On Pre-generalized c*-homeomorphisms in topological spaces

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1. Introduction

Norman Levine introduced the concept of semi-continuous function in 1963. In 1980, Jain introduced totally continuous functions. In 2011, S.S. Benchalli and Umadevi I Neeli introduced the concept of semi-totally continuous functions in topological spaces. H. Maki et al. introduced and investigated generalized homeomorphisms and gc-homeomorphisms. R. Devi et al. introduced and studied semi-generalized homeomorphisms and generalized semi-homeomorphisms. In this paper, we introduce pre-generalized c*-homeomorphisms in topological spaces and study their basic properties.

Section 2 deals with the preliminary concepts. In section 3, pre-generalized c*-homeomorphisms in topological spaces are introduced and their basic properties are studied.

2. Preliminaries :

Throughout this paper X denotes a topological space on which no separation axiom is assumed. For any subset A of X, cl(A) denotes the closure of A, int(A) denotes the interior of A, pcl(A) denotes the pre-closure of A and bcl(A) denotes the b-closure of A. Further X\A denotes the complement of A in X. The following definitions are very useful in the subsequent sections.

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Definition: 2.1 A subset $A$ of a topological space $X$ is called

i. a semi-open set if $A \subseteq \text{cl}(\text{int}(A))$ and a semi-closed set if $\text{int}(\text{cl}(A)) \subseteq A$.

ii. a pre-open set if $A \subseteq \text{int}(\text{cl}(A))$ and a pre-closed set if $\text{cl}(\text{int}(A)) \subseteq A$.

Definition: 2.2 A subset $A$ of a topological space $X$ is said to be a $c^*$-open set if $\text{int}(\text{cl}(A)) \subseteq A \subseteq \text{cl}(\text{int}(A))$.

Definition: 2.3 A subset $A$ of a topological space $X$ is called

i. a generalized pre-regular closed set (briefly, gpr-closed) if $\text{pcl}(A) \subseteq H$ whenever $A \subseteq H$ and $H$ is regular-open in $X$.

ii. a weakly closed set (briefly, w-closed) (equivalently, $\tilde{g}$ -closed) if $\text{cl}(A) \subseteq H$ whenever $A \subseteq H$ and $H$ is semi-open in $X$.

The complements of the above mentioned closed sets are their respectively open sets.

Definition: 2.4 A subset $A$ of a topological space $X$ is said to be a generalized $c^*$-closed set (briefly, gc*-closed set) if $\text{cl}(A) \subseteq H$ whenever $A \subseteq H$ and $H$ is $c^*$-open. The complement of the gc*-closed set is gc*-open.

Definition: 2.5 A subset $A$ of a topological space $X$ is said to be a pre-generalized $c^*$-closed set (briefly, pgc*-closed set) if $\text{pcl}(A) \subseteq H$ whenever $A \subseteq H$ and $H$ is $c^*$-open. The complement of the pgc*-closed set is pgc*-open.

Definition: 2.6 A function $f : X \to Y$ is called

i. totally-continuous if the inverse image of every open subset of $Y$ is clopen in $X$.

ii. strongly-continuous if the inverse image of every subset of $Y$ is clopen subset of $X$.

iii. semi-totally continuous if the inverse image of every semi-open subset of $Y$ is clopen in $X$.

iv. gpr-continuous if inverse image of every closed subset of $Y$ is gpr-closed in $X$.

v. w-continuous (equivalently, $\tilde{g}$ -continuous) if inverse image of every closed subset of $Y$ is w-closed in $X$.

Definition: 2.7 A function $f : X \to Y$ is said to be a $\tilde{g}$ -open map if $f(U)$ is $\tilde{g}$ -open in $Y$ for every open set $U$ of $X$.

Definition: 2.8 A function $f : X \to Y$ is said to be a generalized $c^*$-open (briefly, gc*-open) map if $f(U)$ is gc*-open in $Y$ for every open set $U$ of $X$.

Definition: 2.9 A function $f : X \to Y$ is said to be a pre-generalized $c^*$-open (briefly, pgc*-open) map if $f(U)$ is pgc*-open in $Y$ for every open set $U$ of $X$.

Definition: 2.10 Let $X$ and $Y$ be two topological spaces. A function $f : X \to Y$ is called a generalized $c^*$-continuous (briefly, gc*-continuous) function if $f^{-1}(V)$ is gc*-closed in $X$ for every closed set $V$ of $Y$.

Definition: 2.11 Let $X$ and $Y$ be two topological spaces. A function $f : X \to Y$ is called a pre-generalized $c^*$-continuous (briefly, pgc*-continuous) function if $f^{-1}(V)$ is pgc*-closed in $X$ for every closed set $V$ of $Y$.

Definition: 2.12 A bijective function $f : X \to Y$ is called a $\tilde{g}$ -homeomorphism if $f$ is both $\tilde{g}$ -continuous and $\tilde{g}$ -open.

Definition: 2.13 A bijective function $f : X \to Y$ is said to be generalized $c^*$-homeomorphism (briefly, gc*-homeomorphism) if $f$ is both gc*-continuous and gc*-open map.
3. Pre-generalized c*-homeomorphisms:

In this section, we introduce pre-generalized c*-homeomorphisms and study their basic properties.

**Definition:** 3.1 A bijective function $f : X \rightarrow Y$ is said to be pre-generalized c*-homeomorphism (briefly, pgc*-homeomorphism) if $f$ is both pgc*-continuous and pgc*-open map.

**Example:** 3.2 Let $X = \{a,b,c\}$ and $Y = \{1,2,3\}$. Then, clearly $\tau = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ is a topology on $X$ and $\sigma = \{\emptyset, \{1\}, Y\}$ is a topology on $Y$. Define $f : X \rightarrow Y$ by $f(a) = 1$, $f(b) = 3$, $f(c) = 2$. Then $f$ is both pgc*-continuous and pgc*-open map. Therefore, $f$ is a pgc*-homeomorphism.

**Proposition:** 3.3 Let $X, Y$ be topological spaces. Then every homeomorphism is a pgc*-homeomorphism.

**Proof:** Let $f : X \rightarrow Y$ be a homeomorphism. Then $f$ is both continuous and open map. By Proposition 3.4[10], $f$ is pgc*-continuous and by Proposition 4.4[9], $f$ is a pgc*-open map. Therefore, $f$ is pgc*-homeomorphism.

The converse of Proposition 3.3 need not be true which can be verified from the following example.

**Example:** 3.4 In Example 3.2, the image of the open set $\{b\}$ in $X$ is $\{3\}$, which is not open in $Y$. Therefore, $f$ is not homeomorphism.

**Proposition:** 3.5 Let $X$ be a topological space. Then every $\hat{g}$-homeomorphism is a pgc*-homeomorphism.

**Proof:** Let $f : X \rightarrow Y$ be a $\hat{g}$-homeomorphism. Then $f$ is both $\hat{g}$-continuous and $\hat{g}$-open map. By Proposition 3.4 [10], $f$ is pgc*-continuous. Also, by Proposition 4.6 [9], $f$ is a pgc*-open map. Therefore, $f$ is pgc*-homeomorphism.

The converse of Proposition 3.5 need not be true as seen from the following example.

**Example:** 3.6 In Example 3.2, the function $f : X \rightarrow Y$ is a pgc*-homeomorphism. But the inverse image of the closed set $\{2,3\}$ in $Y$ under $f$ is $\{b,c\}$, which is not a $\hat{g}$-closed set in $X$. Therefore, $f$ is not a $\hat{g}$-continuous function. Hence $f$ is not a $\hat{g}$-homeomorphism.

**Proposition:** 3.7 Let $X$ be a topological space. Then every gc*-homeomorphism is a pgc*-homeomorphism.

**Proof:** Let $f : X \rightarrow Y$ be a gc*-homeomorphism. Then $f$ is both gc*-continuous and gc*-open map. By Proposition 3.4[10], $f$ is pgc*-continuous. Since every gc*-open map is pgc*-open map, we have $f$ is a pgc*-open map. Therefore, $f$ is a pgc*-homeomorphism.

The following example shows that the converse of Proposition 3.7 need not be true.

**Example:** 3.8 Let $X = \{a,b,c,d,e\}$ and $Y = \{1,2,3,4,5\}$. Then, clearly $\tau = \{\emptyset, \{a,b\}, \{c,d\}, \{a,b,c,d\}, X\}$ is a topology on $X$ and $\sigma = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}, Y\}$ is a topology on $Y$. Define $f : X \rightarrow Y$ by $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, $f(d) = 4$, $f(e) = 5$. Then $f$ is a pgc*-homeomorphism. But $f$ is not a gc*-homeomorphism, since the inverse image of the closed set $\{4\}$ in $Y$ under $f$ is $\{d\}$, which is not a gc*-closed set in $X$. Therefore, $f$ is not a gc*-homeomorphism.

The composition of two pgc*-homeomorphisms need not be a pgc*-homeomorphism. For example, let $X = \{a,b,c\}$, $Y = \{1,2,3\}$ and $Z = \{p,q,r\}$. Then, clearly $\tau = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ is a topology on $X$ and $\sigma = \{\emptyset, \{1\}, Y\}$ is a topology on $Y$. Define $f : X \rightarrow Y$ by $f(a) = 1$, $f(b) = 3$, $f(c) = 2$ and define $g : Y \rightarrow Z$ by $g(1) = q$, $g(2) = p$, $g(3) = r$. Then $f$ and $g$ are pgc*-homeomorphisms. Consider the closed set $\{r\}$ in $Z$. Then $(g \circ f)^{-1}(\{r\}) = f^{-1}(g^{-1}(\{r\})) = f^{-1}(\{3\}) = \{b\}$, which is not a pgc*-closed set in $X$. Therefore, $g \circ f$ is not a pgc*-homeomorphism.

**Proposition:** 3.9 Let $X, Y, Z$ be topological spaces. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are homeomorphisms, then $g \circ f : X \rightarrow Z$ is a pgc*-homeomorphism.

**Proof:** Assume that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are homeomorphisms. Then $f$ and $g$ are both continuous and
open maps. By Proposition 3.10, \( g \circ f \) is a pgc*-continuous function. Also, by Proposition 4.9, \( g \circ f \) is a pgc*-open map. Hence \( g \circ f \) is a pgc*-homeomorphism.

**Proposition: 3.10** Let \( X, Y \) be topological spaces. If \( f : X \rightarrow Y \) is strongly continuous and image of every subset of \( X \) is a clopen subset of \( Y \), then \( f \) is pgc*-homeomorphism.

**Proof:** Let \( f : X \rightarrow Y \) be a strongly continuous function. Then by Proposition 3.4, \( f \) is a pgc*-continuous function. Now, let \( U \) be a open set in \( X \). By our assumption, \( f(U) \) is a clopen in \( Y \). By Proposition 3.7, \( f(U) \) is gc*-open in \( Y \). This implies, \( f(U) \) is pgc*-open in \( Y \). Therefore, \( f \) is a pgc*-open map. Hence \( f \) is a pgc*-homeomorphism.

**Proposition: 3.11** Let \( X, Y \) be topological spaces. If \( f : X \rightarrow Y \) is a semi-totally continuous function and image of every semi-open subset of \( X \) is clopen in \( Y \), then \( f \) is pgc*-homeomorphism.

**Proof:** Let \( f : X \rightarrow Y \) be a semi-totally continuous function. Then by Proposition 3.4, \( f \) is a pgc*-continuous function. Now, let \( U \) be a open set in \( X \). Then \( U \) is semi-open in \( X \). By our assumption, \( f(U) \) is a clopen in \( Y \). By Proposition 3.7, \( f(U) \) is gc*-open in \( Y \). This implies, \( f(U) \) is pgc*-open in \( Y \). Therefore, \( f \) is a pgc*-open map. Hence \( f \) is a pgc*-homeomorphism.

**Proposition: 3.12** Let \( X, Y \) be topological spaces. If \( f : X \rightarrow Y \) is a totally continuous function and image of every open subset of \( X \) is clopen in \( Y \), then \( f \) is pgc*-homeomorphism.

**Proof:** Let \( f : X \rightarrow Y \) be a totally continuous function. Then by Proposition 3.4, \( f \) is a pgc*-continuous function. Now, let \( U \) be a open set in \( X \). By our assumption, \( f(U) \) is a clopen in \( Y \). By Proposition 3.7, \( f(U) \) is gc*-open in \( Y \). This implies, \( f(U) \) is pgc*-open in \( Y \). Therefore, \( f \) is a pgc*-open map. Hence \( f \) is a pgc*-homeomorphism.

**Proposition: 3.13** Let \( X, Y \) be topological spaces. If \( f : X \rightarrow Y \) is a pgc*-homeomorphism, then \( f \) is gpr-continuous and image of every closed subset of \( X \) is gpr-closed in \( Y \).

**Proof:** Assume that \( f \) is a pgc*-homeomorphism. Then \( f \) is both pgc*-continuous and pgc*-open map. Then by Proposition 3.6, \( f \) is gpr-continuous. Now, let \( V \) be a closed set in \( Y \). Since \( f \) is a pgc*-open map, by Proposition 4.3, \( f(V) \) is a pgc*-closed set in \( Y \). Therefore, by Proposition 3.15, \( f(V) \) is gpr-closed in \( X \). Hence the proof.

**Proposition: 3.14** Let \( X, Y \) be a topological space. A bijective function \( f : X \rightarrow Y \) is a pgc*-homeomorphism if and only if \( f \) is pgc*-continuous and \( f^{-1} : Y \rightarrow X \) is pgc*-continuous.

**Proof:** Assume that \( f \) is a pgc*-homeomorphism. Then \( f \) is pgc*-continuous and pgc*-open map. By Proposition 3.8, \( f^{-1} : Y \rightarrow X \) is a pgc*-continuous function. Conversely, assume that \( f \) is pgc*-continuous and \( f^{-1} \) is pgc*-continuous. Then by Proposition 3.8, \( f : X \rightarrow Y \) is a pgc*-open map. Hence \( f \) is a pgc*-homeomorphism.

**Proposition: 3.15** Let \( X, Y \) be topological spaces. If \( f : X \rightarrow Y \) is pgc*-homeomorphism and \( g : Y \rightarrow Z \) is totally-continuous and if \( g(U) \) is pgc*-open for every pgc*-open set \( U \) in \( Y \), then \( g \circ f : X \rightarrow Z \) is pgc*-homeomorphism.

**Proof:** Let \( V \) be an open set in \( Z \). Then \( g^{-1}(V) \) is clopen in \( Y \). This implies, \( g^{-1}(V) \) is open in \( Y \). Since \( f \) is pgc*-continuous, we have \( f^{-1}(g^{-1}(V)) \) is pgc*-open. That is, \( (g \circ f)^{-1}(V) \) is pgc*-open in \( X \). Therefore, \( g \circ f \) is pgc*-continuous. Let \( U \) be an open set in \( X \). Then \( f(U) \) is pgc*-open in \( Y \). This implies, \( g(f(U)) \) is pgc*-open in \( Z \). That is, \( (g \circ f)(U) \) is pgc*-open in \( Z \). Therefore, \( g \circ f \) is pgc*-open map. Hence \( g \circ f \) is pgc*-homeomorphism.
**Proposition: 3.16** Let $X, Y$ and $Z$ be topological spaces. If $f : X \to Y$ is pgc*-homeomorphism and $g : Y \to Z$ is semi-totally continuous and if $g(U)$ is pgc*-open for every pgc*-open set $U$ in $Y$, then $g \circ f : X \to Z$ is pgc*-homeomorphism.

**Proof:** Let $V$ be an open set in $Z$. Then $V$ is semi-open in $Z$. This implies, $g^{-1}(V)$ is clopen in $Y$. Since $f$ is pgc*-continuous, we have $f^{-1}(g^{-1}(V))$ is pgc*-open. That is, $(g \circ f)^{-1}(V)$ is pgc*-open in $X$. Therefore, $g \circ f$ is pgc*-continuous. Let $U$ be an open set in $X$. Then $f(U)$ is pgc*-open in $Y$. This implies, $g(f(U))$ is pgc*-open in $Z$. That is, $(g \circ f)(U)$ is pgc*-open in $Z$. Therefore, $g \circ f$ is pgc*-open map. Hence $g \circ f$ is pgc*-homeomorphism.

**Proposition: 3.17** Let $X, Y$ and $Z$ be topological spaces. If $f : X \to Y$ is both open and strongly-continuous and $g : Y \to Z$ is pgc*-homeomorphism, then $g \circ f : X \to Z$ is pgc*-homeomorphism.

**Proof:** Let $V$ be an open set in $Z$. Then $g^{-1}(V)$ is pgc*-open in $Y$. Since $f$ is strongly-continuous, we have $f^{-1}(g^{-1}(V))$ is clopen in $X$. That is, $(g \circ f)^{-1}(V)$ is pgc*-open in $X$. Therefore, $g \circ f$ is pgc*-continuous. Let $U$ be an open set in $X$. Then $f(U)$ is open in $Y$. This implies, $g(f(U))$ is pgc*-open in $Z$. That is, $(g \circ f)(U)$ is pgc*-open in $Z$. Therefore, $g \circ f$ is pgc*-open map. Hence $g \circ f$ is pgc*-homeomorphism.

**Conclusion**

In this paper we have introduced pgc*-homeomorphisms in topological spaces. Also, we have studied the relationship between pgc*-homeomorphism and other continuous functions already exist.

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