Chiral-coupling-assisted refrigeration in trapped ions

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Received 27 January 2023, revised 14 March 2023
Accepted for publication 23 March 2023
Published 19 April 2023

Abstract

Trapped ions can be cooled close to their motional ground state, which is imperative in implementing quantum computation and quantum simulation. Here, we theoretically investigate the capability of light-mediated chiral couplings between ions to enable a superior cooling scheme exceeding the single-ion limit of sideband cooling. Under asymmetric driving, the target ion manifests the chiral-coupling-assisted refrigeration at the price of heating others, where its steady-state phonon occupation outperforms the lower bound set by a single ion. We further explore the optimal operation conditions for the refrigeration where a faster rate of cooling can still be sustained. Under an additional nonguided decay channel, a broader parameter regime emerges to support the superior cooling and carries over into the reciprocal coupling, suppressing the heating effect instead. Our results present a tunable resource of collective chiral couplings which can help surpass the bottleneck of the cooling procedure and open up new possibilities in applications of trapped-ion-based quantum computation and simulation.

Keywords: laser cooling, chiral coupling, sideband cooling, atomic arrays

(Some figures may appear in colour only in the online journal)

1. Introduction

Trapped-ion quantum computation [1] has reached a level of large-scale architecture [2–4], where a high-performance universal quantum computer can be envisioned. In this scalable trapped-ion quantum computer, parallel zones of interactions and fast transport of ions can be integrated with high-fidelity gate operations [5, 6] in multiple small quantum registers. One of the bottlenecks in achieving this feat is the cooling procedure [3, 7, 8] which aims to prepare the system in its motional ground state. Two commonly used cooling schemes for ions are sideband [9–12] and electromagnetically induced-transparency cooling [13–18]. Reaching the many-body ground state of ions is also essential in ensuring genuine quantum operations on these ionic registers, which can further enable simulations of other quantum many-body systems [19, 20].

When multiple ions are involved in the cooling process, collective spin–phonon correlations arise owing to multiple scattering of light and recoil momentum [8, 21], leading to effective dipole–dipole interactions between ions [22, 23]. This collective interaction [24] is ubiquitous in any light–matter-interacting quantum interface [25], which can manifest a giant frictional force for atoms in an optical cavity [26] or form optically bound pairs of atoms in free space [27, 28]. The reciprocal nature of these light-induced dipole–dipole interactions can be further modified and controlled in an atom–waveguide interface [29–34], making the chiral quantum optical setup [35–50] a novel scheme for exploration of motional refrigeration in optomechanical systems [51, 52].

Here we consider an ionic chain tightly confined in harmonic trapping potentials under the sideband cooling scheme
and with collective chiral couplings, as shown in figure 1. The chiral couplings between ions are employed to host spin-exchange hopping and nonreciprocal decay channels, where \( \gamma_L \neq \gamma_R \). The effective coupling can be achieved either by moving the ions close to the waveguide [53] where the guided modes mediate the long-range chiral couplings [44] or by utilizing a chiral photonic quantum link in free space [54]. This setup leads to an unexplored territory of distinct heat exchange processes in cold ions. We emphasize that implementing chiral coupling with waveguide-mediated interactions in ions is a challenging experimental task, as charges on the dielectric waveguide may disturb the trapping potential [53] or may induce significant motional heating due to the electric field noise from the inherent losses of the waveguide material [55, 56]. This compromises the performance of quantum operations essential in quantum computations and other ion-based quantum technologies. Nevertheless, ongoing efforts are under development to understand better the surface charge distribution and the effect of dielectric materials on the ion’s motion, which can help reduce the unwanted heating and design an optimized trapped-ion system.

In this article, we propose a novel cooling scheme that relaxes the assumption of a single-particle spontaneous emission process. In essence, the intrinsic dissipation channel does not induce correlations between composite systems, and therefore many-atom cooling behavior can be attributed simply to single-atom results. In contrast, we introduce the resonant dipole–dipole interactions between atoms, which are universal in many light–matter-interacting systems. Considering a one-dimensional atomic array subject to one-dimensional reservoir as in an atom–waveguide interface, we are able to further modify the dissipation process and its directionality, which allows tailored collective spin-exchange couplings and new parameter regimes for superior cooling performance. This results from the buildup and the dominance of spin-exchange process within the composite systems over the sideband cooling in a single ion, which enables further heat removal. Furthermore, an extra nonguided channel we include can open a new paradigm to mitigate the heating effect at the reciprocal coupling, in essence to reduce the spin–phonon correlations which are otherwise more significant in heating. The tunable resource of collective chiral couplings we apply here can facilitate the motional ground state of ions and further push forward a large-scale and universal quantum computer employing trapped ions.

One of the crucial observations in our cooling scheme is the asymmetric driving condition. Under this condition, one of the ions in a one-dimensional atomic chain, the target ion, is driven with a relatively higher laser intensity, and the rest of them are the refrigerant ions acting as a reservoir of spin excitations and deexcitations for the target ion. With an additional asymmetry introduced in the nonreciprocal coupling strengths of \( \gamma_L \) and \( \gamma_R \), these ions further allow directional spin-exchange interactions, leading to an asymmetric heat transfer. This is the essence of the refrigeration effect in multiple ions mediated with chiral couplings. As for the requirement of an asymmetric driving condition, as long as we can sufficiently couple the refrigerant ions and the target ion by different intensities of laser fields, say only a fraction of one tenth or less for the refrigerant ones, we are safe in the superior cooling regime. Therefore, it does not matter how precisely the coupling rates are tuned as long as the asymmetric driving condition is satisfied. In our scheme, it would only require a relatively strong laser field on the target ion with weaker fields on the rest of the refrigerant ions in experiments to achieve our superior cooling performance.

The paper is organized as follows. In section 2, we introduce the Hamiltonian of sideband cooling in composite ions with chiral couplings. In section 3, we present that light-mediated chiral couplings between ions enable a superior cooling scheme to sideband cooling of a single ion. We find that the chiral-coupling-assisted refrigeration of the target ion can be feasible at a price of heating the other residual ones. In section 4, we calculate the cooling dynamics and obtain the cooling rates. We investigate the effect of nonguided modes and multi-ion enhancement of cooling in section 5. In section 6, we discuss the anomalous heating from ion traps and possible operations of our cooling scheme in quantum computation architecture. The appendix presents the detailed calculations of the steady-state phonon occupation in the target ion.

2. Theoretical model

We consider a generic model of \( N \) trapped ions with mass \( m \) under standing wave sideband cooling [57] with chiral couplings in Lindblad forms [42]. The time evolutions of the density matrix \( \rho \) of \( N \) ions with quantized motional states \( |n⟩ \) and an
internal ground (|g⟩) and excited states (|e⟩) can be described by (h = 1):

$$\frac{d\rho}{dt} = -i[H_{LD} + H_L + H_R, \rho] + \mathcal{L}_L[\rho] + \mathcal{L}_R[\rho],$$

(1)

where $H_{LD}$ for the sideband cooling in the Lamb–Dicke (LD) regime (in the first order of LD parameter $\eta$) reads:

$$H_{LD} = -\Delta \sum_{i=1}^{N} a_i^\dagger a_i + \nu \sum_{i=1}^{N} a_i^\dagger a_i + \frac{1}{2} \sum_{i=1}^{N} \eta \Omega \left( a_i + a_i^\dagger \right),$$

(2)

and the coherent and dissipative chiral couplings in the zeroth order of $\eta$ are, respectively,

$$H_{L(R)} = -\frac{\gamma_{L(R)}}{2} \sum_{\mu < \nu} \sum_{\nu} \left( e^{i\kappa (r_{\mu} - r_{\nu})} \sigma_\mu^a \sigma_\nu^b - H.c. \right),$$

(3)

and

$$\mathcal{L}_{L(R)}[\rho] = -\frac{\gamma_{L(R)}}{2} \sum_{\mu, \nu} e^{i\kappa (r_{\mu} - r_{\nu})} \left( \sigma_\mu^a \rho \sigma_\nu^b + \rho \sigma_\nu^b \sigma_\mu^a - 2\sigma_\nu^b \sigma_\mu^a \right).$$

(4)

The laser Rabi frequency is $\Omega$, with a detuning $\Delta = \omega_L - \omega_{eg}$ denoting the difference between its central ($\omega_L$) and atomic transition frequencies ($\omega_{eg}$), and the dipole operators are $\sigma_\mu^a \equiv |e⟩_\mu \langle g|$ with $\sigma_\mu^a = (\sigma_\mu^a)\dagger$. $\nu$ is the harmonic trap frequency with creation $a_i$ and annihilation operators $a_i$ in the Fock space of phonons $|n⟩$, and the LD parameter is $\eta = k_L/\sqrt{2m\nu}$ with $k_L \equiv \omega_L/c, k_L$ denotes the wave vector in the guided mode that mediates chiral couplings $\gamma_{L(R)}$, and we can use $\zeta \equiv k_L(r_{\mu+1} - r_{\mu})$ to quantify the light-induced dipole–dipole interactions associated with the relative positions of trap centers $r_{\mu}$ and $r_{\nu}$.

The Lindblad forms in equation (1) take into account spin-exchange processes between ions with nonreciprocal and long-range dipole–dipole interactions, and we use a normalized decay rate $\gamma = \gamma_R + \gamma_L$ to characterize the timescale of the system dynamics. In the sideband cooling with $\eta \Omega, \gamma \ll \nu$ and the resolved sideband condition of $\Delta = -\nu$, the steady-state (st) phonon occupation in the case of a single ion can then be calculated as $\langle n⟩_{st} \equiv \text{tr} (\rho_{st} a_i^\dagger a_i) \propto (\gamma/\nu)^2$ with a cooling rate of $O((\gamma^2 \Omega^2)/\gamma)$ [12, 57] in the weak field regime. This means that $\gamma$ determines the lower bound of phonon occupation, and a rate to reach this near motional ground state can be much smaller than $\gamma$. Next we explore the distinct cooling mechanism with the collective dipole–dipole interaction between every other ion in the sideband cooling scheme, where a superior cooling regime can be identified under an asymmetric driving condition $\Omega_1 \neq \Omega_2$ on different ions.

### 3. Chiral-coupling-assisted cooling

We first demonstrate the chiral-coupling-assisted refrigeration in the case of two ions, which represent the essential element of interacting quantum registers. Whether there is refrigeration in these ions or not lies in their steady-state phonon occupations compared to their respective single-ion results without chiral couplings. We obtain the steady-state solutions by solving $d\rho/dt = 0$ in equation (1), which is equivalent to finding the right eigenvector $\rho_{st}$ with zero eigenvalue of the Lindblad map, that is, $\mathcal{L}[\rho_{st}] = 0$ obtained from time-evolving solutions of $\rho(t) = e^{\mathcal{L}t} [\rho(t = 0)]$ [58]. The steady-state solution of $\rho_{st}$ is also called the null space of the Lindblad map,

$$\rho_{st} = \text{Null}(\mathcal{L}),$$

(5)

under the constraint of probability conservation $\text{Tr}(\rho_{st}) = 1$. The complete Hilbert space involves intrinsic spin and external motional degrees of freedom, which we denote as $|\alpha, n⟩_\mu$, where $\alpha \in \{g, e\}$ denotes the ground and excited states for the $\mu$th ion and $n$ denotes the phonon number of phononic Fock states. Here we restrict $n \in \{0, 1\}$, which is valid when the dominant phononic Fock state is in the vicinity of the motional ground state. We note that for computing the null space of the Lindblad map we convert the density matrix to Fock–Liouville space [59], which has a dimension equal to $4^N N$ in our case, which leads to a computation complexity $O(4^N N)$ by using singular-value-decomposition algorithms. This suggests a challenging, if not impossible, task of numerically simulating the case for $N = 4$.

In figure 2, we numerically obtain the steady-state properties with a phonon number up to $n = 1$, which is sufficient in the LD regime where $\langle n⟩_{st} \ll 1$. We use the normalized steady-state phonon occupation $\bar{n}_i \equiv \langle n⟩_{st} / \langle n⟩_{0}$ to represent the cooling performance by comparing the results of respective single ions in a single-ion calculation versus $\gamma_R$, a right-propagating decay rate defined in equation (4) or schematically seen in figure 1. The phonon occupation $\langle n⟩_{st}$ for a single ion has been calculated as $\langle n⟩_{st} \approx (\gamma/4\nu)^2 + (\sigma L_{\Omega}/\nu)^2/8$ under a weak or strong field regime [12], and we also obtain them numerically in the bottom plots of figure 2 as a reference. The chiral-coupling-assisted cooling of the target ion (first ion) can be seen in the regions of $\bar{n}_j < 1$ in figure 2(a) under asymmetric driving. This is more evident when the driving field on the target ion is weaker as shown in figure 2(b). For a symmetric driving condition, the refrigeration phenomenon never takes place. We also explore the effect of light-induced dipole–dipole interaction in figure 2(c), where a superior cooling emerges at $\zeta$ close to $\pi$ or $2\pi$. We find that the phonon occupation of the second ion is always larger than the phonon occupation in a single-ion calculation, where the second ion acts as the refrigerant ion that always heats up while it cools the target ion. Under an asymmetrical driving condition, the refrigerant ion acts as a reservoir of spin excitations and deexcitations for the target ion. Therefore, the asymmetry between $\gamma_L$ and $\gamma_R$ further allows directional spin-exchange interactions, leading to an asymmetric heat transfer. We note that $\langle n⟩_{st}$ retrieves the
single-ion results when $\gamma_R/\gamma = 1$ and 0 for the target and refrigerant ions, respectively. This results from the unidirectional coupling regime where spin-exchange couplings are forbidden, and thus spin–phonon correlations do not play a role in determining the steady-state properties.

In figures 2(a) and (c), we find a moderate cooling performance of $\bar{n}_1 \lesssim 0.9$, which can be further pushed to below 0.2 when $\Omega_1$ is made weaker in figure 2(b). To understand the superior cooling parameter regimes in figure 2(b), we investigate specifically the cooling performance in the target ion by tracing over the phononic degrees of freedom of the refrigerant ion. Considering the perturbations of $\gamma^2$ and $\eta^2\Omega^2_1$ on an equal footing, we obtain the steady-state phonon occupation of the target ion by truncating to their first orders,

$$\langle n_1 \rangle_{st} \approx \frac{\gamma^2}{4\nu^2} \left( \frac{1}{2} - \frac{\gamma^2}{\gamma} \right)^2 + \frac{\eta^2\Omega^2_1}{8\nu^2} \times \left( \frac{\eta^2\Omega^2_1 + 2\gamma^2}{\eta^2\Omega^2_1 + 8\gamma^2(1/2 - \gamma_R/\gamma)^2} \right),$$

which we calculate in detail in appendix. The excess heating for both the target and refrigerant ions shown in the subplots of figures 2(d) and (e) can be attributed to collective spin-exchange interactions especially under reciprocal couplings, contrary to the nonreciprocal couplings that can redirect the heat transfer between these two ions. This excess heating can as well be revealed in equation (6) for the target ion, where under the reciprocal coupling condition, the second bracket of equation (6) reaches its maximum and gives rise to the heating effect. The boundary that determines $\bar{n}_1 = 1$ from equation (6) gives $\gamma_R = \gamma/2 \pm \sqrt{3}\eta\Omega_1/2\sqrt{\gamma}$, which delineates the onset of superior cooling and agrees well with numerical simulations in figure 2(b). The linear dependence of $\gamma_R$ and $\Omega_1$ in the boundary indicates that excess cooling behavior happens symmetrically to the reciprocal coupling regime with a linear dependence on the driving field. This shows a competition between the laser driving field and the intrinsic spontaneous emission rate, where excess cooling emerges when $\eta\Omega_1 \lesssim (2\gamma_R - \gamma)$. This also represents the dominance of the spin-exchange process over the sideband cooling, which leads to a superior cooling performance. As for the symmetric dependence of $\gamma_R$ in the $\langle n_1 \rangle_{st}$ at small driving fields in figure 2(e), this can be explained again by treating the refrigerant ion as a reservoir for spin-exchange interactions under the asymmetric driving condition. The process of spin excitations and deexcitations of the target ion by spin-exchanging with the refrigerant ion effectively involves both the coupling strengths of $\gamma_R$ and $\gamma_L$, that is $\propto (\gamma_R/\gamma - 1/2)$($\gamma_L/\gamma - 1/2$), which leads to the symmetry in $\gamma_R$ or $\gamma_L$ with respect to $\gamma/2$. Under the condition of unidirectional coupling when $\gamma_R = \gamma$, $\langle n_1 \rangle_{st}$ again retrieves the single-ion result $\langle n_1 \rangle^s_{st}$ as expected.

We further identify three local extreme points in equation (6) as $\gamma_R/\gamma = 0.5$ for one maximum $\langle n_1 \rangle^\text{max}_{st} = \gamma^2/(4\nu^2) + \eta^2\Omega^2_1/(8\nu^2)$ which is always larger than $\langle n_1 \rangle^s_{st}$, and two equal minimums with corresponding values of $\gamma_R^\text{min}$,

$$\langle n_1 \rangle^\text{min}_{st} = \frac{\eta\Omega_1}{8\nu^2} \sqrt{\eta^2\Omega^2_1 + 2\gamma^2} - \frac{\eta^2\Omega^2_1}{32\nu^2},$$

$$\gamma_R^\text{min} = \frac{\gamma}{2} \pm \frac{1}{2} \sqrt{\eta\Omega_1 \left[ \eta^2\Omega^2_1 + 2\gamma^2 \right] - \frac{\eta^2\Omega^2_1}{2}}.$$

Interestingly, the local minimum $\langle n_1 \rangle^\text{min}_{st}$ indicates a ‘mixing’ effect of the driving field and the intrinsic decay rate, which
ion can achieve, however, suffers from an extremely slow cooling rate $(\propto\Omega_1^2)$. Next we show that the cooling rate of the target ion under chiral couplings, determined by a fitted overall timescale, can still surpass the single-ion case, but a longer time is needed to reach the steady state owing to a small $\eta\Omega_1$.

4. Cooling rate

In numerically simulating the time dynamics of the phonon occupations for both ions as shown in figure 4, we assume the initial state of the trapped ions in a thermal state $|\nu\rangle = \sum_{n=0}^{\infty} \frac{n^0_n}{(n_0+1)^{n+1}} |g,n\rangle$, where $n_0$ is an average phonon number for both ions. We use $n_0 \lesssim 1$ with a finite truncation of the motional states to guarantee the convergence in numerical simulations. To quantify the cooling behaviors, we use an exponential fit for the timescale to reach $|n_1\rangle$ with a function of $ae^{-bt} + \langle n_1 \rangle a$ for arbitrary constants $a$ and $b$. We then obtain the corresponding cooling rate $W = b$, which generally gives an overall timescale of the cooling process.

In figures 4(a) and (b), we show the fitted cooling rates comparing the respective single-ion results and corresponding time evolutions in figures 4(c) and (d). The different panels of figures 4(c) and (d) correspond to the time evolutions with the parameter regimes in figures 4(a) and (b), respectively, where we have chosen the cooling and heating cases of the target ion in the upper and lower panels as comparisons. For the refrigerant ion, the cooling rate does not change significantly and behaves similarly to the single-ion case with a rate $\propto\Omega_2^2/\gamma_2$, showing a rather prolonged time dynamics owing to an asymmetric setting of the driving fields. Meanwhile, a faster cooling rate emerges for the target ion when $\gamma_R \approx 0.85$ and $\Omega_1/\nu \lesssim 1.5$, as shown in figure 4(b). The time region when the target ion surpasses the single-ion limit can be seen in figures 4(c) and (d), where the refrigeration effect shows up at a later stage than the single-ion case. The time for establishing refrigeration appears approximately 10 times longer than the time for a single ion to reach its steady state.

The slow rates of $W$ in figure 4(a) at $\gamma_R/\gamma \sim 0.5$ reflect a delay from multiple exchanges of spin excitations and phonon occupations, while a retrieved rate as in a single ion emerges again in the unidirectional coupling regime. As $\Omega_1$ increases in figure 4(b), both cooling rates approach their respective single-ion cases, which depend on $\gamma/(2(1+n_0))$ bounded by $\Gamma$ [12]. The slow cooling rates in the reciprocal coupling regime can be attributed to a lack of directionality in dissipation. This leads to a slow spread of spin diffusion [61, 62] and associated stagnant removal of phonon, in addition to the buildup of spin–spin correlations owing to the collective nature of nonreciprocal couplings between these constituent atoms. We note as well that the reciprocal coupling regime allows a more significant interference in spin populations, which is highly related to the multiple reflections and transmissions in spin exchanges.
before they relax as time evolves. This could be one of the reasons why the system takes a longer time to reach the steady state in figure 4(a).

5. Effect of nonguided decay and multi-ion case

Here we introduce an additional nonguided mode on top of the guided nonreciprocal couplings. This move our system away from a strong coupling regime but closer to a realistic setting, where unwanted decays can be unavoidable [45]. The nonguided decay rate $\gamma_{ng}$ can simply be cast into equation (1) in the form of

$$\mathcal{L}_{ng} = -\frac{\gamma_{ng}}{2} \sum_{\mu=1}^{N} \left( \sigma_{\mu}^{\dagger} \sigma_{\mu} \rho + \rho \sigma_{\mu}^{\dagger} \sigma_{\mu} - 2 \sigma_{\mu} \rho \sigma_{\mu}^{\dagger} \right).$$

The parameter, $\beta \equiv \gamma / (\gamma + \gamma_{ng})$, can quantify the crossover from a strong coupling ($\beta = 1$) to a purely noninteracting regime ($\beta = 0$).

As shown in figure 5, we find a broader parameter regime of $\beta$ that can sustain better cooling performance where $n_1 < 1$ and further reduce its local minimum of phonon occupations. More surprisingly, the heating behavior at the reciprocal coupling of $\gamma_{R}/\gamma = 0.5$ can be suppressed and turned to cooling instead with $\beta \lesssim 0.9$. This is manifested as well in the case of three ions under asymmetric driving, where the target ion can still present superior cooling behavior with an even lower $\tilde{n}_1^{\text{min}}$ using two refrigerant ions. The crescent-like region of low $\tilde{n}_1$ in the case of two ions can be analyzed by tracing over the refrigerant ion’s motional states. An analytical prediction of the local minimums, which results from a quartic equation of $\beta^2 (\gamma_{R}/\gamma)^2$ in appendix section ‘Minimal phonon occupation of the target ion’, is shown on top with this crescent-like region. This leads to two local minimums for a fixed and finite $\beta$ and a continuation of $\tilde{n}_1^{\text{min}}$ at $\beta = 1$ toward the parameter regimes of $\beta < 1$ and $\gamma_{R} = 0.5\gamma$, which provide a route to superior cooling even under a finite $\gamma_{ng}$. The reason why the superior cooling can be allowed here might be due to the extra dissipative channel that mitigates the effect of reciprocal couplings. This extra dimension of nonguided mode provides the possibility for the composite system to explore between the regimes with highly correlated spin–phonon couplings at $\gamma_{R} = \gamma_{L}$ with $\beta = 1$ and purely noninteracting ones at $\beta = 0$. Since the cooling performance of the target ion reduces to the single-ion result at $\beta = 0$, naturally and as expected a superior cooling regime would emerge in between for a finite $\beta$. We can also attribute these new parameter regions for cooling to a reduction of spin–spin correlations, which are otherwise more evident in the heating regime as shown in figures 2 and 3. Essentially, the role of the nonguided mode here makes the composite system less susceptible to the collective spin-exchange

![Figure 4](image-url)

Figure 4. Cooling rates $W$ of the target and refrigerant ions. The condition for the initial thermal ensemble of ions is taken as $n_0 = 0.7$ and a truncation of phonon number to $n = 4$. All cooling rates of the target (blue-△) and refrigerant ions (red-●) are compared to their respective single-ion results $W_{i}$ (dashed lines), dependent on (a) $\gamma_{R}$ with $\Omega_{1}/\nu = 1$ and (b) $\Omega_{1}$ with $\gamma_{R}/\gamma = 0.85$, where both plots take $\Omega_{2}/\Omega_{1} = 0.1$ and $\xi = 2\pi$. The corresponding time evolutions of phonon occupations (blue- and red-solid lines) in (a) and (b) are shown in (c) and (d), respectively, for $\gamma_{R}/\gamma = 0.85$, 0.5, and $\Omega_{1}/\nu = 0.2, 3.2$, in the upper and lower plots. The respective single-ion results (dashed lines) are plotted for comparison. The refrigeration effect initiates before and after the time $\sim 10^4 \nu^{-1}$ in (c) and (d), yellow-shaded areas. The $\gamma$ is set to be the same as in figure 2, and the inset plots in (c) and (d) are normalized $\tilde{n}_1$ for an identification of the time crossing $\tilde{n}_1 = 1$ when cooling initiates and sustains.

![Figure 5](image-url)

Figure 5. Nonguided mode in cooling the target ion. The nonguided decay rate $\gamma_{ng}$ is introduced in the cases of two and three ions with an equal interparticle distance at $\xi = 2\pi$, where $\beta \equiv \gamma / (\gamma + \gamma_{ng})$ indicates the portion of decay to the guided mode. Similar shading color is used in respective lower panels as in figure 2 with the parameters of $\Omega_{2}/\Omega_{1} = 0.1, \Omega_{1} = 1\nu$, and $\gamma = 0.1\nu$. The upper panels present some cuts in the lower ones at $\beta = 1$ (solid), $\beta = 0.8$ (dash-dotted), and $\beta = 0$ (dashed). A dashed line in the lower plot of the two-ion case represents a local minimum predicted from an analytical derivation in appendix.
interactions which are augmented the most in the reciprocal coupling regime.

For the case of multiple ions under asymmetric driving, we are able to take the partial trace of the motional degrees of freedom in the refrigerant ions by assuming the laser driving strengths on them are small enough. This leads to a reduced Hilbert space spanned by complete internal and motional states of the target ion and only the internal states of refrigerant ions. Although the relative location of the target ion to other refrigerant ions can matter as seen from equations (3) and (4) under chiral couplings, we have checked that the configuration of the target ion in an ionic periodic array of $N = 3$ is irrelevant under the asymmetric driving condition, that is, $\langle n_1 \rangle_{\text{at}}$ is the same for the target ion at the end or in the middle of the chain when the interparticle separation is chosen as $\xi = 2\pi$. Therefore, we consider that the target ion locates at the leftmost site of an $N$-ion chain without loss of generality.

We proceed by keeping the density matrix elements whose leading terms are up to the order of $\gamma^2/\nu^2$ and $\eta^{2}\Omega^2/\nu^2$. We find that they can be selected by the following two rules. One is the Hamming distance between the specific density matrix element and that of the many-body ground state (e.g. $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$ for the three-ion case) is not greater than two. The other is that the row and column indices of the density matrix elements can only contain at most one excited state, where $|e\rangle$ and $|n = 1\rangle$ are treated as excited states. However, there is an exception for $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$ which should be included since it represents the population in the $|e1\rangle$ state, which is $O(\eta^{2}\Omega^2)$ due to the driving on the target ion. With these conditions, we find that the following relationships still hold as in equation (A2),

$$\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} = \frac{\eta^{2}\Omega^2}{16\nu^2} \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}},$$

$$\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} = \frac{i\gamma\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}}{8\nu^2} \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}},$$

$$\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} = \frac{-4\nu - i\Gamma}{16\nu^2} \Omega^{2} \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}},$$

where the first two indices in the row and column indices represent the internal and motional state of the target ion, and the $(i + 2)$th index stands for the internal state of the $i$th refrigerant ion for $i \in [1, N - 1]$.

Next, we construct the multi-ion generalization of equation (A3). Here we categorize these undetermined density matrix elements according to the indices of the target ion as follows,

$$B_i = \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} = -\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}.$$  

$$C_i = \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} = \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}.$$  

$$D_{ij} = \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} = \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}.$$  

where $D_{ij} = D_{ji}$. The above represent spin–phonon and spin–spin correlations between refrigerant and target ions, and spin–spin correlations within refrigerant ones, respectively. Combining the above variables with other undetermined variables, such as $A = \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} = -\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$, $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$, and $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$, we obtain the following coupled equations:

$$0 = -2i\nu \Omega \Lambda - 2i \gamma \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}},$$

$$0 = \Gamma B_i + 2\gamma R A + 2\gamma R \sum_{j=1}^{N-1} B_j + 2\gamma L \sum_{j=i+1}^{N-1} B_j + i\eta \Omega \Delta_i,$$

$$0 = 2\Gamma C_i + 2\gamma R \sum_{j=1}^{N-1} C_j + 2\gamma L \sum_{j=i+1}^{N-1} C_j + 2\gamma L \sum_{j=1}^{N-1} D_{ij} + i\eta \Omega \Delta_i + 2\gamma R \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}},$$

$$0 = \Gamma D_{ij} + \gamma R \sum_{k=1}^{N-1} D_{ik} + \gamma L \sum_{k=i+1}^{N-1} D_{ik} + \gamma R \sum_{k=i+1}^{N-1} D_{ik} + \gamma L \sum_{k=1}^{N-1} D_{ik} + \gamma R (C_i + C_j),$$

$$0 = 2\Gamma \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} + 4\gamma L \sum_{j=1}^{N-1} C_j + 2i\nu \Omega \Lambda,$$

where we have $N(N+3)/2$ variables, i.e. $A$, $B_i$, $C_i$, $D_{ij}$ ($i \leq j$, real symmetric matrix), and $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$. They can be solved numerically in terms of $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$, or equivalently $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$, and finally we obtain $\rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}}$ from

$$0 = i\eta \Omega \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} - \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} - \rho_{\xi_1 \xi_2 \xi_3}^{\text{ggg}} + \Gamma A + i\nu \Omega \Lambda,$$

which presents the potentiality in multi-ion-assisted cooling via collective chiral couplings.
Within experimental reach, ion-surface separation, which can be as low as 0.01 K s, estimated bound 13 K s, an anomalous heating rate can be made much smaller than the bound that would compromise our cooling scheme. Again, the heating rate as \( \eta \approx 10^{3} \) s, which is feasible in several typical platforms. The parameters used here are \( \eta = 0.04, \Omega = 1 \nu, \) and \( \Gamma = 0.1 \nu. \)

6. Discussion and conclusion

We have shown theoretically that the chiral couplings introduced in the trapped-ion system enable a better cooling performance than a single ion in sideband cooling. This light-mediated chiral coupling between ions manifests a resource with capability to achieve a superior cooling scheme that surpasses the lower bound of the steady-state phonon occupation a single ion can allow. The chiral-coupling-assisted refrigeration in two and three ions can be useful in a large-scale quantum computer composed of multiple small entities of ions without compromising the cooling rates. When \( \gamma/2\pi = 20 \) MHz is used in our results, it gives a cooling time of \( 10^{7}(\nu^{-1}) \) within 100 \( \mu s \), which is feasible in several typical platforms of \( ^{9}\text{Be}^{+} \) [21], \(^{40}\text{Ca}^{+} \) [63], \(^{172}\text{Yb}^{+} \) [64], or \(^{171}\text{Yb}^{+} \) ions [15].

In conclusion, our results present a distinctive control over the motional ground states with tunable chiral couplings and provide new insights in getting around the cooling barrier in trapped-ion-based applications of quantum computation and simulation. Last but not least, the scheme we consider here can also be implemented with optical tweezers in a scalable ion crystal for high-performance gate operations [65, 66].

We note that an anomalous heating is unavoidable in ion traps owing to the electric field noise from the electrode surfaces. The anomalous heating could be an issue in our new cooling scheme when it becomes the dominating factor. This, however, can be lessened by lowering the electrode temperature [67], applying surface plasma cleaning [68], or increasing the axial trap frequency with higher trapping heights [67, 69]. Considering \( \gamma/2\pi = 20 \) MHz for the decay rate again, we estimate that a 10^{-3} phonon number gives a temperature \( T \approx 1.3 \times 10^{-3} \) K [7]. Within a cooling time of 100 \( \mu s \), we can further estimate the comparable anomalous heating rate as \( T/(100 \mu s) \approx 13 \) K s^{-1}, which sets the lower bound that would compromise our cooling scheme. Again, the anomalous heating rate can be made much smaller than the estimated bound \( 13 \) K s^{-1} by tuning the axial frequency and ion-surface separation, which can be as low as 0.01 K s^{-1} and within experimental reach [69].

Finally, for quantum computation protocols using our cooling scheme with multiple ions, we resort to the trapped-ion quantum charge-coupled device as quantum computer architecture [3]. In a similar way to using parallel interaction zones, our multi-ion cooling scheme can be implemented in parallel as well, which would prepare the target ions close to the motional ground state even in the case of two ions. This coincides with the design using a small-ion crystal, which offers a better performance in state preparation or gate operation owing to its high controllability. We then can collect all the target ions into the interaction zone after the cooling procedure via adiabatic ion transport. Presumably within a small-ion crystal, we can save some error and time budget in quantum computation from our proposed scheme. For more ions, as shown in figure 6(b), the cooling performance saturates as \( N \) increases, and these many ions would experience unexpected heating owing to system complexities of electric field noise or laser field fluctuations. As for design of small-ion quantum registers, our multi-ion enhancement in cooling could be compromised, but it is still good to know that already a reasonable superior cooling performance can be achieved with fewer than three or four ions in our scheme. Essentially, our cooling scheme offers an alternative method to get around the cooling protocol bottleneck, which helps to improve the quantum computation architecture.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

We acknowledge support from the Ministry of Science and Technology (MOST), Taiwan, under the Grant No. MOST-109-2112-M-001-035-MY3. We are also grateful for support from TG 1.2 and TG 3.2 of NCTS and inspiring discussions with G-D Lin.

Figure 6. Steady-state phonon occupation number of the target ion as a function of \( \beta \) and \( \gamma/R/\gamma \) for the multi-ion case. (a) Numerically calculated \( \bar{n}_{1} \) under asymmetric driving and interparticle distances chosen as multiples of 2\( \pi \). (b) Numerically calculated global minimum of \( \bar{n}_{1}^{\text{min}} \). The parameters used here are \( \eta = 0.04, \Omega = 1 \nu, \) and \( \Gamma = 0.1 \nu. \)
Appendix. Analytical form of the steady-state occupation of the target ion

In chiral-coupling-assisted cooling of two ions, the Hilbert space dimension is 16 with 256 coupled linear equations, which hardly gives insightful results analytically. To explore the optimal condition for the target ion in the steady state, we perform partial trace to the refrigerant ion with respect to the motional degree of freedom (\(a_{z}\)), which diminishes the dimension of the Hilbert space to 8. This is valid if the laser driving strength of the refrigerant ion is much smaller than that of the target ion. In this appendix, we replace \(\Omega_{1}\) by \(\Omega\) for simplicity, and we define the total decay rate as \(\Gamma = \gamma_{R} + \gamma_{L} + \gamma_{R}\), which is fixed by the intrinsic decay rate of ion \(\Gamma\).

Now the dynamics of this system can be determined by the reduced density matrix \(Tr_{a_{z}}(\rho)\). Since we focus on solving \(\langle n_{1}\rangle_{st}\), the number of equations required can be further reduced to 20. These equations generally involve the steady-state density matrix elements of \(\rho_{\mu_{1}m_{1};\mu_{2}m_{2}} = \langle \mu_{1},m_{1};\mu_{2} | Tr_{a_{z}}(\rho) | \mu_{1},m_{1};\mu_{2} \rangle\). In the resolved sideband cooling under the Lamb–Dicke regime where \(\Delta = -\nu\) along with the condition \(\epsilon_{id} = 1\), we can take advantage of the fact that \(\rho_{00\theta0\theta}\) is \(O(1)\), and \(\gamma^{2}/\nu^{2}\) and \(\eta_{4}\) are much smaller than one. As a result, we neglect those density matrix elements whose leading term is higher than second order, such as \(\rho_{10\theta1\theta}, \rho_{01\theta0\theta}, \rho_{01\theta1\theta}, \rho_{10\theta1\theta}, \rho_{01\theta1\theta}, \rho_{11\theta0\theta}, \rho_{11\theta1\theta}\), and \(\rho_{11\theta1\theta}\). This leads to

\[
0 = \eta\Omega(\rho_{00\theta1\theta} - \rho_{10\theta0\theta}) + 2\gamma(\rho_{00\theta1\theta} + \rho_{01\theta0\theta}),
\]

\[
0 = -\eta\Omega(\rho_{10\theta0\theta} - 2i\nu\rho_{00\theta0\theta} - 2i\nu\rho_{10\theta0\theta} - 2i\nu\rho_{01\theta0\theta}),
\]

\[
0 = \eta\Omega(\rho_{00\theta0\theta} - \rho_{11\theta0\theta}) - 2i\nu(\rho_{00\theta0\theta} + \rho_{01\theta0\theta}) - 2i\nu\rho_{01\theta0\theta},
\]

\[
0 = -2i\nu(\rho_{01\theta0\theta} + \rho_{10\theta0\theta}) - 2i\nu\rho_{11\theta0\theta},
\]

\[
0 = \eta\Omega(\rho_{00\theta0\theta} - \rho_{11\theta1\theta}) + i\gamma\eta_{4}^{2}\Omega^{2}/\nu^{2},
\]

(A1)

where we have used the following relationships,

\[
\rho_{11\theta0\theta} = \frac{\eta_{4}^{2}\Omega^{2}}{16\nu^{2}}\rho_{00\theta0\theta} - \frac{i\gamma}{8\nu}\rho_{00\theta0\theta},
\]

\[
\rho_{11\theta1\theta} = \frac{\eta_{4}^{2}\Omega^{2}}{16\nu^{2}}\rho_{00\theta0\theta} - \frac{i\gamma}{8\nu}\rho_{00\theta0\theta},
\]

\[
\rho_{11\theta1\theta} = \frac{4\nu - i\Gamma}{16\nu^{2}}\rho_{00\theta0\theta}.
\]

(A2)

Finally we have the following density matrix elements expressed in terms of \(\rho_{00\theta0\theta}\),

\[
\rho_{00\theta1\theta} = -\rho_{10\theta0\theta} = -i\Gamma\frac{\eta\Omega}{16\nu^{2}}\rho_{00\theta0\theta},
\]

\[
\rho_{00\theta1\theta} = \rho_{01\theta0\theta} = -\frac{\Gamma}{2\gamma\nu}\rho_{00\theta0\theta},
\]

\[
\rho_{01\theta0\theta} = \rho_{00\theta1\theta} = -\frac{\Gamma}{2\gamma\nu}\rho_{00\theta0\theta},
\]

\[
\rho_{00\theta1\theta} = \frac{\gamma^{2}}{16\nu^{2}} - \gamma\gamma_{L} + \frac{\nu\eta_{4}^{2}\Omega^{2}}{8},
\]

\[
\rho_{00\theta1\theta} = \frac{\gamma^{2}}{16\nu^{2}} - \gamma\gamma_{L} + \frac{\nu\eta_{4}^{2}\Omega^{2}}{8},
\]

\[
\rho_{10\theta1\theta} = \frac{\gamma^{2}}{16\nu^{2}} - \gamma\gamma_{L} + \frac{\nu\eta_{4}^{2}\Omega^{2}}{8},
\]

\[
\rho_{10\theta1\theta} = \frac{\gamma^{2}}{16\nu^{2}} - \gamma\gamma_{L} + \frac{\nu\eta_{4}^{2}\Omega^{2}}{8},
\]

\[
\rho_{10\theta0\theta} = \frac{\gamma^{2}}{16\nu^{2}} - \gamma\gamma_{L} + \frac{\nu\eta_{4}^{2}\Omega^{2}}{8},
\]

(A3)

The steady-state occupation for the target ion can therefore be derived as \(\rho_{00\theta0\theta} \approx 1\)

\[
\langle n_{1}\rangle_{st} = \rho_{10\theta1\theta} + \rho_{11\theta0\theta} + \rho_{11\theta1\theta} + \rho_{11\theta1\theta},
\]

\[
= \frac{\Gamma^{2}}{16\nu^{2}} + \frac{\eta_{4}^{2}\Omega^{2}}{8\nu^{2}} - \gamma\gamma_{L}^{2}/4\nu^{2},
\]

\[
+ \frac{\eta_{4}^{2}\Omega^{2}}{8\nu^{2}} + \frac{2\gamma^{2}}{8\nu^{2}} - \gamma\gamma_{L}^{2}/4\nu^{2},
\]

(A4)

where the first two terms are the steady-state phonon occupation of a single-ion cooling, and the remaining terms are the modifications arising from the chiral couplings. A comparison between the prediction from equation (A4) and the numerical simulation is shown in figure 7. The blue dashed lines represent the numerical results without partial tracing out the refrigerant ion’s motional degree of freedom, and the blue solid lines show our analytical results. The blue solid lines display a mild deviation from the numerical result on the side \(\gamma_{R} < 0.5\) since the simulation results include the influence of finite laser driving of the refrigerant ion, which causes the asymmetry of the \(\langle n_{1}\rangle_{st} - \gamma_{R}\) curve.

Minimal phonon occupation of the target ion

From equation (A4), the minimal phonon occupation of target ion can be obtained as
tem no longer allows the optimal minimal

\[ \langle n \rangle_{\text{st}} = \langle n \rangle_{\text{st}}^r - \frac{1}{32\nu^2} \left( \sqrt{\eta^2\Omega^2 + 2\Gamma^2} - 2\eta\Omega \right)^2, \]

where the minimum can occur when

\[ \gamma R|_{\beta} = \frac{1}{8} \left( \Omega^2 + 2\Gamma^2 \pm 2\eta\Omega \sqrt{\eta^2\Omega^2 + 2\Gamma^2} \right). \]  

Due to the constraint on \( \gamma R|_{\beta} \), i.e. \( 0 \leq \gamma R|_{\beta} \leq \beta^2\Gamma^2/4 \), \( \gamma R|_{\beta} \) in equation (A6) can be ruled out. Since the other solution \( \gamma R|_{\beta} \) does not always satisfy the lower bound of \( \gamma R|_{\beta} \), the condition under which \( \langle n \rangle_{\text{st}} \) can be minimized is

\[ 2\Gamma^2 \geq 3\eta^2\Omega^2 \]  

\[ 2\beta^2\Gamma^2 \geq \eta^2\Omega^2 + 2\Gamma^2 - 2\eta\Omega \sqrt{\eta^2\Omega^2 + 2\Gamma^2}. \]  

Consequently, the best performance predicted in equation (A5) still persists in the presence of nonguided decay if the total coupling efficiency \( \beta \) satisfies

\[ \beta \geq \beta_0 = \sqrt{\frac{\eta\Omega}{2} - \frac{1}{2}} \left( \frac{\eta\Omega}{\Gamma^2} \right). \]  

We choose four representative cases in figure 7 to show the emergence of \( \langle n \rangle_{\text{st}} \) at different \( \beta \), where equation (A7) is satisfied. The horizontal dashed and dotted line are the references for the single-ion cooling limit and the minimal phonon occupation predicted by equation (A5). For each \( \beta \in (\beta_0, 1] \), we find that there are two \( \gamma_{R, \text{min}} \) corresponding to \( \langle n \rangle_{\text{st}, \text{min}} \), and they are located at

\[ \gamma_{R, \text{min}} = \frac{1}{2} \beta^2 \Gamma \pm \frac{1}{2} \left( \beta^2 - 1 \right) \Gamma^2 - \frac{\eta^2\Omega^2}{2} + \eta\Omega \sqrt{\eta^2\Omega^2 + 2\Gamma^2}. \]  

In particular, the two \( \gamma_{R, \text{min}} \) approach \( \gamma_R = 0.5 \) as \( \beta \) decreases from 1, and they coalesce at the point \( \beta = \beta_0 \) which is shown in figure 7(c). Once \( \beta < \beta_0 \), as shown in figure 7, the system no longer allows the optimal minimal \( \langle n \rangle_{\text{st}} \) predicted by equation (A5), and the minimal \( \langle n \rangle_{\text{st}} \) gradually regresses to the single-ion cooling limit.

Superior cooling parameter regime

We now try to find the superior cooling parameter regime from equation (A4). It can be shown that \( \langle n \rangle_{\text{st}} \) exceeds \( \langle n \rangle_{\text{st}}^r \) when

\[ 3\eta^2\Omega^2 > 2\Gamma^2 - 8\gamma R|_{\beta} \]  

and the superior cooling parameter regime \( (\langle n \rangle_{\text{st}} < \langle n \rangle_{\text{st}}^r) \) corresponds to

\[ 3\eta^2\Omega^2 < 2\Gamma^2 - 8\gamma R|_{\beta}. \]

We note that there is a constraint: \( 8\gamma R|_{\beta} \leq 2\beta^2\Gamma^2 \). This means that \( \langle n \rangle_{\text{st}} \) can exceed \( \langle n \rangle_{\text{st}}^r \) only when

\[ \beta^2 \geq 1 - \frac{3\eta^2\Omega^2}{2\Gamma^2}, \]

and the boundary of the superior cooling parameter regime is determined by

\[ \gamma_R = \frac{1}{2} \beta \Gamma \pm \frac{1}{2} \left( \beta^2 - 1 \right) \Gamma^2 + \frac{3}{2} \eta^2\Omega^2. \]  

However, for the strong field regime such that \( \Gamma^2 < 3\eta^2\Omega^2/2 \), every configuration of \( \beta \) and \( \gamma R|_{\beta} \) results in \( \langle n \rangle_{\text{st}} > \langle n \rangle_{\text{st}}^r \) according to equation (A11). Thus, we can only achieve the superior cooling parameter regime when equation (A7) holds, under which \( \beta \) and \( \gamma_R \) can be tuned to realize the best performance in equation (A5).

Cooling without nonguided decay (\( \beta = 1 \))

To discuss the chiral-coupling-assisted cooling with the ideal chiral coupling (\( \beta = 1 \)), we can adopt the result of equation (A4) by setting \( \gamma = \Gamma = \gamma_R + \gamma_L \), which leads to equation (6) in the main text

\[ \langle n \rangle_{\text{st}} = \left( \frac{\gamma_R - \gamma_L}{16\nu^2} \right)^2 + \left( 1 + \frac{8\gamma R|_{\beta}}{\eta^2\Omega^2 + 2(\gamma_R - \gamma_L)^2} \right) \]

\[ \times \frac{\eta^2\Omega^2}{8\nu^2}. \]  

(A15)
With the constraint $0 \leq (\gamma_R - \gamma_L)^2 \leq \gamma^2$, there are three values of $\gamma_R - \gamma_L$ that determine the local extremes of $\langle n_1 \rangle_{st}$:

$$\gamma_R - \gamma_L \big|_{\max} = 0, \quad (A16)$$

$$\gamma_R - \gamma_L \big|_{\min} = \pm \sqrt{-\frac{\eta^2 \Omega^2}{2} + \eta \Omega \sqrt{\eta^2 \Omega^2 + 2 \gamma^2}}. \quad (A17)$$

Here, equation (A16) corresponds to the local maximum of $\langle n_1 \rangle_{st}$:

$$\langle n_1 \rangle_{\max} = \frac{\gamma^2}{4 \mu^2} + \frac{\eta^2 \Omega^2}{8 \mu^2}, \quad (A18)$$

and equation (A17) corresponds to the same local minimum of $\langle n_1 \rangle_{st}$:

$$\langle n_1 \rangle_{\min} = \frac{\eta \Omega}{8 \mu^2} \sqrt{\eta^2 \Omega^2 + 2 \gamma^2} - \frac{\eta^2 \Omega^2}{32 \mu^2}. \quad (A19)$$

In addition, the superior cooling parameter regime ($\langle n_1 \rangle_{st} < \langle n_1 \rangle_{\text{opt}}$) is given by equation (A12) at $\gamma = \Gamma = \gamma_R + \gamma_L$:

$$3 \eta \Omega^2 < 2 (\gamma_R - \gamma_L)^2, \quad (A20)$$

which is a straight line in a $\Omega$ versus $\gamma_R$ plot as shown in figure 2(b).

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