On conformal higher spins in curved background

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Received 5 October 2016
Accepted for publication 5 December 2016
Published 23 February 2017

Abstract

We address the question of how to represent an interacting action for a tower of conformal higher spin fields in a form covariant with respect to a background metric. We use the background metric to define the star product which plays a central role in the definition of the corresponding gauge transformations. By analogy with the kinetic term in the 4-derivative Weyl gravity action expanded near an on-shell background one expects that the kinetic term in such an action should be gauge-invariant in a Bach-flat metric. We demonstrate this fact to first order in expansion in powers of the curvature of the background metric. This generalizes the result of arXiv:1404.7452 for spin 3 case to all conformal higher spins. We also comment on a possibility of extending this claim to terms quadratic in the curvature and discuss the appearance of background-dependent mixing terms in the quadratic part of the conformal higher spin action.

Keywords: higher spins, conformal theory, curved background

1. Introduction

Conformal higher spin (CHS) field models are $s > 2$ generalizations of Maxwell ($F_{\mu\nu}^2$) and Weyl ($C_{\mu\nu\lambda\rho}^2$) theories [1–4]. While they have higher-derivative $\partial^{2j}$ kinetic terms and thus are formally non-unitary they have a remarkable feature of describing pure spin $s$ states off-shell, i.e. have maximal spin $s$ gauge symmetry consistent with locality.

The free CHS action in flat 4-dimensional space may be written as

$$S_s = \int d^4x \ h_s P_s \ \partial^{2s} h_s = \int d^4x \ (-1)^s \ C_s C_s,$$

(1.1)
where \( h_s = (h_{\mu_1...\mu_s}) \) is a totally symmetric tensor and \( P_i = (P_{\mu_1...\mu_i}) \) is the transverse projector which is traceless and symmetric within \( \mu \) and \( \nu \) groups of indices. This action is thus invariant under a combination of differential (generalized reparametrizations) and algebraic (generalized Weyl) gauge transformations: \( \delta h_s = \partial \epsilon_{s-1} + \eta_2 \omega_{s-2} \) (here \( \eta_2 \) is a flat metric and \( \epsilon \) and \( \omega \) are parameter tensors). \( C_i = (C_{\mu_1...\mu_i\nu_1...\nu_i}) \) is the gauge-invariant field strength or generalized Weyl tensor.

The theory containing an infinite tower of CHS fields \( h_s \) (\( s = 0, 1, 2,... \)) is a non-trivial interacting field theory with an action that can be defined as a local part of an induced action [3–6]. Explicitly, one may start with a free CFT of \( N \) scalar fields \( \int d^4x \phi_\mu \partial^2 \phi_\mu \) which has the on-shell conserved and traceless spin \( s \) currents \( J_s = \phi_s^\mu J_s^\mu \) and consider the generating functional for correlation functions of these currents

\[
\Gamma[h] = N \log \det \Delta(h), \quad \Delta(h) = -\partial^2 + \sum_s h_s J_s. \tag{1.2}
\]

Here \( h_s(x) \) are source fields which have linearized gauge symmetries implied by the on-shell conservation and tracelessness of the currents \( J_s \). The UV logarithmically divergent part of (1.2) is local, has the required linearized gauge symmetries and expanded in \( h_s \) starts with (1.1) as its quadratic term. The coefficient of the logarithmic divergence (or, equivalently, the \( \lambda^0 \) Seeley coefficient in the small \( \lambda \) expansion of the heat kernel of the operator \( \Delta(h) \)) can thus be taken as a definition of the full CHS action, i.e.

\[
S[h] = N \left[ \log \det \Delta(h) \right]_{\log \lambda} \sim N \Tr e^{-\lambda \Delta(h)} \big|_{\lambda = 0}. \tag{1.3}
\]

In this particular construction \( N \) plays the role of the square of the inverse coupling constant which, in general, can be arbitrary. A discussion of some cubic and quartic terms in this action appeared in [5, 7, 8].

This CHS theory has a close connection to AdS/CFT but has also several remarkable features on its own. On general grounds, the theory \( S_{\text{CHS}} \sim \int d^4x h_0^2 + F_{\mu\nu}^2 + C_{\mu\nu\lambda\rho}^2 + \ldots \) with dimensionless coupling constant should be renormalizable—the gauge symmetries should fix the local action uniquely. The central question is the absence of anomalies, in particular, the Weyl anomaly. It was found in [6, 9] that the one-loop \( a \)-coefficient of Weyl anomaly of the \( d = 4 \) CHS theory vanishes under a particular prescription (which should be consistent with the underlying symmetries, see also [10, 11]) for summation over spins. The same was found also for the one-loop conformal anomaly \( c \)-coefficient [9–12] under the assumption that contributions to the conformal anomaly from higher derivative CHS operators on Ricci

3 According to vectorial AdS/CFT this induced action should follow also from the massless higher spin theory in the AdS\(_5\) bulk upon computing it on the solution of the equations of motion with \( h_s \) setting the boundary conditions for the 5d massless higher spin fields.

4 One gets \( \Gamma[h] = N \sum h_s K_s + O(k^4) \), where \( K_s \sim N^{-1} < J_s(x)J_s(x') > \sim P_i |x - x'|^{-\lambda_s - 2} \sim P_i \partial^2 \delta^{(4)}(x - x') \log \lambda + \ldots \). Let us note that to get diagonal kinetic terms for all CHS fields one needs to apply a certain field redefinition required to make the algebraic Weyl-type symmetry manifest [4].

5 Since the dimension of \( h_s \) is \( 2 - s \) and the theory is scale invariant the \( h^m (m = 3, 4,...) \) interaction vertex containing fields of spins \( s_i \) (\( i = 1, \ldots, m \)) involves \( k = 4 + \sum_{i=1}^{m} (k_i - 2) \) derivatives. Thus the coupling to the dimension 0 spin 2 field (conformal graviton) is special: one may add an arbitrary number of \( h_2 \) factors in the vertex without increasing the number of derivatives.
flat background factorize\textsuperscript{6}. As the Weyl symmetry is one of the CHS gauge symmetries, this is an indication that the same anomaly cancellation may apply to all algebraic CHS symmetries.

The CHS theory has also the vanishing Casimir energy on $R \times S^3$ \cite{14} and zero total number of degrees of freedom (trivial flat-space partition function) \cite{11} which is a reflection of the large underlying gauge symmetry of this theory. The global part of this symmetry also strongly constrains the S-matrix involving exchanges of the CHS fields implying that it should be trivial \cite{7, 8}.

The action (1.3) is naturally defined as an expansion in powers of $h_\mu$ fields near flat space. It can thus be interpreted as a higher spin interacting classical conformal field theory. One may then wonder if it may admit a reparametrization and Weyl covariant generalization to a curved background which is known to exist for the standard low-spin ($s=1, 2$) cases. Assuming that the $s=2$ field $h_{\mu\nu}$ may be interpreted as the conformal graviton, one may ask if the action (1.3) can be rewritten (after some field redefinitions) as an expansion near a curved background $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$.

Here we will restrict attention to terms in the CHS action that are quadratic in the fields $h_\mu$ but all-order in the background metric $g_{\mu\nu}$ and address the question which background geometries admit a consistent (gauge-covariant) propagation of $h_\mu$. It follows from the flat-space conformal invariance that the CHS field can be consistently defined on any conformally flat background $\sigma(x) \equiv \eta_{\mu\nu}$. In the case of an arbitrary $\sigma(x)$ the form of the generic spin $s$ kinetic operator is not known explicitly but can be reconstructed, in principle, by a $\sigma$-dependent rescaling of the field (assuming there exists a Weyl-invariant generalization of the flat-space action (1.1))\textsuperscript{7}. In the case of a homogeneous conformally-flat space (4-sphere or AdS or dS or $R \times S^3$) the CHS kinetic operator is known and can be represented as a product of second-order differential operators \cite{9, 13–16}. The question is whether the CHS fields can be consistent on non-conformally-flat backgrounds with non-vanishing Weyl tensor and what are the conditions on the Weyl tensor for this to happen.

For $s=1$ (Maxwell) and $s=2$ (Weyl) cases the CHS kinetic terms admit the well-known generalizations to a non-trivial background metric $g_{\mu\nu}$. For $s=1$ we get no constraints on $g_{\mu\nu}$ while for $s=2$ the invariance of the quadratic term in the Weyl action $\int d^4x \sqrt{g} C_{\mu\nu\lambda\rho}^2$ expanded in $h_{\mu\nu}$ (with $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$) gives a special $\nabla^4 + \ldots$ kinetic operator \cite{1, 14, 17}. This operator is covariant under the gauge transformations $\delta g_{\mu\nu} = \nabla_{(\mu} \epsilon_{\nu)} + \nabla_{(\mu} \epsilon_{\nu)} + \omega g_{\mu\nu}$ provided $g_{\mu\nu}$ is an on-shell background for the Weyl theory, i.e. is Bach-flat,

$$B_{\mu\nu} = 0, \quad B_{\mu\nu} \equiv \nabla^\rho \nabla_\rho P_{\mu\nu} = \nabla^\rho \nabla_\rho P_{\mu\nu} = P^{\alpha\beta\gamma\delta} C_{\mu\alpha\nu\beta\gamma\delta}, \quad P_{\mu\nu} \equiv \frac{1}{2} \left( R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu} \right)$$ \hspace{1cm} (1.4)

\textsuperscript{6}The computation of the one-loop conformal anomaly $c$-coefficient in the CHS theory in \cite{9} was based on two assumptions:

(i) the CHS action obtained as a UV divergent part of the induced action in near-flat space expansion can be reformulated (using a field redefinition) in such a way that at least quadratic kinetic terms in generic curved metric background are reparametrization and Weyl covariant;

(ii) the higher derivative kinetic operators $\nabla^2 + \ldots$ while not factorizing, in general, into products of $\nabla^2 + \ldots$ operators on a Ricci-flat background \cite{13} (as they do on AdS or on the sphere) still contribute to the $c$-anomaly in the same way as if they were factorizing. The reason is that the terms with derivatives of the curvature tensor that obstruct the factorization can not contribute to the $C^2_{\text{anomaly}}$ conformal anomaly on dimensional grounds.

\textsuperscript{7}Alternatively, including some auxiliary and Stueckelberg fields one can reformulate the CHS action in a manifestly conformal form for which rewriting in a generic conformally-flat background amounts to just picking an appropriate $\omega(d, 2)$-connection and conformal compensator.
For $s = 3$ CHS field this question was addressed in [13] where the corresponding covariant $\nabla^6 + \ldots$ kinetic operator was found to linear order in the background curvature tensor and was shown to be gauge-invariant on Bach-flat backgrounds (to first order in the curvature).

A goal of the present paper is to make a step towards a covariant description of all CHS fields on curved Bach-flat (or, in particular, Ricci-flat) backgrounds. Our starting point will be an equivalent definition of the non-linear CHS action (1.3) based on an effective particle Hamiltonian associated with the operator $\Delta(h)$ in (1.2) [4] that makes the full non-linear symmetry of the theory more explicit.

In section 2 we shall review the definition of the particle Hamiltonian coupled to the CHS fields following [4, 5]. Its quantization leads to a quadratic scalar action in CHS background that has gauge invariances inherited from the freedom in the definition of the particle dynamics. In section 3 we shall suggest a procedure of how to define the scalar action in a way covariant with respect to a background metric and having the required gauge symmetries. Then the corresponding CHS action can be again defined as a UV singular part of the induced action found after integrating out the scalar field.

In section 4 we shall analyse the expansion of this action in powers of the CHS fields and the consistency conditions of this expansion using perturbation theory in powers of the curvature of the background metric. Section 5 will contain some concluding remarks. In appendix A we shall review the Fedosov-type approach to covariant formulation of first-quantized particle dynamics that plays important role in our definition of the CHS gauge transformations in a non-trivial background. In appendix B we shall make some general comments on the structure of Weyl invariants built out of the curvature and its covariant derivatives.

2. Particle Hamiltonian in CHS background and expansion near flat space

Before developing a covariant approach to CHS fields let us briefly recall how their gauge transformations and gauge-invariant action arise from the coordinate-dependent quantized particle approach [4, 18].

2.1. Gauge transformations

Let us start with a space-time manifold with coordinates $x^\mu$ and introduce the momenta $p_\mu$ conjugate to $x^\mu$. We will interpret functions of $(x, p)$ which are smooth in $x$ and polynomial in $p$ as symbols of differential operators acting on ‘wave functions’ of $x$. The $*$-product will denote the operator composition in terms of (Weyl) symbols$^8$

$$* = \exp \left[ \frac{h}{2} (\overleftarrow{\partial} \frac{\partial}{\partial x^\mu} - \overrightarrow{\partial} \frac{\partial}{\partial p_\mu} \partial x^\mu) \right].$$

Let us consider a generic relativistic particle Hamiltonian generalizing the free one $H_0$

$$H(x, p) = H_0(x, p) + h(x, p),$$

$$h(x, p) = \sum_{x=0}^\infty h^{\mu_1 \ldots \mu_r}(x) p_{\mu_1} \ldots p_{\mu_r}$$

and subject it to the following gauge transformations [4, 18]$^9$

$^8$ Here $h$ is a formal parameter that can be always set to 1. Note also that we shall use $\mu, \nu, \ldots$ for coordinate indices and $a, b, \ldots$ for tangent space indices.

$^9$ Here the commutator $[,]$ and anticommutator $\{,\}$ are defined with respect to the above $*$-product.
\[ \delta H = \hbar^{-1} \{ H, \epsilon(x, p) \} + \{ H, \omega(x, p) \}, \]  
\[ (2.3) \]

where \( \epsilon, \omega \) are unconstrained symbols interpreted as gauge parameters. They induce the linearized CHS gauge transformations of the coefficient fields \( h_i \).

These gauge symmetries have a simple interpretation [19, 20] in the context of a constrained system \( T_a(x, p) = 0 \) where the symbols \( T_a(x, p) \) are subject to the 1st class condition \( [T_a, T_b] = U_{ab}^* \ast T_c \). A given constrained system can be described by an equivalent set of constraints: an infinitesimal equivalence relation \( T_a \sim T_a + \hbar^{-1} T_\chi \) corresponds to an infinitesimal canonical transformation while the equivalence relation \( T_a \sim T_a + \chi^0 \ast T_b \) corresponds to an infinitesimal redefinition of the constraints (which preserves the constraint surface). Then the space of gauge-inequivalent configurations is a moduli space of constrained systems that have fixed number of 1st class constraints and satisfy certain extra conditions (e.g. belong to a vicinity of certain vacuum \( H_0 \)).

To relate this to (2.3) let us consider the case of just one constraint \( T \equiv H \) and identify parameters as \( \epsilon = \chi = -\frac{\hbar}{2} \lambda, \omega = \frac{1}{2} \lambda \). Then the gauge transformations (2.3) are the natural equivalence transformations of the constrained system describing a relativistic particle. The ‘vacuum’ (quadratic in \( p \)) choice of \( H \)

\[ H_0 = g(x, p) \equiv -\frac{1}{2} g^{\mu\nu}(x) \ p_\mu p_\nu \]  
\[ (2.4) \]

is the standard Hamiltonian of a particle in a gravitational background. In this case the linearized gauge transformations (2.3) read as

\[ \delta h = \hbar^{-1} \{ H_0, \epsilon(x, p) \} + \{ H_0, \omega(x, p) \} = p_\mu g^{\mu
u} \frac{\partial}{\partial x^\nu} \epsilon - p_\lambda \left(\frac{1}{2} \partial_\lambda g^{\mu\nu} p_\nu \frac{\partial}{\partial p_\mu} \epsilon - \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \omega \right) + O(\hbar). \]  
\[ (2.5) \]

Let \( \tilde{H}(x, \frac{\partial}{\partial x}) \) be a differential operator associated to the symbol \( H(x, p) \) (assumed to be such that \( \tilde{H} \) is formally hermitian). As we are using the Weyl symbols this means that \( H \) is real if \( x^\mu \) is real and \( p_\mu \) is imaginary. Then the complex scalar action defined as

\[ S[\phi, h] = \int d^d x \ \phi(x) \tilde{H}(x, \frac{\partial}{\partial x}) \phi(x) \]  
\[ (2.6) \]

is invariant under the transformations (2.3) provided at the same time \( \phi \) transforms as

\[ \delta \phi = - \epsilon (H^{-1} \tilde{\partial} + \tilde{\omega}) \phi, \quad e^\epsilon = - \epsilon, \quad \omega^\lambda = \omega. \]  
\[ (2.7) \]

As follows from (2.3) and properties of the Weyl star product, one can consistently put to zero all fields \( h_i \) of odd spins appearing in (2.2) along with the gauge parameters \( \epsilon, \omega \) of even/odd degree in \( p_\mu \). On top of this there is a consistent truncation to a system where all fields with \( s > 2 \) and their associated gauge parameters are set to zero. This is due to the fact that the elements which are at most linear in \( p_\mu \) form a Lie subalgebra of a Weyl star-product algebra. The
above two truncations can be combined, resulting in a system for fields of spins 0 and 2 only. Furthermore, the spin 0 field can also be eliminated. Apart from the above consistent truncation to spins \( \leq 2 \) one can not get a gauge invariant action depending only on a finite number of fields \( h_s \).

### 2.2. Conserved currents

For \( H \) in (2.2) and (2.4) the action (2.6) may be written as

\[
S = \int d^4x \left[ \phi^* \phi \mathcal{H}_0 \phi + \phi^* \phi \right].
\]

The condition of its invariance under (2.3) combined with \( \delta \phi = -h^{-1} \varepsilon \phi \) takes the form

\[
\int d^4x \left( (\mathcal{H}_0, \varepsilon \phi), + [h_s, \varepsilon \phi] \right) \frac{\delta S}{\delta h_s} = 0.
\]

Introducing a generating function for conserved currents (here \( u^\mu \) is an auxiliary constant vector)

\[
J = \sum_s \frac{1}{s!} u^{\mu_1} \cdots u^{\mu_s} \frac{\delta S}{\delta h^{\mu_1} \cdots \mu_s},
\]

the quadratic in the fields term in (2.9) may be written as

\[
\int d^4x \left( (\mathcal{H}_0, \varepsilon \phi), J \right) - 2(\varepsilon \phi)^* \mathcal{H}_0 \phi = 0,
\]

where \( \langle \, , \rangle \) denotes a natural inner product (contraction of indices) between polynomials in \( p_\mu \) and polynomials in \( u^\mu \). For \( \mathcal{H}_0 = -\frac{1}{2} \eta^{\mu \nu} p_\mu p_\nu \equiv -\frac{1}{2} p^2 \) one gets the usual on-shell conservation condition for the currents

\[
\eta^{\mu \nu} \frac{\partial}{\partial u^\mu} \frac{\partial}{\partial x^\nu} J = 0.
\]

Applying analogous arguments to the second gauge invariance with the parameter \( \omega \) in (2.3) results in the generalized on-shell tracelessness condition for the currents:

\[
\int d^4x \left( (\mathcal{H}_0, \omega \phi), J \right) - 2(\omega \phi)^* \mathcal{H}_0 \phi = 0.
\]

For \( \mathcal{H}_0 = -\frac{1}{2} p^2 \) one gets the ‘deformed’ tracelessness condition

\[
(\eta^{\mu \nu} \frac{\partial}{\partial u^\mu} \frac{\partial}{\partial x^\nu} - \frac{1}{2} h^2 \square) J = 0.
\]

Redefining the components of \( J \) one can make them strictly traceless \[4\] but it is not always useful to perform this redefinition explicitly.

More generally, taking a variational derivative of (2.9) with respect to \( \varepsilon \) and not decomposing the result according to the homogeneity in the fields leads to a nonlinear generalization of the conservation condition (2.12) which now involves the fields \( h_s \). Analogous arguments apply to gauge invariance under the transformations with the parameter \( \omega \) leading to a nonlinear version of (2.14).

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14 For instance, \((1, 1) = 1, (\omega^\mu, p_\mu) = \delta^\mu_\nu \), etc. Note that then \((\omega^\nu)^\mu = \frac{\partial}{\partial \omega^\mu} \), etc.
3. Covariant form of the scalar field action in CHS background

To generalize the above discussion to a curved background we shall first consider a covariant framework for a relativistic particle quantization. One can naturally define a quantization of the cotangent bundle over a curved spacetime in a geometrically covariant way. This covariant description is based on the metric $g_{\mu\nu}$ and the metric connection and includes (see also appendix A):

- **star product**: for any functions of $x^\mu, p_\nu$ which are smooth in $x$ and polynomial in $p$ there is a well-defined (and unique under some extra natural conditions) associative $\star$-product. The $\star$-commutator (anti-commutator) carries odd (resp. even) homogeneity in $p$.

- **state space**: space of functions of $x^\mu$ equipped with the natural inner product $\langle \phi, \chi \rangle = \int d^4x \sqrt{g} \phi^*(x)\chi(x)$.

- **symbol map**: there is a well-defined map $\widehat{f}$ from functions of $x, p$ (symbols) to differential operators acting on functions of $x^\mu$ (‘wave functions’) such that $\hat{f}_1 \star \hat{f}_2 = \hat{f}_1 \hat{f}_2$.

The operator associated to $f(x, p)$ is denoted by $\mathcal{R}(g,f) = \hat{f}(x, \frac{\partial}{\partial x})$. Here $g$ indicates the dependence on the background metric, i.e. on the covariant derivative and the curvature built out of it (see appendix A and more specifically (A.13) and propositions A.2 and A.3). The real symbols correspond to hermitian operators.

Let us redefine the spin 0 part of $h(x, p)$ in (2.2) by the scalar curvature of the metric and write $H$ in (2.2) and (2.4) as

$$H(x, p) = g(x, p) + \mathcal{R}(x) + h(x, p), \quad g(x, p) = -\frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu, \quad \mathcal{R} = \gamma R,$$

where $R$ is the scalar curvature and $\gamma$ is a numerical coefficient. The covariant version of the scalar field action (2.6) and (2.8) then reads

$$S[g, h, \phi] = \int d^4x \sqrt{g} \phi^*(g(x, p) + \mathcal{R}(x))\phi + \int d^4x \sqrt{g} \phi^* h(x, p)\phi,$$

where we have explicitly separated the $h$-independent term. The coefficient $\gamma$ in (3.3) is chosen so that the first term in the above action is the standard action of the conformally coupled scalar. Note that this action depends on $g_{\mu\nu}$ also through the metric connection entering the symbol map.

By construction, this action is invariant under the covariant version of (2.3) and (2.7)$^{16}$

$$\delta h = [g + \mathcal{R} + h, \epsilon], \quad \delta \phi = -([g + \mathcal{R} + h, \omega])\phi,$$

where $h$ is the scalar field.

$$\delta h = [g + \mathcal{R} + h, \epsilon], \quad [g + \mathcal{R} + h, \omega] = h.$$

Let us stress that the background field $g_{\mu\nu}(x)$ is not affected by this gauge transformation. However, there are hidden gauge transformations of the action (3.4) related to redefinition of $g_{\mu\nu}$ and $h_2 = (h_{\mu\nu})$ which do not change their sum $g + h$ modulo relevant redefinition of the

$^{15}$ For example, if $f(x, -p) = f(x, p)$ and $u(x, -p) = u(x, p)$ then $[f, u](x, -p) = -[f, u](x, p)$.

$^{16}$ Below we set $\hbar = 1$ for notational simplicity.
symbol map (which depends on $g^{\mu \nu}$). For further analysis it is useful to employ two extra types of symmetries which have natural geometrical meaning and are, in fact, certain combinations of (3.5) and (3.6) and redefinitions of $g^{\mu \nu}$ and $h^{\mu \nu}$.

First, given that the action (3.4) contains only covariant objects, it is invariant under the diffeomorphisms generated by a vector field $\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$ with $h^{\mu \nu}$, transforming as tensors, i.e. under

$$\delta g = L_\xi g, \quad \delta h = L_\xi h, \quad \delta \phi = \xi \phi.$$  (3.7)

Second, the $h$-independent term in (3.4) is the action of a conformally coupled scalar and hence it is invariant under the usual Weyl symmetry $\delta \omega g^{\mu \nu} = 2 \omega g^{\mu \nu}$, $\delta \omega \phi = (\frac{d}{2} - 1) \omega \phi$, where $\omega_0$ is $p$-independent. The second term in (3.4) can also be made invariant by setting $\delta_\omega h = 2 \omega h + \delta'_{\omega} h$, where

$$\delta'_{\omega} h = [\omega_{0}, h]_s - 2 \omega h + \delta''_{\omega} h.$$  (3.8)

Here $\delta''_{\omega} h$ takes into account the variation of the symbol map under the variation of the metric. More precisely, because the map between the operators and the symbols is one-to-one, one can always trade a variation of the symbol map for an appropriate variation $\delta''_{\omega} h$ of $h$. If we denote by $\mathcal{R}(g, h)$ the operator associated by the symbol map to the symbol $h$ then

$$\mathcal{R}(g + 2 \omega_0 g, h) = \mathcal{R}(g, h + \delta''_{\omega} h) + O(\omega_0^2).$$  (3.9)

One can then represent the variation of $h$ as $\delta_h h = [\omega_{0}, h]_s + \delta''_{\omega} h = 2 \omega h + \delta'_{\omega} h$. It follows from the structure of the star product and the symbol map that $\delta''_{\omega} h$ is linear in the CHS fields and only depends on $h_s$ with $s_1 > s_0$.

We conclude that the action (3.4) has an infinitesimal symmetry which is the direct analog of the usual Weyl transformations

$$\delta g = 2 \omega g^{\mu \nu}, \quad \delta h = 2 \omega h + \delta'_{\omega} h, \quad \delta \phi = (\frac{d}{2} - 1) \omega \phi.$$  (3.10)

It will be called the deformed Weyl symmetry in what follows.

4. Covariant expansion of the CHS action in a non-trivial metric

Starting with the covariant version (3.4) of the scalar field action minimally coupled to the CHS fields one can integrate out the scalar $\phi$ and extract the local log-divergent part $S[g, h]$ of the resulting induced action as in (1.3). This local term is invariant under the $h$-field part (3.5) of the gauge symmetries (3.5) and (3.6) as well as under the symmetries (3.7) and (3.10) of

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17The only nontrivial point is to check this for $\delta''_{\omega} h$. The terms in the $\delta''_{\omega} h$ involving $s$ derivatives of $\phi$ have the structure $h^{\mu_1 \cdots \mu_t} \partial_{\mu_1} \cdots \partial_{\mu_t} \phi$ where contractions of indices between the two groups may be via $\delta$-symbol, $\Gamma^{\mu_5}_{\mu_1 \mu_2}$ and the curvature and its derivatives. It is clear that any such contraction can be nonvanishing only for $t \geq s$ (note the number of upper and lower indices in $\Gamma$, $R$ etc), and, moreover, at $t = s$ one can have the leading contribution where only $\delta$-symbols are employed in the contraction. Furthermore, the variations of the above expression under the change of $g^{\mu \nu}$ and the respective change of the connection, curvature, etc, can be compensated by the variation of $h_s$. This way one finds the compensating transformation $\delta''_{\omega} h$ which, by construction, is proportional to $h_s$ with $s > s_0$. For instance, $\delta''_{\omega} h$ contains the terms such as $h^{\mu_1 \cdots \mu_t} \partial_{\mu_1} \cdots \partial_{\mu_t} \phi$, $\delta'_{\omega} h^{\mu_1 \cdots \mu_t} \partial_{\mu_1} \cdots \partial_{\mu_t} \phi$, $\delta'_{\omega} h^{\mu_1 \cdots \mu_t} \partial_{\mu_1} \cdots \partial_{\mu_t} \phi$ as well as further terms involving $h_s, h_{s-1}$, etc. Here, $\delta''_{\omega}$ acts only on the Levi-Civita connection coefficients and the respective curvature and denotes their variation under the Weyl transformation.
the original scalar action and thus provides a natural definition of the CHS action $S[g, h]$ in a general metric background.

### 4.1. Expansion of the CHS action

Let us specify to the case of $d = 4$ and consider the expansion of $S[g, h]$ in powers of $h_s$:

\[
S[g, h] = S[g] + S_1[g, h] + S_2[g, h] + \ldots, \tag{4.1}
\]

\[
S_1[g, h] = \sum_s \int d^4x \sqrt{g} K_{\mu_1\ldots\mu_4} h^{\mu_1\ldots\mu_4}, \quad S_2 = \sum_s \int h_s \mathcal{O}_s[g] h_s, \ldots \tag{4.2}
\]

Here we ignore total derivatives and hence $K_s = (K_{\mu_1\ldots\mu_s})$ can be assumed to be a local function of the metric $g$. The diffeomorphisms (3.7) transform $g$ and $h_s$ through themselves. Under the deformed Weyl transformations (3.10) $g$ gets rescaled while $h_t$ transforms into $h_t$ with $t \geq s$. As $S[g]$ must be invariant under both diffeomorphisms and the usual Weyl transformations of the metric $g$ and is local, it should be the standard $C^2_{\mu l \nu \rho}$ Weyl action (see the discussion of Weyl invariants in appendix B).

As the diffeomorphism and the deformed Weyl symmetries are homogeneous in $h$, the linear in $h$ term $S_1$ must be invariant on its own. Thus $K_{\mu_1\ldots\mu_4}[g]$ should be a tensor under the diffeomorphisms and should vanish (or give a total derivative) if $g_{\mu
u}$ is flat. The fact that the flat-space CHS action has no terms linear in $h$ is clear directly from (1.2) and (1.3). Indeed, as $h_t$ has mass dimension $2 - s$ and the CHS action is local and dimensionless, $K_s$ in $S_1$ should have dimension $2 + s$, i.e. it should have a structure $\nabla^2 + R\nabla^s + \ldots + R^{s+2\nu l \nu \rho}$, where $R$ is the curvature. We shall ignore the leading highest derivative term as it gives a total derivative in (4.2).

Let us note that for a flat $g_{\mu
u}$ background the quadratic in $h_s$ term is not manifestly diagonal before one performs the algebraic redefinition of the fields (that takes care of the traces of the fields, i.e. is related to the algebraic part of the gauge transformations [4]). For a non-trivial $g_{\mu
u}$ one will face a more serious non-diagonality issue due to terms involving the curvature of $g_{\mu
u}$ that mix fields of different spin; this is related to the differential part of the gauge transformations.

Suppose that $g^{\mu \nu} = g_0^{\mu \nu}(x)$ and $h_s = 0$ (for all $s$) is a particular solution of the equations corresponding to $S[g, h]$. The necessary and sufficient conditions for that are (ignoring total derivative terms)

\[
\frac{\delta S[g]}{\delta g} \bigg|_{g=g_0} = 0, \quad K_{\mu_1\ldots\mu_s}[g_0] = 0. \tag{4.3}
\]

Thus $g_0$ should be Bach-flat and $K_s$ should vanish on a Bach-flat background. Then the expansion of (4.1) near this solution reads as

\[
S[g_0, h] = S[g_0] + S_2[g_0, h] + \ldots, \tag{4.4}
\]

where we set to zero the perturbation $\delta h_t$ of $g_t$ itself. As $g_t$ is not affected by the gauge transformations (3.5) and (3.6) the term $S_2[g_0, h]$ should be invariant under the linearized version of (3.5), i.e.

\[
\delta h_s = \left[ (g_0 + \mathcal{R}_s \omega_s + A_s) \right]_s, \tag{4.5}
\]

Here $A_s$ denotes the projection to spin $s$ of the generating function $A(x, p)$, i.e. the term of homogeneity $s$ in $p_\nu$. 

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It follows from the structure of the star-product that one can consistently put to zero all the fields with $s > s_0$ along with $\epsilon$ parameters of homogeneity $> s_0 - 1$ and $\omega$ of homogeneity $> s_0 - 2$ in $p$. Hence $S_d[g_0, h_1]_{s_0 > s > 0}$ is invariant under the linearized gauge transformations (4.5) with $\epsilon$ of degree $< s_0$ and $\omega$ of degree $< s_0 - 1$.

Let us now show the vanishing of $K_d[g_0]$ in Bach-flat background at least to first order in the background curvature. This will generalize the result of [13] that the $s = 3$ kinetic operator is gauge-invariant in a Bach-flat background to linear order in the curvature.

4.2. Conditions for the vanishing of the linear fluctuation term

Let us study the consequences of the deformed Weyl symmetry (3.10) for the structure of $K_{\mu_1 \ldots \mu_s}$. It is useful to introduce the following notation for the transformation of $K_s$ under $\delta g^{\mu \nu} = 2 \omega g^{\mu \nu}$:

$$
\delta'_{\omega} K_{\mu_1 \ldots \mu_s} = \delta_\omega K_{\mu_1 \ldots \mu_s} - 2 \omega \delta K_{\mu_1 \ldots \mu_s}.
$$

Consider the variation of $\sqrt{g} K_d h_1$ under the deformed Weyl transformation. Since $\delta' h_0$ (defined in (3.8)) does not depend on $h_0$, the only term proportional to $h_0$ is $\delta' K_0 h_0$ and hence $\delta' K_0 = 0$. This implies that $K_0$ is Weyl invariant with weight $-2$ (i.e. behaves like $g^{\mu \nu}$) but there are no such non-trivial invariants (see appendix B).

Let us proceed by induction. Assuming that we have shown that $K_r = 0$ for $r < s$, let us consider the variation of $\sqrt{g} K_d h_s$ under the deformed Weyl transformation. Concentrating on the terms in the variation proportional to $h_s$ gives

$$
(\delta'_{\omega} K) h_s + \sum_{l=0}^{\infty} K_l(\delta' h_l) \bigg|_{h_l=0, t=s, s} = 0.
$$

Note that $(\delta' h_l)$ $= 0$ for $l \geq s$. Taking this and $K_l = 0$ for $l < s$ into account one finds $\delta'_{\omega} K_s = 0$ and hence $K_s$ is a tensor which is Weyl invariant of weight $-2$.

Next, let us consider the gauge transformations (3.5). The gauge variation of $S_1$ under the inhomogeneous part $(g + R, \epsilon)_{s_1} + \{g + R, \omega_s\}$ of (3.5) should vanish. Setting to zero all the fields of spin $> s$ and the associated gauge parameters one gets

$$
\int d^4x \sqrt{g} K_{\mu_1 \ldots \mu_s} \nabla^{(\mu_s \epsilon \mu_1 \ldots \mu_{s-1})} = 0, \quad \int d^4x \sqrt{g} K_{\mu_1 \ldots \mu_s} g^{(\mu_s \mu_1 \mu_2 \ldots \mu_{s-1})} = 0.
$$

This leads to

$$
\text{Tr} K_s = 0, \quad \nabla^{\mu_1} K_{\mu_2 \ldots \mu_s} = 0,
$$

i.e. $K_s$ should have the same properties as a covariantly-conserved traceless current. The relations (4.8) and (4.9) have direct generalizations to $d > 4$ dimensions.

We have thus shown that under the induction assumption $K_s$ should be a Weyl invariant tensor of weight 2 which is also traceless and covariantly conserved. The totally symmetric traceless tensors of Weyl weight $-2$ are called in the math literature as ‘admissible invariants’. For $s = 3$ these invariants are known explicitly [22]: for $s = 1$ any invariant vanishes; for $s = 2$ it is proportional to the Bach tensor; for $s = 3$ it is proportional to the Eastwood–Dighton tensor (see appendix B for more details):

$$
E_{\mu_1 \mu_2} \rightarrow E_{ABCA | EDF} = \Psi_{ABCD} \nabla^{\delta D E} \Psi_{ABCD} = \Psi_{AFCD} \nabla^{\delta D F} \Psi_{AFCD}.
$$

(4.10)
Here we resorted to the spinor conventions where $\Psi_{ABCD}$ and $\Psi^{ABCD}$ are Weyl spinors corresponding to (anti)self-dual components of the Weyl tensor.

For general $s$, one may consider a special case of 4d Bach-flat backgrounds with Weyl tensor which is (anti)selfdual. It is easy to see that in this case $K_s$ must vanish (see appendix B) and hence CHS are consistent.

Considering generic backgrounds, let us restrict attention to terms in $K_s$ which are linear in the curvature or Weyl tensor $C$. Then we will have $\nabla^2 C$ like terms where $s$ indices are symmetrised and 4 indices are contracted by $g^{\mu\nu}$. Since $\nabla^2 C$ should be a totally symmetric tensor and since $C$ has 4 indices and is traceless, two of the derivatives should act on $C$ itself. Such terms should vanish on a Bach-flat background.

It would be important to extend the above argument of the vanishing of $K_s$ beyond the linear in curvature terms. Let us make few comments that may be useful for an attempt to prove this. As the metric $g$ should actually be a background for the spin 2 field $h$, there should be a hidden gauge symmetry which transforms $g$ and $h$ in such a way that their sum $g + h$ remains invariant while all other fields $h_s$ also transform to compensate for the change of the symbol map. This symmetry may be useful to eliminate some unwanted terms. Another remark is that we are dealing with the CHS theory involving an infinite set of fields but so far made use of only some of the gauge symmetries that preserve the subspace of field configurations where only a finite collection of fields are non-vanishing. In particular, for the above arguments to work it is enough to compute the CHS action as the divergent part of the scalar effective action (1.3) with $h_r = 0$ for $r > s$. This way one may avoid subtleties related to the fact that the full space of CHS fields is infinite-dimensional. Finally, let us mention that the entire construction can probably be made more geometrical by employing the conformally equivariant quantization which is known [23, 24] for generic conformal manifolds.

4.3. Gauge invariance of spin-s quadratic term to first order in curvature

As we have argued above, $K_s[g_0]$ must vanish at least up to terms of second order in the curvature\(^{18}\) if the background metric $g_0$ is Bach-flat. The gauge invariance of the complete action (4.1) at the zeroth and the first order in $h$ gives

$$\int \sqrt{g} K_s \delta_s^0 h_s = 0, \quad \int \sqrt{g} K_s \delta_s^1 h_s + \int \sqrt{g} \frac{\delta S_2}{\delta h_s} \delta_s^0 h_s = 0,$$

(4.11)

where $\delta_s^0$ and $\delta_s^1$ denote the leading and the linear in $h$ parts of the gauge transformation. As $K_s \sim C^2$ (here $C$ denotes the Weyl tensor and its Weyl-covariant derivatives, see appendix B) the second equality implies

$$\delta_{\epsilon,\omega}^0 S_2[g_0, h] = O(R^2),$$

(4.12)

where $\delta_{\epsilon,\omega}^0$ denotes the gauge transformation linearized around $g = g_0$, $h_s = 0$. To zeroth order in the curvatures the gauge transformations are explicitly

$$\delta_{\epsilon,\omega}^0 h_s = (p_j \nabla^j) e_{s-1}, \quad \delta_{\epsilon,\omega}^0 h_s = -\frac{1}{2} g^{\mu\nu} p_\mu p_\nu \omega_{s-2} - \frac{1}{4} g^{\mu\nu} \nabla_\mu \nabla_\nu \omega_s.$$  

(4.13)

We now set to zero all the fields with $s > s_0$ and their associated gauge parameters. For $s = s_0$ (4.13) then gives the exact linearized gauge transformation. Indeed, the curvature contributions may only affect fields with the spins lower than $s_0$. Note also that the second term in the expression for $\delta_{\epsilon,\omega}^0 h_s$ vanishes for $s = s_0$. Moreover, to zeroth order in the curvature

\(^{18}\) By curvature terms we always mean the products of Riemann tensor and its covariant derivatives.
this term can be removed [4] by the field and the gauge parameter redefinition. Upon this redefinition the gauge transformation takes the standard diagonal form, with the usual derivatives replaced by the covariant ones. This implies that to zeroth order in the curvature the term $S_2[g_0, h]$ is just a direct sum of the standard quadratic actions for all spins $1, 2, \ldots, s_0$.

Let us now include terms of first order in the curvature. Because to zeroth order in curvature $S_2[g_0, h]$ is diagonal (does not contain terms mixing different spins) the gauge invariance implies that the quadratic in $h_0$ term in $S_2[g_0, h]$ is just a direct sum of the standard quadratic actions for all spins $s_0$. Let us now assume that the background metric is chosen such that both the Bach tensor (1.4) and the Eastwood–Dighton tensor (4.10) vanish, i.e. $B_{\mu\nu} = 0 = E_{\mu\nu\rho}$. For an algebraically-general Weyl tensor this implies that the metric is conformally Einstein [22, 25]. Unfortunately, the vanishing of $K_3$ in (4.2) and (4.9) does not directly imply that the spin 3 CHS field kinetic term is always consistent (i.e. gauge-invariant) on such a background. Taking $\epsilon = \epsilon(x)$ and extracting the linear in $h_3$ contribution in the $h_4$ variation in the second equation in (4.11) one gets:

$$\int d^4x \sqrt{g} K_4[h_3, \epsilon_1 h_1] + \int d^4x \sqrt{g} h_1 \frac{\delta^2 S_2}{\delta h_0 h_1} \delta_0^0 h_1 + \int d^4x \sqrt{g} h_1 \frac{\delta^2 S_2}{\delta h_0 h_3} \delta_0^0 h_3 = 0.$$  \hfill (4.14)

Thus if $K_4$ were nonvanishing beyond the leading order in curvature, our argument would not in general imply that the spin 1 plus spin 3 system is consistent on its own. Below we shall assume that this is not the case, i.e. all $K_i$ vanish on Bach-flat background to all orders in curvature expansion.

It is clear from the structure of (4.14) that on a non-trivial background the spin 3 field may mix with the spin 1 in the quadratic term $S_2$ in (4.1). To understand the reason for this mixing let us go back to the discussion in sections 2 and 3 and consider the linearized gauge transformations around the vacuum Hamiltonian $H_0 = -\frac{1}{2} \eta^{\mu\nu} p_\mu p_\nu = -\frac{1}{2} \eta^{ab} p_a p_b$ in (2.4).

\[ \delta h_0 = \epsilon(x) \gamma_0 + 2 \omega h_0 + \frac{1}{2} \eta^{ab} \nabla_a \nabla_b \omega. \]

Using the transformation law of the scalar curvature $\delta R = 2 \omega R + \eta^{ab} \nabla_a \nabla_b \omega$ under Weyl transformations of the metric $g \rightarrow g \omega$ one finds that $h_0 = h_0 - \gamma R$ transforms homogeneously: $\delta h_0 = \epsilon(x) h_0 + 2 \omega h_0$. It follows from the above transformation law that one can consistently put $h_0$ to zero in the scalar field action

$$\int d^4x \sqrt{g} \phi'(\phi + h_0) = \int d^4x \sqrt{g} \phi'(-\nabla^2 + \gamma R + h_0) \phi.$$
As was noted above, it follows from the structure of the star-product that the linearized gauge
transformations with parameters $\varepsilon$ and $\omega$ of degree $s-1$ and $s-2$ respectively can only affect
the fields of spins $s, s-2, s-4, \ldots$. Thus the simplest nontrivial system is that of spins 1 and 3.
For $s=1$ field the gauge transformations are standard. For $s=3$ the parameters are $\epsilon^{ab}p_cpp_p$ and $\omega^{a}p_a$.
Let us first consider the gradient-like transformation

$$
\delta \eta = -\nabla^\ast \varepsilon \varepsilon^{hp pp pp pp p}, \quad \delta \omega = -\nabla^\ast \varepsilon \varepsilon^{hp pp pp p} \omega^{a}p_a.
$$

(4.15)

where we projected to the spin 3 component and disregarded the contribution from the back-
ground scalar curvature. This is thus a natural covariantisation of the flat-space gradient
gauge transformation. However, the transformation generated by $\epsilon^{cd}p_c p_d$ gives also a non-zero
contribution to the spin 1 sector:

$$
\delta \eta = -\nabla^\ast \varepsilon \varepsilon^{hp pp pp p} R^{ab} \varepsilon^{cd}p_c p_d.
$$

(4.16)

Thus the linearized gauge transformations with parameters $\epsilon^{ab}$ and $\omega^a$ will act on spin 1 field as

$$
\delta h_a = \frac{4}{3} R_{ab} \varepsilon^d \nabla^e e^{bl} \varepsilon - \frac{1}{4} \nabla^2 \omega_a.
$$

(4.17)

While the second term here can be removed by a field redefinition (the ‘dressing map’ of [4])
h_a \rightarrow h_a + e \nabla^l h_{ab} b^l, the first term is non-trivial.

The presence of the $\epsilon^{bl}$ term in (4.17) implies that the standard Maxwell $\partial h_0 \partial h_1$ term in the
quadratic action $S_2$ can not be invariant under such transformation. As a result, we should then expect $h_0 h_3$ mixing, i.e. non-diagonal terms like $R \nabla h_0 \nabla h_3 + R \nabla \nabla h_1 + RR h_0 h_1$ that should compensate for the variation of the quadratic in $h_1$ term under the $h_3$ gauge transformation in

(4.17)

As we have seen above, to first order in the curvature the mixing terms in the action like $h_0 h_3$
one do not affect the gauge invariance of the quadratic term $S_2$ under the transformation with
parameters $\epsilon_{s-1}$ and $\omega_{s-2}$ (so there is no contradiction with [13] where quadratic in $h_3$ term in
the action was constructed to linear order in the curvature by imposing the condition of gauge
invariance). However, to second order in the curvature the mixing terms can not be neglected.
Then it is natural to expect that in general only a system of all spins $s, s-2, s-4, \ldots$ can be
well-defined on a sufficiently curved background.

The presence of non-diagonal terms in $S_2$ on curved background is thus expected in general
and deserves further study.

20 Let us note that the need for spin 1 field to transform under the spin 3 gauge transformations when the back-
ground is not conformally flat can be concluded directly from the study of the conservation condition of a
complex scalar spin 3 current on a curved (e.g. Ricci-flat) background (Roiban and Tseytlin, unpublished). If
$J_3 = \phi \nabla \ldots \nabla \phi + \ldots$ then on a Ricci-flat background and using $\nabla^2 \phi = 0$ one can show that $\nabla^2 J_3 \sim C_{ab} \nabla^2 F^a$ implying that to have the $h_0 h_3 + h_0 \omega_3$ coupling term to be invariant under spin 3 gauge transformations
$\delta h_0 = \nabla^a \omega_a + \ldots$ one is to modify the spin 1 transformation by $C^{ab} \nabla^a \varepsilon_{ab}$. term.

21 Note that in the constant curvature space where $R_{abcd} = \lambda (\eta_{bc} - \eta_{bd} - \eta_{bd} + \eta_{bc})$ the first term in (4.17) takes the form
$\delta h_a = \lambda (\nabla^a \xi_{b} - \nabla^b \xi_{a} - \nabla^b \xi_{a} + \eta^{b} (\nabla^a \omega_a) )$ and hence can be removed by a combination of field redefinition and
gauge parameter redefinition. The same should be true also for general conformally-flat metrics.
5. Conclusions

In this paper we addressed the question of covariant description of conformal higher spin fields in a non-trivial background. The standard definition of the CHS action (1.3) gives an expansion near flat space and thus is not generally covariant. Given that the spin 2 CHS field should have a natural interpretation of a conformal graviton, one expects that there should be a possibility to rewrite this action in a manifestly covariant form with the spin 2 part represented by the non-linear Weyl action\(^22\).

We suggested a way to define the CHS action in a covariant way by using the background metric to define the star product in the associated particle dynamics and thus in the definition of the gauge transformations.

As is well known, the quadratic term in an action expanded near its classical solution should have linearized gauge invariance. For example, the quadratic 4-derivative operator in the Weyl action expanded near Bach-flat background is consistent, i.e. has the standard reparametrization invariance (which is fixed by a background gauge in quantum computations). The same was previously found to be true to linear order in the curvature expansion for the conformal spin 3 operator in a Bach-flat metric [13]. Here we generalized this fact to any conformal higher spin field and commented on a possibility of extending this claim to terms quadratic in the curvature. We also pointed out the presence of curvature-dependent mixing terms in the quadratic part of the conformal higher spin action expanded in a non-trivial background.

Acknowledgments

We would like to thank M Beccaria, R Metsaev, R Roiban, E Skvortsov and M Taronna for useful discussions of related questions. MG also wishes to thank N Boulanger for a useful discussion of Weyl invariants. AAT is grateful to S. Kuzenko for useful discussions and the hospitality during visit of the University of Western Australia. This work was supported by the Russian Science Foundation grant 14-42-00047. The work of AAT was also supported by the ERC Advanced grant No.290456, the STFC Consolidated grant ST/L00044X/1 and by the Australian Research Council, project No. DP140103925.

Appendix A. Covariant quantization in Fedosov-type approach: quantum version of normal coordinate expansion

Let us recall how to perform quantization on the cotangent bundle in generic coordinates. Let \(x^\mu\) be coordinates on the base manifold and \(p_\mu\) their conjugate momenta. The canonical Poisson bracket reads as \(\{x^\mu, p_\nu\} = \delta^\mu_\nu\). We would like to define quantization compatible with a given Riemannian metric \(g_{\mu\nu}(x)\). Let us introduce frame field \(e^a_\mu\) and Lorentz connection \(\omega^a_\mu\) such that\(^23\)

\[
\nabla e^a_\mu = 0, \quad \omega^b_\mu \Gamma^a_{bc} + \omega^a_\mu \eta_{ba} = 0, \quad g_{\mu\nu} = \epsilon^a_\mu \epsilon^b_\nu \eta_{ab}. \tag{A.1}
\]

In what follows we will use the coordinates \(x^\mu\), \(p_\mu = \epsilon^a_\mu p_a\) on the cotangent bundle.

Let us introduce extra variables \(y^a\) which are coordinates on the tangent spaces and the star product

\(22\) It should be noted that a possibility to rewrite the action for an infinite set of fields in a manifestly covariant and local way is not a priori obvious. For a somewhat related discussion in the string theory context see [26].

\(23\) We use convention \(\nabla (T^a_\mu) = dx^\nu \omega^a_\mu T^a_\nu, \quad \nabla (R^a_\mu) = \partial_\mu^\nu R^a_\nu + \omega^a_\mu \omega^b_\nu - (\mu \rightleftarrows \nu).\) In particular,

\[
\nabla^2 f(x, y, p) = \frac{1}{2} dx^\mu dx^\nu R^a_\mu R^a_\nu f(x, y, p).
\]
\[ \circ = \exp \left[ \frac{\hbar}{2} \left( \frac{\partial}{\partial y^a} \frac{\partial}{\partial p_b} - \frac{\partial}{\partial p_a} \frac{\partial}{\partial y^b} \right) \right] \]  

(A.2)

**Proposition A.1.** Given \( \omega \), there exist a nonlinear connection whose covariant derivative (acting on forms with values in functions of \( y, p \)) has the form

\[ D = dx^\mu \frac{\partial}{\partial x^\mu} + h^{-1} \mathbb{e}_a^\mu p_\mu + \omega^b_\mu \frac{\partial}{\partial p_a} p_\mu, \quad \cdot_0 + h^{-1} \mathbb{r} \cdot, \quad \mathbb{r} = y^a y^b dx^\mu r_{\mu ab}(x, y, p) \]  

(A.3)

and for any \( f(x, y, p) \) satisfies

\[ D D f(x, y, p) = 0. \]  

(A.4)

Under the extra condition \( \mathbb{e}_a^\mu \frac{\partial}{\partial x^\mu} \mathbb{r} = 0 \) this connection is unique and is such that \( \mathbb{r} \) is linear in \( p_a \).

**Proof.** The proof is based on using suitable degree of homogeneity in \( y \) plus and \( \hbar \) and acyclicity of the differential \( \delta = dx^\mu \frac{\partial}{\partial x^\mu} \) in nonzero form-degree. \( \square \)

Note that by construction \( D \) differentiates \( \cdot \)-product. If \( \mathbb{r} \) is linear in \( p_a \) it also satisfies the Poisson bracket version of the flatness condition (i.e. coincides with its classical limit). In what follows we assume that \( D \) is minimal (satisfies \( \mathbb{e}_a^\mu \frac{\partial}{\partial x^\mu} \mathbb{r} = 0 \)). Terms of degree 4 and less read explicitly as

\[ \mathbb{r} = dx^\mu \left[ \frac{1}{3} R_{\mu cd}^a p_d y^a y^b + (\ldots) \nabla_d R_{\mu cd}^a p_d y^b y^d + (\ldots) \nabla_d R_{\mu cd}^a y^b y^d y^e + \ldots \right] + dx^\mu (R_{\mu cd}^a R_{\mu cd}^b y^b y^d y^e + \ldots) + \ldots \]  

(A.5)

Here the first line contain terms linear in curvature and its covariant derivatives.

**Proposition A.2.** For any \( f(x, p) \) there exist a unique \( \tilde{f}(x, y, p) \) such that

\[ D \tilde{f} = 0, \quad \tilde{f}_{y=0} = f. \]  

(A.6)

Moreover, if \( D \) is a unique connection such that \( \mathbb{e}_a^\mu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial \chi^a} \mathbb{r} = 0 \) then for \( f(x, -p) = \pm f(x, p) \) the associated \( \tilde{f} \) also satisfies \( \tilde{f}(x, -p) = \pm \tilde{f}(x, p) \). More precisely, for \( f \) of homogeneity \( s \in p_a \), \( \tilde{f} \) contains terms of homogeneity \( s, s - 2, s - 4, \ldots \)

**Proof.** \( \tilde{f} \) is constructed iteratively in the degree of homogeneity in \( y \) and \( \hbar \). For \( D \) special \( \mathbb{r} \) is linear in \( p_a \) so that the star commutator may only reduce the homogeneity in \( p_a \) by an even number. \( \square \)

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24 We shall denote by (\ldots\) some numerical coefficients precise values of which is not relevant for our discussion.

25 Note that expansion in homogeneity in curvatures \( \mathbb{r} = \sum_{n=1}^\infty \mathbb{r}_n \) is well defined and the flatness condition decomposes as

\[ \nabla_{\mathbb{r}_n} - \delta_{\mathbb{r}_n} + \frac{1}{2} \sum_{\mu, k, l, k, i, k, 0} [\mathbb{r}_n, \mathbb{r}_n] = 0 \]
It follows that the space of all functions in $x$, $p$ is isomorphic to covariantly constant functions depending in addition on $y$-variables. Below and in the main text we need the following example: if $\eta = \frac{1}{2} \eta^{ab} p_a p_b$, then

$$\tilde{\eta} = \frac{1}{2} \eta^{ab} p_a p_b + \frac{1}{6} R_{bca}^{a} p^b y^c y^d + \text{(terms of degree } > 2)$$  \tag{A.7}

This is related to the expansion in normal coordinates if one identifies $y^\mu$ as normal coordinates around $\mu$. Note that, in general, terms independent of momenta may appear but they are of order $h^2$. For a general element $f = f^{ab}(x) p_a p_b$ quadratic in $p_a$ one has

$$\tilde{f} = f^{ab}(x) p_a p_b + y^a \nabla_a f^{bc} p_b p_c + \frac{1}{2} y^a y^b \nabla_a \nabla_b f^{cd} p_c p_d + \ldots R_{bca}^{a} f^{bc} y^c y^d p_b p_c$$

$$+ \frac{2}{3} h^2 R_{bca}^{a} \nabla_a f^{cd} y^b + \ldots$$ \tag{A.8}

where dots denote terms of total degree higher than 2. For a linear one

$$\tilde{f} = f^{a}(x) p_a + y^b \nabla_b f^{ab} p_a + \frac{1}{2} y^a y^b \nabla_a \nabla_b f^{ab} p_c + \ldots R_{bca}^{a} f^{bc} y^c y^d p_b + \text{(deg } \geq 3 \text{ terms})$$ \tag{A.9}

The above construction gives the covariant product on contangent bundle: using the above propositions we may define

$$f * g := (\tilde{f} \circ \tilde{g})\big|_{y=0}.$$ \tag{A.10}

The consistency of this definition follows from the fact that for any $\tilde{f}, \tilde{g}$ satisfying $D\tilde{f} = D\tilde{g} = 0$ one has $D(\tilde{f} \circ \tilde{g}) = 0$. The above construction of the star product is a version of that of [27] which in turn has its origin in the Fedosov quantization [29].

As an example let us compute explicitly the transformation of the spin 1 under the transformation generated by $\epsilon^{ab} p_a p_b$:

$$\delta(h^a p_a) = \left[ -\frac{1}{2} \eta^{ab} p_a p_b, \epsilon^{cd} R_{bca}^{a} \right]_{\frac{h}{2}} = -\frac{4}{3} R_{bca}^{a} \nabla_a \epsilon^{cd} p_b.$$ \tag{A.11}

Next, let us describe the representation space in a covariant way. Let $\rho$ denote a map that sends Weyl symbol $f(y, p)$ into the respective operator in coordinate representation (i.e. on functions of $y$). For instance, $\rho(y p_b) = -\frac{1}{2} h (\frac{\partial}{\partial y^a} + \frac{\partial}{\partial \phi} \phi^a) y_b$.

**Proposition A.3.** For any wave function $\phi(x)$ there exist a unique lift $\tilde{\phi}(x, y)$ satisfying $h [\nabla + \rho(\epsilon^{ab} p_a + R)] \tilde{\phi} = 0$ and $\tilde{\phi}|_{y=0} = \phi$.

To illustrate this, let us explicitly evaluate the lift up to terms of degree 3:

$$\tilde{\phi} = \phi + y^a \nabla_a \phi + \frac{1}{2} y^a y^b \nabla_a \nabla_b \phi + \ldots$$ \tag{A.12}

The action of the operator $\tilde{f} (x, \frac{\partial}{\partial y})$ with symbol $f(x, p)$ on the wave function $\phi(x)$ is defined by

$$\tilde{f} \phi = (\rho(\tilde{f} )\Phi)|_{y=0}.$$ \tag{A.13}

Note that by construction 1 acts as an identity operator and $(\tilde{f} * \tilde{g}) = \tilde{f} \tilde{g}$ (because $\rho$ is a representation map). This way we have constructed a covariant symbol map that sends functions.
of $x, p$ to differential operators on $x$. Note that the map is solely expressed in terms of covariant
derivatives, frame field, and curvature (along with its covariant derivatives). This shows that
although the map is written in terms of local coordinates and local frame it does not depend
on the choice of coordinates and the frame.

The above technique allows to reformulate the relations (2.3 and (2.7) in manifestly coor-
dinate-independent terms. In so doing the component fields entering $H(x, p)$ transform as ten-
sors under a change of coordinates. By a suitable field redefinition one can also achieve that
they transform homogeneously under the linearized gauge transformations (see the end of this
appendix for spin 2 case).

Let us now discuss the inner product. The minimal choice is
\[
\langle \phi, \chi \rangle = \int d^4x \sqrt{g} \; \phi^*(x) \chi(x)
\]  
(A.14)
The question is how to identify (anti)hermitian operators at the level of symbols.

**Proposition A.4.** Real (imaginary) symbols correspond to hermitian (antihermitian)
operators.

**Proof.** First of all we show that for $f(x, p)$ real (imaginary) the respective lift $\tilde{f}(x, p, y)$ is also
real (imaginary). Let us for definiteness consider real $f$. It is enough to assume all coefficients to
be real so that $f(x, p)$ contains only even powers of $p$. By inspecting the recursive construction
of $\tilde{f}$ we see that odd powers of $p$ can not appear as well as imaginary coefficients (we assume
that the metric, frame field and connection are real). Finally, because $\tilde{f}$ is real $\rho(\tilde{f})$ is formally
hermitian when represented on wave functions of $y$ where the conjugation rules are
\[
(\tilde{\phi})^\dagger = \tilde{\phi}^* \quad \text{and} \quad (\tilde{\chi})^\dagger = -\frac{\partial}{\partial y^a}
\]
in this case dependence on $x^a$ is irrelevant and as before $x^{a\dagger} = x^a$.

It is enough to check this statement for operators whose symbols are of zeroth and first
order in $p$. Indeed, such operators generate the entire algebra. For $f = f(x)$ the statement is
obvious. For $f = \phi^a(x) p_a$ we have (this is just a rewriting of (A.9)
\[
\tilde{f}(x, p, y) = \phi^a(x, y) p_a, \quad \phi^a = \phi^a + y^b \nabla_b \phi^a + O(y^2)
\]  
(A.15)
Because $\rho(f)$ is formally antihermitean on wave functions of $y$ we have
\[
\int d^4x \sqrt{g} \; \phi^* \tilde{f} \chi = \int d^4x \sqrt{g} \; \left( \phi^* \rho(\tilde{f}) \chi \right)_{y=0}
\]
\[
= \frac{1}{2} \int d^4x \sqrt{g} \left( \phi^* \left( \frac{\partial}{\partial y^a} \phi_a + \frac{\partial}{\partial y^a} \phi^a \right) \chi \right)_{y=0}
\]
\[
= -\int d^4x \sqrt{g} \left( \rho(\phi^a) \phi^a \chi \right)_{y=0} + \int d^4x \sqrt{g} \left( \frac{\partial}{\partial y^a} (\phi^* \phi^a) \chi \right)_{y=0}.
\]  
(A.16)
Using that $(\frac{\partial}{\partial y^a} X)_{y=0} = \nabla_a Y$ for some $Y$, where $X$ is $\phi^a$ or $\phi^a$ or $\phi^a$, the integrand of the last term can be rewritten as
\[
\sqrt{g} \nabla_a X^a = \frac{\partial}{\partial y^a} (\phi^a) \chi
\]
and hence the integral vanishes under the standard assumptions.

To summarize, we have constructed a covariant (independent of the choice of local coor-
dinates) description of quantum mechanics on the cotangent bundle. We thus have all the
required ingredients: representation space, inner product, operators, symbols and symbol-map.
Appendix B. Weyl invariants

Let us briefly recall some known results on the structure of the conformal and diffeomorphism invariants. More precisely, we are interested in (tensor valued) local functions of the metric and its derivatives (see (4.2)) that transform covariantly under the diffeomorphisms and Weyl transformations. It turns out that a candidate invariant is a polynomial in

\[ g_{\mu\nu} C_{\mu\nu\rho\sigma} D_{\alpha\beta} C_{\mu\nu\rho\sigma}, \ldots \]  

(B.1)

with indices properly contracted by \( g_{\mu\nu} \). Here \( C_{\mu\nu\rho\sigma} \) is the Weyl tensor and \( D_{\alpha\beta} \) denotes a Weyl-covariant derivative related to the so-called Thomas D-derivative. In general, such polynomial is not invariant under Weyl transformations as in contrast to \( g_{\mu\nu} \) and \( C_{\mu\nu\rho\sigma} \) the transformation law of \( \ldots D_{\alpha\beta} C_{\mu\nu\rho\sigma} \) may involve a gradient of the Weyl parameter \( \omega_0 \). Hence the invariance condition imposes extra constraints on the structure of the polynomial. For more details we refer to [28] and references there.

Let us analyze the necessary condition for a rank \( s \) tensor-valued local function to be diffeomorphism covariant and Weyl covariant with weight \( w \). Taking into account that \( g_{\mu\nu} \) has Weyl weight \(-2\) while \( g_{\mu\nu} \) and \( C_{\mu\nu\rho\sigma} \) have weight \( 2 \), we get

\[ 2n_g + 4n_C + n_D - 2n^s = s, \quad -2n_g - 2n_C + 2n^s = -w, \]  

(B.2)

where \( n_g, n_C, n_D \) and \( n^s \) denote, respectively, the numbers of \( g_{\mu\nu}, C, D \) and \( g^{\mu\nu} \) factors in a polynomial. The first equation counts indices while the second counts the Weyl weight. As a consequence, we have

\[ 2n_C + n_D = -w + s. \]  

(B.3)

Consider, for example, a scalar invariant which is an integrand of \( \int \mathcal{S}_0[g] = \int d^4x \sqrt{g} L_0 \). One finds that \( L_0 \) is Weyl invariant of weight \(-4\) for which (B.3) has two solutions \( n_C = 1, n_D = 2 \) and \( n_C = 2, n_D = 0 \). The first one gives zero (as \( C \) is traceless) so one ends up with \( L_0 = C^2 \), i.e. the well-known Weyl gravity Lagrangian.

Next, let us consider a rank-one tensor \( K_{\mu} \) of weight \( w = -2 \) appearing in (4.2). Then we have only one nontrivial solution: \( n_C = 1, n_D = 1 \). It should again vanish as here at least two indices of the Weyl tensor should be contracted with the metric.

For a polynomial \( g_{\mu\nu} C_{\mu\nu\rho\sigma} \) with \( s = 2 \), \( w = -2 \) we have two solutions: \( n_C = 2, n_D = 0 \) and \( n_C = 1, n_D = 2 \). The latter one necessarily contains two derivatives contracted with the indices of \( C \) and hence should vanish on a Bach-flat background. The former can be brought to the following form

\[ k g_{\mu\nu} C_{\alpha\beta} C^{\alpha\beta} + k_2 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}. \]  

(B.4)

Imposing the tracelessness \( (k_1 = -1/4k_2) \) and covariant conservation conditions (4.8) this can be shown to vanish on a Ricci-flat background using \( \nabla_{[\alpha} C_{\beta\gamma]} = 0 \); Weyl-covariance implies that same should be true on a Bach-flat background.

The analysis for \( s > 2 \) becomes rather involved. Considerable simplification can be achieved by employing the spinor formalism in 4d. In this approach the self-dual (anti-self-dual) component of \( C \) is represented by the totally symmetric spinor \( \Psi_{ABCD} (\Psi_{A'B'C'D'} \) where \( A = 1, 2 \) \( \langle A' = 1, 2 \rangle \). The invariant contractions of indices are performed with the help of the antisymmetric tensor \( \epsilon^{AB} \) or \( \epsilon^{A'B'} \). In particular, the Minkowski metric \( \eta_{\mu\nu} \) in spinorial notations reads as \( \eta_{\mu'A'B'} = 2\epsilon_{A'B'} \epsilon^{A'B'} \) (for a concise exposition see, e.g. [30] and refs. therein).

26 The first D-derivative of Weyl tensor is the same as the ordinary covariant derivative , i.e. \( D_{\alpha\beta} = \nabla_{\alpha} C_{\beta\rho\sigma} \).
For example, for \( s = 2 \), by writing the spinorial counterpart of (B.4) one finds that the second term necessarily vanishes so that the tracelessness condition implies that the first term vanishes as well.

For \( s = 3 \) we have \( K_{\mu
u\rho} \) which, according to (B.3), can not have terms of order higher than 2 in Weyl tensor and its Weyl-covariant derivatives. As the linear in \( C_{\mu
u\rho\sigma} \) term vanishes on Bach-flat background let us concentrate on the quadratic contribution which should involve only one covariant derivative. In the spinorial approach \( K_{\mu
u\rho\sigma} \) is described by \( K_{ABCD} \) to which only the following terms may contribute

\[
\Psi_{ABCD} \nabla_{EE} \Psi_{ABCD}, \quad \Psi_{ABCD} \nabla_{EE} \Psi_{ABCD},
\]

where the indices are contracted with the \( \epsilon \)-tensors. It is clear that there is only one inequivalent contraction that leaves 3 + 3 free indices. It results in the following general expression:

\[
n_1 \Psi_{ABCD} \nabla^{DI} \Psi_{ABCD} + n_2 \Psi_{ABCD} \nabla^{DI} \Psi_{ABCD}.
\]

The Weyl covariance of \( n_2 \epsilon_{AB} \) implies that \( n_2 = -n_1 \) in which case the above expression is proportional to the Eastwood–Dighton tensor \( E_{\mu
u\rho\sigma} \) in (4.10). It is known to vanish for the metric conformal to the Einstein one. Note that the Eastwood–Dighton tensor is by construction trace-free and its divergence is proportional to the Bach tensor and thus vanishes on a Bach-flat background.

Let us note that all \( K_i \) vanish in the special case of Bach-flat 4d backgrounds with self dual (or antselfdual) Weyl tensor. In this case \( \Psi_{ABCD} = 0 \) so that \( K_i \) is build out of \( \Psi_{ABCD} = 0, \epsilon_{AB}, \epsilon_{AB}, \nabla_{AB}. \) Moreover, \( \epsilon \) may only enter to contract indices because \( K_i \) should be totally symmetric so that primed indices may only originate from covariant derivatives so that (B.3) implies \( nC = 1 \) and hence \( K_i \) should be proportional to the Bach tensor.

Indeed, in this case \( K_3 \) and \( K_4 \) vanish just on the basis of the index structure.

Finally, let us list two useful relations in spinorial notations. The Bianchi identity for the Weyl tensor reads

\[
\nabla^A \Psi_{ABCD} = \nabla^A \Phi_{CDAB} - 2\epsilon_{B[C} \nabla_{D]A} \Lambda,
\]

where \( \Phi_{ABCD} \) is the trace-free Ricci spinor and \( \Lambda \) is a multiple of the scalar curvature. The Bach tensor is given by

\[
B_{AB} = 2(\nabla^C A \nabla^D B + \Phi^{CD} AB) \Psi_{ABCD} = 2(\nabla^C A \nabla^D B + \Phi^{CD} AB) \Psi_{ABCD}.
\]

\[
\text{References}
\]

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