EVOLUTION EQUATIONS AND ANGULAR ORDERING AT SMALL $x$†

M. SCORLETTI
Dipartimento di Fisica, Università di Milano and INFN, sezione di Milano
via Celoria 16, 20133 Milano, ITALY

This talks examines the effect of angular ordering on the small-$x$ evolution of the unintegrated gluon distribution, and discusses the characteristic function for the CCFM equation.

1 Introduction

Angular ordering is an important feature of perturbative QCD with a deep theoretical origin and many phenomenological consequences. It is the result of destructive interference: outside angular ordered regions amplitudes involving soft gluons cancel. This property is quite general, and it is present in both time-like processes, such as $e^+e^-$ annihilation, and in space-like processes, such as deep inelastic scattering (DIS). Moreover it is valid in the regions both of large and small $x$, in which $x$ is the registered energy fraction in the $e^+e^-$ fragmentation function or the Bjorken variable in the DIS structure function. Due to the universality of angular ordering one has a unified leading order description of all hard processes involving coherent soft gluon emission.

The detailed analysis of angular ordering in multi-parton emission at small $x$ and in the related virtual corrections shows that to leading order the initial-state gluon emission can be formulated as a branching process in which angular ordering is taken into account both in real emissions and virtual corrections.

In DIS, angular ordering is essential for describing the structure of the final state, but not for the gluon density at small $x$. This is because in the resummation of singular terms of the gluon density, there is a cancellation between the real and virtual contributions. As a result, to leading order the small-$x$ gluon density is obtained by resumming $\ln x$ powers coming only from IR singularities, and angular ordering contributes only to subleading corrections.

The calculation of the gluon density by resummation of $\ln x$ powers without

†Talk presented at the “Conference on Perspectives in Hadronic Physics”, Trieste, 12-16 May 1997
angular ordering was done 20 years ago\textsuperscript{[1]} and leads to the BFKL equation, which is an evolution equation for $F(x, k)$, the unintegrated gluon density at fixed transverse momentum $k$:

\[
x \frac{\partial F(x, k)}{\partial x} = \bar{\alpha}_S \int \frac{d^2 q}{\pi q^2} [F(x, |k + q|) - \theta(k - q) F(x, k)]
\]

where $\bar{\alpha}_S = \frac{4\pi}{\alpha_S}$. $F(x, k)$ is related to the small-$x$ part of the gluon structure function $F(x, Q)$ by

\[
F(x, Q) = \int d^2 k \ F(x, k) \theta(Q - k) .
\]

In this talk, as a first step of a systematic study of multi-parton emission in DIS, the effect of angular ordering on the small-$x$ evolution of the gluon structure function is studied\textsuperscript{[3]} with both analytical and numerical techniques.

## 2 Evolution equation for gluon density

In this section we recall the basic ingredients used to build the coherent branching equation for the gluon density at small $x$.

The evolution of the gluon density can be described (Fig. 1) as a multi-branching process involving only gluons, since gluons dominate the small-$x$ region. The emission process takes place in the angular ordered region given by $\theta_i > \theta_{i-1}$ with $\theta_i$ the angle of the emitted gluon $q_i$ with respect to the incoming gluon $k_0$. In terms of the emitted transverse momenta $q_i$ this region is given by

\[
\theta_i > \theta_{i-1} , \quad \Rightarrow \quad q_i > z_{i-1} q_{i-1} .
\]

The branching distribution for the emission of gluon $i$ reads

\[
dP_i = \frac{d^2 q_i}{\pi q_i^2} \frac{\bar{\alpha}_S}{z_i} \Delta(z_i, q_i, k_i) \ \theta(q_i - z_{i-1} q_{i-1}) ,
\]

where

\[
\ln \Delta(z_i, q_i, k_i) = - \int_{z_i}^1 dz' \frac{\bar{\alpha}_S}{z'} \int \frac{d^2 q'}{q'^2} \ \theta(k_i - q') \ \theta(q' - z' q_i)
\]

is the form factor which resums important virtual corrections for small $z_i$\textsuperscript{[4]}. The branching (4) — which includes angular ordering (3) both in the real and the virtual emissions — is accurate to leading IR order\textsuperscript{[5]}.\textsuperscript{[6]}

\textsuperscript{[1]}The usual Sudakov form factor is not included in the single-branching kernel, since it is cancelled by soft emissions.
The “non-Sudakov” form factor (5) has a simple probabilistic interpretation. It corresponds to the probability for having no radiation of gluons with energy fraction \( x' = z' x_{i-1} \) in the region \( x_i < x' < x_{i-1} \), and with a transverse momentum \( q' \) smaller than the total emitted transverse momentum \( k_i \) and with an angle \( \theta' > \theta_i \). The two boundaries in \( q' \) are due to coherence in the exchanged gluon (\( k > q' \)) and in the emitted one (\( \theta' > \theta_i \Rightarrow q' > z' q_i \)).

Angular ordering provides a lower bound on transverse momenta, so that no collinear cutoff is needed other than a small virtuality for the first incoming gluon. On the other hand, in order to deduce a recurrence relation for the inclusive distribution in the last gluon with fixed \( x = x_n \) and \( k = k_n \) one has to introduce an additional dependence on a momentum variable \( p \). That variable corresponds to the transverse momentum associated with the maximum available angle \( \bar{\theta} \) for the last emission, which in DIS is settled by the angle of the quarks produced in the boson-gluon fusion. The dependence on \( p \) is through

\[
\theta_n < \bar{\theta} \quad \Rightarrow \quad z_n q_n < p, \tag{6}
\]

where \( p \simeq x E \bar{\theta} \) and \( x E \) is the energy of the \( n \)-th gluon, which undergoes the hard collision at the scale \( Q \).

The distribution for emitting \( n \) initial state gluons is defined as

\[
A^{(n)}(x, k, p) = \int \prod_{i=1}^{n} dP_i \theta(p - z_n q_n) \delta(k^2 - k_n^2) \delta(x - x_n), \tag{7}
\]
so that the fully inclusive gluon density

\[ A(x, k, p) = \sum_{n=0}^{\infty} A^{(n)}(x, k, p), \]  

satisfies the equation (CCFM equation)

\[ A(x, k, p) = A^{(0)}(x, k, p) + \int \frac{d^2q}{\pi q^2} dz \frac{\bar{\alpha}_S}{z} \Delta(z, q, k) \theta(p - zq) A \left( \frac{x}{z}, |k + q|, q \right), \]  

where the inhomogeneous term \( A^{(0)}(x, k, p) \) is the distribution for no gluon emission.

It can be proved that the gluon density \( A(x, k, p) \) becomes independent of \( p \) for \( p \to \infty \). Indeed, neglecting the \( p \)-dependence in \( A(x, k, p) \) corresponds to neglecting angular ordering. In this case the transverse momenta have no lower bound, and we need to introduce a collinear cutoff \( \mu \) to avoid singularities. We then modify the branching distribution in (4) and the virtual corrections (5) by the substitution \( \theta(q_i - z_{i-1}q_{i-1}) \to \theta(q_i - \mu) \) and \( \theta(q' - z'q) \to \theta(q' - \mu) \) respectively. The modified branching distribution reads

\[ dP^{(0)}(i) = \frac{d^2q_i}{\pi q_i^2} dz_i \frac{\bar{\alpha}_S}{z_i} \Delta^{(0)}(z_i, k_i) \theta(q_i - \mu), \]  

with the form factor

\[ \ln \Delta^{(0)}(z, k) = - \int_{z}^{1} dz' \frac{\bar{\alpha}_S}{z'} \int dq' \frac{q'^2}{q^2} \theta(k - q') \theta(q' - \mu). \]

The gluon density \( F(x, k) \) (in this case there is no dependence on the “maximum angle” \( p \)) satisfies the following recurrence relation:

\[ F(x, k) = F^{(0)}(x, k) + \int \frac{d^2q}{\pi q^2} dz \frac{\bar{\alpha}_S}{z} \Delta^{(0)}(z, k) \theta(q - \mu) F \left( \frac{x}{z}, |k + q| \right). \]  

Although every branching factor (10) is divergent in the limit of vanishing \( \mu \), collinear singularities cancel in the inclusive sum which defines the structure function. As a consequence, the limit \( \mu \to 0 \) can be safely performed in (12) which — in this limit — prove to be equivalent to the BFKL equation (3). In spite of the very different behaviour in the collinear region of the branching distributions (4) and (11), the neglecting of angular ordering has no effect on the structure functions at leading order.

This is no longer true for exclusive quantities. Here collinear singularities do not cancel any more, and angular ordering becomes essential to control the
structure of singularities. A fixed cutoff $\mu$ regulates the collinear divergence which is present, but gives the wrong final state properties. The elimination of a large fraction of the small-transverse-momentum emissions indeed means that angular ordering has a big effect on the final state.

2.1 Properties of gluon distributions

The major task of this talk is to examine the corrections to structure function evolution that arise from angular ordering, which is expected to be part of the full NLO contribution.

As is well known, the BFKL equation has eigensolutions (strictly speaking eigensolutions of the equation without an inhomogeneous term and with no upper limit in the $z$ integral) of the form:

$$x\mathcal{F}(x, k) = x^{-\omega} \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right) \gamma$$

(13)

where the exponents $\omega$ and $\gamma$ are related through the characteristic function $\chi$

$$1 = \frac{\bar{\alpha}_S}{\omega} \chi(\gamma), \quad \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma),$$

(14)

with the QCD coupling $\alpha_S$ taken as a fixed parameter.

For a general initial condition the asymptotic behaviour of $\mathcal{F}(x, k)$ at small $x$ is determined by the leading singularity of $\gamma \left( \frac{\bar{\alpha}_S}{\omega} \right)$ in the $\omega$-plane, which is located at $\gamma_c = \gamma \left( \frac{\bar{\alpha}_S}{\omega} \right) = \frac{1}{2}$, giving $\omega_c = \bar{\alpha}_S \chi \left( \frac{1}{2} \right) = 4\bar{\alpha}_S \ln 2$.

The analytic treatment of the CCFM equation is more complicated than that of the BFKL equation because the gluon density contains one extra parameter, $p$. By analogy, we take the eigensolutions of (9) in the form

$$x\mathcal{A}(x, k, p) = x^{-\omega} \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right) \tilde{\gamma} \ G \left( \frac{p}{k} \right),$$

(15)

where $\tilde{\gamma}$ and $\omega$ are related through the unknown CCFM characteristic function $\tilde{\chi}$

$$1 = \frac{\bar{\alpha}_S}{\omega} \tilde{\chi}(\tilde{\gamma}, \alpha_S),$$

(16)

and the function $G \left( \frac{p}{k} \right)$ takes into account angular ordering, parameterising the unknown dependence on $p$.

\[b\] The renormalisation group dependence of $\alpha_S$ on a scale is an effect which goes beyond the leading order contribution.\[4]
For $0 < \tilde{\gamma} < 1$ fixed, one obtains a coupled pair of equations for $G$ and $\tilde{\chi}$:

$$p \partial_p G \left( \frac{p}{k} \right) = \bar{\alpha} S \int_p \frac{d^2 q}{\pi q^2} \left( \frac{p}{q} \right)^{\bar{\alpha} S} \Delta \left( \frac{p}{q}, q, k \right) G \left( \frac{q}{|k + q|} \right) \left( \frac{|k + q|}{k^2} \right)^{\tilde{\gamma} - 1} ,$$

with the initial condition $G(\infty) = 1$, and

$$\tilde{\chi} = \int \frac{d^2 q}{\pi q^2} \left( \frac{|k + q|^2}{k^2} \right)^{\tilde{\gamma} - 1} G \left( \frac{q}{|k + q|} \right) - \theta(k - q) G \left( \frac{q}{k} \right) .$$

(18)

By putting $G = 1$ in this last equation, one notes that $\tilde{\chi}$ becomes just the BFKL characteristic function (14). Since $1 - G \left( \frac{q}{k} \right)$ is formally of order $\alpha S$, this demonstrates that angular ordering has a next-to-leading effect on structure function evolution, and one can also shows that the first corrections to the small-$x$ anomalous dimension are of the form $\bar{\alpha}^3 / \omega^2$.

Though a number of asymptotic properties of the function $G \left( \frac{q}{k} \right)$ have been determined, it has not so far been possible to obtain its full analytic form.

3 Numerical results

In this section we summarise the main results of the numerical analysis carried out both for BFKL and CCFM equations in order to gain further insight into the (subleading) effects of angular ordering on the structure function.

Fig. 2 shows the results for $\tilde{\chi}$ compared to the BFKL characteristic functions as a function of $\tilde{\gamma}$ for various $\alpha S$. The difference $\delta \chi = \chi - \tilde{\chi}$ is positive, increases with $\tilde{\gamma}$, and increases with $\alpha S$. Moreover we find $\delta \chi \sim \bar{\alpha} \tilde{\gamma}$ for $\tilde{\gamma} \rightarrow 0$ ($\bar{\alpha} S$ small and fixed) and $\delta \chi \sim \bar{\alpha} S$ for $\bar{\alpha} S \rightarrow 0$ ($\tilde{\gamma}$ small and fixed). This implies that the next-to-leading correction to the gluon anomalous dimension coming from angular ordering is of order $\alpha^3 S / \omega^2$.

With respect to the BFKL case, the position of the minimum of the characteristic function $\tilde{\chi}$ gets shifted to the right, the value of the minimum is lowered and — in contrast to the BFKL case — there is no longer even a divergence at $\gamma = 1$. This behaviour of $\tilde{\chi}$ reduces the exponent $\omega_c$ of the small-$x$ growth of the structure function, in accordance with the fact that angular ordering reduces the phase space for evolution.

In Fig. 3a and 3b we plot as a function of $\alpha S$ the values $\tilde{\gamma}_c$ and $\tilde{\chi}_c$ with $\tilde{\chi}_c$ the minimum of $\tilde{\chi}$ and $\tilde{\gamma}_c$ its position. As expected the differences compared to the BFKL values $\chi_c = 4 \ln 2$ and $\gamma_c = \frac{1}{2}$ are of order $\bar{\alpha} S$.  
Fig. 2: The characteristic functions with and without angular ordering; $\tilde{\chi}(\gamma, \alpha_S)$ and $\chi(\gamma)$ are plotted as functions of $\gamma$.

Fig. 3c shows the second derivative, $\tilde{\chi}''_c$, of the characteristic function at its minimum; this quantity is important phenomenologically because the diffusion in $\ln k$ is inversely proportional to $\sqrt{\tilde{\chi}''_c}$. From this result, one can therefore conclude that the inclusion of angular ordering significantly reduces the diffusion compared to the BFKL case.

The loss of symmetry under $\gamma \to 1 - \gamma$ relates to the loss of symmetry between small and large scales: while in BFKL regions of small and large momenta are equally important, in the CCFM case angular ordering favours instead the region of larger $k$. However, at each intermediate branching, the region of vanishing momentum is still reachable for $x \to 0$, so that the evolution still contains non-perturbative components.

4 Final state distributions

The inclusion of angular ordering is expected to have relevant effects when simple exclusive quantities, associated with one-gluon inclusive distributions, are considered. Indeed, the cancellation between real emissions and virtual
Figure 3: (a) The value of the minimum of the characteristic function, $\tilde{\chi}_c$, as a function of $\alpha_S$. (b) The position of the minimum of the characteristic function, $\tilde{\gamma}_c$, as a function of $\alpha_S$. (c) The second derivative of the characteristic function, $\tilde{\chi}'',_c$, at its minimum, as a function of $\alpha_S$. 
corrections — which in the angular ordering equation for the inclusive structure function reconstruct at leading level the BFKL solution — is no longer guaranteed for the modified kernel which enter the evolution equations for associated distribution.

Although the analysis of this subject is far from being completed, preliminary calculations confirm that both the shapes and the normalisations of final state quantities are sensitive to the phase spaces reduction associated with angular ordering.

Fig. 4a shows the distribution of the number of initial state gluons emitted. As expected from the different behaviour in the collinear region, BFKL branching has more emissions and a broader tail with respect to the CCFM case.

Fig. 4b shows the $p_T$-distribution in rapidity. As expected, angular ordering suppress the radiation in the central and high rapidity regions.

Acknowledgements

This research was carried out in collaboration with G. Bottazzi, G. Marchesini and G.P. Salam and supported in part by the Italian MURST.

Figure 4: (a) Distribution of number of emission with $q > q_0 = 1$GeV, for DGLAP, CCFM and BFKL evolution to $x = 5 \times 10^{-5}$, $k = 5$ GeV, $\alpha_S = 0.2$. (b) Transverse momentum flow in the hadronic centre of mass frame as a function of the rapidity $\eta^*$ for evolution to $x = 2 \times 10^{-4}$, $k = 3$ GeV, $\alpha_S = 0.2$ (the proton direction is to the left).
References

[1] A.H. Mueller, *Phys. Lett.* B 104, 161 (1981); B.I. Ermolaev and V.S. Fadin, JETP Lett. 33 (1981) 285; Yu.L. Dokshitzer, V.S. Fadin and V.A. Khoze, *Z. Phys.* C 15, 325 (1982); A. Bassetto, M. Ciafaloni and G. Marchesini and A.H. Mueller, *Nucl. Phys.* B 207, 189 (1982); A. Bassetto, M. Ciafaloni and G. Marchesini, *Phys. Rep.* 100, 201 (1983); Yu.L. Dokshitzer, V.A. Khoze, S.I. Troyan and A.H. Mueller, *Basics of Perturbative QCD* (Editions Frontieres, Paris, 1991).

[2] Ya.A. Azimov, Yu.L. Dokshitzer, S.I. Troyan and V.A. Khoze, *Phys. Lett.* B 165, 147 (1985); R.K. Ellis, G. Marchesini and B.R. Webber, *Nucl. Phys.* B 286, 643 (1987); L.V. Gribov, Yu.L. Dokshitzer, S.I. Troyan and V.A. Khoze, Sov. Phys. JETP 68 (88) 1303; S Catani, Yu.L. Dokshitzer and B.R. Webber, *Phys. Lett.* B 322, 263 (1994).

[3] M. Ciafaloni, *Nucl. Phys.* B 296, 249 (1987); S. Catani, F. Fiorani and G. Marchesini, *Phys. Lett.* B 234, 339 (1990); S. Catani, F. Fiorani and G. Marchesini, *Nucl. Phys.* B 336, 18 (1990).

[4] G. Marchesini, *Nucl. Phys.* B 445, 49 (1995); G. Marchesini, in *Proceedings of the Workshop "QCD at 200TeV"* ed. L. Ciffarelli and Yu.L. Dokshitzer (Plenum press, New York, 1992).

[5] L.N. Lipatov, Yad. Fiz. 23 (1976) 642 [Sov. J. Phys. 23 (1976) 338]; E.A. Kuraev, L.N. Lipatov and V.S. Fadin, Zh. Eksp. Teor. Fiz. 72 (1977) 373 [Sov. Phys. JETP 45 (1977) 199]; Ya. Balitskii and L.N. Lipatov, Yad. Fiz. 28 (1978) 1597 [Sov. J. Nucl. Phys. 28 (1978) 822]. L.V. Gribov, E.M. Levin and M.G. Ryskin, *Phys. Rep.* 100, 1 (1983).

[6] G. Bottazzi, G. Marchesini, G.P. Salam and M. Scorletti, preprint IFUM 552-FT, [hep-ph/9702418](http://arxiv.org/abs/hep-ph/9702418) (Milano, 1997).

[7] L.N. Lipatov and V.S. Fadin, *Nucl. Phys.* B 406, 259 (1993); V.S. Fadin and L.N. Lipatov, *Nucl. Phys.* B 477, 767 (1996); V. Del Duca, *Phys. Rev.* D 54, 989 (1996); V. Del Duca, *Phys. Rev.* D 54, 4474 (1996); S. Catani and F. Hautmann, *Phys. Lett.* B 315, 157 (1993); S. Catani and F. Hautmann, *Nucl. Phys.* B 427, 475 (1994); G. Camici and M. Ciafaloni, Preprint DFF/264/01/97, [hep-ph/9701303](http://arxiv.org/abs/hep-ph/9701303); G. Camici and M. Ciafaloni, Preprint DFF/260/11/96, [hep-ph/9612233](http://arxiv.org/abs/hep-ph/9612233); G. Camici and M. Ciafaloni, *Phys. Lett.* B 386, 341 (1996).

[8] S. Catani, F. Fiorani, G. Marchesini and G. Otrani, *Nucl. Phys.* B 361, 645 (1991).