GENERIC IRS IN FREE GROUPS, AFTER BOWEN

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ABSTRACT. Let $E$ be a measure preserving equivalence relation, with countable equivalence classes, on a standard Borel probability space $(X, \mathcal{B}, \mu)$. Let $(\mathcal{E}, d_u)$ be the (Polish) full group endowed with the uniform metric. If $F_r = \langle s_1, \ldots, s_r \rangle$ is a free group on $r$-generators and $\alpha \in \text{Hom}(F_r, \mathcal{E})$ then the stabilizer of a $\mu$-random point $\alpha(F_r)_x \triangleleft F_r$ is a random subgroup of $F_r$ whose distribution is conjugation invariant. Such an object is known as an invariant random subgroup or an IRS for short. Bowen’s generic model for IRS in $F_r$ is obtained by taking $\alpha$ to be a Baire generic element in the Polish space $\text{Hom}(F_r, \mathcal{E})$. The lean aperiodic model is a similar model where one forces $\alpha(F_r)$ to have infinite orbits by imposing that $\alpha(s_1)$ be aperiodic.

In this setting we show that for $r < \infty$ the generic IRS $\alpha(F_r)_x \triangleleft F_r$ is of finite index a.s. if and only if $E = E_0$ is the hyperfinite equivalence relation. For any ergodic equivalence relation we show that a generic IRS coming from the lean aperiodic model is co-amenable and core free. Finally, we consider the situation where $\alpha(F_r)$ is highly transitive on almost every orbit and in particular the corresponding IRS is supported on maximal subgroups. Using a result of Le-Maître we show that such examples exist for any aperiodic ergodic $E$ of finite cost. For the hyperfinite equivalence relation $E_0$ we show that high transitivity is generic in the lean aperiodic model.

1. INTRODUCTION

Let $\Gamma$ be a countable group, $\{0,1\}^\Gamma$ the space of all subsets of $\Gamma$ with the compact Tychonoff topology. The collection of all subgroups $\text{Sub}(\Gamma) \subset \{0,1\}^\Gamma$ is closed and therefore a compact metrizable space in its own right. The induced topology on $\text{Sub}(\Gamma)$ is known as the Chabauty topology (see [Cha50]). The group $\Gamma$ acts, continuously, from the left, by conjugation on $\text{Sub}(\Gamma)$.

**Definition 1.1.** An invariant random subgroup, or IRS for short, of $\Gamma$ is a $\Gamma$ invariant Borel probability measure on $\text{Sub}(\Gamma)$. We will also say that $\Delta \in \text{Sub}(\Gamma)$ is an invariant random subgroup and write $\Delta \triangleleft \Gamma$ to signify that such an invariant probability measure has been fixed and that $\Delta$ is a random subgroup chosen according to this distribution. We write IRS($\Gamma$) and IRS$^e(\Gamma)$ to denote the collection of all (resp. all ergodic) invariant measures on $\text{Sub}(\Gamma)$.

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Note that IRS(Γ) can be viewed as a simplex in the dual Banach space $C(\text{Sub}(\Gamma))^*$ and IRS(Γ) is the collection of extreme points of this simplex.

A similar notion can be defined in the setting of a locally compact group $G$ - see [ABB+, ABB+11]. In this setting $\text{Sub}(G)$ denotes the collection of all closed subgroups of $G$ which is again a compact metrizable space. During the last few years IRS turned out to be surprisingly useful in a wide array of mathematical branches see [AGV13a, AGV13b, ABB+, BG04, AL07, Bowb, Ver12, Ver10, Can, Bowa, BGKb, BGKa, GM, KN13, TTD, SZ94, Bek07, PT, CPa, CPb, HT, TD, Cre]. Topological analogues of IRS were introduced in [GW14].

**Example 1.2.** Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of $\Gamma$ on a probability space. Then there is a natural map $\Phi : X \rightarrow \text{Sub}(\Gamma)$ defined by $x \mapsto \Phi(x) = \Gamma_x$. Since the action is measure preserving and the map $\Phi$ is $\Gamma$ equivariant, the image of the measure $\Phi_\ast(x)\mu \in \text{IRS}(\Gamma)$. In short one can just say that $\Gamma_x \triangleleft \Gamma$ for every probability preserving action as above. In fact in [AGV13a, Proposition 13] it is shown that this example is universal in the sense that every $\nu \in \text{IRS}(\Gamma)$ can be obtained by such a construction whenever $\Gamma$ is finitely generated. This was generalized to locally compact groups in [ABB+].

The oldest, and one of the deeper results to date concerning IRS, due to Stuck-Zimmer [SZ94] is the full classification of IRS($G$) where $G$ is a higher rank semi-simple Lie group, or a lattice thereof. In both cases every $\mu \in \text{IRS}^e(G)$ is supported on an orbit (i.e. a conjugacy class) either of a finite central subgroup or of a discrete group $H$ of finite co-volume - when $G$ is a Lie group $H < G$ is a lattice and when $G$ is a lattice $H$ is just a finite index subgroup. Other structure results single out groups in which the space IRS($\Gamma$) is small are due to Vershik [Ver10] ($G = S_\infty^r$, the infinite finitary permutation group), Peterson-Thom [PT] ($\text{PSL}_n(k)$ where $k$ is a countable field).

Let $F_r = \langle S \rangle$ be the non-abeian free group on $r$-generators $S = \{s_1, s_2, \ldots, s_r\}$. In contrary to the above results, Lewis Bowen [Bowa] studies IRS($F_r$) using a variety of methods and finds a very rich structure.

**1.1. Bowen’s model of generic IRS in $F_r$.** One of the methods that Bowen introduces is that of a *generic IRS*. We give only a short survey of this method and refer the readers to the original paper [Bowa] (and also [Bowb]) for more details. We use the word *generic* in the Baire category sense of the word. A generic model for an IRS will consist of a Polish (= metrizable, separable and complete) topological space $I$ together with a map $f : I \rightarrow \text{IRS}(F_r)$. Given such a model we will look for properties of the IRS $f(x)$ that hold for every $x$ in a residual (e.g. a dense $G_\delta$) subset of $I$.

The group $A := \text{Aut}(X, \mathcal{B}, \mu)$ itself has a natural Polish structure coming from the weak topology. Thus the space $\text{Hom}(F_r, A) \cong A^r$ is also Polish. As explained in Example 1.2 above to any $\alpha \in \text{Hom}(F_r, A)$ one can associate
the IRS $\alpha(\Gamma)_x \triangleleft \Gamma$. But as it turns out, for a residual set of $\alpha$’s, this IRS is almost surely trivial. Bowen’s idea was to fix in advance a Borel equivalence relation $E \subset X \times X$ with countable classes. Assume that $\mu$ is $E$ invariant and that $E$ is aperiodic, in the sense of the following:

**Definition 1.3.** An equivalence relation $[F]$ is called periodic if its equivalence classes are finite a.s. It is called aperiodic if its equivalence classes are infinite a.s.

With such an equivalence relation we can associate its full group:

**Definition 1.4.** Given an equivalence relation $E$ as above let $[E]$ denote the full group of this equivalence relation:

$$[E] = \{ g : X \to X \mid g \text{ is a Borel isomorphism and } xEgx \forall x \in X \}.$$ 

endowed with the uniform metric

$$d(\phi, \psi) = \mu(\{ x \in X \mid \phi(x) \neq \psi(x) \}).$$

The uniform metric can be defined on the whole group $\text{Aut}(X, \mathcal{B}, \mu)$ but it gives rise to a non-separable topological group. When restricted to the full group $[E]$ the uniform metric gives rise to a Polish group structure as proven in [Kec10, Proposition 3.2].

Once this polish group has been fixed we can define the following Polish space:

$$\text{Hom}(F_r, [E]) \cong [E]^r,$$

$$d_u(\alpha, \beta) = \sup_{i=1...r} \{ d_u(\alpha(s_i), \beta(s_i)) \} \alpha, \beta \in \text{Hom}(F_r, [E]). \quad (1.1)$$

Once the equivalence relation has been fixed, the Gaboriau-Levitt theory of cost plays an important part. $\text{Cost}(E)$ is an invariant associated with the equivalence relation. A famous Theorem of Gaboriau [Gab00, Corollaire 1] shows that the cost of the orbit-equivalence relation coming from a free action of $F_r$ is exactly $r$. Thus if $r > \text{Cost}(E)$ the action given by any $\phi : F_r \to [E]$ cannot be essentially free and consequently the associated IRS cannot be a.s. trivial.

1.2. Main results. In many of our theorems we address the question of when the IRS arising from a generic $\alpha \in \text{Hom}(F_r, [E])$ is large. The word large here is considered in various different meanings of the word. One obvious sense in which this might occur is that the IRS would be of finite index, or in other words if the $\alpha(F_r)$-orbit equivalence relation is periodic. As it turns out, this depends on whether the equivalence relation is hyperfinite or not.

**Definition 1.5.** A Borel equivalence relation $E$ is called hyperfinite if it can be expressed as the ascending union of sub-equivalence relations $E = \cup_n F_n$ where $F_n \subset F_{n+1}$ and each $F_n$ has finite equivalence classes.
It is a famous result of Orenstein-Weiss [OW80] that every action of an amenable group gives rise to a hyperfinite equivalence relation.

**Theorem 1.6.** Let $E$ be a Borel, measure preserving equivalence relation with countable equivalence classes on a standard probability space $(X, \mathcal{B}, \mu)$. For $\alpha \in \text{Hom}(F_r, [E])$ consider the associated IRS $\alpha(F_r) \lhd F_r$ and its index $I_\alpha := [F_r : \alpha(F_r)_x]$ - an integer valued random variable. Then for a generic $\alpha \in \text{Hom}(F_r, [E])$ the following holds:

1. **For** $r = \infty$. $\alpha(F_\infty)$ spans the equivalence relation. In particular $I_\alpha = \infty$ a.s.

2. **For** $2 \leq r < \infty$. The following conditions are equivalent:
   - $E$ is hyperfinite.
   - $I_\alpha < \infty$ a.s.

One of the main reasons for considering generic models for IRS is as a rich source of examples. As such finite index IRS are not interesting. It is therefore desirable to find a generic model that forces the IRS to be of infinite index. The most natural way to do this is to set $\text{Hom}^\infty(F_r, [E]) := \{\alpha \in \text{Hom}(F_r, [E]) | [F_r : \alpha(F_r)_x] = \infty \text{ a.s.}\}$. This turns out to be a $G_\delta$ subset of $\text{Hom}(F_r, [E])$ and therefore, even though it might be meager, it is a Polish space in its own right and can serve as a model for infinite index IRS. We adopt a different model which is perhaps less natural but much easier to work with. We force the action to be non-periodic by demanding that the first generator $S := \alpha(s_1)$ already be non-periodic.

**Definition 1.7.** For every $r$ and every aperiodic, measure preserving equivalence relation $E$ on a standard probability space we define the lean aperiodic model to be

$I = I(r, E) = \{\alpha \in \text{Hom}(F_r, [E]) | \alpha(s_1) \text{ is aperiodic}\} \subset \text{Hom}(F_r, [E])$.

**Remark 1.8.** Theorem [Kec10, Theorem 3.6] says that whenever $E$ is ergodic the set $I_e := \{\alpha \in I | \alpha(s_1) \text{ is ergodic}\}$ is residual in $I$. Thus in this case any generic statement in $I_e$ is true also in $I$ and vice versa.

Just like amenability is a very natural generalization of finiteness, co-amenability, in the sense of the definition below, generalizes the notion of finite index.

**Definition 1.9.** A subgroup $\Delta < \Gamma$ is called coamenable if one and hence all of the following equivalent conditions hold: (i) there is a $\Gamma$ invariant mean on $\Gamma/\Delta$; (ii) there exists a sequence of Følner sets $F_n \subset \Gamma/\Delta$ such that $\lim_{n \to \infty} |\gamma F_n \Delta F_n| / |F_n| = 0$ for every $\gamma \in \Gamma$; (iii) for every continuous affine action of $\Gamma$ on a compact convex subspace $C$ of a locally convex space, the existence of a $\Delta$ fixed point implies the existence of a $\Gamma$ fixed point.

Our next theorem says that for ergodic equivalence relations co-amenability is generic in the lean aperiodic model.
**Theorem 1.10.** Let $E$ be an ergodic, measure preserving equivalence relation with countable equivalence classes on a standard Borel probability space $(X, \mathcal{B}, \mu)$. Let $I$ be the lean aperiodic model defined above. Then for a residual set of $\alpha \in I$ the associated IRS $\alpha(F_r)_x \ll F_r$ is co-amenable a.s.

**Remark 1.11.** If $E = E_0$ is the hyperfinite equivalence relation then it was proved by Vadim Kaimanovich in [Kai97] that every $\alpha \in \text{Hom}(F_r, [E_0])$ gives rise to a coamenable IRS $\alpha(F_r)_x \ll F_r$.

Here are a few additional ways in which a subgroup can be large.

**Definition 1.12.** Let $\Gamma$ be a countable group. A subgroup $\Delta < \Gamma$ is called co highly-transitive or co-HT for short, if the permutation action $\Gamma \curvearrowright \Gamma/\Delta$ is transitive on ordered $k$-tuples of points for every $k \in \mathbb{N}$.

**Definition 1.13.** The core of a subgroup $\Delta < \Gamma$ is $\text{Core}_\Gamma(\Delta) = \cap_{\gamma \in \Gamma} \gamma \Delta \gamma^{-1}$ - the largest normal subgroup that it contains. If $\text{Core}_\Gamma(\Delta) = \langle e \rangle$ we say that $\Delta$ is core free.

**Definition 1.14.** A subgroup $\Delta < \Gamma$ is called profinitely dense (resp. pro-dense) if $\Delta N = \Gamma$ for every finite index (resp. for every non-trivial) normal subgroup $N < \Gamma$.

Clearly every pro-dense subgroup is in particular profinitely dense. It is well known that the following implications hold for a subgroup $\Delta < \Gamma$.

- co-HT $\Rightarrow$ Maximal of infinite index $\Rightarrow$ profinitely dense
- co-HT and core free $\Rightarrow$ Maximal and core free $\Rightarrow$ pro dense

For the core of a generic IRS we have the following

**Theorem 1.15.** Let $E$ be an aperiodic measure preserving equivalence relation with countable equivalence classes on a standard Borel probability space $(X, \mathcal{B}, \mu)$. Let $\alpha$ be a generic element in the lean aperiodic model $I$. Then the associated IRS $\alpha(F_r)_x \ll F_r$ is core free almost surely.

For every aperiodic Borel equivalence relation $E$ there exist $\alpha \in \text{Hom}(F_r, [E])$ s.t. the associated IRS is co-HT a.e. To see this note that every such $E$ contains an ergodic aperiodic hyperfinite subequivalence relation $E_0$ ([KM04, Lemma 23.2]). From 1.17 below, there is an abundance of elements in $\text{Hom}(F_r, [E_0])$ s.t. the associated IRS is highly transitive a.e. But of course each such element is also an element of $\text{Hom}(F_r, [E])$. It seems more interesting to construct actions that span the whole equivalence relation, whose associated IRS are co-HT a.s. This can be done, albeit not in a generic construction. A very elegant paper of François Le-Maître [LM] proves that if $r > \text{Cost}(E)$, for an ergodic equivalence relation $E$, then there exists a representation $\alpha \in \text{Hom}(F_r, [E])$ with a dense image. As a more or less direct consequence of his result we can show that for every ergodic equivalence relation $E$ on $(X, \mathcal{B}, \mu)$ with $\text{Cost}(E) < r$ there exists a measure preserving action of $F_r$ on $X$ generating $E$ and acting HT on a.e. $E$-class:
Theorem 1.16. Let $E$ be an ergodic measure preserving equivalence relation with countable equivalence classes on a standard Borel probability space $(X, \mathcal{B}, \mu)$. Assume that $\text{Cost}(E) < r$ then there exists an action $\alpha \in \text{Hom}(F_r, [E])$ that generates $[E]$, and whose associated IRS $\alpha(F_r)_x \ll F_r$ is almost surely co highly transitive.

We do not know if a generic IRS in the lean aperiodic model is co-HT for general equivalence relations. However for the hyperfinite equivalence relation we can prove that this is the case.

Theorem 1.17. Let $E_0$ be the ergodic aperiodic hyperfinite equivalence relation and $I$ the corresponding lean aperiodic model. For a generic $\alpha \in I$ the associated IRS $\alpha(F_r)_x \ll F_r$ is co-HT.

Remark 1.18. If $H < F_r$ is a subgroup and $\Delta \ll F_r$ it is possible to restrict the IRS to the subgroup $\Delta \cap H \ll H$. A slightly more general consequence can be proved, treating also the restriction of a generic IRS to many subgroups of $F_r$. We mention this in passing, but chose not to include the proof as it is slightly more technical on the one hand and seems incomplete as it treats only very special subgroups on the other hand. If you are interested, prove it or find the proof commented out in the texfile.

Theorem 1.19. In the setting of Theorem 1.17, for a generic $\alpha$ and for a subgroup $H < F_r$. If $H$ is a non-cyclic group that contains a conjugate of a power of $s_1$ then $\Delta \cap H \ll H$ is co-HT in $H$. We chose not to include the proof of this more general statement because it is slightly more technical and it is not clear to us that if it is really useful.

2. Basic lemmata and definitions

Lemma 2.1. Let $E$ be an ergodic equivalence relation on a standard Borel probability space $(X, \mathcal{B}, \mu)$ and let $\sigma \in [E]$. Then given any $A \in \mathcal{B}$ and $\tau \in [E]$ there exists $\sigma_1 \in [E]$ such that $\sigma_1|_A = \tau|_A$ and $d_u(\sigma, \sigma_1) < 2\mu(A)$.

Proof. Let

$$\sigma_1(x) = \begin{cases} \tau(x) & x \in A \\ \sigma(x) & x \notin A \cup \sigma^{-1}\tau(A) \\ \eta(x) & \text{otherwise} \end{cases}$$

(2.1)

where $\eta \in [E]$ is some element taking $\sigma^{-1}\tau A \setminus A$ to $\sigma(A) \setminus \tau(A)$ the existence of which is guaranteed by the fact that $[E]$ is ergodic and that these two sets have the same measure. $\square$

Lemma 2.2. The substitution map $I \to [E]$ given by $\alpha \mapsto \alpha(\gamma)$ for some fixed $\gamma \in F_r$ is continuous. In fact $d_u(\alpha(\gamma), \beta(\gamma)) \leq \ell(\gamma)d_u(\alpha, \beta)$, $\forall \alpha, \beta \in I$, where $\ell(\gamma)$ is the word length of $\gamma$ with respect to the standard set of generators.
We prove this by induction on \( \ell = \ell(\gamma) \). Let \( V(\gamma) = \{ x \in X \mid \alpha(\gamma, x) \neq \beta(\gamma, x) \} \). By definition of the uniform metric \( \mu(V(s)) \leq d_u(\alpha, \beta), \forall s \in S \sqcup S^{-1} \) which means that the lemma is true for \( \ell = 1 \). Assume now that \( \gamma = \gamma' \) s where \( s \in S \sqcup S^{-1} \) and \( \gamma' \in F_r \) with \( \ell(\gamma') = \ell - 1 \),

\[
V(\gamma) \subset V(s) \cup (\alpha(s))^{-1}(V(\gamma'))
\]

and the lemma follows. \( \square \)

We recall that a Schreier graph of \( F_r \) is a \( 2r \)-regular rooted graph \( (Z, z_0) = (V, E, z_0) \) with a labeling of the oriented edges \( \iota : E \to S \sqcup S^{-1} \) satisfying the two properties \( \iota(1) = \iota(e)^{-1} \) and \( \iota^{-1}(Lk(v)) = S \sqcup S^{-1} \) \( \forall v \in V \). To every \( \alpha \in \text{Hom}(F_r, [E]) \) and \( x \in X \) we can associate the corresponding Schreier graph \( \text{Sch}(\alpha, x) = (V, E, x) \) with vertex set \( V = \alpha(F_r) \cdot x \) and an edge labeled \( s \) connecting \( v \) and \( sv \) for every \( v \in V \) and \( s \in S \sqcup S^{-1} \). By an isomorphism of Schreier graphs we mean a graph isomorphism that respects the base vertex and the edge coloring. Let \( S\text{gr}(r) \) be the collection of isomorphism classes of \( F_r \)-Schreier graphs with the natural topology that turns it into a compact metrizable space. Namely \( d((Z, z_0), (Y, y_0)) = e^{-\max\{ r \in \mathbb{N} \mid B_r(Z, z_0) \sqsubseteq B_r(Y, y_0) \}} \).

**Lemma 2.3.** For a subset \( \Xi \subset S\text{gr}(r) \) we define

\[
\tilde{\Xi} = \{ \alpha \in \text{Hom}(F_r, [E]) \mid \text{Sch}(\alpha, x) \in \Xi \text{ for } \mu \text{ a.e. } x \in X \}.
\]

Then \( \tilde{\Xi} \) is \( G_\delta \) in \( \text{Hom}(F_r, [E]) \) whenever \( \Xi \) is \( G_\delta \) in \( S\text{gr}(r) \).

**Proof.** It is enough to prove the statement if \( \Xi \) is open. Indeed if \( \Xi = \cap_i \Xi_i \) is a countable intersection of open sets \( \Xi_i \) then \( \tilde{\Xi} = \cap_i \tilde{\Xi}_i \). So we will assume that \( \Xi \) is open. In a metric space this just means a union of open balls \( \Xi = \bigcup_{j \in J} B_{f_j}(R_j) \), where \( F = \{ f_j \mid j \in J \} \subset S\text{gr} \) is a set of Schreier graphs, \( B_{f_j}(R_j) \) denotes the set of all Schreier graphs \( f \) such that the ball of radius \( R_j \) in \( f \) centred at its base point is isomorphic to the ball of radius \( R_j \) in \( f_j \) centred at its base point, and \( \{ R_j \mid j \in J \} \) a corresponding collection of radii. (Note that every ball in \( S\text{gr}(r) \) can be written as \( B_{f_j}(R_j) \)). Set

\[
\tilde{\Xi}(\epsilon) = \{ \alpha \in \text{Hom}(F_r, [E]) \mid \mu \{ x \in X \mid \text{Sch}(\alpha, x) \in \Xi \} > 1 - \epsilon \}
\]

Since \( \tilde{\Xi} = \cap_{m \in \mathbb{N}} \tilde{\Xi}(1/m) \), it is enough to prove that \( \tilde{\Xi}(\epsilon) \) is open for all \( \epsilon > 0 \). Now for every \( \alpha \in \tilde{\Xi}(\epsilon) \) one can find a finite subset \( JF \subset J \) such that \( \mu \{ x \in X \mid \text{Sch}(\alpha, x) \in B_{f_j}(R_j) \text{ for some } j \in JF \} > 1 - \eta > 1 - \epsilon \), for some \( \eta < \epsilon \). Let \( R = \max \{ R_j \mid j \in JF \} \) be the maximal of these radii.
For $\beta$ close enough to $\alpha$, say $d_u(\alpha, \beta) < \delta$, consider the set
\[
\{ x \in X \mid \text{Sch}(\beta, x) \in B_{f_j}(R_j) \text{ for some } j \in JF \}.
\]
We want to show that if $\delta$ is sufficiently small then this set still has a measure larger than $1 - \epsilon$. Let $\Omega = B_{F_r}(2R + 1) = \{ \gamma \in F_r \mid \ell(\gamma) < 2R + 1 \}$. Consider the set
\[
B = \{ x \in X \mid \alpha(\gamma, x) = \beta(\gamma, x) \forall \gamma \in \Omega \}
\]
Applying Lemma 2.2 to each $\gamma \in \Omega$ separately and taking the union bound we see that
\[
\mu(B) \geq 1 - \delta(2R + 1)|\Omega| \geq 1 - \delta(2R + 1)(2R)^{2R+1} > 1 - \eta'.
\]
for some $\eta' < \epsilon - \eta$, where the last inequality is easily attained by choosing $\delta$ sufficiently small. For every $x \in B$ the two balls in the Schreier graphs are isomorphic $B_{\text{Sch}(\alpha,x)}(R) \cong B_{\text{Sch}(\beta,x)}(R)$. Indeed two elements of $\gamma, \gamma' \in \Gamma$:
\begin{itemize}
  \item Represent the same vertex in the Schreier graph $\text{Sch}(\alpha, x)$ if and only if $\alpha(\gamma^{-1}\gamma', x) = x$.
  \item Represent two vertices connected by an edge labeled $s$ if and only if $\alpha(\gamma^{-1}s\gamma', x) = x$.
\end{itemize}
But if $\ell(\gamma), \ell(\gamma') < R$ then all of the above discussion implies that these two quarries would yield the exact same results if we apply $\beta$ instead of $\alpha$. This finishes the proof, as clearly $\mu(\{ x \in X \mid \text{Sch}(\beta, x) \in B_{f_j}(R_j) \text{ for some } j \in JF \}) > 1 - \eta - \eta' > 1 - \epsilon$. \hfill $\square$

3. Periodicity vs. hyperfinitness

In this section we prove Theorem 1.6. We start with a lemma of independent interest.

**Lemma 3.1.** Let $\alpha_n, \alpha \in \text{Hom}(F_r, \text{Aut}(X, B, \mu))$ with $\lim_n \alpha_n \overset{u}{\rightarrow} \alpha$ in the uniform topology. If the orbits of $\alpha_n(F_r)$ are a.s. finite for every $n$ then the equivalence relation spanned by $\alpha(F_r)$ is hyperfinite.

**Proof.** Let $E$ denote the equivalence relation induced by $\alpha(F_r)$. We would like to show that there is a co-null subset $Y$ of $X$ s.t. the restriction of $E$ to $Y$ is hyperfinite. We may suppose, passing to a subsequence of $\{\alpha_n\}$ if necessary, that $d_n := d_u(\alpha_n, \alpha) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$, where $d_u$ denotes the uniform distance as usual. Let $s_1, \ldots, s_r$ be the fixed set of generators of $F_r$. Let $E_n'$ denote the equivalence relation generated by $\alpha_n$. For every $n$ consider the subset $A'_n := \{ x \in X \mid \alpha_n(s_i)(x) = \alpha(s_i)(x), \ 0 \leq i \leq r \}$. Clearly $\mu(A'_n) > 1 - \frac{1}{2^n}$. Define $A_n := A'_n \setminus \bigcup_{k=n+1}^{\infty} (X \setminus A'_k)$. Then $\{A_n\}$ is an ascending sequence of Borel subsets of $X$, $\mu(A_n) > 1 - \frac{1}{2^n}$, and $\alpha_n(s_i)(x) = \alpha(s_i)(x)$ for all $x \in A_n$ and $1 \leq i \leq r$. Let $E_n$ be the Borel equivalence relation defined as $E_n|A_n$ on $A_n$ and as the identity relation on $X \setminus A_n$. Then $\{E_n\}_{n=1}^{\infty}$ is an ascending sequence of periodic equivalence
relations. Fix $\gamma \in F_r$, and suppose $\gamma = s_{i_1}s_{i_2}\ldots s_{i_k}$. For a given $n \in \mathbb{N}$, we have

\[
\mu \left( \{ x \in X \mid \langle x, \alpha(\gamma)x \rangle \notin E_n \} \right) \\
\leq \mu \left( \{ x \in X \mid \langle x, \alpha(s_{i_1})x \rangle \notin E_n \lor \langle \alpha(s_{i_1})x, \alpha(s_{i_1}, s_{i_2})x \rangle \notin E_n \lor \ldots \lor \langle \alpha(s_{i_2} \ldots s_{i_k})x, \alpha(\gamma)x \rangle \notin E_n \} \right) \\
\leq \sum_{j=1}^{k} \mu \left( \{ x \in X \mid \langle \alpha(s_{i_1} \ldots s_{i_k})x, \alpha(s_{i_1} \ldots s_{i_k})x \rangle \notin E_n \} \right) \\
\leq \sum_{j=1}^{k} \mu \left( \{ x \in X \mid \langle \alpha(s_{i_1})x \rangle \notin E_n \} \right) \\
\leq \sum_{j=1}^{k} \mu \left( X \setminus A_n \cup X \setminus s_{i_j}^{-1}A_n \right) \leq 2k \cdot \frac{1}{2^{n-1}}
\]

Let $B_{\gamma} := \{ x \in X \mid \langle x, \alpha(\gamma)x \rangle \notin E_n, \forall n \in \mathbb{N} \}$. Then by the calculation above $\mu(B_{\gamma}) = 0$ and $\mu(\cup_{\gamma \in F_r} B_{\gamma}) = 0$. Let $Y := X \setminus \cup_{\gamma \in F_r} B_{\gamma}$. Then $\mu(Y) = 1$ and $E_{\gamma} \subset (\cup E_n)_{\gamma}$, so the restriction of $E$ to the co-null subset $Y$ is hyperfinite, proving the lemma. \hfill $\square$

**Remark 3.2.** Note that in the lemma above we may take any finitely generated group instead of $F_r$, and we may also assume $\alpha_n$ hyperfinite for all $n \in \mathbb{N}$ instead of periodic. (The last assertion is due to the fact that an ascending union of hyperfinite equivalence relations is hyperfinite.) Thus, the subset of the space of actions of a f.g. group consisting of actions that generate a hyperfinite equivalence relation is closed in the uniform topology.

**Proof.** (of Theorem 1.6) We start with the case $r = \infty$. Let

\[
\text{SPAN} = \{ \alpha \in \text{Hom}(F_\infty, [E]) \mid \alpha(F_\infty) \text{ spans the equivalence relation } E \}.
\]

By the Feldman-Moore theorem [FM77], [KM04, Theorem 1.3] we can assume that the equivalence relation $E$ is generated by a different action $\beta \in \text{Hom}(F_\infty, [E])$. Fixing this action $\beta$ once and for all it is clear that $\text{SPAN} = \bigcap \text{SPAN}(l, \Omega)$, where the intersection is over all finite subsets $\Omega \subset F_\infty$, all $l \in \mathbb{N}$ and

\[
\text{SPAN}(l, \Omega) = \left\{ \alpha \in \text{Hom}(F_\infty, [E]) \mid \mu \{ x \in X \mid \forall \omega \in \Omega, \exists \gamma \in F_\infty \text{ s.t. } \alpha(\gamma)x = \beta(\omega)x \} > 1 - \frac{1}{l} \right\}.
\]

We claim that every $\text{SPAN}(l, \Omega)$ is open and hence $\text{SPAN}$ is $G_\delta$. Indeed if $\alpha \in \text{SPAN}(l, \Omega)$ then in fact it is so by virtue of only finitely many $\gamma \in F_r$. Namely for such $\alpha$ we find a finite set $\Gamma \subset F_r$ such that

\[
\mu \{ x \in X \mid \forall \omega \in \Omega, \exists \gamma \in \Gamma \text{ s.t. } \alpha(\gamma)x = \beta(\omega)x \} > 1 - \frac{1}{l}.
\]

If $d_n(\alpha, \alpha')$ is small enough the same will be true for $\alpha'$. Which proves that $\text{SPAN}(l, \Omega)$ is open.

We turn to demonstrate density of $\text{SPAN}(l, \Omega)$. Fix a basic open set in $\text{Hom}(F_\infty, [E])$, of the form $U = U(n; U_1, U_2, \ldots U_n) = \{ \alpha \in \text{Hom}(F_\infty, [E]) \mid \alpha(s_i) \in U_i \}$ . It is clear that $U \cap \text{SPAN} \neq \emptyset$, as one can always choose $\{ \alpha(s_i) \mid i > N \}$ in such a way that these span the whole equivalence relation. Thus we have
shown that $\text{SPAN} \subset \text{Hom}(F_{\infty}, [E])$ is a dense $G_\delta$ proving statement (1) of the Theorem.

From now on we focus on finite $r$. As the collection of finite graphs is clearly open in $S^{\text{gr}}$, Lemma 2.3 implies that the set

$$\text{Fin}(r) = \{ \alpha \in \text{Hom}(F_r, [E]) \mid \text{almost all } \alpha(F_r) - \text{orbits are finite} \}$$

is $G_\delta$. In order to prove claim (2) of the theorem we have to show that for every $1 < r \in \mathbb{N}$, $\text{Fin}(r) \subset \text{Hom}(F_r, [E])$ is dense if and only if $E$ is hyperfinite.

Assume first that $E$ is hyperfinite and let us realize it as an ascending union of finite equivalence relations $E = \bigcup F_j$. Setting

$$X(n, \gamma) = \{ x \in X \mid \alpha(\gamma)xF_nx \},$$

it is clear that $X(n, \gamma^{-1}) = \alpha(\gamma)(X(n, \gamma))$ and that we can choose some $N \in \mathbb{N}$ such that

$$\mu(X(N, s)) > 1 - \epsilon \quad \forall s \in S \sqcup S^{-1}.$$

For every $s \in S \sqcup S^{-1}$ we define

$$\beta(s)(x) = \begin{cases} 
\alpha(s)(x) & \text{if } x \in X(N, s) \\
\alpha(s^{-\xi(x,s)}x) & x \in X(N, s^{-1}) \setminus X(N, s) \\
x & \text{otherwise}
\end{cases}$$

Where $\xi(x, \gamma) \in \mathbb{N}$ is defined to be the minimal integer such that $\alpha(\gamma^{-\xi(x,\gamma)})x \notin X(N, \gamma^{-1})$. It is easy to check that this is well defined and gives us the desired $\beta$.

Conversely, assume that the periodic representations are dense in $\text{Hom}(F_r, [E])$. By Lemma 3.1, every element of $\text{Hom}(F_r, [E])$ generates a hyperfinite sub-equivalence relation of $E$. By the Feldman-Moore theorem [FM77], [KM04, Theorem 1.3] we can assume that $E$ is generated by the action of a group $\Gamma$ generated by $\{ \phi_1, \phi_2, \ldots \}$. Let $E_n \subset E$ be the equivalence relation generated by the action of $\langle \phi_1, \ldots, \phi_n \rangle$. By the above observation $E_2$ is hyperfinite (being generated by an element of $\text{Hom}(F_r, [E])$). Now we argue by induction that $E_n$ is hyperfinite for every $n$. Indeed if $E_n$ is hyperfine then it is generated by the action of a single element $\langle \psi_n \rangle < [E]$ and $E_{n+1}$ is therefore generated by the action of the two generated group $\langle \psi_n, \phi_{n+1} \rangle$, so that the above observation still applies. As $E$ is the ascending union of the $E_n$ it follows that $E$ is also hyperfinite, as claimed.

\[\square\]

4. Amenability

This section is dedicated to the proof of Theorem 1.10. We wish to show that the set:

$$\text{Amm} := \{ \alpha \in I \mid \alpha(\Gamma)x < \Gamma \text{ is coamenable for almost all } x \in X \}$$

is residual in $I$. 

Proof. We will prove that this set is a dense $G_δ$. The $G_δ$ condition follows, using Lemma 2.3, from the fact that the collection of Schreier graphs that contain a $1/l$-Følner set is open in $\mathcal{G}$.

Now for density. Given any $α ∈ I$ and $ε > 0$ we seek $β ∈ \mathcal{A}$ such that $d_α(α, β) < ε$. The idea of the proof uses a variation on a construction by Vadim Kaimanovich [Kai97] further developed in [KM04, Theorem 9.7]. Let us define $S := (s_1)$, and note that by ergodicity and Remark 1.8 we may assume $S$ is ergodic. Let $A ⊆ X$ be a subset s.t. $μ(A) < \frac{ε}{2r}$, and let $\{A_n\}_{n=1}^∞$ be a Borel partition of $A$ (s.t. $μ(A_n) > 0$ for infinitely many $n ∈ \mathbb{N}$). The restriction of $E$ to $A_n$ is aperiodic and so by [KM04, proposition 7.4] we may choose a finite sub-equivalence relation $F_n ⊂ E |_{A_n}$ s.t. every equivalence class in $F_n$ has cardinality $n$. For every $F_n$ choose a transversal $T_n$ and let $T := \bigcup_{n=1}^∞ T_n$. Such a transversal exists because $F_n$ has finite classes, take for instance a Borel linear ordering on $A_n$ and let the transversal be the minimal element in each class. We now define $β$ as follows: Let $β(s_1)$ be an element of $[E]$ that generates $F_n$ on each $A_n$, and s.t. $d_α(α(s_1), β(s_1)) < \frac{ε}{r}$, constructed as in 2.1. Using the same Lemma, for $1 ≤ i < r$ let $β(s_i)$ be such that $d_α(α(s_i), β(s_i)) < \frac{ε}{i}$ and $β(s_i)$ acts as the identity on $A \setminus T$. In addition, for $y ∈ S^{-1}(A \setminus T) \setminus (A \setminus T)$ let $β(s_1)(y)$ be defined to be $S^k(y)$ where $k ∈ \mathbb{N}$ is the least positive natural number s.t. $S^k(y) ∉ A \setminus T$. By ergodicity, for a.e. $y ∈ X \setminus A$ the $S$ orbit of $y$ intersects $T_n$ for infinitely many $n ∈ \mathbb{N}$. But the definition of $β(s_1)$ insures that if $S^m(y) ∈ T_n$, for some $m ∈ \mathbb{N}$, then for some natural $k$ we have $(β(s_1))^k(y) ∈ T_n$. Thus for a.e. $y ∈ X$ the orbit of $y$ under $β(F_n)$ contains an equivalence class of $F_n$ for infinitely many $n ∈ \mathbb{N}$. Consequently, we may choose the sequence of Følner sets to be those $F_n$ equivalence classes.

\[5.0\]

5. CO-HT SUBGROUPS AFTER LE-MAÎTRE

Theorem 1.16 follows directly from the theorem of Le-Maître on the existence of an $α ∈ \text{Hom}(F_n, [E])$ with a dense image, coupled with the following Proposition whose proof relies on the Lemma that follows it.

Proposition 5.1. Let $Δ < [E]$ be a dense subgroup of the full group of a Borel equivalence relation $E$ on $(X, μ)$. Then $Δ$ acts highly transitively on a.e. equivalence class of $E$.

Lemma 5.2. Let $Γ$ be a group acting on a standard Borel measure space $(X, μ)$ by measure preserving automorphisms. Let $γ_1, \ldots, γ_n ∈ Γ$ be a set of elements of the group. Let $\text{Supp}(γ_i) = \{x ∈ X | γ_i x ≠ x \}$, and let $A_0 ⊆ \bigcap_i \text{Supp}(γ_i)$. Suppose $μ(A_0) > 0$. Then there exists a Borel subset $A ⊂ A_0$ s.t. $μ(A) > 0$ and $μ(A \cap γ_i A) = 0$, $∀ 1 ≤ i ≤ n$.

Proof. (Of the lemma) The proof is by induction. Assume we have found $A_i ⊂ A_0$ with positive measure and such that $μ(A_∩ γ_j A) = 0$, $∀ 1 ≤ j ≤ i$. If there exists a subset $B ⊂ A_i$ such that $μ(B \setminus γ_i+1 B) > 0$ then we may choose $A_{i+1} = B \setminus γ_i+1 B$. But if no such $B$ exists then $γ_i+1 B = B$ modulo null
sets for any subset of the Borel set \( A_i \). This implies that \( \gamma_{i+1} \) acts trivially on \( A_i \) in contradiction to the assumption that \( A_i \subset A_0 \subset \text{Supp}(\gamma_{i+1}) \). This completes the proof of the lemma.

Proof. (of the Proposition) The lemma implies that for almost every \( x \in \cap_i \text{Supp}(\gamma_i) \) there exists a subset \( A \) as in the lemma, with \( x \in A \). (Indeed, if there were a subset of positive measure in \( \cap_i \text{Supp}(\gamma_i) \) such that there was no such subset for any \( x \) this would contradict the lemma). Now let \( G \) be a group that generates \( E \). Let \( x \in X \) and \( g_1, \ldots, g_n \) be such that \( g_1 x, \ldots, g_n x \) are all distinct. This is equivalent to saying that \( x \in \cap_i \text{Supp}(g_i^{-1} g_j) \). For ease of reference let’s denote \( S_{g_1, \ldots, g_n} := \cap \text{Supp}(g_i^{-1} g_j) \). Thus, almost every \( x \in S_{g_1, \ldots, g_n} \) is contained in a Borel subset \( A \subset S_{g_1, \ldots, g_n} \) as in the lemma.

In particular, the sets \( g_i A \) are pairwise disjoint. As \( X \) is a standard Borel space we may assume that \( A \) belongs to some countable collection \( \{A_k\}_{k=1}^\infty \).

Given a permutation \( \tau \in S_n \), define an element \( h_k \) of the full group \([E]\) by \( h_k A_k = g_{\tau(i)} g_i^{-1} A_k \). (On the rest of \( X \), \( h_k \) may be defined arbitrarily). By definition, \( \forall x \in A_k \), \( h_k g_k x = g_{\tau(i)} x \). Let \( \delta_k \in \Delta \) be such that \( d_u(\delta_k, h_k) < \frac{\epsilon}{2^k} \), where \( d_u \) denotes the uniform distance. Then we see that for every \( x \) in a subset of \( S_{g_1, \ldots, g_n} \) of measure at least \( \mu(S_{g_1, \ldots, g_n}) - \epsilon \) there exists an element \( \delta \in \Delta \) such that \( \delta g_i x = g_{\tau(i)} x \). As \( \epsilon \) may be chosen to be arbitrarily small, this implies that for a.e. \( x \in S_{g_1, \ldots, g_n} \), there exists \( \delta \in \Delta \) as above.

In order to finish the proof we would like to show that for a.e. \( x \in X \), for every \( n \in \mathbb{N} \), and any set \( \{x_1, \ldots, x_n\} \subset [x]_E \) of distinct elements, and any \( \tau \in S_n \), there exists \( \delta \in \Delta \) s.t. \( \delta x_i = x_{\tau(i)} \). Suppose \( x \) is a ”bad” element, i.e. one for which the above property does not hold. Then there exist distinct elements \( \{x_1, \ldots, x_n\} \subset [x]_E \), (where we may assume \( x_i = g_i x \) for \( g_i \in G \), and some \( \tau \in S_n \) s.t. one cannot find an element of \( \Delta \) that induces \( \tau \) on this set. This implies that \( x \) belongs to some null subset \( N_{\tau, g_1, \ldots, g_n} \subset S_{g_1, \ldots, g_n} \).

But one can easily see that there is only a denumerable set of these subsets as one varies \( n \in \mathbb{N} \), \( \tau \in S_n \) and \( g_1, \ldots, g_n \in G \). Thus the union of all the sets \( N_{\tau, g_1, \ldots, g_n} \) is also a null set, and the proposition is proven.

Note that the proof does not use ergodicity. Le-Maitre announces in [LM] results on more general equivalence relations (i.e. not necessarily ergodic). For example, for equivalence relations generated by a free action of a free group on \( r \) generators it is claimed that the full group is topologically generated by \( r + 1 \) elements. Obviously, the same conclusion can be drawn regarding to highly transitive actions.

6. Higher Transitivity

Here we prove Theorem 1.17. Fix our equivalence relation to be the ergodic hyperfinite equivalence relation \( E_0 \). We want to show that the set \( \text{HT} = \{ \alpha \in \mathcal{I} \mid \alpha(\Gamma) \text{ is highly transitive on almost all equivalence classes} \} \), is dense \( G_\delta \)
Proof. Let \( \sigma \in [E_0] \) be an automorphism generating the equivalence relation. Clearly \( HT = \bigcap_{m \in \mathbb{N}, \tau \in S_m} HT(\tau) \) where

\[
HT(\tau) = \{ \alpha \in I \mid \text{for a.e. } x \in X, \exists \gamma \in F, \text{ such that } \alpha(\gamma, \sigma^i x) = \sigma^i x, \forall 0 \leq i \leq m-1 \}.
\]

so by Baire’s theorem it is enough to show that the set \( HT(\tau) \) is a dense \( G_\delta \) subset of \( I \) for every \( m \in \mathbb{N} \) and every \( \tau \in S_m \). The \( G_\delta \) claim follows directly from Lemma 2.3.

To prove density of \( HT(\tau) \), let \( \alpha \in I \) and \( \epsilon > 0 \) be given, we seek \( \beta \in HT(\tau) \) with \( d_u(\alpha, \beta) < \epsilon \). By [Kec10, Theorem 3.4] the conjugacy class of any aperiodic element in \( [E] \) is uniformly dense within the set of aperiodic elements of \( [E] \), hence after replacing \( \alpha \) by a very close representation and conjugating we may assume that \( \alpha(s_1) = \sigma \).

Now use Lemma 5.2 to find a set \( O \) of measure \( 0 < \mu(O) < \epsilon/2m \) such that \( \{ \sigma^i O \mid 0 \leq i \leq m-1 \} \) are pairwise disjoint. By Lemma 2.1, we can find an element \( \beta \in I \) such that \( \beta(s_2, \sigma^i x) = \sigma^i x \forall x \in O \) and such that \( d_u(\alpha, \beta) < \epsilon \). Now by ergodicity for almost every \( x \in X \) one can find an \( n \in \mathbb{N} \) such that \( \sigma^n x \in O \). We choose \( n = n(x) \) to be the minimal natural number satisfying this property. Now clearly

\[
\beta(s_1^{-n(x)} s_2 s_1^n(x), \sigma^i x) = \sigma^i x, \text{ for a.e. } x \in X, \text{ and } \forall 0 \leq i \leq m-1.
\]

Proving that \( \beta \in HT(\tau) \). This completes the proof of higher transitivity, using Baire’s theorem. \( \square \)

7. Core free

This section is dedicated to the proof of Theorem 1.15. Clearly a subgroup with a non-trivial core contains some non-trivial conjugacy class. It is enough to fix a conjugacy class \( C_g = \{ \gamma g \gamma^{-1} \mid \gamma \in I \} \) for some \( g \neq e \) and show that the collection

\[
CF(g) = \{ \alpha \in I \mid \alpha(F_r) x \not\supset C(g) \text{ for } \mu \text{-almost all } x \in X \} = \{ \alpha \in I \mid \alpha(g) \text{ acts non trivially on } \mu \text{-almost every orbit of } \alpha \}
\]

is residual, as

\[
CF = \{ \alpha \in I \mid \alpha(F_r) x \text{ is a.s. core free} \} = \bigcap_{\beta \in F_r \setminus \{e\}} CF(g)
\]

The fact that \( CF(g) \) is \( G_\delta \) follows directly from Lemma 2.3.

Given \( \alpha \in I \) and \( \epsilon > 0 \) we look for \( \beta \in I \) such that \( d_u(\alpha, \beta) < \epsilon \) and \( \beta \in CF(g) \). Namely we want \( \beta(g) \) to act non-trivially on almost all \( \beta \)-orbits. The strategy to prove this is to define such \( \beta \) with \( \beta(s_1) = \alpha(s_1) \) and make sure that \( \beta(g) \) acts non-trivially on almost all \( \langle \alpha(s_1) \rangle = \langle \beta(s_1) \rangle \) orbits. Recall that in the lean aperiodic model \( \alpha(s_1) = \beta(s_1) \) is, by assumption aperiodic. Let us set \( \sigma = \alpha(s_1) = \beta(s_1) \).

Since \( CF(s_1^l) = I, \forall 0 \neq l \in \mathbb{Z} \) we can assume to begin with that \( g \notin C(s_1^l) \) for any such \( l \). Also since we are interested only in the conjugacy class of \( g \) we may assume that \( g \) can be represented by a cyclicly reduced word
\[ g = w_s w_{s-1} \ldots w_1 \] where \( w_i \in S \sqcup S^{-1} \). By our assumption that \( g \not\in C(s'_1) \), not all letters are \( s_1 \) or \( s_1^{-1} \). Using the Rokhlin Lemma, (or 5.2), we can find a set \( O \in \mathcal{B} \) with \( \mu(O) < \epsilon/2(s + 1) \) such that the collection of sets \( O, \sigma O, \sigma^2 O \ldots, \sigma^s O \) are disjoint. Let us find some permutation \( \tau \in S_{s+1} \) with the property that:

- whenever \( w_i = s_1 \) then \( \tau(i) = \tau(i - 1) + 1 \)
- whenever \( w_i = s_1^{-1} \) then \( \tau(i) = \tau(i - 1) - 1 \).

We denote the disjoint union of these sets by \( W = \bigcup_{i=0}^{s} \sigma^i(O) = \bigcup_{i=0}^{s} \sigma^{\tau(i)}(O) \).

There is an obvious element \( \xi \in [E] \) that permutes the sets \( \{\sigma^i(O) \mid 0 \leq i \leq s\} \) according to the new cyclic order induced on them by the permutation \( \tau \). Namely

\[
\xi(x) = \begin{cases} 
\sigma^{\tau(i+1)-\tau(i)}(x) & \text{if } x \in \sigma^{\tau(i)}(O) \\
x & \text{if } x \not\in W
\end{cases}
\]

Where \( \tau(i + 1) \) is understood as \( \tau(0) \) when \( i = s \). Now using Lemma 2.1 we can find some \( \beta \in I \) such that \( d_u(\alpha, \beta) < \epsilon \) but with the property that

\[
\beta(w_i)(x) = \xi(x) \quad \forall x \in \sigma^{\tau(i-1)}(O) \quad (7.1)
\]

There are only two things that one has to verify. First that there are no contradictions in the instructions prescribed in Equation 7.1 which follows directly from the fact that the word was assumed to be reduced and hence we are prescribing the actions of these elements on distinct sets. The second is that whenever \( w_i \in \{s_1, s_1^{-1}\} \) the prescribed action actually coincides with that of \( \sigma, \sigma^{-1} \). But the permutation \( \tau \) was chased exactly in order to accommodate that.

Now we can conclude the theorem. Things were arranged in such a way that \( \beta(g)(\sigma^{\tau(0)}(O)) = \sigma^{\tau(s)}(O) \). And since, by ergodicity, \( \sigma^{\tau(0)}(O) \) intersects almost every orbit, we are done.

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