A TROPICAL ANALOGUE OF FAY’S TRISECANT IDENTITY
AND THE ULTRA-DISCRETE PERIODIC TODA LATTICE

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Abstract. We introduce a tropical analogue of Fay’s trisecant identity for a special
family of hyperelliptic tropical curves. We apply it to obtain the general solution of the
ultra-discrete Toda lattice with periodic boundary conditions in terms of the tropical
Riemann’s theta function.

1. Introduction

1.1. Background and main results. Fay’s trisecant identity is an important and special
identity satisfied by Riemann’s theta functions for the Jacobian of a curve [1, 12], and plays
a crucial role in studying classical integrable systems. For instance, Fay’s trisecant identity
gives a solution to the Hirota-Miwa equation [2, 11] from which many soliton equations
derive.

Recently a tropical analogue of Riemann’s theta function was introduced in some con-
texts ([8, 10], in [14] the tropical Riemann’s addition formula is introduced). The first aim
of this paper is to introduce a tropical version of Fay’s trisecant identity (Theorem 2.4).
Due to a technical difficulty, our result is restricted to the case of some special hyperelliptic
tropical curves, but we expect that it also holds for general tropical curves.

On the other hand, in [4] we studied the ultra-discrete periodic Toda lattice (UD-pToda)
and proposed a method to deal with integrable cellular automata via the tropical algebraic
geometry. Since the UD-pToda is closely related to an important soliton cellular automata
called the box and ball system [15], it is regarded as a fundamental object in the studies
of integrable automata. The UD-pToda is defined by the following piecewise-linear map

\[ Q_{n+1}^t = \min[W_n^t, Q_n^t - X_n^t], \quad W_{n+1}^t = Q_{n+1}^t + W_n^t - Q_{n+1}^t, \]

with \( X_n^t = \min_{k=0, \ldots, g} \left[ \sum_{l=1}^{k} (W_{n-l}^t - Q_{n-l}^t) \right] \),

(1.1)
on the phase space:

\[ \mathcal{T} = \{(Q, W) = (Q_1, \ldots, Q_g, W_1, \ldots, W_g) \in \mathbb{R}^{2(g+1)} \mid \sum_{n=1}^{g+1} Q_n < \sum_{n=1}^{g+1} W_n \}. \]  

(1.2)

Here we fix \( g \in \mathbb{Z}_{>0} \) and assume periodicity \( Q_{n+g+1}^t = Q_n^t \) and \( W_{n+g+1}^t = W_n^t \). This
system has a tropical spectral curve \( \Gamma \), and we conjectured that its general isolevel set is
isomorphic to the tropical Jacobian \( J(\Gamma) \) of \( \Gamma \) (the cases of \( g = 1, 2, 3 \) were proved). It is
expected that the solution for the UD-pToda is written in terms of the tropical Riemann’s
theta function associated with $\Gamma$ as the classical cases, which is another aim of this paper. For this purpose we transform (1.1) into an equation for the quasi-periodic function $T^t_n$,

$$T^t_{n+2} + T^t_n = \min_{k=0,\ldots,g} \left[ kL + 2T^t_{n-k} + T^t_{n+1} + T^t_n - (T^t_{n-k+1} + T^t_{n-k}) \right],$$

(1.3)

via

$$W^t_n = L + T^t_{n-1} + T^t_{n+1} - T^t_n + T^t_{n+1} + C_g,$$

$$Q^t_n = T^t_{n-1} + T^t_{n+1} - T^t_n - T^t_n + C_g.$$ (1.4)

Here $L$ and $C_g$ are determined by $\{Q^t_n, W^t_n\}_{n=1,\ldots,g+1}$ and preserved by the map (1.1). We show that the equation (1.3) is essentially equivalent to a tropical bilinear equation (Proposition 3.4):

$$T^t_{n-1} + T^t_{n+1} = \min \left[ 2T^t_n, T^t_{n-1} + T^t_{n+1} + L \right],$$

(1.5)

which is turned out to be a particular case of Fay’s trisecant identity for the tropical Riemann’s theta function (Corollary 2.13). Finally we obtain the general solution for the UD-pToda in terms of the tropical Riemann’s theta function (Theorem 3.5).

In the following three subsections, we introduce some fundamental notions related to tropical geometry used in this article and the background results on the relation between the discrete periodic Toda lattice (D-pToda) and the UD-pToda.

1.2. Tropical Jacobian.

**Definition 1.1.** A finite connected graph $\Sigma \hookrightarrow \mathbb{R}^2$ is called a (plane) tropical curve if the weight $w_e \in \mathbb{Z}_{>0}$ is defined for each edge $e$ and the following is satisfied:

(i) The tangent vector of each edge is rational.

(ii) For each vertex $v$, let $e_1, \ldots, e_n$ be the oriented edges outgoing from $v$. Then the primitive tangent vectors $\xi_{e_k}$ of $e_k$ satisfy $\sum_{k=1}^n w_{e_k} \xi_{e_k} = 0$.

Further, a tropical curve $\Sigma$ is called smooth if the following is satisfied:

(iii) All the weights are 1.

(iv) Each vertex $v$ is 3-valent and the primitive tangent vectors satisfy $|\xi_{e_i} \wedge \xi_{e_j}| = 1$ for $i \neq j \in \{1,2,3\}$.

Let $\tilde{\Gamma}$ be a smooth tropical curve, $\Gamma := \tilde{\Gamma} \backslash \{\text{infinite edges}\}$ be the maximal compact subset of $\tilde{\Gamma}$, $g$ be the genus of $\Gamma$, i.e. $g = \dim H_1(\Gamma, \mathbb{Z})$, and $B_1, \ldots, B_g$ be a basis of $H_1(\Gamma, \mathbb{Z})$.

Following [10] §3.3, we equip $\Gamma$ with the structure of a metric graph. For points $x, y$ on some edge $e$ in $\Gamma$, we define the weighted distance $d(x, y)$ by

$$d(x, y) = \frac{\| x - y \|}{\| \xi_e \|},$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^2$. With this distance the tropical curve $\Gamma$ becomes a metric graph. The metric on $\Gamma$ defines a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the space of paths on $\Gamma$ as follows: for a non-self-intersecting path $\rho$, set $\langle \rho, \rho \rangle := \text{length}_q(\rho)$, and extending it to any pairs of paths bilinearly.
Definition 1.2. \[6.1\] The tropical Jacobian of $\Gamma$ is a $g$ dimensional real torus defined as

$$J(\Gamma) = H_1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z}) \simeq \mathbb{R}^g / K \mathbb{Z}^g.$$ 

Here $K \in M_g(\mathbb{R})$ are given by

$$K_{ij} = \langle B_i, B_j \rangle. \quad (1.6)$$

Since $\langle \cdot, \cdot \rangle$ is nondegenerate, $K$ is symmetric and positive definite. In particular, $J(\Gamma)$ is called principally polarized.

Fix $P_0 \in \Gamma$. Let $\text{Div}(\Gamma)$ be the set of divisors on $\Gamma$. Following [10] we define the tropical Abel-Jacobi map $\eta : \text{Div}(\Gamma) \to J(\Gamma)$ by

$$\sum_i k_i P_i \mapsto \sum_i k_i \int_{P_0}^{P_i} := \sum_i k_i(\langle \Gamma_{P_0}^{P_i}, B_1 \rangle, \cdots, \langle \Gamma_{P_0}^{P_i}, B_g \rangle), \quad (1.7)$$

where $k_i$ is a nonzero integer and $\Gamma_{P_0}^{P_i}$ is a path from $P_0$ to $P_i$ on $\Gamma$. For example, the divisor $P_2 - P_1$ is mapped to $\eta(P_2 - P_1) = \int_{P_0}^{P_2} = (\langle \Gamma_{P_0}^{P_2}, B_1 \rangle, \cdots, \langle \Gamma_{P_0}^{P_2}, B_g \rangle)$.

1.3. Ultra-discrete limit. The ultra-discrete limit (UD-limit) links discrete dynamical systems to cellular automata, and algebraic curves to tropical curves. This limit is also called tropicalization. We define a map $\log_\varepsilon : \mathbb{R}_{>0} \to \mathbb{R}$ with an infinitesimal parameter $\varepsilon > 0$ by

$$\log_\varepsilon : x \mapsto -\varepsilon \log x. \quad (1.8)$$

For $x > 0$, we define $X \in \mathbb{R}$ by $x = e^{-\frac{X}{\varepsilon}}$. Then the limit $\varepsilon \to 0$ of $\log_\varepsilon(x)$ converges to $X$. The procedure $\lim_{\varepsilon \to +0} \log_\varepsilon$ with the scale transformation as $x = e^{-\frac{X}{\varepsilon}}$ is called the UD-limit.

We summarize this procedure in more general setting:

Proposition 1.3. For $A, B, C \in \mathbb{R}$ and $k_a, k_b, k_c > 0$, set

$$a = k_a e^{-\frac{A}{\varepsilon}}, \quad b = k_b e^{-\frac{B}{\varepsilon}}, \quad c = k_c e^{-\frac{C}{\varepsilon}}.$$ 

Then the UD-limit of the equations

(i) $a + b = c,$ \hspace{1cm} (ii) $ab = c,$ \hspace{1cm} (iii) $a - b = c$

yields the followings:

(i) $\min[A, B] = C,$ \hspace{1cm} (ii) $A + B = C,$

(iii) \begin{align*}
A = C & \quad (\text{if } A < B, \text{ or } A = B \text{ and } k_a > k_b) \\
\text{contradiction} & \quad (\text{otherwise})
\end{align*}.
1.4. From D-pToda lattice to UD-pToda. We briefly review the D-pToda and the way to obtain the UD-pToda. Fix \( g \in \mathbb{Z}_{>0} \). The \((g+1)\)-periodic Toda lattice of discrete time \( t \in \mathbb{Z} \) is given by the rational map on the phase space \( \mathcal{U} = \{ (I_1, \cdots, I_{g+1}, V_1, \cdots, V_{g+1}) \} \simeq \mathbb{C}^{2(g+1)} \):

\[
I_{n+1}^t = I_n^t + V_n^t - V_{n-1}^t, \quad V_{n+1}^t = \frac{I_n^t + V_n^t}{I_n^{t+1}},
\]

where we assume the periodicity \( I_{n+g+1}^t = I_n^t \) and \( V_{n+g+1}^t = V_n^t \). This system has the \((g+1)\) by \((g+1)\) Lax matrix:

\[
L^t(y) = \begin{pmatrix}
  a_1^t & 1 & (-1)^g b_1^t \\
  b_2^t & a_2^t & 1 \\
  \ddots & \ddots & \ddots \\
  (-1)^g y & b_g^t & a_g^t \\
  b_{g+1}^t & b_{g+1}^t & 1 \\
\end{pmatrix}
\]

(1.10)

Here \( y \in \mathbb{C} \) is a spectral parameter, and we set \( a_i^t = I_{n+i}^t + V_{n+i}^t, b_i^t = I_n^t V_n^t \). The evolution (1.9) preserves the algebraic curve \( \gamma \) given by \( f(x,y) = y \det(x I_{g+1}^t + L^t(y)) = 0 \). When we fix a polynomial \( f(x,y) \):

\[
f(x,y) = y^2 + y(x^{g+1} + c_g x^g + \cdots + c_1 x + c_0) + c_{-1},
\]

the isolevel set \( \mathcal{U}_c \) for (1.9) is

\[
\mathcal{U}_c = \{(I_n^t, V_n^t)_{n=1,\cdots,g+1} \in \mathcal{U} \mid y \det(x I_{g+1}^t + L^t(y)) = f(x,y)\}.
\]

It is known that for generic \( f(x,y) \), \( \mathcal{U}_c \) is isomorphic to an affine part of the Jacobian \( \text{Jac}(\gamma) \) of \( \gamma \).

Eq. (1.9) are rewritten as [7]

\[
I_n^{t+1} = V_n^t + I_n^t - \frac{1 - \prod_{i=1}^{g+1} V_n^t}{1 + \prod_{i=1}^{g+1} I_n^t},
\]

\[
V_n^{t+1} = \frac{I_n^t + V_n^t}{I_n^{t+1}}.
\]

(1.12)

Under the condition \( \prod_{i=1}^{g+1} I_n^t > \prod_{i=1}^{g+1} V_n^t \), the UD-limit of (1.12) with the scale transformation \( I_n^t = e^{-\frac{\alpha t}{W t}}, V_n^t = e^{-\frac{\alpha t}{W t}} \) gives the UD-pToda (1.11). The UD-pToda preserves the tropical curve \( \tilde{\Gamma} \subset \mathbb{R}^2 \) given by [4]:

\[
\tilde{\Gamma} = \{(X,Y) \in \mathbb{R}^2 \mid F(X,Y) \text{ is not smooth}\},
\]

where \( F(X,Y) = \min[2Y, Y + \min[(g+1)X, gX + C_g], \cdots, X + C_1, C_0], C_{-1}\).

(1.13)

Here \( C_i \)'s are regarded as tropical polynomials on \( \mathcal{T} \). For generic \( (Q,W) \in \mathcal{T}, C = (C_{-1}, C_0, \cdots, C_g) \in \mathbb{R}^{g+2} \) satisfies

\[
C_{-1} > 2C_0, \quad C_i + C_{i+2} > 2C_{i+1} (i = 0, \cdots, g - 2), \quad C_{g-1} > 2C_g,
\]

(1.14)

and we refer it as the generic condition. With this condition, \( \tilde{\Gamma} \) becomes a smooth tropical curve. We show the shape of \( \tilde{\Gamma} \) with the basis \( B_i \)'s of \( H_1(\Gamma, \mathbb{Z}) \) at Figure 4.
1.5. **Content.** In §2, we introduce a tropical analogue of Fay’s trisecant identity (Theorem 2.4) and obtain the bilinear form of the UD-pToda (1.5) as its particular case. In §3, we study the relation between the UD-pToda (1.1) and the bilinear form (1.5) (Lemma 3.2), and obtain the general solutions in terms of the tropical Riemann’s theta function (Theorem 3.5). Appendix A is devoted to prove Theorem 2.6 which is a key to Theorem 2.4.

1.6. **Notations.** We use the following notations of vectors in $\mathbb{R}^g$:

- $\vec{g} = (g, g-1, \ldots, 1)$,
- $\mathbf{e}_i$: the $i$-th vector of standard basis of $\mathbb{R}^g$,
- $\mathbb{I} = (1, 1, \ldots, 1) = \mathbf{e}_1 + \cdots + \mathbf{e}_g$,
- $\mathbb{I}_k = (1, \ldots, 1, 0, \ldots, 0) = \mathbf{e}_1 + \cdots + \mathbf{e}_k$.

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2. **Fay’s trisecant identity and its tropicalization**

2.1. **Fay’s trisecant identity for hyperelliptic curves.** Let $\gamma$ be the hyperelliptic curve given by $v^2 = \prod_{i=1}^{2g+2} (u - u_i)$, which defines the two-sheeted covering $u_\pm$ of $u$ with branches $[u_{2k+1}, u_{2k+2}]$ ($k = 0, 1, 2, \ldots, g$). Choose the basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ of $H_1(\gamma, \mathbb{Z})$ as usual as
(a) \( a_k \) goes the circuit around the branch \([u_{2k+1}, u_{2k+2}]\) on \( u_+ \),

(b) \( b_k \) goes on the upper half of \( u_+ \) from \([u_1, u_2]\) to \([u_{2k+1}, u_{2k+2}]\) and goes on the lower half of \( u_- \) from \([u_{2k+1}, u_{2k+2}]\) to \([u_1, u_2]\).

Let \( \omega_1, \ldots, \omega_g \) be a basis of the holomorphic differentials \( H^0(\gamma, \Omega^1) \) normalized so that the period matrix with respect to \( a_1, \ldots, a_g, b_1, \ldots, b_g \) has the form \((I, \Omega)\), where \( I \) is the \( g \times g \) identity matrix and \( \Omega \in \text{definite.} \) We write \( \text{Jac}(\gamma) \)

\[
\theta[\alpha, \beta](z) := \exp\{\pi \sqrt{-1}(\beta \Omega \beta + 2\beta(z + \alpha))\} \theta(z + \Omega \beta + \alpha)
\]

for \( \alpha, \beta \in \mathbb{R}^g \) and \( z \in \mathbb{C}^g \), where Riemann’s theta function \( \theta(z) \) is

\[
\theta(z) = \theta[0, 0](z) = \sum_{m \in \mathbb{Z}^g} \exp\{\pi \sqrt{-1}(m \Omega m + 2m z)\}. \tag{2.1}
\]

We set \( Q_i = (u_i, 0) \in \gamma \), and take \( Q_1 \) as a base point of the Abel-Jacobi map,

\[
\eta : \text{Div}(\gamma) \rightarrow \text{Jac}(\gamma) ; \quad \sum_i k_i P_i \mapsto \sum_i k_i \left( \int_{Q_1}^P \omega_j \right)_{j=1, \ldots, g}, \tag{2.2}
\]

where \( k_i \in \mathbb{Z} \), \( P_i \in \gamma \). Let \( K_\gamma \) be the canonical divisor on \( \gamma \) and \( \phi \) be the hyperelliptic involution of \( \gamma \) (interchanging the two sheets \( u_\pm \)). One sees that \( \Delta := -Q_1 + Q_3 + Q_5 + \cdots + Q_{2g+1} \in \text{Pic}^{g-1}(\gamma) \) is a theta characteristic (i.e. \( 2\Delta = K_\gamma \) in \( \text{Pic}^{2g-2}(\gamma) \)) and that \( D := P + \phi(P) \in \text{Pic}^2(\gamma) \) for \( P \in \gamma \) satisfies \( \eta(D) = 0 \). Via the Abel-Jacobi map \((2.2)\), the theta characteristics correspond to the half-periods of \( \text{Jac}(\gamma) \). For instance we have

\[
\eta(\Delta) = \frac{1}{2} \left( \begin{array}{cccc} g & g - 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\end{array} \right)_\Omega,
\]

where \( \left( \begin{array}{cc} \alpha \\ \beta \end{array} \right)_\Omega = \alpha I + \beta \Omega \in \mathbb{C}^g \) for \( \alpha, \beta \in \mathbb{R}^g \).

For \( m = 0, 1, \ldots, \left[ \frac{g+1}{2} \right] \), let \( \{i_1, \ldots, i_{g+1-2m}\} \) be a subset of \( \{1, 2, \ldots, 2g + 2\} \). The following is known \([1, \text{pp 12-15}]\): for \( m = 0 \), \( \eta(\Delta + D - \sum_{k=1}^{g+1} Q_{i_k}) \) are the non-singular even half-periods, while for \( m = 1 \), \( \eta(\Delta - \sum_{k=1}^{g+1} Q_{i_k}) \) are the non-singular odd half-periods, and for \( m > 1 \), \( \eta(\Delta - (m - 1)D - \sum_{k=1}^{g+1-2m} Q_{i_k}) \) are the even (odd) singular half-periods of multiplicity \( m \) when \( m \) is even (odd). By using the formulae:

\[
\eta(Q_2) = \frac{1}{2} \left( \begin{array}{cccc} 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\end{array} \right)_\Omega, \quad \eta(Q_{2k+1}) = \frac{1}{2} \left( \begin{array}{cccc} 0 & 
\cdots & 0 & k \\
0 & \cdots & 0 & 1 \\
\end{array} \right)_\Omega, \quad \eta(Q_{2k+2}) = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 1 \\
\end{array} \right)_\Omega
\]

we obtain the following:

**Proposition 2.1.** For a hyperelliptic curve \( \gamma \) and a half integer vector \( \beta(\neq 0) \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g \),

set \( \alpha \in \frac{1}{2} \mathbb{Z}^g \) as

\[
\alpha = (0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0),
\]

\[
\left( \begin{array}{cccc}
\alpha \\
\beta \\
\end{array} \right)_\Omega = \frac{1}{2} \left( \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\end{array} \right)_\Omega.
\]
where \( \beta_i = \frac{1}{2} \). Then \( \left( \frac{\alpha}{\beta} \right)_\Omega \) is a non-singular odd half-period of \( \text{Jac}(\gamma) \).

**Proof.** It is elementarily shown by using the fact that singular half-periods have at least two nonzero entries in \( \alpha \).

Fix a non-singular odd half-period \( \left( \frac{\alpha}{\beta} \right)_\Omega \), and denote the corresponding (non-singular odd) theta characteristic by \( \delta \). Define the half order differential \( h_\delta(x) \) on \( \gamma \) by

\[
h_\delta^2(x) = \sum_{i=1}^9 \frac{\partial \theta[\alpha, \beta]}{\partial z_i}(0)\omega_i(x).
\]

Here \( h_\delta(x) \) is the holomorphic section of the line bundle corresponding to \( \delta \). Then the prime form is defined by

\[
E(x, y) = \frac{\theta[\alpha, \beta](\eta(y - x))}{h_\delta(x)h_\delta(y)}
\]

for \( x, y \in \gamma \). We do not use \( h_\delta(x) \) in this paper except in the following theorem. See [1] or [12] for general settings other than the hyperelliptic case.

**Theorem 2.2.** [1] eq.(45)] Let \( \gamma \) be a hyperelliptic curve, \( \theta(z) \) be the Riemann’s theta function (2.1), and \( E(x, y) \) be the prime form (2.3). Then for \( P_1, P_2, P_3, P_4 \) in the universal covering space of \( \gamma \) and \( z \in \mathbb{C}^g \), the formula

\[
\theta(z + \int_{P_1}^{P_3})\theta(z + \int_{P_2}^{P_4})E(P_3, P_2)E(P_1, P_4)
\]

\[
+ \theta(z + \int_{P_2}^{P_4})\theta(z + \int_{P_1}^{P_3})E(P_3, P_1)E(P_4, P_2)
\]

\[
= \theta(z + \int_{P_1+P_2}^{P_3+P_4})\theta(z)E(P_3, P_4)E(P_1, P_2)
\]

holds, where \( \int_{P_i}^{P_j} \) denotes \( \eta(P_j - P_i) \).

By eliminating the common denominator, we have

\[
\theta(z + \int_{P_1}^{P_3})\theta(z + \int_{P_2}^{P_4})\theta[\alpha, \beta](\int_{P_3}^{P_2})\theta[\alpha, \beta](\int_{P_1}^{P_4})
\]

\[
+ \theta(z + \int_{P_2}^{P_4})\theta(z + \int_{P_1}^{P_3})\theta[\alpha, \beta](\int_{P_3}^{P_1})\theta[\alpha, \beta](\int_{P_4}^{P_2})
\]

\[
= \theta(z + \int_{P_1+P_2}^{P_3+P_4})\theta(z)\theta[\alpha, \beta](\int_{P_3}^{P_4})\theta[\alpha, \beta](\int_{P_1}^{P_2}).
\]

**2.2. Tropical analogue of Fay’s identity.** For a positive definite symmetric matrix \( K \in M_g(\mathbb{R}) \) and \( \beta \in \mathbb{R}^g \) we define

\[
g_\beta(m, Z) = \frac{1}{2} mKm^\perp + m(Z + \beta K)^\perp \quad (Z \in \mathbb{R}^g, \ m \in \mathbb{Z}^g),
\]

and write the tropical Riemann’s theta function as

\[
\Theta(Z) = \min_{m \in \mathbb{Z}^g} g_0(m, Z) \quad (Z \in \mathbb{R}^g).
\]
Let us introduce a generalization of the tropical Riemann’s theta function:

\[ \Theta[\beta](Z) := \frac{1}{2} \beta K \beta^\perp + \beta Z + \min_{m \in \mathbb{Z}^g} q_\beta(m, Z). \]

We write \( \arg_{m \in \mathbb{Z}^g} q_\beta(m, Z) \) for \( m \in \mathbb{Z}^g \) where \( q_\beta(m, Z) \) takes the minimum value.

**Proposition 2.3.** The function \( \Theta[\beta](Z) \) satisfies the following properties:

(i) the periodicity in \( \beta \):

\[ \Theta[\beta + 1](Z) = \Theta[\beta](Z) \quad (1 \in \mathbb{Z}^g), \]

(ii) the quasi-periodicity in \( Z \):

\[ \Theta[\beta](Z + 1K) = \frac{1}{2} \beta K \beta^\perp - \beta Z + \Theta[\beta](Z) \quad (1 \in \mathbb{Z}^g), \]

(iii) the symmetry in \( \beta \) and \( Z \):

\[ \Theta[\beta][-Z] = \Theta[-\beta](Z). \]

(iv) If \( \beta \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g \), then \( \Theta[\beta](Z) \) is an even function with respect to \( Z \) and \( \beta \).

**Proof.** (i) and (ii) Replace \( m \) by \( m - 1 \) in

\[ \Theta[\beta + 1](Z) = \frac{1}{2} (\beta + 1)K(\beta + 1)^\perp + (\beta + 1)Z + \min_{m \in \mathbb{Z}^g} \left[ \frac{1}{2} mKm^\perp + m(Z + (\beta + 1)K)^\perp \right] \]

and

\[ \Theta[\beta](Z + 1K) = \frac{1}{2} \beta K \beta^\perp + \beta(Z + 1K)^\perp + \min_{m \in \mathbb{Z}^g} \left[ \frac{1}{2} mKm^\perp + m(Z + 1K + \beta K)^\perp \right]. \]

(iii) Replace \( m \) by \(-m\) in

\[ \Theta[\beta][-Z] = \frac{1}{2} \beta K \beta^\perp - \beta Z + \min_{m \in \mathbb{Z}^g} \left[ \frac{1}{2} mKm^\perp + m(-Z + \beta K)^\perp \right]. \]

(iv) By (i) and (ii), if \( \beta \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g \), we have

\[ \Theta[\beta][-Z] = \Theta[-\beta](Z) = \Theta[-\beta + 2\beta](Z) = \Theta[\beta](Z). \]

\[ \square \]

In the rest of this section, let us restrict ourselves to the case of a family of hyper-elliptic tropical curves given by \( (1.13) \) with the generic condition \( (1.14) \). We define \( L, \lambda_0, \lambda_1, \ldots, \lambda_g \) and \( p_1, \ldots, p_g \) by

\[ L = C_{-1} - 2(g + 1)C_g, \quad \lambda_0 = C_g, \]

\[ \lambda_i = C_{g-i} - C_{g-i+1}, \quad p_i = L - 2 \sum_{j=1}^{g} \min_{1 \leq i \leq g} \lambda_i - \lambda_0 \lambda_j - \lambda_0 \quad (1 \leq i \leq g). \]

(2.5)

We set \( \bar{\lambda} = (\lambda_1 - \lambda_0, \ldots, \lambda_g - \lambda_{g-1}) \). Due to the condition \( (1.14) \) one sees

\[ \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_g, \quad 0 < p_g < p_{g-1} < \cdots < p_1, \quad 2 \sum_{i=1}^{g} (\lambda_i - \lambda_0) < L, \]

\[ \bar{g} \cdot \bar{\lambda} = \sum_{i=1}^{g} (\lambda_i - \lambda_0) = C_0 - (g + 1)C_g. \]

(2.6)
We show the maximal compact subset $\Gamma$ of $\tilde{\Gamma}$ in Figure 2 where a scalar on each edge denotes its length $d$. The period matrix $K = (K_{ij})$ for $\Gamma$ becomes

$$K_{ij} = \begin{cases} p_{i-1} + p_i + 2(\lambda_i - \lambda_{i-1}) > 0, & \text{where } p_0 = L \quad (i = j) \\ -p_i < 0 & (j = i + 1) \\ -p_j < 0 & (i = j + 1) \\ 0 & \text{(otherwise)} \end{cases} \quad (2.7)$$

**Theorem 2.4.** (Tropical analogue of Fay’s trisecant identity) Let $\tilde{\Gamma}$ be a smooth tropical curve given by (1.13) with the generic condition (1.14).

For $\beta \in \frac{1}{2} \mathbb{Z}^g (\beta \neq 0 \mod \mathbb{Z}^g)$, set $\alpha \in \frac{1}{2} \mathbb{Z}^g$ as

$$\alpha = (0, \cdots, 0, -\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0),$$

where $\beta_j = 0$ for $1 \leq j \leq i - 1$ and $\beta_i \neq 0$. For $P_1, P_2, P_3, P_4$ on the universal covering space of $\Gamma$, we define the sign $s_i \in \{\pm 1\}$ $(i = 1, 2, 3)$ as $s_i = (-1)^{k_i}$, where

$$k_1 = 2\alpha \cdot \left( \arg_{m \in \mathbb{Z}^g} q_\beta(m, \int_{P_2}^{P_3} m) + \arg_{m \in \mathbb{Z}^g} q_\beta(m, \int_{P_3}^{P_4} m) \right),$$

$$k_2 = 2\alpha \cdot \left( \arg_{m \in \mathbb{Z}^g} q_\beta(m, \int_{P_3}^{P_4} m) + \arg_{m \in \mathbb{Z}^g} q_\beta(m, \int_{P_4}^{P_1} m) \right),$$

$$k_3 = 1 + 2\alpha \cdot \left( \arg_{m \in \mathbb{Z}^g} q_\beta(m, \int_{P_4}^{P_1} m) + \arg_{m \in \mathbb{Z}^g} q_\beta(m, \int_{P_1}^{P_2} m) \right).$$
For $Z \in \mathbb{R}^q$, set $F_1, F_2, F_3 \in \mathbb{R}$ as

$$F_1 = \Theta(Z + \int_{P_3}^{P_4}) + \Theta(Z + \int_{P_2}^{P_4}) + \Theta[\beta](\int_{P_3}^{P_4}) + \Theta[\beta](\int_{P_1}^{P_4}),$$

$$F_2 = \Theta(Z + \int_{P_2}^{P_4}) + \Theta(Z + \int_{P_1}^{P_4}) + \Theta[\beta](\int_{P_3}^{P_4}) + \Theta[\beta](\int_{P_2}^{P_4}),$$

$$F_3 = \Theta(Z + \int_{P_1+P_2}^{P_4}) + \Theta(Z) + \Theta[\beta](\int_{P_3}^{P_4}) + \Theta[\beta](\int_{P_2}^{P_4}).$$

Then, the formula

$$F_i = \min[F_{i+1}, F_{i+2}]$$

holds if $s_i = \pm 1, s_{i+1} = s_{i+2} = \mp 1$ for $i \in \mathbb{Z}/3\mathbb{Z}$.

**Remark 2.5.** (i) The case of $s_1 = s_2 = s_3$ does not occur.

(ii) The sign $s_i$ is not determined when the corresponding $\text{arg}_{m \in \mathbb{Z}^q} q_\beta(m, \int_{P_j}^{P_k})$ is not unique. Sometimes it is possible to move the points $P_j$'s slightly so that $s_i$ is determined. But it can not be done always. (See Example 2.12)

The following theorem is the key to the proof of Theorem 2.4, which links the Abel-Jacobi map on $\gamma$ to that on $\Gamma$:

**Theorem 2.6.** By the UD-limit with the scale transformation

$$|x| = e^{-X/\varepsilon}, \quad |y| = e^{-Y/\varepsilon}, \quad z = -\frac{Z}{2\pi \sqrt{-1} \varepsilon}, \quad \Omega_{ij} = -\frac{K_{ij}}{2\pi \sqrt{-1} \varepsilon},$$

the Abel-Jacobi map (2.2) on $\gamma$ becomes the tropical Abel-Jacobi map (1.7) on $\Gamma$ as

$$\eta : \text{Div}(\Gamma) \to J(\Gamma); \quad \sum_i m_i P_i \mapsto \sum_i m_i ((\Gamma_{P_i, \tilde{B}_j})_{j=1, \ldots, g}),$$

where $P_0 \in \Gamma$ is a base point and $\tilde{B}_j = B_{g-j+1} + B_{g-j+2} + \cdots + B_g$. In this limit, $\tilde{K}$ becomes $\tilde{K}_{ij} = (\tilde{B}_i, \tilde{B}_j)$.

We will prove it analytically as a variation of the result in [6]. Since it is straightforward but tedious, we give it in the appendix.

**Remark 2.7.** (i) The cycles $\tilde{B}_j$'s are obtained from $B_j$'s by the base change as $(\tilde{B}_1, \ldots, \tilde{B}_g) = (B_1, \ldots, B_g) T$ with

$$T = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in M_g(\mathbb{C}).$$

Thus the tropical Abel-Jacobi map (1.7) is obtained from the complex Abel-Jacobi map (2.2) through the UD-limit.

(ii) We have $\tilde{K} = TKT$ which corresponds to the base change of $H_1(\gamma, \mathbb{Z})$ as $(a_i)_i \mapsto (a_i)_i T$ and $(b_i)_i \mapsto (b_i)_i T^{-1} = (b_{g-i+1} - b_{g-i})_i$. Thus $\alpha I + \beta \Omega$ of Theorem 2.4 corresponds to $\alpha T + \beta T^{-1} \Omega$ which is a nonsingular odd half period of $\text{Jac}(\gamma)$ from Proposition 2.1.
Proof of Theorem 2.4. By changing the basis of $H_1(\gamma, \mathbb{Z})$ from $(a_i), (b_i)$ to $(a_i), T, (b_i), T^{-1}$, the limit of $(b_i), T^{-1}$ becomes $(B_i)$. By the scale transformation of $z$ and $\Omega$ as (2.9), the theta function $\theta[\alpha, \beta](z)$ becomes

\[
\theta[\alpha, \beta](z) = e^{2\pi \sqrt{T} \beta \alpha \perp} \exp\{-\frac{1}{\varepsilon} \left( \frac{1}{2} \beta K \beta \perp + \beta Z \perp \right) \} \\
\times \sum_{m \in \mathbb{Z}^d} e^{2\pi \sqrt{T} m \alpha \perp} \exp\{-\frac{1}{\varepsilon} \left( \frac{1}{2} m K m^\perp + m (Z + K \beta \perp) \right) \}.
\]

(2.10)

Step 1: Since $e^{4\pi \sqrt{T} \beta \alpha \perp}$ is $\pm 1$ is a common factor and $e^{2\pi \sqrt{T} m \alpha \perp} = \pm 1$, we can set

\[
f_1 := e^{-4\pi \sqrt{T} \beta \alpha \perp} \theta(z + \int_{P_1} f_1) \theta(z + \int_{P_2} f_1) \theta[\alpha, \beta](\int_{P_1} f_1 + \int_{P_2} f_1) = f_1^+ - f_1^-,
\]

(2.11)

where we take $f_1^+ > 0$ and $f_1^- > 0$ as the part of $e^{2\pi \sqrt{T} m \alpha \perp} = 1$ and of $e^{2\pi \sqrt{T} m \alpha \perp} = -1$ respectively. Similarly we set

\[
f_2 := e^{-4\pi \sqrt{T} \beta \alpha \perp} \theta(z + \int_{P_1} f_2) \theta(z + \int_{P_2} f_2) \theta[\alpha, \beta](\int_{P_1} f_2 + \int_{P_2} f_2) = f_2^+ - f_2^-,
\]

\[
f_3 := e^{-4\pi \sqrt{T} \beta \alpha \perp} \theta(z + \int_{P_1} f_3) \theta(z \theta[\alpha, \beta](\int_{P_1} f_3 + \int_{P_2} f_3) = f_3^+ - f_3^-.
\]

Then $f_1^+ + f_2^+ + f_3^+ = f_1^- + f_2^- + f_3^-$ holds from (2.3). We write $F_i^\pm$ for the UD-limit of $f_i^\pm$. Then we obtain $F_i = \min[F_i^+, F_i^-]$ and $\min[F_1^+, F_2^+, F_3^+] = \min[F_1^-, F_2^-, F_3^-]$. Now we have the following cases:

(i) If $F_1^+ < F_1^-, F_2^+ < F_2^-$ and $F_3^+ < F_3^-$, then $\min[F_1, F_2] = F_3$.
(ii) If $F_1^+ > F_1^-, F_2^+ > F_2^-$ and $F_3^+ > F_3^-$, then $\min[F_1, F_2] = F_3$.
(iii) If $F_1^+ < F_1^-, F_2^+ > F_2^-$ and $F_3^+ < F_3^-$, then $\min[F_2, F_3] = F_1$.
(iv) If $F_1^+ > F_1^-, F_2^+ < F_2^-$ and $F_3^+ > F_3^-$, then $\min[F_2, F_3] = F_1$.
(v) If $F_1^+ < F_1^-, F_2^+ > F_2^-$ and $F_3^+ > F_3^-$, then $\min[F_3, F_1] = F_2$.
(vi) If $F_1^+ > F_1^-, F_2^+ < F_2^-$ and $F_3^+ < F_3^-$, then $\min[F_3, F_1] = F_2$.
(vii) If $F_1^+ < F_1^-, F_2^+ < F_2^-$ and $F_3^+ > F_3^-$, then $f_1^+ + f_2^+ + f_3^+ > f_1^- + f_2^- + f_3^-$ for sufficiently small $\varepsilon$, which is a contradiction.
(viii) If $F_1^+ > F_1^-, F_2^+ > F_2^-$ and $F_3^+ < F_3^-$, then $f_1^+ + f_2^+ + f_3^+ < f_1^- + f_2^- + f_3^-$ for sufficiently small $\varepsilon$, which is a contradiction.

Step 2: We check the definition of $s_i$. By (2.10) and (2.11), $s_1 = 1$ means

\[
e^{\pi \sqrt{T} \alpha \cdot \text{Arg}_{m \in \mathbb{Z}^d} \cdot \text{Im}(m_f(P_2))} \cdot e^{\pi \sqrt{T} \alpha \cdot \text{Arg}_{m \in \mathbb{Z}^d} \cdot \text{Im}(m_f(P_1))} = 1,
\]

and thus $F_1 = F_1^+$. Similarly we have $F_1 = F_1^-$ if $s_1 = -1$, $F_2 = F_2^+$ if $s_2 = 1$, $F_2 = F_2^-$ if $s_2 = -1$, $F_3 = F_3^+$ if $s_3 = -1$ and $F_3 = F_3^-$ if $s_3 = 1$.

From Steps 1 and 2 the claim follows.

\[
\square
\]

2.3. Bilinear equation of Toda type. We return to the definition of the tropical Riemann’s theta function and investigate the fundamental regions. We define the fundamental region $D_m$ of $\Theta(Z)$ as $D_m = \{ Z \in \mathbb{R}^d \mid \Theta(Z) = \frac{1}{2} m K m^\perp + m Z^\perp \}$, which is explicitly
Figure 3. Fundamental regions of $g = 2$ ($\mathbf{Z} = (Z_1, Z_2)$)

written as

$$D_m = \{ \mathbf{Z} \in \mathbb{R}^g \mid -l\mathbf{Z} \perp \leq lK(m + \frac{1}{2}l) \perp \text{ for any } l \in \mathbb{Z}^g \}.$$ 

Note that $\text{arg}_{m \in \mathbb{Z}^g} q_0(m, \mathbf{Z}) = m$ if and only if $\mathbf{Z}$ is in the interior of $D_m$. See Figure 3 for the $g = 2$ case.

We easily see the following:

**Lemma 2.8.** The period matrix $K$ (2.7) satisfies the following properties:

(i) $IK = p_g e_g + Le_1 + 2\lambda$,  
(ii) $\sum_{j=1}^{g} K_{ij} > 0$,  
(iii) $\bar{g}K = (g + 1)L e_1$.

**Lemma 2.9.** $D_m$ is written as

$$D_m = \{ \mathbf{Z} \in \mathbb{R}^g \mid -l\mathbf{Z} \perp \leq lK(m + \frac{1}{2}l) \perp \text{ for any } l \in \mathbb{Z}^g \}. \quad (2.12)$$

Proof. Since $D_m = D_0 - mK$ from the definition, it is enough to show for $D_0$. We show that if $\mathbf{Z} \in \mathbb{R}^g$ satisfies

$$-l\mathbf{Z} \perp \geq \frac{1}{2}lK1 \perp$$

for some $l \in \mathbb{Z}^g$, then there exists $l' = \pm(e_j + e_{j+1} + \cdots + e_k)$ for $1 \leq j \leq k \leq g$, which satisfies

$$-l'\mathbf{Z} \perp > \frac{1}{2}l'K(l') \perp. \quad (2.13)$$

For a vector $\mathbf{v} \in \mathbb{R}^g$, let $\mathbf{v} \geq 0$ denote that all elements of $\mathbf{v}$ are greater than or equal to zero.

This lemma is shown by checking the following three claims (a)-(c):

(a) There exists $l' \geq 0$ or $l' \leq 0$ which satisfies (2.13). Decompose $l$ as $l = l_+ + l_-$ such
that $1_+ \geq 0$, $1_- \leq 0$ and $1_+ \cdot 1_- = 0$. Then we have

$$-(1_+ + 1_-)Z^\perp > \frac{1}{2}(1_+ + 1_-)K(1_+ + 1_-)^\perp = \frac{1}{2}(1_+ K1_+^\perp + 1_- K1_-^\perp) + 1_+ K1_-^\perp.$$  

Since $K_{ij} \leq 0$ for $i \neq j$, $\frac{1}{2}1_i K1_i^\perp \geq 0$ holds. We have

$$-(1_+ + 1_-)Z^\perp > \frac{1}{2}(1_i K1_i^\perp + 1_- K1_-^\perp)$$

and hence $-1_i Z^\perp > \frac{1}{2}1_i K1_i^\perp$ or $-1_- Z^\perp > \frac{1}{2}1_- K1_-^\perp$.

(b) If $1 \geq 0$ or $1 \leq 0$, then there exists $l' \in \pm \{0, 1\}^g$ which satisfies (2.13). Without loss of generality we can assume that $1 \geq 0$. For simplicity we also assume that $l = (l_1, l_2, \ldots, l_g)$ satisfies $l_1 \geq l_2 \geq \cdots \geq l_g$. One can prove similarly in other cases. We use the induction on $k := l_1$. Assume that if $l_i^* \leq k - 1$ for $l^*$ satisfying the above assumptions for $l$, then there exists $l' \in \{0, 1\}^g$ satisfying (2.13). We set a natural number $r$ as $l_1 = l_2 = \cdots = l_r > l_{r+1} \geq \cdots \geq l_g$ and a vector $l^*$ as $l^* = l - (l_1 - l_{r+1})II_r$. Then $l$ is decomposed as $l = (l_1 - l_{r+1})II_r + l^*$ and we have

$$-1Z^\perp > \frac{1}{2}1l_1 K1^\perp$$

$$= \frac{1}{2}(l_1 - l_{r+1})^2 II_r K1^\perp + \frac{1}{2} l^* K(l^*)^\perp + (l_1 - l_{r+1})II_r K(l^*)^\perp,$$

where

$$\II_r K(l^*)^\perp = \II_r \cdot \begin{pmatrix} l_{r+1}1_{11} + \cdots + l_{r+1}1_{1r} + l_{r+1}1_{1r+1} + \cdots + l_g1_{1g} \\ \vdots \\ l_{r+1}1_{r1} + \cdots + l_{r+1}1_{rr} + l_{r+1}1_{rr+1} + \cdots + l_g1_{rg} \\ \vdots \\ \vdots \\ \geq l_{r+1}((1_{11} + \cdots + 1_{1g}) + \cdots + (1_{r1} + \cdots + 1_{rg})) \\ \geq 0. \end{pmatrix}$$

Thus we obtain

$$-(l_1 - l_{r+1})II_r + l^*)Z^\perp > \frac{1}{2}(l_1 - l_{r+1})^2 II_r K1^\perp + \frac{1}{2} l^* K(l^*)^\perp,$$

and therefore

$$-\II_r Z^\perp > \frac{1}{2} \II_r K1^\perp \quad \text{or} \quad -l^* Z^\perp > \frac{1}{2} l^* K(l^*)^\perp$$

since $l_1 - l_{r+1} \geq 1$. If the former holds, we obtain the claim by setting $l' = \II_r$. Otherwise, since $l_i^* \leq k - 1$, there exists $l' \in \{0, 1\}^g$ satisfying (2.13) by the induction hypothesis.

(c) Suppose $1 \geq 0$ or $1 \leq 0$ and $l_s = 0$ for some $1 < s < g$, and decompose $l$ as $l = l_L + l_R$ in such a way that $(l_L)_i = 0$ for $s \leq i \leq g$ and $(l_R)_i = 0$ for $1 \leq i \leq s$. Then $l' = l_L$ or $l' = l_R$ satisfies (2.13). We can assume $l \geq 0$. We have

$$-(l_L + l_R)Z^\perp > \frac{1}{2}(l_L + l_R)K(l_L + l_R)^\perp$$

$$= \frac{1}{2} l_L K1_L^\perp + \frac{1}{2} l_R K1_R^\perp + 1_L K1_R^\perp.$$
Proposition 2.10. The tropical Riemann’s theta function satisfies

$$ \min[2\Theta(Z + \lambda), \Theta(Z - Le_1) + \Theta(Z + Le_1 + 2\lambda) + L] = \Theta(Z + 2\lambda) + \Theta(Z). $$

Proof. Take $\beta = -\frac{1}{2}I$ and

$$ P_1 = (\lambda, \frac{1}{2}L + \frac{1}{2}p_g + (g + 1)C_g), \quad P_2 = (\lambda, (g + 1)C_g), $$

$$ P_3 = (\lambda, L + (g + 1)C_g), \quad P_4 = (\lambda, \frac{1}{2}L - \frac{1}{2}p_g + (g + 1)C_g). $$

Then we have

$$ \int_{P_1}^{P_2} = \int_{P_1}^{P_3} = \lambda, \quad \int_{P_2}^{P_3} = -Le_1, $$

$$ \int_{P_1}^{P_4} = Le_1 + 2\lambda, \quad \int_{P_3}^{P_4} = \int_{P_1}^{P_4} = Le_1 + \lambda. $$

We use the following lemma which will be proved after this proof.

Lemma 2.11. Set $\beta = -\frac{1}{2}I$. Then the followings are satisfied:

(i) the point $Z = \beta K$ is on the boundary $\partial D_0$;
(ii) all of $Z = \beta K + nLe_1 + t\lambda$ ($n = 0, 1, t = 0, 1, 2$) except $\beta K$ are in the interior of $D_0$;
(iii) if $\beta K + v$ is in the interior of $D_0$, then $\beta K - v$ is in the interior of $D_1$.

From (ii) and (iii), we have $\arg_{m \in \mathbb{Z}^g} q_{\beta}(m, nLe_1 + t\lambda) = 0$ and $\arg_{m \in \mathbb{Z}^g} q_{\beta}(m, -nLe_1 - t\lambda) = 1$ for $n = 0, 1, t = 0, 1, 2$ except for $n = t = 0$. Further, from the definition of $\Theta[\beta](Z)$, we have the following:

$$ \Theta[\beta](\int_{P_1}^{P_2}) = -\frac{1}{2}L + \frac{1}{2}\beta K\beta^\perp, $$

$$ \arg_{m \in \mathbb{Z}^g} q_{\beta}(m, \int_{P_1}^{P_2}) = 0. $$

$$ \Theta[\beta](\int_{P_1}^{P_4}) = -\frac{1}{2}L - (\lambda_\beta - \lambda_0) + \frac{1}{2}\beta K\beta^\perp, $$

$$ \arg_{m \in \mathbb{Z}^g} q_{\beta}(m, \int_{P_1}^{P_4}) = 0. $$

$$ \Theta[\beta](\int_{P_3}^{P_4}) = \Theta[\beta](\int_{P_1}^{P_3}) = -\frac{1}{2}(\lambda_\beta - \lambda_0) + \frac{1}{2}\beta K\beta^\perp, $$

$$ \arg_{m \in \mathbb{Z}^g} q_{\beta}(m, \int_{P_3}^{P_4}) = 1, $$

$$ \Theta[\beta](\int_{P_3}^{P_4}) = \Theta[\beta](\int_{P_1}^{P_3}) = -\frac{1}{2}(\lambda_\beta - \lambda_0) + \frac{1}{2}\beta K\beta^\perp, $$

$$ \arg_{m \in \mathbb{Z}^g} q_{\beta}(m, \int_{P_3}^{P_4}) = 0. $$

Thus $s_1 = 1$, $s_2 = 1$ and $s_3 = -1$ hold. Substituting them into the tropical Fay’s identity (2.8), we obtain the claim. □
Proof of Lemma 2.11. In this proof we use

\[ f_m(Z, l) := 1Z^\perp + 1K(m + \frac{1}{2}l)^\perp, \]

which is “r.h.s - l.h.s” of the conditional equation for \( D_m \).

(i) For \( l = I \) and \( m = 0 \), we have \( f_0(\beta K, I) = (-1)(\frac{1}{2}KI)^\perp + I(\frac{1}{2}KI)^\perp = 0 \). For \( l \in \{0,1\}^g \setminus \{0, I\} \), we have

\[ f_0(\beta K, l) = \frac{1}{2}1KI^\perp + 1K\beta^\perp = -\frac{1}{2}1K(I - 1)^\perp = -\frac{1}{2}\sum_{i \neq j} \varepsilon_{ij} K_{ij}, \]

where \( \varepsilon_{ij} = 1 \) if “\( l_i = 1 \) and \( l_j = 0 \)” and otherwise \( \varepsilon_{ij} = 0 \). Since \( \varepsilon_{i,i\pm 1} \) is not zero by the assumption, \( f_0(\beta K, I) \) is positive. For \( l \in \{0, -1\}^g \setminus \{0, I\} \), we have

\[ f_0(\beta K, l) = \frac{1}{2}lKI^\perp - \frac{1}{2}KI^\perp \geq \frac{1}{2}lKI^\perp > 0. \]

Thus we see \( f_0(\beta K, l) > 0 \) for \( l \in \{0,1\}^g \cup \{0, -1\}^g \) \setminus \{0, I\} \) and \( f_0(\beta K, I) = 0 \). This implies \( \beta K \in \partial D_0 \) from Lemma 2.9.

(ii) Set \( Z = \beta K + nL^1 + t^\perp \lambda \) (\( n = 0, 1, t = 0, 1, 2 \)). For \( l = I \), we have

\[ f_0(Z, I) = \frac{1}{2}I^\perp + I(\beta K + nL^1 + t^\perp \lambda)^\perp = I(nL^1 + t^\perp \lambda) \geq 0, \quad \text{(by (i))}, \]

where the equality holds iff \( n = t = 0 \). For \( l \in \{0,1\}^g \setminus \{0, I\} \), we have

\[ f_0(Z, l) = \frac{1}{2}lKI^\perp + l(\beta K + nL^1 + t^\perp \lambda)^\perp > I(nL^1 + t^\perp \lambda) \geq 0, \quad \text{(by (i))}. \]

From Lemma 2.8 we have

\[ \beta K + nL^1 + t^\perp \lambda = -\frac{1}{2}p_g e_g + (n - \frac{1}{2})L^1 + (t - 1)^\perp \lambda \]

\[ = -\beta K - p_g e_g - (1 - n)L^1 - (2 - t)^\perp \lambda. \]

Thus, for \( l \in \{0, -1\}^g \setminus \{0\} \), \( f_0(Z, l) \) becomes

\[ \frac{1}{2}lKI^\perp + l(-\beta K - p_g e_g - (1 - n)L^1 - (2 - t)^\perp \lambda)^\perp \]

\[ > -l(p_g e_g + (1 - n)L^1 + (2 - t)^\perp \lambda) \]

\[ \geq 0 \quad (n = 0, 1, t = 0, 1, 2). \]

Then the claim follows from Lemma 2.9.

(iii) When \( \beta K + v \) is in the interior of \( D_0 \), \( f_0(\beta K + v, l) = \frac{1}{2}lKI^\perp + l(\beta K + v)^\perp > 0 \) holds for all \( l \in \mathbb{Z}^g \). Thus we see that \( f_1(\beta K - v, -1) > 0 \) for all \( l \in \mathbb{Z}^g \), since

\[ f_1(\beta K - v, -1) = -lK(\frac{1}{2}I - 1)^\perp - l(\beta K - v)^\perp = f_0(\beta K + v, l), \]

for all \( l \in \mathbb{Z}^g \). \( \square \)

Example 2.12 (Counter example). Set as \( g = 2, C_2 = 0, L = 11, \lambda_1 = 2, \lambda_2 = 3, \beta = (-\frac{1}{2}, 0) \) and take \( P_1, P_2, P_3, P_4 \) as Proposition 2.10 then

\[ K = \begin{pmatrix} 18 & -3 \\ -3 & 6 \end{pmatrix}, \quad P_1 = (3, 6), P_2 = (0, 0), P_3 = (0, 11), P_4 = (3, 5), \]

\[ \int_{P_1}^{P_4} = \int_{P_2}^{P_4} = (2, 1), \quad \int_{P_2}^{P_4} = (-11, 0), \quad \int_{P_1}^{P_4} = (15, 2), \quad \int_{P_4}^{P_1} = \int_{P_4}^{P_2} = (13, 1), \]

\[ 15 \]
hold, and thus $\beta K + \int_{P_1}^n = (6,3.5)$ is on the boundary $D_{(0,-1)} \cap D_{(-1,-1)}$. Therefore we cannot determine the sign $s_1$, while at $Z = (0,0)$ we obtain that

\begin{align*}
F_1 & = \Theta(2,1) + \Theta(2,1) + \Theta(2,1) + \Theta(2,1.5) + \Theta(2,1.5) + \Theta(2,1) + \Theta(2,1) + \Theta(15,2) + \Theta(15,2) + \Theta(15,2) + \Theta(15,2) = -9 \\
F_2 & = \Theta(-11,0) + \Theta(15,2) + \Theta(15,2) + \Theta(15,2) + \Theta(-11,0.5) = -7.5 \\
F_3 & = \Theta(4,2) + \Theta(0,0) + \Theta(4,2.5) + \Theta(4,2.5) + \Theta(13,1) + \Theta(4,2.5) = -8.5,
\end{align*}

and thus Fay’s type identities do not hold in this case.

From Proposition 2.10 we have the following.

**Corollary 2.13.** For $Z_0 \in \mathbb{R}^g$, the function $T_n^t$ given by

\[ T_n^t = \Theta(Z_0 - nL_1 + \lambda t) \quad (2.14) \]

satisfies the tropical bilinear equation (1.3).

3. Solution of UD-pToda

3.1. $\tau$-function for UD-pToda. Fix a positive integer $g$. The UD-pToda is defined by the piecewise-linear map

$\varphi_T : (Q_n, W_n^t)_{n=1,\ldots,g+1} \mapsto (Q_{n+1}^t, W_{n+1}^t)_{n=1,\ldots,g+1}$

given by (1.1) on the phase space $T$ (1.2). The map $\varphi_T$ preserves the tropical polynomials $C_i(Q,W)$ ($i = -1,0,\ldots,g$) on $T$. Fix $C = (C_{-1}, C_0, \cdots, C_g) \in \mathbb{R}^{g+2}$ as (1.14), and define the tropical curve $\Gamma$ (1.13) and the isolevel set $T_C$ as

\[ T_C = \{(Q,W) \in T \mid C_i(Q,W) = C_i \ (i = -1,0,\cdots,g)\}. \quad (3.1) \]

See [H] for a detail of $C_i(Q,W)$.

Let $S_t \ (t \in \mathbb{Z})$ be a subset of infinite dimensional space:

\[ S_t = \{T_n^t \in \mathbb{R} \mid n \in \mathbb{Z}\}, \]

where $T_n^t$ has a quasi-periodicity; i.e. $T_{n+g+1}^t = T_n^t + c_n^t$, where $c_n^t$ satisfies

\[ c_n^t = an + bt + c \quad (3.2) \]

for some $a,b,c \in \mathbb{R}$. Fix $L \in \mathbb{R}$ such that

\[ 2b - a < (g + 1)L \quad (3.3) \]

and define a map $\varphi_S$ from $S_t \times S_{t+1}$ to $S_{t+1} \times S_{t+2}$ as $\varphi_S : (T_n^t, T_{n+1}^t)_{n \in \mathbb{Z}} \mapsto (T_{n+1}^t, T_{n+2}^t)_{n \in \mathbb{Z}}$ with

\[ T_{n+2}^t = 2T_{n+1}^t - T_n^t + X_{n+1,t}^{(g)} \quad (3.4) \]

where we define a function on $S_t \times S_{t+1}$:

\[ X_{n,t}^{(k)} = \min_{j=0,\ldots,k} \left[ jL + 2T_{n+j}^{t+1} + T_n^t + T_{n-1}^t - (2T_{n-1}^{t+1} + T_{n-j}^t + T_{n-j-1}^t) \right], \quad (3.5) \]

for $k \in \mathbb{Z} \geq 0$. Note that it follows from (3.2) and (3.3) that $2c_{n+1}^t - c_n^t - c_{n+1}^t < (g + 1)L$ for all $n \in \mathbb{Z}$. The function $X_{n,t}^{(k)}$ has the following properties:
Lemma 3.1. (i) $X^{(k)}_{n,t}$ satisfies a recursion relation:

$$2T_{n-1}^{t+1} + X^{(k)}_{n,t} = \min \left[2T_{n-1}^{t+1}, L + 2T_{n-2}^{t+1} + T_n^t - T_{n-2}^t + X^{(k-1)}_{n-1,t} \right], \text{ for } k \geq 1. \quad (3.6)$$

(ii) $X^{(g+1)}_{n,t} = X^{(g)}_{n,t}$.

Proof. (i) It is shown just by the definition of $X^{(k)}_{n,t}$.

(ii) For simplicity we rewrite $(3.5)$ as $X^{(k)}_{n,t} = \min_{j=0,\ldots,k}[a_j]$, where

$$a_j = jL + 2T_{n-j-1}^{t+1} + T_n^t + T_{n-1}^t - (2T_{n-1}^{t+1} + T_{n-j}^t + T_{n-j-1}^t).$$

By making use of $(3.2)$ we obtain

$$a_{g+1} = (g+1)L + 2T_{n-g-2}^{t+1} + T_n^t + T_{n-1}^t - (2T_{n-1}^{t+1} + T_{n-g-1}^t + T_{n-g-2}^t) = (g+1)L + c_{n-g-2}^t + c_{n-g-1}^t - 2c_{n-g-2}^{t+1} > 0.$$ At the same time we have $a_0 = 0$. Thus we obtain

$$X^{(g+1)}_{n,t} = \min\{X^{(g)}_{n,t}, a_{g+1}\} = X^{(g)}_{n,t}.$$

□

Let $\sigma_t$ be a map $\sigma_t : S_t \times S_{t+1} \to \mathcal{T}$ given by $(1.4)$. It is easy to check that $\sigma_t$ is well-defined where $(3.2)$ assures the periodicity of $(Q_n^t, W_n^t)_n$, and $(3.3)$ assures the condition $\sum_{n=0}^{g+1} Q_n^t < \sum_{n=0}^{g+1} W_n^t$ of the phase space $\mathcal{T}$.

Lemma 3.2. The relation $\sigma_t(X^{(g)}_{n,t}) = X^t_n$ holds, and the following diagram is commutative:

$$\begin{array}{ccc}
S_t \times S_{t+1} & \xrightarrow{\sigma_t} & \mathcal{T} \\
\downarrow \phi_S & & \downarrow \phi_T \\
S_{t+1} \times S_{t+2} & \xrightarrow{\sigma_{t+1}} & \mathcal{T}
\end{array} \quad (3.7)
$$

Proof. By direct calculation we can check $\sigma_t(X^{(g)}_{n,t}) = X^t_n$. To check $\phi_T \circ \sigma_t = \sigma_{t+1} \circ \phi_S$, it is enough to calculate $Q^t_{n+1}$ in the image of each map. We have

$$Q^t_{n+1} = \min\{W_n^t, Q_n^t - X^t_n\}$$

$$= \min\{L + T_{n-1}^{t+1} + T_n^t - T_{n-1}^t, T_{n-1}^t + T_n^t - T_{n-1}^t - T_n^t - X^{(g)}_{n,t} \} \quad \text{(by (1.4))},$$

for $\phi_T \circ \sigma_t$, and for $\sigma_{t+1} \circ \phi_S$,

$$Q^t_{n+1} = T_{n-1}^{t+2} + T_n^{t+1} - T_{n-2}^{t+1}$$

$$= 2T_n^{t+1} + X^{(g)}_{n+1,t} + (T_{n-1}^{t+1} - T_n^t - T_{n-1}^t - T_n^{t+1}) \quad \text{(by (3.4))}$$

$$= 2T_n^{t+1} + X^{(g+1)}_{n+1,t} + \left(-X^{(g)}_{n,t} - T_{n-1}^t - T_n^{t+1} + T_{n-1}^t \right) \quad \text{(by Lemma 3.1(ii) and (3.4))}$$

$$= \min\{2T_n^{t+1}, L + 2T_n^{t+1} + T_{n+1}^t - T_{n-1}^t + X^{(g)}_{n,t} \} \quad \text{(by (3.6)).}$$

It is easy to see these two expressions of $Q^t_{n+1}$ coincide. □

From Lemma 3.1 and Lemma 3.2 the next proposition immediately follows.

Proposition 3.3. If $\{T_n^t\}_{t,n} \in \prod_t S_t$ satisfies $(3.4)$, then it satisfies $(1.5)$. 

Proof.

\[
\min[2T^t_{n+1}, T^t_{n+1} + T^t_{n+1}] = \min[2T^t_{n+1}, (2T^t_{n+1} + X^{(g)}_{n+1}) + T^t_{n+1} + L]
\]
\[
= 2T^t_{n+1} + X^{(g+1)}_{n+1} = 2T^t_{n+1} + X^{(g)}_{n+1, t}
\]
\[
= T^t_{n+1} + T^t_{n+1}
\]

\[
\Box
\]

Conversely the following proposition holds, which proved after Theorem 3.5 is proved.

**Proposition 3.4.** Let \( \{T^t_n\}_{n,t} \in \prod \mathbb{S}_t \) satisfy (1.5) and \( A_t \) denote a set:

\[
\{ n \in \mathbb{Z} \mid T^t_n + T^t_{n+2} = 2T^t_n, \ i.e. \ 2T^t_{n+1} \leq T^t_{n+1} + T^t_{n+1} + L \}.
\]

Then we have the followings:

(i) \( n \in A_t \iff n + g + 1 \in A_t \), (ii) \( A_t \neq \emptyset \), (iii) (3.4) holds for all \( n, t \in \mathbb{Z}^2 \).

The following is the main result of this section.

**Theorem 3.5.** Let \( \iota_t : \mathbb{R}^{g} \to \mathbb{S}_t \times \mathbb{S}_{t+1} \) be a map:

\[
Z_0 \mapsto (T^t_n = \Theta(Z_0 - nLe_1 + t\bar{\lambda}), T^t_{n+1} = \Theta(Z_0 - nLe_1 + (t+1)\bar{\lambda}))_{n \in \mathbb{Z}}.
\]

Then the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{R}^g & \xrightarrow{\iota_t} & \mathbb{S}_t \times \mathbb{S}_{t+1} \\
\downarrow \text{id.} & & \downarrow \varphi_{\mathbb{S}} \quad \sigma_t \\
\mathbb{R}^g & \xrightarrow{\iota_t+1} & \mathbb{S}_{t+1} \times \mathbb{S}_{t+2} \\
\end{array}
\]

\[
\Downarrow \varphi_{\mathbb{T}} \quad \Downarrow \sigma_{t+1} \\
\mathbb{T}_C
\]

In short, for any \( Z_0 \in \mathbb{R}^g \), \( T^t_n = \Theta(Z_0 - nLe_1 + t\bar{\lambda}) \) satisfies (3.4) and gives a solution of (1.1) through (1.4).

**Proof of Theorem 3.5.** We first check that \( T^t_n = \Theta(Z_0 - nLe_1 + t\bar{\lambda}) \) satisfies the quasi-periodicity (3.2) and (3.3). From Prop. 2.3 and Lemma 2.8 (iv), we have

\[
T^t_{n+g+1} = T^t_n + c^t_n, \quad c^t_n = an + bt + c = \bar{g} \cdot (Z_0 - nLe_1 + t\bar{\lambda} - \frac{1}{2}gK)^t.
\]

Then from (2.6) we obtain

\[
a = -gL, \ b = 2 \sum_{i=1}^{g} (\lambda_i - \lambda_0), \ c = \bar{g} \cdot Z_0 - \frac{1}{2}g(g+1)L,
\]
\[
(g+1)L - 2b + a = (g+1)L - 2 \sum_{i=1}^{g} (\lambda_i - \lambda_0) + gL = p_g > 0.
\]

Due to Prop. 3.4 (iii), the left part of the diagram (3.8) is commutative. Since \( \Theta(Z) \) is associated to \( \Gamma \), the image of \( \sigma_t \) is in \( \mathbb{T}_C \), while the commutativity of the right part of (3.8) follows from Lemma 3.2.

\[
\Box
\]

**Remark 3.6.** If \( A_t = \emptyset \), the map \("\sigma_t+1 \circ \varphi_{\mathbb{S}} \circ \sigma_t^{-1}\"\) induces a map given by \((Q^t_n, W^t_{n+1})_{n=1, \ldots, g+1} \mapsto (Q^t_n = W^t_{n}, W^t_{n+1} = Q^t_{n+1})_{n=1, \ldots, g+1} \), which does not preserve the inequality \( \sum_n Q^t_n < \sum_n W^t_n \).
Proof of Proposition 3.4.

(i) \( n \in A_t \Leftrightarrow T_n^t + T_n^{t+2} = T_{n-1}^t + T_{n+1}^t + L \)
\( \Leftrightarrow (T_{n+g+1}^t - t^1_n) + (T_{n+g+1}^{t+2} - c_n^{t+2}) = (T_{n+g}^t - c_{n-1}^t) + (T_{n+g+2}^t - c_{n+1}^t) + L \)
\( \Leftrightarrow T_{n+g+1}^t + T_{n+g+1}^{t+2} = T_{n+g}^t + T_{n+g+2}^t + L \)
\( \Leftrightarrow n + g + 1 \in A_t. \)

(ii) Assume that \( T_n^t \) satisfies the quasi-periodicity \((5.2)\) and \( A_t = \emptyset. \) Then we have
\( T_n^t + T_n^{t+2} = T_{n-1}^t + T_{n+1}^t + L \) for all \( n, \)
\( \Rightarrow T_{n+g+1}^t - T_{n-1}^t = T_{n+g+1}^t - T_{n+g+1}^t - (g + 1)L \)
\( \Rightarrow -a + 2b + c = c + (g + 1)L \)
\( \Leftrightarrow 2b - a = (g + 1)L. \)

This contradicts the claim of the quasi-periodicity.

(iii) From (i) and (ii), it is enough to show that if \( n \in A_t \) then \( T_n^{t+2} = 2T_n^{t+1} - T_n^t + X_n^{(g)} \) holds for all \( s \geq 0. \) We show it by induction on \( s. \)

In case \( s = 0. \) Due to \((1.5),\)
\( T_n^{t+2} \leq 2T_n^{t+1} - T_n^t \)
\( T_n^{t+2} \leq L + T_n^t + T_n^{t+2} - T_n^t \)

hold for all \( j. \) Taking the sum of \((3.11)\) for \( j = 0, 1, \ldots, k - 1 \) and using \((3.10),\) we have
\( T_n^{t+2} \leq kL + T_n^t + T_n^{t+2} - T_n^t \)
\( \leq kL + (2T_n^{t+1} - T_n^t) + T_n^t + T_n^{t+2} - T_n^t \)

for all \( k \geq 0, \) where \( X_n^{(g)} = \min_{k=0,1,\ldots,g}[a_k^1], \) and thus, \( T_n^{t+2} \leq 2T_n^{t+1} - T_n^t + X_n^{(g)} \) holds. Since \( T_n^{t+2} = 2T_n^{t+1} - T_n^t \) from the assumption \( n \in A_t \) and \( X_n^{(g)} \leq a_0^1 = 0, \) we have \( X_n^{(g)} = 0, \) which yields the claim for \( s = 0. \)

Assume \( T_n^{t+2} = 2T_n^{t+1} - T_n^t + X_n^{(g)} \) holds for some \( s \geq 0. \)
\( T_n^{t+2} = 2T_n^{t+1} - T_n^t + \min[0, T_n^{t+2} + T_n^{t+2} + L - 2T_n^{t+1} + T_n^t \]
\( = 2T_n^{t+1} - T_n^t + \min[0, (2T_n^{t+1} - T_n^t) + T_n^{t+2} + L - 2T_n^{t+1} + T_n^t \]
\( = 2T_n^{t+1} - T_n^t + X_n^{(g+1)} \) (by Lemma 3.1)
\( = 2T_n^{t+1} - T_n^t + X_n^{(g)} \)

Thus, we have the claim holds for \( s + 1. \)

\[ \square \]

Corollary 3.7. We write \( \varphi_{\tilde{\lambda}} \) for the translation on the Jacobian \( J(\Gamma) \) as \( \varphi_{\tilde{\lambda}} : [Z_0] \mapsto [Z_0 + \tilde{\lambda}]. \) Let \( \iota_{\sigma} : J(\Gamma) \to \mathcal{T}_C \) be the map induced by \( \sigma_t \circ \iota_t : \mathbb{R}^g \to \mathcal{T}. \) Then the following
form holds.\[\omega\]

We choose the homology basis \(\Theta\) so that the zero locus of \(\Delta(u)\) becomes the standard form of a hyperelliptic curve:

\[\gamma^1 : \quad v^2 = \Delta(u)^2 - 4c_{-1}.\]

It is known that \(\Delta(u)\) has special properties [16, 17]:

(i) the zeros of \(\Delta(u) = \prod_{i=0}^g (u_i - u)\) are simple and positive, ordered as

\[0 < u_0 < u_1 < \cdots < u_g.\]  \hfill (A.2)

(ii) The zeros of \(\Delta(u)^2 - 4c_{-1} = \prod_{i=0}^g (u - u_i)(u - u_i)\) are positive and they can be ordered as

\[0 < u_0^{-1} < u_1^{-1} < u_1^1 < u_2^{-1} < \cdots < u_g^{-1} < u_1^1 < u_2^1 < \cdots < u_g^1.\]

(iii) We have \(u_i^1 < u_i < u_i^{-1}\).

Further, since \(|c_{-1}|\) is very small when we consider the case of the UD-limit of \(\gamma\), we can assume that the zero locus of \(\Delta(u)^2 - 4c_{-1}\) are simple and positive, i.e.

\[0 < u_0^{-1} < u_0^1 < u_1^{-1} < u_1^1 < \cdots < u_g^{-1} < u_g^1\]  \hfill (A.3)

holds.

Take two-sheeted covering \(u_+, u_-\) of \(u\) with branches \([u_j^{-1}, u_j^1]\) \((j = 0, 1, 2, \ldots, g)\) and choose the homology basis \(a_1, \ldots, a_g, b_1, \ldots, b_g\) and the basis of the holomorphic differentials \(\omega_1, \ldots, \omega_g \in H^0(\gamma, \Omega^1)\) as usual as in Section 2.1 i.e. \(\omega_j\)'s are written in the form

\[\omega_j = \frac{w_{j,g-1}u^{g-1} + w_{j,g-2}u^{g-2} + \cdots + w_{j,0}}{u} du \quad (w_{j,k} \in \mathbb{C})\]

and satisfy \((\int_{a_i} \omega_j)_{i,j} = I, (\int_{b_i} \omega_j)_{i,j} = \Omega\).

**Lemma A.1.** When \(C_i\) satisfy the generic condition \(1.1\), for \(0 \leq i < i + 2 \leq j \leq g\),

\[C_i + C_j > C_{i+1} + C_{j-1}\]

holds.

---

\(^1\) We proved this conjecture after submitting the present paper [5]. The proof needs rather complicated combinatorial discussion.
Proof. Consider the sum of the equations $C_{i+k} + C_{i+k+2} > 2C_{i+k+1}$ for $k = 0, 1, \ldots, g - 2$. 

Proposition A.2. By the UD-limit with the scale transformation 
\begin{align}
|u| &= e^{-x}, \ |y| = e^{y}, \ u_j = e^{-\frac{X_j}{2}}, \ u_j^{\pm1} = e^{-\frac{X_j^{\pm1}}{2}}, \ c_i = e^{-\frac{C_i}{2}}, \\
\end{align}

the followings holds.
(i) $X_j X_j^{-1} X_j^{+1} = C_j - C_{j+1}$.
(ii) The limit of cycle $b_i$ is $\bar{B}_i = B_{g-i+1} + B_{g-i+2} + \cdots + B_g$. 

Figure 4. Spectral curve with $\Delta(u) = \prod_{k=1}^{g}(u - (2k - 1))$
Remark A.4. By the scale transformation (A.4), the integral paths will consider by the variables \( X \) \( X \) \( X \) \( X \) \( X \) variables \( \exp(\cdot) \) also transformed. Although the integral paths converge to zero in the variables \( X \), \( \Delta(u) = 0 \) and \( \Delta(u)^2 - 4c_{-1} = 0 \) are respectively written as

\[
e^{-\frac{C_0}{\varepsilon}} + e^{-\frac{2X + C_2}{\varepsilon}} + \cdots + e^{-\frac{(g-1)X}{\varepsilon}} = e^{-\frac{X + C_1}{\varepsilon}} + e^{-\frac{3X + C_3}{\varepsilon}} + \cdots + e^{-\frac{gX + C_g}{\varepsilon}}
\]

and

\[
\left( e^{-\frac{2C_0}{\varepsilon}} - 4e^{-\frac{C_{-1}}{\varepsilon}} \right) + \left( 2e^{-\frac{3X + C_2 + C_3}{\varepsilon}} + e^{-\frac{2X + 2C_1}{\varepsilon}} \right) + \cdots + e^{-\frac{(2g+1)X + C_g}{\varepsilon}}.
\]

By Lemma [A.1], taking the UD-limit of both sides, we have

\[
\min[C_0, 2X + C_2, \ldots, (g + 1)X] = \min[X + C_1, 3X + C_3, \ldots, gX + C_g]
\]

and

\[
\min[2C_0, 2X + 2C_1, \ldots, (2g + 2)X] = \min[X + C_0 + C_1, 3X + C_1 + C_2, \ldots, (2g + 1)X + C_g].
\]

By solving these, we obtain the claim. \( \square \)

(ii) It is easily shown.

Proposition A.3. Set \( \omega_j^0 \) as

\[
\omega_j^0 = \frac{1}{2\pi \sqrt{-1}} \left\{ \frac{1}{u - u_j} - \frac{1}{u - u_0} \right\} du
\]

for \( j = 1, \ldots, g \) and define \( u_{j,k} \in \mathbb{C} \) by

\[
\omega_j^0 = \frac{u_{j,g-1}u^{g-1} + u_{j,g-2}u^{g-2} + \cdots + u_{j,0}}{\Delta(u)} du.
\]

Set \( \tilde{\omega}_j \) as

\[
\tilde{\omega}_j = \frac{u_{g-1}u^{g-1} + u_{g-2}u^{g-2} + \cdots + u_{0}}{\sqrt{\Delta(u)^2 - 4c_{-1}}} du
\]

and take the scale transformation (A.4), then

\[
\lim_{\varepsilon \to +0} \int_{a_i} \tilde{\omega}_j = \lim_{\varepsilon \to +0} \int_{a_i} \omega_j^0 = \delta_{i,j} \quad (A.5)
\]

hold.

Proof. Since \( 2C_0 < C_{-1} \) from (1.14), we have \( \lim_{\varepsilon \to +0} \tilde{\omega}_j / \omega_j^0 = 1 \). Further, by the residue theorem, \( \int_{a_i} \omega_j^0 = \delta_{i,j} \) holds for any \( \varepsilon > 0 \). \( \square \)

By this proposition, denoting the normalized 1-form on \( \gamma^1 \) as \( \omega_j \), we have \( \lim_{\varepsilon \to +0} \omega_j / \omega_j^0 = 1 \).

Remark A.4. By the scale transformation (A.4), the integral paths \( a_i \) and \( b_i \) are also transformed. Although the integral paths converge to zero in the variables \( u(= \exp(-X/\varepsilon)) \), \( u_t(= \exp(-X_t/\varepsilon)) \), they converge to nonzero paths of finite length in the variables \( X, X_t \). Since we can exchange the order of the UD-limit and the integration, we will consider by the variables \( X \)'s in the following.
Proposition A.5. Suppose that there exist the limits \( \lim_{\varepsilon \to +0} \log_\varepsilon (u_a, y_a) = (X_a, Y_a) \) and \( \lim_{\varepsilon \to +0} \log_\varepsilon (u_b, y_b) = (X_b, Y_b) \).

(i) If \((X_a, Y_a)\) and \((X_b, Y_b)\) are on the same edge of \(\Gamma\), and \(X_{i+1} < X_a, X_b < X_i\) for some \(i\), then

\[
-2\pi \sqrt{1-\varepsilon} \lim_{\varepsilon \to +0} \int_{u_a}^{u_b} \omega_j = \begin{cases} 0 & (j \leq i) \\
(X_a - X_b) & (j > i \text{ and } Y_a, Y_b \geq \frac{1}{2} C_1) \\
-(X_a - X_b) & (j > i \text{ and } Y_a, Y_b \leq \frac{1}{2} C_1) \end{cases} \tag{A.6}
\]
holds.

(ii) If \(X_a = X_b = X_i\) for some \(i\), then

\[
-2\pi \sqrt{1-\varepsilon} \lim_{\varepsilon \to +0} \int_{y_a}^{y_b} \omega_j = \begin{cases} -(Y_a - Y_b) & (i = 0) \\
(Y_a - Y_b) & (i = j) \\
0 & (i \neq 0, j) \end{cases} \tag{A.7}
\]
holds.

Remark A.6. The sign of \(\omega_j\) is changed by passing through the branches \([u_k^{-1}, u_k^{+1}]\). At the branch point \(u = u_k^{+1}\), we have \(v = 0\) and \(y = -\frac{1}{2} \Delta(u) = \pm \sqrt{c-1}\), where the sign is + if and only if \(u = u_k^{+1}\) \((k\) is odd\) or \(u = u_k^{-1}\) \((k\) is even\).

Proof. For simplicity, we omit the sign of \(\omega_j\) if there is no possibility of misunderstanding. Since \(\lim_{\varepsilon \to 0} \omega_j / \omega_j^0 = 1\), the integrals are not changed by substituting \(\omega_j\) by \(\omega_j^0\).

(i) Without loss of generality we can assume \(Y_a, Y_b \geq \frac{1}{2} C_1\), which corresponds to the sign of \(\omega_j\) is +.

\[
\lim_{\varepsilon \to +0} -2\pi \sqrt{1-\varepsilon} \int_{u_a}^{u_b} \omega_j^0 = \lim_{\varepsilon \to +0} -2\pi \sqrt{1-\varepsilon} \int_{u_a}^{u_b} \frac{1}{2\pi \sqrt{1-\varepsilon}} \left\{ \frac{1}{u - u_j} - \frac{1}{u - u_0} \right\} du \\
= -\lim_{\varepsilon \to +0} \varepsilon \left[ \log \left\{ \frac{(u_b - u_j)}{(u_a - u_j)} \right\} - \log \left\{ \frac{(u_b - u_0)}{(u_a - u_0)} \right\} \right] \\
= \begin{cases} (X_b - X_a) - (X_b - X_a) = 0 & (j \leq i) \\
X_a - X_b & (j > i) \end{cases} \tag{A.7}
\]

\(\lim_{\varepsilon \to +0} \log_\varepsilon (u - u') = X\) if \(u > u' > 0\).

(ii) We divide to two cases (ii-1) \(Y_a, Y_b \leq \frac{1}{2} C_1\) or \(Y_a, Y_b \geq \frac{1}{2} C_1\) and (ii-2) \(Y_a < \frac{1}{2} C_1 < Y_b\) or \(Y_a > \frac{1}{2} C_1 > Y_b\). (ii-1) Without loss of generality we can assume \(Y_a, Y_b \geq \frac{1}{2} C_1\).

Substituting

\[
du = -\frac{2g + \Delta(u)}{y \Delta'(u)} dy,
\]

\[
\Delta'(u) = \left( \prod_{k=0}^{g} (u - u_k) \right)' = \Delta(u) \sum_{k=0}^{g} \frac{1}{u - u_k}
\]
and

\[
\Delta(u) = -\frac{y^2 + c-1}{y},
\]
into \(\omega_j^0\), we have

\[
\omega_j^0 = \frac{1}{2\pi \sqrt{1-\varepsilon}} \left( \frac{1}{u - u_j} - \frac{1}{u - u_0} \right) \left( \sum_{k=0}^{g} \frac{1}{u - u_k} \right)^{-1} \left( -\frac{1}{y} + \frac{2y}{y^2 + c-1} \right) dy.
\]
Here, if \( \lim_{\varepsilon \to +0} \log_\varepsilon u = X_i \),
\[
\frac{1}{2\pi \sqrt{-1}} \left( \frac{1}{u - u_j} - \frac{1}{u - u_0} \right) \left( \sum_{k=0}^{g} \frac{1}{u - u_k} \right)^{-1} = \begin{cases} 
\frac{-1}{2\pi \sqrt{-1}} & (i = 0) \\
\frac{1}{2\pi \sqrt{-1}} & (i = j) \\
0 & (i \neq 0, j).
\end{cases}
\]

Thus we have
\[
-2\pi \sqrt{-1} \varepsilon \lim_{\varepsilon \to +0} \int_{y_a}^{y_b} \omega_j = \begin{cases} 
\lim_{\varepsilon \to +0} \varepsilon [\log(y + \frac{C_{-1}}{y})]_{y_a}^{y_b} & (i = 0) \\
\lim_{\varepsilon \to +0} -\varepsilon [\log(y + \frac{C_{-1}}{y})]_{y_a}^{y_b} & (i = j) \\
-\min[Y_b, C_{-1} - Y_a] + \min[Y_a, C_{-1} - Y_b] = -(Y_a - Y_b) & (i = 0) \\
\min[Y_b, C_{-1} - Y_b] - \min[Y_a, C_{-1} - Y_a] = (Y_a - Y_b) & (i = j) \\
0 & (i \neq 0, j).
\end{cases}
\]

(ii-2) Without loss of generality we can assume \( Y_a < \frac{1}{2} C_{-1} < Y_b \). The ultra-discrete limit of \( |y| = \sqrt{c_{-1}} \) is \( Y = \frac{1}{2} C_{-1} \). From (ii-1) and Remark A.6

\[
-2\pi \sqrt{-1} \varepsilon \lim_{\varepsilon \to +0} \int_{y_a}^{y_b} \frac{\omega_j}{\omega_j} = -2\pi \sqrt{-1} \varepsilon \lim_{\varepsilon \to +0} \left( \int_{y_a}^{\frac{1}{2} \Delta(u)} \omega_j + \int_{\frac{1}{2} \Delta(u)}^{y_b} \omega_j \right) = \begin{cases} 
-\left( \frac{1}{2} C_{-1} - Y_a \right) + \left( \frac{1}{2} C_{-1} - Y_b \right) = Y_a - Y_b & (i = 0) \\
\left( \frac{1}{2} C_{-1} - Y_a \right) - \left( \frac{1}{2} C_{-1} - Y_b \right) = -Y_a + Y_b & (i = j) \\
0 & (i \neq 0, j).
\end{cases}
\]

\( \square \)

Theorem 2.6 immediately follows from Proposition A.5 and Proposition A.2 (ii). Actually, by Proposition A.5 Theorem 2.6 holds if the points \( P \) and \( Q \) are on a common edge of \( \Gamma \). General case is shown by the additivity of the Abel-Jacobi map \( \eta \) for paths. By this fact together with Proposition A.2 (ii), the UD-limit of \( \Omega_{ij} \) becomes \( \tilde{K}_{ij} = (\tilde{B}_i, \tilde{B}_j) \).

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