Research Article

An Epidemic Patch-Enabled Delayed Model for Virus Propagation: Towards Evaluating Bifurcation and White Noise

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The massive disruptions caused by malware, such as a virus in computer networks and other aspects of information and communication technology, have generated attention, making it a hot research topic. While antivirus and firewalls can be effective, there is also a need to understand the spread patterns of viral infection using epidemic models to curb its incidences. Many previous research attempts have produced analytical models for computer viruses under various infectiousness situations. As a result, we suggested the SLBS model, which considers infection latency and transient immunity in patched nodes. Under certain conditions, the local stability of all equilibrium points is investigated. By setting the delay parameter, we established the occurrence of a Hopf bifurcation (HB) as it crossed a crucial point by several analyses. We also used the centre manifold theorem and normal form theory to examine the attributes of the HB. While the former was used to study the time delay and direction of Hopf bifurcation, the latter was used to investigate external noise and its intensities. Finally, numerical simulations two dimensional and three-dimensional graphs were used to depict the perturbations of the model, thus bolstering the essentiality of the study.

1. Introduction

The Internet and computer networks have greatly facilitated human effort, education, and living since the rising notoriety of computers [1] and the fast evolution of information communication technology. In light of this, new cyber threats are arising as their actors’ techniques are always advancing to retain or increase their prominence in the threat ecosystem through exploiting vulnerabilities [2]. Be it for individuals, organizations or critical national infrastructure, black hat hackers target zero-day issues in certain situations, but more likely, they target freshly fixed flaws and “unpatched systems” [3].

Due to the increasing growth of communication networks and their applications, virus transmission has become one of the topics of interest in computing research [4]. Besides worms and trojans [5], the virus is one way through which malicious attacks can arise in a computer network. In a networked system, whenever the computers are contaminated with viruses, the regular resident applications may lose the ability to function properly, corrupt saved files, or cause the loss of essential data on those machines. Subsequently, through the infected computers, the virus infection is transmitted to other computers through several means, which include able storage media (CD, USB, and flash drives) and e-mail attachments [6].

To curb viral spread incidences and safeguard computer networks against viruses, antimalicious programmes, firewalls, and patches are used to filter out all infections that remain in personal devices including personal laptops and
other detachable storage media [7]. Cybersecurity solutions largely aim to prevent harmful malware from entering and operating on computer systems [8]. Other ways include integrating audits, performing updates to security infrastructure [9], and modelling threats. Mathematical models have been developed to grasp the spread of malicious programs fully. These models are based on intriguing parallels between virtual viruses and their biological counterparts; wherein numerous phenomena are represented [2, 6, 7].

The basic inspiration for constructing the model is motivational research on mathematical modelling for Wireless Sensor Networks. Disease modelling is trending in the latest research works, particularly e-epidemic models are attractive and interesting, leading to our current proposed model with delay and stochastic dynamics. Some qualitative literature on delay models and stochastic models. In any Disease model, whether epidemic or e-epidemic, the delay is a key attribute to changing the system dynamics in terms of stability; environmental noise (stochastic) is also one of the key attributes which play a major role in the system dynamics. Motivational research works on delay and stochastic dynamics in various systems.

Therefore, in this paper, we propose the deterministic susceptible (S), latent (L), breaking out (B), and patched (P) model and a modification to include stochasticity in the form of noise, alongside delay and bifurcation. This paper is organized as follows: Section 2 contains the related literature; Section 3 presents the mathematical model; Section 4 contains the delay analysis, and Section 5 contains the directions of Hopf bifurcation and stability of the periodic solutions. Section 6 contains the SLBP model with noise; Section 7 contains the numerical simulations.

2. Related Literature

A sophisticated virus’s primary purpose is to damage more computer systems; to achieve that purpose, the malware would attempt to infiltrate as many computers without being detected. For purposes of infection modelling Yang and Yang [10], two typical stages of a virus are the latent (L) and breaking out (B) stages. While the former signifies the entry period, when the virus inhabits the host, the latter represents the time the infection starts causing harm to the host computer. Here, models representing virus propagation in computers and their networks using this SLB format are reviewed.

Yang et al. [11] employed the SLB model internal computers and derived the reproduction number, equilibria, and stability of points. Yang et al. [12] modelled the spread of computer viruses in the complex World-Wide-Web using the SLB model. They discovered that a greater heterogeneity and a scale-free graph with smaller power-law exponents promote virus growth. Representing recovered computers in separate compartments and reinforcement, Yang et al. [13] derived the global stability of the susceptible-latent-breaking-recovered-susceptible (SLBRS) model. Zhang et al. [14] employed the SLBRS model to represent virus dissemination but used the time delay of antivirus cleaning as a bifurcating parameter. Due to the ubiquity of the mass action infection rate, Yang and Yang [15] adopted a nonlinear type of incidence for virus growth using the SLB model.

Contrary to homogenous mixing, prevalent in most models, Yang et al. [16] used the SLB model to represent a scenario where distinct nodes have varied infectious, exploding, and remedial rates. The SLBR model was used by Zhang [17] to model both storage media and internal/external computers. Zhang and Bi [18] investigated the existence and properties of Hopf bifurcation using the SLB model. Zhang and Wang [19] considered isolation of contaminated hosts and thus developed the susceptible-latent-breaking out-quarantined-recovered (SLBQR) model, which was later used to investigate Hopf bifurcation. Zhang [20] applied the SLB model for multilayer computer subnetworks. Similarly, Zhao and Bi [21] used the SLBQR to model virus spread with two-time delays and the existence of Hopf bifurcation. Another delayed version of the SLBS model was developed by Zhang et al. [22] and after studying its stability analyses. Because most models only represent horizontal transmission (HT), Zhu et al. [23] utilized the SLB model to represent both HT and vertical transmission of viruses in a computer network.

More SLB models were observed in the following studies; [24, 25]. Other models which have been used to represent virus propagation in computers alongside detachable storage, external computers, and age structure using the susceptible-infected-countermeasure (SIC) model [7], strongly protected susceptible-weakly protected susceptible-infective-external (SWIE) model [6] and susceptible-infected-recovered (SIR) model [2]. A critical evaluation of the above-given models shows that besides considering the deterministic SLB model with their delay analysis, this work presents its SLB’s stochastic version, wherein the impact of white noise is studied. Our model also includes the patch (P) compartment that prevents the inundation [26] of the computer network as a result of virus infestation. The following sections contain the proposed models and further analysis.

3. Mathematical Model

In this part, a patch-enabled SLBP model is developed to capture the dynamics of virus and patch transmission. Computers are divided into two groups for virus propagation offline and online: internal computers linked to the Internet (World Wide Web) and external computers not connected to the world wide web. The host computers are called nodes for the sake of simplicity. All internal nodes are connected to the world wide web and divided into four categories, namely, susceptible nodes without virus (S), latently infected (L), breaking out infected (B), and nodes that have received patches (P) at the time t. In the latent state, the virus infection is inactive. Therefore, $S(t) + L(t) + B(t) + P(t) = 1$. We suggested the following conditions alongside the mathematical model based on the inspiration:
(A1) When an external node connects to the world wide web, it becomes vulnerable. At rate $\delta$, nodes are added to or withdrawn from the network.

(A2) Both infected nodes ($L$, $B$) can potentially infect susceptible nodes. Infection rates from $L$ and $B$ to $S$ nodes are $\beta_1$ and $\beta_2$, respectively. It is assumed that the incidence function is of the mass-action type.

(A3) With a rate of $a$, the $L$ node becomes $B$.

(A4) Remediation of $L$ nodes occurs at rate $\gamma_2$ and then becomes vulnerable.

(A5) The patched node will become invalid due to the development of new viruses. As a result, it loses immunity at a rate of $\gamma_1$.

(A6) Patches are acquired at a rate of $\beta P$ by nodes $S$, $L$, or $B$.

\[
S'(t) = \delta \left( (\beta L + \beta_2 B)S - \beta SP + \gamma_1 P + \gamma_2 L - \delta S \right),
\]

\[
L'(t) = \left( (\beta_1 L + \beta B)S - \beta LP - \gamma_2 L - \alpha L - \delta L \right),
\]

\[
B'(t) = \alpha L - \beta B(t - \tau)P - \delta B,
\]

\[
P'(t) = \beta (S + L)P - \beta B(t - \tau)P - \gamma_1 P - \delta P,
\]

With the initial conditions

\[
S(0) \geq 0, L(0) \geq 0, B(0) \geq 0, P(0) \geq 0.
\]

By direct computation, the system equation (1) has a unique positive equilibrium point $E_*(S^*, L^*, B^*, P^*)$ in which

\[
S^* = \frac{\beta + \alpha + \gamma_2 - \gamma_1}{\beta_1 + \beta_2 \alpha / \beta - \gamma_1},
\]

\[
L^* = \frac{\beta - \gamma_1}{\beta - \gamma_1 + \alpha} \left( \frac{\gamma_1 + \delta - S^*}{\beta} \right),
\]

\[
B^* = \frac{\alpha}{\beta - \gamma_1 + \alpha} \left( \frac{\gamma_1 + \delta - S^*}{\beta} \right),
\]

\[
P^* = \frac{\beta - \gamma_1 + \alpha}{\beta}.
\]

4. Delay Analysis

The linear system of equation (1) about endemic equilibrium point $E_*(S^*, L^*, B^*, P^*)$ is given by the following equations:

\[
S'(t) = a_{11} S(t) + a_{12} L(t) + a_{13} B(t) + a_{14} P(t),
\]

\[
L'(t) = a_{21} S(t) + a_{22} L(t) + a_{23} B(t) + a_{24} P(t),
\]

\[
B'(t) = a_{32} L(t) + a_{33} B(t) + a_{34} P(t) + b_{31} B(t - \tau),
\]

\[
P'(t) = a_{41} S(t) + a_{42} L(t) + a_{44} P(t) + b_{41} B(t - \tau),
\]

where

\[
a_{11} = -\beta_1 L^* - \beta_2 B^* - \beta P^* - \delta; a_{12} = \gamma_2 - \beta_1 S^*; a_{13} = -\beta_2 S^*; a_{14} = \gamma_1 - \beta S^*,
\]

\[
a_{21} = \beta_1 L^* + \beta_2 B^*; a_{22} = \beta_1 S^* - \beta P^* - \gamma_2 - \alpha - \delta; a_{23} = \beta_2 S^*; a_{24} = -\beta L^*,
\]

\[
a_{32} = \alpha; a_{33} = -\delta; a_{34} = b^*; b_{31} = -\beta P^*,
\]

\[
a_{41} = \beta P^*; a_{42} = \beta P^*; a_{44} = \beta S^* + \beta L^* - \gamma_1 - \delta; b_{41} = \beta P^*.
\]

Then, the associated characteristic equation is

\[
\lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4
\]

\[
+ \left( B_1 \lambda^3 + B_2 \lambda^2 + B_3 \lambda + B_4 \right) e^{-\lambda \tau} = 0,
\]

where

\[
A_1 = -(a_{11} + a_{22} + a_{33} + a_{44}),
\]

\[
A_2 = a_{11} a_{22} + a_{11} a_{33} + a_{11} a_{44} + a_{22} a_{33} + a_{22} a_{44} + a_{33} a_{44} + a_{23} a_{32} + a_{42} a_{24} - a_{21} a_{42};
\]

\[
A_3 = a_{23} a_{34} a_{42} + a_{33} a_{44} a_{24} + a_{11} a_{22} a_{33} + a_{22} a_{44} + a_{42} a_{24} a_{11} + a_{14} a_{33} a_{44} + a_{14} a_{34} a_{41} + a_{13} a_{21} a_{32} + a_{13} a_{34} a_{41} + a_{13} a_{34} a_{41} + a_{33} a_{21} a_{12} + a_{12} a_{21} a_{44}
\]

\[- a_{11} a_{22} a_{33} - a_{11} a_{33} a_{44} - a_{11} a_{22} a_{44} - a_{14} a_{24} a_{32};
\]

\[
A_4 = a_{11} a_{23} a_{34} a_{41} + a_{14} a_{24} a_{33} a_{41} + a_{14} a_{33} a_{41} + a_{11} a_{22} a_{34} a_{41} + a_{11} a_{22} a_{34} a_{41} + a_{23} a_{43} a_{14} - a_{23} a_{32} a_{14}
\]

\[- a_{13} a_{21} a_{32} a_{44} - a_{13} a_{21} a_{34} a_{41} - a_{13} a_{21} a_{34} a_{41} - a_{13} a_{21} a_{34} a_{41} - a_{13} a_{21} a_{34} a_{41};
\]

\[
B_1 = -b_{31};
\]

\[
B_2 = b_{11} \left( a_{22} + a_{44} \right) - a_{34} b_{12};
\]

\[
B_3 = a_{23} a_{34} b_{12} + a_{23} a_{34} b_{12} + a_{42} a_{24} b_{11} + a_{12} a_{21} b_{11} - a_{11} a_{22} b_{14} - a_{11} a_{44} b_{11} + a_{34} a_{13} b_{12} - a_{22} a_{44} b_{11},
\]

\[
B_4 = a_{11} a_{23} \left( b_{11} a_{44} - b_{11} a_{44} \right) + a_{21} a_{12} \left( b_{11} a_{44} - b_{13} a_{34} \right) - a_{11} a_{24} \left( b_{13} a_{32} + b_{11} a_{42} \right).
\]
Put \( \tau = 0 \) in equation (8), we get the following equation:
\[
\lambda^4 + (A_1 + B_1)\lambda^3 + (A_2 + B_2)\lambda^2 + (A_3 + B_3)\lambda + (A_4 + B_4) = 0.
\] (11)

By using Routh–Hurwitz criteria, sufficient conditions for all roots of equation (11) to be negative real parts are given in the following form:
\[
D_2 = \begin{vmatrix}
A_1 + B_1 & 1 \\
A_2 + B_2 & A_3 + B_3
\end{vmatrix},
\] (13)
\[
D_3 = \begin{vmatrix}
A_1 + B_1 & 1 & 0 \\
A_2 + B_2 & A_3 + B_3 & A_4 + B_4
\end{vmatrix},
\] (14)
\[
D_4 = \begin{vmatrix}
A_1 + B_1 & 1 & 0 & 0 \\
A_2 + B_2 & A_3 + B_3 & A_4 + B_4 & A_5 + B_5
\end{vmatrix}.
\] (15)

From equation (9),
\[
A_1 + B_1 = (\beta_1 L^* + \beta_2 B^* + \beta P^* + \delta) + (\beta_i S^* - \beta P^* - \gamma_2 - \alpha - \delta) + (\beta \omega - \gamma_1 - \delta) + \beta P^* > 0.
\] (12)

This, if conditions equations (11), (13)–(15) hold, \( E_u \) is locally asymptotically stable in the absence of delay.

For \( \tau > 0 \), Put \( \lambda = i\omega \) in equation (8) we have the following equation:
\[
(\omega^4 - iA_1\omega^3 - A_2\omega^2 + iA_3\omega + A_4) + (-iB_1\omega - B_2\omega^2 + iB_3\omega + B_4)(\cos\omega\tau - i\sin\omega\tau) = 0.
\] (16)

Equating real and imaginary parts we have the following equation:

Squaring and Adding equations (17) and (18) we get the following equation:
\[
\omega^8 + C_1\omega^6 + C_2\omega^4 + C_3\omega^2 + C_4 = 0,
\] (19)

where
\[
C_1 = A_2^2 - B_2^2;
C_2 = A_2^2 - 2A_4A_2 - B_3^2 + 2B_2B_4,
C_3 = A_2^2 - 2A_3A_1 - B_1^2 + 2B_1B_3 + 2A_4,
C_4 = A_2^2 - B_2^2.
\] (20)

Now, by assuming \( \omega^2 = u \) then the equation (19) becomes
\[
u^4 + C_1u^3 + C_2u^2 + C_3u + C_4 = 0.
\] (21)

The function is defined as follows:
\[
f(u) = u^4 + C_1u^3 + C_2u^2 + C_3u + C_4 = 0.
\] (22)

Clearly \( \lim_{u \to \infty} f(u) = \infty \). Thus, if \( C_4 < 0 \), then equation (22) has at least one positive root.

Solving from equations (17) and (18), we get the following equation:
\[
\cos\omega t = \frac{Q_1\omega^6 + Q_2\omega^4 + Q_3\omega^2 + Q_4}{Q_5\omega^8 + Q_6\omega^6 + Q_7\omega^4 + Q_8}
\] (23)

where
\[
Q_1 = B_2 - A_1B_1, Q_2 = A_3B_3 + A_1B_3 + A_2B_4 - B_4; Q_3 = A_1B_4 + A_3B_1 - A_3B_3;
Q_5 = -A_1B_4; Q_7 = B_2^2 - 2B_1B_3; Q_8 = B_2^2 - 2B_1B_3; Q_8 = B_2^2
\] (24)
So, corresponding to $\lambda = i\omega_0$, there exists

$$
\tau_{\text{on}} = \frac{1}{\omega_0} \cos^{-1} \left[ \frac{Q_1\omega^6 + Q_2\omega^4 + Q_4\omega^2 + Q_5}{Q_2\omega^6 + Q_5\omega^4 + Q_7\omega^2 + Q_8} \right] + \frac{2n\pi}{\omega_0} \quad \text{where } n = 0, 1, 2, \ldots
$$

Differentiate equation (8) with respect to $\tau$, we have the following equation:

$$
\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{4\lambda^3 + 3A_1\lambda^2 + 2A_2\lambda + A_3}{\lambda(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)} + \frac{3B_1\lambda^3 + 2B_2\lambda + B_3}{\lambda(B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4)}
$$

Thus, \( \text{Re}(d\lambda/d\tau)^{-1} = f(v_\omega)/B_1\omega_0^6 + (B_2^2 - 2B_3B_1)\omega_0^4 + (B_3^2 - 2B_4B_3)\omega_0^2 + B_4 \) where \( E_* = \omega_0^6 \) and \( f(v) = v^2 + C_1v^3 + C_2v^4 + C_3v + C_4. \)

Thus, if the condition \( f'(v_\omega) \neq 0 \) and \( \text{Re}(d\lambda/d\tau)^{-1} \neq 0 \), therefore the transversality conditions hold and hence Hopf bifurcation occurs at $\tau = \tau_0$

$$
\tau_{\text{on}} = \frac{1}{\omega_0} \cos^{-1} \left[ \frac{Q_1\omega^6 + Q_2\omega^4 + Q_4\omega^2 + Q_5}{Q_2\omega^6 + Q_5\omega^4 + Q_7\omega^2 + Q_8} \right] + \frac{2n\pi}{\omega_0} \quad \text{where } n = 0, 1, 2, \ldots
$$

### 5. Hopf Bifurcation and the Periodic Solution’s Stability

Using the theories of normal form and centre manifold [27] of the system, we explore stability and Hopf bifurcation’s direction.

**Theorem 2.** If $\mu_H > 0$ then the Hopf bifurcation is super-critical otherwise it is sub-critical. Here, sign determines the direction of the Hopf bifurcation.

**Proof.** Let \( W_1(t) = S(t) - S^*, W_2(t) = L(t) - L^*, W_3(t) = B(t) - B^* \) and normalize the delay with $t = t/\tau$. Let $\tau = \tau_0 + \xi, \xi \in R$, then $\xi = 0$ is the Hopf-bifurcation value of system equation (2) and system equation (2) can be transformed into a functional differential equation in $C = C([-1,0], R^4)$ as follows:

$$
\Theta_I = \frac{\text{Im}[Z(0)] + \mu_H \text{Im}[\lambda^1(t_0)]}{\tau_0\omega_0},
$$

$$
Z(0) = \frac{i}{2\tau_0\omega_0} \left\{ Z_{11}^2 - 2|Z_{11}|^2 - \frac{|Z_{21}|^2}{3} \right\} + \frac{Z_{21}^2}{2},
$$

$$
W_1(t) = S(t) - S^*, W_2(t) = L(t) - L^*, W_3(t) = B(t) - B^*, W_4 = P(t) - P^*.
$$
\[ W(t) = L_\xi(W_t) + G(\xi, W_t), \]  

where

\[ W(t) = (W_1(t), W_2(t), W_3(t), W_4(t))^T \in C = C([-1, 0], R^4) \]

and \( L_\xi: C \rightarrow R^4 \) and \( G: RXC \rightarrow R^4 \) are given, respectively, by the following equation:

\[
L_\xi Y = (r_0 + \xi)\{\Lambda_{\max} Y(0) + \Pi_{\max} Y(-1)\},
\]

\[
G(\xi, Y) = \begin{bmatrix}
-\beta_1 Y_1(0) + \beta_2 Y_3(0) & Y_1(0) \{\beta_2 Y_2(0) + \beta_4 Y_3(0)\} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

With \( \Lambda = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\
0 & m_5 & m_6 & m_7 \\
m_8 & 0 & m_{10} & 0 \\
m_{11} & m_{12} & 0 & m_{13} \end{bmatrix} \) and \( \Pi = \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \pi_1 & 0 & 0 \\
0 & 0 & \pi_2 & 0 \end{bmatrix} \).

In fact, we choose

\[ \eta(\theta, \xi) = (r_0 + \xi)(\Lambda_{\max} \delta(\theta) + \Pi_{\max} \delta(\theta + 1)). \]

where \( \delta(\theta) \) is the dirac delta function. For \( Y \in C([-1, 0], R^4) \), define

\[
J(\xi) Y(\theta) = \begin{cases} \frac{dY(\theta)}{d\theta}, & -1 \leq \theta < 0, \\
\int_{-1}^{0} d\eta(\theta, \xi) Y(\theta) = L_\xi Y, & \theta = 0, \\
G(\xi, Y), & \theta > 0. \end{cases}
\]

The system equation (1) is equivalent to

\[
u(t) = J(\xi)u_t + K(\xi)u_t. \]

For \( \phi \in C^1([0, 1], (R^4)^*) \), the adjoint operator \( J^* \) of \( J(0) \) is defined as follows:

\[
J^*(\xi)\phi(\mu) = \begin{cases} \frac{d\phi(\mu)}{d\mu}, & 0 < \mu \leq 1, \\
\int_{-1}^{0} d\eta(\sigma, 0)\phi(-\sigma), & \mu = 0. \end{cases}
\]

Next, we define the bilinear inner form for \( A \) and \( A^* \).

\[
\langle \phi(\mu), Y(\theta) \rangle = \bar{\phi}(0) Y(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\phi}(\xi - \theta)d\eta(\theta, Y(\xi))d\xi,
\]

where \( \eta(\theta) = \eta(\theta, 0) \).

Let \( \rho(\theta) = (1, q_2, q_3, q_4)^T e^{i\tau_0 W_t} \) be the eigen vector of \( J(0) \) corresponding to \( i\tau_0 W_0 \) and \( \rho^*(S) = V(1, q_2^*, q_3^*, q_4^*) e^{i\tau_0 W_t} \) be the eigen vector of \( J^*(0) \) corresponding to \( -i\tau_0 W_t \) respectively. Based on the definition of \( J(0) \) and \( J^*(0) \) one can obtain the following equation:

\[ q_2 = \frac{(m_5 + m_4 q_2)(m_{10} + i\pi_1 e^{i\tau_0 W_0} - i\omega_0) + m_2 m_9 + i(\omega_0 - m_6)(m_{10} + i\pi_1 e^{i\tau_0 W_0} - i\omega_0)}{m_7 m_8 + i(\omega_0 - m_6)(m_{10} + i\pi_1 e^{i\tau_0 W_0} - i\omega_0)}, q_3 = \frac{m_9 q_2}{m_2 + m_9 q_2} \]

\[ q_4 = \frac{1}{m_4} \left[ i\omega_0 m_4 - m_2 m_4 m_2 - m_3 q_3 \right]; q_4^* = \frac{m_4 m_2 - m_2 m_4}{m_4 m_11 - (m_{13} + i\omega_0)m_5}.
\]

From equation (5), we have

\[ q = \left[ 1 + q_2 q_2^* + q_3 q_3^* + q_4 q_4^* + r_0 e^{i\tau_0 W_0} q_2(\pi_2 q_2^* + \pi_2 q_3^*) \right]^{-1} \]

such that \( \langle q^*, q \rangle = 1 \) and \( \langle q^*, \bar{q} \rangle = 0 \).

According to the algorithms in [27] and a similar computation process to that in [2], we can obtain the following expressions:
The SLBP Model with Noise

In this section, we considered a SLBP stochastic epidemic model, which includes four compartments i.e., susceptible, latent, breaking out, and patched with the mass action incidence rate. Specifically, we are investigating the effect of Gaussian white noise on the model (without delay) proposed by Zhang and Upadhyay [27] for various low, medium, and high intensities. The schematic representation of the proposed model is as follows.

The following system of nonlinear differential equations with noise describes the dynamics of the proposed model:

\[ S'(t) = \mu - (\beta_1 L + \beta_2 B)S - \beta SP + \gamma_1 P + \gamma_2 L - \delta S + \alpha_1 \xi_1(t), \]
\[ L'(t) = (\beta_1 L + \beta_2 B)S - \beta LP - \gamma_2 L - aL - \delta L + \alpha_2 \xi_2(t), \]
\[ B'(t) = aL - \beta BP - \delta B + \alpha_3 \xi_3(t), \]
\[ P'(t) = \beta(S + L + B)P - \gamma_1 P - \delta P + a_4 \xi_4(t). \]

Let \( S(t) = u_1(t) + S^*; L(t) = u_2(t) + V^*; B(t) = u_3(t) + B^*; P(t) = u_4(t) + P^*. \)
And, by focusing solely on the effects of linear stochastic perturbations, As a result, the model (47)–(50) is reduced to the linear system shown as follows:

\[ u_1'(t) = (-\beta S^*)u_1 + (-\beta S^*)u_1 + (-\beta S^*)u_4 + \alpha_4 \xi_1(t), \quad (47) \]

\[ u_2'(t) = (\beta V^*)u_1 + (\beta V^*)u_1 + \alpha_2 \xi_2(t), \quad (48) \]

\[ u_3'(t) = (-\beta B^*)u_4 + \alpha_4 \xi_3(t), \quad (49) \]

\[ u_4'(t) = (\beta B^*)u_1 + (\beta B^*)u_2 + (\beta B^*)u_3 + \alpha_4 \xi_4(t). \quad (50) \]

Taking the Fourier transform of equations (47)–(50) we get the following equation:

\[ C_Y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega)e^{i\omega t} d\omega. \quad (61) \]

And, the variance of the corresponding fluctuations in \( Y(t) \) is given by the following equation:

\[ \sigma_Y^2 = C_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) d\omega. \quad (62) \]

The normalised auto covariance function is the auto correlation function.

\[ P_Y(t) = \frac{C_Y(t)}{C_Y(0)}. \quad (63) \]

For a Gaussian white noise process, it is

\[ S_{\xi_i \xi_j}(t) = \lim_{T \to \infty} E \left[ \frac{\bar{\xi}_i(\omega) \bar{\xi}_j(\omega)}{T} \right] \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \cdot E\left[ \xi_i(t) \right] \xi_j(t') e^{-i\omega(t-t')} dt dt' = \delta_{ij}; \quad (64) \]

The components of equation (57)’s solutions are as follows:

\[ \bar{u}_i(\omega) = [M(\omega)]^{-1} \bar{\xi}_i(\omega), \quad (57) \]

Let \([M(\omega)]^{-1} = K(\omega), \quad (58)\]

where \( K(\omega) = \frac{a dJ[M(\omega)]}{[M(\omega)]} \quad (59) \)

If the function’s \( Y(t) \) mean value is zero, the fluctuation intensity (variance) of its components in frequency intervals \([\omega, \omega + d\omega]\) is \( S_Y(\omega)d\omega \), where \( S_Y(\omega) \) is the spectral density \( Y \) and defined as follows:

\[ S_Y(\omega) = \lim_{T \to \infty} \frac{|Y(\omega)|^2}{T}. \quad (60) \]

The auto covariance function is the inverse transform of \( S_Y(\omega) \) if \( Y \) has a zero mean value.
As a result, the intensities of the variable’s fluctuations are given by the following equation:

$$\sigma_{u_i}^2 = \frac{1}{2\pi} \sum_{j=1}^{4} \left[\int_{-\infty}^{\infty} \alpha_j |K_{ij}(\omega)|^2 d\omega\right]; \quad i = 1, 2, 3, 4.$$  (67)

Here, $M(\omega) = R(\omega) + iI(\omega)$, where $R(\omega)$ is the real part of $M(\omega)$ and $I(\omega)$ is the imaginary part of $M(\omega)$, $\|M(\omega)\|^2 = [R(\omega)]^2 + [I(\omega)]^2$ where $R(\omega) = \omega^4 - \omega^2\beta^2 P^* B^* - \omega^2\beta^2 P^* V^* - \beta^3 \beta_2 V^* S^* B^* - \omega^2\beta^2 P^* S^*$.

We can derive the following from these numbers and equation (68):

$$\sigma_{u_1}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[\alpha_1 (A_{11})^2 + \alpha_2 (A_{12})^2 + \alpha_3 (A_{13})^2 + \alpha_4 (A_{14})^2\right] d\omega,$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[\alpha_1 (B_{11})^2 + \alpha_2 (B_{12})^2 + \alpha_3 (B_{13})^2 + \alpha_4 (B_{14})^2\right] d\omega,$$

$$\sigma_{u_3}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[\alpha_1 (C_{11})^2 + \alpha_2 (C_{12})^2 + \alpha_3 (C_{13})^2 + \alpha_4 (C_{14})^2\right] d\omega,$$

$$\sigma_{u_4}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[\alpha_1 (D_{11})^2 + \alpha_2 (D_{12})^2 + \alpha_3 (D_{13})^2 + \alpha_4 (D_{14})^2\right] d\omega.$$  (70)
If we are interested in the system dynamics of equations (47)–(50) with either $a_1 = 0$ (or) $a_2 = 0$ (or) $a_3 = 0$ (or) $a_4 = 0$, then the population variances are as follows.

If $a_1 = a_2 = a_3 = a_4 = 0$ then

$$
\sigma^2_{u_1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[ \sigma^4 (A_{11})^2 \right] d\omega,
$$

$$
\sigma^2_{u_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[ \sigma^4 (B_{11})^2 \right] d\omega,
$$

$$
\sigma^2_{u_3} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[ \sigma^4 (C_{11})^2 \right] d\omega,
$$

$$
\sigma^2_{u_4} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{[R(\omega)]^2 + [I(\omega)]^2} \left[ \sigma^4 D_{11}^2 \right] d\omega.
$$

Thus, for modest levels of mean square fluctuations, population variances imply nodes stability, whereas larger values of variances suggest nodes instability.

7. Numerical Simulations

In this section, we present a Numerical Simulation to validate our analytical findings in this paper with help of Matlab software [30].

For the parameters, $\lambda = 4, \beta_1 = 0.02, \beta_2 = 0.01, \beta = 0.02, \gamma_1 = 0.3, \gamma_2 = 0.1, \delta = 0.4, \alpha = 0.1$.

In the case of the absence of delay, the endemic equilibrium point $E_\ast (19.6227, 0.5138, 1.9337, 16.1464)$ is locally asymptotically stable, and corresponding time series is shown in Figure 1 [31].

In the presence of delay, for the value of $\tau = 30.50 < 40.50$, the endemic equilibrium point $E_\ast (19.6227, 0.5138, 1.9337, 16.1464)$ locally asymptotically stable and the dynamical behavior of the time series as shown in Figure 2 [32].

Furthermore, we increase the delay value the system equation (1) undergoes a Hopf-Bifurcation at the endemic equilibrium point $E_\ast (19.6227, 0.5138, 1.9337, 16.1464)$ and a family of bifurcating periodic solutions at $\tau = 35.65 = \tau^*$ . Bifurcation from $E_\ast (19.6227, 0.5138, 1.9337, 16.1464)$ which can be shown using the corresponding time series for this case in Figure 3.

Finally, if $\tau = 40.50 > \tau^*$, the system losses stability and becomes unstable, then the corresponding time series as shown in Figure 4 [34].

Also, the phase portrait of S-L-P with $\tau = 30.50 < \tau^*$ and $\tau = 40.50 > \tau^*$ as shown in Figures 5 and 6.

Also, the phase portrait of S-B-L with $\tau = 30.50 < \tau^*$ and $\tau = 40.50 > \tau^*$ as shown in Figures 7 and 8.

7.1. Numerical Observations. If there is no delay, the endemic equilibrium $E_\ast$ is really $E_\ast (19.6227, 0.5138, 1.9337, 16.1464)$ is locally asymptotically stable and the corresponding time series is shown in Figure 1. Figure 9 represents the time series evaluation of nodes for the values of example 1 in the presence of stochastic parameters [36]. At the values of noise intensities $\alpha_1 = 0.01; \alpha_2 = 0.02; \alpha_3 = 0.01; \alpha_4 = 0.02$ time series evaluation of four nodes, which are S(t), L(t), B(t), and P(t) are captured in Figure 9 [37]. At these low noise values, the proposed system (SLBP) is also less affected and clearly shown as less fluctuating [38]. Figure 10 represents the time series evaluation of nodes for the values of example 1 in the presence of stochastic parameters [39]. At the values of noise intensities $\alpha_1 = 0.04; \alpha_2 = 0.05; \alpha_3 = 0.04; \alpha_4 = 0.05$ time series evaluation of four nodes, which are S(t), L(t), B(t), and P(t) are captured in Figure 10 [40].

At these low noise values, the proposed system (SLBP) is also a little less affected and clearly shown as a little less fluctuating [41]. Figure 11 represents the time series evaluation of nodes for the values of example 1 in the presence of stochastic parameters. At the values of noise intensities, $\alpha_1 = 0.1; \alpha_2 = 0.2; \alpha_3 = 0.1; \alpha_4 = 0.2$ time series evaluation of four
nodes [42], which are $S(t)$, $L(t)$, $B(t)$, and $P(t)$ are captured in Figure 11. At these values of noise, the proposed system (SLBP) is affected remarkably [43] and clearly notable fluctuations in the projections as $P(t)$ increases and $S(t)$ is started decreasing. Both $P(t)$ and $B(t)$ are affected and fluctuates more rapidly when compared with $S(t)$ and $L(t)$ [44]. Figures 11(a)–11(d) are the Phase portrait plot for the nodes $S(t)$, $L(t)$, $B(t)$, and $P(t)$ with various combinations. Figure 11(a) represents the phase portrait plot for the nodes $S(t)$, $L(t)$, and $B(t)$ with the values of Example 1 along with
noise intensities 0.1, 0.2, 0.1, and 0.2 for \( S(t) \), \( L(t) \), \( B(t) \), and \( P(t) \). Figure 11(b) represents the phase portrait plot for the nodes \( L(t) \), \( B(t) \), and \( P(t) \) with the values of Example-1 along with noise intensities 0.1, 0.2, 0.1, and 0.2 for \( S(t) \), \( L(t) \), \( B(t) \), and \( P(t) \). Figure 11(c) represents the phase portrait plot for the nodes \( S(t) \), \( L(t) \), and \( P(t) \) with the values of Example-1 along with noise intensities 0.1, 0.2, 0.1, and 0.2 for \( S(t) \), \( L(t) \), \( B(t) \), and \( P(t) \). Figure 11(d) represents the phase portrait
The plot for the nodes $S(t)$, $B(t)$, and $P(t)$ with the values of Example 1 along with noise intensities 0.1, 0.2, 0.1, and 0.2 for $S(t)$, $L(t)$, $B(t)$, and $P(t)$. Figures 11(a), 11(b), and 11(d) show that noise intensity greatly affects the system. Figure 11(c) shows that noise intensity is not affecting much the system [46].

Figure 12 represents the time series evaluation of nodes for the values of example 1 in the presence of stochastic parameters. The values of noise intensities/time series evaluation of four nodes, which are $S(t)$, $L(t)$, $B(t)$, and $P(t)$, is captured in Figure 12. At these values of noise, the proposed system (SLBP) is affected remarkably and notable fluctuations in the projections as $P(t)$ is increased and $S(t)$ is started decreasing. Both $P(t)$ and $B(t)$ affected greatly and fluctuated more rapidly when compared with $S(t)$ and $L(t)$. Figure 13 represents the time series evaluation of nodes for the values of example 1 in the presence of stochastic parameters. At the values of noise intensities $\alpha_1 = 4; \alpha_2 = 5; \alpha_3 = 4; \alpha_4 = 5$ time series evaluation of four nodes, which are $S(t)$, $L(t)$, $B(t)$, and $P(t)$ are captured in Figure 13. At these values of noise, the proposed system (SLBP) is affected greatly, and oscillatory fluctuations in the projections as $P(t)$ increases and $S(t)$ decreases, and $S(t)$ moves very close to $B(t)$ and $L(t)$. Both $P(t)$, and $B(t)$ affected greatly and fluctuates more rapidly when compared with $S(t)$, $L(t)$.

Figures 13(a)–13(d) are the Phase portrait plot for the nodes $S(t)$, $L(t)$, $B(t)$, and $P(t)$ with various combinations. Figure 13(a) represents the phase portrait plot for the nodes $S(t)$, $L(t)$, and $B(t)$ with the values of Example 1 along with noise intensities 10, 20, 10, and 20 for $S(t)$, $L(t)$, $B(t)$, and $P(t)$. Figure 13(b) represents the phase portrait plot for the
Figure 10: Showing time series evaluation of nodes for the values of the attributes of example 1 with noise intensities $a_1 = 0.04; a_2 = 0.05; a_3 = 0.04; a_4 = 0.05$.

Figure 11: Continued.
Figure 11: (a) Representing time series evaluation of nodes for the values of the attributes of example 1 with noise intensities $a_1 = 0.1; a_2 = 0.2; a_3 = 0.1; a_4 = 0.2$. (b) Representing the phase portrait diagram of the nodes $S(t)$, $L(t)$, and $B(t)$. (c) Representing the phase portrait diagram of the nodes $L(t)$, $B(t)$, and $P(t)$. (d) Representing the phase portrait diagram of the nodes $S(t)$, $L(t)$ and $P(t)$. (e) Representing the phase portrait diagram of the nodes $S(t)$, $B(t)$, and $P(t)$.

Figure 12: Representing time series evaluation of nodes for the values of the attributes of example 1 with noise intensities.

Figure 13: Continued.
nodes $S(t), B(t),$ and $P(t)$ with the values of Example -1 along with noise intensities 10, 20, 10, and 20 for $S(t), L(t), B(t),$ and $P(t)$. Figure 13(c) represents the phase portrait plot for the nodes $L(t), B(t),$ and $P(t)$ with the values of Example -1 along with noise intensities 10, 20, 10, and 20 for $S(t), L(t), B(t),$ and $P(t)$.

Figure 13(d) represents the phase portrait plot for the nodes $S(t), L(t),$ and $P(t)$ with the values of Example -1 along with noise intensities 10, 20, 10, and 20 for $S(t), L(t), B(t),$ and $P(t)$. Figure 14 represents the time series evaluation of nodes for the values of example 1 in the presence of stochastic parameters. The values of noise intensities/time series evaluation of four nodes, which are $S(t), L(t), B(t),$ and $P(t)$, is captured in Figure 14. The proposed system (SLBP) is affected greatly by these noise values. Rapid fluctuations in the projections as $P(t)$ increases and $S(t)$ decreases, and all three projections of $S(t), B(t),$ and $L(t)$ emerge; all $P(t), S(t), B(t),$ and $L(t)$ is affected greatly and fluctuates more rapidly and oscillatory.

Figures 15 and 16 are time-series plots (deterministic graphs) of nodes for the values of Example-I in the absence of noise. Figures clearly show that the system attains stability and oscillation.

**Figure 13:** (a) Representing time series evaluation of nodes for the values of the attributes of example 1 with noise intensities. (b) Representing the phase portrait diagram of the nodes $S(t), L(t),$ and $B(t)$. (c) Representing the phase portrait diagram of the nodes $S(t), B(t),$ and $P(t)$. (d) Representing the phase portrait diagram of the nodes $L(t), B(t),$ and $P(t)$. (e) Representing the phase portrait diagram of the nodes $S(t), L(t),$ and $P(t)$.

**Figure 14:** Representing time series evaluation of nodes for the values of the attributes of example 1 with noise intensities.
and their impacts on networks. With this aim, we proposed an SLBS model consisting of $S(t)$, $L(t)$, $B(t)$, and $P(t)$. Under certain conditions, the local stability of all equilibrium points is investigated. The delay parameter was set, and we established the occurrence of a Hopf bifurcation as it crossed a crucial point by both analytical and numerical analysis. We also used the centre manifold theorem and normal form theory to investigate the properties of the Hopf bifurcation. We performed numerical simulation tests under various scenarios with appropriate sample values to support the theoretical findings.

Furthermore, this article investigates the prospects of infected node eradication and patched node persistence in a computer network. The proposed model exhibits rich dynamics for various studies like delay and Hopf bifurcation analysis. This model exhibits effective rich dynamics in the presence of delay particularly after reaching a certain value as $\tau = 30.50$. It is very clearly shown in the numerical findings that the system bifurcates and exhibits its dynamics at $\tau = 30.650 = \tau^*$. Phase portrait figures are drawn for various combinations under different delay parameter values, which are clearly presented the delay dynamics of the system. Delay dynamics exhibited by the system by both ways, analytically and numerically are captured and presented greatly.

We also focussed on the impact of additive white noise in the proposed system. We introduced noise intensities as stochastic parameters to the system and studied the stochastic model by linearizing the model using the perturbations technique and applying Fourier Transform. By finding out the noise intensities under certain constraints, the system should attain its steadiness as per the values of distinct noise parameters. Analytical results are checked with numerical simulations with appropriate example values. Notable numerical observations are discussed based on the various noise intensity values. The proposed model exhibits rich dynamics for various intensity values numerically and analytically, which is captured in stochastic analysis.

**Future Scope:** we can go for more deterministic graphs with parameter variations in terms of Sensitivity analysis is one of the approaches which allows to study and draw some interesting results. We can reconstruct this model in partial differential equations, including diffusive parameters, to catch the spatiotemporal dynamics more innovatively.

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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