Sequential Detection of Market shocks using Risk-averse Agent Based Models

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Abstract—This paper considers a statistical signal processing problem involving agent based models of financial markets which at a micro-level are driven by socially aware and risk-averse trading agents. These agents trade (buy or sell) stocks by exploiting information about the decisions of previous agents (social learning) via an order book in addition to a private (noisy) signal they receive on the value of the stock. We are interested in the following: (1) Modelling the dynamics of these risk averse agents, (2) Sequential detection of a market shock based on the behaviour of these agents.

Structural results which characterize social learning under a risk measure, CVaR (Conditional Value-at-risk), are presented and formulation of the Bayesian change point detection problem is provided. The structural results exhibit two interesting properties: (i) Risk averse agents herd more often than risk neutral agents (ii) The stopping set in the sequential detection problem is non-convex. The framework is validated on data from the Yahoo! Tech Buzz game dataset and it is revealed that (a) The model identifies the value changes based on agent’s trading decisions, (b) Reasonable quickest detection performance is achieved when the agents are risk-averse.

Index Terms—conditional value at risk (CVaR), social learning filter, market shock, quickest detection, agent based models, monotone Bayesian update

I. INTRODUCTION

Financial markets evolve based on the behaviour of a large number of interacting entities. Understanding the interaction of these agents is therefore essential in statistical inference from financial data. This motivates the study of “agent based models” for financial markets. Agent based models are useful for capturing the global behaviour of highly interconnected financial systems by simulating the behaviour of the local interacting systems [1], [2], [3], [4]. Unlike standard economic models which emphasize the equilibrium properties of financial markets, agent based models stress local interactions and out-of-equilibrium dynamics that may not reach equilibrium in the long run [5]. Agent based models are commonly used to determine the conditions that lead a group of interacting agents to form an aggregate behaviour [6], [7], [8], [9] and to model stylized facts like correlation of returns and volatility clustering [10], [11]. Agent based models have also been used model anomalies that the standard approaches fail to explain like “fat tails”, absence of simple arbitrage, gain/loss asymmetry and leverage effects [12], [13].

In this paper, we are interested in developing agent based models for studying global events in financial markets where the underlying value of the stock experiences a jump change (shock). Market shocks are known to affect stock market returns [14], cause fluctuations in the economy [15] and necessitate market making [16]. Therefore detecting shocks is essential and when the interacting agents are acting based on private signals and complete history of other agents’ trading decisions, it is non-trivial [17].

The problem of market shock detection in the presence of social learning considered in this paper is different from a standard signal processing (SP) problem in the following ways:

1) Agents (or social sensors) influence the behaviour of other agents, whereas in standard SP sensors typically do not affect other sensors.
2) Agents reveal quantized information (decisions) and have dynamics, whereas in standard SP sensors are static with the dynamics modelled in the state equation.
3) Standard SP is expectation centric. In this paper we use coherent risk measures which generalizes the concept of expected value and is much more relevant in financial applications. Such coherent risk measures [18] are now widely used in finance to model risk averse behaviour. Properties 1 and 2 above are captured by social learning models. Such social learning models, where agents face fixed prices, are considered in [19], [20], [21], [9]. They show that after a finite amount of time, an informational cascade takes place and all subsequent agents choose the same action regardless of their private signal. Models where agents act sequentially to optimize local costs (to choose an action) and are socially aware were considered in [7], [22]. This paper considers a similar model, but, in order to incorporate property 3 above (risk averse behaviour), we will replace the classical social learning model of expected cost minimizers to that of risk averse minimizers. The resulting risk-averse social learning filter has several interesting (and unusual) properties that will be discussed in the paper.

Main Results and Organization

Section II presents the social learning agent based model and the market observer’s objective for detecting shocks. The formulation involves the interaction of local and global decision makers. Individual agents perform social learning and the market maker seeks to determine if the underlying asset value has changed based on the agent behaviour. The shock in the asset value changes at a phase distributed time (which generalizes geometric distributed change times). The problem of market shock detection considered in this paper is different from the classical Bayesian quickest detection [23].
where, local observations are used to detect the change. Quickest detection in the presence of social learning was considered in [17] where it was shown that making global decisions (stop or continue) based on local decisions (buy or sell) leads to discontinuous value function and the optimal policy has multiple thresholds. However, unlike [17] which deals with expected cost, we consider a more general measure to account for the local agents’ attitude towards risk.

It is well documented in various fields like economics and behavioural economics, psychology [27] that people prefer a certain but possibly less desirable outcome over an uncertain but potentially larger outcome. To model this risk aversion, commonly used risk measures [18] are Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), Entropic risk measure and Tail value at risk; see [28]. We consider social learning under CVaR risk measure. CVaR [29] is an extension of VaR that gives the total loss given a loss event and is a coherent risk measure [18]. In this paper, we choose CVaR risk measure as it exhibits the following properties [18], [29]: (i) It associates higher risk with higher cost. (ii) It ensures that risk is not a function of the quantity purchased, but arises from the stock. (iii) It is convex. CVaR as a risk measure has been used in solving portfolio optimization problems [30], [31], credit risk optimization [32], order execution [33] and also to optimize an insurer’s product mix [34]. For an overview of risk measures and their application in finance, see [28].

Section III provides structural results which characterize the social learning under CVaR risk measure and its properties. We show that, under reasonable assumptions on the costs, the trading decisions taken by socially aware and risk-averse agents are ordinal functions of their private observations and monotone in the prior information. This implies that the Bayesian social learning follows simple intuitive rules. The change point detection problem is formulated as a Market Observer seeking to detect a shock in the stock value (modelled as a Markov chain) by balancing the natural trade-off between detection delay and false alarm.

Section IV discusses the unusual properties exhibited by the CVaR social learning filter and explores the link between local and global behaviour in agent based models for detection of market shocks. We show that the stopping region for the sequential detection problem is non-convex; this is in contrast to standard signal processing quickest detection problems where the stopping set is convex. Similar results were developed in [17], [35], [36].

Finally, Section V discusses an application of the agent based model and change detection framework in a stock market data set. We use a data set from Tech Buzz Game which is a stock market simulation launched by Yahoo! Research and O’Reilly Media to gain insights into forecasting high-tech events and trades. Tech Buzz uses Dynamic parimutuel markets (DPM) as its trading mechanism. DPMs are known to provide accurate predictions in field studies on price formation in election stock markets [37], mechanism design for sales forecasting [38] and betting in sports markets [39], [40].

II. CVaR Social Learning Model and Market Observer’s Objective

This section presents the Bayesian social learning model and defines the objective of the market observer. As will be shown later in Section III the model results in ordinal decision making thereby mimicking human behavior and the risk measure captures a trader’s attitude towards risk.

A. CVaR Social Learning Model

The market micro-structure is modelled as a discrete time dealer market motivated by algorithmic and high-frequency tick-by-tick trading [41]. There is a single traded stock or asset, a market observer and a countable number of trading agents. The asset has an initial true underlying value $x_0 \in \mathcal{X} = \{1, 2, \ldots, X\}$. The market observer does not receive direct information about $x \in \mathcal{X}$ but only observes the public buy/sell actions of agents, $a_k \in \mathcal{A} = \{1(\text{buy}), 2(\text{sell})\}$. The agents themselves receive noisy private observations of the underlying value $x$ and consider this in addition to the trading decisions of the other agents visible in the order book [42], [43], [44], [45]. At a random time, $\tau^0$ determined by the transition matrix $P$, the asset experiences a jump change in its value to a new value. The aim of the market observer is to detect the change time (global decision) with minimal cost, having access to only the actions of these socially aware agents. Let $y_k \in \mathcal{Y} = \{1, 2, \ldots, Y\}$ denote agent $k$’s private observation. The initial distribution is $\pi_0 = (\pi_0(i), i \in \mathcal{X})$ where $\pi_0(i) = \mathbb{P}(x_0 = i)$.

The agent based model has the following dynamics:

1. **Shock in the asset value**: At time $\tau^0 > 0$, the asset experiences a jump change (shock) in its value due to exogenous factors. The change point $\tau^0$ is modelled by a phase type (PH) distribution. The family of all PH-distributions forms a dense subset for the set of all distributions $\{F_{n, \nu}, n \geq 1\}$ to approximate $F$ uniformly over $[0, \infty)$. The PH-distributed time $\tau^0$ can be constructed via a multi-state Markov chain $x_t$ with state space $\mathcal{X} = \{1, \ldots, X\}$ as follows: Assume state ‘1’ is an absorbing state and denotes the state after the jump change. The states $2, \ldots, X$ (corresponding to beliefs $e_2, \ldots, e_X$) can be viewed as a single composite state that $x$ resides in before the jump. So $\tau^0 = \inf\{k : x_k = 1\}$ and the transition probability matrix $P$ is of the form

$$P = \begin{bmatrix} 1 & 0 \\ P_{(X-1) \times 1} & \bar{P}_{(X-1) \times (X-1)} \end{bmatrix}$$

(1)

1 A risk measure $\varphi : \mathcal{L} \rightarrow \mathbb{R}$ is a mapping from the space of measurable functions to the real line which satisfies the following properties: (i) $\varphi(0) = 0$. (ii) If $S_1, S_2 \in \mathcal{L}$ and $S_1 \leq S_2$ a.s then $\varphi(S_1) \leq \varphi(S_2)$. (iii) If $a \in \mathbb{R}$ and $S \in \mathcal{L}$, then $\varphi(S + a) = \varphi(S) + a$. The risk measure is coherent if in addition $\varphi$ satisfies: (iv) If $S_1, S_2 \in \mathcal{L}$, then $\varphi(S_1 + S_2) \leq \varphi(S_1) + \varphi(S_2)$. (v) If $a \geq 0$ and $S \in \mathcal{L}$, then $\varphi(aS) = a\varphi(S)$. The expectation operator is a special case where subadditivity is replaced by additivity.

2 The market observer could be the securities dealer (investment bank or syndicate) that underwrites the stock which is later traded in a secondary market.
The distribution of the absorption time to state 1 is

\[ \nu_0 = \pi_0(1), \quad \nu_k = \pi_k \bar{P}^{k-1} P, \quad k \geq 1, \] 

where \( \pi_0 = [\pi_0(2), \ldots, \pi_0(X)]^T \). The key idea is that by appropriately choosing the pair \((\pi_0, P)\) and the associated state dimension \(X\), one can approximate any given discrete distribution on \([0, \infty)\) by the distribution \(\{\nu_k, k \geq 0\}\); see [46, pp.240-243]. The event \(\{x_k = 1\}\) means the change point has occurred at time \(k\) according to PH-distribution \(\mathcal{F}(x)\). In the special case when \(x\) is a 2-state Markov chain, the change time \(\tau^0\) is geometrically distributed.

2. **Agent’s Private Observation**: Agent \(k\)’s private (local) observation denoted by \(y_k\) is a noisy measurement of the true value of the asset. It is obtained from the observation likelihood distribution as,

\[ B_{xy} = \mathbb{P}(y_k = y|x_k = x) \tag{3} \]

3. **Private Belief update**: Agent \(k\) updates its private belief using the observation \(y_k\) and the prior public belief \(\pi_{k-1}(i) = \mathbb{P}(X = i|a_1, \ldots, a_{k-1})\) as the following Hidden Markov Model update

\[ \eta_k = \frac{B_{y_k} P \pi_{k-1}}{1 B_{y_k} P \pi_{k-1}} \tag{4} \]

4. **Agent’s trading decision**: Agent \(k\) executes an action \(a_k \in \mathcal{A} = \{1(\text{buy}), 2(\text{sell})\}\) myopically to minimize its cost. Let \(c(i, a)\) denote the cost incurred if the agent takes action \(a\) when the underlying state is \(i\). Let the local cost vector be

\[ c = [c(1, a) \, c(2, a) \ldots c(X, a)] \tag{5} \]

The costs for different actions are taken as

\[ c(i, j) = p_j - \beta_{ij} \quad \text{for} \quad i \in \mathcal{X}, j \in \mathcal{A} \tag{6} \]

where \(\beta_{ij}\) corresponds to the agent’s demand. Here demand is the agent’s desire and willingness to trade at a price \(p_j\) for the stock. Here \(p_1\) is the quoted price for purchase and \(p_2\) is the price demanded in exchange for the stock. We assume that the price is the same during the period in which the value changes. As a result, the willingness of each agent only depends on the degree of uncertainty on the value of the stock.

**Remark.** The analysis provided in this paper straightforwardly extends to the case when different agents are facing different prices like in an order book [42], [43]. [45]. For notational simplicity we assume the cost are time invariant.

The agent considers measures of risk in the presence of uncertainty in order to overcome the losses incurred in trading. To illustrate this, let \(c(x, a)\) denote the loss incurred with action \(a\) while at unknown and random state \(x \in \mathcal{X}\). When an agent solves an optimization problem involving \(c(x, a)\) for selecting the best trading decision, it will take into account not just the expected loss, but also the “riskiness” associated with the trading decision.

\[ a. \quad \text{The agent therefore chooses an action } a_k \text{ to minimize the CVaR measure} \tag{3} \]

\[ a_k = \arg\min_{a \in \mathcal{A}} \left\{ \text{CVaR}_\alpha(c(x, a)) \right\} \tag{7} \]

\[ = \arg\min_{a \in \mathcal{A}} \left\{ \min_{z \in \mathbb{R}} \left\{ z + \frac{1}{\alpha} \mathbb{E}_{y_k} [\max\{c(x, a) - z, 0\}] \right\} \right\} \]

Here \(\alpha \in (0, 1]\) reflects the degree of risk-aversion for the agent (the smaller \(\alpha\) is, the more risk-averse the agent is). Define

\[ \mathcal{H}_k := \sigma\text{- algebra generated by } (a_1, a_2, \ldots, a_{k-1}, y_k) \tag{8} \]

\(\mathbb{E}_{y_k}\) denotes the expectation with respect to private belief, i.e, \(\mathbb{E}_{y_k} = \mathbb{E}[\cdot|\mathcal{H}_k]\) when the private belief is updated after observation \(y_k\).

5. **Social Learning and Public belief update**: Agent \(k\)’s action is recorded in the order book and hence broadcast publicly. Subsequent agents and the market observer update the public belief on the value of the stock according to the social learning Bayesian filter as follows

\[ \pi_k = T^{\pi_{k-1}}(\pi_{k-1}, a_k) = \frac{R_{a_k}^{\pi_{k-1}} P' \pi_{k-1}}{1 R_{a_k}^{\pi_{k-1}} P' \pi_{k-1}} \tag{9} \]

Here, \(R_{a_k}^{\pi_{k-1}} = \text{diag}(\mathbb{P}(a_k|\pi_{k-1} = \pi))\), \(\pi = \mathbb{P}(a_k|\pi_{k-1} = \pi)\) and \(\mathbb{P}(a_k|\pi_{k-1} = \pi)\) is a probability distribution over \(\mathcal{A}\).

Note that \(\pi_k\) belongs to the unit simplex \(\Pi(X) \triangleq \{\pi \in \mathbb{R}^X : 1^T \pi = 1, 0 \leq \pi \leq 1 \text{ for all } i \in \mathcal{X}\} \)

6. **Market Observer’s Action**: The market observer (security dealer) seeks to achieve quickest detection by balancing delay with false alarm. At each time \(k\), the market observer chooses action \(u_k \in \mathcal{U} = \{1(\text{stop}), 2(\text{continue})\}\)

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Here ‘Stop’ indicates that the value has changed and the dealer incorporates this information before selling new issues to investors. The formulation presented considers a general parametrization of the costs associated with detection delay and false alarm costs. Define

\[ \mathcal{G}_{\pi} := \sigma\text{- algebra generated by } (a_1, a_2, \ldots, a_{k-1}, a_k) \tag{11} \]

**i) Cost of Stopping**: The asset experiences a jump change (shock) in its value at time \(\tau^0\). If the action \(u_k = 1\) is chosen before the change point, a false
alarm penalty is incurred. This corresponds to the event
\( \bigcup_{i \geq 2} \{ x_k = i \} \cap \{ u_k = 1 \} \). Let \( f_i I(x_k = i, u_k = 1) \)
denote the cost of false alarm in state \( i, i \in X \) with \( f_i \geq 0 \). The expected false alarm penalty is
\[
C(\pi_k, u_k = 1) = \sum_{i \in X} f_i E[I(x_k = i, u_k = 1) | G_k]
\]
where \( f = (f_1, \ldots, f_X) \) and it is chosen with increasing elements, so that states further from ‘1’ incur higher penalties. Clearly, \( f_1 = 0 \).

\( \textbf{ii) Cost of delay:} \) A delay cost is incurred when the event \( \{ x_k = 1, u_k = 2 \} \) occurs, i.e. even though the state changed at \( k \), the market observer fails to identify the change. The expected delay cost is
\[
C(\pi_k, u_k = 2) = d E[I(x_k = i, u_k = 1) | G_k]
\]
where \( d > 0 \) is the delay cost and \( e_1 \) denotes the unit vector with 1 in the first position.

\[ J^*(\pi_0) = \inf_{\mu \in \Pi} J_{\mu}(\pi_0) \]

The sequential detection problem (16) can be viewed as a function
\[ J^*(\pi_0) = \inf_{\mu \in \Pi} J_{\mu}(\pi_0) \]
where \( T^*(\pi, a) = R^{*\pi^2} \) is the CVaR-social learning filter. The market maker chooses its action at each time \( k \), \( u_k = \mu(\pi_k) \in \{1, \text{stop}, 2, \text{continue}\} \) for the optimal policy of the market observer
\[ \mu^*(\pi) = \arg\min_{\mu \in \Pi} J_{\mu}(\pi) \]

\[ V(\pi) = \min \left\{ C(\pi, 1), C(\pi, 2) + \rho \sum_{a \in A} V(T^*(\pi, a)) \sigma(\pi, a) \right\} \]

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would classify this cost as “high”. Also credit rating agencies use ordinal symbols such as AAA, AA, A.

**Theorem 1.** Under (A1) and (A2), the action $a^*(\pi, y)$ made by each agent is increasing and hence ordinal in $y$ for any prior belief $\pi$.

Under (A2), $a^*(\pi, y)$ is increasing in $\pi$ with respect to the monotone likelihood ratio order (Definition 7 in the appendix).

The proof is given in the appendix. Theorem 1 says that agents exhibit monotone ordinal behaviour. The condition that $a^*(\pi, y)$ is monotone in the observation $y$ is required to characterize the local decision matrices on different regions in the belief space which is stated next.

**Theorem 2.** Under (A1) and (A2), there are at most $Y + 1$ distinct local decision likelihood matrices $R^\pi$ and the belief space $\Pi(X)$ can be partitioned into the following $Y + 1$ polytopes:

- $\mathcal{P}^\pi_0 = \{\pi \in \Pi(X) : H(1, 1) - H(1, 2) \geq 0\}$
- $\mathcal{P}^\pi_0 = \{\pi \in \Pi(X) : H(l, 1) - H(l - 1, 1) - H(l - 1, 2) < 0\}$
- $\mathcal{P}^\pi_Y = \{\pi \in \Pi(X) : H(Y, 1) - H(Y, 2) < 0\}$

Also, the matrices $R^\pi$ are constant on each of these polytopes.

The proof is given in the appendix. Theorem 2 is required to specify the policy for the market observer. Indeed it leads to unusual behavior (non-convex) stopping regions in quickest detection as described in Section IV-B.

### IV. Social Learning and Change Detection for Risk-Averse Agents

This section illustrates the properties of the risk-averse social learning filter which leads to a non-convex value function and therefore non-convex stopping set of quickest detection.

#### A. Social Learning Behavior of Risk Averse Agents

The following discussion highlights the relation between risk-aversion factor $\alpha$ and the regions $\mathcal{P}^\pi_l$. For a given risk-aversion factor $\alpha$, Theorem 2 shows that there are at most $Y + 1$ polytopes on the belief space. It was shown in [17] that for the risk neutral case with $X = 2$, and $P = I$ (the value is a random variable) the intervals $\mathcal{P}^\alpha_1$ and $\mathcal{P}^\alpha_1$ correspond to the herding region and the interval $\mathcal{P}^\alpha_2$ corresponds to the social learning region. In the herding region, the agents take the same action as the belief is frozen. In the social learning region there is observational learning. However, when the agents are optimizing a more general risk measure (CVaR), the social learning region is different for different risk-aversion factors. The social learning region for the CVaR is measure is shown in Fig. [1]. The following parameters were chosen:

$$B = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.9 \end{bmatrix}, c = \begin{bmatrix} 1 & 2 \\ 3 & 0.5 \end{bmatrix}.$$  

It can be observed that the width of the social learning region decreases as $\alpha$ decreases. This can be interpreted as risk-averse agents showing a larger tendency to go with the crowd rather than “risk” choosing the other action. With the same $B$ and $c$ parameters, but with transition matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.9 \end{bmatrix}$$

the social learning region is shown in Fig. [2]. From Fig. [2] it is observed that when the state is evolving and when the agents are sufficiently risk-averse, social learning region is very small. It can be interpreted as: agents having a strong risk-averse attitude don’t prefer to “learn” from the crowd; but rather face the same consequences, when $P \neq I$.

#### B. Nonconvex Stopping Set for Market Shock Detection

We now illustrate the solution to the Bellman’s stochastic dynamic programming equation [17], which determines the optimal policy for quickest market shock detection, by considering an agent based model with two states. Clearly the agents (local decision makers) and market observer interact – the local decisions $a_k$ taken by the agents determines the public belief.
Relations $\mu^*(\pi)$ and hence determines decision $u_k$ of the market observer via (14).

From Theorem 2, the polytopes $P_1^{\alpha}, P_2^{\alpha}$ and $P_3^{\alpha}$ are subsets of $[0,1]$. Under (A1) and (A2), $P_3^{\alpha} = [0,\pi^{**}(2)], P_2^{\alpha} = [\pi^{**}(2), \pi^{*}(2)], P_1^{\alpha} = [\pi^{*}(2), 1]$, where $\pi^{**}$ and $\pi^{*}$ are the belief states at which $H^o(2, 1) = H^o(2, 2)$ and $H^o(1, 1) = H^o(1, 2)$ respectively. From Theorem 2 and (17), the value function can be written as,

$$V(\pi) = \min\{C(\pi), C(\pi, 2) + \rho V(\pi) I(\pi \in P_1^{\alpha}) + \rho \sum_{a \in A} V(I(\pi, a)) \sigma(\pi, a) I(\pi \in P_2^{\alpha}) + \rho V(\pi) I(\pi \in P_3^{\alpha})\}$$

The explicit dependence of the filter on the belief $\pi$ results in discontinuous value function. The optimal policy in general has multiple thresholds and the stopping region in general is non-convex.

Example: Fig. 3 displays the value function and optimal policy for a toy example having the following parameters:

$$B = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0.06 & 0.94 \end{bmatrix}, c = \begin{bmatrix} 1 & 2 \\ 2.5 & 0.5 \end{bmatrix}.$$

The parameters for the market observer are chosen as: $d = 1.25, f = [0, 3], \alpha = 0.8$ and $\rho = 0.9$.

From Fig. 3 it is clear that the market observer has a double threshold policy and the value function is discontinuous. The double threshold policy is unusual from a signal processing point of view. Recall that $\pi(2)$ depicts the posterior probability of no change. The market observer “changes its mind” - it switches from no change to change as the posterior probability of change decreases! Thus the global decision (stop or continue) is a non-monotone function of the posterior probability obtained from local decisions in the agent based model. The example illustrates the unusual behaviour of the social learning filter.

V. Dataset Example

A. Tech Buzz Game and Model:

To validate the framework, we consider the data from Tech Buzz Game, a stock market simulation launched by Yahoo! Research and O’Reilly Media to gain insights into forecasting high-tech events and trades. The game consisted of multiple sub-markets trading stocks of contemporary rival technologies. At each trading instant, the traders (or players) had access to the search “buzz” around each of the stocks. The buzz was an indicator of the number of users scouring Yahoo! search on the stock over the past seven days, as a percentage of all searches in the same market. Thus, if searches for the stock named “SKYPE” make up 80 percent of all Yahoo! searches in the telecommunication application software market, then SKYPE’s buzz score is 80. The buzz scores of all technologies within a market always add up to 100. The dataset was chosen to demonstrate the framework as the trading information was made available by Yahoo!

The stock market simulation is modelled as follows. The state $X$ is chosen to represent value of the stock, with $X=1$ indicating a high valued stock and $X=2$ indicating a stock of low value. It is well known that if the perceived value is more then it is going to be popular. Hence, the noisy observations are taken to be the buzz scores which are a proxy for the popularity of the stock [50]. For tractability, is assumed that all agents have the same attitude toward risk, i.e. $\alpha$ is the same for all agents. The agents choose to buy ($a=1$) or sell ($a=2$) depending on the cost and the belief updated using the buzz score. On each day the stock is traded, we consider only the agent which buys or sells maximum shares and record its trading decision (positive or negative values in the dataset) as buying or selling a unit of stock. This is reasonable assumption in the sense that the agent trading maximum shares (“big players” in finance) will significantly influence the public belief.

B. Dataset

The buzz score for the stocks, SKYPE and IPOD trading in markets VOIP and PORTMEDIA respectively, was obtained for the period from April 1, 2005 to July 27, 2005 from the Yahoo! Dataset[5]. Value of the stock is basically a function of the payout of the stock and its dividend [50]. The payout and dividend are directly proportional to the buzz score. Using the buzz score, value of the stock was calculated with the method suggested in [50]. Space discretized value of the stocks SKYPE and IPOD during the period is shown in the Fig. 4 and Fig. 7 along with the scaled buzz score.

1) Quickest detection for SKYPE: By eyeballing the data in Fig. 4 it is seen that the value changed during the month of June. To apply the quickest detection protocol, we consider a window from May 17 to June 8. It is observed that the price was (almost) constant during this period with a value close to $13 per stock. The trading decisions (along with the value) and the public belief during this period are shown in Fig. 5[6].
The local and global costs for the market observer were chosen as:

\[ c = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 2 \end{bmatrix}, \quad d = 0.8 \]

The model parameters were chosen as follows:

\[ B = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0.04 & 0.96 \end{bmatrix} \]

The choice of the parameters for the observation matrix \( B \) was motivated by the experimental evidence provided in [51], that when there is social learning “alone”, the trading rate was 71% based on peer effects. Since the local decision likelihood matrix \( R^\pi = B \) in the social learning region (in our model), the parameters were so chosen. Parameters in the transition matrix \( P \) were chosen to reflect the time window considered, as \( \mathbb{E}\{\tau_0\} = 25 \).

It was observed that the state changed on June 6 and for a risk-aversion factor of \( \alpha = 0.45 \), it was detected on June 7. The value function and the optimal policy for the market observer are shown in Fig. 6. The stopping set corresponds to \( \pi(2) \in [0, 0.354] \). The regions \( \pi(2) \in [0, 0.34] \) and \( \pi(2) \in [0.76, 1] \) correspond to the regions where social learning is absent. It can be observed that the value function is discontinuous.

2) *Quickest detection for IPOD*: From Fig. 7, it is seen that the value changed during April and July. To apply the quickest detection protocol, we consider a window from July 2 to July 10. It is observed that the price was (almost) constant during this period with a value close to $17 per stock. The trading decisions (along with the value) and the public belief during this period are shown in Fig. 8.

The local and global costs for the market observer were chosen as:

\[ c = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 1.8 \end{bmatrix}, \quad d = 0.95 \]

The model parameters were chosen as follows
the reasons for this problem to be non-trivial - the stopping region is in general non-convex; and it also accounted for the agents’ risk attitude by considering a coherent risk measure, CVaR, instead of the expected value measure in the agents’ optimization problem. Results which characterize the structural properties of social learning under the CVaR risk measure were provided and the importance of these results in understanding the global behaviour was discussed. It was observed that the behaviour of these risk-averse agents is, as expected, different from risk neutral agents. Risk averse agents herd sooner and don’t prefer to “learn” from the crowd, i.e, social learning region is smaller the more risk-averse the agents are. Finally, the model was validated using a financial dataset from Yahoo! Tech Buzz game. From a signal processing point of view, the formulation and solutions are non-standard due to the three properties described in Section I.

In current work, we are interested in determining structural results for the optimal change detection. Structural results for POMDPs were developed in [36], [52] and it is worthwhile extending these results to the current framework. When the market observer incurs a measurement cost, quickest change detection often has a monotone policy as described in [53]. It is of interest to generalize these results to the current setup.

APPENDIX A
PRELIMINARIES AND DEFINITIONS:

Definition 1. MLR Ordering [54] (≥_r): Let π_1, π_2 ∈ Π(X) be any two belief state vectors. Then π_1 ≥_r π_2 if

\[ \pi_1(i)\pi_2(j) \leq \pi_2(i)\pi_1(j), \quad i < j, \quad j \in \{1, \ldots, X\}. \]

Definition 2. First-Order Stochastic Dominance (≥_s): Let π_1, π_2 ∈ Π(X) be any two belief state vectors. Then π_1 ≥_s π_2 if

\[ \sum_{i=1}^{X} \pi_1(i) \geq \sum_{i=1}^{X} \pi_2(i) \quad \text{for} \quad j \in \{1, \ldots, X\}. \]

Lemma 3. [54] π_2 ≥ s π_1 iff for all v ∈ V, v’π_2 ≤ v’π_1, where V denotes the space of X-dimensional vectors v, with non-increasing components, i.e., v_1 ≥ v_2 ≥ \ldots v_X.

Lemma 4. [54] π_2 ≥ s π_1 iff for all v ∈ V, v’π_2 ≥ v’π_1, where V denotes the space of X-dimensional vectors v, with non-decreasing components, i.e., v_1 ≤ v_2 ≤ \ldots v_X.

Let Π(X) = {π ∈ R^X : 1^T_Xπ = 1, 0 ≤ π(i) ≤ 1 for all i ∈ X}

Definition 3. Submodular function [49]: A function f : Π(X) × \{1, 2\} → R is submodular if

\[ f(\pi, u) - f(\pi, \bar{u}) \leq f(\bar{\pi}, u) - f(\bar{\pi}, \bar{u}), \quad \text{for} \quad \bar{u} \leq u, \pi \geq r, \bar{\pi}. \]

Definition 4. Single Crossing Condition [49]: A function g : Y × A → R satisfies a single crossing condition in (y, a) if

\[ g(y, a) - g(y, \bar{a}) \geq 0 \Rightarrow g(\bar{y}, a) - g(\bar{y}, \bar{a}) \geq 0 \]

for \( \bar{a} > a \) and \( \bar{y} > y \). For any such function g,

\[ a^*(y) = \arg\min_a g(y, a) \text{ is increasing in } y. \]
Theorem 5. \[\text{If } f : \Pi(X) \times \{1, 2\} \to \mathbb{R} \text{ is sub-modular, then there exists a } u^*(\pi) = \arg\min_{u \in \{1, 2\}} f(\pi, u) \text{ satisfying,}\]
\[\bar{\pi} \succeq_{r} \pi \Rightarrow u^*(\bar{\pi}) \leq u^*(\pi)\]

APPENDIX B
PROOFS

The following lemmas are required to prove Theorem 4 and Theorem 2. The results will be proved for general state and observation spaces having two actions.

Lemma 6. \text{For a finite state and observation alphabet, } \arg\min_{z \in \mathbb{R}} \{z + \frac{1}{\alpha} \mathbb{E}_y[\max\{c(x, a) - z, 0\}]\} \text{ is equal to } c(i, a) \text{ for some } i \in \{1, 2, \ldots, X\}.

Proof. \text{Let } \eta_y \text{ be the belief update (p.m.f) with observation } y, \text{ i.e., } \eta_y(i) = \mathbb{P}_y(x = i). \text{ Let } F_y(x) \text{ denote the cumulative distribution function. For simplicity of notation, let } h_y(z) = z + \frac{1}{\alpha} \mathbb{E}_y[\max\{c(x, a) - z, 0\}]. \text{ The extremum of } h_y(z) \text{ is attained where the derivative is zero. It is obtained as follows.}\]
\[
h_y'(z) = 1 + \frac{1}{\alpha} \mathbb{E}_y[\max\{c(x, a) - z, 0\}] = 1 + \frac{1}{\alpha} \mathbb{P}_y(x = i, a > z).\]

Also, \[h_y''(z) = \frac{1}{\alpha} \frac{d}{dz} (F_y(z)) \text{ and therefore } h_y''(z) \geq 0. \text{ We have, } \arg\min_{z \in \mathbb{R}} \{h_y(z)\} = \{z : \mathbb{P}_y(c(x, a) > z) = \alpha\}. \text{ Since } X \text{ is a random variable, } c(x, a) \text{ is a random variable with realizations } c(i, a) \text{ for } i \in \{1, \ldots, X\}. \text{ Hence } z = c(i, a) \text{ for some } i \in \{1, 2, \ldots, X\}.\]

The result of Lemma 6 is similar to Proposition 8 in [53]. It was shown in [48] that \[\eta_{y+1} \succeq_r \eta_y. \text{ Also, MLR dominance implies first order dominance, i.e., } \eta_{y+1} \succeq_r \eta_y.\]

Lemma 7. \text{Let } l \text{ be the index such that } \arg\min_{z \in \mathbb{R}} \{h_y(z)\} = c(l, a) \text{ and } k \text{ be the index such that } \arg\min_{z \in \mathbb{R}} \{h_{y+1}(z)\} = c(k, a). \text{ For all } y \in \{1, 2, \ldots, Y\}, \text{ } k \geq l.

Proof. \text{Proof is by contradiction. From lemma (6), we have } F_y(c(l, a)) = 1 - \alpha \text{ and } F_{y+1}(c(k, a)) = 1 - \alpha. \text{ Suppose } l > k. \text{ We know that } F_{y+1}(z) \text{ is a monotone function in } z. \text{ Since } l > k, F_{y+1}(c(l, a)) > 1 - \alpha. \text{ But, by definition of first order stochastic dominance, } F_y(z) \geq F_{y+1}(z) \text{ for all } z. \text{ Therefore, } F_y(c(l, a)) \geq F_{y+1}(c(l, a)) > 1 - \alpha, \text{ a contradiction.}\]

From lemma 6 and equation (19), we have
\[
H^\alpha(y, 2) = c(l, 2) + \frac{1}{\alpha} \sum_{i=1}^{l-1} \eta_y(i)(c(i, 2) - c(l, 2)),
\]
\[
H^\alpha(y+1, 2) = c(k, 2) + \frac{1}{\alpha} \sum_{i=1}^{k-1} \eta_{y+1}(i)(c(i, 2) - c(k, 2))
\]

Lemma 8. \text{\textbf{H}^\alpha(y, 2) \geq \textbf{H}^\alpha(y+1, 2) if } \alpha \geq 1 - \mathbb{P}_y(x = X).\]

Proof. From the definitions of \(H^\alpha(y, 2)\) and \(H^\alpha(y+1, 2)\) we have,
\[
H^\alpha(y, 2) - H^\alpha(y+1, 2) = c(l, 2) - c(k, 2) + \frac{1}{\alpha} \sum_{i=1}^{l-1} \eta_y(i)(c(i, 2) - c(l, 2)) + \frac{1}{\alpha} \sum_{i=1}^{k-1} \eta_{y+1}(i)(c(k, 2) - c(i, 2)) \geq c(l, 2) - c(k, 2) + \frac{1}{\alpha} \sum_{i=1}^{l-1} \eta_y(i)(c(i, 2) - c(l, 2)) + \frac{1}{\alpha} \sum_{i=1}^{k-1} \eta_{y+1}(i)(c(k, 2) - c(i, 2)) \geq c(l, 2) - c(k, 2) + \frac{1}{\alpha} \Gamma^\alpha \eta_y
\]
\[
\text{Equation (22) follows from lemma 3 and can be simplified as}
\]
\[
H^\alpha(y, 2) - H^\alpha(y+1, 2) \geq c(l, 2) - c(k, 2) + \frac{1}{\alpha} \sum_{i=1}^{l-1} \eta_y(i)(c(k, 2) - c(l, 2)) + \frac{1}{\alpha} \sum_{i=1}^{k-1} \eta_{y+1}(i)(c(k, 2) - c(i, 2)) \geq c(l, 2) - c(k, 2) - \frac{1}{\alpha} \Gamma^\alpha \eta_y
\]
\[
\text{where } \Gamma \text{ is such that } \Gamma_i = c(l, 2) - c(k, 2) \text{ for } i = 1, \ldots, l - 1 \text{ and } \Gamma_i = c(i, 2) - c(k, 2) \text{ for } i = l, \ldots, k - 1. \text{ Clearly, } \Gamma_i \geq 0 \text{ and decreasing. Right hand side of inequality attains its maximum when } k = X \text{ and } l = 1 \text{ and } \Gamma_i = c(l, 2) - c(k, 2) \text{ for all } i. \text{ Therefore, we have}
\]
\[
H^\alpha(y, 2) - H^\alpha(y+1, 2) \geq c(l, 2) - c(k, 2) - \frac{1}{\alpha} \Gamma^\alpha \eta_y
\]
\[
\geq (c(l, 2) - c(k, 2)) - \frac{1}{\alpha} (c(l, 2) - c(k, 2))(1 - \mathbb{P}_y(x = X))
\]
\[
\text{After rearrangement we have,}
\]
\[
H^\alpha(y, 2) - H^\alpha(y+1, 2) \geq \frac{\alpha - (1 - \mathbb{P}_y(x = X))}{\alpha} (c(l, 2) - c(k, 2))
\]
\[
\text{Since } \alpha \geq 1 - \mathbb{P}_y(x = X) \text{ and } (c(l, 2) - c(k, 2)) \geq 0 \text{ (follows from lemma 7 and assumption (A2)), we have } H^\alpha(y, 2) \geq H^\alpha(y+1, 2).\]

From Lemma 8 and (19), we have
\[
H^\alpha(y, 1) = c(l, 1) + \frac{1}{\alpha} \sum_{i=1}^{x} \eta_y(i)(c(i, 1) - c(l, 1)),
\]
\[
H^\alpha(y+1, 1) = c(k, 1) + \frac{1}{\alpha} \sum_{i=1}^{x} \eta_{y+1}(i)(c(i, 1) - c(k, 1))
\]

Lemma 9. \text{\textbf{H}^\alpha(y+1, 1) \geq \textbf{H}^\alpha(y, 1) if } \alpha \geq 1 - \mathbb{P}_y(x = X).\]
Proof. From the definitions of $H^\alpha(y+1,1)$ and $H^\alpha(y,1)$ we have,

$$H^\alpha(y+1,1) - H^\alpha(y,1) = c(k,1) - c(l,1) + \frac{1}{\alpha} \sum_{i=k+1}^{X} \eta_{y+1}(i)(c(i,1) - c(k,1)) - \frac{1}{\alpha} \sum_{i=l+1}^{X} \eta_y(i)(c(i,1) - c(l,1)) \geq c(k,1) - c(l,1) + \frac{1}{\alpha} \sum_{i=k+1}^{X} \eta_{y+1}(i)(c(i,1) - c(k,1)) - \frac{1}{\alpha} \sum_{i=l+1}^{X} \eta_{y+1}(i)(c(i,1) - c(l,1))$$

Equation (23) follows from lemma 4 and can be simplified as

$$H^\alpha(y+1,1) - H^\alpha(y,1) \geq (c(k,1) - c(l,1)) - \frac{1}{\alpha} (c(k,1) - c(l,1))(1 - \mathbb{P}_{y+1}(x = X))$$

After rearrangement we have,

$$H^\alpha(y+1,1) - H^\alpha(y,1) \geq \frac{\alpha - (1 - \mathbb{P}_{y+1}(x = X))}{\alpha} (c(k,1) - c(l,1))$$

(24)

Since $\alpha \geq 1 - \mathbb{P}_{y+1}(x = X)$ and $c(k,1) - c(l,1) \geq 0$ (follows from lemma 7 and assumption (A2)), we have $H^\alpha(y+1,1) \geq H^\alpha(y,1)$.

Lemma 10. Let $\alpha \geq (1 - \mathbb{P}_y(x = X))$. The function $H^\alpha(y,a)$ satisfies the single crossing condition i.e.

$$(H^\alpha(y,1) - H^\alpha(y,2)) \geq 0 \Rightarrow (H^\alpha(y+1,1) - H^\alpha(y+1,2)) \geq 0$$

Proof. Assume $(H^\alpha(y,1) - H^\alpha(y,2)) \geq 0$. We have,

$$H^\alpha(y,1) - H^\alpha(y,2) \geq 0 \Rightarrow H^\alpha(y,1) - H^\alpha(y+1,2) \geq 0$$

Equation (25) follows from lemma 8 Also,

$$H^\alpha(y,1) - H^\alpha(y+1,2) \geq 0 \Rightarrow H^\alpha(y+1,1) - H^\alpha(y+1,2) \geq 0$$

Equation (26) follows from Lemma 9.
