CUSPIDAL DISCRETE SERIES
FOR SEMISIMPLE SYMMETRIC SPACES

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Abstract. We propose a notion of cusp forms on semisimple symmetric spaces.
We then study the real hyperbolic spaces in detail, and show that there exists both cuspidal and non-cuspidal discrete series. In particular, we show that all the spherical discrete series are non-cuspidal.

1. Introduction

The main purpose of this paper is to initiate a generalization of Harish-Chandra’s notion of cusp forms for a real semisimple Lie group $G$ to the more general case of a semisimple symmetric space $G/H$. In Harish-Chandra’s work on the Plancherel formula for $G$ the fact that all discrete series are cuspidal plays an important role. However, in the established generalizations to $G/H$ (see [11], [10], [9], [7], [8]), cuspidality plays no role and, in fact, is not defined at all.

We propose a notion of cuspidal discrete series for semisimple symmetric spaces $G/H$ in general, and we show by explicit calculations on the real hyperbolic spaces $SO(p,q+1)/SO(p,q)$ that the notion is meaningful in that case. The notion agrees with the standard one of Harish-Chandra for the discrete series of $G$, but in contrast to the situation for $G$, it is not true in general that all discrete series are cuspidal. Our main result determines exactly which discrete series representations for $SO(p,q+1)/SO(p,q)$ are cuspidal. If $p \geq q-1$, all discrete series representations are cuspidal, but if $p < q-1$, there is a non-empty and finite family of non-cuspidal discrete series.

The notion of cuspidality relates to integral geometry on the symmetric space by using integration over a certain unipotent subgroup $N^* \subset G$. The definition of $N^*$ is given in Section 2. The map $f \mapsto \int_{N^*} f(nH) \, dn$, which maps functions on $G/H$ to functions on $G/N^*$, is a kind of Radon transform for $G/H$. A discrete series subspace of $L^2(G/H)$ is said to be cuspidal if it is annihilated by this transform (assuming the convergence of the integral on an appropriate dense subspace of $L^2(G/H)$). In the group case $G \simeq G \times G/G$, we have $N^* = N \times \{e\}$, where $N$ corresponds to a minimal parabolic subgroup of $G$, and thus the Radon transform of a function $f$ on $G$ is the function $\int_{N^*} f(nx^{-1}) \, dn$ on $G \times G$. It follows that the annihilation by this transform agrees with Harish-Chandra’s cuspidality condition for the minimal parabolic subgroup.

It is clear that certain discrete series for $G/H$, which are spherical (that is, they contain the trivial $K$-type), cannot be cuspidal since they contain functions taking only positive values. Obviously a positive function cannot be annihilated by integration over any subgroup $N^* \subset G$. The present investigation shows that for the hyperbolic spaces all spherical discrete series are non-cuspidal, but also that in general there exist non-cuspidal, non-spherical discrete series. The non-spherical non-cuspidal discrete series are given by odd functions on the real hyperbolic space.
which means that they do not descend to functions on the projective hyperbolic space.

The first section of the paper describes in more detail the suggested program for general symmetric spaces and motivates our study of the hyperbolic spaces. The hyperbolic spaces are treated in the following sections. Apart from the motivation, this treatment is to a large extend independent of the general theory. The definition of \( N^+ \) and the generalized notion of cuspidality was introduced by the second author in lectures at Oberwolfach (2001). The results presented in this paper were announced in [18].

2. A GENERAL NOTION OF CUSPIDALITY

We first recall from Harish-Chandra [13] the notion of cuspidality for \( G \) and its relation to the Plancherel decomposition. Let \( G \) be a connected semisimple real Lie group with finite center (or more generally, reductive of Harish-Chandra’s class), and let \( K \subset G \) be a maximal compact subgroup with corresponding Cartan involution \( \theta \). Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) denote the corresponding decomposition of the Lie algebra, and let \( \mathfrak{a} \subset \mathfrak{p} \) be a maximal abelian subspace.

Let \( \mathcal{C}(G) \) denote the Schwartz space for \( G \), which is dense in \( L^2(G) \). By definition, a cusp form on \( G \) is a function \( f \in \mathcal{C}(G) \) such that

\[
(1) \quad \int_N f(xny) \, dn = 0,
\]

for all parabolic subgroups \( P = MAN \subset G \), and all \( x, y \in G \) (the integral converges absolutely for all \( f \in \mathcal{C}(G) \)). Let \( L^2_{\text{ds}}(G) \) denote the sum of all the discrete series representations in \( L^2(G) \). It is both left and right invariant, and the intersection \( \mathcal{C}_{\text{ds}}(G) = L^2_{\text{ds}}(G) \cap \mathcal{C}(G) \) is a dense subspace.

**Theorem 2.1** (Harish-Chandra). \( \mathcal{C}_{\text{ds}}(G) \) is exactly the space of cusp forms. It is non-zero if and only if \( G \) and \( K \) have equal rank.

The Plancherel decomposition splits \( L^2(G) \) into a finite sum of series, each of which is related to a particular cuspidal parabolic subgroup \( P = MAN \) (that is, \( \text{rank } M = \text{rank } M \cap K \)). The splitting can be accomplished as follows.

Let \( \theta_1, \ldots, \theta_r \) be a complete (up to conjugation) set of \( \theta \)-stable Cartan subalgebras in \( \mathfrak{g} \), and let \( \mathfrak{a}_i = \theta_i \cap \mathfrak{p} \) for \( i = 1, \ldots, r \). For each \( i = 1, \ldots, r \), let \( P_i \) be a parabolic subgroup with Langlands decomposition \( M_i A_i N_i \) such that \( A_i = \exp \mathfrak{a}_i \).

We can arrange that \( \mathfrak{a}_1 = \mathfrak{a} \), then \( P_1 \) is a minimal parabolic subgroup.

We now define \( \mathcal{C}_i(G) \subset \mathcal{C}(G) \) as the set of functions \( f \in \mathcal{C}(G) \) for which

- \( f \) is orthogonal to all \( h \in \mathcal{C}(G) \) with \( \int_{N_i} h(xny) \, dn = 0 \), for all \( x, y \in G \).
- \( \int_{N} f(xny) = 0 \), for all \( x, y \in G \), for all cuspidal parabolic subgroups some conjugate of which is properly contained in \( P_i \).

In particular, for \( i = 1 \) the second condition is vacuous, and \( \mathcal{C}_1(G) \), which is called the most-continuous series, is just the orthocomplement of space of functions annihilated by all integrals \( \int_{N_1} g(xny) \, dn \). On the other hand, for rank \( G = \text{rank } K \), we can arrange that \( \mathfrak{a}_r = \{ 0 \} \) and \( N_r = \{ e \} \). Then for \( i = r \) the first condition is vacuous, and \( \mathcal{C}_r(G) \) is the space \( \mathcal{C}_{\text{ds}}(G) \) of cusp forms.

**Theorem 2.2** (Harish-Chandra). The following is an orthogonal direct sum

\[
\mathcal{C}(G) = \bigoplus_{i=1}^r \mathcal{C}_i(G).
\]

In Harish-Chandra’s Plancherel decomposition, each piece \( \mathcal{C}_i(G) \) (or its closure in \( L^2(G) \)) is further decomposed into generalized principal series representations induced from \( P_i \).
Let now $G/H$ be a semisimple symmetric space, that is, the homogeneous space of $G$ with a subgroup $H$ satisfying $G^* \subset H \subset G^*$, where $\sigma : G \to G$ is an involution, $G^*$ the group of its fixed points, and $G^*_1$ the identity component of this group. The problem of obtaining the Plancherel decomposition for $L^2(G/H)$ has been solved (see the references cited in the introduction). In general terms the outcome is similar to what was described above for $L^2(G)$. In particular, discrete series occur if and only if $G/H$ and $K/K \cap H$ have equal rank.

One can also define a Schwartz space $\mathcal{C}(G/H)$ for $G/H$, and again (see [7], Theorem 23.1) there is a finite decomposition

$$
\mathcal{C}(G/H) = \oplus_i \mathcal{C}_i(G/H)
$$

where each piece decomposes as a direct integral of representations induced from a particular parabolic subgroup. However, the pieces in this decomposition are defined representation theoretically. The motivation behind this paper was to study the following problem.

**Question:** Is there a description of the $\mathcal{C}_i(G/H)$ through integrals over subgroups $N$ (or $N/N \cap H$), similar to that for $G$? In particular, can the discrete series be characterized through some reasonable definition of cusp forms?

Recall that a minimal $\sigma$-stable parabolic subgroup $P_{\sigma\text{-min}}$ is obtained as follows. Let $g = h \oplus q$ be the decomposition according to $\sigma$. We may assume that $\sigma$ and $\theta$ commute, and can arrange that $a$ is $\sigma$-invariant and that $a_q := a \cap q$ is maximal abelian in $p \cap q$. The set $\Sigma$ of non-zero weights of $a_q$ in $g$ is a root system, and $P_{\sigma\text{-min}} = MAqN$ is determined from a choice $\Sigma^+$ of positive roots. Here $A_q = \exp a_q$, and $MA_q$ is its centralizer. The most-continuous series for $G/H$, which is a basic summand in (2), is induced from a parabolic subgroup of this form (see [2]). More generally, the representations in $\mathcal{C}_i(G/H)$ are induced from a (not necessarily minimal) $\sigma$-$\theta$-stable parabolic subgroup $P_i$ (see [8], Theorem 10.9).

It would be tempting to apply the unipotent radicals $N_i$ of the $P_i$ in a definition of cusp forms on $G/H$:

$$
\int_{N_i} f(gnH) \, dn = 0 \quad (g \in G, P_i \neq G).
$$

In the group case, where $G$ is considered as a symmetric space for $G \times G$, the $\sigma\theta$-stable parabolic subgroups of $G \times G$ are of the form $P \times \theta(P)$, where $P \subset G$ is parabolic, and thus the integral (3) becomes an integral over both $N$ and $\theta(N)$. Hence (3) differs substantially from Harish-Chandra’s definition (4) in this case. Furthermore, although (3) does converge in the group case (see [19], Lemma 15.8.1), this is not the case for general symmetric spaces. An example is provided below in Lemma 4.1 (see however [15] for the special case of $L^1$-discrete series for $G/H$).

Based on these observations one is lead to look for integrals over different subgroups, and the following approach was suggested by the second author. We first fix a system of positive roots for $a$ in $g$, such that $\Sigma^+$ consists of the non-zero restrictions to $a_q$. Since $\Sigma^+$ was already given, this only amounts to a choice of positive roots for the root system of pure $a_h$-roots, that is, the roots of $a$ which vanish on $a_q$. On $a_q$ an ordering is determined by $\Sigma^+$. On $a_h$ we choose an ordering which is compatible with the positive pure roots. More precisely, these orderings can be attained by choosing elements $X_q \in a_q$ and $X_h \in a_h$ such that $\alpha(X_q) > 0$ for all $\alpha \in \Sigma^+$, and $\beta(X_h) > 0$ for all positive pure $a_h$-roots $\beta$. Furthermore, we request of $X_h$ that $\alpha(X_h) \neq 0$ for all roots of $a$ with non-zero $a_q$-restriction. Then $X_q$ and $X_h$ determine the corresponding notions of positivity for elements in $a_q^*$ and $a_h^*$. Notice that the notion which results from the choice of $X_h$ is in general not unique.

We now define the following subspaces (in fact subalgebras) of the Lie algebra $\mathfrak{n}$ of $N$:
$n_+ = \sum_\beta g^\beta$, where $\beta$ is a root with $\beta|_{a_q} > 0$ and $\beta|_{a_h} > 0$.
$n_- = \sigma(\theta(n_-)) = \sum_\beta g^\beta$, where $\beta$ is a root with $\beta|_{a_q} > 0$ and $\beta|_{a_h} < 0$.
$n_0 = \sum_\beta g^\beta$, where $\beta$ is a root with $\beta|_{a_q} > 0$ and $\beta|_{a_h} = 0$.

Then $n = n_+ \oplus n_0 \oplus n_-$, and

$$n^* := n_+ \oplus n_0 = \sum_{\beta|_{a_q} > 0, \beta|_{a_h} \geq 0} g^\beta$$

is a subalgebra. Let $n^{**} = n_-$ such that $n = n^* \oplus n^{**}$. Similarly, let $N^* = \exp(n^*)$ and $N^{**} = \exp(n^{**})$, then $N = N^* N^{**}$. Notice that $N^*$ intersects trivially with $H$, since this is the case already for $N$.

The suggestion is to replace $N$ by $N^*$ in (4) and consider integrals of the form

$$\int_{N^*} f(gn^*H) \, dn^*$$

in a possible definition of cusp forms on $G/H$. It is easily seen that in the group case, the integrals [3] amount exactly to those in Harish-Chandra’s original integral [1] for minimal parabolic subgroups.

It is useful also to view $N^*$ as a quotient of the nilpotent part of a particular minimal parabolic subgroup $P_1$ of $G$. For this purpose we define

$$n_1 = n_+ \oplus n_0 \oplus \theta(n_-) \oplus \sum_\beta g^\beta,$$

with the final summation over the positive pure $a_h$-roots. It follows from the maximality of $\theta_q$, that the sum $\sum_\beta g^\beta$ in (5) is contained in $\mathfrak{h}$. Using this and the fact that $\sigma: n_+ \to \theta(n_-)$ is bijective, one concludes easily for the Lie algebra $n_1 = n^* \oplus (n_1 \cap \mathfrak{h})$.

Let $N_1 = \exp(n_1)$, then the following holds similarly.

**Lemma 2.3.** The mapping $(n_1, n_2) \mapsto n_1 n_2$ is a diffeomorphism of $N^* \times (N_1 \cap H)$ onto $N_1$.

**Proof.** The map $(n_1, n_2) \mapsto n_1 n_2$ is clearly injective. By [14, Lemma VI 5.2], it is a diffeomorphism onto an open subset $N^*(N_1 \cap H)$ of $N_1$ containing $N_0$. Let $a_t = \exp(tX_0)$. Then $\lim_{t \to 0} a_t^{-1} n_1 a_t \in N_0$ for all $n_1 \in N_1$, whence $a_t^{-1} n_1 a_t \in N^*(N_1 \cap H)$ for $t$ sufficiently large. Since both $N^*$ and $(N_1 \cap H)$ are normalized by $a_t$, it follows that $n_1 \in N^*(N_1 \cap H)$. Therefore $N_1 = N^*(N_1 \cap H)$, and the result follows.

We thus have

$$N^* \simeq N_1 / N_1 \cap H.$$
hyberbolic spaces investigated in the present paper, this extra condition is always fulfilled.

Assuming that \( Rf \) is well defined for \( f \in \mathcal{C}(G/H) \), one can define the cuspidal discrete series for \( G/H \) to consist of those discrete series for which the corresponding functions in \( \mathcal{C}(G/H) \) are annihilated by \( R \).

We shall need a result about the relation between \( R \) and invariant differential operators on \( G/H \). We let \( P_{\sigma-\min} = MA_qN \) be as above. Since
\[
g = n \oplus (m \cap q) \oplus a_q \oplus h,
\]
we can define a map
\[
\mu : \mathbb{D}(G/H) \to \mathbb{D}(M/M \cap H) \otimes \mathbb{D}(A_q)
\]
by \( \mu(D) = T(D_0) \), where
\[
u - u_0 \in nU(g) + U(g)h,
\]
and \( u \in U(g)^H, u_0 \in U(m)^{M \cap H} \otimes U(a_q) \) are elements that represent \( D \) and \( D_0 \), and where \( T(D_0) = a^{-\rho}D_0 \circ a^\rho \) (see for example [2], p. 341). The map is independent of the choice of positive system for \( a_q \).

Let \( m_{nc} \) be the ideal in \( m \) generated by \( m \cap p \). It follows from maximality of \( a_q \) that \( m_{nc} \subset h \). The complementary ideal \( m_c \) is contained in \( l \) and centralizes \( a \). Let \( M_c \subset M \) be the corresponding analytic subgroup. Using the decomposition
\[
m = m_c \oplus m_{nc},
\]
and the fact that \( m_{nc} \subset h \), we see that
\[
\mathbb{D}(M/M \cap H) \simeq \mathbb{D}(M_c/M_c \cap H).
\]
Therefore, we may as well regard \( \mu \) as a map
\[
\mu : \mathbb{D}(G/H) \to \mathbb{D}(M_c/M_c \cap H) \otimes \mathbb{D}(A_q).
\]

We denote by \( \rho, \rho^\ast, \rho^{\ast\ast}, \rho_1 \in a^* \) half the sum of the roots of \( n, n^*, n^{\ast\ast} \), and \( n_1 \) respectively (with multiplicities). Then \( \rho = \rho^\ast + \rho^{\ast\ast} \) and
\[
\rho_1|_{a_q} = (\rho^\ast - \rho^{\ast\ast})|_{a_q}.
\]
Let \( f \) be a smooth function on \( G/H \), such that the defining integral of \( Rf \) allows the application of right derivatives by all elements from \( U(g) \).

**Lemma 2.4.** Let
\[
Af(ma) := a^{\rho_1}Rf(ma),
\]
for \( m \in M_c \) and \( a \in A_q \). Then
\[
A(Df) = \mu(D)Af,
\]
for \( D \in \mathbb{D}(G/H) \).

**Proof.** Notice first that since \( M_cA_q \) centralizes \( a \), it preserves \( N^* \) in the adjoint action. Moreover, the pull-back of the invariant measure \( dn^* \) by the action of \( ma \in M_cA_q \) is \( a^{-2\rho} dn^* \). It follows that
\[
Af(ma) = a^{\rho_1} \int_{N^*} f(man^*H) dn^* = a^{-\rho} \int_{N^*} f(n^*maH) dn^*.
\]

Let \( u \) and \( u_0 \) be as above, and note that as remarked above we may assume \( u_0 \in U(m_c + a_q) \). We shall prove that (5) implies
\[
u - u_0 \in n^*U(g) + U(g)h,
\]
from which the desired property of \( A(Df)(ma) \) then follows by application of \( u \) from the right in the last expression in (11).

By Poincaré-Birkhoff-Witt \( u - u_0 \), modulo \( U(g)h \), is a sum of terms of the form \( X_1 \cdots X_kY_1 \cdots Y_l \) where \( X_1, \ldots, X_k \) are root vectors in \( n \), say for roots \( \alpha_1, \ldots, \alpha_k, \ldots, \alpha_l \).
and $Y_1, \ldots, Y_l$ belong to $(\mathfrak{m} \cap q) + a_q$. We arrange that the basis elements $X_i$ for $\mathfrak{n}$ are ordered such that roots of $\mathfrak{n}^*$ come first. Since $u - u_0$ commutes with $a_h$, it follows from the uniqueness of the expression that $\alpha_1 + \cdots + \alpha_k$ vanishes on $a_h$ for all non-zero terms. As the roots of $\mathfrak{n}^*$ are strictly negative on some element in $a_h$, it follows that in each non-zero contribution at least one root vector $X_i$ must belong to $\mathfrak{n}^*$.

Notice that if $f$ is an eigenfunction of the Laplace operator $L$ on $G/H$, then it follows from Lemma 2.4 that $Af$ is an eigenfunction for $\mu(L)$ on $M_c A$ with the same eigenvalue. The operator $\mu(L)$ is explicitly determined in [5], Lemma 5.3. In particular, if $M_c \subset H$, it follows that if $Lf = cf$, then

$$ (L_A - \rho^2)(Af) = cAf, $$

on $A$. Here $L_A$ is the (Euclidean) Laplace operator on $A$, normalized compatibly with the normalization of $L$. When we define $L$ to be the image of the Casimir element in $\mathcal{U}(g)$, this means that correspondingly $L_A$ is the image of the Casimir element in $\mathcal{U}(a)$.

3. Notation and definitions for real hyperbolic spaces

In this and the following sections $G = \text{SO}(p, q + 1)_c$ denotes the identity component of $\text{SO}(p, q + 1)$ and $H = \text{SO}(p, q)_c$ the identity component of $\text{SO}(p, q)$, embedded in the upper left corner of $G$ as the stabilizer of $x_0 = e_{p+q+1} \in \mathbb{R}^{p+q+1}$. Throughout we assume $p, q \geq 1$. Then $G/H$ is a non-Riemannian symmetric space.

The corresponding involution $\sigma$ of $G$ is obtained from conjugation by the diagonal matrix $(1, \ldots, 1, -1)$. The fixed point group $G^\sigma$ has two components, $H$ and $H_c$, where $c \in G$ is the diagonal matrix $(1, \ldots, 1, -1, -1)$.

It is known that $G/H$ is simply connected except for $q = 1$, where $G/H$ has an infinite-folded covering. This means that for $q = 1$ we can get a somewhat more general result by going to coverings.

The map $G \ni g \mapsto gx_0$ induces an identification of $G/H$ with the real hyperbolic space $X = X_{p,q}$, defined by the equation

$$ x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q+1}^2 = -1 $$

in $\mathbb{R}^{p+q+1}$. Likewise $G/G^\sigma$ is identified with the projective real hyperbolic space $\mathbb{P}X$, in which antipodal points $x$ and $-x$ are identified.

The group $K = K_1 \times K_2 = \text{SO}(p) \times \text{SO}(q + 1) \subset G$ is a maximal compact subgroup, of which the corresponding Cartan involution will be denoted $\theta$. We define one-parameter abelian subgroups $A = \{a_t\} \subset G$ and $T = \{k_\theta\} \subset K_2$ by

$$ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\
0 & I_{p+q-1} & 0 \\
\sinh t & 0 & \cosh t \end{pmatrix}, $$

and

$$ k_\theta = \begin{pmatrix} I_p & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
0 & 0 & I_{q-1} & 0 \\
0 & -\sin \theta & 0 & \cos \theta \end{pmatrix}, $$

where $I_j$ denotes the identity matrix of size $j$. Then

$$ k_\theta a_t x_0 = (\sinh t, 0, \ldots, 0; \sin \theta \cosh t, 0, \ldots, 0; \cos \theta \cosh t). $$

The semicolon, which will be used again later, indicates the separation of the first $p$ from the last $q + 1$ coordinates. The generalized Cartan decomposition $G = KAH$ holds and gives rise to the use of polar coordinates on $X$:

$$ K \times \mathbb{R} \ni (k, t) \mapsto k a_t H \in X. $$
In this case we have in addition that $K_1 \subset H$ and $K_2 = (K_2 \cap H)T(K_2 \cap H)$, where $K_2 \cap H = \text{SO}(q)$ centralizes $A$. Hence

(15) \[ K = (K \cap H)T(K_2 \cap H) \quad \text{and} \quad G = (K \cap H)TAH. \]

In particular, we shall deal with functions $f$ on $G/H$ which are $K \cap H$-invariant from the left. It follows that such a function is uniquely determined by the values $f(k_0a_\mu H)$ for $(\theta, t) \in [0, 2\pi] \times \mathbb{R}$. Notice, that the antipodal point corresponding to $k_0a_\mu H$ is $k_{0+ie^{-i\theta}}a_{-1}H$.

The $K$-types with a $K \cap H$-fixed vector are generated by the $K \cap H$-bi-invariant zonal spherical functions on $K$. In the present case the zonal spherical functions on $\text{SO}(q+1)/\text{SO}(q)$ are given by

\[ k_\theta \mapsto R_\mu(\cos \theta), \]

when $q > 1$, where $R_\mu$ is a particular Gegenbauer polynomial of degree $\mu \in \mathbb{Z}^+$ (i.e. $\mu \in \mathbb{Z}$ and $\mu \geq 0$). We normalize these by $R_\mu(1) = 1$, and note that in particular $R_1(\cos \theta) = \cos \theta$. When $q = 1$, we also allow the integer $\mu$ to be negative, and replace $R_\mu(\cos \theta)$ by

\[ k_\theta \mapsto e^{i\mu \theta}. \]

It follows that a function $f$ on $G/H$ which is $K$-finite of irreducible type $\mu$ and $K \cap H$-invariant, must be of the form

(16) \[ f(k_0a_\mu) = R_\mu(\cos \theta)f(a_t), \]

respectively \[ f(k_0a_\mu) = e^{i\mu \theta}f(a_t). \]

In the study of discrete series on semisimple symmetric spaces $G/H$, one often needs the following general fact:

**Lemma 3.1.** Let $\mathcal{V}$ be an irreducible, non-zero closed invariant subspace of $L^2(G/H)$ or $C^\infty(G/H)$. Then $\mathcal{V}$ contains a function $f$ with the properties:

(a) \[ f(e) = 1, \]

(b) \[ f \text{ is an eigenfunction for the Casimir operator on } G/H, \]

(c) \[ f \text{ is } K \cap H \text{-invariant and of some irreducible } K \text{-type } \mu. \]

Since $\mathcal{V}$ is irreducible, it is generated by this element $f$.

The number $\mu$ is related to the highest weight of the $K$-type as follows. Let $T \in \mathfrak{k}$ be the infinitesimal generator of $k_\theta$, then $i\mu$ is the value of the highest weight on $T$ (with a suitable choice of positive restricted roots for $\mathfrak{k}$).

### 3.1. Parametrization of discrete series.

We define

(17) \[ \rho = \frac{1}{2}(p + q - 1) \quad \text{and} \quad \rho_c = \frac{1}{2}(q - 1), \]

and for $\lambda > 0$

\[ \mu_\lambda = \lambda + \rho - 2\rho_c. \]

We first consider the case $q > 1$. It is known (see for example [11], Section 8) that the discrete series for the hyperbolic space $G/H$ is parametrized by the set of positive numbers $\lambda$ such that $\mu_\lambda \in \mathbb{Z}$. The representations that arise from the construction for general semisimple symmetric spaces in [11] are exactly those for which $\mu_\lambda \geq 0$. The remaining discrete series (called ‘exceptional’ in [11]), correspond to those (finitely many) parameters $\lambda > 0$ for which $\mu_\lambda < 0$. The exceptional parameters exist if and only if $q > p + 3$. The discrete series representation with parameter $\lambda$ descends to the projective space $\mathbb{P}X$ if and only if $\mu_\lambda$ is even. For general semisimple symmetric spaces, the full discrete series (including the exceptional ones) is described in [10].

For $q = 1$, where $\rho_c = 0$, we parametrize the discrete series by $\lambda \in \mathbb{R} \setminus \{0\}$ such that $|\lambda| + \rho \in \mathbb{Z}$. In this case we have $\mu_\lambda = \lambda + \rho \geq 1$ for $\lambda > 0$, whereas for $\lambda < 0$ we define

\[ \mu_\lambda = \lambda - \rho \leq -1. \]
There are no exceptional discrete series. We note that for $q = 1$ every $\lambda \neq 0$ defines a relative discrete series for the infinite covering space of $G/H$.

We return to the general situation $q \geq 1$, and describe the discrete series which are spherical, according to \cite{newref}. Spherical discrete series exist if and only if $q > p + 1$, and in this case, the representation with parameter $\lambda$ is spherical if and only if $\mu_\lambda \leq 0$ and even.

The discrete series parameter $\lambda$ is related to the eigenvalue of the Laplace-Beltrami operator $\Delta$ of $G/H$ on the corresponding representation space in $L^2(G/H)$. More precisely, we have $\Delta f = (\lambda^2 - \rho^2) f$, for functions $f$ in this space (with suitable normalization of $\Delta$), and we can explicitly describe the discrete series by a generating function of the form \cite{newref} as follows.

**Proposition 3.2.** Let $\lambda \in \mathbb{R} \setminus \{0\}$ be a discrete series parameter (if $q > 1$, this implies in particular that $\lambda > 0$). The corresponding discrete series representation $T_\lambda$ has a $K \cap H$-invariant generating function of the following form

(i) For $q > 1$ and $\mu_\lambda \geq 0$,

$$\psi_{\lambda}(k_\theta a_t) = R_{\mu_\lambda}(\cos \theta)(\cosh t)^{-\lambda - \rho}.$$  

For $q = 1$ and $\mu_\lambda \in \mathbb{Z}$, or for all $\lambda$ for the relative discrete series for the universal covering,

$$\psi_{\lambda}(k_\theta a_t) = e^{i\mu_\lambda \theta}(\cosh t)^{-|\lambda| - \rho}.$$  

(ii) For $q > p + 1$, $\mu_\lambda = -n < 0$ and

(I) $n = 2m$ even,

$$\xi_{\lambda}(k_\theta a_t) = P_\lambda(\cosh^2 t)(\cosh t)^{-\lambda - \rho - 2m}.$$  

(II) $n = 2m - 1$ odd,

$$\xi_{\lambda}(k_\theta a_t) = \cos \theta P_\lambda(\cosh^2 t)(\cosh t)^{-\lambda - \rho - 2m},$$

where in each case $P_\lambda$ is a polynomial of degree $m$.

**Proof.** The expressions for the generating functions can be derived from known explicit formulas with hypergeometric functions for the $K$-finite functions on $G/H$, see \cite{oldref}, p. 1864, or \cite{newref}, p. 403. However, we prefer to give an alternative proof which relates more directly to general theory.

For (i) we refer to \cite{newref}, formula (8.11), and for (ii) we refer to \cite{newref}, where explicit expressions are determined for the generating functions of the exceptional discrete series. It follows from \cite{newref}, Theorem 5.1, that the following function generates the discrete series with parameter $\lambda$:

(I) $n = 2m$ even,

$$\xi_{\lambda}(k_\theta a_t) = \phi_{n,m}(\sinh^2 t),$$

(II) $n = 2m - 1$ odd,

$$\xi_{\lambda}(k_\theta a_t) = \cos \theta \cosh t \phi_{n,m}(\sinh^2 t).$$

Here $\phi_{n,m}$ is the function on $\mathbb{R}^+$ defined by

\begin{equation}
\phi_{n,m}(s^2) = [\omega^m(1 + x^2)^{n-\rho}];_{x=(s,0,...,0)},
\end{equation}

where $x \in \mathbb{R}^p$, and where $\omega$ denotes the Laplace operator on $\mathbb{R}^p$.

Note that $\xi_{\lambda}$ differs from the function constructed in \cite{newref} by being $K \cap H$-invariant. In the notation of \cite{newref}, the $K \cap H$-invariant generating function is $\int_{K \cap H} \xi_{\lambda}(kg) dk$.

In order to prove the proposition it now suffices to show for each relevant pair $(n, m)$ that there exists a polynomial $P$ of degree $m$ such that

\begin{equation}
\phi_{n,m}(s^2) = P(s^2)(1 + s^2)^{n-2m-\rho}, \quad (s \in \mathbb{R})
\end{equation}
for all \( s \in \mathbb{R} \). The expression (19) is derived from (18) by successive use of the following lemma. Note that \( n - \rho_c = -\lambda = \rho + \rho_c < -\frac{p}{2} \).

**Lemma 3.3.** Let \( Q \) be a polynomial of degree \( d \) and let \( \nu \in \mathbb{R} \). Then there exists a polynomial \( P \) of degree \( \leq d + 1 \) such that

\[
\omega(Q(x^2)(1 + x^2)^\nu) = P(x^2)(1 + x^2)^{\nu - 2}, \quad \forall x \in \mathbb{R}.
\]

If \( \nu + d \neq 0 \) and \( \nu + d \neq -\frac{e}{2} \) then \( \deg P = d + 1 \)

**Proof.** The existence of the polynomial \( P \) is an easy computation. The statement about its degree follows from the fact that

\[
Q(x^2)(1 + x^2)^\nu = (x^2)^{\nu + d} + \text{lower order terms},
\]

since \( \omega(x^\mu) = \mu(\mu + p - 2)x^{\mu - 2} \) for all \( \mu \in \mathbb{R} \).

\[
\square
\]

4. A UNIPOTENT SUBGROUP

Define

\[
n_{u,v} = \exp(Z_{u,v}) \in G, \quad Z_{u,v} = \begin{pmatrix} 0 & u & v & 0 \\ -u^t & 0 & 0 & u^t \\ v^t & 0 & 0 & -v^t \\ 0 & u & v & 0 \end{pmatrix} \in \mathfrak{g},
\]

where \( u \in \mathbb{R}^{p-1} \) and \( v \in \mathbb{R}^q \) are considered as rows, and \( u^t, v^t \) are the corresponding columns. If \( Y \) denotes the infinitesimal generator of \( a_t \), then \( [Y, Z_{u,v}] = Z_{u,v} \) for all \( u, v \). The matrices \( Z_{u,v} \) span a commutative subalgebra \( \mathfrak{n} \) of \( \mathfrak{g} \), and the subgroup \( N = \exp(\mathfrak{n}) \) is the unipotent radical of a minimal \( \sigma \theta \)-stable parabolic subgroup \( P \) in \( G \). In particular, we note that \( N \cap H \) is trivial.

Easy calculations show

\[
n_{u,v}x_0 = \left( \frac{1}{2}(u^2 - v^2), u; -v, 1 + \frac{1}{2}(u^2 - v^2) \right),
\]

and

\[
a_s n_{u,v}x_0 = (\sinh s + \frac{1}{2}e^s(u^2 - v^2), u; -v, \cosh s + \frac{1}{2}e^s(u^2 - v^2)),
\]

for all \( s \in \mathbb{R} \), where \( u^2 = u \cdot u \) and \( v^2 = v \cdot v \) as usual.

**Lemma 4.1.** Assume \( p > 1 \) and \( p + q > 3 \). Then there exists a non-negative \( K \)-invariant Schwartz function \( f \in C(G/H) \) for which the integral \( \int_N f(n) \, dn \) diverges.

Assume in addition \( q > p + 1 \). Then the integral diverges for the \( K \)-invariant discrete series function \( f = \psi_\lambda \) where \( \lambda = \frac{1}{2}(q - p - 1) \) (see Proposition 3.2(i)).

**Proof.** Let \( f(k_a H) = (\cosh t)^{-p - \nu} \), with \( \nu > 0 \), then \( f \in C(G/H) \) (see the remark after Theorem 3.1). Using (13) and (20), we find

\[
\int_N f(n) \, dn = \int_{\mathbb{R}^q} \int_{\mathbb{R}^{p-1}} f(n_{u,v}) \, du \, dv
\]

\[
= \int_{\mathbb{R}^q} \int_{\mathbb{R}^{p-1}} (v^2 + (1 + \frac{1}{2}(u^2 - v^2))^{\frac{1}{2}(p + \nu)}) \, du \, dv
\]

\[
= \int_0^\infty \int_0^\infty (y^2 + (1 + \frac{1}{2}(x^2 - y^2))^{\frac{1}{2}(p + \nu)}) \, x^{p-2}y^{q-1} \, dx \, dy.
\]

In particular, for \( 1 \leq y \leq x \leq y + 1 \), we have \( 0 \leq x^2 - y^2 = (x - y)(x + y) \leq 2y + 1 \), and hence

\[
y^2 + (1 + \frac{1}{2}(x^2 - y^2))^{\frac{1}{2}(p + \nu)} \leq 10y^2.
\]
Hence
\[ \int_N f(n) \, dn \geq \int_1^\infty \int_y^{y+1} (10y^2)^{-\frac{1}{2}(p+\nu)} \rho x^p y^{q-1} \, dx \, dy \geq C \int_1^\infty y^{-(p+\nu)+p+q-3} \, dy, \]
with \( C > 0 \). This integral diverges when \( \nu \leq \frac{1}{2}(p + q - 3) \).

If \( q > p + 1 \) and \( \lambda = \frac{1}{2}(q - p - 1) \), then the function \( f \) defined above with \( \nu = \lambda \) is exactly \( \psi_\lambda \). The integral diverges since in this case \( \lambda = \frac{1}{2}(q - p - 1) \leq \frac{1}{2}(p + q - 3) \). \( \square \)

If \( p + q \leq 3 \), it is likely that the \( N \)-integral converges for all Schwartz functions, but we shall not consider this question here.

Motivated by Section 2, we now define the following subgroup of \( N \). Let
\[ u = (u_1, \ldots, u_{p-1}) \in \mathbb{R}^{p-1} \quad \text{and} \quad v = (v_q, \ldots, v_1) \in \mathbb{R}^q. \]
It is convenient to number the entries of \( v \) from right to left as indicated. It is not difficult to verify that the following agrees with (14).

**Definition 4.2.** Let \( N^* \subset N \) be the \( \max(p - 1, q) \)-dimensional subgroup
\[ N^* = \{ n_{u,v} \mid u \in \mathbb{R}^{p-1}, v \in \mathbb{R}^q, u_j = v_j \text{ for } j = 1, \ldots, l \}, \]
where \( l = \min(p - 1, q) \).

We want to integrate \( K \cap H \)-invariant functions on \( G/H \) over sets of the form \( a_s N^* \), where \( s \in \mathbb{R} \), with respect to Haar measure of \( N^* \). For this purpose we shall need the following, which is easily deduced from (14) and (21).

The relation
\[ (K \cap H) k_\theta a_t H = (K \cap H) a_s n_{u,v} H \]
implies
\[ \cosh^2 t = v^2 + \left( \cosh s + \frac{1}{2}e^s \right) (u^2 - v^2) \]
and
\[ \cos \theta = \left( \cosh s + \frac{1}{2}e^s \right) / \cosh t \]
for all \( \theta, t, s, u \) and \( v \).

When \( p = 1 \), the value of \( t \), including its sign, can also be determined by
\[ \sinh t = \sinh s + \frac{1}{2}e^s (-v^2), \]
whereas if \( p > 1 \) the sign is redundant since by (14) the double cosets \( (K \cap H) k_\theta a_t H \)
and \( (K \cap H) k_\theta a_{-t} H \) are identical. We assume in this case that \( t \geq 0 \).

With these relations between \( (s, u, v) \) and \( (\theta, t) \), we have \( f(a_s n_{u,v} H) = f(k_\theta a_t H) \)
for \( K \cap H \)-invariant functions \( f \) on \( G/H \).

We assume now \( n_{u,v} \in N^* \). Then the expression \( u^2 - v^2 \) simplifies. We distinguish the two cases:

**A.** \( p > q \). Then \( u = (v_1, \ldots, v_q, u') \), where \( u' \in \mathbb{R}^{p-1-q} \). We put \( x = \|u'\| \) and \( y = \|v\| \), and obtain
\[ \cosh^2 t = y^2 + \left( \cosh s + \frac{1}{2}e^s x^2 \right)^2. \]
and
\[ \cos \theta = \left( \cosh s + \frac{1}{2}e^s x^2 \right) / \cosh t. \]

For the integration over \( N^* \) we use polar coordinates for \( u' \) and \( v \) with
\[ \alpha = p - 2 - q, \quad \beta = q - 1. \]

We conclude that the measure on \( N^* \) can be normalized such that for a \( K \cap H \)-invariant function,
\[ \int_{N^*} f(a_s n^* H) \, dn^* = \int_0^\infty \int_0^\infty f(k_\theta a_t H) \, x^\alpha y^\beta \, dx \, dy. \]
where $t = t(s, x, y) \geq 0$ is determined by (22) and $\theta = \theta(s, x, y)$ by (23).

Note that in the degenerate case $p - 1 = q$, we have $u' = 0$. Hence $x = 0$ in (22), and the right hand side of (25) has to be interpreted without the integration over $x$.

B. $q \geq p$. Then $v = (u', u_{p-1}, \ldots, u_1)$, where $v' \in \mathbb{R}^{q-p+1}$. We put $x = \|u\|$ and $y = \|v'\|$, and obtain

$$\cosh^2 t = x^2 + y^2 + (\cosh s - \frac{1}{2} e^s y^2)^2.$$  

and

$$\cos \theta = (\cosh s - \frac{1}{2} e^s y^2)/\cosh t.$$

We use polar coordinates for $u$ and $v'$ with

$$\alpha = p - 2, \quad \beta = q - p.$$  

Then (25) holds with $t = t(s, x, y) \geq 0$ determined by (26) and $\theta = \theta(s, x, y)$ by (27).

In the degenerate case $p = 1$, the sign of $t$ is determined by

$$\sinh t = \sinh s - \frac{1}{2} e^s (y^2).$$

Note also that in this case $u = 0$, so that $x = 0$ in (26), and again (25) is interpreted without integration over $x$.

To summarize, let us define for $s, x, y \in \mathbb{R}$

$$\Theta(s, x, y) = \begin{cases} y^2 + (\cosh s + \frac{1}{2} e^s x^2)^2, & (p > q) \\ x^2 + y^2 + (\cosh s - \frac{1}{2} e^s y^2)^2, & (q \geq p). \end{cases}$$

Then in particular for a $K$-invariant function on $G/H$, we can write

$$\int_{N^*} f(a_n n^* H) \, dn^* = \int_0^\infty \int_0^\infty F(\Theta(s, x, y)) x^\alpha y^\beta \, dx \, dy,$$

where $F$ is the function $F(\cosh^2 t) = f(a_1 H)$ on $\mathbb{R}_+$, and $\alpha$ and $\beta$ are given by either (27) or (28).

We have normalized the measure on $N^*$ such that this integral equation is valid without a constant.

5. Main results for real hyperbolic spaces

For functions on $G/H$ we define, assuming convergence,

$$Rf(g) = \int_{N^*} f(gn^* H) \, dn^*, \quad (g \in G),$$

in accordance with the definition of the Radon transform in Section 2.

We shall be particularly interested in the values of $Rf$ on the elements $a_s$. For simplicity we write

$$Rf(s) = Rf(a_s) = \int_{N^*} f(a_n n^* H) \, dn^*, \quad (s \in \mathbb{R}).$$

For $K$-invariant functions this integral is explicitly expressed in (30).

Referring to Lemma 2.4, we find

$$\rho_1 = \begin{cases} \frac{1}{2}(p - q - 1) & \text{if } p > q \\ \frac{1}{2}(q - p + 1) & \text{if } q \leq p, \end{cases}$$

and recall that $A_f(a) = a^{\rho_1} Rf(a)$. It follows from (13) that if $Lf = (\lambda^2 - \rho^2)f$, then $(d/ds)^2 A_f = \lambda^2 A_f$, and hence $Rf$ is a linear combination

$$Rf(s) = C_1 e^{(-\rho_1 + \lambda)s} + C_2 e^{(-\rho_1 - \lambda)s}.$$  

(31)
Theorem 5.1. Let $f$ be a continuous function on $G/H$, and assume there exists a constant $C > 0$, such that

$$ |f(ka_t)| \leq C (\cosh t)^{-p} (1 + \log(\cosh t))^{-2}, $$

for all $t \in \mathbb{R}$, $k \in K$.

(i) Convergence. The defining integral of $Rf(s)$ converges absolutely for all $s \in \mathbb{R}$.

(ii) Compact support. If $f(ka_t) = 0$ for all $k \in K$ and $|t| \geq t_0 > 0$, then

$$ Rf(s) = 0, $$

A: for all $|s| \geq t_0$ if $p > q$,

B: for all $s \leq -t_0$ if $p \leq q$.

(iii) Decay. Let $N \geq 2$, and put

$$ \|f\|_N = \sup_{t \in \mathbb{R}, k \in K} (\cosh t)^p (1 + \log(\cosh t))^{N} |f(ka_t)|. $$

Assume $\|f\|_N < \infty$. There exists a constant $C_N > 0$ (independent of $f$), such that

$$ e^{\rho \|s\|} |Rf(s)| \leq C_N \|f\|_N (1 + |s|)^{-N(N-2)}, $$

a: for all $s \in \mathbb{R}$ if $p \geq q$,

b: for all $s \leq 0$ if $p < q$.

(iv) Limits. Assume that $p < q$. The function $e^{s}Rf(s)$ is bounded on $\mathbb{R}^+$. If $f$ is $K$-invariant, then the limit $\lim_{s \to \infty} e^{s}Rf(s)$ exists, and if in addition $f$ is positive and not identically zero, then this limit is positive.

Note the difference between the conditions for A, B versus a, b.

We also remark that a Schwartz function $f \in \mathcal{C}(G/H)$ by definition, see [1], Definition 2.1, satisfies the growth conditions $\|Df\|_N < \infty$, for all $D \in \mathcal{D}(G/H)$ and all $N \in \mathbb{N}$, with $\|\cdot\|_N$ defined by (32). In particular, $f \in \mathcal{C}(G/H)$ satisfies (32).

Theorem 5.2. Let $\lambda \neq 0$ be a discrete series parameter, and let $f$ be the generating function of Proposition 5.1.

1. If $\lambda > 0$ and $\mu_\lambda > 0$, then $Rf = 0$. Likewise if $\lambda < 0$ and $\mu_\lambda < 0$.

2. If $\lambda > 0$ and $\mu_\lambda \leq 0$, then $Rf(s) = Ce^{(r_\lambda + \lambda)s}$ ($s \in \mathbb{R}$), for some $C \neq 0$.

Notice that the second statement in (1) is only relevant when $q = 1$. In this case (2) never occurs, and we always have $Rf = 0$. This is also the case for all relative discrete series parameters (see Subsection 5.1).

In conclusion, we have the following characterization of the cuspidal and non-cuspidal discrete series.

Theorem 5.3. Let $\lambda \neq 0$ be a discrete series parameter.

1. Let $q > 1$. Then the discrete series representation $T_\lambda$ is cuspidal if and only if $\mu_\lambda > 0$.

2. Let $q \leq p + 1$. Then all discrete series are cuspidal (if $q = 1$, then all the relative discrete series are also cuspidal).

3. All spherical discrete series for $G/H$ are non-cuspidal. These representations exist if and only if $q > p + 1$.

4. There exist non-spherical non-cuspidal discrete series if and only if $q > p + 3$. These representations do not descend to discrete series of the real projective hyperbolic space.

This follows easily from Theorem 5.2 and the description of the discrete series in Subsection 5.1.
6. Proofs

The proofs are based on the following two lemmas.

**Lemma 6.1.** For each \( s \in \mathbb{R} \), there exist numbers \( a, b > 0 \) such that
\[
\Theta(s, x, y) \geq \begin{cases} 
    y^2 + ax^4 + b & \text{if } p > q \\
    x^2 + ay^4 + b & \text{if } q \geq p,
\end{cases}
\]
for all \( x, y \in \mathbb{R} \). If \( p > q \) or \( s \leq 0 \), then the numbers \( a, b \) can be chosen as follows
\[
a = \frac{1}{4} e^{2s}, \quad b = \cosh^2 s.
\]

**Proof.** The statements for \( p > q \) are straightforward from (29). Assume \( p \leq q \).
From (29) we obtain
\[
\Theta(s, x, y) = x^2 + \frac{1}{4} e^{2s}(y^2 - 1)^2 + \frac{1}{2} y^2 + \frac{1}{2} + \frac{1}{4} e^{-2s},
\]
we finally see that for \( s \geq 0 \)
\[
\Theta(s, x, y) \geq x^2 + \frac{1}{4} (y^2 - 1)^2 + \frac{1}{2} y^2 + \frac{1}{2} = x^2 + \frac{1}{4} y^4 + \frac{3}{4}.
\]

**Lemma 6.2.** Let \( a, b, c, d \) and \( \gamma \) be \( > 0 \), then
\[
\int_{0}^{\infty} \int_{0}^{\infty} (1 + x^a + y^b)^{-\gamma} x^{c-1} y^{d-1} \, dx \, dy < \infty,
\]
if
\[
\frac{c}{a} + \frac{d}{b} < \gamma.
\]
Furthermore
\[
\int_{0}^{\infty} \int_{0}^{\infty} (1 + x^a + y^b)^{-\gamma} x^{c-1} y^{d-1} (1 + \log(1 + x^a + y^b))^{-\delta} \, dx \, dy < \infty,
\]
if \( \delta > 1 \) and
\[
\frac{c}{a} + \frac{d}{b} = \gamma.
\]

**Proof.** Easy. \( \square \)

**Proof of Theorem 5.1** We may assume that \( f \) is \( K \)-invariant, since otherwise we can replace it by the continuous function \( gH \mapsto \sup_{k \in K} |f(kgH)| \). The defining integral (30) of \( Rf(s) \) is bounded by the following integral
\[
|Rf(s)| \leq \int_{0}^{\infty} \int_{0}^{\infty} x^{\alpha} y^{\beta} \Theta(s, x, y)^{-\rho/2} (1 + \log \Theta(s, x, y))^{-2} \, dx \, dy,
\]
with the values of \( \alpha, \beta, \) and \( \rho \) from (24), (28), and (17). The convergence of this integral is an easy consequence of the preceding lemmas. Note that the logarithmic term is needed for the convergence. This proves (i).

It is seen from Lemma 6.1 that if \( p > q \) or \( s \leq 0 \), then
\[
\Theta(s, x, y) \geq \cosh^2 s.
\]
We can thus conclude that if \( f(a_t) = 0 \) for all \( |t| \geq t_0 \), then the integrand in (30) vanishes for \( |s| \geq t_0 \) if \( p > q \), and for \( s \leq -t_0 \) if \( p \leq q \). This proves (ii).

We now study the asymptotic behavior of \( Rf(s) \) as \( s \to \pm \infty \). The following arguments have to be adapted slightly in the degenerate cases \( p = q + 1 \) and \( p = 1 \), where there is no \( x \)-integral.
Assume first that $p > q$. Recall that $\alpha = p - 2 - q$ and $\beta = q - 1$, and that
\begin{equation}
Rf(s) = \int_0^\infty \int_0^\infty F(\Theta(s, x, y)) x^\alpha y^\beta \, dx \, dy.
\end{equation}

From Lemma 6.1, we have
\begin{equation}
\Theta(s, x, y) \geq y^2 + \frac{1}{4} e^{2s} x^4 + \cosh^2 s,
\end{equation}
and by the definition of $\|f\|_N$, we have
\begin{equation}
|F(\Theta)| \leq C \|f\|_N \Theta^{-\rho/2} (1 + \log(\Theta))^{-N}.
\end{equation}

We insert this bound in (35), replace $\Theta(s, x, y)$ by the lower bound, and substitute $\frac{1}{4} e^{2s} x^4 = \cosh(s) \xi^2$ and $y = \cosh(s) \eta$, so that
\begin{equation}
y^2 + \frac{1}{4} e^{2s} x^4 + \cosh^2 s = (1 + \eta^2 + \xi^4) \cosh^2 s.
\end{equation}

Simplifying by the relation $\frac{1}{4} (\alpha + 1) + \beta + 1 = \rho$, we finally obtain
\begin{equation}
|Rf(s)| \leq C \|f\|_N e^{-\frac{1}{2}(\alpha+1)s} \int_0^\infty \int_0^\infty (1 + \eta^2 + \xi^4)^{-\rho/2} (1 + \log(\xi))^{-N} \eta^\beta \xi^\alpha \, d\eta \, d\xi,
\end{equation}
where $\Theta$ is the omitted argument in the logarithm. The logarithmic term is bounded above by a constant times
\begin{equation}
(1 + \log(1 + \eta^2 + \xi^4))^{-2} (1 + |s|)^{-(N-2)}.
\end{equation}

With Lemma 6.2 statement (iii)a follows, except for the case $p = q$.

We now assume $p \leq q$. Then $\alpha = p - 2$ and $\beta = q - p$. For $s \leq 0$, we use the estimate
\begin{equation}
\Theta(s, x, y) \geq x^2 + \frac{1}{4} e^{2s} y^2 + \cosh^2 s,
\end{equation}
from Lemma 6.1. Proceeding as before, with the roles of $x$ and $y$ interchanged, we obtain (iii)b and the negative direction of the remaining case in (iii)a.

If instead we use the estimate
\begin{equation}
\Theta(s, x, y) \geq x^2 + \frac{1}{4} e^{2s} (y^2 - 1)^2,
\end{equation}
which follows from (34), and substitute $x = \frac{1}{2} e^s |y^2 - 1| \xi$, we obtain for $s > 0$
\begin{equation}
|Rf(s)| \leq C e^{-s/2} (1 + s)^{N-2}.
\end{equation}

Here $C \in [0, \infty]$ is given by
\begin{equation}
C = \int_0^{\infty} (1 + \xi^2)^{-\rho/2} \xi^\alpha \, d\xi \int_0^\infty |y^2 - 1|^{(p-q-1)/2} (1 + \log(|y|))^{-2} \eta^\beta \, dy,
\end{equation}
where argument in the logarithm is the maximum of 1 and $(y^2 - 1)^2$. The integral is finite precisely when $p = q$, and in this case we thus obtain the desired bound in the positive direction. This finishes the proof of (iii).

Assume $p < q$. Recall that
\begin{equation}
\Theta(s, x, y) = x^2 + y^2 + (\cosh s - \frac{1}{2} e^s y^2)^2.
\end{equation}
Let $v = -\sinh s + 1/2 e^s y^2$, then $y^2 = 1 + 2 e^{-s} v - e^{-2s}$ and
\begin{equation}
\Theta(s, x, y) = 1 + x^2 + v^2.
\end{equation}

With this substitution, we find
\begin{equation}
Rf(s) = e^{-s} \int_0^\infty \int_{-\cosh s}^\infty F(1 + x^2 + v^2)(1 + 2 e^{-s} v - e^{-2s})^{(\beta-1)/2} x^\alpha \, dv \, dx,
\end{equation}
with the following upper bound, for $s \geq 0$, since $\beta \geq 1$:
\begin{equation}
|Rf(s)| \leq C e^{-s} \int_0^\infty \int_{-\infty}^\infty \frac{(1 + x^2 + v^2)^{-\rho/2}}{[1 + \log(1 + x^2 + v^2)]^2} (1 + 2|v|)^{-(\beta-1)/2} x^\alpha \, dv \, dx < +\infty.
\end{equation}
Applying Lebesgue’s theorem, we get
\[ \lim_{s \to \infty} e^s Rf(s) = \int_0^{\infty} \int_{-\infty}^{\infty} F(1 + x^2 + v^2)x^s dvdx \]
from which (iv) follows. \[ \square \]

Before we proceed, we note that if \( f \) satisfies a sharper decay than (32), we can improve on the decay of \( Rf(s) \) expressed in (iii).

**Lemma 6.3.** Let \( f \) be a continuous function on \( G/H \), and assume that
\[ \sup_{t \in \mathbb{R}, k \in K} (\cosh t)^{\rho + \gamma} |f(ka)| < \infty \]
for some \( \gamma > 0 \). Then for each \( \epsilon > 0 \) there exists a constant \( C > 0 \), such that
\[ e^{\rho + 1} |f(s)| \leq C(\cosh s)^{-\gamma + \epsilon}, \]

- \( a \): for all \( s \in \mathbb{R} \) if \( p \geq q \),
- \( b \): for all \( s \leq 0 \) if \( p < q \).

**Proof.** Replace (36) by
\[ |F(\Theta)| \leq C\Theta^{-(\rho + \gamma)/2} \]
in the proof of (iii) above and proceed as before. \[ \square \]

**Proof of Theorem 5.2.** Let \( \lambda \) be a discrete series parameter, and let \( f \) be the generating function of Proposition 3.2. We know, see (31), that \( Rf \) will be improved on the decay of \( Rf(ka) \) expressed in (iii).

We shall first establish conclusion (1) in the theorem. Here we use the following lemma. The vanishing of \( Rf \) will be established by showing that, for some \( \alpha \in \mathbb{R} \) and all \( k \in K \), \( e^{\alpha s} Rf(ka) \) decays to 0 as \( s \to -\infty \), and that it is bounded in the other direction.

**Lemma 6.4.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a linear combination of exponential functions with real exponents. If \( \phi(s) \) is bounded as \( s \to \infty \), and decays to 0 as \( s \to -\infty \), then \( \phi = 0 \).

**Proof.** The maximal exponent is \( \leq 0 \) and the minimal exponent is \( > 0 \).

For simplicity we assume in what follows that \( p > 1 \). The arguments for the case \( p = 1 \) are similar. Notice that if \( q = 1 \) and \( \lambda < 0 \), then by definition \( \psi_\lambda \) equals the complex conjugate of \( \psi_{-\lambda} \). Hence \( R\psi_\lambda \) is the complex conjugate of \( R\psi_{-\lambda} \). Therefore, to prove Theorem 5.2 (1), we may assume \( \lambda > 0 \).

We first assume \( \mu_\lambda > 0 \). It follows from the expression for \( f = \psi_\lambda \) in Proposition 3.2 (i) that
\[ |f(ka)| \leq C(\cosh t)^{-\lambda - \rho}, \]
for all \( k \in K \) and \( t \in \mathbb{R} \).

For \( p \geq q \), it follows from Theorem 5.1 (iii) a that
\[ e^{\rho + 1} Rf(ka) \to 0 \]
in both directions \( s \to \pm \infty \), thus \( Rf(ka) = 0 \), for all \( s \in \mathbb{R} \) and \( k \in K \), and hence \( Rf = 0 \).

For \( p < q \), it follows from Lemma 6.3 that \( e^{(\rho + 1)s} Rf(ka) \) is bounded for \( s \to -\infty \), for all \( \epsilon > 0 \). Since \( \rho_1 - \lambda = -\mu_\lambda + 1 < 1 \), we can choose \( \epsilon = \frac{1}{2} (1 - \rho_1 + \lambda) \), and conclude that
\[ e^{\epsilon s} Rf(ka) = e^{\epsilon s} e^{(\rho_1 - \lambda)s} Rf(ka) \to 0 \]
for \( s \to -\infty \). On the other hand, from Theorem 5.1 (iv), we infer that \( e^{\epsilon s} Rf(ka) \) is bounded at \( +\infty \). Thus again \( Rf = 0 \). This proves Theorem 5.2 (1).
Assume now that $\mu_\lambda = 0$. This only happens when $p + 1 < q$. From Lemma 6.3 we see that $e^{\rho_1 s} Rf(s) \to 0$ as $s \to -\infty$. We know that $e^{\rho_1 s} Rf(s)$ is a linear combination of the exponential functions $e^{\lambda s}$ and $e^{-\lambda s}$, but because of the limit relation at $-\infty$, only the first one can occur. Hence $Rf(s)$ is a multiple of $e^{(-\rho_1 + \lambda) s} = e^{-s}$.

Since $f \geq 0$, we conclude from Theorem 5.1(iv) that $\lim_{s \to -\infty} e^{s} Rf(s) = C \neq 0$. Hence $Rf(s) = Ce^{-s}$.

Assume finally that $\mu_\lambda < 0$, then $p + 3 < q$ and $f = \xi_\lambda$ in Proposition 3.2(ii). From the expressions in (ii), we infer immediately

$$|\xi_\lambda(k\theta a_t)| \leq C(\cosh t)^{-(\lambda + \rho)}.$$  

Furthermore, since $P_\lambda$ has degree $m$, the limit

$$\lim_{t \to \infty} (\cosh t)^{\lambda + \rho} \xi_\lambda(a_t)$$

exists and is non-zero.

We first consider the even case $n = 2m$, where $\xi_\lambda$ is $K$-spherical. We shall need the following lemma.

**Lemma 6.5.** Assume $p < q$. Let $f$ be a $K$-invariant continuous function on $G/H$,

and assume that there exists $\gamma > 0$ such that

$$\sup_{t \in \mathbb{R}} (\cosh t)^{p + \gamma} |f(a_t)| < \infty,$$

and such that

$$\lim_{t \to \infty} (\cosh t)^{p + \gamma} f(a_t)$$

exists and is non-zero. Then

$$\lim_{s \to -\infty} e^{(\rho_1 - \gamma) s} Rf(s)$$

exists and is non-zero.

**Proof.** Recall that for $p < q$, we have

$$\Theta(s, x, y) = x^2 + y^2 + (\cosh s - \frac{1}{2} e^s y^2)^2.$$  

We make the substitutions

$$v = \frac{1}{2}(1 + e^{2s}(y^2 - 1)), \quad u = e^s x,$$

and find

$$y^2 = e^{-2s}(2v + e^{2s} - 1), \quad \cosh s = \frac{1}{2} e^s y^2 = e^{-s}(1 - v),$$

so that

$$\Theta(s, x, y) = e^{-2s}(u^2 + v^2) + 1.$$  

Hence $Rf(s)$ equals

$$e^{-s(\alpha + \beta + 2)} \int_0^\infty \int_{(1-e^{2s})/2} F(1 + e^{-2s}(u^2 + v^2))(2v + e^{2s} - 1)^{(\beta - 1)/2} u^\alpha \, dv \, du.$$  

Note that $\alpha + \beta + 2 = q$. For $s$ sufficiently large negative we obtain a uniform bound

$$e^{qs}|Rf(s)| \leq C \int_0^\infty \int_{(1-e^{2s})/2} [1 + e^{-2s}(u^2 + v^2)]^{-(\gamma + \rho)/2}(2v + e^{2s} - 1)^{(\beta - 1)/2} u^\alpha \, dv \, du \leq Ce^{s(\gamma + \rho)} \int_0^\infty \int_{1/4}^\infty (u^2 + v^2)^{-(\gamma + \rho)/2} e^{(\beta - 1)/2} u^\alpha \, dv \, du \leq +\infty.$$  

It follows that we can apply Lebesgue's theorem to the limit

$$\lim_{s \to -\infty} e^{(\rho_1 - \gamma) s} Rf(s) = \lim_{s \to -\infty} e^{-(\gamma - \rho + q) s} Rf(s).$$
Since $r^{\gamma+\rho} F(r^2)$ allows a non-zero limit $c$ for $r \to \infty$, it follows that
$$e^{-(\gamma+\rho)s} F(1 + e^{-2s}(u^2 + v^2)) \to c(u^2 + v^2)^{-(\gamma+\rho)/2}$$
for $s \to -\infty$. We conclude that $e^{-(\gamma-\rho+\rho)} Rf(s)$ tends to
$$c \int_0^\infty \int_{1/2}^\infty (u^2 + v^2)^{-(\gamma+\rho)/2}(2v - 1)^{(\beta-1)/2}u^\alpha \, dv \, du \neq 0$$
as $s \to -\infty$.

Proceeding with the proof of Theorem 5.2, we recall that $f = \xi$. It follows from (27) that $\cos \theta P_{\lambda}(\cosh^2 t)(\cosh t)^{-\lambda-\rho-2m}$.

Since $f$ is not $K$-invariant, Lemma 6.3 is not directly applicable. However, we shall adapt its proof. It follows from (27) that $\cos \theta \cosh t = \cos s - \frac{1}{2} e^{-s} y^2$, and hence
$$Rf(s) = \int_0^\infty \int_0^\infty x^\alpha y^\beta \cosh s - \frac{1}{2} e^s y^2) \, Q(s, x, y) \, dx \, dy,$$
where
$$Q(s, x, y) = P_{\lambda}(\Theta(s, x, y))(\Theta(s, x, y))^{-\frac{1}{2}(\lambda+\rho+1)-m}.$$ 

As in the proof of Lemma 6.3, we perform the substitutions (41). By application of (42), we see
$$e^{(\gamma+1)s} Rf(s) = \int_0^\infty \int_{(1-e^2)^2/2}^\infty u^\alpha (2v + e^{2s} - 1)(\beta-1)/2(1-v) \, \tilde{Q}(s, u, v) \, dv \, du,$$
where
$$\tilde{Q}(s, u, v) = P_{\lambda}(1 + e^{-2s}(u^2 + v^2))(1 + e^{-2s}(u^2 + v^2))^{-\frac{1}{2}(\lambda+\rho+1)-m}.$$ 

This gives the domination for $s$ sufficiently large negative $e^{(\gamma+1)s} Rf(s)$|
$$\leq C \int_0^\infty \int_{(1-e^2)^2/2}^\infty |1 - v| (1 + e^{-2s}(u^2 + v^2))^{-(\lambda+\rho+1)/2}(2v + e^{2s} - 1)^{(\beta-1)/2} u^\alpha \, dv \, du$$
$$\leq C e^{s(\gamma+\rho+1)} \int_0^\infty \int_{1/4}^\infty (u^2 + v^2)^{-(\lambda+\rho+1)/2}v^{(\beta+1)/2}u^\alpha \, dv \, du < +\infty.$$ 

Again we can apply Lebesgue’s theorem and obtain
$$\lim_{s \to +\infty} e^{(1-\mu)s} Rf(s) = c \int_0^\infty \int_{1/2}^\infty (1 - v)(u^2 + v^2)^{-(\lambda+\rho+1)/2}u^\alpha (2v - 1)^{(\beta-1)/2}dv \, du,$$
with $c = \lim_{r \to \infty} r^{-\lambda+\rho+1}\phi_{n,m}(r^2) \neq 0$.

In order to prove that $Rf(s) = C e^{(\mu-\lambda)s}$, with $C \neq 0$, we argue as before. We only need to establish that the following integral is non-zero:
$$I := \int_0^\infty \int_{1/2}^\infty (1 - v)(u^2 + v^2)^{-(\lambda+\rho+1)/2}u^\alpha (2v - 1)^{(\beta-1)/2}dv \, du.$$ 

We rewrite this by setting $u = vx$ and $2v - 1 = y$. Up to a power of 2 we obtain
$$\int_0^\infty (1 + x^2)^{-(\lambda+\rho+1)/2}x^\alpha \, dx \int_0^\infty (1 - y)(\beta-1)/2(y + 1)^{-(\lambda+\rho+\alpha)}dy,$$
in which the first integral is clearly finite and non-zero.
We now use the formula
\[ \int_0^\infty y^{k-1}(1 + y)^{-l}dy = B(k, l - k) = \frac{\Gamma(k)\Gamma(l - k)}{\Gamma(l)}, \]
valid for \(0 < k < l\). It follows that for \(0 < k < l - 1\)
\[ \int_0^\infty (1 - y)y^{k-1}(1 + y)^{-l}dy = (l - 2k - 1)\frac{\Gamma(k)\Gamma(l - k - 1)}{\Gamma(l)}. \]
Hence the second integral in (44) is zero if and only if
\[ (\lambda + \rho - \alpha) - (\beta + 1) - 1 = 0. \]
With the current values of \(\alpha, \beta, \lambda\) and \(\rho\), we have \(\lambda + \rho - \alpha - \beta - 1 = -n\), and hence \(I \neq 0\). This finishes the proof of Theorem 5.2. \(\square\)

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