How accurately can the microcanonical ensemble describe small isolated quantum systems?

Tatsuhiko N. Ikeda\textsuperscript{1} and Masahito Ueda\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan
\textsuperscript{2}Center for Emergent Matter Science (CEMS), RIKEN, Wako, Saitama 351-0198, Japan

(Dated: March 30, 2015)

We numerically investigate quantum quenches of a nonintegrable hard-core Bose-Hubbard model to test the accuracy of the microcanonical ensemble in small isolated quantum systems. We show that, in a certain range of system size, the accuracy increases with the dimension of the Hilbert space \( D \) as \( 1/D \). We ascribe this rapid improvement to the absence of correlations between many-body energy eigenstates as well as to the eigenstate thermalization. Outside of that range, the accuracy is found to scale as \( 1/\sqrt{D} \) and improves algebraically with the system size.

PACS numbers: 05.30.-d, 03.65.-w

Introduction. — The microcanonical ensemble (ME) is the fundamental working hypothesis in statistical mechanics for the description of equilibrium states of an isolated system [1]. However, the range of applicability and the accuracy of the ME have yet to be fully understood. This problem due originally to von Neumann’s seminal work [2,4] has seen resurgence of interest recently [3,0] partly because isolated quantum systems have been realized using ultracold atoms [7–10] and thus the verification of the hypothesis has now become an issue of practical importance.

The understanding of why the ME can describe equilibrium states in the thermodynamic limit (TDL) has advanced considerably in recent years. Even under unitary time evolution, an effective stationary state can appear due to dephasing between many-body energy eigenstates [11–14], where physical quantities are obtained by the weighted average of expectation values over individual energy eigenstates (see Eq. (3)) with the weights determined by the initial condition. On the other hand, it is empirically known that the ME, which dictates the equal-weight average of those expectation values (see Eq. (1)), well describes the physical quantities in the effective stationary state, regardless of the initial condition. Recent studies have suggested that this success of the ME derives from the fact that quantum states with close eigenenergies typically emulate the same thermal state in the TDL. This scenario has been formulated mathematically [2,4,15] and verified by uniform random samplings of states in a narrow energy shell [16–18]. The eigenstate thermalization hypothesis (ETH) [20–22] is a popular representation of this scenario, which states that expectation values of a macroscopic observable for energy eigenstates (eigenstate expectation values, EEVs) give the same value in the TDL. The ETH has been supported by several numerical studies in various nonintegrable models [24–30].

While these studies concern why the ME works in the TDL, we address the question of how accurately it does in small systems that can now be tested experimentally [31–33]. As shown below, the ETH gives, in small systems, an upper bound on the accuracy of the ME and the bound scales as \( 1/\sqrt{D} \), where \( D \) is the dimension of the Hilbert space [24, 29]. However, the accuracy itself has not yet been explored in concrete numerical experiments.

In this paper, we numerically investigate the accuracy of the ME for quantum quenches in a one-dimensional nonintegrable model of hard-core bosons (HCBs) with the lengths of \( L = 15, 18, 21 \) and 24. We show that, for some quench magnitudes, the accuracy improves proportionally to \( 1/D \), which is much better than the upper bound given by the ETH mentioned above. We argue that the \( 1/D \) scaling implies that, only by quantum quenches in nonintegrable systems, no correlations are induced between many-body eigenstates. Finally, we argue that, as we increase the system size with the quench magnitude fixed, we encounter three different scalings of the accuracy, which are proportional to (i) \( 1/\sqrt{D} \) (ETH regime), (ii) \( 1/D \) (no-correlation regime), and (iii) \( L^{-\alpha} \) with \( \alpha > 0 \) (algebraic regime).

Formulation of the problem. — We begin by formulating the problem in a general setup. We consider an isolated quantum system described by a time-independent Hamiltonian \( \hat{H} \). Let \( \{ |E_n| \}_n \) be the set of the eigenstates of \( \hat{H} \) with eigenenergies \( \{ E_n \}_n \) : \( \hat{H} |E_n \rangle = E_n |E_n \rangle \) for \( n = 1, 2, \ldots, D \), where \( D (\gg 1) \) is the dimension of the Hilbert space. An initial pure state \( |\psi_{\text{initial}} \rangle \) evolves in time as

\[
|\psi(t)\rangle = \sum_{n=1}^{D} c_n e^{-iE_n t} |E_n \rangle
\]

with \( c_n \equiv \langle n | \psi_{\text{initial}} \rangle \), where the Planck constant is set to unity throughout this paper. We assume that the energy gaps \( \{ E_n - E_m \} \) are all different [11, 12] and the effective dimension \( D_{\text{eff}} \), which is the effective number of the energy eigenstates involved in the initial state, is much greater than unity:

\[
d_{\text{eff}} \equiv \frac{1}{\sum_n |c_n|^2} \gg 1.
\]

Then the time-dependent expectation value of a few-body observable \( \hat{O} \), or \( \langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_{n,m=1}^{D} c_n c^*_m \hat{O}(E_n - E_m) \), evolves in time as

\[
\langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_{n,m=1}^{D} c_n c^*_m e^{-i(E_n - E_m) t} \hat{O}(E_n - E_m).
\]
\[ \sum_{m,n} c_m c_n e^{i(E_n - E_m)t} \langle E_n | \hat{O} | E_m \rangle \] is most of the times close to the diagonal ensemble average \[22\]

\[ \langle \hat{O} \rangle_{DE} \equiv \sum_{n=1}^{D} |c_n|^2 O_n, \] (3)

since the off-diagonal \((m \neq n)\) contributions cancel each other, where \(O_n \equiv \langle E_n | \hat{O} | E_n \rangle\) is called an eigenstate expectation value (EVE) and \(|c_n|^2\) represents the energy distribution. In other words, Eq. (3) describes the physical quantities in the effective stationary state. We note that Eq. (3) explicitly depends on the microscopic details of the initial state.

The expectation value \(\langle \hat{O} \rangle_{DE}\) can be calculated approximately from the ME that involves only two parameters, i.e., the central energy \(E_C\) and the energy width \(\delta\), by

\[ \langle \hat{O} \rangle_{ME} \equiv N_{E_C,\delta}^{-1} \sum_{n \in I(E_C, \delta)} O_n, \] (4)

where \(I(E_C, \delta) \equiv \{ n | E_n \in [E_C - \delta, E_C + \delta]\}\), and the normalization factor \(N_{E_C,\delta} = \sum_{n \in I(E_C, \delta)} 1\) gives the number of the energy eigenstates in the energy window \([E_C - \delta, E_C + \delta]\). Thus we define the error of the ME by

\[ \text{Error of ME} \equiv \langle \hat{O} \rangle_{ME} - \langle \hat{O} \rangle_{DE} \] (5)

and numerically investigate it in a concrete model.

**Model.**—We use a one-dimensional model of HCBs with the nearest- and next-nearest-neighbor hopping and interaction. The Hamiltonian is given by

\[ \hat{H}^{(u)} = \sum_{i=1}^{L} \left[ -(\hat{b}_{i+1}^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{b}_{i+1}) + u\hat{n}_i\hat{n}_{i+1} \right] \]
\[ + \sum_{i=1}^{L} \left[ -(\hat{b}_{i+2}^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{b}_{i+2}) + \hat{n}_i\hat{n}_{i+2} \right], \] (6)

where the periodic boundary conditions are imposed and \(\hat{b}_i^\dagger\) (\(\hat{b}_i\)) is the annihilation (creation) operator of a HCB on site \(i\) with \([\hat{b}_i, \hat{b}_j] = [\hat{b}_i^\dagger, \hat{b}_j^\dagger] = [\hat{b}_i, \hat{b}_j^\dagger] = 0\) for \(i \neq j\), \(\hat{b}_i^2 = (\hat{b}_i^\dagger)^2 = 0\) and \([\hat{b}_i, \hat{b}_i^\dagger] = 1\), and \(\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i\). The total number \(N\) of HCBs is conserved and assumed to be \(N = L/3\) in our numerical experiment.

For simplicity, the nearest- and next-nearest-neighbor hopping and the next-nearest-neighbor interaction energies are set to unity \[34\] in Eq. (6). In the following, we consider quantum quenches by suddenly changing the parameter \(u\) from 0 to \(u_f\). In these quenches, \(\hat{H}^{(u)}\) is nonintegrable and the energy level spacings obey the Wigner-Dyson statistics \[33\] due to the next-nearest-neighbor contributions \[30\].

Due to the translational invariance of our model, the Hilbert space is decomposed into \(L\) sectors labeled by the translational momentum \(P = 2\pi m/L\) \((m = 0, 1, \ldots, L-1)\), and we take the Hilbert space with \(P = 2\pi L\), which is not decomposed into smaller sectors. The dimension \(D\) of this sector at each system size is shown in Table I and the energy eigenstates in the sector are denoted by \(|E_n^{(u)}\rangle\) \(\equiv \hat{H}^{(u)}|E_n^{(u)}\rangle\) \(\equiv \hat{H}^{(u)}|E_n^{(u)}\rangle\). Meanwhile, we calculate the ME average of \(\langle \hat{O} \rangle\) by setting \(E_C\) and \(\delta\) so that \(\langle \hat{O} \rangle_{DE} \approx \langle \hat{O} \rangle_{ME}\).

**Protocol of our numerical experiment.**—We consider a quantum quench where the initial state \(|\psi_{\text{initial}}\rangle\) is an eigenstate \(|E_n^{(0)}\rangle\) of \(\hat{H}^{(0)}\). The time evolution is governed by \(\hat{H}^{(u_f)}\) and an effective stationary state is eventually reached where the expectation value of a few-body observable \(\hat{O}\) is given by the diagonal ensemble average \[3\] with \(c_n = \langle E_n^{(u_f)} \rangle |E_n^{(0)}\rangle\). Meanwhile, we compare our results with the ETH, which is believed to hold in the middle of the spectrum [27, 28].

We examine the scaling of the error \(\delta = \sum_{n=1}^{D} |\langle \hat{O} \rangle_{DE} - \langle \hat{O} \rangle_{ME}|\) obtained for each ini-
FIG. 1. (Color Online) The distribution of the errors of the ME (Eq. (5)) obtained for each of the initial states in the quench of \( u_f = 0.4 \) for (a) \([\text{D}]\) \( L = 18 \) (solid) and 21 (dotted) and (b) \([\text{D}]\) \( L = 24 \) (solid) for \( \hat{O} = \hat{O}_1 \) \([\text{D}]\). The dashed curves in the panels (b) and (d) are the least squares fits of the distributions with the Gaussian distributions. We note that the horizontal and vertical axes are scaled by the factors shown at the left-top and right-bottom corners of the panels.

Figures (a) and (b) show that, for \( \hat{O}_1 \), the errors become more concentrated around zero as the system size increases. Figures (c) and (d) show that the distribution of the errors for \( \hat{O}_2 \) behaves similarly to that for \( \hat{O}_1 \). Since the forms of the distribution functions depend slightly on individual observables, we do not look into them further. Instead, we discuss, in the following, how fast the width of the distribution vanishes by looking into the root mean square (RMS) of the errors, which we call the accuracy of the ME.

The first main result of this paper is that, in the quench of \( u_f = 0.4 \), the accuracy is proportional to \( 1/D \) for both \( \hat{O}_1 \) and \( \hat{O}_2 \) as shown in Fig. 2(a), where the accuracy is plotted against \( D \) with the error bars representing the estimation errors. By conducting the least squares fits of the accuracy with a function \( f(D) = AD^B \), we find the exponent \( B \) to be \(-1.00^{+0.06}_{-0.04}\) for \( \hat{O} = \hat{O}_1 \) and \(-1.01^{+0.05}_{-0.04}\) for \( \hat{O} = \hat{O}_2 \) with 95% confidence, which implies the \( 1/D \) scaling of the accuracy. We have also conducted similar analyses for various quench magnitudes and obtained the \( 1/D \) scaling for the range of \( 0.2 \leq u_f \leq 0.75 \) as shown in Fig. 2(b). The discrepancies of the exponents from -1 seen for \( 0 \leq u_f \leq 0.2 \) and \( u_f \geq 0.75 \) will be addressed later, and we here focus on the implications of the \( 1/D \) scaling of the accuracy.

We rephrase the \( 1/D \) scaling in terms of the number \( N \) of HCBs in our model with a general filling factor \( \nu \equiv N/L \), which has been fixed to be 1/3 in the above discussions. The dimension \( D \) is approximately given by \( D \approx (b/N)/L \approx 10^{0.5N} \), where \( \gamma(\nu) = -\nu^{-1}[\nu \log_{10} \nu + (1 - \nu) \log_{10}(1 - \nu)] \) is a monotonically decreasing function giving, in particular, \( \gamma(1/4) = 0.9768 \cdots \approx 1 \). Thus the \( 1/D \) scaling implies that, at 1/4 filling, the accuracy improves by one order of magnitude as we increase the number of HCBs by one.

Two ETH upper bounds on the accuracy.— Before discussing the underlying mechanism for the \( 1/D \) scaling of the accuracy, we derive from the ETH two upper bounds on the accuracy of the ME by assuming that the energy distribution \( |c_n|^2 \) is localized. Then we point out that the upper bounds decrease proportionally to \( 1/\sqrt{D} \), implying that the ETH alone cannot explain the \( 1/D \) scaling.

First, we note that the EEVs \( O_n \) of a few-body observable \( \hat{O} \), in nonintegrable systems are known to behave as

\[
O_n = f(E_n/L) + \delta O_n, \tag{7}
\]

where \( f(x) \) is a smooth function and \( \delta O_n \) represents random fluctuations around it \([\text{R}]\). Correspondingly, the error of the ME is decomposed into two parts:

\[
\langle \hat{O} \rangle_{\text{DE}} - \langle \hat{O} \rangle_{\text{ME}} = \Delta O_{\text{sys}} + \Delta O_{\text{rand}}, \tag{8}
\]

where

\[
\Delta O_{\text{sys}} = \sum_n |c_n|^2 f(E_n/L) - f_{ME}, \tag{9}
\]

\[
\Delta O_{\text{rand}} = \sum_n |c_n|^2 \delta O_n \tag{10}
\]

with \( f_{ME} \equiv N_{E_C,\delta}^{-1} \sum_{n \in f(E_C,\delta)} f(E_n/L) \). Here we have assumed that \( N_{E_C,\delta}^{-1} \sum_{n \in f(E_C,\delta)} S_n = 0 \) because the local average of the random fluctuation is negligibly small.

Second, we note that \( \Delta O_{\text{sys}} \) is negligible if the energy distribution \( |c_n|^2 \) is localized. In fact, by the Taylor expansion of \( f(E_n/L) \) up to the second order, we have \( \Delta O_{\text{sys}} \approx 0 \) because we choose \( E_C \)
and \( \delta \) so that \( \langle \psi_{\text{initial}} | \hat{H}(u_f)^{\dagger} | \psi_{\text{initial}} \rangle \approx \langle \hat{H}(u_f) \rangle_{\text{ME}} \) and \( \langle \psi_{\text{initial}} | \hat{H}(u_f)^{\dagger} 2 | \psi_{\text{initial}} \rangle \approx \langle \hat{H}(u_f)^{\dagger} \rangle_{\text{ME}} \).

Thus, if the energy distribution \( |c_n|^2 \) is sufficiently localized, the error of the ME is dominated by \( \Delta O_{\text{rand}} \), which is bounded from above strictly by \( \Delta O \equiv \max_n |\delta O_n| \) and roughly by the standard deviation \( \sigma_O \) of \( \{\delta O_n\}_n \). We note that \( \Delta O \) and \( \sigma_O \) should be calculated in an energy window where \( |c_n|^2 \)'s are significantly weighted instead of the entire spectrum. In the following discussion, we take the window to be the microcanonical one for simplicity. We call both \( \Delta O \) and \( \sigma_O \) the ETH upper bounds because they are commonly used as the indicators of the ETH in the strong and weak senses, respectively, which imply \( \Delta O \to 0 \) and \( \sigma_O \to 0 \) in the TDL 22 29 30 31.

It has recently been shown that \( \sigma_O \) approaches zero proportionally to \( 1/\sqrt{D} \) in nonintegrable spin systems 23 24 40. We have also obtained the \( 1/\sqrt{D} \) scaling for both ETH indicators \( \Delta O \) and \( \sigma_O \) as illustrated in Fig. 3. These results imply that the ETH alone cannot explain the much greater improvement in the accuracy of the ME proportional to \( 1/D \) found in our numerical study.

The underlying mechanism for the \( 1/D \) scaling.— The second main result is that yet another indicator

\[
\tilde\sigma_O \equiv \frac{\sigma_O}{\sqrt{d_{\text{eff}}}}
\]

(11)
can describe the accuracy as illustrated in Fig. 3. This indicator involves, in addition to \( \sigma_O \), an extra suppression factor \( 1/\sqrt{d_{\text{eff}}} \). Since \( d_{\text{eff}} \) represents the effective number of nonzero terms on the RHS of Eq. (10), the extra suppression factor \( 1/\sqrt{d_{\text{eff}}} \) implies that there is little correlation between the terms.

It has been proposed as an alternative to the ETH 23 41 42 that the absence of correlation between the energy distribution \( |c_n|^2 \) and the EEV \( O_n \) suppresses the error of the ME. Indeed this is the case for our setup since the initial states are chosen independently of the observables \( \hat{O}_1 \) and \( \hat{O}_2 \). However, this mechanism alone cannot explain why the accuracy is so good as Eq. (11) if \( |c_n|^2 \)'s are correlated with each other.

Let us derive Eq. (11) by introducing the "no-correlation model" in which there are no correlations between \( |c_n|^2 \)'s except for \( \sum_n |c_n|^2 = 1 \) and \( \sum_n |c_n|^2 = 1/d_{\text{eff}} \). To be more specific, we define \( |c_n|^2 = x_n \) by introducing positive random variables \( x_1, \ldots , x_D \) that independently obey the same probability distribution \( p(x) \) whose first and second moments are \( 1/D \) and \( 1/(d_{\text{eff}} D) \), respectively. Then, we obtain \( (\Delta O_{\text{rand}})^2 = (d_{\text{eff}}^{-1} - 1/D)\sigma_O^2 \), where \( \overline{\cdots} \) denotes the statistical average with the probability distribution \( P(x_1, x_2, \ldots , x_D) \equiv p(x_1)p(x_2) \cdots p(x_D) \) and we have used \( 1/D \sum_n \delta O_n = 0 \) and \( 1/D \sum_n (\delta O_n)^2 = \sigma_O^2 \) [43]. This result reduces to Eq. (11) when \( d_{\text{eff}}/D \ll 1 \), which holds in our numerical study.

Our result that the no-correlation model explains the \( 1/D \) scaling of the accuracy implies that we cannot induce correlations between \( |c_n|^2 \)'s through a sudden change of a single parameter. While we can control the total energy by changing the parameter, we cannot manipulate individual many-body eigenstates whose landscape changes drastically from one to the neighboring one for nonintegrable systems.

Discussions.— First, we speculate how the \( 1/D \) scaling of the accuracy implies that we cannot induce correlations between \( |c_n|^2 \)'s through a sudden change of a single parameter. While we can control the total energy by changing the parameter, we cannot manipulate individual many-body eigenstates whose landscape changes drastically from one to the neighboring one for nonintegrable systems.
scaling at the large quench seen in Fig. 2(b).

Second, we speculate how the $1/D$ scaling changes in smaller system sizes. As we decrease the system size with the quench magnitude held fixed, the quench energy becomes smaller than the energy level spacings and we enter the regime where $d_{\text{eff}} \sim 1$. Thus, for a given quench magnitude, there exists $L_{\text{lower}}$ below which the $1/D$ scaling disappears. In this case, Eq. (11) reduces to the indicator of the ETH in the weak sense and the accuracy is proportional to $1/\sqrt{D}$. This crossover between the $1/D$ scaling and the $1/\sqrt{D}$ scaling is seen for $u_f = 0.05$ as shown in Fig. 2(b). Thus, the lack of mixing of numerous energy eigenstates is the cause for the deviation from the $1/D$ scaling at the small quench as seen in Fig. 2(b).

Thus, we find three regimes of the system size for a given quench magnitude: (i) $L < L_{\text{lower}}$ where the accuracy is described by the ETH and proportional to $1/\sqrt{D}$, (ii) $L_{\text{lower}} < L < L_{\text{upper}}$ where the accuracy is proportional to $1/D$ due to the absence of correlations between the many-body eigenstates, and (iii) $L_{\text{upper}} < L$ where the accuracy improves only algebraically with $L$ because $\Delta E_{\text{sys}}$ dominates in the error of the ME rather than $\Delta E_{\text{rand}}$ (see Eq. (8)).

Our findings imply that there exists an exponential enhancement of the accuracy in the small system sizes and the ME can describe equilibrium states quite accurately even in small isolated quantum systems. In fact, the accuracy of the ME reaches $10^{-4}$ or even better in the system with only 8 HCBs on 24 sites as shown in Figs. 3 and 4.

Conclusions.— We have numerically investigated the accuracy of the ME in interaction quenches for a nonintegrable hard-core Bose-Hubbard model. We have found a regime where the accuracy improves proportionally to $1/D$ (see Fig. 2). This rapid improvement of the accuracy implies that quenching a single parameter cannot induce correlations between the numerous many-body eigenstates since they depend nontrivially on the parameter in nonintegrable systems. As we increase the system size, there are three regimes where the accuracy improves (i) proportionally to $1/\sqrt{D}$ (ETH regime), (ii) proportionally to $1/D$ (no-correlation regime), and (iii) algebraically with $L$. Due to the regimes (i) and (ii), where the accuracy improves exponentially with $L$, the ME can describe the equilibrium states quite accurately even in small systems.

Acknowledgements.— Fruitful discussions with Shunsuke Furukawa, Kohaku H. Z. So, and Tomohiro Shiota are gratefully acknowledged. We also thank Hyungwon Kim to have given helpful comments on the manuscript. This work was supported by KAKENHI 26287088, a Grant-in-Aid for Scientific Research on Innovation Areas “Topological Quantum Phenomena” (KAKENHI 22103005), and the Photon Frontier Network Program, from MEXT of Japan. T. N. I. acknowledges the JSPS for financial support (Grant No. 248408).

[1] L. Landau and E. Lifshitz, *Statistical Physics*, v. 5 (Elsevier Science, 1996).
[2] J. von Neumann, Eur. Phys. J. H 35, 201 (2010).
[3] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Proc. R. Soc. A Math. Phys. Eng. Sci. 466, 3203 (2010).
[4] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Eur. Phys. J. H 35, 173 (2010).
[5] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalatore, Rev. Mod. Phys. 83, 863 (2011).
[6] V. Yukalov, Laser Phys. Lett. 8, 485 (2011).
[7] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature 440, 900 (2006).
[8] S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Nat. Phys. 8, 325 (2012).
[9] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Science 337, 1318 (2012).
[10] T. Langen, R. Geiger, M. Kuhnert, B. Rauer, and J. Schmiedmayer, Nat. Phys. 9, 640 (2013).
[11] H. Tasaki, Phys. Rev. Lett. 80, 1373 (1998).
[12] P. Reimann, Phys. Rev. Lett. 101, 190403 (2008).
[13] N. Linden, S. Popescu, A. Short, and A. Winter, Phys. Rev. E 79, 061103 (2009).
[14] A. J. Short, New J. Phys. 13, 055009 (2011).
[15] H. Tasaki, arXiv:1003.5424 (2010).
[16] A. Sugita, RIMS Kokyuroku 1507, 147 (2006).
[17] S. Popescu, A. J. Short, and A. Winter, Nat. Phys. 2, 754 (2006).
[18] S. Goldstein, J. Lebowitz, R. Tumulka, and N. Zanghì, Phys. Rev. Lett. 96, 050403 (2006).
[19] P. Reimann, Phys. Rev. Lett. 99, 160404 (2007).
[20] J. Deutsch, Phys. Rev. A 43, 2046 (1991).
[21] M. Srednicki, Phys. Rev. E 50, 888 (1994).
[22] M. Rigol, V. Dunjko, and M. Olshanii, Nature 452, 854 (2008).
[23] M. Rigol and M. Srednicki, Phys. Rev. Lett. 108, 110601 (2012).
[24] M. Rigol, Phys. Rev. A 80, 053607 (2009).
[25] M. Rigol and L. F. Santos, Phys. Rev. A 82, 8 (2010).
[26] L. F. Santos and M. Rigol, Phys. Rev. E 82, 031130 (2010).
[27] W. Beugeling, R. Moessner, and M. Haque, Phys. Rev. E 89, 042112 (2014).
[28] H. Kim, T. N. Ikeda, and D. A. Huse, Phys. Rev. E 90, 052105 (2014).
[29] R. Steinigeweg, A. Khodja, H. Niemeyer, C. Gogolin, and J. Gemmer, Phys. Rev. Lett. 112, 130403 (2014).
[30] A. Khodja, R. Steinigeweg, and J. Gemmer, Phys. Rev. E 91, 012120 (2015).
[31] F. Serrano, G. Zürn, T. Löwe, T. B. Ottenstein, A. N. Wenz, and J. Jochim, Science 332, 336 (2011).
[32] G. Zürn, A. N. Wenz, S. Murmann, A. Bergschneider, T. Löwe, and J. Jochim, Phys. Rev. Lett. 111, 175302 (2013).
[33] A. N. Wenz, G. Zürn, S. Murmann, I. Brouzos, T. Löwe, and J. Jochim, Science 342, 457 (2013).
We have confirmed that the results shown in this paper do not qualitatively change if the next-nearest-neighbor hopping and interaction energies are greater than 0.1, where the Wigner-Dyson-like statistics is seen.

O. Bohigas, M. Giannoni, and C. Schmit, Phys. Rev. Lett. 52, 1 (1984).

L. F. Santos and M. Rigol, Phys. Rev. E 81, 036206 (2010).

We regard the mean square of our data as the sum of the squared mean and the variance, which are unbiasedly estimated. Then, with the chain rule, we obtain the error of the mean square from those of the mean and the variance, which are estimated from the variances of their unbiased estimators.

M. Srednicki, J. Phys. A. Math. Gen. 29, L75 (1996).

G. Biroli, C. Kollath, and A. M. Läuchli, Phys. Rev. Lett. 105, 250401 (2010).

T. N. Ikeda, Y. Watanabe, and M. Ueda, Phys. Rev. E 87, 012125 (2013).

A. Peres, Phys. Rev. A 30, 504 (1984).

T. N. Ikeda, Y. Watanabe, and M. Ueda, Phys. Rev. E 84, 021130 (2011).

In general, $D^{-1} \sum_{n=1}^{D} (\delta O_n)^2$ may be different from $\sigma_0^2$, which is defined locally rather than in the entire spectrum. Even in this case, unless $|c_n|^2$ covers the entire spectrum, we can repeat a similar discussion and justify Eq. (11) by defining the non-correlation model on the effective support of $|c_n|^2$. 

T. N. Ikeda, Y. Watanabe, and M. Ueda, Phys. Rev. E 84, 021130 (2011).