SPACE OF INITIAL VALUES OF A MAP
WITH A QUARTIC INVARIANT

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Abstract

We compactify and regularise the space of initial values of a planar map with a quartic invariant and use this construction to prove its integrability in the sense of algebraic entropy. The system has certain unusual properties, including a sequence of points of indeterminacy in \( \mathbb{P}^1 \times \mathbb{P}^1 \). These indeterminacy points lie on a singular fibre of the mapping to a corresponding QRT system and provide the existence of a one-parameter family of special solutions.

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1. Introduction

We consider the birational maps
\[
\varphi: (x, y) \mapsto \left( \frac{x(3hy - 1)}{4h^2x^2 + 2h^2y^2 + hy - 1}, \frac{4hx^2 - hy^2 - y}{4h^2x^2 + 2h^2y^2 + hy - 1} \right),
\]
\[
\psi: (x, y) \mapsto \left( -\frac{x(3hy + 1)}{4h^2x^2 + 2h^2y^2 - hy - 1}, -\frac{4hx^2 - hy^2 + y}{4h^2x^2 + 2h^2y^2 - hy - 1} \right),
\]
with \( h \neq 0 \), whose actions preserve the rational function
\[
H(x, y) = \frac{x^2(x^2 - y^2)}{(1 - h^2y^2)(1 - 8h^2x^2 - h^2y^2)},
\]
in the sense that \( H(\varphi(x, y)) = H(x, y) \) and \( H(\psi(x, y)) = H(x, y) \). The maps \( \varphi \) and \( \psi \) are inverses in \( \mathbb{P}^1 \times \mathbb{P}^1 \), that is, \( \varphi \circ \psi = \psi \circ \varphi = \text{Id} \). Their iteration leads to a dynamical system, whose geometric properties are considered in this paper. For simplicity, we sometimes refer to one of them, \( \varphi \), as the map and describe \( H \) as an invariant of the map.

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The maps (1.1) preserve the following volume form (found by Celledoni et al. [4]):

\[ \omega(x, y) = \frac{dx \wedge dy}{x(x^2 - y^2)} \implies \omega(\varphi(x, y)) = \omega(\psi(x, y)) = \omega(x, y). \] (1.3)

Due to this measure-preserving property, the possession of the invariant \( H \) suggests that \( \varphi \) is integrable [8]. However, almost all the known integrable maps in the plane have biquadratic invariants, while the invariant \( H(x, y) \) is evidently quartic.

The maps \( \varphi \) and \( \psi \) are clearly undetermined at certain points; for example, \( \varphi \) becomes \( 0/0 \) as \( (x, y) \to (0, -1/h) \). We blow up \( \mathbb{P}^1 \times \mathbb{P}^1 \) at such points and consider the lifted maps on the resulting surface, called the initial value space. We show that the initial value space of the maps (1.1) possesses geometric properties that are not usually seen in other known integrable systems. Another motivation of our study was the reconciliation of the unusual characteristics of this example with the observation that this map is transformable (via a birational mapping) to a Quispel–Roberts–Thompson (QRT) map, which has a biquadratic invariant. (This transformation was found by Van der Kamp et al. [22].)

To state our results, we first recall that the degree \( \deg(\varphi) \) of a rational map \( \varphi : (x, y) \mapsto P(x, y)/Q(x, y) \) is given by \( \max(\deg(P), \deg(Q)) \). Here, the degree of a polynomial of several variables is defined to be the sum of individual degrees in each variable, that is, \( \deg(P) = \deg_x(P) + \deg_y(P) \). A crucial concept in the study of the dynamics of the map is the degree of the \( n \)th iterate, \( d_n = \deg(\varphi^{(n)}) \), as \( n \to \infty \). The algebraic entropy of the map is defined by \( \lim_{n \to \infty} \frac{\log(d_n)}{n} \).

The main new result of this paper is an algebro-geometric proof that the maps (1.1) possess quadratic degree growth. Consequently, the algebraic entropy vanishes. The proof is carried out by constructing the space of initial values, on which the maps are not automorphisms but become analytically stable (see Definition 3,1). The surface contains an infinite sequence of indeterminacy points in \( \mathbb{P}^1 \times \mathbb{P}^1 \), which are spurious [5] as their blow-up is not necessary to compute the growth of the map, but are connected to the existence of a special one-parameter family of solutions. A geometric proof of linear degree growth was carried out for linearisable maps by Takenawa et al. [20], while an infinite sequence of (spurious) blow-ups was shown to exist for linearisable maps by Hay et al. [7]. To the best of our knowledge, the maps (1.1) form the first nonlinearisable example where an infinite number of blow-ups appears to be needed.

1.1. Main result. We refer the reader to terminology defined in Section 3. The main result of this paper is summarised in the following theorem.

Theorem 1.1.

(a) There exists an algebraic surface \( S \), obtained by a composition of blow-ups \( \pi : S \to \mathbb{P}^1 \times \mathbb{P}^1 \), on which the maps (1.1) are lifted to analytically stable maps \( \varphi^*, \psi^* : S \to S \).

(b) There exists an infinite sequence of points \( \{\theta_k\}_{k \in \mathbb{Z}, 0} \) generated by blowing up two indeterminacy points \( \theta_{\pm 1} \), corresponding to the ill-posed initial values of a one-parameter family of solutions with initial conditions \( (x_0, y_0) = (0, a) \).
(c) The degree of the Nth iterate of the maps (1.1) is given by
\[
d_N = \frac{2}{3} N^2 - \frac{2}{9} \cos \left( \frac{2\pi N}{3} \right) + \frac{11}{9}.
\]
(1.4)

(d) The invariant \( H \) in (1.2) corresponds to the lowest-degree nontrivial element of
the eigenspace relative to the eigenvalue 1 of either \( \varphi^* \) or \( \psi^* \).

The proof of this theorem is presented in Section 3. The peculiarity of such an
example is that the two lifted maps \( \varphi^* \) and \( \psi^* \) are not automorphisms. This occurred in
other examples of maps with linear growth [7, 20] but, to the best of our knowledge,
this is the first time that this occurrence appears in a map with quadratic growth.
We mention that the notion of analytical stability was used to compute the growth
of integrable maps in four dimensions in [2].

1.2. Background. The geometric theory of discrete integrable systems has a long
history (see [5] for a summary). Its relationship with elliptic curves is the foundation
of the study of QRT maps [16, 17, 21]. It also underlies the geometric formulation
of discrete Painlevé equations [18], which are expressed through Cremona transforma-
tions of a regular algebraic surface called the space of initial values.

In recent years a procedure called Kahan–Hirota–Kimura (KHK) discretisation has
become a popular way of producing integrable discrete equations from systems of
integrable ordinary differential equations. This procedure was presented first by Kahan
in a series of unpublished lecture notes [11] as a method to obtain better numerical
approximations. The interest of the integrable systems community in this procedure
arose after it was proved to produce an integrable discretisation of the Lagrange top
[13] (see [14]).

From the results of [3], it follows that the KHK discretisation of a two-dimensional
system with a cubic Hamiltonian is always integrable. This observation led
to the consideration of a nonstandard quadratic system in [4], whose KHK
discretisation possesses a higher-degree polynomial invariant. In this paper, we
consider the KHK discretisation of one of the two systems discovered independently
in [4, 15].

The maps (1.1) form a discretisation of the octahedral reduced Nahm system [9, 14].
We note that a broader class of maps containing (1.1) was presented in [22], where
it was shown that the invariant (1.2) can be reduced to an invariant of QRT type,
dropping its total degree by four through a fractional linear transformation. However,
the corresponding pencil factors to give a QRT pencil times a multiple singular fibre
given by \( x^4 = 0 \).

1.3. Outline of the paper. The plan of the paper is as follows. The transformation
of the maps (1.1) to a QRT system is given in Section 2. Section 3 is devoted to the
proof of Theorem 1.1. Finally, in Section 4, we summarise our results and discuss open
questions.
2. The relation to a QRT system

In this section we recall the relation between the maps (1.1) and QRT systems [16, 17] provided by [22].

The proof in [22] can be summarised as follows. Consider the change of variables

\[ x = \frac{2}{u + v}, \quad y = -\frac{u - v}{h(u + v)}. \]  \hspace{1cm} (2.1)

Then the invariant (1.2) is mapped to

\[ K(h) = \frac{1}{4h^2} \frac{(u - v - 2h)(u - v + 2h)}{uv(8h^2 - uv)} , \]  \hspace{1cm} (2.2)

which is the ratio of two symmetric biquadratic polynomials, hence an invariant of QRT type. The fact that the quartic invariant (1.2) reduces to the biquadratic invariant (2.2) through the fractional linear transformation (2.1) means that some factorisation is happening. We will show below that this phenomenon appears clearly if instead of the rational invariants we consider the associated covariant pencils of curves.

Under transformation (2.1), the maps (1.1) become

\[ \tilde{\phi} : (u, v) \mapsto \left(v, -\frac{8h^2 - uv}{2u - v}\right), \quad \tilde{\psi} : (u, v) \mapsto \left(\frac{8h^2 - uv}{u - 2v}, u\right). \]  \hspace{1cm} (2.3)

The maps (2.3) are symmetric QRT maps and define a second-order recurrence relation.

These results were presented in [22]. Now we consider the reduction from a quartic to a biquadratic invariant. Consider the pencil of curves associated to (1.2):

\[ p(x, y) = x^2(x^2 - y^2) + \varepsilon_0(1 - h^2y^2)(1 - 8h^2x^2 - h^2y^2). \]  \hspace{1cm} (2.4)

Here \( \varepsilon_0 \in \mathbb{P}^1 \) is the parameter of the pencil. Consider now the pencil associated to the invariant (2.2):

\[ q(u, v) = (u - v - 2h)(u - v + 2h) + 4\varepsilon_0 h^2 uv(8h^2 - uv). \]  \hspace{1cm} (2.5)

Applying the coordinate transformation (2.1), we obtain

\[ p(u, v) = -\frac{4q(u, v)}{h^2(u + v)^4} = -\frac{x^4}{4h^2 q(u, v)}. \]  \hspace{1cm} (2.6)

Therefore, the two pencils are equivalent except on the one-dimensional singular multiple fibre \( x = 0 \). This is what causes the bi-degree to drop from (4, 4) for \( p \) to (2, 2) for \( q \).

3. Space of initial values of maps (1.1)

In this section we prove Theorem 1.1. We start by constructing a rational surface \( S \), which is obtained by blowing up a sequence of points of indeterminacy \( p_i \), through monoidal transformations \( \pi_i : S_i \rightarrow S_{i-1} \) with centre \( p_i \), where \( S_0 = \mathbb{P}^1 \times \mathbb{P}^1 \). The set \( S_i \) is called the strict transform of \( S_{i-1} \) and the map \( \varphi \) is said to be lifted to \( S_i \). Each \( \pi_i \)
induces an isomorphism of $S_i - \pi_i^{-1}(p_i)$ onto $S_{i-1} - p_i$ [6, Section V.3]. Here the set $\pi_i^{-1}(p_i)$ is isomorphic to $\mathbb{P}$ and is called an exceptional line. The composition of all monoidal transformations $\pi_i$ will be denoted by $\pi: S \to \mathbb{P}^1 \times \mathbb{P}^1$.

The following definition is equivalent to the standard definition of analytical stability given in [5, 20]. (The equivalency is stated as a remark in [20, Section 3.3].)

**Definition 3.1** [20]. Suppose we are given a map $\varphi: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, lifted to an initial value space $S$, by a sequence of blow-ups $\pi: S \to \mathbb{P}^1 \times \mathbb{P}^1$. If the lifted map $\varphi^*: S \to S$ does not give rise to any iteration sequence of the form

$$
\mathcal{E} \xrightarrow{\varphi} p_1 \xrightarrow{\varphi} p_2 \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} p_K \xrightarrow{\varphi} \mathcal{E}',
$$

where $\mathcal{E}$, $\mathcal{E}'$ are one-dimensional sub-varieties and $p_1, p_2, \ldots, p_K$ is a finite set of points, then the map $\varphi^*$ is called **analytically stable**.

Heuristically an analytically stable map is a map which does not possess any periodic singularity pattern.

Let the affine charts of $\mathbb{P}^1 \times \mathbb{P}^1$ be given by $(x, y)$, $(1/x, y)$, $(x, 1/y)$, and $(1/x, 1/y)$, where the lines at infinity correspond to $1/x = 0$ and $1/y = 0$, respectively. Suppose $(x_0, y_0)$ is a point of indeterminacy of $\varphi$. Its blow-up is given by

$$
(x, y) \leftarrow \left( x - x_0, \frac{y - y_0}{x - x_0} \right) \bigcup \left( x - x_0, \frac{y - y_0}{y - y_0} \right).
$$

(3.2)

The map $\varphi$ is indeterminate at the points $(0, -1/h)$, $(0, 1/2h)$, $(1/3h, 1/3h)$, $(-1/3h, 1/3h)$ and $(\infty, \infty)$. We obtain the following sequence of blow-ups. (For simplicity, we state only one coordinate chart at each step of (3.2) and indicate the centre and exceptional line of each blow-up above and below the arrow corresponding to the monoidal transformation.)

$$
(x, y) \xrightarrow{(0,-1/h)}_{E_1} (x, \frac{y + 1/h}{x}) \xrightarrow{(0,0)}_{E_2} (x, \frac{y + 1/h}{x^2});
$$

(3.3a)

$$
(x, y) \xrightarrow{(1/3h,-1/3h)}_{E_3} \left( x - \frac{1}{3h}, \frac{y + 1/3h}{x - 1/3h} \right);
$$

(3.3b)

$$
(x, y) \xrightarrow{(-1/3h,-1/3h)}_{E_4} \left( x + \frac{1}{3h}, \frac{y + 1/3h}{x + 1/3h} \right);
$$

(3.3c)

$$
(x, y) \xrightarrow{(\infty,\infty)}_{E_5} \left( \frac{1}{x}, \frac{1}{y} \right).
$$

(3.3d)

We will consider the point $(0, 1/2h)$ separately later. The images of the exceptional lines $E_4$ and $E_{13}$ under $\varphi$ are given by

$$
\varphi^*(E_3) = \left( \frac{1}{h}, -\frac{1}{h} \right), \quad \varphi^*(E_4) = \left( -\frac{1}{h}, \frac{1}{h} \right),
$$

(3.4)

indicating additional points of indeterminacy.
This implies that we need to blow up the corresponding image points:

\[
(x, y) \leftarrow_{E_6} \left( x, \frac{y - 1/h}{x - 1/h} \right) \quad (3.5a)
\]

\[
(x, y) \leftarrow_{E_7} \left( x, \frac{y + 1/h}{x + 1/h} \right) \quad (3.5b)
\]

The images of the exceptional lines \(E_6\) and \(E_7\) under \(\varphi\) are

\[
\varphi^*(E_6) = \left( -\frac{1}{h}, \frac{1}{h} \right), \quad \varphi^*(E_7) = \left( 1, \frac{1}{h} \right) \quad (3.6)
\]

Again, we blow up the corresponding image points:

\[
(x, y) \leftarrow_{E_8} \left( x, \frac{y - 1/h}{x - 1/h} \right) \quad (3.7a)
\]

\[
(x, y) \leftarrow_{E_9} \left( x, \frac{y + 1/h}{x + 1/h} \right) \quad (3.7b)
\]

The images of the exceptional lines \(E_8\) and \(E_9\) are now one-dimensional curves. This implies that the lifted map \(\varphi^*\) obtained by blowing up all the above points satisfies Definition 3.1 and is analytically stable.

Now we consider the map \(\psi\), which is indeterminate at the points \((0, 1/h), (0, -1/2h), (1/3h, -1/3h)\) and \((-1/3h, -1/3h)\). The two maps share the indeterminate point at \((\infty, \infty)\), but it is automatically resolved under the transformation (3.3d). As above, we obtain the following sequence of blow-ups:

\[
(x, y) \leftarrow_{E_{10}} \left( x, \frac{y - 1/h}{x} \right) \left( x, \frac{y - 1/h}{x^2} \right) \quad (3.8a)
\]

\[
(x, y) \leftarrow_{E_{12}} \left( x, \frac{y - 1/3h}{x - 1/3h} \right) \quad (3.8b)
\]

\[
(x, y) \leftarrow_{E_{12}} \left( x, \frac{y + 1/3h}{x + 1/3h} \right) \quad (3.8c)
\]

We will consider the point \((0, -1/2h)\) later. The images of the resulting exceptional lines are one-dimensional, showing that the lifted map \(\psi^*\) after blowing up all these points is analytically stable (see Definition 3.1).

**Remark 3.2.** Consider the set of indeterminacy of a map \(\varphi\) defined by

\[
\mathcal{I}(\varphi) = \{ p \in \mathbb{P}^1 \times \mathbb{P}^1 : \varphi \text{ is indeterminate at } p \}.
\]

A space \(S\) is called a *space of initial values* when the indeterminacy set of the lift of \(\varphi\) to \(S\) is empty [18]. We denote the space \(\mathbb{P}^1 \times \mathbb{P}^1\) blown up at points given in Equations (3.3), (3.5), (3.7), (3.8) by \(S\) and use it as a basis for our arguments below. We will see below that \(S\) is not strictly a space of initial values. However, for our purposes, it is sufficient to consider the maps \(\varphi, \psi\) on \(S\) where they are analytically stable.
The only remaining points to consider are \( \theta_{\pm 1} := (0, \pm 1/2h) \). Blowing up \( \theta_{\pm 1} \) gives

\[
(x, y) \xleftarrow{\theta_{\pm 1}} \Theta_{\pm 1} \left( x, \frac{y + 1/2h}{x} \right),
\]

where \( \Theta_{\pm 1} \) are the corresponding exceptional lines. The images of these exceptional lines are

\[
\psi^*(\Theta_1) = \left( 0, \frac{1}{4h} \right) =: \theta_2, \quad \varphi^*(\Theta_{-1}) = \left( 0, -\frac{1}{4h} \right) =: \theta_{-2}.
\]

(3.10)

Blowing up these two points \( \theta_{\pm 2} \) leads to exceptional lines whose images are points again. We can show inductively that iterating this process leads to an infinite sequence of points \( \theta_k := (0, 1/(2kh)), k \in \mathbb{Z} \setminus \{0\} \), with

\[
(x, y) \xleftarrow{\theta_k} \Theta_k \left( x, \frac{y - 1/2hk}{x} \right),
\]

(3.11)

for which the following relations hold:

\[
\varphi^*(\Theta_k) = \theta_{k-1}, \quad \psi^*(\Theta_k) = \theta_{k+1}, \quad k \in \mathbb{Z} \setminus \{\pm 1, 0\}.
\]

(3.12)

This implies the existence of infinite sequences of images under \( \varphi^* \) and \( \psi^* \):

\[
\Theta_{-1} \xrightarrow{\varphi^*} \theta_{-2} \xrightarrow{\varphi^*} \theta_{-3} \xrightarrow{\varphi^*} \theta_{-4} \xrightarrow{\varphi^*} \theta_{-5} \xrightarrow{\varphi^*} \cdots,
\]

(3.13a)

\[
\Theta_1 \xrightarrow{\psi^*} \theta_2 \xrightarrow{\psi^*} \theta_3 \xrightarrow{\psi^*} \theta_4 \xrightarrow{\psi^*} \theta_5 \xrightarrow{\psi^*} \cdots,
\]

(3.13b)

which never terminate on a one-dimensional sub-variety. This violates the requirement of analytical stability.

In conclusion, the maps \( \varphi^* \) and \( \psi^* \) are analytically stable on \( S \) defined in Remark 3.2. This proves statement (a) of Theorem 1.1. A schematic representation of the surface \( S \) obtained after performing all the blow-ups is given in Figure 3.1.

We now prove claim (b) of Theorem 1.1. Consider the orbit of the one-parameter family of initial points \( (x_0, y_0) = (0, a) \):

\[
\varphi(0, a) = \left( 0, \frac{a}{1 - 2ah} \right), \quad \varphi^2(0, a) = \left( 0, \frac{a}{1 - 4ah} \right),
\]

(3.14a)

\[
\varphi^3(0, a) = \left( 0, \frac{a}{1 - 6ah} \right), \quad \varphi^4(0, a) = \left( 0, \frac{a}{1 - 8ah} \right), \quad \cdots,
\]

\[
\psi(0, a) = \left( 0, \frac{a}{1 + 2ah} \right), \quad \psi^2(0, a) = \left( 0, \frac{a}{1 + 4ah} \right),
\]

(3.14b)

\[
\psi^3(0, a) = \left( 0, \frac{a}{1 + 6ah} \right), \quad \psi^4(0, a) = \left( 0, \frac{a}{1 + 8ah} \right). \quad \cdots
\]

By induction, it follows that the orbit of \( (x_0, y_0) = (0, a) \) is given by

\[
\left\{\left( 0, \frac{a}{1 - 2akh} \right)\right\}_{k \in \mathbb{Z}}.
\]

(3.15)
However, the sequence (3.15) is ill-defined on the points $(0, 1/2kh), k \in \mathbb{Z} \setminus \{0\}$. Hitting one of this points will result in the orbit falling into one of the orbits given in equation (3.13). This proves (b).

Now consider statement (c) of Theorem 1.1. The Picard lattice of the algebraic surface $\pi: S \to \mathbb{P}^1 \times \mathbb{P}^1$ is given by the 15-dimensional $\mathbb{Z}$-module

$$\text{Pic}(S) = \mathbb{Z}H_x + \mathbb{Z}H_y + \sum_{i=1}^{13} \mathbb{Z}E_i.$$  \hspace{1cm} (3.16)

The maps $\varphi^*$ and $\psi^*$ have the following linear actions on $\text{Pic}(S)$:

$$\varphi^*: \begin{pmatrix} H_x \\ H_y \\ E_1, \\ E_2, E_3, E_4, \\ E_5, \\ E_6, E_7, E_8, E_9, \\ E_{10}, E_{11}, \\ E_{12}, \\ E_{13} \end{pmatrix} \mapsto \begin{pmatrix} 2H_x + 2H_y - E_3 - E_4 - 2E_5 - E_{10} \\ 2H_x + 2H_y - E_3 - E_4 - 2E_5 - E_{10} - E_{11} \\ H_x + H_y - E_3 - E_4 - E_5 \\ H_x - E_5, E_6, E_7, \\ 2H_x + 2H_y - E_3 - E_4 - 2E_5 - E_{10} - E_{11} \\ E_9, E_{10}, E_{12}, E_{13} \\ E_{11} \\ E_{12} \\ E_{13} \end{pmatrix}, \hspace{1cm} (3.17a)$$

$$\psi^*: \begin{pmatrix} H_x \\ H_y \\ E_1, \\ E_2, E_3, E_4, \\ E_5, \\ E_6, E_7, E_8, E_9, \\ E_{10}, E_{11}, \\ E_{12}, \\ E_{13} \end{pmatrix} \mapsto \begin{pmatrix} H_x + H_y - E_3 - E_5 - E_{10} \\ H_x + H_y - E_4 - E_{10} \end{pmatrix}.$$  \hspace{1cm} (3.17b)
where, for each integer

\[\psi^*:\begin{pmatrix} H_x \\ H_y \\ E_1, E_2, \\ E_3, \\ E_4, \\ E_5, \\ E_6, E_7, E_8, E_9, \\ E_{10}, \\ E_{11}, E_{12}, E_{13}\end{pmatrix} \rightarrow \begin{pmatrix} 2H_x + 2H_y + E_1 + 2E_5 - E_{12} - E_{13} \\ 2H_x + 2H_y + E_1 - E_2 + 2E_5 - E_{12} - E_{13} \\ E_{10}, E_{11} \\ H_x + H_y + E_1 - E_5 - E_{12} \\ H_x + H_y - E_1 - E_5 - E_{13} \\ H_x + H_y + E_2 - 2E_5 - E_{12} - E_{13} \\ E_3, E_4, E_7, E_6 \\ H_x + H_y - E_5 - E_{12} - E_{13} \\ H_x - E_5, E_8, E_9 \end{pmatrix}. \] (3.17b)

Iterating the map \(\psi^*\), we obtain

\[(\psi^*)^N(H_x) = \alpha_N H_x + \alpha_N H_y + \cdots, \quad (\psi^*)^N(H_y) = (\alpha_N - \beta_N) H_x + \alpha_N H_y + \cdots, \] (3.18)
where, for each integer \(N\),

\[\alpha_N = \frac{2}{3} N^2 + \frac{11}{9} - \frac{2}{9} \cos \left(\frac{2N\pi}{3}\right), \quad \beta_N = \frac{1}{3} + \frac{2}{3} \cos \left(\frac{2N\pi}{3}\right). \] (3.19)

Takenawa [19, Section 6] showed that the coefficients of \(H_x\) and \(H_y\) arising in the iterated image of a map provide a geometric estimate of the degree of the map. In particular, the degree \(d_N\) of the \(N\)th iterate of \(\psi\) is realized by a curve in \(\mathbb{P}^1 \times \mathbb{P}^1\). This proves formula (1.4). The same considerations apply for the map \(\psi^*\), which also provides the degree growth (3.20). This proves statement (c). Clearly \(d_N \sim N^2\) as \(N \rightarrow \infty\) and this shows that the algebraic entropy of the maps (1.1) vanishes.

Finally, we prove statement (d) of Theorem 1.1. To this end, we consider the eigenspace of the eigenvalue \(\lambda = 1\) of the map \(\varphi^*\) or \(\psi^*\) on the Picard lattice. This eigenspace of \(\varphi^*\) is generated by the following elements:

\[D_1 = H_x + H_y - E_4 - E_5 - E_7 - E_8 - E_{12}, \] (3.21a)
\[D_2 = H_x + H_y - E_3 - E_5 - E_6 - E_9 - E_{13}, \] (3.21b)
\[D_3 = H_y - E_1 - E_2 - E_{10} - E_{11}, \] (3.21c)
\[D_4 = E_1 - E_3 - E_6 - E_9 + E_{10} - E_{13}. \] (3.21d)

To find an invariant, we search for a linear combination \(D = n_1 D_1 + n_2 D_2 + n_3 D_3 + n_4 D_4\) which is realized by a curve in \(\mathbb{P}^1 \times \mathbb{P}^1\). From a numerical search, we find that a suitable set of values is \((n_1, n_2, n_3, n_4) = (1, 3, 0, -2)\). That is,

\[D = 4H_x + 4H_y - 2E_1 - E_3 - E_4 - 4E_5 - E_6 - E_7 - E_8 - E_9 - 2E_{10} - E_{12} - E_{13}. \] (3.22)
The corresponding curve is given by a multiple of the quartic pencil (2.4). A similar computation holds for the map $\psi^*$ and gives the same invariant (3.22). These serve to prove statement (d) and conclude our proof of Theorem 1.1.

4. Conclusion

We have constructed an algebraic surface $S$ by resolving (or blowing up) a sequence of points in $\mathbb{P}^1 \times \mathbb{P}^1$ given by singularities of the two maps $\varphi$ and $\psi$ (1.1). We showed that the lifted maps are analytically stable on $S$ but are not automorphisms of $S$. We used this construction to prove that the growth of degrees in the orbits generated by the maps is quadratic in the step $N$ (see formula (1.4)), thus proving the integrability of the maps (1.1) in the sense of algebraic entropy [1, 19].

In this study, we encountered a spurious infinite sequence of exceptional lines $\{\Theta_k\}_{k \in \mathbb{Z}\{0\}}$ which are mapped to points under $\varphi$ and $\psi$. We showed that this sequence gives two orbits, which never terminate on a variety of codimension 1. However, the corresponding sequence of indeterminacy points $\{\theta_k\}_{k \in \mathbb{Z}\{0\}}$ lies entirely on the singular fibre $\{x = 0\}$, which plays a special role in the transformation from the pencil $p$ (2.4), covariant under (1.1), to the pencil $q$ (2.5), covariant under the associated QRT map (2.3). The quartic pencil $p$ factors into four factors of the vertical line $\{x = 0\}$ and the biquadratic pencil $q$. Although not needed to compute the algebraic entropy of the map, this sequence of indeterminate points turn out to correspond to inadmissible initial values characterising a one-parameter family of solutions.

The occurrence of such solutions (in $\mathbb{P}^1 \times \mathbb{P}^1$) shows that spurious indeterminacy points can also appear in integrable examples. See [7] for several examples of this occurrence in the linearisable case.

While revising this paper, we became aware of the preprint [23], which considered the integrability of the two-dimensional version of the maps presented in [15]. These include the example we presented in this paper, with a different compactification of $\mathbb{C}^2$, namely $\mathbb{P}^2$. The results presented in [23] agree with those in our paper.

Our results lead to interesting open problems. One such problem concerns possible de-autonomisation of maps arising from KHK discretisation. Such a de-autonomisation is likely to be one of the discrete Painlevé equations described in [18]. See also [10, 12] for a comprehensive list of discrete Painlevé equations and the symmetry groups of the associated elliptic surfaces on $\mathbb{P}^1 \times \mathbb{P}^1$.

The surface $S$ constructed in Section 3 is not minimal, as there exist lines like $\{x \pm y = 0\}$ which have self-intersection $-3$. However, finding the corresponding minimal model is a nonalgorithmic and nontrivial task, which we will address in future works. A final open question is whether there exist partial difference (or lattice) equations which give such KHK maps as periodic or generalised reductions.

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