Wigner-function description of EPR experiment

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Abstract. We provide a detailed description of the EPR paradox (in the Bohm version) for a two qubit-state in the discrete Wigner function formalism. We compare the probability distributions for two qubit relevant to simultaneously-measurable observables (computed from the Wigner function) with the probability distributions representing two perfectly-correlated classic particles in a discrete phase-space. We write in both cases the updating formulae after a measure, thus obtaining a mathematical definition of classic collapse and quantum collapse. We study, with the EPR experiment, the joint probability distributions of Alice’s and Bob’s qubit before and after the measure, analyzing the non-local effects. In particular, we give a more precise definition of locality, which we call m-locality: we show that quantum systems may violate this kind of locality, thus preserving, in an EPR-like argument, the completeness of Quantum Mechanics.

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1. Introduction

The Wigner function is a quasi-probability distribution which can be used, in alternative to the density-matrix formalism, to represent quantum states. The main advantage in the application of Wigner function is that it behaves similarly to classical probability distributions from several points of view. For quantum states with infinite dimensional Hilbert-spaces, the Wigner function has become a standard part of considerations. For finite dimensional Hilbert-spaces, the Wigner-function formalism was first investigated by Wootters [1] and later a more geometric definition has been introduced [2]. For a list of references, see also [3]. The discrete Wigner function has shown to be useful in several applications: the investigation of coherent states in a finite-dimensional basis [4], the definition of Q-functions and other propensities [5], the development of number-phase Wigner functions [6], the quantum tomography [7], the quantum computation [8], the theory of quantum games [9], the quantum teleportation [10] and the study of entanglement [3].

The EPR paradox [11] plays a central role both in the discussions about the interpretation of quantum mechanics and in many quantum information applications. The original aim of the EPR paradox was to entail the incompleteness of Quantum Mechanics. More recently, EPR-type experiments, championed by Aspect et al. [12], are often interpreted as empirical evidence for the existence of the ”quantum non-locality”. These non-local effects are basic in all the main applications of Quantum Information Theory. The EPR paradox is a paradox in the following sense: if one takes quantum mechanics and adds some seemingly reasonable conditions (referred to as ”locality”, ”realism”, and ”completeness”), then one obtains a contradiction. However, quantum mechanics by itself does not appear to be internally inconsistent, nor does it contradict relativity.

The paper is organized as follows: in section (2) we give a brief review of some elements of finite dimensional Wigner-function formalism (for a more detailed review, see [1, 3]) and we describe in terms of Wigner function the measurement process. In section (3) the entire EPR paradox is discussed in terms of discrete Wigner function, while in subsection (3.1) the concept of non-locality is discussed and a more precise definition is purposed. Finally in section (4) conclusions are drawn.

2. The Wigner function and the measurement process

According to the definition given in [1], the discrete Wigner function relevant to a density matrix \( \hat{\rho} \) in a \( N = r^n \)-dimensional Hilbert space \( H \) (with \( r \) prime number and \( n > 0 \)) is a real function \( W(\alpha) \), whose arguments \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_i, ..., \alpha_n) \) represent the coordinate in the whole phase space, each \( \alpha_i = (q_i, p_i) \) being the coordinate relevant to the \( i \)-th subsystem. We can also consider the coordinate \( \alpha \) as a couple \( (q, p) \) where each coordinate is a multi-label, that is for example \( q = (q_1, q_2, ..., q_k, ..., q_n) \). The explicit form of the Wigner function is \( W(\alpha) = \frac{1}{N} tr[\hat{\rho} \hat{A}(\alpha)] \), where the operators \( \hat{A}(\alpha) \)'s are
the discrete phase-point operators, forming a complete basis of the hermitian operators acting on a $N = r^n$-dimensional Hilbert space. When necessary, we will write the label $W_\rho(\alpha)$, evidencing that the Wigner function is relevant to the density matrix $\hat{\rho}$, or $W_i(\alpha)$ where $i = a, b, \ldots$ labels different steps. The general definition of $\hat{A}(\alpha)$ is given in [1], while we are interested in the Wigner function relevant to a system of $N$ qubits.

In the particular case of $n = 1$ (single qubit), the phase space is the set of points $\alpha = (q, p)$, where $q, p = 0, 1$ are the discrete phase-space coordinates (discrete position and discrete momentum), while the discrete phase-point operators are written in terms of Pauli matrices as $[1] \hat{A}(\alpha) = \frac{1}{2} [I + (-1)^q \sigma_z + (-1)^p \sigma_x + (-1)^{q+p} \sigma_y]$. Consistently with the definition of phase space in terms (of cartesian product) of constituent subspaces [1], phase-point operators $\hat{A}(\alpha)$ are defined as tensor product of phase-point operators relevant to the corresponding subsystems: $\hat{A}(\alpha) = \hat{A}(\alpha_1) \otimes \hat{A}(\alpha_2) \otimes \ldots \otimes \hat{A}(\alpha_n)$. In the two qubit case, the Wigner function can be written explicitly as $W(q_1, q_2, p_1, p_2) = \frac{1}{4} tr[\hat{\rho} \hat{A}(q_1, p_1) \otimes \hat{A}(q_2, p_2)]$.

The Wigner function has an important property, called the inner-product rule:

$$tr(\hat{\rho} \hat{\sigma}) = N \sum_\alpha W_\rho(\alpha) W_\sigma(\alpha).$$

(1)

The phase-space operators $\hat{Q}$ and $\hat{P}$, relevant to the discrete phase-space coordinates $q$ and $p$, are defined in terms of the sets of states $\{|q\} \text{ and } \{|p\}$ (forming two orthonormal basis of $H$ and connected by $|p\rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} e^{i\frac{2\pi}{N} q p} |q\rangle$) as [10]

$$\hat{Q} = \sum_{q=0}^{N-1} q |q\rangle \langle q| \ , \ \hat{P} = \sum_{p=0}^{N-1} p |p\rangle \langle p| .$$

(2)

Since the EPR experiment involves a measure on an entangled state, we describe the measurement process within the discrete Wigner function picture. In general, quantum systems exist in a superposition of basis states, and evolve according to the time-dependent Schrödinger equation (a process included in all interpretations of Quantum Mechanics). For example, we consider a quantum state $|s\rangle$ in a finite dimensional Hilbert space and an observable $\hat{X}$, with a discrete set of eigenvalues $\{x\}$ and eigenvectors $\{|x\} \text{.}$

Suppose we measure the observable $\hat{X}$: we could get, as a measurement result, the value $\bar{x}$ (with probability $|\langle s|\bar{x}\rangle|^2$). After the measurement process, the final state is $|\bar{x}\rangle$. The wave-function collapse is represented by this “jump” to one of the basis states. After the collapse, the system begins to evolve again according to the Schrödinger equation. In the Copenhagen interpretation [13], the collapse is a real process by which quantum systems evolve according to the laws of quantum mechanics. We first analyze the analogue of a collapse situation in the case of a classic probability distribution, where the effect of a measurement is evidenced by the following:

**Property A:** let us consider a set of independent observable quantities $X, Y, Z, \ldots \text{, with discrete values } \{x, y, z, \ldots\}$; we suppose that there is a generic initial joint probability distribution $P_a(x, y, z, \ldots)$, representing the probability of measuring contemporarily the values $(x, y, z, \ldots)$. If the measurement of $X$ gives a particular result $\bar{x}$, then the joint
probability distribution becomes (classic-collapse updating formula)

\[ P_{b}(x, y, z, ...) = \delta(x, \bar{x})P_{a}(y, z, ...|\bar{x}) , \]

which evidences that, for \( x \neq \bar{x} \), \( P_{b} \) is null (since we have measured \( \bar{x} \)), while for \( x = \bar{x} \) the final distribution is the conditional probability \( P(y, z, ...|\bar{x}) = P_{a}(\bar{x}, y, z, ...)P_{a}(\bar{x}) \). Note that \( P_{a}(\bar{x}) \) is a marginal distribution, obtained by summing \( P_{a}(\bar{x}, y, z, ...) \) over the variables \( (y, z, ...) \) and \( \delta \) is the Kröneker symbol. In the special case of a probability distribution of one variable \( P_{a}(x) \) with measurement result \( \bar{x} \), the final probability distribution is simply \( P_{b}(x) = \delta(x, \bar{x}) \): after the measure, we know with certainty that the observable \( X \) has the value \( \bar{x} \).

In the quantum context all the probabilities can be computed from \( W(\alpha) \). We first recall an important property of Wigner function, which can be derived from equation (1), valid for the two qubit case (but easily extendible for \( n \) qubit):

**Property B:** given a two qubit state \( \hat{\rho} \) and the corresponding Wigner function \( W(\alpha) \), from the couples of commuting operators \( (\hat{X}_{1}, \hat{Y}_{2}) \) (with \( \hat{X}, \hat{Y} = \hat{Q}, \hat{P} \)) we can write the probability \( P(x_{1}, y_{2}) = \langle x_{1}, y_{2}|\hat{\rho}|x_{1}, y_{2} \rangle \) of measuring contemporarily the values \( (x_{1}, y_{2}) \).

By writing the Wigner functions \( W_{[x_{1}, y_{2}]} = W_{x_{1}}W_{y_{2}} \), we have that

\[ W(q_{1}, p_{1}, q_{2}, p_{2}) = \sum_{x_{1}, y_{1}=0,1} P(x_{1}, y_{2})W_{[x_{1}, y_{2}]}(q_{1}, p_{1}, q_{2}, p_{2}) \]

\[ P(x_{1}, y_{2}) = 2 \sum_{\alpha} W_{[x_{1}, y_{2}]}(q_{1}, p_{1}, q_{2}, p_{2}) W(q_{1}, p_{1}, q_{2}, p_{2}) . \]  

Equation (1) can be used in the case of a measure of observable \( \hat{X}_{1} \) with result \( \bar{x}_{1} \): the initial distribution \( P_{a} \) is subjected to a classic collapse, inducing on the initial state \( W_{a} \), a quantum collapse thought the updating formula:

\[ W_{b}(\alpha) = \sum_{x_{1}, y_{1}} P_{b}(x_{1}, y_{2})W_{[x_{1}, y_{2}]}(\alpha) = \sum_{x_{1}, y_{1}} \delta(x_{1}, \bar{x}_{1})P_{a}(y_{2}|\bar{x}_{1})W_{x_{1}y_{2}}(\alpha) . \]

It is important to note that the probability distributions computed from (6) can violate the updating formula (3) of the classic case. Equation (5) shows that in the case of the observables \( \hat{Q} \) and \( \hat{P} \), defined by (2), the related probabilities can be computed by summing \( W(\alpha) \) in the discrete phase-space over vertical and horizontal lines respectively. In fact the Wigner functions associated to states \(|q\rangle \) and \(|p\rangle \) have graphics of vertical and horizontal lines (see (1)).

We now compare the quantum and classic collapse in the simplest example: given a single classic particle in a discrete phase space with \( q, p = 0, 1 \), with a starting probability distribution \( P_{a}(q, p) \), if we measure \( Q \) with value \( \bar{q} \), by using equation (3) we have that the collapse produces \( P_{b}(q, p) = \delta(q, \bar{q})P_{a}(p|\bar{q}) \). In the corresponding quantum case, a single-qubit state, the probability distribution (5) involves only \( \hat{Q} \) or \( \hat{P} \) (since they are non-commuting operators). If we measure \( \hat{Q} \) with value \( \bar{q} \), the starting probability distribution \( P_{a}(q) \) becomes \( P_{b}(q) = \delta(q, \bar{q}) \) and thus, through equation (6), the final Wigner function is \( W_{b}(q, p) = \frac{1}{2}\delta(q, \bar{q}) \) (the Wigner function associated to \(|\bar{q}\rangle\)). Thus \( P_{b}(p) = \frac{1}{2} \) (complete indetermination), different from the classic result \( P_{b}(p) = P_{a}(p|\bar{q}) \).
3. The EPR argument for a two-qubit system

The EPR argument [11] can be applied both in the classic and in the quantum case, by considering two non-interacting and correlated systems in two spatially-separated laboratories. In the first Alice can perform two different measures: we are interested in which effects can have these measures on Bob’s particle (second laboratory). In [11] three hypothesis are considered as valid: 1) the reality: if we can predict the value of an observable without perturbing in any way the system, we have an element of reality. For example, if we have the probability distribution \( P(x) = \delta(x, \tilde{x}) \) without performing any measure on the system, then the observable \( X \) is an element of reality. 2) the locality: given two distant and non-interacting (in a certain time-interval) systems, the change in physical properties of one system can not have direct influence on the other in the same interval. 3) the completeness: the description given by Quantum Mechanics is complete.

First we consider the classic case, consisting in a two-particle system with discrete observables \( Q_i, P_j \) (with \( i, j = 1, 2 \)) which can be all measured contemporarily. The classic version of the EPR state is a perfectly correlated two-particle state, whose probability distribution (with \( q_i, p_j \in \{0, 1\} \)) is

\[
P_a(q_1, p_1, q_2, p_2) = \frac{1}{4} \delta(q_1, q_2) \delta(p_1, p_2).
\]

(7)

In the EPR experiment, Alice can measure on the first particle \( Q_1 \) or \( P_1 \): we suppose that Alice measures \( Q_1 \) with result \( \tilde{q}_1 \), which entails, from equation (3), the probability distribution

\[
P_b(q_1, p_1, q_2, p_2) = \frac{1}{2} \delta(q_1, \tilde{q}_1) \delta(q_2, \tilde{q}_1) \delta(p_1, p_2).
\]

(8)

The measure of Alice on her particle produces a change in the probability distribution relevant to Bob’s particle, as evidenced by \( P_b(q_2) = \delta(q_2, \tilde{q}_1) \). Nevertheless, since the two particles are spatially separated and non-interacting, from hypothesis 2) there is no physical change in Bob’s system, and the only change is in the knowledge of \( Q_2 \): thus \( Q_2 \) is an element of reality. If Alice chooses to measure \( P_1 \) and obtains \( \tilde{p}_1 \), the probability distribution becomes

\[
P_c(q_1, p_1, q_2, p_2) = \frac{1}{2} \delta(p_1, \tilde{p}_1) \delta(p_2, \tilde{p}_1) \delta(q_1, q_2).
\]

(9)

Once again, the correlations of the particles allow us to entail that \( p_2 = \tilde{p}_1 \), without measuring it and thus that also \( P_2 \) is an element of reality. From the two measures, we conclude that both \( Q_2 \) and \( P_2 \) are elements of reality. Alice can also perform the two measurements consecutively on her particle, and from equation (3) the distribution is

\[
P_d(q_1, p_1, q_2, p_2) = \delta(q_1, \tilde{q}_1) \delta(q_2, \tilde{q}_1) \delta(p_1, \tilde{p}_1) \delta(p_2, \tilde{p}_1).
\]

(10)

Let us now consider the quantum case: the Wigner function corresponding to the EPR state \( |\Psi+\rangle = (|0, 0\rangle + |1, 1\rangle)/\sqrt{2} \) (written in the base \( |0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |1, 1\rangle \)),
which results to be maximally entangled, is

\[ W_a(q_1, p_1, q_2, p_2) = \frac{1}{8} - \frac{1}{4} \delta(q_1 \oplus q_2, 1) \delta(p_1 \oplus p_2, 1) \]  
(11)

\[ W^T_a(q_1, p_1, q_2, p_2) = \frac{1}{4} \delta(q_1, q_2) \delta(p_1, p_2) , \]  
(12)

where the symbol \( \oplus \) denotes a sum mod2 and (12) is the Wigner function of the partially transposed state, which is identical to the classic distribution (7) (see [2] and [10]): functions (11) and (12) have the same marginal distributions. From the initial state (11) we can write the following probability distributions

\[ P_a(q_1, q_2) = \frac{1}{2} \delta(q_1, q_2) , \]

\[ P_a(q_1, p_2) = \frac{1}{2} \delta(p_1, p_2) , \]  
(13)

\[ P_a(p_1, p_2) = \frac{1}{2} \delta(p_1, p_2) , \]

\[ P_a(p_1, q_2) = \frac{1}{4} , \]  
(14)

related to the couple of observables \( (\hat{Q}_1, \hat{Q}_2) \), \( (\hat{Q}_1, \hat{P}_2) \) and \( (\hat{P}_1, \hat{P}_2) \), \( (\hat{P}_1, \hat{Q}_2) \), respectively. If Alice measures \( \hat{Q}_1 \) with value \( \tilde{q}_1 \), we have from equations (13) and (3) that \( P_b(q_1, q_2) = \delta(q_1, \tilde{q}_1) \delta(q_2, \tilde{q}_1) \). This is an effect of the correlations encoded in the EPR state: without any measure, we perfectly know not only \( \hat{Q}_1 \) but also \( \hat{Q}_2 \). Since the second system is spatially separated and non-interacting with the first, from the hypothesis of locality we have that the second system can not be perturbed by Alice’s measurement: thus \( \hat{Q}_2 \) is an element of reality. The state resulting after the measure is

\[ W_b(q_1, q_2, p_1, p_2) = \frac{1}{4} \delta(q_1, \tilde{q}_1) \delta(q_2, \tilde{q}_1) . \]  
(15)

The main difference from the classic case (8) is that function (15) does not involve \( \hat{P}_1 \) and \( \hat{P}_2 \), and thus the correlation between them has been completely erased. If Alice measures \( \hat{P}_1 \) with result \( \tilde{p}_1 \), with similar arguments and by using equations (14), we deduce that \( \hat{P}_2 \) is an element of reality, while \( \hat{Q}_2 \) is undetermined after the measurement:

\[ W_c(q_1, q_2, p_1, p_2) = \frac{1}{4} \delta(p_1, \tilde{p}_1) \delta(p_2, \tilde{p}_1) . \]  
(16)

Function (16) does not involve \( \hat{Q}_1 \) and \( \hat{Q}_2 \), and thus the correlation between them has been completely erased. The two measures performed by Alice allow to obtain two different elements of reality, as written in the distributions (15) and (16), relevant to two non-commuting operators. The EPR conclusion is that, since hypotheses 1) and 2) should be considered valid, the assumption 3) is wrong and the standard description of Quantum Mechanics is incomplete. In the EPR argument is implicit the idea that the context of the first measure does not influence the second: for example, that the measure of \( \hat{Q}_1 \) does not erase the correlation between \( \hat{P}_1 \) and \( \hat{P}_2 \). This is true in the classic case, where the two measures are independent, but not in quantum mechanics, as can be seen from the updated functions (15) and (16): thus the two measures are not independent and the EPR conclusions must be considered in a different way.
3.1. The non-locality and the Wigner function

From the updated functions (15) and (16) we can deduce that the measurements performed by Alice have influenced Bob’s qubit in a non-classic way, thus determining the differences in the final results. In the EPR argument the hypothesis of locality analyzes the change in physical properties of Bob’s qubit as a direct influence of Alice’s measure. It can be easily shown that all the probability distributions of Bob’s qubit \( P(q_2) \) and \( P(p_2) \) change after Alice’s measurs following the classic updating formula (3), that is under the conditional probability formula. This is sufficient to preserve the no-communication theorem [14] (fundamental for special relativity) and to ensure no instantaneous information transfer. On the contrary, we have evidenced that important differences from the classic case arise in the quantum systems when we consider joint probabilities of observables of the first and second qubit. We thus propose a more precise definition of locality for a probability distribution under a measurement process, which we call measure-locality or m-locality:

two distant and non-interacting systems (in a certain time-interval) are measure-local (or m-local) when the measurement of physical properties of the first system influences any probability distribution of simultaneously-measurable observables of the two systems, in the same time interval, according to the updating formula (3).

In the classic case, if Alice measures \( Q_1 \) we have through equation (7) that all the distributions \( P(x_1, y_2) \) obey (3), like for example \( P_a(p_1, p_2) = P_b(p_1, p_2) = \frac{1}{2} \delta(p_1, p_2) \). In the quantum case, we have after measuring \( \hat{Q}_1 \) and through equations (13) and (15),

\[
P_a(p_1, p_2) = \frac{1}{2} \delta(p_1, p_2) \rightarrow P_b(p_1, p_2) = \frac{1}{4},
\]

which violates the classic updating formula (3): the quantum system shows a non-m-locality. If we change, in the EPR argument, the hypothesis 2) with the m-locality, we have that the completeness of Quantum Mechanics is preserved. Quantum systems are local (according to Einstein’s definition), but non-m-local.

4. Conclusions

We have described the EPR argument in terms of discrete Wigner function: the use of a pseudo-probability function helped us to directly compare the classic and the quantum situations. We have written the explicit updating formulae (3) and (4) for a measurement in the classic and quantum case respectively, defining explicitly the concept of classic collapse and quantum collapse in terms of probabilities. We have evidenced through the Wigner function that the effect of a measure on a quantum system is to erase certain correlations. This has been evidenced by directly comparing the corresponding updated functions (8,9) and (15,16) in the EPR argument. We have concluded that, as an effect of the quantum collapse (6), the two measures performed by Alice can not be performed independently and thus they are contextual and somehow non-local. We have pointed out that the marginal probabilities of Bob’s qubit change according to the classic updating formula, and thus do not violate Einstein’s locality principle. We
have written a more precise definition of locality, which refers to the joint probability distributions of the two qubit, called \textit{m-locality}. We have shown through equation (17) that in the EPR argument the joint probability distributions of Alice and Bob manifest a non-m-locality. We have purposed to modify the EPR argument by changing the locality hypothesis with the m-locality, thus removing the contradiction.

We notice that the anomalous change in the correlations between subsystems before and after the measurement can be used to define a new measure of entanglement, with a direct operational meaning: partial results have been found on this respect, to be presented in a separate paper. Other correlated considerations and results are contained in [3] (Local Uncertainty Relation).

[1] Wootters W K 1987 \textit{Ann. Phys.} \textbf{176} 1

[2] Wootters W K 2003 \textit{preprint} quant-ph/0306135 v4 Gibbons K S, Hoffman M J and Wootters W K 2004 \textit{Phys. Rev. A} \textbf{70} 062101 Wootters W K 2004 \textit{IBM Journal of Research and Development} \textbf{48} 99

[3] Franco R and Penna V, 2006 \textit{J. Phys. A} Math. Gen. \textbf{39} 5907-5919

[4] Buzek V, Wilson-Gordon A D, Knight P L, and Lai W K, \textit{Phys. Rev. A} \textbf{45}, 8079 (1992).

[5] Opatrny T, Buzek V, Bajer J, and Drobny G, \textit{Phys. Rev. A} \textbf{52}, 2419 (1995).

[6] J A Vaccaro and D T Pegg, \textit{Phys. Rev. A} \textbf{41}, 5156 (1990).

[7] Leonhardt U, \textit{Phys. Rev. Lett.} \textbf{74}, 4101 (1995).

[8] Galvao E F 2005 \textit{Phys. Rev. A} \textbf{71} 042302

[9] Paz J P, Roncaglia A J and Saraceno M 2005 \textit{Phys. Rev. A} \textbf{72} 012309

Miquel C, Paz J P, Saraceno M 2002 \textit{Phys. Rev. A} \textbf{65} 062309

[10] Koniorczyk M, Buzek V, and Janszky J 2001 \textit{Phys. Rev. A} \textbf{64} 034301

[11] Einstein A, Podolsky B and Rosen N 1935 \textit{Phys. Rev.} \textbf{47}(10) 777-780.

[12] Aspect A, Dalibard J and Roger G \textit{Phys. Rev. Lett.} \textbf{49}, 1804-1807 (1982)

Aspect A, Grangier P, Roger G \textit{Phys. Rev. Lett.} \textbf{49}, 91-94 (1982)

[13] Bohr N 1935 \textit{Nature} \textbf{136} 1025-1026.

Heisenberg W 1930 The Physical Principles of the Quantum Theory. Trans. Carl Eckhart and F.C. Hoyt. New York: Dover

[14] Peres A 2004 \textit{Rev.Mod.Phys.} \textbf{76} 93