MODULI OF QUIVER REPRESENTATIONS FOR EXCEPTIONAL COLLECTIONS ON SURFACES

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Abstract. Suppose $S$ is a smooth projective surface over an algebraically closed field $k$, $\mathcal{L} = \{L_1, \ldots, L_n\}$ is a full strong exceptional collection of line bundles on $S$. Let $Q$ be the quiver associated to this collection. One might hope that $S$ is the moduli space of representation of $Q$ with dimension vector $(1, \ldots, 1)$ for a suitably chosen stability condition $\theta: S \cong M_\theta$. In this paper, we show that this is the case for many surfaces with such collections.

1. Introduction

Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic 0. Recall that objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ in the bounded derived category of coherent sheaves $\mathcal{D}^b(\text{coh}(X))$ forms a full exceptional collection if

1. $\text{Hom}(\mathcal{E}_i, \mathcal{E}_i[m]) = k$ if $m = 0$ and is 0 otherwise;
2. $\text{Hom}(\mathcal{E}_i, \mathcal{E}_j[m]) = 0$ for all $m \in \mathbb{Z}$ if $j < i$;
3. The smallest triangulated subcategory of $\mathcal{D}^b(\text{coh}(X))$ containing $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is itself.

An exceptional collection is strong if in addition: $\text{Hom}(\mathcal{E}_i, \mathcal{E}_j[m]) = 0$ for all $i, j$ if $m \neq 0$.

In this paper we are only concerned with the case when $X$ is a smooth projective surface, the objects $\mathcal{E}_i$ are line bundles and the exceptional collection is strong. In this situation, we can consider the finite dimensional associative algebra

$$\mathcal{A} = \text{End}(\oplus_{i=1}^n \mathcal{E}_i)$$

it is well know that there is an exact equivalence of derived categories

$$\mathbf{R}\text{Hom}(\oplus_{i=1}^n \mathcal{E}_i, -): \mathcal{D}^b(\text{coh}(X)) \to \mathcal{D}^b(\text{mod-}\mathcal{A})$$

whose inverse is given by

$$(-) \otimes^L (\oplus_{i=1}^n \mathcal{E}_i): \mathcal{D}^b(\text{mod-}\mathcal{A}) \to \mathcal{D}^b(\text{coh}(X))$$

This gives a non-commutative interpretation of the derived category of $X$. We note that when we input the structure sheaf $\mathcal{O}_x$ of a close point $x \in X$ into the first functor, we
obtain:
\[
\mathbb{R} \text{Hom}(\bigoplus_{i=1}^{n} E_i, O_x) = \text{Hom}(\bigoplus_{i=1}^{n} E_i, O_x) = \bigoplus_{i=1}^{n} (E_i^\vee)_x
\]

Note since \( A^\text{op} = \text{End}(\bigoplus_{i=1}^{n} (E_i^\vee)) \) is a finite dimensional algebra, there exist a bound quiver \((Q, I)\) such that giving an \( A^\text{op} \)-module is equivalent to giving a representation of \((Q, I)\).

King\[16]\ proved that when restricted by a stability condition \( \theta \), the moduli space \( M_{\theta}(\alpha) \) of semistable representations of \((Q, I)\) with any dimension vector \( \alpha \) is a projective scheme.

In our situation, for each point \( x \in X \), one can associate the representation \( \bigoplus_{i=1}^{n} (E_i^\vee)_x \) of \( A^\text{op} \), which has dimension vector \((1, \ldots, 1)\). This provides a tautological map
\[
T_0 : X \to \text{Rep}_{(1,\ldots,1)}(Q)
\]
to the moduli stack of representation of \((Q, I)\) with dimension vector \((1, \ldots, 1)\). Following \[2\], we note \( T_0 \) induce a tautological rational map
\[
T(x) = \text{Hom}(\bigoplus_{i=1}^{n} E_i, O_x) = \bigoplus_{i=1}^{n} (E_i^\vee)_x
\]
if \( x \in X \) is in the domain of \( T \). It is natural to wonder when \( T \) is a morphism and the relation between \( M_\theta \) and \( X \) for various \( \theta \). In particular, one can ask if it is possible to choose \( \theta \) so that \( T \) is defined everywhere on \( X \) and is an isomorphism.

To study this question, it is necessary to understand the full strong exceptional collection of line bundles on surfaces. It is conjectured that the derived category of a smooth projective surface admits a full exceptional collection of line bundles if and only if the surface is rational. In the paper \[12\], L.Hille and M.Perling introduced an operation called augmentation on exceptional collection of line bundles which we recall next:

Let \( S_0 \) be a smooth surface and let \( \pi : S \to S_0 \) be a blowup of a point \( p \in S_0 \) with the corresponding \((-1)\)-curve \( E \subset S \). Let
\[
\{O_{S_0}(D_1), \ldots, O_{S_0}(D_n)\}
\]
be a collection of line bundles on \( S_0 \). For some \( 1 \leq i \leq n \) consider the collection
\[
\{O_S(\pi^*D_1+E), \ldots, O_S(\pi^*D_{i-1}+E), O_S(\pi^*D_i), O_S(\pi^*D_i+E), O_SX(\pi^*D_{i+1}), \ldots, O_S(\pi^*D_n)\}
\]
The collection \( (1.2) \) is called an augmentation of the collection \( (1.1) \). If collection \( (1.2) \) is a full strong exceptional collection on \( S \), then \( (1.1) \) is also such. The converse is not true in general but is often true under some assumptions on collection \( (1.1) \). See \[7\] Prop. 2.18 and Prop 2.27 in our paper for weak del Pezzo surfaces. A collection is called a
standard augmentation if it is obtained by a series of augmentations from a collection on $\mathbb{P}^2$ or a Hirzebruch surface.

The main result of this paper is the following theorem:

**Theorem 1.3 (Main Theorem).** Let $S = S_n \to S_{n-1} \to \ldots \to S_1 \to S_0 = \mathbb{P}^2$ be a sequence of surfaces obtained by blowing up from $\mathbb{P}^2$ such that irreducible components of exceptional divisors on those surfaces have self-intersection number either $-2$ or $-1$. Let $\{O_S(D_1), \ldots, O_S(D_n)\}$ be a full strong exceptional collection of line bundles on $S_n$ via standard augmentations from $\mathbb{P}^2$ satisfying

$$(1.4) \quad -K_S - (D_n - D_1) = H - E_1 - \ldots - E_k$$

where $E_i$ are exceptional divisors and $0 \leq k \leq 3$. Let $\mathcal{A} = \text{End}\left( \bigoplus_{i=1}^n O_S(D_i) \right)$ be the endomorphism algebra. Let $(Q, I)$ be a bound quiver for $\mathcal{A}^{\text{op}}$. Then one can choose (many) stability conditions $\theta$ such that the coarse moduli space $M_\theta$ of $\theta$-semistable representations of $(Q, I)$ with dimension vector $(1, \ldots, 1)$ is a fine moduli space, and the tautological rational map:

$$T : S \dasharrow M_\theta$$

is an isomorphism.

Note from now on if a surface $S$ is obtained by consecutive blow ups from $\mathbb{P}^2$, we call total transformations of the exceptional curves from the blow ups exceptional divisors.

There are many rational surfaces satisfying the assumptions in this theorem, i.e. the irreducible components of exceptional divisors have self-intersection number either $-2$ or $-1$. A particular interesting class is so called weak del Pezzo surface. Recall that a weak del Pezzo surface is a smooth projective surface $S$ such that $K_S^2 > 0$ and $-K_S$ is numerically effective. Alternatively, $S$ is the minimal resolution of singularities on a singular del Pezzo surface having only rational double points as singularities. Thus, $C^2 \geq -2$ for any curve $C \subset S$. The assumptions automatically holds. In [8], A.Elagin and the second author proved that any full strong exceptional collection of line bundles on weak del Pezzo surface with $K_S^2 \geq 3$ comes from standard augmentations from $\mathbb{P}^2$, while it is proven that this is false for many weak del Pezzo surface with $K_S^2 = 2$. A simple criterion of being strong exceptional for exceptional collection of line bundles on weak del Pezzo surface was also established [8 Theorem 4.7]. Thus, it is very easy to construct strong exceptional collections of line bundles satisfying the technical condition (1.4) stated in the main theorem above. Indeed, we verified using computer that there are a lot of such collections on a general rational surface. As a very special case, A.Elagin and V.Lunts showed that every full strong exceptional collection of line bundles on del Pezzo surface is elementary augmentations from that on $\mathbb{P}^2$ [7 Theorem 1.4]. Thus, the main theorem implies:

**Corollary 1.5.** Every smooth (weak)del Pezzo surface $S$ of degree $\geq 3$ is a moduli space of stable representations (with respect to a suitable stability condition $\theta$) for a bound quiver
$(Q, I)$ of a full strong exceptional collection of line bundles on $S$. Indeed, the rational map:

$$T : S \to M_\theta$$

is an isomorphism.

Remark 1.6. Not all the weak del Pezzo surface admits full strong exceptional collection of line bundles. It is proved in [8] that weak del Pezzo surface admits full strong exceptional collection of line bundles if and only if it was blown up from $F_n, n = 0, 1, 2$ at most two steps. Also note that not all the (weak)del Pezzo surface are obtained by blowing up from $P^2$, say $F_0, F_2$. In [17], A.King showed that every Hirzebruch surface is a moduli space of stable representation of quiver for full strong exceptional collection of line bundles on it.

The proof of the main theorem uses induction on standard augmentations. Suppose $S_0$ is a surface and $\mathcal{L}_0$ is a full strong exceptional collection on $S_0$ satisfying the assumptions in Main Theorem. Assume $S$ is the blow up of $S_0$ at a point and $\mathcal{L}$ is a full strong exceptional collection obtained by standard augmentation. Under the induction hypothesis that there exists $\theta_0$ so that $T_0 : S_0 \to M_{\theta_0}$ is an isomorphism. We first show how to find stability condition $\theta$ for the bound quiver of $\mathcal{L}$. Second, we construct a morphism $f : M_\theta \to M_{\theta_0}$ between moduli spaces with our choice of stability condition. Then we show that the induced morphism between moduli spaces is actually the blow up at a point and the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{T} & M_\theta \\
\downarrow{\pi} & & \downarrow{f} \\
S_0 & \xrightarrow{T_0} & M_{\theta_0}
\end{array}
$$

is commutative.

By induction, we actually proved the following result

**Corollary 1.7.** Let $S_n \to S_{n-1} \to \ldots \to S_1 \to S_0 = P^2$ be a sequence of blowing up among weak del Pezzo surfaces. Then, there exists a sequence of stability conditions $\theta_n, \theta_{n-1}, \ldots, \theta_1, \theta_0$ such that $S_i \cong M_{\theta_i}(Q_i)$ for each $i$, where $Q_i$ is the quiver section of a full strong exceptional collection of line bundles on $S_i$. This means that contracting $(-1)$-curves on $S_i$, or in another word, Minimal Model Program for those rational surfaces is realized by the sequence of blowing up maps between moduli spaces of representations of quivers.

There are many related results in the direction of realizing varieties as moduli spaces. Bergman-Proudfoot [2] proved that if a variety $X$ admits a full strong exceptional collection and the character is 'great' (a base-point-freeness condition), then $X$ is a connected component of $M_\theta$. Our result differs from theirs in the sense that we proved a stronger result (isomorphism) for a collection of varieties. Craw-Smith [4] showed that if $X$ is a projective toric variety, then we can choose a collection of globally generated line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ and a canonical choice of weight $\theta$, so that the moduli space $M_\theta$ is isomorphic
to the variety $X$. Craw-Winn [5] extended this result to all Mori Dream spaces. Note all weak del Pezzo surfaces are Mori Dream spaces [3], [18]. Our results are different from theirs in the sense that our collection of line bundles has natural categorical meaning, and our method applies to many rational surfaces with $K_X^2 < 0$. Furthermore, it is interesting to note that Y. Toda proved a similar statement in [19, Corollary 1.4]. But there, he realized any smooth surface as moduli space of Bridgeland stable objects in derived category instead of quiver moduli. The behaviour of the space of Bridgeland stability conditions induced by the augmentations of exceptional collection of line bundles will be addressed in our future work.

This paper is organized as follows: In the next section we recall some definitions and preliminary results that will be useful for later sections. In Section 3, we prove a few results on how standard augmentation affects the quiver of sections. In Section 4, we discuss the case of $S = \mathbb{P}^2$, which is the base case of induction. The last three sections are devoted to the proof of the main theorem.

Notations and Conventions.

- If $\vec{v}$ is a vector of size $n$, then we use $v_i$ to denote the $i$-th entry of the vector for $1 \leq i \leq n$.
- If $\mathcal{L}$ is a line bundle on a scheme $S$ and $s \in H^0(X, \mathcal{L})$ is a section, we use $\text{div}(s)$ to denote the divisor of zeros of $s$.

2. Preliminaries

2.1. Divisors on Surfaces. In this section, we prove some useful properties on lo/slo divisors on rational surfaces. Let $S$ be a rational smooth projective surface over an algebraically closed field $k$ of characteristic zero. Let $K_S$ be the canonical divisor on $S$. Let $d = K^2_S$ be the degree of $S$, further we always assume that $d > 0$. The Picard group $\text{Pic}(S)$ of $S$ is a finitely generated abelian group of rank $10 - d$. It is equipped with the intersection form $(D_1, D_2) \mapsto D_1 \cdot D_2$ which has signature $(1, 9 - d)$. For a divisor $D$ on $S$, we will use the following shorthand notations:

$$H^i(D) := H^i(S, O_S(D)), \quad h^i(D) = \dim H^i(D), \quad \chi(D) = h^0(D) - h^1(D) + h^2(D).$$

By the Riemann-Roch formula, one has

$$\chi(D) = 1 + \frac{D \cdot (D - K_S)}{2}.$$  

The following notions are introduced by Hille and Perling in [12, Definition 3.1].

**Definition 2.1.** A divisor $D$ on $S$ is **numerically left-orthogonal** if $\chi(-D) = 0$ (or equivalently $D^2 + D \cdot K_S = -2$).

A divisor $D$ on $S$ is **left-orthogonal** (or briefly lo) if $h^i(-D) = 0$ for all $i$.

A divisor $D$ on $S$ is **strong left-orthogonal** (or briefly slo) if $h^i(-D) = 0$ for all $i$ and $h^i(D) = 0$ for $i \neq 0$. 


Definition 2.2. We call $D$ an $r$-class if $D$ is numerically left-orthogonal and $D^2 = r$.

Motivation: if $C \subset S$ is a smooth rational irreducible curve, then the class of $C$ in $Pic(S)$ is an $r$-class where $r = C^2$.

If $C$ is an irreducible reduced curve on $S$ and $r = C^2$, it is said that $C$ is an $r$-curve. An $r$-curve is negative if $r < 0$.

The following propositions are easy consequences of Riemann-Roch formula, see [12, Lemma 3.3] or [7, Lemma 2.10, Lemma 2.11].

Proposition 2.3. Let $D$ be a numerically left-orthogonal divisor on $S$. Then
\[
\chi(D) = D^2 + 2 = -D \cdot K_S.
\]

Proposition 2.4. Suppose $D_1, D_2$ are numerically left-orthogonal divisors on $S$. Then $D_1 + D_2$ is numerically left-orthogonal if and only if $D_1D_2 = 1$. If that is the case, then
\[
\chi(D_1 + D_2) = \chi(D_1) + \chi(D_2) \quad \text{and} \quad (D_1 + D_2)^2 = D_1^2 + D_2^2 + 2.
\]

Proposition 2.5. Let $S$ be a weak del Pezzo surface $S$ of degree $d$, $d \leq 7$. Let $D$ be a left-orthogonal divisor on $S$, denote $r = D^2$. Suppose $0 \leq r \leq d - 2$, then the base set $Bs|D|$ of the linear system $|D|$ is a union of $-2$-curves.

We need several lemmas for the proof.

Lemma 2.6. Let $D$ be a divisor on a rational surface of degree $d$. Then the following conditions are equivalent
\begin{enumerate}
  \item $D$ is an $r$-class where $r = d - 2$;
  \item $D = -K_S + R$ where $R$ is a $(−2)$-class.
\end{enumerate}

Proof. Simple calculations.

Lemma 2.7. Let $S$ be a rational surface of degree $d$, let $D$ be a $(d - 2)$-class on $S$. Then $D \cdot C \geq 0$ for any $-1$-class $C$ on $S$ unless $d = 1$ and $C = D$.

Proof. By Lemma 2.6 $D = -K_S + R$ where $R$ is a $-2$-class on $S$. Consider a sequence of $d - 1$ (arbitrary) blow-ups $Y = S$ such that $Y$ is a surface of degree 1. There is an embedding $Pic(S) \rightarrow Pic(Y)$ and we will identify $Pic(S)$ with its image in $Pic(Y)$. By Lemma 2.6 $C = -K_Y + S$ where $S$ is a $-2$-class on $Y$. We have $K_Y = K_S + E_1 + \ldots + E_{d-1}$ where $E_i$ denote exceptional curves of the blow-ups. It follows that $R \cdot K_Y = 0$ and $R$ is a $-2$-class on $Y$. Now we have
\[
D \cdot C = (-K_S + R)(-K_Y + S) = (-K_Y + (E_1 + \ldots + E_{d-1}) + R)(-K_Y + S) =
K_Y^2 + (E_1 + \ldots + E_{d-1})C + RS.
\]
Note that $(E_1 + \ldots + E_{d-1})C = 0$ because $C \in Pic(S)$. Further, $R, S \in K_Y^1 \subset Pic(Y)$ and the intersection form on $K_Y^1$ is negative definite. It follows that $|RS| \leq \sqrt{|R^2||S^2|} = 2$, therefore $RS \geq -2$. We get
\[
D \cdot C = K_Y^2 + RS = 1 + RS \geq -1.
\]
Suppose we have got an equality $D \cdot C = -1$, then $RS = -2$. Consequently, $R = S$ and $K_Y = S - C = R - C \in \text{Pic}(S)$ what implies that $S = Y$, $d = 1$ and $C = D$. \hfill $\square$

**Proof of Proposition 2.5.** First we consider the special case $r = d - 2$. Let us check that $|D|$ is non-empty for any $(d - 2)$-class $D$. Indeed, $h^2(D) = h^0(K_S - D) = 0$ because $(K_S - D)(-K_S) = -d + DK_S = -d - 2 - (d - 2) = -2d < 0$. Thus $h^0(D) \geq h^0(D) - h^1(D) = \chi(D) = d > 0$.

We claim that there exists a sequence $R_1, \ldots, R_m$ of irreducible $-2$-curves with the following properties. Denote $D_0 = D$ and $D_k = D_{k-1} - R_k$ for $1 \leq k \leq m$. Then

1. $R_k \subset Bs|{D_{k-1}}|$;
2. $R_k \cdot D_{k-1} = -1$;
3. $D_k$ is a left-orthogonal $(d - 2)$-class;
4. if $r \geq 0$: $|D_m|$ is nonempty and has no base points; if $r = -1$: $D_m$ is irreducible.

We construct this sequence of $R_i$ by induction. Suppose for some $k$ the sequence $R_1, \ldots, R_k$ is constructed such that conditions (i)-(iii) are satisfied.

Suppose for any $-2$-curve $R$ we have $R \cdot D_k \geq 0$. We claim that condition (iv) holds. We may assume that $D_k \geq 0$. Since $D_k$ is left-orthogonal, it is a sum of smooth rational curves $E_i$ with $E_i^2 \geq -2$. By assumption, for any $-2$-curve $R$ we have $R \cdot D_k \geq 0$. Further, suppose that for any $-1$-curve $C$ we have $C \cdot D_k \geq 0$. Obviously, for any $E_i$ with $E_i^2 \geq 0$ and for any $C$ which is not a component of $D_k$, one has $E_i \cdot D_k \geq 0$ and $C \cdot D_k \geq 0$. Therefore $D_k$ is nef. By [Derenthal], Lemma 7, the linear system $|D_k|$ has no base points. It means that $r = D_k^2 \geq 0$, and (4) holds. Now suppose that for some $-1$-curve $C$ we have $C \cdot D_k < 0$. By Lemma 2.7 it follows that $d = 1$ (thus $r = -1$) and $D_k = C$ is a $-1$-curve. Again, (iv) holds, so we terminate the process.

Now suppose that for some $-2$-curve $R$ we have $R \cdot D_k < 0$. It follows that $R$ is a base component of $|D_k|$. Denote $P_k = D_k + K_S$, by Lemma 2.6 the divisor $P_k$ is a $-2$-class. One has $R \cdot D_k = R \cdot (P_k - K_S) = R \cdot P_k$. Since $R$ and $P_k$ are $-2$-classes, we get $R \cdot P_k \geq -2$, and $R \cdot P_k = -2$ only if $R = P_k$. If $R \cdot D_k = R \cdot P_k = -2$, then $h^2(-D_k) = h^0(K_S + D_k) = h^0(P_k) = h^0(R) = 1$. It is impossible because $D_k$ is lo. Thus, $R \cdot D_k = -1$. We let $R_{k+1} = R$, $D_{k+1} = D_k - R_{k+1}$. It is easy to check that $D_{k+1}$ is an $(d - 2)$-class. The next element of sequence $R_i$ is constructed.

It remains to note that this process cannot be infinite: all divisors $D_k$ are different because no sum $\sum_{i=1}^k R_i$ of effective curves can be zero. But the number of $(d - 2)$-classes in $\text{Pic}(S)$ is finite. Thus, the required sequence is constructed. Finally, $|D| = R_1 + \ldots + R_m + |D_m|$. If $r \geq 0$, then the linear system $|D_m|$ has no base points, therefore $Bs|D| = R_1 + \ldots + R_m$.

Now we treat the general case $-1 \leq r \leq d - 2$. This case is reduced to the case $r = d - 2$ by taking a certain blow-up of $S$. Let $p: \bar{S} \to S$ be the blow-up of $d - 2 - r$ general points on $S$. Degree of $\bar{S}$ is $d - (d - 2 - r) = r + 2 \geq 1$. Since the points are general, $\bar{S}$ is a weak del Pezzo surface. Let $\tilde{D} = p^*D$, then $\tilde{D}$ is left-orthogonal and $(\tilde{D})^2 = D^2 = r = \deg(\bar{S}) - 2$. If $r \geq 0$, then the linear system $|\tilde{D}|$ has a union of $-2$-curves as a base set. It follows that
no points on $Bs|D|$ are blown up by $p$ (otherwise $Bs|\tilde{D}| = p^*(Bs|D|)$ would contain at least one $-1$-curve). Therefore $Bs|D| \cong p^*(Bs|D|) = Bs|\tilde{D}|$ and $Bs|D|$ is also a union of $-2$-curves.

Let $S_0$ be a rational surface which is a blow up of $\mathbb{P}^2$, let $p : S_0 \to \mathbb{P}^2$ be the natural morphism. Let $S$ be surface obtained from blowing up $S_0$ at a point $P$. We use $\pi : S \to S_0$ to denote the projection morphism. Assume for all exceptional divisors on $S, S_0$, each irreducible component has self-intersection number either $-2$ or $-1$.

**Lemma 2.8.** The linear system $|p^*H|$ on $S_0$ has no base points. The point $P$ is not a base point of $|p^*H - E_1|$ for any exceptional divisor $E_1$ on $S_0$.

**Proof.** Let $s \in S_0$, choose a line $l$ in $\mathbb{P}^2$ that does not pass through $p(s)$. Let $D$ be the corresponding divisor. Then $p^*(D)$ does not pass through $s$.

Let $P' = p(E_1)$. If $P \neq P'$, we choose a line $l$ in $\mathbb{P}^2$ so that it passes through $P'$ but does not pass through $P$. Let $D$ be the corresponding divisor, then $D - E_1$ is effective, but $D - E_1$ does not pass through $P$.

If $P = P'$, we notice $p^{-1}(P') - E_1$ has a collection of $\mathbb{P}^1$ as irreducible components, all of whose self intersection number is $-2$. Moreover, $p^{-1}(P') - E_1$ is the base locus of the linear system $|p^*H - E_1|$. But then $P \notin p^{-1}(P') - E_1$, since otherwise blowing up $P$ would result in a smooth rational curve in $S$ with self intersection number $-3$, which contradicts our assumption. Thus $P$ is not a base point of $|p^*H - E_1|$.

We provide alternative proof for the above two results in the situation when both $S$ and $S_0$ are weak del Pezzo surfaces.

**Proof.** $p^*H$ is a nef divisor on $S_0$, thus $|p^*H|$ is base point free by [Derenthal]. By Proposition 2.6, the base locus of $|p^*H - E_1|$ is a union of $(-2)$-curves, thus $P$ can not be the base point of it, otherwise, there will be $(-3)$-curve on $S$ contradicts the fact that $S$ is a weak del Pezzo surface.

**Corollary 2.9.** In the setting above, there is an effective divisor $D$ in the linear system $|(\pi p)^*H|$ such that $D - E$ is not effective. In particular

$$h^0(S, (\pi p)^*H - E) = h^0(S, (\pi p)^*H) - 1 = 2$$

There is an effective divisor $D'$ in the linear system $|(\pi p)^*H - \pi^*(E_1)|$ such that $D' - E$ is not effective. In particular

$$h^0(S, (\pi p)^*H - E - E_1) = h^0(S, (\pi p)^*H - E_1) - 1 = 1$$

**Lemma 2.10.** Let $F$ be an exceptional divisor, then the natural map

$$H^0(S, H) \otimes H^0(S, H - F) \to H^0(S, 2H - F)$$

is surjective.
**Proof.** Note $h^0(S, H - F) = 2$. Pick two linear functions $f, g$ such that

$$H^0(S, H - F) = \text{span}(f, g)$$

We can find another linear function $h$ so that

$$H^0(S, H) = \text{span}(f, g, h)$$

then $H^0(S, H) \otimes H^0(S, H - F) = \text{span}(f^2, fg, g^2, fh, gh)$ which is a vector space of dimension five. Since $h^0(2H - F) = 5$, we see the natural inclusion is surjective. □

**Lemma 2.11.** Let $E_1, E_2$ be two exceptional divisors. If neither $E_1 - E_2$ nor $E_2 - E_1$ is effective, then the natural map

$$H^0(S, H - E_1) \otimes H^0(S, H - E_2) \to H^0(S, 2H - E_1 - E_2)$$

is surjective.

**Proof.** If $S$ is weak del Pezzo, note that $H - E_1$ is a nef divisor, thus $O(H - E_1)$ is globally generated. Then the result of the lemma follows directly from Gallego and Purnaprajna [GP99, p. 154].

In general, by Corollary 2.9, $h^0(S, H - E_1 - E_2) = 1$, let the corresponding linear function be $f$. Then we can choose linear function $g, h$ such that

$$H^0(S, H - E_1) = \text{span}(f, g)$$

$$H^0(S, H - E_2) = \text{span}(f, h)$$

By our assumption, $g, h$ are linearly independent, then the image of $H^0(S, H - E_1) \otimes H^0(S, H - E_2)$ contains $f^2, fg, fh, gh$, which span a 4 dimensional vector space. Since $h^0(S, 2H - E_1 - E_2) = 4$, we see the natural map is surjective. □

**Lemma 2.12.** Let $E_1, E_2, E$ be exceptional divisors. Suppose $E_1 - E_2$ is a strong left orthogonal divisor and $E_1 - E > 0$, then the natural map

$$H^0(S, H - E_1) \otimes H^0(S, H - E) \to H^0(S, 2H - E_1 - E)$$

is not surjective, the image has dimension 3 and can span $H^0(S, 2H - E_1 - E)$ along with an effective divisor $D$ with the property that $D - E_2 > 0$.

**Proof.** By Corollary, $h^0(S, H - E_1 - E) = 1$. Let the corresponding linear function be $f$. Suppose $H^0(S, H - E_1) = \text{span}(f, g)$. Since $E_1 - E > 0$, $H^0(S, H - E) = \text{span}(f, g)$. Hence the image of $H^0(S, H - E_1) \otimes H^0(S, H - E)$ is $\text{span}(f^2, fg, g^2)$, which has dimension 3.

Since $E_1 - E_2$ is strong left orthogonal, $H^0(S, H - E_1) \neq H^0(S, H - E_2)$ when considered as subspace of $H^0(S, H)$. Suppose $h$ is a linear function such that $h \in H^0(S, H - E_2) \setminus H^0(S, H - E_1)$, then $H^0(S, H) = \text{span}(f, g, h)$, and $fh \in H^0(S, 2H - E_1 - E)$. Since $h$ passes through $E_2$, we see $\text{div}(fh) - E_2 > 0$.

**Lemma 2.13.** Suppose $h^0(S, H - E_1 - E_2 - E_3) = 0$, $E_1 - E > 0$ and $E_1$ and $E_2$ do not share any irreducible components. Then $h^0(S, H - E - E_2 - E_3) = 0$. 
Proof. If $S$ is weak del Pezzo, this directly follows from the effectiveness criterion of divisors on weak del Pezzo surfaces \[8\] Lemma A.5: $H - E - E_2 - E_3 = (H - E_1 - E_2 - E_3) + (E_1 - E)$, where $E_1 - E > 0$ but $H - E_1 - E_2 - E_3$ is not effective.

In general, since $h^0(S, H - E_1 - E_2 - E_3) = 0$, $h^0(S, H - E_1 - E_3) = 1$. By Lemma 2.8 $h^0(S, H - E_3 - E_3) = 1$. Since $E_1 - E > 0$, we have the short exact sequence

$$0 \to \mathcal{O}_S(-(E_1 - E)) \to \mathcal{O}_S \to \mathcal{O}_{E_1 - E} \to 0$$

Twisting this by $H - E_3 - E$, we obtain

$$0 \to \mathcal{O}_S(H - E_1 - E_3)) \to \mathcal{O}_S(H - E - E_3) \to \mathcal{O}_{E_1 - E}(H - E - E_3) \to 0$$

hence we have the inclusion

$$H^0(H - E_1 - E_3) \hookrightarrow H^0(H - E - E_3)$$

By looking at dimensions, we see this is an isomorphism. Let $D$ be the unique effective divisor linearly equivalent to $H - E_1 - E_3$, $D - E_2$ is not effective. Also $D + (E_1 - E)$ is the unique effective divisor linearly equivalent to $H - E - E_3$. Since $E_1$ and $E_2$ share no components, $D + (E_1 - E) - E_2$ is not effective. Hence $h^0(S, H - E - E_2 - E_3) = 0$. \[\square\]

2.2. Exceptional Toric systems and standard augmentations. We recall the important notion of a toric system, introduced by Hille and Perling in \[12\]. For a sequence $(O_S(D_1), \ldots, O_S(D_n))$ of line bundles one can consider the infinite sequence (called a helix) $(O_S(D_i))$, $i \in \mathbb{Z}$, defined by the rule $D_{k+n} = D_k - K_S$. From Serre duality it follows that the collection $(O_S(D_1), \ldots, O_S(D_n))$ is exceptional (resp. numerically exceptional) if and only if any collection of the form $(O_S(D_{k+1}), \ldots, O_S(D_{k+n}))$ is exceptional (resp. numerically exceptional). One can consider the $n$-periodic sequence $A_k = D_{k+1} - D_k$ of divisors on $S$. Following Hille and Perling, we will consider the finite sequence $(A_1, \ldots, A_n)$ with the cyclic order and will treat the index $k$ in $A_k$ as a residue modulo $n$. Vice versa, for any sequence $(A_1, \ldots, A_n)$ one can construct the infinite sequence $(O_S(D_i)), i \in \mathbb{Z}$, with the property $D_{k+1} - D_k = A_k \mod n$.

**Definition 2.14** (See \[12\] Definitions 3.4 and 2.6). A sequence $(A_1, \ldots, A_n)$ in $Pic(S)$ is called a toric system if $n = \text{rank} K_0(S)$ and the following conditions are satisfied (where indexes are treated modulo $n$):

- $A_i \cdot A_{i+1} = 1$;
- $A_i \cdot A_j = 0$ if $j \neq i, i \pm 1$;
- $A_1 + \ldots + A_n = -K_S$.

Note that a cyclic shift $(A_k, A_{k+1}, \ldots, A_n, A_1, \ldots, A_{k-1})$ of a toric system $(A_1, \ldots, A_n)$ is also a toric system. Also, note that by our definition any toric system has maximal length.

**Example 2.15.** Let $Y$ be a smooth projective toric surface. Its torus-invariant prime divisors form a cycle, denote them $T_1, \ldots, T_n$ in the cyclic order. Then $(T_1, \ldots, T_n)$ is a toric system on $Y$. 
Definition 2.16. Given an exceptional collection of line bundles $\mathcal{O}_S(D_1), \ldots, \mathcal{O}_S(D_n)$, the corresponding toric system is defined to be a collection of line bundles $(A_1, \ldots, A_n)$ where

$$A_i = \begin{cases} D_{i+1} - D_i & \text{if } i = 1, 2, \ldots, n-1 \\ -K_S - (D_n - D_1) & \text{if } i = n. \end{cases}$$

The following is proved in [12, Lemma 3.3], see also [7, Propositions 2.8 and 2.15].

Proposition 2.17. A sequence $(A_1, \ldots, A_n)$ in $\text{Pic}(S)$ is a toric system if and only if $n = \text{rank} K_0(S)$ and the corresponding collection $\mathcal{O}_S(D_1), \ldots, \mathcal{O}_S(D_n)$ is numerically exceptional.

For the future use we make the following remark, see the proof of Proposition 2.7 in [12].

Remark 2.18. For any toric system $(A_1, \ldots, A_n)$ the elements $A_1, \ldots, A_n$ generate $\text{Pic}S$ as abelian group.

A toric system $(A_1, \ldots, A_n)$ is called exceptional (resp. strong exceptional) if the corresponding collection $(\mathcal{O}_S(D_1), \ldots, \mathcal{O}_S(D_n))$ is exceptional (resp. strong exceptional). Note that exceptional toric systems are stable under cyclic shifts while strong exceptional toric systems are not in general.

A toric system $(A_1, \ldots, A_n)$ is called cyclic strong exceptional if the collection $(\mathcal{O}_S(D_{k+1}), \ldots, \mathcal{O}_S(D_{k+n}))$ is strong exceptional for any $k \in \mathbb{Z}$. Equivalently: if all cyclic shifts $(A_k, A_{k+1}, \ldots, A_n, A_1, \ldots, A_{k-1})$

are strong exceptional.

Notation: for a toric system $(A_1, \ldots, A_n)$ denote

$$A_{k,k+1,\ldots,l} = A_k + A_{k+1} + \ldots + A_l.$$

We allow $k > l$ and treat $[k, k+1, \ldots, l] \subset [1, \ldots, n]$ as a cyclic segment. Note that $A_{k,l}$ is a numerically left-orthogonal divisor with

$$A_{k,l}^2 + 2 = \sum_{i=k}^{l}(A_i^2 + 2).$$

Remark 2.20. If for a toric system $A$ one has $A_i^2 \geq -2$ for all $i$, then one has $A_{k,l}^2 \geq -2$ for any cyclic segment $[k, k+1, \ldots, l] \subset [1, \ldots, n]$.

Remark 2.21. If $A_{k,l}$ is a strong left-orthogonal divisor, then $h^0(A_{k,l}) = A_{k,l}^2 + 2 = \sum_{i=k}^{l}(A_i^2 + 2) = \sum_{i=k}^{l}A_i^2 + 2(l - k + 1)$.

The following theorem is proved in [S]

Theorem 2.22. Let $A = (A_1, \ldots, A_n)$ be a toric system on a weak del Pezzo surface $S$. Suppose $A_i^2 \geq -2$ for $1 \leq i \leq n - 1$. Then:
(1) $A$ is exceptional if and only if the following holds: for any cyclic segment $[k \ldots l] \subset \{1 \ldots n\}$ such that one of the next conditions holds:
   
   (a) $1 \leq k \leq l \leq n - 1$ and $A^2_{k..l} = -2$,
   (b) $k > l$ and $A^2_{k..n..l} = A^2_n \leq -2$
   
   the divisor $A_{k..l}$ is left-orthogonal (or equivalently: the divisor $A_{k..l}$ is not anti-effective).

(2) $A$ is strong exceptional if and only if $A$ is exceptional and the following holds: for any $1 \leq k \leq l \leq n - 1$ such that $A^2_{k..l} = -2$, the divisor $A_{k..l}$ is strong left-orthogonal (or equivalently: the divisor $A_{k..l}$ is not effective nor anti-effective).

Suppose moreover that $A^2_i \geq -2$ for all $i$. Then

(3) $A$ is exceptional if and only if the following holds: for any cyclic segment $[k \ldots l] \subset \{1 \ldots n\}$ such that $A^2_{k..l} = -2$, the divisor $A_{k..l}$ is left-orthogonal (or equivalently: the divisor $A_{k..l}$ is not anti-effective).

(4) $A$ is cyclic strong exceptional if and only if the following holds: for any cyclic segment $[k \ldots l] \subset \{1 \ldots n\}$ such that $A^2_{k..l} = -2$, the divisor $A_{k..l}$ is strong left-orthogonal (or equivalently: the divisor $A_{k..l}$ is not effective nor anti-effective).

2.2.1. Augmentations. Following Hille and Perling [12], we define augmentations. They provide a wide class of explicitly constructed toric systems.

Let $A' = (A'_1, \ldots, A'_n)$ be a toric system on a surface $S'$, and let $p: S \to S'$ be the blow up of a point with the exceptional divisor $E \subset S$. Denote $A_i = p^*A'_i$. Then one has the following toric systems on $S$:

- $\text{augm}_{p,1}(A') = (E, A_1 - E, A_2, \ldots, A_{n-1}, A_n - E)$;
- $\text{augm}_{p,m}(A') = (A_1, \ldots, A_{m-2}, A_{m-1} - E, E, A_m - E, A_{m+1}, \ldots, A_n)$ for $2 \leq m \leq n$;
- $\text{augm}_{p,n+1}(A') = (A_1 - E, A_2, \ldots, A_{n-1}, A_n - E, E)$.

Toric systems $\text{augm}_{p,m}(A')$ ($1 \leq m \leq n + 1$) are called elementary augmentations of toric system $(A'_1, \ldots, A'_n)$.

**Proposition 2.23** (See [7, Proposition 3.3]). In the above notation, let $A$ be a toric system on $S$ such that $A_m = E$ for some $m$. Then $A = \text{augm}_{p,m}(A')$ for some toric system $A'$ on $S'$.

**Definition 2.24.** A toric system $A$ on $S$ is called a standard augmentation if $S$ is a Hirzebruch surface or $A$ is an elementary augmentation of some standard augmentation. Equivalently: $A$ is a standard augmentation if there exists a chain of blow-ups

$$S = S_n \xrightarrow{p_n} S_{n-1} \to \ldots \to S_1 \xrightarrow{p_1} S_0$$

where $S_0$ is the Hirzebruch surface and

$$A = \text{augm}_{p_n, k_n}(\text{augm}_{p_{n-1}, k_{n-1}}(\ldots \text{augm}_{p_1, k_1}(A') \ldots))$$

for some $k_1, \ldots, k_n$ and a toric system $A'$ on $S_0$. In this case we will say that $A$ is a standard augmentation along the chain $p_1, \ldots, p_n$. 
Remark 2.25. To be more accurate, one should add that (the unique) toric system on $\mathbb{P}^2$ is also considered as a standard augmentation. To simplify the forthcoming definitions and statements, we will ignore this issue.

Proposition 2.26 (See [7], Proposition 2.21). Let $A = \text{augm}_k(A')$. Then

1. $A$ is exceptional if and only if $A'$ is exceptional;
2. if $A$ is strong exceptional then $A'$ is strong exceptional;
3. if $A$ is cyclic strong exceptional then $A'$ is cyclic strong exceptional.

The following proposition gives a very simple algorithm to generate (cyclic)strong exceptional toric systems on weak del Pezzo surfaces by augmentations.

Proposition 2.27. Let $S$ be weak del Pezzo surface and $A$ is toric system on $S, A = \text{augm}_{p,m}(A')$. Where $A'$ is toric system on $S'$ and $\pi : S \to S$ is blow up at $P \in S'$ and $E \in S$ is the correspondent exceptional divisor. Then

1. $A$ is strong exceptional iff $A'$ is strong exceptional and the point $P$ is not a base point of the linear system $|A_k + A_{k+1} + \ldots + A_l|$ of divisors on $S'$ for any $1 \leq k \leq l \leq n - 1$, such that $k \leq m, m - 1 \leq l$ and $(A_k' + A_{k+1}' + \ldots + A_l')^2 \leq -1$, i.e. if $F \in |A_k + A_{k+1} + \ldots + A_l|$, $F - E$ is not an effective $(-2)$-class and the $(F - E)^2 \geq -2$.
2. $A$ is cyclic strong exceptional iff $A'$ is cyclic strong exceptional and the point $P$ is not a base point of the linear system $|A_k' + A_{k+1}' + \ldots + A_l'|$ of divisors on $S'$ for any $1 \leq k \leq l$ in any cyclic interval of length $\leq n - 1$, such that $(A_k' + A_{k+1}' + \ldots + A_l')^2 \leq -1$, i.e. if $F \in |A_k' + A_{k+1}' + \ldots + A_l'|$, $F - E$ is not an effective $(-2)$-class and the $(F - E)^2 \geq -2$.

Proof. By proposition 2.18 of [7], $A$ is (cyclic)strong exceptional iff $A'$ is (cyclic)strong exceptional and point $P$ is not a base point of the linear system $|A_k' + A_{k+1}' + \ldots + A_l'|$ of divisors on $S'$ for any $1 \leq k \leq l \leq n - 1$ (when $k \leq l$ in cyclic segment of length $\leq n - 1$). Note that $(A_k' + A_{k+1}' + \ldots + A_l')^2 \geq -2$ and $D := A_k' + A_{k+1}' + \ldots + A_l'$ is strong left-orthogonal. If $A'$ is cyclic strong exceptional toric system, $-2 \leq D^2 \leq d - 2(d = K_S^2)$. By Proposition 2.25 $Bs|D|$ is a union of $(-2)$-curves if $0 \leq D^2 \leq d - 2$ Then $P$ can not be the base point of $|D|$, otherwise, there is $(-3)$-curve on $S = Bl_p S'$ which contradicts to the assumption that $S$ is weak del Pezzo surface. If $A'$ is strong exceptional toric system, $-2 \leq D^2$. If $D^2 \geq 0$, it is enough to show that $\pi^*(D) - E$ as sum of left-orthogonal divisors of $A$ on $S$ is strong left-orthogonal. Note that $(\pi^*(D) - E)^2 = D^2 - 1 \geq -1$ and it is left-orthogonal. Thus, it is strong left-orthogonal divisor. If $D^2 = -2$, then any point $P$ would be the base point of $|D|$, thus the point $P$ can not be augmented at a slo divisor with self intersection number $-2$. If $D^2 = -1$, then $Bs|D|$ is a union of $(-1)$-curves an a chain of $(-2)$-curves(by Proposition 8.1 and [7]). Thus $P$ is a base point of $Bs|D|$ means that $F - E$ is an effective $(-2)$-class, where $F \in |D|$. Hence, exceptional toric system $A'$ is not strong exceptional. Conversely, if $P$ is not base point of $|D|$, $F - E$ is a sum of effective $(-2)$-class and a slo $(-2)$-divisor. By Lemma A.2 in [8], it is not effective. \qed
2.3. Quivers and quiver representations. A quiver $Q$ is given by two sets $Q_{vx}$ and $Q_{ar}$, where the first set is the set of vertices and the second is the set of arrows, along with two functions $s, t : Q_{ar} \rightarrow Q_{vx}$ specifying the source and target of an arrow. The path algebra $kQ$ is the associative $k$-algebra whose underlying vector space has a basis consists of elements of $Q_{ar}$. The product of two basis elements is defined by concatenation of paths if possible, otherwise 0. The product of two general elements is defined by extending the above linearly. A bound quivers is a pair $(Q, I)$. Here $Q$ is a quiver and $I$ is a two sided ideal of $kQ$ generated by elements of the form $\sum_{i=1}^{n} k_i p_i$, where $k_i \in k^*$ and $p_i$ are paths with same heads and same tails for $i \in \{1, \ldots, n\}$. We simply use $Q$ to denote this pair when the existence of $I$ is understood.

Let $Q$ be a quiver. A quiver representation $R = (R_v, r_a)$ consists of a vector space $R_v$ for each $v \in Q_{vx}$ and a morphism of vector spaces $r_a : R_{s(a)} \rightarrow R_{t(a)}$ for each $a \in Q_{ar}$. For a bound quivers $(Q, I)$, a representation $R = (R_v, r_a)$ is same as above, with the additional condition that

$$\sum_{i=1}^{n} k_i r_{p_i} = 0$$

if $\sum_{i=1}^{n} k_i p_i$ is a generator of $I$. A subrepresentation of $R$ is a pair $R' = (R'_v, r'_a)$ where $R'_v$ is a subspace of $R_v$ for each $v \in Q_{vx}$ and $r'_a : R'_{s(a)} \rightarrow R'_{t(a)}$ a morphism of vector spaces for each $a \in Q_{ar}$ such that

$$r'_a = r_a |_{R'_v}$$

and

$$(2.28) \quad r_a(R'_{s(a)}) \subset R'_{t(a)}$$

Thus we have the commutative diagram

$$
\begin{array}{ccc}
R_i' & \xrightarrow{r'_a} & R_j' \\
\downarrow{\iota_i} & & \downarrow{\iota_j} \\
R_i & \xrightarrow{r_a} & R_j
\end{array}
$$

for any arrow $a$ from $i$ to $j$. We use $R' \subset R$ to denote that $R'$ is a subrepresentation of $R$.

If the vertices of a quiver has a natural ordering, as it will be the case when we discuss quiver of sections of an exceptional collection of line bundles, we define dimension vector $\vec{d}$ so $d_i$ is the dimension of the vector space $R_i$ at that vertex. We call the set of vertices where $R_v$ has positive dimension the support of $R$.

In this paper, we are particularly interested in representations with dimension vector $\mathbf{1} = (1, \ldots, 1)$. Notice when $R$ is a representation with dimension vector $\mathbf{1}$, and $R' \subset R$, all the inclusion maps

$$\iota_k : R'_k \rightarrow R_k$$
are either zero map or identity. We prove the following easy lemma:

**Lemma 2.29.** Let \((Q, I)\) be a bound quivers whose vertices are label by \(\{0, 1, 2, \ldots, n\}\) and \(R\) be a representation of \(Q\) with dimension vector \(1\). Then any subrepresentation \(R'\) is determined by its dimension vector \(\vec{d}\). Moreover, a vector \(\vec{d}\) of size \(n + 1\) with entries 0 and 1 is the dimension vector of a subrepresentation of \(R\) if and only if \(r_a = 0\) for all \(a \in Q_{ar}\) with \(d_{s(a)} = 1\) and \(d_{t(a)} = 0\).

**Proof.** Since \(\dim R_i = 1\), its subspaces are determined by dimensions. Moreover, we see the morphism of subspaces \(r'_a\) are restrictions of \(r_a\), hence the dimension vector \(\vec{d}\) determines \(R'_i\) for all \(i \in Q_{vx}\).

Given any vector \(\vec{d}\) as in the second part of the lemma, it is the dimension of a vector subspace if (2.28) is satisfied. Note (2.28) is always true unless for arrows with \(d_{s(a)} = 1\) and \(d_{t(a)} = 0\), in which case we must have \(r_a = 0\). \(\square\)

**Remark 2.30.** We will use Lemma 2.29 to construct subrepresentations. We will do so by prescribing dimension of the subrepresentation at each vertex, and check the vanishing of arrows as in Lemma 2.29 along the way, until we have the whole dimension vector, which determines the subrepresentation.

**Remark 2.31.** If \(R, T\) are both representations with dimensional vector \(1\) and \(R\) has more arrows with value 0, i.e. \(t_a = 0\) implies \(r_a = 0\) for \(a \in Q_{ar}\), then \(R\) has more subrepresentations than \(T\), more precisely, if \(T\) has a subrepresentation with dimension vector \(\vec{d}\), then \(S\) also has a subrepresentation with the same dimension vector.

### 2.4. Moduli space of semistable representations of a quiver.

Given a bound quivers \((Q, I)\), a weight is an element \(\theta \in \mathbb{Z}^N\) where \(N = |Q_{ex}|\) such that \(\sum_{i=1}^{N} \theta_i = 0\). Let \(\theta = (\theta_1, \ldots, \theta_N)\) be a weight, we defined its toric form to be

\[(-\theta_1, -\theta_1 - \theta_2, \ldots, -\theta_1 - \theta_2 - \ldots - \theta_{N-1}) \in \mathbb{Z}^{n-1}\]

It is an easy exercise to see that one can recover a weight from its toric form.

**Definition 2.32.** A weight is admissible if every entry of its toric form is a positive integer.

For a weight \(\theta\), the weight function is defined by:

\[\theta(S) = \sum_{i=1}^{N} d_i \theta_i\]

where \(S\) is a representation of \(Q\) and \(d_i\) and \(\theta_i\) are the \(i\)-th entries of \(\vec{d}\) and \(\theta\) respectively. We recall the definition of semi-stability:

**Definition 2.33.** A representation \(R\) is \(\theta\)-semistable if for any subrepresentation \(R' \subset R\)

\[\theta(R') \geq 0\]

\(R\) is \(\theta\)-stable if all the above inequalities are strict.
We restrict our attention to $R$ with dimension vector $\mathbf{1}$. Given a bound quivers $(Q, I)$, we can associate to it an affine scheme $\text{Rep}(Q)$ called the representation scheme of $(Q, I)$. The coordinate ring of this affine scheme is the quotient of $k[a \in Q_{ar}]$ by the ideal $J$ which is generated by generators $\sum_{i=1}^{n} k_ip_i$ of $I$ treated as elements in the above polynomial ring. It is obvious from the definition that closed points of representation scheme are in 1-to-1 correspondence with representations of $Q$ with dimension vector $\mathbf{1}$. For a weight $\theta$, the set of $\theta$-semistable representations forms an open subscheme $\text{Rep}(Q)^{SS}_{\theta}$ of $\text{Rep}(Q)$, the set of $\theta$-stable representations forms an open subscheme $\text{Rep}(Q)^{S}_{\theta}$ of $\text{Rep}(Q)^{SS}_{\theta}$.

The group $(k^*)^{Q_{vx}}$ acts by incidence on $\text{Rep}(Q)$, in other words, it acts by $(g \cdot a) = g_{t(a)}r_ag_{s(a)}^{-1}$. Apparently, the diagonal subgroup $k^*_{\text{diag}}$ of $(k^*)^{Q_{vx}}$ consisting of elements of the form $(k, k, \ldots, k)$ for $k \in k^*$ acts trivially on $\text{Rep}(Q)$. So it is natural to only consider the action of $\text{PGL}(1) := (k^*)^{Q_{vx}}/k^*_{\text{diag}}$.

**Definition 2.34.** Two representations of dimension vector $\mathbf{1}$ are isomorphic if they are in the same orbit under the action of $\text{PGL}(1)$.

Give a weight $\theta$, the moduli space of $\theta$-semistable representation with dimension vector $\mathbf{1}$ is the GIT quotient

$$M_{\theta} = \text{Rep}(Q)\sslash_{\theta}\text{PGL}(1) = \text{Rep}(Q)^{SS}_{\theta}\sslash\text{PGL}(1)$$

We mention a few facts about $M_{\theta}$. For details, the readers are referred to [16]. An equivalent definition of $M_{\theta}$ is to consider the graded ring

$$B_{\theta} = \bigoplus_{r \geq 0} B(r\theta)$$

where $B(r\theta)$ is $r\theta$-semi-invariant functions in the coordinate ring of $\text{Rep}(Q)$. Then the GIT quotient is defined as

$$M_{\theta} = \text{Proj}(B_{\theta})$$

From this definition, it is easy to see that $M_{\theta}$ is a reduced projective scheme. Note if all $\theta$-semistable representations are $\theta$-stable, i.e. $\text{Rep}(Q)^{SS}_{\theta} = \text{Rep}(Q)^{S}_{\theta}$, then $M_{\theta}$ is the fine moduli space of $\theta$-stable representations, in particular, the closed points of $M_{\theta}$ are in 1-to-1 correspondence with the isomorphism classes of $\theta$-stable representations. We now give an easy criterion for obtaining fine moduli spaces as above.

**Lemma 2.35.** With the notions above, if for any proper nonempty subset $P$ of $Q_{vx}$, we have $\sum_{i \in P} \theta_i \neq 0$, then any semistable representation $R$ is in fact stable. In particular, $M_{\theta}$ is a fine moduli space.

**Proof.** If $R$ is strictly semistable, then there exist a proper nonzero subrepresentation $R'$ such that $\theta(R') = \sum_{i \in \text{supp}(R')} \theta_i = 0$, but this cannot happen given the conditions in the statement. □
2.5. Quivers of Sections. The main reference for this section is Craw-Smith\cite{Craw-Smith} and Craw-Winn\cite{Craw-Winn}. We mention that our indexing is different since we are concerned with the quiver with path algebra $\mathcal{A}^{\text{op}}$ instead of $\mathcal{A}$ as in the introduction.

Let $L = \{L_1, L_2, \ldots, L_n\}$ be a collection of line bundles on a projective variety $X$. For $1 \leq i, j \leq n$, we call a section $s \in H^0(X, L_j^\vee \otimes L_i)$ irreducible if $s$ does not lie in the images of the multiplication map

$$H^0(X, L_j^\vee \otimes L_k) \otimes_k H^0(X, L_k^\vee \otimes L_i) \to H^0(X, L_j^\vee \otimes L_i)$$

for $k \neq i, j$.

Definition 2.36. The quiver of sections of the collection $L$ on $X$ is defined to be a quiver with vertex set $Q_{\text{ex}} = \{1, \ldots, n\}$ and where the arrows from $i$ to $j$ corresponds to a basis of irreducible sections of $H^0(X, L_{(n+1)−j} \otimes L_{(n+1)−i})$.

We mention one of the basic properties of a quiver of sections.

Lemma 2.37. \cite{Craw-Smith} The quiver of sections $Q$ is connected, acyclic and $1 \in Q_{\text{ex}}$ is the unique source.

The quiver of sections only include information about the sections in $H^0(X, L_{(n+1)−j} \otimes L_{(n+1)−i})$, but left relations between them behind. We now define a two sided ideal

Definition 2.38. Let $I_L$ be a two sided ideal in $kQ$.

$$I_L = (\sum_{k=1}^{N} a_k p_k | p_k \text{ are paths from } i \text{ to } j \text{ and } \sum_{k=1}^{N} a_k p_k \text{ represents } 0 \text{ in } H^0(X, L_{(n+1)−j} \otimes L_{(n+1)−i}))$$

We call the pair $(Q, I_L)$ the bound quiver of sections of the collection $L$.

Proposition 2.39. \cite{Craw-Smith} \cite{Craw-Winn} The quotient algebra $kQ/I_L$ is isomorphic to $\mathcal{A}^{\text{op}} = \text{End}_{\mathcal{O}_X}(\oplus_{i=1}^{n} L_i^\vee)$ and for $1 \leq i, j \leq n$, we have $e_j(kQ/I_L)e_i \cong H^0(X, L_{(n+1)−j} \otimes L_{(n+1)−i})$.

Remark 2.40. Suppose $L = \{L_1, L_2, \ldots, L_n\}$ is an exceptional collection of line bundles on a projective variety $X$, and $\mathcal{T}S = \{A_1, \ldots, A_n\}$ is the corresponding toric system. If we define $\mathcal{T}S^{\text{op}} = \{B_1, \ldots, B_n\}$ by

$$B_i = \begin{cases} A_{n−i} & \text{if } i = 1, 2, \ldots, n−1 \\ A_n & \text{if } i = n. \end{cases}$$

Note $\mathcal{T}S^{\text{op}}$ is also an exceptional toric system (it remains strong if $\mathcal{T}S$ is), and for $1 \leq i < j \leq n$, we have $e_j(kQ/I_L)e_i \cong H^0(X, B_i + \ldots + B_{j−1})$. In fact, $\mathcal{T}S^{\text{op}}$ is the corresponding toric system of $\{L_1^\vee, L_{n−1}^\vee, \ldots, L_n^\vee\}$

Given any weight $\theta$ for $(Q, I_L)$, we can consider the moduli space of semistable representations $M_\theta$. There is a tautological rational map

$$T : X \dashrightarrow M_\theta$$
so that if $T$ is defined at $x$, then

$$T(x) = \bigoplus_{i=0}^{n} (L_{i}^\vee)_{x}$$

Moreover, $T$ is defined at $x$ if $\bigoplus_{i=0}^{n} (L_{i}^\vee)_{x}$ can be represented by a $\theta$-semistable representation.

3. Augmentation for quiver of sections

In this section we discuss properties of quivers of sections coming from full strong toric systems, and define the operation of augmentation between them.

Let $S_0$ be a rational surface, $\mathcal{T}S_0 = \{A_1, \ldots, A_n\}$ a full strong toric system on $S_0$. Let $\pi : S \rightarrow S_0$ be a blow up of $S_0$ at a single point $P$ with the exceptional curve $E$. Let

$$\mathcal{T}S = \{\pi^*(A_1), \ldots, \pi^*(A_{n-1-k}), \pi^*(A_{n-k})-E, E, \pi^*(A_{n-k+1})-E, \pi^*(A_{n-k+2}), \ldots, \pi^*(A_n)\}$$

be a full strong toric system obtained from standard augmentation of $\mathcal{T}S_0$ at position $n-1-k$.

Note if $\mathcal{T}S_{0}^{op} = \{B_1, B_2, \ldots, B_{n-1}, B_n\}$

$$= \{A_{n-1}, A_{n-2}, \ldots, A_2, A_1, A_n\}$$

then

$$\mathcal{T}S^{op} = \{\pi^*(B_1), \ldots, \pi^*(B_{k-2}), \pi^*(B_{k-1})-E, E, \pi^*(B_k) - E, \pi^*(B_{k+1}), \ldots, \pi^*(B_n)\}$$

$$= \{\pi^*(A_{n-1}), \ldots, \pi^*(A_{n-k+2}), \pi^*(A_{n-k+1})-E, E, \pi^*(A_{n-k})-E, \pi^*(A_{n-k-1}), \ldots, \pi^*(A_1), \pi^*(A_n)\}$$

Remark 3.1. We label the vertices of $Q_0$ by $1, 2, \ldots, n$. And label the vertices of $Q$ by $1, \ldots, k-1, k, k', k+1, \ldots, n$. In this way, we obtain an 1-to-1 correspondence between the vertices of $Q$ and $Q_0$ except at the place where the augmentation is performed.

Definition 3.2. We use $e$ to denote the unique arrow from $k$ to $k'$ corresponding to the unique section of the line bundle $E$.

Note we have $\pi_*(\mathcal{O}_S) = \mathcal{O}_{S_0}$, hence by adjunction and projection formula:

$$\text{Hom}_{\mathcal{O}_S}(\pi^*(L_1), \pi^*(L_2)) = \text{Hom}_{\mathcal{O}_{S_0}}(L_1, \pi_\#\pi^*L_2)$$

$$= \text{Hom}_{\mathcal{O}_{S_0}}(L_1, \pi_\#\mathcal{O}_{S_0} \otimes L_2)$$

$$= \text{Hom}_{\mathcal{O}_{S_0}}(L_1, L_2)$$

This computation shows the section quiver of $\mathcal{T}S$ is same as the section quiver of $\mathcal{T}S_0$ when we look at parts which are either between vertex 1 to $k-1$ or between vertex $k+1$ to $n$.

Remark 3.3. From now on we omit $\pi^*$ when no confusion will be caused.
We now analyze the arrows from vertex $i$ to vertex $k'$. where $i < k$.

**Lemma 3.4.** If $i < k$, we have a natural embedding

$$H^0(S, B_i + \ldots + B_{k-1} - E) \hookrightarrow H^0(S, B_i + \ldots + B_{k-1})$$

and $H^0(S, B_i + \ldots + B_{k-1})$ is spanned by the image of $H^0(S, B_i + \ldots B_{k-1} - E)$ and an element which represents a divisor $f$ such that $\text{div}(f) - E$ is not effective.

**Proof.** Consider the standard short exact sequence

$$0 \rightarrow \mathcal{O}_S(-E) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_E \rightarrow 0$$

Twist this exact sequence by $\mathcal{O}_S(B_i + \ldots + B_{k-1})$, and note $E \cdot (B_i + \ldots + B_{k-1}) = 0$, we obtain

$$0 \rightarrow \mathcal{O}_S(B_i + \ldots + B_{k-1} - E) \rightarrow \mathcal{O}_S(B_i + \ldots + B_{k-1}) \rightarrow \mathcal{O}_E \rightarrow 0$$

This induce long exact sequence

$$0 \rightarrow H^0(S, B_i + \ldots + B_{k-1} - E) \rightarrow H^0(S, B_i + \ldots + B_{k-1}) \rightarrow H^0(E, \mathcal{O}_E) \rightarrow H^1(S, B_i + \ldots + B_{k-1} - E)$$

This gives the inclusion in the statement. Using the formula in preliminary, we see

$$h^0(S, B_i + \ldots + B_{k-1} - E) + 1 = h^0(S, B_i + \ldots + B_{k-1})$$

Hence there is an effective divisor $F$ linearly equivalent to $B_i + \ldots + B_{k-1}$ such that $F - E$ is not effective. The section corresponding to $F$ and $H^0(S, B_i + \ldots B_{k-1} - E)$ spans $H^0(S, B_i + \ldots + B_{k-1})$ by looking at the dimension. \[\square\]

The computation at the beginning of this section tells us that the arrows from vertex $i$ to vertex $k'$ in $Q$ are in 1-to-1 correspondence with the arrows from vertex $i$ to vertex $k$ in $Q_0$. This lemma tells us in addition that all but one arrows from vertex $i$ to vertex $k'$ in $Q$ comes from arrows from vertex $i$ to vertex $k$ composed with $e$, while there is a unique arrow from vertex $i$ to vertex $k'$ which is not a multiple of $e$.

Using exact same argument, we obtain mirror result for arrows from vertex $k$ to vertex $j$ where $j > k$.

**Lemma 3.5.** If $j > k$, we have a natural embedding

$$H^0(S, -E + B_k + \ldots B_{j-1}) \hookrightarrow H^0(S, B_k + \ldots + B_{j-1})$$

and $H^0(S, B_k + \ldots + B_{j-1})$ is spanned by the image of $H^0(S, -E + B_k + \ldots B_{j-1})$ and an element $g$ such that $\text{div}(g) - E$ is not effective.

**Definition 3.6.** If $i < k$, we fix an arrow in $Q$ from $i$ to $k'$ which corresponds to a section

$$f \in H^0(S, B_i + \ldots + B_{k-1})$$

such that $\text{div}(f) - E$ is not effective as in Lemma 3.4. Denote this arrow by $u_i$. We note $u_i$ is not the composition of an arrow from $i$ to $k$ with $e$. We use $w_i$ to denote the arrow in $Q_0$ from $i$ to $k$ that represents the same section as $f$ under the isomorphism

$$H^0(S, \pi^*B_i + \ldots + (\pi^* B_{k-1} - E) + E) = H^0(S_0, B_i + \ldots + B_{k-1})$$
If \( j > k \), we fix an arrow from \( k \) to \( j \) which corresponds to the section

\[ g \in H^0(S, B_k + \ldots + B_{j-1}) \]

such that \( \text{div}(g) - E \) is not effective as in Lemma 3.5. We denote this arrow by \( u_j \). Note \( u_j \) is not a composition \( e \) with an arrow from \( k' \) to \( j \). We use \( w_j \) denote the arrow in \( Q_0 \) from \( i \) to \( k \) that represents the same section as above.

Finally we analyze arrows in \( Q \) from vertex \( i \) to vertex \( j \) where \( i < k < j \).

\[ B_i + \ldots + (B_{k-1} - E) + E + (B_k - E) + \ldots + B_j = B_i + \ldots B_j - E. \]

**Lemma 3.7.** If \( i < k < j - 1 \), we have a natural embedding

\[ H^0(S, B_i + \ldots + B_{j-1} - E) \hookrightarrow H^0(S, B_i + \ldots + B_{j-1}) \]

and \( H^0(S, B_i + \ldots + B_{j-1}) \) is spanned by the image of \( H^0(S, B_i + \ldots + B_{j-1} - E) \) and the product of element \( f \) in \( H^0(S, B_i + \ldots + B_{k-1}) \) with element \( g \) in \( H^0(S, B_{k+1} + \ldots + B_{j-1}) \) with the additional property that \( fg \) does not vanish at \( P \), hence not in the image of \( H^0(S, B_i + \ldots + B_{j-1} - E) \). Hence \( fg \) and \( H^0(S, B_i + \ldots + B_{j-1} - E) \) span

\[ H^0(S, B_i + \ldots + B_{j-1}) \]

by looking at the dimension. \( \square \)

This lemma tells us that if \( i < k < j \), then all but one arrows from vertex \( i \) to vertex \( j \) in \( Q_0 \) corresponds to arrows from vertex \( i \) to \( j \) in \( Q \).

4. **Example:** \( \mathbb{P}^2 \)

In this section we consider the case when \( S = \mathbb{P}^2 \). This case is on the one hand a motivating example and on the other hand the base case of induction by augmentation for the next section.

We use \( H \) to denote a divisor corresponding to the hyperplane line bundle \( \mathcal{O}_{\mathbb{P}^2}(1) \). We have the unique full cyclic strong toric system \( \mathcal{TS} = \{ H, H, H \} \) and \( \mathcal{TS}^{op} = \{ H, H, H \} \). The corresponding quiver of sections is

\[
1 \iff 2 \iff 3
\]
We call the three arrows from 0 to 1 by $x_1, y_1, z_1$, the three arrows from 1 to 2 by $x_2, y_2, z_2$, where $x_i, y_i, z_i$ represents the linear functions $x, y, z$ in $H^0(\mathbb{P}^2, H)$. Then we have relation:

\[
\begin{align*}
y_2x_1 &= x_2y_1 \\
y_2z_1 &= z_2y_1 \\
z_2x_1 &= z_1x_2
\end{align*}
\]

We denote this quiver with relation by $Q$.

**Theorem 4.1.** For $m, n \in \mathbb{Z}_{>0}$, $m \neq n$, if $\theta$ has toric form $(m, n)$, i.e. $\theta = (-m, m-n, n)$, then $T : \mathbb{P}^2 \rightarrow M_\theta$ is an isomorphism.

**Proof.** Take $s = [a : b : c] \in \mathbb{P}^2$, then $a, b, c$ cannot be 0 simultaneously. Consider the isomorphism class $\mathcal{O}(-H)_s \oplus \mathcal{O}(-2H)_s \oplus \mathcal{O}(-3H)_s$ of representations of $Q$. By change of bases for each of the three summands, we can find $R \in \mathcal{O}(-H)_s \oplus \mathcal{O}(-2H)_s \oplus \mathcal{O}(-3H)_s$ so that

\[
\begin{align*}
x_1 &= a \\
y_1 &= b \\
z_1 &= c
\end{align*}
\]

By Lemma 2.29, then only possible nontrivial proper subrepresentations of $R$ has dimension vector $(0, 1, 1)$ or $(0, 0, 1)$, in both cases, if we use $S$ to denote the subrepresentation, we can see $\theta(S) > 0$ using the fact that $m, n > 0$. Thus $T$ is a morphism, i.e it is defined on all of $\mathbb{P}^2$.

Now consider the natural map

\[
f_k : B(k\theta) \rightarrow H^0(\mathbb{P}^2, k(mH + nH))
\]

for $k \geq 0$. Note $B(k\theta)$ has a basis \{ $x_1^{p_1}y_1^{q_1}z_1^{r_1}x_2^{p_2}y_2^{q_2}z_2^{r_2}$ \}$_{p_1+q_1+r_1=km,p_2+q_2+r_2=kn}$ and this map maps $x_1^{p_1}y_1^{q_1}z_1^{r_1}x_2^{p_2}y_2^{q_2}z_2^{r_2}$ to $x^{p_1+p_2}y^{q_1+q_2}z^{r_1+r_2}$. Using the relations, one easily sees $f_n$ is an isomorphism for each $k \geq 0$ and $(f_k)_{k \geq 0}$ gives a isomorphism of graded rings

\[
f : \bigoplus_{k \geq 0} B(k\theta) \rightarrow \bigoplus_{k \geq 0} H^0(\mathbb{P}^2, k(mH + nH))
\]

Since the tautological map $T$ is obtained from $f$ by taking Proj of both sides, we see $T$ is an isomorphism.

**Remark 4.2.** It is clear that for $m, n \in \mathbb{Z}_{>0}$, $m \neq n$ and $\theta = (-m, m-n, n)$, $\theta$ is a weight that satisfies the condition in Lemma 2.35.
5. Choice of weight

The proof of Main Theorem is by induction. We fix some notation here. $S$ is a rational surface admitting a full strong exceptional collection of line bundles $\{L_1, \ldots, L_n\} := \{O_S(D_1), \ldots, O_S(D_n)\}$ obtained from standard augmentation from $\mathbb{P}^2$. By definition of standard augmentation, we know there is a rational surface $S_0$ and a full strong toric system $TS_0$ on it such that the toric system of $\{L_1, \ldots, L_n\}$,

$$TS = \{\pi^*(A_1), \ldots, \pi^*(A_{n-k}) - E, E, \pi^*(A_{n-k+1}) - E, \ldots, \pi^*(A_n)\}$$

is a standard augmentation of $TS_0 = \{A_1, A_2, \ldots, A_n\}$. We also assume that $S$ satisfies the condition that all irreducible components of exceptional divisors have self intersection $\geq -2$, this will imply the same property for $S_0$. We show that if there exist weight $\theta_0$ such that $T_0 : S_0 \to M_{\theta_0}$ is an isomorphism, then we can find a weight $\theta$ so that $T : S \to M_{\theta}$ is an isomorphism.

Remark 5.1. We observe that under the assumption (1.4), if

$$TS_0^{\text{op}} = \{B_1, B_2, \ldots, B_{n-1}, B_n\}$$

$$= \{A_{n-1}, A_{n-2}, \ldots, A_2, A_1, A_n\}$$

then

$$TS^{\text{op}} = \{\pi^*(B_1), \ldots, \pi^*(B_{k-1}) - E, E, \pi^*(B_k) - E, \ldots, \pi^*(B_n)\}$$

$$= \{\pi^*(A_{n-1}), \ldots, \pi^*(A_{n-k}) - E, E, \pi^*(A_{n-k+1}) - E, \ldots, \pi^*(A_1), \pi^*(A_n)\}$$

is a full strong toric system obtained by standard augmentation from $TS^{\text{op}}$ at position $k$.

We will always use weights that are admissible. Suppose $\theta_0$ is an admissible weight, let its toric form be $(b_1, b_2, \ldots, b_n)$. We define $\theta$ in its toric form as

$$(2b_1, 2b_2, \ldots, 2b_{k-1}, (2b_{k-1} + 2b_k - 1), 2b_k, \ldots, 2b_{n-1})$$

, i.e. the exceptional divisor $E$ in the toric system $TS^{\text{op}}$ is given weight $2b_{k-1} + 2b_k - 1$. Note if $k = 1$, then the toric form of $\theta$ is

$$(2b_1 - 1, 2b_1, 2b_2, \ldots, 2b_{n-1})$$

. If $k = n$, the toric form of $\theta$ is

$$(2b_1, 2b_2, \ldots, 2b_{n-1}, 2b_{n-1} - 1)$$

. One can think of these two cases as a natural extension of the general case by thinking $b_k = 0$ if $k < 1$ or $k > n$. It is clear that $\theta$ defined in this way is also admissible.

The motivation for such a choice is that we will choose in a fashion that $\sum_{i=1}^{n-1} b_iB_i$ is a very ample divisor on $S_0$, and our new weight will correspond to divisor $2\sum_{i=1}^{n-1} b_i\pi^*(B_i) - E$, which can be easily verified to be very ample on $S$ using Nakai-Moishezon criterion. We believe that any weight whose toric form gives a very ample divisor should cut out a moduli space isomorphic to $S$, although in this paper we only manage to prove some very
special choices.
The following proposition along with Lemma 2.35 shows some nice properties our choice of \( \theta \) enjoys:

**Proposition 5.2.** With the above assumption, let \( \theta_0 \) whose toric form is \((b_1, \ldots, b_{n-1})\) be a character satisfying the conditions in Lemma 2.35 on \( S_0 \), then \( \theta \) defined as above, whose toric form is

\[
(2b_1, 2b_2, \ldots, 2b_{k-1}, (2b_{k-1} + 2b_k - 1), 2b_k, \ldots, 2b_{n-1})
\]
also satisfies the conditions in Lemma 2.35.

**Proof.** We use combinatorics to verify that \( \theta \) satisfy the condition in the above lemma. We prove the proposition for \( 1 < k < n \), the cases when \( k = 1 \) or \( k = n \) are handled exactly the same. Note

\[
\theta_0 = (-b_1, b_1 - b_2, \ldots, b_{n-2} - b_{n-1}, b_{n-1})
\]
and

\[
\theta = (-2b_1, 2(b_1-b_2), \ldots, 2(b_{k-2}-b_{k-1}), 1-2b_k, 2b_{k-1}-1, 2(b_k-b_{k+1}), \ldots, 2(b_{n-2}-b_{n-1}), 2b_{n-1})
\]
With the assumption on \( \theta \), any subset of entries of \( \theta \) not involving \( 1 - 2b_k \) and \( 2b_{k-1} - 1 \) cannot sum up to 0. If the subset only contains on of the above two entries, the sum cannot be 0 due to parity. If the subset contains both entries, since \( 1-2b_k + 2b_{k-1} - 1 = 2(b_{k-1}-b_k) \), the sum cannot be 0 again due to the assumption on \( \theta_0 \). Thus \( \theta \) satisfies the assumption of Lemma 2.35. \( \square \)

**Remark 5.3.** This proposition implies both \( M_\theta \) and \( M_{\theta_0} \) are fine moduli space of stable representations.

**Remark 5.4.** For the rest of this paper, when \( \theta \) and \( \theta_0 \) show up together, we always assume \( \theta \) is obtained from \( \theta_0 \) from the above procedure.

6. **Construction of morphism between moduli spaces**

We start by constructing a morphism between representation schemes, i.e \( F : \text{Rep}(Q) \to \text{Rep}(Q_0) \). Since both \( \text{Rep}(Q) \) and \( \text{Rep}(Q_0) \) are affine schemes, it suffice to construct a \( \mathbb{k} \)-algebra homomorphism between their coordinate rings. We notice for each \( a \in Q_{0,ar} \) from \( i \) to \( j \), it corresponds to an element \( s \) in \( \text{Hom}_{\mathcal{O}_{S_0}}(L_i, L_j) \), it is natural to consider mapping it to the element in \( \mathbb{k}[b \in Q_{ar}]/J \) which corresponds to \( \pi^*s \in \text{Hom}_{\mathcal{O}_{S}}(\pi^*L_i, \pi^*L_j) \). The nontrivial part lies in the fact that such an element is not always of the form \( b \) for \( b \in Q_{ar} \).

**Theorem 6.1.** There is a natural morphism

\[
F : \text{Rep}(Q) \to \text{Rep}(Q_0)
\]

**Proof.** We define a \( \mathbb{k} \)-algebra homomorphism \( \phi : \mathbb{k}[a \in Q_{0,ar}]/J_0 \to \mathbb{k}[b \in Q_{ar}]/J \) as follows: For \( a \in Q_{0,ar} \), if \( t(a) < k \) or \( s(a) > k \), by the computation at the beginning of Section 3, we see there is a unique arrow \( b \) in \( Q \) corresponding to \( a \), define \( \phi(a) = b \).
If \( t(a) = k \), then there is a unique arrow \( b \) in \( Q \) with \( t(b) = k' \) corresponding to \( a \), define \( \phi(a) = b \). The case when \( s(a) = k \) is handled similarly.

If \( s(a) < k \) and \( t(a) > k \), by Lemma 3.7, all but one such arrows represent sections whose divisor of zero contains \( E \) as a component, and corresponds to a unique arrow \( b \) in \( Q \), define \( \phi(a) = eb \). For the unique arrow \( w \) that represents a section \( s \) such that \( \text{div}(s) - E \) is not effective, there is not arrow in \( Q \) corresponding to it. By the proof of Lemma 3.4, we see the natural choice is to set \( \phi(u) = u_{s(w)}u_{t(w)} \).

It remains to check \( \phi \) is well-defined, i.e., elements in \( J_0 \) gets mapped into \( J \) by \( \phi \). It suffices to show we can choose a collection of generators of \( I_0 \), such that when we consider the generators as elements in \( k[a \in Q_{0,a}] \), their image under \( \phi \) lies in \( J \). The natural choice of generators consists of \( \sum_{i=1}^{n} k_i p_i \) where \( k_i \in k^* \) and \( p_i \)’s are pairwise different paths that share the same source \( i \) and target \( j \) such so that \( \sum_{i=1}^{n} k_i p_i \) corresponds to 0 in \( H^0(S_0, B_{i+1} + \ldots + B_j) \). By our choice above, it is clear that if either \( i \geq k \) or \( j \leq k \), \( \phi(\sum_{i=1}^{n} k_i p_i) = 0. \) If \( i < k < j \), then the unique arrow cannot show up in \( \sum_{i=1}^{n} k_i p_i \) with nonzero coefficient, as it represents the unique dimension in \( H^0(S_0, B_{i+1} + \ldots + B_j) \) that does not lie in the span of the rest of the arrows. Thus \( \phi(\sum_{i=1}^{n} k_i p_i) \) is a multiple of \( e \). By our construction \( \phi(\sum_{i=1}^{n} k_i p_i) \) corresponds to 0 in \( H^0(X, B_{i+1} + \ldots + B_j) \) (Caution: This does not imply \( \phi(\sum_{i=1}^{n} k_i p_i) \) is in \( J \)). The fact that \( \phi(\sum_{i=1}^{n} k_i p_i) \) is a multiple of \( e \) implies it is in the image of the inclusion \( H^0(X, B_{i+1} + \ldots + B_j - E) \hookrightarrow H^0(X, B_{i+1} + \ldots + B_j) \), so \( \phi(\sum_{i=1}^{n} k_i p_i)/e \) corresponds to 0 in \( H^0(X, B_{i+1} + \ldots + B_j - E) \), hence is a generator of \( J \). So \( \phi(\sum_{i=1}^{n} k_i p_i) = (\phi(\sum_{i=1}^{n} k_i p_i)/e)e \in J. \)

**Corollary 6.2.**

\[
F : \text{Rep}(Q) \to \text{Rep}(Q_0)
\]

is a surjective morphism.

**Proof.** Given \( R_0 \in \text{Rep}(Q_0) \), we set \( r_\circ = 1 \) and the proof of Theorem 6.1 immediately determines the values of other arrows. The fact that these values comes from a representations follows from the fact that \( R_0 \) is.

The next proposition shows \( F \) respects the PGL(1)-action.

**Proposition 6.3.** Let \( R_1, R_2 \) be two representations of \( Q \) with dimension vector \( \mathbf{1} \). Suppose \( R_1 \sim R_2 \), via the element \( (g_1, \ldots, g_{k-1}, g_k, g_{k'}, g_{k+1}, \ldots, g_n) \), then \( F(R_1) \sim F(R_2) \).

**Proof.** From the construction of \( F \), one directly check the element

\[
(g_1g_k, g_2g_k, \ldots, g_{k-1}g_k, g_kg_{k'}, g_{k+1}g_{k'}, \ldots, g_ng_k)
\]

provides the equivalence. \( \square \)

We now consider the interaction between \( F \) and stability conditions. We first make an easy but important remark.

**Remark 6.4.** The PGL(1)-action is compatible with stability, i.e. if \( R_1 \sim R_2 \) and \( R_1 \) is \( \theta \)-semistable, then so is \( R_2 \).
We let $U = \text{Rep}(Q) - \mathbf{V}(e)$, this is the open subset of $\text{Rep}(Q)$ consisting of representations where the value of the arrow $e$ is not 0.

**Proposition 6.5.** Suppose $F(R) = R'$, $R \in U$, $R$ is $\theta$-stable, then $R'$ is $\theta_0$-stable.

**Proof.** Note if 

$$\theta_0 = (-b_1, b_1 - b_2, \ldots, b_{n-2} - b_{n-1}, b_{n-1})$$

as in Proposition 5.2 then

$$\theta = (-2b_1, 2(b_1-b_2), \ldots, 2(b_{k-2} - b_{k-1}), 1-2b_k, 2b_{k-1}-1, 2(b_k-b_{k+1}), \ldots, 2(b_{n-2} - b_{n-1}), 2b_{n-1})$$

We mention that $(1 - 2b_k) + (2b_{k-1} - 1) = 2(b_{k-1} - b_k)$. 

Given $S' \subset R'$, $\vec{d}$ be the dimension vector of $S'$. We claim that we can find a subrepresentation $S$ of $R$ whose dimension vector $\vec{d}$ is as follows:

1. If $i < k$, $d_i = d'_i$.
2. If $j > k$, $d_j = d'_j$.
3. $d_k = d_{k'} = d'_k$

We now verify $\vec{d}$ indeed gives a subrepresentation of $R$ using Lemma 2.29. Let $a$ be an arrow from $i$ to $j$ such that $d_i = 1$ and $d_j = 0$, we want to show $r_a = 0$.

If $j < k$, then $d'_i = 1$, $d'_j = 0$ and by construction of $F$, we see $r'_a = r_a = 0$.

If $j = k$, then $d'_i = 1$, $d'_k = 0$. Then there is an arrow $a'$ from $i$ to $k$ in $Q_0$ such that $r'_a = r_a r_e$ by the construction of $F$. Since $R \in U$, $r_e \neq 0$. But $r'_a = 0$, so we must have $r_a = 0$.

If $j = k'$, then $d'_i = 1$, $d'_k = 0$, there is an arrow $a'$ in $Q_0$ such that $r_a = r'_a = 0$.

If $j > k$, and $i = k, k'$ or $i > k$, the above arguments applies. When $i < k$, then there is an arrow $a'$ in $Q_0$ such that $0 = r'_a = r_a r_e$ again as $r_e \neq 0$, we get $r_a = 0$.

So a subrepresentation $S$ of $R$ with dimension vector exists. Now it is clear by our choice of dimension vector

$$\theta(S) = 2\theta_0(S')$$

Since $R$ is $\theta$-stable, $\theta(S) > 0$, thus $\theta_0(S') > 0$. Apply this argument for all $S' \subset R'$, we see $R' = F(R)$ is $\theta_0$-stable.

**Proposition 6.6.** Suppose $R_1, R_2 \in U$, and $F(R_1) \sim F(R_2)$ under the action of $(g_1, \ldots, g_n)$, then $R_1 \sim R_2$.

**Proof.** Let $e_i$ denote the value of $e$ in $R_i$ for $i = 1, 2$, then $e_1 e_2 \neq 0$. Again by the construction of $F$, one directly checks that

$$\left(g_1 e_2, g_2 e_2, \ldots, g_{k-1} e_2, g_k e_1, g_k e_2, g_{k+1} e_1, \ldots, g_n e_1\right)$$

provides the equivalence. 

Now we turn to representations in the set $\mathbf{V}(e)$. We first introduce a definition that will be used the proposition to come.
Lemma 6.7. Let $R \in V(e)$, we can define a subrepresentation $S^L$ of $R$ using the by setting $\dim(S^L_k) = \dim(S^L_j) = 1$ for $j > k$, $\dim(S^L_k') = 0$. For $i < k$:

$$\dim(S^L_i) = \begin{cases} 1 & \text{if and only if } r_{ui} = 0 \\ 0 & \text{if and only if } r_{ui} \in k' \end{cases}$$

We call this subrepresentation the left standard subrepresentation of $R$. Similarly, we can define the right standard subrepresentation $S^R$ by setting $\dim(S^R_i) = \dim(S^R_k') = 1$ for $i < k$, $\dim(S^R_k) = 0$, and for $j > k$:

$$\dim(S^R_j) = \begin{cases} 1 & \text{if and only if } r_{uj} = 0 \\ 0 & \text{if and only if } r_{uj} \in k' \end{cases}$$

Proof. We prove the lemma for left standard subrepresentation. The right case is left as an exercise.

By Lemma 2.29, to make $S^L$ as in the statement well-defined. We need

1. $r_e = 0$.
2. $r_{ui} = 0$ if $i < k$ and $\dim(S^L_i) = 1$.
3. $r_a = 0$ for any arrow with $t(a) < k$, $\dim(S^L_{s(a)}) = 1$ and $\dim(S^L_{t(a)}) = 0$.

(1) is satisfied since $R \in V(e)$. (2) is satisfied by definition of $S^L$. Let $a$ be an arrow as in (3). Since $\dim(S^L_{s(a)}) = 1$, we have $r_{s(a)} = 0$. By Lemma 3.4, all arrows from $s(a)$ to $k'$ has value 0. Now $u_{t(a)} \circ a$ is an arrow from $s(a)$ to $k'$ whose value is $u_{t(a)}r_a = 0$. But $\dim(S^L_{t(a)}) = 0$ implies $u_{t(a)} \neq 0$, so $r_a = 0$. \qed

The next proposition provides important structural properties of representations with $r_e = 0$. The proof uses case by case arguments and we suggest the readers to skip the proof during first read.

Any full strong exceptional toric system coming from standard augmentation from $\mathbb{P}^2$ satisfying (1.4) is of the form \{\ldots, H - \Delta, \ldots, H - \Delta', \ldots, H - \Delta''\} where $\Delta, \Delta', \Delta''$ are (possibly empty) sum of at most 3 exceptional divisors, and the terms in \ldots are single exceptional divisor $E_1$ or $E_1 - E_2$ where $E_1 - E_2$ is not effective. Note all divisors in the above toric system are required to be slo except the last one. We will use these restrictions when we exhaust all possible forms of toric systems for $TS^{op}$ (see Remark 5.1).

Proposition 6.8. Suppose $R \in V(e)$ and $R$ is $\theta$-stable, then for all $i < k$

$$r_{ui} \neq 0$$

Also for all $j > k$,

$$r_{uj} \neq 0.$$
Proof. We first assume that there exist $i < k$ such that the value of $u_i$ is zero. We may further assume that $i$ is the largest vertex with such property, i.e. for all $i < l < k$, $r_{u_l} \not= 0$. Our set-up for the bad representation $S$ will be

$$\dim(S_i) = \dim(S_k) = \dim(S_j) = 1$$

for all $j > k$ and

$$\dim(S_{k'}) = \dim(S_h) = 0$$

for all $i < h < k$. The dimension of $S_t$ with $t < i$ will be determined later. Note (4.10) and (4.11) together requires

1. $r_e = 0$
2. $r_a = 0$ if $s(a) = i$ and $t(a) = k'$
3. $r_b = 0$ if $s(b) = i$ and $t(b) = h$ for $i < h < k$

Firstly, $r_e = 0$ since $R \in V(e)$. Secondly, since all but one arrow form $i$ to $k'$ is the composition of $e$ with some arrow from $i$ to $k$, if we use $a$ to denote such an arrow and $a'$ to denote the corresponding arrow from $i$ to $k$, we see $r_a = r_{a'}r_e = 0$. The only remaining arrow is $u_i$, whose value is assumed to be 0. Lastly, if there exist arrow $b$ with $s(b) = i$, $t(b) = h$ such that $r_b \not= 0$, then $u_h \circ b$ is an arrow from $i$ to $k'$ so that $r_{u_h \circ b} = r_{u_h}r_b \not= 0$. This contradicts (2).

We remark here that by now the parts of $S$ that has already be fixed coincides with $S^L$. In the following, we will sometimes take $S^L$ as $S$.

Now we determine $\dim(S_l)$ for $l < i$. We distinguish different cases according to what $B_i$ is.

Case 1: $B_i = E_1$ for some exceptional curve $E_1$. We have 3 subcases for what $B_{i-1}$ is.

- If $i = 1$, i.e. the only bad arrow is $u_1$. In this case, we have already determined $S$. We compute

$$\theta(S) = (-2b_1) + (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2(b_{n-2} - b_{n-1}) + 2b_n - 1 - 2b_1 < 0$$

- If $B_{i-1} = H - E_1 - \Delta$, where $\Delta$ is a (possibly empty) sum of exceptional divisors other than $E$ and $E_1$. In this case $b_{i-1} < b_i$, we can let $\dim(S_l) = 0$, for all $l < i$. Note this choice does not require any additional vanishing of values of arrows from $R$ by Lemma 2.29. We compute

$$\theta(S) = 2(b_{i-1} - b_i) + (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2(b_{n-2} - b_{n-1}) + 2b_n - 1 + 2(b_{i-1} - b_i) < 0$$

- If $B_{i-1} = E_2 - E_1$. Again in this case $b_{i-1} < b_i$. Hence the exact same choice of dimensions as in the previous case yields

$$\theta(S) < 0$$
Case 2: $B_i = E_1 - E_2$ for some exceptional curves $E_1, E_2$. We have four sub-cases for what $B_{i-1}$ is:

- $B_{i-1} = H - E_1 - \Delta$, where $\Delta$ is a (possibly empty) sum of exceptional divisors other than $E, E_1, E_2$. In this case $b_{i-1} < b_i$, so we can find a non-stable subrepresentation as in Case 1 by letting $\dim(S_l) = 0$ for all $l < i$.
- $B_{i-1} = F - E_1$, where $F$ is an exceptional curve, then again in this case we $b_{i-1} < b_i$, so we can find a non-stable subrepresentation as in Case 1 by letting $\dim(S_l) = 0$ for all $l < i$.
- $B_{i-1} = E_2$. We use $e_2$ to denote the arrow that represents the 1-dimensional vector space $H^0(S, E_2)$.

If $u_i \circ e_2$ is not a multiple of $e$, then

$$u_i \circ e_2 = m u_{i-1} + e \circ \text{(arrow from } i-1 \text{ to } k)$$

where $m \neq 0$. Taking the value we see $r_{u_{i-1}} = 0$, then values of all arrows from $i-1$ to $k'$ are 0. It is also easy to see that the values of all arrows from $i-1$ to $j$ are 0 for $i-1 < j < k$ using the arguments when we proved (3) in the first paragraph of this proof. Thus we can set $\dim(S_i) = \dim(S_{i-1}) = 1$ and set all the remaining dimensions to be 0. We compute:

$$\theta(S) = 2(b_{i-2} - b_{i-1}) + 2(b_{i-1} - b_i) + (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2b_{n-1}
= 1 + 2(b_{i-2} - b_i)$$

Now either $i - 2 = 0$, which indicates $b_{i-2} = 0$ or $B_{i-2} = H - E_1 - E_2 - \Delta$, in which case $b_{i-2} < b_i$. Hence $\theta(S) < 0$.

If $u_i \circ e_2$ is a multiple of $e$, then $E_2 - E > 0$. If $r_{u_{i-1}} = 0$, then we can still use the subrepresentation in the previous paragraph, otherwise, we have four sub-cases:

1. $\mathcal{TS}^{op} = \{E_2, E_1 - E_2, \ldots\}$. Note we have already determined all $\dim(S_i)$ except for $S_1$. We take $\dim(S_1) = 0$. Note such an $S$ is always well-defined. Then

$$\theta(S) = 1 + 2(b_1 - b_2)$$

But $b_1 - b_2 = -2^{N-1}$ where $N$ is the number of blow up performed after obtaining $E_2$. Thus $\theta(S) < 0$.

2. $\mathcal{TS}^{op} = \{H - E_1 - E_2, E_2, E_1 - E_2, \ldots\}$. In this case $i = 3$ and by assumption $r_{u_3} = 0$ and $r_{u_2} \neq 0$. We have $B_1 + B_2 = H - E_1$. By Lemma 2.9 there is an arrow from 1 to 3 whose corresponding divisor does not contain $E$ as a component, we denote that arrow by $u_{13}$. Then $u_3 \circ u_{13}$ is not a multiple of $e$. Thus as above we obtain $r_{u_1} = 0$. We let $S = S^L$, then $\dim(S_1) = 1$ and $\dim(S_2) = 0$.

$$\theta(S) = 1 + 2(-b_1 + b_2 - b_3)$$

Again, notice $-b_1 + b_2 - b_3 = -2^{N-1}$ where $N$ is the number of blow up performed after obtaining $E_2$. Thus $\theta(S) < 0$. 


(3) \( \mathcal{TS}^{op} = \{ E_3, H - E_1 - E_2 - E_3, E_2, E_1 - E_2, \ldots \} \). Note \( B_2 + B_3 = H - E_1 - E_3 \). In this case \( i = 4 \) and by assumption \( r_{u_1} = 0 \) and \( r_{u_3} \neq 0 \). Since \( H - E_1 - E_2 - E_3 \) is strong left orthogonal
\[
\begin{align*}
\dim(S, H - E_1 - E_2 - E_3) &= 0 \\
\dim(S, H - E_2 - E_3) &= 1
\end{align*}
\]
Since \( E_2 - E > 0, E_1 - E_2 \) is slo, by Lemma 2.13, \( \dim(S, H - E - E_1 - E_3) = 0 \). This implies if we denote the unique arrow from 2 to 4 by \( e \), \( E \) is strong left orthogonal
\[
\begin{align*}
u_1 - \delta &= 0. \text{ Then } \dim(S_1) = \dim(S_2) = 1 \text{ and } \dim(S_3) = 0, \text{ thus } \\
\theta(S) &= 1 + 2(-b_2 + b_3 - b_4)
\end{align*}
\]
Since \( b_2 - b_3 + b_4 = 2^{N-1} \) where \( N \) is the number of blow up performed after obtaining \( E_2 \), hence \( \theta(S) < 0 \).

(4) \( \mathcal{TS}^{op} = \{ H - F - E_1 - E_2, E_2, E_1 - E_2, F - E_1, \ldots \} \). In this case \( i = 3 \). By assumption, \( r_{u_3} = 0 \) and \( r_{u_2} \neq 0 \), then \( E_2 - E > 0 \). By Lemma 2.13, \( r_{u_1} = 0 \). We let \( S = S^L \), then \( \dim(S_1) = 1 \) and \( \dim(S_2) = 0 \). Thus
\[
\begin{align*}
\theta(S) &= -2b_1 + 2(b_2 - b_3) + 1 \\
&= 1 - 2(b_1 - b_2 + b_3)
\end{align*}
\]
Since \( b_1 - b_2 + b_3 = 2^{N-1} \) where \( N \) is the number of blow up performed after obtaining \( E_2 \), hence \( \theta(S) < 0 \).

• \( B_{i-1} = E_2 - E_3 \), then we must have \( B_{i-2} = E_3 \) (see preliminary). If \( i - 3 = 0 \), then we can think of \( b_{i-3} = 0 \), so \( b_2 = b_{i-1} < b_i = b_3 \), we can set \( \dim(S_i) = 0 \) for all \( i < 3 \) and obtain \( \theta(S) < 0 \) as before. Otherwise \( \mathcal{S} = \{ H - E_1 - E_2 - E_3, E_3, E_2 - E_3, E_1 - E_2, \ldots \} \). Let \( S = S^L \). If \( r_{u_2} = 0 \), then \( \dim(S_2) = 1 \).
\[
\begin{align*}
\theta(S) &= \delta_1(-2b_1) + 2(b_1 - b_2) + \delta_2(2b_2 - 2b_3) + 2(b_3 - b_4) + 1 \\
&= 2(b_1 - b_2) + 2(b_2 - b_3) + 2(b_3 - b_4) + 1 < 0
\end{align*}
\]
If \( r_{u_2} \neq 0 \). Then \( u_4 \circ e_2 \) is a multiple of \( e \), thus \( E_2 - E > 0 \). Then use same argument in the previous case, we can see that \( r_{u_1} = 0 \). Moreover, since \( E_2 - E_3 \) is a strong left orthogonal divisor, \( E_2 \) and \( E_3 \) share no irreducible components, thus \( E \) is not a component of \( E_3 \). This implies \( u_3 \circ e_3 \) is not a multiple of \( e \), then
\[
\theta(S) < 2(b_1 - b_2) + 2(b_2 - b_3) + 2(b_3 - b_4) + 1 < 0
\]
where \( m \neq 0 \). Since \( r_{u_2} \neq 0 \ r_{e_3} \neq 0 \), we have \( r_{u_3} \neq 0 \), thus in the standard left subrepresentation \( S \), we have \( \dim(S_1) = 1 \), and \( \dim(S_2) = \dim(S_3) = 0 \). So
\[
\theta(S) = -2b_1 + 2(b_3 - b_4) + 1 \\
= 2(-b_1 + b_3 - b_4) + 1 \\
= -2(2^{N-1}) + 1
\]
where \( N \) is the number of blow up performed after obtaining \( E_2 \), hence \( \theta(S) < 0 \).

Case 3: \( B_i = H - \Delta \). We have four sub-cases:

- \( i = 1 \), by the same computation as in 1st sub-case in Case 1, we can find subrepresentation \( S \subset R \) so that \( \theta(S) < 0 \).
- \( B_{i-1} = H - \Delta' \). There are three possibilities for \( \Delta' \):
  - If \( \Delta' = \emptyset \), then \( i = 2 \) and \( T S^{op} = \{H, H - \Delta, \ldots\} \). Since \( E \) is not a base component of \( H \), there exist an arrow \( a \) from 0 to 1 such that the corresponding divisor does not contain \( E \) as a component. Then we can take \( u_2 \circ a \) is not a multiple of \( e \), thus as before we can deduce \( r_{u_1} = 0 \). We let \( S = S^L \), then \( \dim(S_1) = 1 \).
    \[
    \theta(S) = -2b_1 + 2(b_3 - b_2) + (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2b_{n-1} \\
    = 1 - 2b_2 \\
    < 0
    \]
  - If \( \Delta' = E_1 \) for some exceptional divisor \( E_1 \), then \( i = 3 \) or 4.
    If \( i = 3 \), then \( B_1 = E_1 \) and \( T S^{op} = \{E_1, H - E_1, H - \Delta, \ldots\} \). We let \( S = S^L \). Note \( E_1 + (H - E_1) = H \). Using same argument above we see that the standard subrepresentation has \( \dim(S_1) = 1 \). We already know \( \dim(S_3) = 1 \). Now \( \dim(S_2) = 1 \) if \( r_{e_2} = 0 \), in this case:
    \[
    \theta(S) = -2b_1 + 2(b_3 - b_2) + 2(b_2 - b_3) + (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2b_{n-1} \\
    = 1 - 2b_3 \\
    < 0
    \]
    and \( \dim(S_2) = 0 \) if \( r_{e_2} \neq 0 \), in this case
    \[
    \theta(S) = -2b_1 + 2(b_2 - b_3) + (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2b_{n-1} \\
    = 1 + 2(b_2 - b_1) - 2b_3
    \]
  - Note \( (b_2 - b_1) = 2^{N-1} \) where \( N \) is the number of blow-up required to get to \( S \) after blowing-up \( E_1 \), while \( b_3 = 2^{N-1}c \) where \( c > 1 \). Hence \( \theta(S) < 0 \).
If \( i = 4 \), then \( B_2 = E_1 - E_2 \), \( B_1 = E_2 \) and \( T S^{op} = \{ E_2, E_1 - E_2, H - E_1, H - \Delta, \ldots \} \). Use the same argument in the first case, we see \( r_{u_1} = 0 \). So

\[
\theta(S) = -2b_1 + 2\delta_1(b_1 - b_2) + 2\delta_2(b_2 - b_3) + 2(b_3 - b_4) + 1
\]

where \( \delta_i \) take values 0, 1 for all \( i \). Note in this case \( b_1 < b_2, b_2 < b_3 \), thus

\[
\theta(S) < 2(-b_1 + b_3 - b_4) + 1 = 1 + 2(b_3 - b_1) - 2b_4
\]

Then \( \theta(S) < 0 \) by the same argument as when \( i = 2 \).

If \( \Delta' = E_1 + E_2 \), then \( i = 4 \), \( B_2 = E_2 \), \( B_1 = E_1 - E_2 \) and \( T S^{op} = \{ E_1 - E_2, E_2, H - E_1 - E_2, H - \Delta, \ldots \} \). We take \( S = S^L \). Since \( B_2 + B_3 = H - E_1 \) and by Lemma 2.9, \( E \) is not a base component of \( H - E_1 \), we can find an arrow \( a \) from 2 to 4 whose corresponding effective divisor does not have \( E \) as a component, then \( u_4 \circ a \) is not a multiple of \( e \). Hence \( r_{u_2} = 0 \) and \( \dim(S_2) = 1 \). Note \( B_1 + B_2 + B_3 = H - E_2 \), apply the same argument, we get \( r_{u_1} = 0 \) and \( \dim(S_1) = 1 \). Since \( E_1 - E_2 \) is slo, either \( E_1 - E \) is not effective or \( E_2 - E \) is not effective, then either \( u_3 \circ e_1 \) is not a multiple of \( e \) or \( u_3 \circ e_2 \) is not a multiple of \( e \). In the first case, we have

\[
r_{u_3}r_{e_2} = mr_{u_1} + 0
\]

where \( m \neq 0 \). Since \( r_{e_2} \neq 0 \) and \( r_{u_3} = 0 \). Same argument applies to the second case and we see \( r_{u_3} \) is always 0. So \( \dim(S_3) = 1 \). We obtain

\[
\theta(S) = 1 - 2b_4 < 0
\]

If \( \Delta' = E_1 + E_2 + E_3 \), then \( i = 5 \) and \( T S^{op} = \{ E_1 - E_2, E_2 - E_3, E_3, H - E_1 - E_2 - E_3, H - \Delta, \ldots \} \). By assumption, \( r_{u_5} = 0 \). We take \( S = S^L \).

Suppose \( r_{u_4} = 0 \). If \( E \) is not a component of \( E_1, E_2 \) or \( E_3 \), then neither of \( u_4 \circ e_3 \), \( u_4 \circ e_2 \) and \( u_4 \circ e_1 \) is a multiple of \( e \), hence as before \( r_{u_i} = 0 \) for \( i = 1, 2, 3, 4 \). Hence \( \dim(S_1) = \dim(S_2) = \dim(S_3) = \dim(S_4) = 1 \)

\[
\theta(S) = -2b_1 + 2(b_1 - b_2) + \ldots + 2(b_4 - b_5) + 1
\]

\[
= 1 - 2b_5
\]

< 0

Otherwise, since \( E_1 - E_2, E_2 - E_3, E_1 - E_3 \) are all strong left orthogonal, \( E \) can only be the component of one of them. If \( E_3 - E > 0 \), then by Lemma 2.13, \( E \) is not a component of \( H - E_1 - E_2 \), hence there is an arrow \( a \) from 3 to 5 such that \( u_5 \circ a \) is not a multiple of \( e \). Thus \( r_{u_5} = 0 \). Since \( E \) is not a component of \( E_1 \) and \( E_2 \), we can use the argument above to show \( r_{u_1} = r_{u_2} = 0 \), hence again \( \dim(S_1) = \dim(S_2) = \dim(S_3) = \dim(S_4) = 1 \) and

\[
\theta(S) = 1 - 2b_5 < 0
\]

The same argument applies to the cases when \( E_1 - E > 0 \) or \( E_2 - E > 0 \).

Suppose \( r_{u_4} \neq 0 \). Let \( p : S \to \mathbb{P}^2 \) be the projection. Since \( E_1 - E_2, E_2 - E_3, E_1 - E_3 \) are all strong left orthogonal, \( p(E_1), p(E_2) \) and \( p(E_3) \) are three distinct points
on $\mathbb{P}^2$. Since $H - E_1 - E_2 - E_3$ is strong left orthogonal, $p(E_1), p(E_2), p(E_3)$ does not lie on the same line. Note
\[
B_1 + B_2 + B_3 + B_4 = H - E_2 - E_3 \\
B_2 + B_3 + B_4 = H - E_1 - E_3 \\
B_3 + B_4 = H - E_1 - E_2
\]

$E$ is a component of $H - E_i - E_j$ is equivalent to $p(E)$ is on the line passing through $p(E_i)$ and $p(E_j)$ for $i \neq j$, then $E$ cannot be the base component of all three linear systems. Hence
\[
\theta(S) = -2\delta_1 b_1 + 2\delta_2 (b_1 - b_2) + 2\delta_3 (b_2 - b_3) + 2(b_4 - b_5) + 1
\]
where $\delta_i$ takes value 0 or 1 for all $i \in \{1, 2, 3\}$. Note if $\delta_1 = 0$, then $r_{u_0} \neq 0$. Let $u_{15}$ be the unique arrow from 1 to 5 representing the unique section of $H - E_2 - E_3$, then $u_5 \circ u_{15}$ is an arrow from 1 to $k'$. If $E$ is not a base component of $|H - E_2 - E_3|$, then $u_5 \circ u_{15}$ is not a multiple of $e$, contradicting $r_{u_1} \neq 0$. Hence $\delta_1 = 0$ implies $E$ is a base component of $H - E_2 - E_3$. Apply the similar argument to the remaining $\delta$'s, we see $\delta_1, \delta_2, \delta_3$ cannot vanish simultaneously.

Since $b_1 < b_2$, $b_2 < b_3$, and $\delta_i$ cannot all be 0 for $i \in \{1, 2, 3\}$
\[
\theta(S) \leq \max\{1 - 2(b_1 - b_4 + b_5), 1 + 2(b_1 - b_2 + b_4 - b_5), 1 + 2(b_2 - b_3 + b_4 - b_5)\}
\]
The three values we are taking maximum of is obtained when setting exactly one of $\delta_i$ to be 1. $b_1 - b_4 = 2^{N_1 - 1}$ where $N_1$ is the number of blow up performed after obtaining $E_1$, while $2^{N_1} | b_5$, hence $1 - 2(b_1 - b_4 + b_5) < 0$. $(b_1 - b_2 + b_4) = 2^{N_2 - 1}$ where $N_2$ is the number of blow up performed after obtaining $E_2$, while $2^{N_2} | b_5$, hence $1 + 2(b_1 - b_2 + b_4 - b_5) < 0$. Similarly, one shows $1 + 2(b_2 - b_3 + b_4 - b_5) < 0$, so $\theta(S) < 0$.

- $B_{i-1} = E_1 - E_2$. We let $S = S^L$. Here are the possible situations:

  (1) $i = 3$, then $T S^{op} = \{E_2, E_1 - E_2, H - \Delta \ldots \}$. Note $b_1 - b_2 < 0$ and $b_2 - b_3 < 0$. We take the standard subrepresentation. Thus
\[
\theta(S) = -2\delta_1 b_1 + 2\delta_2 (b_1 - b_2) + 2(b_2 - b_3) \\
\quad + (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2b_{n-1} \\
\quad = 1 - 2\delta b_1 + 2\delta_2 (b_1 - b_2) + (b_2 - b_3) \\
\quad < 1 + 2(b_2 - b_3)
\]
where $\delta_i$ is 0 or 1 for all $i$. Then it is clear $\theta(S) < 0$.

  (2) $i = 4$ with $T S^{op} = \{E_3, E_2 - E_3, E_1 - E_2, H - \Delta \ldots \}$. This case can be handled using the same method in (1).

  (3) $i = 4$ with $T S^{op} = \{H - E_1 - E_2, E_2, E_1 - E_2, H - \Delta \ldots \}$. By assumption $r_{u_4} = 0$. We let $S = S^L$. Note $B_1 + B_2 + B_3 = H - E_2$, by Lemma 2.9, $E$ is not a base component of $|H - E_2|$, hence we can find an arrow $a$ from 1 to
4 such that $u_4 \circ a$ is not a multiple of $e$, hence $r_{u_1} = 0$ and $\dim(S_1) = 1$. If $\dim(S_2) = 1$, then
\[
\theta(S) = -2b_1 + 2(b_1 - b_2) + 2\delta(b_2 - b_3) + 2(b_3 - b_4)
+ (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2b_n
\]
\[
\theta(S) = 1 - 2b_2 + 2\delta(b_2 - b_3) + 2(b_3 - b_4)
\]
where $\delta$ is either 0 or 1. Notice $b_2 > b_3$, so
\[
\theta(S) < 1 - 2b_2 + 2(b_2 - b_3) + 2(b_3 - b_4) = 1 - 2b_4 < 0
\]
If $\dim(S_2) = 0$, then as above we see $E_1 - E > 0$. Since $E_1 - E_2$ is strong left orthogonal, $E$ is not a component of $E_2$. Thus $u_3 \circ e_2$ is not a multiple of $e$. Since $r_{u_2} \neq 0$ and $r_{e_2} \neq 0$, $r_{u_3} \neq 0$. Thus $\dim(S_3) = 0$. Hence
\[
\theta(S) = -2b_1 + 2(b_3 - b_4) + 1
\]
\[
2(-b_1 + b_3 - b_4) = -2N
\]
where $N$ is the number of blow ups performed after obtaining $E_1$. So $\theta(S) = 1 - 2N < 0$.

(4) \(i = 5\) with $T_5^{op} = \{E_3, H - E_1 - E_2 - E_3, E_2, E_1 - E_2, H - \Delta \ldots\}$. We take $S = S^L$. Since $B_1 + B_2 + B_3 + B_4 = H - E_2$, by Lemma 2.9 $E$ is not a base component of $|H - E_2|$, hence there is an arrow $a$ from 1 to 5 such that $u_5 \circ a$ is not a multiple of $e$, thus $r_{u_1} = 0$ and $\dim(S_1) = 1$. If $r_{u_3} = 0$, then $\dim(S_3) = 1$.
\[
\theta(S) = -2b_1 + 2\delta_2(b_1 - b_2) + 2(b_2 - b_3) + 2\delta_3(b_3 - b_4) + 2(b_4 - b_5)
+ (1 - 2b_{k+1}) + 2(b_{k+1} - b_{k+2}) + \ldots + 2b_n
\]
\[
\theta(S) = 1 - 2b_1 + 2(b_2 - b_3) + 2\delta_2(b_1 - b_2) + 2\delta_3(b_3 - b_4) + 2(b_4 - b_5)
\]
Then
\[
\theta(S) = \begin{cases} 
1 - 2b_5 & \text{if } \delta_2 = \delta_3 = 1 \\
1 + 2(b_2 - b_1 - b_5) & \text{if } \delta_2 = 0 \text{ and } \delta_3 = 1 \\
1 + 2(b_4 - b_3 - b_5) & \text{if } \delta_2 = 1 \text{ and } \delta_3 = 0 \\
1 - 2(b_1 + b_5) + 2(b_2 - b_3 + b_4) & \text{if } \delta_2 = \delta_3 = 0 
\end{cases}
\]
Clearly $1 - 2b_5 < 0$. $b_2 - b_1 = 2N_3 - 1$ where $N_3$ is the number of blow ups performed after obtaining $E_3$, but $2N_3|a_5$, thus $1 + 2(b_2 - b_1 - b_5) < 0$. $b_4 < b_3$, hence $1 + 2(b_4 - b_3 - b_5) < 0$. $2(b_4 - b_3 + b_5) = 2N_2$ where $N_2$ is the number of blow ups performed after obtaining $E_2$, but $2N_2|a_5$, hence $1 + 2(b_4 - b_3 - b_5) < 0$.

If $r_{u_4} \neq 0$, then $\dim(S_3) = 0$, and $E_1 - E > 0$. As in (3), we see $u_4 \circ e_2$ is not a multiple of $e$. This implies $r_{u_4} \neq 0$, so $\dim(S_4) = 0$. We claim $\dim(S_2) = 1$. It suffices to show if $u_{25}$ is the unique arrow from 2 to 5, then $u_5 \circ u_{25}$ is not a multiple of $e$. Now $\text{div}(u_{25})$ is an effective divisor linear equivalent to $B_2 + B_3 + B_4 = H - E_2 - E_3$. It suffices to show $\text{div}(u_{25}) - E$ is not effective,
and we can accomplish this by showing \( h^0(S, H - E - E_2 - E_3) = 0 \). Since \( E_1 - E > 0 \), this follows from Lemma 2.13. Then

\[
\theta(S) = -2b_1 + 2(b_1 - b_2) + 2(b_4 - b_5) + 1 \\
= 1 - 2(b_2 - b_4 + b_5)
\]

Since \( 2(b_2 - b_4 + b_5) = 2^{N_1} \) where \( N_1 \) is the number of blow ups performed after obtaining \( E_1 \), \( \theta(S) < 0 \).

(5) \( i = 5 \) with \( T \mathcal{S}^{op} = \{ H - E_1 - E_2 - E_3, E_2 - E_3, E_1 - E_2, H - \Delta \ldots \} \). Let \( p : S \rightarrow \mathbb{P}^2 \) be the projection. Since \( E_1 - E_2, E_2 - E_3, E_1 - E_3 \) are all strong left orthogonal, \( p(E_1), p(E_2) \) and \( p(E_3) \) are three distinct points on \( \mathbb{P}^2 \). Since \( H - E_1 - E_2 - E_3 \) is strong left orthogonal, \( p(E_1), p(E_2), p(E_3) \) does not lie on the same line. Note

\[
B_1 + B_2 = H - E_1 - E_2 \\
B_1 + B_2 + B_3 = H - E_1 - E_3 \\
B_1 + B_2 + B_3 + B_4 = H - E_2 - E_3
\]

\( E \) is a component of \( H - E_i - E_j \) is equivalent to \( p(E) \) is on the line passing through \( p(E_i) \) and \( p(E_j) \) for \( i \neq j \), then \( E \) cannot be the base component of all three linear systems, then if \( r_{u_3} = r_{u_4} = 0 \), there is an arrow \( a_i \) from 1 to \( i \) for some \( i \in \{3, 4, 5\} \) such that \( u_i \circ a_i \) is not a multiple of \( e \), hence \( r_{u_i} = 0 \).

Suppose \( r_{u_2} = 0 \), then \( \dim(S_2) = 1 \). If moreover \( r_{u_3} = r_{u_4} = 0 \), then as above \( r_{u_1} = 0 \) and we have \( \dim(S_i) = 1 \) for \( i = 1, 2, 3, 4, 5 \). So

\[
\theta(S) = 1 - 2b_5 < 0
\]

Otherwise

\[
\theta(S) = -2\delta_1b_1 + 2(b_1 - b_2) + 2\delta_2(b_2 - b_3) + 2\delta_3(b_3 - b_4) + 2(b_4 - b_5) \\
< 2(b_1 - b_2) + 2\delta_2(b_2 - b_3) + 2\delta_3(b_3 - b_4) + 2(b_4 - b_5)
\]

where \( \delta_i \) takes values 0 or 1 and \( \delta_2 \) and \( \delta_3 \) cannot both be 1, then

\[
\theta(S) < \begin{cases} 
1 + 2(b_1 - b_2 + b_4 - b_5) & \text{if } \delta_2 = \delta_3 = 0. \\
1 + 2(b_1 - b_3 + b_4 - b_5) & \text{if } \delta_2 = 1 \text{ and } \delta_3 = 0. \\
1 + 2(b_1 - b_2 + b_3 - b_5) & \text{if } \delta_2 = 0 \text{ and } \delta_3 = 1.
\end{cases}
\]

As before, we can easily verify all three numbers are negative, so \( \theta(S) < 0 \).

Suppose \( r_{u_2} \neq 0 \), then \( E_1 - E > 0 \) and \( E \) is not a component of \( |H - E_2 - E_3| \). Thus if \( u_{15} \) is the unique arrow from 1 to 5, we have \( u_5 \circ u_{15} \) is not a multiple of \( e \), hence \( r_{u_5} = 0 \) and \( \dim(S_1) = 1 \). Moreover, \( E \) is not a component of neither \( E_2 \) or \( E_3 \). Thus \( u_3 \circ e_3 \) is not a multiple of \( e \). Since \( r_{u_2} \neq 0 \), \( r_{u_3} \neq 0 \).
Similarly, \( r_{u_4} \neq 0 \). Thus

\[
\theta(S) = -2b_1 + 2(b_4 - b_5) + 1
\]

\[
= 1 - 2(b_1 - b_4 + b_5)
\]

Note \( 2(b_1 - b_4 + b_5) = 2^N \) where \( N \) is the number of blow ups performed after obtaining \( E_1 \), hence \( \theta(S) < 0 \).

- \( B_{i-1} = E_1 \) for some exceptional divisor \( E_1 \). In all the following sub-cases, we let \( S = S^L \).

1. \( \mathcal{TS}^{op} = \{E_1, H - \Delta \ldots\} \) or \( \mathcal{TS}^{op} = \{E_2 - E_1, E_1, H - \Delta \ldots\} \) or \( \mathcal{TS}^{op} = \{E_3 - E_2, E_2 - E_1, E_1, H - \Delta \ldots\} \). In all these cases, \( b_i < b_{i+1} \), we can consider the subrepresentation \( S \) with \( \dim(S_j) = 0 \) for all \( j < i \). Then

\[
\theta(S) = 1 - 2(b_i - b_{i+1}) < 0
\]

where \( \delta \) is either 0 or 1. Note \( b_1 < b_2 \), so \( \theta(S) \leq -2b_1 + 2(b_2 - b_3) + 1 = 1 - 2(b_1 - b_2 + b_3) = 1 - 2^N \) where \( N \) is the number of blow ups performed after obtaining \( E_1 \). Thus \( \theta(S) < 0 \).

2. \( \mathcal{TS}^{op} = \{H - E_1, E_1, H - \Delta \ldots\} \). Since \( B_1 + B_2 = H \), \( E \) is not a base component of \( |H| \), there is an arrow \( a \) from 1 to 3 such that \( u_3 \circ a \) is not a multiple of \( e \), thus \( r_{u_3} = 0 \) and \( \dim(S_1) = 1 \). Thus

\[
\theta(S) = -2b_1 + 2\delta(b_1 - b_2) + 2(b_2 - b_3) + 1
\]

\[
= -2(b_2 - b_3) + 1 + 2\delta(b_2 - b_3)
\]

Since \( b_2 - b_3 < 0 \), \( \theta(S) \leq -2(b_2 - b_3 + b_4) + 1 = 1 - 2^N \) where \( N \) is the number of blow up performed after obtaining \( E_1 \), hence \( \theta(S) < 0 \).

3. \( \mathcal{TS}^{op} = \{E_2, H - E_1 - E_2, E_1, H - \Delta \ldots\} \). \( B_2 + B_3 = H - E_2 \), thus there exist an arrow \( a \) from 1 to 3 such that \( u_3 \circ a \) is not a multiple of \( e \). Hence \( r_{u_2} = 0 \) and \( \dim(S_2) = 1 \). Similarly, since \( B_1 + B_2 + B_3 = H \), we can show \( \dim(S_1) = 1 \). Hence

\[
\theta(S) = -2b_1 + 2(b_1 - b_2) + 2\delta(b_2 - b_3) + 2(b_3 - b_4) + 1
\]

\[
= -2(b_2 - b_3) + 1 + 2\delta(b_2 - b_3)
\]

Since \( b_2 - b_3 < 0 \), \( \theta(S) \leq -2(b_2 - b_3 + b_4) + 1 = 1 - 2^N \) where \( N \) is the number of blow up performed after obtaining \( E_1 \), hence \( \theta(S) < 0 \).

4. \( \mathcal{TS}^{op} = \{E_3, E_2 - E_3, H - E_1 - E_2, E_1, H - \Delta \ldots\} \). This case can be dealt with using the arguments in (3).

5. \( \mathcal{TS}^{op} = \{E_2 - E_3, E_3, H - E_1 - E_2 - E_3, E_1, H - \Delta \ldots\} \). Use the argument in (3), we see \( \dim(S_1) = \dim(S_2) = 1 \). Suppose \( \dim(S_3) = 1 \), then

\[
\theta(S) = -2b_1 + 2(b_1 - b_2) + 2\delta(b_2 - b_3) + 2(b_3 - b_4) + 2(b_4 - b_5) + 1
\]

where \( \delta \) is 0 or 1. Notice \( b_2 > b_3 \), hence \( \theta(S) < 1 - 2b_5 < 0 \).

Suppose \( \dim(S_4) = 0 \), then \( E_1 - E > 0 \). Since \( E_2 - E_3 \) is strong left orthogonal, \( E_1 \) cannot be share component with both of them. By Lemma 2.13, \( u_{35} \) is the unique arrow from 3 to 5, then \( u_5 \circ u_{35} \) is not a multiple of \( e \), thus \( r_{u_3} = 0 \).
and \( \dim(S_3) = 1 \). Thus
\[
\theta(S) = 1 - 2(b_3 - b_4 + b_5)
\]

\( 2(b_3 - b_4 + b_5) = 2^N \) where \( N \) is the number of blow ups performed after obtaining \( E_1 \), hence \( \theta(S) < 0 \). As before, one can easily check all cases give \( \theta(S) < 0 \).

(6) \( TS^{op} = \{ H - E_2, E_2 - E_1, E_1, H - \Delta \} \) As before, we can find an arrow \( a \) from 1 to 4 such that \( u_4 \circ a \) is not a multiple of \( e \), hence \( \dim(S_1) = 1 \). Note \( \dim(S_3) = 0 \) implies \( E_1 - E > 0 \). Similarly, \( \dim(S_2) = 0 \) implies \( E_2 - E > 0 \). Since \( E_2 - E_1 \) is strong left orthogonal, \( E_1 \) and \( E_2 \) share no irreducible components. So \( \dim(S_2) \) and \( \dim(S_3) \) cannot be both 0. We compute
\[
\theta(S) = \begin{cases} 
1 - 2b_4 & \text{if } \dim(S_1) = \dim(S_2) = 1 \\
1 - 2(b_2 - b_3 + b_4) & \text{if } \dim(S_1) = 1 \text{ and } \dim(S_2) = 0 \\
1 - 2(b_1 - b_2 + b_4) & \text{if } \dim(S_1) = 0 \text{ and } \dim(S_2) = 1
\end{cases}
\]

One can easily check all cases give \( \theta(S) < 0 \).

(7) \( TS^{op} = \{ E_3, H - E_2 - E_3, E_2 - E_1, E_1, H - \Delta \} \) This case is similar to (6). One checks \( \dim(S_1) = \dim(S_2) = 1 \), then follow the argument in (6).

(8) \( TS^{op} = \{ E_4, E_3 - E_4, H - E_2 - E_3, E_2 - E_1, E_1, H - \Delta \} \). This case again is similar to (6). One checks \( \dim(S_1) = \dim(S_2) = \dim(S_3) = 1 \), then follow the argument in (6).

(9) \( TS^{op} = \{ E_3 - E_4, E_4, H - E_2 - E_3 - E_4, E_2 - E_1, E_1, H - \Delta \} \). In this case \( i = 6 \). Note \( B_1 + B_2 + \ldots + B_5 = H - E_4 \), by Lemma 2.39, there is an arrow \( a \) from 1 to 6 such that \( u_6 \circ a \) is not a multiple of \( e \), thus \( r_{u_4} = 0 \). Similarly, \( B_2 + \ldots + B_5 = H - E_3 \), we obtain \( r_{u_2} = 0 \) via the same argument. So \( \dim(S_1) = \dim(S_2) = 0 \). Note \( E_1 - E_2 \) is strong left orthogonal, then \( E \) cannot be component for both of them. Thus either \( u_6 \circ e_1 \) is not a multiple of \( e \) or \( u_6 \circ e_2 \) is not a multiple of \( e \) or both. So \( \dim(S_4) \) and \( \dim(S_5) \) cannot both be zero. Note \( b_2 > b_3 \), then
\[
\theta(S) = -2b_1 + 2(b_1 - b_2) + 2\delta(b_2 - b_3) + 2\delta_1(b_3 - b_4) + 2\delta_2(b_4 - b_5) + 2(b_5 - b_6) + 1
\]
\[
< -2b_2 + 2(b_2 - b_3) + 2\delta_1(b_3 - b_4) + 2\delta_2(b_4 - b_5) + 2(b_5 - b_6) + 1
\]
Since \( \dim(S_4) \) and \( \dim(S_5) \) cannot both be zero is the same as saying \( \delta_1 \) and \( \delta_2 \) cannot both be zero, we obtain

\[
\theta(S) < \begin{cases} 
1 - 2(b_3 - b_4 + b_6) & \text{if } \delta_1 = 0 \text{ and } \delta_2 = 1, \\
1 - 2(b_4 - b_5 + b_6) & \text{if } \delta_1 = 1 \text{ and } \delta_2 = 0, \\
1 - 2b_6 & \text{if } \delta_1 = \delta_2 = 1.
\end{cases}
\]

Again one can easily verify all three numbers are negative, so \( \theta(S) < 0 \).

So we have exhausted all possible situations when \( r_{ui} = 0 \) for some \( j < k \). The case for \( i > k \) can be handled using the arguments above by symmetry and using right standard subrepresentations. This finishes the proof. \( \square \)

The following Proposition generalizes Proposition 6.5.

**Proposition 6.12.** Suppose \( F(R) = R' \), \( R \) is \( \theta \)-stable, then \( R' \) is \( \theta_0 \)-stable.

**Proof.** If \( R \in U \), the result follows from Proposition 6.5. Suppose \( R \in V(e) \), we will use the same recipe of the proof of the Proposition 6.5. Let \( S' \subset R' \), \( \vec{d} \) be the dimension vector of \( S' \), we claim there is a subrepresentation \( S \) of \( R \) whose dimension vector \( \vec{d} \) specifies as follows:

1. If \( i < k \), \( d_i = d'_i \).
2. If \( j > k \), \( d_j = d'_j \).
3. \( d_k = d_{k'} = d'_k \).

To prove the claim, we check the conditions in Lemma 2.29. Let \( a \) be an arrow from \( i \) to \( j \) such that \( d_i = 1 \) and \( d_j = 0 \). If \( j < k \), then \( d'_i = 1 \), \( d'_j = 0 \) and by construction of \( F \), if \( a' \) is the corresponding arrow in \( Q_0 \) to \( a \), we see \( r_{a'} = r_a = 0 \).

If \( j = k \) or \( j = k' \), then \( d'_k = 0 \), \( d'_i = 1 \). By Proposition 6.8, \( r_{ui} \neq 0 \). Thus in \( R' \), \( r_{ui} \neq 0 \), this leads to a contradiction. So if \( d_k = 0 \) or \( d_{k'} = 0 \), we must have \( d_i = 0 \) for all \( i < k \).

If \( j > k \), then if in addition \( i > k \), then \( d'_i = 1 \) and \( d'_j = 0 \). So there is an arrow \( a' \) in \( Q_0 \) such that \( r_a = r_{a'} = 0 \) by construction of \( F \). If \( i < k \), then \( d'_i = 1 \) and \( d'_j = 0 \). By Lemma 3.7 there is an arrow in \( Q_0 \) from \( i \) to \( j \) whose value is \( r_{ui} \), contradiction. Similarly, if \( i = k' \) or \( i = k \), then \( d'_k = 1 \) and \( d'_j = 0 \), we reach a contradiction following the argument in the previous paragraph.

So a subrepresentation \( S \) of \( R \) with dimension vector exists. Now it is clear that

\[
\theta(S) = 2\theta_0(S')
\]

Since \( R \) is \( \theta \)-stable, \( \theta(S) > 0 \), thus \( \theta_0(S') > 0 \). Apply this argument for all \( S' \subset R' \), we see \( R' = F(R) \) is \( \theta_0 \)-stable. \( \square \)
7. Analysis of the fibre

Let \( C \) denote the closed subscheme of \( M_{\theta} \) containing stable orbits of representations with \( r_e = 0 \), i.e.

\[ C = V(e)^S // \text{PGL}(1) \]

where \( V(e)^S \) is the open subscheme of \( V(e) \) consisting of stable representations.

**Theorem 7.1.**

\[ F(V(e)^S) \in T_0(P) \]

In other words, the image of a stable representation in \( V(e) \) under \( F \) lies in the isomorphism class \( T_0(P) \in M_{\theta_0} \).

**Proof.** Let \( R \) be a representation of \( Q \) such that \( r_e = 0 \), then by results in Section 3, the construction of \( F \) and Proposition 6.8, \( F(R) \) satisfies the following properties:

1. For any \( i < k \), all the arrows in \( H^0(S_0, B_i + \ldots + B_{k-1}) \) passing through \( P \) have value 0.
2. For each \( i < k \), the unique arrow \( w_i \) in \( H^0(S_0, B_i + \ldots + B_{k-1}) \) not passing through \( P \) have nonzero value.
3. For any \( j > k \), all the arrows in \( H^0(S_0, B_k + \ldots + B_{j-1}) \) passing through \( P \) have value 0.
4. For each \( j > k \), the unique arrow \( w_j \) in \( H^0(S_0, B_k + \ldots + B_{j-1}) \) not passing through \( P \) have nonzero value.

Claim: All representations of \( Q_0 \) satisfying the above properties are in the same \( \text{PGL}(1) \) orbit.

Let \( R' \) be such a representation, then by letting \( g_k = 1 \) and choose suitable \( g_l \) for \( l \neq k \) and replace \( R' \) by \( g \cdot R \) we can assume \( r'_{w_i} = 1 \) for all \( i \). We now show the value of all other arrows only depends on \( Q_0 \), instead of \( R' \).

Let \( a \) be an arrow in \( Q \). If \( t(a) < k \), then \( w_{t(a)} \circ a \) is an arrow from \( s(a) \) to \( k \). Using the property above, we can see that:

\[
    r'_{a} = \begin{cases} 
        0 & \text{if } \text{div}(a) \text{ passes through } P \\
        1 & \text{if } \text{div}(a) \text{ does not pass through } P 
    \end{cases}
\]

Similarly, we can get the value for arrows with \( s(a) < k \).

If \( s(a) < k \) and \( t(a) > k \). By Lemma 3.7 if the passes through \( P \), then its value is 0, otherwise 1. This shows all such \( R' \) satisfying the above properties are isomorphic to a representation \( R \) whose values of arrows are given by for all \( i \)

\[
    r_{w_i} = 1
\]

and for other arrows

\[
    r_{a} = \begin{cases} 
        0 & \text{if } \text{div}(a) \text{ passes through } P \\
        1 & \text{if } \text{div}(a) \text{ does not pass through } P 
    \end{cases}
\]
By Proposition 6.12, the unique orbit containing $F(V(e)^S)$ is $\theta_0$-stable, so it corresponds to a point on $S_0$. On the other hand, it is clear that the any representative in the isomorphism class $T_0(P) = \bigoplus_{i=0}^{n}(L_i^\gamma)_P$ satisfies the all the above properties, so the image must be in $T_0(P)$. □

**Proposition 7.2.** If $R$ is a representation in $\operatorname{Rep}(Q)^S_\theta$ such that $F(R) \in T_0(P)$ and $r_e \neq 0$, then $R$ is not $\theta$-stable.

**Proof.** Note $F(R)$ satisfies the 4 conditions in the proof of the previous theorem. Using the construction of $F$ and the fact that $r_e \neq 0$, we see that $R$ satisfies:

1. For any $i < k$, all arrows in from $i$ to $k$ have value 0.
2. For any $j > k$, all arrows from $k'$ to $j$ have value 0.
3. For any $i < k < j$, all arrows from $i$ to $j$ have value 0.

Note the last property uses the same argument in the last paragraph of proof of Theorem. Hence we see that $R$ has a subrepresentation $S$ defined by:

$$\dim(S_1) = \dim(S_2) = \ldots = \dim(S_{k-1}) = \dim(S_k') = 1$$
$$\dim(S_k) = \dim(S_{k+1}) = \ldots = \dim(S_n) = 0$$

We compute:

$$\theta(S) = -2b_0 + 2(b_0 - b_1) + \ldots + 2(b_{k-2} - b_{k-1}) + (2b_{k-1} - 1)$$
$$= -1$$

This shows $R$ is not $\theta$-stable. □

**Corollary 7.3.** The natural morphism

$$F : \operatorname{Rep}(Q) \to \operatorname{Rep}(Q_0)$$

descends to a projective morphism

$$f : M_\theta \to M_{\theta_0}$$

which fits into a commutative diagram

$$\begin{array}{ccc}
S & \xrightarrow{T} & M_\theta \\
\downarrow\pi & & \downarrow f \\
S_0 & \xrightarrow{T_0} & M_{\theta_0}
\end{array}$$

In other words, the above diagram is commutative wherever $T$ is defined.

**Proof.** By Proposition 6.12 we can restrict the domain of $F$ to get

$$F_0 : \operatorname{Rep}(Q)^S_{\theta} \to \operatorname{Rep}(Q_0)^S_{\theta_0}$$

Composed with the projection map $\operatorname{Rep}(Q)^S_{\theta} \to M_{\theta_0}$ of the geometric quotient, we obtain a morphism

$$F_1 : \operatorname{Rep}(Q)^S_{\theta} \to \operatorname{Rep}(Q_0)^S_{\theta_0}/\text{PGL}(1) = M_{\theta_0}$$
Proposition 6.3 shows $F_1$ descends to the quotient, and give morphism
\[ f : M_\theta \to M_{\theta_0} \]
Since both $M_\theta, M_{\theta_0}$ are projective schemes, $f$ is a projective morphism.

Suppose the tautological rational map $T$ is defined at a point $s \in S$ and $s$ does not lie on the exceptional curve $E$, then we know
\[ T(s) = \bigoplus_{i=1}^{n-k} (\pi^*\mathcal{O}(-D_i))_s \oplus (\pi^*\mathcal{O}(-D_{n-k+1} + E))_s \oplus (\pi^*\mathcal{O}(-D_{n-k+1}))_s \oplus \bigoplus_{i=n-k+2}^{n} (\pi^*\mathcal{O}(-D_i + E))_s \]
is $\theta$-stable. Since $s \notin E$, there exist an neighbourhood $V$ of $s$ such that $\pi|_V = id$. We note that by projection formula
\[ A_0 = \text{End}(\bigoplus_{i=1}^{n} \mathcal{O}(-D_i)) \]
\[ = \text{End}(\bigoplus_{i=1}^{n} \pi^*(\mathcal{O}(-D_i))) \]
Recall the construction of $F$, in the definition of ring homomorphism, for an arrow $a$ in $Q_0$ from $i$ to $j$ representing an section $s$ in $\text{Hom}_{\mathcal{O}_{S_0}}(L_i, L_j)$, we pulled it back by $\pi^*$ to get an element in $\text{Hom}_{\mathcal{O}_{S_0}}(\pi^*L_i, \pi^*L_q)$, then find the corresponding arrow or multiple of arrows using the Lemmas in section 3. Hence tracing the definition of $F$ and the descent we have
\[ f(T(s)) = \bigoplus_{i=1}^{n} (\pi^*\mathcal{O}(-D_i))_s \]
whose $A_0$-module structure is given by (7.4) and (7.5). Since $\pi = id$ near $s$, we see $\bigoplus_{i=1}^{n} (\pi^*\mathcal{O}(-D_i))_s$ as an $A_0$-module is isomorphic to $\bigoplus_{i=1}^{n} (\mathcal{O}(-D_i))_{\pi(s)} = T_0(\pi(s))$.

Suppose $T$ is defined at a point $s \in E$, then $T(s)$ is $\theta$-stable and $T(s) \in V(e)$. By Theorem 3.1 we see $f(T(s)) = T_0(P) = T_0(\pi(s))$. \qed

**Proposition 7.6.** The tautological map $T$ is defined on a nonempty open subset of $S \setminus E$. In particular,
\[ M_\theta \neq \emptyset \]

**Proof.** Consider a representation $R$ in the isomorphism class $\bigoplus_{i=1}^{n-k} (\pi^*\mathcal{O}(-D_i))_s \oplus (\pi^*\mathcal{O}(-D_{n-k+1} + E))_s \oplus \bigoplus_{i=n-k+2}^{n} (\pi^*\mathcal{O}(-D_i + E))_s$ of representations of $(Q, I)$. For any arrow $a$ in $Q$, $r_a = 0$ if $s \in \text{div}(a)$. Since $\text{div}(a)$ is a divisor for all arrows and there are finitely many arrows $a$ in $Q$, there exist an open set $U \subset S$ such that for any $s \in U$, $r_a \neq 0$ for all arrows $a \in Q_{ar}$. We take $s \in U$. By Remark 2.31 this representation $\bigoplus_{i=1}^{n-k} (\pi^*\mathcal{O}(-D_i))_s \oplus (\pi^*\mathcal{O}(-D_{n-k+1} + E))_s \oplus (\pi^*\mathcal{O}(-D_{n-k+1}))_s \oplus \bigoplus_{i=n-k+2}^{n} (\pi^*\mathcal{O}(-D_i + E))_s$ will be the best candidate to be stable.

Let $s_0 = \pi(s)$, then $R' = F(R) \not\in \bigoplus_{i=1}^{n} (\mathcal{O}(-D_i))_{s_0}$. By definition of $F$, $r'_{b} \neq 0$ for all arrows $b \in Q_{0, ar}$. Since $M_{\theta_0} \neq \emptyset$, by Remark 2.31 $R'$ is $\theta_0$-semistable. Now let $S$ be a
subrepresentation of $R$. We first assume that $\dim(S_k) = \dim(S_{\nu'})$. We claim we can find a subrepresentation $S' \subset R'$ such that $\dim(S'_{i}) = \dim(S_{i})$ for all $i \in \{1, 2, \ldots, n\}$. To check $S'$ is well defined, by Lemma 2.29, it suffices to show for any $i < j$ with $\dim(S'_{i}) = 1$ and $\dim(S'_{j}) = 0$, there are no arrow from $i$ to $j$. By definition of $S'$, $\dim(S_i) = 1$ and $\dim(S_j) = 0$. By our choice of $s$, this is only possible when there are no arrows from $i$ to $j$. Since $E^2 = -1$, either $j < k$ or $i > k$, in both cases the arrows from $i$ to $j$ in $Q$ are in bijection with arrows from $i$ to $j$ in $Q_0$, hence there is no arrows from $i$ to $j$ in $Q_0$. Hence $S'$ with the prescribed dimension is well-defined and we obtain

$$\theta(S) = 2\theta_0(S')$$

Since $R'$ is stable and $S' \subset R'$, we have $\theta_0(S') > 0$, so $\theta(S) > 0$.

Suppose $\dim(S_k) \neq \dim(S_{\nu'})$, then either $\dim(S_k) = 1$ and $\dim(S'_{k}) = 0$, but this is not possible since such dimensions require $r_e = 0$, or $\dim(S_k) = 0$ and $\dim(S'_{k}) = 1$, in this case we claim either $\dim(S_i) = 0$ for all $i < k$ or $\dim(S_j) = 1$ for all $j > k$.

If the claim is not true, the there exist $i < k < j$ such that $\dim(S_i) = 1$ and $\dim(S_j) = 0$. By Lemma, there is an arrow $a$ from $i$ to $j$, hence $r_a = 0$, contradicting our choice of $s$.

If $\dim(S_i) = 0$ for all $i < k$, then we see that $S^\#$ defined by $\dim(S^\#_{i}) = \dim(S_{i})$ for all $i$ except $\dim(S^\#_{k}) = 0$ is also a subrepresentation of $R$. Note $\theta(S) = \theta(S^\#) + (2a_k - 1) > \theta(S^\#)$. Since $\dim(S^\#_{k}) = \dim(S^\#_{k'}) = 0$, we can apply the first part of this proof to show $\theta(S^\#) > 0$. Hence $\theta(S) > 0$.

If $\dim(S_j) = 1$ for all $j > k$, then we see that $S^\#$ defined by $\dim(S^\#_{l}) = \dim(S_{l})$ for all $l$ except $\dim(S^\#_{k}) = 1$ is also a subrepresentation of $R$. Note $\theta(S) = \theta(S^\#) - (1 - 2b_{k-1}) > \theta(S^\#)$. Since $\dim(S^\#_{k}) = \dim(S^\#_{k'}) = 1$, we can apply the first part of this proof to show $\theta(S^\#) > 0$. Hence $\theta(S) > 0$.

Thus we have shown that the tautological rational map $T$ is defined on $U$, which is a nonempty open subset of $S \backslash E$. Moreover, $M_\theta$ contains the image of $T|_U$, hence has to be nonempty.

\textbf{Corollary 7.7.} $f$ is a surjective morphism, $T$ is defined on $S \backslash E$. Moreover, $f$ induces an isomorphism between $M_\theta \backslash C$ and $S_0 - \{P\}$ and $T$ induces an isomorphism between $S \backslash E$ and $M_\theta \backslash C$.

\textbf{Proof.} By Proposition 7.6 $T$ is defined on $U \subset S \backslash E$. By Corollary 7.3 $T_0(\pi(U))$ is in the image of $f$. Note $T_0(\pi(U))$ is an open dense subset of $M_\theta$. Since $f$ is proper, the image is all of $M_\theta$.

We claim $T$ is defined on $S \backslash E$. Let $s \in S \backslash E$, and let $s_0 = \pi(s)$. We identify $S_0$ and $M_\theta_0$ using $T_0$, then

$$T_0(s_0) = \bigoplus_{i=1}^{n}(\mathcal{O}(-D_i))_{s_0}$$
lies in the image of $f$ since $f$ is surjective. Pick a representation $R$ of $Q$ so that $f([R]) = T_0(s_0)$. Now consider the $A$-module
\[
\bigoplus_{i=1}^{n-k} (\pi^*O(-D_i))_s \oplus (\pi^*O(-D_{n-k+1} + E))_s \oplus (\pi^*O(-D_{n-k+1}))_s \oplus \bigoplus_{i=n-k+2}^n (\pi^*O(-D_i + E))_s
\]
which we need to prove to be $\theta$-stable. Take a basis for each of the 1-dimensional direct summands and apply $F$, we see from the proof of Corollary 7.3 that
\[
F\left(\bigoplus_{i=1}^{n-k} (\pi^*O(-D_i))_s \oplus (\pi^*O(-D_{n-k+1} + E))_s \oplus (\pi^*O(-D_{n-k+1}))_s \oplus \bigoplus_{i=n-k+2}^n (\pi^*O(-D_i + E))_s\right)
\]
is isomorphic to $\bigoplus_{i=1}^n (O(-D_i))_s$. By Lemma 6.6, this shows
\[
\bigoplus_{i=1}^{n-k} (\pi^*O(-D_i))_s \oplus (\pi^*O(-D_{n-k+1} + E))_s \oplus (\pi^*O(-D_{n-k+1}))_s \oplus \bigoplus_{i=n-k+2}^n (\pi^*O(-D_i + E))_s \sim R
\]
. Thus $T$ is defined at $s$. This applies to all $s \in S \setminus E$, so the domain of $T$ contains $S \setminus E$.

Restricting to $S \setminus E$, we have commutative diagram
\[
\begin{array}{ccc}
S \setminus E & \xrightarrow{T|_{S \setminus E}} & M_\theta \setminus C \\
\downarrow{\pi|_{S \setminus E}} & & \downarrow{f|_{M_\theta \setminus C}} \\
S_0 \setminus \{P\} & \xrightarrow{T_0|_{S_0 \setminus \{P\}}} & M_{\theta_0} \setminus \{P\}
\end{array}
\]
Since $f(V(e) = \{T_0(P)\}$, and $f$ is surjective, $f|_{M_\theta \setminus C}$ is also surjective. By Lemma 6.6, $f|_{M_\theta \setminus C}$ is also injective, so $f|_{M_\theta \setminus C}$ is a bijection. This in turn imply $T|_{S \setminus E}$ is surjective, since $T_0|_{S_0 \setminus \{P\}} \circ \pi|_{S \setminus E}$ is an isomorphism. Since $S \setminus E$ is irreducible, so is $M_\theta \setminus C = T|_{S \setminus E}(S \setminus E)$. In particular, $M_\theta \setminus C$ is connected and $f|_{M_\theta \setminus C}$ is a bijection mapping it to a normal variety. So we can apply Zariski Main Theorem to conclude that $f|_{M_\theta \setminus C}$ is an isomorphism. Then $T|_{S \setminus E}$ is an isomorphism from the commutative diagram above. \[\square\]

We summarize what we know about $C$ so far:

- $C$ is nonempty since $M_\theta$ is projective.
- $C$ is proper since $f$ is proper and $C = f^{-1}(P)$ by Theorem 7.1 and Proposition 7.2.

**Theorem 7.8.** In the setting as the beginning of this section, we have $C \cong \mathbb{P}^1$
The idea of this proof is to reduce the $\text{PGL}(1)$ action on $V(e)$ to the scaling action of $k^*$ on $\mathbb{A}^2 - 0$, thus showing the quotient is $\mathbb{P}^1$.

Let $R \in V(e)$, by Proposition 6.8, $r_{ui} \neq 0$ for all $i$. Let

$$g_\lambda = \left( r_{u_1}, r_{u_2}, \ldots, r_{u_{k-1}}, \lambda, \frac{\lambda}{r_{u_{k+1}}}, \ldots, \frac{\lambda}{r_{u_n}} \right) \in \text{PGL}(1)$$

where $\lambda \in k^*$ is arbitrary, then $g_\lambda \cdot R \in V(e)$ satisfies $r_{ui} = 1$ for all $i$. Moreover if $a \in Q_{ar}$, and $t(a) < k$, then $u_{t(a)} \circ a$ is an arrow from $s(a)$ to $k'$, thus if $u_{t(a)} \circ a$ is a multiple of $e$, which is equivalent to $\text{div}(a)$ passes through $P$, then $r_a = 0$, otherwise $r_a \neq 0$. In the second case, we can replace $a$ by a scalar multiple of it and assume $r_a = 1$. Thus

$$r_a = \begin{cases} 0 & \text{if } a \text{ passes through } P \\ 1 & \text{if } a \text{ does not pass through } P \end{cases}$$

Similarly, all arrows from with $s(a) > k$ have their values determined. We call such a representation in the normal form. A normalized representation $R \in V(e)$ is determined by the values of the following three types of arrows

i. Indecomposable arrows $a$ with $t(a) = k$.

ii. Indecomposable arrows $a$ with $s(a) = k'$.

iii. Indecomposable arrows $a$ with $s(a) < k$ and $t(a) > k'$.

We note the subset of normalized representation in $V(e)$ is an algebraic subset, more specifically, it is $V(e, u_1 - 1, \ldots, u_n - 1)$.

For $\lambda \in k^*$, let $\lambda$ acts on $V(e, u_1 - 1, \ldots, u_n - 1)$ by $g_\lambda$ as above, then it is clear from the definition of geometric quotient

\begin{equation}
V(e, u_1 - 1, \ldots, u_n - 1) / k^* = V(e)^S / / \text{PGL}(1)
\end{equation}

Remark 7.10. From now on, we denote $V(e, u_1 - 1, \ldots, u_n - 1)$ by $W$ for simplicity of presentation.

**Lemma 7.11.** A representation $R \in V(e)$ satisfying $u_i = 1$ for all $i$ is determined by the value of two arrows, in other words

$$V(e, u_1 - 1, \ldots, u_n - 1) \cong \mathbb{A}^2$$

**Proof.** Part 1 Suppose the two line divisors are on same side of $E$. Without loss of generality, we can assume they are both on the left of $E$.

Case 1: The second line divisor is of the form $H - E$. If $\mathcal{TS}^{\text{op}} = \{H, H - E, E, H - E\}$.

In this case, the undertermined arrows are of type i. Now $h^0(S, H - E) = 2$, $h^0(S, 2H - E) = 5$ and by Lemma 2.11 all arrows from vertex 1 to vertex 3 are decomposable. So the value of the two arrows from 2 to 3 determines the representation. Clearly the possible values forms an algebraic set $\mathbb{A}^2$.

If $\mathcal{TS}^{\text{op}} = \{E_1, H - E_1, H - E, E, H - E - E_1\}$, then $k = 4$. In this case, the undertermined arrows are of type i. Again $h^0(S, H - E) = 2$. Note $B_1 + B_2 = E_1 + (H - E_1) = H$, hence using the same argument as above, we see all arrows from vertex 1 to vertex 4 are decomposable. If $E_1 - E$ is not effective, by Lemma 2.11, all arrows from 2 to 4
are decomposable, thus the representation is determined by the value of the two arrows \(a, b\) from 3 to 4. If \(E_1 - E > 0\), then without loss of generality, we can assume \(E_1\) is the blow up at \((0 : 0 : 1)\) and \(P\) is the point on \(E_1\) representing the direction \(x + y\). Then \(H^0(S, H - E) = \text{span}(x, y)\), so we let the two arrows from 3 to 4 to represent the line \(x\) and \(y\) and denote them by \(a\) and \(b\). \(H^0(S, H - E_1) = \text{span}(x, y)\), while \(H^0(2H - E_1 - E) = \text{span}(x^2, xy, y^2, z(x + y))\). Denote the arrow representing \(z(x + y)\) by \(c\), and the arrow in \(H^0(S, E_1 + (H - E_1))\) representing \(z\) by \(d\), then \(c \circ e_1 = a \circ d + b \circ d\). Since \(r_{e_1} = 0, r_d \neq 0\), we obtain \(r_a + r_b = 0\). To sum up, the representations are determined by \(r_a\) and \(r_c\).

If \(TS^{op} = \{E_2, E_1 - E_2, H - E_1, H - E, E, H - E - E_1 - E_2\}\), we can apply the arguments in the previous subcase.

If \(TS^{op} = \{E_1 - E_2, E_2, H - E_1 - E_2, H - E, E, H - E - E_1\}\). Now \(h^0(S, B_4) = 2\). \(h^0(S, B_3) = h^0(S, H - E_1 - E_2) = 1\). Also, \(h^0(S, B_3 + B_4) = 3\). We see the unique arrow from 3 to 4 composed with the two arrows from 4 to 5 gives two distinct arrows from 3 to 5, thus there is an indecomposable arrow \(c\) from 3 to 5. Since \(E_1 - E_2\) is strong left orthogonal, \(P\) cannot be on both \(E_1\) and \(E_2\). If \(P\) is not on \(E_1\), then \(e_1 \circ c\) represents an element in \(H^0(S, H - E_1 + H - E)\). Note \(r_{e_1} = 1\), then by Lemma 2.11 \(c \circ e_1\) can be decomposed as composition of arrows from 1 to 4 with arrows from 4 to 5, the value of \(r_c\) is again a fixed function of the two values of sections in \(H^0(S, B_3)\).

For arrows from 1, 2 to 5, we use Lemma 2.11 and Lemma 2.12 to see all of them must be decomposable. So the values of the two arrows from 4 to 5 gives the desired \(\mathbb{A}^2\).

If \(TS^{op} = \{E_1 - E_2, E_2 - E_3, E_3, H - E_1 - E_2 - E_3, H - E, E, H - E - E_1\}\). In this case we only have arrows of type i. There are two arrows of type \(i\) from 5 to 6, which we call \(f_1\) and \(f_2\) and two arrows of type \(i\) from 4 to 6, which we call \(g_1\) and \(g_2\). We need to show there are two independent values among these four arrows. Unfortunately the previous arguments do not extend very well due to the fact that we have more arrows. On the other hand, the fact that \(E_1 - E_2, E_2 - E_3\) and \(H - E_1 - E_2 - E_3\) are all slo allow us to assume without loss of generality that \(E_1, E_2, E_3\) is obtained from blowing up \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\) respectively. It remains to discuss the position of \(E\) (or equivalently \(P\)) and carry out concrete computation. We have the following subcases:

- \(P\) is not neither of the line \(x, y, z\). In this case, we can assume without loss of generality \(P = [1 : 1 : 1]\).
- \(P\) is on one of the lines \(x, y, z\) but not on any exceptional curve. In this case, we can assume \(P = [1 : 1 : 0]\).
- \(P\) is on an exceptional curve, but represents a generic direction. In this case we can assume \(P\) is on \(E_1\) representing the direction \(y + z\).
- \(P\) is on an exceptional curve representing a special direction. In this case we assume \(P\) is on \(E_1\) representing the direction \(y\).

We will carry out the computation in the first subcase and leave the similar computation in other subcases as exercise. In the first subcase, we can let \(f_1\) represent the line \(x - z\), \(f_2\) represent the line \(y - z\), \(g_1\) represent the quadric \(x(y - z)\) and \(g_2\) represent the quadric
$y(x - z)$. We let $u_{04}$ represent the line $x$, $u_{14}$ represent $y$ and $u_{24}$ represent the line $z$, $u_{4}$ represent $x$ and $u_{3}$ represent $yx$. By looking at arrow from 1, 2, 3 to 6, we obtain three relations:

\[
\begin{align*}
    f_2 \circ u_{15} &= g_1 \circ e_1 \\
    f_1 \circ u_{25} &= g_2 \circ e_2 \\
    (f_1 - f_2) \circ u_{35} &= (g_2 - g_1) \circ e_3
\end{align*}
\]

From the first two equations, we see the values of $g_1, g_2$ are determined by those of $f_1$ and $f_2$ since $r_{e_2} \neq 0, r_{e_3} \neq 0$. It remains to check the third equation does not give extra information on the values of $f_1, f_2, g_1, g_2$. Using the following relations

\[
\begin{align*}
    u_5 \circ u_{25} &= u_4 \circ e_2 \\
    u_4 \circ e_1 &= u_5 \circ u_{15} + e \circ (f_2 - f_1) \circ e \\
    u_5 \circ u_{35} &= u_4 \circ e_3 - e \circ g_1 \circ e_3
\end{align*}
\]

and the fact that $r_e = 0$, we obtain

\[
\begin{align*}
    (r_{f_1} - r_{f_2}) r_{u_{35}} &= r_g r_{e_2} r_{u_{35}} / r_{u_{25}} - r_g r_{e_1} r_{u_{35}} / r_{u_{15}} \\
    &= r_{g_2} r_{u_3} r_{u_{35}} / r_{u_4} - r_{g_1} (r_{u_3} r_{u_{35}} / r_{u_4}) \\
    &= (r_{g_2} - r_{g_1}) r_{u_4}
\end{align*}
\]

hence the third equation does not give new restrictions and the value of $f_1$ and $f_2$ determines the representation.

Case 2: The second line divisor is of the form $H - E - E_1$. If $T S^{\text{op}} = \{H, H - E - E_1, E, E_1 - E, H - E_1\}$. Since $E_1 - E$ is strong left orthogonal, $p(E_1) \neq p(E)$. Without loss of generality, we can assume $E$ is obtained from blowing up at $[0 : 0 : 1]$ and $E_1$ is obtained from blowing up at $[0 : 1 : 0]$. $h^0(S, B_2) = h^0(S, H - E - E_1) = 1$, and $h^0(S, B_1 + B_2) = h^0(S, 2H - E_1 - E) = 3$. Then we see that the three arrows from 1 to 2 composed with the unique arrow $a$ from 2 to 3 gives three arrows from 1 to 3. So there is an indecomposible arrow $c$ from 1 to 3. Let $a$ represent the line $x$ and we can take $c$ to represent the quadratic function $yz$. We claim the value of $a, c$ determines the representation. It suffices to check arrows of type iii. There exist an indecomposible arrow $d$ from 2 to 4, which we can take to represent the line $y$. Now take an arrow $b$ from 1 to 2 representing $z$, note $\text{div}(z) - E$ is not effective, so $r_b = 1$. Also, $d \circ b = e_1 \circ c$. Hence the value of $d$ is determined by that of $c$. Since $B_1 = H \text{ Bnd } B_2 + B_3 + B_4 = H - E$, all arrows from 1 to 4 are compositions of arrow from 1 to 2 and 2 to 4 by Lemma [2.10]. So the claim is proved.

If $T S^{\text{op}} = \{E_2, H - E_2, H - E - E_1, E, E_1 - E, H - E_1 - E_2\}$, then $h^0(S, B_3) = 1$, we call the corresponding arrow $a$, $h^0(S, B_2) = 2$ and there is an unique indecomposible arrow $b$ from 2 to 4. Similarly, there is an unique indecomposible arrow from 1 to 3. Without loss of generality, we let the two arrows from 2 to 3 be $f$ and $g$, and the unique arrow from 1 to 3 be $h$. And all but one arrow from 1 to 4 are composition of $e_2$ with some arrow, we call the remaining arrow $d$. If $a$ does not pass through $E_2$, i.e $\text{div}(a) - E_2$ is not effective,
then $a \circ h$ is not a multiple of $e_2$, so we can take $d = a \circ h$. Otherwise, $d$ is indecomposable. Note
\[
h^0(S, B_1 + B_2 + B_3) = h^0(S, 2H - E - E_1) = 4
\]
and since $d$ is indecomposable, there is a linear relation between the four decomposable arrows
\[
m_1a \circ h + m_2a \circ f \circ e_2 + m_3a \circ g \circ e_2 + m_4b \circ e_2 = 0
\]
Note $m_4 \neq 0$ since otherwise we get $a \circ (m_1h + m_2f \circ e_2 + m_3g \circ e_2) = 0$ which implies $(m_1h + m_2f \circ e_2 + m_3g \circ e_2) = 0$, contradicting our choice of $f, g, h$. Similarly, one can show $m_1 \neq 0$. If $r_{e_2} \neq 0$, then $r_a$ is determined by $r_a$. If $r_{e_2} = 0$, then we have $m_1r_a = 0$. We claim this implies $r_a = 0$. It suffices to show $r_h = 0$. If otherwise, then all arrows from 1 to 3 will have value 0, this implies $E$ is a base component of the linear system $|B_1 + B_2| = |H|$, which is absurd.

For arrows of type iii, we see there is an indecomposable arrow $\alpha$ from 3 to 5. Note $B_1 + B_2 = H$, so we can take a linear combination $n_1h + n_2f \circ e_2 + n_3g \circ e_2$ which represents a line $l$ so that $\text{div}(l) - E_1 > 0$ but $\text{div}(l) - E$ is not effective. This is possible since $E_1 - E$ is strong left orthogonal, then $n_1r_h + n_2f \circ e_2 + n_3g \circ e_2 \neq 0$ and $\alpha \circ n_1h + n_2f \circ e_2 + n_3g \circ e_2$ is a multiple of $e_1$. Hence we see the value of $\alpha$ is determined by those of $a, b$ and $c$. We claim there are no indecomposable arrows from 2 to 5. $h^0(S, (H - E_2 + H - E - E_1 + E + E_1 - E) = 4$, where there is a three dimensional subspace $S_1$ which is represented by the composition of arrows from 2 to 4 composed with $e_1$. Apply Corollary 2.9 to $H - E_1 - E_2$, we see $\text{div}(f) - E_1$ and $\text{div}(g) - E_1$ cannot both be effective. Without loss of generality, we assume $\text{div}(f) - E_1$ is not effective. Then since $\text{div}(\alpha) - E_1$ is not effective, then $\alpha \circ f$ is not a multiple of $e_1$, hence together with $S_1$ they span the whole space. Hence there is no indecomposable arrows from 2 to 5. There is no indecomposable arrow from 1 to 5 by Lemma 2.10.

There are two more subcases which are handled using the exact same method as previous subcase, we list them here.

- $\mathcal{TS}^{op} = \{E_2 - E_3, E_3, H - E_2 - E_3, H - E - E_1, E, E_1 - E, H - E_1\}$
- $\mathcal{TS}^{op} = \{E_2 - E_3, E_3 - E_4, E_4, H - E_2 - E_3 - E_4, H - E - E_1, E, E_1 - E, H - E_1\}$

If $\mathcal{TS}^{op} = \{E_3, E_2 - E_3, H - E_2, H - E - E_1, E, E_1 - E, H - E_1 - E_2 - E_3\}$. Since $E_2 - E_3$ is a strong left orthogonal divisor, we see $E_2 - E_3$ is not an effective divisor. Hence $H^0(S, H - E_2)$ and $H^0(S, H - E_3)$ are not the same 2-dimensional subspace of $H^0(S, H)$. Without loss of generality, we assume $H^0(S, H - E_2)$ is spanned by linear functions $f, g$, and $H^0(S, H - E_3)$ is spanned by linear functions $f, h$. Using similar argument as above, we see that there are three indecomposable arrows from $i$ to $k$ for $i < k$, they are $a$ which is the unique arrow from 4 to 5, $b$ the unique indecomposable arrow from 3 to 5, and $c$ from 2 to 5. $h^0(S, B_1 + \ldots + B_4) = h^0(S, 2H - E_1 - E) = 4$, then there is an unique
nontrivial linear relation between the five arrows:

(7.12) \[ m_1 a \circ f \circ e_2 + m_2 a \circ g \circ e_2 + m_3 a \circ h \circ e_3 + m_4 c \circ e_3 + m_5 b \circ e_2 = 0 \]

If \( m_4 r_{e_3} \neq 0 \) or \( m_5 r_{e_2} \neq 0 \), then the above equation gives a nontrivial relation between \( r_a, r_b \) and \( r_c \). Otherwise, we have \( m_4 r_{e_3} = m_5 r_{e_2} = 0 \). Note \( E_2 - E_3 \) is strong left orthogonal, we cannot have \( e_2 = e_3 = 0 \). Using argument in the previous case, we cannot have \( m_4 = m_5 = 0 \). Now if \( m_4 = r_{e_2} = 0 \), then \( E_2 - E > 0 \) and \( r_{e_3} = 1 \), then equation (7.12) implies

\[ m_3 r_b r_a = 0 \]

If \( m_3 = 0 \), then we can factor \( e_2 \) in equation (7.12) and get a contradiction. Moreover, by our choice of \( f, g, h \), \( \text{div}(h) - E \) is not effective. Hence \( r_b = 1 \), thus \( r_a = 0 \). If \( m_5 = r_{e_3} = 0 \), then \( E_3 - E > 0 \) and \( r_{e_2} = 1 \), then equation (7.12) gives

\[ m_1 r_f r_a + m_2 r_g r_a = 0 \]

By our choice \( \text{div}(f) - E > 0 \) and \( \text{div}(g) - E \) is not effective. So we have \( m_2 r_g r_a = 0 \). If \( m_2 = 0 \), then

(7.13) \[ m_1 a \circ f \circ e_2 + m_3 a \circ h \circ e_3 + m_5 b \circ e_2 = 0 \]

It is easy to see that \( m_1, m_2, m_5 \) cannot all be 0, so (7.13) is a nontrivial linear relation. But \( b \) is not a multiple of \( a \), so \( m_5 = 0 \), then \( f \circ e_2 = f \circ e_3 \) is linear dependent to \( h \circ e_3 \), which is a contradiction. Hence \( m_2 \neq 0 \), thus \( m_2 r_g r_a = 0 \) implies \( r_a = 0 \). For arrows of type iii, we follow the arguments in the previous case and show their values are determined by those of \( a, b \) and \( c \). Hence we can pick two of \( r_a, r_b, r_c \) whose values are independent and determined the remaining one, and obtain \( \mathbb{A}^2 \).

If \( T S^{op} = \{ H - E_1, E_1, H - E - E_1, E, H - E \} \). This can be dealt with using arguments similar to that of subcase 2 in Case 2.

If \( T S^{op} = \{ H - E_1 - E_2, E_2, E_1 - E_2, H - E - E_1, E, H - E \} \)

or

\( T S^{op} = \{ H - E_1 - E_2 - E_3, E_3, E_2 - E_3, E_1 - E_2, H - E - E_1, E, H - E \} \)

, these cases can be solved using arguments similar to subcase 2 in Case 1.

If \( T S^{op} = \{ E_2, H - E_1 - E_2, E_1, H - E - E_1, E, H - E - E_2 \} \). We can still use arguments above, except that we have to consider different situations regarding the position of \( E_1, E_2, E_3 \), this case is left as an exercise. The few remaining cases are all solved using similar methods we have seen. We list them here and omit the details

- \( T S^{op} = \{ E_3, E_2 - E_3, H - E_1 - E_2, E_1, H - E - E_1, E, H - E - E_3 \} \)
- \( T S^{op} = \{ E_2 - E_3, E_3, H - E_1 - E_2 - E_3, E_1, H - E - E_1, E, H - E - E_2 \} \)
- \( T S^{op} = \{ H - E_1 - E_2 - E_3, E_3, E_2 - E_3, E_1 - E_2, H - E - E_1, E, H - E \} \)
- \( T S^{op} = \{ E_3, H - E_1 - E_2 - E_3, E_2, E_1 - E_2, H - E - E_1, E, H - E - E_3 \} \)
- \( T S^{op} = \{ E_3, H - E_1 - E_2 - E_3, E_2, E_1 - E_2, H - E - E_1, E, H - E - E_3 \} \)
Case 3: The second line divisor is of the form \( H - E - E_1 - E_2 \). If \( \mathcal{T}S^{op} = \{ H - E_1, E_1 - E_2, H - E - E_1 - E_2, E, H - E \} \). The we only need to check arrows of type i. Note \( k = 5 \), then arrows of type i consist of an indecomposable arrow from 3 to 5, and an indecomposable arrow from 2 to 5, so the representation is determined by the value of these two arrows.

If \( \mathcal{T}S^{op} = \{ E_3, H - E_1 - E_3, E_1 - E_2, E_2, H - E - E_1 - E_2, E, H - E - E_3 \} \). Then \( k = 6 \) and we only need to check arrows of type i. As in the previous case, we have one indecomposable arrow \( a \) from 4 to 6 and one \( b \) from 3 to 6. As in the previous case, there is no indecomposable arrow from 1 to 6. We are done if there is no indecomposable arrow from 2 to 6.

If there is an indecomposable arrow \( c \) from 2 to 6, then after multiplying replacing \( a \) by a scalar multiple of it, we have \( a \circ u_{24} = b \circ u_{23} \). Note

\[
\begin{align*}
div(a) & \in H^0(S, H - E_1 - E) \\
div(u_{24}) & \in H^0(S, H - E_3 - E_2) \\
div(b) & \in H^0(S, H - E_2 - E) \\
div(u_{23}) & \in H^0(S, H - E_3 - E_1)
\end{align*}
\]

\( r_{u_{24}} = 0 \) if and only if \( div(u_{24}) - E > 0 \) and \( r_{u_{23}} = 0 \) if and only if \( div(u_{23}) - E > 0 \). We see \( r_{u_{24}} = r_{u_{23}} = 0 \) cannot happen as it would imply \( H - E_1 - E_2 - E_3 - E \) has a section which contradicts the fact that \( H - E - E_1 - E_2 \) is so. Thus either \( r_a = r_b \) or \( r_a = 0 \) or \( r_b = 0 \). Thus in the case when there is an indecomposable arrow \( c \) from 2 to 6. We can choose the value of \( c \) with either \( a \) or \( b \) to determine the representation. There are a few other cases that can be handled using the arguments in the previous two cases, we list them here:

- \( \mathcal{T}S^{op} = \{ E_4, E_3 - E_4, H - E_1 - E_3, E_1 - E_2, E_2, H - E - E_1 - E_2, E, H - E - E_3 - E_4 \} \)
- \( \mathcal{T}S^{op} = \{ E_3 - E_4, E_4, H - E_1 - E_3 - E_4, E_1 - E_2, E_2, H - E - E_1 - E_2, E, H - E - E_3 - E_1 \} \)

If \( \mathcal{T}S^{op} = \{ H, H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E \} \). We note \( E_1 - E, E_2 - E_1, E_2 - E \) are all strong left orthogonal, moreover \( H - E - E_1 - E_2 \) is strong left orthogonal, then \( E_1, E_2, E_3 \) are obtained from blowing up three points on \( \mathbb{P}^2 \) which do not lie on the same line. We let \( l_1, l_2, l \) denote the line that goes through \( E, E_2, E_1 \) and \( E_1, E_2 \) respectively. We use \( a_1, a_2, a \) to denote the three arrows from 1 to 2 corresponding to \( l_1, l_2, l \), then \( r_a = 1 \) and \( r_{a_1} = r_{a_2} = 0 \). Now there are three indecomposable arrows from 1 to 3, \( b_1, b_2, b \) which we can take to represent \( l_2 + l, l_1 + l, \) and \( l_1 + l_2 \) respectively. Moreover, note \( u_1 \) represents the line \( l \). There is an indecomposable arrow from 1 to 3 which we call \( u_{24} \) representing \( l_1 \), an indecomposable arrow from 1 to 4 which we call \( u_{25} \) representing \( l_2 \). It is easy to see there are no other arrows of type iii. Then

\[
\begin{align*}
u_{24} \circ a & = e_1 \circ b_2 \\
u_{25} \circ a & = e_2 \circ b_1
\end{align*}
\]
This shows \( r_{u_{24}} = r_b \) and \( r_{u_{25}} = r_b \).

Moreover, we have

\[
u_{24} \circ a_2 = e_1 \circ b
\]

divides \( r_b = 0 \). Thus a representation is determined by the value of \( b_1 \) and \( b_2 \).

If \( T S^{op} = \{ E_3, H - E_3, H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 - E_3 \} \). Note in this case \( p(E_1), p(E_2), p(E) \) are three distinct points on \( \mathbb{P}^2 \) which do not lie on the same line.

This case is essentially the same as the previous case, one only need to start by writing the arrows from 1, 2 to 3 as linear combinations of \( l, l_1, l_2 \) and arrows from 1, 2 to 4 by linear combination of \( l_1 + l_2, l_1 + l, l_2 + l \) defined in the previous case.

If \( T S^{op} = \{ E_4, E_3 - E_4, H - E_3, H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 - E_3 - E_4 \} \) or \( T S^{op} = \{ E_3 - E_4, E_4, H - E_3 - E_4, H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 \} \) or \( T S^{op} = \{ E_3 - E_4, E_4 - E_5, E_5, H - E_3 - E_4 - E_5, H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 \} \), the same idea applies. We start by noting \( p(E_1), p(E_2), p(E) \) are three distinct points on \( \mathbb{P}^2 \) which do not lie on the same line. The we proceed to show that all the values of type \( i, iii \) arrows are determined by the value of \( l + l_1 \) and \( l + l_2 \) as above. Hence we can pick two linearly independent arrows and they will determine the representation.

If \( T S^{op} = \{ H - E_2, E_2, H - E - E_1 - E_2, E, E_1 - E, H - E_1 \} \). There are two arrows of type \( i, f_1 \) from 2 to 4 and \( f_2 \) from 1 to 4. There are no arrows of type \( ii \). There is an arrow \( g \) from 3 to 5 of type \( iii \). Note \( B_1 + B_2 = H \), if we let the two arrows from 1 to 2 be \( a_1, a_2 \), since \( E_1 - E \) is slo, we can find a linear combination \( w = m_1e_2 \circ a_1 + m_2e_2 \circ a_2 + m_3u_{13} \) so that it represents a line going through \( E_1 \) but not \( E \). Then \( r_w \neq 0 \). Moreover, we can find constants \( n_1, n_2, n_3 \in \mathbb{C} \)

\[
g \circ w = n_1e_1 \circ f_1 \circ a_1 + n_2e_1 \circ f_1 \circ a_2 + n_3e_1 \circ f_2
\]

this shows the value of \( g \) is determined by those of \( f_1, f_2 \).

There is no arrow of type \( iii \) from 1 to 5. If there is an arrow \( h \) of type \( iii \) from 2 to 5, then \( e_1 \circ f_1 \) and \( g \circ e_2 \) represent the same line, so we can assume \( e_1 \circ f_1 = g \circ e_2 \). Let \( v = m_1a_1 + m_2a_2 \) represents a line in the subspace \( H^0(S, H - E_1 - E_2) \subseteq H^0(S, H - E_2) \), then this line cannot go through \( E \) since \( H - E_1 - E_2 - E \) is slo. Since \( E_1 - E \) is slo, we have \( r_v \neq 0 \). Then as before, we can find constants \( n_1, n_2, n_3 \in \mathbb{C} \) so that

\[
h \circ v = n_1e_1 \circ f_1 \circ a_1 + n_2e_1 \circ f_1 \circ a_2 + n_3e_1 \circ f_2
\]

Thus the values of \( g, h \) are determined by those of \( f_1, f_2 \).

A similar case is \( T S^{op} = \{ E_3, H - E_2 - E_3, E_2, H - E - E_1 - E_2, E, E_1 - E, H - E_1 - E_3 \} \).

All arrows of type \( iii \) can be handled as above. As before, we will have two type \( i \) arrow, \( f_1 \) from 3 to 5, \( f_2 \) from 2 to 5. The difference here is we may have an third arrow \( f_3 \) of type \( i \), which is from 1 to 5. This happens when there is a nontrivial linear relation:

\[
m_1f_1 \circ u_{13} + m_2f_2 \circ e_3 + m_3f_1 \circ u_{23} \circ e_3 = 0
\]
As before we can easily see \( m_2 \neq 0, m_1 \neq 0 \). If \( r_{e_3} = 1 \), then \( r_{f_1} \) determines \( f_2 \), in this case \( r_{f_1}, r_{f_3} \) determine the representation. Otherwise \( r_{e_3} = 0 \), taking the value of the equation gives

\[
m_1 r_{f_1} r_{u_{13}} = 0
\]

Note \( u_{13} \) represents an section in \( H^0(S, H - E_2) \) that is not in \( H^0(S, H - E_2 - E_3) \). Since \( E_3 - E > 0 \), we can easily see \( r_{u_{13}} = 1 \), so \( r_{f_1} = 0 \). So in this case \( r_{f_2} \) and \( r_{f_3} \) determine the representation.

Another similar case is \( T S^{op} = \{ E_4, E_3 - E_4, H - E_2 - E_3, E_2, H - E - E_1 - E_2, E, E_1 - E, H - E_1 - E_3 - E_4 \} \). Again the type iii arrows can be dealt as in previous cases. In this case we will have type i arrows \( f_1 \) from 4 to 6, \( f_2 \) from 3 to 6, \( f_3 \) from 2 to 6. If \( h^0(S, H - E_2 - E_3 - E_4) = 0 \), note \( h^0(S, B_1 + \ldots + B_5) = h^0(S, 2H - E - E_1 - E_2) = 3 \), so there is a nontrivial linear relation

\[
m_1 f_1 \circ u_{24} \circ e_4 + m_2 f_1 \circ u_{34} \circ e_3 + m_3 f_2 \circ e_3 + m_4 f_3 \circ e_4 = 0
\]

If \( m_3 r_{e_3} \neq 0 \), then the value of \( f_2 \) is determined by \( f_1 \) and \( f_3 \). If \( m_4 r_{e_4} \neq 0 \), then \( r_{f_3} \) is determined by \( r_{f_1} \) and \( r_{f_2} \). If \( m_3 r_{e_3} = m_4 r_{e_4} = 0 \), note \( E_3 - E_4 \) is slo, so \( r_{e_3}, r_{e_4} \) cannot both be 0. If \( r_{e_3} = 0 \), then \( m_4 = 0 \) and

\[
m_1 r_{f_1} \circ u_{24} = 0
\]

Note if \( r_{u_{24}} = 0 \), then \( h^0(S, H - E_2 - E_3 - E_4 - E) = 1 \). It is not hard to see this will imply \( h^0(S, H - E_2 - E_3 - E_4) = 1 \) since \( E_3 - E > 0 \), which leads to a contradiction. So \( r_{f_1} = 0 \) in this case. The case when \( r_{e_4} = 0 \) is handled similarly. So the values of \( f_1, f_2, f_3 \) are linearly related by 1 relation in this case and we can choose two independent values to get \( A^2 \).

If \( h^0(S, H - E_2 - E_3 - E_4) = 1 \), then \( u_{24} \circ e_4 = u_{34} \circ e_3 \) and there is an indecomposable arrow from 1 to 4. Then there is a nontrivial linear relation

\[
n_1 f_1 \circ u_{24} \circ e_4 + n_2 f_1 \circ u_{14} + n_3 f_2 \circ e_3 + n_4 f_3 \circ e_4 = 0
\]

and we can use similar arguments as above to get our result.

There are four cases that are similar, we list them here

- \( T S^{op} = \{ H - E_2 - E_3, E_3, E_2 - E_3, H - E - E_1 - E_2, E, E_1 - E, H - E_1, H \} \)
- \( T S^{op} = \{ E_4, H - E_2 - E_3 - E_4, E_3, E_2 - E_3, H - E - E_1 - E_2 - E, E_1 - E, H - E_1 - E_3 \} \)
- \( T S^{op} = \{ E_3 - E_4, E_4, H - E_2 - E_3 - E_4, E_2, H - E - E_1 - E_2 - E, E_1 - E, H - E_1 - E_3 \} \)
- \( T S^{op} = \{ H - E_2 - E_3 - E_4, E_4, E_3 - E_4, E_2 - E_3, H - E - E_1 - E_2 - E, E_1 - E, H - E_1 - E_1, \} \)

Case 4: The second line divisor is of the form \( H - E_1 \) where \( E_1 \neq E \).

If \( T S^{op} = \{ H, H - E_1, E_1 - E, E, H - E - E_1 \} \). There are only two arrows of type i, both from 2 to 4. There are no arrows of type ii or iii. Hence the value of the two arrows from 2 to 4 determines the representation.

If \( T S^{op} = \{ E_2, H - E_2, H - E_1, E_1 - E, E, H - E_1 - E_2 - E \} \). Again there are no arrows of type ii or iii. For type i arrows, we have two arrows from 3 to 5. When \( E_2 - E \)
is not effective, there are no arrows of type i from 2 to 5 and as before no arrows from 1 to 5 of type i. So we are done as before.

When $E_2 - E$ is effective. There is an arrow $h$ of type i from 2 to 5. Without loss of generality, we can assume $p(E) = [0 : 0 : 1]$. We let the two arrows $f_1, f_2$ from 3 to 5 represent $x, y$ and two arrows $g_1, g_2$ from 2 to 3 represent $x, y$. Then we have

$$f_1 \circ g_2 = f_2 \circ g_1$$

Note the value of $g_2, g_1$ cannot both be zero, hence the value of one of $f_1, f_2$ is determined by the other in this case. Hence the value of $h$ and one of $f_1, f_2$ determines the representation. A few cases follow the same argument, we list them here and omit the details:

- $\mathcal{TS}^{op} = \{E_2 - E_3, E_3, H - E_2 - E_3, H - E_1, E_1 - E, E, H - E_1 - E_2 - E\}$
- $\mathcal{TS}^{op} = \{E_2 - E_3, E_3 - E_4, E_4, H - E_2 - E_3 - E_4, H - E_1, E_1 - E, E, H - E_1 - E_2 - E\}$
- $\mathcal{TS}^{op} = \{H, H - E_1, E_1 - E_2, E_2 - E, E, H - E_1 - E_2 - E\}$

Case 5: The second line divisor is of the form $H - E_1 - E_2$ where $E_1, E_2$ are different from $E$. In all the subcases, there will be no arrows of type ii or iii, so we only need to analyze arrows of type i.

If $\mathcal{TS}^{op} = \{H - E_2, E_2, H - E_1 - E_2, E_1 - E, E, H - E_1 - E\}$. In this case there are only two arrows of type i, one from 2 to 5 and the other from 3 to 5. So the representation is determined by the two values of arrows of type i.

The case $\mathcal{TS}^{op} = \{H - E_2, E_2, H - E_1 - E_2, E_1 - E_4, E_4, E, H - E_1 - E - E_4\}$ can be solved using the same arguments.

If $\mathcal{TS}^{op} = \{E_3, H - E_2 - E_3, E_2, E_1 - E, E, H - E_1 - E - E_3\}$. This case is similar to the previous case, we have two arrows $g, f$ of type i from 3, 4 to 6. But we might have an arrow $h$ from 2 to 6. If this is the case, then there exist $m_1, m_2 \in \mathbb{C}$, so that

$$g \circ u_{23} = m_1 f \circ e_2 \circ u_{23} + m_2 f \circ u_{24}$$

If $r_{u_{23}} \neq 0$, then the value of $g$ is determined by that of $f$. Otherwise we have

$$m_2 r_f \circ r_{u_{24}} = 0$$

We can check as before $m_2 \neq 0$ and $r_{u_{24}} \neq 0$, then $r_f = 0$. So when $h$ exists, its value along with one of $f, g$ determines the representation.

If $\mathcal{TS}^{op} = \{H - E_2 - E_3, E_3, E_2 - E_3, H - E_1 - E_2, E_1 - E, E, H - E - E_1\}$. There are two arrows from type i, $f$ from 3 to 6 and $g$ from 4 to 6. There are no arrows from 1 to 6 of type 1. If there are no arrows of type i $h$ from 2 to 6, then we are done. Otherwise, we have

$$f \circ e_3 = g \circ e_2$$

Since $E_2 - E_3$ is slo, either $r_{e_2} \neq 0$ or $r_{e_3} \neq 0$. So in this case the value of $h$ together with one of $f, g$ determines the representation.

The case $\mathcal{TS}^{op} = \{H - E_2 - E_3, E_3, E_2 - E_3, H - E_1 - E_2, E_1 - E_4, E_4 - E, E, H - E - E_1 - E_4\}$ can be solved using the same arguments.
If $\mathcal{T}^{\text{op}} = \{H - E_2 - E_3 - E_4, E_4, E_3 - E_4, E_2 - E_3, H - E_1 - E_2, E_1 - E, E, H - E - E_1\}$. In this case, there are three arrows of type i, f from 3 to 7, g from 4 to 7 and h from 5 to 7. Using the fact that $E_2 - E_3, E_3 - E_4$ are slo, we see there are no more arrows of type i. Moreover, we have exactly one linear relation

$$m_4 f \circ e_4 + m_3 g \circ e_3 + m_2 h \circ e_2 = 0$$

where at least two of $m_4, m_3, m_2$ do not vanish and at least two of $r_{e_4}, r_{e_3}, r_{e_2}$ do not vanish, so as before, we can choose two of the independent $f, g, h$ to determine the representation.

The case $\mathcal{T}^{\text{op}} = \{H - E_2 - E_3 - E_4, E_4, E_3 - E_4, E_2 - E_3, H - E_1 - E_2, E_1 - E_5, E_5 - E, E, H - E - E_1 - E_4\}$ and $\mathcal{T}^{\text{op}} = \{E_4, H - E_2 - E_3 - E_4, E_3, E_2 - E_3, H - E_1 - E_2, E_1 - E_5, E_5 - E, E, H - E - E_1 - E_4\}$ can be solved using similar arguments.

Case 6: The second line divisor is of the form $H - E_1 - E_2 - E_3$ where $E_1, E_2, E_3$ are different from $E$.

If $\mathcal{T}^{\text{op}} = \{H - E_2, E_2 - E_3, E_3, H - E_1 - E_2 - E_3, E_1 - E, E, H - E_1 - E\}$. There are no arrows of type ii and iii. There are two arrows of type i, one from 2 to 6 and one from 3 to 6. The fact that there are no arrows of type i from 1 to 6 follows from the fact that $H - E - E_2 - E_3$ is slo. Then the values of the two arrows determine the representation.

The case $\mathcal{T}^{\text{op}} = \{H - E_2, E_2 - E_3, E_3, H - E_1 - E_2 - E_3, E_1 - E_4, E_4 - E, E, H - E - E_1 - E_4\}$.

If $\mathcal{T}^{\text{op}} = \{E_4, H - E_2 - E_4, E_2 - E_3, E_3, H - E_1 - E_2 - E_3, E_1 - E, E, H - E_1 - E_4 - E\}$. There are two arrows of type i, f from 3 to 7 and g from 4 to 7. As in above there are no arrows of type i from 1 to 7. If there are no arrows from 2 to 7, then we are done. Otherwise, there is an arrow $h$ of type i from 2 to 7, then we have

$$g \circ u_{24} = f \circ u_{23}$$

Since $H - E - E_2 - E_3$ is slo, the values of $u_{24}$ and $u_{23}$ cannot both vanish, so we obtain a nontrivial relation between $r_f$ and $r_g$. Hence $r_h$ and one of $r_g, r_f$ determine the representation.

If $\mathcal{T}^{\text{op}} = \{E_4 - E_5, E_5, H - E_2 - E_4 - E_5, E_2 - E_3, E_3, H - E_1 - E_2 - E_3, E_1 - E, E, H - E_1 - E - E_4\}$. In this case there are three arrows of type i, we handle them using the arguments as in $\mathcal{T}^{\text{op}} = \{H - E_2 - E_3 - E_4, E_4, E_3 - E_4, E_2 - E_3, H - E_1 - E_2, E_1 - E, E, H - E - E_1\}$ in Case 5.

**Part 2** Suppose the two line divisors are on different side of $E$. We will use symmetry to reduce the case discussed.

We first consider the cases when $E$ is adjacent to the two line divisors in the toric system. The easiest case is $\mathcal{T}^{\text{op}} = \{H - E, E, H - E, H\}$. This comes from blowing up a point on $\mathbb{P}^2$. Without loss of generality, we assume $P = [0 : 0 : 1]$. Then the two sections of $H - E$ are $x, y$. We denote the two arrows from 1 to 2 by $x_1, y_1$, the unique indecomposable arrow from 1 to 2' by $z_1$, the unique indecomposable arrow from 2 to 3 by $z_2$ and the two arrows from 2' to 3 by $x_2, y_2$. Note $r_{z_1} = r_{z_2} = 1$. Moreover, we have the
Thus the values of \(x_1,y_1\) determine the value of \(x_2,y_2\). By Lemma \(2.10\) there are no arrows of type iii. So the value of \(x_1, x_2\) provides the desired \(A^2\).

If \(TS^\text{op} = \{E_1, H - E - E_1, E, H - E, H - E_1\}\). There are two arrows of type i, one is from 1 to 3, which we call \(f_2\), another is from 2 to 3, which we call \(f_1\). Then \(f_1\) represents the unique section of \(H - E - E_1\). There are two arrows from 3’ to 4 of type ii. We denote them by \(g_2, g_2\). Note \(f_2, g_2\) also represents sections of \(H - E\). Moreover, \(u_1, u_4\) both represent sections in \(H^0(S, H)\) which do not lie in the subspace \(H^0(S, H - E)\), we let them represent the same section. So we can choose \(f_2, g_2\) so that

\[
\begin{align*}
f_2 \circ u_1 &= u_4 \circ f_1 \circ e_1 \\
g_2 \circ u_1 &= u_4 \circ g_1
\end{align*}
\]

Thus the value of \(f_1, g_1\) determines the value of \(f_2, g_2\). Moreover, it is easy to check there are no type iii arrows from 1 to 4. For arrows from 2 to 4, one can easily check \(f_2 \circ e \circ f_1, g_2 \circ e \circ f_1, u_3 \circ f_1, g_2 \circ u_1\) spans all the arrows from 2 to 4, so there are no type iii arrows. Hence the value of \(f_1, g_1\) gives the desired \(A^2\).

If \(TS^\text{op} = \{E_1 - E_2, E_2, H - E - E_1 - E_2, E, H - E, H - E_1\}\). There are two arrows of type i, one is from 1 to 4, denoted by \(f_1\), which represents the unique section of \(H - E - E_2\). The other is from 2 to 4, denoted by \(g_1\), which represents the unique sections of \(H - E - E_1\). Note since \(E_1 - E_2\) and \(H - E - E_1 - E_2\) are strong left orthogonal, \(f_1, g_1\) does not represent the same line. When embedded, they span the two dimensional vector space \(H^0(S, H - E)\). So we can assume the two arrows \(f_2, g_2\) from 4’ to 5 represents the same lines as \(f_1, g_1\) respectively. Note \(u_2\) represents a line in \(|H - E_1| \subset |H|\), and \(u_1\) represents a line in \(|H - E_2| \subset |H|\). Since \(f_1, f_2, g_1, g_2\) represents the same line respectively, there exists constants \(m_i, n_i\) where \(i = 1, 2, 3\), so that

\[
\begin{align*}
g_2 \circ u_2 &= m_1 u_5 \circ g_1 + m_2 f_2 \circ e \circ g_1 + m_3 g_2 \circ e \circ g_1 \\
f_2 \circ u_1 &= n_1 u_5 \circ f_1 + n_2 f_2 \circ e \circ f_1 + n_3 g_2 \circ e \circ f_1
\end{align*}
\]

Hence the value of \(f_2, g_2\) is determined by those of \(f_1, g_1\).

For arrows of type iii, we can easily see there are no such arrows from 1, 2 to 5 following arguments in the previous case. There is an arrow \(h\) from 3 to 5 of type iii. Since \(E_1 - E_2\) is slo, \(E\) cannot be a component of both \(E_1\) and \(E_2\). Suppose \(E_1 - E\) is not effective, and \(r_{e_1} = 1\), then \(h \circ e_1\) is an arrow from 1 to 5. Again by the argument at the end of previous case, we see it is in the span of \(f_2 \circ e \circ f_1, g_2 \circ e \circ f_1, u_5 \circ f_1, g_2 \circ u_1, f_2 \circ u_1\). So the value of \(h\) is determined by the values of arrow type i,iii. The case when \(E_2 - E\) is not effective can be handled similarly.

If \(TS^\text{op} = \{E_2, E_1 - E_2, H - E - E_1, E, H - E, H - E_1 - E_2\}\). Note \(h^0(S, B_3) = h^0(S, H - E - E_1) = 1\), and \(h^0(S, B_2 + B_3) = h^0(S, H - E - E_2) = 1\), their corresponding
arrows are clearly of type i. We call the one from 3 to 4 by $f_1$ and the one from 2 to 4 by $g_1$. Suppose $h^0(S, H - E - E_1 - E_2) = 0$, since $E_1 - E_2$ is a strong left orthogonal divisor, the composition $f_1 \circ e_1$ and $g_1 \circ e_2$ represents two linearly independent sections of $H - E$ and there is no indecomposable arrow from 1 to 4. We claim that the values of $f_1$ and $g_1$ gives the desired $A^2$. There are two arrows from 4' to 5, we denote by $f_2, g_2$, representing the same sections as $f \circ e_1$ and $g \circ e_2$. Note $u_1, u_5$ both represents a section of $H$ that is not in the span of $f_1, g_1$, so we may as we assume they represent the same section. Thus we clearly have the following relations:

$$u_5 \circ f_1 \circ e_1 = f_2 \circ u_1$$

$$u_5 \circ g_1 \circ e_2 = g_2 \circ u_1$$

Hence the values of $f_1, g_1$ determine those of $f_2, g_2$.

Suppose $h^0(S, H - E - E_1 - E_2) = 1$, then $f_1 \circ e_1 = g_1 \circ e_2$ and there is an arrow $h$ of type i from 1 to 4. Since $E_1 - E_2$ is slo, either $r_{e_1} = 1$ or $r_{e_2} = 1$. If $r_{e_1} = 1$, then $r_{f_1} = r_{g_1} r_{e_2}$. This time we can denote the two arrows from 4' to 5 $h, g_1$ and let them denote the same line as $g_1 \circ e_2$ and $h$, then same argument as above shows the value of $h, g_1$ will determine the value of all arrows of type i,ii.

In both cases, we use the arguments in previous case to see there are no arrows of type iii.

If $T S^{op} = \{E_3, E_2 - E_3, E_1 - E_2, H - E_1 - E, E, H - E, H - E_1 - E_2 - E_3\}$. This case is almost the same as the one above. We only illustrate the new features. There are one arrow each from 2, 3, 4 to 5, all indecomposable, which we denote by $f, g, h$. Then $f \circ e_1$, $g \circ e_2$, $h \circ e_3$ all represents sections in $H - E$. First suppose $h^0(S, H - E_1 - E_2 - E_3) = 0$. Since $E_1 - E_2, E_2 - E_3, H - E_1 - E_2 - E_3$ are all strong left orthogonal, $f \circ e_1$, $g \circ e_2$, $h \circ e_3$ span $H^0(S, H - E)$. Since $h^0(S, H - E) = 2$, we have a nontrivial linear relation

$$m_1 f \circ e_1 + m_2 g \circ e_2 + m_3 h \circ e_3 = 0$$

where at least two of $m_1, m_2, m_3$ are nonzero. Moreover, using the fact that $E_1 - E_2, E_1 - E_3, E_2 - E_3$ are all strong left orthogonal, $E$ can be a component of at most one of them. So at least two of $r_{e_i} = 1$ for $i = 1, 2, 3$. To conclude, equation (7.14) gives a nontrivial linear relation between the values of $f, g, h$. We can now choose two linearly independent ones to give us the $A^2$, and follow the arguments in previous cases to show the values of all type ii,iii arrows are determined by the two chosen values.

The case when $h^0(S, H - E_1 - E_2 - E_3) = 1$ is handled similarly.

If $T S^{op} = \{E_1, H - E - E_1, E, H - E - E_2, E_2, H - E_1 - E_2\}$. Note this collection is symmetric around $E$, so we make an extra assumption that if $P$ is on either $E_1$ or $E_2$ or maybe both, we make sure $P$ is on $E_1$ by a possible reverse of toric system. There is a unique arrow from 2 to 3, which we denote by $f_1$. There is an unique indecomposable arrow from 1 to 3, which we denote by $f_2$. Similarly, we denote the indecomposable arrow from 3' to 4 and 5 by $g_1, g_2$ respectively. Note $B_1 + B_2 + B_3 = B_3 + B_1 + B_3 = H$. We fix a line $l \in H^0(S, H) - H^0(S, H - E)$. Then there is a linear combination of arrows from 1
of arrows from 3 to 5
\[ w_4 = b_1 g_2 \circ e + b_2 e_2 \circ g_1 \circ e + b_3 e_2 \circ u_4 + b_4 u_5 \]
such that \( w_4 \) represents the line \( l \) and \( r_{w_4} = 1 \). Then we can find \( m_1, m_2, n_1, n_2 \in \mathbb{C} \) so that
\[
(7.15) \quad m_1 w_4 \circ f_2 + m_2 w_4 \circ f_1 \circ e_1 = e_2 \circ g_1 \circ w_0
\]
\[
(7.16) \quad n_1 w_4 \circ f_2 + n_2 w_4 \circ f_1 \circ e_1 = g_2 \circ w_0
\]
Now if \( E_2 - E \) is not effective, then the above two relations show that the values of \( g_1, g_2 \) are determined by those of \( f_1 \) and \( f_2 \). It is easy to check the only possible arrows of type iii are from 2 to 4. If there are no such arrows, then the values of \( f_1, f_2 \) will determine the representation. If such an arrow \( h \) exists (there can be at most one), note we have decomposable arrows: \( u_4 \circ f_1, g_1 \circ e \circ f_1 \) and \( g_1 \circ u_2 \), then the line \( H - E - E_1 \) must be the same as \( H - E - E_2 \). Suppose moreover \( E_1 - E \) is not effective, then \( h \circ f_1 \) is an arrow from 1 to 4. By Lemma 2.11, since \( E_2 - E \) is slo, all arrows from 1 to 4 can be written as a composition of an arrow from 1 to 3 with an arrow from 3 to 4. Hence the value of \( h \) is determined by \( f_1 \) and \( f_2 \). Suppose \( E_1 - E > 0 \), then \( H^0(S, H - E_1) \) and \( H^0(S, H - E) \) are the same one-dimensional subspace of \( H^0(S, H) \). So we can choose \( f_2 \) and \( u_2 \) to represent the same line. Thus \( e \circ f_2 = u_2 \circ e_1 \). Hence \( r_{f_2} = 0 \). Then the value of \( h \) and \( f_1 \) determine the representation and gives the \( \mathbb{A}^2 \) in this case.

If \( E_2 - E > 0 \), with our convention, \( E_1 - E > 0 \). Hence either \( E_1 - E_2 \) is effective or \( E_2 - E_1 \) is effective. Without loss of generality, we assume \( E_1 - E_2 \geq 0 \). Then one readily check \( H^0(S, H - E - E_1) \) and \( H^0(S, H - E - E_2) \) are the same one-dimensional subspace of \( H \). Similarly, so we can set \( f_1 \) and \( g_1 \) to represent the same linear function in \( H^0(S, H) \). We can set \( f_2, g_2, u_2, u_5 \) all to represent the same line. We can set \( u_1, u_5 \) represent same functions respectively. Then
\[
\begin{align*}
  u_5 \circ f_2 &= g_2 \circ u_1 \\
u_4 \circ f_1 &= g_1 \circ u_2
\end{align*}
\]
Thus the values of \( f_1, f_2 \) determine those of \( g_1, g_2 \). Now we take a look at arrows of type iii. Again, they can be only from 2 to 4 and in our case we will have one, which we denote by \( h \). As above, we have \( e \circ f_2 = u_2 \circ e_1 \), thus \( r_{f_2} = r_{g_2} = 0 \). Then the value of \( h \) and \( f_1 \) determine the representation and gives the \( \mathbb{A}^2 \) in this case.

There are some cases that can be solved using mainly the method in the above example, combining with arguments in the first few examples in part 2, we list them here and omit the details:

- \( \{ E_2, E_1 - E_2, H - E - E_1, E, H - E - E_3, E_3, H - E_1 - E_2 - E_3 \} \)
• \( \{ E_1 - E_2, E_2, H - E - E_1 - E_2, E, H - E - E_3, E_3, H - E_1 - E_3 \} \)
• \( \{ E_1 - E_2, E_2, H - E - E_1 - E_2, E, H - E - E_3, E_3 - E_4, E_4, H - E_1 - E_3 - E_4 \} \)
• \( \{ E_1 - E_2, E_2, H - E - E_1 - E_2, E, H - E - E_3 - E_4, E_4, E_3 - E_4, H - E_1 - E_3 \} \)

Now we consider the case when \( E, E_1 - E \) is a segment of the toric system. Note this also covers the possibility of \( E_1 - E, E \) by symmetry.

The easiest case here is \( \mathcal{T} S^{op} = \{ H - E_1 - E, E, E_1 - E, H - E_1, H \} \). There are two arrows \( f_1, f_2 \) from 2' to 4, both indecomposable, we denote them by \( f_1, f_2 \). Since \( E_1 - E \) is strong left orthogonal, without loss of generality, we assume \( E_1 \) comes from blowing up at \([0 : 0 : 1]\) and \( P = [0 : 1 : 0] \). Hence we can let \( f_1 \) represent \( x \) and \( f_2 \) represent \( z \).

There is a unique arrow from 1 to 2 (clearly indecomposable), which we denote by \( g \) and an indecomposable arrow from 1 to 3, which we denote by \( h \). We let \( g \) represent the linear function \( x \) and \( h \) represent the linear function \( z \). Moreover, we denote the two arrows from 3 to 4 by \( l_1 \) and \( l_2 \), and let them represent \( x, y \) respectively. Also notice we can let \( u_1 \) represent \( y \). Then we have

\[
\begin{align*}
    l_2 \circ h &= f_2 \circ u_1 \\
    l_2 \circ e_1 \circ g &= f_1 \circ u_1 
\end{align*}
\]

Since \( y \) does not vanish on \( P \), we see \( r_{l_2} = 1 \). Also \( r_{e_1} = 1 \). Hence the values of \( g, h \) are determined by those of \( f_1 \) and \( f_2 \). One can then easily check there are no indecomposable arrows from 1 to 4, hence the values of \( f_1 \) and \( f_2 \) will give us the \( \mathbb{A}^2 \).

If \( \mathcal{T} S^{op} = \{ H - E_1 - E, E, E_1 - E, H - E_1 - E_2, E_2, H - E_2 \} \), suppose \( h^0(S, H - E - E_1 - E_2) = 0 \), there are four indecomposable arrows whose values are undetermined. We call the arrow from 2' to 4 and 5 by \( a \) and \( b \) respectively. We call the unique indecomposable arrow from 1 to 3 by \( c \), the one from 1 to 2 by \( d \). We let \( u_{34} \) denote the unique arrow from 3 to 4. By our assumption, \( r_{u_{34}} = 1 \). Since \( h^0(S, B_1 + B_2 + B_3 + B_4) = h^0(S, 2H - E - E_1 - E_2) = 3 \), there exists constants \( m_i \) for \( i = 1, 2, 3, 4 \) so that

\[
\begin{align*}
    m_1 u_{34} \circ e_1 \circ d + m_2 a \circ e \circ d + m_3 a \circ u_1 + m_4 u_{34} \circ c = 0 
\end{align*}
\]

Note as before, \( m_1, m_4 \) cannot both be 0 and \( m_2, m_3 \) cannot both be 0. Taking the value and notice \( r_{e_1} = 1 \), we have

\[
\begin{align*}
    m_1 r_d + m_3 r_a + m_4 r_c = 0 
\end{align*}
\]

Note \( B_1 + B_2 + B_3 = H - E \) and \( B_4 + B_5 = H - E_1 \), so by Lemma 2.11, all arrows from 1 to 5 is a composition of arrows from 1 to 3 with arrows from 3 to 5, then

\[
\begin{align*}
    u_1 \circ b = n_1 u_{35} \circ c + n_2 e_2 \circ u_{34} \circ c + n_3 u_{35} \circ e_1 \circ d + n_4 e_2 \circ u_{34} \circ e_1 \circ d
\end{align*}
\]

for \( n_i \in \mathbb{C} \). Hence the value of \( b \) is determined by those of \( a, c, d \). Thus by equation 7.17 we can find two values among \( a, c, d \) to determine the representation.

Suppose \( h^0(S, H - E - E_1 - E_2) = 1 \), we call the unique section \( l \). In this case \( r_{u_{34}} = 0 \). Note now we can let \( a, d, u_{34} \) all represent the line \( l \). Take a line \( l_1 \) in the linear system \( H - E_1 \) that is not \( l \), then we can let \( u_1 \) and \( u_{35} \) to represent \( l_1 \). Similarly, take a line \( l_2 \) in
Thus the value of $f$ that is not $l$, we can let $b, c$ to represent $l_2$. Then $u_{35} \circ c = b \circ u_1$. Hence $r_b = r_c$. Moreover we have

$$e_2 \circ a \circ u_1 = u_{35} \circ e_1 \circ d$$

Since $r_{u_{35}} = 1$ and $r_{e_1} = 1$, the value of $d$ is determined by $a$. Hence the representation is determined by the values of $a, c$.

This conclude the case at hand. We list a few cases that can be dealt with using the method introduced above (only more complicated since we have more exceptional curves):

- $TS^{op} = \{ E_2, H - E - E_1 - E_2, E, E_1 - E, H - E_1, H - E_2 \}$
- $TS^{op} = \{ E_2, H - E - E_1 - E_2, E, E_1 - E, H - E_1 - E_3, E_3, H - E_2 - E_3 \}$
- $TS^{op} = \{ E_2, H - E - E_1 - E_2, E, E_1 - E, H - E_1 - E_3 - E_4, E_4, E_3 - E_4, H - E_2 - E_3 \}$
- $TS^{op} = \{ E_2, H - E - E_1 - E_2, E, E_1 - E, H - E_1 - E_3 - E_4, E_4, H - E_2 - E_3 - E_1 \}$
- $TS^{op} = \{ H - E - E_1, E, E_1 - E, H - E_1 - E_2 - E_3, E_3, E_2 - E_3, H - E_2 \}$
- $TS^{op} = \{ H - E - E_1, E, E_1 - E, H - E_1 - E_2 - E_3, E_3 - E_4, H - E_2 - E_3 - E_1 \}$

Lastly we consider the case when $E, E_1 - E, E_2 - E_1$ is a segment of the toric system. Note this also covers the case of $E_2 - E_1, E_1 - E, E$ by symmetry.

The easiest case is $TS^{op} = \{ H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2, H \}$. Then undetermined indecomposable arrows are the two arrows from $2'$ to 5, which we call $f_1$ and $f_2$, the arrow $g$ from 1 to 3 and the arrow $h$ from 1 to 4. Note $g$ represents the unique section in $H^0(S, H - E - E_2)$, $h$ represents the unique section in $H^0(S, H - E - E_1)$. Since $E_2 - E_1$ and $H - E - E_1 - E_2$ are both strong left orthogonal, the two linear functions $h, g$ represent spans $H^0(S, H - E)$. On the other hands, $f_1, f_2$ represents sections in $H^0(S, H - E)$. So we can assume $f_1$ represents the same function as $g$, $f_2$ represents the same function as $h$.

Note arrows from 3 to 5 represents sections in $H^0(S, H - E_1)$, which contains the subspace $H^0(S, H - E_1 - E_2)$. So we can choose an arrow $l_1$ from 3 to 5 such that it represents the same function as $u_1$. Hence we have the relation:

$$l_1 \circ g = f_1 \circ u_1$$

Since $r_{u_1} = 1$, we see the value of $g$ determines that of $f_1$. In fact here one can verify that $r_{l_1} = 1$, so $f, g_1$ have the same value. Similarly, one can find an arrow $l_2$ from 4 to 5 such that

$$l_2 \circ h = f_2 \circ u_1$$

Thus the value of $f_2$ is determined by that of $h$. So the values of $g, h$ give the $A^2$.

If $TS^{op} = \{ H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 - E_3, E_3, H - E_3 \}$. Then undetermined indecomposable arrows consists of one arrow from $2'$ to 5, which we call $f_1$, one arrow from $2'$ to 6, which we call $f_2$, the arrow $g$ from 1 to 3 and the arrow $h$ from 1 to 4.
Note $h^0(S, A_1 + \ldots + A_4) = h^0(S, 2H - E_1 - E_2 - E_3 - E) = 2$, then there is a linear relation
\[m_1 f_1 \circ u_1 + m_2 u_{35} \circ g + m_3 \circ u_{45} \circ h\]
where $m_1, m_2, m_3 \in \mathbb{C}$ and at least two of them are nonzero. Taking the values we obtain
\[(7.19) \quad m_1 r_{f_1} + m_2 r_{u_{35}} r_g + m_3 r_{u_{45}} r_h = 0\]
Moreover, we can check at least one of $r_{u_{35}}, r_{u_{45}}$ is nonzero, or we will get $h^0(S, H - E_1 - E_2 - E) = 1$, which contradicts the fact that it is slo. So equation (7.18) gives nontrivial linear relation between $r_{f_1}, r_g, r_h$.

Note $h$ represents the unique line in $|H - E - E_1|$, $u_0$ represents the unique line in $|H - E_1 - E_2|$. Since $B_1 + B_5 = H - E_2$, we can find $n_1, n_2 \in \mathbb{C}$ such that $n_1 e_3 \circ u_{45} + n_2 u_{46}$ represents the same line as $u_1$ and thus have value 1. Similarly, we can find $n_3, n_4 \in \mathbb{C}$ such that $n_3 e_3 \circ f_1 + n_4 f_2$ represents the same line as $h$. So we have
\[(7.19) \quad (n_1 e_3 \circ u_{45} + n_2 u_{46}) \circ h = (n_3 e_3 \circ f_1 + n_4 f_2) \circ u_1\]
thus
\[r_h = n_3 r_{e_3} r_{f_1} + n_4 r_{f_2}\]
So we can choose two independent values in equation (7.18) to determine the representation.

Two of the remaining cases can be solved using similar arguments. We list them here and omit the details:
- $TS_{op} = \{H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 - E_3, E_3 - E_4, E_4, H - E_3 - E_4\}$
- $TS_{op} = \{H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 - E_3, E_3 - E_4, E_4 - E_5, E_5, H - E_3 - E_4 - E_5\}$

If $TS_{op} = \{H - E - E_1 - E_2, E, E_1 - E, E_2 - E_1, H - E_2 - E_3 - E_4, E_4, E_3 - E_1, H - E_3\}$. There are five undetermined- indecomposable arrows in this case, $f_1$ from 2 to 6, $f_2$ from 2' to 7, $g_1$ from 1 to 3, $g_2$ from 1 to 4 and $h$ from 1 to 5. Since $E_1 - E, E_2 - E_1, E_2 - E, H - E - E_1 - E_2$ are all slo, we see the three lines represented by $u_1, g_1, g_2$ are mutually distinct and spans $|H|$. Suppose $E_3 - E > 0$, since $E_3 - E_4$ is slo, $E_4 - E$ is not effective. In this case we can see $u_{36}, g_2$ represents the same line, $u_{46}, g_1$ represents the same line. Thus
\[u_{36} \circ g_1 = u_{46} \circ g_2\]
Note if $r_{u_{46}} = r_{u_{36}} = 0$, then $|H - E - E_1 - E_2 - E_3|$ has a section, which is absurd. So if $r_{u_{46}} = 1$, the value of $g_2$ determined $g_1$ and vice versa. Without loss of generality, we assume $r_{u_{46}} = 1$, the case when $r_{u_{46}} = 0$ and $r_{u_{36}} = 1$ can be dealt similarly. Since $H - E_2 - E_3 - E_4, H - E_1 - E_3 - E_4$ are slo, we have $h$ and $u_{36} \circ g_1$ represent two different quadratic functions. Since $h^0(S, B_1 + \ldots + B_5) = 2$, we have
\[f_1 \circ u_1 = m_1 u_{36} \circ g_1 + m_2 e_4 \circ h\]
So the values of $f_1$ is determined by $g_1$ and $h$. 
Since $E_4 - E$ is not effective, it is easy to check that $u_{37} \circ g_1$ and $u_{47} \circ g_2$ spans $H^0(S, B_1 + \ldots + B_6)$, thus there exists $n_1, n_2$ so that

$$f_2 \circ u_0 = n_1 u_{37} \circ g_1 + n_2 u_{47} \circ g_2$$

Thus the value of $f_2$ is determined by $g_1$. So when $E_3 - E > 0$ and $r_{u_{46}} = 1$, the representation is determined by $g_1$ and $h$. Similar arguments handles when $r_{u_{46}} = 0$ and $r_{u_{36}} = 1$. The case when $E_4 - E > 0$ is similar.

The remaining case is when $E_3 - E, E_4 - E$ are not effective. We claim in this case $u_{36} \circ g_1, u_{46} \circ g_2$ span $H^0(S, B_1 + \ldots + B_6)$. If not, then we must have $g_1, u_{46}$ represent the same line and $g_2, u_{36}$ represent the same line. If $g_1 \in H^0(S, H - E - E_2), u_{46} \in H^0(S, H - E_2 - E_3)$, if they are the same line, then $h^0(S, H - E_2 - E_3 - E) = 1$ since $E_2 - E, E_3 - E$ are slo. Similarly, using $g_2, u_{36}$ we get $h^0(S, H - E_1 - E_3 - E) = 1$. But both of these 1-dimensional vector spaces are the same subspace of $H^0(S, H - E - E_3)$. there is a line passing through $E, E_1, E_2$, contradicting the fact $H - E - E_1 - E_2$ is slo. As above, we can show the value of $u_{36}, u_{46}$ cannot vanish simultaneously. We have

$$f_1 \circ u_1 = m_1 u_{36} \circ g_1 + m_2 u_{46} \circ g_2 \circ h = n_1 u_{36} \circ g_1 + n_2 u_{46} \circ g_2$$

Taking the value we see the values of $f_1, g$ are determined by those of $g_1, g_2$.

Similarly, one can show $u_{37} \circ g_1, u_{47} \circ g_2$ span $H^0(S, B_1 + \ldots B_6)$ and the value of $f_2$ is determined by those of $g_1, g_2$.

By now we have discussed all possible cases, and this concludes the proof. \qed

**Proof of Theorem 7.8.** By our reduction before, we see

$$C = V(e) \mathbb{S} / \text{PGL}(1) = V(e, u_1 - 1, \ldots, u_n - 1)^S / \mathbb{K}^*$$

We note $C \neq \emptyset$ implies $V(e, u_1 - 1, \ldots, u_n - 1)^S \neq \emptyset$. Let $a, b$ be the two arrows in the above lemma. We claim if $R \in V(e, u_1 - 1, \ldots, u_n - 1)^S$ satisfies $r_a = r_b = 0$, $R$ is not $\theta$-stable. For such an $R$, we note it satisfies

i. For any $i < k$, all arrows in from $i$ to $k$ have value 0.

ii. For any $j > k$, all arrows from $k'$ to $j$ have value 0.

iii. For any $i < k < j$, all arrows from $i$ to $j$ have value 0.

Hence we can use the subrepresentation of $R$ as in Prop 7.2 and show $R$ is not stable. For any arrow $g$ of type i, ii or iii, we realize from the proof of Lemma, that

$$r_g = m_a r_a + m_b r_b$$

where $m_a, m_b \in \mathbb{C}$ and depends only on $(Q, I)$. Thus we see $V(e, u_1 - 1, \ldots, u_n - 1)^S$ is of the complement of a finite collection of lines through origin in $\mathbb{A}^S$. If the collection is nonempty, then $C = V(e, u_1 - 1, \ldots, u_n - 1)^S / \mathbb{K}^*$ is the complement of finite nonzero number of points in $\mathbb{P}^1$, which contradicts the fact that $C$ is proper. Hence the collection of lines is empty and $C \cong \mathbb{P}^1$. \qed

**Corollary 7.20.** Let $a, b$ be as above. A representation $R \in V(e)$ satisfying $r_u \neq 0$ for all $i$ is unstable if and only if $r_a = r_b = 0$. 
Proof. Given a representation $R$ satisfying the above property, we can find in its orbit $R'$ under the $\text{PGL}(1)$ action such that $r'_{u_i} = 1$ for all $i$. Since $R$ is stable if and only if $R'$ is, we assume $r_{u_i} = 1$ for all $i$.

In the last paragraph of proof of Theorem 7.8, we proved that the stable locus of $V(e, u_1 - 1, \ldots, u_n - 1) = \mathbb{A}^2 \setminus \{(0, 0)\}$ where the point $(0, 0)$ corresponds to representation the unique representation in $V(e, u_1 - 1, \ldots, u_n - 1)$ with $r_a = r_b = 0$. Thus $R$ is unstable if and only if $r_a = r_b = 0$. \hfill $\square$

**Proposition 7.21.** If $R \in V(e)$, then $\text{Ext}^1_A(R, R) = k^2$.

*Proof.* The theorem follows from multiple steps of reduction. Since the dimension of $\text{Ext}^1$ only depends on equivalence classes of representations, we may assume $R$ to be in the standard form, i.e., $r_{u_i} = 1$ for all $i$. Let

$$0 \rightarrow R \rightarrow \mathcal{E} \rightarrow R \rightarrow 0$$

be any extension. To give such an extension is equivalent to give for each vertex $i$, an extension

$$(7.22) \quad 0 \rightarrow R_i \overset{\iota_i}{\rightarrow} \mathcal{E}_i \overset{p_i}{\rightarrow} R_i \rightarrow 0$$

and for each arrow $a$ from $i$ to $j$, a linear transformation $\epsilon_a : \mathcal{E}_i \rightarrow \mathcal{E}_j$ such that the following diagram is commutative:

$$\begin{array}{cccccc}
0 & \rightarrow & R_i & \overset{\iota_i}{\rightarrow} & \mathcal{E}_i & \overset{p_i}{\rightarrow} & R_i & \rightarrow & 0 \\
 & \downarrow{r_a} & \downarrow{\epsilon_a} & \downarrow{r_a} & \downarrow{r_a} & & & & \\
0 & \rightarrow & R_j & \overset{\iota_j}{\rightarrow} & \mathcal{E}_j & \overset{p_j}{\rightarrow} & R_j & \rightarrow & 0
\end{array}$$

Note similar to the action of $GL(\bar{1})$ on representations, the group $GL(\bar{2}) = (GL(2))^N$ where $N$ is the number of vertices in $Q$ acts on the set of self-extensions of $R$ in the following way: Let $M = (M_0, M_1, \ldots, M_n) \in GL(\bar{2})$ by

$$M \cdot \epsilon_a = M_{t(a)} \epsilon_a M_{s(a)}^{-1}$$

for any arrow $a$, and for all $i$,

$$M \cdot \iota_i = M_i \iota_i$$

$$M \cdot p_i = p_i M_i^{-1}$$

Note all the operation are matrix multiplications on the right hand sides of the above three equations. Two self-extensions of $R$, $E, E'$ are isomorphic if they are in the same orbit of the $GL(\bar{2})$ action. It is standard that the isomorphism classes of self extensions form a vector space and we need to show it has dimension 2.
Let $E$ be such an extension, first we perform a base change, i.e an action of some $M \in GL(\mathbb{2})$ such that

\begin{align*}
\eta_l(x) &= (x, 0) \\
p_l(x, y) &= y
\end{align*}

for all $l \in \{0, 1, \ldots, n\}$ Using the commutative diagram above, we see now

\[ \epsilon_a = \begin{bmatrix} r_a & \lambda_a \\ 0 & r_a \end{bmatrix} \]

Hence the orbit of all self-extensions of $R$ under the action of $GL(\mathbb{2})$ is isomorphic to the action of self-extensions of $R$ satisfying equations (7.23) and (7.24) for all $l$, under the action of a subgroup $H \subset GL(\mathbb{2})$, where the $i$-th component of an element $M$ of $H$ has the form

\[ M_i = \begin{bmatrix} 1 & c_i \\ 0 & 1 \end{bmatrix} \]

Notice all the matrix in our consideration now are upper-triangular matrices with identical diagonal elements. So the diagonal subgroup

\[ D = (M_1, M_1, \ldots, M_1) \subset H \]

acts trivially. Second, for all $i < k$, we have $r_{u_i} = 1$, thus

\[ \epsilon_{u_i} = \begin{bmatrix} 1 & \lambda_{u_i} \\ 0 & 1 \end{bmatrix} \]

For any given

\[ M'_k = \begin{bmatrix} 1 & c_k' \\ 0 & 1 \end{bmatrix} \]

for each $i < k$ one can always choose an unique $M_i$ depending on $M_k'$ such that

\[ \epsilon'_{u_i} = M_k' \epsilon_{u_i} M_i^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

For any given

\[ M_k = \begin{bmatrix} 1 & c_k \\ 0 & 1 \end{bmatrix} \]

we can do the same thing to $u_j$ with $j > k$. Hence the orbit self-extensions of $R$ under the action of $GL(\mathbb{2})$ is isomorphic to the set of extensions satisfying equations (7.23) and (7.24) and $\epsilon_{u_i} = I_2$ for all $i$, under the action of $k^2$, where $(a, b)$ acts on the extensions by setting

\[ M_k = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \]
\[ M_{k'} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \]

and find \( M_i \) for other \( i \) as above.

Now let \( f, g \) be the two arrows in \( Q \) that whose values gives the \( A^2 \) in the above proof. Then either \( r_f \neq 0 \) or \( r_g \neq 0 \). Without loss of generality, we can let \( r_f \neq 0 \). Then one can adjust \( M_k \) and \( M_{k'} \) so that \( e_f = I_2 \). Noting the fact that the diagonal subgroup acts trivially, the set of isomorphism classes of self-extension of \( R \) is isomorphic to the set of self-extensions of \( R \) that satisfies equations (7.23), (7.24), \( \epsilon_{u_i} = I_2 \) for all \( i \) and \( e_f = I_2 \). Now we count the dimension of such extensions. We claim \( \lambda_e \) and \( \lambda_g \) determines the extension.

If an arrow \( c \) from \( i \) to \( j \) satisfies \( j < k \). Then either \( u_j \circ c = m_c u_i + e \circ (\sum_{l=1}^{L} n_la_l) \) for some \( m_c \neq 0, n_l \in C \) and \( a_l \) arrow from \( i \) to \( k \), or \( u_j \circ c = e \circ (\sum_{l=1}^{L} n_la_l) \) is a multiple of \( e \). In the first case, we have

\[
\epsilon_c = m_c I_2 + \epsilon_e (\sum_{l=1}^{L} n_l \epsilon a_l) = \begin{bmatrix} m_c \sum_{l=1}^{L} (n_l \epsilon a_l) \lambda_e \\ 0 \end{bmatrix}
\]

Notice \( \epsilon_c \) is determined by \( \lambda_e, R \) and \( Q \). In the second case, we have

\[
I_2 \epsilon_c = \epsilon_e (\sum_{l=1}^{L} n_l \epsilon a_l)
\]

Thus

\[
\epsilon_c = \begin{bmatrix} 0 \\ 0 \sum_{l=1}^{L} (n_l \epsilon a_l) \lambda_e \end{bmatrix}
\]

So all the linear transformations are determined by \( \lambda_e, R \) and \( Q \) except for arrows of the form i,ii and iii.

Now any indecomposable arrow \( h \) of type i,ii,iii other than \( f, g \) satisfies a linear relation involving \( f, g \), determining \( r_h \). We claim the linear relation also determines the \( e_h \). We illustrate this using an example:

Consider the case when \( TS^{op} = \{ E_3, E_2 - E_3, E_1 - E_2, H - E_1 - E, E, H - E, H - E_1 - E_2 - E_3 \} \). There are one arrow each from 2,3,4 to 5', all indecomposable, which we denote by \( f, g, h \). Then \( f \circ e_1, g \circ e_2, h \circ e_3 \) all represents sections in \( H \). Moreover, it is clear that they represents sections in the image of the natural embedding \( H^0(S, H - E) \hookrightarrow H^0(S, H) \), so we have a linear relation:

(7.25)
\[
a f \circ e_1 + b g \circ e_2 + c h \circ e_3 = 0
\]

Note \( P \) can be only on at most on of \( E_1, E_2, E_3 \), without loss of generality we assume \( P \) is not on \( E_3 \). Then as in the proof, \( r_h \) is determined by \( r_f \) and \( r_g \). Here then

\[
\epsilon_h = c^{-1} (a \epsilon_f + b \epsilon_g)
\]
So $\epsilon_h$ is determined by $\epsilon_f$ and $\epsilon_g$. Note $a, b$ are determined by $Q$, not by the representation $R$. In conclusion, any isomorphism class of self-extension of $R$ contains a unique normal form, which is determined by $\lambda_e$ and $\lambda_g$. Thus $\text{Ext}^1_A(R, R) = k^2$.

**Proof of Main Theorem.** Set theoretically,

$$M_\theta = U \cup C$$

where $U \cong S_0 \setminus P$ by Corollary 7.7 and $C \cong \mathbb{P}^1$ by Theorem 7.8. Since $M_\theta$ is projective, it is connected and of dimension 2. Again since $U \cong S_0 \setminus P$, $M_\theta$ is smooth in the open set $U$. For any point $m \in C$, let $R$ be a representation in this isomorphism class, we notice by Proposition 7.21

$$\dim(T_m M_\theta) = \dim \text{Ext}^1_A(R, R) = 2$$

thus $M_\theta$ is smooth along $C$ also. Thus $M_\theta$ is a smooth, connected projective scheme of dimension 2, so $M_\theta$ is a surface. Moreover, since $M_\theta$ has a dense open set isomorphic to $S_0 \setminus P$ whose complement is isomorphic to $\mathbb{P}^1$, it is the blow up of $S_0$ at $P$ and

$$f : M_\theta \to M_{\theta_0}$$

is the blow up morphism. Consider the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{T} & M_\theta \\
\downarrow \pi & & \downarrow f \\
S_0 & \xrightarrow{t_0} & M_{\theta_0}
\end{array}
$$

and the composition

$$T_0 \circ \pi : S \to M_{\theta_0}$$

which is a birational morphism. $(T_0 \circ \pi)^{-1}$ is defined except at the point $T_0(P)$. Hence by the universal property of blow up, there exists a birational morphism $T' : S \to M_\theta$ so that

$$
\begin{array}{ccc}
S & \xrightarrow{T'} & M_\theta \\
\downarrow \pi & & \downarrow f \\
S_0 & \xrightarrow{t_0} & M_{\theta_0}
\end{array}
$$

It is clear that $T'|_{S \setminus E}$ coincides with $T|_{S \setminus E}$. Since $S$ is a variety and $S \setminus E$ is a nonempty open subset, the rational map $T$ is defined on all of $S$, hence

(7.26)

$$T : S \to M_\theta$$

is a morphism. Now since both $S$ and $M_\theta$ are isomorphic to blow up of $S_0$ at $P$, and $T$ is birational, $T$ is an isomorphism. □
Proof of Corollary 1.5. Let $S$ be a smooth del Pezzo surface of degree $d \geq 3$. Suppose $S$ is not $\mathbb{P}^1 \times \mathbb{P}^1$, then $S$ is obtained from blowing up $9 - d$ points on $\mathbb{P}^2$ in general position (in any order). Thus by Main Theorem, it suffices to find a full strong exceptional collection of line bundles on $S$ obtained from standard augmentation from $\mathbb{P}^2$ satisfying (1.4). See the Table 1 below for examples: It is easy to check that they are standard augmentations.

For $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, the result follows from [16].

\[\square\]

Table 1. Strong exceptional collection of line bundles on Del Pezzo surfaces satisfying (1.4)

| Degree $= K_S^2$ | Strong exceptional toric systems $E_{i,k} := \sum_{j=i}^{k} E_i$ |
|------------------|---------------------------------------------------------------|
| 8                | $H - E_1, E_1, H - E_1, H$                                   |
| 7                | $H - E_{12}, E_2, E_1 - E_2, H - E_1, H$                      |
| 6                | $H - E_{123}, E_3, E_2 - E_3, E_1 - E_2, H - E_1, H$         |
| 5                | $H - E_{123}, E_3, E_2 - E_3, E_1 - E_2, H - E_{14}, E_4, H - E_4$ |
| 4                | $H - E_{123}, E_3, E_2 - E_3, E_1 - E_2, H - E_{145}, E_5, E_4 - E_5, H - E_4$ |
| 3                | $E_4, E_2 - E_4, H - E_{125}, E_5, E_5 - E_1 H - E_{136}, E_6, E_3 - E_6, H - E_{234}$ |

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8. Appendix A: Standard Augmentation of strong exceptional toric system in the strong sense by A. Elagin and S. Zhang

In [8], the second author and Alexey Elagin showed that every full strong exceptional toric system on weak del Pezzo surfaces of degree $\geq 3$ is standard augmentation in the mild sense. In this section, we provide a stronger result which is useful for our purpose.

Definition 8.1. A strong exceptional toric system $A = (A_1, \ldots, A_n)$ on $S$ is called a standard augmentation up to reordering in the strong sense if $S$ is a Hirzebruch surface or $A$ can be obtained by several trivial mutations with $1 \leq k \leq n - 1$ from a strong exceptional toric system $B$ on $S$ which is an elementary augmentation with $1 \leq m \leq n - 1$ of some strong exceptional toric system $B'$ where $B'$ is a standard augmentation up to reordering in the strong sense.

Remark 8.2. The difference between standard augmentation in the strong sense and mild sense is that the mutations of toric systems are only allowed among divisors $A_i$ with $1 \leq i \leq n - 1$ and the cyclic shifts are not allowed in the definition of standard augmentations in
the strong sense. In general, we have a strong exceptional toric system is an augmentation up to reordering in the strong sense $\implies$ in the mild sense $\implies$ in the weak sense.

**Theorem 8.3.** Any full strong exceptional toric system on weak del Pezzo surface $S$ of $K_S^2 \geq 5$ is standard augmentation in the strong sense. The same statement holds for $3 \leq K_S^2 \leq 4$ except for cyclic strong exceptional toric systems.

For strong exceptional toric system of type II-VI on $S$, there is no difference between standard augmentation in the strong sense and mild sense, the proof is given in [8]. Here we give a proof for strong exceptional toric system of the first kind for $K_S^2 \geq 6$ cases to illustrate the idea, for degree 3, 4, 5 cases, reader who is interested in can check [22].

Denote the set of $(-1)$-classes on $S$ by $I(S)$. We treat $I(S)$ as a graph: two vertices $D_1 \neq D_2 \in I(S)$ are connected by $m$ edges if $D_1 \cdot D_2 = m$. Note that the graph $I(S)$ depends only on deg$(S)$. If one takes $S$ to be a del Pezzo surface, then any $(-1)$-class corresponds to an irreducible curve, and we have $D_1 \cdot D_2 \geq 0$. Therefore the above definition makes sense. Also note that $I(S)$ has no multiple edges for deg$(S) \geq 3$. Denote by $I^{irr}(S) \subset I(S)$ the full subgraph of irreducible $(-1)$-classes and by $I^{red}(S) \subset I(S)$ the complement of $I^{irr}(S)$. These subgraphs depend on the surface $S$. Clearly, $I(S) = I^{red}(S) \sqcup I^{irr}(S)$.

We argue by decreasing induction in deg$(S)$. Induction base is the case of Hirzebruch surfaces. For a toric system $A = (A_1, \ldots, A_n)$ on a surface $S$, define a subset $I^F(S, A) \subset I(S)$:

$$I^F(S, A) = \{A_{k \ldots l} \in I(S)| 1 \leq k \leq l \leq n - 1\}.$$  

We will check the following

**Claim.** For any strong exceptional toric system $A$ on $S$ the set $I^F(S, A)$ cannot be a subset of $I^{red}(S)$.

Therefore there exists a $-1$-class $A_{k \ldots l}$ with $1 \leq k \leq l \leq n - 1$ which is an exceptional curve $E$. Since divisor $A_{k \ldots l}$ is a $-1$-class, there exists $m, k \leq m \leq l$, such that

$$A_m^2 = -1 \quad \text{and} \quad A_k^2 = \ldots = A_{m-1}^2 = A_{m+1}^2 = \ldots = A_l^2 = -2.$$  

Consider the toric system

$$B = \text{mut}_{m-1}\text{mut}_{m-2} \ldots \text{mut}_k(\text{mut}_{m+1}\text{mut}_{m+2} \ldots \text{mut}_l(A)).$$

It is strong exceptional and $B_m = A_{k \ldots l}$. Therefore $B$ is an elementary augmentation $\text{augm}_m(B')$ of some strong exceptional toric system on the blow-down of $E$. By induction, $B'$ is an augmentation up to reordering in the strong sense. Therefore $A$ also is such.

8.1. **Degree 7.** Let $S$ be a weak del Pezzo surface of degree 7. There are two such surfaces. The first one is a blow-up of $\mathbb{P}^2$ at two points, denote it $S_7$. It is a del Pezzo surface. The second one is a blow-up of $F_1$ at a point on $-1$-curve, denote it $S_{7, A_1}$. Both surfaces are toric. We have

$$I(S) : \quad E_1 \quad \text{---} \quad H_{12} \quad \text{---} \quad E_2 ; \quad R(S) = \{\pm(E_1 - E_2)\}.$$
On $S_7$ all three $-1$-classes are irreducible, on $S_{7,A_1}$ classes $E_2$ and $L_{12}$ are irreducible and $E_1$ is reducible. In both cases $|I^{red}(S)| \leq 1$. There is a unique $-2$-curve $E_1 - E_2$ on $S_{7,A_1}$.

Let $A = (A_1, \ldots, A_5)$ be a strong exceptional toric system. Suppose $A$ is of the first kind with $A_1$ is an exceptional curve. Therefore $A$ is of the first kind. Then up to cyclic shifts and a symmetry $A^2 = (A_1^2, \ldots, A_5^2) = (0, 0, -1, -1, -1)$ or $(0, 1, -1, -1, -2)$. Suppose $A$ is of the second type, then up to a symmetry we have $A^2 = (0, k, -1, -1, -k - 1)$ where $k \geq 2$.

In cases $(0, 0, -1, -1, -1)$ and $(0, k, -1, -1, -k - 1)$ we have $|I^F(S, A)| = 2$ or $3$, thus $I^F(S, A)$ cannot be a subgraph of $I^{red}(S)$. In the case $(0, 1, -1, -1, -2)$ and $S = S_7$ we have $|I^F(S, A)| = 1$ or $2$ and $I^{red}(S) = \emptyset$, again $I^F(S, A)$ cannot be a subgraph of $I^{red}(S)$. Suppose $A^2$ is $(0, 1, -1, -1, -2)$ up to cyclic shifts and symmetries and $S = S_{7,A_1}$. Note that $(E_1 - E_2)$ are the only $-2$-classes and they are not slo. Hence $A_2^2 = -2$ and again $|I^F(S, A)| = 2$.

Note that we have actually proved the following: there exists $i, 1 \leq i \leq 4$, such that $A_i$ is an exceptional curve. Therefore $A$ is an augmentation (without any reordering).

Remark 8.4. The proof above means that for any full strong exceptional toric system of the first kind with $A_n$ of the form $H - \sum_{i=1}^m E_i$ on a weak del Pezzo surface $S$ of degree 7, Theorem 1.1 applies.

8.2. Degree 6. Let $S$ be a weak del Pezzo surface of degree 6. We consider only non-toric surfaces because toric were studied by [HP]. There are two non-toric weak del Pezzo surfaces of degree 6. The first one is the blow-up of $\mathbb{P}^2$ at three point lying on a line, denote it by $S_{6,A_1}$. The second one is constructed as follows: we blow-up a general point on the $-1$-curve $E_2$ on the surface $S_{7,A_1}$ of degree 7. Denote this surface by $S_{6,A_2}$.

We have

$$I(S) : E_1 \quad H_{12} \quad E_2$$

$$H_{13} \quad E_3 \quad H_{23}$$

and

$$R(S) = \{ \pm(E_1 - E_2), \pm(E_1 - E_3), \pm(E_2 - E_3), \pm H_{123} \}$$

For $S_{6,A_1}$ we have

$I^{irr} = \{ E_1, E_2, E_3 \}$, $R^{eff} = \{ H_{123} \}$, $R^{slo} = \{ \pm(E_1 - E_2), \pm(E_2 - E_3), \pm(E_1 - E_3) \}$.

For $S_{6,A_2}$ we have

$I^{irr} = \{ H_{12}, E_3 \}$, $R^{eff} = \{ E_1 - E_2, E_2 - E_3, E_1 - E_3 \}$, $R^{slo} = \{ \pm H_{123} \}$.

Let $A = (A_1, \ldots, A_6)$ be a strong exceptional toric system on $S$. 
8.2.1. A is of the first type. Up to the symmetry, $A^2$ is one of the following (see [8 Proposition 5.1]):

(8.5) $(-1, -1, -1, -1, -1, -1)$
(8.6) $(-1, -1, -2, -1, -1, 0)$
(8.7) $(-1, -1, 0, -1, -1, -2)$
(8.8) $(0, -1, -1, -2, -1, -1)$
(8.9) $(-2, -1, 0, -1, -1)$
(8.10) $(0, 0, -2, -1, -2, -1)$
(8.11) $(0, -2, -1, -2, -1, 0)$
(8.12) $(-2, -1, -2, -1, 0, 0)$
(8.13) $(-1, -2, -1, 0, 0, -2)$
(8.14) $(-2, -1, 0, 0, -2, -1)$
(8.15) $(-1, 0, 0, -2, -1, -2)$
(8.16) $(1, -2, -1, -2, -2, 0)$
(8.17) $(-2, -1, -2, -2, 0, 1)$
(8.18) $(-1, -2, -2, 0, 1, -2)$
(8.19) $(-2, -2, 0, 1, -2, -1)$
(8.20) $(-2, 0, 1, -2, -1, -2)$
(8.21) $(0, 1, -2, -1, -2, -2)$

Recall that $|I^{\text{red}}(X_{6,A_1})| = 3$ and $|I^{\text{red}}(S_{6,A_2})| = 4$.
In case (1) we have $|I^F(S, A)| = 5$, so we are done.
In case (2) we have $|I^F(S, A)| = 6$, so we are done.
In case (3) we have $|I^F(S, A)| = 4$, so we are done for $S = S_{6, A_1}$. Suppose $S = S_{6, A_2}$ and $I^F(S, A) = I^{\text{red}}(S)$. Then

$$\{A_{56}, A_{61}\} = I(S, A) \setminus I^F(S, A) = I^{\text{irr}}(A) = \{L_{12}, E_3\}.$$

It follows that $A_5 - A_1 = A_{56} - A_{61} = \pm H_{123}$. The $-2$-class $A_6$ is lo, therefore $A_6 = \pm L_{123}, E_1 - E_2, E_1 - E_3$ or $E_2 - E_3$. Suppose $A_6 = \pm H_{123}$ then we have $0 = A_6(A_5 - A_1) = \pm H_{123}^2 = \pm 2$, a contradiction. Suppose $A_6 = E_1 - E_2, E_1 - E_3$ or $E_2 - E_3$. Then $A_6 E_3 = 0$ or $1$. But $E_3 = A_{56}$ or $A_{61}$ and by properties of a toric system $A_6 A_{56} = A_6 A_{61} = -1$, a contradiction.

In case (4) we have $I^F(S, A) = \{A_2, A_3, A_{34}, A_{45}, A_5\}$, $|I^F(S, A)| = 5$, so we are done.
In case (5) we have

$$I^F(S, A) : \quad A_{12} \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_5.$$ 

Therefore $I^F(S, A)$ is not a subgraph of $I^{\text{red}}(S)$.
Cases (6)-(8), (10), (12), (13) and (15)-(17) are impossible. There are two \(-2\)-classes \(A_i\) and \(A_j\) with \(1 \leq i < j \leq 5\) which are slo and such that \(A_iA_j = 0\). But such classes do not exist on \(S_{6,A_1}\) and \(S_{6,A_2}\).

Case (9) is similar to (3). We have \(|I^F(S, A)| = 4\), so we are done for \(S = S_{6,A_1}\). Suppose \(S = S_{6,A_2}\) and \(I^F(S, A) = I^{red}(S)\). Then

\[
\{A_{612}, A_{61}\} = I(S, A) \setminus I^F(S, A) = I^{irr}(A) = \{H_{12}, E_3\}.
\]

It follows that \(A_2 = \pm H_{123}\). The \(-2\)-class \(A_6\) is lo and orthogonal to \(A_2\), therefore \(A_6 = E_1 - E_2, E_1 - E_3\) or \(E_2 - E_3\). Then \(A_6E_3 = 0\) or 1. But \(E_3 = A_{612}\) or \(A_{61}\) and by properties of a toric system \(A_6A_{612} = A_6A_{61} = -1\), a contradiction.

In case (11) we have

\[
I^F(S, A) : A_1 \quad A_5 \quad A_{45}.
\]

For \(S = S_{6,A_2}\) the graph \(I^F(S, A)\) is not a full subgraph of \(I^{red}(S)\). Hence we may assume \(S = S_{6,A_1}\) and \(I^F(S, A) = I^{red}(S) = \{H_{12}, H_{13}, H_{23}\}\). Divisors \(A_4\) and \(A_6\) are two orthogonal lo \(-2\) classes. It follows that one of them is \(H_{123}\) and the other is \(E_i - E_j\). Since \(H_{123}\) is not slo, we have \(A_6 = H_{123}\). Now we have \(1 = A_5A_6 = H_{ij}H_{123} = -1\), a contradiction.

Case (14) is similar to (11). We have

\[
I^F(S, A) : A_1 \quad A_{12} \quad A_{123}.
\]

For \(S = S_{6,A_2}\) the graph \(I^F(S, A)\) is not a full subgraph of \(I^{red}(S)\). Hence we may assume \(S = S_{6,A_1}\) and \(I^F(S, A) = I^{red}(S) = \{H_{12}, H_{13}, H_{23}\}\). Divisors \(A_2\) and \(A_6\) are two orthogonal lo \(-2\) classes. It follows that one of them is \(H_{123}\) and the other is \(E_i - E_j\). Since \(L_{123}\) is not slo, we have \(A_6 = H_{123}\). Now we have \(1 = A_1A_6 = H_{ij}H_{123} = -1\), a contradiction.

Remark 8.22. The proof means that for any full strong exceptional toric system of the first kind with \(A_n = H - \sum_{i=1}^{m} A_i\), after performing several mutations of toric systems within \(1 \leq i \leq n - 1\), Theorem 1.1 can be applied.

Remark 8.23. We modified the algorithms in [8, Appendix, A.1] to search for all full strong exceptional toric systems of the first kind with \(A_n\) which is naively augmentation from that on \(\mathbb{P}^2\). It turns out that the majority of full strong exceptional toric systems on each weak del Pezzo surface of degree \(\geq 3\) is such. This means that for most full strong exceptional toric system on each weak del Pezzo surface of degree \(\geq 3\), Theorem 1.1 applies.

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