SOME OBSTRUCTIONS ON THE SUBGROUPS OF THE BRIN-THOMPSON GROUPS AND A SELECTION OF TWISTED BRIN-THOMPSON GROUPS

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Abstract. Motivated by Burillo, Cleary and Röver’s summary on obstructions of subgroups of Thompson’s group \( V \), we explored the higher dimensional version of the groups, Brin-Thompson groups \( nV \) and \( SV \), a class of infinite dimensional Brin-Thompson groups and an easy class of the twisted version of the Brin-Thompson groups \( SV_G \) with some certain condition. We found that they have similar obstructions as Thompson’s group \( V \) on the torsion subgroups and a selection of the interesting Baumslag-Solitor groups are excluded as the subgroups of \( SV \) and \( SV_G \). We also discuss a little on the group \( \mathcal{S}V \), an even larger class relaxing some of the ”finiteness condition” and observe that some of the restrictions on subgroups disappear.

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Thompson’s groups $F$, $T$ and $V$ were first introduced from a logic aspect by Richard Thompson and later turned out to be very counter-intuitive examples of groups. These groups were generalised to many different infinitely families, among which the higher dimensional Thompson groups or the Brin-Thompson groups defined by Brin [9] in the beginning of this century are one of the interesting families of generalisations of Thompson’s groups that has not yet been fully explored. The higher dimensional Thompson groups can be roughly described as the group of the self-homeomorphisms of the product of $n$ copies of the Cantor sets or the Cantor spaces.

These groups are proved to be simple, finitely presented [9] and have the finiteness property $FP_{\infty}$ [16]. They also contain many interesting class of subgroups such as Right-angled Artin groups ($RAAGs$) and Right-angled Coxeter groups ($RACGs$), the products of $Z$ with more freedom compare to Thompson’s groups [3]. Belk and Matucci [6] have generalised this family to some larger class further, the twisted Brin-Thompson groups and deduced finiteness property $FP_{\infty}$ and remarkable simplicity results of these groups [18].

This work is motivated by Burillo, Cleary and Röver’s summary [11] on the constraints to the subgroups of Thompson’s group $V$. We investigate the higher dimensional versions from a more combinatorial description of the groups and found some constraints to the subgroups of $nV$ and some “easy” version of $nV_C$ namely, infinite torsion subgroups and certain Baumslag-Solitar group are not inside, which is comparable to the constraints to the usual Thompson’s groups.

Moreover, the result give some indications on the embedding problems of the groups into the Brin-Thompson groups and a relative simple version of the twisted Brin-Thompson’s groups. We will give a more precise description in the following subsection.

1.1. Thompson’s group $V$. We first introduce the group $V$ briefly as follows.

Definition 1.1 (Thompson’s group $V$). $V$ is the group of the right-continuous bijections from the unit interval $[0,1]$ to itself which are differentiable except at finitely many dyadic breakpoints, such that the slope of each subinterval is the powers of $2$.

From the definition, we see that the interval $[0,1]$ is divided so that the subintervals are only in the form $\left[\frac{\ell}{2^k}, \frac{\ell+1}{2^k}\right]$ where $\ell + 1 \leq 2^k$ and $k, \ell \in \mathbb{N} \cup \{0\}$. We call such intervals dyadic intervals. The dyadic intervals in the unit interval $[0,1]$ can be identified with the set of all binary words in $\{0,1\}$. We view $C \times C$ as a subset of the unit square $I^2 = [0,1] \times [0,1]$.

1.2. Dyadic blocks, Brin-Thompson groups and Basics. Now we want to generalise the group $V$ to some higher dimensional families. We consider the “dyadic blocks” which are called “patterns” in Brin’s definition and called “dyadic brick” in Belk, Matucci and Zaremskiy’s definition [6]. The dyadic blocks can be seen as the generalisations of the dyadic intervals.

We first follow Brin’s original description [9] and zoom into the case of $2V$. We take the Cantor set $C$ as described in the previous paragraph and identify the points of $C$ with the set of all binary words in $\{0,1\}$. We view $C \times C$ as a subset of the unit square $I^2 = [0,1] \times [0,1]$. 

1. Introduction
A 2-dyadic block (which are called “pattern” in Brin’s original definition) is identified with $I^2$ having a finite set of the rectangles in it with pairwise disjoint, non-empty interiors, with sides parallel to the sides of $I^2$ and whose union is all of $I^2$. The trivial 2-dyadic block is the square $I^2$ itself.

We denote this trivial 2-dyadic blocks by $B(\epsilon, \epsilon)$ where $\epsilon$ represent the empty binary word and each of the rectangles inside $B(\epsilon, \epsilon)$ by $B(\omega_0, \omega_1)$ where $\omega_0, \omega_1 \in \{0, 1\}^*$ are some binary words of finite length associated with the construction below.

Define inductively general 2-dyadic blocks by dividing $I^2$ into rectangles with horizontal or vertical lines each at one step such that the sides of the rectangles are the finitely many dyadically subdivided intervals.

When we divide $I^2$ by a horizontal line segment, we denote the rectangle on the top by $B(\epsilon, 0)$ and the rectangle at the bottom by $B(\epsilon, 1)$. Here the first coordinate in this representation of the dyadic blocks somehow corresponding to the subdivision in the horizontal part by dyadic line segments. The notation for the rectangles constructed by the vertical subdivision are $B(0, \epsilon)$ and $B(1, \epsilon)$. 

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**Figure 1.**

**Figure 2. The trivial pattern**

**Figure 3. The partitions**
Each rectangle \( B(\cdot, \cdot) \) corresponds to a closed and open subset of \( C \times C \) and \( I^2 \) corresponds to the union of the all the rectangles which is \( C \times C \). Here we assign to each rectangle whose union form \( B(\epsilon, \epsilon) \), a number \( i \in \mathbb{N} \) independent of the subdivision and denote the rectangle by \( B_i(\cdot, \cdot) \).

Now we associate the Cantor space \( C^2 \) with the dyadic block description as follows: take \( \varphi_i = (\varphi_i(1), \varphi_i(2)) \) where \( \varphi_i(1) \) and \( \varphi_i(2) \) are finite binary strings \( \{0, 1\}^* \). We denote by

\[
B_i(\varphi) = \{(\varphi_i(1)\kappa(1), \varphi_i(2)\kappa(2)) \mid (\kappa(1), \kappa(2)) \in C^2\}
= B(\varphi(1), \varphi(2))
\]

one of the rectangles in the subdivision and denote by \( B(\varphi) = \{B_i(\varphi)\} \) the 2-dyadic block with finitely many rectangles as disjoint subdivided regions associated to \( \varphi \) where \( i \in \{0, \ldots, m-1\} \) and \( m \) is the label of the rectangles in the 2-dyadic blocks. \( B(\varphi) \) is called the cone on \( \varphi \) in Zaremsky’s definition [18] associated to \( C^2 \) by canonical homeomorphism

\[
h_\varphi : C^2 \mapsto B(\varphi)
\]

taking the pair \((\kappa(1), \kappa(2))\) to \((\varphi_i(1)\kappa(1), \varphi_i(2)\kappa(2))\) for all \( i \) and \( \kappa(1), \kappa(2) \) are some infinite strings in \( C \).

**Figure 4.** The numbering the patterns

For \( B(\varphi) \) and \( B(\psi) \), two labeled 2-dyadic blocks with the same number of rectangles, we label the rectangles as mentioned above and then obtain a self-homeomorphism of \( C \times C \) mapping each open or closed set in \( C \times C \) associated with rectangles having the same labelings affinely.

The meaning of the word “affinely” here can be interpreted as “affine homotheties” of rectangles or, more precisely, affine maps preserving the horizontal and vertical directions: when we regard a trivial 2-dyadic block \([0, 1] \times [0, 1] \) as a square embedded in the Euclidean plane \( \mathbb{R}^2 \) with vertices \((0, 0), (0, 1), (1, 0) \) and \((1, 1)\), then partitions defined above are the line segments embedded in the square with length \( \frac{1}{2^k} \) for some integer \( k \).

The rectangle \( B_i(\varphi) \) maps to \( B_i(\psi) \) affinely means that there is an affine linear transformation that maps every point \( B_i(\varphi) \) in the source to every point in \( B_i(\psi) \) in the target. We call these rectangles the sub 2-dyadic blocks or subblocks in the rest of the paper.

This affine map can be regarded as a collection of prefix replacement maps taking each subblocks \( \{B_i(\varphi(1), \varphi(2))\} \) in the source to each subblocks \( \{B_i(\psi(1), \psi(2))\} \) in the target together with canonical homeomorphisms, they provide the self homeomorphisms we want, i.e.

\[
h_\psi \circ h_\varphi^{-1} : B(\varphi) \mapsto B(\psi).
\]

**Definition 1.2 (Brin [8]).** The set of such self-homeomorphisms of \( C \times C \) described above together with the composition as the binary operation forms the group \( 2V \).
The higher dimensional version of the groups $nV$ can be defined similarly. The group elements are acting on $C^n$ as prefix-replacement maps in $[3, 1, 4]$.

1.3. Twisted Brin-Thompson groups. We follow the definition from Belk et.al. [5, 18] to define the twisted version of the group $2V$ by introducing the twist homeomorphism induced by the group action. Let $G$ be a group acting on the set $\{1, 2\}$ by permutation, i.e. $G \mapsto S_2$ where $S_2$ denotes the symmetric group of order 2.

We define the twisted version of the group $2V$ by introducing the twist homeomorphism induced by the group action. Let $G$ be a group acting on the set $\{1, 2\}$ by permutation, i.e. $G \mapsto S_2$ where $S_2$ denotes the symmetric group of order 2.

Take $\gamma_1, \gamma_2 \in G$, define the basic twists induced by the action of $\gamma_i$ on $\{1, 2\}$ to be

$$\tau_{\gamma_i} : B(\varphi) \mapsto B(\psi)$$

that maps each rectangle

$$(\varphi(1), \varphi(2))$$

to rectangles

$$(\gamma_i(\varphi(1)), \gamma_i(\varphi(2))) = (\varphi(1)\gamma_i(\varphi(1)), \varphi(2)\gamma_i(\varphi(2)))$$

taking subblocks

$$B_i(\varphi) = \{(\varphi(1), \varphi(2)) \mid (\varphi(1), \varphi(2)) \in C^2\}$$

to

$$B_i(\psi) = \{(\varphi(1)\gamma_i(\varphi(1)), \varphi(2)\gamma_i(\varphi(2))) \mid (\varphi(1), \varphi(2)) \in C^2\}$$

for $i \in \mathbb{N}$. Then the twisted version of the self-homeomorphisms can be defined as follows,

$$h_\psi \circ \tau_{\gamma_i} \circ h_\varphi^{-1} : B(\varphi) \mapsto B(\psi).$$

**Definition 1.3** (Belk, Zaremsky [5]). The set of such self-homeomorphisms of $C \times C$ described above together with the composition as the binary operation forms the group $2V_G$.

This definition again can be generalised to $nV_G$ by considering the action of a group $G$ on larger sets with $n$ elements or even an infinite set.

1.4. Some known results. Many of the known results on Brin-Thompson groups $nV$ are, somehow, motivated by the study on the embedding problems of the groups or finding out the structure of the conjugacy classes and the centralisers of the groups. It has been shown that this family of generalisation of Thompson’s groups has interesting and rich geometry: The groups $nV$ are proved to be simple, to be finitely presented [9] and more generally have finiteness property $FP_\infty$ [16]. They also contain many interesting classes of subgroups such as $RAAG$s, $RACG$s and many groups constructed from direct products and wreath products of infinite cyclic groups $\mathbb{Z}$ [2, 8].

Moreover, Brin-Thompson groups also contain a large class of the groups called the rational groups according to [3] which provide some indication on the “coarse” hyperbolic-type properties of the groups.

The isomorphism problem has been investigated by [4] with the result that, for any $n \neq m$, $nV$ and $mV$ are not isomorphic. A selection of the results on the embedding problems and subgroup distortion are as follows: Some $RAAG$s and $RACG$s can be embedded into $nV$, Bleak, Belk, Matucci; Kato [3, 15] while $\mathbb{Z}^{n+1} \ast \mathbb{Z}$ does not embed into $nV$ [13]; When considering the wreath
product instead of the directly product, we have $\mathbb{Z}^n \times \mathbb{Z}^n$ embeds into $2V$; The groups $F, T, V$ are distorted in $2V$.

From a more combinatorial perspective, Belk, Bleak and Hyde [1] proved that these groups have solvable word problem, but unsolvable torsion problem which is provided as the first concrete examples of the groups described in Arzhantseva, Lafonts, Minasyanin [1].

In addition, as is mentioned previously, Belk and Matucci [6] have constructed and investigated the class of twisted Brin-Thompson groups and proved simplicity results and the finiteness properties for the class with some restrictions. A more recent paper by Zaremsky [18] provide more details on the motivation and relations to other problems. Some of the above results lead us to the following investigation on the torsion elements in $2V$.

2. Obstructions on subgroups of Brin-Thompson groups

2.1. Torsion local finiteness. Thompson’s group $F$ is torsion free while the groups $T$, and $V$ have torsion elements, $T$ contains all the finite cyclic groups and $V$ has all finite groups inside it.

As is mentioned in the Introduction, Burillo, Cleary, Röver [11] provided a list of the obstructions for the subgroups of Thompson’s groups $V$. One of properties that $V$ has, is the torsion local finiteness which provides some hints on what the torsion elements in $V$ should be like and should interact with other torsion elements. Torsion local finiteness of $V$ excludes infinite torsion subgroups such as Grigorchuck groups and provide an answer to the Burnside problems.

In addition, torsion local finiteness is a property that can be passed to subgroups, thus, we expect $2V$ (and even $nV$ for general positive integer $n$) to have the same property. We first state the definition of torsion locally finite as follows,

**Definition 2.1** (Torsion local finiteness). A group $G$ is said to be torsion locally finite if all its finitely generated torsion subgroups are finite groups.

2.2. Torsion elements. Inspired by the fact that torsion elements in $V$ can be represented by identical tree pairs, we consider one of the most obvious combinatorial representatives that a torsion element in $2V$ can be: an identical 2-dyadic block pair.

**Lemma 2.2.** A 2-dyadic block pair with two identical 2-dyadic blocks represent a torsion element in $2V$.

**Proof.** Since the number of the subblocks in each 2-dyadic block in the pair is finite, the group element acting on the 2-dyadic block pairs just permutes the labelings on the subblocks. Each subblock has a finite orbit, hence the 2-dyadic block pair representing the element has finite order. □

For an identical 2-dyadic block pair, we can obtain an identical rooted finite tree pair that correspond to the identical block pair depicted in Figure 5. Here, the fact that we can interchange the tree pairs and 2-dyadic blocks pairs indicates that the torsion elements in the group $2V$ have similar dynamics as the ones in $V$.

We describe more precisely how we obtain an identical tree pair from an identical dyadic block pair as follows: Since we define a 2-dyadic block inductively by adding horizontal and vertical line segments which are the partition line segments in $I^2$, we can naturally describe a partial order
on these line segments, which is on the set of the horizontal and vertical dyadic line segments and the length of these line segments in the set form a monotonically decreasing sequence.

Nevertheless, there is one particular case that one needs to be careful: there is a dyadic block that can be produced by different sets of the partition lines and hence the two different set of the partition line segments have slight different partial order three line segments, but this difference is not going to have effect on partial order of other line segments in the set.

In Figure 6, the two sets of the partition line segments provide the same 2-dyadic block, but the block on the left is partitioned by one vertical partition line segment with length 1 first and then by two horizontal partition line segments with length $\frac{1}{2}$ from the left to the right, while the one on the right is partitioned by one horizontal partition line segment with length 1 first and then two vertical partition line segments with length $\frac{1}{2}$ from the top to the bottom subblocks.

With the partial order defined, we can build rooted binary trees from dyadic blocks by associating 2-carets (a vertex attached with two edges) to the partition line segments. The product of the unit interval $I^2$ corresponding to $C^2$ (also called the trivial pattern) can be associated to the root of a binary tree. We next attach a 2-caret to the root, when we have either a horizontal or a vertical partition line segment in the dyadic block with length 1, then for the vertices on the other end of the two edges of the 2-caret, we call them leaves of a 2-caret, corresponding to the two subblocks after the partition.

We attach 2-carets to the leaves corresponding to the previously constructed subblocks inductively according to the partial order of the partition line segments.

The partial order on the partition line segments also provides an order on how we are expanding the blocks which influences the partial ordering on the subblocks. In the case in Figure 6, the dyadic block on the left is first divided into two subblocks, the left and right subblocks,
then the two subblocks are further divided into two subblocks, top and bottom each. While the dyadic block on the right is first subdivided into top and bottom subblocks and then left and right subblocks further. So there is a different “hierarchy” in the set of the dyadic blocks and subblocks.

**Remark 2.3.** For $2V$, we can also associate a 2-coloured tree pair with the 2-dyadic block pair which preserves the information of the forms of the partition line segments. Some more precise descriptions can be referred to [10].

Thus, we have associated an identical 2-dyadic block pair with an identical rooted binary tree pair by the above construction.

**Lemma 2.4.** For an element $\phi \in 2V$ represented by a pair of 2-dyadic blocks $(D, R, \sigma)$ where $\sigma$ permutes the labelings of the subblocks, if there exists some integer $k > 0$ such that $\phi^k$ can be represented by identical 2-dyadic blocks, then $\phi$ is torsion.

*Proof.* This follows immediately from Lemma 2.2 that $\phi$ is torsion. □

**Proposition 2.5.** Let $\alpha \in 2V$ be a torsion element and let $(A, B, \sigma)$ be the reduced 2-dyadic block pair representing $\alpha$, then there is a 2-dyadic block pair $(D, R, \beta)$ representing $\alpha$ such that $D$ and $R$ are identical 2-dyadic blocks. In addition, $A, B \subset D = R$.

*Proof.* We are going to use a slightly modified argument of [12, The proof of Prop6.1]. Let $\alpha \in 2V$ be a torsion element and $(A, B, \sigma)$ be the reduced 2-dyadic block pair representing $\alpha$ as in the statement of the proposition. Let $(P_A, P_B)$ be the pair of the set of the dyadic partition segments in $A$ and $B$, respectively and use the notation $\alpha(P_A) = P_B$. We denote by $(P_A^k, P_B^k) = (P_A, P_B)^k$ for general positive integer $k$, the pair of partition line segments in $(A, B, \sigma)^k$, calculated without reduction.

Here we introduce two similar notion: the expansion of a single set of the partitioned line segments induced by the expansion of a single 2-dyadic block; the minimal expansion of a partition line segment pair: when we perform the multiplication of two 2-dyadic block pairs, the first step is to make the target 2-dyadic block pairs of the former pair and the source 2-dyadic block pairs of the latter pair identical, we call this 2-dyadic block pairs the joint expansion, and the minimal 2-dyadic block pairs (the 2-dyadic block with minimal number of the partition line segment) that we can obtain the minimal joint expansion.

We focus on the changes in the pair of the set of the partition line segments and try to prove that $P_B^{k+1}$ is an expansion of $P_B^k$ by induction.

If $k = 2$, we have

$$(P_A^2, P_B^2) = (P_A, P_B)^2 = (P_A, P_B)(P_A, P_B)$$

and

$$P_A^2 = P_B^2,$$

$$\alpha(P_A^2) = P_B^2.$$

The multiplication can be interpreted as

$$(P_A, P_B)(P_A, P_B) = (\alpha^{-1}(P_E), P_E)(P_E, \alpha(P_E))$$

where $E$ is the minimal joint expansion. So we have

$$P_A^2 = P_B^2 = \alpha(P_E) \supset P_E.$$
Now we suppose by the hypothesis that

\[ P^k_B \supset P^{k-1}_B \]

for \( k < m \). We calculate the following,

\[ (A, B, \sigma)^{k-1}(A, B, \sigma), \]

we have

\[ (P^{k-1}_A, P^{k-1}_B)(P_A, P_B) = (\alpha^{-(k-1)}(P_{E_{k-1}}), P_{E_{k-1}})(P_{E_{k-1}}, \alpha(P_{E_{k-1}})) = (P^k_A, P^k_B), \]

where \( P_{E_{k-1}} \) is the minimal joint expansion of \( P_{E_{k-1}} \) and \( P_A \).

\[ (A, B, \sigma)^k(A, B, \sigma) = (\alpha^{-k}(E_k), E_k, \sigma_k)(E_k, \alpha(E_k), \sigma) = (A, B, \sigma)^{k+1} \]

induces

\[ (P^k_A, P^k_B)(P_A, P_B) = (\alpha^{-k}(P_{E_k}), P_{E_k})(P_{E_k}, \alpha(P_{E_k})) = (P^{k+1}_A, P^{k+1}_B) \]

where \( P_{E_k} \) is the minimal expansion of \( P^{k-1}_B \) and \( P_A \). Since \( P_{E_k} \) is an expansion of both \( P^k_B \) and \( P_B \), and in particular is an expansion of \( P_{E_{k-1}} \) by the induction hypothesis. Also since \( P_{E_{k-1}} \) is a minimal expansion of \( P_{E_{k-1}} \) and \( P_A \), \( P_{E_k} \) is an expansion of \( P_{E_{k-1}} \). Thus, \( P_{E_{k+1}} = \alpha(P_{E_k}) \supset \alpha(P_{E_{k-1}}) = P_{B_k} \). By the assumption, there is some positive integer \( m \) such that \( \alpha^m = 1 \) and we have \( (A, B, \sigma)^m = id, P^m_A = P^m_B \).

\[ (A, B, \sigma)^m = (\alpha^{-(m-1)}(E_{m-1}), E_{m-1}, f(E_{m-1})) = (A, B)^{m-1}(A, B). \]

induces the following,

\[ (P^m_A, P^m_B)(P_A, P_B) = (\alpha^{-(m-1)}(P_{E_{m-1}}), P_{E_{m-1}})(P_{E_{m-1}}, \alpha(P_{E_{m-1}})) = (P^{m-1}_A, P^{m-1}_B)(P_A, P_B). \]

Since \( P_{E_{m-1}} = \alpha(P_{E_{m-1}}) \) is an expansion of \( P_{E_{m-1}} \), and \( P_{E_{m-1}} \) is the minimal joint expansion of both \( P_{E_{m-1}} \) and \( P_A \), \( \alpha(P_{E_{m-1}}) \) is an expansion \( P_{E_{m-1}} \). Also since \( \alpha(P_{E_{m-1}}) \) and \( P_{E_{m-1}} \) are the partition line segments in the target and source 2-dyadic blocks of \( \alpha^{m-1} \), which indicates that the sets have the same number. Hence \( P_{E_{m-1}} = \alpha(P_{E_{m-1}}) \) and they are identical. \( \square \)

**Remark 2.6** (A remark by Bleak). Let \((\mathcal{D}, \mathcal{R}, \sigma)\) be a pair of non-identical 2-dyadic blocks representing a torsion element \( \alpha \in 2V \). Take \( \alpha^n = (\mathcal{D}, \mathcal{R}, \sigma^n) = (\mathcal{D}^n, \mathcal{R}^n, \sigma_n) \) calculated without reductions. There is a positive number \( m \), such that taking powers of \( \alpha \) and applying the multiplication of the 2-dyadic block pairs without reduction produces identical 2-dyadic block pairs \( f^m = (\mathcal{D}^m, \mathcal{R}^m, \sigma_m) \) such that \( \mathcal{D}^m = \mathcal{R}^m = \mathcal{D}' = \mathcal{R}' \) where \((\mathcal{D}', \mathcal{R}', \sigma)\) representing \( \alpha \) can be deduced from \((\mathcal{D}, \mathcal{R}, \sigma)\), by adding finitely many dyadic line segments to each dyadic block, i.e. \( \mathcal{D} \subset \mathcal{D}' \) and \( \mathcal{R} \subset \mathcal{R}' \).

**Proposition 2.7.** For an element \( \phi \in 2V \) represented by the reduced 2-dyadic block pairs \((\mathcal{D}, \mathcal{R}, \sigma)\) where \( \sigma \) permutes the numberings of the rectangles, \( \phi \) is torsion if and only if there exists an integer \( k > 0 \) such that \( \phi^k \) can be represented by identical patterns.

**Proof.** This is summarised from the Lemma 2.4 and Proposition 3.1. \( \square \)

This resembles the combinatorial properties of torsion elements and torsion subgroups in \( V \) [12] that torsion elements can be represented by identical tree pairs and these pairs reveal the dynamics of the elements.
2.2.1. Torsion subgroups. Now we consider the torsion subgroups inside $2V$.

**Lemma 2.8.** Let $a, b \in 2V$ be torsion elements, then any torsion element $g \in 2V$ generated by $a, b$ embeds into the same $V$ inside $2V$ as $a$ and $b$, namely, there exists an embedding $i : V \hookrightarrow 2V$ such that $a, b, g \in i(V)$. 

**Proof.** Assume that $a, b \in 2V$ are torsion elements, represented by 2-dyadic block pairs $(A_s, A_t, \sigma_a)$ and $(B_s, B_t, \sigma_b)$, respectively. Any finite group element $g_\omega$ represented by some finite word $\omega$ in $\{a, b\}$ can be represented by some identical 2-dyadic block pairs $(G_s, G_t, \sigma_g)$, and hence induce a pair of identical trees $(\hat{G}_s, \hat{G}_t, \sigma'_g)$ as constructed above. Then $A_s, A_t, B_s, B_t$ are sub 2-dyadic blocks of the $G_s(G_t)$ in the pair $(\hat{G}_s, \hat{G}_t)$, which can be represented by subtrees of $\hat{G}_s(\hat{G}_t)$. (This indicates that all three elements $a, b$ and $g_\omega$ act on the same infinite binary tree induced from the pattern as partial automorphism) as elements with finite order. Hence $a, b$ and $g_\omega$ can be embedded into the same $V$ inside $2V$. \qed

From the preceding lemma we could see that, when the product of the torsion elements are torsion, then these torsion elements and their products act on the “same infinite binary tree” induced from the 2-dyadic block. This resembles the correspondence in Figure 5 and we want to generalise this idea further.

**Proposition 2.9.** Let $a, b \in 2V$ are torsion elements, if the subgroup $\langle a, b \rangle$ in $2V$ is torsion, then there exists an embedding $i : V \hookrightarrow 2V$ such that $\langle a, b \rangle < i(V)$. 

**Proof.** The idea is to repeatedly use the argument in the preceding lemma. Again, we let $a, b \in 2V$ be the torsion elements in $2V$ represented by pattern pairs $(A_s, A_t, \sigma_a)$ and $(B_s, B_t, \sigma_b)$, respectively.

We start by considering the torsion elements as the reduced finite words in $\{a, b\}$ and we represent them by aligning the 2-dyadic blocks pairs. For instance, for the reduced word $abb$, we compose them as

$$(A_s, A_t, \sigma_a)(B_s, B_t, \sigma_b)(B_s, B_t, \sigma_b)(B_s, B_t, \sigma_b),$$

and we abuse the notation a little by eliminating the permutation sign so that we have the following

$$(A_s, A_t)(B_s, B_t)(B_s, B_t)(B_s, B_t).$$

Then we compose these 2-dyadic block pairs in turns from the left to right (the order does not really matter), and apply finitely many elementary expansions to the pairs while computing by multiplication, then we obtain

$$(A_s, A_t)(B_s, B_t)(B_s, B_t)(B_s, B_t) = (A'_s, B'_t),$$

where $(A'_s, B'_t)$ represents an identical pair of 2-dyadic blocks such that all 2-dyadic blocks pairs representing their subwords are sub 2-dyadic block pairs of $(A'_s, B'_t)$. These 2-dyadic block pairs induce tree pairs representing these elements.

$$(A'_s, B'_t)(B'_s, B'_t)(B'_s, B'_t) = (A'_s, B'_t)$$

Hence all elements representing subwords of $abb$ are inside the same $V$ in $2V$. 

Now, we extend the above example to a more generalised inductive argument. By treating the above example as the base case, we prove the hypothesis by inducting on the word length $k$ of the reduced words $\omega_k$ in $\{a, b\}$ representing elements in $2V$. Suppose for elements $g_{\omega_k}$ where $\omega_k \in \{a, b\}$ represented reduced word of word length less than or equal to $k$, all elements
represented by its subwords are embedded in the same $V$, then any reduced words build from \( \omega_k \) are in the following form \( \omega_{k+1} = \omega_k l \), or \( \omega_{k+1} = l \omega_k \) where \( l \in \{a,a^{-1},b,b^{-1}\} \). We assume that \( \omega_{k+1} \) is reduced and by the construction the element \( g_{\omega_{k+1}} \) is of finite order. By aligning the 2-dyadic block pairs as in the above case, we again conclude that the word \( \omega_{k+1} \) can be represented by a pair of identical 2-dyadic blocks and each 2-dyadic block of the pair includes 2-dyadic blocks in the pairs representing \( \omega_k \) and \( l \).

Thus the elements \( g_{\omega_{k+1}} \) with reduced word length \( k+1 \) are in the same \( V \) as the previous elements with shorter word length and we have that the group \( \langle a,b \rangle \) can be embedded into the same \( V \) in \( 2V \).

\[ \square \]

**Theorem 2.10.** Every finitely generated torsion subgroup embeds into the same \( V \) inside \( 2V \).

*Proof.* We generalise the argument in the Proposition 2.9 by taking more than two generators and again inducting on the word length of the reduced words, we will have the desired result. \[ \square \]

**Theorem 2.11.** \( 2V \) is torsion locally finite.

*Proof.* This follows from Theorem 2.10 and since \( V \) is torsion locally finite, then torsion subgroups in \( 2V \) are finite ones. \[ \square \]

**Corollary 2.12.** All torsion elements in \( 2V \) can either be embedded into \( V \) or be the roots of some torsion elements in \( V \) inside \( 2V \).

*Proof.* This is simply because \( 2V \) does not have any extra torsion groups. \[ \square \]

**Remark 2.13** \((nV_m \text{ for any number } n \text{ and } m)\). This argument can, in fact, be generalised to higher dimensional Brin-Thompson groups \( nV_m \) for positive integer \( n \) and \( m \). Thus the Brin-Thompson groups \( nV_m \) do not contain infinite torsion groups in general.

2.3. **Constrains on certain Baumslag-Solitor subgroups.**

2.3.1. **Infinite order elements.** Having had focused on the elements with finite order in \( 2V \), we now turn to the ones with infinite order in this section.

Before we explore these elements in \( 2V \), we recall that for a group \( G \), an element \( h \in G \) is said to be a root of \( g \), if there is another element \( g \in G \), such that \( h^t = g \) for some \( t \in \mathbb{N} \).

2.3.2. **Some quantitative notion.** Let \( f \in 2V \) be a torsion-free element represented by a (not necessarily reduced) pair of 2-dyadic blocks \((S,T,\sigma)\), where \( S \) and \( T \) are non-identical, namely, \( S \neq T \).

From some discussion in the previous section (Proposition 2.7), we know that pairs of 2-dyadic blocks representing infinite order elements are never going to be identical.

Let us define some quantitative notion for the later argument. Let \((S_i,T_i,\sigma_i)\) denote the non-reduced 2-dyadic block pair representing \( f^i \), namely, \((S,T,\sigma)\) composing to itself \( i \) times without reductions, where \( i \in \mathbb{N} \) whereas \((S'_i,T'_i,\sigma'_i)\) be the reduced 2-dyadic block pair representing \( f^i \).

Let \( T_{pi} \) denote the number of the partition line segments in \( S_i \) or in \( T_i \), and \( T_{pi}' \) denote the number of the partition line segments in \( S'_i \) or in \( T'_i \). Let \( C_{pi} \) denote the number of the common partition line segments in the pair \((S_i,T_i)\), and \( D_{pi} = T_{pi} - C_{pi} \) be the difference.
Let $I_i$ denote the number of the increased partition line segments from $S$ to $S_i$,

$$I_i \leq T_{p_{i}} - T_{p_{i}} \leq \sum_{n=1}^{i} D_{p_n} \text{ (at least } \left\lfloor \frac{i}{2^n} \right\rfloor).$$

Finally, we let $R(i)$ be the number of the reduction $T_{p_{i}} - T_{p_{i}}$.

**Remark 2.14.** Note that when we consider the number of the partition line segments in the 2-dyadic block pairs, we only count the number of the partition line segments in each block of the block pairs. Also, we talk about the common partition line segments in two 2-dyadic blocks $S$ and $T$ we intend to say that two line segments are in the same position with the same length, we do not take “partial” common partitions line segments into account.

**Example 2.15** (A baby case). Let $f$ and $(S, T, \sigma)$ represent $f$ and other quantitative notion as above, here we again abuse the notation of the 2-dyadic block pair to use a simplified version $(S, T)$. For $f^2$, we have $f^2 = (S, T)(S, T) = (f^{-1}(S \setminus T) + S, T + f(T \setminus S))$.

We want to investigate these quantitative notion to see how they change during the composition and, in particular, we want to know when the reduction of the total number of the partition line segments occurs, therefore we consider the following three cases:

1. if $R(f^2) < I_2 = D_{p_2}$, then $T_{p_2} > T_{p_1}$, which means that the total number of the partition lines increases and no reduction occurs.

2. if $R(f^2) = D_{p_2}$ then we usually have $C_{p_2} < C_{p_1}$ however, in this special case, the element $f$ is has finite order. by the following combinatorial argument: for the partition line segments in $S$ and $T$, we take $C$ to be the common partition line segments, let $P_1$ denote the partition line segments in $S \setminus T$ and let $P_2$ denote the partition line segments in $T \setminus S$. Then $f^2$ can be represented by $f^{-1}(P_1) + C, P_2 + C + f(P_2)$, since the partition line segment sets $P_1$ and $P_2$ are disjoint and the four sets $f^{-1}(P_1)$, $P_1$, $P_2$ and $f(P_2)$ all have the same number of partition line segments, $f^{-1}(P_1) + C, P_2 + C + f(P_2)$ has to be an identical pair and hence is torsion which violets the assumption.

3. if $R(f^2) > D_{p_2}$, then $T_{p_2} < T_{p_1}$. We let $T_{p_1} - T_{p_2} = \epsilon$, then $C_{p_1} - C_{p_2} \leq \epsilon$ then $D_{p_2} \leq D_{p_1}$.

**Remark 2.16.** We discuss, in particular, the third condition in the baby case 2.15, if the number of the reduced partition line segments exceeds the number of the increased partition line segments, the number of the common partition line segments reduce from $C_{p_1}$ to $C_{p_2}$ (for $f^2$) and $D_{p_2}$ keeps unchanged, i.e. $D_{p_2} = D_{p_1}$, then $(f^2)^2$ will have a larger number of increase in the number of the partition line segments. There is an $n \in \mathbb{N}$ such that there are no reductions occur for pattern pair representing $f^n$.

**Proposition 2.17.** Let $f \in 2V$ be a torsion-free element represented by a (reduced) pair of 2-dyadic blocks $(S, T)$, $S \neq T$, the number of the reduced partition line segments $R(i)$ will not exceed the number of the increased partition line segments $I_i$ for some $i \in \mathbb{N}$.

**Proof.** Now we generalise some of the ideas in the baby case to the following argument.

Let $f$ and $(S, T)$ and all other quantitative notion be as above and let the reduction of the 2-dyadic block pairs first occur at $f^n$ for some $n \in \mathbb{N}, n \geq 3$. 
Assume that the reduction in the number of the partition line segments first occurs at $f^n$, again, we consider the following three cases,

1. If $R(f^n) < I_n = \sum_{j=1}^n D_{p_j} = n D_{p_1}$, then $T_{p_1} < T_{p_n}$ and $C_{p_n} < C_{p_1}$, hence $D_{p_1} < D_{p_n}$.

2. If $R(f^n) = I_n = \sum_{j=1}^n D_{p_j} = n D_{p_1}$, again we consider combinatorially the following product, $(S_{n-1}, T_{n-1})(S, T)$. By assumption, we know that no reduction occur at $(S_{n-1}, T_{n-1})$, and the reduction first occurs for the pair $(S_{n-1}, T_{n-1})(S, T) = (S_{n-1} + f^{-n+1}(T_{n-1})(S), f(S \setminus T_{n-1}) + T)$. However, the difference in the partition line segments $D_{p_n} - D_{p_{n-1}}$ is far less than reduce partition line segments $R(f^n)$ which yields a contradiction.

3. If $R(f^n) > I_n = \sum_{j=1}^n D_{p_j} = n D_{p_1}$, then $C_{p_n} < C_{p_1}$ and $T_{p_1} - T_{p_n} \geq C_{p_1} - C_{p_n}$, i.e. $D_{p_1} \geq D_{p_n}$.

Hence the difference between the number total partition line segments can only change according to the results in the first and the third condition. In the third condition, we again analyse $R((f^n)^2)$, if the $D_{p^n}$ keep decreasing when the value $n$ increases, then we will ultimately obtain a identical 2-dyadic blocks which means that $f$ is torsion, so this can not happen.

This leaves that only the first condition is valid, the number of the partition lines will grow along when the multiple increases. \hfill \Box

**Theorem 2.18.** An element $g \in 2V$ of infinite order does not have arbitrarily large root, i.e. there is a bound on $t \in \mathbb{N}$ for which there may exist $h \in 2V$ with $g = h^t$.

**Proof.** Let $h, g \in 2V$ be torsion-free elements represented by a pair of reduced 2-dyadic blocks where $g = h^t$, and let $\phi(i), \psi(j)$ denote the number of partition lines in the reduced 2-dyadic block pairs representing $h^t$ and $g^t$. By the Proposition 2.17 $\phi(i)$ is monotonic increase after some $n > i$, $i \in \mathcal{N}$ and the number $\phi(i)$ is always finite and $\psi(j)$ has the same criterion, $t$ can not be arbitrarily large. \hfill \Box

The above result provide a method for classifying the group element in $2V$ by the “asymptotic” behaviour defined intuitively by the rate of the increase in the number of the partition line segments. This reflects some of the dynamics in the group elements. In addition, by the same argument, we could say that any Brin-Thompson $nV_m$ for $n, m \in \mathbb{N}$ has the above properties.

**Remark 2.19** (Röver[17]). This result rules out some certain Baumslag-Solitar groups as the subgroups of $2V$.

**Remark 2.20** (Embedding problems). This result also indicates that the additive group $\mathbb{Q}$ does not embed into $2V$ which is an interesting comparison to a result of Belk and Matucci [8] that a group $T$ of the central extension of Thompson’s group $T$ having $\mathbb{Q}$s embedded as a subgroup inside it and [8] has further generalisations constructed to these groups.

3. Constrains on subgroups in twisted Brin-Thompson groups

We now focus on the twisted version of Brin-Thompson groups. The action of the group $G$ on the set $S$ in the class of twisted Brin-Thompson group $SV_G$ enlarges the group and may change some of the properties, though properties such as the finiteness property of the groups are preserved.
Following the definition of Belk and Zaremsky [18], we consider several classes of $SV_G$ depending on the action of $G$ on the set $S$.

**Theorem 3.1.** If we take $S$ to be finite, then $SV_G$ is torsion locally finite and has restrictions on certain Baumslag-Solitar groups.

*Proof.* We can consider the action of $G$ on $S$ as permutation on the “partitions” of the dyadic blocks. When $S$ is finite, it is obvious that the arguments in both the proof of Theorem 2.10 and Theorem 2.18 apply to the group $SV_G$ for any group $G$. □

3.1. **The group $\mathcal{SV}$ for infinite set $\mathcal{S}$.** When we take an infinite set $\mathcal{S}$, the group $\mathcal{SV}_G$ will be more complicated regardless of the group action, thus, we first consider the non-twisted case, then the group $\mathcal{SV}$ contains all of $nV$ when $\mathcal{S}$ is a set of infinite order.

The group $SV$ considered in [9, 5] are subgroups of $\mathcal{SV}$ that can be regarded as some kind of “Arfinification” of the group $nV$ with some extra restrictions on the number of partitions and they includes all $nV$s as subgroups

Let $f \in \mathcal{SV}$, then $f$ can be represented by $(B(\varphi), B(\psi))$ a pair of infinite dimensional blocks $[0, 1]^N$ (corresponding to the product of infinitely many Cantor sets $C^N$) with dyadic subdivisions, with elementary expansions and collapses defined similarly. Moreover, there is still this partial order on partitioning the blocks. Then the group operation can be defined as the multiplication on blocks and the group structure follows.

Then Lemma 2.4 and Proposition 2.7 do not hold for the general $\mathcal{SV}$ any more, since, for instance, we have elements in $\mathcal{SV}$ that can be represented by identical pair having permutations of infinite order.

**Lemma 3.2.** For an element $\phi \in \mathcal{SV}$ where $\mathcal{S}$ is a countable set represented by the reduced 2-dyadic block pairs $(D, R, \sigma)$ where $\sigma$ permutes the numberings of the subblocks (rectangles), $\phi$ is of finite order then the pair can be represented by an identical pair. For the converse, the pair represents a torsion element then there exist integers $k_1, k_2 > 0$ such that $\phi^{k_1, k_2}$ can be represented by identical patterns and $\phi^{k_1, k_2}$ is the identity.

*Proof.* This lemma is a stricter version of Proposition 2.7. The former part is obvious from the argument in the proof of Proposition 3.1 and the second condition excludes the case that identical pair having infinite order permutation. □

The latter condition in the if part in the preceding Lemma indicate that there exists some more exotic torsion elements and we now closely look at some of these torsion elements in $SV$ for countable infinite set $S$. We take an element $f \in \mathcal{SV}$ to have the identical block pair representation $(B(\varphi), B(\psi))$ such that the block pairs are constructed as follows:

$$B(\varphi) = \{B_i(\varphi)\}, B(\psi) = \{B_i(\psi)\}$$

for $i \in \mathbb{N}$ are two countable infinite sets where each block has infinite dimension. Take $[0, 1]^N$ to be $B(\varepsilon, \varepsilon, \cdots)$, we partition the first interval $[0, 1]$ into half to obtain two subblocks $B(0, \varepsilon, \cdots)$, and $B(1, \varepsilon, \cdots)$. Next, we take the subblocks and divide the two subblocks $B(0, \varepsilon, \cdots)$, $B(1, \varepsilon, \cdots)$ into the following four blocks $B(0, 0, \cdots)$, $B(0, 1, \cdots)$ and $B(1, 0, \cdots), B(1, 1, \cdots)$. For the next step, we take the last subblock $B(1, 1, \cdots)$, and partition it to obtain $B(1, 1, 0, \cdots)$ and $B(1, 1, 1, \cdots)$. Then we take the last subblock $B(1, 1, 1, \cdots)$ and repeat the process of partitioning the blocks all over again and we will ultimately obtain an infinite sequence of the
partitioned subblocks whose union form $[0,1]^N$. Here we can associate the partitions (which are the dyadic subblocks) to an infinite binary tree according to the order of the partitions as described in Subsection 2.2. The identical block pairs $(B(\varphi), B(\psi))$ can now be represented by a pair of infinite binary trees. We now label the subblocks as follows,

$$B_i(\varphi) = \{ \varphi_i(1)\kappa(1), \varphi_i(2)\kappa(2), \ldots | (\kappa(1), \kappa(2), \cdots) \in C^N \}$$

so that we have the following collections,

$$B_1(\varphi) = \{ 0\kappa(1), 0\kappa(2), \epsilon\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_2(\varphi) = \{ 0\kappa(1), 1\kappa(2), \epsilon\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_3(\varphi) = \{ 1\kappa(1), 0\kappa(2), \epsilon\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_4(\varphi) = \{ 1\kappa(1), 1\kappa(2), \epsilon\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_5(\varphi) = \{ 1\kappa(1), 1\kappa(2), 0\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_6(\varphi) = \{ 1\kappa(1), 1\kappa(2), 1\kappa(3), \epsilon\kappa(4), \epsilon\kappa(5), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_7(\varphi) = \{ 1\kappa(1), 1\kappa(2), 1\kappa(3), 1\kappa(4), \epsilon\kappa(5), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

for $B_i(\psi)$ we have

$$B_1(\varphi) = \{ 0\kappa(1), 1\kappa(2), \epsilon\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_2(\varphi) = \{ 0\kappa(1), 0\kappa(2), \epsilon\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_3(\varphi) = \{ 1\kappa(1), 0\kappa(2), 1\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_4(\varphi) = \{ 1\kappa(1), 1\kappa(2), 0\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_5(\varphi) = \{ 1\kappa(1), 1\kappa(2), 1\kappa(3), \epsilon\kappa(4), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_6(\varphi) = \{ 1\kappa(1), 1\kappa(2), 1\kappa(3), 1\kappa(4), \epsilon\kappa(5), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

$$B_7(\varphi) = \{ 1\kappa(1), 1\kappa(2), 1\kappa(3), 1\kappa(4), 1\kappa(5), \cdots | (\kappa(1), \kappa(2), \kappa(3), \kappa(4) \ldots) \in C^N \}$$

which can be interpreted alternatively by tree pairs with the labelings on the leaves as follows:

$$\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$$

$$\{2, 1, 4, 1, 5, 7, 6, 9, \cdots\}.$$ 

It is obvious that $f$ is of finite order.

**Theorem 3.3.** The first Grigorchuck group embeds into $\mathcal{I}V$ for a countable infinite set $\mathcal{I}$.

**Proof.** The elements constructed above is one of the generators and we can construct the other three similarly by constructing identical block pairs that can be represented by infinite binary tree pairs.

Hence we can conclude that:

**Corollary 3.4.** The group $\mathcal{I}V$ is not torsion locally finite when $\mathcal{I}$ is a countable infinite set.
Let \( f \in \mathcal{I}V \) be a torsion-free element in \( \mathcal{I}V \) represented by a (not necessarily reduced) pair of \( \mathcal{I} \)-dyadic blocks \((S, T, \sigma)\), where \( S \) and \( T \) are non-identical, namely, \( S \neq T \). Proposition 2.7 in the previous section does not hold any more, since we have the following.

**Proposition 3.5.** An element \( g \in \mathcal{I}V \) (where \( \mathcal{I} \) is a countable infinite set) of infinite order does have arbitrarily large roots, i.e., there is no bound on \( t \in \mathbb{N} \) for which there may exist \( h \in \mathcal{I}V \) with \( g = h^t \).

**Proof.** This follows from the existence of identical block pairs representing infinite ordered elements and hence the existence of infinite permutation group as a subgroups. \( \square \)

**Remark 3.6.** Proposition 3.5 takes away the restrictions on the Baumslag-Solitar subgroups in \( SV \), though there is evidence that general Baumslag-Solitar subgroups are not very likely to appear in Thompson-like groups.

The general \( \mathcal{I}V \) is a rather large groups and contains all different infinite groups acting on trees. We now consider again a slightly smaller class \( SV \), the large Brin-Thompson groups.

### 3.2. The groups \( SV \) and \( SV_G \)

We adopt the notation that we had in Subsection 1.2, we take \( \mathcal{I} \) to be the infinite countable set, \( C^\mathcal{I} \) to be the product of \(|\mathcal{I}| \) copies of the Cantor set \( C \). We take the following block notation,

\[
B(\psi) = \{B_i(\varphi)\}_{i \in \mathbb{N}} = \{\{B(\varphi(s_{i1}), \varphi(s_{i2}), \ldots, \varphi(s_{i\ell}))\mid \{s_{ik}\}_{k \in \mathbb{N}} \subset \mathcal{I}\}
\]

\[
= \{(\varphi_1(s_1)\kappa_1(1), \varphi_1(s_2)\kappa_1(2), \varphi_1(s_1)\kappa_1(i_1), \ldots, \varphi_1(s_{i_2})\kappa_1(i_2), \ldots, \varphi_1(s_{i_\ell})\kappa_1(i_\ell)) \mid \kappa(1), \kappa(2), \ldots, \kappa(i_1), \ldots, \kappa(i_\ell), \ldots \in \mathcal{I}^\mathcal{I}, s_j \in \mathcal{I}\}
\]

such that \( \varphi(s_j) = \emptyset \) for all but finitely many \( s_j \in \mathcal{I} \subset \mathcal{I} \).

and we have

\[
\bigcup_{i=1}^{\mathbb{N}} B_i(\varphi) = C^\mathcal{I} = C^\mathcal{I}
\]

and the map

\[
h_\varphi : C^\mathcal{I} \mapsto B(\varphi)
\]

to identify the Cantor spaces and the partitioned blocks and eventually,

\[
h_\psi \circ h_\varphi^{-1} : B(\varphi) \mapsto B(\psi).
\]

provides group elements for \( SV \) in the sense of Brin, Belk and Matucci. Brin proved that \( SV \) is finitely presented and Belk Matucci further constructed \( SV_G \) and proved that it is finitely generated under some conditions depending on \( G \) and the action of \( G \) on the set \( S \).

The extra condition posed on the group elements of \( SV \) ensures that the codimension one partition blocks are distributed in finitely many dimensions \(|S|\) in the \(|\mathcal{I}| \) dimension in each block in the pair representing a group element. Here \( S \subset \mathcal{I} \) is taken to be some finite set.

We adopt the following quantitative notation: let \((S_i, T_i, \sigma_i)\) denote the non-reduced \( \mathcal{I} \)-dyadic block pair representing \( f^i \in SV \); namely, \( f = (\mathcal{S}, T, \sigma) \) composing to itself \( i \) times without reductions, where \( i \in \mathbb{N} \) whereas \((S'_i, T'_i, \sigma'_i)\) be the reduced \( \mathcal{I} \)-dyadic block pair representing \( f^i \); The notion \( T_{p_i}, T'_{p_i}, C_{p_i}, \) and \( D_{p_i} = T_{p_i} - C_{p_i} \) are defined as in Subsection 2.3.2.
Proposition 3.7. For any element $f = (S, T, \sigma) \in SV$ as described above, the number of the codimensional one partition blocks in both $S$ and $T$ are finite.

Proof. This follows from the extra condition posed on the set. Let

$S = B(\varphi) = \{B_i(\varphi)\}_{i \in \mathbb{N}} = \{\{B(\varphi(s_{i_1}), \varphi(s_{i_2}), \ldots, \varphi(s_{i_k}))\}\}

= \{(\varphi(\kappa(1)), \varphi(\kappa(2)), \varphi(\kappa(i_1)), \ldots, \varphi(s_{i_k}), \kappa(\kappa(i_2)), \ldots) \mid (\kappa(1), \kappa(2), \ldots, \kappa(i_1), \ldots, \kappa(i_2), \ldots) \in C^\varphi, s_j \in F\}$

such that $\varphi(s_{j_k}) \neq \emptyset$ for finitely many $s_{j_k} \in S \subset F \}_{i \in \mathbb{N}}$.

We suppose that $B(\varphi)$ contains infinitely many codimensional one partitions, and hence infinitely many subblocks without partitions inside and whose union is the block $\bigcup_{i=1}^{\infty} B(\varphi_i) = C^S = S$. There exists some subblock $B_j(\varphi(\kappa(1)), \varphi(\kappa(2)), \ldots)$ in $B(\varphi)$ such that there are infinitely many non trivial words $\varphi(s_j)$ which violates the definition. The same applies to $T$. \qed

Hence for an element $f$ of the group $SV$ in the sense of Belk and Zaremsky in the block pair $(S, T, \sigma_f)$, we have that $(S, T)$ and $(T \setminus S)$ are equal, thus $D_{p_f}$ is well defined and finite, this can be seen obviously from the construction of the block pairs for elements in $nV$.

Theorem 3.8. The group $SV$ is torsion locally finite and has no certain Baumslag-Solitar subgroups.

Proof. The former statement can be proved by a similar argument as in the proof of Proposition 2.17 and the latter part can be proved by a similar argument in the proof of Proposition 2.9 \qed

3.3. On twisted Brin-Thompson groups $SV_G$. Last but not the least, since $SV$ is such a large group that one can hardly put hand on, we consider the twisted case and more precisely the actions are considered to be “oligomorphic” as in [6, 18].

For a group $G$ acting on the set $S$ in the definition of the group $SV_G$, the action of the group elements $g \in G$ are permuting each $[0, 1]$ in on of the subblocks inside the $[0, 1]^\infty$. Then if the action $g$ on these infinite dimensional block pairs has infinite orbits, the element $g$ is not of finite order.

Definition 3.9 (Oligomorphic action). A group $G$ acting on a set $\Omega$, is said to be oligomorphic if for each $k > 1$, G has only finite orbits on the $k$-element set of $\Omega$.

Theorem 3.10. For infinite set $S$ with $G$ acting on $S$ oligomorphically the group $SV_G$ is torsion locally finite and excludes certain Baumslag-Solitar subgroups.

Proof. The idea behind is similar as the case of $SV$, here the group action just permutes the “edges” of the subblocks up to parallelism. while it does not effect the two constraints on the subgroups in the statement above. \qed

Remark 3.11. Oligomorphic action generalises the notion of transitive action. The twisted Brin-Thompson groups $SV_G$ have finiteness properties $FP_\infty$ and are simple, quasi-isometry classes of the group $G$ [6, 18]. Now we know that the extra condition posed on the construction of $SV$ preserve the torsion local finiteness and it provides larger class of interesting subgroups.
without changing the quasi-isometry class. What are the restrictions on the subgroups of these groups? What will happen to the properties on the subgroups of $SV_G$ when we consider other group action or rather if the group action affects any of these restrictions still remain mysterious.

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