A method for calculating average electric polarizability density of arbitrary small aperture

Ying Kang¹, Zihua Zhao²,³,⁴ and Bin Li³

¹ Science and Technology on Electronic Information Control Laboratory, Chengdu 610036, China
² School of Electronic and Information Engineering, Beihang University, Beijing, China
³ Guiyang Military Representative Office of the Chinese navy
⁴ E-mail: Firstky@sina.com.cn

Abstract. Most of the existing analytical methods for calculating the average electric polarizability density of arbitrary shapes and small apertures based on the hypergeometric function. They are not easy to follow since strong mathematical background knowledge are required. The disadvantage of these methods lies in their complexity. In this thesis, a new analytical formula is proposed to significantly simplify the calculation without losing much accuracy. It is crucial to improve the efficiency of calculating aperture coupling. The new method for calculating average polarizability only need to calculate the integration of a simple function along edge of the aperture.

1. Introduction

In microwave work, it is often important to know the effect of a small hole in a cavity upon the oscillation of that cavity. For instance, two cavities may be coupled by a small hole in their common boundary and we wish to know the characteristic frequencies and the phase relations for the oscillation of the coupled system [1]. Hence many scholars have been attracted by this problem since about eighty years ago. In 1944, the famous theory of diffraction of small holes is proposed by H.A. Bathe. According to Bathe’s theory, the diffraction of electromagnetic radiation by a circular small hole in a perfectly conducting plane screen can be expressed as the radiation of fictitious magnetic charge and currents in the hole. What pity is, only circular small hole can be calculated by Bathe’s method due to the average polarizability density of non-circular aperture is very difficult to calculate.

Since then, many researcher devoted to find the average polarizability density of all kinds of small holes. For example, Cohn S B [2] developed an electrolytic-tank method of measurement that has been used to obtain extensive data on the electric polarizability of apertures of various practical shapes By this method, Cohn S B measured the average electric polarizability density of cross, rosette, rectangular, round slot and dumbbell apertures. Although the average electric polarizability density of apertures of arbitrary shape can be obtained by Cohn’s method, the measurement procedure is very inconvenient due to the electrolytic-tank and the metal plane with aperture is required. Hence, some researchers turn to developing numerical or analytical method. Based on a method of moments approach, Okon et al [3] developed a numerical procedure, which can compute the magnetic and electric average polarizabilities density of electrically small apertures of arbitrary shape. Fabrikant [4] developed an analytical solution to the problem of electrical average polarizability density of
arbitrarily shaped small apertures. His method based on an integral representation for the reciprocal distance between two points obtained earlier by him. Specific formulas are derived for an aperture having the shape of a polygon, a rectangle, a rhombus, a cross, a circular sector and a circular segment. All the formulas are checked against the solutions known in the literature, and their accuracy is confirmed. But the inconvenience of the method proposed by Fabrikant is that these formulas are very complex for one who does not major mathematics. Even someone who does not major in mathematics nearly cannot fully understand these formulas.

In this paper, a relative simple formula is proposed for calculating average polarizability density of arbitrary small aperture.

2. The definition of average polarizability density

The electric dipole moment and magnetic dipole moment is defined to express the diffraction field of small aperture in a thin metal plane. The electric dipole moment \( \vec{P}_e \) is defined as [1]

\[
\vec{P}_e = \frac{1}{2} \int_{\text{aperture}} \vec{r} \times \vec{K}(\vec{r}')\,d\vec{r}'
\]

where \( \vec{K}(\vec{r}') \) is the magnetic current density. The magnetic dipole moment is defined as

\[
\vec{P}_m = \frac{1}{\mu} \int_{\text{aperture}} \vec{r}' \eta(\vec{r}')\,d\vec{r}'
\]

where \( \eta(\vec{r}') \) is the magnetic charge density. The diffraction field \( \vec{E}_d(\vec{r}) \) produced by electric dipole moment is:

\[
\vec{E}_d(\vec{r}) = \frac{\vec{P}_e}{2\pi\varepsilon_0} \left( k^2 \hat{z} + \nabla \frac{\partial}{\partial z} \right) \frac{e^{-j\mu|\vec{r}|}}{|\vec{r}|}
\]

where \( \hat{z} \) is the normal direction of the metal plane, \( K \) is the wave number. The diffraction field \( \vec{E}_m(\vec{r}) \) produced by magnetic dipole moment is:

\[
\vec{E}_m(\vec{r}) = \frac{-j\mu}{2\pi} \left( \nabla \frac{\partial}{\partial z} \right) \frac{e^{-j\mu|\vec{r}|}}{|\vec{r}|} \times \vec{P}_m
\]

According to Equ.(1) and Equ.(2), both of the electric dipole moment and magnetic dipole moment are decided by the area and shape of the aperture. Furthermore, the electric dipole moment and magnetic dipole moment have relation with the incident field \( \vec{E}_i \) and \( \vec{H}_i \). Therefore, in order study the relation between the field coupling character and aperture’s shape and area, the electric polarizability vector \( \vec{a}_e \) and the magnetic polarizability tensor \( \vec{a}_m \) are defined as follow:

\[
\vec{P}_e = \vec{a}_e \vec{E}_i \times \hat{n}
\]

\[
\vec{P}_m = \vec{a}_m \vec{H}_i
\]

Where \( \hat{n} \) is the norm direction of aperture. According to Equ.(5) and Equ.(6), if we have known \( \vec{P}_e \) and \( \vec{P}_m \), then \( \vec{a}_e \) and \( \vec{a}_m \) can be calculated. But only several special shape aperture can be calculated according to Equ.(1) and Equ.(2) due to the \( \vec{K}(\vec{r}') \) and \( \eta(\vec{r}') \) is difficult to obtain. The analytical solution of \( \vec{a}_e \) and \( \vec{a}_m \) for circular and ellipse is given in table 1 as follow [5].

In table 1, \( \vec{a}_w = \alpha_w \hat{x} + \alpha_w \hat{y} \), the long axis and short axis of the ellipse is \( 2l \) and \( 2w \) ( \( l \geq w \) ) respectively. The \( e \) is Eccentricity of the ellipse, which equals \( \sqrt{1-(w/l)^2} \). The two integrations \( K(e) = \int_0^\pi \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} \) and \( E(e) = \int_0^\pi \sqrt{1-e^2\sin^2\theta} \,d\theta \) is the first and second kind of complete elliptic integrals, respectively.
Table 1. The $\tilde{\alpha}_e$ and $\tilde{\alpha}_m$ of circular and ellipse.

|                      | circular                  | ellipse                           |
|----------------------|---------------------------|-----------------------------------|
| $|\tilde{\alpha}_e|$ | $2a^3/3$                  | $\pi w^3/3E(e)$                   |
| $|\tilde{\alpha}_m|$ | $4a^3/3$                  | $\pi e^3/(3(K(e) - E(e)))$        |
| $|\tilde{\alpha}_e|$ | $4a^3/3$                  | $\pi e^3/(3(L_w^3 E(e) - K(e)))$  |

Although the definition of electric polarizability and magnetic polarizability can eliminate the influence of incident field, but their values still are decided by both the shape and area of aperture. In order to study the relation between the aperture’s coupling characters and its shape, the average electric polarizability $\tau_e$ and the average magnetic polarizability $\bar{\mu}_m$ is defined by researchers as follow [1]:

$$\tilde{\alpha}_e = \tau_e s^2$$  \hspace{1cm} (7)

$$\tilde{\alpha}_m = \bar{\mu}_m s^2$$  \hspace{1cm} (8)

Where $s$ is the area of a small aperture. Both the average electric polarizability $\tau_e$ and the average magnetic polarizability $\bar{\mu}_m$ is dimensionless. Therefore $\tau_e$, $\tau_m$, $\bar{\mu}_m$ has no relation with the area of aperture. In other words, if two apertures have the same shape but have different area, their $\tau_e$, $\bar{\mu}_m$ will be equal respectively. The average electric polarizability density of circular aperture is $\tau_e = \frac{4}{3\pi \sqrt{3}}$.

3. The new formula

According to above analysis, the key point of calculating the diffraction field of small holes is to obtain the average electric polarization density and the average magnetic polarization rate density. However, it is very difficult to solve the average polarizability density strictly for a small hole with arbitrary shape. Although Fabricant developed an analytical solution based on hypergeometric function to solve the average polarizability of a small aperture, the method is very complex. Furthermore, the solution of Fabricant’s method is not an exact result. When the ratio between the length and width of an aperture is relatively large, the results from Fabricant’s method are quite different from the experimental results. A new analytical method is proposed in this paper. The new method can easily calculate the average electric polarizability density with satisfying accuracy. The new method is given as follow.

As shown in figure 1, the outline of the small aperture on the infinite conductive screen is $\rho(\theta)$. In order to minimize the fluctuation of $\rho(\theta)$, the origin of the polar coordinates is arranged at the gravity center of the aperture.

Figure 1. The arbitrary shape aperture.

According to the definition of the average electric polarizability density, for the aperture shown in figure 1, we get:

$$\alpha = \tau_{av} \cdot s_0^2$$  \hspace{1cm} (9)
Where $\alpha$ is the electric polarizability, $\tau_{av}$ is the unknown average electric polarizability density, $s_0$ is the area of the aperture. Now, we take place the original aperture with a circular aperture. The center of the circular aperture is at the gravity center of the original aperture, and the radius of the circular is $\rho(\theta_i = 0)$. Then the electric polarizability of the circular aperture is:

$$\alpha_i = \tau_c \cdot (\rho^3(\theta_i))^\frac{3}{2}$$  \hspace{1cm} (10)

Where $\tau_c$ is the average electric polarizability density of circular aperture, and $\tau_c = \frac{4}{3\pi}\sqrt{\pi}$.

Similarly, we repeatedly replace the original aperture with a serial of circular apertures which have the radius $\rho(\theta_i)$, $\rho(\theta_i)$ until $\rho(\theta_i = 2\pi)$ ($0 = \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$). All center of the circular aperture is at the gravity center of the original aperture. Then the electric polarizability of these circular apertures can be calculated as follow:

$$\alpha_i = \tau_c \cdot (\rho^3(\theta_i))^\frac{3}{2}$$  \hspace{1cm} (11)

$$\alpha_2 = \tau_c \cdot (\rho^3(\theta_2))^\frac{3}{2}$$  \hspace{1cm} (12)

$$\vdots$$

$$\alpha_n = \tau_c \cdot (\rho^3(\theta_n))^\frac{3}{2}$$  \hspace{1cm} (13)

Add all the above equations, and then get:

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \tau_c \cdot (\rho^3(\theta_i))^\frac{3}{2}$$  \hspace{1cm} (14)

Obviously, some of the $\alpha_i$ are greater than $\alpha$ and some of the $\alpha_i$ are less than $\alpha$. Then we treat the average value of $\alpha_i$ as $\alpha$, get:

$$\frac{1}{n} \sum_{i=1}^{n} \alpha_i \approx \alpha = \tau_{av} \cdot s_0^\frac{3}{2} \approx \frac{1}{n} \sum_{i=1}^{n} \tau_c \cdot (\rho^3(\theta_i))^\frac{3}{2}$$  \hspace{1cm} (15)

Let $n \rightarrow \infty$, then Eq.(15) turns into:

$$\alpha = \tau_{av} \cdot s_0^\frac{3}{2} = \frac{\sqrt{\pi}}{2} \int_0^{2\pi} \rho^3(\theta)d\theta$$  \hspace{1cm} (16)

According to Eq.(16), the average electric polarizability of the original aperture:

$$\tau_{av} \approx \frac{\sqrt{\pi}}{2s_0^\frac{3}{2}} \int_0^{2\pi} \rho^3(\theta)d\theta \approx \frac{2s_0^\frac{3}{2}}{3\pi} \int_0^{2\pi} \rho^3(\theta)d\theta$$  \hspace{1cm} (17)

In order to express than Eq.(17) is simpler than Fabricant’s equation, the Fabricant’s equation [4] is given as follow,

$$\tau_{av} = \frac{1}{s_0} \int_{0}^{2\pi} \int_{0}^{x(\theta)} w(r,\theta) r dr d\theta$$  \hspace{1cm} (18)

where

$$w(r,\theta) = \frac{\delta}{\rho(\theta)} \left[ \rho^3(\theta) - r^3 \right]^\frac{1}{2}$$  \hspace{1cm} (19)

$$\delta = \frac{4\pi}{\sqrt{s_0}} \sum_{n=1}^{\infty} i^n \int_{0}^{\infty} \frac{x^4}{(r^2 - x^2)^2} \int_{0}^{2\pi} \frac{\rho^3(\theta_i) - x^3}{\rho^3(\theta_i)} \cdot F \left( 2 - \frac{1}{2}, \frac{1}{2}, 1 - \frac{x^2}{\rho^3(\theta_i)}, \exp \left[ i(n(\theta - \theta_i)) \right] \right) d\theta_i$$  \hspace{1cm} (20)
In Eq. (20), the function \( F \left( 2 - \frac{|b|}{2}, 1, 1, \frac{x^2}{\rho^2 (\theta_b)} \right) \) is so called hypergeometric function. The hypergeometric function is very complex and too difficult for one who does not major mathematics. Some results calculated by Eq. (17) are given in Table 2. The relative error is also shown in Table 2. According to Table 2, the results from the proposed method is very close to the Fabricant’s method.

| Aperture                        | Fabricant’s method | Proposed method | Relative error |
|---------------------------------|--------------------|-----------------|----------------|
| Triangle                        | 0.2251             | 0.2147          | 4.62%          |
| Square                          | 0.2357             | 0.2269          | 3.73%          |
| Regular pentagon                | 0.2380             | 0.2298          | 3.45%          |
| Rectangle (the ratio of length to width is 5) | 0.1462             | 0.1421          | 2.80%          |

4. Conclusions

In this paper, a new analytical solution is proposed for solving the average electric polarizability of arbitrary small aperture. This method is very simple compared to the Fabricant’s method due to the proposed method is only need to integrate an elementary function. On the other hand, although the proposed method is very simple, its accuracy is only slightly lower than the Fabricant’s method.

References

[1] Bethe H A 1944 Theory of Diffraction by Small Holes *Phys Rev* 66(7-8): 163-182
[2] Cohn S B 1952 The Electric Polarizability of Apertures of Arbitrary Shape *Proceedings of the Ire* 40(9): 1069-1071
[3] Okon E E, Harrington R F 1981 The Polarizabilities of Electrically Small Apertures of Arbitrary Shape *IEEE Transactions on Electromagnetic Compatibility* EMC-23(4): 359-366
[4] Fabrikant V I 1987 Electrical polarizability of small apertures: analytical approach *International Journal of Electronics* 62(4): 533-545
[5] Collin R 1960 Field Theory of Guided Waves *Field theory of guided waves* McGraw-Hill 50-5