An alternative way of deriving the Finite Difference Time Domain method (FDTD) for simulating the dynamics of electromagnetic waves in matter for one dimensional systems with grid spacing and material properties that vary with position is presented. This alternative derivation provides useful insight into the physics of electromagnetic waves in matter and the properties of FDTD. The method uses d’Alembert’s splitting of waves into forward and backward pulses of arbitrary shape to solve the one-dimensional wave equation. Constant velocity of waves in dispersionless dielectric materials, partial reflection and transmission at boundaries between materials with different indices of refraction, and partial reflection, transmission, and attenuation through conducting materials are derived in the process of deriving the method so that real physics is learned simultaneously with the numerics. The traditional FDTD equations are reproduced from a model of wave pulses partially reflecting, transmitting, and attenuating through regions of constant current. The alternative method therefore shows the physical effects of the finite difference approximation in a way that is easy to visualize. The method allows for easy derivation of some results that are more complicated to derive with the traditional method, such as showing that FDTD is exact for dielectrics when the ratio of the spacial and temporal grid spacing is the wave speed, incorporating reflectionless boundary conditions, and showing the method retains second order accuracy when the grid spacing varies with position and the material parameters make sudden jumps across layer boundaries.
I. INTRODUCTION

A. Background and Motivation

The Finite Difference Time Domain (FDTD) method is a very efficient and popular way to computationally solve Maxwell’s equations for the dynamics of electromagnetic waves in matter. Because it is a real time method, it is efficient for simulations requiring a broad range of frequencies such as femtosecond pulses and atomic systems treated semi-classically. Despite its utility and popularity, the method is not covered in the standard physics Electricity and Magnetism (E&M) textbooks. This is understandable since the typical E&M curriculum is dense with both fundamental insights, mathematical frameworks, and important technical applications. This structure leaves little space for a parallel development of numerical recipes. The usual and natural derivation of FDTD is to replace the partial derivatives in Maxwell’s equations by finite difference equations. Although simple and straightforward, this does not, by itself, teach any of the physics of electromagnetic waves in matter, and so the subject does not fit naturally into the E&M curriculum and is often relegated to a specialized computational methods course. This is regrettable since numerical simulations and data processing have taken on increased prominence in technical fields due to the availability of computational power. Numerics is now not a specialized field, but part of the basic toolbox for many researchers. Given the increased importance of computation, some educators are integrating numerical recipes into physics courses and texts using user friendly computer languages designed for visualization such as VPython and Glowscript. One way to simultaneously accommodate the time and content constraints so that simulation methods can be taught in the E&M course is teaching the numerical method in such a way that it also communicates the essential physics. In this paper, we show an alternative derivation of FDTD for one dimensional systems that connects the numerics more clearly to the physics of electromagnetic waves in matter. Constant velocity of waves in dispersionless dielectric materials, partial reflection and transmission at boundaries between materials with different indices of refraction, and reflection and attenuation through conducting materials are explained in the process of deriving the method so that real physics is learned simultaneously with the numerics. No restriction to single frequency, or understanding of complex numbers, trigonometric functions, or exponential functions is required to understand these results. We believe that this alternative derivation of FDTD can contribute to understanding of the interaction of fields with matter in a way that compliments knowledge learned from studying the interaction of matter with fields of a single frequency.

The method uses a discretization of d’Alembert’s splitting of waves into forward and backward pulses of arbitrary shape to solve Maxwell’s equations for the dynamics of electromagnetic waves in one-dimensional media. The traditional FDTD equations are reproduced from a model of wave pulses partially reflecting, transmitting, and attenuating through regions of constant current. The alternative method therefore shows the physical effects of the finite difference approximation in a way that is easy to visualize. It also allows for easy derivation of some results that are more complicated with the traditional FDTD derivation such as showing that FDTD is exact for dielectrics when the ratio of the spacial and temporal grid spacings is the wave speed, incorporating reflectionless boundary conditions, and showing the method retains second order accuracy when the grid spacing varies with position and the material parameters make sudden jumps across layer boundaries.

B. Structure of this Paper

In Section II we derive d’Alembert’s method for Maxwell’s equations and apply it to a homogeneous dispersionless dielectric. Solving the system exactly, we show that waves pulses move at constant speed without changing their shape. In Section III we apply the method to a system composed of layers of dispersionless dielectrics, with arbitrary changes in the index of refraction across layers. We show that such systems have exact solutions when the layers are of equal optical path length, as is the case for distributed Bragg reflectors. We also derive an exact formula for the partial reflection and transmission of wave pulses across boundaries between layers of differing material properties and layer lengths and derive exact formulas for reflection-less boundary conditions at system boundaries. In Section IV we derive a model of wave pulses partially reflecting, transmitting, and attenuating through regions of constant current to capture the physics of wave pulses moving through layered materials with arbitrary material parameters in each layer, including conductors and dielectrics with dispersion, and grid spacing arbitrarily varying with position. In Section V we demonstrate the model for the case of a simple conductor and develop a succinct formula for the partial reflection, transmission, and attenuation of waves pulses through a conductor. In Section VI we show that our method is equivalent to FDTD. In Section VII we show that the method is second order accurate for grid spacings arbitrarily varying with position and for arbitrary jumps in material parameters across layer boundaries.
II. WAVE PULSE APPROACH FOR SOLVING MAXWELL’S EQUATIONS IN ONE DIMENSIONAL SYSTEMS

d’Alembert’s method is a simple solution to the one-dimensional wave equation usually encountered in a first class on partial differential equations or the physics of waves. d’Alembert solved the wave equation by transforming to variables representing wave pulses that move forward and backwards through the media. In this section, using d’Alembert’s method, we rewrite Maxwell’s equations for electromagnetic waves in media in terms of wave pulses and use them to derive an exact solution for wave pulses moving through a dispersionless dielectric. Maxwell’s equation for electromagnetic waves in matter are

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$$

$$\frac{1}{\mu} \nabla \times \mathbf{B} = \epsilon \partial_t \mathbf{E} + \sigma \mathbf{E} + \mathbf{J}_p.$$  

$\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively. The material parameters $\mu$, $\epsilon$, and $\sigma$ are the magnetic permeability, electric permittivity, and electrical conductivity respectively. $\epsilon \partial_t \mathbf{E}$, incorporates the displacement current and polarization current. $\sigma \mathbf{E}$ is the conduction current. $\mathbf{J}_p$ is the remaining part of the current that includes any more complicated dependencies on the electric field, usually through a differential equation. In the simplest model of a dielectric material $\sigma = 0$ and $\mathbf{J}_p = 0$. For the simplest model of a conductor $\mathbf{J}_p = 0$.

We consider only one dimensional systems in this paper, by which we mean that the system is homogeneous and infinite and all fields are non-varying in two of the spacial dimensions but can vary in the remaining spacial dimension, so that we can take advantage of d’Alembert’s solution to the one-dimensional wave equation. This allows us to consider all of the physics of wave pulses interacting with matter at normal incidence. Also, for simplicity, only constant $\mu$ is considered and only waves of one polarization are considered. For electromagnetic waves polarized in the $y$ direction and moving in the $x$ direction through materials that have constant $\mu$ and with other material properties varying only in the $x$ direction, Maxwell’s equations simplify to

$$\partial_x E_y(x, t) = -\partial_t B_z(x, t)$$

$$\partial_x B_z(x, t) = -\mu (\epsilon - \frac{1}{\mu v^2}) \partial_t E_y(x, t) - \mu \sigma E_y(x, t) - \mu J_{py}(x, t)$$

In what follows we will suppress the dimensional indices and $x$ and $t$ variables on the field and current components.

The variables that one chooses to represent physical phenomena can make some physics clear and other physics obscure. Maxwell’s equations in terms of $E$ and $B$ fields clearly show that a changing magnetic field creates an electric field and a changing electric field creates a magnetic field. However, it is not immediately obvious by looking at these equations that wave pulses move through space with constant velocity. But we can see that explicitly if we change to d’Alembert’s wave pulse variables

$$F = \frac{E + vB}{2},$$

$$G = \frac{E - vB}{2},$$

$$E = F + G,$$

$$vB = F - G.$$  

where the physical meaning of $v$, which has units of velocity, will be determined momentarily. Taking the partial derivatives of $F$ and $G$ we arrive at

$$\partial_t F + v \partial_x F = -\frac{\mu v^2}{2} J$$

$$\partial_t G - v \partial_x G = -\frac{\mu v^2}{2} J$$

where

$$J = (\epsilon - \frac{1}{\mu v^2}) \partial_t E + \sigma E + \mathbf{J}_p$$
Using the directional derivatives,
\[ \partial_R = \partial_t + v \partial_x, \]
\[ \partial_L = \partial_t - v \partial_x, \]
in which we can think of \( \partial_R \) as a forward change in time and a spatial change to the right and \( \partial_L \) as a forward change in time and a spatial change to the left, Maxwell’s equations for waves in matter become
\[ \partial_R F = -\frac{\mu v^2}{2} J, \]
\[ \partial_L G = -\frac{\mu v^2}{2} J. \]

If we know the values of the \( F \) and \( G \) fields at a given position and time, we can determine the values at a different position and time by integrating the equations along their paths. We integrate the equation for \( F \) forward through time and to the right. We integrate the equation for \( G \) forward through time and to the left.

\[ F(x_i + v(t_f - t_i), t_f) = F(x_i, t_i) - \frac{\mu v^2}{2} \int_{t_i}^{t_f} d\tau J(x_i + v(\tau - t_i), \tau) \]
\[ G(x_i - v(t_f - t_i), t_f) = G(x_i, t_i) - \frac{\mu v^2}{2} \int_{t_i}^{t_f} d\tau J(x_i - v(\tau - t_i), \tau) \]

These equations are exact. \( F \) is a wave pulse that moves to the right with speed \( v \) and \( G \) is a wave pulse that moves to the left with speed \( v \). \( F \) and \( G \) interact through the current \( J \). For the special case of a simple dielectric with speed \( v \) chosen so that \( \epsilon\mu v^2 = 1 \) then \( J = 0 \) and the exact wave pulse equations simplify to
\[ F(x' + v(t - t')), t) = F(x', t'), \]
\[ G(x' - v(t - t')), t) = G(x', t'). \]

We choose a grid spacing so that \( \Delta x = v\Delta t \). We measure \( t \) in units of \( \Delta t \) and \( x \) in units of \( v\Delta t \). With these units, the equations become
\[ F_{x+1}^{t+1} = F_x^t, \]
\[ G_{x-1}^{t+1} = G_x^t. \]

where \( x \) and \( t \) are the integer spatial and temporal grid coordinates. These equations look like a finite difference approximation but they are exact, and, as shown in Figure 1 completely express the physics that \( F \) wave pulses move to the right and \( G \) wave pulses move to the left at constant speed \( v = (\mu\epsilon)^{-1/2} \) through a simple dielectric material without changing their shape. If a different grid spacing was chosen so that \( \Delta t \neq (\mu\epsilon)^{-1/2}\Delta t \) then the current would not be zero, the update equations would not be exact, and there would be anomalous dispersion from the mismatch of the grid with the physical wave velocity. We, therefore, call the first term in Equation (11), the grid current.

Boundary conditions, including reflectionless boundary conditions with an incoming rightward moving pulse are easy in this formalism. The condition at the left end at position \( x = 1 \) is satisfied by \( F_1^t = P_t^i \), where \( P_t^i \) represents the amplitude of the incoming pulse at time \( t \), and the condition at the right end at position \( x = N \) is satisfied by \( G_N^t = 0 \).

In the next section we apply d’Alembert’s solution to Maxwell’s equations to exactly determine the fields in a system of layered dispersionless dielectrics. In the process we derive the exact formula for partial reflection and transmission at boundaries between materials with differing indices of refraction.

### III. EXACT RESULTS FOR LAYERED DISPERSIONLESS DIELECTRICS

#### A. Transport Through Layers of the Same Optical Path Length

In the previous section, we derived exact results for a homogeneous dielectric, meaning a dielectric with a permittivity independent of position. In this section we determine exact results for an inhomogenous dielectric, meaning a
FIG. 1. F wave pulses move to the right and G wave pulses move to the left at constant speed through a simple dielectric material without changing their shape. Electric and magnetic fields are determined by adding and subtracting F and G wave pulses. For example the electric field in the second position at time \( t=1 \) is \( F_E + G_C \) and the magnetic field is \( \frac{(F_E - G_C)}{v} \). Times are shown in units of \( \Delta t \).

dielectric where the permittivity is a function of the \( x \) coordinate. Our solution is exact provided that two conditions are satisfied. The first is that the grid spacing is small enough that the permittivity is approximately constant throughout each grid spacing. This is a common numerical approximation. We call the space between grid points ‘layers’ and refer to the constant permittivity in the \( x \)th layer as \( \epsilon_x \). Our second condition is less common. It is that the grid current be zero, meaning that the \( x \)th layer has length \( \Delta x_x = v_x \Delta t \), where \( \epsilon_x \mu v_x^2 = 1 \). This means that the time it takes for light to travel through each layer is the same for each layer even if their permittivities, and therefore their speeds of light differ. Dielectric layers with a faster speed of light will be longer and dielectric layers with a slower speed of light will be shorter so that light takes the same time to travel through each layer. In other words, the optical path lengths are the same. There are important layered systems, such as Bragg reflectors\(^{20-23}\), where this condition is exactly satisfied, and which motivated the development of this method.

As shown in Figure 2, for systems where each layer has the same optical path length, after a time \( \Delta t \) each F pulse moves across one layer to the right and each G pulse moves across one layer to the left without changing their amplitude. The equations that exactly describe this transport of light across simple dielectric layers are

\[
F_{x+}^{t+1/2} = F_{x-}^{t-1/2},
\]

\[
G_{x-}^{t+1/2} = G_{x+}^{t-1/2},
\]

where here and throughout the rest of this paper, grid times are measured in units of \( \Delta t \), grid positions are measured in units of \( \Delta x_x = v_x \Delta t \), the time \( t \) occurs when the two pulses cross in the center of each layer, and we refer to the right end of layer \( x \) as \( x+ \) and the left end of layer \( x \) as \( x- \). For a Bragg reflector, each layer spacing in the simulation can equal the entirety of an actual material layer. We next derive what happens to a pulse when it crosses from one layer to another.
B. Reflection and Transmission through Layer Boundaries

F and G wave pulses partially reflect and partially transmit through the boundaries between layers. Both $x+$ and $x+1−$ correspond to practically the same position, $x+1/2$, at the boundary layer $x$ and layer $x+1$ but the values of the pulses $F_{x+}$ and $F_{x+1−}$ and of pulses $G_{x+}$ and $G_{x+1−}$ can be different. Integrating Maxwell’s equations across the $x+1/2$ layer boundary at time $t+1/2$ obtains

$$E_{x+1-}^{t+1/2} - E_{x+}^{t+1/2} = - \int_{x+}^{x+1-} dx \partial_t B(x, t + \frac{1}{2}),$$ \hspace{1cm} (24)

$$B_{x+1-}^{t+1/2} - B_{x+}^{t+1/2} = - \int_{x+}^{x+1-} dx \left( \mu(x) \partial_t E(x, t + \frac{1}{2}) + \mu \sigma(x) E(x, t + \frac{1}{2}) + \mu J_p(x, t + \frac{1}{2}) \right).$$ \hspace{1cm} (25)

Noting that these integrals go to zero as the distance between $x+$ and $x+1−$ goes to zero demonstrates that the $E$ and $B$ fields are continuous across layer boundaries. The continuity of the electric field across the $x+1/2$ layer boundary determines the following relationship between $F$ and $G$ wave pulses across the boundary.

$$E_{x+1/2}^{t+1/2} = F_{x+}^{t+1/2} + G_{x+}^{t+1/2} = F_{x+1−}^{t+1/2} + G_{x+1−}^{t+1/2}$$ \hspace{1cm} (26)

Defining $n_x = \frac{c}{v_x}$, which is the index of refraction for layer $x$, the continuity of the magnetic field across the the same boundary determines another relationship between the $F$ and $G$ wave pulses.

$$cB_{x+1/2}^{t+1/2} = n_x(F_{x+1−}^{t+1/2} - G_{x+1−}^{t+1/2}) = n_{x+1}(F_{x+1−}^{t+1/2} - G_{x+1−}^{t+1/2})$$ \hspace{1cm} (27)

Combining these so that wave pulses coming out of a boundary are determined by the wave pulses going into a boundary we have

$$F_{x+1−}^{t+1/2} = \frac{2n_x}{n_x + n_{x+1}} F_{x+}^{t+1/2} + \frac{n_{x+1} - n_x}{n_x + n_{x+1}} G_{x+1−}^{t+1/2}$$ \hspace{1cm} (28)
which, as shown in Figure 3, captures all of the physics of transmission and reflection at boundaries between dielectric layers of different indices of refraction. Such a boundary attenuates a transmitted pulse, as indicated by the factor multiplying the first term on the right-hand side (RHS) that is less than one, causing some of a pulse to reflect and causes the reflected pulse to undergo a 180 degree phase change if the layer reflected from has a lower index of refraction, as indicated by the second term on the RHS. As the difference in the two indices increases the proportion of the wave that is reflected increases. As the differences of the two indices approach zero, the wave pulse is transmitted without any reflection or attenuation. Boundary conditions at the two ends of the system are derived by taking the appropriate limit of the layer boundary equations. For example, reflectionless boundary conditions at both ends with an incoming pulse entering on the leftmost end are satisfied with

\[ F_1^{t+1/2} = p_{t+1/2}, \quad G_{N_+}^{t+1/2} = 0. \] (30, 31)

Equations (28, 29), with the appropriate initial and boundary conditions such as Eq. (30, 31) are exact update formulas for one dimensional systems with simple dielectric layers in which each layer has equal optical path length, such as distributed Bragg reflectors.

Eq. (28, 29) are usually derived for plane waves of definite frequency but they remain true for pulses of any shape and for boundaries between any two materials, including conductors and dielectrics with dispersion, because the only physics used to derive them is the continuity of the electric and magnetic fields across boundaries. Likewise, the reflectionless boundary conditions such as Eq. (30, 31), remain the same regardless of the system’s interior. In cases where the materials are not dispersionless dielectrics, or where the grid current is not zero, \( n_x = \frac{V_c}{V} = \frac{c \Delta t}{\Delta x} \) is a property of the grid, not necessarily related to the phase or group wave velocity in any way.

FIG. 3. F and G wave pulses partially reflect and partially transmit through layer boundaries.
IV. THE CONSTANT CURRENT MODEL

When the system being simulated includes conducting layers, or dielectrics with dispersion, or it is not possible to set the grid spacing so that the grid current is zero, the update Eq. (22,23) will no longer be correct. We present a piece-wise constant current model for such systems, in which, as seen in Figure 4, the current is constant within each layer but changes in time intervals of $\Delta t$. This is a reasonable approximation provided the layer lengths and $\Delta t$ are small. We will later show that the model is equivalent to FDTD.

We return to our exact results, equations (16,17), for materials for which the current $J$ is not zero. Measuring $t$ in units of $\Delta t$, $x$ in units of $\Delta x$, and using + and - notation for wave pulses moving end to end across the $x$th layer obtains,

$$F_{x+}^{t+1/2} = F_{x-}^{t-1/2} - \frac{\mu v_x^2}{2} \int_{t-1/2}^{t+1/2} d\tau J(x + v(\tau - t), \tau),$$

(32)

$$G_{x-}^{t+1/2} = G_{x+}^{t-1/2} - \frac{\mu v_x^2}{2} \int_{t-1/2}^{t+1/2} d\tau J(x - v(\tau - t), \tau).$$

(33)

If the current was constant and equal to $J^t_x$ everywhere in layer $x$ from $x-$ to $x+$ and in time from $t - 1/2$ to $t + 1/2$ then the above equations are easily exactly determined. The wave pulses at one end of a boundary depend on the other end via,

$$F_{x+}^{t+1/2} = F_{x-}^{t-1/2} - \frac{\mu v_x^2 \Delta t}{2} J^t_x,$$

(34)

$$G_{x-}^{t+1/2} = G_{x+}^{t-1/2} - \frac{\mu v_x^2 \Delta t}{2} J^t_x.$$  

(35)
The above equations require us to know the current at the center of each layer. The current at the center of each layer is dependent on the electric field at the center of the layer, which we find by integrating the F and G equation from the layer boundaries at time \( t - \frac{1}{2} \Delta t \) to their centers at time \( t \),

\[
F^t_z = F^t_{z-} - \frac{\mu v^2 \Delta t}{4} J^t_z,
\]

\[
G^t_z = G^t_{z+} - \frac{\mu v^2 \Delta t}{4} J^t_z.
\]

Adding the equations together produces

\[
E^t_z = F^t_{z-} + G^t_{z+} - \frac{\mu v^2 \Delta t}{2} J^t_z.
\]

It will be convenient to express the wave pulses at the boundaries in terms of the electric field at the center instead of the current. This can be done by replacing the current term in Eq. (34,35) with its dependence on \( E^t_z \) from Eq.(38),

\[
- \frac{\mu v^2 \Delta t}{2} J^t_z = E^t_z - F^t_{z-} - G^t_{z+}.
\]

The resulting update equations for F and G at the boundaries are

\[
F^{t+1/2}_{z+} = E^t_z - G^{t-1/2}_{z+},
\]

\[
G^{t+1/2}_{z+1-} = E^t_{z+1} - F^{t-1/2}_{z+1-}.
\]

Reflectionless boundary conditions for pulses at positions \( x = 1 \) and \( x = N \) are determined from Eq.(40,41) together with Eq.(30,31),

\[
F^{t+1/2}_{1-} = E^t_1 - P^{t-1/2},
\]

\[
G^{t+1/2}_{N+1-} = E^t_N - P^{t-1/2},
\]

\[
F^{t+1/2}_{N+} = E^t_N,
\]

\[
G^{t+1/2}_{N+} = 0.
\]

Equations (28,29,30,31,32,35), form a complete set of equations for the wave pulses in the constant-current model. To proceed further we produce explicit formulas for the constant currents \( J^t_z \).

V. ABSORPTION AND PARTIAL REFLECTION THROUGH A CONDUCTOR

We now show how the constant current model illuminates the physics of waves through a conductor. Choosing layer lengths such that the grid current is zero, \( \epsilon_x \mu v^2 = 1 \), and considering a simple conductor, \( J_p = 0 \), the current reduces to

\[
J^t_z = \sigma_x E^t_x
\]

Plugging this \( J^t_z \) into the equation for \( E^t_z \) obtains,

\[
E^t_z = F^{t-1/2}_{z-} + G^{t-1/2}_{z+} - \frac{\mu v^2 \Delta t}{2} \sigma_x E^t_z + \mu v^2 \Delta t \sigma_x.
\]

Solving for \( E^t_z \) we have

\[
E^t_z = \frac{F^{t-1/2}_{z-} + G^{t-1/2}_{z+}}{1 + \frac{\mu v^2 \Delta t \sigma_x}{2}}.
\]
Using this $E_x^t$ in the equation for the wave pulses at the boundaries, Eq. (40,41), we have

$$F_{x+1/2} = \frac{1}{1 + \frac{\nu v \sigma_x \Delta t}{2}} F_{x-1/2} - \frac{\nu^2 \sigma_x \Delta t}{2} G_{x+1/2}$$  \hspace{1cm} (49)$$

$$G_{x-1/2} = \frac{1}{1 + \frac{\nu v \sigma_x \Delta t}{2}} G_{x+1/2} - \frac{\nu^2 \sigma_x \Delta t}{2} F_{x-1/2}$$  \hspace{1cm} (50)

FIG. 5. Wave pulses partially reflect and partially transmit through a conducting layer.

These equations show all of the qualitative effects of conductivity. As shown in Figure 5, conducting current attenuates a pulse, as indicated by the factor multiplying the first term on the RHS that is less than one, and causing some of a pulse to reflect with a 180 degree phase change, as indicated by the second term on the RHS of each equation. As the strength of the conductivity goes to infinity all of the wave is reflected and no transmission occurs. As the conductivity goes to zero, the wave pulse is transmitted without any attenuation or reflection.

In the next section we extend these results for more general currents of the form Eq.(11), including when the grid current is not zero, in the process we demonstrate equivalence with FDTD.

VI. EQUIVALENCE OF THE CONSTANT CURRENT MODEL AND FDTD

A. The Electric Field at Layer Centers

We incorporate constant currents that depend on the time derivative of the electric field, which occurs when the grid current and/or $J_p$ is not zero, by integrating in time the formula for $J$, Eq.(11), from $t - 1/2$ to $t + 1/2$ at position $x$, the layer center,

$$\int_{t-1/2}^{t+1/2} d\tau J(x, \tau) = (\epsilon_x - \frac{1}{\mu v_x^2}) \int_{t-1/2}^{t+1/2} d\tau \partial_t E(x, \tau) + \sigma_x \int_{t-1/2}^{t+1/2} d\tau E(x, \tau) + \int_{t-1/2}^{t+1/2} d\tau J_p(x, \tau).$$  \hspace{1cm} (51)
The first term on the RHS is integrated exactly. All other terms are evaluated via the constant current model, or equivalently, the average end point approximation of the integral, which is accurate to second order in $\Delta t$,

$$\frac{\Delta t \mu J_x^t + J_x^{t-1}}{2} = (\epsilon_x - \frac{1}{\mu \varepsilon_x^2})(E_x^t - E_x^{t-1}) + \Delta t \sigma_x E_x^t \frac{E_x^t + E_x^{t-1}}{2} + \Delta t \frac{J_{px}^t + J_{px}^{t-1}}{2} + O((\Delta t)^3). \quad (52)$$

This expression for $J_x^t$ in terms of $J_x^{t-1}$ can be used in Eq. (38).

Alternatively, we can remove any explicit dependence of $J$ in the past, so that we do not need to keep it in memory as a separate variable. To do this, consider the electric field at layer center $x$ at time $t$ and transport its component wave pulses backwards in time to reproduce Eq. (38).

We derive the update equation for the magnetic field at layer boundaries and system boundaries by combining the equations for wave pulse propagation within a layer, Eq. (40,41), with the equations of continuity of electric and magnetic fields across boundaries, Eq. (42,43), keeping in mind that the wave velocity on either side of a boundary may be different.

$$E_x^t - E_x^{t-1} = v_x \left( B_{x+}^{t-\frac{1}{2}} - B_{x-}^{t-\frac{1}{2}} \right) - \frac{\mu \varepsilon_x^2 \Delta t}{2} (J_x^t + J_x^{t-1}). \quad (55)$$

Placing the expression for the current, Eq. (52), into the expression for the difference of the electric fields, Eq. (55), and dividing by $v_x^2 = (\frac{\Delta x}{\Delta t})^2$ obtains

$$\mu \varepsilon_x (E_x^t - E_x^{t-1}) = -\Delta t \frac{B_{x+}^{t-\frac{1}{2}} - B_{x-}^{t-\frac{1}{2}}}{\Delta x} - \Delta t \sigma_x E_x^t \frac{E_x^t + E_x^{t-1}}{2} - \Delta t \mu \varepsilon_x J_x^t + J_x^{t-1}, \quad (56)$$

which is the standard FDTD update formula for the electric field at layer centers. We can also derive an update formula for the magnetic field at layer centers, but that is not necessary for showing equivalence with FDTD which only evaluates electric fields at layer centers and magnetic fields at layer boundaries.

### B. The Magnetic Field at Layer Boundaries

We derive the update equation for the magnetic field at layer boundaries and system boundaries by combining the equations for wave pulse propagation within a layer, Eq. (41,42), with the equations of continuity of electric and magnetic fields across boundaries, Eq. (26,27), keeping in mind that the wave velocity on either side of a boundary may be different.

Subtracting Eq. (40) for layer $x$ and Eq. (41) for layer $x+1$ obtains,

$$F_{x+}^{t+1/2} - G_{x+1}^{t+1/2} = E_x^t - E_{x+1}^t - G_{x+}^{t-\frac{1}{2}} + F_{x+}^{t-\frac{1}{2}}. \quad (57)$$

Using the relationship between fields and wave pulses at boundaries, Eq. (26,27), the magnetic field at time $t + 1/2$ at layer boundary $x + 1/2$ in terms of wave pulses entering the boundary is

$$\frac{v_x + v_{x+1}}{2} B_{x+1}^{t+1/2} = F_{x+}^{t+1/2} - G_{x+1}^{t+1/2} \quad (58)$$

and the magnetic field at time $t - 1/2$ at layer boundary $x + 1/2$ in terms of wave pulses leaving the boundary is

$$\frac{v_x + v_{x+1}}{2} B_{x+1}^{t-1/2} = F_{x+}^{t-1/2} - G_{x+1}^{t-1/2} \quad (59)$$

Replacing wave pulse differences in Eq. (57) with B fields, and dividing by $\frac{1}{2} (v_x + v_{x+1}) = \frac{1}{2} \left( \frac{\Delta x}{\Delta t_x} + \frac{\Delta x}{\Delta t_{x+1}} \right)$ produces the simplest generalization of the traditional FDTD equation for the magnetic field between layers with variable grid spacing,

$$B_{x+1}^{t+\frac{1}{2}} - B_{x+}^{t-\frac{1}{2}} = -\Delta t \frac{E_x^{t+1/2} - E_x^t}{\frac{1}{2} (\Delta x_x + \Delta x_{x+1})}. \quad (60)$$
Reflectionless boundary conditions for the magnetic fields at the ends of the system are determined from Eq. (42–45),

\[ v_1 B_{1+1/2}^t = F_{1-}^{t+1/2} - G_{1-}^{t+1/2} = P^{t+1/2} - E_1^t + P^{t-1/2}, \]  

(61)

\[ v_N B_{N+1/2}^t = F_{N+}^{t+1/2} - G_{N+}^{t+1/2} = E_N^t. \]  

(62)

The results of this section show that the traditional FDTD method is equivalent to the d’Alembert’s method with the piece-wise constant current. This provides a physical model of FDTD inexactness. Everything that is true about one method is true about the other. In particular, the FDTD method is exact for dispersionless dielectrics where each layer is of equal optical path length, and it is exact for a model system composed of conducting layers and dielectrics with dispersion with piece-wise constant current in each layer.

In the next section we show that the update equations are accurate to second order.

VII. PROOF OF SECOND ORDER ACCURACY

In this section we show that the constant current model, and equivalently, the traditional FDTD method with the simplest generalization to variable grid spacing, is second order accurate, regardless of how the grid spacing varies with position and regardless of the presence of sharp jumps in the values of the material parameters across layers. This is important since both variable grid spacing and jumps in material parameters occur in one-dimensional photonic crystals, such as Bragg reflectors. Traditional proofs of second order accuracy of the FDTD equations for constant grid spacing and constant material parameters rely on centered differences across boundaries and/or the midpoint or endpoints approximations to the line integral across boundaries. However, a centered difference approximation taken across such a boundary in a photonic crystal is not an automatically second order accurate approximation of the spatial derivative. Neither are the midpoint and endpoint average integral approximations when integrated across a boundary. The accuracy of these operations are apparently lowered to first order because the position of the boundary is not at the midpoint between layer centers and jumps in material parameters cause discontinuities in some of the spacial derivatives of the fields at the boundaries. In order to prove second order accuracy regardless of the details of the differences between the two layers at a boundary, we retain the exact formulas across the boundaries (28-29) and make all of our approximations within layers.

We return to our exact results, equations (42-43), for materials for which the current \( J \) is not zero and approximate the current in layer \( x \) by the first terms of its Taylor expansion at the layer center position \( x \) at time \( t \),

\[ J(x + v(\tau - t), \tau) = J^t_x + (\tau - t)\partial_R J^t_x + O((\tau - t)^2), \]  

(63)

\[ J(x - v(\tau - t), \tau) = J^t_x + (\tau - t)\partial_L J^t_x + O((\tau - t)^2), \]  

(64)

to arrive at

\[ F_{x+1/2}^{t+1/2} = F_{x-1/2}^{t-1/2} - \frac{\mu v_x^2}{2} \int_{t-1/2}^{t+1/2} d\tau (J^t_x + (\tau - t)\partial_R J^t_x + O((\tau - t)^2)), \]  

(65)

\[ G_{x-1/2}^{t+1/2} = G_{x+1/2}^{t-1/2} - \frac{\mu v_x^2}{2} \int_{t-1/2}^{t+1/2} d\tau (J^t_x + (\tau - t)\partial_L J^t_x + O((\tau - t)^2)). \]  

(66)

Integrating these expressions for \( J \) across layer \( x \) we get

\[ F_{x+1/2}^{t+1/2} = F_{x-1/2}^{t-1/2} - \frac{\mu v_x^2 \Delta t}{2} J^t_x + O((\Delta t)^3), \]  

(67)

\[ G_{x-1/2}^{t+1/2} = G_{x+1/2}^{t-1/2} - \frac{\mu v_x^2 \Delta t}{2} J^t_x + O((\Delta t)^3). \]  

(68)

The \( \partial_R J^t_x \) and \( \partial_L J^t_x \) terms drop out in agreement with the constant current model. But we are not so lucky when we integrate half way across a layer.

\[ F_x^t = F_x^{t-\frac{1}{2}} - \frac{\mu v_x^2 \Delta t}{4} J^t_x + \frac{\mu v_x^2 (\Delta t)^2}{8} \partial_R J^t_x + O((\Delta t)^3), \]  

(69)
\[ G^t_x = \theta^t_{x+\frac{1}{2}} - \frac{\mu v^2 \Delta t}{4} J^t_x + \frac{\mu v^2 (\Delta t)^2}{8} \partial_L J^t_x + O((\Delta t)^3). \]  

(70)

Each of these equations has a second order term proportional to \( \partial_R J^t_x \) or \( \partial_L J^t_x \). We now show that these terms are actually third order if they were zero at the initial time when the simulation began. We show the method explicitly for the first equation by subtracting the same equation one time step in the past,

\[ F^t_x \Rightarrow F^{t-1}_x \]

The last term on the RHS is third order in \( \Delta t \). This is because the expression in parenthesis is first order in \( \Delta t \), which can be shown by

\[ \partial_R J^t_x - \partial_R J^{t-1}_x = \int_{t-1}^t d\tau \partial_x \partial_R J^\tau_x = \Delta t \partial_x \partial_R J^{t-1}_x + O((\Delta t)^3). \]  

(72)

The RHS was determined by presuming that since \( J \) is linearly dependent on \( E \) and the material properties do not change in time, the integrand changes smoothly. Therefore, the second order accurate midpoint approximation to the boundary. Subsuming the \( \partial_R J^t_x \) term into the error and moving some other terms around we get

\[ F^t_x - F^{t-1}_x + \frac{\mu v^2 \Delta t}{4} J^t_x = F^{t-1}_x - F^{t-2}_x + \frac{\mu v^2 \Delta t}{4} J^{t-1}_x + O((\Delta t)^3). \]  

(73)

The left and right sides are the same except for the time, indicating that each side is equal to the same quantity, which is constant in time, where ‘constant’ in this context means differing by terms third order in \( \Delta t \). If the initial conditions are that there are no fields and currents at the initial time, the RHS at time \( t = 0 \) is zero because each term on the RHS is individually zero. The constant will remain zero to second order accuracy for each time step, even when fields pass through the system,

\[ F^t_x - F^{t-1}_x + \frac{\mu v^2 \Delta t}{4} J^t_x = 0 + O((\Delta t)^3). \]  

(74)

We have managed to dispense with the troublesome \( \partial_R J \) term through an appeal to field-less and current-less initial conditions in the bulk of the material. The same procedure works with the second equation to dispense with the term proportional to \( \partial_L J \).

Our update equations for the pulses at the center are now

\[ F^t_x = F^{t-1}_x - \frac{\mu v^2 \Delta t}{4} J^t_x + O((\Delta t)^3), \]  

(75)

\[ G^t_x = G^{t-1}_x - \frac{\mu v^2 \Delta t}{4} J^t_x + O((\Delta t)^3). \]  

(76)

These equations are the same that one would derive exactly for the constant current model. The result of this section is that the constant current model, and therefore the traditional FDTD equations with the simplest generalization to variable grid spacing, are accurate to second order, even when the grid spacing varies with position and the material parameters make sudden jumps between layers. The price we have paid for this is the insistence that the initial conditions for the system must be field-less and current-less, which is true in the system interior for most simulations. The update equation for the magnetic field at the left-most system boundary, Eq. (71), where the initial field values are not necessarily zero, is only guaranteed to first order accuracy. Even so, the global accuracy of the method is preserved when the numerical treatment at the boundary is one order less accurate than the interior as has been shown by Gustafson.24

VIII. SUMMARY

The Finite Difference Time Domain method for numerically simulating electromagnetic waves in matter for one dimensional systems was derived using d’Alembert’s solution to the wave equation. The method utilizes forward and backward moving wave pulse variables instead of electric and magnetic field variables. Constant velocity of waves in
dispersionless dielectric materials, Eq. (22-23), partial reflection and transmission at boundaries between materials with different indices of refraction, Eq. (28-29), and reflection and attenuation through conducting materials, Eq. (49-50) were derived in the process of deriving the method so that real physics can be learned simultaneously with the numerics. The traditional FDTD equations were derived from a piece-wise constant current model showing the physical effects of the finite difference approximation in a way that is easy to visualize. d’Alembert’s method allowed for easy derivation of some results that are more complicated with finite differences alone such as showing that FDTD is exact for dielectrics when the ratio of the spatial and temporal grid spacings is the wave speed, Eq. (22-23), deriving reflectionless boundary conditions, Eq. (42-45, 61-62), and showing the method retains second order accuracy for grid spacing that varies with position and when the material parameters make sudden jumps across boundaries, Eq. (72-76).

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