SOCLE DEGREES OF FROBENIUS POWERS

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In honor of Phillip Griffith, on the occasion of his retirement.

Abstract. Let $k$ be a field of positive characteristic $p$, $R$ be a Gorenstein graded $k$-algebra, and $S = R/J$ be an artinian quotient of $R$ by a homogeneous ideal. We ask how the socle degrees of $S$ are related to the socle degrees of $F^e_R(S) = R/J^q$. If $S$ has finite projective dimension as an $R$-module, then the socles of $S$ and $F^e_R(S)$ have the same dimension and the socle degrees are related by the formula:

$$D_i = q d_i - (q - 1) a(R),$$

where $d_1 \leq \cdots \leq d_\ell$ and $D_1 \leq \cdots \leq D_\ell$ are the socle degrees $S$ and $F^e_R(S)$, respectively, and $a(R)$ is the $a$-invariant of the graded ring $R$, as introduced by Goto and Watanabe. We prove the converse when $R$ is a complete intersection.

Let $(R, m)$ be a Noetherian graded algebra over a field of positive characteristic $p$, with irrelevant ideal $m$. We usually let $R = P/C$ with $P$ a polynomial ring, and $C$ a homogeneous ideal. Let $J$ be an $m$-primary homogeneous ideal in $R$. Recall that if $q = p^e$, then the $e$th Frobenius power of $J$ is the ideal $J^q$ generated by all $i^q$ with $i \in J$. The basic question is:

**Question.** How do the degrees of the minimal generators of $(J^q : m)/J^q$ vary with $q$?

The largest of the degrees of a generators of the socle $(J : m)/J$ will be called the *top socle degree* of $R/J$. The question of finding a linear bound for the top socle degree of $R/J^q$ has been considered by Brenner in [2] from a different point of view; his main motivation there is finding inclusion-exclusion criteria for tight closure.

The answer to the Question is well-known (although not explicitly stated in existing literature) in the case when $J$ has finite projective dimension; see Observation 1.7. We prove that the converse holds when $R = P/C$ is a complete intersection.
Theorem. Let $k$ be a field of positive characteristic $p$, $q = p^e$ for some positive integer $e$, $P$ be a positively graded polynomial ring over $k$, and $R = P/C$ be a complete intersection ring with $C$ generated by a homogeneous regular sequence. Let $\mathfrak{m}$ be the maximal homogeneous ideal of $R$, $J$ be a homogeneous $\mathfrak{m}$-primary ideal in $R$, and $I$ be a lifting of $J$ to $P$. Let $\ell$ be the dimension of the socle $(J : \mathfrak{m})/J$ of $R/J$ and $d_1, \ldots, d_\ell$ be the degrees of the generators of the socle. Then the following statements are equivalent:

(a) $\text{pd}_R R/J < \infty$,

(b) the socle $(J[q] : \mathfrak{m})/J[q]$ of $R/J[q]$ has dimension $\ell$ and the degrees of the generators are $qd_i - (q-1)a$, for $1 \leq i \leq \ell$, where $a$ denotes the $a$-invariant of $R$,

(c) $(C + I)[q] : (C[q] : C) = C + I[q]$, and

(d) $I[q] \cap C = (I \cap C)[q] + CI[q]$.

Of course, the general question

Question. How do the socle degrees of Frobenius powers $J[q]$ encode homological information about the ideals $J[q]$?

remains wide open and very compelling.

1. Preliminary notions.

Let $k$ be a field of positive characteristic $p$. We say that the ring $R$ is a graded $k$-algebra if

$$R \text{ is non-negatively graded, } R_0 = k, \text{ and } R \text{ is finitely generated as a ring over } k.$$  

(1.1)

Every ring that we study in this paper is a graded $k$-algebra. In particular, “Let $P$ be a polynomial ring” means $P = k[x_1, \ldots, x_n]$, for some $n$, and each variable has positive degree. Every calculation in this paper is homogeneous: all elements and ideals that we consider are homogeneous, all ring or module homomorphisms that we consider are homogeneous of degree zero. If $r$ is a homogeneous element of the ring $R$, then $|r|$ is the degree of $r$. The graded $k$-algebra $R$ has a unique homogeneous maximal ideal

$$\mathfrak{m} = \mathfrak{m}_R = R_+ = \bigoplus_{i > 0} R_i;$$

furthermore, $R$ has a unique graded canonical module $K_R$, which is equal to the graded dual of the graded local cohomology module $H^n_m(R)$, where $n$ is the Krull dimension of $R$; that is,

$$K_R = \text{Hom}_R(H^n_m(R), E_R),$$
for $E_R = \text{Hom}_k(R,R/m)$ the injective envelope of $R/m$ as a graded $R$-module. (See, for example [5, Def. 2.1.2].) The $a$-invariant of $R$ is defined to be

$$a(R) = -\min\{m \mid (K_R)_m \neq 0\} = \max\{m \mid (\text{H}_m^n(R))_m \neq 0\}.$$  

The definition of the $a$-invariant is rigged so that if $R$ is a Gorenstein graded $k$-algebra, then $K_R = R(a(R))$. When the ring $R$ is Cohen-Macaulay, there are many ways to compute $a(R)$. The main tool for these calculations, Proposition 1.2 below, may be found as Proposition 2.2.9 in [5] or Proposition 3.6.12 in [3]. The functor $\text{Ext}^c$ is underlined, as in [5], to emphasize the graded nature of this functor. All of our homological calculations are homogeneous; however, we will not take the trouble to underline every graded object.

**Proposition 1.2.** If $R \to S$ is a graded surjection of graded $k$-algebras, and $R$ is Cohen-Macaulay, then

$$K_S = \text{Ext}^c_R(S,K_R),$$

where $c = \dim R - \dim S$. In particular, if $S = R/C$ and the ideal $C$ is generated by the homogeneous regular sequence $f_1, \ldots f_c$, then

$$K_{R/C} = \frac{K_R}{C} \left( \sum_{i=1}^c |f_i| \right).$$

**Corollary 1.3.**

(a) If $P$ is the polynomial ring $k[x_1, \ldots , x_n]$, then $a(P) = -\sum_{i=1}^n |x_i|.$

(b) If $R$ is the complete intersection ring $P/C$ where $P$ is the polynomial ring $k[x_1, \ldots , x_n]$ and $C$ is the ideal in $P$ generated by the homogeneous regular sequence $f_1, \ldots f_c$, then $a(R) = \sum_{i=1}^c |f_i| - \sum_{i=1}^n |x_i|.$

(c) If $R \to S$ is a surjection of graded Cohen-Macaulay $k$-algebras, and $S$ has finite projective dimension as an $R$-module, then $a(S) = a(R) + N$, where $N$ is the largest back twist in the minimal homogeneous resolution of $S$ by free $R$-modules. In other words, if

$$0 \to \bigoplus_i R(-b_{c,i}) \to \cdots \to \bigoplus_i R(-b_{1,i}) \to R \to S \to 0$$

is the minimal homogeneous resolution of $S$ by free $R$-modules, then $N = \max_i \{b_{c,i}\}$.

**Definition.** If $S$ is an artinian graded $k$-algebra, then the socle of $S$,

$$\text{soc} S = 0 : m_S = \{ s \in S \mid sm_S = 0 \} ,$$

is a finite dimensional graded $k$-vector space: $\text{soc} S = \bigoplus_{i=1}^\ell k(-d_i)$. We refer to the numbers $d_1 \leq d_2 \leq \cdots \leq d_\ell$ as the **socle degrees of $S$**.
Observation 1.4. Let $R$ be an artinian Gorenstein graded $k$-algebra with socle degree $\delta$, and $J$ be a homogeneous ideal of $R$. If the socle degrees of $R/J$ are $\{d_i\}$, then the minimal generators of $\text{ann} J$ have degree $\{\delta - d_i\}$.

Proof. Choose minimal generators $g_1, \ldots, g_s$ of $\text{ann} J$. Gorenstein duality implies that $\text{ann}(g_1, \ldots, \hat{g}_i, \ldots, g_s) \not\subseteq \text{ann}(g_i)$; and thus, for each $i$, we can choose an element $u_i \in \text{ann}(g_1, \ldots, \hat{g}_i, \ldots, g_s)$, which represents a generator for the socle of $R/\text{ann}(g_i)$. The ideals $J$ and $\text{ann}(g_1, \ldots, g_s)$ are equal and the socle of $R/\text{ann}(g_1, \ldots, g_s)$ is minimally generated by $u_1, \ldots, u_s$. On the other hand, $u_i g_i$ generates the socle of $R$, so the degree of $u_i$ is equal to $\delta - |g_i|$. □

Proposition 1.5. If $S$ is an artinian graded $k$-algebra and $d_1 \leq \cdots \leq d_\ell$ are the socle degrees of $S$, then the minimal generators of the canonical module $K_S$ have degrees $-d_\ell \leq \cdots \leq -d_1$.

Proof. Let $P = k[x_1, \ldots, x_n]$ be a polynomial ring which maps onto $S$. One may compute the degrees of the generators of $K_S$ as well the socle degrees of $S$ in terms of the back twists in the minimal homogeneous resolution of $S$ as a $P$-module:

$$0 \to \bigoplus_i P(-b_{n,i}) \to \cdots \to \bigoplus_i P(-b_{1,i}) \to P \to S \to 0.$$ 

The canonical module $K_S$ is equal to $\text{Ext}_P^n(S, K_P)$, where $K_P = P(a(P))$ and $a(P) = -\sum_{i=1}^n |x_i|$. It follows that the minimal homogeneous resolution of $K_S$ is

$$0 \to P(a(P)) \to \cdots \to \bigoplus_i P(a(P) + b_{n,i}) \to K_S \to 0;$$

and therefore, the minimal generators of $K_S$ (over either $S$ or $P$) have degrees $\{-a(P) - b_{n,i}\}$. On the other hand, one may compute $\text{Tor}_n^P(S, P/\mathfrak{m}_P)$ in each coordinate (see for example [6, Lemma 1.3]) in order to conclude that

$$\bigoplus_i k(-b_{n,i}) = \text{Tor}_n^P(S, k) = \text{soc} S(a(P)).$$

Thus, the socle degrees of $S$ are equal to $\{a(P) + b_{n,i}\}$. □

Corollary 1.6. Let $R \to S$ be a surjection of graded $k$-algebras with $S$ artinian, and $R$ Gorenstein. If $\text{pd}_R S$ finite, then the socle degrees of $S$ are $\{b_i + a(R)\}$, where the back twists in the minimal homogeneous resolution of $S$ by free $R$-modules are $\{b_i\}$. 

Proof. We know, from Proposition 1.2, that $K_S = \text{Ext}^d_R(S, K_R)$, with $K_R = R(a(R))$; and therefore,

$$0 \to R(a(R)) \to \cdots \to \bigoplus_i R(a(R) + b_i) \to K_S \to 0$$

is a minimal resolution of $K_S$ and the minimal generators of $K_S$ as an $R$-module, or as an $S$-module, have degrees $\{ -a(R) - b_i \}$. Apply Proposition 1.5. \qed

Let $R$ be a graded $k$-algebra. We write $\phi^n R$ to represent the ring $R$ endowed with an $R$-module structure given by the $e^{th}$ iteration of the Frobenius endomorphism $\phi_R: R \to R$. (If $r$ is a scalar in $R$ and $s$ is a ring element in $\phi^n R$, then $r \cdot s$ is equal to $r^q s \in \phi^n R$, for $q = p^e$.) The Frobenius functor $F_R^n(\_\_) = \_ \otimes_R^e \phi^n R$ is base change along the homomorphism $\phi_R$. If $g$ is a matrix with entries in $R$, then $g^{[q]}$ is the matrix in which each entry of $g$ is raised to the power $q$. If $G_1$ is the free module $\bigoplus_{i=1}^m R(-b_i)$, then $G_1^{[q]}$ is the free module $\bigoplus_{i=1}^m R(-qb_i)$. If $g$ is a matrix and $g: G_2 \to G_1$ is a map of free $R$-modules, then

$$g^{[q]} : G_2^{[q]} \to G_1^{[q]}$$

is a very clean way to write $(g: G_2 \to G_1) \otimes_R^e \phi^n R$. If $J$ is the of $R$-ideal $(a_1, \ldots, a_m)$, then $J^{[q]}$ is the $R$-ideal $(a_1^q, \ldots, a_m^q)$. The Frobenius functor is always right exact; so, in particular, $F_R^n(R/J) = R/J^{[q]}$.

Observation 1.7. Let $k$ be a field of positive characteristic $p$, $R \to S$ be a surjection of graded $k$-algebras in the sense of (1.1), with $R$ Gorenstein and $S$ artinian. If $S$ has finite projective dimension as an $R$-module, then the socles of $S$ and $F_R^n(S)$ have the same dimension; furthermore, if the socle degrees of $S$ and $F_R^n(S)$ are given by

$$d_1 \leq d_2 \leq \cdots \leq d_\ell \quad \text{and} \quad D_1 \leq D_2 \leq \cdots \leq D_\ell,$$

respectively, then

$$D_i = qd_i - (q - 1)a(R),$$

for all $i$.

Proof. Consider the minimal homogeneous resolution $F$ of $S$ by free $R$-modules. We know, from the Theorem of Peskine and Szpiro [7, Theorem 1.7] that $F_R^n(F) = F^{[q]}$ is the minimal homogeneous resolution of $F_R^n(S)$. If back twists of $F$ are $\{b_i \mid 1 \leq i \leq \ell \}$, then the back twists of $F^{[q]}$ are $\{qb_i\}$. Use Corollary 1.6 to see that $L = \ell$, $d_i = b_i + a(R)$, and $D_i = qb_i + a(R)$, for all $i$. \qed

We prove the converse of Observation 1.7 under the assumption that $R$ is a complete intersection. Our main result is the following statement.
Theorem 1.8. Let \( k \) be a field of positive characteristic \( p \), \( R \to S \) be a surjection of graded \( k \)-algebras in the sense of (1.1), with \( R \) a complete intersection and \( S \) artinian. Let \( e \) be a positive integer, \( q = p^e \), and \( d_1 \leq \cdots \leq d_\ell \) be the socle degrees of \( S \). If the socle of \( F^e_R(S) \) has the same dimension as the socle of \( S \), and the socle degrees of \( F^e_R(S) \) are given by \( D_1 \leq \cdots \leq D_\ell \), with
\[
D_i = q d_i - (q - 1) a(R),
\]
for all \( i \), then
\[
\text{Tor}_1^R(S, \phi^e_R) = 0.
\]

The plan of attack 1.9. We express \( R = P/C \), where \( P \) is the polynomial ring \( P = k[x_1, \ldots, x_n] \), each variable has positive degree, and \( C \) is a homogeneous Gorenstein ideal in \( P \) of grade \( c \). Let \( I \) be a homogeneous \( m_P \)-primary ideal in \( P \) with \( S = R/IR \). Let \( T = P/I \).

In Corollary 2.2, we convert numerical information about the socle degrees of \( S \) and \( F^e_R(S) \) into numerical information about \( \text{Tor}_c^P(K_T, R) \) and \( \text{Tor}_c^P(K_{F^e_T}, R) \). In Proposition 3.1, the numerical information about \( \text{Tor}_c \)'s is converted into the statement
\[
\text{Tor}_1^R(M \otimes_P R, \phi^e_R) = 0,
\]
where \( M \) is the the \((c-1)\)-syzygy of the \( P \)-module \( K_T \). This homological statement is expressed as a statement about ideals:
\[
(C^{[q]} + I^{[q]}): (C^{[q]}:C) = C + I^{[q]}
\]
in Proposition 4.1. In Proposition 5.1 we deduce
\[
I^{[q]} \cap C = (I \cap C)^{[q]} + CI^{[q]}.
\]
This result is equivalent to
\[
\text{Tor}_1^R(S, \phi^e_R) = 0,
\]
as is recorded in Proposition 2.4.

We would like to prove that Theorem 1.8 continues to hold after one replaces the hypothesis that \( R \) is a complete intersection with merely the hypothesis that \( R \) is Gorenstein. Three of our five steps (2.2, 4.1, and 2.4) work when \( R \) is Gorenstein. The arguments that we use in the other two steps (3.1 and 5.1) require that \( R \) be a complete intersection; although in Proposition 6.2 we prove the ideal theoretic version of (3.1) under the hypothesis that \( R \) is Gorenstein and F-pure. At any rate, if \( R \) is a complete intersection and the conclusion of Theorem 1.8 holds, then the Theorem of Avramov and Miller [1] (see also [4]) guarantees that \( S \) has finite projective dimension as an \( R \)-module. We are very curious to know if some form of the Avramov-Miller result:
\[
\text{Tor}_1^R(M, \phi^e_R) = 0 \implies \text{pd}_R M < \infty,
\]
for finitely generated \( R \)-modules \( M \), can be proven when \( R \) is Gorenstein, but not necessarily a complete intersection.
2. Convert socle degrees into degrees of generators of Tor-modules.

**Lemma 2.1.** Adopt the notation of 1.9. If the socle degrees of \( S \) are

\[ \{d_i \mid 1 \leq i \leq \ell \}, \]

then the minimal generators of \( \text{Tor}^P_c(K_T(-a(P)), R) \) have degrees

\[ \{a(R) - d_i \mid 1 \leq i \leq \ell \}. \]

**Proof.** Let \( G \) be the minimal homogeneous resolution of \( R \) by free \( P \)-modules. Corollary 1.3 (c) tells us that \( G_c = P(a(P) - a(R)) \). It follows that

\[
\text{Tor}^P_c(K_T(-a(P)), R) = H_c(K_T(-a(P)) \otimes_P G) = \{\alpha \in K_T(-a(R)) \mid C\alpha = 0\} = \text{Hom}_P(R, K_T(-a(R)).
\]

On the other hand, we have a surjection \( T \to S \); so Proposition 1.2 guarantees

\[ K_S = \text{Hom}_T(S, K_T) = \text{Hom}_P(R, K_T). \]

Thus,

\[ K_S(-a(R)) = \text{Hom}_P(R, K_T(-a(R))) = \text{Tor}^P_c(K_T(-a(P)), R). \]

Apply Proposition 1.5. □

Lemma 2.1 also applies when the ideal \( I \) is replaced by the ideal \( I^{[q]} \); and consequently, if the socle degrees of \( F^e_R(S) \) are \( \{D_i \mid 1 \leq i \leq L\} \), then the minimal generators of \( \text{Tor}^P_c(K_{F^e_R(T)}(-a(P)), R) \) have degrees \( \{a(R) - D_i \mid 1 \leq i \leq L\} \). Our conversion is complete.

**Corollary 2.2.** Retain the notation of 1.9. Assume that the socles of \( S \) and \( F^e_R(S) \) have the same dimension. Let

\[ d_1 \leq \cdots \leq d_\ell \quad \text{and} \quad D_1 \leq \cdots \leq D_\ell \]

be the socle degrees of \( S \) and \( F^e_R(S) \), respectively; and

\[ \gamma_1 \leq \cdots \leq \gamma_\ell \quad \text{and} \quad \Gamma_1 \leq \cdots \leq \Gamma_\ell, \]

be the minimal generator degrees of

\[ \text{Tor}^P_c(K_T(-a(P)), R) \quad \text{and} \quad \text{Tor}^P_c(K_{F^e_R(T)}(-a(P)), R), \]

 respectively.
respectively. Then

\[ D_i = qd_i - (q - 1)a(R) \text{ for all } i \iff \Gamma_i = q\gamma_i \text{ for all } i. \]

In the notation of 1.9, let \( \mathbb{F} \) be the minimal homogeneous resolution of \( K_T(-a(P)) \). The proof of Proposition 1.5 shows that \( F_F^p(\mathbb{F}) = \mathbb{F}[q] \) is the minimal homogeneous resolution of \( K_{F_F^p(T)}(-a(P)) \) (rather than some other shift of \( K_{F_F^p(T)} \)); hence,

\[
F_F^p(K_T(-a(P))) = K_{F_F^p(T)}(-a(P)),
\]

\[
\text{Tor}_i^P(K_T(-a(P)), R) = H_i(\mathbb{F} \otimes_P R), \text{ and}
\]

\[
\text{Tor}_i^P(F_F^p(K_T(-a(P))), R) = H_i(\mathbb{F}[q] \otimes_P R),
\]

for all \( i \). We focus on \( i = c \). Let \( M \) be the \((c - 1)\)-syzygy of \( K_T(-a(P)) \). The beginning of the minimal homogeneous resolution of \( M \) is

\[
\cdots \rightarrow \mathbb{F}_{c+1} \rightarrow \mathbb{F}_c \rightarrow \mathbb{F}_{c-1} \rightarrow M \rightarrow 0.
\]

One may calculate \( \text{Tor}_1^R(M \otimes_P R, \phi_R^e) \) as

\[
\frac{H_c(\mathbb{F}[q] \otimes_P R)}{(H_c(\mathbb{F} \otimes_P R))[q]},
\]

as described in Observation 2.3. The conclusion of Corollary 2.2 tells us that if \([\bar{z}_1], \ldots, [\bar{z}_\ell] \) is a minimal generating set for \( H_c(\mathbb{F} \otimes_P R) \), then the elements \([z_1[q]], \ldots, [z_\ell[q]] \), in \( H_c(\mathbb{F}[q] \otimes_P R) \), have the correct degrees to be a minimal generating set for \( H_c(\mathbb{F}[q] \otimes_P R) \). In Proposition 3.1 we prove that, when \( R \) is a complete intersection, then \([\bar{z}_1[q]], \ldots, [\bar{z}_\ell[q]] \) do indeed generate \( H_c(\mathbb{F}[q] \otimes_P R) \).

**Observation 2.3.** Let \( P \rightarrow R \) be a surjection of graded \( k \)-algebras, with \( P \) a polynomial ring, and let \( M \) be a finitely generated graded \( P \)-module. Then there is an exact sequence of graded \( R \)-modules:

\[
F^e_R(\text{Tor}_1^P(M, R)) \rightarrow \text{Tor}_1^P(F_F^p(M), R) \rightarrow \text{Tor}_1^R(M \otimes_P R, \phi_R^e) \rightarrow 0.
\]

**Proof.** Let \( (N, n) \) be the minimal homogeneous resolution of \( M \) by free \( P \)-modules. The functor \( F_F^p(\_ \_ \_ ) \) is exact; so, \( N[q] \) is the minimal homogeneous resolution of \( F_F^p(M) \) by free \( P \)-modules; and \( \text{Tor}_1^P(F_F^p(M), R) \) is equal to

\[
H_1(F_F^p(N) \otimes_P R) = H_1(F_R^e(N \otimes_P R)).
\]
The functors $F^c_P(\_ \otimes_P R)$ and $F^c_R(\_ \otimes_P R)$ are equal because the homomorphisms

$$
P \xrightarrow{\phi^c_P} P \quad \text{quot. map} \quad \xrightarrow{\phi^c_R} R \quad \text{quot. map}
$$

commute. Let $\bar{-}$ denote the functor $\_ \otimes_P R$. Select elements $z_1, \ldots, z_\ell$ of $\mathbb{N}_1$ so that $\bar{z}_1, \ldots, \bar{z}_\ell$ are cycles in $\mathbb{N} \otimes_P R$ and the homology classes $[\bar{z}_1], \ldots, [\bar{z}_\ell]$ are a minimal generating set for $H_1(\mathbb{N} \otimes_P R) = \operatorname{Tor}_1^P(M, R)$. It is clear that $z_i^{[q]}$ is an element of $F^c_\mathbb{N}(\mathbb{N}_1)$ with $z_i^{[q]} = z_i^{[q]}$ a cycle in $F^c_\mathbb{N}(\mathbb{N}) \otimes_P R = F^c_R(\mathbb{N} \otimes_P R)$, for each $i$.

The technique of killing cycles tells us that

$$
\mathbb{M} : \quad \mathbb{N}_2 \oplus \bigoplus_{i=1}^\ell R(-|z_i|) \xrightarrow{\bar{n}_2 \bar{z}_1 \ldots \bar{z}_\ell} \bar{\mathbb{N}}_1 \xrightarrow{\bar{n}_1} \bar{\mathbb{N}}_0 \to \mathbb{M} \to 0
$$

is the beginning of a homogeneous resolution of $\mathbb{M}$ by free $R$-modules. It follows that

$$
\operatorname{Tor}_1^R(\mathbb{M}, \phi^c_R) = H_1(\mathbb{M} \otimes_R \phi^c_R) = \frac{H_1(F^c_R(\mathbb{N}))}{([\bar{z}_1], \ldots, [\bar{z}_\ell])} \xrightarrow{\phi^c_R} \frac{\operatorname{Tor}_1^P(F^c_\mathbb{N}(\mathbb{N}), R)}{([\bar{z}_1], \ldots, [\bar{z}_\ell])}. \quad \square
$$

The following result is an application of the technique of Observation 2.3.

**Proposition 2.4.** Let $R = P/C$ and $T = P/I$, where $I$ and $C$ are homogeneous ideals in the polynomial ring $P$. Then

$$
\operatorname{Tor}_1^R(T \otimes_P R, \phi^c_R) = \frac{I^{[q]} \cap C}{(I \cap C)^{[q]} + I^{[q]}C}.
$$

**Proof.** Apply Observation 2.3 to see that

$$
\operatorname{Tor}_1^R(T \otimes_P R, \phi^c_R) = \frac{H_1(\mathbb{N}^{[q]} \otimes_P R)}{([\bar{z}_1], \ldots, [\bar{z}_\ell])},
$$

where $(\mathbb{N}, \mathbb{n})$ is the minimal homogeneous resolution of $T$ by free $P$-modules, $\bar{-}$ is the functor $\_ \otimes_P R$, and $z_1, \ldots, z_\ell$ are elements of of $\mathbb{N}_1$ with $[\bar{z}_1], \ldots, [\bar{z}_\ell]$ a minimal generating set for $H_1(\mathbb{N} \otimes_P R) = \operatorname{Tor}_1^P(P/P, C) = \frac{I \cap C}{I \cap C}$.

Observe that $I \cap C = (\mathbb{n}_1(z_1), \ldots, \mathbb{n}_1(z_\ell)) + IC$. Observe also, that $H_1(\mathbb{N}^{[q]} \otimes_P R) = \operatorname{Tor}_1^P(P/P, C) = \frac{I^{[q]} \cap C}{I^{[q]} \cap C}$. The isomorphism $H_1(\mathbb{N}^{[q]} \otimes_P R) \to \frac{I^{[q]} \cap C}{I^{[q]} \cap C}$ carries $[\bar{z}_i]^{[q]}$ to the class of $(\mathbb{n}_1(z_i))^{[q]}$. \quad \square
3. Degree considerations concerning Tor₁.

Proposition 3.1 is a general statement that says that if the degrees of the minimal generators of
\[ \text{Tor}^1_P(M, R) \quad \text{and} \quad \text{Tor}^1_P(F^e_P(M), R) \]
are related in the appropriate manner, then \( M \otimes_P R \) has finite projective dimension as an \( R \)-module. When the notation of 1.9 and the hypothesis of Theorem 1.8 are in effect, then, as we saw after Corollary 2.2, Proposition 3.1 may be applied with \( M \) equal to the \((c-1)\)-syzygy of \( K_T(−a(P)) \).

**Proposition 3.1.** Let \( P \rightarrow R \) be a surjection of graded \( k \)-algebras, with \( P \) a polynomial ring and \( R \) a complete intersection, and let \( M \) be a finitely generated graded \( P \)-module. Suppose that the minimal generators of \( \text{Tor}^1_P(M, R) \) have degrees \( \{γ_i | 1 ≤ i ≤ ℓ\} \). If the minimal generators of \( \text{Tor}^1_P(F^e_P(M), R) \) have degrees \( \{qγ_i | 1 ≤ i ≤ ℓ\} \), then \( \text{Tor}^1_R(M \otimes_P R, φ^e_R) = 0 \).

**Proof.** Inflation of the base field \( k \rightarrow K \) gives rise to faithfully flat extensions \( P \rightarrow P \otimes_k K \) and \( R \rightarrow R \otimes_k K \). Consequently, we may assume that \( k \) is a perfect field. Let \( C \) be the ideal in \( P \) with \( R = P/C \), and let \( f_1, \ldots, f_c \) be a homogeneous regular sequence in \( P \) that generates \( C \). We retain the notation from the proof of Observation 2.3. So, \( \mathbb{N} : \ N_2 \xrightarrow{n_2} N_1 \xrightarrow{n_1} N_0 \rightarrow M \rightarrow 0 \)
is the beginning of the minimal homogeneous resolution of \( M \) by free \( P \)-modules, \( - \) is the functor \( \_ \otimes_P R \), and \( z_1, \ldots, z_ℓ \) are elements of \( N_1 \) with \([z_1], \ldots, [z_ℓ] \) a minimal generating set for \( H_1(\overline{N}) = \text{Tor}^1_P(M, R) \). We will prove that \([z_1[q], \ldots, [z_ℓ[q]] \)
generates \( H_1(\overline{N}[q]) = \text{Tor}^1_P(F^e_P(M), R) \).

For each integer \( δ \), let \( V_δ \) be the vector space
\[ V_δ = \frac{H_1(\overline{N}[q])}{W'_δ + W''_δ}, \]
where \( W'_δ \) is the \( P \)-submodule
\[ \sum_{δ<i} \left[H_1(\overline{N}[q])\right]_i, \]
of \( H_1(\overline{N}[q]) \) and \( W''_δ \) is the \( P \)-submodule of \( H_1(\overline{N}[q]) \) generated by
\[ \sum_{i<δ} \left[H_1(\overline{N}[q])\right]_i. \]
Let $X$ be a homogeneous minimal generating set for the $P$-module $H_1(\bar{N}[q])$. Let $X_\delta$, be the subset of $X$ which consists of those generators which have degree equal to $\delta$. Notice that the set $X_\delta$ is a basis the vector space $V_\delta$ over $k$. Let

$$Y_\delta = \{ [\bar{z}_i^{[q]}] \in H_1(\bar{N}[q]) \mid 1 \leq i \leq \ell \text{ and } |z_i^{[q]}| = \delta \}.$$ 

Our hypothesis guarantees that the dimension of the vector space $V_\delta$ is exactly equal to the number of elements in $Y_\delta$. Once we prove that the elements of $Y_\delta$ are linearly independent in $V_\delta$, then we will know that $Y_\delta$ is a basis for $V_\delta$ and the elements of $Y_\delta$ are part of a minimal generating set of $H_1(\bar{N}[q])$.

We work by induction on $\delta$ starting with $\delta = 0$ and then looking at successively higher values of $\delta$. When $\delta$ is small, then $Y_\delta$ is the empty set, $V_\delta$ is zero, and everything is fine. Now we work on the inductive step. There is nothing to do unless $\delta$ is a multiple of $q$. Relabel the $z_i$, if necessary, and select the integer $\lambda$ so that $z_1, \ldots, z_\lambda$ have degree less than $\delta/q$, and $z_{\lambda+1}, \ldots, z_\ell$ have degree at least $\delta/q$. Consider a non-trivial $k$-linear combination of of the elements of $Y_\delta$. The field $k$ is closed under the taking of $q$th roots; so this linear combination is equal to $[\bar{z}^{[q]}]$ where $\bar{z}$ is a non-trivial $k$-linear combination of those $z_i$ that have degree equal to $\delta/q$. So

(3.2) $[\bar{z}]$ is not zero in $\frac{H_1(\bar{N})}{P([\bar{z}_1], \ldots, [\bar{z}_\lambda])}$.

We assume that

(3.3) $[\bar{z}^{[q]}] = 0$ in $\frac{H_1(\bar{N}[q])}{W''_\delta}$;

and we will show that this assumption leads to a contradiction. Keep in mind that the induction hypothesis guarantees that

(3.4) $W''_\delta = P([\bar{z}_1^{[q]}], \ldots, [\bar{z}_\lambda^{[q]}])$.

We know that $\bar{z}$ and $\bar{z}_1, \ldots, \bar{z}_\lambda$ all are cycles in $\bar{N}$; so

(3.5) $n_1z \in CN_0$ and $n_1z_i \in CN_0$ for $1 \leq i \leq \lambda$.

We introduce a notational convenience. Let

$$Q_2 = \mathbb{N}_2 \oplus \bigoplus_{i=1}^\lambda P(-|z_i|)$$

and let $q_2 : Q_2 \to \mathbb{N}_1$ be the map

$$q_2 = [n_2 \quad z_1 \quad \ldots \quad z_\lambda].$$
Of course, the map
\[ q_2^{[q]} : \mathbb{Q}_2^{[q]} \to \mathbb{N}_1^{[q]} \]
also now has meaning. Assumption 3.3, together with the induction hypothesis (3.4), tells us that
\[ z^{[q]} \in \text{im} \cdot q_2^{[q]} + C \mathbb{N}_1^{[q]}, \]
which is the base case for the following induction. We prove that if \( 1 \leq t \leq c(q - 1) \), then
\[
(3.6) \quad z^{[q]} \in \text{im} \cdot q_2^{[q]} + C^{[q]} \mathbb{N}_1^{[q]} + C^t \mathbb{N}_1^{[q]} \quad \implies \quad z^{[q]} \in \text{im} \cdot q_2^{[q]} + C^{[q]} \mathbb{N}_1^{[q]} + C^{t+1} \mathbb{N}_1^{[q]}.
\]
As soon as (3.6) is established, then
\[ z^{[q]} \in \text{im} \cdot q_2^{[q]} + C^{[q]} \mathbb{N}_1^{[q]} \]
because \( C^{c(q-1)+1} \subseteq C^{[q]} \). Now the proof is complete because we apply Observation 3.7 to
\[ z^{[q]} \in \text{im} \cdot q_2^{[q]} f_1 I \ldots f_c I \]
to conclude that
\[ z \in \text{im} \cdot q_2^{[q]} f_1 I \ldots f_c I, \]
and this violates (3.2).

We prove (3.6). For each \( c \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_c) \) of non-negative integers, with \( \alpha_i < q \) for all \( i \), and \( \sum \alpha_i = t \), there exists \( y_\alpha \in \mathbb{N}_1^{[q]} \) such that
\[ z^{[q]} - \sum_{\alpha} f_1^{\alpha_1} \ldots f_c^{\alpha_c} y_\alpha \in C^{[q]} \mathbb{N}_1^{[q]} + \text{im} \cdot q_2^{[q]}.
\]
Fix a \( c \)-tuple \( \alpha \). Apply \( n_1^{[q]} \) and use (3.5) to see that
\[ f_1^{\alpha_1} \ldots f_c^{\alpha_c} n_1^{[q]} y_\alpha \in (f_1^{\alpha_1+1}, \ldots, f_c^{\alpha_c+1}) \mathbb{N}_1^{[q]}.
\]
It follows, from the fact that \( f_1, \ldots, f_c \) is a regular sequence, that
\[ n_1^{[q]} y_\alpha \in C \mathbb{N}_1^{[q]}.
\]
So, \( \bar{y}_\alpha \) is a one-cycle, of degree less than \( \delta \), in \( \mathbb{N}^{[q]} \). The induction hypothesis (3.4) tells us that
\[ y_\alpha \in \text{im} \cdot q_2^{[q]} + C \mathbb{N}_1^{[q]}, \]
and (3.6) is established. \( \Box \)

We close this section with a quick application of the flatness of the Frobenius functor for regular rings.
**Observation 3.7.** Let $P$ be a polynomial ring, $q : Q_2 \to Q_1$ be a homomorphism of graded $P$-modules, and $z$ be an element of $Q_1$. If $z^{[q]}$ is in the image of $q^{[q]}$, then $z$ is in the image of $q$.

**Proof.** Let $\tilde{Q}_2$ be the graded free module $Q_2 \oplus P(-|z|)$ and $\tilde{q}$ be the map of graded free modules

$$\tilde{q} = [q \ z] : \tilde{Q}_2 \to Q_1.$$ 

The hypothesis ensures the existence of $h \in Q_2^{[q]}$ with $[-h \ 1]$ is in the kernel of $\tilde{q}^{[q]}$. The Frobenius functor $\_ \otimes_P \phi_P$ is flat; so, there exist $[t_{bi}]$ in ker $\tilde{q}$ and $a_i \in P$ such that $\sum a_i [t_{bi}]^{[q]} = [-h \ 1]$. Degree considerations tell us that $b_i$ is a unit, for some $i$. For this $i$, we have $q(\frac{t_{bi}}{b_i}) = z$. □

4. **We interpret Tor$_1$ of the syzygy in terms of ideals.**

Recall, from the beginning of section 3, that if the notation of 1.9 and the hypotheses of Theorem 1.8 are in effect, then Tor$_1^R(M \otimes_P R, \phi_R) = 0$, where $M$ is the $(c-1)$-syzygy of $K_T(-a(P))$. In this section, we interpret this Tor-module in terms of ideals. Our interpretation continues to hold even when the hypotheses of Theorem 1.8 are not in effect.

**Proposition 4.1.** Adopt the notation of 1.9 and let $M$ be the $(c-1)$-syzygy in the minimal homogeneous resolution of $K_T(-a(P))$ by free $P$-modules. Then

$$\text{Tor}_1^R(M \otimes_P R, \phi_R) = \text{Hom}_P \left( \frac{(C^{[q]} + I^{[q]}): (C^{[q]}: C)}{C + I^{[q]}}, \frac{P}{A^{[q]}(N)} \right),$$

where $A \subseteq I$ is a homogeneous $m_P$-primary Gorenstein ideal and $N$ is equal to $a(P/A^{[q]}) - a(R)$; furthermore,

$$\text{Tor}_1^R(M \otimes_P R, \phi_R) = 0 \iff \frac{(C^{[q]} + I^{[q]}): (C^{[q]}: C)}{C + I^{[q]}} = 0.$$

**Proof.** In our argument, we assume that $A \subseteq I$ have the same grade without assuming that these ideals are $m_P$-primary, and we prove that

$$\text{Tor}_1^R(M \otimes_P R, \phi_R) = \frac{A^{[q]}: (C + I^{[q]})}{(A: (C + I))^{[q]}(C^{[q]}: C) + A^{[q]}(N)}.$$ 

Once (4.3) is established, then, when $A$ is a $m_P$-primary ideal, Gorenstein duality may be employed to show that to the right side of (4.3) and the right side of (4.2) are both equal to

$$\frac{A^{[q]}: (C + I^{[q]})}{A^{[q]}: ((C + I)^{[q]}: (C^{[q]}: C))}(N).$$
Let \((G, g_\bullet)\) be the minimal homogeneous resolution of \(R\) by free \(P\)-modules and \(y_\bullet : G^{[q]} \to G\) be a map of complexes which lifts the natural quotient map \(P/C^{[q]} \to R\). The ideal \(C\) is Gorenstein of grade \(c\), so \(G_c = P(a(P) - a(R))\). The map \(y_c : G^{[q]}_c \to G_c\) is multiplication by \(y\), for some element \(y\) in \(P\) of degree \((q - 1)(a(R) - a(P))\). We know, from linkage theory, that

\[
C^{[q]} : C = (y, C^{[q]}).
\]

Use the surjections \(P_A \to T\) and \(P_{A[q]} \to F^c(T)\) to calculate

\[
K_T = \frac{A \cdot I(A)}{A} (a(P/A) - a(R)) \quad \text{and} \quad K_{F^c(T)} = \frac{A[q] \cdot I^{[q]}(a(P/A^{[q]}))}{A^{[q]}(a(P/A^{[q]}))}.
\]

It follows that \(H_c(K_T(-a(P)) \otimes_P G)\) is equal to

\[
\{ \alpha \in \frac{A \cdot I(A)}{A} (a(P/A) - a(R)) \mid \alpha C = 0 \} = \frac{A \cdot (I + C)}{A} (a(P/A) - a(R))
\]

and

\[
H_c(K_{F^c(T)}(-a(P)) \otimes_P G) = \frac{A[q] \cdot (I^{[q]} + C)}{A^{[q]}} (N).
\]

Let \((F, f)\) be the minimal homogeneous resolution of \(K_T(-a(P))\) by free \(P\) modules. We saw in Observation 2.3 that

\[
\text{Tor}_1^R(M \otimes_P R, \phi^c_R) = \frac{H_c(F^{[q]} \otimes_P R)}{([\bar{z}_1^{[q]}], \ldots, [\bar{z}_\ell^{[q]}])},
\]

where \(z_1, \ldots, z_\ell\) are elements in \(F_c\) with \([\bar{z}_1], \ldots, [\bar{z}_\ell]\) a minimal generating set for \(H_c(F \otimes_P R)\). We use the isomorphisms

\[
H_c(F \otimes_P R) = H_c(Tot(F \otimes_P G)) = H_c(K_T(-a(P)) \otimes_P G) = \frac{A \cdot (I + C)}{A} (a(P/A) - a(R))
\]

and

\[
H_c(F^{[q]} \otimes_P R) = H_c(Tot(F^{[q]} \otimes_P G)) = H_c(K_{F^c(T)}(-a(P)) \otimes_P G) = \frac{A[q] \cdot (I^{[q]} + C)}{A^{[q]}} (N)
\]

(4.4)

(4.5)

(4.6)

\((f_i \otimes 1)(w_i) = (1 \otimes g_{c-i+1})(w_{i-1})\).
In particular, $w_0$ is an element of $\mathbb{F}_0 \otimes \mathbb{G}_c$. The isomorphism (4.4) sends the homology class $[\overline{w}_c]$ to the image of $w_0$ in $A_{A:(C+I^q)}(a(P/A) - a(R))$. In a similar manner, if $W_c$ is an element of $\mathbb{F}^q_c$ with $f^q_c(W_c) \in C \mathbb{F}^q_{c-1}$, then the isomorphism (4.5) sends the homology class $[\overline{W}_c]$ in $H_c(\mathbb{F}^q \otimes_R R)$ to the image of $W_0$ in $A_{A^{[q]}_{A:(C+I^q)}}(N)$, where $W_i \in \mathbb{F}^q_i \otimes \mathbb{G}_{c-i}$ and

\[(4.7) \quad (f^q_i \otimes 1)(W_i) = (1 \otimes g_{c-i+1})(W_{i-1}).\]

We finish the argument by showing that the submodule $([z_1^q], \ldots, [z_e^q])$ of $H_c(\mathbb{F}^q \otimes_R R)$ is sent to the submodule $A_{A:(C+I^q)}[y + A^{[q]}_{A:(C+I^q)}(N)]$ of $A_{A^{[q]}_{A:(C+I^q)}}(N)$ under the isomorphism (4.5). Let $\overline{w}_c$ be a cycle in $\mathbb{F} \otimes_R$, for some element $w_c$ of $\mathbb{F}_c$. We are given the family $\{w_i\}$ with $w_i \in \mathbb{F}_i \otimes \mathbb{G}_{c-i}$ such that (4.6) holds. If $w_i = \sum u_{i,j} \otimes v_{c-i,j}$ with $u_{i,j} \in \mathbb{F}_i$ and $v_{c-i,j} \in \mathbb{G}_{c-i}$, then let

$$W_i = \sum_j u_{i,j}^q \otimes y_{c-i}(v_{c-i,j}^q) \in \mathbb{F}^q_i \otimes \mathbb{G}_{c-i}.$$ 

A short calculation shows that that (4.7) holds for $\{W_i\}$ and we conclude that if $a$ in $A_{A:(C+I^q)}(a(P/A) - a(R))$ is the image of the homology class $[\overline{w}_c]$ under the isomorphism (4.4), then $ya^q$ in $A_{A^{[q]}_{A:(C+I^q)}}(N)$ is the image of the homology class $[\overline{w}_c^q]$ under the isomorphism (4.5). \qed

5. The key calculation.

In section 3, we proved that if the socle hypothesis of Theorem 1.8 is in effect and $M$ is the $(c - 1)$-syzygy of $K_T$ as a $P$-module, then $\text{Tor}_1^R(M \otimes_R R, \phi \mathbb{R}) = 0$. Our goal is to prove that $\text{Tor}_1^R(T \otimes_R R, \phi \mathbb{R}) = 0$. Homological arguments in sections 2 and 4 connect these Tor-modules to quotients of ideals. In the present section we show how information about the Tor-module of $M$ gives information about the Tor-module of $T$, when $R$ is a complete intersection.

**Proposition 5.1.** Let $P$ be a regular ring of positive characteristic $p$, and let $C$ and $I$ be ideals in $P$. Assume that $C$ is generated by the regular sequence $f_1, \ldots, f_c$ and that

$$(C + I)^q[y] = C + I^q,$$

where $y = (f_1 \cdots f_c)^{q-1}$. Then

$$I^q \cap C = (I \cap C)^q + CI^q.$$
Proof. Notice that $C^q\colon C = (y) + C^q$. Take $\xi \in I^q \cap C$. We prove that if $1 \leq t \leq c(q - 1)$, then

$$\xi \in C^t + C^q + CI^q \implies \xi \in C^{t+1} + C^q + CI^q. \tag{5.2}$$

Of course, we know that the hypothesis of (5.2) holds for $t = 1$. Once we have established (5.2), then, since $C^{c(q-1)+1} \subseteq C^q$, we know that

$$\xi \in I^q \cap C^q + CI^q = (I \cap C)^q + CI^q,$$

because the Frobenius functor on $P$ is flat. Now we prove (5.2). Write $\xi$ as an element of $C^q + CI^q$ plus

$$\sum_{\alpha} b_\alpha f_1^{\alpha_1} \cdots f_c^{\alpha_c},$$

where $\alpha = (\alpha_1, \ldots, \alpha_c)$ varies over all $c$-tuples of non-negative integers with $\alpha_i < q$ for all $i$ and $\sum_{i=1}^c \alpha_i = t$. Fix an index $\alpha$. Observe that

$$f_1^{q-\alpha_1-1} \cdots f_c^{q-\alpha_c-1} \xi$$

is equal to $b_\alpha y$ plus an element of $C^q + I^q$. The hypothesis tells us that $b_\alpha$ is in $C + I^q$; (5.2) is established, and the proof is complete. \qed

6. The Gorenstein F-pure case.

The question of whether the conclusion of Theorem 1.8 holds when $R$ is Gorenstein is still open. In this section, we include partial results in this direction. Recall that the ring $R$ of positive prime characteristic $p$ is F-pure if whenever $J$ is an ideal of $R$ and $x$ is an element of $R$ with $x \not\in J$, then $x^q \not\in J^q$ for all $q = p^e$.

First note that the top socle degree (tsd) of a Frobenius power is always at least equal to the “expected” top socle degree:

**Proposition 6.1.** Let $k$ be a field of positive characteristic $p$, $R \to S$ be a surjection of graded $k$-algebras with $S$ artinian. Assume that either $R$ is a complete intersection or $R$ is Gorenstein and F-pure. If $d$ is the top socle degree of $S$, then the top socle degree of $F^e_R(S)$ is at least $qd - (q - 1)a(R)$.

**Proof.** Write $S = R/J$, with $J \subset R$ an $m$-primary ideal, where $m$ is the unique homogeneous maximal ideal of $R$.

We first assume that $R$ is Gorenstein and F-pure. Let $a$ be an $m$-primary ideal of $R$, generated by a regular sequence, with $a \subset J$. Let $g_1, \ldots, g_s$, with $|g_1| \leq \cdots \leq |g_s|$, be elements in $R$ which represent a minimal generating set for $(a:J)/a$. The hypothesis that $R$ is F-pure ensures that $g_1^q, \ldots, g_s^q$ represents a minimal generating
set for \((g_1^q, \ldots, g_s^q, a[q]) / a[q]\); hence, the minimum degree among non-zero elements of \((g_1^q, \ldots, g_s^q, a[q]) / a[q]\) is \(q|g_1|\). Observation 1.4 yields that

\[
\text{tsd} \frac{R}{J} = \text{socle degree} \frac{R}{a} - |g_1| \quad \text{and} \quad \text{tsd} \frac{R}{a[q]} : (g_1^q, \ldots, g_s^q) = \text{socle degree} \frac{R}{a[q]} - q|g_1|.
\]

The \(R\)-module \(R/a\) has finite projective dimension so

\[
\text{socle degree} \frac{R}{a[q]} = q \text{socle degree} \frac{R}{a} - (q - 1)a(R).
\]

Duality gives \(J = a : (g_1, \ldots, g_s)\). It follows that \(J[q] \subseteq a[q] : (g_1^q, \ldots, g_s^q)\); and therefore,

\[
\text{tsd} \frac{R}{J[q]} \geq \text{tsd} \frac{R}{a[q]} : (g_1^q, \ldots, g_s^q) = \text{socle degree} \frac{R}{a[q]} - q|g_1| \quad = \quad \text{socle degree} \frac{R}{a[q]} - q \left( \text{socle degree} \frac{R}{a} - \text{tsd} \frac{R}{J} \right) \quad = \quad q \text{tsd} \frac{R}{J} - (q - 1)a(R).
\]

The proof is complete if \(R\) is Gorenstein and F-pure. Throughout the rest of the argument, \(R\) is a complete intersection. We begin by reducing to the case where \(J\) is an irreducible ideal. Assume, for the time being, that the result has been established for irreducible ideals. Let \(J = J_1 \cap \cdots \cap J_n\), with each \(J_i\) irreducible.

Recall that \(\text{tsd} R/J\) is the largest integer \(d\) with \(R_d \not\subseteq J\). It follows that that the \(\text{tsd} R/J\) is equal to the maximum of the set \(\{\text{tsd} R/J_k\}\). Fix a subscript \(k\) with \(\text{tsd} R/J = \text{tsd} R/J_k\). We know that \(J[q] \subseteq J[k][q]\); and therefore,

\[
q \text{tsd} \frac{R}{J} - (q - 1)a(R) = q \text{tsd} \frac{R}{J_k} - (q - 1)a(R) \leq \text{tsd} \frac{R}{J[q]} \leq \text{tsd} \frac{R}{J[q]}
\]

Henceforth, the ideal \(J\) is irreducible. Write \(R = P/C\), where \(P\) is a polynomial ring and the ideal \(C\) is generated by the homogeneous regular sequence \(f_1, \ldots, f_c\). Let \(I\) be the pre-image of \(J\) in \(P\). In particular, \(C \subseteq I\). The rings \(R/J = P/I\) and \(P/I[q]\) are Gorenstein, so Observation 1.4 gives

\[
\text{socle degree}(\frac{P}{I[q]}) - M = \text{tsd}(\frac{P}{I[q]+C}),
\]

where \(M\) is the least degree among homogeneous non-zero elements of \(\frac{I[q]+C}{I[q]}\). The \(P\)-module \(P/I\) has finite projective dimension; so

\[
q \text{socle degree}(\frac{P}{I}) - (q - 1)a(P) = \text{socle degree}(\frac{P}{I[q]}).
\]

Recall the formula \(a(P/C) = a(P) + \sum_{i=1}^c |f_i|\). The inequality

\[
q \text{socle degree}(\frac{P}{I}) - (q - 1)a(P/C) \leq \text{tsd}(\frac{P}{I[q]+C})
\]
is equivalent to the inequality
\[ M \leq (q - 1)(|f_1| + \cdots + |f_c|). \]

We establish the most recent inequality. There exists an integer \( t \), with \( 0 \leq t \leq c(q - 1) \), such that \( C^t \not\subseteq I^{[q]} \); but \( C^{t+1} \subseteq I^{[q]} \). Thus, some element \( f_1^{t_1} \cdots f_c^{t_c} \), with \( \sum t_i = t \) and \( 0 \leq t_i \leq q - 1 \) for all \( i \), of \( C^t \) is a non-zero element of \( \frac{I^{[q]}_C}{I^{[q]}_A} \); and therefore,
\[ M \leq |f_1^{t_1} \cdots f_c^{t_c}| \leq (q - 1)(|f_1| + \cdots + |f_c|). \, \square \]

The next result shows that we can get most of the way through the proof of Theorem 1.8 under the assumption that \( R \) is Gorenstein and F-pure. Only the last step (the result of Proposition 5.1) is still missing.

**Proposition 6.2.** Let \( k \) be a field of positive characteristic \( p \), \( R \to S \) be a surjection of graded \( k \)-algebras with \( R \) Gorenstein and \( S \) artinian. Assume, in addition, that \( R \) is F-pure. Assume that \( d_1 \leq \cdots \leq d_\ell \) are the socle degrees of \( S \) and the socle of \( F^\ell_R(S) \) has the same dimension as the socle of \( S \), with degrees of the generators given by \( D_1 \leq D_2 \leq \cdots \leq D_\ell \), with
\[ D_i = qd_i - (q - 1)a(R), \]
for all \( i \). Let \( R = P/C \), and \( S = R/IR \), with \( P \) a polynomial ring, \( I \subset P \). Then we have
\[ (C^{[q]} + I^{[q]}):(C^{[q]}:C) = C + I^{[q]}. \]

**Proof.** Let \( A = C + (x_1, \ldots, x_d) \), where the images of \( x_1, \ldots, x_d \) are a system of parameters in \( R \). Let \( K = A:(I + C) \), so that we also have \( I + C = A:K \). We have
\[ (I^{[q]} + C^{[q]}):(C^{[q]}:C) = (A^{[q]}:K^{[q]}):(C^{[q]}:C) = (A^{[q]}:(C^{[q]}:C)):K^{[q]}. \]

We claim that \( A^{[q]}:(C^{[q]}:C) = A^{[q]} + C \). We see this by looking at the comparison map of resolutions induced by the projection \( P/A^{[q]} \to P/(A^{[q]} + C) \). If \( F \) is the resolution of \( P/C \), and \( \mathbb{K} \) is the Koszul complex on \( x_1, \ldots, x_d \), then the resolution of \( P/A^{[q]} \) is given by \( F^{[q]} \otimes \mathbb{K}^{[q]} \), the resolution of \( P/(A^{[q]} + C) \) is given by \( F \otimes \mathbb{K}^{[q]} \), and the comparison map between them is given by the comparison map \( F^{[q]} \to F \), tensored with \( \mathbb{K}^{[q]} \). Thus, the last map is multiplication by an element of \( P \) which represents the generator of \( (C^{[q]}:C)/C^{[q]} \).

It follows that \( (I^{[q]} + C^{[q]}):(C^{[q]}:C) = (A^{[q]} + C):K^{[q]} \). It is clear that
\[ (A^{[q]} + C):K^{[q]} \supseteq I^{[q]} + C. \]

We next show that the rings defined by these ideals have the same socle degrees. Let \( \delta \) and \( \Delta \) be the socle degrees of the Gorenstein rings \( \frac{P}{A^{[q]} + C} \) and \( \frac{P}{A^{[q]}} \), respectively.
The $P/C$-module $P/A$ has finite projective dimension; so $\Delta = q\delta - (q - 1)a(P_C)$. Let $g_1, \ldots, g_s$ be elements of $K$ which represent a minimal generating set for $\frac{K}{A}$. Observation 1.4 gives that the socle degrees of $\frac{P}{I+P}$ are $\{\delta - |g_i|\}$. So, our hypothesis tells us that the socle degrees of $\frac{P}{I[q]+C}$ are

$$\{q(\delta - |g_i|) - (q - 1)a(P_C)\} = \{\Delta - q|g_i|\}.$$  

It is clear that $g_1^q, \ldots, g_s^q$ represents a generating set for $\frac{K[q]+C}{A[q]+C}$. The hypothesis that $\frac{P_C}{C}$ is F-pure guarantees that $g_1^q, \ldots, g_s^q$ is a minimal generating set of $\frac{K[q]+C}{A[q]+C}$. Observation 1.4 yields that the socle degrees of $\frac{P}{(A[q]+C):K[q]}$ are exactly the same as the socle degrees of $\frac{P}{I[q]+C}$; and therefore, the result of [6] shows that $I[q] + C = (A[q] + C):K[q] = (I[q] + C[q]):(C[q]:C)$. □

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