Generalized Hermite polynomials in superspace as eigenfunctions of the supersymmetric rational CMS model

Patrick Desrosiers
Département de Physique, de Génie Physique et d’Optique,
Université Laval,
Québec, Canada, G1K 7P4.

Luc Lapointe†
Instituto de Matemática y Física,
Universidad de Talca,
Casilla 747, Talca, Chile.

Pierre Mathieu‡
Département de Physique, de Génie Physique et d’Optique,
Université Laval,
Québec, Canada, G1K 7P4.

May 2003

Abstract

We present an algebraic construction of the orthogonal eigenfunctions of the supersymmetric extension of the rational Calogero-Moser-Sutherland model with harmonic confinement. These eigenfunctions are the superspace extension of the generalized Hermite (or Hi-Jack) polynomials. The conserved quantities of the rational supersymmetric model are related to their trigonometric relatives through a similarity transformation. This leads to a simple expression between the corresponding eigenfunctions: the generalized Hermite superpolynomials are written as a differential operator acting on the corresponding Jack superpolynomials. As an aside, the maximal superintegrability of the supersymmetric rational Calogero-Moser-Sutherland model is demonstrated.
Contents

1 Introduction 3

2 The srCMS model and symmetric functions in superspace 5

2.1 The srCMS Hamiltonian ........................................... 5
2.2 Symmetric superfunctions ........................................... 6
2.3 Monomial symmetric superfunctions ......................... 7
2.4 Jack superpolynomials ........................................... 7
2.5 Power-sum superpolynomials ................................... 8

3 Conserved quantities 8

3.1 Trigonometric Case ............................................. 9
3.2 Rational case .................................................. 9

4 From Jack to Hermite polynomials in superspace 11

4.1 Algebraic construction and triangularity .................. 11
4.2 Orthogonality .................................................. 12

5 Maximal superintegrability 13

6 Other bases of eigenfunctions 14

7 Conclusion 15

References 16
1 Introduction

Operator constructions of the eigenfunctions of the rational Calogero-Moser-Sutherland\(^1\) model with confinement (rCMS) \(^2\),

\[ \mathcal{H} = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \sum_{1 \leq i < j \leq N} \beta(\beta - 1) \frac{\beta_i - \beta_j}{x_{ij}^2}, \tag{1} \]

were first initiated by Perelomov \[^2\], who tried to obtain creation operators \(B_n^\dagger\) satisfying \([\mathcal{H}, B_n^\dagger] = n\omega B_n^\dagger\) in order to obtain the excited states as \(|\lambda\rangle = \prod_{i=1}^{N} (B_{n_i}^\dagger)^{\lambda_i}|0\rangle\). However, his technique only allowed him to obtain explicitly the first few \(B_n^\dagger\). A complete set of creation operators was later presented in \[^3\], using the powerful (and at the time, rather new) Dunkl-operator formalism. An independent construction (later found to be related to the former in a simple way \[^4\]) was also presented in \[^5\], in which the creation operators were expressed directly in terms of the Lax operators. However, the resulting eigenfunctions are not orthogonal.

The first few orthogonal eigenfunctions were constructed in \[^6\] by the simultaneous diagonalization of the Hamiltonian and the first non-trivial conserved operator. In this way, no pattern ever showed up. Shortly after, a quite different approach to the construction of the rCMS eigenfunctions was presented in \[^7\]. It consisted in the extension to the rational case of the operator method found in \[^8\] for building the excited states (the Jack polynomials) of the trigonometric CMS (tCMS) model. That this operator method can be translated directly to the rational case is certainly remarkable and actually rather surprising at first sight.\(^2\) The first few orthogonal functions constructed in \[^6\] were recovered using this operator method. In the \(\omega \to \infty\) limit, the rCMS eigenfunction indexed by the partition \(\lambda\) and written \(J_{\omega \lambda}\) reduces to the corresponding Jack polynomial \(J_\lambda\):

\[ \lim_{\omega \to \infty} J_{\omega \lambda} = J_\lambda. \tag{2} \]

Because of this hidden connection with the Jack polynomials, the orthogonal rCMS eigenfunctions have been called the Hi-Jack polynomials.\(^3\)

In parallel to these developments in physics, the very same functions were studied in mathematics under the name of generalized Hermite polynomials. Indeed, when \(N = 1\), the eigenfunctions of the rCMS Hamiltonian are precisely the Hermite polynomials. For \(\beta = 1/2\), the generalized Hermite polynomials have first been introduced in \[^11\]. The general case was treated by Lasalle in \[^12\]. There, the generalized Hermite polynomials are defined uniquely (up to normalization) from their eigenfunction characterization (in terms of a differential operator equivalent to the rCMS Hamiltonian without the ground-state contribution) and their triangular decomposition in terms of Jack polynomials. Lasalle also found a remarkably simple operational relation between the Jack polynomials.

\(^1\)We use the qualitative ‘Calogero-Moser-Sutherland’ to describe the generic class of models that includes the models studied by Calogero and Sutherland in the quantum case and by Moser in the classical case. In this work, we only treat the quantum case, and in this context the name of Moser is often omitted. The rationale for its inclusion is due to the fundamental importance of the Lax formulation in the quantum case, which is a direct extension of the classical one that he introduced. Since the quantum rational model with confinement was studied by Calogero, what we call the rCMS model is often referred to as the Calogero model.

\(^2\)There is of course an extra parameter in the rational case, namely \(\omega\). In the absence of confinement (\(\omega = 0\)), the rational case can be recovered as a limiting case of the trigonometric model. The above situation is thus a priori unexpected.

\(^3\)More recently, a second orthogonal basis has been found in \[^9\], building on the work \[^10\].
and the generalized Hermite/Hi-Jack polynomials:
\[ J_{\lambda}^{\omega} = e^{-\Delta/4\omega}J_{\lambda}, \]
where \( \Delta \) is some operator to be defined below. This provides a direct one-to-one correspondence between Jack polynomials and their ‘higher’ relatives and also, in principle, a computational tool for constructing the \( J_{\lambda}^{\omega} \)’s. A rather extensive analysis of their properties is presented in \[14, 15\].

In this work, we present a very natural extension of these polynomials, by constructing their superspace analogues. The resulting generalized Hermite superpolynomials (also called the generalized Hermite polynomials in superspace) reduce to the usual generalized Hermite polynomials in the zero-fermion sector (the fermion-sector label ranging from 0 to \( N \)). The generalized Hermite superpolynomials are eigenfunctions of the supersymmetric tCMS (srCMS) model first introduced in \[15\]. Our main result is to provide the direct supersymmetric lift of the Lasalle’s operational definition. The generalized Hermite superpolynomials are then defined in terms of the the Jack superpolynomials, which are the same eigenfunctions of the supersymmetric extension of the tCMS model \[18, 19, 20\].

As pointed out in the conclusion, there are other ways of constructing the generalized Hermite superpolynomials. However, in this work, we focus on the construction that is closely tied to the underlying physical Hamiltonian.

The article is organized as follows. Some relevant background material is presented in section 2. In particular, we give the definition of the srCMS Hamiltonian as well as that of the symmetric superfunctions. We then display three bases for the ring of superfunctions, the monomial symmetric superfunctions (called the supermonomials), the Jack superpolynomials and the power-sum superpolynomials.

Sections 3 and 4 are concerned with the superextension of Lasalle’s construction. This is embedded within a broader analysis of the srCMS algebraic structure centered on the Dunkl operator and the relationship between the srCMS and stCMS conserved quantities. In particular, it is shown that all srCMS conserved quantities can be readily obtained as ‘gauge transformations’ of the stCMS ones. As an offshoot of these algebraic considerations, we establish the superintegrability of the srCMS model in Section 5 and display two other bases of eigenfunctions in Section 6.

A summary of our main results, concluding remarks as well as a listing of some natural extensions are collected in Section 7.

---

\(^4\) A later derivation of this formula was given by Sogo in \[13\].

\(^5\) Reference \[15\] also mentions two unpublished manuscripts, one by Macdonald and the other by Lasalle, where further properties of these generalized Hermite polynomials are presented.

\(^6\) An algebraic construction of the eigenstates that generalizes Perelomov’s was proposed in \[15\], but only the first non-trivial creation operators was explicitly written. Later, Ghosh \[17\] obtained all fermionic and bosonic operators using a similarity transformation from the srCMS model to supersymmetric harmonic oscillators. His set of solutions was however over-complete. Note that superpartitions, which provide the proper solution labelling, were not known at the time.

\(^7\) The relationship between the three articles \[18, 19, 20\] is the following. In \[18\], we construct eigenfunctions of the stCMS Hamiltonian that decompose triangulaly in the supermonomial basis. These eigenfunctions were called Jack superpolynomials (and denoted \( J_{\lambda} \)) because they reduce to Jack polynomials in the zero-fermion sector. In \[19\], the coefficients in their triangular decomposition have been computed explicitly, leading to a determinantal expression for the \( J_{\lambda} \)’s. The \( J_{\lambda} \)’s diagonalize the \( N \) bosonic conserved charges that reduce to the tCMS conserved quantities in absence of fermionic variables \[20\]. However they are not orthogonal. To construct orthogonal combinations of the \( J_{\lambda} \)’s, we needed to find the eigenfunctions of the second tower of bosonic conserved charges. This was done in \[20\]. The resulting orthogonal eigenfunctions are written \( J_{\lambda} \). These are the superpolynomials that, from our point of view, deserve to be called the Jack superpolynomials. From now on, we thus use the name ‘Jack superpolynomials’ or equivalently, ‘Jack polynomials in superspace’, for the \( J_{\lambda} \)’s of \[20\]. A survey of these three articles is presented in \[21\].
2 The srCMS model and symmetric functions in superspace

In this section, we first define the srCMS model, and then introduce the essential quantities, such as superpartitions and bases of superpolynomials, that will be needed to construct its eigenfunctions.

2.1 The srCMS Hamiltonian

The Hamiltonian of the srCMS model reads [16]:

\[ H = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \sum_{1 \leq i < j \leq N} \frac{\beta(\beta - 1 + \theta_{ij}\theta_{ij}^\dagger)}{x_{ij}^2} - \frac{N}{2}[\beta(N - 1) - \omega] \]  

(4)

where

\[ x_{ij} = x_i - x_j, \quad \theta_{ij} = \theta_i - \theta_j \quad \text{and} \quad \theta_{ij}^\dagger = \partial \theta_i - \partial \theta_j. \]  

(5)

The \( \theta_i \)'s, with \( i = 1, \cdots, N \), are anticommuting (Grassmannian or fermionic) variables. The commuting (bosonic) variables \( x_j \) are real. The parameters \( \beta \) and \( \omega \) also belong to the real field.

The model is supersymmetric since it can be written as the anticommutation of two charges,

\[ H = \frac{1}{2}\{Q, Q^\dagger\} \quad \text{with} \quad Q^2 = (Q^\dagger)^2 = 0, \]  

(6)

where the srCMS fermionic operators are

\[ Q = \sum_{j=1}^{N} \theta_j \left( \partial x_j + \omega x_j - \beta \sum_{1 \leq k \leq N, k \neq j} \frac{1}{x_{jk}} \right), \]  

\[ Q^\dagger = \sum_{j=1}^{N} \partial \theta_j \left( -\partial x_j + \omega x_j - \beta \sum_{1 \leq k \leq N, k \neq j} \frac{1}{x_{jk}} \right). \]  

(7)

As observed in [22], the term \( 1 - \theta_{ij}\theta_{ij}^\dagger \) is simply a fermionic exchange operator

\[ \kappa_{ij} \equiv 1 - \theta_{ij}\theta_{ij}^\dagger = 1 - (\theta_i - \theta_j)(\partial \theta_i - \partial \theta_j), \]  

(8)

whose action on a function \( f(\theta_i, \theta_j) \) is

\[ \kappa_{ij} f(\theta_i, \theta_j) = f(\theta_j, \theta_i) \kappa_{ij}. \]  

(9)

This provides a substantial simplification in that the model can be studied using the exchange-operator formalism [23].

We now remove the contribution of the ground-state wave function \( \psi_0 \) from the Hamiltonian \( \mathcal{H} \). The unique square integrable function satisfying \( Q\psi_0 = 0 \) is

\[ \psi_0(x) = \prod_{1 \leq j < k \leq N} (x_{jk})^\beta e^{-\frac{1}{2}\omega \|x\|^2}, \quad \|x\|^2 = \sum_{i=1}^{N} x_i^2, \]  

(10)

with \( \omega > 0 \) and \( \beta > -1/2. \) We restrict \( \beta \) to be a positive integer: odd or even for antisymmetric (fermionic) or symmetric (bosonic) ground state respectively. The transformed (or gauged) Hamiltonian

\[ \mathcal{H} \equiv \psi_0(x)^{-1} \mathcal{H} \psi_0(x), \]  

(11)

\[ ^8\text{When the coupling constant } \beta \leq -1/2, \text{ the supersymmetry is spontaneously broken [15]. In this article, only the supersymmetric phase is considered (} \beta > -1/2). \]
reads
\[
\bar{H} = 2\omega \sum_i (x_i \partial_i + \theta_i \partial_{\theta_i}) - \sum_i \partial_i^2 - 2\beta \sum_{i<j} \frac{1}{x_{ij}} (\partial_i - \partial_j) + 2\beta \sum_{i<j} \frac{1}{x_{ij}^2} (1 - \kappa_{ij}),
\]
(12)
where $\partial_i = \partial_{x_i}$. The generalized Hermite polynomials in superspace will be orthogonal eigenfunctions of this Hamiltonian.

The supersymmetric Hamiltonian $\bar{H}$ is self-adjoint with respect to the physical scalar product given by
\[
\langle F(x, \theta), G(x, \theta) \rangle_{\beta, \omega} = \prod_j \left( \int_{-\infty}^{\infty} dx_j \int d\theta_j \theta_j \right) \prod_{k \leq l} (x_{kl})^{2\beta} e^{-\omega \|x\|^2} F(x, \theta)^* G(x, \theta),
\]
(13)
where $F$ and $G$ are arbitrary functions. The complex conjugation $^*$ is defined such that
\[
(\theta_1 \cdots \theta_m)^* \theta_1 \cdots \theta_m = 1 \quad \text{and} \quad x_j^* = x_j.
\]
(14)
In other words, $\theta_j^*$ behaves as $\theta_j^* = \partial_{\theta_j}$. The integration over the Grassmannian variable is the standard Berezin integration, i.e.,
\[
\int d\theta = 0, \quad \int d\theta \theta = 1.
\]
(15)

### 2.2 Symmetric superfunctions

The Hamiltonian $\bar{H}$ leaves invariant the space of symmetric superpolynomials, $P^{S_N}$, invariant under the simultaneous action of $\kappa_{ij}$ and $K_{ij}$, where $K_{ij}$ is the exchange operator acting on the $x_i$ variables:
\[
K_{ij} f(x_i, x_j) = f(x_j, x_i) K_{ij}.
\]
(16)
A polynomial $f$ thus belongs to $P^{S_N}$ if it is invariant under the action of the product
\[
K_{ij} = \kappa_{ij} K_{ij},
\]
(17)
that is, if $K_{ij} f = f$ for all $i, j$. Con As argued in [18], superpartitions provide a labelling of symmetric superpolynomials. We recall that a superpartition $\Lambda$ in the $m$-fermion sector is a sequence of integers composed of two standard partitions separated by a semicolon: $\Lambda = (\Lambda^a; \Lambda^s)$. The first partition, $\Lambda^a$, is associated to an antisymmetric function and has thus distinct parts. While the second one, $\Lambda^s$, is associated to a symmetric function and is a standard partition. To be more specific,

**Definition 1.** Let $\Lambda$ be a sequence of $N$ integers $\Lambda_j \geq 0$ that decomposes into two parts separated by a semi-colon:
\[
\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N) \equiv (\Lambda^a; \Lambda^s).
\]
(18)
Then, $\Lambda$ is a superpartition iff $\Lambda^a = (\Lambda_1, \ldots, \Lambda_m)$ with $\Lambda_i > \Lambda_{i+1} \geq 0$ for $i = 1, \ldots, m-1$ and $\Lambda^s = (\Lambda_{m+1}, \ldots, \Lambda_N)$ with $\Lambda_j \geq \Lambda_{j+1} \geq 0$ for $j = m+1, \ldots, N-1$.

---

\(^9\)From now on, $\sum_i$ means $\sum_{i=1}^{N}$, $\sum_{i<j}$ means $\sum_{1 \leq i < j \leq N}$, $\sum_{j \neq i}$ means $\sum_{1 \leq j \leq N, j \neq i}$, etc.
Notice that in the zero-fermion sector, the semicolon is usually omitted and $\Lambda$ reduces then to $\Lambda^s$. We denote the degree of a superpartition and its fermionic number respectively by:

$$|\Lambda| = \sum_{i=1}^{N} \Lambda_i, \quad \text{and} \quad \Lambda = m.$$  \hspace{1cm} (19)

To every superpartition $\Lambda$, we associate a unique composition $\Lambda_c$ obtained by deleting the semi-colon, i.e., $\Lambda_c = (\Lambda_1, \ldots, \Lambda_N)$. Finally, the partition rearrangement in non-increasing order of the entries of $\Lambda$ is denoted $\Lambda^\star$.

We now introduce three bases for the space of symmetric superfunctions.

### 2.3 Monomial symmetric superfunctions

The monomial symmetric superfunctions [18], superanalogues to the monomial symmetric functions, are defined as follows:

$$m_\Lambda(x, \theta) = m_{(\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N)}(x, \theta) = \sum_{\sigma \in S_N} \theta^{(1, \ldots, m)} x^{\sigma(\Lambda)},$$  \hspace{1cm} (20)

where the prime indicates that the summation is restricted to distinct terms, and where

$$x^{\sigma(\Lambda)} = x_1^{\Lambda_{\sigma(1)}} \cdots x_m^{\Lambda_{\sigma(m)}} x_{m+1}^{\Lambda_{\sigma(m+1)}} \cdots x_N^{\Lambda_{\sigma(N)}} \quad \text{and} \quad \theta^{(1, \ldots, m)} = \theta_{\sigma(1)} \cdots \theta_{\sigma(m)}.$$  \hspace{1cm} (21)

For technical manipulations, it is convenient to reexpress the symmetric group action on the variables using exchange operators. To a reduced decomposition $\sigma = \sigma_i \cdots \sigma_{i_n}$ of an element $\sigma$ of the symmetric group $S_N$, we associate $K_\sigma$, which stands for $K_{i_1 \cdots i_n}$. The monomial superfunction $m_\Lambda$ can thus be rewritten as

$$m_\Lambda = \frac{1}{f_\Lambda} \sum_{\sigma \in S_N} K_\sigma \left( \theta_1 \cdots \theta_m x^\Lambda \right), \quad f_\Lambda = f_{\Lambda^s} = n_{\Lambda^s(0)}! n_{\Lambda^s(1)}! n_{\Lambda^s(2)}! \cdots,$$  \hspace{1cm} (22)

with $n_{\Lambda^s(i)}$ the number of $i$'s in $\Lambda^s$, the symmetric part of $\Lambda = (\Lambda^s; \Lambda^s)$.

When the bosonic variables lay on a unit circle in the complex plane, that is, $x_j \in \mathbb{C}$ and $x_j^* = 1/x_j$ for all $j$, the monomials are orthogonal with respect to the following scalar product:

$$\langle A(x, \theta), B(x, \theta) \rangle = \prod_j \left( \frac{1}{2\pi i} \int \frac{dx_j}{x_j} \int d\theta_j \theta_j \right) [A(x, \theta)^* B(x, \theta)],$$  \hspace{1cm} (23)

where the complex conjugation of the fermionic variables was defined in (14).

### 2.4 Jack superpolynomials

The generators of the second basis are the superanalogues of the Jack polynomials first introduced in [18] and whose definition has been clarified in [19] and [20] (see also [21] for a pedagogical presentation). The Jack superpolynomials $J_\Lambda$ are defined to be the unique orthogonal eigenfunctions of the (gauged) stCMS Hamiltonian $H_2$ that decompose triangularly (with respect to a given ordering) in terms of supermonomials.

To clarify this definition we need to introduce a partial ordering on superpartitions and a scalar product. We first recall the usual dominance ordering on two partitions $\lambda$ and $\mu$ of the same degree: $\lambda \leq \mu$ iff $\lambda_1 + \ldots + \lambda_k \leq \mu_1 + \ldots + \mu_k$ for all $k$. The partial ordering on superpartitions is similar.
2.5 Power-sum superpolynomials

**Definition 2.** The dominance ordering $\leq$ is such that $\Omega \leq \Lambda$ if either $\Omega^* < \Lambda^*$, or $\Omega^* = \Lambda^*$ and $\Omega_1 + \ldots + \Omega_k \leq \Lambda_1 + \ldots + \Lambda_k$ for all $k$.

Note that this ordering is simply the usual ordering on compositions $[2 4]$.

Next, we introduce the physical scalar product of the stCMS model:

$$\langle A(x, \theta), B(x, \theta) \rangle_\beta = \prod_j \left( \frac{1}{2\pi i} \oint dx_j \right) \left( \prod_{k \neq l} \left( 1 - \frac{x_k}{x_l} \right)^\beta A(x, \theta) B(x, \theta) \right),$$

where the variable $x_j$ represents the $j^{th}$ particle’s position on the unit circle in the complex plane; this means $x_j^* = 1/x_j$. We recall again that the complex conjugation of the fermionic variables was defined in $[14]$.

The simplest definition of the Jack polynomial in superspace is the following.

**Definition 3.** $[27]$ The Jack polynomials $J_\Lambda = J_\Lambda(x, \theta; 1/\beta)$ in superspace are the unique functions in $P_{SN}$ such that

$$\langle J_\Lambda, J_\Omega \rangle_\beta \propto \delta_{\Lambda,\Omega} \quad \text{and} \quad J_\Lambda = m_\Lambda + \sum_{\Omega < \Lambda} t_{\Lambda,\Omega}(\beta)m_\Omega.$$

2.5 Power-sum superpolynomials

The space of superpolynomials is generated by $2N$ algebraically independent variables: $N$ commutative $x_i$’s and $N$ anticommutative $\theta_i$’s. Equivalently, $2N$ independent and symmetric quantities can generate the space $P_{SN}$ of superpolynomials invariant under the $K_{ij}$’s. An obvious choice for the $2N$ independent quantities is the following $[21]$:

$$p_n = \sum_i x_i^n = m_{\langle n \rangle} \quad \text{and} \quad q_{n-1} = \sum_i \theta_i x_i^{n-1} = m_{\langle n-1, 0 \rangle},$$

where $n = 1, \ldots, N$. The $p_n$’s are the standard power sums while the $q_{n-1}$’s are their fermionic counterparts.

The power-sum basis in superspace, denoted $\{p_\Lambda\}_\Lambda$, is constructed from the generalized product of power sums:

$$p_\Lambda = q_{\Lambda_1} \cdots q_{\Lambda_m} p_{\Lambda_{m+1}} \cdots p_{\Lambda_N}, \quad m = \bar{\Lambda}.$$  

In the next section, we generalize (i.e., deform via the parameter $\omega$) the three preceding bases of $P_{SN}$; they will thus become bases of eigenfunctions for the srCMS Hamiltonian $\hat{H}$.

3 Conserved quantities

The simplest way to define the srCMS conserved operators is from the stCMS’s ones using a mapping from the trigonometric to the rational case. Trigonometric conserved quantities are constructed using Dunkl operators, denoted $D_j$, and a projection onto the space of symmetric superpolynomials, denoted $|P_{SN}|$. 
### 3.1 Trigonometric Case

The trigonometric Dunkl operators are the following differential operators \[25, 26\]:

\[
D_j = x_j \partial_j + \beta \sum_{k<j} \frac{x_j}{x_{jk}} (1 - K_{jk}) + \beta \sum_{k>j} \frac{x_k}{x_{jk}} (1 - K_{jk}) - \beta (j - 1) \quad j = 1, \ldots, N.
\] (28)

They satisfy the degenerate Hecke relations:

\[
K_{i,i+1} D_{i+1} - D_i K_{i,i+1} = \beta \quad \text{and} \quad K_{j,j+1} D_i = D_i K_{j,j+1} \quad (i \neq j, j + 1). \tag{29}
\]

The 4\(N\) conserved charges are obtained via the projection of the following quantities:

\[
\mathcal{H}_n = \sum_{i=1}^N D_i^n, \\
Q_{n-1} = \sum_{w \in S_N} K_w \theta_1 D_1^{n-1}, \\
Q_1^{n-1} = \sum_{w \in S_N} K_w \frac{\partial}{\partial \theta_1} D_1^{n-1}, \\
I_{n-1} = \sum_{w \in S_N} K_w \theta_1 \frac{\partial}{\partial \theta_1} D_1^{n-1},
\] (30)

where \(n = 1, 2, \ldots, N\). Specially, upon projection onto \(P^S_N\) (that is, when limiting their action to functions belonging to \(P^S_N\)), they become 4\(N\) conserved operators of the stCMS model. However, only the 2\(N\) bosonic ones are in involution:

\[
[H_m|_{P^S_N}, H_n|_{P^S_N}] = [H_m|_{P^S_N}, I_{n-1}|_{P^S_N}] = [I_{m-1}|_{P^S_N}, I_{n-1}|_{P^S_N}] = 0. \tag{31}
\]

### 3.2 Rational case

We now define the rational Dunkl operators (without the ground state contribution) \[27\]:

\[
D_j = \frac{\partial}{\partial x_j} + \beta \sum_{k \neq j} \frac{1}{x_{jk}} (1 - K_{jk}).
\] (32)

They satisfy

\[
[D_i, D_j] = 0 \quad \text{and} \quad K_{ij} D_i = D_j K_{ij}. \tag{33}
\]

These operators are not self-adjoint with respect to the physical scalar product \[13\]. The Hermitian adjoint of \(D_j\) is

\[
D_j^\dagger = 2\omega x_j - D_j. \tag{34}
\]

Note that \(K_{ij}^\dagger = K_{ij}\) since the measure in \[13\] is \(S_N\)-invariant. We can construct a set of \(N\) commuting and self-adjoint operators as follows:\[11\]

\[
D_j^\omega = \frac{1}{2\omega} D_j D_j^\dagger + \beta \sum_{k \geq j} K_{jk} - \beta (N - 1) = \frac{1}{2\omega} D_j D_j^\dagger - \beta \sum_{k < j} K_{jk} - \beta (N - 1) - 1,
\] (35)

\[\text{These operators are also called Cherednik operators and denoted } \xi_j \text{ by some authors (cf. } [24]).\]

\[\text{In essence, } D_j^\omega \text{ corresponds to the operator } d_j \text{ defined in } [4], \text{ plus the exchange term } \beta \sum_{k > j} K_{jk}. \text{ This deformation was inspired by } [26].\]
for $j = 1, \ldots, N$. The importance of the $D^\omega_j$'s becomes more transparent if we rewrite them as

$$D^\omega_j = D_j - \frac{1}{2\omega}D_j^2$$

(36)

(we stress that on the r.h.s., there is a $D_j$ defined in 28 and a $D_j$) which implies

$$\lim_{\omega \to \infty} D^\omega_j = D_j.$$  

(37)

This result suggests that we can construct $2N$ Hermitian and commuting quantities by simply ‘$\omega$-deforming’ the set $\{H_n, I_{n-1}\}$. We thus define

$$H^\omega_n = \sum_j (D^\omega_j)^n,$$

$$I^\omega_{n-1} = \sum_{w \in S_N} K_w \theta_1 \partial \theta_1 (D^\omega_1)^{n-1}.$$  

(38)

**Lemma 4.** The rational operators $\{H^\omega_n, I^\omega_{n-1}\}$ are related to their trigonometric counterparts $\{H_n, I_{n-1}\}$ by the following similarity transformations:

$$H^\omega_n = e^{-\frac{\Delta}{4\omega}} H_n e^{\frac{\Delta}{4\omega}} \quad \text{and} \quad I^\omega_{n-1} = e^{-\frac{\Delta}{4\omega}} I_{n-1} e^{\frac{\Delta}{4\omega}},$$

(39)

where $n = 1, \ldots, N$ and

$$\Delta = \sum_j D_j^2.$$  

(40)

**Proof.** Since $D_j$ and $\Delta$ commute, we first remark that

$$[D^\omega_j, \Delta] = [D_j, \Delta] = -2D_j^2,$$

(41)

Then, using the Baker-Campbell-Hausdorff formula, we get

$$e^{-\frac{\Delta}{4\omega}} D_j e^{\frac{\Delta}{4\omega}} = D_j - \frac{1}{4\omega}[\Delta, D_j] + \frac{1}{2!}(\frac{1}{4\omega})^2[\Delta, [\Delta, D_j]] - \cdots = D_j - \frac{1}{2\omega} D_j^2 = D^\omega_j,$$

(42)

which implies the relation 39.

The operator $\Delta$ preserves the space $P^{S_N}$ of symmetric superpolynomials. Therefore, the similarity transformation is not affected by the projection operation onto the space of symmetric superpolynomials. More explicitly, if $F(D_j)$ and $G(D_j)$ are operators satisfying $K_{ij}F = F$ and $K_{ij}G = G$, then

$$e^{-\frac{\Delta}{4\omega}} F e^{\frac{\Delta}{4\omega}} |_{p s_n} = e^{-\frac{\Delta}{4\omega}} |_{p s_n} F |_{p s_n} e^{\frac{\Delta}{4\omega}} |_{p s_n},$$

(43)

and therefore,

$$[F, G] |_{p s_N} = 0 \quad \implies \quad \left[ e^{-\frac{\Delta}{4\omega}} F e^{\frac{\Delta}{4\omega}} |_{p s_n}, e^{-\frac{\Delta}{4\omega}} G e^{\frac{\Delta}{4\omega}} |_{p s_n} \right] = 0.$$  

(44)

This observation and Lemma 4 prove the following proposition.

**Proposition 5.** For $n, m \in \{1, 2, \ldots, N\}$, the set of operators $\{H^\omega_n, I^\omega_{n-1}\}$ defined by equations 28 or 29 is such that

$$[H^\omega_m |_{p s_N}, H^\omega_n |_{p s_N}] = [H^\omega_m |_{p s_N}, I^\omega_{n-1} |_{p s_N}] = [I^\omega_{m-1} |_{p s_N}, I^\omega_{n-1} |_{p s_N}] = 0.$$  

(45)
We stress that the supersymmetric Hamiltonian defined in (12) can be rewritten as
\[ \tilde{H} = 2 \omega \mathcal{H}_1^\omega |_{P^SN} + 2 \omega I_0^\omega |_{P^SN}. \] (46)
Furthermore, the fermionic operators
\[ Q_1^\omega |_{P^SN} = e^{-\frac{i}{4} \Delta} Q_1 e^{\frac{i}{4} \Delta} |_{P^SN} \quad \text{and} \quad Q_1^{\omega \dagger} |_{P^SN} = e^{-\frac{i}{4} \Delta} Q_1^{\dagger} e^{\frac{i}{4} \Delta} |_{P^SN} \] (47)
are essentially the supersymmetric generators (7) without the ground state contribution.

As we will see in the next section, the generalized Hermite superpolynomials will be orthogonal eigenfunctions of the \( 2N \) quantities \( \mathcal{H}_n^\omega |_{P^SN} \) and \( I_{n-1}^\omega |_{P^SN}, n = 1, \ldots, N. \)

### 4 From Jack to Hermite polynomials in superspace

#### 4.1 Algebraic construction and triangularity

The bijection established in (39) between the srCMS and stCMS conserved quantities allows us to construct generalized Hermite superpolynomials directly from the Jack superpolynomials. The Jack polynomials in superspace are the common eigenfunctions of the \( \mathcal{H}_n^\omega \)'s and \( I_{n-1}^\omega \)'s, that is,
\[ \mathcal{H}_n^\omega J_\Lambda = \epsilon_{\lambda,n} J_\Lambda \quad \text{and} \quad I_{n-1}^\omega J_\Lambda = \epsilon_{\lambda,n-1} J_\Lambda, \] (48)
where the eigenvalues are such that
\[ \lim_{\beta \to 0} \epsilon_{\lambda,n} = \sum_{i=1}^{N} \Lambda_i^n \quad \text{and} \quad \lim_{\beta \to 0} \epsilon_{\lambda,n-1} = \sum_{i=1}^{\Lambda} \Lambda_i^{n-1}. \] (49)
Note the following two special cases
\[ \epsilon_{\lambda,1} = |\Lambda| \quad \text{and} \quad \epsilon_{\lambda,0} = \overline{\Lambda}. \] (50)
Given that the stCMS and srCMS conserved operators are related by a similarity transformation, i.e.,
\[ \mathcal{H}_n^\omega = e^{-\frac{i}{4} \Delta} \mathcal{H}_n e^{\frac{i}{4} \Delta} \quad \text{and} \quad I_{n-1}^\omega = e^{-\frac{i}{4} \Delta} I_{n-1} e^{\frac{i}{4} \Delta}, \] (51)
the eigenfunctions of the srCMS charges, defined by
\[ \mathcal{H}_n^\omega J_\Lambda^\omega = \epsilon_{\lambda,n} J_\Lambda^\omega \quad \text{and} \quad I_{n-1}^\omega J_\Lambda^\omega = \epsilon_{\lambda,n-1} J_\Lambda^\omega, \] (52)
are directly found to be
\[ J_\Lambda^\omega = J_\Lambda(x, \theta; 1/\beta, \omega) = e^{-\frac{i}{4} \Delta} J_\Lambda(x, \theta; 1/\beta). \] (53)

**Definition 6.** The generalized Hermite polynomials in superspace are given by
\[ J_\Lambda^\omega = e^{-\frac{i}{4} \Delta} J_\Lambda. \] (54)

We stress that this definition makes sense since, as it is proved in [29], the action of the operator \( \Delta \) on monomials is finite and triangular when \( |\Lambda| < \infty \) and \( N < \infty \). Therefore, the relation between the \( J_\Lambda^\omega \)'s and the \( J_\Lambda \)'s is bijective. More precisely, \( \Delta \) has degree \(-2\) in \( x \) and \( 0 \) in \( \theta \). Given that \( \Delta \lambda_{\Lambda} \) or \( \Delta J_\Lambda \) has a finite polynomial decomposition, we may write
\[ J_\Lambda^\omega = e^{-\frac{i}{4} \Delta} J_\Lambda = J_\Lambda + \sum_{\Omega < \Lambda} A_{\Lambda \Omega}(\beta, \omega, N) J_\Omega, \] (55)
where the ordering \( \leq_u \) is defined as follows.
4.2 Orthogonality

Definition 7. The u-ordering is such that $\Omega \preceq_u \Lambda$ if either $\Omega = \Lambda$, or $|\Omega| = |\Lambda| - 2n$ for $n = 1, 2, 3, \ldots$

The action of the rational conserved charges on the Jack superpolynomials is thus triangular.

Lemma 8. The $\mathcal{H}_n^\omega$’s and the $\mathcal{T}_{n-1}^\omega$’s act triangularly on the $J_\Lambda$’s:

\[ \mathcal{H}_n^\omega J_\Lambda = \varepsilon_{\Lambda,n} J_\Lambda + \sum_{\Omega \prec_u \Lambda} B_{\Lambda\Omega} J_\Omega \quad \text{and} \quad \mathcal{T}_{n-1}^\omega J_\Lambda = \varepsilon_{\Lambda,n-1} J_\Lambda + \sum_{\Omega \prec_u \Lambda} C_{\Lambda\Omega} J_\Omega, \]

where $B_{\Lambda\Omega}$ and $C_{\Lambda\Omega}$ are some coefficients in $\beta$, $\omega$ and $N$.

4.2 Orthogonality

The triangular action of the conserved operators on the Jack superpolynomial basis and their hermiticity are essential properties: they imply the orthogonality of the generalized Hermite superpolynomials.

Theorem 9. The generalized Hermite polynomials, $J_\Lambda^\omega$, form the unique basis of $P_{SN}$ such that

\[ \langle J_\Lambda^\omega, J_\Omega^\omega \rangle_{\beta,\omega} \propto \delta_{\Lambda,\Omega} \quad \text{and} \quad J_\Lambda^\omega = J_\Lambda + \sum_{\Omega \prec_u \Lambda} w_{\Lambda\Omega}(\beta,\omega,N) J_\Omega. \]

Proof. The generalized Hermite superpolynomials are triangular and monic (see (55)). To show their orthogonality, we simply need to see that they are eigenfunctions (see (52)) of the complete set of self-adjoint operators $\{\mathcal{H}_n^\omega, \mathcal{T}_{n-1}^\omega\}$. The completeness follows from (58) which implies that the set

\[ \left\{ (\varepsilon_{\Lambda,1}, \varepsilon_{\Lambda,2}, \ldots, \varepsilon_{\Lambda,N}; \varepsilon_{\Lambda,0}, \varepsilon_{\Lambda,1}, \ldots, \varepsilon_{\Lambda,N-1}) \mid |\Lambda| = n, \; \underline{\Lambda} = \underline{m} \right\} \]

does not contain identical multiplets (when $\beta$ is viewed as a formal parameter). There is uniqueness because the Gram-Schmidt orthonormalization procedure ensures that there is at most one orthonormal family with a given triangularity.\footnote{The fact that our order is partial makes the condition stronger. That is, given a partial order, it is not sure that an orthonormal basis triangular with respect to that order exist. But if it exists, it is unique.}

Theorem 9 can be interpreted as another definition of the generalized Hermite polynomials in superspace. The following theorem gives a third definition of the generalized Hermite polynomials: as triangular eigenfunctions of the supersymmetric Hamiltonian $\mathcal{H}_1^\omega$.

Theorem 10. The generalized Hermite polynomials, $J_\Lambda^\omega$, form the unique basis of $P_{SN}$ such that

\[ \mathcal{H}_1^\omega J_\Lambda^\omega = |\Lambda| J_\Lambda^\omega \quad \text{and} \quad J_\Lambda^\omega = J_\Lambda + \sum_{\Omega \prec_u \Lambda} w_{\Lambda\Omega}(\beta,\omega,N) J_\Omega. \]

Proof. The $J_\Lambda^\omega$’s are triangular and eigenfunctions of $\mathcal{H}_1^\omega$ with the right eigenvalue (see (55) and (52)). Uniqueness is proved as follows. We suppose the existence of $\tilde{J}_\Lambda^\omega$ satisfying equation (59), so that

\[ J_\Lambda^\omega - \tilde{J}_\Lambda^\omega = \sum_{\Omega \prec_u \Lambda} D_{\Lambda\Omega} J_\Omega = \sum_{i=1}^n D_{\Lambda\Omega(i)} J_{\Omega(i)}, \]

where $D_{\Lambda\Omega}$ and $D_{\Lambda\Omega(i)}$ are coefficients in $\beta$, $\omega$ and $N$. The completeness follows from (58) which implies that the set

\[ \left\{ (\varepsilon_{\Lambda,1}, \varepsilon_{\Lambda,2}, \ldots, \varepsilon_{\Lambda,N}; \varepsilon_{\Lambda,0}, \varepsilon_{\Lambda,1}, \ldots, \varepsilon_{\Lambda,N-1}) \mid |\Lambda| = n, \; \underline{\Lambda} = \underline{m} \right\} \]
where $\Omega^{(1)} \preceq \omega \ldots \preceq \omega \Omega^{(n)} < \Lambda$ and $\preceq_u$ stands for a total ordering compatible with $\leq_u$. On the one hand,

$$H^w_n(J^x_n - \bar{J}^x_n) = \varepsilon_{\Lambda,1}(J^x_n - \bar{J}^x_n) = \varepsilon_{\Lambda,1} \sum_{i=1}^n D_{\Lambda,\Omega(i)} J_{\Omega(i)} = \varepsilon_{\Lambda,1} D_{\Lambda\Omega(n)} J_{\Omega(n)} + \ldots,$$

(61)

where ‘…” represents lower terms with respect to the ordering $\preceq_u$. On the other hand, using Lemma 8, we get

$$H^w_n(J^x_n - \bar{J}^x_n) = \sum_{i=1}^n D_{\Lambda\Omega(i)} \left( \varepsilon_{\Omega(i),1}(J_{\Omega(i)} + \sum_{\Gamma \leq \Omega(i)} E_{\Omega(i)} J_{\Gamma}) \right) = \varepsilon_{\Omega(n),1} D_{\Lambda\Omega(n)} J_{\Omega(n)} + \ldots$$

(62)

Equations (61) and (62) imply $\varepsilon_{\Lambda,1} = \varepsilon_{\Omega(n),1}$. However, this is impossible since $\varepsilon_{\Lambda,1} = |\Lambda|$ and $\varepsilon_{\Omega(n),1} = |\Omega^{(n)}| < |\Lambda|$. The function $J^x_n$ is thus unique.

5 Maximal superintegrability

As an offshoot of our analysis of the srCMS conserved quantities, we will now demonstrate that this model is superintegrable. A bosonic Hamiltonian model with $N$ variables $x_i$ is superintegrable if it possesses more than $N$ functionally independent conserved quantities. The maximum number of such quantities is $2N - 1$ while only $N$ can be in involution at the same time (cf. [28]). A fermionic extension of a model that contains also $N$ Grassmannian variables $\theta_i$ is superintegrable if it has more than $2N$ conserved and independent operators. It is maximally superintegrable if the total number of operators that commute with the Hamiltonian is $4N - 2$.

The supersymmetric generalization of the rational CMS model without harmonic confinement ($\omega = 0$) is maximally superintegrable. This was shown in [21] using the Dunkl-operator formalism. When $\omega \neq 0$, the superintegrability is even simpler to prove. It uses a bijection between the srCMS model and the free supersymmetric model, whose Hamiltonian is

$$H = \{Q, Q^\dagger\} = \sum_j x_j \partial_j = H_1,$$

(63)

with the fermionic charges formally defined as

$$Q = \sum_j \theta_j (x_j \partial_j)^{1/2} \quad \text{and} \quad Q^\dagger = \sum_j \partial_{\theta_j} (x_j \partial_j)^{1/2}.$$

(64)

The conserved and independent operators are simply

$$H_n = \sum_j (x_j \partial_j)^n, \quad I_{n-1} = \sum_j \theta_j \partial_{\theta_j} (x_j \partial_j)^{n-1}, \quad n = 1, \ldots, N,$$

$$J_n = H_n L_0 - H_1 L_{n-1}, \quad K_{n-1} = I_n M_0 - I_1 M_{n-1}, \quad n = 1, \ldots, N - 1,$$

(65)

where $L_n$ and $M_n$ are defined as

$$L_{n-2} = \sum_j \ln |x_j (x_j \partial_j)^{n-1}| \quad \text{and} \quad M_{n-2} = \sum_j \ln |x_j \theta_j \partial_{\theta_j} (x_j \partial_j)^{n-1}|,$$

(66)

if $n \geq 1$. The commutativity of the $H_n$’s and the $I_{n-1}$’s is immediate:

$$[H_m, H_n] = [H_m, I_{n-1}] = [I_{m-1}, I_{n-1}] = 0.$$

(67)
Now, the transformation (71) implies that other sets of eigenfunctions.

Nevertheless, we stick to metric and the free model respectively.

As we showed in the beginning of Section 3, the srCMS Hamiltonian \( H_1 \) and the free Hamiltonian \( H = H_1 = H_1 \) (recall that \( H_1 \) denotes the first trigonometric conserved quantity) are related by a similarity transformation:

\[
H_1 = e^{-\frac{\Delta}{\omega}} H_1 e^{\frac{\Delta}{\omega}} |_{p_{SN}} = e^{-\frac{\Delta}{\omega}} H_1 e^{\frac{\Delta}{\omega}} |_{p_{SN}}. 
\]

This means in particular that, instead of \( \omega \)-deforming the trigonometric conserved quantities \( H_n \) to obtain the Hamiltonians \( H_n^\omega \) of the srCMS model (cf. Section 3.1), one can construct another set of rational conserved operators \( H_n^\omega \) by transforming the free ones.\(^{13}\) More generally, equations (70) and (71) give an algebraic construction of all srCMS conserved operators.

**Proposition 11.** The srCMS model is maximally superintegrable; an independent set of conserved operators can be constructed as follows:

\[
\begin{align*}
H_n^\omega |_{p_{SN}} &= e^{-\frac{\Delta}{\omega}} H_n e^{\frac{\Delta}{\omega}} |_{p_{SN}}, \\
J_n^\omega |_{p_{SN}} &= e^{-\frac{\Delta}{\omega}} J_n e^{\frac{\Delta}{\omega}} |_{p_{SN}}, \\
K_n^\omega |_{p_{SN}} &= e^{-\frac{\Delta}{\omega}} K_n e^{\frac{\Delta}{\omega}} |_{p_{SN}}.
\end{align*}
\]

### 6 Other bases of eigenfunctions

The preceding construction of commuting operators in the srCMS model furnishes directly two other sets of eigenfunctions.

Indeed, let the superpolynomial \( m_\Lambda^\omega = m_\Lambda(x, \theta; \omega, \beta) \) and the power-sum product \( p_\Lambda^\omega = p_\Lambda(x, \theta; \omega, \beta) \) defined by the following operational transformation:

\[
m_\Lambda^\omega = e^{-\frac{\Delta}{\omega}} m_\Lambda \quad \text{and} \quad p_\Lambda^\omega = e^{-\frac{\Delta}{\omega}} p_\Lambda.
\]

We stress that these superpolynomials are two-parameter deformations of the monomial and power-sum basis respectively. In that sense, a more precise notation certainly would be \( m_\Lambda^{\omega, \beta} \) and \( p_\Lambda^{\omega, \beta} \). Nevertheless, we stick to \( m_\Lambda^\omega \) and \( p_\Lambda^\omega \) to lighten the notation and because the polynomials satisfy, like the \( D_j^\omega \)'s and the \( J_n^\omega \)'s, the following limiting relation:

\[
\lim_{\omega \to \infty} m_\Lambda^\omega = m_\Lambda(x, \theta), \quad \lim_{\omega \to \infty} p_\Lambda^\omega = p_\Lambda(x, \theta).
\]

Now, the transformation (71) implies that

\[
\begin{align*}
H_n^\omega m_\Lambda^\omega &= \left( \sum_j A_j^n \right) m_\Lambda^\omega, \quad \Gamma_{n-1}^\omega m_\Lambda^\omega = \left( \sum_{j=1}^{n-1} A_j^{n-1} \right) m_\Lambda^\omega, \\
H_n^\omega p_\Lambda^\omega &= \left( \sum_j A_j^n \right) p_\Lambda^\omega, \quad \Gamma_{n-1}^\omega p_\Lambda^\omega = \left( \sum_{j=1}^{n-1} A_j^{n-1} \right) p_\Lambda^\omega.
\end{align*}
\]

\(^{13}\)There is no algebraic relation between the sets \( \{ H_n^\omega, J_n^\omega \} \) and \( \{ H_n^\omega, I_{n-1}^\omega \} \), obtained from the trigonometric and the free model respectively.
Proof. A direct calculation gives \( \Delta \).

**Theorem 12.** The set \( \{m^\omega_\Lambda \}_\Lambda \) is the unique basis of \( P^{SN} \) such that

\[
\mathcal{H}^\omega m^\omega_\Lambda = |\Lambda| m^\omega_\Lambda \quad \text{and} \quad m^\omega_\Lambda = m_\Lambda + \sum_{\Omega<\omega,\Lambda} y_{\Lambda\Omega}(\beta, \omega; N) m_\Omega. \tag{75}
\]

Similarly, \( \{p^\omega_\Lambda \}_\Lambda \) is the only basis of \( P^{SN} \) such that

\[
\mathcal{H}^\omega p^\omega_\Lambda = |\Lambda| p^\omega_\Lambda \quad \text{and} \quad p^\omega_\Lambda = p_\Lambda + \sum_{\Omega<\omega,\Lambda} z_{\Lambda\Omega}(\beta, \omega; N) p_\Omega. \tag{76}
\]

**Proof.** See Theorem 10 \( \square \)

Notice that the functions \( p^\omega_\Lambda \) generalize, in the superspace, the solutions published in \( \mathcal{H}^\omega \) and independently in \( \mathcal{H}^\omega \). When \( \omega = 1 \) and the \( \theta_i \)'s are replaced by the \( \theta_i \)'s, the solutions \( p^\omega_\Lambda = e^{-\Delta/4\omega} p_\Lambda \) are equivalent to the algebraic constructions of Ghosh \( \mathcal{H}^\omega \) (even though he did not use a superpartition labelling).

We end this section by writing each \( \omega \)-deformed superpolynomial as a non-deformed superpolynomial, now expressed in terms of \( N \) differential operators rather than the \( x_i \)'s, acting on the identity. In the zero-fermion sector, these algebraic expressions reduce to those obtained in \( \mathcal{H}^\omega \).

**Proposition 13.** Let \( \hat{D}^\dagger_i = D^\dagger_i/2\omega \) and \( \hat{D}^\dagger = (\hat{D}^\dagger_1, \ldots, \hat{D}^\dagger_N) \), where \( D^\dagger_i \) is the Dunkl operator defined in \( \mathcal{H}^\omega \). Then,

\[
J^\omega_\Lambda = J_\Lambda(\hat{D}^\dagger; \theta; 1/\beta) \cdot 1, \quad m^\omega_\Lambda = m_\Lambda(\hat{D}^\dagger, \theta) \cdot 1, \quad \text{and} \quad p^\omega_\Lambda = p_\Lambda(\hat{D}^\dagger, \theta) \cdot 1. \tag{77}
\]

**Proof.** A direct calculation gives \( [\Delta, x_j] = 2D_i \), hence (using again the Baker-Campbell-Hausdorff formula)

\[
e^{\pm i\Delta} D^\dagger_i e^{-\pm i\Delta} = e^{\pm i\Delta} (2\omega x_i - D_i) e^{-\pm i\Delta} = 2\omega x_i. \tag{78}
\]

Thus, using \( \Delta \cdot 1 = 0 \), we get

\[
m_\Lambda(\hat{D}^\dagger, \theta) \cdot 1 = m_\Lambda \left( e^{-\pm i\Delta} x e^{\pm i\Delta}, \theta \right) \cdot 1 = e^{-\pm i\Delta} m_\Lambda(x, \theta) e^{\pm i\Delta} \cdot 1 = m^\omega_\Lambda(x, \theta), \tag{79}
\]

and similar relations hold for \( J_\Lambda(\hat{D}^\dagger; \theta; 1/\beta) \cdot 1 \) and \( p_\Lambda(\hat{D}^\dagger, \theta) \cdot 1 \). \( \square \)

### Conclusion

In this work, we have defined the generalized Hermite superpolynomials as the eigenfunctions of the srCMS model that decompose triangularly in terms of Jack superpolynomials. The algebraic construction given in Section 4.1 amounts to defining the generalized Hermite superpolynomials \( J^\omega_\Lambda \) from the Jack polynomials \( J_\Lambda \) as follows:

\[
J^\omega_\Lambda = e^{-\Delta/4\omega} J_\Lambda. \tag{80}
\]
This almost immediately implies their orthogonality. Albeit elegant, a drawback of this construction should be pointed out: the underlying computation depends on the number \( N \) of variables and does not directly express the \( J_\Lambda \)'s in terms of the Jack polynomials \( J_\lambda \). Note also that the \( J_\Lambda \)'s are not stable with respect to the number of variables.

In the same way as the Jack polynomials can be viewed as a one-parameter deformation of the monomial basis that preserves their orthogonality, the generalized Hermite superpolynomials are thus obtained from a step further orthogonality-preserving deformation:

\[
m_\Lambda \overset{\beta}{\rightarrow} J_\Lambda \overset{\omega}{\rightarrow} J_{\Lambda'}.
\]

(81)

Recall that the variables \( x_j \) are the ‘position vectors’: they are constrained to the unit circle for the \( m_\Lambda \)'s and the \( J_\Lambda \)'s while they lie on an infinite line for the \( J_{\Lambda'} \)'s. The weight function also changes from one scalar product to the other. Consequently, the scalar product is modified at each step in Eq. (81).

As reviewed in Section 2.2, a symmetric polynomial in superspace can be decomposed in terms of functions built out of a monomial in fermionic variables times a polynomial of the bosonic variables having mixed symmetry property. This indicates that a given type of superpolynomial could be reconstructed by an appropriate ‘supersymmetrization’ of the corresponding non-symmetric version of the polynomial multiplied by suitable fermionic monomials. We showed in [20] that the Jack superpolynomials can indeed be constructed in this way. This route provides another method for constructing the generalized Hermite superpolynomials out of the non-symmetric generalized Hermite polynomials defined in [24].

Still another construction is presented in [29], in the spirit of [30] and [19]. It furnishes a different construction that expresses the generalized Hermite superpolynomials in determinantal form.

An immediate extension of this work is to study in detail the continuous-spectrum limit \( \omega = 0 \): this transforms the generalized Hermite superpolynomials into generalized Bessel functions in superspace. Another very natural generalization is to construct the eigenfunctions of the supersymmetric CMS models rational or trigonometric defined for other root systems. At first sight the lift to other root systems appear to be mostly a technical problem. One can however identify a more difficult problem along these ‘extension lines’. It is the construction of the supersymmetric version of the relativistic CMS models and their eigenfunctions, the Macdonald superpolynomials.

Finally, let us stress that as an aside of our study, we have clarified the integrability structure of the srCMS model by expressing its conserved charges in terms of a similarity transformation of the stCMS ones or the free ones. Moreover, we have constructed supplementary conserved quantities whose existence makes the model superintegrable.

Acknowledgments. This work was supported by NSERC and FONDECYT (Fondo Nacional de Desarrollo Científico y Tecnológico) grant #1030114. L.L. wishes to thank Luc Vinet for his support at the early stages of this work and P.D. is grateful to the Fondation J.A.-Vincent for a student fellowship.

References

[1] F. Calogero, Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. 12 (1971), 419 ; J. Moser, Three integrable Hamiltonian systems connected with isospectral deformations, Adv. Math. 16 (1975), 197 ; B. Sutherland, Exact results for a quantum many body problem in one-dimension, Phys. Rev. A4 (1971), 2019 ; M. A. Olshanetsky and A. M. Perelomov, Quantum integrable systems related to Lie algebras, Phys. Rept. 94 (1983), 313.
[2] A. M. Perelomov, *Algebraic approach to the solution of a one-dimensional model of N interacting particles*, Theo. Math. Phys. 6 (1971), 263.

[3] L. Brink, T.H. Hanson and M.A. Vassiliev, *Explicit solution of the N-body Calogero problem*, Phys. Lett. B 286 (1992) 109; hep-th/9206049.

[4] H. Ujino and M. Wadati, *Correspondence between the QISM and the exchange operator formalism*, J. Phys. Soc. Jap. 64 (1995), 4121.

[5] H. Ujino and M. Wadati, *On the quantum Calogero model and the W-algebra*, J. Phys. Soc. Jap. 63 (1994), 3583.

[6] H. Ujino and M. Wadati, *Orthogonal symmetric polynomials assosciated with the quantum Calogero model*, J. Phys. Soc. Jap. 64 (1995), 2703.

[7] H. Ujino and M. Wadati, *Algebraic construction of the eigenstates for the second conserved operator of the quantum Calogero model*, J. Phys. Soc. Jap. 65 (1996), 653; *Orthogonality of the Hi-Jack polynomials associated with the Calogero model*, J. Phys. Soc. Jap. 66 (1997), 345.

[8] L. Lapointe and L. Vinet, *A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture*, Internat. Math. Res. Notices, 9 (1995), 419; *Exact operator solution of the Calogero-Sutherland model*, Commun. Math. Phys., 178 (1996), 425; q-alg/9509003.

[9] H. Ujino and M. Wadati, *Algebraic construction of a new symmetric orthogonal basis for the Calogero model*, J. Phys. Soc. Jap. 67 (1998), 1.

[10] N. Gurappa and P. P Panigrahi, *Equivalence of the Calogero-Sutherland model to free harmonic oscillators*, Phys. Rev. B 59 (1999), R2490; cond-mat/9710035.

[11] A. T. James *Special functions of matrix and simple arguments in statistics*, in *Theory and applications of special functions*, ed. R. Askey, Academic Press (1975), 497.

[12] M. Lasalle, *Polynômes de Hermite généralisés*, C.R. Acad. Sci. Paris, t. 313, série I (1991), 579.

[13] K. Sogo, *A simple derivation of multivariable Hermite and Legendre polynomials*, J. Phys. Soc. Jap. 65 (1996), 3097.

[14] J. F. van Diejen, *Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement*, Commun. Math. Phys. 188 (1997), 467.

[15] T. H. Baker and P. J. Forrester, *The Calogero-Sutherland model and generalized polynomials* Commun. Math. Phys. 188 (1997), 1175; solv-int/9608004.

[16] D. Z. Freedman and P. F. Mende, *An exactly solvable N particle system in supersymmetric quantum mechanics*, Nucl. Phys. B344 (1990), 317.

[17] P. K. Ghosh, *Super-Calogero-Moser-Sutherland systems and free super-oscillators: A mapping*, Nucl. Phys. B595 (2001), 519; hep-th/0007208.

[18] P. Desrosiers, L. Lapointe and P. Mathieu, *Supersymmetric Calogero-Moser-Sutherland models and Jack superpolynomials*, Nucl. Phys. B606 (2001), 547; hep-th/0103178.

[19] P. Desrosiers, L. Lapointe and P. Mathieu, *Jack superpolynomials, superpartition ordering and determinantal formulas*, Commun. Math. Phys. 233, 383 (2003); hep-th/0105107.

[20] P. Desrosiers, L. Lapointe and P. Mathieu, *Jack polynomials in superspace*, to appear in Commun. Math. Phys.; hep-th/0209074.
[21] P. Desrosiers, L. Lapointe and P. Mathieu, *Supersymmetric Calogero-Moser-Sutherland models: superintegrability structure and eigenfunctions*, to appear in the proceedings of the Workshop on superintegrability in classical and quantum systems, ed. P Winternitz, CRM series, Springer; hep-th/0210190.

[22] B. S. Shastry and B. Sutherland, *Superlax pairs and infinite symmetries in the 1/r^2 system*, Phys. Rev. Lett. **70** (1993), 4029; cond-mat/9212029.

[23] A. P. Polychronakos, *Exchange operator formalism for integrable systems of particles*, Phys. Rev. Lett. **69** (1992), 703; hep-th/9202057. J. A. Minahan and A. P. Polychronakos, *Integrable systems for particles with internal degrees of freedom*, Phys. Lett. **B302** (1993), 299; hep-th/9206046.

[24] T. H. Baker and P. J. Forrester, *The Calogero-Sutherland model and polynomials with prescribed symmetry*, Nucl. Phys. **B492** (1997), 682; solv-int/9609010.

[25] I. Cherednik, *A unification of Knizhnik-Zamalodchikov and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411.

[26] D. Bernard, M. Gaudin, F. D. Haldane and V. Pasquier, *Yang-Baxter equation in long range interacting system*, J. Phys. **A26** (1993), 5219; hep-th/9301084.

[27] C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Am. Math. Soc. **311** (1989), 1.

[28] V. B. Kuznetsov, *Hidden symmetry of the quantum Calogero-Moser system*, Phys. Lett. **A218** (1996), 212; solv-int/9509001.

[29] P. Desrosiers, L. Lapointe and P. Mathieu, *Explicit formulas for the Generalized Hermite polynomials in superspace*, in preparation.

[30] L. Lapointe, A. Lascoux and J. Morse, *Determinantal formula and recursion for Jack polynomials*, Electro. J. Comb. **7** (2000), 467.