Furstenberg Transformations and Approximate Conjugacy

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Abstract

Let $\alpha$ and $\beta$ be two Furstenberg transformations on 2-torus associated with irrational numbers $\theta_1$, $\theta_2$, integers $d_1$, $d_2$ and Lipschitz functions $f_1$ and $f_2$. We show that $\alpha$ and $\beta$ are approximately conjugate in a measure theoretical sense if (and only if) $\theta_1 \pm \theta_2 = 0$ in $\mathbb{R}/\mathbb{Z}$. Closely related to the classification of simple amenable $C^*$-algebras, we show that $\alpha$ and $\beta$ are approximately $K$-conjugate if (and only if) $\theta_1 \pm \theta_2 = 0$ in $\mathbb{R}/\mathbb{Z}$ and $|d_1| = |d_2|$. This is also shown to be equivalent to that the associated crossed product $C^*$-algebras are isomorphic.

1 Introduction

A celebrated result of Giordano, Putnam and Skau ([6]) states that two minimal Cantor systems are strong orbit equivalent if (and only if) the associated crossed product $C^*$-algebras are isomorphic which can also be described by their $K$-theory. Moreover, two such Cantor systems are topological orbit equivalent if part of the $K$-theoretical information (namely the tracial range of the $K_0$-groups) of the associated $C^*$-algebras are (unital) order isomorphic. This note is an attempt to explore its possible analogy in the case that the space is connected.

With the recent rapid development of the classification of amenable simple $C^*$-algebras of stable rank one ([2], [3], [9], [1], [11] and [12], to name few), it becomes possible to apply $C^*$-algebra theory to the study of minimal homeomorphisms on more general spaces. Several versions of approximate conjugacy have been introduced and studied recently (see [14], [15], [21], [10] and [17]). In [10] and [17], minimal homeomorphisms on the product of Cantor set and circle were studied. It is shown that if a set of $K$-theoretical information of the two minimal homeomorphisms on the product of Cantor set and circle are the same then they are approximate $K$-conjugate (and the converse also holds).

One of the reasons that Giordano, Putnam and Skau’s work is so successful is that Cantor set is totally disconnected. Perhaps the orbit equivalence for Cantor minimal systems may

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be viewed as something which lives between measure theory and topology. When the space $X$ is connected, the situation is very different. For example, by a result of Sierpinski, for connected spaces, two topological orbit equivalent minimal homeomorphisms are in fact flip conjugate (see Prop. 5.5 of [20]). Therefore, for connected space, one should not expect that two minimal homeomorphisms are topological orbit equivalent if their associated crossed product $C^*$-algebras have order isomorphic $K$-groups, or if their associated crossed product $C^*$-algebras are isomorphic.

We are interested in the following two questions:

**Q1:** Let $X$ be a (connected) compact metric space and let $\alpha$ and $\beta$ be two minimal homeomorphisms on $X$. Let $A_\alpha$ and $A_\beta$ be the associated crossed product $C^*$-algebras. Suppose that the tracial range of $K_0(A_\alpha)$ and that of $K_0(A_\beta)$ are (unital) order isomorphic. What can one say about the homeomorphisms $\alpha$ and $\beta$?

**Q2:** Let $X$ be a (connected) compact metric space and let $\alpha$ and $\beta$ be two minimal homeomorphisms on $X$. Let $A_\alpha$ and $A_\beta$ be the associated crossed product $C^*$-algebras. Suppose that $A_\alpha$ and $A_\beta$ are isomorphic with additional $K$-theoretical information of $(j_\alpha)_*$ and $(j_\beta)_*$. What can one say about the homeomorphism $\alpha$ and $\beta$?

(See the definition of $j_\alpha$ and $j_\beta$ in [2.6] below. A clarification of this question will be discussed in [2.11].)

Giordano, Putnam and Skau’s results answered both questions, namely, $\alpha$ and $\beta$ are topological orbit equivalent for the Q1 and $\alpha$ and $\beta$ are strong orbit equivalent for Q2 under the assumption that $X$ is the Cantor set.

The results in [15] [21], [10] and [17] suggest that answer to Q2 should be that $\alpha$ and $\beta$ are approximately $K$-conjugate, and for Q1, $\alpha$ and $\beta$ are approximately conjugate in a more measure theoretical sense. However, spaces studied in the above mentioned articles are not connected. It is interesting to see answers to the question Q1 and Q2 for any connected spaces (other than $\mathbb{T}$).

A classical example of minimal homeomorphisms on the 2-torus $\mathbb{T}^2$ was studied by Furstenberg ([4]). Let $\theta$ be an irrational number and $g : \mathbb{T} \to \mathbb{T}$ be a continuous map. The Furstenberg transform $\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ is defined to be $\alpha(\xi, \zeta) = (\xi e^{2\pi i \theta}, \zeta g(\xi))$ for $\xi \in \mathbb{T}$ and $\zeta \in \mathbb{T}$ with $g$ being homotopically non-trivial. One may write that $\alpha = (\xi e^{2\pi i \theta}, \zeta e^{i 2\pi f(\xi)})$, where $d$ is an integer and $f$ is a real continuous function in $C(\mathbb{T})$. Denote this $\alpha$ by $\Phi_{\theta,d,f}$. It is proved that $\alpha$ is always minimal if $d \neq 0$. It is also shown in [4] that if $g$ satisfies the Lipschitz condition then $\alpha$ is also unique ergodic ([4]). This is one of the favorite examples and has been intensively studied (for example, [4], [5], [21], [7], [8] and [22], to name a few).

It was conjectured by R. Ji ([5]) that $\Phi_{0,1,0}$ is conjugate to $\Phi_{0,1,0}$. By considering quasi-discrete spectrum, counter-examples have been constructed by Rouhani ([21]) that $\Phi_{0,1,0}$ may not be flip conjugate to $\Phi_{0,1,0}$.

Let $\alpha = \Phi_{\theta_1, d_1, f}$ and $\beta = \Phi_{\theta_2, d_2, f}$. In this note, we first show that, if $\theta_1 + \theta_2 = 0$ in $\mathbb{R}/\mathbb{Z}$, then $\alpha$ and $\beta$ are approximately conjugate in a measure theoretical sense (see [2.1] below). We also show that the converse is true, i.e., if $\alpha$ and $\beta$ are approximately conjugate in that sense, then $\theta_1 + \theta_2 = 0$ in $\mathbb{R}/\mathbb{Z}$. Let $A_\alpha = C(\mathbb{T}^2) \rtimes_\alpha \mathbb{Z}$ and $A_\beta = C(\mathbb{T}^2) \rtimes_\beta \mathbb{Z}$ be the associated crossed
products. At least in the case that \( f_1 \) and \( f_2 \) satisfy the Lipschitz condition, the ranges of \( K_0(A_\alpha) \) and \( K_0(A_\beta) \) under the tracial map are the same, namely, \( \mathbb{Z} + \mathbb{Z}(\theta_1) \) (we still assume that \( \theta_1 \pm \theta_2 = 0 \) in \( \mathbb{R}/\mathbb{Z} \)). This result seems closer to that of the topological orbit equivalence in the Cantor minimal systems.

It has been recently proved that (\cite{12}), in the unique ergodic cases, \( A_\alpha \) and \( A_\beta \) are unital simple \( C^* \)-algebras with tracial rank zero. Therefore, by the classification of unital simple amenable \( C^* \)-algebras with tracial rank zero (see \cite{12}), \( A_\alpha \) and \( A_\beta \) are isomorphic as \( C^* \)-algebras if and only if \( \theta_1 \pm \theta_2 = 0 \) in \( \mathbb{R}/\mathbb{Z} \) and \( |d_1| = |d_2| \). We will show that \( \alpha \) and \( \beta \) are approximately \( K \)-conjugate if and only if \( \theta_1 \pm \theta_2 = 0 \) in \( \mathbb{R}/\mathbb{Z} \) and \( |d_1| = |d_2| \). In the process, we also show that, when \( f_1 - f_2 \) is in a dense subset of real part of \( C(\mathbb{T}) \), \( \Phi_{\theta,d,f_1} \) and \( \Phi_{\theta,d,f_2} \) are actually conjugate (see \cite{13} below).

The results in this note are very special. However, it is our hope that this special case will lead us to more interesting answers to the questions \( Q_1 \) and \( Q_2 \) and serves as an invitation for further exploration.

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2 The Main results

Definition 2.1. Let \( X \) and \( Y \) be two compact metric spaces and let \( \alpha : X \to X \) and \( \beta : Y \to Y \) be two minimal homeomorphisms. Let \( T_\alpha \) and \( T_\beta \) be sets of \( \alpha \)-invariant and \( \beta \)-invariant normalized Borel measures, respectively. We say that \( \alpha \), \( \beta \) are approximately conjugate in the sense (M1) if there exist two sequences of homeomorphisms \( \sigma_n : X \to Y \) and \( \gamma_n : Y \to X \) and affine homeomorphisms \( \Lambda_1 : T_\alpha \to T_\beta \) and \( \Lambda_2 : T_\beta \to T_\alpha \) such that

\[
\lim_{n \to \infty} \sup_{\nu \in T_\beta} \nu(\{ y \in Y : \text{dist}(\sigma_n \alpha \circ \sigma_n^{-1}(y), \beta(y)) \geq a \}) = 0, \tag{e 2.1}
\]

\[
\lim_{n \to \infty} \sup_{\mu \in T_\alpha} \mu(\{ x \in X : \text{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) \geq a \}) = 0 \tag{e 2.2}
\]

for all \( a > 0 \),

\[
\Lambda_1(\mu)(S) = \mu(\sigma_n^{-1}(S)), \quad \text{and} \quad \Lambda_2^{-1}(\nu)(G) = \nu(\gamma_n^{-1}(G)) \quad n = 1, 2, \ldots \tag{e 2.3}
\]

for all Borel sets \( S \subset Y \), \( G \subset X \) and for all \( \mu \in T_\alpha \), \( \nu \in T_\beta \).

A couple of remarks are in order.

(1) Suppose that there exists a homeomorphism \( \sigma : X \to Y \) such that \( \sigma \circ \alpha \circ \sigma^{-1} = \beta \). Define \( \Lambda : T_\alpha \to T_\beta \) by \( \Lambda(\mu)(S) = \mu(\sigma^{-1}(S)) \) for all Borel sets \( S \subset Y \). It should be noted that

\[
\Lambda(\mu) = \mu(\sigma^{-1} \circ \beta) \tag{e 2.4}
\]

\[
= \mu(\sigma^{-1} \circ \beta \circ \sigma \circ \sigma^{-1}(S))
\]

\[
= \mu(\alpha \circ \sigma^{-1}(S)) = m(\sigma^{-1}(S)) = \Lambda(\mu(S))
\]
for all Borel sets $S \subset Y$. So $\Lambda(\mu) \in T_\beta$. In particular, conjugate homeomorphisms are approximately conjugate in the sense (M1).

In general, a sequence homeomorphisms $\{\sigma_n\}$ does not preserve measures even though both (e 2.1) and (e 2.2) hold. One could have $\lim_{n \to \infty} \mu(\sigma_n(S)) = 0$. Here we require, in the definition, that $\{\sigma_n\}$ has some consistent information on measure spaces. So in the definition 2.1, one should view that the conditions in (e 2.3) are important part of the definition.

(2) In the definition of 2.1, put $E_n(a) = \{y \in Y : \text{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}(y), \beta(y)) \geq a\}$ for $a > 0$. Put $S_n(a) = \{x \in X : \text{dist}(\sigma_n \circ \alpha(x), \beta \circ \sigma_n(x)) \geq a\}$. Then $\sigma_n^{-1}(E_n(a)) = S_n(a)$.

By (e 2.1) and (e 2.4), $\lim_{n \to \infty} \sup_{\mu \in T_\alpha} \mu(S_n(a)) = 0$.

(3) It is an easy exercise that approximately conjugacy in the sense (M1) is an equivalence relation among minimal homeomorphisms.

**Definition 2.2.** Let $\theta$ be an irrational number and $g : \mathbb{T} \to \mathbb{T}$ be a continuous function. A Furstenberg transform is a map $\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $\alpha((\xi, \zeta)) = (\xi e^{i2\pi \theta}, \zeta g(\xi))$ for $\xi \in \mathbb{T}$ and $\zeta \in \mathbb{T}$ with $g$ being of degree $d \neq 0$ (the winding number is $d$). There is a real function $f \in C(\mathbb{T})$ such that $g(\xi) = \xi^d \exp(i2\pi f(\xi))$ for $\xi \in \mathbb{T}$. We will call $\alpha$ the Furstenberg transform associated with irrational number $\theta$, integer $d$ and function $f$. It will also be denoted by $\Phi_{\theta,d,f}$.

We are interested in the case that $(\mathbb{T}^2, \alpha)$ is uniquely ergodic. It is known that $\alpha$ is always minimal (see [3]). It is also shown in [3] that $(\mathbb{T}^2, \alpha)$ is uniquely ergodic if $g$ has Lipschitz property (or $f$ is Lipschitz). The unique invariant measure is the product of the normalized Lebesgue measure $m_\mathbb{T} = m \times m$. We fix the following metric on $\mathbb{T}^2$:

$$\text{dist}((\xi, \zeta), (\xi', \zeta')) = \sqrt{|\xi \xi' - 1|^2 + |\zeta \zeta' - 1|^2},$$

where $\xi, \xi', \zeta, \zeta' \in \mathbb{T}$.

We will keep these notation throughout this note.

**Theorem 2.3.** Let $\alpha = \Phi_{\theta_1,d_1,f_1}, \beta = \Phi_{\theta_2,d_2,f_2} : \mathbb{T}^2 \to \mathbb{T}^2$ be unique ergodic Furstenberg transforms. Then the following are equivalent:

(1) $[\theta_1 \pm \theta_2] = 0$ in $\mathbb{R}/\mathbb{Z}$;

(2) $\alpha$ and $\beta$ are approximately conjugate in the sense (M1),


Theorem 2.4. Let \( \alpha = \Phi_{\theta_1, a_1, f_1}, \beta = \Phi_{\theta_2, d_2, f_2} : \mathbb{T}^2 \to \mathbb{T}^2 \) be unique ergodic Furstenberg transforms. Then each condition (1) or (2) in \((\text{x})\) is also equivalent to the following:

(3) There are sequences of Borel equivalences \( \{\sigma_n\} \) and \( \{\gamma_n\} \) such that

\[
\lim_{n \to \infty} \sup \{ \text{dist}(\sigma_n \circ \sigma_n^{-1}(y), \beta(y)) : y \in Y \} = 0,
\]

\[
\lim_{n \to \infty} \sup \{ \text{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) : x \in X \} = 0,
\]

if one does not insists that \( \sigma_n \) and \( \gamma_n \) to be continuous everywhere. More precisely, we have the following

**Definition 2.5.** Let \( A \) be a stably finite unital \( C^* \)-algebra and let \( T(A) \) be the tracial state space. Denote by \( Tr \) the usual (non-normalized) trace on \( M_k \). Define \( \rho : K_0(A) \to \text{Aff}(T(A)) \) by \( \rho([p]) = \tau(p) \) for projections in \( M_k(A) \), where \( \tau = t \otimes Tr \) and \( t \in T(A) \).

**Definition 2.6.** Let \( X \) be a compact metric space and let \( \alpha : X \to X \) be a minimal homeomorphism. Then the transformation group \( C^* \)-algebra, the crossed product, \( C(X) \rtimes_\alpha Z \) will be denoted by \( A_\alpha \).

We will use \( j_\alpha : C(X) \to A_\alpha \) for the natural embedding.

For a unital \( C^* \)-algebra \( A \), we denote by \( ad(u) = u^* au \) for all \( a \in A \).

We fix a unitary \( u_\alpha \) so that \( ad u_\alpha j_\alpha(f) = j_\alpha(f \circ \alpha) \) for all \( f \in C(X) \). It should be noted that there are other choices for such \( u_\alpha \). For example, if \( z \in C(X) \) is a unitary then \( w = u_\alpha j_\alpha(z) \) is another choice. In fact,

\[
ad w(j_\alpha(f)) = j_\alpha(z)^* u_\alpha^*(j_\alpha(f)) u_\alpha j_\alpha(z) = j_\alpha(z^*(f \circ \alpha) z) = j_\alpha(f \circ \alpha)
\]

for all \( f \in C(X) \).
Remark 2.7. Let $X = \mathbb{T}^2$ and let $\alpha = \Phi_{\theta,d,f}$ be a Furstenberg transformation with Lipschitz $f$. Then $A_\alpha$ is a unital simple $C^*$-algebra with a unique tracial state.

It is computed (see Example 4.9 of [23]) that $K_0(A_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ with $\rho(K_0(A_\alpha)) = \mathbb{Z} \oplus \mathbb{Z}(\theta) \subset \mathbb{R}$ and
\[ K_0(A_\alpha)_+ = \{ m_1 + m_2 + m_3 \in \mathbb{Z}^3 : m_1 + m_3 \theta > 0 \text{ or } m_1 = m_2 = m_3 = 0 \}, \]
where the first two copies of $\mathbb{Z}$ is identified with the image of $K_0(C(\mathbb{T}^2))$ under the embedding $(j_\alpha)_0$, and $K_1(A_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus d\mathbb{Z} \oplus \mathbb{Z}$, where $d\mathbb{Z} \oplus \mathbb{Z}$ is the image of $K_1(C(\mathbb{T}^2))$ under $(j_\alpha)_1$. Let $z_1, z_2 : C(\mathbb{T}^2) \to \mathbb{T}$ be the functions defined by $z_1((\xi, \zeta)) = \xi$ and $z_2((\xi, \zeta)) = \zeta$ ($\xi, \zeta \in \mathbb{T}$). Then $(j_\alpha)_1([z_1])$ is the standard generator of $d\mathbb{Z} \oplus \mathbb{Z}$ and $(j_\alpha)_1([z_2])$ is the standard generator of $\mathbb{Z}$.

Let $X$ be a compact metric space, and let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. It is certainly desirable to have two sequences of homeomorphisms $\{\sigma_n\}$ and $\{\gamma_n\}$ on $X$ so that
\[
\lim_{n \to \infty} \sup \{ \text{dist}(\sigma_n \circ \sigma_n^{-1}(x), \beta(x)) : x \in X \} = 0 \quad \text{and} \quad (e \ 2.11)
\]
\[
\lim_{n \to \infty} \sup \{ \text{dist}(\gamma_n \circ \gamma_n^{-1}(x), \alpha(x)) : x \in X \} = 0 \quad (e \ 2.12)
\]

However, when a sequence of maps (such as $\{\sigma_n\}$ and $\{\gamma_n\}$) involved, one also expects that the maps have something in common. At least in the Cantor set case, without any consistency on the conjugating maps $\{\sigma_n\}$ and $\{\gamma_n\}$, (2.11) and (2.12) are not so interesting (see [15]). Even though one should not expect these maps will converge in any meaningful way, one hopes that some information about these maps are independent of $n$. For example, one would like to require that both sequences preserve the measures as in (2.11) as well as in (3) of Theorem 2.4. To be more topologically interesting, one may require that both sequences preserve some topological data. For example, in [15], approximate $K$-conjugacy requires that both sequences preserve $K$-theory (in the crossed products). We use the following definition in this note:

Definition 2.8. Let $X$ be a compact metric space, and let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Two homeomorphisms $\alpha$ and $\beta$ are said to be \textit{approximately $K$-conjugate} if there exist two sequences of homeomorphisms $\{\sigma_n\}$ and $\{\gamma_n\}$ on $X$ such that (2.11) and (2.12) hold, and there exists an isomorphism $\phi : A_\alpha \to A_\beta$ and sequences of unitaries $u_n \in A_\beta$ and $v_n \in A_\alpha$ such that
\[
\lim_{n \to \infty} \| \text{ad } u_n \circ \phi(j_\alpha(f)) - j_\beta(f \circ \sigma_n^{-1}) \| = 0 \quad (e \ 2.13)
\]
for all $f \in C(X)$ and
\[
\lim_{n \to \infty} \| \text{ad } u_n \circ \phi(u_n) - u_\beta z_n \| = 0, \quad (e \ 2.14)
\]
where $z_n \in U_0(A_\beta)$ such that
\[
\lim_{n \to \infty} \| z_n j_\beta(f) - j_\beta(f) z_n \| = 0 \quad (e \ 2.15)
\]
for all \( f \in C(X) \), and

\[
\lim_{n \to \infty} \| \text{ad} v_n \circ \phi(j_{\beta}(f)) - j_{\alpha}(f \circ \gamma_n^{-1}) \| = 0 \tag{e 2.16}
\]

for all \( f \in C(X) \) and

\[
\lim_{n \to \infty} \| \text{ad} v_n \circ \phi(u_{\beta}) - u_{\alpha} y_n \| = 0, \tag{e 2.17}
\]

where \( y_n \in U_0(A_{\alpha}) \) such that

\[
\lim_{n \to \infty} \| y_n j_{\beta}(f) - j_{\alpha}(f) y_n \| = 0 \tag{e 2.18}
\]

for all \( f \in C(X) \).

**Remark 2.9.** (1) It should be noted that (e 2.11 and (e 2.13) imply both (e 2.14) and (e 2.15). In fact one can take \( z_n = u_{\beta}(u_{\alpha} u_n) \). At least from the discussion that leads to (e 2.10), one sees that the unitaries \( z_n \) and \( y_n \) can not be omitted.

(2) The existence of the unitaries \( \{ u_n \} \) and \( \{ v_n \} \) implies that \( \{ \sigma_n \} \) and \( \{ \gamma_n \} \) preserve the invariant measures. Note also if \( p \in M_k(C(X)) \) is a projection, then \([j_{\beta}(p \circ \sigma_n^{-1})] = [j_{\beta}(p)] \) in \( K_0(A_{\beta}) \) for all large \( n \). In fact, \( \{ \sigma_n \} \) and \( \{ \gamma_n \} \) preserve the ordered \( K \)-theory (independent of \( n \)) and beyond.

Condition (1) in Theorem 2.3 implies that \( \rho(A_{\alpha}) \) and \( \rho(A_{\beta}) \) are (unital) order isomorphic. In the case that \( X \) is the Cantor set, by a Giordano, Putnam and Skau’s theorem, the condition that \( \rho(A_{\alpha}) \) and \( \rho(A_{\beta}) \) are unital order isomorphic is equivalent to that \( \alpha \) and \( \beta \) are topological orbit equivalent. Similar conclusion is not possible for connected space, as mentioned earlier that two topological orbit equivalent minimal homeomorphisms on connected spaces are flip conjugate. As we stated in the introduction, topological orbit equivalence for minimal Cantor systems seems to be something between measure theory and topology. The conclusion of Theorem 2.3 also has both topological and measure theoretical favor. It should be noted that one can not make maps \( \{ \sigma_n \} \) and \( \{ \gamma_n \} \) in (3) of Theorem 2.3 to be homeomorphisms in general as the next theorem states. It should also be mentioned that the approximate conjugacy in the sense of (5) in Theorem 2.10 is rather weak relation for minimal Cantor system (see [15]). However, this notion plays completely different role for homeomorphisms on the connected spaces.

It is recent proved (118) that for a unique ergodic Furstenberg transformation \( \alpha \) the associated crossed product \( A_\alpha \) has tracial rank zero. So classification theorem (see [11] and [12]) can be applied. In particular, if \( \alpha = \Phi_{\theta_1,d_1,f_1} \) and \( \beta = \Phi_{\theta_2,d_2,f_2} \), then \( A_\alpha \cong A_\beta \) is and only if \( \theta_1 \pm \theta_2 = 0 \) in \( \mathbb{R}/\mathbb{Z} \) and \( |d_1| = |d_2| \).

**Theorem 2.10.** Let \( \alpha, \beta : \mathbb{T}^2 \to \mathbb{T}^2 \) be two unique ergodic Furstenberg transforms associated with irrational numbers \( \theta_1, \theta_2 \) and integers \( d_1, d_2 \in \mathbb{Z} \setminus \{0\} \), respectively.

Then the following are equivalent.

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Remark 2.11. It should be pointed out, in general, (1) or (4) do not imply that \( S \) for all Borel subsets are approximately K-conjugate. See, for example, Example 9.2 of [16]. Let \((\alpha, \beta)\) compact CW -complex and let 
\[
\tau \in \text{distinct values in the range of } r, \quad \rho \text{ the case that } \tau
\]
and 
\[
\text{Then: } \min_{n \to \infty} \sup_{x \in \mathbb{T}^2} \{ \text{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}(x), \beta(x)) : x \in \mathbb{T}^2 \} = 0, \quad (e 2.19)
\]
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{T}^2} \{ \text{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) : x \in \mathbb{T}^2 \} = 0, \quad (e 2.20)
\]
\[
m_2(\sigma_n(S)) = m_2(S) \quad \text{and} \quad m_2(\gamma_n(S)) = m_2(S) \quad (e 2.21)
\]
for all Borel subsets \( S \subset \mathbb{T}^2 \).

Theorem 3.1. Let \( X \) be an infinite compact metric space, and let \( h : X \to X \) be a minimal homeomorphism. Let \( Y \subset X \) be closed and have nonempty interior. For \( y \in Y \) set \( r(y) = \min\{ n \geq 1 : h^n(y) \in Y \} \). Then \( \sup_{y \in Y} r(y) < \infty \). Let \( n(0) < n(1) < n(2) < \cdots < n(l) \) be the distinct values in the range of \( r \), and for \( 0 \leq k \leq l \) set 
\[
Y_k = \{ y \in Y : r(y) = n(k) \} \quad \text{and} \quad Y_k^\circ = \text{int}(\{ y \in Y : r(y) = n(k) \}).
\]

Then:
The sets $h^j(Y_k^o)$, for $0 \leq k \leq l$ and $1 \leq j \leq n(k)$, are disjoint.

(2) $\bigcup_{k=1}^{n} \bigcup_{j=1}^{n(k)} h^j(Y_k) = X$. 

(3) $\bigcup_{k=0}^{n} h^k(Y_k) = Y$.

**Lemma 3.2.** Let $\alpha : \mathbb{T} \to \mathbb{T}$ be a minimal homeomorphism. Let $n > 1$ be an integer. Then there are finitely many pairwise disjoint open arks $\{J_i = (c_i, d_i) : i = 1, 2, ..., k\}$ of $\mathbb{T}$ such that

1. $\alpha^j(J_i)$ are pairwise disjoint for $0 \leq j \leq h(i) - 1$ and $i = 1, 2, ..., k$;
2. $n \leq h(i)$, $1 \leq i \leq k$;
3. $\mathbb{T} \setminus \bigcup_{i=1}^{k} \bigcup_{j=0}^{h(i)-1} \alpha^j(J_i)$ is a set of finite many points.

**Proof.** Fix $x \in \mathbb{T}$ and fix an integer $n > 1$. Since $\alpha$ is minimal, there is a closed arc $Y$ containing $x$ such that $\alpha^j(J)$ are pairwise disjoint for $0 \leq j \leq n$. Set

$$r(y) = \min\{m \geq 1 : \alpha^m(y) \in Y\}.$$  

Applying 3.1 we obtain $n(0) < n(1) < \cdots n(l)$ and $Y_0, Y_1, ..., Y_l$ and $Y_0^o, Y_1^o, ..., Y_l^o$ as in Lemma 3.1. Note that $n(0) \geq n$.

Let $X_k = \{y \in Y : r(y) = n(k)\}$, $k = 0, 1, 2, ..., l$ and let $\Omega = \text{int} Y$. Note that $X_0 = Y_0$. Let

$$V_1 = \alpha^{n(0)}(\Omega) \cap \Omega.$$  

Since $\Omega$ is a non-empty ark, so is $V_1$. Set $S_1 = \alpha^{-n(0)}(V_1)$. Then $S_1$ is an open sub-ark of $Y$. Note that $S_1 = Y_0^o$.

Let

$$V_2 = \alpha^{n(1)}(\Omega) \cap \Omega.$$  

Then $V_2$ is a non-empty open ark. Set $S_2 = \alpha^{-n(1)}(V_2)$. Then $Y_0^o = S_2 \setminus X_0$. This shows that $X_1$ is a union of finitely many arcs (possibly neither open nor closed).

Let $V_3 = \alpha^{n(2)}(\Omega) \cap \Omega$. Then $V_3$ is a non-empty open ark. Set $S_3 = \alpha^{-n(2)}(V_3)$. Then $Y_2^o = S_3 \setminus (X_1 \cup X_2)$. So $X_2$ is a union of finitely many arks.

By induction, we conclude that all $X_k$ are union of finitely many arcs. It follows that $Y_k^o$ is a union of finitely many open arks and $Y_k$ is the closure of $Y_k^o$. Note that $\mathbb{T} \setminus \bigcup_{k=0}^{n} \bigcup_{j=1}^{n(k)} \alpha^j(Y_k^o)$ contains only finitely many points. The lemma then follows. \qed

A similar lemma like below for the Cantor set appeared in the proof of 4.4 of [21]. The following is a version of that for the circle. Note the function $\omega$ can not be made to be continuous on the whole space as in the Cantor set case.

**Lemma 3.3.** Let $\{I_1, I_2, ..., I_k\}$ be finitely many disjoint open arks of $\mathbb{T}$ and let $\alpha : \mathbb{T} \to \mathbb{T}$ be a homeomorphism such that $\alpha^j(I_i)$ are pairwise disjoint for $0 \leq j \leq h(i) - 1$ and $1 \leq i \leq k$. Let $F,G : \mathbb{T} \to \mathbb{T}$ be continuous maps. Then, for any $\varepsilon > 0$, and for each $i$, there are $\{I_i^{(s)} \subseteq I_i : s = 1, 2, ..., m(i)\}$ disjoint open arks and there is a continuous maps $\omega : S \to \mathbb{T}$, where $S = \bigcup_{j=0}^{h(i)-1} \bigcup_{s=1}^{m(i)} \alpha^j(I_i^{(s)})$, such that
\( \omega(x) = 0 \) if \( x \in \cup_{s,i} I_j^{(s)} \) \hfill \\
(2) \hfill \\
\left| [F(\alpha^j(x)) + \omega(\alpha^j(x))] - [\omega(\alpha^{j+1}(x)) + G(\alpha^j(x))] \right| < \frac{1}{h(i)} \quad \text{(e 3.22)} \]

for all \( x \in I_i^{(s)}, s = 1, 2, \ldots, m(i), j = 0, 1, \ldots, h(i) - 2 \) (we identify \( \mathbb{T} \) with \( \mathbb{R}/\mathbb{Z} \)) and

\( \left| [F(\alpha^{h(i)-1}(x)) + \omega(\alpha^{h(i)-1}(x))] - [G(\alpha^{h(i)-1}(x))] \right| < \frac{1}{h(i)} \) \hfill \\
\text{(e 3.23)} \]

for all \( x \in I_i^{(s)}, s = 1, 2, \ldots, m(i), i = 1, 2, \ldots, k, \) and

(4) \( I_i \setminus \cup_{s=1}^m I_j^{(s)} \) contains only finitely many points.

Moreover, on the closure of each \( I_i^{(s)} \), \( \omega \) can be extended to be a continuous function and (1), (2) and (3) remain true for \( x \) in the left-closed arks of \( I_i^{(s)} \) if inequalities are replaced by “\( \leq \)”.

Proof. We identify \( \mathbb{T} \) with \( \mathbb{R}/\mathbb{Z} \). Define

\[ \kappa(x) = \sum_{i=0}^{h(i)-1} (F(\alpha^j(x)) - G(\alpha^j(x))) \] \hfill \\
\text{(e 3.24)} \]

for \( x \in I_i, 1 \leq i \leq k \).

Let \( K = \sum_{i=1}^k h(i) \). One can break \( I_i \) into a disjoint union of finitely many arks so that the image of \( \kappa \) on each sub-ark is a subset of a proper closed subset of \( \mathbb{T} \). Thus, there are, for each \( i \), pairwise disjoint open sub-arcs \( I_i^{(s)} \subset I_i, s = 1, 2, \ldots, m(i) \), such that \( \kappa|_{I_i^{(s)}} \) has image contained in a proper subset of \( \mathbb{T} \) such that

\[ I_i \setminus \cup_{s=1}^m I_i^{(s)} \]

contains only finitely many points, \( i = 1, 2, \ldots, k \). So (4) now holds.

Define \( \tilde{\kappa}(x) \) as follows

\[ \tilde{\kappa}(x) = \kappa(x) + \mathbb{Z} \quad \text{and} \quad -1 \leq \tilde{\kappa}(x) \leq 1 \] \hfill \\
\text{(e 3.25)} \]

for \( x \in I_i^{(s)}, 1 \leq s \leq m(i), 1 \leq i \leq k \).

Put \( S = \cup_{i=1}^k \cup_{s=1}^m I_i^{(s)} \). Define \( \eta : S \to \mathbb{T} \) as follows:

\[ \eta(\alpha^j(x)) = -\frac{j}{h(i)} \tilde{\kappa}(x) + \mathbb{Z} \] \hfill \\
\text{(e 3.26)} \]

for \( x \in I_i^{(s)}, j = 0, 1, \ldots, h(i) - 1, 1 \leq i \leq k \). Since the image of \( \kappa \) on each \( I_i^{(s)} \) is a proper subset of \( \mathbb{T} \), we see that \( \eta \) is continuous.
Put $\Omega_0 = \sum_{i=1}^{k} \cup_{s=1}^{m(i)} I_i^{(s)}$. Define $\omega : S \to T$ as follows:

$$\omega(x) = 0$$  \hspace{1cm} (e 3.27)

if $x \in \Omega_0$, and

$$\omega(\alpha^j(x)) = \eta(\alpha^j(x)) + \sum_{l=0}^{j-1} \left[ F(\alpha^l(x)) - G(\alpha^l(x)) \right]$$  \hspace{1cm} (e 3.28)

for $x \in I_i^{(s)}$, $j = 1, 2, ..., h(i) - 1$, $1 \leq s \leq m(i)$, $1 \leq i \leq k$.

If $x \in I_i^{(s)}$,

$$|F(x) + \omega(x)| - |\omega(\alpha(x)) + G(x)| \leq |F(x) - \frac{1}{h(i)}(\kappa(x) + (F(x) - G(x)) + G(x))| < \frac{1}{h(i)}$$  \hspace{1cm} (e 3.29)

for $1 \leq s \leq m(i)$ and $1 \leq i \leq k$.

If $x \in \alpha^j(I_i^{(s)})$ for $j = 1, 2, ..., h(i) - 2$,

$$|F(\alpha^j(x)) + \omega(x)| - |\omega(\alpha^{j+1}(x)) + G(\alpha^j(x))| \leq |\eta(\alpha^j(x)) - \eta(\alpha^{j+1}(x))| < \frac{1}{h(i)}$$  \hspace{1cm} (e 3.30)

for $1 \leq i \leq k$.

We verify that, in $\mathbb{R}/\mathbb{Z}$,

$$|F(\alpha^{h(i)-1}(x)) + \omega(\alpha^{h(i)-1}(x))| - |G(\alpha^{h(i)-1}(x))|$$

$$= |\kappa(x) - \frac{h(i) - 1}{h(i)} \kappa(x)| < \frac{1}{h(i)}$$  \hspace{1cm} (e 3.31)

for all $x \in I_i$, $1 \leq i \leq k$.

We also note that the last statement follows easily since $F$ and $G$ are continuous functions.  \hfill $\square$

**Lemma 3.4.** Let $\sigma((\xi, \zeta)) = (\sigma_1(\xi, \zeta), \sigma_2(\xi, \zeta))$ be a Borel equivalence from $T^2$ to $T^2$ such that

$$m_2(\sigma(S)) = m_2(\sigma(S))$$  \hspace{1cm} (e 3.35)

for all Borel set $S$.

(1) Then, for any Borel set $S_1, S_2 \subset T$,

$$m(\sigma_1((S_1, \zeta))) = m(S_1) \quad \text{and} \quad m(\sigma_2((\xi, S_2))) = m(S_2)$$  \hspace{1cm} (e 3.36)

for almost all $\zeta \in T$ and almost all $\xi \in T$, 11
(2) If there exists a closed subset $F \subset T$ with $m(F) = 0$ such that $\sigma_1(-, \zeta)$ is continuous on $T$ for all $\zeta \in T$, then for each $\zeta \in T$,

$$\sigma_1(\xi, \zeta) = \xi g_1(\zeta)$$

(e 3.37)

for all $\xi \in T$ or

$$\sigma_1(\xi, \zeta) = \xi g_1(\zeta)$$

(e 3.38)

for all $\xi \in T$.

(3) If $\sigma$ is continuous, then there are continuous maps $g_1 : T \to T$ such that

$$\sigma_1(\xi, \zeta) = \xi g_1(\zeta)$$

(e 3.39)

for all $\xi, \zeta \in T$ or

$$\sigma_1(\xi, \zeta) = \xi g_1(\zeta)$$

(e 3.40)

for all $\xi, \zeta \in T$.

Proof. (1) Let $S_1 \subset T$ be a Borel subset. Suppose that there exists a measurable subset $E_1 \subset T$ with positive measure such that

$$\int_{S_1} \sigma_1(\xi, \zeta) d\xi \neq m(S_1)$$

(e 3.41)

for all $\zeta \in E_1$. Put

$$E_1^+ = \{ \zeta \in E_1 : \int_{S_1} \sigma_1(\xi, \zeta) d\xi > m(S_1) \} \quad \text{and} \quad E_1^- = \{ \zeta \in E_1 : \int_{S_1} \sigma_1(\xi, \zeta) d\xi < m(S_1) \}.$$

If $m(E_1^+) > 0$, then

$$m_2(S_1 \times E_1^+) = \int_{S_1 \times E_1^+} \sigma_1(\xi, \zeta) d\xi d\zeta > \int_{E_1^+} m(S_1) d\zeta = m_2(S_1 \times E_1^+).$$

(e 3.42)

If $m(E_1^-) > 0$, then

$$m_2(S_1 \times E_1^-) = \int_{S_1 \times E_1^-} \sigma_1(\xi, \zeta) d\xi d\zeta < \int_{E_1^-} m(S_1) d\zeta = m_2(S_1 \times E_1^-).$$

(e 3.43)

Neither could be true. The proof for the variable $\zeta$ is the same.

(2) Applying part (1), we have a measurable set $E \subset T$ with $m(E) = m(T) = 1$ such that

$$m(\sigma_1((S, \zeta))) = m(S)$$

(e 3.44)
for all Borel subsets $S \subset T$ and $\zeta \in E$. Thus, if $\zeta \in E \cap (T \setminus F)$, by (3.44), it is well known that either
\[ \sigma_1(\xi, \zeta) = \xi g_1(\zeta) \] (e 3.45)
for all $\xi \in T$ and for some $g_1(\zeta) \in T$ or
\[ \sigma_1(\xi, \zeta) = \bar{\xi} g_1(\zeta) \] (e 3.46)
for all $\xi \in T$ and for some $g_1(\zeta) \in T$.

(3) This part follows immediately from (2). By considering the subset $\{(1, \zeta) : \zeta \in T\}$ and applying (3.37) and (3.38), we conclude that $\sigma_1(1, \zeta) = \xi g_1(\zeta)$ is a continuous function.

Then, by continuity of $\sigma$, (3) follows.

The proof of Theorem 2.4

Proof. Let $\sigma((\xi, \zeta)) = (\xi, \zeta)$. Then $\sigma : T^2 \to T^2$ is a homeomorphism and $\sigma^{-1} = \sigma$. One has
\[ \sigma^{-1} \circ \Phi_{\theta, 1, 0} \sigma((\xi, \zeta)) = \sigma^{-1}(\xi e^{2\pi i \theta}, \zeta) \]
\[ = (\xi^{-i 2\pi \theta}, \zeta^{-1}) \]
for all $\xi, \zeta \in T$. It follows that $\Phi_{\theta, 1, 0}$ and $\Phi_{-\theta, -1, 0}$ are conjugate.

We will show that if $\theta_1 \pm \theta_2 = 0$ in $\mathbb{R}/\mathbb{Z}$, then (3) in 2.4 holds. For the convenience, we will say $\alpha$ and $\beta$ are approximately conjugate in the sense (M2) if (3) holds. In this part of the proof, we will identify $T$ with $\mathbb{R}/\mathbb{Z}$.

Let $\theta \in (0, 1)$ be an irrational number. Since $\Phi_{\theta, 1, 0}$ and $\Phi_{-\theta, -1, 0}$ are conjugate as shown above, it suffices to show that $\alpha$ and $\beta$ are approximately conjugate in the sense (M2) if
\[ \alpha = \Phi_{\theta, d_1, f_1} \] and $\beta = \Phi_{\theta, d_2, f_2}$.

Let $\epsilon > 0$. Choose $n > 0$ so that $1/n < \epsilon$. Let $J_1, J_2, \ldots, J_k$ be the open arks provided by Lemma 3.2 with the integer $n$ and $(\alpha(t) = t + \theta$ for $t \in \mathbb{R}/\mathbb{Z}$) Put $F(\xi) = \xi^{d_1} \exp(i \pi f_1(\xi))$ and $G(\xi) = \xi^{d_2} \exp(i \pi f_2(\xi))$ for $\xi \in T$. Let $\omega$ be the function in Lemma 3.3 with $\alpha(t) = t + \theta$ and $I_i = J_i, i = 1, 2, \ldots, k$. By extending $\omega$ continuously on the left-sided closed arks $J_i^{(s)}$ for each $s$ and $i$, as the last part of Lemma 3.3, one has
\[ |[F(x) + \omega(x)] - [\omega(x + \theta) + G(x)]| \leq \epsilon \]
for all $x \in \mathbb{R}/\mathbb{Z}$. Define
\[ \sigma((x, t)) = (x, t + \omega(x)) \] for $t, x \in \mathbb{R}/\mathbb{Z}$.

Therefore,
\[ m_2(\sigma(S)) = m_2(S) \]
for all Borel set $S \subset T^2$.  

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Moreover, $\sigma$ is continuous except finitely many circles (with the form of $B \times T$, where $B$ is a finite subset of $T$).

Now, by (3.49),

$$\text{dist}(\alpha \circ \sigma((x, t)), \sigma \circ \beta((x, t))) \leq \|F(x) + \omega(x) - [\omega(x + \theta) + G(x)]\| < \varepsilon$$

(e 3.50)

for all $x, t \in \mathbb{R}/\mathbb{Z}$.

For the converse, let $\theta_1$ and $\theta_2$ be two irrational numbers such that $|\theta_1 \pm \theta_2| \neq 0$ in $\mathbb{R}/\mathbb{Z}$.

Since we have shown that $\Phi_{\theta_1,d_1,f_1}$ and $\Phi_{\theta_1,1,0}$ are approximately conjugate in the sense (M2) above, and $\Phi_{\theta_2,d_2,f_2}$ and $\Phi_{\theta_2,1,0}$ are approximately conjugate in the sense (M2) above, respectively, it suffices to show that $\Phi_{\theta_1,1,0}$ and $\Phi_{\theta_2,1,0}$ are not approximately conjugate in the sense (M2).

Put $\alpha = \Phi_{\theta_1,1,0}$ and $\beta = \Phi_{\theta_2,1,0}$. Put

$$a = |e^{2\pi(\theta_1 - \theta_2)} - 1| > 0 \quad \text{and} \quad b = |e^{2\pi(\theta_1 + \theta_2)} - 1| > 0.$$  

(e 3.51)

Let $\varepsilon > 0$ such that $\varepsilon < \min\{a/2, b/2\}$. Suppose that there exists $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$\sup\{\text{dist}(\alpha \circ \sigma(x), \sigma \circ \beta(x)) : x \in \mathbb{T}^2\} < \varepsilon$$

(e 3.52)

and $m_2(\sigma(S)) = m_2(S)$ for all Borel sets $S \subset \mathbb{T}^2$.

By (3.4) there exists a Borel set $E \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus E) = 0$ and for each $\zeta \in E$,

$$\sigma_1((\xi, \zeta)) = \xi g_1(\zeta), \quad \text{or} \quad \sigma_1((\xi, \zeta)) = \bar{\xi} g_1(\zeta),$$

(e 3.53)

for all $\xi \in \mathbb{T}$. Thus, for $\zeta \in E$, by (e 3.52), we have

$$|\xi g_1(\zeta) e^{i2\pi \theta_1} - \xi g_1(\zeta) e^{i2\pi \theta_2}| < \varepsilon,$$

(e 3.54)

$$|\bar{\xi} g_1(\zeta) e^{i2\pi \theta_1} - \bar{\xi} g_1(\zeta) e^{-i2\pi \theta_2}| < \varepsilon,$$

(e 3.55)

$$|\xi g_1(\zeta) e^{i2\pi \theta_1} - \xi g_1(\zeta) e^{-i2\pi \theta_2}| < \varepsilon$$

or

(e 3.56)

$$|\bar{\xi} g_1(\zeta) e^{i2\pi \theta_1} - \bar{\xi} g_1(\zeta) e^{i2\pi \theta_2}| < \varepsilon$$

(e 3.57)

for all $\xi \in \mathbb{T}$.

Choose $\xi = 1$, for all $\zeta \in E$, one computes that either

$$a = |g_1(\zeta) e^{2i\pi \theta_1} - e^{2i\pi \theta_2} g_1(\zeta)| < \varepsilon$$

(e 3.58)

$$b = |g_1(\zeta) e^{2i\pi \theta_1} - e^{-2i\pi \theta_2} g_1(\zeta)| < \varepsilon$$

(e 3.59)

By (e 3.51), neither is possible. \qed
The Proof of Theorem 2.3

Proof. (1) ⇒ (2): We will modify the relevant part of the proof of Theorem 2.4. Let \( \varepsilon > 0 \). By the proof of Theorem 2.4 there exists a finite subset \( B \subset \mathbb{T} \) and a function \( \omega : \mathbb{T} \to \mathbb{T} \) which is continuous on \( \mathbb{T} \setminus B \) such that

\[
\text{dist}(\sigma \circ \alpha((x,t)), \beta \circ \sigma((x,t))) < \varepsilon
\]

for all \( x,t \in \mathbb{R}/\mathbb{Z} \), where \( \sigma((x,t)) = (x, t + \omega(x)) \) for all \( x,t \in \mathbb{R}/\mathbb{Z} \). There is an open subset \( G \subset \mathbb{T} \) such that \( B \subset G \) and

\[
m(G) < \varepsilon
\]

where \( \omega \) is a continuous function \( \omega_0 \) from \( \mathbb{Z} \) to \( \mathbb{R}/\mathbb{Z} \) such that

\[
\omega_0(x) = \omega(x) \quad \text{for} \quad x \in \mathbb{T} \setminus G.
\]

Now define \( \sigma_0((x,t)) = (x, t + \omega_0(x)) \) for \( x,t \in \mathbb{R}/\mathbb{Z} \). Then

\[
\{(x,t) : \text{dist}(\sigma_0 \circ \alpha((x,t)), \beta \circ \sigma_0((x,t))) \geq \varepsilon\} \subset \mathbb{T} \times G.
\]

It follows that

\[
m_2(\{(x,t) : \text{dist}(\sigma_0 \circ \alpha((x,t)), \beta \circ \sigma_0((x,t))) \geq \varepsilon\}) < \varepsilon.
\]

This proves (1) ⇒ (2).

To see (2) ⇒ (1), let \( \theta_1 \) and \( \theta_2 \) be two irrational numbers such that \( \overline{\theta_1 + \theta_2} \neq \mathbb{Q} \) in \( \mathbb{R}/\mathbb{Z} \).

Since we have shown that \( \Phi_{\theta_1,\theta_1}, \Phi_{\theta_2,\theta_2} \) and \( \Phi_{\theta_1,1,0}, \Phi_{\theta_2,1,0} \) are approximately conjugate in the sense of (M1), and \( \Phi_{\theta_1,\theta_2}, \Phi_{\theta_2,\theta_1} \) are approximately conjugate in the sense of (M1), respectively, it suffices to show that \( \Phi_{\theta_1,1,0}, \Phi_{\theta_2,1,0} \) are not approximately conjugate in the sense of (M1).

Put \( \alpha = \Phi_{\theta_1,1,0} \) and \( \beta = \Phi_{\theta_2,1,0} \). Suppose that \( \sigma : \mathbb{T}^2 \to \mathbb{T}^2 \) is a homeomorphism such that

\[
m_2(\sigma(S)) = m_2(S).
\]

Write \( \sigma((\xi,\eta)) = (\sigma_1(\xi,\eta), \sigma_2(\xi,\eta)) \). It follows from (3) of \ref{3.4} that

\[
either \sigma_1(\xi,\eta) = \xi g_1(\eta), \quad \text{or} \quad \sigma_1(\xi,\eta) = \xi g_2(\eta)
\]

for all \( \xi, \eta \in \mathbb{T} \), where \( g_1 : \mathbb{T} \to \mathbb{T} \) is a continuous map. Put

\[
a = |e^{2\pi i(\theta_1 - \theta_2)} - 1| > 0 \quad \text{and} \quad b = |e^{2\pi i(\theta_1 + \theta_2)} - 1| > 0.
\]

Let \( 0 < \varepsilon < \min\{1/4, a/4, b/4\} \). Let \( z \in C(\mathbb{T}) \) be defined by \( z((\xi,\eta)) = \xi \) for \( \xi \in \mathbb{T} \). If \( \alpha \) and \( \beta \) are approximately conjugate in the sense (M1), then there exists \( \sigma \) described above such that

\[
\int_{\mathbb{T}} \int_{\mathbb{T}} |z(\alpha \circ \sigma(\xi,\eta) - z(\sigma \circ \beta(\xi,\eta)))| d\eta d\xi < \varepsilon^4/4.
\]
One then computes that
\[
\int_T |z(\alpha \circ \sigma(\zeta, \zeta)) - z(\sigma \circ \beta(\zeta, \zeta))|d\xi < \varepsilon^2
\] (e 3.69)
for all \(\zeta \in E\), where \(E\) is a measurable set such that \(m(E) > 1 - \varepsilon\).

We now assume that \(\sigma_1(\xi, \zeta) = \xi g_1(\zeta)\) for all \(\xi, \zeta \in \mathbb{T}\). For \(\zeta \in E\),
\[
\varepsilon^2 > \int_T |\xi g_1(\zeta)e^{i2\pi\theta_1} - \xi e^{i2\pi\theta_2}g_1(\zeta)|d\xi \quad \text{ (e 3.70)}
\]
\[
= \int_T |g_1(\zeta)e^{i2\pi(\theta_1 - \theta_2)} - g_1(\zeta)|d\xi. \quad \text{ (e 3.71)}
\]

By considering the constant function \(F_1(\zeta) = g_1(\zeta)e^{i2\pi(\theta_1 - \theta_2)}\) and the function \(F_2(\zeta) = g_1(\zeta)\) and by translating by \(\zeta\), we obtain
\[
\int_E |g_1(\zeta)e^{i2\pi(\theta_1 - \theta_2)} - g_1(\zeta)|d\xi < \varepsilon^2 \quad \text{ (e 3.72)}
\]
for all \(\zeta \in E\). Fix \(\zeta_0 \in E\) and let
\[
E_{\zeta_0} = \{\xi \in |g_1(\zeta_0)e^{i2\pi(\theta_1 - \theta_2)} - g_1(\xi)| < \varepsilon\}.
\]
Then by (e 3.72), we compute that \(m(E_{\zeta_0}) > 1 - \varepsilon\). Therefore \(m(E_{\zeta_0} \cap E) > 0\). If \(\zeta_1 \in E_{\zeta_0} \cap E\), by (e 3.72), we have that
\[
\varepsilon^2 > \int_T |g_1(\zeta_1)e^{i2\pi(\theta_1 - \theta_2)} - g_1(\zeta)|d\xi \quad \text{ (e 3.73)}
\]
\[
\geq |g_1(\zeta_0)e^{i2\pi(\theta_1 - \theta_2)} - g_1(\zeta_1)e^{i4\pi(\theta_1 - \theta_2)}|
\quad \text{ (e 3.74)}
\]
\[
- |g_1(\zeta_0)e^{i4\pi(\theta_1 - \theta_2)} - g_1(\zeta_1)e^{i2\pi(\theta_1 - \theta_2)}| - \int_T |g_1(\zeta_0)e^{i2\pi(\theta_1 - \theta_2)} - g_1(\xi)|d\xi \quad \text{ (e 3.75)}
\]
\[
> |e^{i2\pi(\theta_1 - \theta_2)} - 1| - \varepsilon - \varepsilon^2 > a/2. \quad \text{ (e 3.76)}
\]
By the choice of \(\varepsilon\), this is impossible.

Now we assume that \(\sigma_1(\xi, \zeta) = \xi g_1(\zeta)\) for all \(\xi, \zeta \in \mathbb{T}\). As above, one has
\[
\varepsilon^4 > \int_T |\xi g_1(\zeta)e^{i2\pi\theta_1} - \xi e^{-i2\pi\theta_2}g_1(\zeta)|d\xi \quad \text{ (e 3.77)}
\]
\[
= \int_T |g_1(\zeta)e^{i2\pi(\theta_1 + \theta_2)} - g_1(\zeta)|d\xi. \quad \text{ (e 3.78)}
\]

The same argument used above leads us to
\[
\varepsilon^2 > |e^{i2\pi(\theta_1 + \theta_2)} - 1| - \varepsilon - \varepsilon^2 > \frac{b}{2}. \quad \text{ (e 3.79)}
\]
This would violate the choice of \(\varepsilon\).
4 Approximate $K$-Conjugacy

Lemma 4.1. Let $\theta \in [0, 1]$ be an irrational number and let
\[ V = \{ a \sin kt + b \cos mt : a, b \in \mathbb{R}, k, m \in \mathbb{Z}, t \in [0, 2\pi] \}. \]
Then, for every $f \in V$, there exists $g \in V$ such that
\[ f(t) = g(t) - g(t + \theta) \quad \text{for all} \quad t \in [0, 2\pi]. \] (e 4.80)

Proof. Put
\[ V_0 = \{ f(t) = g(t) - g(t + \theta) : g \in V \}. \]
It is clear that $V_0$ is a (real) vector space.

One has two elementary inequalities: for any integer $n > 1$,
\[ | \sum_{k=1}^{n} \sin k\theta | = \left| \frac{\cos \frac{\theta}{2} - \cos (n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}} \right| \leq \frac{1}{\sin \frac{\theta}{2}} \] (e 4.81)
and similarly
\[ | \sum_{k=1}^{n} \cos k\theta | \leq \frac{1}{\sin \frac{\theta}{2}}. \] (e 4.82)

Now, for any integer $m \in \mathbb{Z}$,
\[ | \sum_{k=0}^{n} (mt + k\theta) | = | \cos mt (\sum_{k=0}^{n} \sin k\theta) + \sin mt (\sum_{k=0}^{n} \cos k\theta) | \] (e 4.83)
\[ \leq | \sum_{k=0}^{n} \sin k\theta | + | \sum_{k=0}^{n} \cos k\theta | \] (e 4.84)
\[ \leq 1 + \frac{2}{\sin \frac{\theta}{2}} \] (e 4.85)
for all $t \in \mathbb{R}$. Now, since $t \mapsto t + \theta$ ($t \in \mathbb{R}/\mathbb{Z}$) is a minimal homeomorphism on $\mathbb{T}$, by a lemma of Furstenberg (Lemma 5.2 of [1]), there is $g \in C(\mathbb{T})$ (real) such that $\sin mt = g(t) - g(t + \theta)$ (for $t \in [0, 2\pi]$). It follows that $\sin mt \in V_0$. Similarly, $\cos mt \in V_0$. Since $V_0$ is a real vector space, $V \subset V_0$. \hfill \Box

The above lemma can be proved directly by some trigonometric identities and the function $g$ in the proof may be chosen to be in $V$. 

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**Lemma 4.2.** Let \( \theta \in [0, 1] \) be an irrational number and let \( f \in C(\mathbb{T}) \) be a real function. Then, for any \( \varepsilon > 0 \), there exists a continuous map \( g : \mathbb{T} \to \mathbb{T} \) such that
\[
|\exp(i2\pi f(\xi))g(\xi)e^{i2\pi \theta}) - 1| < \varepsilon
\]
for all \( \xi \in \mathbb{T} \).

Moreover, if \( d \neq 0 \) is an integer, \( g \) may be chosen to have the form
\[
g(\xi) = \xi^{kd} \exp(i2\pi g_0(\xi)),
\]
where \( k \in \mathbb{Z} \) and \( g_0 \in C(\mathbb{T}) \) is a real function.

**Proof.** Note that \( \mathbb{Z} + \mathbb{Z}(\theta) \) is dense in \( \mathbb{R} \). Thus \( \mathbb{Z} + \mathbb{Z}(\theta) \) is dense in \( \mathbb{R}/\mathbb{Z} \). Therefore, for any \( a \in \mathbb{R} \), there exists an integer \( k \in \mathbb{Z} \) such that
\[
|e^{i2\pi a}e^{-i2\pi k\theta} - 1| < \varepsilon \quad (e.4.87)
\]
Hence
\[
|e^{i2\pi a}e^{i2\pi kt}e^{-i2\pi k(t+\theta)} - 1| = |e^{i2\pi a}e^{-i2\pi k\theta} - 1| < \varepsilon \quad (e.4.88)
\]
for all \( t \in [0, 1] \).

Let \( V \) be as in the proof of 4.1 and let \( f_0 \in V \). Applying 4.1 and choose a real \( g_0 \in C(\mathbb{T}) \) such that
\[
f_0(t) = g_0(t + \theta) - g_0(t) \quad (e.4.89)
\]
for all \( t \in \mathbb{R}/\mathbb{Z} \). Let \( f = a + f_0 \) and \( g(t) = \exp(i2\pi(kt + g_0(t))) \) for \( t \in \mathbb{R}/\mathbb{Z} \). Then
\[
|\exp(i2\pi f(t))g(t)g(t+\theta) - 1| \quad (e.4.90)
\]
\[
= |\exp(i2\pi f(t))\exp(i2\pi(kt + g_0(t))\exp(-i2\pi[k(t+\theta) + g_0(t+\theta)]) - 1| \quad (e.4.91)
\]
\[
= |\exp(i2\pi(a + f_0))\exp(i2\pi(-k\theta + g_0(t) - g_0(t+\theta))) - 1| \quad (e.4.92)
\]
\[
= |\exp(i2\pi(a - k\theta)) - 1| < \varepsilon \quad (e.4.93)
\]
for all \( t \in \mathbb{R}/\mathbb{Z} \).

By the Stone–Weierstrass theorem, the set of real trigonometric polynomials is dense in the real part of \( C(\mathbb{T}) \). Thus the first part of the lemma follows.

To see the last part of the lemma, we only need to note that \( d\theta \) is also an irrational number and \( \mathbb{Z}d\theta \) is dense in \( \mathbb{R}/\mathbb{Z} \).

**Proof of 2.10**

**Proof.** That (2) ⇔ (4) follows the computation in Example 4.9 of [23] (see 2.7) and (1) ⇔ (4) follows from the classification theorem in [12] as mentioned at the end of 2.9. It is also clear that (3) ⇒ (5).

It remains to show (2) ⇒ (3) and (5) ⇒ (2).
We will first show (2) ⇒ (3).

As in the proof of 2.3, Φ_{θ,d,0} and Φ_{−θ,−d,0} are conjugate. Define σ((ξ, ζ)) = (ξ, ζ). Then σ : T^2 → T^2 is a homeomorphism and σ^{-1} = σ. One check that

\[ \sigma^{-1} \circ \Phi_{θ,d,0} \circ \sigma((ξ, ζ)) = \sigma^{-1}((ξe^{2iπθ}, ζ)) = (ξe^{2iπθ}, ζ^{-1}) = \Phi_{θ,−d,0}(ξ, ζ) \] (e 4.95)

for all ξ, ζ ∈ T. Therefore Φ_{θ,d,0} and Φ_{−θ,−d,0} are conjugate. Combining with the fact that Φ_{θ,d,0} and Φ_{−θ,−d,0} are conjugate that mentioned above, we conclude that Φ_{θ,d,0} and Φ_{−θ,−d,0} are conjugate. Thus, to complete the proof, it suffices to show that α = Φ_{θ,d,f,1} and β = Φ_{θ,d,f,2} are approximately K-conjugate for any real continuous Lipschitz functions f_1 and f_2.

It follows from Theorem 4.6 of [18] that both A_α and A_β have tracial rank zero. By the K-theory computation in 2.7, there is an order isomorphism

κ : (K_0(A_α), K_0(A_α)_+, [1_{A_α}], K_1(A_α)) → (K_0(A_β), K_0(A_β)_+, [1_{A_β}], K_1(A_β))

such that κ([u_α]) = [u_β]. By the classification theorem [12], there exists a unital isomorphism φ : A_α → A_β such that

\[ [φ] = κ. \]

Let f(ξ) = f_2(ξ) − f_1(ξ) for ξ ∈ T. Fix δ > 0. By applying [12] we obtain

\[ g(ξ) = ξ^{kd} \exp(2iπg_0(ξ)) \] (e 4.96)

for ξ ∈ T, where g_0 ∈ C(T) is a real function such that

\[ |\exp(2iπf(ξ))g(ξ)\bar{g(ξ)e^{2iπθ}}| < δ \] (e 4.97)

for all ξ ∈ T. Define

\[ σ((ξ, ζ)) = (ξ, ζg(ξ)) \] (e 4.98)

for all (ξ, ζ) ∈ T^2. Then

\[ \sigma \circ α((ξ, ζ)) = (ξe^{2iπθ}, ζg(ξ)e^{2iπf_1(ξ)}(ξe^{2iπθ})) \] and

\[ β \circ σ((ξ, ζ)) = (ξe^{2iπθ}, ζg(ξ)e^{2iπf_2(ξ)}) \] (e 4.100)

for all ξ, ζ ∈ T. Using (e 4.100), we estimate that

\[ \text{dist}(σ \circ α((ξ, ζ)), β \circ σ((ξ, ζ))) < δ \] (e 4.101)

for all (ξ, ζ) ∈ T^2.

Note that σ is homotopic to Φ_{θ,kd,g_0}. As the computation in Example 4.9 of [23] (see [27]), we have σ_{α_0} = id_{A_0} on K_0(C(T^2)) and σ_{α_1} on K_1(C(T^2)) ≅ Z ⊕ Z is represented by the matrix

\[ \begin{pmatrix} 1 & kd \\ 0 & 1 \end{pmatrix} \] .
It induces the identity map from $\mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}$ onto $\mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}$. Therefore

$$h_{\ast i} = (\phi \circ j_{\alpha})_{\ast i} = (j_{\beta})_{\ast i}, \; i = 0, 1,$$

where $h : C(T^2) \to A_\beta$ is defined by $h(f) = \phi \circ j_{\alpha}(f \circ \sigma)$ for $f \in C(T^2)$. Since $K_i(C(T^2))$ is free, we, in fact, have that

$$[h] = [\phi \circ j_{\alpha}] = [j_{\beta}] \text{ in } KL(C(T^2), A_\beta).$$

We also note that

$$\tau \circ h(f) = \tau \circ \phi \circ j_{\alpha}(f) = \tau \circ j_{\beta}(f)$$

for all $f \in C(T^2)$, where $\tau$ is the unique tracial state on $A_\beta$.

Let

$$F_1 = \mathcal{F} \cup \{f \circ \sigma^{-1} : f \in \mathcal{F}\} \cup \{f(\sigma \circ \alpha \sigma^{-1}) : f \in \mathcal{F}\}.$$

By (e 4.103) and (e 4.104), and by Theorem 3.4 of [14], there exists a unitary $W \in A_\beta$ such that

$$W j_{\beta}(f) W^* \approx_{\varepsilon/3} \phi \circ j_{\alpha}(f \circ \sigma) \text{ on } F_1.$$

In particular, if $f \in \mathcal{F}$,

$$W^* \phi \circ j_{\alpha}(f) W \approx_{\varepsilon/3} j_{\beta}(f \circ \sigma^{-1}) \text{ and }$$

$$W^* \phi \circ j_{\alpha}(f(\sigma \circ \alpha)) W \approx_{\varepsilon/3} j_{\beta}(f(\sigma \circ \alpha \circ \sigma^{-1})).$$

Therefore

$$W^* \phi(u_{\alpha}^*) W j_{\beta}(f) W^* \phi(u_{\alpha}) W \approx_{\varepsilon/3} W^* \phi(u_{\alpha}^*) \phi \circ j_{\alpha}(f \circ \sigma) \phi(u_{\alpha}) W$$

$$= W^* \phi \circ j_{\alpha}(f \circ \sigma \circ \alpha) W$$

$$\approx_{\varepsilon/3} j_{\beta}(f \circ \sigma \circ \alpha \circ \sigma^{-1})$$

for all $f \in \mathcal{F}$. It follows that, with sufficiently small $\delta$,

$$\text{ad}(W^* \phi(u_{\alpha}) W) \circ (j_{\beta}(f)) \approx_{\varepsilon} j_{\beta}(f \circ \beta)$$

for all $f \in \mathcal{F}$. Put $z = u_{\beta}^*(W^* \phi(u_{\alpha}) W)$. Then, since $[\phi(u_{\alpha})] = [u_{\beta}]$, $z \in U_0(A_\beta)$ and

$$j_{\beta}(f) z \approx_{\varepsilon} z j_{\beta}(f)$$

for all $f \in \mathcal{F}$. This shows that $\alpha$ and $\beta$ are approximately $K$-conjugate.
Now we consider (5) \(\Rightarrow\) (2). Suppose that \(\alpha = \Phi_{\theta_1,d_1,f_1}\) and \(\beta = \Phi_{\theta_2,d_2,f_2}\) are two Furstenberg transformations. Suppose that there exist sequences of homeomorphisms \(\{\sigma_n\}\) and \(\{\gamma_n\}\) on \(T^2\) such that
\[
\lim_{n \to \infty} \sup \{ \text{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}((\xi,\zeta)), \beta((\xi,\zeta))) : (\xi,\zeta) \in T^2 \} = 0 \tag{e 4.113}
\]
\[
\lim_{n \to \infty} \sup \{ \text{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}((\xi,\zeta)), \alpha((\xi,\zeta))) : (\xi,\zeta) \in T^2 \} = 0, \tag{e 4.144}
\]
\[
m_2(\sigma_n(S)) = m_2(S) \quad \text{and} \quad m_2(\gamma_n(S)) = m_2(S) \tag{e 4.155}
\]
for all Borel sets \(S \subset T^2\). It follows from (2) that \(\vartheta_1 \pm \vartheta_2 = 0\) in \(\mathbb{R}/\mathbb{Z}\).

Write
\[
\sigma_n((\xi,\zeta)) = (G_1^{(n)}((\xi,\zeta)), G_2^{(n)}((\xi,\zeta))
\]
for all \((\xi,\zeta) \in T^2\). It follows from (3), (4) that there are continuous maps \(g_1, g_2 : T \to T\) such that
\[
G_1^{(n)}((\xi,\zeta)) = \tilde{\zeta}g_1^{(n)}(\zeta) \quad \text{and} \quad G_2^{(n)}((\xi,\zeta)) = \tilde{\zeta}g_2^{(n)}(\zeta) \tag{e 4.166}
\]
for all \(\xi, \zeta \in T\), where \(\tilde{\zeta} = \xi \) for all \(\xi \in T\) or \(\tilde{\zeta} = \xi\) for all \(\zeta \in T\), \(\tilde{\zeta} = \zeta\) for all \(\zeta \in T\) or \(\tilde{\zeta} = \zeta\) for all \(\zeta \in T\). We have
\[
\sigma_n \circ \alpha((\xi,\zeta)) = (\tilde{\xi}e^{\pm 2i\pi \theta_1}g_1^{(n)}(\xi)\xi \xi f_1(\xi), \tilde{\zeta} \xi \xi e^{\pm 2i\pi f_1(\xi)}g_2^{(n)}(\xi)\xi e^{2i\pi \theta_1}) \tag{e 4.177}
\]
and
\[
\beta \circ \sigma_n((\xi,\zeta)) = (\tilde{\xi}g_1^{(n)}(\xi)\xi \xi e^{2i\pi \theta_2}, \tilde{\zeta}g_2^{(n)}(\xi)\xi \xi f_2(\xi)\xi e^{2i\pi f_2(\xi)}g_1^{(n)}(\zeta)) \tag{e 4.188}
\]
for all \((\xi,\zeta) \in T\).

We compute that, for all sufficiently large \(n\),
\[
|\xi^{\pm d_1 \pm d_2} e^{2i\pi (f_2(\tilde{\xi}g_1^{(n)}(\zeta)) \pm f_1(\xi))} g_2^{(n)}(\xi e^{2i\pi \theta_2}) g_1^{(n)}(\zeta) \xi^{d_2} - 1| < 1/2 \tag{e 4.119}
\]
for all \(\xi, \zeta \in T\). Fix \(\zeta \in T\) and let \(\xi\) vary in \(T\). It follows that, for fixed \(\zeta \in T\),
\[
|\xi^{\pm d_1 \pm d_2} e^{2i\pi (f_2(\xi)\xi g_1^{(n)}(\xi)) \pm f_1(\xi)} g_2^{(n)}(\xi e^{2i\pi \theta_2}) g_1^{(n)}(\zeta) \xi^{d_2}|
\]
is homotopically trivial as a unitary in \(C(T)\). Since \(g_2^{(n)}(\xi)g_1^{(n)}(\xi e^{2i\pi \theta})\) is homotopically trivial, (e 4.111) implies that \(\xi^{\pm d_1 \pm d_2}\) is homotopically trivial. However, that can only happen when \(|d_1| = |d_2|\).

\[\Box\]

**Corollary 4.3.**

\[V_1 = \{m_1 \theta + m_2 + a \cos m_2 t + b \sin m_4 t : m_1, m_2, m_3, m_4 \in \mathbb{Z}, a, b \in \mathbb{R}, t \in [0, 2\pi]\}.\]

Let \(\alpha = \Phi_{\theta,d,f_1}\) and \(\beta = \Phi_{\theta,d,f_2}\) such that \(f_1 - f_2 \in V_1\). Then \(\alpha\) and \(\beta\) are conjugate. Note that \(V_1\) is dense in the real part of \(C(T)\).
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