is $O(\text{poly}(\varepsilon^{-1} \log n)(n/t^{1/3} + m^{3/2}/t))$.

We can easily boost the success probability by repeating and taking the median estimate. The query complexity can be made $O(\text{poly}(\varepsilon^{-1} \log n) \min(m, n/t^{1/3} + m^{3/2}/t))$. As we shall see, the algorithm begins by quickly obtaining an accurate estimate of $m$. If the algorithm ever makes more queries than $m$, we can pause, query the entire graph and store it. Now, we run the remainder of the algorithm on this stored version.

### 1.1 Previous work

For the sake of clarity, we suppress any dependences on the approximation parameter $\varepsilon$ or on $\log n$ using the notation $O^*(\cdot)$. The result of Eden, Levi, and Ron [ELR15] (denoted ELR) can be traced back to ideas of Feige [Fei06] and Goldreich and Ron [GR08], who provided sublinear algorithms for estimating the average degree. Feige shows that averaging the degree of $O^*(n/\sqrt{m})$ uniform random vertices suffices to give a $(2 + \varepsilon)$-approximation of the average degree. His primary tool was a new tail bound for sums of independent random variables with unbounded variance. Goldreich and Ron subsequently give (Feige also observes this in the journal version) a substantially simpler proof, and the cost of a worse dependence on $\varepsilon$. Extending the analysis, they provide a $(1 + \varepsilon)$-approximation with the same query complexity when the algorithm is also allowed neighbor queries. Gonen, Ron, and Shavitt [GRS11] build on these ideas to count stars, and ELR goes further to count triangles.

We note that there is a significant leap required to perform triangle counting. Since the query model allows the algorithm to directly get degrees, it is plausible that one can estimate moments of the degree sequence. Triangle counting is a much harder beast, since mere knowledge of degrees provides no (obvious) help.

The high level approach in ELR, building on Goldreich-Ron and Gonen et al, is vertex bucketing. We bin the vertices in $O(\log n)$ buckets, by rounding their triangle count (or degree, in previous work) down to the nearest power of $(1 + \varepsilon)$. With respect to the buckets, each edge or triangle has only $\text{poly}(\log n)$ different possibilities, or “types”. So we can apply concentration bounds for each type of edge/triangle, take a union bound over the different types. Of course, the challenge is now in counting triangles of a given type. There are numerous issues caused by small buckets and determining the type of a triangle, which lead to a fairly complex algorithm and analysis.

### 1.2 Main ideas

Our approach goes via a different route. Start with the exact triangle counter of Chiba and Nishizeki [CN85]. Simply enumerate over all edges, and count the number of triangles $t_e$ containing each edge $e$. For edge $e = (u, v)$, it suffices to compute the intersection of neighbor lists of $u$ and $v$. This can be done in $\min(d_u, d_v)$ time, leading to an overall bound of $\sum_{e=(u,v)} \min(d_u, d_v)$. By bounds on the arboricity (or degeneracy) of graphs, this expression can be bounded by $O(m^{3/2})$.

How about implementing this algorithm using sampling? Pick a u.a.r. (uniform at random) edge $e$ and let $u$ denote the endpoint of lower degree. Pick a u.a.r. neighbor $w$ of $u$. If $e$ and $w$ form a triangle, set $X = d_u$. Else, $X = 0$. The expected value of $X$ is precisely $3t/m$, so one hopes that $O(m/t)$ samples suffices to get concentration. But
the variance of $X$ could be too large, so we simply use more samples for “high variance edges”. After picking $e$ with endpoint $u$, if $d_u$ is large, we sample more neighbors $w_1, w_2, \ldots$ and take the average of the corresponding $X$ values. We set the number of these neighbors to be $d_u/\sqrt{m}$, and can show that $\text{var}(X) \leq \sqrt{mE[X]}$. So $O(m^{3/2}/t)$ samples suffice for concentration. But the time per sample has potentially gone up from constant to expected $m^{-1}\sum_{e=(u,v)} \min(d_u, d_v)/\sqrt{m}$. By Chiba-Nishizeki’s bound, this is still $O(1)$ and we are done!

Our problem is not yet solved, because we need to sample a uniform random edge, a query that is not allowed. We can sample some set $S$ of vertices, query all their degrees, and can sample a u.a.r. edge incident to $S$ in $O(1)$ time. We could perform the above algorithm on edges incident to $S$, but we required the number of triangles incident to $S$ to concentrate well. This is suspiciously similar to the average degree questions of Feige and Goldreich-Ron. Using a stripped-down version of Feige’s analysis, we can argue that sampling $O(n/t^{1/3})$ vertices is enough, with a painful caveat. This only provides a $3$-approximation to $t$ (analogous to Feige’s $2$-approximation for average degree). To get down to $(1 + \varepsilon)$, we need to sample a uniform triangle incident to $S$, determine the number triangles incident to these vertices, and then weight the triangle accordingly. (ELR also perform the sampling of $n/t^{1/3}$ vertices and explicitly move from the $3$ to $(1 + \varepsilon)$-approximation. In this analysis, we go straight to the $(1 + \varepsilon)$-approximation.) The number of triangles incident to a vertex $v$ can be estimated by the algorithm in the previous paragraph, by sampling edges incident to $v$. With some care, all of this can be put together in $O^*(n/t^{1/3} + m^{3/2}/t)$ time.

As an aside, we also give a simpler analysis of a $(1 + \varepsilon)$-approximation $O^*(n/\sqrt{m})$ algorithm for the average degree, first shown by Goldreich-Ron [GR08]. We defer this to the end of the paper.

2 Preliminaries

We use $d_v$ for the degree of vertex $v$, $\delta(v)$ for the set of edges incident to $v$, and $\Gamma(v)$ for the neighborhood of $v$. We use $t_e$ for the number of triangles incident to edge $e$, and set $t_v = \sum_{e \in \delta(v)} t_e$. Note that the latter is twice the number of triangles incident to $v$. The set of triangles incident to $e$ is $T_e$. We use $c, c_1, \ldots$ to denote sufficiently large constants.

Our initial description of the algorithm will use the value of $m$ and $t$ to decide how much to sample. This (somewhat circular) assumption is easily removed by doing a geometric search on $m$ and $t$, and is explained at the end of our proof. We use $\varepsilon$ to denote the approximation parameter.

3 Heavy and light vertices

Roughly speaking, a heavy vertex is one with either high degree or many triangles. By “high”, we mean in the top $t^{1/3}$ values. For the main algorithm, it will be important to distinguish such heavy vertices efficiently.
Lemma 2

1. If $d_v > 2m/(\varepsilon t)^{1/3}$, output heavy.
2. Repeat for $i = 1, 2, \ldots, c \log n$:
   (a) Repeat for $j = 1, 2, \ldots, (4/\varepsilon^2)(m^{3/2}/t) = s$:
      i. Select u.a.r. edge $e \in \delta(v)$, and let $u$ be endpoint with smaller degree.
      ii. Repeat for $k = 1, 2, \ldots, [d_u/\sqrt{m}]$:
         A. Pick u.a.r. neighbor $w$ of $u$.
         B. If $e$ with $w$ forms a triangle, set $Z_k = d_u$, else $Z_k = 0$.
   iii. Set $Y_j = \sum_k Z_k/d_u/\sqrt{m}$.
3. If median of $X_i$s is greater than $t^{2/3}/\varepsilon^{1/3}$, output heavy, else output light.

We have three nested loops, with loop variables $i, j, k$ respectively. We reference these as “iteration $i$”, “iteration $j$”, and “iteration $k$”.

Lemma 2. For any iteration $i$, $\Pr[|X_i - t_v| > \varepsilon \max(t_v, td_v/m)] < 1/4$.

Proof. Fix an iteration $j$, and let $e_j$ denote the edge chosen in the $j$th iteration, with $u_j$ as the smaller degree endpoint. We use $\mathcal{E}_j$ to denote the event of $e_j$ being chosen. The probability of finding a triangle in any iteration $k$ is $t_e/d_u$. Hence, $E[Z_k|\mathcal{E}_j] = (t_e/d_u) \cdot d_u = t_e$, and $\text{var}(Z_k|\mathcal{E}_j) \leq E[Z_k^2|\mathcal{E}_j] \leq d_u E[Z_k|\mathcal{E}_j]$. By linearity of expectation, $E[Y_j|\mathcal{E}_j] = t_e$ and by independence, $\text{var}(Y_j|\mathcal{E}_j) \leq d_u E[Y_j]/[d_u/\sqrt{m}] \leq \sqrt{m}E[Y_j|\mathcal{E}_j]$. The conditioning can be removed to yield $E[Y_j] = \sum_{e \in \delta(v)} t_e/d_u = t_v/d_u$ and $\text{var}(Y_j) \leq \sqrt{m}E[Y_j]$.

By Chebyshev’s inequality on $Y = \sum_j Y_j/s$, $\Pr[|\bar{Y} - t_v/d_v| > \varepsilon \max(t_v/d_v, t/m)]$ is at most

$$\frac{\text{var}(\bar{Y})}{\varepsilon^2 \max(t_v/d_v, t/m)^2} \leq \frac{\sqrt{m}(t_v/d_v)}{\varepsilon^2(4/\varepsilon^2)(m^{3/2}/t) \cdot (t_v/d_v) \cdot (t/m)} = 1/4$$

Lemma 3. The following hold with probability $> 1 - 1/n$ over all calls of heavy. If $v$ is declared light, then $t_v \leq 2t^{2/3}/\varepsilon^{1/3}$. If $v$ is declared heavy, then $d_v > 2m/(\varepsilon t)^{1/3}$ or $t_v > t^{2/3}/2\varepsilon^{1/3}$.

Proof. It is more convenient to prove the contrapositive statements. Obviously, if $d_v \leq 2m/(\varepsilon t)^{1/3}$, then $v$ is heavy. So assume that $d_v \geq 2m/(\varepsilon t)^{1/3}$. Then, $td_v/m \leq 2t^{2/3}/\varepsilon^{1/3}$. Suppose $t_v > 2t^{2/3}/\varepsilon^{1/3}$. By Lemma 2, for any iteration $i$, $\Pr[|X_i - t_v| > \varepsilon t_u] < 1/4$. By a standard Chernoff bound, the median of the $c \log n$ $X_i$s will be greater than $t^{2/3}/\varepsilon^{1/3}$ with probability at least $1 - 1/n^2$, and $v$ is declared heavy.

Suppose $t_v \leq t^{2/3}/2\varepsilon^{1/3}$. By Lemma 2, $\Pr[|X_i - t_v| > \varepsilon(2t^{2/3}/\varepsilon^{1/3})] < 1/4$. By Chernoff again, the median will be less than $t^{2/3}/\varepsilon^{1/3}$ with probability at least $1 - 1/n^2$, and $v$ is light. A union bound over all $v$ completes the argument.

Obviously, there is an upper bound on the number of vertices with either high degree or many triangles.

Corollary 4. The number of heavy vertices is at most $3(\varepsilon t)^{1/3}$.

The following is where the degeneracy bounds come into play. We give a direct, self-contained proof, but note the connection to Chiba-Nishizeki’s bound.

Lemma 5. The expected runtime of heavy$(v)$ is $O^*(m^{3/2}/t)$. 
Lemma 3. Lemma 3
Lemma 6.
is at most 2
is at most 9
But the number of triangles with weight 0 is at most this expression is 0. By interchanging summations,
Proof. Define indicator \( \chi(e, \Delta) \) for triangle \( \Delta \) containing edge \( e \). Consider a triangle \( \Delta \) that contains \( \ell \neq 0 \) light vertices. Then \( \sum_{v \in L} \sum_{e \in \delta(v)} \chi(e, \Delta) \) is exactly \( 2\ell \), which is \( 1/wt(\Delta) \). If \( \ell = wt(\Delta) = 0 \), this expression is 0. By interchanging summations,
\[
\sum_{v \in L} \sum_{e \in \delta(v)} wt(T_v) = \sum_{\Delta} wt(\Delta) \sum_{v \in L} \sum_{e \in \delta(v)} \chi(e, \Delta) = t - |\{\Delta|wt(\Delta) = 0\}|.
\]
But the number of triangles with weight 0 is at most \( \left( \frac{|H|}{3} \right) \), which by Corollary 4 is at most \( 9\varepsilon t \).

We define \( wt(T) = \sum_{\Delta \in T} wt(\Delta) \) for any set \( T \) of triangles. Abusing notation, define \( wt(v) = \sum_{e \in \delta(v)} wt(T_v) \) for \( v \in L \), and \( wt(v) = 0 \) for \( v \in H \).

Theorem 7. Let \( s \geq (c \log(n/\varepsilon)/\varepsilon^3)n/t^{1/3} \). Sample \( s \) u.a.r. vertices \( v_1, v_2, \ldots, v_s \). Then \( \mathbb{E}[\sum_{i \leq s} wt(v_i)/s] \in [t(1 - 9\varepsilon)/n, t/n] \) and \( \Pr[\sum_{i \leq s} wt(v_i)/s < t(1 - 10\varepsilon)/n] < \varepsilon^2/n \).

Proof. The expectation holds by Lemma 6. Let \( Y \) denote the random variable \( \sum_{i \leq s} wt(v_i)/s \). Note that \( wt(v) \leq t_v \), which by Lemma 3 is at most \( 2t_v^{2/3}/\varepsilon^{1/3} \). A multiplicative Chernoff bound tells us that \( \Pr[Y < \mathbb{E}[Y](1 - \varepsilon)] < \exp(-\varepsilon^2 s \mathbb{E}[Y]/3(2t_v^{2/3}/\varepsilon^{1/3})) \). Algebra on the exponent:
\[
\frac{\varepsilon^2 s \mathbb{E}[Y]}{6t_v^{2/3}/\varepsilon^{1/3}} \geq \frac{c \log(n/\varepsilon)(n/\varepsilon t^{1/3}) \cdot t/(2n)}{6t_v^{2/3}/\varepsilon^{1/3}} = \Omega(\log(n/\varepsilon)).
\]

\[\square\]
4 The full algorithm

We come to the main algorithm. It is convenient to define a procedure \texttt{estimate} uses the values of \(m\) and \(t\), and only has a lower tail bound. Later, we employ Markov and geometric search to get the bonafide algorithm that estimates \(t\).

\begin{table}[h]
\begin{center}
\begin{tabular}{|l|}
\hline
\textbf{estimate} \hline
1. Sample \(s_1 = c_1\epsilon^{-3}\log(n/\epsilon)nt^{-1/3}\) u.a.r. vertices. Call this multiset \(S\). \\
2. Set up a data structure to sample vertices in \(S\) proportional to degree. \\
3. Repeat for \(i = 1, 2, \ldots, s_2 = c_2\epsilon^{-4}(\log^2 n)m^{3/2}t^{-1}\) times: \\
   \hspace{0.5cm} (a) Sample \(v \in S\) proportional to \(d_v\) and sample u.a.r. \(e \in \delta(v)\). Let \(u\) be lower degree endpoint. \\
   \hspace{0.5cm} (b) If \(d_u \leq \sqrt{m}\), set \(R = 1\) with probability \(d_u/\sqrt{m}\) and set \(R = 0\) otherwise. If \(d_u > \sqrt{m}\), set \(R = \lfloor d_u/\sqrt{m} \rfloor\). \\
   \hspace{0.5cm} (c) Repeat for \(j = 1, 2, \ldots, R\):
      \hspace{1cm} i. Pick u.a.r. neighbor \(w\) of \(u\).
      \hspace{1cm} ii. If \(e\) with \(w\) does not form triangle, set \(Z_j = 0\).
      \hspace{1cm} iii. If \(e\) with \(w\) forms triangle \(\Delta\): call \texttt{heavy} for all vertices in \(\Delta\). If \(v\) is heavy, set \(Z_j = 0\). Otherwise, set \(Z_j = \max(d_u, \sqrt{m})wt(\Delta)\). \\
   \hspace{0.5cm} (d) Set \(Y_i = \sum_j Z_j / R\). (If \(R = 0\), set \(Y_i = 0\).) \\
4. Output \(X = n(\sum_{v \in S} d_v)(\sum_i Y_i)/s_1s_2\) \\
\hline
\end{tabular}
\end{center}
\end{table}

**Theorem 8.** \(E[X] \in [t(1-2\epsilon), t]\) and \(\Pr[X < t(1-20\epsilon)] < 3\epsilon/\log n\).

**Proof.** There are three “levels” of randomness. First is the choice of \(S\), the second level is the choice of \(e\) (Step 3a), and finally the \(Z_j\)s. For analyzing the randomness in any level, we condition on the previous levels. It is helpful to break the proof up using claims.

**Claim 9.** Condition of some \(S\) being chosen, and let \(d_S = \sum_{v \in S} d_v\). \(E[Y_i|S] = d_S^{-1} \sum_{v \in S} wt(v)\) and \(\text{var}(Y_i|S) \leq \sqrt{mE[Y_i|S]}\).

**Proof.** This is similar to the argument in Lemma 2. Let vertex \(v_i\) and edge \(e_i\) with lower degree endpoint \(u_i\) be chosen in iteration \(i\). We simply refer to this event by \(E_i\), so the conditioning is over \(S\) and \(E_i\). (We use \(\mathbb{1}\) for the indicator for event \(E_i\).)

If \(d_{u_i} \leq \sqrt{m}\), \(E[Y_i|S, E_i] = (d_{u_i}/\sqrt{m}) d_S^{-1} \mathbb{1}_{v_i \in L} \sum_{T_{e_i} \in T} \mathbb{1}_{v_i \in L} \sum_{T_{e_i} \in T} \sqrt{m} \cdot wt(\Delta) = \mathbb{1}_{v_i \in L} wt(T_{e_i})\).

Since the maximum value of \(Y\) in this case is at most \(\sqrt{m}\), \(\text{var}(Y_i|S, E_i) \leq \sqrt{mE[Y_i|S, E_i]}\).

We prove the same bound if \(d_{u_i} > \sqrt{m}\). So \(E[Z_j|S, E_i] = d_{u_i}^{-1} \mathbb{1}_{v_i \in L} \sum_{T_{e_i} \in T} \mathbb{1}_{v_i \in L} wt(\Delta) = \mathbb{1}_{v_i \in L} wt(T_{e_i})\).

As before, \(\text{var}(Z_j|S, E_i) \leq d_{u_i} E[Z_j|S, E_i]\).

By linearity of expectation \(E[Y_i|S, E_i] = \mathbb{1}_{v_i \in L} wt(T_{e_i})\).

By independence of the \((Z_j|S, E_i)\) variables, \(\text{var}(Y_i|S, E_i) \leq \sqrt{mE[Y_i|S, E_i]}\).

We remove the conditioning on \(E_i\):

\[E[Y_i|S] = \sum_{v \in S} \frac{d_v}{d_S} \sum_{e \in \delta(v)} \mathbb{1}_{v_i \in L} wt(T_{e_i}) = d_S^{-1} \sum_{v \in S} \sum_{\substack{e \in \delta(v) \in \delta(v)}} \mathbb{1}_{v_i \in L} wt(T_{e_i})\]

We also have \(\text{var}(Y_i|S) \leq \sqrt{mE[Y_i|S]}\).

Hence, \(E[X|S] = nd_S E[Y_i|S]/s_1 = n \sum_{v \in S} wt(v)/|S|\). By Theorem 7, the expectation over \(S\) (which yields \(E[X]\)) is in \([t(1-9\epsilon), t]\).

We will call \(S\) \textit{good} if \(\sum_{v \in S} wt(v)/s_1 \geq t(1-10\epsilon)/n\). By Theorem 7, this happens with probability at least \(1 - \epsilon^2/n\). We call \(S\) \textit{great} if, in addition to being good, \(d_S = \sum_{v \in S} d_v \leq \)
Theorem 8. The expected value, over $S$, of $d_S$ is $s_1(2m/n)$. By the Markov bound and the union bound, the probability that $s$ is great is at least $1 - 2\varepsilon/\log n$.

We apply Chebyshev’s inequality on $Y = \sum_i Y_i/s_2$, conditioned over $S$. We get that $\Pr[|Y| > \varepsilon]/E[Y|S]| > \varepsilon/E[Y|S]$ is at most

$$\frac{\text{var}(Y|S)}{\varepsilon^2E[Y|S]^2} \leq \frac{\sqrt{mE[Y|S]}}{\varepsilon^2(2\varepsilon^2 - \log^2 n)m^{3/2}/t - 1} = \frac{\varepsilon^2}{c_2(2\log^2 n)(m/t)E[Y|S]}.$$

Note that $E[Y|S] = d^{S}_{-1}\sum_{v}\text{wt}(v)$, which for great $S$ is at least $(t/4n)/(\log n/\varepsilon)$. Hence, $\Pr[|Y| > \varepsilon/E[Y|S]| \leq \varepsilon/\log n$. Conditioned on $S$, $X$ is just a scaling of $Y$. So we get $\Pr[|X| > \varepsilon/E[X|S]| \leq \varepsilon/\log n$. Note that $E[X|S] = n\sum_{v}\text{wt}(v)/s_1$, which for great $S$ is at least $t(1 - 10\varepsilon)$. Hence, for great $S$, $\Pr[X|S < t(1 - 20\varepsilon)/n] \leq \varepsilon/\log n$. The probability of $S$ not being great is at most $2\varepsilon/\log n$. We apply the union bound to remove the conditioning, so $\Pr[X < t(1 - 20\varepsilon)/n] \leq 3\varepsilon/\log n$. \qed

Theorem 10. The expected running time of estimate is $O^*(n/t^{1/3} + m^{3/2}/t)$.

Proof. The sampling of $S$ is done in $O^*(n/t^{1/3})$ time. Let us compute the expected number of triangles found. In iteration $i$, we are basically picking a u.a.r. edge of $G$. Conditioned on choosing edge $e$, the expected number of triangles found is at most $2(d_u/\sqrt{m})(d_v/\sqrt{m}) = 2t/\sqrt{m}$. Averaging over edge $e$, the expected number of triangles found in a single iteration is at most $2t/m^{3/2}$. There are $O^*(m^{3/2}/t)$ iterations, leading to grand total of $O^*(1)$ expected triangles. Thus, there are $O^*(1)$ expected calls to heavy, each taking $O^*(m^{3/2}/t)$ time by Lemma 5.

The time required to generate the $Z_i$s is also $O^*(m^{3/2}/t)$ by an argument identical to that in the proof of Lemma 5. \qed

Theorem 11. There exists a $O^*(n/t^{1/3} + m^{3/2}/t)$ algorithm that provides an estimate for the triangle count in $[(1 - \varepsilon)t, (1 + \varepsilon)t]$ with probability at least $5/6$.

Proof. First, we need to estimate $m$. This can be done with suitably high probability by the algorithm of Goldreich-Ron [GR08] in $O^*(n/\sqrt{m})$ time. (We give an independent proof in the next section of Theorem 13, which gives the desired algorithm.) Since $t \leq m^{3/2}$, this can be absorbed in the $O^*(n/t^{1/3})$ bound.

We perform a geometric search for $t$, by guessing its value as $n^3, n^3/2, n^3/2^2, \ldots$. We run the procedure of Theorem 8 independently $\varepsilon^{-1}\log \log n$ times, and take the minimum estimate. By Markov $\Pr[X \leq (1 + \varepsilon)t] > \varepsilon/2$. With probability at least $1 - 1/\log^2 n$, the minimum estimate is at most $(1 + \varepsilon)t$.

If the estimate is larger than the current guess of $t$, we halve the guess for $t$. Eventually, we reach an appropriate guess where the estimates of Theorem 8 to kick in. At the stage, the minimum of $\varepsilon^{-1}\log \log n$ estimates is at most $(1 + \varepsilon)t$ and at least $(1 - \varepsilon)t$ with probability at least $1 - 1/\log n$ (we rescale $\varepsilon$ from Theorem 8). A union bound over all errors completes the proof. \qed

5 A (1 + $\varepsilon$)-approximation for the average degree

We impose a total order $\succ$ on the vertices, where $u \prec v$ if $d_u < d_v$ or, $d_u = d_v$ and the ID of $u$ is less than that of $v$. (The latter is just an arbitrary but consistent tie-breaking rule). We use $d^*_v$ to denote the number of neighbors of $v$ “higher” than $v$ according to $\prec$.

Define random variable $X$ as follows. Pick u.a.r. vertex $v$, then pick u.a.r. vertex $u \in \Gamma(v)$. If $v \prec u$, $X = 2d_u$, else $X = 0$. We use $d$ to denote $2m/n$, the average degree.
Theorem 12. Suppose $s \geq \lfloor c\log(\varepsilon^{-1}\log n)\varepsilon^{-2}\rfloor n / \sqrt{m}$. Let $X_1, \ldots, X_s$ be i.i.d. as $X$ (defined above), and $\overline{X} = \sum_i X_i / s$. Then $E[\overline{X}] = \overline{d}$ and $Pr[\overline{X} < (1 - \varepsilon)\overline{d}] < 1 / \varepsilon^2 \log^2 n$.

Proof. The expectation bound is direct. $E[\overline{X}] = n^{-1} \sum_v (d_v^+/d_v) \cdot 2d_v = 2\sum_v d_v^+ / n = \overline{d}$. Let $v_1, v_2, \ldots, v_s$ be the u.a.r. vertices corresponding to $X_1, \ldots, X_s$. Set $k = \sqrt{m}$, so let $S_k$ denote the highest $k$ vertices, according to $\prec$. First, some properties of $S_k$. For all $v \in S_k$, $d_v^+ \leq k$. On the other hand, for all $v \notin S_k$, $d_v \leq 2m / k$. (If not, the total sum of degrees in $S_k$ exceeds $2m$.)

Define random variable $Y_i$ as: $Y_i = X_i$ if $v_i \notin S_k$ and 0 otherwise. Denoting $\overline{Y} = \sum_i Y_i / s$, note that $\overline{X} \geq \overline{Y}$, so it suffices to provide a lower tail for $\overline{Y}$. Observe that $E[Y] = 2n^{-1} \sum_{v \notin S_k} d_v^+ = 2n^{-1} (m - \sum_{v \in S_k} d_v^+) \geq 2n^{-1} (m - k^2) \geq (1 - \varepsilon)\overline{d}$.

We apply a standard multiplicative Chernoff bound on $\overline{Y} = \sum_i Y_i / s$, noting that $Y_i \in [0, 2m / k]$. So $Pr[\overline{Y} \geq (1 - \varepsilon)E[\overline{Y}]] \leq \exp(-\varepsilon^2 sE[\overline{Y}] / 3(2m / k))$. and going through the motions,

$$\varepsilon^2 sE[\overline{Y}] / 3(2m / k) \geq c(1 - \varepsilon) \log(\varepsilon^{-1} \log n)\overline{d}n / 6m = \Omega(\log(\varepsilon^{-1} \log n))$$

By a sufficiently large choice of $c$, $Pr[\overline{Y} \geq (1 - \varepsilon)^2\overline{d}] \leq 1 / \varepsilon^3 \log^2 n$. Rescale $\varepsilon$ to complete the proof. \qed

To get an upper tail bound, we simply use Markov on $\overline{X}$. Of course, we do not have a bonafide algorithm, since the value of $m$ is required to choose $s$. But a simple geometric search for $m$ wraps up the whole proof. This is a repeat of the proof of Theorem 11.

Theorem 13. There exists a $O(\varepsilon^{-3.5} \log(\varepsilon^{-1} \log n) n / \sqrt{m})$ algorithm that provides an estimate in $[(1 - \varepsilon)\overline{d}, (1 + \varepsilon)\overline{d}]$ with probability $> 5/6$.

Proof. We perform a geometric search for $m$, by guessing its value as $n^2, n^2 / 2, n^2 / 2^2, \ldots$. Consider the estimate of Theorem 12 (for some guessed value of $m$). By Markov, $Pr[\overline{X} \leq (1 + \varepsilon)\overline{d}] > \varepsilon / 2$. Basically, we run the procedure of Theorem 12 independently $\varepsilon^{-1} \log \log n$ times and take the minimum estimate. With probability at least $1 - 1 / \log^2 n$, the estimate is at most $(1 + \varepsilon)\overline{d}$.

This yields an estimate for $m$ as well. If the estimate is larger than the current guess, we halve the guess for $m$. Eventually, we reach a guess such that $s$ is large enough (in Theorem 12). At the stage, the minimum of $\varepsilon^{-1} \log \log n$ estimates is at most $(1 + \varepsilon)\overline{d}$ and at least $(1 - \varepsilon)\overline{d}$ with probability at least $1 - 1 / \log n$. A union bound over all errors completes the proof. \qed

We note that this proof is very similar to Feige's $(2 + \varepsilon)$-approximation. His proof is more involved and uses tighter arguments to get the dependence on $\varepsilon$ down to $\varepsilon^{-1}$.

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