Dynamical invariants for variable quadratic Hamiltonians

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Abstract
We consider linear and quadratic integrals of motion for general variable quadratic Hamiltonians. Fundamental relations between the eigenvalue problem for linear dynamical invariants and solutions of the corresponding Cauchy initial value problem for the time-dependent Schrödinger equation are emphasized. An eigenfunction expansion of the solution of the initial value problem is also found. A nonlinear superposition principle for generalized Ermakov systems is established as a result of decomposition of the general quadratic invariant in terms of the linear ones.

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1. Introduction
Quantum systems with variable quadratic Hamiltonians are called generalized harmonic oscillators (see, for example, [2, 14, 15, 28, 39, 53, 84, 85] and references therein). They have attracted substantial attention over the years in view of their great importance in many advanced quantum problems. Examples include coherent states and uncertainty relations [33], [53–55], [57], Berry’s phase [2, 3, 28, 39, 62], asymptotic and numerical methods [36, 58, 61, 63], quantization of mechanical systems [22–25], [34, 35] and Hamiltonian cosmology [4, 26, 27, 29], charged particle traps [52] and motion in uniform magnetic fields [6, 10, 15, 37, 43, 44, 48, 55], molecular spectroscopy [16, 53] and polyatomic molecules in varying external fields, crystals through which an electron is passing and exciting the oscillator modes, and other interactions of the modes with external fields [25]. Quadratic Hamiltonians have particular applications in quantum electrodynamics because the electromagnetic field can be represented as a set of forced harmonic oscillators [3, 15, 25, 60, 73]. Nonlinear oscillators play a central role in the novel theory of Bose–Einstein condensation [11]. From a general point of view, the dynamics of gases of cooled atoms in a magnetic trap at very low temperatures can be described by an effective equation for the condensate wavefunction known as the Gross–Pitaevskii (or nonlinear Schrödinger) equation [30–32], [68].

In this paper, we consider the one-dimensional time-dependent Schrödinger equation
\[ \frac{i}{\hbar} \frac{\partial \psi}{\partial t} = H \psi, \]  
with general variable quadratic Hamiltonians of the form
\[ H = a(t) p^2 + b(t) x^2 + c(t) px + d(t) xp, \quad p = -i \frac{\partial}{\partial x}, \] 
where \( a(t), b(t), c(t) \) and \( d(t) \) are real-valued functions of time \( t \) only (see, for example, [6–9], [14], [23–25], [38, 50, 59], [76–79], [84, 85] for a general approach and some elementary solutions). The corresponding Green function, or Feynman’s propagator, can be found as follows [6, 78]:
\[ \psi = \frac{1}{2\pi i \mu_0(t)} e^{i(\omega_0(t)x^2 + \beta_0(t)xy + \gamma_0(t)y^2)}, \] 
where
\[ \omega_0(t) = \frac{1}{4a(t)} \mu_0(t) - \frac{c(t)}{2a(t)}; \] 
\[ \beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}; \quad \lambda(t) = \exp \left( \int_0^t (c(s) - d(s)) ds \right); \] 
\[ \gamma_0(t) = \frac{\alpha(t) \lambda^2(t)}{\mu_0(t) \mu_0'(t)} + \frac{c(0)}{2a(0)} + 4 \int_0^t \frac{a(s) \sigma(s) \lambda^2(s)}{\mu_0'(s)^2} ds, \]
and the function $\mu_0(t)$ satisfies the characteristic equation
\[ \mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0, \]  
(1.7)
with
\[ \tau(t) = \frac{a'}{a} + 2c - 2d, \quad \sigma(t) = ab - cd + \frac{c}{2} \left( \frac{a'}{a} - \frac{c'}{c} \right) \]  
(1.8)
subject to the initial data
\[ \mu_0(0) = 0, \quad \mu_0'(0) = 2\alpha(0) \neq 0. \]  
(1.9)
(More details can be found in [6, 78].) Then by the superposition principle the solution of the Cauchy initial value problem can be presented in an integral form
\[ \psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t) \psi(y)dy, \quad \lim_{t \to 0} \psi(x,t) = \psi(x) \]  
(1.10)
for a suitable initial function $\varphi$ on $\mathbb{R}$ (a rigorous proof is given in [78] and uniqueness is analyzed in [8]; other forms of solution are provided by (4.20) and (7.11)).

A detailed review on dynamical symmetries and quantum integrals of motion for the time-dependent Schrödinger equation can be found in [15, 53] (see also an extensive list of references therein). In this paper, which is a continuation of the recent paper [8], a natural connection between the linear and quadratic integrals of motion for general variable quadratic Hamiltonians is established. As a result, a nonlinear superposition principle for the corresponding Ermakov systems, known as Pinney’s solution, is obtained. We also pay special attention to fundamental relations between the linear dynamical invariants of Dodonov, Malkin, Man’ko and Trifonov and solutions of the Cauchy initial value problem (see original works [14, 15, 53, 56]). In addition, this initial value problem is explicitly solved in terms of the quadratic invariant eigenfunction expansion, which seems to be missing in the available literature in general.

This paper is organized as follows. We start from the standard definitions of the dynamical symmetry and quantum integrals of motion, introducing also some elementary tools, in the next two sections. Then, in section 4, we describe all linear dynamical invariants for variable quadratic Hamiltonians and determine their actions on the solutions of the corresponding time-dependent Schrödinger equation. In section 5 all quadratic quantum integrals of motion are characterized in terms of solutions of the generalized Ermakov equation and a detailed proof is given. Their decomposition into products of the linear invariants is derived in section 6, and explicit actions on the solutions are established in section 7. The last section deals with a related nonlinear superposition principle for Ermakov’s equations and some computational details are provided in appendices A and B.

2. Dynamical symmetry

In this paper we elaborate on the following property.

Lemma 1. If
\[ i\frac{\partial \psi}{\partial t} = H \psi, \quad i\frac{\partial O}{\partial t} + O H - H^\dagger O = 0, \]  
(2.1)
then the function $\psi_1 = O\psi$ satisfies the time-dependent Schrödinger equation
\[ i\frac{\partial \psi_1}{\partial t} = H^\dagger \psi_1, \]  
(2.2)
where $H^\dagger$ is the adjoint of the Hamiltonian $H$.

When $H = H^\dagger$, this property is taken as a definition of the dynamical symmetry of the time-dependent Schrödinger equation (1.1) (see, for example, [15, 48, 53] and references therein). At the same time one has to deal with non-self-adjoint Hamiltonians in the theory of dissipative quantum systems (see, for example, [7, 13, 35, 81, 82] and references therein) or when using separation of variables in an accelerating frame of reference for a charged particle moving in a uniform variable magnetic field [6].

Proof. Partial differentiation
\[ i\frac{\partial \psi_1}{\partial t} = i\frac{\partial}{\partial t}(O\psi) = \frac{\partial}{\partial t}O\psi + iO\frac{\partial \psi}{\partial t} = (H^\dagger O - O H)\psi + O H \psi = H^\dagger \psi_1, \]  
(2.3)
provides a direct proof. □

Definition of the dynamical symmetry is usually given in terms of solutions of the same equation. A simple modification helps with the non-self-adjoint quadratic Hamiltonians.

Lemma 2. The wavefunctions $\psi$ and $\chi$, related by
\[ \psi = \left(e^{\int_{-\infty}^{0} (c-d)ds} O \right) \chi, \]  
(2.4)
are solutions of the same Schrödinger equation (1.1)–(1.2), if the operator $O$ satisfies the hypothesis of lemma 1.

Proof. The simplest dynamical invariant, or an operator with the property (2.1), is given by
\[ O_0 = O_0 (c, d) = e^{\int_{-\infty}^{0} (c-d)ds} I, \]  
(2.5)
where $I = \text{id}$ is the identity operator. (More details are provided in section 4.) Apply lemma 1 twice in the following order,
\[ \psi = O_0 (d, c) (O \chi), \]  
(2.6)
and use $(H^\dagger)^\dagger = H$ to complete the proof. □

Examples occur throughout the paper.

3. Differentiation of operators and dynamical invariants

Following lemma 1, we define the time derivative of an operator $O$ as follows:
\[ \frac{dO}{dt} = \frac{\partial O}{\partial t} + \frac{1}{i} \left( O H - H^\dagger O \right), \]  
(3.1)
where $H^\dagger$ is the adjoint of the Hamiltonian operator $H$. (This formula is a simple extension of the well-known
expression [12, 37, 60, 73] to the case of a non-self-adjoint Hamiltonian [7]. By definition,

$$\frac{dO}{dt} = \frac{\partial O}{\partial t} + \frac{1}{i} (OH - H^\dagger O) = 0 \quad (3.2)$$

for any dynamical invariant. (In what follows we assume that the invariant \(O\) does not involve time differentiation.)

This derivative is a linear operator

$$\frac{d}{dt} (c_1 O_1 + c_2 O_2) = c_1 \frac{dO_1}{dt} + c_2 \frac{dO_2}{dt} \quad (3.3)$$

and the product rule takes the form

$$\frac{d}{dt} (O_1 O_2) = \frac{\partial (O_1 O_2)}{\partial t} + \frac{1}{i} ((O_1 O_2)H - H^\dagger (O_1 O_2))$$

$$= \frac{dO_1}{dt} O_2 + O_1 \frac{dO_2}{dt} + iO_1 (H - H^\dagger) O_2. \quad (3.4)$$

For the general quadratic Hamiltonian, equation (1.2), one obtains

$$\frac{d}{dt} (O_1 O_2) = \frac{dO_1}{dt} O_2 + O_1 \frac{dO_2}{dt} + (c - d) O_1 O_2 \quad (3.5)$$

and by definition (3.1)

$$\frac{d}{dt} \left( e^{i \int_0^t (c - d) ds} O_1 O_2 \right) = e^{i \int_0^t (c - d) ds} \left( \frac{dO_1}{dt} O_2 + O_1 \frac{dO_2}{dt} \right)$$

$$+ (\alpha + 1)(c - d) e^{i \int_0^t (c - d) ds} O_1 O_2. \quad (3.6)$$

If \(\alpha = -1\), we finally obtain

$$\frac{d}{dt} \left( e^{-i \int_0^t (c - d) ds} O_1 O_2 \right) = e^{-i \int_0^t (c - d) ds} \left( \frac{dO_1}{dt} O_2 + O_1 \frac{dO_2}{dt} \right). \quad (3.7)$$

This implies that if the operators \(O_1\) and \(O_2\) are dynamical invariants, namely

$$\frac{dO_1}{dt} = \frac{\partial O_1}{\partial t} + \frac{1}{i} (O_1 H - H^\dagger O_1) = 0, \quad (3.8)$$

$$\frac{dO_2}{dt} = \frac{\partial O_2}{\partial t} + \frac{1}{i} (O_2 H - H^\dagger O_2) = 0, \quad (3.9)$$

then their modified product,

$$E = e^{-i \int_0^t (c - d) dt} O_1 O_2, \quad (3.10)$$

is also a dynamical invariant:

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + \frac{1}{i} (EH - H^\dagger E) = 0. \quad (3.11)$$

In section 6, this property allows us to describe connections between linear and quadratic dynamical invariants of the time-dependent Hamiltonian (1.2).

4. Linear integrals of motion

All invariants of the form

$$P = A(t) p + B(t) x + C(t), \quad \frac{dP}{dt} = 0, \quad (4.1)$$

(we call them the Dodonov–Malkin–Man’ko–Trifonov invariants; see, for example, [14, 15, 53, 56] and references therein) for the general variable quadratic Hamiltonian (1.2) can be found in the following fashion. Use of the differentiation formula (3.1) results in the system [8]

$$A' = 2c(t) A - 2a(t) B, \quad (4.2)$$

$$B' = 2b(t) A - 2d(t) B, \quad (4.3)$$

$$C' = (c(t) - d(t)) C. \quad (4.4)$$

The last equation is explicitly integrated and elimination of \(B\) and \(A\) from (4.2) and (4.3), respectively, gives the second-order equations

$$A'' = \left( \frac{a'}{a} + 2c - 2d \right) A' + 4 \left( ab - cd + \frac{c}{2} \left( \frac{a'}{a} - \frac{c'}{c} \right) \right) A = 0, \quad (4.5)$$

$$B'' = \left( \frac{b'}{b} + 2c - 2d \right) B' + 4 \left( ab - cd - \frac{d}{2} \left( \frac{b'}{b} - \frac{d'}{d} \right) \right) B = 0. \quad (4.6)$$

The first is simply the characteristic equation (1.7)–(1.8). (It also coincides with the Ehrenfest theorem [18, 17] when \(c \leftrightarrow d\).)

Thus all linear quantum invariants are given by

$$P = A(t) p + \frac{2c(t) A(t) - A'(t)}{2a(t)} x$$

$$+ C_0 \exp \left( \int_0^t (c(s) - d(s)) ds \right), \quad (4.7)$$

where \(A(t)\) is a general solution of equation (4.5) depending upon two parameters and \(C_0\) is the third constant. The study of spectra of the linear dynamical invariants allows one to solve the Cauchy initial value problem [14, 15, 53, 56].

Theorem 1. (Eigenvalue problem for linear invariants). If

$$P(t) = \mu(t) p + \frac{2c(t) \mu(t) - \mu'(t)}{2a(t)} x, \quad (4.8)$$

then for any solution \(A = \mu(t)\) of the characteristic equation (1.7)–(1.8), we have

$$P(t) K(x, y, t) = \beta(0) \mu(0) \lambda(t) y \ K(x, y, t). \quad (4.9)$$

The eigenfunctions are bounded solutions of the time-dependent Schrödinger equation (1.1) given by

$$K(x, y, t) = \frac{1}{\sqrt{2\pi \mu(t)}} e^{i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2)}, \quad (4.10)$$

where \(\lambda(t) = \exp\left( \int_0^t (c(s) - d(s)) ds \right)\) and

$$\mu(t) = 2\mu(0) \mu_0(t)(\alpha(0) + \gamma_0(t)), \quad (4.11)$$
We have constructed in [48, 59] the statement of lemma which takes care of the first case (we have verified once again the Man’ko–Trifonov invariant (1.2) corresponds to a formal inversion operator in general). The reader can easily connect initial conditions of two solutions (4.20) and (4.22) with the help of an analogue of the Fourier transform (see also [77] for a formal inversion operator in general).

5. Quadratic dynamical invariants

All quantum quadratic integrals of motion are given by

\[ E = A(t) p^2 + B(t) x^2 + C(t) (p x + x p), \quad \frac{dE}{dt} = 0, \]

and can be found as follows [8].

Lemma 3. The quadratic dynamical invariants of the Hamiltonian (1.2) have the form

\[
E(t) = \left[ \left( s p + \frac{(c - d) k - k'}{2a} x \right)^2 + \frac{C_0}{k^2} x^2 \right] \times \exp \left( \int_0^t (c - d) ds \right),
\]
where $C_0$ is a constant and function $\kappa(t)$ is a solution of the following nonlinear auxiliary equation:

$$\kappa'' - \frac{a'}{a} \kappa' + \left[4ab + \left(\frac{a'}{a} - c + d \right) (c + d) - c' - d' \right] \kappa = C_0 \frac{(2a)^2}{\kappa^3}. \quad (5.3)$$

The structure of quadratic invariants is once again in perfect agreement with lemma 2.

**Proof.** In a general form this result has been established in [8] (see also [48, 80, 85] for important earlier articles). A somewhat different and more direct proof is presented here. It is sufficient to show that the corresponding linear system

$$A' + 4aC - (3c + d)A = 0, \quad (5.4)$$
$$B' - 4bC + (c + 3d)B = 0, \quad (5.5)$$
$$C' + 2(aB - bA) - (c - d)C = 0 \quad (5.6)$$

has the following general solution:

$$A(t) = e^t \exp \left( \int_0^t (c - d) \, ds \right), \quad (5.7)$$
$$B(t) = \left( \frac{\kappa' - (c + d)\kappa}{2a} + \frac{C_0}{\kappa} \right) \exp \left( \int_0^t (c - d) \, ds \right), \quad (5.8)$$
$$C(t) = \frac{\kappa (c + d)\kappa - \kappa'}{2a} \exp \left( \int_0^t (c - d) \, ds \right), \quad (5.9)$$

where $C_0$ is a constant and the function $\kappa(t)$ satisfies the nonlinear auxiliary equation (5.3).

Indeed the substitution

$$A(t) = \tilde{A}(t) \lambda(t), \quad B(t) = \tilde{B}(t) \lambda(t), \quad C(t) = \tilde{C}(t) \lambda(t), \quad (5.10)$$

where $\lambda(t) = \exp \left( \int_0^t (c - d)ds \right)$, transforms the original system into a more convenient form:

$$\tilde{A}' + 4a\tilde{C} - 2(c + d)\tilde{A} = 0, \quad (5.11)$$
$$\tilde{B}' - 4b\tilde{C} + 2(c + d)\tilde{B} = 0, \quad (5.12)$$
$$\tilde{C}' + 2(a\tilde{B} - b\tilde{A}) = 0. \quad (5.13)$$

Letting

$$\tilde{A} = k^2 \quad \text{and} \quad \tilde{\lambda}' = 2\kappa \tilde{\lambda}', \quad (5.14)$$

in the first equation (5.11), we obtain

$$\tilde{C} = \frac{k}{2a} \left( (c + d)\kappa - \kappa' \right)$$
$$= -e^{(c+d)\kappa} \frac{k}{2a} \left( ke^{-f(c+d)dr} \right). \quad (5.15)$$

Then from the third equation (5.13),

$$\tilde{B} = \frac{b}{a} k^2 + \frac{1}{2a} \left[ \frac{\kappa' - (c + d)\kappa}{2a} \right]$$
$$= \frac{b}{a} k^2 + \frac{1}{2a} \left( e^{f(c+d)dr} \frac{k}{2a} \left( ke^{-f(c+d)dr} \right) \right). \quad (5.16)$$

and substitution of (5.15)–(5.16) into (5.12) gives

$$\frac{d}{dt} \left[ 4abk^2 \mu^2 + \frac{d}{dt} \left( \mu \left( \frac{k}{2a} \frac{d\mu}{dt} \right) \right) \right] + 8abk^2 \mu \left( \frac{k}{2a} \frac{d\mu}{dt} \right) = 0 \quad \text{(5.17)}$$

in the following temporary notations:

$$\mu = \kappa e^{-f(c+d)dr}, \quad k = \frac{1}{2a} e^{2f(c+d)dr}. \quad (5.18)$$

Using

$$\mu(t) = y(\tau), \quad \frac{d\mu}{dt} = \frac{dy}{d\tau}, \quad q = 4abk^2 = \frac{b}{a} e^{4f(c+d)dr}, \quad (5.19)$$

one obtains

$$\frac{d}{d\tau} \left[ \frac{d^2y}{dy^2} + \frac{d}{dy} \left( \frac{dy}{d\tau} \right) \right] + 2q \frac{dy}{d\tau} = 0 \quad (5.20)$$

or

$$y \left( y'' + qy \right) + 3y' \left( y'' + qy \right) = 0 \quad (5.21)$$

(see also [48] for an important special case). By an integrating factor,

$$\frac{d}{d\tau} \left[ y^3 \left( y'' + qy \right) \right] = 0, \quad y'' + qy = \frac{C_0}{y^3}, \quad (5.22)$$

and a back substitution with the help of

$$\frac{d^2y}{d\tau^2} = \frac{e^{3f(c+d)dr}}{2a} \left[ \kappa'' - \frac{a'}{a} \kappa' + \left( \frac{a'}{a} - c - d \right) (c + d) - c' - d' \right] \kappa \quad (5.23)$$

results in the required first integral of the system,

$$\frac{d}{dt} \left[ \frac{k^3}{(2a)^2} \left( \frac{\kappa'' - \frac{a'}{a} \kappa'}{2a} + \left( \frac{a'}{a} - c - d \right) (c + d) - c' - d' \right) \kappa \right] = 0, \quad (5.24)$$

which gives our auxiliary equation (5.3).

The last term in (5.16) can be transformed as follows:

$$\frac{1}{2a} \frac{d}{dt} \left[ e^{f(c+d)dr} \frac{k}{2a} \frac{d}{dt} \left( ke^{-f(c+d)dr} \right) \right]$$
$$= e^{-2f(c+d)dr} \frac{d}{dr} \left( \frac{dy}{d\tau} \right) = e^{-2f(c+d)dr} \left( yy'' + (y')^2 \right)$$
$$= e^{-2f(c+d)dr} \left[ \left( \frac{dy}{d\tau} \right)^2 - qy^2 + \frac{C_0}{y^2} \right]$$
$$= \left( \frac{\kappa' - (c + d)\kappa}{2a} \right)^2 - \frac{b}{a} k^2 + \frac{C_0}{\kappa^2}, \quad (5.25)$$

in view of (5.18)–(5.19) and (5.22). We have also utilized a convenient identity

$$\frac{dy}{d\tau} = \frac{1}{2a} \left( \frac{dk}{dt} - (c + d)\kappa \right) e^{f(c+d)dr}. \quad (5.25)$$
Thus
\[
\hat{B} = \left( \frac{\kappa' - (c + d)\kappa}{2\alpha} \right)^2 + \frac{C_0}{\kappa^2}
\] (5.26)
and the proof is complete.

The case \(a = 1/2\), \(b = \omega^2(t)/2\) and \(c = d = 0\) corresponds to the familiar Ermakov–Lewis–Riesenfeld invariant \[21, 43–45, 48\]; see also \[46, 47\]. (The corresponding classical invariant in general is discussed in \[80, 85\]). Many examples of completely integrable generalized harmonic oscillators and their integrals of motion are given in \[8\].

For a positive constant, \(C_0 > 0\), the quantum dynamical invariant (5.2) can be presented in the standard harmonic oscillator form \[7, 8, 70, 71\]:
\[
E = \frac{\omega(t)}{2} \left( \hat{a}^\dagger(t) \hat{a}(t) + \hat{a}^\dagger(t) \hat{a}(t) \right),
\] (5.27)
where
\[
\omega(t) = \omega_0 \exp \left( \int_0^t (c - d) \, ds \right), \quad \omega_0 = 2\sqrt{C_0} > 0,
\] (5.28)
\[
\hat{a}(t) = \left( \frac{\sqrt{\omega_0}}{2\kappa} - i \frac{\kappa' - (c + d)\kappa}{2a\sqrt{\omega_0}} \right) x + \frac{\kappa}{\sqrt{\omega_0}} \frac{\partial}{\partial x} \quad \text{(5.29)}
\]
\[
\hat{a}^\dagger(t) = \left( \frac{\sqrt{\omega_0}}{2\kappa} + i \frac{\kappa' - (c + d)\kappa}{2a\sqrt{\omega_0}} \right) x - \frac{\kappa}{\sqrt{\omega_0}} \frac{\partial}{\partial x} \quad \text{(5.30)}
\]
and \(\kappa\) is a real-valued solution of the nonlinear auxiliary equation (5.3). Here the time-dependent annihilation \(\hat{a}(t)\) and creation \(\hat{a}^\dagger(t)\) operators satisfy the canonical commutation relation:
\[
\hat{a}(t) \hat{a}^\dagger(t) - \hat{a}^\dagger(t) \hat{a}(t) = 1. \quad \text{(5.31)}
\]
The oscillator-type spectrum of the dynamical invariant \(E\) can be obtained now in a standard way by using the Heisenberg–Weyl algebra of the raising and lowering operators (a ‘second quantization’ \[48\], the Fock states):
\[
\hat{a}(t) \Psi_n(x, t) = \sqrt{n} \Psi_{n-1}(x, t), \quad \hat{a}^\dagger(t) \Psi_n(x, t) = \sqrt{n+1} \Psi_{n+1}(x, t),
\] (5.32)
\[
E(t) \Psi_n(x, t) = \omega(t) \left( n + \frac{1}{2} \right) \Psi_n(x, t). \quad \text{(5.33)}
\]
The corresponding orthonormal time-dependent eigenfunctions are given by
\[
\Psi_n(x, t) = \exp \left( i \frac{\kappa' - (c + d)\kappa}{4ak} x^2 \right) v_n(\xi),
\] (5.34)
in terms of Hermite polynomials \[64\]:
\[
v_n = C_n e^{-\varepsilon^2/2} H_n(\xi), \quad \xi = \varepsilon x,
\] (5.35)
where
\[
|C_n|^2 = \frac{\varepsilon}{\sqrt{2\pi n!}}, \quad \varepsilon^2 = \frac{\omega_0}{2\kappa^2} = \frac{\sqrt{C_0}}{\kappa^2}.
\] (5.36)
Their relation with the original Cauchy initial value problem is discussed in section \(7\) (see theorem 2). In addition the \(n\)-dimensional oscillator wavefunctions form a basis of the irreducible unitary representation of the Lie algebra of the non-compact group \(\text{SU}(1, 1)\) corresponding to the discrete positive series \(D^\pm_n\) (see \[59, 64, 75\]). Operators (5.29)–(5.30) allow us to extend these group-theoretical properties to the general quadratic dynamical invariant (5.27).

When \(C_0 = 0\), the dynamical invariant (5.2) is given as the square of a linear invariant up to a simple factor, which resembles the case of a free particle. If \(C_0 < 0\), one deals with the Hamiltonian of a linear repulsive ‘oscillator’.

### 6. Relation between linear and quadratic invariants

By lemma 3, the operators \(p^2, x^2\) and \(px + xp\) form a basis for all quadratic invariants. Here we take two linearly independent solutions, say \(\mu_1 = A_1\) and \(\mu_2 = A_2\), of equations (1.7) and (4.5) and consider the corresponding Dodonov–Malkin–Man’ko–Trifonov invariants (4.7):
\[
P_1 = A_1 p + B_1 x, \quad P_2 = A_2 p + B_2 x.
\] (6.1)
Introducing the following quadratic invariants
\[
E_1 = P_1^2 e^{-\int_0^t (c - d) \, ds}, \quad E_2 = P_2^2 e^{-\int_0^t (c - d) \, ds},
\] (6.2)
with the help of (3.10) as another basis, one gets
\[
E = C_1 E_1 + C_2 E_2 + C_3 E_3
\] (6.3) for some constants \(C_1, C_2\) and \(C_3\). As a result, the following operator identity holds
\[
\left[ \left( \kappa p + \frac{(c + d)\kappa - \kappa'}{2a} x \right)^2 + \frac{C_0}{\kappa^2} x^2 \right] \exp \left( \int_0^t (c - d) \, ds \right)
\] (6.4)
where
\[
A_1 = \mu_1, \quad B_1 = \frac{2c\mu_1 - \mu_1'}{2a}, \quad A_2 = \frac{2c\mu_2 - \mu_2'}{2a}.
\] (6.5)
Thus, we obtain
\[
\kappa^2 = (C_1 \mu_1^2 + C_2 \mu_2^2 + 2C_3 \mu_1 \mu_2) \exp \left( -2 \int_0^t (c - d) \, ds \right)
\] (6.6)
as a relation between solutions of the nonlinear auxiliary equation (5.3) and the linear characteristic equation (4.5). In addition, the substitution
\[
\mu_1 = \kappa_1 \exp \left( \int_0^t (c - d) \, ds \right), \quad \mu_2 = \kappa_2 \exp \left( \int_0^t (c - d) \, ds \right)
\] (6.7)
transforms the characteristic equation (4.5) into our auxiliary equation (5.3) with \(C_0 = 0\). Finally, a general solution of the nonlinear equation is given by the following ‘operator law of cosines’:

\[
\kappa^2(t) = C_1\kappa_1^2(t) + C_2\kappa_2^2(t) + 2C_3\kappa_1(t)\kappa_2(t) \quad (6.8)
\]

in terms of two linearly independent solutions \(\kappa_1\) and \(\kappa_2\) of the homogeneous equation. The constant \(C_0\) is related to the Wronskian of two linearly independent solutions \(\kappa_1\) and \(\kappa_2\):

\[
C_1C_2 - C_3^2 = \frac{(2a)^2}{W^2(\kappa_1, \kappa_2)} \quad , \quad W(\kappa_1, \kappa_2) = \kappa_1\kappa_2^\prime - \kappa_1^\prime\kappa_2 \quad (6.9)
\]

(more details are given in appendix A). This is a well-known nonlinear superposition property of the so-called Ermakov systems (see, for example, [10, 18, 21], [40–42], [45, 51, 65, 67, 69, 74] and references therein). Here we have obtained this ‘nonlinear superposition principle’ (or Pinney’s solution) in an operator form by multiplication and addition of the linear dynamical invariants together with an independent characterization of all quantum quadratic invariants, which seems to be missing, in general, in the available literature (see also [43, 45] for an important classical case). An extension is given in the last section.

It is worth noting, in conclusion, that the linear invariants of Dodonov, Malkin, Man’ko and Trifonov [14, 15, 53, 56] can be presented as follows,

\[
P_1 = \left(\kappa_1 p + \frac{c+d}{2a} x - \kappa_1^\prime\right) \exp\left(\int_0^t (c - d) \, ds\right),
\]

\[
P_2 = \left(\kappa_2 p + \frac{c+d}{2a} x - \kappa_2^\prime\right) \exp\left(\int_0^t (c - d) \, ds\right),
\]

(6.10)

(6.11)

in terms of two linearly independent solutions, \(\kappa_1\) and \(\kappa_2\), of the homogeneous equation (5.3) when \(C_0 = 0\). Comparing these expressions with the form of the quadratic invariant (5.2) at \(C_0 = 0\) (no ‘potential’, a ‘free particle’), when the operator square is complete, one can treat the linear invariants as ‘operator square roots’ [12] of the special quadratic invariants (see also lemma 2 regarding a convenient common factor).

7. Quadratic invariants and Cauchy initial value problem

Our decomposition (6.3) of the quantum quadratic invariant in terms of products of the linear ones not only results in the Pinney solution (6.8)–(6.9) of the corresponding generalized Ermakov system (5.3) in a form of an ‘operator law of cosines’, but also provides a somewhat better understanding, with the help of lemma 1 and properties of the linear invariants discussed in section 4, of how the quadratic invariants act on solutions of the original time-dependent Schrödinger equation. Indeed by (4.20) for two different solutions, say \(A_1 = \mu_1\) and \(A_2 = \mu_2\), of the characteristic equation (4.5), we have, in operator form,

\[
\psi(x, t) = K_1(t) [\chi_1(y)] = K_2(t) [\chi_2(y)] \quad (7.1)
\]

in view of uniqueness of the Cauchy initial value problem (see, for example, [8, 78]). Then by (4.22),

\[
P_1 \psi = e^{\int_0^t (c-d)ds} K_1(\psi_1), \quad P_2 \psi = e^{\int_0^t (c-d)ds} K_2(\psi_2)
\]

and

\[
E \psi = e^{-\int_0^t (c-d)ds} \left[ C_1 P_1^2 \psi + C_2 P_2^2 \psi + C_3 (P_1 P_2 + P_2 P_1) \psi \right] = e^{\int_0^t (c-d)ds} \left[ C_1 K_1(y^2 \chi_1) + C_2 K_2(y^2 \chi_2) \right] + C_3 [P_1 (K_2(\chi_2)) + P_2 (K_1(\chi_1))].
\]

(7.2)

(7.3)

We have

\[
K_2(\chi_2) = K_1(\varphi_1), \quad \varphi_1 = K_1^{-1}(0) [K_2(0) (\chi_2)], \quad K_1(\chi_1) = K_2(\varphi_2), \quad \varphi_2 = K_2^{-1}(0) [K_1(0) (\chi_1)]
\]

with the help of an analogue of the Fourier transform. Therefore,

\[
P_1 (K_2(y \chi_2)) = e^{\int_0^t (c-d)ds} K_1(\psi_1), \quad P_2 (K_1(y \chi_1)) = e^{\int_0^t (c-d)ds} K_2(\psi_2),
\]

and finally

\[
E \psi = e^{\int_0^t (c-d)ds} \left[ C_1 K_1(y^2 \chi_1) + C_2 K_2(y^2 \chi_2) \right] + C_3 [K_1(\psi_1) + K_2(\psi_2)],
\]

(7.4)

where each term satisfies the corresponding Schrödinger equation. By the superposition principle, we arrive at a new solution in complete agreement with our lemma 1. The corresponding initial data follow from (7.4) at \(t = 0\) and the time-evolution operator (1.10) can be applied (see also equation (7.17) for an eigenfunction expansion).

On the other hand, one can always expand a square-integrable solution (4.20) of the Cauchy initial value problem in the standard form

\[
\psi(x, t) = \int_{-\infty}^{\infty} K(x, y, t) \chi(y) \, dy = \sum_{n=0}^\infty c_n(t) \Psi_n(x, t),
\]

(7.5)

where

\[
c_n(t) = \int_{-\infty}^{\infty} \Psi_n^*(x, t) \psi(x, t) \, dx \quad (7.6)
\]

(we use the asterisk for complex conjugate) by the Riesz–Fisher theorem [70–72] due to completeness of the eigenfunctions (5.34)–(5.36) at all times. Then by the Fubini theorem,

\[
c_n(t) = \int_{-\infty}^{\infty} \chi(y) \left( \int_{-\infty}^{\infty} \Psi_n^*(x, t) K(x, y, t) \, dx \right) \, dy
\]

(7.7)
and the second integral can be evaluated with the help of equations (4.10), (5.34)–(5.36) and (B.1) as follows:

$$\int_{-\infty}^{\infty} \psi_n^*(x,t)K(x,y,t)\,dx$$

$$= 
\frac{1}{(\sqrt{2\pi}e^{\frac{1}{2}x})^2} \left( \frac{\sqrt{C_0}}{C_0} \right) \left( \kappa \left( \frac{\kappa}{\kappa + \kappa'} \right) \right)^{1/4} \right) \times \exp \left( i \int_0^t (d-c) \, ds \right) \times \exp \left( i \left[ \kappa \left( \frac{\kappa}{\kappa + \kappa'} \right) \right] \left( \frac{d}{\kappa} \right) \times \exp \left( -\beta^2(0) \kappa^2(0) \sqrt{C_0} \left( \frac{\kappa}{\kappa + \kappa'} \right) \left( \frac{d}{\kappa} \right) \right) \times \exp \left( -\beta^2(0) \kappa^2(0) \sqrt{C_0} \left( \frac{\kappa}{\kappa + \kappa'} \right) \left( \frac{d}{\kappa} \right) \right) \times \frac{H_n}{\sqrt{C_0} \left( \frac{\kappa}{\kappa + \kappa'} \right) \left( \frac{d}{\kappa} \right)} \right) \right), \quad (7.8)$$

Here \( \kappa_1 = \mu \exp(\int_0^t (d-c) \, ds) \) and \( \kappa \) are the corresponding solutions of auxiliary equation (5.3) with \( C_0 = 0 \) and \( C_0 \neq 0 \), respectively, with the Wronskian \( W(\kappa_1, \kappa) = \kappa_1 \kappa' - \kappa_1' \kappa \) and

$$\tan \varphi = \frac{\kappa_1 \kappa' - \kappa_1' \kappa}{\kappa_1 \kappa} \quad (7.9)$$

$$C_0 \left( \frac{\kappa_1}{\kappa} \right)^2 + \left( \frac{\kappa_1 \kappa' - \kappa_1' \kappa}{2\kappa} \right)^2 = \text{constant} \quad (7.10)$$

by (A.4). (The computational details are left to the reader; our equation (7.10) is equivalent to the classical Ernbrook invariant [21].) We may choose \( \beta(0) \kappa_1(0) = 1 \) and arrive at the following result.

**Theorem 2.** (Eigenfunction expansions). Solution of the Cauchy initial value problem (4.20) in \( L^2(\mathbb{R}) \) can be obtained as an infinite series of multiples of the quadratic invariant eigenfunctions (5.34):

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n(t) \psi_n(x,t), \quad (7.11)$$

where the time-dependent coefficients are given by

$$c_n(t) = i^n \left( \frac{\delta}{\sqrt{\pi}2^n n!} \right)^{1/2} \exp \left( \frac{i}{2} \int_0^t (d-c) \, ds \right) \times \int_{-\infty}^{\infty} \exp \left( i \xi y^2 \right) e^{-\delta y^2/2H_n(\hat{\delta}y)} \chi(y) \, dy. \quad (7.12)$$

Here \( \kappa_1(t) \) and \( \kappa(t) \) are real-valued solutions of the homogeneous and non-homogeneous auxiliary equations (5.3), respectively, with the Wronskian \( W(t) = \kappa_1 \kappa' - \kappa_1' \kappa \). The phases \( \varphi(t) \) and \( \gamma(t) \) are determined in terms of these solutions as follows:

$$\varphi = \arctan \left( \frac{\kappa_1 \kappa' - \kappa_1' \kappa}{\kappa_1 \kappa} \right), \quad \frac{d\varphi}{dt} = 2\sqrt{C_0} \frac{a}{\kappa'}, \quad (7.13)$$

$$\frac{dy}{dt} + \frac{a}{\kappa'} = 0 \quad (7.14)$$

and

$$\delta = -\frac{\kappa_1}{\sqrt{C_0}} \left( \frac{\kappa_1}{\kappa_1 + \kappa'} \right) > 0, \quad (7.15)$$

$$\xi = \gamma + \frac{\delta^2}{2\sqrt{C_0}} \left( \frac{\kappa_1 \kappa' - \kappa_1' \kappa}{2\kappa} \right) \quad (7.16)$$

are constants. A spectral decomposition of the quadratic invariant \( E \) in the space of \( L^2 \)-solutions is given by

$$E(t)\psi(x,t) = \omega(t) \sum_{n=0}^{\infty} c_n(t) \left( n + \frac{1}{2} \right) \psi_n(x,t), \quad (7.17)$$

where the ‘frequency’ \( \omega(t) \) is defined by (5.28).

It is worth noting that

$$\frac{d}{dt} \left[ \gamma + \frac{\delta^2}{2\sqrt{C_0}} \frac{W}{\kappa_1} \right] = 0 \quad (7.18)$$

by (7.14) and (A.6), which means that the phase factor (7.16) in front of \( y^2 \) in the second integral of equation (7.12) is indeed a constant. The second equation (7.13) follows from (7.18) with the help of (7.9) and (7.14).

Finally by choosing in equations (7.11)–(7.12) special (square) integrable initial data of the form

$$\chi_m(y) = \exp \left( -i\xi y^2 \right) e^{-\delta y^2/2H_m(\delta y)}, \quad (7.19)$$

we conclude in view of the orthogonality property of Hermite polynomials that the time-dependent wavefunctions

$$\psi_m(x,t) = D_m \exp \left( \frac{1}{2} \int_0^t (d-c) \, ds \right) \times e^{-i(\alpha+1/2)\psi(t)} \psi_n(x,t) \quad (7.20)$$

are particular solutions of the Schrödinger equation (1.1)–(1.2) for arbitrary constants \( D_m \):

$$i \frac{\partial \psi_m}{\partial t} = H \psi_m, \quad E \psi_m = \omega \left( m + \frac{1}{2} \right) \psi_m. \quad (7.21)$$

They are also eigenfunctions of the quadratic invariant \( E \) corresponding to the discrete ‘spectrum’:

$$\langle E \rangle_m = \int_{-\infty}^{\infty} \psi_m^* E \psi_m \, dx = |D_m|^2 \omega_0 \left( m + \frac{1}{2} \right), \quad (7.22)$$

see (5.33). (It can be verified by direct substitution; the details are left to the reader.) The explicit wavefunctions (7.20) are derived here without separation of the variables with the aid
of our theorem 2 and certain variants of the Ermakov invariant. If \(|D_n| = 1\), solution (7.11) takes the form
\[
\psi(x, t) = \sum_{n=0}^{\infty} i^n e^{-i(n+1/2)\psi(0)} \psi_n(x, t) \int_{-\infty}^{\infty} \psi_n^*(y, 0) \chi(y) \, dy
\]
(7.23)
in terms of the time-dependent wavefunctions (7.20) provided that
\[
\epsilon(0) = \frac{C_0^{1/4}}{\kappa(0)} = \delta, \quad \frac{(c(0) + d(0)) \kappa(0) - \kappa'(0)}{4aD(0) \kappa(0)} = \xi
\]
(7.24)
(see also [85] for the path-integral method). The traditional operator approach (for the parametric oscillator) is presented in [48] and/or elsewhere (see also [1, 39]).

8. A general nonlinear superposition principle for Ermakov’s equations

The Pinney superposition formula (6.8)–(6.9) allows one to construct solutions of the nonlinear auxiliary equation (5.3) in terms of given solutions of the corresponding linear equation. In general, we take two different solutions, say \(\kappa_1\) and \(\kappa_2\), of the generalized Ermakov equations (5.3) with \(C_0^{1/2} \neq 0\) and consider two quadratic invariants:
\[
E_1(t) = \left[ \left( \kappa_1 p + \frac{(c + d) \kappa_1 - \kappa_1'}{2a} x \right)^2 + \frac{C_0^{(i)}}{\kappa_1^2} x^2 \right] \lambda(t),
\]
(8.1)
\[
E_2(t) = \left[ \left( \kappa_2 p + \frac{(c + d) \kappa_2 - \kappa_2'}{2a} x \right)^2 + \frac{C_0^{(2)}}{\kappa_2^2} x^2 \right] \lambda(t).
\]
(8.2)

Their arbitrary linear combination,
\[
D_1 E_1(t) + D_2 E_2(t) = E(t)
\]
(8.3)
\((D_1 \text{ and } D_2 \text{ are constants}),\) is also a quadratic invariant given by (5.2) for a certain solution \(\kappa\) of the nonlinear auxiliary equation (5.3). Thus the following operator identity holds,
\[
\left( k^2 \frac{(c + d) k - \kappa'}{2a} x \right)^2 + \frac{C_0}{k^2} x^2
= D_1 \left[ \left( \kappa_1 p + \frac{(c + d) \kappa_1 - \kappa_1'}{2a} x \right)^2 + \frac{C_0^{(i)}}{\kappa_1^2} x^2 \right]
+ D_2 \left[ \left( \kappa_2 p + \frac{(c + d) \kappa_2 - \kappa_2'}{2a} x \right)^2 + \frac{C_0^{(2)}}{\kappa_2^2} x^2 \right],
\]
(8.4)
and in a similar fashion we arrive at a general nonlinear superposition principle for solutions of Ermakov’s equations (5.3):
\[
k^2(t) = D_1 \kappa_1^2(t) + D_2 \kappa_2^2(t),
\]
(8.5)
where the constant \(C_0\) is given by
\[
C_0 = -C_0^{(i)} D_1^2 - C_0^{(2)} D_2^2 = D_1 D_2 \left[ \frac{W^2 (\kappa_1, \kappa_2)}{(2a)^2} \right.
+ C_0^{(i)} \frac{\kappa_2^2}{\kappa_1^2} + C_0^{(2)} \frac{\kappa_2^2}{\kappa_2^2} \right]
\]
(8.6)
with \(W(\kappa_1, \kappa_2)\) being the Wronskian of two solutions (see appendix A). One can also derive this property by adding the corresponding solutions (5.7)–(5.9) of the original linear system (5.4)–(5.6) or with the help of the Pinney formula (6.8)–(6.9). The details are left to the reader.

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Appendix A. Required transformations

Equating coefficients in front of the operators \(p^2, x^2\) and \(p x + x p\) in (6.4)–(6.5) with the help of (6.7), one obtains
\[
k^2 = C_1 \kappa_1^2 + C_2 \kappa_2^2 + 2C_3 \kappa_1 \kappa_2,
\]
(A.1)
\[
\left( \frac{(c + d) \kappa - \kappa'}{2a} x \right)^2 + \frac{C_0}{k^2}
= C_1 \left[ \frac{(c + d) \kappa_1 - \kappa_1'}{2a} \right]^2 + C_2 \left[ \frac{(c + d) \kappa_2 - \kappa_2'}{2a} \right]^2
+ 2C_3 \left[ \frac{(c + d) \kappa_1 - \kappa_1'}{2a} \right] \left[ \frac{(c + d) \kappa_2 - \kappa_2'}{2a} \right],
\]
(A.2)
\[
k^2 = C_1 \kappa_1 \left( \frac{(c + d) \kappa_1 - \kappa_1'}{2a} \right) + C_2 \kappa_2 \left( \frac{(c + d) \kappa_2 - \kappa_2'}{2a} \right)
+ C_3 \left( \frac{(c + d) \kappa_1 - \kappa_1'}{2a} \right) \left( \frac{(c + d) \kappa_2 - \kappa_2'}{2a} \right)
\]
(A.3)
respectively. Multiply (A.2) by (A.1) and use (A.3) in the left-hand side in order to obtain (6.9) as a result of elementary but rather tedious calculations.

A general nonlinear superposition principle for Ermakov’s equations, given by (8.5)–(8.6), is derived from the quadratic invariant identity (8.4) in a similar fashion. In addition, the constant in the right-hand side of (8.6) (see also (7.10)) can be verified by direct differentiation as follows:
\[
\frac{d}{dt} \left[ \frac{W}{2a} \right]^2 + C_0^{(i)} \left( \frac{k_1}{\kappa_1} \right)^2 + C_0^{(2)} \left( \frac{\kappa_2}{\kappa_2} \right)^2
= \frac{2W}{(2a)^2} \left( W' - \frac{a'}{a} W \right) + 2W \left( C_0^{(1)} \frac{k_2}{\kappa_1} - C_0^{(2)} \frac{k_2}{\kappa_2} \right)
= \frac{2W}{(2a)^2} \left( W' - \frac{a'}{a} W \right)
- \frac{2W}{(2a)^2} \left( \frac{k_1 k_2' - k_1' k_2}{a} \right) \left( \kappa_1 \kappa_2 - \kappa_1' \kappa_2' \right) = 0
\]
(A.4)
with the help of auxiliary equations (5.3). (It is a natural generalization of the classical Ermakov invariant [21].) En route, we derive an identity:

\[
\frac{1}{2a} \left( W - 2a \right) + C_{(1)}^{(2)} \frac{\kappa_2}{\kappa_1} - C_{(1)}^{(2)} \frac{\kappa_1}{\kappa_2} = 0, \tag{A.5}
\]

which may be considered as an extension of the familiar Abel theorem, occurring when \( C_{(1)}^{(2)} = C_0^{(2)} = 0 \), to the nonlinear Ermakov equations. Another identity,

\[
\frac{d}{dr} \left[ \frac{\kappa_2}{\kappa_1} \left( W - 2a \right) \right] = 2a \left( \frac{W}{2a} \right)^2 - C_{(1)}^{(2)} \left( \frac{\kappa_1}{\kappa_2} \right)^2 + C_{(2)}^{(2)} \left( \frac{\kappa_1}{\kappa_2} \right)^2, \tag{A.6}
\]

has been useful in section 7 with \( C_{(1)}^{(2)} = 0 \), \( C_{(1)}^{(2)} = C_0 \) and \( \kappa_2 = \kappa \) when the right-hand side reduces to a multiple of the Ermakov invariant (see equation (7.18)).

**Appendix B. An integral evaluation**

In section 7 (see equation (7.8)), we use the following integral,

\[
\int_{-\infty}^{\infty} e^{-\lambda^2(y - \gamma)^2} \, H_n(ay) \, dy = \frac{\sqrt{\pi}}{\lambda^{n+1}} \left( \lambda^2 - \alpha^2 \right)^{n/2} H_n \left( \frac{\lambda ax}{(\lambda^2 - \alpha^2)^{1/2}} \right), \quad \text{Re} \, \lambda^2 > 0, \tag{B.1}
\]

which is equivalent to equation (30) on p 195 of vol 2 of [19] (the Gauss transform of Hermite polynomials) or equation (17) on p 290 of vol 2 of [20].

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