Problems on One Way Road Networks

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Abstract

Let $OWRN = (W_x, W_y)$ be a One Way Road Network where $W_x$ and $W_y$ are the sets of directed horizontal and vertical roads respectively. $OWRN$ can be considered as a variation of directed grid graph. The intersection of the horizontal and vertical roads are the vertices of $OWRN$ and any two consecutive vertices on a road is connected by an edge.

In this work, we analyze the problem of collision free Traffic configuration in an $OWRN$. A traffic configuration is a two-dimensional set of vehicles at a time instant. A traffic configuration is a two-tuple $TC = (OWRN, C)$, where $C$ is a set of cars travelling on a pre-defined path. We prove that finding a maximum cardinality subset $C_{sub} \subseteq C$ such that $TC = (OWRN, C_{sub})$ is collision-free, is NP-hard.

Lastly we investigate the properties of connectedness, shortest paths in an $OWRN$.

1 Introduction

The rapid development in the existing motor vehicle technology has led to the increase in demand of automated vehicles, which are in themselves capable of various decision activities such as motion-controlling, path planning etc. This has motivated many to address a large number of algorithmic and optimisation problems. The 1939 paper by Robbins [2], which gives the idea of orientable graphs and the paper by Masayoshi et al. [3], are to a certain extent an inspiration to formulating our graph network. The work by Dasler and Mount [1], which basically considers motion coordination of a set of vehicles at a traffic-crossing (intersection), has been a huge motivation and a much closer approach to that of ours. But unlike their work, we consider a much simpler version of a grid graph and mainly concentrate on analysing essential properties, and deriving suitable algorithms and structures to have a collision-free movement of traffic in the given graph network. Further, our work also extends in mentioning the only possible shortest path configurations in our defined graph. We now discuss some definitions and notations that are referred to in the rest of the paper.

1.1 Definitions and Notations

A road is a directed line, which is either parallel to X-axis ($X_i$) or Y-axis ($Y_i$) and it is uniquely defined by its direction and distance from the corresponding parallel axis. Here direction is the constraint which restricts the movement on the road.

Formally, a road is defined as a 2-tuple, $X_i = (d_i, x_i)$, $Y_j = (d_j, y_j)$, where $i, j$ are the indices with respect to their parallel axis, $d_k$ is the direction of the road i.e., $d_k \in \{0, 1\}$ (where 0 represents $-ve$ direction and 1 represents $+ve$ direction of the respective axis), and $x_i$ is the distance of the road $X_i$ from X-axis, similarly for $y_j$.

We define a One Way Road Network ($OWRN$) as a network with a set of $n$ horizontal and $m$ vertical Roads. Formally a $OWRN$ is a 2-tuple, $OWRN = (W_x, W_y)$, where, $W_x = \{X_1, X_2, X_3, \ldots , X_n\}$, $W_y = \{Y_1, Y_2, Y_3, \ldots , Y_m\}$.

A junction or vertex $v_{ij}$ is defined as the intersection of $X_i$ and $Y_j$. Formally, $v_{ij} \in (W_x \times W_y)$.

An edge is a connection between two adjacent vertices of a road and all the edges on a road are in the same direction as that of the road.

The boundary roads of a OWRN are the furthest and nearest roads from the X-axis and Y-axis, i.e., $X_1, X_n, Y_1$ and $Y_m$. In this paper we term each vertex on the boundary roads as boundary vertex, i.e., all the vertices with degrees 2 and 3.

A vehicle $c_r$ is defined as a 3-tuple $⟨t, s, P⟩$, where $t$ is the starting time of the vehicle, $s$ is the speed of the vehicle (throughout the journey), and $P$ is the path to be travelled by the vehicle. Formally, $c_r = (t, s, P)$.

A path $P_r$ of a vehicle $c_r$ is defined as the ordered set of vertices through which it traverses the OWRN, Formally, $P = \{v_1, v_2, v_3, \ldots , v_l\}$, $\forall i, v_i \in (W_x \times W_y)$, and $\forall i, 0 < i < l, v_i \rightarrow v_{i+1}$.

A traffic configuration is defined as a collection of vehicles over a OWRN. Formally a $TC$ is a 2-tuple, $TC = \langle OWRN, C \rangle$.
Lemma 1. is evident from the following lemmas. So a \( ovarian \) traffic configuration is a TC without any collisions.

2 Results

Before considering the traffic configuration problem, we define the connectivity of a OWRN.

2.1 Connectivity of a One Way Road Network

In this section, we consider a general OWRN of \( n \times m \) roads, and show the conditions for it to be strongly-connected.

The reachability to (and from) the non-boundary vertices is evident from the following lemmas.

Lemma 1. For every non-boundary vertex \( v_{ij}, \) \( 1 < i < n, \) \( 1 < j < m \) there exists \( e, f \) such that we can always reach the boundary vertices \( v_{ie}, v_{fj} \) from \( v_{ij} \).

Proof. We prove this lemma by considering the two roads which intersect to form the vertex \( v_{ij} \).

1. For the road \( X_i, \) if \( d_i = 0 \) then by definition we can reach \( v_{in} \) from \( v_{ij}, \) i.e., \( e = 1. \) Otherwise we can reach \( v_{in} \) from \( v_{ij}, \) i.e., \( e = n. \)

2. For the road \( Y_j, \) if \( d_j = 0 \) then by definition we can reach \( v_{ij} \) from \( v_{ij}, \) i.e., \( f = 1. \) Otherwise we can reach \( v_{mj} \) from \( v_{mj}, \) i.e., \( f = m. \)

From the above conditions we can clearly see that for any non-boundary vertex \( v_{ij}, \) \( \exists e, f \) such that \( v_{ie}, v_{fj} \) are reachable from \( v_{ij}. \)

Lemma 2. For every non-boundary vertex \( v_{ij} \) there exists \( e, f \) such that \( v_{ij} \) is reachable from the boundary vertices \( v_{ie}, v_{fj}. \)

Proof. We prove this lemma by considering the two roads which intersect to form the vertex \( v_{ij} \).

1. For the road \( X_i, \) if \( d_i = 0 \) then by definition we can reach \( v_{in} \) from \( v_{in}, \) i.e., \( e = 1. \) Otherwise we can reach \( v_{in} \) from \( v_{in}, \) i.e., \( e = n. \)

2. For the road \( Y_j, \) if \( d_j = 0 \) then by definition we can reach \( v_{ij} \) from \( v_{mj}, \) i.e., \( f = m. \) Otherwise we can reach \( v_{ij} \) from \( v_{mj}, \) i.e., \( f = 1. \)

From the above conditions we can clearly see that for every non-boundary vertex \( v_{ij}, \) \( \exists e, f \) such that \( v_{ij} \) is reachable from \( v_{ie} \) and \( v_{fj}. \)

Theorem 3. A One Way Road Network is strongly-connected if the boundary roads form a cycle.

Proof. The proof of this theorem follows from Lemma 4 and Lemma 5.

Lemma 4. If all the boundary vertices of a OWRN form a cycle, then it is strongly-connected.

Proof. Consider two vertices \( v_{ij}, v_{kl} \) in a OWRN, to reach from \( v_{ij} \) to \( v_{kl}, \) we have four different possibilities

1. Both boundary vertices: Any boundary vertex is reachable from any other boundary vertex, since they all form a cycle. Therefore a path exists.

2. \( v_{ij} \) non-boundary vertex, \( v_{kl} \) boundary vertex: From Lemma 2 we know that, from any non-boundary vertex \( v_{ij} \) we can always reach a boundary vertex, and from that vertex we can reach \( v_{kl} \) as shown in 1. Therefore a path exists.

3. \( v_{ij} \) boundary vertex, \( v_{kl} \) non-boundary vertex: From Lemma 2 we know that any non-boundary vertex \( v_{kl} \) is always reachable from a boundary vertex, and which in turn is reachable from \( v_{ij} \) as shown in 1. Therefore a path exists.

4. Both non-boundary vertices: From 1, 2 and 3 it is implied that there exists a path in this case too.

Lemma 5. If a given One Way Road Network is strongly-connected, then all the boundary vertices form a cycle.

Proof. Let us assume on the contrary that the boundary vertices do not form a cycle in the OWRN. Then there will exist a boundary vertex of degree 2 such that either both the boundary roads are incoming or outgoing.

1. Both incoming roads: In this case, we will not be able to reach any other vertex from that vertex.

2. Both outgoing roads: In this case, we will not be able to reach that vertex from any other vertex.

So there will exist at least one vertex which is not strongly-connected. Therefore, the OWRN is not strongly-connected.

Hence, by contradiction, we can claim that the boundary vertices of a strongly-connected OWRN will always form a cycle.

2.2 Traffic Configuration

We now define the traffic configuration problem in a connected OWRN.

Problem 1. Given a traffic configuration \( ovarian, C, \) our objective is to find a maximum cardinality subset \( C_{sub}, C_{sub} \subseteq C, \) such that the new traffic configuration \( ovarian, C_{sub}, \) is collision-free.

In the following sections we discuss the hardness of the above problem, and also mention some of the restricted versions of the same.
2.2.1 Hardness of Collision-Free Traffic Configuration

In this section we show that finding a solution to the traffic configuration problem is \textsc{NP-Hard}. For this, we have the following theorem.

**Theorem 6.** Given an undirected graph \( G = (V, E) \), there exists a traffic configuration \((\text{OWRN}, C)\), computable in polynomial-time, such that the cardinality of \textit{Maximum Independent Set} is \( k \) iff the maximum cardinality of \( C_{\text{sub}} \) is \( k \).

To prove this theorem, we reduce Maximum Independent Set problem to the Traffic Configuration problem, which is achieved with the help of the following lemmas and algorithms.

**Lemma 7.** Any complete graph \( K_n \) can be converted to an equivalent traffic configuration \( TC \).

**Proof.** We prove this lemma using proof by construction. The following steps show how to construct a \( TC \) from \( K_n \).

1. We construct a OWRN of \( 2n \times n \) roads, with \( 2n \) horizontal roads and \( n \) vertical roads in which
   
   (a) For the road \( X_i \), \( d_i = \begin{cases} 1 & 1 < i \leq 2n \\ 0 & i = 1 \end{cases} \)
   
   and \( x_i = \begin{cases} 0 & i = 1 \\ x_{i-1} + \delta & 1 < i \leq 2n \end{cases} \)
   
   (b) For the road \( Y_j \), \( d_j = \begin{cases} 0 & 1 < j \leq n \\ 1 & j = 1 \end{cases} \)
   
   and \( y_j = \begin{cases} 0 & j = 1 \\ y_{j-1} + \delta & 1 < j \leq n \end{cases} \)
   
   where \( \delta \) is a numeric constant.

2. The set of vehicles \( C \) is defined as \( \{c_1, c_2, \ldots, c_n\} \) and for each vehicle \( c_i \in C \) we assume
   
   (a) The start time to be 0 and the velocity to be \( \omega \).
   
   (b) \( P_i = \{v_{r1}, v_{(r-1)i}, \ldots, v_{q1}, v_{q(i+1)}, \ldots, v_{qn}\} \), where \( r = n + i - 1, q = n - i + 1 \).
   
   (c) \( c_i = \{0, \omega, P_i\} \)

3. Now we can observe that two vehicles \( \{c_i, c_j\} \in C \) collide at vertex \( v_{(n+1)(j)} \), where \( i < j \).

4. We assume that each node \( l \in K_n \) corresponds to a vehicle \( c_l \) and each edge between two nodes \( \alpha \) and \( \gamma \) in \( K_n \) corresponds to the collision of the respective vehicles \( c_\alpha, c_\gamma \).

\[ \vdash \text{We obtain the corresponding } TC = (\text{OWRN}, C) \text{ of } K_n. \]

\[ \square \]

**Figure 2:** Paths for vehicles \( \{c_1, c_2, \ldots, c_n\} \) in the above \( TC \)

Now to reduce any simple graph \( G \), we first compute the corresponding \( TC \) for the complete graph \( K_n(G) \). We then introduce 4 equi-spaced roads with directions \( \{0, 1, 0, 1\} \) between every two adjacent roads \( X_i, X_{i+1} \) and \( Y_j, Y_{j+1} \), respectively, in the above formed OWRN, the path of each vehicle is to be modified accordingly.

We define method \textsc{Delay}(\( \alpha, \beta, P_i, \Delta \)), where \( \alpha \) and \( \beta \) are the vertices in the path of \( c_i \), and \( \Delta \) is the total number of delays. Which modify the path \( P_i \) to introduce a \( \Delta \) number of small time delays in between the vertices \( \alpha, \beta \), this delay will also be propagated to all the successive vertices of \( \beta \) in \( P_i \).

**Algorithm 1: Delay method**

\[
\begin{align*}
\text{Input: } & P_r = \{\ldots, \alpha, \ldots, \beta, \ldots\}, \text{ no.of delays } \Delta \\
\text{Output: } & P_r \text{ after introduction of } \Delta \text{ delays} \\
\text{procedure } & \textsc{Delay}(\alpha, \beta, P_r, \Delta) \\
\text{if } & \Delta = 0 \\
\text{then return} \\
& \gamma_1, \gamma_2 \text{ are two successive vertices of } \alpha \text{ in } P_r \\
& \epsilon_1 \neq \epsilon_2, \text{ is a vertex } \mid \text{there is an edge } \gamma_1 \rightarrow \epsilon_1 \\
& \epsilon_2 \text{ is a vertex } \mid \epsilon_1 \rightarrow \epsilon_2 \rightarrow \gamma_1 \\
& P_r = \{\ldots, \alpha, \gamma_1, \epsilon_1, \epsilon_2, \gamma_2, \ldots, \beta, \ldots\} \\
& \textsc{Delay}(\beta, \beta, P_r, \Delta - 1) \\
\text{end procedure} \\
\end{align*}
\]

The method \textsc{CollisionVertex}(\( c_i, c_j \))\((i \neq j)\) will return the common vertex through which both the vehicles travel.
between the two collision vertices. The maximum number of delays introduced between the two collision vertices \( \alpha \) and \( \beta \) as defined in the reduction algorithm, will be two.

**Proof.** The proof of this lemma follows from the above stated properties. The number of delays introduced in the path \( P_j \), before collision of vehicles \( c_i \) and \( c_j \) is either \( i, i - 1 \). The number of delays introduced in the path \( P_j \), before collision of vehicles \( c_i+1 \) and \( c_j \) is either \( i + 1, i \).

From the above Lemma and the reduction algorithm, we have the following Lemma

**Lemma 9.** The above Reduction algorithm can be solved using Dynamic Programming approach in polynomial-time \( O(n^2) \), and the space complexity of both TC and OWRN created is \( O(n^2) \).

**Lemma 10.** If \( C_{\text{sub}} \) be any subset of \( C \) in TC such that \( TC_{\text{new}} = (\text{OWRN, C}_{\text{sub}}) \) is collision-free, then \( C_{\text{sub}} \) corresponds to Independent Set of \( G \).

**Proof.** Since \( TC_{\text{new}} \) is collision-free, so no two nodes in the graph \( G \), which corresponds to respective cars in \( C_{\text{sub}} \), consists of an edge. Thus, we can claim that \( C_{\text{sub}} \) corresponds to an independent set in \( G \).

From Lemma 9 we can say that maximum \( C_{\text{sub}} \) corresponds to Maximum Independent Set in \( G \). Now, using Lemma 7 and Lemma 10 we can prove that the traffic configuration problem is \( NP-Hard \).

### 2.2.2 Restricted Version

If we constrain our vehicles to move in a straight line motion, then the corresponding graph to \( TC \) will be a Bipartite Graph. And, Maximum Independent Set of a Bipartite Graph can be computed using Konig’s Theorem and Network-Flow Algorithm in polynomial-time. Hence, the restricted version of the problem is solvable in polynomial-time.
2.3 Shortest Path Properties

Suppose in a city of only One Way Road Network, a person wants to move from one point to another in minimum time. Now, the objective would be to compute the shortest path to the destination in least possible time. Designing efficient algorithms to compute the shortest path in a One Way Road Network would be useful in many applications in the areas of facility location, digital micro-fluidic bio-chips, etc.

The length of the shortest path between two vertices in a OWRN may not be the Manhattan distance. There may be a pair of neighbouring vertices which are the farthest pair of vertices in the OWRN metric. A turn in a path is defined when two consecutive pair of edges are from different roads.

We have the following properties for shortest path between any two vertices in a OWRN:

1. Between any pair of vertices \( u, v \), there exists a shortest path of at most four turns.
2. The upper bound on the length of the shortest path between any pair of vertices \( u, v \) is the perimeter of the boundary of the OWRN.

We observe that any shortest path between every pair of vertices in the OWRN will be a rotationally symmetric to one of the paths shown below:

![Figure 4: Shortest path configurations](image-url)

3 Remarks

We have shown all the possible configurations of the path that connects two vertices in a OWRN. In the future we will extend this work to compute various kind of facility location problems on a OWRN. It will be interesting to investigate the time complexity of one-centre or k-centre problems with respect to OWRN metric. Other interesting problems may be to design an efficient data structure for dynamic maintenance of shortest path in directed grid graphs.

References

[1] Philip Dasler, David M. Mount. On the Complexity of an Unregulated Traffic Crossing [arXiv:1505.00874] \[cs\], May 2015.
[2] H. E. Robbins. A Theorem on Graphs, with an Application to a Problem of Traffic Control The American Mathematical Monthly, 46: 281-283, 1939.
[3] T. Kashiwabara and T. Fujisawa. NP-Completeness of the Problem of Finding a Minimum-Clique-Number Interval Graph Containing a Given Graph as a Subgraph. Proceedings International Conference on Circuits and Systems, 657-660, 1979.
[4] T. Kashiwabara and S. Masuda, K. Nakajima and T. Fujisawa. Generation of maximum independent sets of a bipartite graph and maximum cliques of a circular-arc-graph. Journal of Algorithms, 13, 161-174, 1992.
[5] Masayoshi Kakikura, Jun-ichi Takeno and Masao Mukaidono. A Tour Optimization Problem in a Road Network with One-way Paths. The transactions of the Institute of Electrical Engineers of Japan.C, 98: 257-264, 1978.
[6] Garey, M.R. and Johnson, D.S. Computers and Intractability: A Guide to the Theory of NP-Completeness San Francisco, CA: Freeman, 1979.
[7] K. C. Tan and S. C. Lew. Fashioning one-way urban road network design as a module orientation problem. Urban Transport X. Urban Transport and the Environment in the 21st Century, p. 649-658, 2004-5.
[8] F. Berger and R. Klein. A travellers problem. In Proceedings of the Twenty-sixth Annual Symposium on Computational Geometry, SoCG ’10, page 176182, New York, NY, USA, 2010. ACM.
[9] K. M. Dresner and P. Stone. A multiagent approach to autonomous intersection management. J. Artif. Intell.Res.(JAIR), 31:591656, 2008.
[10] P. Fiorini and Z. Shiller. Motion planning in dynamic environments using velocity obstacles. The International Journal of Robotics Research, 17(7):760772, July 1998.
[11] R. Rajamani. Vehicle Dynamics and Control. Springer Science and Business Media, December 2011.
[12] P. R. Wurman, R. Dandrea, and M. Mountz. Coordinating hundreds of cooperative, autonomous vehicles in warehouses. The AI magazine, 29(1):919, 2008.
[13] J. Yu and S. M. LaValle. Multi-agent path planning and network flow. [arXiv:1204.5717] \[cs\], April 2012.
[14] G. B. Dantzig and J. H. Ramser. The truck dispatching problem. Management Science, 6(1):8091, October 1959.
[15] Marius M. Solomon. Algorithms for the vehicle routing and scheduling problems with time window constraints. Operations Research, 35(2):254265, March 1987.