Aspects of chiral pion–nucleon physics

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Abstract
The next–to–leading order chiral pion–nucleon Lagrangian contains seven finite low–
energy constants. Two can be fixed from the nucleon anomalous magnetic moments
and another one from the quark mass contribution to the neutron–proton mass
splitting. We find a set of nine observables, which to one loop order do only depend
on the remaining four dimension two couplings. These are then determined from a
best fit. We also show that their values can be understood in terms of resonance
exchange related to ∆ excitation as well as vector and scalar meson exchange. In
particular, we discuss the role of the fictitious scalar–isoscalar meson. We also
investigate the chiral expansion of the two P–wave scattering volumes $P_1^−$ and $P_2^+$
as well as the isovector S–wave effective range parameter $b^−$. The one–loop calculation
is in good agreement with the data. The difference $P_1^− − P_2^+$ signals chiral loop
effects in the $\pi N$ P–waves. The calculated D– and F–wave threshold parameters
compare well with the empirical values.

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1 Introduction and summary

Chiral perturbation theory is the tool to systematically investigate the consequences of the spontaneous and explicit chiral symmetry breaking in QCD. S–matrix elements and transition currents of quark operators are calculated with the help of an effective field theory formulated in terms of asymptotically observed fields, the Goldstone bosons and the low–lying baryons. A systematic perturbative expansion in terms of small external momenta and meson masses is possible. We call this double expansion from here on chiral expansion and denote the small parameters collectively by $q$. Beyond leading order, coupling constants not fixed by chiral symmetry appear, the so–called low–energy constants (LECs). For the chiral pion Lagrangian, i.e. the two–flavor case, these were determined more than a decade ago by Gasser and Leutwyler \[1\] by fitting a set of observables calculated at next–to–leading order. In the presence of nucleons, the situation is less satisfactory. At next–to–leading order ($q^2$), seven finite LECs appear \[2\] and 24 at order $q^3$ \[3\], which is the first order where loops can contribute (11 of these are finite, the other 13 are scale–dependent because they are needed to absorb the one–loop divergences). The dimension two pion–nucleon Lagrangian can be written as

$$\mathcal{L}_{\pi N}^{(2)} = \sum_{i=1}^{7} c_i O_i,$$  \hspace{1cm} (1)

with the $O_i$ monomials in the fields of dimension two. At present, no completely systematic evaluation of the LECs $c_i$ exists. In particular, the four LECs called $c_{1,2,3,4}$ related to pion–nucleon scattering have been determined to one loop accuracy in the review \[4\] and to order $O(q^2)$ in \[5\]. The resulting values differ by factors of 1.5. None of these determinations is satisfactory since some of the input data are not very well known or large cancelations between individual terms appear (the best example is the isoscalar $\pi N$ S–wave scattering length $a^+$). Furthermore, if one wants to extract the dimension three LECs, one needs the $c_i$ as input since they enter via $1/m$ suppressed vertices at that order (compare the form of the complete $\tilde{\mathcal{L}}_{\pi N}^{(3)}$ in \[3\]). Clearly, a more stringent determination of these parameters is called for. A reliable determination should also be based on more observables than LECs in order to have some consistency checks. We close this gap in this paper. Without going into details, we will proceed as follows. The LECs $c_6$ and $c_7$ can be directly inferred from the anomalous magnetic moments of the proton and the neutron \[2\]. In the absence of external (pseudo)scalar fields, the operator $O_5$ is only non–vanishing for unequal light quark masses, $m_u \neq m_d$. The corresponding LEC $c_5$ can be extracted from the strong contribution to the neutron–proton mass difference. For the other four coupling constants, we find a set of nine observables which at one–loop order are given entirely in terms of tree graphs with insertions from $\mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)}$ together with their $1/m$–corrections and finite loop contributions. These are very special cases since in general at this order divergences would appear and thus dimension three LECs would be
needed. From a best fit to these nine observables, we are able to determine the LECs $c_{1,2,3,4}$.

Furthermore, in the meson sector it can be shown that the numerical values of the renormalized LECs $L_i' (\mu = M_\rho)$ can be understood to a high degree of accuracy from resonance saturation, i.e. they can be expressed in terms of resonance masses and coupling constants of the low–lying vector $(V)$, axial–vector $(A)$, scalar $(S)$ and pseudoscalar $(P)$ multiplets (the $\eta'$, to be precise) \[\text{I}\] (in some cases, there is also some contribution from tensor mesons \[\text{II}\]). We investigate how well one can understand the numerical values of the $c_i$ in terms of baryonic $(\Delta, N^*, \ldots)$ and mesonic $(S, V, \ldots)$ excitations. In particular, we discuss the role of the fictitious scalar–isoscalar meson and show how such correlated two–pion exchange reveals itself in certain LECs. Since we do not include the $\Delta$ as an active degree of freedom in the effective field theory, it contributes dominantly to some of the LECs as it is expected from the important role this resonance plays in pion–nuclear physics \[\text{III}\]. We have already shown in a series of detailed calculations concerning a variety of reactions in the corresponding threshold regions that it is legitimate to encode the effects of the $\Delta$ in the pertinent LECs, see the review \[\text{IV}\]. Since here we mostly consider threshold parameters (like scattering lengths and effective ranges), this procedure is expected to be sufficiently accurate. It remains to be proven by the authors who include the $\Delta$ as an active degree of freedom that their approach is equally precise (in the threshold region, of course).

The pertinent results of this investigation can be summarized as follows:

(i) We have determined the seven finite low–energy constants of the dimension two chiral pion–nucleon Lagrangian, $\mathcal{L}_{\pi N}^{(2)}$. We have found a set of nine observables that to one–loop order $q^3$ are given entirely in terms of tree graphs including insertions $\sim c_{1,2,3,4}$ and finite loop contributions, but with none from the 24 new LECs of $\mathcal{L}_{\pi N}^{(3)}$. A best fit allows to pin down these LECs. The other three can be determined from the strong neutron–proton mass difference ($c_5$, which is only relevant in the case $m_u \neq m_d$) and from the anomalous magnetic moments of the proton and the neutron ($c_6, c_7$). The resulting values are listed in table 1 in section 4.

(ii) We have shown that the empirical values of the LECs $c_{1,\ldots,4}$ can be understood from resonance exchange. Assuming that $c_1$ is saturated completely by scalar meson exchange, the values for $c_2, c_3$ and $c_4$ can be understood from a combination of $\Delta, \rho$ and scalar meson exchange. It is remarkable that the scalar mass to coupling constant ratio $M_S/\sqrt{g_S}$ needed to saturate the LEC $c_1$ is in perfect agreement with typical ratios obtained in boson–exchange models of the NN force, where the $\sigma$–meson models the strong pionic correlations coupled to nucleons. There is, however, some sizebale uncertainty related to the $\Delta$ contribution as indicated by the ranges given in table 1. Concerning the LECs $c_6$ and $c_7$ related to $\kappa_v$ and $\kappa_s$, we find that the isoscalar and isovector anomalous moments in the chiral limit can be well understood from neutral vector meson exchange. For the LEC $c_5$, resonance saturation can not be used since there is no information on isospin–violating coupling constants.
Having established that resonance saturation can explain the LECs related to pion–nucleon scattering, we have considered the chiral expansion of the P–wave scattering volumes $P_{1}^{−}$ and $P_{2}^{+}$ to order $q^3$. After renormalizing the appearing divergences, the chiral predictions agree at the few percent level with the empirical values. The largest uncertainty comes actually from the $\Delta(1232)$-contribution. The difference $P_{1}^{−} - P_{2}^{+}$ shows the relevance of chiral loops in the $\pi N$ P–waves.

The eight D– and F–wave threshold parameters $a_{l\pm}^\pm$ ($l = 2, 3$) are given to order $q^3$ by lowest order tree and loop graphs only. The calculated values agree nicely with the empirical ones. This investigation is the first systematic attempt to pin down the low–energy constants of the chiral pion–nucleon Lagrangian. Clearly, more precise data are needed to sharpen the determination of the $c_i$. The present work, however, paves the way of fixing a subset of the dimension three LECs enumerated in [3]. For that, a systematic study of $\pi N$ scattering to order $q^3$ should be performed. Such a study has recently been performed by Mojžiš [4].

2 Effective Lagrangian at next–to–leading order

In this section, we briefly review the next–to–leading order pion–nucleon Lagrangian $\mathcal{L}_{\pi N}^{(2)}$ to fix our notation. We work in the path integral formulation of heavy baryon chiral perturbation theory which automatically obeys reparametrization invariance. All details are spelled out in [2] or the review [4]. The pions are collected in the SU(2) matrix $U(x) = u(x)$ and the proton and the neutron in the iso–doublet $N(x)$. With $v_\mu$ the four–velocity of the heavy nucleon fields and $S_\mu$ the covariant spin–operator à la Pauli–Lubanski, $\mathcal{L}_{\pi N}^{(2)}$ takes the form

$$\mathcal{L}_{\pi N}^{(2)} = \bar{N} \left\{ \frac{1}{2 m}(v \cdot D)^2 - \frac{1}{2 m} D \cdot D - \frac{i \bar{g}_A}{2 m} \{S \cdot D, v \cdot u\} + c_1 \mathrm{Tr}(\chi_+) + \left( c_2 - \frac{g_A}{g_\sigma} \right) (v \cdot u)^2 + c_3 u \cdot u \\ + \left( c_4 + \frac{1}{4 \sigma} \right) [S^\mu, S^\nu] u_\mu u_\nu + c_5 \left( \chi_+ - \frac{1}{2} \mathrm{Tr}(\chi_+) \right) \right\} N ,$$

with

$$\chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u ,$$

$$f_{\mu\nu}^+ = u F_{\mu\nu}^L u^\dagger + u^\dagger F_{\mu\nu}^R u ,$$

$$u_\mu = i u^\dagger \nabla_\mu U u^\dagger .$$
Here, $F^{L,R}_{\mu\nu}$ are the non-abelian field strength tensors of external left/right handed vector gauge fields and $v^{(s)}_{\mu}$ is defined analogously in terms of the isosinglet vector field $v^{(s)}$ necessary to generate the full electromagnetic current. $D_\mu$ is the covariant derivative acting on the nucleons and, similarly, $\nabla_\mu$ the one acting on the pions. Furthermore, $\chi = 2B_0\mathcal{M} + \ldots$ with $\mathcal{M} = \text{diag}(m_u, m_d)$ the light quark mass matrix and $B_0 = |\langle 0 |\bar{u}u|0 \rangle|/F^2_\pi$, with $F_\pi = 92.4$ MeV the pion decay constant. Some of the terms in eq. (2) receive $1/m$ corrections from the expansion of the relativistic Dirac $\pi N$ Lagrangian. We have kept these explicitly one reason being that a phenomenological interpretation in terms of resonance exchange can not generate such terms. All parameters appearing are taken to be at their values in the chiral limit, i.e.

$$\hat{Q} = Q [1 + \mathcal{O}(m^2)] ,$$

where $m_q$ denotes any one of the light quark masses or its average. In most cases, one has $\alpha = 1/2$, exceptions being the anomalous isoscalar magnetic moment $\hat{\kappa}_s$ and $c_5$ with $\alpha = 1$ (see below). In what follows, we can identify the nucleon mass and the axial–vector coupling constant with their physical values, $\hat{m} = m_p = 938.27$ MeV and $\hat{g}_A = g_A = 1.26$.

We will now be concerned with the numerical values of the LECs appearing in $\mathcal{L}^{(2)}_{\pi N}$, these are the $c_i$ ($i = 1, \ldots, 5$) as well as $\hat{\kappa}_s$ and $\hat{\kappa}_v$. The machinery to do these calculations is spelled out in detail in [4].

### 3 Calculation of observables

In this section, we calculate various observables to pin down the LECs $c_i$. The $c_{1,2,3,4}$ are all related to pion–nucleon threshold and subthreshold parameters and the much discussed pion–nucleon $\sigma$–term. We consider here only observables which to one loop order $\mathcal{O}(q^3)$ are given by tree graphs including the $c_i$ and finite loop corrections but have no contribution from the 24 LECs of $\mathcal{L}^{(3)}_{\pi N}$.

Consider first a subset of observables which depend on the LECs $c_1$, $c_2$ and $c_3$. We introduce the small parameter $\mu = M_\pi/m$, i.e. the pion to nucleon mass ratio. Our notation concerning the $\pi N$ amplitudes and parameters is identical to the one used by Höhler [10]. Calculation of the $\sigma$–term and the isospin–even scattering amplitude at and below threshold gives four relations (the one–loop contributions to the $\pi N$ scattering amplitude are collected in app. A),

$$\sigma(0) = -4c_1M^2 - \frac{9g^2_{\pi N}M^3_\pi}{64\pi m^2} + \mathcal{O}(M^4) ,$$

$$a_{00}^+ = \frac{2M^2_\pi}{F^2_\pi}\left(c_3 - 2c_1\right) + \frac{g^2_A M^3_\pi}{8\pi F^4_\pi}\left(2g^2_A + \frac{3}{8}\right) + \mathcal{O}(M^4) ,$$

$$a_{01}^+ = -\frac{c_3M_\pi}{16\pi F^2_\pi} + \frac{277}{48} + \mathcal{O}(M^2_\pi) ,$$

$$d_{10}^+ = \frac{2c_2M_\pi}{8\pi F^2_\pi}\left(\frac{5}{4}g^2_A + 1\right) + \mathcal{O}(M^2_\pi) .$$


Note that the formula for \( a_0^{+1} \) was already derived in [4] for the so–called axial polarizability \( \alpha_A = 2a_0^{+1} \). Another relation can be derived from the isospin–even non–spin–flip scattering volume, \( P_1^+ \):

\[
P_1^+ = \frac{4\pi}{3} (1 + \mu) \left( 4a_{33} + 2a_{31} + 2a_{13} + a_{11} \right)
\]

\[
= \frac{2}{F_\pi^2} \left( c_2 \mu - c_3 \right) + \frac{g_{\pi N}^2 \mu}{4m^3} - \frac{g_A^2 M_\pi}{12\pi F_\pi^4} \left( g_A^2 + \frac{77}{32} \right) + \mathcal{O}(M_\pi^2) .
\]

Consider also the real part of isospin–even \( \pi N \) forward amplitude close to threshold,

\[
\text{Re} T^+ (\omega) = 4\pi \frac{\sqrt{s}}{m} (a^+ + q^2) + P_1^+ \bar{q}^2 + \mathcal{O}(\bar{q}^4)
\]

\[
= T^+(M_\pi) + \bar{q}^2 \beta^+ + \mathcal{O}(\bar{q}^4) ,
\]

with

\[
\beta^+ = \frac{1}{2M_\pi} \frac{\partial}{\partial \omega} \text{Re} T^+ (\omega) \bigg|_{\omega=M_\pi} = P_1^+ + 4\pi (1 + \mu) \left( b^+ + \frac{a^+}{2mM_\pi} \right) .
\]

Here, \( \omega \) denotes the pion cms energy. The chiral expansions of the scattering length \( a^+ \) and of the range parameter \( \beta^+ \) take the form

\[
T^+(M_\pi) = 4\pi (1 + \mu) a^+ = \frac{M_\pi^2}{F_\pi^2} \left( -4c_1 + 2c_2 - \frac{g_A^2}{4m} + 2c_3 \right) + \frac{3g_A^2 M_\pi^2}{64\pi F_\pi^4} + \mathcal{O}(M_\pi^4) ,
\]

\[
\beta^+ = \frac{2c_2}{F_\pi^2} (1 + 2\mu) + \frac{g_{\pi N}^2}{4m^3} (1 + 2\mu) - \frac{g_A^2 M_\pi^2}{12\pi F_\pi^4} + \mathcal{O}(M_\pi^2) .
\]

The calculation leading to these results is somewhat tricky. The tree terms are most easily evaluated by considering the relativistic pion–nucleon Lagrangian with the two couplings \( c_2' \) and \( c_2'' \), see ref.[4]. It leads to the forward scattering amplitude

\[
T^+ = (c_2' + c_2'') \frac{s - m^2 - M_\pi^2}{2m^2 F_\pi^2} = 2c_2 \frac{\omega_L}{F_\pi^2} ,
\]

with \( \omega_L \) the pion laboratory energy. Expanding in powers of \( 1/m \) gives the desired result.

The LEC \( c_4 \) appears in the chiral expansion of the isospin–odd spin–flip scattering volume \( P_2^- \) and the subthreshold parameter \( b_{00}^- \),

\[
P_2^- = \frac{4\pi}{3} (1 + \mu) \left( a_{33} - a_{31} - a_{13} + a_{11} \right)
\]

\[
= \frac{2 + \mu}{8mF_\pi^2} + \frac{c_4}{F_\pi^2} (1 + \mu) - \frac{g_A^2 M_\pi}{48\pi F_\pi^4} (2g_A^2 + 3) + \mathcal{O}(M_\pi^2) ,
\]

\[
b_{00}^- = \frac{1}{2F_\pi} \left( 1 + 4mc_4 \right) - \frac{g_A^2 m M_\pi}{8\pi F_\pi^4} (1 + g_A^2) + \mathcal{O}(M_\pi^2) .
\]
In the absence of a precise scheme to separate isospin-violating quark mass effects from the ones of virtual photons for dynamical processes, we use the information on the strong contribution to the neutron–proton mass difference to pin down $c_5$,

$$(m_n - m_p)^{(\text{non-em})} = 4c_5 B_0 (m_u - m_d) + \mathcal{O}(M_\pi^4) = 4c_5 M_\pi^2 \frac{m_u - m_d}{m_u + m_d} + \mathcal{O}(M_\pi^4).$$  \hspace{1cm} (17)

We assume here the standard scenario of spontaneous chiral symmetry breaking, i.e. $B_0 \gg F_\pi$. We remark that the generalized scenario with $B_0 \sim F_\pi$ would lead to a vastly different value of $c_5$. Other observables sensitive to this LEC are the $\pi N$ S-wave scattering lengths (taken not in the isospin limit) for processes involving at least one neutral pion [12].

The anomalous magnetic moments appearing in the dimension two Lagrangian have been calculated in [3, 4].

$$\tilde{\kappa}_s = \kappa_s + \mathcal{O}(M_\pi^2),$$  \hspace{1cm} (18)

$$\tilde{\kappa}_v = \kappa_v + \frac{g_{\pi N \mu}}{4\pi} + \mathcal{O}(M_\pi^2).$$  \hspace{1cm} (19)

These are related to the LECs $c_6$ and $c_7$ used there via

$$c_6 = \tilde{\kappa}_v, \quad c_7 = \frac{1}{2}(\tilde{\kappa}_s - \tilde{\kappa}_v).$$  \hspace{1cm} (20)

There are no one-loop corrections at order $q^2$ to $\kappa_s$ since the spectral functions of the isoscalar electromagnetic form factors start at the three-pion cut, $t_0 = 9M_\pi^2$.

### 4 Determination of the low-energy constants

First, we must fix parameters. We use $g_{\pi N} = 13.4$ and $g_A$ as determined from the Goldberger–Treiman relation, $g_A = g_{\pi N} F_\pi / m = 1.32$. We also have performed fits with the smaller $g_{\pi N} = 13.05$ and thus $g_A = 1.29$. For the $\sigma$-term, we use $\sigma(0) = 45 \pm 8 \text{ MeV}$ [13]. The threshold and subthreshold parameters we take from [10], these are

$a_{00}^+ = (-1.46 \pm 0.10) M_\pi^{-1}$, $b_{00}^- = (10.36 \pm 0.10) M_\pi^{-2}$, $a_{01}^+ = (1.14 \pm 0.02) M_\pi^{-3}$, $d_{10}^+ = (1.12 \pm 0.02) M_\pi^{-3}$, $P_1^+ = (3.01 \pm 0.05) M_\pi^{-3}$, $P_2^- = (1.00 \pm 0.02) M_\pi^{-3}$ and $b^+ = -(44 \pm 7) \cdot 10^{-3} M_\pi^{-3}$. For the isoscalar S-wave scattering length, we use a generous bound $a^+ = (0 \pm 10) \cdot 10^{-3} M_\pi^{-1}$ since the Karlsruhe–Helsinki phase shifts [14] give $a^+ = -(8.3 \pm 3.8) \cdot 10^{-3} M_\pi^{-1}$ where as the new PSI-ETHZ [15] value is small and positive, $a^+ = (0 \ldots 4) \cdot 10^{-3} M_\pi^{-1}$. Consequently, the value for $\beta^+$ follows to be $\beta^+ = (2.36 \pm 0.15) M_\pi^{-3}$, adding the uncertainties of $P_1^+$, $b^+$ and $a^+$ in quadrature. The magnetic moments are known very precisely, for our purpose it suffices to take $\kappa_v = 3.706$ and $\kappa_s = -0.120$. Finally, we need a value for the strong neutron–proton mass difference. This has been evaluated
in great detail in [10] and we thus use \((m_n - m_p)^{\text{non-elm}} = (2.0 \pm 0.3)\) MeV. We remind the reader that the photon cloud contribution as calculated via the Cottingham formula is about 0.8 MeV. The light quark mass ratio has been determined recently by Leutwyler [7], \(m_d/m_u = 1.8\).

With the error bars for the various observables as given above, we obtain as values of the \(c_i\)

\[
c_1 = -1.02 \pm 0.06 \text{ GeV}^{-1}, \quad c_2 = 3.32 \pm 0.03 \text{ GeV}^{-1}, \quad c_3 = -5.57 \pm 0.05 \text{ GeV}^{-1}. \tag{21}
\]

for our central set of parameters. The uncertainties for \(c_{1,2,3}\) refer to the parabolic errors of the MINUIT fitting routine used. We remark that the fit prefers a negative value for \(a^+\) and the \(\sigma\)-term on the large side, \(a^+ = -10.1 \times 10^{-3} M_{\pi}^{-1}, \sigma(0) = 54.9\) MeV. Clearly, the \(\chi^2/\text{dof}\) of 3.03 shows that the input data are not all mutually consistent (to order \(q^2\)). Higher order corrections not yet calculated might remove these discrepancies. In particular, almost half of the total \(\chi^2\) stems from \(P_1^+\), i.e. the error in \(c_2\) and \(c_3\) is certainly larger than the one obtained from the fitting procedure. To get a more realistic estimate of the uncertainties for the various LECs, we have performed a fit were we have increased the uncertainties in all observables to \(\pm 15\%\) leaving \(\sigma(0)\) and the range for \(a^+\) as before. Considering the present status of the low–energy pion–nucleon scattering data basis, we consider such uncertainties as more realistic. For that fit, the \(\chi^2/\text{dof}\) of 0.33 is much better and the resulting values are \(c_1 = -0.93 \pm 0.09 \text{ GeV}^{-1}, c_2 = 3.34 \pm 0.18 \text{ GeV}^{-1},\) and \(c_3 = -5.29 \pm 0.25 \text{ GeV}^{-1}\). These we consider our central values as given in table 1 (the uncertainties are rounded towards the larger side) together with the dimensionless couplings \(c'_i = 2mc_i, i = 1, \ldots, 5\) (the prefactor 2\(m\) appears naturally in the heavy mass expansion). This fit leads to \(\sigma(0) = 47.6\) MeV and \(a^+ = -4.7 \cdot 10^{-3} M_{\pi}^{-1}\). For comparison, the values determined in the review [3] based solely on the input from the \(\sigma\)-term, \(a_{01}^+\) and \(a^+\) from the Karlsruhe–Helsinki analysis, are \(c_1 = -0.87 \pm 0.11 \text{ GeV}^{-1}, c_2 = 3.34 \pm 0.27 \text{ GeV}^{-1}\) and \(c_3 = -5.25 \pm 0.22 \text{ GeV}^{-1}\). If we use the smaller value for \(g_{\pi N} = 13.05\) (i.e. \(g_A = 1.29\) from the GTR), we get \(c_1 = (-1.01 \pm 0.06) \text{ GeV}^{-1}, c_2 = (3.20 \pm 0.03) \text{ GeV}^{-1}\) and \(c_3 = (-5.45 \pm 0.05) \text{ GeV}^{-1}\). The \(\chi^2/\text{dof}\) of 3.74 is considerably worse. This is, however, not due to one observable but almost all of them contribute more to the total \(\chi^2\) compared to the choice \(g_{\pi N} = 13.4\). Again, for the enlarged uncertainties one gets a substantially lower \(\chi^2/\text{dof} = 0.30\) for the values \(c_1 = -0.91 \pm 0.09 \text{ GeV}^{-1}, c_2 = 3.25 \pm 0.18 \text{ GeV}^{-1},\) and \(c_3 = -5.16 \pm 0.25 \text{ GeV}^{-1}\). Note that the tree level prediction \(b_{00} = 2m P_2\) is violated by 30%. With the inclusion of loop effects, however, a consistent value of \(c_4\) can be obtained from both observables. The same is true for the set of seven observables depending on \(c_{1,2,3}\). An omission of the loop corrections results in a ten times larger \(\chi^2/\text{dof}\). For the observables considered here, the loop effects are typically of the order of 30% to 50%, i.e. not small. It is also worth emphasising that we do not quote an uncertainty for \(c_6\) and \(c_7\) since the magnetic moments of the proton and the neutron have been determined with extreme precision. Notice that in [2] a somewhat larger value for \(c_5\) is obtained based on an leading order SU(3) estimate for \((m_n - m_p)^{\text{non-elm}}\).
Table 1: Values of the LECs $c_i$ in GeV$^{-1}$ and the dimensionless couplings $c'_i$ for $i = 1, \ldots, 5$. The LECs $c_{6,7}$ are dimensionless. Also given are the central values (cv) and the ranges for the $c_i$ from resonance exchange as detailed in section 5. The * denotes an input quantity.

| $i$ | $c_i$       | $c'_i$      | $c_{i\text{Res}}^\text{cv}$ | $c_{i\text{Res}}^\text{ranges}$ |
|-----|-------------|-------------|-----------------------------|----------------------------------|
| 1   | $-0.93 \pm 0.10$ | $-1.74 \pm 0.19$ | $-0.9^*$                     | $-$                              |
| 2   | $3.34 \pm 0.20$     | $6.27 \pm 0.38$     | 3.9                          | 2...4                            |
| 3   | $-5.29 \pm 0.25$    | $-9.92 \pm 0.47$    | $-5.3$                       | $-4.5...-5.3$                    |
| 4   | $3.63 \pm 0.10$     | $6.81 \pm 0.19$     | 3.7                          | $3.1...3.7$                      |
| 5   | $-0.09 \pm 0.01$    | $-0.17 \pm 0.02$    | $-$                          | $-$                              |
| 6   | $5.83$            | $-$             | 6.1                          | $-$                              |
| 7   | $-2.98$           | $-$             | $-3.0$                       | $-$                              |

5 Phenomenological interpretation of the low-energy constants

In this section, we will be concerned with the phenomenological interpretation of the values for the LECs $c_i$. For that, guided by experience from the meson sector \[6\], we use resonance exchange. To be specific, consider an effective Lagrangian with resonances chirally coupled to the nucleons and pions. One can generate local pion–nucleon operators of higher dimension with given LECs by letting the resonance masses become very large with fixed ratios of coupling constants to masses. That procedure amounts to decoupling the resonance degrees of freedom from the effective field theory. However, the traces of these frozen particles are encoded in the numerical values of certain LECs. In the case at hand, we can have baryonic ($N^*$) and mesonic ($M$) excitations,

$$c_i = \sum_{N^*=\Delta,R,...} c_{iN^*} + \sum_{M=S,V,...} c_{iM}^M,$$

where $R$ denotes the Roper $N^*(1440)$ resonance. We remark again that the $c_i$ are finite and scale–independent.

We consider first scalar ($S$) meson exchange. The SU(2) $S\pi\pi$ interaction can be written as

$$\mathcal{L}_{\pi S} = S \left[ \bar{c}_m \text{Tr}(\chi_+) + \bar{c}_d \text{Tr}(u\mu u^\mu) \right].$$

From that, one easily calculates the s–channel scalar meson contribution to the invariant amplitude $A(s, t, u)$ for elastic $\pi\pi$ scattering,

$$A^S(s, t, u) = \frac{4}{F_\pi^2(M_\pi^2 - s)} \left[ 2\bar{c}_m M_\pi^2 + \bar{c}_d (s - 2M_\pi^2) \right]^2 + \frac{16\bar{c}_m M_\pi^2}{3F_\pi^2 M_\pi^2} \left[ 2\bar{c}_m M_\pi^2 + \bar{c}_d (3s - 4M_\pi^2) \right].$$
Comparing with the SU(3) amplitude calculated in [18], we are able to relate the $\bar{c}_{m,d}$ to the $c_{m,d}$ of [6] (setting $M_{S_1} = M_{S_8} = M_S$ and using the large–$N_c$ relations $\bar{c}_{m,d} = c_{m,d}/\sqrt{3}$ to express the singlet couplings in terms of the octet ones),

$$\bar{c}_{m,d} = \frac{1}{\sqrt{2}} c_{m,d}, \quad (25)$$

with $|c_m| = 42$ MeV and $|c_d| = 32$ MeV. Assuming now that $c_1$ is entirely due to scalar exchange, we get

$$c_S^1 = -\frac{g_S \bar{c}_m}{M_S^2}. \quad (26)$$

Here, $g_S$ is the coupling constant of the scalar–isoscalar meson to the nucleons, $L_{SN} = -g_S \bar{N}N S$. What this scalar–isoscalar meson is essentially doing is to mock up the strong pionic correlations coupled to nucleons. Such a phenomenon is also observed in the meson sector. The one loop description of the scalar pion form factor fails beyond energies of 400 MeV, well below the typical scale of chiral symmetry breaking, $\Lambda_\chi \simeq 1$ GeV. Higher loop effects are needed to bring the chiral expansion in agreement with the data [19].

Effectively, one can simulate these higher loop effects by introducing a scalar meson with a mass of about 600 MeV. This is exactly the line of reasoning underlying the arguments used here (for a pedagogical discussion on this topic, see [20]). It does, however, not mean that the range of applicability of the effective field theory is bounded by this mass in general. In certain channels with strong pionic correlations one simply has to work harder than in the channels where the pions interact weakly (as demonstrated in great detail in [13]) and go beyond the one loop approximation which works well in most cases. For $c_1$ to be completely saturated by scalar exchange, $c_1 \equiv c_S^1$, we need

$$\frac{M_S}{\sqrt{g_S}} = 180 \text{ MeV}. \quad (27)$$

Here we made the assumption that such a scalar has the same couplings to pseudoscalars as the real $a_0(980)$ resonance. It is interesting to note that the effective $\sigma$–meson in the Bonn one–boson–exchange potential [21] with $M_S = 550$ MeV and $g_S^2/(4\pi) = 7.1$ has $M_S/\sqrt{g_S} = 179$ MeV. This number is in stunning agreement with the value demanded from scalar meson saturation of the LEC $c_1$. With that, the scalar meson contribution to $c_3$ is fixed including the sign, since $c_m c_d > 0$ (see ref.[3]),

$$c_3^S = -2 \frac{g_S \bar{c}_d}{M_S^2} \frac{c_d}{c_m} c_1 = -1.40 \text{ GeV}^{-1}. \quad (28)$$

The isovector $\rho$ meson only contributes to $c_4$. Taking a universal $\rho$–hadron coupling and using the KSFR relation, we find

$$c_4^\rho = \frac{\kappa_\rho}{4m} = 1.63 \text{ GeV}^{-1}. \quad (29)$$
using \( \kappa_p = 6.1 \pm 0.4 \) from the analysis of the nucleon electromagnetic form factors, the process \( \bar{N}N \rightarrow \pi\pi \) \cite{22, 23} and the phenomenological one–boson–exchange potential for the NN interaction.

We now turn to the baryon excitations. Here, the dominant one is the \( \Delta(1232) \). Using the isobar model and the SU(4) coupling constant relation (the dependence on the off–shell parameter \( Z \) has already been discussed in \cite{4}), the \( \Delta \) contribution to the various LECs is readily evaluated,

\[
c^\Delta_2 = -c^\Delta_3 = 2c^\Delta_4 = \frac{g^2_A(m_\Delta - m)}{2[(m_\Delta - m)^2 - M^2_\pi]} = 3.83 \text{ GeV}^{-1}.
\]

(30)

These numbers we consider as our central values. Unfortunately, there is some sizeable uncertainty in these \( \Delta \) contributions. Dropping e.g. the factor \( M^2_\pi \) in the denominator of eq.(30), the numerical value decreases to 2.97 GeV\(^{-1}\). Furthermore, making use of the Rarita–Schwinger formalism and varying the parameter \( Z \), one can get sizeable changes in the \( \Delta \) contributions ( e.g. \( c^\Delta_2 = 1.89, c^\Delta_3 = -3.03, c^\Delta_4 = 1.42 \) in GeV\(^{-1}\) for \( Z = -0.3 \)). From this, we deduce the following ranges: \( c^\Delta_2 = 1.9 \ldots 3.8, c^\Delta_3 = -3.8 \ldots -3.0, c^\Delta_4 = 1.4 \ldots 2.0 \) (in GeV\(^{-1}\)).

The Roper \( N^*(1440) \) resonance contributes only marginally,

\[
c^R_2 = \frac{g^2_R m_\tilde{R}}{8(m^*-m^2)} = 0.05 \text{ GeV}^{-1},
\]

\[
c^R_3 = -\frac{g^2_R \tilde{R}}{16(m^*-m)} = -0.06 \text{ GeV}^{-1},
\]

\[
c^R_4 = \frac{g^2_R \tilde{R}}{8(m^*-m)} = 0.12 \text{ GeV}^{-1},
\]

(31)

using \( \tilde{R} = 0.28 \) as obtained from the partial decay width \( \Gamma(N^* \rightarrow N\pi) \simeq 110 \text{ MeV} \) \cite{3}.

Putting pieces together, we have for \( c_2, c_3 \) and \( c_4 \) from resonance exchange (remember that \( c_1 \) was assumed to be saturated by scalar exchange)

\[
\begin{aligned}
c^{\text{Res}}_2 &= c^\Delta_2 + c^R_2 = 3.83 + 0.05 = 3.88, \\
c^{\text{Res}}_3 &= c^\Delta_3 + c^S_3 + c^R_3 = -3.83 - 1.40 - 0.06 = -5.29, \\
c^{\text{Res}}_4 &= c^\Delta_4 + c^S_4 + c^R_4 = 1.92 + 1.63 + 0.12 = 3.67,
\end{aligned}
\]

(32)

with all numbers given in units of GeV\(^{-1}\). Comparison with the empirical values listed in table 1 shows that these LECs can be understood from resonance saturation, assuming only that \( c_1 \) is entirely given by scalar meson exchange. As argued before, the scalar meson parameters needed for that are in good agreement with the ones derived from fitting NN scattering data and deuteron properties within the framework of a one–boson–exchange model. We stress again that this \( \sigma \)–meson is an effective degree of freedom which
parametrizes the strong $\pi\pi$ correlations (coupled to nucleons) in the scalar–isoscalar channel. It should not be considered a novel degree of freedom which limits the applicability of the effective field theory to a lower energy scale. As pointed out before, there is some sizeable uncertainty related to the $\Delta$ contribution as indicated by the ranges for the $c_i^{\text{Res}}$ in table 1. It is, however, gratifying to observe that the empirical values are covered by the band based on the resonance exchange model.

The LECs $\hat{\kappa}_s = -0.12$ and $\hat{\kappa}_v = 5.83$ can be estimated from neutral vector meson exchange, in particular

\begin{equation}
\hat{\kappa}_s = \kappa_\omega, \quad \hat{\kappa}_v = \kappa_\rho.
\end{equation}

Using e.g. the values from [22], $\kappa_\omega = -0.16 \pm 0.01$ and $\kappa_\rho = 6.1 \pm 0.4$, we see that the isoscalar and isovector anomalous magnetic moments in the chiral limit can be well understood from $\omega$ and $\rho^0$ meson exchange. It is amusing that the isovector pion cloud of the nucleon calculated to one loop allows to explain the observed difference between $\kappa_\rho$ and $\kappa_v$. In strict vector meson dominance these would be equal. It is well known [10] that the low energy part of the nucleon isovector spectral functions can not be understood in terms of the $\rho$–resonance alone.

6 Aspects of pion–nucleon scattering

Having established that resonance saturation works rather well for the dimension two LECs, we proceed to calculate the chiral expansion of the isovector $S$–wave effective range parameter $b^-$ and of the other two $P$–wave $\pi N$ scattering volumes up-to-and-including terms of order $q^3$. Finally, we also work out the $D$– and $F$–wave threshold parameters $a_{l\pm}^\pm$, $l = 2, 3$. Results for the subthreshold parameters which do not receive any contribution from $\mathcal{L}^{(3)}_{\pi N}$ are collected in app. B.

6.1 Chiral expansion of $P$–wave scattering volumes

We consider $P_2^+$, the isoscalar spin-flip scattering volume, and $P_1^-$ related to the isovector spin non-flip amplitude [8].

\begin{align}
P_1^- &= \frac{4\pi}{3} (1 + \mu) \left( -2a_{33} - a_{31} + 2a_{12} + a_{11} \right) = (-2.52 \pm 0.03) M_\pi^{-3}, \quad (34) \\
P_2^+ &= \frac{4\pi}{3} (1 + \mu) \left( -2a_{33} + 2a_{31} - a_{13} + a_{11} \right) = (-2.74 \pm 0.03) M_\pi^{-3}. \quad (35)
\end{align}

Our aim is to see how well the empirical values given in eqs.\,(34,35) can be understood within chiral perturbation theory. For that, we have to account for Born terms, one loop graphs and insertions from $\mathcal{L}^{(3)}_{\pi N}$ (because of the crossing properties of these amplitudes). In contrast to the previous cases, the one–loop contributions are not finite and an appropriate renormalization has to be performed.
Consider first the Born terms. Including all terms, in particular the $\pi\pi\bar{N}N$ Weinberg vertex, the expansion to order $q^3$ gives

\[
P_1^-(\text{Born}) = -\frac{g_{\pi N}^2}{2m^2} \left( \frac{1}{M_{\pi}} + \frac{1}{m} + \frac{3M_{\pi}}{4m^2} \right) + \frac{1}{4mF_\pi^2} \left( 1 + \frac{M_{\pi}}{2m} \right) = -2.22 M_{\pi}^{-3} \quad (36)
\]

\[
P_2^+(\text{Born}) = -\frac{g_{\pi N}^2}{2m^2} \left( \frac{1}{M_{\pi}} + \frac{1}{m} + \frac{M_{\pi}}{4m^2} \right) = -2.29 M_{\pi}^{-3}, \quad (37)
\]

where the numbers refer to our standard set of parameters ($g_{\pi N} = 13.4$). We now turn to the chiral loop corrections at order $q^3$. First, one has to perform the standard coupling constant renormalization, $\bar{g}_{\pi N} \to g_{\pi N}$. We use dimensional regularization and the corresponding renormalization scale $\lambda$ is varied between $M_\rho = 0.77$ GeV and $m^* = 1.44$ GeV. In principle, this scale–dependence would be balanced by the contribution from the LECs. Since we use resonance saturation to estimate these, there remains a small scale–dependent reminder which can not be fixed (compare also [6]). As a check on the one–loop calculation, one verifies that the divergence appearing in $P_1^-$ is canceled by the local counter term $O_1 + O_2$ and the one in $P_2^+$ by the counter term $O_{15}$ of ref.[3]. At the scale $\lambda = m$, we have

\[
P_1^-(\text{Loop}) = -\frac{M_\pi}{48\pi^2 F_\pi^4} \left[ \left( 2g_A^4 + 5g_A^2 + 1 \right) \ln \frac{M_\pi}{\lambda} + \frac{1}{3} g_A^4 + \frac{7}{2} g_A^2 + \frac{11}{2} \right]
\]

\[
= (0.25 \pm 0.05) M_{\pi}^{-3}, \quad (38)
\]

\[
P_2^+(\text{Loop}) = -\frac{g_A^4 M_\pi}{24\pi^2 F_\pi^4} \left( \frac{7}{6} + \ln \frac{M_\pi}{\lambda} \right) = (0.05 \pm 0.02) M_{\pi}^{-3}. \quad (39)
\]

The uncertainty stems from the variation in $\lambda$ as described above. The counter term contribution is estimated from $\Delta$-resonance exchange employing the Rarita-Schwinger formalism,

\[
P_1^-(\Delta) = \frac{g_{\pi N}^2 M_\pi}{m^2 m_\Delta^2} \left[ \frac{m_\Delta^2 (m_\Delta - 2m)}{2(m_\Delta - m)^2} + \frac{2Z - 1}{4} (m + m_\Delta) + Z^2 m_\Delta \right] = -0.35 M_{\pi}^{-3}, \quad (40)
\]

\[
P_2^+(\Delta) = \frac{g_{\pi N}^2 M_\pi}{m^2 m_\Delta^2} \left[ -\frac{m_\Delta m}{2(m_\Delta - m)^2} + \frac{1}{4} - Z^2 \right] = -0.33 M_{\pi}^{-3}, \quad (41)
\]

for the off–shell parameter $Z = -0.3$. Using the non-relativistic isobar model and performing no chiral expansion one finds from the $\Delta$(1232)-resonance,

\[
P_1^-(\Delta) = P_2^+(\Delta) = -\frac{g_{\pi N}^2 M_\pi}{2m^2 [(m_\Delta - m)^2 - M_{\pi}^2]} = -0.58 M_{\pi}^{-3}, \quad (42)
\]

which is almost twice as large as before. Taking the average of both $\Delta$(1232)-estimates and adding uncertainties in quadrature, the chiral predictions to $\mathcal{O}(q^3)$ are

\[
P_1^- = (-2.44 \pm 0.13) M_{\pi}^{-3}, \quad P_2^+ = (-2.70 \pm 0.12) M_{\pi}^{-3}. \quad (43)
\]
The major uncertainty comes here from the $\Delta(1232)$ contribution which seems hard to pin down accurately. Further contributions at $O(M_\pi)$ coming from the Roper resonance and the $\rho$-meson (as calculated in [24]) fall into the error band given in eq.(43). We note that in both cases the Born terms are dominant and $\Delta$ exchange amounts to a 18 and 17 % correction, respectively. The loop correction is very small for $P_2^+$ and roughly $-10\%$ for $P_1^-$. Interestingly the small difference between $P_1^-$ and $P_2^+$ stems mainly from the chiral loops. In tree level calculations [8] and also the Skyrme soliton model [25], $P_1^- - P_2^+ = 4\pi(1 + \mu)(a_{13} - a_{31})$ is actually zero as a consequence of $SU(4)$–spin–flavor symmetry. This quantity therefore serves as an interesting signal for chiral loop effects in the $\pi N$ P–wave amplitudes. Furthermore the chiral expansion of these observables shows a good convergence (as expected for these particular P–waves scattering volumes). The chiral predictions are well within the empirical values for $P_1^-$ and $P_2^+$, however the theoretical uncertainty is larger than the experimental one. We conclude that also these particular $\pi N$ threshold parameters can be understood within heavy baryon chiral perturbation theory.

6.2 The isovector S–wave effective range parameter

The real part of the isovector forward scattering amplitude $T^-$ close to threshold takes a similar form than given in eqs.(10,11) for $T^+$, with

$$\beta^- = \frac{1}{2M_\pi} \frac{\partial}{\partial \omega} \text{Re} T^- (\omega) |_{\omega=M_\pi} = 4\pi(1 + \mu) b^- + \frac{T^- (M_\pi)}{2mM_\pi} + P_1^- .$$

The second term is proportional to the isovector S–wave scattering length, $a^-$, which we already discussed in detail in [26]. We just mention that the prediction given in that paper agrees well with the recent determinations from pionic atoms [15]. Therefore, we will discuss here the isovector S–wave effective range parameter $b^-$, which is smaller and of opposite sign than the isoscalar one, $b^- \sim -0.3 b^+$. We prefer to keep the kinematical factor $4\pi(1 + \mu)$ and thus have to compare with the empirical value [10],

$$4\pi(1 + \mu) b^- = (0.19 \pm 0.09) \cdot M_\pi^{-3}.$$  

The chiral expansion of this quantity takes the following form. From the tree graphs one finds up to order $O(M_\pi)$,

$$4\pi(1 + \mu) b^- \text{ (Born)} = \frac{1}{4F_\pi^2 M_\pi} - \frac{g_A^2}{2mF_\pi^2} + \frac{M_\pi}{16m^2 F_\pi^2} (2 - 5g_A^2)$$

$$= (0.57 - 0.30 - 0.02) \cdot M_\pi^{-3} = 0.25 \cdot M_\pi^{-3} ,$$

with the contributions of the powers $M_\pi^{-1.0.1}$ given separately. One sees that the truncation at order $M_\pi^0$ is already in agreement with the experimental value (which has quite a large
error bar). As a further contribution we only mention the chiral loop correction. After renormalization of the pion decay constant and the $\pi N$ coupling constant it reads,

$$4\pi(1 + \mu) b^-(\text{Loop}) = \frac{M_\pi}{48\pi^2 F_\pi^4} \left[ (5g_A^2 - 8) \ln \frac{M_\pi}{\lambda} + \frac{7}{2} g_A^2 - 1 \right] = 0.04 \cdot M_\pi^{-3},$$

(47)

for $\lambda = m$. Varying $\lambda$ between $M_\rho$ and $m^*$, this number changes by less than 10%. The loop correction is smaller than the experimental uncertainty and this presumably holds for all other order $M_\pi$ counter term contributions. We conclude, that also the value of the isovector S–wave effective range parameter $b^-$ can be understood within heavy baryon CHPT.

### 6.3 D– and F–wave threshold parameters

Finally, we discuss the eight D- and F-wave threshold parameters $a_{l\pm}^\pm$, $l = 2, 3$. To these only the Born (lowest order tree) graphs and two specific one loop graphs contribute at order $q^3$, but no counter terms from $\mathcal{L}^{(2,3)}_{\pi N}$. In the following expressions the loop contributions are the ones carrying the factor $F_\pi^{-4}$. We find that in most cases the chiral loop corrections are quite important to bring the chiral expansion close to the experimental values. The latter are taken from [10]. We also remark that Mojžiš' calculation [9] of these threshold parameters is prior to ours. The chiral expansion up to order $q^3$ reads:

D-wave threshold parameters:

$$a_{2+}^+ = - \frac{g_A^2(2 + \mu)}{120\pi m F_\pi^2 M_\pi^2} + \frac{193g_A^2}{115200\pi^2 F_\pi^4 M_\pi^4} = -1.83 \cdot 10^{-3} M_\pi^{-5}$$

$$a_{2+}^+(\text{exp}) = (-1.8 \pm 0.3) \cdot 10^{-3} M_\pi^{-5}$$

(48)

$$a_{2-}^+ = \frac{g_A^2(2 + \mu)}{480\pi m F_\pi^2 M_\pi^2} + \frac{193g_A^2}{115200\pi^2 F_\pi^4 M_\pi^4} = 2.38 \cdot 10^{-3} M_\pi^{-5}$$

$$a_{2-}^+(\text{exp}) = (2.2 \pm 0.3) \cdot 10^{-3} M_\pi^{-5}$$

(49)

$$a_{2+}^- = \frac{g_A^2(2 + \mu)}{120\pi m F_\pi^2 M_\pi^2} + \frac{1 + g_A^2(7 - 5\pi)}{14400\pi^3 F_\pi^4 M_\pi^4} = 3.21 \cdot 10^{-3} M_\pi^{-5}$$

$$a_{2+}^-(\text{exp}) = (3.2 \pm 0.1) \cdot 10^{-3} M_\pi^{-5}$$

(50)

$$a_{2-}^- = - \frac{g_A^2(2 + \mu)}{480\pi m F_\pi^2 M_\pi^2} + \frac{2 + g_A^2(14 + 15\pi)}{28800\pi^3 F_\pi^4 M_\pi^4} = -0.21 \cdot 10^{-3} M_\pi^{-5}$$

$$a_{2-}^-(\text{exp}) = (0.1 \pm 0.2) \cdot 10^{-3} M_\pi^{-5}$$

(51)
F-wave threshold parameters:

\[ a_{3+}^+ = \frac{g_A^2}{140\pi F_F^2 M_F^3} \left( \frac{1}{m^2} + \frac{73}{5376\pi F_F^2} \right) = 0.29 \cdot 10^{-3} M^{-7} \]

\[ a_{3+}^+(\text{exp}) = (0.42 \pm 0.13) \cdot 10^{-3} M^{-7} \]  

\[ a_{3-}^+ = \frac{g_A^2}{840\pi F_F^2 M_F^3} \left( \frac{73}{896\pi F_F^2} - \frac{1}{m^2} \right) = 0.06 \cdot 10^{-3} M^{-7} \]

\[ a_{3-}^+(\text{exp}) = (0.15 \pm 0.12) \cdot 10^{-3} M^{-7} \]  

\[ a_{3+}^- = \frac{1}{140\pi F_F^2 M_F^3} \left( \frac{2 + g_A^3(18 - 7\pi)}{3360\pi^2 F_F^2} - \frac{g_A^2}{m^2} \right) = -0.20 \cdot 10^{-3} M^{-7} \]

\[ a_{3+}^-(\text{exp}) = (-0.25 \pm 0.02) \cdot 10^{-3} M^{-7} \]  

\[ a_{3-}^- = \frac{1}{840\pi F_F^2 M_F^3} \left( \frac{g_A^2}{m^2} + \frac{3 + g_A^3(27 + 14\pi)}{840\pi^2 F_F^2} \right) = 0.06 \cdot 10^{-3} M^{-7} \]

\[ a_{3-}^-(\text{exp}) = (0.10 \pm 0.02) \cdot 10^{-3} M^{-7} \]  

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\section*{A Pion–nucleon scattering amplitude}

Here, we give explicit closed form expressions for the one-loop contribution to the \( \pi N \)-scattering amplitude. In the center-of-mass (cms) frame the \( \pi N \)-scattering amplitude \( \pi^a(q) + N(p) \rightarrow \pi^b(q') + N(p') \) takes the following form:

\[ T_{\pi N}^{ba} = \delta^{ba} \left[ g^+(\omega, t) + i\vec{\sigma} \cdot (\vec{q}' \times \vec{q}) h^+(\omega, t) \right] + 2\epsilon^{bac} \epsilon \left[ g^-(\omega, t) + i\vec{\sigma} \cdot (\vec{q}' \times \vec{q}) h^-(\omega, t) \right] \]  

(A.1) with \( \omega = v \cdot q = v \cdot q' \) the pion cms energy and \( t = (q - q')^2 \) the invariant momentum transfer squared. \( g^\pm(\omega, t) \) refers to the isoscalar/isovector non-spin-flip amplitude and \( h^\pm(\omega, t) \) to the isoscalar/isovector spin-flip amplitude. After renormalization of the pion decay constant \( F_\pi \) and the pion-nucleon coupling constant \( g_{\pi N} \) one finds the following one-loop contributions to the cms amplitudes \( g^\pm(\omega, t) \) and \( h^\pm(\omega, t) \) at order \( q^3 \):

\[ g^+(\omega, t)_{\text{loop}} = \frac{g_A^2}{32\pi F_F^2} \left\{ -\frac{4\omega^2}{g_A^2} \sqrt{M_\pi^2 - \omega^2} + (M_\pi^2 - 2t) \left[ M_\pi + \frac{2M_\pi^2 - t}{2\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2M_\pi} \right] \right. \\
+ \left. \frac{4g_A^2}{3\omega^2} (2\omega^2 + t - 2M_\pi^2) \left[ (M_\pi^2 - \omega^2)^{3/2} - M_\pi^3 \right] \right\} \]  

(A.2)
\[ h^+(\omega, t)_{\text{loop}} = \frac{g_A^4}{24\pi^2 F_\pi^4} \left\{ -\omega \left( \frac{M_\pi}{\lambda} + \frac{1}{6} \right) - \frac{M_\pi^2}{\omega} + \frac{(M_\pi^2 - \omega^2)^{3/2}}{\omega^2} \arcsin \frac{\omega}{M_\pi} \right\} \tag{A.3} \]

\[ g^-(\omega, t)_{\text{loop}} = \frac{\omega}{48\pi^2 F_\pi^4} \left\{ 3\omega^2 \left( 1 - 2 \ln \frac{M_\pi}{\lambda} \right) - 6\omega \sqrt{M_\pi^2 - \omega^2} \arcsin \frac{\omega}{M_\pi} + g_A^4 (2\omega^2 + t - 2M_\pi^2) \left[ \frac{5}{6} \ln \frac{M_\pi}{\lambda} - \frac{M_\pi^2}{\omega^2} + \frac{(M_\pi^2 - \omega^2)^{3/2}}{\omega^3} \arcsin \frac{\omega}{M_\pi} \right] + \left[ 2M_\pi^2 (1 + 2g_A^2) - \frac{t}{2} (1 + 5g_A^2) \right] \sqrt{1 - \frac{4M_\pi^2}{t} \ln \left( \frac{\sqrt{4M_\pi^2 - t} + \sqrt{-t}}{2M_\pi} \right)} - \frac{t}{2} (1 + 5g_A^2) \ln \frac{M_\pi}{\lambda} + \frac{t}{12} (5 + 13g_A^2) - 2M_\pi^2 (1 + 2g_A^2) \right\} \tag{A.4} \]

\[ h^-(\omega, t)_{\text{loop}} = \frac{g_A^4}{32\pi^2 F_\pi^4} \left\{ -M_\pi + \frac{t - 4M_\pi^2}{2\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2M_\pi} + \frac{4g_A^2}{3\omega^2} \left[ (M_\pi^2 - \omega^2)^{3/2} - M_\pi^3 \right] \right\} \tag{A.5} \]

The analytic continuation above threshold \( \omega > M_\pi \) is done via the formulae

\[ \sqrt{1 - x^2} = -i\sqrt{x^2 - 1}, \quad \arcsin x = \frac{\pi}{2} + i \ln(x + \sqrt{x^2 - 1}) . \tag{A.6} \]

The \( t \)-dependences of the loop-amplitudes \( g^\pm(\omega, t)_{\text{loop}} \) and \( h^\pm(\omega, t)_{\text{loop}} \) show an interesting structure, if one discards terms proportional to \( g_A^4 \). The \( t \)-dependence of \( h^+(\omega, t)_{\text{loop}} \) is then given by \( (2t - M_\pi^2)/(3M_\pi^2 F_\pi^2) \sigma(t)_{\text{loop}} \), with \( \sigma(t) \) the nucleon scalar form factor. Furthermore, the \( t \)-dependence of \( g^-(\omega, t)_{\text{loop}} \) becomes equal to \( \omega/(2F_\pi^2) G^V_E(t)_{\text{loop}} \), with \( G^V_E(t) \) the nucleon isovector electric form factor (normalized to unity). Finally, \( h^-(\omega, t)_{\text{loop}} \) has the same \( t \)-dependence as \( -1/(4mF_\pi^2) G^V_M(t)_{\text{loop}} \), with \( G^V_M(t) \) the nucleon isovector magnetic form factor. The one-loop calculation of these nucleon form factors can be found in [2].

### B Results for some subthreshold parameters

Here, we collect the results for those coefficients of the subthreshold expansion (around \( \nu = t = 0 \)) which to order \( q^3 \) are pure loop effects. The experimental values are taken from [11].
Obviously, only in some cases the one-loop result is in good agreement with the empirical values as deduced from the Karlsruhe–Helsinki (KH) phase shift analysis. Note, however, that recent low energy πN-scattering data from PSI [27] show some disagreement with the KH80 solution of πN dispersion analysis. It therefore seems necessary to redo the πN-dispersion analysis with the inclusion of these new data. A new determination of the subthreshold coefficients is now also called for.

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