On the adiabatic properties of a stochastic adiabatic wall: Evolution, stationary non-equilibrium, and equilibrium states

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The time evolution of the adiabatic piston problem and the consequences of its stochastic motion are investigated. The model is a one-dimensional piston of mass $M$ separating two ideal fluids made of point particles with mass $m \ll M$. For infinite systems it is shown that the piston evolves very rapidly toward a stationary nonequilibrium state with non-zero average velocity even if the pressures are equal but the temperatures different on both sides of the piston. For finite systems it is shown that the evolution takes place in two stages: first the system evolves rather rapidly and adiabatically toward a metastable state where the pressures are equal but the temperatures different; then the evolution proceeds extremely slowly toward the equilibrium state where both the pressures and the temperatures are equal. Numerical simulations of the model are presented. The results of the microscopical approach, the thermodynamical equations and the simulations are shown to be qualitatively in good agreement.

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1. INTRODUCTION

The adiabatic piston problem is a well-known example in thermodynamics which has given rise to continuous controversy for the last 40 years (see [1] for selected references). It was also mentioned by E. Lieb as one of the “problems in statistical mechanics that I would like to see solved” at the StatPhys 20 meeting in 1998 [2]. The problem is the following. The system is a finite cylinder containing two gases separated by an adiabatic movable piston. Initially, the piston is rigidly fixed by a brake and the two gases are in thermal equilibrium characterized by $(U_1, V_1, N_1)$ and $(U_2, V_2, N_2)$ (see Fig. 1). At a certain time $t_0$, the brake is released and the question is to find the final equilibrium state.

Within the framework of thermostatics, one only knows that the final equilibrium state $(U_1(\infty), V_1(\infty), N_1(\infty))$, $(U_2(\infty), V_2(\infty), N_2(\infty))$, corresponds to a maximum of the entropy function $S(U, V) = S_1[1] + S_2[2] = S_1(U, V) + S_2(U_0 - U, V - V_0)$, where $U_0 = U_1 + U_2$, $V_0 = V_1 + V_2$, submitted to the constraints $S_1[1] \geq S_1[1](t_0)$, $S_2[2] \geq S_2[2](t_0)$. The solution of this mathematical problem yields the result that the final pressures must necessarily be equal $p_1(\infty) = p_2(\infty)$ (i.e. mechanical equilibrium), but nothing can be said about the final temperatures $T_1(\infty)$, $T_2(\infty)$ (originally Kubo had arrived at the wrong conclusion $p_1(\infty)/T_1[1](\infty) = p_2(\infty)/T_2[2](\infty)$ [3]). In other words the laws of thermostatics are not sufficient to predict the final equilibrium state. Here started the controversy, because some physicists could not accept this limitation of thermostatics. Callen writes in [4] “the movable adiabatic wall presents a unique problem with subtleties”. The problem is however neither very subtle, nor unique since it appears every
time movable adiabatic walls are present, and they are indeed essential in the axiomatic formulation of thermostatistics (see e.g. [3]).

Recently, with the help of an oversimplified macroscopical model for the gases, it was shown that the adiabatic piston problem, can be solved within the framework of thermodynamics [1]. Equations for the time evolution were derived from the first and second law. They show that the equilibrium state depends in an essential manner on the viscosity of the gases and one has to solve explicitly the time evolution to find the final state, which is however uniquely defined. In particular in the final state the temperatures \( T_{[1]}(\infty) \) and \( T_{[2]}(\infty) \) are different.

Using arguments from the kinetic theory of gases, similar equations were previously obtained by B. Crosignani et al. [4], without the use of the laws of thermodynamics, but neglecting the effects of fluctuations. The conclusion were however the same as the one obtained from thermodynamics, namely \( T_{[1]}(\infty) \neq T_{[2]}(\infty) \). At this point the controversy started again: it was argued that the stochastic motion of the piston, induced by the stochastic forces exerted by the particles of the gases, might lead to a “true” equilibrium state where \( T_{[1]}(\infty) = T_{[2]}(\infty) \) and \( p_{[1]}(\infty) = p_{[2]}(\infty) \). In [3] it was stated “this stochastic motion leads to a violation of the second law” and “this would be equivalent to permit heat to flow although the piston itself remains an insulator”. It was also suggested that this evolution toward the “true” equilibrium state, if it exists, might be so slow that it can never be observed in practice.

The work done by J. L. Lebowitz in 1959 [7] should have given some insight into these questions already a long time ago. Indeed, using a model of non-interacting particles (in one dimension) to describe the gases, J. L. Lebowitz computed for the first time in [7], the heat conductivity of a movable “adiabatic” piston at lowest order in \( \epsilon = \sqrt{m/M} \), where \( m \) is the mass of the gas particles and \( M \) is the mass of the piston. However, we should remark that the model discussed in [3] was slightly different than the one considered here since the piston was constrained to remain in mechanical equilibrium by means of an external force.

This same microscopical model, but without external force, was reintroduced by Piasecki and Gruber in [3]. Starting with the Boltzmann equation, they obtained the exact stationary solution for the infinite one dimensional system, with \( p_{[1]}(t) = p_{[2]}(t) \), \( T_{[1]}(t) = T_{[2]}(t) \). The main conclusion in [3] was that the space asymmetry \( (T_{[1]} \neq T_{[2]}) \) conspires with the stochastic motion to give a stationary non-equilibrium state with non-zero average velocity of the “piston” toward the region of the higher temperature. The same model with \( M = m \) was then considered by Piasecki and Sinai [6] to study the finite system. They were able to show that the piston evolves toward an equilibrium position corresponding to uniform density and uniform distribution of velocities (thus uniform “temperature”), for the particles throughout the system. Simultaneously Gruber and Piasecki investigated the more physical situation \( m \ll M \). Assuming that the solution of the Boltzmann equation has an asymptotic expansion in \( \epsilon = \sqrt{m/M} \), the stationary solution of the infinite system was computed at the order \( \epsilon^2 \).

It was again established that the temperature difference \( (T_{[1]} \neq T_{[2]}) \) induces a macroscopic motion toward the high temperature region despite the absence of any macroscopic force \( (p_{[1]} = p_{[2]} \) ). In particular, it was shown that in this stationary state the piston behaves as a randomly moving particle with kinetic energy \( k_B \sqrt{T_{[1]}}/2 \), and average velocity \( \sqrt{\frac{m}{M}} \sqrt{\frac{\epsilon k_T}{8}} \left( \sqrt{T_{[2]}} - \sqrt{T_{[1]}} \right) \).

In all these recent articles, the physical problems of heat conduction, viscous force, work power, as well as the approach toward the stationary, or the equilibrium, state were not considered. These questions are investigated in the present article.

Although we did not solve the problem mentioned by E. Lieb, which would need a rigorous mechanical many-body approach, the following analysis answers some of the conjectures discussed above and should contribute to a better understanding of the physical mechanisms involved in the time evolution of the adiabatic piston. It raises also some conjectures that one might be able to prove.

The main conclusion of this paper is that the phenomenological macroscopic thermodynamics, the microscopical Boltzmann equation, and the numerical simulations, give results which coincide with a very high degree of accuracy when \( m \ll M \). In other words there is no violation of the second law and no paradoxes involved.

The results of this investigation show that a wall which is “adiabatic” when rigidly fixed (no heat transfer), becomes “conducting” as soon as it is allowed to have a stochastic motion, and this is independent of the fact that its macroscopic velocity is zero or non-zero. This means that in this model heat transfer is associated with the stochastic motion of the piston, while work power is related to the macroscopic (or average) velocity of the piston \( \langle V \rangle \). The time evolution obtained from thermodynamics (where the above conductivity is taken into account), from Boltzmann equation, and from the numerical simulations, shows that the system evolves toward a final state where the temperatures, as well as the pressures, are equal. However, and this is the fundamental point, when \( m \ll M \) the time scale needed to reach this final state is so enormous (about the age of the universe for realistic numbers), that no reasonable person would call such a piston a conductor. More precisely our numerical analysis indicates that as soon as \( m/M \) is small enough, the time evolution proceeds in two stages with time scales which are very different. In the first stage the evolution is adiabatic and proceeds rather rapidly, with or without oscillations, towards a state of mechanical
equilibrium where the pressures become equal, but the temperatures remain different (this is the “quasi-equilibrium state” suggested in [4]). After this quasi-equilibrium situation is reached the piston will slowly drift into the high temperature domain and the system evolves, on a time scale several orders of magnitude larger, with equal pressures and slowly varying temperatures toward the final equilibrium state where temperatures, densities and pressures are equal on both sides of the piston. On the other hand for \( m/M \approx 1 \), the adiabatic piston is a good conductor and thermal equilibrium \( T_{11}(\infty) = T_{22}(\infty) \) is reached rapidly. Notice that on a very short time scale, there exists a stage where the piston behaves as if the system is infinite. In this stage, if the initial pressures are different, the piston is accelerated while if the pressures are initially equal the piston evolves with a (small) constant velocity inside the high temperature fluid.

In the following analysis the microscopical model is a one dimensional system of point particles which do not interact except for purely elastic collisions. It is exactly this model which has been simulated on a computer.

For the analytical analysis we introduce three main assumptions. First we assume that the correlations between the velocities of the fluid particles and the velocity of the piston can be neglected. Then we assume that the velocity distribution functions for the fluid particles on both faces of the piston admit a scaling function with respect to their mass \( m \), as it is the case with Maxwellian distributions. However, because of the simulations (Sec.III), and the exact solution of Piasecki and Sinai [1], we do not assume Maxwellian distribution for the velocity of the fluid particles. The temperatures \( T_{11}(t) \) and \( T_{22}(t) \) on both faces of the piston are then defined by the second moment of the velocity of the fluid particles. Finally we assume that the moments \( \langle V^s \rangle_t \) of the velocity \( V \) of the piston, \( s \geq 1 \), have an asymptotic expansion in powers of \( \epsilon = \sqrt{m/M} \).

To discuss the solutions of the time evolution equations, we shall either assume that the fluids densities and temperatures are constant (in the case of an infinite system), or uniform (in the case of a finite system).

The microscopical model is defined in Sec.II. In Sec.III we use the conservation laws of the problem to derive the dynamical equations for the first and second moment of the piston velocity distribution and give in Sec.IV these equations for arbitrary moments \( \langle V^s \rangle_t \). The relation between the microscopical model and a phenomenological thermodynamic approach is discussed in Sec.V where we derive a microscopical definition of the friction coefficients and the heat conductivity. The analysis of the stationary non-equilibrium, and the equilibrium, states is presented in Sec.VI, while the time evolution is discussed in Sec.VII. The numerical simulations, the comparison with the dynamic equations, and the thermodynamic equations, is the subject of Sec.VIII. Finally, conclusions are given in Sec.IX.

II. MICROSCOPICAL MODEL

We consider the fluids to be perfect gases made of \( N_{[1]} \), respectively \( N_{[2]} \), identical non-interacting point particles of mass \( m \), making purely elastic collisions, and contained in a fixed cylinder of length \( L \). The basis of the cylinder is a circle with area \( A \). The adiabatic piston is a rigid solid (i.e. no internal degree of freedom), with mass \( M \), described by the microscopical variables \((X, V = \frac{dX}{dt})\) (see Fig.1).

We assume that the cylinder is an ideal constraint, which has no physical properties: the collisions of the particles on the cylinder are purely elastic, and there is no friction between the piston and the cylinder.

For \( t \leq 0 \), the piston is fixed \((X = X_0, V = 0)\) and both fluids are in thermal equilibrium, with temperatures \( T_{11} \), respectively \( T_{22} \), described by the Maxwellian distribution functions:

\[
\begin{align*}
\rho_{[1]}(x, v; t \leq 0) &= \rho_{[1]} \phi_{T_{[1]}}(v) \quad \text{for} \quad 0 \leq x < X_0 \\
\rho_{[2]}(x, v; t \leq 0) &= \rho_{[2]} \phi_{T_{[2]}}(v) \quad \text{for} \quad X_0 < x \leq L 
\end{align*}
\]

where

\[
\begin{align*}
\rho_{[1]} &= \frac{1}{A} \frac{N_{[1]}}{X_0} = \frac{1}{A} n_{[1]} \quad \rho_{[2]} = \frac{1}{A} \frac{N_{[2]}}{L - X_0} = \frac{1}{A} n_{[2]} 
\end{align*}
\]

and

\[
\phi_{T}(v) = \sqrt{\frac{m}{2\pi k_B T}} \exp \left( -\frac{mv^2}{2k_B T} \right).
\]

At time \( t = 0 \), the break is released and the piston can move freely. It will thus have a stochastic motion induced by collisions with the fluids and \((X, V)\) become random variables. The problem is to investigate the random motion of the piston. To simplify the discussion, we consider the system to be one-dimensional. In this case the adiabatic piston is just a point particle with mass \( M \).
For $t > 0$, the piston moves freely. Since the collisions are assumed to be purely elastic, then if the masses $M$ and $m$ have velocities $V$ and $v$ before the collision, their velocities after the collision are given by

$$v' = v - \frac{2M}{M + m}(v - V)$$
$$V' = V + \frac{2m}{M + m}(v - V)$$ \hspace{1cm} (4)

The main assumption of our analysis is that the time evolution is such that it is possible to neglect the correlations between the velocity of the piston and the velocities of the fluids particles. This means that we assume that the joint probability distribution to find, at time $t$, the piston at position $X$ with velocity $V$ and a fluid particle at position $X \pm |c|$ with velocity $v$ can be expressed by

$$\Phi_{[1]}(X, V, X, v; t) = \Phi(X, V; t) \lim_{\epsilon \to 0} \rho_{[1]}(X - |c|, v; t) = \Phi(X, V; t)n_{[1]}(t)\phi_{[1]}(v, t)$$
$$\Phi_{[2]}(X, V, X, v; t) = \Phi(X, V; t) \lim_{\epsilon \to 0} \rho_{[2]}(X + |c|, v; t) = \Phi(X, V; t)n_{[2]}(t)\phi_{[2]}(v, t)$$ \hspace{1cm} (5)

where we have introduced the time dependent particle densities on the faces of the piston $n_{[i]}$ and the normalized fluids velocity distribution functions on the faces of the piston $\phi_{[i]}(v, t)$ for $i = 1, 2$.

In the case of an infinite cylinder, we shall consider that $n_{[1]}(t)$ and $n_{[2]}(t)$ are constant, equal to the fluid initial densities, and that $\phi_{[1]}(v, t) = \phi_{[2]}(v)$ ($i = 1, 2$). This assumption is justified if $m \ll M$, since in this case there is a vanishing probability that the piston interact back with the perturbation it causes in the state of the surrounding fluids (no recollision). In other words, the piston always “sees” on both sides the unperturbed initial states.

For the finite system, when $m \ll M$, we expect the average velocity of the piston to be small, and therefore the time needed for a macroscopical but small displacement of the piston will be much larger than the relaxation time of the fluids (if the initial pressures difference is not too large). We shall then assume in Sec.IV that at all time $t$, the fluids on both side of the piston are homogeneous and thus that the distribution functions $\rho_{[i]}(x, v, t)$, $0 < x < X$, and $\rho_{[2]}(x, v, t)$, $X < x < L$, are independent from the position $x$. We then have

$$n_{[1]}(t) = \frac{N_{[1]}}{(X)_t}, \quad n_{[2]}(t) = \frac{N_{[2]}}{L - (X)_t}$$ \hspace{1cm} (6)

and the time-dependent temperatures $T_{[1]}(t)$ and $T_{[2]}(t)$ are defined by the second moment of the fluid velocity.

For the following analysis, we introduce the small parameters

$$\alpha = \frac{2m}{M + m} \quad \text{and} \quad \epsilon = \sqrt{\frac{m}{M}}.$$ \hspace{1cm} (7)

### III. CONSERVATION LAWS

We use the conservation laws present in the problem to determine the dynamical equations for moments of the piston velocity distribution. In particular, the conservation of momentum and energy leads to equations for the first and second moment of the piston velocity distribution.

#### A. Linear momentum

Let $\Pi_{[1]}(t)$ and $\Pi_{[2]}(t)$ denote the linear momentum of the fluids on the left and on the right of the piston at time $t$. Using the collision equations (4) and the assumption (5) the force exerted by the left fluid on the piston is

$$F^{[1] \to p} = - \lim_{\delta t \to 0} \left< \frac{\delta \Pi_{[1]}(t)}{\delta t} \right> = M\alpha \int_{-\infty}^{\infty} dV \Phi_e(V; t) \int_{V}^{\infty} dv n_{[1]}(t)\phi_{[1]}(v, t)(v - V)^2$$ \hspace{1cm} (8)

where $\Phi_e(V; t)$ is the time dependent velocity distribution function of the piston. Similarly the force exerted by the right fluid on the piston is

$$F^{[2] \to p} = - \lim_{\delta t \to 0} \left< \frac{\delta \Pi_{[2]}(t)}{\delta t} \right> = -M\alpha \int_{-\infty}^{\infty} dV \Phi_e(V; t) \int_{-\infty}^{V} dv n_{[2]}(t)\phi_{[2]}(v, t)(v - V)^2.$$ \hspace{1cm} (9)
Therefore, the conservation law of linear momentum
\[
M \frac{d}{dt} \langle V \rangle_t - F^{[1]} - F^{[2]} = M \frac{d}{dt} \langle V \rangle_t + \lim_{\delta t \to 0} \left[ \left\langle \frac{\delta \Pi^{[1]}(t)}{\delta t} \right\rangle + \left\langle \frac{\delta \Pi^{[2]}(t)}{\delta t} \right\rangle \right] = 0
\] (10)
can be written in the following form
\[
\frac{d}{dt} \langle V \rangle_t = \alpha \int_{-\infty}^{\infty} dV \Phi_t(V; t) \int_{0}^{\infty} dv \left[ n^{[1]}(t) \phi^{[1]}(v, t)(v - V)^2 - n^{[2]}(t) \phi^{[2]}(-v, t)(v + V)^2 \right]
- \alpha \int_{-\infty}^{\infty} dV \Phi_t(V; t) \int_{0}^{V} dv (v - V)^2 \left[ n^{[1]}(t) \phi^{[1]}(v, t) + n^{[2]}(t) \phi^{[2]}(v, t) \right].
\] (11)

Using the expansion
\[
\phi^{[i]}(v, t) = \sum_{k=0}^{\infty} \frac{v^k}{k!} \phi^{(k)}_{[i]}(t), \quad (i = 1, 2)
\] (12)
where
\[
\phi^{(k)}_{[i]}(t) = \left. \frac{d}{dv} \phi^{[i]}(v, t) \right|_{v=0}, \quad (i = 1, 2),
\] (13)
we obtain
\[
\frac{d}{dt} \langle V \rangle_t = \alpha \left[ n^{[1]}(t) \langle v^2 \rangle_t^{[1]} - n^{[2]}(t) \langle v^2 \rangle_t^{[2]} \right]
- \langle V \rangle_t 2\alpha \left[ n^{[1]}(t) \langle v \rangle_t^{[1]} + n^{[2]}(t) \langle v \rangle_t^{[2]} \right]
+ \langle V^2 \rangle_t \alpha \left[ n^{[1]}(t) \langle v^0 \rangle_t^{[1]} - n^{[2]}(t) \langle v^0 \rangle_t^{[2]} \right]
- \sum_{k=0}^{\infty} \langle V^{k+3} \rangle_t \frac{2\alpha}{(k + 3)!} \left[ n^{[1]}(t) \phi^{(k)}_{[1]}(t) + n^{[2]}(t) \phi^{(k)}_{[2]}(t) \right]
\] (14)
where for any non negative integers \(s\)
\[
\langle v^s \rangle_t^{[1]} = \int_{0}^{\infty} dv \phi^{[1]}(v, t)v^s,
\langle v^s \rangle_t^{[2]} = \int_{0}^{\infty} dv \phi^{[2]}(-v, t)v^s, \quad \text{and}
\langle V^s \rangle_t = \int_{-\infty}^{\infty} dV \Phi_t(V; t)V^s.
\] (15)

We should insist on the fact that in the definitions of \(\langle v^s \rangle_t^{[i]}\) the integration is on \(v\) positive. In particular \(\langle v^0 \rangle_t^{[2]}\) is not equal to 1.

We should note that the evolution equation (14) for the average velocity of the piston depends on higher moments of the piston velocity. We should also remark that if the term \(n^{[1]}(t) \langle v^2 \rangle_t^{[1]} - n^{[2]}(t) \langle v^2 \rangle_t^{[2]}\) is zero (which means, as we shall see in Sec.\(\mathbb{X}\) that the pressures exerted by the fluids are equal) and if we neglect the fluctuations of the piston velocity \(\langle V^s \rangle_t = \langle V \rangle_t^{s}\) we find the stable equilibrium solution \(\langle V \rangle_t = 0\). The effects of the fluctuations is the main problem to be investigated and will give a very different picture.

**B. Energy**

Similarly, let \(E_{[1]}(t)\) and \(E_{[2]}(t)\) denote the energy of the fluids on the left and on the right of the piston at time \(t\). It is equal to the kinetic energy, since the particles do not interact except for elastic collisions. Using the collision equations (4) and the assumption (5), we have
to obtain the equations for the velocity moments coupled with higher order moments. We derive in this section the infinite set of coupled equations for all the moments if the case of mechanical equilibrium, the initial state \( \langle v \rangle = 0 \) is unstable. Let us remark that we use the Boltzmann’s equation for the piston velocity distribution derived in [8] under the assumption (5).

We thus obtained an equation for the evolution of the second moment of the piston velocity. Let us remark that even in case of mechanical equilibrium where \( n_{[1]}(t) \langle v^3 \rangle_i + n_{[2]}(t) \langle v^3 \rangle_i = 0 \) the initial state \( \langle V^s \rangle_0 = 0 \) is unstable if \( n_{[1]}(t) \langle v^3 \rangle_i + n_{[2]}(t) \langle v^3 \rangle_i \neq 0 \). In conclusion, if we do not neglect the correlations in [4], we see that even in case of mechanical equilibrium, the initial state \( \langle V^s \rangle_0 = 0 \) is unstable and thus the piston will start to move.

IV. EQUATION FOR ARBITRARY MOMENT

In the previous section, we derived equations for the first and second moments of the piston velocity which are coupled with higher order moments. We derive in this section the infinite set of coupled equations for all the moments of the piston velocity.

We use the Boltzmann’s equation for the piston velocity distribution derived in [8] under the assumption (5)

\[
\frac{\partial}{\partial t} \Phi_r(V; t) = \int_{-\infty}^{\infty} dv |V - v| \left[ \Theta(V - v) n_{[1]}(t) \phi_{[1]}(v - (2 - \alpha)(v - V); t) \Phi_r(V + \alpha(v - V); t) \\
+ \Theta(v - V) n_{[2]}(t) \phi_{[2]}(v - (2 - \alpha)(v - V); t) \Phi_r(V + \alpha(v - V); t) \\
- \Theta(v - V) n_{[1]}(t) \phi_{[1]}(v, t) \Phi_r(V, t) - \Theta(V - v) n_{[2]}(t) \phi_{[2]}(v, t) \Phi_r(V, t) \right]
\]

(21)
to obtain the equations for the velocity moments
\[
\frac{d}{dt} \langle V^s \rangle_t = \int_{-\infty}^{\infty} dVV^s \frac{\partial}{\partial t} \Phi_r(V; t) \\
= \left\langle \int_{V}^\infty dv n_{[1]}(t)\phi_{[1]}(v, t) (v - V) \{ [V(1 - \alpha) + \alpha v]^s - V^s \} \right\rangle_t \\
- \left\langle \int_{-\infty}^{V} dv n_{[2]}(t)\phi_{[2]}(v, t) (v - V) \{ [V(1 - \alpha) + \alpha v]^s - V^s \} \right\rangle_t 
\]

where \( \langle A(V) \rangle_t = \int_{-\infty}^{\infty} dV A(V)\Phi_r(V; t) \). This equation can be written in the following form

\[
\frac{d}{dt} \langle V^s \rangle_t = \left\langle \int_{0}^{\infty} dv n_{[1]}(t)\phi_{[1]}(v, t) (v - V) \{ [V(1 - \alpha) + \alpha v]^s - V^s \} \right\rangle_t \\
+ \left\langle \int_{0}^{\infty} dv n_{[2]}(t)\phi_{[2]}(-v, t) (v + V) \{ [V(1 - \alpha) - \alpha v]^s - V^s \} \right\rangle_t \\
- \left\langle \int_{0}^{V} dv (v - V) \{ [V(1 - \alpha) + \alpha v]^s - V^s \} [n_{[1]}(t)\phi_{[1]}(v, t) + n_{[2]}(t)\phi_{[2]}(v, t)] \right\rangle_t. 
\]

Using the expansions (22) and

\[
[V + \alpha(v - V)]^s - V^s = \sum_{q=1}^{s} \frac{s^q}{q!(s-q)!} \alpha^q (v - V)^{q} V^{s-q}, 
\]

we obtain

\[
\frac{d}{dt} \langle V^s \rangle_t = \sum_{r=0}^{\infty} A_{s,r}(t) \langle V^r \rangle_t 
\]

where one has the explicit expression of \( A_{s,r}(t) \) as a function of the parameter \( \alpha \). In the following, it will be more convenient to have the explicit expression of \( A_{s,r}(t) \) as a function of the parameter \( \epsilon = \sqrt{m/M} \).

Motivated by the Maxwellian distribution (23), we introduce the scaled velocity \( w = \sqrt{mv} \) and the corresponding normalized distribution functions \( f_{[i]}(w, t) \)

\[
\phi_{[i]}(v, t) = \sqrt{m} f_{[i]}(\sqrt{mv}, t), \quad \int_{-\infty}^{\infty} dw f_{[i]}(w, t) = 1, \quad (i = 1, 2). 
\]

We then have from Eqs. (15)-(23),

\[
\langle w^q \rangle_t^{[i]} = m^{-q/2} \int_{0}^{\infty} dw f_{[i]}(w; t) w^q = m^{-q/2} \langle w^q \rangle_t^{[i]}, \quad (i = 1, 2) 
\]

and

\[
\phi_{[i]}^{(k)}(t) = m^{(k+1)/2} f_{[i]}^{(k)}(t). 
\]

For any integer \( r \) we introduce the functions \( K_{r}^{[i]}(t) \) and \( K_{r}(t) \) defined in the following manner. For any \( q \geq 0 \), and \( i = 1, 2 \),

\[
K_{r}^{[i]}_{2+q}(t) = M^{-q/2} n_{[i]}(t) \langle w^q \rangle_t^{[i]}, 
\]

\[
K_{r}^{[i]}_{3-q}(t) = M^{(q+1)/2} n_{[i]}(t) f_{[i]}^{(q)}(t), 
\]

\[
K_{r}(t) = K_{r}^{[1]}(t) - (-1)^r K_{r}^{[2]}(t) \quad \text{for} \quad r \geq -2, 
\]

\[
K_{r}(t) = K_{r}^{[1]}(t) + K_{r}^{[2]}(t) \quad \text{for} \quad r \leq -3. 
\]

With these definitions, we have
where the coefficients $A_{s,r}(t)$ are given by

$$A_{s,r}(t) = \frac{\epsilon^{s-r-1}}{(1+\epsilon^2)^s} P_{s,r}(\epsilon^2) K_{s-r-1}(t).$$

(34)

The functions $K_r(t)$, Eqs. (11,22), depend only on the densities and the velocity distribution functions of the fluid particles on both faces of the piston, while $P_{s,r}(\epsilon^2)$ is a polynomial in $\epsilon^2$, of order $\min(r,s-1)$, with constant coefficients, given by

$$P_{s,r}(\epsilon^2) = (1-\epsilon^2)^{r-1} \left[(s+1-r) - \epsilon^2(s+1+r)\right] \frac{2^{s-r} s!}{r!(s-r+1)!}, \quad r \leq s-1;$$

(35)

$$P_{s,s}(\epsilon^2) = \frac{(1-\epsilon^2)^{s-1} [1-\epsilon^2(2s+1)] - (1+\epsilon^2)^{s}}{\epsilon^2};$$

(36)

$$P_{s,s+1}(\epsilon^2) = \frac{(1+\epsilon^2)^{s} - (1-\epsilon^2)^{s}}{\epsilon^2};$$

(37)

$$P_{s,s+2+k}(\epsilon^2) = -2 \sum_{q=0}^{s-1} (-1)^q(2\epsilon^2)^q(1+\epsilon^2)^{s-q-1} \frac{s!(q+2)}{(s-1-q)!(k+q+3)!}, \quad r \geq s+2.$$

(38)

One can verify that Eq.(33) for $s = 1, 2$ are identical to the equations for the first and the second moment ([14, 20]) derived in the preceding section.

V. FROM KINETIC THEORY TO THERMODYNAMICS: FRICTION COEFFICIENT AND HEAT CONDUCTION

In this section we relate the microscopical model to a phenomenological thermodynamic approach. This will suggest microscopical definitions for “pressure”, “viscous force”, “temperature” and “heat flux”.

Using a very primitive model for the fluids, the equations for the time evolution of the piston problem were derived in [1] from the first and second laws of the thermodynamics. They read

$$\frac{dV}{dt} = A(p_{[1]} - p_{[2]}) - (\lambda_{[1]} + \lambda_{[2]})V; \quad V = \frac{dX}{dt}$$

(39)

$$\frac{dS_{[1]}}{dt} = \frac{\lambda_{[1]} V^2}{T_{[1]}} + \frac{\kappa}{T_{[1]}}(T_{[2]} - T_{[1]}),$$

(40)

$$\frac{dS_{[2]}}{dt} = \frac{\lambda_{[2]} V^2}{T_{[2]}} - \frac{\kappa}{T_{[2]}}(T_{[2]} - T_{[1]}).$$

(41)

where $X$ and $V$ are the macroscopic position and velocity of the piston of mass $M$, $\delta MV^2/2$ represents the kinetic energy of the fluids, $A$ is the area of the cylinder, $p_{[1]}, p_{[2]}, \lambda_{[1]}, \lambda_{[2]}, \kappa$ are phenomenological time-dependent functions of the variables $(X, V, S_{[1]}, S_{[2]})$ and denote the pressures, the friction coefficients, and the heat conductivity. $S_{[1]}$ and $S_{[2]}$ denote the entropy of the fluids on the left and on the right of the piston.

For ideal fluids in one dimension, the thermal equation (11) and (12) are equivalent to

$$\frac{1}{2} N_{[1]} k_B \frac{dT_{[1]}}{dt} = \frac{\lambda_{[1]} V^2}{X} - \frac{N_{[1]} k_B T_{[1]}}{X} V + \kappa(T_{[2]} - T_{[1]})$$

(42)

$$\frac{1}{2} N_{[2]} k_B \frac{dT_{[2]}}{dt} = \frac{\lambda_{[2]} V^2}{L - X} - \frac{N_{[2]} k_B T_{[2]}}{L - X} V - \kappa(T_{[2]} - T_{[1]}).$$

(43)

In thermodynamics, the adiabatic piston is defined by $\kappa = 0$. One should remark that in this case ($\kappa = 0$), if the velocity of the piston is sufficiently small so that the term in $V^2$ can be neglected, the evolution of the piston is damped, but reversible in the thermodynamical sense, characterized by the adiabats.
\[ S_{[1]}(t) = S_{[1]} \quad \text{and} \quad S_{[2]}(t) = S_{[2]} \]

i.e.
\[ T_{[1]}X^2 = \text{cte} \quad \text{and} \quad T_{[2]}(L - X)^2 = \text{cte}. \]  

(44)

In particular, the solution \( p_{[1]} = p_{[2]} \), \( V = 0 \) is stable.

To make the connection with the thermodynamical equations (22), we write the microscopic equation for conservation of momentum (11) or (14) in the form
\[ M \left(1 + \frac{m}{M}\right) \frac{d}{dt} \langle V \rangle_t = 2m \left[n_{[1]}(t) \langle v^2 \rangle_t + n_{[2]}(t) \langle v^2 \rangle_t\right] - \langle V \rangle_t 2mn_{[1]}(t) \left[2 \langle v^1 \rangle_t - \langle V \rangle_t \langle v^0 \rangle_t \right] + \frac{1}{\langle V \rangle_t} \int_0^{\langle V \rangle_t} dv \phi_{[1]}(v, t)(v - \langle V \rangle_t)^2 \]
\[ - \langle V \rangle_t 2mn_{[2]}(t) \left[2 \langle v^2 \rangle_t + \langle V \rangle_t \langle v^0 \rangle_t \right] + \frac{1}{\langle V \rangle_t} \int_0^{\langle V \rangle_t} dv \phi_{[2]}(v, t)(v - \langle V \rangle_t)^2 \]
\[ + \left[\langle V^2 \rangle_t - \langle V \rangle_t^2\right] 2m \left[n_{[1]}(t) \langle v^0 \rangle_t + n_{[2]}(t) \langle v^0 \rangle_t\right] - \sum_{k=0}^{\infty} \left[\langle V^{k+3} \rangle_t - \langle V \rangle_t^k\right] \frac{4m}{(k+3)!} n_{[1]}(t) \phi^{(k)}_{[1]}(t) + n_{[2]}(t) \phi^{(k)}_{[2]}(t) \right]. \]

(45)

We are thus led to identify \( \langle V \rangle_t \) with the macroscopic velocity \( V(t) \) of the piston, and to define the “pressure”, the “temperature”, and the “friction coefficient”, on both sides \((i=1, 2)\) of the piston by
\[ p_{[i]}(t) = n_{[i]}(t)k_B T_{[i]}(t) = 2mn_{[i]}(t) \langle v^2 \rangle_t = 2n_{[i]}(t) \langle w^2 \rangle_t = 2MK_{[i]}^2(t) \]

(46)

\[ \lambda_{[i]}(V, t) = 2mn_{[i]}(t) \left[2 \langle v^1 \rangle_t + (-1)^iV \langle v^0 \rangle_t \right] + \frac{1}{\langle V \rangle_t} \int_0^{\langle V \rangle_t} dv \phi_{[i]}(v, t)(v - V)^2 \]

(47)

which gives in particular
\[ \lambda_{[1]}(V = 0) = 4\sqrt{mn_{[i]} \langle w \rangle^2} = 4\epsilon MK_{[i]}, \]

(48)

\[ \lambda_{[1]}(V) + \lambda_{[2]}(V) = M \left[4\epsilon K_{-1} - 2\epsilon^2 K_{-2}V + \mathcal{O} (\epsilon^3V^2)\right], \]

(49)

and shows that the condition \( \lambda_{[i]}(V) \geq 0 \) will be satisfied in the domain considered here where the average piston velocity is much smaller than the average absolute velocity of a fluid particle, \( V \ll \langle v \rangle^i \). Notice that the temperature definition Eq. (16) is consistent with a Maxwellian definition of the temperature. We thus have from Eq. (16) \( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \)
\[ M(1 + \epsilon^2) \frac{d}{dt} \langle V \rangle_t = (p_{[1]}(t) - p_{[2]}(t)) - (\lambda_{[1]}(V, t) + \lambda_{[2]}(V, t)) \langle V \rangle_t + F_{st}(t), \]

(50)

and
\[ F_{[1]}^{[1] \rightarrow p}(t) = -F_{p \rightarrow [1]}(t) = \frac{1}{1 + \epsilon^2} \left[p_{[1]}(t) - \lambda_{[1]}(V, t) \langle V \rangle_t + F_{st}^{[1]}\right] \]

(51)

\[ F_{[2]}^{[2] \rightarrow p}(t) = -F_{p \rightarrow [2]}(t) = \frac{1}{1 + \epsilon^2} \left[-p_{[2]}(t) - \lambda_{[2]}(V, t) \langle V \rangle_t + F_{st}^{[2]}\right] \]

(52)

where the stochastic forces are
\[ F_{st}^{[1]} = \left[\langle V^2 \rangle_t - \langle V \rangle_t^2\right] 2mn_{[1]}(t) \langle v^0 \rangle_t - \sum_{k=0}^{\infty} \left[\langle V^{k+3} \rangle_t - \langle V \rangle_t^k\right] \frac{4m}{(k+3)!} n_{[1]}(t) \phi^{(k)}_{[1]}(t) \]

(53)

\[ F_{st}^{[2]} = -\left[\langle V^2 \rangle_t - \langle V \rangle_t^2\right] 2mn_{[2]}(t) \langle v^0 \rangle_t - \sum_{k=0}^{\infty} \left[\langle V^{k+3} \rangle_t - \langle V \rangle_t^k\right] \frac{4m}{(k+3)!} n_{[2]}(t) \phi^{(k)}_{[2]}(t). \]

and \( F_{st} = F_{st}^{[1]} + F_{st}^{[2]} \).
We now compute the heat flux from one side of the piston to the other side in the microscopic model. This will give us a microscopic definition of the heat conductivity \( \kappa \). We will see that the microscopical equations are compatible with the phenomenological thermodynamical equations in which we take into account a heat flux characterized by a conductivity \( \kappa \neq 0 \) which comes from the fluctuations of the microscopical model. From the first law of thermodynamics

\[
\frac{d}{dt} E_{[1]} = P_{W}^{p \rightarrow [1]} + P_{Q}^{[2] \rightarrow [1]} \tag{54}
\]

where \( P_{W}^{p \rightarrow [1]} \), respectively \( P_{Q}^{[2] \rightarrow [1]} \), denote the power transmitted to the fluid on the left in the form of work, resp. in the form of heat, together with the thermodynamic definition of the work power, i.e.

\[
P_{W}^{p \rightarrow [1]}(t) = F_{p \rightarrow [1]}^{\rightarrow}(t)V(t) \tag{55}
\]

we are led to define, through Eqs.(51,16,18,33), the microscopic heat power, or heat flux, and the microscopic heat conductivity \( \kappa \)

\[
P_{Q}^{[2] \rightarrow [1]}(t) = \left\langle \frac{dE_{[1]}}{dt} \right\rangle - P_{W}^{p \rightarrow [1]}(t) \langle V \rangle_{t}
\]

\[
= -\frac{1}{2} MA_{2,0}^{[1]} + M \sum_{r=0}^{\infty} \left[ A_{1,r}^{[1]} \langle V^{r} \rangle_{t} - \frac{1}{2} A_{2,r+1}^{[1]} \langle V^{r+1} \rangle_{t} \right] 
\]

\[
= M \frac{4 \epsilon^2 K_{1}^{[1]}}{1 + \epsilon^2} \left( \frac{1}{2} A_{2,r+1}^{[1]} \langle V^{r+1} \rangle_{t} \right)
\]

\[
+ M \sum_{r=2}^{\infty} \left[ A_{1,r}^{[1]} \langle V^{r} \rangle_{t} - \frac{1}{2} A_{2,r+1}^{[1]} \langle V^{r+1} \rangle_{t} \right] 
\]

\[
= \kappa(T_{[2]} - T_{[1]}) \tag{56}
\]

where \( A_{i,j} \) are defined by Eq.(34) with \( K_{1}^{[i]}. \)

At order \( O(\epsilon) \), it reduces to

\[
P_{Q}^{[2] \rightarrow [1]} = 4 \epsilon M K_{1}^{[1]} \left[ \langle V^2 \rangle_{t} - \langle V \rangle_{t}^2 \right] - 2 \epsilon M K_{1}^{[1]} + O(\epsilon^2)
\]

\[
= 4 \frac{\sqrt{m}}{M} n_{[1]}(t) M \left[ \langle V^2 \rangle_{t} - \langle V \rangle_{t}^2 \right] - \frac{1}{2} \langle w^3 \rangle_{t} + O(\epsilon^2). \tag{57}
\]

To compare our definition (56) with the heat conductivity obtained in [7], let us consider the case of the infinite system. In this case the coefficients \( K_{1}^{[i]} \) and the densities \( n_{[i]} \) are independent on time. As we shall see in Sec.VI, for any stationary state of the infinite system we have

\[
\langle V^2 \rangle_{\infty} - \langle V \rangle_{\infty} = \frac{K_{1}}{2K_{-1}} + O(\epsilon)
\]

which thus gives for a stationary state

\[
P_{Q}^{[2] \rightarrow [1]} = \frac{2 \sqrt{m}}{M} n_{[1]} n_{[2]} \left[ \langle w^3 \rangle_{[1]} \langle w \rangle_{[2]} - \langle w^3 \rangle_{[2]} \langle w \rangle_{[1]} \right] + O(\epsilon^2). \tag{59}
\]

In particular, for a Maxwellian distribution of the fluid particles, this last equation reads

\[
P_{Q}^{[2] \rightarrow [1]} = \frac{k_{B}}{M} \sqrt{\frac{8 \pi m k_{B}}{3}} n_{[1]} n_{[2]} \sqrt{T_{[1]}^{2}} \frac{\sqrt{T_{[1]}^{1}} T_{[2]}^{2}}{\sqrt{T_{[1]}^{[1]} + n_{[2]} \sqrt{T_{[2]}^{[2]}}}} \left( T_{[2]} - T_{[1]} \right) + O(\epsilon^2). \tag{60}
\]

Therefore the heat conductivity is

\[
\kappa = \frac{k_{B}}{M} \sqrt{\frac{8 \pi m k_{B}}{3}} n_{[1]} n_{[2]} \sqrt{T_{[1]}^{2}} \frac{\sqrt{T_{[1]}^{1}} T_{[2]}^{2}}{\sqrt{T_{[1]}^{[1]} + n_{[2]} \sqrt{T_{[2]}^{[2]}}}} + O(\epsilon^2). \tag{61}
\]
Similarly, for the infinite system the friction coefficient \( \lambda \) can be written as

\[
\lambda(V) = n[V] \sqrt{\frac{8k_B T[V] m}{\pi}} - n[V] m V + n[V] m \sqrt{\frac{2m}{\pi k_B T[V]}} \int_0^V dv \exp \left( - \frac{m v^2}{2k_B T[V]} \right) v^2
\]  

It is interesting to note that for a stationary state of the infinite system and Maxwellian distribution for the fluids, the expression for the friction coefficient \( \lambda(V = 0) \) Eq. (62) and for the heat conductivity \( \kappa \) Eq. (61) are, to lowest order in \( \epsilon \), identical with those first derived by J.L. Lebowitz for a slightly different model \([7]\). In \([7]\) the piston is constrained in a state of mechanical equilibrium (i.e. \( \langle V^k \rangle = 0 \) for all odd integer \( k \)) by mean of an external potential and the heat flux is just given by \( dE/V/\partial t \). In our model there is no external potential and in the stationary state the piston has a constant non-zero average velocity; we thus have to take into account the work power in the definition of the heat flux Eq. (56). We should also remark that in \([7]\) the friction coefficient is introduced via the Ornstein-Uhlenbeck process describing the motion of the piston, while our definition is associated with the mechanical definition of the heat flux Eq. (56). We should also remark that in \([7]\) the friction coefficient is introduced via the Ornstein-Uhlenbeck process describing the motion of the piston, while our definition is associated with the mechanical definition of the heat flux Eq. (56). We should also remark that in \([7]\) the friction coefficient is introduced via the Ornstein-Uhlenbeck process describing the motion of the piston, while our definition is associated with the mechanical definition of the heat flux Eq. (56).

In summary, if initially \( \langle V^r \rangle_{t=0} = 0 \) for all \( r \geq 1 \), then as long as the contributions

\[
\frac{\lambda[V]}{M} \left[ M \langle \langle V^2 \rangle_t - \langle V \rangle_t^2 \rangle - k_B T[V] \right], \quad (i = 1, 2),
\]

as well as the stochastic force \( F_k \), can be neglected, the time evolution Eqs. (54-60) will coincide with the thermodynamic evolution of the adiabatic piston Eq. (42-41) with \( k = 0 \). This will be the case as long as \( \langle V^r \rangle_t \simeq \langle V \rangle_t^r \), and \( \langle V \rangle_t \gg k_B \max(T_{[1]}, T_{[2]})/M \). Furthermore, as long as \( \langle V \rangle_t^2 \ll 1 \) the evolution is described by the adiabats \( S_{[1]}(t) = S_{[1]}, S_{[2]}(t) = S_{[2]} \). Therefore during this first stage the piston evolves according to \([56]\) towards a state of mechanical equilibrium, i.e. \( p_{[1]} \simeq p_{[2]} \), but not thermal equilibrium \( (T_{[1]} \neq T_{[2]}) \). As soon as the pressures are approximately equal, the above assumptions are no longer valid, and as we shall see in Sec. VII the system will evolve toward a state of thermal equilibrium. Numerical simulations will be presented in Sec. VIII to show the time scales on which these evolutions take place.
VI. STATIONARY NON-EQUILIBRIUM AND EQUILIBRIUM STATE

A. Preliminary remarks

We have seen in Sec. IV that the time evolution of the piston is characterized by the equations

\[
\frac{d}{dt} \langle V^s \rangle_t = C_s + \sum_{r=1}^{\infty} A_{s,r} \langle V^r \rangle_t, \quad s \geq 1
\]

(68)

where \( C_s(t) = A_{s,0}(t) \) and for any \( r \geq 0 \)

\[
A_{s,r}(t) = \frac{\epsilon^{s-r-1}}{(1+\epsilon^2)^s} P_{s,r}(\epsilon^2) K_{s-r-1}(t).
\]

(69)

In particular for \( s = 1 \) and \( s = 2 \) we have (where \( \equiv \) denotes “for Maxwellian distributions”),

\[
\frac{d}{dt} \langle V \rangle_t = \frac{1}{M(1+\epsilon^2)} \left[ (p[1] - p[2]) - \lambda \langle V \rangle_t + m(n[1] - n[2]) \langle V^2 \rangle_t + \cdots \right]
\]

(70)

\[
\frac{d}{dt} \langle V^2 \rangle_t = \frac{2}{M(1+\epsilon^2)^2} \left[ (1-2\epsilon^2)(p[1] - p[2]) \langle V \rangle_t - \lambda \langle V^2 \rangle_t + 2\epsilon MK_1 + \cdots \right]
\]

(71)

with

\[
p[1] - p[2] = M2K_0 = k_B(n[1]T[1] - n[2]T[2])
\]

\[
\lambda = M4\epsilon K_{-1} \equiv \sqrt{m} \frac{8k_B}{\pi} \left( n[1]\sqrt{T[1]} + n[2]\sqrt{T[2]} \right)
\]

\[
2MK_1 = \sqrt{\frac{m}{M}} \frac{8k_B}{\pi} k_B \left( n[1]T[1]^{3/2} + n[2]T[2]^{3/2} \right)
\]

\[
2K_{-2} = n[1] - n[2]
\]

(72)

To lowest order, i.e. in the limit \( \epsilon \to 0 \), Eq. (68) together with the definition (16) of the pressure yields

\[
\frac{d}{dt} \langle V^s \rangle_t = 2sK_0(t) \langle V^{s-1} \rangle_t
\]

\[
= s \frac{1}{M} [p[1](t) - p[2](t)] \langle V^{s-1} \rangle_t.
\]

(73)

As discussed in Sec. IV, for an infinite system with \( \epsilon \ll 1 \), the assumptions

\[
n[i](t) = n[i], \quad \phi[i](v,t) = \phi_T(v), \quad (i = 1, 2)
\]

(74)

are justified and give \( p[1](t) = p[1], p[2](t) = p[2] \) constant.

In this case, for the initial conditions

\[
\langle V^s \rangle_{t=0} = \langle (V)_{t=0} \rangle^s
\]

(75)

we obtain at lowest order

\[
\langle V \rangle_t = \langle V \rangle_{t=0} + \frac{1}{M} (p[1] - p[2]) t
\]

\[
\langle V^s \rangle_t = \langle (V) \rangle^s
\]

(76)

and thus there is no stationary state in the infinite system unless \( p[1] = p[2] \) in the limit \( \epsilon \to 0 \).

On the other hand, for the finite system, if the evolution is sufficiently slow and \( \epsilon \ll 1 \), we may assume that the following homogeneity conditions hold:

\[
n[1](t) = \frac{N[1]}{\langle X \rangle_t}, \quad n[2](t) = \frac{N[2]}{L-\langle X \rangle_t}, \quad \langle E[i] \rangle_t = N[i]m \langle v^2 \rangle \langle i \rangle \equiv \frac{1}{2} N[i]k_B T[i](t), \quad (i = 1, 2)
\]

(77)
where we do not assume Maxwellian distribution of the velocities but define the temperatures by the second moments of the fluid velocity distribution Eq.(46). Under this homogeneity condition, the time evolution for \( T_{[i]} \) and \( T_{[2]} \) are therefore

\[
\frac{dT_{[i]}(t)}{dt} = \frac{2}{N_{[i]} k_B} \left( \frac{d}{dt} E_{[i]} \right) = -\frac{M}{N_{[i]} k_B} \sum_{r=0}^{\infty} A_{2,r} \langle V^r \rangle_t, \quad (i = 1, 2). \tag{78}
\]

In the limit \( \epsilon \rightarrow 0 \) we have from Eqs.(78,77,78)

\[
M \frac{d^2}{dt^2} \langle X \rangle_t = k_B \left[ \frac{N_{[1]} \gamma_{[1]} - N_{[2]} \gamma_{[2]}}{\langle X \rangle_t} \right]
\]

\[
\frac{dT_{[1]}(t)}{dt} = \frac{2}{\langle X \rangle_t} T_{[1]} \frac{d}{dt} \langle X \rangle_t
\]

\[
\frac{dT_{[2]}(t)}{dt} = \frac{2}{L - \langle X \rangle_t} T_{[2]} \frac{d}{dt} \langle X \rangle_t
\]

which yields the “adiabats”

\[ T_{[1]}(t) \langle X \rangle_t^2 = \gamma_{[1]} = \text{cte}; \quad T_{[2]}(t)(L - \langle X \rangle_t)^2 = \gamma_{[2]} = \text{cte} \tag{80} \]

and the undamped oscillatory motion

\[
M \frac{d^2}{dt^2} \langle X \rangle_t = k_B \left[ \frac{N_{[1]} \gamma_{[1]} - N_{[2]} \gamma_{[2]}}{\langle X \rangle_t^3 - (L - \langle X \rangle_t)^3} \right]. \tag{81}
\]

In conclusion, to lowest order in \( \epsilon \), there is no approach toward an equilibrium state in the finite system while a stationary state in the infinite system can exist only if the initial pressures are equal \((p_{[1]} = p_{[2]})\). This conclusion is consistent with the previous result, since for finite \( M \) the condition \( \epsilon \rightarrow 0 \) means that \( m \rightarrow 0 \) and thus the friction coefficient and the heat conductivity are both zero.

**B. Stationary state of the infinite system**

In this section, we discuss the stationary states of the infinite system. In this case the densities \( n_{[1]}, n_{[2]} \), and the coefficients \( K_r \) are constant. Moreover we consider the moments \( \langle u^q \rangle \) which appear in the definition of \( K_r \) to be given by the Maxwellian distributions i.e.

\[
\langle u^0 \rangle_{[i]}^2 = \frac{1}{2}, \quad \langle u^1 \rangle_{[i]}^2 = \frac{1}{2\pi} \sqrt{2k_B T_{[i]}},
\]

\[
\langle u^2 \rangle_{[i]} = \frac{1}{2} k_B T_{[i]}, \quad \langle u^3 \rangle_{[i]} = \frac{1}{2\pi} (2k_B T_{[i]})^{3/2}, \quad (i = 1, 2). \tag{82}
\]

The stationary states of the infinite system are given by the solution of the equations

\[ 0 = \sum_{r \geq 0} \epsilon^{s-r-1} P_{s,r} \langle \epsilon^2 \rangle K_{s-r-1} \langle V^r \rangle_\infty. \tag{83} \]

From Eqs.(78,77,78), it follows immediately that

\[
\langle V \rangle_\infty = \frac{p_{[1]} - p_{[2]}}{\lambda} + \frac{m}{\lambda} \left( n_{[1]} - n_{[2]} \right) \langle V^2 \rangle_\infty + \cdots \tag{84}
\]

\[
\langle V^2 \rangle_\infty \left[ 1 - (1 - 2\epsilon^2) \frac{p_{[1]} - p_{[2]}}{\lambda} \left( n_{[1]} - n_{[2]} \right) \right] = (1 - 2\epsilon^2) \left( \frac{p_{[1]} - p_{[2]}}{\lambda} \right)^2 + 2 \frac{\epsilon}{\lambda} M K_1 + \cdots \tag{85}
\]

i.e. at lowest order in \( \epsilon \) we have

\[
\langle V \rangle_\infty = \frac{p_{[1]} - p_{[2]}}{\lambda} + \left( \frac{p_{[1]} - p_{[2]}}{\lambda} \right)^2 \frac{m}{\lambda} k_B \left( \frac{p_{[1]} T_{[1]} - p_{[2]} T_{[2]}}{T_{[1]} T_{[2]}} \right) + \frac{m p_{[1]} T_{[2]} - p_{[2]} T_{[1]}}{M} \frac{T_{[1]} T_{[2]}}{\lambda \sqrt{T_{[1]} T_{[2]}}}, \tag{86}
\]

\[
\langle V^2 \rangle_\infty = \left( \frac{p_{[1]} - p_{[2]}}{\lambda} \right)^2 + \frac{k_B}{M} \sqrt{T_{[1]} T_{[2]}} \tag{87}
\]
which shows that is impossible to have a stationary state with \( \langle V \rangle_{\infty} = 0, p_{[1]} = p_{[2]}, T_{[1]} \neq T_{[2]} \).

In the following we shall assume that the solution of Eq.\((83)\) has an asymptotic expansion in \( \epsilon \)

\[
\langle V' \rangle_{\infty} = \sum_{l=0}^{\infty} \epsilon^l \langle V'^{l} \rangle_{\infty} .
\]  

(88)

Then in the limit \( \epsilon \to 0 \) we must have

\[
K_0 = 0, \text{ i.e. } p_{[1]} = p_{[2]}
\]  

(89)

which can also be seen explicitly with Eqs.\((84,85)\) since \( \lambda = O(\epsilon) \).

To take into account the possibility of non-zero pressure difference for the infinite system, and to have results valid for finite systems where the equilibrium densities might depend on \( \epsilon \), we now consider that the densities \( n_{[1]}, n_{[2]} \) are function of \( \epsilon \)

\[
n_{[i]}(\epsilon) = \sum_{l=0}^{\infty} \epsilon^l n_{[i]}^{(l)} \quad (i = 1, 2)
\]  

(90)

which then implies that the coefficients \( K_r \) are also functions of \( \epsilon \)

\[
K_r(\epsilon) = \sum_{l=0}^{\infty} \epsilon^l K_r^{(l)}
\]  

(91)

The condition \((89)\) for a stationary solution is now

\[
K_0^{(0)} = \lim_{\epsilon \to 0} K_0(\epsilon) = 0
\]  

(92)

i.e.

\[
n_{[1]}^{(0)} T_{[1]} - n_{[2]}^{(0)} T_{[2]} = 0
\]  

(93)

or

\[
\lim_{\epsilon \to 0} |p_{[1]}(\epsilon) - p_{[2]}(\epsilon)| = p_{[1]}^{(0)} - p_{[2]}^{(0)} = 0.
\]  

(94)

Therefore Eq.\((83)\) can be simplified by \( \epsilon \) to give

\[
0 = P_{s,s-1} \left( \frac{K_0}{\epsilon} \right) \langle V^{s-1} \rangle_{\infty} + \sum_{r=0}^{s-2} \epsilon^r \left[ P_{s,s-2-r} K_{1+r} \langle V^{s-2-r} \rangle_{\infty} + P_{s,s+r} K_{-1-r} \langle V^{s+r} \rangle_{\infty} \right] + \sum_{r=0}^{\infty} \epsilon^{s-1+r} P_{s,2s-1+r} K_{-s-r} \langle V^{2s-1+r} \rangle_{\infty}.
\]  

(95)

We then solve \((95)\) at successive order in \( \epsilon \).

At order \( \epsilon^0 \), we have

\[
0 = 2K_0^{(1)} - 4K_0^{(0)} \langle V \rangle_{\infty} \quad (s = 1)
\]

\[
0 = 4K_0^{(1)} \langle V \rangle_{\infty} + 4K_0^{(0)} - 8K_0^{(0)} \langle V^2 \rangle_{\infty} \quad (s = 2)
\]

\[
0 = 2sK_0^{(1)} \langle V^{s-1} \rangle_{\infty} + 2s(s-1)K_1^{(0)} \langle V^{s-2} \rangle_{\infty} - 4sK_0^{(0)} \langle V^s \rangle_{\infty} \quad (s \geq 3)
\]

(96)

which yields

\[
\langle V \rangle_{\infty}^{(0)} = \frac{K_0^{(1)}}{2K_0^{(0)}} = V_0 = \frac{1}{2\sqrt{M}} \sum_{i=1}^{2} \left( \frac{n_{[1]}^{(0)} \langle w^2 \rangle_{[1]}^{[1]} - n_{[2]}^{(0)} \langle w^2 \rangle_{[2]}^{[2]}}{n_{[1]}^{(0)} \langle w \rangle_{[1]}^{[1]} + n_{[2]}^{(0)} \langle w \rangle_{[2]}^{[2]}} \right)
\]

\[
= \sqrt{\frac{\pi k_B}{8M}} T_{[1]} T_{[2]} \lim_{\epsilon \to 0} \frac{p_{[1]}(\epsilon) - p_{[2]}(\epsilon)}{\epsilon p_{[1]}(\epsilon) \sqrt{T_{[2]}} + p_{[2]}(\epsilon) \sqrt{T_{[1]}}},
\]  

(97)
\[
\begin{align*}
\langle V^2 \rangle_{\infty}^{(0)} - \left( \langle V \rangle_{\infty}^{(0)} \right)^2 &= \frac{K_{10}^{(0)}}{2K_{-1}^{(0)}} = \Delta = \frac{1}{2M} \lim_{\epsilon \rightarrow 0} \frac{n_{[1]}(\epsilon) \langle w^3 \rangle_{1}^{[1]} + n_{[2]}(\epsilon) \langle w^3 \rangle_{2}^{[2]}}{n_{[1]}(\epsilon) \langle w \rangle_{1}^{[1]} + n_{[2]}(\epsilon) \langle w \rangle_{2}^{[2]}} \\
&= \frac{k_{B}}{M} \sqrt{T_{[1]}T_{[2]}} \lim_{\epsilon \rightarrow 0} \frac{p_{[1]}(\epsilon)\sqrt{T_{[1]}} + p_{[2]}(\epsilon)\sqrt{T_{[2]}}}{p_{[1]}(\epsilon)\sqrt{T_{[2]}} + p_{[2]}(\epsilon)\sqrt{T_{[1]}}}
\end{align*}
\]  

and

\[
\langle V^s \rangle_{\infty}^{(0)} = \sum_{k=0}^{[s/2]} \frac{s!}{2^k k!(s - 2k)!} V_0^{s-2k} s^k.
\]  

Therefore at order \( \epsilon^0 \), the distribution function for the velocity of the piston is

\[
\Phi^{(0)}(V) = \frac{1}{\sqrt{2\pi \Delta}} \exp \left[ -\frac{(V - V_0)^2}{2\Delta} \right].
\]

Let us remark that we recover Eqs. (94,97) at order zero.

At order \( \epsilon^1 \), introducing the notation

\[
\begin{align*}
k_{r}^{(i)} &= \frac{K_{r}^{(i)}}{2K_{-1}^{(0)}}; \quad k_{0}^{(0)} = 0; \quad k_{1}^{(0)} = V_0; \quad k_{1}^{(1)} = \Delta
\end{align*}
\]

Eq. (95) yields

\[
\begin{align*}
\langle V \rangle_{\infty}^{(1)} &= -2k_{-1}^{(1)}V_0 + k_{0}^{(2)} + k_{-2}^{(0)}(\Delta + V_0^2) \quad (102) \\
\langle V^2 \rangle_{\infty}^{(1)} &= V_0 \langle V \rangle_{\infty}^{(1)} - 2k_{-1}^{(1)}(\Delta + V_0^2) + k_{0}^{(2)}V_0 + k_{1}^{(1)} + k_{-2}^{(0)}V_0(3\Delta + V_0^2) \quad (103) \\
\langle V^s \rangle_{\infty}^{(1)} &= V_0 \langle V^{s-1} \rangle_{\infty}^{(1)} + (s - 1)\Delta \langle V^{s-2} \rangle_{\infty}^{(1)} - 2k_{-1}^{(1)} \langle V^{s-2} \rangle_{\infty}^{(0)} + k_{0}^{(2)} \langle V^{s-2} \rangle_{\infty}^{(0)} + k_{1}^{(1)} + k_{-2}^{(0)} \langle V^{s-1} \rangle_{\infty}^{(0)} + k_{1}^{(0)} \langle V^{s+1} \rangle_{\infty}^{(0)}. \quad (104)
\end{align*}
\]

As one can verify the distribution function at order \( \epsilon^1 \) is

\[
\Phi_c(V) = \frac{1}{\sqrt{2\pi \Delta}} \exp \left[ -\frac{1}{2\Delta} \left( V - \sqrt{\frac{\pi k_{B}}{2M} T_{[1]}T_{[2]}} \frac{p_{[1]}(\epsilon) - p_{[2]}(\epsilon)}{p_{[1]}(\epsilon)\sqrt{T_{[2]}} - p_{[2]}(\epsilon)\sqrt{T_{[1]}}} \right)^2 \right] \times \left\{ 1 + e^{-a_1V_0 + a_2(V^2 - \Delta - V_0^2) + a_3(V^3 - 3\Delta V_0 - V_0^3)} \right\}
\]

where \( V_0, \Delta \) are given by Eqs. (94,98) and

\[
\begin{align*}
a_1\Delta^3 &= k_{-1}^{(1)}V_0\Delta^2 - k_{1}^{(1)}V_0\Delta - \frac{2}{3}k_{1}^{(0)}(\Delta - V_0^2) \quad (106) \\
a_2\Delta^3 &= -k_{-1}^{(1)}\Delta^2 + \frac{1}{2}k_{1}^{(1)}\Delta - \frac{2}{3}k_{1}^{(0)}V_0 \quad (107) \\
a_4\Delta^3 &= \frac{3}{2}k_{1}^{(2)}\Delta^2 - \frac{2}{3}k_{1}^{(0)} \quad (108)
\end{align*}
\]

Let us remark that for \( p_{[1]} = p_{[2]} \) then \( V_0 = 0 \) and we recover the results of Gruber and Piasecki [8]. One can then successively obtain higher order terms for the velocity moments.

In conclusion, for the infinite system we observe a stationary state with non-zero average velocity \( \langle V \rangle_{\infty} \), which depends on \( n_{[1]}, n_{[2]}, T_{[1]}, T_{[2]} \), given at order \( \epsilon \) by Eqs. (94,92).

It is interesting to note that this result present some analogy and differences with those obtained by J.L. Lebowitz in [7], where he considered an adiabatic piston constrained around the origin by an external force: The coefficient \( \Delta \) Eq. (98) is exactly the same as in [7]; however for \( p_{[1]} - p_{[2]} = \mathcal{O}(\epsilon) \neq 0 \) we have a drift \( V_0 \), while in [7] the external force give \( \langle V^{2k+1} \rangle_{\infty}^{(0)} = 0 \) \( (k \geq 0) \).

To conclude the discussion on the stationary states of the infinite system, we investigate under what condition there exists a stationary state with zero average velocity i.e. \( \langle V \rangle_{\infty} = 0 \).
From Eq. \([2]\) it follows immediately that
\[
p_{[1]} - p_{[2]} = \langle n_{[1]} - n_{[2]} \rangle \langle V^2 \rangle_{\infty} + O(\epsilon^3).
\]  
(109)

However one has to check that a stationary solution of Eq. \((83)\) does exist with \(\langle V \rangle_{\infty} = 0\) at all order in \(\epsilon\).

From Eqs. \((84,102)\), \(\langle V \rangle_{\infty} = 0\) up to order \(\epsilon\) iff
\[
K_0^{(0)} = K_0^{(1)} = 0, \quad \text{i.e.} \quad \lim_{\epsilon \to 0} (p_{[1]}(\epsilon) - p_{[2]}(\epsilon))/\epsilon = 0
\]  
(110)

and
\[
K_0^{(2)} + K_{-2}^{(0)} \Delta = 0.
\]  
(111)

Expressed differently, \(\langle V \rangle_{\infty} = 0\) up to order \(\epsilon\) iff
\[
\begin{align*}
n_{[1]}^{(0)} T_{[1]} - n_{[2]}^{(0)} T_{[2]} &= 0 \\
n_{[1]}^{(1)} T_{[1]} - n_{[2]}^{(1)} T_{[2]} &= 0 \\
n_{[1]}^{(2)} T_{[1]} - n_{[2]}^{(2)} T_{[2]} &= -\frac{M}{k_B} \Delta \left[ n_{[1]}^{(0)} - n_{[2]}^{(0)} \right].
\end{align*}
\]  
(112)

This last equation is equivalent to
\[
p_{[1]} - p_{[2]} = -m \left[ n_{[1]}^{(0)} - n_{[2]}^{(0)} \right] \langle V^2 \rangle_{\infty}^{(0)}
\]  
(113)

and, in this case
\[
\langle V^2 \rangle_{\infty}^{(0)} = \Delta = \frac{k_B}{M} \sqrt{T_{[1]} T_{[2]}}
\]  
(114)

\[
\langle V^2 \rangle_{\infty}^{(1)} = k_1^{(1)} - 2k_{-1}^{(1)} \Delta.
\]  
(115)

At order \(\epsilon^2\), we have (taking \(s = 1\))
\[
\langle V \rangle_{\infty}^{(2)} = k_0^{(3)} - 2k_{-1}^{(1)} \langle V \rangle_{\infty}^{(1)} - 2k_{-2}^{(1)} V_0 + k_{-1}^{(1)} \langle V^2 \rangle_{\infty}^{(0)} + k_{-2}^{(0)} \langle V^2 \rangle_{\infty}^{(1)} - \frac{1}{3} k_{-3}^{(0)} \langle V^3 \rangle_{\infty}^{(0)}.
\]  
(116)

Therefore \(\langle V \rangle_{\infty} = 0\) up to order \(\epsilon^2\) iff
\[
K_0^{(3)} + K_{-2}^{(1)} \Delta + K_{-2}^{(0)} (k_1^{(1)} - 2k_{-1}^{(1)} \Delta) = 0
\]  
(117)

Finally at order \(\epsilon^q\), we have (taking \(s = 1\))
\[
0 = 2[K_0]^{(q+1)} - 4[K_{-1} \langle V \rangle_{\infty}]^{(q)} + 2[K_{-2} \langle V^2 \rangle_{\infty}]^{(q-1)} - \sum_{r \geq 3} \frac{4}{r^2} \Delta [K_{-r} \langle V_r \rangle_{\infty}]^{(q-r)}.
\]  
(118)

Therefore \(\langle V \rangle_{\infty} = 0\) up to order \(\epsilon^q\) iff all \(K^{(l)}\) are uniquely determined by the functions obtained at previous order.

In conclusion, under the assumption Eq. \((83)\) and given the density \(n_{[2]}\), and the distributions \(\phi_{[1]}(v)\), \(\phi_{[2]}(v)\), with \(T_{[1]} \neq T_{[2]}\), then the average velocity of the piston in the stationary state is non-zero for all densities \(n_{[1]}\) except for one special value \(n_{[1]} = n_{[1]}(\epsilon)\) for which \(\langle V \rangle_{\infty} = 0\). For this special solution, one has
\[
\begin{align*}
n_{[1]}^{(0)} \langle w^2 \rangle_{[1]} &= n_{[2]} \langle w^2 \rangle_{[2]} \\
n_{[1]}^{(1)} &= 0 \\
n_{[1]}^{(2)} \langle w^2 \rangle_{[1]} &= -[n_{[1]}^{(0)} \langle w^0 \rangle_{[1]} - n_{[2]} \langle w^0 \rangle_{[2]}] \Delta \\
\frac{1}{M} n_{[1]}^{(q)} \langle w^2 \rangle_{[1]} &= -[K_{-2} \langle V^2 \rangle_{\infty}]^{(q-2)} + \sum_{r \geq 3} \frac{2}{r^2} [K_{-r} \langle V_r \rangle_{\infty}]^{(q-r)} \quad (q \geq 3).
\end{align*}
\]  
(119-122)
Therefore at order $\epsilon$, i.e. taking into account (98), Eq.(124) implies at order $\epsilon$

\[
P^2_{Q} = -\frac{1}{\sqrt{M}} 2e \epsilon(\epsilon^2/2) K^2_{-1}(\infty) \langle V^2 \rangle_{\infty} + \epsilon^2 K^2_{-2}(\infty) \langle V^3 \rangle_{\infty} - \sum_{r \geq 4} \epsilon^{r-1} \frac{2}{r!} (r+\epsilon^2r-3) K^r_{-r+1}(\infty) \langle V^r \rangle_{\infty}
\]

(124)

and thus there is a constant heat flux for any distribution $\phi_1(v), \phi_2(v)$ such that $\langle w^3 \rangle_1 \langle w \rangle_2 \neq \langle w^3 \rangle_2 \langle w \rangle_1$. For Maxwellian distribution, this means that there is a constant flux for any $T_{[1]} \neq T_{[2]}$.
Therefore there exists a special non-equilibrium stationary state with zero average velocity of the piston, which is however distinct from the one given in $\text{[1]}$ since in our case $\langle V^{2k+1} \rangle_{\infty}$ is not zero (for $k \geq 1$).

C. Equilibrium state of the finite system

Any equilibrium state of the finite system is necessarily a stationary state with $\langle V \rangle_{\infty} = 0$ and $P^2_{Q} = 0$ where, for $\langle V \rangle_{\infty} = 0$ (see Eq.(10)),

\[
P^2_{Q} = -\frac{2M}{(1+\epsilon^2)^2} \left[ \epsilon K^1_{1}(\infty) - 2\epsilon(1 - \epsilon^2/2) K^1_{-1}(\infty) \langle V^2 \rangle_{\infty} + \epsilon^2 K^2_{-2}(\infty) \langle V^3 \rangle_{\infty} - \sum_{r \geq 4} \epsilon^{r-1} \frac{2}{r!} (r+\epsilon^2r-3) K^r_{-r+1}(\infty) \langle V^r \rangle_{\infty} \right].
\]

(125)

(126)

\[
\text{Assuming Maxwellian distribution for the fluid particles, it implies } T_{[1]}(\infty) = T_{[2]}(\infty) \text{ at order } \epsilon \text{ and in this case we have}
\]

\[
\phi_1(v) = \phi_2(-v).
\]

We conjecture that for any equilibrium state the relation (124) must be satisfied, which then implies $T_{[1]}(\infty) = T_{[2]}(\infty)$.
Assuming this conjecture (126) to hold, Eq.(110) yields first

\[
K^{(i)}_0(\infty) = 0 \text{ for } l = 0, 1
\]

(127)

i.e. $n^{(i)}_{[1]}(\infty) = n^{(i)}_{[2]}(\infty)$ for $l = 0, 1$

(128)

Then, with Eq.(112), Eq.(128) is also valid for $l = 3$, thus

\[
K^{(i)}_r(\infty) = 0 \text{ for all } r \text{ even}, \quad l = 0, 1, 2
\]

(129)

and with (105)

\[
\langle V^{2k+1} \rangle_{\infty}^{(1)} = 0 \text{ for all } k \geq 0.
\]

(130)

Iterating the argument at successive order we arrive at the following conclusion. Under the assumption (88) and the conjecture (126), any equilibrium state of the finite system must satisfy $n_{[1]}(\infty) = n_{[2]}(\infty)$, in other words $p_{[1]}(\infty) = p_{[2]}(\infty)$ and $T_{[1]}(\infty) = T_{[2]}(\infty)$.

Conversely, if $\phi_{[1]}(v) = \phi_{[2]}(-v)$, thus $T_{[1]} = T_{[2]}$, and $n_{[1]}(\infty) = n_{[2]}(\infty)$ then the solution of the stationary equation at order $\epsilon$ is

\[
\langle V^{2k+1} \rangle_{\infty} = 0
\]

\[
\langle V^{2k} \rangle_{\infty} = (2k-1)!\Delta^k, \quad \Delta = \frac{K_{1}(\infty)}{2K_{-1}(\infty)}.
\]

(131)

Therefore at order $\epsilon$ the velocity distribution of the piston is necessarily Maxwellian with temperature $T = M\Delta/k_B$, i.e.
\[ \Phi_{r}(V) = \sqrt{\frac{1}{2\pi\Delta}} \exp \left[ \frac{V^2}{2\Delta} \right]. \]  

(132)

In conclusion the adiabatic piston is stricto senso a conductor since the only equilibrium states are those for which \( T_{1[\infty]} = T_{2[\infty]} \). However the important question to be considered in the next section concerns the time scale necessary to reach this true equilibrium state.

VII. TIME EVOLUTION

In this section we give a qualitative (non rigorous) discussion of the time evolution for the infinite and the finite systems.

A. Infinite systems

As discussed in Sec. I, for infinite systems and \( \epsilon \ll 1 \) the assumptions \( n_{i[1]}(t) = n_{i[2]}, \phi_{i}(v, t) = \phi_{T[i]}(v), i = 1, 2 \) are justified. In this case the functions \( K_{r} \) are constant, and we are led to solve the equation (33)

\[ \frac{d}{dt} V = \epsilon [C + AV], \quad V(t = 0) = 0, \]  

(133)

where

\[ V = \{ \langle V^{s} \rangle_{t} ; s \geq 1 \} \]
\[ C_{s} = \frac{1}{\epsilon} A_{s,0} = 2s\epsilon^{s-1}M^{-\frac{1}{2}} \left[ n_{[1]} \langle w^{s+1} \rangle_{1} + n_{[2]} \langle w^{s+1} \rangle_{2} \right] \]
\[ A_{s,r} = \frac{\epsilon^{[s-r-1]} 1}{(1+\epsilon^{2})^{s-\epsilon}} P_{s,r}(\epsilon^{2}) K_{s-r-1} \]  

(134)

i.e. with \( t' = \epsilon t \)

\[ \frac{d}{dt'} V = C + AV. \]  

(135)

In the limit \( \epsilon \to 0, \) with the condition that

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} K_{0} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [p_{[1]}(\epsilon) - p_{[2]}(\epsilon)] \]  

(136)

exists and is finite, the matrix \( A_{s,r} \) is non-zero only for \( r = s - 2, s - 1, s, \) and its eigenvalues \( \alpha_{n} \) are all negative given by

\[ \alpha_{s} = -s4K_{-1} = -s \frac{\lambda}{\epsilon M}, \quad s \geq 1 \]  

(137)

where the friction coefficient \( \lambda = \lambda(V = 0), \) given by (122), is of order \( \epsilon. \)

We thus obtain at lowest order in \( \epsilon \)

\[ \langle V \rangle_{t}^{(0)} = V_{0} \left( 1 - e^{-\frac{\lambda}{\epsilon M} t} \right) \]
\[ \langle V^{2} \rangle_{t}^{(0)} = V_{0}^{2} \left( 1 - e^{-\frac{\lambda}{\epsilon M} t} \right)^{2} + \Delta \left( 1 - e^{-2\frac{\lambda}{\epsilon M} t} \right) \]  

(138)

with \( V_{0} \) and \( \Delta \) given by Eqs. (97, 98) or (97, 98).

At order \( \epsilon^{2} \) we find for the case \( p_{[1]} = p_{[2]} \)

\[ \langle V \rangle_{t} = \langle V \rangle_{\infty} \left( 1 - e^{-\frac{\lambda}{M(1+\epsilon^{2})} t} \right)^{2} \]  

(139)
\[ \langle V^{2} \rangle_{t} = \langle V^{2} \rangle_{\infty} \left[ 1 - e^{-2\frac{\lambda}{M(1+\epsilon^{2})} t} + 2\epsilon^{2} \left( e^{-\frac{\lambda}{M(1+\epsilon^{2})} t} - e^{-2\frac{\lambda}{M(1+\epsilon^{2})} t} \right) \right] \]  

(140)
with \( \langle V \rangle_\infty \) and \( \langle V^2 \rangle_\infty \) given by Eqs.(86,87) with \( p[1] = p[2] \).

Similarly for \( p[1] \neq p[2] \) we will have another linear combination of \( \exp(-t/\tau_a) \), \( \exp(-2t/\tau_a) \) with

\[
\tau_a = \frac{M + m}{\lambda}.
\]

We notice that taking realistic numbers one finds \( \tau_a \) of the order \( 10^{-1} - 10^{-2} \) which means that the stationary state is reached very rapidly.

### B. Finite systems

For finite systems the equations for the time evolution of \( \langle V \rangle_\infty \) and \( \langle V^2 \rangle_\infty \) are given at order \( \epsilon^2 \) by Eqs.(70,71)

where the densities \( n[i] \) and the coefficients \( K_r[i] \) are now functions of time. Since we do not consider the full many-body problem describing the fluids, we take care of this unknown time dependence by assuming that the fluids satisfy the homogeneity condition (77). Moreover for \( K_r[i] \), \( r = -2, -1, 1 \), we take the expressions (72) obtained with the Maxwellian distributions. From Eq.(73) we obtain at order \( \epsilon \)

\[
\begin{align*}
\frac{d}{dt} T[1] &= -2T[1] \left[ \langle V \rangle_t - \sqrt{m} \sqrt{8\pi k_B T[1]} \langle V^2 \rangle_t + \frac{\sqrt{m}}{M} \sqrt{8k_B T[1]} \right], \\
\frac{d}{dt} T[2] &= -2T[2] \left[ -\langle V \rangle_t - \sqrt{m} \sqrt{8\pi k_B T[2]} \langle V^2 \rangle_t + \frac{\sqrt{m}}{M} \sqrt{8k_B T[2]} \right].
\end{align*}
\]

At order \( \epsilon \) we thus have a system of five O.D.E for the unknown \( \langle X \rangle_t, \langle V \rangle_t, \langle V^2 \rangle_t, T[1] \) and \( T[2] \). At this order we conclude that the system will evolve toward the unique equilibrium state

\[
\begin{align*}
T[1](\infty) &= T[2](\infty) = \frac{N[1]T[1] + N[2]T[2]}{N[1] + N[2]} \quad (144) \\
\langle X \rangle_\infty &= \frac{L N[1]}{N[1] + N[2]} \quad (145) \\
\langle V \rangle_\infty &= 0 \quad (146) \\
\langle V^2 \rangle_\infty &= \frac{k_B N[1]T[1] + N[2]T[2]}{M N[1] + N[2]} \quad (147)
\end{align*}
\]

which is identical to the one obtained in thermodynamics for the conducting piston (\( \kappa \neq 0 \)).

Let us then analyze qualitatively how the evolution toward this final equilibrium state takes place.

Given that at \( t = 0 \), we have \( X = X_0, V = 0, T[1](0) = T[1], T[2](0) = T[2] \), and \( p[1](0) \neq p[2](0) \), then as long as the velocity \( \langle V \rangle_t \) remains small Eqs.(142-144) yield the adiabats

\[
\begin{align*}
T[1](t) \langle X \rangle_t^2 &= T[1]X_0^2 \quad (148) \\
T[2](t) (L - \langle X \rangle_t)^2 &= T[2] (L - X_0)^2 \quad (149)
\end{align*}
\]

and the time evolution for the piston is

\[
\frac{d}{dt} \langle V \rangle_t = \frac{k_B}{M} \left[ N[1]T[1] \langle X_0^2 \rangle_t \langle X \rangle_t^2 - N[2]T[2] \langle L - X_0 \rangle^2 \langle L - \langle X \rangle_t \rangle^2 \right] - \frac{\lambda(t)}{M} \langle V \rangle_t \quad (150)
\]

with \( \lambda(t) \) given by Eq.(72), together with Eqs.(148,149).

The piston evolves thus adiabatically until a time \( t_a \), of the order \( \tau_a \) Eq.(141), where the pressures become equal, i.e.

\[
\frac{N[1]T[1](t_a)}{\langle X \rangle_{t_a}} = \frac{N[2]T[2](t_a)}{L - \langle X \rangle_{t_a}} \quad (151)
\]

which yields
\[
\langle X \rangle_{t_a} = \frac{L}{1 + \left( \frac{N_{1}[T_{1]}(t_a)}{N_{1}[T_{1}]} \right)^{1/3}} \tag{152}
\]

\[
T_{[1]}(t_a) = T_{[1]} \left( \frac{X_{0}}{\langle X \rangle_{t_a}} \right)^2 \tag{153}
\]

\[
T_{[2]}(t_a) = T_{[2]} \left( \frac{L - X_{0}}{L - \langle X \rangle_{t_a}} \right)^2 \tag{154}
\]

\[
p(t_a) = k_{B} N_{[1]} T_{[1]} \frac{X_{0}^2}{\langle X \rangle_{t_a}^3}. \tag{155}
\]

In the next stage, as can be seen from the numerical simulations of Sec. VIII, the pressures will remain approximately constant and equal, \( i.e. \)

\[
p_{[1]}(t) = k_{B} \frac{N_{[1]} T_{[1]}(t)}{\langle X \rangle t} \simeq \text{cte} \tag{156}
\]

\[
p_{[2]}(t) = k_{B} \frac{N_{[2]} T_{[2]}(t)}{L - \langle X \rangle t} \simeq \text{cte} \tag{157}
\]

and

\[
N_{[1]} T_{[1]}(t) [L - \langle X \rangle t] = N_{[2]} T_{[2]}(t) \langle X \rangle t \tag{158}
\]

From Eqs. (156–157) we have

\[
\frac{1}{T_{[1]}} \frac{dT_{[1]}}{dt} = \frac{\langle V \rangle t}{\langle X \rangle t} \tag{159}
\]

\[
\frac{1}{T_{[2]}} \frac{dT_{[2]}}{dt} = \frac{\langle V \rangle t}{L - \langle X \rangle t} \tag{160}
\]

and using Eqs. (142–143) we conclude that

\[
M \langle V^2 \rangle_t = k_{B} \sqrt{T_{[1]}(t) T_{[2]}(t)} \tag{161}
\]

\[
\langle V \rangle_t = \frac{2}{3} \sqrt{m} \frac{8 k_{B}}{\pi} \left[ \sqrt{T_{[2]}(t)} - \sqrt{T_{[1]}(t)} \right]. \tag{162}
\]

In other words, the temperatures \( T_{[1]}(t) \) and \( T_{[2]}(t) \) evolves, but at all time \( t \) we have

\[
\langle V^2 \rangle_t = \langle V^2 \rangle_{\infty} \tag{163}
\]

\[
\langle V \rangle_t = \frac{16}{3 \pi} \langle V \rangle_{\infty} \tag{164}
\]

where \( \langle V \rangle_{\infty} \) and \( \langle V^2 \rangle_{\infty} \) are the stationary values for the infinite system with temperatures \( T_{[i]} = T_{[i]}(t), i = 1, 2. \)

Introducing the expressions (161) and (162) in (142), (143) yields, with \( p_{[1]} = p_{[2]} = p \)

\[
\frac{d}{dt} T_{[1]} = -\frac{2}{3} \frac{p}{N_{[1]} k_{B}} \sqrt{m} \frac{8 k_{B}}{\pi} \left[ \sqrt{T_{[1]}} - \sqrt{T_{[2]}} \right] \tag{165}
\]

\[
\frac{d}{dt} T_{[2]} = \frac{2}{3} \frac{p}{N_{[2]} k_{B}} \sqrt{m} \frac{8 k_{B}}{\pi} \left[ \sqrt{T_{[1]}} - \sqrt{T_{[2]}} \right] \tag{166}
\]

and thus

\[
\frac{d}{dt} (T_{[2]} - T_{[1]}) = -\frac{2}{3} \left( \frac{1}{N_{[1]} k_{B}} + \frac{1}{N_{[2]} k_{B}} \right) \sqrt{m} \frac{8 k_{B}}{\pi} \frac{n_{[1]} k_{B} T_{[1]}^2}{\sqrt{T_{[1]}} + \sqrt{T_{[2]}}} (T_{[2]} - T_{[1]}). \tag{167}
\]
Let us note that for \( p_1 = p_2 \), it follows from Eqs. (162) that

\[
\kappa = \frac{\sqrt{m}}{M} \sqrt{\frac{8kB}{\pi} \frac{n_{[1]} k_B T_{[1]}}{T_{[2]}}} \]

(168)

\[
\lambda_{[1]} + \lambda_{[2]} = \sqrt{m} \sqrt{\frac{8kB}{\pi} n_{[1]} T_{[1]} \sqrt{T_{[1]} + T_{[2]}}} \]

(169)

\[
i.e. \quad \kappa = \frac{\lambda}{M} k_B (\theta + \theta^{-1})^2, \quad \theta = \left( \frac{T_{[1]}}{T_{[2]}} \right)^{1/4}.
\]

(170)

We thus have

\[
\frac{d}{dt}(T_{[2]} - T_{[1]}) = -\frac{2}{3} \left( \frac{1}{N_{[1]} k_B} + \frac{1}{N_{[2]} k_B} \right) \kappa (T_{[2]} - T_{[1]})
\]

(171)

from which follows that the system will reach equilibrium with a relaxation time \( \tau_e \) of the order

\[
\tau_e \simeq \kappa^{-1} \simeq \frac{1}{kB} \frac{M}{\lambda} \simeq \frac{1}{kB} \tau_a.
\]

(172)

Taking realistic numbers we obtain a relaxation time which is several time the age of the universe.

The coefficient \( 2 \left( \frac{1}{N_{[1]} k_B} + \frac{1}{N_{[2]} k_B} \right) \) is clearly the inverse of the specific heat. To understand the origin of the factor \( 1/3 \) we must remember that the piston is moving. Using Eqs. (162, 165) yields

\[
P_Q^{[2] \rightarrow [1]} = \frac{1}{2} N_{[1]} k_B \frac{dT_{[1]}}{dt} + p_{[1]} \langle V \rangle_t = \kappa (T_{[2]} - T_{[1]}) = -P_Q^{[1] \rightarrow [2]}
\]

(173)

as it should.

Let us also note that the equality of pressure \( (158) \), together with the conservation of energy

\[
N_{[1]} T_{[1]}(t) + N_{[2]} T_{[2]}(t) + \frac{M}{k_B} \langle V^2 \rangle_t = N_{[1]} T_{[1]} + N_{[2]} T_{[2]},
\]

(174)

which is exactly satisfied by Eqs. (70, 71, 142, 143) if \( p_{[1]} = p_{[2]} \), gives the relation between position and temperatures \( (156, 157) \)

\[
LN_{[1]} T_{[1]}(t) = \left( N_{[1]} T_{[1]} + N_{[2]} T_{[2]} - \frac{M}{k_B} \langle V^2 \rangle_t \right) \langle X \rangle_t
\]

\[
\simeq \left( N_{[1]} T_{[1]} + N_{[2]} T_{[2]} \right) \langle X \rangle_t
\]

(175)

\[
LN_{[2]} T_{[2]}(t) \simeq \left( N_{[1]} T_{[1]} + N_{[2]} T_{[2]} \right) \left( L - \langle X \rangle_t \right)
\]

(176)

and thus from Eq. (162), or Eq. (163),

\[
\frac{d}{dt} \langle X \rangle_t = 2 \frac{\sqrt{m}}{3 M} \sqrt{\frac{8kB}{\pi}} \left( \frac{k_B \left( N_{[1]} T_{[1]} + N_{[2]} T_{[2]} \right)}{L} \right) \left( \sqrt{\frac{L - \langle X \rangle_t}{N_{[2]}}} - \sqrt{\frac{\langle X \rangle_t}{N_{[1]}}} \right).
\]

(177)

Finally if we compare \( \frac{d}{dt} \langle V \rangle_t \) obtained from Eq. (177) with Eq. (71), we arrive at the conclusion that

\[
p_{[2]} - p_{[1]} = \mathcal{O} \left( e^2 (T_{[2]} - T_{[1]}) \right)
\]

(178)

which justifies our starting point for the second stage.

Let us also remark that for some time interval between the adiabatic evolution Eq. (154), and the approach toward equilibrium with \( p_{[1]} \simeq p_{[2]} \) \( (174) \), there will be an intermediate stage for which \( P_Q^{[1] \rightarrow [2]} + P_Q^{[2] \rightarrow [1]} \) is not zero. During this intermediate evolution the stochastic motion of the piston will have to be introduced in the thermodynamical equations not only with a conductivity coefficient \( \kappa \neq 0 \), but also by means of an internal energy \( U_p \) and and entropy \( S_p \) of the piston with \( U_p = U_p(S_p) \).
VIII. NUMERICAL SIMULATIONS

In order to verify the assumptions on which our previous analysis is based, we made numerical simulations for the finite as well as for the infinite system in one dimension taking $k_B = 1$. The initial state of the fluids particles of mass $m$ is given by Maxwellian distributions of velocities according to Eq. (1). Then the particles and the piston (a particle of mass $M$ with initial coordinate $(X_0, V_0)$) interact only through elastic collisions.

For the infinite system, we simulate the openness of the system with sources of in-going particles very far from the piston position. We compute the average time dependent position $\langle X \rangle_t$ of the piston on $10^3 - 10^4$ different samples. An example of the time evolution of the piston average position is shown on Fig. 2 for the following parameter $m = 1$, $T_1 = 1$, $T_2 = 10$, $n_1 = 1$, $n_2 = 1/10$, i.e. equal pressure on both sides of the piston, and $M = 2, 5, 10, 20, 50, 100$ (from top to bottom). As expected from Eq.(139) we observe that the average position quickly behaves as $\langle X \rangle_t = \langle V \rangle_\infty t$ ($t \gtrsim 10^2$). The relaxation time $\tau_a$ necessary to reach this stationary behavior is too short to be represented on Fig. 2, and is presented on Fig. 3 for $M = 5, 10, 50, 100$.

Remark that $\tau_a$ depends on the ratio $m/M$.

Even if the pressures are equal on both sides of the piston, i.e. $n_1 T_1 = n_2 T_2$, the stationary velocity $\langle V \rangle_\infty$ is non zero and is presented as a function of $M$ on Fig. 4 for the above parameters, as well as for $T_2 = 20$ and $n_2 = 1/20$. The lines on Fig. 4 show the average stationary velocity computed in Sec. VII B up to the fourth order in $\epsilon = \sqrt{m/M}$

$$\langle V \rangle_\infty = \epsilon \langle V \rangle_\infty^{(1)} + \epsilon^3 \langle V \rangle_\infty^{(3)} + \mathcal{O}(\epsilon^5)$$

with (see Eq.(102))

$$\langle V \rangle_\infty^{(1)} = \frac{\sqrt{2\pi}}{4} \frac{1}{\sqrt{M}} \left( \sqrt{T_2} - \sqrt{T_1} \right)$$

and

$$\langle V \rangle_\infty^{(3)} = \frac{\sqrt{2\pi}}{96} \frac{1}{\sqrt{M}} \left[ (16 - 3\pi) \left( \frac{\sqrt{T_2} - \sqrt{T_1}}{\sqrt{T_1} T_2} \right)^3 - 6 \left( \sqrt{T_2} - \sqrt{T_1} \right) \right].$$

One sees that the values obtained at the fourth order agree very well with the numerical results.
FIG. 3. The average position of the piston as a function of time for the infinite system. Magnification of the dotted box on Fig. 2. The parameters are as in Fig. 2 with $M = 5, 10, 50, 100$.

FIG. 4. The stationary average velocity reached by the piston for the infinite system as a function of its mass $M$. The parameters of the simulation are $m = 1, T_{[1]} = 1, n_{[1]} = 1$ while $T_{[2]} = 10, n_{[2]} = 1/10$ (circles) and $T_{[2]} = 20, n_{[2]} = 1/20$ (squares), respectively. The solid, resp. dashed, line shows the theoretical value obtained in the text up to fourth order in $\epsilon$. 

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On Fig. 6 and Fig. 7, we plotted respectively the evolution of the temperature of $\theta$ system which is of the order $O(1)$. The time scale of our simulations has to be compared with the typical time it takes for a given particle to cross the equilibrium solution ($X = 3\times 10^3$) while the dotted lines show the thermodynamical prediction ($X = 84260.5$) for the adiabatic piston ($\kappa = 0$).

Concerning the finite system, we present here a qualitative study of the piston properties as a function of the parameter $\epsilon$. We consider a finite system of size $L = 6 \times 10^5$ with $N[1] = N[2] = 10^5$ and with $X_0 = 10^5$ (thus we have $\nu[1] = 1$, $\nu[2] = 1/5$). The initial temperatures on the left and on the right of the piston are $T[1] = 1$ and $T[2] = 10$. The initial pressure difference is thus $p[1] - p[2] = k_B(n[1]T[1] - n[2]T[2]) < 0$, pushing the piston toward the left side. The time scale of our simulations has to be compared with the typical time it takes for a given particle to cross the system which is of the order $O(10^5)$. We plot on Fig. 5 the time behavior of the piston’s position for different values of $\epsilon = 2^{-1/2}, 10^{-n/2}$ ($n = 1, 2, 3, 4$).

On Fig. 6 and Fig. 7, we plotted respectively the evolution of the temperatures $T[i](t)$, ($i = 1, 2$) and of the pressures for the same parameters. Notice that the temperatures are estimated through the second moment of the fluid particles velocity for all the particles present on the same side of the piston. It is equivalent to the temperatures on the face of the piston only if we suppose that the system is homogeneous. The dashed lines show the equilibrium solution predicted in Sec. VIB where $n[i](\infty) = n[2](\infty) = (N[1] + N[2])/L$, $X[\infty] = LN[1]/(N[1] + N[2])$, $T[1](\infty) = T[2](\infty) = (T[1] + T[2])/2$, $p[1](\infty) = p[2](\infty) = k_B(N[1] + N[2])/(T[1] + T[2])/2L$. On the other hand, the dotted lines show the equilibrium values obtained from a numerical integration of the phenomenological thermodynamical equations [152, 153] with $\kappa = 0$ for the primitive model of an adiabatic piston and where we used $\lambda[i](V) \simeq \lambda[i](V = 0) = n[i](t)\sqrt{8k_BT[i]/m/\pi}$. In this case the system evolves toward a metastable state with $p[1](\infty) = p[2](\infty)$ but $T[1](\infty) \neq T[2](\infty)$. One should note that on the scale of Figs. 6 and 7, one can hardly distinguish the results obtained by numerical integration ($X \simeq 84260.5$, $T[1] \simeq 1.545$, $T[2] \simeq 9.455$ and $p[1] = p[2] \simeq 1.83$) from those obtained by heuristic arguments Eqs. [152, 153], $X \simeq 82200$, $T[1] \simeq 1.48$, $T[2] \simeq 9.32$ and $p[1] = p[2] \simeq 1.80$.

Let us first comment on the Fig. 5 where one sees that the pressures quickly become constant and equal to their final values. The time scale $t_\alpha$ needed to equalize the pressures is roughly independent of the mass ratio $m/M$. This behavior has been used in Sec. VIII B to give a qualitative discussion of the evolution. Consider now the Fig. 6 where one sees that the temperatures reach quickly an intermediate stage characterized by the equality of the pressures (time scale $t_\alpha$) then evolve slowly toward their equilibrium values on a time scale $\tau_\alpha$. The time scale $\tau_\alpha$ depends strongly on the mass ratio $\epsilon^2 = m/M$ and for $\epsilon \ll 1$ we have $\tau_\alpha \gg t_\alpha$. One remarks that, for times covered in our simulations and for $\epsilon^2 \simeq 10^{-4}$ the system behaves as if the piston was “truly” adiabatic ($\kappa = 0$).

We plotted on Fig. 6 and Fig. 7 the velocity distribution for the particles of both the fluids. It is averaged on all particles located on the same side of the piston. The initial conditions are $T[i] = 1$, $T[2] = 10$, $p[1] = p[2] = 1$, $X_0 = V_0 = 0$, $m = 1$ and $M = 10$. The dotted lines show the Maxwellian initial distribution $\phi T[i](v)$. We present these distributions for three different times $t = t_1, t_2$. 
FIG. 6. The temperature of the fluids on both sides of the piston as a function of time. The parameters of the simulation are $m = 1$, $T_{[1]} = 1$, $n_{[1]} = 1$ while $T_{[2]} = 10$, $n_{[2]} = 1/5$. $T_{[1]}(t)$, resp. $T_{[2]}(t)$, is shown for for $M = 2, 10, 10^2, 10^3, 10^4$ (from middle to bottom, resp. to top). The dashed line stands for the equilibrium solution ($T_{[1]}(\infty) = T_{[2]}(\infty) = (T_{[1]} + T_{[2]})/2 = 5.5$) while the dotted lines show the thermodynamical prediction ($T_{[1]}(\infty) \sim 1.545$, $T_{[2]}(\infty) \sim 9.455$) for $\kappa = 0$.

FIG. 7. The pressures of the fluids on both sides of the piston as a function of time. The parameters of the simulation are $m = 1$, $T_{[1]} = 1$, $n_{[1]} = 1$, $p_{[1]} = 1$ while $T_{[2]} = 10$, $n_{[2]} = 1/5$, $p_{[2]} = 2$. $p_{[1]}(t)$, resp. $p_{[2]}(t)$, is shown for for $M = 10, 10^2, 10^3, 10^4$. The dashed line stands for the equilibrium solution which is equal to the thermodynamical prediction ($p_{[1]}(\infty) = p_{[2]}(\infty) = N_{[1]}(T_{[1]} + T_{[2]})/L = 11/6$).
FIG. 8. The particles velocity distribution $\phi_{[1]}(v,t)$ of the fluid [1] at different time for the finite system. The initial conditions are $T_{[1]} = 1$, $T_{[2]} = 10$, $p_{[1]} = p_{[2]} = 1$, $X_0 = V_0 = 0$, $m = 1$ and $M = 10$. The dotted line show the initial distribution $\phi_{T_{[1]}}(v)$ while the dashed line show the expected equilibrium distribution $\phi_{T_{[1]}(\infty)}(v)$ with $T_{[1]}(\infty) = (T_{[1]} + T_{[2]})/2$. The squares show the velocity distribution for $t_1 = 2 \times 10^5$ where the temperature of the fluid is $T_{[1]}(t_1) \simeq 1.6$. The solid line is for $t_2 = 2.5 \times 10^6$ with $T_{[1]}(t_2) \simeq 3.7$ while the circles are for $t_3 = 7.5 \times 10^7$ where $T_{[1]}(t_3) \simeq T_{[1]}(\infty)$.

FIG. 9. The particles velocity distribution $\phi_{[2]}(v,t)$ of the fluid [2] at different time for the finite system. The initial conditions are $T_{[1]} = 1$, $T_{[2]} = 10$, $p_{[1]} = p_{[2]} = 1$, $X_0 = V_0 = 0$, $m = 1$ and $M = 10$. The dotted line show the initial distribution $\phi_{T_{[2]}}(v)$ while the dashed line show the expected equilibrium distribution $\phi_{T_{[2]}(\infty)}(v)$ with $T_{[2]}(\infty) = (T_{[1]} + T_{[2]})/2$. The squares show the velocity distribution for $t_1 = 2 \times 10^5$ where the temperature of the fluid is $T_{[2]}(t_1) \simeq 9.4$. The solid line is for $t_2 = 2.5 \times 10^6$ with $T_{[2]}(t_2) \simeq 7.3$ while the circles are for $t_3 = 7.5 \times 10^7$ where $T_{[2]}(t_3) \simeq T_{[2]}(\infty)$. 
For the time \( t_1 = 2 \times 10^5 \) the system has not reached the relaxation time \( t_a \) and the pressures are not yet constant. At the time \( t_2 \) (\( t_a < t_2 < \tau_e \)), the system has constant pressures but has not reached the equilibrium state. At time \( t_3 > \tau_e \) the system has reached its final equilibrium state. We present on Fig. 10 the time dependence of the fluid temperatures. Vertical lines stand for times \( t_1, t_2, t_3 \). We see that, for times such as \( t < \tau_e \), these distributions are not Maxwellian but develops peaks or wells for small velocities. Moreover these intermediate time distribution are clearly not symmetric. This facts are the reason why we did not suppose Maxwellian distributions for the fluid particles in our previous analytical study. After a sufficiently long time of the order of the equilibrium relaxation time of the system \( \tau_e \), the distributions are very well fitted with the Maxwellian distribution and we conjecture that \( \phi[1](v, \infty) = \phi[2](v, \infty) = \phi[T[1](\infty)](v) \) with \( T[1](\infty) = (T[1] + T[2])/2 = 11/2 \). The dashed lines on Fig. 8 and Fig. 9 show the conjectured equilibrium distribution. This last conjecture is valid only for \( m < M \) as it has been shown that the equilibrium distribution in the case of \( m = M \) is a simple superposition of the two initial Maxwellian [9].

IX. CONCLUSIONS

Several conclusions can be drawn from this investigation on the stochastic motion of a piston which is adiabatic when rigidly fixed.

For infinite systems, it has been shown that very rapidly a stationary state is reached, where the average velocity of the piston is a function of the pressures and temperatures of the fluids on both sides. In particular if the pressures are equal but the temperatures different the final average velocity is non-zero and the piston has a macroscopic motion toward the high temperature region. This result is related to the asymmetry in the fluctuations of the piston, due to the asymmetry of the temperatures on both sides, which in turn induces both a macroscopic force and a heat current from one side to the other. In other words in the stationary state the stochastic piston is a conductor. To obtain a stationary state with zero average velocity it is necessary to compensate the stochastic force by a macroscopic difference in the pressures of the fluids. In this very special state no work is done by the piston on the fluids, but heat is transferred from one side to the other. Moreover this state is not a state of mechanical equilibrium since all the odd moments of the piston velocity, except the first one, are non zero.

For finite systems it has been shown that the system will always evolve toward the equilibrium state predicted for a conducting piston where pressures, temperatures and densities are equal on both sides of the piston, but the time needed to reach this final equilibrium state will depend strongly on the mass ratio \( m/M \) and can reach several time the age of the universe for reasonable numbers. This evolution takes place in two stages. In the first stage the system
reaches rather rapidly (although not as fast as the time needed to reach the stationary state of the infinite system) a state of “metastable equilibrium” where the pressures are equal but the temperatures different. This initial evolution corresponds to an adiabatic evolution (no heat transfer) and the results obtained from the microscopical theory, from the thermodynamic equations, and from the numerical simulations are in good agreement. In the second stage the piston and the temperatures of the fluids evolve in such a way that the pressures remain approximatively constant and equal. If \( m \ll M \) it appears that at all time the average velocity of the piston coincide with the velocity in the stationary state of the infinite system with values of temperatures and pressures given by those of the finite system at that time. This observation can be understood from the fact that the stationary state of the infinite system is reached very rapidly. If \( m \ll M \) the time needed to reach the final state is so large that on the numerical simulations the “metastable equilibrium” state appear to be stable and the piston behaves as an adiabatic piston on the time scale considered. On the other hand for \( m \simeq M \) the final equilibrium state is reached rather rapidly. Moreover it was observed that during the time evolution the velocity distribution of the fluid particles is not Maxwellian and that this distribution evolves slowly toward the equilibrium Maxwellian distribution characterized by the equilibrium temperature.

Finally the conclusions obtained from thermodynamics, from kinetic theory, and from numerical simulations, are all in good agreements. There appears to be no violation of the second law and no paradox involved.

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