Matrix equations of hydrodynamic type as lower-dimensional reductions of Self-dual type $S$-integrable systems

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Abstract

We show that matrix $Q \times Q$ Self-dual type $S$-integrable Partial Differential Equations (PDEs) possess a family of lower-dimensional reductions represented by the matrix $Q \times n_0Q$ quasilinear first order PDEs solved in [29] by the method of characteristics. In turn, these PDEs admit two types of available particular solutions: (a) explicit solutions and (b) solutions described implicitly by a system of non-differential equations. The later solutions, in particular, exhibit the wave profile breaking. Only first type of solutions is available for (1+1)-dimensional nonlinear $S$-integrable PDEs. (1+1)-dimensional $N$-wave equation, (2+1)- and (3+1)-dimensional Pohlmeyer equations are represented as examples. We also represent a new version of the dressing method which supplies both classical solutions and solutions with wave profile breaking to the above $S$-integrable PDEs.

1 Introduction

There are several types of nonlinear equations of Mathematical Physics, which are referred to as integrable Partial Differential Equations (PDEs). We underline three following types of equations:

1. Equations integrable by the Inverse Spectral Transform Method (ISTM), which are also called soliton or $S$-integrable equations. Korteweg-de-Vries equation (KdV) has been discovered as the first representative of this class [1]. Among other methods of study of soliton equations we underline so-called dressing method, which was invented in [2, 3] and developed in set of papers, for instance, [4, 5, 6, 7, 8], see also [9, 10, 11].

2. Nonlinear PDEs linearizable by some direct substitution, or $C$-integrable equations [12, 13, 14, 15, 16, 17]. Mostly remarkable is Hopf substitution allowing to integrate Bürgers type nonlinear PDEs.
3. Quasilinear first order PDEs integrable by the method of characteristics [18] and by generalized hodograph method [19, 20, 21]. The later method is well applicable to (1+1)-dimensional nonlinear PDEs. Investigation of the higher dimensional systems by this method requires special efforts (see, for instance, the method of hydrodynamic reductions, which will be cited below).

Many other method for study of the nonlinear PDEs have been developed. For instance, the algebraic-geometrical approach is well suited for the construction of local solutions [22, 23]. However, only restricted class of global solutions may be described in this way. A method for implicit description of solutions manifold is the method of hydrodynamic reductions [24, 25, 26, 27, 28], where solutions of multidimensional integrable PDEs are described in terms of Riemann invariants of some (1+1)-dimensional first order quasilinear PDEs. The criterion for applicability of such technique to the given nonlinear PDE has been invented. This method allows to implement infinitely many arbitrary functions of single argument into solution of the nonlinear PDE satisfying this criterion.

Our approach to the problem of construction of the particular solutions to the Self-dual type S-integrable PDEs is most similar to the method of hydrodynamic reductions. However, if it is applicable, we are able to describe the bigger solutions space. Namely, number of arguments in the arbitrary functions implemented into solution has no restriction and is defined by the particular nonlinear PDE. But we have no criterion for applicability of this algorithm to the given nonlinear PDE, unlike the method of hydrodynamic reductions.

We exhibit a class of solutions with wave profile breaking to the S-integrable PDEs selected above. This phenomenon is not surprising, since, simultaneously, these solutions are solutions of the well established class of first order quasilinear PDEs in lower dimension, which are known to possess solutions spaces implicitly described by systems of non-differential equations [29, 30]. Remark that solutions with breaking of wave profile to dispersion-less Kadomtsev-Pertviasvili equation (dKP) (which is compatibility condition of appropriate pair of vector fields) have been obtained and described in [31] using ISTM.

In the rest of Introduction we, first of all, recall the method of characteristics applicable to a certain type of matrix first order quasilinear PDEs, Sec.1.1. Then we remind some important properties of the Self-dual type nonlinear S-integrable PDEs, Sec.1.2. Basic novalties appearing in this paper are underlined in Sec.1.3. Finally, we represent the brief contents of the paper in Sec.1.4.

### 1.1 First order quasilinear PDEs integrable by the method of characteristics

As it was mentioned above, we associate the given Self-dual type S-integrable nonlinear PDE with the family of lower dimensional first order matrix PDEs integrable by the method of characteristics. Let us describe this family of PDEs. Throughout this paper we will write superscripts in parentheses in order to distinguish them from powers.

It is well known that the scalar first order PDE

\[
    u_t + \sum_{k=1}^{N} u_{x_k} \rho^{(k)}(u) = \rho(u), \quad u: \mathbb{R}^{n+1} \rightarrow \mathbb{R},
\]

(1)

can be solved by the method of characteristics [18]. For instance, solution \( u(x,t) \) of (1) for
\(\rho = 0\) is defined implicitly by the non-differential equation
\[
u = f \left( x_1 - \rho^{(1)}(u)t, \ldots, x_N - \rho^{(N)}(u)t \right).
\] (2)

Vector generalizations of equation (1), i.e. the systems of several coupled first order scalar PDEs, have been investigated in set of papers [19, 21, 32, 33] using generalized hodograph method in most cases. Recently [29] the matrix generalization of eq.(1) has been solved using algebraic approach:
\[
w_t + \sum_{k=1}^{N} w_{x_k} \rho^{(k)}(w) = \rho(w) + [w, T \tilde{\rho}(w)],
\] (3)

where \(w\) is the unknown \(Q \times Q\) matrix function of the \(N+1\) independent variables \((x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}\), \(T\) is any constant diagonal matrix, \([·, ·]\) is the usual commutator between matrices, and \(\rho^{(k)}, \rho, \tilde{\rho} : \mathbb{R} \to \mathbb{R}, \ k = 1, \ldots, N\) are \(N+2\) arbitrary scalar functions representable by the positive power series:
\[
\rho^{(j)}(z) = \sum_{i \geq 0} \alpha^{(j,i)} z^i, \quad \rho(z) = \sum_{i \geq 0} \alpha^{(i)} z^i, \quad \tilde{\rho}(z) = \sum_{i \geq 0} \tilde{\alpha}^{(i)} z^i,
\] (4)

where \(\alpha^{(j,i)}, \alpha^{(i)}, \tilde{\alpha}^{(i)}\) are scalar constants, so that the quantities \(\rho^{(k)}(w), \ k = 1, \ldots, N, \rho(w)\) and \(\tilde{\rho}(w)\) are well-defined functions of the matrix \(w\).

Remark. We may always apply the following change of variable \(t\) and field \(w\)
\[
\partial_t + \sum_{j=1}^{N} \alpha^{(j;0)} \partial_{x_j} \to \partial_t, \ w \to e^{-\tilde{\alpha}^{(0)} T t} w e^{\tilde{\alpha}^{(0)} T t}
\] (5)

which is equivalent to putting
\[
\alpha^{(j;0)} = 0, \quad \tilde{\alpha}^{(0)} = 0
\] (6)
in the eqs.(4) without loss of generality.

Solutions space to the matrix equation (3) with \(\rho = 0\) is implicitly described by the following algebraic equation:
\[
w_{\alpha \beta} = \sum_{j=1}^{Q} \left( e^{-T \alpha \tilde{\rho}(w)t} F_{\alpha \gamma} \left( x_1 I - \rho^{(1)}(w)t, \ldots, x_N I - \rho^{(N)}(w)t \right) e^{T \gamma \tilde{\rho}(w)t} \right)_{\gamma \beta},
\] (7)

where \(\alpha = 1, \ldots, Q, \beta = 1, \ldots, Q, \ I\) is \(Q \times Q\) identity matrix.

If \(\rho = 0\), then eq.(3) has infinitely many commuting flows. In this case, it is convenient to replace eq.(3) by the next equation
\[
w_{m} + \sum_{k=1}^{N} w_{x_k} \rho^{(mk)}(w) = [w, T^m \tilde{\rho}^{(m)}(w)], \ m = 1, 2, \ldots,
\] (8)

where index \(m\) enumerates commuting flows, \(\rho^{(mk)}\) are arbitrary scalar functions representable by a positive power series
\[
\rho^{(mj)}(z) = \sum_{i \geq 0} \alpha^{(mj;i)} z^i, \quad \tilde{\rho}^{(m)}(z) = \sum_{i \geq 0} \tilde{\alpha}^{(m;i)} z^i,
\] (9)
\(\alpha^{(mj)}\), \(\alpha^{(mi)}\) are scalar constants, \(T^{(m)}\) are diagonal matrices. Note, that the transposed equation

\[
\bar{w}_{\bar{m}} + \sum_{k=1}^{N} \rho^{(mk)}(\bar{w}) \bar{w}_{x_k} = [\bar{\rho}^{(m)}(\bar{w}) T^{(m)}, \bar{w}], \quad \bar{w} = w^T, \quad m = 1, 2, \ldots
\]

is also integrable.

Equation (3) as well as its generalizations whose solutions spaces may be completely described implicitly by the system of non-differential equations will be referred to as PDE0 in this paper for the sake of brevity.

Using either the algebraic manipulations (Sec.2) or the modified version of the dressing method (Sec.3), we will show that PDE0s (8) may be viewed as lower dimensional reductions of the classical Self-dual type \(S\)-integrable PDEs. These PDEs are referred to as PDE1s in this paper.

1.2 Family of PDE1s

PDE1s may be viewed as compatibility conditions of the following overdetermined system of linear PDEs for some spectral function \(V(\lambda; x)\) (spectral system):

\[
V_{\bar{m}}(\lambda; x) + \sum_{i=1}^{N} \lambda^{n_{mi}} V_{x_i}(\lambda; x) + \sum_{i=1}^{\tilde{N}} \lambda^{\tilde{n}_{mi}} V(\lambda; x) T^{(i)} = \sum_{i=0}^{N_{\text{max}}} \lambda^i V(\lambda; x) q^{(mi)}(x), \quad m = 1, 2.
\]

Here all functions are \(Q \times Q\) matrix functions, \(\lambda\) is a complex spectral parameter, \(n_{mi}\) and \(\tilde{n}_{mi}\) are some integers, \(N_{\text{max}}^{(m)} = \max(n_{mi1}, \tilde{n}_{mi2}, i_1 = 1, \ldots, N, i_2 = 1, \ldots, \tilde{N})\); \(q^{(mi)}(x)\) are some matrix functions of \(x = (x_1, x_2, \ldots, x_1, x_2, \ldots)\). Throughout this paper we use Greek letters in the lists of arguments for spectral parameters. The feature of this system is that its LHS is a first order differential operator having coefficients which are polynomial in \(\lambda\) and independent on the vector parameter \(x\), i.e. \(x\)-dependent potentials appear only in the RHS, which is also polynomial in \(\lambda\). Compatibility condition of any pair of eqs.(11) results in PDE1 for the functions \(q^{(mi)}\). Note that eq.(11) uses only multiplication by the spectral parameter \(\lambda\) and does not involve derivatives of the spectral function with respect to the spectral parameter. For this reason such \(S\)-integrable PDEs as dispersion-less Kadomtsev-Petviashvili equation (dKP) and Heavenly equation may not be considered in the framework of the represented dressing algorithm.

Remark, that the spectral system (11) may be replaced by the equivalent one

\[
\partial_{\bar{m}} V(\lambda; x) + \sum_{i=1}^{N} A^{n_{mi}}(\lambda, \nu) \ast \partial_{x_i} V(\nu; x) + \sum_{i=1}^{\tilde{N}} A^{\tilde{n}_{mi}}(\lambda, \nu) \ast V(\nu; x) T^{(i)} = \sum_{i=0}^{N_{\text{max}}} A^i(\lambda, \nu) \ast V(\nu; x) q^{(mi)}(x),
\]

where function \(A(\lambda, \mu)\) replaces multiplication by \(\lambda\) and we define operator \(A^i\) as follows:

\[
A^i = A \ast \cdots \ast A.
\]
Emphasise that \( A \) is independent on \( x \). The eq.(12) reduces to the eq.(11) if \( A(\lambda, \nu) = \lambda \delta(\lambda - \mu)I \), where \( I \) is \( Q \times Q \) identity matrix.

**Example 1: N-wave system.** Linear spectral system reads in the simplest case:

\[
\partial_{t_m} V(\lambda; x) - A(\lambda, \nu) \ast V(\nu; x) T^{(m)} = V(\lambda; x) [T^{(m)}, w^{(0)}(x)], \quad m = 1, 2.
\]  

(14)

Compatibility condition of this system yields the next PDE:

\[
[T^{(1)}, w^{(0)}_{t_2}] - [T^{(2)}, w^{(0)}_{t_1}] + [[T^{(2)}, w^{(0)}], [T^{(1)}, w^{(0)}]] = 0.
\]  

(15)

Most physical applications require reduction \( w^{(0)}_{a \beta} = \bar{w}^{(0)}_{\beta \alpha} \), where \( \alpha \neq \beta \) and ”bar” means complex conjugate. This equation is well known as (1+1)-dimensional N-wave equation [10]. We consider eq.(15) without reductions.

**Example 2: Pohlmeyer equation.** Another example is associated with the following spectral system:

\[
\partial_{t_m} V(\lambda; x) + A(\lambda, \nu) \partial_{x_m} V(\nu; x) = V(\lambda; x) w^{(0)}_{x_m}(x), \quad m = 1, 2.
\]  

(16)

Compatibility condition of this system yields the next PDE:

\[
w^{(0)}_{x_1 t_2} - w^{(0)}_{x_2 t_1} = [w^{(0)}_{x_1}, w^{(0)}_{x_2}],
\]  

(17)

which may be written in a different form:

\[
(J^{-1} J_{t_1})_{x_2} - (J^{-1} J_{t_2})_{x_1} = 0,
\]  

(18)

\[
w^{(0)}_{x_1} = J^{-1} J_{t_1}, \quad w^{(0)}_{x_2} = J^{-1} J_{t_2}.
\]

This is Pohlmeyer equation [34, 35, 36]. Most applications in Physics is associated with its reduction \( J^+ = \pm J \), \( \det J = 1 \), which yields (Anti)Selfdual Yang-Mills equation (ASDYM). This equation has been studied by many authors: [37, 38, 39, 40, 41, 42, 43, 44] (instanton solutions), [45] (merons), [46, 47] (finite gap solutions). (2+1)-dimensional version of ASDYM associated with the reduction

\[
\partial_{x_2} J = \partial_{t_1} J, \quad \partial_{x_2} w^{(0)} = \partial_{t_1} w^{(0)}
\]  

(19)

is also well known:

\[
w^{(0)}_{t_1 t_1} - w^{(0)}_{t_2 x_1} = [w^{(0)}_{t_1}, w^{(0)}_{x_1}] \quad \text{or}
\]  

(20)

\[
(J^{-1} J_{t_1})_{t_1} - (J^{-1} J_{t_2})_{x_1} = 0.
\]

It has been studied in [48, 49] (Initial Value Problem), [50, 51] (localized solutions). Here we will deal with eqs.(17) and (20a) without any reduction for the field \( w^{(0)} \).
1.3 Basic novalties

There are two manifolds of particular solutions to PDE1s. First manifold is associated with the uniquely solvable linear integral equation for some spectral function (Zakharov-Shabat method [52], classical $\bar{\partial}$-problem [53]). We concentrate on the second manifold of solutions which is described implicitly by a system of non-differential equations (see Sec.2). Simultaneously, this manifold is solution manifold to appropriate lower dimensional PDE0. Proper choice of initial data leads to wave profile breaking. Such solutions for Pohlmeyer equation will be discussed. In turn, second manifold of solutions has a sub-manifold of explicit solutions. Regarding $(1+1)$-dimensional $N$-wave equation, only such solutions from the second manifold are available. It is likely that the same conclusion is valid for any other $(1+1)$-dimensional $S$-integrable model.

It is remarkable, that there is a version of the dressing method, which joins both manifolds of solutions to DPE1s, Sec.3. This fact is remarkable, because the second manifold is beyond the scope of the classical dressing method. In addition, this fact demonstrates that two significantly different classes of PDEs ($S$-integrable PDE1s and integrable by the method of characteristics PDE0s) are combined by the dressing method. There are examples of similar combinations. For instance, dressing method combining $S$- and $C$-integrable models has been suggested in [54]; dressing method combining $C$-integrable equations and equations integrable by the method of characteristics has been proposed in [30].

All these examples tell us about flexibility of the dressing method which is important for development of the uniform method for integration of multidimensional nonlinear PDEs.

Usually, we will use notations PDE1($t_{m_1}, t_{m_2}; w^{(0)}$), PDE0($t_m; n_0, w$) and PDE0($t_m; k_0, \tilde{w}$) with arguments reflecting field as well as derivative(s) of field with respect to variable(s) $t_m$ appearing in PDE. Of course, PDE0s and PDE1s involve also derivatives with respect to $x_i$. However, we do not represent them in the lists of arguments. Here $w(x)$ is $n_0Q \times n_0Q$ matrix and $\tilde{w}(x)$ is $k_0Q \times k_0Q$ matrix block functions:

$$w(x) = \begin{bmatrix} w^{(0)}(x) & w^{(1)}(x) & \cdots & w^{(n_0-2)}(x) & w^{(n_0-1)}(x) \\ I & 0 & \cdots & 0 & r^{(1)} \\ 0 & I & \cdots & 0 & r^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & r^{(n_0-1)} \end{bmatrix},$$

$$\tilde{w}(x) = \begin{bmatrix} -\tilde{w}^{(0)}(x) & I & 0 & \cdots & 0 \\ -\tilde{w}^{(1)}(x) & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\tilde{w}^{(k_0-2)}(x) & 0 & 0 & \cdots & I \\ -\tilde{w}^{(k_0-1)}(x) & \tilde{r}^{(1)} & \tilde{r}^{(2)} & \cdots & \tilde{r}^{(k_0-1)} \end{bmatrix}, \quad \tilde{w}^{(0)} = w^{(0)},$$

where $I$ and $0$ are $Q \times Q$ identity and zero matrices, $w^{(i)}$ and $\tilde{w}^{(i)}$ are $Q \times Q$ matrix functions, $n_0$ and $k_0$ are arbitrary integers, $r^{(i)}, \tilde{r}^{(i)}, i = 1, \ldots, n_0 - 1$ are arbitrary constant $Q \times Q$ matrices (if $T^{(i)} = 0$ in eqs.(25,26)) or arbitrary diagonal $Q \times Q$ matrices (if $T^{(i)} \neq 0$). Each PDE1($t_1, t_2; w^{(0)}$) written for the matrix field $w^{(0)}$ and involving derivatives with respect to $t_1$ and $t_2$ possesses a family of PDE0($t_m; n_0, w$) and PDE0($t_m; k_0, \tilde{w}$), $m = 1, 2, n_0, k_0 = 1, 2, \ldots, \text{as lower dimensional reductions.}$ Of course, PDE0($t_1; n_0, w$) is compatible with PDE0($t_2; n_0, w$), as well as PDE0($t_1; n_0, \tilde{w}$) is compatible with PDE0($t_2; n_0, \tilde{w}$).

For instance, in Sec.2 we will derive the following PDE0s associated with $N$-wave equation...
and PDE0s associated with Pohlmeyer equation (Sec.2.1.2):

\[
\begin{align*}
& w_{t_m} + w_{x_m} w = 0, \quad \tilde{w}_{t_m} + \tilde{w}_{x_m} = 0, \quad m = 1, 2.
\end{align*}
\]

If reduction (19) is imposed, then eq.(23) reduces to

\[
\begin{align*}
& w_{t_1} + w_{x_1} w = 0, \quad w_{t_2} + w_{t_1} w = 0, \\
& \tilde{w}_{t_1} + \tilde{w}_{x_1} = 0, \quad \tilde{w}_{t_2} + \tilde{w}_{t_1} = 0.
\end{align*}
\]

General forms of PDE0\((t_m; n_0, w)\) and PDE0\((t_m; k_0, \tilde{w})\), \(m = 1, 2\), which are lower dimensional reductions of some PDE1\((t_1, t_2; w^{(0)})\) are following:

\[
\begin{align*}
& w_{t_m} + \sum_{i=1}^{N} w_{x_i} \rho^{(m_i)}(w) = \sum_{i=1}^{N} [w, T^{(m)}_{n_0} \tilde{\rho}^{(m_i)}(w)], \quad m = 1, 2, \\
& \tilde{w}_{t_m} + \sum_{i=1}^{N} \tilde{\rho}^{(m_i)}(\tilde{w}) \tilde{w}_{x_i} = \sum_{i=1}^{N} [\tilde{\rho}^{(m_i)}(\tilde{w}) \tilde{T}^{(m)}_{k_0}, \tilde{w}], \quad m = 1, 2,
\end{align*}
\]

where functions \(\tilde{\rho}^{(m_i)}\) are defined by the series similar to the eq.(9):

\[\tilde{\rho}^{(m_j)}(z) = \sum_{i=0}^{\infty} \tilde{\alpha}^{(m_j;i)} z^i,\]

and \(\tilde{\alpha}^{(m_j;i)}\) are scalar constants. Remarks on the origin of eqs.(25) and (26) will be given in Sec.3.3, see n.3,5 and n.4,6 respectively. Eq.(26) has the form of transposed equation (25) with replacements

\[
\tilde{w} = w^T, \quad T^{(m)}_{n_0} \rightarrow \tilde{T}^{(m)}_{k_0}.
\]

Because of this similarity, we will deal with PDE1\((t_1, t_2; w^{(0)})\) and PDE0\((t_m; n_0, w)\), \(m = 1, 2\). Regarding PDE0\((t_m; k_0, \tilde{w})\), only some details will be given.

Remark 1. Although any PDE1 is compatibility condition of eqs.(12), we are not able to represent general form of PDE1s explicitly, unlike general form of PDE0, eqs.(25,26).

Remark 2. The eq.(25) defers from the eq.(8) by the sum in the RHS. However, eq.(25) may be treated by the methods developed in [29, 30] for eq.(8).

Remark 3. The eqs.(22,23,25,26) (where \(m = 1, 2\)) have infinitely many commuting flows corresponding to \(m > 2\) in these equations.
1.4 Brief contents

In the next section (Sec.2) we give algebraic description of PDE1(\(t_1, t_2; w^{(0)}\)), PDE0(\(t_m; n_0, w\)) and PDE0(\(t_m; k_0, \tilde{w}\)). Derivation of PDE1 (namely, (1+1)-dimensional N-wave and (3+1)- and (2+1)-dimensional Pohlmeyer equations) associated with appropriate PDE0(\(t_m; n_0, w\)) and PDE0(\(t_m; k_0, \tilde{w}\)), \(m = 1, 2\) is given in Sec.2.1. Associated linear overdetermined systems have been described therein as well. Using slightly modified results of [29, 30] we represent the non-differential matrix equations implicitly describing the solutions spaces to the above PDE0s and associated solutions manifolds to the appropriate PDE1s, Sec.2.2. We derive some manifolds of explicit solutions to N-wave and Pohlmeyer equations and describe solutions with wave profile breaking for (2+1)- and (3+1)-dimensional Pohlmeyer equations.

A new version of the dressing method describing PDE0s and their relations with PDE1s is proposed in Sec.3. We have derived examples of PDE0s and examples of PDE1s (following Sec.2, N-wave and Pohlmeyer equations are taken as examples of PDE1s) together with appropriate linear overdetermined systems (spectral systems), Sec.3.1. Classical solutions manifolds to PDE1s (Sec.3.2.1) and solutions manifolds with wave profile breaking to PDE1s and PDE0s (Sec.3.2.2) have been derived in Sec.3.2.

Although this version of the dressing method is mostly straightforward in spirit of the derivation of PDE1s, it does not supply full solutions spaces to PDE0s, unlike the method of characteristics. The second version of the dressing algorithm exhibits fullness of the solutions spaces to PDE0s, Sec.3.3.

Conclusions are given in Sec.4.

2 Algebraic description of PDE1s and PDE0s

We consider three examples of PDE1(\(t_1, t_2; w^{(0)}\)) (namely, eqs.(15) and (17) together with its (2+1)-dimensional reduction (20a)) and appropriate families of PDE0(\(t_m; n_0, w\)) and PDE0(\(t_m; k_0, \tilde{w}\)), \(m = 1, 2\), in the framework of the algebraic approach, see Secs.2.1.1,2.1.2. The solutions spaces to PDE1(\(t_1, t_2; w^{(0)}\)) and PDE0(\(t_m; n_0, w\)), \(m = 1, 2\), will be investigated in Sec.2.2.

2.1 Derivation of PDE1s and appropriate families of PDE0s

2.1.1 (1+1)-dimensional N-wave equation

It is well known, that any S-integrable equation can be represented as system of two commuting discrete chains. Mostly explicitly this representation is given in Sato approach to integrability, see for instance, [55]. However, any known dressing method gives rise to this representation. We show, that N-wave equation can be derived from the following pair of commuting discrete chains with two discrete parameters:

\[
 w_{t_m}^{(kn)} = w^{(k(n+1))}T^{(m)} - T^{(m)}w^{((k+1)n)} + w^{(k0)}T^{(m)}w^{(0m)}, \quad m = 1, 2, \tag{29}
\]

supplemented by the next non-differential relation among \(w^{(ij)}\):

\[
 w^{((k+1)n)} = w^{(k(n+1))} + w^{(k0)}w^{(0n)} \tag{30}
\]

(see also Sec.3.1.3, eqs.(150,151) with \(s^{(m)} = 0\)). In fact, first of all remark, that eq.(29) gives rise to two alternative discrete chains with single discrete parameter in view of (30). The first
chain follows after putting $k = 0$ and eliminating $w^{(1n)}$ using eq. (30):

$$w_{tm}^{(n)} - [w^{(n+1)}, T(m)] + [T(m), w^{(n)}]w^{(n)} = 0, \ m = 1, 2,$$

where

$$w^{(n)} = w^{(0n)}, \ n = 0, 1, \ldots$$

The second discrete chain follows after putting $n = 0$ and eliminating $w^{(k1)}$ from (29) using (30):

$$\tilde{w}_{tm}^{(k)} - [\tilde{w}^{(k+1)}, T(m)] + \tilde{w}^{(k)}[\tilde{w}^{(0)}, T(m)] = 0, \ m = 1, 2,$$

where

$$\tilde{w}^{(k)} = w^{(k0)}, \ k = 0, 1, \ldots, \tilde{w}^{(0)} = w^{(0)}.$$  

Finally, in order to write PDE1 for the function $w^{(0)}$, we fix $n = 0$ in (31) and eliminate $w^{(1)}$, or fix $k = 0$ in (33) and eliminate $\tilde{w}^{(1)}$. In both cases we will end up with the same equation (15).

Along with PDE1($t_1, t_2; w^{(0)}$) (15) we may derive the family of PDE0($t_m; n_0, w$), $m = 1, 2$, from the eqs. (31) imposing the reduction

$$w^{(n_0)}(x) = \sum_{i=0}^{n_0-1} w^{(i)}(x)r^{(i)},$$

where $n_0$ is an arbitrary integer and $r^{(i)}$ are arbitrary diagonal constant matrices. Similarly, we may derive PDE0($t_m; k_0, \tilde{w}$), $m = 1, 2$, from the eq. (33) with the reduction

$$\tilde{w}^{(k_0)}(x) = \sum_{i=0}^{k_0-1} \tilde{r}^{(i)}\tilde{w}^{(i)}(x), \ \tilde{w}^{(0)} = w^{(0)},$$

where $k_0$ is arbitrary integer and $\tilde{r}^{(i)}$ are arbitrary diagonal constant matrices. Both (35) and (36) are closures of the chains (31) and (33) respectively. However, parameters $r^{(0)}$ and $\tilde{r}^{(0)}$ may be removed from the PDE0s by the shifts of fields $w^{(n_0-1)} + r^{(0)} \rightarrow w^{(n_0-1)}$ and $-\tilde{w}^{(k_0-1)} + \tilde{r}^{(0)} \rightarrow -\tilde{w}^{(k_0-1)}$. Thus hereafter we take

$$r^{(0)} = \tilde{r}^{(0)} = 0$$

without loss of generality. Parameters $n_0, k_0, r^{(i)}$ and $\tilde{r}^{(i)}$ appear in the definitions of $w$ and $\tilde{w}$, eqs. (21). The simplest explicit examples of PDE0($t_m; n_0, w$) and PDE0($t_m; k_0, \tilde{w}$) are following:

$$PDE0(t_m; 1, w) : \begin{align*}
    w_{tm}^{(0)} + [T(m), w^{(0)}]w^{(0)} &= 0, \ m = 1, 2, \\
    w_{tm}^{(1)} + [T(m), w^{(0)}]w^{(1)} &= 0,
\end{align*}$$

$$PDE0(t_m; 2, w) : \begin{align*}
    w_{tm}^{(0)} - [w^{(1)}, T(m)] + [T(m), w^{(0)}]w^{(0)} &= 0, \\
    w_{tm}^{(1)} + [T(m), w^{(0)}]w^{(1)} &= 0, \ m = 1, 2,
\end{align*}$$

$$PDE0(t_m; 1, \tilde{w}) : \begin{align*}
    \tilde{w}_{tm}^{(0)} + \tilde{w}^{(0)}[\tilde{w}^{(0)}, T(m)] &= 0, \\
    \tilde{w}_{tm}^{(1)} + \tilde{w}^{(1)}[\tilde{w}^{(0)}, T(m)] &= 0, \ m = 1, 2,
\end{align*}$$

$$PDE0(t_m; 2, \tilde{w}) : \begin{align*}
    \tilde{w}_{tm}^{(0)} - [\tilde{w}^{(1)}, T(m)] + \tilde{w}^{(0)}[\tilde{w}^{(0)}, T(m)] &= 0, \\
    \tilde{w}_{tm}^{(1)} + \tilde{w}^{(1)}[\tilde{w}^{(0)}, T(m)] &= 0, \ m = 1, 2.
\end{align*}$$
It is not difficult to observe that the above PDE0($t_m; n_0, w$) and PDE0($t_m; k_0, \tilde{w}$) with arbitrary $n_0$ and $k_0$, may be written in the form (22a) and (22b) respectively, where $w$ and $\tilde{w}$ are defined by the formulae (21).

Since the eq. (22b) follows from the eq.(22a) after transposition with replacements (28), it is enough to study one of them, say, eq.(22a). Slightly modifying result of [30], we describe the solutions space to the eq.(22a) with arbitrary integer $n_0$ by the next implicit algebraic equation:

$$w_{\alpha \beta} = \sum_{\gamma=1}^{n_0 Q} \left( e - \frac{\sum_{m=1}^{2} (T_{n_0}^{(m)})_{\alpha \gamma T_m} F_{\alpha \gamma}(w) + \sum_{m=1}^{2} (T_{n_0}^{(m)})_{\gamma \beta T_m} }{F_{\alpha \gamma}(w)} \right)_{\gamma \beta},$$

$$\alpha, \beta = 1, \ldots, n_0 Q,$$

where $F(z_0)$ is $n_0 Q \times n_0 Q$ matrix function of single argument. Details of derivation of this formula by the dressing method are given in Secs.3.2.2 and 3.3. However, the structure of $w$ tells us that $F$ must have the following structure:

$$F_{\alpha \beta}(z_0) = \begin{cases} \text{arbitrary scalar function of arguments,} & \alpha \leq Q \\ \delta_{\alpha \beta} z_0, & \alpha > Q, \end{cases},$$

so that equation (41) becomes an identity for $\alpha > Q$. Thus, the square matrix algebraic equation (41) reduces to the rectangular matrix equation with $\alpha = 1, \ldots, Q$ and $\beta = 1, \ldots, n_0 Q$. The scalar functions of single argument $F_{\alpha \beta}(z_0), \alpha = 1, \ldots, Q, \beta = 1, \ldots, n_0 Q$ are arbitrary. The simplest case $n_0 = 1$ yields $w \equiv w^{(0)}, T_1^{(m)} \equiv T^{(m)}$:

$$w_{\alpha \beta}^{(0)} = \sum_{\gamma=1}^{Q} \left( e - \frac{\sum_{m=1}^{2} T_{3}^{(m)w^{(0)} t_m} F_{\alpha \gamma}(w^{(0)}) + \sum_{m=1}^{2} T_{\gamma}^{(m)w^{(0)} t_m} }{F_{\alpha \gamma}(w^{(0)})} \right)_{\gamma \beta}, \quad \alpha, \beta = 1, \ldots, Q.$$

**Assotiated linear overdetermined system of PDEs.** As it was mentioned above, the linear spectral problem for $N$-wave equation is given by the eqs.(14), where $V(\lambda; x)$ is, generally speaking, a rectangular $lQ \times Q$ matrix function, $l$ is some integer (to anticipate, $l$ is associated with the dimension of the kernel of the integral operator in eq.(114); $l = 2$ in Secs.3.1.3.2). One can extend this spectral system introducing discrete parameter $n$ by the following formulae:

$$A(\lambda, \nu) \ast V^{(n)}(\nu; x) = V^{(n+1)}(\lambda; x) + V^{(0)}(\lambda; x)w^{(n)}(x),$$

$$V_{t_m}^{(n)}(\lambda; x) - A(\lambda, \nu) \ast V^{(n)}(\nu; x)T^{(m)} = V^{(0)}(\lambda; x)[T^{(m)}, w^{(n)}(x)],$$

$$n = 0, 1, 2, \ldots, m = 1, 2.$$

which becomes eq.(14) if $n = 0$ and $V \equiv V^{(0)}$. This extension can be derived formally, for instance, using a version of the dressing method developed in Sec.3, see Sec.3.1.3 eq.(149) with $s^{(m)} = 0$.

Let us consider the reduction (35,37) for the fields $w^{(n)}$, which causes the appropriate reduction for the spectral functions $V^{(n)}$:

$$V^{(n_0)} = \sum_{i=1}^{n_0-1} V^{(i)} p^{(i)}.$$

In view of this reduction, we may introduce the block-vector spectral function

$$V = [V^{(0)} \ldots V^{(n_0-1)}].$$
Then eqs.(44,45) can be written in a compact form:

\[ A(\lambda, \nu) \ast \mathbf{V}(\nu; x) = \mathbf{V}(\lambda; x)w(x) \]
\[ \mathbf{V}_{tm}(\lambda; x) - A(\lambda, \nu) \ast \mathbf{V}(\nu; x)T_{m0}^{(m)} = \mathbf{V}(\lambda; x)[T_{m0}^{(m)}, w(x)], \quad m = 1, 2. \]

An important feature of this system is the eq.(48) which has no derivatives with respect to variables \( t_i \) and \( x_i \). Thus, \( w(x) \) in the RHS of this equation may be treated as a matrix parameter. For this reason, eq.(48) may be explicitly solved, at least, for diagonalizable matrix \( w \) (more complex case, when \( w \) is transformable to the Jordan form, is not considered here):

\[ w(x) = P(x)E(x)P^{-1}(x), \]
\[ P_{\alpha\alpha} = 1, \]

where \( E \) is a diagonal matrix of eigenvalues, \( P \) is a matrix of eigenvectors normalized by the condition (51).

Let \( A \) be in the following form:

\[ A(\lambda, \nu) = \lambda \delta(\lambda - \nu)I_l, \]

where \( I_l \) is \( lQ \times lQ \) identity matrix. Then solution of eq.(48) reads:

\[ \mathbf{V}(\lambda; x)P(x) = \hat{\mathbf{V}}(x)\delta(\lambda I - E), \]

where

\[ \hat{\mathbf{V}} = [\hat{\mathbf{V}}^{(0)} \ldots \hat{\mathbf{V}}^{(n_0-1)}] \]

Thus, \( \mathbf{V}(\lambda; x)P(x) \) is delta-function of \( \lambda \). This is an interesting fact and a non-trivial reduction imposed on the spectral function \( \mathbf{V}(\lambda; x) \). It is remarkable, that there is a dressing algorithm providing consistency of this reduction with eq.(49), see Sec.3.1.3.

There is an important remark. **Remark:** The formula (53) means that

\[ \mathbf{V}(\lambda; x) \to 0 \quad \text{as} \quad \lambda \to \infty \]

for any function \( w(x) \). Such a behaviour of the spectral function is in contradiction with the classical dressing, where the spectral function is not vanishing at infinity. For this reason the solutions described in this section are not observable in the framework of ISTM.

### 2.1.2 Pohlmeyer equation

One can show that Pohlmeyer equation may be derived from the following pair of commuting discrete chains with two discrete parameters:

\[ w_{tm}^{(kn)} + w_{xm}^{((k+1)n)} - w_{xm}^{(kn)}w_{xm}^{(0n)} = 0, \]

supplemented by the eq.(30) (see also Sec.3.1.3, eq.(150,151) with \( s^{(m)} = 1, T^{(m)} = 0 \)). First of all, similar to the Sec.2.1.1, eq.(56) gives rise to two alternative chains with single discreet parameter. The first chain appears after putting \( k = 0 \) and eliminating \( w^{(1n)} \) using eq.(30):

\[ w_{tm}^{(n)} + w_{xm}^{(n+1)} + w_{xm}^{(0n)}w_{xm}^{(n)} = 0, \quad n = 0, 1, 2, \ldots, \quad m = 1, 2, \]
where fields \( w^{(n)} \) are given by the eq.(32). The second discrete chain appears after putting \( n = 0 \) and eliminating \( w^{(k1)} \) using eq.(30):
\[
\dot{w}^{(k)}_{tm} + \ddot{w}^{(k+1)}_{tm} - \dot{w}^{(k)}_{tm} = 0, \quad n = 0, 1, 2, \ldots, \quad m = 1, 2,
\]
where fields \( w^{(n)} \) are defined by the eq.(34). Finally, in order to write PDE1 for \( w^{(0)} \), we put \( n = 0 \) in eq.(57) and eliminate \( w^{(1)} \) or put \( k = 0 \) in eq.(58) and eliminate \( \dot{w}^{(1)} \). In both cases we end up with eq.(17).

Along with PDE1\((t_1, t_2; w^{(0)})\) (17) we may derive PDE0\((t_m; n_0, w)\), \( m = 1, 2 \), imposing the reduction (35,37) to the eq.(57) or PDE0\((t_m; k_0, \dot{w})\), \( m = 1, 2 \), imposing the reduction (36,37) to the eq.(58). The simplest examples are following:

\[
\begin{align*}
PDE0(t_m; 1, w) &: \quad w_{tm}^{(0)} + w_{tm}^{(1)} + w_{tm}^{(0)} w_{tm}^{(0)} = 0, \quad m = 1, 2, \\
PDE0(t_m; 2, w), \quad r^{(1)} = 0 &: \quad w_{tm}^{(0)} + w_{tm}^{(1)} + w_{tm}^{(0)} w_{tm}^{(0)} = 0, \\
PDE0(t_m; 1, \dot{w}) &: \quad \ddot{w}_{tm}^{(0)} - \ddot{w}_{tm}^{(0)} \dot{w}_{tm}^{(0)} = 0, \quad m = 1, 2, \\
PDE0(t_m; 2, \dot{w}), \quad \ddot{r}^{(1)} = 0 &: \quad \ddot{w}_{tm}^{(0)} + \ddot{w}_{tm}^{(1)} - \ddot{w}_{tm}^{(0)} \ddot{w}_{tm}^{(0)} = 0,
\end{align*}
\]

It is not difficult to observe that the above PDE0\((t_m; n_0, w)\) and PDE0\((t_m; k_0, \dot{w})\) with arbitrary \( n_0 \) and \( k_0 \) may be written in the form (23a) and (23b) respectively, where \( w \) and \( \dot{w} \) are defined by the formulae (21).

Similar to the previous section, we represent the detailed consideration only to the eq.(23a). Its solutions space is described by the next non-differential equation [29]:
\[
w_{\alpha\beta} = \sum_{\gamma=1}^{n_0Q} \left( F_{\alpha\gamma}(w, x_1I_{n_0} - wt_1, x_2I_{n_0} - wt_2) \right)_{\gamma\beta},
\]
\[
\alpha, \beta = 1, \ldots, n_0Q.
\]

Details of derivation of this formula by the dressing method are given in Secs.3.2.2 and 3.3. Here, similar to the \( N \)-wave equation, \( F \) must have the structure predicted by the structure of \( w \) given by the eq.(21a):
\[
F_{\alpha\beta}(z_0, z_1, z_2) = \begin{cases} \text{arbitrary scalar function of arguments,} & \alpha \leq Q \\ \delta_{\alpha\beta}z_0, & \alpha > Q \end{cases},
\]
i.e. the square matrix equation (61) reduces to the rectangular one with \( \alpha = 1, \ldots, Q, \ \beta = 1, \ldots, n_0Q \) and arbitrary scalar functions of three arguments \( F_{\alpha\beta}(z_0, z_1, z_2) \). The simplest case \( n_0 = 1 \) yields \( w \equiv w^{(0)} \):
\[
w_{\alpha\beta}^{(0)} = \sum_{\gamma=1}^{Q} \left( F_{\alpha\gamma}(w^{(0)}, x_1I - w^{(0)}t_1, x_2I - w^{(0)}t_2) \right)_{\gamma\beta}, \alpha, \beta = 1, \ldots, Q.
\]

Let reduction (19) be used. Consider PDE0\((t_m; n_0, w)\) (24), \( m = 1, 2 \), corresponding to PDE1\((t_1, t_2; w^{(0)})\) (20). Its solutions space is implicitly described by the next algebraic equation
\[
w_{\alpha\beta} = \sum_{\gamma=1}^{n_0Q} \left( F_{\alpha\gamma}(w, x_1I_{n_0} - wt_1 + w^2t_2) \right)_{\gamma\beta},
\]
\[
\alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0Q
\]
with arbitrary scalar functions of two arguments \(F_{\alpha\beta}(z_0, z_1)\). The simplest case \(n_0 = 1\) yields (\(w \equiv w^{(0)}\)):

\[
w_{\alpha\beta}^{(0)} = \sum_{\gamma=1}^{Q} \left( F_{\alpha\gamma}(w^{(0)}, x_1 I - w^{(0)} t_1 + (w^{(0)})^2 t_2) \right)_{\gamma\beta}, \quad \alpha, \beta = 1, \ldots, Q.
\] (65)

**Associated linear overdetermined system.** The classical overdetermined system for Pohlmeyer equation is given by the eq.(16), where \(V\) is \(lQ \times Q\) matrix function, \(l\) is some integer. System (16) may be extended as follows (such extension may be derived, for instance, using a version of the dressing method developed in Sec.3, see Sec.3.1.3 eq.(149) with \(s^{(m)} = 1, T^{(m)} = 0\)):

\[
V_{tm}^{(n)}(\lambda; x) + A(\lambda, \nu) * V_{xm}^{(n)}(\nu; x) - V^{(0)}(\lambda; x) w_{xm}^{(n)}(x) = 0, \quad m = 1, 2,
\] (66)

where \(V^{(n+1)}\) is related with \(V^{(n)}\) by the eq.(44). Eq.(66) becomes eq.(16) if \(n = 0\) and \(V \equiv V^{(0)}\).

We consider the reduction (35,37,46) and introduce the block-vector spectral function (47), which allows us to write the eq.(66) in a compact form:

\[
V_{tm}^{(n)}(\lambda; x) + A(\lambda, \nu) * V_{xm}^{(n)}(\nu; x) - V(\lambda; x) w_{xm}^{(n)}(x) = 0, \quad m = 1, 2,
\] (67)

while eq.(44) becomes eq.(48). Thus, the spectral equation (48) appears in both spectral systems corresponding to \(N\)-wave and Pohlmeyer equations. This spectral equation is universal equation associated with reduction (46). Suppose that the matrix \(w\) is diagonalizable and representable in the form (50). Let \(A(\lambda, \nu)\) be in the form (52). Then \(V(\lambda; x)\) is given by the eq.(53). The dressing algorithm consistent with reductions (35,37,46) will be represented in Secs.3.1.3 and 3.3.

### 2.2 Solutions space

As before, we assume diagonalizability of \(w\), i.e. \(w\) may be represented in the form (50), and describe two different types of solutions to PDE0\((t_m; n_0, w)\), \(m = 1, 2\), and associated solutions to PDE1\((t_1, t_2; w^{(0)})\):

1. All eigenvalues of \(w\) are independent on \(x\). This leads to the explicit solutions \(w\) without the wave profile breaking. In particular, the solutions in the form of rational function of exponential functions with linear in \(x\) arguments are available. Solitary waves belong to this type.

2. Some or all eigenvalues of \(w\) depend on \(x\). This leads to the solutions \(w\) with wave profile breaking. Such specific behaviour is typical for the systems of hydrodynamic type.

In this section we will use the description of the solutions space to PDE0s given in [29], i.e., instead of the equations (41,61,64) we will consider the appropriate non-differential equations describing the solutions spaces to the first order quasilinear PDEs for \(E\) and \(P\). For PDE0\((t_m; n_0, w)\) written in general form (25) these quasilinear PDEs read:

\[
E_{tm} + \sum_{i=1}^{N} E_{xi} \rho^{(mi)}(E) = 0, \quad m = 1, 2,
\] (68)

\[
P_{tm} + \sum_{i=1}^{N} P_{xi} \rho^{(mi)}(E) = \sum_{i=1}^{\tilde{N}} \left[ P, T_{n_0}^{(i)} \right] \tilde{\rho}^{(mi)}(E), \quad m = 1, 2.
\] (69)
This system is solvable by the method of characteristics [29] giving the following non-differential
equations for $E_\beta$ and $P_{\alpha\beta}$:

\[
E_\beta = F^{(E)}_{\beta} \left( E_\beta, x_1 - \sum_{m=1}^{2} \rho^{(m)}(E_\beta) t_m, \ldots, x_N - \sum_{m=1}^{2} \rho^{(mN)}(E_\beta) t_m \right),
\]

\[
P_{\alpha\beta} = e^{-\sum_{m=1}^{2} \sum_{j=1}^{\tilde{N}} (T_{n_0}^{(j)})_\alpha \tilde{\rho}^{(mj)}(E_\beta) t_m} \sum_{m=1}^{2} \sum_{j=1}^{\tilde{N}} (T_{n_0}^{(j)})_\alpha \tilde{\rho}^{(mj)}(E_\beta) t_m,
\]

\[
x_N - \sum_{m=1}^{2} \rho^{(mN)}(E_\beta) t_m = \sum_{m=1}^{2} \sum_{j=1}^{\tilde{N}} (T_{n_0}^{(j)})_\beta \tilde{\rho}^{(mj)}(E_\beta) t_m,
\]

where $F^{(E)}_{\beta}$ and $F^{(P)}_{\alpha\beta}$ are arbitrary functions of arguments if only there is no restriction on the
structure of $w$. However, unlike [29, 30], we have the particular (rather then general) matrix
structure of $w$, which requires the appropriate particular matrix structure of $P$. In fact, eq.(50) may be written in the form

\[
\sum_{\gamma=1}^{n_0Q} w_{\alpha\gamma} P_{\gamma\beta} = P_{\alpha\beta} E_\beta, \quad \alpha, \beta = 1, \ldots, n_0Q.
\]

Eqs.(72) with $\alpha = 1, \ldots, Q$ and $\beta = 1, \ldots, n_0Q$ should be taken as definitions of $w^{(j)}$, $j = 0, \ldots, n_0 - 1$ in terms of $E$ and $P$. Other equations of the system (72) yield the next linear relations among the elements of $P$:

\[
\sum_{\gamma=1}^{n_0Q} w_{\alpha\gamma} P_{\gamma\beta} = P_{\alpha\beta} E_\beta \Rightarrow P_{(\alpha-Q)\beta} + \sum_{\gamma=(n_0-1)Q}^{n_0Q} r^{(s+1)}_{\tilde{\alpha}\gamma} P_{\gamma\beta} = P_{\alpha\beta} E_\beta,
\]

\[
s = \left\lfloor \frac{\alpha}{Q} \right\rfloor, \quad \tilde{\alpha} = \alpha - sQ, \quad \alpha = Q + 1, \ldots, n_0Q, \quad \beta = 1, \ldots, n_0Q,
\]

where $\lfloor \cdot \rfloor$ means an integer part of number. Eq.(73) with $E_\beta \neq 0$ and arbitrary $r^{(i)}$ may be always solved for the following set of elements $P_{\alpha\beta}$:

\[
P = \{ P_{(\alpha-Q)\beta}, P_{(\alpha-Q)\beta} : \alpha = Q + 1, \ldots, n_0Q, \beta = 1, \ldots, n_0Q, \beta \neq \alpha - Q \}.
\]

Other $n_0^2Q^2 - (n_0 - 1)n_0Q^2 - n_0Q = n_0Q(Q - 1)$ elements of $P$ are defined by the eq.(71) with arbitrary scalar functions $F^{(P)}_{\alpha\beta}$. The case $n_0 > 1$, $E_{\beta_0} = 0$ for some $\beta_0$ results in some restrictions for $r^{(i)}$ so that solution to the eqs.(73) will differ from $P$. This case will not be considered.

All in all, the following algorithm gives either the explicit solutions to PDE0($t_m; n_0, w$) and associated PDE1($t_1, t_2; w^{(0)}$) ($E(x) = \text{const}$) or the non-differential equations ($E(x) \neq \text{const}$) describing the solutions spaces to PDE0($t_m; n_0, w$), $m = 1, 2$, and associated PDE1($t_1, t_2; w^{(0)}$):

1. Fix the arbitrary eigenvalues $E_\beta \neq 0, \alpha = 1, \ldots, n_0Q$, in the form (70).

2. Fix $n_0Q(Q - 1)$ elements of $P_{\alpha\beta} \notin P$ in the form (71).
3. Find other elements of \( P \in \mathcal{P} \) solving the linear system (73) with normalization (51).

4. Find \( w \) using formula (50).

We will describe some solutions to \( N \)-wave and Pohlmeyer equations in Secs.(2.2.1) and (2.2.2) respectively.

Implicit representation of the solutions space by eqs.(70,71) reveals solutions with wave profile breaking. This phenomenon is associated with \( E_\beta(x) \neq \text{const} \) (Pohlmeyer equation). We refer to the break points manifold related with \( E_\beta \) as \( \mathcal{E}_\beta \). We will show that break points manifold for \( P_{\alpha\beta} \) is also \( \mathcal{E}_\beta \). Then the break points manifold corresponding to \( w \) will be \( \mathcal{E} = \cup_{\beta=1}^{n_0} \mathcal{E}_\beta \).

2.2.1 \textit{N-wave equation}

The equations (68,69) corresponding to the eq.(22a) read:

\[
E_{t_m} = 0, \quad P_{t_m} = [P, T^{(m)}]E, \quad m = 1, 2.
\]

Then eq.(41) describing the solutions space to \( N \)-wave equation will be replaced by the next pair of equations (the appropriate variant of the eqs. (70,71)):

\[
E_\alpha = \epsilon_\alpha = \text{const}, \quad \alpha = 1, \ldots, n_0 Q,
\]

\[
P_{\alpha\beta} = e^{-\sum_{m=1}^{2} (\tau^{(m)}_m \alpha \epsilon_{\beta m}(P_0)_{\alpha\beta}) e_{\alpha m}}, \quad \alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0 Q
\]

where \( P_{0\alpha\beta} \) are constants, related by the eq.(73). Only \( n_0 Q(\xi - 1) \) of them may be arbitrary. Appropriate function \( w^{(0)} \) will be rational function of exponents.

An important problem is to exhibit such solutions which have no singularities in domains of all independent variables. Two examples are given below:

1. Let \( \epsilon_\alpha = i \epsilon_\alpha, \epsilon_\alpha \) be real parameters, and require \( \det P \neq 0, t_1 \in \mathbb{R}, t_2 \in \mathbb{R} \). For instance let \( Q = 3, n_0 = 1, P_{023} = P_{032} = 0, \) other elements of \( P_0 \) equal \( i, i^2 = -1 \) \( (P_{0\alpha\beta} = i) \), \( T^{(n)}_\alpha = \alpha^n, \epsilon_\alpha = i (\alpha - 1) \). Corresponding \( P \) reads:

\[
P = \begin{bmatrix}
1 & i e^{i(t_1+4t_2)} & i e^{4i(t_1+4t_2)} \\
i & 1 & 0 \\
i & 0 & 1
\end{bmatrix},
\]

which produces periodic \( w^{(0)} \).

2. Let \( \epsilon_\alpha \) be real parameters, \( \det P \neq 0, t_1 \in \mathbb{R}, t_2 \in \mathbb{R} \). However, unlike the previous example, this is not enough to obtain the bounded solutions, because one has to eliminate the exponential growth of \( w \). Let \( Q = 3, n_0 = 1 \). Exponential growth of \( w^{(0)} \) disappears if one of the eigenvalues is zero and two others have opposite signs. For instance, let \( \epsilon_2 = 0, \epsilon_1 = -\epsilon_3 = 1, T^{(n)}_\alpha = \alpha^n, P_{023} = P_{032} = 0, \) other \( P_{0\alpha\beta} \) equal \( i \). Then \( P \) reads:

\[
P = \begin{bmatrix}
1 & i e^{-(t_1+3t_2)} & i e^{-(t_1+4t_2)} \\
i e^{-(t_1+3t_2)} & 1 & i e^{-(t_1+5t_2)} \\
i e^{2(t_1+4t_2)} & i & 1
\end{bmatrix},
\]

which produces the solitary wave solution \( w^{(0)} \).

An open problem remains how to provide physically important reduction \( w_{\alpha\beta} = \bar{w}_{\beta \alpha} \).
2.2.2 Pohlmeyer equation

Eqs.(68,69) corresponding to the eq.(23a) read:

\[
E_{t_n} + E_{x_n}E = 0, \\
P_{t_n} + P_{x_n}E = 0, \quad n = 1, 2.
\] (79)

The general solution of the eqs.(79) is described by the next pair of non-differential equations (the appropriate variant of the formulae (70,71)):

\[
E_{\beta} = F^{(E)}_{\beta}(E_{\beta}, x_1 - t_1 E_{\beta}, x_2 - t_2 E_{\beta}), \quad (80) \\
P_{\alpha\beta} = F^{(P)}_{\alpha\beta}(E_{\beta}, x_1 - t_1 E_{\beta}, x_2 - t_2 E_{\beta}), \\
\alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0 Q,
\]

where \(F^{(P)}_{\alpha\beta}(z_0, z_1, z_2)\) are related by the system (73), so that only \(n_0 Q (Q - 1)\) of them remain arbitrary functions of three arguments. Similarly, eqs.(68,69) corresponding to the eq.(24a) read:

\[
E_{t_1} + E_{x_1}E = 0, \quad E_{t_2} + E_{t_1}E = 0, \\
P_{t_1} + P_{x_1}E = 0, \quad P_{t_2} + P_{t_1}E = 0.
\] (81)

Solution to this system is implicitly described by the next pair of non-differential equations:

\[
E_{\beta} = F^{(E)}_{\beta}(E_{\beta}, x_1 - t_1 E_{\beta} + t_2 E_{\beta}^2), \\
P_{\alpha\beta} = F^{(P)}_{\alpha\beta}(E_{\beta}, x_1 - t_1 E_{\beta} + t_2 E_{\beta}^2), \\
\alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0 Q,
\] (82)

where \(F^{(P)}_{\alpha\beta}(z_0, z_1)\) are related by the system (73), so that only \(n_0 Q (Q - 1)\) of them remain arbitrary functions of two arguments. System (82) corresponds to the particular choice of \(F^{(E)}\) and \(F^{(P)}\) in (80). Namely,

\[
F^{(E)}(z_0, z_1, z_2) = F^{(E)}(z_0, z_1 - z_0 z_2), \quad F^{(P)}(z_0, z_1, z_2) = F^{(P)}(z_0, z_1 - z_0 z_2), \quad x_2 = 0. \quad (83)
\]

Hereafter we consider the “reduced” formulae (80,82) with

\[
F^{(E)}(z_0, z_1, z_2) = F^{(E)}(z_1, z_2), \quad F^{(P)}(z_0, z_1, z_2) = F^{(P)}(z_1, z_2), \\
F^{(E)}(z_0, z_1) = F^{(E)}(z_1), \quad F^{(P)}(z_0, z_1) = F^{(P)}(z_1). \quad (84)
\]

**On explicit solutions.** As it was mentioned above, the explicit solutions correspond to \(E_{\beta} = \text{const}\). Then the matrix \(P\) is the only object introducing dependence on \(x\) in the field \(w\).

Let \(F^{(P)}_{\alpha\beta}(z)\) be some smooth function vanishing as \(|z| \to \infty\) with single maximum at \(z = z^{(\text{max})}\). The algebraic equations (82) describe two-dimensional surfaces in 3-dimensional space:

\[
x_1 - t_1 E_{\beta} + t_2 E_{\beta}^2 = z^{(\text{max})}, \quad \beta = 1, \ldots, n_0 Q. \quad (85)
\]

Let \(F^{(P)}_{\alpha\beta}(z_1, z_2)\) be smooth functions vanishing as \(|z_1| \to \infty\) and \(|z_2| \to \infty\) with single maximum at \(z_i = z^{(\text{max})}_i, \quad i = 1, 2\). Algebraic equations (80) describe two-dimensional surfaces in 4-dimensional space:

\[
x_1 - t_1 E_{\beta} = z^{(\text{max})}_1, \quad x_2 - t_2 E_{\beta} = z^{(\text{max})}_2, \quad \beta = 1, \ldots, n_0 Q. \quad (86)
\]

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Break points manifolds for wave (82). Consider the solutions (82a) with wave profile breaking assuming that $t_1$ is a physical time and the space variables are $(x_1, t_2)$. The break points manifold $E_\beta$ is defined by the two requirements.

The first requirement is that the derivative of $E_\beta$ in some direction(s) in space $(x_1, t_2)$ tends to infinity. Calculating the partial derivatives

$$
\partial_{x_1} E_\beta = \frac{\partial_z F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2}}{\partial z}, \quad \partial_{t_2} E_\beta = \frac{E_\beta^2 \partial_z F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2}}{\partial z},
$$

we see that all of them tend to infinity when the denominator of the above expressions is zero:

$$
(1 + (t_1 - 2 E_\beta t_2) \partial_z F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2} = 0.
$$

This condition tells us that $d = 1 \neq 0$ on the surface $t_1 - 2 E_\beta t_2 = 0$, so that the only way to have the infinite derivative of $E_\beta$ on this surface is through the infinite derivative of the given function $F^{(E)}(z)$ in accordance with (87). To eliminate this case, hereafter we consider only those functions $F^{(E)}(z)$ which have smooth behaviour and finite derivatives for $z \in \mathbb{R}$. Thus, the necessary condition for the break points manifold is

$$
t_1 - 2 E_\beta t_2 \neq 0.
$$

Note that the derivative of $E_\beta$ is zero in the next direction:

$$
\vec{t} \cdot \nabla E_\beta = 0, \quad \vec{t} = \frac{1}{\sqrt{1 + E_\beta^4}}(E_\beta^2, -1), \quad \nabla = (\partial_{x_1}, \partial_{t_2}).
$$

The second requirement to the break points manifold is that function $E_\beta(x_1, t_1, t_2)$ must change its concavity on this manifold. This requirement is evident in $(1+1)$-dimensional case and is well described in textbooks [18]. It is remarkable, that our $(2+1)$-dimensional case is very similar. To show this, we follow the classical strategy and take $E_\beta$ as independent variable in the neighborhood of the breaking point. There are two possible variants:

1. $x_1$ is a function of $E_\beta, t_2$ and $t_1$, (91)
2. $t_2$ is a function of $E_\beta, x_1$ and $t_1$. (92)

Consider the case (91). All partial derivatives of $x_1$ are finite in the break points manifold (we do not need time-derivative for this analysis):

$$
\partial_{E_\beta} x_1 = \frac{1 + (t_1 - 2 t_2 E_\beta) \partial_z F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2}}{\partial_z F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2}},
$$

$$
\partial_{t_2} x_1 = -E_\alpha^2.
$$

The requirement (88) is equivalent to $\partial_{E_\beta} x_1 = 0$. Similarly, the second derivatives of $x_1$ are following:

$$
\partial_{E_\beta}^2 x_1 = \frac{-2 t_2 (\partial_z F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2})^3 + \partial_z^2 F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2}}{(\partial_z F^{(E)}_\beta(z)|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2})^3},
$$

$$
\partial_{E_\beta} \partial_{t_2} x_1 = -2E_\beta,
$$

$$
\partial_{t_2}^2 x_1 = 0.
We take equation $\partial_{x_1}^2 x_1 = 0$ as the second requirement to the break points manifold, which, in view of (88), reads:

$$
\left. \left( 2t_2 - (t_1 - 2t_2 E_\beta)^3 \partial_z F^{(E)}_\beta(z) \right) \right|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2} = 0. \quad (95)
$$

Eqs.(88,95) are supplemented by the eq.(82a).

It is remarkable, that, starting with eq.(92), we will end up with the same equations (88,95). We do not represent details of appropriate calculations.

The system (82a,88,95) can be solved explicitly:

$$
\mathcal{E}_\beta : \quad t_2 = -\frac{\partial_z^2 F^{(E)}_\beta(z)}{2\left( \partial_z F^{(E)}_\beta(z) \right)^3} \bigg|_{z=\tilde{F}^{(E)}_\beta(E_\beta)},
$$

$$
t_1 = -\frac{\left( \partial_z F^{(E)}_\beta(z) \right)^2 + E_\beta \partial_z^2 F^{(E)}_\beta(z)}{\left( \partial_z F^{(E)}_\beta(z) \right)^3} \bigg|_{z=\tilde{F}^{(E)}_\beta(E_\beta)},
$$

$$
x_1 = t_1 E_\beta - t_2 E_\beta^2 + \tilde{F}^{(E)}_\beta(E_\beta),
$$

where $\tilde{F}^{(E)}_\beta(z)$ is inverse of $F^{(E)}_\beta(z)$: $F^{(E)}_\beta(\tilde{F}^{(E)}_\beta(z)) = z$. This system of three equations describes the break points manifold $\mathcal{E}_\beta$. We can calculate two-component velocity $(\partial_{t_1} x_1, \partial_{t_1} t_2)$ of the break points manifold differentiating the eqs.(82a,88,95) with respect to $t_1$ and assuming that variables $E_\beta, x_1$ and $t_2$ are functions of $t_1$:

$$
\partial_{t_1} E_\beta = \frac{2E_\beta t_2 - t_1}{E_\beta \left( 12t_2 + (t_1 - 2E_\beta t_2)^5 \partial_z^2 F(z) \big|_{z=x_1-t_1 E_\beta+t_2 E_\beta^2} \right)^2}, \quad (97)
$$

$$
\partial_{t_1} x_1 = \frac{E_\beta}{2}, \quad \partial_{t_1} t_2 = \frac{1}{2E_\beta}.
$$

As a simple example, let

$$
F^{(E)}_\beta(z) = - \tanh(z). \quad (98)
$$

Then the systems (96,97) yield:

$$
t_2 = -\frac{E_\beta}{(E_\beta^2 - 1)^2}, \quad t_1 = \frac{1 - 3E_\beta^2}{(E_\beta^2 - 1)^2}, \quad x_1 = \frac{E_\beta - 2E_\beta^3}{(E_\beta^2 - 1)^2} - \text{arctanh}(E_\beta), \quad (99)
$$

$$
\partial_{t_1} E_\beta = \frac{(E_\beta^2 - 1)^3}{2E_\beta (1 + 3E_\beta^2)}, \quad \partial_{t_1} x_1 = \frac{E_\beta}{2}, \quad \partial_{t_1} t_2 = \frac{1}{2E_\beta}. \quad (100)
$$

Equations (99) describe one-dimensional line in 3-dimensional space.

The similar analysis of the break points manifolds for the functions $P_{\alpha\beta}$ shows that these manifolds coincide with $\mathcal{E}_\beta$, i.e. they are described by the same system (88,95) supplemented by both eqs.(82). To obtain this result, we need eq.(82b) where $P_{\alpha\beta}, t_1$ and $t_2$ are taken as independent variable while $x_1$ and $E_\beta$ are considered as functions of them.
Break points manifold for wave (80). Consider the wave profile breaking for the solutions of the eq.(80a) with time \( t_1 \) and space variables \((x_1, x_2, t_2)\). As usual, the first requirement to the break points manifold \( E_\beta \) is that the derivative of \( E_\beta \) tends to infinity in some direction(s) in space \((x_1, x_2, t_2)\). Let us write all partial derivatives:

\[
\frac{\partial_{x_1} E_\beta}{\partial_{x_2} E_\beta} = \frac{\partial z_1 E_\beta}{\partial z_2 E_\beta} \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta}, \quad \frac{\partial_{x_2} E_\beta}{\partial_{x_3} E_\beta} = \frac{\partial z_1 E_\beta}{\partial z_2 E_\beta} \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta},
\]

(101)

\[
\frac{\partial t_1 E_\beta}{\partial t_2 E_\beta} = - \frac{E_\beta \partial z_1 E_\beta}{\partial z_2 E_\beta} \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta}, \quad \partial t_2 E_\beta = - \frac{E_\beta \partial z_2 E_\beta}{\partial z_2 E_\beta} \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta}
\]

\[
d = \left(1 + t_1 \partial z_1 E_\beta(z_1, z_2) + t_2 \partial z_2 E_\beta(z_1, z_2)\right) \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta}
\]

We see that all derivatives tend to infinity if \( d = 0 \):

\[
\left(1 + t_1 \partial z_1 E_\beta(z_1, z_2) + t_2 \partial z_2 E_\beta(z_1, z_2)\right) \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta} = 0
\]

(102)

Note that the derivative of \( E_\beta \) is zero in the direction \( \vec{t} = \frac{1}{\sqrt{1 + E_\beta}}(0, E_\beta, 1) \). The second requirement is that function must change concavity in the break points. To relate this requirement with the second derivative of some coordinate with respect to \( E_\beta \) we follow the usual strategy. Let \( E_\beta \) be independent variable. There are three following cases:

1. \( x_2 \) is a function of \( E_\beta, x_1, t_2 \) and \( t_1 \),
2. \( x_1 \) is a function of \( E_\beta, x_2, t_2 \) and \( t_1 \),
3. \( t_2 \) is a function of \( E_\beta, x_1, x_2 \) and \( t_1 \).

Let us consider the first case in details. One can find the first derivatives of \( x_2 \) differentiating eq.(80) with respect to \( E_\beta, x_1 \) and \( t_2 \) (we do not need the derivative with respect to time \( t_1 \)):

\[
\partial_{E_\beta} x_2 = \frac{1 + t_1 \partial z_1 E_\beta(z_1, z_2) + t_2 \partial z_2 E_\beta(z_1, z_2)}{\partial z_2 E_\beta(z_1, z_2)} \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta},
\]

(103)

\[
\partial_{x_1} x_2 = - \frac{\partial z_1 E_\beta(z_1, z_2)}{\partial z_2 E_\beta(z_1, z_2)} \Big|_{z_1 = z_1 - t_1 E_\beta, z_2 = z_2 - t_2 E_\beta}, \quad \partial_{t_2} x_2 = E_\beta.
\]

The requirement (102) yields \( \partial_{E_\beta} x_2 = 0 \). Similar to the previous paragraph, we take the condition \( \partial_{E_\beta} x_2 = 0 \) as the second requirement defining the break points manifold. The explicit form of this requirement may be found differentiating the eq.(106a) with respect to \( E_\beta \) and using eqs.(106) for the first derivatives of \( x_2 \). One gets in result:

\[
\left[ t_1 \partial_{t_2} F_\beta(z_1, z_2) \right] \left( 2 (1 + t_1 \partial z_1 E_\beta(z_1, z_2)) \partial z_1 \partial_{t_2} F_\beta(z_1, z_2) - t_1 \partial_{z_1} F_\beta(z_1, z_2) \partial_{t_2} F_\beta(z_1, z_2) \right) - \left( \partial_{z_1} F_\beta(z_1, z_2) \right)^2 \partial_{t_2} F_\beta(z_1, z_2) = 0.
\]

(104)
This equation may be given a simpler form using (102) in order to eliminate \( \partial_{z_2} F^E_\beta(z_1, z_2) \) (if \( t_2 \neq 0 \)) or \( \partial_{z_1} F^E_\beta(z_1, z_2) \) (if \( t_1 \neq 0 \)):

\[
\left( t_1^2 \partial_{z_1}^2 F^E_\beta(z_1, z_2) + 2t_1 t_2 \partial_{z_1} \partial_{z_2} F^E_\beta(z_1, z_2) + t_2^2 \partial_{z_2}^2 F^E_\beta(z_1, z_2) \right) \bigg|_{z_1 = s_1 - t_1 E_\beta \atop z_2 = s_2 - t_2 E_\beta} = 0. 
\] (108)

The equations (102,108) must be supplemented by the eq. (80a).

Remark, that starting with the eq.(104) or eq.(105) we will end up with the same two conditions defining the break points manifold, eqs.(102,108).

System (80a,102,108) allows one to write the linear system defining the velocity of the break points manifold. Assuming that variables \( E_\beta, x_1, x_2 \) and \( t_2 \) depend on \( t_1 \) and differentiating eqs.(102,108) with respect to \( t_1 \) we get:

\[
(E_\beta - x'_1) \partial_{z_1} F^E_\beta(z_1, z_2) + (E_\beta t'_2 - x'_2) \partial_{z_2} F^E_\beta(z_1, z_2) = 0, 
\] (109)

\[
\partial_{z_1} F^E_\beta(z_1, z_2) - E_\beta t_2 \partial_{z_2} F^E_\beta(z_1, z_2) - E_\beta t_1 \partial_{z_1} F^E_\beta(z_1, z_2) + 
\]

\[
x'_1 \left( t_2 \partial_{z_1} \partial_{z_2} F^E_\beta(z_1, z_2) + t_1 \partial_{z_1}^2 F^E_\beta(z_1, z_2) \right) + 
\]

\[
x'_2 \left( t_2 \partial_{z_2}^2 F^E_\beta(z_1, z_2) + t_1 \partial_{z_1} \partial_{z_2} F^E_\beta(z_1, z_2) \right) + 
\]

\[
t_2 \left( -E_\beta t_1 \partial_{z_1} F^E_\beta(z_1, z_2) + 2t_1 t_2 \partial_{z_1} \partial_{z_2} F^E_\beta(z_1, z_2) + t_2 \partial_{z_2}^2 F^E_\beta(z_1, z_2) \right) + 
\]

\[
x'_1 \left( t_2 \partial_{z_2} F^E_\beta(z_1, z_2) + t_1 \partial_{z_1} \partial_{z_2} F^E_\beta(z_1, z_2) \right) + 
\]

\[
x'_2 \left( t_2 \partial_{z_1} F^E_\beta(z_1, z_2) + t_1 \partial_{z_1} \partial_{z_2} F^E_\beta(z_1, z_2) \right) + 
\]

where \( z_1 = x_1 - E_\beta t_1, z_2 = x_2 - E_\beta t_2 \). As a simple example, let

\[
F^E_\beta(z_1, z_2) = -\tanh(z_1 + z_2). 
\] (110)

Then eqs.(80a,102,108,109) yield:

\[
x_1 = -x_2, \quad t_2 = 1 - t_1, \quad E_\beta = 0, \quad (111) 
\]

\[
x'_1 = -x'_2, \quad t'_2 = -1, \quad E'_\beta = 0. \quad (112) 
\]

Equations (111) describe two-dimensional surface in 4-dimensional space; breaking starts at fixed value of \( E_\beta; E_\beta = 0 \). Similar analysis shows that \( E_\beta \) is break points manifold corresponding to the functions \( P_{\alpha\beta} \) (see eq.(80b)) as well.
3 New version of the dressing method

We have described some solutions manifolds to well known $S$-integrable models (which were called PDE1$(t_1, t_2; w^{(0)})$ in Sec.2. It is remarkable, that these solutions manifolds cover full solutions spaces to the associated families of the matrix first-order quasilinear PDEs integrable by the method of characteristics (which were called PDE0$(t_m; n_0, w)$, $m = 1, 2$).

The purpose of this section is the construction of such dressing algorithm which $(a)$ would reproduce the solutions manifold to PDE1s available by the classical dressing method and $(b)$ would describe the different solution manifold to PDE1s in spirit of the method of characteristics, i.e. we will derive non-differential equations implicitly describing solutions to PDE1$(t_1, t_2; w^{(0)})$ which are solutions to appropriate PDE0$(t_m; n_0, w)$, $m = 1, 2$, as well. Our consideration is based on the new version of the dressing method, which is subsequent development of the basic ideas of [29, 30, 56, 57, 58]. As a particular result, eqs.(41,61,64) will be obtained.

Let us describe the basic objects appearing in the dressing algorithm. In [30], eq.(3) with $\rho = 0$ has been derived as a simplest example of the implementation of the dressing method based on the following integral homogeneous equation

$$\int_D \Psi(\lambda, \nu; x) U(\nu; x) d\Omega(\nu) \equiv \Psi(\lambda, \nu; x) \ast U(\nu; x) = 0, \quad (113)$$

where $\Psi$ and $U$ are $Q \times Q$ matrix dressing and spectral functions respectively, $\lambda$ and $\nu$ are complex spectral parameters, $\Omega$ is some measure on the complex plane $\nu$, "$\ast$" means integration over $\nu$ and $x = (x_1, x_2, \ldots, t_1, t_2 \ldots)$ is a set of independent variables of nonlinear PDEs.

However, integral equation (113) is not appropriate for our purposes. In order to derive PDE1s together with associated PDE0s, we shall replace the equation (113) with the following integral homogeneous equation

$$\Psi(\lambda, \nu; x) \ast U(\nu, \mu; x) = 0, \quad (114)$$

where $(M + 1)Q \times Q$ matrix spectral function $U$ depends on two spectral parameters; $\Psi$ is $Q \times (M + 1)Q$ dressing function and kernel of the integral operator. Here integer parameter $M = \dim \ker \Psi$. Following the strategy of [30], we assume that the solution of (114) is not unique but may be represented in the form:

$$U(\lambda, \mu; x) = \sum_{j=1}^{M} U^{(h;j)}(\lambda, \nu; x) \ast f^{(j)}(\nu, \mu; x), \quad (115)$$

where$f^{(j)}(\nu, \mu; x)$, $j = 1, \ldots, M$, are arbitrary $Q \times Q$ matrix functions of arguments and $U^{(h;j)}$, $j = 1, \ldots, M$, are $M$ nontrivial linearly independent solutions of the homogeneous equation (114), i.e

$$\sum_{j=1}^{M} U^{(h;j)}(\lambda, \nu; x) \ast S^{(j)}(\nu, \mu; x) = 0 \Rightarrow S^{(j)}(\nu, \mu; x) \equiv 0, \quad j = 1, \ldots, M. \quad (116)$$

This assumption causes the single linear relation among any $M + 1$ independent solutions $U^{(j)}$, $j = 0, \ldots, M$, of eq.(114):

$$U^{(0)}(\lambda, \mu; x) = \sum_{j=1}^{M} U^{(j)}(\lambda, \nu; x) \ast F^{(j)}(\nu, \mu; x), \quad (117)$$
where $F^{(j)}$ are some $Q \times Q$ matrix functions. As we shall see, all $U^{(j)}$ are expressed in terms of the single solution $U$ through some linear operators $L^{(j)}$, either differential or non-differential:

$$U^{(j)}(\lambda, \mu; x) = L^{(j)}(\lambda, \nu) * U(\nu, \mu; x), \quad j = 0, \ldots, M. \tag{118}$$

Thus, eq.(117) represents a linear equation for the spectral function $U$. Besides, we will show that $F^{(j)}$ may be expressed in terms of $U$ using the external $Q \times (M + 1)Q$ dressing matrix function $G(\lambda, \mu; x)$, similar to [30].

To increase dimensionality of the dressing functions (i.e., the number of variables $x_j$ which may be introduced arbitrarily in the dressing functions) and, consequently, to increase the dimensionality of the associated nonlinear PDEs, we introduce the $Q \times Q$ matrix function $A(\lambda, \mu)$ and $(M + 1)Q \times (M + 1)Q$ matrix function $A(\lambda, \mu)$ by the following generalized commutation relation involving $\Psi$:

$$A(\lambda, \nu) * \Psi(\nu, \mu; x) = \Psi(\lambda, \nu; x) * A(\nu, \mu). \tag{119}$$

Similar to eq.(13) we define operators $A^i$ and $A^j$ as follows: $A^i = A \ast \cdots \ast A$, $A^j = A \ast \cdots \ast A$. Consequently, functions $A(\lambda, \mu)$ and $A(\lambda, \mu)$ generate the following set of functions

$$A^{(m)} = \rho^{(m)}(A), \quad A^{(m)} = \rho^{(m)}(A) : \quad \rho^{(m)}(A) * \Psi = \Psi * \rho^{(m)}(A), \tag{120}$$

where scalar functions $\rho^{(m)}(z)$ are representable by a positive power series of $z$, see eq.(9), so that $A^{(m)}$ and $A^{(m)}$ are well defined $Q \times Q$ and $(M + 1)Q \times (M + 1)Q$ matrix functions respectively.

As usual, $x$-dependence of the spectral function $U$ appears through the $x$-dependence of the dressing functions. Let $x$-dependence of $\Psi$ be given by the equation

$$\Psi_{tm}(\lambda, \mu; x) + \sum_{j=1}^{N} A^{(m)}(\lambda, \nu) * \Psi_{x}(\nu, \mu; x) + C^{(m)}(\lambda, \nu) * \Psi(\nu, \mu; x) = \Psi(\lambda, \nu; x) * C^{(m)}(\nu, \mu), \tag{121}$$

where $A^{(m)}$, $A^{(m)}$ do not depend on $x$; $C^{(m)}$ and $C^{(m)}$ are $Q \times Q$ and $(M + 1)Q \times (M + 1)Q$ matrix functions respectively, $A^{(m)} * C^{(m)} * A \neq 0$, $A * C - C * A \neq 0$. Equations (119) and (121) represent the overdetermined linear system for $\Psi$. Compatibility condition of (119) and (121) yields:

$$P^{(m)} * \Psi = \Psi * P^{(m)}, \quad P^{(m)} = A * C^{(m)} - C^{(m)} * A, \quad P^{(m)} = A * C^{(m)} - C^{(m)} * A. \tag{123}$$

In addition, equations (121) with different values of $m$ are compatible only if $P^{(m)} = 0$ and $P^{(m)} = 0$. In other words, if $P^{(m)} \neq 0$ and $P^{(m)} \neq 0$, then derived nonlinear PDEs will not possess commuting flows, at least among PDE0s.

Now we may obtain the set of different solutions to the homogeneous eq. (114) applying $A^{(m)} *$ and $(\partial_{tm} + \sum_{j=1}^{N} A^{(m)} * \partial_{x_j} + C^{(m)} *)$ to (114). One gets

$$\Psi(\lambda, \nu; x) * E^{(j;m)}(\nu, \mu; x) = 0, \quad j = 1, 2, \tag{124}$$
where
\[ E^{(1;m)}(\lambda, \mu; x) = A^m(\lambda, \nu) * U(\nu, \mu; x), \] (125)
\[ E^{(2;m)}(\lambda, \mu; x) = U_{t_m}(\lambda, \mu; x) + \sum_{j=1}^{N} A^{(mj)}(\lambda, \nu) * U_{x_j}(\nu, \mu; x) + C^{(m)}(\lambda, \nu) * U(\nu, \mu; x). \]

Thus, \( E^{(j;m)} \) are desirable solutions of the integral homogeneous equation. The derivation of nonlinear PDEs significantly depends on the value of the parameter \( M \). Hereafter we consider \( M = 1 \).

3.1 Simplest degeneration of the ker \( \Psi: M = 1 \)

In this case eq.(115) reads:
\[ U(\lambda, \mu; x) = U^{(h:1)}(\lambda, \nu; x) * f^{(1)}(\nu, \mu; x). \] (126)

It follows from the above discussion that we have to introduce so-called external dressing \( Q \times 2Q \) matrix function \( G(\lambda, \mu; x) \), whose prescription will be explored in Sec.3.1.1. Let \( G \) be defined by the next overdetermined system of linear equations:
\[ G(\lambda, \nu; x) * A(\nu, \mu) = \hat{A}(\lambda, \nu) * G(\nu, \mu; x) + H_1(\lambda; x)H_2(\mu; x), \] (127)
\[ G_{t_m}(\lambda, \mu; x) + \sum_{j=1}^{N} G_{x_j}(\lambda, \nu; x) * A^{(mj)}(\nu, \mu) - G(\lambda, \nu; x) * C^{(m)}(\nu, \mu) = \] (128)
\[ -\hat{C}^{(m)}(\lambda, \nu) * G(\nu, \mu; x) - \sum_{j=1}^{\hat{N}} T^{(j)}G(\lambda, \nu; x) * \hat{A}^{(mj)}(\nu, \mu), \]

where \( \hat{N} \) is some integer, \( \hat{A} \) and \( H_1 \) are \( \hat{Q} \times Q \), while \( H_2 \) is \( \hat{Q} \times 2Q \) matrix functions; functions \( \hat{A}^{(mj)} = \hat{\rho}^{(mj)}(A) \) are defined by the series (27). Require also
\[ T^{(m)}A = AT^{(m)}, \quad T^{(m)}H_1 = H_1T^{(m)}. \] (129)

Functions \( H_1(\lambda; x) \) and \( H_2(\mu; x) \) are called external dressing functions as well as function \( G(\lambda, \mu; x) \). The compatibility condition of eqs.(127) and (128) yields:
\[ G(\lambda, \nu; x) * P^{(m)}(\nu, \mu) - \hat{P}^{(m)}(\lambda, \nu) * G(\nu, \mu; x) = \] (130)
\[ -L_1^{(m)}H_1(\lambda; x)H_2(\mu; x) - \sum_{j=1}^{N} H_{1x_j}(\lambda; x)H_2(\nu; x) * A^{(mj)}(\nu, \mu) - H_1(\lambda; x)L_2^{(m)}H_2(\mu; x), \]

where
\[ \hat{P}^{(m)} = \hat{A} * \hat{C}^{(m)} - \hat{C}^{(m)} * \hat{A}, \]
\[ L_1^{(m)}H_1(\lambda; x) = H_{1t_m}(\lambda; x) + \hat{C}^{(m)}(\lambda, \nu) * H_1(\nu; x), \]
\[ L_2^{(m)}H_2(\mu; x) = H_{2t_m}(\mu; x) + \sum_{j=1}^{N} H_{2x_j}(\nu; x) * A^{(mj)}(\nu, \mu) + \]
\[ \sum_{j=1}^{\hat{N}} T^{(j)}H_2(\nu; x) * \hat{A}^{(mj)}(\nu, \mu) - H_2(\nu; x) * C^{(m)}(\nu, \mu). \]
In order to derive the eq.(130), we differentiate the eq.(127) with respect to $t_m$ and use eq. (128) to eliminate $G_{t_m}$ and eq.(127) to simplify the result. After all transformations we end up with eq.(130).

The compatibility condition (130) produces equations defining $H_i, i = 1, 2$, in the following way. Eq.(130) must be identical in $\lambda$ and $\mu$. Solving it, we must take into account that two spectral parameters are not separated in the LHS of the eq.(130), while each term in the RHS of this equation separates two spectral parameters. We suggest two different solution of the eq. (130):

1. Let

$$\hat{A} * H_1 = 0,$$

$$P^{(m)} = \rho^{(m_0)}(A), \quad \mathcal{P}^{(m)} = \rho^{(m_0)}(A), \quad \hat{P}^{(m)} = \rho^{(m_0)}(A),$$

$$H_{1t_m} + \tilde{C}^{(m)} * H_1 = 0, \quad H_{1x_j} = 0,$$

$$H_{2t_m} + \sum_{j=1}^{N} H_{2x_j} * A^{(mj)} + \sum_{j=1}^{N} T^{(j)} H_2 * \tilde{A}^{(mj)} - H_2 * C^{(m)} = 0,$$

$$-H_2 * \rho^{(m_0)}(A) * A^{-1},$$

where functions $\rho^{(m_0)}$ are defined by the eq.(9). Then eq.(130) becomes:

$$G * \rho^{(m_0)}(A) - \rho^{(m_0)}(A) * G = H_1 H_2 \rho^{(m_0)}(A) * A^{-1},$$

which is equivalent to the eq.(127) in virtue of the eq.(132a). Compatibility of eqs.(128) with different $m$ requires $P^{(m)} = 0$ and $\hat{P}^{(m)} = 0$. These two conditions provide compatibility of eqs.(132) with different $m$. Otherwise $m$ takes a single value (say, $m = 1$) and assotiated nonlinear PDEs have no commuting flows, at least among PDE0s.

Remark. Eq.(133) requires that positive series for $\rho^{(m_0)}(z)$ start with $z^1$, which corresponds to $\alpha^{(m_0;0)} = 0$ in eq.(9). However, it will be shown in the Sec.(3.1.2) (see a paragraph after the eq.(147)), that this is not a strong restriction.

2. Let

$$\hat{A} * H_1 \neq 0,$$

$$C^{(m)} = \tilde{C}^{(m)} = 0, \quad C^{(m)} = 0,$$

$$H_{1t_m} = 0, \quad H_{1x_j} = 0, \quad j = 1, \ldots, N,$$

$$H_{2t_m} + \sum_{j=1}^{N} H_{2x_j} * A^{(mj)} + \sum_{j=1}^{N} T^{(j)} H_2 * \tilde{A}^{(mj)} = 0.$$

Appropriate nonlinear PDEs possess commuting flows since the eqs. (128) with different $m$ are compatible, as well as eqs.(134) with different $m$. It is evident that the eq.(122) is satisfied for both cases (132) and (134).

3.1.1 Spectral system for $U(\lambda, \mu; x)$

Now we are ready to derive the linear spectral system. Eq. (126) means that any two solutions of the homogeneous equation (114) are linearly dependent. In particular, any solution $E^{(j;m)}$
Thus, eqs.(135,136) become the overdetermined spectral system for the spectral function $U(\lambda, \mu; x)$. Remember that solution of eq.(114) is not unique. To obtain uniqueness we introduce one more equation for the spectral function $U$. This equation can be largely arbitrary, but, in order to derive the simplest nonlinear PDEs, we select the following equation:

$$G(\lambda, \nu; x) * U(\nu, \mu; x) = I\delta(\lambda - \mu),$$  

(137)

so that $U$ is the unique solution of the system (114,137). In other words, the equation (137) fixes the function $f^{(1)}(\lambda, \mu; x)$ in the eq.(126). Now, applying $G*$ to the eqs.(135,136), one gets the following expressions for $\hat{F}$ and $F^{(m)}$:

$$\hat{F}(\nu, \mu; x) = G(\lambda, \nu; x) * E^{(1;1)}(\nu, \mu; x) = \hat{A}(\lambda, \mu) + H_1(\lambda; x)H_2(\nu; x) * U(\nu, \mu; x),$$

(138)

$$F^{(m)}(\nu, \mu; x) = G(\lambda, \nu; x) * E^{(2;m)}(\nu, \mu; x) = \hat{C}^{(m)}(\lambda, \mu) + \sum_{j=1}^N T^{(j)}(\lambda; x) * \hat{A}^{(mj)}(\nu, \nu) * U(\nu, \mu; x) + \sum_{j=1}^N \left( G(\lambda, \nu; x) * A^{(mj)}(\nu, \nu) * U(\nu, \mu; x) \right)_{x_j}, m = 1, 2, \ldots$$

Thus, eqs.(135,136) become the overdetermined spectral system for the spectral function $U(\lambda, \mu; x)$. We consider the particular examples of the spectral systems and of the appropriate nonlinear PDEs in Secs.3.1.2 and 3.1.3.

**Spectral functions depending on single spectral parameter; fields and remarkable reductions.** Eqs.(135,136) depend on two spectral parameters due to the spectral function $U(\lambda, \mu; x)$. However, functions of single spectral parameter appear in the algorithm naturally. These functions are following:

$$V^{(j)}(\lambda; x) = U(\lambda, \mu; x) * \hat{A}^j(\mu, \nu) * H_1(\nu; x),$$

$$W^{(j)}(\mu; x) = H_2(\lambda; x) * \hat{A}^j(\lambda, \nu) * U(\nu, \mu; x).$$

(139)

They satisfy the spectral equations with single spectral parameter which appear after applying $\hat{A}^j * H_1$ and $H_2 * \hat{A}^j *$ to the eqs.(135,136), see Secs.3.1.2,3.1.3.

The dependent variables (or fields) of the nonlinear PDEs are expressed in terms of the spectral and dressing functions by the next formulae:

$$w^{(kn)}(x) = H_2(\lambda; x) * A^k(\lambda, \mu) * V^{(n)}(\mu; x) = W^{(k)}(\mu; x) * \hat{A}^n(\mu, \nu) * H_1(\nu; x).$$

(140)
This definition of the fields suggests the following two types of reductions:

1. \[ \hat{A}^{n_0} \star H_1 = \sum_{j=1}^{n_0-1} \hat{A}^j \star H_1 r^{(j)} \Rightarrow \quad (141) \]

   \[ w^{(kn_0)} = \sum_{j=1}^{n_0-1} w^{(kj)} r^{(j)}, \quad \forall k, \quad k=0, (32) \Rightarrow (35), \]

2. \[ H_2 \star A^{k_0} = \sum_{j=1}^{k_0-1} r^{(j)} H_2 \star A^j \Rightarrow \quad (142) \]

   \[ w^{(kn)} = \sum_{j=1}^{k_0-1} r^{(j)} w^{(jn)}, \quad \forall n = 0, (34) \Rightarrow (36). \]

It will be shown in Sec.(3.2.2) (see text after the eq.(205)), that \( r^{(i)} \) must be scalar constants (remember that these parameters are matrices in the algebraic approach, see the paragraph after the eqs.(21)). We consider only the reduction (141).

In accordance with Introduction, we refer to the nonlinear PDEs corresponding to the reduction (141) as PDE0s, while the nonlinear PDEs without any reduction will be referred to as PDE1s. Although the nonlinear PDEs may be derived for any given functions \( \hat{\rho}^{(mj)} \) and \( \hat{\tilde{\rho}}^{(mj)} \), it is difficult to write the nonlinear PDEs keeping these functions unfixed, in general. The example considered in Sec.3.1.2 is an exception.

### 3.1.2 Case (132). Quasilinear first order matrix equations solvable by the method of characteristics.

In this section, we briefly reproduce results obtained in [29, 30] by a different method. First of all, remark that the eq.(132a) is nothing but the reduction (141) with \( n_0 = 1 \). Thus, the derived equations will be PDE0s. In this case eqs.(135,136) yield:

\[ A \star U = U \star \hat{A} + U \star H_1 H_2 \star U, \quad (143) \]

\[ U_{t_m} + \sum_{j=1}^{N} A^{(mj)} \star U_{x_j} + C^{(m)} \star U = U \star \left[ \hat{C}^{(m)} + \right. \]

\[ \left. \sum_{j=1}^{\tilde{N}} T^{(j)} (\tilde{A}^{(mj)} + H_1 H_2 \star \tilde{A}^{(mj)} \star A^{-1} \star U) + \sum_{j=1}^{N} (H_1 H_2 \star A^{(mj)} \star A^{-1} \star U)_{x_j} \right]. \]

Applying \( *H_1 \) to the eqs.(143) one gets \((V = V^{(0)}, w = w^{(00)})\)

\[ A \star V = V w \quad \Rightarrow \quad A^{(mj)} \star V = V \hat{\rho}^{(mj)}(w), \quad (144) \]

\[ V_{t_m}(\lambda;x) + \sum_{j=1}^{N} A^{(mj)}(\lambda,\nu) \star V_{x_j}(\nu;x) + C^{(m)}(\lambda,\nu) \star V(\nu;x) = \]

\[ V(\lambda;x) \sum_{j=1}^{\tilde{N}} T^{(j)} \hat{\tilde{\rho}}^{(mj)}(w) + V(\lambda;x) \sum_{j=1}^{N} \rho^{(mj)}_{x_j}(w). \]

26
or, substituting the eq. (144) into the eq.(145), one has:

\[ V_{l_m}(\lambda; x) + \sum_{j=1}^{N} V_{xj}(\lambda; x)\rho^{(m_j)}(w) + C^{(m)}(\lambda, \nu) * V(\nu; x) = V(\lambda; x) \sum_{j=1}^{\tilde{N}} T^{(j)}\tilde{\rho}^{(m_j)}(w). \] (146)

After applying \( H_2 \) to this equation one ends up with PDE0:

\[ w_{l_m} + \sum_{j=1}^{N} w_{xj}\rho^{(m_j)}(w) + \rho^{(m_0)}(w) = \left[ w, \sum_{j=1}^{\tilde{N}} T^{(j)}\tilde{\rho}^{(m_j)}(w) \right]. \] (147)

It was mentioned above, that this equation has commuting flows only if \( \rho^{(m_0)} = 0 \). Then \( m = 1, 2, \ldots \) in this equation. Otherwise, \( m \) must be fixed, say, \( m = 1 \).

In accordance with the Remark given after the eq.(133), power series for \( \rho^{(10)}(z) \) must start with \( z^1 \). Otherwise, multiplying eq.(147) by \( w \) from the right and replacing \( t_1 \rightarrow x_{N+1} \) we result in the PDE0 having the form of "stationary" eq.(147) satisfying the condition of the above remark.

Eqs. (144) and (145) has to be taken as the spectral system with the spectral function \( V(\lambda; x) \) corresponding to the nonlinear PDE (147). In [29] this equation has been derived with fixed \( m = 1, \rho^{(l_j)} \equiv \rho^{(j)}, j > 0, \rho^{(10)}(z) \equiv \rho, \tilde{\rho}^{(l_j)} = \tilde{\rho}\delta_{l_1j} \). In addition, \( \rho = 0 \) in [30], where another variant of the dressing method has been introduced. However, both techniques developed in [29] and [30] may be applied to the eq.(147) as well.

### 3.1.3 Case (134). PDE1s and associated PDE0s: N-wave and Pohlmeyer equations

The main feature of the eqs. (134) is that they set \( C^{(m)} = 0 \) and do not force the reduction (141). Nonlinear PDEs corresponding to this case give rise to PDE1s. Then, been imposed, reduction (141) reduces PDE1s to PDE0s.

We study nonlinear PDEs corresponding to the particular choice of the functions \( \rho^{(m)} \) and \( \tilde{\rho}^{(m)} \). For instance, let \( A^{(m_j)} = s^{(m)}A\delta_{mj}, \tilde{A}^{(m_j)} = A\delta_{mj}, s^{(m)} \) are scalar constants. Then the spectral system with two spectral parameters, eqs.(135,136), reads:

\[ A * U = U * \hat{A} + V^{(0)}W^{(0)}, \] \[ U_{t_m} + s^{(m)}A * U_{x_m} = U * \hat{A}T^{(m)} + V^{(0)}T^{(m)}W^{(0)} + s^{(m)}V^{(0)}W_x^{(0)}, \] \[ m = 1, 2. \] (148)

The appropriate spectral system for the spectral functions \( V^{(n)}(\lambda; x) \) with single spectral parameter appears after applying \( \hat{A}^n * H_1, n = 0, 1, \ldots, \) to the eqs.(148):

\[ A(\lambda, \nu; x) * V^{(n)}(\nu; x) = V^{(n+1)}(\lambda; x) + V^{(0)}(\lambda; x)w^{(0n)}(x), \] \[ V_{t_m}^{(n)}(\lambda; x) + s^{(m)}(A(\lambda, \nu) * V_{x_m}^{(m)}(\nu; x) - V^{(0)}(\lambda; x)w^{(0n)}_{x_m}(x)) = V^{(n+1)}(\lambda; x)T^{(m)} + V^{(0)}(\lambda; x)T^{(m)}w^{(0n)}(x), \] \[ m = 1, 2. \] (149)

In principle, applying \( H_2 * \hat{A}^k * \) to the eq.(148), one could get the linear equations for another set of functions of single spectral parameter \( W^{(k)}(\mu; x) \). However, these equations will not be used, so that we do not represent them here.
The system of nonlinear PDEs may be derived applying $H_2 \ast A_k^\ast$ to the eq.(149).

\begin{align}
    w^{(k+1)n} &= w^{(k(n+1))} + w^{(k0)}w^{(0n)} , \\
    w^{(k)\mu} + s^{(m)}(w^{(k+1)\mu} - w^{(k0)}w^{(0\mu)}) = & \\
    w^{(k(n+1))}T^{(m)} - T^{(m)}w^{((k+1)n)} + w^{(k0)}T^{(m)}w^{(0n)} , & m = 1, 2.
\end{align}

Using eq.(150) to eliminate $w^{(k+1)n}$ from the eq.(151) and putting $k = 0$ in the result we get (see eq.(32) for definition of $w^{(n)}$):

\begin{align}
    w^{(n)\mu}_m + s^{(m)}(w^{(n+1)\mu}_m + w^{(0)\mu}_m) - [w^{(n+1)}_m, T^{(m)}] + [T^{(m)}, w^{(0)}_m]w^{(0)} = 0 , \\
    n = 0, 1, \ldots , m = 1, 2.
\end{align}

Fixing $n = 0$ and eliminating $w^{(1)}$ from this system of two PDEs, we get PDE1($t_1, t_2; w^{(0)}$). Reduction (141) yields PDE0($t_m; n_0, w$), $m = 1, 2$:

\begin{align}
    w^{(0)}_m + s^{(m)}w^{(n)}_m w + [T^{(n)}_{m_0}, w]w = 0 , & m = 1, 2.
\end{align}

From another point of view, eliminating $w^{(k(n+1))}$ from the system (151) and putting $n = 0$ one gets:

\begin{align}
    w^{(k)\mu}_m + s^{(m)}(w^{(k+1)\mu}_m - w^{(k0)}w^{(0\mu)}) + [T^{(m)}, w^{(k+1)}_m] + [w^{(k0)}_m, T^{(m)}] = 0 , \\
    k = 0, 1, \ldots , m = 1, 2,
\end{align}

where fields $\tilde{w}^{(n)}$ are defined by the eq.(34). Fixing $k = 0$ and eliminating $\tilde{w}^{(1)}$ from this system of two PDEs, we get the same PDE1($t_1, t_2; w^{(0)}$). Reduction (142) yields PDE0($t_m; k_0, \tilde{w}$), $m = 1, 2$:

\begin{align}
    \tilde{w}^{(0)}_m + s^{(m)}\tilde{w}\tilde{w}^{(n)}_m + [\tilde{w}^{(m)}, T^{(k)}_{k_0}] = 0 , & m = 1, 2.
\end{align}

N-wave and Pohlmeyer equations with appropriate PDE0s and spectral problems follow from eqs.(149-155) with $s^{(m)} = 0$ (see Sec.2.1.1) and $T^{(m)} = 0$, $s^{(m)} = 1$ (see Sec.2.1.2) respectively.

Eq.(154) can be formally obtained by the transposition of the eq.(152) with replacements $w^T \rightarrow -\tilde{w}, T^{(m)} \rightarrow -T^{(m)}$, similarly to the relations between eqs.(33) and (31) and between eqs.(58) and (57). Because of this similarity, we will study only eqs.(152,153).

### 3.2 Solutions space described by the dressing method

The dressing algorithm describing solutions space to PDE1s consists of two basic parts:

1. **The non-evolutionary part**, where we do not need the particular dependence of the dressing functions on $x_i$ and $t_i$. As a result, we derive the system of (non-differential) equations which is valid for any possible $x$-dependence of the dressing functions.

   (a) Solve the eq.(119) as an equation defining $\Psi$.

   (b) Solve the spectral equation (148a) as an equation establishing the restrictions for the spectral function $U$.

   (c) Solve the eq.(127) as an equation for $G$.

   (d) Solve the eqs.(114) and (137) as equations for $U$ taking into account result of n.1b.
2. The evolutionary part, where we presume the special $x$-dependence of the dressing functions.

(a) Solve eq. (121) as an equation defining $x$-dependence of the internal dressing function $\Psi(\lambda, \mu; x)$.

(b) Solve one of the systems (132) or (134) defining $H_i$, $i = 1, 2$. This step defines $x$-dependence of the external dressing function $G(\lambda, \mu; x)$ owing to the eq.(127). Eq. (128) will be automatically satisfied.

Details of the dressing algorithm producing the classical solutions manifold to PDE1s are given in Sec.3.2.1. In order to describe solutions manifold to PDE1s associated with reduction (141) one has to evaluate the same algorithm with eqs.(114, 137,148a) modified by the reduction (141), see Sec.3.2.2 for details.

Before describing all steps of the algorithm in details, we give a description of the particular representation of the dressing and spectral functions which is necessary hereafter. First of all, in order to fit the requirement $M = 1$, we introduce the next block-matrix representation of the functions:

$$
\Psi(\lambda, \mu; x) = \begin{bmatrix}
\psi_0(\lambda, \mu; x) & \psi_1(\lambda, \mu; x)
\end{bmatrix},
U(\lambda, \mu; x) = \begin{bmatrix}
u_0(\lambda, \mu; x) \\
u_1(\lambda, \mu; x)
\end{bmatrix},
$$

$$
V^{(j)}(\lambda; x) = \begin{bmatrix}
v_0^{(j)}(\lambda; x) \\
v_1^{(j)}(\lambda; x)
\end{bmatrix},
W^{(j)}(\mu; x) = \begin{bmatrix}
w_0^{(j)}(\mu; x)
\end{bmatrix},
G(\lambda, \mu; x) = \begin{bmatrix}
g_0(\lambda, \mu; x) & g_1(\lambda, \mu; x)
\end{bmatrix},
H_2(\lambda; x) = \begin{bmatrix}
h_20(\lambda; x) & h_21(\lambda; x)
\end{bmatrix},
H_1(\lambda; x) = h_1(\lambda; x).
$$

Any function in the RHS of formulae (156) is $Q \times Q$ matrix function, so that $\Psi$, $G$ and $H_2$ are $Q \times 2Q$ matrix functions; $U$ and $V^{(j)}$ are $2Q \times Q$ matrix functions; $W^{(j)}$ and $H_1$ are $Q \times Q$ matrix functions. We also have to fix the functions $A(\lambda, \mu)$, $\hat{A}(\lambda, \mu)$ and $\check{A}(\lambda, \mu)$ (compare with eq.(52)):

$$
A(\lambda, \mu) = \hat{A}(\lambda, \mu) = \lambda \delta(\lambda - \mu) I,
A(\lambda, \mu) = \lambda \delta(\lambda - \mu) I_2.
$$

3.2.1 Dressing algorithm: classical solutions manifolds to PDE1s

Non-evolutionary part of the dressing algorithm. We describe all steps of the non-evolutionary part of the algorithm given in the beginning of the section (3.2) for the case (134) without reduction (141).

(1a) Function $\Psi$ is defined by the eq.(119), which reads

$$(\lambda - \mu) \Psi(\lambda, \mu; x) = 0.
$$

Its solution is following:

$$
\Psi(\lambda, \mu; x) = \hat{\Psi}(\lambda; x) \delta(\lambda - \mu),
\hat{\Psi}(\lambda; x) = [\hat{\psi}_0(\lambda; x) \quad \hat{\psi}_1(\lambda; x)],
$$

where $\hat{\psi}_i$, $i = 0, 1$, are $Q \times Q$ matrix functions.

(1b) As it was mentioned above, the main feature of the spectral system described in this paper is the presence of the spectral equation which has no derivatives with respect to $x$, see eq.(148a):

$$
U(\lambda, \mu; x)(\lambda - \mu) = V^{(0)}(\lambda; x)W^{(0)}(\mu; x).
$$

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This suggests us the next representation for $U$:

$$U(\lambda, \mu; x) = \frac{V^{(0)}(\lambda; x)W^{(0)}(\mu; x)}{\lambda - \mu} + U_0(\lambda; x)\delta(\lambda - \mu), \quad U_0(\lambda; x) = \begin{bmatrix} u_{00}(\lambda; x) \\ u_{01}(\lambda; x) \end{bmatrix},$$  \hspace{1cm} (161)

where $u_{0i}$, $i = 0, 1$, are $Q \times Q$ matrix functions.

(1c) Similarly, eq.(127) reads

$$G(\lambda, \mu; x)(\lambda - \mu) = -H_1(\lambda; x)H_2(\mu; x)$$  \hspace{1cm} (162)

which suggests us the next formula for $G$:

$$G(\lambda, \mu; x) = -\frac{H_1(\lambda; x)H_2(\mu; x)}{\lambda - \mu} + G_0(\lambda; x)\delta(\lambda - \mu), \quad G_0(\lambda; x) = [g_{00}(\lambda; x) \ g_{01}(\lambda; x)],$$  \hspace{1cm} (163)

where $g_{0i}$, $i = 0, 1$, are $Q \times Q$ matrix functions and $G_0(\lambda; x)\delta(\lambda - \mu)$ is a solution of the eq.(128).

(1d) Next, substituting the eqs.(161) and (163) into the eq.(137) one gets

\begin{align*}
-H_1(\lambda; x) & \int_D d\nu H_2(\nu; x)V^{(0)}(\nu; x)W^{(0)}(\mu; x) \frac{1}{(\lambda - \nu)(\nu - \mu)} - H_1(\lambda; x) \frac{H_2(\mu; x)U_0(\mu; x)}{\lambda - \mu} + \\
\frac{G_0(\lambda; x)V^{(0)}(\lambda; x)W^{(0)}(\mu; x)}{\lambda - \mu} + G_0(\lambda; x)U_0(\lambda; x)\delta(\lambda - \mu) &= I\delta(\lambda - \mu).
\end{align*}

(164)

This equation must be identity for any $\lambda$ and $\mu$, which suggests us to split the eq.(164) into two matrix equations. The first equation reads:

$$G_0(\lambda; x)U_0(\lambda; x) = I,$$  \hspace{1cm} (165)

which is the system of $Q^2$ scalar equations for $2Q^2$ elements of $U_0$, i.e. underdetermined system of scalar equations. The second equation reads:

$$\frac{1}{\lambda - \mu} [E_1(\lambda; x)W(\mu; x) + H_1(\lambda; x)E_2(\mu; x)] = 0,$$  \hspace{1cm} (166)

$$\begin{align*}
E_1(\lambda; x) &= G_0(\lambda; x)V^{(0)}(\lambda; x) - H_1(\lambda; x) \int_D d\nu \frac{H_2(\nu; x)V^{(0)}(\nu; x)}{\lambda - \nu} - H_1(\lambda; x) \\
E_2(\mu; x) &= - \int_D d\nu \frac{H_2(\nu; x)V^{(0)}(\nu; x)W^{(0)}(\mu; x)}{\nu - \mu} - H_2(\mu; x)U_0(\mu; x) + W^{(0)}(\mu; x)
\end{align*}

In turn, the eq.(166) must be split into two matrix equations:

$$E_1(\lambda; x) = 0,$$  \hspace{1cm} (167)

$$E_2(\mu; x) = 0.$$  \hspace{1cm} (168)

Due to our choice of the last term in the expression for $E_2$, eq.(168) coincides with eq.(161) after application $H_2$ to it, which is necessary condition. The last term in expression for $E_1$ compensates the last term of $E_2$ after substitution in eq.(166).

The matrix equation (167) represents $Q^2$ scalar equations for $2Q^2$ elements of the matrix function $V^{(0)}$, while the matrix equation (168) represents $Q^2$ scalar equations for $Q^2$ elements.
of the matrix function $W^{(0)}$. Thus, matrix equation (167) is underdetermined systems of scalar equations.

The rest of equations for elements of $V^{(0)}$ and $U_0$ follows from the eq.(114) after substitution the eq.(159) for $\Psi$ and the eq.(161) for $U$:

$$\hat{\Psi}(\lambda; x) \frac{V^{(0)}(\lambda; x)W^{(0)}(\mu; x)}{\lambda - \mu} + \hat{\Psi}(\lambda; x)U_0(\lambda; x)\delta(\lambda - \mu) = 0.$$  \hspace{1cm} (169)

Since eq.(169) must be identity for any $\lambda$ and $\mu$, it must be splitted into the following equations for $V^{(0)}$ and $U_0$:

$$\hat{\Psi}(\lambda; x)V^{(0)}(\lambda; x) = 0,$$

$$\hat{\Psi}(\lambda; x)U_0(\lambda; x) = 0.$$  \hspace{1cm} (170) \hspace{1cm} (171)

Each of these matrix equations represents $Q^2$ scalar equations for $2Q^2$ elements of the matrix functions $V^{(0)}$ and $U_0$ respectively. Thus, the system (165,167,168,170,171) is the complete system for elements of $V^{(0)}(\lambda; x)$, $W^{(0)}(\mu; x)$ and $U_0(\lambda; x)$. Having these functions, the function $U(\lambda, \mu; x)$ may be constructed using the formula (161). This ends the non-evolutionary part of the dressing algorithm.

All in all, in order to construct the spectral function $U$, one has to solve

1. the eqs. (165) and (171) for $U_0$,  \hspace{1cm} (172)
2. the eqs. (167) and (170) for $V^{(0)}$,  
3. the eq.(168) for $W^{(0)}$.

After the function $U$ has been constructed, one must use the definition (140) in order to construct the solution to $PDE1(t_1, t_2; w^{(0)})$, i.e. the function $w^{(00)} \equiv w^{(0)}$. However, it is simple to observe, that essentially important for construction of $w^{(0)} = H_2 * V^{(0)}$ are eqs.(167) and (170) defining $V^{(0)}$. Moreover, eq.(167) is equivalent to the classical $\bar{\partial}$-problem for Self-dual type $S$-integrable equations. Let us write this equation in standard form. To simplify derivation, we take

$$H_1(\lambda; x) = I,$$  \hspace{1cm} (173)

which is consistent with eq.(134b). Eq.(170) yields

$$v_1^{(0)}(\lambda; x) = -\hat{\psi}^{-1}_1(\lambda; x)\hat{\psi}_0(\lambda; x)v_0^{(0)}(\lambda; x).$$  \hspace{1cm} (174)

Then eq.(167) gets the next form:

$$\phi(\lambda; x)v_0^{(0)}(\lambda; x) + \int_D d\nu \frac{\chi(\nu; x)v_0^{(0)}(\nu; x)}{\lambda - \nu} = I,$$  \hspace{1cm} (175)

where

$$\phi(\lambda; x) = g_{00}(\lambda; x) - g_{01}(\lambda; x)\hat{\psi}^{-1}_1(\lambda; x)\hat{\psi}_0(\lambda; x),$$

$$\chi(\lambda; x) = -h_{20}(\lambda; x) + h_{21}(\lambda; x)\hat{\psi}^{-1}_1(\lambda; x)\hat{\psi}_0(\lambda; x).$$  \hspace{1cm} (176)
Introduce the new spectral function

\[ v(\lambda; x) = \phi(\lambda; x)v_0^{(0)}(\lambda; x). \]  

Then eq.(175) yields

\[ \int_{\mathcal{D}} d\nu \hat{R}(\lambda, \nu; x)v(\nu; x) = I, \]  

where

\[ \hat{R}(\lambda, \nu; x) = \frac{R(\nu; x)}{\lambda - \nu} + I\delta(\lambda - \nu), \quad R(\nu; x) = \chi(\nu; x)\phi^{-1}(\nu; x). \]  

Here \( R(\nu; x) \) is a new dressing function and \( \hat{R} \) is a new kernel of the integral operator. By construction, dim ker \( \hat{R} = 0 \), so that eq.(178) is uniquely solvable for \( v \). Linear PDE for \( R \) follows from the linear PDEs for the functions \( \hat{\psi}_i, g_0i \) and \( h_{2i}, i = 0,1 \). These PDEs are eqs.(121), (128) and (134c), which read (taking into account eq.(134a)):

\[
\begin{align*}
\left( \hat{\psi}_i(\lambda; x) \right)_{t_m} + \sum_{j=1}^{N} \rho^{(mj)}(\lambda) \left( \hat{\psi}_i(\lambda; x) \right)_{x_j} &= 0, \\
\left( g_{0i}(\lambda; x) \right)_{t_m} + \sum_{j=1}^{N} \rho^{(mj)}(\lambda) \left( g_{0i}(\lambda; x) \right)_{x_j} + \sum_{j=1}^{N} T^{(j)} \bar{\rho}^{(mj)}(\lambda) g_{0i}(\lambda; x) &= 0, \\
\left( h_{2i}(\lambda; x) \right)_{t_m} + \sum_{j=1}^{N} \rho^{(mj)}(\lambda) \left( h_{2i}(\lambda; x) \right)_{x_j} + \sum_{j=1}^{N} T^{(j)} \bar{\rho}^{(mj)}(\lambda) h_{2i}(\lambda; x) &= 0,
\end{align*}
\]

where \( i = 0,1 \). From this system, we obtain the linear PDE for \( R \):

\[ R_{t_m}(\lambda; x) + \sum_{j=1}^{N} \rho^{(mj)}(\lambda) R_{x_j}(\lambda; x) + \sum_{j=1}^{N} \bar{\rho}^{(mj)}(\lambda) [T^{(j)}, R(\lambda; x)] = 0, \]

Eqs.(178,179,182) represent the classical \( \bar{\partial} \)-problem for PDE1s, [52, 53]. Eq.(140) for field \( w^{(0)} \) reduces to the next one:

\[ w^{(0)}(x) = \int_{\mathcal{D}} d\lambda R(\lambda; x)v(\lambda; x). \]

**Evolutionary part of the dressing algorithm.** In this case, nn.2a,2b of the dressing algorithm reduce to the solution of the single eq.(182):

\[ R(\lambda; x) = \int_{\mathcal{D}} dq e^{\sum_{j=1}^{I} \sum_{m=1}^{N} \rho^{(mj)}(\lambda) q_j t_m} - \sum_{j=1}^{I} \sum_{m=1}^{N} T^{(j)} \bar{\rho}^{(mj)}(\lambda) t_m R_0(\lambda, q) e^{\sum_{j=1}^{I} \sum_{m=1}^{N} T^{(j)} \bar{\rho}^{(mj)}(\lambda) t_m}, \]

where \( R_0 \) is arbitrary \( Q \times Q \) matrix function, \( q = (q_1, \ldots, q_N) \), parameters \( q_i \) are complex in general and \( \mathcal{D} \) is some integration region in space of vector parameter \( q \).
N-wave and Pohlmeyer equations correspond to $\rho^{(mj)} = 0$, $\tilde{\rho}^{(mj)}(\lambda; x) = \lambda \delta_{mj}$ and $\rho^{(mj)}(\lambda; x) = \lambda \delta_{mj}$ respectively.

The evident manifold of explicit particular solutions corresponds to the following choice of $R_0$:

$$(R_0(\lambda; q))_{\alpha\beta} = \begin{cases} 
\sum_j c^{(j)}_{a\beta}(q) \delta(\lambda - a_{a\beta}^{(j)}), & \alpha \neq \beta \\
0, & \alpha = \beta 
\end{cases} \quad , \quad \alpha, \beta = 1, \ldots, Q$$

with scalar parameters $a_{a_1a_2}^{(i)} \neq a_{a_3a_4}^{(j)}$ $\forall$ $i, j$ and $(\alpha_1, \alpha_2) \neq (\alpha_3, \alpha_4)$ and arbitrary functions $c^{(j)}_{a\beta}(q)$. Such solutions have the form of fractional rational function of the exponential functions with linear in $x$ arguments. We do not consider these solutions in details. Instead, we concentrate on another manifold of particular solutions in the next subsection.

### 3.2.2 Dressing algorithm: reduction (141) and appropriate solutions to PDE0($t_m; n_0, w$), $m = 1, 2$, and PDE1($t_1, t_2; w^{(0)}$)

**Non-evolutionary part of the dressing algorithm.** We represent the non-evolutionary part of the dressing algorithm describing the solutions manifold to both PDE0($t_m; n_0, w$), $m = 1, 2$, and PDE1($t_1, t_2; w^{(0)}$) associated with reduction (141). Solution of (141a) should be taken in the next form:

$$H_1(\mu; x) = \sum_{j=1}^{n_0} \hat{h}^{(j)}(x) \delta(\mu - b_j), \quad (185)$$

where $\hat{h}^{(i)}(x)$ are diagonal matrix functions and $b_j, j = 1, \ldots, n_0$ are scalar constants solving the next linear algebraic system:

$$b_j^{n_0} = \sum_{i=1}^{n_0-1} b_j^i r^{(i)}, \quad (186)$$

where the parameters $r^{(i)}$ are scalars. Requirement to have scalar $b_i$ and, as a consequence, scalar $r^{(i)}$ is related with transformation from the eq.(204) to the eq.(207), see text after the eq.(205). Eq.(186) has $n_0$ different roots $b_j, j = 1, \ldots, n_0$, which agrees with the summation limits in eq.(185).

We describe the manifold of the particular solutions to PDE1($t_1, t_2; w^{(0)}$) corresponding to the reduction (141) starting with the original system (114,137,148a) modified by the reduction (141) as follows.

Applying $* \hat{A}^j * H_1$ to (114), $j = 0, \ldots, n_0 - 1$, one gets:

$$\Psi * V = 0, \quad (187)$$

where $V$ is $2Q \times n_0 Q$ matrix function (47). Similarly, applying $* \hat{A}^j * H_1$ to the eq.(137) one gets

$$G(\lambda, \nu; x) * V(\nu; x) = \tilde{P}(\lambda; x), \quad (188)$$

where $\tilde{P}$ is a row of $n_0$ blocks

$$\tilde{P} = [H_1 \quad \hat{A} * H_1 \quad \ldots \quad \hat{A}^{n_0-1} * H_1]. \quad (189)$$
Finally, the spectral equation (148a) can be written in the next form after applying \( \star \hat{A}^j \star H_1 \):

\[
A(\lambda, \nu) \star V^{(j)}(\nu; x) = V^{(j+1)}(\lambda; x) + V^{(0)}(\lambda; x)w^{(0j)},
\]

or

\[
A(\lambda, \mu) \star V(\mu; x) = V(\lambda; x)w(x),
\]

where \( w \) is \( n_0Q \times n_0Q \) block matrix (21a).

In order to solve the system (187,188,191) we assume diagonalizability of \( w \), i.e. \( w \) is representable in the form (50). The case when \( w \) is not diagonalizable but reducible to Jordan form is more complicated and will be discussed in different paper. Now we describe all steps of the non-evolutionary part of the dressing algorithm using \( A, \hat{A} \) and \( A \) given by the eqs.(157).

(1a) coincides with the previous case, see eqs.(158,159).

(1b) Eq. (191) after multiplication by \( P \) from the right yelds the eq.(53), which we write in the next form:

\[
\sum_{\gamma=1}^{n_0Q} V_{\alpha\gamma}(\mu; x)P_{\gamma\beta}(x) = \hat{V}_{\alpha\beta}(x)\delta(\mu - E_\beta),
\]

\( \alpha = 1, \ldots, 2Q, \beta = 1, \ldots, n_0Q, \)

where \( \hat{V} \) has the structure given by the eqs.(54) and

\[
\hat{V}^{(i)}(x) = \begin{bmatrix} \hat{v}_0^{(i)}(x) \\ \hat{v}_1^{(i)}(x) \end{bmatrix}, \quad i = 0, \ldots, n_0 - 1.
\]

(1c) coincides with the previous case, see eqs.(162,163).

(1d) Multiplying the eq.(187) by \( P \) from the right, using the eq.(159) for \( \Psi \) and the eq.(192) for \( V \) one gets:

\[
\sum_{\gamma=1}^{2Q} \hat{\Psi}_{\alpha\gamma}(E_\beta; x)\hat{V}_{\gamma\beta}(x) = \sum_{j=0}^{n_0Q} \sum_{\gamma=1}^{2Q} (\hat{\psi}_j)_{\alpha\gamma}(E_\beta; x)(\hat{V}_j)_{\gamma\beta}(x) = 0,
\]

\( \alpha = 1, \ldots, Q, \beta = 1, \ldots, n_0Q, \)

where

\[
\hat{V}_j = [\hat{v}_j^{(0)} \cdots \hat{v}_j^{(n_0-1)}], \quad j = 0, 1.
\]

Eq.(188) reads:

\[
G(\lambda, \nu; x) \star V(\nu; x) = H_1(\lambda; x)\hat{P}(\lambda), \quad \hat{P}(\lambda) = [I \ \lambda I \cdots \lambda^{n_0-1}I].
\]

Multiplying this equation by \( P \) from the right, substituting \( G \) from the eq.(163) and \( V \) from the eq.(192) one gets:

\[
(H_1)_{\alpha}(\lambda; x) \left( -\sum_{\gamma=1}^{2Q} \frac{(H_2)_{\alpha\gamma}(E_\beta; x)}{\lambda - E_\beta} \hat{V}_{\gamma\beta}(x) - \sum_{\gamma=1}^{n_0Q} \hat{P}_{\alpha\gamma}(\lambda)P_{\gamma\beta}(x) \right) + \sum_{j=0}^{1} g_{0j}(\lambda; x)(\hat{V}_j(x))_{\alpha\beta} \delta(\lambda - E_\beta) = 0, \quad \alpha = 1, \ldots, Q, \beta = 1, \ldots, n_0Q.
\]
Eq.(197) consists of terms with separated spectral parameters. Thus, it must be splitted into two equations:

\[
(H_1)_\alpha(\lambda; x) \left( \sum_{\gamma=1}^{2Q} \frac{(H_2)_{\alpha\gamma}(E_\beta; x)\hat{V}_{\gamma\beta}(x)}{\lambda - E_\beta} + \sum_{\gamma=1}^{n_0Q} \hat{P}_{\alpha\gamma}(\lambda)P_{\gamma\beta}(x) \right) = 0, \tag{198}
\]

\[
\sum_{j=0}^{1} g_{0j}(E_\beta; x)(\hat{V}_j(x))_{\alpha\beta} = 0, \quad \alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0Q. \tag{199}
\]

Consider \(H_1\) defined by the eq.(185). Then the equation (198) becomes equivalent to the next one:

\[
\sum_{j=0}^{1} \sum_{\gamma=1}^{2Q} (h_{2j})_{\alpha\gamma}(E_\beta; x)(E_\beta - b_j)^{-1}(\hat{V}_j(x))_{\gamma\beta}(x) - \sum_{\gamma=1}^{n_0Q} \hat{P}_{\alpha\gamma}(b_j)P_{\gamma\beta}(x) = 0. \tag{200}
\]

Eqs. (194, 200) must be viewed as a system of equations for \(\hat{V}\), while eq. (199) is a matrix equation for elements of \(E\) and \(P\). These three equations may be written in terms of \(w\) in the following way. First of all, the eq.(194) relates \(\hat{V}_j, j = 0, 1: \hat{V}_1(x) = -\hat{v}_1^{-1}(E_\beta; x)\hat{v}_0(E_\beta; x)\hat{V}_0(x)\). Substituting this relation into eq.(199) and into eq.(200) multiplied by \((E_\beta - b_j)\) from the right results in

\[
\sum_{\gamma=1}^{2Q} (\psi^{(-)}(E_\beta; x)(\hat{V}_0(x))_{\gamma\beta} = 0, \tag{201}
\]

\[
\alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0Q.
\]

\[
\sum_{\gamma=1}^{Q} h_{\alpha\gamma}^{(-)}(E_\beta; x)(\hat{V}_0(x))_{\gamma\beta}(x) = \sum_{\gamma=1}^{n_0Q} \hat{P}_{\alpha\gamma}(b_j)P_{\gamma\beta}(x)(E_\beta - b_j), \tag{202}
\]

\[
\alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0Q, \quad j = 1, \ldots, n_0,
\]

where

\[
\psi^{(-)}(\lambda; x) = g_{00}(\lambda; x) - g_{01}(\lambda; x)\hat{v}_1^{-1}(\lambda; x)\hat{v}_0(\lambda; x),
\]

\[
h^{(-)}(\lambda; x) = h_{20}(\lambda; x) - h_{21}(\lambda; x)\hat{v}_1^{-1}(\lambda; x)\hat{v}_0(\lambda; x).
\]

Next, eliminating \(\hat{V}_0\) from the eq. (201) using the eq.(202) one gets

\[
\sum_{\gamma_1=1}^{Q} \sum_{\gamma_2=1}^{n_0Q} R_{\alpha\gamma_1}(E_\beta; x)E_\beta^{-1}\hat{P}_{\gamma_1\gamma_2}(b_j)P_{\gamma_2\beta}(x)(E_\beta - b_j) = 0, \tag{204}
\]

\[
\alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0Q.
\]

where

\[
R(\lambda; x) = \psi^{(-)}(\lambda; x)(h^{(-)})^{-1}(\lambda; x)\lambda.
\]

Finally, applying \(P^{-1}\) from the right, using the fact that \(b_j\) are scalars and the evident identity

\[
\hat{P}(b_j)(w - b_jI) \equiv \hat{P}(0) w \tag{206}
\]
one gets the resulting equation in terms of $w$:

$$\left[ \sum_{\gamma=1}^{Q} R_{\alpha\gamma}(w; x) \right]_{\gamma\beta} = 0, \quad \alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0 Q,$$

which is an equation for the elements of $Q \times Q$ matrices $w^{(0)}$, $j = 0, \ldots, n_0 - 1$. This ends the non-evolutionary part of the dressing algorithm.

We see that, along with the solution $U(\lambda, \mu; x)$ to the spectral problem, we have described implicitly the function $w(x)$ (solution to the appropriate PDE0) and, as a consequence, the function $w^{(0)}(x)$ (solution to PDE1). As a simple example, let $n_0 = 1$. Then $\hat{A} \ast H_1 = 0$, $\tilde{P}(k) = I$, $w = w^{(00)}$. The eq. (207) gets the next form:

$$\left[ \sum_{\gamma=1}^{Q} R_{\alpha\gamma}(w; x) \right]_{\gamma\beta} = 0, \quad \alpha, \beta = 1, \ldots, Q. \quad (208)$$

Eq.(207) holds for any PDE1 and associated PDE0. In order to show the equivalence of this equation to the non-differential equations derived in Sec.2 (see eqs.(41,61,64)), we must introduce the particular dependence of $R$ on variables $t_i$ and $x_i$, which will be done in the rest of Sec.3.2.2.

**Evolutionary part of the dressing algorithm.** As we have seen, the structure of eq.(207) does not depend on the particular $x$-dependence of the functions $\psi^{(-)}$ and $h^{(-)}$ defined through the $x$-dependence of the functions $\Psi$, $G_0$ and $H_2$ (see eqs.(121,128,132,134)). Here we describe the $x$-dependence of (207), related with the $x$-dependence of $\Psi$ (eq.(121)), $G_0$ (eq.(128)) and $H_2$ (eqs.(132d) or (134c)). Remark, that evolution of $H_1$ given by the eqs.(132c) or (134b), is not needed in resulting formula (207). Emphasise that index $m$ appearing in the equations of this section is meaningful only in the case (134), when $P^{(m)} = \tilde{P}^{(m)} = 0$ and $P^{(m)} = 0$. Otherwise this index must be fixed indicating that the appropriate PDEs do not possess the commuting flows.

Functions $C^{(m)}$, $C^{(m)}$ and $\hat{C}^{(m)}$ are defined by the eqs.(123,131,132b):

$$C^{(m)}(\lambda, \mu) = \hat{C}^{(m)}(\lambda, \mu) = -\left( \rho(\lambda)\delta(\lambda - \mu) \right)_\lambda I,$$

$$C^{(m)}(\lambda, \mu) = -\left( \rho(\lambda)\delta(\lambda - \mu) \right)_\lambda I_2. \quad (209)$$

Eqs.(121) (in view of eq.(159)) and (128) yield the following equations for the functions $\hat{\psi}_j$ and $\hat{g}_{0j}$ respectively ($j = 0, 1$):

$$\left( \hat{\psi}_1(\lambda; x) \right)_t + \sum_{j=1}^{N} \rho^{(mj)}(\lambda) \left( \hat{\psi}_1(\lambda; x) \right)_{x_j} - \left( \hat{\psi}_1(\lambda; x) \rho^{(m0)}(\lambda) \right)_\lambda = 0, \quad i = 0, 1, \quad (210)$$

$$\left( g_{0i}(\lambda; x) \right)_t + \sum_{j=1}^{N} \rho^{(mj)}(\lambda) \left( g_{0i}(\lambda; x) \right)_{x_j} - \left( g_{0i}(\lambda; x) \rho^{(m0)}(\lambda) \right)_\lambda + \sum_{j=1}^{\tilde{N}} T^{(j)} \tilde{\rho}^{(mj)}(\lambda) g_{0i}(\lambda; x) = 0, \quad i = 0, 1, \quad (211)$$
Combining these equations and taking into account the definition of $\psi^(-)$ (eq.(203a)) we obtain equation for $\psi^(-)$:

$$\psi^(-)(\lambda; x) + \sum_{j=1}^{N} \rho^{(m)}(\lambda) \psi^(-)(\lambda; x) - \left(\psi^(-)(\lambda; x) \rho^{(m)}(\lambda)\right)_{\lambda} +$$

$$\sum_{j=1}^{\tilde{N}} T^{(j)} \tilde{\rho}^{(m)}(\lambda) \psi^(-)(\lambda; x) = 0, \ i = 0, 1.$$

Mostly general equation for $H_2$ is eq.(132d), which may be written in the next form in view of eq.(132b):

$$H_2t_m + \sum_{j=1}^{N} H_{2x_j} * A^{(m)} + \sum_{j=1}^{\tilde{N}} T^{(j)} H_{2} * \tilde{A}^{(m)} - H_{2} * A^{-1} * C^{(m)} * A = 0. \quad (213)$$

Then, in particular, eq.(134c) corresponds to $C^{(m)} = 0$. Introduce the function $H(\lambda; x)$ by the formula:

$$H_2 = H_2 * A. \quad (214)$$

Then the above equation takes the next form after applying $*A^{-1}$:

$$H_2t_m + \sum_{j=1}^{N} H_{2x_j} * A^{(m)} + \sum_{j=1}^{\tilde{N}} T^{(j)} H_{2} * \tilde{A}^{(m)} - H_{2} * C^{(m)} = 0. \quad (215)$$

Function $H_2$ has the structure analogous to $H_2$:

$$H_2(\mu; x) = [h_{20}(\mu; x) \ h_{21}(\mu; x)]. \quad (216)$$

Then eqs.(215) yields:

$$\left(h_{2i}(\lambda; x)\right)_{t_m} + \sum_{j=1}^{N} \rho^{(m)}(\lambda) \left(h_{2i}(\lambda; x)\right)_{x_j} +$$

$$\sum_{j=1}^{\tilde{N}} T^{(j)} \tilde{\rho}^{(m)}(\lambda) h_{2i}(\lambda; x) - \left(h_{2i}(\lambda; x)\right)_{\lambda} \rho^{(m)}(\lambda) = 0, \ i = 0, 1.$$

Combining eqs.(210) and (217), taking into account definition of $h^(-)$ (203b) one gets

$$h_{tm}(-)(\lambda; x) + \sum_{j=1}^{N} \rho^{(m)}(\lambda) h_{x_j}(-)(\lambda; x) +$$

$$\sum_{j=1}^{\tilde{N}} T^{(j)} \tilde{\rho}^{(m)}(\lambda) h(-)(\lambda; x) - \left(h(-)(\lambda; x)\right)_{\lambda} \rho^{(m)}(\lambda) = 0, \ i = 0, 1.$$

where

$$h(-)(\lambda; x) = h_{20}(\lambda; x) - h_{21}(\lambda; x) \tilde{\psi}^{-1}(\lambda; x) \tilde{\psi_0}(\lambda; x), \quad \tilde{h}(-)(\lambda; x) = \tilde{h}(-)(\lambda; x) \lambda. \quad (219)$$
Finally, linear PDE for \( R(\lambda; x) \) defined by the eq. (205) reads:

\[
R_{tm}(\lambda; x) + \sum_{j=1}^{N} \rho^{(mj)}(\lambda) R_{xj}(\lambda; x) + \sum_{j=1}^{\tilde{N}} \tilde{\rho}^{(mj)}(\lambda) [T^{(j)}, R(\lambda; x)] - \left( R(\lambda; x) \rho^{(m0)}(\lambda) \right)_{\lambda} = 0, \quad i = 0, 1.
\]

Thus, evolutionary part of the dressing algorithm, nn.2a,2b, is reduced to the solution of the single eq. (220). We describe two particular examples.

**Solutions to the PDE0s (147).** In this case \( n_0 = 1 \) so that \( w \equiv w^{(0)} \). It has been shown in Sec.3.1.2, that index \( m \) must be fixed. Let \( m = 1 \). Use the following notations:

\[
t \equiv t_{1}, \quad \rho(j) \equiv \rho^{(1j)}, \quad \tilde{\rho}(j) \equiv \tilde{\rho}^{(1j)}. \tag{221}
\]

In general, when \( \rho^{(0)} \neq 0 \), the eq. (220) may be integrated by the method of characteristics. Eq. (220) yields:

\[
\frac{d\lambda}{dt} = -\rho^{(0)}(\lambda), \tag{222}
\]

\[
\frac{dx_{j}}{dt} = \rho^{(j)}(\lambda), \quad j = 1, \ldots, N,
\]

\[
\frac{dR}{dt} = \rho^{(0)}(\lambda) R - \sum_{j=1}^{\tilde{N}} \tilde{\rho}^{(j)}(\lambda) [T^{(j)}, R].
\]

Integrating the eqs. (222) in quadratures we receive:

\[
t = -\int_{\lambda}^{t} \frac{d\lambda}{\rho^{(0)}(\lambda)} + c_{0}, \tag{223}
\]

\[
x_{j} = \int_{t}^{\lambda} \frac{dt}{\rho^{(0)}(\lambda(t))} + c_{j}, \quad j = 1, \ldots, N,
\]

\[
R = e^{-\int_{t}^{\lambda} \sum_{j=1}^{\tilde{N}} T^{(j)} \tilde{\rho}^{(j)}(\lambda(t)) C(c_{0}, \ldots, c_{N}) e^{-\int_{t}^{\lambda} \sum_{j=1}^{\tilde{N}} T^{(j)} \tilde{\rho}^{(j)}(\lambda(t))}}.
\]

where \( c_{i} \) are integration constants and \( C(z_{0}, \ldots, z_{N}) \) is an arbitrary function of \( N+1 \) arguments.

Integration constants \( c_{i} = c_{i}(\varpi, x) \), \( i = 0, \ldots, N \), must be expressed in terms of \( \lambda \) and \( x \) using (223a,b). Let

\[
C(z_{0}, \ldots, z_{N}) = z_{0} - F(z_{0}, \ldots, z_{N}). \tag{224}
\]

Then the eq. (207) yields

\[
(c_{0}(w, x))_{\alpha\beta} = \left. \left( \frac{c_{0}(w, x)}{\rho^{(0)}(\lambda)} \right) \right|_{\lambda \rightarrow w} \left. \sum_{j=1}^{Q} \left[ F_{\alpha\gamma} e^{\int_{t}^{\lambda} \sum_{j=1}^{\tilde{N}} T^{(j)} \tilde{\rho}^{(j)}(\lambda(t))} \left|_{\lambda \rightarrow w} \right. \right]_{\gamma \beta} = 0, \quad \alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_{0} Q.
\]

This equation describes the solutions space to the PDE0 (147) with \( m = 1 \) and notations (221). Of course, the same result may be obtained using the algebraic approach.
Construction of the solutions manifold to the PDE1\((t_1, t_2; w^{(0)})\) and PDE0\((t_m; n_0, w)\), \(m = 1, 2\), derived in Sec.3.1.3. If \(\rho^{(m)} = 0\) (see eq.(134)) then eq.(220) may be solved explicitly:

\[
R_0(\lambda, q) = \int \mathcal{D} q \ e^{i \mathcal{N} \mathcal{P}_j \sum_{m=1}^{2} \rho^{(m)}(\lambda) t_m} R_0(\lambda, q) e^{i \mathcal{N} \mathcal{P}_j \rho^{(m)}(\lambda) t_m},
\]

where \(q = (q_1, \ldots, q_N)\), parameters \(q_i\) are complex in general and integration is over some region of the space of parameter \(q\). The proper choice of \(R_0\) is following:

\[
R_0(\lambda, q) = \lambda - \tilde{F}(\lambda, q),
\]

Then the eq.(207) gives us

\[
\begin{align*}
\alpha \beta & = \sum_{\gamma=1}^{Q} \left[ e^{ - \mathcal{N} \mathcal{P}_j \rho^{(m)}(\lambda) t_m} F_\alpha \left( w, x_1 \mathcal{I}_{n_0} - \sum_{m=1}^{2} \rho^{(m)}(\lambda) t_m, \ldots, \right) \\
\alpha & = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0 Q,
\end{align*}
\]

where we use notation

\[
F(z_0, \ldots, z_N) = \int \mathcal{D} q \tilde{F}(z_0, q_1, \ldots, q_N) e^{i \sum_{j=1}^{N} q_j z_j}.
\]

**Solutions to N-wave equation.** If \(\rho^{(m)} = 0\), \(\rho^{(mj)} = A\delta_{mj}\), then PDEs have derivatives only with respect to \(t_i\) (but not with respect to \(x_i\)). We may take \(F(z_0, z_1, \ldots, z_N) = F(z_0)\) in the eq.(228), so that eq.(228) gives us the algebraic equation (41).

**Solutions to Pohlmeyer equation.** One must substitute \(\rho^{(mj)} = A\delta_{mj}\), \(T^{(i)} = 0\), \(N = 2\) into the eq.(228), which yields the algebraic equation (61) describing solution manifold to the eq.(17). Solution manifold to the eq.(20a) corresponds to the reduction

\[
\tilde{F}(z_0, q_1, q_2) = \tilde{F}(z_0, q_1) \delta(q_2 - z_0 q_1), \quad x_2 = 0,
\]

which yields the algebraic equation (64).

### 3.3 Second version of the dressing method for PDE1\((t_1, t_2; w^{(0)})\) with reduction (35,37).

Disadvantage of the dressing algorithm described above is a poor available solutions space to PDE0s, since all freedom is associated with the function \(R\) which is \(Q \times Q\) matrix function, while \(w\) has \(n_0 Q^2\) scalar fields. However, we know that the algebraic approach provides a fullness of the solutions space. Thus we may ask if there is another version of the dressing method which yields fullness of the solutions space as well.
Such version of the dressing method is based on the integral equation (114) as \( n_0 Q \times n_0 Q \) matrix equation (it was \( Q \times Q \) matrix equation in the above algorithm). For simplicity, we consider only those PDE0s which are associated with PDE1s. Thus, PDE0 (147) with nonzero \( \rho^{(m0)} \) is not the subject of this subsection.

This version of the dressing algorithm is very similar to one discussed in Secs.3.1,3.2. In particular, all equations for dressing and spectral functions remain correct, only matrix dimensionalities must be changed. So we just underline a few features.

1. It follows from the first paragraph of this section, that PDE0(\( t_m; n_0, w \), \( m = 1, 2 \), with any fixed \( n_0 \) is associated with its own integral equation (114).

2. Field

\[
  w = H_2 * U * H_1
\]

must have the structure (21a). As before, \( w^{(0)} \) is a solution of PDE1. However, \( r^{(i)} \) are not necessary scalars unlike Secs.3.1 and 3.2. Since structure (21a) must be supported by the nonlinear PDEs (8), parameters \( r^{(i)} \) should be defined as in the algebraic approach, see a paragraph after the eqs.(21).

3. In order to establish relation between PDE1(\( t_1, t_2; w^{(0)} \)) and PDE0(\( t_m; n_0, w \), \( m = 1, 2 \), we take the reduction (141) in the next form:

\[
  \hat{A} * H_1 = 0 \Rightarrow H_2 * U * \hat{A} * H_1 = 0. \tag{232}
\]

Now the functions \( \mathcal{A}, A \) and \( \hat{A} \) are defined as follows (compare with the formulae (157)):

\[
  \mathcal{A}(\lambda, \mu) = \hat{A}(\lambda, \mu) = \lambda \delta(\lambda - \mu) I_{n_0}, \quad A(\lambda, \mu) = \lambda \delta(\lambda - \mu) I_{2n_0}. \tag{233}
\]

4. Similarly to the n.3, in order to establish relation between PDE1(\( t_1, t_2; w^{(0)} \)) and PDE0(\( t_m; k_0, \bar{w} \), \( m = 1, 2 \), we take the reduction (142) in the next form:

\[
  H_2 * A = 0 \Rightarrow H_2 * A * U * H_1 = 0. \tag{234}
\]

5. Owing to n.3, derivation of PDE0(\( t_m; n_0, w \)) in this section is equivalent to derivation made in Sec.3.1.2 with \( \rho^{(m0)} = 0 \), see resulting equation (147). This fact suggests us the general form of PDE0(\( t_m; n_0, w \), eq.(25).

6. Similarly, owing to n.4, we may derive general form of PDE0(\( t_m; k_0, \bar{w} \), see eq.(26). We do not represent appropriate calculations.

7. Eq.(228) must be replaced by the next equation

\[
  w_{\alpha\beta} = \sum_{\gamma=1}^{n_0 Q} e^{-\frac{2}{\gamma} \sum_{j=1}^{N} T_{\alpha}^{(j)} \rho^{(mj)}(w)t_m} F_{\alpha\gamma}(w, x_1 I_{n_0} - \sum_{m=1}^{2} \rho^{(m1)}(w)t_m, \ldots), \tag{235}
\]

\[
  x_N I_{n_0} - \sum_{m=1}^{2} \rho^{(mN)}(w)t_m \left[ \sum_{\gamma=1}^{n_0 Q} e^{-\frac{2}{\gamma} \sum_{j=1}^{N} T_{\gamma}^{(j)} \rho^{(mj)}(w)t_m} \right]_{\gamma\beta};
\]

\[
  \alpha = 1, \ldots, Q, \quad \beta = 1, \ldots, n_0 Q.
\]

We see that the algorithm discussed in this section provides the same richness of the solutions space as the algebraic approach does. Namely, we have \( n_0 Q^2 \) arbitrary functions \( F_{\alpha\beta}, \alpha = 1, \ldots, Q, \beta = 1, \ldots, n_0 Q, \) and diagonal \( (T^{(j)} \neq 0) \) or arbitrary \( (T^{(j)} = 0) \) matrix parameters \( r^{(j)}, j \geq 1 \).
4 Conclusions

First of all, we have represented the algorithm for the derivation of the system of non-differential equations which implicitly describes the special type of solutions to Self-dual type $S$-integrable equations (11). This is possible due to the relation between $S$-integrable PDEs (PDE1s) and appropriate family of matrix first order quasilinear PDEs (PDE0s).

As we have seen, PDE0s have set of remarkable properties:

1. Manifold of PDE0s is splitted into subclasses. Each subclass joins such PDE0s which are lower dimensional reductions of appropriate PDE1. These PDE0s admit infinitely many commuting flows, similar to PDE1s. Besides, there is a subclass of PDE0s which is not related with PDE1s; we have not identified the commuting flows for this subclass.

2. Any PDE0 admits the spectral system consisting of two linear equations for the spectral function $V(\lambda; x)$. But only one of these equations has derivatives with respect to variables $x_i$ and $t_i$, unlike the spectral systems for PDE1s. This fact gives rise to all following properties written in this list.

3. The solutions space of PDE0s is implicitly described by the system of non-differential equations. This is a typical description for the first order quasilinear equations [19, 20, 21, 29, 30].

4. As we know, PDE1s (like any $S$-integrable PDE) may be written using one of the following procedures: (a) applying $H_2 \ast A^\ast$ and $\ast \hat{A}^\ast \ast H_1$ to the spectral problem and using the definitions of the dependent variables in terms of the spectral and dressing functions, see eq.(140), or (b) as the compatibility conditions of the appropriate spectral problems. However, PDE0s may be derived only using the first procedure.

5. Functions $\rho_m(z)$ and $\tilde{\rho}_m(z)$ may be replaced by $\rho_m(z; x)$ and $\tilde{\rho}_m(z; x)$ respectively, which are representable by positive power series of $z$ with scalar coefficients explicitly depending on $x$. The solutions spaces to such PDEs is implicitly described by the system of the non-differential equations supplemented by the pair of decoupled linear PDEs with non-constant coefficients which, in turn, are integrable by the method of characteristics. We leave this statement without proof.

Although all basic conclusions have been established using the algebraic approach, we represent a version of the dressing method allowing to derive the same system of non-differential equations describing solutions space to PDE0s and PDE1s. In addition, this method reproduces the manifold of classical solutions to PDE1s.

The fact that the dressing method exhibits a connection with the method of characteristics is important for unification of the methods for solving of nonlinear PDEs in multidimensions. It tells us that the dressing method is more flexible in comparison, at least, with some other integration methods mentioned in this paper, such as Inverse Spectral Transform (IST) ($S$-integrability), direct linearization ($C$-integrability) and method of characteristics. In fact, direct linearization may not be applied to both $S$-integrable equations and PDE0s. Similarly, IST may not be applied to PDE0s and to $C$-integrable equations. Method of characteristics was modified to hodograph and generalized hodograph method allowing to study some ($S$-integrable) PDEs by the method of hydrodynamic reductions. However, using this technique we face a problem of the explicit representation of solutions.
On the contrary, although originally the dressing method was invented as a special technique for solving $S$-integrable equations, it was shown in [30] as well as in this paper that the properly modified dressing method works also for PDE0s. It yields exactly the same matrix non-differential equation describing solutions space as method of characteristics does. The fact that the dressing method may be a starting point for solving $C$-integrable equations has been demonstrated in [54] as well as in [30].

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