GENERALIZED EXTENSION OF WATSON’S THEOREM FOR THE SERIES $\binom{3}{2}(1)$

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ABSTRACT. The $\binom{3}{2}$ hypergeometric function plays a very significant role in the theory of hypergeometric and generalized hypergeometric series. Despite that $\binom{3}{2}$ hypergeometric function has several applications in mathematics, also it has a lot of applications in physics and statistics.

The fundamental purpose of this research paper is to find out the explicit expression of the $\binom{3}{2}$ Watson’s classical summation theorem of the form:

$$\binom{3}{2} \begin{bmatrix} a, & b, & c; \\
\frac{1}{2}(a + b + i + 1), & 2c + j; \\
1 \end{bmatrix}$$

with arbitrary $i$ and $j$, where for $i = j = 0$, we get the well known Watson’s theorem for the series $\binom{3}{2}(1)$.

1. INTRODUCTION

The generalized hypergeometric function with $p$ numerator and $q$ denominator parameters is defined by [1]

$$\binom{p}{q} \begin{bmatrix} a_1, ..., a_p; \\
b_1, ..., b_q; \\
z \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},$$

(1.1)

where $(\alpha)_n$ denotes the shifted factorial defined for any complex number $\alpha$, by

$$(\alpha)_n = \left\{ \begin{array}{ll}
\alpha(\alpha + 1) \cdots (\alpha + n - 1); & n = 1, 2, 3, \\
1; & n = 0.
\end{array} \right.$$  

Using the main property $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$, $(\alpha)_n$ can be written as

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}.$$

It should be noted here that whenever hypergeometric and generalized hypergeometric functions summarized to be represented in term of Gamma function, the outcomes are critical from a theoretical and an appropriate point of view. Only some of summation theorems are available in literature, and it is Known as the classical summation theorems such as Gauss, Gauss’s second, Kummer, and Bailey for the series $\binom{2}{1}$, Watson, Dixon, and Whipple for the series $\binom{3}{2}$.

The $\binom{3}{2}$ hypergeometric function plays a very remarkable part in the theory of hypergeometric and generalized hypergeometric series. Despite that $\binom{3}{2}$ hypergeometric function has several applications in mathematics such as in:

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1. Differential Equations: Since the generalized hypergeometric function and the monodromy of generalized hypergeometric function represented solutions of Picard-Fuchs equations, which used to solve many problems in classical mechanics and mathematical physics. For more information about such applications, see [17].

2. Conformal Mapping: Since in [18] de Branges used the inequality

\[ \begin{align*}
\binom{-n, n + \alpha, \frac{1}{2}(\alpha + 1)}{\alpha + 1, \frac{1}{2}(\alpha + 3)}^3_{2} x > 0,
\end{align*} \]

where \( 0 \leq x < 1, \alpha > -2 \) and \( n = 0, 1, 2, \ldots \), to proof the Bieberbach conjecture. Where the proof of this inequality is given in [29], see also [30].

3. Combinatorics and Number Theory: Many combinatorial identities are particular case of hypergeometric identities. For more details about such applications, see [19], also it has a lot of applications in physics and statistics such as:

1. Random Walks: Generalized hypergeometric function and Appell features appear in the assessment of the so-referred to as Watson integrals which symbolize the handiest possible lattice walks. They are also probably useful for the solution of extra complicated constrained lattice stroll issues. For further information about such application see [20].

2. Loop Integrals in Feynman Diagrams: Appell hypergeometric function gave One-loop integrals in Feynman diagrams also extension to two-loop, [21][22].

3. 3j, 6j and 9j Symbols: Since we can use \( \binom{3}{2} \) functions with a unit argument to define the 3j symbols, which assume a critical part in the decay of reducible representations of the turning group into irreducible representations. Also recently, special cases of the 9j symbols are \( \binom{3}{2} \) functions with a unit argument. Many of combinatorial identities are individual cases of hypergeometric identities, see [23][24].

Now, we begin by introducing the classical Watson’s summation theorem \( \binom{3}{2} \) of unit argument [4], which takes the form:

\[ \binom{a, b, c}{\frac{1}{2}(a + b + 1), 2c}^3_{2} 1 \]

\[ = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c + \frac{1}{2})\Gamma\left(\frac{a + b + 1}{2} + \frac{1}{2}\right)\Gamma\left(c - \frac{a + b + 1}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{a + b + 1}{2} + \frac{1}{2}\right)\Gamma\left(c + \frac{1}{2}\right)\Gamma\left(c - \frac{a + b + 1}{2} + \frac{1}{2}\right)} \] (1.2)

where \( \text{Re}(2c - a - b) > -1 \).

In [2], Watson gave the demonstrate of (1.2) when one of the parameters \( a \) or \( b \) is a negative integer, and subsequently was established more generally in the non-terminating case by Whipple in [3].
The standard prove of (1.2) given in [4, p.149] and [5, p.54], depend on the following transformation due to Thomae [6]:

\[
\begin{align*}
3F2 \left[ \begin{array}{ccc}
    a, & b, & c \\
    d, & e
\end{array}; 1 \right] \\
= \frac{\Gamma(d) \Gamma(c) \Gamma(s)}{\Gamma(a) \Gamma(b + s) \Gamma(c + s)} \times 3F2 \left[ \begin{array}{ccc}
    d - a, & e - a, & s \\
    b + s, & c + s
\end{array}; 1 \right]
\end{align*}
\]

where \( s = d + e - a - b - c \), \( d = \frac{1}{2}(a + b + 1) \) and \( e = 2c \).

MacRobert [7] provided an alternative and more interested proof, by using the quadratic transformation for the Gauss’s hypergeometric function [1, Theorem 25, p. 67]:

\[
2F1 \left[ \begin{array}{ccc}
    2a, & 2b, & x \\
    a + b + \frac{1}{2}
\end{array}; x \right] = 2F1 \left[ \begin{array}{ccc}
    a, & b, & 4x(1-x) \\
    a + b + \frac{1}{2}
\end{array}; 4x(1-x) \right]
\]

valid for \(|x| < 1 \) and \(|4x(1-x)| < 1\).

Recently Rathie and Paris [8] gave a basic confirmation of (1.2) that just depends on the Gauss summation theorem for the \(2F1\) hypergeometric function, namely, [9]:

\[
2F1 \left[ \begin{array}{ccc}
    a, & b, & c \\
    c
\end{array}; 1 \right] = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}
\]

where \( Re(c - a - b) > 0 \), while Rakha in [10] gave an extremely straightforward proof of (1.2) by using the Gauss’s second summation theorem:

\[
3F2 \left[ \begin{array}{ccc}
    a, & b, & c \\
    d, & 2b
\end{array}; z \right] \\
= \frac{\Gamma(d)}{\Gamma(c) \Gamma(d - s)} \int_0^1 t^{c-1 - (d - c - 1)} dt 
\]

In 1987, Lavoie [11], obtained the following two summation formulas

\[
3F2 \left[ \begin{array}{ccc}
    a, & b, & c \\
    \frac{1}{2}(a + b + 1), & 2c + 1
\end{array}; 1 \right] \\
= \frac{2^{a+b-2} \Gamma(c + \frac{1}{2}) \Gamma(\frac{a}{2} + \frac{1}{2}) \Gamma(\frac{b}{2} + \frac{1}{2}) \Gamma(\frac{c}{2} - \frac{a}{2} - \frac{b}{2} + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(a) \Gamma(b)}
\]

\[
\times \left\{ \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(c - \frac{a}{2} + \frac{1}{2}) \Gamma(c - \frac{b}{2} + \frac{1}{2})} - \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(c - \frac{a}{2} + 1) \Gamma(c - \frac{b}{2} + 1)} \right\}
\]

(1.3)

provided \( Re(2c - a - b) > -3 \), and
\[
\begin{align*}
3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a+b+1), 2c-1 \end{array} ; 1 \right] \\
= \frac{2^{a+b-2} \Gamma(c-\frac{3}{2}) \Gamma(\frac{a}{2}+\frac{1}{2}) \Gamma(\frac{b}{2}+\frac{1}{2}) \Gamma(c-\frac{a}{2}-\frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(a) \Gamma(b)} \\
\times \left\{ \frac{\Gamma(\frac{a}{2}) \Gamma(\frac{b}{2})}{\Gamma(c-\frac{a}{2}-\frac{1}{2}) \Gamma(c-\frac{b}{2}-\frac{1}{2})} \right\}
\end{align*}
\]

provided \( \text{Re}(2c-a-b) > 1. \)

In 1992, Lavoie et al. in [12], took out explicit expression of the series

\[
3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a+b+i+1), 2c+j \end{array} ; 1 \right]
\]

for \( i, j = 0, \pm 1, \pm 2, \) where at \( i = j = 0 \) we obtain (1.2).

Other remarkable results of such computations, are:

1. Stainislaw [13], in 1997 gave an analytical formula for (1.5) with a fixed \( j \) and arbitrary \( i. \)
2. Kim et al., in [14], have obtained the above result (1.5) for \( j = 0 \) and \( i = 0, \pm 1, \pm 2, \ldots, \pm 5. \)
3. Chu [15], in 2012, investigates the generalized Watson’s series with two extra integer parameters by combining the linearization method with Dougall’s sum for well-poised \( 5F_4 \)-series.
4. in 2013, Rakha et al. in [16], established result (1.5) for \( i = 0, \pm 1, \pm 2, \ldots, \pm 5; j = 0, \pm 1, \pm 2. \)

The major purpose of this paper is to find explicit extensions of the classical Watson’s summation theorem (1.5) for any \( i \) and \( j, \) to have more summations theorems as well as more contiguous relations about the \( 3F_2(1), \) hypergeometric series.

2. Main Results

Let us consider that

\[
f_{i,j}(a,b,c) = 3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{a+b+i+1}{2} \\ \end{array} ; 2c+j \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(\frac{a+b+i+1}{2})_n (2c+j)_n n!}.
\]
It is clear that

\[(2c + j) \, f_{i,j+1}(a, b, c)\]

\[= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n (2c + j)}{(a+b+i+1)_n (2c+j+1)_n n!} \cdot \frac{1}{n!}\]

\[= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n (2c + j + n)}{(a+b+i+1)_n (2c+j+1)_n n!} - \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n n}{(a+b+i+1)_n (2c+j+1)_n n!} \]

\[= (2c + j) \, f_{i,j}(a, b, c) - \frac{2abc}{(a + b + i + 1)(2c + j + 1)} \sum_{n=0}^{\infty} \frac{(a+1)_{n+1} (b+1)_{n+1} (c+1)_{n+1}}{(a+b+i+1)_{n+1} (2c+j+2)_{n+1}} \frac{1}{n!}\]

\[= (2c + j) \, f_{i,j}(a, b, c) - \frac{2abc}{(a + b + i + 1)(2c + j + 1)} f_{i,j}(a + 1, b + 1, c + 1). \quad (2.2)\]

from which, we conclude the following general important main relation:

\[(2c + j) \, f_{i,j+1}(a, b, c)\]

\[= (2c + j) \, f_{i,j}(a, b, c) - \frac{2abc}{(a + b + i + 1)(2c + j + 1)} f_{i,j}(a + 1, b + 1, c + 1). \quad (2.2)\]

So, if we know \(f_{i,0}(a, b, c)\), we can generate \(f_{i,j}(a, b, c)\) for any values of \(i\) and \(j\).

2.1. Special Cases.

(1) When \(i = j = 0\) in (2.2), we obtain

\[\mathbf{3F}_{2} \left[ \begin{array}{ccc} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 \\ \end{array} \right]^{\frac{1}{2}(a+b+3), & 2c + 2}; 1 \]

\[= \mathbf{3F}_{2} \left[ \begin{array}{ccc} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \\ \end{array} \right]^{\frac{1}{2}(a+b+3), & 2c + 2}; 1 \]

\[- \frac{ab}{(2c+1)(a+b+1)} \mathbf{3F}_{2} \left[ \begin{array}{ccc} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+3), & 2c + 2 \\ \end{array} \right]; 1 \]

which appeared in [10] Eq.(3.18), pp. 229], [11] Result (2), pp.269], [12] Result (1),pp.24], [23] Eq.(4.4).pp.12] and [28] Theorem 4, p.147].

(a) In such a case, the result when \(a = 1, b = 1\) and \(c = 1\), appeared in [26] Result 209, p. 459].

(b) In such a case the result when \(a = \frac{2}{3}, b = \frac{4}{3}\) and \(c = 1\), appeared in [27] Eq. 26].
(2) When \( i = 1 \) and \( j = 0 \) in (2.2), we obtain

\[
\begin{align*}
3F_2 \left[ \begin{array}{ccc}
    a, & b, & c \\
    \frac{1}{2}(a + b + 2), & 2c + 1 \\
    \frac{1}{2}(a + b + 2), & 2c
\end{array}; 1 \right] \\
= 3F_2 \left[ \begin{array}{ccc}
    a, & b, & c \\
    a + 1, & b + 1, & c + 1 \\
    \frac{1}{2}(a + b + 4), & 2c + 2
\end{array}; 1 \right] \\
- \frac{ab}{(2c + 1)(a + b + 2)^3} \quad 3F_2 \left[ \begin{array}{ccc}
    a + 1, & b + 1, & c + 1 \\
    \frac{1}{2}(a + b + 4), & 2c + 2
\end{array}; 1 \right]
\end{align*}
\]

which appeared in [16, Eq.(3.18), pp.229], [25, Eq.(4.5), pp.12] and [26, Results 187, 188 & 211, pages 458, 458 & 459], respectively.

In such a case, the results when \( a = \frac{1}{2}, b = 1, c = \frac{3}{4} \); \( a = \frac{1}{2}, b = \frac{3}{8}, c = 1 \) and \( a = b = c = 1 \) appeared in [26, Results 204, 234 & 242, pages 459 & 460], respectively.

(3) When \( i = 2 \) and \( j = 0 \) in (2.2), we obtain

\[
\begin{align*}
3F_2 \left[ \begin{array}{ccc}
    a, & b, & c \\
    \frac{1}{2}(a + b + 3), & 2c + 1 \\
    \frac{1}{2}(a + b + 3), & 2c
\end{array}; 1 \right] \\
= 3F_2 \left[ \begin{array}{ccc}
    a + 1, & b + 1, & c + 1 \\
    \frac{1}{2}(a + b + 4), & 2c + 2
\end{array}; 1 \right] \\
- \frac{ab}{(2c + 1)(a + b + 3)^3} \quad 3F_2 \left[ \begin{array}{ccc}
    a + 1, & b + 1, & c + 1 \\
    \frac{1}{2}(a + b + 4), & 2c + 2
\end{array}; 1 \right]
\end{align*}
\]

which appeared in [16, Eq.3.18, pp.229], [25, Eq.(4.5), pp.12] and [12, Result (1), pp.24].

In such a case, the results when \( a = 1, b = \frac{3}{2}, c = \frac{3}{4} \); \( a = 1, b = 2, c = 1 \) and \( a = \frac{3}{2}, b = \frac{3}{2}, c = 1 \) appeared in [26, Results 204, 234 & 242, pages 459 & 460], respectively.

(4) When \( i = 3 \) and \( j = 0 \) in (2.2), we obtain

\[
\begin{align*}
3F_2 \left[ \begin{array}{ccc}
    a, & b, & c \\
    \frac{1}{2}(a + b + 4), & 2c + 1 \\
    \frac{1}{2}(a + b + 4), & 2c
\end{array}; 1 \right] \\
= 3F_2 \left[ \begin{array}{ccc}
    a + 1, & b + 1, & c + 1 \\
    \frac{1}{2}(a + b + 6), & 2c + 2
\end{array}; 1 \right] \\
- \frac{ab}{(2c + 1)(a + b + 4)^3} \quad 3F_2 \left[ \begin{array}{ccc}
    a + 1, & b + 1, & c + 1 \\
    \frac{1}{2}(a + b + 6), & 2c + 2
\end{array}; 1 \right]
\end{align*}
\]

which appeared in [16, Eq.(3.18), pp.229] and [13, Result(2.23), pp.380].

In such a case, the result when \( a = 1, b = 3 \) and \( c = 1 \) appeared in [26, Result (237), p.460].
(5) When \( i = 4 \) and \( j = 0 \) in (2.2), we obtain

\[
\begin{align*}
3F_2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 5), & 2c + 1 \\
\end{array} ; 1 \right] \\
3F_2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 5), & 2c \\
\end{array} ; 1 \right] \\
- \frac{ab}{(2c + 1)(a + b + 5)} 3F_2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 7), & 2c + 2 \\
\end{array} ; 1 \right]
\end{align*}
\]

which appeared in [16, Eq. (3.18), pp.229].

In such a case, the result when \( a = 4, b = c = 1 \) appeared in [26] Results 240 & 239, page 460.

(6) When \( i = 5 \) and \( j = 0 \) in (2.2), we obtain

\[
\begin{align*}
3F_2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 6), & 2c + 1 \\
\end{array} ; 1 \right] \\
3F_2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 6), & 2c \\
\end{array} ; 1 \right] \\
- \frac{ab}{(2c + 1)(a + b + 6)} 3F_2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 8), & 2c + 2 \\
\end{array} ; 1 \right]
\end{align*}
\]

which appeared in [16] Eq.(3.18), pp.229] and [13] Result (2.22), pp.380].

In such a case, the results when \( a = c = 1, b = 3 \) and \( a = c = 1, b = 5 \) appeared in [25] Results 238 & 241, p. 460, respectively.

(7) When \( i = -1 \) and \( j = 0 \) in (2.2), we obtain

\[
\begin{align*}
3F_2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b), & 2c + 1 \\
\end{array} ; 1 \right] \\
3F_2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b), & 2c \\
\end{array} ; 1 \right] \\
- \frac{ab}{(2c + 1)(a + b)} 3F_2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 2), & 2c + 2 \\
\end{array} ; 1 \right]
\end{align*}
\]

which appeared in [12] Result (1), pp.24], [16] Eq.(3.18), pp.229], [15] Example (11), pp.9] and [28] Theorem (1), pp.143].
(8) When $i = -2$ and $j = 0$ in (2.2), we obtain

$$3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 1), 2c + 1 \end{array} ; 1 \right]$$

$$= 3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 1), 2c \end{array} ; 1 \right]$$

$$- \frac{ab}{(2c + 1)(a + b - 1)} 3F_2 \left[ \begin{array}{ccc} a + 1, b + 1, c + 1 \\ \frac{1}{2}(a + b + 1), 2c + 2 \end{array} ; 1 \right]$$

which appeared in [12, Result(1), pp.24] and [16, Eq. (3.18), pp.229].

(9) When $i = -3$ and $j = 0$ in (2.2), we obtain

$$3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 2), 2c + 1 \end{array} ; 1 \right]$$

$$= 3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 2), 2c \end{array} ; 1 \right]$$

$$- \frac{ab}{(2c + 1)(a + b - 2)} 3F_2 \left[ \begin{array}{ccc} a + 1, b + 1, c + 1 \\ \frac{1}{2}(a + b), 2c + 2 \end{array} ; 1 \right]$$

which appeared in [16, Eq.(3.18), pp.229].

(10) When $i = -4$ and $j = 0$ in (2.2), we obtain

$$3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 3), 2c + 1 \end{array} ; 1 \right]$$

$$= 3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 3), 2c \end{array} ; 1 \right]$$

$$- \frac{ab}{(2c + 1)(a + b - 3)} 3F_2 \left[ \begin{array}{ccc} a + 1, b + 1, c + 1 \\ \frac{1}{2}(a + b - 1), 2c + 2 \end{array} ; 1 \right]$$

which appeared in [16, Eq.(3.18), pp.229].

(11) When $i = -5$ and $j = 0$ in (2.2), we obtain

$$3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 4), 2c + 1 \end{array} ; 1 \right]$$

$$= 3F_2 \left[ \begin{array}{ccc} a, b, c \\ \frac{1}{2}(a + b - 4), 2c \end{array} ; 1 \right]$$

$$- \frac{ab}{(2c + 1)(a + b - 4)} 3F_2 \left[ \begin{array}{ccc} a + 1, b + 1, c + 1 \\ \frac{1}{2}(a + b - 2), 2c + 2 \end{array} ; 1 \right]$$
which appeared in [16] Eq.(3.18), pp.229).

(12) When \(i = 0\) and \(j = 1\) in (2.2), we obtain

\[
\frac{3}{2}F_2\left[ \begin{array}{ccc} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+2 \\ \end{array} \right ; 1
\]

\[
= \frac{3}{2}F_2\left[ \begin{array}{ccc} \frac{a}{2}(a+b+1), & 2c+1 \\ a, & b, & c \\ \end{array} \right ; 1
\]

\[
- \frac{2abc}{(2c+1)(2c+2)(a+b+1)} \frac{3}{2}F_2\left[ \begin{array}{ccc} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+3), & 2c+3 \\ \end{array} \right ; 1
\]

which appeared in [16] Eq.(3.20), pp. 230 and [12] Result (1), pp.24.

In such a case, the results when \(a = \frac{1}{2}, b = 1\) and \(c = \frac{1}{2}\); \(a = \frac{1}{2}, b = 1\) and \(c = \frac{1}{2}\); \(a = \frac{1}{2}, b = 1\) and \(c = \frac{1}{2}\); \(a = \frac{1}{2}, b = 1\) and \(c = \frac{1}{2}\); \(b = 1\) and \(c = \frac{3}{2}\) appeared in [26] Results 123, 130, 136, 137, 160 & 203, pages 456, 457 & 459.

(13) When \(i = 1\) and \(j = 1\) in (2.2), we obtain

\[
\frac{3}{2}F_2\left[ \begin{array}{ccc} a, & b, & c \\ \frac{1}{2}(a+b+2), & 2c+2 \\ \end{array} \right ; 1
\]

\[
= \frac{3}{2}F_2\left[ \begin{array}{ccc} \frac{a}{2}(a+b+2), & 2c+1 \\ a, & b, & c \\ \end{array} \right ; 1
\]

\[
- \frac{2abc}{(2c+1)(2c+2)(a+b+2)} \frac{3}{2}F_2\left[ \begin{array}{ccc} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+4), & 2c+3 \\ \end{array} \right ; 1
\]

which appeared in [12] Result (1), pp. 24], [28] Theorem (5), pp.148 and [15] Example (9), pp.8.

In such a case, the result when \(a = \frac{3}{2}, b = 1\) and \(c = \frac{1}{2}\), appeared in [26] Result 139, p. 456.

(14) When \(i = 2\) and \(j = 1\) in (2.2), we obtain

\[
\frac{3}{2}F_2\left[ \begin{array}{ccc} a, & b, & c \\ \frac{1}{2}(a+b+3), & 2c+2 \\ \end{array} \right ; 1
\]

\[
= \frac{3}{2}F_2\left[ \begin{array}{ccc} \frac{a}{2}(a+b+3), & 2c+1 \\ a, & b, & c \\ \end{array} \right ; 1
\]

\[
- \frac{2abc}{(2c+1)(2c+2)(a+b+3)} \frac{3}{2}F_2\left[ \begin{array}{ccc} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+5), & 2c+3 \\ \end{array} \right ; 1
\]

which appeared in [12] Result (1), pp.24.

In such a case, the result when \(a = \frac{1}{2}, b = 1\) and \(c = \frac{1}{2}\), appeared in [26] Result 139, p. 456.
When } i = 0 \text{ and } j = -1 \text{ in (2.2), we obtain }
\begin{align*}
3F2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 1), & 2c - 1 \\
\end{array} ; 1 \right] \\
= 3F2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 1), & 2c \\
\end{array} ; 1 \right] \\
+ \frac{ab}{(2c - 1)(a + b + 1)} 3F2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 3), & 2c + 1 \\
\end{array} ; 1 \right]
\end{align*}

which appeared in \[12\] Result(1), pp.24, \[16\] Eq.(3.19), pp. 230, \[11\] Result(1), pp. 269 and \[28\] Theorem (7), pp. 152.

In such a case, the result when } a = \frac{1}{2}, \ b = \frac{1}{2} \text{ and } c = 2 \text{ appeared in } [26] Result 172, p.458 & [31] Result(2.9), pp.5.

When } i = 1 \text{ and } j = -1 \text{ in (2.2), we obtain }
\begin{align*}
3F2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 2), & 2c - 1 \\
\end{array} ; 1 \right] \\
= 3F2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 2), & 2c \\
\end{array} ; 1 \right] \\
+ \frac{ab}{(2c - 1)(a + b + 2)} 3F2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 4), & 2c + 1 \\
\end{array} ; 1 \right]
\end{align*}

which appeared in \[12\] Result(1), pp. 24, \[16\] Eq.(3.20), pp. 230, \[15\] Example(10), pp. 8 and \[28\] Theorem (8), pp.153.

When } i = 2 \text{ and } j = -1 \text{ in (2.2), we obtain }
\begin{align*}
3F2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 3), & 2c - 1 \\
\end{array} ; 1 \right] \\
= 3F2 \left[ \begin{array}{ccc}
a, & b, & c \\
\frac{1}{2}(a + b + 3), & 2c \\
\end{array} ; 1 \right] \\
+ \frac{ab}{(2c - 1)(a + b + 3)} 3F2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 4), & 2c + 1 \\
\end{array} ; 1 \right]
\end{align*}

which appeared in \[12\] Result(1), pp.24.

In such a case, the results when } a = 1, \ b = 2 \text{ and } c = 2 \text{ appeared in } [26] Results 243, page 460.
(18) When \( i = 1 \) and \( j = -2 \) in (2.2), we obtain
\[
3F_2 \left[ \begin{array}{ccc}
\frac{1}{2}(a + b + 2), & 2c - 1 & \\
a, & b, & c \\
\end{array} ; 1 \right] \\
= 3F_2 \left[ \begin{array}{ccc}
\frac{1}{2}(a + b + 2), & 2c - 2 & \\
a, & b, & c \\
\end{array} ; 1 \right] \\
- \frac{abc}{(2c-1)(c-1)(a+b+2)} 3F_2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 4), & 2c & \\
\end{array} ; 1 \right]
\]
which appeared in [12, Result(1), pp. 24] and [28, Theorem(8), pp. 153].

(19) When \( i = -1 \) and \( j = -2 \) in (2.2), we obtain
\[
3F_2 \left[ \begin{array}{ccc}
\frac{1}{2}(a + b), & 2c - 1 & \\
a, & b, & c \\
\end{array} ; 1 \right] \\
= 3F_2 \left[ \begin{array}{ccc}
\frac{1}{2}(a + b), & 2c - 2 & \\
a, & b, & c \\
\end{array} ; 1 \right] \\
- \frac{abc}{(2c-1)(c-1)(a+b)} 3F_2 \left[ \begin{array}{ccc}
a + 1, & b + 1, & c + 1 \\
\frac{1}{2}(a + b + 2), & 2c & \\
\end{array} ; 1 \right]
\]
which appeared in [12, Result(1), pp. 24] and [15, Example(12), pp. 9] and [28, Theorem(6), pp. 150].

3. Concluding Remarks
1. Various other special cases of our result can be obtained.
2. Many new identities and relations which obtained from our result are under examinations and will be published later.
3. We have already established in the previous section a recursive relation (2.2), that generalized the extension of Watson summation theorem \( 3F_2(1) \).

**Theorem 3.1.**

\[
f_{i,j}(a, b, c) = (2c + j) f_{i+1,j}(a - 1, b, c) \\
- \frac{2ab}{(a + b + i + 1)(2c + j)} f_{i+1,j-1}(a, b + 1, c + 1)
\]

where \( f_{i,j}(a, b, c) \) is defined as in (2.1).

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References

[1] E. D. Rainville, Special functions, Macmillan, New York, (1960).
[2] G. N. Watson, A note on generalized hypergeometric series, Proc. Lond. Math. Soc. (2), 23 (1925).
[3] F. J. W. Whipple, A group of generalized hypergeometric series: relations between 120 allied series of the type F(a, b, c; e, f), Proc. Lond. Math. Soc. (2), 104-114, 23, (1925).
[4] W.N. Bailey, Generalized hypergeometric series, Cambridge University Press, (1985).
[5] L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge (1966).
[6] J. Thomae, Ueber die funktionen welche durch reihen von der Form dargestellt werden ..., J. Führ Math., 26-73, 87 (1879).
[7] T. M. MacRobert, Functions of complex variables, 5th edition, Macmillan, London, (1962).
[8] A.K. Rathie and R.B. Paris, A new proof of Watson’s theorem for the series 3F2, Appl. Math. Sci., 3, 161-164, No 4 (2009).
[9] M. Abramowitz and I. Stegun (Eds.), Handbook of mathematical functions, St. Dover, New York, pp. 556-557, (1965).
[10] M. A. Rakha, A new proof of the classical Watson’s summation theorem, Applied Mathematics E-Notes, 278-282, 11, (2011).
[11] J.L. Lavoie, Some summation formulas for the series 3F2(1), Math. Comp., 269-274,49, (1987).
[12] J.L. Lavoie, F. Grondin and A.K. Rathie, Generalization of Watson’s theorem on the sum of a 3F2, Indian J. Math., 23-32,34, (1992).
[13] S. Lewanowicz, Generalized Watson’s formula for 3F2(1), J. of Comput. and Appl. Math. (86), pp. 375-386, (1997).
[14] Y.S. Kim, and A. K. Rathie, Application of a generalized form of a Gauss second theorem to the series 3F2, Math. Commun. 16, 481-489, No 2 (2011).
[15] C. Wenchang, Analytical formulae for extended 3F2 - Series of Watson-Whipple-Dixon with two extra integer parameters, Mathematics of Computations 81:277,PP.467-479, (2012).
[16] M. A. Rakha, A. K. Rathie and U. Pandey, On a generalization of contiguous Watson’s theorem for the series 3F2(1), Complex Analysis and Applications '13 (Proc. of International Conference, Sofia, 31 Oct.-2 Nov. 2013)
[17] P. Berglund, P. Candelas and X. de la Ossa, Periods for Calabi-Yau and Landau-Ginzburg Vacua, Nuclear Phys. B 419(2), pp. 352-403, (1994).
[18] De Branges, A proof of the Bieberbach conjecture, Acta Math. 154(1-2), pp. 137-152, (1985).
[19] M. Petkovsek, H. S. Wilf and D. Zeilberger, A = B, Wellesley, MA: A K Peters Ltd. with a separately available computer disk, (1996).
[20] M. N. Barber and B. W. Ninham, Random and Restricted Walks: Theory and Applications, New York: Gordon and Breach, pp.147-148,(1970).
[21] L. G. Cabral-Rosetti and M. A. Sanchis-Lozano, Generalized hypergeometric functions and the evaluation of scalar one-loop integrals in Feynman diagrams. J. Comput. Appl. Math. 115(1-2), pp. 93-99,(2000).
[22] S. Moch, P. Uwe and S. Weinzierl , Nested sums expansion of transcendental functions and multiscalar multiloop integrals. J. Math. Phys. 43(6), pp. 3363-3386,(2002).
[23] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, Quantum Theory of Angular Momentum, Singapore: World Scientific Publishing Co. Inc, (1988).
[24] Frank W. J. Olver, Daniel W. Lozier and et, Nist handbook of mathematical functions, National institute of standards and technology, chapter 34,pp.758-766, (2010).
[25] Y. S. Kim, M. A. Rakha and A. K. Rathie, Extensions of certain classical summation theorems for the series 3F1, 3F2 and 4F3 with applications in Ramanujan’s summations, Int. J. of Math. and Math. Sci., Volume 2010, Article ID 309503, 26 pages.
[26] Prudnikov A.P., Brychkov Yu.A., Marichev O.I. Integrally i ryady. T. 3. Special funkii. Dopолнитель glavy. (2e izd., FML, 2003).
[27] Michael Milgram, On hypergeometric 3F2(1)-A review, (2010)
[28] C. Wenchang and R. R.Zhou, Watson-like formulae for terminating 3F2-series, Springer-Verlag Berlin Heidelberg,(2013).
[29] R. Askey and G. Gasper, Positive Jacobi polynomials sums,II. Amer. J. Math. 98(3), pp.709-737,(1976).
[30] N. D. Kazarinoff, Special functions and the bieberbach conjecture. Amer. Math. monthly 95(8), pp.689-696, (1988).

[31] M. M. Awad, A. O. Mohammed, M. A. Rakha and A. K. Rathie, New series identities for $\frac{1}{\pi}$, Communications of the Korean Mathematical Society, (2017), To Appear.

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