An analog of the Iwasawa conjecture for a compact hyperbolic threefold

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November 6, 2018

Abstract

For a local system on a compact hyperbolic threefold, under a cohomological assumption, we will show that the order of its twisted Alexander polynomial and of the Ruelle L function at \( s = 0 \) coincide. Moreover we will show that their leading constant are also identical. These results may be considered as a solution of a geometric analogue of the Iwasawa conjecture in the algebraic number theory.

1 Introduction

In recent days, it has been recognized there are many similarities between the theory of a number field and one of a topological threefold. In this note, we will show one more evidence, which is “a geometric analog of the Iwasawa conjecture”.

At first let us recall the original Iwasawa conjecture (§). Let \( p \) be an odd prime and \( K_n \) a cyclotomic field \( \mathbb{Q}(\zeta_{p^n}) \). The Galois group \( \text{Gal}(K_n/\mathbb{Q}) \) which is isomorphic to \( \mathbb{Z}/(p^n−1) \times \mathbb{F}_p^* \) by the cyclotomic character \( \omega \) acts on the \( p \)-primary part of the ideal class group \( A_n \) of \( K_n \). By the action of \( \text{Gal}(K_1/\mathbb{Q}) \cong \mathbb{F}_p^* \), it has a decomposition

\[
A_n = \bigoplus_{i=0}^{p-2} A_{n^i},
\]

where we set

\[
A_{n^i} = \{ \alpha \in A_n | \gamma \alpha = \omega(\gamma)^i \alpha \text{ for } \gamma \in \text{Gal}(K_1/\mathbb{Q}) \}.
\]

For each \( i \) let us take the inverse limit with respect to the norm map:

\[
X_i = \lim_{\longleftarrow} A_{n^i}.
\]

If we set \( K_\infty = \cup_n K_n \) and \( \Gamma = \text{Gal}(K_\infty/K_1) \), each \( X_i \) becomes a \( \mathbb{Z}_p[[\Gamma]] \)-module. Since there is an (noncanocal) isomorphism \( \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[s]] \), each \( X_i \)
may be considered as a $\mathbb{Z}_p[[s]]$-module. Iwasawa has shown that it is a torsion $\mathbb{Z}_p[[s]]$-module and let $L^{alg,i}_p$ be its generator, which will be referred as the Iwasawa power series.

On the other hand, let

$$Z_p[[s]] \cong Z_p[[\Gamma]] \xrightarrow{\chi} Z_p$$

be the ring homomorphism induced by $\omega$. For each $0 < i < p - 1$, using the Kummer congruence of the Bernoulli numbers, Kubota-Leopoldt and Iwasawa have independently constructed an element of $L^{ana,i}_p$ which satisfies

$$\chi^r(L^{ana,i}_p) = (1 - p^r)\zeta(-r),$$

for any positive integer $r$ which is congruent $i$ modulo $p - 1$. Here $\zeta$ is the Riemann zeta function. We will refer $L^{ana,i}_p$ as the $p$-adic zeta function. The Iwasawa main conjecture, which has been solved by Mazur and Wiles ([4]) says that ideals in $Z_p[[s]]$ generated by $L^{alg,i}_p$ and $L^{ana,i}_p$ are equal.

Now we will explain our geometric analog of the Iwasawa main conjecture.

It is broadly recognized a geometric substitute for the Iwasawa power series is the Alexander invariant. Let $X$ be a connected finite CW-complex of dimension three and $\Gamma_g$ its fundamental group. In what follows, we always assume that there is a surjective homomorphism

$$\Gamma_g \twoheadrightarrow \mathbb{Z}.$$

Let $X_\infty$ be the infinite cyclic covering of $X$ which corresponds to $\text{Ker} \epsilon$ by the geometric Galois theory and $\rho$ a finite dimensional unitary representation of $\Gamma_g$. Then $H_*(X_\infty, \mathbb{C})$ and $H_*(X_\infty, \rho)$ have an action of $\text{Gal}(X_\infty/X) \cong \mathbb{Z}$, which make them $\Lambda$-modules. Here we set $\Lambda = \mathbb{C}[\mathbb{Z}]$ which is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Suppose that each of them is a torsion $\Lambda$-module. Then due to the results of Milnor ([6]), we know $H^i(X_\infty, \rho)$ is also a torsion $\Lambda$-module for all $i$ and vanishes for $i \geq 3$. Let $\tau^*$ be the action of $t$ on $H^i(X_\infty, \rho)$. Then the Alexander invariant is defined to be the alternating product of the characteristic polynomials:

$$A^*(t) = \frac{\det [t - \tau^* | H^0(X_\infty, \rho)] \cdot \det [t - \tau^* | H^2(X_\infty, \rho)]}{\det [t - \tau^* | H^1(X_\infty, \rho)]}.$$

On the other hand, we will take the Ruelle L-function as a geometric substitute for the $p$-adic zeta function. Let $X$ be a connected closed hyperbolic threefold. Then its fundamental group $\Gamma_g$ may be considered as a torsion-free cocompact discrete subgroup of $\text{PSL}_2(\mathbb{C})$. By the one to one correspondence between the set of loxiodromic conjugacy classes of $\Gamma_g$ and one of closed geodesics.
of $X$, the Ruelle $L$-function is defined to be a product of the characteristic polynomials of $\rho(\gamma)$ over prime closed geodesics:

$$R_\rho(s) = \prod_{\gamma} \det[1 - \rho(\gamma)e^{-s l(\gamma)}].$$

Here $s$ is a complex number and $l(\gamma)$ is the length of $\gamma$. It absolutely converges for $s$ whose real part is sufficiently large. Fried [1] has shown that it is meromorphically continued in the whole plane. Using his results [1], we will show the following theorem.

**Theorem 1.1.** Suppose that $H^0(X_\infty, \rho)$ vanishes and let $\beta$ be the dimension of $H^1(X, \rho)$. Then

$$-2\beta = \text{ord}_{s=0} R_\rho(s) \geq 2 \text{ord}_{t=1} A_\rho^*(s),$$

and the identity holds if the action of $\tau^*$ on $H^1(X_\infty, \rho)$ is semisimple. Moreover if all $H^i(X, \rho)$ vanish, we have

$$|R_\rho(0)| = \delta_\rho |A_\rho^*(1)|^2,$$

where $\delta_\rho$ is a positive constant which can be determined explicitly.

In particular if we make a change of variables:

$$t = s + 1,$$

our theorem implies that two ideals in $\mathbb{C}[[s]]$ which are generated by $R_\rho(s)$ and $A_\rho^*(s)^2$ coincide. Thus it may be considered as a solution of a geometric analog of the Iwasawa main conjecture.

In fact, after a certain modification, **Theorem 1.1** is still true for a complete hyperbolic threefold of a finite volume, which will be discussed in [7].

When $X_\infty$ is homeomorphic to a mapping torus derived a homeomorphism $f$ of a compact Riemannian surface $S$, we can prove a limit formula.

**Theorem 1.2.** Suppose that $X$ is homeomorphic to a mapping torus of $(S, f)$ and that the surjective homomorphism $\epsilon$ is induced from the structure map:

$$X \rightarrow S^1.$$

If $H^0(S, \rho)$ vanishes, we have

$$-2\beta = \text{ord}_{s=0} R_\rho(s) \geq 2 \text{ord}_{t=1} A_\rho^*(s),$$

and the identity holds if the action of $f^*$ on $H^1(S, \rho)$ is semisimple. Moreover if this is satisfied, we have

$$\lim_{t \rightarrow 1} |s^2 R_\rho(s)| = \lim_{t \rightarrow 1} |(t - 1)\beta A_\rho^*(t)|^2 = |\tau_\rho^*(X, \rho)|^2,$$

where $\tau_\rho^*(X, \rho)$ is the (cohomological) Milnor-Reidemeister torsion of $X$ and $\rho$. 


2 The Milnor-Reidemeister torsion and the Alexander invariant

Let Λ = ℂ[t, t^{-1}] be a Laurent polynomial ring of complex coefficients. The following lemma is easy to see.

Lemma 2.1. Let f and g be elements of Λ such that

\[ f = ug, \]

where u is a unit. Then their order at \( t = 1 \) are equal:

\[ \text{ord}_{t=1} f = \text{ord}_{t=1} g. \]

We will recall the Milnor-Reidemeister torsion of a complex (5, 6).

Let \((C, \partial)\) be a bounded complex of free Λ-modules of finite rank whose homology groups are torsion Λ-modules. Suppose that it is given a base \( c_i \) for each \( C_i \). Such a complex will refered as a based complex. We set

\[ C_{\text{even}} = \bigoplus_{i\in\mathbb{Z}/2} C_i, \quad C_{\text{odd}} = \bigoplus_{i\equiv1(2)} C_i, \]

which are free Λ-modules of finite rank with basis \( c_{\text{even}} = \bigoplus_{i\equiv0(2)} c_i \) and \( c_{\text{odd}} = \bigoplus_{i\equiv1(2)} c_i \) respectively. Choose a base \( b_{\text{even}} \) of a Λ-submodule \( B_{\text{even}} \) of \( C_{\text{even}} \) (necessary free) which is the image of the differential and column vectors \( x_{\text{odd}} \) of \( C_{\text{odd}} \) so that

\[ \partial x_{\text{odd}} = b_{\text{even}}. \]

Similarly we take \( b_{\text{odd}} \) and \( x_{\text{even}} \) satisfying

\[ \partial x_{\text{even}} = b_{\text{odd}}. \]

Then \( x_{\text{even}} \) and \( b_{\text{even}} \) are expressed by a linear combination of \( c_{\text{even}} \):

\[ x_{\text{even}} = X_{\text{even}} c_{\text{even}}, \quad b_{\text{even}} = Y_{\text{even}} c_{\text{even}}, \]

and we obtain a square matrix

\[
\begin{pmatrix}
X_{\text{even}} \\
Y_{\text{even}}
\end{pmatrix}.
\]

Similarly the equation

\[ x_{\text{odd}} = X_{\text{odd}} c_{\text{odd}}, \quad b_{\text{odd}} = Y_{\text{odd}} c_{\text{odd}} \]

yields a square matrix

\[
\begin{pmatrix}
X_{\text{odd}} \\
Y_{\text{odd}}
\end{pmatrix}.
\]
Now the Milnor-Reidemeister torsion $\tau_{\Lambda}(C, c.)$ of the based complex $\{C, c.\}$ is defined as

$$\tau_{\Lambda}(C, c.) = \pm \frac{\det \begin{pmatrix} X_{even} \\ Y_{even} \end{pmatrix}}{\det \begin{pmatrix} X_{odd} \\ Y_{odd} \end{pmatrix}}$$ \hspace{1cm} (1)$$

It is known $\tau_{\Lambda}(C, c.)$ is independent of a choice of $b$.

Since $H_i(C)$ are torsion $\Lambda$-modules, they are finite dimensional complex vector spaces. Let $\tau_{i*}$ be the action of $t$ on $H_i(C)$. Then the Alexander invariant is defined to be the alternating product of their characteristic polynomials:

$$A_{C}(t) = \prod_i \det[t - \tau_{i*}]^{(-1)^i}.$$ \hspace{1cm} (2)

Then Assertion 7 of [6] shows the fractional ideals generated by $\tau_{\Lambda}(C, c.)$ and $A_{C}(t)$ are equal:

$$(\tau_{\Lambda}(C, c.)) = (A_{C}(t)).$$

In particular Lemma 2.1 implies

$$\text{ord}_{t=1} \tau_{\Lambda}(C, c.) = \text{ord}_{t=1} A_{C}(t),$$ \hspace{1cm} (3)

and we know

$$\tau_{\Lambda}(C, c.) = \delta \cdot t^k A_{C}(t),$$

where $\delta$ is a non-zero complex number and $k$ is an integer. $\delta$ will be referred as the difference of the Alexander invariant and the Milnor-Reidemeister torsion.

Let $\{(C, \overline{c})\}$ be a bounded complex of a finite dimensional vector spaces over $\mathbb{C}$. If it is given basis $c_i$ and $h_j$ for each $C_i$ and $H_j(C)$ respectively, the Milnor-Reidemeister torsion $\tau_{\mathbb{C}}(C, \overline{c})$ is also defined ([5]). Such a complex will be referred as a based complex again. By definition, if the complex is acyclic, it coincides with (1). Let $(C, c.)$ be a based bounded complex over $\Lambda$ whose homology groups are torsion $\Lambda$-modules. Suppose its annihilator $\text{Ann}_{\Lambda}(H_i(C))$ does not contain $t - 1$ for each $i$. Then

$$(\overline{C}, \overline{\partial}) = (C, c.) \otimes_{\Lambda} \Lambda/(t - 1)$$

is a based acyclic complex over $\mathbb{C}$ with a preferred base $\overline{c}$ which is the reduction of $c.$ modulo $(t - 1)$. This observation shows the following proposition.

**Proposition 2.1.** Let $(C, c.)$ be a based bounded complex over $\Lambda$ whose homology groups are torsion $\Lambda$-modules. Suppose the annihilator $\text{Ann}_{\Lambda}(H_i(C))$ does not contain $t - 1$ for each $i$. Then we have

$$\tau_{\Lambda}(C, c.)|_{t=1} = \tau_{\mathbb{C}}(C, \overline{c})$$
For a later purpose we will consider these dual.

Let \( \{ C', d \} \) be the dual complex of \( \{ C, \partial \} \):

\[
(C', d) = \text{Hom}_\Lambda((C, \partial), \Lambda).
\]

By the universal coefficient theorem we have

\[
H^q(C', d) = \text{Ext}^1_\Lambda(H_{q-1}(C, \partial), \Lambda)
\]

and the cohomology groups are torsion \( \Lambda \)-modules. Moreover the characteristic polynomial of \( H^q(C', d) \) is equal to one of \( H_{q-1}(C, \partial) \). Thus if we define the Alexander invariant \( A_C(t) \) of \( \{ C, d \} \) by the same way as (2), we have

\[
A_C(t) = A_C(t)^{-1}.
\]

3 The Milnor-Reidemeister torsion of a CW-complex of dimension three

Let \( X \) be a connected finite CW-complex and \( \{ c_{i, \alpha} \}_\alpha \) its \( i \)-dimensional cells. We will fix its base point \( x_0 \) and let \( \Gamma \) be the fundamental group of \( X \) with a base point \( x_0 \). Let \( \rho \) be a unitary representation of a finite dimension and \( V_\rho \) its representation space. Suppose that there is a surjective homomorphism

\[
\Gamma \to \mathbb{Z},
\]

and let \( X_\infty \) be the infinite cyclic covering of \( X \) which corresponds to \( \text{Ker} \, \epsilon \) by the Galois theory. Finally let \( \tilde{X} \) be the universal covering of \( X \).

The chain complex \( (C_i(\tilde{X}), \partial) \) is a complex of free \( \mathbb{C}[\Gamma] \)-module of finite rank. We take a lift of \( c_i = \{ c_{i, \alpha} \}_\alpha \) as a base of \( C_i(\tilde{X}) \), which will be also denoted by the same character. Note that such a choice of base has an ambiguity of the action of \( \Gamma \).

Following [2] consider a complex over \( \mathbb{C} \):

\[
C_i(X, \rho) = C_i(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho.
\]

On the other hand, restricting \( \rho \) to \( \text{Ker} \, \epsilon \), we will make a chain complex

\[
C_i(X_\infty, \rho) = C_i(\tilde{X}) \otimes_{\mathbb{C}[[\text{Ker} \epsilon]]} V_\rho,
\]

which has the following description. Let us consider \( \mathbb{C}[\mathbb{Z}] \otimes_{\mathbb{C}} V_\rho \) as \( \Gamma \)-module by

\[
\gamma(p \otimes v) = p \cdot t^{\epsilon(\gamma)} \otimes \rho(\gamma) \cdot v, \quad p \in \mathbb{C}[\mathbb{Z}], \, v \in V_\rho.
\]

Then \( C_i(X_\infty, \rho) \) is isomorphic to a complex ([2 Theorem 2.1]):

\[
C_i(X, V_\rho[\mathbb{Z}]) = C_i(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (\mathbb{C}[\mathbb{Z}] \otimes_{\mathbb{C}} V_\rho).
\]
and we know $C.(X_\infty, \rho)$ is a bounded complex of free $\Lambda$-modules of finite rank. We will fix a unitary base $v = \{v_1, \ldots, v_m\}$ of $V_\rho$ and make it a based complex with a preferred base $c \otimes v = \{c_{i,\alpha} \otimes v_j\}_{\alpha,i,j}$.

In the following we will fix an isomorphism between $\mathbb{C}[\mathbb{Z}]$ and $\Lambda$ and identify them. Note that such an isomorphism is determined modulo $t^k$ ($k \in \mathbb{Z}$). Note that for such a choice there is an ambiguity of sending the generator 1 to $t^{\pm 1}$.

Then by the surjection:

$$\Lambda \rightarrow \Lambda / (t - 1) \cong \mathbb{C},$$

$C.(X_\infty, \rho) \otimes \Lambda$ is isomorphic to $C.(X, \rho)$. Moreover if we take $c \otimes v$ as a base of the latter, they are isomorphic as based complexes.

Let $C.(\tilde{X})$ be the cochain complex of $\tilde{X}$:

$$C.(\tilde{X}) = Hom_{\mathbb{C}[\Gamma]}(C.(\tilde{X}), \mathbb{C}[\Gamma]),$$

which is a bounded complex of free $\mathbb{C}[\Gamma]$-module of finite rank. For each $i$ we will take the dual $c_i = \{c_{i,\alpha}\}_\alpha$ of $c_i = \{c_{i,\alpha}\}_\alpha$ as a base of $C.(\tilde{X})$. Thus $C.(\tilde{X})$ becomes a based complex with a preferred base $c \otimes v = \{c_i \otimes v_j\}_{\alpha,i,j}$. Since $\rho$ is a unitary representation, it is easy to see that the dual complex of $C.(X_\infty, \rho)$ is isomorphic to

$$C.(X_\infty, \rho) = C.(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (\Lambda \otimes \mathbb{C} V_\rho),$$

if we twist its complex structure by the complex conjugation. Also we will make it a based complex by the base $c \otimes v = \{c_i \otimes v_j\}_{\alpha,i,j}$.

Dualizing the exact sequence

$$0 \rightarrow C.(X_\infty, \rho) \xrightarrow{t - 1} C.(X_\infty, \rho) \rightarrow C.(X, \rho) \rightarrow 0$$

in the derived category of bounded complex of finitely generated $\Lambda$-modules, we will obtain a distinguished triangle:

$$C.(X, \rho) \rightarrow C.(X_\infty, \rho) \xrightarrow{t - 1} C.(X_\infty, \rho) \rightarrow C.(X, \rho)[1] \rightarrow .$$

(5)

Here we set

$$C.(X, \rho) = C.(\tilde{X}, \rho) \otimes_{\mathbb{C}[\Gamma]} V_\rho.$$

and in general for a bounded complex $C$, $C.[n]$ denotes its shift, which is defined as

$$C^[n] = C^{i+n}.$$

Note that $C.(X, \rho)$ is isomorphic to the reduction of $C.(X_\infty, \rho)$ modulo $(t - 1)$.

Let $\tau^*$ be the action of $t$ on $H.(X_\infty, \rho)$. Then (5) induces an exact sequence:

$$\rightarrow H^q(X, \rho) \rightarrow H^q(X_\infty, \rho) \xrightarrow{t - 1} H^q(X_\infty, \rho) \rightarrow H^{q+1}(X, \rho) \rightarrow .$$

(6)

In the following, we will assume that the dimension of $X$ is three and that all $H.(X_\infty, \mathbb{C})$ and $H.(X_\infty, \rho)$ are finite dimensional vector spaces over $\mathbb{C}$. The arguments of §4 of [6] will show the following theorem.
Theorem 3.1. (i)

1. For \( i \geq 3 \), \( H^i(X_\infty, \rho) \) vanishes.

2. For \( 0 \leq i \leq 2 \), \( H^i(X_\infty, \rho) \) is a finite dimensional vector space over \( \mathbb{C} \) and there is a perfect pairing:

\[
H^i(X_\infty, \rho) \times H^{2-i}(X_\infty, \rho) \to \mathbb{C}.
\]

The perfect pairing will be referred as the Milnor duality.

Let \( A_{\rho^*}(t) \) and \( A_\rho^*(t) \) be the Alexander invariants of \( C(X_\infty, \rho) \) and \( C^*(X_\infty, \rho) \) respectively. Since the latter complex is the dual of the previous one, (4) implies

\[ A^*_\rho(t) = A_{\rho^*}(t)^{-1}. \]

Let \( \tau^*_\Lambda(X_\infty, \rho) \) be the Milnor-Reidemeister torsion of \( C^*(X_\infty, \rho) \) with respect to the preferred base \( c^* \otimes v \). Because of an ambiguity of a choice of \( c^* \) and \( v \), it is well-defined modulo

\[ \{ zt^n \mid z \in \mathbb{C}, \ |z| = 1, \ n \in \mathbb{Z} \}. \]

Let \( \delta_\rho \) be the absolute value of the difference between \( A^*_\rho(t) \) and \( \tau^*_\Lambda(X_\infty, \rho) \).

Theorem 3.2. The order of \( \tau^*_\Lambda(X_\infty, \rho) \), \( A^*_\rho(t) \) and \( A_{\rho^*}(t)^{-1} \) at \( t = 1 \) are equal. Let \( \beta \) be the order. Then we have

\[
\lim_{t \to 1} |(t-1)^{-\beta} \tau^*_\Lambda(X_\infty, \rho)| = \delta_\rho \lim_{t \to 1} |(t-1)^{-\beta} A^*_\rho(t)| = \delta_\rho \lim_{t \to 1} |(t-1)^{-\beta} A_{\rho^*}(t)^{-1}|.
\]

Note that Theorem 3.1 implies that the Alexander invariant has the following form:

\[
A^*_\rho(t) = \frac{\det [t - \tau^* | H^0(X_\infty, \rho)] \cdot \det [t - \tau^* | H^2(X_\infty, \rho)]}{\det [t - \tau^* | H^1(X_\infty, \rho)]}.
\] (7)

Theorem 3.3. Suppose \( H^0(X_\infty, \rho) \) vanishes. Then we have

\[ \text{ord}_{t=1} A^*_\rho(t) \leq - \dim H^1(X, \rho) \]

and the identity holds if the action of \( \tau^* \) on \( H^1(X_\infty, \rho) \) is semisimple.

Proof. We know by Theorem 3.1 that \( H^2(X_\infty, \rho) \) also vanishes. Moreover the exact sequence (6) shows

\[
0 \to H^1(X, \rho) \to H^1(X_\infty, \rho) \xrightarrow{\tau^*} H^1(X_\infty, \rho),
\]

which implies

\[ \text{ord}_{t=1} \det [t - \tau^* | H^1(X_\infty, \rho)] \geq \dim H^1(X, \rho), \]

and the identity holds if \( \tau^* \) is semisimple. Now the desired result follows from (7).
Let $\tau_c^*(X, \rho)$ be the Milnor-Reidemeister torsion of $C^*(X, \rho)$ with respect to the preferred base $c \otimes v$. By an ambiguity of a choice of $v$, only its absolute value is well-defined.

**Theorem 3.4.** Suppose $H^i(X, \rho)$ vanishes for all $i$. Then we have

$$|\tau_c^*(X, \rho)| = \delta_{\rho} A_\rho^*(1) = \frac{\delta_{\rho}}{|A_{\rho^*}(1)|}$$

**Proof.** The exact sequence (6) and the assumption implies $t - 1$ is not contained in the annihilator of $H^1(X_\infty, \rho)$. Now the theorem will follow from **Proposition 2.1** and **Theorem 3.2**.

In the following, we will specify these arguments to a mapping torus.

Let $S$ be a connected CW-complex of dimension 2 and $f$ its automorphism. Let $X$ be the mapping torus of the pair $S$ and $f$. We will take a base point from $S$ and let $\Gamma_S$ be the fundamental group of $S$ with respect to the point. Suppose $H^0(S, \rho)$ vanishes. (e.g. The restriction of $\rho$ to $\Gamma_S$ is irreducible.) Let

$$\Gamma = \pi_1(X, s_0) \to \mathbb{Z}$$

be the homomorphism induced by the structure map:

$$X \to S^1.$$ 

Then $X_\infty$ is a product of $S$ with the real axis and therefore $C^*(X_\infty, \rho)$ is chain homotopic to $C^*(S, \rho)$. We have an exact sequence of complexes:

$$0 \to C^*(S, \rho)[-1] \to C^*(S, \rho) \to C^*(X, \rho) \to 0. \quad (8)$$

The cells of $S$ defines a base $c_S$ of the chain complex $C(S)$. Thier product with the unit interval and themselves form a base $C(X)$, which will be denoted by $c_X$. Let $c^S$ and $c^X$ be the dual of them. If we fix a unitary base $v = \{v_1, \cdots, v_m\}$, both $C^*(S, \rho)$ and $C^*(X, \rho)$ become based complexes with preferred base $c^S \otimes v$ and $c^X \otimes v$ respectively.

The exact sequence of cohomology groups of (8) may be considered as an acyclic complex:

$$
\begin{array}{cccc}
& 0 & \to & H^0(X, \rho) \to H^0(S, \rho) \\
\text{inc} & H^0(S, \rho) & \to & H^1(X, \rho) \to H^1(S, \rho) \\
\text{inc} & H^1(S, \rho) & \to & H^2(X, \rho) \to H^2(S, \rho) \\
\text{inc} & H^2(S, \rho) & \to & H^3(X, \rho) \to 0,
\end{array}
$$

(9)
which will be denoted by $\mathcal{H}$. Note that this is nothing but (6). Since $H^0(S, \rho)$ which is a subgroup of $H^0(S, \rho)$ vanishes, the Poincaré duality implies $H^2(S, \rho)$ also does. Thus $\mathcal{H}$ is isomorphic to

$$\mathcal{H}_0 = [H^1(X, \rho) \to H^1(S, \rho) \xrightarrow{f^* - 1} H^1(S, \rho) \to H^1(X, \rho)][-4].$$

Choosing basis of $H^1(X, \rho)$ and $H^1(S, \rho)$, we will make $\mathcal{H}_0$ a based acyclic complex.

Now we compute the Milnor-Reidemeister torsion of $C'$. In the following computation, we will assume that the action of $f^*$ on $H^1(S, \rho)$ is semisimple.

Since in general the Milnor-Reidemeister torsion $\tau^*_C(C'[n])$ of a shift of a based bounded complex $C'$ is equal to $\tau^*_C(C')^{(-1)n}$, Theorem 3.2 of [5] implies

$$\tau^*_C(X, \rho) = \tau^*_C(H^0).$$

In particular we know $\tau^*_C(H^0)$ is independent of a choice of basis of $H^1(X, \rho)$ and $H^1(S, \rho)$. We set

$$I = H^1(S, \rho)/(\text{Ker}[f^* - 1]),$$

and let

$$H^1(S, \rho) \xrightarrow{\pi} I$$

be the natural projection. Then $f^* - 1$ induces an isomorphism of $I$ and we have a diagram:

$$\begin{array}{cccccc}
0 & \to & H^1(X, \rho) & \to & H^1(S, \rho) & \xrightarrow{\pi} I & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^1(X, \rho) & \to & H^1(S, \rho) & \xrightarrow{\pi} I & \to & 0.
\end{array} \tag{10}$$

Here the left vertical arrow is the null homomorphism and the middle one is $f^* - 1$. The right vertical arrow is the isomorphism induced by it. The snake lemma yields an exact sequence of acyclic complexes:

$$0 \to \mathcal{F} \to \mathcal{H}_0 \to \mathcal{G} \to 0,$$

where we set

$$\mathcal{F} = [H^1(X, \rho) \xrightarrow{id} H^1(X, \rho) \xrightarrow{\theta} H^1(X, \rho) \xrightarrow{id} H^1(X, \rho)]$$

and

$$\mathcal{G} = [0 \to I \xrightarrow{f^* - 1} I \to 0] = [I \xrightarrow{f^* - 1} I][-1].$$

We choose basis $\mathbf{h}$ and $\mathbf{i}$ of $H^1(X, \rho)$ and $I$ respectively. Then $\mathbf{h}$ and a lift of $\mathbf{i}$ form a base of $H^1(S, \rho)$.
Now we compute the Milnor-Reidemeister torsion of $\mathcal{H}_0$.

**Theorem 3.1** of [5] shows

$$\tau^*_C(\mathcal{H}_0) = \tau^*_C(F) \cdot \tau^*_C(G).$$

Since $\tau^*_C(F) = 1$ and since

$$\tau^*_C(G) = \tau^*_C([I \xrightarrow{f^{-1}} I])^{-1} = (\det|f^* - 1|I)^{-1},$$

we have

$$\tau^*_C(X, \rho) = \tau^*_C(\mathcal{H}_0) = (\det|f^* - 1|I)^{-1}. \quad (11)$$

Note that (7) implies

$$A^*_\rho(t) = \frac{1}{\det[t - \tau^*|H^1(S, \rho)]}.$$ 

Let $\beta$ be the dimension of $H^1(X, \rho)$. Then **Theorem 3.3** and (11) show that the order of $A^*_\rho(t)$ is $-\beta$ and that

$$\lim_{t \to 1} |(t - 1)^\beta A^*_\rho(t)| = |\tau^*_C(X, \rho)|.$$

Thus we have proved the following theorem.

**Theorem 3.5.** Let $f$ be an automorphism of a connected finite CW-complex of dimension two $S$ and $X$ its mapping torus. Let $\rho$ be a unitary representation of the fundamental group of $X$ which satisfies $H^0(S, \rho) = 0$. Suppose that the surjective homomorphism

$$\Gamma \xrightarrow{\tau} \mathbb{Z}$$

is induced from the structure map

$$X \to S^1,$$

and that the action of $f^*$ on $H^1(S, \rho)$ is semisimple. Then the order of $A^*_\rho(t)$ is $-\beta$, where $\beta$ be the dimension of $H^1(X, \rho)$ and

$$\lim_{t \to 1} |(t - 1)^\beta A^*_\rho(t)| = |\tau^*_C(X, \rho)|.$$

In particular we know that $|\tau^*_C(X, \rho)|$ is determined by the homotopy class of $f$. Note that without semisimplicity of $f^*$, we only have

$$\operatorname{ord}_{t=1} A^*_\rho(t) \leq -\beta.$$
Comparison of the Milnor-Reidemeister torsion with the twisted Alexander polynomial

Let $K$ be a knot in the three-dimensional sphere and $X$ its complement. Since the first homology group of $X$ is the infinite cyclic group, there is a surjective homomorphism $\epsilon$ from the fundamental group $\Gamma$ of $X$ to $\mathbb{Z}$. Let $\rho$ be an $m$-dimensional unitary representation of $\Gamma$. We assume the homology groups of $C.(X_\infty, \rho)$ are torsion $\Lambda$-modules. Thus $C.(X_\infty, \rho) \otimes_\Lambda \mathbb{C}(t)$ is an acyclic complex over $\mathbb{C}(t)$ and we can define the twisted Alexander polynomial after Kitano (3).

Let
\[ P(\Gamma) = < x_1, \ldots, x_k | r_1, \ldots, r_{k-1} > \]
be the Wirtinger representation of $\Gamma$ and $\gamma$ the natural projection from the free group $F_k$ with $k$ generators:
\[ F_k \xrightarrow{\gamma} \Gamma. \]
The $\rho$ and $\epsilon$ induces a ring homomorphism:
\[ \mathbb{C}[\Gamma] \xrightarrow{\rho \otimes \epsilon} M_m(\mathbb{C}(t)). \]

Composing this with $\gamma$, we obtain a homomorphism:
\[ \mathbb{C}[F_k] \xrightarrow{\Phi} M_m(\mathbb{C}(t)). \]

Using a CW-complex structure of $X$ which is derived from the Wirtinger representation, the complex $C.(X_\infty, \rho) \otimes_\Lambda \mathbb{C}(t)$ becomes (3 p.438)
\[ 0 \to (\mathbb{C}(t)^{\oplus m})^{\oplus(k-1)} \xrightarrow{\partial_2} (\mathbb{C}(t)^{\oplus m})^{\oplus k} \xrightarrow{\partial_1} \mathbb{C}(t)^{\oplus m} \to 0. \quad \text{(12)} \]
Here differentials are
\[ \partial_2 = \begin{pmatrix}
\Phi\left( \frac{\partial r_1}{\partial x_1} \right) & \cdots & \Phi\left( \frac{\partial r_{k-1}}{\partial x_1} \right) \\
\vdots & \ddots & \vdots \\
\Phi\left( \frac{\partial r_1}{\partial x_k} \right) & \cdots & \Phi\left( \frac{\partial r_{k-1}}{\partial x_k} \right)
\end{pmatrix}, \]
and
\[ \partial_1 = \left( \Phi(x_1 - 1), \ldots, \Phi(x_k - 1) \right), \]
and the derivatives are taken according to the Fox’s free differential calculus. Note that each entry is an element of $M_m(\mathbb{C}(t))$. If we set
\[ a_1 = \begin{pmatrix}
\Phi\left( \frac{\partial r_1}{\partial x_1} \right) & \cdots & \Phi\left( \frac{\partial r_{k-1}}{\partial x_1} \right)
\end{pmatrix}, \]
and
\[ A_1 = \begin{pmatrix}
\Phi\left( \frac{\partial r_1}{\partial x_2} \right) & \cdots & \Phi\left( \frac{\partial r_{k-1}}{\partial x_2} \right) \\
\vdots & \ddots & \vdots \\
\Phi\left( \frac{\partial r_1}{\partial x_k} \right) & \cdots & \Phi\left( \frac{\partial r_{k-1}}{\partial x_k} \right)
\end{pmatrix}, \]
we have
\[ \partial_2 = \begin{pmatrix} a_1 \\ A_1 \end{pmatrix}. \]

Kitano\textsuperscript{[3] Proposition 3.1} has shown that \( \Phi(x_j - 1) \) and \( A_1 \) are invertible and that the torsion of (12), which is equal to the twisted Alexander polynomial \( \Delta_{K, \rho} \) of \( K \), is
\[ \frac{\det A_1}{\det \Phi(x_1 - 1)}. \]

Now let us consider the dual of (12). Since the transpose of \( \partial_2 \) is
\[ \partial_2^t = (a_1^t, A_1^t), \]
and since \( A_1^t \) is invertible, we may take a lift of the standard base \( e^* \) of the dual space of \((\mathbb{C}(t)^{\oplus m})^{\oplus (k-1)}\) as
\[ e^* = \begin{pmatrix} 0 \\ (A_1^t)^{-1}(e^*) \end{pmatrix}. \]

Let \( f^* \) be the standard base of the dual of \( \mathbb{C}(t)^{\oplus m} \). Then the Milnor-Reidemeister torsion of the dual of (12) is
\[ \det(\partial_2^t(f^*), e^*) = \det \begin{pmatrix} \Phi(x_1 - 1)^t & 0 \\ \vdots & \ddots \\ \Phi(x_k - 1)^t & (A_1^t)^{-1} \end{pmatrix} = \frac{\det \Phi(x_1 - 1)}{\det A_1}, \]
which is the inverse of the twisted Alexander polynomial. Thus we have proved the following theorem.

**Theorem 4.1.** Let \( X \) be the complement of a knot \( K \) in the three dimensional sphere and \( \rho \) a unitary representation of its fundamental group. Suppose \( H_i(X_\infty, \rho) \) are finite dimensional complex vector spaces for all \( i \). Then the Milnor-Reidemeister torsion of \( C^1 \left( X_\infty, \rho \right) \otimes_{\Lambda} \mathbb{C}(t) \) is the inverse of the twisted Alexander polynomial \( \Delta_{K, \rho} \) of \( K \).

**Proposition 2.1, Theorem 3.2, Theorem 3.3 and Theorem 4.1 imply**

**Corollary 4.1.** Suppose \( H^0(X_\infty, \rho) \) vanishes. Then we have
\[ \text{ord}_{t=1} \Delta_{K, \rho}(t) = -\text{ord}_{t=1} A_\rho^* (t) \geq \dim H^1(X, \rho), \]
and the identity holds if the action of \( \tau^* \) on \( H^1(X_\infty, \rho) \) is semisimple. Moreover suppose \( H^1(X, \rho) \) vanishes for all \( i \). Then
\[ |\tau^*_C(X, \rho)| = \frac{1}{|\Delta_{K, \rho}(1)|}. \]
5 A special value of the Ruelle L-function

Let $X$ be a compact hyperbolic threefold. Then its fundamental group is identified with a discrete cocompact subgroup $\Gamma$ of $PSL_2(\mathbb{C})$. In particular there is a one to one correspondence between the set of loxodromic conjugacy classes of $\Gamma$ and the set of closed geodesics of $X$. Let

$$\Gamma \rightarrow U(V_\rho)$$

be a finite dimensional unitary representation. Using the correspondence, for a closed geodesic $\gamma$ of $X$, we can define a function:

$$\det[1 - \rho(\gamma)e^{-s l(\gamma)}].$$

Here $s$ is a complex number and $l(\gamma)$ is the length of $\gamma$. Following [1], we will define the Ruelle L-function to be

$$R_\rho(s) = \prod_\gamma \det[1 - \rho(\gamma)e^{-s l(\gamma)}],$$

where $\gamma$ runs through prime closed geodesics. It is known that $R_\rho(s)$ absolutely convergents for $\Re s > 0$.

Let us take a triangulation of $X$ and a unitary base $v = \{v_1, \cdots, v_m\}$ of $V_\rho$. Using them, we make a base of the chain complex $C(X, \rho)$ and its Milnor-Reidemeister torsion $\tau^*_C(X, \rho)$ is defined. It is known that $\tau^*_C(X, \rho)$ is independent of a choice of a triangulation. Moreover its absolute value is also independent of a choice a unitary base.

If we apply Theorem 3 of [1] to our case, we will obtain the following theorem.

**Theorem 5.1.** The Ruelle L-function is meromorphically continued in the whole plane. Its order at $s = 0$ is

$$e = 4 \dim H^0(X, \rho) - 2 \dim H^1(X, \rho).$$

Moreover we have

$$\lim_{s \to 0} |s^{-e} R_\rho(s)| = |\tau^*_C(X, \rho)|^2.$$

Fried has shown the theorem for an orthogonal representation but his proof is still valid for a unitary case.

Suppose there is a surjective homomorphism

$$\Gamma \rightarrow \mathbb{Z},$$

and let $X_\infty$ be the infinite cyclic covering of $X$ which corresponds to $\text{Ker} \epsilon$. Suppose that both $H_i(X_\infty, \mathbb{C})$ and $H_i(X_\infty, \rho)$ are finite dimensional vector spaces for all $i$. Then Theorem 3.3, Theorem 3.4 and Theorem 5.1 implies
Theorem 5.2. 1. Suppose $H^0(X, \rho)$ vanishes. Then we have
\[-2\beta = \text{ord}_{s=0} R_\rho(s) \leq 2\text{ord}_{t=1} A^*_\rho(t),\]
where $\beta$ is $\dim H^1(X, \rho)$. Moreover if the action of $\tau^*$ on $H_1(X_\infty, \rho)$ is semisimple, the identity holds.

2. Suppose $H^i(X, \rho)$ vanishes for all $i$. Then
\[|R_\rho(0)| = |\delta_\rho \cdot A^*_\rho(1)|^2 = \left|\frac{\delta_\rho}{A^*_\rho(1)}\right|^2,\]
where $\delta_\rho$ is the difference of the Alexander invariant and the Milnor-Reidemeister torsion.

When $X$ is a mapping torus of an automorphism $f$ of a compact Riemannian surface $S$, we can say much more. Suppose $\epsilon$ is induced from the structure map:

$X \rightarrow S^1$.

Then Theorem 3.5 implies the following theorem.

Theorem 5.3. Suppose $H^0(S, \rho)$ vanishes. Then
\[-2\beta = \text{ord}_{s=0} R_\rho(s) \leq 2\text{ord}_{t=1} A^*_\rho(t),\]
where $\beta$ is $\dim H^1(X, \rho)$. Moreover suppose that the action of $f^*$ on $H^1(S, \rho)$ is semisimple. Then the identity holds and we have
\[\lim_{s \rightarrow 0} |s^{2\beta} R_\rho(s)| = \lim_{t \rightarrow 1} |(t - 1)^\beta A^*_\rho(t)|^2 = |\tau^*_\rho(X, \rho)|^2.\]

If we make a change of variables:
\[t = s + 1,\]

Theorem 5.2 and Theorem 5.3 show the fractional ideals in the formal power series ring $\mathbb{C}[[s]]$ generated by $R_\rho(s)$ and $A^*_\rho(s)^2$ coincide. Thus our theorems may be considered as a solution of “a geometric Iwasawa conjecture”.

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