Nonlinear Landau damping for the 2d Vlasov-Poisson system with massless electrons around Penrose-stable equilibria

Lingjia Huang∗, Quoc-Hung Nguyen† and Yiran Xu‡

Abstract

In this paper, we prove the nonlinear asymptotic stability of the Penrose-stable equilibria among solutions of the 2d Vlasov-Poisson system with massless electrons.

1 Introduction

This paper is devoted to study the Vlasov-Poisson system of the form:

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f &= 0, \\
E &= -\nabla_x U, \quad -\Delta U + U = \rho - 1 + A(U), \quad \rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v) dv,
\end{align*}
\]

on the whole space \(x \in \mathbb{R}^2, v \in \mathbb{R}^2\), where \(f = f(t, x, v) \geq 0\) is the probability distribution of charged particles in plasma, \(\rho(t, x)\) is the electric charge density, and \(E = E(t, x)\) is electric field and \(A : \mathbb{R} \to \mathbb{R}\) is smooth and satisfies \(A(r) = O(r^2)\) as \(r \to 0\). In particular, the system is for massless electrons or ions when \(A(r) = r + 1 - e^r\). This system was extensively studied ([1–4, 10–13, 16–19, 22, 23, 25, 27, 30, 36–39]), focusing on the global existence, regularity results and longtime behavior of solutions. We are interested in the asymptotic stability of solutions \(f_i\) to (1.1) in the form

\[
f(t, x, v) = \mu(v) + f(t, x, v)
\]

where \(\mu(v)\) is a stable equilibrium with \(\int_{\mathbb{R}^2} \mu(v) dv = 1\) and \(f(t = 0, x, v)\) closes to \(\mu(v)\). So, \(f\) solves the following perturbed system

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu &= -E \cdot \nabla_v f, \\
E &= -\nabla_x U, \quad -\Delta U + U = \rho + A(U), \quad \rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v) dv,
\end{align*}
\]

(1.2)

The following are assumptions on \(\mu\) and \(A\) in this paper.

- **Assumption 1:** \(\mu\) satisfies the Penrose stability condition:

\[
\inf_{\tau \in \mathbb{R}, \xi \in \mathbb{R}^2} \left| 1 - \int_{0}^{\infty} e^{-is} \frac{1}{1 + |\xi|^2} i\xi \cdot \nabla_v \mu(s\xi) ds \right| \geq \check{c},
\]

(1.3)

for some constant \(\check{c} > 0\), \(\nabla_v \mu\) is the Fourier transform of \(\nabla_v \mu\) in \(\mathbb{R}^2\).
for the density $\rho$ pointwise dispersive estimates of the linearized system of (1.2) to obtain the decay estimates of their result is far from optimal. In [20], Han-Kwan, T. Nguyen, and Rousset used a mechanism in Fourier space to control the plasma echo resonance. However, a decay in Bedrossian, Masmoudi, and Mouhot with Sobolev data. Their proof relies on the dispersive asymptotic stability result for the unscreened Vlasov-Poisson system (i.e., $A = 0$) in $\mathbb{R}$ linearized unscreened Vlasov-Poisson equation around suitably stable analytic homogeneous equilibria in $\mathbb{R}$ with Gevrey or analytic data (see also [9,14] for refinements and simplifications). In [6] Bedrossian showed that the results therein do not hold in finite regularity (see also [15]). We note that related mechanisms in the fluid correction.

Before we state our theorem, we need to recall some notations in [24]. For $m \in \mathbb{N}$, and $A : \mathbb{R} \to \mathbb{R}$ is $C^3$ and satisfies

$$\sup_{|r| \leq 1} \left( \frac{|A(r)|}{r^2} + \frac{|A'(r)|}{r} + |A''(r)| + |A'''(r)| \right) \leq C_A,$$

for some constant $C_A > 0$.

Note that a particular example for the Assumption 3 is $A(r) = r + 1 - e^{-r}$ corresponding to the Vlasov–Poisson system with electrons mass, see [10].

Under the Penrose condition (1.3), Landau damping was studied in the breakthrough paper [31] by Mouhot and Villani in the case $\mathbb{R}^d \times \mathbb{R}^d$ with Gevrey or analytic data (see also [9,14] for refinements and simplifications). In [6] Bedrossian showed that the results therein do not hold in finite regularity (see also [15]). We note that related mechanisms in the fluid are the vorticity mixing by shear flows [5,26,29,32]. In the whole space $\mathbb{R}^d \times \mathbb{R}^d$ with $d \geq 3$, it was established for the screened Vlasov-Poisson system (i.e., $A(U) = 0$) in $\mathbb{R}$ by Bedrossian, Masmoudi, and Mouhot with Sobolev data. Their proof relies on the dispersive mechanism in Fourier space to control the plasma echo resonance. However, a decay in time of their result is far from optimal. In [20], Han-Kwan, T. Nguyen, and Rousset used pointwise dispersive estimates of the linearized system of (1.2) to obtain the decay estimates for the density $\rho$ as follows:

$$\sum_{j=0,1} \left[ (1 + t)^j \|\nabla^j \rho(t)\|_{L^1} + (1 + t)^{d+j} \|\nabla^j \rho(t)\|_{L^\infty} \right] \lesssim \varepsilon_0 \log(t + 2), \quad \forall \ t > 0,$$

with $d \geq 3$. Note that (1.3) is optimal up to a logarithmic correction. Also, higher derivatives for the density were established in [34]. Note that the problem (1.2) in dimension $d = 2$ is critical and open. Recently, in [28] Ionescu, Pausader, Wang, and Widmayer proved the first asymptotic stability result for the unscreened Vlasov-Poisson system (i.e., $A(U) = U$) in $\mathbb{R}^3$ around the Poisson equilibrium, see also in [35] for the case of a repulsive point charge. The unscreened case is open for the general equilibria. However, in [7,21] they studied the linearized unscreened Vlasov-Poisson equation around suitably stable analytic homogeneous equilibria in $\mathbb{R}_+^d \times \mathbb{R}_+^d$.

Very recently, authors in [24] established new estimates and cancellations of the kernel to the linearized problem (see Proposition 2.2) and proved the sharp decay estimates for the density $\rho$ in Besov spaces:

$$\sum_{j=0,1} \sum_{p=1,\infty} \left( (1 + t)^j \frac{d(a-1)}{p} \|\nabla^j \rho(t)\|_{L^p} + (1 + t)^{j+a+\frac{d(a-1)}{p}} \|\nabla^j \rho(t)\|_{B^p_{p,\infty}} \right) \lesssim \varepsilon_0, \quad \forall \ t > 0,$$

with $d \geq 3$, for some $a \in (0,1)$. In particular, this implies (1.4) without logarithmic correction.

Our goal in this paper is to extend our work in [24] to dimension $d = 2$.

Before we state our theorem, we need to recall some notations in [24]. For $\gamma \in (0,1)$ and $m \in \mathbb{N}$, and $g : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, we define for $T > 0$,

$$\|g\|_{m+\gamma,T} = \sum_{j=0}^m \sum_{p=1,\infty} \sup_{s \in [0,T]} \left( \langle s \rangle^{\frac{d(a-1)}{p}} \|g(s)\|_{L^p} + \langle s \rangle^{j+\gamma+\frac{d(a-1)}{p}} \|\nabla^j g(s)\|_{B^p_{p,\infty}} \right).$$
Let \( T > 0 \) such that the hypothesis of Corollary 1.4 holds. The proof is omitted as it is analogous to [20, Proof of Corollary 1.1].

Remark 1.2: Our method does not work in the 1d dimension case because the density \( \rho \) is not decaying enough to estimate for characteristics.

Remark 1.3: While finishing our work, we learned that Toan Nguyen was working on this problem and had a similar result to ours. But the two works are independent.

Our main result is as follows.

**Theorem 1.1** Let \( a \in (0, 1) \). There exist \( C_0 > 0 \), \( \varepsilon \in (0, 1) \) such that if \( \|f_0\|_{1+a} \leq \varepsilon \), and \( \liminf_{k \to 0} \|f_0 - f_0^k\|_{1+a} = 0 \) for some sequence \( f_0^k \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \). Then the problem (1.2) has a unique global solution \( f \) with

\[
\|f\|_{1+a} + \|U\|_{1+a} \leq C_0 \|f_0\|_{1+a}.
\]

**Remark 1.4**

As [20 Corollary 1.1], thanks to Theorem 1.1, we also obtain the following scattering property for the solution to (1.2). The proof is omitted as it is analogous to [20 Proof of Corollary 1.1].

**Corollary 1.4** With the same assumptions and notations as in Theorem 1.1, there is a function \( f_\infty \in W_{1, \infty} \) given by

\[
f_\infty(x, v) = f_0(x + Y_\infty(x, v), v + W_\infty(x, v)) + \mu(v + W_\infty(x, v)) - \mu(v),
\]

such that

\[
\sup_{t>0} \|f(t, x + tv, v) - f_\infty(x, v)\|_{L_{p, \infty}} \lesssim \|f_0\|_{1+a}, \quad \|Y_\infty\|_{L_{p, \infty}} + \|W_\infty\|_{L_{p, \infty}} \lesssim \|f_0\|_{1+a}.
\]

The following is a key proposition of this paper. Let \( T > 0 \), \( \mathcal{g} : [0, T] \times \mathbb{R}^2 \to \mathbb{R} \). One considers

\[
E(t, x) = -\nabla(-\Delta + 1)^{-1}(\mathcal{g})(t, x), \quad \|\mathcal{g}\|_{1+a, T} < \infty.
\]

Let \((X_{s,t}(x, v), V_{s,t}(x, v))\) be the flow associated to the vector field \((v, E(t, x))\), i.e.

\[
\begin{cases}
\frac{d}{dt}X_{s,t}(x, v) = V_{s,t}(x, v), & X_{s,s}(x, v) = x, \\
\frac{d}{dt}V_{s,t}(x, v) = E(s, X_{s,t}(x, v)), & V_{s,s}(x, v) = v,
\end{cases}
\]

for any \( 0 \leq s \leq t \leq T \), \((x, v) \in \mathbb{R}^2 \times \mathbb{R}^2 \). Hence,

\[
X_{s,t}(x, v) = x - (t - s)v + Y_{s,t}(x - vt, v), \quad V_{s,t}(x, v) = v + W_{s,t}(x - vt, v),
\]

where

\[
Y_{s,t}(x, v) = \int_s^t (\tau - s)E(\tau, x + \tau v + Y_{\tau,t}(x, v))d\tau,
\]

\[
W_{s,t}(x, v) = -\int_s^t E(\tau, x + \tau v + Y_{\tau,t}(x, v))d\tau.
\]
Proposition 1.5 Let $a \in (0, 1)$, then there exists $\varepsilon_0 \in (0, 1)$ such that for any $\|g\|_{1+a,T} \leq \varepsilon_0$, we have

$$
\sup_{0 \leq s \leq t \leq T} \sup_{\alpha} \left\| \frac{\delta^\alpha_{\gamma} \nabla^\alpha_{\gamma} Y_{s,t}}{|\alpha|^a} \right\|_{L^\infty_{x,v}} + \sup_{0 \leq s \leq t \leq T} \left( s \sup_{\alpha} \left\| \frac{\delta^\alpha_{\gamma} \nabla^\alpha_{\gamma} W_{s,t}}{|\alpha|^a} \right\|_{L^\infty_{x,v}} \right) \lesssim_a \|g\|_{1+a,T}, \quad (1.10)
$$

where

$$
\delta^\alpha_{\gamma} Y_{s,t}(x, v) = Y_{s,t}(x, v) - Y_{s,t}(x, v - \alpha), \quad \delta^\alpha_{\gamma} W_{s,t}(x, v) = W_{s,t}(x, v) - W_{s,t}(x, v - \alpha).
$$

Remark 1.6 Estimate of the first term (in LHS) in (1.10) is critical. It is a key estimate in the proof of Theorem 1.1.

The strategy to prove Theorem 1.1 in this paper is slightly different from that in our previous paper [24]. First, we use the fixed-point theorem with a local in-time norm to prove the local existence result (see Proposition 2.4), then we establish a suitable bootstrap property in (1.3) in our paper [24]. First, we use the fixed-point theorem with a local in-time norm to prove the local existence result (see Proposition 2.4), then we establish a suitable bootstrap property in (1.3) in our paper [24]. We finish the proof of Theorem 1.1 by using a bootstrap argument.

The paper is organized as follows. In section 2, we establish the equivalence of $\rho$ and $f$, then we state our bootstrap property and local existence result; in the last part of this section, we prove Theorem 1.1. In section 3, we prove pointwise estimates of characteristics ($Y_{s,t}, W_{s,t}$) in (1.9). In section 4, we estimate the contribution of the initial data and the reaction term. In Section 5, we prove our bootstrap property and local existence result.

Acknowledgements: Q.H.N. is supported by the Academy of Mathematics and Systems Science, Chinese Academy of Sciences startup fund, and the National Natural Science Foundation of China (No. 12050410257 and No. 12288201) and the National Key R&D Program of China under grant 2021YFA1000800. Q.H.N. also wants to thank Benoit Pausader for his stimulating comments and suggestion to consider the Vlasov-Poisson system with massless electrons.

2 Dynamics of the density and bootstrap argument

In this section, we recast the system (1.2) as a problem for the density $\rho$. Let $(X_{s,t}(x, v), V_{s,t}(x, v))$ be the flow associated to the vector field $(v, E(t, x))$. Then, the solution $f$ of the equation (1.2) is given by

$$
f(t, x, v) = f_0(X_{0,t}(x, v), V_{0,t}(x, v)) - \int_0^t E(s, X_{s,t}(x, v)) \cdot \nabla_v \mu(V_{s,t}(x, v)) ds, \quad (2.1)
$$

(see (1.3) in [24] with $B(t, x, v) = (v, E(t, x))$). Set $g = \rho + A(U)$, so $E = -\nabla_x (-\Delta + 1)^{-1} g$. We denote

$$
\mathcal{I}_f_0(g)(t, x) = \int_{\mathbb{R}^2} f_0(X_{0,t}(x, v), V_{0,t}(x, v)) dv,
$$

$$
\mathcal{R}(g)(t, x) = \int_0^t \int_{\mathbb{R}^2} (E(s, x - (t - s)v) \cdot \nabla_v \mu(v) - E(s, X_{s,t}(x, v)) \cdot \nabla_v \mu(V_{s,t}(x, v))) dv ds.
$$

As [24] (2.5)], we obtain that $\rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v) dv$ is the solution of the following equation:

$$
\rho = G \ast_{(t,x)} (\mathcal{I}(g) + \mathcal{R}(g) + A(U)) + \mathcal{I}(g) + \mathcal{R}(g), \quad (2.3)
$$
where \( G \) is the kernel satisfying:
\[
\tilde{G}(\tau, \xi) = \int_0^{+\infty} e^{-i\tau t} \frac{1}{1 + |\xi|^2} i\xi \cdot \tilde{\nu}_\mu(t\xi) dt,
\]
and \( \tilde{\cdot} \) is the "space-time" Fourier transform on \( \mathbb{R}^2 \times \mathbb{R} \). Thanks to the Penrose condition, one has \( |1 - \tilde{\nu}| \geq \tilde{c} > 0 \), which implies that \( \tilde{G} \) is well defined.

Define for \( T > 0 \)
\[
\mathcal{V}_\varepsilon := \{ h \in L^1 \cap L^\infty(\mathbb{R}^2) : \|h\|_{L^1 \cap L^\infty(\mathbb{R}^2)} \leq \varepsilon \}.
\]
By the standard argument, one has

**Lemma 2.1** Let \( \varepsilon \in (0, 1) \). There exist \( C_0 \geq 1 \) and \( \tilde{\varepsilon}_0 \in (0, 1) \) such that if \( g \in \mathcal{V}_{\tilde{\varepsilon}_0} \), the problem
\[
-\Delta u + u = g + A(u)
\]
has a unique solution \( u \) in \( \mathcal{V}_{c\tilde{\varepsilon}_0} \) satisfying
\[
\sum_{j=0}^{2} \sum_{p=1,\infty} \left( \| (u, A(u))\|_{L^p} + \| \nabla^j (u, A(u))\|_{B^p_{|\xi|}} \right) \leq C_0 \sum_{p=1,\infty} \left( \| g\|_{L^p} + \| g\|_{B^p_{|\xi|}} \right). \tag{2.6}
\]
In addition, we can define a map \( \mathcal{N} : \mathcal{V}_{\tilde{\varepsilon}_0} \rightarrow \mathcal{V}_{c\tilde{\varepsilon}_0} : u = \mathcal{N}(g) \) for any \( \rho \in \mathcal{V}_{c\tilde{\varepsilon}_0} \). The map \( \mathcal{N} \) satisfies
\[
\sum_{j=0}^{2} \sum_{p=1,\infty} \left( \| \mathcal{N}(g_1) - \mathcal{N}(g_2)\|_{L^p} + \| \nabla^j (\mathcal{N}(g_1) - \mathcal{N}(g_2))\|_{B^p_{|\xi|}} \right) \leq C_0 \sum_{p=1,\infty} \left( \| g_1 - g_2\|_{L^p} + \| g_1 - g_2\|_{B^p_{|\xi|}} \right). \tag{2.7}
\]

Thanks to Lemma 2.1, \( g \) and \( U \) in (2.3) can be performed via \( \rho \):
\[
g(t) = \rho(t) + A(\mathcal{N}(\rho)(t)), \quad U(t) = \mathcal{N}(\rho(t)), \tag{2.7}
\]
provided that \( \rho(t) \in \mathcal{V}_{c\tilde{\varepsilon}_0} \). In particular, we can write (2.3) as an equation for the density \( \rho \) in \([0, T]\) when \( \rho(t) \in \mathcal{V}_{c\tilde{\varepsilon}_0} \) for any \( t \in [0, T] \).

The following is the boundedness of the operator \( \mathcal{G} := G_{(t, x)} \) in [24, Theorem 3.9].

**Proposition 2.2** There holds
\[
\| G_{(t, x)} \|_{1+a,T} \leq C \| g\|_{1+a,T}, \tag{2.8}
\]
where \( C = C(\| \cdot \|_{1+a,\mathbb{R}^2}, 1, \xi, \varepsilon) \).

The following is our main bootstrap proposition.

**Proposition 2.3** Let \( a \in (0, 1) \). There exist \( C_1 \geq 1 \), \( \varepsilon_1 \in (0, 1) \) such that if \( \| f_0\|_{1+a} < \infty \) and the problem (1.2) has a unique solution \( (f, U) \) in \([0, T]\) for some \( T < \infty \) with
\[
\| \rho\|_{1+a,T} + \| U\|_{1+a,T} \leq \varepsilon_1. \tag{2.9}
\]
Then, we have
\[
\| \rho\|_{1+a,T} + \| U\|_{1+a,T} \leq C_1 \| f_0\|_{1+a}.
\]
Here is the local well-posedness result.
Proposition 2.4 Let $a \in (0,1)$. There exists $\varepsilon_2 > 0$ such that if $\rho(0,x) = \int_{\mathbb{R}^2} f_0(x,v)dv$ satisfies
\begin{equation}
\sum_{p=1,\infty} ||\rho(0)||_{L^p} + ||\nabla \rho(0)||_{B^p_{\infty}} \leq \varepsilon_2, \tag{2.10}
\end{equation}
and
\begin{equation}
|||f_0|||_{1+a} < \infty, \quad \lim_{\kappa \to 0} |||f_0 - f_0^\kappa|||_{1+a} = 0, \tag{2.11}
\end{equation}
for some sequence $f_0^\kappa \in C_c^\infty(\mathbb{R}_x^2 \times \mathbb{R}_{v}^2)$. Then, the problem (1.2) has a unique local solution $f$ in $[0,T]$ with
\begin{equation}
||\rho||_{1+a,T} + ||U||_{1+a,T} \leq C_2 \left( \sum_{p=1,\infty} ||\rho(0)||_{L^p} + ||\nabla \rho(0)||_{B^p_{\infty}} \right), \tag{2.12}
\end{equation}
and
\begin{equation}
\lim_{t \to T^+} \left( ||\rho - \rho(0)||_{1+a,t} + ||U - U(0)||_{1+a,t} \right) = 0, \tag{2.13}
\end{equation}
for some $T \in (0,1)$ where $C_2 \geq 1$ only depends on $a$ and $C_A$.

With Proposition 2.3 and Proposition 2.4 in hand, we can obtain Theorem 1.1.

Proof of Theorem 1.1. By (1.5), one has
\begin{equation}
\sum_{p=1,\infty} ||\rho(0)||_{L^p} + ||\nabla \rho(0)||_{B^p_{\infty}} \leq C_3 ||f_0||_{1+a}, \tag{2.14}
\end{equation}
for some $C_3 \geq 1$. Let $C_1 \geq 1$, $\varepsilon_1 \in (0,1)$ be in Proposition 2.3 and $C_2 \geq 1$, $\varepsilon_2 \in (0,1)$ be in Proposition 2.4

$T^* = \{ T : \text{the solution } f \text{ of (1.2) exists on } [0,T], \text{ s.t. } ||\rho||_{1+a,T} + ||U||_{1+a,T} \leq M ||f_0||_{1+a}, \}$
with $M = 100C_1C_2C_3$. Assume
\begin{equation}
|||f_0|||_{1+a} \leq \min\{\varepsilon_1, \varepsilon_2\} \tag{2.15}
\end{equation}
So, thanks to Proposition 2.4 the problem (1.2) has a unique local solution $f$ in $[0,T]$ satisfying
\begin{equation}
||\rho||_{1+a,T} + ||U||_{1+a,T} \leq C_2 \left( \sum_{p=1,\infty} ||\rho(0)||_{L^p} + ||\nabla \rho(0)||_{B^p_{\infty}} \right) \tag{2.16}
\end{equation}
This gives $T^* > 0$. Now we suppose that $T^* < \infty$. By (2.15) one has $||\rho||_{1+a,T^*} + ||U||_{1+a,T^*} \leq \varepsilon_1$. So, we can apply Proposition 2.3 to get that
\begin{equation}
||\rho||_{1+a,T^*} + ||U||_{1+a,T^*} \leq C_1 ||f_0||_{1+a}. \tag{2.16}
\end{equation}
In particular,
\begin{equation}
\sum_{p=1,\infty} ||\rho(T^*)||_{L^p} + ||\nabla \rho(T^*)||_{B^p_{\infty}} \leq C_1 ||f_0||_{1+a} \leq \varepsilon_2. \tag{2.16}
\end{equation}
Moreover, by Remark 5.1 one has $||f(T^*)||_{1+a} < \infty$ and $\lim_{\kappa \to 0} ||f(T^*) - f(T^\kappa)||_{1+a} = 0$ for some sequence $f_0^\kappa \in C_c^\infty(\mathbb{R}_x^2 \times \mathbb{R}_{v}^2)$. Hence, by Proposition 2.4 the solution $f$ can be extended from $[0,T^*]$ to $[0,T^* + \delta]$ for some $\delta > 0$ satisfying
\begin{equation}
||\rho||_{1+a,T^*+\delta} + ||U||_{1+a,T^*+\delta} \leq 2(||\rho||_{1+a,T^*} + ||U||_{1+a,T^*}), \tag{2.16}
\end{equation}
since (2.13). Combining this with (2.16) to get that
\begin{equation}
||\rho||_{1+a,T^*+\delta} + ||U||_{1+a,T^*+\delta} \leq 2C_1 ||f_0||_{1+a} < M ||f_0||_{1+a}, \tag{2.17}
\end{equation}
this contradicts to $T^* < \infty$. Therefore, $T^* = \infty$ and the proof is complete.
3 Pointwise estimates of characteristics

In this section, we will prove the following estimates of characteristics that it will be used in the proof of Proposition 2.3 and 2.4.

Proposition 3.1 Let \((Y_{s,t}, W_{s,t})\) be in (1.1). Let \(a \in (0, 1)\), then there exists \(\varepsilon_0 \in (0, 1)\) such that for any \(\|g\|_{1+a,T} \leq \varepsilon_0\), we have the following estimates:

\[
\sup_{0 \leq s \leq t \leq T} \left( \langle s \rangle \|Y_{s,t}\|_{L^\infty_{x,v}} + \langle s \rangle^{1+a} \|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}} + \langle s \rangle^{1+a} \sup_\alpha \|\partial^a_{x} \nabla_x Y_{s,t}\|_{L^\infty_{x,v}} \right)
+ \sup_{0 \leq s \leq t \leq T} \langle s \rangle^a \|\nabla_v Y_{s,t}\|_{L^\infty_{x,v}} \lesssim_a \|g\|_{1+a,T},
\]

(3.1)

\[
\sup_{0 \leq s \leq t \leq T} \left( \langle s \rangle^2 \|W_{s,t}\|_{L^\infty_{x,v}} + \langle s \rangle^{2+a} \|\nabla_x W_{s,t}\|_{L^\infty_{x,v}} + \langle s \rangle^{2+a} \sup_\alpha \|\partial^a_{x} \nabla_x W_{s,t}\|_{L^\infty_{x,v}} \right)
+ \sup_{0 \leq s \leq t \leq T} \langle s \rangle^{1+a} \|\nabla_v W_{s,t}(x,v)\|_{L^\infty_{x,v}} \lesssim_a \|g\|_{1+a,T},
\]

(3.2)

and

\[
\sup_{0 \leq s \leq t \leq T} \sup_\alpha \frac{\|\partial^a_{x} \nabla_x Y_{s,t}\|_{L^\infty_{x,v}}}{|\alpha|^a} \lesssim_a \|g\|_{1+a,T}.
\]

(3.3)

Moreover, for any \(0 \leq s \leq t < T\), we have a \(C^1\) map \((x,v) \mapsto \Psi_{s,t}(x,v)\), which satisfies for all \(x,v \in \mathbb{R}^2\):

\[
X_{s,t}(x, \Psi_{s,t}(x,v)) = x - (t-s)v,
\]

(4.4)

and

\[
\langle s \rangle^2 |\Psi_{s,t}(x,v) - v| + \langle s \rangle^{2+a} |\nabla_x \Psi_{s,t}(x,v)| + \langle s \rangle^{1+a} |\nabla_v (\Psi_{s,t}(x,v)) - v| \lesssim_a \|g\|_{1+a,T}.
\]

(3.5)

Here we use the notations

\[
\partial^a_{x} Y_{s,t}(x,v) = Y_{s,t}(x,v) - Y_{s,t}(x - \alpha, v), \quad \partial^a_{x} Y_{s,t}(x,v) = Y_{s,t}(x,v) - Y_{s,t}(x,v - \alpha).
\]

Clearly, Proposition 3.1 follows from (3.2) and (3.3).

Proof of Proposition 3.1. Thanks to [21] Lemma 4.2 we have for any \(\tau \in [0,T]\),

\[
\langle \tau \rangle^{3} \|E(\tau)\|_{L^\infty} + \langle \tau \rangle^{3+a} \|\nabla_x E(\tau)\|_{L^\infty} + \langle \tau \rangle^{3+a} \|\nabla^2_x E(\tau)\|_{L^\infty} \leq c_0\|g\|_{1+a,T}.
\]

(3.6)

Using the same argument as [21] Proof of Proposition 4.1] and (3.3), we obtain (3.1) and (3.2). Now, we prove (3.3). One has

\[
\sup_\alpha \frac{\|\partial^a_{x} \nabla_x Y_{s,t}\|_{L^\infty_{x,v}}}{|\alpha|^a} \leq \sup_\alpha \frac{1}{|\alpha|^a} \left( \int_s^t \langle \tau - s \rangle \|\nabla_x E(\tau)\|_{L^\infty} \|\partial^a_{x} \nabla_x Y_{s,t}\|_{L^\infty_{x,v}} d\tau \right)
+ \int_s^t \langle \tau - s \rangle \min \left\{ \|\nabla_x E(\tau)\|_{L^\infty}, \|\nabla^2_x E(\tau)\|_{L^\infty} \left( |\alpha| |\tau| + \|\partial^a_{x} Y_{s,t}\|_{L^\infty_{x,v}} \right) \right\} \left( |\tau| + \|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}} \right) d\tau.
\]
By (3.3) and (3.1), one gets

\[
\begin{align*}
\sup_{\alpha} \frac{\|\delta_a^\alpha \nabla v Y_{s,t} \|_{L^\infty_{x,v}}}{|\alpha|^a} & \leq c_0 \int_0^T \frac{\tau-s}{(\tau)^3} \|g\|_{1+a,T} \sup_{s \leq \tau \leq t} \sup_{\alpha} \|\delta_a^\alpha \nabla v Y_{s,t} \|_{L^\infty_{x,v}} \frac{1}{|\alpha|^a} \int_s^t (\tau-s) \min \left\{ \frac{1}{(\tau)^{3+a}}, \frac{|\alpha|}{(\tau)^{2+a}} \right\} d\tau \\
& + c_0 \|g\|_{1+a,T} \frac{1}{10} \sup_{s \leq \tau \leq t} \sup_{\alpha} \|\delta_a^\alpha \nabla v Y_{s,t} \|_{L^\infty_{x,v}} + c_0 \|g\|_{1+a,T} \frac{1}{10} \int_0^\infty \min \{1, |\alpha|/\tau \} d\tau
\end{align*}
\]

for \( \|g\|_{1+a,T} \leq \varepsilon_0 \) small enough. This implies (3.3).
Moreover, similar to [24, Proof of Proposition 4.7], we can construct a \( C^1 \) map \((x,v) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \mapsto \Psi_{s,t}(x,v)\), satisfying (3.4) and (3.5). The proof of Proposition 3.1 is complete. \( \square \)

4 Contribution of the initial data and the reaction term

As a consequence of Proposition 3.1, we have

**Lemma 4.1** Let \( h = h(x,v) : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R} \), define

\[
I_h(g)(t,x) = \int_{\mathbb{R}^2} h(X_{0,t}(x,v),V_{0,t}(x,v)) dv,
\]

where \((X_{s,t},V_{s,t})\) are in Proposition 3.1 with \( \|g\|_{1+a,T} \leq \varepsilon_0 \). Then there holds

\[
\|I_h(g)\|_{1+a,T} \lesssim \|h\|_{1+a}.
\]

The proof of Lemma 4.1 is very similar to [24, Proof of Proposition 5.1], we omit it.

We next turn to the reaction term. For any given \( F : [0,T] \times \mathbb{R}^2 \to \mathbb{R} \) and \( \eta : \mathbb{R}^2 \to \mathbb{R} \), we denote

\[
\mathcal{T}[F,\eta](t,x) = -\mathcal{T}_{NL}[F,\eta](t,x) + \mathcal{T}_L[F,\eta](t,x),
\]

with

\[
\begin{align*}
\mathcal{T}_L[F,\eta](t,x) &= \int_0^t \int_{\mathbb{R}^2} F(s,x-(t-s)v)\eta(v) dv ds, \\
\mathcal{T}_{NL}[F,\eta](t,x) &= \int_0^t \int_{\mathbb{R}^2} F(s,X_{s,t}(x,v))\eta(V_{s,t}(x,v)) dv ds,
\end{align*}
\]

where \((X_{s,t},V_{s,t})\) are in Proposition 3.1 with \( \|g\|_{1+a,T} \leq \varepsilon_0 \). By changing variable, we reformulate it as follows:

\[
\begin{align*}
\mathcal{T}_L[F,\eta](t,x) &= \int_0^t \int_{\mathbb{R}^2} F(s,w + \frac{s}{t}(x-w))\eta\left(\frac{x-w}{t}\right) \frac{dw ds}{t^2}, \\
\mathcal{T}_{NL}[F,\eta](t,x) &= \int_0^t \int_{\mathbb{R}^2} F\left(s,Y_{s,t}(w),\frac{x-w}{t}\right) + w + \frac{s(x-w)}{t} \eta\left(W_{s,t}(w),\frac{x-w}{t}\right) + \frac{x-w}{t} \right) \frac{dw ds}{t^2}.
\end{align*}
\]

We have the following Lemma.
Lemma 4.2 Let $a \in (0,1)$. Let $\eta$ be such that
\[
\sum_{j=0}^{3} |v|^j |\nabla^j \eta(v)| \leq 1. \tag{4.3}
\]

Then,
\[
|T[F, \eta](t, x)| \lesssim_a \|g\|_{1+a, T} \int_0^t \int_{\mathbb{R}^2} |F(s, x - (t-s)v) - t|^{1+a} \frac{dvds}{(s)^{1+a} \langle v \rangle^3}. \tag{4.4}
\]

In particular, for $p = 1, \infty$,
\[
\langle t \rangle^{2(p-1)/p} \|T[F, \eta](t)\|_{L^p} \lesssim_a \|g\|_{1+a, T} \left( \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \|F(s)\|_{L^p} + \sup_{s \in [0,t]} \|F(s)\|_{L^1} \right). \tag{4.5}
\]

(2) For any $0 \leq t \leq T$,
\[
\sum_{p=1,\infty} \langle t \rangle^{2(p-1)/p} \|T[F, \eta](t)\|_{L^p} \lesssim_a \|g\|_{1+a, T} \sum_{p=1,\infty} \sup_{s \in [0,t]} \langle s \rangle^{1+2(p-1)/p} \left( \|F(s)\|_{L^p} + \|F(s)\|_{\hat{F}^a_{p,\infty}} \right). \tag{4.6}
\]

(3) For any $0 \leq t \leq T$,
\[
\sum_{p=1,\infty} \langle t \rangle^{2(p-1)/p} \|\nabla T[F, \eta](t)\|_{L^p} \lesssim_a \|g\|_{1+a, T} \sum_{j=0,1} \sum_{p=1,\infty} \sup_{s \in [0,t]} \langle s \rangle^{1+2(p-1)/p} \left( \|F(s)\|_{\hat{F}^a_{p,\infty}} + \|F(s)\|_{L^1} \right), \tag{4.7}
\]

where
\[
\|g\|_{\hat{F}^a_{p,\infty}} := \| \sup_{\alpha} \frac{|\delta_{\alpha} g(x)|}{|\alpha|^a} \|_{L^p}. \]

**Proof.** Step 1) First of all, we change of variable and obtain
\[
|T[F, \eta](t, x)| = \left| - \int_0^t \int_{\mathbb{R}^2} F(s, x - (t-s)v) \eta(V_{s,t}(x, \Psi_{s,t}(t, x))) \det(\nabla v \Psi_{s,t}(x, v)) dvds 
\right.
\]
\[
+ \int_0^t \int_{\mathbb{R}^2} F(s, x - (t-s)v) \eta(v) dvds \right| 
\]\n\[
\leq \int_0^t \int_{\mathbb{R}^2} |F(s, x - (t-s)v)| |\eta(V_{s,t}(x, \Psi_{s,t}(t, x))) - \eta(v)| dvds 
\]
\[
+ \int_0^t \int_{\mathbb{R}^2} |F(s, x - (t-s)v)| \left| \det(\nabla v \Psi_{s,t}(x, v)) \right| - 1 \right| |\eta(V_{s,t}(x, \Psi_{s,t}(t, x))) - \eta(v)| dvds. \]

Hence, since we have
\[
|\eta(V_{s,t}(x, \Psi_{s,t}(t, x))) - \eta(v)| \leq \int_0^1 |\nabla \eta(\omega V_{s,t}(x, v) + (1 - \omega)v)| V_{s,t}(x, v) - v | d\omega 
\]
\[
\lesssim \sup_{\omega \in [0,1]} \frac{|V_{s,t}(x, v) - v|}{\omega W_{s,t}(x - vt, v) + v} \sim \frac{|W_{s,t}(x - vt, v)|}{\langle v \rangle^3} \lesssim_a \|g\|_{1+a, T} \frac{1}{\langle s \rangle^2 \langle v \rangle^3},
\]

9
and \(|\eta(V_{s,t}(x, \Psi_{s,t}(x, x)))| \lesssim (W_{s,t}(x - t \Psi_{s,t}(t, x), \Psi_{s,t}(t, x)) + v)^{-3} \lesssim (v)^{-3}\), where we use the fact that \(\sup_{0 \leq s \leq t} \|W_{s,t}\|_{L^\infty} \leq 1\) in (3.3). Hence

\[
|T[F, \eta](t, x)| \lesssim_{\alpha} \|g\|_{1+a,T} \int_0^t \int_{\mathbb{R}^2} |F(s, x - (t - s)v)| \frac{1}{\langle s \rangle^{1+a}} \langle v \rangle^3 \, dvds.
\]

This gives (4.4). Moreover,

\[
\|T[F, \eta](t)\|_{L^p} \lesssim_{\alpha} \|g\|_{1+a,T} \int_0^t \int_{\mathbb{R}^2} |F(s, x - (t - s)v)| \frac{1}{\langle v \rangle^{3+2}} \, dvds \|T[F, \eta](s)\|_{L^p} \, ds.
\]

Then, thanks to Lemma 4.3, one gets

\[
\|T[F, \eta](t)\|_{L^p} \lesssim_{\alpha} \|g\|_{1+a,T} \left( \sup_{s \in [0,t]} \|F(s)\|_{L^1} + \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \|F(s)\|_{L^p} \right).
\]

Combining this with

\[
\|T[F, \eta](t)\|_{L^p} \lesssim_{\alpha} \|g\|_{1+a,T} \int_0^t \|F(s)\|_{L^p} \frac{1}{\langle s \rangle^{1+a}} \, ds \lesssim_{\alpha} \|g\|_{1+a,T} \sup_{s \in [0,t]} \|F(s)\|_{L^p},
\]

to obtain (4.5).

Note that we only use \(\sup_v (v)^3 (|\eta(v)| + |\nabla \eta(v)|) \leq 1\) in the proof of (4.1).

**Step 2** We estimate \(\|T[F, \eta](t)\|_{\dot{B}^q_{p,\infty}}\).

2.1) Case \(0 \leq t \leq \min\{1, T\}\). We divide \(\delta_\alpha T[F, \eta](t, x)\) into three terms:

\[
\delta_\alpha T[F, \eta](t, x) = \mathcal{T}_\alpha^1[F, \eta](t, x) + \mathcal{T}_\alpha^2[F, \eta](t, x) + \mathcal{T}_\alpha^3[F, \eta](t, x)
\]

where

\[
\mathcal{T}_\alpha^1[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^2} (-\delta_\alpha F(s, \cdot) (X_{s,t}(x, v))) \eta (V_{s,t}(x, v)) + \delta_\alpha F(s, \cdot) (x - (t - s)v) \eta(v)) \, dvds,
\]

\[
\mathcal{T}_\alpha^2[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^2} \left[ F(s, X_{s,t}(x, v) - \alpha) - F(s, X_{s,t}(x, v) - \alpha) \right] \eta (V_{s,t}(x, v)) \, dvds,
\]

\[
\mathcal{T}_\alpha^3[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^2} F(s, X_{s,t}(x, v) - \alpha) \eta (V_{s,t}(x, v) - \alpha) - \eta (V_{s,t}(x, v)) \right] \, dvds.
\]

For the first term, using (1.3) and (4.4) we have

\[
\|\mathcal{T}_\alpha^1[F, \eta](t)\|_{L^p} = \|T[\delta_\alpha F, \eta](t)\|_{L^p} \lesssim_{\alpha} \|g\|_{1+a,T} \left( \sup_{s \in [0,t]} \|\delta_\alpha F(s)\|_{L^1} + \sup_{s \in [0,t]} \|\delta_\alpha F(s)\|_{L^p} \right)
\]

\[
\lesssim_{\alpha} |\alpha|^2 \|g\|_{1+a,T} \left( \sup_{s \in [0,t]} \|F(s)\|_{\dot{B}^q_{1,\infty}} + \sup_{s \in [0,t]} \|F(s)\|_{\dot{B}^q_{p,\infty}} \right).
\]

Since

\[
|X_{s,t}(x, v) - (X_{s,t}(x, v) - \alpha)| = |Y_{s,t}(x, v - \alpha - tv, v) - Y_{s,t}(x - tv, v)| \leq |\alpha| \|\nabla_x Y_{s,t}\|_{L^\infty},
\]
we can estimate $\mathcal{T}_{\alpha}^3[F, \eta](t, x)$ as follows

$$
\| \mathcal{T}_{\alpha}^3[F, \eta](t) \|_{L^p} \lesssim a^a \int_0^t \int_{\mathbb{R}^2} \| F(s) \|_{L^p} \| \nabla \eta \|_{L^\infty} \left( \sup_{s \in [0, t]} \| F(s) \|_{L^p} \right) \int_0^t \langle s \rangle^{-(1+a)} ds
$$

Note that by (3.2), one has $\sup_{s, t} \| W_{s,t} \|_{L^p_{x,v}} \leq 1$. Then,

$$
\langle \varpi V_{s,t}(x, v) + (1 - \varpi)V_{s,t}(x - \alpha, v) \rangle = \langle v + \varpi W_{s,t}(x - tv, v) + (1 - \varpi)W_{s,t}(x - \alpha - tv, v) \rangle \sim \langle v \rangle.
$$

(4.9)

We can estimate $\mathcal{T}_{\alpha}^3[F, \eta](t)$,

$$
\| \mathcal{T}_{\alpha}^3[F, \eta](t) \|_{L^p} \lesssim a^a \int_0^t \int_{\mathbb{R}^2} \| F(s) \|_{L^p} \| \nabla \eta \|_{L^\infty} \left( \sup_{s \in [0, t]} \| F(s) \|_{L^p} \right) \int_0^t \langle s \rangle^{-(1+a)} ds
$$

We can estimate $\mathcal{T}_{\alpha}^2[F, \eta](t)$,

$$
\| \mathcal{T}_{\alpha}^2[F, \eta](t) \|_{L^p} \lesssim a^a \int_0^t \int_{\mathbb{R}^2} \| F(s) \|_{L^p} \| \nabla \eta \|_{L^\infty} \left( \sup_{s \in [0, t]} \| F(s) \|_{L^p} \right) \int_0^t \langle s \rangle^{-(1+a)} ds
$$

In conclusion, for $0 \leq t \leq \min\{1, T\}$,

$$
\| \mathcal{T}[F, \eta](t) \|_{L^p_{x,v}} \lesssim a^a \left( \sup_{s \in [0, t]} \| F(s) \|_{L^p} + \sup_{s \in [0, t]} \| F(s) \|_{L^p} \right).
$$

(4.10)

2.2) Case $T > 1$ and $1 \leq t \leq T$. Set

$$
Z_1(x) = Y_{s,t}(w, \frac{x - w}{t}) + w + \frac{s(x - w)}{t}, \quad Z_2(x) = W_{s,t}(w, \frac{x - w}{t}) + \frac{x - w}{t},
$$

$$
Z_3(x) = W_{s,t}(w, \frac{x - \alpha - w}{t}) + \frac{x - w}{t}, \quad Z_4(x) = Y_{s,t}(w, \frac{x - w}{t}) + w + \frac{s(x - \alpha - w)}{t}.
$$

(4.11)

We have

$$
\delta \eta T[F, \eta](t, x) = \mathcal{T}_{\alpha}^1[F, \eta](t, x) + \mathcal{T}_{\alpha}^2[F, \eta](t, x) + \mathcal{T}_{\alpha}^3[F, \eta](t, x) + \mathcal{T}_{\alpha}^4[F, \eta](t, x),
$$

where

$$
\mathcal{T}_{\alpha}^1[F, \eta](t, x) = -\int_0^t \int_{\mathbb{R}^2} \delta \mathbb{E} F(s, \cdot) (Z_1(x)) \eta (Z_2(x)) \frac{dudv}{t^2},
$$

$$
+ \int_0^t \int_{\mathbb{R}^2} \delta \mathbb{E} F(s, \cdot)(w + \frac{s}{t}(x - w)) \eta \left( \frac{x - w}{t} \right) \frac{dudv}{t^2},
$$

$$
\mathcal{T}_{\alpha}^2[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^2} F(s, Z_1(x - \alpha)) (\delta \mathbb{E} \eta) (Z_2(x)) \frac{dudv}{t^2},
$$

$$
- \int_0^t \int_{\mathbb{R}^2} F(s, w + \frac{s}{t}(x - w - \alpha)) (\delta \mathbb{E} \eta) \left( \frac{x - w}{t} \right) \frac{dudv}{t^2},
$$

$$
\mathcal{T}_{\alpha}^3[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^2} (F(s, Z_1(x - \alpha)) - F(s, Z_4(x))) \eta (Z_2(x)) \frac{dudv}{t^2},
$$

$$
\mathcal{T}_{\alpha}^4[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^2} F(s, Z_1(x - \alpha)) (\eta (Z_3(x)) - \eta (Z_2(x))) \frac{dudv}{t^2}.
$$
Going back to the variable $v = \frac{x-w}{t}$, and denoting
\[
Z_5(x) = Y_{s,t}(x-tv, v-\frac{\alpha}{t}) + x - (t-s)v - \frac{s\alpha}{t}, \quad Z_6(x) = W_{s,t}(x-tv, v-\frac{\alpha}{t}) + v,
\]
we reformulate it as follows:
\[
T^1_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} \left(-\delta_T F(s, .)(X_{s,t}(x, v)) \eta(V_{s,t}(x, v)) + \delta_T F(s, .)(x - (t-s)v) \eta(v)\right) dv ds,
\]
\[
T^2_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} \left(F(s, X_{s,t}(x, v)) \left(\delta_T \eta\right)(V_{s,t}(x, v)) - F(s, x - (t-s)v) \left(\delta_T \eta\right)(v)\right) dv ds,
\]
\[
T^3_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} \left(F(s, X_{s,t}(x, v) - \frac{s\alpha}{t}) - F(s, Z_5(x)) \eta(V_{s,t}(x, v)) dv ds,
\]
\[
T^4_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} F(s, Z_5(x)) (\eta(V_{s,t}(x, v)) - \eta(Z_6(x))) dv ds.
\]
Note that for $p = 1, \infty$ and $|\alpha| \geq t$,
\[
\langle t \rangle \frac{2(p-1)}{p} + a \frac{\|\delta_T F\|_{L^p}}{|\alpha|^a} \leq 2\langle t \rangle \frac{2(p-1)}{p} + a \frac{\|\delta_T F\|_{L^p}}{|\alpha|^a} \leq 4\langle t \rangle \frac{2(p-1)}{p} + a \frac{\|F\|_{L^p}}{|\alpha|^a}.
\]
So, it is enough to consider $|\alpha| \leq t$.

2.2.1) Estimate $T^1_\alpha[F, \eta](t, x)$.
Applying (4.13) with $\delta_T F$, we have
\[
\|T^1_\alpha[F, \eta](t)\|_{L^p} = \|T[\delta_T F, \eta](t)\|_{L^p} \lesssim a \langle t \rangle \frac{2(p-1)}{p} + a \frac{\|F\|_{L^p}}{|\alpha|^a} \left(\sup_{s \in [0, t]} \langle s \rangle^{2(p-1)} \frac{\|F\|_{L^p}}{|\alpha|^a} + \sup_{s \in [0, t]} \langle s \rangle^{2(p-1)+a} \frac{\|F\|_{B^{\infty}_p}}{|\alpha|^a}\right).
\]
Thus, we can yield
\[
t \frac{2(p-1)}{p} + a \sup_{\alpha} \frac{\|T^1_\alpha[F, \eta](t)\|_{L^p}}{|\alpha|^a} \lesssim a \langle t \rangle \frac{2(p-1)}{p} + a \frac{\|F\|_{L^p}}{|\alpha|^a} \left(\sup_{s \in [0, t]} \langle s \rangle^{2(p-1)} \frac{\|F\|_{L^p}}{|\alpha|^a} + \sup_{s \in [0, t]} \langle s \rangle^{2(p-1)+a} \frac{\|F\|_{B^{\infty}_p}}{|\alpha|^a}\right).
\]

2.2.2) Estimate $T^2_\alpha[F, \eta](t, x)$.
Note that
\[
|\delta_T \eta(v)| + |\nabla_v (\delta_T \eta)(v)| \lesssim \min \left\{ \frac{|\alpha|}{t}, 1 \right\} \left(\frac{1}{\langle v \rangle^3} + \frac{1}{\langle v - \alpha \rangle^3}\right) \lesssim \frac{|\alpha|}{t^a} \frac{1}{\langle v \rangle^3},
\]
since $|\alpha| \leq t$. Thus, by Step 1 we have
\[
\|T^2_\alpha[F, \eta](t)\|_{L^p} = \frac{|\alpha|^a}{t^a} \left\|T[F, t^a \frac{\alpha}{|\alpha|^a} \delta_T \eta(v)]\right\|_{L^p} \lesssim a \frac{|\alpha|^a}{t^a} \langle t \rangle \frac{2(p-1)}{p} + a \frac{\|F\|_{L^p}}{|\alpha|^a} \left(\sup_{s \in [0, t]} \langle s \rangle^{2(p-1)} \frac{\|F\|_{L^p}}{|\alpha|^a} + \sup_{s \in [0, t]} \langle s \rangle^{2(p-1)+a} \frac{\|F\|_{L^p}}{|\alpha|^a}\right),
\]
12
which implies
\[
\frac{2(p-1)}{p} + a \sup_{\alpha} \frac{\| \mathcal{T}_\alpha^3 \|_{L^p}}{\alpha^{\alpha}} \lesssim a \left( \sup_{s \in [0,t]} \frac{\langle s \rangle^{2(p-1)/p} \| F(s) \|_{L^p}}{\| s \|} + \sup_{s \in [0,t]} \| F(s) \|_{L^1} \right).
\]

2.2.3) Estimate $\mathcal{T}_\alpha^3[F, \eta](t, x)$.

\[
|\mathcal{T}_\alpha^3[F, \eta](t, x)| \lesssim c_1 \int_0^t \int_{\mathbb{R}^2} |F(s, X_{s,t}(x, v) - \frac{a\xi}{t}) - F(s, Z_\delta(x))| \, dvds \left( \frac{\langle \xi \rangle}{\langle s \rangle^{3/2}} \right)^3.
\]

Thus, as (4.3), we apply Lemma 4.3 with $H = \sup_z |z|^{-a} |\delta_z F(s, .)|$ and $\varphi(x, v) = Y_{s,t}(x - tv, v + \frac{a\xi}{t}) - \frac{a\xi}{t}$, so

\[
\| \mathcal{T}_\alpha^3[F, \eta](t) \|_{L^p} \lesssim a \left( \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \| F(s) \|_{L^p} + \sup_{s \in [0,t]} \| F(s) \|_{L^1} \right).
\]

Thanks to (4.20)

\[
\| \mathcal{T}_\alpha^3[F, \eta](t) \|_{L^p} \lesssim a \left( \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \| F(s) \|_{L^p} + \sup_{s \in [0,t]} \| F(s) \|_{L^1} \right) + \frac{1}{\langle s \rangle^{1/2}} \int_0^t \| F(s) \|_{L^1} \, ds.
\]

Thanks to (4.19)

\[
\| \mathcal{T}_\alpha^3[F, \eta](t) \|_{L^p} \lesssim a \left( \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \| F(s) \|_{L^p} + \sup_{s \in [0,t]} \| F(s) \|_{L^1} \right).
\]

2.2.4) Estimate $\mathcal{T}_\alpha^4[F, \eta](t, x)$.

One has

\[
|\mathcal{T}_\alpha^4[F, \eta](t, x)| \lesssim a \left( \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \| F(s) \|_{L^p} + \sup_{s \in [0,t]} \| F(s) \|_{L^1} \right).
\]

Then, as (4.3) we apply Lemma 4.3 with $H = F$ and $\varphi(x, v) = Y_{s,t}(x - tv, v - \frac{a\xi}{t}) - \frac{a\xi}{t}$ to get

\[
\| \mathcal{T}_\alpha^4[F, \eta](t) \|_{L^p} \lesssim a \left( \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \| F(s) \|_{L^p} + \sup_{s \in [0,t]} \| F(s) \|_{L^1} \right).
\]
Thus, we obtain
\[ t^{2(p-1)+a} \sup_{\alpha} \| T_{\alpha}^d [F, \eta] (t) \|_{L^p} \lesssim_a \| g \|_{1+a,T} \left( \sup_{s \in [0, t]} \langle s \rangle^{a} \| F(s) \|_{F, \eta} + \sup_{s \in [0, t]} \langle s \rangle^{2(p-1)+a} \| F(s) \|_{F, \eta} \right). \]

Summing up, we conclude that
\[ \sum_{p=1, \infty} t^{2(p-1)+a} \| T[F, \eta](t) \|_{B^p_{p, \infty}} \lesssim_a \| g \|_{1+a,T} \sum_{p=1, \infty} \sup_{s \in [0, t]} \langle s \rangle^{1+2(p-1)/p} \left( \| F(s) \|_{L^p} + \| F(s) \|_{F, \eta} \right). \]

(4.15)

Hence, (4.6) follows from (4.10) and (4.15).

**Step 3)** We estimate \( \| \nabla T[F, \eta](t) \|_{B^p_{p, \infty}} \).

3.1) Case \( 0 \leq t \leq \min\{1, T\} \). We have
\[
\partial_{x_i} T[F, \eta](t, x) = T[\partial_{x_i} F, \eta](t, x) + T^{R,i}[F, \eta](t, x), \quad i = 1, 2,
\]
where
\[
T^{R,i}[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^2} F(s, X_{s,t}(x, v)) \nabla \eta(V_{s,t}(x, v)) \partial_{x_i} W_{s,t}(x - vt, v) dv ds.
\]

By (4.10), one has
\[
\sum_{p=1, \infty} \sum_{i=1, 2} \| T[\partial_{x_i} F, \eta](t) \|_{B^p_{p, \infty}} \lesssim_a \| g \|_{1+a,T} \sum_{p=1, \infty} \sup_{s \in [0, t]} \left( \| \nabla F(s) \|_{L^p} + \| \nabla F(s) \|_{F, \eta} \right).
\]

It is easy to check (see [24] Proof of Proposition 6.6, the second part) that
\[
\sum_{p=1, \infty} \sum_{i=1, 2} \| T^{R,i}[F, \eta](t) \|_{B^p_{p, \infty}} \lesssim_a \| g \|_{1+a,T} \sum_{p=1, \infty} \sup_{s \in [0, t]} \left( \| F(s) \|_{L^p} + \| \nabla F(s) \|_{F, \eta} \right).
\]

Thus, we obtain (4.7) with \( 0 \leq t \leq \min\{1, T\} \).

3.2) Case \( T > 1 \) and \( 1 \leq t \leq T \). We have
\[
\partial_{x_i} T[F, \eta](t, x) = \frac{1}{t} T[\tilde{F}_i, \eta](t, x) + \frac{1}{t} T[F, \partial_i \eta](t, x) + T^{1,i}[F, \eta](t, x) + T^{2,i}[F, \eta](t, x), \quad i = 1, 2,
\]
where \( \tilde{F}_i(s, \cdot) := s \partial_{x_i} F(s, \cdot) \),
\[
T^{1,i}[F, \eta](t, x) := - \int_0^t \int_{\mathbb{R}^2} F \left( s, Y_{s,t}(w, \frac{x - w}{t}) + w + \frac{s(x - w)}{t} \right) \times (\nabla \eta) \left( W_{s,t}(w, \frac{x - w}{t}) + \frac{x - w}{t} \right). (\partial_{v_i} W_{s,t})(w, \frac{x - w}{t}) \, dw ds/t^3,
\]
\[
T^{2,i}[F, \eta](t, x) := - \int_0^t \int_{\mathbb{R}^2} \nabla F \left( s, Y_{s,t}(w, \frac{x - w}{t}) + w + \frac{s(x - w)}{t} \right). (\partial_{v_i} Y_{s,t})(w, \frac{x - w}{t}) \times \eta \left( W_{s,t}(w, \frac{x - w}{t}) + \frac{x - w}{t} \right) \, dw ds/t^3.
\]
Applying (4.15) to $T[\tilde{F}, \eta]$ and $T[F, \partial_t \eta]$ to get

$$
\sum_{p=1, \infty} t^{2(p-1)/p} \|T[\tilde{F}, \eta](t)\|_{L^p_{x} T^a_{\infty}} \lesssim_{a} \|g\|_{1+a, T} \sum_{s=1, \infty} \langle s \rangle^{2(p-1)/p} \left( \|F(s)\|_{L^p} + \|\nabla F(s)\|_{L^p_{x} T^a_{\infty}} \right),
$$

(4.16)

$$
\sum_{p=1, \infty} t^{2(p-1)/p} \|T[F, \partial_t \eta](t)\|_{L^p_{x} T^a_{\infty}} \lesssim_{a} \|g\|_{1+a, T} \sum_{s=1, \infty} \langle s \rangle^{2(p-1)/p} \left( \|F(s)\|_{L^p} + \|\nabla F(s)\|_{L^p_{x} T^a_{\infty}} \right).
$$

(4.17)

Now we estimate $T^{1,i}[F, \eta](t, x)$ and $T^{2,i}[F, \eta](t, x)$. Indeed, Case 1: $|\alpha| \geq t$. Thanks to (3.1), (3.2) and (4.3), one has

$$
\|\delta_{\alpha}T^{1,i}[F, \eta](t)\|_{L^p} + \|\delta_{\alpha}T^{2,i}[F, \eta](t)\|_{L^p} \leq 2 \|T^{1,i}[F, \eta](t)\|_{L^p} + 2 \|T^{2,i}[F, \eta](t)\|_{L^p}
$$

$$
\lesssim a \|g\|_{1+a, T} \sum_{j=0,1} \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \|\nabla^j F(s)\|_{L^1} + \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p}\|\nabla^j F(s)\|_{L^p}.
$$

This implies

$$
\sum_{p=1, \infty} \sum_{k=1,2} \langle t \rangle^{1+a+2(p-1)/p} \|\delta_{\alpha}T^{k,i}[F, \eta](t)\|_{L^p} \lesssim_{a} \|g\|_{1+a, T} \sum_{j=0,1} \sup_{s \in [0,t]} \langle s \rangle^{2(p-1)/p} \|\nabla^j F(s)\|_{L^p}.
$$

(4.18)

Case 2: $|\alpha| \leq t$. As Step 2, thanks to (3.1), (3.2) and (4.3), one has

$$
\|\delta_{\alpha}T^{1,i}[F, \eta](t)\|_{L^p} \lesssim_{a} \frac{|\alpha|^a}{\langle t \rangle^{1+a}} \|g\|_{1+a, T} \int_{0}^{t} \sup_{z} \left| z \right|^{-a} \|\delta_{z} F(s, \cdot)(Y_{s,t}(w, x-w/t)) + w + \frac{s(x-w)}{t} \| \frac{dwdz}{(s\langle x-w \rangle^3 t^2)}
$$

$$
+ \frac{|\alpha|^a}{\langle t \rangle^{1+a}} \|g\|_{1+a, T} \int_{0}^{t} \sup_{z} \left| z \right|^{-a} \|\delta_{z}(\nabla_x F)(s, \cdot)(Y_{s,t}(w, x-w/t)) + w + \frac{s(x-w)}{t} \| \frac{dwdz}{(s\langle x-w \rangle^3 t^2)}.
$$

$$
\|\delta_{\alpha}T^{2,i}[F, \eta](t)\|_{L^p} \lesssim_{a} \frac{|\alpha|^a}{\langle t \rangle^{1+a}} \|g\|_{1+a, T} \int_{0}^{t} \sup_{z} \left| z \right|^{-a} \|\delta_{z}(\nabla_x F)(s, \cdot)(Y_{s,t}(w, x-w/t)) + w + \frac{s(x-w)}{t} \| \frac{dwdz}{(s\langle x-w \rangle^3 t^2)}.
$$

By Lemma 4.3,

$$
\|\delta_{\alpha}T^{1,i}[F, \eta](t)\|_{L^p} \lesssim_{a} \frac{|\alpha|^a}{\langle t \rangle^{2(p-1)/p+1+a}} \|g\|_{1+a, T} \left( \sup_{s \in [0,t]} \langle s \rangle\|F(s)\|_{L^p_{x} T^a_{\infty}} \right)
$$

$$
+ \sup_{s \in [0,t]} \langle s \rangle^{1+2(p-1)/p} \|F(s)\|_{L^p_{x} T^a_{\infty}} + \sup_{s \in [0,t]} \langle s \rangle\|F(s)\|_{L^1} + \sup_{s \in [0,t]} \langle s \rangle^{1+2(p-1)/p} \|F(s)\|_{L^p}.
$$

15
\[ \| \delta_{\alpha} T^{2,i} [F, \eta](t) \|_{L^p} \lesssim_a \frac{|\alpha|^a}{t^{\frac{2(p-1)}{p}}} \| g \|_{1+a,T} \left( \sup_{s \in [0,t]} \langle s \rangle^{1+\frac{2}{p}} \| \nabla F(s) \|_{F_{p,\infty}} \right) \]

\[ + \sup_{s \in [0,t]} \langle s \rangle^{1+\frac{2(p-1)}{p}} \| \nabla F(s) \|_{L^p_{\infty}} + \sup_{s \in [0,t]} \langle s \rangle^{1+\frac{2}{p}} \| \nabla F(s) \|_{L^1} + \sup_{s \in [0,t]} \langle s \rangle^{1+\frac{2(p-1)}{p}} \| \nabla F(s) \|_{L^p}. \]

Thus,

\[ \sum_{p=1,\infty} \sum_{k=1,2} \langle t \rangle^{1+a+\frac{2(p-1)}{p}} \| \delta_{\alpha} T^{k,i} [F, \eta](t) \|_{L^p} \lesssim_a \| g \|_{1+a,T} \sum_{j=0,1} \sum_{p=1,\infty} \sup_{s \in [0,t]} \langle s \rangle^{1+\frac{2(p-1)}{p}} \left( \langle s \rangle^{\frac{2}{p}} \| \nabla^j F(s) \|_{F_{p,\infty}^a} + \| F(s) \|_{L^p} \right), \]

Combining this with (4.16), (4.17) and (4.18) to get

\[ \sum_{p=1,\infty} \langle t \rangle^{1+a+\frac{2(p-1)}{p}} \| \nabla T [F, \eta](t) \|_{B_{p,\infty}^a} \lesssim_a \| g \|_{1+a,T} \sum_{j=0,1} \sum_{p=1,\infty} \sup_{s \in [0,t]} \langle s \rangle^{1+\frac{2(p-1)}{p}} \left( \langle s \rangle^{\frac{2}{p}} \| \nabla^j F(s) \|_{F_{p,\infty}^a} + \| F(s) \|_{L^p} \right), \]

provided that \( 1 \leq t \leq T \) and \( T \geq 1 \). So, we proved (4.17). The proof is complete. \( \blacksquare \)

We used the following lemma in the proof of Lemma 4.2. It is Lemma 8.1 in [24].

**Lemma 4.3** Assume \( H \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) and \( \varphi(x, v) \) satisfies \( \| \nabla_x \varphi(x, v) \|_{L^\infty_{x,v}} \leq \frac{1}{2} \), then for \( p = 1, \infty, 0 \leq s \leq t \) and \( t \geq 0 \) we have

\[ \left\| \int_{\mathbb{R}^2} H (\varphi(x, v) + x - (t-s)v) \right\|_{L^p_v} \lesssim \| H \|_{L^p}. \tag{4.19} \]

Moreover, for \( 0 \leq s \leq \frac{1}{2} \) and \( t \geq 1 \), we also get

\[ \left\| \int_{\mathbb{R}^2} \frac{H (\varphi(x, v) + x - (t-s)v)}{\langle v \rangle^3} \right\|_{L^p_v} \lesssim \frac{1}{t^{\frac{2(p-1)}{p}}} \| H \|_{L^1}. \tag{4.20} \]

## 5 Proof of Proposition 2.3 and Proposition 2.4

First, we prove Propositions 2.3. Assume that the problem (1.2) has a unique solution \((f, U)\) in \([0, T]\) for some \( T < \infty \) with

\[ \| \rho \|_{1+a,T} + \| U \|_{1+a,T} \leq \varepsilon_1. \]

Since \( g = \rho + A(U) \) and Assumption 3,

\[ \| g \|_{1+a,T} \leq \| \rho \|_{1+a,T} + \| A(U) \|_{1+a,T} \leq \varepsilon_1 + C \varepsilon_1^2. \]

Assume that \( \varepsilon_1 + C \varepsilon_1^2 \leq \varepsilon_0 \). One has

\[ \| g \|_{1+a,T} \leq \varepsilon_0. \]

Thus, we can apply Proposition 3.1 with \((g, E) = (\rho + A(U), E)\) and Lemma 4.1 with \( f_0 \) to obtain that

\[ \| \mathcal{I}(g) \|_{1+a,T} \lesssim_a \| f_0 \|_{1+a}. \]
Applying Lemma 2.2 with \((F, \eta) = (E_i, \partial \eta, \mu)\) and thanks to \(\mathcal{R}(\varrho) = \sum_{i=1}^{2} T[E_i, \partial \eta, \mu]\), one gets

\[
||\mathcal{R}(\varrho)||_{1+a,T} \lesssim_\varrho ||\varrho||_{1+a,T} 1 + \frac{2^{(1+a)}}{p} \left(\sum_{j=0,1}^{1} \sup_{s \in [0,T]} (s) \frac{2^{(1+a)}}{p} ||\nabla^2 E(s)||_{F_{p,\infty}} + ||E(s)||_{L^p} \right)
\]

(6.6)

Thus, \(||\varrho||_{1+a,T} \lesssim_\varrho ||\varrho||_{1+a,T} + ||A(U)||_{1+a,T} \lesssim_\varrho ||\varrho||_{1+a,T} + ||U||^{2(1+a)}_{1+a,T} \).

On the other hand, by (2.3) and Proposition 2.2

\[
||\varrho||_{1+a,T} \lesssim_\varrho ||\mathcal{I}(\varrho)||_{1+a,T} + ||\mathcal{R}(\varrho)||_{1+a,T} + ||A(U)||_{1+a,T}.
\]

Thus, we get

\[
||\varrho||_{1+a,T} \lesssim_\varrho ||f_0||_{1+a} + ||\varrho||^{1+a}_{1+a,T} + ||U||^{2(1+a)}_{1+a,T}.
\]

Since \(U = (-\Delta + 1)^{-1}(\rho + A(U))\),

\[
||U||_{1+a,T} \lesssim_\varrho ||\varrho||_{1+a,T} + ||U||^{2}_{1+a,T} \lesssim_\varrho ||\varrho||_{1+a,T} + \varepsilon_1 ||U||_{1+a,T}.
\]

This implies

\[
||U||_{1+a,T} \lesssim_\varrho ||\varrho||_{1+a,T},
\]

provided \(\varepsilon_1 \in (0, \varepsilon_0/4)\) small. Combining this with (5.1) to obtain that

\[
||\varrho||_{1+a,T} \lesssim_\varrho ||f_0||_{1+a} + ||\varrho||^{1+a}_{1+a,T} \lesssim_\varrho ||f_0||_{1+a} + \varepsilon_1 ||\varrho||_{1+a,T}.
\]

Thus,

\[
||\varrho||_{1+a,T} + ||U||_{1+a,T} \lesssim_\varrho ||f_0||_{1+a},
\]

provided \(\varepsilon_1 \in (0, \varepsilon_0/4)\) small. So, we proved Proposition 2.3.

**Remark 5.1** It follows from Proposition 3.1 that \(||f(T)||_{1+a} < \infty\). However, it is easy to check that if there exists \((f_0^\kappa)_{\kappa} \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)\) satisfying \(\lim_{\kappa \to 0} ||f_0 - f_0^\kappa||_{1+a} = 0\), then we can construct \((f_0^\kappa)_{\kappa} \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)\) such that \(\lim_{\kappa \to 0} ||f(T) - f_0^\kappa||_{1+a} = 0\).

Next, we prove Proposition 2.4. Set

\[
S_{\varepsilon_1, T_0} := \{ \vartheta \in (L^1 \cap L^\infty([0,T])) \times \mathbb{R}^2) : ||\vartheta||_{a,T_0} = ||\vartheta 1_{[0,T_0]}(s)||_{a} \leq \varepsilon \},
\]

(5.2)

for \(T_0 \in [0,1]\). Let \(\tilde{\varepsilon}_0\) be in Lemma 2.1 and \(\varepsilon_0 > 0\) be in Proposition 3.1. Thanks to (2.7), we can define a map

\[
\mathcal{J}(\rho) := G*(t,x) (I_{f_0}(\varrho) + \mathcal{R}(\varrho) + A(U)) + I_{f_0}(\varrho) + \mathcal{R}(\varrho), \varrho = \rho + A(U), U = N(\rho),
\]

for \(\rho \in S_{\varepsilon_1, T_0}\).

We will prove that \(\mathcal{J}(\rho)\) has a unique fixed point in \(S_{\varepsilon_1, T_0}\) for some \(\varepsilon_1 \leq \varepsilon_0\) and \(T_0 \in (0,1)\). By (6.6) in Lemma 2.1, for any \(\rho \in S_{\varepsilon_1, T_0}\),

\[
||\varrho||_{a,T_0} \leq ||\varrho||_{a,T_0} \leq \varepsilon_0, \varepsilon_0 \leq \varepsilon_0,
\]

(5.3)

with \(\varepsilon_1 \in (0, \varepsilon_0)\) small. Let \((Y^p_{s,t}, W^p_{s,t})\) be the characteristics in Proposition 3.1 with \(\varrho = \rho + A(\mathcal{N}(\rho))\). Using the same argument as [21] Proof of Proposition 4.1, we can obtain for any \(0 \leq s \leq t \leq T_0\),

\[
\sum_{i=0,1} \frac{||\nabla^i \left(Y^p_{s,t}, W^p_{s,t} \right)||_{L^\infty_{x,v}}}{||\alpha||^a} \lesssim_\varrho T_0 \rho_{a,T_0},
\]

(5.4)

\[
\sum_{i=0,1} \frac{||\nabla^i \left(Y^p_{s,t} - Y^p_{s,t}, W^p_{s,t} - W^p_{s,t} \right)||_{L^\infty_{x,v}}}{||\alpha||^a} \lesssim_\varrho T_0 \rho_1 - \rho_2, \rho_{a,T_0},
\]

(5.5)
for any $\rho, \rho_1, \rho_2 \in S_{\tilde{\varepsilon}, T_0}$. It is easy to obtain from (5.4) that for any $\rho \in S_{\tilde{\varepsilon}, T_0}$,

$$
\|R(\rho + A(N(\rho)))\|_{1+a, T_0} \lesssim_a T_0,
\|I_{f_0}(\rho + A(N(\rho)))\|_{1+a, T_0} \lesssim_a \|f_0\|_{1+a}.
$$

(5.6)

Hence, since $J(\rho) - I_{f_0}(g) = G *_{(t, x)} (I_{f_0}(g) + R(g) + A(U)) + R(g)$,

$$
\|J(\rho) - I_{f_0}(g)\|_{1+a, T_0} \lesssim T_0 + (1 + \|f_0\|_{1+a}) \int_0^{T_0} \|G(t)\|_{L^1} dt \lesssim T_0(1 + \|f_0\|_{1+a}),
$$

(5.7)

where we use [24] Theorem 3.6, (3.18) in the last inequality. Hence

$$
\|J(\rho) - I_{f_0}(g)\|_{1+a, T_0} \leq \|J(\rho) - I_{f_0}(g)\|_{1+a, T_0} + \|I_{f_0} - I_{f_0}(g)\|_{1+a, T_0} + \|I_{f_0}(g) - I_{f_0}(g)\|_{1+a, T_0} + \int_{I_{f_0}(g)} f_0^r(x, v)dv\|_{1+a, T_0} + \int_{I_{f_0}(g)} f_0^r(x, v)dv\|_{1+a, T_0}
$$

(5.8)

In particular,

$$
\sup_{\rho \in S_{\tilde{\varepsilon}, T_0}} \|I_{f_0}(\rho + A(N(\rho)))\|_{1+a, T_0} \lesssim T_0(C(f_0^r) + \|f_0\|_{1+a} + \|f_0 - f_0^r\|_{1+a} + \|\rho(0)\|_{1+a, T_0})
$$

(5.9)

Moreover, as [24] Proof of Theorem 2.2, we get from (3.1), (5.5) and Lemma 2.1 that

$$
\|J(\rho_1) - J(\rho_2)\|_{a, T_0} \lesssim T_0(1 + \|f_0\|_{1+a})\|\rho_1 - \rho_2\|_{a, T_0},
$$

(5.10)

for any $\rho_1, \rho_2 \in S_{\tilde{\varepsilon}, T_0}$. Thanks to (5.9), (5.10), (2.11) and the fixed-point theorem, we obtain that $J$ has a unique fixed point $\rho$ in $S_{\tilde{\varepsilon}, T_0}$ satisfying (2.12) for some $T_0 \in (0, 1)$, $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$ small enough. Clearly, (2.13) follows from (5.8). The proof of the Proposition (2.4) is complete.

References

[1] A. Arsenèv. Global Existence of a Weak Solution of Vlasov system of equations, U.S.S.R. Comp. Math. Math. Phys., 15 (1975), pp. 131-143.

[2] C. Bardos and P. Degond. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. Annales de l’Institut Henri Poincaré C, Analyse non linéaire, 2(2):101-118, 1985.

[3] C. Bardos, F. Golse, T. Nguyen, and R. Sentis. The Maxwell-Boltzmann approximation for ion kinetic modeling. Physics. D, 376/377:94–107, 2018.

[4] J. Batt. Global Symmetric Solutions of the Initial Value Problem of Stellar Dynamics, Journal of Differential Equations, 25 (1977), pp. 342-364.

[5] J. Bedrossian and N. Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. Publ. Math. Inst. Hautes Études Sci., 122:195–300, 2015.

[6] J. Bedrossian. Nonlinear echoes and Landau damping with insufficient regularity. Tunis. J. of Math., Vol. 3 (2021), No. 1, 121–205, DOI: 10.2140/tunis.2021.3.121.
[7] J. Bedrossian, N. Masmoudi and C. Mouhot. Linearized wave-damping structure of Vlasov-Poisson in $\mathbb{R}^3$. \texttt{arXiv:2007.08580}, 2020.

[8] J. Bedrossian, N. Masmoudi and C. Mouhot. Landau damping in finite regularity for unconfined systems with screened interactions. \textit{Communications on Pure and Applied Mathematics, 71}(3): 537-576, 2018.

[9] J. Bedrossian, N. Masmoudi and C. Mouhot. Landau damping: paraproducts and Gevrey regularity. \textit{Annals of PDE, 2}(1): 1-71, 2016.

[10] F. Bouchut. Global weak solution of the Vlasov-Poisson system for small electrons mass. \textit{Comm. Partial Differential Equations, 16}(8-9):1337-1365, 1991.

[11] S.-H. Choi, S.-Y. Ha and H. Lee. Dispersion estimates for the two-dimensional Vlasov–Yukawa system with small data. \textit{Journal of Differential Equations, 250}(1):515–550, 2011.

[12] P. Flynn, Z. Ouyang, B. Pausader, and K.Widmayer. Scattering Map for the Vlasov–Poisson System. \textit{Peking Math J} (2021). https://doi.org/10.1007/s42543-021-00041-x

[13] R. T. Glassey. The Cauchy problem in kinetic theory. \textit{Society for Industrial and Applied Mathematics, Philadelphia, PA, 1996}.

[14] E. Grenier, T. T. Nguyen and I. Rodnianski. Landau damping for analytic and Gevrey data. \texttt{arXiv:2004.05979}, 2020.

[15] E. Grenier, T. T. Nguyen, and I. Rodnianski. Plasma echoes near stable Penrose data. \textit{SIAM J. Math. Anal.} (to appear) preprint \texttt{arXiv:2004.05984}, 2020.

[16] M. Griffin-Pickering and M. Iacobelli. Global well-posedness in 3-dimensions for the Vlasov–Poisson system with massless electrons. \texttt{arXiv:1810.06928}.

[17] Y. Guo and B. Pausader. Global Smooth Ion Dynamics in the Euler-Poisson System. \textit{Commun. Math. Phys.} 303, 89–125 (2011). https://doi.org/10.1007/s00220-011-1193-1.

[18] D. Han-Kwan. Quasineutral limit of the Vlasov-Poisson system with massless electrons. \textit{Communications in Partial Differential Equations, 36}(8):1385-1425, 2011.

[19] D. Han-Kwan and M. Iacobelli. The quasineutral limit of the Vlasov-Poisson equation in Wasserstein metric. \textit{Commun. Math. Sci.}, 15(2):481–509, 2 2017.

[20] D. Han-Kwan, T. T. Nguyen and F. Rousset. Asymptotic stability of equilibria for screened Vlasov–Poisson systems via pointwise dispersive estimates. \textit{Annals of PDE, 7}(2): 1-37, 2021.

[21] D. Han-Kwan, T. Nguyen and F. Rousset. On the linearized Vlasov–Poisson system on the whole space around stable homogeneous equilibria. \textit{Communications in Mathematical Physics, 387}(3): 1405-1440, 2021.

[22] E. Horst. On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation (Parts I and II), \textit{Math. Meth. Appl. Sci.} 3 (1981),pp. 229-248 and 4 (1982), pp. 19-32.

[23] E. Horst. On the asymptotic growth of the solutions of the Vlasov-Poisson system. \textit{Mathematical Methods in the Applied Sciences, 16}(2):75–86, 1993.
[24] L. Huang, Q-H. Nguyen and Y. Xu. Sharp estimates for screened Vlasov-Poisson system around Penrose-stable equilibria in $\mathbb{R}^d$, $d \geq 3$. arXiv:2205.10261.

[25] H.-J. Hwang, A. Rendall, and J.-L. Velázquez. Optimal gradient estimates and asymptotic behaviour for the Vlasov-Poisson system with small initial data. Archive for Rational Mechanics and Analysis, 200(1):313–360, 2011.

[26] A. D. Ionescu and H. Jia. Inviscid damping near the Couette flow in a channel. Communications in Mathematical Physics, 374(3):2015–2096, 2020.

[27] A. Ionescu, B. Pausader, X. Wang and K. Widmayer. On the Asymptotic Behavior of Solutions to the Vlasov–Poisson System. International Mathematics Research Notices, 2021, rnab155, https://doi.org/10.1093/imrn/rnab155

[28] A. Ionescu, B. Pausader, X. Wang and K. Widmayer. Nonlinear Landau damping for the Vlasov-Poisson system in $\mathbb{R}^3$: the Poisson equilibrium. arXiv:2205.04540, 2022.

[29] A. D. Ionescu and H. Jia. Nonlinear inviscid damping near monotonic shear flows. Acta Mathematica (to appear), arXiv:2001.03087, 2020.

[30] P. L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. Inventiones mathematicae, 105(1): 415-430, 1991.

[31] C. Mouhot and C. Villani. On Landau damping. Acta Mathematica, 207(1):29–201, 2011.

[32] N. Masmoudi, W. Zhao. Nonlinear inviscid damping for a class of monotone shear flows in finite channel, arXiv:2001.08564.

[33] Q. H. Nguyen. Quantitative estimates for regular Lagrangian flows with BV vector fields. Communications on Pure and Applied Mathematics, 74(6): 1129-1192, 2021.

[34] T. T. Nguyen. Derivative estimates for screened Vlasov-Poisson system around Penrose-stable equilibria. Kinetic and Related Models, 2020, 13(6): 1193-1218. doi: 10.3934/krm.2020043.

[35] B. Pausader and K. Widmayer. Stability of a point charge for the Vlasov-Poisson system: the radial case. Communications in Mathematical Physics, 385(3):1741–1769, 2021.

[36] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. Journal of Differential Equations, 95(2):281–303, 1992.

[37] J. Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. Communications in Partial Differential Equations, 16(8-9):1313–1335, 1991.

[38] J. Smulevici. Small data solutions of the Vlasov-Poisson system and the vector field method. Annals of PDE, 2(2): 11-55, 2016.

[39] X. Wang. Decay estimates for the 3D relativistic and non-relativistic Vlasov-Poisson systems. arXiv:1805.10837, 2018.