Coherent interaction of multistate quantum systems possessing the Wigner–Majorana and Morris–Shore dynamic symmetries with pulse trains

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Abstract
We present exact analytic formulae which describe the interaction of multistate quantum systems possessing the Wigner–Majorana and Morris–Shore dynamic symmetries with a train of pulses. The pulse train field can be viewed as repeated interactions of the quantum system with the same field and hence the overall propagator is expressed as the matrix power of the single-pulse propagator. Because of the symmetries the multistate dynamics is characterized by intrinsic two-state features, described by one or more pairs of complex-valued Cayley–Klein parameters. This facilitates the derivation of explicit formulae linking the single-step and multi-step propagators. The availability of such analytic relations opens the prospects for a variety of applications with ensembles of qubits, qutrits and generally qudits, e.g. analytic description of coherent pulse-train interactions, coherent amplification of quantum gate errors for accurate quantum gate tomography, dynamical rephasing of inhomogeneously broadened ensembles, quantum sensing of small electric or magnetic fields, etc.

Keywords: Morris–Shore decomposition, pulse train, Raman qubit, tripod system, lambda system, multistate system, Wigner–Majorana symmetry

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum systems with multiple states, as encountered in real physical systems, are generally difficult to treat analytically due to their prohibitively complex dynamics [1, 2]. Yet, some of them allow for such a treatment thanks to an intrinsic two-state behavior. One example is the situation when all but two of the states are far off resonance with any of the driving fields; then adiabatic elimination of all these states leaves us with an effective two-state system and the effect of the other states shows up as ac Stark shifts in the detuning [1, 2].

Interestingly, a modified adiabatic elimination can be performed even when the far-off-resonance condition is alleviated to a near-resonance one [3].

Another example is found in systems with the Morris–Shore (MS) symmetry [4], in which all states can be grouped into two manifolds, such that interactions are allowed between states of different manifolds only, but not within the same manifold. All couplings must share the same time dependence. Moreover, all states within the same manifold should be degenerate (in the rotating-wave approximation), but there could be a nonzero detuning between states from different manifolds; however, it should be the same for all couplings. The MS transformation (MST) casts such multistate systems into a set of independent two-state systems and a number
of decoupled (dark) single states. Such linkages naturally emerge, e.g. in the interaction of two degenerate atomic levels with an elliptically polarized laser field [1, 2, 5–11]. The MST has been generalized to an arbitrary many manifolds of degenerate states [12], and its extensions and applications have been reviewed by Shore [13]. Recently, this transformation has been generalized to unequal detunings [14] and different time dependences of the couplings [15].

A third example of reducible multistate dynamics is found in systems with the Wigner–Majorana (WM) SU(2) symmetry [16, 17]. It arises, e.g. in (radio-frequency) transitions between the magnetic sublevels of a level with a definite angular momentum, such as in Bose–Einstein output couplers [18, 19]. It naturally emerges also in Raman systems subjected to combined external laser field and static magnetic field [20].

While the adiabatic elimination is an approximate method, the other two methods present, in principle, exact reduction of the multistate systems to one or more two-state systems. This reduction allows one to use the two-state quantum control methods to design similar methods in multistate systems [18, 20, 21].

In this paper, we use these analogies between two-state and multistate systems in order to describe analytically the interaction of a multistate system with either the WM or MS symmetry with a train of identical pulses, viewed as a multi-pass interaction of the system with the same field. We derive explicit analytic formulae which express the overall multi-pass propagator in terms of the single-pulse propagator. In addition to the obvious application of using these results in order to develop analytic models of multi-pulse excitation of multistate systems, they present the framework for the development of precise quantum tomography of such systems by coherent error amplification due to the repeated application of the same quantum gate. They allow also for accurate description of dynamical decoupling and quantum sensing with qubits.

The paper is organized as follows. In section 2 we define the problems and the framework of their treatment. Section 3 presents the multi-pass interaction of systems with the WM symmetry, with an explicit example for a three-state system. Section 4 discusses the multi-pass interaction of systems with the MS symmetry, with explicit examples for the general multipod systems and their simplest and most important cases of the Raman three-level system and the tripod system. In section 5, we consider systems which possess both the WM and MS symmetries. Finally, section 6 presents a summary of the results.

2. Case studies

We consider a coherently driven quantum system with $K$ states driven by a train of $N$ identical pulses of duration $T$ each. Its evolution is governed by the time-dependent Schrödinger equation [1, 2] ($\hbar = 1$),

$$i \frac{d}{dt}\Psi(t) = \mathbf{H}(t)\Psi(t). \quad (1)$$

The evolution of the state vector $\Psi(t)$ can be described by the propagator $\mathbf{U}(t,t_0)$ as

$$\Psi(t) = \mathbf{U}(t,t_0)\Psi(t_0). \quad (2)$$

Without loss of generality, we take $t_0 = 0$. If the Hamiltonian $\mathbf{H}(t)$ has the same form for each time interval $[(n-1)T,nT]$ ($n = 1, 2, \ldots, N$), during which the $n$th pulse acts, then the propagator generated by each pulse will be the same, i.e. $\mathbf{U}((n-1)T,nT) = \mathbf{U}(T,0)$. Then

$$\mathbf{U}(NT,0) \equiv [\mathbf{U}(T,0)]^N. \quad (3)$$

We wish to find the $N$-pulse propagator $\mathbf{U}(NT,0)$ in terms of the parameters of the Hamiltonian $\mathbf{U}(T,0)$. This problem has been explicitly solved for a two-state system [22] and we will use and build up on these results here.

The dynamics of a two-state quantum system is governed by the Hamiltonian

$$\mathbf{H}_2(t) = \frac{1}{2} \begin{bmatrix} -\Delta(t) & \Omega(t) \\ \Omega^*(t) & \Delta(t) \end{bmatrix}, \quad (4)$$

where the detuning $\Delta(t)$ and the Rabi frequency $\Omega(t)$ are arbitrary functions of time. Because we have chosen to express the Hamiltonian in the traceless form (4), the propagator has the SU(2) dynamic symmetry,

$$\mathbf{U}_2 = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, \quad (5)$$

where $a$ and $b$ are two complex-valued Cayley–Klein parameters, with $|a|^2 + |b|^2 = 1$; hence $\det \mathbf{U}_2 = 1$. It has been proved [22] that the $N$th power of the SU(2) propagator 5 reads

$$\mathbf{U}_2^N = \begin{bmatrix} a_N & b_N \\ -b_N^* & a_N^* \end{bmatrix}, \quad (6)$$

where

$$a_N = \cos(N\theta) + i \frac{\sin(N\theta)}{\sin\theta} \text{Im}a, \quad (7a)$$

$$b_N = b \frac{\sin(N\theta)}{\sin\theta}, \quad (7b)$$

$$\cos\theta = \Re a. \quad (7c)$$

The relations (7) make it possible to find out how a pulse train affects the two-state system if we know the single-pulse action. There exist a number of analytic single-pulse solutions, which can be generalized using the formulae above to analytic multiple-pulse solutions [22]. The relations (7) also allow for accurate description of dynamical decoupling, which uses repeated application of the same (usually $\pi$) pulse [23, 24], or repetition of a specific sequence of pulses [25]. On the other hand, these relations allow one to precisely find out the action of a single pulse from the action of a train of pulses. This is important, for instance, for measuring small deviations from a desired single-pulse action, e.g. for characterizing a high-fidelity quantum gate [26] for which the admissible error is of the order of $10^{-3}$ or less. It has applications also in the measurement of small frequency shifts, as needed in quantum sensing [27].
The Cayley–Klein parameters $a$ and $b$ obviously depend on $\Delta(t)$ and $\Omega(t)$, i.e. on the model. There are several analytically soluble models for which $a$ and $b$ have exact expressions, e.g. the models of Rabi [28], Landau–Majorana–Stückelberg–Zener [17, 29–31], Rosen–Zener [32], Demkov [33], Nikitin [34], Allen–Eberly–Hioe [21, 35, 36], Bambini–Berman [37], Demkov–Kunike [38–40], etc. For the sake of generality, we are interested in the relations between the single-step parameters $a$ and $b$ and the multi-step parameters $a_N$ and $b_N$. It is important that these relations do not depend on the particular expressions of $a$ and $b$, and hence are not limited to a certain model. Hitherto, such single-pass to multi-pass relations have been known for a two-state system only. Here we extend these results to multistate systems possessing the WM or MS symmetries, which are reducible to one or more two-state systems.

**Wigner–Majorana (WM) symmetry.** It fulfils the requirements of WM decomposition [16, 17], having a reducible dynamic symmetry from SU($K$) to SU(2) [16, 17, 41–43].

**Morris–Shore (MS) symmetry.** The MS Hamiltonian fulfils the requirements of the MS decomposition [4, 12, 14, 44], for which the quantum system is composed by a set of $L$ ground degenerate states and a set of $M$ exited degenerate states. Couplings exist between states from different sets only, but not within the same set. Such a system is reducible to a set of $M$ independent two-state systems and $M−L$ decoupled (dark) states.

Below we consider first the multiple interactions of multistate systems with the WM symmetry, and then systems with the MS symmetry. Following the detailed investigation of these two types of systems, we shall study systems which possess both the WM and MS symmetries.

### 3. Systems with the WM symmetry

#### 3.1. WM propagator

The WM decomposition has been presented in several papers [16, 17, 20, 41–43]. It stems from the rotation group theory for the angular momentum. The Hamiltonian has the tridiagonal form [42]

$$
H_K = \begin{bmatrix}
H_{11} & H_{12} & 0 & \cdots & 0 & 0 \\
H_{21} & H_{22} & H_{23} & \cdots & 0 & 0 \\
0 & H_{32} & H_{33} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & H_{K−1,K−1} & H_{K−1,K} \\
0 & 0 & 0 & \cdots & H_{K,K−1} & H_{KK}
\end{bmatrix}
$$

where the nonzero matrix elements read

$$
H_{kk}(t) = \left( k - \frac{K+1}{2} \right) \Delta(t), \quad (9a)
$$

$$
H_{k+1,k}(t) = H_{k,k+1}(t) = \frac{1}{2} \sqrt{k(k−1)} \Omega(t), \quad (9b)
$$

In terms of the angular momentum quantum numbers $j$ and $m$, we have the relations $K = 2j+1$ and $k = j+1−m$.

The matrix elements of the WM propagator $U_K$ are [16, 42, 43, 45]

$$
U_{kl} = \sum_r \frac{\sqrt{(k−1)!(l−1)!(K−k)!(K−l)!}}{(l−1−r)!(K−k−r)!(r−l+1)!r!} \times a^{K−k−r}b^{l−1−r}b^*(−b^*)^{−l+k}, \quad (10)
$$

where $r$ runs from $r_{min}$ to $r_{max}$, with

$$
r_{min} = \min[0,k+l+1−K], \quad (11a)
$$

$$
r_{max} = \max[k−1,l−1]. \quad (11b)
$$

For $K = 2$ states, the propagator reduces to equation (5).

The propagators for $K = 3$ and $K = 4$ states read

$$
U_3 = \begin{bmatrix}
a^3 & ab\sqrt{2} & b^2 \\
−a^2b\sqrt{3}/2 & |a|^2−|b|^2 & ba\sqrt{2}/2 \\
−b^2 & −a^2b\sqrt{3}/2 & a^2
\end{bmatrix}, \quad (12a)
$$

$$
U_4 = \begin{bmatrix}
a^3 & a^2b\sqrt{3}/2 & b^2 \\
−a^2b\sqrt{3}/2 & |a|^2−|b|^2 & ba\sqrt{2}/2 \\
ab^2\sqrt{3}/2 & b^3/(|b|^2−2|a|^2) & ba^2\sqrt{3} \\
ab^2\sqrt{3}/2 & ba^2\sqrt{3} & a^3
\end{bmatrix}. \quad (12b)
$$

These matrices are also called Wigner $D$-matrices [16]. In 1955 Salwen [43] has shown the relationship between the Wigner and Majorana formulas by using the paper of Bloch and Rabi [41].

#### 3.2. Multi-pass WM propagator

We shall derive the WM propagator of a $K$-state system after $N$ repetitions of the interaction described by the Hamiltonian.
$H_K$ of equation (8) in two alternative manners. First we shall use the analogy of $M$-state to two-state dynamics and then we shall derive it by diagonalization of the propagator $U_M$.  

3.2.1. First approach.  In the first approach, it is important to note that the parameters $a$ and $b$, parameterizing the propagator $U_K$, are the same as the parameters in $U_2$ of equation (5). In other words, if the Hamiltonian $H_2$ of equation (4) generates the propagator $U_2$ of equation (5), then the Hamiltonian $H_K$ of equation (8) generates the propagator $U_K$ with the matrix elements of equation (10). The implication is that we can find the propagator of the $K$-state system with the WM symmetry in two equivalent manners: (a) solve the Schrödinger equation with the $K$-state Hamiltonian $H_K$, or (b) solve the two-state problem with the Hamiltonian $H_2$ and use equation (10) to find the propagator $U_K$. In either cases, the solutions should be identical. In this sense, we say that the WM symmetry admits reduction of the $K$-state system to an effective two-state system.

Now consider a sequence of multiple pulses, each generating the same propagator $U_K$. By the same reasons, given in the preceding paragraph, the multi-pulse propagator for the $K$-state system, i.e. $U_K^K$ can be calculated using the one for the two-state system, $U_2^2$. We thereby conclude that the matrix elements of the $N$-pulse propagator for the $K$-state WM system has the same form as the single-pulse propagator elements of equation (10),

$$U_{kl} = \sum_r \frac{\sqrt{(k-1)!(l-1)!(K-k)!(K-l)!}}{(l-1-r)!(K-k-r)!(r-l+k)!} \times a_N^{K-k-l}(\alpha_K^*)^{l-1-r}b_N^{l-k}(-\beta_N)^{-l+k},$$

with $a_N$ and $b_N$ given by equations (7), and $r$ runs from $r_{\text{min}}$ to $r_{\text{max}}$ given by equations (11).

3.2.2. Second approach.  In the second approach, we find the multi-pulse propagator $U_K^K$ by diagonalizing the single-pulse one $U_K$ (we drop hereafter the subscript $K$ for simplicity),

$$V^lUV = D,$$

and hence

$$U = VDV^l,$$

According to the 3D rotation group theory [45], the diagonal matrix $D$ has the form

$$D = \begin{bmatrix}
    e^{-i(K-1)\theta} & 0 & \cdots & 0 & 0 \\
    0 & e^{-i(K-3)\theta} & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & e^{i(K-3)\theta} & 0 \\
    0 & 0 & \cdots & e^{i(K-1)\theta} & 0 \\
\end{bmatrix},$$

where the phase factors are the eigenvalues of $U$ and $\theta$ is defined by equation (7c). The diagonalizing matrix $V$ is composed by the eigenvectors of $U$. One can show that, quite remarkably, it has the same form (i.e. possesses the WM symmetry) as the propagator $U$ of equation (10), but with different Cayley–Klein parameters $a$ and $v$, instead of $a$ and $b$,

$$V_{kl} = \sum_r \frac{\sqrt{(k-1)!(l-1)!(K-k)!(K-l)!}}{(l-1-r)!(K-k-r)!(r-l+k)!} \times a^{K-k-l}(u^*)^{l-1-r}v^{l-k},$$

with $|u|^2 + |v|^2 = 1$. The relations between the parameters $a$ and $v$ of the diagonalizing matrix and the parameters $a$ and $b$ of the propagator can be found as follows.

Looking at the elements (10) of the propagator $U$, exemplified for $K = 3$ and 4 states in equations (12), it is easy to notice a prominent feature: the elements on the top row are the square roots of the terms in the expansion of $(a^2 + b^2)^{K-1}$, i.e.

$$U_{11} = \left(\frac{K-1}{l-1}\right)^{\frac{1}{2}} a^{K-1}b^{l-1}.$$  

Indeed, due to the factorials in the denominator and the fact that $1/(-n)! = 0$ for any positive integer $n$, for $k = 1$ the only nonzero contribution in the sum is for $r = l = 1$. Similarly, the elements on the bottom row (for $k = K$) are the square roots of the terms in the expansion of $(a^2 - b^2)^{K-1}$, viz.

$$U_{K1} = (-b^*)^{K-1},$$

$$U_{KK} = (a^*)^{K-1}.$$  

Let us now assume that the diagonalizing matrix $V$ is composed of the matrix elements of equation (17). As for $U$, the elements on the top row are the square roots of the terms in the expansion of $(a^2 + v^2)^{K-1}$, i.e.

$$V_{11} = \left(\frac{K-1}{l-1}\right)^{\frac{1}{2}} u^{K-1}v^{l-1},$$

and the elements on the bottom row (for $k = K$) are the square roots of the terms in the expansion of $(a^2 - v^2)^{K-1}$, viz.

$$V_{K1} = \left(\frac{K-1}{l-1}\right)^{\frac{1}{2}} (u^*)^{l-1}(-v^*)^{K-l},$$

because the only nonzero contribution to the sum comes from $r = 0$. In particular, in the corners we have

$$U_{11} = a^{K-1},$$

$$U_{K1} = (-b^*)^{K-1},$$

$$U_{KK} = (a^*)^{K-1}.$$  


The propagator elements of the single-pass and N-pass interaction of the general $K$-state quantum system with the WM symmetry allow one to conduct two types of tasks: (a) given the action of the single interaction find the action of the repeated multiple interactions, and (b) deduce the action of the single interaction by measuring the result of the multiple repetition of this interaction. These can be very useful in designing the best scenarios for coherent amplification of quantum gate errors and hence precise quantum gate tomography, as well as for enhanced quantum sensing of electric and magnetic fields by amplification of frequency shifts [27]. Indeed, having an exact analytic relation between the single-pass and multiple-pass processes enables the accurate determination of tiny gate errors or frequency shifts from the amplified error or shift.

We emphasize that the $K$-state WM system presents a clear benefit in this respect compared to the simple two-state system ($K = 2$): the corner elements of the propagator [see equation (20)] are the $(K - 1)$-st power of the respective elements $a$ and $b$ of the two-state system. If one of the parameters $a$ or $b$ is very small, then the $(K - 1)$-st power will only make it smaller and closer to 0. However the $(K - 1)$-st power of the other parameter (which in modulus should be close to 1) will make it deviate much more strongly from 1 than for $K = 2$. This deviation will be further amplified by using multiple interactions instead of a single one.

Finally, we point out that in the second approach of derivation, we have used the explicit form of the diagonalizing matrix $V$, which is composed of the eigenvectors of the WM propagator $U$, with their elements obeying equations (26). This result can be useful by itself, e.g. for developing adiabatic control approaches [46, 47], or the so-called ‘shortcuts to adiabaticity’ [48].

### 3.2.3. Discussion

The relations derived here between the propagator elements of the single-pass and $N$-pass interaction of the general $K$-state quantum system with the WM symmetry allow one to conduct two types of tasks: (a) given the action of the single interaction find the action of the repeated multiple interactions, and (b) deduce the action of the single interaction by measuring the result of the multiple repetition of this interaction. These can be very useful in designing the best scenarios for coherent amplification of quantum gate errors and hence precise quantum gate tomography, as well as for enhanced quantum sensing of electric and magnetic fields by amplification of frequency shifts [27]. Indeed, having an exact analytic relation between the single-pass and multiple-pass processes enables the accurate determination of tiny gate errors or frequency shifts from the amplified error or shift.

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### 3.3. Example: three-state system

The propagator $U_3$ shown in (12a) was constructed by the equation for the matrix elements $U_{kl}$ of equation (10). The Hamiltonian is defined by equation (8) and it is

$$H_3(t) = \begin{bmatrix} -\Delta(t) & \frac{\Omega(t)}{\sqrt{2}} & 0 \\ \frac{\Omega(t)}{\sqrt{2}} & 0 & \frac{\Omega(0)+\Delta(t)}{\sqrt{2}} \\ 0 & \frac{\Omega(0)+\Delta(t)}{\sqrt{2}} & \Delta(t) \end{bmatrix}.$$  (31)

The system is shown schematically in figure 1. The multi-pass propagator $U_3^N$ can be obtained from the single one $U_3$ by the substitution $a \rightarrow a_N$ and $b \rightarrow b_N$ according to equation (7),

$$U_3^N = \begin{bmatrix} a_N^2 & \sqrt{2}a_Nb_N & b_N^2 \\ -\sqrt{2}a_Nb_N & |a_N|^2 - |b_N|^2 & \sqrt{2}b_Na_N^* \\ b_N^2 & -\sqrt{2}b_Na_N^* & a_N^2 \end{bmatrix}.$$  (32)

Therefore, already for $K = 3$ states we have quadratic powers in the Cayley–Klein parameters and this amplification is further boosted by the application of $N$ interactions.

For example, if the Cayley–Klein parameter $a$ is real then $a^2 = \cos\theta$ is the probability for no transition and


Figure 1. Wigner–Majorana decomposition for the three-state $\Lambda$ system, with the equivalent two-state system.

$p_2 = 1 - a^2 = \sin^2 \theta$ is the transition probability for a single-pass in the two-state system. The $N$-pass transition probability reads

$$p_2^{(N)} = p_2 \frac{\sin^2 N \theta}{\sin^2 \theta} = \sin^2 N \theta. \quad (33)$$

Correspondingly, the transition probability in the three-state WM system from state 1 to state 3 is $p_3 = \sin^4 \theta$, and the $N$-pass transition probability reads

$$p_3^{(N)} = p_3 \frac{\sin^4 N \theta}{\sin^4 \theta} = \sin^4 N \theta. \quad (34)$$

A real $a$ occurs for resonant excitation ($\Delta = 0$) and also when the Rabi frequency is a symmetric function of time while $\Delta$ is anti-symmetric [27].

The results in this example can be used for efficient tomography of qutrit gates as well as for quantum sensing with qutrits. Similar relations can be derived for larger number of states $K$.

4. Systems with the MS symmetry

4.1. Single MS propagator

The MS transformation (MST) [4, 5, 12–14, 44, 49] is a powerful tool for reducing the dynamics of a certain class of multistate systems to the dynamics of one or more two-state systems. The MS system is shown schematically in figure 2. It consists of two sets of states: a ground set with $L$ states and an exited set with $M$ states. There are couplings, quantified by the Rabi frequencies $\Omega_{lm} f(t)$, only between states from different levels ($l = 1, 2, \ldots, L; m = 1, 2, \ldots, M$), and $f(t)$ is a common time dependence of all couplings. All couplings have the same detunings $\Delta(t)$. The Hamiltonian of the MS system can be written as

$$H(t) = \frac{1}{2} \left[ \begin{array}{cc} \Omega_L & f(t) \Omega_L \Delta(t) \\ f(t) \Omega_L^\dagger \end{array} \right], \quad (35)$$

where the constant matrix $\Omega$ is $L \times M$ dimensional,

$$\Omega = \left[ \begin{array}{ccc} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1M} \\
\Omega_{21} & \Omega_{22} & \cdots & \Omega_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{L1} & \Omega_{L2} & \cdots & \Omega_{LM} \end{array} \right], \quad (36)$$

and $\Omega_{lm} (l = 1, 2, \ldots, L; m = 1, 2, \ldots, M)$ are arbitrary complex constants. The MST reduces the multi-state dynamics, which have a Hilbert space dimension of $L + M$ [figure 2 (top)] to a set of $M$ independent two-state systems with additional $d = L - M$ decoupled (dark) states [figure 2 (bottom)]. In the MS basis, the Hamiltonian has the block matrix form

$$\tilde{H}(t) = S^\dagger H(t) S = \frac{1}{2} \left[ \begin{array}{cc} \Omega_L & f(t) \Omega_L \Delta(t) \\ f(t) \Omega_L^\dagger \end{array} \right], \quad (37)$$

where $S$ is a constant unitary matrix, defined by two square unitary matrices $S_L$ and $S_M$ with dimensions of $L$ and $M$, respectively,

$$S = \left[ \begin{array}{cc} S_L & 0 \\
0 & S_M \end{array} \right], \quad SS^\dagger = S^\dagger S = 1_{(L+M)}. \quad (38)$$

Then, by using equations (37) and (38), the transformed coupling matrix $\tilde{\Omega}$ can be expressed as

$$\tilde{\Omega} = S_L \Omega S_M^\dagger \quad (39)$$

The matrices $S_L$ and $S_M$ are defined by the condition that they diagonalize $\Omega_L \Omega$ and $\Omega_M^\dagger$, i.e.
Figure 2. The MS transformation. A multistate system consisting of two coupled sets of degenerate levels is transformed into a set of $M$ independent two-state systems and a set of $d = L - M$ decoupled dark ground states. All couplings have the same time dependence $f(t)$ and the same detunings $\Delta(t)$.

\[
S_L \Omega^\dagger \Omega S_L^\dagger = 1_L, \tag{40a}
\]

\[
S_M \Omega^\dagger \Omega S_M^\dagger = 1_M, \tag{40b}
\]

and are found by solving these equations. The $M$-dimensional square matrix $\Omega^\dagger \Omega$ has $M$ generally nonzero eigenvalues $\lambda_m^2$ ($m = 1, 2, \ldots, M$). The $L$-dimensional square matrix $\Omega^\dagger$ has the same $M$ eigenvalues as $\Omega^\dagger \Omega$ and additional $d = L - M$ zero eigenvalues, corresponding to the dark states.

The transformed Hamiltonian (37) acquires the form

\[
\tilde{H}(t) = \begin{bmatrix}
O_{d \times d} & O_{d \times 2M} \\
O_{2M \times d} & \tilde{H}_c(t)
\end{bmatrix},
\tag{41}
\]

where $\tilde{H}_c(t)$ is formed by four square $M \times M$ matrices,

\[
\tilde{H}_c(t) = \frac{1}{2} \begin{bmatrix}
O & \Lambda f(t) \\
\Lambda f(t) & 2\Delta(t)1
\end{bmatrix}.
\tag{42}
\]
With an appropriate reordering of the states (i.e. the rows and the columns), described by a matrix $\mathbf{R}$, the propagator of the full system in MS basis can be cast into the block matrix form $\mathbf{R}^{-1/2}\tilde{\mathbf{H}}_b(t)\mathbf{R} = \tilde{\mathbf{H}}_b(t)$, with

$$
\tilde{\mathbf{H}}_b(t) = \begin{bmatrix}
\tilde{\mathbf{H}}_1(t) & 0 & \cdots & 0 \\
0 & \tilde{\mathbf{H}}_2(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{\mathbf{H}}_M(t)
\end{bmatrix},
$$

with

$$
\tilde{\mathbf{H}}_m(t) = \frac{1}{2} \begin{bmatrix}
0 & \lambda_m f(t) \\
\lambda_m f(t) & 2\Delta(t)
\end{bmatrix},
$$

being the Hamiltonian describing the $m$th independent two-state system in the MS basis.

Each of these $M$ independent two-state Hamiltonians generates $M$ independent two-state propagators with $M$ different pairs of Cayley–Klein parameters $a_m$ and $b_m$,

$$
\tilde{\mathbf{U}}_m = \begin{bmatrix}
a_m & b_m \\
-b_m e^{-i\delta} & a_m e^{-i\delta}
\end{bmatrix},
$$

where $|a_m|^2 + |b_m|^2 = 1$ and

$$
\delta = \int_0^T \Delta(t) dt
$$

is a common accumulated phase for all $M$ independent systems. The propagator in the reordered MS basis has the same block matrix structure as $\tilde{\mathbf{H}}_b(t)$,

$$
\tilde{\mathbf{U}}_b = \begin{bmatrix}
\tilde{\mathbf{U}}_1 & 0 & \cdots & 0 \\
0 & \tilde{\mathbf{U}}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{\mathbf{U}}_M
\end{bmatrix},
$$

After the reordering operation $\mathbf{R}\tilde{\mathbf{U}}_b\mathbf{R}^{-1} = \tilde{\mathbf{U}}_c$ we find the propagator of the full system in MS basis in a block matrix form of four $M \times M$ square matrices,

$$
\tilde{\mathbf{U}} = \begin{bmatrix}
1_{d \times d} & \mathbf{O}_{d \times 2M} \\
\mathbf{O}_{2M \times d} & \tilde{\mathbf{U}}_c
\end{bmatrix},
$$

where

$$
\tilde{\mathbf{U}}_c = \begin{bmatrix}
\mathbf{A} & \mathbf{B} \\
-B^* e^{-i\delta} & \mathbf{A}^* e^{-i\delta}
\end{bmatrix},
$$

with

$$
\mathbf{A} = \begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_M
\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}
b_1 & 0 & \cdots & 0 \\
0 & b_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_M
\end{bmatrix}.
$$

The original propagator obeys the same transformation as the original Hamiltonian,

$$
\mathbf{U} = \tilde{\mathbf{U}}^S_{SU}.\quad (52)
$$

### 4.2. Multi-pass MS propagator

With the expression (52) for the single propagator $\mathbf{U}$ at hand, we can easily find the $N$-pass propagator by taking the $N$th power of $\mathbf{U}$,

$$
\mathbf{U}^N = \mathbf{S} \mathbf{U}^S \mathbf{S} \mathbf{U}^S \cdots \mathbf{S} \mathbf{U}^S \mathbf{S}^\dagger = \mathbf{S} \mathbf{U}^N \mathbf{S}^\dagger, \quad (53)
$$

where $\mathbf{U}^N$ is the transformed MS propagator. As the MS system is decomposed into $M$ independent two-state systems, then all of them have $M$ independent evolutions, governed by the respective propagators $\tilde{\mathbf{U}}_m$ of equation (46). Indeed, we have from equation (49)

$$
\tilde{\mathbf{U}}^N = \begin{bmatrix}
1_{d \times d} & \mathbf{O}_{d \times 2M} \\
\mathbf{O}_{2M \times d} & \tilde{\mathbf{U}}^N_c
\end{bmatrix}.
$$

We have $\tilde{\mathbf{U}}^N = \mathbf{R}\tilde{\mathbf{U}}^N_b\mathbf{R}^{-1}$, and $\tilde{\mathbf{U}}^N_c$ is readily derived,

$$
\tilde{\mathbf{U}}^N_c = \begin{bmatrix}
\tilde{\mathbf{U}}^N_1 & 0 & \cdots & 0 \\
0 & \tilde{\mathbf{U}}^N_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{\mathbf{U}}^N_M
\end{bmatrix}. \quad (55)
$$

We can find all $\tilde{\mathbf{U}}^N_c$ using equations (5) and (6), and then construct the propagator of the full system.

In order to find $\tilde{\mathbf{U}}^N_m$ we first note that it does not possess the SU(2) symmetry as its determinant is $e^{-i\delta}$, and hence we cannot use equations (5) and (6) directly because they require SU(2) symmetry. Therefore, we represent $\tilde{\mathbf{U}}_m$ as

$$
\tilde{\mathbf{U}}_m = e^{-i\delta/2} \begin{bmatrix}
a_m e^{i\delta/2} & b_m e^{i\delta/2} \\
-b_m e^{-i\delta/2} & a_m e^{-i\delta/2}
\end{bmatrix}, \quad (56)
$$

where the matrix on the right-hand side is now SU(2) symmetric, which allows us to apply relations (5) and (6). We have

$$
\tilde{\mathbf{U}}^N_m = e^{-iN\delta/2} \begin{bmatrix}
a_m e^{i\delta/2} & b_m e^{i\delta/2} \\
-b_m e^{-i\delta/2} & a_m e^{-i\delta/2}
\end{bmatrix}^N. \quad (57)
$$

Then, by introducing the notation

$$
a_m' = a_m e^{i\delta/2}, \quad b_m' = b_m e^{i\delta/2}, \quad (58)
$$
and using equations (5) and (6), we find
\[
\tilde{U}^N_m = \begin{bmatrix}
    a_{mN}' & b_{mN}' \\
    -b_{mN}^*e^{-i\delta} & a_{mN}^*e^{-i\delta}
\end{bmatrix},
\] (59)
where
\[
a_{mN}' = \cos(N\theta_m' + i \text{Im}(a_m')) \sin(N\theta_m') / \sin(\theta_m'),
\] (60a)
\[
b_{mN}' = b_m\sin(N\theta_m') / \sin(\theta_m') e^{-i\delta/2},
\] (60b)
\[
\theta_m' = \arccos(\text{Re} a_m').
\] (60c)

The relations (60), together with (58) give the connection between the single \( U_m \) and the repeated \( \tilde{U}^N_m \) propagators. Thereby we find the multi-pass propagator of the \( M \) MS two-state systems [cf equation (54)],
\[
\tilde{U}^N = \begin{bmatrix}
    A_N & B_N \\
    -B_N^*e^{-i\delta} & A_N^*e^{-i\delta}
\end{bmatrix},
\] (61)
with
\[
A_N = \begin{bmatrix}
    a_{1N}' & 0 & \cdots & 0 \\
    0 & a_{2N}' & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_{MN}'
\end{bmatrix},
\] (62a)
\[
B_N = \begin{bmatrix}
    b_{1N}' & 0 & \cdots & 0 \\
    0 & b_{2N}' & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & b_{MN}'
\end{bmatrix}.
\] (62b)

Next, we find \( \tilde{U}^N \) from equations (54) and (61). Finally, we find the original \( N \)-pass propagator \( U^N \) by the transformation (53), i.e. \( U^N = S^N \tilde{U}^N \).

Because \( S \) is a constant matrix and it appears in both \( U \) of equation (52) and \( U^N \) of equation (53) in the same manner, it follows that \( U^N \) can be obtained from \( U \) by the substitutions \( a_m \rightarrow a_m' \) and \( b_m \rightarrow b_m' \) and \( e^{i\delta} \rightarrow e^{i\delta/2} \), according to the connections (60).

4.3. Special cases: multipod systems

4.3.1. Single-pass multipod propagator. As the first example, we consider the multipod system, shown schematically in figure 3. It consists of \( L \) ground states and a single excited state, \( M = 1 \). All ground states are coupled to the excited state but not between themselves. All couplings have the same time dependence \( f(t) \) and the same detuning \( \Delta(t) \), but their magnitudes can be all different. Therefore, its Hamiltonian fulfills the MS symmetry.

The constant matrix \( \Omega \) of equation (36) has the dimension of \( L \times 1 \) and the Hamiltonian reads
\[
H = \frac{1}{2} \begin{bmatrix}
    0 & 0 & \cdots & 0 & \Omega_1 f(t) \\
    0 & 0 & \cdots & 0 & \Omega_2 f(t) \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 0 & \Omega_L f(t)
\end{bmatrix},
\] (63)
where \( \Omega_i \) (\( i = 1, 2, \ldots, L \)) are arbitrary complex constants. The MST from the original to the MS basis for the Hamiltonian and the propagator reads
\[
\tilde{H} = S^1 H S, \quad \tilde{U} = S^1 U S,
\] (64)
where the constant transformation matrix \( S \) has the form
\[
S = \begin{bmatrix}
    \Omega_1^\dagger & \Omega_2^\dagger & \cdots & \Omega_L^\dagger & 0 \\
    -\Omega_1^\dagger & \Omega_2^\dagger & \cdots & \Omega_L^\dagger & 0 \\
    0 & -\Omega_1^\dagger & \cdots & \Omega_L^\dagger & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 0 & -\Omega_{L-1}^\dagger & \Omega_L^\dagger & 0 \\
    0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix},
\] (65)
with the real constants
\[
X_l = \sqrt{\sum_{k=1}^l |\Omega_k|^2}, \quad (l = 2, 3, \ldots, L).
\] (66)
The MST of equation (64), along with equations (63) and (65), gives the transformed MS Hamiltonian \( \tilde{H}(t) \), which reduces to an effective two-state system,
\[
\tilde{H} = \frac{1}{2} \begin{bmatrix}
    0 & 0 & \cdots & 0 & 0 \\
    0 & 0 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 0 & \Omega f(t) \\
    0 & 0 & \cdots & \Omega f(t) & 2\Delta(t)
\end{bmatrix},
\] (67)
with
\[
\Omega = \sqrt{\sum_{l=1}^L |\Omega_l|^2}.
\] (68)
The MS Hamiltonian (67) generates the propagator
Figure 3. The MS transformation for the multipod system, which consists of $L$ degenerate ground states and a single excited state. The system is transformed into a single two-state system and a set of $L - 1$ decoupled (dark) ground states. All couplings have the same time dependence $f(t)$ and the same detuning $\Delta(t)$.

The original propagator is found by the inverse of equation (64), i.e. $U = SUS^\dagger$, or explicitly,

$$
\tilde{U} = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & a & b \\
0 & 0 & \cdots & 0 & -b^* e^{-i\delta} & a^* e^{-i\delta}
\end{bmatrix},
$$

which has only a pair of Cayley–Klein parameters $a, b$, because all other states in the MS basis are decoupled. The
where \(|\Omega\rangle = \{\Omega_1, \Omega_2, \ldots, \Omega_L\}^T\) and hence \(\langle \Omega | =
\{\Omega_1, \Omega_2, \ldots, \Omega_L\}^\ast\).

### 4.3.3. Multipass propagator for the \(\Lambda\) system

The \(\Lambda\) system is shown in figure 4. It is the most ubiquitous multistate system used, for instance, in stimulated Raman adiabatic passage (STIRAP) [47], as a Raman qubit (formed by the two lower states) [50, 51], and a qutrit [52–54] in quantum information, etc. There is a single condition for the applicability of the MST: the two couplings must share the same time dependence \(f(t)\), as shown in the figure. That means the MST is not applicable to STIRAP, which requires different time dependence of the Rabi frequencies. Nevertheless, the MST is suitable to Raman qubits, with the time-coincident fields. To this end, we note that the MST has been generalized recently to different time dependences [15], but this treatment is approximate.

The MST matrix is

\[
S_\Lambda = \begin{bmatrix}
\frac{\Omega_1}{\Omega_2} & \frac{\Omega_1}{\Omega_2} & 0 \\
-\frac{\Omega_1}{\Omega_2} & \frac{\Omega_1}{\Omega_2} & 0 \\
0 & 0 & 1 
\end{bmatrix},
\]

and the multi-pass propagator reads

\[
U^\Lambda_N = \begin{bmatrix}
1 + (a'_N - 1) \frac{\Omega_1}{\Omega_2} & (a'_N - 1) \frac{\Omega_1}{\Omega_2} & b'_N \frac{\Omega_1}{\Omega_2} \\
(a'_N - 1) \frac{\Omega_1}{\Omega_2} & 1 + (a'_N - 1) \frac{\Omega_1}{\Omega_2} & b'_N \frac{\Omega_1}{\Omega_2} \\
-b'_N \frac{\Omega_1}{\Omega_2} e^{-i\delta_N} & -b'_N \frac{\Omega_1}{\Omega_2} e^{-i\delta_N} & a'_N e^{-i\delta_N}
\end{bmatrix},
\]

where \(\Omega = \sqrt{|\Omega_1|^2 + |\Omega_2|^2}\).

### 4.3.4. Multipass propagator for the tripod system

The tripod system, shown in figure 5, is another popular system because it possesses two dark states, which can serve as a decoherence-free qubit in topologic quantum information [55–57]. It also allows a great flexibility in the creation of arbitrary coherent superposition of the three lower states, and can be used more generally as a very convenient implementation of a qutrit. That said, we note that if the three couplings have the same time dependence, as it is the case in the MST,
then the two dark states decouple from each other and one cannot implement a gate between them. For such a gate, one has to introduce an additional field or fields with a different time dependence. Yet, the MST can serve as a good starting point in this direction. However, the MST in the case of three coincident pulsed fields can be useful in another context—constructing quantum gates between all three ground states, which form a qutrit. Then the MS decomposition allows one to simplify the four-state dynamics and reduce it to a two-state one, which facilitates the search for fields which produce particular qutrit gates.

The MST matrix is

$$
S_T = \begin{bmatrix}
\frac{\Omega_f}{\hbar} & \frac{\Omega_f}{\hbar} & \frac{\Omega_f}{\hbar} & 0 \\
-\frac{\Omega_f}{\hbar} & -\frac{\Omega_f}{\hbar} & -\frac{\Omega_f}{\hbar} & 0 \\
0 & -\frac{\Omega_f}{\hbar} & -\frac{\Omega_f}{\hbar} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
$$

and the multipass propagator is

$$
U_N^T = \begin{bmatrix}
1 + (a_N^f - 1)^2 & (a_N^f - 1)\frac{\Omega_f}{\hbar} & (a_N^f - 1)^2 & b_N^f \frac{\Omega_f}{\hbar} \\
(a_N^f - 1)^2 & 1 + (a_N^f - 1)^2 & (a_N^f - 1)^2 & b_N^f \frac{\Omega_f}{\hbar} \\
(a_N^f - 1)^2 & (a_N^f - 1)^2 & 1 + (a_N^f - 1)^2 & b_N^f \frac{\Omega_f}{\hbar} \\
b_N^f \frac{\Omega_f}{\hbar} e^{-i\delta} & -b_N^f \frac{\Omega_f}{\hbar} e^{-i\delta} & b_N^f \frac{\Omega_f}{\hbar} e^{-i\delta} & 1
\end{bmatrix},
$$

where $\Omega = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}$.

5. A system possessing both WM and MS symmetries

Finally, we consider a system which has simultaneously the WM and MS symmetries. The WM and MS symmetries are compatible only on resonance ($\Delta = 0$), which we shall assume. For nonzero detuning in the WM system, one cannot separate two degenerate sets of states, as required by the MS symmetry.

In order to elucidate the connection between the two symmetries, we shall show how a WM system looks like in the MS basis. The system is shown on figure 6. As above, all Rabi frequencies have the same time dependence $f(t)$. However, contrary to the general MS case when the coupling amplitudes

![Figure 5. Morris–Shore transformation for the tripod system.](image)

\[
\text{Figure 5. Morris–Shore transformation for the tripod system.}
\]



\[
\text{Figure 5. Morris–Shore transformation for the tripod system.}
\]

\[
\text{Figure 5. Morris–Shore transformation for the tripod system.}
\]
Figure 6. A system with WM and MS symmetry. Top: the WM system is represented as two sets of chainwise-connected states with zero detunings. The values of the couplings $\Omega_{k,k+1}$ are given by equation (77). Bottom: the MS transformation reduces the system to $M$ independent two-state MS systems. For $K = 2M + 1$ ($M = 1, 2, \ldots$), in the MS basis there are $M$ two-state systems with couplings given by equation (80c) and a single dark state. For $K = 2M$ (not shown), there is no dark state and the couplings are given by equation (80b).

The system configuration is illustrated in the top part of the figure, with the WM system shown as a series of coupled states labeled $|1\rangle$, $|2\rangle$, $|3\rangle$, $|K-2\rangle$, and $|K\rangle$, connected by coupling terms $\Omega_{12}$, $\Omega_{23}$, $\Omega_{34}$, $\Omega_{K-2,K-1}$, and $\Omega_{K-1,K}$. The MS transformation is applied to this system, reducing it to $M$ independent two-state MS systems, each characterized by a coupling term $\Omega_{0}$ or $\Omega_{0}'$, and a dark state $|\gamma_{1}\rangle$. The transformed states are labeled $|\alpha_{1}\rangle$, $|\alpha_{2}\rangle$, $|\alpha_{3}\rangle$, $|\alpha_{M}\rangle$, $|\beta_{1}\rangle$, $|\beta_{2}\rangle$, $|\beta_{3}\rangle$, $|\beta_{M}\rangle$.

The matrix $\Omega$ representing the WM system is given by:

$$
\Omega = \begin{bmatrix}
\Omega_{12} & 0 & 0 & \cdots & 0 \\
\Omega_{23} & \Omega_{34} & 0 & \cdots & 0 \\
0 & \Omega_{45} & \Omega_{56} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \Omega_{K-3,K-2} & \Omega_{K-2,K-1} \\
0 & 0 & \cdots & 0 & \Omega_{K-1,K}
\end{bmatrix}
$$

The eigenvalues $\lambda_{m}^{2}$ of the $M$-dimensional square matrix $\Omega^{2}$ are $\lambda_{m}^{2} = m_{j}^{2}|\Omega_{0}|^{2}$, i.e. the coupling constants $\lambda_{m}$ of the $M$ independent systems are

$$
\lambda_{m} = m_{j}|\Omega_{0}|,
$$

where $m_{j}$ are the eigenvalues of $\Omega^{2}$ given by equations (80a, 80b, 80c).

The MS Cayley–Klein parameters are

$$
\begin{align*}
a_{mN} &= \cos(N\theta_{m}), \\
b_{mN} &= -i\sin(N\theta_{m}), \\
\theta_{m} &= \frac{m_{j}|\Omega_{0}|}{2} \int_{0}^{T} f(t) \, dt.
\end{align*}
$$

The transformation is achieved by applying the Moris-Shore (MS) transformation, which reduces the system from $|\alpha_{M}\rangle$ to $|\beta_{M}\rangle$ with the help of the MS transformation matrix $M\Omega_{0}$.
which give the connection between the angles $\theta$ and $\theta_m$ of both MS and WM representations

$$\theta_m = m \theta. \quad (82)$$

6. Conclusion

In this paper, we have derived explicit analytic formulae describing the interaction of multistate quantum systems with either the WM SU(2) or the MS symmetry, or both, with a driving field consisting of $N$ identical single-step fields. For a single-step interaction the dynamics of these systems can be reduced to the dynamics of one or more two-state systems. We have used this feature in order to derive the propagators for these two types of systems in terms of the parameters of the two-state propagators and the number of interactions $N$. These relations allows one to readily find out the state of the quantum system for arbitrary initial conditions.

Our results can find applications in the development of quantum control method for multistate systems by using the well-known methods for two-state systems. In particular, one can find exact analytic solutions for the multi-pass dynamics of multistate systems using the well-known single-pass two-state analytic models. Moreover, the results make it possible to estimate the efficiency of various dynamical decoupling sequences for qubits and qudits as well.

On the other hand, the explicit analytic formulae make it possible to develop precise methods for quantum gate tomography of multistate systems (e.g. qutrits, and qudits in general, as well as an ensemble of a few qubits) by repeated application of the quantum gate, which quickly amplifies its infidelity to levels which can be measured very accurately [26, 27]. Then the analytic connections between the single-pass and multi-pass propagator parameters allow one to deduce the single-pass ones from the multi-pass ones. By the same token, our results allow to design novel methods for quantum sensing of small frequency shifts with qubits by amplifying their effect, measuring the amplified signal and then deducing the small frequency shift via the analytic formulae.

Finally, we have assumed that each pulse (or set of pulses) in the multi-step pulse-train interaction is identical. When the pulses are not identical, our formalism does not apply, which is a limitation of the method. On the other hand, this assumption is not unreasonable because a laser or microwave source, under the same conditions, produces the same output pulses. Therefore, it is natural to assume that the pulses are identical. Furthermore, in each interaction step, we have assumed pulses fields with the same time dependence because this is demanded by both WM and MS models, as well as specific relations for the detunings. One might wish to drop these restrictions and consider more general models. However, the systems can only be treated approximately outside the conditions of validity of the WM and MS models. The resulting error stemming from the approximation is unwanted because it may easily exceed the tiny gate error, the dynamical decoupling error, or the quantum sensing signal we wish to measure. Therefore, we restricted ourselves to the exact relations, especially because the conditions for the WM and MS models are easily satisfied in a real experiment.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

We dedicate this paper to the memory of our dear friend Bruce W Shore, who revealed to us the power of the beautiful Morris–Shore transformation, one of the most elegant approaches to understanding the complex dynamics of multistate quantum systems.

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