FERMIONIC FORM AND BETTI NUMBERS

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Abstract. We state a conjectural relationship between the fermionic form [HKOTY] and the Betti numbers of a Grassmannian over a preprojective algebra or, equivalently, of a lagrangian quiver variety.

1. Notation. We fix a graph of type $ADE$ with set of vertices $I$. Let $E$ be a $\mathbb{R}$-vector space with a basis $(\alpha_i)_{i \in I}$ and a positive definite symmetric bilinear form $(,): E \times E \rightarrow \mathbb{R}$ given by $(\alpha_i, \alpha_i) = 2$, $(\alpha_i, \alpha_j) = -1$ if $i, j$ are joined in the graph, $(\alpha_i, \alpha_j) = 0$ if $i \neq j$ are not joined in the graph. Let $(\varpi_i)_{i \in I}$ be the basis of $E$ defined by $(\varpi_i, \alpha_j) = \delta_{i,j}$. For $\xi \in E$ define $i^{\xi}, i^{-\xi}$ in $\mathbb{R}$ by

$$\xi = \sum_i (i^{\xi}) \varpi_i = \sum_i (i^{-\xi}) \alpha_i.$$  

Let $P = \{\xi \in E| i^{\xi} \in \mathbb{Z} \ \forall i \in I\}$, $P^+ = \{\xi \in E| i^{\xi} \in \mathbb{N} \ \forall i \in I\}$. Let $\rho = \sum_i \varpi_i \in P^+$. We consider the usual partial order on $P$:

$$\xi \leq \xi' \Leftrightarrow i^{\xi'} - i^{\xi} \in \mathbb{N} \text{ for all } i.$$  

For $i \in I$ define $s_i : E \rightarrow E$ by $s_i(\xi) = \xi - (\xi, \alpha_i) \alpha_i$. Let $W$ be the (finite) subgroup of $GL(E)$ generated by $\{s_i| i \in I\}$. Let $\mathbb{Z}[P]$ be the group ring of $P$ with obvious basis $(\xi)_{\xi \in P}$. For $\xi \in P^+$ define $V_\xi \in \mathbb{Z}[P]$ by Weyl’s formula

$$\sum_{w \in W} \det(w)[w(\xi + \rho)] = V_\xi \sum_{w \in W} \det(w)[w(\rho)].$$  

2. The fermionic form [HKOTY]. Let $q$ be an indeterminate. For $p, m \in \mathbb{N}$ define

$$\begin{bmatrix} p+m \\ m \end{bmatrix} = \frac{(q^{p+1} - 1)(q^{p+2} - 1) \ldots (q^{p+m} - 1)}{(q-1)(q^2-1) \ldots (q^m-1)} \in \mathbb{Z}[q].$$  

Let $\nu = \{\nu^{(i)}_k \in \mathbb{N}| i \in I, k \geq 1\}$ where all but finitely many $\nu^{(i)}_k$ are zero. Let $\lambda \in P^+$. In [HKOTY, 4.3] a “fermionic form” $M(\nu, \lambda, q)$ (or $M(W, \lambda, q)$ in the

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notation of \textit{loc.cit.} is attached to $\nu, \lambda$. This is a $q$-analogue of an expression which first appeared in Kirillov and Reshetikhin [KR]. For $q = 1$ it conjecturally gives the multiplicities in certain representations of an affine quantum group when restricted to the ordinary quantum group. In [Kl], Kleber rewrites the formula of [KR] in the form of a computationally efficient algorithm. (In his paper, it is assumed that one of the $\nu_k^{(i)}$ is 1 and the other s are 0 but, as he pointed out to me, the same procedure works in general for $q = 1$.)

In the remainder of this note we assume that $\nu_k^{(i)} = 0$ for $i \in I$, $k \geq 2$.

In this case we identify $\nu$ with the element of $P^+$ such that $i \nu = \nu_1^{(i)}$ for all $i$. By definition,

\begin{equation}
(a) \quad M(\nu, \lambda, q) = \sum_{\{m\}} q^{c(\{m\})} \prod_{i \in I, k \geq 1} \left[ \frac{p_k^{(i)} + m_k^{(i)}}{m_k^{(i)}} \right],
\end{equation}

\begin{align*}
c(\{m\}) &= \frac{1}{2} \sum_{i, j \in I} (\alpha_i, \alpha_j) \sum_{k, l \geq 1} \min(k, l) m_k^{(i)} m_l^{(j)} - \sum_{i \in I} \sum_{k \geq 1} \nu m_k^{(i)}, \\
p_k^{(i)} &= i \nu - \sum_{j \in I} (\alpha_i, \alpha_j) \sum_{l \geq 1} \min(k, l) m_l^{(j)},
\end{align*}

where the sum $\sum_{\{m\}}$ is taken over $\{m_k^{(i)} \in \mathbb{N} | i \in I, k \geq 1\}$ satisfying $p_k^{(i)} \geq 0$ for $i \in I, k \geq 1$ and $\sum_{i \in I} \sum_{k \geq 1} \sum_{l \geq 1} k m_k^{(i)} \alpha_i = \nu - \lambda$ for $i \in I$. We rewrite this by extending the method of [Kl] to the $q$-analogue; we obtain

\begin{equation}
(b) \quad M(\nu, \lambda, q) = \sum_{\omega} q^{c(\omega)} \prod_{i \in I, k \geq 1} \left[ \frac{i \omega_k + i \mu_k}{i \mu_k} \right]
\end{equation}

sum over all sequences $\omega$ in $P^+$ of the form

$$\nu = \omega_0 > \omega_1 > \omega_2 > \cdots > \omega_s = \omega_{s+1} = \omega_{s+2} = \cdots = \lambda$$

such that

$$\omega_0 - \omega_1 \geq \omega_1 - \omega_2 \geq \omega_2 - \omega_3 \geq \cdots$$

that is,

$$\mu_k = \omega_{k-1} - 2 \omega_k + \omega_{k+1} \geq 0 \text{ for } k \geq 1, \quad \mu_k = 0 \text{ for } k \gg 0,$$
and
\[ c(\omega) = \frac{1}{2} \sum_{k \geq 1} (X_k, X_k) - (\nu, X_1) \]

where
\[ X_k = \omega_{k-1} - \omega_k \text{ for } k \geq 1. \]
The connection between (a) and (b) is as follows: in terms of the data in (a) we have
\[ \omega_k = \sum_i p_k^{(i)} \omega_i, \quad \mu_k = \sum_i m_k^{(i)} \alpha_i. \]
Since \( \mu_k = X_k - X_{k+1} \) for \( k \geq 1 \), we have for \( i, j \in I \):
\[
\sum_{k,l \geq 1} \min(k,l)m_k^{(i)}m_l^{(j)} = \sum_{k,l \geq 1} \min(k,l)(iX_k - jX_{k+1})(jX_l - jX_{l+1})
\]
\[
= \sum_{k,l \geq 1} \min(k,l)(iX_k(jX_l) - iX_{k+1}(jX_l) - iX_k(jX_{l+1}) + iX_{k+1}(jX_{l+1}))
\]
\[
= \sum_{k,l \geq 1} (\min(k,l) - \min(k-1,l) - \min(k,l-1) + \min(k-1,l-1))iX_k(jX_l)
\]
\[
= \sum_{k \geq 1} iX_k(jX_k),
\]
and
\[
\sum_{k \geq 1} i\nu m_k^{(i)} = \sum_{k \geq 1} i\nu(iX_k - iX_{k+1}) = (^i\nu)(iX_1),
\]
hence \( c(\{m\}) = c(\omega) \).
The following result is stated without proof in [HKOTY].

**Lemma 3.** \( M(\nu, \lambda, q) \in \mathbb{N}[q^{-1}] \).

Let \( \omega \) be as in Sec.2. The product of \( q \)-binomial coefficients in the term corresponding to \( \omega \) is a polynomial in \( q \) of degree
\[
N = \sum_{i \in I; k \geq 1} ^i\omega_k(\mu_k) = \sum_{k \geq 1} (\omega_k, \mu_k)
\]
\[
= \sum_{k \geq 1} (\nu - X_1 - X_2 - \cdots - X_k, X_k - X_{k+1}) = (\nu, X_1) - \sum_{k \geq 1} (X_k, X_k).
\]
It is enough to show that \( c(\omega) + N \leq 0 \). We have
\[
c(\omega) + N = -\frac{1}{2} \sum_{k \geq 1} (X_k, X_k)
\]
and this is clearly \( \leq 0 \).
Lemma 4. Let $\xi \in P^+$ and let $\eta \in P$ be such that $\eta \geq 0$. Then $(\xi, \eta) \geq 0$.

This is obvious.

Lemma 5. If $\nu \geq \lambda$ then $M(\nu, \lambda, q) = q^{-(\nu, \nu)/2 + (\lambda, \lambda)/2} +$ strictly larger powers of $q$.

Let $\omega$ be as in Sec.2. We show that

(a) $c(\omega) \geq -(\nu, \nu)/2 + (\lambda, \lambda)/2$

that is,

$$\frac{1}{2} \sum_{k \geq 1} (X_k, X_k) - (\nu, X_1) - \frac{1}{2} (\nu - \lambda, \nu - \lambda) + (\nu, \nu - \lambda) \geq 0.$$ 

Since $\nu - \lambda = X_1 + X_2 + X_3 + \ldots$, this is equivalent to

(b) $(\nu, X_2 + X_3 + \ldots) - \sum_{1 \leq k < l} (X_k, X_l) \geq 0.$

Applying Lemma 4 to $\xi = \nu - X_1 - X_2 - \cdots - X_{k+1} = \omega_{k+1}$, $\eta = X_{k+1}$, we obtain

$$(\nu - X_1 - X_2 - \cdots - X_{k+1}, X_{k+1}) \geq 0.$$ 

Adding these inequalities over all $k \geq 1$ we obtain

$$\sum_{k \geq 1} (\nu - X_1 - X_2 - \cdots - X_k - X_{k+1}, X_{k+1}) \geq 0,$$

that is,

$$(\nu, X_2 + X_3 + \ldots) - \sum_{1 \leq k < l \geq 1} (X_k, X_l) \geq \sum_{k \geq 2} (X_k, X_k) \geq 0.$$ 

Thus, (b) hence (a) are proved. This proof shows also that the inequality (a) is strict unless $\omega$ satisfies $X_2 = X_3 = \cdots = 0$. If this last condition is satisfied then $\omega$ is the sequence $\nu = \omega^0 \geq \omega^1 = \omega^2 = \cdots = \lambda$ and (a) is an equality. The lemma is proved.

6. Inversion. Define $M^*(\nu, \lambda, q) \in \mathbf{Z}[q^{-1}]$ for any $\nu, \lambda \in P^+$ by the requirement that the matrix $(M^*(\nu, \lambda, q))_{\mu, \lambda}$ is inverse to the matrix $(M(\nu, \lambda, q))_{\mu, \lambda}$ (which is lower triangular with 1 on diagonal). Thus, $M^*(\nu, \nu, q) = 1$ and $\sum_{\lambda \in P^+} M^*(\nu, \lambda, q)M(\lambda, \xi, q) = 0$ for any $\nu > \xi$ in $P^+$. There is some evidence that the matrix $M^*$ is simpler than $M$. For example, in type $A_1$, we have

$$M^*(\nu, \lambda, 1) = (-1)^{(\nu - \lambda)} \binom{i(\lambda + i(\nu - \lambda))}{i(\nu - \lambda)}$$

for any $\nu \geq \lambda$ in $P^+$. 

7. Path algebra. Let $I$ be the set of all sequences $i_1, i_2, \ldots, i_s$ (with $s \geq 1$) in $I$ such that $i_k, i_{k+1}$ are joined for any $k \in [1, s - 1]$. Let $\mathcal{F}$ be the $\mathbb{C}$-vector space spanned by elements $[i_1, i_2, \ldots, i_s]$ corresponding to the various elements of $I$. We regard $\mathcal{F}$ as an algebra in which the product $[i_1, i_2, \ldots, i_s][j_1, j_2, \ldots, j_s]$ is equal to $[i_1, i_2, \ldots, i_s, j_1, j_2, \ldots, j_s]$ if $i_s = j_1$ and is zero, otherwise. For $i \in I$, let $\vartheta_i = \sum [ij]$ where $j$ runs over the elements of $I$ that are joined with $i$. For $i, j \in I$ let $\mathcal{F}_{ij}$ be the subspace of $\mathcal{F}$ spanned by the elements $[i_1, i_2, \ldots, i_s]$ with $i_1 = i, i_s = j$. For $u \in \mathbb{Z}$ let $\mathcal{F}^u$ be the subspace of $\mathcal{F}$ spanned by the elements $[i_1, i_2, \ldots, i_s]$ with $s = u + 1$. (For $u < 0$ we have $\mathcal{F}^u = 0$.) Let $\mathcal{F}_{ij}^u = \mathcal{F}_{ij} \cap \mathcal{F}^u$. We have
\[
\mathcal{F} = \oplus_{i,j} \mathcal{F}_{ij}, \mathcal{F} = \oplus_u \mathcal{F}^u, \mathcal{F} = \oplus_{i,j,u} \mathcal{F}_{ij}^u.
\]
Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{F}$ generated by the elements $\vartheta_i$ ($i \in I$). The quotient algebra $\mathbb{P} = \mathcal{F}/\mathcal{I}$ has finite dimension over $\mathbb{C}$ [GP]. Let $\mathbb{P}_{ij}, \mathbb{P}^u, \mathbb{P}_{ij}^u$ be the image of $\mathcal{F}_{ij}, \mathcal{F}^u, \mathcal{F}_{ij}^u$ in $\mathbb{P}$. We have
\[
\mathbb{P} = \oplus_{i,j} \mathbb{P}_{ij}, \mathbb{P} = \oplus_u \mathbb{P}^u, \mathbb{P} = \oplus_{i,j,u} \mathbb{P}_{ij}^u.
\]
Let $\mathbb{D}$ a finite dimensional $\mathbb{C}$-vector with a given direct sum decomposition $\mathbb{D} = \oplus_{i \in I} \mathbb{D}_i$. Then $\mathbb{D}^\dagger = \oplus_{i,j} \mathbb{P}_{ij} \otimes \mathbb{D}_j$ is a left $\mathbb{P}$-module in an obvious way (in fact a projective $\mathbb{P}$-module of finite dimension over $\mathbb{C}$). Let $\nu = \sum_{i \in I} \dim \mathbb{D}_i \varpi_i \in P^+$. Let Grass$_\mathbb{P}(\mathbb{D}^\dagger)$ be the algebraic variety consisting of all $\mathbb{P}$-submodules of $\mathbb{D}^\dagger$. We have a partition
\[
\text{Grass}_\mathbb{P}(\mathbb{D}^\dagger) = \sqcup_{\xi \in \mathbb{P}} \text{Grass}_\mathbb{P},\xi(\mathbb{D}^\dagger)
\]
where Grass$_\mathbb{P},\xi(\mathbb{D}^\dagger)$ consists of all $\mathbb{P}$-submodules $\mathcal{V}$ such that $\sum_i \dim([i]^{\mathbb{D}^\dagger}/[i]^{\mathcal{V}}) \alpha_i = \nu - \xi$. Then

**Conjecture A.** Let $q^{1/2}$ be an indeterminate. For any $\xi \in \mathbb{P}$ we have

\[
(a) \quad \sum_{s \in \mathbb{N}} \dim H^s(\text{Grass}_\mathbb{P},\xi(\mathbb{D}^\dagger)) q^{s/2} = \sum_{\lambda \in \mathcal{P}_+} (\xi : V_\lambda) q^{(\nu, \nu)/2 - (\xi, \xi)/2} M(\nu, \lambda, q)
\]

where $(\xi : V_\lambda)$ is the coefficient in $\xi$ in $V_\lambda$ and $H^s()$ denotes ordinary cohomology with coefficients in a field.

Since $(\xi : V_\lambda)$ and $(\xi, \xi)$ are $W$-invariant in $\xi$, we see that the right hand side of (a) is $W$-invariant in $\xi$. The analogous property of the left hand side of (a) is known to be true. (See [L2].)

In [L1] it is shown that Grass$_\mathbb{P},\xi(\mathbb{D}^\dagger)$ is isomorphic to a (lagrangian) quiver variety defined in Nakajima [NA] and, conversely, all such quiver varieties are obtained. Thus the conjecture above gives at the same time the Betti numbers of quiver varieties.

Assuming the conjecture, we show that Grass$_\mathbb{P},\xi(\mathbb{D}^\dagger)$ is connected if $(\xi : V_\nu) \neq 0$ (an expected but not yet proved property of quiver varieties). We may assume
that $\xi \in P^+, \xi \leq \nu$. It suffices to show that $\dim H^0(\text{Grass}_{\mathbf{P}, \xi}(\mathbf{D}^\dagger)) = 1$ or that the constant term of
\[ \sum_{\lambda \in P^+} (\xi : V_\lambda)q^{(\nu, \nu)/2 - (\xi, \xi)/2}M(\nu, \lambda, q) \]
is 1. By Lemma 5, the constant term of the term corresponding to $\lambda = \xi$ is 1. Consider now the term corresponding to $\lambda \neq \xi$; we show that its constant term is 0. We may assume that $\lambda \leq \nu$ and $(\xi : V_\nu) \neq 0$ so that $\xi < \lambda$. By Lemma 5, $q^{(\nu, \nu)/2 - (\xi, \xi)/2}M(\nu, \lambda, q)$ is of the form
\[ q^{(\nu, \nu)/2 - (\xi, \xi)/2} - q^{(\nu, \nu)/2 + (\lambda, \lambda)/2} \]
strictly larger powers of $q$.

Since $(\xi, \xi) < (\lambda, \lambda)$ for any $\lambda, \xi$ in $P^+$ such that $\xi < \lambda$, our assertion is established.

The same argument shows that the right hand side of (a) satisfies the conjecture, since the dimension of Grass$_{\mathbf{P}, \xi}(\mathbf{D}^\dagger)$ is known to be equal to $q^{(\nu, \nu)/2 - (\xi, \xi)/2}$.

In the $A_1$ case, the left hand side of (a) is a $q$-binomial coefficient; the right hand side can be computed by results in [Ki],[KSS]; the conjecture holds in this case.

8. The conjecture implies that
\[ (a) \quad \chi(\text{Grass}_{\mathbf{P}}(\mathbf{D}^\dagger)) = \sum_{\lambda \in P^+} \dim(V_\lambda)M(\nu, \lambda, 1) \]
where $\chi$ denotes Euler characteristic and $\dim(V_\lambda) = \sum_{\xi \in P}(\xi : V_\lambda)$. Let $f(\nu)$ (resp. $g(\nu)$) be the left (resp. right) hand side of (a). According to [HKOTY], it is expected that $g(\nu + \nu') = g(\nu)g(\nu')$ for any $\nu, \nu' \in P^+$. The corresponding identity $f(\nu + \nu') = f(\nu)f(\nu')$ is known [L3, 3.20].

9. We can partition $I$ into two disjoint subsets $I^0, I^1$ so that no two vertices in $I^0$ are joined and no two vertices in $I^1$ are joined. For any $u \in \mathbf{Z}$ let
\[ u^\mathbf{D}^\dagger = \bigoplus_{i \in I^0, j \in I}(\mathbf{P}_{ij}^u \oplus \mathbf{P}_{ij}^{u-1}) \otimes \mathbf{D}_j \]
where $\delta \in \{0, 1\}$ is defined by $u = \delta \mod 2$. We have $\mathbf{D}^\dagger = \bigoplus_u u^\mathbf{D}^\dagger$. Consider the $C^*$-action $t, d \mapsto td$ on $\mathbf{D}^\dagger$ with weight $u$ on $u^\mathbf{D}^\dagger$. This action is compatible with the $\mathbf{P}$-module structure in the following sense: $t(pd) = (tp)(td)$ for $t \in C^*, p \in \mathbf{P}, d \in \mathbf{D}^\dagger$ where $tp$ is given by the $C^*$-action on $\mathbf{P}$ (through algebra automorphisms) for which $\mathbf{P}^u$ has weight $u$. Hence we have an induced $C^*$-action on Grass$_{\mathbf{P}, \xi}(\mathbf{D}^\dagger)$ for any $\xi$. Let Grass$'_{\mathbf{P}, \xi}(\mathbf{D}^\dagger)$ be the fixed point set of this $C^*$-action (a smooth variety). It consists of all $\mathcal{V} \in \text{Grass}_{\mathbf{P}, \xi}(\mathbf{D}^\dagger)$ such that $\mathcal{V} = \bigoplus_u (\mathcal{V} \cap u^\mathbf{D}^\dagger)$.
Conjecture B. For any $\xi \in P$ we have

$$\sum_{s \in \mathbb{N}} \dim H^s(\text{Grass}^\prime_{P,\xi}(D^\dagger))q^{s/2} = \sum_{\lambda \in \mathbb{P}^+} (\xi : V_\lambda)M(\nu, \lambda, q^{-1}).$$

One can show that this is equivalent to Conjecture A.

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