On two exactly-solvable one-dimensional Hamiltonians with PT symmetry

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We provide an explanation to the behaviour of the spectra of two exactly-solvable one-dimensional Hamiltonians with PT symmetry proposed earlier. We calculate the branch points at which pairs of eigenvalues coalesce and discuss the perturbation series.

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I. INTRODUCTION

Some time ago Ahmed [1] solved the Schrödinger equation with the PT-symmetric potentials $V_1(x) = x^2/4 + ig|x|x$ and $V_2(x) = |x| + igx$ exactly and found an interesting behaviour of the corresponding point spectra. In both cases there is an infinite number of real eigenvalues when $g = 0$ and as $g$ increases the number of real eigenvalues decreases to just one. At every finite nonzero value of $g$ there is an odd number of real eigenvalues (say $2k+1$) and as $g$ increases this number reduces to $2k - 1, 2k - 3, 2k - 5, \ldots, 1$. Ahmed did not explain this phenomenon satisfactorily and did not identify the critical $g$ values at which each change takes place. However, he proposed to call it scarcity of real discrete eigenvalues.

The purpose of this paper is to provide a more detailed discussion of the behaviour of the spectra of those models. In section II we outline some properties of the perturbation series for a particular class of parameter-dependent Hamiltonians and in section III we discuss the spectra of the models mentioned above. Finally, in section IV we draw conclusions.

II. PERTURBATION SERIES

Let $H(\lambda)$ be a parameter-dependent linear operator and $U$ a linear invertible operator such that $UH(\lambda)U^{-1} = H(-\lambda)$. Therefore, if $\psi_n(\lambda)$ is an eigenfunction of $H(\lambda)$ with eigenvalue $E_n(\lambda)$ then it follows from

$$UH(\lambda)\psi_n(\lambda) = UH(\lambda)U^{-1}U\psi_n(\lambda) = H(-\lambda)U\psi_n(\lambda) = E_n(\lambda)U\psi_n(\lambda),$$

that $U\psi_n(\lambda)$ is proportional to an eigenfunction $\psi_m(-\lambda)$ of $H(-\lambda)$ with eigenvalue

$$E_m(-\lambda) = E_n(\lambda).$$

Here, we are interested in the case that $H(0)$ is an Hermitian operator. Since equation (2) is assumed to be valid for all $\lambda$ we conclude that $E_m(0) = E_n(0)$. Consequently, if the eigenvalues of $H(0)$ are nondegenerate then $m = n$ and $E_n(\lambda) = E_n(-\lambda)$ for all $\lambda$.

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If the eigenvalue $E_n(\lambda)$ can be expanded in a Taylor series about $\lambda = 0$

$$E_n(\lambda) = \sum_{j=0}^{\infty} E_n^{(j)} \lambda^j,$$

(3)

then $E_n^{(2j+1)} = 0$ for all $j = 0, 1, \ldots$. If this expansion converges for $|\lambda| < R_n$ then

$$E_n(ig) = \sum_{j=0}^{\infty} E_n^{(2j)} (-g^2)^j,$$

(4)

is real for all $|g| < R_n$ provided that $g$ and the perturbation coefficients $E_n^{(2j)}$ are real.

### III. TWO EXACTLY-SOLVABLE PT-SYMMETRIC MODELS

As indicated in the Introduction Ahmed\(^1\) briefly discussed two exactly-solvable one-dimensional PT-symmetric models. One of them is given by

$$H(\lambda) = p^2 + \frac{1}{4}x^2 + \lambda|x|x,$$

(5)

where, $[x,p] = i$. In this case the parity transformation

$$PpP = -p, \quad PxP = -x,$$

(6)

yields $PH(\lambda)P = H(-\lambda)$, where $P^{-1} = P$. This Hamiltonian operator satisfies the conditions discussed in the preceding section and is PT-symmetric when $\lambda = ig$, $g$ real, because $V(-x)^* = V(x)$ (see, for example, \(^2\) and references therein).

Ahmed\(^1\) proved that when $\lambda = ig$ the eigenvalues are solutions to the nonlinear equation

$$D(E, g) = \Re \left[ \Gamma \left( \frac{3}{4} - \frac{i}{2} \frac{1}{2g} \right) \Gamma \left( \frac{1}{4} - \frac{i}{2} \frac{1}{2g} \right) \right] = 0,$$

(7)

and from a straightforward numerical calculation he conjectured that the number of real eigenvalues decreases from infinity ($g = 0$) to just one (for a sufficiently large value of $g$).

The reason for this behaviour is that as $g$ increases from $g = 0$ a pair of eigenvalues $E_{2n-1}$ and $E_{2n}$, $n = 1, 2, \ldots$, approach each other, coalesce at $g = g_n > 0$ and become a pair of complex conjugate numbers for $g > g_n$. The only eigenvalue that appears to remain real for all $g$ is $E_0$. This result is consistent with the fact that there is just one real eigenvalue when $V(x) = i|x|x$\(^3\) that is the potential for the strong-coupling limit ($g \to \infty$) of (5). In fact, from the canonical transformation

$$U_\gamma xU_\gamma^{-1} = \gamma x, \quad U_\gamma pU_\gamma^{-1} = \gamma^{-1}p,$$

(8)

with $\gamma = g^{-1/4}$ we easily prove that

$$\lim_{g \to \infty} g^{1/2}U_\gamma xU_\gamma^{-1} = p^2 + i|x|x.$$

(9)

Numerical calculation shows that the critical points $g_n$ decrease with $n$ so that for a given value of $g$ only the pairs of eigenvalues with $g_n < g$ are real. When $g > g_1$ there is only one real eigenvalue ($E_0$) and when $g_{n+1} < g < g_n$
there are $2n + 1$ real eigenvalues. This fact explains why Ahmed obtained an odd number of real eigenvalues for every $g < g_1$. For example, Fig. illustrates the coalescence of the eigenvalues $E_1$ and $E_2$. Ahmed suggests that there is just one real eigenvalue for $g = 2$; however, our calculation shows that there are three real eigenvalues: $E_0 = 1.720857958$, $E_1 = 6.579362071$ and $E_2 = 7.39812626$. He probably missed the two eigenvalues $E_1$ and $E_2$ because they are quite close to each other or due to insufficient accuracy in the calculation.

Equation (7) yields either $E(g)$ or $g(E)$. If we take into account that $\frac{dD}{dE} = 0$ when $E$ and $g$ are linked by $D(E, g) = 0$ then we find that

$$\frac{dg}{dE} = \frac{\partial D/\partial E}{\partial D/\partial g}.$$  \hfill (10)

Therefore, the coalescence points that are given by $\frac{dg}{dE} = 0$ (square-root branch points) can be easily obtained by solving the system of nonlinear equations (see, for example, [4, 5])

$$D(E, g) = 0, \frac{\partial D}{\partial E}(E, g) = 0. \hfill (11)$$

Table shows the first five critical points (note that $g_1 > 2$).

As argued in the preceding section, the application of perturbation theory to the model (5) should yield a $g^2$-series for every eigenvalue $E_n(ig)$. Since both $H(0)$ and $dH(\lambda)/d\lambda = |x|x$ are real, then the coefficients $E_n^{(j)}$ are real. The first three terms of the perturbation series for the first three eigenvalues are:

$$E_0(ig) = \frac{1}{2} + (2 + \ln 2) g^2 + \left[ -7 \ln 2 + 3 (\ln 2)^2 + \frac{1}{3} (\ln 2)^3 - \frac{1}{4} \zeta(3) - 12 + \frac{1}{12} \pi^2 \ln 2 - \frac{1}{12} \pi^2 \right] g^4 + \ldots$$

$$= 0.5000000000 + 2.693147181 g^2 - 15.85255355 g^4 + \ldots \hfill (12)$$

$$E_1(ig) = \frac{3}{2} + (3 + 9 \ln 2) g^2 + \left[ -\frac{81}{4} \zeta(3) - 63 \ln 2 + 54 (\ln 2)^2 - 18 + \frac{27}{4} \pi^2 \ln 2 + 27 (\ln 2)^3 - \frac{9}{2} \pi^2 \right] g^4 + \ldots$$

$$= 1.500000000 + 9.238324625 g^2 - 49.30966898 g^4 + \ldots \hfill (13)$$

$$E_2(ig) = \frac{5}{2} + \left( \frac{5}{2} + 25 \ln 2 \right) g^2 + \left[ -\frac{625}{4} \zeta(3) + 75 \ln 2 + \frac{125}{2} (\ln 2)^2 + \frac{625}{12} \pi^2 \ln 2 - \frac{405}{8} + \frac{625}{3} (\ln 2)^3 - \frac{875}{24} \pi^2 \right] g^4 + \ldots$$

$$= 2.500000000 + 14.82867952 g^2 - 90.5745397 g^4 + \ldots \hfill (14)$$

where $\zeta(k)$ is the zeta function. The radius of convergence of the perturbation series for the pair of eigenvalues $E_{2n-1}$ and $E_{2n}$, $n > 0$ is expected to be $g_n$.

The eigenvalues for the second model

$$H(\lambda) = p^2 + |x| + \lambda x,$$  \hfill (15)
when $\lambda = ig$, are roots of the nonlinear expression

\[ D(E, g) = \nu_1 \mu_2 Ai(E/\mu_2) Ai'(E/\mu_1) - \mu_1 \nu_2 Ai(E/\mu_1) Ai'(E/\mu_2) = 0, \]

\[ \nu_1 = -1 + ig, \nu_2 = 1 + ig, \mu_1 = -(\nu_1^2)^{1/3}, \mu_2 = -(\nu_2^2)^{1/3}. \]  

(16)

The behaviour of the spectrum of the PT-symmetric Hamiltonian (15) is similar to the one discussed above in all relevant aspects except one. In this case the strong-coupling limit is given by the canonical transformation with $\gamma = g^{-1/3}$ and

\[ \lim_{g \to \infty} g^{-2/3} U_\gamma H U_\gamma^{-1} = p^2 + ix. \]  

(17)

Bender and Boettcher proved that this model does not exhibit eigenvalues and that the ground state of $H = p^2 - (ix)^N$ diverges as $N \to 1^+$. This is the most relevant difference with respect to the preceding example.

The first four critical points of the spectrum of (15) are shown in Table II.

IV. CONCLUSIONS

The two PT-symmetric Hamiltonians discussed in the preceding section exhibit critical points $g_n > 0$ that decrease with $n$ (we omit the consideration of the case $g < 0$ because $E(-ig) = E(ig)$). Since it appears that \( \lim_{n \to \infty} g_n = 0 \) the number of real eigenvalues is finite for all $g > 0$. This result is important because it may shed light on the spectra of a family of PT-symmetric multidimensional oscillators with critical points that decrease with the magnitude of the coalescing eigenvalues. Present results suggest that the PT-phase transition for the Hamiltonians (5) and (15) takes place at the trivial Hermitian limit $g = 0$. Such a behaviour commonly takes place in multidimensional problems that exhibit some kind of point-group symmetry. For this reason the results above for one-dimensional models that merely exhibit parity symmetry $PH(0)P = H(0)$ appear to be quite interesting.

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TABLE I: First critical points for the Hamiltonian (5)

| n | \( g_n \) | \( E_{2n-1} = E_{2n} \) |
|---|---|---|
| 1 | 2.3262718829432516237 | 7.5466636499753202587 |
| 2 | 0.380315938036989303399 | 5.954372352940988301 |
| 3 | 0.24202011969988939326 | 7.5185674146488458902 |
| 4 | 0.1851158654901497563 | 9.3051810150783521871 |
| 5 | 0.15263064129841349291 | 11.167144554229643582 |

TABLE II: First critical points for the Hamiltonian (15)

| n | \( g_n \) | \( E_{2n-1} = E_{2n} \) |
|---|---|---|
| 1 | 0.627820757384615047 | 3.3482781381164933022 |
| 2 | 0.35319223654461721210 | 4.7569117414811876143 |
| 3 | 0.2541930500674157648 | 6.05266901706522118370 |
| 4 | 0.20120031566314106331 | 7.2414114087019937638 |

FIG. 1: \( D(E, g) \) vs. \( E \) for \( g = 0.4, 0.8, 1.2, 1.6, 2.0, 2.4, 2.8 \)