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Positive families and Boolean chains of copies of ultrahomogeneous structures

Miloš S. Kurilić and Boriša Kuzeljević

Abstract. A family of infinite subsets of a countable set $X$ is called positive iff it is closed under supersets and finite changes and contains a co-infinite set. We show that a countable ultrahomogeneous relational structure $X$ has the strong amalgamation property iff the set $P(X) = \{A \subset X : A \cong X\}$ contains a positive family. In that case the family of large copies of $X$ (i.e. copies having infinite intersection with each orbit) is the largest positive family in $P(X)$, and for each $R$-embeddable Boolean linear order $L$ whose minimum is non-isolated there is a maximal chain isomorphic to $L \setminus \{\text{min } L\}$ in $\langle P(X), \subset \rangle$.

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1. Introduction

The purpose of this short note is twofold. One is to present some new results about positive families. The other one is to provide a natural context for the recent research from [11–13]. For a countably infinite set $X$, a family $\mathcal{P} \subset P(X)$ is called a positive family on $X$ (see [10]) iff

(P1) $\mathcal{P} \subset [X]^\omega$, 
(P2) $\mathcal{P} \ni A \subset B \subset X \Rightarrow B \in \mathcal{P}$, 
(P3) $A \in \mathcal{P} \land |F| < \omega \Rightarrow A \setminus F \in \mathcal{P}$, 
(P4) $\exists A \in \mathcal{P} \land |X \setminus A| = \omega$.

We regard a positive family $\mathcal{P}$ on $X$ as a suborder of the partial order $\langle [X]^\omega, \subset \rangle$ (isomorphic to $\langle [\omega]^\omega, \subset \rangle$) and important examples of positive families are co-ideals: if $\mathcal{I} \subset P(\omega)$ is an ideal containing the ideal $\text{Fin}$ of finite subsets of $\omega$, then the set $\mathcal{I}^+ := P(\omega) \setminus \mathcal{I}$ of $\mathcal{I}$-positive sets is a positive family. Thus $[\omega]^\omega$ is the largest, while non-principal ultrafilters $\mathcal{U} \subset P(\omega)$ are the minimal positive families of this form. Also, $\mathcal{I}_{\text{owd}}^+ = \{A \subset Q : \text{Int } A \neq \emptyset\}$ and $\mathcal{I}_{\text{linz}}^+ = \{A \subset Q : \mu(\overline{A}) > 0\}$ are positive families on the set of rationals $Q$, where $\overline{S}$, $\text{Int } S$ and $\mu(S)$ denote the $\mathbb{R}$-closure, $\mathbb{R}$-interior
and Lebesgue measure of a subset \( S \) of the real line \( \mathbb{R} \) with the standard topology. Taking a non-maximal filter \( \mathcal{F} \subset \mathcal{P}(\omega) \) which extends the Fréchet filter we obtain a positive family which is not a co-ideal; another such example is the family Dense\((\mathbb{Q})\) from Example 7; see also Theorem 5.

In our notation \( \mathcal{P}(\mathcal{X}) = \{ A \subset \mathcal{X} : A \cong \mathcal{X} \} \) denotes the set of all copies of a structure \( \mathcal{X} \) contained in \( \mathcal{X} \). The class of order types of maximal chains in the poset \( (\mathcal{P}(\mathcal{X}), \subset) \) will be denoted by \( \mathcal{M}_{\mathcal{X}} \). Let \( \mathcal{C}_R \) denote the class of order types of sets of the form \( K \setminus \{ \min K \} \), where \( K \subset \mathbb{R} \) is a compact set such that \( \min K \) is an accumulation point of \( K \). Let \( \mathcal{B}_R \) be the subclass of order types from \( \mathcal{C}_R \) for which the corresponding compact set \( K \) is, in addition, nowhere dense. Main results from \([12,13]\) state that for a countable ultrahomogeneous partial order \( \mathcal{P} \)

\[
\mathcal{M}_{\mathcal{P}} = \begin{cases} 
\mathcal{B}_R, & \text{if } \mathcal{P} \text{ is a countable antichain,} \\
\mathcal{C}_R, & \text{otherwise,}
\end{cases}
\]

while for a countable ultrahomogeneous graph \( \mathcal{G} \) we have

\[
\mathcal{M}_{\mathcal{G}} = \begin{cases} 
\mathcal{B}_R, & \text{if } \mathcal{G} \text{ is a disjoint union of complete graphs,} \\
\mathcal{C}_R, & \text{otherwise.}
\end{cases}
\]

These results suggest that there might be a general theorem describing the classes \( \mathcal{M}_{\mathcal{X}} \). The reason for focusing on ultrahomogeneous structures is that \( \mathcal{M}_{\mathcal{X}} \subset \mathcal{C}_R \) for an ultrahomogeneous \( \mathcal{X} \) (see \([13]\) for example). Still, there are pathological structures even in the class of ultrahomogeneous ones. For example, there are ultrahomogeneous structures without non-trivial copies (see \([8, p. 399]\)). This kind of obstruction does not exist in the class of countable ultrahomogeneous relational structures whose age satisfies the strong amalgamation property (SAP). Recall the following equivalence (see \([8, p. 399]\)): a countable ultrahomogeneous relational structure \( \mathcal{X} \) satisfies SAP if and only if \( X \setminus \mathcal{F} \in \mathcal{P}(\mathcal{X}) \), for each finite \( \mathcal{F} \subset X \).

Section 2 contains results about positive families. The central one is that for a countable ultrahomogeneous relational structure \( \mathcal{X} \), there is a positive family \( \mathcal{P} \) on \( X \) such that \( \mathcal{P} \subset \mathcal{P}(\mathcal{X}) \) if and only if the age of \( \mathcal{X} \) satisfies SAP. From this result in Section 3 we deduce that the structures whose age satisfies SAP provide a natural context for investigating the phenomena we have described above.

**Theorem 1.** If \( \mathcal{X} \) is a countable ultrahomogeneous relational structure whose age satisfies SAP, then \( \mathcal{B}_R \subset \mathcal{M}_{\mathcal{X}} \subset \mathcal{C}_R \).

Since the class \( \mathcal{B}_R \) is quite rich, the previous result shows that many linear orders can be realized as maximal chains in \( \mathcal{P}(\mathcal{X}) \) in that case. For example, the reverse of every countable limit ordinal, or the order type of the Cantor set without \( 0 \). Note also that the countable complete graph \( \mathcal{K}_\omega \) satisfies SAP and that \( \mathcal{M}_{\mathcal{K}_\omega} = \mathcal{B}_R \). On the other hand, the Rado graph \( \mathcal{G}_{Rado} \) also satisfies SAP, but \( \mathcal{M}_{\mathcal{G}_{Rado}} = \mathcal{C}_R \). This implies that it is not possible to narrow the interval of possibilities in Theorem 1. However, we do not know an answer to the following question.

**Question 2.** Is there a countable ultrahomogeneous relational structure \( \mathcal{X} \) whose age satisfies SAP, but such that \( \mathcal{B}_R \subset \mathcal{M}_{\mathcal{X}} \subset \mathcal{C}_R \)?

We assume that the reader is familiar with Fraïssé theory. The theory itself was started in \([5–7]\), while a detailed treatment is given in \([8]\). Besides the mentioned book, \([13]\) is a good reference for all undefined notions. We will only comment on the notion of an orbit. Suppose that \( \mathcal{X} \) is a relational structure and \( F \subset X \) finite. We say that \( x \sim_F y \) iff there is \( g \in \text{Aut}(\mathcal{X}) \) such that \( g \setminus F = \text{id}_F \) and \( g(x) = y \). Clearly, \( \sim_F \) is an equivalence relation, and \( \text{orb}_F(x) \) denotes the class of an element \( x \). The sets \( \text{orb}_F(x) \) are called the orbits of \( \mathcal{X} \). We call a copy \( A \in \mathcal{P}(\mathcal{X}) \) large iff it has infinite intersection with each orbit of \( \mathcal{X} \). For sets \( A \) and \( B \), let \( A \subset^* B \) denote the inclusion modulo finite, i.e. \( A \subset^* B \iff |A \setminus B| < \omega \).
2. SAP, large copies and positive families

**Theorem 3.** If $\mathcal{X}$ is a countable ultrahomogeneous structure $\mathcal{X}$ satisfying SAP, then a copy $A \in \mathcal{P}(\mathcal{X})$ is large iff it intersects each orbit of $\mathcal{X}$.

**Proof.** Suppose that $A$ is a copy of $\mathcal{X}$ intersecting all orbits of $\mathcal{X}$ and that the intersection $A \cap \text{orb}_f(x) = F_1$ is finite, for some finite set $F \subset X$ and some $x \in X \setminus F$. Since $\mathcal{X}$ satisfies SAP we have $|\text{orb}_f(x)| = \omega$ and, thus, we can assume that $x \not\in F_1$. Now, $\text{orb}_{F \cup F_1}(x) \subset \text{orb}_f(x) \setminus F_1$ and, hence, $A \cap \text{orb}_{F \cup F_1}(x) = \emptyset$, which is a contradiction. □

Note that the assumption that $\mathcal{X}$ has SAP can not be removed from the previous theorem, since (trivially) $X$ intersects all orbits of $\mathcal{X}$.

**Theorem 4.** For a countable ultrahomogeneous relational structure $\mathcal{X}$ the following conditions are equivalent:

(a) $\mathcal{X}$ satisfies the strong amalgamation property,

(b) $\mathcal{X}$ has a large copy,

(c) There is a positive family $\mathcal{P}$ on $X$ such that $\mathcal{P} \subset \mathcal{P}(\mathcal{X})$,

(d) There is a co-infinite $A \in \mathcal{P}(\mathcal{X})$ such that $B \subset \mathcal{P}(\mathcal{X})$, whenever $A \subset^* B \subset X$.

**Proof.** (a) $\Rightarrow$ (b). Recall that $\mathcal{X}$ satisfies SAP iff all the orbits of $\mathcal{X}$ are infinite (cf. [2, Theorem 2.15, p. 37]). Then $X$ is a large copy of $\mathcal{X}$. Conversely, if $A$ is a large copy of $\mathcal{X}$, then $A$ witnesses that all orbits of $\mathcal{X}$ are infinite; thus $\mathcal{X}$ satisfies SAP.

(a) $\Rightarrow$ (c). If $\mathcal{X}$ satisfies SAP, then the orbits of $\mathcal{X}$ are infinite and by Bernstein’s Lemma (see [9, Lemma 2, p. 514], with $\omega$ instead of $c$) there are two disjoint sets $A_0, A_1 \subset X$ intersecting all orbits of $\mathcal{X}$, which implies that $A_0, A_1 \in \mathcal{P}(\mathcal{X})$ (see e.g. [14, Theorem 2.3]). By Theorem 3 $A_0$ and $A_1$ are large copies of $\mathcal{X}$ (alternatively, see [14, Theorem 3.2]). Now, $\mathcal{P} := \{ A \in \mathcal{P}(\mathcal{X}) : A_0 \subset^* A \subset [X]^{\omega} \}$ and, since $A_1 \subset X \setminus A_0$, (P4) is true. If $\mathcal{P} \ni A \subset B \subset X$, then $A_0 \subset^* B$. In addition, for each orbit $O$ of $\mathcal{X}$ we have $|A_0 \cap O| = \omega$ and, hence, $|B \cap O| = \omega$, which gives $B \subset \mathcal{P}(\mathcal{X})$ (by [14, Theorem 2.3] again). Thus $B \in \mathcal{P}$ and (P2) is true. If $A \in \mathcal{P}$ and $F \subset X$ is a finite set, then, clearly, $A_0 \subset^* A \setminus F$ and, as above, $A \setminus F \in \mathcal{P}(\mathcal{X})$. Thus $A \setminus F \in \mathcal{P}(\mathcal{X})$, (P3) is true and $\mathcal{P}$ is a positive family indeed.

(c) $\Rightarrow$ (d). If $\mathcal{P} \subset \mathcal{P}(\mathcal{X})$ is a positive family, then by (P4) there is a co-infinite set $A \in \mathcal{P}$ and, hence, $A \in \mathcal{P}(\mathcal{X})$. For $B \subset X$ such that $A \setminus B =: F$ is a finite set, by (P3) we have $\mathcal{P} \ni A \setminus F \subset B$ and, by (P2), $B \in \mathcal{P}$, thus $B \in \mathcal{P}(\mathcal{X})$.

(d) $\Rightarrow$ (a). Suppose that $A \subset X$ is a copy given by (d). Then for each finite set $F \subset X$ we have $A \subset^* X \setminus F$. Thus, by (d), $X \setminus F \in \mathcal{P}(\mathcal{X})$. Now [4, Theorem 2] implies that the structure $\mathcal{X}$ satisfies SAP. □

Now we turn to maximal positive families.

**Theorem 5.** Let $\mathcal{X}$ be a countable ultrahomogeneous relational structure satisfying SAP. If $\mathcal{P}_\text{max} := \{ A \in \mathcal{P}(\mathcal{X}) : \forall B \subset X \ (A \subset^* B \Rightarrow B \in \mathcal{P}(\mathcal{X})) \}$, then

(a) $\mathcal{P}_\text{max}$ is the largest positive family on $X$ contained in $\mathcal{P}(\mathcal{X})$;

(b) $\mathcal{P}_\text{max} = \{ A \in \mathcal{P}(\mathcal{X}) : \forall B \subset X \ (A \subset B \Rightarrow B \in \mathcal{P}(\mathcal{X})) \}$;

(c) $\mathcal{P}_\text{max} = \{ A \subset X : A \text{ intersects all the orbits of } \mathcal{X} \}$;

(d) $\mathcal{P}_\text{max} = \{ A \subset X : A \text{ is a large copy of } \mathcal{X} \}$.

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Example 7. For the rational line, \( \mathbb{Q} \), the orbits are open intervals. Thus \( \mathcal{P}_{\text{max}} = \text{Dense}(\mathbb{Q}) := \{ A \subset \mathbb{Q} : \forall p, q \in \mathbb{Q} \ (p < q \Rightarrow A \cap (p, q) \neq \emptyset) \} \).

This means that the fact that the rational line can be split into countably many disjoint dense sets is a special case of Theorem 3.2 in [14], while the fact that there is a continuum-sized almost disjoint family of dense subsets of the rational line is a special case of Theorem 4.1 in [14].
3. Boolean maximal chains of copies

Here we prove Theorem 1 and present some applications. Let $X$ be a countable ultrahomogeneous relational structure satisfying SAP. As already mentioned, $M_X \subset C_R$ is known (for example, take a look at [13, Theorem 2.2]). The remaining part of the statement follows from the next proposition.

**Theorem 8.** If $X$ is a countable ultrahomogeneous relational structure satisfying SAP, then $B_R \subset M_X$.

**Proof.** Suppose that $L$ is such that $\text{otp}(L) \in B_R$. Let $L' = L \cup \{-\infty\}$ where $\{-\infty\}$ is the minimum of $L'$. By Theorem 3 in [11], $L'$ is isomorphic to an $R$-embeddable complete linear order whose minimum is non-isolated. Since $X$ satisfies SAP, by Theorem 5(d) $P = \{A \subset X : A$ is a large copy of $X\}$ is a positive family contained in $P(X)$. Theorem 3.2 in [14] guarantees that $\bigcap P = \emptyset$. Hence, Theorem 3.6(a) in [12] implies that there is a maximal chain $L$ in $(P(X), \subset)$ isomorphic to $L$. Thus $B_R \subset M_X$. □

**Example 9.** Countable ultrahomogeneous digraphs have been classified by Cherlin [3]. Referring to the list given in [1] and [15], we mention some structures satisfying SAP, i.e. structures to which Theorem 1 can be applied.

- All countable ultrahomogeneous partial orders except the posets $\langle C_n, <_n \rangle$, for $2 \leq n < \omega$, where $C_n = \mathbb{Q} \times n$ and $\langle q_1, k_1 \rangle <_n \langle q_2, k_2 \rangle$ iff $q_1 < q_2$ (thus, $C_n$ is a $\mathbb{Q}$-chain of antichains of size $n$).
- All countable ultrahomogeneous tournaments: the rational line $\mathbb{Q}$; the random tournament $T_\infty$; and the local order $\langle S(2), \rightarrow \rangle$, where $S(2)$ is a countable dense subset of the unit circle, such that no two of its points are antipodal, and $x \rightarrow y$ iff the counterclockwise angle between $x$ and $y$ is less than $\pi$.
- All Henson’s digraphs with forbidden sets of tournaments;
- The digraphs $\Gamma_n$, for $n > 1$, where $\Gamma_n$ is the Fraïssé limit of the amalgamation class of all finite digraphs not embedding the empty digraph of size $n$.
- Two “sporadic” primitive digraphs $S(3)$ and $P(3)$. The digraph $S(3)$ is defined as the local order $S(2)$, but with angle $2\pi/3$. The digraph $P(3)$ has a more complicated definition; it is precisely defined in [3, p. 76].
- The imprimitive digraphs $n * I_\infty$, for $2 \leq n \leq \omega$. The digraph $n * I_\infty$ is obtained from a countable complete $n$-partite graph by randomly orienting its edges.
- The digraph which is a semigeneric variant of $\omega * I_\infty$ with a parity constraint, i.e. it is a countable ultrahomogeneous digraph in which non-relatedness is an equivalence relation and for any two pairs $A_1, A_2$ taken from distinct equivalence classes, the number of edges from $A_1$ to $A_2$ is even.

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