Quantum Fluctuations in Open Pre-Big Bang Cosmology

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Abstract

We solve exactly the (linear order) equations for tensor and scalar perturbations over the homogeneous, isotropic, open pre-big bang model recently discussed by several authors. We find that the parametric amplification of vacuum fluctuations (i.e. particle production) remains negligible throughout the perturbative pre-big bang phase.

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I. INTRODUCTION

The question of whether, in the presence of spatial curvature, the pre-big bang (PBB) scenario \cite{1–3} needs a very large amount of fine-tuning is still a subject of debate \cite{4–10}. Furthermore, Kaloper et al. \cite{8} have argued that, even assuming that the two classical moduli of the open ($K = -1$), homogeneous, isotropic cosmological solution \cite{11,5} lie deeply inside the perturbative region, the unavoidable existence of vacuum quantum fluctuations modifies so drastically the classical behaviour as to prevent the occurrence of an appreciable amount of inflation.

In this paper, as a first step towards addressing this second objection, we carry out a detailed study of quantum fluctuations around the $K = -1$ solution of \cite{11,5}. It is well known \cite{12,13} that quantum fluctuations in a non-spatially flat background are considerably harder to study than the corresponding ones in a flat Universe. Nevertheless, somewhat to our surprise, the corresponding equations can still be integrated exactly in terms of standard hypergeometric functions. The conclusion is that particle production (i.e. the amplification of vacuum fluctuations) is strongly suppressed at very early times because of a cancellation between the effect of a non-vanishing Hubble parameter and the one of spatial curvature. In other words, particle production is proportional to the deviation of the background from its asymptotic Milne form and thus to the time variation of the background dilaton. As a result, particle production remains small through the whole perturbative PBB phase and does not impede the occurrence of PBB inflation.

We will first recall the explicit form of the homogeneous, isotropic, $K = -1$ PBB background we shall be dealing with and derive the general, covariant form of the action to second order in the perturbations. We then solve, successively, the equations for tensor and scalar perturbations. Finally, we discuss the physical implications of our results, and comment on their possible relevance to the issue raised in \cite{8}.

II. THE BACKGROUND AND THE SECOND-ORDER ACTION

Our conventions are such that (after reduction to $D = 4$) the (normalized) string-frame action takes the form

$$
\bar{h}^{-1} S^{(s)} = \frac{1}{2\ell_s^2} \int d^4x \sqrt{-G} e^{-\phi} \left( R(G) + G^\mu_\nu \partial_\mu \phi \partial_\nu \phi + \ldots \right),
$$

(2.1)

where $G_{\mu\nu}$ is the string-frame metric, $\phi$ is the $(D = 4)$ dilaton, $\ell_s$ is the fundamental length scale of string theory and the dots indicate other fields (e.g. a Kalb-Ramond axion field) that will be set to zero hereafter. The above action allows for classical homogeneous, isotropic solutions of the standard Friedmann–Robertson–Walker (FRW) type

$$
ds^2 = a_s^2(\eta) \left( -d\eta^2 + \frac{dr^2}{1 - K r^2} + r^2 d\Omega^2 \right).
$$

(2.2)

As usual there are both post- and pre-big bang solutions coming from a singularity, or going towards it, respectively. For $K = -1$, the PBB-type solution was first given in \cite{11} and then rederived and discussed in \cite{5}. It reads:
\[ a_s(\eta) = L (\cosh \eta)^{\frac{1}{\sqrt{3}}} (\sinh \eta)^{\frac{1}{\sqrt{3}}} \]

\[ \phi(\eta) = -\sqrt{3} \ln(-\tanh \eta) + \phi_{\text{in}}, \quad \eta < 0 , \]

where \( L \) and \( \phi_{\text{in}} \) are a dimensional and a dimensionless integration constant, respectively.

The arbitrariness of \( L \) and \( \phi_{\text{in}} \) reflects the symmetries of the classical problem under a constant shift of the dilaton \( \phi \) and a constant rescaling of the metric \( G_{\mu\nu} \). These are precisely the two parameters to be chosen in an appropriate (fine-tuned) range in order to ensure a sufficient amount of PBB inflation. Indeed, Eq. (2.3) describes a universe that is almost trivial (Milne-like) from \(-\infty < \eta < \mathcal{O}(-1)\), and then inflates with an initial curvature \( \mathcal{O}(L^{-2}) \) and initial coupling \( \mathcal{O}(\exp(\phi_{\text{in}}/2)) \) till it meets, eventually, the strong curvature and/or strong coupling regimes at \( \eta \sim \eta_1 \). The critical value \( \eta_1 \) is easily determined in terms of the integration constants \( L \) and \( \phi_{\text{in}} \):

\[ (-\eta_1) = \max \left( e^{\phi_{\text{in}}/\sqrt{3}}, (\ell_s/L)^{1+1/\sqrt{3}} \right) . \]

It is well known that the study of perturbations is technically simpler in the so-called Einstein frame, defined by \( g_{\mu\nu} = \exp(\phi_{\text{today}} - \phi) G_{\mu\nu} \), and, correspondingly, by the action:

\[ \bar{h}^{-1} S^{(E)} = \frac{1}{2\ell_p} \int d^4 x \sqrt{-g} \left( R(g) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) , \]

where \( \phi_{\text{today}} \) is the present value of the dilaton and \( \ell_p \equiv \sqrt{8\pi G \bar{h}} = \exp(\phi_{\text{today}}/2)/\ell_s \sim 0.1\ell_s \) is the present value of Planck’s length. We will compute perturbations in the Einstein frame and then convert the results back to the original string frame for a physical interpretation.

In the Einstein frame the background equations for a generic FRW universe are given by:

\[ \mathcal{H}' = -\frac{1}{6} \phi'^2, \quad \text{where} \quad \mathcal{H} = \frac{a'}{a} \]

\[ \mathcal{H}^2 + \mathcal{K} = \frac{1}{12} \phi'^2, \quad \phi'' + 2\mathcal{H} \phi' = 0 , \]

where a prime denotes differentiation with respect to the conformal time \( \eta \). For \( \mathcal{K} = -1 \) the solution is just given by rewriting (2.3) in the Einstein frame:

\[ a(\eta) = \ell (-\sinh \eta \cosh \eta)^{\frac{1}{\sqrt{3}}} \]

\[ \phi(\eta) = -\sqrt{3} \ln(-\tanh \eta) + \phi_{\text{in}}, \quad \eta < 0 , \]

where the new modulus \( \ell \), given by \( \ell^2 = L^2 \exp(\phi_{\text{today}} - \phi_{\text{in}}) \), replaces the string-frame classical modulus \( L \).

To estimate quantum fluctuations around (2.7) we first go over to isotropic spatial coordinates \((x,y,z)\) defined by

\[ ^1\text{Although we restrict our attention to the case } \mathcal{K} = -1, \text{ we will occasionally keep } \mathcal{K} \text{ in the formulae for an easy comparison with the spatially-flat case.} \]
\[ r = R \left( 1 + \frac{K}{4} R^2 \right)^{-1}, \quad \text{where} \quad R^2 = x^2 + y^2 + z^2, \quad (2.8) \]

and by the obvious identification of the angular coordinates. In these coordinates the FRW metric takes the generic form

\[ ds^2 = a^2(\eta) \left(-d\eta^2 + \gamma_{ij} dx^i dx^j\right), \quad \text{where} \quad \gamma_{ij} = \delta_{ij} \left(1 + \frac{K}{4} R^2 \right)^{-2}, \quad i, j = 1, 2, 3, \quad (2.9) \]

and generic perturbations are defined by

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \quad \phi = \phi^{(0)} + \delta \phi, \quad (2.10) \]

where a superscript (0) denotes the background solution.

We now consider the form of the action (2.5) up to second-order terms in the fluctuations. The calculations are long but straightforward. After using the background equations (2.6), and after dropping irrelevant boundary terms (total divergences), the result can be expressed covariantly in the form:

\[
\delta^{(2)} S = \frac{1}{2\ell_p^2} \int d^4 x \sqrt{-g} \left[ - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \gamma^{\lambda\sigma} \left( \nabla_\lambda \delta g_{\beta\mu} \nabla_\sigma \delta g_{\nu\alpha} - \nabla_\sigma \delta g_{\mu\nu} \nabla_\lambda \delta g_{\alpha\beta} \\
+ 2 \nabla_\alpha \delta g_{\mu\nu} \nabla_\sigma \delta g_{\beta\lambda} - 2 \nabla_\lambda \delta g_{\beta\mu} \nabla_\nu \delta g_{\alpha\sigma} \right) - g^{\mu\nu} \partial_\mu \delta \phi \partial_\nu \delta \phi \\
+ g^{\mu\nu} g^{\lambda\sigma} \partial_\lambda \phi \delta \phi \nabla_\sigma \delta g_{\mu\nu} - 2 g^{\mu\lambda} g^{\nu\sigma} \partial_\lambda \phi \delta \phi \nabla_\sigma \delta g_{\mu\nu} - 2 g^{\mu\lambda} g^{\nu\sigma} \nabla_\lambda \phi \delta \phi \delta g_{\mu\nu} \right], \quad (2.11)\]

where, to this order, we can replace \( g_{\mu\nu} \) and \( \phi \) by their background expression (2.7), and all covariant derivatives are to be evaluated with respect to the background metric.

### III. Solving the Perturbation Equations

#### A. Tensor Perturbations

Since tensor metric perturbations are automatically gauge-invariant, and decouple from dilatonic perturbations, they are easier to study. They can be defined by

\[ \delta g^{(T)}_{\mu\nu} = \text{diag}(0, a^2 h_{ij}), \quad (3.1) \]

where the symmetric three-tensor \( h_{ij} \) satisfies the transverse-traceless (TT) conditions

\[ \nabla^i h_{ij} = 0, \quad h^{i}_{i} = 0, \quad (3.2) \]

with \( \nabla^i \) denoting the covariant derivative with respect to \( \gamma_{ij} \). Inserting (3.1) into Eq. (2.11), and using (2.6), we easily find:

\[ \delta^{(2)} S^{(T)} = \frac{1}{4\ell_p^2} \int d^4 x \sqrt{-g} \ a^2 \left( h^{ij} h^i_{ij} - \nabla^i h^{ij} \nabla_i h_{ij} - 2 K h^{ij} h_{ij} \right). \quad (3.3) \]
For $K = -1$, tensor perturbations $h_{ij}$ can be expanded in TT tensor pseudospherical harmonics [14] as

$$h_{ij}(\eta, x) = \int d\eta \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h_{nlm}(\eta)(G_{ij}(x))_{lm}^n,$$  

(3.4)

where the tensor harmonics $(G_{ij})_{lm}^n$ satisfy the eigenvalue equation

$$\nabla^2(G_{ij}(x))_{lm}^n = -(n^2 + 3)(G_{ij}(x))_{lm}^n.$$  

(3.5)

Choosing their normalization so that:

$$\int d^3x \sqrt{h} (G_{ij}(x))_{lm}^n (G_{ij}(x))_{l'm'}^n = \delta(n - n') \delta_{ll'} \delta_{mm'},$$  

(3.6)

and inserting (3.4) in (3.3), we obtain

$$\delta^{(2)} S^{(T)} = \frac{1}{4 \ell_P^2} \int d\eta \, dn \, a^2 \sum_{l,m} \left[ (h_{nlm}')^2 - (n^2 + 1)h_{nlm}^2 \right].$$  

(3.7)

Introducing finally the canonical variable

$$u_{nlm} = ah_{nlm},$$  

(3.8)

and using the background equations (2.6), we get:

$$\delta^{(2)} S^{(T)} = \frac{1}{4 \ell_P^2} \int d\eta \, dn \sum_{l,m} \left[ (u_{nlm}')^2 - (n^2 + \frac{1}{12} \phi'^2) u_{nlm}^2 \right],$$  

(3.9)

yielding for $u_{nlm}$ the simple equation

$$u_{nlm}'' + \left( n^2 + \frac{1}{12} \phi'^2 \right) u_{nlm} = 0.$$  

(3.10)

Luckily, for the background (2.7), Eq. (3.10) can be exactly solved in terms of the standard hypergeometric function $F \equiv _2F_1$ [15] by

$$u_N(\eta) = C_1 [\text{csch}^2(2\eta)]^{-\frac{in}{4}} F \left[ \frac{1 - in}{4}, \frac{1 - in}{4}, \frac{2}{2}, -\text{csch}^2(2\eta) \right] + C_2 [\text{csch}^2(2\eta)]^{\frac{in}{4}} F \left[ \frac{1 + in}{4}, \frac{1 + in}{4}, \frac{2 + in}{2}, -\text{csch}^2(2\eta) \right],$$  

(3.11)

where $N$ stands for the collection of indices $(nlm)$ and $C_{1,2}$ are (classically arbitrary) integration constants. In order to correctly normalize the tensor perturbations, the action (3.9) has to be quantized. At early times, $n^2 \gg \phi'^2$, and thus $u$ is a free canonical field. Hence we impose, as $\eta \to -\infty$,

$$u_N(\eta) \rightarrow u_N^{-\infty}(\eta) \equiv \frac{2\ell_P}{\sqrt{n}} e^{-in\eta}.$$  

(3.12)
Using $F[a,b,c,0] = 1$, Eq. (3.12) fixes the integration constants as $|C_1| = 2\ell_P/\sqrt{\bar{n}}$, $C_2 = 0$. The deviation from a trivial plane-wave behaviour can easily be computed from the small argument limit of $F$. We find

$$u_N(\eta) = u_N^\infty(\eta) \left(1 + \alpha_n e^{4\eta - i\beta_n}\right),$$

(3.13)

where $\alpha_n, \beta_n$ are $n$-dependent constants fixed from the Taylor expansion of the hypergeometric function. We note that the correction to the vacuum amplitude dies off as $e^{4\eta}$, i.e. as $t^{-4}$ in terms of cosmic time $t \sim -e^{-\eta}$.

We finally estimate the behaviour of the solution near the singularity, i.e. for $\eta \to 0$, using $[15]

$$F[a,a,c,-\text{csch}^2(2\eta)] \simeq \frac{\Gamma(c)}{\Gamma(a)\Gamma(a + \frac{1}{2})} \left[-2^{2a+1}|\eta|^{2a} \ln |\eta|\right].$$

(3.14)

Then, by virtue of the small $\eta$ behaviour $a \simeq \ell |\eta|^{1/2}$ and of Eq. (3.8), we find

$$|h_N| \simeq 2\sqrt{\frac{2\ell_P}{\pi \ell}} \sqrt{\coth \left(\frac{n\pi}{2}\right) \ln |\eta|}.$$

(3.15)

We shall come back to this result after deriving a similar expression for scalar perturbations.

**B. Scalar perturbations**

Consider now scalar metric-dilaton perturbations defined by $[12]

$$\delta \phi, \quad \delta g^{(S)}_{\mu\nu} = -a^2(\eta) \left(\frac{2\varphi}{\nabla_i B} \nabla_i B + 2(\psi \gamma ij + \nabla_i \nabla_j E)\right).$$

(3.16)

Inserting (3.16) in Eq. (2.11), and making use of (2.4), we find

$$\delta^{(2)} S^{(S)} = \frac{1}{2\ell_P^2} \int d^4x \ a^2(\eta) \sqrt{-g} \left[\left(\delta \phi' \right)^2 - \nabla \delta \phi \cdot \nabla \delta \phi + 6\phi' \delta \phi \psi' - 2\phi \delta \phi' - 2\phi' \delta \phi \nabla^2(B - E') - 12\psi'^2 - 8\nabla \varphi \cdot \nabla \psi + 4(\nabla \psi)^2 - 24\mathcal{H} \varphi \psi' + 12\mathcal{K}(\varphi^2 - \psi^2 + 2\varphi \psi) - 8\nabla \psi' \cdot \nabla B - 8\mathcal{H} \nabla \varphi \cdot \nabla B - 8\mathcal{H} \varphi \nabla^2 E' - 8\psi' \nabla^2 E' + 4\mathcal{K}(B - E') \nabla^2(B - E')\right].$$

(3.17)

In (3.17) the variables $B, \varphi$ do not have time derivatives and thus act as Lagrange multipliers, which provide constraints. These are:

$$0 = \mathcal{C}_B \equiv \phi' \delta \phi - 4 \psi' - 4\mathcal{H} \varphi - 4\mathcal{K}(B - E')$$

$$0 = \mathcal{C}_\varphi \equiv \phi' \delta \phi' - 12\mathcal{K} \varphi + 12\mathcal{H} \psi' - 4(\nabla^2 + 3\mathcal{K}) \psi - 4\mathcal{H} \nabla^2(B - E').$$

(3.18)

Following $[13]$, we introduce the gauge-invariant variable $\Psi$ by

$$\Psi = \frac{4}{\phi'} [\psi + \mathcal{H}(B - E')],$$

(3.19)
and, after inserting the constraints, we recast the action \((3.17)\) in the convenient form

$$
\delta^{(2)} S^{(S)} = \frac{1}{2\ell_P^2} \int d^4x \ a^2 \sqrt{\gamma} (\nabla^2 + 3K) \Psi \left[ \partial_{\eta}^2 - \nabla^2 + 2(\mathcal{H}' + \mathcal{K}) \right] \Psi .
$$

(3.20)

One can now make use of the constraints to eliminate the variable \((B - E')\) from the action \((3.20)\) in terms of \(\varphi, \psi\) and \(\delta \phi\). The latter variables are not independent either, being related by a linear combination of the two constraints \(C_\varphi, C_B\). After its implementation the action \((3.20)\) contains only true degrees of freedoms.

In analogy with the case of tensor perturbations, we introduce a canonical field \(\Psi_c\) and expand it as

$$
\Psi_c \equiv a \Psi = \int d\eta \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Psi_{nlm}(\eta) Q_{nlm}(x),
$$

(3.21)

where \(Q_{nlm}(x)\) are the scalar pseudospherical harmonics, satisfying

$$
\nabla^2 Q_{nlm}(x) = -(n^2 + 1) Q_{nlm}(x)
$$

\[\int d^3x \sqrt{\gamma} Q_{nlm}(x) Q_{n'l'm'}(x) = \delta(n - n') \delta_{ll'} \delta_{mm'} .\]

(3.22)

As a result, \((3.20)\) becomes

$$
\delta^{(2)} S^{(S)} = \frac{1}{2\ell_P^2} \int d\eta \ d\eta' \left[ (\bar{\Psi}_N)^2 - (n^2 - \frac{1}{4} \phi^2) \bar{\Psi}_N^2 \right] , \quad N = (nlm) ,
$$

(3.23)

where \(\bar{\Psi}_N \equiv \sqrt{n^2 + 4} \Psi_N\). The quantity \(\bar{\Psi}_N\) enters the action in a canonical way and therefore its vacuum fluctuations, like those of \(u\), are easily normalized. The equation for \(\bar{\Psi}_N\) is simply

$$
\bar{\Psi}_N'' + (n^2 - \frac{1}{4} \phi^2) \bar{\Psi}_N = 0 ,
$$

(3.24)

so that we must impose, as \(\eta \rightarrow -\infty\),

$$
\bar{\Psi}_N(\eta) \rightarrow \bar{\Psi}_N^{-\infty}(\eta) = \frac{\ell_P}{\sqrt{n}} e^{-in\eta}
$$

$$
\bar{\Pi}_N(\eta) \rightarrow \bar{\Pi}_N^{-\infty}(\eta) = -i \frac{\sqrt{n}}{\ell_P} e^{-in\eta} .
$$

(3.25)

As was the case for tensor perturbations, also Eq. \((3.24)\) can be transformed (for the background \((2.6)\)) into a hypergeometric equation. We find, specifically,

$$
\bar{\Psi}_N(\eta) = \tilde{C}_1 \left[ \text{csch}^2(2\eta) \right]^{-\frac{in}{4}} F\left[ \frac{-1 - in}{4}, \frac{3 - in}{4}, \frac{2 - in}{2}, -\text{csch}^2(2\eta) \right]
$$

$$
+ \tilde{C}_2 \left[ \text{csch}^2(2\eta) \right]^{\frac{in}{4}} F\left[ \frac{-1 + in}{4}, \frac{3 + in}{4}, \frac{2 + in}{2}, -\text{csch}^2(2\eta) \right] ,
$$

(3.26)

where, as before, we have to take \(|\tilde{C}_1| = \ell_P/\sqrt{n}, \tilde{C}_2 = 0\). Corrections to the free plane wave can be easily computed and, again, are suppressed by four powers of \(1/t\):
\[ \Psi_N(\eta) = \Psi_N^{-\infty}(\eta) \left( 1 + \tilde{\alpha}_n e^{4\eta - i\tilde{\beta}_n} \right), \]  
(3.27)

where \( \Psi_N^{-\infty} \) is given by (3.25) and \( \tilde{\alpha}_n, \tilde{\beta}_n \) are \( n \)-dependent constants fixed from the expansion of the hypergeometric function.

To estimate the behaviour of (3.26) near \( \eta \simeq 0 \), we use the formula \[ F[a, a + 1, c, -\text{csch}^2(2\eta)] \simeq \frac{\Gamma(c)}{\Gamma(a + 1)\Gamma(c - a)} \left[ -2^{2a+3} a(a - c + 1)|\eta|^{2(a+1)} \ln |\eta| + 2^{2a}|\eta|^{2a} \right], \]  
(3.28)

and obtain:

\[ |\Psi_N| \simeq \ell_P \sqrt{\frac{n^2 + 1}{2\pi}} \sqrt{\coth \left( \frac{n\pi}{2} \right)} \left( -|\eta|^{3/2} \ln |\eta| + \frac{2}{n^2 + 1}|\eta|^{-1/2} \right). \]  
(3.29)

IV. DISCUSSION

In order to discuss the physical significance of our results it is useful to choose a convenient gauge. In the spatially flat case it was found [16] that the so-called off-diagonal gauge [17,16] was particularly useful in order to suppress the large gauge artifacts present in the more commonly used [12] longitudinal gauge. The off-diagonal gauge is defined by setting \( \psi = E = 0 \) in Eq. (3.16). We shall now see how one can reconstruct the scalar field fluctuation from \( \Psi \) in this gauge.

We first note that, in this gauge, the variables \( \Psi \) and \( B \) are related through (3.19) as:

\[ \Psi = \frac{4HB}{\phi'}. \]  
(4.1)

Using Eq. (3.24) for \( \Psi_N \), as well as (4.1), we can derive the evolution equation for \( B \):

\[ B'' - \nabla^2 B + \left( 2\mathcal{H} - \frac{4\mathcal{K}}{\mathcal{H}} \right) B' - (4\mathcal{H}^2 + 12\mathcal{K}) B = 0, \]  
(4.2)

which agrees with Ref. [16] for \( \mathcal{K} = 0 \). To relate \( \delta \phi \) and \( \Psi \) we first observe that the first of the two constraints (3.18) provides the relation

\[ \phi' \delta \phi = 4(\mathcal{H}\phi + \mathcal{K}B), \]  
(4.3)

while, eliminating \( \delta \phi \) from the two constraints (3.18) and using (4.2), we arrive at a second relation

\[ \varphi = B' + 2\mathcal{H}B. \]  
(4.4)

Combining (4.4) and (4.3), and making use of (4.1), we are finally able to express \( \delta \phi \) directly in terms of \( \Psi \) as
\[ \delta \phi = \Psi' + \frac{\mathcal{K} - \mathcal{H}'}{\mathcal{H}} \Psi, \quad (4.5) \]

implying that \( \delta \phi \) represents, in this gauge, a gauge-invariant object.

It is instructive to compare the \( \mathcal{K} = -1 \) case with the spatially flat one, where the relevant gauge-invariant variable, given by

\[ \psi^{(gi)} = \psi + \frac{H}{\phi'} \delta \phi, \quad (4.6) \]

becomes \( \delta \phi \) itself in the off-diagonal gauge. The canonical field, given by \( v = a \delta \phi \), satisfies the well-known equation [12]:

\[ v'' + \left( n^2 - \frac{z''}{z} \right) v = 0, \quad \text{where} \quad z = \frac{a \phi'}{\mathcal{H}}. \quad (4.7) \]

Even in the presence of spatial curvature, the field \( v \) still plays the role of the canonical field in the far past, when \( \eta \) is large and negative. This can be checked by computing the equation of motion for \( v \) in the presence of curvature. The explicit form of the equation for \( v \) is given by

\[ v'' + A_1 v' + A_2 v = 0, \quad A_1 = -\mathcal{K}^2 \phi'^2 \left[ \mathcal{H}^2 (n^2 - \mathcal{K} + 3\mathcal{K}^2) \right]^{-1}, \]

\[ A_2 = n^2 + \frac{\phi'^2}{12} \left( 1 - \frac{12 \mathcal{K}}{\mathcal{H}^2} \right) - (\mathcal{H} + \frac{3 \mathcal{K}}{\mathcal{H}}) A_1. \quad (4.8) \]

Thus, as long as we are interested in the early-time regime, \( A_1 \) is exponentially small, \( A_2 \to n^2 \), and \( v \) can be treated as the canonical field.

Using Eq. (4.3), the behaviour of \( v \) in the far past follows directly from that of \( \bar{\Psi}_N \), given in Eqs. (3.25), (3.27):

\[ v^{-\infty}(\eta) \equiv \frac{\ell_p}{\sqrt{n}} \frac{\sqrt{2 - in}}{2 + in} e^{-in\eta}, \]

\[ \pi_v^{-\infty}(\eta) \equiv -i \frac{\sqrt{n}}{\ell_p} \sqrt{2 + in} e^{-in\eta}, \quad (4.9) \]

with corrections again suppressed as \( t^{-1} \), i.e.

\[ v(\eta) = v^{-\infty}(\eta) \left( 1 + \hat{\alpha}_n e^{4\eta - i\hat{\beta}_n} \right), \quad (4.10) \]

where \( \hat{\alpha}_n, \hat{\beta}_n \) are \( n \)-dependent constants.

We can study how other variables behave near \( \eta \simeq 0 \) by using their relation to \( \Psi \) in this gauge and the behaviour of \( \Psi \), Eq. (3.23). We easily find:

\[ |B_N| \simeq \frac{\ell_p}{\ell} \sqrt{\frac{n^2 + 1}{2\pi}} \sqrt{\frac{\coth\left( \frac{n\pi}{2} \right)}{n^2 + 4}} \left( -|\eta| \ln |\eta| + \frac{2}{n^2 + 1} |\eta|^{-1} \right), \quad (4.11) \]
while
\[ |\delta \phi_N| \simeq \frac{\ell_P}{\ell} \sqrt{\frac{n^2 + 1}{2\pi}} \sqrt{\frac{\coth(\frac{n\pi}{2})}{n^2 + 4}} \ln |\eta| . \]  

(4.12)

Let us finally compare the energy contained in the quantum fluctuations of the dilaton and that in the classical solution near the singularity. Note that the expansion (3.28) can be trusted only up to some maximum \( n \) for which \( 1 \ll n_{\text{max}} \sim 1/|\eta| \). Consequently, the ratio of the kinetic energy densities near \( |\eta| \simeq 0 \) (up to constant prefactors of \( O(1) \)) becomes
\[
\frac{\mathcal{E}_Q}{\mathcal{E}_C} = \frac{\ell_P^2}{\ell^2} \int_{n_{\text{max}}}^{n_{\text{max}}} \frac{dn}{n} n^3 .
\]  

(4.13)

We can express the above result in terms of the value of the physical Hubble parameter \( H(\eta) \equiv \mathcal{H}/a \) at horizon crossing of the scale \( n \), \( H_{\text{HC}}(n) \), which is easily computed as
\[
H_{\text{HC}}(n) \sim \frac{1}{\eta a} (\eta \sim 1/n) \sim n^{3/2}/\ell .
\]  

(4.14)

Thus (4.13) takes the suggestive form
\[
\frac{\mathcal{E}_Q}{\mathcal{E}_C} = \frac{\ell_P^2}{\ell^2} \int_{n_{\text{max}}}^{n_{\text{max}}} \frac{dn}{n} H_{\text{HC}}^2(n) .
\]  

(4.15)

In general, in order to draw physical conclusions, we should transform back the results to the string frame. However, in our case, this is hardly necessary. Concerning the importance of vacuum fluctuations as \( \eta \to 0 \), we observe that the final result (4.13) expresses the relative importance of quantum and classical fluctuations near the singularity in terms of a frame-independent quantity, the ratio of the effective Planck length to the size of the horizon. Since, by definition of the perturbative dilaton phase, the Hubble radius is always larger than the string scale, we find that the relative importance of quantum fluctuations is always bounded by the ratio \( \ell_P/\ell_s \) which is always less than one in the perturbative phase.

Let us now come to the more subtle issue of the far-past behaviour of tensor and scalar quantum fluctuations. Computations may be done in either frame, since the dilaton is approximately constant in the far past. Our results, expressed in Eqs. (3.13) and (4.10), show that corrections to the trivial quantum fluctuations are of relative order \( \epsilon^4 \eta \sim t^{-4} \), i.e. of order \( t^{-3} \) relative to the (homogeneous) classical perturbation. This suggests that quantum effects do not modify appreciably classical behaviour in the far past, in contrast to the claim made in [8]. This attitude is also supported by the structure of the superstring one-loop effective-action (which is well-defined thanks to the string cutoff). Because of supersymmetry, neither a cosmological term nor a renormalization of Newton’s constant are generated at one-loop, but only terms containing at least four derivatives. As a result, quantum corrections to early-time classical behaviour are of relative order \( t^{-6} \), i.e just like our corrections \((\delta \phi'/\phi')^2\). Note, incidentally, that generating a cosmological constant by quantum corrections would upset completely the whole PBB scenario.

We also see, however, that, as claimed in [8], the leading (free-theory) fluctuations (the 1’s in Eqs. (3.13) and (4.10)) dominate over the homogeneous classical perturbation by one
power of $t$. If taken at face value, they upset classical behaviour at early-enough times, $|t| > \ell^2/\ell P$\cite{8}. The answer to the issue raised in \cite{8} thus appears to depend on whether (zero-point, non-amplified) vacuum quantum fluctuations in the (trivial) Milne background can give physically important effects on the scale of Milne’s Hubble radius $H^{-1} \sim t$. A complete clarification of this point would be certainly desirable.

We stress however that, irrespectively of the final answer to this issue, vacuum fluctuations have the same time dependence as the typical inhomogeneous classical perturbation discussed in \cite{7,10}, but much smaller amplitudes. Indeed, an initial classical state apt to give rise to a pre-big bang event (i.e. to gravitational collapse in the Einstein frame) in a region of space of size $\ell_m \gg \ell P$ must correspond, quantum mechanically, to having parametrically large occupation numbers in certain quantum states \cite{10}. Such a quasi-classical configuration cannot be appreciably affected by quantum fluctuations $O(1)$ in those occupation numbers.

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