THE FRACTIONAL K-METRIC DIMENSION OF GRAPHS

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Let $G$ be a graph with vertex set $V(G)$. For any two distinct vertices $x$ and $y$ of $G$, let $R(x,y)$ denote the set of vertices $z$ such that the distance from $x$ to $z$ is not equal to the distance from $y$ to $z$ in $G$. For a function $g$ defined on $V(G)$ and for $U \subseteq V(G)$, let $g(U) = \sum_{s \in U} g(s)$. Let $\kappa(G) = \min\{|R(x,y)| : x \neq y \text{ and } x, y \in V(G)\}$. For any real number $k \in [1, \kappa(G)]$, a real-valued function $g : V(G) \to [0,1]$ is a $k$-resolving function of $G$ if $g(R(x,y)) \geq k$ for any two distinct vertices $x, y \in V(G)$. The fractional $k$-metric dimension, $\dim_{k}f(G)$, of $G$ is $\min\{g(V(G)) : g$ is a $k$-resolving function of $G\}$. In this paper, we initiate the study of the fractional $k$-metric dimension of graphs. For a connected graph $G$ and $k \in [1, \kappa(G)]$, it’s easy to see that $k \leq \dim_{k}f(G) \leq \frac{k|V(G)|}{\kappa(G)}$; we characterize graphs $G$ satisfying $\dim_{k}f(G) = k$ and $\dim_{k}f(G) = |V(G)|$, respectively. We show that $\dim_{k}f(G) \geq k \dim_{f}(G)$ for any $k \in [1, \kappa(G)]$, and we give an example showing that $\dim_{k}f(G) - k \dim_{f}(G)$ can be arbitrarily large for some $k \in (1, \kappa(G))$; we also describe a condition for which $\dim_{k}f(G) = k \dim_{f}(G)$ holds. We determine the fractional $k$-metric dimension for some classes of graphs, and conclude with two open problems, including whether $\phi(k) = \dim_{k}f(G)$ is a continuous function of $k$ on every connected graph $G$.

1. Introduction

Let $G$ be a finite, simple, undirected, and connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the open neighborhood of $v$ is $N(v) = \{u \in V(G) : d(v,u) = 1\}$.

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V(G) : \forall \{w \in E(G)\}, and the closed neighborhood of v is \( N[v] = N(v) \cup \{v\} \). The degree of a vertex \( v \in V(G) \), denoted by \( \deg(v) \), is \( |N(v)| \); a leaf is a vertex of degree one, and a major vertex is a vertex of degree at least three. The distance between two vertices \( x, y \in V(G) \), denoted by \( d(x, y) \), is the length of a shortest path between \( x \) and \( y \) in \( G \). The diameter, \( \text{diam}(G) \), of a graph \( G \) is \( \max\{d(x, y) : x, y \in V(G)\} \).

The complement of \( G \), denoted by \( \overline{G} \), is the graph whose vertex set is \( V(G) \) and \( xy \in E(\overline{G}) \) if and only if \( xy \notin E(G) \) for \( x, y \in V(G) \). We denote by \( K_n \) and \( P_n \) the complete graph and the path on \( n \) vertices, respectively.

For two distinct vertices \( x, y \in V(G) \), let \( R\{x, y\} = \{z \in V(G) : d(x, z) \neq d(y, z)\} \). A subset \( S \subseteq V(G) \) is called a resolving set of \( G \) if \( |S \cap R\{x, y\}| \geq 1 \) for any two distinct vertices \( x \) and \( y \) in \( G \). The metric dimension, \( \text{dim}(G) \), of \( G \) is the minimum cardinality of \( S \) over all resolving sets of \( G \). Since metric dimension is suggestive of the dimension of a vector space in linear algebra, sometimes a minimum resolving set of \( G \) is called a basis of \( G \). The concept of metric dimension was introduced independently by Slater [28], and by Harary and Melter [20]. Applications of metric dimension can be found in network discovery and verification [7], robot navigation [24], sonar [28], combinatorial optimization [27], chemistry [25], and strategies for the mastermind game [9]. It was noted in [19] that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been extensively studied. For a survey on metric dimension in graphs, see [4, 8]. The effect of the deletion of a vertex or of an edge on the metric dimension of a graph was raised as a fundamental question in graph theory in [8]; the question is essentially settled in [11].

If a minimum number of requisite robots are installed in a network to identify the exact location of an intruder in the network, one malfunctioning robot can lead to failure of detection. Thus, it is natural to build a certain level of redundancy into the detection system. As a generalization of metric dimension, \( k \)-metric dimension was introduced first by Estrada-Moreno et al. [12] and, independently, by Adar and Epstein [11] soon afterwards. Let \( \kappa(G) = \min\{|R\{x, y\}| : x \neq y \text{ and } x, y \in V(G)\} \). For a positive integer \( k \in \{1, 2, \ldots, \kappa(G)\} \), a set \( S \subseteq V(G) \) is called a \( k \)-resolving set of \( G \) if \( |S \cap R\{x, y\}| \geq k \) for any two distinct vertices \( x \) and \( y \) in \( G \). The \( k \)-metric dimension, \( \text{dim}^k(G) \), of \( G \) is the minimum cardinality over all \( k \)-resolving sets of \( G \). It was shown in [12] that \( k \)-metric dimension of a connected graph \( G \) exists for every \( k \in \{1, 2, \ldots, \kappa(G)\} \), and \( G \) is called \( \kappa(G) \)-metric dimensional. For an application of \( k \)-metric dimension to error-correcting codes, see [5]. For other articles on the \( k \)-metric dimension of graphs, see [6, 13, 14].

The fractionalization of various graph parameters has been extensively studied (see [26]). Currie and Oellermann [10] defined fractional metric dimension as the optimal solution to a linear programming problem, by relaxing a condition of the integer programming problem for metric dimension. A formulation of fractional metric dimension as a linear programming problem can be found in [15]. Arumugam and Mathew [2] officially studied the fractional metric dimension of graphs.

\footnote{In fact, the notation of this parameter has been \( \text{dim}_k(G) \) in previous works. However, we rather prefer to use \( \text{dim}^k(G) \) here, to facilitate the notation of \( \text{dim}^k(G) \).}
For a function $g$ defined on $V(G)$ and for $U \subseteq V(G)$, let $g(U) = \sum_{s \in U} g(s)$. A real-valued function $g : V(G) \rightarrow \{0, 1\}$ is a resolving function of $G$ if $g(R\{x, y\}) \geq 1$ for any two distinct vertices $x, y \in V(G)$. The fractional metric dimension, $\operatorname{dim}_f(G)$, of $G$ is $\min\{g(V(G)) : g$ is a resolving function of $G\}$. Notice that $\operatorname{dim}_f(G)$ reduces to $\operatorname{dim}(G)$, if the codomain of resolving functions is restricted to $\{0, 1\}$. For more articles on the fractional metric dimension, as well as the closely related fractional strong metric dimension, of graphs, see [3, 16, 17, 18, 21, 22, 23, 29].

Now, we introduce fractional $k$-metric dimension, which can be viewed as a generalization of $\operatorname{dim}_f(G)$, as well as a fractionalization of $\operatorname{dim}^k(G)$. For any real number $k \in [1, \kappa(G)]$, a real-valued function $h : V(G) \rightarrow [0, 1]$ is a $k$-resolving function of $G$ if $h(R\{x, y\}) \geq k$ for any distinct vertices $x, y \in V(G)$. The fractional $k$-metric dimension, $\operatorname{dim}^k_f(G)$, of $G$ is $\min\{h(V(G)) : h$ is a $k$-resolving function of $G\}$. Note $\operatorname{dim}^1_f(G) = \operatorname{dim}_f(G)$ and that $\operatorname{dim}^k_f(G)$ reduces to $\operatorname{dim}^k(G)$ when the codomain of $k$-resolving functions is restricted to $\{0, 1\}$ and $k \in [1, \kappa(G)]$ is restricted to positive integers.

Next, we recall some results on the fractional metric dimension of graphs. One can easily see that, for any connected graph $G$ of order at least two, $1 \leq \operatorname{dim}_f(G) \leq \frac{|V(G)|}{2}$ (see [2]). For the characterization of graphs $G$ achieving the lower bound, see Theorem 3(a). Regarding the characterization of graphs $G$ achieving the upper bound, the following result is stated in [2] and a correct proof is provided in [21].

**Theorem 1.** [2, 21] Let $G$ be a connected graph of order at least two. Then $\operatorname{dim}_f(G) = \frac{|V(G)|}{2}$ if and only if there exists a bijection $\alpha : V(G) \rightarrow V(G)$ such that $\alpha(v) \neq v$ and $|R\{v, \alpha(v)\}| = 2$ for all $v \in V(G)$.

An explicit characterization of graphs $G$ satisfying $\operatorname{dim}_f(G) = \frac{|V(G)|}{2}$ is given in [3]. We recall the following construction from [3]. Let $\mathcal{K} = \{K_n : n \geq 2\}$ and $\overline{\mathcal{K}} = \{\overline{K}_n : n \geq 2\}$. Let $H[\mathcal{K} \cup \overline{\mathcal{K}}]$ be the family of graphs obtained from a connected graph $H$ by (i) replacing each vertex $u_i \in V(H)$ by a graph $H_i \in \mathcal{K} \cup \overline{\mathcal{K}}$, and (ii) each vertex in $H_i$ is adjacent to each vertex in $H_j$ if and only if $u_i u_j \in E(H)$.

**Theorem 2.** [3] Let $G$ be a connected graph of order at least two. Then $\operatorname{dim}_f(G) = \frac{|V(G)|}{2}$ if and only if $G \in H[\mathcal{K} \cup \overline{\mathcal{K}}]$ for some connected graph $H$.

Now, we recall the fractional metric dimension of some classes of graphs. We begin by recalling some terminologies. Fix a graph $G$. A leaf $u$ is called a terminal vertex of a major vertex $v$ if $d(u, v) < d(u, w)$ for every other major vertex $w$. The terminal degree, $\operatorname{ter}_G(v)$, of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ is an exterior major vertex if it has positive terminal degree. Let $\operatorname{ex}(G)$ denote the number of exterior major vertices of $G$, $\operatorname{ex}_a(G)$ the number of exterior major vertices $u$ with $\operatorname{ter}_G(u) = a$, and $\sigma(G)$ the number of leaves of $G$.

**Theorem 3.**

(a) [23] For any graph $G$ of order $n \geq 2$, $\operatorname{dim}_f(G) = 1$ if and only if $G \cong P_n$.

(b) [29] For a tree $T$, $\operatorname{dim}_f(T) = \frac{1}{2}(\sigma(T) - \operatorname{ex}_1(T))$. 

To view the whole document, please see the text above.
For the Petersen graph \( \mathcal{P} \), \( \dim_f(\mathcal{P}) = \frac{5}{7} \).

For an \( n \)-cycle \( C_n \), \( \dim_f(C_n) = \left\{ \begin{array}{ll} \frac{n}{n-2} & \text{if } n \text{ is even}, \\ \frac{n}{n-1} & \text{if } n \text{ is odd}. \end{array} \right. \)

For the wheel graph \( W_n \) of order \( n \geq 5 \), \( \dim_f(W_n) = \left\{ \begin{array}{ll} 2 & \text{if } n = 5, \\ \frac{3}{2} & \text{if } n = 6, \\ \frac{n+1}{4} & \text{if } n \geq 7. \end{array} \right. \)

If \( B_m \) is a bouquet of \( m \) cycles with a cut-vertex (i.e., the vertex sum of \( m \) cycles at one common vertex), where \( m \geq 2 \), then \( \dim_f(B_m) = m \).

For \( m \geq 2 \), let \( G = K_{a_1,a_2,\ldots,a_m} \) be a complete \( m \)-partite graph of order \( n = \sum_{i=1}^{m} a_i \), and let \( s \) be the number of partite sets of \( G \) consisting of exactly one element. Then

\[
\dim_f(G) = \left\{ \begin{array}{ll} \frac{n-1}{2} & \text{if } s = 1, \\ \frac{n}{s} & \text{otherwise}. \end{array} \right.
\]

For the grid graph \( G = P_s \square P_t \) (\( s, t \geq 2 \)), \( \dim_f(G) = 2 \).

In this paper, we initiate the study of the fractional \( k \)-metric dimension of graphs. For a connected graph \( G \), let \( \kappa(G) = \min\{ |R(x,y)| : x \neq y \text{ and } x,y \in V(G) \} \). The paper is organized as follows. In section 2, we compare \( \dim_f(G) \) with \( \dim^k_f(G) \) for certain \( k \). We prove that \( \dim^k_f(G) \geq k \dim_f(G) \) for any \( k \in [1, \kappa(G)] \); we describe a condition for which \( \dim^k_f(G) = k \dim_f(G) \) holds for all \( k \in [1, \kappa(G)] \). For \( k \in [1, \kappa(G)] \), we show that \( k \leq \dim^k_f(G) \leq \frac{k}{\kappa(G)} |V(G)| \), which implies \( k \leq \dim^k_f(G) \leq |V(G)| \); we characterize graphs \( G \) satisfying \( \dim^k_f(G) = k \) and \( \dim^k_f(G) = |V(G)| \), respectively. We also show an example such that two non-isomorphic graphs \( H_1 \) and \( H_2 \) satisfy \( \dim^k_f(H_1) = \dim^k_f(H_2) \) for all \( k \in [1, \kappa] \), where \( \kappa(H_1) = \kappa(H_2) = \kappa \). In section 3, for \( k \in [1, \kappa(G)] \), we determine the fractional \( k \)-metric dimension of trees, cycles, wheel graphs, the Petersen graph, a bouquet of cycles (i.e., the vertex sum of cycles at one common vertex), complete multi-partite graphs, and grid graphs (i.e., the Cartesian product of two paths). Along the way, we give an example showing that \( \dim^k_f(G) - k \dim_f(G) \) can be arbitrarily large for some \( k \in (1, \kappa(G)] \). We conclude with some open problems.

2. Some general results on fractional \( k \)-metric dimension

We begin with some observations. Two distinct vertices \( x, y \in V(G) \) are called twin vertices if \( N(x) - \{y\} = N(y) - \{x\} \).

**Observation 4.** Let \( G \) be a connected graph and let \( k \in [1, \kappa(G)] \). If two distinct vertices \( x \) and \( y \) are twin vertices in \( G \), then \( R\{x,y\} = \{x,y\} \), and thus \( \kappa(G) = 2 \) and \( g(x) + g(y) \geq k \) for any \( k \)-resolving function \( g \) of \( G \).

**Observation 5.** Let \( G \) be a connected graph.

(a) For \( k \in [1, \kappa(G)] \), \( \dim_f(G) \leq \dim^k_f(G) \).
(b) For $k \in \{1, 2, \ldots, \kappa(G)\}$, $\dim^k_f(G) \leq \dim^k(G)$.

(c) \[2, 12\] For $k \in \{1, 2, \ldots, \kappa(G)\}$, $\dim_f(G) \leq \dim(G) \leq \dim^k(G)$.

Observation 5 provides inequalities between any two graph parameters among $\dim_f(G)$, $\dim(G)$, $\dim^k_f(G)$ for $k \in [1, \kappa(G)]$, and $\dim^k(G)$ for $k \in \{1, 2, \ldots, \kappa(G)\}$, excluding the relation between $\dim(G)$ and $\dim^k_f(G)$. So, it is natural to compare $\dim(G)$ and $\dim^k_f(G)$ for $k \in [1, \kappa(G)]$.

Remark 6. (a) The value of $\dim^k_f(G) - \dim(G)$ can be arbitrarily large, as $G$ varies, for some $k \in \{1, \kappa(G)\}$. For $n \geq 4$, note that $\dim(P_n) = 1 \leq \dim^k_f(P_n)$ for $k \in [1, n - 1]$ and $\dim^{n-1}_f(P_n) = n$ (see Proposition 12). Thus, $\dim^{n-1}_f(P_n) - \dim(P_n) = n - 1$ can be arbitrarily large.

(b) The value of $\dim(G) - \dim^k_f(G)$ can be arbitrarily large, as $G$ varies, for some $k \in [1, \kappa(G)]$. For $n \geq 3$, note that $\dim(K_n) = n - 1$ and $\dim^k_f(K_n) = \frac{kn}{2}$ for $k \in [1, 2]$ (see Proposition 36). Now, let $k \in [1, 2]$; then $\dim(K_n) - \dim^k_f(K_n) = \frac{(2-k)n}{2} - 1$ becomes arbitrarily large as $n \to \infty$.

In light of Observation 5(b), we have the following

**Theorem 7.** The value of $\dim^k(G) - \dim^k_f(G)$ can be arbitrarily large, as $G$ varies, for some $k \in \{1, 2, \ldots, \kappa(G)\}$.

**Proof.** Let $H$ be a connected graph with vertex set $V(H) = \{u_1, u_2, \ldots, u_n\}$, where $n \geq 2$. Let $G$ be the graph obtained from $H$ as follows:

(i) for each $i \in \{1, 2, \ldots, n\}$, add three vertices $a_{i,1}, b_{i,1}, c_{i,1}$ and three edges $u_ia_{i,1}, u_ib_{i,1}, u_ic_{i,1}$;

(ii) for each $i \in \{1, 2, \ldots, n\}$, subdivide the edge $u_ia_{i,1}$ (or $b_{i,1}$ and $c_{i,1}$, respectively) exactly $s - 1$ times so that the edge $u_i a_{i,1}$ (or $b_{i,1}$ and $c_{i,1}$, respectively) in (i) becomes the $u_i - a_{i,1}$ path given by $u_i, a_{i,s} a_{i,s-1}, \ldots, a_{i,1}$ (the $u_i - b_{i,1}$ path given by $u_i, b_{i,s} b_{i,s-1}, \ldots, b_{i,1}$ and the $u_i - c_{i,1}$ path given by $u_i, c_{i,s} c_{i,s-1}, \ldots, c_{i,1}$, respectively).

For each $i \in \{1, 2, \ldots, n\}$, let $T_i$ be the subtree of $G$ consisting of the $u_i - a_{i,1}$ path, the $u_i - b_{i,1}$ path, and the $u_i - c_{i,1}$ path; further, let $P_i^{a}$ be the $a_{i,s} - a_{i,1}$ path, $P_i^{b}$ the $b_{i,s} - b_{i,1}$ path, and $P_i^{c}$ the $c_{i,s} - c_{i,1}$ path. Then, for each $i \in \{1, 2, \ldots, n\}$,

\[
\begin{align*}
R(a_{i,s}, b_{i,s}) &= V(P_i^{a}) \cup V(P_i^{b}), \\
R(a_{i,s}, c_{i,s}) &= V(P_i^{a}) \cup V(P_i^{c}), \\
R(b_{i,s}, c_{i,s}) &= V(P_i^{b}) \cup V(P_i^{c}).
\end{align*}
\]

We determine $\kappa(G)$, $\dim^k(G)$ for $k \in \{1, 2, \ldots, \kappa(G)\}$, and $\dim^k_f(G)$ for $k \in [1, \kappa(G)]$.

**Claim 1:** $\kappa(G) = 2s$.

**Proof of Claim 1.** Let $x$ and $y$ be two distinct vertices of $G$. First, let $x, y \in V(T_i)$ for some $i \in \{1, 2, \ldots, n\}$. If $d(u_i, x) \neq d(u_i, y)$, then $R(x, y) \supseteq V(T_j)$ with $|R(x, y)| \geq \ldots$
Claim 2: For $k \in \{1, 2, \ldots, 2s\}$, $\dim^k(G) = \begin{cases} \frac{3kn}{2} & \text{if } k \text{ is even}, \\ \frac{(3k+1)n}{2} & \text{if } k \text{ is odd}. \end{cases}$

Proof of Claim 2. Let $k \in \{1, 2, \ldots, 2s\}$. First, we show that $\dim^k(G) \geq \frac{3kn}{2}$ for an even $k$, and $\dim^k(G) \geq \frac{(3k+1)n}{2}$ for an odd $k$. Let $S$ be a minimum $k$-resolving set of $G$. For each $i \in \{1, 2, \ldots, n\}$, [1] implies

\[
\begin{align*}
|S \cap R(a_i, b_i)| &= |S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| \geq k, \\
|S \cap R(a_i, c_i)| &= |S \cap V(P^{i,a})| + |S \cap V(P^{i,c})| \geq k, \\
|S \cap R(b_i, c_i)| &= |S \cap V(P^{i,b})| + |S \cap V(P^{i,c})| \geq k.
\end{align*}
\]

Suppose $k$ is even. By summing over the three inequalities in [2], we obtain $|S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| + |S \cap V(P^{i,c})| \geq \frac{3kn}{2}$, and thus $|S \cap V(T_i)| \geq \frac{3kn}{2}$ for each $i \in \{1, 2, \ldots, n\}$. So, $|S \cap V(G)| \geq \sum_{i=1}^{n} \frac{3kn}{2} = \frac{3kn}{2}$, and hence $\dim^k(G) \geq \frac{3kn}{2}$.

Now, suppose $k$ is odd. If $|S \cap V(P^{i,a})| \leq \frac{k}{2}$ and $|S \cap V(P^{i,b})| \leq \frac{k}{2}$ for some $i \in \{1, 2, \ldots, n\}$, then $|S \cap R(a_i, b_i)| = |S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| \leq \frac{k}{2} + \frac{k}{2} = k - 1$, contradicting the assumption that $S$ is a $k$-resolving set of $G$; thus, $|S \cap V(P^{i,a})| \leq \frac{k}{2}$ and $|S \cap V(P^{i,b})| \leq \frac{k}{2}$ for at most one $t \in \{a, b, c\}$ for each $i \in \{1, 2, \ldots, n\}$. Let $|S \cap V(P^{i,c})| \leq \alpha \leq \min\{|S \cap V(P^{i,a})|, |S \cap V(P^{i,b})|\}$ for $i \in \{1, 2, \ldots, n\}$. Then $|S \cap V(P^{i,a})| \geq k - \alpha$ and $|S \cap V(P^{i,b})| \geq k - \alpha$ from [2], and thus $|S \cap V(T_i)| \geq 2k - \alpha \geq 2k - \frac{k}{2} = \frac{3k+1}{2}$. So, $|S \cap V(G)| \geq \sum_{i=1}^{n} \frac{3k+1}{2} = \frac{(3k+1)n}{2}$, and hence $\dim^k(G) \geq \frac{(3k+1)n}{2}$.

Second, we show that $\dim^k(G) \leq \frac{3kn}{2}$ for an even $k$, and $\dim^k(G) \leq \frac{(3k+1)n}{2}$ for an odd $k$. If $k$ is even, let $W_0 = \bigcup_{i=1}^{n} \{a_i, a_i, a_i, \ldots, a_i, \frac{1}{2}\} \cup \{b_i, b_i, b_i, \ldots, b_i, \frac{1}{2}\} \cup \{c_i, c_i, c_i, \ldots, c_i, \frac{1}{2}\}$. If $k$ is odd, let $W_1 = \bigcup_{i=1}^{n} \{a_i, a_i, a_i, \frac{1}{2}\} \cup \{b_i, b_i, b_i, \ldots, b_i, \frac{1}{2}\} \cup \{c_i, c_i, c_i, \frac{1}{2}\}$. Note that $|W_0| = \frac{3kn}{2}$, $|W_1| = \frac{(3k+1)n}{2}$, $|W_0 \cap V(P^{i,a})| = |W_0 \cap V(P^{i,b})| = |W_0 \cap V(P^{i,c})| = \frac{k}{2} = \frac{2k}{4}$ and $|W_1 \cap V(P^{i,a})| = |W_1 \cap V(P^{i,b})| = |W_1 \cap V(P^{i,c})| = \frac{k}{2} = \frac{2k}{4}$ for $i \in \{1, 2, \ldots, n\}$. It suffices to show that $W_0$ (or $W_1$, respectively) is a $k$-resolving set of $G$ when $k$ is even (odd, respectively). Let $x$ and $y$ be distinct vertices of $G$. Suppose $x, y \in V(T_i)$ for some $i \in \{1, 2, \ldots, n\}$. If $d(u_i, x) \neq d(u_i, y)$, then $|W_0 \cap R\{x, y\} \geq |V(T_i)| \geq \frac{k}{2}$ for $j \neq i$ and for $\ell \in \{0, 1\}$. If $d(u_i, x) = d(u_i, y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,c})$ (other cases can be handled similarly), then $|W_0 \cap R\{x, y\} = |W_0 \cap V(P^{i,a}) \cup V(P^{i,c})| \geq k$ for $\ell \in \{0, 1\}$. Now, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i, j \in \{1, 2, \ldots, n\}$; suppose $d(u_i, x) \leq d(u_j, y)$, without loss of generality. Since $R\{x, y\} \supseteq V(T_i)$, $|W_0 \cap R\{x, y\} \geq |W_0 \cap V(T_i)| \geq \frac{3k+1}{2}$ for $\ell \in \{0, 1\}$. So, $W_0$ (or $W_1$, respectively) is a $k$-resolving set of $G$ when $k$ is even (odd, respectively). □
Claim 3: For $k \in [1, 2s]$, $\dim^k_f(G) = \frac{3kn}{2}$.

Proof of Claim 3. Let $k \in [1, 2s]$. First, we show that $\dim^k_f(G) \geq \frac{3kn}{2}$. Let $g : V(G) \to [0, 1]$ be any $k$-resolving function of $G$. From [1], for each $i \in \{1, 2, \ldots, n\}$, we have $g(R\{a_i, b_i, s_i\}) = g(V(P^{i,a})) + g(V(P^{i,b})) \geq k$, $g(R\{a_i, c_i, s_i\}) = g(V(P^{i,a})) + g(V(P^{i,c})) \geq k$, and $g(R\{b_i, c_i, a_i\}) = g(V(P^{i,b})) + g(V(P^{i,c})) \geq k$. By summing over the three inequalities, we obtain $g(V(P^{i,a})) + g(V(P^{i,b})) + g(V(P^{i,c})) \geq \frac{3k}{2}$, and thus $g(V(T_i)) \geq \frac{3k}{2}$ for each $i \in \{1, 2, \ldots, n\}$. So, $g(V(G)) \geq \sum_{i=1}^{n} \frac{3k}{2} = \frac{3kn}{2}$, and hence $\dim^k_f(G) \geq \frac{3kn}{2}$.

Second, we show that $\dim^k_f(G) \leq \frac{3kn}{2}$. Let $h : V(G) \to [0, 1]$ be a function defined by

$$h(v) = \left\{ \begin{array}{ll} 0 & \text{if } \deg(v) \geq 3, \\ \frac{k}{2} & \text{if } \deg(v) \leq 2. \end{array} \right.$$ 

Notice that $h(V(P^{i,a})) = h(V(P^{i,b})) = h(V(P^{i,c})) = \frac{k}{2}$ for each $i \in \{1, 2, \ldots, n\}$, and $h(V(G)) = \frac{3kn}{2}$. It suffices to show that $h$ is a $k$-resolving function of $G$. Let $x$ and $y$ be distinct vertices of $G$. Suppose $x, y \in V(T_i)$ for some $i \in \{1, 2, \ldots, n\}$. If $d(u, x) \neq d(u, y)$, then $h(R\{x, y\}) \geq h(V(T_i)) \geq \frac{3k}{2}$ for $j \neq i$. If $d(u, x) = d(u, y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,b})$ without loss of generality, then $h(R\{x, y\}) = h(V(P^{i,a})) + h(V(P^{i,b})) = k$. Now, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i, j \in \{1, 2, \ldots, n\}$; suppose $d(u, x) \leq d(u, y)$, without loss of generality. Then $h(R\{x, y\}) \geq h(V(T_i)) \geq \frac{3k}{2}$. So, $h$ is a $k$-resolving function of $G$. \qed

By Claims 2 and 3, we see that, for each odd $k \in \{1, 2, \ldots, 2s\}$, $\dim^k(G) - \dim^k_f(G) = \frac{(3k+1)n}{2} - \frac{3kn}{2} = \frac{n}{2}$, which can be arbitrarily large. \qed

Next, we compare the fractional metric dimension and the fractional $k$-metric dimension of graphs.

**Lemma 8.** For any connected graph $G$ and for any $k \in [1, \kappa(G)]$, $\dim^k_f(G) \geq k \dim_f(G)$.

**Proof.** Let $g : V(G) \to [0, 1]$ be a minimum $k$-resolving function of $G$. Then $g(R\{x, y\}) \geq k$ for any two distinct vertices $x, y \in V(G)$. Now, let $h : V(G) \to [0, 1]$ be a function defined by $h(u) = \frac{1}{k}g(u)$ for each $u \in V(G)$. Then $h(R\{x, y\}) = \frac{1}{k}g(R\{x, y\}) \geq 1$ for any distinct vertices $x, y \in V(G)$; thus, $h$ is a resolving function of $G$. So, $h(V(G)) = \frac{1}{k} \dim^k_f(G) \geq \dim_f(G)$, i.e., $\dim^k_f(G) \geq k \dim_f(G)$. \qed

Next, we examine the conditions for which $\dim^k_f(G) = k \dim_f(G)$ holds.

**Lemma 9.** Let $G$ be a connected graph and let $k \in [1, \kappa(G)]$. If there exists a minimum resolving function $g : V(G) \to [0, 1]$ such that $g(v) \leq \frac{1}{k}$ for each $v \in V(G)$, then $\dim^k_f(G) = k \dim_f(G)$ for any $k \in [1, \kappa(G)]$.

**Proof.** Let $g : V(G) \to [0, 1]$ be a minimum resolving function of $G$ satisfying $g(v) \leq \frac{1}{k}$ for each $v \in V(G)$. Let $h : V(G) \to [0, 1]$ be a function defined by $h(v) = kg(v)$ for each $v \in V(G)$. Then $h$ is a $k$-resolving function of $G$: (i) for
each \( v \in V(G) \), \( 0 \leq h(v) = kg(v) \leq 1 \); (ii) for any two distinct \( x, y \in V(G)\), 
\( h(R\{x, y\}) = kg(R\{x, y\}) \geq k \), since \( g(R\{x, y\}) \geq 1 \) by the assumption that \( g \) is a 
resolving function of \( G \). So, \( \dim^k_f(G) \leq h(V(G)) = kg(V(G)) = k \dim_f(G) \). Since 
\( \dim^k_f(G) \geq k \dim_f(G) \) by Lemma 8, \( \dim^k_f(G) = k \dim_f(G) \).

Next, we obtain the lower and upper bounds of \( \dim^k_f(G) \) in terms of \( k, \kappa(G) \), and the order of \( G \).

**Proposition 10.** Let \( G \) be a connected graph of order \( n \). For any \( k \in [1, \kappa(G)] \), 
\( k \leq \dim^k_f(G) \leq \frac{kn}{\kappa(G)} \), where both bounds are sharp.

**Proof.** The lower bound is trivial. For the upper bound, let \( g : V(G) \to [0, 1] \) be a 
function such that \( g(v) = \frac{k}{\kappa(G)} \) for each \( v \in V(G) \). Since \( \kappa(G) = \min\{\{R\{x, y\} : \)
x \neq y \text{ and } x, y \in V(G)\}, g(R\{u, v\}) \geq k \) for any distinct vertices \( u, v \in V(G) \). So, \( g \) 
is a \( k \)-resolving function of \( G \), and hence \( \dim^k_f(G) \leq g(V(G)) = \sum_{i=1}^{n} k \frac{k}{\kappa(G)} = \frac{kn}{\kappa(G)} \).

For the sharpness of the lower bound, see Proposition 12(a); for the sharpness 
of the upper bound, we refer to Proposition 15.

As an immediate consequence of Proposition 10 we have the following.

**Corollary 11.** For a connected graph \( G \) of order \( n \) and for \( k \in [1, \kappa(G)] \), 
\( k \leq \dim^k_f(G) \leq n \).

Next, we characterize graphs \( G \) achieving the lower bound and the upper bound, respectively, of Corollary 11. Let \( \mathcal{R}_\kappa(G) = \bigcup_{x,y \in V(G), x \neq y, |R\{x, y\}| = \kappa} R\{x, y\} \), 
where \( \kappa = \kappa(G) \).

**Proposition 12.** For a connected graph \( G \) of order \( n \geq 2 \) and for \( k \in [1, \kappa(G)] \), 

(a) \( \dim^k_f(G) = k \) if and only if \( G \cong P_n \) and \( k \in [1, 2] \),

(b) \( \dim^k_f(G) = n \) if and only if \( k = \kappa(G) = \kappa \) and \( V(G) = \mathcal{R}_\kappa(G) \).

**Proof.** (a) \((\Leftarrow) \) If \( G \cong P_n \) and \( k \in [1, 2] \), then \( \dim^k_f(G) = k \) by Proposition 15

\((\Rightarrow) \) Suppose that \( \dim^k_f(G) = k \). Since \( \dim^k_f(G) \geq k \dim_f(G) \geq k \) by 
Lemma 8, \( \dim^k_f(G) = k \) implies \( \dim_f(G) = 1 \); thus \( G \cong P_n \) by Theorem 3(a) 
and \( k \in [1, 2] \) from Proposition 15.

(b) \((\Leftarrow) \) Let \( k = \kappa(G) = \kappa \) and \( V(G) = \mathcal{R}_\kappa(G) \). Then, for any vertex \( v \in V(G) \), 
there exist two distinct vertices \( x, y \in V(G) \) such that \( v \in R\{x, y\} \) with 
\( |R\{x, y\}| = \kappa \). Since any \( \kappa \)-resolving function \( g \) of \( G \) must satisfy \( g(R\{x, y\}) \geq \kappa \) 
and \( g(v) \leq 1 \) for each \( v \in V(G) \), \( g(u) = 1 \) for each \( u \in R\{x, y\} \) with 
\( |R\{x, y\}| = \kappa \). Since \( V(G) = \mathcal{R}_\kappa(G) \), \( g(v) = 1 \) for each \( v \in V(G) \). Thus, \( g(V(G)) = n \) and hence 
\( \dim^k_f(G) = n \).
Let \( \dim^k_t(G) = n \). By Proposition 10, \( k = \kappa \). Suppose that \( R_\kappa(G) \subseteq V(G) \) and let \( w \in V(G) - R_\kappa(G) \). Then, for any vertex \( w' \in V(G) - \{w\} \), \(|R\{w, w'\}| \geq \kappa(G) + 1\). If \( h : V(G) \rightarrow [0, 1] \) is a function defined by \( h(w) = 0 \) and \( h(v) = 1 \) for each \( v \in V(G) - \{w\} \), then \( h \) is a \( \kappa \)-resolving function of \( G \) with \( h(V(G)) = n - 1 \), which contradicts the assumption that \( \dim^k_t(G) = n \).

**Remark 13.** Let \( G \in H[K \cup \overline{K}] \) for some connected graph \( H \), as described in Theorem 2. Then \( \kappa(G) = 2 \) and \( V(G) = R_\kappa(G) \); thus \( \dim^2_t(G) = |V(G)| \) by Proposition 12(b). More generally, \( \dim^k_t(G) = k \dim_t(G) = \frac{k}{2}|V(G)| \) for \( k \in [1, 2] \) by Lemma 9, since \( g : V(G) \rightarrow [0, 1] \) defined by \( g(u) = \frac{1}{2} \), for each \( u \in V(G) \), forms a minimum resolving function of \( G \).

We conclude this section with an example showing that two non-isomorphic graphs can have the same \( \kappa \) and identical \( k \)-fractional metric dimension for all \( k \in [1, \kappa] \).

**Remark 14.** There exist non-isomorphic graphs \( H_1 \) and \( H_2 \) such that \( \dim^k_t(H_1) = \dim^k_t(H_2) \) for all \( k \in [1, \kappa] \), where \( \kappa = \kappa(H_1) = \kappa(H_2) \). For example, let \( H_1 \cong K_{2,2,3} \) and \( H_2 \cong K_{3,4} \); then \( \dim^1_t(H_1) = \dim^1_t(H_2) = \frac{7}{2} \) by Theorem 3(g). Since both \( H_1 \) and \( H_2 \) have twin vertices, \( \kappa(H_1) = \kappa(H_2) = 2 \). Also note that a function \( g_i : V(H_i) \rightarrow [0, 1] \) defined by \( g_i(u) = \frac{1}{2} \), for each \( u \in V(H_i) \), forms a minimum resolving function for \( H_i \), where \( i \in \{1, 2\} \). By Lemma 9, \( \dim^k_t(H_1) = \dim^k_t(H_2) = \frac{7}{2} \) for every \( k \in [1, \kappa] \), whereas \( H_1 \not\cong H_2 \).

### 3. The fractional \( k \)-metric dimension of some graphs

In this section, we determine \( \dim^k_t(G) \) for \( k \in [1, \kappa(G)] \) when \( G \) is a tree, a cycle, a wheel graph, the Petersen graph, a bouquet of cycles, a complete multipartite graph, or a grid graph (the Cartesian product of two paths). Along the way, we provide an example showing that \( \dim^k_t(G) - k \dim_t(G) \) can be arbitrarily large for some \( k \in (1, \kappa(G)] \). First, we determine \( \dim^k_t(G) \) when \( G \) is a path.

**Proposition 15.** Let \( P_n \) be an \( n \)-path, where \( n \geq 2 \). Then \( \dim^k_t(P_2) = k \) for \( k \in [1, 2] \) and, for \( n \geq 3 \),

\[
\dim^k_t(P_n) = \begin{cases} 
k & \text{if } k \in [1, 2], \\
2 + (k - 2) \frac{n - 2}{n - 3} & \text{if } k \in (2, n - 1). 
\end{cases}
\]

**Proof.** Let \( P_n \) be an \( n \)-path given by \( u_1, u_2, \ldots, u_n \), where \( n \geq 2 \); then \( \kappa(P_2) = 2 \) and \( \kappa(P_n) = n - 1 \) for \( n \geq 3 \). Since a function \( h \) defined on \( V(P_2) \) by \( h(u_1) = h(u_2) = \frac{1}{2} \) is a minimum resolving function of \( P_2 \), \( \dim^k_t(P_2) = k \dim_t(P_2) = k \) for \( k \in [1, 2] \) by Theorem 3(a) and Lemma 9. So, let \( n \geq 3 \) and we consider two cases.

**Case 1:** \( k \in [1, 2] \). If \( g : V(P_n) \rightarrow [0, 1] \) is a function defined by \( g(u_i) = g(u_{i+1}) = \frac{1}{2} \) and \( g(u_i) = 0 \) for each \( i \in \{2, \ldots, n - 1\} \), then \( g \) is a minimum resolving function...
function of $P_n$: (i) for any two distinct vertices $x, y \in V(P_n)$, $R\{x, y\} \supseteq \{u_1, u_n\}$, and hence $g(R\{x, y\}) \geq g(u_i) + g(u_n) = \frac{1}{2} + \frac{1}{2} = 1$; (ii) $g(V(P_n)) = 1 = \dim_f(P_n)$ by Theorem 3(a). Since $g(u_i) \leq \frac{1}{2} \leq \frac{1}{k}$ for each $i \in \{1, 2, \ldots, n\}$, we have $\dim_f^k(P_n) = k \dim_f(P_n) = k$ for any $k \in [1, 2]$ by Lemma 3 and Theorem 3(a).

Case 2: $k \in (2, n - 1]$. Note that $n \geq 4$ in this case since $\kappa(P_3) = 2$. Let $h : V(P_n) \to [0, 1]$ be a $k$-resolving function of $P_n$. Let $h(u_1) + h(u_n) = b$, then $0 \leq b \leq 2$. Since $R\{u_i, u_{i+2}\} \neq V(P_n) - \{u_{i+1}\}$ for $i \in \{1, 2, \ldots, n - 2\}$, $h(R\{u_i, u_{i+2}\}) = h(V(P_n)) - h(u_{i+1}) \geq k$ for each $i \in \{1, 2, \ldots, n - 2\}$. By summing over the $(n - 2)$ inequalities, we have $(n - 2)h(V(P_n)) - \sum_{j=3}^{n-1} h(u_j) \geq (n - 2)k$, i.e., $(n - 3)h(V(P_n)) + h(u_1) + h(u_n) \geq (n - 2)k$. So, $h(V(P_n)) \geq \frac{(n-2)k-b}{n-3}$ since $n > 3$; note that the minimum of $h(V(P_n))$ is $\frac{(n-2)k-2}{n-3}$ when $b = h(u_1) + h(u_n)$ takes the maximum value 2. Thus, $\dim_f^k(P_n) \geq \frac{(n-2)k-2}{n-3} = 2 + (k - 2)\frac{n-2}{n-3}$.

Now, let $g : V(P_n) \to [0, 1]$ be a function defined by

$$g(u_i) = \begin{cases} 1 & \text{if } i \in \{1, n\}, \\ \frac{k-2}{n-3} & \text{otherwise}. \end{cases}$$

Then $g$ is a $k$-resolving function of $P_n$: (i) $0 < \frac{k-2}{n-3} \leq 1$ for any $k \in (2, n - 1]$; (ii) for any distinct $i, j \in \{1, 2, \ldots, n\}$, $g(R\{u_i, u_j\}) \geq 2 + (n-3)\frac{k-2}{n-3} = k$ since $R\{u_i, u_j\} \supseteq \{u_1, u_n\}$ and $|R\{u_i, u_j\}| \geq n - 1$. So, $\dim_f^k(P_n) \leq g(V(P_n)) = 2 + (k - 2)\frac{n-2}{n-3}$.

Therefore, $\dim_f^k(P_n) = 2 + (k - 2)\frac{n-2}{n-3}$ for $k \in (2, n - 1]$.

Second, we determine $\dim_f^k(T)$ when $T$ is a tree that is not a path. Let $M(T)$ be the set of exterior major vertices of a tree $T$. Let $M_1(T) = \{w \in M(T) : ter_T(w) = 1\}$, $M_2(T) = \{w \in M(T) : ter_T(w) = 2\}$, and $M_3(T) = \{w \in M(T) : ter_T(w) \geq 3\}$; note that $M(T) = M_1(T) \cup M_2(T) \cup M_3(T)$. For any vertex $v \in M(T)$, let $T_v$ be the subtree of $T$ induced by $v$ and all vertices belonging to the paths joining $v$ with its terminal vertices. Let $M^*(T) = M_2(T) \cup M_3(T)$. We recall the following result on $\kappa(T)$.

**Theorem 16.** [12] For a tree $T$ that is not a path,

$$\kappa(T) = \min \bigcup_{w \in M^*(T)} \{d(\ell_i, \ell_j) : \ell_i \text{ and } \ell_j \text{ are two distinct terminal vertices of } w\}.$$ 

We begin by examining $\dim_f^k(T)$ when $T$ is a tree with exactly one exterior major vertex.

**Proposition 17.** Let $T$ be a tree with $ex(T) = 1$. Let $v$ be the exterior major vertex of $T$ and let $\ell_1, \ell_2, \ldots, \ell_a$ be the terminal vertices of $v$ in $T$ (note that $a \geq 3$). Suppose that $d(v, \ell_1) \leq d(v, \ell_2) \leq \ldots \leq d(v, \ell_a)$. Then $\kappa(T) = d(\ell_1, \ell_2)$, and

(a) if $d(v, \ell_1) = d(v, \ell_2)$, then $\dim_f^k(T) = k \dim_f(T) = \frac{ka}{2}$ for $k \in [1, \kappa(T)]$;
(b) if \( d(v, \ell_1) < d(v, \ell_2) \), then
\[
\dim^K(T) = \begin{cases} 
\frac{ka}{T} & \text{for } k \in [1, 2d(v, \ell_1)], \\
\frac{(a-1)k - (a-2)d(v, \ell_1)}{(a-1)k - (a-2)d(v, \ell_1)} & \text{for } k \in (2d(v, \ell_1), \kappa(T)].
\end{cases}
\]

Proof. For each \( i \in \{1, 2, \ldots, a\} \), let \( s_i \) be the neighbor of \( v \) lying on the \( v - \ell_i \) path, and let \( P_i \) denote the \( s_i - \ell_i \) path in \( T \). By Theorem 16, \( \kappa(T) = d(\ell_1, \ell_2) \).

Let \( k \in [1, \kappa(T)] \).

(a) Let \( d(v, \ell_1) = d(v, \ell_2) \). By Lemma 8, \( \dim^K(T) \geq k \dim_f(T) \). We will show that \( \dim^K(T) \leq k \dim_f(T) \). Let \( g : V(T) \to [0, 1] \) be a function defined by
\[
g(u) = \begin{cases} 
0 & \text{if } u = v, \\
\frac{k - d(v, \ell_1)}{d(v, \ell_1)} & \text{for each vertex } u \in V(P_i), \text{ where } i \in \{1, 2, \ldots, a\}.
\end{cases}
\]

Note that, for each \( i \in \{1, 2, \ldots, a\} \), (i) \( g(V(P_i)) = \frac{k}{2} \), (ii) \( 0 \leq \frac{k - d(v, \ell_1)}{d(v, \ell_1)} \leq 1 \) since \( k \leq \kappa(T) = d(\ell_1, \ell_2) \).

If two distinct vertices \( x \) and \( y \) lie on the \( v - \ell_i \) path for some \( i \in \{1, 2, \ldots, a\} \), then \( g(R\{x, y\}) \geq g(V(T) - V(P_i)) \geq g \) since \( a \geq 3 \). For distinct \( i, j \in \{1, 2, \ldots, a\} \), if \( x \in V(P_i) \) and \( y \in V(P_j) \) with \( d(v, x) \neq d(v, y) \), say \( d(v, x) < d(v, y) \) without loss of generality, then \( g(R\{x, y\}) \geq g(V(T) - V(P_i)) \geq k; \) if \( x \in V(P_i) \) and \( y \in V(P_i) \) with \( d(v, x) = d(v, y) \), then \( g(R\{x, y\}) = g(V(P_i) \cup V(P_j)) = k. \) So, \( g \) is a \( k \)-resolving function of \( T \) with \( g(V(T)) = \frac{k}{2} \). Thus \( \dim^K(T) \leq \frac{ka}{T} = k \dim_f(T) \) by Theorem 3(b). Therefore, \( \dim^K(T) = k \dim_f(T) = \frac{ka}{T} \) for \( k \in [1, \kappa(T)] \).

(b) Let \( d(v, \ell_1) < d(v, \ell_2) \), and we consider two cases.

Case 1: \( k \in [1, 2d(v, \ell_1)] \). In this case, the function \( g \) in (3) is a \( k \)-resolving function of \( T \) as shown in the proof for (a); thus, \( \dim^K(T) \leq g(V(T)) = \frac{ka}{T} = k \dim_f(T) \). Since \( \dim^K(T) \geq k \dim_f(T) \) by Lemma 8, \( \dim^K(T) = k \dim_f(T) = \frac{ka}{T} \).

Case 2: \( k \in (2d(v, \ell_1), \kappa(T)] \). Let \( h : V(T) \to [0, 1] \) be a minimum \( k \)-resolving function of \( T \). Note that (i) \( h(V(P_i)) \leq d(v, \ell_1) \) since \( h(u) \leq 1 \) for each \( u \in V(T); \) (ii) for distinct \( i, j \in \{1, 2, \ldots, a\} \), \( h(V(P_i)) + h(V(P_j)) \geq k \) since \( R\{s_i, s_j\} = V(P_i) \cup V(P_j) \). Let \( h(V(P_i)) = \beta \). From \( h(V(P_i)) + h(V(P_j)) \geq k \) for each \( j \in \{2, 3, \ldots, a\} \), \( h(V(P_j)) \geq k - \beta \). So, \( h(V(T)) \geq h(V(P_1)) + \sum_{i=2}^{a} h(V(P_i)) \geq \beta + (a-1)(k-\beta) = (a-1)k - (a-2)\beta \geq (a-1)k - (a-2)d(v, \ell_1) \) since \( \beta \leq d(v, \ell_1) \); thus \( \dim^K(T) \geq (a-1)k - (a-2)d(v, \ell_1) \).

Next, we show that \( \dim^K(T) \leq (a-1)k - (a-2)d(v, \ell_1) \). Let \( g : V(T) \to [0, 1] \) be a function defined by
\[
g(u) = \begin{cases} 
0 & \text{if } u = v, \\
1 & \text{for each vertex } u \in V(P_i), \\
\frac{k - d(v, \ell_1)}{d(v, \ell_1)} & \text{for each vertex } u \in V(P_j), \text{ where } j \in \{2, \ldots, a\}.
\end{cases}
\]

Note that (i) \( g(V(P_1)) = d(v, \ell_1); \) (ii) for \( j \in \{2, 3, \ldots, a\} \), \( g(V(P_j)) = k - d(v, \ell_1) > k - \frac{k}{2} = \frac{k}{2} \) since \( k > 2d(v, \ell_1) \); (iii) \( g(V(T)) = d(v, \ell_1) + (a-1)(k - d(v, \ell_1)) = (a-1)k - (a-2)d(v, \ell_1) \). Also note that \( g \) is a \( k \)-resolving function of
T: (i) \(0 \leq \frac{d(v, x_i)}{d(v, x_j)} \leq \frac{k-d(v, x_i)}{d(v, x_j)} \leq \frac{d(v, x_i)-d(v, x_j)}{d(v, x_j)} = \frac{d(v, x_i)}{d(v, x_j)} \leq 1\) for \(j \in \{2, \ldots, a\}\); (ii) if two distinct vertices \(x\) and \(y\) lie in the same \(v\)-terminal path for some \(i \in \{1, 2, \ldots, a\}\), then 
\[g[R(x, y)] = g(V(T)) - g(V(P')) \geq \min\{k, 2(k-d(v, x_i))\} = k\] since \(a \geq 3\) and \(k-d(v, x_i) > \frac{k}{2}\); (iii) if \(x \in V(P')\) and \(y \in V(P')\) with \(d(v, x) \neq d(v, y)\) for distinct \(i, j \in \{1, 2, \ldots, a\}\), then 
\[g[R(x, y)] = g(V(T)) - V(P') \geq \min\{g(V(T)-V(P')), g(V(T)-V(P'))\} \geq k,\] 
since at most one vertex in the \(e_i - e_j\) path can be at equal distance from both \(x\) and \(y\) in \(T\); (iv) if \(x \in V(P')\) and \(y \in V(P')\) with \(d(v, x) = d(v, y)\) for distinct \(i, j \in \{1, 2, \ldots, a\}\), then 
\[g[R(x, y)] = g(V(T)) + g(V(P')) \geq \min\{k, 2(k-d(v, x_i))\} = k.\] 
Thus, \(\dim^k_T(T) \leq g(V(T)) = (a-1)k - (a-2)d(v, e_i).\)

Therefore, \(\dim^k_T(T) = (a-1)k - (a-2)d(v, e_i).\)

Next, we determine \(\dim^k_T(T)\) for a tree \(T\) with \(ex(T) \geq 1\). We begin with the following lemma, which, besides being useful for Theorem 19, bears independent interest.

**Lemma 18.** Let \(T\) be a tree with \(ex(T) \geq 2\). For \(w \in M^*(T), x \in V(T_w)\) and \(y \in V(T) - V(T_w)\). Then either \(R(x, y) \supseteq V(T_w)\) or \(R(x, y) \supseteq V(T_{w'})\) for some \(w' \in M^*(T) - \{w\}\).

**Proof.** Since \(ex(T) \geq 2\), \(|M^*(T)| \geq 2\). Let \(x \in V(T_w)\) for some \(w \in M^*(T)\). First, suppose \(y \in V(T_{w'})\) for some \(w' \in M^*(T) - \{w\}\). Assume, for contradiction, that there exist \(u \in V(T_w)\) and \(v \in V(T_{w'})\) such that

\[d(x, u) = d(y, u) \text{ and } d(x, v) = d(y, v).\]

Put \(a = d(y, v), b = d(v, w'), c = d(w', w), d = d(w, u), e = d(u, x)\). Let us call a path leading from \(w\) to any of its leaves a “\(w\)-terminal path”. We may assume that \(u\) and \(x\) lie in the same \(w\)-terminal path, since \(d(x, u) = d(y, u)\) implies \(d(x, w) = d(y, w)\) if \(u\) and \(x\) lie in distinct \(w\)-terminal paths. Likewise, we assume \(v\) and \(y\) lie in the same \(w'\)-terminal path. After writing the two equations (4) in terms of components \(a, b, c, d, e\) and simplifying, we obtain \(b + c + d = 0\). This means that \(b = c = d = 0\), since all variables denote (nonnegative) distances. In particular, the distinctness of \(w\) and \(w'\) is contradicted by \(c = 0\).

Now, suppose \(y \notin V(T_w)\) for any \(z \in M^*(T)\); then either \(y \in V(T_{w'})\) for some \(y' \in M_z(T)\) or \(y \notin V(T_{w'})\) for any \(z' \in M(T)\). Note that there exists a vertex \(w' \in M^*(T) - \{w\}\) such that either \(y\) or \(y'\) lies in the \(w - w'\) path in \(T\). Since \(d(s, x) = d(s, y)\) for \(s \in V(T_{w'})\) is equivalent to \(d(w', x) = d(w', y)\), we may assume, for contradiction, that

\[d(x, w') = d(y, w') \text{ and } d(x, t) = d(y, t), \text{ where } t \in V(T_w).\]

Put \(d(y, w') = a + b\) and \(d(w, w') = c + b\), where \(b \geq 0\) denotes the length of the path shared between the \(w\) and \(w'\) path and the \(y\) and \(w'\) path. Similarly, put \(d(w, t) = d\) and \(d(x, t) = e\). As before, we may assume that \(x\) and \(t\) lie in the same \(w\)-terminal path. After simplifying the two equations (5) in terms of \(a, b, c, d, e\), we obtain \(c = d = 0\).

This implies that either \(y \notin V(T_{w'})\) (when \(b > 0\)) or \(w = w'\) (when \(b = 0\)); both possibilities contradict the present assumptions. \(\square\)
Theorem 19. Let $T$ be a tree with $ex(T) \geq 1$. Then $\kappa(T) = \min \{ \kappa(T_v) : v \in M^*(T) \}$ and, for $k \in [1, \kappa(T)]$,

$$\dim^k(T) = k|M_2(T)| + \sum_{v \in M_3(T)} \dim^k(T_v).$$

Proof. If $ex(T) = 1$, then $|M_2(T)| = 0$ and $|M_3(T)| = 1$, and so (6) trivially holds; see Proposition 17 for explicit formulas for $\kappa(T)$ and $\dim^k(T)$. So, let $ex(T) \geq 2$; then $|M^*(T)| \geq 2$. By Theorem 10, $\kappa(T) = \min \{ \kappa(T_v) : v \in M^*(T) \}$. Let $k \in [1, \kappa(T)]$; notice that $\kappa(T) \leq \kappa(T_z)$ for any $z \in M^*(T)$.

First, we show that $\dim^k(T) \geq k|M_2(T)| + \sum_{v \in M_3(T)} \dim^k(T_v)$. For $v \in M_2(T)$, let $N(v) \cap V(T_v) = \{ r_1, r_2 \}$. For $w \in M_3(T)$, let $\ell_1, \ell_2, \ldots, \ell_\sigma$ be the terminal vertices of $w$ with $\text{ter}_T(w) = \sigma$, and let $s_i$ be the neighbor of $w$ lying on the $w - \ell_i$ path, where $i \in \{ 1, 2, \ldots, \sigma \}$; further, let $P^i$ denote the $s_i - \ell_i$ path. Let $g : V(T) \to [0, 1]$ be a minimum $k$-resolving function of $T$. If $M_2(T) \neq \emptyset$, then, for any $v \in M_2(T)$, $R\{ r_1, r_2 \} = V(T_v) \setminus \{ v \}$ and $g(R\{ r_1, r_2 \}) = g(V(T_v) \setminus \{ v \}) \geq k$; thus $\sum_{v \in M_2(T)} g(V(T_v)) \geq \sum_{v \in M_3(T)} k = k|M_2(T)|$. If $M_3(T) \neq \emptyset$, then, for any $w \in M_3(T)$, notice $R\{ s_i, s_j \} = V(P^i) \cup V(P^j)$ for any distinct $i, j \in \{ 1, 2, \ldots, \sigma \}$. This, together with the argument used in the proof of Proposition 17, we have $\sum_{w \in M_3(T)} g(V(T_w)) \geq \sum_{w \in M_3(T)} \dim^k(T_w)$. Thus, we have

$$g(V(T)) \geq \sum_{v \in M_2(T)} g(V(T_v)) + \sum_{v \in M_3(T)} g(V(T_v)) \geq k|M_2(T)| + \sum_{v \in M_3(T)} \dim^k(T_v).$$

Next, we show that $\dim^k(T) \leq k|M_2(T)| + \sum_{v \in M_3(T)} \dim^k(T_v)$. For each $w \in M_3(T)$, let $g_w$ be a minimum $k$-resolving function on $V(T_w)$. For each $w \in M_2(T)$, define a function $h_w$ on $V(T_w)$ such that $h_w(w) = 0$ and $h_w(u) = \frac{1}{|V(T_w)|}$ if $u \neq w$. For $k \in [1, \kappa(T)]$, let $g : V(T) \to [0, 1]$ be the function defined by

$$g(u) = \begin{cases} 
g_w(u) & \text{if } u \in V(T_w), \text{ for } w \in M_3(T), 
h_w(u) & \text{if } u \in V(T_v), \text{ for } w \in M_2(T), 
0 & \text{otherwise.} \end{cases}$$

Note that (i) for each $w \in M_2(T)$, $g(V(T_w)) = h_w(V(T_v) \setminus \{ w \}) = k$; (ii) for each $w \in M_3(T)$, $g(V(T_w)) = g_w(V(T_w)) = \dim^k(T_w) \geq k$; (iii) $g(V(T)) = k|M_2(T)| + \sum_{w \in M_3(T)} \dim^k(T_v)$. It suffices to show $g$ is a $k$-resolving function of $T$. Obviously, $0 \leq g(u) \leq 1$ for each $u \in V(T)$. So, let $x$ and $y$ be distinct vertices of $T$; we will show that $g(R\{ x, y \}) \geq k$. Consider three cases: (1) there is a $w \in M^*(T)$ such that $\{ x, y \} \subseteq V(T_w)$; (2) there is a $w \in M^*(T)$ such that $x \in V(T_w)$ and $y \notin V(T_w)$; (3) $\{ x, y \} \subseteq V(T) \setminus \cup_{w \in M^*(T)} V(T_w)$. In case (1), if $w \in M_3(T)$, then $g(R\{ x, y \}) \geq g_w(R\{ x, y \} \cap V(T_w)) \geq k$, since $g_w$ is a $k$-resolving function on $V(T_w)$; if $w \in M_2(T)$ and $d(x, w) \neq d(y, w)$, then there is a $z \in M^*(T) \setminus \{ w \}$ such that $g(R\{ x, y \}) \geq g(V(T_z)) \geq k$; if $w \in M_2(T)$ and $d(x, w) = d(y, w)$, then $g(R\{ x, y \}) = h_w(V(T_w) \setminus \{ w \}) = k$. In case (2), by Lemma 18, either $R\{ x, y \} \supseteq V(T_w)$ or $R\{ x, y \} \supseteq V(T_w)$ for some $w' \in M^*(T) \setminus \{ w \}$; thus $g(R\{ x, y \}) \geq k$. So,
we consider case (3). Note that $x \in V(T_{x'})$ for some $x' \in M_1(T)$ or $x \notin V(T_2)$ for any $z \in M(T)$; similarly, $y \in V(T_{y'})$ for some $y' \in M_1(T)$ or $y \notin V(T_2')$ for any $z' \in M(T)$. If $\{x, y\} \subseteq V(T_n)$ for some $v \in M_1(T)$, then $d(v, x) \neq d(v, y)$ and there exist distinct $v', v'' \in M^*(T)$ such that $v$ lies on the $v' - v''$ path in $T$; thus $g(R\{x, y\}) \geq g(V(T_n)) + g(V(T_n')) \geq 2k$. If $\{x, y\} \notin V(T_n)$ for any $v \in M(T)$, there exist distinct $w_1, w_2 \in M^*(T)$ such that both $x$ (or $x'$) and $y$ (or $y'$) lie on the $w_1 - w_2$ path in $T$; then $d(w_1, x) = d(w_1, y)$ and $d(w_2, x) = d(w_2, y)$ imply either $x = y$ or $\{x, y\} \subseteq V(T_n)$ for some $s \in M_1(T)$, where both possibilities contradict the present assumptions. Thus $R\{x, y\} \supseteq V(T_m)$ for at least one $i \in \{1, 2\}$, and $g(R\{x, y\}) \geq g(V(T_m)) \geq k$.

Next, we provide an example showing that $\dim_{f}^{k}(G) - \dim_{f}(G)$ can be arbitrarily large for some $k \in [1, \kappa(G)]$.

**Remark 20.** The value of $\dim_{f}^{k}(G) - \dim_{f}(G)$ can arbitrarily large, as $G$ varies, for some $k \in [1, \kappa(G)]$. Let $T$ be a tree with $\text{ex}(T) = 1$. Let $v$ be the exterior major vertex of $T$ and let $\ell_1, \ell_2, \ldots, \ell_\alpha$ be the terminal vertices of $T$ such that $d(v, \ell_j) = 1 < \beta = d(v, \ell_j)$ for each $j \in \{2, 3, \ldots, \alpha\}$, where $\alpha \geq 3$. By Proposition 17, $\kappa(T) = \beta + 1$ and $\dim_{f}^{\beta+1}(T) = (\alpha - 1)(\beta + 1) - (\alpha - 2) = (\alpha - 1)\beta + 1$. Since $\dim_{f}(T) = \frac{\alpha}{2}$ by Theorem 3(b), $\dim_{f}^{\beta+1}(T) - (\beta + 1)\dim_{f}(T) = (\alpha - 1)\beta + 1 - (\beta + 1)\frac{\alpha}{2} = (\frac{\alpha}{2} - 1)(\beta - 1)$ can be arbitrarily large, as $\alpha$ or $\beta$ gets big enough.

Next, we determine the fractional $k$-metric dimension of cycles.

**Proposition 21.** Let $C_n$ be an $n$-cycle, where $n \geq 3$. Then

\begin{equation}
\dim_{f}^{k}(C_n) = \dim_{f}(C_n) = \begin{cases} \frac{kn}{n-2} & \text{if } n \text{ is even and } k \in [1, n-2], \\ \frac{kn}{n-1} & \text{if } n \text{ is odd and } k \in [1, n-1]. \end{cases}
\end{equation}

**Proof.** Note that $\kappa(C_n) = n - 2$ for an even $n$, and $\kappa(C_n) = n - 1$ for an odd $n$. Let $k \in [1, \kappa(C_n)]$. For an even $n \geq 4$, a function $g : V(C_n) \rightarrow [0, 1]$ defined by $g(u) = \frac{1}{n-2}$, for each $u \in V(C_n)$, is a minimum resolving function of $C_n$: (i) $0 < g(u) = \frac{1}{n-2} \leq \frac{1}{2} \leq 1$ since $n \geq 4$; (ii) for distinct $x, y \in V(C_n)$, $|R\{x, y\}| \geq n - 2$, and thus $g(R\{x, y\}) \geq (n - 2)(\frac{1}{n-2}) = 1$; (iii) $g(V(C_n)) = \frac{n}{n-2} = \dim_{f}(C_n)$ by Theorem 3(d). Similarly, for an odd $n \geq 3$, one can easily check that a function $h : V(C_n) \rightarrow [0, 1]$ defined by $h(u) = \frac{1}{n-2}$, for each $u \in V(C_n)$, is a minimum resolving function of $C_n$ satisfying $h(u) \leq \frac{1}{2}$, by Lemma 3 and Theorem 3(d). (1) follows.

**Remark 22.** Note that, for any fixed $k \in [1, \kappa(C_n)]$, $\lim_{n \to \infty} \dim_{f}^{k}(C_n) = k$ by Proposition 27 (c.f. Proposition 12(a)).

Next, we determine the fractional $k$-metric dimension of wheel graphs.

**Proposition 23.** For the wheel graph $W_n$ of order $n \geq 5$,

\begin{equation}
\dim_{f}^{k}(W_n) = \dim_{f}(W_n) = \begin{cases} 2k & \text{if } n = 5 \text{ and } k \in [1, 2], \\ \frac{3k}{2} & \text{if } n = 6 \text{ and } k \in [1, 4], \\ \frac{k(n-1)}{4} & \text{if } n \geq 7 \text{ and } k \in [1, 4]. \end{cases}
\end{equation}
Proof. For $n \geq 5$, the wheel graph $W_n = C_{n-1} + K_1$ is obtained from an $(n - 1)$-cycle $C_{n-1}$ by joining an edge from each vertex of $C_{n-1}$ to a new vertex, say $v$; let the $C_{n-1}$ be given by $u_1, u_2, \ldots, u_{n-1}, u_1$. Note that $\text{diam}(W_n) = 2$ for $n \geq 5$.

Case 1: $n = 5$. Note that $\kappa(W_5) = 2$ since $R\{u_1, u_3\} = \{u_1, u_3\}$. Let $k \in [1, 2]$. Then $g : V(W_5) \to [0, 1]$ be a function defined by $g(v) = 0$ and $g(u_i) = \frac{1}{2}$ for each $i \in \{1, 2, 3, 4\}$. Then $g$ is a minimum resolving function of $W_5$: (i) $0 \leq g(x) \leq \frac{1}{2}$ for each $x \in V(W_5)$; (ii) for distinct $i, j \in \{1, 2, 3, 4\}$, $g(R\{u_i, u_j\}) \geq g(u_i) + g(u_j) = 1$; (iii) for $i \in \{1, 2, 3, 4\}$, $g(R\{v, u_i\}) \geq g(u_i) = 1$, where $u_i \in V(W_5) - N[u_4];$ (iv) $g(V(W_5)) = 2 = \dim_f(W_5)$ by Theorem 3(e). By Lemma 9 and Theorem 3(e), $\dim^k_f(W_5) = k \dim_f(W_5) = 2k$ for $k \in [1, 2]$.

Case 2: $n \geq 6$. First, we show that $\kappa(W_n) = 4$ in this case. For each $i \in \{1, 2, \ldots, n-1\}$, $R\{v, u_i\} = (V(W_n) - N(u_i)) \cup \{v\}$ with $|R\{v, u_i\}| = n - 2 \geq 4$. For distinct $i, j \in \{1, 2, \ldots, n-1\}$, (i) if $u_i, u_j \in E(W_n)$, then $R\{u_i, u_j\} = (N(u_i) \cup N(u_j)) - \{v\}$ with $|R\{u_i, u_j\}| = 4$; (ii) if $u_i, u_j \not\in E(W_n)$ and $|N(u_i) \cap N(u_j)| = 1$, then $R\{u_i, u_j\} = (N[u_i] \cup N[u_j]) - \{v\}$. For each $i \in \{1, 2, \ldots, n-1\}, |R\{u_i, u_j\}| = 4 \text{ and } \kappa(W_n) = 4$ for $n \geq 6$.

Second, we determine $\dim^k_f(W_n)$ for $n \geq 6$. Let $k \in [1, 4]$. For $n = 6$, a function $g : V(W_6) \to [0, 1]$ defined by $g(x) = \frac{1}{2}$, for each $x \in V(W_6)$, is a minimum resolving function of $W_6$: (i) $0 < g(x) \leq \frac{1}{2} \leq 1$ for each $x \in V(W_6)$; (ii) for any distinct $x, y \in V(W_6)$, $g(R\{x, y\}) \geq 4 \left(\frac{1}{2}\right) = 2 = \dim_f(W_6)$ by Theorem 3(e). For $n \geq 7$, let $h : V(W_n) \to [0, 1]$ be a function defined by $h(v) = 0$ and $h(u_i) = \frac{1}{2}$ for each $i \in \{1, 2, \ldots, n-1\}$. Then $h$ is a minimum resolving function of $W_n$: (i) $0 \leq h(x) \leq \frac{1}{2} \leq 1$ for each $x \in V(W_n)$; (ii) for each $i \in \{1, 2, \ldots, n-1\}$, $|R\{v, u_i\}| \geq n - 2 \geq 5$ since $n \geq 7$, and hence $h(R\{v, u_i\}) \geq 4 \left(\frac{1}{2}\right) = 2 = \dim_f(W_n)$ by Theorem 3(e). Therefore, by Lemma 9 and Theorem 3(e), $\dim^k_f(W_n)$ holds for $n \geq 6$ and $k \in [1, 4]$.

Next, we determine the fractional $k$-metric dimension of the Petersen graph.

Proposition 24. For the Petersen graph $P$, $\dim^k_f(P) = k \dim_f(P) = \frac{k}{4}k$ for $k \in [1, 6]$.

Proof. Note that $P$ is 3-regular and vertex-transitive. Since $\text{diam}(P) = 2$, any two distinct vertices in $P$ are either adjacent or at distance two apart.

We first show that $\kappa(P) = 6$. For any distinct $x, y \in V(P)$, $R\{x, y\} = N[x] \cup N[y] - (N(x) \cap N(y))$ and $|R\{x, y\}| = 6$: (i) if $xy \in E(P)$, then $N(x) \cap N(y) = \emptyset$ and $x \in N[y]$ and $y \in N[x]$; (ii) if $xy \not\in E(P)$, then $|N(x) \cap N(y)| = 1$. So, $\kappa(P) = 6$.

Now, let $k \in [1, 6]$. Since $\dim^k_f(P) \geq k \dim_f(P)$ by Lemma 8, it suffices to show that $\dim^k_f(P) \leq k \dim_f(P)$. Let $g : V(P) \to [0, 1]$ be a function defined by $g(v) = \frac{k}{6}$ for each $v \in V(P)$. Since $0 \leq g(v) \leq 1$ for each vertex $v \in V(P)$ and $g(R\{x, y\}) = 6 \left(\frac{k}{6}\right) = k$ for any two distinct $x, y \in V(P)$, $g$ is a $k$-resolving function of $P$. So, $\dim^k_f(P) \leq |V(P)| \left(\frac{k}{6}\right) = \frac{k \dim_f(P)}{6} = \frac{k}{6} \dim_f(P)$ by Theorem 3(e). □
Next, we determine the fractional $k$-metric dimension of a bouquet of cycles.

**Proposition 25.** Let $B_m$ be a bouquet of $m$ cycles $C^1, C^2, \ldots, C^m$ with a cut-vertex (i.e., the vertex sum of $m$ cycles at one common vertex), where $m \geq 2$; further, let $C^1$ be the cycle of the minimum length among the $m$ cycles of $B_m$. Then $\kappa(B_m) = \begin{cases} |V(C^1)| - 1 & \text{if } C^1 \text{ is an odd cycle,} \\ |V(C^1)| - 2 & \text{if } C^1 \text{ is an even cycle,} \end{cases}$ and, for $k \in [1, \kappa(B_m)]$, \[ \dim_f^k(B_m) = k \dim_f(B_m) = km. \]

**Proof.** Let $v$ be the cut-vertex of $B_m$. For each $i \in \{1, 2, \ldots, m\}$, let $C^i$ be given by $v, u_{i,1}, u_{i,2}, \ldots, u_{i,r_i}, v$ and let $P^i = C^i - v$; further, let $P'^i$ be the $u_{i,1} - u_{i,[\frac{r_i}{2}]}$ geodesic. Without loss of generality, let $r_1 \leq r_2 \leq \ldots \leq r_m$.

**Claim 1:** If $C^i$ is an odd cycle, then $\kappa(B_m) = r_1 = |V(C^1)| - 1$; if $C^i$ is an even cycle, $\kappa(B_m) = r_1 - 1 = |V(C^1)| - 2$.

Proof of Claim 1. Let $x$ and $y$ be distinct vertices of $B_m$. First, let $x, y \in V(C^i)$ for some $i \in \{1, 2, \ldots, m\}$. If $d(v, x) \neq d(v, y)$, then $R\{x, y\} \subseteq V(C^j)$ with $|R\{x, y\}| \geq |V(C^j)| = r_j + 1 \geq r_1 + 1$ for $j \neq i$. If $d(v, x) = d(v, y)$ and $C^i$ is an odd cycle, then $R\{x, y\} = V(P^i)$ with $|R\{x, y\}| = r_i \geq r_1$; notice, for an odd cycle $C^i$, $|R\{u_{i,1}, u_{i,r_i}\}| = r_1$. If $d(v, x) = d(v, y)$ and $C^i$ is an even cycle, then $R\{x, y\} = V(P^i) - \{u_{i,[\frac{r_i}{2}]}\}$ with $|R\{x, y\}| = r_i - 1$, where $r_i - 1 \geq r_1$ if $C^i$ is an odd cycle, and $r_i - 1 \geq r_1 - 1$ if $C^i$ is an even cycle; notice, for an even cycle $C^i$, $|R\{u_{i,1}, u_{i,r_i}\}| = r_1 - 1$.

Second, let $x \in V(P^i)$ and $y \in V(P^j)$ for distinct $i, j \in \{1, 2, \ldots, m\}$; let $x \in V(P'^i)$ and $y \in V(P'^j)$, without loss of generality. If $d(v, x) = d(v, y)$, then $R\{x, y\} \subseteq V(P'^i) \cup V(P'^j)$ with $|R\{x, y\}| \geq |\frac{r_i}{2}| + |\frac{r_j}{2}| \geq r_1$. So, let $d(v, x) \neq d(v, y)$, say $d(v, x) < d(v, y)$ without loss of generality; then $d(u, x) \neq d(u, y)$ for each $u \in V(P'^i)$, and $r_j \geq 3$. If $r_j = 3$, then $R\{x, y\} \subseteq V(P'^i) \cup V(P'^j)$ such that $w_t \in V(P')$ with $d(w_t, x) = d(w_t, y)$, where $t \in \{1, 2\}$. In each case, $|R\{x, y\}| \geq |\frac{r_i}{2}| + |\frac{r_j}{2}| \geq r_1$. □

**Claim 2:** For $k \in [1, \kappa(B_m)]$, $\dim_f^k(B_m) = k \dim_f(B_m) = km$.

Proof of Claim 2. Let $k \in [1, \kappa(B_m)]$, and let $h : V(B_m) \to [0, 1]$ be a function defined by

\[ h(u) = \begin{cases} \frac{1}{r_{i,u}} & \text{for each } u \in V(P^i) \text{ if } C^i \text{ is an odd cycle,} \\ \frac{1}{r_{j,u}} & \text{for each } u \in V(P^j) - \{u_{j,[\frac{r_j}{2}]}\} \text{ if } C^j \text{ is an even cycle,} \\ 0 & \text{otherwise.} \end{cases} \]

Note that, for each $i \in \{1, 2, \ldots, m\}$, $h(V(P^i)) = 1$ if $C^i$ is an odd cycle, and $h(V(P^i)) = h(V(P^i) - \{u_{i,[\frac{r_i}{2}]}\}) = 1$ if $C^i$ is an even cycle. We also note that $h$ is a minimum resolving function of $B_m$: (i) $0 \leq h(u) \leq \frac{1}{k} \leq 1$ for each $u \in V(B_m)$; (ii) if $x, y \in V(C^i)$ with $d(v, x) = d(v, y)$, for some $i \in \{1, 2, \ldots, m\}$, then $h(R\{x, y\}) \geq h(V(C^i)) \geq 1$ for $j \neq i$; (iii) if $x, y \in V(C^i)$ with $d(v, x) = d(v, y)$ and $x \neq y$, for some $i \in \{1, 2, \ldots, m\}$, then $h(R\{x, y\}) = h(V(P^i)) = 1$ when
Proposition 28. \(C^n\) is an odd cycle, and \(h(R[x,y]) = h(V(P^n)) - h(u_i, \frac{1}{2}) = 1\) when \(C^n\) is an even cycle; (iv) if \(x \in V(P^n)\) and \(y \in V(P^n)\) for distinct \(i, j \in \{1, 2, \ldots, m\}\), then \(h(R[x,y]) \geq \frac{1}{2} h(V(P^n)) + \frac{1}{2} h(V(P^n)) = 1\) using a similar argument used in the proof of Claim 1; (v) \(h(V(B_m)) = m = \dim_f(B_m)\) by Theorem 3(f). So, by Lemma 9 and Theorem 3(f), \(\dim_k(f(B_m)) = k \dim_f(B_m) = km\) for \(k \in [1, \kappa(B_m)]\). \(\square\)

Next, we determine the fractional \(k\)-metric dimension of complete multipartite graphs.

**Proposition 26.** For \(m \geq 2\), let \(G = K_{a_1, a_2, \ldots, a_m}\) be a complete \(m\)-partite graph of order \(n = \sum_{i=1}^{m} a_i \geq 3\), and let \(s\) be the number of partite sets of \(G\) consisting of exactly one element. Then, for \(k \in [1, 2]\),

\[
\dim_k^f(G) = k \dim_f(G) = \begin{cases} 
\frac{k(n-1)}{2} & \text{if } s = 1, \\
\frac{kn}{2} & \text{otherwise}.
\end{cases}
\]

**Proof.** Let \(V(G)\) be partitioned into \(m\)-partite sets \(V_1, V_2, \ldots, V_m\) with \(|V_i| = a_i\), where \(i \in \{1, 2, \ldots, m\}\). Without loss of generality, let \(a_1 \leq a_2 \leq \ldots \leq a_m\). Note that \(\kappa(G) = 2\): (i) if \(a_m \geq 2\), then, for two distinct \(x, y \in V_m\), \(R[x, y] = \{x, y\}\); (ii) if \(a_m = 1\), then, for \(x \in V_1\) and \(y \in V_2\), \(R[x, y] = \{x, y\}\). Let \(k \in [1, 2]\).

First, let \(s \neq 1\). A function \(g : V(G) \to [0, 1]\) defined by \(g(v) = \frac{1}{2^s}\) for each \(v \in V(G)\), is a minimum resolving function of \(G\): (i) \(0 < g(v) \leq \frac{1}{2} \leq 1\) for each \(v \in V(G)\); (ii) for any distinct vertices \(x, y \in V(G)\), \(g(R[x, y]) \geq g(x) + g(y) = 1\); (iii) \(g(V(G)) = \frac{n}{2} = \dim_f(G)\) by Theorem 3(g). So, \(\dim_k^f(G) = k \dim_f(G) = \frac{kn}{2}\) by Lemma 9 and Theorem 3(g).

Second, let \(s = 1\). Let \(h : V(G) \to [0, 1]\) be a function defined by \(h(u) = 0\) for \(u \in V_1\) and \(h(v) = \frac{1}{2}\) for each \(v \in V(G) - V_1\). Then \(h\) is a minimum resolving function of \(G\): (i) \(0 \leq h(v) \leq \frac{1}{2} \leq 1\) for each \(v \in V(G)\); (ii) for any two distinct vertices \(x, y \in V(G) - V_1\), \(h(R[x, y]) \geq h(x) + h(y) = 1\); (iii) for \(x \in V_1\) and \(y \in V_i \subseteq V(G) - V_1\), \(h(R[x, y]) \geq h(V_i) \geq 1\), where \(i \in \{2, \ldots, m\}\); (iv) \(h(V(G)) = \frac{n-1}{2} = \dim_f(G)\) by Theorem 3(g). So, \(\dim_k^f(G) = k \dim_f(G) = \frac{k(n-1)}{2}\) for \(k \in [1, 2]\) by Lemma 9 and Theorem 3(g).

Now, we consider the fractional \(k\)-metric dimension of grid graphs (i.e., the Cartesian product of two paths). The *Cartesian product* of two graphs \(G\) and \(H\), denoted by \(G \square H\), is the graph with the vertex set \(V(G) \times V(H)\) such that \((u, w)\) is adjacent to \((u', w')\) if and only if either \(u = u'\) and \(ww' \in E(H)\), or \(w = w'\) and \(uu' \in E(G)\). See Figure 3 for the labeling of \(P_{s} \square P_{t}\).

We recall the following result.

**Theorem 27.** For \(s, t \geq 2\), \(\kappa(P_{s} \square P_{t}) = s + t - 2\) and \(\dim_k^f(P_{s} \square P_{t}) = 2k\), where \(k \in \{1, 2, \ldots, s + t - 2\}\).

**Proposition 28.** For \(k \in [1, s + t - 2]\), \(\dim_k^f(P_{s} \square P_{t}) = k \dim_f(P_{s} \square P_{t}) = 2k\), where \(s, t \geq 2\).
Figure 1: Labeling of $P_6 \square P_4$.

Proof. Let $s \geq t \geq 2$, and let $G = P_s \square P_t$ and $L = \{ v \in V(G) : 2 \leq \deg(v) \leq 3 \}$. By Theorem 27, $\kappa(G) = s + t - 2$. Let $k \in [1, s + t - 2]$. Since $\dim^k_f(G) \geq k \dim_f(G) = 2k$ by Lemma 28 and Theorem 39, it suffices to show that $\dim^k_f(G) \leq 2k$. Let $g : V(G) \to [0, 1]$ be a function defined by

$$g(v) = \begin{cases} \frac{k}{s+t-2} & \text{if } v \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(V(G)) = 2k$. We will show that $g$ is a $k$-resolving function of $G$. Clearly, $0 \leq g(v) \leq 1$ for each $v \in V(G)$. Let $x = (a, b)$ and $y = (c, d)$ be two distinct vertices of $G$. We consider two cases.

**Case 1:** $a = c$ or $b = d$. If $a = c$, then $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(i, 1), (i, t)\}$, and thus $g(R\{x, y\}) \geq (2s)(\frac{k}{s+t-2}) \geq k$ since $s \geq t \geq 2$. So, let $b = d$; without loss of generality, let $a < c$. Let $z = (a, \beta) \in L$. Note that (i) if $\alpha < a$, then $d(z, x) < d(z, y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(i, j)\} \cup (\cup_{i=2}^s \{(i, 1), (i, t)\})$; (ii) if $\alpha \geq c$, then $d(z, x) > d(z, y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(s, j)\} \cup (\cup_{i=2}^s \{(i, 1), (i, t)\})$; (iii) if $a < \alpha < c$ and $\beta = 1$, then there exists at most one such $z \in L$ satisfying $d(z, x) = d(z, y)$, since $d(z, x) = d(z, y)$ implies $\alpha - a - b - 1 = c - \alpha - b - 1$, i.e., $2\alpha = a + c$; similarly, if $a < \alpha < c$ and $\beta = t$, then there exists at most one such $z \in L$ satisfying $d(z, x) = d(z, y)$. Thus, $|R\{x, y\} \cap L| \geq 2s + 2t - 6$, and hence $g(R\{x, y\}) \geq (2s + 2t - 6)(\frac{k}{s+t-2}) \geq 2k(\frac{k}{s+t-2}) \geq k$, since $s + t \geq 4$.

**Case 2:** $a \neq c$ and $b \neq d$. Without loss of generality, let $a < c$: further, assume that $b < d$ (the case for $b > d$ can be handled similarly). Let $z' = (\alpha', \beta') \in L$. Note that (i) if $\alpha' \leq a$ and $\beta' = 1$, then $d(z', x) < d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(i, 1)\}$; (ii) if $\alpha' \geq c$ and $\beta' = t$, then $d(z', x) > d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(i, t)\}$; (iii) if $a < \alpha' < c$ (i.e., $c \neq a + 1$) and $\beta' = 1$, then there exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$, since $d(z', x) = d(z', y)$ implies $\alpha' - a + b - 1 = c - \alpha' + b - 1$, i.e., $2\alpha' = a - b + c + d$; similarly, if $a < \alpha' < c$ and $\beta' = t$, there exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$. Likewise, we note that (i) if $\alpha' = 1$ and $\beta' \leq b$, then $d(z', x) < d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(1, j)\}$; (ii) if $\alpha' = s$ and $\beta' \geq d$, then $d(z', x) > d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(s, j)\}$; (iii) if $\alpha' = 1$ and $b < \beta' < d$ (i.e., $d \neq b + 1$), then there
exists at most one such \( z' \in L \) satisfying \( d(z', x) = d(z', y) \), since \( d(z', x) = d(z', y) \) implies \( a - 1 + \beta' - b = c - 1 + d - \beta' \), i.e., \( 2\beta' = -a + b + c + d \); similarly, if \( \alpha' = s \) and \( b < \beta' < d \), then there exists at most one such \( z' \in L \) satisfying \( d(z', x) = d(z', y) \).

So, if \( c = a + 1 \) or \( d = b + 1 \), then \( |R(x, y)\cap L| \geq s + t - 2 \); if \( c \geq a + 2 \) and \( d \geq b + 2 \), then \( |R(x, y)\cap L| \geq a + (s-c+1) + 2(c-a-2) + b + (t-d+1) + 2(d-b-2) \geq s + t - 2 \).

In each case, \( g(R(x, y)) \geq (s + t - 2)(\frac{k}{s+t-2}) = k \).

Thus, in each case, \( g \) is a \( k \)-resolving function of \( G \), and hence \( \dim_{f}^{k}(G) \leq g(V(G)) = 2k \). Therefore, \( \dim_{f}^{k}(G) = k \dim_{f}(G) = 2k \) for \( k \in [1, s + t - 2] \) for \( s \geq t \geq 2 \).

4. Open Problems

We conclude this paper with two open problems.

**Problem 1.** Let \( \phi(k) = \dim_{f}^{k}(G) \) be a function of \( k \), for a fixed \( G \), on domain \([1, \kappa(G)]\). Is \( \phi \) a continuous function of \( k \) on every connected graph \( G \)?

**Problem 2.** Suppose \( \dim_{f}^{k}(G) \) is given by \( \psi(k) \) for integral values of \( k \). When and how can we interpolate \( \psi \) and deduce \( \dim_{f}^{k}(G) \) for any real number \( k \in [1, \kappa(G)] \)?

For example, let \( G = P_{s}\square P_{t} \), where \( s, t \geq 2 \). Then \( \dim_{f}^{k}(G) = 2k \) for integers \( k \in \{1, 2, \ldots, \kappa(G)\} \) by Theorems \([\ref{thm3}], h\), Lemma \([\ref{lem8}], b\), and Theorem \([\ref{thm27}] \). In Proposition \([\ref{prop28}] \), we proved that \( \dim_{f}^{k}(G) = 2k \) for any real number \( k \in [1, \kappa(G)] \), by using Lemmas \([\ref{lem8}], b\) and constructing a \( k \)-resolving function \( g \) on \( V(G) \) with \( g(V(G)) = 2k \) for \( k \in [1, \kappa(G)] \). The construction of \( k \)-resolving function for any real number \( k \in [1, \kappa(G)] \) in determining \( \dim_{f}^{k}(G) \) in Proposition \([\ref{prop28}] \) does not appear to carry to the construction of \( k \)-resolving set for any integral values \( k \in \{1, 2, \ldots, \kappa(G)\} \) in determining \( \dim_{f}^{k}(G) \), and vice versa.

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**REFERENCES**

1. R. Adar, L. Epstein: *The k-metric dimension*. J. Comb. Optim., 34(1) (2017), 1-30.
2. S. Arumugam, V. Mathew: *The fractional metric dimension of graphs*. Discrete Math., 312 (2012), 1584-1590.
3. S. Arumugam, V. Mathew, J. Shen: *On fractional metric dimension of graphs*. Discrete Math. Algorithms Appl., 5 (2013), 1350037.
4. R.F. Bailey, P.J. Cameron: Base size, metric dimension and other invariants of groups and graphs. Bull. London Math. Soc., 43(2) (2011), 209-242.
5. R.F. Bailey, I.G. Yero: Error-correcting codes from k-resolving sets. Discuss. Math. Graph Theory, In Press (2017).
6. A.F. Beardon, J.A. Rodríguez-Velázquez: On the k-metric dimension of metric spaces. Ars Math. Contemp., In Press (2018).
7. Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihal'ák, L.S. Ram: Network discovery and verification. IEEE J. Sel. Areas Commun., 24 (2006), 2168-2181.
8. G. Chartrand, P. Zhang: The theory and applications of resolvability in graphs. A Survey. Congr. Numer., 160 (2003), 47-68.
9. V. Chvátal: Mastermind. Combinatorica, 3 (1983), 325-329.
10. J. Currie, O.R. Oellermann: The metric dimension and metric independence of a graph. J. Combin. Math. Combin. Comput., 39 (2001), 157-167.
11. L. Eroh, P. Feit, C.X. Kang, E.Yi: The effect of vertex or edge deletion on the metric dimension of graphs. J. Comb., 6(4) (2015), 433-444.
12. A. Estrada-Moreno, J.A. Rodríguez-Velázquez, I.G. Yero: The k-metric dimension of a graph. Appl. Math. Inf. Sci., 9(6) (2015), 2829-2840.
13. A. Estrada-Moreno, I.G. Yero, J.A. Rodríguez-Velázquez: The k-metric dimension of corona product graphs. Bull. Malays. Math. Sci. Soc., 39(1) (2016), 135-156.
14. A. Estrada-Moreno, I.G. Yero, J.A. Rodríguez-Velázquez: The k-metric dimension of the lexicographic product of graphs. Discrete Math., 339(7) (2016), 1924-1934.
15. M. Fehr, S. Gosselin, O.R. Oellermann: The metric dimension of Cayley digraphs. Discrete Math., 306 (2006), 31-41.
16. M. Feng, Q. Kong: On the fractional metric dimension of corona product graphs and lexicographic product graphs. Ars Combin., 138 (2018), 249-260.
17. M. Feng, B. Lv, K. Wang: On the fractional metric dimension of graphs. Discrete Appl. Math., 170 (2014), 55-63.
18. M. Feng, K. Wang: On the metric dimension and fractional metric dimension of the hierarchical product of graphs. Appl. Anal. Discrete Math., 7(2) (2013), 302-313.
19. M.R. Garey, D.S. Johnson: Computers and intractability: A guide to the theory of NP-completeness. Freeman, New York, 1979.
20. F. Harary, R.A. Melter: On the metric dimension of a graph. Ars Combin., 2 (1976), 191-195.
21. C.X. Kang: On the fractional strong metric dimension of graphs. Discrete Appl. Math., 213 (2016), 153-161.
22. C.X. Kang, I.G. Yero, E. Yi: The fractional strong metric dimension in three graph products. Discrete Appl. Math., In Press (2018) https://doi.org/10.1016/j.dam.2018.05.051.
23. C.X. Kang, E. Yi: The fractional strong metric dimension of graphs. Lecture Notes in Comput. Sci., 8287 (2013), 84-95.
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24. S. Khuller, B. Raghavachari, A. Rosenfeld: *Landmarks in graphs*. Discrete Appl. Math., 70 (1996), 217-229.

25. D.J. Klein, E. Yi: *A comparison on metric dimension of graphs, line graphs, and line graphs of the subdivision graphs*. Eur. J. Pure Appl. Math., 5(3) (2012), 302-316.

26. E.R. Scheinerman, D.H. Ullman: *Fractional graph theory: A rational approach to the theory of graphs*. John Wiley & Sons, New York, 1997.

27. A. Sebő, E. Tannier: *On metric generators of graphs*. Math. Oper. Res., 29 (2004), 383-393.

28. P.J. Slater: *Leaves of trees*. Congr. Numer., 14 (1975), 549-559.

29. E. Yi: *The fractional metric dimension of permutation graphs*. Acta Math. Sin. (Engl. Ser.), 31 (2015), 367-382.

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