A new initial basis for the simplex method combined with the nonfeasible basis method

Kasitinart Sangngern and Aua-aree Boonperm*

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Thailand

s.kasitinart@gmail.com, aua-aree@mathstat.sci.tu.ac.th*

Abstract The simplex method is a popular method to solve a linear programming problem. It starts by choosing an initial basis associated with the origin point which is far from the optimal point in some problems. In this paper, we introduce a construction of an initial basis for starting the simplex method associated with an extreme point which is close to the optimal solution. This construction involves the angle between the coefficient vector of the objective function and the coefficient vector of each constraint. The small angles will be considered to construct a basis. If the proposed initial basis gives the infeasible solutions, then the nonfeasible basis method is performed to avoid the use of artificial variables. Not only the starting point is close to the optimal point, but also the artificial variable is needless. The results in the computational aspect show that the number of iterations and the running time of our method are less than other methods.

1. Introduction

Mathematical programming is a combination of applied mathematics and algorithms to arrive at optimal or near optimal solutions to complex problems. It is the use of mathematical models to achieve the best possible result from a set of feasible points.

A linear programming problem is a mathematical model which its objective function and all constraints are linear. The popular method for solving a linear programming problem is the simplex method introduced by G B Dantzig [1,2]. This method starts with a basic feasible solution and then it goes to an adjacent extreme point until the optimal solution is found. Although the simplex method is the popular method, it performs slowly when the problem size is large. So, there are many issues that many researchers have attempted to improve the algorithm for solving the linear program such as improving an initial basis, proposing a new pivot rule, developing a new technique for removing redundant constraints [3], developing a new method to solve a specific linear programming problem [4, 5], etc.

One of an important issue for improving the simplex algorithm is to choose the initial basic feasible solution. Since the construction of the feasible basis is difficult, many researchers have attempted to develop the technique to construct an initial basis for the simplex algorithm [6, 7, 8]. The well-known techniques are the Big-M method and the Two-Phase method used to construct the feasible basis. However, these techniques need the artificial variables which increase the larger size of the problem causing more iterations and more running time to solve it. Therefore, the artificial-free technique is an interesting issue, and there are many researches related this technique [9, 10, 11].

In 2009, Nabli [12] proposed the new method, called Nonfeasible basis method (NFB), for initializing the simplex method without using artificial variables, and the condensed tableau is...
introduced. Next, in 2015, Nabli and Chahdour [11] improved their method by developing a pivot rule and presented the method to initialize the simplex algorithm. However, their initial basis is associated with the origin point which is far from the optimal solution.

Although the NFB method was proposed for improving the simplex method without using artificial variables, the chosen initial basis $B$ is not recommended. So, in this paper, we propose a new method that initializes a basis close to the optimal solution by considering the constraint which has the angle between its coefficient vectors close to the objective function in the primal problem. If the chosen basis leads to the infeasibility of dual and primal problems then the nonfeasible basis method is applied.

This paper consists of 5 sections. In section 2, we brief about the condensed tableau and the nonfeasible basis method followed by the explanation of our method in section 3. The computational results are shown in section 4, and the last section is the conclusion.

2. Preliminaries
In this paper, we propose a new initial basis for the simplex algorithm with the condensed tableau and the nonfeasible basis method proposed by Nabli [12] described below.

2.1. Condensed tableau
Consider the following linear programming problem:

$$\begin{align*}
\text{max} & \quad Z = c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}$$

where $x, c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$ and $m \leq n$.

Let $A = [B \quad N]$ where $B \in \mathbb{R}^{m \times m}$ which is a nonsingular matrix, $N \in \mathbb{R}^{m \times (n-m)}$, and $x = [x_B \quad x_N]^T$ where $x_B \in \mathbb{R}^m, x_N \in \mathbb{R}^{n-m}$. Then, the equivalent linear programming problem can be rewritten as below.

$$\begin{align*}
\text{max} & \quad Z = (c_N^T - c_B^T B^{-1} N) x_N = c_B^T B^{-1} b \\
\text{s.t.} & \quad I_m x_B + B^{-1} N x_N = B^{-1} b \\
& \quad x_B, x_N \geq 0.
\end{align*}$$

Since the column of nonbasic variables ($x_N$) will be considered for choosing the entering variable without the consideration of the column of basic variable, the column of basic variables ($x_B$) can be removed from the consideration. Therefore, a new tableau called the condensed tableau is introduced instead the original simplex tableau, it can be written as follows:

| $B^{-1} N$ | $B^{-1} b$ | $x_B$ |
| $c_N^T - c_B^T B^{-1} N$ | $-Z$ | $x_N$ |

The simplex method with the condensed tableau will be used in our method for reducing the size of the problem in each iteration.

2.2. Nonfeasible Basis method
The nonfeasible basis method (NFB) was proposed to construct the feasible basis without using artificial variables and deals with the condensed tableau. So, the size of a matrix of each problem handled by this method is smaller than the original simplex method. The NFB method starts when the
chosen basis $B$ is an infeasible basis $\left( B^\top b \geq 0 \right)$. This method is guaranteed that a basic feasible solution is obtained, and the step of the algorithm is shown in Algorithm 1.

**Algorithm 1: Nonfeasible Basis method (NFB)**

**Input**: $A, b$ and $c$

**Require**: Infeasible basis $B$

$$ Bb = \min \left\{ \left( B^\top b \right), \forall i \right\} $$

While $Bb < 0$ do

$$ k = \arg \min \left\{ Bb \right\}, \beta = B^\top N \text{ and } K = \left\{ j | \beta_{ij} < 0 \right\} $$

If $K = \emptyset$ then

exit/*the feasible domain is empty*/

end if

$$ s = \arg \min \left\{ \beta_{ij} | j \in K \right\} $$

$$ r = \arg \max \left\{ \left( B^\top b \right), \beta_s \left( B^\top b \right) < 0 \text{ and } \beta_s < 0, i = 1, ..., m \right\} $$

basic($r$) $\leftrightarrow$ nonbasic($s$)

**Apply pivoting**

end while

The current basis is feasible.

Apply the simplex method by the desired pivot rule.

Although the NFB method was proposed for improving the simplex method without using artificial variables, the chosen initial basis $B$ is not recommended. So, if the chosen initial basis $B$ associated with an extreme point which is close to the optimal solution is also important. Therefore, in this paper, we suggest the initial basis $B$ which is close to the optimal solution combined with the NFB method.

3. The proposed method

Let us consider the following linear programming problem

$$ \begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*} \quad (3) $$

where $x, c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

Our idea of this research for suggesting an initial basis is obtained from the observation of the optimal solution binding $n$ constraints. Consider the following figure.

![Figure 1](image.png)

**Figure 1.** Optimal solution formed by two constraints which its vectors is close to vector $c$
From Figure 1, we observed that the optimal solution is formed by the intersection of two constraints which these two constraints have its coefficient vectors that are close in the angle to the coefficient of the objective function. So, if we can choose the basic variables associated with these constraints, then the optimal solution can be obtained rapidly.

From the problem (3), to convert the problem to the standard form, the slack variables are added and named $x_i$, $i = n+1, ..., n+m$. So, the number of variables is $n+m$. For the convenience of the selection a basis, the constraints $Ax \geq b$ are named the constraint $i, i = n+1, ..., n+m$, and the nonnegative constraints are named $i, i = 1, ..., n$. Let $\theta_i$ be the angle between the coefficient vector of constraint $i$ and the coefficient vector of the objective function. Note that $\theta_i$ is related to $x_i$ for all $i$.

In our method, we start by computing all $\theta_i$. Then, all angles are ascending sort. For the first $n$ constraints associated with the first $n \theta_i$, the associated variables are not chosen to construct a basis because if the constraint $i$ is chosen, then its slack variable will be zero. So, its slack variable should be one of a nonbasic variable. Next, we define the indices of nonbasic variables and basic variables sets as $NT = \{\text{the indices of first } n \theta_i\}$ and $BT = \{\text{the indices of last } m \theta_i\}$. The step of our algorithm is shown below.

Algorithm 2 : The proposed method

Input : $A, b$ and $c$

Compute $\theta_i = \cos^{-1}\left(\frac{c \cdot g_i}{\|c\|\|g_i\|}\right)$ where $g_i$ is the $i^{th}$ row of the matrix $G = \begin{bmatrix} A \\ -I \end{bmatrix}$, sort $\theta_i$ and arrange their indices into set $BT$ and $NT$.

Let $x_B = \{x_i | i \in BT\}, x_N = \{x_i | i \in NT\}$, and $w^T_N = c_N^T - c_B^T B^{-1} N$.

If ($B^{-1} b \geq 0$) then

- If ($w^T_N \leq 0$) then
  - the optimal solution is $\begin{bmatrix} x_B^* \\ x_N^* \end{bmatrix}^T = \left[ B^{-1} b \quad 0 \right]^T$.
  - Else
    - apply the simplex algorithm with the condensed tableau.
  - Else
    - apply the dual simplex method with the condensed tableau.
Else

- If ($w^T_N \leq 0$) then
  - apply the dual simplex method with the condensed tableau.
Else

- apply the NFB method.

From Algorithm 2, in the last case, both dual and primal solutions are infeasible. We can use any technique such as Two-phase method or Big-M method. However, these methods need to add the artificial variable. So, we choose the nonfeasible basis method for avoiding the use of artificial variables.

4. Computational experiments

For testing the performance of the proposed method, we generated randomly linear programming problems which have the following conditions:

- all constraints are “≤” and $x \geq 0$,
- the $n \times n$ matrix $A$ where $n \in \{10, 20, ..., 300\}$, $a_{ij}, c_i \in [-9, 9]$ and $x_i \in [0, 9]$.

Then, we computed $b$ from $b = Ax$.

For each dimension, we generated randomly ten problems solved by our method, nonfeasible basis method, Two-phase method and Big-M method. After all problems are solved, we compare the average number of iterations and the running time shown in Table 1 and Figure 2.
### Table 1. The average number of iterations and the running time (10 tested LP per dimension)

| n   | PO  | NFB | TP  | BM  | PO  | NFB | TP  | BM  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 10  | 9.9 | 9.9 | 15.1| 13  | 0.0080 | 0.0083 | 0.2035 | 0.0155 |
| 20  | 22.8 | 27  | 39.2| 28.3| 0.0350 | 0.0476 | 0.1186 | 0.0918 |
| 30  | 38.9 | 44.4| 58.1| 47.2| 0.1080 | 0.1508 | 0.3210 | 0.2469 |
| 40  | 59.7 | 73.8| 85.1| 71.3| 0.2269 | 0.3394 | 0.6646 | 0.5323 |
| 50  | 75.6 | 88.9| 100.2| 88.2| 0.4010 | 0.6026 | 1.2065 | 1.0087 |
| 100 | 193.2 | 221.3| 243.1| 213.4| 3.9327 | 5.2347 | 9.4625 | 8.1219 |
| 150 | 344.8 | 408.2| 428.6| 358.8| 15.5093 | 20.5899 | 33.6265 | 28.4792 |
| 200 | 512  | 664.1| 624.4| 519.4| 40.2669 | 57.6458 | 84.9245 | 71.3555 |
| 250 | 766.2 | 922.5| 803.7| 792.2| 94.4698 | 124.5572 | 174.2017 | 152.9996 |
| 300 | 965.5 | 1151.9| 1021.6| 973.9| 175.0189 | 227.2903 | 314.8729 | 266.8288 |
| Sum | 2988.6 | 3612 | 3419.1| 3105.7| 329.9765 | 436.4666 | 619.602 | 529.6802 |

**Figure 2.** The average number of iterations and the running time of four methods

Table 1 shows the computational results of all experiments. Column ‘PO’ represents the average number of iterations and the running time solved by our method for each dimension, column ‘NFB’ represents the average number of iterations and the running time solved by the nonfeasible basic method, column ‘TP’ represents the average number of iterations and the running time solved by the two-phase method and column ‘BM’ represents the average number of iterations and the running time solved by the Big-M method.

The results show that the average number of iterations of our method is 82.74% of NFB method, 87.41% of two-phase method and 96.23% of Big-M method. Furthermore, the average runtime of our method is 75.60% of NFB method, 53.26% of two-phase method and 62.30% of Big-M method.

From the results, we found that our method is more effective than the other methods for the generated problem.

### 5. Conclusion

In this paper, we propose the construction of an initial basis which might be close to the optimal solution combined with the nonfeasible basis method when the chosen basis is an infeasible basis. Since the proposed basis is close to the optimal solution and the nonfeasible basis method does not involve the artificial variable, the average number of iterations and the running time are less than other methods. For future works, we will attempt to find the optimal solution instead nonfeasible basis method.
References

[1] Danzig G B 1951 Activity analysis of production and allocation (T. Koopmans, Ed.) *Maximization of a linear function of variables subject to linear inequality* p 339-347

[2] Danzig G B 1963 *Linear programming and extensions* New Jersey, Princeton: University

[3] Paulraj S and Sumathi P 2010 A Comparative Study of Redundant Constraints Identification Methods in Linear Programming Problems *Mathematical Problems in Engineering* ID 723402

[4] Gao C, Yan C, Zhang Z, Hu Y and Mahadevan S 2014 An amoeboid algorithm for solving linear transportation problem. *Physica A* p 179-186

[5] Zhang X, Zhang Y, Hu Y, Deng Y and Mahadevan S 2013 An adative amoeba algorithm for constrained shortest paths *Expert Systems with Applications* p 7607-7616

[6] Hu J 2007 A note on "an improved initial basis for the simplex algorithm" *Computer & Operations Research* p 3397-3401

[7] Junior H V and Lins M P 2005 An improved initial basis for the Simplex algorithm *Computer & Operations Research* p 1983-1993

[8] Nabli H and Chahdoura S 2015 Algebraic simplex initialization combined with the nonfeasible basis method *European Journal of Operation Research* p 384-391

[9] Boonperm A and Sinapiromsaran K 2013 The artificial-free technique along the objective direction for the simplex algorithm. *2nd International Conference on Mathematical Modeling in Physical Science 2013* (pp. 10.1088/1742-6596/490/1/012193) Prague Czech Republic: IOP Publishing

[10] Pan P 2000 Primal Perturbation Simplex algorithms for Linear programming *Journal of Computational Mathematics* p 587-596

[11] Zionts S 1969 The Criss-Cross Method for Solving Linear Programming Problems *Management Science* p 426-445

[12] Zhang X, Zhang Y, Hu Y, Deng Y and Mahadevan S 2013 An adative amoeba algorithm for constrained shortest paths *Expert Systems with Applications* p 7607-7616