Singularity Propagation for the Gurtin-Pipkin equation

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Abstract

We show that the Dirac delta function in the boundary condition of the Gurtin-Pipkin equation generates a moving delta-function with an exponentially decreasing factor.

1 Introduction

Gurtin and Pipkin in [3] have derived the first time the equation for the heat transfer with finite propagation speed in contrast yo the "classical" heat equation., see [13] where this fact has been was first proved.

Now systems of such type are considered in several fields of physics such as systems with thermal memory [9], viscoelasticity problems [8], and acoustic waves in composite media [7].

We consider the equation of the first order in time on (0, ∞) with zero initial data

\[ \theta_t = \int_0^t k(t-s)\theta_{xx}(s)\,ds, \quad \theta(x,0) = 0 \] (1)

and with nonhomogeneous DBC

\[ \theta(0,t) = u(t), \]

where \( u \) may be a distribution. We suppose that \( k > 0 \). In application \( k \) is a decreasing function.

In the case \( k \) is the Dirac delta \( \delta(t) \) the equation becomes the heat one with the infinite velocity. In the 'opposite' case \( k(t) = \text{Const} \) the equation (11) is, in a fact, an integrated wave equation. In [11], [12] the kernel of the form \( a/(t+\omega)^l \), \( l > 1 \) is considered, what gives a non-autonomous telegraph-equation, which has a self-similar solution. The damped wave equation ( or

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a kind of the telegraph-equation) arises, if we take an exponential \( k(t) = a \exp(-bt) \), see below.

In a sense in this notes we refine the border separating the system with finite and infinite velocity of propagation. In the terms of the Laplace transform the dynamical system has a finite velocity if the image of the kernel devreasing as \( 1/z \) or faster.

Note that in \([1]\) the integral term gives a stronger perturbation than in the system of the second order in time. Also study the spectrum of the systems corroborate this fact, see \([1],[10]\). See also papers about regularity of solutions, \([5],[10]\).

The main result of the notes is that for a positive smooth kernel the main singularity of the solution generated by \( \delta(t) \) in the boundary condition has the form

\[
e^{-\beta x} \delta(t - x/a) \tag{2}
\]

with

\[
a = \sqrt{k(0)}, \quad \beta = -k'(0)/2k(0). \tag{3}
\]

The proof is based on the Laplace transform of the solution and the Paley - Wiener theorem.

The main term \((2)\) coincides with the main term of singularity of the damped wave equation

\[
\theta_{tt} = a^2 \theta_{xx} - b \theta_t, \quad b = 2\alpha\beta. \tag{4}
\]

Also the regularity and partially behavior of the solutions to this equation is the same as regularity of the solutions to \((1)\), see \([5]\). This allows us to say that the \((1)\) is a perturbation of the \((4)\).

Probably, there are some relation with the following result. In \([4]\) it was proved that the sharp control time for second order in time equation with memory is the same as for corresponding telegraph equation.

### 2 Main results

#### 2.1 Finite velocity

We will work with distribution here and suppose that the kernel and the solution does not grow too fast and we can apply the Laplace transform. Apply this transform to \((1)\) and denote the Laplace-image by capital characters.

**Theorem 1** Let \( u(t) = \delta(t) \) and

\[
K(z) = \frac{a^2}{z} + O \left( \frac{1}{z^2} \right). \tag{5}
\]

Then the solution propagates with the velocity \( a \).
Proof:

After Laplace transform we obtain the family of ODEs

\[ z \Theta(x, z) = K(z) \Theta_{xx}(x, z), \quad x > 0, \quad \Theta(0, z) = U(z). \quad (6) \]

For every \( z \) this differential (in \( x \)) equation has constant coefficients. Set

\[ \varphi(z) = \sqrt{z/K(z)} \]

(the main branch). The solution do not increasing exponentially in the right half plane is

\[ \Theta(x, z) = U(z)e^{-\varphi(z)x}. \]

Let the boundary condition be the Dirac delta: \( u(t) = \delta(t) \).

On the case of \( \Theta \)

\[ \varphi(z) = \frac{z}{a} + O(1). \]

We see that for fixed \( x \) the function \( e^{xz/a} \Theta(x, z) \) is bounded in the right half plane. By the Paley-Wiener theorem for distributions, see [6], Th 7.3.1, this gives that the preimage \( \theta(x, t) \) is t-supported on \((0, x/a)\). In the other words, the front of the solution is moving with the velocity \( a \).

Example 2 The idea of the proof is very clear for the wave equation, i.e., for the case \( k(t) = a^2 \). Evidently, the solution is \( \theta = \delta(t - x/a) \). In the Laplace-images we have

\[ K(z) = \frac{a^2}{z}, \quad \varphi(z) = \frac{z}{a}. \]

Therefore the solution to \( (6) \) is

\[ \Theta(x, z) = e^{-\frac{x}{a}z}. \]

The function \( e^{xz/a} \Theta(x, z) \) is 1 here.

Example 3 If \( k = 1/t^\alpha, \quad \alpha < 1 \), then

\[ K = z^{1-\alpha}, \quad \varphi = z^{1-\alpha/2} \]

and the velocity is infinite.
2.2 Propagation of singularities

**Theorem 4** Let \( u(t) = \delta(t) \) and

\[
K(z) = \frac{a^2}{z} - \frac{\beta}{z^2} + \frac{c}{z^3} + o\left(\frac{1}{z^4}\right), \quad a > 0. \tag{7}
\]

Then the solution has the form

\[
\theta(x, t) = e^{-bx/2a} \delta(t - x/a) + p(x)H(t - x/a) + q(x, t), \tag{8}
\]

and \( q(x, t) \) is a function continuous near the front \( t = x/a \).

**Proof:** If we have

\[
F(z) = e^{-\alpha z} \left( a_1 + a_2 \frac{1}{z} + a_3 \frac{1}{z^2} + \ldots \right), \tag{9}
\]

then the preimage is

\[
f(t) = a_1 \delta(t - \alpha) + a_2 H(t - \alpha) + a_3 (t - \alpha)_+ + \ldots.
\]

The idea of the proof is to obtain the expansion \( (9) \) for \( e^{-\varphi(z)x} \). For simplicity we omit calculation of the coefficient \( p \) by the Heaviside function. Find, using \( (7) \), the coefficients in the expression of \( \varphi \)

\[
\varphi(z) = a_1 + a_2 \frac{1}{z} + O\left(\frac{1}{z^2}\right).
\]

We have

\[
\frac{z}{K(z)} = \frac{z^2}{a^2} \left( \frac{1}{1 - \beta/a^2 z + O(1/z^2)} \right).
\]

From here

\[
\varphi = \frac{z}{a} \left[ 1 + \frac{\beta}{2a^2 z} + O\left(\frac{1}{z^2}\right) \right].
\]

Now

\[
\Theta(x, z) = e^{-\varphi(z)x} = e^{-\frac{\beta x}{2a^2} e^{\frac{\beta x}{2a^2}} e^{O(1/z)}} = e^{-\frac{\beta x}{2a^2}} e^{-\frac{\beta x}{2a^2} (1 + O(1/z))}.
\]

Then the preimage of the main term of \( \Theta \) is

\[
e^{-\frac{\beta x}{2a^2}} \delta(t - x/a) = e^{-bx/2a} \delta(t - x/a),
\]

with \( b = \beta a^3/2 \).

\[\blacksquare\]
\subsection{damped wave equation}

Here we find the singularity in the case of $k$ is an exponential. If $k(t) = a^2 e^{-bt}$ (and $K(z) = a^2/(z+b)$), then the differentiation gives a damped (or telegraph) wave equation \cite{2}. Set

$$\theta = e^{-bt/2}y(x,t).$$

The equation for $y$ is

$$y_{tt} = \alpha^2 y_{xx} + \frac{b^2}{4} y$$

with the same initial condition and the boundary condition

$$y(0, t) = e^{bt/2} u(t).$$

Apply the WKB method to find main singularities. The justification can be found in \cite{2}. Let

$$u(t) = \delta(t).$$

Write the solution as

$$y(x, t) = \delta(t - x/a) + p(x) H(t - x/a) + q(x, t)(t - x/a)+.$$

Then the main terms of the LHS are

$$\delta'' + p\delta' + q\delta.$$

And the main terms of the RHS are

$$a^2 \left[ \frac{1}{a^2} \delta'' + \frac{1}{a^2} p\delta' - 2\frac{1}{a} p' \delta + \frac{1}{a^2} q\delta \right] + \frac{b^2}{4} \delta.$$

Matching the coefficients of the singularities we see that the coefficients by $\delta''$ and $\delta'$ coincide for any $p$ and $q$. Comparing the coefficients by $\delta$ we find

$$2a p' = \frac{b^2}{4}.$$

Taking into account the boundary condition we can write

$$p(x) = \frac{b^2}{8a} x.$$ 

Now

$$y(x, t) = \delta(t - x/a) + \frac{b^2}{8a} x H(t - x/a) + q(x, t)(t - x/a)+.$$

And

$$y(x, t) = e^{-bt/2} \delta(t - x/a) + \frac{b^2}{8a} x e^{-bt/2} H(t - x/a) + q(x, t)e^{-bt/2}(t - x/a)+.$$
Because we are interested in the behavior near the front \( x = at \), this can be rewritten as

\[
\theta(x, t) = e^{-bx/2a} \delta(t - x/a) + \frac{b^2}{8a} x e^{-bx/2a} H(t - x/a) + q(x, t)(t - x/a)_+.
\]

In the case of (7) we can obtain similar result with more complicated calculations.

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