HARMONIC EXTENSIONS OF QUASISYMMETRIC MAPS

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Abstract. We study the Dirichlet problem for harmonic maps between hyperbolic disks, under the assumption that the Euclidean harmonic extension of the boundary map is $K$-quasiconformal, with $K < \sqrt{2}$.

1. Statement of the results

Let us denote by $\mathbb{H}^2$ and $\mathbb{D}^2$ the hyperbolic disk and the Euclidean disk respectively and let $\mathbb{S}^1$ be the unit circle. Let $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ be a $C^1$ diffeomorphism. Assume, without loss of generality, that $\Phi$ is sense preserving. The complex distortion of $\Phi$ at $z_0 \in \mathbb{D}^2$ is

$$D_\Phi(z_0) = \frac{|\partial_z \Phi(z_0)| + |\partial_{\bar{z}} \Phi(z_0)|}{|\partial_z \Phi(z_0)| - |\partial_{\bar{z}} \Phi(z_0)|} \geq 1.$$ 

If $K \geq 1$, we say that $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ is $K$-quasiconformal when $D_\Phi(z) \leq K$ holds for every $z \in \mathbb{D}^2$. We say that $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ is quasiconformal if it is $K$-quasiconformal for some $K \geq 1$. A homeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric if for there is a quasiconformal map $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$, such that $\Phi|_{\mathbb{S}^1} = \phi$.

It was conjectured by Schoen in [14] that every quasisymmetric homeomorphism of the circle can be extended to a quasiconformal harmonic map diffeomorphism of the hyperbolic disc onto itself, and that such an extension is unique. This conjecture was generalized to all hyperbolic spaces by Li and Wang in [10]. The uniqueness part of the conjecture has been proved by Li and Tam in [9] for dimension 2 and by Li and Wang [10] for all dimensions. The existence part of the conjecture is still an open problem, and there are only partial results (e.g. see the seminal works [7, 8, 9] that opened a new era for the study of harmonic maps between hyperbolic spaces). Note that in [11], Markovic has provided an interesting partial answer to the conjecture in dimension 2. Furthermore, one of the most important results that far, is contained in a recent article by Markovic [12], where he proves the conjecture in dimension 3.

In the present note we prove the next result, by following the same strategy as in [5] and [9].

Theorem 1. If $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetric homeomorphism, then it has a quasiconformal harmonic extension $u : \mathbb{H}^2 \to \mathbb{H}^2$, provided that the Euclidean harmonic extension $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ of $\phi$ is $K$-quasiconformal with $K < \sqrt{2}$.

Let us add the following two remarks. Firstly, we would like to emphasize that we do not assume any smoothness of the boundary map. Note that we require a uniform bound on the quasiconformal constant, thus in general the extension $\Phi$ is not asymptotically hyperbolic and so the known results (e.g. see the interesting

1991 Mathematics Subject Classification. 58E20.

Key words and phrases. Harmonic maps, hyperbolic spaces, Schoen conjecture, quasiconformal, quasisymmetric.
results in \([3, 5, 7, 8, 9, 10, 16, 17, 18, 19]\) and the references therein) cannot be applied. Secondly, let us point out that there is a necessary and sufficient condition on the boundary map \(\phi\) in order for the Euclidean harmonic extension \(\Phi\) to be quasiconformal. More precisely, according to the result of Pavlović [13, Theorem 1.1], a \(2\pi\)-periodic function \(\psi\) is bi-Lipschitz and the Hilbert transformation of \(\psi'\) is essentially bounded on \(\mathbb{R}\), if and only if the Euclidean harmonic extension of \(\phi = e^{i\psi} : S^1 \to S^1\) is quasiconformal. The condition that \(\psi\) is bi-Lipschitz imply that \(\psi\) (and \(\psi^{-1}\)) is differentiable almost everywhere, thus it may not be smooth.

We use the compact exhaustion method, we construct a sequence of harmonic maps and we prove that there exists a subsequence that converges to the required harmonic extension.

The organization of the paper is as follows. In Section 2 we recall some preliminaries and in Section 3 we give the proof of Theorem 1.

### 2. Preliminaries

The hyperbolic plane \(\mathbb{H}^2\) can be described as the unit disk \(D^2 = \{z \in \mathbb{C} : |z| < 1\}\) equipped with the Poincaré metric

\[
\gamma = 4(1 - |z|^2)^{-2}|dz|^2,
\]

where \(|dz|^2\) is the Euclidean metric on \(\mathbb{C}\). The ideal boundary of the hyperbolic plane can be identified with \(S^1 = \{z \in \mathbb{C} : |z| = 1\}\).

Let \(\nabla_0\) and \(\Delta_0\) denote the Euclidean gradient and Laplacian respectively. The energy density of a map \(u = (f, g) : \mathbb{H}^2 \to \mathbb{H}^2\) is given by

\[
e(u)(z) = \frac{(1 - |z|^2)^2}{(1 - |u|^2(z))^2} \left( |\nabla_0 f|^2(z) + |\nabla_0 g|^2 \right),
\]

and the Jacobian is given by

\[
J(u)(z) = \frac{(1 - |z|^2)^2}{(1 - |u|^2(z))^2} (\partial_x f \partial_y g - \partial_y f \partial_x g).
\]

The energy of \(u\) is given by

\[
E(u) = \int_{\mathbb{H}^2} e(u)(z) \frac{dz}{(1 - |z|^2)^2}.
\]

The tension field of \(u = (f, g)\) is the section of the bundle \(u^{-1}(T\mathbb{H}^2)\) given by

\[
\tau(u) = Tr_\gamma \nabla du,
\]

where \(\gamma\) is the hyperbolic metric.

The equations \(\tau(u) = 0\) are precisely the Euler-Lagrange equations of the energy functional. A map \(u\) that is a solution of these equations is called a harmonic map.

The components of the tension field are given by [5, p.171]

\[
\tau^1(u) = \frac{(1 - |z|^2)^2}{4} \left( \Delta_0 f - \frac{2}{1 - |u|^2}(f(|\nabla_0 f|^2 - |\nabla_0 g|^2) + 2g < \nabla_0 f, \nabla_0 g >) \right),
\]

\[
\tau^2(u) = \frac{(1 - |z|^2)^2}{4} \left( \Delta_0 g - \frac{2}{1 - |u|^2}(g(|\nabla_0 g|^2 - |\nabla_0 f|^2) + 2f < \nabla_0 f, \nabla_0 g >) \right).
\]

The norm of the tension field \(u = (f, g)\) is given by

\[
\| \tau(u) \| = 2 \sqrt{\frac{(\tau^1(u))^2 + (\tau^2(u))^2}{1 - |u|^2}}.
\]
Let $\Phi$ be a $K$–quasiconformal map. Then, for the energy density and the Jacobian of $\Phi$ in complex notation we have that
\[
e(\Phi)(z) = \frac{2(1 - |z|^2)^2}{(1 - |\Phi(z)|^2)^2} \left( |\partial_z \Phi|^2(z) + |\partial_{\overline{z}} \Phi|^2(z) \right),
\]
and
\[
J(\Phi)(z) = \frac{(1 - |z|^2)^2}{(1 - |\Phi(z)|^2)^2} \left( |\partial_z \Phi|^2(z) - |\partial_{\overline{z}} \Phi|^2(z) \right).
\]
If $\Phi$ is $K$–quasiconformal then we find that
\[
(2.1) \quad \frac{2J(\Phi)}{e(\Phi)} \geq \frac{2K}{K^2 + 1} > 0.
\]

If $z = \rho e^{i\theta}$ then the Euclidean harmonic extension $\Phi$ of $\phi$ is given by
\[
(2.2) \quad \Phi(z) = \frac{1}{2\pi} \int_0^{2\pi} P_\rho(\theta - t)\phi(t)dt,
\]
where
\[
P_\rho(\theta) = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos \theta}
\]
is the Poisson kernel.

Then, $\Phi$ is a homeomorphism of $\overline{D}^2$ onto $\overline{D}^2$ and $\Delta_0 \Phi^\alpha = 0$, $\alpha = 1, 2$ on $D^2$. Conversely, every orientation preserving homeomorphism $\Phi : \overline{D}^2 \to \overline{D}^2$, harmonic in $D^2$, can be represented in this form, [1]. From now on, consider $\Phi : \overline{D}^2 \to \overline{D}^2$ to be the Euclidean harmonic extension of $\phi : S^1 \to S^1$, given by the Poisson representation.

3. Proof of the results

We shall employ the compact exhaustion method. More precisely, let $B_R = B_R(o) \subset \mathbb{H}^2$ be the ball of radius $R > 0$ centered at $o = (0, 0) \in \mathbb{H}^2$. By [4], there exists a harmonic map $u_R : B_R \to \mathbb{H}^2$ such that $u_R = \Phi$ on $\partial B_R$, where $\Phi$ is given by (2.2). Let
\[
d_R(z) = r(u_R(z), \Phi(z)),
\]
where $r$ is the distance function of $\mathbb{H}^2$.

Consider $\sigma$ to be the unit speed geodesic, such that $\sigma(0) = u_R(z)$ and $\sigma(d_R) = \Phi(z)$. Next, choose
\[
f_1 = -\frac{d\sigma}{ds}(0) \quad \text{and} \quad \overline{f}_1 = \frac{d\sigma}{ds}(d_R)
\]
and complete these vectors to obtain positively oriented frames $f_1, f_2$ and $\overline{f}_1, \overline{f}_2$ at $u_R(z)$ and $\Phi(z)$ respectively. Consider $e_1, e_2$ to be an orthonormal frame at $z$ in the domain. Let
\[
d\Phi(e_j) = \Phi_j \overline{f}_1 + \Phi_j \overline{f}_2,
\]
and
\[
\hat{e}(\Phi) = (\Phi_1^2)^2 + (\Phi_2^2)^2.
\]
For the energy density we have that
\[
e(\Phi) = (\Phi_1^2)^2 + (\Phi_2^2)^2 + (\Phi_1^2)^2 + (\Phi_2^2)^2.
\]
Note that $\hat{e}(\Phi)$ depends on the local frame while $e(\Phi)$ is independent of the local frame.
Lemma 1. If
\[ \sup_{z \in B^2} \frac{\| \tau(\Phi) \|}{\hat{c}(\Phi)} (z) \leq c_0 < 1, \]
then
\[ d_R \leq 2 \tanh^{-1} c_0. \]

Proof. Set
\[ X_j = du_R(e_j) + d\Phi(e_j) \]
\[ = (u_R)_j^1 f_1 + (u_R)_j^2 f_2 + \Phi_j^1 J_1 + \Phi_j^2 J_2 \in T_{u_R(z)} \mathbb{H}^2 \times T_{\Phi(z)} \mathbb{H}^2. \]

Let \( r_{X_j X_j} \) denote the Hessian of the distance function \( r \). We shall use now an estimate of the Laplacian \( \Delta d_R \). More precisely, according to [3, p.621] (see also [15, p.368]), we have that
\[ \Delta d_R \geq \sum_{j=1}^2 r_{X_j X_j} - \| \tau(\Phi) \|. \]

The Hessian of the distance function can be expressed by Jacobi fields as follows. Let us denote by \( Y_j : [0, d_R] \to T_{u_R(z)} \mathbb{H}^2 \times T_{\Phi(z)} \mathbb{H}^2 \) the Jacobi field along \( \sigma \) with \( Y_j(0) = (u_R)_j^2 f_2 \) and \( Y_j(d_R) = \Phi_j^2 J_2 \), i.e. \( Y_j(0) \) and \( Y_j(d_R) \) are the normal components of \( du_R(e_j) \) and \( d\Phi(e_j) \) respectively.

Let \( \langle \cdot, \cdot \rangle \) denote the hyperbolic inner product. Then, by [6, p.240], we have that
\[ r_{X_j X_j} = \langle Y_j, Y_j \rangle \big|_{d_R} - \langle Y_j(0), Y_j'(0) \rangle. \]

Moreover, following [6] p.241, we obtain the estimate
\[ \langle Y_j, Y_j \rangle \big|_{d_R} \geq \frac{\cosh d_R (|Y_j(0)|^2 + |Y_j(d_R)|^2) - 2|Y_j(0)||Y_j(d_R)|}{\sinh d_R} \]
\[ \geq \frac{(\cosh d_R - 1)}{\sinh d_R} (|Y_j(0)|^2 + |Y_j(d_R)|^2) \]
\[ = \tanh \frac{d_R}{2} (|Y_j(0)|^2 + |Y_j(d_R)|^2) \]
\[ \geq \tanh \frac{d_R}{2} |Y_j(d_R)|^2 \]
\[ = \tanh \frac{d_R}{2} (\Phi_j^2)^2. \]

Thus, as in [9] p.597, we find that the following estimate holds true,
\[ \Delta d_R \geq -\| \tau(\Phi) \| + \hat{c}(\Phi) \tanh \frac{d_R}{2}. \]

Let \( z_R \in B_R \) be the point where the maximum of \( d_R(z) \) is attained. Note that \( z_R \) is in the interior of \( B_R \) because \( d_R(z_0) = 0 \) for every \( z_0 \in \partial B_R \).

By the maximum principle, we find that
\[ \tanh \frac{d_R}{2} \leq \tanh \frac{d_R(z_R)}{2} \leq \frac{\| \tau(\Phi) \|}{\hat{c}(\Phi)} (z_R) \leq \sup_{z \in B^2} \frac{\| \tau(\Phi) \|}{\hat{c}(\Phi)} (z) \leq c_0 < 1, \]
thus
\[ d_R \leq 2 \tanh^{-1} c_0, \]
and the proof of Lemma 1 is complete.

**Lemma 2.** If \( \Phi : \mathbb{D}^2 \to \mathbb{D}^2 \) is a Euclidean harmonic map then
\begin{equation}
\| \tau(\Phi) \| \leq \sqrt{e(\Phi)^2 - 4J^2(\Phi)}.
\end{equation}

**Proof.** Note first that after careful computations we find that
\[
\frac{(1-|z|^2)^4}{(1-|\Phi(z)|^2)^4} \left( |\nabla_0 f, \nabla_0 g|^2 - |\nabla_0 f|^2 |\nabla_0 g|^2 \right) = -J^2(\Phi)
\]
holds true.

Now, taking into account that \( \Phi = (f,g) \) is a Euclidean harmonic map, one can find that
\[
\| \tau(\Phi) \|^2 = \frac{(1-|z|^2)^4}{(1-|\Phi(z)|^2)^4} \left\{ (f(|\nabla_0 f|^2 - |\nabla_0 g|^2) + 2g < \nabla_0 f, \nabla_0 g >)^2 \\
+ (g(|\nabla_0 g|^2 - |\nabla_0 f|^2) + 2f < \nabla_0 f, \nabla_0 g >)^2 \right\}
\]
\[
= \frac{(1-|z|^2)^4}{(1-|\Phi(z)|^2)^4} \left( (|\nabla_0 f|^2 - |\nabla_0 g|^2)^2 + 4|\nabla_0 f, \nabla_0 g >|^2 \right) |\Phi|^2
\]
\[
= \frac{(1-|z|^2)^4}{(1-|\Phi(z)|^2)^4} \left( (|\nabla_0 f|^2 + |\nabla_0 g|^2)^2 + 4 (|\nabla_0 f, \nabla_0 g >|^2 - |\nabla_0 f|^2 |\nabla_0 g|^2) \right) |\Phi|^2
\]
\[
= (e(\Phi)^2 - 4J^2(\Phi)) |\Phi|^2.
\]
Since \( |\Phi| \leq 1 \) we conclude that
\[
\| \tau(\Phi) \| \leq \sqrt{e(\Phi)^2 - 4J^2(\Phi)}.
\]

**Lemma 3.** If \( \Phi : \mathbb{D}^2 \to \mathbb{D}^2 \) then there exists \( \theta \in [0,2\pi) \) such that
\begin{equation}
\hat{e}(\Phi)(z) = \left( \frac{1-|z|^2}{1-|\Phi(z)|^2} \right)^2 \left( \sin^2|\nabla_0 f|^2 + 2 \cos \theta \sin \theta < \nabla_0 f, \nabla_0 g > + \cos^2|\nabla_0 g|^2 \right).
\end{equation}

**Proof.** Consider at \( z_R \) the orthonormal frame
\[
e_1 = \frac{1-|z|^2}{2} \partial_x \quad \text{and} \quad e_2 = \frac{1-|z|^2}{2} \partial_y.
\]
Consider the positively oriented frames \( f_1, f_2 \) and \( \bar{f}_1, \bar{f}_2 \) at \( u_R(z_R) \) and \( \Phi(z_R) \) respectively as in the proof of Lemma 1.

Let \( \Phi = (f,g) \). There exists \( \theta \in [0,2\pi) \) such that
\[
\frac{1-|\Phi(z)|^2}{2} \partial_f = (\cos \theta \bar{f}_1 + \sin \theta \bar{f}_2) \quad \text{and} \quad \frac{1-|\Phi(z)|^2}{2} \partial_g = (-\sin \theta \bar{f}_1 + \cos \theta \bar{f}_2).
\]
Then we observe that
\[
\frac{d\Phi(e_1)}{2} = \frac{1 - |z|^2}{2} (\partial_x f \partial_x + \partial_x g \partial_y)
\]
\[
= \frac{1 - |z|^2}{2} (\partial_x f \partial_x + \partial_x g \partial_y)
\]
\[
= \frac{1 - |z|^2}{1 - |\Phi(z)|^2} (\cos \theta \partial_x f - \sin \theta \partial_x g) f_1
\]
\[
+ \frac{1 - |z|^2}{1 - |\Phi(z)|^2} (\sin \theta \partial_x f + \cos \theta \partial_x g) f_2.
\]

Thus,
\[
\Phi_1 = \frac{1 - |z|^2}{1 - |\Phi(z)|^2} (\cos \theta \partial_x f - \sin \theta \partial_x g)
\]
and
\[
\Phi_2 = \frac{1 - |z|^2}{1 - |\Phi(z)|^2} (\sin \theta \partial_x f + \cos \theta \partial_x g).
\]

Similarly, one can find that
\[
\Phi_1 = \frac{1 - |z|^2}{1 - |\Phi(z)|^2} (\cos \theta \partial_y f - \sin \theta \partial_y g)
\]
and
\[
\Phi_2 = \frac{1 - |z|^2}{1 - |\Phi(z)|^2} (\sin \theta \partial_y f + \cos \theta \partial_y g).
\]

Thus,
\[
\hat{\varepsilon}(\Phi)(z) = \left( \frac{1 - |z|^2}{1 - |\Phi|^2(z)} \right)^2 \left( \sin \theta |\nabla_0 f|^2 + 2 \cos \theta \sin \theta \nabla_0 f, \nabla_0 g > + \cos \theta |\nabla_0 g|^2 \right).
\]

It becomes clear from the above lemma that \(\hat{\varepsilon}(\Phi)(z)\) depends on the local frame. Note that
\[
e(\Phi) = \left( \frac{1 - |z|^2}{1 - |\Phi|^2(z)} \right)^2 (|\nabla_0 f|^2 + |\nabla_0 g|^2),
\]
thus \(e(\Phi)\) is independent of the local frame.

**Corollary 1.** If \(\Phi : \mathbb{D}^2 \to \mathbb{D}^2\), then
\[
(3.5) \quad \hat{\varepsilon}(\Phi) \geq \frac{e(\Phi) - \sqrt{e(\Phi)^2 - 4J^2(\Phi)}}{2}.
\]

**Proof.** We observe from (3.4) that \(\hat{\varepsilon}(\Phi)\) is a quadratic form, restricted on the circle. The maximum and minimum value of the function
\[
F(X, Y) = \left( \frac{1 - |z|^2}{1 - |\Phi|^2(z)} \right)^2 \left( X^2 |\nabla_0 f|^2 + 2XY \nabla_0 f, \nabla_0 g > + Y^2 |\nabla_0 g|^2 \right),
\]
on the circle \(X^2 + Y^2 = 1\), are the eigenvalues of the following matrix
\[
A = \left( \frac{1 - |z|^2}{1 - |\Phi|^2(z)} \right)^2 \begin{bmatrix}
|\nabla_0 f|^2 & \nabla_0 f, \nabla_0 g > \\
< \nabla_0 f, \nabla_0 g > & |\nabla_0 g|^2
\end{bmatrix}.
\]

More precisely, we find that
\[
e(\Phi) - \sqrt{e(\Phi)^2 - 4J^2(\Phi)} \leq \hat{\varepsilon}(\Phi)(z) \leq \frac{e(\Phi) + \sqrt{e(\Phi)^2 - 4J^2(\Phi)}}{2},
\]
and the proof of the corollary is complete. \(\square\)
3.1. End of the proof of Theorem 1. From (3.5) and (3.3) we find that
\[
\|\tau(\Phi)\| \leq \frac{2\sqrt{e(\Phi)^2 - 4J^2(\Phi)}}{e(\Phi) - \sqrt{e(\Phi)^2 - 4J^2(\Phi)}}
\]

Note that since \(\Phi\) is \(K\)-quasiconformal, we take into account (2.1) and we find that
\[
(3.6) \quad \|\tau(\Phi)\| \leq K^2 - 1 < 1,
\]
since we have assumed that \(K < \sqrt{2}\) holds true.

From Lemma 1 and (3.6) we find that
\[
d_R \leq \tanh^{-1}(K^2 - 1) < \infty.
\]
Thus we conclude that a uniform bound of \(d_R\) independent of \(R\) exists.

According to [2, Theorem 5.1], if \(\Phi\) is \(K\)-quasiconformal then there exists a constant \(a(K) > 0\) such that
\[
d(z,w)K - a(K) \leq d(\Phi(z),\Phi(w)) \leq Kd(z,w) + a(K).
\]

Thus, by the triangular inequality, it follows that
\[
(3.7) \quad d_R(u_R(x),u_R(y)) \leq d_R(\Phi(x),u_R(x)) + d_R(\Phi(x),\Phi(y)) + d_R(\Phi(y),u_R(y))
\]
\[\leq c + Kd(x,y) .
\]

We shall now recall the following result [5, Lemma 2.1].

**Lemma 4.** If \(z \in B_R\) is at a distance at least 1 from \(\partial B_R\), then \(e(u_R) \leq C(k)\), where \(k > 0\) is such that \(u_R(B_1) \subset B_k(u_R(z))\).

By (3.7), we have that \(d(z,w) < 1\) implies that \(d(u_R(z),u_R(w)) < c(K)\). So, by Lemma 5 follows that
\[
e(u_R(z)) < C(K),
\]
i.e. the energy density is uniformly bounded for all \(z\) such that \(B_1(z) \subset B_R\).

The uniform bounds on \(d_R(u_R,\Phi)\) and \(e(u_R)\) allow us, as in [5, Sections 3.3-3.4], to apply the Arzela-Ascoli theorem. Thus, we find a subsequence \(R_k\) such that \(u_{R_k}\) converges uniformly on compact sets to a harmonic map \(u\), that is at a bounded distance from \(\Phi\) and has uniformly bounded energy density.

Consequently, we have that
\[
d(u, \Phi) \leq \tanh^{-1}(K^2 - 1) < 1.
\]
Thus, it follows that \(u\) and \(\Phi\) have the same asymptotic boundary \(\phi\).

According to [18, Theorem 13], the energy density of an orientation preserving harmonic map of the hyperbolic disk onto itself is uniformly bounded if and only if the harmonic map is quasiconformal. Thus, \(u\) is a quasiconformal harmonic extension of \(\phi\), and the proof is complete.

**Acknowledgements:** The author would like to thank Michel Marias and Andreas Savas-Halilaj for their stimulating discussions.
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