AN INTERSECTION REPRESENTATION FOR A CLASS OF ANISOTROPIC VECTOR-VALUED FUNCTION SPACES

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Abstract. The main result of this paper is an intersection representation for a class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov [32], which includes weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces. In the special case of the classical Triebel-Lizorkin spaces, the intersection representation gives an improvement of the well-known Fubini property. The motivation comes from the weighted $L_q$-$L_p$-maximal regularity problem for parabolic boundary value problems, where weighted anisotropic mixed-norm Triebel-Lizorkin spaces occur as spaces of boundary data.

1. Introduction

The motivation for this paper comes from [46] on the weighted $L_q$-$L_p$-maximal regularity problem for parabolic boundary value problems, which provides an extension of [21] to the weighted setting.

During the last 25 years, maximal regularity has turned out to be an important tool in the theory of nonlinear PDEs (see e.g. [1, 2, 4, 5, 17, 18, 19, 24, 26, 37, 40, 47, 49, 50, 55, 56, 58, 59]). Maximal regularity means that there is an isomorphism between the data and the solution of the problem in suitable function spaces. Having established maximal regularity for the linearized problem, many nonlinear problems can be treated with tools as the contraction principle and the implicit function theorem (see [56]). Concretely, the concept of maximal regularity has found its application in a great variety of physical, chemical and biological phenomena, like reaction-diffusion processes, phase field models, chemotactic behaviour, population dynamics, phase transitions and the behaviour of two phase fluids, for instance (see e.g. [49, 55, 57, 59]).

In order to elaborate a bit on the $L_q$-$L_p$-maximal regularity problem for parabolic boundary value problems, let us for simplicity consider the heat equation with the Dirichlet boundary condition,

$$\begin{align*}
\partial_t u(x,t) - \Delta u(x,t) &= f(x,t), \quad x \in \Omega, \quad t \in J, \\
u(x',t) &= g(x',t), \quad x' \in \partial \Omega, \quad t \in J, \\
u(x,0) &= u_0(x), \quad x \in \Omega,
\end{align*}$$

(1)

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where \( J = (0,T) \) is a finite time interval and \( \partial \Theta \subset \mathbb{R}^d \) is a \( C^\infty \)-domain with a compact boundary \( \partial \Theta \). In the maximal \( L_q - L_p \)-regularity approach to \((1)\) one is looking for solutions \( u \) in the \textit{maximal regularity space}
\begin{equation}
W^1_q(J; L_p(\partial \Theta)) \cap L_q(J; W^2_p(\partial \Theta)).
\end{equation}

The solution to the \( L_q - L_p \)-maximal regularity problem for \((1)\) is classical in the case \( q = p \) (see \([11]\)). However, it is desirable to have maximal \( L_q - L_p \)-regularity for the full range \( q,p \in (1, \infty) \), as this enables one to treat more nonlinearities. For instance, one often requires large \( q \) and \( p \) due to better Sobolev embeddings, and \( q \neq p \) due to criticality and/or scaling invariance (see e.g. \([26, 37, 38, 55, 57]\)). But the case \( q \neq p \) is much more involved than the case \( q = p \) due to a lack of Fubini in the form of \( L_q[L_p] = L_p[L_q] \) when \( q \neq p \).

The main difficulty in the \( L_q - L_p \)-maximal regularity approach to \((1)\) is the treatment of the boundary inhomogeneity \( g \) in the case \( q \neq p \). In the classical case \( q = p \), \( g \) has to be in the intersection space
\begin{equation}
B^\delta_{p,p}(J; L_p(\partial \Theta)) \cap L_p(J; B^{2\delta}_{p,p}(\partial \Theta)) = W^\delta_p(J; L_p(\partial \Theta)) \cap L_p(J; W^{2\delta}_p(\partial \Theta))
\end{equation}
with \( \delta = 1 - \frac{1}{2p} \), where \( W^\delta_p = B^\delta_p \) a non-integer order Sobolev-Slobodeckii space or Besov space. However, in the general case \( q \) has to be in the intersection space
\begin{equation}
F^\delta_{q,p}(J; L_p(\partial \Theta)) \cap L_q(J; B^{2\delta}_{p,p}(\partial \Theta)), \quad \delta = 1 - \frac{1}{2p},
\end{equation}
where \( F^\delta_{q,p} \) is a Triebel-Lizorkin space. This was established in \([72]\) in the case \( p \leq q \) and extended in \([21]\) to the full range for \( q,p \) in the more general setting of vector-valued parabolic boundary value problems with boundary conditions of Lopatinskii-Shapiro type.

The solution to the \( L_q - L_p \)-maximal regularity problem for \((1)\) in particular yields that the intersection space in \((3)\) is the spatial trace space of the maximal regularity space in \((2)\). However, on the one hand, this maximal regularity space \((2)\) can naturally be identified with the anisotropic mixed-norm Sobolev space
\begin{equation}
W^{(2,1)}_{(p,q)}(\Theta \times J) = \left\{ u \in \mathcal{D}'(\Theta \times J) : \partial_t u \in L_p(\partial \Theta \times J), |\partial_x u| \in L_{(p,q)}(\Theta \times J), |\alpha| \leq 2 \right\},
\end{equation}
where the mixed-norm Lebesgue space
\begin{equation}
L_{(p,q)}(\Theta \times J) = \left\{ f \in L_0(\Theta \times J) : \left( \int_{\Theta} \left( \int_{J} |f(x,t)|^p \, dt \right)^{p/q} \, dx \right)^{1/q} < \infty \right\}
\end{equation}
can be naturally identified with the Lebesgue Bochner space \( L_q(J; L_p(\partial \Theta)) \). On the other hand, in \([34]\) it was shown that the anisotropic mixed-norm Triebel-Lizorkin space \( F^{s,(\frac{1}{p},1)}_{(p,q),p}(\mathbb{R}^{d-1} \times \mathbb{R}) \) naturally occurs as the trace space of the anisotropic mixed-norm Sobolev space \( W^{(2,1)}_{(p,q)}(\mathbb{R}^d \times \mathbb{R}) \). This suggest a link between anisotropic mixed-norm Triebel-Lizorkin spaces and intersection spaces of the form \((3)\).

Such a link was in fact obtained in \([22, Proposition 3.23]\) by comparing the trace result \([33, Theorem 2.2]\) with a trace result from \([8, 9]\): for every \( q,p \in (1, \infty) \), \( a,b \in (0, \infty) \) and \( s \in (0, \infty) \),
\begin{equation}
F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^{d-1} \times \mathbb{R}) = F^{s,b}_{p,p}(\mathbb{R}; L_p(\mathbb{R}^{d-1})) \cap L_q(\mathbb{R}; B^{s/a}_{p,p}(\mathbb{R}^{d-1})).
\end{equation}

It is the goal of this paper to provide a more systematic approach to the intersection representation \((4)\) and obtain more general versions of it, covering the
weighted Banach space-valued setting. In order to do so, we introduce a new class of anisotropic vector-valued function spaces in an axiomatic setting à la Hedberg&Netrusov \[32\], which includes Banach space-valued weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces.

The main result of this paper is an intersection representation for this new class of anisotropic function spaces, from which the following theorem can be obtained as a special case (see Example 5.5):

**Theorem 1.1.** Let \(a, b \in (0, \infty), p, q \in (1, \infty), r \in [1, \infty] \) and \(s \in (0, \infty)\). Then

\[
F^{s,(a,b)}_{(p,q),r}(\mathbb{R}^n \times \mathbb{R}^m) = \mathbb{F}^{s/b}_{q,r}(\mathbb{R}^m; L^p(\mathbb{R}^n)) \cap L^q(\mathbb{R}^m; F^{s/a}_{p,r}(\mathbb{R}^n)),
\]

where, for \(E = L^p(\mathbb{R}^n)\),

\[
\mathbb{F}^{s,b}_{q,r}(\mathbb{R}^m; E) = \{ f \in S'(\mathbb{R}^m; E) : (2^{k \sigma} S_k f)_k \in L^q(\mathbb{R}^n; E[\ell_r(\mathbb{N})]) \}
\]

with \((S_k)_{k \in \mathbb{N}}\) a Littlewood-Paley decomposition of \(\mathbb{R}^m\).

In the case where \(r = r\), Fubini yields \(F^{s/b}_{q,r}(\mathbb{R}^m; L^p(\mathbb{R}^n)) = F^{s/b}_{q,p}(\mathbb{R}^m; L^p(\mathbb{R}^n))\) and \(F^{s/a}_{p,r}(\mathbb{R}^n) = B^{s/a}_{p,p}(\mathbb{R}^n)\), and we obtain an extension of the intersection representation \(1\) to decompositions \(\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m\):

\[
F^{s,(a,b)}_{(p,q),p}(\mathbb{R}^n \times \mathbb{R}^m) = \mathbb{F}^{s/b}_{q,p}(\mathbb{R}^m; L^p(\mathbb{R}^n)) \cap L^q(\mathbb{R}^m; B^{s/a}_{p,p}(\mathbb{R}^n)).
\]

In the special case that \(a = b\) and \(p = q\), the latter can be viewed as a special instance of Fubini property. In fact, the main result of this paper, Theorem 5.1 extends the well-known Fubini property for the classical Triebel-Lizorkin spaces \(F^{s}_p(\mathbb{R}^d)\) (see \[70\], Section 4) and the references given therein, see Remark 5.4. However, as seen in Theorem 1.1, the availability of Fubini is unessential for intersection representations, it should just be thought of as a way to simplify the function spaces that one has to deal with in case of its availability.

**Notation and convention.** We will write: \(\mathbb{N} = \{0, 1, 2, 3, \ldots\}\), \(\hat{f} = \mathcal{F} f\), \(\hat{f} = \mathcal{F}^{-1} f\), \(\mathbb{R}_+ = (0, \infty)\), \(\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Re}(z) > 0\}\), \(\ell^s_r(\mathbb{N}) = \{ (a_n)_{n \in \mathbb{N} \subseteq \mathbb{C}} : \sum_{n=0}^{\infty} 2^{sn} |a_n|^p < \infty\}\). Throughout the paper, we work over the field of complex scalars and fix a Banach space \(X\) and \(\sigma\)-finite measure space \((S, \mathcal{A}, \mu)\).

## 2. Preliminaries

### 2.1. Anisotropy and decomposition.

#### 2.1.1. Anisotropy on \(\mathbb{R}^d\).

An anisotropy on \(\mathbb{R}^d\) is a \(\mathbb{R}^d\) is a real \(d \times d\) matrix \(A\) with \(\sigma(A) \subset \mathbb{C}_+\). An anisotropy \(A\) on \(\mathbb{R}^d\) gives rise to a one-parameter group of expansive dilations \((A_t)_{t \in \mathbb{R}_+}\) given by

\[
A_t = t^A = \exp[A \ln(t)], \quad t \in \mathbb{R}_+,
\]

where \(\mathbb{R}_+\) is considered as multiplicative group.

In the special case \(A = \text{diag}(a)\) with \(a = (a_1, \ldots, a_d) \in (0, \infty)^d\), the associated one-parameter group of expansive dilations \((A_t)_{t \in \mathbb{R}_+}\) is given by

\[
A_t = \exp[A \ln(t)] = \text{diag}(t^{a_1}, \ldots, t^{a_d}), \quad t \in \mathbb{R}_+
\]

Given an anisotropy \(A\) on \(\mathbb{R}^d\), an \(A\)-homogeneous distance function is a Borel measurable mapping \(\rho : \mathbb{R}^d \to [0, \infty)\) satisfying

(i) \(\rho(x) = 0\) if and only if \(x = 0\) (non-degenerate);
The quasi-norm \( \rho \) follows: we put \( \rho(x) \leq c \rho(x + y) \) for all \( x, y \in \mathbb{R}^d \) (quasi-triangle inequality). The smallest such \( c \) is denoted \( c_\rho \).

Any two homogeneous quasi-norms \( \rho_1, \rho_2 \) associated with an anisotropy \( A \) on \( \mathbb{R}^d \) are equivalent in the sense that
\[
\rho_1(x) \sim_{\rho_1, \rho_2} \rho_2(x), \quad x \in \mathbb{R}^d.
\]

If \( \rho \) is a quasi-norm associated associated with an anisotropy \( A \) on \( \mathbb{R}^d \) and \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}^d \), then \( C^{\infty} \mathbb{R}^d \{0\} \). We write
\[
B^A(x, r) := B_{\rho_A}(x, r) = \{ y \in \mathbb{R}^d : \rho_A(x - y) \leq r \}, \quad x \in \mathbb{R}^d, r \in (0, \infty).
\]

We furthermore write \( c_A := c_{\rho_A} \).

Given an anisotropy \( A \) on \( \mathbb{R}^d \), we write
\[
\lambda^A_{\min} := \min\{ \text{Re} (\lambda) : \lambda \in \sigma(A) \}, \quad \lambda^A_{\max} := \max\{ \text{Re} (\lambda) : \lambda \in \sigma(A) \}.
\]

Note that \( 0 < \lambda^A_{\min} \leq \lambda^A_{\max} < \infty \). Given \( \varepsilon \in (0, \lambda^A_{\min}) \), it holds that
\[
t^{\lambda^A_{\min} - \varepsilon} \leq t \varepsilon \left| A_t x \right|, \quad |t| \geq 1,
\]
\[
t^{\lambda^A_{\min} + \varepsilon} \leq t \varepsilon \left| A_t x \right|, \quad |t| \leq 1,
\]
and
\[
t^{1/\lambda^A_{\min} - \varepsilon} \rho_A(x) \leq t \varepsilon \rho_A(tx), \quad |t| \geq 1,
\]
\[
t^{1/\lambda^A_{\min} + \varepsilon} \rho_A(x) \leq t \varepsilon \rho_A(tx), \quad |t| \leq 1.
\]

Furthermore,
\[
\rho_A(x) \lambda^A_{\min} \leq \rho_A(x) \lambda^A_{\min} + \varepsilon, \quad |x| \geq 1,
\]
\[
\rho_A(x) \lambda^A_{\max} \leq \rho_A(x) \lambda^A_{\max} + \varepsilon, \quad |x| \leq 1.
\]

An alternative viewpoint to anisotropy is as follows (see [12] and references given there), which is actually more general. A real \( d \times d \) matrix \( B \) is an expansive dilation if \( \min_{\lambda \in \sigma(B)} |\lambda| > 1 \). A quasi-norm associated with an expansive dilation \( B \) is a Borel measurable mapping \( \rho : \mathbb{R}^n \rightarrow [0, \infty) \) satisfying
\[
(i) \rho(x) = 0 \text{ if and only if } x = 0 \text{ (non-degenerate)};
\]
\[
(ii) \rho(Bx) = |\det(B)|\rho(x) \text{ for all } x \in \mathbb{R}^d, t \in \mathbb{R}_+ \text{ (B-homogeneous)};
\]
\[
(iii) \text{ there exists } c \in [1, \infty) \text{ so that } \rho(x + y) \leq c(\rho(x) + \rho(y)) \text{ for all } x, y \in \mathbb{R}^d \text{ (quasi-triangle inequality).} \]

The smallest such \( c \) is denoted \( c_\rho \).

If \( A \) is an anisotropy on \( \mathbb{R}^d \) and \( \rho \) is an A-homogeneous distance function, then \( B = A_2 = \exp[A \ln(2)] \) is an expansive dilation and \( \rho^B(x) := \rho(x)^{\text{tr}(A)} \) defines a quasi-norm associated with \( B \).
2.1.2. $d$-Decompositions and anisotropy. Let $d = (d_1, \ldots, d_\ell) \in (\mathbb{Z}_{\geq 1})^\ell$ be such that $d = |d|_1 = d_1 + \ldots + d_\ell$. The decomposition
\[
\mathbb{R}^d = \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_\ell}.
\]
is called the $d$-decomposition of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we accordingly write $x = (x_1, \ldots, x_\ell)$ and $x_j = (x_{j,1}, \ldots, x_{j,d_j})$, where $x_j \in \mathbb{R}^{d_j}$ and $x_{j,i} \in \mathbb{R}$ ($j = 1, \ldots, \ell; i = 1, \ldots, d_j$). We also say that we view $\mathbb{R}^d$ as being $d$-decomposed. Furthermore, for each $k \in \{1, \ldots, \ell\}$ we define the inclusion map
\[
\iota_k = \iota_{d,k} : \mathbb{R}^{d_k} \rightarrow \mathbb{R}^n, \ x_k \mapsto (0, \ldots, 0, x_k, 0, \ldots, 0),
\]
and the projection map
\[
\pi_k = \pi_{d,k} : \mathbb{R}^n \rightarrow \mathbb{R}^{d_k}, \ x = (x_1, \ldots, x_\ell) \mapsto x_k.
\]
A $d$-anisotropy is tuple $A = (A_1, \ldots, A_\ell)$ with each $A_j$ an anisotropy on $\mathbb{R}^{d_j}$. A $d$-anisotropy $A$ gives rise to a one-parameter group of expansive dilations $(A_t)_{t \in \mathbb{R}_+}$ given by
\[
A_t x = (A_{1,t} x_1, \ldots, A_{\ell,t} x_\ell), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R}_+,
\]
where $A_{j,t} = \exp[A_j \ln(t)]$. Note that $A^0 := \oplus_{j=1}^\ell A_j$ is an anisotropy on $\mathbb{R}^d$ with $A^0_i = A_i$ for every $i \in \mathbb{R}_+$. We define the $A^0$-homogeneous distance function $\rho_A$ by
\[
\rho_A(x) := \max\{\rho_{A_1 x}, \ldots, \rho_{A_\ell(x)}\}, \quad x \in \mathbb{R}^d.
\]
We write
\[
B^A(x, R) := B_{\rho_A}(x, R), \quad x \in \mathbb{R}^d, \ R \in [0, \infty),
\]
and
\[
B^A(x, R) := B^{A_1}(x_1, R_1) \times \ldots \times B^{A_\ell}(x_\ell, R_\ell), \quad x \in \mathbb{R}^d, \ R \in [0, \infty)^\ell.
\]
Note that $B^A(x, R) = B^A(x, R)$ when $R = (R, \ldots, R)$.

2.2. Quasi-Banach Function Spaces. For the theory of quasi-Banach spaces, or more generally, $F$-spaces, we refer the reader to [35, 36].

Let $Y$ be a vector space and $\kappa \in (0, 1]$. A $\kappa$-norm is a function $\|\cdot\|^\kappa : Y \rightarrow [0, \infty)$ with the following three properties:

(i) **Definiteness.** If $y \in Y$ satisfies $\|y\|^\kappa = 0$, then $y = 0$.

(ii) **Homogeneity.** $\|\lambda y\|^\kappa = |\lambda| \cdot \|y\|^\kappa$ for all $y \in Y$ and $\lambda \in \mathbb{C}$.

(iii) **$\kappa$-triangle inequality.** For all $y, z \in Y$,
\[
\|y + z\|^\kappa \leq \|y\|^\kappa + \|z\|^\kappa.
\]

Note that every $\kappa$-norm is a quasi-norm. The Aoki–Rolewitz theorem [6, 61] says that, conversely, given a quasi-normed space $(Y, \|\cdot\|)$ there exists $\kappa \in (0, 1]$ and an $r$-norm $\||\cdot||$ on $Y$ that is equivalent to $\|\cdot\|$.

Let $Y$ be a quasi-Banach space with a quasi-norm that is equivalent to some $\kappa$-norm, $\kappa \in (0, 1]$. If $(y_n)_n \subset Y$ satisfies $\sum_{n=0}^\infty \|y_n\|^\kappa < \infty$, then $\sum_{n=0}^\infty y_n$ converges in $Y$ and $\| \sum_{n=0}^\infty y_n \|_Y \leq \sum_{n=0}^\infty \|y_n\|_Y$.

Let $(T, \mathcal{F}, \nu)$ be a $\sigma$-finite measure space. A quasi-Banach function space $F$ on $T$ is an order ideal in $L_0(T)$ that has been equipped with a quasi-Banach norm $\||\cdot||$ with the property that $\||f|| = ||f||$ for all $f \in F$.

A quasi-Banach function space $F$ on $T$ has the Fatou property if and only if, for every increasing sequence $(f_n)_{n \in \mathbb{N}}$ in $F$ with supremum $f$ in $L_0(T)$ and $\text{sup}_{n \in \mathbb{N}} ||f_n||_F < \infty$, it holds that $f \in F$ with $||f||_F = \text{sup}_{n \in \mathbb{N}} ||f_n||_F$. 
2.3. Vector-valued Functions and Distributions. As general reference to the theory of vector-valued distributions we mention [3] (and [2] Section III.4)).

Let $G$ be a topological vector space. The space of $G$-valued tempered distributions $S'(\mathbb{R}^d; G)$ is defined as $S'(\mathbb{R}^d; G) := \mathcal{L}(S(\mathbb{R}^d), G)$, the space of continuous linear operators from the Schwartz space $S(\mathbb{R}^d)$ to $G$. In this chapter we equip $S'(\mathbb{R}^d; G)$ with the topology of pointwise convergence. Standard operators (derivative operators, Fourier transform, convolution, etc.) on $S'(\mathbb{R}^d; G)$ can be defined as in the scalar-case.

By a combination of [3, Theorem 1.4.3] and (the proof of) [3, Lemma 1.4.6], the space of finite rank operators $S(\mathbb{R}^d) \otimes G$ is sequentially dense in $S'(\mathbb{R}^d; G)$. Furthermore, as a consequence of the Banach-Steinhaus (see [3, Theorem 2.8]), if $G$ is sequentially complete, then so is $S'(\mathbb{R}^d; G)$.

Let $(T, \mathscr{B}, \nu)$ be a $\sigma$-finite measure space and let $G$ be a topological vector space. We define $L_0(T; G)$ as the space of all $\nu$-a.e. equivalence classes of $\nu$-strongly measurable functions $f : T \to G$. Suppose there is a system $Q$ of semi-quasi-norms generating the topology of $G$. We equip $L_0(T; G)$ with the topology generated by the semi-quasi-norms

$$
\rho_{B, q}(f) := \int_B (q(f) \wedge 1) \, d\nu, \quad B \in \mathscr{B}, \nu(B) < \infty, q \in Q.
$$

This topological vector space topology on $L_0(T; G)$ is independent of $Q$ and is called the topology of convergence in measure. Note that $L_0(T) \otimes G$ is sequentially dense in $L_0(T; G)$ as a consequence of the dominated convergence theorem and the definitions.

If $G$ is an $F$-space, then $L_0(T; G)$ is an $F$-space as well. Here we could for example take $G = L_{r, d, \text{loc}}(\mathbb{R}^d; X)$ with $r \in (0, \infty]^d$ and $X$ a Banach space, where

$$
L_{r, d, \text{loc}}(\mathbb{R}^d) = \{ f \in L_0(\mathbb{R}^d) : f 1_B \in L_{r, d}(\mathbb{R}^d), B \subset \mathbb{R}^d \text{ bounded Borel} \}
$$

and

$$
L_{r, d}(\mathbb{R}^d) = L_{r_1}(\mathbb{R}^{d_1}) \cdots [L_{r_1}(\mathbb{R}^{d_1})] \cdots
$$

Let $X$ be a Banach space. Then $L_0(T) \otimes S'(\mathbb{R}^d) \otimes X$ is sequentially dense in both of $L_0(T; S'(\mathbb{R}^d); X)$ and $S'(\mathbb{R}^d; L_0(T; X))$, while the two induced topologies on $L_0(T) \otimes S'(\mathbb{R}^d) \otimes X$ coincide. Therefore, we can naturally identify

$$
L_0(T; S'(\mathbb{R}^d); X) \cong S'(\mathbb{R}^d; L_0(T; X)).
$$

A function $g : T \to X^*$ is called $\sigma(X^*, X)$-measurable (or $X$-weakly measurable) if $\langle x, g \rangle : T \to \mathbb{C}$ is measurable for all $x \in X$. We denote by $L^0(T; X^*, \sigma(X^*, X))$ the vector space of all $\mu$-a.e. equivalence classes of $\sigma(X^*, X)$-measurable functions $g : T \to X^*$.

As $\{ \langle x, g \rangle : x \in B_X \}$ is order bounded in the Dedekind complete $L_0(T)$ for all $g \in L_0(T; X^*, \sigma(X^*, X))$, we may define the abstract norm $\nu : L_0(T; X^*, \sigma(X^*, X)) \to L_0(T)$ by

$$
\nu(g) := \sup \{ ||\langle x, g \rangle|| : x \in B_X \} \quad (g \in L_0(T; X^*, \sigma(X^*, X)));\]

see [53]. Note that $L_0(T; X^*) \subset L_0(T; X^*, \sigma(X^*, X))$ and that $\nu(g) = ||g||_{X^*}$ for all $g \in L_0(T; X^*)$. 

We equip \( L_0(T; X^*, \sigma(X^*, X)) \) with the topology generated by the system of semi-quasi-norms
\[
\rho_B(f) := \int_B (\vartheta(f) \wedge 1) d\nu, \quad B \in \mathcal{B}, \nu(B) < \infty.
\]

For a Banach function space \( E \) on \( T \) we define \( E(X^*, \sigma(X^*, X)) \) by
\[
E(X^*, \sigma(X^*, X)) := \{ f \in L_0(T; X^*, \sigma(X^*, X)) : \vartheta(f) \in E \}.
\]
Endowed with the norm
\[
\|f\|_{E(X^*, \sigma(X^*, X))} := \|\vartheta(f)\|_E,
\]
\( E(X^*, \sigma(X^*, X)) \) becomes a Banach space.

Let \( E \) be a Banach function space on \( T \) with an order continuous norm. Then (see \[24\])
\[
[E(X)]^* = E^\times(X^*, \sigma(X^*, X))
\]
under the natural pairing, where \( E^\times \) is the Köthe dual of \( E \) given by
\[
E^\times = \{ g \in L_0(T) : \forall f \in E, fg \in L_1(T) \}, \quad \|g\|_{E^\times} = \sup_{f \in E, \|f\|_E \leq 1} \int_T fg \, d\nu.
\]
Moreover, if \( X^* \) has the Radon–Nykodým property with respect to \( \nu \), then
\[
[E(X)]^* = E^\times(X^*, \sigma(X^*, X)) = E^\times(X^*).
\]

3. Definitions and Basic Properties

Suppose that \( \mathbb{R}^d \) is \( \delta \)-decomposed with \( \delta \in (\mathbb{Z}_{\geq 1})^\ell \) and let \( \mathbf{A} = (A_1, \ldots, A_\ell) \) be a \( \delta \)-anisotropy. Let \( X \) be a Banach space, \((S, \mathcal{A}, \mu)\) a \( \sigma \)-finite measure space, \( \varepsilon_+, \varepsilon_- \in \mathbb{R} \) and \( r \in (0, \infty)^l \).

For \( j \in \{1, \ldots, \ell\} \), we define the maximal function operator \( M_{r_j; [\varepsilon; j]}^{A_j} \) on \( L_0(S \times \mathbb{R}^d) \) by
\[
M_{r_j; [\varepsilon; j]}^{A_j}(f)(s, x) := \sup_{\delta > 0} \int_{B_{2\delta}} |f(s, x + t_{[\varepsilon; j]}y_j)| \, dy_j.
\]
We define the maximal function operator \( M_r^{\mathbf{A}} \) by iteration:
\[
M_r^{\mathbf{A}}(f) := M_{r_1; [\varepsilon; 1]}^{A_1}(M_{r_2; [\varepsilon; 2]}^{A_2}(\ldots (M_{r_\ell; [\varepsilon; \ell]}^{A_\ell}(f)) \ldots).
\]

The following definition is an extension of \[32\] Definition 1.1.1] to the anisotropic setting with some extra underlying measure space \((S, \mathcal{A}, \mu)\). The extra measure space provides the right setting for intersection representations, see Section \[5\].

**Definition 3.1.** We define \( S(\varepsilon_+, \varepsilon_-, \mathbf{A}, r, (S, \mathcal{A}, \mu)) \) as the set of all quasi-Banach function spaces \( E \) on \( \mathbb{R}^d \times \mathbb{N} \times S \) with the Fatou property for which the following two properties are fulfilled:
\begin{enumerate}[(a)]
\item \( S_+, S_- \in \mathcal{B}(E) \), the left respectively right shift on \( \mathbb{N} \times S \), with
\[
\|(S_+)^k\|_{\mathcal{B}(E)} \lesssim 2^{\varepsilon_- k} \quad \text{and} \quad \|(S_-)^k\|_{\mathcal{B}(E)} \lesssim 2^{\varepsilon_+ k}, \quad k \in \mathbb{N}.
\]
\item \( M_r^{\mathbf{A}} \) is bounded on \( E \):
\[
\|M_r^{\mathbf{A}}(f_n)\|_E \lesssim \|(f_n)\|_E, \quad (f_n) \in E.
\]
\end{enumerate}

We similarly define \( S(\varepsilon_+, \varepsilon_-, \mathbf{A}, r) \) without the presence of \((S, \mathcal{A}, \mu)\), or equivalently, \( S(\varepsilon_+, \varepsilon_-, \mathbf{A}, r) = S(\varepsilon_+, \varepsilon_-, \mathbf{A}, r, \{0\}, \{\emptyset, \{0\}\}, \#) \).
Remark 3.2. Note that $\varepsilon_+ \leq \varepsilon_-$ when $E \neq \{0\}$, which can be seen by considering $(S_+)^k \circ (S_-)^k$, $k \in \mathbb{N}$.

Remark 3.3. Note that

$$S(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \subset S(\varepsilon_+, \varepsilon_-, A, \bar{r}, (S, \mathcal{A}, \mu)), \quad r \geq \bar{r}.$$ 

Example 3.4. Let us provide some examples of $E \in S(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu))$. Condition (b) in Definition 3.1 can be covered by means of the lattice Hardy-Littlewood maximal function operator: if $F$ is a UMD Banach function space on $S$, $A$ an anisotropy, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^d, A)$ then (see [11, 25, 37, 62])

$$Mf(x) := \sup_{\delta > 0} \int_{B^A(x, \delta)} |f(y)| \, dy$$

defines a bounded sublinear operator on $L_p(\mathbb{R}^d, w; F) = L_p(\mathbb{R}^d, w)[F]$. The latter induces a bounded sublinear operator on $L_p(\mathbb{R}^d, w)[F[\ell_\infty]]$ in the natural way. Let us furthermore remark that the mixed-norm space $F^r$ of two UMD Banach function spaces $F$ and $G$ is again a UMD Banach function space (see [62, page 214]).

This leads to the following examples of:

(i) Let $p \in (0, \infty)^\ell$, $q \in (0, \infty]$, $w \in \coprod_{j=1}^\ell A_\infty(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. If $r \in (0, \infty)^\ell$ is such that $r_j < p_1 \wedge \cdots \wedge p_\ell \wedge q$ for $j = 1, \ldots, \ell$ and $w \in \coprod_{j=1}^\ell A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$, then

$$E = L_p(\mathbb{R}^d, w)[\ell_q^r(N)] \subset S(s, s, A, r).$$

(ii) Let $p \in (0, \infty)^\ell$, $q \in (0, \infty]$, $w \in \coprod_{j=1}^\ell A_\infty(\mathbb{R}^{d_j}, A_j)$ and $s \in \mathbb{R}$. If $r \in (0, \infty)^\ell$ is such that $r_j < p_1 \wedge \cdots \wedge p_j \wedge q$ for $j = 1, \ldots, \ell$ and $w \in \coprod_{j=1}^\ell A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$, then

$$E = \ell_q^r(N)[L_p(\mathbb{R}^d, w)] \subset S(s, s, A, r).$$

(iii) Let $p \in (0, \infty)^\ell$, $q \in (0, \infty]$ and $w \in \coprod_{j=1}^\ell A_\infty(\mathbb{R}^{d_j}, A_j)$, $s \in \mathbb{R}$ and $F$ a quasi-Banach function space on $S$. If $r \in (0, \infty)^\ell$ is such that $r_j < p_1 \wedge \cdots \wedge p_j + q$ for $j = 1, \ldots, \ell$ and $w \in \coprod_{j=1}^\ell A_{p_j/r_j}(\mathbb{R}^{d_j}, A_j)$ and $F_{\text{max}}^r$ is a UMD Banach function space,

$$F^r := \{f \in L_0(S) : |f|^{1/r} \in F\}, \quad ||f||_{F^r} := |||f||_{F}^{1/r},$$

then

$$E = L_p(\mathbb{R}^d, w)[F[\ell_q^r(N)]] \subset S(s, s, A, r, (S, \mathcal{A}, \mu)).$$

For a quasi-Banach function space $E$ on $\mathbb{R}^d \times \mathbb{N} \times S$ we define the quasi-Banach function space $E^A_\otimes$ on $S$ by

$$||f||_E^A_\otimes := ||1_{B^{A}(0,1) \otimes 0} \otimes f||_E, \quad f \in L_0(S).$$

For a quasi-Banach function space $E$ on $\mathbb{R}^d \times \mathbb{N} \times S$ we define the quasi-Banach function space $E_\otimes$ on $S$ by

$$||f||_E_\otimes := ||1_{B^{A}(0,1) \otimes 0} \otimes f||_E, \quad f \in L_0(S).$$

Let $p \in (0, \infty)^\ell$ and $w : [1, \infty)^\ell \to (0, \infty)$. We define the quasi-Banach function space

$$(6) \quad B_A^{p,w} := \{f \in L_0(S) : \sup_{R \in [1, \infty)^\ell} w(R) ||f||_{L_p, \ell((B^A(0,R))} < \infty\}$$
Lemma 3.5. Let \( B \) be the quasi-Banach function space \( B^{p,q} \) introduced in [60] (see [60]).

\[
w_{A,q}(R) := R^{-\tau(A)q^{-1}} = \prod_{j=1}^{\ell} R_j^{-\tau(A_j)/q_j}, \quad R \in [1, \infty)^\ell.
\]

The quasi-Banach function space \( B^{p,q} \) will be convenient to formulate some of the estimates we will obtain. Note that, if \( p \in [1, \infty) \), then

\[
B^{p,q}(X) \hookrightarrow L^p(d\mu) \hookrightarrow L^{q,p}(d\mu) \hookrightarrow L^{q,p}(d\mu) \hookrightarrow L^{q,p}(d\mu).
\]

Lemma 3.5. Let \( E \in S(\epsilon_+, \epsilon_-, A, r, (S, \mathcal{A}, \mu)) \) and \( \lambda \in (-\infty, \epsilon_+) \). For \( F = (f_n)_{n \in E} \) and \( g := \sum_{n=0}^{\infty} 2^{-n\lambda}|f_n| \) we have

\[
\|g\|_{E^w} \lesssim \|F\|_{E}.
\]

Moreover, \( g \in E^w_A[B^{r,w,A,r}] \hookrightarrow E^w_A[L_r, d\mu(\mathbb{R}^d)] \) with

\[
\|g\|_{E^w_A[B^{r,w,A,r}]} \lesssim \|F\|_{E}.
\]

Remark 3.6. Suppose that \( \epsilon_+ > 0 \) and \( \lambda \in (0, \epsilon_+) \) in Lemma 3.5. Let \( \kappa \in (0, 1] \) with \( \kappa \leq \tau_{\min} \) be such that \( \|\cdot\|_E \) is equivalent to a \( \kappa \)-norm. Then, in particular,

\[
\sum_{n=0}^{\infty} 2^{-n\lambda \kappa} \|2^n f_n\|_{E^w_A[B^{r,w,A,r}]} \lesssim \|F\|_{E}.
\]

Remark 3.7. Let \( E \in S(\epsilon_+, \epsilon_-, A, r, (S, \mathcal{A}, \mu)) \). Similarly to the proof of Lemma 3.5 (but simpler) it can be shown that

\[
E \hookrightarrow E^w_A[B^{r,w,A,r}].
\]

Proof of Lemma 3.5. This can be shown similarly to [32] Lemma 1.1.4. Let us just provide the details for (5). As \(|B^{A_j}(x_j, R_j)| \approx R_j^{\tau(A_j)/r_j}, \quad j = 1, \ldots, \ell \), for any \( x \in \mathbb{R}^d \) and \( R \in (0, \infty)^\ell \), we have

\[
1_{B^A(0,R)} \otimes \|g\|_{L_{r,A}(B^A(0,R))} \lesssim \prod_{j=1}^{\ell} R_j^{\tau(A_j)/r_j} M^{A}(g), \quad R \in [1, \infty)^\ell.
\]

Therefore,

\[
1_{B^A(0,1)} \otimes w_{A,r}(R)\|g\|_{L_{r,A}(B^A(0,R))} \lesssim M^{A}(g), \quad R \in [1, \infty)^\ell,
\]

so that

\[
1_{B^A(0,1)} \otimes \|g\|_{B^{w,A,r}} \lesssim M^{A}(g).
\]

It thus follows that

\[
\|g\|_{E^w_A[B^{r,w,A,r}]} = \|1_{B^A(0,1)} \times \{0\} \otimes \|g\|_{B^{r,w,A,r}}\|_E \lesssim \|M^{A}(\delta_{0,n}g)_n\|_E.
\]

Using the boundedness of \( M^{A} \) on \( E \) in combination with (7) we obtain the desired estimate (8).
Definition 3.8. Suppose that $\epsilon_+, \epsilon_- > 0$ and let $E \in \mathcal{S}(\epsilon_+, \epsilon_-, A, r, (S, \mathcal{A}, \mu))$. We define $YL^A(E; X)$ as the space of all $f \in L_0(S; L_{r,d,\text{loc}}(\mathbb{R}^d; X))$ which have a representation

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; L_{r,d,\text{loc}}(\mathbb{R}^d; X))$$

with $(f_n)_n \subset L_0(S; S'(\mathbb{R}^d; X))$ satisfying the Fourier support condition

$$\text{supp } \hat{f}_n \subset B^A(0, 2^{n+1}), \quad n \in \mathbb{N},$$

and $(f_n)_n \in E(X)$. We equip $YL^A(E; X)$ with the quasinorm

$$\|f\|_{YL^A(E; X)} := \inf \|\langle f_n \rangle\|_{E(X)},$$

where the infimum is taken over all representations as above.

Definition 3.9. Suppose that $\epsilon_+, \epsilon_- > 0$ and let $E \in \mathcal{S}(\epsilon_+, \epsilon_-, A, r, (S, \mathcal{A}, \mu))$. We define $\overline{YL}^A(E; X)$ as the space of all $f \in L_0(S; L_{r,d,\text{loc}}(\mathbb{R}^d; X))$ for which there exists $(g_n)_n \in E_+$ such that, for all $x^* \in X^*$, $(f, x^*)$ has a representation

$$\langle f, x^* \rangle = \sum_{n=0}^{\infty} \langle f_n, x^* \rangle \quad \text{in} \quad L_0(S; L_{r,d,\text{loc}}(\mathbb{R}^d))$$

with $(f_n, x^*)_n \subset L_0(S; S'(\mathbb{R}^d))$ satisfying the Fourier support condition

$$\text{supp } \hat{f}_n \subset B^A(0, 2^{n+1}), \quad n \in \mathbb{N},$$

and the domination $|f_n, x^*_n| \leq ||x^*|||g_n|$. We equip $\overline{YL}^A(E; X)$ with the quasinorm

$$\|f\|_{\overline{YL}^A(E; X)} := \inf \|\langle g_n \rangle\|_{E},$$

where the infimum is taken over all $(g_n)_n$ as above.

Remark 3.10. Suppose that $\epsilon_+, \epsilon_- > 0$ and let $E \in \mathcal{S}(\epsilon_+, \epsilon_-, A, r, (S, \mathcal{A}, \mu))$. Then the following statements hold:

(i) $YL^A(E; X) \subset \overline{YL}^A(E; X)$ with induced norm.

(ii) Let $f \in YL^A(E; X)$ with $(f_n)_n$ as in Definition 3.8 with $\|(f_n)_n\|_{E(X)} \leq 2\|f\|_{YL^A(E; X)}$. Let $\tilde{r} \in (0, \infty)^f$ be such that

$$E \in \mathcal{S}(\epsilon_+, \epsilon_-, A, \tilde{r}, (S, \mathcal{A}, \mu)).$$

Then, by Remark 3.6 as

$$E^A_0(B^{\tilde{r}, wA, r}_A(X)) \subset L_0(S; L_{\tilde{r},d,\text{loc}}(\mathbb{R}^d; X)) \subset L_0(S; L_{r,\text{loc}}(\mathbb{R}^d; X)),$$

there is the convergence $f = \sum_{n=0}^{\infty} f_n$ in $E^A_0(B^{\tilde{r}, wA, r}_A(X))$ with

$$\|f\|_{E^A_0(B^{\tilde{r}, wA, r}_A(X))} \leq \|(f_n)_n\|_{E(X)} \leq 2\|f\|_{YL^A(E; X)}.$$

In particular, $YL^A(E; X)$ does not depend on $r$ and

$$YL^A(E; X) \hookrightarrow E^A_0(B^{\tilde{r}, wA, r}_A(X)).$$

(iii) Let $f \in \overline{YL}^A(E; X)$ with $(g_n)_n \in E_+$ and $(f, x^*)_n \in E_+$ as in Definition 3.9 with $\|(g_n)_n\|_{E} \leq 2\|f\|_{\overline{YL}^A(E; X)}$. Let $\tilde{r} \in (0, \infty)^f$ satisfy (9). Then $\|f\|_X \leq \sum_{n=0}^{\infty} g_n$, so that $f \in E^A_0(B^{\tilde{r}, wA, r}_A(X)) \subset L_0(S; L_{\tilde{r},d,\text{loc}}(\mathbb{R}^d; X))$ with

$$\|f\|_{E^A_0(B^{\tilde{r}, wA, r}_A(X))} \leq \|(g_n)_n\|_{E} \leq 2\|f\|_{\overline{YL}^A(E; X)}.$$

by Remark 3.6 By (11) it furthermore holds that
\[ \langle f, x^* \rangle = \sum_{n=0}^{\infty} f_{x^*,n} \quad \text{in} \quad L_0(S; L_{\tilde{\tau}, \tilde{\varepsilon}, \text{loc}}(\mathbb{R}^d)). \]

Therefore, \( \tilde{Y}L^A(E; X) \) does not depend on \( r \) and
\[ \tilde{Y}L^A(E; X) \hookrightarrow E_{\otimes}(B^{r,w_A,r}(X)). \]

**Definition 3.11.** Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \). We define \( Y^A(E; X) \) as the space of all \( f \in L_0(S; S'(\mathbb{R}^d; X)) \) which have a representation
\[ f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L_0(S; S'(\mathbb{R}^d; X)) \]
with \((f_n)_n \subset L_0(S; S'(\mathbb{R}^d; X))\) satisfying the Fourier support condition
\[ \supp \hat{f}_0 \subset \overline{B^A}(0, 2) \]
\[ \supp \hat{f}_n \subset \overline{B^A}(0, 2^{n+1}) \setminus B^A(0, 2^{n-1}), \quad n \geq 1, \]
and \((f_n)_n \in E(X). We equip \( Y^A(E; X) \) with the quasinorm
\[ ||f||_{Y^A(E; X)} := \inf ||(f_n)||_{E(X)}, \]
where the infimum is taken over all representations as above.

**Proposition 3.12.** Suppose that \( \varepsilon_+, \varepsilon_- > 0 \) and let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)). \) Then \( YL^A(E; X) \) and \( \overline{Y}L^A(E; X) \) are quasi-Banach spaces with
\[ YL^A(E; X) \subset \overline{Y}L^A(E; X) \hookrightarrow E_{\otimes}(B^{r,w_A,r}(X)), \]
where \( YL^A(E; X) \) is a closed subspace of \( \overline{Y}L^A(E; X). \)

**Proof.** By Remark 3.10
\[ (10) \quad YL^A(E; X), \overline{Y}L^A(E; X) \hookrightarrow E_{\otimes}(B^{r,w_A,r}(X)). \]
That \( YL^A(E; X) \subset \overline{Y}L^A(E; X) \) with \( ||f||_{YL^A(E; X)} = ||f||_{\overline{Y}L^A(E; X)} \) for all \( f \in YL^A(E; X) \) follows easily from the definitions. So it remains to be shown that \( YL^A(E; X) \) and \( \overline{Y}L^A(E; X) \) are complete.

Let us first treat \( YL^A(E; X). \) To this end, let the subspace \( E(X)_A \) of \( E(X) \) be defined by
\[ E(X)_A := \left\{ (f_n)_n \in E(X) : f_n \in L_0(S; S'(\mathbb{R}^d; X)), \supp \hat{f}_n \subset \overline{B^A}(0, 2^{n+1}) \right\} \]
By Lemma 3.5
\[ \Sigma : E(X)_A \rightarrow E_{\otimes}(L_r(\mathbb{R}^d, w))(X) \rightarrow L_0(S; L_{r, \tilde{\varepsilon}, \text{loc}}(\mathbb{R}^d; X)), \quad (f_n)_n \mapsto \sum_{n=0}^{\infty} f_n \]
is a well-defined continuous linear mapping. As
\[ YL^A(E; X) \simeq E(X)_A/\ker(\Sigma) \quad \text{isometrically}, \]

it suffices to show that \( E(X)_A \) is complete.
In order to show that $E(X)_A$ is complete, we prove that it is a closed subspace of the quasi-Banach space $E(X)$. Put $w(x) := \prod_{j=1}^{\ell} (1 + \rho_A(x_j))^{tr(A_j)/\tau_j}$. Then it is enough to show that, for each $k \in \mathbb{N}$,

$$E(X)_A \rightarrow L_0(S; BC(\mathbb{R}^d, w; X)), \ (f_n)_n \mapsto f_k,$$

continuously, where $BC(\mathbb{R}^d, w; X) = \{h \in C(\mathbb{R}^d; X) : wh \in L_{\infty}(\mathbb{R}^d; X)\}$. Indeed, $BC(\mathbb{R}^d, w; X) \hookrightarrow S'(\mathbb{R}^d; X)$.

In order to establish \[(\ref{eq:11})\], let $(f_n) \in E(X)_A$. By Corollary A.2

$$\sup_{z \in B^A(0,2^{-n})} ||f_n||_X \lesssim M^A_r(||f_n||_X)(x),$$

so that

$$||f_n(x)||_X \lesssim \inf_{z \in B^A(0,2^{-n})} M^A_r(||f_n||_X)(x+z) \lesssim 2^{ntr(A) \cdot r^{-1}} ||M^A_r(||f_n||_X)||_{L_{r^{-1}}(BA(x,2^{-n}))}.$$ 

For $R \in [1, \infty)^\ell$ we can thus estimate

$$\sup_{z \in B^A(0,R)} ||f_n(x)||_X \lesssim 2^{ntr(A) \cdot r^{-1}} ||M^A_r(||f_n||_X)||_{L_{r^{-1}}(BA(0,\min(cA+1)R))} \lesssim 2^{ntr(A) \cdot r^{-1}} \inf_{z \in B^A(0,R)} ||M^A_r(||f_n||_X)||_{L_{r^{-1}}(BA(0,\min(cA+1)R))} \lesssim 2^{ntr(A) \cdot r^{-1}} R^{\text{tr}(A) \cdot r^{-1}} \inf_{z \in B^A(0,R)} M^A_r(||f_n||_X)(z).$$

The latter implies that

$$1_{BA(0,R)} \otimes ||f_n||_{L_{\infty}(BA(0,R); X)} \lesssim 2^{ntr(A) \cdot r^{-1}} R^{\text{tr}(A) \cdot r^{-1}} ||M^A_r(||f_n||_X)||_{L_{r^{-1}}(BA(0,\min(cA+1)R))}$$

for $R \in [1, \infty)^\ell$. It thus follows that

$$||f_n||_{E^A_\oplus(L_{\infty}(BA(0,R); X))} \leq \left\| 1_{BA(0,R)} \otimes ||f_n||_{L_{\infty}(BA(0,R); X)} \right\|_E \lesssim 2^{ntr(A) \cdot r^{-1}} R^{\text{tr}(A) \cdot r^{-1}} \left( ||(\delta_0,k M^A_r(||f_n||_X))k ||_E \right) \lesssim 2^{ntr(A) \cdot r^{-1} - \epsilon} R^{\text{tr}(A) \cdot r^{-1}} ||(h_k)k ||_E.$$ 

Let us finally prove that $\hat{Y}^A \rightarrow (E; X)$ is complete. To this end, let $\kappa \in (0,1]$ with $\kappa \leq r_{\text{min}}$ be such that $|| \cdot ||_E$ is equivalent to a $\kappa$-norm. Then $|| \cdot ||_{\hat{Y}^A(E; X)}$ and $|| \cdot ||_{E^A_\oplus(L_{\tau}(\mathbb{R}^d, w)))}$ are equivalent to $\kappa$-norms as well. It suffices to show that, if $(f^{(k)})_{k \in \mathbb{N}} \subset \hat{Y}^A \rightarrow (E; X)$ satisfies $\sum_{k=0}^{\infty} ||f^{(k)}||_{\hat{Y}^A(E; X)} < \infty$, then $\sum_{k=0}^{\infty} f^{(k)}$ is a convergent series in $\hat{Y}^A \rightarrow (E; X)$. So fix such a $(f^{(k)})_{k \in \mathbb{N}}$. As a consequence of \[(\ref{eq:11})\],

$$\sum_{k=0}^{\infty} ||f^{(k)}||_{E^A_\oplus(L_{\tau}(\mathbb{R}^d, w))} \lesssim \sum_{k=0}^{\infty} ||f^{(k)}||_{\hat{Y}^A(E; X)} < \infty.$$ 

As $E^A_\oplus[L_{\tau}(\mathbb{R}^d, w))]$ is a quasi-Banach space with a $\kappa$-norm, $\sum_{k=0}^{\infty} f^{(k)}$ converges to some $F \in E^A_\oplus[L_{\tau}(\mathbb{R}^d, w)]$. To finish the proof, we show that $F \in \hat{Y}^A \rightarrow (E; X)$ with convergence $F = \sum_{k=0}^{\infty} f^{(k)}$ in $\hat{Y}^A \rightarrow (E; X)$. 

For each $k \in \mathbb{N}$ there exists $(g^{(k)}_{n})_{n} \in E_{+}$ with $||(g^{(k)}_{n})_{n}||_{E} \leq 2||f^{(k)}||_{\tilde{Y}L^{A}(E;X)}$ such that, for every $x^{*} \in X^{*}$, $(f^{(k)}, x^{*})$ has the representation

$$(f^{(k)}, x^{*}) = \sum_{n=0}^{\infty} f_{x^{*},n}^{(k)} \text{ in } L_{0}(S; L_{r,\delta,\text{loc}}(\mathbb{R}^{d}))$$

for some $(f_{x^{*},n}^{(k)})_{n} \in E_{A}$ with $|f_{x^{*},n}^{(k)}| \leq ||x^{*}||g^{(k)}_{n}$. By Remark 3.10

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} ||f_{x^{*},n}^{(k)}||_{E_{0}^{\infty}[L_{r}(\mathbb{R}^{d}, w)]} \lesssim \sum_{k=0}^{\infty} ||f^{(k)}||_{L_{r}^{A}(E;X)} < \infty.$$ 

As $E_{0}^{\infty}[L_{r}(\mathbb{R}^{d}, w)] \hookrightarrow L_{0}(S; L_{r,\delta,\text{loc}}(\mathbb{R}^{d})) \hookrightarrow L_{0}(S \times \mathbb{R}^{d})$ is a quasi-Banach space with a $\kappa$-norm, we thus find that $F = \sum_{n=0}^{\infty} f_{x^{*},n} \in L_{0}(S; L_{r,\delta,\text{loc}}(\mathbb{R}^{d}))$ with $F_{x^{*},n} := \sum_{k=0}^{\infty} f_{x^{*},n}^{(k)}$ in $L_{0}(\mathbb{R}^{d} \times S)$ satisfying $|F_{x^{*},n}| \leq \sum_{k=0}^{\infty} |f_{x^{*},n}^{(k)}| \leq ||x^{*}|| \sum_{k=0}^{\infty} g^{(k)}_{n}$. As $E_{A}$ is a closed subspace of the quasi-Banach function space $E$ on $\mathbb{R}^{d} \times \mathbb{N} \times S$ with $\kappa$-norm, it follows from

$$\sum_{k=0}^{\infty} ||(f_{x^{*},n}^{(k)})_{n}||_{E} \leq ||x^{*}|| \sum_{k=0}^{\infty} ||f^{(k)}||_{\tilde{Y}L^{A}(E;X)} < \infty$$

that $(F_{x^{*},n})_{n} = \sum_{k=0}^{\infty} f_{x^{*},n}^{(k)}$ in $E$ and thus that $(F_{x^{*},n})_{n} \in E_{A}$. Moreover, $G_{n} := \sum_{k=0}^{\infty} g^{(k)}_{n}$ defines $(G_{n})_{n} \in E_{+}$ with

$$||(G_{n})_{n}||_{E} \leq \sum_{k=0}^{\infty} ||(g^{(k)}_{n})_{n}||_{E} \leq 2 \sum_{k=0}^{\infty} ||f^{(k)}||_{\tilde{Y}L^{A}(E;X)}$$

and $|F_{x^{*},n}| \leq ||x^{*}||G_{n}$. This shows that $F \in \tilde{Y}L^{A}(E;X)$ with convergence $F = \sum_{k=0}^{\infty} f^{(k)}$ in $\tilde{Y}L^{A}(E;X)$.

The content of the following proposition is a Littlewood-Paley characterization for $Y^{A}(E;X)$. Before we state it, we first need to introduce the set $\Phi^{A}(\mathbb{R}^{d})$ of all $A$-anisotropic Littlewood-Paley sequences $\varphi = (\varphi_{n})_{n} \in \mathbb{N}$.

**Definition 3.13.** For $0 < \gamma < \delta < \infty$ we define $\Phi^{A}_{\gamma, \delta}(\mathbb{R}^{d})$ as the set of all sequences $\varphi = (\varphi_{n})_{n} \in \mathcal{S}(\mathbb{R}^{d})$ that can be constructed in the following way: given $\varphi_{0} \in \mathcal{S}(\mathbb{R}^{d})$ satisfying

$$0 \leq \varphi_{0} \leq 1, \quad \varphi_{0}(\xi) = 1 \text{ if } \rho_{A}(\xi) \leq \gamma, \quad \varphi_{0}(\xi) = 0 \text{ if } \rho_{A}(\xi) \geq \delta,$$

$(\varphi_{n})_{n} \geq 1 \subset \mathcal{S}(\mathbb{R}^{d})$ is obtained through

$$\varphi_{n} = \varphi_{1}(A_{2^{-n+1}} \cdot) - \varphi_{0}(A_{2^{-n}} \cdot) - \varphi_{0}(A_{2^{-n+1}} \cdot), \quad n \geq 1.$$

We define $\Phi^{A}(\mathbb{R}^{d}) := \bigcup_{0 \leq \gamma < \delta < \infty} \Phi^{A}_{\gamma, \delta}(\mathbb{R}^{d})$.

Let $\varphi = (\varphi_{n})_{n} \in \Phi^{A}_{\gamma, \delta}(\mathbb{R}^{d})$. Then $\sum_{n=0}^{\infty} \varphi_{n} = 1$ in $\mathcal{E}(\mathbb{R}^{d})$ with

$$\text{supp } \varphi_{0} \subset \{ \xi : \rho_{A}(\xi) \leq \gamma \}, \quad \text{supp } \varphi_{n} \subset \{ \xi : 2^{n-1}\gamma \leq \rho_{A}(\xi) \leq 2^{n}\delta \}, \quad n \geq 1.$$
Proposition 3.14. Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \) and \( \varphi = (\varphi_n)_{n \in \mathbb{N}} \in \Phi^A(\mathbb{R}^d) \) with associated sequence of convolution operators \((S_n)_{n \in \mathbb{N}}\). Then
\[
Y^A(E; X) = \{ f \in L_0(S; S'(\mathbb{R}^d; X)) : (S_n f)_n \in E(X) \}
\]
with
\[
||f||_{Y^A(E; X)} \approx ||(S_n f)_n||_{E(X)}.
\]

Before we go to the proof of Proposition 3.14, let us first consider:

Example 3.15. In the following three points we let the notation be as in Example 3.4(i), Example 3.4(ii) and Example 3.4(iii), respectively. We define:

(i) \( F_{p,q}^s(\mathbb{R}^d, w; X) := Y^A(E; X) \) for \( E = L_p(\mathbb{R}^d, w)[\ell_q^d(\mathbb{N})] \);
(ii) \( B_{p,q}^s(\mathbb{R}^d, w; X) := Y^A(E; X) \) for \( E = \ell_q^d(\mathbb{N})[L_p(\mathbb{R}^d, w)] \);
(iii) \( \mathbb{F}^s_{p,q}(\mathbb{R}^d, w; F; X) := Y^A(E; X) \) for \( E = L_p(\mathbb{R}^d, w)[F(\ell_q^d(\mathbb{N})] \).

Restricting to special cases we find, in view of Proposition 3.14, \( B \)- and \( F \)-spaces that have been studied in the literature:

- In case \( \ell = 1, w = 1 \) and \( X = \mathbb{C} \), \( F_{p,q}^s(\mathbb{R}^d, w; X) \) and \( B_{p,q}^s(\mathbb{R}^d, w; X) \) reduce to the anisotropic Besov and Triebel-Lizorkin spaces considered in e.g. [20, 23]. The latter are special cases of the anisotropic spaces from the more general [7, 12, 13] by taking \( 2^A \) as the expansive dilation in the approach there.
- In case \( \ell = d, A = \text{diag}(a) \) with \( a \in (0, \infty) \), \( w = 1 \) and \( X = \mathbb{C} \), \( F_{p,q}^s(\mathbb{R}^d, w; X) \) and \( B_{p,q}^s(\mathbb{R}^d, w; X) \) reduce to the anisotropic mixed-norm Besov and Triebel-Lizorkin spaces considered in e.g. [33, 54].
- In case \( A = (a_1 I_d, \ldots, a_l I_d) \) with \( a \in (0, \infty) \), \( F_{p,q}^s(\mathbb{R}^d, w; X) \) and \( B_{p,q}^s(\mathbb{R}^d, w; X) \) reduce to the anisotropic weighted mixed-norm Besov and Triebel-Lizorkin spaces considered in [12, 16].
- In case \( \ell = 1 \) and \( A = I, F_{p,q}^s(\mathbb{R}^d, w; X) \) and \( B_{p,q}^s(\mathbb{R}^d, w; X) \) reduce to the weighted Besov and Triebel-Lizorkin spaces considered in e.g. [16, 14, 13, 25, 29, 30, 41, 44, 45, 69] (\( X = \mathbb{C} \)) and [31, 52, 53] (\( X \) a general Banach space). In the case \( w = 1 \) these further reduces to the classical Besov and Triebel-Lizorkin spaces (see e.g. [64, 68, 69]).

- In case \( \ell = 1, A = I, p \in (1, \infty), q \in [1, \infty], w = 1, F \) is a UMD Banach function space and \( X = \mathbb{C} \), \( F_{p,q}^s(\mathbb{R}^d, w; F; X) \) reduces to a special case of the generalized Triebel-Lizorkin spaces considered in [39].

- In case \( \ell = 1, A = I, p \in (1, \infty), q = 2, w \in A_p(\mathbb{R}^d), F \) is a UMD Banach function space and \( X \) is a Hilbert space, \( F_{p,q}^s(\mathbb{R}^d, w; F; X) \) coincides with the weighted Bessel potential space \( H_p^s(\mathbb{R}^d, w; F(X)) \) (which can be seen as a special case of [53, Proposition 3.2]).

The proof of Proposition 3.14 basically only consists of proving the estimate in the following lemma. We have extracted it as a lemma as it is interesting on its own. A consequence of the lemma for instance is that the Fourier support condition in Definition 3.11 could be slightly modified.

Lemma 3.16. Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \), \( c \in (1, \infty) \) and \( \varphi = (\varphi_n)_{n \in \mathbb{N}} \in \Phi^A(\mathbb{R}^d) \) with associated sequence of convolution operators \((S_n)_{n \in \mathbb{N}}\). For all \( f \in L_0(S; S'(\mathbb{R}^d; X)) \) which have a representation
\[
f = \sum_{n=0}^{\infty} f_n \text{ in } L_0(S; S'(\mathbb{R}^d; X))
\]
with \((f_n)_n \subset L_0(S; S'(\mathbb{R}^d; X))\) satisfying the Fourier support condition

\[
\text{supp} \hat{f}_0 \subset \overline{B}^A(0, c) \\
\text{supp} \hat{f}_n \subset \overline{B}^A(0, c2^n) \setminus B^A(0, c^{-1}2^n), \quad n \geq 1,
\]

there is the estimate

\[
\|(S_n f)_n\|_{E(X)} \lesssim \|(f_n)_n\|_{E(X)}.
\]

**Proof.** This can be established as in [42 Lemma 5.2.10] (also see [68 Section 2.3.2] and [71 Section 15.5]), using a combination of Corollary A.2 and Lemma A.3 □

**Proof of Proposition 3.14** Let \(f \in Y^A(E; X)\). Take \((f_n)_n\) as in Definition 3.13 with \(\|(f_n)_n\|_{E(X)} \lesssim 2\|f\|_{Y^A(E; X)}\). Lemma 3.16 (with \(c = 2\)) then gives

\[
\|(S_n f)_n\|_{E(X)} \lesssim \|(f_n)_n\|_{E(X)} \lesssim 2\|f\|_{Y^A(E; X)}.
\]

For the reverse direction, let \((f_n)_n \subset L_0(S; S'(\mathbb{R}^d; X))\) be such that \((S_n f)_n \in E(X)\).

Pick \(\psi = (\psi_n)_{n \in \mathbb{N}} \in \Phi^A(\mathbb{R}^d)\) such that

\[
\text{supp} \hat{\psi}_0 \subset \overline{B}^A(0, 2), \quad \text{supp} \hat{\psi}_n \subset \overline{B}^A(0, 2^{n+1}) \setminus B^A(0, 2^{n-1}), \quad n \geq 1,
\]

and let \((T_n)_{n \in \mathbb{N}}\) denote the associated sequence of convolution operators. Then

\[
\text{supp} \hat{T}_0 f \subset \overline{B}^A(0, 2), \quad \text{supp} \hat{T}_n f \subset \overline{B}^A(0, 2^n) \setminus B^A(0, 2^{n-1}), \quad n \geq 1,
\]

Picking \(c \in (1, \infty)\) such that

\[
\text{supp} \hat{\phi}_0 \subset \overline{B}^A(0, c), \quad \text{supp} \hat{\phi}_n \subset \overline{B}^A(0, c2^n) \setminus B^A(0, c^{-1}2^n), \quad n \geq 1,
\]

we furthermore have

\[
\text{supp} \hat{S}_0 f \subset \overline{B}^A(0, c), \quad \text{supp} \hat{S}_n f \subset \overline{B}^A(0, c2^n) \setminus B^A(0, c^{-1}2^n), \quad n \geq 1.
\]

As \(f = \sum_{n=0}^{\infty} S_n f \in L_0(S; S'(\mathbb{R}^d; X))\), Lemma 3.16 gives

\[
\|(T_n f)_n\|_{E(X)} \lesssim \|(S_n f)_n\|_{E(X)}.
\]

Since \(f = \sum_{n=0}^{\infty} S_n f \in L_0(S; S'(\mathbb{R}^d; X))\) with (13), it follows that \(f \in Y^A(E; X)\) with

\[
\|f\|_{Y^A(E; X)} \lesssim \|(T_n f)_n\|_{E(X)} \lesssim \|(S_n f)_n\|_{E(X)}.
\]

**Theorem 3.17.** Let \(E \in S(\varepsilon_+ , \varepsilon_-, A, r, (S, \mathcal{A}, \mu))\). Suppose that \(\varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - 1)_+\). Then

\[
\text{Y}_L^A(E; X) \hookrightarrow E_0^A(B_{A}^{1, wA, r^{-1}}(X)) \hookrightarrow L_0(S; L_{1, r, \mathcal{D}, \text{loc}}(\mathbb{R}^d; X))
\]

and

\[
Y^A(E; X) \hookrightarrow E_0^A(B_{A}^{1, wA, r^{-1}}(X)) \hookrightarrow S'(\mathbb{R}^d; E_0^A(X))
\]

\[
\hookrightarrow S'(\mathbb{R}^d; L_0(S; X)) = L_0(S; S'(\mathbb{R}^d; X))
\]

and there is the identity

\[
Y^A(E; X) = YL^A(E; X) = \text{Y}_L^A(E; X).
\]

We will use the following lemma in the proof of Theorem 3.17.
Lemma 3.18. Let the notations and assumptions be as in Theorem 3.17. Let $c \in (0, \infty)$. If 
\[(f_n)_n \in E(X)_A, \infty \] 
then $\sum_{n \in \mathbb{N}} f_n$ is a convergent series in $L_0(S; B^{1,w,A,r\vee 1}_A(X))$ with

$$\left\| \sum_{n=0}^{\infty} f_n \right\|_{E_A(B^{1,w,A,r\vee 1}_A(X))} \leq \left\| \sum_{n=0}^{\infty} \| f_n \|_X \right\|_{E_A(B^{1,w,A,r\vee 1}_A(X))} \approx \| (f_n)_n \|_{E(X)}.$$ 

Proof. It suffices to prove the second estimate. We may without loss of generality assume that $r \in (0, 1]^\ell$. Choose $\kappa > 0$ such that $E_A^2$ has a $\kappa$-norm. For simplicity of notation we only present the case $\ell = 2$, the general case being the same. For simplicity we furthermore restrict ourselves to the case $c = 1$.

Let $(f_n)_n \in E(X)_A$. Let $R \in [1, \infty)^2$. As a consequence of the Paley-Wiener-Schwartz theorem,

$$\tilde{\mathcal{E}}_{B^{1}_A,(0,2^r)}(\mathbb{R}^d, X) \hookrightarrow C^\infty(\mathbb{R}^{d_2}, \tilde{\mathcal{E}}_{B^{1}_A,(0,2^r)}(\mathbb{R}^{d_2}, X)) \cap C^\infty(\mathbb{R}^{d_1}, \tilde{\mathcal{E}}_{B^{1}_A,(0,2^r)}(\mathbb{R}^{d_1}, X)).$$

In particular, as in [12] we find that

\[(17) \quad \| f_n(x_1, z_2) \|_X \lesssim (2^n R_1)^{\text{tr}(A_1)/r_1} M_{A_1}^{A_1}(r_1, |x_1|; |z_2|) \| f_n \|_X(y_1, z_2) \]

for all $x_1, y_1 \in B^{A_1}(0, R_1)$ and $z_1 \in \mathbb{R}^{d_1}$, and

\[(18) \quad \| f_n(x_2, z_2) \|_X \lesssim (2^n R_2)^{\text{tr}(A_2)/r_2} M_{A_2}^{A_2}(r_2, |x_2|; |z_2|) \| f_n \|_X(y_1, z_2) \]

for all $x_2, y_2 \in B^{A_2}(0, R_2)$ and $z_2 \in \mathbb{R}^{d_2}$.

Then, for $z \in B^{A_1}(0, R)$,

\[
\int_{B^{A_1}(0, R)} \| f_n(x) \|_X \, dx
\]

\[
= \int_{B^{A_2}(0, R_2)} \int_{B^{A_1}(0, R_1)} \| f_n(x_1, x_2) \|_X \, dx_1 \, dx_2
\]

\[
\approx (2^n R_1)^{\text{tr}(A_1)/r_1} \int_{B^{A_2}(0, R_2)} M_{A_1}^{A_1}(r_1, |x_2|; |z_2|) \| f_n \|_X(y_1, z_2) \]

\[
\lesssim 2^{n \text{tr}(A_1)(1/r_1 - 1)} R_1^{\text{tr}(A_1)/r_1} \int_{B^{A_2}(0, R_2)} M_{A_1}^{A_1}(r_1, |x_2|; |z_2|) \| f_n \|_X(y_1, z_2) \]

\[
\lesssim 2^{n \text{tr}(A_1)(1/r_1 - 1)} R_1^{\text{tr}(A_1)/r_1} \int_{B^{A_2}(0, R_2)} M_{A_1}^{A_1}(r_1, |x_2|; |z_2|) \| f_n \|_X(y_1, z_2) \]

This implies that

$$1_{B^A(0, R)} \otimes \int_{B^A(0, R)} \sum_{n=0}^{\infty} \| f_n \|_X \, dx \lesssim R_1^{\text{tr}(A_1)/r_1 - 1} \int_{B^{A_2}(0, R_2)} \sum_{n=0}^{\infty} 2^n \| f_n \|_X \lesssim R_1^{\text{tr}(A_1)/r_1 - 1} \sum_{n=0}^{\infty} 2^n \| f_n \|_X.$$
Since $\varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - 1)_+$, it follows that

$$
\| \sum_{n=0}^{\infty} \| f_n \|_X \|_{E_A^A(B_\Lambda^{1,w,A^{\ast}\vee 1})} \lesssim \| ([M^A]^{1\|([f_n]\|_X)n})_E \lesssim \| (f_n)\| E(X).$$

Proof of Theorem 3.17. We may without loss of generality assume that $r \in (0, 1]^t$. As $L_0(S; B_\Lambda^{1,w,A^{\ast}\vee 1}(X)) \hookrightarrow L_0(S; S'(\mathbb{R}^d; X))$, the first inclusion in (15) follows from Lemma 3.18. So in (15) it remains to prove the second inclusion. To this end, let us first note that $S(\mathbb{R}^d) \hookrightarrow B(B_\Lambda^{1,w,A^{\ast}\vee 1}(X), X), \phi \mapsto \langle \cdot, \phi \rangle$.

This induces $S(\mathbb{R}^d) \hookrightarrow B(E_A^A(B_\Lambda^{1,w,A^{\ast}\vee 1}(X)), E_A^A(X)), \phi \mapsto \langle \cdot, \phi \rangle$.

Therefore, $f \mapsto [\phi \mapsto \langle f, \phi \rangle]$ is a continuous linear operator from $E_A^A(B_\Lambda^{1,w,A^{\ast}\vee 1}(X))$ to $L(S(\mathbb{R}^d); E_A^A(X))$, which is a reformulation of the required inclusion.

As $L_0(S; B_\Lambda^{1,w,A^{\ast}\vee 1}) \hookrightarrow L_0(S; L_{r,d,loc}(\mathbb{R}^d))$, the inclusion

$$
Y^A(E) \hookrightarrow E_A^A(B_\Lambda^{1,w,A^{\ast}\vee 1})
$$

follows from Lemma 3.18. We thus get a continuous bilinear mapping

$$
\bar{Y}L^A(E, X) \times X^* \rightarrow YL^A(E) \hookrightarrow L_0(S; S'(\mathbb{R}^d)), (f, x^*) \mapsto \langle f, x^* \rangle.
$$

and a continuous linear mapping

$$
(19) \quad \bar{Y}L^A(E, X) \rightarrow L_0(S; S'(\mathbb{R}^d, X^{**})), f \mapsto T_f,
$$

defined by

$$
\langle x^*, T_f(\phi) \rangle := \langle f, x^* \rangle(\phi), \quad \phi \in S(\mathbb{R}^d), x^* \in X^*.
$$

Let us now show that $f \mapsto T_f$ (19) restricts to a bounded linear mapping

$$
(20) \quad \bar{Y}L^A(E, X) \rightarrow Y^A(E; X^{**}), f \mapsto T_f.
$$

To this end, let $f \in \bar{Y}L^A(E, X)$ and put $F := T_f$. Let $(g_n)_n$ and $(f_{x^*,n})_{x^*,n}$ be as in Definition 3.9 with $\|\| (g_n)_n\|_E \leq 2 \|f\| \bar{Y}L^A(E, X)$. It will convenient to put $g_n := 0$ and $f_{x^*,n} := 0$ for $n \in \mathbb{Z}_{<0}$. By Lemma 3.18 as $(f_{x^*,n})_n \in E_A$ and $B_\Lambda^{1,w,A^{\ast}\vee 1} \hookrightarrow S'(\mathbb{R}^d)$,

$$
\langle f, x^* \rangle = \sum_{k=0}^{\infty} f_{x^*,k} \quad \text{in} \quad L_0(S; B_\Lambda^{1,w,A^{\ast}\vee 1}) \hookrightarrow L_0(S; S'(\mathbb{R}^d)), \quad x^* \in X^*.
$$

Now let $(S_n)_n \in \mathbb{N}$ be as in Proposition 3.14. There exists $h \in \mathbb{N}$ independent of $f$ such that $S_n f_{x^*,k} = 0$ for all $x^* \in X^*, n \in \mathbb{N}$ and $k \in \mathbb{Z}_{<0}$. Let $x^* \in X^*$. Then

$$
\langle x^*, S_n F \rangle = S_n \langle x^*, F \rangle = S_n \langle f, x^* \rangle = S_n \sum_{k=0}^{\infty} f_{x^*,k} = \sum_{k=0}^{\infty} S_n f_{x^*,k}
$$

$$
= \sum_{k=0}^{\infty} S_n f_{x^*,k} = \sum_{k=0}^{\infty} S_n f_{x^*,k+n-h}.
$$
with convergence in $L_0(S; S'([R^d]))$. Together with Corollary [A.6] this implies the pointwise estimates
\[
|\langle x^*, S_n F \rangle| \leq \sum_{k=0}^{\infty} |S_n f_{x^*, k+n-h}| \leq \sum_{k=0}^{\infty} 2^{(k-h)+\text{tr}(A)\cdot(r-1)-1} M^A_r(f_{k+n-h, x^*})
\]
\[
\leq \||x^*|| \sum_{k=0}^{\infty} 2^{(k-h)+\text{tr}(A)\cdot(r-1)-1} M^A_r(g_{k+n-h}).
\]
Taking the supremum over $x^* \in X^*$ with $||x^*|| \leq 1$, we obtain
\[
||S_n F||_{X^*} \leq \sum_{k=0}^{\infty} 2^{(k-h)+\text{tr}(A)\cdot(r-1)-1} M^A_r(g_{k+n-h}).
\]

Picking $\kappa > 0$ such that $E$ has a $\kappa$-norm, we find that
\[
||S_n F||_{E(X^*)} = \left(\sum_{k=0}^{\infty} ||S_n f||_{X^*}^n \right)^{1/\kappa}
\]
\[
\leq \sum_{k=0}^{\infty} 2^{(k-h)+\text{tr}(A)\cdot(r-1)-1} \||M^A_r(g_{k+n-h})||_{E}^\kappa
\]
\[
\leq \sum_{k=0}^{\infty} 2^{(k-h)+\text{tr}(A)\cdot(r-1)-1} \||M^A_r(g_{k+n-h})||_{E}^\kappa.
\]
Since
\[
||M^A_r(g_{k+n-h})||_{E} = \|(g_{k+n-h})_n\|_E \lesssim \begin{cases} \|(S_-)^{h-k}(g_n)_n\|_E, & k \leq h, \\ \|(S_+)^{h-k}(g_{k+n-h})_n\|_E, & k \geq h,
\end{cases}
\]
\[
\lesssim \left(2^{-\varepsilon+(\varepsilon-h)_+} + 2^{\varepsilon+(h-k)_+} \right)\|(g_n)_n\|_E
\]
\[
\lesssim 2^{-\varepsilon+(\varepsilon-h)_+} ||f||_{\overline{Y}^A_r(E; X)}
\]
for all $k \in \mathbb{N}$, it follows that
\[
||S_n F||_{E(X^*)} \lesssim \sum_{k=0}^{\infty} 2^{(k-h)+\text{tr}(A)\cdot(r-1)-1-\varepsilon} \||f||_{\overline{Y}^A_r(E; X)}.\n\]

As $\varepsilon_+ > \text{tr}(A) \cdot (r-1)$, we find that $||S_n F||_{E(X^*)} \lesssim ||f||_{\overline{Y}^A_r(E; X)}$ and thus $F \in Y^A_r(E; X^*)$ with $||F||_{Y^A_r(E; X^*)} \lesssim ||f||_{\overline{Y}^A_r(E; X)}$ (see Proposition [3.14]).

So we obtain the desired (20).

Next we prove that
\[
\overline{Y}^A_r(E; X) \hookrightarrow Y^A_r(E; X).
\]

So let $f \in \overline{Y}^A_r(E; X)$. A combination of (20) and (16) gives that $F := T_f \in L_0(S; X^{**})$. Since $f \in L_0(S; L_{r,d,\text{loc}}([R^d]; X))$ with $\langle x^*, F \rangle = \langle f, x^* \rangle$ for every $x^* \in X^*$, it follows that
\[
f = F \in L_0(S; B^{1,wA, r=1}_A(X^{**})) \cap L_0(S; L_{r,d,\text{loc}}([R^d]; X)) \subset L_0(S; B^{1,wA, r=1}_A(X)).
\]

Therefore, by boundedness of (20),
\[
\overline{Y}^A_r(E; X) \hookrightarrow \{g \in Y^A_r(E; X^{**}) : g \in L_0(S; S'(R^d, X))\} = Y^A_r(E; X).
\]

For a quasi-Banach function space $E$ on $[R^d \times N \times S$ and a number $\sigma \in [0, \infty)$ we define the quasi-Banach function space $E^\sigma$ on $[R^d \times N \times S$ by
\[
||(f_n)_n||_{E^\sigma} := ||(2^{n\sigma} f_n)_n||_E, \quad (f_n)_n \in L_0([R^d \times N \times S]).
\]
Note that $E^n \in S(\varepsilon_+\sigma, \varepsilon_-\sigma, A, r, (S, \mathcal{A}, \mu))$ when $E \in S(\varepsilon_+\sigma, \varepsilon_-\sigma, A, r, (S, \mathcal{A}, \mu))$.

**Proposition 3.19.** Let $E \in S(\varepsilon_+\sigma, \varepsilon_-\sigma, A, r, (S, \mathcal{A}, \mu))$ and $\sigma \in \mathbb{R}$. Let $\psi \in \Theta_M(\mathbb{R}^d)$ be such that $\psi(\xi) = \rho_A(\xi)$ for $\rho_A(\xi) \geq 1$ and $\psi(\xi) \neq 0$ for $\rho_A(\xi) \leq 1$. Then $\phi(D) \in \mathcal{L}(L_0(S; S'(\mathbb{R}^d, X)))$ restricts to an isomorphism

$$\phi(D) : Y^A(E^n; X) \xrightarrow{\cong} Y^A(E; X).$$

**Proof.** Using Proposition 3.14 and Lemma A.3, this can be proved as [42, Lemma 5.2.28] (also see [68, Theorem 2.3.8]).

**Lemma 3.20.** Let $V$ be a quasi-normed space continuously embedded into a complete topological vector space $W$. Suppose that $V$ has the Fatou property with respect to $W$, i.e. for all $(v_n)_{n \in \mathbb{N}} \subset V$ the following implication holds:

$$\liminf_{n \to \infty} v_n = v \text{ in } W, \liminf_{n \to \infty} ||v_n||_V < \infty \implies v \in V, ||f||_V \leq \liminf_{n \to \infty} ||f_n||_V.$$  

Then $V$ is complete.

**Proof.** Suppose that $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $V$. Then, on the one hand, $\lim_{n \to \infty} ||v_n||_V \leq \sup_n ||v_n||_V < \infty$. On the other hand, $(v_n)_{n \in \mathbb{N}}$ is also a Cauchy sequence in the complete topological vector space $W$ because of $V \hookrightarrow W$, whence converges to some $v \in W$. By the Fatou property of $V$ with respect to $W$, $v \in V$. To finish the proof we show that we also have convergence $v_n \xrightarrow{n \to \infty} v$ with respect to the quasi-norm of $V$. To this end, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $||v_l - v_k||_V \leq \varepsilon$ for all $l, k \geq N$. Then, for all $k \geq N$, it holds that $v_l - v_k \in E$, $\liminf_{l \to \infty} ||v_l - v_k||_V \leq \varepsilon$ and $v_l - v_k \xrightarrow{l \to \infty} v - v_k$ in $W$. So applying, for each $k \geq N$, the Fatou property of $V$ (with respect to $W$) to the sequence of differences $(v_l - v_k)_{l \geq N}$ we obtain that $||v_l - v_k||_V \leq \varepsilon$ for all $l \geq N$.

**Proposition 3.21.** Let $E \in S(\varepsilon_+\sigma, \varepsilon_-\sigma, A, r, (S, \mathcal{A}, \mu))$. Then

$$Y^A(E; X) \hookrightarrow S'(\mathbb{R}^d; E^A_0(X)) \hookrightarrow S'(\mathbb{R}^d; L_0(S; X)) = L_0(S; S'(\mathbb{R}^d; X))$$  

and $Y^A(E; X)$, when equipped with an equivalent quasi-norm from Proposition 3.14, has the Fatou property with respect to $L_0(S; S'(\mathbb{R}^d; X))$. As a consequence, $Y^A(E; X)$ is a quasi-Banach space.

**Proof.** The chain of inclusions follow from a combination of Theorem 3.17 and Proposition 3.19.

In order to establish the Fatou property, suppose that $Y^A(E; X)$ has been equipped with an equivalent quasi-norm from Proposition 3.14. Let $f_k \to f$ in $L_0(S; S'(\mathbb{R}^d; X))$ with $\liminf_{k \to \infty} ||f_k||_{Y^A(E; X)} < \infty$. Then

$$S_nf = \lim_{k \to \infty} S_nf_k \text{ in } L_0(S; \Theta_M(\mathbb{R}^d; X)) \hookrightarrow L_0(S; L_{1, loc}(\mathbb{R}^d; X)) \hookrightarrow L_0(\mathbb{R}^d \times S; X),$$

so that

$$(S_nf)_{n \in \mathbb{N}} = \lim_{k \to \infty} (S_nf_k)_{n \in \mathbb{N}} \text{ in } L_0(\mathbb{R}^d \times S; X).$$

By passing to a suitable subsequence we may without loss of generality assume that $(S_nf_k)_{n \in \mathbb{N}} \to (S_nf)_{n \in \mathbb{N}}$ pointwise a.e. as $k \to \infty$. Using the Fatou property of $E$, we find

$$||f||_{Y^A(E; X)} = ||(||S_nf||_X)_n||_E = ||\liminf_{k \to \infty} (||S_nf_k||_X)_n||_E \leq \liminf_{k \to \infty} ||(||S_nf_k||_X)_n||_E = \liminf_{k \to \infty} ||f_k||_{Y^A(E; X)}. \quad \square$$
4. Difference Norms

In this section we derive several estimates for \(YL^A(E; X)\) and \(\tilde{Y}L^A(E; X)\). The main interest lies in the estimates involving differences, as these form the basis for the intersection representation in Section 5.

4.1. **Some notation.** Let \(E\) be a Banach space. For each \(M \in \mathbb{N}_{\geq 1}\) and \(h \in \mathbb{R}^d\) we define difference operator \(\Delta^M_h\) on \(L_0(\mathbb{R}^d; X)\) by \(\Delta^M_h := (L_h - I)^M = \sum_{i=0}^{M} (-1)^i \binom{M}{i} L_{(M-i)h}\), where \(L_h\) denotes the left translation by \(h\):

\[
\Delta^M_h f = \sum_{i=0}^{M} (-1)^i \binom{M}{i} f(\cdot + (M-i)h), \quad f \in L_0(\mathbb{R}^d; X).
\]

For \(N \in \mathbb{N}\) we denote by \(\mathcal{P}_N\) the space of polynomials of degree at most \(N\) on \(\mathbb{R}^d\). We write \(\mathcal{P}_N(Q) \subset \mathcal{P}_N\) for the subset of polynomials having rational coefficients.

Let \(M \in \mathbb{N}_{\geq 1}\). Let \(F = L_{p,d} = L_{p,d}(\mathbb{R}^d)\) with \(p \in (0, \infty)^d\). Let \(B \subset \mathbb{R}^d\) be a bounded Borel set of non-zero measure. For \(f \in L_0(\mathbb{R}^d)\) we define

\[
\mathcal{E}_M(f, B, F) := \inf_{\pi \in \mathcal{P}_{N-1}} \| (f - \pi)1_B \|_F = \inf_{\pi \in \mathcal{P}_{N-1}(Q)} \| (f - \pi)1_B \|_F
\]

and

\[
\mathcal{E}_M(f, B, F) := \frac{\mathcal{E}_M(f, B, F)}{\mathcal{E}_M(1, B, F)}.
\]

We define the collection of dyadic anisotropic cubes \(\{Q^A_{n,k}\}_{(n,k) \in \mathbb{Z} \times \mathbb{Z}^d}\) by

\[
Q^A_{n,k} := A_{2^n} \left( (0,1)^d + k \right).
\]

For \(b \in (0, \infty)\) we define \(\{Q^A_{n,k}(b)\}_{(n,k) \in \mathbb{Z} \times \mathbb{Z}^d}\) by

\[
Q^A_{n,k}(b) := A_{2^n} \left( (0,1)^d(b) + k \right),
\]

where \((0,1)^d(b)\) is the cube concentric to \((0,1)^d\) with sidelength \(b\):

\[
(0,1)^d(b) := \left[ \frac{1+b}{2}, \frac{1-b}{2} \right)^d.
\]

We furthermore define the corresponding families of indicator functions \(\{\chi^A_{n,k}\}_{(n,k) \in \mathbb{Z} \times \mathbb{Z}^d}\) and \(\{\chi^A_{n,k}(b)\}_{(n,k) \in \mathbb{Z} \times \mathbb{Z}^d}\):

\[
\chi^A_{n,k} := 1_{Q^A_{n,k}} \quad \text{and} \quad \chi^A_{n,k}(b) := 1_{Q^A_{n,k}(b)}.
\]

**Definition 4.1.** Let \(E \in \mathcal{S}(\mathcal{F}, r, (S, \mathcal{A}, (s_n^A)_{n \in \mathbb{N}}))\). We define \(y^A(E)\) as the space of all \((s_{n,k})_{(n,k) \in \mathbb{N} \times \mathbb{Z}^d} \subset L_0(S)\) for which \((\sum_{k \in \mathbb{Z}^d} s_{n,k} \chi^A_{n,k})_{n \in \mathbb{N}} \in E\). We equip \(y^A(E)\) with the quasi-norm

\[
\|(s_{n,k})_{(n,k) \in \mathbb{N} \times \mathbb{Z}^d}\|_{y^A(E)} := \left\| \left( \sum_{k \in \mathbb{Z}^d} s_{n,k} \chi^A_{n,k} \right)_n \right\|_E.
\]

**Definition 4.2.** Let \(F\) be a quasi-Banach function space on the \(\sigma\)-finite measure space \((T, \mathcal{B}, \nu)\). We define \(\mathcal{F}_M(X; F)\) as the space of all \(\{F_x\}_{x^* \in X^*} \subset L_0(T)\) for which there exists \(G \in F_+\) such that \(\|F_x\| \leq \|x^*\|G\). We equip \(\mathcal{F}_M(X; F)\) with the quasi-norm

\[
\|(F_x)_{x^*}\|_{\mathcal{F}_M(X^*; F)} := \inf \|G\|_F,
\]

where the infimum is taken over all majorants \(G\) as above.
In the special case that \( F = E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \) in the above definition, it will be convenient to view \( \mathcal{F}_M(X^*: E) \) as the space of all \( \{g_{x^*, n}(x^*, n) \in X^* \times N \subset L_0(S) \) for which there exists \( (g_n)_n \in E_+ \) such that \( |g_{x^*, n}| \leq \|x^*\|g_n \), equipped with the quasi-norm
\[
\|\{g_{x^*, n}(x^*, n)\|\mathcal{F}_M(X^*: E) := \inf \|g_n\|_E,
\]
where the infimum is taken over all majorants \( (g_n)_n \) as above.

Note that the corresponding properties from Definition 3.1 for \( \mathcal{F}_M(X^*: E) \) are inherited from \( E \).

**Definition 4.3.** Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \). We define \( \widetilde{\mathcal{F}}^A(E; X) \) as the space of all \( (s_{x^*, n,k}(x^*, n,k) \in X^* \times \mathbb{N} \times \mathbb{Z}^d \subset L_0(S) \) for which \( (\sum_{k \in \mathbb{Z}^d} s_{x^*, n,k} \chi_{n,k})_n \in \mathcal{F}_M(X^*: E) \). We equip \( \widetilde{\mathcal{F}}^A(E; X) \) with the quasi-norm
\[
\|(s_{x^*, n,k})(n,k)\|\widetilde{\mathcal{F}}^A(E; X) := \left\|\left( \sum_{k \in \mathbb{Z}^d} s_{x^*, n,k} \chi_{n,k} \right)_n \right\|\mathcal{F}_M(X^*: E).
\]

**4.2. Statements of the results.**

**Theorem 4.4.** Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \) and suppose that \( \varepsilon_+, \varepsilon_- > 0 \). Let \( p \in (0, \infty) \) and \( M \in \mathbb{N} \) satisfy \( \varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - p^{-1}) \) and \( M\lambda_{\text{min}}^A > \varepsilon_- \). Given \( f \in L_0(S; L_{p,d}(\mathbb{R}^d; X)) \), consider the following statements:

(i) \( f \in Y^A(E; X) \).

(ii) There exist \( (s_{n,k}(n,k))_n \in Y^A(E) \) and \( (b_{n,k}(n,k))_n \in \mathcal{S}_0(S; C^M([-1, 2]^d)) \) with \( |||b_{n,k}|||_{C^M} \leq 1 \) such that, setting \( a_{n,k} := b_{n,k}(A_{2^k} \cdot -k) \), \( f \) has the representation
\[
(22) \quad f = \sum_{(n,k) \in \mathbb{N} \times \mathbb{Z}^d} s_{n,k} a_{n,k} \quad \text{in} \quad L_0(S; L_{p,d,\text{loc}}(\mathbb{R}^d; X)).
\]

(iii) \( f \in E_0(X) \cap L_0(S; L_{p,d,\text{loc}}(\mathbb{R}^d; X)) \) and \( (d_A^p(f)_n)_{n \geq 1} \in E(\mathbb{N}_{\geq 1}) \), where
\[
d_A^p(f)_n := 2^{n\text{tr}(A)p^{-1}} \|z \mapsto \Delta_z^M f\|_{L_{p,d}(B^A(0,2^{-n}); X)}, \quad n \in \mathbb{N}.
\]

Then \( \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iii) \). Moreover, there are the following estimates:
\[
||f||_{E_0(X)} + ||(d_A^p(f))_n\|_{E(\mathbb{N}_{\geq 1})} \lesssim ||f||_{Y^A(E; X)} \sim ||(s_{n,k}(n,k))_n||_{\mathcal{F}^A(E)}.
\]

**Theorem 4.4** is partial extension of [32 Theorem 1.1.14], which is concerned with \( YL(E) \) with \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, I, r) \). That result actually extends completely to the anisotropic scalar-valued setting \( Y^A(E) \) with \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r) \). However, in the general Banach space-valued case there arises a difficulty due to the unavailability of the Whitney inequality [32 (1.2.2)/Theorem A.1] (see [73, 74]) and the derived Lemma 4.11. We overcome this issue in Theorem 4.5 by extending [32 Theorem 1.1.14] to \( Y^A(E; X) \). This was actually the motivation for introducing the space \( Y^A(E; X) \), which is connected to \( Y^A(E; X) \) and \( Y^A(E; X) \) through Theorem 4.5.

**Theorem 4.5.** Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \) and suppose that \( \varepsilon_+, \varepsilon_- > 0 \). Let \( p \in (0, \infty) \) and \( M \in \mathbb{N} \) satisfy \( \varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - p^{-1}) \) and \( M\lambda_{\text{min}}^A > \varepsilon_- \). Given \( f \in L_0(S; L_{p,d}(\mathbb{R}^d; X)) \), consider the following statements:

(I) \( f \in \overline{Y^A(E; X)} \).

(II) There exist \((s_{x^*,n,k})_{(n,k)} \in \tilde{\mathcal{F}}(A; X)\) and \((b_{x^*,n,k})_{(x^*,n,k) \in X^* \times \mathbb{N} \times \mathbb{Z}^d} \subset L_0(S; C^M([-1,2]^d))\) with \(\|b_{x^*,n,k}\|_{C^M} \leq 1\) such that, setting \(a_{x^*,n,k} := b_{x^*,n,k}(A_{2n} - k)\), for all \(x^* \in X^*\), \((f, x^*)\) has the representation
\[
\langle f, x^* \rangle = \sum_{(n,k) \in \mathbb{N} \times \mathbb{Z}^d} s_{x^*,n,k}a_{x^*,n,k} \quad \text{in} \quad L_0(S; L_{p,d,loc}(\mathbb{R}^d)).
\]

(III) \(f \in E_0(X) \cap L_0(S; L_{p,d,loc}(\mathbb{R}^d; X))\) and
\[
\{d_{m,x^*,n}^A(f)\}_{(x^*,n) \in X^* \times \mathbb{N} \geq 1} \in \mathcal{F}(A; E(N \geq 1)),
\]
where
\[
d_{m,x^*,n}^A(f) := 2^{\text{tr}(A) \cdot p^{-1}} \|z \mapsto \Delta_{x^*}^M(f)(z)\|_{L_{p,d}(BA(0,2^{-n}))}, \quad n \in \mathbb{N}.
\]

(IV) \(f \in E_0(X)\) and there is \(\{\pi_{x^*,n}^A\}_{(x^*,n) \in X^* \times \mathbb{N} \geq 1} \in \mathcal{P}_{d-1}\) such that
\[
g_{x^*,n} := \sum_{k \in \mathbb{Z}^d} \|\langle f, x^* \rangle - \pi_{x^*,n}^A1_{Q_{n,k}^A(1)}\|_{E(N \geq 1)}, \quad n \geq 1,
\]
satisfies \(\{g_{x^*,n}\}_{(x^*,n) \in X^* \times \mathbb{N} \geq 1} \in \mathcal{F}(A; E(N \geq 1))\).

For \(f \in L_0(S; L_{r,d,loc}(\mathbb{R}^d; X))\) it holds that \(\text{[V]} \Rightarrow \text{[I]} \Rightarrow \text{[III]} \& \text{[IV]}\) with corresponding estimates
\[
\|f\|_{E_0(X)} + \|d_{m,x^*,n}^A(f)\|_{\mathcal{F}(X^*; E)} + \|E_{m,x^*,n}^A(f)\|_{(x^*,n) \in X^* \times \mathbb{N} \geq 1} \|\mathcal{F}(X^*; E(N \geq 1))\|
\leq \|f\|_{Y_{L,A}(E; X)} \approx \|(s_{x^*,n,k})_{(x^*,n,k)}\|_{\mathcal{F}(E)}
\leq \|f\|_{E_0(X)} + \|\{g_{x^*,n}\}_{(x^*,n) \in X^* \times \mathbb{N} \geq 1} \|\mathcal{F}(X^*; E(N \geq 1))\|
\leq \|f\|_{E_0(X)} + \|\{d_{m,x^*,n}^A(f)\}_{(x^*,n) \in X^* \times \mathbb{N} \geq 1} \|_{E(N \geq 1)}.
\]

Moreover, for \(f\) of the form \(f = \sum_{i \in I} 1_{S_i} \otimes f[i]\) with \((S_i)_{i \in I} \subset \mathcal{A}\) a countable family of mutually disjoint sets and \((f[i])_{i \in I} \in L_{r,d,loc}(\mathbb{R}^d; X)\), it holds that \(\text{[I]}, \text{[III]}, \text{[IV]}, \text{[III]}\) and \(\text{[IV]}\) are equivalent statements and there are the corresponding estimates
\[
\|f\|_{Y_{L,A}(E; X)} \approx \|f\|_{E_0(X)} + \|\{d_{m,x^*,n}^A(f)\}_{(x^*,n) \in X^* \times \mathbb{N} \geq 1} \|_{\mathcal{F}(X^*; E(N \geq 1))}
\approx \|f\|_{E_0(X)} + \|\{E_{m,x^*,n}^A(f)\}_{n \geq 1} \|_{E(N \geq 1)}.
\]

**Corollary 4.6.** Let \(E \in S(\varepsilon_+, \varepsilon-, A, r, (S, \mathcal{A}, \mu))\) and suppose that \(\varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - 1)_+\). Let \(p \in (0, \infty)\) and \(M \in \mathbb{N}\) satisfy \(\varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - p^{-1})\) and \(M \lambda_{A, \text{min}} > \varepsilon_-.\) Then, for each \(f \in L_0(S; L_{r,d,loc}(\mathbb{R}^d; X))\) of the form \(f = \sum_{i \in I} 1_{S_i} \otimes f[i]\) with \((S_i)_{i \in I} \subset \mathcal{A}\) a countable family of mutually disjoint sets and \((f[i])_{i \in I} \in L_{r,d,loc}(\mathbb{R}^d; X)\),
\[
\|f\|_{Y_{L,A}(E; X)} \approx \|f\|_{E_0(X)} + \|d_{m,n}^A(f)\|_{E(N \geq 1)}.
\]
Theorem 4.7. Let \( E \in S(\varepsilon_+, \varepsilon_-, A, 1, (S, \mathcal{A}, \mu)) \) and suppose that \( \varepsilon_+, \varepsilon_- > 0 \). Let \( p \in [1, \infty]^\ell \) and \( M \in \mathbb{N} \) satisfy \( \varepsilon_+ > \text{tr}(A) \cdot (1 - p^{-1}) \) and \( MA_{\text{min}}^A > \varepsilon_- \). Write
\[
I_{M,n}^A(f) := 2^{n} \int_{B^A(0, 2^{-n})} \Delta_n^M f \, dz, \quad f \in L_0(S; L_{1, \text{loc}}(\mathbb{R}^d; X)).
\]
Then
\[
\|f\|_{Y^A_{E(X); X}} \approx \|f\|_{Y^A_{E(X); X}} \approx \|f\|_{\overline{Y}^A_{E(X); X}}
\]
for all \( f \in E_0(X) \hookrightarrow E_i \hookrightarrow E^A_{\overline{B}_A^r, w \mathcal{A}, r}(X) \) (see Remark [Y7]).

Remark 4.8. Recall from Example 3.15 that, in case \( \ell = 1, A = I, p \in (1, \infty), q = 2, w \in A_p(\mathbb{R}^d), F \) is a UMD Banach function space and \( X \) is a Hilbert space, \( E^A_{p,q}(\mathbb{R}^d; w; F; X) \) coincides with the weighted vector-valued Bessel potential space \( H^r_p(\mathbb{R}^d; w; F(X)) \). Theorem 4.7 thus in particular gives a difference norm characterization for \( H^r_p(\mathbb{R}^d; w; F(X)) \) (cf. [13] Remark 4.10).

Proposition 4.9. Let \( E \in S(\varepsilon_+, \varepsilon_-, A, 1, (S, \mathcal{A}, \mu)) \) and suppose that \( \varepsilon_+, \varepsilon_- > 0 \). Let \( c \in \mathbb{R} \). Let \( p \in (0, \infty]^\ell \) and \( M \in \mathbb{N} \) satisfy \( \varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - p^{-1}) \) and \( M > \varepsilon_- \). Then
\[
\|\{(q^a_{M,c,n,f})_n\}_{n \in \mathbb{N}}\|_{E(X); X} \lesssim \|f\|_{Y^A_{E(X); X}}, \quad f \in L_0(S; L_{r,d}(\mathbb{R}^d; X)),
\]
and
\[
\|\{(q^a_{M,c,x^*,n,f})_{(x^*,n)}\} \|_{\overline{Y}^A_{E(X); X}} \lesssim \|f\|_{\overline{Y}^A_{E(X); X}}, \quad f \in L_0(S; L_{r,d}(\mathbb{R}^d; X)),
\]
where
\[
d^A_{M,c,n,f}(f) := 2^{n} \|\Delta_n^M f\|_{L_{p,1}(B^A(0, 2^{-n}; X))}
\]
and
\[
d^A_{M,c,x^*,n,f}(f) := 2^{n} \|\Delta_n^M (f, x^*)\|_{L_{p,1}(B^A(0, 2^{-n}; X))}.
\]

4.3 Some lemmas.

Lemma 4.10. Let \( E \in S(\varepsilon_+, \varepsilon_-, A, 1, (S, \mathcal{A}, \mu)) \). Put \( C := \max_{x \in [0, 1]^d} \rho_A(x) \in [1, \infty) \). Then, for each \((n,k) \in \mathbb{N} \times \{0,1,\ldots,d\}
\]
\[
\|s_{n,k}\|_{E^A_{\overline{B}_A^r, w \mathcal{A}, r}} \lesssim (C + \rho_A(k))^{\text{tr}(A) \cdot r^{-1}}, \quad (n,k) \in \mathbb{N} \times \mathbb{Z}^d.
\]

Proof. Fix \((i,l) \in \mathbb{N} \times \mathbb{Z}^d \). By Remark 3.7 \( E_i \hookrightarrow E^A_{\overline{B}_A^r, w \mathcal{A}, r} \), so that
\[
\|s_{i,l}\|_{E^A_{\overline{B}_A^r, w \mathcal{A}, r}} = \|s_{i,l} \chi^A_{i,l}\|_{E^A_{\overline{B}_A^r, w \mathcal{A}, r}} \lesssim \|s_{i,l} \chi^A_{i,l}\|_{E},
\]
\[
\leq \left\|\sum_{k \in \mathbb{Z}^d} s_{n,k} \chi^A_{i,l} \right\|_{E} = \|(s_{n,k})_{(n,k)}\|_{E^A_{\overline{B}_A^r, w \mathcal{A}, r}}.
\]

Let \( R = (R, \ldots, R) \in [1, \infty)^\ell \) be given by \( R := c_A(C + \rho_A(l)) \). Then
\[
\rho_A(x+l) \leq c_A(\rho_A(x) + \rho_A(l)) \leq c_A(C + \rho_A(l)) = R \leq 2^l R, \quad x \in [0, 1]^d.
\]
Therefore,
\[
\text{supp}(\chi^A_{i,l}) = A^2_{2^{-l}}([0, 1]^d + l) \subset B^A(0, R).
\]
As a consequence,
\begin{equation}
[c_A(C + \mu_A(l))]^{-\text{tr}(A)^-1} \leq \|\chi_{i,l}^A\|_{L_{r,c}(\mathbb{R}^d)} \leq ||\chi_{i,l}^A||_{B_{r,c}^\infty A,\tau}
\end{equation}

Observing that $||\chi_{i,l}^A||_{L_{r,c}(\mathbb{R}^d)} = c_{i,A,\tau}$, a combination of (23) and (24) gives the desired result. □

**Lemma 4.11.** Let $p \in (0, \infty]$ and $M \in \mathbb{N}_{\geq 1}$. Then there is a constant $C = C_{M,p,d}$ such that: if $f \in L_{p,\text{loc}}(\mathbb{R}^d)$ and $Q = A_\lambda([0,1]^d + b)$ with $\lambda \in (0, \infty)$ and $b \in \mathbb{R}^d$, then there is $\pi \in \mathcal{P}_{M-1}$ satisfying (with the usual modification if $p = \infty$):
\[
|f - \pi|_Q \leq C \left( \int_{B^A(0,\lambda)} |\Delta_z^M f|^p dz \right)^{1/p} + C \left( \int_{B^A(0,\lambda)} \int_{Q(2)} |\Delta_z^M f|^p dy dz \right)^{1/p}.
\]

**Proof.** The case $\lambda = 1$ is contained in [32] Lemma 1.2.1, from which the general case can be obtained by a scaling argument. □

From Lemma 4.12 to Corollary 4.14 we will actually only use Corollary 4.14 in the scalar-valued case in the proof of Theorem 4.5. However, although the scalar-valued case is easier, we have decided to present it in this way as it could be useful for potential extensions of Theorem 4.4 along these lines. In the latter the main obstacle is Lemma 4.11.

We write $\mathcal{P}_{N}(X) \cong X^{M_{N,d}}$, where $M_{N,d} : = \# \{ \alpha \in \mathbb{N}^d : |\alpha| \leq M \}$, for the space of $X$-valued polynomials of degree at most $N$ on $\mathbb{R}^d$.

**Lemma 4.12.** Let $(T, \mathcal{B}, \nu)$ a measure space, $F \subset L_2(T)$ a finite dimensional subspace, $\mathcal{E} \subset L_0(T; X)$ a topological vector space with $F \otimes X \subset \mathcal{E}$ such that
\[
\mathbb{F} \times X \rightarrow \mathcal{E}, \ (p, f) \mapsto f \otimes x,
\]
and
\[
\mathbb{F} \times \mathcal{E} \rightarrow L_1(T; X), \ (f, g) \mapsto fg,
\]
are well-defined bilinear mappings that are continuous with respect to the second variable. Then $\mathbb{F} \otimes X$ is a complemented subspace of $\mathcal{E}$.

**Proof.** Choose an orthogonal basis $b_1, \ldots, b_n$ of the finite dimensional subspace $\mathbb{F}$ of $L_2(T)$. Then
\[
\pi : \mathcal{E} \rightarrow \mathbb{E}, \ g \mapsto \sum_{i=1}^n \left[ \int_T b_i(t) g(t) d\nu(t) \right] \otimes b_i,
\]
is a well-defined continuous linear mapping on $\mathcal{E}$, which is a projection onto the linear subspace $\mathbb{F} \otimes X \subset \mathcal{E}$. □

**Corollary 4.13.** If $\mathbb{E}$ in Lemma 4.12 is an $F$-space, then so is $(\mathbb{F} \otimes X, \tau_{\mathbb{F}})$. As a consequence, if $\tau$ is a topological vector space topology on $\mathbb{F} \otimes X$ with $(\mathbb{F} \otimes X, \tau_{\mathbb{F}}) \hookrightarrow (\mathbb{F} \otimes X, \tau)$, then the latter is in fact a topological isomorphism.

**Corollary 4.14.** Let $B = [-1,2]^d, N \in \mathbb{N}$ and $q \in [1, \infty)$. Set $B_{n,k} := A_{2^{-n}}(B+k)$ for $(n,k) \in \mathbb{N} \times \mathbb{Z}^d$. Then
\[
||\pi(A_{2^{-n}} \cdot +k)||_{C^0_B(B;\mathcal{X})} \lesssim 2^{ntr(A^k)/q} ||\pi||_{L_q(B_{n,k}; \mathcal{X})}, \quad \pi \in \mathcal{P}_{N}(X, (n,k) \in \mathbb{N} \times \mathbb{Z}^d).}
\]
Proof. Let us first note that a substitution gives
$$
||\pi(A_{2^n} \cdot +k)||_{L_q(B;X)} = 2^{ntr(A^{\otimes}/q)}||\pi||_{L_q(B_{n,k};X)},
$$
while $\pi(A_{2^n} \cdot +k) \in P_N^d(X)$. Applying Corollary 4.13 to $F = P_N^d$, viewed as finite dimensional subspace of $L_2(B)$, and $E = C_0^N(B; X)$ and $\tau$ the topology on $P_N(X) = F \otimes X$ induced from $L_q(B; X)$, we obtain the desired result. 

Lemma 4.15. Let $q, p \in (0, \infty)$, $q \leq p$, $b \in (0, \infty)$ and $M \in \mathbb{N}_{\geq 1}$. Let $f \in L_{p, \text{loc}}(\mathbb{R}^d)$ and let $\{\pi_{n,k}\}_{(n,k) \in \mathbb{N} \times \mathbb{Z}^d} \subset P_{M-1}^d$ such that
$$
||f - \pi_{n,k}||_{L_q(Q_n^{A_{n,k}}(b))} \leq 2\mathcal{E}_M(f, Q_n^{A_{n,k}}(b), L_q),
$$
and let $\{\phi_{n,k}\}_{(n,k) \in \mathbb{N} \times \mathbb{Z}^d} \subset L_\infty(\mathbb{R}^d)$ be such that $\text{supp} \phi_{n,k} \subset Q_n^{A_{n,k}}(b)$, $\sum_{k \in \mathbb{Z}^d} \phi_{n,k} = 1$, and $||\phi_{n,k}||_{L_\infty} \leq 1$. Then, for $\{f_n\}_{n \in \mathbb{N}} \subset L_0(S)$ defined by
$$
f_n := \sum_{k \in \mathbb{Z}^d} \pi_{n,k} \phi_{n,k},
$$
there is the convergence $f = \lim_{n \to \infty} f_n$ almost everywhere and in $L_{p, \text{loc}}$. 

Proof. This can be proved as in [32] Lemma 1.2.3. 

Lemma 4.16. Let $E \in S(\mathbb{R}^d)$, $A, r, (S, \mathcal{A}, \mu)$, $b \in (0, \infty)$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $p \in (0, \infty)^d$ satisfy $\varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - p^{-1})$. Define the sublinear operator
$$
T_p^A : L_0(S)^{N \times \mathbb{Z}^d} \rightarrow L_0(S; [0, \infty])^{N \times \mathbb{Z}^d}, \quad (s_{n,k})_{(n,k)} \mapsto (t_{n,k})_{(n,k)},
$$
by
$$
t_{n,k} := 2^{ntr(A) \cdot p^{-1}} \left| \sum_{m,l} s_{m,l} A_{m,l} \right|_{L_{p,d}},
$$
and the sum is taken over all indices $(m,l) \in \mathbb{N} \times \mathbb{Z}^d$ such that $Q_{m,l}^A \subset Q_{n,k}^A(b)$ and $m \geq n$. Then $T_p^A$ restricts to a bounded sublinear operator on $y^A(E)$. 

Proof. Let $(s_{n,k})_{(n,k)} \in y^A(E)$ and $(t_{n,k})_{(n,k)} = T_p^A((s_{n,k})_{(n,k)}) \in L_0(S; [0, \infty])^{N \times \mathbb{Z}^d}$. We need to show that $||||t_{n,k}||_{y^A(E)} \lesssim ||(s_{n,k})||_{y^A(E)}$. Here we may without loss of generality assume that $s_{n,k} \geq 0$ for all $(n,k)$. 

Set
$$
\delta := \frac{1}{2} \left( \varepsilon_+ - \text{tr}(A) \cdot (r^{-1} - p^{-1}) \right) \in (0, \infty).
$$
Define
$$
g_m := \sum_{l \in \mathbb{Z}^d} s_{m,l} A_{m,l} \in L_0(S), \quad m \in \mathbb{N}.
$$
Then
$$
t_{n,k} \leq 2^{ntr(A) \cdot p^{-1}} \left| \sum_{m=n}^\infty g_m \right|_{L_{p,d}(Q_{n,k}^A)}.
$$
As the the right-hand side is increasing in $p$ by Hölder’s inequality, it suffices to consider the case $p \geq r$. 

Several applications of the elementary embedding
$$
\ell_{q_0}^s(\mathbb{N}) \rightarrow \ell_{q_1}^s(\mathbb{N}), \quad s_0 > s_1, q_0, q_1 \in (0, \infty),
$$
in combination with Fubini yield that

\[(26) \quad \left\| \sum_{m=n}^{\infty} g_m \right\|_{L_{p,f}(Q_{n,k}^A(b))} \lesssim \sum_{m=n}^{\infty} 2^{(m-n)\delta} \|g_m\|_{L_{p,f}(Q_{n,k}^A(b))}.
\]

In order to estimate the summands on the right-hand side of (26), we will use the following fact. Let \((T_1, \mathcal{B}_1, \nu_1), \ldots, (T_\ell, \mathcal{B}_\ell, \nu_\ell)\) be \(\sigma\)-finite measure spaces and let \(I_1, \ldots, I_\ell\) be countable sets. Put \(T = T_1 \times \ldots \times T_\ell\) and \(I = I_1 \times \ldots \times I_\ell\). Let \((c_i)_{i\in I} \subset \mathbb{C}\) and, for each \(j \in \{1, \ldots, \ell\}\), let \((A_{ij})_{i \in I_j} \subset \mathcal{B}_j\) be a sequence of mutually disjoint sets. Then

\[(27) \quad \left\| \sum_{i \in I} c_i 1_{A_1^{(i)} \times \ldots \times A_{\ell}^{(i)}} \right\|_{L_{p}(T)} \leq \left( \sup_{j \in I} \left[ \frac{1}{r_j} \right] \left( \left\| A_{ij}^{(j)} \right\|_{L_{r_j}^p} \right)^{p_j/p_i} \right)^{1/r_i}
\]

Indeed,

\[
\left\| \sum_{i \in I} c_i 1_{A_1^{(i)} \times \ldots \times A_{\ell}^{(i)}} \right\|_{L_{p}(T)} = \left( \sum_{i \in I} \left| A_{ij}^{(i)} \right| \left( \sum_{i_1 \in I_1} \left| A_{i_1}^{(1)} \right| \left| c_i \right|^{r_1} \right)^{r_2/r_1} \right)^{1/r_2}
\]

Let us now use the above fact to estimate \(\|g_m\|_{L_{p,f}(Q_{n,k}^A(b))}\):

\[
\|g_m\|_{L_{p,f}(Q_{n,k}^A(b))} \leq \left\| \sum_{l \in \mathbb{Z}^d, Q_{m,l}^A(b) \neq \emptyset} s_{m,l} \chi_{m,l}^A \right\|_{L_{p,f}(\mathbb{R}^d)} 
\]

(27)

\[
\leq 2^{-mtr(A) \cdot (p^{-1} - r^{-1})} \left\| \sum_{l \in \mathbb{Z}^d, Q_{m,l}^A(b) \neq \emptyset} s_{m,l} \chi_{m,l}^A \right\|_{L_{p,f}(\mathbb{R}^d)} 
\]

\[
\leq 2^{-mtr(A) \cdot (p^{-1} - r^{-1})} \|g_m\|_{L_{p,f}(Q_{n,k}^A(b+2))} \leq 2^{(m-n)\delta + mtr(A) \cdot (p^{-1} - r^{-1})} \|g_m\|_{L_{p,f}(Q_{n,k}^A(b+2))}
\]

(28)

Putting (25), (26) and (28) together, we obtain

\[
t_{n,k} \chi_{n,k}^A \leq \sum_{m=n}^{\infty} 2^{(m-n)\delta + mtr(A) \cdot (p^{-1} - r^{-1})} \|g_m\|_{L_{p,f}(Q_{n,k}^A(b+2))} \chi_{n,k}^A.
\]
Lemma 4.19. Let \( \text{Proof.} \) This can be proved in the same way as [32, Lemma 1.2.6].

sequence of measurable functions on \( \mathbb{R} \) converges almost everywhere, and in \( m < n \)

We may of course without loss of generality assume that \( \text{Proof.} \)

Define the sublinear operator

\[
\sum_{m=1}^{\infty} 2^{(m-n)(\varepsilon_+ - \delta)} M_r^A (g_m).
\]

Since

\[
\left( \sum_{m=1}^{\infty} 2^{(m-n)(\varepsilon_+ - \delta)} M_r^A (g_m) \right)_{n \in \mathbb{N}} = \sum_{i=0}^{\infty} 2^{i(\varepsilon_+ - \delta)} (S_+)^i M_r^A [ (g_n)_{n \in \mathbb{N}} ],
\]

it follows that \( (t_{n,k}) \in y^A(E) \) with

\[
\| (t_{n,k}) \|_{y^A(E)}^\kappa = \left\| \left( \sum_{k \in \mathbb{Z}^d} t_{n,k} \lambda_{n,k}^A \right) \right\|_E^\kappa
\]

\[
\lesssim \sum_{i=0}^{\infty} 2^{\kappa i (\varepsilon_+ - \delta)} \| (S_+)^i M_r^A [ (g_n)_{n \in \mathbb{N}} ] \|_E^\kappa
\]

\[
\lesssim \sum_{i=0}^{\infty} 2^{-ni(d)} \| (g_n)_{n \in \mathbb{N}} \|_E^\kappa \lesssim \| (g_n)_{n \in \mathbb{N}} \|_E^\kappa
\]

\[
= \| (s_{n,k}) \|_{y^A(E)}^\kappa,
\]

where \( \kappa \) is such that \( E \) has a \( \kappa \)-norm.

\[\square\]

Corollary 4.17. Let \( E \in S(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \) and suppose that \( \varepsilon_+, \varepsilon_- > 0 \). Let \( p \in (0, \infty]^d \) satisfy \( \varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - p^{-1}) \). Given \( (s_{n,k})_{n \in \mathbb{N}} \in y^A(E) \), set \( g_n = \sum_{k \in \mathbb{Z}^d} s_{n,k} \lambda_{n,k}^A \). Then \( \sum_{n=0}^{\infty} | g_n | \) in \( L_0(S; L_{p,1,\text{loc}}(\mathbb{R}^d)) \) and the series \( \sum_{n=0}^{\infty} g_n \) converges almost everywhere, and in \( L_0(S; L_{p,1,\text{loc}}(\mathbb{R}^d)) \) (when \( p \in (0, \infty)^d \)).

Proof. This follows from [30], see [32 Corollary 1.2.5] for more details. \[\square\]

Lemma 4.18. Let \( E \in S(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)), b \in (0, \infty) \) and \( \lambda \in (\varepsilon_-, \infty) \).

Define the sublinear operator

\[
T_\lambda : L_0(S)^{n \times \mathbb{Z}^d} \rightarrow L_0(S; [0, \infty])^{n \times \mathbb{Z}^d}, \quad (s_{n,k})_{n \in \mathbb{N}} \mapsto (t_{n,k})_{n \in \mathbb{N}},
\]

by

\[
t_{n,k} := \sum_{m,l} 2^{(n-m)\lambda} | s_{m,l} |,
\]

the sum being taken over all indices \( (m,l) \in \mathbb{N} \times \mathbb{Z}^d \) such that \( Q_{m,l}(b) \supset Q_{n,k}^A \) and \( m < n \). Then \( T_\lambda \) restricts to a bounded sublinear operator from \( y^A(E) \) to \( y^A(E) \).

Proof. This can be proved in the same way as [32 Lemma 1.2.6]. \[\square\]

Lemma 4.19. Let \( r \in (0,1]^d \) and \( \rho \in (0,1) \) satisfy \( \rho < r_{\text{min}} \). Let \( (\gamma_n)_{n \in \mathbb{N}} \) be a sequence of measurable functions on \( \mathbb{R}^d \) satisfying

\[
0 \leq \gamma_n(x) \leq (1 + 2^n \rho A(x))^{-\text{tr}(A^\rho)}/\rho.
\]

If \( (s_{n,k})_{n \in \mathbb{N}} \in L_0(S)^{n \times \mathbb{Z}^d} \), \( g_n = \sum_{k \in \mathbb{Z}^d} s_{n,k} \lambda_{n,k}^A \) and \( h_n = \sum_{k \in \mathbb{Z}^d} | s_{n,k} | \gamma_n(\cdot - A_{2^{-n}k}) \), then

\[
h_n \lesssim M_r^A (g_n), \quad n \in \mathbb{N}.
\]

Proof. We may of course without loss of generality assume that \( r = (r, \ldots, r) \) with \( r \in (0,1] \). Now the statement can be established as in [32 Lemma 1.2.7]. \[\square\]
Lemma 4.20. Let $M \in \mathbb{N}$, $\lambda \in (0, \infty)$ and $\Phi \in C^M(\mathbb{R}^d; X)$ be such that
\[ (1 + \rho_A(x))^{\lambda}||D^\beta \Phi(x)||_X \leq 1, \quad x \in \mathbb{R}^d, |\beta| \leq M, \]
and let $\Psi \in S(\mathbb{R}^d)$ be such that $\Psi \perp \mathcal{P}_d$. Set $\Psi_t := t^{-1}(\Phi(x) - t \cdot x)$ for $t \in (0, \infty)$. Then, given $\varepsilon \in (0, \lambda_{\min}^A)$,
\[ ||\Phi \ast \Psi_t(x)||_X \leq t(\lambda_{\min}^A - \varepsilon)M \frac{t(1 + \rho_A(x))^{\lambda}}{(1 + \rho_A(x))^{\lambda + \varepsilon}}, \quad x \in \mathbb{R}^d, t \in (0, 1]. \]

Proof. As $\Psi$ is a Schwartz function, there in particularly exists $C \in (0, \infty)$ such that
\[ ||\Psi(x)|| \leq C(1 + \rho_A(x))^{-\lambda}(1 + |x|)^{-(d+M+1)}, \quad x \in \mathbb{R}^d. \]
The desired inequality can now be obtained as in [22] Lemma 1.2.8. \hfill \Box

Lemmas 4.21 and 4.22 are the corresponding versions of Lemmas 4.16 and 4.18 respectively, for $\tilde{y}^A(E; X)$ instead of $y^A(E; X)$.

Lemma 4.21. Let $E \in S(\varepsilon_+, \varepsilon_-, A, r, (S, \alpha', \mu))$, $b \in (0, \infty)$ and suppose that $\varepsilon_+, \varepsilon_- > 0$. Let $p \in (0, \infty)^d$ satisfy $\varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - p^{-1})$. Define the sublinear operator
\[
T_p^A : L_0(S)^{X^* \times \mathbb{N} \times \mathbb{Z}^d} \rightarrow L_0(S; [0, \infty])^{X^* \times \mathbb{N} \times \mathbb{Z}^d}, \quad (s_{x^*, n, k})_{(x^*, n, k)} \mapsto (t_{x^*, n, k})_{(x^*, n, k)},
\]
by
\[
t_{x^*, n, k} := 2^{n \cdot \text{tr}(A) \cdot p^{-1}} \left| \sum_{m, l} s_{x^*, m, l}A_{m, l} \right|_{L_{p, d}}
\]
and the sum is taken over all indices $(m, l) \in \mathbb{N} \times \mathbb{Z}^d$ such that $Q^A_{m, l} \subset Q^A_{n, k}(b)$ and $m \geq n$. Then $T_p^A$ restricts to a bounded sublinear operator on $\tilde{y}^A(E)$.

Proof. Let $\delta \in (0, \infty)$ be as in the proof of Lemma 4.16. Let $(s_{x^*, n, k})_{(x^*, n, k)} \in \tilde{y}^A(E)$ and $(t_{x^*, n, k})_{(x^*, n, k)} = T_p^A[(s_{x^*, n, k})_{(x^*, n, k)}]$ in $L_0(S; [0, \infty])^{X^* \times \mathbb{N} \times \mathbb{Z}^d}$. Define
\[
g_{x^*, m} := \sum_{l \in \mathbb{Z}^d} s_{x^*, m, l}A_{m, l} \in L_0(S), \quad m \in \mathbb{N}.
\]
Then $(g_{x^*, m})_{(x^*, m)} \in \mathcal{F}_M(X^*; E)$ with $|||g_{x^*, m}|||_{\mathcal{F}_M(X^*; E)} = |||s_{x^*, n, k}|||_{\mathcal{F}_M(X^*; E)}$ is the same. So there exists $g_{m, n} \in E_+$ with $|||g_{m, n}||| \leq 2|||s_{x^*, n, k}|||_{\mathcal{F}_M(X^*; E)}$ such that $|g_{x^*, m}| \leq ||x^*||g_{m}$. By [29] from the proof of Lemma 4.16
\[
t_{x^*, n, k}A_{n, k} \leq \sum_{m=n}^{\infty} 2^{(m-n)(\varepsilon_+ - \delta)} M^A_{\rho}(g_{x^*, m})
\]
As in proof of Lemma 4.16 we find that $(t_{x^*, n, k})_{(x^*, n, k)} \in \tilde{y}^A(E; X)$ with
\[ |||t_{x^*, n, k}|||_{\tilde{y}^A(E; X)} \leq 2|||g_{m}||| \leq 2|||s_{x^*, n, k}|||_{\mathcal{F}_M(X^*; E)}. \hfill \Box
\]

Lemma 4.22. Let $E \in S(\varepsilon_+, \varepsilon_-, A, r, (S, \alpha', \mu))$, $b \in (0, \infty)$ and $\lambda \in (\varepsilon_-, \infty)$. Define the sublinear operator
\[
T_{\lambda} : L_0(S)^{X^* \times \mathbb{N} \times \mathbb{Z}^d} \rightarrow L_0(S; [0, \infty])^{X^* \times \mathbb{N} \times \mathbb{Z}^d}, \quad (s_{x^*, n, k})_{(x^*, n, k)} \mapsto (t_{x^*, n, k})_{(x^*, n, k)},
\]
by
\[ t_{x^*,n,k} := \sum_{m,l} 2^{n(n-m)} |s_{x^*,m,l}|, \]
the sum being taken over all indices \((m,l) \in \mathbb{N} \times \mathbb{Z}^d\) such that \(Q_{m,l}(b) \supset Q_{n,k}^A\) and \(m < n\). Then \(T_{\lambda}\) restricts to a bounded sublinear operator on \(\tilde{y}_A^A(E;X)\).

**Proof.** This can be proved in the same way as \cite[Lemma 1.2.6]{32}.

**Lemma 4.23.** Let \(E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, A, 1, (S, \mathcal{A}, \mu))\) and let \(k \in L_{1, c}(\mathbb{R}^d)\) fulfill the Tauberian condition
\[ |\hat{k}(\xi)| > 0, \quad \xi \in \mathbb{R}^d, \frac{\varepsilon}{2} < \rho_A(\xi) < 2\varepsilon, \]
for some \(\varepsilon \in (0, \infty)\). Let \(\psi \in \mathcal{S}(\mathbb{R}^d)\) be such that \(\text{supp} \hat{\psi} \subset \{\xi : \varepsilon \leq \rho_A(\xi) \leq B\}\) for some \(B \in (\varepsilon, \infty)\). Define \((k_n)_{n \in \mathbb{N}}\) and \((\psi_n)_{n \in \mathbb{N}}\) by \(k_n := 2^{ntr(A^0)} k(A_{2^n} \cdot)\) and \(\psi_n := 2^{ntr(A^0)} \psi(A_{2^n} \cdot)\). Then
\[ ||(\psi_n * f)_n||_{E(X)} \lesssim ||(k_n * f)_n||_{E(X)}, \quad f \in L_0(S; L_{1, \text{loc}}(\mathbb{R}^d; X)). \]

**Proof.** Pick \(\eta \in C_c^\infty(\mathbb{R}^d)\) with \(\text{supp} \eta \subset B^A(0, 2\varepsilon)\) and \(\eta(\xi) = 1\) for \(\rho_A(\xi) \leq \frac{3\varepsilon}{2}\). Define \(m \in \mathcal{S}(\mathbb{R}^d)\) by \(m(\xi) := (\eta(\xi) - \eta(A_2 \xi)) |\hat{k}(\xi)|^{-1}\) if \(\frac{\varepsilon}{2} \leq \rho_A(\xi) \leq 2\varepsilon\) and \(m(\xi) := 0\) otherwise; note that this gives a well-defined Schwartz function on \(\mathbb{R}^d\) because \(\eta - \eta(A_2 \cdot)\) is a smooth function supported in the set \(\{\xi : \frac{\varepsilon}{2} < \rho_A(\xi) < 2\varepsilon\}\) on which the function \(\hat{k} \in BUC_\infty(\mathbb{R}^d)\) does not vanish. Define \((m_n)_{n \in \mathbb{N}}\) by \(m_n := m(A_{2^{-n} \cdot})\). Then, by construction,
\[ \sum_{l=n}^{n+N} m_l k_l(\xi) = \eta(A_2^{-(n+N)} \xi) - \eta(A_2^{-(n+1)} \xi) = 1 \]
for \(2^n \varepsilon \leq \rho_A(\xi) \leq 2^{n+N-1} \varepsilon\), \(n \in \mathbb{N}\), \(N \in \mathbb{N}\). Since \(\text{supp} \hat{\psi}_n \subset \{\xi : 2^n \varepsilon \leq \rho_A(\xi) < 2^n B\}\) for every \(n \in \mathbb{N}\), there thus exists \(N \in \mathbb{N}\) such that \(\sum_{l=n}^{n+N} m_l k_l \equiv 1\) on \(\text{supp} \hat{\psi}_n\) for all \(n \in \mathbb{N}\). For each \(n \in \mathbb{N}\) we consequently have
\[ \psi_n = \psi_n * \left( \sum_{l=n}^{n+N} \tilde{m}_l \ast k_l \right) = \sum_{l=n}^{n+N} \psi_n \ast \tilde{m}_l \ast k_l = \sum_{l=0}^{N} \psi_n \ast \tilde{m}_{n+l} \ast k_{n+l}. \]
As \(\psi, m \in \mathcal{S}(\mathbb{R}^d)\), we obtain the pointwise estimate
\[ ||\psi * f||_{X} \lesssim \sum_{l=0}^{N} ||\psi_n \ast \tilde{m}_{n+l} \ast k_{n+l} \ast f||_{X} \lesssim \sum_{l=0}^{N} M^A(M^A(||k_{n+l} \ast f||_X)). \]
It follows that
\[ ||(\psi_n \ast f)_n||_{E(X)} \lesssim \sum_{l=0}^{N} ||(M^A(M^A(||k_{n+l} \ast f||_X)))_n||_{E(X)} \lesssim \sum_{l=0}^{N} ||(k_{n+l} \ast f)_n||_{E(X)} \lesssim 2^{-\varepsilon+l} ||(k_{n} \ast f)_n||_{E(X)} \lesssim ||(k_{n} \ast f)_n||_{E(X)}. \]
4.4. Proofs of the results in Section 4.2

Proof of Theorem 4.4

Proofs of the results in Section 4.2.

Let \((f_n)_n\) be as in Definition 3.8 with \(||(f_n)_n||_{E(X)} \leq 2||f||_{Y L^A(E;X)}\). For each \((n,k) \in \mathbb{N} \times \mathbb{Z}^d\), we put

\[
\tilde{a}_{n,k} := \omega(A_{2^n} (\cdot - A_{2^n} k)) f_n, \quad s_{n,k} := ||\tilde{a}_{n,k}(A_{2^{-n}} \cdot)||_{C^M_b(\mathbb{R}^d;X)},
\]

and

\[
a_{n,k} := \frac{\tilde{a}_{n,k}}{s_{n,k}} 1\{s_{n,k} \neq 0\}.
\]

Note that

\[
|s_{n,k}| = ||\tilde{a}_{n,k}(A_{2^{-n}} \cdot)||_{C^M_b(\mathbb{R}^d;X)} = ||\omega(\cdot \cdot - k) f_n (A_{2^{-n}} \cdot)||_{C^M_b(\mathbb{R}^d;X)} \\
\lesssim ||\omega(\cdot \cdot - k)||_{C^M_b(\mathbb{R}^d)} ||f_n (A_{2^{-n}} \cdot)||_{C^M_b([-1,2]^d + k;X)} \\
\lesssim \sup_{|\alpha| \leq M} \sup_{y \in [-1,2]^d + k} ||D^\alpha [f_n (A_{2^{-n}} \cdot)](y)||_X
\]

Given \(x \in Q^A_{n,k}\) and \(\hat{x} = A_{2^n} x \in [0,1]^d + k\), for \(y \in [-1,2]^d + k\) we can write

\[
y = \hat{x} + z \text{ with } z = y - \hat{x} = (y - k) - (\hat{x} - k) \in [-1,2]^d - [0,1]^d,
\]

so, in particularly, \(\rho_A(z) \leq C_d\).

Combining the above and subsequently applying Lemma 4.1 to \(f_n (A_{2^{-n}} \cdot)\), whose Fourier support satisfies supp \(\mathcal{F} [f_n (A_{2^{-n}} \cdot)] \subset B^A(0,2)\), we find

\[
||s_{n,k}||_{\mathfrak{y}^A(E)} \lesssim \sup_{|\alpha| \leq M} \rho_A(z) \leq C_d \\
\lesssim M^A_{\mathfrak{y}} ||f_n (A_{2^{-n}} \cdot)||_X \cdot (A_{2^n} x) = M^A_{\mathfrak{y}} ||f_n||_X (x)
\]

for \(x \in Q^A_{n,k}\). Therefore, \((s_{n,k}(n))_{n,k} \in \mathfrak{A}(E)\) with

\[
||s_{n,k}(n)||_{\mathfrak{y}^A(E)} \lesssim ||(M^A_{\mathfrak{y}_n} ||f_n||_X)_n||_{E(X)} \lesssim ||(f_n)_n||_{E(X)} \leq 2||f||_{Y L^A(E;X)}.
\]

Finally, the convergence (22) follows from Corollary 4.17 and the observation that

\[
f = \sum_{n=0}^\infty f_n = \sum_{n=0}^\infty \sum_{k \in \mathbb{Z}^d} s_{n,k} a_{n,k} \text{ in } L^0(S; L_{r,\delta,\text{loc}}(\mathbb{R}^d;X)).
\]

\((\text{ii}) \Rightarrow (\text{iii})\): Set \(g_n := \sum_{k \in \mathbb{Z}^d} s_{n,k} \chi_{n,k}^A \) for \(n \in \mathbb{N}\). For \(n \in \mathbb{N}\), set \(f_n := 0\) and \(g_n := 0\). Pick \(\kappa \in (0,1)\) such that \(E\) has a \(\kappa\)-norm. Pick \(\varepsilon \in (0,\lambda^A_{\text{min}})\) such that \((\lambda^A_{\text{min}} - \varepsilon)M > \varepsilon\). Pick \(\lambda \in (0,\infty)\) such that \(\text{tr}(A^B)/\lambda < r_{\text{min}} \wedge 1\). Pick \(\psi = (\psi_n)_{n \in \mathbb{N}} \in \Phi^A(\mathbb{R}^d)\) such that

\[
\text{supp} \, \hat{\psi} \subset B^A(0,2), \quad \text{supp} \, \hat{\psi} \subset B^A(0,2^{n+1}) \setminus B^A(0,2^{n-1}), \quad n \geq 1,
\]

and set \(\Psi_n := 2^{n\text{tr}(A^B)} \hat{\psi}_0 (A_{2^n} \cdot)\) for each \(n \in \mathbb{N}\). Note that

\[
a_{n,k} * \Psi_n = [b_{n,k} * \Psi](A_{2^n} \cdot - k)
\]

and

\[
a_{n,k} * \psi_m = [b_{n,k} * \psi_{m-n}](A_{2^n} \cdot - k), \quad n < m.
\]
An application of Lemma [4.20] thus yields that

\[(31) \quad \|a_{n,k} * \Psi_n(x)\|_X \lesssim \frac{1}{(1 + 2^n \rho_A(x - A_{2^{-n}k}))^A} \]

and

\[(32) \quad \|a_{n,k} * \psi_m(x)\|_X \lesssim \frac{2^{-(m-n)(A_{\text{min}} - \epsilon)M}}{(1 + 2^n \rho_A(x - A_{2^{-n}k}))^A}, \quad n < m.\]

Now put

\(\tilde{a}_{n,k,m} := \begin{cases} 
    a_{n,k} * \Psi_n, & n = m, \\
    a_{n,k} * \psi_m, & n < m.
\end{cases}\)

Let \(L_M(\mathbb{R}^d; X)\) denote the Fréchet space of all equivalence classes of strongly measurable \(X\)-valued functions on \(\mathbb{R}^d\) that are of polynomial growth; this space can for instance be described as

\[L_M(\mathbb{R}^d; X) := \{ f \in L_0(\mathbb{R}^d; X) : \forall \phi \in S(\mathbb{R}^d), \phi f \in L(\mathbb{R}^d; X) \}.\]

Using Lemma [4.11] together with the support condition of the \(a_{n,k}\) and \(\|a_{n,k}\|_{L(\mathbb{R}^d; X)} \leq 1\), it can be shown that the series \(\sum_{k \in \mathbb{Z}^d} s_{n,k}a_{n,k}\) converges in \(L_0(S; L_M(\mathbb{R}^d; X))\).

Since \(L_M(\mathbb{R}^d; X) \hookrightarrow S'(\mathbb{R}^d; X)\) and convolution gives rise to a separately continuous bilinear mapping \(S \times S' \to \mathcal{O}_M\), it follows that

\[(33) \quad f_{n,m} := \sum_{k \in \mathbb{Z}^d} s_{n,k} \tilde{a}_{n,k,m} = \left( \sum_{k \in \mathbb{Z}^d} s_{n,k}a_{n,k} \right) * \begin{cases} 
    \Psi_n, & n = m, \\
    \psi_m, & n < m, \quad \text{in} \quad L_0(S; \mathcal{O}_M(\mathbb{R}^d; X))
\end{cases}\]

for each \(n, m \in \mathbb{N}\) with \(m \geq n\).

It will be convenient to define

\(f_{n,m}^+ := \sum_{k \in \mathbb{Z}^d} |s_{n,k}| \|\tilde{a}_{n,k,m}\|_X, \quad n, m \in \mathbb{N}, m \geq n.\)

By a combination of (31), (32) and Lemma [4.19]

\[f_{m-l,m}^+ \lesssim 2^{-l(A_{\text{min}} - \epsilon)M} M_r^A(g_{m-l}), \quad m, l \in \mathbb{N}, m \geq l.\]

From this it follows that

\[(34) \quad \|f_{m-l,m}^+\|_{E(\mathbb{N} \geq l)} \lesssim \begin{cases} 
    2^{-l(A_{\text{min}} - \epsilon)M} \|(M_r^A(g_{m-l}))_{m \geq l}\|_{E(\mathbb{N} \geq l)}, & \text{if } M_r^A(g_{m-l}) \text{ is finite on } S_{E(\mathbb{N} \geq l)}.
\end{cases}\]

Therefore, by Lemma [3.5] and the assumption \((A_{\text{min}} - \epsilon)M > \epsilon -\),

\[\sum_{l=0}^{\infty} \sum_{m=l}^{\infty} f_{m-l,m}^+ = \sum_{l=0}^{\infty} \sum_{m=l}^{\infty} \sum_{k \in \mathbb{Z}^d} |s_{m-l,k}| \|\tilde{a}_{m-l,k,m}\|_X\]

belongs to \(E_A^d[\mathcal{O}_r^{M_{\text{max}}}] \hookrightarrow L_0(S; L_{r,\text{loc}}(\mathbb{R}^d))\). By Lebesgue domination this implies that \(\sum_{l=0}^{\infty} \sum_{m=l}^{\infty} \sum_{k \in \mathbb{Z}^d} s_{m-l,k} \tilde{a}_{m-l,k,m}\) converges unconditionally in the space \(L_0(S; L_{r,\text{loc}}(\mathbb{R}^d; X))\). In particular,

\[\sum_{l=0}^{\infty} \sum_{m=l}^{\infty} \sum_{k \in \mathbb{Z}^d} s_{m-l,k} \tilde{a}_{m-l,k,m} = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} s_{n,k} \sum_{m=n}^{\infty} \tilde{a}_{n,k,m} \quad \text{in} \quad L_0(S; L_{r,\text{loc}}(\mathbb{R}^d; X)).\]
Since
\[ a_{n,k} = \lim_{N \to \infty} \Psi_N * a_{n,k} = \lim_{N \to \infty} \sum_{m=n}^{N} \bar{a}_{n,k,m} \text{ in } L_0(S; L_1(\mathbb{R}^d; X)), \]
and since \( f \) has the representation (22), it follows that
\[ f = \sum_{l=0}^{\infty} \sum_{m=l}^{\infty} s_{m-l,k} \bar{a}_{m-l,k,m} \text{ in } L_0(S; L_{r,\alpha}^{\theta},\text{loc}(\mathbb{R}^d; X)). \]
Combining the latter with (33), we find
\[ f = \sum_{l=0}^{\infty} \sum_{m=l}^{\infty} f_{m-l,m} \text{ in } L_0(S; L_{r,\alpha}^{\theta},\text{loc}(\mathbb{R}^d; X)). \]
Note that
\[ \|(f_{m-l,m})_{m \geq l}\|_{L(E; X)} \leq 2^{-l((\lambda_{\min}^{A} - \varepsilon)M - \varepsilon - )}\|(s_{n,k})_{(n,k)}\|_{y^{A}(E)} \]
by (34). Since
\[ \text{supp } f_{m-l,m} \subseteq \begin{cases} \text{supp } \hat{\Psi}_m, & l = 0, \\ \text{supp } \hat{\psi}_m, & l \geq 1, \end{cases} \subset \hat{B}^A(0, 2^{m+1}), \quad m \geq l, \]
it follows that (see Remark 3.10)
\[ F_l := \sum_{m=l}^{\infty} f_{m-l,m} \text{ in } L_0(S; L_{r,\alpha}^{\theta},\text{loc}(\mathbb{R}^d; X)), \]
defines an element of \( YL^A(E; X) \) with
\[ \|F_l\|_{YL^A(E; X)} \lesssim 2^{-l((\lambda_{\min}^{A} - \varepsilon)M - \varepsilon - )}\|(s_{n,k})_{(n,k)}\|_{y^{A}(E)}. \]
As \((\lambda_{\min}^{A} - \varepsilon)M > \varepsilon_-,\) we find that \( F := \sum_{l=0}^{\infty} F_l \in YL^A(E; X) \) with
\[ \|F\|_{YL^A(E; X)} \lesssim \|(s_{n,k})_{(n,k)}\|_{y^{A}(E)}. \]
But \( f = F \) in view of (35) and \( YL^A(E; X) \hookrightarrow L_0(S; L_{r,\alpha}^{\theta},\text{loc}(\mathbb{R}^d; X)) \) (see Remark 3.10), yielding the desired result.

(iii) \( \Rightarrow \) (iv): We will write down the proof in such a way that the proof of Proposition 4.9 only requires a slight modification. Combining the estimate corresponding to (ii) \( \Rightarrow \) (i) with \( YL^A(E; X) \hookrightarrow E_0(X), \) we find
\[ \|f\|_{E_0(X)} \lesssim \|(s_{n,k})_{(n,k)}\|_{y^{A}(E)}. \]
So let us focus on the remaining part of the required inequality. To this end, fix \( c \in \mathbb{R} \) and choose \( R \in [1, \infty) \) such that
\[ \rho_A(tz) \leq R\rho_A(z), \quad z \in \mathbb{R}^d, t \in [0, |c| + M]. \]
Put
\[ d_{M,c,n}^{A,p}(f) := 2^{ntr(A) - p - 1}\|z \to L_{cz} \Delta_z^M f\|_{L_{p,\alpha}(B^A(0, 2^{-n}); X)}; \quad n \in \mathbb{N}. \]
Now let \( f \) has a representation as in (ii) and write \( h_n := \sum_{k \in \mathbb{Z}^d} s_{n,k} a_{n,k}. \) Then
\[ d_{M,c,n}^{A,p}(f)(x) \lesssim 2^{ntr(A) - p - 1}\|z \to \sum_{m=0}^{n-1} \|L_{cz} \Delta_z^M h_m(x)\|_{X}\|_{L_{p,\alpha}(B^A(0, 2^{-n}))}. \]
where the last sum is taken over all \((m, l)\) such that \(Q^{A}_{m, l}(3)\) intersects \(B^{A}(x, R^{-2}n)\) and \(m \geq n\). From this it follows that

\[
2^{ntr(A)\cdot p^{-1}} \left\| z \mapsto \sum_{m=n}^{\infty} \| L_{c_{m}} \Delta_{z}^{M} h_{m}(x) \|_{L_{p, i}(B^{A}(0, 2^{-n}))} \right\| \quad \leq \sum_{k \in \mathbb{Z}^{d}} 2^{ntr(A)\cdot p^{-1}} \left\| \sum_{m \geq k} \| s_{m,l} \|_{X} 1_{Q_{m,l}^{A}(3)} \|_{L_{p, i}(B^{A}(x, R^{-2}n))} \right\|,
\]

where the sum is taken over all \((m, l)\) such that \(Q^{A}_{m, l}(3)\) intersects \(B^{A}(x, R^{-2}n)\) and \(m \geq n\). From this it follows that

\[
2^{ntr(A)\cdot p^{-1}} \left\| z \mapsto \sum_{m=n}^{\infty} \| L_{c_{m}} \Delta_{z}^{M} h_{m}(x) \|_{X} \right\|_{L_{p, i}(B^{A}(0, 2^{-n}))} \quad \leq \sum_{k \in \mathbb{Z}^{d}} 2^{ntr(A)\cdot p^{-1}} \left\| \sum_{m \geq k} \| s_{m,l} \|_{X} 1_{Q_{m,l}^{A}(3)} \|_{L_{p, i}(B^{A}(x, R^{-2}n))} \right\|,
\]

where the sum is taken over all \((m, l)\) such that \(Q^{A}_{m, l}(3)\) intersects \(Q_{m,k}^{A}(3\bar{R})\) and \(m \geq n\). In order to estimate the first term in (30), note that

\[
\Delta_{z}^{M} h_{m}(x) = \int_{[0, 1]^{M}} D^{M} h_{m}(x + (t_{1} + \ldots + t_{M})z)(z, \ldots, z) \, d(t_{1}, \ldots, t_{M})
\]

and thus that

\[
\| \Delta_{z}^{M} h_{m}(x) \|_{X} \leq \sup_{t \in [0, M]} \| D^{M} h_{m}(x + tz)(z, \ldots, z) \|_{X}
\]

\[
= \sup_{t \in [0, M]} \| D^{M}[h_{m} \circ A_{2^{-m}}](A_{2^{m}x + tA_{2^{m}z}})(A_{2^{m}z}, \ldots, A_{2^{m}z}) \|_{X}
\]

\[
\leq \sup_{t \in [0, M]} \sup_{|\alpha| \leq M} \| D^{|\alpha|}[h_{m} \circ A_{2^{-m}}](A_{2^{m}x + tA_{2^{m}z}}) \|_{X} |A_{2^{m}z}|^{M},
\]

from which it follows that

\[
\| L_{c_{m}} \Delta_{z}^{M} h_{m}(x) \|_{X} \leq \sup_{t \in [0, M]} \sup_{|\alpha| \leq M} \| D^{|\alpha|}[h_{m} \circ A_{2^{-m}}](A_{2^{m}x + (c + t)A_{2^{m}z}}) \|_{X} |A_{2^{m}z}|^{M}
\]

\[
\leq \sup_{y \in B^{A}(0, R_{A}(z))} \sup_{|\alpha| \leq M} \| D^{|\alpha|}[h_{m} \circ A_{2^{-m}}](A_{2^{m}[x + y]}) \|_{X} |A_{2^{m}z}|^{M}.
\]
Given \( \varepsilon \in (0, \lambda A A_{\min}) \), for \( m \in \{0, \ldots, n-1\} \) and \( z \in B^A(0,2^{-n}) \) this gives

\[
\|L_{cz} \Delta_{z}^{M} h_{m}(x)\|_{X} \lesssim \varepsilon \sup_{y \in B^A(0,R2^{-n})} \sup_{|\alpha| \leq M} \| D^{\alpha}[h_{m} \circ A_{2^{-m}}(A_{2^{m}}[x+y])] \|_{X} \rho_{A}(A_{2^{m}}z)^{(\lambda_{A_{\min}}-\varepsilon)M} \\
\lesssim \sup_{y \in B^A(0,R2^{-n})} \sup_{|\alpha| \leq M} \| D^{\alpha}[h_{m} \circ A_{2^{-m}}(A_{2^{m}}[x+y])] \|_{X} 2^{(\lambda_{A_{\min}}-\varepsilon)M(m-n)}.
\]

Since

\[
\|D^{\alpha}[h_{m} \circ A_{2^{-m}}](A_{2^{m}}[x+y])\|_{X} \leq \sum_{t \in \mathbb{Z}^{d}} \|s_{m,t}\|_{X} 1_{[-1,2]^{d}+t}(A_{2^{m}}[x+y]) \\
\leq \sum_{t \in \mathbb{Z}^{d}} \|s_{m,t}\|_{X} 1_{Q_{m,t}^{A}(3)}(x+y),
\]

it follows that

\[
2^{ntr(A)\cdot p^{-1}} \|z \mapsto \sum_{m=0}^{n-1} \|L_{cz} \Delta_{z}^{M} h_{m}(x)\|_{X} \|_{L_{p,\ell}(B^A(0,2^{-n}))} \\
\lesssim \varepsilon \sum_{m=0}^{n-1} \sup_{z \in B^A(0,2^{-n})} \|L_{cz} \Delta_{z}^{M} h_{m}(x)\|_{X} 2^{(\lambda_{A_{\min}}-\varepsilon)M(m-n)} \\
\lesssim \sum_{m,l} 2^{(\lambda_{A_{\min}}-\varepsilon)M(m-n)} \|s_{m,t}\|_{X},
\]

where the last sum is taken over all \((m, l)\) such that \(Q_{m,t}^{A}(3)\) intersects \(B^A(x, R2^{-n})\) and \(m < n\). From this it follows that

\[
(38) \quad 2^{ntr(A)\cdot p^{-1}} \|z \mapsto \sum_{m=0}^{n-1} \|L_{cz} \Delta_{z}^{M} h_{m}(x)\|_{X} \|_{L_{p,\ell}(B^A(0,2^{-n}))} \lesssim \sum_{m,l} \|s_{m,t}\|_{X},
\]

where the last sum is taken over all \((m, l)\) such that \(Q_{m,t}^{A}(3) \supset Q_{n,k}^{A}(3)\) and \(m < n\).

A combination of \([30], [37]\), Lemma \[4.10\] \[38\] and Lemma \[4.18\] give the desired result.

\[\Box\]

**Proof of Theorem 4.4.** The chain of implications \( \mathbb{I} \iff \mathbb{II} \iff \mathbb{III} \) with corresponding estimates for \( f \in L_{0}(S; L_{r,d}(\mathbb{R}^{d}; X)) \) can be obtained in the same way as Theorem 4.3 with some natural modifications; in particular, Lemmas 4.16 and 4.18 need to be replaced with Lemmas 4.21 and 4.22 respectively. Furthermore, \( \mathbb{II} \iff \mathbb{IV} \) can be done in the same way as \[22\] Theorem 1.1.14\, similarly to the implication \( \mathbb{III} \iff \mathbb{III} \) (see the proof of \( \mathbb{II} \iff \mathbb{III} \) in Theorem 4.3).

Fix \( q \in (0, \infty) \) with \( q \leq r_{min} \land p_{min} \) and let \( \mathbb{V}^{*}_{q} \) be the statements \( \mathbb{II} \) and \( \mathbb{IV} \), respectively, in which \( p \) gets replaced by \( q := (q, \ldots, q) \in (0, \infty)^{\ell} \). Then, clearly, \( \mathbb{III} \iff \mathbb{III}^{*}_{q} \) and \( \mathbb{IV} \iff \mathbb{IV}^{*}_{q} \).

To finish this proof, it suffices to establish the implication \( \mathbb{V} \iff \mathbb{IV}^{*}_{q} \) for \( f \in L_{0}(S; L_{r,d}(\mathbb{R}^{d}; X)) \) and the implications \( \mathbb{III}^{*}_{q} \iff \mathbb{V} \) and \( \mathbb{IV}^{*}_{q} \iff \mathbb{III} \) for \( f \) of the form \( f = \sum_{i \in I} 1_{S_{i}} \otimes f_{i} \) with \( (S_{i})_{i \in I} \subset \mathcal{M} \) a countable family of mutually disjoint sets and \( (f_{i})_{i \in I} \subset L_{r,d,loc}(\mathbb{R}^{d}; X) \).

\( \mathbb{V} \iff \mathbb{IV}^{*}_{q} \). For this implication we just observe that, for \( x \in Q_{n,k}^{A} \) and \( n \geq 1 \),

\[
\mathcal{E}_{M,x^{+},n}(f)(x) \lesssim \mathcal{E}_{M}((f, x^{+}), Q_{n,k}^{A}(3), L_{q}) \lesssim M_{q}^{A}(g_{x^{+},n})(x) \lesssim M_{r}^{A}(g_{x^{+},n})(x).
\]
(III) \( q \Rightarrow (V) \) for \( f \) of the form \( f = \sum_{i \in I} 1_{S_i} \otimes f^{[i]} \) with \( (S_i)_{i \in I} \subset \mathcal{A} \) a countable family of mutually disjoint sets and \( (f^{[i]})_{i \in I} \in L_{r,d,\text{loc}}(\mathbb{R}^d; X) \): By Lemma 4.11 for each \( i \in I \) and \((x^*, n, k) \in X^* \times \mathbb{N}_{\geq 1} \times \mathbb{Z}^d \) there exists a \( \pi_{x^*, n, k} \in \mathcal{P}_{M-1}^d \) such that

\[
\left\langle f^{[i]}, x^* \right\rangle - \pi_{x^*, n, k}^{[i]} \left\| Q_{x^*, n, k}^{A, q}(3) \right\| \lesssim \left\| \sum_{i \in I} 1_{S_i} \otimes f^{[i]} \right\|_{L_q(Q_{x^*, n, k}^{A, q}(b) \cap Q_{x^*, n, k}^{A, q}(b), L_q)}^{1/q}.
\]

Defining \( \pi_{x^*, n, k} \in L_0(S; \mathcal{P}_{M-1}^d) \) by \( \pi_{x^*, n, k} := \sum_{i \in I} 1_{S_i} \otimes \pi_{x^*, n, k}^{[i]} \), we obtain

\[
\left\langle f, x^* \right\rangle - \pi_{x^*, n, k} \left\| Q_{x^*, n, k}^{A, q}(3) \right\| \lesssim \left\| \sum_{i \in I} 1_{S_i} \otimes f^{[i]} \right\|_{L_q(Q_{x^*, n, k}^{A, q}(b) \cap Q_{x^*, n, k}^{A, q}(b), L_q)}^{1/q}.
\]

Since

\[
\# \{k \in \mathbb{Z}^d : x \in Q_{x^*, n, k}^{A, q}(3) \} \lesssim 1, \quad x \in \mathbb{R}^d, n \in \mathbb{N},
\]

it follows that

\[
\left\| \{g^{[i]} \}_{i \in I} \right\|_{L_q(Q_{x^*, n, k}^{A, q}(3) \cap Q_{x^*, n, k}^{A, q}(3), L_q)} \lesssim \left\| \sum_{i \in I} 1_{S_i} \otimes f^{[i]} \right\|_{L_q(Q_{x^*, n, k}^{A, q}(b) \cap Q_{x^*, n, k}^{A, q}(b), L_q)}^{1/q}.
\]

(IV) \( q \Rightarrow (III) \) for \( f \) of the form \( f = \sum_{i \in I} 1_{S_i} \otimes f^{[i]} \) with \( (S_i)_{i \in I} \subset \mathcal{A} \) a countable family of mutually disjoint sets and \( (f^{[i]})_{i \in I} \in L_{r,d,\text{loc}}(\mathbb{R}^d; X) \): Let \( \omega \in C_c([-1, 2]^d) \) be such that

\[
\sum_{k \in \mathbb{Z}^d} \omega(x - k) = 1, \quad x \in \mathbb{R}^d,
\]

and put \( \omega_{n, k} := \omega(A^{2^n} \cdot -k) \) and \( Q_{x^*, n, k}^{\omega} := A_{2^{-n}}([-1, 2]^d + k) \) for \((n, k) \in \mathbb{N} \times \mathbb{Z}^d \); so \( \text{supp} \omega_{n, k} \subset Q_{x^*, n, k}^{\omega} \). Define

\[
I_{n, k} := \{i \in \mathbb{Z}^d : Q_{x^*, n, k}^{\omega} \cap Q_{x^*, n-1, i} \neq \emptyset\}, \quad (n, k) \in \mathbb{N}_{\geq 1} \times \mathbb{Z}^d.
\]

Then \( \#I_{n, k} \lesssim 1 \) and there exists \( b \in (1, \infty) \) such that

\[
Q_{x^*, n, k}^{\omega} \subset Q_{x^*, n, k}^{A, q}(b) \cap Q_{x^*, n-1, l}^{A, q}(b), \quad l \in I_{n, k}, (n, k) \in \mathbb{N}_{\geq 1} \times \mathbb{Z}^d.
\]

Furthermore, there exists \( n_0 \in \mathbb{N}_{\geq 1} \) such that

\[
Q_{x^*, n, k}^{A, q}(b) \cup Q_{x^*, n-1, l}^{A, q}(b) \subset B^{A}(x, 2^{-(n-n_0)}), \quad x \in Q_{x^*, n, k}^{\omega}, (n, k) \in \mathbb{N} \times \mathbb{Z}^d.
\]

For each \( i \in I \), let us pick \( \pi_{x^*, n, k}^{[i]} \) such that \( \pi_{x^*, n, k}^{[i]}(x^*) \in \mathcal{P}_{M-1}^d \) with the property that

\[
\left\| \{f^{[i]} \}_{i \in I} \right\|_{L_q(Q_{x^*, n, k}^{A, q}(b) \cap Q_{x^*, n, k}^{A, q}(b), L_q)} \lesssim \left\| \sum_{i \in I} 1_{S_i} \otimes f^{[i]} \right\|_{L_q(Q_{x^*, n, k}^{A, q}(b) \cap Q_{x^*, n, k}^{A, q}(b), L_q)}^{1/q}.
\]

and put \( \pi_{x^*, n, k} := \sum_{i \in I} 1_{S_i} \otimes \pi_{x^*, n, k}^{[i]} \in L_0(S; \mathcal{P}_{M-1}^d) \). Define

\[
\omega_{n, k} := \begin{cases} 
\omega_{n, k} \prod_{l \in I_{n-1, l}} \omega_{n-1, l} \mid \pi_{x^*, n-1, l} \mid, & n > n_0, \\
\omega_{n, k} \pi_{x^*, n, k}, & n = n_0, \\
0, & n < n_0.
\end{cases}
\]

Let \( x^* \in X^* \) and \((n, k) \in \mathbb{N}_{\geq n_0 + 1} \times \mathbb{Z}^d \). Let \( l \in I_{n, k} \). For \( x \in Q_{x^*, n, k}^{\omega} \) we can estimate

\[
\left\| \pi_{x^*, n, k} - \pi_{x^*, n-1, l} \right\|_{L_q(Q_{x^*, n, k}^{\omega})} \lesssim \left\| \left\langle f, x^* \right\rangle - \pi_{x^*, n, k} \right\|_{L_q(Q_{x^*, n, k}^{A, q}(b))}.
\]
implying
\[
\|(\tilde{\pi}_{x,n,k} - \pi_{x,n-1,l})(A_{2^{-n}} - + k)\|_{C^k([-1,2]^m)} \\
\lesssim 2^{ntr(A^\nu)/q}E^M(f,x) + B^A(x,2^{-(n-n_0)}), L_q)
\]
in view of Corollary 4.14. Since \#I_{n,k} \leq 1, it follows that
\[
\|u_{x,n,k}(A_{2^{-n}} - + k)\|_{C^k([-1,2]^m)} \lesssim E^M(f,x) + B^A(x,2^{-(n-n_0)}), L_q)
\]
\[
(42)
\]
For \(n = n_0\) we similarly have
\[
\|u_{x,n_0,k}(A_{2^{-n}} - + k)\|_{C^k([-1,2]^m)} \lesssim \|f, x\|_{L_q(B^A(x,1))}
\lesssim \|x\| \|M^A_q(\|f\|_x)(x)
\lesssim \|x\| \|M^A_q(\|f\|_x)(x)
\]n \in Q_{n,k}^n.
\]
Define \(s_{x,n} := u_{x,n,k}(A_{2^{-n}} - + k)\|_{C^k([-1,2]^m)},
\]
\[
a_{x,n,k} = \left\{ \begin{array}{ll}
\frac{u_{x,n,k}}{s_{x,n,k}}, & s_{x,n,k} \neq 0, \\
0, & s_{x,n,k} = 0,
\end{array} \right.
\]
and \(b_{x,n,k} := u_{x,n,k}(A_{2^{-n}} - + k).\) Then \(b_{x,n,k} \in C^k([-1,2]^d)\) with \|b_{x,n,k}\|_{C^k} \leq 1 and \((s_{x,n,k}, (x,n,k)) \in \tilde{y}^A(E;X)\) with
\[
\|(s_{x,n,k})(x,n,k)\|_{\tilde{y}^A(E;X)} \lesssim \|M^A_Q(\|f\|_x)(E_0)
+ \|\{E^M_{x,n} = 0\}(f)\}_{x \in X_{\geq n_0}} \|_{\mathcal{B}(X_e;E(N_{\geq n_0} + 1))}
\lesssim \|f\|_{E_0(X)} + 2^{x-n_0} \|\{E^M_{x,n} = 0\}(f)\}_{x \in X_{\geq n_0}} \|_{\mathcal{B}(X_e;E(N_{\geq 1}))}
\]
Note that, for \(n \geq n_0 + 1,
\]
\[
\sum_{k \in \mathbb{Z}^d} s_{x,n,k} a_{x,n,k} = \sum_{k \in \mathbb{Z}^d} u_{x,n,k}
\]
\[
= \sum_{k \in \mathbb{Z}^d} \pi_{x,n,k} \omega_{x,n,k} \sum_{l \in \mathbb{Z}^d} \omega_{n,l} - \sum_{k \in \mathbb{Z}^d} \omega_{n,k} \sum_{l \in \mathbb{Z}^d} \pi_{x,n-1,l} \omega_{n-1,l}
\]
\[
= \sum_{k \in \mathbb{Z}^d} \pi_{x,n,k} \omega_{x,n,k} - \sum_{l \in \mathbb{Z}^d} \pi_{x,n-1,l} \omega_{n-1,l}.
\]
In combination with Lemma 4.15 and an alternating sum argument, this implies that
\[
(f,x) = \sum_{n=0}^\infty \sum_{k \in \mathbb{Z}^d} s_{x,n,k} a_{x,n,k} \in L_0(S; L_{q,loc}(\mathbb{R}^d)).
\]
The required convergence finally follows from this with an argument as in (the last part of) the proof of the implication \(\text{(a)} \Rightarrow \text{(b)}\) in Theorem 4.13. \(\square\)
Proof of Corollary 4.6. This is an immediate consequence of Theorems 3.17, 4.5 and the observation that
\[ \| (d_M^A f)_{n \geq 1} \|_{E(N_{\geq 1}; X)} \leq \| (d_M^A f)_{n \geq 1} \|_{E(N_{\geq 1}; X)}. \]
\[ \square \]

Proof of Theorem 4.7. The estimates
\[ \| f \|_{Y^A(E; X)} \leq \| f \|_{E_0(X)} \]
follow from Theorem 3.17. Combining the inclusion
\[ Y^A(E; X) \subset E_0(X) \]
with the estimate corresponding to the implication \( \text{Theorem 4.3} \) gives
\[ \| f \|_{E_0(X)} + \| (d_M^A f)_{n \geq 1} \|_{E(N_{\geq 1}; X)} \lesssim \| f \|_{Y^A(E; X)}. \]
As it clearly holds that
\[ \| I_{M,n}^A(f) \|_{X} \leq d_M^A(f), \quad n \in \mathbb{N}, \]
it remains to be shown that
\[ \| f \|_{Y^A(E; X)} \lesssim \| f \|_{E_0(X)} + \| (I_{M,n}^A(f))_{n \geq 1} \|_{E(N_{\geq 1}; X)}. \]

Put \( K := 1_{B^A(0,1)}(2^{-n}, f), \quad n \in \mathbb{N}, \)
\[ K_A(t, f) := t^{-\text{tr}(A^\oplus)} K_A^M(A_{t-1} \cdot) * f + (-1)^M \tau \phi(t f), \quad t \in (0, \infty). \]
Note that
\[ I_{M,n}^A(f) = K_A^M(2^{-n}, f), \quad n \in \mathbb{N}. \]

As \( K_A^M(0) = \sum_{i=0}^{M-1} (-1)^i \binom{M}{i} \hat{K}_{[M-i]^{-1}}, \)
where \( K_A(t, f) := t^d K(-t \cdot) \) for \( t \in (0, \infty). \)
Furthermore, put
\[ K_A^M(t, f) := t^{-\text{tr}(A^\oplus)} K_A^M(A_{t-1} \cdot) * f + (-1)^M \tau \phi(t f), \quad t \in (0, \infty). \]

So there exists \( N \in \mathbb{N} \) such that \( k := 2^N \phi 1_{B^A(0,1)}(2^{-n}, f) \)
\[ I_{M,n}^A(f) \] satisfies
\[ |K_A^M(x)| \geq \frac{c}{2}, \quad x \in \mathbb{R}^d, \quad \frac{c}{2} < \rho A(x) < 2c, \]
for \( \delta := 2^N c > 0. \) Let \( \varphi = (\phi_n)_{n \in \mathbb{N}} \in \Phi^A(\mathbb{R}^d) \) be such that \( \text{supp} \hat{\varphi}_1 \subset \{ x : 2c \leq \rho A(x) \} \) (see Definition 3.13). Let \( (k_n)_{n \in \mathbb{N}} \) be defined by \( k_n := 2^N \phi 1_{B^A(0,1)}(2^{-n}, f). \)
Then, by construction,
\[ k_n * f = K_A^M(2^{-(n+N)}, f) - K_A^M(2^{-n}, f) \]
\[ I_{M,n}^A(f) \]
An application of Lemma 4.23 thus yields that
\[ \| (\varphi_n * f)_{n \geq 1} \|_{E(N_{\geq 1}; X)} \lesssim \| (k_n * f)_{n \geq 1} \|_{E(N_{\geq 1}; X)} \]
\[ \lesssim \| (I_{M,n}^A(f))_{n \geq 1} \|_{E(N_{\geq 1}; X)} + \| (I_{M,n}^A(f))_{n \geq 1} \|_{E(N_{\geq 1}; X)} \]
\[ \lesssim (2^{-\varepsilon+N} + 1) \| (I_{M,n}^A(f))_{n \geq 1} \|_{E(N_{\geq 1}; X)} \]
\[ \| \varphi_n * f \|_{X} \lesssim M^A(\| f \|_{X}), \]
As it clearly holds that
\[ \| \varphi_n * f \|_{E_0(X)} \lesssim \| f \|_{E_0(X)}. \]
A combination of Proposition 5.13, 40 and 47 finally gives \( 44 \).
5. AN INTERSECTION REPRESENTATION

Let \( E \in S(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \) with \( \varepsilon_+, \varepsilon_- > 0 \). Let \( J \) be a nonempty subset of \( \{1, \ldots, \ell\} \), say \( J = \{j_1, \ldots, j_k\} \) with \( 1 \leq j_1 \leq \ldots \leq j_k \leq \ell \). Put \( d_J = (d_{j_1}, \ldots, d_{j_k}) \), \( d_J := |d_J|_1 \) \( A_J := (A_{j_1}, \ldots, A_{j_k}) \), \( r_J := (r_{j_1}, \ldots, r_{j_k}) \) and

\[
(S_J, \mathcal{A}_J, \mu_J) := (\mathbb{R}^{d-d_J}, B(\mathbb{R}^{d-d_J}), \lambda^{d-d_J}) \otimes (S, \mathcal{A}, \mu)
\]

Furthermore, define \( E_{[d_J]} \) as the quasi-Banach space \( E \) viewed as quasi-Banach function space on the measure space \( \mathbb{R}^{d_J} \times \mathbb{N} \times S_J \). Then

\[
E_{[d_J]} \in S(\varepsilon_+, \varepsilon_-, A_J, r_J, (S_J, \mathcal{A}_J, \mu_J))
\]

By Remark 3.10,

\[
\tilde{Y}L^A(E; X) \hookrightarrow E^A_\otimes (B_1(\nu) \subset (X)) \hookrightarrow L_0(S; L_{r, d, \text{loc}}(\mathbb{R}^d; X)).
\]

In the same way,

\[
\tilde{Y}L^A(E_{[d]}; X) \hookrightarrow E^A_\otimes (B_1(\nu) \subset (X)) \hookrightarrow L_0(S; L_{r, d, \text{loc}}(\mathbb{R}^d; X)),
\]

In particular, it makes sense to compare \( \tilde{Y}L^A_{[d]}(E_{[d]}; X) \) with \( \tilde{Y}L^A(E; X) \).

**Theorem 5.1.** Let \( E \in S(\varepsilon_+, \varepsilon_-, A, r, (S, \mathcal{A}, \mu)) \) with \( \varepsilon_+, \varepsilon_- > 0 \). Let \( \{J_1, \ldots, J_L\} \) be a partition of \( \{1, \ldots, \ell\} \).

(i) There is the estimate

\[
\|f\|_{\tilde{Y}L^A_{[d]}(E_{[d]}; X)} \leq \|f\|_{\tilde{Y}L^A(E; X)}, \quad l \in \{1, \ldots, L\},
\]

for all \( f \in L_0(S; L_{r, d, \text{loc}}(\mathbb{R}^d; X)) \).

(ii) There is the estimate

\[
\|f\|_{\tilde{Y}L^A(E; X)} \lesssim \sum_{l=1}^L \|f\|_{\tilde{Y}L^A_{[d]}(E_{[d]}; X)}
\]

for all \( f \in L_0(S; L_{r, d, \text{loc}}(\mathbb{R}^d; X)) \) of the form \( f = \sum_{i \in I} 1_{S_i} \otimes f(x) \) with \( \{S_i\}_{i \in I} \subset \mathcal{A} \) a countable family of mutually disjoint sets and \( (f(x))_{i \in I} \in L_{r, d, \text{loc}}(\mathbb{R}^d; X) \).

In particular, in case \( (S, \mathcal{A}, \mu) \) is atomic,

\[
\tilde{Y}L^A(E; X) = \bigcap_{l=1}^L \tilde{Y}L^A_{[d]}(E_{[d]}; X)
\]

with an equivalence of quasi-norms.

**Proof.** Let us start with (i). Fix \( l \in \{1, \ldots, L\} \) and write \( J := J_l \). Let \( f \in \tilde{Y}L^A(E; X) \). Let \( \epsilon > 0 \). Choose \((g_0)_n\) and \((f_{x, n})_{(x, n)}\) as in Definition 3.9 with \( \|(g_0)_n\|_E \leq (1 + \epsilon) \|f\|_{\tilde{Y}L^A(E; X)} \). As \( f_{x, n} \in L_0(S; S'(\mathbb{R}^d)) \) with \( \text{supp} \tilde{f}_{x, n} \subset B^A(0, 2^{n+1}) \), we can naturally view \( f_{x, n} \) as an element of \( L_0(S_J; S'(\mathbb{R}^{d-J})) \) with \( \text{supp} f_{x, n} \subset B^A(0, 2^{n+1}) \). Since

\[
L_0(S; L_{r, d, \text{loc}}(\mathbb{R}^d)) \hookrightarrow L_0(S_J; L_{r, d, \text{loc}}(\mathbb{R}^{d-J})),
\]

we have

\[
\|f\|_{\tilde{Y}L^A(E; X)} \lesssim \sum_{l=1}^L \|f\|_{\tilde{Y}L^A_{[d]}(E_{[d]}; X)}
\]

for all \( f \in L_0(S; L_{r, d, \text{loc}}(\mathbb{R}^d; X)) \) of the form \( f = \sum_{i \in I} 1_{S_i} \otimes f^{(i)} \) with \( \{S_i\}_{i \in I} \subset \mathcal{A} \) a countable family of mutually disjoint sets and \( (f^{(i)})_{i \in I} \in L_{r, d, \text{loc}}(\mathbb{R}^d; X) \).
it follows that \( f \in \widetilde{Y}_L^{A_j}(E_{[x,j]}; X) \) with
\[
||f||_{\widetilde{Y}_L^{A_j}(E_{[x,j]}; X)} \lesssim ||(g_n)_n||_{E_{[x,j]}} = ||(g_n)_n||_E \leq (1 + \epsilon)||f||_{\widetilde{Y}_L^{A_j}(E; X)}.
\]

Let us next treat (iii). We may without loss of generality assume that \( J_l = \{ l \} \) for each \( l \in \{1, \ldots, \ell \} \). We will write \( E_{[x,j]} = E_{[x,j]} \).

Let \( f \in \bigcap_{j=1}^{\ell} \widetilde{Y}_L^{A_j}(E_{[x,j]}; X) \) be of the form \( f = \sum_{i \in I} 1_{S_i} \otimes f^{[i]} \) with \( (S_i)_{i \in I} \subset \mathcal{A} \) a countable family of mutually disjoint sets and \((f^{[i]})_{i \in I} \in L_{r,d, loc}(\mathbb{R}^d; X)\). In order to establish the desired inequality, we will combine the estimate corresponding to the implication (iii) \( \Rightarrow \) (i) from Theorem 4.5 for the space \( \widetilde{Y}_L^{A_j}(E; X) \) with the estimates from Proposition 4.9 for each of the spaces \( \widetilde{Y}_L^{A_j}(E_{[x,j]}; X) \). To this end, pick \( M \in \mathbb{N} \) with \( M\Lambda_{\min} > \varepsilon^- \). Now, let us define \((g_{x^*,n})_{(x^*,n) \in X^* \times \mathbb{N}} \) and \((g_{c,x^*,n,j})_{(x^*,n) \in X^* \times \mathbb{N}}, j \in \{1, \ldots, \ell \} \) and \( c \in \mathbb{R} \), by
\[
g_{x^*,n} := \begin{cases} d_{0,x^*,0}^{A_r}(f), & n = 0, \\
d_{0,x^*,0}^{A_r}(f), & n \geq 1, \\
d_{\ell M,x^*,n}^{A_r}(f), & \end{cases}
\]
and
\[
g_{c,x^*,n,j} := \begin{cases} d_{\ell j,0}^{A_r}(f), & n = 0, \\
d_{\ell j,0}^{A_r}(f), & n \geq 1, \\
d_{\ell M,x^*,n}^{A_r}(f), & \end{cases}
\]
where the notation is as in Theorem 4.10 and Proposition 4.9.

For \( n = 0 \) we have
\[
g_{x^*,0} = d_{0,x^*,0}^{A_r}(f) \leq \bigg[ \bigcap_{i=2}^{\ell} M_{[x,i]}^{A_1, r_1} \bigg] \bigg( d_{0,x^*,0}^{A_r}(f) \bigg) \leq M_r^{A_r} \bigg[ d_{0,x^*,0}^{A_r}(f) \bigg] = M_r^{A_r} \bigg[ g_{c,x^*,0,1} \bigg], \quad c \in \mathbb{R}.
\]

Now let \( n \geq 1 \). We will use the following elementary fact (cf. 7.1.4.16): there exist \( C \in (0, \infty), K \in \mathbb{N} \) and \( \{c_j^{[k]}\}_{j=1,...,\ell} \subset \mathbb{R} \) such that
\[
|\Delta_x^{M \Lambda} h(x)| \leq C K \sum_{k=0}^K \sum_{j=1}^{\ell} |\Delta_j^{M \Lambda} h(x) + \sum_{i=1}^{\ell} c_i^{[k]} L_{[x,i]}(x)|
\]
for all \( h \in L_0(\mathbb{R}^d) \). Applying this pointwise in \( S \) to \( \langle f, x^* \rangle \), we find that
\[
g_{x^*,n} = d_{\ell M,x^*,n}^{A_r}(f) = 2^{ntr(A)} ||z \mapsto \Delta_x^{M \Lambda} \langle f, x^* \rangle||_{L_{r,d}(B(A,0,2^{-n}))}
\]
\[
\lesssim \sum_{k=0}^K \sum_{j=1}^{\ell} 2^{ntr(A)} ||z \mapsto \prod_{l=1}^{\ell} L_{c_l^{[k]} L_{[x,l]}(x)} \Delta_j^{M \Lambda} \langle f, x^* \rangle||_{L_{r,d}(B(A,0,2^{-n}))}
\]
\[
\lesssim \sum_{k=0}^K \sum_{j=1}^{\ell} 2^{ntr(A)} ||z \mapsto L_{c_j^{[k]} L_{[x,j]}(x)} \Delta_j^{M \Lambda} \langle f, x^* \rangle||_{L_{r,j}(B(A,0,2^{-n}))}
\]
\[
\lesssim \sum_{k=0}^K \sum_{j=1}^{\ell} M_r^{A_r} \bigg[ 2^{ntr(A)} ||z \mapsto L_{c_j^{[k]} L_{[x,j]}(x)} \Delta_j^{M \Lambda} \langle f, x^* \rangle||_{L_{r,j}(B(A,0,2^{-n}))} \bigg]
\]
\[
(49)
\]
\[
= \sum_{k=0}^K \sum_{j=1}^{\ell} M_r^{A_r} \bigg[ d_{[x,j]}^{[k], A_j, r_j} \bigg] \bigg( g_{c_j^{[k]}, x^*, n,j} \bigg) = \sum_{k=0}^K \sum_{j=1}^{\ell} M_r^{A_r} \bigg[ g_{c_j^{[k]}, x^*, n,j} \bigg].
\]
A combination of (18) and (19) gives

\[ g_{x^*, n} \leq \sum_{k=0}^{K} \sum_{j=1}^{\ell} M_{r}^{A} \left[ g_{E_{k}, x^*, n}^{i} \right](f) = \sum_{k=0}^{K} \sum_{j=1}^{\ell} M_{r}^{A} \left[ g_{E_{k}, x^*, n}^{j} \right](f) \]

for all \((x^*, n) \in X^* \times \mathbb{N}\). Therefore,

\[ \| \{ g_{x^*, n} \}(x^*, n) \|_{\mathcal{F}(X^*; E)} \leq \sum_{k=0}^{K} \sum_{j=1}^{\ell} \| \{ M_{r}^{A} \left[ g_{E_{k}, x^*, n}^{j} \right] \}(x^*, n) \|_{\mathcal{F}(X^*; E)} \]

\[ \leq \sum_{k=0}^{K} \sum_{j=1}^{\ell} \| \{ j_{E_{k}, x^*, n}^{j} \}(x^*, n) \|_{\mathcal{F}(X^*; E)} \]

\[ = \sum_{k=0}^{K} \sum_{j=1}^{\ell} \| \{ j_{E_{k}, x^*, n}^{j} \}(x^*, n) \|_{\mathcal{F}(X^*; E)} \]

The desired result now follows from a combination of Theorem 4.5 and Proposition 1.9

As an immediate corollary to Theorems 3.17 and 5.1 we have:

**Corollary 5.2.** Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \mathbf{A}, r, (S, \mathcal{A}, \mu)) \) with \( \varepsilon_+ > 0 \) and \((S, \mathcal{A}, \mu)\) atomic. Let \( \{ J_1, \ldots, J_L \} \) be a partition of \( \{ 1, \ldots, \ell \} \). If \( \varepsilon_+ > \text{tr}(A) \cdot (r^{-1} - 1)_{+}, \)
then

\[ Y^{A}(E; X) = YL^{A}(E; X) = YL^{A}(E; X) = \bigcap_{l=1}^{L} YL^{A}(E_{[i]: J_l}; X) \]

\[ = \bigcap_{l=1}^{L} YL^{A}(E_{[i]: J_l}; X) = \bigcap_{l=1}^{L} YL^{A}(E_{[i]: J_l}; X) \]

with an equivalence of quasi-norms.

**Theorem 5.3.** Let \( E \in \mathcal{S}(\varepsilon_+, \varepsilon_-, \mathbf{A}, 1, (S, \mathcal{A}, \mu)) \) with \( \varepsilon_+ > 0 \). Let \( \{ J_1, \ldots, J_L \} \) be a partition of \( \{ 1, \ldots, \ell \} \). Then

\[ Y^{A}(E; X) = YL^{A}(E; X) = YL^{A}(E; X) = \bigcap_{l=1}^{L} YL^{A}(E_{[i]: J_l}; X) \]

\[ = \bigcap_{l=1}^{L} YL^{A}(E_{[i]: J_l}; X) = \bigcap_{l=1}^{L} YL^{A}(E_{[i]: J_l}; X) \]

with an equivalence of quasi-norms.

*Proof.* In view of Theorem 3.17, this can be proved in exactly the same way as Theorem 5.1 using Theorem 4.7 instead of Theorem 4.5.

**Remark 5.4.** In light of Example 3.15, the intersection representation

\[ (50) \quad Y^{A}(E; X) = \bigcap_{l=1}^{L} YL^{A}(E_{[i]: J_l}; X) \]

from Corollary 5.2 and Theorem 5.3 extends the well-known Fubini property for the classical Triebel-Lizorkin spaces \( F^{s}_{p,q}(\mathbb{R}^d) \) (see [70] Section 4) and the references given therein. It also covers Theorem 1.1 and thereby (4), the intersection
representation from [24] Proposition 3.23. The intersection representation [42] Proposition 5.2.38 for anisotropic weighted mixed-norm Triebel-Lizorkin is a special case as well. Furthermore, it suggests an operator sum theorem for generalized Triebel-Lizorkin spaces in the sense of [39].

**Example 5.5.** Let us state the intersection representation [50] from Corollary 5.2 and Theorem 5.3 for some concrete choices of $E$ (see Examples 3.4 and 3.15) for the case that $\ell = 2$ with partition $\{\{1\}, \{2\}\}$ of $\{1, 2\}$.

1. Let $p \in (0, \infty)^2$, $q \in (0, \infty]$, $w \in A_p(\mathbb{R}^d, A_1) \times A_p(\mathbb{R}^d, A_2)$ and $s \in \mathbb{R}$. Pick $r \in (0, \infty)^2$ such that $r_1 < p_1 \cap q$ and $r_2 < p_2 \cap q$. If $s > \text{tr}(A) \cdot (r^{-1} - 1)_+$, then
   \[
   F^{s, A, 2}_{p, q}(\mathbb{R}^d, w; X) = F^{s, A, 2}_{p, q}(\mathbb{R}^d, w_2; L^2_{p_1}(\mathbb{R}^d, w_1); X) \\
   \cap L^2_{p_2}(\mathbb{R}^d, w_2; F^{s, A, 1}_{p_1, q}(\mathbb{R}^d, w_1); X).
   \]

2. Similarly to Proposition 3.14 we have the following Littlewood-Paley decomposition description for $F^{s, A, 2}_{p, q}(\mathbb{R}^d, w_2; L^2_{p_1}(\mathbb{R}^d, w_1); X)$ and
   \[
   Y^A(L^2_{p_2}(\mathbb{R}^d, w_2); L^2_{p_1}(\mathbb{R}^d, w_1); X) \\
   = F^{s, A, 2}_{p, q}(\mathbb{R}^d, w_2; L^2_{p_1}(\mathbb{R}^d, w_1); X) \cap L^2_{p_2}(\mathbb{R}^d, w_2; B^{s, A, 1}_{p_1, q}(\mathbb{R}^d, w_1); X)).
   \]

6. **Duality**

**Definition 6.1.** Let $E \in S(\varepsilon, \varepsilon, A, r, (S, \sigma, \mu))$. We define $Y^A(E, X^*, \sigma(X^*, X))$ as the space of all $f \in S'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)))$ which have a representation
   \[
   f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad S'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)))
   \]
   with $(f_n) \subset S'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)))$ satisfying the Fourier support condition
   \[
   \text{supp} \hat{f}_0 \subset B^A(0, 2) \\
   \text{supp} \hat{f}_n \subset B^A(0, 2^{n+1}) \setminus B^A(0, 2^{n-1}), \quad n \in \mathbb{N},
   \]
   and $(f_n) \in E(X)$. We equip $Y^A(E, X^*, \sigma(X^*, X))$ with the quasinorm
   \[
   ||f||_{Y^A(E, X^*, \sigma(X^*, X))} := \inf ||(f_n)||_{E(X^*, \sigma(X^*, X))},
   \]
   where the infimum is taken over all representations as above.

Similarly to Proposition 3.13 we have the following Littlewood-Paley decomposition description for $Y^A(E, X^*, \sigma(X^*, X))$:

**Proposition 6.2.** Let $E \in S(\varepsilon, \varepsilon, A, r, (S, \sigma, \mu))$. Let $\varphi = (\varphi_n)_{n \in \mathbb{N}} \in \Phi^A(\mathbb{R}^d)$ with associated sequence of convolution operators $(S_n)_{n \in \mathbb{N}}$. Then
   \[
   Y^A(E, X^*, \sigma(X^*, X)) \\
   = \{ f \in S'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X)) : (S_n f)_{n \in \mathbb{N}} \in E(X^*, \sigma(X^*, X)) \}
   \]
   with
   \[
   ||f||_{Y^A(E, X^*, \sigma(X^*, X))} \approx ||(S_n f)_{n \in \mathbb{N}}||_{E(X^*, \sigma(X^*, X))}.
   \]
Using the description from the above proposition it is easy to see that
\[ Y^A(E; X^*) = Y^A(E; X^*, \sigma(X^*, X)) \cap S'(\mathbb{R}^d; L_0(S; X)) \]
with an equivalence of quasinorms.

**Theorem 6.3.** Let \( E \in S(\varepsilon_+, \varepsilon_-, A, (S, \mathcal{A}, \mu)) \) be a Banach function space with an order continuous norm such that \( E^\times \in S(-\varepsilon_-, -\varepsilon_+, A, 1, (S, \mathcal{A}, \mu)) \). Assume that there exists a Banach function space \( F \) on \( S \) with an order continuous norm such that \( S(\mathbb{R}^d; F(X)) \rightarrow Y^A(E; X) \). Viewing
\[ [Y^A(E; X)]^* \rightarrow S'(\mathbb{R}^d; [F(X)]^*) = S'(\mathbb{R}^d; F^\times(X^*, \sigma(X^*, X))) \rightarrow S'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X))) \]
via the natural pairing, we have
\[ [Y^A(E; X)]^* = Y^A(E^\times; X^*, \sigma(X^*, X)). \]
Consequently, if \( X^* \) has the Radon-Nikodým property with respect to \( \mu \), then
\[ Y^A(E^\times; X^*) = [Y^A(E; X)]^* \rightarrow S'(\mathbb{R}^d; F^\times(X^*)) \rightarrow S'(\mathbb{R}^d; L_0(S; X^*)). \]

**Remark 6.4.** Note that the duality result from Theorem 6.3 together with some of the intersection representations from Section 5 can be used to obtain sum representations, see Section 6.

Let \( E \in S(\varepsilon_+, \varepsilon_-, A, 1, (S, \mathcal{A}, \mu)) \) be a Banach function space. By Remark 3.7 we then have
\[ E_i \hookrightarrow E_\otimes(B^{1, w A^1}_A) \hookrightarrow E_\otimes[A^{1, w A^1}_A], \]
from which it follows that
\[ E_i(X^*, \sigma(X^*, X)) \hookrightarrow E_\otimes[A^{1, w A^1}_A](X^*, \sigma(X^*, X)) \hookrightarrow S'(\mathbb{R}^d; E_\otimes(A^\times, \sigma(X^*, X))) \rightarrow S'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X))). \]

**Lemma 6.5.** Let \( E \in S(\varepsilon_+, \varepsilon_-, A, 1, (S, \mathcal{A}, \mu)) \) be a Banach function space and let \( Z \) be a Banach space with \( Z \hookrightarrow L_0(S; X^*, \sigma(X^*, X)) \). Let \( \varphi = (\varphi_n)_{n \in \mathbb{N}} \in \Phi^A(\mathbb{R}^d) \) with associated sequence of convolution operators \( (S_n)_{n \in \mathbb{N}} \) be such that
\[ \supp \hat{\varphi}_0 \subset B^A(0, 2), \quad \supp \hat{\varphi}_n \subset B^A(0, 2^{n+1}) \backslash B^A(0, 2^{n-1}), \quad n \in \mathbb{N}. \]
Then
\[ Y^A(E; X^*, \sigma(X^*, X)) \cap S'(\mathbb{R}^d; Z) \]
\[ = \left\{ f \in S'(\mathbb{R}^d; Z) : \exists (f_k)_k \in E(X^*, \sigma(X^*, X)), f = \sum_{k=0}^{\infty} S_k f_k \text{ in } S'(\mathbb{R}^d; Z) \right\} \]
with
\[ \| f \|_{Y^A(E; X^*, \sigma(X^*, X))} \leq \inf \| (f_k)_k \|_{E(X^*, \sigma(X^*, X))}. \]

**Proof.** Given \( f \in Y^A(E; X^*, \sigma(X^*, X)) \cap S'(\mathbb{R}^d; Z) \), let \( f_k := T_k f \), where \( T_k := S_{k-1} + S_k + S_{k+1} \). Then \( S_k f_k = S_k f \), so \( f = \sum_{k=0}^{\infty} S_k f_k \) in \( S'(\mathbb{R}^d; Z) \). From
\[ \| (f_k, x) \| = |T_k(S_k f, x)| \leq M^A(S_k f, x) \leq M^A \vartheta(S_k f), \quad x \in B_X, \]

it follows that \( \vartheta(f_k) \leq M^A \vartheta(S_k f) \). Using that \( M^A \) is bounded on \( E \), we find
\[ \| (f_k)_k \|_{E(X^*, \sigma(X^*, X))} \leq \| (S_k f)_k \|_{E(X^*, \sigma(X^*, X))} \overset{[63]}{=} \| f \|_{Y^A(E; X^*, \sigma(X^*, X))}. \]
For the converse, let \( f = \sum_{k=0}^{\infty} S_k f_k \) in \( S'(\mathbb{R}^d; Z) \) with \( (f_k)_k \in E(X^*, \sigma(X^*, X)) \). Then
\[
|\langle S_k f_k, x \rangle| = |S_k(f_k, x)| \lesssim M^A(f_k, x) \leq M^A \vartheta(f_k), \quad x \in B_X,
\]
so that \( \vartheta(S_k f) \lesssim M^A \vartheta(f_k) \). In view of
\[
f = \sum_{k=0}^{\infty} S_k f_k \quad \quad S'(\mathbb{R}^d; Z) \hookrightarrow S'(\mathbb{R}^d; L_0(S; X^*, \sigma(X^*, X))),
\]
we have the boundedness of \( M^A \) on \( E \), it follows that \( f \in Y^A(E; X^*, \sigma(X^*, X)) \) with
\[
||f||_{Y^A(E; X^*, \sigma(X^*, X))} \lesssim ||(S_k f_k)_k||_{E(X^*, \sigma(X^*, X))} \lesssim ||(f_k)_k||_{E(X^*, \sigma(X^*, X))}.
\]

\[\square\]

**Proof of Theorem 6.3.** By assumption and Proposition 3.21
\[
S(\mathbb{R}^d; F(X)) \hookrightarrow Y^A(E; X) \hookrightarrow S'(\mathbb{R}^d; E^A_0(X)),
\]
from which it follows that \( F \rightarrow E^A_0 \), implying in turn that \( E^A_0 \hookrightarrow F^\times \). On the other hand it holds that \( [E^A_0]^{\times} \hookrightarrow [E^A_0]^{\times} \). Therefore, \( [E^\times]^{\times} \hookrightarrow F^\times \). By (a variant of) Proposition 1.21 we thus obtain
\[
Y^A(E^\times; X^*, \sigma(X^*, X)) \hookrightarrow S'(\mathbb{R}^d; [E^A_0]^{\times}(X^*, \sigma(X^*, X))) \hookrightarrow S'(\mathbb{R}^d; F^\times(X^*, \sigma(X^*, X))).
\]

So we can use Lemma 6.3 with \( Z = F^\times(X^*, \sigma(X^*, X)) \) to describe \( Y^A(E^\times; X^*, \sigma(X^*, X)) \). Let us equip \( Y^A(E; X) \) with an equivalent norm from Proposition 3.14. Then
\[
\iota : Y^A(E; X) \rightarrow E(X), \quad f \mapsto (S_k f)_k
\]
defines an isometric linear mapping. By order continuity of \( E \) and \( F \), there are the natural identifications
\[
[E(X)]^* = E^\times(X^*, \sigma(X^*, X)) \quad \quad \text{and} \quad \quad [F(X)]^* = F^\times(X^*, \sigma(X^*, X)).
\]
As \( S(\mathbb{R}^d; F(X)) \hookrightarrow Y^A(E; X) \), we may thus view
\[
[Y^A(E; X)]^* \hookrightarrow S'(\mathbb{R}^d; [F(X)]^*) \Rightarrow S'(\mathbb{R}^d; F^\times(X^*, \sigma(X^*, X))).
\]

Denoting the adjoint of \( \iota \) by \( j \), we thus obtain the following commutative diagram:
\[
\begin{array}{ccc}
E^\times(X^*, \sigma(X^*, X)) & \overset{T}{\longrightarrow} & S'(\mathbb{R}^d; F^\times(X^*, \sigma(X^*, X))) \\
\downarrow & & \uparrow j \\
E^\times(X^*, \sigma(X^*, X)) / \ker j & \overset{j}{\cong} & [Y^A(E; X)]^*
\end{array}
\]
Here \( T \) is explicitly given by
\[
T(f_k)_k = \sum_{k=0}^{\infty} S_k f_k \quad \quad \text{in} \quad S'(\mathbb{R}^d; F^\times(X^*, \sigma(X^*, X))),
\]
which can be seen by testing against \( \phi \in S(\mathbb{R}^d; F(X)) \):
\[
\langle T(f_k)_k, \phi \rangle = \langle (f_k)_k, \iota \phi \rangle = \langle (f_k)_k, (S_k \phi)_k \rangle = \sum_{k=0}^{\infty} \langle f_k, S_k \phi \rangle = \sum_{k=0}^{\infty} \langle S_k f_k, \phi \rangle.
\]
The desired result follows by an application of Lemma 6.3 with \( Z = F^\times(X^*, \sigma(X^*, X)) \) (recall (53)).
\[\square\]
7. A Sum Representation

In this section we combine the intersection representation for \( Y^A(E; X) \) from Theorem 5.3 and the duality result Theorem 6.3 with the following fact on duality for intersection spaces: given an interpolation couple of Banach spaces \((Y, Z)\) for which \(Y \cap Z\) is dense in both \(Y\) and \(Z\), it holds that \((X^*, Y^*)\) is an interpolation couple of Banach space and

\[
[Y \cap Z]^* = Y^* + Z^*, \quad [X + Y]^* = X^* \cap Y^*,
\]

hold isometrically under the natural identifications (see [39, Theorem I.3.1]).

We let the notation be as in Section 6.

**Corollary 7.1.** Let \( E \in S(\varepsilon_+; S, \mathcal{A}, r; (S, \mathcal{A}, \mu)) \) be a Banach function space with an order continuous norm such that \( E^* \in S(-\varepsilon_-; S, \mathcal{A}, 1; (S, \mathcal{A}, \mu)) \) with \( \varepsilon_+, \varepsilon_- < 0 \). Suppose that \( X \) is reflexive. Let \( F \) be a Banach function space on \( S \) with an order continuous norm such that \( S(\mathbb{R}^d; F) \overset{d}{\hookrightarrow} Y^A(E; X) \). Let \( \{J_1, \ldots, J_L\} \) be a partition of \( \{1, \ldots, \ell\} \) and, for each \( l \in \{1, \ldots, L\} \), let \( F_l \) be a Banach function space on \( S_{J_l} \) with an order continuous norm such that \( S(\mathbb{R}^d; F(X)) \overset{d}{\hookrightarrow} S(\mathbb{R}^{d-d_{J_l}}; F_l(X)) \overset{d}{\hookrightarrow} Y^A_{J_l}(E_{[d; J_l]}; X) \). Then

\[
Y^A(E; X) = \sum_{l=1}^L Y^A_{J_l}(E_{[d; J_l]}; X)
\]

with an equivalence of norms.

**Proof.** This follows from a combination of Theorem 6.3, Theorem 6.3, and the fact that the Radon–Nikodým property is implied by reflexivity.

**Appendix A. Some Maximal Function Inequalities**

Suppose that \( \mathbb{R}^d \) is \( d \)-decomposed with \( d \in (\mathbb{Z}_{\geq 1})^\ell \) and let \( \mathcal{A} = (A_1, \ldots, A_d) \) be a \( d \)-anisotropy.

**Lemma A.1** (Anisotropic Peetre’s inequality). Let \( X \) be a Banach space, \( r \in (0, \infty)^\ell \), \( K \subset \mathbb{R}^d \) a compact set and \( N \in \mathbb{N} \). For all \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq N \) and \( f \in \mathcal{S}'(\mathbb{R}^d; X) \) with \( \text{supp} (\hat{f}) \subset K \), there is the pointwise estimate

\[
\sup_{x \in \mathbb{R}^d} \left| \frac{\|D^\alpha f(x + z)\|_X}{\prod_{j=1}^\ell (1 + \rho_{A_j}(z_j))^{\text{tr}(A_j)/r_j}} \right| \leq \sup_{x \in \mathbb{R}^d} \left| \frac{\|f(x + z)\|_X}{\prod_{j=1}^\ell (1 + \rho_{A_j}(z_j))^{\text{tr}(A_j)/r_j}} \right| \lesssim \left[ M^A_\ell (\|f\|_X) \right](x), \quad x \in \mathbb{R}^d.
\]

**Proof.** This can be obtained by combining the proof of [34, Proposition 3.11] (which is actually only a reference to [63, Theorem 1.6.4], the two-dimensional case that easily extends to arbitrary dimensions) for the case \( d = 1 \) with the proof of [13, Lemma 3.4] for the case \( \ell = 1 \).
Corollary A.2. Let $X$ be a Banach space and $r \in (0, \infty)^\ell$. For all $f \in S'(\mathbb{R}^d; X)$ and $R \in (0, \infty)^\ell$ with $\text{supp}(\hat{f}) \subset B^A(0, R)$, there is the pointwise estimate
\[ f^*(A, r, R; x) \lesssim_{A, r} [M^A_r(||f||_X)](x), \quad x \in \mathbb{R}^d. \]

Proof. By a dilation argument it suffices to consider the case $R = 1$, which is contained in Lemma A.1. \hfill \Box

Lemma A.3. Let $X$ and $Y$ be Banach spaces. For all $(M_n)_{n \in \mathbb{N}} \subset \mathcal{F}L^1(\mathbb{R}^d; B(X, Y))$, $(R^{(n)})_{n \in \mathbb{N}} \subset (0, \infty)^\ell$, $c \in [1, \infty)$ and $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}^{-1}L^c(\mathbb{R}^d; X)$, there is the pointwise estimate
\[
\left| \left| \mathcal{F}(M_n f_n)(x) \right| \right|_Y \lesssim c \sum_{j=1}^\ell \lambda_j \sup_{k \in \mathbb{N}} \int_{\mathbb{R}^d} \|M_n(A R^{(n)} y_j)\|_{B(X, Y)} \prod_{j=1}^\ell (1 + \rho_{A_j}(y_j))^{\lambda_j} \, dy \\
\quad \cdot \sup_{x, z \in \mathbb{R}^d} \|f_n(x + z)\|_X \prod_{j=1}^\ell (1 + c R^{(n)} j \rho_{A_j}(y_j))^{\lambda_j}.
\]

Proof. This can be shown as the pointwise estimate in the proof of [42 Proposition 3.4.8], which was in turn based on [51 Proposition 2.4]. \hfill \Box

The following proposition is an extension of [33 Proposition 13.13] to our setting, which is in turn a version of the pointwise estimate of pseudo-differential operators due to Marschall [48]. In order to state it, we first need to introduce the anisotropic mixed-norm homogeneous Besov space $\dot{B}^{p, \alpha}_{\beta, \tau}(\mathbb{R}^d; Z)$.

Let $Z$ be a Banach space, $p \in (1, \infty)^\ell$, $q \in (0, \infty]$ and $s \in \mathbb{R}$. Fix $(\phi_k)_{k \in \mathbb{Z}} \subset S(\mathbb{R}^d)$ that satisfies $\phi_k = \hat{\psi}(A_{2^{-k}} \cdot) - \hat{\psi}(A_{2^{-(k+1)}} \cdot)$ for some $\hat{\psi} \in \mathcal{F}C_0^\infty(\mathbb{R}^d)$ with $\hat{\psi} \equiv 1$ on a neighbourhood of 0. Then $\dot{B}^{p, \alpha}_{\beta, \tau}(\mathbb{R}^d; Z)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which
\[
||f||_{\dot{B}^{p, \alpha}_{\beta, \tau}(\mathbb{R}^d; Z)} := \left\langle (2^{sk} \phi_k * f)_{k \in \mathbb{Z}} \right\rangle_{L_p(Z) L_q(\mathcal{S}'(\mathbb{R}^d); Z)} < \infty.
\]

Proposition A.4. Let $X$ and $Y$ be Banach spaces and $r \in (0, 1)^\ell$. Put $\tau := \tau_{\min} \in (0, 1]$. For all $b \in S(\mathbb{R}^d; B(X, Y))$, $u \in \mathcal{S}(\mathbb{R}^d; X)$, $c \in (0, \infty)$ and $R \in [1, \infty)$ with $\text{supp}(b) \subset B^A(0, c)$ and $\text{supp}(u) \subset B^A(0, c R)$, there is the pointwise estimate
\[
||b(D) u(x)||_Y \lesssim_{A, r} (c R)^{tr(A)(r^{-1} - 1)} ||b||_{L^1(\mathbb{R}^d; B(X, Y))} [M^A_r(||u||_X)](x)
\]
for each $x \in \mathbb{R}^d$.

In the proof of Proposition A.4 we will use the following lemma.

Lemma A.5. Let $X$ be a Banach space and $p, q \in (0, \infty)^\ell$ with $p \leq q$. For every $f \in \mathcal{S}(\mathbb{R}^d; X)$ and $R \in (0, \infty)^\ell$ with $\text{supp}(\hat{f}) \subset B^A(0, R)$,
\[
||f||_{L^q(p, \tau, \mathbb{R}^d; X)} \lesssim_{p, q, d} \prod_{j=1}^\ell R_j^{tr(A)(\frac{1}{p_j} - \frac{1}{q_j})} ||f||_{L^p(p, \tau, \mathbb{R}^d; X)}
\]

Proof. By a scaling argument we may restrict ourselves to the case $R = 1$. Now pick $\phi \in S(\mathbb{R}^d)$ with $\hat{\phi} \equiv 1$ on $B^A(0, 1)$. Then $f = \phi * f$ and the desired inequality follows from an iterated use of Young’s inequality for convolutions. \hfill \Box
Proof of Proposition \([A.4]\): It holds that
\[
b(D)u(x) = \int_{\mathbb{R}^d} b(y)u(x - y) \, dy, \quad x \in \mathbb{R}^d.
\]

For fixed \(x \in \mathbb{R}^d\), by the quasi-triangle inequality for \(\rho_A\) (with constant \(c_A\)),
\[
supp \left( \mathcal{F}[y \mapsto \hat{b}(y)u(x - y)] \right) \subset B_A(0, c) + B_A(0, cR) \subset B_A(0, cA(R + 1)c).
\]
Therefore,
\[
\|b(D)u(x)\|_Y \leq \|y \mapsto \hat{b}(y)u(x - y)\|_{L^1(\mathbb{R}^d)}
\]
\[
\lesssim (cA(R + 1)c)^{\sum_{j=1}^d \text{tr}(A_j)} \|y \mapsto \hat{b}(y)u(x - y)\|_{L_{r,c}(\mathbb{R}^d)}
\]
\[
\lesssim (Rc)^{\sum_{j=1}^d \text{tr}(A_j)} \|y \mapsto \hat{b}(y)u(x - y)\|_{L_{r,c}(\mathbb{R}^d)},
\]
where we used Lemma \([A.5]\) for the second estimate.

Let \((\hat{\phi}_k)_{k \in \mathbb{Z}}\) be as in the definition of the anisotropic homogeneous Besov space \(B_{p,q}^{s,A}\) as given preceding the proposition. Then \(\sum_{k=-\infty}^\infty \hat{\phi}_k(-\cdot) = 1\) on \(\mathbb{R}^d \setminus \{0\}\), so that
\[
\|\hat{b}u(x - \cdot)\|_{L_{r,c}(\mathbb{R}^d)} \leq \left( \sum_{k \in \mathbb{Z}} \|\hat{\phi}_k(-\cdot) \hat{b}u(x - \cdot)\|_{L_{r,c}(\mathbb{R}^d)}^{\tau} \right)^{1/\tau}.
\]

Since
\[
\sup_{y \in \mathbb{R}^d} \|\hat{\phi}_k(-y)\|_{\mathcal{B}(X,Y)} \leq \|\mathcal{F}^{-1} [\hat{\phi}_k(-\cdot) \hat{b}]\|_{L^1(\mathbb{R}^d;\mathcal{B}(X,Y))} = (2\pi)^{-d}\|\mathcal{F}^{-1} [\hat{\phi}_k \hat{b}]\|_{L^1(\mathbb{R}^d;\mathcal{B}(X,Y))}
\]
and \(\sup (\hat{\phi}_k) \subset B^A(0, 2^{k+1})\), it follows from a combination of \((55)\) and \((56)\) that
\[
\|b(D)u(x)\|_Y \lesssim (Rc)^{\sum_{j=1}^d \text{tr}(A_j)} \left( \sum_{k \in \mathbb{Z}} \|\hat{\phi}_k(-\cdot) \hat{b}u(x - \cdot)\|_{L_{r,c}(\mathbb{R}^d)}^{\tau} \right)^{1/\tau}
\]
\[
\lesssim (Rc)^{\sum_{j=1}^d \text{tr}(A_j)} \left( \sum_{k \in \mathbb{Z}} \left[ 2^{k \sum_{j=1}^d \text{tr}(A_j) \frac{1}{L_{r,c}}} \|\mathcal{F}^{-1} [\hat{\phi}_k \hat{b}]\|_{L^1} \right]^{\tau} \right)^{1/\tau}
\]
\[
\lesssim \sup_{k \in \mathbb{Z}} 2^{-k(1+\text{tr}(A)) \frac{1}{L_{r,c}}} \|1_{B^A(0,2^{k+1})}u(x - \cdot)\|_{L_{r,c}(\mathbb{R}^d)}
\]
\[
\leq (Rc)^{\sum_{j=1}^d \text{tr}(A_j)} \left[ \|b\|_{B^A_{1,\tau}(\mathbb{R}^d;\mathcal{B}(X,Y))} \sum_{j=1}^d \text{tr}(A_j) \frac{1}{L_{r,c}} \right] \left[ M^A_r(\|f\|_X)(x) \right].
\]

\[\square\]

Corollary A.6. Let \(X\) and \(Y\) be Banach spaces, \(r \in (0,1]^d\) and \(\psi \in C^\infty_c(\mathbb{R}^d;\mathcal{B}(X,Y))\). Put \(\psi_k := \psi(2^{-k-1})\) for each \(k \in \mathbb{N}\). Then, for all \((f_k)_{k \in \mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)\) with \(\text{supp } f_k \subset B^A(0, r2^k)\) for some \(r \in [1,\infty)\), there is the pointwise estimate
\[
\|\psi_k(D)f_k(x)\|_Y \lesssim r^{\text{tr}(A)-(r-1)} \left[ M^A_r(\|f\|_X) \right](x), \quad x \in \mathbb{R}^d.
\]

Proof. Let \(c \in [1,\infty)\) be such that \(\text{supp } (\psi) \subset B^A(0, c)\). Applying Proposition \([A.4]\) to \(b = \psi_k\), \(u = f_k\) and \(R = r2^k\), we find that
\[
\|\psi_k(D)f_k(x)\|_Y \lesssim (cr2^k)^{\text{tr}(A)-(r-1)} \|\psi_k\|_{B^A_{1,\tau}(\mathbb{R}^d;\mathcal{B}(X,Y))} \left[ M^A_r(\|f\|_X) \right](x).
\]
Observing that
\[
\left| \frac{1}{B_{1,1}(\mathbb{R}^d)} \sum_{j=1}^{k} \text{tr}(A_j) \| \frac{k}{r_j} A \| (\mathbb{R}^d) \right| = 2^{-k} \| \frac{k}{r_j} A \| (\mathbb{R}^d),
\]
we obtain the desired estimate.

\[\square\]

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