Thermodynamics of higher dimensional topological charged AdS black branes in dilaton gravity

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Abstract

In this paper, we study topological AdS black branes of $(n+1)$-dimensional Einstein-Maxwell-dilaton theory and investigate their properties. We use the area law, surface gravity and Gauss law interpretations to find entropy, temperature and electrical charge, respectively. We also employ the modified Brown and York subtraction method to calculate the quasilocal mass of the solutions. We obtain a Smarr-type formula for the mass as a function of the entropy and the charge, compute the temperature and the electric potential through the Smarr-type formula and show that these thermodynamic quantities coincide with their values which are calculated through using the geometry. Finally, we perform a stability analysis in the canonical ensemble and investigate the effects of the dilaton field and the size of black brane on the thermal stability of the solutions. We find that large black branes are stable but for small black brane, depending on the value of dilaton field and type of horizon, we encounter with some unstable phases.
I. INTRODUCTION

The discovery of a close relationship between the nature of quantum gravity and the thermodynamics of black holes has been one of the most important developments in general relativity in the past decades. Strong motivation for studying thermodynamics of black holes originates from the fact that they have a very natural thermodynamic description. For example, black holes have an entropy and temperature related to their horizon area and surface gravity, respectively, and also one can investigate their thermal stability. With the appearance of the anti-de Sitter/conformal field theory correspondence (AdS/CFT) [1], such black holes in asymptotically AdS space become even more interesting since one can gain some significant relations between the thermodynamical properties of the AdS black holes and the dual conformal field theory [2–4].

On the other hand, it is a general belief that in four dimensions the topology of the event horizon of an asymptotically flat stationary black hole is uniquely determined to be the two-sphere $S^2$ [5, 6]. Hawking’s theorem requires the integrated Ricci scalar curvature with respect to the induced metric on the event horizon to be positive [5]. This condition applied to two-dimensional manifolds determines uniquely the topology. The “topological censorship theorem” of Friedmann, Schleich and Witt is another indication of the impossibility of non-spherical horizons [7, 8]. However, when the asymptotic flatness of spacetime is violated, there is no fundamental reason to forbid the existence of static or stationary black holes with nontrivial topologies. It has been shown that for asymptotically AdS spacetime, in the four-dimensional Einstein-Maxwell theory, there exist black hole solutions whose event horizons may have zero or negative constant curvature and their topologies are no longer the two-sphere $S^2$. The properties of these black holes are quite different from those of black holes with usual spherical topology horizon, due to the different topological structures of the event horizons. Besides, the black hole thermodynamics is drastically affected by the topology of the event horizon. It was argued that the Hawking-Page phase transition [9] for the Schwarzschild-AdS black hole does not occur for locally AdS black holes whose horizons have vanishing or negative constant curvature, and they are thermally stable [10].

The studies on the topological black holes have been carried out extensively in many aspects (see e.g. [11–15]). In this paper we shall consider topological black branes in the presence of dilaton and electromagnetic fields in all higher dimensions. The action of the $(n + 1)$-
dimensional \((n \geq 3)\) Einstein-Maxwell-dilaton gravity can be written as

\[
I = -\frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} (R - K(\Phi) - V(\Phi) + \mathcal{L}(\Phi, F)),
\]

(1)

where \(R\) is the Ricci scalar, \(K(\Phi) = 4(\nabla\Phi)^2/(n - 1)\) is a kinetic term, \(V(\Phi)\) is a potential term for the dilaton field \(\Phi\), and \(\mathcal{L}(\Phi, F) = -e^{-4\alpha\Phi/(n-1)}F_{\mu\nu}F^{\mu\nu}\) is a coupled Lagrangian between scalar dilaton and electromagnetic fields. In \(\mathcal{L}(\Phi, F)\), \(\alpha\) is an arbitrary constant governing the strength of the coupling between the dilaton and the Maxwell field, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic field tensor and \(A_\mu\) is the electromagnetic potential. While \(\alpha = 0\) corresponds to the usual Einstein-Maxwell-scalar theory, \(\alpha = 1\) indicates the dilaton-electromagnetic coupling that appears in the low energy string action in Einstein’s frame.

Some attempts have been made to explore various solutions of Einstein-Maxwell-dilaton gravity. The dilaton field couples in a nontrivial way to other fields such as gauge fields and results into interesting solutions for the background spacetime [16, 17]. These scalar coupled black hole solutions [16, 17], however, are all asymptotically flat. It was argued that with the exception of a pure cosmological constant, no dilaton-de Sitter or anti-de Sitter black hole solution exists with the presence of only one Liouville-type dilaton potential [18]. In the presence of one or two Liouville-type potentials, black hole spacetimes which are neither asymptotically flat nor (A)dS have been explored by many authors (see e.g. [19–23]).

Recently, the “cosmological constant term” in the dilaton gravity has been found by Gao and Zhang [24]. With an appropriate combination of three Liouville-type dilaton potentials, they obtained the static dilaton black hole solutions which are asymptotically (A)dS in four and higher dimensions. In such a scenario AdS spacetime constitutes the vacuum state and the black hole solution in such a spacetime becomes an important area to study [1]. For an arbitrary value of \(\alpha\) in AdS spaces the form of the dilaton potential \(V(\Phi)\) in \(n+1\) dimensions is chosen as [24]

\[
V(\Phi) = \frac{2\Lambda}{n(n-2+\alpha^2)^2} \left\{ -\alpha^2 \left[ (n+1)^2 - (n+1)\alpha^2 - 6(n+1) + \alpha^2 + 9 \right] e^{-4(n-2)\Phi/[(n-1)\alpha]} \\
+ (n-2)^2(n-\alpha^2)e^{4\alpha\Phi/(n-1)} + 4\alpha^2(n-1)(n-2)e^{-2\Phi(n-2-\alpha^2)/[(n-1)\alpha]} \right\},
\]

(2)

where \(\Lambda\) is the cosmological constant. The motivations for studying such dilaton black holes with nonvanishing cosmological constant originate from supergravity theory. Gauged supergravity theories in various dimensions are obtained with negative cosmological constant.
in a supersymmetric theory. In addition, it has been shown that one may consider a Big Bang model of the Universe in the presence of dilaton field and presented Liouville-type potential which can mimic the matter (including dark matter) and dark energy. This model predict age of the Universe, transition redshift, Big Bang nucleosynthesis and evolution of dark energy agree with current observations \[25\]. Also, this type of potential can be obtained when a higher dimensional theory is compactified to four dimensions, including various super gravity models \[26\] (see also \[27\] for a recent discussion of these aspects). In particular, for special values of coupling constant, \(\alpha\), this potential reduce to the supersymmetry potential of Gates and Zwiebach in string theory \[26\].

For later convenience we redefine \(\Lambda = -\frac{n(n - 1)}{2l^2}\), where \(l\) is the AdS radius of spacetime. It is clear the cosmological constant is coupled to the dilaton in a very nontrivial way. In the absence of the dilaton field action \((\Pi)\) reduces to the action of Einstein-Maxwell gravity with cosmological constant. Considering this type of dilaton potential, one can extract successfully the AdS solutions of Einstein–Maxwell-dilaton gravity \[28, 29\].

The rest of this paper is outlined as follows. In the next section, we consider the field equations of Einstein-Maxwell-dilaton gravity and present the \((n+1)\)-dimensional topological AdS black brane solutions and investigate their properties. In section \[\text{III}\], we obtain the conserved and thermodynamic quantities of the solutions and verify the validity of the first law of black brane thermodynamics. We perform a stability analysis in the canonical ensemble and disclose the effect of the dilaton field on the thermal stability of the solutions in section \[\text{IV}\]. Conclusions are drawn in the last section.

II. FIELD EQUATIONS AND SOLUTIONS

Varying action \((\Pi)\) with respect to the metric tensor \(g_{\mu\nu}\), the dilaton field \(\Phi\) and the electromagnetic potential \(A_\mu\), the equations of motion are obtained as

\[
R_{\mu\nu} = \frac{4}{n-1} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) + 2 e^{-\frac{4\alpha\Phi}{(n-1)}} F_{\mu\eta} F_{\nu}^{\eta} - \frac{g_{\mu\nu}}{(n-1)} \mathcal{L}(\Phi, F),
\]

\[
\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{\alpha}{2} \mathcal{L}(\Phi, F),
\]

\[
\nabla_\mu \left( e^{-\frac{4\alpha\Phi}{(n-1)}} F^{\mu\nu} \right) = 0.
\]
Here we want to obtain the \( (n+1) \)-dimensional static solutions of Eqs. (3), (4) and (5). We assume that the metric has the following form

\[
ds^2 = -N^2(\rho)f^2(\rho)dt^2 + \frac{d\rho^2}{f^2(\rho)} + \rho^2 R^2(\rho)d\Omega_{n-1}^2, \tag{6}
\]

where

\[
d\Omega_{n-1}^2 = \left\{ \begin{array}{ll}
d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 + \sinh^2 \theta_1 \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = -1 \\
\sum_{i=1}^{n-1} d\phi_i^2 & k = 0
\end{array} \right.
\]

represents the line element of an \( (n-1) \)-dimensional hypersurface with constant curvature \( (n-1)(n-2)k \). It is notable that positive curvature horizon \( (k = 1) \) has been investigated in \[29\]. Here \( N(\rho), f(\rho) \) and \( R(\rho) \) are functions of \( \rho \) which should be determined. First of all, the Maxwell equations \[5\] can be integrated immediately, where all the components of \( F_{\mu\nu} \) are zero except \( F_{t\rho} \)

\[
F_{t\rho} = N(\rho)\frac{qe^{4\alpha\Phi/(n-1)}}{(\rho R)^{n-1}}, \tag{7}
\]

where \( q \), an integration constant, is the charge parameter of the black brane. Our aim here is to construct exact, \( (n+1) \)-dimensional topological AdS black brane solutions of Eqs. \[3\]-\[5\], with the dilaton potential \[2\] for an arbitrary dilaton coupling parameter \( \alpha \) and investigate their properties. Using metric \[6\] and the Maxwell field \[7\], one can show that the system of equations \[3\]-\[11\] have solutions of the form

\[
N^2(\rho) = \Upsilon^{-\gamma(n-3)}, \tag{8}
\]

\[
f^2(\rho) = \frac{\rho^2}{l^2} \Upsilon^{(n-2)\gamma} + \left[ k - \left( \frac{c}{\rho} \right)^{n-2} \right] \Upsilon^{1-\gamma}, \tag{9}
\]

\[
\Phi(\rho) = \frac{n-1}{4} \sqrt{\gamma(2 + 2\gamma - n\gamma)} \ln \Upsilon, \tag{10}
\]

\[
R^2(\rho) = \Upsilon^{\gamma}, \tag{11}
\]

\[
\Upsilon = 1 - \left( \frac{b}{\rho} \right)^{n-2}.
\]

Here \( c \) and \( b \) are integration constants and the constant \( \gamma \) is

\[
\gamma = \frac{2\alpha^2}{(n-2)(n-2+\alpha^2)}. \tag{12}
\]

The charge parameter \( q \) is related to \( b \) and \( c \) by

\[
q^2 = \frac{(n-1)(n-2)^2}{2(n-2+\alpha^2)} c^{n-2} b^{n-2}. \tag{13}
\]
For $\alpha \neq 0$ the solutions are not real for $0 < \rho < b$ and therefore we should exclude this region from the spacetime. For this purpose we introduce the new radial coordinate $r$ as

$$r^2 = \rho^2 - b^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + b^2} dr^2.$$

(14)

With this new coordinate, the above metric becomes

$$ds^2 = -N^2(r)f^2(r)dt^2 + \frac{r^2dr^2}{(r^2 + b^2)f^2(r)} + (r^2 + b^2)R^2(r)d\Omega^2_{n-1},$$

(15)

where the coordinates $r$ assumes the values $0 \leq r < \infty$, and $N^2(r)$, $f^2(r)$, $\Phi(r)$ and $R^2(r)$ are now given as

$$N^2(r) = \Gamma^{-\gamma(n-3)},$$

(16)

$$f^2(r) = \frac{r^2 + b^2}{l^2} \Gamma^{(n-2)\gamma} + \left[ k - \left( \frac{c}{\sqrt{r^2 + b^2}} \right)^{n-2} \right] \Gamma^{1-\gamma},$$

(17)

$$\Phi(r) = \frac{n-1}{4} \sqrt{\gamma(2 + 2\gamma - n\gamma)} \ln \Gamma,$$

(18)

$$R^2(r) = \Gamma^\gamma,$$

(19)

$$\Gamma = 1 - \left( \frac{b}{\sqrt{r^2 + b^2}} \right)^{n-2}.$$

A. Properties of the solutions:

It is notable to mention that these solutions are valid for all values of $\alpha$. When $(\alpha = 0 = \gamma)$, these solutions describe the $(n+1)$-dimensional asymptotically AdS Reissner-Nordstrom black branes. For $b = 0$ ($\Gamma = 1$), the charge parameter $q$ and the scalar field $\Phi(r)$ vanish and our solutions reduce to the solutions of Einstein gravity in the presence of cosmological constant.

Here we should discuss singularity(ies). After some algebraic manipulation, one can show that the Kretschmann and the Ricci scalars in $(n+1)$-dimensions are finite for $r \neq 0$, but in the vicinity of $r = 0$, we have

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} \propto r^{-4(n-2)\left[ 1 + (a^2 + n - 2)^{-1} \right]},$$

$$R \propto r^{-2(n-2)\left[ 1 + (a^2 + n - 2)^{-1} \right]},$$

and thus they diverge at $r = 0$ ($\rho = b$). Thus, $r = 0$ is a curvature singularity. It is worthwhile to note that the scalar field $\Phi(r)$ and the electromagnetic field $F_{\mu\nu}$ become zero...
FIG. 1: \( f^2(\rho) \) versus \( \rho \) for \( b = 1, \ c = 1.5, \ l = 0.5, \ n = 5 \) and \( \alpha = 0 \) (timelike singularity). \( k = 0 \) (bold line), and \( k = -1 \) (dashed line).

FIG. 2: \( f^2(r) \) versus \( r \) for \( b = 1, \ c = 2.5, \ l = 0.5 \) and \( n = 5 \). \( \alpha = 0.1 \) (solid line: naked singularity), \( \alpha = \alpha_{\text{ext}} \) (bold line: extreme black brane), \( \alpha = 0.8 \) (dashed line: null singularity with two horizons), and \( \alpha = 1.2 \) (dotted line: null singularity with one horizon), where \( \alpha_{\text{ext}} = 0.59, 0.67 \) for \( k = -1, 0 \), respectively.

as \( r \to \infty (\rho \to \infty) \). One should note that for nonzero \( \alpha \) the singularity may be null, while it is timelike for \( \alpha = 0 \) (see Figs. 1 and 2). Also, figure 2 shows that depending on the metric parameters, these real solutions may be interpreted as black brane solutions with inner and outer horizons, an extreme black brane or naked singularity.
III. THERMODYNAMICS OF ADS DILATON BLACK BRANE

In this section we intend to study thermodynamics of topological dilaton black branes in the background of AdS spaces. First of all we focus on entropy. The entropy of the dilaton black hole typically satisfies the so-called area law of the entropy which states that the entropy of the black hole is one-quarter of the event horizon area \( S = \frac{b^{n-1} \Gamma_+^{(n-1)/2}}{4(1 - \Gamma_+)^{(n-1)/(n-2)}} \), \( (20) \)
where \( \Gamma_+ = \Gamma(r = r_+) \). It is notable to mention that in contrast with the higher derivative gravities that may lead to negative entropy \( \Gamma_+ \leq 1 \), the presented entropy is positive definite \( (0 \leq \Gamma_+ \leq 1) \). The Hawking temperature of the dilaton black brane on the outer horizon \( r_+ \), may be obtained through the use of the definition of surface gravity,
\[
T_+ = \frac{1}{2\pi} \sqrt{-\frac{1}{2} \left( \nabla_{\mu} \chi \nabla^\nu \chi \right) = \sqrt{r_+^2 + b^2 \frac{(N^2 f^2)'}{4\pi N r}} \bigg|_{r=r_+}, \quad (21) \]
where \( \chi = \partial_t \) is the killing vector and a prime stands for the derivative with respect to \( r \). Finding the radius of outer horizon in terms of the parameters of the metric function is not possible analytically, and therefore we obtain the constant \( c \) in terms of \( b, \alpha \) and \( r_h \) by solving \( f(r_h) = 0 \), where \( r_h \) is the radius of inner or outer horizon of the black brane. Substituting \( c \) into Eq. \( (21) \), one obtains
\[
T_h = \frac{b(n - 2) \Gamma_h^{1-\gamma(n-1)/2}}{4\pi (1 - \Gamma_h)^{1/(n-2)}} \left\{ \frac{[\gamma(n - 1) - 1]}{l^2 \Gamma_h^{2-\gamma(n-1)}} - \frac{(n - 1) [\gamma(n - 2) - 2]}{(n - 2)l^2 \Gamma_h^{1-\gamma(n-1)}} + \frac{k(1 - \Gamma_h)^{2/(n-2)}}{b^2} \right\}, \quad (22) \]
where \( \Gamma_h = \Gamma(r = r_h) \). The equation \( T_h = 0 \) has one real root for \( k = 0 \):
\[
r_{ext} = b \left\{ \left( \frac{(n - 1) [2 - \gamma(n - 2)]}{n} \right)^{2/(n-2)} - 1 \right\}^{1/2}, \quad (23) \]
while it may have two real roots \( (r_{1ext} \text{ and } r_{2ext}) \) for \( k = -1 \). These roots are the radius of the extreme black branes. We are interested in the thermodynamics of event horizon, \( r_+ \), of the black branes and therefore we consider \( T_+ \geq 0 \). The negative values of \( T_h \) associated
to the temperature of inner horizon. The radius of event horizon $r_+ \geq r_{\text{ext}}$ for $k = 0$, and $r_+ \leq r_{1\text{ext}}$ or $r_+ \geq r_{2\text{ext}}$ for $k = -1$. These facts can be seen in Figs. 3 - 6, which show the temperature versus $r_+$ in various dimensions. In order to be more clear, we discuss these figures for $k = 0$ and $k = -1$, separately.

$k = 0$:

As one can see in Figs. 3 - 6 there exist extreme black brane with radius $r_{\text{ext}}$ provided $\alpha < \alpha_{\text{ext}}$, where

$$\alpha_{\text{ext}}^2 = \frac{n - 2}{n} \left( n - 2 + 2(n - 1) \left[ \left( 1 + \frac{r_{\text{ext}}^2}{b^2} \right)^{1-n/2} - 1 \right] \right). \quad (24)$$
FIG. 5: T versus $r_+$ for $b = 0.2, l = 0.5$ and $\alpha = 5$, $n = 4$ (solid line), $n = 5$ (bold line), and $n = 6$ (dashed line).

FIG. 6: T versus $r_+$ for $b = 0.2, l = 0.5$ and $n = 4$, $\alpha = 0.5$ (solid line), $\alpha = 1.5$ (bold line), and $\alpha = 2.5$ (dashed line).

That is the dilaton field removes the existence of extreme black branes.

$k = -1$:

For black branes with hyperbolic horizon and medium values of $\alpha$, numerical analysis shows that the equation $T_h = 0$ has two real roots and therefore one can have both small ($r_+ \leq r_{1ext}$) and large ($r_+ \geq r_{2ext}$) black branes. This can be seen on Figs. 3 - 6. Concerning the metric function $f(r)$ and temperature $T$, Figs. 2 - 6 one can find that for small values of $\alpha$ they behave like charged-AdS black branes while for large values of coupling constant $\alpha$, they are approximately Schwarzschild-AdS black branes.

Inserting solutions (9)-(11) in Eq. (7), with considering the new coordinate (14), the
electromagnetic field can be simplified as
\[ F_{tr} = \frac{q}{(r^2 + b^2)^{(n-1)/2}}. \]  

(25)

As one can see from Eq. (25), in the background of AdS universe, the dilaton field does not exert any direct influence on the matter field \( F_{tr} \), however, the dilaton field modifies the geometry of the spacetime as it participates in the field equations. This is in contrast to the solutions presented in [19–22]. The solutions of Ref. [19–22] are neither asymptotically flat nor (A)dS and the gauge field crucially depends on the scalar dilaton field.

The electric charge of the black brane per unit volume, \( Q \), can be found by calculating the flux of the electromagnetic field at infinity (Gauss theorem), obtaining
\[ Q = \frac{1}{4\pi} \int_{\rho \to \infty} d^{n-1}x \sqrt{-g} F_{t\rho} = \frac{\Omega_{n-1}}{4\pi} q. \]  

(26)

Let us return to Eq. (25). The gauge potential \( A_t \) corresponding to the electromagnetic field (25) can be easily obtained as
\[ A_t = -\frac{q}{(n-2)(r^2 + b^2)^{(n-2)/2}}. \]

The electric potential \( U \), measured at infinity with respect to the horizon, is defined by [33]
\[ U = A_\mu \chi^\mu \bigg|_{r \to \infty} - A_\mu \chi^\mu \bigg|_{r = r_+}, \]  

(27)

where \( \chi = \partial_t \) is the null generator of the horizon. Therefore, the electric potential may be obtained as
\[ U = \frac{q}{(n-2)(r^2 + b^2)^{(n-2)/2}}. \]  

(28)

The quasilocal mass of the dilaton AdS black hole can be calculated through the use of the subtraction method of modified Brown and York (BY) [34]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. In order to use the BY method the metric should have the form
\[ ds^2 = -W(\mathcal{R}) dt^2 + \frac{d\mathcal{R}^2}{V(\mathcal{R})} + \mathcal{R}^2 d\Omega^2. \]  

(29)

Thus, we should write the metric (6) in the above form. To do this, we perform the following transformation [35]:
\[ \mathcal{R} = \rho Y^{\gamma/2}. \]
It is a matter of calculations to show that the metric (6) may be written as (29) with the following $W$ and $V$:

$$W(\mathcal{R}) = N^2(\rho(\mathcal{R})) f^2(\rho(\mathcal{R})), $$
$$V(\mathcal{R}) = f^2(\rho(\mathcal{R})) \left( \frac{d\mathcal{R}}{d\rho} \right)^2 = \Upsilon(\gamma-2) \left[ 1 + \frac{1}{2} (\gamma(n-2) - 2) (1 - \Upsilon) \right]^2 f^2(\rho(\mathcal{R})).$$

The background metric is chosen to be the metric (29) with

$$W_0(\mathcal{R}) = V_0(\mathcal{R}) = f_0^2(\rho(\mathcal{R})) = k + \frac{\rho^2}{l^2} + \begin{cases} 
- \frac{2b_0\alpha^2}{(1+\alpha^2)l^2} + \frac{b_0^2\alpha^4}{(1+\alpha^2)^2l^2} & \text{for } n = 3 \\
- \frac{b_0^2\alpha^2}{(2+\alpha^2)l^2} & \text{for } n = 4 \\
0 & \text{for } n \geq 5
\end{cases} \quad (30)$$

As one can see from the above equation, the solutions for $n = 3$ and $n = 4$ have not "exact" asymptotic AdS behavior. Because of this point, we cannot use the AdS/CFT correspondence to compute the mass. Indeed, for $n \geq 5$ the metric is exactly asymptotically AdS, while for $n = 3, 4$ it is approximately asymptotically AdS. This is due to the fact that if one computes the Ricci scalar then it is not equal to $-n(n+1)/l^2$. It is well-known that the Ricci scalar for AdS spacetime should have this value (see e.g. [36]). Also, the metrics with $f_0^2(\rho)$ given by Eq. (30) for $n = 3$ and $n = 4$ do not satisfy the Einstein equation with the cosmological constant, while an AdS spacetime should satisfy the Einstein equation with cosmological constant. On the other side, at large $\rho$, the metric behaves as $\rho^2$ in all dimensions and therefore we used the word "approximately" asymptotically AdS.

To compute the conserved mass of the spacetime, we choose a timelike Killing vector field $\xi$ on the boundary surface $\mathcal{B}$ of the spacetime (29). Then the quasilocal conserved mass can be written as

$$\mathcal{M} = \frac{1}{8\pi} \int_{\mathcal{B}} d^2\varphi \sqrt{\sigma} \left\{ (K_{ab} - K h_{ab}) - (K^0_{ab} - K^0 h^0_{ab}) \right\} n^a \xi^b, \quad (31)$$

where $\sigma$ is the determinant of the metric of the boundary $\mathcal{B}$, $K_{ab}$ is the extrinsic curvature of the background metric and $n^a$ is the timelike unit normal vector to the boundary $\mathcal{B}$. In the context of counterterm method, the limit in which the boundary $\mathcal{B}$ becomes infinite ($\mathcal{B}_\infty$) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. Although the explicit function $f(\rho(\mathcal{R}))$ cannot be obtained, but at large $\mathcal{R}$ this can be done. Thus, one can calculate the mass per unit volume through the use of the above modified Brown and York formalism as

$$M = \frac{n-1}{16\pi} \left[ \epsilon^{n-2} + k \left( \frac{n-2-\alpha^2}{n-2+\alpha^2} \right) l^{n-2} \right]. \quad (32)$$
In the absence of a non-trivial dilaton field \((\alpha = 0)\), this expression for the mass reduces to the mass of the \((n + 1)\)-dimensional asymptotically AdS black brane.

Finally, we check the first law of thermodynamics for the black brane. In order to do this, we obtain the mass \(M\) as a function of extensive quantities \(S\) and \(Q\). Using the expression for the charge, the mass and the entropy given in Eqs. \((26)\), \((32)\) and \((20)\), we can obtain a Smarr-type formula per unit volume as

\[
M(S, Q) = \frac{(n - 1)}{16\pi} \left[ \frac{32\pi^2(n - 2 + \alpha^2)Q^2b^{2-n}}{(n - 1)(n - 2)^2} + k \left( \frac{n - 2 - \alpha^2}{n - 2 + \alpha^2} \right) b^{n-2} \right],
\]

where \(b = b(Q, S)\). One may then regard the parameters \(S\) and \(Q\) as a complete set of extensive parameters for the mass \(M(S, Q)\) and define the intensive parameters conjugate to \(S\) and \(Q\). These quantities are the temperature and the electric potential

\[
T = \left( \frac{\partial M}{\partial S} \right)_Q = \left( \frac{\partial M}{\partial b} \right)_Q \left( \frac{\partial b}{\partial r_+} \right)_Q,
\]

\[
U = \left( \frac{\partial M}{\partial Q} \right)_S = \left( \frac{\partial M}{\partial b} \right)_S \left( \frac{\partial b}{\partial r_+} \right)_S,
\]

where

\[
\left( \frac{\partial b}{\partial r_+} \right)_Q = -\left( \frac{\partial Z}{\partial b} \right)_Q,
\]

\[
Z = \left[ k - \frac{32\pi^2(n - 2 + \alpha^2)Q^2}{(n - 1)(n - 2)^2b^{2n-4}} \left( \frac{b}{r_+} \right)^{n-2} \right] \left[ 1 - \left( \frac{b}{r_+} \right)^{n-2} \right]^{1-\gamma(n-2)}
\]

\[
- \frac{r_+^2}{l^2} \left[ 1 - \left( \frac{b}{r_+} \right)^{n-2} \right]^\gamma.
\]

Straightforward calculations show that the intensive quantities calculated by Eqs. \((34)\) and \((35)\) coincide with Eqs. \((22)\) and \((28)\). Thus, these thermodynamics quantities satisfy the first law of black brane thermodynamics,

\[
dM = TdS + UdQ.
\]

**IV. STABILITY IN THE CANONICAL ENSEMBLE**

Finally, we study the thermal stability of the solutions in the canonical ensemble. In particular, we will see that the scalar dilaton field makes the solution unstable. The stability
of a thermodynamic system with respect to small variations of the thermodynamic coordinates is usually performed by analyzing the behavior of the entropy $S(M, Q)$ around the equilibrium. The local stability in any ensemble requires that $S(M, Q)$ be a convex function of the extensive variables or its Legendre transformation must be a concave function of the intensive variables. The stability can also be studied by the behavior of the energy $M(S, Q)$ which should be a convex function of its extensive variable. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of $M(S, Q)$ with respect to its extensive variables $X_i$, $H^M_{X_iX_j} = [\partial^2 M/\partial X_i \partial X_j]$. In our case the mass $M$ is a function of entropy and charge. The number of thermodynamic variables depends
FIG. 9: $10^{-2}(\partial^2 M/\partial S^2)_Q$ (solid line) and $T$ (dashed line) versus $\alpha$ for $b = 0.2$, $n = 4$, $l = 1$ and $r_+ = 0.1$.

FIG. 10: $10^2(\partial^2 M/\partial S^2)_Q$ (solid line) and $T$ (dashed line) versus $\alpha$ for $b = 0.2$, $n = 4$, $l = 1$ and $r_+ = 5$.

on the ensemble that is used. In the canonical ensemble, the charge is a fixed parameter and therefore the positivity of the $(\partial^2 M/\partial S^2)_Q$ is sufficient to ensure local stability. In Figs. [7-10] we show the behavior of the $(\partial^2 M/\partial S^2)_Q$ as a function of the coupling constant parameter $\alpha$ for different values of the size of black brane $r_+$. In order to investigate the stability of black branes, we plot both $(\partial^2 M/\partial S^2)_Q$ and $T$ in one single figure for various values of $r_+$ or $\alpha$. Of course, one should note that in these figures, only the positive values of temperature associated to the event horizon of the black branes, and negative values of temperature belong to inner horizon which we are not interested in. We discuss these figures for $k = 0$ and $k = -1$, separately.
FIG. 11: $F^{\text{off}}$ versus $r_+$ for $b = 0.2$, $n = 4$, $l = 1$, $k = 0$, $\alpha = 1.2$ and $T = 0.42$.

$k = 0$:

As we discussed in the last section, small black brane with a radius $r_+$ exist when $\alpha > \alpha_{\text{ext}}$. Figures 7 and 8 show that these small black branes are stable provided $\alpha > \alpha_{\text{crit}}$. On the other side, large black branes are stable as one may see in Figs. 9 and 10.

$k = -1$:

Figures 7 and 8 show that small black branes exist only for medium values of $\alpha$ ($\alpha_{\text{ext}} < \alpha < \alpha_{\text{2ext}}$), but they are unstable. On the other side, large black branes are stable as one may see in Figs. 9 and 10.

In order to confirm the stability analysis of the black branes, one can also use the generalized free energy [38]

$$F^{\text{off}}(b, \alpha, r_+, T) = M(b, \alpha, r_+) - T S_+(b, \alpha, r_+),$$

which applies to any value of $r_+$ with a fixed temperature $T$. This off-shell free energy reduces to the on-shell free energy at $T = T_+$:

$$F = M - T_+ S_+,$$

which is the Legendre transform of $M$ with respect to $S$. As an example of using the stability analysis by off-shell free energy, we plot $F^{\text{off}}$ versus $r_+$ for $k = 0$, $\alpha = 1.2$ and $\alpha = 4$. These are plotted in Figs. 11 and 12, which show that the black brane is unstable for $\alpha = 1.2$, while it is stable for $\alpha = 4$. These figures confirm the stability analysis of Fig. 8. One may also confirm other stability analysis given in this section.
FIG. 12: $F^\text{off}$ versus $r_+$ for $b = 0.2$, $n = 4$, $l = 1$, $k = 0$, $\alpha = 4$ and $T = 0.0001$.

V. CONCLUSIONS

In $(n + 1)$-dimensions, when the $(n - 1)$-sphere of black hole event horizons is replaced by an $(n - 1)$-dimensional hypersurface with zero or negative constant curvature, the black hole is referred to as a topological black brane. The construction and analysis of these exotic black branes in AdS space is a subject of much recent interest. This is primarily due to their relevance for the AdS/CFT correspondence. In this paper, we further generalized these exotic black brane solutions by including a dilaton and the electrodynamic fields in the action. We obtained a new class of $(n + 1)$-dimensional ($n \geq 3$) topological black brane solutions in Einstein-Maxwell-dilaton gravity in the background of AdS spaces. Indeed, the dilaton potential plays a crucial role in the existence of these black brane solutions, as the negative cosmological constant does in the Einstein-Maxwell theory. In the absence of a dilaton field ($\alpha = \gamma = 0$), our solutions reduce to the $(n + 1)$-dimensional topological black brane solutions presented in [14]. We computed the entropy, temperature, charge, mass, and electric potential of the topological dilaton black branes and found that these quantities satisfy the first law of thermodynamics.

For the thermodynamical analysis of the black branes, we divide the black branes into three classes. The black brane is said to be "small" when the radius of event horizon is much smaller than the parameter $b$, it is called "medium" if $r_+$ and $b$ have the same order and it is called "large" for large values of $r_+$ with respect to $b$. The radius of event horizon is always larger or equal to $r_{\text{ext}}$ for the black branes with flat horizon, while it is in the range
$r_+ < r_{\text{ext}}$ or $r_+ > r_{\text{ext}}$ for hyperbolic black branes. We found that the existence of $r_{\text{ext}}$ for $k = 0$, and $r_{\text{ext}}$ for $k = -1$ depends on the value of $\alpha$. For the case of $k = 0$ with large values of $\alpha$ ($\alpha > \alpha_{\text{ext}}$), one can have black brane with any size. This feature shows that dilaton change the thermodynamics of black branes drastically. In this case, the temperature goes to infinity as $r_+$ goes to zero for $n > 4$, while it approaches to a constant for $n = 4$. For the case of $k = -1$, one may have small black branes provided $\alpha_{\text{ext}} < \alpha < \alpha_{\text{crit}}$, while large black branes exist for any value of $\alpha$. Again, this is a drastic change in the properties of the solutions because of the dilaton field. We analyzed the thermal stability of the solutions in the canonical ensemble by finding a Smarr-type formula and considering $(\partial^2 M/\partial S^2)_Q$ for the charged topological dilaton black brane solutions in $(n+1)$ dimensions. We showed that for the case of $k = 0$, small black branes are unstable for $\alpha_{\text{ext}} < \alpha < \alpha_{\text{crit}}$, while they are stable for $\alpha > \alpha_{\text{crit}}$. Also, in this case large black branes are stable for arbitrary $\alpha$. For the case of $k = -1$, small black branes are unstable, while large ones are stable. That is dilaton with the potential given in Eq. (2), changes the stability of the black branes drastically.

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[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998);
    E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998);
    S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998).
[2] M. Hennigson and K. Skenderis, J. High Energy Phys. 07, 023 (1998);
    R. B. Mann, Phys. Rev. D 60, 104047 (1999);
    R. Emparan, C. V. Johnson and R. Meyers, Phys. Rev. D 60, 104001 (1999);
    R. B. Mann, Phys. Rev. D 61, 084013 (2000);
    S.Y. Hyun, W.T. Kim and J. Lee, Phys. Rev. D 59, 084020 (1999);
    V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208, 413 (1999);
    S. N. Solodukhin, Phys. Rev. D 62, 044016 (2000);
P. Kraus, F. Larsen and R. Siebelink, Nucl. Phys. B 563, 259 (1999);  
A. M. Awad and C. V. Johnson, Phys. Rev. D 61, 084025 (2000).

[3] S. Nojiri and S. D. Odintsov, Phys. Lett. B 521, 87 (2001); Erratum-ibid. B 542, 301 (2002);  
S. Nojiri and S. D. Odintsov, Phys. Rev. D 66, 044012 (2002);  
S. Nojiri, S. D. Odintsov and S. Ogushi, Int. J. Mod. Phys. A 17, 4809 (2002);  
S. Nojiri and S. D. Odintsov, Int. J. Mod. Phys. A 18, 2001 (2003).

[4] E. Witten, Adv. Theor. Math. Phys. 2, 505 (1998);  
S. Das and R. B. Mann, J. High Energy Phys. 08, 033 (2000);  
A. Strominger, J. High Energy Phys. 10, 034 (2001);  
A. Strominger, J. High Energy Phys. 11, 049 (2001);  
V. Balasubramanian, P. Horova and D. Minic, J. High Energy Phys. 05, 043 (2001);  
D. Klemm, Nucl. Phys. B 625, 295 (2002);  
V. Balasubramanian, J. D. Boer and D. Minic, Phys. Rev D 65, 123508 (2002);  
S. Nojiri and S. D. Odintsov, Phys. Lett. B 519, 145 (2001);  
S. Nojiri and S. D. Odintsov, J. High Energy Phys. 12, 033 (2001);  
S. Nojiri and S. D. Odintsov, Phys. Lett. B 523, 165 (2001);  
S. Nojiri and S. D. Odintsov, Phys. Lett. B 528, 169 (2002);  
T. Shiromizu, D. Ida and T. Torii, J. High Energy Phys. 11, 010 (2001);  
R. G. Cai, Phys. Lett. B 525, 331 (2002);  
R. G. Cai, Nucl. Phys. B 628, 375 (2002);  
R. G. Cai, Y. S. Myung and Y. Z. Zhang, Phys. Rev. D 65, 084019 (2002);  
R. Bousso, A. Maloney and A. Strominger, Phys. Rev. D 65, 104039 (2002);  
A. M. Ghezelbash and R. B. Mann, J. High Energy Phys. 01, 005 (2002);  
M. H. Dehghani, Phys. Rev. D 65, 104003 (2002);  
M. H. Dehghani and R. B. Mann, Phys. Rev. D 64, 044003 (2002);  
M. H. Dehghani, Phys. Rev. D 65, 104030 (2002);  
M. H. Dehghani, Phys. Rev. D 65, 124002 (2002);  
A. M. Ghezelbash, D. Ida, R. B. Mann and T. Shiromizu, Phys. Lett. B 535, 315 (2002).

[5] S. W. Hawking and G. F. Ellis. The large scale structure of spacetime. Cambridge University Press, Cambridge, England, (1973).  

[6] S. W. Hawking. Commun. Math. Phys. 25, 152 (1972).
[7] J. L. Friedman, K. Schleich and D. M. Witt, Phys. Rev. Lett. 71, 1486 (1993).
[8] J. L. Friedman, K. Schleich, and D. M. Witt, Phys. Rev. Lett. 75, 1872 (1995).
[9] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).
[10] D. Birmingham, Class. Quant. Gravit. 16, 1197 (1999).
[11] J. P. S. Lemos, Phys. Lett. B 353, 46 (1995).
[12] R. G. Cai and Y. Z. Zhang, Phys. Rev. D 54, 4891 (1996);
    L. Vanzo, Phys. Rev. D 56, 6475 (1997).
[13] D. R. Brill, J. Louko, and P. Peldan, Phys. Rev. D 56, 3600 (1997).
[14] R. G. Cai and K. S. Soh, Phys. Rev. D 59, 044013 (1999).
[15] R. G. Cai, J.Y. Ji, and K. S. Soh, Phys. Rev. D 57, 6547 (1998).
[16] G. W. Gibbons and K. Maeda, Nucl. Phys. B 298, 741 (1988);
    T. Koikawa and M. Yoshimura, Phys. Lett. B 189, 29 (1987);
    D. Brill and J. Horowitz, Phys. Lett. B 262, 437 (1991).
[17] D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. D 43, 3140 (1991);
    R. Gregory and J. A. Harvey, Phys. Rev. D 47, 2411 (1993);
    G. T. Horowitz and A. Strominger, Nucl. Phys. B 360, 197 (1991).
[18] S. J. Poletti, D. L. Wiltshire, Phys. Rev. D 50, 7260 (1994);
    S. J. Poletti, J. Twamley and D. L. Wiltshire, Phys. Rev. D 51, 5720 (1995);
    S. Mignemi and D. L. Wiltshire, Phys. Rev. D 46, 1475 (1992).
[19] K. C. K. Chan, J. H. Horne and R. B. Mann, Nucl. Phys. B 447, 441 (1995).
[20] G. Clement, D. Gal’tsov and C. Leygnac, Phys. Rev. D 67, 024012 (2003).
[21] A. Sheykhi, M. H. Dehghani, N. Riazi, Phys. Rev. D 75, 044020 (2007);
    A. Sheykhi, M. H. Dehghani, N. Riazi and J. Pakravan Phys. Rev. D 74, 084016 (2006);
    A. Sheykhi, N. Riazi, Phys. Rev. D 75, 024021 (2007).
[22] A. Sheykhi, Phys. Rev. D 76, 124025 (2007);
    A. Sheykhi, Phys. Lett. B 662, 7 (2008).
[23] M. H. Dehghani, J. Pakravan and S. H. Hendi, Phys. Rev. D 74, 104014 (2006);
    M. H. Dehghani, et al., J. Cosmol. Astropart. Phys. 02, 020 (2007);
    S. H. Hendi, J. Math. Phys. 49, 082501 (2008).
[24] C. J. Gao, S. N. Zhang, Phys. Lett. B 605, 185 (2005);
    C. J. Gao, S. N. Zhang, Phys. Lett. B 612 127 (2005).
[25] C. J. Gao and S. N. Zhang, arXiv:astro-ph/0605682.

[26] E. Radu, D. H. Tchrakian, Class. Quant. Grav. 22, 879 (2005).

[27] S. B. Giddings, Phys. Rev. D 68, 026006 (2003).

[28] A. Sheykhi, Phys. Rev. D 78, 064055 (2008);
    A. Sheykhi, Phys. Lett. B 672, 101 (2009);
    A. Sheykhi, M. Allahverdizadeh, Phys. Rev. D 78, 064073 (2008).

[29] A. Sheykhi, M. H. Dehghani and S. H. Hendi, Phys. Rev. D 81, 084040 (2010).

[30] J. D. Beckenstein, Phys. Rev. D 7, 2333 (1973);
    S. W. Hawking, Nature (London) 248, 30 (1974);
    G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).

[31] C. J. Hunter, Phys. Rev. D 59, 024009 (1999);
    S. W. Hawking, C. J Hunter and D. N. Page, Phys. Rev. D 59, 044033 (1999);
    R. B. Mann Phys. Rev. D 60, 104047 (1999).

[32] M. Cvetic, S. Nojiri and S. D. Odintsov, Nucl. Phys. B 628, 295 (2002); J. E. Lidsey, S. Nojiri
    and S. D. Odintsov, J. High Energy Phys. 0206, 026 (2002).

[33] M. Cvetic and S. S. Gubser, J. High Energy Phys. 04, 024 (1999);
    M. M. Caldarelli, G. Cognola and D. Klemm, Class. Quant. Gravit. 17, 399 (2000).

[34] J. Brown and J. York, Phys. Rev. D 47, 1407 (1993);
    J.D. Brown, J. Creighton, and R. B. Mann, Phys. Rev. D 50, 6394 (1994).

[35] M. H. Dehghani and A. Bazrafshan, Int. J. Mod. Phys. D 19, 293 (2010).

[36] S. Weinberg, Gravitation and cosmology, (John Wiley 1972, Chapter 15).

[37] S. S. Gubser and I. Mitra, J. High Energy Phys. 08, 018 (2001).

[38] Y. S. Myung, Phys. Lett. B 645, 369 (2007);
    L. Cappiello and W. Muck, Phys. Lett. B 522, 139 (2001).