Disjoint axis-parallel segments without a circumscribing polygon

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Abstract

We construct a family of 17 disjoint axis-parallel line segments in the plane that do not admit a circumscribing polygon.

1 Introduction

For any family $S$ of closed segments in the plane, denote by $V(S)$ the set of endpoints of the segments in $S$. A simple polygon $P$ is a circumscribing polygon of $S$ if the vertex set of $P$ is $V(S)$, and every segment in $S$ is either an edge or an internal diagonal in $P$.

Grünebaum [3] constructed a family $S_1$ of six disjoint segments with four distinct slopes that do not admit a circumscribing polygon. Recently, Akitaya et al. [1] constructed a family $S_3$ of nine disjoint segments with three distinct slopes that do not admit a circumscribing polygon, and asked whether every family of disjoint axis-parallel segments in the plane, not all in a line, admit a circumscribing polygon. In this note, we show that the family $S_2$ of 17 disjoint axis-parallel segments illustrated in Figure 1 do not admit a circumscribing polygon.

Figure 1: 17 disjoint axis-parallel segments in a centrally symmetric configuration inside a $[-11, 11] \times [-8, 8]$ grid.

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2 The proof

To prove that \( S_2 \) does not admit a circumscribing polygon, our main tool is the following proposition [2, Lemma 2.1] which is repeatedly used by Akitaya et al. [1] in proving that the family \( S_3 \) they constructed does not admit a circumscribing polygon. For any polygon \( P \), denote by \( \text{conv}(P) \) the convex hull of \( P \).

**Proposition 1.** For any simple polygon \( P \), the vertices in \( \text{conv}(P) \) must appear in the same circular order in both \( P \) and \( \text{conv}(P) \).

Refer to Figure 2 for a magnified illustration of \( S_2 \) with labels. Suppose for contradiction that \( S_2 \) admits a circumscribing polygon \( P \).

Since the eight vertices \( a, b, c, d, e, f, g, h \) are on the convex hull of \( P \), it follows by Proposition 1 that the four segments \( ha, bc, de, fg \) must be edges in \( P \). Then \( P \) is the alternating concatenation of these four edges and four paths \( a \rightarrow b, c \rightarrow d, e \rightarrow f, g \rightarrow h \).

We say that a path visits a segment if it goes through at least one endpoint of the segment. Since the segments in \( S \) are in a centrally symmetric configuration, we can assume without loss of generality that at least one of the two paths \( a \rightarrow b \) and \( c \rightarrow d \) visits the segment \( mn \). In the following we assume that this path is \( a \rightarrow b \). The other case, that this path is \( c \rightarrow d \), is similar.

Let \( P_{ab} \) be the simple polygon obtained by closing the path \( a \rightarrow b \) with the edge \( ba \). Suppose that \( a \rightarrow b \) does not visit any segment to the right of \( mn \). Then it must visit both endpoints of \( ij \). Note that \( a, b, i, j \) and \( \{m, n\} \cap V(P_{ab}) \) are all on \( \text{conv}(P_{ab}) \). Thus it follows by Proposition 1 that \( ij \) must be an internal diagonal of \( P_{ab} \) and hence an external diagonal of \( P \). Since \( ij \) cannot be an external diagonal of the circumscribing polygon \( P \), \( a \rightarrow b \) must visit at least one other segment to the right of \( mn \). Indeed, due to the strategic position of \( kl \), \( a \rightarrow b \) must visit either \( k \) or \( l \) or both.

We claim that \( a \rightarrow b \) must visit \( l \). Suppose the contrary. Then \( a \rightarrow b \) must visit \( i, j, k, l \), and cannot visit any segment to the right of \( kl \). Then \( a, b, j, k, i \) are on \( \text{conv}(P_{ab}) \), and it again follows by Proposition 1 that...
ij is external, a contradiction.

We have shown that $a \rightarrow b$ visits $l$. We claim that $a \rightarrow b$ must not visit any segment to the right of $kl$. Suppose the contrary, and let $z \in \{p, q, r, s, t, u, v, w, x, y\}$ be the rightmost endpoint that $a \rightarrow b$ visits, breaking ties arbitrarily. Then $a \rightarrow b$ must visit both $k$ and $l$, and $a, b, k, z, l$ are on $\text{conv}(P_{ab})$. Then by Proposition 1, $kl$ is external, a contradiction. Also, since $a \rightarrow b$ visits $l$, $c \rightarrow d$ cannot visit any segment to the right of $kl$ either.

In summary, the endpoints $\{p, q, r, s, t, u, v, w, x, y\}$ must be visited by $e \rightarrow f$ and $g \rightarrow h$. We claim that $g \rightarrow h$ does not visit $xy$. Suppose the contrary. Let $z \in \{x, y, w, v, k\}$ be the lowest endpoint that $g \rightarrow h$ visits. Let $P_{gh}$ be the simple polygon obtained by closing the path $g \rightarrow h$ with the edge $hg$. Then $g \rightarrow h$ must visit both endpoints $u$ and $t$, and $g, u, z, t, h$ are all on $\text{conv}(P_{gh})$. Then by Proposition 1, $ut$ is external, a contradiction.

Since $g \rightarrow h$ does not visit $xy$, $e \rightarrow f$ must visit $xy$. Let $P_{ef}$ be the simple polygon obtained by closing the path $e \rightarrow f$ with the edge $fe$.

We claim that $e \rightarrow f$ must visit $u$. Suppose the contrary. Then $e \rightarrow f$ may still visit $t$ but not any segment above $tu$. Let $z \in \{x, y, t\}$ be the highest endpoint that $e \rightarrow f$ visits. Then $e \rightarrow f$ must visit both endpoints $w$ and $v$, and $e, w, z, v, f$ are all on $\text{conv}(P_{ef})$. Then by Proposition 1, $wv$ is external, a contradiction.

We have shown that $e \rightarrow f$ must not visit any segment above $u$. Suppose the contrary, and let $z \in \{p, q, r, s\}$ be the highest endpoint that $e \rightarrow f$ visits. Then $e \rightarrow f$ must visit both $t$ and $u$, and $e, t, z, u, f$ are on $\text{conv}(P_{ef})$. Then by Proposition 1, $tu$ is external, a contradiction.

In summary, the endpoints $\{p, q, r, s\}$ must be visited by $g \rightarrow h$. In addition, the endpoint $t$ may or may not be visited by $g \rightarrow h$.

Finally, let $z \in \{r, s, t\}$ be the leftmost endpoint that $g \rightarrow h$ visits. Then $g \rightarrow h$ must visit both $p$ and $q$, and $g, q, z, p, h$ are on $\text{conv}(P_{gh})$. Then by Proposition 1, $pq$ is external, a contradiction.

Therefore, our initial assumption that $S_2$ admits a circumscribing polygon $P$ does not hold. The proof is now complete.

3 An open question

Akitaya et al. [1] proved that it is NP-hard to decide whether a given family of disjoint segments admit a circumscribing polygon. Is this decision problem still NP-hard on disjoint axis-parallel segments?

References

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