Thermodynamic and Mechanic Consideration on the stability of Anti-symmetric Schaefer’s equation

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Abstract. Schaefer’s equation relates an interaction between population of fishes and the number of units of fishing effort. The population growth of fishes is reduced by the number of units of fishing effort, while the population growth of units of fishing effort depends on the existence of fishes. This paper aims to examine the stability of an anti-symmetric Schaefer’s equation through thermodynamic and mechanic procedure using a formula of entropy production near equilibrium which is recognized as Onsager’s relation. The results confirm that entropic approach (thermodynamics) and dissipative approach (mechanics) are usable to be applied as Lyapunov’s procedure in examining the stability of systems of differential equations.

1. Introduction

Schaefer’s equation. Originally interaction between a school’s population of fish \( (N_1) \) and the number of units of fishing effort \( (N_2) \) is described by [1] as follows:

\[
\begin{align*}
\frac{dN_1}{dt} &= kN_1(L - N_1) - k_1N_1N_2 \\
\frac{dN_2}{dt} &= k_2N_2(N_1 - b)
\end{align*}
\]

(1)

Where \( L \) stands for an environmental carrying capacity to support the population growth of fish. Multiplying \( k \) and \( L \) form a coefficient \( (r = kL) \) of population growth of fish. In the second term of upper equation (1), \( k_1N_1N_2 \) represents the rate of catching and consequently reducing the fish population. In the first term lower equation (1), \( k_2N_1N_2 \) represents the rate of population growth of fishery. The value of \( b \) should be far less than the value of \( L \) in order to recover sooner from disturbances by fishing efforts to maintain system’s sustainability of fishes and fisheries. Schaefer’s equation in the form of anti-symmetric appearance required the condition that:

\[ q = k_1 = k_2 \]

(2)

By dividing the upper by the lower equation in (1) using condition (2), we have:

\[
\frac{dN_1}{dN_2} = \frac{kN_1(L - N_1) - qN_1N_2}{qN_2(N_1 - b)} = \frac{N_1[k(L - N_1) - qN_2]}{qN_2(N_1 - b)}
\]

(3)

Observing the value of \( (dN_1/dN_2) \) through equation (3), Schaefer [1] stated that:
\[\begin{align*}
N_1 > b \quad &\text{and} \quad qN_2 > k(L - N_1) \quad \text{then} \quad (dN_1/dN_2) < 0 \\
N_1 > b \quad &\text{and} \quad qN_2 < k(L - N_1) \quad \text{then} \quad (dN_1/dN_2) > 0 \\
N_1 < b \quad &\text{and} \quad qN_2 > k(L - N_1) \quad \text{then} \quad (dN_1/dN_2) > 0 \\
N_1 < b \quad &\text{and} \quad qN_2 < k(L - N_1) \quad \text{then} \quad (dN_1/dN_2) < 0
\end{align*}\]  (4)

Schaefer found that \(N_1\) and \(N_2\) appear as functions of time in some damped oscillatory functions, where the former fluctuates in the vicinity of \(N_1^* = b\), and the latter fluctuates in the vicinity of \(N_2^* = (k/q)(L - b)\). Both curves are approaching the crossing point between \(N_1^* = Q_4 = b\) and \(N_2^* = Q_2 = (k/q)(L - b)\) as a point of stable equilibrium.

**Objective and method.** The equation (1) has been confirmed mathematically by Schaefer that its behaviour tends toward a stable equilibrium point. This paper tries to search its stability through thermodynamic and mechanic consideration which is the main question to be discussed. The aim of this study is to examine the applicability of thermodynamic consideration on the stability of an anti-symmetric Schaefer’s equation through examining the production of entropy which was applied by [2] based on the work of [3], and the examination will be compromised in transformed formulation to Lyapunov function.

2. **Thermodynamic open systems**

Bertalanffy [4] that a living system should be treated as open thermodynamic systems. The upper equation in (1) is a general transport equation in which the first term has a role to import materials rich in exergy which can avoid the increase of entropy which cannot be averted in closed systems. It gives that the population of \(N_1\) will increase with time due to this factor, where the coefficient \((r = kL)\) is its birth coefficient and has a role to import materials rich in exergy and to receive negative entropy from the environment. This importation of a negative entropy is not unlimited, but there is limitation of it by the value of \(L\) which stands for the carrying capacity of the environment. While the second term of (1) is reducing part to \(N_1\) as units of fishing efforts, but acts as its growing factor to the population of fishery, \(N_2\). Both populations maintain themselves in steady state exchange between the system and the environment, and are approaching a point of stable equilibrium.

Prigogine [5] ensured that there is a thermodynamic function \(S\) called entropy which increases monotonically until it reaches its maximum at the state of thermodynamic equilibrium, in which its formulation can be expressed as:

\[
\frac{ds}{dt} \geq 0
\]  (5)

Extending this thought into open system that required increasing entropy, first we think that internal system representing irreversible interaction system between \(N_1\) and \(N_2\) should has monotonically increasing entropy as:

\[
\frac{ds_{\text{int}}}{dt} \geq 0
\]  (6)

This increasing entropy should be counterbalanced by negative entropy which is imported from the environment with the amount of:

\[
\frac{ds_{\text{ext}}}{dt} < 0
\]  (7)

The sum of these two entropy-productions will be:

\[
\frac{ds_{\text{ext}}}{dt} + \frac{ds_{\text{int}}}{dt} = \frac{ds}{dt}
\]  (8)

Equation (8) could be negative, positive or zero depending on continual existence, continual disappearance or steady state of the system being considered.
In this relation, Wolfgang et al [6] explained equation (8) further in a process of metabolism. According to Yourgrau, during the period of growth, \( \frac{dS_{\text{ext}}}{dt} \) exceeds \( \frac{dS_{\text{int}}}{dt} \) in absolute value, so that \( \frac{dS}{dt} \) becomes negative. When the steady state is established, \( \frac{dS}{dt} = 0 \), a positive \( \frac{dS_{\text{int}}}{dt} \) is exactly counterbalanced by a negative value of \( \frac{dS_{\text{ext}}}{dt} \). Based on these two processes, we are able to think that when a positive value of \( \frac{dS_{\text{int}}}{dt} \) is greater than a negative value of \( \frac{dS_{\text{ext}}}{dt} \) in absolute values, it makes \( \frac{dS}{dt} \) becomes positive and means that the carrying capacity \( L \) of the environment has been exceeded, and the system goes into the period of decay into disappearance.

3. Production of entropy.

Chakrabarti [2] divided the rate of entropy-production \( \frac{dS_{\text{int}}}{dt} \) into two parts: the first is the rate of entropy production per individual species-\( j \); and the second \( \bar{P} \) is the rate of ecological entropy-production which is connected with the change of population densities and is the evolutionary part, playing crucial role in the study of stability and evolution of the ecosystem.

\[
P = \frac{dS_{\text{int}}}{dt} = \sum_j N_j P_j^0 + \frac{dS_{\text{int}}}{dt} = \sum_j N_j P_j^0 + \bar{P}
\]  

Based on Prigogine and Onsager as pioneer thinkers, Chakrabarti [2] proposed an expression for entropy-production as macroscopic thermodynamics of irreversible processes in local equilibrium. The expression derived from Schaefer’s anti-symmetric equation as follows:

\[
\bar{P} = \frac{dS_{\text{int}}}{dt} = \sum_j J_j \chi_j < 0
\]  

In equation (10), \( J_j \) is irreversible thermodynamic fluxes, and \( \chi_j \) is irreversible thermodynamic forces and those terms are related into the form of Onsager’s phenomenological expression:

\[
J_i = \sum_j \ell_{ij} \chi_j
\]  

Wolfgang et al [6] mentioned that equation (11) may be referred as thermodynamic equation of motion, but it is more customary to designate them as phenomenological equations of thermodynamics. Correspondingly, the constants \( \ell_{ij} \) are known as phenomenological coefficients. Prigogine ensured that the formula of (11) can only be established in some neighbourhood of equilibrium. This neighbourhood defines the region of local equilibrium.

Chakrabarti [2] interpreted local equilibrium based on Le Chatelier-Braun principle. One of many various statement of the principle in the list explained by [7] encountered in a statement as follows:

‘Whenever stress is placed on any system in a state of equilibrium, the system will always react in a direction which will tend to counteract the applied stress.’

Considered a transition from the stationary state at \( N_1^* = Q_1 \) and \( N_2^* = Q_2 \) to a state \( N_1 \) and \( N_2 \) which leads to an assumption that the deviation \( \delta N = N_i - Q_i \) is to be small [2]. This assumption makes the entropy-production reduces to the form:

\[
\delta^2 S_{\text{int}} = \sum_i \frac{(\delta N_i)^2}{Q_i}
\]  

The value of (12) is positive definite and can be applied to study local stability of the stationary state. So the criteria of stability of the stationary state \((Q_1, Q_2)\) are:

\[
\sum_i \frac{(\delta N_i)^2}{Q_i} > 0
\]  

\[
\frac{d}{dt} \left( \sum_i \frac{(\delta N_i)^2}{Q_i} \right) < 0
\]
The second condition is necessary in order that the deviation from the stationary state cannot grow in time so that the system will remain within the domain of local equilibrium around \((Q_1, Q_2)\) which clearly apparent if a phase plane.

4. Transformed formulation of anti-symmetric Schaefer’s equation

Considering minor modification to equation (1) stated above, and by using (2) we will have the anti-symmetric Schaefer’s equation in the form as written below:

\[
\begin{align*}
\frac{Q_1 \, d}{N_1 \, dt} \left(\frac{N_1}{Q_1}\right) &= kL - kQ_1 \left(\frac{N_1}{Q_1}\right) - qQ_2 \left(\frac{N_2}{Q_2}\right) \\
\frac{Q_2 \, d}{N_2 \, dt} \left(\frac{N_2}{Q_2}\right) &= qQ_1 \left(\frac{N_1}{Q_1}\right) - qQ_2
\end{align*}
\]  

(14)

Each variables in equation (14) will be transformed using the following relations:

\[
\begin{align*}
(N_1/Q_1) &= e^{x_1}; & (N_2/Q_2) &= e^{x_2} \\
x_1 &= \ln \left(\frac{N_1}{Q_1}\right); & x_2 &= \ln \left(\frac{N_2}{Q_2}\right)
\end{align*}
\]  

(15)

Using (15), equation (14) can be written as:

\[
\begin{align*}
\dot{x}_1 &= -kQ_1(e^{x_1} - 1) - qQ_2(e^{x_2} - 1) \\
\dot{x}_2 &= qQ_1(e^{x_1} - 1)
\end{align*}
\]  

(16)

In the case of no limitation for the value of carrying capacity or if the value of \(L \rightarrow \infty\), the equation (16) will take the form as the famous Lotka-Voltera equation:

\[
\begin{align*}
\dot{x}_1 &= -qQ_2(e^{x_2} - 1) \\
\dot{x}_2 &= qQ_1(e^{x_1} - 1)
\end{align*}
\]  

(17)

By dividing the first by the second of equation (17), we have:

\[
\frac{dx_1}{dx_2} = \frac{-qQ_2(e^{x_2-1})}{qQ_1(e^{x_1-1})} \text{ or } qQ_1(e^{x_1-1})dx_1 + qQ_2(e^{x_2-1})dx_2 = 0
\]  

(18)

Integrating equation (18) in the transformed variable: from 0 to \(x_i\) or in the primitive variable from \(Q_i\) to \(N_i\), then we will have a constant of integration, later it will be called as total energy \(E\) for the equation (17) as a constant of integration:

\[
\begin{align*}
E &= Q_1 \int_0^{x_1}(e^{x_1} - 1)dx_1 + Q_2 \int_0^{x_2}(e^{x_2} - 1)dx_2 \\
E &= Q_1(e^{x_1} - x_1 - 1) + Q_2(e^{x_2} - x_2 - 1)
\end{align*}
\]  

(19)

The Lotka-Voltera equation (17) will give curves of oscillatory function without damping with the total energy \(E\) (19) which is time-independent.

Equation (16) can be rewritten using (11), where thermodynamical forces are written as:

\[
\begin{align*}
\dot{x}_1 &= -kx_1 - qx_2 \\
\dot{x}_2 &= qx_1
\end{align*}
\]  

or

\[
\begin{align*}
\dot{x}_1 &= \ell_{11}x_1 + \ell_{21}x_2 \\
\dot{x}_2 &= \ell_{12}x_1 + \ell_{22}x_2
\end{align*}
\]  

(20)

Four coefficients in (20) \(\ell_{11} = -k; \ell_{21} = -q; \ell_{12} = q; \text{ and } \ell_{22} = 0\) are called as Onsager’s phenomenological coefficients, where \(\chi_i = Q_i(e^{x_i} - 1)\) is used in formulating (20).
Entropy’s production will be calculated through Onsager’s formulation as a product between thermodynamic fluxes and thermodynamic forces. For the non-equilibrium thermodynamic model of the system, defines the thermodynamic flux (or flow) as the intrinsic rate of population growth as follows[2]:

$$J_i = \dot{x}_i ; \ i = 1,2$$  \hfill (21)

Using the corresponding thermodynamical forces which have been defined previously in \eqref{20}, the rate of entropy’s production as formulated previously in \eqref{10} will be:

$$\tilde{P} = \chi_1J_1 + \chi_2J_2$$  \hfill (22)

Using \eqref{21} and \eqref{22}, the rate of entropy production is derived as follows:

$$\tilde{P} = \chi_1[-k\chi_1 - q\chi_2] + \chi_2[q\chi_1]$$  \hfill (23a)

$$\tilde{P} = -k\chi_1^2 - q\chi_1\chi_2 + q\chi_1\chi_2$$  \hfill (23b)

Finally, we have that the produced entropy is decreasing with time:

$$\tilde{P} = -kQ_1^2(e^{x_1} - 1)^2 \leq 0$$  \hfill (24a)

Equation \eqref{24} informs us that the ecological entropy-production is decreasing towards the value of $x_1 \approx 0$ or in the primitive variable towards the value of $N_1 \approx b$. According to the limit of growth, population of $N_1$ could be extended actually towards $N_1 \approx L$, where $L > b$. But beyond the value of $b$, the disturbed population of fishes by harvesting becomes threatening the sustainability of the fishes and fishery’s system. Expressing \eqref{24} in the primitive variables will be:

$$\tilde{P} = -k(N_1 - Q_1)^2 = -k(N_1 - b)^2 \leq 0$$  \hfill (24b)

Again, equation \eqref{24} confirms that the anti-symmetric Schaefer’s equation is thermodynamically stable which confirms the Le Chatelier-Braun principle. The value of ecological entropy production, $\tilde{P}$, tends to be zero at equilibrium point $N_1 \approx b$ as depicted in figure 1.

![Figure 1](image-url)  

*Figure 1.* The curve of ecological entropy production tends to be zero at $b = 150$. 

5. **Lyapunov’s procedure as mechanical energy approach**

Equation (20) is manipulated to be linearized equations by using approximation of \( e^x - 1 \approx x \), we have:

\[
\begin{align*}
\dot{x}_1 &= -k Q_1 x_1 - q Q_2 x_2 \\
\dot{x}_2 &= q Q_1 x_1
\end{align*}
\]  

(25)

Differentiate (25) to time, it becomes:

\[
\begin{align*}
\ddot{x}_1 + k Q_1 \dot{x}_1 + q^2 Q_1 Q_2 x_1 &= 0 \\
\ddot{x}_2 + k Q_1 \dot{x}_2 + q^2 Q_1 Q_2 x_2 &= 0
\end{align*}
\]  

(26)

Each component \( x_1 \) and \( x_2 \) will behave as oscillatory function of time. The second terms in equation (26) contribute to be a damped term to the system of oscillation. Its characteristic equation will give the roots for the solution to (26), that is:

\[
\begin{align*}
\lambda_{1,2} &= -\frac{1}{2} k Q_1 \pm \frac{1}{2} k Q_1 \sqrt{1 - \frac{4q^2 Q_2}{k^2 Q_1}}; \quad Q_1 = b; \quad Q_2 = (k/q)(L - b) \\
\lambda_{1,2} &= -\frac{1}{2} k b \pm \frac{1}{2} k b \sqrt{1 - \frac{4q}{k} \left(\frac{L}{b} - 1\right)}
\end{align*}
\]  

(27a)

(27b)

To maintain both equations in (26) are expressing fluctuating behaviour and decreasing by factor of \( \exp[-kt/2] \) with time, the value under the root should be negative, hence it specifies the value of \( b \) as follows:

\[
1 - \frac{4q}{k} \left(\frac{L}{b} - 1\right) < 0 \quad \text{or} \quad \left(1 + \frac{k}{4q}\right) b < L
\]  

(27c)

Damped oscillatory function for primitive variables \( N_1 \) and \( N_2 \) are plotted in a phase space as depicted in figure 2.

![Equilibrium point at \( N_1=150 \) and \( N_2=45 \)](image)

**Figure 2.** Numerical values of \( L = 600; b = 150; q = 0.025; \) and \( k = 0.0025 \) are used to plot a phase plane of \( N_1 \) and \( N_2 \). The fluctuation tends to the equilibrium point at \( N_1 = 150 \) and \( N_2 = \frac{k}{q}(L - b) = 45 \).
Following [8] and [9] the Lagrangians for equation (26) can be formulated based on the characteristics equations of (27a, b and c) as follows:

\[ \mathcal{L} = e^{kbt} \left\{ \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} \omega^2 (x_1^2 + x_2^2) \right\} ; \omega^2 = q^2 Q_1 Q_2 = kq \left( \frac{L}{b} - 1 \right) \]

\[ \mathcal{L} = e^{kbt} \mathcal{L} \quad ; \quad \mathcal{L} = \left\{ \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} \omega^2 (x_1^2 + x_2^2) \right\} \quad (28) \]

The related mechanical energy \( E \) or Hamiltonian \( \mathcal{H} \) to Lagrangian \( \mathcal{L} \) in (28) is as follows:

\[ \mathcal{H} = \left\{ \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} \omega^2 (x_1^2 + x_2^2) \right\} \quad (29) \]

Euler-Lagrange equations to produce (26) are as follows:

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad \text{or} \quad \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) - \frac{\partial \mathcal{L}}{\partial x_i} = - \frac{\partial \mathcal{F}}{\partial x_i} \quad (30) \]

Physical term \( \mathcal{F} \) stands for friction as dissipative terms to the system under consideration which is formulated by

\[ \mathcal{F} = \frac{1}{2} kQ_1 (\dot{x}_1^2 + \dot{x}_2^2) \quad (31) \]

Lyapunov’s procedure uses a formulation of dissipative energy function given by [10] using (29), (30) and (31) as follows:

\[ \frac{dE}{dt} = -2 \mathcal{F} = -k Q_1 (\dot{x}_1^2 + \dot{x}_2^2) < 0 \quad (32) \]

Since \( kQ_1 \) is a positive constant, \( \dot{E} \) is a negative quantity, so that \( E \) is decreasing function with time. As long as there are values for thermodynamic fluxes, \( E \) can never become negative, but attains its stable rest point at disappearance of \( f_1 = \dot{x}_1 \) and \( f_2 = \dot{x}_2 \) or at the crossing between the line of \( N_1 = b \) and \( N_2 = (k/q)(L - b) \). According to equation (31), when \( L < b \) there will be no growth for the fishery population \( (N_2) \), but for the fishes \( (N_1) \) rich their maximum population without catching by fishing efforts. Equation (32) confirm that the anti-symmetric Schaefer’s equation is also mechanically stable and supports biological stability.

6. Conclusions

The main question is concerning the applicability of thermodynamic and mechanic consideration on the stability of an anti-symmetric Schaefer’s equation. The results are as follows:

a) To keep profitable investment in fishing needs to monitor and to control the population of fishes. It means that \( N_1 \) should not exceed the value of \( b \) which should be less than the value of limited resources as environmental carrying capacity, \( L \), and the population of \( N_2 \) should not exceed the value of \( (k/q)(L - b) \). The crossing point \( (N_1 = b, N_2 = (k/q)(L - b)) \) is a position having thermodynamical stability and supports economic stability. This approach as has been shown by equation (24) which is also can be used as Lyapunov’s procedure due to decreasing value with time: \( \dot{P} = \frac{d}{dt} S_{in} = -k Q_1 (e^{n-1})^2 \leq 0 \).

b) Decreasing value of total mechanical energy is confirmed by the existence of dissipative function expressed by equation (30) that depends on thermodynamical fluxes \( f_1 = \dot{x}_1 \) and \( f_2 = \dot{x}_2 \). Using Lyapunov’s procedure, equation (31) confirms the anti-symmetric Schaefer’s equation is mechanically stable towards the crossing point \( (N_1 = b, N_2 = (k/q)(L - b)) \) and supports biological stability.

c) Entropic approach (thermodynamics) and dissipative approach (mechanics) are usable to be applied as Lyapunov’s procedure in examining the stability of systems of differential equations.
d) Both equations (24b) and (27c) are confirming the usable area of Le Chatelier-Braun principle in interaction between population of fishes and fisheries. The former supports its fluctuating behaviour near equilibrium, and the latter supports damped oscillatory functions toward the point of equilibrium.

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