Asymptotics for a special solution of the thirty fourth Painlevé equation

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Abstract

In a previous paper we studied the double scaling limit of unitary random matrix ensembles of the form
\[ Z_{n,N}^{-1} | \det M |^{2\alpha} e^{-N \text{Tr} V(M)} dM \]
with \( \alpha > -1/2 \). The factor \( | \det M |^{2\alpha} \) induces critical eigenvalue behaviour near the origin. Under the assumption that the limiting mean eigenvalue density associated with \( V \) is regular, and that the origin is a right endpoint of its support, we computed the limiting eigenvalue correlation kernel in the double scaling limit as \( n, N \to \infty \) such that \( n^{2/3} (n/N - 1) = O(1) \) by using the Deift–Zhou steepest descent method for the Riemann–Hilbert problem for polynomials on the line orthogonal with respect to the weight \( |x|^{2\alpha} e^{-NV(x)} \). Our main attention was on the construction of a local parametrix near the origin by means of the \( \psi \)-functions associated with a distinguished solution \( u_\alpha \) of the Painlevé XXXIV equation. This solution is related to a particular solution of the Painlevé II equation, which, however, is different from the usual Hastings–McLeod solution. In this paper we compute the asymptotic behaviour of \( u_\alpha(s) \) as \( s \to \pm \infty \). We conjecture that this asymptotics characterizes \( u_\alpha \) and we present supporting arguments based on the asymptotic analysis of a one-parameter family of solutions of the Painlevé XXXIV equation which includes \( u_\alpha \). We identify this family as the family of tronquée solutions of the thirty fourth Painlevé equation.

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1. Introduction and statement of results

1.1. A unitary random matrix ensemble

For \( n \in \mathbb{N}, N > 0 \) and \( \alpha > -1/2 \), consider the unitary random matrix ensemble

\[
Z_{n,N}^{-1} \det M \left| e^{2N \text{Tr} V(M)} \right| \, dM,
\]

on the space \( \mathcal{M}(n) \) of \( n \times n \) Hermitian matrices \( M \), where \( V \) is real analytic and satisfies

\[
\lim_{x \to \pm\infty} \frac{V(x) \log(x^2 + 1)}{x} = +\infty.
\]

(1.2)

Suppose also that the equilibrium measure \( \mu_V \) for \( V \) is regular [6], and that 0 is a right endpoint of the support of \( \mu_V \).

In the paper [15] we computed the limiting eigenvalue correlation kernel in the double scaling limit as \( n, N \to \infty \) such that \( n^{2/3} (n/N - 1) = O(1) \). We proved that for each \( \alpha > -1/2 \), there is a one-parameter family of limiting kernels \( K^{\text{edge}}_u(x, y; s) \) depending on a double scaling parameter \( s \) where

\[
s = -c_V L, \quad L = \lim n^{2/3} (n/N - 1)
\]

with a certain \( V \)-dependent constant \( c_V > 0 \), see theorem 1.1 in [15].

The kernels \( K^{\text{edge}}_u(x, y; s) \) are associated with a special solution of the equation number XXXIV from the list of Painlevé and Gambier [11],

\[
u'' = 4u^2 + 2su + \frac{(u')^2 - (2\alpha)^2}{2u}.
\]

(1.3)

In particular, we showed how the relevant solution \( u(s) \) of (1.3) (which we denote by \( u_\alpha(s) \)) can be obtained from a solution of a model RH problem, which we describe in the next subsection. In terms of the solution of this model RH problem the kernel is given by

\[
K^{\text{edge}}_u(x, y; s) = \frac{\psi_2(x; s) \psi_1(y; s) - \psi_1(x; s) \psi_2(y; s)}{2\pi i (x - y)}
\]

where

\[
\begin{pmatrix}
\psi_1(x; s) \\
\psi_2(x; s)
\end{pmatrix} = \begin{cases}
\Psi_{u,+}(x; s) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{for } x > 0, \\
\Psi_{u,-}(x; s) \begin{pmatrix} e^{-u\pi i} \\ e^{u\pi i} \end{pmatrix}, & \text{for } x < 0,
\end{cases}
\]

see theorem 1.2 in [15].

1.2. The model RH problem

The model RH problem is posed on a contour \( \Sigma \) in an auxiliary \( z \)-plane, consisting of four rays \( \Sigma_1 = \{ \arg z = 0 \}, \Sigma_2 = \{ \arg z = 2\pi/3 \}, \Sigma_3 = \{ \arg z = \pi \} \) and \( \Sigma_4 = \{ \arg z = -2\pi/3 \} \) with orientation as shown in figure 1. As usual in RH problems, the orientation defines a + and a − side on each part of the contour, where the + side is on the left when traversing the contour according to its orientation. For a function \( f \) on \( \mathbb{C} \setminus \Sigma \equiv \Omega \), we use \( f_\pm \) to denote its limiting values on \( \Sigma \) taken from the ± side, provided such limiting values exist. The contour \( \Sigma \) divides the complex plane into four sectors \( \Omega_j \) also shown in the figure.
The model RH problem reads as follows.

**Riemann–Hilbert problem for $\Psi_\alpha$**

(a) $\Psi_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, for $z \in \Sigma_1$.

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{2\pi i} & 1 \end{pmatrix}, \text{ for } z \in \Sigma_2,$$

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ for } z \in \Sigma_3,$$

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\pi i} & 1 \end{pmatrix}, \text{ for } z \in \Sigma_4.$$

(c) $\Psi_\alpha(z) = (I + O(\frac{1}{z})) z^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\frac{1}{2\sqrt{2}} z^{1/2 + i/2} \sigma_3}$ as $z \to \infty$.

Here $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix.

(d) $\Psi_\alpha(z) = O\left(\frac{|z|^\alpha}{|z|^\alpha} \frac{|z|^{\alpha-\sigma_3}}{|z|^{\alpha-\sigma_3}}\right)$ as $z \to 0$, if $-1/2 < \alpha < 0$ and $\alpha \geq 0$.

$$\Psi_\alpha(z) = \begin{cases} O\left(\frac{|z|^\alpha}{|z|^\alpha} \frac{|z|^{\alpha-\sigma_3}}{|z|^{\alpha-\sigma_3}}\right) & \text{as } z \to 0 \text{ with } z \in \Omega_1 \cup \Omega_4, \\ O\left(\frac{|z|^{\alpha-\sigma_3}}{|z|^{\alpha-\sigma_3}} \frac{|z|^{-\alpha}}{|z|^{-\alpha}}\right) & \text{as } z \to 0 \text{ with } z \in \Omega_2 \cup \Omega_3, \\ \end{cases} \text{if } \alpha \geq 0.$$

Here, and in what follows, the $O$-terms are taken entrywise. Note that the RH problem depends on a parameter $s$ through the asymptotic condition at infinity. If we want to emphasize the dependence on $s$ we will write $\Psi_\alpha(z; s)$ instead of $\Psi_\alpha(z)$.

The model RH problem is uniquely solvable for every $\alpha > -1/2$ and $s \in \mathbb{R}$ (for details see [15, proposition 2.1]).
All solutions of (1.3) are meromorphic in the complex plane. The special solution of relevance in [15] is characterized by the following result.

**Theorem 1.1.** Assume $\alpha > -1/2$. Let $\Psi_\alpha(z; s)$ be the solution of the model RH problem and write

$$\Psi_\alpha(z; s) = \left( I + \frac{m_\Psi(s)}{z} + O \left( \frac{1}{z^2} \right) \right) z^{-\alpha/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-(\frac{1}{2}z^{3/2} + cz^{1/2})\sigma_3}$$

(1.4)

as $z \to \infty$. Then

$$u_\alpha(s) = -\frac{s}{2} - i \frac{d}{ds} [m_\Psi(s)]_{12}$$

(1.5)

exists and satisfies (1.3). The function (1.5) is a global solution of (1.3) (i.e. it does not have poles on the real line).

Moreover, $u_\alpha$ is also given by

$$u_\alpha(s) = i \lim_{z \to 0} \left[ z \left( \frac{d}{dz} \Psi_\alpha(z) \right) \Psi_\alpha^{-1}(z) \right]_{12}.$$  

(1.6)

**Proof.** Expression (1.6) follows from lemma 3.2 in [15]. The remaining statements of theorem 1.1 are contained in theorem 1.4 of [15]. \boxed{}

### 1.3. Main result

The aim of this paper is an analysis of the asymptotic behaviour of the special solution $u_\alpha(s)$, described in theorem 1.1, as $s \to \pm \infty$. Our main result is the following.

**Theorem 1.2.** Let $u_\alpha(s)$ be the solution of (1.3) given in theorem 1.1. Then,

$$u_\alpha(s) = \frac{\alpha}{\sqrt{s}} + O(s^{-2}), \quad \text{as } s \to +\infty$$

(1.7)

and

$$u_\alpha(s) = \frac{\alpha}{\sqrt{-s}} \cos \left( \frac{4}{3} (-s)^{3/2} - \alpha \pi \right) + O(s^{-2}), \quad \text{as } s \to -\infty.$$  

(1.8)

In the sections 2 and 3, we provide a proof of our main result, theorem 1.2. This is accomplished by using the Deift–Zhou steepest descent method for RH problems [4, 7].

In the case at hand it consists of constructing a sequence of invertible transformations $\Psi_\alpha \mapsto A_\alpha \mapsto B_\alpha \mapsto C_\alpha \mapsto D_\alpha$, where the matrix-valued function $D_\alpha$ is close to the identity as $s \to \pm \infty$. By following the above transformations asymptotics for $\Psi_\alpha$ and thus, in view of (1.5) and (1.6), for $u_\alpha(s)$ may be derived.

### 1.4. RH problem for Painlevé XXXIV

In this paper we are mainly concerned with the special solution $u_\alpha(s)$. The analysis of the general solution of the Painlevé XXXIV equation (1.3) can also be performed via the nonlinear steepest descent method applied to the following generalization of the RH problem above.
Riemann–Hilbert problem for the general solution of PXXXIV

(a) $\Psi_{a,+}(z) = \Psi_{a,-}(z) \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}$, for $z \in \Sigma_1$,
$\Psi_{a,+}(z) = \Psi_{a,-}(z) \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$, for $z \in \Sigma_2$,
$\Psi_{a,+}(z) = \Psi_{a,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \Sigma_3$,
$\Psi_{a,+}(z) = \Psi_{a,-}(z) \begin{pmatrix} 1 & 0 \\ b_4 & 1 \end{pmatrix}$, for $z \in \Sigma_4$.

(b) $\Psi_{a,+}(z) = \Psi_{a,-}(z) \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-(\frac{1}{3}z + z^{1/2})/\sqrt{2}}$ as $z \to \infty$.

(c) $\Psi_{a}(z) = (I + O(\frac{1}{z})) z^{-a/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\left(\frac{1}{3}z + z^{1/2}\right)/\sqrt{2}}$ as $z \to \infty$.

(d) If $\alpha \notin \mathbb{N}$, then

$$
\Psi_{a}(z) = B(z) \begin{pmatrix} z^a & 0 \\ 0 & z^{-a} \end{pmatrix} E_j, \quad \text{for } z \in \Omega_j,
$$

where $B$ is analytic. If $\alpha \in \frac{1}{2} + \mathbb{N}_0$, then there exists a constant $\kappa$ such that

$$
\Psi_{a}(z) = B(z) \begin{pmatrix} z^a & \kappa e^{\alpha \log z} \\ 0 & z^{-a} \end{pmatrix} E_j, \quad \text{for } z \in \Omega_j,
$$

where $B$ is analytic.

The complex numbers $b_1, b_2, b_4$ (the Stokes multipliers) and the constant invertible matrices $E_j, j = 1, 2, 3, 4$ (the connection matrices) form the RH data. They satisfy certain general constraints which, in particular, yield the following cyclic relation for the parameters $b_j$:

$$
b_1 + b_2 + b_4 - b_1 b_2 b_4 = 2 \cos(\alpha \pi).
$$

Except for the special case,

$$
2\alpha = n, \quad b_1 = b_2 = b_4 = (-1)^n, \quad n \in \mathbb{N},
$$

when the solution of the RH problem is given in fact in terms of the Airy functions, the connection matrices $E_j$ are determined (up to inessential left diagonal or upper triangular factors) by $\alpha$ and the Stokes multipliers $b_j$.

In the formulation of the general RH problem we keep the previous notation $\Psi_{a}(z)$ for its solution. Formulae (1.5) and (1.6) for the solution of equation (1.3) are still valid, although, of course, the function $u_\alpha(s)$ for an arbitrary choice of the monodromy parameters $b_j$ might have poles on the real line.

The case of our special interest in this paper corresponds to the choice,

$$
b_1 = 1, \quad b_2 = e^{2\alpha \pi i}, \quad b_4 = e^{-2\alpha \pi i},
$$
of the Stokes parameters $b_j$. In section 2 we treat the case $s \to +\infty$, which turns out to be the easier case. In section 3 we deal with the more involved case $s \to -\infty$. The main technical issue is the necessity to construct an extra, in comparison with the $+\infty$ case, parametrix with Bessel functions.

It is well known (see, e.g. [11]) that the Painlevé XXXIV equation (1.3) can be in fact transformed to the Painlevé II equation. We discuss in detail some aspects of this transformation, relevant to our analysis, in the appendix. In particular, we note that the second

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4 We refer to [10, chapter 5] for more details on the setting of the general RH problems for the Painlevé equations.
Painlevé function which is associated with the special Painlevé XXXIV solution \( u_\alpha(s) \) we are studying here is not the familiar in random matrix \([3, 22, 23]\), and string \([21]\) theories Hastings–McLeod function. In addition, we show that although the asymptotic behaviour of \( u_\alpha(s) \) as \( s \to +\infty \) can be extracted from the already known asymptotics of the second Painlevé transcendent, the asymptotic behaviour of \( u_\alpha(s) \) as \( s \to -\infty \) needs indeed a separate analysis.

In the appendix we also discuss the question of the uniqueness of the solution \( u_\alpha(s) \). In fact, we show that there is a one-parameter family of solutions of equation (1.3) with the asymptotics (1.7). This family is characterized by the choice,

\[
\begin{align*}
    b_2 &= e^{\alpha \pi i}, \\
    b_4 &= e^{-\alpha \pi i}, \\
    b_1 &= \text{arbitrary complex number},
\end{align*}
\]

(1.14)
of the monodromy data \( b_j \). At the same time, we conjecture that the asymptotic condition (1.8) fixes the solution uniquely. We present the arguments in support of this conjecture which are based on certain observations related to the asymptotic investigation of the general Painlevé XXXIV RH problem.

We conclude the introduction by mentioning that the Painlevé XXXIV equation has also appeared in several other physical applications. In fact, in the already mentioned paper \([21]\) it was the Painlevé XXXIV image of the Hastings–McLeod Painlevé II solution which showed up and not the solution itself.

2. Proof of theorem 1.2: asymptotics as \( s \to +\infty \)

2.1. First transformation \( \Psi_\alpha \mapsto A_\alpha \)

For the proof of theorem 1.2 we start from the RH problem for \( \Psi_\alpha \) given in section 1.2. Introduce

\[
A_\alpha(z; s) = s^{\sigma_3/4} \Psi_\alpha(sz; s), \quad z \in \mathbb{C} \setminus \Sigma.
\]

(2.1)

It is then easy to see that \( A_\alpha \) satisfies a RH problem similar to that for \( \Psi_\alpha \). In the following we often suppress the \( s \)-dependence of functions whenever it is understood.

**Riemann–Hilbert problem for \( A_\alpha \)**

- (a) \( A_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2 \times 2} \) is analytic.
- (b) \( A_\alpha \) has the same jumps on \( \Sigma \) as those of \( \Psi_\alpha \).
- (c) \( A_\alpha(z) = (I + O(1/z))z^{-\sigma_3/4} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-s/2(z^{3/2} + z^{1/2} \sigma_3)} \) as \( z \to \infty \).
- (d) \( A_\alpha \) has the same behaviour near 0 as that of \( \Psi_\alpha \).

From (1.6) and (2.1) it follows that

\[
u_\alpha(s) = \frac{i}{\sqrt{s}} \lim_{z \to 0} \left[ z \left( \frac{d}{dz} A_\alpha(z) \right) A^{-1}_\alpha(z) \right]_{12}.
\]

(2.2)

2.2. Second transformation \( A_\alpha \mapsto B_\alpha \)

Let us put

\[
t = s^{3/2},
\]

(2.3)

which is the large parameter in the RH problem. Consider figure 2.
Define

\[
B_\alpha(z) = \begin{cases} 
A_\alpha(z), & \text{for } z \in I \cup III \cup IV \cup VI, \\
A_\alpha(z) \left( \frac{1}{e^{2\alpha\pi i}} 0 \right), & \text{for } z \in II, \\
A_\alpha(z) \left( -\frac{1}{e^{-2\alpha\pi i}} 0 \right), & \text{for } z \in V.
\end{cases}
\]

(2.4)

Let \( \tilde{\Sigma} = \Sigma - 1 \), i.e., \( \tilde{\Sigma} \) is \( \Sigma \) translated to the left by one. Then

\[
\tilde{\Sigma} = \bigcup_{j=0}^{4} \tilde{\Sigma}_j,
\]

(2.5)

where the disjoint contours \( \tilde{\Sigma}_j \) are oriented as in figure 2.

**Riemann–Hilbert problem for** \( B_\alpha \)

(a) \( B_\alpha : \mathbb{C} \setminus \tilde{\Sigma} \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) \( B_{\alpha,+}(z) = B_{\alpha,-}(z) \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \), for \( z \in \tilde{\Sigma}_0 = (0, \infty) \),

\[
B_{\alpha,+}(z) = B_{\alpha,-}(z) \left( \frac{e^{2\alpha\pi i}}{0} \right), \text{for } z \in \tilde{\Sigma}_1 = (-1, 0),
\]

\[
B_{\alpha,+}(z) = B_{\alpha,-}(z) \left( \frac{1}{e^{2\alpha\pi i}} 0 \right), \text{for } z \in \tilde{\Sigma}_2,
\]

\[
B_{\alpha,+}(z) = B_{\alpha,-}(z) \left( 0 \frac{1}{0} \right), \text{for } z \in \tilde{\Sigma}_3 = (-\infty, -1),
\]

\[
B_{\alpha,+}(z) = B_{\alpha,-}(z) \left( 1 \frac{e^{-2\alpha\pi i}}{1} \right), \text{for } z \in \tilde{\Sigma}_4.
\]

(c) \( B_\alpha(z) = (I + O(1/z))z^{-\sigma_3/4} \frac{1}{\sqrt{\pi}} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) e^{-\left( \frac{1}{4} z^{3/2} + i \pi/2 \right)} \) as \( z \to \infty \).

(d) If \( \alpha < 0 \), then \( B_\alpha(z) = O \left( \frac{|z|^\alpha}{|z|^\alpha} \right) \) as \( z \to 0 \).

If \( \alpha \geq 0 \), then \( B_\alpha(z) = O \left( \frac{|z|^\alpha}{|z|^{-\alpha}} \right) \) as \( z \to 0 \).
From (2.2) and (2.4) it follows that
\[ u_\alpha(s) = \frac{i}{\sqrt{s}} \lim_{z \to 0} \left[ z \left( \frac{d}{dz} B_\alpha(z) \right) B^{-1}_\alpha(z) \right]_{12}. \] (2.6)

2.3. Third transformation $B_\alpha \mapsto C_\alpha$

We next introduce the $g$-function
\[ g(z) = \left\{ \begin{array}{ll} \frac{1}{4} \frac{z}{\sqrt{2}} + O(\frac{z}{\sqrt{2}}), & \text{as } z \to \infty. \end{array} \right. \] (2.8)

By a straightforward computation
\[ g(z) - \left( \frac{1}{4} \frac{z}{\sqrt{2}} + \frac{z}{\sqrt{2}} \right) = \frac{1}{4} \frac{z}{\sqrt{2}} + O(\frac{z}{\sqrt{2}}), \quad \text{as } z \to \infty. \] (2.8)

Define
\[ C_\alpha(z) = \left( \begin{array}{cc} 1 & 0 \\ \frac{-i\tau/4}{1} & 1 \end{array} \right) B_\alpha(z) e^{i\sigma} \] (2.9)

Then $C_\alpha$ satisfies the following RH problem.

Riemann–Hilbert problem for $C_\alpha$

(a) $C_\alpha : \mathbb{C} \setminus \tilde{\Sigma} \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $C_{\alpha+}(z) = C_{\alpha-}(z) \left( \begin{array}{cc} 1 & e^{-2i\sigma} \\ 0 & 1 \end{array} \right)$, for $z \in \tilde{\Sigma}_0$.

(c) $C_{\alpha}(z) = (I + O(1/z))z^{-\sigma/4} \left( \begin{array}{cc} 1 & i \\ 1 & 1 \end{array} \right)$ as $z \to \infty$.

(d) If $\alpha < 0$, then $C_\alpha(z) = O \left( \begin{array}{cc} |z|^{\alpha} & |z|^{\alpha} \\ |z|^{\alpha} & |z|^{\alpha} \end{array} \right)$ as $z \to 0$.

Note that the prefactor $\left( \begin{array}{cc} 1 & 0 \\ -i\tau/4 & 1 \end{array} \right)$ in the definition (2.9) of $C_\alpha$ is needed for the asymptotic condition (c) in the RH problem. The prefactor does not affect the $12$ entry and so does not influence the computation of $u_\alpha$ via formula (2.6).

Thus by (2.9)
\[ \left[ z \left( \frac{d}{dz} B_\alpha(z) \right) B^{-1}_\alpha(z) \right]_{12} = \left[ z \left( \frac{d}{dz} C_\alpha(z) B^{-1}_\alpha(z) \right) \right]_{12} - t g'(z) \left[ z C_\alpha(z) \sigma_3 C^{-1}_\alpha(z) \right]_{12}. \] (2.10)
In view of item (d) in the RH problem for $C_{\alpha}$ (and the fact that $\text{det} C_{\alpha} \equiv 1$) we have that $zC_{\alpha}(z)C_{\alpha}^{-1}(z) \to 0$ as $z \to 0$. Therefore, the second term in the right-hand side of (2.10) vanishes as $z \to 0$ and it follows by (2.6) that

$$u_{\alpha}(s) = \frac{i}{\sqrt{s}} \lim_{z \to 0} z \left[ \left( \frac{d}{dz} C_{\alpha}(z) \right) C_{\alpha}^{-1}(z) \right]_{12}.$$  \hfill (2.11)

### 2.4. Construction of parametrices

#### 2.4.1. Global parametrix $P_{\alpha}^{(\infty)}$

Away from the point $-1$ we expect that $C_{\alpha}$ should be well approximated by the solution $P_{\alpha}^{(\infty)}$ of the following RH problem.

**Riemann–Hilbert problem for $P_{\alpha}^{(\infty)}$**

(a) $P_{\alpha}^{(\infty)} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $P_{\alpha, +}^{(\infty)}(z) = P_{\alpha, -}^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in (-\infty, -1),$

$P_{\alpha, +}^{(\infty)}(z) = P_{\alpha, -}^{(\infty)}(z) \begin{pmatrix} e^{2\pi i} & 0 \\ 0 & e^{-2\pi i} \end{pmatrix}$, for $z \in (-1, 0).$

(c) $P_{\alpha}^{(\infty)}(z) = (I + O(1/z))z^{-\sigma_3/4} 1/\sqrt{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty.$

It should be noted that $P_{\alpha}^{(\infty)}$ does not depend on $s$.

We seek $P_{\alpha}^{(\infty)}$ in the form

$$P_{\alpha}^{(\infty)}(z) = F_{\alpha}(z)z^{\sigma_3},$$  \hfill (2.12)

where $F_{\alpha}$ is analytic in $\mathbb{C} \setminus (-\infty, -1]$. Clearly then the jump is correct on $(-1, 0)$. A straightforward computation shows that in order to have the correct jump also on $(-\infty, -1)$ we may take

$$F_{\alpha}(z) = E(z+1)^{-\sigma_i/4} 1/\sqrt{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} (\delta_{\alpha}(z))^{\sigma_3},$$  \hfill (2.13)

where $E$ is a constant prefactor and

$$\delta_{\alpha}(z) = \exp \left( -\frac{\alpha}{\pi} (z+1)^{1/2} \int_{1}^{\infty} \frac{\log t}{\sqrt{1-t(t+z)}} \, dt \right).$$  \hfill (2.14)

Using the residue theorem and a contour deformation argument, it is also straightforward to see that

$$\int_{1}^{\infty} \frac{\log t}{\sqrt{t-1(t+z)}} \, dt = \pi \frac{\log z}{(z+1)^{1/2}} + \pi \int_{0}^{1} \frac{dt}{\sqrt{1-t(t+z)}}$$

$$= \pi \frac{\log z}{(z+1)^{1/2}} + \pi \frac{\log (\frac{(z+1)^{1/2} + 1}{(z+1)^{1/2} - 1})}{(z+1)^{1/2}}.$$  \hfill (2.15)

Hence,

$$P_{\alpha}^{(\infty)}(z) = E(z+1)^{-\sigma_i/4} 1/\sqrt{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} (\delta_{\alpha}(z))^{\sigma_3} z^{\sigma_3}$$

$$= E(z+1)^{-\sigma_i/4} 1/\sqrt{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( \frac{(z+1)^{1/2} + 1}{(z+1)^{1/2} - 1} \right)^{-\sigma_3}.$$  \hfill (2.16)
In order to satisfy the asymptotic condition (c) of the RH problem we should take

\[
E = \begin{pmatrix} 1 & 0 \\ 2\alpha i & 1 \end{pmatrix}.
\]  

(2.17)

Note that

\[
\frac{d}{dz} \log \left( \frac{(z+1)^{1/2} + 1}{(z+1)^{1/2} - 1} \right) = -\frac{1}{z(z+1)^{1/2}}
\]

from which it follows after straightforward calculations from (2.16) and (2.17) that

\[
\lim_{z \to 0} z \left( \frac{d}{dz} P_a^{(\infty)}(z) \right) \left( P_a^{(\infty)}(z) \right)^{-1} = E \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} E^{-1}
\]

\[
= -i \alpha \begin{pmatrix} -2\alpha i & 1 \\ -1 + 4\alpha^2 & 2\alpha i \end{pmatrix}.
\]  

(2.18)

2.4.2. Local parametrix $P_a^{(-1)}$. The global parametrix $P_a^{(\infty)}$ will not be a good approximation to $C_a$ near the point $-1$. Let $U^{(-1)}$ be a small open disc around $-1$ of radius $< 1$. We seek a local parametrix $P_a^{(-1)}$ defined in $U^{(-1)}$ which satisfies the following.

**Riemann–Hilbert problem for $P_a^{(-1)}$**

(a) $P_a^{(-1)} : U^{(-1)} \setminus \Sigma \rightarrow \mathbb{C}^{2 \times 2}$ is continuous and analytic on $U^{(-1)} \setminus \Sigma$.

(b) $P_a^{(-1)}(z) = P_a^{(-1)}(z) v_{C_a}(z)$ for $z \in \Sigma \cap U^{(-1)}$, where $v_{C_a}$ denotes the jump matrix for $C_a$ (the contour having the same orientation as $\Sigma$).

(c) $P_a^{(-1)}(z) \left( P_a^{(\infty)}(z) \right)^{-1} = I + O(\frac{1}{t})$, as $t \to \infty$, uniformly for $z \in \partial U^{(-1)} \setminus \Sigma$.

We seek $P_a^{(-1)}$ in the form

\[
P_a^{(-1)}(z) = \tilde{P}_a^{(-1)}(z)e^{g(z)\gamma_0},
\]

(2.19)

where $\tilde{P}_a^{(-1)}$ satisfies the following RH problem with constant jumps.

**Riemann–Hilbert problem for $\tilde{P}_a^{(-1)}$**

(a) $\tilde{P}_a^{(-1)} : U^{(-1)} \setminus \Sigma \rightarrow \mathbb{C}^{2 \times 2}$ is continuous and analytic on $U^{(-1)} \setminus \Sigma$.

(b) $\tilde{P}_a^{(-1)}(z) = \tilde{P}_a^{(-1)}(z) \begin{pmatrix} e^{2\pi i} & 0 \\ 0 & e^{-2\pi i} \end{pmatrix}$, for $z \in \Sigma_1 \cap U^{(-1)}$,

$\tilde{P}_a^{(-1)}(z) = \tilde{P}_a^{(-1)}(z) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, for $z \in \Sigma_2 \cap U^{(-1)}$,

$\tilde{P}_a^{(-1)}(z) = \tilde{P}_a^{(-1)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \Sigma_3 \cap U^{(-1)}$,

$\tilde{P}_a^{(-1)}(z) = \tilde{P}_a^{(-1)}(z) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, for $z \in \Sigma_4 \cap U^{(-1)}$.

(c) $\tilde{P}_a^{(-1)}(z) = P_a^{(\infty)}(z) \left( I + O(\frac{1}{t}) \right)e^{-g(z)\gamma_0}$, as $t \to \infty$, uniformly for $z \in \partial U^{(-1)} \setminus \Sigma$.

A solution to this RH problem can be constructed in terms of Airy functions. The standard Airy parametrix is posed in an auxiliary $\zeta$-plane and satisfies the following RH problem for a contour $\Sigma$ as in figure 1.
\textbf{Riemann–Hilbert problem for }Φ^{(\text{Ai})}\text{ (34th Painlevé equation)}

(a) $Φ^{(\text{Ai})} : \mathbb{C} \setminus \Sigma \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) $Φ_+^{(\text{Ai})} = Φ_+^{(\text{Ai})}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$, on $Σ_1$, $Φ_-^{(\text{Ai})} = Φ_-^{(\text{Ai})}(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix})$, on $Σ_2 \cup Σ_4$, $Φ^{(\text{Ai})}_+ = Φ^{(\text{Ai})}_-(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$, on $Σ_3$.

(c) $Φ^{(\text{Ai})}(ζ) = ζ^{-σ_1/4} (I + O(ζ^{-3/2})) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} e^{-\frac{1}{2} e^{2πi/3}}$ as $ζ \rightarrow \infty$.

The solution is built out of the functions $y_\alpha(ζ) = Ai(ζ)$, $y_1(ζ) = ω Ai(ωζ)$, $y_2(ζ) = ω^2 Ai(ω^2ζ)$, $ω = e^{2πi/3}$, and takes the following form

$$
\begin{align*}
Φ^{(\text{Ai})}_+ &= \begin{pmatrix} y_1 & -y_2 \\ -y_1 & y_2 \end{pmatrix} \text{ in } Ω_2, & Φ^{(\text{Ai})}_- &= \begin{pmatrix} y_0 & -y_2 \\ y_0 & -y_2 \end{pmatrix} \text{ in } Ω_1, \\
Φ^{(\text{Ai})}_+ &= \begin{pmatrix} -y_2 & y_1 \\ -y_2 & y_1 \end{pmatrix} \text{ in } Ω_3, & Φ^{(\text{Ai})}_- &= \begin{pmatrix} y_0 & y_1 \\ y_0 & y_1 \end{pmatrix} \text{ in } Ω_4.
\end{align*}
$$

(2.20)

Then we put

$$Φ^{(\text{Ai})}_+(ζ) = \sqrt{2π} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} Φ^{(\text{Ai})}_-(ζ) e^{±2πiσ_3}, \quad \text{for } ± \text{Im } ζ > 0, \quad (2.21)$$

and $Φ^{(\text{Ai})}_+$ satisfies the following RH problem.

\textbf{Riemann–Hilbert problem for }Φ^{(\text{Ai})}_+

(a) $Φ^{(\text{Ai})}_+ : \mathbb{C} \setminus \Sigma \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) $Φ^{(\text{Ai})}_+^{(\text{Ai})} = Φ^{(\text{Ai})}_+^{(\text{Ai})}(\begin{pmatrix} e^{2πi} & 1 \\ 0 & e^{-2πi} \end{pmatrix})$, on $Σ_1$, $Φ^{(\text{Ai})}_-^{(\text{Ai})} = Φ^{(\text{Ai})}_-^{(\text{Ai})}(\begin{pmatrix} 1 & 0 \\ e^{2πi} & 1 \end{pmatrix})$, on $Σ_2$, $Φ^{(\text{Ai})}_+^{(\text{Ai})} = Φ^{(\text{Ai})}_-^{(\text{Ai})}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$, on $Σ_3$, $Φ^{(\text{Ai})}_-^{(\text{Ai})} = Φ^{(\text{Ai})}_+^{(\text{Ai})}(\begin{pmatrix} e^{-2πi} & 1 \\ 1 & 0 \end{pmatrix})$, on $Σ_4$.

(c) $Φ^{(\text{Ai})}_+(ζ) = ζ^{-σ_1/4} (I + O(ζ^{-3/2})) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} e^{±2πiσ_3} e^{-\frac{1}{2} e^{2πi/3}}$ as $ζ \rightarrow \infty$ with $± \text{Im } ζ > 0$.

Define

$$\hat{P}^{(\text{Ai})}_a(z) = E_a(z) Φ^{(\text{Ai})}_a(s(z + 1)), \quad \text{for } z \in U^{(\text{Ai})} \setminus \Sigma, \quad (2.22)$$

where $E_a$ is analytic in $U^{(\text{Ai})}$. Then $\hat{P}^{(\text{Ai})}_a$ has the correct jumps. In order to satisfy the matching condition in the RH problem we take $E_a$ in the following way:

$$E_a(z) = P_a^{(\text{Ai})}(z) e^{±2πiσ_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (s(z + 1))^{σ_1/4}. \quad (2.23)$$
It is a straightforward computation to verify that \( E_\alpha \) extends as an analytic function in \( U^{-1} \).

Combining (2.19), (2.22) and (2.23), we see that

\[
P_{\alpha}^{-1}(z) = P_{\alpha}^{(\infty)}(z)e^{\frac{\sqrt{2}}{\sqrt{2}} \left( \frac{1}{-i} \right) (s(z+1))^{\frac{1}{2}} \Phi_{\alpha}^{(A)}(s(z+1))e^{\frac{\sqrt{2}}{\sqrt{2}} (s(z+1) + 1)}},
\]

which completes the construction of the local parametrix \( P_{\alpha}^{-1} \).

2.5. Fourth transformation \( C_\alpha \mapsto D_\alpha \)

Define now the final transformation

\[
D_\alpha(z) = \begin{cases} 
C_\alpha(z) \left( P_{\alpha}^{-1}(z) \right)^{-1}, & \text{for } z \in U^{-1} \setminus \Sigma, \\
C_\alpha(z) \left( P_{\alpha}^{(\infty)}(z) \right)^{-1}, & \text{for } z \in \mathbb{C} \setminus (U^{-1} \cup \Sigma).
\end{cases}
\]

Since \( C_\alpha \) and \( P_{\alpha}^{(\infty)} \) have the same jumps on \( (-\infty, -1) \setminus U^{-1} \), and \( C_\alpha \) and \( P_{\alpha}^{-1} \) have the same jumps on \( U^{-1} \cap \Sigma \), we have that \( D_\alpha \) is analytic across these contours. What remains are jumps for \( D_\alpha \) on the contour \( \Sigma_D \) shown in figure 3. Indeed, \( D_\alpha \) satisfies the following RH problem.

**Riemann–Hilbert problem for \( D_\alpha \)**

(a) \( D_\alpha : \mathbb{C} \setminus \Sigma_D \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) \( D_{\alpha,+}(z) = D_{\alpha,-}(z) v_{D_\alpha}(z) \) for \( z \in \Sigma_D \), where

\[
v_{D_\alpha} = \begin{cases} 
P_{\alpha}^{(\infty)} (P_{\alpha}^{-1})^{-1}, & \text{on } \partial U^{-1}, \\
P_{\alpha}^{(\infty)} v_{C_\alpha} (P_{\alpha}^{(\infty)})^{-1}, & \text{on } \Sigma_D \setminus \partial U^{-1}.
\end{cases}
\]

(c) \( D_\alpha(z) = I + O(1/z) \) as \( z \to \infty \).

The subscripts \( \pm \) in \( P^{(\infty)} \) are only relevant for the segment of the horizontal part of the contour to the left of 0.

2.6. Conclusion of the proof of (1.7)

The jump matrix satisfies

\[
v_{D_\alpha}(z) = I + O(1/t), \quad \text{as } t \to \infty,
\]

uniformly on the circle \( \partial U^{-1} \). In addition,

\[
v_{D_\alpha}(z) = I + O(e^{-c|z|+1}), \quad c > 0,
\]

Figure 3. Contour \( \Sigma_D \) in the RH problem for \( D_\alpha \).
uniformly on $\Sigma_D \setminus \partial U^{(-1)}$. In a standard way (see, e.g. [15]) one shows that
\[
D_a(z) = I + O\left(\frac{1}{t(1+|z|)}\right), \quad \text{as } t \to \infty, \tag{2.28}
\]
uniformly for $z \in \mathbb{C} \setminus \Sigma_D$.

Finally, we have by (2.24) and the fact that $D_a(z)$ and $\frac{d}{dz} D_a(z)$ remain bounded as $z \to 0$,
\[
\lim_{z \to 0} z \left( \frac{d}{dz} C_a(z) \right) C_a^{-1}(z) = D_a(0) \lim_{z \to 0} z \left( \frac{d}{dz} P_a^{(\infty)}(z) \right) \left( P_a^{(\infty)}(z) \right)^{-1} D_a^{-1}(0)
\]
so that in view of (2.11) and (2.18)
\[
u_a(s) = \frac{\alpha}{\sqrt{-s}} \left[ D_a(0) \left( \begin{array}{cc}
-2\alpha i & 1 \\
-1 + 4\alpha^2 & 2\alpha i
\end{array} \right) D_a^{-1}(0) \right]_{12}, \tag{2.29}
\]
Inserting (2.28) with $z = 0$ into (2.29) and recalling that $t = s^{3/2}$, we obtain (1.7).

3. Proof of theorem 1.2: asymptotics as $s \to -\infty$

For the asymptotics as $s \to -\infty$ we also perform a sequence of transformations of the model RH problem $\Psi_a \mapsto A_a \mapsto B_a \mapsto C_a \mapsto D_a$, but the transformations are different from the ones we performed for $s \to +\infty$. Thus $A_a$, $B_a$, $C_a$ and $D_a$ will now have a different meaning which hopefully does not lead to any confusion. We assume throughout this section that $s < 0$.

3.1. First transformation $\Psi_a \mapsto A_a$

Similar to (2.1), we introduce
\[
A_a(z; s) = (-s)^{n/4} \Psi_a(-sz; s), \quad z \in \mathbb{C} \setminus \Sigma. \tag{3.1}
\]
The $A_a$-RH problem reads as follows (the $s$-dependence is, as usual, suppressed).

**Riemann–Hilbert problem for $A_a$**

(a) $A_a : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps of $A_a$ on $\Sigma$ are the same as those of $\Psi_a$.

(c) $A_a(z) = (I + O\left(\frac{1}{z}\right)) z^{-\sigma_1/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-i(\frac{3}{2}z^{3/2} - \frac{1}{2}z^1)}$ as $z \to \infty$.

(d) $A_a$ has the same behaviour near 0 as $\Psi_a$ has.

Here, the large positive parameter $t$ is defined by the equation (cf (2.3))
\[
t = (-s)^{3/2}. \tag{3.2}
\]

Using (1.6) together with (3.1) we can express $u_a$ in terms of $A_a$ as follows:
\[
u_a(s) = -\frac{i}{\sqrt{-s}} \lim_{z \to 0} \left[ z \left( \frac{d}{dz} A_a(z) \right) A_a^{-1}(z) \right]_{12}, \tag{3.3}
\]
3.2. Second transformation $A_\alpha \mapsto B_\alpha$

An important difference comparing with the previous case is that a step analogous to the $B_\alpha$-step is skipped. That is, our next step will be the $g$-function ‘dressing’.

Put (cf (2.7))

$$g(z) = \frac{2}{3}(z - 1)^{3/2}, \quad -\pi < \arg(z - 1) < \pi. \quad (3.4)$$

Note that, as before,

$$g(z) - \left(\frac{2}{3}z^{3/2} - z^{1/2}\right) = \frac{1}{4}z^{-1/2} + O(z^{-3/2}), \quad \text{as } z \to \infty. \quad (3.5)$$

Define

$$B_\alpha(z) = \left(\begin{array}{cc} 1 & 0 \\ \frac{it}{4} & 1 \end{array}\right) A_\alpha(z) e^{tg(z)\sigma_3}. \quad (3.6)$$

Then, $B_\alpha$ satisfies the following RH problem.

**Riemann–Hilbert problem for $B_\alpha$**

(a) $B_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $B_{\alpha,+}(z) = B_{\alpha,-}(z) \left(\begin{array}{cc} 1 & e^{-2tg(z)} \\ 0 & 1 \end{array}\right)$, for $z \in (1, \infty)$,

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \left(\begin{array}{cc} e^{-t(g(z)-g_1(z))} & 1 \\ 0 & e^{t(g(z)-g_1(z))} \end{array}\right), \quad \text{for } z \in (0, 1),$$

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \left(\begin{array}{cc} 1 & 0 \\ 2\pi i e^{2g(z)} & 1 \end{array}\right), \quad \text{for } z \in \Sigma_2,$$

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \quad \text{for } z \in \Sigma_3,$$

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \left(\begin{array}{cc} 1 & 0 \\ e^{2\pi i e^{2g(z)}} & 1 \end{array}\right), \quad \text{for } z \in \Sigma_4.$$

(c) $B_\alpha(z) = \left(1 + O\left(\frac{1}{z}\right)\right) z^{-\sigma_{1/4}} \left(\begin{array}{cc} 1 & i \\ i & 1 \end{array}\right)$ as $z \to \infty$.

(d) The behaviour of $B_\alpha(z)$ as $z \to 0$ is the same as that of $\Psi_\alpha(z)$.

We emphasize, that the contour $\Sigma$ is now *the same* as in the original $\Psi_\alpha$-problem.

From the transformation (3.6) it follows that

$$\left[\frac{d}{dz} A_\alpha(z) A_\alpha^{-1}(z)\right]_{12} = \left[\frac{d}{dz} B_\alpha(z) B_\alpha^{-1}(z)\right]_{12} - tg'(z) \left[B_\alpha(z) \sigma_3 B_\alpha^{-1}(z)\right]_{12}. \quad (3.7)$$

From part (d) in the RH problem satisfied by $B_\alpha$ we can deduce that

$$B_\alpha(z) \sigma_3 B_\alpha^{-1}(z) = \begin{cases} O(|z|^{1/2}) & \text{if } -1/2 < \alpha < 0, \\ O(1) & \text{if } \alpha \geq 0, \end{cases}$$

as $z \to 0$. It follows that we can forget about the second term in the right-hand side of (3.7) and we obtain from (3.3) and (3.7) that

$$u_\alpha(s) = \frac{i}{\sqrt{\alpha}} \lim_{z \to 0} \left(z \left[\frac{d}{dz} B_\alpha(z) B_\alpha^{-1}(z)\right]_{12}\right). \quad (3.8)$$
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Figure 4. Images of the rays $\Sigma_2$ and $\Sigma_4$ under the mapping $z \mapsto w = g(z) = \frac{4}{3} (z - 1)^{1/2}$.

### 3.3. Third transformation $B_\alpha \mapsto C_\alpha$

In order to proceed further, we need to analyse the structure of the sign of $\text{Re} \, g(z)$. The first observation is trivial:

\[
g(z) = \frac{4}{3} |z - 1|^{3/2} \geq 0, \quad z \in (1, \infty).
\] (3.9)

Next, we note that the function $w = g(z)$ performs a conformal mapping of the upper half plane to the sector $0 < \arg w < 3\pi/2$. Under this mapping, the domain $\frac{\pi}{3} < \arg(z - 1) < \pi$ becomes the left half plane $\text{Re} \, w < 0$, and the ray $\Sigma_2$ transforms to a simple smooth curve $g(\Sigma_2)$ which ends at $-2i/3$, lies entirely in the left half plane and behaves for large $z \in \Sigma_2$ as

\[
\text{Re} \, w \sim -\frac{4}{3} |z|^{1/2}, \quad \text{Im} \, w \sim -\frac{1}{\sqrt{3}} |z|^{1/2},
\]

see figure 4. Similarly, the function $w = g(z)$ performs a conformal mapping of the lower half plane to the sector $-3\pi/2 < \arg w < 0$. Under this mapping, the domain $-\pi < \arg(z - 1) < -\frac{\pi}{3}$ becomes the left half plane $\text{Re} \, w < 0$, and the ray $\Sigma_4$ is mapped to the mirror image of $g(\Sigma_2)$ with respect to the real axis, see again figure 4. Therefore, there exists a constant $c > 0$ such that

\[
\text{Re} \, g(z) \leq -c |z - 1| \leq 0, \quad z \in \Sigma_2 \cup \Sigma_4.
\] (3.10)

We also note that the function

\[
h(z) := g_-(z) - g_+(z), \quad z \in (0, 1),
\] (3.11)

admits analytic continuation into the domains $\Omega_u$ and $\Omega_d$ indicated in figure 5. Indeed we have,

\[
h(z) = -2g(z), \quad z \in \Omega_u
\] (3.12)

and

\[
h(z) = 2g(z), \quad z \in \Omega_d.
\] (3.13)

The above indicated characterization of the conformal mapping generated by the function $g(z)$ yields the inequalities

\[
\text{Re} \, h(z) > 0, \quad z \in \Omega_u
\] (3.14)

and

\[
\text{Re} \, h(z) < 0, \quad z \in \Omega_d.
\] (3.15)
Estimates (3.9) and (3.10) imply that the jump matrices on the rays \((1, \infty), \Sigma_2 \) and \(\Sigma_4\) and away from the end points are close to the identity matrix, while the estimates (3.14) and (3.15) suggest to ‘open the lenses’ around the interval \((0, 1)\).

Noting that
\[
\left( e^{-t(g(z) - g_{\ast}(z))} \begin{pmatrix} 1 & 0 \\ 0 & e^{t(g(z) - g_{\ast}(z))} \end{pmatrix} \right) = \left( e^{-t h(z)} \begin{pmatrix} 1 & 0 \\ 0 & e^{t h(z)} \end{pmatrix} \right)
\]
\[
= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),
\]
where
\[
C_a(z) = \begin{cases} 
B_a(z), & \text{for } z \notin \Omega_u \cup \Omega_d, \\
B_a(z) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), & \text{for } z \in \Omega_u, \\
B_a(z) \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), & \text{for } z \in \Omega_d.
\end{cases}
\]

We use \(\Sigma_u\) and \(\Sigma_d\) to denote the curves which, in conjunction with the interval \([0, 1]\), make the boundary of the lenses \(\Omega_u\) and \(\Omega_d\), respectively. The curves \(\Sigma_u\) and \(\Sigma_d\) together with their orientation are indicated in figure 5. Let \(\Sigma_C\) denote the contour \(\Sigma\) augmented by the arcs \(\Sigma_u\) and \(\Sigma_d\). Then, \(C_a\) satisfies the following RH problem.

**Riemann–Hilbert problem for \(C_a\)**

(a) \(C_a : \mathbb{C} \setminus \Sigma_C \rightarrow \mathbb{C}^{2 \times 2}\) is analytic.

(b) \(C_{a, +}(z) = C_{a, -}(z) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \text{for } z \in (1, \infty), \)
\(C_{a, +}(z) = C_{a, -}(z) \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \text{for } z \in (0, 1) \cup \Sigma_3, \)
\(C_{a, +}(z) = C_{a, -}(z) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \text{for } z \in \Sigma_u \cup \Sigma_d, \)
\(C_{a, +}(z) = C_{a, -}(z) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \text{for } z \in \Sigma_2, \)
\(C_{a, +}(z) = C_{a, -}(z) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \text{for } z \in \Sigma_4. \)
(c) \( C_\alpha(z) = (I + O\left(\frac{1}{z}\right)) z^{-\sigma_3/4} \left[ \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right] \) as \( z \to \infty \).

(d) \( C_\alpha(z) = O\left(\frac{1}{|z|^\alpha} \right) \) as \( z \to 0 \), if \( \frac{1}{2} < \alpha < 0 \) and

\[
C_\alpha(z) = \begin{cases} 
O\left(\frac{|z|^{-\alpha}}{|z|^{-\alpha}}\right) & \text{as } z \to 0 \text{ with } z \in (\Omega_1 \cup \Omega_4) \setminus (\Omega_a \cup \Omega_d), \\
O\left(\frac{|z|^{-\alpha}}{|z|^{-\alpha}}\right) & \text{as } z \to 0 \text{ with } z \in \Omega_2 \cup \Omega_3 \cup \Omega_a \cup \Omega_d,
\end{cases}
\]

if \( \alpha \geq 0 \).

When formulating the jump conditions across the lenses boundaries, i.e. on the curves \( \Sigma_1 \cup \Sigma_4 \), we have replaced the function \( h(z) \) by the function \( g(z) \) according to relations (3.12) and (3.13). The contours for the \( C_\alpha \)-RH problem are depicted in figure 5.

To express \( u_\alpha \) in terms of \( C_\alpha \) we can use the same formula (3.8) but with the understanding that \( z \to 0 \) from outside the lens. Thus

\[
u_\alpha(z) = \frac{i}{\sqrt{-s}} \lim_{\gamma \to 0^+} \left[ z \left( \frac{d}{dz} C_\alpha(z) \right) C_{\nu}^{-1}(z) \right]_{12}.
\]  

(3.18)

3.4. Construction of parametrices

3.4.1. Global parametrix \( P^{(\infty)} \). Away from the points 0 and 1 the jump matrices on \((1, \infty), \Sigma_2, \Sigma_4, \Sigma_a, \Sigma_d \) all tend to the identity matrix as \( t \to +\infty \) at an exponential rate. Therefore, away from the points 0 and 1 we expect that \( C_\alpha \) should be well approximated by the solution \( P^{(\infty)} \) of the following RH problem with the only non-trivial jump across the ray \((-\infty, 1)\).

**Riemann–Hilbert problem for \( P^{(\infty)} \)**

(a) \( P^{(\infty)} : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) \( P^{(\infty)}_+(z) = P^{(\infty)}_-(z) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \) for \( z \in (-\infty, 1) \).

(c) \( P^{(\infty)}(z) = (I + O\left(\frac{1}{z}\right)) z^{-\sigma_3/4} \left[ \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right] \) as \( z \to \infty \).

It should be noted that now \( P^{(\infty)} \) is clearly independent of both \( s \) and \( \alpha \).

This Riemann–Hilbert problem is even simpler than the corresponding problem for the case of positive \( s \), and its solution is obviously given by the formula (cf (2.16))

\[
P^{(\infty)}(z) = (z - 1)^{-\sigma_3/4} \left[ \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right], \quad -\pi < \arg(z - 1) < \pi.
\]  

(3.19)

Near the points 0 and 1 the parametrix \( P^{(\infty)} \) cannot be expected to represent the asymptotics of \( C_\alpha \). Our next task is to construct the parametrix solutions near the points mentioned.

3.4.2. Local parametrix \( P^{(1)} \). We begin with the point \( z = 1 \). We will see that the form of parametrix at this point is very similar to the parametrix near the point \( z = -1 \) in the previous case of positive \( s \).

Let \( U^{(1)} \) be a small open disc around 1. We seek a parametrix \( P^{(1)} \) defined in \( U^{(1)} \) which satisfies the following RH problem.
Riemann–Hilbert problem for $P^{(1)}$

(a) $P^{(1)}(z) \in \mathbb{C}^{2\times 2}$ is continuous and analytic on $U^{(1)} \setminus \Sigma_C$.
(b) $P^{(1)}_+(z) = v_C(z)P^{(1)}_-(z) v_C(z)^{-1}$ for $z \in \Sigma_C \cap U^{(1)}$.
(c) $P^{(1)}(z) (P^{(\infty)}(z))^{-1} = I + O(\frac{1}{t})$ as $t \to \infty$, uniformly for $z \in \partial U^{(1)} \setminus \Sigma_C$.

Similar to the case of positive $s$, we use $v_C$ to denote the jump matrix in the RH problem for $C_\alpha$. Comparing this Riemann–Hilbert problem with the one for the function $P^{(0)}$ from the previous section, we see that the solution $P^{(1)}$ can be given in terms of the matrix function $\Phi^{(1)}_\alpha$ defined by equation (2.21) and evaluated at $\alpha = 0$. Indeed, we propose the following form for the function $P^{(1)}$ (cf. (2.19), (2.22)):

$$P^{(1)}(z) = (-s)^{\sigma_3/4} \Phi^{(1)}_0(-s(z-1)) e^{\sigma_3 i(z)\eta},$$
for $z \in U^{(1)} \setminus \Sigma_C$, (3.20)

where $\Phi^{(1)}_0 := \Phi^{(0)}_{\alpha=0}$. We note that the relevant left multiplier $E(z)$ is chosen to be a scaling factor $(-s)^{\sigma_3/4}$, and that this is exactly what the matrix function $E_\alpha(z)$ from (2.23) reduces to if $\alpha = 0$ and if we replace $s(z+1)$ by $-s(z-1)$ as is appropriate in the present situation.

Defined by (3.20), the function $P^{(1)}(z)$ has obviously the correct jumps. Moreover, when $z$ belongs to the boundary of $U^{(1)}$ and $-s$ is large, the function $\Phi_0(-s(z-1))$ in the right-hand side of (3.20) can be replaced by its asymptotics. Therefore, we have

$$P^{(1)}(z) = (z-1)^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + O\left(\frac{1}{-s(z-1)^{3/2}}\right) \right)$$

as $t \to \infty$, uniformly for $z \in \partial U^{(1)} \setminus \Sigma_C$, which yields the matching condition needed. The parametrix in the neighbourhood of the point $z = 1$ is then constructed. Note that it does not depend on $\alpha$.

3.5. Local parametrix $P^{(0)}_a$

We now pass to the analysis of the neighbourhood of the point $z = 0$. Let $U^{(0)}$ be a small open disc around 0. We seek a parametrix $P^{(0)}_a$ defined in $U^{(0)}$ which satisfies the following RH problem.

Riemann–Hilbert problem for $P^{(0)}_a$

(a) $P^{(0)}_a(z) \in \mathbb{C}^{2\times 2}$ is continuous and analytic on $U^{(0)} \setminus \Sigma_C$.
(b) $P^{(0)}_+(z) = P^{(0)}_-(z) v_C(z)$ for $z \in \Sigma_C \cap U^{(0)}$.
(c) $P^{(0)}_a(z) (P^{(\infty)}_{\alpha=0}(z))^{-1} = I + O(\frac{1}{t})$ as $t \to \infty$, uniformly for $z \in \partial U^{(0)} \setminus \Sigma_C$.
(d) $P^{(0)}_a$ has the same behaviour near 0 as $C_\alpha$.

The contours and the jump matrices in the RH problem for $P^{(0)}_a$ are depicted in figure 6.

We take $P^{(0)}_a$ in the form

$$P^{(0)}_a(z) = \Phi^{(0)}_{\alpha=0}(z) e^{\sigma_3 i(z)\eta},$$

and we see that $\Phi^{(0)}_{\alpha=0}$ should satisfy a RH problem with jumps that are indicated in figure 7. The jump matrices for $\Phi^{(0)}_{\alpha=0}$ are constant along the six different pieces.

We now first construct the solution $\Phi^{(0)}_{\alpha}$ of a model RH problem with the same constant jumps on six infinite rays in an auxiliary $\zeta$-plane, and then we put

$$\Phi^{(0)}_{\alpha}(z) = E_\alpha(z)\Phi^{(0)}_{\alpha}(t(z)).$$

(3.23)
Asymptotics for a special solution of the thirty fourth Painlevé equation

\begin{equation}
\begin{pmatrix}
1 & 0 \\
e^{2\pi i/3} & 1
\end{pmatrix}
\begin{pmatrix}
\Sigma_2 \\
\Sigma_4
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
e^{2\pi i/3} & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\end{equation}

**Figure 6.** Contours and jump matrices for the RH problem for $P_{0}^{(0)}$ (magnified picture).

\begin{equation}
\begin{pmatrix}
1 & 0 \\
e^{2\pi i/3} & 1
\end{pmatrix}
\begin{pmatrix}
\Sigma_2 \\
\Sigma_4
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\end{equation}

**Figure 7.** Contours and jump matrices for the RH problem for $\tilde{P}_{0}^{(0)}$ (magnified picture).

where $f$ is given by

\begin{equation}
f(z) = \frac{2}{3} - \frac{2i}{3}(z - 1)^{3/2}, \quad 0 < \arg(z - 1) < 2\pi,
\end{equation}

and $E_0(z)$ is an analytic prefactor that will be chosen later. The function $f(z)$ is analytic in $U^{(0)}$. Moreover, $f(z) = z + \cdots$ and therefore it defines a conformal map in the neighbourhood $U^{(0)}$.

After performing a slight contour deformation, we may and do assume that the six contours $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_u$ and $\Sigma_d$ are mapped by $f$ into six rays. The exact relation between the functions $f(z)$ and $g(z)$ is given by the formula

\begin{equation}
f(z) = \frac{2}{3} - i \begin{cases}
g(z), & \text{Im } z > 0, \\
-g(z), & \text{Im } z < 0.
\end{cases}
\end{equation}

We construct $\Phi^{(\text{Bes})}_{u}$ by relating it to a model RH problem constructed by Vanlessen [24] and also used in [20], whose solution we denote here by $\Phi^{(\text{Bes})}_{u}$.

**Riemann–Hilbert problem for $\Phi^{(\text{Bes})}_{u}$**

(a) $\tilde{\Phi}^{(\text{Bes})}_{u} : C \setminus \Gamma \to \mathbb{C}^{2 \times 2}$ is analytic, where $\Gamma$ is the union of the eight half rays shown in figure 8, namely,

\begin{equation}
\Gamma = \{ \zeta \in \mathbb{C} \mid \arg \zeta \in \{0, \pm\pi/3, \pm\pi/2, \pm2\pi/3, \pi\}\}.
\end{equation}

(b) $\tilde{\Phi}^{(\text{Bes})}_{u,-} = \tilde{\Phi}^{(\text{Bes})}_{u,+} v_{\Phi^{(\text{Bes})}_{u}}$ on $\Gamma$, where the constant jump matrices $v_{\Phi^{(\text{Bes})}_{u}}$ are indicated in figure 8.
Figure 8. Contours and jump matrices for the RH problem for $\Phi_{1(\text{Bes})}^\alpha$.

(d) $\Phi_{a(\text{Bes})}^\alpha(\zeta) = O\left(\frac{\zeta^{\alpha}}{\zeta^{\alpha}}\right)$ as $\zeta \to 0$, if $-1/2 < \alpha < 0$ and

$\Phi_{a(\text{Bes})}^\alpha(\zeta) = \begin{cases} O\left(\frac{\zeta^{\alpha}}{\zeta^{\alpha}}\right) & \text{as } \zeta \to 0, \zeta \in \mathbb{C} \setminus \Gamma \text{ with } \pi/3 < |\arg \zeta| < 2\pi/3, \\ O\left(\frac{\zeta^{\alpha}}{\zeta^{\alpha}}\right) & \text{as } \zeta \to 0, \zeta \in \mathbb{C} \setminus \Gamma \text{ with } \zeta \text{ elsewhere,} \end{cases}$

if $\alpha > 0$.

We did not specify the asymptotic condition. A solution of the RH problem for $\Phi_{a(\text{Bes})}^\alpha(\zeta)$ was given in terms of Bessel functions of orders $\alpha \pm 1/2$. There is a different expression in each sector. Let it suffice here to mention the solution in the sector $\pi/2 < |\arg \zeta| < 2\pi/3$. We give it in a form that is different from the one in [24], where the modified Bessel functions $I_{\alpha \pm 1/2}$ and $K_{\alpha \pm 1/2}$ are used. We state it here in terms of the usual Bessel functions $J_{\alpha \pm 1/2}$ and the Hankel function of the first kind $H_{\alpha \pm 1/2}^{(1)}$, as follows:

$\Phi_{a(\text{Bes})}^\alpha(\zeta) = \sqrt{\pi} e^{-\frac{1}{4}\pi i \zeta^{1/2}} \begin{pmatrix} J_{\alpha + 1/2}(\zeta) & H_{\alpha + 1/2}^{(1)}(\zeta) \\ J_{\alpha - 1/2}(\zeta) & H_{\alpha - 1/2}^{(1)}(\zeta) \end{pmatrix}, \quad \pi/2 < |\arg \zeta| < 2\pi/3. \quad (3.26)$

The asymptotics as $\zeta \to \infty$ in this sector is

$\Phi_{a(\text{Bes})}^\alpha(\zeta) = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} + \frac{\alpha}{2\zeta} \begin{pmatrix} -i(\alpha + 1) & \alpha + 1 \\ -(\alpha - 1) & i(\alpha - 1) \end{pmatrix} + O\left(\frac{1}{\zeta^{2}}\right) \right) \times e^{\frac{1}{2}\pi i \sigma_3} e^{\frac{1}{2}\pi i \sigma_3} e^{-i \zeta \sigma_3}, \quad \pi/2 < |\arg \zeta| < 2\pi/3. \quad (3.27)$

Then we define

$\Phi_{a(\text{Bes})}^\alpha(\zeta) = e^{-\left(\frac{i}{2}\pi i \sigma_3 + \frac{1}{2}\pi i \sigma_3\right)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \Phi_{a(\text{Bes})}^\alpha(\zeta) \times \begin{cases} e^{\pi i \sigma_3}, & \text{if } \text{Re } \zeta > 0, \text{Im } \zeta > 0, \\ 1, & \text{if } \text{Re } \zeta < 0, \\ e^{-\pi i \sigma_3}, & \text{if } \text{Re } \zeta > 0, \text{Im } \zeta < 0. \end{cases} \quad (3.28)$

It is then easy to check that $\Phi_{a(\text{Bes})}^\alpha(\zeta)$ is analytic across $i\mathbb{R}$ and has the jump matrices indicated in figure 7, but of course on the contour $\Gamma \setminus i\mathbb{R}$. The behaviour at 0 remains unaffected by the above transformation, while the constant prefactor $e^{-\left(\frac{i}{2}\pi i \sigma_3 + \frac{1}{2}\pi i \sigma_3\right)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ is chosen so that we obtain the precise asymptotics as $\zeta \to \infty$ given in item (c) below.
Riemann–Hilbert problem for $\Phi^{(\text{Bes})}_0$

(a) $\Phi^{(\text{Bes})}_0 : C \setminus (\Gamma \setminus i\mathbb{R}) \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $\Phi^{(\text{Bes})}_{a^+} = \Phi^{(\text{Bes})}_{a^-} v_{a^\alpha}$ on $\Gamma \setminus i\mathbb{R}$, where the constant jump matrices $v_{a^\alpha}$ are indicated in figure 7.

(c) $\Phi^{(\text{Bes})}_0(\xi) = \left( I + O\left( \frac{1}{\xi} \right) \right) e^{-i\xi\alpha}$ as $\xi \to \infty$ with $\Im \xi > 0$, and

$$\Phi^{(\text{Bes})}_0(\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left( I + O\left( \frac{1}{\xi} \right) \right) e^{-i\xi\alpha} \quad \text{as } \xi \to \infty \text{ with } \Im \xi < 0.$$

(d) $\Phi^{(\text{Bes})}_0(\xi) = O\left( \frac{\xi^\alpha}{|\xi|^\alpha} \right)$ as $\xi \to 0$, if $-1/2 < \alpha < 0$ and

$$\Phi^{(\text{Bes})}_0(\xi) = \begin{cases} O\left( \frac{\xi^\alpha}{|\xi|^\alpha} \right) & \text{as } \xi \to 0, \xi \in C \setminus \Gamma \text{ with } \pi/3 < |\arg \xi| < 2\pi/3, \\ O\left( \frac{|\xi|^{-\alpha}}{|\xi|^{-\alpha}} \right) & \text{as } \xi \to 0, \xi \in C \setminus \Gamma \text{ with } \xi \text{ elsewhere,} \end{cases}$$

if $\alpha \geq 0$.

As noted before, we now put

$$P^{(0)}_a(z) = E_a(z) \Phi^{(\text{Bes})}_0(i \xi(z)) e^{g(z)\xi(z)},$$

where $E_a$ is still to be determined. Then, for fixed $z \in \partial U^{(0)}$ we have as $t \to \infty$,

$$P^{(0)}_a(z) = E_a(z) \Phi^{(\text{Bes})}_0(i \xi(z)) e^{g(z)\xi(z)}$$

$$= \begin{cases} E_a(z) \left( I + O\left( 1/t \right) \right) e^{-2i\sigma_3/3}, & \text{Im } z > 0, \\ E_a(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left( I + O\left( 1/t \right) \right) e^{2i\sigma_3/3}, & \text{Im } z < 0, \end{cases}$$

$$= \begin{cases} E_a(z) e^{-2i\sigma_3/3} \left( I + O\left( 1/t \right) \right), & \text{Im } z > 0, \\ E_a(z) e^{2i\sigma_3/3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left( I + O\left( 1/t \right) \right), & \text{Im } z < 0. \end{cases}$$

(3.29)

To match this with $P^{(\infty)}(z)$ for $z \in \partial U^{(0)}$ we choose

$$E_a(z) = \begin{cases} P^{(\infty)}(z) e^{2i\sigma_3/3}, & \text{Im } z > 0, \\ P^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{2i\sigma_3/3}, & \text{Im } z < 0. \end{cases}$$

(3.30)

which is indeed analytic in $U^{(0)}$. This completes the construction of the local parametrix $P^{(0)}_a$.

3.6. Fourth transformation $C_a \mapsto D_a$

In the final transformation we put

$$D_a(z) = \begin{cases} C_a(z) \left( P^{(1)}_a(z) \right)^{-1}, & \text{for } z \in U^{(1)} \setminus \Sigma_C, \\ C_a(z) \left( P^{(0)}_a(z) \right)^{-1}, & \text{for } z \in U^{(0)} \setminus \Sigma_C, \\ C_a(z) \left( P^{(\infty)}(z) \right)^{-1}, & \text{for } z \in C \setminus (U^{(1)} \cup U^{(0)} \cup (-\infty, 1)). \end{cases}$$

(3.31)

By construction, the only jumps that remain for $D_a$ are across the circles $\partial U^{(1)}$ and $\partial U^{(0)}$ and the parts of the arcs $\Sigma_a$ and $\Sigma_d$ and the rays $[1, \infty)$, $\Sigma_2$, and $\Sigma_4$ which lie outside of the neighbourhoods $U^{(1)}$ and $U^{(0)}$. We shall denote this remaining contour as $\Sigma_D$; it is depicted in figure 9. The Riemann–Hilbert problem for $D_a$ is set on this contour.
Figure 9. Contour $\Sigma_D$ in the RH problem for $D_\alpha$.

Riemann–Hilbert problem for $D_\alpha$

(a) $D_\alpha : \mathbb{C} \setminus \Sigma_D \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $D_{\alpha,+}(z) = D_{\alpha,-}(z) v_{D_\alpha}(z)$ for $z \in \Sigma_D$, where

$$v_{D_\alpha} = \begin{cases} p^{(\infty)} (p^{(1)})^{-1}, & \text{on } \partial U^{(1)}, \\ p^{(\infty)} (p^{(0)})^{-1}, & \text{on } \partial U^{(0)}, \\ p^{(\infty)} v_{\alpha} (p^{(\infty)})^{-1}, & \text{on } \Sigma_D \setminus (\partial U^{(1)} \cup \partial U^{(0)}). \end{cases}$$

(c) $D_{\alpha}(z) = I + O(1/z)$ as $z \to \infty$.

Due to the matching conditions of the Riemann–Hilbert problems for $P^{(1)}$ and $P^{(0)}$, we have that

$$v_{D_\alpha}(z) = I + O \left( \frac{1}{t} \right),$$

uniformly on the circles $\partial U^{(1)}$ and $\partial U^{(0)}$. Simultaneously,

$$v_{D_\alpha}(z) = I + O \left( e^{-ct|z|} \right), \quad c > 0,$$

uniformly on $\Sigma_D \setminus (\partial U^{(1)} \cup \partial U^{(0)})$. Hence, as before,

$$D_{\alpha}(z) = I + O \left( \frac{1}{t(1 + |z|)} \right), \quad \text{as } t \to +\infty,$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_D$.

3.7. Conclusion of the proof of (1.8)

The main remaining step in the proof of (1.8) is to express $u_\alpha$ in terms of $D_\alpha$. The result of the calculations is contained in the next lemma.

**Lemma 3.1.** For every $s < 0$, we have

$$u_\alpha(s) = \frac{\alpha}{\sqrt{-s}} \left[ D_\alpha(0) \begin{pmatrix} i \sin \theta & \cos \theta \\ -\cos \theta & -i \sin \theta \end{pmatrix} D_\alpha^{-1}(0) \right]_{12}. \quad (3.36)$$

where

$$\theta = 4t/3 - \pi \alpha. \quad (3.37)$$

and $t = (-s)^{3/2}$ as before.
Theorem 1.2 follows immediately from the lemma and (3.35). Indeed from (3.35) we find that $D_\alpha(0) = I + O((-s)^{-3/2})$ as $s \to -\infty$. Hence by (3.36) and (3.37)
\[
u_\alpha(s) = \frac{\alpha}{\sqrt{-s}} \left( \cos \theta + O((-s)^{-3/2}) \right) = \frac{\alpha}{\sqrt{-s}} \cos \left( \frac{4}{3}(-s)^{3/2} - \pi \alpha \right) + O(s^{-2})
\]
as $s \to -\infty$, which is (1.8).

So it remains to prove lemma 3.1.

**Proof.** Take $z \in U(0)$ with $\text{Im} z > 0$ and outside of the lenses. Then we have by (3.31), (3.22) and (3.23), that
\[
C_\alpha(z) = D_\alpha(z)P_{\alpha}^{(0)}(z) = D_\alpha(z)E_\alpha(z)\Phi_1^{(\text{Bes})}(tf(z))e^{j\nu(z)\sigma_3}.
\]
In the computation of the limit of $z \left[ \left( \frac{d}{dz} C_\alpha(z), C_\alpha^{-1}(z) \right) \right]_{12}$ as $z \to 0$, the only term that will contribute is the one we get by taking the derivative of $\Phi_1^{(\text{Bes})}(tf(z))$. This easily follows from the fact that $D_\alpha$ and $E_\alpha$ are analytic at 0.

Since, by (3.30) and (3.19), we have
\[
E_\alpha(0) = e^{-\sqrt{2}i/4} \frac{1}{\sqrt{2}} \left( \begin{array}{c}
i \\
i \\
i \end{array} \right) e^{2i\nu_\alpha/3}
\]
it then follows from (3.18) and (3.38) that
\[
u_\alpha(s) = \frac{i}{\sqrt{-s}} \left[ D_\alpha(0)e^{-i\nu_\alpha/4} \frac{1}{\sqrt{2}} \left( \begin{array}{c}
i \\
i \\
i \end{array} \right) e^{2i\nu_\alpha/3}
\times \left( \lim_{z \to 0} z \left( \frac{d}{dz} \Phi^{(\text{Bes})}_\alpha(tf(z)) \right) \left( \Phi^{(\text{Bes})}_\alpha(tf(z)) \right)^{-1} \right)
\times e^{-2i\nu_\alpha/3} \frac{1}{\sqrt{2}} \left( \begin{array}{c}
i \\
-1 \\
1 \end{array} \right) e^{i\nu_\alpha/4} D_{\alpha}^{-1}(0) \right]_{12}.
\]

Putting $\zeta = tf(z)$ we get
\[
\frac{d}{d\zeta} \Phi^{(\text{Bes})}_\alpha(tf(z)) = tf'(z) \frac{d}{d\zeta} \Phi^{(\text{Bes})}_\alpha(\zeta).
\]
Noting that $f(0) = 0$ and $f'(0) \neq 0$, we find
\[
\frac{tf'(z)z}{\zeta} = \frac{zf'(z)}{f(z)} \to 1
\]
as $z \to 0$. Therefore,
\[
\lim_{z \to 0} z \left( \frac{d}{dz} \Phi^{(\text{Bes})}_\alpha(tf(z)) \right) \left( \Phi^{(\text{Bes})}_\alpha(tf(z)) \right)^{-1} = \lim_{\zeta \to 0} \zeta \left( \frac{d}{d\zeta} \Phi^{(\text{Bes})}_\alpha(\zeta) \right) \left( \Phi^{(\text{Bes})}_\alpha(\zeta) \right)^{-1}.
\]

From definition (3.28) of $\Phi^{(\text{Bes})}_\alpha$ we find that for $\text{Im} \zeta > 0$,
\[
\left( \frac{d}{d\zeta} \Phi^{(\text{Bes})}_\alpha(\zeta) \right) \left( \Phi^{(\text{Bes})}_\alpha(\zeta) \right)^{-1}
= e^{-\sqrt{2}i/2+i\pi/4}\sigma_3 \frac{1}{\sqrt{2}} \left( \begin{array}{c}
i \\
i \\
i \end{array} \right) \left( \left( \frac{d}{d\zeta} \Phi^{(\text{Bes})}_\alpha(\zeta) \right) \left( \Phi^{(\text{Bes})}_\alpha(\zeta) \right)^{-1} \right)
\times \frac{1}{\sqrt{2}} \left( \begin{array}{c}
i \\
-1 \\
1 \end{array} \right) e^{i\pi/2+i\pi/4}\sigma_3.
\]
Recall that $\Phi_a^{(\text{Bes})}$ is built out of Bessel functions. From the differential-difference relations satisfied by the Bessel functions (see formula (9.1.27) in [1])

$$J'_\nu(\zeta) = J_{\nu-1}(\zeta) - \frac{\nu}{\zeta} J_\nu(\zeta),$$

$$J'_{\nu-1}(\zeta) = -J_{\nu+1}(\zeta) + \frac{\nu}{\zeta} J_\nu(\zeta),$$

and similar ones for the Hankel functions, it easily follows from (3.26) that

$$\frac{d}{d\zeta} \Phi_a^{(\text{Bes})}(\zeta) = \begin{pmatrix} -\alpha/\zeta & 1 \\ -1 & \alpha/\zeta \end{pmatrix} \Phi_a^{(\text{Bes})}(\zeta).$$

Thus

$$\lim_{\zeta \to 0} \zeta \left( \frac{d}{d\zeta} \Phi_a^{(\text{Bes})}(\zeta) \right) \Phi_a^{(\text{Bes})}(\zeta) = -\alpha \sigma_3. \quad (3.43)$$

Combining (3.43) with (3.40), (3.41), (3.42) we get with $\theta = 4t/3 - \pi \alpha$,

$$u_a(s) = \frac{i}{\sqrt{-1}} \left[ D_a(0) e^{-i\sigma_3 s/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{i\sigma_3 s/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right] \times (-\alpha \sigma_3) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-i\sigma_3 s/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{i\sigma_3 s/4} D_a^{-1}(0) \right]_{12}^1,$$

which after straightforward calculation reduces to (3.36). This completes the proof of the lemma.

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Appendix A. Relation to the Painlevé II equation and the uniqueness question

We start with reviewing the general facts concerning the relation between the thirty fourth and the second Painlevé equations.

Let $q(s)$ be a solution of the second Painlevé equation with parameter $\nu$,

$$q'' = 2q^3 + sq - \nu. \quad (A.1)$$

Then, the function $u(s)$ defined by the formulae

$$u(s) = 2^{-1/3} U(-2^{1/3} s), \quad U(s) = q^2(s) + q'(s) + \frac{s}{2} \quad (A.2)$$

satisfies the thirty fourth equation (1.3) with the parameter

$$\alpha = \frac{\nu}{2} - \frac{1}{4}$$

(see [11]; see also [2, 15]). The inverse transformation is given by the formulae

$$q(s) = -2^{-1/3} Q(-2^{-1/3} s), \quad Q(s) = \frac{u' - 2\alpha}{2u}. \quad (A.3)$$
Moreover, the equations (see [2, 19]),

\[ \Psi_{\alpha}(z; s) = \left( \begin{array}{cc} 1 & 0 \\ \eta(s) & 1 \end{array} \right) z^{-\sigma_1/4} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) e^{\pi i \nu/4} \Psi_{2\nu+1/2}(w; -2^{1/3} e^{-\pi i \sigma_3/4}, \text{A.4}) \]

\[ \eta(s) = -2^{1/3} \left( [m_{FN}(s)]_{11} + [m_{FN}(s)]_{12} \right), \]

where \( w = e^{\pi i/2} 2^{-1/3} e^{1/2} \) with \( \text{Im } w > 0 \), establish the relation between the solution \( \Psi_{\alpha}(z; s) \) of the general Painlevé XXXIV RH problem formulated in section 1.4 and the solution \( \Psi_{\nu, \alpha}(w; s) \) of the RH problem associated with the Painlevé II equation [9] with the parameter \( \nu = 2\alpha + 1/2 \). In (A.4), \( m_{FN}(s) \) denotes the first matrix coefficient of the expansion \( \Psi_{\nu, \alpha}(w; s) \) at \( w = \infty \),

\[ \Psi_{\nu, \alpha}(w; s) = \left( I + \frac{m_{FN}(s)}{w} \right) e^{-i \left( \frac{4}{3} w^{3/2} + \frac{w}{3} \right) \sigma_3} \text{A.5} \]

as \( w \to \infty \). We use here the Flaschka–Newell [9] form of the Painlevé II RH problem whose setting we will now recall (for details see [10, chapter 5]).

The general Painlevé II RH problem involves three complex constants \( a_1, a_2, a_3 \) satisfying (cf (1.11))

\[ a_1 + a_2 + a_3 + a_1 a_2 a_3 = -2i \sin \nu \pi, \text{A.6} \]

and certain connection matrices \( E_{FN}^{\nu} \). Let \( S_j = \{ w \in \mathbb{C} \mid \frac{2j-3}{6} \pi < \arg w < \frac{2j-1}{6} \pi \} \) for \( j = 1, \ldots, 6 \), and let \( \Sigma_{FN} = \mathbb{C} \setminus \bigcup_j S_j \). Then \( \Sigma_{FN} \) consists of six rays \( \Sigma_{FN}^j = \{ w \in \mathbb{C} \mid \arg w = \frac{2j-1}{6} \pi \} \) for \( j = 1, \ldots, 6 \), all chosen oriented towards infinity as in figure 10.

The RH problem is the following.

(a) \( \Psi_{\nu, \alpha}^{FN} : \mathbb{C} \setminus \Sigma_{FN} \to \mathbb{C}^{2 \times 2} \) is analytic,

(b) \( \Psi_{\nu, \alpha}^{FN} = \Psi_{\nu, \alpha}^{FN} \left( \begin{array}{cc} 1 & 0 \\ a_1 & 1 \end{array} \right) \) on \( \Sigma_{FN}^1 \),

\[ \Psi_{\nu, \alpha}^{FN} = \Psi_{\nu, \alpha}^{FN} \left( \begin{array}{cc} 1 & a_2 \\ 0 & 1 \end{array} \right) \) on \( \Sigma_{FN}^2 \),

\[ \Psi_{\nu, \alpha}^{FN} = \Psi_{\nu, \alpha}^{FN} \left( \begin{array}{cc} 1 & 0 \\ a_3 & 1 \end{array} \right) \) on \( \Sigma_{FN}^3 \),

\[ \Psi_{\nu, \alpha}^{FN} = \Psi_{\nu, \alpha}^{FN} \left( \begin{array}{cc} 1 & a_1 \\ 0 & 1 \end{array} \right) \) on \( \Sigma_{FN}^4 \).
\[
\psi_{\nu}^{\text{FN}}(z) = \psi_{\nu}^{\text{FN}} \left( \begin{array}{c} 1 \\ a_2 \\ 1 \\ 0 \end{array} \right) \text{ on } \Sigma_{\nu}^{\text{FN}},
\psi_{\nu}^{\text{FN}}(z) = \psi_{\nu}^{\text{FN}} \left( \begin{array}{c} 1 \\ a_3 \\ 0 \\ 1 \end{array} \right) \text{ on } \Sigma_{\nu}^{\text{FN}}.
\]

(c) \( \psi_{\nu}^{\text{FN}}(w) = (I + O(1/w))e^{-iw^2 + o(w)} \) as \( w \to \infty \).

(d) If \( \nu - \frac{1}{2} \notin \mathbb{N} \), then

\[
\psi_{\nu}^{\text{FN}}(w) = (I + O(1/w))e^{-iw^2 + o(w)} E_{\nu}^{\text{FN}}, \quad \text{for } w \in S_{\nu},
\]
where \( B \) is analytic. If \( \nu \in \frac{1}{2} + \mathbb{N} \), then there exists a constant \( \kappa \) such that

\[
\psi_{\nu}^{\text{FN}}(w) = B(w) \left( \begin{array}{c} w^\nu \\ 0 \\ \kappa w^\nu \log w \\ w^{-\nu} \end{array} \right) E_{\nu}^{\text{FN}}, \quad \text{for } w \in S_{\nu},
\]
where \( B \) is analytic.

Except for the special case,

\[
\nu = \frac{1}{2} + n, \quad a_1 = a_2 = a_3 = i(-1)^{n+1} \quad n \in \mathbb{N},
\]
when the solution of the RH problem is given in fact in terms of the Airy functions, the connection matrices \( E_{\nu}^{\text{FN}} \) are determined (up to inessential left diagonal or upper triangular factors) by \( \nu \) and the Stokes multipliers \( a_j \). In the special case the solution is parametrized by the one non-trivial entry of the connection matrix \( E_{\nu}^{\text{FN}} \). We refer to [10, chapters 5 and 11] for more details on the setting and the analysis of the Painlevé II RH problem (see also our paper [15], where we review these results in the notation we use here).

Equation (A.4) implies the following relation between the Painlevé XXXIV and Painlevé II Stokes parameters ([2, 19]; see also [15]),

\[
b_1 = ia_2, \quad b_2 = ia_3, \quad b_3 = ia_1,
\]
where we use the \( b_j \) as in section 1.4. Therefore, taking into account (1.13), we conclude that the second Painlevé function which is related to the special solution \( u_{\alpha}(s) \) of the thirty fourth Painlevé equation studied in this paper corresponds to the choice of the Stokes multipliers,

\[
a_1 = e^{-\nu \pi i}, \quad a_2 = -i, \quad a_3 = -e^{\nu \pi i}.
\]
This is different from the choice,

\[
a_1 = e^{-\nu \pi i}, \quad a_2 = 0, \quad a_3 = -e^{\nu \pi i},
\]
corresponding, as it is shown in [10, chapter 11] to the generalized (\( \nu \neq 0 \)) Hastings–McLeod solution of the second Painlevé equation, i.e. the solution which is characterized by the following asymptotic conditions:

\[
q_{\text{HM}}(s) = \sqrt{-s^2 + O(s^{-1})}, \quad \text{as } s \to -\infty
\]
and

\[
q_{\text{HM}}(s) = \frac{s}{s} + O(s^{-4}), \quad \text{as } s \to +\infty.
\]
see also [3].

The RH problem which is used in [10, chapter 11] differs by a simple gauge transformation from the Flaschka–Newell RH problem. Indeed one has that \( \psi_{\nu}^{\text{FN}} = e^{-i\pi \nu} \psi_{\nu}^{\text{FN}}(z) \), where \( \psi(z) \) is the solution of the RH problem from [10, chapter 11]. This in turn implies that the Flaschka–Newell monodromy parameters \( a_j \) are related to the monodromy parameters \( s_j \) from [10, chapter 11], via the equations, \( a_1 = i s_1, \quad a_2 = -i s_2, \) and \( a_3 = i s_3 \).
It follows from (A.11), however, that both solutions—the Hastings–McLeod solution and the one corresponding to $u_\alpha(s)$, belong to the same one-parameter family of solutions of the Painlevé II equation which is characterized by the following choice of the Stokes multipliers:
\begin{equation}
\begin{aligned}
a_1 &= e^{-\nu \pi i}, & a_3 &= -e^{\nu \pi i}, \\
\end{aligned}
\end{equation}
and the Stokes multiplier $a_2$ is a free parameter of the family. It is shown in [18] (see also [10, chapter 11]) that this family is exactly the classical family of the so called 
tronquée
 solutions, i.e. the solutions all of which exhibit the same behaviour (A.13) at $-\infty$. In fact, for every 
tronquée
 solution the behaviour (A.13) can be extended to a full asymptotic series and it holds in the whole sector $2\pi / 3 < \arg s < 4\pi / 3$,
\begin{equation}
q_{\text{tronq}}(s) \sim \sqrt{-\frac{s}{2}} \sum_{n=0}^{\infty} c_n (-s)^{-3n/2}, \quad c_0 = 1,
\end{equation}
as $s \to -\infty$, $\arg s \equiv \pi + \arg (-s) \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$,
where all the coefficients $c_n$ are uniquely determined by the substitution into the Painlevé II equation (i.e. the series is the same for every solution from the family).

The above made observation suggests that one can obtain the asymptotic statement (1.7) of theorem 1.2 directly from (A.16) using relation (A.2). Indeed, the first two terms of (A.16) read
\begin{equation}
q_{\text{tronq}}(s) = \sqrt{-\frac{s}{2}} - \frac{\nu}{2s} + O(s^{-5/2}),
\end{equation}
as $s \to -\infty$, $\arg s \equiv \pi + \arg (-s) \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.
Substituting this (differentiable!) asymptotics into (A.2) we indeed arrive at (1.7).

The fact that the asymptotics (1.7) holds for a one-parameter family of the solutions of the Painlevé XXXIV equation (1.3) can also be deduced from the direct analysis of the Painlevé XXXIV RH problem. Denote the one-parameter family of solutions of Painlevé XXXIV corresponding to the RH data,
\begin{equation}
b_2 = e^{2\alpha \pi i}, \quad b_4 = e^{-2\alpha \pi i}, \quad b_1 = b \in \mathbb{C},
\end{equation}
as
\begin{equation}
u^{\text{tronq}}(s) \equiv u_\alpha^{(\text{tronq})}(s|b).
\end{equation}
Note that the cyclic relation (1.11) is valid identically for $b_1$ if $b_2$ and $b_4$ are as in (A.18). It is not difficult to see that exactly the same sequence of transformation as the one we used in section 2 in the analysis of the RH problem in the case $s \to +\infty$ can be performed for any value of $b$. In the final $D_\alpha$ RH problem the only difference is in the jump matrix $v_{D_\alpha}$ on the segment of the horizontal part of the jump contour depicted in figure 3 which is to the right of 0. That is, instead of
\begin{equation}
v_{D_\alpha} = P^{(\infty)}_\alpha(z) \begin{pmatrix} \frac{e^{-2gz(z)}}{1} \\ 0 \end{pmatrix} \begin{pmatrix} P^{(\infty)}_\alpha(z) \end{pmatrix}^{-1} \equiv I + O \left( e^{-c|z|+|1|} \right), \quad \text{for } z > 0,
\end{equation}
we now have
\begin{equation}
v_{D_\alpha} = P^{(\infty)}_\alpha(z) \begin{pmatrix} \frac{be^{-2gz(z)}}{1} \\ 0 \end{pmatrix} \begin{pmatrix} P^{(\infty)}_\alpha(z) \end{pmatrix}^{-1} \equiv I + O \left( be^{-c|z|+|1|} \right), \quad \text{for } z > 0.
\end{equation}
6 We refer to [10, chapter 11] for more on the asymptotic analysis of the 
tronquée
 solutions of the second Painlevé equation. In particular, the reader can find there an alternative parametrization (and its explicit relation to $a_2$) of the solutions via the coefficients of the oscillatory terms of the asymptotics on the boundary rays.
In other words, the only difference in \( v_{D_a} \) is in the exponentially small error. Hence the estimates (2.26) and (2.27) are valid for all \( b \) and lead to the same asymptotic behaviour (1.7) of the solution \( u_a^{\text{tronq}}(s) \) for all \( b \). In fact, every solution from the family has the same asymptotic series representation,

\[
u_{a}^{\text{tronq}}(s) \sim \frac{\alpha}{\sqrt{s}} + \sum_{n=1}^{\infty} d_{n} s^{-\frac{n+1}{2}}, \quad \text{as } s \to +\infty,
\]

(A.19)

with the coefficients \( d_{n} \) uniquely determined by the substitution of the series into equation (1.3) (see [15] for the explicit recurrence relation for \( d_{n} \)).

The asymptotic behaviour (A.14) of the Hastings–McLeod solution at \(+\infty\) is also shared by another one-parameter family of tronquée solutions. The corresponding RH parametrization is (see [13]; see also [10, chapter 11]),

\[
a_{2} = 0, \quad a_{1} + a_{5} = -2i \sin \nu \pi.
\]

(A.20)

The Painlevé II function which corresponds to the thirty fourth Painlevé function \( u_{a}(s) \) obviously does not belong to this family and hence does not behave as (A.14) when \( s \to +\infty \). However, the leading term of its behaviour as \( s \to +\infty \) is known ([17]; see also [10, chapter 10]). Unfortunately, the leading term is not enough to derive the corresponding asymptotics as \( s \to -\infty \) of the Painlevé XXXIV function \( u_{a}(s) \). Indeed, the leading asymptotics of \( q(s) \) as \( s \to +\infty \) is of the form

\[
q(s) \sim \sqrt{\frac{s}{2}} \cot \left( \frac{\sqrt{2}}{3} s^{3/2} + \chi \right).
\]

(A.21)

(the phase \( \chi \) is known) and it cancels out in the right-hand side of equation (A.2). Moreover, the solution \( q(s) \), as it follows from (A.21), has poles on the positive real \( s \)-axis while the function \( u_{a}(s) \) is smooth for all real \( s \). This means that the reduction to the Painlevé II equation is not the best way to study the asymptotics of the Painlevé XXXIV function \( u_{a}(s) \) as \( s \to -\infty \). It is better to proceed via the direct analysis of the Painlevé XXXIV RH problem for \( \Psi_{a} \), as we did in section 3 of this paper.

Let us now consider the question of the uniqueness of the solution \( u_{a}(s) \). To this end, let us analyse what effect on the constructions of section 3 would be produced by the passing to the general tronquée solution \( u_{a}^{\text{tronq}}(s) \), i.e. by lifting the restriction \( b_{1} = 1 \). We already saw that the considerations and results of section 2 are not affected. The situation with section 3, i.e. with the analysis of the Painlevé XXXIV RH problem as \( s \to -\infty \) is different. In what follows, we shall analyse the tronquée-RH problem assuming that \( b > 0 \).

(A.22)

There are no changes in the basic three transformations of section 3, except that the factorization (3.16) now is

\[
\begin{pmatrix}
    e^{-i\theta(z)} & b \\
    0 & e^{i\theta(z)}
\end{pmatrix} = \begin{pmatrix}
    1 & 0 \\
    0 & b
\end{pmatrix} \begin{pmatrix}
    0 & b \\
    -b^{-1} & 0
\end{pmatrix} \begin{pmatrix}
    1 & 0 \\
    b^{-1} e^{-i\theta(z)} & 0
\end{pmatrix},
\]

(A.23)

\footnote{In principle, it is possible to use the inverse formula (A.3) and obtain the asymptotic of \( u_{a}(s) \) by integrating (A.21). However, with this approach we face the problem of evaluation of the constant of integration and, once again, one has to take special care of the poles of the Painlevé II function.}
and the transformation (3.17) is modified accordingly. It leads us to the following RH problem for the matrix function $C_{a}(z)$.

(a) $C_{a}: \mathbb{C} \setminus \Sigma_{c} \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $C_{a+}(z) = C_{a-}(z) \begin{pmatrix} 1 & b \sqrt{e^{-2\pi i g(z)}} \\ 0 & 1 \end{pmatrix}$, for $z \in (1, \infty)$,

$C_{a+}(z) = C_{a-}(z) \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$, for $z \in (0, 1)$,

$C_{a+}(z) = C_{a-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \Sigma_{3}$,

$C_{a+}(z) = C_{a-}(z) \begin{pmatrix} 1 & 0 \\ b^{-1} e^{2\pi i g(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_{a} \cup \Sigma_{d}$,

$C_{a+}(z) = C_{a-}(z) \begin{pmatrix} 1 & 0 \\ e^{2\pi i g(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_{2}$,

$C_{a+}(z) = C_{a-}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\pi i g(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_{1}$.

(c) $C_{a}(z) = (I + O(1)) z^{-\sigma_{1}/4} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

(d) $C_{a}(z) = O \begin{pmatrix} |z|^\alpha \\ |z|^\alpha \end{pmatrix} |z|^\alpha$ as $z \to 0$, if $-1/2 < \alpha < 0$; and

$C_{a}(z) = \begin{cases} O \begin{pmatrix} \alpha \alpha \\ \alpha \alpha \end{pmatrix} |z|^{-\alpha} & \text{as } z \to 0 \text{ with } z \in (\Omega_{1} \cup \Omega_{4}) \setminus (\Omega_{a} \cup \Omega_{d}), \\ O \begin{pmatrix} 1 \\ \beta \end{pmatrix} & \text{as } z \to 0 \text{ with } z \in \Omega_{1} \cup \Omega_{2} \cup \Omega_{3} \cup \Omega_{a} \cup \Omega_{d}, \end{cases}$ if $\alpha \geq 0$.

The contour for this RH problem is the same as before, i.e. the one depicted in figure 5.

We can at once make two important observations. Firstly, the neighbourhood of the point $z = 0$ will contribute to the asymptotic analysis (as, in fact, it has in the case $b = 1$) and hence we should expect a change in the asymptotics (1.8) and appearance in it of an explicit dependence of the parameter $b$. Secondly, the inequality $b \neq 1$ yields serious alterations in the constructions of the parametrices $P^{(\infty)}$ and $P^{(b)}$.

The RH problem for the global parametrix $P^{(\infty)}$ now reads as

(a) $P^{(\infty)}: \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $P_{a}^{(\infty)}(z) = P_{a-}^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in (-\infty, 0)$,

$P_{a}^{(\infty)}(z) = P_{a-}^{(\infty)}(z) \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$, for $z \in (0, 1)$,

(c) $P_{a}^{(\infty)}(z) = (I + O(1)) z^{-\sigma_{1}/4} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

This RH problem can still be solved explicitly. In fact, it is now similar to the global parametrix from section 2. The solution is given by the equation

$P^{(\infty)}(z) = E(z - 1)^{-\sigma_{1}/4} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \frac{(z - 1)^{1/2} + i}{(z - 1)^{1/2} - i}^{\beta\sigma_{1}}$ \hspace{1cm} (A.24)

where

$\beta = \frac{i}{2\pi} \log b, \quad E = \begin{pmatrix} 1 \\ 0 \\ \beta \end{pmatrix}$. 


Figure 11. Contours and jump matrices for the RH problem for \( \Phi_{\alpha,\beta}^{(\text{CHF})} \).

and the branches of the arguments are fixed by the inequalities,

\[ -\pi < \arg(z - 1) < \pi, \quad 0 < \arg \left( \frac{(z - 1)^{1/2} + i}{(z - 1)^{1/2} - 1} \right) < \pi. \]

The construction of the local parametrix \( P_{\alpha}^{(0)} \) now involves, instead of the Bessel model RH problem \( \Phi_{\alpha}^{(\text{Bes})} \), the function \( \Phi_{\alpha,\beta}^{(\text{CHF})} \) which satisfies a RH problem with the same contour \( \Gamma \) as \( \Phi_{\alpha}^{(\text{Bes})} \), and with the same jump matrices in the left half plane, see figure 7, while in the right half plane one has the new jump matrices,

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
e^{2\pi i \beta} & 0 \\
0 & e^{-2\pi i \beta}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
e^{2\pi i \beta} & 0 \\
0 & e^{2\pi i \beta}
\end{pmatrix},
\]

on the rays \( \arg \zeta = \pi/3 \), \( \arg \zeta = 0 \) and \( \arg \zeta = -\pi/3 \), respectively. The jump contour is as shown in figure 7, but on straight rays extending to infinity.

The function \( \Phi_{\alpha,\beta}^{(\text{CHF})} \), in turn, admits the following representation:

\[
\Phi_{\alpha,\beta}^{(\text{CHF})}(\zeta) = \Phi_{\alpha,\beta}^{(\text{CHF})}(\zeta),
\]

where the matrix function \( \Phi_{\alpha,\beta}^{(\text{CHF})}(\zeta) \) is the solution of the RH problem depicted in figure 11.

It is shown in [5], see also [14], that the RH problem for \( \Phi_{\alpha,\beta}^{(\text{CHF})} \) supplemented by the proper representation at \( \zeta = 0 \) (inherited from (1.9) and (1.10)) and the asymptotic condition,

\[
\Phi_{\alpha,\beta}^{(\text{CHF})}(\zeta) = I + O\left(\frac{1}{\zeta}\right),
\]

as \( \zeta \to \infty, \quad 0 < \arg \zeta < \pi/2 \), is uniquely solvable; moreover, it admits an explicit solution in terms of the confluent hypergeometric functions \( \psi(a, c; \zeta) \) with the parameters

\[
a = \alpha + \beta, \quad c = 1 + 2\alpha.
\]

Indeed, the solution \( \Phi_{\alpha,\beta}^{(\text{CHF})} \) is described by the following formulae.
Define on the complex plane, cut along the negative imaginary axis, the matrix function

$$
\Psi_0(\zeta) := \begin{pmatrix}
2^\alpha \zeta^\alpha \psi(\alpha + \beta, 1 + 2\alpha, 2e^{\frac{i\pi}{2}} \zeta)e^{i\pi(\beta + \frac{\zeta}{2})}e^{-\zeta} \\
-2^{-\alpha} \zeta^{-\alpha} \psi(1 - \alpha + \beta, 1 - 2\alpha, 2e^{\frac{i\pi}{2}} \zeta)e^{i\pi(\beta - \frac{\zeta}{2})} \frac{\Gamma(1 + \alpha + \beta)}{\Gamma(\alpha - \beta)} \\
-2^\alpha \zeta^\alpha \psi(1 + \alpha - \beta, 1 + 2\alpha, 2e^{-\frac{i\pi}{2}} \zeta)e^{i\pi(\beta + \frac{\zeta}{2})} \frac{\Gamma(1 + \alpha - \beta)}{\Gamma(\alpha + \beta)} \\
2^{-\alpha} \zeta^{-\alpha} \psi(-\alpha - \beta, 1 - 2\alpha, 2e^{-\frac{i\pi}{2}} \zeta)e^{-\frac{i\pi}{2} \zeta} e^{e^{-\pi i} \alpha}
\end{pmatrix},
$$

(A.27)

where the branches of the multi-valued functions in the right-hand side of the equation (including the confluent hypergeometric function) are fixed by the condition,

$$
-\frac{\pi}{2} < \arg \zeta < \frac{3\pi}{2}.
$$

We use I–VIII to denote the eight sectors as in figure 11. The function $\tilde{\Phi}_{a,b}^{(\text{CHF})}(\zeta)$ is then given by the equations

$$
\tilde{\Phi}_{a,b}^{(\text{CHF})}(\zeta) = 2^b \alpha e^{\frac{\pi i}{2} \beta} \begin{pmatrix}
ed^{-i\pi(2\alpha + \beta)} & 0 \\
0 & e^{i\pi(2\alpha + \beta)}
\end{pmatrix}
\begin{pmatrix}
0 & -e^{-\pi i/\beta} \\
e^{\pi i/\beta} & 0 \\
ed^{2\pi i \alpha} & -e^{-\pi i/\beta} \\
e^{\pi i/\beta} & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
e^{\pi i(\beta - 2\alpha)} & 1 \\
e^{\pi i(\beta - \alpha)} & 0 \\
e^{\pi i(\beta - \alpha)} & e^{-\pi i \alpha}
\end{pmatrix}
\Psi_0(\zeta)
$$

(A.28)

It is a straightforward though a bit involved, calculation to check that the function $\tilde{\Phi}_{a,b}^{(\text{CHF})}(\zeta)$ defined by (A.27) and (A.28) does indeed satisfy the jump conditions indicated in figure 11. By a direct calculation, one can also establish the following asymptotic behaviour of $\tilde{\Phi}_{a,b}^{(\text{CHF})}(\zeta)$.
as $\zeta \to \infty$:
\[
\tilde{\Phi}_{\alpha,\beta}^{(\text{CHF})}(\zeta) = \left( I + \frac{m_\Phi}{\zeta} + O\left( \frac{1}{\zeta^2} \right) \right) \zeta^{-\beta_\sigma} e^{-i\zeta^\sigma_1}
\times \begin{cases}
I & \text{if } 0 < \arg \zeta < \frac{\pi}{2}, \\
\epsilon^{i\pi\sigma_1} & \text{if } \frac{\pi}{2} < \arg \zeta < \pi,
\end{cases}
\begin{pmatrix}
0 & -e^{i\pi(\alpha+\beta)} \\
e^{-i\pi(\alpha+\beta)} & 0
\end{pmatrix}
\begin{cases}
0 & \text{if } \pi < \arg \zeta < \frac{3\pi}{2}, \\
e^{-i\pi\beta} & 0
\end{cases}
\begin{pmatrix}
0 & -e^{-i\pi\beta} \\
e^{i\pi\beta} & 0
\end{pmatrix}
\begin{cases}
0 & \text{if } -\frac{\pi}{2} < \arg \zeta < 0,
\end{cases}
\]

where
\[
m_\Phi = \begin{pmatrix}
\frac{1}{2}(s^2 - \alpha^2) & -\frac{1}{2} \Gamma(1+\alpha-\beta)e^{4\pi i\alpha+\pi i}\beta \\
n\frac{1}{2} \Gamma(1+\alpha+\beta)e^{-4\pi i\alpha-\pi i}\beta & \frac{1}{2}(\alpha^2 - \beta^2)
\end{pmatrix}.
\]

The asymptotic formulae (A.29) indicate that the local parametrix $P_{\alpha,\beta}^{(0)}(\zeta)$ again takes the form
\[
P_{\alpha,\beta}^{(0)}(\zeta) = E_{\alpha,\beta}(\zeta) \Phi_{\alpha,\beta}^{(\text{CHF})}(tf(\zeta))e^{\Phi(\zeta)},
\]
where the change-of-variable function $f(\zeta)$ is exactly the same as before, i.e. as in the case $b = 1$, while the holomorphic at the $\zeta = 0$ matrix-valued function $E_{\alpha,\beta}(\zeta)$ is defined by the formula which is slightly more complicated than the previous equations (3.30). Indeed, this time we have
\[
E_{\alpha,\beta}(\zeta) = \begin{cases}
P^{(\infty)}(\zeta)(tf(\zeta))^{\beta_\sigma} e^{2i\pi/3} e^{-i\pi(\alpha+\pi/2)} e^{-i\zeta} e^{-i\zeta}, & \text{Im } \zeta > 0, \\
P^{(\infty)}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (tf(\zeta))^{\beta_\sigma} e^{2i\pi/3} e^{-i\pi(\alpha+\pi/2)} e^{-i\zeta}, & \text{Im } \zeta < 0,
\end{cases}
\]
with $0 < \arg f(\zeta) < 2\pi$, and, in particular,
\[
E_{\alpha,\beta}(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 2\beta - 1 & 2\beta + 1 \end{pmatrix} (4t)^{\beta_\sigma} e^{\frac{4\pi}{3} - i\pi(\frac{2\beta}{3} - \frac{\pi}{4})}.
\]

Having constructed the local parametrix at $\zeta = 0$, the further arguments are identical to the ones we have used in the case $b = 1$. As a result, we arrive at the following representation for the solution $C_{\alpha}(\zeta)$ to the ‘master’ $C$-RH problem (cf (3.38)).
\[
C_{\alpha}(\zeta) = D_{\alpha,\beta}(\zeta) E_{\alpha,\beta}(\zeta) \Phi_{\alpha,\beta}^{(\text{CHF})}(tf(\zeta))e^{\Phi(\zeta)},
\]
where $D_{\alpha,\beta}(\zeta) = I + O((-s)^{3/2})$ as $s \to -\infty$. Hence, similarly to the case $b = 1$, we have from (3.18) and (A.34) that
\[
u(s) = \frac{i}{\sqrt{-s}} \left[ E_{\alpha,\beta}(0) \lim_{\zeta \to 0} \left( \begin{pmatrix} \frac{d}{d\zeta} \Phi_{\alpha,\beta}^{(\text{CHF})}(\zeta) \Phi_{\alpha,\beta}^{(\text{CHF})}(t_\sigma f(\zeta)) \end{pmatrix} \right) \right]_{12}^{-1} + O(s^{-2})
= \frac{i}{\sqrt{-s}} \left[ E_{\alpha,\beta}(0) \lim_{\zeta \to 0} \left( \begin{pmatrix} \frac{d}{d\zeta} \Phi_{\alpha,\beta}^{(\text{CHF})}(\zeta) \Phi_{\alpha,\beta}^{(\text{CHF})}(t_\sigma f(\xi)) \end{pmatrix} \right) \right]_{12}^{-1} + O(s^{-2})
= \frac{i}{\sqrt{-s}} \left[ E_{\alpha,\beta}(0) \lim_{\zeta \to 0} \left( \begin{pmatrix} \frac{d}{d\zeta} \Phi_{\alpha,\beta}^{(\text{CHF})}(\zeta) \Phi_{\alpha,\beta}^{(\text{CHF})}(t_\sigma f(\xi)) \end{pmatrix} \right) \right]_{12}^{-1} + O(s^{-2}),
\]
as \( s \to -\infty \). The last equality follows from the fact that the matrix function \( \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \) differs from the matrix function \( \Phi_{a,\beta}^{\text{CHF}}(\zeta) \) only by a non-singular piecewise constant right matrix multiplier, see (A.25).

The function \( \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \) solves the Riemann–Hilbert problem whose jump conditions are depicted in figure 11, the asymptotic behaviour as \( \zeta \to \infty \) is indicated in (A.26), and the branching singularity at \( \zeta = 0 \) is of the type described in (1.9) and (1.10). Indeed, if \( a - \frac{1}{2} \notin \mathbb{N}_0 \), then

\[
\tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) = B(\zeta) \begin{pmatrix} \zeta^a & 0 \\ 0 & \zeta^{-a} \end{pmatrix} Q, \quad (A.36)
\]

where \( B \) is analytic at \( \zeta = 0 \). If \( a \in \frac{1}{2} + \mathbb{N}_0 \), then there exists a constant \( \kappa \) such that

\[
\tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) = B(\zeta) \begin{pmatrix} \zeta^a & \kappa \zeta^a \log \zeta \\ 0 & \zeta^{-a} \end{pmatrix} Q, \quad (A.37)
\]

where \( B \) is again analytic at \( \zeta = 0 \). The right matrix multipliers \( Q \) in (A.36) and (A.37) are piecewise constant matrix functions; in fact, they are constant in the eight sectors I–VIII. The matrices \( Q \) can be written down explicitly using either the general algebraic properties of the Riemann–Hilbert problem or the explicit formulae (A.27) and (A.28) for the function \( \Phi_{a,\beta}^{\text{CHF}}(\zeta) \). However, we will not do this. The important feature of the matrices \( Q \) as well as of all the jump matrices of the \( \tilde{\Phi}_{a,\beta}^{\text{CHF}} \)-RH problem is that they are constant with respect to \( \zeta \). Therefore, we can exploit the standard arguments of the theory of integrable systems (see, e.g. [8, 16]; see also [10], chapters 2 and 3) and conclude that

\[
\left( \frac{d}{d\zeta} \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \right) \left( \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \right)^{-1} = -i\zeta \sigma_3 + \frac{A_{-1}}{\zeta}, \quad (A.38)
\]

where

\[
A_{-1} = -\beta \sigma_3 + i[\sigma_1, m_\beta] = \left[ \begin{array}{cc} -\beta & \frac{\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+\beta)} e^{4\pi i \sigma_3 + i\pi \beta} \\ \frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha-\beta)} e^{-4\pi i \sigma_3 - i\pi \beta} & \beta \end{array} \right]. \quad (A.39)
\]

Indeed, the \( \zeta \)-independence of all the jump matrices implies that the function\(^8\),

\[
A(\zeta) := \left( \frac{d}{d\zeta} \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \right) \left( \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \right)^{-1},
\]

is analytic on \( \mathbb{C} \setminus \{0\} \). Moreover, from the behaviour of \( \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \) at \( \zeta = 0 \) (see (A.36) and (A.37)) and at \( \zeta = \infty \) (see (A.29)), we conclude that \( A(\zeta) \) is in fact a rational function which has the only simple pole at \( \zeta = 0 \) and such that \( A(\infty) = i\sigma_3 \). Hence, by Liouville’s theorem,

\[
A(\zeta) = -i\zeta \sigma_3 + \frac{A_{-1}}{\zeta}.
\]

This proves (A.38). Equation (A.39) is obtained by substituting the asymptotic expansion (A.29) into the left-hand side of (A.38) and equating the terms of the order\(^9\) \( O(\zeta^{-1}) \).

Relation (A.38) allows us to rewrite the asymptotic formula (A.35) as

\[
u(s) = \frac{i}{\sqrt{-s}} \left[ E_{a,\beta}(0) A_{-1} E_{a,\beta}^{-1}(0) \right] + O(s^{-2}), \quad s \to -\infty. \quad (A.40)
\]

\(^8\) All the jump matrices of the \( \tilde{\Phi}_{a,\beta}^{\text{CHF}} \)-RH problem have the unit determinant. Together with the asymptotic condition (A.26) this yields the identity, \( \det \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) = 1 \) and therefore the holomorphic invertability of the matrix \( \tilde{\Phi}_{a,\beta}^{\text{CHF}}(\zeta) \) for all \( \zeta \).

\(^9\) Of course, equation (A.38) can be deduced via the direct differentiation of (A.27) and use of the classical confluent hypergeometric equation. This derivation, however, is much more involved than the one we just used.
Plugging in here (A.33) and (A.39) yields the asymptotic equation,
\[
u(s) = \frac{i}{\sqrt{-s}} \left[ \beta - \frac{i}{2} \frac{\Gamma(1 + \alpha - \beta)}{\Gamma(\alpha + \beta)} e^{i \theta(s)} - \frac{i}{2} \frac{\Gamma(1 + \alpha + \beta)}{\Gamma(\alpha - \beta)} e^{-i \theta(s)} \right] + O\left(\frac{1}{s^2}\right),
\]
\[s \to -\infty,\tag{A.41}\]
where
\[\theta(s) = \frac{4t}{3} - 2i \beta \log 8t - \alpha \pi, \quad t = (-s)^{3/2}.\]

Taking into account that we have assumed that \(b > 0\) and hence that \(\beta\) is pure imaginary, we transform (A.41) into our final asymptotic representation for the (real-valued) tronquée solutions of the thirty fourth Painlevé equation as \(s \to -\infty\),
\[
u^{\text{tronquée}}(s) = \frac{\beta_0 \sqrt{-s}}{\sqrt{-s}} + \sqrt{\frac{\alpha^2 + \beta_0^2}{2}} \cos \left(\frac{4}{3}(-s)^{3/2} - 3\beta_0 \log(-s) + \chi\right) + O(s^{-2}), \tag{A.42}\]
where
\[\beta_0 \equiv i \beta = -\frac{1}{2\pi} \log b,\]
and the phase \(\chi\) is given explicitly in terms of \(\beta_0\) and \(\alpha\),
\[\chi = -\alpha \pi - 6\beta_0 \log 2 + \arg(\alpha + i\beta_0) + 2 \arg \Gamma(\alpha + i\beta_0). \tag{A.43}\]

**Remark A.1.** The above asymptotic analysis of the Painlevé XXXIV Riemann–Hilbert problem does not need the condition \(b > 0\) and hence the pure imaginary \(\beta\). The necessary restriction is, in fact,
\[|\text{Re } \beta| < \frac{1}{2}.\]

With this restriction, we again arrive at the more general formula (A.41), but with the change in the error term
\[O\left(\frac{1}{s^2}\right)\]
to the term
\[O\left(\frac{1}{s^{2-3|\text{Re } \beta|}}\right).
\]

The asymptotic analysis of the general tronquée solution which we just performed makes a strong case in favour of the fact that the solution \(u_\alpha(s)\) is characterized uniquely by the asymptotic conditions of theorem 1.2. Indeed, using the transformation formula (A.3), we conclude that every tronquée solution of Painlevé XXXIV, i.e. one behaving at \(s \to +\infty\) as (1.7), maps to a tronquée solution of Painlevé II. This means that the Painlevé II Stokes multipliers must be as in (A.15), because otherwise the asymptotics is oscillatory for \(s \to -\infty\) and elliptic in the sectors \(2\pi/3 < \arg s < \pi\) and \(\pi < \arg s < 4\pi/3\) (see [12, 17]). This in turn means that the asymptotic condition (1.7) selects the tronquée family (A.18) of solutions. As we have already shown, for the positive \(b\), in fact for all \(b\) such that \(|\arg b| < \pi\) it must be exactly \(b = 1\) in order for the solution to behave at \(-\infty\) as it is indicated in (1.8). It is natural to expect that in the case \(b < 0\) the ‘minus infinity’ asymptotics of the solution is different from (1.8).
Asymptotics for a special solution of the thirty fourth Painlevé equation

As mentioned in the introduction, we actually conjecture that the asymptotics (1.8) alone fixes the solution uniquely. The supporting arguments are based on the fact that the substitution of the asymptotic ansatz (A.42) into the formula (A.3) yields the asymptotic representation for the Painlevé II function,

\[ q(s) \sim \sqrt{\frac{s}{2}} \cot \left( \frac{\sqrt{2}}{3} s^{3/2} - \frac{3}{2} \beta_0 \log s + \hat{\chi} \right) \quad \text{as} \ s \to \infty. \]  

(A.44)

These asymptotics are consistent with the general results of [17] for real-valued solutions of Painlevé II which provide the explicit formulae relating the Painlevé II monodromy data \( a_1, a_2 \) and \( a_3 \) and the asymptotic parameters \( \beta_0 \) and \( \hat{\chi} \). This, in turn, allows us to establish a one-to-one correspondence between the Painlevé XXXIV monodromy data \( b_1, b_2 \) and \( b_4 \) and the asymptotic parameters \( \beta_0 \) and \( \chi \). Hence the uniqueness of the real-valued solution of the Painlevé XXXIV equation (1.3) with the asymptotic condition (1.8).

References

[1] Abramowitz M and Stegun I 1992 Handbook of Mathematical Functions (New York: Dover) (Reprint of the 1972 edition)
[2] Bleher P, Bolibruch A, Its A and Kapaev A Linearization of the P34 equation of Painlevé–Gambier unpublished manuscript
[3] Claey s T, Kuijlaars A B J and Vanlessen M 2008 Multi-critical unitary random matrix ensembles and the general Painlevé II equation Ann. Math. 167 601–42
[4] Deift P 1999 Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach (Courant Lecture Notes 3) (NY: New York University)
[5] Deift P, Its A and Krasovsky I 2009 Asymptotics of Toeplitz, Hankel, and Toeplitz–Hankel determinants with Fisher–Hartwig singularities arXiv:0905.0443
[6] Deift P, Kriecherbauer T, McLaughlin KT-R, Venakides S and Zhou X 1999 Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory Commun. Pure Appl. Math. 52 1335–425
[7] Deift P and Zhou X 1993 A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation Ann. Math. 137 295–368
[8] Faddeev L D and Takhtajan L A 1987 Hamiltonian Methods in the Theory of Solitons (Berlin: Springer)
[9] Flaschka H and Newell A C 1980 Monodromy and spectrum-preserving deformations I Commun. Math. Phys. 76 65–116
[10] Fokas A S, Its A R, Kapaev A A and Novokshenov V Yu 2006 Painlevé Transcendents: The Riemann–Hilbert Approach (AMS Mathematical Surveys and Monographs vol 128) (Providence, RI: American Mathematical Society)
[11] Ince E L 1944 Ordinary Differential Equations (New York: Dover)
[12] Its A R and Kapaev A A 2000 The irreducibility of the second Painlevé equation and the isomonodromy method Toward the Exact WKB Analysis of Differential Equations, Linear or Non-linear ed C J Howls et al (Kyoto: Kyoto University Press) pp 209–22
[13] Its A R and Kapaev A A 2003 Quasi-linear Stokes phenomenon for the second Painlevé transcendent Nonlinearity 16 363–86
[14] Its A and Krasovsky I 2008 Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump Integrable Systems and Random Matrices (Contemporary Mathematics vol 458) ed J Baik et al (Providence RI: American Mathematical Society) pp 215–48
[15] Its A R, Kuijlaars A B J and Ostensson J 2008 Critical Edge Behavior in Unitary Random Matrix Ensembles and the Thirty Fourth Painlevé Transcendent Int. Math. Res. Notices 2008 article ID rnm017, 67pp
[16] Jimbo M, Miwa T and Ueno K 1980 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients Physica D 2 306–52
[17] Kapaev A A 1992 Global asymptotics of the second Painlevé transcendent Phys. Lett. A 167 356–62
[18] Kapaev A A 2004 Quasi-linear Stokes phenomenon for the Hastings–McLeod solution of the second Painlevé equation arXiv:nlin.SI/0410009
[19] Kapaev A A and Hubert E 1999 A note on the Lax pairs for Painlevé equations J. Phys. A: Math. Gen. 32 8145–56
[20] Kuijlaars A B J and Vanlessen M 2003 Universality for eigenvalue correlations at the origin of the spectrum Commun. Math. Phys. 243 163–91
[21] Seiberg N and Shih D 2005 Flux vacua and branes of the minimal superstring J. High Energy Phys. JHEP 1(2005)38
[22] Tracy C A and Widom H 1994 Level-spacing distributions and the Airy kernel Commun. Math. Phys. 159 151–74
[23] Tracy C A and Widom H 2002 Airy kernel and Painlevé II Isomonodromic Deformations and Applications in Physics ed J Harnad and A Its (CRM Proceedings and Lecture Notes vol 31) (Providence, RI: American Mathematical Society) pp 85–96
[24] Vanlessen M 2003 Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight J. Approx. Theory 125 196–237