TOWARDS THE COHOMOLOGY OF MODULI SPACES OF HIGHER GENUS STABLE MAPS

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Abstract. We prove that the orbifold desingularization of the moduli space of stable maps of genus $g = 1$ recently constructed by Vakil and Zinger has vanishing rational cohomology groups in odd degree $k < 10$.

1. Let $\overline{M}_{g,n}(\mathbb{P}^r, d)$ be the moduli space of stable maps of degree $d$ from $n$-pointed curves of genus $g$ to the projective space $\mathbb{P}^r$. For a comprehensive introduction to this fascinating geometric object we refer to the classical text [7], which not only contains a careful construction of Kontsevich moduli spaces but also outlines their crucial application to Gromov-Witten theory.

For $g = 0$ it turns out that $\overline{M}_{0,n}(\mathbb{P}^r, d)$ carries a natural structure of smooth orbifold, in particular its rational cohomology is well-defined and well-behaved. Indeed, in the last few years there has been a flurry of research about the cohomological properties of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ after the first steps moved in [2]: it is worth mentioning at least the contributions [1], [3], [20], [21], [22], [16], [17], [18], [19], [9], [4], [5], [6].

For higher genus $g > 0$, instead, $\overline{M}_{g,n}(\mathbb{P}^r, d)$ can be arbitrarily singular (see [24]) and it is in general nonreduced with several components of exceptional dimension. As a consequence, a cohomological investigation of the underlying topological spaces has never been addressed and it seems to be completely out of reach. However, at least in the case $g = 1$, the beautiful construction performed in [25] has recently opened a new frontier to the research in the field.

Namely, in [25] it is shown that the closure $\overline{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d)$ of the stratum $\mathcal{M}^0_{1,n}(\mathbb{P}^r, d)$ corresponding to stable maps with smooth domain allows an orbifold desingularization

$$\overline{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d)$$

which can be explicitly described as a sequence of blow-ups along smooth centers. Moreover, the natural $(\mathbb{C}^*)^{r+1}$-action on $\overline{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d)$

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lifts to $\widetilde{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d)$ and the corresponding fixed loci are explicitly determined. As a by-product, the rational cohomology groups

$$H^k(\widetilde{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d), \mathbb{Q})$$

are well-defined and can be approached via localization techniques.

In the present paper we take this direction and prove the following result:

**Theorem 1.** We have $H^k(\widetilde{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d), \mathbb{Q}) = 0$ for every $n \geq 0$, $r \geq 1$, and odd $k < 10$.

We point out that the natural surjective map

$$\widetilde{\mathcal{M}}^0_{1,n}(\mathbb{P}^r, d) \to \mathcal{M}_{1,n}$$

together with the well-known fact that $H^{11}(\mathcal{M}_{1,11}, \mathbb{Q}) \neq 0$ (see, for instance, [23], proof of Corollary 4.7, or [10], Proposition 2) show that in the above statement the bound on the degree $k$ is sharp.

Our proof is based on a suitable version of the Bialynicki-Birula decomposition for singular varieties ([15] and [20]), which reduces the claim of Theorem 1 to the analogous claim on the fixed locus. Next, a careful analysis of its connected components allows us to conclude thanks to previous results on the cohomology of the moduli spaces of stable curves of genus $g \leq 1$ ([13] and [8]).

We work over the field $\mathbb{C}$ of complex numbers.

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2.

We begin by recalling some more or less well-known facts about the Bialynicki-Birula decomposition for orbifolds.

**Proposition 1.** Let $X$ be a smooth projective orbifold with a $\mathbb{C}^*$-action, let $F$ be the fixed locus and let $F_i$ denote its connected components. Then

(i) $X$ is the disjoint union of locally closed subvarieties $S_i$ such that every $S_i$ retracts onto the corresponding $F_i$ and

$$\overline{S_i} \subseteq \bigcup_{j \geq i} S_j$$

(ii) the Betti numbers of $X$ are

$$h^m(X) = \sum_i h^{m-2n_i}(F_i)$$

where $n_i$ is the codimension of $S_i$. 
Proof. (i) follows from [15], (1.14) and (1.15), pp. 388-389; (ii) follows from (i) and [20], Lemma 6 (i).

We also need the following technical result: please refer to [25], Section 2.1, for all non standard definitions and notation.

Lemma 1. Fix \( N \in \mathbb{N} \). Let \( Y \) be a smooth (orbi)variety and let \( \{X_1, \ldots, X_n\} \) be a properly intersecting collection of subvarieties of \( Y \) such that \( \bigcap_{i \in I} X_i \) is smooth for every \( I \subseteq \{1, \ldots, n\} \) and \( H^k(\bigcap_{i \in I} X_i, \mathbb{Q}) = 0 \) for every odd \( k < N \). Let \( Y_1 := Y \) and for \( i = 1, \ldots, n \) let \( f_i : Y_{i+1} \to Y_i \) be the blow-up of the proper transform of \( X_i \) in \( Y_i \).

(i) If \( X_j^i \) denotes the proper transform of \( X_i \) in \( Y_j \), then \( \bigcap_{i \in I} X_j^i \) is smooth for every \( j \geq 1 \) and every \( I \subseteq \{j \leq i \leq n\} \) and \( H^k(\bigcap_{i \in I} X_j^i, \mathbb{Q}) = 0 \) for every odd \( k < N \).

(ii) If moreover \( H^k(Y, \mathbb{Q}) = 0 \) for every odd \( k < N \), then \( H^k(Y_i, \mathbb{Q}) = 0 \) for every odd \( k < N \) and for every \( i \leq n + 1 \).

Proof. (i) By induction on \( j \), the case \( j = 1 \) holding by assumption. We have
\[
\bigcap_{i \in I} X_j^i = \bigcap_{i \in I} \text{Pr}_{X_{j-1}^i}^{-1}X_{j-1}^i = \text{Pr}_{X_{j-1}^i}^{-1}\bigcap_{i \in I} X_{j-1}^i = \text{Bl}_Z X
\]
with \( Z := \bigcap_{i \in I} X_{j-1}^i \cap X_{j-1}^i \) and \( X := \bigcap_{i \in I} X_i^{j-1} \). Here the first equality holds by definition, the second one by [25], Lemma 2.3 (2), and the third one by [25], Lemma 2.3 (1).

Now, it is well-known how to compare cohomology groups after a blow-up along a smooth subvariety of an orbifold (see [11], p. 605 and Proposition on p. 606, and [14], footnote on p. 514, for the orbifold case).

In particular, it follows that if both \( H^k(X, \mathbb{Q}) = 0 \) and \( H^k(Z, \mathbb{Q}) = 0 \) for odd \( k < N \), then \( H^k(\text{Bl}_Z X, \mathbb{Q}) = 0 \) for odd \( k < N \). Hence in our case we obtain by induction both the smoothness and the vanishing result.

(ii) By induction on \( i \), the case \( i = 1 \) holding by assumption. We have
\[
Y_{i+1} = \text{Bl}_{X_i} Y_i
\]
with \( H^k(Y_i, \mathbb{Q}) = 0 \) for odd \( k < N \) by induction and \( H^k(X_i, \mathbb{Q}) = 0 \) for odd \( k < N \) by (i). Hence the claim follows from the cohomological properties of blow-ups recalled above.

Proof of Theorem 1. By [25], Theorem 1.1, \( \tilde{\mathcal{M}}_{0, n}^0(\mathbb{P}^r, d) \) is a smooth projective orbifold and the natural \((\mathbb{C}^*)^{r+1}\)-action on the projective space \( \mathbb{P}^r \) lifts to an action on \( \tilde{\mathcal{M}}_{0, n}^0(\mathbb{P}^r, d) \). By Proposition 1(ii), in order to prove Theorem 1 it is enough to show that the odd cohomology of the fixed loci vanishes in degree \( k < 10 \).
By [25], Section 1.4, such fixed loci are either

\[(1) \prod \mathcal{M}_{g,m} \]

with \( g \leq 1 \) and \( m \in \mathbb{N} \) (see [25], p. 15, and [12], p. 541), or

\[(2) \prod \mathcal{M}_{0,i} \times \mathbb{P}^j \times \tilde{\mathcal{M}}_{1,I,J} \]

with \( i, j \in \mathbb{N} \) (see [25], p. 19), where \( \tilde{\mathcal{M}}_{1,I,J} \) is constructed in [25], pp. 26–27. Roughly speaking, \( \tilde{\mathcal{M}}_{1,I,J} \) is obtained from the moduli space \( \mathcal{M}_{1,I\cup J} \) of stable curves of genus 1 by successively blowing up certain subvarieties \( \mathcal{M}_{1,\rho} \) (where \( \rho \) is a suitable index) and their proper transforms (see [25], p. 25). We just point out a crucial fact: from the definition of \( \mathcal{M}_{1,\rho} \) (see [25], p. 23) it follows that

\[ \mathcal{M}_{1,\rho} = \mathcal{M}_{1,h} \times \prod \mathcal{M}_{0,k} \]

with \( h, k \in \mathbb{N} \). More generally, if \( \bigcap \mathcal{M}_{1,\rho_{\alpha}} = \emptyset \) for some subset of indices \( \{\rho_{\alpha}\} \) then

\[(3) \bigcap \mathcal{M}_{1,\rho_{\alpha}} = \mathcal{M}_{1,s} \times \prod \mathcal{M}_{0,t} \]

with \( s, t \in \mathbb{N} \).

Now, recall that all odd degree cohomology of \( \overline{\mathcal{M}}_{0,n} \) vanishes by Keel’s results (see [13]) and that \( H^k(\overline{\mathcal{M}}_{1,n}, \mathbb{Q}) = 0 \) for odd \( k < 10 \) according to Getzler (see [8] and [10], eq. (4)). Hence the required vanishing trivially follows for fixed loci of type (1) and is reduced to the case of \( \tilde{\mathcal{M}}_{1,I,J} \) for fixed loci of type (2). On the other hand, by (3) we can apply Lemma (1) (ii) to our situation with \( N = 10, Y = \overline{\mathcal{M}}_{1,I\cup J}, X_i = \mathcal{M}_{1,\rho_i} \), and \( Y_{n+1} = \tilde{\mathcal{M}}_{1,I,J} \), so once again Getzler’s claims allow us to conclude and the proof is over.

\[ \square \]

References

[1] K. Behrend and A. O’Halloran: On the cohomology of stable map spaces. Invent. Math. 154 (2003), no. 2, 385–450.
[2] G. Bini and C. Fontanari: On the cohomology of \( \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, d) \). Commun. Contemp. Math. 4 (2002), no. 4, 751–761.
[3] G. Bini and C. Fontanari: Calculating cohomology groups of \( \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \). Ann. Mat. Pura Appl. (4) 182 (2003), no. 3, 337–344.
[4] J. A. Cox: An additive basis for the Chow ring of \( \overline{\mathcal{M}}_{0,2}(\mathbb{P}^r, 2) \). Pre-print math.AG/0501322.
[5] J. A. Cox: A presentation for the Chow ring of \( \overline{\mathcal{M}}_{0,2}(\mathbb{P}^r, 2) \). Pre-print math.AG/0504575.
[6] J. A. Cox: A presentation for the Chow ring \( A^*(\overline{\mathcal{M}}_{0,2}(\mathbb{P}^1, 2)) \) Pre-print math.AG/0505112.
[7] W. Fulton and R. Pandharipande: Notes on stable maps and quantum cohomology. Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[8] E. Getzler: The semi-classical approximation for modular operads. Comm. Math. Phys. 194 (1998), no. 2, 481–492.

[9] E. Getzler and R. Pandharipande: The Betti numbers of $\overline{M}_{0,n}(r,d)$. J. Algebraic Geom. 15 (2006), no. 4, 709–732.

[10] T. Graber and R. Pandharipande: Constructions of nontautological classes on moduli spaces of curves. Michigan Math. J. 51 (2003), no. 1, 93–109.

[11] P. Griffiths and J. Harris: Principles of algebraic geometry. Pure and Applied Mathematics. Wiley-Interscience (John Wiley and Sons), New York, 1978.

[12] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow: Mirror symmetry. Clay Mathematics Monographs, 1. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003.

[13] S. Keel: Intersection theory of moduli space of stable $n$ pointed curves of genus zero. Trans. Amer. Math. Soc. 330 (1992), no. 2, 545–574.

[14] Y.-H. Kiem and J. Li: Desingularizations of the moduli space of rank 2 bundles over a curve. Math. Ann. 330 (2004), no. 3, 491–518.

[15] F. Kirwan: Intersection homology and torus actions. J. Amer. Math. Soc. 1 (1988), no. 2, 385–400.

[16] A. Mustata and M. A. Mustata: Intermediate Moduli Spaces of Stable Maps. Preprint [math.AG/0405569]

[17] A. Mustata and M. A. Mustata: The Chow ring of $\overline{M}_{0,m}(n,d)$. Preprint [math.AG/0507464]

[18] A. Mustata and M. A. Mustata: Universal relations on stable map spaces in genus zero. Preprint [math.AG/0607431]

[19] A. Mustata and M. A. Mustata: Tautological rings of stable map spaces. Preprint [math.AG/0607432]

[20] D. Oprea: Tautological classes on the moduli spaces of stable maps to projective spaces. Preprint [math.AG/0404284]

[21] D. Oprea: Divisors on the moduli spaces of stable maps to flag varieties and reconstruction. J. Reine Angew. Math. 586 (2005), 169–205.

[22] D. Oprea: The tautological rings of the moduli spaces of stable maps to flag varieties. J. Algebraic Geom. 15 (2006), no. 4, 623–655.

[23] M. Pikaart: An orbifold partition of $\overline{M}_g^{\text{arb}}$. The moduli space of curves (Texel Island, 1994), 467–482, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[24] R. Vakil: Murphy’s law in algebraic geometry: badly-behaved deformation spaces. Invent. Math. 164 (2006), no. 3, 569–590.

[25] R. Vakil and A. Zinger: A Desingularization of the Main Component of the Moduli Space of Genus-One Stable Maps into $\mathbb{P}^n$. Preprint [math.AG/0603353]

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