WDVV Equations and Seiberg-Witten theory

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We present a review of the results on the associativity algebras and WDVV equations associated with the Seiberg-Witten solutions of $N = 2$ SUSY gauge theories. It is mostly based on the integrable treatment of these solutions. We consider various examples of the Seiberg-Witten solutions and corresponding integrable systems and discuss when the WDVV equations hold. We also discuss a covariance of the general WDVV equations.

1 What is WDVV

More than two years ago N. Seiberg and E. Witten \[1\] proposed a new way to deal with the low-energy effective actions of $N = 2$ four-dimensional supersymmetric gauge theories, both pure gauge theories (i.e. containing only vector supermultiplet) and those with matter hypermultiplets. Among other things, they have shown that the low-energy effective actions (the end-points of the renormalization group flows) fit into universality classes depending on the vacuum of the theory. If the moduli space of these vacua is a finite-dimensional variety, the effective actions can be essentially described in terms of a system with finite-dimensional phase space (\# of degrees of freedom is equal to the rank of the gauge group), although the original theory lives in a many-dimensional space-time. These effective theories turn out to be integrable. Integrable structures behind the Seiberg-Witten (SW) approach has been found in \[2\] and later examined in detail for different theories in \[3\]–\[13\].

The second important property of the SW framework which merits the adjective ”topological” has been more recently revealed in the series of papers \[14\]–\[19\] and has much to do with the associative algebras. Namely, it turns out that the prepotential of SW theory satisfies a set of Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. These equations have been originally presented in \[20\] (in a different form, see below)

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i$$

where $F_i$’s are matrices with the matrix elements that are the third derivatives of the unique function $F$ of many variables $a_i$’s (prepotential in the SW theory) parameterizing a moduli space:

$$\left( F_i \right)_{jk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k}, \quad i, j, k = 1, ..., n$$

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Although generally there is a lot of solutions to the matrix equations (1), it is extremely non-trivial task to express all the matrix elements through the only function $F$. In fact, there have been only known the two different classes of the non-trivial solutions to the WDVV equations, both being intimately related to the two-dimensional topological theories of type A (quantum cohomologies [21]) and of type B ($N = 2$ SUSY Landau-Ginzburg (LG) theories that were investigated in a variety of papers, see, for example, [22] and references therein). Thus, the existence of a new class of solutions connected with the four-dimensional theories looks quite striking. It is worth noting that both the two-dimensional topological theories and the SW theories reveal the integrability structures related to the WDVV equations. Namely, the function $F$ plays the role of the (quasiclassical) $\tau$-function of some Whitham type hierarchy [22, 2, 23].

In this brief review, we will describe the results of papers [14]-[19] that deal with the structure and origin of the WDVV equations in the SW theories and, to some extent, with their general properties. To give some insight of the general structure of the WDVV equations, let us consider the simplest non-trivial examples of $n = 3$ WDVV equations in topological theories. The first example is the $N = 2$ SUSY LG theory with the superpotential $W'(\lambda) = \lambda^3 - q$ [22]. In this case, the prepotential reads as

$$F = \frac{1}{2}a_1a_2^2 + \frac{1}{2}a_1^3a_3 + \frac{q}{2}a_2a_3^2$$  \hspace{1cm} (3)

and the matrices $F_i$ (the third derivatives of the prepotential) are

$$F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & q \\ 0 & q & 0 \end{pmatrix}.  \hspace{1cm} (4)$$

One can easily check that these matrices do really satisfy the WDVV equations (1).

The second example is the quantum cohomologies of $\mathbb{C}P^2$. In this case, the prepotential is given by the formula [21]

$$F = \frac{1}{2}a_1a_2^2 + \frac{1}{2}a_1^3a_3 + \sum_{k=1}^{\infty} \frac{N_ka_3^{3k-1}}{(3k-1)!}e^{ka_2}$$  \hspace{1cm} (5)

where the coefficients $N_k$ (describing the rational Gromov-Witten classes) counts the number of the rational curves in $\mathbb{C}P^2$ and are to be calculated. Since the matrices $F$ have the form

$$F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & F_{222} & F_{223} \\ 0 & F_{223} & F_{233} \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_{222} & F_{233} \\ 0 & F_{233} & F_{333} \end{pmatrix}  \hspace{1cm} (6)$$

the WDVV equations are equivalent to the identity

$$F_{333} = F_{222}^2 - F_{222}F_{233}$$  \hspace{1cm} (7)

which, in turn, results into the recurrent relation defining the coefficients $N_k$:

$$\frac{N_k}{(3k-4)!} = \sum_{a+b=k} \frac{a^2b(3b-1)b(2a-b)}{(3a-1)!(3b-1)!} N_a N_b.$$  \hspace{1cm} (8)

\textsuperscript{1}We tried to make this review self-consistent. Some related points can be found in other talks presented at the Workshop, in particular, delivered by A.Marshakov and A.Morozov.
The crucial feature of the presented examples is that, in both cases, there exists a constant matrix $F$. Following [22], one can consider it as a flat metric on the moduli space. In fact, in its original version, the WDVV equations have been written in a slightly different form, that is, as the associativity condition of some algebra. We will discuss this later, and now just remark that, having distinguished (constant) metric $\eta \equiv F_1$, one can naturally rewrite (1) as the equations

$$C_iC_j = C_jC_i$$

(9)

for the matrices $C_i \equiv \eta^{-1}F_i$, i.e. $(C_i)^2_{lk} = \eta_{ij}F_{ik}$. Formula (9) is equivalent to (1) with $j = 1$. Moreover, this particular relation is already sufficient [15, 16] to reproduce the whole set of the WDVV equations (1). Indeed, since $F_i = F_1C_i$, we obtain

$$F_iF_j^{-1}F_k = F_1(C_iC_k^{-1}C_j)$$

(10)

which is obviously symmetric under the permutation $i \leftrightarrow j$. Let us also note that, although the WDVV equations can be fulfilled only for some specific choices of the coordinates $a_i$ on the moduli space, they still admit any linear transformation. This defines the flat structures on the moduli space, and we often call $a_i$ flat coordinates.

In fact, the existence of the flat metric is not necessary for (1) to be true, how we explain below. Moreover, the SW theories give exactly an example of such a case, where there is no distinguished constant matrix. This matrix can be found in topological theories because of existence their field theory interpretation where the unity operator is always presented.

## 2 Perturbative SW prepotentials

Before going into the discussion of the WDVV equations for the complete SW prepotentials, let us note that the leading perturbative part of them should satisfy the equations (1) by itself (since the classical quadratic piece does not contribute into the third derivatives). In each case it can be checked by the straightforward calculation. On the other hand, if the WDVV equations are fulfilled for perturbative prepotential, it is a necessary condition for them to hold for complete prepotential.

The perturbative prepotential can be obtained from the one-loop field theory calculations. To this end, let us note that there are two origins of masses in $N = 2$ SUSY YM (SYM) models: first, they can be generated by vacuum values of the scalar $\phi$ from the gauge supermultiplet. For a supermultiplet in representation $R$ of the gauge group $G$ this contribution to the prepotential is given by the analog of the Coleman-Weinberg formula (from now on, we omit the classical part of the prepotential from all expressions):

$$F_R = \pm \frac{1}{4} \text{Tr}_R \phi^2 \log \phi,$$

and the sign is “+” for vector supermultiplets (normally they are in the adjoint representation) and “-” for matter hypermultiplets. Second, there are bare masses $m_R$ which should be added to $\phi$ in (11). As a result, the general expression for the perturbative prepotential is

$$F = \frac{1}{4} \sum_{\text{vector multiplets}} \text{Tr}_A(\phi + M_nI_A)^2 \log(\phi + M_nI_A) -$$

$$- \frac{1}{4} \sum_{\text{hyper multiplets}} \text{Tr}_R(\phi + m_RI_R)^2 \log(\phi + m_RI_R) + f(m)$$

(12)
where the term $f(m)$ depending only on masses is not fixed by the (perturbative) field theory but can be read off from the non-perturbative description, and $I_R$ denotes the unit matrix in the representation $R$.

As a concrete example, let us consider the $SU(n)$ gauge group. Then, say, perturbative prepotential for the pure gauge theory acquires the form

$$F_{\text{pert}}^V = \frac{1}{4} \sum_{ij} (a_i - a_j)^2 \log (a_i - a_j)$$

This formula establishes that when v.e.v.'s of the scalar fields in the gauge supermultiplet are non-vanishing (perturbatively $a_r$ are eigenvalues of the vacuum expectation matrix $\langle \phi \rangle$), the fields in the gauge multiplet acquire masses $m_{rr'} = a_r - a_{r'}$ (the pair of indices $(r, r')$ label a field in the adjoint representation of $G$). In the $SU(n)$ case, the eigenvalues are subject to the condition $\sum_i a_i = 0$. Analogous formula for the adjoint matter contribution to the prepotential is

$$F_{\text{pert}}^A = -\frac{1}{4} \sum_{ij} (a_i - a_j + M)^2 \log (a_i - a_j + M)$$

while the contribution of the fundamental matter reads as

$$F_{\text{pert}}^F = -\frac{1}{4} \sum_i (a_i + m)^2 \log (a_i + m)$$

Similar formulas can be obtained for the other groups. The eigenvalues of $\langle \phi \rangle$ in the first fundamental representation of the classical series of the Lie groups are

$$B_n (SO(2n+1)) : \{a_1, ..., a_n, 0, -a_1, ..., -a_n\};$$

$$C_n (Sp(n)) : \{a_1, ..., a_n, -a_1, ..., -a_n\};$$

$$D_n (SO(2n)) : \{a_1, ..., a_n, -a_1, ..., -a_n\}$$

while the eigenvalues in the adjoint representation have the form

$$B_n : \{\pm a_j; \pm a_j \pm a_k\}; j < k \leq n$$

$$C_n : \{\pm 2a_j; \pm a_j \pm a_k\}; j < k \leq n$$

$$D_n : \{\pm a_j \pm a_k\}, j < k \leq n$$

Analogous formulas can be written for the exceptional groups too. The prepotential in the pure gauge theory can be read off from the formula (17) and has the form

$$B_n : \quad F_0 = \frac{1}{4} \sum_{i,i}(a_i - a_j)^2 \log (a_i - a_j) + (a_i + a_j)^2 \log (a_i + a_j) + \frac{1}{2} \sum_i a_i^2 \log a_i;$$

$$C_n : \quad F_0 = \frac{1}{4} \sum_{i,i}(a_i - a_j)^2 \log (a_i - a_j) + (a_i + a_j)^2 \log (a_i + a_j) + 2 \sum_i a_i^2 \log a_i;$$

$$D_n : \quad F_0 = \frac{1}{4} \sum_{i,i}(a_i - a_j)^2 \log (a_i - a_j) + (a_i + a_j)^2 \log (a_i + a_j)$$

The perturbative prepotentials are discussed in detail in [15]. In that paper is also contained the proof of the WDVV equations for these prepotentials. Here we just list some results of [15] adding some new statements recently obtained.
i) The WDVV equations always hold for the pure gauge theories $F^{\text{pert}} = F^{\text{pert}}_V$ (including the exceptional groups)\footnote{The rank of the group should be bigger than 2 for the WDVV equations not to be empty, thus for example in the pure gauge $G_2$-model they are satisfied trivially. It have been checked in \cite{3,22} (using MAPLE package) that the WDVV equations hold for the perturbative prepotentials of the pure gauge $F_4$, $E_6$ and $E_7$ models. Note that the corresponding non-perturbative SW curves are not obviously hyperelliptic.}. In fact, in \cite{15} it has been proved that, if one starts with the general prepotential of the form

$$ F = \frac{1}{4} \sum_{i,j} \left( \alpha_- (a_i - a_j)^2 \log (a_i - a_j) + \alpha_+ (a_i + a_j)^2 \log (a_i + a_j) \right) + \frac{\eta}{2} \sum_i a_i^2 \log a_i \tag{19} $$

the WDVV hold iff $\alpha_+ = \alpha_-$ or $\alpha_+ = 0$, $\eta$ being arbitrary. A new result due to A.Veselov \cite{22} shows that this form can be further generalized by adding boundary terms. For this latter case, however, we do not know the non-perturbative extension.

ii) If one considers the gauge supermultiplets interacting with the $n_f$ matter hypermultiplets in the first fundamental representation with masses $m_\alpha$

$$ F^{\text{pert}} = F^{\text{pert}}_V + rF^{\text{pert}}_F + Kf_F(m) \tag{20} $$

(where $r$ and $K$ are some undetermined coefficients), the WDVV equations do not hold unless $K = r^2/4$, the masses are regarded as moduli (i.e. the equations \cite{1} contain the derivatives with respect to masses) and

$$ f_F(m) = \frac{1}{4} \sum_{\alpha,\beta} \left( (m_\alpha - m_\beta)^2 \log (m_\alpha - m_\beta) \right) \tag{21} $$

for the $SU(n)$ gauge group and

$$ f_F(m) = \frac{1}{4} \sum_{\alpha,\beta} \left( (m_\alpha - m_\beta)^2 \log (m_\alpha - m_\beta) + (m_\alpha + m_\beta)^2 \log (m_\alpha + m_\beta) \right) + $$

$$ + \frac{r(r+s)}{4} \sum_\alpha m_\alpha^2 \log m_\alpha \tag{22} $$

for other classical groups, $s = 2$ for the orthogonal groups and $s = -2$ for the symplectic ones.

Note that at value $r = -2$ the prepotential \cite{20} can be considered as that in the pure gauge theory with the gauge group of the higher rank $\text{rank}G + n_f$. At the same time, at value $r = 2$, like $a_i$'s lying in irrep of $G$, masses $m_\alpha$'s can be regarded as lying in irrep of some $\tilde{G}$ so that if $G = A_n$, $C_n$, $D_n$, $\tilde{G} = A_n$, $D_n$, $C_n$ accordingly (this is nothing but the notorious (gauge group $\leftrightarrow$ flavor group) duality, see, e.g. \cite{20}). These correspondences "explain" the form of the mass term in the prepotential $f(m)$.

iii) The set of the perturbative prepotentials satisfying the WDVV equations can be further extended. Namely, one can consider higher dimensional SUSY gauge theories \cite{9,10,11,12}, in particular, $5d$ theories compactified onto the circle of radius $R$, so that in four-dimensions it can be seen as a gauge theory of the infinitely many vector supermultiplets with masses $M_k = \pi k/R$. Then, the perturbative prepotentials in the pure gauge $SU(n)$ theory of such a type reads as

$$ F^{\text{pert}} = \frac{1}{4} \sum_{i,j} \left( \frac{i}{3} a_{ij}^3 + \frac{1}{2} \text{Li}_3 \left( e^{-2iRa_{ij}} \right) \right) - \frac{in}{6} \sum_{i>j>k} a_i^3 \tag{23} $$

where $a_{ij} \equiv a_i - a_j$ and $\text{Li}_3(x)$ is the standard three-logarithm function. The first sum in this expression tends to the usual logarithmic prepotential $F^{\text{pert}}_V$ as $R \to 0$, while the second
one vanishes. It deserves mentioning that the second (cubic) term do not come from any
field theory calculation, but corresponds to the Chern-Simons term Tr \( A \wedge F \wedge F \) in the field
theory Lagrangian \[27\]. It is similar to the \( U^3 \)-terms of the perturbative prepotential \( F_{\text{pert}} \) of the
heterotic string \[28\]. The presence of these terms turns to be absolutely crucial for the WDVV
equations to hold. Further details on this case, and on the prepotential with fundamental
hypermultiplets included can be found in \[12\].

One can also consider other classical groups. Then, the perturbative prepotentials acquire
the form \( (18) \) with all \( x^2 \log x \) substituted by \( \frac{1}{3} x^3 + \frac{1}{2} \text{Li}_3 \left( e^{-2iRx} \right) \). One can easily check, along
the line of \[15\] that these prepotentials satisfy the WDVV equations.

iv) If in the 4\( d \) theory the adjoint matter hypermultiplets are presented, i.e. \( F_{\text{pert}} = F_{V_{\text{pert}}} + F_{A_{\text{pert}}} + f_A(m) \), the WDVV equations never hold. At the same time, the WDVV equations
are fulfilled for the theory with matter hypermultiplets in the symmetric/antisymmetric square
of the fundamental representation\[19\] iff the masses of these hypermultiplets are equal to zero.

v) Our last example \[19, 11\] has the most unclear status, at the moment. It corresponds to
the pure gauge 5\( d \) theory with higher, \( N = 2 \) SUSY in five dimensions. Starting with such a
five dimensional model one may obtain four dimensional \( N = 2 \) SUSY models (with fields only
in the adjoint representation of the gauge group) by imposing non-trivial boundary conditions
on half of the fields:

\[ \phi(x_5 + R) = e^{2i\epsilon} \phi(x_5). \]  \( (25) \)

If \( \epsilon = 0 \) one obtains \( N = 4 \) SUSY in four dimensions, but when \( \epsilon \neq 2\pi n \) this is explicitly broken
to \( N = 2 \). The low-energy mass spectrum of the four dimensional theory this time contains
two towers of Kaluza-Klein modes:

\[ M = \frac{\pi n}{R} \quad \text{and} \quad M = \frac{\epsilon + \pi n}{R}, \quad N \in \mathbb{Z}. \]  \( (26) \)

The prepotential for the group \( SU(n) \) (\( i = 1 \ldots n \)) should be

\[ F_{\text{pert}} = \frac{1}{4} \sum_{i,j} \sum_{n=-\infty}^{\infty} \left\{ (Ra_{ij} + \pi n)^2 \log (Ra_{ij} + \pi n) \\
- (Ra_{ij} + \pi n - \epsilon)^2 \log (Ra_{ij} + \pi n - \epsilon) \right\} = \frac{1}{4} \sum_{i,j} f(a_{ij}) \]  \( (27) \)

with

\[ f(a) = \text{Li}_3 \left( e^{-2iRa} \right) - \text{Li}_3 \left( e^{-2i(Ra+\epsilon)} \right) \]  \( (28) \)

The prepotential \( (27) \) satisfies the WDVV equations iff \( \epsilon = \pi \) \[19\]. Moreover, it gives
the most general solution for a general class of perturbative prepotentials \( F_{\text{pert}} \) assuming the
functional form

\[ F = \sum_{\alpha \in \Phi} f(\alpha \cdot a), \]  \( (29) \)

\(^{3}\)These hypermultiplets contribute to the prepotential

\[ F_S = -\frac{1}{4} \sum_{i < j} (a_i + a_j + m)^2 \log(a_i + a_j + m) \]

\[ F_{AS} = -\frac{1}{4} \sum_{i < j} (a_i + a_j + m)^2 \log(a_i + a_j + m) \]  \( (24) \)
where the sum is over the root system \( \Phi \) of a Lie algebra. The mystery about this prepotential is that, on one hand, it never satisfies the WDVV equations unless \( \epsilon = \pi \) and we do not know if the corresponding complete non-perturbative prepotential satisfies the WDVV even for \( \epsilon = \pi \). On the other hand, in the limit \( R \to 0 \) and \( \epsilon \sim mR \) for finite \( m \), (when the mass spectrum \( \text{(26)} \) reduces to the two points \( M = 0 \) and \( M = m \)), the theory is the four dimensional YM model with \( N = 4 \) SUSY softly broken to \( N = 2 \), i.e. includes the adjoint matter hypermultiplet. It corresponds to item \textbf{iv)} when the WDVV always do \textit{not} hold. By all these reasons, this exceptional case deserves further investigation.

From the above consideration of the WDVV equations for the perturbative prepotentials, one can learn the following lessons:

- **masses are to be regarded as moduli**

- **as an empiric rule, one may say that the WDVV equations are satisfied by perturbative prepotentials which depend only on the pairwise sums of the type \((a_i \pm b_j)\), where moduli \(a_i\) and \(b_j\) are either periods or masses\(^4\). This is the case for the models that contain either massive matter hypermultiplets in the first fundamental representation (or its dual), or massless matter in the square product of those. Troubles arise in all other situations because of the terms with \(a_i \pm b_j \pm c_k \pm \ldots\) (The inverse statement is wrong – there are some exceptions when the WDVV equations hold despite the presence of such terms – e.g., for the exceptional groups.)

Note that, for the non-UV-finite theories with the perturbative prepotential satisfying the WDVV equations, one can add one more parameter to the set of moduli – the parameter \( \Lambda \) that enters all the logarithmic terms as \( x^2 \log x/\Lambda \). Then, some properly defined WDVV equations still remain correct despite the matrices \( F_{ij}^{-1} \) no longer exist (one just needs to consider instead of them the matrices of the proper minors) \[29\] – see footnote 9 in sect.6.

### 3 Associativity conditions

In the context of the two-dimensional LG topological theories, the WDVV equations arose as associativity condition of some polynomial algebra. We will prove below that the equations in the SW theories have the same origin. Now we briefly remind the main ingredients of this approach in the standard case of the LG theories.

In this case, one deals with the chiral ring formed by a set of polynomials \( \{ \Phi_i(\lambda) \} \) and two co-prime (i.e. without common zeroes) fixed polynomials \( Q(\lambda) \) and \( P(\lambda) \). The polynomials \( \Phi \) form the associative algebra with the structure constants \( C^k_{ij} \) given with respect to the product defined by modulo \( P' \):

\[
\Phi_i \Phi_j = C^k_{ij} \Phi_k Q' + (\ast) P' \rightarrow C^k_{ij} \Phi_k Q' \tag{30}
\]

the associativity condition being

\[
(\Phi_i \Phi_j) \Phi_k = \Phi_i (\Phi_j \Phi_k), \tag{31}
\]

i.e.

\[
C_i C_j = C_j C_i, \quad (C_i)_k^j = C^j_{ik} \tag{32}
\]

\(^4\)This general rule can be easily interpreted in D-brane terms, since the interaction of branes is caused by strings between them. The pairwise structure \((a_i \pm b_j)\) exactly reflects this fact, \(a_i\) and \(b_j\) should be identified with the ends of string.
Now, in order to get from these conditions the WDVV equations, one needs to choose properly the flat moduli \[22\]:

\[a_i = -\frac{n}{i(n-i)} \text{res} \left( P^{n/i} dQ \right), \quad n = \text{ord}(P)\]  

(33)

Then, there exists the prepotential whose third derivatives are given by the residue formula

\[F_{ijk} = \text{res}_{P'=0} \frac{\Phi_i \Phi_j \Phi_k}{P'}\]  

(34)

On the other hand, from the associativity condition \[32\] and the residue formula \[34\], one obtains that

\[F_{ijk} = (C_i)_j^l F_{Q'lk}, \quad \text{i.e.} \quad C_i = F_i F_{Q'}^{-1}\]  

(35)

Substituting this formula for \(C_i\) into \[32\], one finally reaches the WDVV equations in the form

\[F_i G^{-1} F_j = F_j G^{-1} F_i, \quad G \equiv F_{Q'}\]  

(36)

The choice \(Q' = \Phi_i\) gives the standard equations \[4\]. In two-dimensional topological theories, there is always the unity operator that corresponds to \(Q' = 1\) and leads to the constant metric \(F_{Q'}\).

Thus, from this short study of the WDVV equations in the LG theories, we can get three main ingredients necessary for these equations to hold. These are:

- associative algebra
- flat moduli (coordinates)
- residue formula

We will show that in the SW theories only the first ingredient requires a non-trivial check, while the other two are automatically presented due to proper integrable structures.

### 4 SW theories and integrable systems

Now we turn to the WDVV equations emerging within the context of the SW construction \[1\] and show how they are related to integrable system underlying the corresponding SW theory. The most important result of \[1\], from this point of view, is that the moduli space of vacua and low energy effective action in SYM theories are completely given by the following input data:

- Riemann surface \(\mathcal{C}\)
- moduli space \(\mathcal{M}\) (of the curves \(\mathcal{C}\))
- meromorphic 1-form \(dS\) on \(\mathcal{C}\)

How it was pointed out in \[2, 13, 14\], this input can be naturally described in the framework of some underlying integrable system. Let us consider a concrete example – the \(SU(n)\) pure
gauge SYM theory that is associated with the periodic Toda chain with \( n \) sites. This integrable system is entirely given by the Lax operator

\[
L(w) = \begin{pmatrix}
  p_1 & e^{q_1-q_2} & w \\
  e^{q_1-q_2} & p_2 & \vdots \\
  \vdots & \vdots & \ddots \\
  \frac{1}{w} & \cdots & p_n
\end{pmatrix}
\] (37)

The Riemann surface \( \mathcal{C} \) of the SW data is nothing but the spectral curve of the integrable system, which is given by the equation

\[
\det (L(w) - \lambda) = 0
\] (38)

Taking into account (37), one can get from this formula the equation

\[
w + \frac{1}{w} = P(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i), \quad \sum \lambda_i = 0
\] (39)

where the ramification points \( \lambda_i \) are Hamiltonians (integrals of motion) parameterizing the moduli space \( \mathcal{M} \) of the spectral curves. The replace \( Y \equiv w - 1/w \) transforms the curve (39) to the standard hyperelliptic form \( Y^2 = P^2 - 4 \), the genus of the curve being \( n - 1 \).

The same integrable system, i.e. the periodic Toda chain can be alternatively rewritten in terms of the \( 2 \times 2 \) Lax matrices \( \mathcal{L}_i \) each associated with the site of the chain:

\[
\mathcal{L}_i = \begin{pmatrix}
  \lambda + p_i & e^{q_i} \\
  -e^{-q_i} & 0
\end{pmatrix}
\] (40)

The Lax operator \( \mathcal{L}_i \) can be considered as an "infinitesimal" transfer matrix that shifts from the \( i \)-th to the \( i + 1 \)-th site of the chain

\[
\mathcal{L}_i(\lambda) \Psi_i(\lambda) = \Psi_{i+1}(\lambda)
\] (41)

where \( \Psi_i(\lambda) \) is the two-component Baker-Akhiezer function.

One also needs to consider proper boundary conditions. In the \( SU(n) \) case, they are periodic. The periodic boundary conditions are easily formulated in terms of the Baker-Akhiezer function and read as

\[
\Psi_{i+N_\alpha}(\lambda) = w \Psi_i(\lambda)
\] (42)

where \( w \) is a free parameter (diagonal matrix). The Toda chain with these boundary conditions can be naturally associated with the Dynkin diagram of the group \( A_{n-1}^{(1)} \).

One can also introduce the transfer matrix shifting \( i \) to \( i + n \)

\[
T(\lambda) \equiv \mathcal{L}_n(\lambda) \ldots \mathcal{L}_1(\lambda)
\] (43)

Now the periodic boundary conditions are encapsulated in the spectral curve equation

\[
\det(T(\lambda) - w \cdot 1) = 0
\] (44)

or

\[
w^2 - \text{Tr}T(\lambda)w + \det T(\lambda) = 0
\] (45)
This curve coincides with (39), since \( \det T(\lambda) = 1 \) for the Toda Lax operators (40).

The last important ingredient of the construction is the meromorphic 1-form \( dS = \lambda \frac{dw}{w} = \lambda \frac{dP}{w} \). From the point of view of the Toda chain, it is just the shortened action ”pdq” along the non-contractible contours on the Hamiltonian tori. Its defining property is that the derivatives of \( dS \) with respect to the moduli (ramification points) are holomorphic differentials on the spectral curve.

After this concrete example, we are ready to describe how the SW data emerge within a more general integrable framework and then discuss more on the concrete examples of the SW construction. As before, we start with the theories without matter hypermultiplets. First, we introduce bare spectral curve \( E \) that is torus \( y^2 = x^3 + g_2 x^2 + g_3 \) for the UV finite SYM theories with the associated holomorphic 1-form \( d\omega = dx/y \). This bare spectral curve degenerates into the double-punctured sphere (annulus) for the asymptotically free theories: \( x \to w + 1/w, \ y \to w - 1/w, \ d\omega = dw/w \). On this bare curve, there are given either a matrix-valued Lax operator \( L(x, y) \) if one considers an extension of the (37) Lax representation, or another matrix Lax operator \( L_i(x, y) \) associated with an extension of the representation (40) and defining the transfer matrix \( T(x, y) \). The corresponding dressed spectral curve is defined either from the formula \( \det(L - \lambda) = 0 \), or from \( \det(T - w) = 0 \).

This spectral curve is a ramified covering of \( E \) given by the equation

\[
P(\lambda; x, y) = 0
\]

In the case of the gauge group \( G = SU(n) \), the function \( P \) is a polynomial of degree \( n \) in \( \lambda \).

Thus, the moduli space \( \mathcal{M} \) of the spectral curve is given just by coefficients of \( P \). The generating 1-form \( dS \cong \lambda d\omega \) is meromorphic on \( C \) (”\( \cong \)” denotes the equality modulo total derivatives).

The prepotential and other ”physical” quantities are defined in terms of the cohomology class of \( dS \):

\[
a_i = \oint_{A_i} dS,
\]

\[
a_i^D \equiv \frac{\partial F}{\partial a_i} = \oint_{B_i} dS,
\]

\[
A_i \circ B_j = \delta_{ij}.
\]

The first identity defines here the appropriate flat moduli, while the second one – the prepotential. The defining property of the generating differential \( dS \) is that its derivatives w.r.t. moduli give holomorphic 1-differentials. In particular,

\[
\frac{\partial dS}{\partial a_i} = d\omega_i
\]

and, therefore, the second derivative of the prepotential w.r.t. \( a_i \)’s is the period matrix of the curve \( C \):

\[
\frac{\partial^2 F}{\partial a_i \partial a_j} = T_{ij}
\]

The latter formula allows one to identify prepotential with logarithm of the \( \tau \)-function of the Whitham hierarchy [22, 23]: \( F = \log \tau \).

So far we reckoned without matter hypermultiplets. In order to include them, one just needs to consider the surface \( C \) with punctures. Then, the masses are proportional to residues of \( dS \).
Table. SUSY gauge theories $\Longleftrightarrow$ integrable systems correspondence

| SUSY physical theory | Pure gauge SYM theory, gauge group $G$ | SYM theory with fund. matter | SYM theory with adj. matter |
|----------------------|-------------------------------------|-------------------------------|-----------------------------|
| $4d$                 | Toda chain for the dual affine $\hat{G}^\vee$ | $XXX$ spin chain             | Calogero-Moser system       |
| $5d$                 | Relativistic Toda chain              | $XXZ$ spin chain              | Ruijsenaars-Schneider model |
| $6d$                 | $XYZ$ spin chain                     |                               |                             |
| Comments             | There are two Lax representations     | Spherical bare curve          | Elliptic bare curve         |

at the punctures, and the moduli space has to be extended to include these mass moduli. All other formulas remain in essence the same (see [13] for more details).

By the present moment, the correspondence between SYM theories and integrable systems is built through the SW construction in most of known cases that are collected in the table[1].

This table reflects several possible generalizations of the periodic Toda chain. First of all, one can extend the representation in terms of the ”large” Lax matrices (37). It naturally leads to the system whose potential is doubly-periodic function of coordinates. This system is the Calogero-Moser system. It is associated with the elliptic bare spectral curve and generating 1-form $dS = \lambda d\xi$, $\xi$ being the coordinate on the bare torus. On physical side, this system corresponds to including the adjoint matter hypermultiplets [8].

The second possible extension is generalization of the $2 \times 2$ Lax representation of the Toda chain (39). This leads to the $XXX$ spin chain with the cylindrical bare spectral curve and the same generating 1-form $dS = \lambda dw/w$. Physically, this system describes fundamental matter hypermultiplets included [4, 5].

Now, each of these two systems can be further generalized to higher dimensional, $5d$ and $6d$ SYM gauge theories [10]-[12], with the target space (the fifth and sixth dimensions) being accordingly a cylinder and a torus. Within the integrable framework, this means putting momenta of the system onto the cylinder (Hamiltonians periodic in momenta) and the torus (Hamiltonians doubly-periodic in momenta) respectively. In the generating 1-form one needs just to substitute respectively $\lambda \rightarrow \log \lambda$ in $5d$ and $\lambda \rightarrow \zeta$ in $6d$, $\zeta$ being the coordinate on the target space torus. At the same time, the bare spectral curve associated with coordinate dependence is not changed with this extension.

In a word, adding adjoint matter makes the coordinate dependence (and the bare spectral curve) elliptic, while coming to higher dimensions provides trigonometric and elliptic momentum dependencies (and the corresponding target space).

---

5In the table we considered only the classical groups.
Let us discuss a little the dressed spectral curves. In the adjoint matter case, the spectral curve is non-hyperelliptic, since the bare curve is elliptic. Therefore, it can be described as some covering of the hyperelliptic curve. We do not go into further details here, just referring to [15, 11], since the WDVV equations do not hold, at least, in the standard form (1), with the elliptic bare curve (see below).

Instead, we describe the dressed spectral curves for the 4d theories without adjoint matter with the classical gauge groups in more explicit terms. Let us note that in all these cases the curves are hyperelliptic, since all of them follow from some $2 \times 2$ Lax representation and, therefore, are the spectral curves of the form (15).

More concretely, for the SYM gauge theory with the gauge group $G$, one should consider the integrable system given by this $2 \times 2$ Lax representation on the Dynkin diagram for the corresponding dual affine algebra $\hat{G}^{\vee}$ [7].

The spectral curves can be described by the general formula

$$P(\lambda, w) = 2P(\lambda) - w - \frac{Q(\lambda)}{w}$$

(50)

Here $P(\lambda)$ is the characteristic polynomial of the algebra $G$ itself for all $G \neq C_n$, i.e.

$$P(\lambda) = \det(G - \lambda I) = \prod_i(\lambda - \lambda_i)$$

(51)

where determinant is taken in the first fundamental representation and $\lambda_i$'s are the eigenvalues of the algebraic element $G$. For the pure gauge theories with the classical groups [3], $Q(\lambda) = \lambda^{2s}$ and

$$A_{n-1}: \quad P(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i), \quad s = 0;$$

$$B_n: \quad P(\lambda) = \lambda \prod_{i=1}^{n}(\lambda^2 - \lambda_i^2), \quad s = 2;$$

$$C_n: \quad P(\lambda) = \prod_{i=1}^{n}(\lambda^2 - \lambda_i^2) - \frac{2}{\lambda^2}, \quad s = -2;$$

$$D_n: \quad P(\lambda) = \prod_{i=1}^{n}(\lambda^2 - \lambda_i^2), \quad s = 2$$

(52)

For exceptional groups, the curves arising from the characteristic polynomials of the dual affine algebras do not acquire the hyperelliptic form although the WDVV equations seem to be still fulfilled.

In order to include $n_F$ massive hypermultiplets in the first fundamental representation one can just change $\lambda^{2s}$ for $Q(\lambda) = \lambda^{2s} \prod_{i=1}^{n_F}(\lambda - m_i)$ if $G = A_n$ and for $Q(\lambda) = \lambda^{2s} \prod_{i=1}^{n_F}(\lambda^2 - m_i^2)$ if $G = B_n, C_n, D_n$ [29, 30, 7].

Note that the 5d theories can be also described by the same curves but by different 1-forms $dS$ [12].

5 WDVV equations in SW theories

As we already discussed, in order to derive the WDVV equations along the line used in the context of the LG theories, we need three crucial ingredients: flat moduli, residue formula and

\label{5}

\footnote{In the symplectic case, the curve can be easily recast in the form with polynomial $P(\lambda)$ and $s = 0$.}
associative algebra. However, the first two of these are always contained in the SW construction provided the underlying integrable system is known. Indeed, one can derive (see [15]) the following residue formula

\[ F_{ijk} = \text{res}_{d\omega=0} \frac{d\omega_i d\omega_j d\omega_k}{d\omega d\lambda} \]  (53)

where the proper flat moduli \( a_i \)'s are given by formula (47). Thus, the only point to be checked is the existence of the associative algebra. The residue formula (53) hints that this algebra is to be the algebra \( \Omega^1 \) of the holomorphic differentials \( d\omega_i \). In the forthcoming discussion we restrict ourselves to the case of pure gauge theory, the general case being treated in complete analogy.

Let us consider the algebra \( \Omega^1 \) and fix three differentials \( dQ, d\omega, d\lambda \in \Omega^1 \). The product in this algebra is given by the expansion

\[ d\omega_i d\omega_j = C^{k}_{ij} d\omega_k dQ + (\ast) d\omega + (\ast) d\lambda \]  (54)

that should be factorized over the ideal spanned by the differentials \( d\omega \) and \( d\lambda \). This product belongs to the space of quadratic holomorphic differentials:

\[ \Omega^1 \cdot \Omega^1 \in \Omega^2 \cong \Omega^1 \cdot (dQ \oplus d\omega \oplus d\lambda) \]  (55)

Since the dimension of the space of quadratic holomorphic differentials is equal to \( 3g - 3 \), the l.h.s. of (54) with arbitrary \( d\omega_i \)'s is the vector space of dimension \( 3g - 3 \). At the same time, at the r.h.s. of (54) there are \( g \) arbitrary coefficients \( C^{k}_{ij} \) in the first term (since there are exactly so many holomorphic 1-differentials that span the arbitrary holomorphic 1-differential \( C^{k}_{ij} d\omega_k \)), \( g - 1 \) arbitrary holomorphic differentials in the second term (one differential should be subtracted to avoid the double counting) and \( g - 2 \) holomorphic 1-differentials in the third one. Thus, totally we get that the r.h.s. of (54) is spanned also by the basis of dimension \( g + (g - 1) + (g - 2) = 3g - 3 \).

This means that the algebra exists in the general case of the SW construction. However, generally this algebra is not associative. This is because, unlike the LG case, when it was the algebra of polynomials and, therefore, the product of the two belonged to the same space (of polynomials), product in the algebra of holomorphic 1-differentials no longer belongs to the same space but to the space of quadratic holomorphic differentials. Indeed, to check associativity, one needs to consider the triple product of \( \Omega^1 \):

\[ \Omega^1 \cdot \Omega^1 \cdot \Omega^1 \in \Omega^3 = \Omega^1 \cdot (dQ^2 \oplus \Omega^2 \cdot d\omega \oplus \Omega^2 \cdot d\lambda) \]  (56)

Now let us repeat our calculation: the dimension of the l.h.s. of this expression is \( 5g - 5 \) that is the dimension of the space of holomorphic 3-differentials. The dimension of the first space in expansion of the r.h.s. is \( g \), the second one is \( 3g - 4 \) and the third one is \( 2g - 4 \). Since \( g + (3g - 4) + (2g - 4) = 6g - 8 \) is greater than \( 5g - 5 \) (unless \( g \leq 3 \)), formula (56) does not define the unique expansion of the triple product of \( \Omega^1 \) and, therefore, the associativity spoils.

The situation can be improved if one considers the curves with additional involutions. As an example, let us consider the family of hyperelliptic curves: \( y^2 = Pol_{2g+2}(\lambda) \). In this case, there is the involution, \( \sigma: y \to -y \) and \( \Omega^1 \) is spanned by the \( \sigma \)-odd holomorphic 1-differentials \( \frac{y^{i-1} d\omega}{y^i}, i = 1, \ldots, g \). Let us also note that both \( dQ \) and \( d\omega \) are \( \sigma \)-odd, while \( d\lambda \) is \( \sigma \)-even. This means that \( d\lambda \) can be only meromorphic on the surface without punctures (which is, indeed, the case in the absence of mass hypermultiplets). Thus, \( d\lambda \) omits from formula (54) that acquires the form

\[ \Omega^2_+ = \Omega^1_- \cdot dQ \oplus \Omega^1 \cdot d\omega \]  (57)
where we expanded the space of holomorphic 2-differentials into the parts with definite \( \sigma \)-parity:
\[
\Omega^2 = \Omega^2_+ \oplus \Omega^2_-,
\]
which are manifestly given by the differentials \( \frac{x_i - 1}{y^2} (dx)^2 \), \( i = 1, \ldots, 2g - 1 \) and \( \frac{x_i - 1}{y} (dx)^2 \), \( i = 1, \ldots, g - 2 \) respectively. Now it is easy to understand that the dimensions of the l.h.s. and r.h.s. of (57) coincide and are equal to 2\( g - 1 \).

Analogously, in this case, one can check the associativity. It is given by the expansion
\[
\Omega^3 = \Omega^1_+ \cdot (dQ)^2 \oplus \Omega^2_+ \cdot d\omega
\]
where both the l.h.s. and r.h.s. have the same dimension: \( 3g - 2 = g + (2g - 2) \). Thus, the algebra of holomorphic 1-differentials on hyperelliptic curve is really associative. This completes the proof of the WDVV equations in this case.

Now let us briefly look at cases when there exist the associative algebras basing on the spectral curves discussed in the previous section. First of all, it exists in the theories with the gauge group \( A_n \), both in the pure gauge 4\( d \) and 5\( d \) theories and in the theories with fundamental matter, since, in accordance with the previous section, the corresponding spectral curves are hyperelliptic ones of genus \( n \).

The theories with the gauge groups \( SO(n) \) or \( Sp(n) \) are also described by the hyperelliptic curves. The curves, however, are of higher genus \( 2n - 1 \). This would naively destroy all the reasoning of this section. The arguments, however, can be restored by noting that the corresponding curves (see (52)) have yet another involution, \( \rho : \lambda \rightarrow -\lambda \). This allows one to expand further the space of holomorphic differentials into the pieces with definite \( \rho \)-parity: \( \Omega^1_- = \Omega^1_{-\cdot} \oplus \Omega^1_{-+} \) etc. so that the proper algebra is generated by the differentials from \( \Omega^1_{-\cdot} \).

One can easily check that it leads again to the associative algebra.

Consideration is even more tricky for the exceptional groups, when the corresponding curves are looking non-hyperelliptic. However, additional symmetries should allow one to get associative algebras in these cases too.

There are more cases when the associative algebra exists. First of all, these are 5\( d \) theories, with and without fundamental matter \([14]\). One can also consider the SYM theories with gauge groups being the product of several factors, with matter in the bi-fundamental representation \([31]\). These theories are described by \( SL(p) \) spin chains \([5]\) and the existence of the associative algebra in this case has been checked in \([32]\).

The situation is completely different in the adjoint matter case. In four dimensions, the theory is described by the Calogero-Moser integrable system. Since, in this case, the curve is non-hyperelliptic and has no enough symmetries, one needs to include into consideration both the differentials \( d\omega \) and \( d\lambda \) for algebra to exist. However, under these circumstances, the algebra is no longer to be associative how it was demonstrated above. This can be done also by direct calculation for several first values of \( n \) (see \([13]\)). This also explains the lack of the perturbative WDVV equations in this case (see sect.2).

6 Covariance of the WDVV equations

After we have discussed the role of the (generalized) WDVV equations in SYM gauge theories of the Seiberg-Witten type, let us briefly describe the general structure of the equations themselves. We look at them now just as at some over-defined set of non-linear equations for a
function (prepotential) of \(r\) variables \((t^i)\) (times), \(F(t^i), i = 1, \ldots, r\), which can be written in the form \((36)\)

\[
F_i G^{-1} F_j = F_j G^{-1} F_i,
\]

\[
G = \sum_{k=1}^{r} \eta^k F_k, \quad \forall i, j = 1, \ldots, r \quad \text{and} \quad \forall \eta^k(t)
\]  

(59)

\(F_i\) being \(r \times r\) matrices \((F_i)_{jk} = F_{ijk} = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}\) and the "metric" matrix \(G\) is an arbitrary linear combination of \(F_k\)'s, with coefficients \(\eta^k(t)\) that can be time-dependent.

The WDVV equations imply consistency of the following system of differential equations \([17]\):

\[
\left( F_{ijk} \frac{\partial}{\partial t^l} - F_{ijl} \frac{\partial}{\partial t^k} \right) \psi^j(t) = 0, \quad \forall i, j, k
\]  

(60)

Contracting with the vector \(\eta^l(t)\), one can also rewrite it as

\[
\frac{\partial \psi^j}{\partial t^k} = C_{jk}^i D \psi^j, \quad \forall i, j
\]  

(61)

where

\[
C_k = G^{-1} F_k, \quad G = \eta^i F_i, \quad D = \eta^l \partial_l
\]  

(62)

(note that the matrices \(C_k\) and the differential \(D\) depend on choice of \(\{\eta^l(t)\}\), i.e. on choice of the metric \(G\) and \((59)\) can be rewritten as

\[
[C_i, C_j] = 0, \quad \forall i, j
\]  

(63)

As we already discussed, the set of the WDVV equations \((59)\) is invariant under linear change of the time variables with the prepotential unchanged \([15]\). According to the second paper of

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8We deliberately choose different notations for these variables, \(t\) instead of \(a\) in the gauge theories, in order to point out more general status of the discussion.

9We already discussed in sect.2 that one can add to the set of times (moduli) in the WDVV equations the parameter \(\Lambda\) \([22]\). In this case, the prepotential that depends on one extra variable \(t^0\) \(\equiv \Lambda\) can be naturally considered as a homogeneous function of degree 2:

\[
\mathcal{F}(t^0, t^1, \ldots, t^r) = (t^0)^2 F(t^i/t^0),
\]

see \([23]\) for the general theory. As explained in \([29]\), the WDVV equations \((59)\) for \(F(t^i)\) can be also rewritten in terms of \(\mathcal{F}(t^i)\):

\[
\mathcal{F}_I \hat{G}^{-1} \mathcal{F}_J = \mathcal{F}_J \hat{G}^{-1} \mathcal{F}_I, \quad \forall I, J = 0, 1, \ldots, r; \{\eta^K(t)\}
\]

where this time \(\mathcal{F}_I\) are \((r + 1) \times (r + 1)\) matrices of the third derivatives of \(\mathcal{F}\) and

\[
\mathcal{G} = \sum_{k=0}^{r} \eta^K \mathcal{F}_K, \quad \hat{G}^{-1} = (\det \mathcal{G}) G^{-1}
\]

Note that the homogeneity of \(\mathcal{F}\) implies that \(t^0\)-derivatives are expressed through those w.r.t. \(t^i\), e.g.

\[
t^0 \mathcal{F}_{0ij} = -\mathcal{F}_{ijk} t^k, \quad t^0 \mathcal{F}_{00i} = \mathcal{F}_{ikl} t^k t^l, \quad t^0 \mathcal{F}_{000} = -\mathcal{F}_{klm} t^k t^l t^m
\]

etc.

Thus, all the "metrics" \(\mathcal{G}\) are degenerate, but \(\hat{G}^{-1}\) are non-degenerate. One can easily reformulate the entire present section in terms of \(\mathcal{F}\). Then, e.g., the Baker-Akhiezer vector-function \(\psi(t)\) should be just substituted by the manifestly homogeneous (of degree 0) function \(\psi(t'/t^0)\). The extra variable \(t^0\) should not be mixed with the distinguished "zero-time" associated with the constant metric in the 2d topological theories which generically does not exist (when it does, see comment 2.3 below, we identify it with \(t'\)).
and especially to [13], there can exist also non-linear transformations which preserve the WDVV structure, but they generically change the prepotential. In [18], it is shown that such transformations are naturally induced by solutions of the linear system (60):

\[ t^i \rightarrow \tilde{t}^i = \psi^i(t), \]
\[ F(t) \rightarrow \tilde{F}(\tilde{t}), \]

so that the period matrix remains intact:

\[ F_{ij} = \frac{\partial^2 F}{\partial t^i \partial t^j} = \frac{\partial^2 \tilde{F}}{\partial \tilde{t}^i \partial \tilde{t}^j} \equiv \tilde{F}_{ij} \]

(65)

Now let us make some comments.

1. As explained in [17], the linear system (60) has infinitely many solutions. The "original" time-variables are among them: \( \psi^i(t) = t^i \).

2. Condition (65) guarantees that the transformation (64) changes the linear system (60) only by a (matrix) multiplicative factor, i.e. the set of solutions \( \{ \psi^i(t) \} \) is invariant of (64). Among other things this implies that successively applying (64) one does not produce new sets of time-variables.

3. We already discussed that, in the case of 2d topological models [21, 22, 33], there is a distinguished time-variable, say, \( t^r \), such that all \( F_{ijk} \) are independent of \( t^r \):

\[ \partial_i \partial_j F_{ijk} = 0 \quad \forall i, j, k = 1, \ldots, r \] (66)

(equivalently, \( \partial_i \partial_k F_{rjk} = 0 \). Then, one can make the Fourier transform of (60) with respect to \( t^r \) and substitute it by the system

\[ \frac{\partial}{\partial \tilde{t}^j} \tilde{\psi}^i_z = zC^i_{jk} \tilde{\psi}^k_z, \quad \forall i, j \] (67)

where \( \tilde{\psi}^k_z(t^1, \ldots, t^{r-1}) = \int_{t^1, \ldots, t^{r-1}, t^r} e^{z_t^r} dt^r \). In this case, the set of transformations (64) can be substituted by a family, labeled by a single variable \( z \):

\[ t^i \rightarrow \tilde{t}^i_z = \tilde{\psi}^i_z(t) \] (68)

In the limit \( z \to 0 \) and for the particular choice of the metric, \( \hat{G} = F_r \), one obtains the particular transformation

\[ \frac{\partial \tilde{t}^i}{\partial \tilde{t}^j} = \hat{C}^i_{jk} h^k, \quad h^k = \text{const}, \] (69)

discovered in [13]. (Since \( \hat{C} = \partial_j \hat{C} \), one can also write \( \hat{C}^i_{jk} = \hat{C}^i_{kj} h^k \), \( \hat{C}^i_{j} = (F_r^{-1})^{il} F_{lk} \).)

4. Parameterization like (69) can be used in the generic situation (64) as well (i.e. without distinguished \( t^r \)-variable and for the whole family (64)), the only change is that \( h^k \) is no longer a constant, but a solution to

\[ (\partial_j - DC_j) h^k = 0 \] (70)

(\( h^k = D\psi^k \) is always a solution, provided \( \psi^k \) satisfies (61)).

Note also that, although we have described a set of non-trivial non-linear transformations which preserve the structure of the WDVV equations (59), the consideration above does not prove that all such transformations are of the form (64), (59). Still, (64) is already unexpectedly large, because (59) is an over-defined system and it could seem to be very restrictive, if to have any solutions at all.
Concluding remarks

To conclude this short review, let us emphasize that a lot of problems have to be solved before we get any real understanding of what the WDVV structure means. We already mentioned the problem of lack of the WDVV equations for the Calogero-Moser system. The way to resolve this problem might be to construct higher associativity conditions like it has been done by E.Getzler in the elliptic case \cite{34}, that is to say, for the elliptic Gromov-Witten classes. The other kind of problem is that the WDVV equations in the type A topological theories themselves do still wait for the explanation in terms of associative algebras.

All these problems are to be resolved in order to establish to what extent there is a really deep reason for the WDVV equations to emerge in topological and Seiberg-Witten theories. Note that the latter two are connected through the Whitham hierarchies (see, e.g., \cite{35}, the second paper in \cite{24} and the review by A.Morozov at the Workshop) and there are also tight connections of the Whitham hierarchies with the WDVV equations \cite{22}.

The other problem is more on the structure of the WDVV equations themselves: up to now, we do not understand what is the class of solutions to the equations and how wide it is, having some particular examples in hands. And maybe even more obscure are the origins and implications of the covariance of the WDVV equations.

At last, we still have no any clear understanding of connections between the WDVV and integrable structures. What we know is rather a set of random observations.

All these problems appeal for better understanding before we could put the associative algebras and WDVV equations on any solid footing.

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