Quadratic Constraints for Local Stability Analysis of Quadratic Systems

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Abstract—This paper proposes new quadratic constraints (QCs) to bound a quadratic polynomial. Such QCs can be used in dissipation inequalities to analyze the stability and performance of nonlinear systems with quadratic vector fields. The proposed QCs utilize the sign-indefiniteness of certain classes of quadratic polynomials. These new QCs provide a tight bound on the quadratic terms along specific directions. This reduces the conservatism of the QC bounds as compared to the QCs in previous work. Two numerical examples of local stability analysis are provided to demonstrate the effectiveness of the proposed QCs.

I. INTRODUCTION

Quadratic systems are an important class of nonlinear dynamics. A generic nonlinear system can be approximated by a quadratic system through a Taylor series expansion. This improves the approximation compared to linearization [1]. Further, some systems are directly modeled by quadratic dynamics: e.g., fluid flows governed by the incompressible Navier-Stokes equations. These dynamics are quadratic, and linear analysis is often insufficient due to significant nonlinear effects [2]. Furthermore, quadratic systems can model complex nonlinear behavior such as chaos [3] and limit cycle oscillations [4]. Thus, approaches to analyze quadratic systems can benefit scientific and engineering applications.

Dissipation inequalities can be used to analyze many dynamical system properties, such as stability, reachability, and robustness [5]. The analysis approach generally involves searching for a valid storage function that certifies the dissipativity. The certification can often be posed as a convex optimization problem, such as a semi-definite program (SDP). These convex optimization problems can be solved efficiently, enabling system analysis and control design algorithms [6].

Quadratic constraint (QC) is a modeling framework that abstracts a nonlinearity as a quadratic inequality of the input and output of the functions [7]. QCs allow one to analyze nonlinear systems through dissipation inequalities [8] at the expense of conservatism due to abstraction. For quadratic polynomials, a few local QCs are proposed in the literature in the context of region of attraction (ROA) analysis for fluid systems. QCs were derived in [9] and [10] to bound a quadratic polynomial in a spherical local region. These QCs were further generalized to an ellipsoidal local region in [11]. Recently, [12] proposed QCs to capture the interaction of quadratic polynomials in a hyperrectangle. These works pursued QC-based approaches over prevailing sum-of-squares optimization techniques [13] in order to achieve scalable algorithms for large-dimensional systems.

In this paper, we explore the function landscape of quadratic polynomials and proposed new QCs to tighten the description along the direction which the function equals to zero. These QCs reduce the conservatism of QCs presented in [11], and can also generalize the QCs proposed in [12]. Finally, the effectiveness of the proposed QCs are investigated with two numerical example with ROA estimation problems.

II. PROBLEM FORMULATION

A. Quadratic Nonlinear System

There are \( m = \frac{n^2 + n}{2} \) quadratic monomials that can be constructed from \( x \in \mathbb{R}^n \). Let \( z : \mathbb{R}^n \rightarrow \mathbb{R}^m \) denote the function that constructs the vector of such monomials:

\[
z(x) = \begin{bmatrix} x_1^2 & x_1 x_2 & \ldots & x_2 x_3 & \ldots & x_n^2 \end{bmatrix}^T,
\]

Note that any homogeneous quadratic function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is a linear combination of quadratic monomials. In other words, if \( \phi(x) = x^T Q x \) for some matrix \( Q = Q^T \in \mathbb{R}^{n \times n} \) then there exists \( b \in \mathbb{R}^m \) such that \( \phi(x) = b^T z(x) \). Note that the matrix \( Q \) can be constructed from the Hessian of \( \phi \): \( Q = \frac{1}{2} \nabla^2 \phi \). Both forms for a quadratic function (expressed as \( x^T Q x \) or \( b^T z(x) \)) will be used throughout the paper.

Consider a quadratic polynomial system of the form:

\[
\dot{x}(t) = Ax(t) + Bz(x(t)),
\]

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \). The system can have multiple equilibrium points in general but we focus on \( x_e = 0 \). We assume \( A \) is Hurwitz so that \( x_e = 0 \) is locally asymptotically stable. Other equilibrium points can be shifted to the origin via a coordination transformation to get the same form of quadratic system (2) as shown in [14].

The Lur’e decomposition [1] poses the system (2) as:

\[
\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t)
\]

\[
w(t) = z(x(t)).
\]

The Lur’e decomposition separates the linear time-invariant dynamics from the quadratic nonlinearity as shown in Fig. 1. This decomposition enables one to analyze the quadratic system using dissipation inequality with QCs [8].

This research was sponsored by the US Army Research Office and was accomplished under Grant Number W911NF-20-1-0156. The work of Maziar S. Hemati was supported in part by the Air Force Office of Scientific Research under award numbers FA9550-21-1-0106 and FA9550-21-1-0434, the National Science Foundation under award number CBET-1943988, and the Office of Naval Research under award number N00014-22-1-2029.

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*This research was sponsored by the US Army Research Office and was accomplished under Grant Number W911NF-20-1-0156. The work of Maziar S. Hemati was supported in part by the Air Force Office of Scientific Research under award numbers FA9550-21-1-0106 and FA9550-21-1-0434, the National Science Foundation under award number CBET-1943988, and the Office of Naval Research under award number N00014-22-1-2029.
B. Existing Local Quadratic Constraints

The effect of the nonlinearity $z(x)$ can be bounded in a local region $\mathcal{D} \subset \mathbb{R}^n$ using quadratic constraints (QCs). These QCs take the following form:

$$\begin{bmatrix} x \\ w \end{bmatrix}^T M_i \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \quad \forall w = z(x), x \in \mathcal{D},$$

where $M_i \in \mathbb{R}^{(n+m) \times (n+m)}$ and $i = 1, \ldots, k$. Note that the subscript $i$ is an indexing number, since a nonlinearity can satisfy multiple QCs. Throughout the paper, we consider a local ellipsoidal region of the form $\mathcal{D} = \mathcal{E}_\alpha := \{ x : x^T E x \leq \alpha^2 \}$, where $E = E^T \in \mathbb{R}^{n \times n}$ is a positive definite matrix and $\alpha \in \mathbb{R}$ is a positive scalar.

We summarize two types of QCs that have been developed in the literature. These form the foundation of our new QCs presented in Section III. The QC in Lemma 1 below is stated in [11] and generalizes results in [10].

**Lemma 1 (Cauchy–Schwarz QC).** Let the local ellipsoidal region $\mathcal{E}_\alpha$ be given. A nonlinearity $\phi(x) = x^T Q x$ satisfies the following local QC:

$$\begin{bmatrix} x \\ \phi(x) \end{bmatrix}^T \begin{bmatrix} \alpha^2(QE^{-1}Q) & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ \phi(x) \end{bmatrix} \geq 0 \forall x \in \mathcal{E}_\alpha. \quad (5)$$

The Cauchy-Schwarz QC (CSQC) is named as it involves Cauchy-Schwarz in the bounding process. This QC (5) can be re-written in the form shown in (4). Specifically, express the quadratic nonlinearity in the form $\phi(x) = b^T w$ and substitute $[\phi(x)] = [b \phi(x)]^T$ into (5). Lemma 1 provides a constraint for an arbitrary quadratic function on an ellipsoid. The next lemma provides a bound for products of quadratic functions with special structure. It was originally proposed in Section IV-A of [12].

**Lemma 2.** Let the local region $\mathcal{E}_\alpha$ and two quadratic function $\phi_1(x)$ and $\phi_2(x)$ be given. If $\phi_1(x) \phi_2(x) = x_i^2 x_j x_k$ with $j \neq k$, then the QC holds:

$$\begin{bmatrix} x \phi_1(x) \\ \phi_2(x) \end{bmatrix}^T \begin{bmatrix} \alpha^2(E^{-1})_{ii}c_{e_i} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \phi_1(x) \\ \phi_2(x) \end{bmatrix} \geq 0 \forall x \in \mathcal{E}_\alpha, \quad (6)$$

where $(E^{-1})_{ii}$ is the $(i, i)$ entry of $E^{-1}$, $c = e_j + e_k$, and $e_j, e_k \in \mathbb{R}^n$ are standard basis vectors.

A similar variable substitution can be used to re-write (6) in the form of (4). The QC in [12] was formulated using a hyperrectangle for the local region. Lemma 2 is a variation stated using an ellipsoid $\mathcal{E}_\alpha$ for the local region. This causes a slight difference in the coefficient matrix in the QC. Section III-C will present a more general QC (with proof) which includes Lemma 2.

C. Local Stability Condition with QC and Lyapunov Stability

The QCs (4) can be used to formulate a Lyapunov condition for local stability analysis. Here, we illustrate the approach with a condition to estimate the region of attraction (ROA) for the system in (2). The ROA of an equilibrium $x_e = 0$ is defined as the set of initial conditions for which the solution $x(t)$ of (2) asymptotically converges to the equilibrium. The next theorem from [11] provides a matrix inequality condition that gives a spherical ROA estimate of the system (2).

**Theorem 1.** Let $E = E^T > 0$ and $\alpha > 0$ be given. Moreover, assume the nonlinearity $z(\cdot)$ in the system (3) satisfies a set of QCs (4). If $\exists P = P^T \in \mathbb{R}^{n \times n}, r > 0$ and $\xi_1, \ldots, \xi_k \in \mathbb{R}$ such that:

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \sum_{i=1}^{k} \xi_i M_i \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix} \quad (7a)$$

$$\frac{1}{\alpha^2} E \preceq P \preceq \frac{1}{r^2} I \quad (7b)$$

$$\xi_i \geq 0 \text{ for } i = 1, \ldots, k, \quad (7c)$$

then $x_e = 0$ is a locally asymptotically stable equilibrium. Moreover, $\{ x : x^T x \leq r^2 \}$ is a ROA estimate of system (2).

**Proof.** The proof relies on standard Lyapunov stability arguments [1] combined with QCs [7]. A proof is given in [11] and [10] but is briefly summarized here for completeness. Define the Lyapunov function $V(x) := x^T P x$. Inequality (7b) implies that $V$ is positive definite. Left/right multiply (7a) by $[x(t)^T \ w(t)^T]$ and its transpose to show:

$$\frac{d}{dt} V(x(t)) + \sum_{i=1}^{k} \xi_i \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T M_i \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} < -\epsilon \| x(t) \|^2_2.$$

This implies $\frac{d}{dt} V(x(t)) \leq -\epsilon \| x(t) \|^2_2$ for any $x(t) \in \mathcal{E}_\alpha$, since $\xi_i$ and the QCs are non-negative for any $x(t) \in \mathcal{E}_\alpha$. The equilibrium $x_e = 0$ is locally asymptotically stable by Lyapunov stability theory [1].

Inequality (7b) implies that the level set $\{ x : x^T P x \leq 1 \}$ is contained in $\mathcal{E}_\alpha$. Thus $x(t)$ converges to $x_e$ for any initial condition in the level set $\{ x : x^T P x \leq 1 \}$, i.e. the level-set is contained in the region of attraction. Finally, inequality (7b) implies that the spherical set $\{ x : x^T x \leq r^2 \}$ is contained in the level set $\{ x : x^T P x \leq 1 \}$.

Note that Lyapunov stability condition can be viewed as a special case of dissipation inequality. Similar conditions as in Theorem 1 can be formulated for other system properties, such as reachability, robustness, and performance.

D. Conservatism of Existing QCs

The QCs bound the effect of the nonlinearity in the local region. This enables the estimate of the ROA of (2) (or other system properties) via Lyapunov or dissipation inequality.
conditions. However, if the QC bounds the nonlinearity too “loosely” then the analysis condition will be conservative. The remainder of this section provides an example to illustrate this issue. This motivates the construction of new QCs in Section III.

Here, we present that the CSQC (5) fails to tightly bound a quadratic function $x^\top Q x$ where $Q$ is sign-indefinite. To illustrate, consider the case $\phi(x) = x_1 x_2$ in the local region of a unit sphere ($E = I$ and $\alpha = 1$). The function $\phi(x)$ corresponds to the matrix $Q = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$. The CSQC (5) on $\phi(x)$ corresponds to the inequality:

$$0.25(x_1^2 + x_2^2) \geq \phi(x)^2 \quad \forall x^\top x \leq 1. \quad (8)$$

Fig. 2 visualizes each side of the inequality (8). The landscape of the right-hand side ($\phi(x)^2$, green surface) has peaks along the directions $x_1 = \pm x_2$ and valleys along $\phi(x) = 0$ ($x_1 = 0$ and $x_2 = 0$). Note that the left side ($0.25(x_1^2 + x_2^2)$, blue surface) provides a tight upper bound of the green surface along the peaks. However, the blue surface provides a loose bound along the valleys of the green surface.

These landscape properties (peaks and valleys) are an inherent feature of a sign-indefinite quadratic function $\phi(x)$, as they have multiple directions for which $\phi(x) = 0$. Hence, the loose bound of QCSC (5) does not depend on the shape and the size of the ellipsoidal region $E_n$. Furthermore, this looseness will potentially lead to conservative analysis. The next section proposes new QCs to reduce the conservatism.

### III. Local QCs on Quadratic Nonlinearities

New QCs are introduced in Section III-A and III-B to reduce the conservatism of CSQC (5) by capturing the landscape properties of sign-indefinite quadratic functions. Specifically, QCs are presented for sign-indefinite quadratic function $x^\top Q x$ with $Q$ being rank-2 and rank-3. Furthermore, the method derived in Section III-A is applied to generalize the QC (6) in Section III-C.

#### A. QCs on Rank-2 Sign-indefinite Quadratic Functions

The CSQC in (8) provides a bound for $\phi(x) = x_1 x_2$ on the unit sphere. Consider the following alternative bound:

$$x_1^2 \geq \phi(x)^2 \quad \forall x^\top x \leq 1. \quad (9)$$

This is a valid QC since $x_1^2 - \phi(x)^2 = (1 - x_2^2)x_1^2$ and $x_2^2 \leq x^\top x \leq 1$. Fig. 3 illustrates that the left side of (9) ($x_1^2$, red surface) provides an upper bound of the right side ($\phi(x)^2$, green surface). Furthermore, this QC is specifically tight along the direction $x_1 = 0$. Similarly, the inequality

$$x_2^2 \geq \phi(x)^2 \quad \forall x^\top x \leq 1 \quad (10)$$

is also a valid QC in the unit sphere. The left side of QC (10) corresponds to a similar surface as the red surface with 90-degree rotation, i.e., it tightly bounds $\phi(x)^2$ along the direction $x_2 = 0$. Note that the two QCs (9) and (10) are each tight on one valley of $\phi(x)^2$. QCs (9) and (10) together with the CSQC (8) tightly bound the peaks and valleys of the quadratic function $\phi(x) = x_1 x_2$.

Here, we generalize the above QCs beyond quadratic monomials $\phi(x) = x_1 x_2$ to any quadratic function having a similar function landscape. The following lemmas establish the proposed constraints using the largest eigenvalue of $Q$, denoted as $\lambda_{max}(Q)$.

**Lemma 3.** Let $E = E^\top > 0$, $\alpha > 0$, and matrix $Q = Q^\top \in \mathbb{R}^{n \times n}$ be given. Assume $\lambda_{max}(Q) > 0$. Then:

$$\alpha^2 \lambda_{max}(Q) = \max_{x \in \mathbb{R}^n} x^\top Q x,$$

where $\hat{Q} = E^{-\frac{1}{4}} Q E^{-\frac{1}{4}}$. Furthermore, if $Q = c c^\top$ for some nonzero $c \in \mathbb{R}^n$, then $\lambda_{max}(\hat{Q}) = c^\top E^{-1} c > 0$.
Proof. Define $y = \frac{1}{\alpha} E^{\frac{1}{2}} x$ and $\tilde{Q} = E^{-\frac{1}{2}} Q E^{-\frac{1}{2}}$ so that the constrained optimization problem becomes:

$$\max_{y' \in S^1} \alpha^2 y^\top \tilde{Q} y.$$ 

Note that $\lambda_{\max}(Q) > 0$ implies $\lambda_{\max}(\tilde{Q}) > 0$ (Theorem 4.5.8 [15]). Hence, the problem corresponds to finding the largest eigenvalue of $\tilde{Q}$ (Theorem 4.2.2 [15]). Furthermore, if $Q = cc^\top$ is an outer product of a vector $c \in \mathbb{R}^n$, then $\tilde{Q}$ is rank-1 and has an eigenvector $E^{-\frac{1}{2}} c$ with associated eigenvalue $c^\top E^{-1} c > 0$.

Lemma 4. Let $E = E^\top > 0$, $\alpha > 0$ and vectors $c_1, c_2 \in \mathbb{R}^n$ be given. The quadratic function $\phi(x) = x^\top Q x$ with $Q = \frac{1}{2}(c_1 c_2^\top + c_2 c_1^\top)$ satisfies the following inequality:

$$\begin{bmatrix} x \cr \phi(x) \end{bmatrix}^\top \begin{bmatrix} \alpha^2 W & 0 \\
0 & -1 \end{bmatrix} \begin{bmatrix} x \\
\phi(x) \end{bmatrix} \geq 0 \quad \forall x \in \mathcal{E}_\alpha$$

with $W = (c_1^\top E^{-1} c_1) c_2 c_2^\top$ or $(c_2^\top E^{-1} c_2) c_1 c_1^\top$.

Proof. Define $y = \frac{1}{\alpha} E^{\frac{1}{2}} x$ and $\tilde{Q} = E^{-\frac{1}{2}} Q E^{-\frac{1}{2}}$ so that the constrained optimization problem becomes:

$$\max_{y' \in S^1} \alpha^2 y^\top \tilde{Q} y.$$ 

Note that $\lambda_{\max}(Q) > 0$ implies $\lambda_{\max}(\tilde{Q}) > 0$ (Theorem 4.5.8 [15]). Hence, the problem corresponds to finding the largest eigenvalue of $\tilde{Q}$ (Theorem 4.2.2 [15]). Furthermore, if $Q = cc^\top$ is an outer product of a vector $c \in \mathbb{R}^n$, then $\tilde{Q}$ is rank-1 and has an eigenvector $E^{-\frac{1}{2}} c$ with associated eigenvalue $c^\top E^{-1} c > 0$.

Inequalities (9) and (10) are examples of Rank-2 Valley QCs (13) with $E = I, \alpha = 1$, and $Q = [\mathbf{0} \mid \mathbf{1}]$.

B. QCs on Rank-3 Sign-indefinite Quadratic Functions

The concept of Rank-2 Valley QCs (13) can be extended to more general quadratic functions. Here, we consider a quadratic function $x^\top Q x$, where $Q$ is rank 3 with two positive and one negative eigenvalue. Specifically, let $(\lambda_1, c_1)$ be the eigenpairs of $Q$ for $i = 1, 2, 3$ with $\lambda_1, \lambda_2$ being positive and $\lambda_3$ being negative. Then $Q$ can be written as $\frac{1}{2}(c_1 c_1^\top + c_2 c_2^\top) + c_3 c_3^\top$, where $c_1 = \sqrt{\lambda_1} v_1 + \sqrt{\lambda_3} v_3, c_2 = \sqrt{\lambda_2} v_2 - \sqrt{\lambda_3} v_3$ and $c_3 = \sqrt{\lambda_3} v_3$. Note that $c_1, c_2$ are nonzero, linearly independent vectors with $c_3$ orthogonal to $c_1$ and $c_2$. These facts are shown in Appendix II of [16]. The next theorem provides QCs for nonlinearity of this form.

Theorem 3 (Rank-3 Valley QC). Let $c_1, c_2, c_3 \in \mathbb{R}_n$ be three nonzero, linearly independent vectors with $c_3$ orthogonal to $c_1$ and $c_2$. Define the quadratic function $\phi : \mathbb{R}^n \to \mathbb{R}$ as $\phi(x) = x^\top Q x$ where $Q = \frac{1}{2}(c_1 c_1^\top + c_2 c_2^\top) + c_3 c_3^\top$. There exists $b \in \mathbb{R}^m$ such that $\phi(x) = b^\top w$. Moreover, $\phi$ satisfies the following local QCs:

$$\begin{bmatrix} x \\
w \end{bmatrix}^\top \begin{bmatrix} \alpha^2(W + \gamma c_3 c_3^\top) & 0 \\
0 & -bb^\top \end{bmatrix} \begin{bmatrix} x \\
w \end{bmatrix} \geq 0 \quad \forall x \in \mathcal{E}_\alpha$$

with $W = (c_1^\top E^{-1} c_1) c_2 c_2^\top$ or $(c_2^\top E^{-1} c_2) c_1 c_1^\top$ and $\gamma = \lambda_{\max}(E^{-\frac{1}{2}} (2Q - c_3 c_3^\top) E^{-\frac{1}{2}})$.

Proof. Note that $\phi(x) = (c_1^\top x)(c_1^\top x) + (c_2^\top x)^2$. Lemma 3 and 4 imply that the inequality below holds for all $x \in \mathcal{E}_\alpha$:

$$\phi(x)^2 = (c_1^\top x)^2 (c_1^\top x) + (c_2^\top x)^2 \leq \alpha^2 x^\top W x + \alpha^2 \gamma c_3 c_3^\top x \quad \forall x \in \mathcal{E}_\alpha.$$ 

Substitute $\phi(x) = b^\top w$ and re-arrange the inequality into the quadratic form to obtain (14)

Note that the class of quadratic functions in Theorem 3 is exactly the class with $Q$ being rank 3 with two positive and one negative eigenvalue [16]. The Rank-3 Valley QCs (14) tighten the characterization of this class of quadratic function by the CSQC (5). Furthermore, $Q$ can be alternatively written as $\frac{1}{2}(c_1 c_1^\top + c_2 c_2^\top) + c_3 c_3^\top$, where $c_1 = \sqrt{\lambda_1} v_1 + \sqrt{\lambda_3} v_3, c_2 = \sqrt{\lambda_2} v_2 - \sqrt{\lambda_3} v_3$ and $c_3 = \sqrt{\lambda_3} v_3$. Hence, there exist four Rank-3 Valley QCs for a nonlinearity with rank-3 sign-indefinite matrix $Q$.

If a quadratic function $x^\top Q x$, where $Q$ has exactly one positive eigenvalue and exactly two negative eigenvalues, then the QCs (14) with $Q = -Q$ are valid QCs with identical proof. Also, Theorem 3 recovers Theorem 2 if we choose $c_3 = 0$ in (14) for the special case when $Q$ being rank 2.

C. QC on the Cross-product of Monomials

Section III-A and III-B consider QCs to bound the effect of a single quadratic function. This section considers QCs to bound the cross-product of two monomials. This generalizes the QC (6) developed previously in [12]. The next lemma provides an upper and lower bound on the cross-product.
Lemma 5. Let \( w_p \) and \( w_q \) be quadratic monomials such that the cross-product has the form \( w_p w_q = x_i^2 x_j x_k \) with \( j \neq k \). The following inequalities hold:

\[
-\alpha^2 (x_j - x_k)^2 \leq 2w_p w_q \leq \alpha^2 (x_j + x_k)^2.
\]  

(15)

Proof. Note that \( 2x_i^2 x_j x_k = x_i^2 ((x_j + x_k)^2 - x_j^2 - x_k^2) \) and hence \( 2w_p w_q \leq x_i^2 (x_j + x_k)^2 \). Similarly, \( 2w_p w_q = x_i^2 (x_j^2 + x_k^2 - (x_j - x_k)^2) \) and hence \( 2w_p w_q \geq -x_i^2 (x_j - x_k)^2 \). \qed

The next theorem utilizes the bounds (15) to provide QCs on the cross-product of monomials with the method developed in Section III-A.

Theorem 4 (Cross-Product QC). Let \( w_p \) and \( w_q \) be quadratic monomials such that their cross-product has the form \( w_p w_q = x_i^2 x_j x_k \) with \( j \neq k \). Then the cross-product satisfies the following four QCs:

\[
\begin{bmatrix}
  x \\
  w
\end{bmatrix}
W
\begin{bmatrix}
  x \\
  w
\end{bmatrix}
\geq 0 \quad \forall x \in E_\alpha \quad \quad (16a)
\]

where \( E_\alpha \) is fixed in all comparisons. The index \( i \) can be arbitrary in Lemma 5 and Theorem 4, including \( j \) or \( k \). Note that the case \( j = k \) is excluded, as the relaxation (15) is not necessary. The corresponding QCs for the case \( w_p w_q = x_i^2 x_k^2 \) can be formed as:

\[
\begin{bmatrix}
  x \\
  w
\end{bmatrix}
W
\begin{bmatrix}
  x \\
  w
\end{bmatrix}
\geq 0 \quad \forall x \in E_\alpha \quad \quad (17)
\]

The above bounding procedure in Lemma 5 and Theorem 4 can be extended to a cross-product of general quadratic functions with similar structures. However, more specific conditions and the resulting QCs require dedicated study for the particular quadratic functions of interest.

IV. NUMERICAL EXAMPLES

The proposed QCs are illustrated via an ROA estimation problem. The analysis algorithm is adopted from [11] with simplification detailed in the following paragraph. Note that the proposed QC can also be incorporated into any algorithm utilizes QC, e.g. [12]. The intention of this section is to compare the effectiveness of newly introduced QCs to the existing QCs without involving advanced algorithms for this particular analysis. Hence, comparison against the full algorithms in [11] and [12] is not provided.

The largest ROA estimation is obtained by maximizing \( r \) over \( P \), \( r \), \( \xi \) subject to constraints (7) in Theorem 1. This optimization problem is an SDP for given \( E \) and hence the optimal \( r^* \) can be solved efficiently. The largest \( r^* \) is computed over a grid of \( \alpha \) with a given shape local region \( E \). In this paper, \( E = I \) is fixed in all comparisons. The results can be improved by iteratively updating \( E \) as in [11].

The existing CSQC (5) from literature [11] serves as the baseline analysis. It is compared against results that incorporate the proposed Rank-2 Valley QCs (13), Rank-3 Valley QCs (14), and Cross-Product QCs (16).

A 2-state system and a 3-state system are investigated. Both examples were implemented in MATLAB with CVX [17] and the SDP solver MOSEK [18]. The implementation is made available online*. Note that the effectiveness of proposed work depends on the specific dynamics. For example, the four-state shear flow problem discussed in [10], [11], [12] is not included as the new QCs provide only small improvement on this example.

A. 2-state Example

Consider the quadratic nonlinear system [14]:

\[
\begin{bmatrix}
  d \\
  \frac{dx_1}{dt} \\
  \frac{dx_2}{dt}
\end{bmatrix} = \begin{bmatrix}
  -50 & -16 \\
  13 & -9
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} + \begin{bmatrix}
  13.8 \\
  5.5
\end{bmatrix} x_1 x_2.
\]

(18)

The system has one stable equilibrium at the origin. By simulating trajectories, the phase portrait (Fig. 4) indicates the largest spherical ROA has a radius about \( r^* \approx 4.95 \) with the unstable region at the upper-right.

The nonlinearity \( x_1 x_2 \) in system (18) can be bounded by both the CSQC (5) and Rank-2 Valley QCs (13). The ROA estimation is performed with two sets of QCs. Set 1 applies CSQC (5) on the nonlinearity \( x_1 x_2 \). Set 2 applies both CSQC (5) and Rank-2 Valley QCs (13) on \( x_1 x_2 \).

*Source code is available at https://github.com/SCLiao47/ValleyQC_ROA titled ValleyQC_ROA on GitHub.com.
The QC analysis results are visualized in Fig. 4. Set 2 \( (r^*_2 = 3.5224) \) gives a less conservative estimate than Set 1 \( (r^*_1 = 2.7355) \) by incorporating the Rank-2 Valley QCs (13). The results illustrate the effectiveness of Rank-2 Valley QCs (13) on the quadratic nonlinearity in (18).

**B. 3-state Example**

Consider the 3-state system \( \dot{x} = Ax + Bw \):

\[
A = \begin{bmatrix}
-1 & -1 & -1 \\
-1 & -1 & -6 \\
-1 & -6 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -6 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix},
\]

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}^T, \quad w = \begin{bmatrix}
x_1^2 \\
x_2 x_3 \\
x_2^3 \\
x_3 x_1 \\
x_2^2 x_3 \\
x_3^2
\end{bmatrix}^T.
\]

By numerically solving trajectories, the spherical ROA estimate of the system has an upper bound \( r^* \approx 2.4283 \), where there exists an initial condition not converging to \( x = 0 \). The largest spherical ROA estimate has a radius smaller than \( r \).

The analysis is performed with different sets of QCs. The CSQC (5) is applied to each monomial and \( b_i^j w \) where \( i = 1, 2, 3 \), where \( b_i^j \) is the \( i \)-th row vector of the matrix \( B \). The Rank-2 Valley QCs (13) are applied to each sign-definite monomial \( w_i, b_i^j w \) and \( b_i^j w \). The Rank-3 Valley QCs (14) are applied on \( b_i^j w \). The Cross-Product QCs (16) are applied on each pair of monomials satisfying the conditions.

**TABLE I** summarizes the setting of analysis and the results for eight sets of QCs. Each of the Set 2, 3, and 4 gives a less conservative result than Set 1. The results indicate that each of the proposed QCs improved the analysis individually. Furthermore, Set 5, 6, and 7 show that the analysis result could be improved by including multiple proposed QC into the analysis. Lastly, Set 5 and 8 give the least conservative estimation among all sets. The two analysis are the same up to the numerical tolerance of the solver. While this might imply adding Cross-Product QCs does not improve the analysis, this could be because of the specific system (19) and stability condition used. Another system or stability condition could have different results.

**TABLE I**

| Set # | CSQC | Rank-2 | Rank-3 | Cross-Product | QC # | \( r^* \) |
|-------|------|--------|--------|---------------|------|---------|
| Set 1 | ✓    | ✓      |        |               | 9    | 0.7173  |
| Set 2 | ✓    | ✓      | ✓      |               | 19   | 1.2041  |
| Set 3 | ✓    | ✓      | ✓      |               | 13   | 0.8487  |
| Set 4 | ✓    | ✓      | ✓      |               | 63   | 0.7900  |
| Set 5 | ✓    | ✓      | ✓      |               | 23   | 1.3365  |
| Set 6 | ✓    | ✓      | ✓      |               | 73   | 1.2468  |
| Set 7 | ✓    | ✓      | ✓      |               | 67   | 0.8846  |
| Set 8 | ✓    | ✓      | ✓      |               | 77   | 1.3365  |

**V. Conclusions**

In this work, we proposed new quadratic constraints to reduce conservatism in the analysis of quadratic systems using dissipation inequalities. The proposed QCs exploit the property of sign-definite quadratic polynomials to tighten the bound along with the QC previously derived in [11]. The effectiveness of the proposed QCs is illustrated by successfully enlarging ROA estimations in two numerical examples. Future work includes applying the QCs to other system analysis problems and investigating the computational scalability of the proposed method.

**ACKNOWLEDGMENT**

The authors would like to thank Talha Mushtaq and Diganta Bhattacharjee for valuable discussion.

**References**

[1] H. K. Khalil, *Nonlinear systems; 3rd ed.* Upper Saddle River, NJ: Prentice-Hall, 2002.

[2] P. J. Schmid and D. S. Henningson, *Stability and transition in shear flows*, vol. 142. Springer Science & Business Media, 2000.

[3] E. N. Lorenz, “Deterministic nonperiodic flow,” *Journal of atmospheric sciences*, vol. 20, no. 2, pp. 130–141, 1963.

[4] N. Kuznetsoy, O. Kuznetsoyova, and G. Leonov, “Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system,” *Differential equations and dynamical systems*, vol. 21, no. 1, pp. 29–34, 2013.

[5] M. Arcak, C. Meissen, and A. Packard, *Networks of dissipative systems: compositional certification of stability, performance, and safety*. Springer, 2016.

[6] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. SIAM, 1994.

[7] A. Megretski and A. Rantzer, “System analysis via integral quadratic constraints,” *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 819–830, 1997.

[8] P. Seiler, “Stability analysis with dissipation inequalities and integral quadratic constraints,” *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1704–1709, 2014.

[9] A. Kalur, P. Seiler, and M. S. Hemati, “Nonlinear stability analysis of transitional flows using quadratic constraints,” *Physical Review Fluids*, vol. 6, no. 4, p. 044401, 2021.

[10] C. Liu and D. F. Gayme, “Input-output inspired method for permissible perturbation amplitude of transitional wall-bounded shear flows,” *Phys. Rev. E*, vol. 102, p. 063108, Dec 2020.

[11] A. Kalur, T. Mushtaq, P. Seiler, and M. S. Hemati, “Estimating regions of attraction for transitional flows using quadratic constraints,” *IEEE Control Systems Letters*, 2021.

[12] L. F. Toso, R. Drummond, and S. R. Duncan, “Regional stability analysis of transitional fluid flows,” *IEEE Control Systems Letters*, 2022.

[13] P. A. Parrilo, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. California Institute of Technology, 2000.

[14] F. Amato, C. Cosentino, and A. Merola, “On the region of asymptotic stability of nonlinear quadratic systems,” in *2006 14th Mediterranean Conference on Control and Automation*. pp. 1–5, IEEE, 2006.

[15] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge university press, 2012.

[16] S. Liao, M. S. Hemati, and P. Seiler, “Quadratic constraints for local stability analysis of quadratic systems,” *arXiv:2209.03565*, 2022.

[17] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming, version 2.1,” *http://cvxr.com/cvx*, Mar. 2014.

[18] MOSEK ApS, *The MOSEK optimization toolbox for MATLAB manual. Version 9.0.*, 2019.