Convergence of discrete-time Kalman filter estimate to continuous-time estimate for systems with unbounded observation

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Abstract In this article, we complement recent results on the convergence of the state estimate obtained by applying the discrete-time Kalman filter on a time-sampled continuous-time system. As the temporal discretization is refined, the estimate converges to the continuous-time estimate given by the Kalman–Bucy filter. We shall give bounds for the convergence rates for the variance of the discrepancy between these two estimates. The contribution of this article is to generalize the convergence results to systems with unbounded observation operators under different sets of assumptions, including systems with diagonalizable generators, systems with admissible observation operators, and systems with analytic semigroups. The proofs are based on applying the discrete-time Kalman filter on a dense, numerable subset on the time interval $[0, T]$ and bounding the increments obtained. These bounds are obtained by studying the regularity of the underlying semigroup and the noise-free output.

Keywords Kalman filter · Infinite-dimensional systems · Boundary control systems · Temporal discretization · Sampled data

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1 Introduction

The minimum variance state estimate for linear systems with Gaussian noise processes is given by the continuous-time Kalman filter. However, for obvious reasons, in a practical implementation, the continuous-time system is often first discretized, and then the discrete-time Kalman filter is used on the discretized system. The objective of this article is to expand the recent results presented in [1] by the author on the convergence of the state estimate given by the discrete-time Kalman filter on the sampled system to the continuous-time estimate. The convergence results were shown for finite-dimensional systems and infinite-dimensional systems with bounded observation operators. The expansion in this paper covers systems with unbounded observation operators and systems whose dynamics are governed by an analytic semigroup. In particular, we shall show convergence rate estimates for the variance of the discrepancy between the discrete- and continuous-time estimates.

We study systems whose dynamics are given by

\[
\begin{align*}
    dz(t) &= Az(t)dt + Bdu(t), \quad t \in \mathbb{R}^+, \\
    dy(t) &= Cz(t)dt + dw(t), \\
    z(0) &= x
\end{align*}
\]

where \( A : \mathcal{X} \to \mathcal{X}, B : \mathcal{U} \to \mathcal{X}, \) and \( C : \mathcal{X} \to \mathcal{Y} \). The Hilbert spaces \( \mathcal{X}, \mathcal{U} = \mathbb{R}^q, \) and \( \mathcal{Y} = \mathbb{R}^r \) are called the state space, the input space, and the output space, respectively. The mapping \( A \) is the generator of a \( C_0 \)-semigroup \( e^{At} \) on \( \mathcal{X} \) with domain \( D(A) \), \( B : \mathbb{R}^q \to \mathcal{X} \) is the control operator, and \( C : \mathcal{X} \to \mathbb{R}^r \) is called the observation operator. Dynamics equations (1) are given in the form of stochastic differential equations, see [19] by Øksendal for background. The input and output noise processes \( u \) and \( w \) are assumed to be \( q \)- and \( r \)-dimensional Brownian motions with incremental covariance matrices \( Q \geq 0 \) and \( R > 0 \), respectively. Without loss of generality, we assume that there is no deterministic input, as it can always be removed by the usual techniques. However, the article is concluded with a discussion on the discretization of a deterministic input. The initial state \( x \) is assumed to be a Gaussian random variable with mean \( m \) and covariance \( P_0 \), denoted \( x \sim N(m, P_0) \), and \( u, w, \) and \( x \) are assumed to be mutually independent.

The purpose of this paper is to study the discrepancy of the discrete- and continuous-time state estimates, defined by

\[
\hat{z}_{T,n} := \mathbb{E}(z(T) \mid \{y(i\Delta T_n)\}_{i=1}^n) \quad \text{and} \quad \hat{z}(T) := \mathbb{E}(z(T) \mid \{y(s), s \leq T\}),
\]

respectively, and in particular, find convergence rate estimates for the variance \( \mathbb{E}\left(\|\hat{z}_{T,n} - \hat{z}(T)\|_{\mathcal{X}}^2\right) \) as \( n \to \infty \) when the observation operator \( C \) is not bounded, which typically occurs when we get a pointwise or a boundary measurement from the computational domain of a system whose dynamics are governed through a partial differential equation. However, we do assume that \( C \in \mathcal{L}(D(A), \mathcal{Y}) \).

The state estimates \( \hat{z}_{T,n} \) and \( \hat{z}(T) \) are obtained by the Kalman(–Bucy) filter—provided that the continuous-time Kalman filter equations are solvable. The Kalman
filter was originally presented by Kalman [15] for discrete-time systems and by Kalman and Bucy [16] for continuous-time systems. The infinite-dimensional generalization has been treated, for example, by Falb [10], by Bensoussan [4], by Curtain and Pritchard [6], and by Horowitz [14]. Of course the infinite-dimensional setting gives rise to many technical issues, such as unbounded control and observation operators and the solvability of the corresponding Riccati differential equations. These problems are tackled, for example, by Da Prato and Ichikawa [7] and Flandoli [11].

In the results of this paper, it is assumed that the temporal discretization can be done perfectly, so that the only error source is the sampling of the continuous-time output signal. We refer to review article [13] by Goodwin et al. for a discussion on the sampling of continuous-time systems and in particular [21] by Salgado et al. for a study on the sampled data Riccati equations and the Kalman filter. In practice, approximative numerical schemes are used for solving both the state estimate \( \hat{z}_{T,n} \) and the corresponding error covariance. For a discussion on this topic, see [3] by Axelsson and Gustafsson and [12] by Frogerais et al. treating nonlinear systems.

In Sect. 2, we introduce the ingredients for the proofs of our results. The main idea is to apply the discrete-time Kalman filter on a dense, numerable subset of the interval \([0, T]\). This way we obtain a martingale that starts from the discrete-time estimate \( \hat{z}_{T,n} \) and converges almost surely to the continuous-time estimate \( \hat{z}(T) \). We shall then find bounds for the increments of this martingale. These bounds are obtained by studying the regularity of the semigroup \( e^{At} \) and in particular, the smoothness of the noise-free output \( Ce^{At}x \) for \( x \in X \). As was noted in [1] and as seen later in the proof of Theorem 5, the effects of the input noise process \( u \) and the initial state \( x \) can be treated separately. Therefore we will first derive several results with different assumptions on the system considering just the effect of the initial state in Sect. 3. Finally, in Sect. 4, we will consider the effect of the input noise as well as the effect of using discretized deterministic input in the discrete-time Kalman filter. The input noise effect is shown with the assumption of admissibility of the observation operator \( C \).

Notation and standing assumptions

We denote by \( \{e_k\}_{k=1}^\infty \subset D(A) \) an orthonormal basis for the state space \( X \). The operator \( A \) generates a strongly continuous semigroup that is bounded by \( \|e^{At}\|_{\mathcal{L}(X)} \leq \mu \) for \( t \in [0, T] \).

2 Background

The idea of the proofs is exactly the same as in [1], but here we need to deal with many more technical issues. That is, we define a dense, numerable subset of the time interval \([0, T]\) and apply the discrete-time Kalman filter in this subset. Then we compute an upper bound for each increment in the state estimate and finally sum up these bounds. So let us define the time points \( t_j \) for \( j = 1, 2, \ldots \) through the dyadic division
The time point definition is illustrated in Fig. 1. Then define $T_j := \{t_i\}_{i=1}^{j}$ and the $\mathcal{X}$-valued martingale $\tilde{z}_j = \mathbb{E}(z(T)|\{y(t), t \in T_j\})$. Define also the shorthand notation $y(T_j) = \{y(t), t \in T_j\}$. Now it holds that $\tilde{z}_n = \tilde{z}_{T,n}$, and as discussed in [1, Section 2.1], as $j \to \infty$, the martingale $\tilde{z}_j$ converges almost surely strongly to $\tilde{z}(T)$. The idea in the proofs in this paper is to find upper bounds for the increments $\tilde{z}_{j+1} - \tilde{z}_j$, for $j \geq n$.

As the martingale $\tilde{z}_j$ is square integrable, we have the telescope identity shown in [1, Lemma 1] for $L, N \in \mathbb{N}$ with $L \geq N$:

$$
\mathbb{E} \left( \|\tilde{z}_L - \tilde{z}_N\|^2_\mathcal{X} \right) = \sum_{j=N}^{L-1} \mathbb{E} \left( \|\tilde{z}_{j+1} - \tilde{z}_j\|^2_\mathcal{X} \right).
$$

We remark that setting $N = n$ and letting $L \to \infty$ gives

$$
\mathbb{E} \left( \|\tilde{z}(T) - \tilde{z}_{T,n}\|^2_\mathcal{X} \right) = \sum_{j=n}^{\infty} \mathbb{E} \left( \|\tilde{z}_{j+1} - \tilde{z}_j\|^2_\mathcal{X} \right).
$$

Let us then establish an expression for one increment $\mathbb{E} \left( \|\tilde{z}_{j+1} - \tilde{z}_j\|^2_\mathcal{X} \right)$. Say $[\xi, \xi_1]$ is a jointly Gaussian random variable in some product space. Denote $\hat{\xi}_1 := \mathbb{E}(\xi|\xi_1)$ and $P_1 := \text{Cov} \left[ \hat{\xi}_1 - \xi, \hat{\xi}_1 - \xi \right]$. Then say

$$
\xi_2 = H \xi + w
$$

where $H$ is a bounded linear operator and $w$ is a Gaussian random variable with mean zero and covariance $R > 0$ and $w$ is independent of $[\xi, \xi_1]$. Then it holds that

$$
\hat{\xi}_2 := \mathbb{E}(\xi|\{\xi_1, \xi_2\}) = \hat{\xi}_1 + P_1 H^* (HP_1 H^* + R)^{-1} (\xi_2 - H \hat{\xi}_1).
$$
Then
\[ \hat{\xi}_2 - \hat{\xi}_1 = P_1 H^*(H P_1 H^* + R)^{-1} (H (\xi - \hat{\xi}_1) + w) \]
from which it directly follows that
\[ \text{Cov} \left[ \hat{\xi}_2 - \hat{\xi}_1, \hat{\xi}_2 - \hat{\xi}_1 \right] = P_1 H^*(H P_1 H^* + R)^{-1} H P_1, \]
and further,
\[ \mathbb{E} \left( \left\| \hat{\xi}_2 - \hat{\xi}_1 \right\|^2 \right) = \text{tr} \left( P_1 H^*(H P_1 H^* + R)^{-1} H P_1 \right). \quad (6) \]

The basic ingredients for the proofs in this paper are now presented, namely the martingale \( \tilde{z}_j \) defined as
\[ \mathbb{E} (z(T) \mid y(T_j)) \]
with \( T_j \) defined in (3), the telescope identity (4), and Eq. (6) for the increment norm.

Later we sometimes need the assumption that \( x \in D(A) \) almost surely. With Gaussian random variables this means that \( x \) is actually a \( D(A) \)-valued random variable.

**Proposition 1** Let \( \xi \) be an \( \mathcal{X} \)-valued Gaussian random variable s.t. \( \xi \in \mathcal{X}_1 \) almost surely where \( \mathcal{X}_1 \subset \mathcal{X} \) is another Hilbert space with continuous and dense embedding. Then \( \xi \) is an \( \mathcal{X}_1 \)-valued Gaussian random variable.

**Proof** Pick \( h \in \mathcal{X}_1 \). We intend to show that \( \langle \xi, h \rangle_{\mathcal{X}_1} \) is a real-valued Gaussian random variable. For \( h \in \mathcal{X}_1 \) there exists \( h' \in \mathcal{X}'_1 \), the dual space of \( \mathcal{X}_1 \), s.t.
\[ \langle \xi, h \rangle_{\mathcal{X}_1} = \langle \xi, h' \rangle_{(\mathcal{X}_1, \mathcal{X}'_1)} \]
and further, there exists a sequence \( \{h_i\}_{i=1}^\infty \subset \mathcal{X} \) such that
\[ \langle \xi, h' \rangle_{(\mathcal{X}_1, \mathcal{X}'_1)} = \lim_{i \to \infty} \langle \xi, h_i \rangle_{\mathcal{X}}. \]
Now \( \langle \xi, h_i \rangle_{\mathcal{X}} \) is a pointwise converging sequence of Gaussian random variables and so the limit is also Gaussian. \( \square \)

Fernique’s theorem [8, Theorem 2.6] can be applied to note that if \( \xi \) is an \( \mathcal{X}_1 \)-valued Gaussian random variable then \( \xi \in L^p(\Omega; \mathcal{X}_1) \) for any \( p > 0 \). In particular, \( \mathbb{E} \left( \|\xi\|_{\mathcal{X}_1}^2 \right) < \infty \) and if \( A \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}) \) then \( A\xi \) is an \( \mathcal{X} \)-valued Gaussian random variable.

**3 Convergence results without input noise**

Assume now that there is no input noise in system (1), that is, \( u = 0 \). The output is then given by
\[ y(t) = C \int_0^t e^{As} x \, ds + w(t) \]
where \( w \) is a Brownian motion. In a sense, the output is parameterized by the initial state \( x \), and therefore we also define the martingale \( \tilde{x}_j := \mathbb{E}(x \mid y(T_j)) \). Then in the absence of input noise it holds that \( \tilde{z}_j = e^{AT} \tilde{x}_j \).

In order to use (6) to compute one increment \( \mathbb{E} \left( \|\tilde{z}_{j+1} - \tilde{z}_j\|_{\mathcal{X}}^2 \right) \), we need to consider how to take into account an intermediary observation \( y(t_{j+1}) \) in the state estimate. Obviously the noise process value \( w(t_{j+1}) \) is not independent of the measurements \( y(t_i) \) with \( i = 1, \ldots, j \) as required in order to use (6). As in [1], the dependence of
the noise term \( w(t_{j+1}) \) on \( y(t_i) \) with \( i = 1, \ldots, j \) is removed by subtracting from \( y(t_{j+1}) \) the linear interpolant \( \frac{1}{2} \left( y(t_{j+1} - h) + y(t_{j+1} + h) \right) \) where \( h = \frac{T}{2K_n} \) with the corresponding \( K \) (see (3) and Fig. 1). Note that \( t_{j+1} \pm h \in T_j \). When there is no input noise, this new output is obtained as

\[
\hat{y}_{j+1} = C_h(t_{j+1})x + \tilde{w}_{j+1}
\]

(7)

where \( C_h(t) \) is defined for \( x \in \mathcal{X} \) and \( t \geq h \) by

\[
C_h(t)x := \frac{C}{2} \left( \int_{t-h}^{t} e^{As}x \, ds - \int_{t}^{t+h} e^{As}x \, ds \right),
\]

(8)

and \( \tilde{w}_{j+1} \sim N \left( 0, \frac{k}{2} R \right) \) is independent of \( y(t_i) \) and hence of \( \hat{y}_i \) for \( i = 1, \ldots, j \). Note that we can first define \( C_h(t)x \) only for \( x \in D(A) \). However, it holds that

\[
\left\| \int_{t}^{t+h} e^{As}x \, ds \right\|_{D(A)}^2 = \left\| \int_{t}^{t+h} e^{As}x \, ds \right\|_{\mathcal{X}}^2 + \| A \int_{t}^{t+h} e^{As}x \, ds \|_{\mathcal{X}}^2
\]

\[
\leq h^2 \mu^2 \| x \|_{\mathcal{X}}^2 + \left\| \left( e^{A(t+h)} - e^{At} \right) x \right\|_{\mathcal{X}}^2 \leq \left( h^2 + 4 \right) \mu^2 \| x \|_{\mathcal{X}}^2.
\]

Therefore, we can uniquely extend \( C_h(t) \) to a continuous operator from \( \mathcal{X} \) to \( \mathcal{Y} \), and

\[
\| C_h(t) \|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} \leq \mu \sqrt{h^2 + 4 \| C \|_{\mathcal{L}(D(A), \mathcal{Y})}}.
\]

In addition, using \( C e^{At} \xi = C e^{At} \xi + \int_t^s C e^{A(r)} \xi \, dr \) in (8) yields a useful bound

\[
\| C_h(t) \xi \|_{\mathcal{Y}} \leq \frac{1}{2} \left( \int_{t-h}^{t} \left\| C e^{Ar} \xi \right\|_{\mathcal{Y}} \, dr \, ds + \int_{t}^{t+h} \int_s^t \left\| C e^{Ar} \xi \right\|_{\mathcal{Y}} \, dr \, ds \right)
\]

(9)

provided that \( C e^{Ar} \xi \in L^1(t - h, t + h; \mathcal{Y}) \).

Now form (7) is exactly as (5) and so (recalling \( \tilde{z}_j = e^{At} \tilde{x}_j \)) we can use (6) to obtain

\[
\mathbb{E} \left( \| \tilde{z}_{j+1} - \tilde{z}_j \|_{\mathcal{X}}^2 \right) = \text{tr} \left( e^{AT} P_j C_h(t_{j+1})^* \right. \times \left( C_h(t_{j+1}) P_j C_h(t_{j+1})^* + \frac{h}{2} R \right)^{-1} C_h(t_{j+1}) P_j e^{AT} \).
\]

Finally we are able to present the main lemma, which links the convergence of the state estimate to properties of the operator \( C_h(t) \) and the smoothness of the output \( y \). This lemma serves as the basis for all the proofs of our main theorems when there is no input noise.

**Lemma 1** Let \( \mathcal{X}_1 \) be a dense Hilbert subspace of \( \mathcal{X} \) with a continuous embedding. Assume that there exist \( M > 0 \) and \( k > 1 \) such that for any \( K \in \mathbb{N} \) and \( \xi \in \mathcal{X}_1 \), it holds that

\[ \]
\[
\sum_{j=2^{k-1}n+1}^{2^{k}n} \| C_{j}(t) \xi \|_{Y}^2 \leq M h^{k} \| \xi \|_{X_1}^2
\]  

(10)

where \( h = \frac{T}{2^{k}n} \). Assume also that \( x \in X_1 \) almost surely and \( u = 0 \). Then for \( \hat{z}_{T,n} \) and \( \hat{z}(T) \) defined in (2), it holds that

\[
\mathbb{E} \left( \| \hat{z}_{T,n} - \hat{z}(T) \|_{X}^2 \right) \leq 2M \mathbb{E} \left( \| x \|_{X_1}^2 \right) \mathbb{E} \left( \| \hat{z}_{T,n} - z(T) \|_{X_1}^2 \right) \frac{T^{k-1}}{(2^{k-1} - 1) \min(eig(R))} n^{k-1}.
\]

Note that \( \mathbb{E} \left( \| x \|_{X_1}^2 \right) \) is well defined and finite by Proposition 1. A strict a priori result is obtained by replacing \( \mathbb{E} \left( \| \hat{z}_{T,n} - z(T) \|_{X_1}^2 \right) \) with \( \mu^2 \mathbb{E} \left( \| x \|_{X_1}^2 \right) \). The proof of Lemma 1 is essentially the same as the proof of [1, Theorem 1] (see also the discussion about infinite-dimensional setup in the proof of [1, Theorem 3]), and it is omitted here.

### 3.1 Diagonalizable main operator

We proceed to prove a convergence result for systems with unbounded observation operator \( C \)—provided that \( A \) is (unitarily) diagonalizable. The proof is based on Lemma 1. To get a useful bound for \( \| C_{k} \xi \|_{Y}^2 \), some assumptions on the degree of unboundedness of \( C \) and the spectral asymptotics of \( A \) are required.

**Theorem 1** Let \( \hat{z}_{T,n} \) and \( \hat{z}(T) \) be as defined above in (2). Denote by \( \{ \lambda_{k} \}_{k=1}^{\infty} \subset \mathbb{C} \) the spectrum of \( A \) ordered so that \( |\lambda_{k}| \) is non-decreasing and let \( \{ e_{k} \}_{k=1}^{\infty} \subset D(A) \) be the corresponding set of eigenvectors that give an orthonormal basis for \( X \). Make the following assumptions on \( x, A, \) and \( C \):

(i) \( x \in D(A) \) almost surely;
(ii) There exists \( \delta > 1/2 \) such that

\[
\lim_{k \to \infty} \frac{|\lambda_{k}|}{k^{\beta}} = \begin{cases} 
0 & \text{when } \beta > \delta, \\
\infty & \text{when } \beta < \delta;
\end{cases}
\]

(iii) There exists \( \gamma \in [0, 1) \) such that \( 2\gamma + 1/\delta < 2 \) and

\[
\sup_{k} \frac{\| Ce_{k} \|_{Y}^{\gamma}}{|\lambda_{k}|^{\gamma}} < \infty.
\]

Then the following holds:

- If \( \lim_{k \to \infty} \frac{|\lambda_{k}|}{k^{\beta}} = \Gamma \in (0, \infty) \), then

\[
\mathbb{E} \left( \| \hat{z}_{T,n} - \hat{z}(T) \|_{X}^2 \right) \leq \frac{MT^{3-2\gamma-1/\delta}}{n^{2-2\gamma-1/\delta}}
\]

where the constant \( M \) is given below in (12).
– If either this limit does not exist, or it is 0 or \( \infty \), then for all \( \epsilon \in (0, \delta - \frac{1}{2}) \)

\[
E \left( \left\| \hat{z}_{T,n} - \hat{z}(T) \right\|^2 \right) \leq \frac{M_\epsilon T^{3-2\gamma-1/(\delta+\epsilon)}}{n^{2-2\gamma-1/(\delta-\epsilon)}}
\]

where the \( \epsilon \)-dependent constant \( M_\epsilon \) is given below also in (12) but with different, \( \epsilon \)-dependent parameters (see the last paragraph of the proof).

**Proof** Assume first that \( \lim_{k \to \infty} \frac{\| \lambda_k \|}{k^2} = \Gamma \in (0, \infty) \). Denote \( \xi = \sum_{k=1}^\infty \alpha_k e_k \in \mathcal{D}(A) \) which is equivalent to \( \sum_{k=1}^\infty |\lambda_k|^2 \alpha_k^2 \leq \infty \). Now

\[
C_h \xi = \sum_{k=1}^\infty \frac{\alpha_k}{2} \left( \int_{t-h}^{t} e^{\lambda_k s} \, ds - \int_{t}^{t+h} e^{\lambda_k s} \, ds \right) C e_k.
\]  

(11)

For the term inside parentheses, we have

\[
\left| \int_{t-h}^{t} e^{\lambda_k s} \, ds - \int_{t}^{t+h} e^{\lambda_k s} \, ds \right| \leq h^2 \sup_{s \geq 0} \left| \frac{d}{ds} e^{\lambda_k s} \right| \leq \mu h^2 |\lambda_k|,
\]

since \( \| e^{At} \|_{L(X)} \leq \mu \). On the other hand, computing the integrals yields

\[
\left| \int_{t-h}^{t} e^{\lambda_k s} \, ds - \int_{t}^{t+h} e^{\lambda_k s} \, ds \right| \leq \frac{4\mu}{|\lambda_k|}.
\]

Now the idea is to bound the sum in (11) by using the first bound for small \( k \) and the latter for large \( k \). Define the index \( n(h) := \lceil h^{-1/\delta} \rceil \) for splitting the sum to get

\[
\| C_h \xi \|_Y \leq \sum_{k=1}^{n(h)} \frac{|\alpha_k|}{2} \mu \| C e_k \|_Y |\lambda_k| h^2 + \sum_{k=n(h)+1}^\infty |\alpha_k| \| C e_k \|_Y \frac{2\mu}{|\lambda_k|} =: (I) + (II).
\]

We then proceed to find upper bounds for the two parts. Using the Cauchy–Schwartz inequality and denoting \( \hat{\Gamma} := \sup_k \frac{|\lambda_k|}{k^2} \) gives

\[
(I) \leq \mu \frac{h^2}{2} \left( \sum_{k=1}^{n(h)} \frac{\alpha_k^2}{2} \| C e_k \|_Y^2 |\lambda_k|^{2-2\gamma} \right)^{1/2} \left( \sum_{k=1}^{n(h)} |\lambda_k|^{2\gamma} \right)^{1/2}
\]

\[
\leq \frac{\mu h^2 \hat{\Gamma}^{\gamma/2}}{2} M_L \left( \sum_{k=1}^{n(h)} k^{2\gamma \delta} \right)^{1/2}
\]
where \( M_1 = \left( \sum_{k=1}^{n(h)} \alpha_k^2 \| C e_k \|_2 \| \lambda_k \|_2 \right)^{1/2} \). The sum inside the parentheses can be bounded from above by the integral \( \int_0^{n(h)+1} x^{2\gamma \delta} \mathrm{d}x \) to get

\[
(I) \leq \frac{\mu h^{2\Gamma \gamma}}{2\sqrt{2\gamma \delta} + 1} M_1 \sqrt{(n(h) + 1)^{2\gamma \delta + 1}} \leq \frac{3^\delta \mu \Gamma \gamma}{2\sqrt{2\gamma \delta} + 1} M_1 h^{2\gamma - \frac{1}{2\delta}}
\]

where the last row follows from the facts that

\[
\sqrt{(n(h) + 1)^{2\gamma \delta + 1}} \leq \sqrt{(h^{-1/\delta} + 2)^{2\gamma \delta + 1}} = (1 + 2h^{1/\delta})^{\gamma \delta + \frac{1}{2}} h^{-\frac{1}{2\delta}} \leq 3^\delta h^{-\frac{1}{2\delta}}
\]

if \( h \leq 1 \), and that \( 2\gamma \delta + 1 > 1 \).

For the second part, assume \( \| \lambda_k \| \geq \Gamma k^\delta \) for \( k \geq n(h) + 1 \) where \( \Gamma = 0.9 \Gamma \), for example. Again, using the Cauchy–Schwartz–Schwartz inequality yields

\[
(II) \leq 2\mu \left( \sum_{k=n(h)+1}^{\infty} \alpha_k^2 \| C e_k \|_2 \| \lambda_k \|_2 \right)^{1/2} \left( \sum_{k=n(h)+1}^{\infty} \frac{1}{\| \lambda_k \|_2^{4-2\gamma}} \right)^{1/2}
\]

\[
\leq \frac{2\mu}{\Gamma^{2-\gamma}} M_{II} \left( \sum_{k=n(h)+1}^{\infty} \frac{1}{k^{(4-2\gamma)\delta}} \right)^{1/2}
\]

where \( M_{II} = \left( \sum_{k=n(h)+1}^{\infty} \alpha_k^2 \| C e_k \|_2 \| \lambda_k \|_2 \right)^{1/2} \). Now the sum inside the parentheses can be bounded from above by the integral \( \int_{n(h)}^{\infty} \frac{1}{x^{(4-2\gamma)\delta}} \mathrm{d}x \). Note that our assumptions on \( \gamma \) and \( \delta \) imply \( (4 - 2\gamma)\delta > 2 \). So we get

\[
(II) \leq \frac{2\mu M_{II}}{\Gamma^{2-\gamma} \sqrt{(4-2\gamma)\delta - 1}} \left( \frac{1}{n(h)^{(4-2\gamma)\delta - 1}} \right)^{1/2} \leq \frac{2\mu}{\Gamma^{2-\gamma}} M_{II} h^{2\gamma - \frac{1}{2\delta}}
\]

where in the last inequality we have used \( n(h) \geq h^{-1/\delta} \).

Combining the bounds gives

\[
\| C h^2 \xi \|_2^2 \leq 2((I)^2 + (II)^2)
\]

\[
\leq 2\mu^2 \left( M_I^2 + M_{II}^2 \right) \max \left( \frac{9^\delta \Gamma^{2\gamma}}{4}, \frac{4}{\Gamma^{4-2\gamma}} \right) h^{4-2\gamma - 1/\delta}
\]

\[
\leq 2\mu^2 \| A \xi \|_X^2 \max_k \| C e_k \|_2^2 \frac{4}{\| \lambda_k \|_2^{2\gamma}} \max \left( \frac{9^\delta \Gamma^{2\gamma}}{4}, \frac{4}{\Gamma^{4-2\gamma}} \right) h^{4-2\gamma - 1/\delta}
\]
where we have used
\[ M_1^2 + M_2^2 = \sum_{k=1}^{\infty} \alpha_k^2 \| C e_k \|_Y^2 | \lambda_k |^{2-2\gamma} \leq \sum_{k=1}^{\infty} | \lambda_k |^2 \alpha_k^2 \sup_j \| C e_j \|_Y^2. \]

Note that we assumed that we could choose, for example, \( \tilde{\Gamma} = 0.9 \Gamma \). In some sense this is not our choice but we need to make sure that the “original” \( h = \frac{T}{2n} \) is small enough so that \( n(T/(2n)) = \left( \frac{2n}{T} \right)^{1/\delta} \) is such that there exists \( \tilde{\Gamma} > 0 \) for which \( \frac{\lambda_k^2}{k^\delta} \geq \tilde{\Gamma} \) for \( k \geq n(T/(2n)) \).

To get a bound for the sum in (10), we simply multiply the bound obtained for \( \| C h \xi \|^2_Y \) by \( 2^{K-1}n = \frac{T}{2n} \) to get
\[
\sum_{j=2^{K-1}n+1}^{2^K n} \| C h(t_j) \xi \|^2_Y \leq \sup_k \frac{\| C e_k \|_Y^2}{| \lambda_k |^{2\gamma}} \max \left( \frac{\gamma \delta^{2\gamma}}{4}, \frac{4}{\tilde{\Gamma}^{4-2\gamma}} \right) \mu^2 T \| \xi \|^2_{D(A)} h^{3-2\gamma-1/\delta}
\]
and so the result follows by Lemma 1 with
\[
M = \frac{2\mu \mathbb{E} \left( \| \hat{z}_{T,n} - z(T) \|_{\mathcal{X}}^2 \right) \mathbb{E} \left( \| x \|^2_{D(A)} \right) \sup_k \frac{\| C e_k \|_2^2}{| \lambda_k |^{2\gamma}} \max \left( \frac{\gamma \delta^{2\gamma}}{4}, \frac{4}{\tilde{\Gamma}^{4-2\gamma}} \right) .
\]

(12)

In the case that \( \lim_{k \to \infty} \frac{\lambda_k}{k^\delta} = 0 \), \( \infty \), or it does not exist, some modifications are required to the bounds of (I) and (II). In the bound for (I), \( \delta \) needs to be replaced by \( \delta + \epsilon \) and then \( \hat{\Gamma}_\epsilon = \sup_k \frac{\lambda_k}{k^{\delta+\epsilon}} < \infty \). In the bound for (II), \( \delta \) needs to be replaced by \( \delta - \epsilon \) and then \( \hat{\Gamma}_\epsilon = \inf_{k \geq n(\delta+\epsilon) + 1} \frac{\lambda_k}{k^{\delta+\epsilon}} > 0 \).

The assumption (iii) in the theorem differs from our minimal assumption \( C \in \mathcal{L}(\mathcal{D}(A), \mathcal{Y}) \) which is equivalent to \( \frac{\| C e_k \|_Y}{| \lambda_k |} \in l^2 \) for unitarily diagonalizable \( A \). It is possible to construct a system for which \( C \in \mathcal{L}(\mathcal{D}(A), \mathcal{Y}) \) but (iii) does not hold.

**Remark 1** Theorem 1 can be extended to \( \gamma < 0 \). In that case, when determining the bounds for (I) and (II), the computations are carried out as if \( \gamma \) were zero. This eventually leads to a bound \( \mathbb{E} \left( \| \hat{z}_{T,n} - \hat{z}(T) \|_{\mathcal{X}}^2 \right) \leq \frac{MT^{3-1/\delta}}{n^{2-1/\delta}} \). Note that if assumption (iii) holds for \( \gamma < -\frac{1}{2\delta} \) then \( C \) is actually bounded.

**Example 1** Consider the 1D wave equation with Dirichlet boundary conditions written as a first-order system
\[
\begin{aligned}
\frac{d}{dt} \begin{bmatrix} z_1(x, t) \\ z_2(x, t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix} \begin{bmatrix} z_1(x, t) \\ z_2(x, t) \end{bmatrix}, & x \in [0, 1], \\
z_1(0, t) &= z_1(1, t) = 0, \\
z_1(x, 0) &= z_{1,0}(x), & z_2(x, 0) &= z_{2,0}(x) \\
dy(t) &= \left[ \frac{\partial}{\partial x} z_1(0, t) \right] dt + dw(t)
\end{aligned}
\]
with state space \( \mathcal{X} = H_0^1[0, 1] \times L^2(0, 1) \) and \( \mathcal{D}(A) = H_0^2[0, 1] \times H_0^1[0, 1] \). The eigenvalues of \( A \) are

\[
\lambda_{2k-1} = i\pi k \quad \text{and} \quad \lambda_{2k} = -i\pi k
\]

and the corresponding eigenvectors

\[
e_{2k-1} = \begin{bmatrix} \frac{1}{\pi k} \sin(\pi k x) \\ i \sin(\pi k x) \end{bmatrix} \quad \text{and} \quad e_{2k} = \begin{bmatrix} \frac{1}{\pi k} \sin(\pi k x) \\ -i \sin(\pi k x) \end{bmatrix}.
\]

It holds that

\[
\lim_{k \to \infty} \frac{\lambda_k}{k} = \frac{\pi}{2}, \quad \text{and} \quad \|Ce_k\|_{\mathbb{R}^2} = \sqrt{2}
\]

and so the assumptions of Theorem 1 are satisfied with \( \delta = 1 \) and \( \gamma = 0 \), implying convergence rate \( E\left(\|\hat{z}_{T,n} - \hat{z}(T)\|^2_X\right) \leq \frac{MT^2}{n} \).

The convergence was studied numerically using the eigenbasis consisting of the first 800 eigenvectors (that is, Fourier components \( \sin(\pi k x) \) up to \( k = 400 \)). The convergence was observed at three different times \( T = 0.25, 0.5, \) and 1, when the time discretization step was \( \Delta t = \frac{0.25}{n} \) with \( n = 1, 2, 4, 8, 10, 20, 25, 40, 50, 100 \). The “continuous-time” limit was obtained by an overkill discretization with \( n = 1000 \). The measurement noise covariance was \( R = I \). The initial state was

\[
\begin{align*}
z_{1,0}(x) &= \sum_{k=1}^{400} \frac{\alpha_k}{\pi k} \sin(\pi k x) \\
z_{2,0}(x) &= \sum_{k=1}^{400} \frac{\beta_k}{\pi k} \sin(\pi k x)
\end{align*}
\]

where \( \alpha_k, \beta_k \sim N\left(0, \frac{1}{k+0.01k^2}\right) \), implying \( \begin{bmatrix} z_{1,0} \\ z_{2,0} \end{bmatrix} \in \mathcal{D}(A) \) almost surely. The convergence plots for \( E\left(\|\hat{z}_{T,n} - \hat{z}(T)\|^2_X\right) \) for three different values of \( T \), estimated from 10 replicates, are shown in logarithmic scale in Fig. 2. The convergence plots for times 0.25 and 0.5 suggest that the convergence result obtained in Theorem 1 is correct and sharp. At time 1, it seems that with smaller time steps the errors are so small that the made approximations start to play a significant role in the convergence estimates.

### 3.2 Admissible observation operator

In the next result, we assume that the observation operator \( C \) is admissible in the sense of Weiss [24]. One good example of systems that satisfy assumption (iii) in the following theorem is provided by scattering passive boundary control systems, see article [18] by Malinen and Staffans. For a more extensive background, we refer to [22] by Staffans.

**Theorem 2** Let \( \hat{z}_{T,n} \) and \( \hat{z}(T) \) be as defined above in (2) and \( u = 0 \). Make the following assumptions:

(i) \( x \in \mathcal{D}(A) \) almost surely;
Fig. 2 Convergence plot in logarithmic scale at three different times. The gridlines correspond to $O(n^{-1})$. The plots from top to bottom correspond to times $T = 0.25, 0.5$, and $1$

(ii) The orthonormal basis $\{e_k\} \subset X$ is such that $e_k \in D(A^2)$ for every $k \in \mathbb{N}$ and there exists $\delta > 1/2$ such that for $\xi = \sum_{k=1}^{\infty} \alpha_k e_k$ the norm given by $\sqrt{\sum_{k=1}^{\infty} k^{2\delta} \alpha_k^2}$ is equivalent to the $D(A)$-norm and $\sqrt{\sum_{k=1}^{\infty} k^{4\delta} \alpha_k^2}$ is equivalent to the $D(A^2)$-norm;

(iii) The observation operator is admissible, that is, for any $T \geq 0$ there exists $H_T \geq 0$, such that $\|Ce^{A(t)}x\|_{L^2((0,T);Y)} \leq H_T \|x\|_X$.

Then

$$\mathbf{E}\left(\|\hat{z}_{T,n} - \hat{z}(T)\|_{X}^2\right) \leq \frac{M(T) T^{2-1/2\delta}}{n^{1-1/2\delta}}$$

with $M(T) = \frac{2\mathbf{E}\left(\|\hat{z}_{T,n} - z(T)\|_{X}^2\right)\mathbf{E}\left(\|x\|_{D(A)}^2\right)}{(2^{1-1/2\delta}-1) \min(eig(R))} \max \left( \frac{3^{2\delta+1} T\mu |C|^2_{L^2(D(A),Y)}}{8\delta+4}, \frac{H_T}{2\delta-1} \right)$.

Proof In this proof, the aforementioned norms are used in $D(A)$ and $D(A^2)$. We need to utilize the global output bound $\|Ce^{A(t)}x\|_{L^2((0,T);Y)} \leq H_T \|x\|_X$. To this end, define a stacked operator $\hat{C}_h := [C_h(h), C_h(3h), \ldots, C_h(T-h)]^T$ for $h = \frac{T}{2k\mu}$ mapping to a product space $Y^{2k-1}$. Then the sum on the left hand side of (10) is obtained as $\|\hat{C}_h \xi\|_{Y^{2k-1}}^2$. In this proof, $[a_i]_{i=1}^N$ is used to denote an augmented vector with $N$ components $a_i$. 

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The proof proceeds similarly as the proof of Theorem 1 but the sum in (11) is split using the index \( n(h) = \lceil h^{-1/(2\delta)} \rceil \) to get \( \| \hat{C}_h \xi \|_{\mathcal{Y}^{2K-1-n}} \leq (I) + (II) \) where

\[
(I) = \left\| \sum_{k=1}^{n(h)} \frac{\alpha_k}{2} \left[ C \int_{(2j-2)h}^{(2j-1)h} e^{As} e_k \, ds - C \int_{(2j-1)h}^{2jh} e^{As} e_k \, ds \right] \right\|_{\mathcal{Y}^{2K-1-n}}^{2K-1-n} 
\]

\[
\leq \sum_{k=1}^{n(h)} \frac{|\alpha_k|}{2} \sqrt{\frac{T}{2h}} h^2 \mu \| C \|_{\mathcal{L}(\mathcal{D}(A), \mathcal{Y})} k^{2\delta}
\]

\[
\leq \sqrt{\frac{T h^3}{8}} \mu \| C \|_{\mathcal{L}(\mathcal{D}(A), \mathcal{Y})} \left( \sum_{k=1}^{n(h)} k^{2\delta} \alpha_k^2 \right)^{1/2} \left( \sum_{k=1}^{n(h)} k^{2\delta} \right)^{1/2}
\]

where the first inequality is obtained using (9) with \( t = (2j-1)h \) and bounding the derivative of \( C e^{Ar} e_k \) by

\[
\left\| C e^{Ar} e_k \right\|_{\mathcal{Y}} \leq \mu \| C \|_{\mathcal{L}(\mathcal{D}(A), \mathcal{Y})} \| e_k \|_{\mathcal{D}(A^2)} = \mu \| C \|_{\mathcal{L}(\mathcal{D}(A), \mathcal{Y})} k^{2\delta}
\]

and noting that \( 2^{K-1-n} = \frac{T}{2h} \).

For the remaining part, it holds that

\[
(II) = \left\| \sum_{k=n(h)+1}^{\infty} \frac{\alpha_k}{2} \left[ C \int_{(2i-2)h}^{(2i-1)h} e^{As} e_k \, ds - C \int_{(2i-1)h}^{2ih} e^{As} e_k \, ds \right] \right\|_{\mathcal{Y}^{2K-1-n}}^{2K-1-n} 
\]

\[
\leq \sum_{k=n(h)+1}^{\infty} \frac{|\alpha_k|}{2} \sqrt{2h} H_T \leq \sqrt{\frac{h}{2}} H_T \left( \sum_{k=n(h)+1}^{\infty} k^{2\delta} \alpha_k^2 \right)^{1/2} \left( \sum_{k=n(h)+1}^{\infty} k^{-2\delta} \right)^{1/2}
\]

since it holds that

\[
\left\| C \int_{(2i-2)h}^{(2i-1)h} e^{As} e_k \, ds \right\|_{\mathcal{Y}} \leq \sqrt{2h} \| C e^{As} e_k \|_{L^2((0,T);\mathcal{Y})}^{2K-1-n} 
\]

\[
\leq \left( \sum_{i=1}^{2K-1-n} \left( \int_{(2i-2)h}^{2ih} \| C e^{As} e_k \|_{\mathcal{Y}} \, ds \right)^2 \right)^{1/2} \leq \sqrt{2h} \| C e^{As} e_k \|_{L^2((0,T);\mathcal{Y})}
\]

where the last inequality follows from the Cauchy–Schwartz inequality. Finally, the sums \( \sum_{k=1}^{n(h)} k^{2\delta} \) and \( \sum_{k=n(h)+1}^{\infty} k^{-2\delta} \) are bounded by integrals of \( x^{2\delta} \) and \( x^{-2\delta} \), respectively, as in the proof of Theorem 1 and then the result is obtained by Lemma 1 and

\[
\| \hat{C}_h \xi \|_{\mathcal{Y}^{2K-1-n}}^2 \leq 2((I)^2 + (II)^2).
\]
3.3 Analytic semigroup

In this section, we show the convergence estimate when $A$ is the generator of an analytic semigroup. One result is first shown without additional assumptions for bounded and unbounded observation operator $C$. Then we assume further that $-A$ is a sectorial operator in $\mathcal{X}$ which enables us to work with non-integer powers $(-A)^{\eta}$ for $\eta \geq 0$. An example of such case is provided by heat equation treated below in Example 2.

An important tool here is that for analytic semigroups it holds that

$$
\left\| A^\kappa e^{At} \right\|_{L(\mathcal{X})} \leq \frac{c(\kappa)}{t^\kappa}, \quad t > 0, \ \kappa \in \mathbb{N}
$$

(see [23, Theorem 3.3.1]). Using this to bound the derivative of $Ce^{At} \xi$, that is $CA e^{At} \xi$, given by (9),

$$
\left\| Ch(t)\xi \right\|_Y \leq \frac{c(1)}{2(t-h)} \left\| \xi \right\|_X h^2, \quad t > h.
$$

(14)

It is possible to replace $\left\| C \right\|_{L(\mathcal{X}, Y)}$ by $\left\| C \right\|_{L(\mathcal{D}(A), Y)}$ in the inequality, if $\left\| \xi \right\|_X$ is correspondingly replaced by $\left\| \xi \right\|_{\mathcal{D}(A)}$.

**Theorem 3** Let $\hat{z}_{T,n}$ and $\hat{z}(T)$ be as defined above in (2). Assume $A$ is the generator of an analytic $C_0$-semigroup and assume either

(i) $C \in L(\mathcal{X}, Y)$, or
(ii) $C \in L(\mathcal{D}(A), Y)$ and $x \in \mathcal{D}(A)$ almost surely.

Then

$$
\mathbb{E}\left( \left\| \hat{z}_{T,n} - \hat{z}(T) \right\|_X^2 \right) \leq \frac{MT}{n}
$$

where $M = \frac{2C_{L(\mathcal{X}, \mathcal{Y})}}{\min(\text{sig}(R))} \left( \mu^2 + \frac{c(1)^2\pi^2}{96} \right) \mathbb{E}\left( \left\| \hat{z}(T) - \hat{z}_{T,n} \right\|_X^2 \right) \mathbb{E}\left( \left\| x \right\|_X^2 \right)$ in case (i). For case (ii), replace $\mathcal{X}$ by $\mathcal{D}(A)$ in $\left\| C \right\|_{L(\mathcal{X}, \mathcal{Y})}$ and $\mathbb{E}\left( \left\| x \right\|_X^2 \right)$.

**Proof** The proofs for the two cases are identical so only the case (i) is presented. Case (ii) follows by the aforementioned norm replacements.

The proof is based on Lemma 1, so we need to find a bound for

$$
\sum_{j=2^{k-1}n+1}^{2^k n} \left\| Ch(t_j)\xi \right\|_Y^2 = \sum_{l=1}^{2^k-1} \left\| Ch((2l - 1)h)\xi \right\|_Y^2.
$$

(15)

For $l = 1$, we use $\left\| Ch(h)\xi \right\|_Y \leq h\mu \left\| C \right\|_{L(\mathcal{X}, \mathcal{Y})} \left\| \xi \right\|_X$ which is clear from the definition of $Ch(t)$ in (8). For $l > 1$, we use (14) where the denominator becomes $2((2l - 1)h - h)$.
\[ h = 4h(l-1) \] and so,

\[
\sum_{j=2^{k-1}n+1}^{2^{k}n} \|Ch(t_j)\|_{\mathcal{Y}}^2 \leq \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}^2 \|\xi\|_{\mathcal{X}}^2 \left( \mu^2 + \frac{c(1)^2}{16} \sum_{l=2}^{2^{k-1}n} \frac{1}{(l-1)^2} \right) h^2 \\
\leq \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}^2 \|\xi\|_{\mathcal{X}}^2 \left( \mu^2 + \frac{c(1)^2\pi^2}{96} \right) h^2.
\]

The result follows by Lemma 1. \(\Box\)

One more case is treated where \(A\) is as before and, in addition, \(-A\) is a sectorial operator, see [2, Section 3.8] for definitions. Then it is possible to define non-integer powers \((-A)^{\eta}\) where \(\eta \in \mathbb{R}\), and spaces \(\mathcal{D}((-A)^{\eta})\) equipped with the corresponding graph norm. Also (13) holds then for non-integer \(\kappa \geq 0\) if \(A\) is replaced by \(-A\), see [23, Thm. 3.3.3]. In particular, if \(A\) is strictly negative definite, then \(-A\) is sectorial. This type of systems is also studied in [7] and [11].

**Theorem 4** Let \(\hat{z}_{T,n}\) and \(\hat{z}(T)\) be as defined above in (2). Assume that \(A\) is the generator of an analytic \(C_0\)-semigroup and, in addition, \(-A\) is a sectorial operator. Then assume \(C \in \mathcal{L}(\mathcal{D}((-A)^{\nu}),\mathcal{Y})\) and \(x \in \mathcal{D}((-A)^{\nu})\) almost surely where \(\nu \in \mathbb{R}\) and \(\eta \in \mathbb{R}\) are such that \(|\eta - \nu| < 1/2\). Then\(^1\)

\[
\mathbb{E}\left( \|\hat{z}_{T,n} - \hat{z}(T)\|_{\mathcal{X}}^2 \right) \leq \frac{MT^{1+2(\eta-\nu)}}{n^{1+2(\eta-\nu)}}
\]

where \(M\) is given below in (17).

**Proof** This is done exactly as the proof of Theorem 3. Just the bounds for \(\|Ch((2l-1)h)\|_{\mathcal{Y}}\) in summation (15) are computed differently. To begin with, we note that (13) with non-integer \(\kappa = 1 - \eta + \nu\) yields,

\[
\left\| CAe^{At}\xi \right\|_{\mathcal{Y}} \leq \left\| C\right\|_{\mathcal{L}(\mathcal{D}((-A)^{\nu}),\mathcal{Y})} \|\xi\|_{\mathcal{D}((-A)^{\nu})} \frac{c(1 - \eta + \nu)}{t^{1-\eta+\nu}}.
\]

When treating the term with \(l = 1\) in (15), the cases \(\nu \geq \eta\) and \(\nu < \eta\) have to be considered separately. First for \(\nu \geq \eta\),

\[
\left\| Ce^{At}\xi \right\|_{\mathcal{Y}} = \left\| C(-A)^{-\nu}(-A)^{\nu-\eta}e^{At}(-A)\eta\xi \right\|_{\mathcal{Y}} \\
\leq \left\| C\right\|_{\mathcal{L}(\mathcal{D}((-A)^{\nu}),\mathcal{Y})} \|\xi\|_{\mathcal{D}((-A)^{\nu})} \frac{c(\nu - \eta)}{t^{\nu-\eta}}.
\]

\(^1\) This result extends to \(\eta - \nu = 1/2\) in which case the convergence rate is \(O(T^2n^{-2} \ln n)\).
Then directly by the definition of $C_h$ in (8) (recalling $1 + \eta - \nu > 0$),

$$
\|C_h(h)\xi\|_Y \leq \|C\|_{\mathcal{L}(\mathcal{D}((-A)^\nu),\mathcal{Y})} \|\xi\|_{\mathcal{D}((-A)^\eta)} c(\nu - \eta) \int_0^{2h} \frac{1}{s^{\nu-\eta}} \, ds
\leq \|C\|_{\mathcal{L}(\mathcal{D}((-A)^\nu),\mathcal{Y})} \|\xi\|_{\mathcal{D}((-A)^\eta)} \frac{c(\nu - \eta)}{1 + \eta - \nu} (2h)^{1+\eta-\nu}.
$$

For $\nu < \eta < 1 + \nu$, use (9) with $t = h$. Using (16) to bound the derivative norm $\|CAe^{At}\xi\|_Y$ and computing the integrals yields

$$
\|C_h(h)\xi\|_Y \leq \|C\|_{\mathcal{L}(\mathcal{D}((-A)^\nu),\mathcal{Y})} \|\xi\|_{\mathcal{D}((-A)^\eta)} \frac{c(1 - \eta + \nu)}{2(\eta - \nu) 1+\eta-\nu} 2^1(\eta - \nu) - \frac{2}{h^{1+\eta-\nu}} \leq \|C\|_{\mathcal{L}(\mathcal{D}((-A)^\nu),\mathcal{Y})} \|\xi\|_{\mathcal{D}((-A)^\eta)} \frac{2 \ln 2 c(1 - \eta + \nu) h^{1+\eta-\nu}}{1 + \eta - \nu}.
$$

For the terms with $l > 1$, it holds by (9) and (16), that

$$
\|C_h((2l - 1)h)\xi\|_Y \leq \frac{h^2}{(2h)^{1+\eta-\nu} (l - 1)^{1-\nu}} \|\xi\|_{\mathcal{D}((-A)^\eta)}.
$$

Now summing up the bounds for $\|C_h((2l - 1)h)\xi\|_Y^2$ for $l = 1, \ldots, 2^{n-1}2$ and using $\sum_{l=1}^{\infty} \frac{1}{(l-2)^{(2l-\eta)}} \leq \frac{2^{-2(n(\nu-\eta))}}{1 - 2(n - \eta)}$ yields a bound for the sum in (10), and finally Lemma 1 can be used to get the result with

$$
M = \frac{2 \|C\|^2_{\mathcal{L}(\mathcal{D}((-A)^\nu),\mathcal{Y})}}{(2h)^{1+\eta-\nu} (l - 1)^{1-\nu}} \mathbb{E} \left( \|z(T) - z_{T,n}\|^2_{\mathcal{X}} \right) \frac{\mathbb{E} \left( \|x\|^2_{\mathcal{D}((-A)^\nu)} \right)}{\min(\text{eig}(R))} \times \left( M_{\nu,\eta} + \frac{c(1 - \eta + \nu)^2 (2 - 2(\eta - \nu))}{2^{1-2(\nu-\eta)} 1 - 2(\eta - \nu)} \right)
$$

(17)

where the term with $l = 1$ gives

$$
M_{\nu,\eta} = \begin{cases} 
\frac{4\ln 2 c(1 - \eta + \nu)^2}{(1 + \eta - \nu)^2} & \text{if } \eta > \nu, \\
\frac{2^{2-n(\nu-\eta)} c(\nu - \eta)^2}{(1 + \eta - \nu)^2} & \text{if } \eta \leq \nu.
\end{cases}
$$

**Example 2** Consider the 1D heat equation

$$
\begin{align*}
\frac{\partial}{\partial t} z(x, t) &= \rho \frac{\partial^2}{\partial x^2} z(x, t), \quad x \in [0, 1], \\
z(0, t) &= z(1, t) = 0, \\
z(x, 0) &= z_0,
\end{align*}
\begin{align*}
\frac{dy(t)}{dt} &= \begin{bmatrix}
\frac{\partial}{\partial x} z(0, t) \\
-\frac{\partial}{\partial x} z(1, t)
\end{bmatrix} \, dt + dw(t)
\end{align*}
$$
Fig. 3 Convergence plot in logarithmic scale at five different times. The gridlines correspond to $O(n^{-2})$. The plots from top to bottom correspond to times $T = 0.25, 0.5, 1, 2.5,$ and $5$.

with state space $\mathcal{X} = L^2(0, 1)$ and $\mathcal{D}(A) = H_0^2(0, 1)$. Assume $z_0 \in \mathcal{D}(A)$ almost surely. Now the spectrum of $A$ is $\{-\pi^2 k^2\}$ and the corresponding eigenvectors are $e_k = \sin(\pi k x)$. Then it is easy to see that the assumptions of Theorem 1 are satisfied with $\delta = 2$ and $\gamma = 1/2$ and thus the theorem implies convergence rate $O(n^{-1/2})$ for $E\left(\|\hat{z}_{T,n} - \hat{z}(T)\|_\mathcal{X}^2\right)$. The assumptions of Theorem 2 are satisfied with $\delta = 2$ implying convergence rate $O(n^{-3/4})$, and Theorem 3 clearly implies convergence rate $O(n^{-1})$ but we can do even better.

Denoting $z = \sum_{k=1}^{\infty} \alpha_k e_k$ we have $\|z\|_{\mathcal{D}((-A)^\nu)}^2 = \sum_{k=1}^{\infty} k^{4\nu} \alpha_k^2$. For the output it holds that

$$\|Cz\|^2 = 2 \left( \sum_{k=1}^{\infty} \pi k \alpha_k \right)^2 \leq 2\pi \sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} \sum_{k=1}^{\infty} k^{3+\epsilon} \alpha_k^2$$

from which it can be deduced that $C \in L(\mathcal{D}((-A)^\nu), \mathcal{Y})$ for $\nu > 3/4$. Now Theorem 4 implies convergence rate $O(n^{-3/2+\epsilon})$ for $E\left(\|\hat{z}_{T,n} - \hat{z}(T)\|_{\mathcal{X}}^2\right)$ with $\epsilon > 0$—of course, with a multiplicative constant that tends to infinity as $\epsilon \to 0$.

The convergence was studied numerically in the same way as in Example 3.1, that is, using a function basis consisting of the first 400 Fourier components $\sin(\pi k x)$. The convergence was observed at five different times $T = 0.25, 0.5, 1, 2.5,$ and $5$, when the time discretization step was $\Delta t = \frac{0.25}{n}$ with $n = 1, 2, 4, 8, 10, 20, 25, 40, 50, 100$. Other used parameters were $\rho = 0.001$, and the measurement noise covariance was $R = I$. The initial state (in the Fourier coordinates) was drawn from $N(0, P)$ where $P = \text{diag} \left\{ \left( \frac{1}{(k+0.001k^2)^2} \right)_{k=1}^{400} \right\}$, satisfying $z_0 \in \mathcal{D}((-A)^\eta)$ for $\eta < 1$ almost surely. The convergence plot for $E\left(\|\hat{z}_{T,n} - \hat{z}(T)\|_{\mathcal{X}}^2\right)$ estimated from 10 replicates is shown in logarithmic scale in Fig. 3. The plot suggests that the convergence might in
Lemma 2

For fixed $j$, it holds that was assumed to be bounded, and the proof follows the same outline.

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In this section, we shall first study the effect of the input noise $u$ in (1), and later, in Sect. 4.1, we study the effect of the zero-order hold discretization of a deterministic input. The process noise is studied only under the assumption of an admissible observation operator $C$. The main theorem is a generalization of [1, Theorem 2] where $C$ was assumed to be bounded, and the proof follows the same outline.

In the cases without input noise, the state $z(t)$ was parameterized by the initial state through $z(t) = e^{At}x$. Now we define the solution operator $S(t)$ through

$$S(t) : [x, u] \mapsto e^{At}x + \int_0^t e^{A(t-s)}Bdu(s).$$

Formally we first define $S(t)$ for $u \in H^1(0, T; \mathcal{U})$ and then extend it for Brownian motion using a Wiener integral. Then the solution to (1) is given by $z(t) = S(t)[x, u]$ and the conditional expectation over a given sigma algebra $\sigma$ by $\mathbb{E}(z(t)|\sigma) = S(t)\mathbb{E}([x, u]|\sigma)$. In the following theorem, we consider virtually estimating $[x, u]$ using the outputs $y(t_j)$ and then the state estimate is obtained by $S(T)$. The well-posedness of $S(t)[0, \tilde{u}_j]$ for $\tilde{u}_j := \mathbb{E}(u|[y(t)_i, i = 1, \ldots, j])$ is established next.

Lemma 2 For fixed $j$, it holds that $\tilde{u}_j(\cdot) \in H^1(0, T; \mathcal{U})$.

Proof For fixed $t \in [0, T]$ and $j \in \mathbb{N}$, it holds that

$$\mathbb{E}(u(t)|y(T_j)) = \text{Cov}[u(t), y(T_j)] \text{Cov}[y(T_j), y(T_j)]^{-1} y(T_j).$$

Here $\text{Cov}[y(T_j), y(T_j)]^{-1} y(T_j)$ is a well-defined finite-dimensional vector, and so we only need to differentiate $\text{Cov}[u(t), y(T_j)] \in \mathcal{L}(\mathbb{Y}^j, \mathcal{U})$ which is equivalent to the differentiability of its adjoint, $\text{Cov}[y(T_j), u(t)]$ which is a block operator with components $\text{Cov}[y(t_i), u(t)]$ for $i = 1, \ldots, j$. We show that each block has a bounded derivative.

Since

$$y(t_i) = C \int_0^{t_i} e^{As}x \, ds + C \int_0^{t_i} \int_0^s e^{A(s-r)}Bdu(r) \, ds + w(t_i)$$

$$= C \int_0^{t_i} e^{As}x \, ds + C \int_0^{t_i} \int_r^{t_i} e^{A(s-r)}Bds \, du(r) + w(t_i),$$

it holds that

$$\text{Cov}[y(t_i), u(t)] = C \int_0^{t_i} \int_r^{t_i} e^{A(s-r)}BQds \, dr$$
and therefore
\[
\frac{d}{dt} \text{Cov} [y(t_i), u(t)] = \begin{cases} \int_{t_i}^{t} C e^{A(s-t)} B Q ds, & \text{if } t < t_i, \\ 0, & \text{if } t \geq t_i \end{cases}
\]
which is bounded for \( t, t_i \in [0, T] \), concluding the result. \( \square \)

We are now ready to proceed to the main result of the paper.

**Theorem 5** Assume that \( C \) is an admissible observation operator, that is, for all \( T \geq 0 \) there exists \( H_T \geq 0 \) such that \( \int_0^T \| C e^{A t} x \|_X^2 \, dt \leq H_T^2 \| x \|_X^2 \). In addition, assume that one set of assumptions in Theorems 1–4 are satisfied. Then for \( \hat{z}_{T,n} \) and \( \hat{z}(T) \) defined in (2) it holds that

\[
\mathbb{E} \left( \| \hat{z}_{T,n} - \hat{z}(T) \|_X^2 \right) \leq \frac{M_1(T)}{n} + \frac{M_2(T)}{n^2} + \text{err}_x
\]

where \( M_1(T) \) and \( M_2(T) \) are given below in (19) and \( \text{err}_x \) is given by respective Theorems 1–4.

**Proof** The proof follows exactly the same outline as the proof of [1, Theorem 2], and therefore we will only outline the deviations from that proof. The most significant difference is in the treatment of the two terms

\[
(I) := \frac{C}{2} \int_0^{t_j+1} \int_{t_j+1}^{t_j+1+h} (t_j + h - s) A e^{A(s-r)} B ds du(r)
\]

and

\[
(II) := \frac{C}{2} \int_{t_j+1}^{t_j+1+h} \int_{r}^{t_j+1} (t_j + h - s) A e^{A(s-r)} B ds du(r).
\]

For the first term, it holds that

\[
\text{Cov} [(I), (I)] = \frac{1}{4} \int_0^{t_j+1} \int_{t_j+1}^{t_j+1+h} \int_{t_j+1}^{t_j+1+h} (t_j + h - s)(t_j + h - r) \\
\times C e^{A(s-t)} AB Q B^* A^* e^{A^*(r-t)} C^* dr ds dt,
\]

and from this,

\[
\mathbb{E} \left( \| (I) \|_Y^2 \right) = \text{tr} \left( \text{Cov} [(I), (I)] \right)
\leq \frac{\text{tr}(Q)}{4} \int_0^{t_j+1} \left( \int_{t_j+1}^{t_j+1+h} (t_j + h - s) \| C e^{A(s-t)} AB \|_{L(\mathcal{U},\mathcal{Y})} ds \right)^2 dt
\leq \frac{h^3}{12} \text{tr}(Q) \int_0^{t_j+1} \| C e^{A(s-t)} AB \|_{L(\mathcal{U},\mathcal{Y})}^2 ds dt
\]
where the second inequality follows by the Cauchy–Schwartz inequality. Now make a change of variables \( \tau = s - t \) and note that for a fixed \( s \in [t_{j+1}, t_{j+1} + h] \), it holds that \( \tau \in [s - t_{j+1}, s] \subset [0, T] \), yielding finally

\[
\mathbb{E}\left( \| (I) \|_{\mathcal{Y}}^2 \right) \leq \frac{h^4}{12} \text{tr}(Q) \left\| Ce^{As} AB \right\|_{L^2(0, T; L(\mathcal{U}, \mathcal{Y}))}^2
\]

where \( \left\| Ce^{As} AB \right\|_{L^2(0, T; L(\mathcal{U}, \mathcal{Y}))}^2 \leq H_T^2 \left\| B \right\|_{L(\mathcal{U}, \mathcal{D}(A))}^2 \) due to the admissibility assumption.

For the second term, we have

\[
\text{Cov} [(II), (II)] = \frac{1}{4} \int_{t_{j+1}}^{t_{j+1} + h} \int_t^{t_{j+1} + h} \int_t^{t_{j+1} + h} (t_{j+1} + h - s)(t_{j+1} + h - r) \times Ce^{A(s-t)} AB Q B^* A^* e^{A^*(r-t)} C^* dr \, ds \, dt
\]

and, as with the first term,

\[
\mathbb{E}\left( \| (II) \|_{\mathcal{Y}}^2 \right) = \text{tr}(\text{Cov} [(II), (II)]) \leq \frac{h^4}{12} \text{tr}(Q) \left\| Ce^{As} AB \right\|_{L^2(0, T; L(\mathcal{U}, \mathcal{Y}))}^2.
\]

In (I), \( r \in [0, t_{j+1}] \) and in (II), \( r \in [t_{j+1}, t_{j+1} + h] \), and thus they are independent. Therefore,

\[
\mathbb{E}\left( \| (I) + (II) \|_{\mathcal{Y}}^2 \right) \leq \frac{h^4}{6} \text{tr}(Q) \left\| Ce^{As} AB \right\|_{L^2(0, T; L(\mathcal{U}, \mathcal{Y}))}^2.
\]

Another small difference compared to [1, Theorem 2] is the bound used for the feedthrough term:

\[
\mathbb{E}\left( \left\| \frac{C}{2} \int_{t_{j+1}}^{t_{j+1} + h} B(u(t) - u(t_{j+1})) dt \right\|_{\mathcal{Y}}^2 \right) \leq \frac{h^3}{12} \left\| C \right\|_{L(\mathcal{D}(A), \mathcal{Y})}^2 \left\| B \right\|_{L(\mathcal{U}, \mathcal{D}(A))}^2 \text{tr}(Q).
\]

With these changes the increment bound becomes

\[
\mathbb{E}\left( \| \hat{z}_{j+1} - z_j \|_{\mathcal{X}}^2 \right) \leq \frac{2h^3}{\min(\text{eig}(R))} \left\| Ce^{As} AB \right\|_{L^2(0, T; L(\mathcal{U}, \mathcal{Y}))}^2 \text{tr}(Q) \mathbb{E}\left( \| \hat{z}_{T,n} - z(T) \|_{\mathcal{X}}^2 \right) + \frac{h^2}{\min(\text{eig}(R))} \left\| C \right\|_{L(\mathcal{D}(A), \mathcal{Y})}^2 \left\| B \right\|_{L(\mathcal{U}, \mathcal{D}(A))}^2 \text{tr}(Q) \mathbb{E}\left( \| \hat{z}_{T,n} - z(T) \|_{\mathcal{X}}^2 \right) + \text{incr}_{[x, j+1]}
\]

where \( \text{incr}_{[x, j+1]} \) is the contribution of the initial state \( x \) obtained from one of Theorems 1–4.
This bound for the increment $E\left(\|\tilde{z}_{j+1} - \tilde{z}_j\|^2_\mathcal{X}\right)$ is multiplied by $\frac{T}{2h}$ to get an upper bound for the increments corresponding to one value $h$, that is, one $K$ (see Fig. 1):

$$\sum_{j=2^{K-1}n}^{2^Kn-1} E\left(\|\tilde{z}_{j+1} - \tilde{z}_j\|^2_\mathcal{X}\right) \leq \frac{M_1(T)}{2^Kn} + 3M_2(T)(2^Kn)^2 + \text{incr}_{[x,j+1]}$$

with

$$\begin{cases}
M_1(T) = \frac{T^2\text{tr}(Q)}{2\min(\text{eig}(R))} \|C\|^2_\mathcal{L}(\mathcal{D}(A),\mathcal{Y}) \|B\|^2_\mathcal{L}(\mathcal{D}(A)) \|eA\|^2_\mathcal{X} \|\text{incr}_{[x,j+1]}\|_\mathcal{X}, \\
M_2(T) = \frac{T^3\text{tr}(Q)}{3\min(\text{eig}(R))} H_T^2 \|B\|^2_\mathcal{L}(\mathcal{D}(A)) E\left(\|\tilde{z}_{T,n} - z(T)\|^2_\mathcal{X}\right).
\end{cases}$$

Now summing up (18) for $K = 1, 2, \ldots$ yields the result. \hfill \Box

### 4.1 Deterministic input discretization

As mentioned before, deterministic input can be included, and by superposition, it does not have any effect on the error if it is done accurately. However, in the discrete-time Kalman filter implementation, the deterministic input is often sampled using zero-order hold, for example. Let us end the article by a formal discussion on the effect of the zero-order hold sampling, as opposed to the continuous signal used in the continuous-time filter.

In the presence of a deterministic input $\bar{u}$, the dynamics of the continuous-time filter are given by

$$d\hat{z}(t) = (A\hat{z}(t) + B\bar{u}(t))dt + K(t)(dy(t) - C\hat{z}(t)dt).$$

Using the zero-order hold discretization, the discrete-time filter dynamics are given by

$$\tilde{z}_j = A_d\tilde{z}_{j-1} + B_d\bar{u}_j + K_j(y_j - C_d\tilde{z}_{j-1})$$

where $A_d = e^{A\Delta t}$, $B_d = \Delta t B$, $C_d = C \int_0^{\Delta t} e^{As}ds$, and $\bar{u}_j = \bar{u}(t_{j-1})$. The filters are, of course, initialized from the same state.

The continuous-time filter can be accurately discretized using the semigroup perturbation formula [9, Corollary III.1.7] [5, Theorem 3.9] to get

$$\hat{z}(t_j) = A_d\hat{z}(t_{j-1}) - \int_{t_{j-1}}^{t_j} e^{A(t_j-s)}K(r)CT(r, t_{j-1})\hat{z}(t_{j-1})ds$$

$$+ \int_{t_{j-1}}^{t_j} T(t_j, s)(B\bar{u}(s)ds + K(s)dy(s))$$

\footnote{Not considering solvability issues of the associated Riccati differential equation.}
where $T(t, s)$ is the time evolution operator generated by $A - K(\cdot)C$. Consider then the discrepancy $e(j) := \tilde{z}(t_j) - \tilde{z}_j$. By superposition, it can be presented as a sum of components arising from the measurement $y$ and the discretized input $\bar{u}$, that is, denote $e(j) = e_y(j) + e_{\bar{u}}(j)$. All the theorems in this article give bounds for $\mathbb{E}(\|e_y\|_X^2)$ and all these results remain valid even with the presence of another error source.

In this section, we concentrate on the remaining discrepancy $e_{\bar{u}}(j)$ whose dynamics are given by

$$e_{\bar{u}}(j) = (A_d - K_j C_d) e_{\bar{u}}(j - 1) + (K_j C_d - \tilde{K}_j) \tilde{z}(t_{j-1}) + \int_{t_{j-1}}^{t_j} T(t, s) B \tilde{u}(s) ds - \Delta t (I - K_j C_d) B \bar{u}(t_{j-1})$$

(21)

where $\tilde{K}_j$ is obtained from (20),

$$\tilde{K}_j := \int_{t_{j-1}}^{t_j} e^{A(t_j-s)} K(s) C T(s, t_{j-1}) ds.$$  

(22)

Assuming $B \in \mathcal{L}(\mathcal{U}, \mathcal{D}(A))$, it holds that $T(t, s) B = B + r(B)$ where the residual satisfies $\|r(B)\|_{\mathcal{L}(\mathcal{U}, X)} = O(\Delta t)$, and similarly $\|K_j C_d B\|_{\mathcal{L}(\mathcal{U}, X)} = O(\Delta t)$. Therefore, the last term in (21) can be bounded by

$$\left\| \int_{t_{j-1}}^{t_j} T(t, s) B \tilde{u}(s) ds - \Delta t (I - K_j C_d) B \bar{u}(t_{j-1}) \right\|_X \leq \left\| \int_{t_{j-1}}^{t_j} T(t, s) B(\bar{u}(s) - \bar{u}(t_{j-1})) ds \right\|_X + O(\Delta t^2)$$

where the first term is of order $O(\Delta t^{3/2})$ assuming $\bar{u} \in H^1[0, T]$, or $O(\Delta t^2)$ assuming $\bar{u}$ is Lipschitz continuous.

Finally, what is left is to find a bound for the second term on the RHS in (21). To this end, recall that the discrete-time Kalman gain is given by

$$K_j = A_d P_{j-1} C^*_d (C_d P_{j-1} C^*_d + \Delta t R)^{-1}$$

where the error covariance is given by the recursion

$$P_j = A_d P_{j-1} A^*_d + \Delta t B Q B^* - A_d P_{j-1} C^*_d (C_d P_{j-1} C^*_d + \Delta t R)^{-1} C^*_d P_{j-1} A^*_d.$$

The continuous-time Kalman gain is $K(t) = P(t) C^* R^{-1}$ where $P(t)$ is the solution of the Riccati differential equation

$$\frac{d}{dt} P(t) = A P(t) + P(t) A^* + B Q B^* - P(t) C^* R^{-1} C P(t).$$
Now if \( z_0 \in \mathcal{D}(A) \) almost surely, it means that \( P_0 \) is a continuous operator from \( \mathcal{D}(A)’ \), the dual of \( \mathcal{D}(A) \) (using the state space \( \mathcal{X} \) as a pivot space), to \( \mathcal{D}(A) \). Since \( B \in \mathcal{L}(\mathcal{U}, \mathcal{D}(A)) \), this is true for \( P(t) \) for all \( t \). From the Riccati differential equation, we get a useful integral equation for \( t \geq s \)

\[
P(t) = e^{A(t-s)}P(s)e^{A^*(t-s)} + \int_s^t e^{A(t-r)}(BQB^* - P(r)C^*R^{-1}CP(r))e^{A^*(t-r)}dr.
\]

Now substituting this with \( s = t_j - 1 \) into \( K(t) = P(t)C^*R^{-1} \), and this, in turn to (22) yields

\[
\tilde{K}_j = \int_{t_j - 1}^{t_j} A_d P(t_j - 1) e^{A^*(s - t_j - 1)} + \int_{t_j - 1}^s e^{A(t_j - r)}
\]

\[
\times (BQB^* - P(r)C^*R^{-1}CP(r))e^{A^*(s-r)}dr \right] C^*R^{-1}CT(s, t_j - 1)ds.
\]

Since \( P(t) \in \mathcal{L}(\mathcal{D}(A)’, \mathcal{D}(A)) \), the \( \mathcal{L}(\mathcal{D}(A), \mathcal{X}) \)-norm of the second term is of order \( \mathcal{O}(\Delta t^2) \). The second term on the RHS in (21) is then

\[
(K_jC_d - \tilde{K}_j)\hat{z}(t_j - 1)
\]

\[
= A_d(P(t_j - 1) - P_j - 1) \int_{t_j - 1}^{t_j} e^{A^*(s - t_j - 1)} C^*R^{-1}CT(s, t_j - 1)\hat{z}(t_j - 1)ds
\]

\[
+ A_dP_j - 1 \left[ \int_{t_j - 1}^{t_j} e^{A^*(s - t_j - 1)} C^*R^{-1}CT(s, t_j - 1)ds
\]

\[
- C_d^* \left( \frac{1}{\Delta t} C_d P_j - 1 C_d^* + R \right)^{-1} \frac{C_d}{\Delta t} \right)\hat{z}(t_j - 1) + \mathcal{O}(\Delta t^2).
\]

It holds that \( \|P(t_j - 1) - P_j - 1\|_{\mathcal{L}(\mathcal{X})} \leq \text{tr}(P_j - 1 - P(t_j - 1)) \), and in fact, \( \text{tr}(P_j - 1 - P(t_j - 1)) = \mathbb{E}\left(\|e_\gamma(j)\|^2_{\mathcal{X}}\right) = \mathcal{O}(\Delta t^\beta) \). For this, we have computed bounds in the theorems of this article and the value \( \beta \) depends on the exact theorem. Further, in the last term, it holds that \( \left\| \left( \frac{1}{\Delta t} C_d P_j - 1 C_d^* + R \right)^{-1} - R^{-1} \right\|_{\mathcal{L}(\mathcal{Y})} = \mathcal{O}(\Delta t) \), and \( \left\| CT(s, t_j - 1)\hat{z}(t_j - 1) - \frac{C_d}{\Delta t}\hat{z}(t_j - 1) \right\|_{\mathcal{Y}} = \mathcal{O}(\Delta t) \) assuming \( \|\hat{z}(t_j - 1)\|_{\mathcal{D}(A)} \) is uniformly bounded. This holds if \( \hat{z}(0) \in \mathcal{D}(A) \) since \( B \in \mathcal{L}(\mathcal{U}, \mathcal{D}(A)) \). Then, it holds that

\[
\| (K_jC_d - \tilde{K}_j)\hat{z}(t_j - 1) \|_{\mathcal{X}} = \mathcal{O}(\Delta t^{\beta + 1}).
\]

Combining (21) with the obtained bounds yields \( \|e_\gamma(j)\| = \mathcal{O}(\Delta t^{\min(1, \beta)}) \), provided that \( \hat{u} \) is sufficiently smooth (e.g., Lipschitz continuous). If \( \hat{u} \in H^1[0, T] \) but not Lipschitz continuous, then \( \min(1, \beta) \) is replaced by \( \min(1/2, \beta) \).
Since $\bar{u}$ is deterministic, it holds that $\mathbb{E}(\|e(j)\|_{\mathcal{X}}^2) = \mathbb{E}(\|e_y(j)\|_{\mathcal{X}}^2) + \|\bar{u}(j)\|_{\mathcal{X}}^2$. In conclusion, although the input discretization introduces an additional error source, it does not affect the convergence rate (cf. Theorem 5), provided that the input is smooth enough.

5 Discussion

In this paper, we extended the convergence results presented by the author in [1]. There the convergence rate estimates for $\mathbb{E}(\|\hat{z}_{T,n} - \hat{z}(T)\|_{\mathcal{X}}^2)$ were shown for finite-dimensional systems and infinite-dimensional systems with bounded observation operator $C$. Now convergence rate estimates were found for systems with unbounded observation operators with some additional assumptions on the system operators. Firstly, a result was shown for systems with diagonalizable main operators $A$. In this case, some additional assumptions were needed, including a slightly nonstandard assumption on the output operator (assumption (iii) in Theorem 1). In the problems arising from PDEs on one-dimensional spatial domains, this is not a big problem but unfortunately with more complicated systems, finding a suitable $\gamma$ might be practically impossible. The spectral asymptotics, on the other hand, is an extensively studied field—so much so that it has even been a subject of a few books, such as [17] by Levendorskiì and [20] Safarov and Vassiliev. Theorem 2 treats the case with assuming essentially just the admissibility of the observation operator $C$. Two results were shown for systems with analytic semigroups, in which case the other technical assumptions were not needed. The effect of the input noise was studied in Theorem 5, which extends the corresponding earlier result [1, Theorem 2] to admissible observation operators.

In all results of the paper—except for the analytic semigroup case without input noise—the convergence rate estimates are of the form $\frac{MT^{k+1}}{\eta^k} = MT \Delta t^k$, meaning that the estimates deteriorate as $T$ grows. In general, this cannot be completely avoided because there is no guarantee that the output sampling does not cause essential loss of information. A result where the bound would not deteriorate as $T$ grows could be possible under some additional assumptions (like exponential stability of the system), but the long time behavior should be anyway studied by comparing the solutions of the corresponding discrete- and continuous-time algebraic Riccati equations.

We remark that in this paper, as well as in [1], it has been assumed that the time discretization can be done perfectly and the only error source is the time-sampling of the output signal $y$ that is defined in continuous time. Further research would be needed to estimate the error caused by approximate discretization schemes.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.
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