Distance Expanding Random Mappings,
Thermodynamic Formalism,
Gibbs Measures,
and
Fractal Geometry

Volker Mayer
Bartłomiej Skorulski
Mariusz Urbański

Université de Lille I, UFR de Mathématiques, UMR 8524 du CNRS,
59655 Villeneuve d’Ascq Cedex, France
E-mail address: volker.mayer@math.univ-lille1.fr
Web: math.univ-lille1.fr/~mayer

Departamento de Matemáticas, Universidad Católica del Norte,
Avenida Angamos 0610, Antofagasta, Chile
E-mail address: bskorulski@ucn.cl

Department of Mathematics, University of North Texas, Denton,
TX 76203-1430, USA
E-mail address: urbanski@unt.edu
Web: www.math.unt.edu/~urbanski
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CHAPTER 1

Introduction

In this paper we develop the thermodynamical formalism for measurable expanding random mappings. This theory then is applied in the context of conformal expanding random mappings where we deal with the fractal geometry of fibers.

Distance expanding maps have been introduced for the first time in Ruelle’s monograph [16]. A systematic account of the dynamics of such maps, including the thermodynamical formalism and the multifractal analysis, can be found in [15]. One of the main features of this class of maps is that their definition does not require any differentiability or smoothness condition. Distance expanding maps comprise symbol systems and expanding maps of smooth manifolds but go far beyond. This is also a characteristic feature of our approach.

In this paper we define measurable expanding random maps. The randomness is modeled by an invertible ergodic transformation \( \theta \) of a probability space \((X, \mathcal{B}, m)\). We investigate the dynamics of compositions

\[
T^n_x = T_{\theta^{n-1}(x)} \circ \cdots \circ T_x, \ n \geq 1,
\]

where the \( T_x : J_x \to J_{\theta(x)} \) \((x \in X)\) is a distance expanding mapping. These maps are only supposed to be measurably expanding in the sense that their expanding constant \( \gamma_x > 1 \) is measurable.

In so general setting we first build the thermodynamical formalism for arbitrary Hölder continuous potentials \( \phi_x \). We show, in particular, the existence, uniqueness and ergodicity of a family of Gibbs measures \( \{\nu_x\}_{x \in X} \). Following ideas of Kifer [11], these measures are first produced in a pointwise manner and then we carefully check their measurability.

Our results contain those in [1] and in [11] (see also the expository article [13]) if in the latter one assumes an expanding property. Throughout the entire paper where it is possible we avoid, in hypotheses, absolute constants. Our feeling is that in the context of random systems all (or at least as many as possible) absolute constants appearing in deterministic systems should become measurable functions. With this respect the thermodynamical formalism developed in here represents also, up to our knowledge, new achievements in the theory of random symbol dynamics or smooth expanding random maps acting on Riemannian manifolds.

Our approach to the thermodynamical formalism stems primarily from the classical method presented by Bowen in [3] and undertaken by Kifer in [11]. Unlike recent trends to employ here the method of Hilbert metric (as for example in [7], [12], [18], [17]). Developing it in the context of random dynamical systems we demonstrate that it works well and does not lead to too complicated (at least to our taste) technicalities. We do not need any Markov partitions or (even auxiliary) symbol dynamics. The measurability issue mentioned above results from convergence of the Perron-Frobenius operators. We show that this convergence is
exponential, which implies exponential decay of correlations. These results precede investigations of a pressure function \( x \mapsto P_x(\varphi) \) which satisfies the property

\[
\nu_{\theta(x)}(T_x(A)) = e^{P_x(\varphi)} \int_A e^{-\varphi} d\nu_x
\]

where \( A \) is any measurable set such that \( T_x|_A \) is injective. The integral, against the measure \( m \) on the base \( X \), of this function is a central parameter \( \mathcal{E}P(\varphi) \) of random systems called the \textit{expected pressure}. If the potential \( \varphi \) depends analytically on parameters, we show that the expected pressure also behaves real analytically. We would like to mention that, contrary to the deterministic case, the spectral gap methods do not work in the random setting. Our proof utilizes the concept of complex cones introduced by Rugh in \([17]\), and this is the only place, where we use the projective metric.

We then apply the above results mainly to investigate fractal properties of fibers of \textit{conformal random systems}. They include Hausdorff dimension, Hausdorff and packing measures, as well as multifractal analysis. First, we establish a version of Bowen’s formula (obtained in a somewhat different context in \([2]\)) showing that the Hausdorff dimension of almost every fiber \( J_x \) is equal to \( h \), the only zero of the expected pressure \( \mathcal{E}P(\varphi_t) \), where \( \varphi_t = -t \log |f'| \) and \( t \in \mathbb{R} \). Then we analyze the behavior of \( h \)-dimensional Hausdorff and packing measures. It turned out that the random dynamical systems split into two categories. Systems from the first category, rather exceptional, behave like deterministic systems. We call them, therefore, \textit{quasi-deterministic}. For them the Hausdorff and packing measures are finite and positive. Other systems, called \textit{essentially random}, are rather generic. For them the \( h \)-dimensional Hausdorff measure vanishes while the \( h \)-packing measure is infinite. This, in particular, refutes the conjecture stated by Bogenschütz and Ochs in \([2]\) that the \( h \)-dimensional Hausdorff measure of fibers is always positive and finite. In fact, the distinction between the quasi-deterministic and the essentially random systems is determined by the behavior of the Birkhoff sums

\[
P^n_x(\varphi) = P^{n-1}_x(\varphi) + \ldots + P_x(\varphi)
\]

of the pressure function for potential \( \varphi_h = -h \log |T'| \). If these sums stay bounded then we are in the quasi-deterministic case. On the other hand, if these sums are neither bounded below nor above, the system is called essentially random. The behavior of \( P^n_x \) is often governed by stochastic theorems such as the law of the iterated logarithm whenever it holds. This is the case for our primary examples, namely conformal DG-systems and classical conformal random systems. We are then in position to state that the quasi-deterministic systems correspond to rather exceptional case where the asymptotic variance \( \sigma^2 = 0 \). Otherwise the system is essential.

The fact that Hausdorff measures in the Hausdorff dimension vanish has further striking geometric consequences. Namely, almost all fibers of an essential conformal random system are not bi-Lipschitz equivalent to any fiber of any quasi-deterministic or deterministic conformal expanding system. In consequence almost every fiber of an essentially random system is not a geometric circle nor even a piecewise analytic curve. We then show that these results do hold for many explicit random dynamical systems, such as conformal DG-systems, classical conformal random systems, and, perhaps most importantly, Brück and Bürger polynomial
systems. As a consequence of the techniques we have developed, we positively answer the question of Brück and Bürger (see [5] and Question 5.4 in [4]) of whether the Hausdorff dimension of almost all naturally defined random Julia set is strictly larger than 1. We also show that in this same setting the Hausdorff dimension of almost all Julia sets is strictly less than 2.

Concerning the multifractal spectrum of Gibbs measures on fibers, we show that the multifractal formalism is valid, i.e. the multifractal spectrum is Legendre conjugated to a temperature function. As usual, the temperature function is implicitly given in terms of the expected pressure. Here, the most important, although perhaps not most strikingly visible, issue is to make sure that there exists a set $X_{ma}$ of full measure in the base such that the multifractal formalism works for all $x \in X_{ma}$. Furthermore, for uniformly expanding conformal random systems we also deduce real analyticity of the multifractal spectrum.
CHAPTER 2

Expanding Random Maps

This chapter provides the definition of expanding random maps. These are central objects of our work. We also introduce the spaces of continuous and Hölder continuous functions and define transfer operators, the main tool for our investigation. We prove here only few technical results.

1. Preliminaries

Suppose \((X, \mathcal{F}, m, \theta)\) is a measure preserving dynamical system with invertible and ergodic map \(\theta : X \to X\) which is referred to as the base map. Assume further that a mapping \(X \ni x \mapsto J_x\) is given where \((J_x, \varrho_x)\) is a compact metric space such that

\[
\text{diam}_{\varrho_x}(J_x) \leq 1.
\]

Let

\[
J = \bigcup_{x \in X} \{x\} \times J_x.
\]

We will denote by \(B_x(z, r)\) the ball in the space \((J_x, \varrho_x)\) centered at \(z \in J_x\) and with radius \(r\), and frequently, for ease of notation, we will write \(B(y, r)\) for \(B_x(z, r)\), where \(y = (x, z)\).

Suppose finally that for every \(x \in X\) a continuous mapping

\[
T_x : J_x \to J_{\theta(x)}
\]

is defined. We consider the skew-product map \(T : J \to J\) defined by the formula

\[
T(x, z) = (\theta(x), T_x(z)).
\]

For every \(n \geq 0\) we define the map

\[
T^n_x := T_{\theta^{n-1}(x)} \circ \ldots \circ T_x : J_x \to J_{\theta^n(x)}
\]

and we have that \(T^n(x, y) = (\theta^n(x), T^n_x(y))\). We will frequently use the notation

\[
x_n = \theta^n(x)
\]

for all integers \(n\).

2. Expanding Random Maps

The first part of the paper and, in particular, the whole thermodynamical formalism is developed for expanding random maps which are defined in this section.

A map \(T : J \to J\) is called a expanding random map if the mappings \(T_x : J_x \to J_{\theta(x)}\) are continuous, open, and surjective, and there exist a function \(X \ni x \mapsto \eta_x \in \mathbb{R}_+\) and a real number \(\xi > 0\) such that following conditions hold.

Uniform Openness. \(T_x(B_x(z, \eta_x)) \supset B_{\theta(x)}(T_x(z), \xi)\) for every \((x, z) \in J\).
2. EXPANDING RANDOM MAPS

Measurably Expanding. There exists a measurable function \( \gamma : X \to (1, +\infty) \), \( x \mapsto \gamma_x \) such that for \( m \)-a.e. \( x \in X \),
\[
\varrho_{\theta(x)}(T_x(z_1), T_x(z_2)) \geq \gamma_x \varrho_x(z_1, z_2) \quad \text{whenever} \quad \varrho(z_1, z_2) < \eta_x, \ z_1, z_2 \in J_x.
\]

Measurability of the Degree. The function
\[
x \mapsto \deg(T_x) := \sup_{y \in J_{\theta(x)}} \# T_x^{-1}(\{y\})
\]
is measurable.

Topological Exactness. There exists a measurable function \( x \mapsto n_\xi(x) \) such that for \( m \)-a.e. \( x \in X \)
\[
(2.1) \quad T_x^{n_\xi(x)}(B_x(z, \xi)) = J_{\theta^n \xi(x)}(z) \quad \text{for every} \quad z \in J_x.
\]

Note that the measurably expanding condition implies that \( T_x|_{B(z, \eta_x)} \) is one-to-one for every \((x, z) \in J\). Together with the compactness of the spaces \( J_x \) it yields the numbers \( \deg(T_x) \) to be finite. Therefore the supremum in the condition of measurability of the degree is in fact a maximum.

3. Visiting sequences

Let \( F \in \mathcal{F} \) be a set with a positive measure. Define the sets
\[
V_{+F}(x) := \{ n \in \mathbb{N} : \theta^n(x) \in F \}
\]
and
\[
V_{-F}(x) := \{ n \in \mathbb{N} : \theta^{-n}(x) \in F \}.
\]
The set \( V_{+F}(x) \) is called a visiting sequence (of \( F \) at \( x \)). Then the set \( V_{-F}(x) \) is just a visiting sequence for \( \theta^{-1} \) and we also call it a backward visiting sequence. By \( n_j(x) \) we denote the \( j \)th-visit in \( F \) at \( x \). Since \( m(F) > 0 \), by Birkhoff’s Ergodic Theorem we have that
\[
m(X_{+F}^\prime) = m(X_{-F}^\prime) = 1
\]
where
\[
X_{+F}^\prime := \left\{ x \in X : V_{+F}(x) \text{ is infinite and } \lim_{j \to 0} \frac{n_{j+1}(x)}{n_j(x)} = 1 \right\}
\]
and \( X_{-F}^\prime \) is defined analogously. The sets \( X_{+F}^\prime \) and \( X_{-F}^\prime \) are respectively called forward and backward visiting for \( F \).

Let \( \Psi(x, n) \) be a formula which depends on \( x \in X \) and \( n \in \mathbb{N} \). We say that \( \Psi(x, n) \) holds in a visiting way, if there exists \( F \) with \( m(F) > 0 \) such that, for \( m \)-a.e. \( x \in X_{+F}^\prime \) and all \( j \in \mathbb{N} \), the formula \( \Psi(\theta^n(x), n_j(x)) \) holds, where \((n_j(x))_{j=0}^\infty\) is the visiting sequence of \( F \) at \( x \).

We also say that \( \Psi(x, n) \) holds in an exhaustively visiting way, if there exists a family \( F_k \in \mathcal{F} \) with \( \lim_{k \to \infty} m(F_k) = 1 \) such that, for all \( k, m \)-a.e. \( x \in X_{+F_k}^\prime \), and all \( j \in \mathbb{N} \), the formula \( \Psi(\theta^n(x), n_j(x)) \) holds, where \((n_j(x))_{j=0}^\infty\) is the visiting sequence of \( F_k \) at \( x \).

Now, let \( f_n : X \to \mathbb{R} \) be a sequence of measurable functions. We write that
\[
s\lim_{n \to \infty} f_n = f
\]
if that there exists a family \( F_k \in \mathcal{F} \) with \( \lim_{k \to \infty} m(F_k) = 1 \) such that, for all \( k \) and \( m \)-a.e. \( x \in X_{+F_k}^\prime \) and all \( j \in \mathbb{N} \),
\[
\lim_{j \to \infty} f_{n_j}(x) = f(x)
\]
Moreover, there exists a measurable function almost every such that for \( \nu \) and \( \nu \) is continuous, \( T \)

For every \( n \) \( \in \mathbb{N} \) we can always find finite numbers \( b_1, \ldots, b_n \) such that the essential set \( F \) for \( g_1, \ldots, g_k \) with constants \( b_1, \ldots, b_n \) has the measure \( m(F) \geq 1 - \epsilon \).

4. Remarks on Expanding Random Map

If it does not lead to misunderstanding we will frequently identify \( J_x \) and \( \{x\} \times J_x \).

The conditions of uniform openness and measurably expanding imply that, for every \( y = (x,z) \in J \) there exists a unique continuous inverse branch \( T_y^{-1} : B_{\vartheta(x)}(T(y),\xi) \rightarrow B_x(y,\eta_x) \) of \( T_x \) sending \( T_x(z) \) to \( z \). By the measurably expanding property we have

\[
\varrho(T_y^{-1}(z_1), T_y^{-1}(z_2)) \leq \gamma_x^{-1} \varrho(z_1, z_2)
\]

for \( z_1, z_2 \in B_{\vartheta(x)}(T(y), \xi) \), and

\[
T_y^{-1}(B_{\vartheta(x)}(T(y), \xi)) \subset B_x(y, \gamma_x^{-1} \xi) \subset B_x(y, \xi).
\]

Hence, for every \( n \geq 0 \), the composition

\[
T_y^{-n} = T_y^{-1} \circ T_y^{-1} \circ \cdots \circ T_y^{-1}(y) : B_{\vartheta^n(x)}(T^n(y), \xi) \rightarrow B_x(y, \xi)
\]

is well-defined and has the following properties. The function

\[
T_y^{-n} : B_{\vartheta^n(x)}(T^n(y), \xi) \rightarrow B_x(y, \xi)
\]

is continuous,

\[
T^n \circ T_y^{-n} = \text{Id}_{B_{\vartheta^n(x)}(T^n(y), \xi)},
\]

and

\[
\varrho(T_y^{-n}(z_1), T_y^{-n}(z_2)) \leq \gamma_x^{-1} \varrho(z_1, z_2)
\]

for \( z_1, z_2 \in B_{\vartheta^n(x)}(T^n(y), \xi) \), where

\[
\gamma_x^n = \gamma_x \gamma_{\vartheta(x)} \cdots \gamma_{\vartheta^{n-1}(x)}.
\]

Moreover,

\[
T_x^{-n}(B_{\vartheta^n(x)}(T^n(y), \xi)) \subset B_x(y, (\gamma_x^n)^{-1} \xi) \subset B_x(y, \xi).
\]

**Lemma 2.1.** For every \( r > 0 \), there exists a measurable function \( x \mapsto n_r(x) \) such that for \( m \)-a.e. \( x \in X \)

\[
T_{x}^{n_r(x)}(B_x(z, r)) = J_{\vartheta^{n_r(x)}(x)} \quad \text{for every} \ z \in J_x.
\]

Moreover, there exists a measurable function \( X \ni x \mapsto j(x) \in \mathbb{N} \) such that for almost every \( x \in X \)

\[
T_{x}^{j(x)}(B_{x-j(x)}(z, \xi)) = J_x \quad \text{for every} \ z \in J_{x-j(x)}.
\]
2. EXPANDING RANDOM MAPS

Proof. In order to prove the first statement, consider $\gamma_0 > 1$ and the essential set $F$ for $-\gamma$ with constant $-\gamma_0$. Then for $x \in X'_+ F$, the limit

$$\lim_{n \to \infty} (\gamma^n_x)^{-1} = 0.$$ 

Define

$$X_{+F,k} := \{x \in X'_+ F : (\gamma^n_x)^{-1} \xi < r\}.$$ 

Then $X_{+F,k} \subset X_{+F,k+1}$ and $\bigcup_{k \in \mathbb{N}} X_{+F,k} = X'_+ F$. By measurability of $x \mapsto \gamma_x$, $X_{+F,k}$ is a measurable set. Hence the function

$$X'_+ F \ni x \mapsto n_r(x) := \min\{k : x \in X_{+F,k}\} + n_x(x)$$

is finite and measurable. By (4.5) and (2.1),

$$\text{Define } \alpha \in (0,1]. \text{ By } \mathcal{H}^\alpha(J_x) \text{ we denote the space of Hölder continuous with an exponent } \alpha \text{ functions on } J_x \text{ with an exponent } \alpha \text{. This means that } \varphi_x \in \mathcal{H}^\alpha(J_x) \text{ if and only if } \varphi_x \in \mathcal{C}(J_x) \text{ and } v(\varphi_x) < \infty \text{ where}

$$v_{\alpha}(\varphi_x) := \inf\{H_x : |\varphi(z_1) - \varphi(z_2)| \leq H_x \varphi_x'(z_1, z_2)\}.$$

where the infimum is taken over all $z_1, z_2 \in J_x$ with $\varphi(z_1, z_2) \leq \eta$.

A function $\varphi \in \mathcal{C}^1(J)$ is called Hölder continuous with an exponent $\alpha$ provided that there exists a measurable function $H : X \to [1, +\infty)$, $x \mapsto H_x$, such that $\log H \in L^1(m)$ and such that $v_{\alpha}(\varphi_x) \leq H_x$ for a.e. $x \in X$. We denote the space of all Hölder functions with fixed $\alpha$ and $H$ by $\mathcal{H}^\alpha(J,H)$ and the space of all $\alpha$–Hölder functions by $\mathcal{H}^\alpha(J) = \bigcup_{H \geq 1} \mathcal{H}^\alpha(J,H)$.

6. Transfer operator

For every function $g : J \to \mathbb{C}$ and a.e. $x \in X$ let

$$S_n g_x = \sum_{j=0}^{n-1} g_x \circ T_x^j,$$

and, if $g : X \to \mathbb{C}$, then $S_n g = \sum_{j=0}^{n-1} g \circ \theta^j$.

Let $\varphi$ be a function in the Hölder space $\mathcal{H}^\alpha(J)$. For every $x \in X$, we consider the transfer operator

$$\mathcal{L}_x = \mathcal{L}_{\varphi,x} : \mathcal{C}(J_x) \to \mathcal{C}(J_{\theta(x)})$$
given by the formula
\begin{equation}
\mathcal{L}_x g_z(w) = \sum_{T_x(z) = w} g_x(z) e^{\varphi_z(z)}, \quad w \in \mathcal{J}_\theta(x).
\end{equation}

It is obviously a positive linear operator and it is bounded with the norm bounded above by
\begin{equation}
\|\mathcal{L}_x\|_\infty \leq \text{deg}(T_x) \exp(\|\varphi\|_\infty).
\end{equation}

This family of operators gives rise to the global operator \(\mathcal{L} : \mathcal{C}(\mathcal{J}) \to \mathcal{C}(\mathcal{J})\) defined as follows
\begin{equation}
(\mathcal{L} g)_x(w) = \mathcal{L}_{\theta^{-1}(x)} g_{\theta^{-1}(x)}(w).
\end{equation}

For every \(n > 1\) and a.e. \(x \in X\), we denote
\begin{equation}
\mathcal{L}^n_x := \mathcal{L}_{\theta^{n-1}(x)} \circ \ldots \circ \mathcal{L}_x : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta^n(x)}).
\end{equation}

Note that
\begin{equation}
\mathcal{L}^n_x g_z(w) = \sum_{z \in T_x^{-n}(w)} g_x(z) e^{S_n \varphi_z(z)}, \quad w \in \mathcal{J}_{\theta^n(x)},
\end{equation}

where \(S_n \varphi_z(z)\) has been defined in (6.1). The dual operator \(\mathcal{L}^*_x\) maps \(C^*(\mathcal{J}_{\theta(x)})\) into \(C^*(\mathcal{J}_x)\).

7. Distortion Properties

\textbf{Lemma 2.2.} Let \(\varphi \in \mathcal{H}_0(\mathcal{J}, H)\), let \(n \geq 1\) and let \(y = (x, z) \in \mathcal{J}\). Then
\begin{equation}
|S_n \varphi_x(T_y^{-n}(w_1)) - S_n \varphi_x(T_y^{-n}(w_2))| \leq \varrho^\alpha(w_1, w_2) \sum_{j=0}^{n-1} H_{\theta_i(x)}(\gamma_{\theta_i(x)}^{n-j})^{-\alpha}
\end{equation}

for all \(w_1, w_2 \in B(T_y(z), \xi)\).

**Proof.** We have by (4.3) and Hölder continuity of \(\varphi\) that
\begin{align*}
|S_n \varphi_x(T_y^{-n}(w_1)) - S_n \varphi_x(T_y^{-n}(w_2))| &
\leq \sum_{j=0}^{n-1} |\varphi_x(T_x^j(T_y^{-n}(w_1))) - \varphi_x(T_x^j(T_y^{-n}(w_2)))| \\
&
= \sum_{j=0}^{n-1} \left|\varphi_x(T_x^{-j}(T_y^{-n}(w_1))) - \varphi_x(T_x^{-j}(T_y^{-n}(w_2)))\right| \\
&
\leq \sum_{j=0}^{n-1} \varrho^\alpha(T_x^{-j}(T_y^{-n}(w_1)), T_x^{-j}(T_y^{-n}(w_2)) H_{\theta_i(x)} \\
&
\leq \varrho^\alpha(w_1, w_2) \sum_{j=0}^{n-1} H_{\theta_i(x)}(\gamma_{\theta_i(x)}^{n-j})^{-\alpha}.
\end{align*}

\(\square\)

Set
\begin{equation}
Q_x := Q_x(\mathcal{H}) = \sum_{j=1}^{\infty} H_{\theta^{-j}(x)}(\gamma_{\theta^{-j}(x)}^{j})^{-\alpha}.
\end{equation}
Lemmas 2.3. The function $x \mapsto Q_x$ is measurable and m-a.e. finite. Moreover, for every $\varphi \in \mathcal{H}^\alpha(J, H)$,

$$|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| \leq Q_{\theta^n(x)} \varphi^\alpha(w_1, w_2)$$

for all $n \geq 1$, a.e. $x \in X$, every $z \in J_x$ and every $w_1, w_2 \in B(T^n(z), \xi)$ and where again $y = (x, z)$.

Proof. The measurability of $Q_x$ follows directly from (7.1).

Because of Lemma 2.2, we are only left to show that $Q_x$ is m-a.e. finite. Let $F$ be an essential set for $-\log(\gamma - 1)$. Then there exists $G > 1$ such that $\gamma_x \geq G$ for all $x \in F$. Then, using Birkhoff’s Ergodic Theorem for $\theta^{-1}$, we get that

$$\liminf_{j \to \infty} \frac{1}{j} \sum_{k=0}^{j-1} \log \gamma_{\theta^{-j}(x)} \geq \chi := m(F) \log(G)$$

for m-a.e. $x \in X$. Therefore, there exists a measurable function $C_\gamma : X \to [1, +\infty)$ m-a.e. finite such that

$$(7.2) \quad C_\gamma^{-1}(x)e^{j\chi/2} \leq \gamma_{\theta^{-j}(x)}$$

for all $j \geq 0$ and a.e. $x \in X$. Moreover, since $\log H \in L^1(m)$ it follows again from Birkhoff’s Ergodic Theorem that

$$\lim_{j \to \infty} \frac{1}{j} \log H_{\theta^{-j}(x)} = 0$$

for a.e. $x \in X$. Therefore, there exists a measurable function $C_H : X \to [1, +\infty)$ such that

$$(7.3) \quad H_{\theta^{-j}(x)} \leq C_H(x) e^{j\alpha \chi/4} \quad \text{and} \quad H_{\theta^{-j}(x)} \leq C_H(x) e^{j\alpha \chi/4}$$

for all $j \geq 0$ and a.e. $x \in X$. Then, for a.e. $x \in X$, all $n \geq 0$ and all $a \geq j \geq n - 1$, we have

$$H_{\theta^n(x)} = H_{\theta^{-(n-j)}(\theta^a(x))} \leq C_H(\theta^a(x)) e^{(n-j)\alpha \chi/4}.$$ 

Therefore, still with $x_n = \theta^n(x)$,

$$Q_{x_n} = \sum_{j=0}^{n-1} H(x_j)(\alpha(x_j)^{n-j} - \alpha) \leq \sum_{j=0}^{n-1} C_H(x_n) e^{(n-j)\alpha \chi/4} C_\alpha(x_{n-1}) e^{-\alpha(n-j)\chi/4}$$

$$\leq C_\alpha(x_{n-1}) C_H(x_n) \sum_{j=0}^{n-1} e^{-\alpha(n-j)\chi/4} \leq C_\alpha(x_{n-1}) C_H(x_n)(1 - e^{-\chi/4})^{-1}.$$ 

Hence

$$Q_x \leq C_\alpha(\theta^{-1}(x)) C_H(x)(1 - e^{-\alpha \chi/4})^{-1} < +\infty.$$ 

□
CHAPTER 3

The RPF–theorem

In this chapter we will establish a version of Ruelle-Perron-Frobenius (RPF) Theorem. Notice that this quite substantial fact is proved without any measurable structure on the space $J$. In particular, we do not address measurability issues of $\lambda_x$ and $q_x$. In order to obtain this measurability we will need and will impose a natural measurable structure on the space $J$. This will also enable us to get appropriate uniqueness of $\lambda$ and $q$, and consequently uniqueness of Gibbs measures.

1. Formulation of Theorems

Let $T : J \to J$ be an expanding random map. Denote by $\mathcal{M}(J_x)$ the set of all Borel probability measures on $J_x$. A family of measures $\{\mu_x\}_{x \in X}$ such that $\mu_x \in \mathcal{M}(J_x)$ is called $T$–invariant if $\mu_x \circ T_{\theta(x)}^{-1} = \mu_{\theta(x)}$ for a.e. $x \in X$.

The main results proved in this chapter are listed below.

**Theorem 3.1.** Let $\varphi \in \mathcal{H}^a(J)$ and let $L = \mathcal{L}_\varphi$ be the associated transfer operator. Then the following holds.

1. There exists a unique family $\{\nu_x\}_{x \in X}$ of probability measures ($\nu_x \in \mathcal{M}(J_x)$) such that, for $m$-a.e. $x \in X$,
   \[ L^*_x \nu_{\theta(x)} = \lambda_x \nu_x \]
   for some $\lambda_x \in \mathbb{R}$. Then $\lambda_x = \nu_{\theta(x)}(L_x 1)$ and $\text{supp}(\nu_x) = J_x$.

2. There exists a unique function $q \in C^0(J)$ such that for a.e. $x \in X$,
   \[ L_x q_x = \lambda_x q_{\theta(x)}. \]

Moreover, $q_x \in \mathcal{H}^a(J_x)$ for a.e. $x \in X$.

3. The family of measures $\{\mu_x := q_x \nu_x\}_{x \in X}$ is $T$-invariant.

**Theorem 3.2.**

1. Let $\tilde{\varphi}_x = \varphi_x + \log q_x - \log q_{\theta(x)} \circ T - \log \lambda_x$ where $x \mapsto q_x$. Denote $\tilde{\mathcal{L}} := \mathcal{L}_{\tilde{\varphi}}$. Then, for a.e. $x \in X$ and all $g_x \in C(J_x)$,
   \[ \tilde{\mathcal{L}}^n_x g_x \xrightarrow{n \to \infty} \int g_x q_x \text{d}\nu_x. \]

2. Let $\hat{\varphi}_x = \varphi_x - \log \lambda_x$. Denote $\hat{\mathcal{L}} := \mathcal{L}_{\hat{\varphi}}$. There exist a constant $B < 1$ and a measurable function $A : X \to (0, \infty)$ such that for every function $g \in C^0(J)$ with $g_x \in \mathcal{H}^a(J_x)$ there exists a measurable function $A_g : X \to (0, \infty)$ for which
   \[ \|(\hat{\mathcal{L}}^n g)_x - \left( \int g_{\theta^{-n}(x)} \text{d}\nu_{\theta^{-n}(x)} \right) q_x\|_{\infty} \leq A_g(\theta^{-n}(x)) A(x) B^n \]
   for a.e. $x \in X$ and every $n \geq 1$. 

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3. The RPF-Theorem

(3) There exists $B < 1$ and a measurable function $A' : X \to (0, \infty)$ such that for every $f_{\theta^n(x)} \in L^1(\mu_{\theta^n(x)})$ and every $g_x \in \mathcal{H}^\alpha(J_x)$,

$$|\mu_x((f_{\theta^n(x)} \circ T^n_x)g_x) - \mu_{\theta^n(x)}(f_{\theta^n(x)})\mu_x(g_x)| \leq \mu_{\theta^n(x)}(|f_{\theta^n(x)}|)A'(\theta^n(x)) \left( \int |g_x|d\mu_x + 4\frac{\nu_\alpha(g_xq_x)}{Q_x} \right) B^n.$$ 

A collection of measures $\{\mu_x\}_{x \in X}$ such that $\mu_x \in \mathcal{M}(J_x)$ is called a Gibbs family for $\varphi \in \mathcal{H}^\alpha(J)$ provided that there exists a measurable function $D_\varphi : X \to [1, +\infty)$ and a function $x \mapsto P_x$, called the pseudo-pressure function, such that

$$\frac{(D_\varphi(x)D_\varphi(\theta^n(x)))^{-1}}{\exp(S_n\varphi(y) - S_nP_x)} \leq D_\varphi(x)D_\varphi(\theta^n(x))$$

for every $n \geq 0$, a.e. $x \in X$ and every $z \in J_x$ with $y = (x, z)$. Towards proving uniqueness type result for Gibbs families we introduce the following concept.

Measurability of Cardinality of Covers. There exists a measurable function $X \ni x \mapsto a_x \in \mathbb{N}$ such that for every $x \in X$ there exists a finite sequence $w_1^x, \ldots, w_{a_x}^x \in J_x$ such that

$$\bigcup_{j=1}^{a_x} B(w_j, \xi) = J_x.$$ 

In the case of random compact subsets of a Polish space (see Section 5) this condition is guaranteed by Lemma 4.11.

Theorem 3.3. The collections $\{\nu_x\}_{x \in X}$ and $\{\mu_x\}_{x \in X}$ are Gibbs families. Moreover, if $J$ satisfies the condition of measurability of cardinality of covers and if $\{\nu'_x\}_{x \in X}$ is a Gibbs family, then $\nu'_x$ and $\nu_x$ are equivalent for almost every $x \in X$.

2. Frequently used auxiliary measurable functions

Some technical measurable functions appear throughout the paper so frequently that, for convenience of the reader, we decided to collect them in this section together. However, the reader may skip this part now without any harm and come back to it when it is appropriately needed.

First, define

$$D_\varphi(x) := (\deg T^n_x)^{-1} \exp(-2\|S_n\varphi_x\|_\infty)$$

with $n = n(\xi)$ being the index given by the topological exactness condition (cf. (2.1)). Then, let $j = j(x)$ be the number given by Lemma 2.1 and define

$$C_\varphi(x) := e^{Q_{j-\gamma}} \deg(T^n_{j-\gamma}) \max \left\{ \exp(2\|S_k\varphi_{x-k}\|_\infty) : 0 \leq k \leq j \right\} \geq 1.$$ 

Now let $s > 1$. Put

$$C_{\min}(x) := e^{-sQ_x} e^{-\|S_j\varphi_{x-j}\|_\infty} \leq 1$$

and

$$C_{\max}(x) := e^{sQ_x} \deg(T^n_x) \exp(2\|S_n\varphi_x\|_\infty)$$

where $n := n(\xi)$. Then we define

$$\beta_x(s) := \frac{C_{\min}(x)}{C_\varphi(x)} \inf_{r \in [0, \xi]} \frac{1 - \exp\left( -(s - 1)H_{x-1}\gamma^{\nu_\alpha}\right)}{1 - \exp(-2sQ_xr^\alpha)}.$$
3. Transfer Dual Operators

In order to prove Theorem 3.1 we fix a point \( x_0 \in X \) and, as the first step, we reduce the base space \( X \) to the orbit \( O_{x_0} = \{ \theta^n(x_0), n \in \mathbb{Z} \} \).

The motivation for this is that then we can deal with a sequentially topological compact space on which the transfer (or related) operators act continuously. Our conformal measure then can be produced, for example, by the methods of the fixed point theory, similarly as in the deterministic case.

Removing a set of measure zero, if necessary, we may assume that this orbit is chosen so that all the involved measurable functions are defined and finite on the points of \( O_{x_0} \). For every \( x \in O_{x_0} \), let \( \varphi_x \in C(J_x) \) be the continuous potential of the transfer operator \( L_x : C(J_x) \to C(J_{\theta(x)}) \) which has been defined in (6.2).

Proposition 3.4. There exists \( (\nu_x)_{x \in O_{x_0}} \in \mathcal{P}(O_{x_0}) \) such that
\[
L_x^* \nu_{\theta(x)} = \lambda_x \nu_x \quad \text{for every} \quad x \in O_{x_0},
\]
where
\[
(3.1) \quad \lambda_x := L_x^*(\nu_{\theta(x)})(1) = \nu_{\theta(x)}(L_x 1).
\]

Proof. Let \( C^*(J_x) \) be the dual space of \( C(J_x) \) equipped with the weak* topology. Consider the product space
\[
\mathcal{D}(O_{x_0}) := \prod_{x \in O_{x_0}} C^*(J_x)
\]
with the product topology. The origin has a local basis of the sets of the form
\[
\prod_{x \in O_{x_0}} B_x
\]
where for finitely many \( x \), \( B_x \) is a ball in \( C^*(J_x) \) centered at origin and the rests of \( B_x \) are equal to \( C^*(J_x) \). Therefore, \( \mathcal{D}(O_{x_0}) \) is a locally convex topological space.

Now, let \( \mathcal{M}^1(J_x) \) be the compact subset of \( C^*(J_x) \) containing all probability measures. Then the set
\[
\mathcal{P}(O_{x_0}) := \prod_{x \in O_{x_0}} \mathcal{M}^1(J_x)
\]
is also compact. A simple observation is that the map \( \Psi_x : \mathcal{M}^1(J_{\theta(x)}) \to \mathcal{M}^1(J_x) \) defined by
\[
\Psi_x(\nu_{\theta(x)}) = \frac{L_x^* \nu_{\theta(x)}}{L_x^* \nu_{\theta(x)}(1)}
\]
is weakly continuous. Consider then the global map $\Psi : \mathcal{P}(O_{x_0}) \to \mathcal{P}(O_{x_0})$ given by

$$
\nu = (\nu_x)_{x \in O_{x_0}} \mapsto \Psi(\nu) = (\Psi_x \nu_{\theta(x)})_{x \in O_{x_0}}.
$$

Weak continuity of the $\Psi_x$ implies continuity of $\Psi$ with respect to the coordinate convergence. Since the space $\mathcal{P}(O_{x_0})$ is a compact subset of a locally convex topological space, we can apply the Schauder-Tychonoff fixed point theorem to get $\nu \in \mathcal{P}(O_{x_0})$ fixed point of $\Psi$, i.e.

$$
\mathcal{L}^* \nu_{\theta(x)} = \lambda_x \nu_x \quad \text{where} \quad \lambda_x = \mathcal{L}^* \nu_{\theta(x)}(1) = \nu_{\theta(x)}(\mathcal{L}_x(1))
$$

for every $x \in O_{x_0}$.

\[\square\]

\textbf{Remark 3.5.} Set

$$
\lambda^n_x := \lambda_x \lambda_{\theta(x)} \cdots \lambda_{\theta^{n-1}(x)}.
$$

For the iterated dual operator $(\mathcal{L}^n_x)^* = \mathcal{L}^*_x \circ \cdots \circ \mathcal{L}^*_{\theta^{n-1}(x)}$ we get from Proposition 3.4 that

$$
(\mathcal{L}^n_x)^* \nu^n_{\theta(x)} = \lambda^n_x \nu_x
$$

and (3.1) implies that

$$
\inf_{y \in J_x} e^{\mathcal{L}^{n}(y)\nu_{\theta(x)}} \leq \lambda_x \leq \|\mathcal{L}_x\|_{\infty}.
$$

A straightforward adaptation of the proof of Proposition 2.2 in [8] leads to the following statement equivalent to Proposition 3.3.

\textbf{Lemma 3.6.} For every $n \geq 0$ there exists a finite partition $\{A_k\}$ of $J_x$ into measurable sets such that $T^n|_{A_k}$ is a measurable isomorphism. In addition, for every measurable set $A \subset J_x$,

$$
\nu_{\theta^n(x)}(T^n(A)) \leq \sum_k \nu_{\theta^n(x)}(T^n(A \cap A_k)) = \lambda^n_x \int_A e^{-S_n \varphi} d\nu_x.
$$

If $T^n|_A$ is one-to-one, then

$$
\nu_{\theta^n(x)}(T^n(A)) = \lambda^n_x \int_A e^{-S_n \varphi} d\nu_x.
$$

Here is one more useful estimation.

\textbf{Lemma 3.7.} For every $x \in O_{x_0}$ and $n \geq 1$,

$$
\inf_{z \in J_x} \exp \left( S_n \varphi_x(z) \right) \leq \frac{\lambda^n_x}{\deg(T^n_x)} \leq \sup_{z \in J_x} \exp \left( S_n \varphi_x(z) \right).
$$

Moreover, for every $z \in J_x$ and every $r > 0$,

$$
\nu_x(B(z, r)) \geq D(x, r),
$$

where

$$
D(x, r) := (\deg(T^n_x))^{-1} \inf_{z \in J_x} \exp \left( \inf_{a \in B(z, r)} S_n \varphi_x(a) - \sup_{b \in B(z, r)} S_n \varphi_x(b) \right)
$$

with $N = n_r(x)$ being the index given by Lemma 2.1. It follows that the set $J_x$ is a topological support of $\nu_x$. In particular,

$$
\nu_x(B(z, \xi)) \geq D_\xi(x)
$$

where $D_\xi(x)$ is defined by (2.1).
Proof. Since
\[ \nu_{\theta^n}(z)(\mathcal{L}_x^n 1) = ((\mathcal{L}_x^n)^n \nu_{\theta^n}(z))(1) = \lambda_x^n \nu_x(1) = \lambda_x^n, \]
we get that
\[ \inf_{z \in J_x} \exp \left( S_n \varphi_x(z) \right) \leq \frac{\lambda_x^n}{\deg(T_x^n)} \leq \sup_{z \in J_x} \exp \left( S_n \varphi_x(z) \right). \]
Now fix an arbitrary \( z \in J_x \) and \( r > 0 \). Put \( n = n_r(x) \) (see Lemma 2.1). Then, by Lemma 3.6
\[ \nu_x(B(z, r)) \lambda_x^n \sup_{a \in B(z, r)} e^{-S_n \varphi_x(a)} \geq \lambda_x^n \int_{B(z, r)} e^{-S_n \varphi_x} d\nu_x \geq 1. \]
Thus,
\[ \nu_x(B(z, r)) \geq (\lambda_x^n)^{-1} \exp \left( \inf_{a \in B(z, r)} S_n \varphi_x(a) \right) \]
\[ \geq (\deg(T_x^n))^{-1} \exp \left( \inf_{a \in B(z, r)} S_n \varphi_x(a) - \sup_{b \in B(z, r)} S_n \varphi_x(b) \right). \]
\[ \square \]

4. Invariant density

Consider now the normalized operator \( \hat{\mathcal{L}} \) given by
\[ \hat{\mathcal{L}}_x = \lambda_x^{-1} \mathcal{L}_x, \quad x \in X. \]

Proposition 3.8. For every \( x \in O_{\theta^0}, \) there exists a function \( q_x \in H^\alpha(J_x) \) such that
\[ \hat{\mathcal{L}}_x q_x = q_x(\theta(x)) \quad \text{and} \quad \int_{J_x} q_x d\nu_x = 1. \]
In addition, \( q_x(z_1) \leq \exp\{Q_x \varphi^n(z_1, z_2)\} q_x(z_2) \) for all \( z_1, z_2 \in J_x \) with \( \varphi(z_1, z_2) \leq \xi, \)
and
\[ 1/C_x(x) \leq q_x \leq C_x(x), \]
where \( C_x \) was defined in (2.2).

In order to prove this statement we first need a good estimate for the iterates of the normalized operator applied to the constant function 1.

Lemma 3.9. For all \( w_1, w_2 \in J_x \) and \( n \geq 1 \)
\[ \frac{\mathcal{L}_x^n 1(w_1)}{\mathcal{L}_x^n 1(w_2)} = \frac{\mathcal{L}_x^n 1(w_1)}{\mathcal{L}_x^n 1(w_2)} \leq C_x(x), \]
where \( C_x \) is given by (2.2). If in addition \( \varphi(w_1, w_2) \leq \xi, \) then
\[ \frac{\mathcal{L}_x^n 1(w_1)}{\mathcal{L}_x^n 1(w_2)} \leq \exp\{Q_x \varphi^n(w_1, w_2)\}. \]
Moreover,
\[ 1/C_x(x) \leq \hat{\mathcal{L}}_x^n 1(w) \leq C_x(x) \quad \text{for every} \quad w \in J_x \quad \text{and} \quad n \geq 1. \]
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**Proof.** First, (4.4) immediately follows from Lemma 2.3. Notice also that
\[
\exp \left( Q_x \theta^n (w_1, w_2) \right) \leq \exp Q_x
\]
since diam(\( J_x \)) \leq 1. The global version of (4.5) can be proved as follows. If \( n = 0, \ldots, j(x) \), then for every \( w_1, w_2 \in J_x \),
\[
\mathcal{L}^n_{x-n} 1(w_1) \leq \frac{\deg(T^n_{x-n}) \exp(\|S^n \varphi x_{-n}\|_{\infty}) \mathcal{L}^n_{x-n} 1(w_2)}{\exp(-\|S^n \varphi x_{-n}\|_{\infty})} \leq C_\varphi(x) \mathcal{L}^n_{x-n} 1(w_2).
\]
Next, let \( n > j := j(x) \). Take \( w'_1 \in T_{-j}^n (w_1) \) such that
\[
e^{S_j \varphi(w')} \mathcal{L}^{n-j}_{x-n} 1(w'_1) = \sup_{y \in T_{-j}^n (w_1)} \left( e^{S_j \varphi(y)} \mathcal{L}^{n-j}_{x-n} 1(y) \right)
\]
and \( w'_2 \in T_{-n}^j (w_2) \) such that \( q_{x-}(w'_1, w'_2) \leq \xi \). Then, by (4.4) and (4.6),
\[
\mathcal{L}^n_{x-n} 1(w_1) = \mathcal{L}^j_{x-j} (\mathcal{L}^{n-j}_{x-n} 1)(w_1)
\]
\[
\leq \deg(T^n_{x-n}) e^{S_j \varphi(w')} \mathcal{L}^{n-j}_{x-n} 1(w'_1)
\]
\[
\leq \deg(T^n_{x-n}) e^{S_j \varphi(w')} e^{Q_x-n} \mathcal{L}^{n-j}_{x-n} 1(w'_2)
\]
\[
\leq C_\varphi(x) \mathcal{L}^{n}_{x-n} 1(w_2).
\]
This shows (4.6). By Proposition 3.3,
\[
\int_{J_x} \mathcal{L}^n_{x-n} 1(1) d\nu_x = \int_{J_x} 1 d\nu_{x-n} = 1,
\]
we can find \( w, w' \in J_x \) such that \( \mathcal{L}^n_{x-n} 1(w) \leq 1 \) and \( \mathcal{L}^n_{x-n} 1(w') \geq 1 \). Therefore, by the already proved part of this lemma, we get, for every \( w \in J_x \) and every \( n \geq 1 \), that
\[
1/C_\varphi(x) \leq \mathcal{L}^n_{x-n} 1(w) \leq C_\varphi(x).
\]
\( \square \)

**Proof of Proposition 3.8.** Let \( x \in \mathcal{O}_{x_0} \). Then by Lemma 3.4, for every \( k \geq 0 \) and all \( w_1, w_2 \in J_x \) with \( g(w_1, w_2) \leq \xi \), we have that
\[
| \mathcal{L}^k_{x-k} 1(w_1) - \mathcal{L}^k_{x-k} 1(w_2) | \leq C_\varphi(x) 2Q_x \theta^n (w_1, w_2)
\]
and \( 1/C_\varphi(x) \leq \mathcal{L}^k_{x-k} 1 \leq C_\varphi(x) \). It follows that the sequence
\[
q_{x,n} := \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k_{x-k} 1, \quad n \geq 1,
\]
is equicontinuous for every \( x \in \mathcal{O}_{x_0} \). Therefore, there exists a sequence \( n_j \to \infty \) such that \( q_{x,n_j} \to q_x \) uniformly. For \( k \geq 0 \) we define \( q_{x,k,n_j} = \mathcal{L}^k_{x-k} q_{x} \). For \( k < 0 \), we proceed inductively. Let \( n_{j,k} \to \infty \) be a sequence such that \( q_{x-k,n_{j,k}} \to q_{x-k} \) uniformly. Then the sequence \( n_{j,k} - 1 \) has a subsequence \( n_{j,k+1} \), such that \( q_{x-k-1,n_{j,k+1}} \to q_{x-k-1} \) uniformly. Since
\[
\mathcal{L}^k_{x-k-1} (q_{x-k-1,n-1}) - q_{x-k,n}
\]
\[
= \frac{1}{n} \mathcal{L}^k_{x-k-1} \left( \sum_{i=0}^{n-2} \mathcal{L}^i_{x-k-1} (1_{x-k-i}) \right) - \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}^k_{x-k-1} (1_{x-k-i})
\]
In fact all elements of \( \Lambda \) are \( T \)-invariant. So, if \( f \) is \( \nu \)-invariant, then \( f \) belongs to \( H \) and \( \mu \) can be defined by \( (2.1) \). Then, we provide sufficient measurability conditions for these fiber measures \( \nu \) and \( \mu \) to be integrable to produce global measures projecting on \( X \) to \( m \). The measure \( \mu \) defined by \( (2.1) \) is then \( T \)-invariant.

\[ \frac{\lambda_{i}}{\mu_{x}} \Theta \left( \mu_{x} \right) \]

3.10. Remark. Consider now the potential \( \hat{\phi} = \varphi + \log q_{x} - \log q_{x} \circ T - \log \lambda_{x} \) and the associated transfer operator

\[ L_{x} := L_{\hat{\phi},x} \]

\[ \mu_{x} := q_{x} \nu_{x}. \]

3.11. Remark. In Chapter 4 we provide sufficient measurability conditions for these fiber measures \( \nu \) and \( \mu \) to be integrable to produce global measures projecting on \( X \) to \( m \). The measure \( \mu \) defined by \( (2.1) \) is then \( T \)-invariant.

5. Levels of Positive Cones of Hölder Functions

For \( s \geq 1 \), set

\[ \Lambda_{s}^{x} = \left\{ g \in C(\mathcal{J}_{x}) : g \geq 0, \nu_{x}(g) = 1 \text{ and } g(w_{1}) \leq e^{sQ_{x} \phi(x,w_{1})} g(w_{2}) \right\} \]

\[ \text{for all } w_{1}, w_{2} \in \mathcal{J}_{x} \text{ with } g(w_{1}, w_{2}) \leq \xi. \]
Lemma 3.12. If \( g \geq 0 \) and for all \( w_1, w_2 \in J_x \) with \( g(w_1, w_2) \leq \xi \), we have
\[
g(w_1) \leq e^{sQ_x e^\alpha(w_1, w_2)} g(w_2),
\]
then
\[
v_\alpha(g) \leq sQ_x (\exp(sQ_x \xi^\alpha)) \xi^\alpha \|g\|_\infty.
\]

Proof. Let \( w_1, w_2 \in J_x \) be such that \( g(w_1, w_2) \leq \xi \). Without loss of generality we may assume that \( g(w_1) > g(w_2) \). Then \( g(w_1) > 0 \) and therefore, because of our hypothesis, \( g(w_2) > 0 \). Hence, we get
\[
\frac{|g(w_1) - g(w_2)|}{|g(z_2)|} = \frac{g(w_1)}{g(w_2)} - 1 \leq \exp\left(sQ_x e^\alpha(w_1, w_2)\right) - 1.
\]
Then
\[
|g(w_1) - g(w_2)| \leq sQ_x (\exp(sQ_x \xi^\alpha)) g^\alpha(w_1, w_2) \|g\|_\infty.
\]

Hence the set \( \Lambda^*_x \) is a level set of the cone defined in (2.1), that is
\[
\Lambda^*_x = C^*_x \cap \{g : v_\alpha(g) = 1\}.
\]
In addition, in the following lemma we show that this set is bounded in \( H^a(J) \).

Lemma 3.13. For a.e. \( x \in X \) and every \( g \in \Lambda^*_x \), we have \( \|g\|_\infty \leq C_{\text{max}}(x) \), where \( C_{\text{max}} \) is defined by (2.2).

Proof. Let \( g \in \Lambda^*_x \) and let \( z \in J_x \). Since \( g \geq 0 \) we get
\[
\int_{B(z, \xi)} g \, d\nu_z \leq \int_{J_x} g \, d\nu_x = 1,
\]
and therefore there exists \( b \in B(z, \xi) \) such that
\[
g(b) \leq \frac{1}{\nu_x(B(z, \xi))} \leq \frac{1}{D_\xi(x)}
\]
where the latter inequality is due to Lemma 3.11. Hence
\[
g(z) \leq e^{sQ_x e^\alpha(b, z)} g(b) \leq e^{sQ_x} \frac{e^{Q_x}}{D_\xi(x)} \leq C_{\text{max}}(x).
\]

A kind of converse to Lemma 3.12 is given by the following.

Lemma 3.14. If \( g \in H^a(J_x) \) and \( g \geq 0 \), then
\[
\frac{g + v_\alpha(g)/Q_x}{v_\alpha(g)/Q_x} \in \Lambda^1_x.
\]

Proof. Consider the function \( h = g + v_\alpha(g)/Q_x \). In order to get the inequality from the definition of \( \Lambda^*_x \), we take \( z_1, z_2 \in J_x \). If \( h(z_1) \leq h(z_2) \) then this inequality is trivial. Otherwise \( h(z_1) > h(z_2) \), and therefore
\[
\frac{h(z_1)}{h(z_2)} - 1 = \frac{|h(z_1) - h(z_2)|}{|h(z_2)|} \leq \frac{v_\alpha(g) e^\alpha(z_1, z_2)}{v_\alpha(g)/Q_x} = Q_x e^\alpha(z_1, z_2).
\]

An important property of the sets \( \Lambda^*_x \) is their invariance with respect to the normalized operator \( \tilde{\mathcal{L}}_x = \lambda_x^{-1} L_x \).
Lemma 3.15. Let \( g \in \Lambda^s_x \). Then, for every \( n \geq 1 \),
\[
\frac{\tilde{L}^n_x g(w_1)}{\tilde{L}^n_x g(w_2)} \leq \exp \left( sQ_{x_n}g^s(w_1, w_2) \right), \quad w_1, w_2 \in \mathcal{J}_{g^n(x)} \text{ with } g(w_1, w_2) \leq \xi.
\]
Consequently \( \tilde{L}^n_x(\Lambda^s_x) \subset \Lambda^s_{g^n(x)} \) for a.e. \( x \in X \) and all \( n \geq 1 \).

Notice that the constant function \( 1 \in \Lambda^s_x \) for every \( s \geq 1 \). For this particular function our distortion estimation was already proved in Lemma 3.9.

Proof. First, let \( i = j(x) \). Since
\[
\int_{\mathcal{J}_{g_{x^{-1}}}^i} g \, dv_{x_{x^{-1}}} = 1,
\]
there exists \( a \in \mathcal{J}_{g_{x^{-1}}}^i \) such that \( g(a) \geq 1 \). By definition of \( j(x) \), for any point \( w \in \mathcal{J}_x \), there exists \( z \in T_{x^{-1}}(x) \cap B(a, \xi) \). Therefore
\[
\tilde{L}^i_{x_{x^{-1}}} g(w) \geq e^{S_i g_{x_{x^{-1}}}^i(z)} g(z) \geq e^{S_i g_{x_{x^{-1}}}^i(z)} e^{-sQ_x g(a)} \geq C_{\min}(x).
\]
The case \( i > j(x) \) follows from the previous one, since \( \tilde{L}^{i-j(x)}_{x_{x^{-1}}} g_{x_{x^{-1}}} \in \Lambda_{x^{-1}(x)} \).

6. Exponential Convergence of Transfer Operators

Lemma 3.16. For every \( i \geq j(x) \) and \( g \in \Lambda^s_{x_{x^{-1}}} \),
\[
\tilde{L}^i_{x_{x^{-1}}} g \geq C_{\min}(x),
\]
where \( C_{\min} \) is given by (3.3).

Proof. First, let \( i = j(x) \). Since
\[
\int_{\mathcal{J}_{g_{x^{-1}}}^i} g \, dv_{x_{x^{-1}}} = 1,
\]
there exists \( a \in \mathcal{J}_{g_{x^{-1}}}^i \) such that \( g(a) \geq 1 \). By definition of \( j(x) \), for any point \( w \in \mathcal{J}_x \), there exists \( z \in T_{x^{-1}}(x) \cap B(a, \xi) \). Therefore
\[
\tilde{L}^i_{x_{x^{-1}}} g(w) \geq e^{S_i g_{x_{x^{-1}}}^i(z)} g(z) \geq e^{S_i g_{x_{x^{-1}}}^i(z)} e^{-sQ_x g(a)} \geq C_{\min}(x).
\]
The case \( i > j(x) \) follows from the previous one, since \( \tilde{L}^{i-j(x)}_{x_{x^{-1}}} g_{x_{x^{-1}}} \in \Lambda_{x^{-1}(x)} \).
PROOF. By Lemma 3.10, we have
\[ \tilde{\varphi}_x g_{x,n} \geq C_{\min}(x). \]
Then by (3.2) for all \( w, z \in Y_x \) with \( q_x(z, w) < \xi \),
\[
\begin{align*}
\beta_x \left( \exp \left( sQ_x g_x^0(z, w) \right) q_x(z) - q_x(w) \right) & \leq \beta_x \left( \exp \left( sQ_x g_x^0(z, w) \right) \right) q_x(z) \\
& \leq \beta_x \left( \exp \left( sQ_x g_x^0(z, w) \right) - \exp \left( - sQ_x g_x^0(z, w) \right) \right) C_{\varphi}(x) \\
& \leq \beta_x \left( \exp \left( -2sQ_x g_x^0(z, w) \right) \right) \exp \left( sQ_x g_x^0(z, w) \right) \\
& \leq \left( \exp \left( sQ_x g_x^0(z, w) \right) - \exp \left( (sQ_x - H_{x-1} \gamma_{x-1}^{-\alpha}) g_x^0(z, w) \right) \right) \tilde{\varphi}_{x-1} g_{x-1}(z) \\
& \leq \left( \exp \left( sQ_x g_x^0(z, w) \right) - \exp \left( (sQ_{x-1} + H_{x-1} \gamma_{x-1}^{-\alpha}) g_x^0(z, w) \right) \right) \tilde{\varphi}_{x-1} g_{x-1}(z) \cdot \tilde{\varphi}_{x-1} g_{x-1}(w).
\end{align*}
\]
Since by (5.2), for \( h \in \Lambda^s_{x-1} \),
\[
\begin{align*}
\tilde{\varphi}_{x-1} h(z) & \leq \exp \left( (sQ_{x-1} + H_{x-1}) \gamma_{x-1}^{-\alpha} g_x^0(z, w) \right) \tilde{\varphi}_{x-1} h(w), \\
\tilde{\varphi}_{x-1} g_{x-1}(z) & \leq \exp \left( (sQ_{x-1} + H_{x-1}) \gamma_{x-1}^{-\alpha} g_x^0(z, w) \right) \tilde{\varphi}_{x-1} g_{x-1}(w).
\end{align*}
\]
Then we have that
\[
\begin{align*}
\beta_x \left( \exp \left( sQ_x g_x^0(z, w) \right) q_x(z) - q_x(w) \right) & \leq \exp \left( sQ_x g_x^0(z, w) \right) \tilde{\varphi}_{x-1} g_{x-1}(z) - \tilde{\varphi}_{x-1} g_{x-1}(w)
\end{align*}
\]
and then
\[
\begin{align*}
\tilde{\varphi}_{x-1} g_{x-1}(w) - \beta_x q_x(w) & \leq \exp \left( sQ_x g_x^0(z, w) \right) \left( \tilde{\varphi}_{x-1} g_{x-1}(z) - \beta_x q_x(z) \right).
\end{align*}
\]
Moreover, \( \beta_x q_x \leq C_{\min}(x) \leq \tilde{\varphi}_{x-1} g_{x-1} \). Hence the function
\[
h_x := \frac{\tilde{\varphi}_{x-1} g_{x-1} - \beta_x q_x}{1 - \beta_x} \in \Lambda^s_x.
\]
\[
\square
\]
We are now ready to establish the first result about exponential convergence.

PROPOSITION 3.18. Let \( s > 1 \) and let \( g \in \mathcal{F} \rightarrow \mathbb{R} \). There exist \( B < 1 \) and a measurable function \( A : X \rightarrow (0, \infty) \) such that for a.e. \( x \in X \) and for every \( N \geq 1 \), if \( g_{x,N} \in \Lambda^s_{x-N} \), then
\[
\| (\tilde{\varphi}^N g)_x - q_x \|_\infty = \| \tilde{\varphi}^N_{x-N} g_{x-N} - q_x \|_\infty \leq A(x) B^N.
\]

PROOF. Fix \( x \in X \). Put \( g_n := g_{x,n}, \beta_n := \beta_{x,n}, A^s_n := \Lambda^s_{x-n} \) and \( (\tilde{\varphi}^n g)_x := (\tilde{\varphi}^n g)_{x,n} \). Let \( (i(n))_{n=1}^{\infty} \) be a sequence of integers such that \( i(n + 1) \geq j(x_{-S(n)}) \), where \( S(n) = \sum_{k=1}^{n} i(k), n \geq 1 \), and where \( S(0) = 0 \). If \( g_{-S(n)} \in \Lambda^s_{-S(n)} \), then Lemma 3.17 yields the existence of a function \( h_{n-1} \in \Lambda^s_{-S(n-1)} \) such that
\[
\begin{align*}
(\tilde{\varphi}^n g)_{-S(n-1)} & = \beta_{-S(n-1)} q_{-S(n-1)} + (1 - \beta_{-S(n-1)}) h_{n-1} \\
& = (1 - (1 - \beta_{-S(n-1)})) q_{-S(n-1)} + (1 - \beta_{-S(n-1)}) h_{n-1}.
\end{align*}
\]
Since
\[
\left( \tilde{\mathcal{L}}^{i(n)+i(n-1)} \right)_{-S(n-2)} = \\
= \left( \tilde{\mathcal{L}}^{i(n-1)} \left( \tilde{\mathcal{L}}^{i(n)} g \right) \right)_{-S(n-2)} \\
= \left( \tilde{\mathcal{L}}^{i(n-1)} \left( \beta_{-S(n-1)} q_{-S(n-1)} + (1 - \beta_{-S(n-1)}) h_{n-1} \right) \right)_{-S(n-2)} \\
= \beta_{-S(n-1)} q_{-S(n-2)} + (1 - \beta_{-S(n-1)}) \left( \tilde{\mathcal{L}}^{i(n-1)}(h_{n-1}) \right)_{-S(n-2)}
\]
it follows again from Lemma 3.17 that there is \( h_{n-2} \in \Lambda^s_{-S(n-2)} \) such that
\[
\left( \tilde{\mathcal{L}}^{i(n)+i(n-1)} \right)_{-S(n-2)} = \\
= \beta_{-S(n-1)} q_{-S(n-2)} + (1 - \beta_{-S(n-1)}) \left( \beta_{-S(n-2)} q_{S(n-2)} + (1 - \beta_{-S(n-2)}) h_{n-2} \right) \\
= \left( 1 - (1 - \beta_{-S(n-2)})(1 - \beta_{-S(n-1)}) \right) q_{S(n-2)} + (1 - \beta_{-S(n-2)})(1 - \beta_{-S(n-1)}) h_{n-1}. \\
\]
It follows now by induction that there exists \( h \in \Lambda^s_x \) such that
\[
\left( \tilde{\mathcal{L}}^{S(n)} g \right)_x = \left( \tilde{\mathcal{L}}^{i(n)+...+i(1)} g \right)_x = (1 - \Pi^{(n)}_x) q_x + \Pi^{(n)}_x h \\
\]
where we set
\[
\Pi^{(n)}_x = \prod_{k=0}^{n-1} (1 - \beta_{x,-S(k)}).
\]
Since \( h \in \Lambda^s_x \), we have \( |h| \leq C_{\max}(x) \). Therefore,
\[
(6.1) \quad \left| \left( \tilde{\mathcal{L}}^{S(n)} g \right)_x - \left( 1 - \Pi^{(n)}_x \right) q_x \right| \leq C_{\max}(x) \Pi^{(n)}_x,
\]
if \( g_{-S(n)} \in \Lambda^s_{-S(n)} \).

By measurability of \( \beta \) and \( j \) one can find \( M > 0 \) and \( J \geq 1 \) such that the set
\[
G := \{ x : \beta_x \geq M \text{ and } j(x) \leq J \}
\]
has a positive measure larger than or equal to 3/4. Now, we will show that for a.e. \( x \in X \) there exists a sequence \((n_k)_{k=0}^\infty \) of non-negative integers such that \( n_0 = 0 \), for \( k > 0 \), we have that \( x_{-Jn_k} \in G \), and
\[
(6.3) \quad \# \{ n : 0 \leq n < n_k \text{ and } x_{-Jn} \in G \} = k - 1.
\]
Indeed, applying Birkhoff’s Ergodic Theorem to the mapping \( \theta^{-J} \) we have that for almost every \( x \in X \),
\[
\lim_{n \to \infty} \frac{\# \{ 0 \leq m \leq n - 1 : \theta^{-Jm}(x) \in G \}}{n} = \mathcal{E}(1_G|\mathcal{I}_J)(x),
\]
where \( \mathcal{E}(1_G|\mathcal{I}_J) \) is the conditional expectation of \( 1_G \) with respect to the \( \sigma \)-algebra \( \mathcal{I}_J \) of \( \theta^{-J} \)-invariant sets. Note that if a measurable set \( A \) is \( \theta^{-J} \)-invariant, then set \( \cup_{j=0}^{J-1} \theta^j(A) \) is \( \theta^{-1} \)-invariant. If \( m(A) > 0 \), then from ergodicity of \( \theta^{-1} \) we get that \( m(\cup_{j=0}^{J-1} \theta^j(A)) = 1 \), and then by invariantness of the measure \( m \), we
conclude that \( m(A) \geq 1/J \). Hence we get that for almost every \( x \) the sequence \( n_k \) is infinite and

\[
(6.4) \quad \lim_{k \to \infty} \frac{k}{n_k} \geq \frac{3}{4J}.
\]

Fix \( N \geq 0 \) and take \( l \geq 0 \) so that \( Jn_l \leq N \leq Jn_{l+1} \). Define a finite sequence \( (S(k))_{k=1}^l \) by \( S(k) := Jn_k \) for \( k < l \) and \( S(l) := N \), and observe that by \( (6.4) \), we have \( N \leq Jn_{l+1} \leq 4J^2l \). Then \( (6.4) \) and \( (6.5) \) give

\[
\| \hat{L}^N \alpha \|_\infty \leq \left\| \hat{L}^N \alpha \right\|_\infty + \| \varphi(x) \|_\infty \leq (1 - \frac{\alpha}{\phi})^{\| \varphi(x) \|_\infty}.
\]

This establishes our proposition with \( B = \sqrt{4J - M} \) and

\[
A(x) := \max\{2C_{\text{max}}(x)B^{-Jk^*}, (C_{\varphi}(x) + C_{\text{max}}(x))\},
\]

where \( k^*_x \) is a measurable function such that for all \( k \geq k^*_x \), we have

\[
\frac{k}{n_k} \geq \frac{1}{2J}.
\]

\( \square \)

3. THE RPF–THEOREM

LEMMA 3.19. Let \( s > 1 \) and let \( g : J \to \mathbb{R} \) be any function such that \( g_x \in \mathcal{H}^s(J) \). Then, with the notation of Proposition 3.18, we have

\[
\left\| \hat{L}^n \alpha \right\|_\infty \leq C_{\varphi}(\theta^n(x)) \left( \int |g_x| d\mu_x + 4 \frac{\nu_\alpha(g_x)}{Q_x} \right) A(\theta^n(x))B^n.
\]

PROOF. Fix \( s > 1 \). First suppose that \( g_x \geq 0 \). Consider the function

\[
h_x = \frac{g_x + \nu_\alpha(g_x)}{\Delta_x} \quad \text{where} \quad \Delta_x := \nu_x(g_x) + \nu_\alpha(g_x)/Q_x.
\]

It follows from Lemma 3.14 that \( h_x \) belongs to the set \( \Lambda^*_x \) and from Proposition 3.18 we have

\[
(6.5) \quad \left\| \hat{L}^n \alpha \right\|_\infty \leq \left\| \Delta_x \hat{L}^n \alpha \right\|_\infty \leq \left\| \Delta_x \hat{L}^n h_x - \frac{\nu_\alpha(g_x)}{Q_x} \hat{L}^n 1_x \right\|_\infty \leq \left( \Delta_x + \frac{\nu_\alpha(g_x)}{Q_x} \right) A(\theta^n(x))B^n.
\]
Then applying this inequality for $g_x q_x$ and using (6.2) we get
\[
\left\| \hat{L}_x^n g_x - \left( \int g_x \, d\mu_x \right) 1_{\theta^n(x)} \right\|_\infty \leq \\
\leq \frac{1}{q_{\theta^n(x)}} \cdot \left\| \hat{L}_x^n (g_x q_x) - \left( \int g_x q_x \, d\mu_x \right) q_{\theta^n(x)} \right\|_\infty \\
\leq C_{\hat{L}}(\theta^n(x)) \left( \int g_x \, d\mu_x + 2 v_{\alpha}(g_x q_x) \right) A(\theta^n(x)) B^n.
\]
So, we have the desired estimate for non-negative $g_x$. In the general case we can use the standard trick and write $g_x = g_x^+ - g_x^-$, where $g_x^+, g_x^- \geq 0$. Then the lemma follows. \qed

The estimate obtained in Lemma 3.19 is a bit inconvenient for it depends on the values of a measurable function, namely $C_{\hat{L}} A$, along the positive $\theta$-orbit of $x \in X$. In particular, it is not clear at all from this statement that the item (1) in Theorem 3.2 holds. In order to remedy this flaw, we prove the following proposition.

**Proposition 3.20.** For $m$-a.e. $x \in X$ and every $g_x \in \mathcal{C}(\mathcal{J}_x)$, we have
\[
\left\| \hat{L}_x^n g_x - \left( \int g_x \, d\mu_x \right) 1_{\theta^n(x)} \right\|_\infty \longrightarrow 0.
\]

**Proof.** First of all, we may assume without loss of generality that the function $g_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$ since every continuous function is a limit of a uniformly convergent sequence of Hölder functions. Now, let $A > 0$ be sufficiently big such that the set
\[
(6.6) \quad X_A = \{ x \in X ; A(x) \leq A \}
\]
has positive measure. Notice that, by ergodicity of $m$, some iterate of a.e. $x \in X$ is in the set $X_A$. Then by Poincaré recurrence theorem and ergodicity of $m$, for a.e. $x \in X$, there exists a sequence $n_j \to \infty$ such that $\theta^{n_j}(x) \in X_A$, $j \geq 1$. Therefore we get, for such an $x \in X_A$, from Lemma 3.19 that
\[
(6.7) \quad \left\| \hat{L}_x^n g_x - \left( \int g_x \, d\mu_x \right) 1_{\theta^{n_j}(x)} \right\|_\infty \leq \left( \int g_x \, d\mu_x + 4 v_{\alpha}(g_x q_x) \right)^{-1} A B^{n_j}
\]
for every $j \geq 1$. Finally, to pass from the subsequence $(n_j)$ to the sequence of all natural numbers we employ the monotonicity argument that already appeared in Walters paper [19]. Since $\hat{L}_x 1_x = 1_{\theta(x)}$, we have for every $w \in \mathcal{J}_{\theta(x)}$ that
\[
\inf_{z \in \mathcal{J}_x} g_x(z) \leq \sum_{z \in T_{\psi}(w)} g_x(z) e^{\psi(z)} \leq \sup_{z \in \mathcal{J}_x} g_x(z).
\]
Consequently the sequence
\[
(M_{n,x})_{n=0}^\infty = \left( \sup_{w \in \mathcal{J}_{\theta^{n_j}(x)}} \hat{L}_x^n g_x(w) \right)_{n=0}^\infty
\]
is weakly decreasing. Similarly we have a weakly increasing sequence
\[
(m_{n,x})_{n=0}^\infty = \left( \inf_{w \in \mathcal{J}_{\theta^{n_j}(x)}} \hat{L}_x^n g_x(w) \right)_{n=0}^\infty.
\]
The proposition follows since, by (6.7), both sequences converge on the subsequence $(n_j)$. \qed
7. Exponential Decay of Correlations

The last remaining unproven part of Theorem 3.2 is the item (3). We will prove it in the following proposition. For a function $f_x \in L^1(\mu_x)$ we denote its $L^1$-norm with respect to $\mu_x$ by

$$\|f_x\|_1 := \int |f_x| d\mu_x.$$

**Proposition 3.21.** There exists a $\theta$-invariant set $X' \subset X$ of full $m$-measure such that, for every $x \in X'$, every $f_{\theta^n(x)} \in L^1(\mu_{\theta^n(x)})$ and every $g_x \in \mathcal{H}^n(\mathcal{F}_x),$

$$|\mu_x((f_{\theta^n(x)} \circ T^n_x)g_x) - \mu_{\theta^n(x)}(f_{\theta^n(x)}g_x)| \leq A_x(g_x, \theta^n(x))B^n\|f_{\theta^n(x)}\|_1$$

where

$$A_x(g_x, \theta^n(x)) := C_x(\theta^n(x))\left(\int |g_x| d\mu_x + 4\frac{\nu_{\theta^n(x)}g_x}{Q_x}\right)A(\theta^n(x)).$$

**Proof.** Set $h_x = g_x - \int g_x d\mu_x$ and note that by (4.12) and (4.11) we have that

$$\mu_x((f_{\theta^n(x)} \circ T^n_x)g_x) - \mu_{\theta^n(x)}(f_{\theta^n(x)}g_x) =$$

$$(\theta^n)_{\mu_{\theta^n(x)}}(f_{\theta^n(x)}(T^n_xh_x)) - \mu_{\theta^n(x)}(f_{\theta^n(x)}g_x)$$

$$= \mu_{\theta^n(x)}(f_{\theta^n(x)}(T^n_xh_x)) - \mu_{\theta^n(x)}(f_{\theta^n(x)}g_x)$$

$$= \mu_{\theta^n(x)}(f_{\theta^n(x)}(T^n_xh_x)).$$

Since Lemma 3.19 yields

$$\|\hat{T}_x^n h_x\|_\infty \leq A_x(g_x, \theta^n(x))B^n,$$

it follows from (7.1) that

$$|\mu_x((f_{\theta^n(x)} \circ T^n_x)g_x) - \mu_{\theta^n(x)}(f_{\theta^n(x)}g_x)| \leq$$

$$\leq \int |f_{\theta^n(x)}(T^n_xh_x)| d\mu_{\theta^n(x)}$$

$$\leq A_x(g_x, \theta^n(x))B^n\int |f_{\theta^n(x)}| d\mu_{\theta^n(x)}.$$

□

**Corollary 3.22.** Let $f_{\theta^n(x)} \in L^1(\mu_{\theta^n(x)})$ and $g_x \in L^1(\mathcal{F}_x)$, where $x \in X'$ and $X'$ is the set given by Lemma 3.21. If $\|f_{\theta^n(x)}\|_1 \neq 0$ for all $n$, then

$$\frac{|\mu_x((f_{\theta^n(x)} \circ T^n_x)g_x) - \mu_{\theta^n(x)}(f_{\theta^n(x)}g_x)|}{\|f_{\theta^n(x)}\|_1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

**Remark 3.23.** Note that if $\|f_{\theta^n(x)}\|_1$ grows subexponentially, then

$$\mu_x((f_{\theta^n(x)} \circ T^n_x)g_x) - \mu_{\theta^n(x)}(f_{\theta^n(x)}g_x) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

This is for example the case if $x \mapsto \log \|f_x\|_1$ is $m$-integrable since Birkhoff’s Ergodic Theorem implies that $(1/n)\log \|f_{\theta^n(x)}\|_1 \rightarrow 0$ for a.e. $x \in X$. 


8. Uniqueness

Lemma 3.24. The family of measures \( x \mapsto \nu_x \) is uniquely determined by condition \((1.1)\).

Proof. Let \( \{ \nu_x \}_{x \in X} \) be a family of measures that satisfies \((1.1)\). Then
\[
\lambda_x = \lambda_x \nu_x(1_x) = \mathcal{L}_x^b \nu_{\varphi(x)}(1_x) = \nu_{\varphi(x)}(\mathcal{L}_x 1).
\]

Let
\[
\hat{\mathcal{L}}^n_x g_x(w) = \sum_{z \in T_{-n}^x(w)} g_x(z) \exp(S_n \hat{\varphi}(z)) = (\lambda_x^n q_{\varphi^n(x)}(w))^{-1} (\mathcal{L}_x^n g_x q_x)(w).
\]

Then, by Proposition 3.20 for a.e. \( x \in X \) and for all sequences \( w_n, z_n \in J_{\varphi^n(x)} \), we have
\[
\lim_{n \to \infty} \frac{\mathcal{L}_x^n g_x(w_n)}{\nu_x(1)} = \lim_{n \to \infty} \frac{\mathcal{L}_x^n (g_x/q_x)(w_n)}{\nu_x(1/q_x)(z_n)} = \frac{\nu_x(g_x)}{\nu_x(1)} = \nu_x(g_x).
\]

Hence the measures \( \nu_x \) are determined uniquely. \( \square \)

Lemma 3.25. There exists a unique function \( q \in \mathcal{C}^0(J) \) that satisfies \((1.2)\) for a.e. \( x \in X \).

Proof. The uniqueness follows from Proposition 3.18 since it shows that
\[
\lim_{n \to \infty} \hat{\mathcal{L}}^n_x 1 = \frac{\nu_{\varphi^n(x)}(1)}{\lambda_{x-n}^n} \quad \text{in the week* topology}.
\]

Note that \((8.1)\) can be reformulated. Define a sequence of measures in \( J_x \) by
\[
\nu_{x,n} := \sum_{y \in T_{-n}^x(w_n)} e^{S_n \varphi_x(y)} \delta_y / \mathcal{L}_x^n 1(w_n)
\]
where \( \delta_y \) is the Dirac measure. Then
\[
\nu_{x,n} \overset{n \to \infty}{\longrightarrow} \nu_x
\]
in the weak* topology.

9. Pressure function

The pressure function is defined by the formula
\[
x \mapsto P_x(\varphi) := \log \lambda_x.
\]

If it does not lead to misunderstanding, we will also denote the pressure function by \( P_x \). It is important to note that this function is generally non-constant, even for a.e. \( x \in X \). Actually, if the pressure function is a.e. constant, then the random map shares many properties with a deterministic system. This will be explained in detail in Chapter 7.

Note that it follows from \((8.1)\) and \((3.1)\) that
\[
\lambda_x = \nu_{\varphi(x)}(\mathcal{L}_x 1) = \lim_{n \to \infty} \frac{\mathcal{L}_x^{n+1} 1(w_{n+1})}{\mathcal{L}_{\varphi^n(x)} 1(w_{n+1})}.
\]
Then we can obtain an alternative definition of $P_x(\varphi)$, namely

$$
(9.2) \quad P_x(\varphi) = \log(\nu_{\theta(x)}(L_x1)) = \lim_{n \to \infty} \frac{1}{n} \log \frac{L_{x}^{n+1}1(w_{n+1})}{\theta(w_{n+1})} \nu_{\theta(x)}1(w_{n+1})
$$

where, for every $n \in \mathbb{N}, w_n$ is an arbitrary point from $J_{\theta^n(x)}$.

**Lemma 3.26.** For m.a.e. $x \in X$

$$
\lim_{n \to \infty} \frac{1}{n} S_n P_{x^{-n}} - \frac{1}{n} \log L_{x^{-n}}1_{x^{-n}}(w_n) = 0
$$

and every sequence $w_n \in J_x$.

**Proof.** By (4.2) and Proposition 3.18 we have that, for every $w \in J_x$ and every $n \in \mathbb{N}$,

$$
\frac{1}{C_\varphi(x)} - A(x) B^n \leq \frac{L_{x^{-n}}1_{x^{-n}}(w)}{\lambda_{x^{-n}}} \leq C_\varphi(x) + A(x) B^n.
$$

Then

$$
\log \left( \frac{1}{C_\varphi(x)} - A(x) \right) \leq \log L_{x^{-n}}1_{x^{-n}}(w) - \log \lambda_{x^{-n}} \leq \log \left( C_\varphi(x) + A(x) \right).
$$

**Lemma 3.27.** Then, for m.a.e. $x \in X$,

$$
\lim_{n \to \infty} \frac{1}{n} S_n P_{x} - \frac{1}{n} \log L_{x}1_{x}(y_n) = 0
$$

for every sequence $y_n \in J_{x^n}$.

**Proof.** Using Egorov’s Theorem and Lemma 3.26 we have that then for each $\delta > 0$ there exists a set $F_\delta$ such that $m(X \setminus X_\delta) < \delta$,

$$
\frac{1}{n} S_n P_{x^{-n}} - \frac{1}{n} \min_{y \in J_{x^n}} \log L_{x^{-n}}1_{x^{-n}}(y) \to 0 \quad \text{as } n \to \infty
$$

and

$$
\frac{1}{n} S_n P_{x^{-n}} - \frac{1}{n} \max_{y \in J_{x^n}} \log L_{x^{-n}}1_{x^{-n}}(y) \to 0 \quad \text{as } n \to \infty
$$

uniformly on $F_\delta$. Then the lemma follows from Birkhoff’s Ergodic Theorem.

**Lemma 3.28.** If there exist $g \in L^1(m)$ such that $\log \|L_x1\|_\infty \leq g(x)$, then

$$
\lim_{n \to \infty} \left\| \frac{1}{n} S_n P_x - \frac{1}{n} \log L_x^1 \right\|_\infty = 0.
$$

**Proof.** Let $F := F_\delta$ be the set from the proof of Lemma 3.27, let $x \in X_\delta$ and let $(n_j)$ be the visiting sequence. Let $j$ be such that $n_j < n \leq n_{j+1}$. Then, for $y \in J_{\theta^n(x)}$,

$$
(9.3) \quad \log L_x^1(y) \leq \log \|L_x^1\| + S_{n-n_j}g(\theta^{n_j}(x)).
$$

Now, let $h(x) := \|\varphi_x\|_\infty$. Since by (3.24) $-\log \lambda_x \leq \|\varphi_x\|_\infty$,

$$
-\log \lambda_x = -\log \lambda_x - \log \lambda_x^{n_j} \leq S_{n_j} P_x + S_{n-n_j} h(\theta^{n_j}(x)).
$$

Then by (3.23)

$$
\frac{1}{n} S_n P_x - \frac{1}{n} \log L_x^1(y_{x^n}) \leq \frac{1}{n_j} S_{n_j} P_x - \frac{1}{n_j} \log L_x^1(y_{x_{n_j}}) + \frac{1}{n} S_{n-n_j} (g+h)(\theta^{n_j}(x)).
$$
On the other hand, for $y \in \mathcal{F}_{\theta^n(x)}$, 
\[
\log \mathcal{L}_x^n \mathbf{1}(y) \geq \log \mathcal{L}_x^{n} \mathbf{1}(T_{\theta^n(x)}^{n-1} y) - S_{n} g(\theta^n(x))
\]
and by (3.2), 
\[
\log \lambda^n_x = \log \lambda^n_{x} - \log \lambda^n_{x} + \leq \log \|\mathcal{L}_x^{n} \mathbf{1}\| + S_{n} h(\theta^n(x)).
\]
Then the lemma follows by Birkhoff’s Ergodic Theorem.

\section{10. Gibbs property}

\begin{lemma}
Let $w \in \mathcal{F}_x$, set $y = (x,w)$ and let $n \geq 0$. Then 
\[
e^{-Q^n_{\theta^n(x)}} (D_\epsilon(\theta^n(x))) \leq \frac{\nu_{x}(T_{\theta^n(x)}(B(T^n(y),\xi)))}{\exp(S_n \varphi(y) - S_n P_\epsilon(\varphi))} \leq e^{-Q^n_{\theta^n(x)}}.
\]
\end{lemma}

\begin{proof}
Fix an arbitrary $z \in \mathcal{F}_x$ and set $y = (x,z)$. Then by Lemma 2.3 and Lemma 3.6 
\[
\frac{\nu_{x}(T_{\theta^n(x)}(B(T^n(y),\xi)))}{\exp(S_n \varphi(y) - S_n P_\epsilon(\varphi))} \leq \frac{\lambda_{n} \text{sup}_{z \in T^{-n}_{\theta^n(x)}} (B(T^n(y),\xi)) e^{S_n \varphi(z')}}{\lambda_{n}^n \exp(S_n \varphi(y))} \leq e^{-Q^n_{\theta^n(x)}}.
\]

On the other hand 
\[
\frac{\nu_{x}(T_{\theta^n(x)}(B(T^n(y),\xi)))}{\exp(S_n \varphi(y) - S_n P_\epsilon(\varphi))} \geq \frac{\lambda_{n} \text{inf}_{z \in T^{-n}_{\theta^n(x)}} (B(T^n(y),\xi)) e^{S_n \varphi(z')}}{\lambda_{n}^n \exp(S_n \varphi(y))} \geq \nu_{\theta^n(x)}(B(T^n(y),\xi)) e^{-Q^n_{\theta^n(x)}}.
\]

The lemma follows by (3.5).
\end{proof}

\begin{lemma}
Let $T : \mathcal{J} \to \mathcal{J}$ satisfy the condition of measurability of cardinality of covers and let $\{\nu_i, x\}$, where $i = 1,2$, be two Gibbs families with pressure functions $x \mapsto P_i, x$. Then, for a.e. $x$, the measures $\nu_{i, x}$ and $\nu_{2, x}$ are equivalent and 
\[
\lim_{k \to \infty} \frac{1}{n_k} S_{n_k} P_{1, x} = \lim_{k \to \infty} \frac{1}{n_k} S_{n_k} P_{2, x} = \lim_{k \to \infty} \frac{1}{n_k} S_{n_k} P_x
\]
where $(n_k) = (n_k(x))$ is the visiting sequence of an essential set.
\end{lemma}

\begin{proof}
Let $A$ be compact subset of $\mathcal{F}_x$ and let $\delta > 0$. By regularity of $\nu_{2, x}$ we can find $\epsilon > 0$ such that 
\[
(1.1) \quad \nu_{2, x}(B_x(A, \epsilon)) \leq \nu_{2, x}(A) + \delta.
\]
Now, let $N_x$ be a measurable function such that $\xi(N_x^{-1}) \leq \epsilon/2$. Set 
\[
A_{n}^i := \{y \in T_{x}^{-n}(y_{x}^i) : A \cap T_{y}^{-n}(B(y_{x}^i, \xi)) = \emptyset\}.
\]
Let $Z$ be a $L, N, D, D$-essential set of $a_x, N_x, D_1, D_2$ and let $(n_k) = (n_k(x))$ be the visiting sequence of $Z$. Fix $k \in \mathbb{N}$ and put $n = n_k(x)$. Then we have 
\[
A \subset \bigcup_{j=1}^{a_{x_n}} \bigcup_{y \in A_{n}^i} T_{y}^{-n}(B(y_{x}^i, \xi)) \subset B_x(A, \epsilon).
\]
Then it follows that

$$
\nu_{1,x}(A) \leq \sum_{j=1}^{a_{x_n}} \sum_{y \in A^j_k} \nu_{1,x} \left( T_y^{-n} B(y_{x_n}^j, \xi) \right)
$$

$$
\leq D_1(x)D \sum_{j=1}^{L} \sum_{y \in A^j_k} \exp(S_n \varphi(y) - S_n P_{1,x}(\varphi))
$$

(10.2)

Then by (10.1) and again by (10.3)

$$
\nu_{1,x}(A) \leq D_1(x)D \exp(S_n P_{2,x} - S_n P_{1,x}) \sum_{j=1}^{a_{x_n}} \sum_{y \in A^j_k} \exp(S_n \varphi(y) - S_n P_{2,x}(\varphi))
$$

$$
\leq D_1(x)D_2(x)D^2 \exp(S_n P_{2,x} - S_n P_{1,x}) \sum_{j=1}^{a_{x_n}} \sum_{y \in A^j_k} \nu_{2,x} \left( T_y^{-n} B(y_{x_n}^j, \xi) \right)
$$

$$
\leq D_1(x)D_2(x)D^2 L \exp(S_n P_{2,x} - S_n P_{1,x}) \nu_{2,x}(B(A, \varepsilon))
$$

$$
\leq D_1(x)D_2(x)D^2 L \exp(S_n P_{2,x} - S_n P_{1,x})(\nu_{2,x}(A + \delta),
$$

since for $y \neq y'$ such that $y, y' \in T_x^{-n}(y_{x_n}^j)$, we have that

$$
T_y^{-n} B(y_{x_n}^j, \xi) \cap T_{y'}^{-n} B(y_{x_n}^j, \xi) = \emptyset.
$$

Hence the difference $S_n P_{2,x} - S_n P_{1,x}$ is bounded from below by some constant, since otherwise taking $A = J_x$ we would obtain that $\nu_{1,x}(J_x) = 0$ on a subsequence of $(n_k)$ in (10.3). Similarly, exchanging $\nu_{1,x}$ with $\nu_{2,x}$ we obtain that $S_n P_{1,x} - S_n P_{2,x}$ is bounded from above. Then, letting $\delta$ go to zero, we have that $\nu_{1,x}$ and $\nu_{2,x}$ are equivalent.

Note that

$$
\exp(-S_n P_{1,x}) L_x^\varphi(y_n) = \sum_{y \in T_x^{-n}(y_n)} e^{S_n \varphi(y) - S_n P_{1,x}}
$$

$$
\leq D_1(x)D \sum_{y \in T_x^{-n}(y_n)} \nu_{1,x} \left( T_y^{-n} B(y_n, \xi) \right) \leq D_1(x)D \nu_{1,x}(J_x) = D_1(x)D.
$$

Then

$$
\frac{1}{n} \log L_x^\varphi(y_n) - \frac{1}{n} \log(D_1(x)D) \leq \frac{1}{n} S_n P_{1,x}.
$$

On the other hand, by (10.2), on the same subsequence

$$
1 = \nu_1^J(J_x) \leq D_1(x)D L \sum_{y \in T_x^{-n}(y_n)} e^{S_n \varphi(y) - S_n P_{1,x}}
$$

for some $y_n \in \{ y_{x_n}^1, \ldots, y_{x_n}^{a_{x_n}} \}$. Therefore, using Lemma 3.27 and the Sandwich Theorem, we have that, for $x \in X'_Z \cap X'_P$,

$$
\lim_{k \to \infty} \frac{1}{n_k} S_{n_k} P_{1,x} = \lim_{k \to \infty} \frac{1}{n_k} S_{n_k} P_x.
$$
Remark 3.31. Note that we cannot expect that $P_{1,x} = P_{2}(\varphi)$ $m$-almost surely. In order to see that, take any measurable function $x \mapsto g_x$. Then, for
$$P_{1,x} := P_{2}(\varphi) + g_x - g_{\theta(x)},$$
by Lemma 3.29
$$e^{g_x}e^{-Q_{g^n(x)} - g_{g^n(x)}}(D_{\xi}(g^n(x)))^{-1} \leq \frac{\nu_x(T^{-n}_y(B(T^n(y),\xi)))}{\exp(S_{\xi}(y) - S_{\xi}P_{1,x})} \leq e^{g_x}e^{Q_{g^n(x)} - g_{g^n(x)}}.$$
In this chapter we study measurability of the objects produced in the previous chapter. We do not know, for example, whether the family of measures $\nu_x$ represents the disintegration of a global Gibbs state $\nu$ with marginal $m$ on the fibered space $\mathcal{J}$. Therefore, we define abstract measurable expanding random maps for which the above measurabilities of $\lambda_x$, $q_x$, $\nu_x$, and $\mu_x$ are deduced from assumptions. Then, we can construct a Borel probability invariant ergodic measure on $\mathcal{J}$ for the skew-product transformation $T$ with Gibbs property and study the corresponding expected pressure.

Our settings are related to those of smooth expanding random mappings of one fixed Riemannian manifold from [11] and those of random subshifts of finite type whose fibers are subsets of are subsets of $\mathbb{N}^d$ from [1]. One possible extension of these works is to consider expanding random transformations on subsets of a fixed Polish space. A general framework for this was, in fact, prepared by Crauel in [6] although he did not study expanding maps. In Section 5 we show how Crauel’s random compact subsets of Polish spaces fit into our general framework and, therefore, our settings comprise all these options and go beyond.

The issue of measurability of $\lambda_x$, $q_x$, $\nu_x$, and $\mu_x$ does not seem to have been treated with care in the literature. As a matter of fact, it was not quite clear to us even for symbol dynamics or random expanding systems of smooth manifolds until, very recently, when Kifer’s paper [12] has appeared to take care of these issues.

1. Measurable Expanding Random Maps

Let $T : \mathcal{J} \to \mathcal{J}$ be a general expanding random map. Define

$$\pi_X : \mathcal{J} \ni (x, y) \mapsto x \in X.$$ 

Let $\mathcal{B} := \mathcal{B}_\mathcal{J}$ be a $\sigma$-algebra on $\mathcal{J}$ such that

1. $\pi_X$ and $T$ are measurable,
2. for every $A \in \mathcal{B}$, $\pi_X(A) \in \mathcal{F}$,
3. $\mathcal{B}|_{\mathcal{J}_x}$ is the Borel $\sigma$-algebra on $\mathcal{J}_x$.

By $L^0_{\mathcal{B}}(\mathcal{J})$ we denote the set of all $\mathcal{B}_\mathcal{J}$-measurable functions and by $C^0_{\mathcal{B}}(\mathcal{J})$ the set of all $\mathcal{B}_\mathcal{J}$-measurable functions $g$ such that $g_x \in C(\mathcal{J}_x)$.

**Lemma 4.1.** If $g \in C^0_{\mathcal{B}}(\mathcal{J})$, then $x \mapsto \|g_x\|_{\infty}$ is measurable.

**Proof.** The proof is a consequence of [2]. Indeed, let $(G_n)$ be an increasing approximation of $|g|$ by step functions. So let

$$G_n = \sum_{k=1}^{m} a_k 1_{A_k},$$
where \((a_k)\) is an increasing sequence of non-negative real numbers, and \(A_k\) are \(\mathcal{B}_J\)-measurable. Then, define
\[
X_m := \pi_X(A_m) \quad \text{and} \quad X_k := \pi_X(A_k) \setminus \bigcup_{j=k+1}^{m} \pi_X(A_j)
\]
where \(k = 1, \ldots, m - 1\). Let
\[
H_n(x) := \sum_{k=0}^{m} a_k \mathbb{1}_{X_k}(x) = \sup_{y \in \mathcal{J}_x} G_n(x, y).
\]
Then the sequence \((H_n)\) is increasing and pointwise converges to the function \(x \mapsto \|g_x\|_\infty\).

Then, \(L^1_m(\mathcal{J})\) is, by definition, the set of all \(g \in L^0_m(\mathcal{J})\), such that
\[
\int \|g_x\|_\infty dm(x) < \infty.
\]
We also define
\[
C^1_m(\mathcal{J}) := C^0_m(\mathcal{J}) \cap L^1_m(\mathcal{J})
\]
and
\[
\mathcal{H}^\alpha_m(\mathcal{J}) := C^1_m(\mathcal{J}) \cap \mathcal{H}^\alpha(\mathcal{J}).
\]
By \(\mathcal{M}^1(\mathcal{J})\) we denote the set of probability measures and by \(\mathcal{M}^1_m(\mathcal{J})\) its subset consisting of measures \(\nu'\) such that there exists a system of fiber measures \(\{\nu'_x\}_{x \in X}\) with the property that for every \(g \in L^1_m(\mathcal{J})\), the map
\[
x \mapsto \int_{\mathcal{J}_x} g_x \, d\nu'_x
\]
is measurable and
\[
\int_{\mathcal{J}} g \, d\nu' = \int_{X} \int_{\mathcal{J}_x} g_x \, d\nu'_x \, dm(x).
\]
Then
\[
(1.1) \quad m = \nu' \circ \pi_X^{-1}
\]
and the family \(\{\nu'_x\}_{x \in X}\) is the canonical system of conditional measures of \(\nu'\) with respect to the measurable partition \(\{\mathcal{J}_x\}_{x \in X}\) of \(\mathcal{J}\). It is also instructive to notice that in the case when \(\mathcal{J}\) is a Lebesgue space then \(1.1\) implies that \(\nu' \in \mathcal{M}^1_m(\mathcal{J})\).

The measure \(\mu' \in \mathcal{M}^1(\mathcal{J})\) is called \(T\)-invariant if \(\mu' \circ T^{-1} = \mu'\). If \(\mu' \in \mathcal{M}^1_m(\mathcal{J})\), then, in terms of the fiber measures, clearly \(T\)-invariance equivalently means that the family \(\{\mu'_x\}_{x \in X}\) is \(T\)-invariant; see Section \(1\) for the definition of \(T\)-invariance of a family of measures.

Fix \(\varphi \in \mathcal{H}^1_m(\mathcal{J})\). Then the general expanding random map \(T : \mathcal{J} \to \mathcal{J}\) is called a measurable expanding random map if the following conditions are satisfied.

Measurability of the Transfer Operator. The transfer operator is measurable i.e. for every \(g \in C^0_m(\mathcal{J})\), \(Lg \in C^0_m(\mathcal{J})\).

Integrability of the Logarithm of the Transfer Operator. The function \(X \ni x \mapsto \log \|L_x \mathbb{1}_x\|_\infty\) belongs to \(L^1(m)\).

We shall now provide a simple, easy to verify, sufficient condition for integrability of the logarithm of the transfer operator.

Lemma 4.2. If \(\log(\deg(T_x)) \in L^1(m)\), then \(x \mapsto \log \|L_x \mathbb{1}_x\|_\infty\) belongs to \(L^1(m)\).
2. MEASURABILITY

Proof. Recall that
\[ e^{-\|\varphi_x\|_\infty} \leq \sum_{T_y(z)=w} e^{\varphi_y(z)} \leq \deg(T_x) e^{\|\varphi_x\|_\infty}. \]
Hence
\[ -\|\varphi_x\|_\infty \leq \log \| L_x \|_\infty \leq \log(\deg(T_x)) + \|\varphi_x\|_\infty. \]
□

2. Measurability

Now, we assume that \( T : \mathcal{J} \to \mathcal{J} \) is a measurable expanding random map. In particular, the operator \( \mathcal{L} \) is measurable. Armed with these assumptions, we come back to the families of Gibbs states \( \{\nu_x\}_{x \in X} \) and \( \{\mu_x\}_{x \in X} \) whose pointwise construction was given in Theorem 3.1. Since we have already established good convergence properties, especially the exponential decay of correlations, it will follow rather easily that these families form in fact conditional measures of some measures \( \nu \) and \( \mu \) from \( \mathcal{M}_m^1(\mathcal{J}) \). As an immediate consequence of item 3 of Theorem 3.1, we get that the probability measure \( \mu \) is invariant under the action of the map \( T : \mathcal{J} \to \mathcal{J} \). All of this is shown in the following lemmas.

Lemma 4.3. For every \( g \in L^1(\mathcal{J}) \), the maps \( x \mapsto \nu_x(g_x) \) is measurable.

Proof. It follows from (8.1) that
\[ \lim_{n \to \infty} \| L^n_x g_x \|_\infty / \| L^n_x 1 \|_\infty = \nu_x(g_x). \]
Then measurability of \( x \mapsto \nu_x(g_x) \) is a direct consequence of measurability of the transfer operator. □

This lemma enables us to introduce the probability measure \( \nu \) on \( \mathcal{J} \) given by the formula
\[ \nu(g) = \int_X \int_{\mathcal{J}_x} g_x d\nu_x dm(x). \]
This measure, therefore, belongs to \( \mathcal{M}_m^1(\mathcal{J}) \).

Lemma 4.4. The map \( X \ni x \mapsto \lambda_x \in \mathbb{R} \) is measurable and the function \( q : \mathcal{J} \ni (x, y) \mapsto q_x(y) \) belongs to \( L_m^0(\mathcal{J}) \).

Proof. Since \( \nu \in \mathcal{M}_m^1(\mathcal{J}) \), measurability of \( \lambda \)'s follows from the formula (8.2) and measurability of the transfer operator. Then measurability of \( \lambda \)'s again together with measurability of the transfer operator and (8.2) imply measurability of \( q \). □

From this lemma and Lemma 4.3 it follows that we can define a measure \( \mu \) by the formula
\[ \mu(g) = \int_X \int_{\mathcal{J}_x} q_x g_x d\nu_x dm(x). \]
3. The expected pressure

The pressure function of a measurable expanding random map has the following important property.

**Lemma 4.5.** The pressure function $X \ni x \mapsto P_x(\varphi)$ is integrable.

**Proof.** It follows from the definition of the transfer operator, that

$$-\|\varphi_x\|_{\infty} \leq \log \nu_\theta(x)(L_x^1) \leq \log \|L_x^1\|_{\infty}. \quad (3.1)$$

Then, by (3.1) and integrability of the logarithm the transfer operator, the function $P_x(\varphi)$ is bounded above and below by integrable functions, hence integrable. □

Therefore, the expected pressure of $\varphi$ given by

$$\mathcal{E}P(\varphi) = \int_X P_x(\varphi)dm(x)$$

is well-defined.

The equality (8.1) yields alternative formulas for the expected pressure. In order to establish them, observe that by Birkhoff’s Ergodic Theorem

$$\mathcal{E}P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \lambda^n_x \text{ for a.e. } x \in X. \quad (3.2)$$

In addition, by (3.1),

$$\lambda^n_x = \lambda^n_x \nu_x(1) = \nu_{\theta^n(x)}(L^n_x(1)).$$

Thus, it follows that

$$\frac{1}{n} \log \lambda^n_x = \lim_{k \to \infty} \frac{1}{n} \log \frac{L^{k+n}_x \mathbf{1}_x(w_{k+n})}{L^n_x \mathbf{1}_{\theta^n(x)}(w_{k+n})}.$$

However, by Lemma 3.28 we can get even more interesting formula.

**Lemma 4.6.** For every $\varphi \in \mathcal{H}_m^\alpha(J)$ and for almost every $x \in X$

$$\mathcal{E}P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log L^n_x \mathbf{1}(w_n)$$

where the points $w_n \in J_{\theta^n(x)}$ are arbitrarily chosen.

4. Ergodicity of $\mu$

**Proposition 4.7.** The measure $\mu$ is ergodic.

**Proof.** Let $B$ be a measurable set such that $T^{-1}(B) = B$ and, for $x \in X$, denote by $B_x$ the set $\{y \in J_x : (x, y) \in B\}$. Then we have that $T^{-1}(B_{\theta(x)}) = B_x$.

Now let

$$X_0 := \{x \in X : \mu_x(B_x) > 0\}.$$ 

This is clearly a $\theta$-invariant subset of $X$. We will show that, if $m(X_0) > 0$, then $\mu_x(B_x) = 1$ for a.e. $x \in X_0$. Since $\theta$ is ergodic with respect to $m$, this implies ergodicity of $T$ with respect to $\mu$.

Define a function $f$ by $f_x := \mathbf{1}_{B_x}$. Clearly $f_x \in L^1(\mu_x)$ and $f_{\theta^n(x)} \circ T^n_x = f_x$ $m$-a.e. Let $x \in X' \cap X_0$, where $X'$ is given by Lemma 3.21. Let $g_x$ be a function from $L^1(J_x)$ with $\int g_x d\mu_x = 0$. Then using (7.2) we obtain that

$$\mu_x((f_{\theta^n(x)} \circ T^n_x)g_x) \to 0 \text{ as } n \to \infty.$$
Consequently
\[
\int_{B_x} g_x \, d\mu_x = 0.
\]
Since this holds for every mean zero function \(g_x \in L^1(J_x)\), we have that \(\mu_x(B_x) = 1\) for every \(x \in X' \cap X_0\). This finishes the proof of ergodicity of \(T\) with respect to the measure \(\mu\).

A direct consequence of Lemma 3.30 and ergodicity of \(T\) is the following.

**Proposition 4.8.** The measure \(\mu \in M^1_m(J)\) is a unique \(T\)-invariant measure satisfying (1.3).

5. Random Compact Subsets of Polish Spaces

Suppose that \((X, \mathcal{F}, m)\) is a complete measure space. Suppose also that \((Y, \rho)\) is a Polish space which is normalized so that \(\text{diam}(Y) = 1\). Let \(\mathcal{B}_Y\) be the \(\sigma\)-algebra of Borel subsets of \(Y\) and let \(\mathcal{K}_Y\) be the space of all compact subsets of \(Y\) topologized by the Hausdorff metric. Assume that a measurable mapping \(X \ni x \mapsto J_x \in K_Y\) is given.

Following Crauel [6, Capter 2], we say that a map \(X \ni x \mapsto Y_x \subset Y\) is measurable if, and only if, for every \(y \in Y\), the map \(x \mapsto d(y, Y_x)\) is measurable, where
\[
d(y, Y_x) := \inf\{d(y, y_x) : y_x \in Y_x\}.
\]

This map is also called a random set. If every \(Y_x\) is closed (res. compact), it is called a closed (res. compact) random set. With this terminology \(X \ni x \mapsto J_x \subset Y\) is a compact random set (see [6, Remark 2.16, p. 16]).

Closed random sets have the following important properties (cf. [6, Proposition 2.4 and Theorem 2.6]).

**Theorem 4.9.** Suppose that \(X \ni x \mapsto Y_x\) is a closed random set such that \(Y_x \neq \emptyset\).

(a) For all open set \(V \subset Y\), the set \(\{x \in X : Y_x \cap V \neq \emptyset\}\) is measurable.

(b) The set
\[
\mathcal{J} := \text{graph}(x \mapsto Y_x) := \{(x, y_x) : x \in X \text{ and } y_x \in Y_x\}
\]
is a measurable subset of \(X \times Y\) i.e. \(\mathcal{J}\) is a subset of \(\mathcal{F} \otimes \mathcal{B}_Y\), the product \(\sigma\)-algebra of \(\mathcal{F}\) and \(\mathcal{B}_Y\).

(c) For every \(n\), there exists a measurable function \(X \ni x \mapsto y_{x,n} \in Y_x\) such that
\[
Y_x = \text{cl}\{y_{x,n} : n \in \mathbb{N}\}.
\]

In particular, there exists a measurable map \(X \ni x \mapsto y_x \in Y_x\).

Note that item (b) implies that \(\mathcal{J}\) is a measurable subset of \(X \times Y\). Let \(\mathcal{B}_{\mathcal{J}} := \mathcal{F} \otimes \mathcal{B}_Y|_{\mathcal{J}}\). Then by Theorem 2.12 from [6] we get that for all \(A \in \mathcal{B}_{\mathcal{J}}, \pi_X(A) \in \mathcal{F}\).

Now, let \(X \ni x \mapsto Y_x\) be a compact random set and let \(r > 0\) be a real number. Then every set \(Y_x\) can be covered by some finite number \(a_x = a_x(r) \in \mathbb{N}\) of open balls with radii equal to \(r\). Moreover, by Lebesgue’s Covering Lemma, there exits \(R_x = R_x(r) > 0\) such that every ball \(B(y_x, R_x)\) with \(y_x \in Y_x\) is contained in a ball from this cover. As we prove below, we can actually choose \(a_x\) and \(R_x\) in a
measurable way. Hence, since \( x \mapsto J_x \) is a compact random set, the hypothesis measurability of cardinality of covers (see Section 1 just before Theorem 3.3) does not need to be assumed but can be proved.

In the proof of Lemma 4.11 we will use the following Proposition 2.1 from [6, p. 15].

**Proposition 4.10.** For compact random set \( x \mapsto Y_x \) and for every \( \varepsilon \), there exists a (non-random) compact set \( Y_{\varepsilon} \subset Y \) such that
\[
m(\{x \in X : Y_x \subset Y_{\varepsilon}\}) \geq 1 - \varepsilon.
\]

**Lemma 4.11.** There exists a measurable set \( X'_{a} \subset X \) of full measure \( m \) such that for every \( r > 0 \) and every positive integer \( k \), there exists measurable functions \( X'_{a} \ni x \mapsto y_{x,k} \in Y_x \) and there exist measurable functions \( X'_{a} \ni x \mapsto a_{x} \in \mathbb{N} \) and \( X'_{a} \ni x \mapsto R_{x} \in \mathbb{R}_{+} \) such that for every \( x \in X'_{a} \),
\[
\bigcup_{k=1}^{a_{x}} B_{x}(y_{x,k},r) \supset Y_x,
\]
and for every \( y_{x} \in Y_x \), there exists \( k = 1, \ldots, a_{x} \) for which
\[
B_{x}(y_{x}, R_{x}) \subset B_{x}(y_{x,k},r).
\]

**Proof.** For \( n \in \mathbb{N} \) let \( Y_{1/n} \subset Y \) be a compact set given by Proposition 4.10. Then the set \( X_{n} := \{x \in X : Y_x \subset Y_{1/n}\} \) is measurable and has the measure \( m(X_{n}) \) greater or equal to \( 1 - 1/n \). Define
\[
X'_{a} := \bigcup_{n \in \mathbb{N}} X_{n}.
\]
Then \( m(X'_{a}) = 1 \).

Let \( \{y_{n} : n \in \mathbb{N}_{+}\} \) be a dense subset of \( Y \). Since \( Y_{1/n} \) is compact, there exists a positive integer \( a(n) \) such that
\[
\bigcup_{k=1}^{a(n)} B(y_{k},r/2) \supset Y_{1/n}.
\]
Define a function \( X'_{a} \ni x \mapsto a_{x} \), by \( a_{x} = a(n) \) where
\[
n := \min\{k : x \in X_{k}\}.
\]
The measurability of \( X_{n} \) gives us the required measurability of \( x \mapsto a_{x} \).

Let \( \{y_{k} : k \in \mathbb{N}\} \) be a countable dense set of \( Y \) and \( m \in \mathbb{N} \). For every \( k \in \mathbb{N} \) define a function \( x \mapsto G_{x,k} \) by
\[
G_{x,k} = \begin{cases} 
B(y_{k},r/2) & \text{if } Y_x \cap B(y_{k},r/2) \neq \emptyset \\
Y_x & \text{otherwise}.
\end{cases}
\]
Since, by Theorem 4.9 (a), the set
\[
\{x \in X : Y_x \cap B(y_{k},r/2) \neq \emptyset\}
\]
is measurable, it follows that \( X \ni x \mapsto G_{x,k} \) is a closed random set. Hence, by Theorem 4.9 (c), there exists a measurable selection \( X \ni x \mapsto y_{x,k} \in G_{x,k} \). Note that, if \( y_{x,k} \in B(y_{k},r/2) \), then \( B(y_{k},r/2) \subset B(y_{x,k},r) \). Therefore, by (5.1),
\[
\bigcup_{k=1}^{U_x} B(y_{x,k},r) \supset Y_{1/n} \supset Y_x.
\]
for all \( x \in X_n \).

Finally, for \( x \in X_n \), let \( R_x > 0 \) be a real number such that, for \( y \in Y_{1/n} \), there exists \( k = 1, \ldots, U(n) \) for which

\[
B(y, R_x) \subset B(y_k, r/2) \subset B(y_{x,k}, r).
\]

Then \( X'_U \ni x \mapsto R_x \in \mathbb{R}_+ \) is also measurable. \( \square \)
CHAPTER 5

Uniformly Expanding Random Map

In this chapter, somewhat against our general philosophy, but with agreement with the existing tradition (see for example [1], [11] and [7]), we introduce the class of maps for which the functions $x \mapsto \gamma_x$, $x \mapsto \deg(T_x)$ and $x \mapsto n_\xi(x)$ are uniformly bounded. We introduce these assumptions in order to be in position to apply probability laws such as the Law of Iterated Logarithm or the Central Limit Theorem. These are needed for our study of geometric/fractal properties of conformal random expanding systems (see Chapter 7, Chapter 8 and Chapter 9).

1. Uniformly Expanding Random Map

Let $T : J \to J$ be a measurable expanding random map. The map $T : J \to J$ is called a uniformly expanding random map provided that

1. $\gamma_* := \inf_{x \in X} \gamma_x > 1$,
2. $\deg(T) := \sup_{x \in X} \deg(T_x) < \infty$,
3. $n_\xi(*) := \sup_{x \in X} n_\xi(x) < \infty$.

Then by $C^\infty(J)$ we denote the space of $\mathcal{B}$-measurable mappings $g : J \to \mathbb{R}$ with $g_x : J_x \to \mathbb{R}$ continuous such that $\sup_{x \in X} \|g_x\|_{\infty} < \infty$. For $H_0 \geq 0$, by $\mathcal{H}_\alpha^\star(J, H_0)$ we denote the space of all functions $\phi$ in $\mathcal{H}_\alpha^{m}(J) \cap C^\infty_{m}(J)$ such that all of $H_x$ are bonded above by $H_0$. Let

$$\mathcal{H}_\alpha^\star(J) = \bigcup_{H_0 \geq 0} \mathcal{H}_\alpha^\star(J, H_0).$$

For $\phi \in \mathcal{H}_\alpha^{\star}(J, H_0)$ we put

$$Q := H_0 \sum_{j=1}^{\infty} \gamma^{-\alpha j} = \frac{H_0 \gamma^{-\alpha}}{1 - \gamma^{-\alpha}}.$$

Then Lemma 2.3 takes on the following form.

**Lemma 5.1.** For every $\phi \in \mathcal{H}_\alpha^{\star}(J, H_0)$,

$$|S_n \phi_x(T_y^{-n}(w_1)) - S_n \phi_x(T_y^{-n}(w_2))| \leq Q_0^\alpha(w_1, w_2)$$

for all $n \geq 1$, all $x \in X$, every $z \in J_x$ and every $w_1, w_2 \in B(T^n(z), \xi)$ and where $y = (x, z)$.

In this paper, whenever we deal with uniformly expanding random maps, we always assume that potentials belong to $\mathcal{H}_\alpha^{\star}(J)$. Hence all the functions $C_\alpha(x)$, $C_{\max}(x)$, $C_{\min}(x)$ and $\beta_x$ defined respectively by (2.2), (2.4), (2.3) and (2.5) are uniformly bounded on $X$. Therefore, there exists $A \in \mathbb{R}$ such that $A(x) \leq A$ for all $x \in X$, where $A(x)$ is the function from Proposition 3.18. In particular, we can prove the following.
LEMMA 5.2. There exists a constant $A_\lambda$ such that, for $x \in X$ and all $y_1, y_2 \in J_{x_{n_0}}$

$$\left| \frac{\mathcal{L}_x^n(1)(y_1)}{\mathcal{L}_x^n(1)(y_1)} - \lambda_x \right| \leq A_\lambda B^n.$$

PROOF. It follows from Proposition 3.18 that

$$|\hat{\mathcal{L}}_x(\hat{1})(y_1) - \hat{\mathcal{L}}_x(1)(y_2)| \leq 2AB^{n-1}.$$ 

Then by Lemma 5.3 and 4.4

$$\left| \frac{\mathcal{L}_x^n(1)(y_1)}{\mathcal{L}_x^n(1)(y_2)} - \lambda_x \right| \leq \frac{2AB^{-1}B^n\lambda_x}{\mathcal{L}_x^n(1)(y_2)} \leq A_\lambda B^n$$

for some constant $A_\lambda x$-independent. \hfill \Box

2. Continuity of lambdas

This section has a somewhat more special character. We topologized the base space $X$ and we introduce a concept, called the global neighborhood property, which enables us to gain continuity of the function $x \mapsto \lambda_x$. We assume that $T : J \to J$ is a uniformly expanding random map.

Since a uniform limit of continuous function is continuous, the following lemma is a direct consequence of Lemma 5.2.

LEMMA 5.3. Let $X$ be a compact metric set and $\theta : X \to X$ be continuous. Suppose that, for every $n$ and for every $x \in X$ there exists a neighborhood $W$ of $x$ and a function $W \ni x \mapsto y_n(x) \in Y_n(x)$ such that the map

$$W \ni x \mapsto \mathcal{L}_x^n(1)(y_n(x)) \in \mathbb{R}$$

is continuous. Then $x \mapsto \lambda_x$ is continuous.

We say that a uniform expanding random map satisfies the global neighborhood property if, and only if, the following conditions hold.

(a) For every $x \in X$, $J_x \subset Y$ and $\theta_x = \theta|_{J_x}$, where $(Y, \theta)$ is a Polish space which is normalized so that $\text{diam}(Y) = 1$.

(b) There exists $U \subset Y$ with the property that, for every $x \in X$ we can find $V_x \subset U$ such that $T_x(V_x) = U$.

(c) $J_x = \bigcap_{n \geq 0} T_x^{-n}(V_x)$

(d) If $y_1, y_2 \in U$ and $\theta(y_1, y_2) \leq \eta$, then

$$\theta(T_x(y_1), T_x(y_2)) \geq \gamma \theta(y_1, y_2).$$

(e) The function $(x, y) \mapsto T_x^{-1}(y) \in \mathcal{K}(U)$ is continuous in the Hausdorff metric.

(f) For every $y \in U$ the map $x \mapsto \varphi_x(y)$ is continuous.

LEMMA 5.4. If $T : J \to J$ satisfies the global neighborhood property, then the function $x \mapsto \mathcal{L}_x^n(1)(y)$ is continuous.

PROOF. First, we prove that condition (e), implies continuity of the function $(x, y) \mapsto \# T_x^{-1}(y)$. In fact, let $(x, y)$ be the point of discontinuity of this function. Let $\xi' \leq \xi$ be such that for all $w_1, w_2 \in T_x^{-1}(y)$, $B(w_1, \xi') \cap B(w_2, \xi') = \emptyset$ if $w_1 \neq w_2$. Let $\varepsilon > 0$ be such that $d_H(T_x^{-1}(y), T_x^{-1}(y')) \leq \xi'/2$, if $\theta((x, y), (x', y')) \leq \varepsilon$, where $d_H$ denotes the Hausdorff metric. Choose such $(x', y')$ that which has a different
number of preimages then \((x, y)\). Without loss of generality we can assume that it is bigger. Then there exists \(w \in T_x^{-1}(y)\) and
\[
(2.1) \quad w'_1, w'_2 \in T_x^{-1}(y') \cap B(w, \xi')
\]
such that \(w_1 \neq w_2\). But the condition \((d)\) gives us that \(T_{x'}(w'_1) \neq T_{x'}(w'_2)\), which is a clear contradiction with \((2.1)\).

Let \(x \in X\), let \(W\) be a neighborhood of \(x\) and \(X \ni x' \mapsto y(x') \in U\) be a continuous function. From what was proved before we have that for every \(w \in T_x^{-1}(y(x))\), there exists a continuous function \(x' \mapsto z_w(x')\) such that \(T_{x'}(z_w(x')) = y(x')\), \(z_w(x) = w\) and
\[
T_{x'}^{-1}(y(x')) = \{z_w(x') : w \in T_x^{-1}(y(x))\}.
\]
Therefore we get that for every \(y \in U\), every \(x'\) close to \(x\) and every \(w \in T_x^{-n}(y)\), there exists a continuous function \(x' \mapsto z_w(x)\) such that \(T_{x'}^n(z_w(x)) = y\), \(z_w(x) = w\) and
\[
T_{x'}^{-n}(y) = \{z_w(x') : w \in T_x^{-n}(y(x))\}.
\]
Then the lemma follows from the continuity of \(\varphi_x\). \(\Box\)

From this lemma and Lemma 5.3, we get the following proposition.

**Proposition 5.5.** The function \(x \mapsto \lambda_x\) is continuous.
CHAPTER 6

The pressure as a function of a parameter

In this chapter we deal with two parameter families of potentials. Firstly, by careful close look at the measurable bounds obtained in Chapter 3, we will be able to show that the theorems from that chapter can be proved to hold for every parameter and almost every $x$ (common for all parameters). Then, we will use this fact to show that many properties of the pressure as a function of a parameter known to hold in the deterministic case, continue to be true also in the random case.

1. The pressure as a function

In this section we only assume that $T : Z \to Z$ is a measurable expanding random map. Let $\varphi^{(1)}, \varphi^{(2)} \in H_m(J)$ and let $t = (t_1, t_2) \in \mathbb{R}^2$. Put

$$(1.1) \quad |t| := \max\{|t_1|, |t_2|\} \quad \text{and} \quad t^* := \max\{1, |t|\}.$$ 

Set

$$\varphi_t := t_1 \varphi^{(1)} + t_2 \varphi^{(2)}$$

and

$$(1.2) \quad \varphi := |\varphi^{(1)}| + |\varphi^{(2)}|.$$ 

Fix $\alpha > 0$ and a measurable log-integrable function $H : X \to [0, +\infty)$ such that $\varphi^{(1)}, \varphi^{(2)} \in H^\alpha_m(J, H)$. Then for all $x \in X$ and all $y_1, y_2 \in J_x$, we have

$$|\varphi_{t,x}(y_2) - \varphi_{t,x}(y_1)| = |t_1(\varphi^{(1)}_x(y_2) - \varphi^{(1)}_x(y_1)) + t_2(\varphi^{(2)}_x(y_2) - \varphi^{(2)}_x(y_1))|$$

$$\leq H_x |t_1| |\rho^\alpha_x(y_2, y_1)| + H_x |t_2| |\rho^\alpha_x(y_2, y_1)|$$

$$\leq 2|t| H_x |\rho^\alpha_x(y_2, y_1)|.$$ 

Therefore

$$\varphi_t \in H^\alpha_m(J, 2|t| H) \subset H^\alpha_m(J, 2t^* H).$$

Also, for all $x \in X$ and all $y \in J_x$, we have

$$|S_n \varphi_{t,x}(y)| = |t_1 S_n \varphi^{(1)}_x(y) + t_2 S_n \varphi^{(2)}_x(y)|$$

$$\leq |t_1| |S_n \varphi^{(1)}_x(y)| + |t_2| |S_n \varphi^{(2)}_x(y)|$$

$$\leq |t| (|S_n \varphi^{(1)}_x(y)| + |S_n \varphi^{(2)}_x(y)|)$$

$$\leq |t| S_n \varphi_x(y)$$

$$\leq |t| ||S_n \varphi_x||_\infty.$$ 

Therefore,

$$(1.3) \quad ||S_n \varphi_{t,x}||_\infty \leq |t| ||S_n \varphi_x||_\infty \leq t^* ||S_n \varphi_x||_\infty.$$
Concerning the potential \( \varphi \), we get
\[
|\varphi_x(y_2) - \varphi_x(y_1)| = \left| (|\varphi_x^{(1)}(y_2)| - |\varphi_x^{(1)}(y_1)|) + (|\varphi_x^{(2)}(y_2)| - |\varphi_x^{(2)}(y_1)|) \right|
\]
\[
\leq \left| |\varphi_x^{(1)}(y_2)| - |\varphi_x^{(1)}(y_1)| \right| + \left| |\varphi_x^{(2)}(y_2)| - |\varphi_x^{(2)}(y_1)| \right|
\]
\[
\leq |\varphi_x^{(1)}(y_2) - \varphi_x^{(1)}(y_1)| + |\varphi_x^{(2)}(y_2) - \varphi_x^{(2)}(y_1)|
\]
\[
\leq 2H_x\rho_x^\alpha(y_2, y_1).
\]
Thus
\[
\varphi \in \mathcal{H}_m^\alpha(J, 2H).
\]
Denote by \( C_t, C_{t, \max}, C_{t, \min}, D_{\xi, t} \) and \( \beta_t(s) \), the respective functions associated to the potential \( \varphi_t \) as in Section 2. If the index \( t \) is missing, these numbers, as usually, refer to the potential \( \varphi \) given by (1.2). Using (1.3) and (1.4), we then immediately get
\[
D_{\xi, t}(x) \geq D_{\xi, \varphi}^t,
\]
\[
C_t(x) \leq \exp(Q_x(2t^*H)) \max_{0 \leq k \leq j} \left\{ \exp(2t^*||S_k\varphi_{x-1}||_{\infty}) \right\}
\]
\[
\leq \left( \exp(Q_x(2H)) \max_{0 \leq k \leq j} \left\{ \exp(2||S_k\varphi_{x-1}||_{\infty}) \right\} \right)^{t^*} = C_t^t,
\]
\[
C_{t, \min}(x) \geq \exp(-Q_x(2t^*H)) \exp(-2t^*||S_n\varphi_x||_{\infty}) = C_{\min}(x)^t,
\]
\[
C_{t, \max}(x) = \exp(Q_x(2t^*H)) \deg(T_x^\alpha) \exp(2t^*||S_n\varphi_x||_{\infty}) \leq C_{\max}(x)^t,
\]
and therefore,
\[
\beta_{t, x}(s) \geq \left( \frac{C_{\min}(x)}{C_{\varphi}(x)} \right)^{t^*} \frac{(s-1)2t^*H_{x-1}^\gamma_{x-1}^\alpha}{4t^*sQ_x}
\]
\[
= \left( \frac{C_{\min}(x)}{C_{\varphi}(x)} \right)^{t^*} \frac{(s-1)H_{x-1}^\gamma_{x-1}^\alpha}{2sQ_x}
\]
\[
\geq \left( \frac{C_{\min}(x)}{C_{\varphi}(x)} \right)^{t^*} \left( \frac{(s-1)H_{x-1}^\gamma_{x-1}^\alpha}{2sQ_x} \right)^{t^*}
\]
\[
= \beta_t^t(s).
\]
Finally we are going to look at the function \( A(x) \) and the constant \( B \) obtained in Proposition 5.13. We fix the set
\[
G := \{ x : \beta_x \geq M \text{ and } j(x) \leq J \}
\]
as defined by (6.2). Note that by (11), for \( x \in G \) we have,
\[
\beta_{x, t} \geq M^{t^*}.
\]
Denote by \( G_{x, t} \) the corresponding visiting set for backward iterates of \( \theta \), and by \( (n_k)_{-\infty}^\infty \) the corresponding visiting sequence. In particular
\[
\lim_{k \to -\infty} \frac{k}{n_k} \geq \frac{3}{4J}
\]
Putting
\[ B_t = \frac{\nu^2}{1 - M^t}. \]
and
\[ A_t(x) := \max\{2C_{\text{max}}(x)B_t^{-j_k}, C_{\varphi}(x) + C_{\text{max}}(x)\}, \]
as an immediate consequence of Proposition 3.18 and its proof along with our estimates above, we obtain the following.

**Proposition 6.1.** For every \( t \in \mathbb{R}^2 \), for every \( x \in \mathcal{G}'_x \), and every \( g_x \in \Lambda_{t,x}^* \)
\[ \| \mathcal{L}_{x_{-n},t}g_{x_{-n}} - q_{t,x} \|_\infty \leq A_t(x)B_t^n. \]
More generally, if \( g_x \in \mathcal{H}^\alpha(J_x) \), then
\[ \| \hat{\mathcal{L}}_{t,x}^n g_x - \left( \int g_x d\mu_{t,x} \right) I \|_\infty \leq C_{\varphi}^*(\theta^n)(x) \left( \int |g_x| d\mu_{t,x} + 4 \frac{\nu_\alpha(g_xq_x)}{t_sQ_x} A_t((\theta^n)(x))B_t^n. \]
In here and in the sequel, by \( q_{t,x} \), \( \Lambda_{t,x}^* \) and \( \mathcal{L}_{t,x} \) denote the respective objects for the potential \( \varphi_t \).

**Remark 6.2.** It follows from the estimates of all involved measurable functions, that, for \( R > 0 \) and \( t \in \mathbb{R} \) such that \( |t| \leq R \), the functions \( A_t \) and \( B_t \) in Proposition 6.1 can be replaced by \( A_{\max(|R,1)} \) and \( B_{\max(|R,1)} \) respectively.

Now, let us look at Proposition 3.20. Similarly as with the set \( G \), we consider the set \( X_A \) defined by \( 6.6 \) with \( A(x) \) generated by \( \varphi \). So, if \( x \in X_A \), then \( A_t(x) \leq A_t \) for some finite number \( A_t \) which depends on \( t \). Denote by \( X_{A,+} \) the corresponding visiting set intersected with \( G'_x \). Therefore, the following is a consequence of the proof of Proposition 6.20 and the formula \( 6.20 \).

**Proposition 6.3.** For every \( R > 0 \), every \( x \in X_{A,+} \), and every \( g_x \in C(J_x) \) we have that
\[ \lim_{n \to \infty} \sup_{|t| \leq R} \left\{ \| \hat{\mathcal{L}}_{t,x}^n g_x - \left( \int g_x d\mu_{t,x} \right) I \theta^n(x) \|_\infty \right\} = 0 \]
and, for all \( x \in X_{A,+} \) and all \( t \in \mathbb{R}^2 \),
\[ \nu_{x,t,n} \xrightarrow{n \to \infty} \nu_{t,x} \]
in the weak* topology, where
\[ \nu_{x,t,n} := \sum_{y \in T_x^n(w_n)} e^{S_{n\varphi_t}(y)} \delta_y \]
and, as usually, \( \delta_y \) is the Dirac measure supported at \( y \).
Moreover, we obtain the following consequence of Lemma 3.29 and \( 1.5 \).

**Lemma 6.4.** There exist a set \( X' \subset X \) of full measure, and a measurable function \( X \ni x \mapsto D_1(x) \) with the following property. Let \( x \in X' \), let \( w \in J_x \) and let \( n \geq 0 \). Put \( y = (x,w) \). Then
\[ (D_1(\theta^n(x)))^{-t_*} \leq \frac{\nu_{t,x}(T_x^n(B(T^n(y),\xi))))}{\exp(S_{n\varphi_t}(y) - S_nP_x(\varphi_t))} \leq (D_1(\theta^n(x)))^t, \]
for all \( t \in \mathbb{R}^2 \).
For all $t \in \mathbb{R}^2$ set
\[ \mathcal{E}P(t) := \mathcal{E}P(\varphi_t). \]

We now shall prove the following.

**Lemma 6.5.** The function $\mathcal{E}P : \mathbb{R}^2 \to \mathbb{R}$ is convex, and therefore, continuous. There exists a measurable set $X'_\mathcal{E}$ such that for all $x \in X'_\mathcal{E}$ and all $t \in \mathbb{R}^2$, the limit
\[ (1.11) \lim_{n \to \infty} \frac{1}{n} \log L^n_{t,x} \mathbb{I}(w_n) \]
exists and represents a convex function, whence continuous. Since for all $t \in \mathbb{Q}^2$ this continuous function is equal to the continuous function $\mathbb{R}^2 \ni t \mapsto \mathcal{E}P(t)$, we conclude that for all $x \in X'_\mathcal{E}$ and all $t \in \mathbb{R}^2$, we have
\[ \lim_{n \to \infty} \frac{1}{n} \log L^n_{t,x} \mathbb{I}(w_n) = \mathcal{E}P(t). \]

We are done.
Lemma 6.6. Fix $t_2 \in \mathbb{R}$ and assume that there exist measurable functions $L : X \ni x \mapsto L_x \in \mathbb{R}$ and $c : X \ni x \mapsto c_x > 0$ such that
\begin{equation}
S_n \varphi_{x,1}(z) \leq -nc_x + L_x
\end{equation}
for every $z \in J_x$ and every $n \geq 1$. Then the function $\mathbb{R} \ni t_1 \mapsto \mathcal{E}(t_1, t_2) \in \mathbb{R}$ is strictly decreasing and
\begin{equation}
\lim_{t_1 \to +\infty} \mathcal{E}(t_1, t_2) = -\infty \quad \text{and} \quad \lim_{t_1 \to -\infty} \mathcal{E}(t_1, t_2) = +\infty \quad m - a.e.
\end{equation}

Proof. Fix $x \in X'$. Let $t_1 < t_1'$. Then by (1.13)
\begin{align*}
\sum_{z \in T_{x}^{-n}(w_n)} \exp(S_n \varphi(t_1, t_2)(z)) &= \\
&= \sum_{z \in T_{x}^{-n}(w_n)} \exp(t_1 S_n \varphi_1(z)) \exp(t_2 S_n \varphi_2(z)) \\
&= \sum_{z \in T_{x}^{-n}(w_n)} \exp(t_1 S_n \varphi_1(z)) \exp(t_2 S_n \varphi_2(z)) \exp((t_1 - t_1') S_n \varphi_1(z)) \\
&\geq \sum_{z \in T_{x}^{-n}(w_n)} \exp(t_1 S_n \varphi_1(z)) \exp(t_2 S_n \varphi_2(z)) \exp((t_1 - t_1')(L_x - nc_x)) \\
&= \sum_{z \in T_{x}^{-n}(w_n)} \exp(S_n \varphi(t_1, t_2)(z)) \exp((t_1' - t_1)(nc_x - L_x))
\end{align*}
Therefore,
\begin{align*}
\frac{1}{n} \log \left( \sum_{z \in T_{x}^{-n}(w_n)} \exp(S_n \varphi(t_1, t_2)(z)) \right) &
\geq \frac{1}{n} \log \left( \sum_{z \in T_{x}^{-n}(w_n)} \exp(S_n \varphi(t_1, t_2)(z)) \right) + (t_1' - t_1)(c_x - L_x/n).
\end{align*}
Hence, passing to the subsequence $(n_j)$ and letting $n_j \to \infty$, we get
\begin{equation}
p_x(t_1, t_2) \geq p_x(t_1', t_2) + (t_1' - t_1)c_x.
\end{equation}
It directly follows from this inequality that the function $t_1 \mapsto p_x(t_1, t_2)$ is strictly decreasing and
\begin{equation}
\lim_{t_1 \to +\infty} p_x(t_1, t_2) = -\infty \quad \text{and} \quad \lim_{t_1 \to -\infty} p_x(t_1, t_2) = +\infty.
\end{equation}
Hence, by Lemma 6.5 the function $t_1 \mapsto \mathcal{E}(t_1, t_2)$ has the same properties. \hfill \square

2. Real cones

Let $\mathcal{H}_x := \mathcal{H}_{\mathbb{R}, x} := \mathcal{H}^s(J_x)$ and let $\mathcal{H}_{C, x} := \mathcal{H}_{\mathbb{R}, x} \oplus i \mathcal{H}_{\mathbb{R}, x}$ its complexification.
\begin{equation}
\mathcal{C}^*_{x} := \mathcal{C}^*_{\mathbb{R}, x} := \{ g \in \mathcal{H}_x : g(w_1) \leq e^{sQ_x \varphi(w_1, w_2)} g(w_2) \text{ if } g(w_1, w_2) \leq \xi \}.
\end{equation}
Whenever it is clear what we mean by $s$, we also denote this cone by $\mathcal{C}_x$.

By $\mathcal{C}^+_x$ we denote the subset of all non-zero functions from $\mathcal{C}^*_{x}$. For $l \in (\mathcal{H}_x)^*$, the dual space of $\mathcal{H}_x$, we define
\begin{align*}
K(\mathcal{C}^*_{x}, l) &:= \sup_{g \in \mathcal{C}^*_{x}} \frac{||l||a ||g||_{a}}{||l, g||}.
\end{align*}
Then the aperture of \( C^*_x \) is

\[
K(C^*_x) := \inf \{ K(C^*_x, l) : l \in (H_x)^*, l \neq 0 \}.
\]

**Lemma 6.7.** \( K(C^*_x) < \infty \). This property of a cone is called an outer regularity.

**Proof.** Let \( w_k \in Y_x, k = 0, \ldots, N \) be such that

\[
\bigcup_{k=1}^{L_x} B(w_k, \xi) = Y_x.
\]

Define

\[
(2.2) l_0(g) := \sum_{k=1}^{L_x} g(w_k).
\]

Then by Lemma 3.12 we have that

\[
\|l_0\|_\alpha \leq \left( sQ_x (\exp(sQ_x \xi^\alpha)) + 1 \right) \|g\|_\alpha
\]

Note that \( \|l_0\|_\alpha = L_x \), since \( l_0(g) \leq L_x \|g\|_\alpha \leq L_x \|g\|_\alpha \) and \( l_0(1) = L_x = L_x \|1\|_\alpha \).

Hence

\[
(2.3) \frac{\|l_0\|_\alpha \|g\|_\alpha}{(l_0, g)} \leq K_x^* := L_x \left( sQ_x (\exp(sQ_x \xi^\alpha)) + 1 \right) \exp(sQ_x \xi^\alpha).
\]

Let

\[
s' x := \frac{sQ_x \gamma^{-\alpha}_{x-1} + H_{x-1} \gamma^{-\alpha}_{x-1}}{Q_x}
\]

By (5.3) for \( s > 1, s' x < s \). Moreover, like in (5.2) we have the following.

**Lemma 6.8.** Let \( g \in C^*_x \) and let \( w, w' \in J_{\theta(x)} \) with \( \varrho(w_1, w_2) \leq \xi \). Then, for \( y \in T^{-1}_{x}(w_1) \)

\[
(2.4) \frac{e^{S_{x}(y)}}{e^{S_{x}(T^{-1}_{x}(w_2))}} \frac{g(y)}{g(T^{-1}_{y}(w_2))} \leq \exp \left\{ s' \theta(x) Q_{\theta(x)} \gamma^\alpha(w_1, w_2) \right\}.
\]

Consequently

\[
L_x g(w_1) \leq \exp \left\{ s' \theta(x) Q_{\theta(x)} \gamma^\alpha(w_1, w_2) \right\}.
\]

**Lemma 6.9.** There is a measurable function \( C_R : X \to (0, \infty) \) such that

\[
\frac{L^*_x, g(w)}{L^*_x, g(z)} \leq C_R(x) \text{ for every } i \geq j(x) \text{ and } g \in C^*_z.
\]

**Proof.** First, let \( i = j(x) \). Let \( a \in T^{-1}_{x}(z) \) be such that

\[
e^{S_{x}(a)} g(a) = \sup_{y \in T^{-1}_{x}(z)} e^{S_{x}(y)} g(y).
\]
By definition of \( j(x) \), for any point \( w \in J_x \) there exists \( b \in T_{x^{-i}}(w) \cap B(a, \xi) \). Therefore
\[
\mathcal{L}_{x^{-i}}^i g(w) \geq e^{S_i \varphi_{x^{-i}}(b)} g(z)
\]
\[
\geq \exp(S_i \varphi_{x^{-i}}(b) - S_i \varphi_{x^{-i}}(a)) e^{S_i \varphi_{x^{-i}}(a)} e^{-sQ_x} g(a)
\]
\[
\geq \exp(-2\|S_j(x)\varphi_{x^{-j}(x)}\|_{\infty} - sQ_x) \frac{\deg(T^j_{x^{-j}})}{\deg(T^j_{x^{-j}})} \mathcal{L}_{x^{-j}}^i g(z)
\]
\[
\geq (C_R(x))^{-1} \mathcal{L}_{x^{-i}}^i g(z)
\]
where
\[
C_R(x) := \left( \exp \left( \frac{-sQ_x - 2\|S_j(x)\varphi_{x^{-j}(x)}\|_{\infty}}{\deg(T^j_{x^{-j}})} \right) \right)^{-1} \leq 1.
\]
The case \( i > j(x) \) follows from the previous one, since \( \mathcal{L}_{x^{-i}}^{i-j(x)} g_{x^{-i}} \in C_{x^{-j(x)}}^s \).

Let \( s > 1 \) and \( s' < s \). Define
\[
\tau_x := \tau_{x,s,s'} := \sup_{r \in (0, \xi]} \frac{1 - \exp \left( -(s+s')Q_x r^\alpha \right)}{1 - \exp \left( -(s-s')Q_x r^\alpha \right)} \leq \frac{s+s'}{s-s'}.
\]

**Lemma 6.10.** For \( g_x, f_x \in C_x^s \),
\[
\tau_x \sup_{y \in J_x} \frac{|g_x(y)|}{f_x(y)} f_x - g_x \in C_{\bar{R},x}^s.
\]

**Proof.** For all \( w, z \in J_x \) with \( g_x(z, w) < \xi \),
\[
\tau_x \|g_x/f_x\|_{\infty} \left( \exp \left( sQ_x g_x^0(z, w) \right) f_x(z) - f_x(w) \right)
\]
\[
\geq \tau_x \|g_x/f_x\|_{\infty} \left( \exp \left( sQ_x g_x^0(z, w) \right) - \exp \left( s'Q_x g_x^0(z, w) \right) \right) f_x(z)
\]
\[
\geq \left( \exp \left( sQ_x g_x^0(z, w) \right) - \exp \left( -s'Q_x g_x^0(z, w) \right) \right) g_x(z)
\]
\[
\geq \exp \left( sQ_x g_x^0(z, w) \right) g_x(z) - g_x(w).
\]
Then
\[
\exp \left( sQ_x g_x^0(z, w) \right) \left( \tau_x \|g/f\|_{\infty} f_x(z) - g_x(z) \right) \geq \tau_x \|g/f\|_{\infty} f_x(w) - g_x(w).
\]

We say that \( g_x \in C_x^s \) is **balanced** if
\[
(2.7) \quad \frac{f_x(y_1)}{f_x(y_2)} \leq C_R(x)
\]
for all \( y_1, y_2 \in J_x \).

Let \( g_x, f_x \in C_x^s \). Put
\[
\beta_{x,s}(f_x, g_x) := \inf \{ \tau > 0 : \tau f_x - g_x \in C_x^s \}
\]
and define the **Hilbert projective distance** \( \text{Pdist} : C_x^s \times C_x^s \rightarrow \mathbb{R} \) by the formula
\[
\text{Pdist}_x(f_x, g_x) := \text{Pdist}_{x,s}(f_x, g_x) := \log(\beta_{x,s}(f_x, g_x) \cdot \beta_{x,s}(g_x, f_x)).
\]
Let
\[
\Delta_x := \text{diam}_{\bar{R}}(\mathcal{L}_{x^{-j}}^i (C_{x^{-j}(\mathbb{R}))}^s).
\]
Then and the second is Lemma 9.3 from [17].

\[ \alpha, \beta, \gamma > 0 \implies \exists \text{ corresponding complex Perron-Frobenius operators} C \]

Denote also by \( C \). Define \( \text{Pdist}_x(f_x, g_x) \leq 2 \log \left( \frac{s + s'}{s - s'} \cdot C_R(x) \right) \).

It follows that

\[ \Delta_x \leq 2 \log \left( \frac{s + s'}{s - s'} \cdot C_R(x) \right) \]

3. Canonical complexification

Motivated by the work of Rugh [17] we now extend real cones to complex ones.

Define

\[ C_x^+ := \{ l \in (\mathcal{H}_x)^* : l|_{c_x} \geq 0 \}, \]

\[ C_{c,x}^+ := \{ g \in \mathcal{H}_{c,x} : \forall l_1, l_2 \in C_x, \Re(l_1, g) \Re(l_2, g) \geq 0 \}. \]

Denote also by \( C_{c,x}^+ \) the set of all \( g \in C_{c,x}^+ \) such that \( g \neq 0 \).

There are other equivalent definition of \( C_{c,x}^* \). The first one is called polarization identity by Rugh in [17] Proposition 5.2.

**Proposition 6.12 (Polarization identity).**

\[ C_{c,x}^* = \{ a(f^* + ig^*) : f^* + g^* \in C_{c,x}^* \} \]

In our case we can also define \( C_{c,x}^* \) as follows. Let \( g(w, w') < \xi \). Define

\[ l_{w, w'} (g) := g(w) - \Re \sigma Q \Re \sigma (w, w') \]

and

\[ F_x := \{ l_{w, w'} : g(w, w') < \xi \} \subset C_{c,x}^* \]

Then

\[ C_x^* = \{ g \in \mathcal{H}_x : \forall l \in F_x, l(g) \geq 0 \}. \]

Later in this section we following two facts about geometry of complex numbers.

The first one is obvious and the second is Lemma 9.3 from [17].

**Lemma 6.13.** Given \( c_1, c_2 > 0 \) there exist \( p_1, p_2 > 0 \) such that if \( s_0 := c_1 p_2 \) and \( Z \in \{ re^{i \theta} : 1 \leq s_0, |u| \leq 2p_1 + 2s_0 \}, \)

then there exist \( \alpha, \beta, \gamma > 0 \) such that

\[ \Re Z \geq \alpha, \quad \Re Z \leq \beta, \quad \Im Z \leq \gamma \quad \text{and} \quad \gamma c_2 \leq \alpha. \]

**Lemma 6.14.** Let \( z_1, z_2 \in \mathbb{C} \) be such that \( \Re z_1 > \Re z_2 \) and define \( u \in \mathbb{C} \) though

\[ e^{i \Im z_1} u = \frac{e^{z_1} - e^{z_2}}{e^{\Re z_1} - e^{\Re z_2}}. \]

Then

\[ |\text{Arg } u| \leq \frac{|\Im (z_1 - z_2)|}{|\Re (z_1 - z_2)|} \quad \text{and} \quad 1 \leq |u|^2 \leq 1 + \left( \frac{|\Im (z_1 - z_2)|}{|\Re (z_1 - z_2)|} \right)^2. \]

Let \( \varphi = \Re \varphi + i \Im \varphi \) be such that \( \Re \varphi, \Im \varphi \in \mathcal{H}^\alpha (J) \). We now consider the corresponding complex Perron-Frobenius operators \( L_{x, \varphi} \) defined as follows

\[ L_{x, \varphi} g_x (w) = \sum_{T_x(z) = w} e^{\varphi(z)} g_x(z), \quad w \in J_{\theta(x)}. \]
LEMMA 6.15. Let $w, w', z, z' \in \mathcal{F}_x$ such that $\varrho(w, w') < \xi$ and $\varrho(z, z') < \xi$. Then, for all $g_1, g_2 \in L^2_{x, \mathbb{R}}$,

$$
\frac{l_{w, w'}(L_{x, \varphi}g_1)l_{z, z'}(L_{x, \varphi}g_2)}{l_{w, w'}(L_{x, \text{Re} \varphi}g_1)l_{z, z'}(L_{x, \text{Re} \varphi}g_2)} = Z
$$

where

$$
Z \in A_x := \{re^{iu} : 1 \leq r \leq 1 + s_0^2, |u| \leq 2|| \Im \varphi||_{\infty} + 2s_0\}.
$$

and

$$
s_0 := \frac{v_0(\Im \varphi)\gamma_x^{-\alpha}}{(s - s'_{\theta(x)})Q_{\theta(x)}}
$$

PROOF. For $y \in T_x^{-1}(w)$, by $y'$ we denote $T_y^{-1}(w')$. Then for $g \in L^2_x$

$$
l_{w, w'}(L_{x, \varphi}g) := L_{x, \varphi}g(w) - e^{-sQ_x \theta^\alpha(w, w')}L_{x, \varphi}g(w')
$$

$$
= \sum_{y \in T_x^{-1}(w)} e^{\varphi(y)}g(y) - e^{-sQ_x \theta^\alpha(w, w')}e^{\varphi(y')}g(y')
$$

$$
= \sum_{y \in T_x^{-1}(w)} n_y(\varphi, g),
$$

where

$$
n_y(\varphi, g) := e^{\varphi(y)}g(y) - e^{-sQ_x \theta^\alpha(w, w')}e^{\varphi(y')}g(y').
$$

Define implicitly $u_y$ so that

$$
n_y(\Re \varphi, g)e^{i \Im \varphi(y)}u_y = n_y(\varphi, g).
$$

Put $z_1 := \varphi(y) + \log g(y)$ and $z_2 := -sQ_x \theta^\alpha(w, w') + \varphi(y') + \log g(y')$. Then

$$
e^{i \Im z_1}u_y = \frac{e^{z_1} - e^{z_2}}{e^{\Re z_1} - e^{\Re z_2}}
$$

By [222]

$$
\Re \varphi(y) - \log g(y) - (\Re \varphi(y') + \log g(y')) \geq -s'_{\theta(x)}Q_{\theta(x)}\theta^\alpha(w_1, w_2).
$$

Then

$$
\Re(z_1 - z_2) \geq (s - s'_{\theta(x)})Q_{\theta(x)}\theta^\alpha(w_1, w_2).
$$

We also have that

$$
|\Im(z_1 - z_2)| \leq v_0(\Im \varphi)\gamma_x^{-\alpha} \theta^\alpha(w_1, w_2),
$$

since $\Im(z_1 - z_2) = \Im \varphi(y) - \Im \varphi(y')$. Therefore, by Lemma 6.13

$$
|\text{Arg } u_y| \leq s_0 := \frac{v_0(\Im \varphi)\gamma_x^{-\alpha}}{(s - s'_{\theta(x)})Q_{\theta(x)}}
$$

and

$$
1 \leq |u_y|^2 \leq 1 + s_0^2.
$$

Since

$$
l_{w, w'}(L_{x, \varphi}g) = \sum_{y \in T_x^{-1}(w)} n_y(\varphi, g) = \sum_{y \in T_x^{-1}(w)} e^{i \Im \varphi(y)}u_y n_y(\Re \varphi, g),
$$

$$
\frac{l_{w, w'}(L_{x, \varphi}g)}{l_{w, w'}(L_{x, \text{Re} \varphi}g)} = Z
$$
where
\[ Z \in A_x := \{ re^{iu} : 1 \leq r \leq 1 + s_0^2, |u| \leq 2\| \text{Im } \varphi \|_{\infty} + 2s_0 \}. \]

Similarly
\[ \frac{l_{w,w}(L_x,\varphi g_1)l_z,z(L_x,\varphi g_2)}{l_{w,w}(L_x,\text{Re } \varphi g_1)l_z,z(L_x,\text{Re } \varphi g_2)} = Z \]
for possibly another \( Z \in A_x \).

Let \( p_1, p_2 \) be the real numbers given by Lemma 6.13 with
\[ c_1 = \frac{\gamma_x^{-\alpha}}{(s - s_x')Q_x} \quad \text{and} \quad c_2 = \cosh \frac{\Delta_x}{2} \]

Having Lemma 6.15, Lemma 6.13 and Lemma 6.11 the following proposition is a consequence of the proof of Theorem 6.3 in [17].

**Proposition 6.16.** Let \( j = j(x) \). If
\[ \| \text{Im } S_j \varphi_{x,j} \|_{\infty} \leq p_1 \quad \text{and} \quad v_\alpha(\text{Im } S_j \varphi_{x,j}) \leq p_2, \]
then
\[ L^1_{j,x}(C^\alpha_{C_x},C_x) \subset C^\alpha_{C_x}. \]

Let \( l_0 \) (the functional defined by (2.2)). Then by Lemma 5.3 in [17] we get
\[ K := K(C^\alpha_{C_x},l_0) := \sup_{g \in C^\alpha_{C_x}} \frac{|\langle l_0,|g|_\alpha \rangle|}{|\langle l_0,g \rangle|} \leq K_x := 2\sqrt{2}K'_x \]
where \( K'_x \) is defined by [23]. By \( l \) we denote the functional which is a normalized version of \((1/L_x)l_0\). So \( ||l||_\alpha = 1 \). Then, for every \( g \in C^\alpha_{C_x} \),
\[ 1 \leq \frac{||g||_\alpha}{\langle l,g \rangle} \leq K_x. \]

4. The pressure is real-analytic

We are now in position to prove the main result of this chapter. Here, we assume that \( T : \mathcal{J} \to \mathcal{J} \) is uniformly expanding random map. Then there exists \( j \in \mathbb{N} \) such that \( j(x) = j \) for all \( x \in X \). Without loss of generality we assume that \( j = 1 \).

**Theorem 6.17.** Let \( t_0 = (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( R > 0 \). Assume that the following conditions are satisfied.

(a) For every \( x \in X \) and every \( w \in \mathcal{J}_x \), \( D(t_0,R) \ni z \mapsto \varphi_{z,x}(w) \in \mathbb{C} \) is holomorphic, where
\[ D(t_0,R) := \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \forall k |z_k - t_k| < R \}. \]

(b) For \( z \in \mathbb{R}^n \cap D(t,R) \), \( \varphi_{z,x} \in H_{\mathbb{R},x} \).

(c) For all \( z \in D(t_0,R) \) and all \( x \in X \), there exists \( H \) such that
\[ ||\varphi_{z,x}||_\alpha \leq H. \]

(d) For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( z \in D(t_0,\delta) \) and all \( x \in X \),
\[ ||\text{Im } \varphi_{z,x}||_\alpha \leq \varepsilon. \]

Then the function \( D(t_0,R) \cap \mathbb{R}^n \ni z \mapsto EP(\varphi_{z,x}) \) is real-analytic.
4. THE PRESSURE IS REAL-ANALYTIC

PROOF. Since we assume that the measurable constants are uniform for \( x \in X \) we get that from Proposition 6.16 and condition (d) that there exists \( r > 0 \) such that, for all \( z \in \mathcal{D} \) and all \( x \in X \),

\[
\mathcal{L}_{z,x} (\mathcal{C}_{z}^{\alpha} \subset \mathcal{C}_{z}^{\alpha}.
\]

Then by (5.1),

\[
\frac{||\mathcal{L}_{z,x}^{n} (1)||_{\alpha}}{L_{e}(\mathcal{L}_{z,x}^{n} (1))} \leq K.
\]

Therefore, by Montel Theorem, the family \( \mathcal{L}_{z,x}^{n} (w) \) is normal. Since, for all \( z \in \mathbb{R}^{n} \cap \mathcal{D} \) and all \( x \in X \) we have that

\[
\frac{\mathcal{L}_{z,x}^{n} (1) (w)}{l_{e}(\mathcal{L}_{z,x}^{n} (1))} \overset{n \to \infty}{\longrightarrow} \frac{q_{z,x} (w)}{l_{e}(q_{z,x})},
\]

we conclude that there exists an analytic function \( z \mapsto g_{z,x} (w) \) such that

\[
(4.1) \quad \frac{\mathcal{L}_{z,x}^{n} (1) (w)}{l_{e}(\mathcal{L}_{z,x}^{n} (1))} \overset{n \to \infty}{\longrightarrow} g_{z,x} (w).
\]

Since, in addition,

\[
\mathcal{L}_{e} \left( \frac{\mathcal{L}_{z,x}^{n} (1) (w)}{l_{e}(\mathcal{L}_{z,x}^{n} (1))} \right) = \frac{\mathcal{L}_{z,x}^{n+1} (1) (w)}{l_{e}(\mathcal{L}_{z,x}^{n+1} (1))} \cdot l_{e} (g_{z,x}),
\]

we therefore get that

\[
\mathcal{L}_{e} \left( \frac{\mathcal{L}_{z,x}^{n} (1) (w)}{l_{e}(\mathcal{L}_{z,x}^{n} (1))} \right) \overset{n \to \infty}{\longrightarrow} l_{e} (g_{z,x}) g_{x,z}.
\]

Thus, using (4.1), again, we obtain,

\[
\mathcal{L}_{z,x} (g_{z,x}) = l_{e} (\mathcal{L}_{z,x} (g_{z,x})) g_{x,z}.
\]

As for all \( z \in \mathcal{D} \cap \mathbb{R}^{n} \),

\[
g_{z,x} = \frac{q_{z,x}}{l_{e}(q_{z,x})} = \frac{L_{e} q_{z,x}}{\sum_{k=0}^{N} q_{z,x}(w_{k})},
\]

we conclude that,

\[
(4.2) \quad l_{e} (\mathcal{L}_{z,x} g_{z,x}) = l_{e} (\mathcal{L}_{z,x} \frac{q_{z,x}}{l_{e}(q_{z,x})}) = \lambda_{z,x} \frac{l_{e} (q_{z,x})}{l_{e}(q_{z,x})}.
\]

By the very definitions

\[
l_{e} (\mathcal{L}_{z,x} g_{z,x}) = (1/L_{e}) \sum_{k=1}^{L_{e}} l_{e}(\mathcal{L}_{z,x} g_{z,x}(w_{k})
\]

and

\[
\mathcal{L}_{z,x} g_{z,x}(w) = \sum_{y \in T_{l_{e}^{-1}(w)}^{e}} \varphi_{z,x}(y) g_{z,x}(y).
\]

Denote \( g_{z,x}(w) \) by \( F(z) \) and \( \varphi_{z,x}(w) \) by \( G(z) \). Then, for \( z = (z_{1}, \ldots, z_{n}) \in D(t_{0}, r/2), \) and \( \Gamma(u) = z + ((r/2) e^{2\pi i u_{1}}, \ldots, (r/2) e^{2\pi i u_{n}}), \) where \( u = (u_{1}, \ldots, u_{n}) \in [0, 2\pi]^{n}, \) by the Cauchy Integral Formula,

\[
\left| \frac{\partial F}{\partial z_{k}}(z) \right| = \left| \frac{1}{(2\pi i)^{2}} \int_{\Gamma} \frac{F(\xi)}{(\xi_{1} - z_{1}) \cdots (\xi_{k} - z_{k})^{2} \cdots (\xi_{2} - z_{2})} d\xi \right| \leq 2K/r
\]
for $k = 1, \ldots, n$. Similarly we obtain that
\[
\left| \frac{\partial G}{\partial z_k}(z) \right| \leq 2H/r
\]
for $k = 1, \ldots, n$. Then, for $k = 1, \ldots, n$,
\[
\left| \frac{\partial e^{\phi_{z,x}(y)} g_{z,x}(y)}{\partial z_k} \right| = \left| \frac{\partial e^{\phi_{z,x}(w)} g_{z,x}(y)}{\partial z_k} + e^{\phi_{z,x}(y)} \frac{\partial g_{z,x}(y)}{\partial z_k} \right| 
\leq (2H/r)e^{HK} + e^{H}(2K/r).
\]
It follows that there exists $C_g$ such that for all $x \in X$,
\[
|\partial l_x(L_{t_0,x}g_{t_0,x})| \leq C_g.
\]
Using (4.2) we obtain that
\[
C_{\varphi}^{-1} \leq q_{t_0,x}(y) \leq C_{\varphi}
\]
and then
\[
C_{\varphi}^{-1} \leq l_x(q_{t_0,x}(y)) \leq C_{\varphi}
\]
for all $x \in X$. Moreover, it follows from Lemma 3.7 that
\[
\lambda_{t_0,x} \geq \exp(-\|\varphi_{t_0,x}\|_{\infty}).
\]
Then
\[
z_0 := l_{x_1}(L_{t_0,x}g_{t_0,x}) = \lambda_{t_0} \frac{l_{x_1}(q_{t_0,x})}{l_{x}(q_{t_0,x})} \geq \exp(-\sup_{x \in X} \|\varphi_x\|_{\infty})C_{\varphi}^{-2} > 0.
\]
Hence, by (4.3), there exists $r_1 > 0$ so small that
\[
l_{x_1}(L_{z,x}g_{z,x}) \in D(z_0, z_0/2)
\]
for all $z \in D(t_0, r_1)$. Therefore, for all $x \in X$ we can define the function
\[
D(t_0, r_1) \ni z \mapsto \log l_{x_1}(L_{z,x}g_{z,z}) \in \mathbb{C}.
\]
Now consider the holomorphic function
\[
z \mapsto \int \log l_{x_1}(L_{z,x}g_{z,x})dm(x).
\]
Since the measure $m$ is $\theta$-invariant, by (4.2)
\[
\int \log l_{x_1}(L_{z,x}g_{z,x})dm(x) = \int \log \lambda_{z,x} \frac{l_{x_1}(q_{z,x})}{l_{x}(q_{z,x})}dm(x)
= \int \log \lambda_{z,x}dm + \int l_{x_1}(q_{z,x})dm - \int l_{x}(q_{z,x})dm(x)
= \int \log \lambda_{z,x}dm = EP(\varphi_t)
\]
for $z \in D(t_0, r_1) \cap \mathbb{R}^n$. Therefore the function $D(t_0, r_1) \cap \mathbb{R}^n \ni z \mapsto EP(\varphi_z)$ is real-analytic. □
5. Derivative of the pressure

Now, let $T : \mathcal{J} \to \mathcal{J}$ be uniformly expanding random map. Throughout the section, we assume that $\varphi \in \mathcal{H}_m(\mathcal{J})$ is a potential such that there exist measurable functions $L : X \ni x \mapsto L_x \in \mathbb{R}$ and $c : X \ni x \mapsto c_x > 0$ such that

\begin{equation}
S_n \varphi(x) \leq -nc_x + L_x
\end{equation}

for every $z \in \mathcal{J}_x$ and $n$ and $\psi \in \mathcal{H}_m(\mathcal{J})$. For $t \in \mathbb{R}$, define

$\varphi_t := t\varphi + \psi$.

Let $R > 0$ and let $|t_0| \leq R/2$. Since we are in the uniform case, it follows from Remark 6.2 that there exist constants $A_R$ and $B_R$ such that, for $t \in [-R, R]$,

\begin{equation}
\left\| \frac{L_{1,n} g_x}{g_{\alpha_1}(x)} - \left( \int g_x d\nu_{t,x} \right) \right\|_{\infty} \leq \left( \|g_x\|_{\infty} + 2^p(g_x) \right) \frac{A_R B_R}{Q}.
\end{equation}

**Proposition 6.18.**

\[
\frac{d\mathcal{E}(t)}{dt} = \int \varphi_x d\mu_t dm(x) = \int \varphi d\mu_t.
\]

**Proof.** Assume without loss of generality that $|t| \leq R/2$ for some $R > 0$. Let $x \mapsto y(x) \in Y_x$ be a measurable function and let

\[
\mathcal{E}(t, n) := \frac{1}{n} \log \mathcal{L}_{t,n}^n(y(x)) dm(x).
\]

Then $\lim_{n \to \infty} \mathcal{E}(t, n) = \mathcal{E}(t)$ by Lemma 4.6. Fix $x \in X$ and put $y_n := y(x_n)$. Observe that

\[
\frac{d\mathcal{L}_{t,n}^n 1_x(y_n)}{dt} = \sum_{y \in T_{x,n}^{-1}(y_n)} e^{S_n(\varphi_x)(y)} S_n \varphi_x(y)
\]

\[
= \sum_{j=0}^{n-1} \sum_{y \in T_{x,n}^{-1}(y_n)} e^{S_n(\varphi_y)(y)} \varphi_{x,j}(T^j_{x,y}) = \sum_{j=0}^{n-1} \mathcal{L}_{t,x}^n(\varphi_{x,j} \circ T^j_{x,y})(y_n).
\]

Since

$S_n(\varphi_{x,j})(y) = S_j(\varphi_{x,j})(y) + S_{n-j}(\varphi_{x,j})(T^j_{x,y})$

we have that

$\mathcal{L}_{t,x}^n(\varphi_{x,j} \circ T^j_{x,y})(y(x_n)) = \mathcal{L}_{x,j}^{n-j}(\varphi_{x,j} \mathcal{L}_{t,x}^j 1_x)(y(x_n))$

Then by a version of Leibniz integral rule (see for example [14], Proposition 7.8.4 p. 40)

\[
\frac{d\mathcal{E}(t, n)}{dt} = \int \frac{\frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}_{x,j}^{n-j}(\varphi_{x,j} \mathcal{L}_{t,x}^j 1_x)(y(x_n))}{\mathcal{L}_{t,x}^n 1_x(y_n)} dm(x).
\]

Observe that

$\mathcal{L}_{x,j}^{n-j}(\varphi_{x,j} \mathcal{L}_{t,x}^j 1_x)(y_n) = \lambda_x^n \mathcal{L}_{x,j}^{n-j}(\varphi_{x,j} \mathcal{L}_{t,x}^j 1_x)(y_n)$

and

$\mathcal{L}_{t,x}^n 1_x(y_n) = \lambda_x^n \mathcal{L}_{t,x}^n 1_x(y_n)$.

Then

\begin{equation}
\frac{\mathcal{L}_{t,x}^n(\varphi_{x,j} \circ T^j_{x,y})(y_n)}{\mathcal{L}_{t,x}^n 1_x(y_n)} = \frac{\mathcal{L}_{x,j}^{n-j}(\varphi_{x,j} \mathcal{L}_{t,x}^j 1_x)(y_n)}{\mathcal{L}_{t,x}^n 1_x(y_n)}.
\end{equation}
The function \( \varphi \), \( \tilde{L}^j_{t,x} 1_x \) is uniformly bounded. So does its Hölder variation. Therefore it follows from [5.22], that there exists a constant \( A_R \) and \( B_R \) such that

\[
\left\| \tilde{L}^j_{x,t} \left( \varphi_j \tilde{L}^j_{t,x} 1_x \right)(y_n)/q_{x_n} - \left( \int \varphi_j \tilde{L}^j_{t,x} 1_x d\nu_t \right) \right\| \leq A_R B_R^{n-j}
\]

and

\[
\left\| \tilde{L}^n_{t,x}(1_x)(y_n)/q_{x_n} - 1_{x_n} \right\| \leq A_R B_R^n.
\]

From this by [5.3] it follows that

\[
\frac{\int \varphi_j \tilde{L}^j_{t,x} 1_x d\nu_{t,x} - A_R B_R^{n-j}}{1 + A_R B_R^n} \leq \frac{\mathcal{L}^n_{t,x}(\varphi_j \circ T^j_t)(y_n)}{1 - A_R B_R^n},
\]

Since \( m \) is \( \theta \)-invariant, we have that

\[
\int \int \varphi_j \tilde{L}^j_{t,x} y^j 1_x d\nu_{t,x} dm(x) = \int \int \varphi_j \tilde{L}^j_{x_{-j},t} 1_{x_{-j}} d\nu_{t,x} dm(x).
\]

Hence, for large \( n \),

\[
\frac{\int \int \varphi_j \left( \frac{1}{n} \sum_{j=0}^{n-1} \tilde{L}^j_{x_{-j},t} 1_{x_{-j}} \right) d\nu_{t,x} dm(x) - \frac{1}{n} \sum_{j=0}^{n-1} \left( A_R B_R^{n-j} \right)}{1 + A_R B_R^n} \leq \frac{\text{dEP}(\varphi^t, n)}{dt} \leq \frac{\int \int \varphi_j \left( \frac{1}{n} \sum_{j=0}^{n-1} \tilde{L}^j_{x_{-j},t} 1_{x_{-j}} \right) d\nu_{t,x} dm(x) - \frac{1}{n} \sum_{j=0}^{n-1} \left( A_R B_R^{n-j} \right)}{1 - A_R B_R^n}.
\]

Therefore

\[
\lim_{n \to \infty} \frac{\text{dEP}(t, n)}{dt} = \int \varphi_x d\mu_t^x dm(x)
\]

uniformly for \( t \in [-R, R] \).
CHAPTER 7

Fractal Structure of Conformal Expanding Random Repellers

In this chapter we deal with conformal expanding random maps. We prove an appropriate version of Bowen’s Formula, which asserts that the Hausdorff dimension of almost every fiber $J_x$, denoted throughout the paper by HD, is equal to a unique zero of the function $t \mapsto \mathcal{E}P(t)$. We also show that typically Hausdorff and packing measures on fibers respectively vanish and are infinite. A simple example of such a phenomenon is a Random Cantor Set described at the beginning of Chapter 9. Later in that chapter the reader will find more refined and general examples of Random Conformal Systems notably Classical Random Expanding Systems, Brück and Bürger Polynomial Systems and DG-Systems.

1. Conformal Expanding Random Maps; Bowen’s Formula

**Definition 7.1.** Let the ambient space $Y$ be a smooth Riemannian manifold and assume that we deal with mappings $f_x : J_x \rightarrow J_{\theta(x)}$ that can be extended to a neighborhood of $J_x$ in $Y$ to conformal $C^{1+\alpha}$ mappings. If in addition

$$f : (x, z) \mapsto (\theta(x), f_x(z))$$

is a measurably expanding random map then we call the random map conformal expanding random map.

By $|f_x'(y)|$ we denote the norm of the derivative of $f_x$ and by $||f_x'||_{\infty}$ its supremum over $z \in J_x$. Since the system is expanding we have

$$|f_x'| \geq \gamma_x \text{ for a.e. } x \in X,$$

where $|f_x'|$ is the similarity factor of the derivative $f_x'$.

For every $t \in \mathbb{R}$ we consider the potential

$$\varphi_t(x, z) = -t \log |f_x'(z)|.$$

The associated topological pressure $P(\varphi_t)$ will be denoted $P(t)$. Let

$$\mathcal{E}P(t) = \int_X P_x(t)dm(x)$$

be its expected value with respect to the measure $m$. In view of $\text{[11]}$, it follows from Lemma $6.6$ that the function $t \mapsto \mathcal{E}P(t)$ has a unique zero. Denote it by $h$. The result of this subsection is the following version of Bowen’s formula, identifying the Hausdorff dimension of almost all fibers with the parameter $h$.

**Theorem 7.2 (Bowen’s Formula).** The parameter $h$, i.e. the zero of the function $t \mapsto \mathcal{E}P(t)$, is m-a.e. equal to the Hausdorff dimension $\text{HD}(J_x)$ of the fiber $J_x$. 

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PROOF. Let \((\nu_{x,h})_{x \in X}\) be the measures produced in Theorem 3.1 for the potential \(\varphi_h\). Fix \(x \in X\) and \(z \in J_x\) and set again \(\gamma = (x, z)\). For every \(r \in (0, \xi]\) let \(k = k(z, r)\) be the largest number \(n \geq 0\) such that

\[
B(z, r) \subset f_y^{-n}(B(f_x^n(z), \xi)).
\]

By the expanding property this inclusion holds for all \(0 \leq n \leq k\) and \(\lim_{r \to 0} k(z, r) = +\infty\). Fix such an \(n\). By Lemma 3.26

\[
\nu_{x,h}(B(z, r)) \leq \nu_{x,h}(f_y^{-n}(B(f_x^n(z), \xi))) \leq \exp(hQ_{\theta^n(x)}) \left(\left| (f_x^n)'(z)\right|^{-1}\right)^n \exp(-P_{x}^n(h)).
\]

On the other hand,

\[
B(z, r) \not\subset f_y^{-\left\lfloor s+1 \right\rfloor}(B(f_x^{s+1}(z), \xi))
\]

for every \(s \geq k\). But, since by Lemma 2.3,

\[
B(z, \exp(-Q_{\theta^{s+1}(x)}\xi^n)||(f_x^{s+1})'(z)|^{-1}) \subset f_y^{-\left\lfloor s+1 \right\rfloor}(B(f_x^{s+1}(z), \xi)),
\]

we get

\[
\exp(-Q_{\theta^{s+1}(x)}\xi^n)|(f_x^{s+1})'(z)|^{-1} \leq r
\]

and

\[
|(f_x^{s})'(z)|^{-1} \leq \xi^{-1} \exp(Q_{\theta^{s}(x)}\xi^n)r.
\]

Inserting this to (1.3) we obtain,

\[
\nu_{x,h}(B(z, r)) \leq \xi^{-h} \exp(hQ_{\theta^n(x)}) \exp(hQ_{\theta^{s+1}(x)}\xi^n)r^h \cdot \exp(-P_{x}^n(h)) \left(\left| (f_x^n)'(z)\right|\right|^h.
\]

or equivalently

\[
\frac{\log \nu_{x,h}(B(z, r))}{\log r} \geq h + \frac{hQ_{\theta^{s}(x)}\xi^n}{\log r} + \frac{hQ_{\theta^{s+1}(x)}\xi^n}{\log r} + \frac{-h \log \left(\left| (f_x^{s+1-n})'(f_x^n(z))\right|\right)}{\log r} + \frac{-h \log \xi}{\log r} + \frac{-P_{x}^n(h)}{\log r}.
\]

Our goal is to show that

\[
\liminf_{r \to 0} \frac{\log \nu_{x,h}(B(z, r))}{\log r} \geq h \quad \text{for a.e.} \quad x \in X \quad \text{and all} \quad z \in J_x.
\]

Since the function \(x \mapsto Q_x\) is measurable and almost everywhere finite, there exists \(M > 0\) such that \(m(A) > 0\), where

\[
A = \{x \in X : Q_x \leq M\}.
\]

Fix \(n = n_k \geq 0\) to be the largest integer less than or equal to \(k\) such that \(\theta^n(x) \in A\) and \(s = s_k\) to be the least integer greater than or equal to \(k\) such that \(\theta^{s+1}(x) \in A\).

It follows from Birkhoff’s Ergodic Theorem that

\[
\lim_{k \to \infty} s_k/n_k = 1.
\]

Of course if for \(k \geq 1\) we take any \(r_k > 0\) such that \(k(z, r_k) = k\), then \(\lim_{k \to \infty} r_k = 0\).

Now, note that by (1.3), the formula

\[
f_y^{-n}(B(f_x^n(z), \xi)) \subset B(z, \exp(Q_{\theta^{s}(x)}\xi^n)|(f_x^n)'(z)|^{-1}) \xi
\]


yields
\[ r \leq \exp(Q_{\theta^n}(x)\xi^\alpha)|(f_x^n)'(z)|^{-1}\xi. \]

Equivalently,
\[ -\log r \geq \log |(f_x^n)'(z)| - \xi^\alpha Q_{\theta^n}(x) - \log \xi. \]

Since \( \log |(f_x^n)'(z)| \geq \log \gamma^\alpha_n \) and since the function \( x \mapsto \log \gamma_x \) is integrable, \( \chi = \int \log \gamma dm > 0 \), we thus get from Birkhoff’s Ergodic Theorem that for a.e. \( x \in X \) and all \( r > 0 \) small enough (so \( k \) and \( n_k \) and \( s_k \) large enough too)

\[ (1.7) \]
\[ -\log r \geq \frac{\chi}{2} n \geq \frac{\chi}{3} s. \]

Remember that \( \theta^n(x) \in A \) and \( \theta^{*+1}(x) \in A \). We thus obtain from \( (1.6) \) that

\[ (1.8) \]
\[ \lim_{r \to 0} \frac{\log \nu_{x,h}(B(z,r))}{\log r} \geq h - 3h \lim \sup_{k \to \infty} \frac{1}{s} \log \left( \left| (f_x^{n+1}(z))^n(f_x^n(z)) \right| \right) - 2 \frac{1}{n} P^n_x(h). \]

for a.e. \( x \in X \) and all \( z \in J_x \). But as \( \int P^n_x(h) dm(x) = 0 \), we have by Birkhoff’s Ergodic Theorem that

\[ (1.9) \]
\[ \lim_{n \to \infty} \frac{1}{n} P^n_x(h) = 0. \]

Also, since the measure \( \mu_h \) is \( f \)-invariant, it follows from Birkhoff’s Ergodic Theorem that there exists a measurable set \( X_0 \subset X \) such that for every \( x \in X_0 \) there exists at least one (in fact of full measure \( \mu_{x,h} \)) \( x \in J_x \) such that

\[ \lim_{j \to \infty} \frac{1}{j} \log \left| (f_x^j)'(z_x) \right| = \hat{\chi} := \int \log |f_x^j(z)| d\mu_h(x,z) \in (0, +\infty). \]

Hence, remembering that \( \theta^n(x) \) and \( \theta^{*+1}(x) \) belong to \( A \), we get

\[ \lim_{k \to \infty} \frac{1}{s} \log \left( \left| (f_x^{n+1}(z))^n(f_x^n(z)) \right| \right) = \lim_{k \to \infty} \frac{1}{s} \log \left( \left| (f_x^{n+1}(z))^n(f_x^n(z)) \right| \right) - \lim \inf_{k \to \infty} \frac{1}{s} \log \left| (f_x^n)'(z_x) \right| = \hat{\chi} - \hat{\chi} = 0. \]

Inserting this and \( (1.9) \) to \( (1.8) \) we get that

\[ (1.10) \]
\[ \lim_{r \to 0} \frac{\log \nu_{x,h}(B(z,r))}{\log r} \geq h. \]

Keep \( x \in X \), \( z \in J_x \) and \( r \in (0, \xi] \). Now, let \( l = l(z,r) \) be the least integer \( \geq 0 \) such that

\[ (1.11) \]
\[ f_x^{n-l}(B(f_x^l(z),\xi)) \subset B(z,r). \]

Then, by Lemma \( [3,29] \)

\[ \nu_{x,h}(B(z,r)) \geq \nu_{x,h}(B(f_x^l(z),\xi)) \]
\[ \geq D_1(\theta^l(x)) \exp(-Q_{\theta^l}(x))(f_x^l)'(z)|^{-l} exp(-P_x^l(h)). \]

On the other hand

\[ f_x^{n-l}(B(f_x^{n-l}(z),\xi)) \not\subset B(z,r). \]
But, since
\[ f_y^{-\left((-1)^{j-1}\right)}(B(f_{x-1}^{-1}(z), \xi)) \subset B(y, \exp(Q_{\theta^{-1}}(x)(\xi))|(f_{x-1}^{-1})'(z)|^{-1}\xi), \]
we get
\[ r \leq \xi \exp(Q_{\theta^{-1}}(x)(\xi))|(f_{x-1}^{-1})'(y)|^{-1}. \]
Thus
\[ |(f_{x-1}^{-1})'(z)|^{-1} \geq \xi^{-1} \exp(-Q_{\theta^{-1}}(x)(\xi))r. \]
Inserting this to (1.12) we obtain,
\[ (1.14) \quad \nu_{x,h}(B(z, r)) \geq \xi^{-h} D_1(\theta^i(x)) e^{-Q_{\theta^{-1}}(x)}|(f_{x-1}^{-1})'(z)|^{-h} \exp(-h Q_{\theta^{-1}}(x)(\xi))r^h \exp(-P_x^i(h)). \]
Now, given any integer \( j \geq 1 \) large enough, take \( R_j > 0 \) to be the least radius \( r > 0 \) such that \( f_y^{-1}(B(f_{x-1}^{-1}(z), \xi)) \subset B(z, r) \). Then \( l(y, R_j) = j \). Since the function \( Q \) is measurable and almost everywhere finite, and \( \theta \) is a measure-preserving transformation, there exist a set \( \Gamma \subset X \) with positive measure \( m \) and a constant \( E > 0 \) such that \( Q_x \leq E, D_1(x) \leq E \) and \( Q_{\theta^{-1}}(x) \leq E \) for all \( x \in \Gamma \). It follows from Birkhoff’s Ergodic Theorem and ergodicity of the map \( \theta : X \to X \) that there exists a measurable set \( X_1 \subset X \) with \( m(X_1) = 1 \) such that for every \( x \in X_1 \) there exists an unbounded increasing sequence \( (j_i)_{i=1}^\infty \) such that \( \theta^j(x) \in \Gamma \) for all \( i \geq 1 \).

Formula (1.13) then yields
\[ -\log R_j \geq -E \xi^a + \log \xi + \log |(f_{x-1}^{-1}(z))| \geq -E \xi^a + \log \xi + \log \gamma_j^{-1} \geq \frac{X}{2} j_i, \]
where the last inequality was written because of the same argument as (1.7) was, intersecting also \( X_1 \) with an appropriate measurable set of measure 1. Now we get from (1.14) that
\[ \frac{\log \nu_{x,h}(B(z, R_j))}{\log R_j} \leq h + \frac{2 \log E}{\chi j_i} - \frac{2E}{\chi j_i} \frac{1}{x} j_i \log \|(f_{x-1}^{-1})'(z)|\|_\infty - \frac{2h \xi^a E}{\chi j_i} - \frac{2h \log \xi}{\chi j_i} - \frac{1}{2} \frac{P_x^i(h)}{j_i}. \]
Noting that \( \int_X P_x(t)dm(x) = 0 \) and applying Birkhoff’s Ergodic Theorem, we see that the last term in the above estimate converges to zero. Also \( \frac{1}{j_i} \log \|(f_{\theta^{-1}}(x))'\|_\infty \) converges to zero because of Birkhoff’s Ergodic Theorem and integrability of the function \( x \mapsto \log \|f_x\|_\infty \). Since all the other terms obviously converge to zero, we thus get for a.e. \( x \in X \) and all \( z \in \mathcal{J}_x \), that
\[ \liminf_{r \to 0} \frac{\log \nu_{x,h}(B(z, r))}{\log r} \leq \liminf_{i \to \infty} \frac{\log \nu_{x,h}(B(z, R_{j_i}))}{\log R_{j_i}} \leq h. \]
Combining this with (1.10), we obtain that
\[ \liminf_{r \to 0} \frac{\log \nu_{x,h}(B(z, r))}{\log r} = h \]
for a.e. \( x \in X \) and all \( z \in \mathcal{J}_x \). This gives that \( \text{HD}(\mathcal{J}_x) = h \) for a.e. \( x \in X \). We are done. \( \Box \)
2. Conformal uniformly expanding random maps; Hausdorff and Packing Measures

A conformal measurable random map \( f : J \to J \) which is uniformly expanding is simply called conformal uniformly expanding. However, in order to investigate the finer fractal structure of \( J \), we also need a mild information about the asymptotic behavior of Birkhoff’s sums \( P^n_x(h) \). The latter will be guaranteed for instance by the Law of Iterated Logarithm or even by the Central Limit Theorem.

**Definition 7.3.** We call a conformal random map \( f \) essentially random if and only if the following two conditions are satisfied.

(a) \( f \) is conformal uniformly expanding.

(b) For \( m \)-a.e. \( x \in X \),

\[
\limsup_{n \to \infty} P^n_x(h) = +\infty \quad \text{and} \quad \liminf_{n \to \infty} P^n_x(h) = -\infty,
\]

where \( h \) is the Bowen’s parameter coming from Theorem 7.2.

A conformal random map is called quasi-deterministic if, and only if, it satisfies condition (a) and

(c) there exists \( A > 0 \) such that for \( m \)-almost all \( x \in X \) and all \( n \geq 0 \),

\[-A \leq P^n_x(h) \leq A.\]

**Remark 7.4.** Because of ergodicity of the transformation \( \theta : X \to X \), for a conformal random map to be essential it suffices to know that the condition (b) above is satisfied for a set of points \( x \in X \) with a positive measure \( m \).

**Remark 7.5.** If the number

\[
\sigma^2(P(h)) = \lim_{n \to \infty} \frac{1}{n} \int \left( S_n(P(h)) \right)^2 dm > 0
\]

and if the Law of Iterated Logarithm holds, i.e. if

\[
-\sqrt{2\sigma^2(P(h))} = \liminf_{n \to \infty} \frac{P^n_x(h)}{\sqrt{n \log \log n}} \leq \limsup_{n \to \infty} \frac{P^n_x(h)}{\sqrt{n \log \log n}} = \sqrt{2\sigma^2(P(h))} \quad m-a.e.,
\]

then our conformal random map is essential. It is essential even if only the Central Limit Theorem, i.e. if

\[
m \left( \left\{ x \in X : \frac{P^n_x(h)}{\sqrt{n}} < r \right\} \right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-s^2/2\sigma^2(P(h))} ds.
\]

**Remark 7.6.** If there exists a bounded everywhere defined measurable function \( u : X \to \mathbb{R} \) such that \( P_x(h) = u(x) - u \circ \theta(x) \) (i.e. if \( P(h) \) is a coboundary) for all \( x \in X \), then our system is quasi-deterministic.

For every \( \alpha > 0 \) let \( \mathcal{H}_\alpha \) refer to the \( \alpha \)-dimensional Hausdorff measure and let \( \mathcal{P}_\alpha \) refer to the \( \alpha \)-dimensional packing measure. Recall that a Borel probability measure
μ defined on a metric space M is geometric with an exponent α if and only if there exist $A \geq 1$ and $R > 0$ such that

$$A^{-1} r^\alpha \leq \mu(B(z, r)) \leq A r^\alpha$$

for all $z \in M$ and all $0 \leq r \leq R$. The most significant basic properties of geometric measures are the following.

- The measures $\mu$, $\mathcal{H}^\alpha$, and $\mathcal{P}^\alpha$ are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity.
- $0 < \mathcal{H}^\alpha(M), \mathcal{P}^\alpha(M) < +\infty$.
- $\text{HD}(M) = h$.

The main result of this section is the following.

**Theorem 7.7.** Suppose $f : J \to J$ is a conformal random map.

(a) If the system $f : J \to J$ is essential, then $\mathcal{H}^h(J_x) = 0$ and $\mathcal{P}^h(J_x) = +\infty$ for m-a.e. $x \in X$.

(b) If, on the other hand, the system $f : J \to J$ is quasi-deterministic, then for every $x \in X$,

- (b1) $\nu_x^h$ is a geometric measure with exponent $h$.
- (b2) The measures $\nu_x^h$, $\mathcal{H}^h|_{J_x}$, and $\mathcal{P}^h|_{J_x}$ are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of $x \in X$ and $y \in J_x$.
- (b3) $0 < \mathcal{H}^h(J_x), \mathcal{P}^h(J_x) < +\infty$.
- (b4) $\text{HD}(J_x) = h$.

**Proof.** Part (a). Remember that by its very definition $\mathcal{E}P(h) = \int P_x(h) dm(x) = 0$. By Definition 7.3 there exists a measurable set $X_1$ with $m(X_1) = 1$ such that for every $x \in X_1$ there exists an increasing unbounded sequence $(n_j)_{j=1}^\infty$ (depending on $x$) of positive integers such that

$$\lim_{j \to \infty} P_x^{n_j}(h) = -\infty.$$  

Since we are in the uniformly expanding case, the formula (1.12) from the proof of Theorem 7.2 (Bowen's Formula) takes on the following simplified form

$$\nu_x(B(z, r)) \geq D^{-1} r^h \exp(-P_x^{(z, r)}(h))$$

with some $D \geq 1$ and all $z \in J_x$. Since the map is uniformly expanding, for all $j \geq 1$ large enough, there exists $r_j > 0$ such that $l(z, r_j) = n_j$. So disregarding finitely many terms, we may assume without loss of generality, that this is true for all $j \geq 1$. Clearly,

$$\lim_{j \to \infty} r_j = 0.$$  

It thus follows from (2.2) that

$$\nu_{x, h}(B(z, r_j)) \geq D^{-1} r_j^h \exp(-P_x^{n_j}(h))$$
for all \( x \in X_1 \), all \( z \in J_x \) and all \( j \geq 1 \). Therefore, by (2.1),

\[
\limsup_{r \to 0} \frac{\nu_{x,h}(B(z,r))}{r^h} \geq \limsup_{j \to \infty} \frac{\nu_{x,h}(B(z,r_j))}{r_j^h} \geq D^{-1} \limsup_{j \to \infty} \exp\left(-P_x^{n_j}(h)\right) = +\infty.
\]

Thus \( \mathcal{H}^h(J_x) = 0 \).

The proof for packing measures is similar. By Definition 7.3 there exists a measurable set \( X_2 \) with \( m(X_2) = 1 \) such that for every \( x \in X_2 \) there exists an increasing unbounded sequence \( (s_j)_{j=1}^{\infty} \) (depending on \( x \)) of positive integers such that

\[
\lim_{j \to \infty} P_x^{s_j}(h) = +\infty.
\]

Since we are in the expanding case, formula (1.5) from the proof of Theorem 7.2 (Bowen’s Formula), applied with \( s = k(z,r) \), takes on the following simplified form.

\[
(2.4) \quad \nu_{x,h}(B(z,r)) \leq Dr^h \exp\left(-P_x^{k(z,r)}(h)\right)
\]

with \( D \geq 1 \) sufficiently large, all \( x \in X_2 \) and all \( z \in J_x \). By our uniform assumptions, for all \( j \geq 1 \) large enough, there exists \( R_j > 0 \) such that \( k(z,R_j) = s_j \).

Clearly, \( \lim_{j \to \infty} R_j = 0 \).

It thus follows from (2.4) that

\[
\nu_{x,h}(B(z,r_j)) \leq DR_j^h \exp\left(-P_x^{s_j}(h)\right)
\]

for all \( x \in X_2 \), all \( z \in J_x \) and all \( j \geq 1 \). Therefore, using (2.3), we get

\[
\liminf_{r \to 0} \frac{\nu_{x,h}(B(z,r))}{r^h} \leq \liminf_{j \to \infty} \frac{\nu_{x,h}(B(z,R_j))}{R_j^h} \leq D \liminf_{j \to \infty} \exp\left(-P_x^{s_j}(h)\right) = 0.
\]

Thus \( \mathcal{P}^h(J_x) = +\infty \). We are done with part (a).

Suppose now that the map \( f : J \to J \) is quasi-deterministic. It then follows from Definition 7.3 and (2.2) along with (2.4), that for every \( x \in X \) and for every \( r > 0 \) small enough independently of \( x \in X \), we have

\[
(AD)^{-1}r^h \leq \nu_{x,h}(B(y,r)) \leq ADr^h, \quad x \in X, \quad z \in J_x.
\]

This means that each \( \nu_{x,h}, x \in X \), is a geometric measure with exponent \( h \). Consequently, \( \mathcal{H}^h|_{J_x}, \mathcal{P}^h|_{J_x} \), and \( \nu_{x,h} \) are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of \( x \in X \) and \( z \in J_x \), \( 0 < \mathcal{H}^h(J_x), \mathcal{P}^h(J_x) < +\infty \), and \( \text{HD}(J_x) = h \) for all \( x \in X \). We are done. \( \square \)

As a straightforward consequence of this theorem we get a corollary transparently stating that essential conformal random systems are entirely new objects, drastically different from deterministic self-conformal sets.

**Corollary 7.8.** Suppose that conformal random map \( f : J \to J \) is essential. Then for \( m \text{-a.e. } x \in X \) the following hold.
(1) The fiber $J_x$ is not bi-Lipschitz equivalent to any deterministic nor quasi-deterministic self-conformal set.

(2) $J_x$ is not a geometric circle nor even a piecewise smooth curve.

(3) If $J_x$ has a non-degenerate connected component (for example if $J_x$ is connected), then $h = \text{HD}(J_x) > 1$.

(4) Let $d$ be the dimension of the ambient Riemannian space $Y$. Then $\text{HD}(J_x) < d$.

**Proof.** Item (1) follows immediately from Theorem 7.7(a) and (b). Item (3) from Theorem 7.7(a) and the observation that $\mathcal{H}^1(W) > 0$ whenever $W$ is connected. Since (3) obviously implies (2), we are done. □
CHAPTER 8

Multifractal analysis

The second direction of our study of fractal properties of conformal random expanding repellers is to investigate the multifractal spectrum of Gibbs measures on fibers. We show that the multifractal formalism is valid. It seems that it is impossible to do it with a method inspired by the proof of Bowen’s formula since one gets full measure sets for each real \( \alpha \) and not one full measure set \( X_{ma} \) such that for all \( x \in X_{ma} \), the multifractal spectrum of the Gibbs measure on the fiber over \( x \) is given by the Legendre transform of a temperature function which is independent of \( x \in X_{ma} \). In order to overcome this problem we work out a different proof in which we minimize the use Birkhoff’s Ergodic Theorem and instead we base the proof on the definition of Gibbs measures and the behavior of the Perron-Frobenius operator. In this point we were partially motivated by the approach presented in Falconer’s book.

Another issue we would like to bring up here is real analyticity of the multifractal spectrum. We establish it assuming that the system is uniformly expanding and we apply the real-analitycity results proven for the expected pressure in Section 4.

1. Concave Legendre Transform

Let \( \varphi \in H_m(J) \) be such that \( EP(\varphi) = 0 \). Fix \( q \in \mathbb{R} \). We will not use the function \( q_x \) and therefore this will not cause any confusion. Define auxiliary potentials

\[
\varphi_{q,x,t}(y) := q(\varphi_x(y) - P_x(\varphi)) - t \log |f_x'(y)|.
\]

By Lemma 6.5, the function \( \mathbb{R}^2 \ni (q,t) \mapsto EP(q,t) := EP(\varphi_{q,t}) \in \mathbb{R} \) is convex. Moreover, since \( \log |f_x'(y)| \geq \log \gamma_x > 0 \), it follows from Lemma 6.9 that for every \( q \in \mathbb{R} \) there exists a unique \( T(q) \in \mathbb{R} \) such that

\[
EP(\varphi_{q,T(q)}) = 0.
\]

The function \( q \mapsto T(q) \) defined implicitly by this formula is referred to as the temperature function. Put

\[
\varphi_q := \varphi_{q,T(q)}
\]

By \( D_T \) we denote the set of differentiability points of the temperature function \( T \). By convexity of \( EP \), for \( \lambda \in (0, 1) \),

\[
EP(\lambda q_1 + (1 - \lambda)q_2, \lambda T(q_1) + (1 - \lambda)T(q_2)) \leq \lambda EP(q_1, T(q_1)) + (1 - \lambda)EP(q_2, T(q_2)) = 0.
\]

Since \( t \mapsto EP(\lambda q_1 + (1 - \lambda)q_2, t) \) is decreasing,

\[
T(\lambda q_1 + (1 - \lambda)q_2) \leq \lambda T(q_1) + (1 - \lambda)T(q_2).
\]
Hence the function \( q \mapsto T(q) \) is convex and continuous. Furthermore, it follows from its convexity that the function \( T \) is differentiable everywhere but a countable set, where it is left and right differentiable.

Define

\[
L(T)(\alpha) := \inf_{-\infty < q < \infty} (\alpha q + T(q)),
\]

where

\[
\alpha \in \text{Dom}(L) = \left[ \lim_{q \to -\infty} -T'(q^-), \lim_{q \to \infty} -T'(q^+) \right].
\]

We call \( L \) the concave Legendre transform. This transform is related to the (classical) Legendre transform \( L \) by the formula \( L(T)(\alpha) = -L(T)(-\alpha) \). The transform \( L \) sends convex functions to concave ones and, if \( q \in D_T \), then

\[
L(T)(-T'(q)) = -T'(q)q + T(q).
\]

**Lemma 8.1.** Let \( q \in D_T \). Then for every \( \varepsilon > 0 \) there exists \( \delta \varepsilon > 0 \), such that, for all \( \delta \in (0, \delta \varepsilon) \), we have

\[
\mathcal{E}P((1 + \delta)q, T(q) + (qT'(q) + \varepsilon)\delta) < 0
\]

and

\[
\mathcal{E}P((1 - \delta)q, T(q) + (-qT'(q) + \varepsilon)\delta) < 0.
\]

**Proof.** Since the temperature function \( T \) is differentiable at the point \( q \), we may write

\[
T(q + \delta q) = T(q) + T'(q)\delta q + o(\delta).
\]

for all \( \delta > 0 \) sufficiently small, say \( \delta \in (0, \delta_\varepsilon^{(1)}) \). So,

\[
T(q) + (qT'(q) + \varepsilon)\delta - T((1 + \delta)q) = \varepsilon \delta + o(\delta) > 0.
\]

Then, in virtue of Lemma 6.6, we get that

\[
\mathcal{E}P((1 + \delta)q, T(q) + (qT'(q) + \varepsilon)\delta) < \mathcal{E}P((1 + \delta)q), T((1 + \delta)q)) = 0,
\]

meaning that the first assertion of our lemma is proved. The second one is proved similarly producing a positive number \( \delta_\varepsilon^{(2)} \). Setting then \( \delta_\varepsilon = \min\{\delta_\varepsilon^{(1)}, \delta_\varepsilon^{(2)}\} \) completes the proof. \( \square \)

### 2. Multifractal Spectrum

Let \( \mu \) be the invariant Gibbs measure for \( \varphi \) and let \( \nu \) be the \( \varphi \)-conformal measure. For every \( \alpha \in \mathbb{R} \) define

\[
K_\varepsilon(\alpha) := \left\{ y \in \mathcal{J}_x : d_{\mu_x}(y) := \lim_{r \to 0} \frac{\log \mu_x(B(y, r))}{\log r} = \alpha \right\},
\]

and

\[
K_\varepsilon' := \left\{ y \in \mathcal{J}_x : \text{the limit } \lim_{r \to 0} \frac{\log \mu_x(B(y, r))}{\log r} \text{ does not exist} \right\}.
\]

This gives us the multifractal decomposition

\[
\mathcal{J}_x := \bigcup_{\alpha \geq 0} K_\varepsilon(\alpha) \cup K_\varepsilon'.
\]

The multifractal spectrum is the family of functions \( \{g_{\mu_x}\}_{x \in X} \) given by the formulas

\[
g_{\mu_x}(\alpha) := \text{HD}(K_\varepsilon(\alpha)).
\]
The function $d_{\mu_x}(y)$ is called the local dimension of the measure $\mu_x$ at the point $y$. Since for $m$ almost every $x \in X$ the measures $\mu_x$ and $\nu_x$ are equivalent with Radon-Nikodym derivatives uniformly separated from 0 and infinity (though the bounds may and usually do depend on $x$), we conclude that we get the same set $K_x(\alpha)$ if in its definition the measure $\mu_x$ is replaced by $\nu_x$. Our goal now is to get a "smooth" formula for $g_{\mu_x}$.

Let $\mu_q$ and $\nu_q$ be the measures for the potential $\varphi_q$ given by Theorem 3.1. The main technical result of this section is this.

**Proposition 8.2.** For every $q \in D_T$ there exists a measurable set $X_{ma} \subset X$ with $m(X_{ma}) = 1$ and such that, for every $x \in X_{ma}$, and all $q \in D_T$, we have

$$g_{\mu_q}(-T'(q)) = -qT'(q) + T(q).$$

**Proof.** Firstly, by Lemma 6.4, for every $0 < R \leq \xi$ there exists a measurable function $D_R : X \to (0, +\infty)$ such that for all $q \in \mathbb{R}$, all $x \in X$, all $y \in \mathbb{J}_x$, and all integers $n \geq 0$, we have

$$D_R^{-q^*}(\theta^n(x)) \leq \nu_{q,x}(f^{-n}(B(f^n(y), R))) \leq D_R^q(\theta^n(x)),$$

where $q^* := (q, T(q))^*$ as defined in (1.1). In what follows we keep the notation from the proof of Theorem 7.2. The formulas (1.2) and (1.11) then give for every $j \geq l$ and every $0 \leq i \leq k$, that

$$D_{\xi}^{-q^*}(\theta^l(x))^{-1} \exp(q(S_j \varphi(y) - \bar{P}_x^l(\varphi))) |(f^j_x)'(y)|^{-T(q)} \leq \nu_{q,x}(B(y, r)) \leq D_{\xi}^q(\theta^l(x)) \exp(q(S_j \varphi(y) - \bar{P}_x^l(\varphi))) |(f^j_x)'(y)|^{-T(q)}.$$

By $Q_x$ we denote the measurable function given by Lemma 2.3 for the function $-\log |f'|$. Let $X_0$ be an essential set for the functions $X \ni x \mapsto R_x$, $x \ni x \mapsto a(x)$, $x \ni Q_x$, and $X \ni x \mapsto D_\xi(x)$ with constants $\bar{R}$, $\bar{a}$, $\bar{Q}$ and $D_\xi$. Let $(n_j)_{j}^\infty$ be the positively visiting sequence for $X_0$ at $x$. Let $X'_\xi$ be the set given by Lemma 6.5 for potentials $\phi_{q,t}$, $q, t \in \mathbb{R}^2$. Let

$$X'_\xi := X_{\xi} \cap X_{+X_0}.$$ 

Let us first prove the upper bound on $g_{\mu_x}(-T'(q))$. Fix $x \in X'_\xi$. Fix $\epsilon_1 > 0$. For every $j \geq 1$ let $\{w_k(x_{n_j}) : 1 \leq k \leq a(x_{n_j})\}$ be a $\xi$ spanning set of $\mathbb{J}_{x_{n_j}}$. As $EP(\phi_q) = 0$, it follows from Lemma 6.6 that $\gamma := \frac{1}{2}EP(\phi_{q,T(q)+\epsilon_1}) < 0$. So, in virtue of Lemma 5.3 there exists $C \geq 1$ such that

$$\mathcal{L}_{\phi_{q,T(q)+\epsilon_1}}(w_k(x_{n_j})) \leq Ce^{-\gamma n_j}$$

for all $j \leq 1$ and all $k = 1, 2, \ldots, a(\theta^{n_j}(x_{n_j})) \leq \bar{a}$. Now, fix an arbitrary $\epsilon_2 \in \mathbb{R}$ such that $q\epsilon_2 \geq 0$. For every integer $l \geq 1$ let

$$K_x(\epsilon_2, l) = \left\{ y \in K_x(-T'(q)) : -T'(q) - \frac{1}{2} |\epsilon_2| \leq \frac{\log \nu_x(B(y, r))}{\log r} \leq -T'(q) + \frac{1}{2} |\epsilon_2| \right\}$$

for all $0 < r \leq 1/l$.

Note that

$$K_x(-T'(q)) = \bigcup_{l=1}^{\infty} K_x(\epsilon_2, l).$$
Let
\[ \Gamma_{n_j}(x) = \left\{ z \in \bigcup_{k=1}^{o(x_n_j)} f_{x^{-n_j}}(w_k(x_{n_j})) : K_x(\varepsilon_2, l) \cap f_{x^{-n_j}}(B(f^{n_j}(z), \xi/2)) \neq \emptyset \right\}. \]

Then
\[ (2.5) \quad K_x(\varepsilon_2, l) \subset \bigcup_{z \in \Gamma_{n_j}(x)} f_{x^{-n_j}}(B(f^{n_j}(z), \xi/2)). \]

For every \( z \in \Gamma_{n_j}(x) \), say \( z \in f_{x^{-n_j}}(w_k(x_{n_j})) \), choose \( \hat{z} \in K_x(\varepsilon_2, l) \cap f_{x^{-n_j}}(B(w_k(x_{n_j}), \xi/2)). \)

Then \( B(w_k(x_{n_j}), \xi/2) \subset B(f^{n_j}(z), \xi) \), and therefore
\[ f_{x^{-n_j}}(B(w_k(x_{n_j}), \xi/2)) \subset f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)). \]

It follows from this and (2.5) that
\[ (2.6) \quad K_x(\varepsilon_2, l) \subset \bigcup_{z \in \Gamma_{n_j}(x)} f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)). \]

Put
\[ r^{(1)}_j(\hat{z}) = \hat{Q}^{-1}|(f_{x^{n_j}})^{\prime}(\hat{z})|^{-1} \quad \text{and} \quad r^{(2)}_j(\hat{z}) = \hat{Q}|(f_{x^{n_j}})^{\prime}\prime(\hat{z})|^{-1} \]

We then have
\[ B(\hat{z}, r^{(1)}_j(\hat{z})) \subset f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)) \subset B(\hat{z}, r^{(2)}_j(\hat{z})). \]

Therefore, assuming \( j \geq 1 \) to be sufficiently large so that the radii \( r^{(1)}_j(\hat{z}) \) and \( r^{(2)}_j(\hat{z}) \) are sufficiently small, particularly \( \leq 1/l \), we get
\[ \frac{\log \nu_x\left( f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)) \right)}{-\log |(f_{x^{n_j}})^{\prime}(\hat{z})|} \leq \frac{\log \nu_x\left( B(\hat{z}, \hat{Q}^{-1}|(f_{x^{n_j}})^{\prime}(\hat{z})|^{-1}) \right)}{-\log |(f_{x^{n_j}})^{\prime}(\hat{z})|} \leq \frac{\log \nu_x\left( B(\hat{z}, r^{(1)}_j(\hat{z})) \right)}{\log(r^{(1)}_j(\hat{z})) + \log \hat{Q}} \leq -T(q) + |\varepsilon_2|. \]

and
\[ \frac{\log \nu_x\left( f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)) \right)}{-\log |(f_{x^{n_j}})^{\prime}(\hat{z})|} \geq \frac{\log \nu_x\left( B(\hat{z}, \hat{Q}|(f_{x^{n_j}})^{\prime}(\hat{z})|^{-1}) \right)}{-\log |(f_{x^{n_j}})^{\prime}(\hat{z})|} \geq \frac{\log \nu_x\left( B(\hat{z}, r^{(2)}_j(\hat{z})) \right)}{\log(r^{(2)}_j(\hat{z})) - \log \hat{Q}} \geq -T(q) - |\varepsilon_2|. \]

Hence,
\[ |q|\left( \log \nu_x\left( f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)) \right) - (T(q) + |\varepsilon_2|) \log |(f_{x^{n_j}})^{\prime}(\hat{z})| \right) \leq 0 \]

and
\[ |q|\left( \log \nu_x\left( f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)) \right) - (T(q) - |\varepsilon_2|) \log |(f_{x^{n_j}})^{\prime}(\hat{z})| \right) \geq 0. \]

So, in either case (as \( \varepsilon_2 q > 0 \)),
\[ -q\left( \log \nu_x\left( f_{x^{-n_j}}(B(f^{n_j}(\hat{z}), \xi)) \right) - (T(q) - |\varepsilon_2|) \log |(f_{x^{n_j}})^{\prime}(\hat{z})| \right) \leq 0. \]
or equivalently,
\[ \nu^{-q}_s \left( f^{n_s}_z(B(f^{n_s}_z(z), \xi)) \right) = q^{T(q) - \varepsilon_2 q} \leq 1. \]

Put \( t = -qT(q) + T(q) + \varepsilon_1 + \varepsilon_2 q \). Using (2.7) and (2.3) we can then estimate as follows.

\[ \sum_{z \in \Gamma_{n_s}(x)} \text{diam}^{-qT(q)+T(q)+\varepsilon_1+\varepsilon_2 q} \left( f^{n_s}_z(B(f^{n_s}_z(z), \xi)) \right) = \]

\[ = \sum_{z \in \Gamma_{n_s}(x)} \text{diam}^{T(q)+\varepsilon_1} \left( f^{n_s}_z(B(f^{n_s}_z(z), \xi)) \right) \text{diam}^{-qT(q)+\varepsilon_2 q} \left( f^{n_s}_z(B(f^{n_s}_z(z), \xi)) \right) \]

\[ \leq \sum_{z \in \Gamma_{n_s}(x)} \langle \hat{Q}^{-1} \rangle^{1/2} \langle f^{n_s}_z \rangle^{1/2} \sum_{z \in \Gamma_{n_s}(x)} \exp \left( q(S_{n_s} \varphi(z) - P_{n_s}^z(\varphi)) \right) - (T(q) + \varepsilon_1) \log \left( \langle f^{n_s}_z \rangle^{1/2} \right) \]

\[ \leq \langle \hat{Q}^{-1} \rangle^{2t} e^{\hat{Q}_0} \sum_{z \in \Gamma_{n_s}(x)} \langle \hat{Q}^{-1} \rangle^{2t} \sum_{z \in \Gamma_{n_s}(x)} \exp \left( q(S_{n_s} \varphi(z) - P_{n_s}^z(\varphi)) \right) - (T(q) + \varepsilon_1) \log \left( \langle f^{n_s}_z \rangle^{1/2} \right) \]

\[ \leq \langle \hat{Q}^{-1} \rangle^{2t} e^{\hat{Q}_0} \sum_{z \in \Gamma_{n_s}(x)} \langle \hat{Q}^{-1} \rangle^{2t} \sum_{z \in \Gamma_{n_s}(x)} \exp \left( q(S_{n_s} \varphi(z) - P_{n_s}^z(\varphi)) \right) - (T(q) + \varepsilon_1) \log \left( \langle f^{n_s}_z \rangle^{1/2} \right) \]

\[ \leq \langle \hat{Q}^{-1} \rangle^{2t} e^{\hat{Q}_0} \sum_{z \in \Gamma_{n_s}(x)} \langle \hat{Q}^{-1} \rangle^{2t} \sum_{z \in \Gamma_{n_s}(x)} \exp \left( q(S_{n_s} \varphi(z) - P_{n_s}^z(\varphi)) \right) \]

\[ \leq \langle \hat{Q}^{-1} \rangle^{2t} e^{\hat{Q}_0} a(x_{n_s}) \sum_{k=1} \mathcal{L}_{\phi_{T(q)+\varepsilon_1}, x \, \mathcal{I}(w_{n_s}(x_{n_s}))} \]

\[ \leq C\langle \hat{Q}^{-1} \rangle^{2t} e^{\hat{Q}_0} a(x_{n_s}) e^{-\gamma_{n_s}} \leq C\langle \hat{Q}^{-1} \rangle^{2t} e^{\hat{Q}_0} a e^{-\gamma_{n_s}}. \]

Letting \( j \to \infty \) and looking also at (2.9), we thus conclude that \( \mathcal{H}^d(K_\varepsilon(x_2, l)) = 0 \). In virtue of (2.4) this implies that \( \mathcal{H}^d(K_{x_2}(-T(q))) = 0 \). Since \( \varepsilon_1 > 0 \) and \( \varepsilon_2 q > 0 \) were arbitrary, it follows that

\[ g_{\mu_{x}}(-T(q)) = \text{HD}(K_{x}(-T(q))) \leq -qT(q) + T(q). \]

Let us now prove the opposite inequality. For every \( s \geq 1 \) let \( s_- \) be the largest integer in \([0, s - 1]\) such that \( \theta^{-s_-}(x) \in X_\ast \) and let \( s_+ \) be the least integer in \([s + 1, +\infty)\) such that \( \theta^{s_+}(x) \in X_\ast \). It follows from (2.2) applied with \( j = l_+ \) and \( i = k_- \), that (2.3) is true with \( s + 1 \) replaced by \( k_+ \), and (1.13) is true with \( l - 1 \) replaced by \( l_- \), that

\[ \log \nu_{q,x}(B(y, r)) \leq \frac{-q^* \log \hat{D}_x + q(S_{n_s} \varphi(y) - P_{n_s}^z(\varphi)) - T(q) \log \left( \langle f^{n_s}_z \rangle^{1/2} \right)}{\log \xi + \xi^a \hat{Q} - \log \left( \langle f^{n_s}_z \rangle^{1/2} \right)}. \]
and
\[
\frac{\log \nu_{q,x}(B(y, r))}{\log r} \geq q^* \log \hat{D}_\xi + q \left( S_{k,-}(\varphi) - \hat{P}_{x^-}^k(\varphi) \right) - T(q) \log |(f_{x^+}^k)'(y)|. 
\]

Hence,
\[
(2.9) \quad \limsup_{r \to 0} \frac{\log \nu_{q,x}(B(y, r))}{\log r} \leq \limsup_{n \to \infty} \left( q \frac{P_{x^+}^n(\varphi) - S_{n,x}(\varphi)}{\log |(f_{x^-}^n)'(y)|} \right) + T(q) \limsup_{n \to \infty} \frac{\log |(f_{x^+}^n)'(y)|}{\log r}
\]
and
\[
(2.10) \quad \liminf_{r \to 0} \frac{\log \nu_{q,x}(B(y, r))}{\log r} \geq \liminf_{n \to \infty} \left( q \frac{P_{x^+}^n(\varphi) - S_{n,x}(\varphi)}{\log |(f_{x^-}^n)'(y)|} \right) + T(q) \liminf_{n \to \infty} \frac{\log |(f_{x^+}^n)'(y)|}{\log r}.
\]

Now, given \( \varepsilon > 0 \) and \( \delta_{\varepsilon} > 0 \) ascribed to \( \varepsilon \) according to Lemma 5.1, fix an arbitrary \( \delta \in (0, \delta_{\varepsilon}] \). Set
\[
\phi^{(1)} = \phi^{(1)}_{\varepsilon, \delta} = \phi(1+\delta)q,T(q)+(-qT'(q)+\varepsilon)\delta \] \exp\((-1+\delta)P(\phi_q))
\]
and
\[
\phi^{(2)} = \phi^{(2)}_{\varepsilon, \delta} = \phi(1-\delta)q,T(q)+(-qT'(q)+\varepsilon)\delta \] \exp\((-1+\delta)P(\phi_q)).
\]
Since
\[
\mathcal{E}P(\phi^{(1)}) = \mathcal{E}P(\phi(1+\delta)q,T(q)+(-qT'(q)+\varepsilon)\delta) = (1+\delta) \int P(\phi_q) dm
\]
and
\[
\mathcal{E}P(\phi^{(2)}) = \mathcal{E}P(\phi(1-\delta)q,T(q)+(-qT'(q)+\varepsilon)\delta) = (1-\delta) \int P(\phi_q) dm
\]
and
\[
\mathcal{E}P(\phi^{(1)}) = \mathcal{E}P(\phi(1+\delta)q,T(q)+(-qT'(q)+\varepsilon)\delta) = (1+\delta) \int P(\phi_q) dm
\]
it follows from Lemma 8.1 and Lemma 6.5 there exists \( \kappa(q, \varepsilon, \delta) \in (0, 1) \) such that for all \( k = 1, 2 \), and all \( n \geq 1 \) sufficiently large, we have
\[
\frac{1}{n} \log \mathcal{L}_\phi^{n(1)}(1)(w) \leq \log \kappa
\]
for all \( x \in X^+ \) and all \( w \in J_{\sigma^n(x)} \). Equivalently,
\[
(2.11) \quad \mathcal{L}_\phi^{n(1)}(1)(w) \leq \kappa^n.
\]
Now, for all \( x \in X^+_\), all \( j \geq 1 \), all \( 1 \leq k \leq a(\theta^{n_j}(x) \leq \alpha \), and all \( z \in f_{x}^{-n_j}(w_k(x_{n_j})) \), define
\[
A(z) := \{ y \in f_{x}^{-n_j}(B(w_k(x_{n_j}), \xi)) : B(f_{y}^{n_j}(y), R) \subset B(w_k(x_{n_j}), \xi) \}.
\]
Note that
\[
(2.12) \quad \bigcup_{k=1}^{a(x_{n_j})} \bigcup_{z \in f_{x}^{-n_j}(w_k(x_{n_j}))} A(z) = J(x).
\]
Fix any \( q \in D_T \) and set
\[ \Delta_x = \sup_{0 < \delta \leq \delta_x} \left\{ \max\{((1 + \delta)q, T(q) + (qT'(q) + \varepsilon)\delta)^*, ((1 - \delta)q, T(q) + (-qT'(q) + \varepsilon)\delta)^*\} \right\}. \]

Let \( x \in X'_+ \). Set
\[
M := \exp(\hat{\mathcal{Q}}\delta(-qT'(q) + T(q) - \varepsilon)).
\]

Then, using Lemma 2.3 (for the potential \((x, z) \mapsto \log |f_x'(z)|, f_x'(z)\), and (2.11), we obtain

\[
\nu_{q, x}(\{ y \in J_x : \nu_{q, x}(f_x^{-n} y \cdot (B(f_x^n y), R)) \geq |(f_x^n y)'(y)|^{(-qT'(q) + T(q) + \varepsilon)} \}) = \\
= \nu_{q, x}(\{ y \in J_x : \nu_{q, x}(f_x^{-n} y \cdot (B(f_x^n y), R)) |(f_x^n y)'(y)|^{-qT'(q) + T(q) - \varepsilon} \geq 1 \}) = \\
= \nu_{q, x}(\{ y \in J_x : \nu_{q, x}^\delta(f_x^{-n} y \cdot (B(f_x^n y), R)) |(f_x^n y)'(y)|^{-qT'(q) + T(q) - \varepsilon} \geq 1 \}) \leq \\
\int_{A(z)} \mu_{q, x}(f_x^{-n} y \cdot (B(f_x^n y), R))) |(f_x^n y)'(y)|^{-qT'(q) + T(q) - \varepsilon} d\nu_{q, x}(y) \\
\leq \sum_{k=1} \sum_{z \in f_x^{-n} y \cdot (w_k(x_n))} \mu_{q, x}^\delta(f_x^{-n} y \cdot (B(w_k(x_n), \xi))) \\
\leq M \sum_{k=1} \sum_{z \in f_x^{-n} y \cdot (w_k(x_n))} \nu_{q, x}^\delta(f_x^{-n} y \cdot (B(w_k(x_n), \xi))) |(f_x^n y)'(z)|^{-qT'(q) + T(q) - \varepsilon} \nu_{q, x}(f_x^{-n} y \cdot (B(w_k(x_n), \xi))) \\
\leq M \sum_{k=1} \sum_{z \in f_x^{-n} y \cdot (w_k(x_n))} \nu_{q, x}^{1+\delta}(f_x^{-n} y \cdot (B(w_k(x_n), \xi))) |(f_x^n y)'(z)|^{-qT'(q) + T(q) - \varepsilon} \mathcal{D}_x^\Delta \sum_{k=1} \sum_{z \in f_x^{-n} y \cdot (w_k(x_n))} \exp \left( (1 + \delta)q \{ S_{n_0} \phi(z) - P_{x}^{n_0} (\phi(z)) - (1 + \delta)P_{x}^{n_0} (\phi(z)) \} \right) \\
\cdot |(f_x^n y)'(z)|^{-qT'(q) + T(q) - \varepsilon} \exp(-1 + \delta)P_{x}^{n_0} (\phi(z))) \\
= \mathcal{D}_x^\Delta \sum_{k=1} \sum_{z \in f_x^{-n} y \cdot (w_k(x_n))} \exp \left( (1 + \delta)q \{ S_{n_0} \phi(z) - P_{x}^{n_0} (\phi(z)) - (1 + \delta)P_{x}^{n_0} (\phi(z)) \} \right) \\
\cdot |(f_x^n y)'(z)|^{-qT'(q) + T(q) - \varepsilon} \exp(-1 + \delta)P_{x}^{n_0} (\phi(z))) \]
Therefore,

\[ \sum_{j=1}^{\infty} \nu_{q,x}\left( \{ y \in \mathcal{J}_x : \mu_{q,x}(f_{y}^{-n_j}(B(f^{n_j}(y), R))) \geq |(f_{y}^{n_j})'(y)|^{-(qT'(q)+T(q)+\varepsilon)} \} \right) < +\infty. \]

Hence, by the Borel-Cantelli Lemma, there exists a measurable set \( \mathcal{J}^q_{1,\varepsilon,x} \subset \mathcal{J}_x \) such that \( \nu_{q,x}(\mathcal{J}^q_{1,\varepsilon,x}) = 1 \) and

\[ \# \left\{ j \geq 1 : \nu_{q,x}\left( \{ y \in \mathcal{J}_x : \mu_{q,x}(f_{y}^{-n_j}(B(f^{n_j}(y), R))) \geq |(f_{y}^{n_j})'(y)|^{-(qT'(q)+T(q)+\varepsilon)} \} \right) \right\} < \infty. \]

Arguing similarly, with the function \( \phi^{(1)} \) replaced by \( \phi^{(2)} \), we produce a measurable set \( \mathcal{J}^q_{2,\varepsilon,x} \subset \mathcal{J}_x \) such that \( \nu_{q,x}(\mathcal{J}^q_{2,\varepsilon,x}) = 1 \) and

\[ \# \left\{ j \geq 1 : \nu_{q,x}\left( \{ y \in \mathcal{J}_x : \mu_{q,x}(f_{y}^{-n_j}(B(f^{n_j}(y), R))) \right. \left. \leq |(f_{y}^{n_j})'(y)|^{-(qT'(q)+T(q)+\varepsilon)} \} \right) \right\} < \infty. \]

Set

\[ \mathcal{J}^q_x = \bigcap_{n=1}^{\infty} \mathcal{J}^q_{1,1/n,x} \cap \mathcal{J}^q_{2,1/n,x}. \]

Then \( \nu_{q,x}(\mathcal{J}^q_x) = 1 \) and, it follows from (2.13) and (2.1), that for all \( y \in \mathcal{J}^q_x \), we have

\[ \lim_{j \to \infty} \frac{q(P_{n_j}(y) - S_{n_j}y)}{\log |(f_{y}^{n_j})'(y)|} = -qT'(q) \]

Since \( \lim_{n \to \infty} \frac{n}{n_j} = 1 \), it thus follows from (2.3) and (2.10) that

\[ d_{\nu_{q,x}}(y) = -qT'(q) + T(q), \]

and (recall that \( \nu_{1,x} = \nu_x \) and \( T(1) = 0 \))

\[ \lim_{r \to 0} \frac{\log \nu_x(B(x,r))}{\log r} = -T'(q) \]

for all \( y \in \mathcal{J}^q_x \). As the latter formula implies that \( \mathcal{J}^q_x \subset K(-T'(q)) \), and as \( \nu_{q,x}(\mathcal{J}^q_x) = 1 \), applying (2.10), we get that

\[ g_{\mu_x}(-T'(q)) = \text{HD}(K_x(-T'(q))) \geq \text{HD}(\mathcal{J}^q_x) = -qT'(q) + T(q). \]

Combining this formula with (2.8) completes the proof. \( \square \)

As an immediate consequence of this proposition we get the following theorem.

**Theorem 8.3.** Suppose that \( f(x,z) = (\theta(x), f_z(z)) \) is a conformal random expanding map. Then the Legendre conjugate, \( g : \text{Range}(-T') \to [0, +\infty) \), to the temperature function \( \mathbb{R} \ni q \to T(q) \) is differentiable everywhere except a countable set of points, call it \( D_T^* \), and there exists a measurable set \( X_{ma} \subset X \) with \( m(X_{ma}) = 1 \) such that for every \( \alpha \in D_T^* \) and every \( x \in X_{ma} \), we have

\[ g_{\mu_x}(\alpha) = g(\alpha). \]
3. Multifractal spectrum for uniformly expanding random maps

Now, as in Section 5.1, we assume that we deal with a conformal uniform random expanding map. In particular, the essential infimum of $\gamma_x$ is larger than some $\gamma > 1$ and functions $H_x, n_x(x), j(x)$ are finite. In addition, we have that there exist constants $L$ and $c > 0$ such that

$$S_n \varphi_y \leq -nc + L$$

for every $y \in J_x$ and $n$ and $\mathcal{E} \mathcal{P}(\varphi) = 0$. With these assumptions we can get the following property of the function $T$.

**Proposition 8.4.** Suppose that $f: J \rightarrow J$ is a conformal uniformly random expanding map. Then the temperature function $T$ is real-analytic and for every $q$, we have

$$T' = \frac{\int_J \varphi d\mu_q}{\int_J \log |f'| d\mu_q} < 0.$$  

**Proof.** The potentials

$$\varphi_{q,x,t}(y) := q(\varphi_{x}(y) - P_{x}(\varphi)) - t \log |f'_{x}(y)|,$$

extend by the the same formula to holomorphic functions $\mathbb{C} \times \mathbb{C} \ni (q,t) \mapsto \varphi_{q,x,t}(y)$. Since these functions are in fact linear, we see that the assumptions of Theorem 6.17 are satisfied, and therefore the function $\mathbb{R} \times \mathbb{R} \ni (q,t) \mapsto \mathcal{E} \mathcal{P}(q,t)$ is real-analytic. Since $|f'_{x}(y)| > 0$, in virtue of Proposition 6.18 we obtain that

$$\frac{\partial \mathcal{E} \mathcal{P}(q,t)}{\partial t} = -\int_J \log |f'_{x}| d\mu_{q,x,t} dm(x) < 0.$$

Hence, we can apply the Implicit Function Theorem to conclude that the temperature function $\mathbb{R} \ni q \mapsto T(q) \in \mathbb{R}$, satisfying the equation,

$$\mathcal{E} \mathcal{P}(q, T(q)) = 0,$$

is real-analytic. Hence,

$$0 = \frac{d \mathcal{E} \mathcal{P}(\varphi)}{dq} = \left. \frac{\partial \mathcal{E} \mathcal{P}(q,t)}{\partial q} \right|_{t=T(q)} + \left. \frac{\partial \mathcal{E} \mathcal{P}(q,t)}{\partial t} \right|_{t=T(q)} T'(q).$$

Then

$$T'(q) = \left. \frac{\partial \mathcal{E} \mathcal{P}(q,t)}{\partial q} \right|_{t=T(q)} = -\frac{\int_J (\varphi_{x} - P_{x}) d\mu_{q,x} dm(x)}{\int_J \log |f'_{x}| d\mu_{q,x} dm(x)}$$

$$= \frac{\int_J \varphi_{x} d\mu_{q,x} dm(x) - \int_J P_{x} dm(x)}{\int_J \log |f'_{x}| d\mu_{q,x} dm(x)} = \frac{\int_J \varphi d\mu_q}{\int_J \log |f'| d\mu_q}.$$ 

So, we obtain (3.2). It follows, in particular, that

$$T'(q) < 0,$$

since by (3.1), the integral $\int_J \varphi d\mu_q$ is negative. □

Combining this proposition with Proposition 8.2 we get the following result which concludes this section.
Theorem 8.5. Suppose that $f : J \to J$ is a conformal uniformly random expanding map. Then the Legendre conjugate, $g : \text{Range}(-T') \to [0, +\infty)$, to the temperature function $\mathbb{R} \ni q \mapsto T(q)$ is real-analytic, and there exists a measurable set $X_{ma} \subset X$ with $m(X_{ma}) = 1$ such that for every $\alpha \in \text{Range}(-T')$ and every $x \in X_{ma}$, we have

$$g_{\mu_x}(\alpha) = g(\alpha).$$
CHAPTER 9

Examples

In this chapter we continue our investigations of geometric properties of conformal random expanding maps. We analyze a rather large class of examples. They appear as similarity maps with different contraction rates, the so-called G-systems, classical random systems, classical random conformal systems, complex dynamical systems and, perhaps most importantly, Brück and Bürgers polynomial systems. Here as a consequence of the techniques we have developed, in particular, we positively answer the question of Brück and Bürgers (see [5] and Question 5.4 in [4]) of whether the Hausdorff dimension of almost all (most) naturally defined random Julia sets is strictly larger than 1. We also show that in this same setting the Hausdorff dimension of almost all Julia sets is strictly less than 2.

1. Random Cantor Set

Here is our first example of an essentially random system. Define

\[ f_0(x) = 3x \mod 1 \] for \( x \in [0, 1/3] \cup [2/3, 1] \)

and

\[ f_1(x) = 4x \mod 1 \] for \( x \in [0, 1/4] \cup [3/4, 1] \).

Let \( X = \{0, 1\}^\mathbb{Z} \), \( \theta \) be the shift transformation and \( m \) be the standard Bernoulli measure. For \( x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in X \) define \( f_x = f_{x_0} \), \( f^n_x = f_{\theta^{n-1}(x)} \circ f_{\theta^{n-2}(x)} \circ \ldots \circ f_x \) and

\[ J_x = \bigcap_{n=0}^{\infty} (f^n_x)^{-1}([0, 1]). \]

The skew product map defined on \( \bigcup_{x \in X} J_x \) by the formula

\[ f(x, y) = (\theta(x), f_x(y)) \]

generates a conformal random expanding system. We shall show that this system is essential. To simplify the next calculation, we define recurrently:

\[ \xi_x(1) = \begin{cases} 3 & \text{if } x_0 = 0 \\ 4 & \text{if } x_0 = 1 \end{cases}, \quad \xi_x(n) = \xi_{\theta^{n-1}(x)}(1) \xi_x(n - 1). \]

Consider the potential \( \varphi^t \) defined by the formula

\[ \varphi^t_x = -t \log \xi_x(1). \]

Then

\[ S_n \varphi^t_x = -t \log \xi_x(n). \]

Let \( C_n \) be a cylinder of the order \( n \) that is \( C_n \) is a subset of \( J_x \) of diameter \( (\xi_x(n))^{-1} \) such that \( f^n_x|_{C_n} \) is one-to-one and onto \( J_{\theta^n(x)} \). We can project the measure \( m \) on \( J_x \) and we call this measure \( \mu_x \). In other words, \( \mu_x \) is such a measure that all
cylinders of level \( n \) have the measure \( 1/2^n \). Then by Low of Large Numbers for \( m \)-almost every \( x \)
\[
\lim_{n \to \infty} \frac{\log \mu_x(C_n)}{\log \text{diam}(C_n)} = \frac{\log 2}{(1/n) \log \xi_x(n)} = \frac{\log 4}{\log 12} =: h. 
\]
Therefore the Hausdorff dimension of \( J_x \) is for \( m \)-almost every \( x \) constant and equal to \( h \). Next note that
\[
(1.1) \quad \frac{\mu_x(C_n)}{\text{diam}(C_n)^h} = \exp(-S_n P_x)
\]
where
\[
P_x := \log 2 - h \log \xi_x(1).
\]
This will give us the value of the Hausdorff and packing measure. So let \( Z_0, Z_1, \ldots \) be independent random variables, each having the same distribution such that the probability of \( Z_n = \log 2 - h \log 3 \) is equal to the probability of \( Z_n = \log 2 - h \log 4 \) and is equal to \( 1/2 \). The expected value of \( Z_n, E P \), is zero and its standard deviation \( \sigma > 0 \). Then the Law of the Iterated Logarithm tells us that the following equalities
\[
\liminf_{n \to \infty} \frac{Z_1 + \ldots + Z_n}{\sqrt{n \log \log n}} = -\sqrt{2\sigma}
\]
and
\[
\limsup_{n \to \infty} \frac{Z_1 + \ldots + Z_n}{\sqrt{n \log \log n}} = \sqrt{2\sigma}
\]
hold with probability one. Then, by (1.1),
\[
\limsup_{n \to \infty} \frac{\mu_x(C_n)}{\text{diam}(C_n)^h} = \infty \quad \text{and} \quad \liminf_{n \to \infty} \frac{\mu_x(C_n)}{\text{diam}(C_n)^h} = 0
\]
for \( m \)-almost every \( x \). In particular, the Hausdorff measure of almost every fiber \( J_x \) vanishes and the packing measure is infinite. Note also that the Hausdorff dimension of fibers is not constant as clearly \( \text{HD}(J_{0 \sigma}) = \log 2/\log 3 \), whereas \( \text{HD}(J_{1 \sigma}) = \log 2/\log 4 = 1/2 \).

2. Random expanding maps on smooth manifold

Let \( M \) be an \( n \)-dimensional compact Riemannian manifold and let \( I \) be a set equipped with a probabilistic measure \( m_0 \). With every \( a \in I \) we associate a differentiable expanding transformation \( f_a \) of \( M \) into itself. Put \( X = I^\mathbb{Z} \) and let \( m \) be the product measure induced by \( m_0 \). For \( x = \ldots a_{-1}a_0a_1 \ldots \) consider \( \varphi_x := -\log |\det f_a| \). We assume that all our assumptions are satisfied. Then the measure \( \nu = \text{vol}_M \), where \( \text{vol}_M \) is the normalized Riemannian volume on \( M \) is the fixed point of the operator \( \mathcal{L}_{x, \varphi}^\infty \) with \( \lambda_x = 1 \). Let \( q_x \) be the function given by Theorem 3.1 and let \( \mu_x \) be the measure determined by \( d\mu_x/d\nu_x = q_x \).

We write \( I^\mathbb{Z} = I^{-N} \times I^N \) where points from \( I^{-N} \) we denote by \( x^- = \ldots a_{-2}a_{-1} \) and from \( I^N \) by \( x^+ = a_0a_1 \ldots \). Then \( x^-x^+ \) means \( x = \ldots a_{-1}a_0a_1 \ldots \) Note that \( q_x \) does not depend on \( x^+ \), since nor does \( \mathcal{L}_{x, \varphi}^{a, \infty} \). Then we can write \( q_x^- := q_x \) and \( \mu_x^- := \mu_x \). Since \( \mu_x(g \circ f_a) = \mu_{\varphi(x)} \),
\[
(2.1) \quad \mu_x^-(g \circ f_a) = \mu_x^-(g)
\]
for every \( a \in I \).
Define a measure $\mu^*$ by $d\mu^* = d\mu_x - dm^-(x^-)$ where $m^-$ is the product measure on $I^{-\mathbb{N}}$. Then by (2.1)
\[
\int \mu^*(g \circ f_a) dm_0(a) = \int \mu_x^-(g \circ f_a) dm^-(x^-)
= \int \int \mu_x^-(g) dm^-(x^-) dm_0(a) = \mu^*(g).
\]
Therefore, $\mu^*$ is a stationary measure.

3. Classical Expanding Random Systems

We shall now describe in detail a class of expanding random maps that because of their "random independence" can be considered as classical objects of probability theory. Let $I$ be an arbitrary set and let $m_0$ be an arbitrary probability measure on $I$. Let $m = m_\infty^0$ and $m_+$ be the product measures respectively on $I^\mathbb{Z}$ and $I^\mathbb{N}$.

Denote by $\sigma$ the shift map on $I^\mathbb{Z}$ and $I^\mathbb{N}$ alike. Both measures $m$ and $m_+$ are shift-invariant and ergodic.

**Definition 9.1.** A measurable expanding random map $T : J \to J$ is called a classical expanding random system if $I$ is a bounded metric space and the following five properties are satisfied.

(a) The base map $\theta : X \to X$ is equal to the shift map $\sigma : I^\mathbb{Z} \to I^\mathbb{Z}$ with the product invariant measure $m = m_\infty^0$.
(b) For every $a \in I$ there exists a map $T_a : Y \to Y$ such that for every $\omega \in I^\mathbb{Z},$
\[
T_{\omega_0}(J_\omega) \subset J_{\sigma(\omega)} \quad \text{and} \quad T_{\omega_0}|_{J_\omega} = T_\omega.
\]
(c) The map $T : J \to J$ has the global neighborhood property.
(d) The potential $\phi : J \to \mathbb{R}$ depends only on 0th coordinate, meaning that if $\omega_0 = \tau_0$ and $y \in J_\omega \cap J_\tau$, then $\phi(\omega, y) = \phi(\tau, y)$
(e) The map $I^\mathbb{Z} \ni \omega \mapsto L_{\phi, \omega}$ is continuous.

We say that a function $g : I^\mathbb{Z} \to \mathbb{R}$ is past independent if $g(\omega) = g(\tau)$ for any $\omega, \tau \in I^\mathbb{Z}$ with $|\omega_0^\infty| = |\tau_0^\infty|$. Fix $\kappa \in (0, 1)$ and for every function $g : I^\mathbb{Z} \to \mathbb{R}$ set
\[
v_\kappa(g) = \sup_{n \geq 0} \{v_{\kappa,n}(g)\},
\]
where
\[
v_{\kappa,n}(g) = \kappa^{-n} \sup \{|g(\omega) - g(\tau)| : |\omega_0^n| = |\tau_0^n|\}
\]
Denote by $H_\kappa$ the space of all bounded Borel measurable functions $g : I^\mathbb{Z} \to \mathbb{R}$ for which $v_\kappa(g) < +\infty$. Note that all functions in $H_\kappa$ are past independent. Let $\mathbb{Z}_-$ be the set of negative integers. If $I$ is a metrizable space and $d$ is a bounded metric on $I$, then the formula
\[
d_+(\omega, \tau) = \sum_{n=0}^{\infty} 2^{-n} d(\omega_n, \tau_n)
\]
defines a pseudo-metric on $I^\mathbb{Z}$, and for every $\tau \in I^\mathbb{Z}$, the pseudo-metric $d_+$ restricted to $\{\tau\} \times \mathbb{N}$, becomes a metric which induces the product (Tychonoff) topology on $\{\tau\} \times \mathbb{N}$.
THEOREM 9.2. Suppose that $T : \mathcal{J} \to \mathcal{J}$ and $\phi : I^Z \to \mathbb{R}$ form a classical expanding random system. Let $\lambda : I^Z \to (0, +\infty)$ be the corresponding function coming from Theorem 5.11. Then both functions $\lambda$ and $P(\phi)$ belong to $\mathcal{H}_e$ with some $\kappa \in (0, 1)$, and both are continuous with respect to the pseudo-metric $d_+$.

PROOF. Let $y$ be the global point resulting from Definition 9.1(c). Fix $n \geq 0$ and $\omega, \tau \in I^Z$ with $\omega_0^n = \tau_0^n$. By Lemma 3.2, we have
\[
\left| \frac{L_{\lambda}^{n+1}(y)}{L_{\phi}^{n}(\omega)} - \frac{L_{\lambda}^{n+1}(y)}{L_{\phi}^{n}(\tau)} \right| \leq A \kappa^n \text{ and } \left| \frac{L_{\lambda}^{n+1}(y)}{L_{\phi}^{n}(\omega)} - \lambda \right| \leq A \kappa^n
\]
with some constants $A > 0$ and $\kappa \in (0, 1)$. Since, by our assumptions, $L_{\lambda}^{n+1}(y) = L_{\tau}^{n+1}(y)$ and $L_{\phi}^{n}(\omega) = L_{\phi}^{n}(\tau)$, we conclude that $|\lambda_\omega - \lambda_\tau| \leq 2A \kappa^n$. So,
\[
v_\kappa(\lambda) \leq 2A.
\]
Since, by Lemma 5.3, the function $\lambda : I^Z \to (0, +\infty)$ is continuous, it is therefore bounded above and separated from zero. In conclusion, both functions $\lambda$ and $P(\phi)$ belong to $\mathcal{H}_e$ with some $\kappa \in (0, 1)$, and both are continuous with respect to the pseudo-metric $d_+$.

COROLLARY 9.3. Suppose that $T : \mathcal{J} \to \mathcal{J}$ and $\phi : I^Z \to \mathbb{R}$ form a classical expanding random system. Then the number (asymptotic variance of $P(\phi)$)
\[
\sigma^2(P(\phi)) = \lim_{n \to \infty} \frac{1}{n} \int \left( S_n(P(\phi)) - n\mathbb{E} P(\phi) \right)^2 dm \geq 0
\]
even exists, and the Law of Iterated Logarithm holds, i.e.
\[
-\sqrt{2\sigma^2(P(\phi))} = \liminf_{n \to \infty} \frac{P_n^\phi - n\mathbb{E} P(\phi)}{\sqrt{n \log \log n}} \leq \limsup_{n \to \infty} \frac{P_n^\phi - n\mathbb{E} P(\phi)}{\sqrt{n \log \log n}} = \sqrt{2\sigma^2(P(\phi))} \quad m - a.e.
\]

PROOF. Let $p_0 : I^Z \to I$ be the canonical projection onto the 0th coordinate and let $\mathcal{G} = p_0^{-1}(B)$, where $B$ is the $\sigma$-algebra of Borel sets of $I$. We want to apply Theorem 1.11.1 from [15]. Condition (1.11.6) is satisfied with the function $\phi$ (object being here as in Theorem 1.11.1 and by no means our potential!) identically equal to zero since $|m(A \cap B) - m(A)m(B)| = 0$ for every $A \in \mathcal{G}_0^m := \mathcal{G} \cap \sigma^{-1}(\mathcal{G}) \cap \ldots \sigma^{-m}(\mathcal{G})$ and $B \in \mathcal{G}_n^\infty = \cap_{j=0}^{\infty} \sigma^{-j}(\mathcal{G})$, whenever $n > m$. The integral $\int |P(\phi)|^2 + \beta dm$ is finite (for every $\delta > 0$) since, by Theorem 9.2, the pressure function $P(\phi)$ is bounded. This then implies that for all $n \geq 1$, $|P(\phi)(\omega) - \mathbb{E}(P(\phi)|\mathcal{G}_n^\infty)(\omega)| \leq v_\kappa(P(\phi))\kappa^n$, where $v_\kappa(P(\phi)) < +\infty$. Therefore,
\[
\int |P(\phi) - \mathbb{E}(P(\phi)|\mathcal{G}_n^\infty)| dm \leq v_\kappa(P(\phi))\kappa^n,
\]
whence condition (1.11.7) from [15] holds. Finally, $P(\phi)$ is $\mathcal{G}_n^\infty$-measurable, since $P(\phi)$ belonging to $\mathcal{H}_e$ is past independent. We have thus checked all the assumptions of Theorem 1.11.1 from [15] and, its application yields the existence of the asymptotic variance of $P(\phi)$ and the required Law of Iterated Logarithm to hold. \qed
Proposition 9.4. Let \( g \in H_{\kappa} \). Then \( \sigma^2(g) = 0 \) if and only if there exists \( u \in C((\text{supp}(m_0))^2) \) such that \( g - m(g) = u - u \circ \sigma \) holds throughout \((\text{supp}(m_0))^2\).

Proof. Denote the topological support of \( m_0 \) by \( S \). The implication that the cohomology equation implies vanishing of \( \sigma^2 \) is obvious. In order to prove the other implication, assume without loss of generality that \( m(g) = 0 \). Because of Theorem 2.51 from \([10]\) there exists \( u \in L_2(m) \) independent of the past (as so is \( g \)) such that

\[
(3.1) \quad g = u - u \circ \sigma
\]

in the space \( L_2(m) \). Our goal now is to show that \( u \) has a continuous version and \((3.1)\) holds at all points of \( S^Z \). In view of Lusin’s Theorem there exists a compact set \( K \subset S^Z \) such that \( m(K) > 1/2 \) and the function \( u|_K \) is continuous. So, in view of Birkhoff’s Ergodic Theorem there exists a Borel set \( B \subset S^Z \) such that \( m(B) = 1 \), for every \( \omega \in B \), \( \sigma^{-n}(\omega) \in K \) with asymptotic frequency \( > 1/2, u \) is well-defined on \( \bigcup_{n=-\infty}^{\infty}\sigma^{-n}(B) \), and \((3.1)\) holds on \( \bigcup_{n=-\infty}^{\infty}\sigma^{-n}(B) \). Let \( Z_- = \{1, -2, \ldots\} \) and let \( \{m_\tau\}_{\tau \in f^-} \) be the canonical system of conditional measures for the partition \( \{\{\tau \times F^n\}_\tau \}_{\tau \in f^-} \) with respect to the measure \( m \). Clearly, each measure \( m_\tau \), projected to \( F^N \), coincides with \( m_+ \). Since \( m(B) = 1 \), there exists a Borel set \( F \subset S^Z \) such that \( m_-(F) = 1 \) and \( m_\tau(B \cap \{\tau \times F^N\}) = 1 \) for all \( \tau \in F \), where \( m_- \) is the infinite product measure on \( S^Z \). Fix \( \tau \in F \) and set \( Z = p_B(B \cap \{\{\tau \times F^N\}) \), where \( p_B : I^Z \to I^N \) is the natural projection from \( I^Z \) to \( I^N \). The property that \( m_\tau(B \cap \{\{\tau \times F^N\}) = 1 \) implies that \( \mathbb{Z} = \mathbb{S}^N \). Now, it immediately follows from the definitions of \( Z \) and \( B \) that for all \( x, y \in Z \) there exists an increasing sequence \( (n_k)_{k=1}^\infty \) of positive integers such that \( \sigma^{-n_k}(\tau x), \sigma^{-n_k}(\tau y) \in K \) for all \( k \geq 1 \). For every \( 0 < q \leq n_k \) we have from \((3.1)\) that

\[
(3.2) \quad \sum_{j=0}^{n_k-q} \left( g(\sigma^j(\sigma^{-n_k}(\tau y))) - g(\sigma^j(\sigma^{-n_k}(\tau x))) \right) + \\
+ \sum_{j=n_k-q+1}^{n_k} \left( g(\sigma^j(\sigma^{-n_k}(\tau y))) - g(\sigma^j(\sigma^{-n_k}(\tau x))) \right) = \\
= (u(\sigma^{-n_k}(\tau y)) - u(\sigma^{-n_k}(\tau x)) + (u(\tau x) - u(\tau y)).
\]

Since \( g \in H_{\kappa} \), we have

\[
(3.3) \quad \sum_{j=0}^{n_k-q} \left| g(\sigma^j(\sigma^{-n_k}(\tau y))) - g(\sigma^j(\sigma^{-n_k}(\tau x))) \right| \leq \\
\leq \sum_{j=0}^{n_k-q} |g(\sigma^j(\sigma^{-n_k}(\tau y))) - g(\sigma^j(\sigma^{-n_k}(\tau y)))| \\
\leq \sum_{j=0}^{n_k-q} v_{\kappa}(g) n_k^{n_k-j} \leq v_{\kappa}(g)(1 - \kappa)^{-1} \kappa^q.
\]

Now, fix \( \varepsilon > 0 \). Take \( q \geq 1 \) so large that

\[
(3.4) \quad v_{\kappa}(g)(1 - \kappa)^{-1} \kappa^q < \varepsilon/2.
\]

Since the function \( g : I^Z \to \mathbb{R} \) is uniformly continuous with respect to the pseudometric \( d \), there exists \( \delta > 0 \) such that \( |g(b) - g(a)| < \frac{\varepsilon}{2} \) whenever \( d(a, b) < \delta \).
Assume that \( d(x, y) < \delta \) (so \( d(\sigma^{-i}(\tau x), \sigma^{-i}(\tau y)) < \delta \) for all \( i \geq 0 \)) It then follows from (3.2)-(3.4) that for every Frobenius operators induced by potentials \( \phi \) if the conditions (a)-(c) of Definition classical include that \( \lim_{k \to \infty} |u_{\tau x} - u_{\tau y}| \leq v(n)(1 - \kappa)^{-1}n^q + q\varepsilon \) on the dense set \( W \). We start with the following definition.

This section is a continuation of the previous one and its application to geometry.

Since \( d \leq \sigma^{-nk}(\tau x), \sigma^{-nk}(\tau y) \in K \) for all \( k \geq 1 \), since \( \lim_{k \to \infty} d(\sigma^{-nk}(\tau x), \sigma^{-nk}(\tau y)) = 0 \), and since the function \( u \), restricted to \( K \), is uniformly continuous, we conclude that \( \lim_{k \to \infty} |u(\sigma^{-nk}(\tau y)) - u(\sigma^{-nk}(\tau x))| = 0 \). Hence, by (3.5), we get that \( |u(\tau x) - u(\tau y)| = 0 \). This shows that the function \( u \) is uniformly continuous (with respect to the metric \( d \)) on the set

\[
W = \bigcup_{\tau \in F} B \cap \{ \{ \tau \} \times \mathbb{N} \}
\]

Since \( W = S^\mathbb{Z} \) (as \( m(W) = 1 \)) and since \( u \) is independent of the past, we conclude that \( u \) extends continuously to \( S^\mathbb{Z} \). Since both sides of (3.1) are continuous functions, and the equality in (3.1) holds on the dense set \( W \cap \sigma^{-1}(W) \), we are done.

4. Classical Conformal Expanding Random Systems

This section is a continuation of the previous one and its application to geometry. We start with the following definition.

**Definition 9.5.** A conformal expanding random map \( f : \mathcal{J} \to \mathcal{J} \) is called classical if the conditions (a)-(e) of Definition 9.1 are satisfied and the Perron-Frobenius operators induced by potentials \( \phi_i = -t \log |f'|, t \in \mathbb{R} \), are continuous (condition (e) holds). The condition (d) of Definition 9.1 is then automatically satisfied. So, \( f \) (with potentials \( \phi_i \)) is classical in the sense of Definition 9.1.

**Theorem 9.6.** Suppose \( f : \mathcal{J} \to \mathcal{J} \) is a classical conformal expanding random system. Then the following hold.

(a) The asymptotic variance \( \sigma^2(P(h)) \) exists.

(b) If \( \sigma^2(P(h)) > 0 \), then the system \( f : \mathcal{J} \to \mathcal{J} \) is essential, \( \mathcal{H}^h(\mathcal{J}_x) = 0 \) and \( \mathcal{P}^h(\mathcal{J}_x) = +\infty \) for \( m \)-a.e. \( x \in \mathbb{I}^z \).

(c) If, on the other hand, \( \sigma^2(P(h)) = 0 \), then the system \( f : \mathcal{J} \to \mathcal{J} \), reduced in the base to the topological support of \( m \) (equal to \( \text{supp}(m_0)^z \)), is quasideterministic, and then for every \( x \in \text{supp}(m) \), we have:

(c1) \( \nu_x^h \) is a geometric measure with exponent \( h \).

(c2) The measures \( \nu_x^h, \mathcal{H}^h|\mathcal{J}_x, \text{ and } \mathcal{P}^h|\mathcal{J}_x \) are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of \( x \in \mathbb{I}^z \) and \( y \in \mathcal{J}_x \).

(c3) \( 0 < \mathcal{H}^h(\mathcal{J}_x), \mathcal{P}^h(\mathcal{J}_x) < +\infty \).

(c4) \( \text{HD}(\mathcal{J}_x) = h \).
PROOF. It follows from Corollary [9.3] that the asymptotic variance \( \sigma^2(P(h)) \) exists. Combining this corollary (the Law of Iterated Logarithm) with Remark [7.3] we conclude that the system \( f : J \to J \) is essential. Hence, item (b) follows from Theorem [9.7](a). If, on the other hand, \( \sigma^2(P(h)) = 0 \), then the system \( f : J \to J \), reduced in the base to the topological support of \( m \) (equal to supp\((m_0)^2\)), is quasi-deterministic because of Proposition [9.4]. Theorem [9.2] \( (P(h) \in \mathcal{H}_e) \), and Remark [7.4] items (c1)-(c4) follow now from Theorem [7.7][b1]-[b4]. We are done. \( \square \)

As a consequence of this theorem we get the following.

THEOREM 9.7. Suppose \( f : J \to J \) is a classical conformal expanding random system. Then the following hold.

(a) Suppose that for every \( x \in I^2 \), the fiber \( J_x \) is connected. If there exists at least one \( w \in \text{supp}(m) \) such that \( \text{HD}(J_w) > 1 \), then \( \text{HD}(J_x) > 1 \) for \( m \)-a.e. \( x \in I^2 \).

(b) Let \( d \) be the dimension of the ambient Riemannian space \( Y \). If there exists at least one \( w \in \text{supp}(m) \) such that \( \text{HD}(J_w) < d \), then \( \text{HD}(J_x) < d \) for \( m \)-a.e. \( x \in I^2 \).

PROOF. Let us proof first item (a). By Theorem [9.6](a) the asymptotic variance \( \sigma^2(P(h)) \) exists. If \( \sigma^2(P(h)) > 0 \), then by Theorem [9.6](a) the system \( f : J \to J \) is essential. Thus the proof is concluded in exactly the same way as the proof of Theorem [7.8](3). If, on the other hand, \( \sigma^2(P(h)) = 0 \), then the assertion of (a) follows from Theorem [9.6](c4) and the fact that \( \text{HD}(J_w) > 1 \) and \( w \in \text{supp}(m) \).

Let us now prove item (b). If \( \sigma^2(P(h)) > 0 \), then, as in the proof of item (a), the claim is proved in exactly the same way as the proof of Theorem [7.8](4). If, on the other hand, \( \sigma^2(P(h)) = 0 \), then the assertion of (b) follows from Theorem [9.6](c4) and the fact that \( \text{HD}(J_w) < d \) and \( w \in \text{supp}(m) \). We are done. \( \square \)

5. Complex Dynamics and Brück and Bürgers Polynomial Systems

We now want to describe some classes of examples coming from complex dynamics. They will be classical conformal expanding random systems as well as \( G \)-systems defined later in this section. Indeed, having a sequence of rational functions \( F = \{f_n\}_{n=0}^\infty \) on the Riemann sphere \( \hat{C} \) we say that a point \( z \in \hat{C} \) is a member of the Fatou set of this sequence if and only if there exists an open set \( U_z \) containing \( z \) such that the family of maps \( \{f_n|_{U_z}\}_{n=0}^\infty \) is normal in the sense of Montel. The Julia set \( J(F) \) is defined to be the complement (in \( \hat{C} \)) of the Fatou set of \( F \). For every \( k \geq 0 \) put \( F_k = \{f_{k+n}\}_{n=0}^\infty \) and observe that

\[
J(F_{k+1}) = f_k(J(F_k)).
\]

Now, consider the maps \( f_c(z) = f_d,c(z) = z^d + c, d \geq 2 \). Notice that for every \( \varepsilon > 0 \) there exists \( \delta_\varepsilon > 0 \) such that if \( |c| \leq \delta_\varepsilon \), then \( f_c(\overline{B}(0,\varepsilon)) \subset \overline{B}(0,\varepsilon) \). Consequently, if \( \omega \in \overline{B}(0,\varepsilon) \mathbb{Z} \), then

\[
J(\{f_\omega\}_{n=0}^\infty) \subset \{z \in \mathbb{C} : |z| \geq \varepsilon \}
\]

and

\[
|f'_\omega(z)| \geq d\varepsilon^{d-1}
\]
for all $z \in \mathcal{J}(\{f_{\omega+k,n}\}_{n=0}^{\infty})$. Let
\[ \delta(d) = \sup\{\delta_\varepsilon : \varepsilon > \frac{d-1}{\sqrt{d}}\}. \]
Fix $0 < \delta < \delta(d)$. Then there exists $\varepsilon > \frac{d-1}{\sqrt{d}}$ such that $\delta < \delta_\varepsilon$. Therefore, by (5.2),
\[ |f'_{\omega_k}(z)| \geq d\varepsilon^{d-1} \]
for all $\omega \in B(0,\delta)^Z$, all $k \geq 0$ and all $z \in \mathcal{J}(\{f_{\omega+k,n}\}_{n=0}^{\infty})$. A straight calculation (1), p. 349) shows that $\delta(2) = 1/4$. Keep $0 < \delta < \delta(d)$ fixed. Let
\[ \mathcal{F}_{d,\delta} = \{f_{d,c} : c \in B(0,\delta)\}. \]
Consider an arbitrary ergodic measure-preserving transformation $\theta : X \to X$. Let $m$ be the corresponding invariant probability measure. Let also $H : X \to \mathcal{F}_{d,\delta}$ be an arbitrary measurable function. Set $f_{d,x} = H(x)$ for all $x \in X$. For every $x \in X$ let $J_x$ be the Julia set of the sequence $\{f_{d,n}(x)\}_{n=0}^{\infty}$, and then $\mathcal{J} = \bigcup_{x \in X} J_x$. Note that, because of (5.1), $f_{d,x}(J_x) = J_{\theta(x)}$. Thus, the map
\[ f_{d,\delta,\theta,H}(x, y) = (\theta(x), f_{d,x}(y)) \]
defines a skew product map in the sense of Section 1 of our paper. In view of (5.4), when $\theta : X \to X$ is invertible, $f_{d,\delta,\theta,H}$ is a distance expanding random system, and, since all the maps $f_x$ are conformal, $f_{d,\delta,\theta,H}$ is a conformal measurably expanding system in the sense of Definition 7.1. As an immediate consequence of Theorem 7.2 we get the following.

**Theorem 9.8.** Let $\theta : X \to X$ be an invertible measurable map preserving a probability measure $m$. Fix an integer $d \geq 1$ and $0 < \delta < \delta(d)$. Let $H : X \to \mathcal{F}_{d,\delta}$ be an arbitrary measurable function. Finally, let $f_{d,\delta,\theta,H}$ be the distance expanding random system defined by formula (5.4). Then for almost all $x \in X$ the Hausdorff dimension of the Julia set $J_x$ is equal to the unique zero of the expected value of the pressure function.

**Theorem 9.9.** For the conformal measurably expanding systems $f_{d,\delta,\theta,H}$ defined in Theorem 9.8 the multifractal theorem, Theorem 8.4 holds.

We now define and deal with Brück and Bürger polynomial systems. We still keep $d \geq 2$ and $0 < \delta < \delta(d)$ fixed. Let $X = B(0,\delta)^Z$ and let $\theta : B(0,\delta)^Z \to B(0,\delta)^Z$ to be the shift map denoted in the sequel by $\sigma$. Consider any Borel probability measure $m_0$ on $B(0,\delta)$ which is different from $\delta_0$, the Dirac $\delta$ measure supported at 0. Define $H : X \to \mathcal{F}_{d,\delta}$ by the formula $H(\omega) = f_{d,\omega_0}$. The corresponding skew-product map $f_{d,\delta} : \mathcal{J} \to \mathcal{J}$ is then given by the formula
\[ f_{d,\delta}(\omega, z) = (\sigma(\omega), f_{d,\omega_0}(z)) = (\sigma(\omega), z^d + \omega_0), \]
and $f_{d,\delta}(\omega, z) = z^d + \omega_0$ acts from $\mathcal{J}_\omega$ to $\mathcal{J}_{\sigma(\omega)}$, where $\mathcal{J}_\omega = \mathcal{J}(\{f_{d,\omega,n}\}_{n=0}^{\infty})$. Then $f : \mathcal{J} \to \mathcal{J}$ is called Brück and Bürger polynomial systems. Clearly, $f : \mathcal{J} \to \mathcal{J}$ is a classical conformal expanding random system.
In [4] Brück speculated on page 365 that if \( \delta < 1/4 \) and \( m_0 \) is the normalized Lebesgue measure on \( \overline{B}(0, \delta) \), then \( \text{HD}(\mathcal{J}_\omega) > 1 \) for \( m_+ \)-a.e. \( \omega \in \overline{B}(0, \delta)^\mathbb{N} \) with respect to the skew-product map
\[
(\omega, z) \mapsto (\sigma(\omega), z^2 + \omega_0).
\]
In [5] this problem was explicitly formulated by Brück and Bürger as Question 5.4. Below (Theorem 9.10) we prove a more general result (with regard to the measure on \( \overline{B}(0, \delta) \) and the integer \( d \geq 2 \) being arbitrary), which contains the positive answer to the Brück and Bürger question as a special case. In [4] Brück also proved that if \( \delta < 1/4 \) and the above skew product is considered then \( \lambda_2(\mathcal{J}_\omega) = 0 \) for all \( \omega \in \overline{B}(0, \delta)^\mathbb{N} \), where \( \lambda_2 \) denotes the planar Lebesgue measure on \( \mathbb{C} \). As a special case of Theorem 9.10 below we get a partial strengthening of Brück’s result saying that \( \text{HD}(\mathcal{J}_\omega) < 2 \) for \( m_+ \)-a.e. \( \omega \in \overline{B}(0, \delta)^\mathbb{N} \). Our results are formulated for the product measure \( m \) on \( \overline{B}(0, \delta)^\mathbb{Z} \), but as \( m_+ \) is the projection from \( \overline{B}(0, \delta)^\mathbb{Z} \) to \( \overline{B}(0, \delta)^\mathbb{N} \) and as the Julia sets \( \mathcal{J}_\omega, \omega \in \overline{B}(0, \delta)^\mathbb{Z} \) depend only on \( \omega \begin{array}{@{}c@{}}^{+}\infty\end{array} \), i.e. on the future of \( \omega \), the analogous results for \( m_+ \) and \( \overline{B}(0, \delta)^\mathbb{N} \) follow immediately. Proving what we have just announced, note that if \( \omega_0 \in \text{supp}(m_0) \setminus \{0\} \), then
\[
\text{HD}(\mathcal{J}_{\omega_0}^\infty) = \text{HD}(\mathcal{J}(f_{\omega_0})) \in (1, 2)
\]
(the equality holds already on the level of sets: \( \mathcal{J}_{\omega_0}^\infty = \mathcal{J}(f_{\omega_0}) \)), and by [5], all the sets \( \mathcal{J}_\omega, \omega \in \overline{B}(0, \delta)^\mathbb{Z} \), are Jordan curves. Hence, since \( f : \mathcal{J} \to \mathcal{J} \) is a classical conformal expanding random system, as an immediate application of Theorem 9.7 we get the following.

**Theorem 9.10.** If \( d \geq 2 \) is an integer, \( 0 < \delta < \delta(d) \) the skew-product map \( f_{d, \delta} : \mathcal{J} \to \mathcal{J} \) is given by the formula
\[
f_{d, \delta}(\omega, z) = (\sigma(\omega), f_{d, \omega_0}(z)) = (\sigma(\omega), z^d + \omega_0),
\]
and \( m_0 \) is an arbitrary Borel probability measure on \( \overline{B}(0, \delta) \), different from \( \delta_0 \), the Dirac \( \delta \) measure supported at 0, then for \( m \)-almost every \( \omega \in \overline{B}(0, \delta)^\mathbb{Z} \) we have
\[
1 < \text{HD}(\mathcal{J}_\omega) < 2.
\]

### 6. DG-Systems

We now want to discuss another class of expanding random maps. This is the setting from [7]. We want to describe this setting now. So, suppose that \( X_0 \) and \( Z_0 \) are compact metric spaces, \( \theta_0 : X_0 \to X_0 \) and \( T_0 : Z_0 \to Z_0 \) are open topologically exact distance expanding maps in the sense as in [115]. We assume that \( T_0 \) is a skew-product over \( Z \), i.e. for every \( x \in X_0 \) there exists a compact metric space \( \mathcal{J}_x \) such that \( Z_0 = \bigcup_{x \in X_0} \{x\} \times \mathcal{J}_x \) and the following diagram commutes
\[
\begin{array}{ccc}
Z_0 & \xrightarrow{T_0} & Z_0 \\
\downarrow \pi & & \downarrow \pi \\
X_0 & \xrightarrow{\theta_0} & X_0
\end{array}
\]
where \( \pi(x, y) = x \) and the projection \( \pi : Z_0 \to X_0 \) is an open map. Additionally, we assume that there exists \( L \) such that
\[
(6.1) \quad d_{X_0}(\theta_0(x), \theta_0(x')) \leq L d_X(x, x')
\]
for all $x \in X$ and that there exists $\xi_1 > 0$ such that, for all $x, x'$ satisfying $d_{X_0}(x, x') < \xi_1$ there exist $y, y'$ such that
\begin{equation}
(6.2) \quad d((x, y), (x', y')) < \xi.
\end{equation}

We then refer to $T_0 : Z_0 \to Z_0$ and $\theta_0 : X_0 \to X_0$ as a DG system. Note that $T_0(\{x\} \times J_x) \subset \{\theta_0(x)\} \times J_{\theta_0(x)}$ and this gives rise to the map $T_x : J_x \to J_{\theta_0(x)}$.

Since $T_0$ is distance expanding, conditions uniform openness, measurably expanding measurability of the degree, topological exactness (see Chapter 2) hold with some constants $\gamma_x \geq \gamma > 1$, $\deg(T_x) \leq N_1 < +\infty$ and the number $n_r = n_r(x)$ in fact independent of $x$. Scrutinizing the proof of Remark 2.9 in [7] one sees that Lipschitz continuity (Denker and Gordin assume differentiability) suffices for it to go through and Lipschitz continuity is incorporated in the definition of expanding maps in [15]. Now assume that $\phi : Z \to \mathbb{R}$ is a H\"older continuous map. Then the hypothesis of Theorems 2.10, 3.1, and 3.2 from [7] are satisfied. Their claims are summarized in the following.

\textbf{Theorem 9.11.} Suppose that $T_0 : Z_0 \to Z_0$ and $\theta_0 : X_0 \to X_0$ form a DG system and that $\phi : Z \to \mathbb{R}$ is a H\"older continuous potential. Then there exists a H\"older continuous function $P(\phi) : X_0 \to \mathbb{R}$, a measurable collection $\{\nu_x\}_{x \in X_0}$ and a continuous function $q : Z_0 \to [0, +\infty)$ such that
\begin{enumerate}
\item $\nu_{\theta_0(x)}(A) = \exp(P_x(\phi)) \int_A e^{-\phi} \, d\nu_x$ for all $x \in X_0$ and all Borel sets $A \subset J_x$ such that $T_x|_A$ is one-to-one.
\item $\int_{J_x} q_x \, d\nu_x = 1$ for all $x \in X_0$.
\item Denoting for every $x \in X_0$ by $\mu_x$ the measure $q_x \nu_x$ we have
\begin{equation}
\sum_{w \in \theta_0^{-1}(x)} \mu_w(T_w^{-1}(A)) = \mu_x(A)
\end{equation}
\end{enumerate}
for every Borel set $A \subset J_x$.

This would mean that we got all the objects produced in Section 4 of our paper. However, the map $\theta_0 : X_0 \to X_0$ need not be, and apart from the case when $X_0$ is finite, is not invertible. But to remedy this situation is easy. We consider the projective limit (Rokhlin’s natural extension) $\theta : X \to X$ of $\theta_0 : X_0 \to X_0$. Precisely,
\[ X = \{(x_n)_{n \leq 0} : \theta_0(x_n) = x_{n+1} \forall n \leq -1 \} \]
and
\[ \theta((x_n)_{n \leq 0}) = (\theta_0(x_n))_{n \leq 0}. \]

Then $\theta : X \to X$ becomes invertible and the diagram
\begin{equation}
\begin{array}{ccc}
X & \underset{\theta}{\longrightarrow} & X \\
| & & | \\
\downarrow & & \downarrow \\
X_0 & \underset{\theta_0}{\longrightarrow} & X_0
\end{array}
\end{equation}
commutes, where $p((x_n)_{n \leq 0}) = x_0$. If in addition, as we assume from now on, the space $X$ is endowed with a Borel probability $\theta_0$-invariant ergodic measure $m_0$,.
then there exists a unique $\theta$-invariant probability measure measure $m$ such that $m \circ \pi^{-1} = m_0$. Let

$$Z := \bigcup_{x \in X} \{x\} \times \mathcal{J}_{x_0}.$$  

We define the map $T : Z \to Z$ by the formula

$$T(x, y) = (\theta(x), T_{x_0}(y))$$

and the potential $X \ni x \mapsto \phi(x_0)$ from $X$ to $\mathbb{R}$. We keep for it the same symbol $\phi$. Clearly the quadruple $(T, \theta, m, \phi)$ is a Hölder fiber system as defined in Section 2 of our paper. It follows from Theorem 9.11 along with the definition of $z$ a commutativity of the diagram (6.3) for $x \in X$ all the objects $P_x(\phi) = P_{x_0}(\phi)$, $\lambda_x = \exp(P_x(\phi))$, $q_x = q_{x_0}$, $\nu_x = \nu_{x_0}$, and $\mu_x = \mu_{x_0}$ enjoy all the properties required in Theorem 3.1 and Theorem 3.2; in particular they are unique. From now on we assume that the measure $m$ is a Gibbs state of a Hölder continuous potential on $X$ (having nothing to do with $\phi$ or $P(\phi)$; it is only needed for the Law of Iterated Logarithm to hold). We call the quadruple $(T, \theta, m, \phi)$ DG*-system.

The following Hölder continuity theorem appeared in the paper [7]. We provide here an alternative proof under weaker assumptions.

**Theorem 9.12.** If $d_X(x, x') < \xi$, then $|\lambda_x - \lambda_{x'}| \leq H d_X^\xi(x, x')$.

**Proof.** Let $n$ be such that

$$(6.4) \quad d_X(\theta^{2n-1}(x), \theta^{2n-1}(x')) < \xi_1 \quad \text{and} \quad d_X(\theta^{2n}(x), \theta^{2n}(x')) \geq \xi_1.$$  

Let $z \in T^{-2n+1}(y)$ and $z' \in T^{-2n+1}(y')$. Then for all $k = 0, \ldots, n-1$

$$|\varphi(T^k(z)) - \varphi(T^k(z'))| \leq C d^\alpha(T^k(z), T^k(z')) \leq C \gamma^{-\alpha n} \gamma^{-\alpha(n-k-1)} \xi.$$  

Then

$$|S_n \varphi(z) - S_n \varphi(z')| \leq \frac{C \xi \gamma^{-\alpha n}}{1 - \gamma^{-\alpha}}.$$  

Put $C' := C \xi / (1 - \gamma^{-\alpha})$. Then

$$\left| \log \frac{\mathcal{L}_x^n(\mathbb{I}(w))}{\mathcal{L}_{\theta x}^n(\mathbb{I}(w'))} \right| \leq C' \gamma^{-\alpha n}$$

and

$$\left| \log \frac{\mathcal{L}_{\theta x}^{n-1}(\mathbb{I}(w))}{\mathcal{L}_{\theta x}^{n-1}(\mathbb{I}(w'))} \right| \leq C' \gamma^{-\alpha n}.$$  

Then

$$(6.5) \quad \left| \log \frac{\mathcal{L}_x^n(\mathbb{I}(w))}{\mathcal{L}_{\theta x}^{n-1}(\mathbb{I}(w))} - \log \frac{\mathcal{L}_x^n(\mathbb{I}(w'))}{\mathcal{L}_{\theta x}^{n-1}(\mathbb{I}(w'))} \right| \leq 2 C' \gamma^{-\alpha n}.$$  

Let $\alpha' := (\alpha \log \gamma) / (2 \log L)$. Then by (6.4)

$$\gamma^{-\alpha n} = L^{-2n \alpha'} \leq \frac{(d(\theta^{2n}(x), \theta^{2n}(x')))^{\alpha'}}{\xi_1^{\alpha'} L^{-2n \alpha' \xi_1}} \leq \frac{(d(x, x'))^{\alpha'}}{\xi_1^{\alpha'}}.$$  

Then (6.5) finishes the proof. \hfill \square
Since the map $\theta_0 : X_0 \to X_0$ is expanding, since $m$ is a Gibbs state, and since $P(\phi) : X_0 \to \mathbb{R}$ is Hölder continuous, it is well-known (see [15] for example) that the following asymptotic variance exists

$$
\sigma^2(P(\phi)) = \lim_{n \to \infty} \frac{1}{n} \int \left( S_n(P(\phi)) - nE_P(\phi) \right)^2 dm.
$$

The following theorem of Livsic flavor is (by now) well-known (see [15]).

**Theorem 9.13.** Suppose $(T, \theta, m, \phi)$ is a DG*-system. Then the following are equivalent.

(a) $\sigma^2(P(\phi)) = 0$.

(b) The function $P(\phi)$ is cohomologous to a constant in the class of real-valued continuous functions on $X$ (resp. $X_0$), meaning that there exists a continuous function $u : X \to \mathbb{R}$ (resp. $u : X_0 \to \mathbb{R}$) such that $P(\phi) - (u - u \circ \theta)$ (resp. $P(\phi) - (u - u \circ \theta_0)$) is a constant.

(c) The function $P(\phi)$ is cohomologous to a constant in the class of Hölder continuous functions on $X$ (resp. $X_0$), meaning that there exists a Hölder continuous function $u : X \to \mathbb{R}$ (resp. $u : X \to \mathbb{R}$) such that $P(\phi) - (u - u \circ \theta)$ (resp. $P(\phi) - (u - u \circ \theta_0)$) is a constant.

(d) There exists $R \in \mathbb{R}$ such that $P_n^x(\phi) = nR$ for all $n \geq 1$ and all periodic points $x \in X$ (resp. $X_0$).

As a matter of fact such theorem is formulated in [15] for non-invertible $(\theta_0)$ maps only but it also holds for the Rokhlin’s natural extension $\theta$. The following theorem follows directly from [15] and Theorem 9.11 (Hölder continuity of $P(\phi)$).

**Theorem 9.14.** (the Law of Iterated Logarithm) If $(T, \theta, m, \phi)$ is a DG*-system and $\sigma^2(P(\phi)) > 0$, then

$$
-\sqrt{2\sigma^2(P(\phi))} = \liminf_{n \to \infty} \frac{P_n^x(\phi) - nE_P(\phi)}{\sqrt{n \log \log n}} \leq \limsup_{n \to \infty} \frac{P_n^x(\phi) - nE_P(\phi)}{\sqrt{n \log \log n}} = \sqrt{2\sigma^2(P(\phi))} \quad m - a.e.
$$

7. Conformal DG*-Systems

Now we turn to geometry. This section dealing with, below defined, conformal DG*-systems is a continuation of the previous one in the setting of conformal systems. We shall show that these systems naturally split into essential and quasi-deterministic, and will establish their fractal and geometric properties. Suppose that $(f_0, \theta_0)$ is a DG-system endowed with a Gibbs measure $m_0$ at the base. Suppose also that this system is a random conformal expanding repeller in the sense of Section 6.1 and that the function $\phi : Z \to \mathbb{R}$ given by the formula

$$
\phi(x, y) = -\log |f'_x(y)|,
$$

is Hölder continuous.

**Definition 9.15.** The corresponding system $(f, \theta, m) = (f, \theta, m, \phi)$ (with $\theta$ the Rokhlin natural extension of $\theta_0$ as described above) is called conformal DG*-system.
For every \( t \in \mathbb{R} \) the potential \( \phi_t = t \phi \), considered in Section 6.2, is also Hölder continuous. As in Section 6.2 denote its topological pressure by \( P(t) \). Recall that \( h \) is a unique solution to the equation \( EP(t) = 0 \). By Theorem 7.2 (Bowen’s Formula) \( \text{HD}(J_x) = h \) for \( m \)-a.e. \( x \in X \). As an immediate consequence of Theorem 7.7, Theorem 9.14, and Remark 7.6, we get the following.

**Theorem 9.16.** Suppose \((f, \theta, m) = (f, \theta, m, \phi)\) is a random conformal \( DG^*\)-system.

(a) If \( \sigma^2(P(h)) > 0 \), then the system \((f, \theta, m)\) is essential, and then \( \mathcal{H}^h(J_x) = 0 \) and \( \mathcal{P}^h(J_x) = +\infty \).

(b) If, on the other hand, \( \sigma^2(P(h)) = 0 \), then \((f, \theta, m) = (f, \theta, m, \phi)\) is quasi-deterministic, and then for every \( x \in X \), we have

(b1) \( \nu_x^h \) is a geometric measure with exponent \( h \).

(b2) The measures \( \nu_x^h \), \( \mathcal{H}^h|_{J_x} \), and \( \mathcal{P}^h|_{J_x} \) are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of \( x \in X \) and \( y \in J_x \).

(b3) \( 0 < \mathcal{H}^h(J_x), \mathcal{P}^h(J_x) < +\infty \).

(b4) \( \text{HD}(J_x) = h \).

Exactly as Corollary 7.8 is a consequence of Theorem 7.7, the following corollary is a consequence of Theorem 9.16

**Corollary 9.17.** Suppose \((f, \theta, m) = (f, \theta, m, \phi)\) is a conformal \( DG^*\)-system and \( \sigma^2(P(h)) > 0 \). Then the system \((f, \theta, m)\) is essential, and for \( m \)-a.e. \( x \in X \) the following hold.

1. The fiber \( J_x \) is not bi-Lipschitz equivalent to any deterministic nor quasi-deterministic self-conformal set.
2. \( J_x \) is not a geometric circle nor even a piecewise smooth curve.
3. If \( J_x \) has a non-degenerate connected component (for example if \( J_x \) is connected), then \( h = \text{HD}(J_x) > 1 \).
4. Let \( d \) be the dimension of the ambient Riemannian space \( Y \). Then \( \text{HD}(J_x) < d \).

Now, in the same way as Theorem 9.7 is a consequence of Theorem 9.6, Corollary 9.17 yields the following.

**Theorem 9.18.** Suppose \((f, \theta, m) = (f, \theta, m, \phi)\) is a conformal \( DG^*\)-system. Then the following hold.

(a) Suppose that for every \( x \in X \), the fiber \( J_x \) is connected. If there exists at least one \( w \in \text{supp}(m) \) such that \( \text{HD}(J_w) > 1 \), then \( \text{HD}(J_x) > 1 \) for \( m \)-a.e. \( x \in I^2 \).

(b) Let \( d \) be the dimension of the ambient Riemannian space \( Y \). If there exists at least one \( w \in X \) such that \( \text{HD}(J_w) < d \), then \( \text{HD}(J_x) < d \) for \( m \)-a.e. \( x \in X \).
We end this subsection and the entire section with a concrete example of a conformal DG*-system. In particular, the three above results apply to it. Let $X := S^1_{\delta_d} = \{z \in \mathbb{C} : |z| = \delta\}$. Fix an integer $k \geq 2$. Define the map $\theta_0 : X \to X$ by the formula

$$\theta_0(x) = \delta^{1-k}x^k.$$ 

Then $\theta'_0(x) = k\delta^{1-k}x^{k-1}$, and therefore $|\theta'_0(x)| = k \geq 2$ for all $x \in X$. The normalized Lebesgue measure $\lambda_0$ on $X$ is invariant under $\theta_0$. Define the map $H : X \to F_d$ by setting $H(x) = f_x$. Then

$$f_{\theta_0,H,0}(x,y) = (k\delta^{1-k}x^{k-1},y^d + x).$$

Note that $(f_{\theta_0,H,0,\theta_0,\lambda_0})$ is a uniformly conformal DG-system and let $(f_{\theta,H,\theta},\lambda)$ be the corresponding random conformal $G$-system, both in the sense of Chapter 7. Theorem 9.16, Theorem 9.18, and Corollary 9.17 apply.

**Theorem 9.19.** If $(f_{\theta_0,H,0,\theta_0,\lambda_0})$ is the random conformal DG*-system described above, then

(f) $\mathcal{H}^h(J_x) = 0$ and $\mathcal{P}^h(J_x) = +\infty$ for $\lambda$-a.e. $x \in X$ if only $(f_{\theta_0,H,0,\theta_0,\lambda_0})$ is essential. In consequence, for $\lambda$-a.e. $x \in X$ the Julia set $J_x$ is not bi-Lipschitz equivalent to any deterministic self-conformal set. Furthermore, $J_x$ is not a geometric circle nor even a piecewise smooth curve. In fact, if $J_x$ is connected (it suffices to have a non-degenerate connected component) then

$$h = \text{HD}(J_x) > 1.$$ 

(g) If, on the other hand, $(f_{\theta_0,H,0,\theta_0,\lambda_0})$ is quasi-deterministic, then for every $x \in X$,

(g1) $\nu^h_x$ is a geometric measure with exponent $h$.

(g2) The measures $\nu^h_x$, $\mathcal{H}^h|_{J_x}$, and $\mathcal{P}^h|_{J_x}$ are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of $x \in X$ and $y \in J_x$.

(g3) $0 < \mathcal{H}^h(J_x), \mathcal{P}^h(J_x) < +\infty$.

(g4) $\text{HD}(J_x) = h$. 


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