On a Property of Congruence Lattices of Slim, Planar, Semimodular Lattices

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ON A PROPERTY OF CONGRUENCE LATTICES OF SLIM, PLANAR, SEMIMODULAR LATTICES

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Abstract. In a 2021 paper with Gábor Czédli, we introduced and verified the Three-pendant Three-crown Property, 3P3C, for congruence lattices of slim (no \( M_3 \) sublattice), planar, semimodular lattices. The proof is very long; in part, because it relies on Czédli’s 2021 paper on lamps.

This paper verifies 3P3C using the Swing Lemma, an elementary and short approach.

1. Introduction

In my joint paper with G. Czédli [4], we defined the Three-pendant Three-crown Property (3P3C Property), as follows:

The ordered set \( R_3 \) of Figure 1 has no cover-preserving embedding into the ordered set of join-irreducible congruences of \( K \), \( J(\text{Con} K) \).

The following is the result of the same paper.

3P3C Theorem. Let \( K \) be a slim, planar, semimodular lattice. Then \( \text{Con} K \) satisfies the 3P3C Property.

In this paper, we provide a short and elementary proof, utilizing the Swing Lemma of my paper [6].

This paper is largely self-contained. Apart from some very elementary concepts (semimodular lattice, complemented lattice, interval, etc.), Sections 2 and 3 present all the concepts and results we need, sometimes informally.

For more extensive background, download Part I of the book [10], see arXiv:2104.06539
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2. Some basic concepts and results

2.1. Join-irreducible congruences. Let $L$ be a finite lattice. In $L$, let us call an interval $[a, b]$ prime, if $[a, b]$ has only two elements, namely, $a$ and $b$. A prime interval $[a, b]$ of $L$ is represented by an edge $E$ in a diagram of $L$. We will use the terms prime interval and edge interchangeably.

If $[a, b]$ is a prime interval, then $\text{con}(a, b)$, the smallest congruence collapsing it, is a join-irreducible congruence, and conversely. Similarly, for edges.

2.2. Two acronyms. Let $L$ be a planar lattice. The lattice $L$ is slim, if it has no $M_3$ sublattice. The acronym SPS stands for slim, planar, semimodular, as in SPS lattices.

Following my joint paper with E. Knapp [11], a planar semimodular lattice $L$ is rectangular, if its left boundary chain has exactly one doubly-irreducible element, $c_l$, and its right boundary chain has exactly one doubly-irreducible element, $c_r$, and these elements are complementary, that is,

\begin{align*}
    c_l \lor c_r &= 1, \\
    c_l \land c_r &= 0.
\end{align*}

The acronym SR stands for slim rectangular, as in SR lattices.

2.3. Forks. Let $L$ be an SPS lattice. Following G. Czédli and E. T. Schmidt [5], we introduce the concept of inserting a fork into $L$ at $C$, where $C$ is an interval of $L$ that is a covering square. We start with the ordered set $F$, the fork, as pictured in the first diagram of Figure 2 and the SPS lattice $L = C_3 \times C_4$, with the top interval of $L$ as $C$. We place $F$ into $C$ as in the second diagram of Figure 2. We obtain a lattice but it is not semidistributive, so the left bottom element of the fork requires an additional element, left and down from it, to correct, and symmetrically. The lattice we obtain, as in the third diagram of Figure 2, is still not semidistributive, so the left bottom element of the new fork similarly requires an additional element to correct.

It is easy to see that we obtain an SPS lattice.

![Figure 2. Inserting a fork](image-url)
2.4. **Representation theorems.** In this branch of lattice theory, a *representation theorem* (abbreviated as RT) is a statement of the form: every finite distributive lattice can be represented as the congruence lattice of a lattice of some type. The first RT found is the following.

**Basic Representation Theorem.** *Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite lattice \( L \).*

This is due to R. P. Dilworth from around 1944 (see the book [1]). The Basic RT was not published until 1962 in my joint paper with E. T. Schmidt [14].

With a finite distributive lattice \( D \), we can associate the ordered set \( P \) of join-irreducible elements; also, from the ordered set \( P \), we can easily reconstruct \( D \). This may reduce a complex finite distributive lattice \( D \) to a much smaller ordered set \( P \) of simple structure, see Figure 3 for an illustration.

![Figure 3. A finite distributive lattice \( D \) and the ordered set of join-irreducible elements \( P \) of \( D \)](image)

So we may rephrase the Basic Representation Theorem as follows.

**Basic Representation Theorem’.** *Every finite ordered set \( P \) can be represented as the ordered set, \( J(\text{Con } L) \), of join-irreducible congruences of a finite lattice \( L \).*

The first specialized RT was in the same paper. Recall that a finite lattice \( L \) is *sectionally complemented*, if every ideal of \( L \) is a complemented lattice, that is, for all \( b \leq a \in L \), there exists a \( c \in L \) such that \( a = b \lor c \) and \( 0 = b \land c \).

**RT for sectionally complemented lattices.** *Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite sectionally complemented lattice \( L \).*

2.5. **Two-cover Theorem.** The topic we make a contribution to in this paper started with the following result in my paper [8]. Let \( K \) be an SPS lattice with at least three elements. A finite ordered set \( P \) satisfies the **Two-Cover Condition**, if any element of \( P \) has at most two covers.
Two-cover Theorem. The ordered set of join-irreducible congruences of an SPS lattice $K$ satisfies the Two-Cover Condition.

3. The Swing Lemma

The crucial tool employed in this paper is the Swing Lemma, introduced in this section. Let $K$ be an SPS lattice. The $S_7$ lattice is illustrated in Figure 4. A peak $S_7$ sublattice (peak sublattice, for short) $S$ is an $S_7$ sublattice, whose top three edges are covering. See Figure 4 for examples.

![Figure 4. The $S_7$ lattice and two peak sublattices $S(a, b, c)$; in the second example, the elements of the peak sublattice are black filled](image)

For the prime intervals $p, q$ of a slim, planar, semimodular lattice (SPS lattice) $L$, we define a binary relation: $p$ swings to $q$, written as $p \bowtie q$, if $1_p = 1_q$, this element covers at least three elements, and $0_q$ is neither the left-most nor the right-most element covered by $1_p = 1_q$, see Figure 5 for two examples. If $0_p$ is either the left-most or the right-most element covered by $1_p = 1_q$, then we call the swing external, in formula, $p \bowtie q$. Otherwise, the swing is internal, in formula, $p \bowtie q$. Figure 5 shows an external swing, $p \bowtie q$, and an internal swing, $p \bowtie q$.

![Figure 5. Swings, $p \bowtie q$](image)

This paper is a series of applications of the following result (see my paper [6]).

Swing Lemma. Let $L$ be an SPS lattice and let $p$ and $q$ be distinct prime intervals in $L$. Then $q$ is collapsed by $\text{con}(p)$ iff there exists a prime interval $r$ and a sequence of pairwise distinct prime intervals

$$r = r_0, r_1, \ldots, r_n = q$$

(1)
such that $p$ is up perspective to $r$ and $s_i$ is down perspective to or swings to $r_{i+1}$ for $i = 0, \ldots, n - 1$. In addition, the sequence (1) is descending:

\[
1_{r_0} \geq 1_{r_1} \geq \cdots \geq 1_{r_n}.
\]

Recall that $p$ is up perspective to $r$, if $1_p \lor 0 = 1_q$ and $1_p \land 0 = 0_p$. We define down perspective dually.

Most of the time, we do not need the full Swing Lemma, only the following two special cases (which, combined, prove the Swing Lemma).

**Equality Lemma.** Let $p$ and $q$ be prime intervals of $K$. Then the equality

\[
\text{con}(p) = \text{con}(q)
\]

holds iff there exist prime intervals $s$ and $t$ in $K$, such that

\[
p \uparrow \sim s, \ s \in \sim t, \ t \dn \sim q,
\]

see Figure 6.

![Figure 6. Illustrating the Equality Lemma, con(p) = con(q)](image)

**Covering Lemma.** Let $p$ and $q$ be prime intervals of $K$. The covering

\[
\text{con}(q) \prec \text{con}(p)
\]

holds in $J(\text{Con} K)$ iff there exist prime intervals $r$, $s$, $t$, $u$, $v$ in $K$, such that

\[
p \uparrow \sim r, \ r \in \sim s, \ s \dn \sim t, \ t \ex \sim u, \ u \dn \sim q.
\]

In (4), we may have

\[
p = r, \ r = s, \ s = t, \ u = q,
\]

or any combination thereof. Moreover, there are prime intervals, $t, u$ and a peak sublattice $S(t, u)$ in $K$ with $\text{col}(u) = \text{col}(q)$ and $\text{col}(t) = \text{col}(p)$.

For every $\text{con}(q) \prec \text{con}(p)$, the Covering Lemma provides a peak sublattice. This gives an approach to the proof of the 3P3C Theorem: the covers in the ordered set $R_3$ of Figure 1 give us the peak sublattices $S_1, S_2, \ldots, S_{12}$. I have not been able to use this approach; 12 peak sublattices are too many to handle. The key to the proof is the V-Lemma, which cuts the 12 peak sublattices to 6 (really, to 3).
4. The Problem

In my paper [8], I raised the following.

**Problem.** Characterize congruence lattices of SPS lattices.

This paper is a contribution to it.

G. Czédli maintains a list of related papers (mostly by Czédli and myself), 56 as of this writing, see

http://www.math.u-szeged.hu/~czedli/m/listak/publ-psml.pdf

5. Preparing for the proof

The 3P3C theorem will be proved in Section 6. This section prepares for the proof by introducing four relations on the set of join-irreducible congruences of an SR lattice $K$.

5.1. The V-relation. Let $K$ be an SR lattice. The set of colors of $K$ is defined as the set of join-irreducible congruences of $K$, that is, $J(\text{Con} K)$. We define the ternary relation $V$ on the colors of $K$ as follows: $V(a, b, c)$ holds in $K$, if $a, b, c$ are three distinct colors and $a \prec b, c$ (the first the smallest). We call this relation the V-relation. We denote by $K^\wedge$ those elements of $K$ that cover three or more elements.

**Figure 8.** The V-relation on the colors of $K$, $J(\text{Con} K)$
V-lemma.

(i) If \( V(a, b, c) \) holds in \( K \), then there is a peak sublattice \( S(a, b, c) \), where \( a \) colors the middle edge, \( b, c \) color the other two top edges.

(ii) Let \( S(a, b, c) \) be a peak sublattice of \( K \), as illustrated in Figure 4, where \( a \) colors the middle edge, \( b, c \) color the other two top edges of \( S(a, b, c) \). Then \( V(a, b, c) \) holds in \( K \).

(iii) We make the following assumptions:
   (a) \( V(a, b, c) \) holds in \( K \);
   (b) \( S(p, q) \) is a peak sublattice of \( K \), where \( u \) is the middle edge, as in Figure 7;
   (c) \( \text{col}(u) = a \) and \( \text{col}(t) = b \).

Then \( V(a, b, c) \mapsto 1_{S(p, q)} \) maps the set of \( V \)-relations of \( K \) onto the subset \( K^A \) of \( K \).

Proof. If \( V(a, b, c) \) holds in \( K \), then \( a \prec b \), so by the Covering Lemma, there are the prime intervals \( t, u \) and the peak sublattice \( S(p, q) \) in \( K \) with \( \text{col}(u) = a \), \( \text{col}(t) = b \).

Let \( t' \) be the third top prime interval of \( S(p, q) \) and let \( b' = \text{col}(t') \). Then \( a \prec b, c, b' \) and \( b \neq c \). By the Two-cover Theorem, \( c = b' \). So the sublattice \( S(t, u) \) is the peak sublattice \( S(p, q) \) we required.

(ii) and (iii) are trivial. \( \square \)

5.2. The \( W \)-relation. The \( W \)-relation, \( W \), in \( K \) is a 6-ary relation on the set of colors of \( K \); the relation \( V(a, b, c, d, e, f) \) holds in \( K \), if \( V(a, b, c) \), \( V(d, c, f) \), \( V(e, b, f) \) hold in \( K \), see the diagram on the left of Figure 11.

5.2.1. The \( W \)-relation, Version 1. In this variant, we get \( c = e \) by applying the Equality Lemma as in the top diagram of Figure 9. The dotted edges of the top diagram in Figure 9 are covering and they are all distinct except maybe for the two internal edges at the unit element. Figure 11 shows the smaller variant of the top diagram in which these two edges are equal. The smaller variant will be used in the subsequent diagrams; the full diagram is too big to draw.

5.2.2. The \( W \)-relation, Version 2. As shown in Figure 9.

5.3. The 3C-relation. The 3-Crown-relation, 3C, in \( K \) is also a 6-ary relation on the set of colors of \( K \). The relation \( V(a, b, c, d, e, f) \) holds in \( K \), if

\[
V(a, b, c), \ V(d, c, f), \ V(e, b, f)
\]

hold in \( K \), see the diagram on the left of Figure 11.
5.3.1. The 3C-relation, Version 1. \(V(a, b, c), V(d, c, f)\) form a W-relation, so we get the top half of the diagram of the lattice in Figure 11 (small variant).

In addition, \(V(e, b, f)\) holds in \(K\). So there is a covering square colored by \(b\) and \(f\), and we get \(V(e, b, f)\) by having a fork (see Section 2.3) in the covering square. There is only one covering square in \(K\) colored by \(b, f\), at the bottom of
the diagram. So we have a fork in there, the middle edge colored by $e$. In larger
eexamples, there may be many covering squares in $K$ colored by $b, f$, we pick one.

5.3.2. The 3C-relation, Version 2. We need a cover-preserving SPS extension of
the bottom lattice of Figure 9 establishing the W-relation, Version 2, with a cover-

preserving square colored by $b$ and $f$, see Figure 12. The elements of this square
are black-filled. We then insert a fork (see Section 2.3 into this square.
6. Proving the 3P3C Theorem

In my joint paper [12] with E. Knapp, we proved that an SPS lattice \( K \) has a congruence-preserving rectangular extension \( \hat{K} \). So to verify the 3P3C Theorem for an SPS lattice \( K \), it is sufficient to verify it for \( \hat{K} \). Therefore, we can assume that \( K \) is an SR lattice.

We want to prove that the ordered set \( R_3 \) has no cover-preserving embedding into \( J(\text{Con} \ K) \). The relation \( 3C(a, b, c, d, e, f) \) holds in \( K \), as well, as the three relations

\[
V(x, a, f), \ V(y, e, c), \ V(z, b, d).
\]

6.1. **Version 1.** There is an edge \( y \) that was obtained by inserting a fork in a covering square colored by \( c \) and \( e \). But there is no such covering square possible because the \( c \) colored edges and the \( e \) colored edges run parallel. This completes the proof for Version 1.

6.2. **Version 2.** In Figures 12, let \( I \) denote the ideal generated by the black-filled element. All the nontrivial congruence classes of the color \( a \) are in \( I \). All the nontrivial congruence classes of the color \( f \) are outside of \( I \). But \( x \leq a \) and \( x \leq f \), a contradiction. This completes the proof for Version 2.

We have now completed the proof of the 3P3C Theorem.

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