ON C*-ALGEBRAS RELATED TO CONSTRAINED REPRESENTATIONS OF A FREE GROUP

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Abstract. We consider representations of the free group $F_2$ on two generators such that the norm of the sum of the generators and their inverses is bounded by $\mu \in [0, 4]$. These $\mu$-constrained representations determine a C*-algebra $A_\mu$ for each $\mu \in [0, 4]$. When $\mu = 4$ this is the full group C*-algebra of $F_2$. We prove that these C*-algebras form a continuous bundle of C*-algebras over $[0, 4]$ and calculate their $K$-groups.

1. Introduction

The aim of this paper is to study certain family of C*-algebras related to representations of a free group with a given bound for the norm of the sum of the generating elements.

Let $\Gamma$ be a discrete group. If we consider different sets of unitary representations of $\Gamma$ (all representations in this paper are unitary ones), they lead to different group C*-algebras of $\Gamma$. For example, the full group C*-algebra of $\Gamma$, denoted by $C^*(\Gamma)$, is the closure of the group ring $\mathbb{C}[\Gamma]$ with respect to the norm induced by the universal representation (or, equivalently, by all representations); while the reduced group C*-algebra of $\Gamma$, denoted by $C^*_r(\Gamma)$, is the closure of $\mathbb{C}[\Gamma]$ with respect to the norm induced by the regular representation. Here we consider some special classes of representations for the free group on two generators in order to obtain the corresponding C*-algebras. These classes are related to the special element $x$ of the group ring — the sum of all generators and their inverses, sometimes called an averaging operator. This element plays an important role in research related to groups and their C*-algebras. For example, amenability of $\Gamma$ is equivalent to $\|\lambda(x)\| = n$, where $\lambda$ is the regular representation of $\Gamma$ (we use the same notation for representations of groups and of their group rings and C*-algebras) and $n$ is the number of summands in $x$ (twice the number of generators). Property (T) for $\Gamma$ is equivalent to existence of a spectral gap near $n$ in the spectrum of $\pi(x)$ for the universal representation $\pi$ [1].

Let $u$ and $v$ denote the two generators of the free group $F_2$. Then $x = u + u^{-1} + v + v^{-1} \in \mathbb{C}[\Gamma]$.

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Definition 1.1. For \( \mu \in [0, 4] \), a representation \( \pi : F_2 \to U(H_x) \) is called a \( \mu \)-

\[ \|\pi(x)\| = \|\pi(u) + \pi(u)^* + \pi(v) + \pi(v)^*\| \leq \mu. \]

Given \( 0 \leq \mu \leq 4 \), the assignment \( u, v \mapsto (\mu/4 \pm i\sqrt{1-(\mu/4)^2})I, \) where \( I \) is

the identity operator on any (in particular, one-dimensional) Hilbert space, gives rise to a \( \mu \)-constrained representation of \( F_2 \) for any \( \mu \in [0, 4] \). This shows that \( \mu \)-constrained representations exist. Actually there are abundant. For example, all representations of \( F_2 \) are 4-constrained ones, in which case there is actually no constraint at all. Moreover, if \( \pi \) is a \( \mu \)-constrained representation and \( \mu \leq \mu' \leq 4 \), then \( \pi \) is also a \( \mu' \)-constrained one. Let \( \Pi_\mu \) denote the set of all \( \mu \)-constrained representations, then \( \Pi_\mu \subseteq \Pi_{\mu'} \) if \( 0 \leq \mu_1 \leq \mu_2 \leq 4 \). The one-dimensional example above shows also that if \( \mu_1 \neq \mu_2 \) then \( \Pi_{\mu_1} \) is strictly smaller than \( \Pi_{\mu_2} \). Note that \( \Pi_4 \) consists of all representations of \( F_2 \).

As in the case of \( C^*(F_2) \), we first define a (semi)norm \( \| \cdot \|_\mu \) over \( C[F_2] \) induced by \( \Pi_\mu \) and then complete \( C[F_2] \) with respect to \( \| \cdot \|_\mu \), thus obtaining the corresponding group \( C^* \)-algebra \( A_\mu \).

Definition 1.2. For \( a \in C[F_2], \mu \in [0, 4] \), set

\[ \|a\|_\mu := \sup_{\pi \in \Pi_\mu} \|\pi(a)\|. \]

Remark 1.3. Since \( \|a\|_\mu \leq \|a\|_4 = \|a\|_{\text{max}} \), where \( \| \cdot \|_{\text{max}} \) is the norm on \( C[F_2] \) induced by the universal representation, it is clear that \( \| \cdot \|_\mu \) is bounded. Moreover, \( \| \cdot \|_{\mu_1} \leq \| \cdot \|_{\mu_2} \) if \( 0 \leq \mu_1 \leq \mu_2 \leq 4 \). As we use only unitary representations, this is a \( C^* \)-semimnorm.

Set \( N_\mu = \{a \in C[F_2] : \|a\|_\mu = 0\} \) and complete \( C[F_2]/N_\mu \) with respect to \( \| \cdot \|_\mu \) (which is already a norm there). Let us denote this completion by \( A_\mu \). It is obviously a \( C^* \)-algebra for any \( \mu \in [0, 4] \). Our aim is to study the family of \( C^* \)-algebras \( A_\mu \).

Remark 1.4. Note that \( A_\mu \) can be defined as a universal \( C^* \)-algebra generated by two unitaries, \( u, v \) satisfying a single relation \( \|u + u^* + v + v^*\| \leq \mu \).

Proposition 1.5. For any \( 0 \leq \mu_1 \leq \mu_2 \leq 4 \), the identity map on \( C[F_2] \) extends to a surjective \( * \)-homomorphism \( A_{\mu_2} \to A_{\mu_1} \).

Proof. Since \( N_{\mu_2} \subseteq N_{\mu_1} \), the identity map on \( C[F_2] \) gives rise to a map \( C[F_2]/N_{\mu_2} \to C[F_2]/N_{\mu_1} \), which extends to a \( * \)-homomorphism from \( A_{\mu_2} \) to \( A_{\mu_1} \) by continuity. Since the range of this \( * \)-homomorphism is dense in \( A_{\mu_1} \), it is surjective.

Note that \( A_4 \) is isomorphic to the full group \( C^* \)-algebra \( C^*(F_2) \). Later on we shall give a description for \( A_0 \). For \( 2\sqrt{3} \leq \mu \leq 4 \) the identity map on \( C[F_2] \) extends to a surjective \( * \)-homomorphism from \( A_\mu \) to the reduced group \( C^* \)-algebra \( C_r^*(F_2) \) [4].

The aim of this paper is to study the family of \( C^* \)-algebras \( A_\mu \). In the next section we show that this family is a continuous bundle of \( C^* \)-algebras and then we identify \( A_0 \) as a certain amalgamated free product. Finally, following Cuntz [2], we calculate the \( K \)-theory groups for \( A_\mu \) and show that they don’t depend on \( \mu \).
2. Continuity of $A_\mu$

If $\mu_1$ is close to $\mu_2$ then one would expect that $A_{\mu_1}$ and $A_{\mu_2}$ are close to each other. In other words, there is some kind of “continuity” of $A_\mu$ with respect to $\mu$. In order to characterize such “continuity”, we use the notion of continuous bundle of $C^*$-algebras due to Dixmier.

Let $I$ be a locally compact Hausdorff space and let $\{A(x)\}_{x \in I}$ be a family of $C^*$-algebras. Denote by $\prod_{x \in I} A(x)$ the set of functions $a = a(x)$ defined on $I$ and such that $a(x) \in A(x)$ for any $x \in I$.

**Definition 2.1 (3).** Let $A \subset \prod_{x \in I} A(x)$ be a subset with the following properties:

(i) $A$ is a $*$-subalgebra in $\prod_{x \in I} A(x)$,
(ii) for any $x \in I$ the set $\{a(x) : a \in A\}$ is dense in the algebra $A(x)$,
(iii) for any $a \in A$ the function $x \mapsto \|a(x)\|$ is continuous,
(iv) let $a \in \prod_{x \in I} A(x)$, if for any $x \in I$ and for any $\varepsilon > 0$ one can find such $\alpha' \in A$ such that $\|a(x) - \alpha'(x)\| < \varepsilon$ in some neighborhood of the point $x$, then one has $a \in A$.

Then the triple $(A(x), I, A)$ is called a continuous bundle of $C^*$-algebras.

Let $I = [0,4]$, $A = C(I, C^*(F_2))$ and let $B = \{f \in A : \|f(\mu)\|_\mu = 0, \forall \mu \in I\}$. It is clear that $B$ is a closed ideal of $A$, with the quotient map $q : A \to A/B$. Define the map $\iota : A/B \to \prod_{x \in I} A_\mu$ by $\iota(b)(\mu) = q_\mu(a(\mu))$, where $b \in A/B$ and $a \in A$ such that $b = q(a)$, $q_\mu : C^*(F_2) \to A_\mu$ is the quotient map. It is simple to check that $\iota$ is well-defined and injective, so from now on we treat $A/B$ as a subalgebra of $\prod_{x \in I} A_\mu$. In order to prove that $(A_\mu, I, A/B)$ is a continuous bundle of $C^*$-algebras, we need some lemmas.

**Lemma 2.2.** For any $a \in C^*(F_2)$, the function $N_a : I \to \mathbb{R}_+$ defined by $\mu \mapsto \|a\|_\mu$ is continuous.

**Proof.** Given any fixed $\mu_0 \in I$, we will prove that $N_a$ is continuous at $\mu_0$ in two steps: $N_a$ is left and right continuous at $\mu_0$, respectively.

**Step 1.** Note that $N_a$ is a non-decreasing function, so $l = \lim_{\mu \to \mu_0^-} N_a(\mu)$ exists. Assume that $l < N_a(\mu_0)$, then there must exist a representation $\pi$ of $F_2$ such that $\|\pi(u + u^* + v + v^*)\| = \mu_0$ and $l < \|\pi(a)\| \leq N_a$.

Let us first give a family of Borel functions $\{f_t : S^1 \to S^1\}_{t \in [0,1]}$ as follows:

$$f_t(e^{i\theta}) = \begin{cases} e^{i\arccos((1-t)\cos \theta)}, & \theta \in [0,\pi] \\ e^{-i\arccos((1-t)\cos \theta)}, & \theta \in (-\pi,0) \end{cases}$$

Applying Borel functional calculus of $f_t$ to $\pi(u)$ and $\pi(v)$, we get a new representation $\pi_t$ of $F_2$ which is defined by $u \mapsto f_t(\pi(u))$ and $v \mapsto f_t(\pi(v))$, and $\{\pi_t\}_{t \in [0,1]}$ is a continuous family of representations. Since $f_t(z) + f_t(\overline{z}) = (1-t)(z + \overline{z})$, we have $\pi_t(u + u^* + v + v^*) = f_t(\pi(u)) + f_t(\pi(u^*)) + f_t(\pi(v)) + f_t(\pi(v^*)) = (1-t)(u + u^* + v + v^*)$. Hence $\|\pi_t(u + u^* + v + v^*)\| < \mu_0$. Meanwhile, $\|\pi_t(a)\|$ varies also continuously, which contradicts the assumption.

**Step 2.** Assume that, for some $a \in \mathbb{C}[F_2]$, $N_a$ is not continuous at $\mu_0$ from the right, i.e., $N_a(\mu_0) < \lim_{\mu \to \mu_0^+} N_a(\mu) = r$. Then there must exist a family of representations $\{\pi_n : F_2 \to \mathcal{U}(\mathcal{H}_n)\}_{n \in \mathbb{N}}$ such that $\{\|\pi_n(u + u^* + v + v^*)\|\}_{n \in \mathbb{N}}$ is a decreasing sequence convergent to $\mu_0$ and $\lim_{n \to \infty} \|\pi_n(a)\| = r$. 
Let $\prod_{n \in \mathbb{N}} B(\mathcal{H}_n)$ be the $C^*$-algebra of all sequences $b = (b_1, b_2, \ldots, b_n) \in B(\mathcal{H}_n)$, such that $\|b\| := \sup_{n \in \mathbb{N}} \|b_n\| < \infty$. Let $\oplus_{n \in \mathbb{N}} B(\mathcal{H}_n)$ be the ideal of $\prod_{n \in \mathbb{N}} B(\mathcal{H}_n)$ that consists of sequences $(b_1, b_2, \ldots)$ such that $\lim_{n \to \infty} \|b_n\| = 0$. Then $\prod_{n \in \mathbb{N}} B(\mathcal{H}_n)/\oplus_{n \in \mathbb{N}} B(\mathcal{H}_n)$ is a quotient $C^*$-algebra. By Gelfand-Naimark-Segal theorem, there exists a faithful representation $\rho: \prod_{n \in \mathbb{N}} B(\mathcal{H}_n)/\oplus_{n \in \mathbb{N}} B(\mathcal{H}_n) \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Let $q : \prod_{n \in \mathbb{N}} B(\mathcal{H}_n) \to \prod_{n \in \mathbb{N}} B(\mathcal{H}_n)/\oplus_{n \in \mathbb{N}} B(\mathcal{H}_n)$ be the canonical quotient map. Note that, if $b = (b_1, b_2, \ldots) \in \prod_{n \in \mathbb{N}} B(\mathcal{H}_n)$, $\|q(b)\| = \limsup_{n \to \infty} \|b_n\|.

Let $\pi_\infty$ be the representation of $F_2$ defined by $u \mapsto (\rho \circ q)(u)$ and $v \mapsto (\rho \circ q)(v)$. Then $\|\pi_\infty(a + u + v + v^*)\| = \mu_0$, so $\pi_\infty$ is a $\mu_0$-constrained representation of $F_2$. But $\|\pi_\infty(a)\| = \limsup_{n \to \infty} \|\pi_n(a)\| = r > \|a\|_{\mu_0}$, which is a contradiction.

\[ \square \]

Recall that $B = \{ f \in C(I, C^*(F_2)) : \|f(\mu)\|_{\mu} = 0 \text{ for any } \mu \in I \}$. 

**Lemma 2.3.** Set $I_\mu = \{ a \in C^*(F_2) : \|a\|_{\mu} = 0 \}$ Then $\{ g(\mu_0) : g \in B \} = I_{\mu_0}$ for any $\mu_0 \in I$.

**Proof.** From the definition of $B$, it is obvious that $\{ f(\mu_0) : f \in B \} \subseteq I_{\mu_0}$, thus we just need to prove the converse inclusion. An easy observation implies that it suffices to prove this inclusion for positive elements of $I_{\mu_0}$.

Let $a \in I_{\mu_0}$ be positive. We have to find $g \in B$ such that $g(\mu_0) = a$. Define a family of continuous functions by

$$f_\mu(t) = \begin{cases} 0, & t \in (-\infty, \|a\|_\mu] \\ t - \|a\|_\mu, & t \in ([\|a\|_\mu, \infty). \end{cases}$$

As $\|a\|_\mu = 0$ for $\mu \leq \mu_0$, so $f_\mu(a) = a$ for $\mu \leq \mu_0$. It follows from Lemma 2.2 that $f_\mu$ is continuous in $\mu$. Define a function $g : I \to C^*(F_2)$ by $g(\mu) = f_\mu(a)$. Then $g \in A = C(I, C^*(F_2))$ and $g(\mu_0) = a \in I_{\mu_0}$.

Let $q_\mu : C^*(F_2) \to C^*(F_2)/I_{\mu_0} \cong A_\mu$ denote the quotient map. As $\|q_\mu(a)\| = \|a\|_\mu$, so $q_\mu(f_\mu(a)) = f_\mu(q_\mu(a)) = 0$, thus $g(\mu) = f_\mu(a) \in I_{\mu}$, hence $g \in B$.

\[ \square \]

Since we have treated $A/B$ as a subalgebra of $\prod_{\mu \in I} A_\mu$, for any $b \in A/B$, besides the quotient norm, we can also treat $b$ as a function defined on $I$ and take the supremum norm. The following lemma asserts that these two norms coincide.

**Lemma 2.4.** Let $a \in A$, $b = q(a) \in A/B$. Set

$$\|b\|_1 = \inf_{g \in B} \|a + g\| = \inf_{g \in B} \sup_{\mu \in I} \|a(\mu) + g(\mu)\|$$

and

$$\|b\|_2 = \sup_{\mu \in I} \inf_{g \in B} \|a(\mu) + g(\mu)\| = \sup_{\mu \in I} \|a(\mu)\|_\mu \text{ (by Lemma 2.3)}. \]

Then $\|b\|_1 = \|b\|_2$.

**Proof.** This follows from uniqueness of a $C^*$-norm on the $C^*$-algebra $A/B$.

\[ \square \]

**Theorem 2.5.** $(A_\mu, I, A/B)$ is a continuous bundle of $C^*$-algebras.
Proof. Let us check the conditions from the definition of a continuous bundle of $C^*$-algebras one by one.

(i) and (ii) are obviously satisfied and \( \{a(\mu) : a \in A/B\} \) equals \( A_\mu \).

For any \( b \in A/B \) with \( b = q(a) \) where \( a \in A \), given \( \mu, \mu' \in I \),
\[
\|b(\mu')\|_{\mu'} - \|b(\mu)\|_{\mu} \\
\leq \max_{\nu \in \{ \mu, \mu' \}} \|a(\mu') - a(\mu)\|_{\nu} \\
\leq \max_{\nu \in \{ \mu, \mu' \}} \|a(\mu')\|_{\nu} - \|a(\mu)\|_{\nu} + \max_{\nu \in \{ \mu, \mu' \}} \|a(\mu)\|_{\nu} - \|a(\mu)\|_{\nu} \\
\leq \|a(\mu') - a(\mu)\|_{\mu} + \|a(\mu)\|_{\mu'} - \|a(\mu)\|_{\mu}.
\]

If \( \mu' \) is close to \( \mu \) then \( \|a(\mu') - a(\mu)\|_{\max} \) is small because the function \( \mu \mapsto a(\mu) \) is continuous, and \( \|a(\mu)\|_{\mu'} - \|a(\mu)\|_{\mu} \) is small due to Lemma 2.2 therefore, the map \( \mu \mapsto \|b(\mu)\|_{\mu} \) is continuous, i.e., (iii) is satisfied.

Suppose \( z \in \prod_{\mu \in I} A_\mu \) such that for every \( \mu \in I \) and every \( \varepsilon > 0 \), there exists an \( b \in A/B \) such that \( \|z(\mu) - b(\mu)\| \leq \varepsilon \) in some neighborhood \( U_\mu \) of \( \mu \). Thus we obtain an open covering \( \{U_\mu\}_{\mu \in I} \) of \( I \). Let \( \{U_1\}_{i=1}^p \) be its finite sub-covering and let \( (\eta_1, \ldots, \eta_p) \) be a continuous partition of unity in \( I \) subordinate to the covering \( \{U_i\}_{i=1}^p \). Then
\[
\|z(\mu) - \eta_1(\mu)b_1(\mu) - \cdots - \eta_p(\mu)b_p(\mu)\| \leq \varepsilon, \text{ for any } \mu \in I,
\]
or equivalently,
\[
\|z - \eta_1b_1 - \cdots - \eta_pb_p\| \leq \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, \( \eta_pb \) belongs to \( A/B \) and \( A/B \) is norm closed, we have \( z \in A/B \). So (iv) is satisfied.

\[\square\]

3. \( A_0 \) as an amalgamated free product

Here we identify \( A_0 \) as an amalgamated product of \( C^* \)-algebras.

Recall that, given \( C^* \)-algebras \( A_1, A_2 \) and \( B \), the amalgamated free product is the \( C^* \)-algebra, denoted \( A_1 \ast_B A_2 \), together with embeddings \( j_i : A_i \rightarrow A_1 \ast_B A_2 \), satisfying \( j_1 \circ i_1 = j_2 \circ i_2 \), such that the following holds: if \( \phi_k : A_k \rightarrow A_k \), \( k = 1, 2 \), are \( * \)-homomorphisms with \( \phi_1 \circ i_1 = \phi_2 \circ i_2 \) then there is a unique \( * \)-homomorphism \( \phi : A_1 \ast_B A_2 \rightarrow A \) such that \( \phi \circ j_k = \phi_k \), \( k = 1, 2 \). The \( * \)-homomorphism \( \phi \) induced by \( \phi_1 \) and \( \phi_2 \) will sometimes be denoted by \( \phi_1 \ast_B \phi_2 \).

Let \( p : S^1 \rightarrow [-1,1] \) be the projection of the circle \( x^2 + y^2 = 1 \) onto the \( x \) axis. It induces an inclusion \( i_1 : C[-1,1] \rightarrow C(S^1) \) such that \( i_1(\text{id}) = z + \bar{z} \), where \( z = x + iy \) is the coordinate on \( S^1 \) and \( \text{id} \) is the identity function on \( C[-1,1] \). Let \( \tau : C[-1,1] \rightarrow C[-1,1] \) be the flip automorphism, which changes the orientation of the interval and is given by \( \text{id} \mapsto -\text{id} \). Set \( i_2 = i_1 \circ \tau \). Then \( i_2(\text{id}) = -(w + \bar{w}) \).

The inclusions \( i_1 \) and \( i_2 \) of \( C[-1,1] \) into \( C(S^1) \) give us the amalgamated free product \( D = C(S^1) \ast_{C[-1,1]} C(S^1) \).

Lemma 3.1. \( C^* \)-algebras \( A_0 \) and \( D \) are isomorphic.

Proof. Recall that \( A_0 \) is a universal \( C^* \)-algebra generated by two unitaries, \( u \) and \( v \), with a single relation \( u + u^* = -(v + v^*) \).
Let \( \tilde{u}, \tilde{v} \) be generators for the two copies of \( C(S^1) \). Define \( \varphi_k : C(S^1) \to A_0, k = 1, 2 \), by \( \varphi_1(\tilde{u}) = u, \varphi_2(\tilde{v}) = v \). Then \( \varphi_1 \circ i_1 = \varphi_2 \circ i_2 \), hence the maps \( \varphi_k \) give rise to a \( * \)-homomorphism \( D \to A_0 \).

Using universality of \( A_4 \), we can construct a \( * \)-homomorphism \( \psi : A_4 \to D \) by setting \( \psi(u) = \tilde{u} * 1, \psi(v) = 1 * \tilde{v} \). Note that \( A_0 \) is the quotient of \( A_4 \) under a single relation \( u + u^* = -(v + v^*) \), and \( \psi(u + u^*) = -\psi(v + v^*) \), therefore, \( \psi \) factorizes through \( A_0 \), thus giving a \( * \)-homomorphism from \( A_0 \) to \( D \).

The two \( * \)-homomorphisms \( D \to A_0 \) and \( A_0 \to D \) are obviously inverse to each other, hence the two \( C^* \)-algebras are isomorphic. 

Now we may apply the \( K \)-theory exact sequence for amalgamated free products due to Cuntz [2]:

\[
\begin{align*}
K_0(C[-1, 1]) &\xrightarrow{(i_1, i_2)} K_0(C(S^1)) \oplus K_0(C(S^1)) \xrightarrow{j_1 - j_2} K_0(A_0) \\
K_1(A_0) &\xrightarrow{j_1 - j_2} K_1(C(S^1)) \oplus K_1(C(S^1)) \xrightarrow{(i_1, i_2)} K_1(C[-1, 1])
\end{align*}
\]

**Corollary 3.2.** (i) \( K_0(A_0) \cong \mathbb{Z} \) and is generated by the class \([1]\) of unit element;

(ii) \( K_1(A_0) \cong \mathbb{Z}^2 \) and is generated by \([u]\) and \([v]\), which are considered as elements of the first and the second copy of \( C(S^1) \) respectively.

4. \( K \)-Groups of \( A_\mu \)

In [2], J. Cuntz proved that \( K_0(C^*(F_2)) \cong \mathbb{Z}, K_1(C^*(F_2)) \cong \mathbb{Z}^2 \). Here we use his method to calculate the \( K \)-groups for \( A_\mu, 0 \leq \mu < 4 \).

**Remark 4.1.** From Corollary 3.2 we can get some information about \( K \)-groups of \( A_\mu(0 < \mu < 4) \). Since the quotient map \( A_4 \to A_0 \) factorizes through \( A_\mu \) and induces an isomorphism in \( K \)-theory, we may conclude that \( K_*(A_\mu) \) contains \( K_*(A_4) \) as a direct summand.

In Section 1 we show that \( A_\mu \) possesses certain continuity with respect to \( \mu \), together with the fact that the \( K \)-groups of \( A_0 \) and \( C^*(F_2) \) are the same, it would be reasonable to conjecture that all \( A_\mu(0 \leq \mu \leq 4) \) have the same \( K \)-groups. Below we give a proof of this conjecture. The idea of the proof is taken from [2] (cf. Appendix in [5]): to construct a homotopy between the universal representation of \( F_2 \) and the trivial representation. But the trivial representation is not constrained for any \( \mu < 4 \), so we have to replace it by some other representation.

**Theorem 4.2.** The quotient map \( A_4 \to A_\mu \) induces an isomorphism of their \( K_\ast \)-groups.

**Proof.** Let \( B = C(S^1 \vee S^1) \) be the \( C^* \)-algebra of continuous functions on the wedge \( S^1 \vee S^1 \) of two circles. This is the algebra of pairs of functions \( (f, g), f, g \in C(S^1) \) such that \( f(1) = g(1) \), where \( 1 \in S^1 \) is the common point of the two circles (we consider the circle as the subset of the complex plane given by \( |z| = 1 \)). Then \( K_0(B) \cong \mathbb{Z}, K_1(B) \cong \mathbb{Z}^2 \).
Set $\alpha(z) = -\text{Re } z + i |\text{Im } z|$. Since $|\alpha(z)| = 1$, this is a function from $S^1$ to itself with the trivial winding number (equivalently, the trivial homotopy class). Note that $\text{Re}(z + \alpha(z)) = 0$.

For each $\mu$ we define $\ast$-homomorphisms $\phi : A_\mu \to M_2(B)$ and $\psi : B \to M_2(A_\mu)$ as follows,

Set $\psi : (z, 1) \mapsto \begin{pmatrix} u & 0 \\ 0 & \tau \end{pmatrix}$; $(1, z) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$.

Note that $\psi$ defines a $\ast$-homomorphism from $B$ to $M_2(A_\mu)$ for $\mu = 4$, hence one can pass to the quotient to obtain a $\ast$-homomorphism to $M_2(A_\mu)$ for arbitrary $\mu$.

Set $\phi : u \mapsto \begin{pmatrix} (z, 1) & (0, 0) \\ (0, 0) & (-1, \alpha(z)) \end{pmatrix}$; $v \mapsto \begin{pmatrix} (\alpha(z), -1) & (0, 0) \\ (0, 0) & (1, z) \end{pmatrix}$.

Note that $\phi(u + u^* + v + v^*) = \begin{pmatrix} (0, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{pmatrix}$, so $\phi$ is well-defined as a $\ast$-homomorphism from $A_0$ to $M_2(B)$. Then it is well-defined for any $\mu$.

For the composition $\psi \circ \phi : A_\mu \to M_4(A_\mu)$, one has

$$(\psi \circ \phi)(u) = \begin{pmatrix} u & 0 \\ 0 & \alpha(v) \end{pmatrix}; \quad (\psi \circ \phi)(v) = \begin{pmatrix} \alpha(u) & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix}.$$ 

Set

$$V_t = \begin{pmatrix} \cos t & 0 & 0 & \sin t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & 0 \\ -\sin t & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha(u) & 0 \\ 0 & -1 \\ \cos t & 0 & 0 & -\sin t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & 0 \\ \sin t & 0 & 1 & 0 \end{pmatrix},$$

$t \in [0, \pi/2]$. Then one can define a homotopy of $\ast$-homomorphisms $\lambda_t : A_\mu \to M_4(A_\mu)$ by

$$\lambda_t(u) = \psi \circ \phi(u); \quad \lambda_t(v) = V_t.$$ 

Indeed, direct calculation shows that

$$\|\lambda_t(x)\| = \left\| \begin{pmatrix} \sin^2 t \cdot x & 0 & 0 & \sin t \cos t \cdot x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin t \cos t \cdot x & 0 & 0 & -\sin^2 t \cdot x \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sin^2 t & \sin t \cos t \\ \sin t \cos t & -\sin^2 t \end{pmatrix} \right\| \cdot \|x\| = \sin t \cdot \|x\| \leq \|x\| \leq \mu,$$ 

where $x = u + u^* + v + v^*$, hence $\lambda_t$ is continuous for any $t \in [0, \pi/2]$. 

Then $\lambda_0$ and $\lambda_{\pi/2}$ induce the same map for the $K$-theory. At the end-points one has $\lambda_0 = \psi \circ \phi$ and $\lambda_{\pi/2} = \text{id}_{A_\mu} \oplus \tau_1 \oplus \tau_2 \oplus \tau_3$, where $\tau_1(u) = 1_{A_\mu}$, $\tau_1(v) = -1_{A_\mu}$; $\tau_2(u) = -1_{A_\mu}$, $\tau_2(v) = 1_{A_\mu}$; $\tau_3(u) = \alpha(v)$, $\tau_3(v) = \alpha(u)$. 

Let $\tau : A_\mu \to A_\mu$ be a $\ast$-homomorphism given by $\tau(u) = \tau(v) = i \cdot 1_{A_\mu}$. The formulas $u_t = -t \text{Re } v + i \sqrt{1 - t^2 (\text{Re } v)^2}$, $v_t = -t \text{Re } u + i \sqrt{1 - t^2 (\text{Re } u)^2}$, $t \in [0, 1]$, provide a homotopy connecting $\tau_3$ and $\tau$. Similarly, $\tau_1$ and $\tau_2$ are homotopic to $\tau$ due to the homotopies $u_t = (\pm \cos t + i \sin t) \cdot 1_{A_\mu}$, $v_t = (\mp \cos t + i \sin t) \cdot 1_{A_\mu}$, $t \in [0, \pi/2]$. All these homotopies satisfy the constraint $\|u_t + u_t^* + v_t + v_t^*\| \leq \mu$ when $\|u + u^* + v + v^*\| \leq \mu$. 

Thus, for the induced maps in $K_\ast$-groups one has $(\psi \circ \phi)_\ast = \text{id}_{K_\ast(A_\mu)} + 3\tau_\ast$, or, equivalently,

$$\text{id}_{K_\ast(A_\mu)} = (\psi \circ \phi)_\ast - 3\tau_\ast.$$ 

Note that $\tau$ factorizes through $C$: $\tau: A_\mu \to C \to A_\mu$. Therefore, for $K_1$, the map $\tau_*: K_1(A_\mu) \to K_1(A_\mu)$ is zero (as $K_1(C) = 0$), so $\text{id}_{K_1(A_\mu)} = (\psi \circ \phi)_\ast$.

For any $\mu \in (0, 4)$, consider the commuting diagram

\[
\begin{array}{ccc}
K_1(A_4) & \xrightarrow{\phi} & K_1(B) \\
\downarrow & & \downarrow \\
K_1(A_\mu) & \xrightarrow{\psi} & K_1(A_\mu) \\
\downarrow & & \downarrow \\
K_1(A_0) & \xrightarrow{\psi} & K_1(A_0)
\end{array}
\]

where the diagonal arrows are isomorphisms and the compositions of the vertical arrows are identity maps. The latter implies that the map $K_1(A_4) \to K_1(A_\mu)$ is injective. If it is not surjective, there would exist some element in $K_1(A_\mu)$ that doesn’t come from $K_1(A_4)$, but this contradicts $\text{id}_{K_1(A_\mu)} = (\psi \circ \phi)_\ast$. Thus, the map $K_1(A_4) \to K_1(A_\mu)$ induced by the quotient map is an isomorphism.

As the map $\tau_*: K_0(A_\mu) \to K_0(A_\mu)$ is not trivial, the case of $K_0$ is slightly more difficult, and we deal with it below.

Recall that $\tau$ factorizes through $C$. Let $\rho: A_\mu \to C$ denote the character such that $\tau = \iota \circ \rho$, where $\iota: C \to A_\mu$ is the canonical inclusion of scalars, $\iota(\lambda) = \lambda \cdot 1_{A_\mu}$. Then $\rho(u) = \rho(v) = i$.

Note that the composition $K_0(C) \xrightarrow{\iota_*} K_0(A_\mu) \xrightarrow{\rho_*} K_0(C)$ is the identity map on $K_0(C)$. Thus $K_0(A_\mu) = K_0(C) \oplus \ker \tau_*$.  

Let $\sigma: B \to C$ be a $\ast$-homomorphism defined by $\sigma((z, 1)) = \sigma((1, z)) = i$. Then $\sigma((-1, \alpha(z))) = \alpha(i) = i$. Therefore, $\sigma(\phi(u)) = \sigma(\phi(v)) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, hence $\sigma \circ \phi = \rho \oplus \rho$.

Let $p \in K_0(A_\mu)$, $p \in \ker \rho_* = \ker \tau_*$. Then $(\sigma \circ \phi)_\ast(p) = 2\rho_\ast(p) = 0$. As $\sigma_*: K_0(B) \to K_0(C)$ is an isomorphism, so $p \in \ker \phi_*$.  

Since $\text{id}_{K_0(A_\mu)} = (\psi \circ \phi)_\ast - 3\tau_\ast$,  

$$p = (\psi \circ \phi)_\ast(p) - 3\tau_\ast(p) = \psi_\ast(\phi_\ast(p)) - 3\tau_\ast(\rho_\ast(p)) = 0,$$

hence $\ker \tau_* = 0$, $K_0(A_\mu) \cong \mathbb{Z}$ (generated by $[1_{A_\mu}]$).

\[\square\]

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