Intermediate spectral statistics in the many–body localization transition

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Spectral statistics of systems that undergo many–body localization transition are studied. An analysis of the gap ratio statistics from the perspective of inter- and intra-sample randomness allows us to pinpoint differences between transitions in random and quasi-random disorder, showing the effects due to Griffiths rare events for the former case. It is argued that the transition for a random disorder exhibits universal features that are identified by constructing an appropriate model of intermediate spectral statistics which is a generalization of the family of short-range plasma models. The model incorporates the inter- and intra-sample fluctuations and faithfully reproduces level spacing distributions as well as number variance during the transition from ergodic to many–body localized phase. In particular, it grasps the critical level statistics which arise at disorder strength for which the fluctuations are the strongest.

Many-body localization (MBL) seems to be the most robust manifestation of ergodicity breaking in the quantum world attracting enormous interest (for recent reviews see [1, 2] as well as a topical issue of Annalen der Physik [3]). From the early days of MBL systems were often characterized by level spacing distributions known from random matrix [4] and quantum chaos studies [5]. It has been realized [6] that the level unfolding (necessary to obtain a unit mean density of states) is a tricky procedure that, done naively, may affect the results. Instead a dimensionless ratio of consecutive energy levels gaps (referred as the gap ratio) was introduced [6]. It is defined as $r_n = \min\{\delta_n, \delta_{n-1}\}/\max\{\delta_n, \delta_{n-1}\}$ where $\delta_n = E_{n+1} - E_n$ is an energy difference between two consecutive levels.

The average gap ratio, $\bar{r}$, is different for fully extended systems (in the following we shall concentrate on the gaussian orthogonal ensemble (GOE) for time-reversal invariant systems) $\bar{r}_{\text{GOE}} \approx 0.53$ and for localized systems $\bar{r}_{\text{Poi}} \approx 0.39$ [6]. That property was used by many authors in attempts to localize the transition [7–13]. It has been possible to obtain analytic predictions for distributions of $r$ both for GOE (in a simplified small matrix approach) and for the Poisson random sequence [14].

The latter limit seems highly relevant as it has been found that for MBL systems an extensive set of local integrals of motion exists making these systems integrable (LIOM) depends on the disorder realization. Therefore, averaging over disorder implies averaging over different sets of LIOMs. Thus, contrary to system specific chaotic to integrable transitions, one may argue that ergodic to many-body localized transition may have universal statistical features. Especially, as it is to some extent successfully described by renormalization group picture [19, 20]. On the other hand, it has been postulated that the universality class of the transition depends on the disorder type [21, 22] identifying intra-sample randomness as the dominant feature for quasiperiodic disorder (QPD) while the inter-sample randomness being essential for purely random disorder (RD). Those important observations were made studying the entanglement entropy behavior.

The aim of this letter is twofold. Firstly, we show that a proper analysis of gap ratio statistics allows us to get similar insight on the randomness of system in MBL transitions as the entanglement entropy [21] in a simpler way. Secondly, this analysis, as a by–product gives hints on the construction of universal statistics for MBL transition which we provide generalizing earlier attempts [23, 24].

The gap ratio analysis. The usual way of calculating the mean gap ratio $\bar{r}$ is to average the $r_n$ variable over a certain number of energy levels getting a mean gap ratio for one sample $r_S = \langle r_n \rangle_S$. Then, the mean gap ratio is obtained by averaging of $r_S$ over disorder realizations $\overline{r} = \langle r_S \rangle_{d_S}$. While, as mentioned above $\overline{r}$ obtained in this way reflects the character of eigenstates of the system [6–12] a part of information encoded in the $r_n$ variables is necessarily lost. Let us examine $P(r_S)$ – the distribution of the $r_S$ – it provides a direct information about variations of the $r_S$ for different disorder realizations. As an example we consider the XXZ spin chain with additional
GOE and Poisson behaviors. The tail of the distribution indicates that QPD reproduces GOE statistics only approximately; (b) the distributions $P(r_s)$ for random disorder (RD) and QPD; (c) the inter-sample variance $V_S$ for RD and QPD; (d) the intra-sample variance $V_I$ for QPD and RD during the transition.

Figure 1: Top: (a) the distributions $P(r_s)$ for quasi-periodic disorder (QPD) of strength $W$. Dashed lines give limiting GOE and Poisson behaviors. The tail of the $W = 1$ distribution indicates that QPD reproduces GOE statistics only approximately; (b) the distributions $P(r_s)$ for random disorder (RD). Bottom: (c) the inter-sample variance $V_S$ for RD and QPD; (d) the intra-sample variance $V_I$ for QPD and RD.

next-nearest-neighbors coupling (similar to that of [21])

$$H = J \sum_i \tilde{\sigma}_i \cdot \tilde{\sigma}_{i+1} + W \sum_i \cos(2\pi \zeta i + \phi) \sigma_i^z + J_1 \sum_i \sigma_i^z \sigma_{i+2}^z,$$

where $\zeta = (\sqrt{5} - 1)/2$ (the golden ratio) and $\phi$ is a fixed phase for a given disorder realization (leading to QPD) or is random on each lattice site (leading to RD with the same on-site distribution, as in the QRD case) [21]. We fix $J = 1$ as the energy unit and we study the case of $J_1 = J$ first. For the system size $L = 16$ we consider sequences of $N = 400$ consecutive eigenvalues from the middle of the spectrum yielding a collection of $r_s$ values for $n_{\text{dis}} = 2000$ disorder realizations. The resulting distributions, $P(r_s)$, for different disorder strengths $W$ are shown in Fig. 1.

Had all $r_n$ been independent of each other the distribution of $r_s = \sum_{n=1}^{N} r_n / N$ should be Gaussian with width determined by the variance of the $r_n$ distribution and proportional to $1/\sqrt{N}$. Despite the correlations – particularly strong for GOE – the $P(r_s)$ are Gaussian in the limiting cases of GOE and Poisson statistics. Surprisingly, the $P(r_s)$ distributions remain Gaussian for QPD across the transition.

In a striking contrast, the distributions in the RD case become strongly asymmetric with enlarged variance in the transition region. This reflects the inter-sample randomness importance for the RD and is a clear, nice manifestation of the existence of rare Griffiths regions [25–28]: for samples with $\bar{r}$ close to GOE there exist realizations of disorder leading to $r_S$ close to Poisson limit. Similarly, on a localized side for $\bar{r}$ close to integrable limit there are rare events with $r_S$ values close to GOE value. The stark difference in the $P(r_s)$ distributions between the RD and QPD cases can be quantified by calculating a variance: $V_S = \langle r_s^2 - \bar{r}_S^2 \rangle_{\text{dis}}$. As Fig. 1c) shows, the inter-sample variance $V_S$ has a clear peak in the MBL transition for the RD whereas it varies only slightly for the QPD.

Consider now the variance $v_I$ of the $r_S$ variable, $v_I = \langle r_s^2 - \bar{r}_S \rangle$. Averaged over disorder realizations $V_I = \langle v_I \rangle_{\text{dis}}$, it provides information about fluctuations of $r_n$ within a single spectrum of the system at a certain disorder strength – characterizing intra-sample randomness. As could be expected from the long range correlations of GOE, it is small for GOE and conversely, it is maximal for Poissonian spectrum. Fig. 1d) shows that it behaves similarly for QPD and RD interpolating between the values for GOE and Poisson statistics. The intra-sample randomness is larger for QPD than for RD which can be expected from the correlations between adjacent sites in the QPD case. The transition is sharper for the system with QPD, implying that it is less affected by finite size effects [21].

Seeing that the distribution $P(r_s)$ and the variances $V_S$ and $V_I$ provide a valuable information about the randomness at the MBL transition, let us switch our attention to the more standard Heisenberg chain case taking $J_1 = 0$ in Eq. (1) and assuming random uniform disorder so that $\cos(2\pi \zeta i + \phi)$ is exchanged by $h_i \in [-1, 1]$ in Eq. (1). Despite the fact that the distribution of disorder is different and the studied model contains now nearest neighbor couplings only, the $P(r_s)$ behaves quite similarly to the case shown in Fig. 1b) revealing strong asymmetry and broadening across the transition. Particularly, the broader distributions in the transition regime suggest that one may use the maximal variance $V_S$ as an indicator of the transition point.

A standard finite size scaling of different quantities can be performed assuming $W \rightarrow (W - W_C)^{1/\nu}$. For $\bar{r}$ such an analysis has been performed already [9, 29] with the data collapsing to a single curve. Similar scaling may be used for the variance $V_S$. Observe that both the position of the maximum as well as its value depend on the system size – Fig. 2(a). If, together with the rescaling of the disorder strength, the variance $V_S$ is rescaled according to $V_S \rightarrow \bar{V}_S = (V_S - V_{GOE})/L^\kappa$ (where $V_{GOE}$ is the inter-sample variance for GOE) the data for various system sizes collapse onto a single curve – Fig. 2c) for the exponents $\nu = 0.95(10)$, $\kappa = 1.2(1)$ and the critical disorder strength $W_{C} = 3.5(1)$. The scaling of the $V_S$ will necessarily cease to work for larger system sizes as the support of the $P(r_s)$ distribution is limited by $\tau_{\text{poi}}$ and $\tau_{\text{GOE}}$. On the other hand, the critical disorder strength $W_C = 3.5(1)$ and the exponent $\nu = 0.95(10)$ are in nice agreement with results of [9]. A similar finite size scal-
ing may be performed for the intra-sample variance \( V_I \) with the same \( W_C \) and \( \nu \) – Fig. 2(d). It is notable that all three measures \( \tau \), \( V_S \) and \( V_I \) scale in a very similar manner. Being interconnected they still provide different insights into physics of the system during the MBL transition.

**Critical level statistics.** After the finite size analysis we can identify the critical statistics. We assume that it can be extracted from data for a system of size \( L \) for disorder strength \( W_L \) that maximizes the inter-sample variance \( V_S \), e.g. \( W_L = 2.7 \) for \( L = 16 \). The finite size analysis assures that in the thermodynamic limit \( L \to \infty \) \( W_L \to W_C = 3.5(1) \). The critical statistics obtained in this way is presented in Fig. 4. It is almost system size independent within the available system sizes. To get the critical level spacing distribution \( P(s) \) and the number variance \( \Sigma^2(L) \) we had to unfold the spectra carefully (as described in [30]). Only then one may compare the data with a theoretical model. Previous attempts used either a mean field plasma model [23] or different variants of critical statistics [24] known from single particle studies [31–33].

We consider a family of short–range plasma models (SRPMs) [34] describing eigenvalue distributions with logarithmic interactions only among a finite number \( h \) of neighboring eigenvalues. This model interpolates between GOE statistics for which \( h \to \infty \) and the Poisson statistics for which the eigenvalues are uncorrelated (hence \( h = 0 \)). The SRPM have exponential tails of the level spacings distributions \( P(s) \propto \exp\left( -(h\beta + 1)s \right) \) and asymptotically linear number variance \( \Sigma^2(L) \propto L/(h\beta + 1) \). However, the \( P(r_S) \) distribution for critical level statistics has a broad and slightly asymmetric shape while the distribution for SRPM is a narrow Gaussian denoted with the dotted line in Fig. 3. Thus, the pure SRPM cannot account for the inter-sample variance correctly. This also means that the exponential tail of the critical level spacing distribution is not reproduced accurately – similar holds for other quantities e.g. the number variance \( \Sigma^2(L) \).

As we show in detail in [30], the level statistics during the whole ergodic–MBL transition may be obtained by modifying SRPMs – let us dub our procedure as a weighted SRPM (wSRPM). The basic feature of the transition observed above is the broad \( P(r_S) \) distribution (reflecting Griffiths regions and probably to some extent also finite size effects). An attempt to model such a situation by a simple random matrix model seems fruitless. Instead, we construct an ensemble which is a mixture of different SRPMs. Let \( P_c^S(E_1,\ldots,E_N) \) be a joint probability distribution function (JPDF) of all eigenvalues for SRPM characterized by \( h \) and \( \beta \). A JPDF for wSRPM statistics is obtained as

\[
P_{wSRPM}(E_1,\ldots,E_N) = \sum_i c_i P_{h_i}^\beta(E_1,\ldots,E_N) \tag{2}
\]

where \( h_i \) and \( \beta_i \) range over an appropriate set of values and \( c_i \) are weight coefficients (\( \sum_i c_i = 1 \)). The level spacing distribution for wSRPM is a linear combination of level spacing distributions for SRPMs which are its ingredients: \( P_{wSRPM}(s) = \sum_i c_i P_{h_i}^\beta(s) \). Other quantities for wSRPM such as \( n \)-level correlation functions and the number variance \( \Sigma^2(L) \) are also linear combinations of appropriate quantities for SRPM which enter the JPDF of wSRPM.

The wSRPM model, defined above, is dependent on a large number of parameters – one needs to specify the \( h_i, \beta_i \) sets as well as the weight coefficients \( c_i \). Each of the SRPMs has its \( P_c^S(r_S) \) distribution which is approximately Gaussian centered around a mean gap ratio \( \tau^\beta_h \) which depends on \( h \) and \( \beta \). In order to get the \( \{(h_i,\beta_i,c_i)\} \) set of parameters it suffices to fit the \( P(r_S) \) distribution at a given point of MBL transition by \( P_{wSRPM}(r_S) = \sum_{i=1}^{t_{max}} c_i P_{h_i}^\beta(r_S) \). In this way the coefficients \( \{(h_i,\beta_i,c_i)\}_{i=1}^{t_{max}} \) are fixed. Already for \( t_{max} = 4 \) this procedure allows for finding wSRPM model parameters which describe faithfully the bulk and the exponential tail of the level spacing distribution as well as grasp the non-local correlations in the spectrum which are reflected by the number variance \( \Sigma^2(L) \).

For a system close to the ergodic regime, we find that level statistics is a mixture of a dominant SRPM with \( \beta = 1 \) and varying \( h > 1 \) (which diverges as one approaches the ergodic regime) with small contribution of SRPMs with smaller \( h \) and \( \beta \) that account for disorder realizations for which the system has more localized properties. Upon approaching the MBL transition,
The interactions between eigenvalues get more local and \( h_i = \max \{ h_i \} \) decreases (but still \( \beta_i = \max \{ \beta_i \} = 1 \)). Finally, as one gets closer to the MBL regime, the appropriate leading distribution is based on SRPM with \( \beta_i < 1 \) and \( h_i = 1 \) with certain admixtures of more localized statistics as well as SRPMs with longer range interactions are present accounting again for the rare events. Those contributions get gradually smaller and in the MBL phase there are no correlations between eigenvalues – \( \beta_i = 0 \).

The critical level statistics presented in Fig. 4 corresponds to a mixture of SRPM characterized interactions among a few neighboring eigenvalues \( (h_i = 3) \) with SRPMs with local eigenvalue correlations. Note each of the four contributions was essential to recover the broad \( P(r_S) \) distribution (Fig. 3). Moreover, the presence of all of the admixtures is absolutely vital to faithfully reproduce the number variance. The values of spectral compressibility \( \chi \) defined by the linear large \( L \) behavior of the number variance \( \Sigma^2(L) \propto \chi L \) together with the average gap ratios \( \bar{r} \) are shown in Tab. I. This quantities are in good agreement with the predictions of the wSRPM \( r_{wSRPM} \) and \( \chi_{wSRPM} \). The data suggest that the remaining small deviation in the spectral compressibility \( \chi \) is probably a finite size effect.

A few remarks are in order. First, observe that the information about the admixture of systems with stronger localization properties is contained in the tail of \( P(s) \) showing its significance. Let us also note that the relation between the exponential tail of \( P(s) \) and the slope of number variance existing for the standard SRPM no longer holds if one considers the wSRPM – this allows the latter to fit the XXZ data with such a precision. Finally, although the set of parameters \( \{(h_i, \beta_i, c_i)\} \) needed to specify the wSRPM is large, it is completely fixed by shape \( P(r_S) \) distribution – in that sense the procedure of finding appropriate wSRPM is straightforward.

**Conclusions and beyond.** The gap ratio analysis demonstrates that more than just an overall information about the crossover between ergodic and MBL regimes can be obtained from the \( r_n \) variables. The considered inter- and intra-sample variances \( V_S \) and \( V_I \) reflect nicely the differences between RD and QPD universality classes. Furthermore, the \( P(r_S) \) distribution quantifies the intersample fluctuations of a system undergoing MBL transition and gives a particularly clear demonstration of the Griffiths regime. On the other hand, it hints how to formulate the wSRPM model of spectral statistics across the MBL transition for the random disorder. The relevant ensemble is a mixture of short-range plasma models [34] allowing us to reproduce both the short-range (spacings, gap ratios) and the long range (number variance) spectral correlations. It is also interesting to find that the MBL transition for the QPD case cannot be described within this model. It supports the claim of [21] that the transitions for RD and QPD are of different universality classes. The ensemble that reproduces QPD MBL transition is yet to be identified.

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| \( L \) | \( W \) | \( \bar{r} \) | \( \chi \) |
|---|---|---|---|
| 14 | 2.62 | 0.4528(4) | 0.545(9) |
| 16 | 2.7 | 0.4537(5) | 0.587(5) |
| 18 | 2.8 | 0.4569(7) | 0.605(4) |
| 18 | 2.8 | 0.643(19) | 0.4556(21) |

Table I: The average gap ratio \( \bar{r} \) and spectral compressibility \( \chi \) for the XXZ spin chain at disorder strength which corresponds to \( W_S \) at \( L \rightarrow \infty \). For comparison, the predictions of wSRPM \( r_{wSRPM} \) and \( \chi_{wSRPM} \) are displayed.
[1] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B 90, 174202 (2014), URL http://link.aps.org/doi/10.1103/PhysRevB.90.174202.

[2] R. Nandkishore and D. A. Huse, Ann. Rev. Cond. Mat. Phys. 6, 15 (2015).

[3] Annalen der Physik 529 (2017), ISSN 1521-3889, URL http://dx.doi.org/10.1002/andp.201770051.

[4] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. 110, 084101 (2013), URL http://link.aps.org/doi/10.1103/PhysRevLett.110.084101.

[5] P. Sierant and J. Zakrzewski, New Journal of Physics 18, 043032 (2018), URL http://stacks.iop.org/1367-2630/20/i=4/a=043032.

[6] J. Janarek, D. Delande, and J. Zakrzewski, Phys. Rev. B 97, 155133 (2018), URL http://link.aps.org/doi/10.1103/PhysRevB.97.155133.

[7] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Phys. Rev. Lett. 110, 084101 (2013), URL http://link.aps.org/doi/10.1103/PhysRevLett.110.084101.

[8] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. 111, 127201 (2013), URL http://link.aps.org/doi/10.1103/PhysRevLett.111.127201.

[9] J. Z. Imbrie, Phys. Rev. Lett. 117, 027201 (2016), URL https://link.aps.org/doi/10.1103/PhysRevLett.117.027201.

[10] H. Friedrich and H. Wintgen, Physics Reports 183, 37 (1989), ISSN 0370-1573, URL http://www.sciencedirect.com/science/article/pii/037015738990121X.

[11] M. C. Gutzwiller, J. Math. Phys. 12, 343 (1971).

[12] A. C. Potter, R. Vasseur, and S. A. Parameswaran, Phys. Rev. X 5, 031033 (2015), URL https://link.aps.org/doi/10.1103/PhysRevX.5.031033.

[13] R. Vosk, D. A. Huse, and E. Altman, Phys. Rev. X 5, 031032 (2015), URL https://link.aps.org/doi/10.1103/PhysRevX.5.031032.

[14] V. Khemani, D. N. Sheng, and D. A. Huse, Phys. Rev. Lett. 119, 075702 (2017), URL https://link.aps.org/doi/10.1103/PhysRevLett.119.075702.

[15] S.-X. Zhang and H. Yao, ArXiv e-prints (2018), 1805.05958.

[16] M. Serbyn and J. E. Moore, Phys. Rev. B 93, 041424 (2016), URL http://link.aps.org/doi/10.1103/PhysRevB.93.041424.

[17] C. L. Bertrand and A. M. García-García, Phys. Rev. B 94, 144201 (2016), URL https://link.aps.org/doi/10.1103/PhysRevB.94.144201.

[18] R. B. Griffiths, Phys. Rev. Lett. 23, 17 (1969), URL https://link.aps.org/doi/10.1103/PhysRevLett.23.17.

[19] T. Vojta, J. Low Temp. Phys. 161, 299 (2010).

[20] K. Agarwal, S. Gopalakrishnan, M. Knap, M. Müller, and E. Demler, Phys. Rev. Lett. 114, 160401 (2015), URL https://link.aps.org/doi/10.1103/PhysRevLett.114.160401.

[21] K. Agarwal, E. Altman, E. Demler, S. Gopalakrishnan, D. A. Huse, and M. Knap, 529 (2016).

[22] K. Kudo and T. Deguchi, Phys. Rev. B 93, 041424 (2016), URL https://link.aps.org/doi/10.1103/PhysRevB.93.041424.

[23] K. Kudo and T. Deguchi, Phys. Rev. B 97, 220201 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.97.220201.

[24] P. Sierant and J. Zakrzewski, Weighted models for level statistics across the many-body localization transition (2018), arXiv: 1808.02795, 1808.02795, URL http://arxiv.org/abs/1808.02795.

[25] V. E. Kravtsov and K. A. Muttalib, Phys. Rev. Lett. 79, 1913 (1997), URL https://link.aps.org/doi/10.1103/PhysRevLett.79.1913.

[26] S. M. Nishigaki, Phys. Rev. E 59, 2853 (1999), URL https://link.aps.org/doi/10.1103/PhysRevE.59.2853.

[27] A. M. García-García and J. J. M. Verbaarschot, Phys. Rev. E 67, 046104 (2003), URL https://link.aps.org/doi/10.1103/PhysRevE.67.046104.

[28] E. Bogomolny, E., Gerland, U., and Schmit, C., Eur. Phys. J. B 19, 121 (2001), URL https://doi.org/10.1007/s100510170357.