Conformally equivariant quantization and symbol maps associated with $n$-ary differential operators on weighted densities

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Abstract We are interested in the study of the space of $n$-ary differential operators denoted by $D_{\lambda,\mu}$ where $\Delta = (\lambda_1, ..., \lambda_n)$ acting on weighted densities from $\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2} \otimes ... \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_{\mu}$ as a module over the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$. As a consequence, we prove the existence and the uniqueness of a canonical conformally equivariant symbol map from $D_{\lambda,\mu}$ to the corresponding space of symbols as well for the explicit expression of the associated quantization map.

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1 Introduction

The quantization is a concept that comes from physics. The quantization of a classical system whose phase space is a symplectic manifold, consists in the construction of a Hilbert space $H$ and a correspondence between classical and quantum observables. Let $M$ be a smooth manifold, $T^*M$ the cotangent bundle on $M$ and $\mathcal{S}(M)$ the space of smooth functions on $T^*M$ polynomial on the fibers, which the is usually called the space of symbols of differential operators. The standard quantization procedure consists of constructing a map $Q$ between the space $\text{Pol}(T^*M)$ of polynomials on $T^*M$ and the space $\mathcal{D}(M)$ of linear differential operators on $M$ called a quantization map. The inverse $\sigma = Q^{-1}$ is thus called a symbol map. Generally, there is no quantization and symbol maps equivariant with respect to the action the Lie algebra $\text{Vect}(M)$ of vector fields on $M$ (or the group $\text{Diff}(M)$ of diffeomorphisms of $M$) on the two spaces $\mathcal{D}(M)$ and $\text{Pol}(T^*M)$. Thus, we restrict ourselves to equivariant symbols and quantization maps with respect to the action of a given subalgebra of $\text{Vect}(M)$.

More precisely, Let for every $\lambda \in \mathbb{C}$, $\mathcal{F}_{\lambda}(M)$ and $D_{\lambda,\mu}(M)$ stand for the space of tensor densities of degree $\lambda$ on $M$ and the space of linear differential operators from $\mathcal{F}_{\lambda}(M)$ to $\mathcal{F}_{\mu}(M)$ ($\lambda, \mu \in \mathbb{C}$) respectively. These spaces are naturally modules over the Lie algebra $\text{Vect}(M)$. The

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space of symbols corresponding to $\mathcal{D}_{\lambda,\mu}(M)$ is there for $\mathcal{S}_\delta(M) = \mathcal{S}(M) \otimes \mathcal{F}_\delta(M)$ where $\delta = \mu - \lambda$, there is a filtration

$$\mathcal{D}^0_{\lambda,\mu} \subset \mathcal{D}^1_{\lambda,\mu} \subset \cdots \mathcal{D}^k_{\lambda,\mu} \subset \cdots$$

and the associated module $\mathcal{S}_\delta(M) = gr(\mathcal{D}_{\lambda,\mu})$ is graded by the degree of polynomials:

$$\mathcal{S}_\delta^0 \subset \mathcal{S}_\delta^1 \subset \cdots \mathcal{S}_\delta^k \subset \cdots$$

The problem of equivariant quantization is the quest for a quantization map:

$$Q_{\lambda,\mu}: \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$$

that commutes with the action of a given Lie subalgebra of $\text{Vect}(M)$. In other words, it amounts to an identification of these two spaces which is canonical with respect to the geometric on $M$. The inverse of the quantization map.

$$\sigma_{\lambda,\mu} := (Q_{\lambda,\mu})^{-1}$$

is called symbol map.

The concept of equivariant quantization over $\mathbb{R}^n$ was introduced by P. Lecomte and V. Ovsienko in [16]. In this seminal work, they considered spaces of differential operators acting between densities and the Lie algebra of projective vector fields over $\mathbb{R}^n$, $\text{sl}(n + 1)$. In this situation, they showed the existence and uniqueness of an equivariant quantization. This result was generalized in many references (see for instance [7], [14]). In [15], P. Lecomte generalized the problem of equivariant quantization by defining the problem of natural invariant quantization on arbitrary manifolds. Finally in [3], [4], [5], [6], [10], [12], [13], [19], [20], [21], [22], the authors proved the existence of such quantizations by using different methods in more and more general contexts.

Recently, several papers dealt with the problem of equivariant quantizations in the context of supergeometry: the papers [17] and [23] exposed and solved respectively the problems of the $\mathfrak{pgl}(p + 1|q)$-equivariant quantization over the superspace $\mathbb{R}^{p|q}$ and of the $\mathfrak{osp}(p + 1; q + 1|2r)$-equivariant quantization over $\mathbb{R}^{p+q+2r}$, whereas in [18], the authors define the problem of the natural and projectively invariant quantization on arbitrary supermanifolds and show the existence of such a map. In [11], [24], [25] the problem of equivariant quantizations over the supercircles $S^{1|1}$ and $S^{1|2}$ endowed with canonical contact structures was considered, these quantizations are equivariant with respect to Lie superalgebras $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ of contact projective vector fields respectively.

In [2], for the $S^{1|1}$-case, we were interested in the study of the space $\mathcal{D}_{\lambda_1,\lambda_2,\mu}$ of bilinear differential operators from $\mathfrak{f}_{\lambda_1} \otimes \mathfrak{f}_{\lambda_2} \to \mathfrak{f}_{\mu}$. For almost all values $(\lambda_1, \lambda_2, \mu)$, we prove the existence and the uniqueness (up to normalization) of a projectively, i.e., $\mathfrak{osp}(1|2)$-equivariant symbol map between $\mathcal{D}_{\lambda_1,\lambda_2,\mu}$ and the corresponding space of symbols $\mathcal{S}_{\lambda_1,\lambda_2,\mu}$ and calculate the explicit expressions of the symbol and the associated quantization maps.

Our motivation in this work is the generalization of the results proved [2]. Namely we consider the superspace $\mathcal{D}_{\lambda_1,\lambda_2,\mu}$ of $n$-ary differential operators $A : \mathfrak{f}_{\lambda_1} \otimes \mathfrak{f}_{\lambda_2} \otimes \cdots \otimes \mathfrak{f}_{\lambda_n} \to \mathfrak{f}_{\mu}$, where $\mathfrak{f}_{\lambda}, \lambda \in \mathbb{C}$, is the space of tensor densities on the supercircle $S^{1|1}$ of degree $\lambda$. The analogue, in the super setting, of the projective algebra $\text{sl}(2)$ is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$, which is the smallest simple Lie superalgebra, can
be realized as a subalgebra of $\text{Vect}_C(S^{1|1})$. Naturally, the Lie superalgebra $\text{Vect}_C(S^{1|1})$, and therefore $\mathfrak{osp}(1|2)$, act on $D_{\Delta\mu}$, the $\mathfrak{osp}(1|2)$-module $D_{\Delta\mu}$ is filtered as:

$$D_{\Delta\mu}\subset D_{\Delta\mu}^{\frac{1}{2}}\subset D_{\Delta\mu}^{\frac{3}{2}}\subset \cdots \subset D_{\Delta\mu}^{k-\frac{1}{2}}\subset D_{\Delta\mu}^{k}\subset \cdots.$$ 

The graded module $\text{gr}(D_{\Delta\mu})$, also called the space of symbols and denoted by $S_{\Delta\mu}$, depends only on the shift, $\delta = \mu - |\lambda|$, $|\lambda| = \lambda_1 + \cdots + \lambda_n$, of the weights. Moreover, as a $\text{Vect}_C(S^{1|1})$-module, $S_{\Delta\mu}$ is decomposed as $\bigoplus_{k \in \mathbb{N}} S^k_{\Delta\mu}$ where

$$S^k_{\Delta\mu} = \bigoplus_{\ell=0}^{2k} D_{\Delta\mu}^\ell / D_{\Delta\mu}^{\ell-\frac{1}{2}} = \bigoplus_{\ell=0}^{2k} \mathfrak{F}_{\delta-\frac{\ell}{2}},$$

$\mathfrak{F}_{\delta-\frac{\ell}{2}}$ stands for the sum $\bigoplus \mathfrak{F}_{\delta-\frac{\ell}{2}}$ where $\mathfrak{F}_{\delta-\frac{\ell}{2}}$ is counted $\left(\ell + n - 1\right)$ times.

Moreover, we prove that, if $\delta = \mu - |\lambda| = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots, k$, then $D_{\Delta\mu}^{k}$ is isomorphic to $S^k_{\Delta\mu}$ as an $\mathfrak{osp}(1|2)$-module. This isomorphism, called a conformally equivariant symbol map, is unique (once we fix a principal symbol). Explicit expressions of the normalized symbol and its inverse, the conformally equivariant quantization map, are given.

## 2 The main definitions

In this section, we recall the main definition and facts related to the geometry of the supercircle $S^{1|1}$. (See for instance [1], [9], [11])

### 2.1 Geometry of the supercircle $S^{1|1}$

The supercircle $S^{1|1}$ is the simplest supermanifold of dimension 1|1 generalizing $S^1$. In order to fix notation, let us give here the basic definitions of geometric objects on $S^{1|1}$. We define the supercircle $S^{1|1}$ by describing its graded commutative algebra of functions which we denote by $C^\infty(S^{1|1})$ and which is constituted by the elements

\[ F = f_0(x) + \theta f_1(x), \tag{2.1} \]

where $x$ is an arbitrary parameter on $S^1$ (the even variable), $\theta$ is the odd variable ($\theta^2 = 0$) and $f_0$, $f_1$ are $C^\infty$ complex valued functions. We denote by $F'$ the derivative of $F$ with respect to $x$, i.e, $F'(x, \theta) = f'_0(x) + \theta f'_1(x)$.

### 2.2 Vector fields and differential forms

Let $\text{Vect}(S^{1|1})$ be the superspace of vector fields on $S^{1|1}$:

\[ \text{Vect}_C(S^{1|1}) = \left\{ F_0 \partial_x + F_1 \partial_\theta \mid F_i \in C^\infty(S^{1|1}) \right\}, \tag{2.2} \]

where $\partial_\theta$ (resp. $\partial_x$) means the partial derivative $\frac{\partial}{\partial \theta}$ (resp. $\frac{\partial}{\partial x}$).

Let $\Omega^1(S^{1|1})$ be the rank 1|1 right $C^\infty(S^{1|1})$-module with basis $dx$ and $d\theta$, we interpret it as
the right dual over $C^\infty(S^{1|1})$ to the left $C^\infty(S^{1|1})$-module $\Vect_C(S^{1|1})$, by setting
\[ \langle \partial_y_i, dy_j \rangle = \delta_{ij} \text{ for } y = (x, \theta). \] The space $\Omega^1(S^{1|1})$ is a left module over $\Vect_C(S^{1|1})$, the action being given by the Lie derivative:
\[ \langle X, L_Y(\alpha) \rangle := \langle [X, Y], \alpha \rangle \]

### 2.3 Lie superalgebra of contact vector fields

The standard contact structure on $S^{1|1}$ is defined as a codimension 1 non-integrable distribution $\langle \mathcal{D} \rangle$ on $S^{1|1}$, i.e., a subbundle in $TS^{1|1}$ generated by the odd vector field
\[ \mathcal{D} = \partial_\theta - \theta \partial_x, \quad (2.3) \]
This contact structure can be equivalently defined as the kernel of the differential 1-form
\[ \alpha = dx + \theta d\theta. \quad (2.4) \]
These vector fields satisfy the condition
\[ \mathcal{D}^2 = (-1)^j \mathcal{D}^2 = (-1)^j \partial^j_x, \forall j \in \mathbb{N}. \quad (2.5) \]
where $D = \partial_\theta + \theta \partial_x$.

One can easily check the super Leibniz formula:
\[ \mathcal{D}^j \circ F = \sum_{i=0}^{j} \binom{j}{i}_s (-1)^{i+j} \mathcal{D}^{(j-i)}(F) \mathcal{D}^{j-i}, \quad (2.6) \]
where the notions $\binom{j}{i}_s$ and $| \cdot |$ stand respectively for the super combination defined by
\[ \binom{j}{i}_s = \begin{cases} \binom{j}{i} & \text{if } i \text{ is even or } j \text{ is odd} \\ 0 & \text{otherwise}. \end{cases} \quad (2.7) \]
and for the parity function ($[x]$ denotes the integer part of a real number $x$).

A vector field $X$ is said to be contact if it preserves the contact distribution, i.e.,
\[ [X, \mathcal{D}] = F_X \mathcal{D}, \quad (2.8) \]
where $F_X \in C^\infty_S(S^{1|1})$ is a function depending on $X$.

We denote by $\mathcal{K}(1)$ the Lie superalgebra of contact vector fields on $S^{1|1}$. It is well-known that every contact vector field can be expressed, for some function $f \in C^\infty_S(S^{1|1})$, by (see [11]):
\[ X_f = -f \mathcal{D}^2 + \frac{1}{2} D(f) \mathcal{D}. \quad (2.9) \]
The vector field $X_f$ is said to be the contact vector field with contact Hamiltonian $f$. One checks that
\[ \mathcal{L}_{X_f} \alpha = f' \alpha , \quad [X_f, \mathcal{D}] = -\frac{1}{2} f' \mathcal{D} \]
The contact bracket is defined by \([X_f, X_g] = X_{\{f,g\}}\). The space \(C_\infty^\infty(S^{1|1})\) is thus equipped with a Lie superalgebra structure isomorphic to \(K(1)\). The explicit formula can be easily calculated:

\[
\{f, g\} = fg' - f'g + \frac{1}{2}(-1)^{|f||g|+1}D(f)D(g).
\]  

(2.10)

The action of \(K(1)\) on \(C_\infty^\infty(S^{1|1})\) is defined by:

\[
\mathcal{L}_X(g) = fg' + \frac{1}{2}D(f)\mathcal{D}(g) = fg' + \frac{1}{2}(-1)^{|f|+1}\mathcal{D}(f)\mathcal{D}(g).
\]  

(2.11)

2.4 The orthosymplectic Lie superalgebra \(osp(1|2)\)

If we identify \(S^1\) with \(\mathbb{RP}^1\) with homogeneous coordinates \((x_1 : x_2)\) and choose the affine coordinate \(x = x_1/x_2\), the vector fields

\[
\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}
\]

are globally defined and correspond to \(\mathcal{L}\). The standard projective structure on \(\mathbb{RP}^1\). In this adapted coordinate the action of the subalgebra \(sl(2)\) of the Lie algebra \(\text{Vect}(S^1)\):

\[
sl(2) = \text{Span}\left(\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\right)
\]

is well defined.

Similarly, we consider the orthosymplectic Lie superalgebra \(osp(1|2)\) as a subalgebra of \(K(1)\):

\[
osp(1|2) = \text{Span}(X_1 = \partial_x, X_x = x\partial_x + \frac{1}{2}\theta\partial_\theta, X_{x^2} = x^2\partial_x + x\theta\partial_\theta, X_\theta = \frac{1}{2}D, X_{x\theta} = \frac{1}{2}xD).
\]  

(2.12)

The space of even elements :

\[
(osp(1|2))_0 = \text{Span}(X_1, X_x, X_{x^2})
\]  

(2.13)

is isomorphic to \(sl(2)\), the space of odd elements is two dimensional:

\[
(osp(1|2))_1 = \text{Span}(X_\theta = D, X_{x\theta} = xD).
\]  

(2.14)

The new commutation relations are

\[
[X_1, X_{x^2}] = 2X_x, \quad [X_\theta, X_\theta] = \frac{1}{2}X_1, \quad [X_x, X_1] = -X_1, \\
[X_x, X_{x^2}] = X_{x^2}, \quad [X_{x^2}, X_{x\theta}] = \frac{1}{2}X_{x^2}, \quad [X_x, X_{x\theta}] = -X_{x\theta}, \\
[X_{x^2}, X_\theta] = -\frac{1}{2}X_\theta, \quad [X_1, X_{x\theta}] = X_\theta, \quad [X_1, X_\theta] = 0, \\
[X_x, X_x\theta] = \frac{1}{2}X_{x\theta}, \quad [X_{x^2}, X_x\theta] = 0, \quad [X_{x\theta}, X_\theta] = \frac{1}{2}X_x.
\]

As in the \(S^1\) case, there exist adapted coordinates \((x, \theta)\) for which the \(osp(1|2)\)-action is well defined (see [11] for more details).
2.5 The space of weighted densities on $S^{1|1}$

In the super setting, by replacing $dx$ by the 1-form $\alpha$, we get analogous definition for weighted densities, i.e., we define the space of $\lambda$-densities as

$$\mathfrak{F}_\lambda = \left\{ F\alpha^\lambda \mid F \in C^\infty_\mathbb{C}(S^{1|1}) \right\}.$$ (2.15)

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty_\mathbb{C}(S^{1|1})$.

For contact vector field $X_F$, define a one-parameter family of first order differential operator on $C^\infty_\mathbb{C}(S^{1|1})$

$$L^\lambda_{X_F} = X_F + \lambda F', \lambda \in \mathbb{C}. \quad (2.16)$$

One easily checks that the map $X_F \mapsto L^\lambda_{X_F}$ is a homomorphism of Lie superalgebra, i.e., $[L^\lambda_{X_F}, L^\lambda_{X_G}] = L^\lambda_{[X_F, X_G]}$ for every $\lambda$. Thus $\mathfrak{F}_\lambda$ becomes a $\mathcal{K}(1)$-module on $C^\infty_\mathbb{C}(S^{1|1})$. Evidently, the Lie derivative of the density $G\alpha^\lambda$ along the vector field $X_F$ in $\mathcal{K}(1)$ is given by:

$$L^\lambda_{X_F}(G\alpha^\lambda) = (X_F(G) + \lambda F'G)\alpha^\lambda. \quad (2.17)$$

Explicitly, if we put $F = f_0(x) + f_1(x)\theta$, $G = g_0(x) + g_1(x)\theta$,

$$L^\lambda_{X_F}(G) = L^\lambda_{f_0\partial_x}(g_0) + \frac{1}{2} f_1 g_1 + \left( L^\lambda_{f_0\partial_x}(g_1) + \lambda g_0 f'_1 + \frac{1}{2} g'_0 f_1 \right) \theta. \quad (2.18)$$

2.6 Multilinear differential operators on weighted densities

We fix a natural number $n$. In order to avoid clutter, we have found that it is convenient to use the notations of [4]:

- Denote by $i$ either the $n$-tuple $(i_1, \cdots, i_n)$ or the indices $i_1, \cdots, i_n$, as, for instance, $a_i = a_{i_1, \cdots, i_n}$. The difference should be discernable from the context.
- Denote by $|i|$ the sum $\sum_{j=1}^n i_j$.
- Denote $1_i := (0, \cdots, 0, 1, 0, \cdots, 0)$, where 1 is in the $i$-th position.
- Denote by $\mathfrak{S}_\lambda^{(i)} = \oplus \mathfrak{F}_\lambda$ where $\mathfrak{F}_\lambda$ is counted $\binom{i+n-1}{n-1}$ times.
- $\bigotimes_{i=1}^n D^i := D^{i_1} \otimes D^{i_2} \otimes \cdots \otimes D^{i_n}$.
- Throughout the text, we use the classical convention $\sum_{i=1}^n c_i = 0$.

Obviously, $\forall \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$, $\mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_2} \otimes \cdots \otimes \mathfrak{F}_{\lambda_n}$ also a $\mathcal{K}(1)$-module with the action

$$L^\lambda_{X_F}(\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n) = \sum_{p=1}^n (-1)^{F_p}([\sum_{i=1}^{p-1} |\Phi_i|] \Phi_1 \otimes \Phi_2 \otimes \cdots \otimes L^\lambda_{X_F}(\Phi_p) \otimes \cdots \otimes \Phi_n). \quad (2.19)$$
Since $\mathcal{D}^2 = -D^2 = -\partial_x$, every differential operator $A \in \mathcal{D}_{\Delta\mu}$ can be expressed in the form (see (11))

$$A = \sum_{\ell=0}^{2k} \sum_{|j|=\ell} a_{ij}(x,\theta) \mathcal{D}^j \otimes \mathcal{D}^2 \otimes \cdots \otimes \mathcal{D}^n$$  \hspace{1cm} (2.20)

where the coefficients $a_{ij}$ are smooth functions on $S^{11}$ and $\ell \in \mathbb{N}$. That is, for all $F_1 = f_1 \alpha^{\lambda_1} \in \mathcal{F}_{\lambda_1}, F_2 = f_2 \alpha^{\lambda_2} \in \mathcal{F}_{\lambda_2}, \ldots, F_n = f_n \alpha^{\lambda_n} \in \mathcal{F}_{\lambda_n},$

$$A(F_1 \otimes F_2 \otimes \cdots \otimes F_n) = \left( \sum_{\ell=0}^{2k} \sum_{|j|=\ell} a_{ij}(x,\theta) (-1)^{(|F_1| + |F_2| + \cdots + |F_n|)} \mathcal{D}^j (f_1) \mathcal{D}^2 (f_2) \cdots \mathcal{D}^n (f_n) \right) \alpha^\mu.$$  \hspace{1cm} (2.21)

Moreover, if $A \in \mathcal{D}_{\Delta\mu}^k$ then $\ell = 2k$. For short, we will write the operator $A$ as:

$$A = \sum_{\ell=0}^{2k} \sum_{|j|=\ell} a_{ij} \mathcal{D}^j.$$  \hspace{1cm} (2.22)

Where $\mathcal{D}^j = \mathcal{D}^j \otimes \mathcal{D}^2 \otimes \cdots \otimes \mathcal{D}^n$. Thus, we consider a family of $\mathcal{K}(1)$-actions on the superspace of multilinear differential operators $\mathcal{D}_{\Delta\mu} := \text{Hom}_\text{diff}(\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}, \mathcal{F}_\mu)$:

$$\mathcal{L}_{X_F}^\lambda \mu(A) = \mathcal{L}_F^\mu \circ A - (-1)^{|A||F|} A \circ \mathcal{L}_{X_F}^\lambda \mu.$$  \hspace{1cm} (2.23)

### 2.7 Explicit formulas for the action of $\mathcal{K}(1)$ on $\mathcal{D}_{\Delta\mu}^k$

Let us calculate explicitly the action $\mathcal{K}(1)$ on the superspace $\mathcal{D}_{\Delta\mu}^k$. Given a differential operator $A = \sum_{\ell=0}^{2k} \sum_{|j|=\ell} a_{ij} \mathcal{D}^j \otimes \mathcal{D}^2 \otimes \cdots \otimes \mathcal{D}^n \in \mathcal{D}_{\Delta\mu}^k$ and $X_F, F \in C^\infty(S^{11})$, an arbitrary contact vector field.

**Proposition 2.1.** The natural action of $\mathcal{K}(1)$ on $\mathcal{D}_{\Delta\mu}^k$ is given by:

$$\mathcal{L}_{X_F}^\lambda \mu(A) = \sum_{\ell=0}^{2k} \sum_{|j|=\ell} a_{ij} X_F^j \mathcal{D}^j \otimes \mathcal{D}^2 \otimes \cdots \otimes \mathcal{D}^n$$  \hspace{1cm} (2.24)

where:

$$a_{ij} = \mathcal{L}_{X_F}^\lambda \mu(a_{ij}) - \sum_{r=1}^{2k-|j|} (-1)^r (|F| + |a_{ij+r1}|) \left[ \binom{r+i_1}{r+2} s - \frac{1}{2} (-1)^{i_1} \binom{r+i_1}{r+1} s \right]$$

$$+ \lambda_1 \left( \binom{r+i_1}{r} s \right) a_{ij+r1} - \sum_{r=1}^n \sum_{t=2}^{2k-|j|} (-1)^r (|F| + |a_{ij+r1}| + |i_1+i_2+\cdots+i_{r-1}|) \left[ \binom{r+i_1}{r+2} s \right]$$

$$- \frac{1}{2} (-1)^{i_1} \left( \binom{r+i_1}{r+1} s \right) + \lambda_1 \left( \binom{r+i_1}{r} s \right) a_{ij+r1}.$$  \hspace{1cm} (2.25)
Proof. Let $\phi_1 = \varphi_1 \alpha_1 \in \mathfrak{F}_{\lambda_1}$, $\phi_2 = \varphi_2 \alpha_2 \in \mathfrak{F}_{\lambda_2}$, \ldots, $\phi_n = \varphi_n \alpha_n \in \mathfrak{F}_{\lambda_n}$. Upon using (2.16), (2.19) and (2.23), we get

$$
\mathfrak{L}_{X^\mu}^\lambda(A)\left(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n\right) = \mathfrak{L}_{X^\mu}^\lambda\left(A\left(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n\right)\right) - \left(-1\right)^{|A|}F \left(\mathfrak{L}_{X^\mu}^\lambda\left(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n\right)\right) - \cdots - \\
\left(-1\right)^{|A|\left(|F| + |\varphi_1|\right)} A\left(\phi_1 \otimes \mathfrak{L}_{X^\mu}^\lambda\left(\phi_2 \otimes \cdots \otimes \phi_n\right)\right) - \cdots - \\
\left(-1\right)^{|A|\left(|F| + |\varphi_1| + \cdots + |\varphi_{n-1}|\right)} A\left(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \mathfrak{L}_{X^\mu}^\lambda\left(\phi_n\right)\right)
$$

$$
= \sum_{\ell=0}^{2k} \sum_{i_1 + i_2 + \cdots + i_n = \ell} F\left(-1\right)^{\sum_{j=1}^{n-1} \left|\varphi_j\right|\left(i_j + 1 + \cdots + i_n\right) a_1 D^{i_1} \left(\varphi_1\right) D^{i_2} \left(\varphi_2\right) \cdots D^n \left(\varphi_n\right)\right) + \\
\frac{1}{2} D(F) \overline{D}\left(-1\right)^{\sum_{j=1}^{n-1} \left|\varphi_j\right|\left(i_j + 1 + \cdots + i_n\right) a_1 D^{i_1} \left(\varphi_1\right) D^{i_2} \left(\varphi_2\right) \cdots D^n \left(\varphi_n\right)\right) + \\
\mu F'\left(-1\right)^{\sum_{j=1}^{n-1} \left|\varphi_j\right|\left(i_j + 1 + \cdots + i_n\right) a_1 D^{i_1} \left(\varphi_1\right) D^{i_2} \left(\varphi_2\right) \cdots D^n \left(\varphi_n\right)\right) - \\
\left(-1\right)^{|A|}F \left(-1\right)^{\sum_{j=1}^{n-1} \left|\varphi_j\right|\left(i_j + 1 + \cdots + i_n\right) \left(-1\right)^{|F|\left(i_2 + \cdots + i_n\right)} a_1 \right) \\
D^{i_1} \left(F' \varphi_1' + \frac{1}{2} D(F) \overline{D}(\varphi_1) + \lambda_1 F' \varphi_1\right) D^{i_2} \left(\varphi_2\right) \cdots D^n \left(\varphi_n\right) - \\
\left(-1\right)^{|A|\left(|F| + |\varphi_1|\right)} \left(-1\right)^{\sum_{j=1}^{n-1} \left|\varphi_j\right|\left(i_j + 1 + \cdots + i_n\right) \left(-1\right)^{|F|\left(i_2 + \cdots + i_n\right)} a_1 \right) D^{i_1} \left(\varphi_1\right) D^{i_2} \left(F' \varphi_2' + \frac{1}{2} D(F) \overline{D}(\varphi_2) + \lambda_2 F' \varphi_2\right) D^{i_3} \left(\varphi_3\right) \cdots D^n \left(\varphi_n\right) - \\
\left(-1\right)^{|A|\left(|F| + |\varphi_1| + \cdots + |\varphi_{n-1}|\right)} \left(-1\right)^{\sum_{j=1}^{n-1} \left|\varphi_j\right|\left(i_j + 1 + \cdots + i_n\right) \left(-1\right)^{|F|\left(i_2 + \cdots + i_n\right)} a_1 \right) D^{i_1} \left(\varphi_1\right) D^{i_2} \left(\varphi_2\right) D^{i_3} \left(\varphi_3\right) \cdots D^n \left(\varphi_n\right) \left(F' \varphi_n' + \frac{1}{2} D(F) \overline{D}(\varphi_n) + \lambda_n F' \varphi_n\right) \right) \alpha^\mu.
$$

Using the super Leibniz formula (2.25) and by writing (2.1) in the form

$$
\mathfrak{L}_{X^\mu}^\lambda(A)\left(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n\right) = \\
\sum_{\ell=0}^{2k} \sum_{i_1 + i_2 + \cdots + i_n = \ell} \left(-1\right)^{\sum_{j=1}^{n-1} \left|\varphi_j\right|\left(i_j + 1 + \cdots + i_n\right) a_1 X^\mu D^{i_1} \left(\varphi_1\right) D^{i_2} \left(\varphi_2\right) \cdots D^n \left(\varphi_n\right)\right) \alpha^\mu,
$$

By identification, we get easily the formulas (2.25). \hfill \square

2.8 Space of symbols of multilinear differential operators.

Consider the graded $\mathcal{K}(1)$-module $\text{gr}(D_{\Delta^\mu}^k)$ associated with the filtration

$$
D_{\Delta^\mu}^0 \subset D_{\Delta^\mu}^1 \subset D_{\Delta^\mu}^2 \subset \cdots \subset D_{\Delta^\mu}^{k-\frac{1}{2}} \subset D_{\Delta^\mu}^k \subset \cdots
$$

(2.26)
i.e, the direct sum
\[
\text{gr}(\mathcal{D}_{\Delta \mu}) = \bigoplus_{k=0}^{\infty} \mathcal{D}^k_{\Delta \mu}/\mathcal{D}^{k-\frac{1}{2}}_{\Delta \mu}
\]
We call this \( \mathcal{K}(1) \)-module the \textit{space of symbols of multilinear differential operators} and denote it \( S_{\Delta \mu} \).

The quotient module \( \mathcal{D}^k_{\Delta \mu}/\mathcal{D}^{k-\frac{1}{2}}_{\Delta \mu}, k \in \frac{1}{2} \mathbb{N}, \) can be decomposed into \( \mathcal{F}_{\delta - \frac{k}{2}} \) components that transform under coordinates change as \( \delta - \frac{k}{2} \) densities, where \( \delta = \mu - |\lambda| \).

Therefore, the multiplication of these components by any non-singular matrix \( \varpi \) gives rise to a \( \mathcal{K}(1) \)-invariant isomorphism called \textit{a principal symbol map}
\[
\sigma_{\varpi}^{pr} : \mathcal{D}^k_{\Delta \mu}/\mathcal{D}^{k-\frac{1}{2}}_{\Delta \mu} \longrightarrow \mathcal{F}_{\delta - \frac{k}{2}} \oplus \mathcal{F}_{\delta - \frac{k}{2}} \oplus \cdots \oplus \mathcal{F}_{\delta - \frac{k}{2}} \quad (n_k \text{ copies}).
\]

The space of symbols of order \( \leq k, k \in \frac{1}{2} \mathbb{N}, \) is
\[
S^k_{\Delta \mu} = \bigoplus_{\ell=0}^{2k} \mathcal{D}^\ell_{\Delta \mu}/\mathcal{D}^{\ell-\frac{1}{2}}_{\Delta \mu}
\]

The \( \mathcal{K}(1) \)-module \( S^k_{\mu-|\lambda|} \) depends only on the shift, \( \delta \), of the weights and not on \( \mu, \lambda_1, \lambda_2 \cdots \lambda_n \) independently. Moreover, for every \( k \in \frac{1}{2} \mathbb{N}, \) we have
\[
S^k_{\mu-|\lambda|} = S^k_{\delta} = \bigoplus_{\ell=0}^{2k} \mathcal{D}^\ell_{\Delta \mu}/\mathcal{D}^{\ell-\frac{1}{2}}_{\Delta \mu} = \bigoplus_{\ell=0}^{2k} \mathcal{F}_{\delta - \frac{\ell}{2}},
\]
here the notation \( \mathcal{F}_{\lambda}^{(i)}, i \in \mathbb{N} \) and \( \lambda \in \mathbb{C}, \) stands for the sum \( \bigoplus \mathcal{F}_{\lambda} \) where \( \mathcal{F}_{\lambda} \) is counted \( \binom{i+n-1}{n-1} \) times.

Thanks to the isomorphism (2.28), an element \( P \) of \( S^k_{\delta} \) can be written in a unique way in the form
\[
P = \alpha^{\delta} \sum_{\ell=0}^{2k} \sum_{\mu=\ell} a_{\mu}(x, \theta) \alpha^{-\frac{\mu}{2}}
\]
where \( a_{\mu} \) are arbitrary functions in \( C^\infty(S^{1|1}). \)

As the orthosymplectic superalgebra \( \mathfrak{osp}(1|2) \) is a subalgebra \( \mathcal{K}(1) \), the space of symbols \( S_{\delta} \) can be viewed as an \( \mathfrak{osp}(1|2) \)-module.

3 \( \mathfrak{osp}(1|2) \)-equivariant symbol and quantization maps.

We restrict the \( \mathcal{K}(1) \)-module structures to the particular subalgebra \( \mathfrak{osp}(1|2) \) and look for \( \mathfrak{osp}(1|2) \)-isomorphisms between \( \mathcal{D}_{\Delta \mu} \) and \( S_{\delta} \). We fix a principal symbol map \( \sigma_{\varpi}^{pr} \) as in (2.28), where \( \varpi \) is a non singular matrix.

\textbf{Definition 3.1.} A symbol map is a a linear bijection
\[
\sigma_{\varpi}^{pr} : \mathcal{D}_{\Delta \mu} \rightarrow S_{\delta}
\]
such that the highest-order term of \( \sigma_{\varpi}^{pr}(A) \), where \( A \in \mathcal{D}_{\Delta \mu} \), coincides with the principal symbol \( \sigma_{\varpi}^{pr}(A) \). Hence, the inverse map, \( Q = (\sigma_{\varpi}^{pr})^{-1} \), will be called a quantization map.
The problem of existence and uniqueness of $\mathfrak{osp}(1|2)$-equivariant symbol (and so quantization) map can be tackled once the symbol map $\sigma_{pr}^{\varpi}$ is fixed.

The first main result of this paper is the following:

**Theorem 3.2.** if $\delta$ is non-resonant, i.e., $\delta = \mu - |\lambda| \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, k$ then, $\mathcal{D}_\Delta^{k\mu}$ and $\mathcal{S}_\delta^k$ are $\mathfrak{osp}(1|2)$-isomorphic through the family of $\mathfrak{osp}(1|2)$-equivariant maps $\sigma_{pr}^{\varpi}$ defined by:

$$
\sigma_{pr}^{\varpi}(A) = \alpha^2 \sum_{p=0}^{2k} \sum_{|j|=p} \sum_{\ell=|s|} \varpi_{\frac{\ell}{2}}^2 D^{\ell-p}(a_{\frac{s}{2}}) \alpha^{-\frac{|s|}{2}} \quad (3.2)
$$

where $A = \sum_{|j|=p} a_j \mathcal{T}_1 \otimes \mathcal{T}_2 \otimes \ldots \mathcal{T}_n \in \mathcal{D}_\Delta^{k\mu}$ and $\varpi_{\frac{\ell}{2}}^2$ are constants given by the induction formula

$$
(-1)^{\ell-p} \left( \left[ \frac{-1}{2} \right] + (1 - (-1)^{\ell-p})(\delta - \frac{1}{2}) \right) \varpi_{\frac{\ell}{2}}^2 - \left( \left[ \frac{1}{2} \right] + (1 - (-1)^{s_1}) \lambda_1 \right) \varpi_{\frac{1}{2}}^{s_1-1} - \sum_{j=2}^{n} (-1)^{s_1+s_2+\ldots+s_{j-1}} \left( \left[ \frac{s_j}{2} \right] + (1 - (-1)^{s_j}) \lambda_j \right) \varpi_{\frac{s_j}{2}}^{s_j-1} = 0. \quad (3.3)
$$

If $\varpi$ is the identity map, we obtain the "normalized" symbol map $\sigma_{pr}^{I\mu}$ given by the rule

$$
\sigma_{pr}^{I\mu}(A) = \alpha^2 \sum_{p=0}^{2k} \sum_{|j|=p} \sum_{\ell=|s|} \gamma_{\frac{\ell}{2}}^2 D^{\ell-p}(a_{\frac{s}{2}}) \alpha^{-\frac{|s|}{2}} \quad (3.4)
$$

such that

$$
\gamma_{\frac{\ell}{2}} = (-1)^{(\frac{\ell+1}{2})} \prod_{t=2}^{n} \frac{(-1)^{\varphi(t)} \left( \varphi(t) \right)_{s_t-i_t}}{\Xi_{s_t,i_t}(\lambda_t)} \Xi_{s_t,i_t}(\lambda_t) \quad (3.5)
$$

where the functions $\varphi, \psi$ and $\Xi$ are defined by

$$
\varphi(t) = \sum_{j=1}^{t-1} s_t - i_t, \quad \psi(t) = \sum_{j=1}^{t-1} s_j(s_j+1-i_j+1), \quad \Xi_{s_t,i_t}(\lambda_t) = \left( \frac{\left[ \frac{1}{2} \right]}{\left[ \frac{\lambda_t}{2} \right]} \right) \left( \frac{2\lambda_t + \left[ \frac{\lambda_t-1}{2} \right]}{2(s_t-i_t)+1(-1)^{i_t}} \right)
$$

and the notation $\left( \begin{array}{c} \nu \\ q \end{array} \right)$ stands for the binomial coefficient given by $\left( \begin{array}{c} \nu \\ q \end{array} \right) = \frac{\nu!}{q!(\nu-q)!}$.

Moreover, once the principal symbol is fixed, the symbol map $\sigma_{pr}^{\varpi}$ is unique.

**Proof.** We begin the proof by proving the $\mathfrak{osp}(1|2)$-equivariance of the map $\sigma_{pr}^{I\mu}$. Indeed, Let $X = X_F \in \mathcal{K}(1)$. We have

$$
\sigma_{pr}^{I\mu}(\mathcal{D}_\Delta^{k\mu}(A)) = \alpha^2 \sum_{p=0}^{2k} \sum_{|j|=p} \gamma_{\frac{\ell}{2}}^2 D^{\ell-p}(a_{\frac{s}{2}}) \alpha^{-\frac{|s|}{2}} \quad (3.6)
$$

Then, we readily see that

$$
\gamma_{\frac{\ell}{2}}^2 = \sum_{\ell=p}^{2k} \sum_{|s|=\ell} \gamma_{\frac{\ell}{2}}^2 D^{\ell-p}(a_{\frac{s}{2}}), \quad p = |j|.
$$
Thanks to the proposition 2.1 for all $0 \leq p = |\underline{\alpha}| \leq k$, we get

$$\bar{\sigma}^{X}_{\underline{\alpha}} = \sum_{\ell = p}^{2k} \sum_{|\underline{s}|=\ell} (-1)^{|\underline{a}|} |\ell-p| (\ell-p)|\ell-p+1|/2 \gamma_{\underline{a}}^{\ell-p}(a^{X}_{\underline{\alpha}})$$

$$= \sum_{\ell = p}^{2k} \sum_{|\underline{s}|=\ell} (-1)^{|\underline{a}|} |\ell-p| (\ell-p)|\ell-p+1|/2 \gamma_{\underline{a}}^{\ell-p}(a^{X}_{\underline{\alpha}})$$

$$- \sum_{m=1}^{2k-|\underline{a}|} \sum_{n=1}^{n} \sum_{\ell = p}^{2k} (-1)^{m(|F|+|\underline{a}|)} \left( \left( \frac{m+s_1}{m+2} \right)^{s} - \frac{1}{2} (1)^{s} \left( \frac{m+s_1}{m+1} \right)^{s} + \lambda_{1} \left( \frac{m+s_1}{m} \right)^{s} \right) \bar{D}^{n}(F') a^{X}_{\underline{a}+m1}$$

Thus

$$\bar{\sigma}^{X}_{\underline{\alpha}} - \left[ \sum_{\ell = p}^{2k} \sum_{|\underline{s}|=\ell} (-1)^{|\underline{a}|} |\ell-p| (\ell-p)|\ell-p+1|/2 \gamma_{\underline{a}}^{\ell-p}(a^{X}_{\underline{\alpha}}) \right]$$

$$+ \left( \sum_{s=1}^{\ell-p} \right) \bar{D}(F')(F') \bar{D}^{\ell-p-1}(a^{X}_{\underline{\alpha}})$$

$$+ \sum_{j=2}^{n} \left( \sum_{l=1}^{s_1} \gamma_{\underline{a}}^{\ell-p-1} \left( \left( \frac{s_1}{3} \right)^{s} + \frac{1}{2} (1)^{s} \left( \frac{s_1}{2} \right)^{s} + \lambda_{1} \left( \frac{s_1}{3} \right)^{s} \right) \right) \bar{D}(F')(F') \bar{D}^{\ell-p-1}(a^{X}_{\underline{\alpha}})$$

(higher terms in $\bar{D}^{n}(F')$, $n \geq 2$).

Now, through a simple calculation, one can check out that the scalars $\gamma_{\underline{a}}^{\ell}$ satisfies the relationship

$$(-1)^{\ell-p} \Upsilon(\delta - \frac{\ell}{2}, \ell - p) \gamma_{\underline{a}}^{\ell} - \Upsilon(\lambda_{1}, s_{1}) \gamma_{\underline{a}}^{\ell-1} \Upsilon(\lambda_{1}, s_{1}) \gamma_{\underline{a}}^{\ell-1} j = 0$$

where, for $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, we put

$$\Upsilon(\lambda, m) = \frac{1}{2} \left( \frac{m}{2} + (1 - (-1)^{m}) \lambda \right).$$

Since, the term in $\bar{D}(F')$ vanishes, we can clearly see that the map $\sigma_{\Delta, \mu}^{d}$ $\text{osp}(1|2)$-equivariant. Now, we can easily adapt the proof of locality given in [11] for the uniaxial case to our case and then use the locality property of an $\text{osp}(1|2)$-equivariant symbol map. Therefore, in addition, from the expression of the ”normalized” symbol map $\sigma_{\Delta, \mu}^{d}$ we can suppose that a general symbol map $\sigma_{\Delta, \mu}^{d}$ can be written as

$$A = \sum_{p=0}^{2k} \sum_{|\underline{\alpha}|=p} a^{X}_{\underline{\alpha}} \bar{D}^{1} \otimes \bar{D}^{2} \otimes \ldots \otimes \bar{D}^{n} \mapsto \delta^{\frac{2k}{p}} \sum_{p=0}^{2k} \sum_{|\underline{\alpha}|=p} \sum_{\ell = p}^{2k} \sum_{|\underline{s}|=\ell} \alpha^{\frac{2k}{p}} \bar{D}^{\ell-p}(a^{X}_{\underline{\alpha}}) \alpha^{\frac{|\underline{\alpha}|}{p}}.$$  (3.6)

Obviously, to get the condition of $\text{osp}(1|2)$-equivariance, it is sufficient to impose invariance with respect to the vector fields $D = 2X_{\theta}$ and $x D = 2X_{x \theta}$ to meet the whole condition $\text{osp}(1|2)$-equivariance. Thus we have:
a) A symbol map \( \sigma_{\lambda}^{1d} \) commutes with the action of \( D \) if and only if the coefficients \( \varpi_{\lambda}^{1d} \) are constants (i.e., do not depend on \( x, \theta \)),

b) A symbol map \( \sigma_{\lambda}^{1d} \) commutes with the action of \( xD \) if and only if the coefficients \( \varpi_{\lambda}^{1d} \) satisfy the induction formula \( (3.3) \).

If \( \delta = \mu - |\lambda| \) is non-resonant, i.e., \( \delta = \mu - |\lambda| \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, k \), then, it is easy to see that the solution of the equation \( (3.3) \) and once the principal symbol \( \sigma^{1d} \) where \( \varpi = (u_{\frac{i}{2}}, |\vec{y}|=2k \) is fixed, the symbol map \( \sigma_{\lambda}^{1d} \) is unique.

**Remark 3.3.** We can write the symbol map \( \sigma_{\lambda}^{1d} \) as in [13], Theorem 5.6. Indeed, Let \( A = a_{\frac{i}{2}} \mathcal{D}^{\frac{i}{2}} \otimes \mathcal{D}^{\frac{i}{2}} \otimes \ldots \otimes \mathcal{D}^{\frac{i}{2}} \in \mathcal{D}_{\lambda}^{1d} \) and \( |\vec{y}| = 2k \), then

\[
\sigma_{\lambda}^{1d}(A) = \alpha^{\delta} \sum_{|\vec{y}|=0}^{2k} \sum_{|\vec{s}|=\ell} \chi_{\lambda}^{\delta} D^{\ell}\left( a_{\frac{i}{2}} \alpha^{-|\vec{y}|^2} \right)
\]

where

\[
\chi_{\lambda}^{\delta} = \left\{ \begin{array}{ll}
(1) \prod_{\ell=2}^{n} \left( \frac{\Gamma(t)}{\Gamma(t-1)} \right) \chi_{s_{t} \cdot s_{t}}(\lambda_{1}) & \text{if } i_{t} \geq s_{t}, t \in \{1, 2, ..., n\}, \\
0 & \text{otherwise}
\end{array} \right.
\]

where \( \Gamma(t) = \sum_{j=1}^{t} s_{j} \) and \( \Delta(t) = \sum_{j=1}^{t-1} s_{j} s_{j+1} \).

Now, by a direct computation, one can easily check the following explicit formula for the quantization map \( Q_{\lambda}^{1d} \):

**Proposition 3.4.** The quantization map \( Q_{\lambda}^{1d} \), i.e., the inverse of the symbol map \( \sigma_{\lambda}^{1d} \) given in theorem 3.2 associates to a polynomial \( P = \alpha^{\delta} \sum_{|\vec{y}|=\ell} \sum_{|\vec{s}|=\ell} \beta_{\lambda}^{\delta} \alpha^{-\frac{|\vec{y}|^2}{2}} \in S^{1d} \) the differential operator

\[
Q_{\lambda}^{1d}(P) = \sum_{\ell=0}^{2k} \sum_{|\vec{y}|=\ell} \beta_{\lambda}^{\delta} D^{\ell-s} \left( a_{\frac{i}{2}} \right),
\]

where

\[
\beta_{\lambda}^{\delta} = \left\{ \begin{array}{ll}
(-1)^{\left( \left\lfloor \frac{p-1}{2} \right\rfloor \right)} \prod_{t=2}^{n} \left( \frac{\Gamma(t)}{\Gamma(t-1)} \right) \chi_{s_{t} \cdot s_{t}}(\lambda_{1}) & \text{if } \ell = |\vec{y}| > p = |\vec{s}| \\
\beta_{\lambda}^{\delta} = \gamma_{\lambda}^{\delta} \alpha^{-\frac{|\vec{y}|^2}{2}} & \text{if } \ell = |\vec{s}| = |\vec{y}|
\end{array} \right.
\]

12
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