On modeling weakly stationary processes

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Abstract
In this article, we show that general weakly stationary time series can be modeled applying Gaussian subordinated processes. We show that, for any given weakly stationary time series \((z_t)_{t \in \mathbb{N}}\) with given equal one-dimensional marginal distribution, one can always construct a function \(f\) and a Gaussian process \((X_t)_{t \in \mathbb{N}}\) such that \((f(X_t))_{t \in \mathbb{N}}\) has asymptotically the same marginal distributions and the same autocovariance function as \((z_t)_{t \in \mathbb{N}}\). Consequently, we obtain asymptotic distributions for the mean and autocovariance estimators by using the rich theory on limit theorems for Gaussian subordinated processes. This highlights the role of Gaussian subordinated processes in modeling general weakly stationary time series. We compare our approach to standard linear models, and show that our model is more flexible and requires weaker assumptions.

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1 Introduction

Time series models are of great significance in numerous areas of applications, e.g. finance, climatology and signal processing, to name just a few. Central limit theorems play an important role in statistical inference. However, due to dependencies, it is challenging to obtain central limit theorems under general time series models. Moreover, from practical point of view, obtaining central limit theorem is not enough. It is also important to study how fast the convergence takes place, i.e. how far one is from the limiting distribution.
A simple generalisation of the classical central limit theorem is the central limit theorem for $M$-dependent sequence of random variables. That is, the elements in the sequence are independent, if their indices are far away from each other. For general time series with arbitrary dependence structure, the problem becomes more subtle, and it might happen that the limiting distribution is not Gaussian and/or that one has to use different scaling than the classical $\sqrt{T}$, where $T$ is the sample size. Thus, a natural approach to the problem is to study limiting distributions of properly scaled averages of stationary processes with a given autocovariance structure. What happens on the limit is dictated by the dependence structure of the time series. If the dependence is weak enough, then central limit theorem is obtained. See a recent book [28] for a comprehensive introduction to the topic and [18] for functional central limit theorem. Another option is to impose mixing conditions. Limit theorems for strong mixing processes are studied, e.g. in [11, 15, 23]. However, specific mixing conditions are often more than difficult to verify.

If we consider stationary time series models, two general classes, linear processes and Gaussian subordinated processes, are applied extensively in different fields. The class of univariate linear processes consists of stationary processes $(z_t)_{t \in \mathbb{N}}$ of the form

$$z_t = \sum_{j=-\infty}^{\infty} \phi_j \xi_{t-j},$$

where the coefficients $\phi_j$ satisfy some specific assumptions and $(\xi_j)_{j \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables. For example, this class covers stationary ARMA-models with I.I.D. errors. For theory of such processes together with central limit theorems we refer to [9] as well as to more recent articles [19, 29] studying limit theorems of linear processes. Finally, we mention [10], where Berry-Esseen type bounds are derived for linear processes and [20, 21], where estimation of the mean and the autocovariances is studied in the case of long-memory and heavy-tailed linear processes.

The class of univariate Gaussian subordinated processes consists of stationary processes $(z_t)_{t \in \mathbb{N}}$ of the form $z_t = f(X_t)$, where $(X_t)_{t \in \mathbb{N}}$ is a $d$-variate stationary Gaussian process and $f$ is a given function. It is usually assumed that $f(X_0) \in L^2$. Central limit theorems for such time series date back to Breuer and Major [8] and the topic has been studied extensively. Indeed, for Gaussian subordinated processes central and non-central limit theorems have been studied at least in [1, 2, 3, 4, 13, 16]. Motivated by real-life applications, the non-central limit theorems have been studied mostly in the case of long-memory processes. In this case one has to use stronger normalisation, and the limiting distribution is Gaussian only if the so-called Hermite rank of the function $f$ is 1. More generally, in this case, the properly scaled
average of $z_t$ converges towards a Hermite process of order $k$, where $k$ is the Hermite rank of $f$. These central and non-central limit theorems have been considered in statistical applications for long-memory processes at least in [13] (empirical process and U-statistics), [12] (change point tests), [22] (estimation of scale and autocovariances in Gaussian setup), and [17] (Whittle estimator).

In addition to the study of long-memory case and non-central limit theorems, the central limit theorems for Gaussian subordinated stationary processes have emerged again to the center of mathematical community’s interest. The reason behind this is that it has been observed that Stein’s method and Malliavin calculus suit together admirably well — straightforwardly giving new tools to study central limit theorems for Gaussian subordinated processes. For recent developments on the topic, we refer to the articles [24, 25] and to the monograph [26]. Also, a stronger version of the Breuer-Major theorem was proven in [27]. It was proven that, in addition to the convergence in distribution, the convergence towards normal random variable holds even in stronger topologies, such as Kolmogorov or Wasserstein distance. Moreover, the authors also provided Berry-Esseen type bounds in these metrics. Finally, we mention [6], where the result was generalised to cover non-stationary Gaussian fields.

In this article, we consider univariate weakly stationary time series $(z_t)_{t \in \mathbb{N}}$. We study the asymptotic behavior of the traditional mean and autocovariance estimators under the assumption of equal one-dimensional marginal distributions* of $(z_t)_{t \in \mathbb{N}}$. Our main contribution is to show that for any weakly stationary time series $(z_t)_{t \in \mathbb{N}}$ with some given autocovariance structure and with some given equal one-dimensional marginal distributions, one can always construct a univariate Gaussian process $(X_t)_{t \in \mathbb{N}}$ and a function $f$ such that $(f(X_t))_{t \in \mathbb{N}}$ has, asymptotically, the same autocovariance structure and the same one-dimensional marginal distributions as $(z_t)_{t \in \mathbb{N}}$. Relying on that, we complement the above mentioned works on limit theorems in the case of Gaussian subordination. There exists a rich literature on the topic, and we propose to model time series directly with $(f(X_t))_{t \in \mathbb{N}}$. In comparison to the above mentioned literature, where the model is assumed to be $(f(X_t))_{t \in \mathbb{N}}$, we start with a given weakly stationary time series with equal one-dimensional marginals, and we construct a function $f$ and a Gaussian process $(X_t)_{t \in \mathbb{N}}$ such that $(f(X_t))_{t \in \mathbb{N}}$ is a suitable model for $(z_t)_{t \in \mathbb{N}}$. We obtain limiting normal distributions for the traditional mean and autocovariance estimators for any time series within our model that has absolutely summable autocovariance function. In addition, we show that within our model, as desired, the function $f$ does have Hermite rank equal to 1. Indeed, Hermite rank equal to 1 ensures that even in the long-memory case,

*By one-dimensional marginal distributions we refer to the distributions of $z_t$ for fixed time indices $t$.
the limiting distribution is normal. We compare our approach and results to
the existing literature including comparison to the theory of linear processes
that are covered by our model. Note that, our model is not limited to, but
covers e.g. stationary ARMA-models. We observe that the assumptions
that are usually posed in the literature for obtaining limiting normal distri-
bution, are clearly stronger than the assumptions we require. In addition,
our assumption of summable covariance function is rather intuitive, as well
as easily verified, compared to, e.g. complicated assumptions on the coeffi-
cients $\phi_j$ of linear processes. These results highlight the applicability of
Gaussian subordinated processes in modeling weakly stationary time series.

The rest of the article is organised as follows. In section 2 we recall some
basic definitions and the Breuer-Major theorem. Section 3 is devoted to our
main results. We study general weakly stationary processes, and consider
asymptotic behavior of the traditional mean and covariance estimators. In
Section 4 we give some concluding remarks and compare our approach to
the existing literature.

2 Preliminaries

In this section we review some basic definitions and fundamental results that
are later applied in Section 3. We start by recalling the definition of weak
stationarity.

**Definition 2.1.** Let $(z_t)_{t \in \mathbb{N}}$ be a stochastic process. Then $(z_t)_{t \in \mathbb{N}}$ is *weakly stationary* if for all $t, s \in \mathbb{N},$

(i) $E z_t = \mu < \infty,$

(ii) $E z_t^2 = \sigma^2 < \infty,$ and

(iii) $Cov(z_t, z_s) = r(t - s)$ for some function $r.$

We now recall Hermite polynomials and the Hermite ranks of functions.
The Hermite polynomials $H_k$ are defined recursively as follows:

$$H_0(x) = 1, H_1(x) = x, \text{ and } H_{k+1}(x) = xH_k(x) - kH_{k-1}(x).$$

The $k$th Hermite polynomial $H_k$ is clearly a polynomial of degree $k.$ Moreover, it is well-known that Hermite polynomials form an orthogonal basis of the Hilbert space of functions $f$ satisfying

$$\int_{-\infty}^{\infty} [f(x)]^2 e^{-x^2} dx < \infty,$$
or equivalently, $E[f(X)]^2 < \infty$, where $X \sim N(0, 1)$. Every $f$ that belongs to that Hilbert space has a Hermite decomposition

$$f(x) = \sum_{k=0}^{\infty} \alpha_k H_k(x),$$

and for $X \sim N(0, 1)$, $Y \sim N(0, 1)$, we have that

$$E[f(X)f(Y)] = \sum_{k=0}^{\infty} k! \alpha_k^2 [Cov(X,Y)]^k.$$

**Definition 2.2 (Hermite rank).** Let $(X_t)_{t \in \mathbb{N}}$, $X_t = (X_t^{(1)}, X_t^{(2)}, ..., X_t^{(d)})$, be a $d$-dimensional stationary Gaussian process. Let $f : \mathbb{R}^d \to \mathbb{R}$, $f(X_t) \in L^2$. The function $f$ is said to have Hermite rank $q \geq 1$ with respect to $X_t$, if

$$E[(f(X_t) - E[f(X_t)])^{p_m}(X_t)] = 0 \text{ for all polynomials } p_m : \mathbb{R}^d \to \mathbb{R} \text{ that are of degree } m \leq q - 1, \text{ and if there exists a polynomial } p_q \text{ of degree } q \text{ such that } E[(f(X_t) - E[f(X_t)])p_q(X_t)] \neq 0.$$

**Remark 2.1.** Note that the Hermite rank of a function $f$ is the smallest number $q \geq 1$ such that $\alpha_q \neq 0$ in decomposition (2.1).

We next recall the Breuer-Major theorem [8].

**Theorem 2.1.** [8, Theorem 1] Let $(X_t)_{t \in \mathbb{N}}$, $X_t = (X_t^{(1)}, X_t^{(2)}, ..., X_t^{(d)})$, be a $d$-dimensional stationary Gaussian process. Assume that $f : \mathbb{R}^d \to \mathbb{R}$, $f(X_t) \in L^2$, has a Hermite rank $q \geq 1$. Denote

$$r_{X}^{k,i}(\tau) = E[X_{\tau}^{(k)}X_{0}^{(i)}].$$

If

$$\sum_{i=0}^{\infty} |r_{X}^{k,i}(\tau)|^q < \infty, \quad \forall k, i = 1, 2, \ldots, d,$$

then $\sigma^2 = Var[f(X_0)] + 2 \sum_{t=1}^{\infty} Cov[f(X_0), f(X_t)]$ is well-defined and

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} [f(X_t) - E[f(X_t)]] \xrightarrow{d} N(0, \sigma^2),$$

as $T \to \infty$.

A stronger version of Theorem 2.1 was proven in a recent article [27]. It was shown that the convergence holds even in stronger topologies than the convergence in distribution, e.g. the convergence holds in Wasserstein distance and in Kolmogorov distance. Furthermore, applying Theorem 2.1 of [27], it is possible to study the rate of convergence. Obviously, one could...
apply these results in our setting as well, but for a general function $f$, the bounds are rather complicated. Thus the rate of convergence should be studied case by case. However, for an interested reader, we refer to [27].

It is known that also a functional version of Theorem 2.1 holds, i.e.

$$
\frac{1}{\sqrt{T}} \sum_{t=1}^{[nT]} [f(X_t) - \mathbb{E}f(X_t)]
$$

converges weakly towards $\sigma$ times a Brownian motion in the Skorokhod space. Theorem 2.1 is refined also for long-range dependent sequences, where the summability condition does not hold. One then has to use other normalizations than the classical $T^{-\frac{1}{2}}$. Furthermore, the limiting process in the functional version is the so-called $q$th Hermite process. For more details, see [4] and the references therein.

3 Modeling general weakly stationary time series

Let $(z_t)_{t \in \mathbb{N}}$ be a given weakly stationary univariate time series with an expected value $\mu = \mathbb{E}[z_t]$ and a given autocovariance function $r_z(\tau) = \mathbb{E}[z_t z_0] - \mu^2$. Assume that the one-dimensional marginals of $(z_t)_{t \in \mathbb{N}}$ are all equal. By equal one-dimensional marginal distributions we mean that the distribution of $z_t$ is the same for all time indices $t$. The corresponding one-dimensional variable is denoted by $z$, and its cumulative distribution function is denoted by $F_z$. We are interested in the mean and the autocovariance estimators given by

$$m_z = \frac{1}{T} \sum_{t=1}^{T} z_t,$$

and

$$\hat{r}_z(\tau) = \frac{1}{T} \sum_{t=1}^{T-\tau} \left[ z_t - m_z \right] \left[ z_{t+\tau} - m_z \right].$$

For simplicity, we divide by $T$ instead of $T - \tau$. Consequently, the estimators $\hat{r}_z(\tau)$ are only asymptotically consistent. On the other hand, in this case, the sample autocovariance function preserves the desired property of positive semidefiniteness. Obviously the asymptotic behaviour of $\hat{r}_z(\tau)$ is the same as for

$$\tilde{r}_z(\tau) = \frac{1}{T - \tau} \sum_{t=1}^{T-\tau} \left[ z_t - m_z \right] \left[ z_{t+\tau} - m_z \right].$$

Finally, for the case $\mu = 0$, a simpler version

$$\tau_z(\tau) = \frac{1}{T} \sum_{t=1}^{T-\tau} z_t z_{t+\tau}$$
is often used.  

The use of $r_z(\tau)$ is justified by the following simple lemma which states that asymptotically the difference between $\hat{r}_z(\tau)$ and $r_z(\tau)$ is negligible.

**Lemma 3.1.** Assume that $m_z = m_z(T) \to \mu$ in probability, as $T \to \infty$. Then

$$\hat{r}_z(\tau) = r_z(\tau) - [m_z(T)]^2 + O_p(T^{-1}).$$

**Proof.** We have that

$$\hat{r}_z(\tau) = \frac{1}{T} \sum_{t=1}^{T-\tau} z_t z_{t+\tau} - m_z \frac{1}{T} \sum_{t=1}^{T-\tau} z_t - m_z \frac{1}{T} \sum_{t=1}^{T-\tau} z_{t+\tau} + m_z^2 \frac{T-\tau}{T},$$

$$= r_z(\tau) - m_z^2 + R_T,$$

where

$$R_T = m_z \frac{T}{T} \sum_{t=T-\tau+1}^{T} z_t + m_z \frac{T}{T} \sum_{t=1}^{T} z_t - \frac{T}{T} m_z^2$$

for $\tau \geq 1$ and $R_T = 0$ for $\tau = 0$. Now, since $(z_t)_{t \in \mathbb{N}}$ has finite second moments, the both sums in $R_T$ are bounded in probability. Similarly, the last term is $O_p(T^{-1})$, as $T \to \infty$. 

We next present the main results of this paper.

**Theorem 3.1.** Let $(z_t)_{t \in \mathbb{N}}$ be a weakly stationary process with an autocovariance function $r_z(\tau) \geq 0$. Assume that $(z_t)_{t \in \mathbb{N}}$ has equal one-dimensional distribution functions $F_z$. Then there exists a function $f$ and a stationary Gaussian process $(X_t)_{t \in \mathbb{N}}$ with autocovariance function $r_X(\tau)$ such that $f(X_t)$ has the same autocovariance function and the same one-dimensional marginal distributions as the process $(z_t)_{t \in \mathbb{N}}$.

**Proof.** Without loss of generality, we may and will assume that $Ez_t = 0$ and $Ez_t^2 = 1$. Let $\Phi$ denote the distribution function of the standard normal distribution. Then, from the standard inverse method, it is clear that $X = \Phi^{-1}(F_z(z))$ has standardized normal distribution. Denote

$$F_z^{-1}(y) = \inf_x \{F_z(x) \geq y\}.$$ 

Now $F_z^{-1}(\Phi(X))$ is distributed as $z$, and hence we may set $f(\cdot) = F_z^{-1}(\Phi(\cdot))$. Furthermore, since $z \in L^2$, we also have $f(X) \in L^2$. Thus $f$ has a Hermite decomposition

$$f(x) = \sum_{j=0}^{\infty} \alpha_j H_j(x).$$
Since $Ez_k = 0$, we also set $Ef(X_\tau) = \alpha_0 = 0$. Hence

$$E[f(X_0)f(X_\tau)] = \sum_{j=1}^{\infty} j! \alpha_j^2 [r_X(\tau)]^j.$$  

For $\beta \in [-1, 1]$, define a function $g(\beta) = \sum_{j=1}^{\infty} j! \alpha_j^2 \beta^j$ and set $\gamma = \min_{\beta \in [-1,1]} g(\beta)$.

As

$$Ef(X_0)^2 = \sum_{j=1}^{\infty} j! \alpha_j^2 = 1,$$

we observe that $g$ is continuous. Hence, by intermediate value theorem, for any number $r_z(\tau) \in [\gamma, 1]$ there exists $\beta_\tau \in [-1,1]$ such that $g(\beta_\tau) = r_z(\tau)$.

Thus it suffices to choose a Gaussian process $X$ with covariance function

$$r_X(\tau) = \beta_\tau.$$

**Remark 3.1.** We emphasize that we are only claiming that the one-dimensional distributions of $(F^{-1}((\Phi(X_t))))_{t \in \mathbb{N}}$ are equal to the one-dimensional distributions of $(z_t)_{t \in \mathbb{N}}$. (The multidimensional distributions are not necessarily the same.) We also stress that we do not use the assumption of non-negativity of $r_z(\tau)$ in the proof of the theorem. In fact, we proved that $(F^{-1}((\Phi(X_t))))_{t \in \mathbb{N}}$ can have the same autocovariance function that $(z_t)_{t \in \mathbb{N}}$ provided that $\gamma \leq r_z(\tau)$ for all $\tau \in \mathbb{N}$. This is clearly the case if $r_z(\tau) \geq 0$. If $r_z(\tau) < 0$, it might happen that $r_z(\tau) < \gamma$. However, asymptotically, we can always find the corresponding $r_x(\tau)$ provided that $r_z(\tau) \to 0$.

Note that the construction in the proof of Theorem 3.1 is not unique. Indeed, this follows from the fact that $r_z(\tau)$ is a polynomial (possibly of infinite order) of $r_X(\tau)$. Thus one has to construct the underlying Gaussian process carefully. We address this issue in the following theorem.

**Theorem 3.2.** Let $(z_t)_{t \in \mathbb{N}}$ be a weakly stationary process with equal one-dimensional marginals and an autocovariance function $r_z(\tau)$. Let $f$ be a function and let $(X_t)_{t \in \mathbb{N}}$ be a stationary Gaussian process with autocovariance function $r_X(\tau)$ such that $(f(X_t))_{t \in \mathbb{N}}$ has the same autocovariance function and the same one-dimensional marginals as $(z_t)_{t \in \mathbb{N}}$. Then the Gaussian process $(X_t)_{t \in \mathbb{N}}$ can be chosen in a way that there exist constants $c, C$, independent of $\tau$, such that

$$c |r_X(\tau)|^q \leq |r_z(\tau)| \leq C |r_X(\tau)|^q, \quad \forall \tau \in \mathbb{N},$$  

where $q$ is the Hermite rank of $f$.  

8
Before proving Theorem 3.2, we remark that given the asymptotics of the autocovariance \(r_X(\tau)\), the term \(|r_X(\tau)|^q\) determines the asymptotics of the autocovariance of \((f(X_t))_{t \in \mathbb{N}}\) [7 p. 223]. Here our aim is to do the opposite; given the autocovariance \(r_z(\tau)\) we construct \((X_t)_{t \in \mathbb{N}}\) such that \((f(X_t))_{t \in \mathbb{N}}\) has the autocovariance function \(r_z(\tau)\) and the inequality (3.1) holds.

Proof. Using the notation of the proof of Theorem 3.1, we have

\[
r_z(\tau) = \sum_{k=q}^{\infty} k! \alpha_k^2 r_X(\tau)^k.
\]

This implies

\[
|r_z(\tau)| \leq |r_X(\tau)|^q \sum_{k=q}^{\infty} k! \alpha_k^2 = r_z(0)|r_X(\tau)|^q,
\]

proving the right-hand side of (3.1). For the left-hand side, first note that for \(r_z(\tau) \geq 0\) we can always choose \(r_X(\tau) \in [0, 1]\). In this case

\[
|r_X(\tau)|^q \leq C \sum_{k=q}^{\infty} k! \alpha_k^2 r_X(\tau)^k = Cr_z(\tau).
\]

Let now \(r_z(\tau) < 0\). Then also \(r_X(\tau) < 0\), and hence

\[
r_z(\tau) = \sum_{k \text{ even}} k! \alpha_k^2 |r_X(\tau)|^k - \sum_{k \text{ odd}} k! \alpha_k^2 |r_X(\tau)|^k.
\]

Since \(r_z(\tau) < 0\), we have

\[
|r_z(\tau)| = \sum_{k \text{ odd}} k! \alpha_k^2 |r_X(\tau)|^k - \sum_{k \text{ even}} k! \alpha_k^2 |r_X(\tau)|^k. \tag{3.2}
\]

If \(|r_z(\tau)| \geq \epsilon\) for some \(\epsilon > 0\), then

\[
|r_X(\tau)|^q \leq \frac{|r_X(\tau)|^q}{|r_z(\tau)|} |r_z(\tau)| \leq \epsilon^{-1} |r_z(\tau)|.
\]

Let now \(r_z(\tau) < 0\) be very small. Then also \(r_X(\tau)\) can be assumed to be very small. Indeed, by continuity, we can always assume that \(r_X(\tau) \to 0\) whenever \(r_z(\tau) \to 0\).

Assume first that \(q\) is odd. Then the claim is equivalent to

\[
|r_X(\tau)|^q + C \sum_{k \text{ even}} k! \alpha_k^2 |r_X(\tau)|^k \leq C \sum_{k \text{ odd}} k! \alpha_k^2 |r_X(\tau)|^k
\]

for some \(C > 0\). Dividing with \(|r_X(\tau)|^q\) and rearranging terms yields

\[
C \sum_{k \text{ even}} k! \alpha_k^2 |r_X(\tau)|^{k-q} \leq C - 1 + \sum_{k > q \text{ odd}} k! \alpha_k^2 |r_X(\tau)|^{k-q}.
\]
As all the terms on the left side converge to zero as \( r_X(\tau) \to 0 \), the inequality follows trivially for large enough \( C > 0 \).

Assume then that \( q \) is even. Then it follows from (3.2) that

\[
\frac{|r_z(\tau)|}{|r_X(\tau)|^q} = \sum_{k \text{ odd}} k!\alpha_k^2 |r_X(\tau)|^{k-q} - \sum_{k \text{ even}} k!\alpha_k^2 |r_X(\tau)|^{k-q}.
\]

As here the first term is small for \( r_X(\tau) \) small while the second one is close to a negative constant, it follows that for \( q \) even there do not exist indices \( \tau \) such that \( r_z(\tau) \) is arbitrarily close to zero. This finishes the proof. \( \square \)

**Remark 3.2.** In the above proof, we constructed the function \( r_X(\tau) \) such that (3.1) holds and \( f(X_t) \) has the same autocovariance and the same marginals as \((z_t)_{t \in \mathbb{N}}\). In general it is not clear whether the function \( r_X(\tau) \) is an autocovariance function. However, it is clear that there exists a Gaussian process such that the autocovariance \( r_X(\tau) \) decays asymptotically as \( r_z(\tau)^{\frac{1}{q}} \). Moreover, for any \( |r_z(\tau)| \geq \epsilon \), the inequality (3.1) is trivially true.

**Remark 3.3.** We remark that if one constructs the Gaussian process \((X_t)_{t \in \mathbb{N}}\) such that (3.1) holds, then uncorrelatedness of two values \( z_t \) and \( z_{t+\tau} \) implies that \( r_X(\tau) = 0 \). As \( X \) is Gaussian, it follows that \( X_t \) and \( X_{t+\tau} \) are independent. This in turn forces \( z_t \) and \( z_{t+\tau} \) to be independent. However, this clearly is not a problem if uncorrelatedness holds only for finite number of indices. And asymptotically, we focus on time series \((z_t)_{t \in \mathbb{N}}\) where \( r_z(\tau) \) does not vanish. If \( r_z(\tau) = 0 \) for all large enough \( \tau \), then one can apply central limit theorems for \( M \)-dependent sequences.

Theorems 3.1 and 3.2 justify that for weakly stationary processes \((z_t)_{t \in \mathbb{N}}\) it is reasonable to assume that

\[
z_t = f(X_t), \tag{3.3}
\]

where \((X_t)_{t \in \mathbb{N}}\) is a stationary Gaussian sequence, and where the inequality (3.1) holds. In this case the following Corollary on the central limit theorem for the mean estimator follows directly from Theorem 2.1.

**Corollary 3.1.** Let \((z_t)_{t \in \mathbb{N}} = \) be given by (3.3). Assume that the inequality (3.1) holds and that

\[
\sum_{t=0}^{\infty} |r_z(\tau)| < \infty.
\]

Then

\[
\sqrt{T} [m_z - \mu] \to N(0, \sigma^2),
\]

where \( \sigma^2 = \mathbb{E} z_0^2 + 2 \sum_{t=1}^{\infty} r_z(t) \).
By the above result an absolutely summable covariance function is sufficient in order to obtain asymptotic normality for the mean estimator whatever the Hermite rank of $f$ is. However, in order to obtain asymptotic normality for covariance estimators one needs to study Hermite rank of $g(X_t, X_{t+\tau}) = f(X_t)f(X_{t+\tau})$. Unfortunately, in general the Hermite rank of $f(X_t)f(X_{t+\tau})$ might be larger or smaller than the Hermite rank of $f$.

Example 3.1. Let $f(x) = x$. Then $f$ has Hermite rank 1, while $[f(x)]^2 = x^2$ has Hermite rank 2.

Example 3.2. Let $f(x) = H_2(x)$. Then $f$ has Hermite rank 2 as well as $[f(x)]^2 = x^4 - 2x^2 + 1$.

Example 3.3. Let $f(x) = H_3(x) + H_2(x)$. Then $f$ has Hermite rank 2, while $[f(x)]^2 = x^6 + 2x^5 - 5x^4 - 8x^3 + 7x^2 + 6x + 1$ has Hermite rank 1.

Hermite rank 1 makes the mean and the autocovariance estimators stable, and one usually obtains Gaussian limits with suitable normalisations. (See [4] for detailed discussion on the stability in the case of Hermite rank 1.) Furthermore, in our model an absolutely summable autocovariance function of $(z_t)_{t \in \mathbb{N}}$ is sufficient for obtaining asymptotic normality of the mean and the covariance estimators, provided that $f$ has Hermite rank 1. Luckily, in our model we can always assume that this is the case.

Proposition 3.1. Let $F$ be an arbitrary distribution function. Then

$$f(\cdot) = F^{-1}(\Phi(\cdot))$$

has Hermite rank 1.

Proof. In order to prove the claim we have to show that

$$\mathbb{E}[f(X)X] \neq 0$$

for $X \sim \mathcal{N}(0, 1)$. We have

$$\int_{-\infty}^{\infty} F^{-1}(\Phi(x)) x e^{-\frac{x^2}{2}} \, dx$$

$$= \int_{-\infty}^{0} F^{-1}(\Phi(x)) x e^{-\frac{x^2}{2}} \, dx + \int_{0}^{\infty} F^{-1}(\Phi(x)) x e^{-\frac{x^2}{2}} \, dx$$

$$= -\int_{0}^{\infty} F^{-1}(\Phi(-x)) x e^{-\frac{x^2}{2}} \, dx + \int_{0}^{\infty} F^{-1}(\Phi(x)) x e^{-\frac{x^2}{2}} \, dx$$

$$= \int_{0}^{\infty} [F^{-1}(\Phi(x)) - F^{-1}(\Phi(-x))] x e^{-\frac{x^2}{2}} \, dx.$$
for all $x \geq 0$. Furthermore, the inequality is strict for large enough $x$, giving
\[ \mathbb{E}[f(X)X] = \int_0^\infty \left[ F^{-1}(\Phi(x)) - F^{-1}(\Phi(-x)) \right] xe^{-\frac{x^2}{2}} \, dx > 0. \]

Remark 3.4. We stress that while $z = F^{-1}(\Phi(X))$ has distribution $F$, in general it is not true that $F^{-1}(\Phi(X))X$ is distributed as $zX$. For example, if $z = g(X)$ with suitable $g$ the distribution of $g(X)X$ is not the same as the distribution of $F^{-1}(\Phi(X))X$. A simple example of such case is $\chi^2(1)$ distribution, where $g(x) = x^2$ but $F^{-1}(\Phi(x))^2 \neq x^2$. Clearly, $g(X)$ has Hermite rank 2 while $F^{-1}(\Phi(x))$ has Hermite rank 1. This fact highlights our proposal to model $z$ with $z = F^{-1}(\Phi(X))$ directly. It is also worth to note that if $g(x)$ is bijective, then the distributions of $F^{-1}(\Phi(X))X$ and $g(X)X$ are equal.

We next consider asymptotic behaviour of autocovariance estimators. For that we naturally need the existence of finite fourth moments.

Theorem 3.3. Let $(z_t)_{t \in \mathbb{N}}$ be given by
\[ z_t = F^{-1}_z(\Phi(X_t)) \]
Assume that the inequality (3.1) holds, $\mathbb{E}z_t^4 = c < \infty$, and assume that
\[ \sum_{\tau=1}^\infty |r_z(\tau)| < \infty. \tag{3.4} \]
Then
\[ \sqrt{T}[m_z - \mu] \to N(0, \sigma^2) \tag{3.5} \]
with $\sigma^2 = \text{Var}(z_0) + 2 \sum_{\tau=1}^\infty r_z(\tau)$, and for any $k \geq 0$
\[ \sqrt{T}[\hat{r}_z(0) - r_z(0), \hat{r}_z(1) - r_z(1), \ldots, \hat{r}_z(k) - r_z(k)] \to N(0, \Sigma), \tag{3.6} \]
where $\Sigma = (\Sigma_{ij})$, $i, j = 0, 1, \ldots, k$ is given by
\[ (\Sigma)_{ij} = \text{Cov}(z_0 z_i, z_0 z_j) + 2 \sum_{\tau=1}^\infty \text{Cov}(z_\tau z_{i+\tau}, z_0 z_j). \]

Proof. The convergence (3.5) is in fact the statement of Corollary 3.1. For the convergence (3.6), first note that without loss of generality and for the sake of simplicity, we may and will assume that $\mu = 0$ and use the estimators $\hat{r}_z(\tau)$ instead. Indeed, the general case then follows easily from (3.5), Lemma
and the Slutsky’s theorem. In order to prove (3.6) we have to show that, for any \( n \geq 1 \) and any \((\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n\), the linear combination

\[
\sqrt{T} \sum_{k=0}^{n} \alpha_k \left[ r_z(k) - r_z(k) \right],
\]

converges towards a Gaussian random variable. We define an \( n+1 \)-dimensional stationary Gaussian process \( \mathbf{X}_t = (X_t, X_{t+1}, \ldots, X_{t+n}) \) and a function

\[
F(\mathbf{X}_t) = \sum_{k=0}^{n} \alpha_k \left[ f(X_t)f(X_{t+k}) - r_z(k) \right],
\]

where \( f(\cdot) = F_z^{-1}(\Phi(\cdot)) \). With this notation we have

\[
\sqrt{T} \sum_{k=0}^{n} \alpha_k \left[ r_z(k) - r_z(k) \right] = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F(\mathbf{X}_t) + R(T),
\]

where

\[
R(T) = \frac{1}{\sqrt{T}} \sum_{k=0}^{n} \alpha_k \sum_{t=T-k+1}^{T} [z_t z_{t+k} - r_z(k)].
\]

Since \( \mathbb{E}z_t^4 = c < \infty \), it follows from Cauchy-Schwarz inequality that \( F(\mathbf{X}) \in L^2 \). Thus assumption (3.4) together with Theorem 2.1 implies that

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} F(\mathbf{X}_t) \to N(0, \sigma^2).
\]

For the term \( R(T) \), we observe that the sum

\[
\sum_{k=0}^{n} \alpha_k \sum_{t=T-k+1}^{T} [z_t z_{t+k} - r_z(k)]
\]

is bounded in \( L^2 \), and hence \( R(T) \to 0 \) in probability. Thus the convergence of any linear combination of the form (3.7) towards a normal random variable follows directly from Slutsky’s theorem. Finally, the covariance matrix \( \Sigma \) is derived by considering convergence of

\[
\sqrt{T} [\hat{r}_z(i) - r_z(i) + \hat{r}_z(j) - r_z(j)]
\]

together with Theorem 2.1 and by direct computations. \( \square \)
4 Discussion

In this article, we argued why it is advantageous to model weakly stationary time series using Gaussian subordinated processes. Under this construction, we are able to provide central limit theorems for the standard mean and autocovariance estimators. Furthermore, even functional versions of the central limit theorems and Berry-Esseen type bounds in different metrics are available. In our modeling approach \((z_t)_{t \in \mathbb{N}} = (f(X_t))_{t \in \mathbb{N}}\), the Hermite rank of the function \(f\) is equal to 1. This is especially useful in the case of long memory processes as the limiting distribution is normal if and only if the Hermite rank of \(f\) is equal to 1. In general, one needs to be more cautious with the estimators, when analysing long memory processes, as the autocovariance function is not absolutely summable. As a simple example, let \((X_t)_{t \in \mathbb{N}}\) be a centered Gaussian process with \(r_X(\tau) \sim \tau^{2H-2}\), where \(H \in (\frac{1}{2}, \frac{3}{4})\). Then

\[
E \left[ \sqrt{T} m_X \right]^2 \sim \sum_{k=1}^{T} j^{2H-2} \sim T^{2H-1} \rightarrow \infty.
\]

On the other hand, the Hermite rank of \(X_tX_{t+\tau}\) is 2 implying asymptotic normality of the autocovariance estimators \(\hat{r}_z(\tau)\) with rate \(\sqrt{T}\).

We end this paper by comparing our approach to the existing literature. Linear processes of the form

\[
z_t = \sum_{j=0}^{\infty} \phi_j \xi_{t-j},
\]

where \((\xi_t)_{t \in \mathbb{Z}}\) is an independent and identically distributed sequence, are widely applied models for stationary time series. To obtain central limit theorems for the mean and the autocovariance estimators, conditions on the coefficients \((\phi_j)_{j \in \mathbb{Z}}\) are required. A sufficient condition for obtaining central limit theorems is

\[
\sum_{j=0}^{\infty} |\phi_j| < \infty
\]

(4.1) together with \(E|\xi_t|^4 < \infty\) (see Theorem 7.1.2. and Theorem 7.2.1. in [9]). As the sequence \((\xi_t)_{t \in \mathbb{Z}}\) is independent and identically distributed, it follows that the one-dimensional marginals of the process are equal. Hence linear processes are covered by our modeling approach. Moreover, it is easy to see that \(E|\xi_t|^4 < \infty\) implies \(E|z_t|^4 < \infty\), and \((4.1)\) is strictly stronger than the assumption of absolutely summable autocovariance function. Thus our modeling approach is more flexible as it requires weaker assumptions.
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