On the Dimension of Unimodular Discrete Spaces
Part II: Relations with Growth Rate

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August 9, 2018

Abstract

The notions of unimodular Minkowski and Hausdorff dimensions are defined in [5] for unimodular random discrete metric spaces. The present paper is focused on the connections between these notions and the polynomial growth rate of the underlying space. It is shown that bounding the dimension is closely related to finding suitable equivariant weight functions (i.e., measures) on the underlying discrete space. The main results are unimodular versions of the mass distribution principle and Billingsley’s lemma, which allow one to derive upper bounds on the unimodular Hausdorff dimension from the growth rate of suitable equivariant weight functions. Also, a unimodular version of Frostman’s lemma is provided, which shows that the upper bound given by the unimodular Billingsley lemma is sharp. These results allow one to compute or bound both types of unimodular dimensions in a large set of examples in the theory of point processes, unimodular random graphs, and self-similarity. Further results of independent interest are also presented, like a version of the max-flow min-cut theorem for unimodular one-ended trees.

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1 Introduction

This paper is the second in a series of three ([5], the current paper, and [6]), which are referred to as Part I, II and III below. The present paper uses the definitions and symbols of Part I. A list of notation is provided in Table 1 at the end of the paper to ease the reading. For cross referencing the definitions and results of Part I, the prefix ‘I.’ is used. For example, Definition I.2.1 refers to Definition 2.1 in Part I.

Part I introduced the notion of unimodular random discrete metric space and two notions of dimension for such spaces, namely the unimodular Minkowski dimension (Section I.3.1 in Part I) and the unimodular Hausdorff dimension (Section I.3.3).

The present paper is centered on the connections between these dimensions and the growth rate of the space, which is the polynomial growth rate of \( |N_r(o)| \), where \( N_r(o) \) represents the closed ball of radius \( r \) centered at the origin and \( |N_r(o)| \) is the number of points in this ball.

Section 2 is focused on the basic properties of these connections. It is first shown that the upper and lower polynomial growth rates of \( |N_r(o)| \) (i.e., \( \limsup \) and \( \liminf \) of \( \log(|N_r(o)|)/\log r \) as \( r \to \infty \)) provide upper and lower bound for
the unimodular Hausdorff dimension, respectively. This is a discrete analogue of Billingsley’s lemma (see e.g., [9]). A discrete analogue of the mass distribution principle is then provided, which is useful to derive upper bounds on the unimodular Hausdorff dimension. In the particular case of a point-stationary point process equipped with the Euclidean metric, it is also shown that the unimodular Minkowski dimension is bounded from above by the polynomial decay rate of $\mathbb{E}[1/#N_n(o)]$. Weighted versions of these inequalities, where a weight is assigned to each point, are also presented.

Section 3 is devoted to examples. It continues the main example section of Part II and introduces further examples. It is shown there that the bounds established in Section 2 are very useful for calculating the unimodular dimensions in the main instances of unimodular discrete spaces discussed in Part I, namely point processes, random graphs and self similar random discrete sets.

Section 4 gives a unimodular analogue of Frostman’s lemma. Roughly speaking, this lemma states that there is a weight function such that the upper bound in Billingsley’s lemma is sharp. This lemma provides a powerful tool to study the unimodular Hausdorff dimension, in particular, to assess the dimension of subspaces and product spaces. In the Euclidean case, another proof of the unimodular Frostman lemma is provided using a unimodular version of the max-flow min-cut theorem, which is of independent interest.

## 2 Connections to Growth Rate

Let $D$ be a discrete space and $o \in D$. The upper and lower (polynomial) growth rates of $D$ are

$$\text{growth}(\#N_r(o)) = \limsup_{r \to \infty} \log \#N_r(o) / \log r,$$

$$\text{growth}(\#N_r(o)) = \liminf_{r \to \infty} \log \#N_r(o) / \log r.$$

$D$ has finite polynomial growth if $\text{growth}(\#N_r(o)) < \infty$. If the upper and lower growth rates are equal, the common value is called the growth rate of $D$.

For $v \in D$, one has $N_r(o) \subseteq N_{r+c}(v)$ and $N_r(v) \subseteq N_{r+c}(o)$, where $c := d(o,v)$. This implies that $\text{growth}(\#N_r(o))$ and $\text{growth}(\#N_r(v))$ do not depend on the choice of the point $o$.

In various situations in this paper, some weight in $\mathbb{R}_{\geq 0}$ can be assigned to each point of $D$. In these cases, it is natural to redefine the growth rate by considering the weights; i.e., by replacing $\#N_r(o)$ with the sum of the weights of the points in $N_r(o)$. This will be formalized below using the notion of equivariant processes of Subsection I.2.6.

In the following definition, weights should be defined for all discrete spaces $D$. However, if a random pointed discrete space $[D,o]$ is considered, it is enough to define weights in almost every realization (see Subsection I.2.6 for more on the matter). Also, given $D$, the weights are allowed to be random.
Definition 2.1. An equivariant weight function \( w \) is an equivariant process (Definition I.2.6) with values in \( \mathbb{R}^{\geq 0} \). For a discrete space \( D \) and \( v \in D \), the (random) value \( w(v) := w_D(v) \) is called the weight of \( v \). Also, for \( S \subseteq D \), let

\[
    w(S) := w_D(S) := \sum_{v \in S} w(v).
\]

The last equation shows that one could also call \( w \) an equivariant measure.

Assume \([D,o]\) is a unimodular discrete space (Subsection I.2.5). Lemma I.2.28 shows that \([D,o;w_D]\) is a random pointed marked discrete space and is unimodular. Also, one can let \( w \) be undefined for a class of discrete spaces, as long as \( w_D \) is almost surely defined.

In the following, the term ‘\( w_D(\cdot) \) is non-degenerate (i.e., not identical to zero) with positive probability’ means that

\[
    \mathbb{P} \left[ \exists v \in D : w_D(v) \neq 0 \right] > 0. \tag{2.1}
\]

In the case when \([D,o]\) is unimodular, Lemma I.2.30 implies that the above condition is equivalent to

\[
    \mathbb{E}[w(o)] > 0.
\]

Also, the term ‘\( w_D(\cdot) \) is non-degenerate a.s.’ means that

\[
    \mathbb{P} \left[ \exists v \in D : w_D(v) \neq 0 \right] = 1.
\]

2.1 Unimodular Mass Distribution Principle

Theorem 2.2 (Mass Distribution Principle). Let \([D,o]\) be a unimodular discrete space.

(i) Let \( \alpha, c, M > 0 \) and assume there exists an equivariant weight function \( w \) such that the weight of the ball with center \( o \) and radius \( r \) satisfies

\[
    \forall r \geq M : w(N_r(o)) \leq cr^\alpha, \quad \text{a.s.} \quad \tag{2.2}
\]

Then, \( \mathcal{H}_M^\alpha(D) \) defined in (I.3.3) satisfies

\[
    \mathcal{H}_M^\alpha(D) \geq \frac{1}{c} \mathbb{E}[w(o)].
\]

(ii) If in addition to (2.2), \( w_D(\cdot) \) is non-degenerate with positive probability (see (2.1)), then

\[
    \text{udim}_H(D) \leq \alpha.
\]

Proof. Let \( R \) be an arbitrary equivariant covering such that \( R(\cdot) \in \{0\} \cup [M, \infty) \) a.s. By the assumption on \( w \), \( R(o)^\alpha \geq \frac{1}{c} w(N_R(o)) \) a.s. Therefore,

\[
    \mathbb{E}[R(o)^\alpha] \geq \frac{1}{c} \mathbb{E}[w(N_R(o))]. \tag{2.3}
\]
By letting $g(u,v) := w(v)1_{v \in N_R(u)}$, one gets $g^+(o) = w(N_R(o))$. Also, $g^-(o) = w(o) \sum_{v \in D} 1_{o \in N_R(u)} \geq w(o)$ a.s., where the last inequality follows from the fact that $R$ is a covering. Therefore, the mass transport principle I.2.3 implies that $E[w(N_R(o))] \geq E[w(o)]$ (recall that by convention, $N_R(o)$ is the empty set when $R(o) = 0$). So by (2.3), one gets $E[w(N_R(o))] \geq \frac{1}{c} E[w(o)]$. Since this holds for any $R$, one gets that $\mathcal{H}_M^\alpha(D) \geq \frac{1}{c} E[w(o)]$ and the first claim is proved.

If, with positive probability, $w_D(\cdot)$ is non-degenerate, then Lemma I.2.30 implies that $w(o) > 0$ with positive probability. So $E[w(o)] > 0$. Therefore, $\mathcal{H}_1^\alpha(D) > 0$ and the second claim is proved.

**Example 2.3.** Theorem 2.2, applied to the counting measure $w \equiv 1$, implies that $\dim u \mathcal{H}(\delta Z_k) \leq k$ and $\mathcal{M}^k(\delta Z_k) \geq (2/\delta)^k$. As already proved in Propositions I.3.24 and I.3.38, equality holds in both.

**Remark 2.4.** The assumption (2.2) in Theorem 2.2 implies a uniform polynomial bound on the size of all balls by the fact that what holds a.s. at the root, holds a.s. at all points (Lemma I.2.30).

In practice, this assumption is not as applicable as its continuum counterpart (Lemma 1.9 in [9]), except for some simple examples. The unimodular Billingsley lemma in the next subsection will actually be more useful. See also Lemma 2.5 below.

### 2.2 Unimodular Billingsley Lemma

The main result of this subsection is Theorem 2.8. It is based on Lemmas 2.5 and 2.6 below. Lemma 2.5 is a stronger version of the mass distribution principle (Theorem 2.2).

**Lemma 2.5 (An Upper Bound).** Let $[D,o]$ be a unimodular discrete space.

(i) If $c \geq 0$ and $w$ is an equivariant weight function such that

$$\limsup_{r \to \infty} \frac{w(N_r(o))}{r^\alpha} \leq c, \text{ a.s.,}$$

then

$$\mathcal{H}_M^\alpha(D) \geq \frac{1}{2c} E[w(o)].$$

(ii) In addition, if $w_D(\cdot)$ is non-degenerate with positive probability (see (2.1)), then $\dim_H(D) \leq \alpha$.

**Proof.** Let $c' > c$ be arbitrary. The assumption implies that $\sup\{r \geq 0 : w(N_r(o)) > c'r^\alpha\} < \infty$ a.s. For $m \geq 1$, let

$$A_m := \{v \in D : \forall r \geq m : w(N_r(v)) \leq c'r^\alpha\},$$

which is an increasing sequence of equivariant subsets. Since $c' > c$,

$$\lim_{m \to \infty} P[o \in A_m] = 1.$$  \hspace{1cm} (2.4)
Let $R$ be an equivariant covering such that $R(\cdot) \in \{0\} \cup [m, \infty)$ a.s. One has

$$E[R(o)^\alpha] \geq E[R(o)^\alpha 1_{\{N_R(o) \cap A_m \neq \emptyset\}}].$$  \tag{2.5}$$

If $N_R(o) \cap A_m \neq \emptyset$, then $R(o) \neq 0$ and hence $R(o) \geq m$. In the next step, assume that this is the case. Let $v$ be an arbitrary point in $N_R(o) \cap A_m$. By the definition of $A_m$, one gets that for all $r \geq m$, $w(N_r(v)) \leq c' r^\alpha$. Since $N_R(o) \subseteq N_{2R(o)}(v)$, it follows that $w(N_R(o)) \leq w(N_{2R(o)}(v)) \leq 2^\alpha c' R(o)^\alpha$. Therefore, (2.5) gives

$$E[R(o)^\alpha] \geq \frac{1}{2^\alpha c'} E \left[ w(N_R(o)) 1_{\{N_R(o) \cap A_m \neq \emptyset\}} \right].$$  \tag{2.6}$$

By letting $g(u, v) = w(v) 1_{\{v \in N_R(u)\}} 1_{\{N_R(u) \cap A_m \neq \emptyset\}}$, one gets that $g^+(o) = w(N_R(o)) 1_{\{N_R(o) \cap A_m \neq \emptyset\}}$. Also, since there is a ball $N_R(u)$ that covers $o$ a.s., one has $g^-(o) \geq w(o) 1_{\{o \in A_m\}}$ a.s. Therefore, the mass transport principle (1.2.3) and (2.6) imply that

$$E[R(o)^\alpha] \geq \frac{1}{2^\alpha c'} E \left[ w(o) 1_{\{o \in A_m\}} \right].$$

This implies that $H_\alpha(\infty(D)) \geq H^\alpha_m(D) \geq \frac{1}{2^\alpha c'} E \left[ w(o) 1_{\{o \in A_m\}} \right]$. Using (2.4) and letting $m$ tend to infinity gives $M_\alpha^\alpha(D) \geq \frac{1}{2^\alpha c'} E \left[ w(o) \right]$. Since $c' > c$ is arbitrary, the first claim is proved.

Part (ii) of the claim is proved by arguments similar to those in Theorem 2.2 \hfill \square

**Lemma 2.6 (A Lower Bound).** Let $[D, o]$ be a unimodular discrete space, $\alpha \geq 0$ and $c > 0$. Let $w$ be an arbitrary equivariant weight function such that $E[w(o)] < \infty$.

(i) If $\exists \gamma > 0 : \forall r \geq r_0 : w(N_r(o)) \geq cr^\alpha$ a.s., then \udim_M(D) \geq \alpha.

(ii) If growth (w(N_r(o))) \geq \alpha a.s., then \udim_H(D) \geq \alpha.

(iii) If $\lim_{n \to \infty} \inf_{r \to n} P[w(N_r(o)) \leq \delta r^\alpha] = 0$, then \udim_H(D) \geq \alpha.

(iv) If $\lim_{n \to \infty} \exp \left( - \frac{w(N_r(o))}{n^\alpha} \right) \geq \alpha$, then \udim_M(D) \geq \alpha.

**Proof.** The proofs of the first two parts are very similar. So the second part is proved first.

(iii) Let $\beta$, $\gamma$ and $\kappa$ be such that $\gamma < \beta < \kappa < \alpha$. Fix $n \in \mathbb{N}$. Let $S = S_D$ be the equivariant subset obtained by selecting each point $v$ with probability $1 \wedge (n^{-\beta} w(v))$ independently. Let $R_n(v) = n$ if $v \in S_D$, $R_n(v) = 1$ if $N_n(v) \cap S_D \neq \emptyset$, and $R_n(v) = 0$ otherwise. Then $R_n$ is an equivariant covering. It is shown below that $E[R_n(o)^\gamma] \to 0$. 


Let $M := \sup\{r \geq 0 : w(N_r(o)) < r^\alpha\}$. By the assumption, $M < \infty$ a.s. One has

$$E[R_n(o)] = n^n P[o \in S_D] + P[N_n(o) \cap S_D = \emptyset]$$

$$= n^n E[1 \wedge n^{-\beta} w(o)] + E\left[\prod_{v \in N_n(o)} (1 - (1 \wedge n^{-\beta} w(v)))\right]$$

$$\leq n^n E[w(o)] + E[\exp(-n^{-\beta} w(N_n(o)))]$$

$$= n^n E[w(o)] + E[\exp(-n^{-\beta} w(N_n(o))) | M < n] P[M < n]$$

$$+ E[\exp(-n^{-\beta} w(N_n(o))) | M \geq n] P[M \geq n]$$

$$\leq n^n E[w(o)] + \exp(-n^{\alpha-\beta}) + P[M \geq n].$$

Therefore, $E[R_n(o)] \to 0$. It follows that $udim_H(D) \geq \gamma$. Since $\gamma$ is arbitrary, this implies $udim_H(D) \geq \alpha$.

\((\text{I})\) Only a small change is needed in the above proof. For $n \geq n_0$, let $R_n(v) = n$ if either $v \in S_D$ or $N_n(v) \cap S_D = \emptyset$, and let $R_n(v) = 0$ otherwise. Note that $R_n$ is a covering by balls of equal radii. By the same computations and the assumption $M \leq n_0$, one gets

$$P[R_n(o) \neq 0] \leq n^{-\beta} E[w(o)] + \exp(-n^{\alpha-\beta}),$$

which is of order $n^{-\beta}$ for large $n$. This implies that $udim_M(D) \geq \beta$. Since $\beta$ is arbitrary, one gets $udim_M(D) \geq \alpha$ and the claim is proved.

\((\text{II})\) Let $\beta < \alpha$ be arbitrary. It will be proved below that there is a sequence $r_1, r_2, \ldots$ such that $E[\exp(-r_n^{-\beta} w(N_r(o)))] \to 0$. If so, by a slight modification of the proof of Part (II), one can find a sequence of equivariant coverings $R_n$ such that $E[R_n(o)] < \infty$ and (II) is proved.

Let $\epsilon > 0$ be arbitrary. By the assumption, there is $\delta > 0$ and $r \geq 1$ such that $P[w(N_r(o)) \leq \delta r^\alpha] < \epsilon$. So

$$E[\exp(-r^{-\beta} w(N_r(o)))] \leq E[\exp(-r^{-\beta} w(N_r(o)) | w(N_r(o)) > \delta r^\alpha]$$

$$+ P[w(N_r(o)) \leq \delta r^\alpha]$$

$$\leq \exp(-\delta r^{\alpha-\beta}) + \epsilon.$$

Note that for fixed $\epsilon$ and $\delta$ as above, $r$ can be arbitrarily large. Now, choose $r$ large enough for the right hand side to be at most $2\epsilon$. This shows that $E[\exp(-r^{-\beta} w(N_r(o)))]$ can be arbitrarily small and the claim is proved.

\((\text{III})\) As before, let $R_n(v) = n$ if either $v \in S_D$ or $N_n(v) \cap S_D = \emptyset$, and let $R_n(v) = 0$ otherwise. The calculations in the proof of part (II) show that

$$P[R_n(o) \neq 0] \leq n^{-\beta} E[w(o)] + E[\exp(-n^{-\beta} w(N_n(o)))] .$$

Now, the assumption implies the claim.

\(\square\)

**Remark 2.7.** The assumption in part (III) of Lemma 2.6 is equivalent to the condition that a subsequence of the family of random variables $w(N_r(o))/r^\alpha$
converges to zero in probability (as \( r \to \infty \)). Also, from the proof of the lemma, one can see that this assumption is equivalent to
\[
\liminf_{n \to \infty} \mathbb{E} \left[ \exp \left( - \frac{w(N_n(o))}{n^\alpha} \right) \right] = 0.
\]

**Theorem 2.8** (Unimodular Billingsley Lemma). Let \([D, o]\) be a unimodular discrete metric space. For all equivariant weight functions \(w\) such that
\[
0 < \mathbb{E} [w(o)] < \infty,
\]
one has

(i) If the upper and lower growth rates of \(D\) are almost surely constant (e.g., when \([D, o]\) is ergodic), then, almost surely,
\[
\text{growth}(w(N_r(o))) \leq \text{udim}_H(D) \leq \text{growth}(w(N_r(o))) \leq \text{growth}(\mathbb{E}[w(N_r(o))])
\]

(ii) In general,
\[
\text{ess inf } \left( \text{growth}(w(N_r(o))) \right) \leq \text{udim}_H(D) \leq \text{ess inf } \left( \text{growth}(w(N_r(o))) \right) \leq \text{growth}(\mathbb{E}[w(N_r(o))])
\]

**Proof.** The first part is implied by the second one. So it is enough to prove the second part.

The first inequality is implied by part (ii) of Lemma 2.6. The last inequality is implied by Lemma A.1. For the second inequality, assume that \(w(N_r(o)) \leq r^\alpha\) for large \(r\); i.e., \(\limsup_r w(N_r(o))/r^\alpha \leq 1\) a.s. Now, Lemma 2.5 implies that \(\text{udim}_H(D) \leq \alpha\). This proves the result.

**Corollary 2.9.** Under the assumptions of Theorem 2.8 if \(\text{growth}(w(N_r(o)))\) exists and is constant a.s., then
\[
\text{udim}_H(D) = \text{growth}(w(N_r(o))).
\]

Subsection 3.5.1 below provides an example where \(\text{growth}(\cdot) \neq \text{growth}(\cdot)\).

**Remark 2.10.** In fact, the assumption \(\mathbb{E}[w(o)] < \infty\) in Theorem 2.8 is only needed for the lower bound while the assumption \(\mathbb{E}[w(o)] > 0\) is only needed for the upper bound. These assumptions are also necessary as shown below. For example, assume \(\Phi\) is a point-stationary point process in \(\mathbb{R}\) (see Example I.2.18). For \(v \in \Phi\), let \(w(v)\) be the sum of the distances of \(v\) to its next and previous points in \(\Phi\). This equivariant weight function satisfies \(w(N_r(v)) \geq 2r\) for all \(r\), and hence \(\text{growth}(w(N_r(o))) \geq 1\). But \(\text{udim}_H(\Phi)\) can be strictly less than 1 as shown in Subsection 3.3.1.

Also, the condition that \(w_D\) is non-degenerate a.s. is trivially necessary for the upper bound.
Remark 2.11. In many examples, it is enough to consider \( w \equiv 1 \) in Billingsley’s lemma (i.e., \( w(N_r(o)) = \#N_r(o) \)). Examples where other weight functions are used are 2-ended trees (Subsection 3.1.1), point-stationary point processes (Proposition 2.17), and embedded spaces (Subsection 4.5).

Remark 2.12. A converse to the unimodular Billingsley lemma is the unimodular Frostman lemma, which will be discussed in Section 4.

Remark 2.13. Without the assumption of part (i) of Theorem 2.8, the claim is still valid for the sample Hausdorff dimension of \( D \), which will be discussed in Part III.

Here is a first application of the unimodular Billingsley lemma.

Proposition 2.14. Let \( [G, o] \) be a unimodular random graph equipped with the graph-distance metric. If \( G \) is infinite almost surely, then \( \text{udim}_M(G) \geq 1 \) and else, \( \text{udim}_M(G) = \text{udim}_H(G) = 0 \).

Proof. If \( G \) is infinite a.s., then for \( w_G \equiv 1 \), one has \( w(N_r(o)) \geq r \) for all \( r \). So part (i) of Lemma 2.6 implies the first claim.

For all discrete spaces \( G \), let \( w_G(\cdot) \equiv 1 \) if \( G \) is finite and \( w_G(\cdot) \equiv 0 \) otherwise. One has \( \text{growth}(w(N_r(o))) = 0 \). If \( G \) is finite with positive probability, then \( \mathbb{E}[w(o)] > 0 \). Therefore, the unimodular Billingsley lemma (Theorem 2.8) implies that \( \text{udim}_H(G) = 0 \), which in turn implies the second claim.

Remark 2.15. The upper bound in Theorem 2.8 is analogous to Billingsley’s lemma in the continuum setting (see e.g., Lemma 3.1 in [9]). It is interesting that there is no need to assume \( [D, o] \) is embedded in \( \mathbb{R}^k \) or the bounded subcover property holds (see Remark 3.2 in [9]). Note also that \( \text{growth}(w(N_r(o))) \) does not depend on the origin in contrast to the analogous term in the continuum version.

2.3 Bounds for Point Processes

The next results use the following equivariant covering. Let \( \varphi \) be a discrete subset of \( \mathbb{R}^k \) equipped with the \( l_\infty \) metric and \( r \geq 1 \). Let \( C := C_r := [0, r)^k \), \( U := U_r \) be a point chosen uniformly at random in \( -C \), and consider the partition \( \{C + U + z : z \in r\mathbb{Z}^k\} \) of \( \mathbb{R}^k \) by cubes. Then, for each \( z \in r\mathbb{Z}^k \), choose a random element in \( (C + U + z) \cap \varphi \) independently (if the intersection is nonempty). The distribution of this random element should depend on the set \( (C + U + z) \cap \varphi \) in a translation-invariant way (e.g., choose with the uniform distribution or choose the least point in the lexicographic order). Let \( R = R_\varphi \) assign the value \( r \) to the selected points and zero to the other points of \( \varphi \). As in Example I.3.16, one can show that \( R \) is an equivariant covering. Also, each point is covered at most \( 3^k \) times. So \( R \) is \( 3^k \)-bounded (Definition I.3.12).

Theorem 2.16 (Minkowski Dimension in the Euclidean Case). Let \( \Phi \) be a point-stationary point process in \( \mathbb{R}^k \) and assume the metric in \( \Phi \) is equivalent
to the Euclidean metric. Then, for all equivariant weight functions \( w \) such that \( w(0) > 0 \) a.s., one has

\[
\text{udim}_M(\Phi) = \text{decay} \left( \mathbb{E} \left[ \frac{w(0)}{w(C_r + U_r)} \right] \right) \leq \text{growth} \left( \mathbb{E} \left[ \frac{w(N_r(0))}{w(0)} \right] \right),
\]

where \( U_r \) is a uniformly at random point in \(-C_r\) independent of \( \Phi \) and \( w \).

Proof. By Theorem I.3.41, one may assume the metric on \( \Phi \) is the \( l_\infty \) metric without loss of generality. Given any \( r > 0 \), consider the equivariant covering \( R \) described above, but when choosing a random element of \((C_r + U_r + z) \cap \varphi\), choose point \( v \) with probability \( \frac{w(v)}{w(C_r + U_r + z)} \) (conditioned on \( w(\varphi) \)). One gets

\[
P[0 \in R] = \mathbb{E} \left[ \frac{w(0)}{w(C_r + U_r)} \right].
\]

As mentioned above, \( R \) is equivariant and uniformly bounded (for all \( r > 0 \)). So Lemma I.3.13 implies both equalities in the claim. The inequalities are implied by the facts that \( w(C_r + U_r) \leq w(N_r(0)) \) and

\[
\mathbb{E} \left[ \frac{w(0)}{w(N_r(0))} \right] \mathbb{E} \left[ w(N_r(0)) \right] \geq \mathbb{E} \left[ \sqrt{w(0)} \right]^2 > 0,
\]

which is implied by the Cauchy-Schwartz inequality.

In many examples, the case where \( w(\cdot) \equiv 1 \) is used. An example where the decay rate of \( \mathbb{E} \left[ 1/\#N_n(o) \right] \) is strictly smaller than the growth rate of \( \mathbb{E} \left[ \#N_n(o) \right] \) can be found in Example III.5.2 for suitable parameters. However, this example is not ergodic (see Remark I.3.28).

Proposition 2.17. If \( \Phi \) is a point-stationary point process in \( \mathbb{R}^k \) and the metric on \( \Phi \) is equivalent to the Euclidean metric, then \( \text{udim}_H(\Phi) \leq k \).

Proof. One may assume the metric on \( \Phi \) is the \( l_\infty \) metric without loss of generality. Let \( C := [0, 1)^k \) and \( U \) be a random point in \(-C \) chosen uniformly. For all discrete subsets \( \varphi \subseteq \mathbb{R}^k \) and \( v \in \varphi \), let \( C(v) \) be the cube containing \( v \) of the form \( C + U + z \) (for \( z \in \mathbb{Z}^k \)) and \( w(\varphi)(v) := 1/\#(\varphi \cap C(v)) \). Now, \( w \) is an equivariant weight function. The construction readily implies that \( w(N_r(o)) \leq (2r + 1)^k \). Moreover, by \( w \leq 1 \), one has \( \mathbb{E} \left[ w(0) \right] < \infty \). Therefore, the unimodular Billingsley lemma (Theorem 2.8) implies that \( \text{udim}_H(\Phi) \leq k \).

Proposition 2.18. If \( \Psi \) is a stationary point process in \( \mathbb{R}^k \) and \( \Phi \) is its Palm version, then

\[
\text{udim}_M(\Phi) = \text{udim}_H(\Phi) = k.
\]

Moreover,

\[
\mathcal{M}^k(\Phi) = 2^k \rho(\Psi),
\]

where \( \rho(\Psi) \) is the intensity of \( \Psi \).
Notice that if $\Phi \subseteq \mathbb{Z}^k$, the claim is implied by Theorem I.3.48. The general case is treated below.

Proof. For the first claim, by Proposition 2.17, it is enough to prove that $\text{udim}_M(\Phi) \geq k$. Let $\Psi'$ be a shifted square lattice independent of $\Psi$ (i.e., $\Psi' = \mathbb{Z}^k + U$, where $U \in [0, 1)^k$ is chosen uniformly, independently of $\Psi$). Let $\Psi'' := \Psi \cup \Psi'$. Since $\Psi''$ is a superposition of two independent stationary point processes, it is a stationary point process itself (see e.g., [10]). By letting $p := \rho(\Psi)/\rho(\Psi) + 1)$, the Palm version $\Phi''$ of $\Psi''$ is obtained by the superposition of $\Phi$ and an independent stationary lattice with probability $p$ (heads), and the superposition of $\mathbb{Z}^k$ and $\Psi$ with probability $1 - p$ (tails).

For all $n \in \mathbb{N}$, there exists a disjoint $n$-covering of $\mathbb{Z}^k$ (Example I.3.15). In both cases (heads or tails) above, one can consider this covering as a random subset of the shifted lattice. It is easy to see that it provides an equivariant $(n+1)$-covering of $\Phi''$ (note that by enlarging the balls, all of $\mathbb{R}^k$ is covered). Also, the probability of having a ball centered at the origin is $(1 - p)(2n+1)^{-k}$. It follows that $\text{udim}_M(\Phi''') \geq k$.

Note that $\Phi'''$ has two natural equivariant subsets which, after conditioning to contain the origin, have the same distributions as $\Phi$ and $\mathbb{Z}^k$ respectively. Therefore, one can use Theorem I.3.48 to deduce that $\text{udim}_M(\Phi) \geq \text{udim}_M(\Phi''') = k$. Therefore, Proposition 2.17 implies that $\text{udim}_M(\Phi) = \text{udim}_M(\Phi''') = k$.

Also, by using Theorem I.3.48 twice, one gets $\mathcal{M}^k(\Phi) = p\mathcal{M}^k(\Phi''')$ and $\mathcal{M}^k(\mathbb{Z}^k) = (1 - p)\mathcal{M}^k(\Phi''')$. Therefore,

$$\mathcal{M}^k(\Phi) = \frac{p}{1 - p}\mathcal{M}^k(\mathbb{Z}^k) = 2^k \rho(\Psi),$$

where the last equality is by Proposition I.3.38. So the claim is proved.

The last claim of Proposition 2.18 suggests the following, which is verified when $k = 1$ in the next proposition.

**Conjecture 2.19.** If $\Phi$ is a point-stationary point process in $\mathbb{R}^k$ which is not the Palm version of any stationary point process, then $\mathcal{M}^k(\Phi) = 0$.

**Proposition 2.20.** Conjecture 2.19 is true when $k = 1$.

Proof. Denote $\Phi$ as $\Phi = \{S_n : n \in \mathbb{Z}\}$ such that $S_0 = 0$ and $S_n < S_{n+1}$ for each $n$. Then, the sequence $(S_{n+1} - S_n)_n$ is stationary under shifting the indices (see Example I.2.18). The assumption that $\Phi$ is not the Palm version of a stationary point process is equivalent to $\mathbb{E}[S_1] = \infty$ (see [11] or Proposition 6 of [17]). Indeed, if $\mathbb{E}[S_1] < \infty$, then one could bias the probability measure by $S_1$ (Definition I.B.1) and then shift the whole process by $-U$, where $U \in [0, S_1]$ is chosen uniformly and independently.

Since $\mathbb{E}[S_1] = \infty$, Birkhoff’s pointwise ergodic theorem [20] implies that $\lim_n (S_1 + \cdots + S_n)/n = \infty$. This in turn implies that $\lim_r \# N_r(0)/r = 0$. Therefore, Lemma 2.5 gives that $\mathcal{M}^1(\Phi) = \infty$; i.e., $\mathcal{M}^1(\Phi) = 0$. \qed
2.4 Connections to Birkhoff’s Pointwise Ergodic Theorem

This subsection discusses a corollary of the unimodular Billingsley lemma. The reader may skip it at first reading.

The following corollary of the unimodular Billingsley lemma is of independent interest. Note that the statement does not involve dimension.

**Theorem 2.21.** Let \([D, o]\) be a unimodular discrete space. For any two equivariant weight functions \(w_1\) and \(w_2\), if \(E[w_1(o)] < \infty\) and \(w_2(\cdot)\) is non-degenerate a.s., then

\[
\text{growth}(w_1(N_r(o))) \leq \text{growth}(w_2(N_r(o))), \quad \text{a.s.}
\]

In particular, if \(w_1(N_r(o))\) and \(w_2(N_r(o))\) have well defined growth rates, then their growth rates are equal.

Note that the condition \(E[w_1(o)] < \infty\) is necessary as shown in Remark 2.10.

**Proof.** Let \(\epsilon > 0\) be arbitrary and

\[
A := \{[D, o] \in D_\ast : \text{growth}(w_1(N_r(o))) > \text{growth}(w_2(N_r(o))) + \epsilon\}.
\]

It can be seen that \(A\) is a measurable subset of \(D_\ast\). Assume \(P([D, o] \in A) > 0\).

Denote by \([D', o']\) the random pointed discrete space obtained by conditioning \([D, o]\) on \(A\). Since \(A\) does not depend on the root (i.e., if \([D, o] \in A\), then \(\forall v \in D : [D, v] \in A\)), by a direct verification of the mass transport principle (I.2.2), one can show that \([D', o']\) is unimodular. So by using the unimodular Billingsley lemma (Theorem 2.8) twice, one gets

\[
\text{ess inf} \left(\text{growth}(w_1(N_r(o')))) \right) \leq \text{udim}_H(D') \leq \text{ess inf} \left(\text{growth}(w_2(N_r(o')))) \right).
\]

By the definition of \(A\), this contradicts the fact that \([D', o'] \in A\) a.s. So \(P([D, o] \in A) = 0\) and the claim is proved.

**Remark 2.22.** Theorem 2.21 is a generalization of a weaker form of Birkhoff’s pointwise ergodic theorem as explained in the following.

(i) If \(D = \mathbb{Z}\), then \(w_1\) and \(w_2\) are stationary under the shift \(n \mapsto n - 1\) (see Example I.2.18). Therefore, Birkhoff’s pointwise ergodic theorem (see e.g., [20]) implies that \(\lim \frac{w_1(N_r(o))}{w_2(N_r(o))} = \frac{E[w_1(0)]}{E[w_2(0)]}\) a.s.

This equation is stronger than the claim of Theorem 2.21 in this case.

(ii) If \([D, o]\) is a point-stationary point process in \(\mathbb{R}\), then the above argument still holds by using stationarity under shifting the origin to its next point (see Example I.2.18).

(iii) If \([D, o]\) is the Palm version of a stationary point process, the cross-ergodic theorem (see e.g., [4]) also gives the property stated in (i).

Note that amenability is not assumed in Theorem 2.21. But the claim is weaker since nothing can be said about \(\lim \frac{w_1(N_r(o))}{w_2(N_r(o))}\).
3 Examples

The structure of this section is analogous to that of Section I.4. It provides further results on the examples introduced there. Some new examples are also presented.

3.1 General Unimodular Trees

3.1.1 Unimodular Two-Ended Trees

Here, another proof is given for the fact that the Minkowski and Hausdorff dimensions of any unimodular two-ended tree are equal to one (Theorem I.4.2) using the unimodular Billingsley lemma.

Let \([T, o]\) be a unimodular two-ended tree. For all two-ended trees \(T\) and \(v \in T\), let \(w_T(v) = 1\) if \(v\) belongs to the trunk of \(T\) and 0 otherwise. It can be seen that \(w\) is an equivariant process (Definition I.2.21). Let \(c(v)\) be the distance of \(v\) to the trunk of \(T\). For \(n \in \mathbb{N}\) larger than \(c(v)\), one has \(w(N_n(v)) = 2(n - c) + 1\). Therefore, the unimodular Billingsley lemma (Theorem 2.8) implies that \(\text{udim}_H(T) \leq 1\). On the other hand, Proposition 2.14 implies that \(\text{udim}_M(T) \geq 1\). Therefore, \(\text{udim}_M(T) = \text{udim}_H(T) = 1\).

3.1.2 Unimodular Trees with Infinitely Many Ends

The following conjecture is of independent interest beyond its connections to the dimension. The authors believe it is new.

**Conjecture 3.1.**

(i) There is no unimodular random tree with polynomial growth and infinitely many ends a.s.

(ii) There is no equivariant metric (Definition I.3.40) on the \(k\)-regular tree \(T_k\) (for \(k \geq 3\)) that has polynomial growth.

By the unimodular Billingsley lemma (Theorem 2.8), this conjecture is implied by Conjectures I.4.7 and I.4.8.

A stronger form of part (i) of the above conjecture is that there is no unimodular random tree with infinitely many ends a.s. such that \(\text{growth}(N_r(o)) < \infty\).

The following example and proposition provide special cases where Conjecture I.4.8 is known to be true. Another example is provided in Subsection 3.2.1 below.

In the following, the ball of radius \(r\) under the metric \(d'\) and centered at \(v\) is denoted by \(N'_r(v)\).

**Example 3.2.** By assigning i.i.d. random lengths to the edges on the \(k\)-regular tree \(T_k\) (as in Example I.2.24), one gets an equivariant metric on \(T_k\), denoted by \(d'\). The claim is that for \(k \geq 3\), one has \(\text{udim}_M(T_k, d') = \text{udim}_H(T_k, d') = \infty\), which implies that Conjecture I.4.8 holds in this case.

By Theorem I.3.41, one can replace the distribution of the lengths with another distribution which is stochastically larger. So one can assume the distribution
of the lengths is non-lattice without loss of generality. Therefore, Theorem 21.1 of [14] for supercritical age-dependent branching processes implies that the limit of \( \#N'_r(o)e^{-\alpha r} \) as \( r \to \infty \) exists a.s. and is positive for some \( \alpha > 0 \). Therefore, Lemma 2.6 implies that \( \text{udim}_M(T_k,d') = \text{udim}_H(T_k,d') = \infty \) and the claim is proved.

Recall from Subsection I.4.1.4 that an equivariant metric \( d' \) is said to be generated by equivariant edge lengths (or a geodesic metric) if for every path \( v_1v_2\ldots v_k \), one has \( d'(v_1,v_k) = \sum_{i=1}^{k-1} d'(v_i,v_{i+1}) \).

**Proposition 3.3.** Let \( d' \) be an equivariant metric on the 3-regular tree \( T_3 \) which is generated by equivariant edge lengths. If the random variable \( \sum_{v \sim o} d'(o,v) \) has finite moment of order \( \alpha \), then \( \text{udim}_M(T_3,d') \geq \alpha \). In particular, if it has finite moments of any order, then \( \text{udim}_M(T_3,d') = \text{udim}_H(T_3,d') = \infty \).

**Proof.** It is enough to assume \( d'(x,y) \geq 1 \) for all \( x \sim y \) since increasing the edge lengths does not increase the dimension (Theorem I.3.41). Consider the following equivariant weight function on \( T_3 \):

\[
w(u) := C \sum_{v \sim u} d'(u,v)^\alpha,
\]

where \( C \) is a constant such that

\[
\forall x \in [0,1] : Cx^\alpha + (1-x)^\alpha \geq \frac{1}{2}.
\]

It is easy to see that such a \( C \) exists. Now, Lemma 3.4 below, which is a deterministic result, implies that \( w(N'_r(o)) \geq r^\alpha \) a.s. for every \( r \geq 0 \). Also, the assumption on \( d' \) implies that \( w(o) \) has finite mean. So Lemma 2.6 implies that \( \text{udim}_M(T_3,d') \geq \alpha \) and the claim is proved.

The following lemma is used in the last proof.

**Lemma 3.4.** Let \( \alpha < \infty \) and \((T,o)\) be a deterministic rooted tree such that \( \text{deg}(o) \geq 2 \) and \( \text{deg}(v) \geq 3 \) for all \( v \neq o \). Let \( d' \) be a metric on \( T \) which is generated by a function on the edges such that \( d'() \geq 1 \). Define the weight function \( w \) on \( T \) and the constant \( C \) by (3.1) and (3.2) respectively. Then, for all \( r \geq 0 \), one has \( w(N'_r(o)) \geq r^\alpha \).

**Proof.** For \( r \geq 0 \), let \( f(r) \) be the infimum value of \( w(N'_r(o)) \) for all trees with the stated conditions. So one should prove \( f(r) \geq r^\alpha \). The claim is true for \( r = 0 \). Also, if \( 0 < r < 1 \), one has \( N'_r(o) = \{o\} \) and the claim is trivial. The proof uses induction on \( |r| \). Assume that \( r \geq 1 \) and for all \( s < |r| \), one has \( f(s) \geq s^\alpha \). For \( y \sim o \), let \( T_y \) be the connected component containing \( y \) when the edge \( (o,y) \) is removed. It can be seen that \( [T_y,y] \) satisfies the conditions of
the lemma. Therefore, one obtains

\[
\begin{align*}
w(N'_r(o)) &= w(o) + \sum_{y: y \sim o} w(N'_{r-d'(o,y)}(T_y, y)) \\
&\geq w(o) + \sum_{y: y \sim o} f(r - d'(o,y)) \\
&\geq \sum_{y: y \sim o} \left[ C d'(o,y)^\alpha + (r - d'(o,y))^\alpha \right] \\
&\geq \deg(o) \cdot \min_{0 \leq x \leq r} \{ Cx^\alpha + (r - x)^\alpha \} \\
&\geq \deg(o)r^\alpha/2 \\
&\geq r^\alpha,
\end{align*}
\]

where the third line is by the definition of \(w(o)\) in (3.1) and the induction hypothesis, the fifth line is due to (3.2), and the last line is by the assumption \(\deg(o) \geq 2\). This implies that \(f(r) \geq r^\alpha\) and the induction claim is proved. \(\square\)

The following is a slight generalization of Proposition 3.3, which will be used in Subsection 3.2.3.

**Lemma 3.5.** Let \([T, o]\) be a unimodular tree in which the degree of every vertex is at least 3 a.s. Let \(d'\) be an equivariant metric on \([T, o]\) which is generated by equivariant edge lengths. For \(u \in T\), let \(w(u)\) be the third minimum number in the multi-set \(\{d'(u,v)^\alpha : v \sim u\}\). If \(E[w(o)] < \infty\), then \(\udim_M(T) \geq \alpha\).

**Proof.** By Lemma 2.6, it is enough to show that \(\forall r \geq 0 : w(N'_r(o)) \geq \frac{1}{C} r^\alpha\), where \(C\) is defined by (3.2). Notice that this is a deterministic claim. So consider a realization \((T,o)\) of \([T, o]\). Define the subtree \((T',o)\) of \(T\) by adding vertices recursively as follows. First, add 3 neighbors of \(o\) which are closest to \(o\) (under the metric \(d')\) to \(T'\). Then, recursively, for every newly added vertex \(u\), add 2 neighbors of \(u\) which are closest to \(u\) and not already added. It is straightforward that \(T'\) is a 3-regular tree and \(w(u) \geq \sum_i d'(u,v)^\alpha\), where the sum is over the neighbors of \(u\) in \(T'\). So Lemma 3.4 implies that \(w(N'_r(T', o)) \geq \frac{1}{C} r^\alpha\). Hence, \(w(N'_r(T, o)) \geq \frac{1}{C} r^\alpha\). So the claim is proved. \(\square\)

### 3.2 Instances of Unimodular Trees

#### 3.2.1 Unimodular Galton-Watson Trees

The Galton-Watson tree and the unimodular Galton-Watson tree \([1]\) are recalled in what follows. They differ from the Eternal Galton-Watson tree of Subsection I.4.2.3. Let \(\mu = (\mu_0, \mu_1, \ldots)\) be a probability measure on \(\mathbb{Z}_{\geq 0}\). Start from a single vertex \(o\). For each newly added vertex \(v\), add an independent random number of new vertices (called offspring of \(v\)) with distribution \(\mu\) and connect them to \(v\). By repeating this process, a random tree is constructed which is the Galton-Watson tree with offspring distribution \(\mu\). The unimodular Galton-Watson tree \([T, o]\) is constructed similarly with the difference that
the offspring distribution of the origin is different from that of the other vertices: It has for distribution \( \hat{\mu} = (\frac{n}{m} p_n)_n \), where \( m \) is the mean of \( \mu \) (assumed to be finite). It is shown in [1] that \( [T, o] \) is a unimodular random tree.

In what follows, the trivial case \( p_1 = 1 \) is excluded. If \( m \leq 1 \), then \( T \) is finite a.s.; i.e., there is extinction a.s. Therefore, Proposition 1.3.23 implies that \( \text{udim}_H(T) = 0 \). So assume the supercritical case, namely \( m > 1 \). If \( p_0 > 0 \), then \( T \) is finite with positive probability. So \( \text{udim}_H(T) = 0 \) for the same reason. Nevertheless, one can condition on non-extinction as follows.

**Proposition 3.6.** Let \( [T, o] \) be a supercritical unimodular Galton-Watson tree conditioned on non-extinction. Then,

\[
\text{udim}_M(T) = \text{udim}_H(T) = \infty.
\]

**Proof.** The result for the Hausdorff dimension is followed from the unimodular Billingsley lemma (Theorem 2.8) and the Kesten-Stigum theorem [16], which implies that \( \lim_{n \to \infty} N_n(o) m^{-n} \) exists and is positive a.s.

Computing the Minkowski dimension is more difficult. By part (iv) of Lemma 2.6, it is enough to prove that \( E[(1 - n^{-\alpha} d_n(o))] \) has infinite decay rate for every \( \alpha \geq 0 \). Denote by \( \tilde{T}, \tilde{o} \) the Galton-Watson tree with the same parameters. Using the fact that \( N_n(o) \) is stochastically larger than \( N_{n-1}(\tilde{o}) \), one gets that it is enough to prove the last claim for \( \tilde{T}, \tilde{o} \).

For simplicity, the proof is given for the case \( p_0 = 0 \) only. The general case can be proved with similar arguments and by using the decomposition theorem of supercritical Galton-Watson trees (see e.g., Theorem 5.28 of [18]). By this assumption, the probability of extinction is zero.

Let \( f(s) := \sum_{n} p_n s^n \) be the generating function of \( \mu \). By classical results of the theory of branching processes, for all \( s \leq 1 \),

\[
E[s^{d_n(o)}] = f^{(n)}(s),
\]

where \( d_n(o) := N_n(o) - N_{n-1}(\tilde{o}) \) and \( f^{(n)}(s) \) is the \( n \)-fold composition of \( f \) with itself. Let \( a > 0 \) be fixed and \( g(s) := \frac{a s}{s + a + 1} \) (such functions are frequently used in the literature on branching processes; see, e.g., [3]). One has \( f(0) = g(0) = 0, f(1) = g(1) = 1, f'(1) = m > 1, g'(1) = (1 + a)/a \), and \( f \) is convex. Therefore, \( a \) can be chosen large enough such that \( f(s) \leq g(s) \) for all \( s \in [0, 1] \). So

\[
f^{(n)}(s) \leq g^{(n)}(s) = \frac{a^n s}{a^n + (a + 1)^n (1 - s)},
\]

where the last equality can be checked by induction. Therefore,

\[
f^{(n)}(1 - n^{-\alpha}) \leq \frac{a^n}{a^n + n^{-\alpha} (a + 1)^n},
\]

It follows that decay \( f^{(n)}(1 - n^{-\alpha}) \) = \( \infty \). So the above discussion gives that \( E[(1 - n^{-\alpha} N_n(o))] \) has infinite decay rate and the claim is proved. \( \square \)
3.2.2 Unimodular Eternal Galton-Watson Trees

Unimodular eternal Galton-Watson (EGW) trees were introduced in Subsection I.4.2.3. The following theorem complements Theorem I.4.15.

**Theorem 3.7.** Let \([T, o]\) be a unimodular eternal Galton-Watson tree. If the offspring distribution has finite variance, then

\[
\text{udim}_M(T) = \text{udim}_H(T) = 2.
\]

**Proof.** Theorem I.4.15 proves that \(\text{udim}_M(T) = 2\). So it remains to prove \(\text{udim}_H(T) \leq 2\). By the unimodular Billingsley lemma (Theorem 2.8), it is enough to show that

\[
\mathbb{E}[\#N_n(o)] \leq cn^2
\]

for a constant \(c\).

Recall from Subsection I.4.1.3 that \(F(v)\) represents the parent of vertex \(v\) and \(D(v)\) denotes the subtree of descendants of \(v\). Write \(N_n(o) = Y_0 \cup Y_1 \cup \cdots \cup Y_n\), where \(Y_n := N_n(o) \cap D(o)\) and \(Y_i := N_n(o) \cap D(F^{n-i}(o)) \setminus D(F^{n-i-1}(o))\) for \(0 \leq i < n\). By the explicit construction of EGW trees in [7], \(Y_n\) is a critical Galton-Watson tree up to generation \(n\). Also, for \(0 \leq i < n\), \(Y_i\) has the same structure up to generation \(i\), except that the distribution of the first generation is size-biased minus one (i.e., \((np_{n+1})_n\) with the notation of Subsection 3.2.1). So the assumption of finite variance implies that the first generation in each \(Y_i\) has finite mean, namely \(m'\). Now, one can inductively show that \(\mathbb{E}[\#Y_n] = n\) and \(\mathbb{E}[\#Y_i] = im',\) for \(0 \leq i < n\). It follows that \(\mathbb{E}[\#N_n(o)] \leq (1 + m')n^2\) and the claim is proved. 

3.2.3 The Poisson Weighted Infinite Tree

The Poisson Weighted Infinite Tree (PWIT) is defined as follows (see e.g., [2]). It is a rooted tree \([T, o]\) such that the degree of every vertex is infinite. Regarding \(T\) as a family tree with progenitor \(o\), the edge lengths are as follows. For every \(u \in T\), the set \(\{d(u, v) : v \text{ is an offspring of } u\}\) is a Poisson point process on \(\mathbb{R}^\geq 0\) with intensity function \(x^k\), where \(k > 0\) is a given integer. Moreover, for different vertices \(u\), the corresponding Poisson point processes are jointly independent. See for example [2] for more details. It is also shown there that the PWIT is unimodular. Notice that although each vertex has infinite degree, the PWIT is boundedly-finite as a metric space.

**Proposition 3.8.** The PWIT satisfies

\[
\text{udim}_M(PWIT) = \text{udim}_H(PWIT) = \infty.
\]

**Proof.** Denote the neighbors of \(o\) such that \(d(o, v_1) \leq d(o, v_2) \leq \cdots \) by \(v_1, v_2, \ldots\). It is straightforward that all moments of \(d(o, v_3)\) are finite. Therefore, Proposition 3.3 and Lemma 3.5 imply that \(\text{udim}_M(T) = \infty\). This proves the claim.

3.3 Examples associated with Random Walks

As in Subsection I.4.3, consider the simple random walk \((S_n)_{n \in \mathbb{Z}}\) in \(\mathbb{R}^k\), where \(S_0 = 0\) and the jumps \(S_n - S_{n-1}\) are i.i.d.
The following complements Theorem I.4.17.

Theorem 3.9. Let \( \Phi := \{S_n\}_{n \in \mathbb{Z}} \) be the image of a simple random walk \( S \) in \( \mathbb{R} \), where \( S_0 := 0 \). Assume the jumps \( S_n - S_{n-1} \) are positive a.s.

(i) If \( \beta := \text{decay}(\mathbb{P}[S_1 > r]) \) exists, then
\[
\text{udim}_M(\Phi) = \text{udim}_H(\Phi) = 1 \land \beta.
\]

(ii) \( \text{udim}_M(\Phi) \geq 1 \land \text{decay}(\mathbb{P}[S_1 > r]) \).

(iii) \( \text{udim}_H(\Phi) \leq 1 \land \text{decay}(\mathbb{P}[S_1 > r]) \).

Proof. The claims concerning the Minkowski dimension are proved in Theorem I.4.17. So it is enough to prove part (iii). Since \( \Phi \) is a point-stationary point process in \( \mathbb{R} \) (see Subsection I.4.3.1), Proposition 2.17 implies that \( \text{udim}_H(\Phi) \leq 1 \). Now, assume \( \text{decay}(\mathbb{P}[S_1 > r]) < \beta \). Then, there exists \( c > 0 \) such that \( \mathbb{P}[S_1 > r] > cr^{-\beta} \) for all \( r \geq 1 \). By using Lemma A.2 twice, for the positive and negative parts of the random walk, one can prove that there exists \( C < \infty \) and a random number \( r_0 > 0 \) such that for all \( r \geq r_0 \), one has \( \#N_r(\alpha) \leq Cr^\beta \log \log r \) a.s. Therefore, the unimodular Billingsley lemma (Theorem 2.8) implies that \( \text{udim}_H(\Phi) \leq \beta + \epsilon \) for every \( \epsilon > 0 \), which in turn implies that \( \text{udim}_H(\Phi) \leq \beta \).

The following theorem complements Theorem I.4.18. It is readily implied by Theorem 3.9 above.

Theorem 3.10. Let \( \Psi \) be the zero set of the symmetric simple random walk on \( \mathbb{Z} \) with uniform jumps in \( \{\pm 1\} \). Then,
\[
\text{udim}_M(\Psi) = \text{udim}_H(\Psi) = \frac{1}{2}.
\]

Example 3.11 (Infinite Hausdorff Measure). In Theorem 3.9, assume \( \mathbb{P}[S_1 > r] = 1/\log r \) for large enough \( r \). Then, part (i) of the theorem implies that \( \text{udim}_H(\Phi) = 0 \). However, since \( \Phi \) is infinite a.s., it has infinite 0-dimensional Hausdorff measure (Proposition I.3.37).

Example 3.12 (Zero Hausdorff Measure). In Theorem 3.9, assume \( \mathbb{P}[S_1 > r] = 1/r \) for large enough \( r \). Then, part (i) of the theorem implies that \( \text{udim}_H(\Phi) = 1 \). Since \( \mathbb{E}[S_1] = \infty \), \( \Phi \) is not the Palm version of any stationary point process (see Proposition 2.20). Therefore, Proposition 2.20 implies that \( M^1(\Phi) = 0 \).
3.3.2 The Graph of the Simple Random Walk

The graph of the random walk \((S_n)_{n \in \mathbb{Z}}\) is

\[ \Psi := \{(n, S_n) : n \in \mathbb{Z}\} \subseteq \mathbb{R}^{k+1}. \]

Notice that \(\Psi\) is the image of the simple random walk with jumps \((1, S_n - S_{n-1})\). Since no point is visited more than once, the arguments in Subsection I.4.3.1 show that \(\Psi\) is a point-stationary point process. Hence, \([\Psi, 0]\) is unimodular.

Since \(\Psi \cap [-n, n]^{k+1}\) has at most \(2n + 1\) elements, the mass distribution principle (Theorem 2.2) implies that \(\text{udim}_H(\Psi) \leq 1\). In addition, if \(S_1\) has finite first moment, then the strong law of large numbers implies that \(\lim n^{-1} S_n = \mathbb{E}[S_1]\). This implies that \(\lim inf_n \text{udim}_H(\Psi \cap [-n, n]^{k+1}) = 0\). Therefore, the unimodular Billingsley lemma (Theorem 2.8) implies that \(\text{udim}_H(\Psi) \geq 1\). Hence, \(\text{udim}_H(\Psi) = 1\).

Below, the focus is on the case \(k = 1\) and on the following metric:

\[ d((x, y), (x', y')) := \max\{\sqrt{|x - x'|}, |y - y'|\}. \tag{3.3} \]

Theorem I.3.41 implies that, by considering this metric, unimodularity is preserved and dimension is not decreased. Under this metric, the ball \(N_n(0)\) is \(\Psi \cap [-n^2, n^2] \times [-n, n]\). Note that the whole \(\mathbb{Z}^2\) has an equivariant disjoint \(n\)-covering similar to the one in Subsection 2.3. By using Theorem 2.8 one readily obtains that \(\mathbb{Z}^2\) has Minkowski and Hausdorff dimension 3 under this metric.

**Proposition 3.13.** If the jumps are \(\pm 1\) uniformly, under the metric (3.3), the graph \(\Psi\) of the simple random walk satisfies

\[ \text{udim}_M(\Psi) = \text{udim}_H(\Psi) = 2. \]

**Proof.** Let \(n \in \mathbb{N}\). The ball \(N_n(0)\) has at most \(2n^2 + 1\) elements. So the mass distribution principle (Theorem 2.2) implies that \(\text{udim}_H(\Psi) \leq 2\). For the other side, let \(C\) be the equivariant disjoint covering of \(\mathbb{Z}^2\) by translations of the rectangle \([-n^2, n^2] \times [-n, n]\) (similar to Example I.3.15). For each rectangle \(\sigma \in C\), select the right-most point in \(\sigma \cap \Psi\) and let \(S = S_{\Psi}\) be the set of selected points. By construction, \(S\) gives an \(n\)-covering of \(\Psi\) and it can be seen that it is an equivariant covering. Let \(\sigma_0\) be the rectangle containing the origin. By construction, \(0 \in S\) if and only if it is either on a right-edge of \(\sigma_0\) or on a horizontal edge of \(\sigma_0\) and the random walk stays outside \(\sigma_0\). The first case happens with probability \(1/(2n^2 + 1)\). By classical results concerning the hitting time of random walks, one can obtain that the probability of the second case lies between two constant multiples of \(n^{-2}\). It follows that \(\mathbb{P}[0 \in S]\) lies between two constant multiples of \(n^{-2}\). Therefore, \(\text{udim}_M(\Psi) \geq 2\). This proves the claim. \(\square\)

**Remark 3.14.** One can generalize the above proposition by allowing \(S_n \in \mathbb{R}^k\) and assuming that the jumps have zero mean and finite second moments.
3.4 A Drainage Network Model

Let \([T,o]\) be the one-ended tree in Subsection I.4.5. Recall that the set of vertices is the even lattice \(\{(x,y) \in \mathbb{Z}^2 : x + y \mod 2 = 0\}\), and the parent \(F(x,y)\) of vertex \((x,y)\) is \((x \pm 1, y - 1)\), where the sign is chosen uniformly and independently. The following theorem complements Theorem I.4.22.

**Theorem 3.15.** Under the graph-distance metric, one has

\[
\text{udim}_M(T) = \text{udim}_H(T) = \frac{3}{2}.
\]

**Proof.** Theorem I.4.22 proves that \(\text{udim}_M(T) = \frac{3}{2}\). So it is enough to prove \(\text{udim}_H(T) \leq \frac{3}{2}\). To use the unimodular Billingsley lemma, an upper bound on \(E[\#N_n(o)]\) is derived. Let \(e_{k,l} := \#(F^{-k}(F^l(o)) \setminus F^{-(k-l-1)}(F^{l-1}(o)))\) be the number of descendants of order \(k\) of \(F^l(o)\) which are not a descendant of \(F^{l-1}(o)\) (for \(l = 0\), let it be just \(\#F^{-k}(o)\)). One has \(\#N_n(o) = \sum_{k,l} e_{k,l}1\{k+l \leq n\}\). It can be seen that \(E[e_{k,l}]\) is equal to the probability that two independent paths of length \(k\) and \(l\) starting both at \(o\) do not collide at another point. Therefore, \(E[e_{k,l}] \leq c(k \land l)^{-\frac{1}{2}}\) for some \(c\) and all \(k, l\). This implies that (in the following, \(c\) is updated at each step to a new constant without changing the notation)

\[
E \left[ \sum_{k,l \geq 0} e_{k,l}1\{k+l \leq n\} \right] \leq \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c(k \land l)^{-\frac{1}{2}}(n - k)
\]

\[
\leq cn \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} k^{-\frac{1}{2}}
\]

\[
\leq cn^\frac{3}{2}.
\]

The above inequalities imply that \(E[\#N_n(o)] \leq cn^\frac{3}{2}\) for some \(c\) and all \(n\). Therefore, the unimodular Billingsley lemma (Theorem 2.8) implies that \(\text{udim}_H(T) \leq \frac{3}{2}\). So the claim is proved. \(\square\)

3.5 Self Similar Unimodular Spaces

In this subsection, two examples are presented which have some kind of self-similarity heuristically, but do not fit into the framework of Subsection I.4.6.

3.5.1 Unimodular Discrete Spaces Defined by Digit Restriction

Let \(J \subseteq \mathbb{Z}_{\geq 0}\). For \(n \geq 0\), consider the set of numbers with expansion \((a_n a_{n-1} \ldots a_0)\) in base 2 such that \(a_i = 0\) for every \(i \notin J\). As in the definition of the unimodular discrete Cantor set (Subsection I.4.6.1), one can shift this set randomly and take a limit to obtain a unimodular discrete space. This can be constructed in the following way as well: Let \(T_0 := \{0\}\). If \(n \in J\), let \(T_{n+1} := T_n \cup (T_n \pm 2^n)\), where the sign is chosen i.i.d., each sign with probability \(1/2\). If \(n \notin J\), let \(T_{n+1} := T_n\). Finally, let \(\Psi := \cup_n T_n\).
The upper and lower asymptotic densities of $J$ in $\mathbb{Z}^{\geq 0}$ are defined by $\overline{d}(J) := \limsup_n \frac{1}{n} J_n$ and $\underline{d}(J) := \liminf_n \frac{1}{n} J_n$, where $J_n := |J \cap \{0, \ldots, n\}|$.

**Proposition 3.16.** Almost surely,

$$\text{udim}_H(\Psi) = \overline{\text{udim}}_M(\Psi) = \text{growth}(|N_n(o)|) = \overline{d}(J),$$

$$\text{udim}_M(\Psi) = \underline{\text{udim}}_M(\Psi) = \text{growth}(|N_n(o)|) = \underline{d}(J).$$

**Proof.** Let $n \geq 0$ be given. Cover $T_n$ by a ball of radius $2^n$ whose center the minimal element of $T_n$. By the same recursive definition, one can cover $T_{n+1}$ by either 1 or 2 balls of the same radius. Continuing the recursion, an equivariant $2^n$-covering $R_n$ is obtained. It is straightforward to see that $P[R_n(o) > 0] = 2^{-J_n}$. Since these coverings are uniformly bounded (Definition I.3.12), Lemma I.3.13 implies that $\overline{\text{udim}}_M(\Psi) = \overline{d}(J)$ and $\underline{\text{udim}}_M(\Psi) = \underline{d}(J)$. One has $|T_m| = 2^{J_m}$. (3.4)

This implies that $|N_{2^n}(o)| \leq 2^{J_n+1}$. One can deduce that $\text{growth}(|N_n(o)|) \leq \overline{d}(J)$. So the unimodular Billingsley lemma (Theorem 2.8) gives $\text{udim}_H(\Psi) \leq \overline{d}(J)$. This proves the claim.

### 3.5.2 Randomized Discrete Cantor set

Let $0 \leq p \leq 1$ and $b > 1$. The random Cantor set in $\mathbb{R}^k$ [11] (see also [9]) is defined by $\Lambda_k(b, p) := \cap_n E_n$, where $E_n$ is defined by the following random algorithm: Let $E_0 := [0, 1]^k$. For each $n \geq 0$ and each $b$-adic cube of edge length $b^{-n}$ in $E_n$, divide it into $b^k$ smaller $b$-adic cubes of edge length $b^{-n-1}$. Keep each smaller $b$-adic cube with probability $p$ and delete it otherwise independently from the other cubes. Let $E_{n+1}$ be the union of the kept cubes. It is shown in Section 3.8 of [9] that $\Lambda_k(b, p)$ is empty for $p \leq b^{-k}$ and otherwise, has dimension $k + \log_b p$ conditioned on being non-empty. This section proposes a unimodular discrete analogue of the random Cantor set.

For each $n \geq 0$, let $K_n$ be the set of lower left corners of the $b$-adic cubes forming $E_n$. It is easy to show that $K_n$ tends to $\Lambda_k(b, p)$ a.s. under the Hausdorff metric.

**Proposition 3.17.** Let $K'_n$ denote the random set obtained by biasing the distribution of $K_n$ by $\#K_n$ (Definition I.B.1). Let $\alpha'_n$ be a point chosen uniformly at random in $K_n$.

(i) The distribution of $[b^n K'_n, \alpha'_n]$ converges to some unimodular discrete space $[\hat{K}, \hat{o}]$.

(ii) If $p < b^{-k}$, then $\hat{K}$ is finite a.s., hence, $\text{udim}_H(\hat{K}) = 0$ a.s.

(iii) If $p \geq b^{-k}$, then $\hat{K}$ is infinite a.s. and

$$\text{udim}_H(\hat{K}) = \text{udim}_M(\hat{K}) = k + \log_b p, \quad a.s.$$
Note that in contrast to the continuum analogue [15], for \( p = b^{-k} \), the set is non-empty and even infinite, though still zero dimensional. Also, for \( p < b^{-k} \) the set is non-empty as well.

To prove the above proposition, the following construction of \( \hat{K} \) will be used. First, consider the usual nested sequence of partitions \( \Pi_n \) of \( \mathbb{Z}^k \) by translations of the cube \( \{0, \ldots, b^n - 1\}^k \), where \( n \geq 0 \). To make it stationary, shift each \( \Pi_n \) randomly as follows. Let \( a_0, a_1, \ldots \in \{0, 1, \ldots, b-1\}^k \) be i.i.d. uniform numbers and let \( U_n = \sum_{i=0}^{\infty} a_i b^i \in \mathbb{Z}^k \). Shift the partition \( \Pi_n \) by the vector \( U_n \) to form a partition denoted by \( \Pi'_n \). It is easy to see that \( \Pi'_n \) is a nested sequence of partitions.

**Lemma 3.18.** Let \( (\Pi'_n) \) be the stationary nested sequence of partitions of \( \mathbb{Z}^k \) defined above. For each \( n \geq 0 \) and each cube \( C \in \Pi'_n \) that does not contain the origin, with probability \( 1 - p \) (independently for different choices of \( C \)), mark all points in \( C \cap \mathbb{Z}^k \) for deletion. Then, the set of the unmarked points of \( \mathbb{Z}^k \), pointed at the origin, has the same distribution as \( [\hat{K}, \hat{o}] \) defined in Proposition 3.17.

**Proof of Lemma 3.18** Let \( \Phi \) be the set of unmarked points in the algorithm. For \( n \geq 0 \), let \( \Gamma_n \) be the cube in \( \Pi'_n \) that contains the origin. It is proved below that \( \Gamma_n \cap \Phi \) has the same distribution as \( b^n(\Gamma_n - \hat{o}_n) \). This easily implies the claim.

Let \( A_n \subseteq [0, 1]^k \) be the set of possible outcomes of \( \hat{o}_n \). One has \( \#A_n = b^{kn} \). For \( v \in A_n \), it is easy to see that the distribution of \( b^n(\Gamma_n - \hat{o}_n) \), conditioned on \( \hat{o}_n = v \), coincides with the distribution of \( \Gamma_n \cap \Phi \) conditioned on \( \Gamma_n = b^n([0, 1]^k - v) \). So it remains to prove that \( \mathbb{P}[\hat{o}_n = v] = \mathbb{P}[\Gamma_n = b^n([0, 1]^k - v)] \), which is left to the reader.

Here is another description of \( \hat{K} \). The nested structure of \( \bigcup_n \Pi'_n \) defines a tree as follows. The set of vertices is \( \bigcup_n \Pi'_n \). For each \( n \geq 0 \), connect (the vertex corresponding to) every cube in \( \Pi'_n \) to the unique cube in \( \Pi'_{n+1} \) that contains it. This tree is the canopy tree (Subsection I.4.2.1) with offspring cardinality \( N := b^k \), except that the root (the cube \( \{0\} \) is always a leaf. Now, keep each vertex with probability \( p \) and remove it with probability \( 1 - p \) in an i.i.d. manner. Let \( T \) be the connected component of the remaining graph that contains the root. Conditioned on the event that \( T \) is infinite, \( \hat{K} \) corresponds to the set of leaves in the connected component of the root.

**Proof of Proposition 3.17** The unimodular Billingsley lemma is used to get an upper bound on the Hausdorff dimension. For this \( \mathbb{E}[\#N_{b^n}(\hat{o})] \) is studied. Consider the tree \( [T, \hat{o}] \) defined above and obtained by the percolation process on the canopy tree with offspring cardinality \( N := b^k \). Let \( C \) be any cube in \( \Pi'_n \) that does not contain the origin. Note that the subtree of descendants of \( C \) in the percolation cluster (conditioned on keeping \( C \)) is a Galton-Watson tree with binomial offspring distribution with parameters \( (N, p) \). Classical results on branching processes say \( \mathbb{E}\left[\#C \cap \hat{K} | \Pi'_n\right] = pm^t \), where \( m := pb^k \). So the
construction implies that

\[ E \left( \#C_n \cap \hat{K} \right) = 1 + p(N - 1) \left( m^{n-1} + m^{n-2} + \cdots + 1 \right). \]

For \( m > 1 \), the latter is bounded by \( lm^n \) for some constant \( l \) not depending on \( n \). Note that \( N_{b^n}(o) \) is contained in the union of \( C_n \) and \( 3^k - 1 \) other cubes in \( \Pi'_n \).

It follows that \( E \left( \#N_{b^n}(o) \right) \leq l'm^n \), where \( l' = l + (3^k - 1)p \). So the unimodular Billingsley lemma (Theorem 2.8) implies that \( \text{udim}_H(\hat{K}) \leq k + \log_b p \). The claim for \( m = 1 \) and \( m < 1 \) are similar.

Consider now the Minkowski dimension. By the above part of the proof, it is enough to assume \( m > 1 \). Let \( n \geq 0 \) be given. By considering the partition \( \Pi'_n \) by cubes, one can construct a \( b^n \)-covering \( R_n \) as in Theorem 2.16. This covering satisfies

\[ \mathbb{P} \left[ R_n(o) \geq 0 \right] = E \left[ \frac{1}{\#C_n \cap \hat{K}} \right]. \]

Let \( [T', o'] \) be the eternal Galton-Watson tree of Subsection I.4.2.3 with binomial offspring distribution with parameters \( (N, p) \). By regarding \( T' \) as a family tree, it is straightforward that \( [T, o] \) has the same distribution as the part of \( [T', o'] \), up to the generation of the root (see [7] for more details on eternal family trees). Therefore, Lemma 5.7 of [7] implies that

\[ E \left[ \frac{1}{\#C_n \cap \hat{K}} \right] = m^{-n} \mathbb{P} \left[ h(o') \geq n \right]. \]

Since \( m > 1 \), \( \mathbb{P} \left[ h(o') \geq n \right] \) tends to the non-extinction probability of the descendants of the root, which is positive. By noticing the fact that the radii of the balls are \( b^n \) and the covering is uniformly bounded, one gets that \( \text{udim}_M(\hat{K}) = \log_b m = k + \log_b p \).

Finally, it remains to prove that \( \hat{K} \) is infinite a.s. when \( p = b^{-k} \). In this case, consider the eternal Galton-Watson tree \( [T', o'] \) as above. Proposition 6.8 of [7] implies that the generation of the root is infinite a.s. This proves the claim. \( \square \)

### 3.6 Cayley Graphs

Recall the discussion on Cayley graphs from Subsection I.4.8. The following theorem connects the dimension of Cayley graphs to growth rate of groups.

**Theorem 3.19.** For a finitely generated group \( H \) with polynomial growth rate \( \alpha \in [0, \infty] \), one has

\[ \text{udim}_M(H) = \text{udim}_H(H) = \alpha. \]

Moreover, \( \alpha \) is either an integer or infinity.

**Proof.** Gromov’s theorem [13] implies that \( \alpha \) is either an integer or infinity.

First, assume \( \alpha < \infty \). The result of Bass [8] implies that there are constants \( c, C > 0 \) such that \( \forall r \geq 1 : cr^\alpha < \#N_r(o) \leq Cr^\alpha \), where \( o \) is an arbitrary
element of $H$. So the mass distribution principle (Theorem 2.2) and part \(1\) of Lemma 2.6 imply that $\text{udim}_M(H) = \text{udim}_H(H) = \alpha$.

Second, assume $\alpha = \infty$. The result of [22] shows that for any $\beta < \infty$, $\#N_r(o) > r^\beta$ for sufficiently large $r$ (see (1.10) in [22]). Therefore, part \(1\) of Lemma 2.6 implies that $\text{udim}_M(H) \geq \beta$. Hence, $\text{udim}_M(H) = \text{udim}_H(H) = \infty$ and the claim is proved. □

4 Frostman’s Theory

This section provides a unimodular version of Frostman’s lemma and some of its applications. In a sense to be made precise later, this lemma gives converses to the mass distribution principle and the unimodular Billingsley lemma (see in particular, Corollary 4.3 below). It is a powerful tool to prove theoretical results regarding the unimodular Hausdorff dimension. For example, it is used in this section to derive inequalities for the dimension of product spaces and embedded spaces (Subsections 4.4 and 4.5). It is also the basis of many of the results in Part III.

The general case is presented in Subsection 4.1. A slightly improved bound is given for point-stationary point processes with a different proof (Subsection 4.3). The latter relies on a max-flow min-cut theorem for unimodular one-ended trees, which is presented in Subsection 4.2 and is of independent interest.

4.1 Unimodular Frostman Lemma

The statement of the unimodular Frostman lemma requires the definition of weighted Hausdorff content as follows. For this, the following mark space is needed. Let $\Xi$ be the set of functions $c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ which are positive in finitely many points; i.e., $c^{-1}((0, \infty))$ is a finite set. Let $\Xi'$ be the set of compact subsets of $\mathbb{R}^2$. By identifying $c \in \Xi$ with the finite set $\{(x, c(x)) : x \in \mathbb{R}_{\geq 0}, c(x) > 0\}$, one can identify $\Xi$ with a subset of $\Xi'$. It is well known that $\Xi'$ is a complete separable metric space under the Hausdorff metric (see e.g., [10]). So one can define the notion of $\Xi'$-valued equivariant processes as in Definition I.2.21. Therefore, $\Xi$-valued equivariant processes also make sense.

Consider a unimodular discrete space $[D, o]$ with distribution $\mu$. An equivariant weighted collection of balls $c$ is a $\Xi$-valued equivariant process (the term ‘weighted’ refers to the weighted sums in the following and should not be confused with equivariant weight functions of Definition 2.1). For $v \in D$, the reader can think of the value $c_r(v) := c(v)(r)$, if positive, to indicate that there is a ball in the collection, with radius $r$, centered at $v$, and with cost (or weight) $c_r(v)$. Note that extra randomness is allowed in the definition. A ball-covering $R$ can be regarded a special case of this construction by letting $c_r(v)$ be 1 when $r = R(v)$ and 0 otherwise.

Definition 4.1. Let $f : D_+ \rightarrow \mathbb{R}$ be a measurable function and $M \geq 1$. An
equivariant weighted collection of balls $c$ is called a $(f, M)$-covering if
\begin{equation}
\forall v \in D : f(v) \leq \sum_{u \in D} \sum_{r \geq M} c_r(u) 1_{\{v \in N_r(u)\}}, \quad a.s.,
\end{equation}
\begin{equation}
\forall v \in D : f(v) \leq \sum_{u \in D, r \geq M: v \in N_r(u)} c_r(u), \quad a.s.,
\end{equation}
where $f(v) := f[D, v]$ for $v \in D$. For $\alpha \geq 0$, define
$$\xi_M^\alpha(f) := \inf \left\{ \mathbb{E} \left[ \sum_r c_r(o)r^{\alpha} \right] : c \text{ is a } (f, M)-\text{covering} \right\}.$$  
Also, let
$$\xi_\infty^\alpha(f) := \lim_{M \to \infty} \xi_M^\alpha(f).$$

It is straightforward that every equivariant ball-covering of Definition I.3.21 gives a $(1,1)$-covering, where 1 is regarded as the constant function $f \equiv 1$ on $D$. This gives
\begin{equation}
\xi_M^\alpha(1) \leq \mathcal{H}_M^\alpha(D). \tag{4.3}
\end{equation}

Also, by considering the case $c_M(v) := f(v) \lor 0$, one can see that if $f \in L_1(D, \mu)$, then
$$\xi_M^\alpha(f) \leq M^\alpha \mathbb{E} [f(o) \lor 0] < \infty.$$  
Recall that a deterministic equivariant weight function is given by a measurable function $w : D \to \mathbb{R}^\geq 0$ (see Example I.2.23). In the following theorem, to be consistent with the setting of the paper, the following notation is used: $w(u) := w([D, u])$ for $u \in D$, and $w(N_r(v)) = \sum_{u \in N_r(u)} w(u)$.

**Theorem 4.2** (Unimodular Frostman Lemma). Let $[D, o]$ be a unimodular discrete space, $\alpha \geq 0$ and $M \geq 1$.

(i) There exists a bounded weight function $w : D \to \mathbb{R}^\geq 0$ such that, almost surely,
\begin{equation}
\forall v \in D, \forall r \geq M : w(N_r(v)) \leq r^\alpha \tag{4.4}
\end{equation}
and
$$\mathbb{E} [w(o)] = \xi_M^\alpha(1).$$

(ii) Given a non-negative function $h \in L_1(D, \mu)$, the last condition can be replaced by
$$\mathbb{E} [w(o)h(o)] = \xi_M^\alpha(h).$$

(iii) In the setting of (iii), if $D$ has finite $\alpha$-dimensional Hausdorff measure and $h \not\equiv 0$, then $w[D, o] \not\equiv 0$ with positive probability.  

The proof is given later in this subsection.
Corollary 4.3. For all unimodular discrete spaces $[D,o]$ and all $\epsilon > 0$, there exists a deterministic equivariant weight function $w$ such that 
\[
\overline{\text{growth}}(w(N_r(o))) - \epsilon \leq \text{udim}_H(D) \leq \overline{\text{growth}}(w(N_r(o))).
\]
In addition, if $D$ has finite $\text{udim}_H(D)$-dimensional Hausdorff measure, then $w$ can be chosen such that
\[
\text{udim}_H(D) = \overline{\text{growth}}(w(N_r(o))).
\]

Proof. If $\text{udim}_H(D) = \infty$, then let $w(\cdot) \equiv 1$. In this case, the claim follows from the unimodular Billingsley lemma (Theorem 2.8). So assume $\text{udim}_H(D) < \infty$ and let $\alpha := \text{udim}_H(D) + \epsilon$. One has $M^\alpha(D) = 0$ (Lemma I.3.36). So, by part (iii) of the unimodular Frostman lemma, the function $w$ in the lemma is not identical to zero. Therefore, the unimodular Billingsley lemma implies that $\text{udim}_H(D) \leq \overline{\text{growth}}(w(N_r(o)))$. On the other hand, (4.4) implies that $\overline{\text{growth}}(w(N_r(o))) \leq \alpha = \text{udim}_H(D) + \epsilon$. This proves the claim. \[\square\]

Remark 4.4. One can show that (4.4) implies that 
\[
E[w(o)] \leq \xi^k_M(1), \quad E[w(o)h(o)] \leq \xi^k_M(h).
\]

Therefore, the (deterministic) weight function $w$ given in the unimodular Frostman lemma is a maximal equivariant weight function satisfying (4.4) (it should be noted that the converse of this claim is not true). The proof is similar to that of the mass distribution principle (Theorem 2.2) and is left to the reader.

Conjecture 4.5. One has $\mathcal{H}^\alpha_M(D) = \xi^\alpha_M(1)$.

Example 4.6. Assume $[D,o] := [\mathbb{Z}^k,0]$ is equipped with the $l_\infty$ metric and let $M \in \mathbb{N}$. By the proof of Proposition I.3.38, one can see that
\[
\xi^k_M(1) \leq \mathcal{H}^k_M(\mathbb{Z}^k) \leq \left(\frac{M}{2M+1}\right)^k.
\]

Let $w(\cdot) = \left(\frac{M}{2M+1}\right)^k$. This weight function satisfies (4.4) for $\alpha = k$ and also $E[w(o)] \geq \xi^k_M(1)$. Therefore, by Remark 4.4 above, $E[w(o)] = \xi^k_M(1)$. So $w$ satisfies the claim of the unimodular Frostman lemma. Note that Conjecture 4.5 holds in this case.

Example 4.7. Assume $D$ is $\mathbb{Z}$ with probability $\frac{1}{2}$ and $\mathbb{Z}^2$ with probability $\frac{1}{2}$ (see Example I.3.27). Given $M \in \mathbb{N}$, let $w[\mathbb{Z},0] := \frac{M}{2M+1}$ and $w[\mathbb{Z}^2,0] := 0$.

As in the previous example, one can show that $w$ satisfies the claim of the unimodular Frostman lemma and Conjecture 4.5 holds in this case.

There are very few examples where the function $w$ given by the unimodular Frostman lemma can be explicitly computed. However, in some of the examples, a function $w$ satisfying (4.4) can be found. For example, this is the case in two-ended trees (see Subsection 3.1.1).

The following lemma is needed to prove Theorem 4.2.
Lemma 4.8. The function $\xi_M^\alpha : L_1(\mathcal{D}_*, \mu) \to \mathbb{R}$ is continuous. In fact, it is $M^\alpha$-Lipschitz; i.e.,

$$|\xi_M^\alpha(f_1) - \xi_M^\alpha(f_2)| \leq M^\alpha \mathbb{E} [|f_1(o) - f_2(o)|].$$

Proof. Let $c$ be an equivariant weighted collection of balls satisfying (4.2) for $f_1$. Intuitively, add a ball of radius $M$ at each point $v$ with cost $|f_2(v) - f_1(v)|$. More precisely, let $c'_r(v) := c_r(v)$ for $r \neq M$ and $c'_M(v) := c_M(v) + |f_2(v) - f_1(v)|$. This definition implies that $c'$ satisfies (4.2) for $f_2$. Also,

$$\xi_M^\alpha(f_2) \leq \mathbb{E} \left[ \sum_i c'_i(o) \right] = \mathbb{E} \left[ \sum_v c_r(o) \right] + M^\alpha \mathbb{E} [|f_2(o) - f_1(o)|].$$

Since $c$ was arbitrary, one obtains

$$\xi_M^\alpha(f_2) \leq \xi_M^\alpha(f_1) + M^\alpha \mathbb{E} [|f_2(o) - f_1(o)|],$$

which implies the claim. \qed

The following proof uses the ideas of Thm 8.17 of [19].

Proof of Theorem 4.2. Part (i) is implied by Part (ii) which is proved below. It is easy to see that $\xi_M^\alpha(tf) = t \xi_M^\alpha(f)$ for all $f$ and $t \geq 0$ and also

$$\xi_M^\alpha(f_1 + f_2) \leq \xi_M^\alpha(f_1) + \xi_M^\alpha(f_2)$$

for all $f_1, f_2$. Let $h \in L_1(\mathcal{D}_*, \mu)$ be given. By the Hahn-Banach theorem (see Theorem 3.2 of [21]), there is a linear functional $l : L_1(\mathcal{D}_*, \mu) \to \mathbb{R}$ such that

$$l(h) = \xi_M^\alpha(h)$$

and

$$-\xi_M^\alpha(-f) \leq l(f) \leq \xi_M^\alpha(f), \quad \forall f \in L_1.$$

Since $l$ is sandwiched between two functions which are continuous at 0 and are equal at 0, one gets that $l$ is continuous at 0. Since $l$ is linear, this implies that $l$ is continuous. Since the dual of $L_1(\mathcal{D}_*, \mu)$ is $L_\infty(\mathcal{D}_*, \mu)$, one obtains that there is a function $w \in L_\infty(\mathcal{D}_*, \mu)$ such that

$$l(f) = \mathbb{E} [f(o)w(o)], \quad \forall f \in L_1.$$

Note that if $f \geq 0$, then $\xi_M^\alpha(-f) = 0$ and so $l(f) \geq 0$. This implies that $w(o) \geq 0$ a.s. (otherwise, let $f(o) := 1_{\{w(o)<0\}}$ to get a contradiction). Consider a version of $w$ which is nonnegative everywhere. The claim is that $w$ satisfies the requirements of (i).

Let $r \geq M$ be fixed. For a discrete space $D$, let $S := S_D := \{v \in D : w(N_r(v)) > r^\alpha \}$. Define $f_r(v) := \#N_r(v) \cap S$. By the definition of $S_D$, one has

$$\mathbb{E} [w(N_r(o))1_{\{o \in S\}}] \geq r^\alpha \mathbb{P} [o \in S].$$

(4.5)
Moreover, if $P[o \in S] > 0$, then the inequality is strict. On the other hand, by the mass transport principle for the function $(v, u) \mapsto w(u)1_{\{v \in S\}}1_{\{u \in N_r(v)\}}$, one gets

$$E[w(N_r(o))1_{\{o \in S\}}] = E[w(o)\#N_r(o) \cap S] = E[w(o)f_r(o)] \leq \xi_M^\alpha(f_r) \leq r^\alpha P[o \in S],$$

where the last inequality is implied by considering the following weighted collection of balls for $f_r$: put balls of radius $r$ with cost 1 centered at the points in $S$. More precisely, let $c_r(v) := 1_{\{v \in S\}}$ and $c_s(v) := 0$ for $s \neq r$. It is easy to see that this satisfies (4.2) for $f_r$, which implies the last inequality by the definition of $\xi_M^\alpha(\cdot)$. Thus, equality holds in (4.5). Hence, $P[o \in S] = 0$; i.e., $w(N_r(o)) \leq r^\alpha$ a.s. Lemma I.2.30 implies that almost surely, $\forall v \in D : w(N_r(v)) \leq r^\alpha$. So the same holds for all rational $r \geq M$ simultaneously. By monotonicity of $w(N_r(v))$ w.r.t. $r$, one gets that the latter almost surely holds for all $r \geq M$ as desired.

Also, one has

$$E[w(o)h(o)] = l(h) = \xi_M^\alpha(h).$$

Thus, $w$ satisfies the desired requirements.

To prove (iii), assume $h \neq 0$ and $M^\alpha(D) < \infty$. By Lemma I.3.35, one has $H_M^\alpha(D) > 0$. So Lemma 4.9 below implies that $\xi_M^\alpha(h) > 0$. Now, the above equation implies that $w$ is not identical to zero. \hfill \Box

The above proof uses the following lemma.

**Lemma 4.9.** Let $[D, o]$ be a unimodular discrete metric space.

(i) By letting $b := \xi_1^\alpha(1)$, one has

$$b \leq H_1^\alpha(D) \leq b + b|\log b|.$$

(ii) Let $0 \neq h \in L_1(D, \mu)$ be a non-negative function. For $M \geq 1$, one has

$$H_M^\alpha(D) \leq \inf_{a \geq 0} \left\{ M^\alpha E\left[e^{ah(o)}\right] + a\xi_M^\alpha(h) \right\}.$$

(iii) In addition, $\xi_M^\alpha(h) = 0$ if and only if $H_M^\alpha(D) = 0$.

**Proof.** [1]. By considering the cases where $c(\cdot) \in \{0, 1\}$, the first inequality is easily obtained from the definition of $\xi_1^\alpha(1)$. In particular, this implies that $b \leq 1$. The second inequality is implied by part (ii) by letting $h(\cdot) := 1$ and $a := -\log b \geq 0$. 

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Let \( b' > \xi_M^a(h) \) be arbitrary. So there exists an equivariant weighted collection of balls \( c \) that satisfies (4.2) for \( h \) and

\[
\mathbb{E} \left[ \sum_{r \geq M} c_r(o)r^\alpha \right] \leq b'.
\]

Next, given \( a \geq 0 \), define an equivariant covering \( R \) as follows. For each \( v \in D \) and \( r \geq M \) such that \( c_r(v) > 0 \), put a ball of radius \( r \) at \( v \) with probability \( ac_r(v) \land 1 \). Do this independently for all \( v \) and \( r \) (one should condition on \( D \) first). If more than one ball is put at \( v \), keep only the one with maximum radius.

Let \( S \) be the union of the chosen balls. For \( u \in D \setminus S \), put a ball of radius \( M \) at \( u \). This gives an equivariant covering, namely \( R \), by balls of radii at least \( M \).

Then, one can easily get

\[
\mathbb{E} \left[ R(o) \right]^a \leq M^a \mathbb{P} [o \notin S] + \mathbb{E} \left[ \sum_{r \geq M} (ac_r(o) \land 1)r^\alpha \right] \leq M^a \mathbb{P} [o \notin S] + ab'.
\] (4.6)

To bound \( \mathbb{P} [o \notin S] \), consider a realization of \([D, o]\). First, if for some \( v \in D \) and \( r \geq M \), one has \( ac_r(v) > 1 \) and \( o \in N_r(v) \), then \( o \) is definitely in \( S \). Second, assume this is not the case. By (4.2), one has \( \sum_{u \in D} \sum_{r \geq M} c_r(u)1_{\{o \in N_r(u)\}} \geq h(o) \). This implies that the probability that \( o \notin S \) in this realization is

\[
\prod_{(v, r) : o \in N_r(v)} (1 - ac_r(v)) \leq \exp \left( - \sum_{(v, r) : o \in N_r(v)} ac_r(v) \right) \leq e^{-ah(o)}.
\]

In both cases, one gets \( \mathbb{P} [o \notin S] \leq \mathbb{E} \left[ e^{-ah(o)} \right] \). Thus, (4.6) implies that

\[
\mathbb{E} \left[ R(o) \right]^a \leq M^a \mathbb{E} \left[ e^{-ah(o)} \right] + ab'.
\]

Since \( a \geq 0 \) and \( b' > b \) are arbitrary, the claim follows.

Assume \( \xi_M^a(h) = 0 \). By letting \( a \to \infty \) and using the first claim, one obtains that \( \mathcal{H}_M^a(D) = 0 \). Conversely, assume \( \mathcal{H}_M^a(D) = 0 \). The first inequality in part (ii) gives that \( \xi_M^a(a) = 0 \) for any constant \( a \). Therefore, \( \xi_M^a(h) \leq \xi_M^a(a) + \xi_M^a((h - a) \lor 0) \) \( \leq M^a \mathbb{E} [(h - a) \lor 0] \). By letting \( a \) tend to infinity, one gets \( \xi_M^a(h) = 0 \).

4.2 Max-Flow Min-Cut Theorem for Unimodular One-Ended Trees

This subsection provides a version of the max-flow min-cut theorem for unimodular one-ended trees. This result is used in the next subsection for a Euclidean version of the unimodular Frostman lemma, but is of independent interest.

The max-flow min-cut theorem is a celebrated result in the field of graph theory (see e.g., [12]). In its simple version, it studies the minimum number of edges in a cut-set in a finite graph; i.e., a set of edges the deletion of which disconnects two given subsets of the graph. A generalization of the theorem
in the case of trees is obtained by considering cut-sets separating a given finite subset from the set of ends of the tree. This generalization is used to prove a version of Frostman’s lemma for (continuum) sets in the Euclidean space (see e.g., [9]).

This subsection presents an analogous result for unimodular one-ended trees. It discusses cut-sets separating the set of leaves from the end of the tree. Since the tree has infinitely many leaves a.s. (see e.g., [7]), infinitely many edges are needed in any such cut-set. Therefore, cardinality cannot be used to study minimum cut-sets. The idea is to use unimodularity for a quantification of the size of a cut-set.

Let \([T, o, c]\) be a unimodular marked one-ended tree with mark space \( \mathbb{R}^{\geq 0} \). Assume the mark \( c(e) \) of each edge \( e \) is well defined and call it the conductance of \( e \). Let \( L \) be the set of leaves of \( T \). As in Subsection I.4.1.3, let \( F(v) \) be the parent of vertex \( v \) and \( D(v) \) be the descendants subtree of \( v \). The new vocabulary used in the following definitions is that of the max-flow min-cut theorem.

**Definition 4.10.** A legal equivariant flow on \([T, c]\) is an equivariant way of assigning extra marks \( f(\cdot) \in \mathbb{R} \) to the edges (see Definition I.2.21 and Remark I.2.34), such that almost surely,

\[
\begin{align*}
(i) & \quad 0 \leq f(e) \leq c(e), \\
(ii) & \quad f(v, F(v)) = \sum_{w \in F^{-1}(v)} f(w, v). \\
\end{align*}
\tag{4.7}
\]

Also, an equivariant cut-set is an equivariant subset \( \Pi \) of the edges of \([T, c]\) that separates the set of leaves \( L \) from the end in \( T \).

The reader can think of the value \( f(v, F(v)) \) as the flow from \( v \) to \( F(v) \). So (4.7) can be interpreted as conservation of flow at the vertices except the leaves. Also, the leaves are regarded as the sources of the flow.

Since the number of leaves is infinite a.s., the sum of the flows exiting the leaves might be infinite. In fact, it can be seen that unimodularity implies that the sum is always infinite a.s. The idea is to use unimodularity to quantify how large is the flow. Similarly, in any equivariant cut-set, the sum of the conductances of the edges is infinite a.s. Unimodularity is also used to quantify the conductance of an equivariant cut-set. These are done in Definition 4.12 below.

Note that extra randomness is allowed in the above definition. Since each edge of \( T \) can be uniquely represented as \((v, F(v))\), the following convention is helpful.

**Convention 4.11.** For the vertices \( v \) of \( T \), the symbols \( f(v) \) and \( c(v) \) are used as abbreviations for \( f(v, F(v)) \) and \( c(v, F(v)) \), respectively. Also, by \( v \in \Pi \), one means that the edge \((v, F(v))\) is in \( \Pi \).
Definition 4.12. The norm of the legal equivariant flow $f$ is defined as

$$|f| := \mathbb{E} \left[ f(o) 1_{\{o \in L\}} \right].$$

Also, for the equivariant cut-set $\Pi$, define

$$c(\Pi) := \mathbb{E} \left[ c(o) 1_{\{o \in \Pi\}} \right] = \mathbb{E} \left[ \sum_{w \in F^{-1}(o)} c(w) 1_{\{w \in \Pi\}} \right],$$

where the last equality follows from the mass transport principle (I.2.3).

An equivariant cut-set $\Pi$ is called **equivariantly minimal** if there is no other equivariant cut-set which is a subset of $\Pi$ a.s. Also, it is **almost surely minimal** if in almost every realization, there is no subset of $\Pi$ that separates the leaves from the end. The following lemma shows that these definitions are equivalent.

**Lemma 4.13.** An equivariant cut-set is equivariantly minimal if and only if it is almost surely minimal.

**Proof.** Let $\Pi$ be an equivariant cut-set. If $\Pi$ is almost surely minimal, then it is also equivariantly minimal by definition. Conversely, assume $\Pi$ is equivariantly minimal but not almost surely minimal. Call an edge $e'$ **above** an edge $e$ if $e'$ separates $e$ from the end. Call an edge $e \in \Pi$ **bad** if there is an edge of $\Pi$ above $e$. Let $\Pi'$ be the set of bad edges of $\Pi$. Let $\Pi''$ be the set of lowest edges in $\Pi'$; i.e., the edges $e \in \Pi'$ such that there is no other edge of $\Pi'$ below $e$. It can be seen that the assumption implies that $\Pi''$ is nonempty with positive probability. Now, it can be seen that $\Pi \setminus \Pi''$ is an equivariant cut-set, which contradicts the minimality of $\Pi$. \[\square\]

**Lemma 4.14.** If $f$ is a legal equivariant flow and $\Pi$ is an equivariant cut-set, then $|f| \leq c(\Pi)$. Moreover, if the pair $(f, \Pi)$ is equivariant, then

$$|f| \leq \mathbb{E} \left[ f(o) 1_{\{o \in \Pi\}} \right] \leq c(\Pi).$$

In addition, if $\Pi$ is minimal, then equality holds in the left inequality.

**Proof.** One can always consider an independent coupling of $f$ and $\Pi$ (as in the proof of Theorem 2.2). So assume $(f, \Pi)$ is equivariant from the beginning. Note that the whole construction (with conductances, the flow and the cut-set) is unimodular (Lemma I.2.28). For every leaf $v \in L$, let $\tau(v)$ be the first ancestor of $v$ such that $(v, F(v)) \in \Pi$. Then, send mass $f(v)$ from each leaf $v$ to $\tau(v)$. By the mass transport principle (I.2.3), one gets

$$\mathbb{E} \left[ f(o) 1_{\{o \in L\}} \right] = \mathbb{E} \left[ \sum_{v \in \tau^{-1}(o)} f(v) 1_{\{o \in \Pi\}} \right] \leq \mathbb{E} \left[ \sum_{v \in D(o) \cap L} f(v) \right] = \mathbb{E} \left[ f(o) 1_{\{o \in \Pi\}} \right].$$
where the last equality holds because $f$ is a flow. Moreover, if $\Pi$ is minimal, then, by Lemma 4.13, the above inequality becomes an equality and the claim follows.

The main result is the following converse to the above lemma.

**Theorem 4.15** (Max-Flow Min-Cut for Unimodular One-Ended Trees). For a unimodular marked one-ended tree $[T, o; c]$ equipped with conductances $c$ as above, if $c$ is bounded on the set of leaves, then

$$\max_f |f| = \inf_{\Pi} c(\Pi),$$

where the maximum is over all legal equivariant flows $f$ and the infimum is over all equivariant cut-sets $\Pi$.

**Remark 4.16.** The claim of Theorem 4.15 is still valid if the probability measure (the distribution of $[T, o; c]$) is replaced by any (possibly infinite) measure $P$ on $D^*$ supported on one-ended trees, such that $P(o \in L) < \infty$ and the mass transport principle (I.2.3) holds. The same proof works for this case as well. This will be used in Subsection 4.3.

**Proof of Theorem 4.15**. For $n \geq 1$, let $T_n$ be the sub-forest of $T$ obtained by keeping only vertices of height at most $n$ in $T$. Each connected component of $T_n$ is a finite tree which contains some leaves of $T$. For each such component, namely $T'$, do the following: if $T'$ has more than one vertex, consider the maximum flow on $T'$ between the leaves and the top vertex (i.e., the vertex with maximum height in $T'$). If there is more than one maximum flow, choose one of them randomly and uniformly. Also, choose a minimum cut-set in $T'$ randomly and uniformly. Similarly, if $T'$ has a single vertex $v$, do the same for the subgraph with vertex set $\{v, F(v)\}$ and the single edge adjacent to $v$. By doing this for all components of $T_n$, a (random) function $f_n$ on the edges and a cut-set $\Pi'_n$ are obtained (by letting $f_n$ be zero on the other edges). $\Pi'_n$ is always a cut-set, but $f_n$ is not a flow. However, $f_n$ satisfies (4.7) for vertices of $T_n \setminus L$ except the top vertices of the connected components of $T_n$. Also, it can be seen that $f_n$ and $\Pi'_n$ are equivariant.

For each component $T'$ of $T_n$, the set of leaves of $T'$, excluding the top vertex, is $L \cap T'$. So the max-flow min-cut theorem of Ford-Fulkerson [12] (see e.g., Theorem 1.2 in Chapter 3 of [9]) gives that, for each component $T'$ of $T_n$, one has

$$\sum_{v \in L \cap T'} f_n(v) = \sum_{e \in \Pi'_n \cap T'} c(e).$$

If $u$ is the top vertex of $T'$, let $h(u)$ be the common value in the above equation. By using the mass transport principle (I.2.3) for each of the two representations of $E[h(o)]$, one can obtain

$$E[f_n(o)1_{o \in L}] = E[h(o)] = E[c(o)1_{o \in \Pi'_n}] = c(\Pi'_n).$$

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Since $0 \leq f_n(\cdot) \leq c_n(\cdot)$, one can see that the distributions of $f_n$ are tight (the claim is similar to Lemma I.B.3 and is left to the reader). Therefore, there is a sequence $n_1, n_2, \ldots$ and an equivariant process $f'$ such that $f_n \rightarrow f'$ (weakly). It is not hard to deduce that $f'$ is a legal equivariant flow. Also, since $f'(o)$ and $1_{\{o \in L\}}$ are continuous functions of $[T, o; f']$ and their product is bounded (by the assumption on $c$), one gets that

$$|f'| = \mathbb{E}[f'(o)1_{\{o \in L\}}] = \lim_i \mathbb{E}[f_{n_i}(o)1_{\{o \in L\}}] = \lim_i c(\Pi_{n_i}).$$

Therefore,

$$\max_f |f| \geq \inf \Pi c(\Pi).$$

Note that the maximum of $|f|$ is attained by the same tightness argument as above. So Lemma 4.14 implies that equality holds and the claim is proved. \[\square\]

### 4.3 A Unimodular Frostman Lemma for Point Processes

In the Euclidean case, another form of the unimodular Frostman lemma is given below. Its proof is based on the max-flow min-cut theorem of Subsection 4.2. As will be seen, the claim implies that in this case, Conjecture 4.5 holds up to a constant factor (Corollary 4.18). However, the weight function obtained in the theorem needs extra randomness.

**Theorem 4.17.** Let $\Phi$ be a point-stationary point process in $\mathbb{R}^k$ endowed with the Euclidean metric, and let $\alpha \geq 0$. Then, there exists an equivariant weight function $w$ on $\Phi$ such that, almost surely,

$$\forall v \in \Phi, \forall r \geq 1 : w(N_r(v)) \leq r^\alpha$$  \hspace{1cm} (4.8)

and

$$\mathbb{E}[w(0)] \geq 3^{-k} H_1^\alpha(\Phi).$$  \hspace{1cm} (4.9)

In particular, if $H_1^\alpha(\Phi) > 0$, then $w(0)$ is not identical to zero.

**Proof.** Let $b > 1$ be an arbitrary integer (e.g., $b = 2$). For every $n \in \mathbb{Z}$, let $Q_n$ be the stationary partition of $\mathbb{R}^k$ by translations of the cube $[0, b^n)^k$ as in Subsection 2.3. Consider the nested coupling of these partitions for $n \in \mathbb{Z}$ (i.e., every cube of $Q_n$ is contained in a cube of $Q_{n+1}$ for every $n \in \mathbb{Z}$) independent of $\Phi$. For all points $v \in \Phi$ and $n \in \mathbb{Z}$, let $q_n(v)$ be the cube in $Q_n$ that contains $v$ and let

$$m(v) := 1 \land \max\{n \in \mathbb{Z} : q_n(v) \cap \Phi = \{v\}\}.$$  

Let $T_0$ be the tree whose vertices are the cubes in $\cup_n Q_n$ and the edges are between all pairs of nested cubes in $Q_n$ and $Q_{n+1}$ for all $n$. Let $T \subseteq T_0$ be the subtree consisting of the cubes $q_n(v)$ for all $v \in \Phi$ and $n \geq m(v)$. The set $L$ of the leaves of $T$ consists of the cubes $q_{m(v)}(v)$ for all $v \in \Phi$. Let $\sigma := q_{m(0)}(0)$.

By the natural bijection between $L$ and $\Phi$ and the point-stationarity of $\Phi$, it can be seen that $[L, \sigma]$ is unimodular (with the metric induced from $\Phi$). Also,
the same holds for the metric on $L$ induced by the graph-distance metric on $T$ (see Theorem 1.3.41). In addition, it can be seen that, in $T$, the mass transport principle (I.2.2) holds for functions $g$ supported on the leaves; i.e., $g_T(u, v) = 0$ if either $u \notin L$ or $v \notin L$ (note that $g$ can depend on the whole graph $L$ and hence, the latter is stronger than the unimodularity of $[L, \sigma]$). Therefore, by Theorem 5 in [17], one can see that the following (possibly infinite) measure on $D_*$ makes $T$ unimodular:

$$P[A] := E \left[ \sum_{n \geq m(0)} \frac{1}{e_n} 1_A[T, q_n(0)] \right],$$

(4.10)

where $e_n$ is the number of leaves of $T$ below $q_n(0)$; i.e., $e_n = \# q_n(0) \cap \Phi$. In words, choose the root among $q_{m(0)}(0), q_{m(0)+1}(0), \ldots$ with the measure $q_n(0) \mapsto \frac{1}{e_n}$ (which is not necessarily a probability measure).

Let $E$ denote the integral operator w.r.t. the measure $P$. For any equivariant flow $f$ on $T$, the norm of $f$ w.r.t. the measure $P$ (see Remark 4.16) satisfies

$$|f| = E[f \cdot 1_L] = E \left[ \sum_{n \geq m(0)} \frac{1}{e_n} f(q_n(0)) 1_{\{q_n(0) \in L\}} \right] = E[f(\sigma)],$$

where the second equality is by (4.10).

Consider the conductance function $c(\tau) := 1 \vee b^\alpha(n)$ for all cubes $\tau$ of edge length $b^n$ in $T$ and all $n$. Therefore, Theorem 4.15 and Remark 4.10 imply that the maximum of $E[f(\sigma)]$ over all equivariant legal flows $f$ on $[T, \sigma]$ is attained (note that $[T, \sigma]$ is not unimodular, but the theorem can be used for $P$). Denote by $f_0$ the maximum flow. Let $w$ be the weight function on $\Phi$ defined by $w(v) = \delta f_0(q_{m(v)}(v))$, for all $v \in \Phi$, where $\delta := (b+1)^{-k}$. The claim is that $w$ satisfies the requirements (4.8) and (4.9).

Since $f_0$ is a legal flow, it follows that for every cube $\sigma \in T$, one has

$$w(\sigma) = \delta f_0(\sigma) \leq \delta c(\sigma) = \delta(1 \vee b^{\alpha(n)}).$$

Each cube $\sigma$ of edge length $r \in [b^n, b^{n+1})$ in $\mathbb{R}^k$ can be covered with at most $(b+1)^k$ cubes of edge length $b^n$ in $T_0$. If $n \geq 0$, the latter are either in $T$ or do not intersect $\Phi$. So the above inequality implies that $w(\sigma) \leq r^\alpha$. So (4.8) is proved for $w$.

To prove (4.9), given any equivariant cut-set $\Pi$ of $T$, a covering of $\Phi$ can be constructed as follows: For each cube $\sigma \in \Pi$ of edge length say $n$, let $\tau(\sigma)$ be one of the points in $\sigma \cap \Phi$ chosen uniformly at random and put a ball of radius $1 \vee b^\alpha$ centered at $\tau(\sigma)$. Note that this ball contains $\sigma$. Do this independently for all cubes in $T$. If a point in $\Phi$ is chosen more than once, consider only the largest radius assigned to it (which might be infinite). It can be seen that this
gives an equivariant covering of $\Phi$, namely $R$. Since $\sigma_n$ has edge length $b_n + m(0)$, one has

$$
E[R(0)^\alpha] \leq E \left[ \sum_{n \geq 0} (1 \vee b_n^\alpha) 1_{\{q_n(0) \in \Pi\}} 1_{\{\tau(q_n(0))\}} \right]
$$

$$
= E \left[ \sum_{n \geq 0} \frac{1}{e_n} (1 \vee b_n^\alpha) 1_{\{q_n(0) \in \Pi\}} \right].
$$

On the other hand, by (4.10), one can see that

$$
c(\Pi) = E \left[ \sum_{n \geq 0} \frac{1}{e_n} c(q_n(0)) 1_{\{q_n(0) \in \Pi\}} \right] = E \left[ \sum_{n \geq 0} \frac{1}{e_n} (1 \vee b_n^\alpha) 1_{\{q_n(0) \in \Pi\}} \right].
$$

Therefore, $E[R(0)^\alpha] \leq c(\Pi)$. So $\mathcal{H}_I^\alpha(\Phi) \leq c(\Pi)$. Since $\Pi$ is an arbitrary equivariant cut-set, by the unimodular max-flow min-cut theorem established above (Theorem 4.15) and the maximality of the flow $f_0$, one gets that

$$
\mathcal{H}_I^\alpha(\Phi) \leq |f_0| = E[f_0(\sigma)] = \delta^{-1}E[w(0)].
$$

So the claim is proved.

The following corollary shows that in the setting of Theorem 4.17, the claim of Conjecture 4.5 holds up to a constant factor (compare this with Lemma 4.9).

**Corollary 4.18.** For all point-stationary point processes $\Phi$ in $\mathbb{R}^k$ endowed with the Euclidean metric and all $\alpha \geq 0$,

$$
3^{-k} \mathcal{H}_I^\alpha(\Phi) \leq \xi_1^\alpha(\Phi) \leq \mathcal{H}_I^\alpha(\Phi).
$$

**Proof.** The claim is directly implied by (4.3), Theorem 4.17 and Remark 4.4. □

### 4.4 Application: Dimension of Product Spaces

Let $[D_1, o_1]$ and $[D_2, o_2]$ be independent unimodular discrete metric spaces. By considering any of the usual product metrics; e.g., the sup metric or the $p$ product metric, the **independent product** $[D_1 \times D_2, (o_1, o_2)]$ makes sense as a random pointed discrete space. It is not hard to see that the latter is also unimodular (see also Proposition 4.11 of [1]).

**Proposition 4.19.** Let $[D_1 \times D_2, (o_1, o_2)]$ represent the independent product of $[D_1, o_1]$ and $[D_2, o_2]$ defined above. Then,

$$
\text{udim}_H(D_1) + \text{udim}_M(D_2) \leq \text{udim}_H(D_1 \times D_2) \leq \text{udim}_H(D_1) + \text{udim}_H(D_2).
$$

(4.11)
Proof. By Theorem I.3.41, one can assume the metric on $D_1 \times D_2$ is the sup metric without loss of generality. So $\mathcal{N}_r(v_1, v_2) = \mathcal{N}_r(v_1) \times \mathcal{N}_r(v_2)$.

The upper bound is proved first. For $i = 1, 2$, let $\alpha_i > \dim_H(D_i)$ be arbitrary. By the unimodular Frostman lemma (Theorem I.22), there is a non-negative measurable functions $w_i$ on $D_i$ such that

$$\forall v \in D_i : \forall r \geq 1 : w_i(\mathcal{N}_r(v)) \leq r^\alpha, a.s.$$ 

In addition, $w_i(o_i) \neq 0$ with positive probability. Consider the equivariant weight function $w$ on $D_1 \times D_2$ defined by

$$w(v_1, v_2) := w_1[D_1, v_1] \times w_2[D_2, v_2].$$

It is left to the reader to show that $w$ is an equivariant weight function. One has $w(\mathcal{N}_r(v_1, v_2)) = w_1(\mathcal{N}_r(v_1))w_2(\mathcal{N}_r(v_2)) \leq r^{\alpha_1 + \alpha_2}$. Also, by the independence assumption, $w(o_1, o_2) \neq 0$ with positive probability. Therefore, the mass distribution principle (Theorem 2.2) implies that $\dim_H(D_1 \times D_2) \leq \alpha_1 + \alpha_2$. This proves the upper bound.

For the lower bound in the claim, let $\alpha < \dim_H(D_1)$, $\beta < \dim_M(D_2)$ and $\epsilon > 0$ be arbitrary. It is enough to find an equivariant covering $R$ of $D_1 \times D_2$ such that $\mathbb{E}[R(o_1, o_2)^{\alpha + \beta}] < \epsilon$. One has $\text{decay}(\lambda_r(D_2)) > \beta$, where $\lambda_r$ is defined in (I.3.1). So there is $M > 0$ such that $\forall r \geq M : \lambda_r(D_2) < r^{-\beta}$. So for every $r \geq M$, there is an equivariant $r$-covering of $D_2$ with intensity less than $r^{-\beta}$.

On the other hand, since $\alpha < \dim_H(D_1)$, one has $\mathcal{H}_M^\alpha(D_1) = 0$ (by Lemma I.3.35). Therefore there is an equivariant covering $R_1$ of $D_1$ such that $\mathbb{E}[R_1(o_1)^{\beta}] < \epsilon$ and $\forall v \in D_1 : R_1(v) \in \{0\} \cup [M, \infty) a.s.$ Choose the extra randomness in $R_1$ independently from $[D_2, o_2]$. Given a realization of $[D_1, o_1]$ and $R_1$, do the following: Let $v_1 \in D_1$ such that $R_1(v_1) \neq 0$ (and hence, $R_1(v_1) \geq M$). One can find an equivariant subset $S_{v_1}$ of $D_2$ that gives a covering of $D_2$ by balls of radius $R_1(v_1)$ and has intensity less than $R_1(v_1)^{-\beta}$. Do this independently for all $v_1 \in D_1$. Now, for all $(v_1, v_2) \in D_1 \times D_2$, define

$$R(v_1, v_2) := \begin{cases} R_1(v_1) & \text{if } R_1(v_1) \neq 0 \text{ and } v_2 \in S_{v_1}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, $R$ is a covering of $D_1 \times D_2$ and it can be seen that it is an equivariant covering. Also, given $[D_1, o_1]$ and $R_1$, the probability that $o_2 \in S_{o_1}$ is less than $R_1(o_1)^{-\beta}$. So one gets

$$\mathbb{E}[R(o_1, o_2)^{\alpha + \beta}] = \mathbb{E}\left[\mathbb{E}[R(o_1, o_2)^{\alpha + \beta} | [D_1, o_1], R_1]\right]$$

$$< \mathbb{E}[R_1(o_1)^{\alpha + \beta} R_1(o_1)^{-\beta}]$$

$$= \mathbb{E}[R_1(o_1)^{\alpha}]$$

$$< \epsilon.$$ 

So the claim is proved. \qed
The following examples provide instances where the inequalities in (4.11) are strict.

**Example 4.20.** Assume \([D_1, o_1]\) and \([D_2, o_2]\) are unimodular discrete spaces such that \(\text{udim}_M(G_1) < \text{udim}_H(G_1)\) and \(\text{udim}_M(G_2) = \text{udim}_H(G_2)\). By Proposition 4.19 one gets

\[
\text{udim}_H(G_1 \times G_2) \geq \text{udim}_H(G_1) + \text{udim}_M(G_2) > \text{udim}_H(G_2) + \text{udim}_M(G_1).
\]

So by swapping the roles of the two spaces, an example of strict inequality in the left hand side of (4.11) is obtained.

**Example 4.21.** Let \(J\) be a subset of \(\mathbb{Z}_{\geq 0}\) such that \(d(J) = 1\) and \(d(J) = 0\) simultaneously (see Subsection 3.5.1 for the definitions). Let \(\Psi_1\) and \(\Psi_2\) be defined as in Subsection 3.5.1 corresponding to \(J\) and \(\mathbb{Z}_{\geq 0} \setminus J\) respectively. Proposition 3.16 implies that \(\text{udim}_H(\Psi_1) = \text{udim}_H(\Psi_2) = 1\). On the other hand, (3.4) implies that

\[
\#N_2^1(o_1 \times o_2) \leq 2^{n+1} \times 2^{(n+1-J_n)+1} = 2^{n+3}.
\]

This implies that \(\text{growth}(N_r(o)) \leq 1\). So the unimodular Billingsley lemma (Theorem 2.8) implies that \(\text{udim}_H(D_1 \times D_2) \leq 1\) (in fact, equality holds by Theorem 4.25 below). So the rightmost inequality in (4.11) is strict here.

### 4.5 Application: Dimension of Embedded Spaces

It is natural to think of \(\mathbb{Z}\) as a subset of \(\mathbb{Z}^2\). However, \([\mathbb{Z}, 0]\) is not an equivariant subspace of \([\mathbb{Z}^2, 0]\) as defined in Definition I.2.29. By the following definition, \([\mathbb{Z}, 0]\) is called *embeddable in* \([\mathbb{Z}^2, 0]\). The dimension of embedded subspaces is studied in this subsection, which happens to be non-trivial.

**Definition 4.22.** Let \([D_0, o_0]\) and \([D, o]\) be random pointed discrete spaces. An embedding of \([D_0, o_0]\) in \([D, o]\) is a (not necessarily unimodular) random pointed marked discrete space \([D', o'; m]\) with mark space \(\{0, 1\}\) such that

(i) \([D', o']\) has the same distribution as \([D, o]\).

(ii) \(m(o') = 1\) a.s.

(iii) By letting \(S := \{v \in D' : m(v) = 1\}\) equipped with the induced metric from \(D'\), \([S, o']\) has the same distribution as \([D_0, o_0]\).

If in addition, \([D_0, o_0]\) is unimodular, then \([D', o'; m]\) is called an *equivariant embedding* if

(iv) The mass transport principle holds on \(S\); i.e., (I.2.3) holds for functions \(g(u, v) := g(D', u, v; m)\) such that \(g(u, v)\) is zero when \(m(u) = 0\) or \(m(v) = 0\).

If an embedding (resp. an equivariant embedding) exists as above, \([D_0, o_0]\) is called *embeddable* (resp. *equivariantly embeddable*) in \([D, o]\).
It should be noted that \([D', o'; m] \) is not an equivariant process on \(D \) in general.

**Example 4.23.** The following are instances of Definition 4.22.

(i) \([\mathbb{Z}^n, 0]\) is equivariantly embeddable in \([\mathbb{Z}^m, 0]\) for \(m \geq n\).

(ii) A point-stationary point process in \(\mathbb{Z}^k\) (pointed at 0) is equivariantly embeddable in \([\mathbb{Z}^k, 0]\).

(iii) Let \([D_0, o_0] := [\mathbb{Z}, 0]\) and \([D, o] := [\mathbb{Z}^2, 0]\) equipped with the sup metric. Consider \(m : \mathbb{Z}^2 \to \{0, 1\}\) which is equal to one on the boundary of the positive cone. Then, \([\mathbb{Z}^2, 0; m]\) is an embedding of \([\mathbb{Z}, 0]\) in \([\mathbb{Z}^2, 0]\), but is not an equivariant embedding since it does not satisfy (iv).

(iv) Let \(H\) be a finitely generated group equipped with the graph-distance metric of an arbitrary Cayley graph over \(H\) (see subsection I.4.8). Then, any subgroup of \(H\) is equivariantly embeddable in \(H\).

It should be noted that there are examples where \([D_0, o_0]\) is embeddable in \([D, o]\) and both are unimodular, but the former is not equivariantly embeddable in the latter.

**Remark 4.24.** \([D_0, o_0]\) is embeddable in \([D, o]\) if and only if there exists a coupling of them such that in almost every realization, the former is a pointed subspace of the latter (to show this, one should be cautious about the automorphisms). In this case, it is also feasible to call \([D_0, o_0]\) stochastically dominated by \([D, o]\) by considering the inclusion relation between pointed metric spaces. Being equivariantly embeddable seems to be difficult to state in this way.

Here is the main result of this subsection followed by some conjectures and problems.

**Theorem 4.25.** If \([D_0, o_0]\) and \([D, o]\) are unimodular discrete spaces and the former is equivariantly embeddable in the latter, then

\[
\text{udim}_H(D) \geq \text{udim}_H(D_0),
\]

\[
\xi_M^\alpha(D, 1) \leq \mathcal{H}_M^\alpha(D_0),
\]

for all \(\alpha \geq 0\) and \(M \geq 1\), where \(\xi_M^\alpha\) is defined in Definition 4.1.

**Proof.** First, assume (4.13) holds. For \(\alpha > \text{udim}_H(D)\), one has \(\mathcal{H}_M^\alpha(D) > 0\) (Lemma I.3.35). Therefore, Lemma 4.9 implies that \(\xi_M^\alpha(D, 1) > 0\). Hence, (4.13) implies that \(\mathcal{H}_M^\alpha(D_0) > 0\), which implies that \(\text{udim}_H(D_0) \leq \alpha\). So it is enough to prove (4.13).

By the unimodular Frostman lemma (Theorem 4.2), there is a bounded function \(w : D \to \mathbb{R}^{\geq 0}\) such that \(\mathbb{E}[w(o)] = \xi_M^\alpha(D, 1)\), and almost surely, \(w(N_r(o)) \leq r^\alpha\) for all \(r \geq M\). Assume \([D', o'; m]\) is an equivariant embedding as in Definition 4.22. For \(x \in D'\), let \(w'(x) := w(D') := w(D', x)\). Consider the random pointed marked discrete space \([S, o'; w']\) obtained by restricting \(w'\) to \(S\).
Below, it will be proved to be unimodular. Assuming this, since \([S, o']\) has the same distribution as \([D_0, o_0]\), Proposition I.B.2 gives an equivariant process \(w_0\) on \(D_0\) such that \([S, o'; w']\) has the same distribution as \([D_0, o_0; w_0]\). According to the above discussion, one has

\[
\forall r \geq M : w'(N_r(S, o')) \leq w'(N_r(D', o')) \leq r^\alpha, \quad \text{a.s.}
\]

This implies that \(w_0(N_r(o_0)) \leq r^\alpha\) a.s. Therefore, Theorem 2.2 implies that \(E[w_0(o_0)] \leq H^\alpha_M(D_0)\). One the other hand, one has

\[
E[w_0(o_0)] = E[w'(o')] = E[w(o)] = \xi_M^\alpha(D, 1),
\]

where the last equality is by the assumption on \(w\). This implies that \(H^\alpha_M(D_0) \geq \xi_M^\alpha(D, 1)\) and the claim is proved.

It remains to prove that \([S, o'; w']\) is unimodular. Since the mass transport principle holds on \(S\) (Definition 4.22), one can show as in Lemma I.2.28 that the mass transport principle (I.2.3) holds for functions \(g(u, v) := g(D', u, v; (w', m))\) that are zero except when \(m(u) = m(v) = 1\). This implies the mass transport principle for \([S, o'; w']\). So \([S, o; w']\) is unimodular and the claim is proved.

It is natural to expect that an embedded space has a smaller Hausdorff measure. This is stated in the following conjecture.

**Conjecture 4.26.** Under the setting of Theorem 4.25, one has

\[
\mathcal{M}^\alpha(D) \geq \mathcal{M}^\alpha(D_0), \quad \forall \alpha > 0.
\]

Note that in the case \(\alpha = 0\), the conjecture is implied by Proposition I.3.37. Also, in the general case, the conjecture is implied by (4.13) and Conjecture 4.5.

**Problem 4.27.** Does the claim of Theorem 4.25 hold if \([D_0, o_0]\) is non-equivariantly embeddable in \([D, o]\)?

As a partial answer, if \(\text{growth}(\#N_r(o))\) exists, then (4.12) holds. This is proved as follows:

\[
\text{udim}_H(D_0) \leq \text{ess inf} \text{growth}(\#N_r(o_0))
\]

\[
\leq \text{ess inf} \text{growth}(\#N_r(o))
\]

\[
= \text{ess inf} \text{growth}(\#N_r(o))
\]

\[
= \text{udim}_H(D),
\]

where the first inequality and the last equality are implied by the unimodular Billingsley lemma (Theorem 2.8).

**Remark 4.28.** Another possible way to prove Theorem 4.25 and Conjecture 4.26 is to consider an arbitrary equivariant covering of \(D_0\) and try to extend it to an equivariant covering of \(D\) by adding some balls (without adding a ball centered at the root). More generally, given an equivariant processes \(Z_0\) on \(D_0\), one might try to extend it to an equivariant process on \(D\) without changing the mark of the root. But the latter is not always possible. A counter example is when \([D_0, o_0]\) is \(K_2\) (the complete graph with two vertices), \([D, o]\) is \(K_3\), \(Z_0(o_0) = \pm 1\) chosen uniformly at random, and the mark of the other vertex of \(D_0\) is \(-Z_0(o_0)\).
A Appendix

Lemma A.1. Let \((X_n)_{n=1}^{\infty} \geq 0\) be a monotone sequence of random variables. Then almost surely,

\[
\text{growth}(X_n) \leq \text{growth}(E[X_n]), \tag{A.1}
\]
\[
\text{growth}(X_n) \leq \text{growth}(E[X_n]). \tag{A.2}
\]

Moreover, if \(\sum_n \frac{\text{var}(X_n)}{E[X_n]^2} < \infty\), then

\[
\text{growth}(X_n) = \text{growth}(E[X_n]).
\]

Proof. The claims will be proved assuming \(0 \leq X_1 \leq X_2 \leq \cdots\). The non-increasing case can be proved with minor changes. To prove (A.1), let \(\alpha\) and \(\beta\) be arbitrary such that \(\text{growth}(E[X_n]) < \beta < \alpha\). So there is a constant \(c\) such that \(E[X_n] \leq cn^\beta\) for all \(n \geq 1\). Let \(M := \max\{n : X_n > n^\alpha\}\), with the convention \(\max\emptyset := 0\). Below, it will be shown that \(M < \infty\) a.s. Assuming this, it follows that \(\text{growth}(X_n) \leq \alpha\) a.s. By considering this for all \(\alpha\) and \(\beta\), (A.1) is implied.

Now, it is proved that \(M < \infty\) a.s. With an abuse of notation, the constant \(c\) is updated in each step without changing the symbol.

\[
P[M \geq n] = P[\exists k \geq n : X_k > k^\alpha]
\leq \sum_{j=0}^{\infty} P[\exists k : n2^j \leq k \leq n2^{j+1}, X_k > k^\alpha]
\leq \sum_{j=0}^{\infty} P[X_{n2^{j+1}} > (n2^j)^\alpha]
\leq \sum_{j=0}^{\infty} \frac{E[X_{n2^{j+1}}]}{(n2^j)^\alpha}
\leq \sum_{j=0}^{\infty} c(n2^{j+1})^\beta / (n2^j)^\alpha
\leq \sum_{j=0}^{\infty} c(n2^j)^{\beta-\alpha}
\leq cn^{\beta-\alpha}.
\]

The RHS is arbitrarily small for large \(n\). This implies that \(M < \infty\) a.s. and (A.1) is proved.

To prove (A.2), assume \(\text{growth}(E[X_n]) < \beta\). So there is a constant \(c\) and a sequence \(n_1 < n_2 < \cdots\) such that \(E[X_{n_i}] \leq cn_i^\beta\) for all \(i\). Define a sequence \(Y_1, Y_2, \ldots\) by \(Y_j = X_{n_i}\), where \(i = i(j)\) is such that \(n_i \leq j < n_{i+1}\). Now, (A.1) gives

\[
\limsup_j \frac{\log Y_j}{\log j} \leq \limsup_j \frac{\log E[Y_j]}{\log j} \quad \text{a.s.}
\]
Note that for \( n_i \leq j < n_{i+1} \), one has \( \log \mathbb{E}[Y_j] / \log j \leq \log \mathbb{E}[X_{n_i}] / \log n_i \). So the above inequality implies
\[
\limsup_j \frac{\log Y_j}{\log j} \leq \limsup_i \frac{\log \mathbb{E}[X_{n_i}]}{\log n_i} \leq \beta \text{ a.s.,}
\]
where the last inequality holds by the choice of the subsequence \((n_i)_i\). On the other hand, since \( Y_{n_i} = X_{n_i} \) for all \( i \) and \( Y_j \) is constant on \( j \in [n_i, n_{i+1}) \), one has
\[
\limsup_j \frac{\log Y_j}{\log j} = \limsup_i \frac{\log \mathbb{E}[X_{n_i}]}{\log n_i} \geq \liminf_n \frac{\log X_n}{\log n}.
\]
The above two inequalities show that \( \liminf_n \log X_n / \log n \leq \beta \) a.s., which implies the claim.

For the third claim, assume similarly that \( \mathbb{E}[X_n] \geq c'n^{\alpha'} \). Similar to above, it is enough to show that \( M' < \infty \) a.s., where \( M' := \max\{n : X_n < n^{\alpha'}\} \) and \( \alpha' < \beta' \) is arbitrary.

\[
\mathbb{P}[M' \geq n] = \mathbb{P}\left[ \exists k \geq n : X_k < k^{\alpha'} \right] \\
\leq \sum_{j=0}^{\infty} \mathbb{P}\left[ \exists k : n2^j \leq k \leq n2^{j+1}, X_k < k^{\alpha'} \right] \\
\leq \sum_{j=0}^{\infty} \mathbb{P}\left[ X_{n2^j} < (n2^{j+1})^{\alpha'} \right] \\
\leq \sum_{j=0}^{\infty} \mathbb{P}\left[ |X_{n2^j} - \mathbb{E}[X_{n2^j}]| > \frac{1}{2}\mathbb{E}[X_{n2^j}] \right] \\
\leq \sum_{j=0}^{\infty} \frac{4 \text{var}(X_{n2^j})}{\mathbb{E}[X_{n2^j}]^{\alpha'}}
\]
where the third inequality holds for large \( n \) and fixed \( \alpha' \) and the last inequality is by Chebyshev’s inequality. The assumptions imply that the last term tends to zero as \( n \) tends to infinity. So the claim is proved.

**Lemma A.2.** Let \( X, X_1, X_2, \ldots \) be a non-negative i.i.d. sequence and \( t > 0 \) be such that \( \mathbb{P}[X > r] \geq cr^{-t} \) for large enough \( r \). Let \( S_n := X_1 + \cdots + X_n \). Then there exists \( C < \infty \) such that almost surely,
\[
\exists n : \forall k \geq n : S^{-1}(k) \leq Ck^t \log \log k.
\]

**Proof.** First, one has
\[
\mathbb{P}\left[ S^{-1}(n) \geq m \right] = \mathbb{P}[S_m \leq n] \leq \mathbb{P}[\forall i \leq m : X_i \leq n] = \mathbb{P}[X \leq n]^m \leq (1 - cn^{-t})^m \leq e^{-cmn^{-t}}. \quad (A.3)
\]
Let $C := 2^{t+1}/c$ and $\psi(x) := Cx \log \log x$, Therefore, for large $n$, one has

\[
\mathbb{P} \left[ \exists k \geq n : S^{-1}(k) > \psi(k) \right] = \mathbb{P} \left[ \max_{k \geq n} \frac{S^{-1}(k)}{\psi(k)} > 1 \right]
\leq \sum_{j=0}^{\infty} \mathbb{P} \left[ \max_{n2^j \leq k < n2^{j+1}} \frac{S^{-1}(k)}{\psi(k)} > 1 \right]
\leq \sum_{j=0}^{\infty} \mathbb{P} \left[ S^{-1}(n2^{j+1}) > \psi(n2^j) \right]
\leq \sum_{j=0}^{\infty} e^{-2 \log \log(n2^j)}
\leq \sum_{j=0}^{\infty} \frac{1}{(j \log 2 + \log n)^2}.
\]

It is clear that the sum in the last term is convergent. Therefore, dominated convergence implies that the right hand side tends to zero as $n \to 0$. This proves the claim.

Table 1: List of definitions and symbols from Part I.

| Symbol | Description | Reference |
|--------|-------------|-----------|
| $\Psi$ | unimodular discrete space | I.2.13 |
| $\chi$ | equivariant process | I.2.21 |
| $\chi_r$ | equivariant $r$-covering | I.3.1 |
| $\chi_{\chi_r}$ | equivariant covering | I.3.21 |
| $\land$ and $\lor$ | minimum and maximum binary operators | |
| $\# A$ | number of elements in set $A$ | |
| $\text{growth}(f)$ | $\lim \sup_{r \to \infty} \log f(r)/\log r$ | I.2.1 |
| $\text{growth}(f)$ | $\lim \inf_{r \to \infty} \log f(r)/\log r$ | I.2.1 |
| $\text{growth}(f)$ | $\lim_{r \to \infty} \log f(r)/\log r$ | I.2.1 |
| $\text{decay}(f)$ | $-\text{growth}(f)$ | I.2.1 |
| $\text{decay}(f)$ | $-\text{growth}(f)$ | I.2.1 |
| $\text{decay}(f)$ | $-\text{growth}(f)$ | I.2.1 |
| $D$ | a discrete metric space, with elements $u, v, \ldots$ | |
| $N_r(D, v)$ | closed $r$-neighborhood of $v \in D$, with $N_0(v) := \emptyset$ | |

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### Continuation of Table II

| Symbol | Description | Reference |
|--------|-------------|-----------|
| \( N_r(v) \) | \( N_r(D, v) \) for \( v \in D \) | I.2.9 |
| \( D_* \) | set of equivalence classes of pointed discrete spaces | |
| \( D'_* \) | as above for the marked case | |
| \([D,o]\) | the equivalence class containing \((D,o)\) | |
| \([D,o]\) | a random rooted discrete space | |
| \([D,o;m]\) | a random rooted marked discrete space | I.2.11 |
| \( g^+(o) \) | \( \sum_v g(o,v) \), outgoing mass from \( o \) | |
| \( g^-(o) \) | \( \sum_v g(v,o) \), incoming mass to \( o \) | |
| \( \rho_D(S) \) | \( \mathbb{P}[o \in S_D] \), intensity of the equivariant subset \( S \) | I.2.29 |
| \( \text{udim}_M(D) \) | upper unimodular Minkowski dimension | I.3.2 |
| \( \text{udim}_M(D) \) | lower unimodular Minkowski dimension | I.3.2 |
| \( \text{udim}_M(D) \) | unimodular Minkowski dimension | I.3.2 |
| \( \mathcal{H}_M^{\alpha}(D) \) | \( \alpha \)-dimensional Hausdorff content of \( D \) | I.3.3 |
| \( \mathcal{H}_M^{\alpha}(D) \) | same, for coverings with radii in \( \{0\} \cup [M, \infty) \) | I.3.3 |
| \( \text{udim}_H(D) \) | unimodular Hausdorff dimension of \( D \) | I.3.22 |
| \( \mathcal{M}_\alpha(D) \) | \( \alpha \)-dimensional Hausdorff measure of \( D \) | I.3.34 |

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