Abstract

We introduce a new, demand-driven variant of Spector’s bar recursion in the spirit of the Berardi-Bezem-Coquand functional of [4]. The bar recursion takes place over finite partial functions, where the control parameter $\varphi$, used in Spector’s bar recursion to terminate the computation at sequences $s$ satisfying $\varphi(\hat{s}) < |s|$, now acts as a guide for deciding exactly where to make bar recursive updates, terminating the computation whenever $\varphi(\hat{u}) \in \text{dom}(u)$. We begin by examining the computational strength of this new form of recursion. We prove that it is primitive recursively equivalent to the original Spector bar recursion, and thus in particular exists in the same models. Then, in the main part of the paper, we show that demand-driven bar recursion can be used to give an alternative functional interpretation of classical countable choice. We use it to extract a new bar recursive program from the proof that there is no injection from $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}$, and this turns out to be both more intuitive and for many inputs more efficient that the usual program obtained using Spector bar recursion.

1 Introduction

In 1962 C. Spector extended Gödel’s functional (Dialectica) interpretation of classical arithmetic [9] to full classical analysis by proving that the functional interpretation of countable choice, and hence full arithmetical comprehension, could be realized by a novel form of recursion which has come to be known as Spector’s bar recursion [17]. Since then, this seminal work has been extended in several ways, and in particular a number of novel variants of bar recursion have been devised to give computational interpretations to classical analysis in new settings, to the extent that bar recursion, in one form or another, is one of the most recognisable methods of giving a computational interpretation to mathematical analysis.

Spector’s original aim was to extend Gödel’s proof of the (relative) consistency of Peano arithmetic to classical analysis. For this purpose, bar recursion is very well suited, allowing us to elegantly and easily expand the soundness of the Dialectica interpretation to incorporate the double negation shift and thus classical countable choice. However, in recent decades applications of proof interpretations such as the Dialectica and modified realizability have moved away from foundational concerns and towards the more practical issue of extracting computational content from proofs. In line with this shift of emphasis comes an increasing interest in how the modes of computation assigned to non-constructive principles behave.

From the perspective of program extraction, bar recursion seems somewhat arbitrary and potentially inefficient. The defining characteristic of bar recursion is that it carries out computations over some well-founded tree of finite sequences $s$, always making recursive calls in a sequential manner over extensions $s \ast (x)$ of these finite sequences. This strict adherence to sequentiality means that in practice, when constructing an approximation to a choice sequence using bar recursion, elements of the approximation are always computed in order, even if we do not require knowledge of all of them.

The purpose of this paper is to introduce a new, demand-driven alternative to Spector’s bar recursion, in which the order of the recursive calls is not fixed but rather directly controlled by its parameters. We prove that this new form of recursion is capable of realizing the Dialectica interpretation of countable choice, and
moreover argue that it is superior to Spector’s original bar recursion because the manner in which it constructs approximations to choice sequences is much more sensitive to its environment. We illustrate this with an example in which we extract realizers from the classical proof that there is no injection from \( \mathbb{N} \to \mathbb{N} \) to \( \mathbb{N} \). In this case the program based on our demand-driven recursion has a much more intuitive behaviour than that based on Spector bar recursion, and significantly outperforms the latter on a small sample of concrete inputs.

Our variant of bar recursion is in some ways similar to the realizer of countable choice proposed by Berardi et al. in \[4\], now often called the BBC-functional. In both cases the recursion is carried out in a ‘symmetric’, based on Spector bar recursion, and significantly outperforms the latter on a small sample of concrete inputs. In this case the program based on our demand-driven recursion has a much more intuitive behaviour than that example in which we extract realizers from the classical proof that there is no injection from \( \mathbb{N} \to \mathbb{N} \) to \( \mathbb{N} \).

Moreover argue that it is superior to Spector bar recursion because the manner in which it constructs approximations to choice sequences is much more sensitive to its environment. We illustrate this with an example in which we extract realizers from the classical proof that there is no injection from \( \mathbb{N} \to \mathbb{N} \) to \( \mathbb{N} \). In this case the program based on our demand-driven recursion has a much more intuitive behaviour than that based on Spector bar recursion, and significantly outperforms the latter on a small sample of concrete inputs.

While several alternatives to bar recursion have been proposed in the context of realizability interpretations of classical analysis, including the aforementioned BBC functional, in this paper we give the first realizer for the Dialectica interpretation of countable choice that is not strictly based on Spector’s bar recursion. We hope that our new method of interpreting choice will contribute towards making the Dialectica interpretation more intuitive and efficient in its role in extracting computational content from proofs.

### 1.1 Preliminaries

The basic formal theory we work over is fully extensional\(^1\) Heyting arithmetic \( \text{E-HA}^\omega \) in all finite types, whose quantifier-free fragment is Gödel’s system \( T \) of primitive recursive functionals (see \[12\] \[18\] for full details). The finite types \( \mathcal{T} \) are typically defined using the following inductive rules

\[
\mathcal{T} ::= \mathbb{N} \mid X \times Y \mid X \to Y
\]

We expand these basic types with a several standard ‘definable’ types, including the unit \( 1 \) and Boolean \( \mathbb{B} = \{0, 1\} \) types, finite sequence types \( X^* \) and co-product types \( X + Y \). We consider partial sequences \( \mathbb{N} \to X \) to be objects of type \( \mathbb{N} \to X + 1 \), where \( 1 \) denotes an ‘undefined’ value.

Finally, we consider one slightly non-standard type: the type \( X^\dagger \) of finite partial functions, that is partial functions defined at only finitely many points. This type can easily be simulated as in \[3\] by \( (\mathbb{N} \times X)^* \), although to minimise syntax we treat \( X^\dagger \) as primitive and avoid any further details of how it can be precisely encoded using the usual types.

For both partial and finite partial sequences \( u \) we write \( \text{dom}(u) \subseteq \mathbb{N} \) to denote the domain of \( u \), and write \( n \in \text{dom}(u) \) whenever \( u \) is defined at \( n \). Note that membership of \( \text{dom}(u) \) is always a decidable predicate. Moreover, there exists a computable functional \( X^\dagger \to \mathbb{N}^* \) for each finite partial function \( u \) encodes its finite domain as a sequence of natural numbers (if \( u \) is encoded in the type \( (\mathbb{N} \times X)^* \) this functional would simply be the first projection).

We write \( x : X \) or \( x^X \) to signify “\( x \) is an object of type \( X \)”, and sometimes write \( Y^X \) instead of \( X \to Y \). The term \( 0_X : X \) denotes the usual inductively defined zero object of each type \( X \), used as a canonical representative of \( X \) - we use the convention that \( 0_{X^\dagger} = \emptyset \) and \( 0_{X^\dagger} = 0 \). In addition to the basic constructors for dealing with the finite types, \( \text{E-HA}^\omega \) contains variables and quantifiers for all types, the predicate \( =_{\text{at}} \) and axioms for equality between integers, induction over arbitrary formulas, combinators which allow us to carry out \( \lambda \)-abstraction, and primitive recursors \( R_X \) for each type which satisfy

\[
\begin{align*}
R_X^{\omega}(0) &=_x y \\
R_X^{\omega}(n+1) &=_x z_x(R_X^{\omega}(n)).
\end{align*}
\]

\(^1\)As it is well-known, the Dialectica interpretation does not validate full extensionality. The system we are describing here is the one we use to verify the bar recursive interpretation of countable choice, and also for our inter-definability results.
In particular, the recursor of type $\mathbb{N}$ allows us to carry out definition by cases, and also assign characteristic functions to all quantifier-free formulas of E-HA$^\omega$. Finally, higher-type equality $=_X$ for arbitrary $X$ is defined inductively in terms of $=_{\mathbb{N}}$, and is treated as fully extensional via the axioms

$$\forall f^{X\rightarrow^\omega}, x^X, y^X (x =_X y \rightarrow f(x) =_Y f(y)).$$

### 1.2 Notation

We make use of the following notational conventions:

- **Finite sequence constructors.** For $s : X^*$, $|s|$ denotes the length of $s$. We use $\langle x_1, \ldots, x_n \rangle : X^*$ to denote the finite sequence constructor. Hence, we write $\langle \rangle$ for the empty sequence and $\langle x \rangle$ for the singleton sequence containing only $x$.

- **Finite sequence concatenation.** Given two finite sequences $s, t : X^*$ we write $s * t$ for their concatenation. For a finite sequence $s : X^*$ and an object $x : X$ we often write $s * x$ for $s * \langle x \rangle$. We also write $s * \alpha : X^\alpha$ to denote the concatenation of the finite sequence $s : X^*$ with an infinite sequence $\alpha : X^\omega$.

- **Initial finite sequence.** Given an $\alpha : X^\omega$ we write $[\alpha](n) = (\alpha(0), \ldots, \alpha(n-1))$ for the finite initial segment of length $n$ of the infinite sequence $\alpha$.

- **Finite partial function constructors.** We use $\emptyset$ for the finite partial function with empty domain, and $(n, x)$ for the finite partial function defined only at point $n$ with value $x$.

- **Ordering of finite partial functions.** Given two finite partial functions $u, v : X^\dagger$ we write $u \subseteq v$ if the domain of $v$ contains the domain of $u$, and $u$ and $v$ coincide on the domain of $u$.

- **Merging finite partial functions.** Given two finite partial functions $u, v : X^\dagger$ we write $u @ v$ to denote the “union” of the two partial functions, where we given priority to the values of $u$ when $u$ and $v$ are both defined at some common point. For a finite partial function $u : X^\dagger$ and an object $x : X$ we write $s @ (n, x)$ for $s @ (n, x)$, to stress the view that this is an update of $u$ with the new value $x$ at point $n$. Of course, if $s$ is already defined at point $n$ then $s @ (n, x) = s$.

- **Canonical extensions.** The canonical extension $\hat{s} : X^\omega$ of the finite sequence $s : X^*$ is defined by $\hat{s}(i) = s_i$ for $i < |s|$ and else $\hat{s}(i) = 0_X$. The canonical extension $\hat{u} : X^\omega$ of a finite partial function $u : X^\dagger$ is defined analogously. Given a function $q : X^\omega \rightarrow R$ we also talk about its canonical extension, and write $\hat{q}$ for the function $\hat{q}(s) = q(\hat{s})$ so that $\hat{q}$ can be either of type $X^* \rightarrow R$ or $X^\dagger \rightarrow R$.

- **Partial application.** As a generalisation of currying, given a function $q : X^\omega \rightarrow R$ and an $s : X^*$, we write $q_s : X^\dagger \rightarrow R$ for the function defined by $q_s(\alpha) = q(s * \alpha)$.

### 2 A demand-driven variant of Spector’s bar recursion

We begin by introducing our novel variant of bar recursion, together with a series of important definitions and recursion-theoretic groundwork that will be needed in the remainder of the paper. In particular we postpone any mention of the Dialectica interpretation until later in the paper. Let us first recall the defining equation of Spector’s general bar recursor $\text{BR}_{X,R}$, which is given by

$$\text{BR}^{\phi, \beta, \varphi}_{X,R} (s^X) =_R \begin{cases} 
\beta(s) & \text{if } \varphi(\hat{s}) < |s| \\
\phi_s(\lambda x^X. \text{BR}^{\phi, \beta, \varphi}(s * x)) & \text{otherwise}
\end{cases}$$
where the parameters have type \( \phi: X^* \to (X \to R) \to R \), \( b: X^* \to R \) and \( \varphi: X^\mathbb{N} \to \mathbb{N} \) (we will always omit these parameters \( \phi, b, \varphi \) from the superscript when there is no danger of ambiguity) and \( X, R \) range over arbitrary types. Just as normal primitive recursion forms a computational analogue of induction, bar recursion can be viewed as a computational analogue of the principle of bar induction:

\[
\text{BI} : \forall \alpha X^\mathbb{N} \exists n P(\langle \alpha \rangle(n)) \land \forall X^\mathbb{N} (\forall x P(t * x) \to P(t)) \to P(\langle \rangle),
\]

where \( P \) is some predicate over finite sequences. Bar induction follows classically from dependent choice, and it plays an important role in intuitionistic mathematics precisely as a constructive version of dependent choice (a more detailed discussion of this principle from an intuitionistic perspective can be found in [18]). Intuitionistic concerns aside, bar induction is an extremely useful tool for reasoning about bar recursion.

The parameter \( \varphi \) acts as a ‘control’ for \( \text{BR}^{b,\varphi} \), whole role is to ensure that at some point the recursive calls stop. Therefore Spector’s bar recursor is well-founded only if each control parameter eventually satisfies \( \varphi(\bar{\delta}) < |s| \) for arbitrary sequences of inputs. We call this requirement Spector’s condition, which can be formulated more precisely as

\[
\text{Spec}_\alpha : \forall \varphi X^\mathbb{N} \exists n \forall \alpha X^\mathbb{N} \exists n(\bar{\varphi}(\langle \alpha \rangle(n)) < n).
\]

As demonstrated by Howard using a trick attributed to Kreisel, Spec must be valid in any model of bar recursion.

**Theorem 2.1** (Howard/Kreisel [13]). \[\text{HA}^\omega + (\text{BR}) \vdash \text{Spec} \].

For this reason, \( \text{BR} \) is in general not well-defined in full type structure of all set-theoretic functionals. However, it is well known to exist in most continuous type-structures (such as the Kleene/Kreisel continuous functionals [11], [13]), and somewhat surprisingly it also exists in the discontinuous type structure of strongly majorizable functionals [7]. It is not too difficult to see that Spec is valid in continuous models, since these satisfy the following sequential continuity principle:

\[
\text{Cont} : \forall \varphi X^\mathbb{N} \exists n \forall \alpha X^\mathbb{N} \exists \beta(\langle \alpha \rangle(N) \equiv \beta[N]) \to \varphi(\alpha) \equiv \beta(\varphi(\beta)),
\]

and therefore if \( N \) is a point of continuity on \( \varphi \) and \( \alpha \) then \( \bar{\varphi}(\langle \alpha \rangle(N)) < N' \) holds for \( N' := \max(N, \varphi(\alpha) + 1) \). In fact, to show that \( \text{BR} \) defines a total continuous functional, we can argue by bar induction on the predicate

\[
P(t) \equiv \text{BR}^{b,\varphi}(s * t) \text{ is total}
\]

where \( \varphi, b, \varphi \) and \( s \) are arbitrary total arguments. Given an infinite sequence \( \alpha: X^\mathbb{N} \) it is clear by Spec that \( \bar{\varphi}(s * \langle \alpha \rangle(n)) < |s| + n \) for some \( n \) and therefore \( \text{BR}^{b,\varphi}(s * \langle \alpha \rangle(n)) = b(s * \langle \alpha \rangle(n)) \) is total. Clearly the bar induction step \( \forall X^\mathbb{N} (\forall x P(t * x) \to P(t)) \) holds and thus we obtain \( P(\langle \rangle) \) and therefore totality of \( \text{BR}^{b,\varphi}(s) \). A broadly similar but somewhat more involved application of bar induction proves that \( \text{BI} \) exists in the majorizable functionals (see [7], [12]).

To summarise, the basic idea behind Spector’s bar recursion is that any sequence of recursive calls made by \( \text{BR} \) eventually hits a bar \( s \) at which the condition \( \varphi(\bar{\delta}) < |s| \) holds and therefore \( \text{BR}(s) \) is assigned a value \( b(s) \). These values propagate backwards along the tree of recursive calls ensuring that \( \text{BR} \) is defined everywhere.

One can view bar recursion as an instance of a more general form of backward recursion in which the main argument is some element with finite domain (here a finite sequence \( s \)), and recursive calls are made by extending the domain of this argument (in this case by extending the argument with one element \( s \ast x \)). From this perspective we see that bar recursion is quite constrained in that the domain of its input is always an initial segment of the natural numbers. This has two obvious disadvantages. Firstly, the implicit dependence on the ordering of the natural numbers makes it unclear how to generalise \( \text{BR} \) to carry out recursion over a wider class of arguments indexed by finite domains. Secondly, adherence to sequentiality means that precise values of the control functional \( \varphi \) are never required: all that matters is whether or not \( \varphi(\bar{\delta}) < |s| \), or in other words, whether
or not \( \varphi(\hat{\alpha}) \) is within the domain of already computed values (as we show in Section 4, it is this condition that is important for giving a computation interpretation to countable choice).

It is natural, then, to ask whether there is an alternative to bar recursion which still terminates on inputs \( u \) with \( \varphi(\hat{\alpha}) \) in the domain of \( u \), but which search for these points in a more flexible way, taking into account information provided by \( \varphi \). This is the idea behind our variant of bar recursion, which we call symmetric bar recursion. The symmetric bar recursor \( sBR_{X,R} \) is given by the defining equation

\[
sBR_{X,R}^h,\varphi(u : X^i) =_R \begin{cases} b(u) & \text{if } \hat{\varphi}(u) \in \text{dom}(u) \\ \phi_h(\lambda x . X) \cdot sBR_{X,R}^h,\varphi(u \oplus (\hat{\varphi}(u), x)) & \text{otherwise} \end{cases}
\]

where now the parameters have type \( \varphi : X^i \to (X \to R) \to R, b : X^i \to R \) and \( \varphi : X^i \to \mathbb{N} \). Recall that the operation \( \oplus \) indicates the extension of the partial function \( u \) with one more piece of information, analogous to the extension of finite partial functions in the defining equation of \( BR \). The crucial difference is that this extension can potentially take place at any point \( n \in \mathbb{N} \setminus \text{dom}(u) \), and so we are no longer restricted to making recursive calls in a sequential fashion. However, this additional freedom requires us to carefully justify the totality of \( sBR \), as its recursive calls are not easily seen to be well-founded. In Definition 2.6 below we give a corresponding symmetric bar induction principle which can be used to reason about \( sBR \). But first we need the following important definition:

**Definition 2.2** (\( \varphi \)-thread of \( u \): \( X^i \) or \( \alpha : X^3 \)). Given \( \varphi : X^i \to \mathbb{N} \), for \( u : X^i \) we call the \( \varphi \)-thread of \( u \) the sequence \( \{u\}_\varphi(i) : X^i \) inductively defined as

\[
\begin{align*}
\{u\}_\varphi(0) & := \emptyset \\
\{u\}_\varphi(i + 1) & := \begin{cases} 
\{u\}_\varphi(i) \oplus (n_{\varphi,i}, u(n_{\varphi,i})) & \text{if } n_{\varphi,i} \in \text{dom}(u) \\
\{u\}_\varphi(i) & \text{otherwise}
\end{cases}
\end{align*}
\]

where \( n_{\varphi,i} := \hat{\varphi}(\{u\}_\varphi(i)) \). Similarly, for \( \alpha : X^3 \) define the \( \varphi \)-thread of \( \alpha \) to be the sequence

\[
\begin{align*}
\{\alpha\}_\varphi(0) & := \emptyset \\
\{\alpha\}_\varphi(i + 1) & := (n_{\varphi,i}, \alpha(n_{\varphi,i}))
\end{align*}
\]

where \( n_{\varphi,i} := \hat{\varphi}(\{\alpha\}_\varphi(i)) \).

The intuition here is that \( \varphi \) works as a control function, that directs the next update, given what has been updated so far. For a sequence that has been constructed by using \( \varphi \) to guide its sequence of update, its \( \varphi \)-thread should be the whole sequence.

**Remark 2.3.** In what follows we will frequently just write \( \{u\}(i) \) when \( \varphi \) is clear from the context.

**Definition 2.4** (\( \varphi \)-threads). Let \( \varphi : X^i \to \mathbb{N} \). We say that a finite partial function \( u : X^i \) is a \( \varphi \)-thread if it equals its \( \varphi \)-thread. This can be expressed formally by the decidable predicate

\[
S_\varphi(u) := \forall n \in \text{dom}(u) \exists i \leq |\text{dom}(u)| (n \in \text{dom}(\{u\}_\varphi(i))).
\]

**Lemma 2.5.** A finite partial function \( u \) satisfies \( S_\varphi(u) \) iff for all \( i \leq |\text{dom}(u)| \),

\[
\{u\}_\varphi(i) = (n_{\varphi,0}, x_0) \oplus (n_{\varphi,1}, x_1) \oplus \ldots \oplus (n_{\varphi,i-1}, x_{i-1})
\]

where \( n_{\varphi,i} \in \text{dom}(u) \) are all distinct, and \( x_j = u(n_{\varphi,j}) \). In particular

\[
S_\varphi(u) \Rightarrow u = \{u\}_\varphi(l) = (n_{\varphi,0}, x_0) \oplus \ldots \oplus (n_{\varphi,l-1}, x_{l-1})
\]

where \( l = |\text{dom}(u)| \).
Proof. First assume \(S_\varphi(u)\), and set \(l := |\text{dom}(u)|\). We prove by induction on \(i \leq l\). If \(i = 0\) then \([u]_\varphi(0) = \emptyset\) by definition. Now, for some \(i < l\), assume that \([u]_\varphi(i) = (n_{\varphi,0}, x_0) \oplus \ldots \oplus (n_{\varphi,i-1}, x_{i-1})\) for distinct \(n_{\varphi,j}\) and \(x_j = u(n_{\varphi,j})\). If \(n_{\varphi,i} \notin \text{dom}(u)\) or if \(n \in \{n_{\varphi,0}, \ldots, n_{\varphi,i-1}\}\) then, by the definition of the \(\varphi\)-thread of \(u\), we would have that \([u]_\varphi(i-1) = [u]_\varphi(i) = \ldots = [u]_\varphi(l)\), and hence \(|\text{dom}([u]_\varphi(l))| < l\), contradicting our assumption \(S_\varphi(u)\).

Therefore, we must have \(n_{\varphi,i} \in \text{dom}(u)\) for \(n_{\varphi,0}, \ldots, n_{\varphi,i-1}\) and

\[
[u]_\varphi(i + 1) = (n_{\varphi,0}, x_0) \oplus \ldots \oplus (n_{\varphi,i-1}, x_{i-1}) \oplus (n_{\varphi,i}, x_i)
\]

for \(x_i = u(n_{\varphi,i})\), where the \(n_{\varphi,j}\) must be distinct for \(j \leq i\). For the other direction, \(u = [u](l)\) for \(l = |\text{dom}(u)|\) clearly implies \(S_\varphi(u)\).

Definition 2.6. Let us write \(\forall u \in S_\varphi A(u)\) as an abbreviation for \(\forall u(S_\varphi(u) \rightarrow A(u))\). The principle \(s\text{BI}\) of symmetric bar induction is given by

\[
s\text{BI} : \forall \varphi^{X^\omega} \exists \iota \exists P(\exists \alpha(n)) \wedge \forall u \in S_\varphi([\varphi(\dot{u}) \notin \text{dom}(u) \wedge \exists x \exists \neg P(u \oplus (\varphi(\dot{u}), x))] \rightarrow P(u) \rightarrow P(\emptyset))
\]

where \(P\) is an arbitrary predicate on \(X^\omega\).

Theorem 2.7. \(\text{PA}^{\omega} + \text{DC} \vdash \text{sBI}\).

Proof. Fix some \(\varphi\) and suppose for contradiction that the premise of \(s\text{BI}\) is true but \(\neg P(\emptyset)\) holds. The second premise of \(s\text{BI}\) is classically equivalent to

\[
\forall u \in S_\varphi(\neg P(u) \rightarrow [\varphi(\dot{u}) \notin \text{dom}(u) \wedge \exists x \exists \neg P(u \oplus (\varphi(\dot{u}), x))]).
\]

Hence, by dependent choice, there exists a sequence \(u_0, u_1, \ldots\) of finite partial functions satisfying

\[
u_0 = \emptyset \quad \text{and} \quad u_{i+1} = u_i \oplus (n_i, x_i)
\]

where \(n_i = \varphi(\dot{u}_i) \notin \text{dom}(u_i)\) and \(S_\varphi(u_i)\) and \(\neg P(u_i)\) for all \(i\). Now, by classical countable choice there exists a function \(\alpha : X^{\omega}\) satisfying

\[
\alpha(n) := \begin{cases} u_i(n) & \text{where } i \text{ is the least such that } n \in \text{dom}(u_i), \text{if it exists} \\ \emptyset_X & \text{otherwise.} \end{cases}
\]

We first show by induction that \([\alpha]_\varphi(i) = u_i\) for all \(i\). First, \([\alpha](0) = \emptyset\) by definition. Now, assuming that \([\alpha](i) = u_i\), we have \(\dot{\varphi}(\alpha(i)) = \varphi(\dot{u}_i) = n_i \in \text{dom}(u_{i+1})\), and therefore \([\alpha](i + 1) = [\alpha](i) \oplus (n_i, \alpha(n_i)) = u_i \oplus (n_i, u_{i+1}(n_i)) = u_{i+1}\). But by the first premise of \(s\text{BI}\) there exists some \(n\) such that \(P([\alpha](n))\), which implies \(P(u_n)\), a contradiction. \(\square\)

Symmetric bar recursion is well-founded only if arbitrary control functionals eventually satisfy \(\dot{\varphi}(u) \in \text{dom}(u)\) for large enough arguments \(u\). Put formally, this requirement can be seen as a symmetric analogue of \(\text{Spec}\), namely

\[
s\text{Spec}_X : \forall \varphi^{X^\omega} \exists \alpha^{X^\omega} \exists n(\dot{\varphi}([\alpha]_\varphi(n)) \in \text{dom}([\alpha]_\varphi(n))).
\]

In fact, analogously to \([10]\), we can prove that \(s\text{Spec}\) must be valid in any model of \(s\text{BR}\).

Proposition 2.8. Define the term \(\theta_{\varphi, \alpha}\) in \(\text{E-HA}^{\omega} + (s\text{BR})\) with free variables \(\alpha : X^{\omega}\) and \(\varphi : X^{\omega} \rightarrow \mathbb{N}\) by

\[
\theta_{\varphi, \alpha}(u_X) = \begin{cases} 0 & \text{if } \varphi(\dot{u}) \in \text{dom}(u) \\ 1 + \theta_{\varphi, \alpha}(u \oplus (\varphi(\dot{u}), \alpha(\varphi(\dot{u})))) & \text{otherwise.} \end{cases}
\]

Then, provably in \(\text{E-HA}^{\omega} + (s\text{BR})\), we have \(\dot{\varphi}([\alpha]_\varphi(n)) \in [\alpha]_\varphi(n)\) for some \(n \leq \theta_{\varphi, \alpha}(\emptyset)\).
Proof. Let $\beta i := \theta_\varphi^\prime((\alpha)_{\varphi}(i))$. By definition of $\theta_\varphi^\prime$, we have

$$
\beta i = \begin{cases} 
0 & \text{if } \hat{\varphi}((\alpha)_{\varphi}(i)) \in \text{dom}((\alpha)_{\varphi}(i)) \\
1 + \beta(i + 1) & \text{otherwise.}
\end{cases}
$$

First note that, by the definition of $\beta$, we have $(\ast) \beta i \neq 0$ if $\hat{\varphi}((\alpha)_{\varphi}(i)) \notin \text{dom}((\alpha)_{\varphi}(i))$. By induction on $i$, using $(\ast)$, it is easy to show

$$\forall i(\forall j \leq i (\beta j \neq 0) \rightarrow \forall j \leq i (\beta j = 1 + \beta(j + 1))).$$

By another induction on $i$, using the above fact, we obtain

$$\forall i(\forall j \leq i (\beta j \neq 0) \rightarrow \beta 0 = i + \beta i).$$

Therefore, setting $i = \beta 0$ we have $\forall j \leq \beta 0(\beta j \neq 0) \rightarrow \beta 0 = \beta 0 + \beta(\beta 0)$. This clearly implies that $\beta j = 0$ for some $j \leq \beta 0$, which, by $(\ast)$, is equivalent to

$$\exists j \leq \beta 0(\hat{\varphi}((\alpha)_{\varphi}(j)) \in \text{dom}((\alpha)_{\varphi}(j))).$$

That concludes the proof since $\beta 0 = \theta_\varphi^\prime(0)$. \hfill \square

We now make our first link between symmetric bar recursion and Spector’s bar recursion via their corresponding axioms $\mathfrak{sSpec}$ and $\mathfrak{Spec}$.

**Theorem 2.9.** $\text{E-HA}^\omega + \mathfrak{sSpec}_{\times\mathbb{B}} \vdash \mathfrak{Spec}_{\times}$

**Proof.** Given $\alpha : X^\omega$ and $\varphi : X^\omega \rightarrow \mathbb{N}$ let us define $\tilde{\alpha} : (X \times \mathbb{B})^\omega$ and $\theta : (X \times \mathbb{B})^\omega \rightarrow \mathbb{N}$ as

$$\tilde{\alpha}(n) := (\alpha(n), 1),$$

$$\theta(\beta) := \mu i \leq \varphi(\lambda k.\pi_0(\beta k))(\pi_1(\beta i) = \mathbb{B} = 0).$$

Intuitively, we are using the booleans to indicate whether a position is “defined” (i.e., equal 1) or not. Hence, the functional $\theta$ returns the first undefined position less than $\varphi(\lambda k.\pi_0(\beta k))$. By $\mathfrak{sSpec}$ there exists some $N$ such that

(i) $\tilde{\theta}([\tilde{\alpha}]_\varphi(N)) \in \text{dom}([\tilde{\alpha}]_\varphi(N))$. Without loss of generality let $N$ be the least such value.

We will show that $\hat{\varphi}([\alpha](N)) < N$. But first we claim that

(ii) $\forall m \leq N(\text{dom}([\tilde{\alpha}]_\varphi(m)) = \{0, \ldots, m - 1\})$ and $\forall m < N(\hat{\varphi}([\alpha](m)) \geq m)$.

The proof of (ii) is by induction on $m$. If $m = 0$ the claim is trivial. For the induction step fix $m$ and assume

(IH1) $\text{dom}([\tilde{\alpha}]_\varphi(m - 1)) = \{0, \ldots, m - 2\}$, and hence $\hat{\varphi}(\lambda k.\pi_0([\tilde{\alpha}]_\varphi(m - 1)(k))) = \hat{\varphi}([\alpha](m - 1))$

(IH2) $\hat{\varphi}([\alpha](m - 1)) \geq m - 1$.

We first show that if $m \leq N$ then $\text{dom}([\tilde{\alpha}]_\varphi(m)) = \{0, \ldots, m - 1\}$. By the induction hypothesis (IH1) and the definition of $[\tilde{\alpha}]_\varphi(m)$ it is enough to show that $\hat{\theta}([\tilde{\alpha}]_\varphi(m - 1)) = m - 1$. This indeed follows by the definition of $\theta$ since, by (IH1), $m - 1$ is the first point which is undefined in $[\tilde{\alpha}]_\varphi(m - 1)$ and, by (IH1) and (IH2), $m - 1 \leq \hat{\varphi}(\lambda k.\pi_0([\tilde{\alpha}]_\varphi(m - 1)(k)))$. We then show, assuming $m < N$, that $\hat{\varphi}([\alpha](m)) \geq m$ as follows: For the sake of a contraction assume $\hat{\varphi}([\alpha](m)) < m$. Hence, from $\text{dom}([\tilde{\alpha}]_\varphi(m)) = \{0, \ldots, m - 1\}$ just shown, it follows

$$m > \hat{\varphi}([\alpha](m)) = \hat{\varphi}(\lambda k.\pi_0([\tilde{\alpha}]_\varphi(m)(k))) \geq \hat{\theta}([\tilde{\alpha}]_\varphi(m))$$

and hence $\hat{\theta}([\tilde{\alpha}]_\varphi(m)) \in \text{dom}([\tilde{\alpha}]_\varphi(m))$, contradicting the minimality of $N$ in (i); concluding the proof of (ii).

By (ii), taking $m = N$, we obtain
Furthermore, primitive recursively define the functional \( \phi([a]_\alpha(N)) = \phi(\lambda k. \pi_0((\tilde{a})_\alpha(N)(k))) \).

Putting (iv) and (v) together and unwinding the definition of \( \theta \) we have

\[
\mu i \leq \tilde{\phi}([a](N))(\pi_1((\tilde{a})_\alpha(N)(i)) = \emptyset) \quad \overset{(iv)}{=} \quad \begin{cases} 
\mu i \leq \tilde{\phi}([a](N))(\pi_1((\tilde{a})_\alpha(N)(i)) = \emptyset) \quad \overset{(iv)}{=} \quad \hat{\theta}((\tilde{a})_\alpha(N)) < N,
\end{cases}
\]

where \( n = \Psi \varphi \alpha \).

**Proof.** Applying modified realizability to the proof of Theorem 2.9, and using Proposition 2.8 to prove the Skolemized variant of sSpec using sBR.

**Theorem 2.11.** WE-PA^\omega + AC_0 + Spec_{X^1} \vdash sSpec_{X}.

**Proof.** Fix \( \alpha : X^{\mathbb{N}} \) and \( \varphi : X^{\mathbb{N}} \rightarrow \mathbb{N} \) and (in WE-PA^\omega + AC_0) define the sequence \((i_n)_{n \in \mathbb{N}}\) as

\[ i_n := \begin{cases} 
\hat{i} & \text{where } i \text{ is the least such that } n \in \text{dom}([\alpha]_\varphi(i)) \\
0 & \text{if no such } i \text{ exists.}
\end{cases} \]

Using \((i_n)_{n \in \mathbb{N}}\) we can define the sequence \( \tilde{\alpha} : (X^1)^{\mathbb{N}} \) as

\[ \tilde{\alpha}(n) := \begin{cases} 
[\alpha]_\varphi(i_n) & \text{if } n \in \text{dom}([\alpha]_\varphi(i_n)) \\
\emptyset & \text{otherwise.}
\end{cases} \]

Furthermore, primitive recursively define the functional \( \theta : (X^1)^{\mathbb{N}} \rightarrow \mathbb{N} \) by \( \theta(\beta) := \tilde{\phi}(d(\beta)) \) where

\[ d(\beta) := \begin{cases} 
\beta(i)(j) & \text{for least } i \leq j \text{ such that } j \in \text{dom}(\beta(i)) \\
\text{undefined} & \text{otherwise.}
\end{cases} \]

Let us begin by observing that the sequence \(([\alpha]_\varphi(k))_{k \in \mathbb{N}}\) is monotone, in the sense that

\[
(i) \quad [\alpha]_\varphi(k) \subseteq [\alpha]_\varphi(k + 1), \quad \text{where } \subseteq \text{ denotes extension of finite partial functions.}
\]

So, applying Spec to \( \theta \) and \( \tilde{\alpha} \) we obtain an \( N \) such that

\[
(ii) \quad \hat{\theta}((\tilde{\alpha})_\alpha(N)) < N.
\]

Let \( i_m = \max[i_0, \ldots, i_{N-1}] \), for some \( m < N \). We will prove that \( n_m \in \text{dom}([\alpha]_\varphi(i_m)) \). Let \( \text{emb}(u) \) denote the embedding of finite partial sequences into the type of arbitrary partial sequences. We first claim that

\[
(iii) \quad [\tilde{a}]_\alpha(N)(n) \subseteq [\alpha]_\varphi(i_m), \quad \text{for all } n.
\]
If \( n \geq N \), recalling that \( \theta_{X'} = \emptyset \), then \([\tilde{\alpha}](N)(n) = \emptyset\) and the result is obvious. If \( n < N \) and \( n \notin \text{dom}(\alpha) \), then \([\tilde{\alpha}](N)(n) = \tilde{\alpha}(n) = \emptyset\), and again the result is obvious. Finally, if \( n < N \) and \( n \in \text{dom}(\alpha) \), then \([\tilde{\alpha}](N)(n) = \tilde{\alpha}(n) = \{\alpha\}_\varphi(i_n)\). By (i) we have \([\alpha]_{\varphi}(i_n) \subseteq \{\alpha\}_\varphi(i_m)\), since \( i_n \leq i_m \).

From (iii) and the definition of \( d \), we claim that

\[
(iv \ d([\tilde{\alpha}](N)) = \{\alpha\}_\varphi(i_m), \text{i.e. the two partial functions are equal when defined.}
\]

We consider two cases. First, if \( k \notin \text{dom}(d([\tilde{\alpha}](N))) \), then, by the definition of \( d \) we have that \( k \notin \text{dom}(\{\alpha\}_\varphi(i_n)) \) for all \( n < N \). Hence, \( k \notin \text{dom}(\{\alpha\}_\varphi(i_m)) \). The second case is when \( d([\tilde{\alpha}](N))(k) = [\tilde{\alpha}](N)(n)(k) \), where \( n \leq k \) is the least such that \( k \in \text{dom}([\tilde{\alpha}](N))(n) \). But by (iii) we have that \([\tilde{\alpha}](N)(n)(k) = [\alpha]_{\varphi}(i_m)(k) \) and hence \( d([\tilde{\alpha}](N))(k) = [\alpha]_{\varphi}(i_m)(k) \).

Point (iv) above clearly implies

\[
(v \ d([\tilde{\alpha}](N)) = \text{emb}((\alpha)_{\varphi}(i_n)).
\]

Recall that \( n_k \) is defined as \( \hat{\varphi}(\alpha)_{\varphi}(k) \). Hence, by (ii) and (v) and weak extensionality we get

\[
n_{i_m} := \hat{\varphi}(\alpha)_{\varphi}(i_m) = \hat{\varphi}(\text{emb}((\alpha)_{\varphi}(i_m))) = \hat{\varphi}(d([\tilde{\alpha}](N))) = \hat{\theta}([\tilde{\alpha}](N)) < N,
\]

i.e. \( n_{i_m} < N \). If \( n_{i_m} \notin \text{dom}(\alpha) \), then \( n_{i_m} \in \text{dom}(\alpha) \) and hence \( i_{i_m} = i_m + 1 \). But this contradicts the construction of \( i_m \) since \( n_{i_m} < N \) implies \( i_m \geq i_{i_m} = i_m + 1 \).

\[\square\]

**Corollary 2.12.** There is a term \( \Delta \) of E-HA\(\omega \) + BR such that

\[E\text{-HA}\omega \cup BR \vdash \hat{\varphi}(\alpha)_{\varphi}(n) \in \text{dom}(\alpha)_{\varphi}(n),\]

where \( n = \Delta \varphi \alpha \).

**Proof.** Follows directly from Theorem 2.11 and the facts that

- WE-PA\(\omega \) + AC\(\omega \) has a Dialectica interpretation in E-HA\(\omega \) + BR (Spector’62).
- E-HA\(\omega \) + BR + Spec (Howard/Kreisel’68).

Note that it is essential that full extensionality was not used in the proof of Theorem 2.11 since the Dialectica interpretation does not validate full extensionality.

\[\square\]

In the next section we expand the ideas in the proofs of Theorems 2.9 and 2.11 to show that the recursors BR and sBR themselves are primitive recursively equivalent. This will also serve the purpose of a formal proof that sBR defines a total continuous functional. Nevertheless, we are already able to give at least an informal argument that sBR defines a total continuous functional, using sBI.

Firstly, since the total continuous functionals validate Spec and countable choice, they also validate sSpec by Theorem 2.9. Now define the predicate \( P(u) \) by

\[P(u) \equiv sBR^{\phi, \varphi}(v @ u)\] if total,

where \( \phi, \beta, \varphi \) and \( v \) are arbitrary total arguments, and carry out the symmetric bar induction argument relative to the functional \( \tilde{\varphi}: X^\omega \to \mathbb{N} \) given by \( \tilde{\varphi}(f) := \varphi(v @ f) \). Firstly, given a function \( \alpha \) let \( n \) be the point given by sSpec. Then clearly sBR of sBR(\( \alpha \)\( \omega \))(n)) is defined. The second hypothesis of sBI follows since for arbitrary u, sBR(\( v @ u \)) is well-defined unless \( \tilde{\varphi}(v @ u) = \varphi(v @ \tilde{u}) = \tilde{\varphi}(u) \notin \text{dom}(v @ u) \), which implies in particular that \( \tilde{\varphi}(u) \notin \text{dom}(u) \). But in that case we can assume that for all x, sBR(\( v @ [u \oplus (\tilde{\varphi}(\tilde{u}), x)] \)) = sBR(\( v @ u \oplus (\tilde{\varphi}(\tilde{u}), x) \)) is well-defined, and so sBR(\( v @ u \)) is also well-defined. Therefore we derive \( P(0) \), and thus totality of sBR\(^{\phi, \varphi}(v) \).
3 Equivalence of BR and sBR over E-HA$^\omega$

In this section we prove that BR and sBR are primitive recursively equivalent. This is the most technical part of the paper, but is entirely self-contained and as such the reader can skip straight ahead to Section 4 if they wish. The most difficult direction – the definability of sBR from BR – can be carried out in E-HA$^\omega$ + sBI and hence (by Theorem 2.7) in E-HA$^\omega$ + DC, and thus as an immediate consequence we prove that sBI exists in any model of E-PA$^\omega$ + DC which also validates BI. In particular, sBI exists in both the Kleene/Kreisel total continuous functional and the strongly majorizable functionals.

**Theorem 3.1.** BR is primitive recursively definable from sBR, provably in E-HA$^\omega$.

**Proof.** Suppose we are given parameters $\phi: X^* \rightarrow (X \rightarrow R) \rightarrow R$, $b: X^* \rightarrow R$ and $\varphi: X^N \rightarrow \mathbb{N}$ for BR$^X_R$. Then there is a term $\Phi^{\phi,b,\varphi}$ primitive recursive in sBR$_{X \times \mathbb{N}, R}$, which satisfies the defining equation of BR$^{\phi,b,\varphi}$. We consider the booleans $\mathbb{B}$ to be the set $\{0, 1\}$ and take 0 to the the canonical zero element $0_\mathbb{B}$, and so naturally $0_{X \times \mathbb{B}} = (0_x, 0_\mathbb{B})$. Define the map $\eta: X^* \rightarrow (X \times \mathbb{B})^*$ by

$$(\eta s)(n) := \begin{cases} (s_n, 1) & \text{if } n < |s| \\ \text{undefined} & \text{otherwise,} \end{cases}$$

so that $\text{dom}(\eta s) = \{0, 1, \ldots, |s| - 1\}$; and conversely the map $\eta': (X \times \mathbb{B})^* \rightarrow X^*$ by $|\eta' u| = N + 1$ where $N$ is the maximum element of $\text{dom}(u)$, and

$$(\eta' u)_i := \begin{cases} \pi_0(u(i)) & \text{if } i \in \text{dom}(u) \\ 0_x & \text{otherwise} \end{cases}$$

where $\pi_0: X \times \mathbb{B} \rightarrow X$ is the first projection. Note that $\eta' \eta s = s$ for all $s: X^*$. Now, define parameters $\tilde{\phi}, \tilde{b}$ and $\tilde{\varphi}$ for sBR$_{X \times \mathbb{B}, R}$ by

$$\tilde{\varphi}(\alpha^{(X \times \mathbb{B})^*}) := \mu \leq \varphi(\pi_0 \circ \alpha)[\pi_1 \alpha(i) =_B 0]$$

$$\tilde{b}(\mu^{(X \times \mathbb{B})^*}) := b(\eta' u)$$

$$\tilde{\phi}_\mu(p^{X \times \mathbb{B} \rightarrow R}) := \phi_{\mu, \alpha(x)}(\lambda x \in X. p(x, 1)))$$

In the definition of $\tilde{\varphi}$, $\mu$ denotes the primitive recursively definable bounded search operator that returns the least $i \leq \varphi(\pi_0 \circ \alpha)$ satisfying the decidable predicate $\pi_1 \alpha(i) = 0$, and $\varphi(\pi_0 \circ \alpha)$ if no such $i$ exists. We claim that $\Phi^{\phi,b,\varphi}(s) := s\text{BR}^{\phi,b,\varphi}(\eta s)$ satisfies the defining equation of BR$^{\phi,b,\varphi}(s)$. To prove this, first note that

(i) $\pi_0 \circ \tilde{\eta} s = X \times \mathbb{B}$, and

(ii) $(\pi_1 \circ \tilde{\eta} s)(i) =_B 1$, for $i < |s|$ and 0 otherwise.

Therefore

(iii) $\tilde{\varphi}(\tilde{\eta} s) = \mu \leq \varphi(\pi_0 \circ \tilde{\eta} s)[(\pi_1 \circ \tilde{\eta} s)(i) =_B 0] \overset{(i)}{=} \mu \leq \varphi(\tilde{s})[\pi_1 \circ \tilde{\eta} s](i) = 0 \overset{(i)}{=} \begin{cases} \varphi(\tilde{s}) & \text{if } \varphi(\tilde{s}) < |s| \\ |s| & \text{if } \varphi(\tilde{s}) \geq |s| \end{cases}$

Since $\text{dom}(\eta s) = \{0, \ldots, |s| - 1\}$ point (iii) above implies the equivalence

(iv) $\varphi(\tilde{s}) < |s| \iff \tilde{\varphi}(\tilde{\eta} s) \in \text{dom}(\eta s)$. 

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Therefore, if \( \varphi(\bar{s}) < |s| \) then \( \bar{\varphi}(\bar{s}s) \in \text{dom}(\eta s) \) and hence
\[
\Phi(s) = s\text{BR}^{\bar{s},\bar{\varphi},\varphi}(\eta s) = b(\eta s) = b(\eta' s) = b(s)
\]
and, if \( \varphi(\bar{s}) \geq |s| \) then \( \bar{\varphi}(\bar{s}s) \notin \text{dom}(\eta s) \) and hence
\[
\Phi(s) = s\text{BR}^{\bar{s},\bar{\varphi},\varphi}(\eta s)
\]
where for the penultimate equality one easily verifies that \( \eta s \oplus (|s|, (x, 1)) = \eta(s * x) \). □

The basic idea behind the preceding proof is that \( \text{BR}_{X,R} \) can be defined from a single instance of \( \text{sBR}_{X,R} \) of (essentially) the same type, in which the symmetric control functional \( \varphi \) is designed to be ‘stubborn’ and always search for the least undefined point to update, thereby simulating Spector’s bar recursion. A similar idea does not seem to work in the opposite direction, however, since Spector’s bar recursion is inherently less flexible than symmetric bar recursion. Instead, to define \( \text{sBR}_{X,R} \) we resort to an instance of Spector’s bar recursion of strictly higher-type, and the resulting construction is somewhat more intricate.

**Theorem 3.2.** Symmetric bar recursion is primitive recursively definable from Spector’s bar recursion, provably in \( \text{E-HA}^\omega + \text{(BR)} + \text{sBI} \).

The main body of the proof of Theorem 3.2 is contained in Lemma 3.3 where for syntactic reasons we first define a slightly simpler (but obviously equivalent) form of \( \text{sBR} \) that only accepts as input partial functions whose domains can be recursively constructed in the parameter \( \varphi \).

**Lemma 3.3.** Let \( \Psi \) have the defining equation
\[
\Psi^{\phi,b,\varphi}(u) := \begin{cases} 
0_R & \text{if } \neg S_{\varphi}(u) \\
b(u) & \text{if } \varphi(\bar{u}) \in \text{dom}(u) \\
\phi_u(\lambda x . \Psi^{\phi,b,\varphi}(u \oplus (\varphi(\bar{u}), x))) & \text{otherwise.} 
\end{cases}
\]

Then \( \Psi \) is primitive recursively definable from Spector’s bar recursion, provably in \( \text{E-HA}^\omega \).

**Proof.** Suppose we are given parameters \( \phi : X^\dag \to (X \to R) \to R \), \( b : X^\dag \to R \) and \( \varphi : X^{\omega} \to \mathbb{N} \) for \( \text{sBR}_{X,R} \). Then there is a term \( \Phi^{\phi,b,\varphi} \) primitive recursive in \( \text{BR}_{X,R} \), where \( Y := X^\dag \times (X^\dag \to (X \to R)) \), that satisfies the defining equation of \( \Psi \).

First, we need some definitions. Suppose \( u,v : X^\dag \) and \( x : X \). Then \( \langle u \rangle(n) * x @ v : X^\dag \) denotes the finite partial function given by
\[
\langle u \rangle(n) * x @ v(i) = \begin{cases} 
u(i) & \text{if } i < n \\
x & \text{if } i = n \\
u(i) & \text{otherwise.}
\end{cases}
\]
Next, define the ‘diagonal’ functional $d: Y^* \rightarrow X^*$ by

$$
d(s)(j) := \begin{cases} 
(\pi_0 s_i)(j) & \text{for least } i \leq j \text{ such that } j \in \text{dom}(\pi_0 s_i) \\
\text{undefined} & \text{otherwise}
\end{cases}
$$

where $\pi_0$ denotes the first projection. The function $d$ returns a particular kind of ‘union’ of the finite partial functions $\pi_0 s_i$, where in the event that the $\pi_0 s_i$ are defined at $j$ for more than one index $i \leq j$, the value at the least index is chosen. Because $s$ is a finite sequence the resulting partial function $d(s)$ must also have finite domain. Similarly, define an infinitary diagonal function $d^{\infty}: Y^{\infty} \rightarrow X^{\infty}$ by

$$
d^{\infty}(\alpha)(j) := \begin{cases} 
(\pi_0 \alpha_i)(j) & \text{for least } i \leq j \text{ such that } j \in \text{dom}(\pi_0 \alpha_i) \\
0_X & \text{otherwise.}
\end{cases}
$$

This function $d^{\infty}$ returns a similar union of the infinite sequence of partial function $\pi_0 \alpha_i$ to return a partial function with potentially infinite domain, and then embeds this partial function in $X^{\infty}$ by assigning the canonical value $0_X$ to undefined elements.

We now define parameters $\tilde{\phi}, \tilde{b}$ and $\tilde{\varphi}$ for $\text{BR}_{Y,R}$ in terms of $\phi, b$ and $\varphi$. First, let

$$
\tilde{\phi}(s^{Y \rightarrow R}) := \begin{cases} 
p(d(s), 0_{X^* \rightarrow (X \rightarrow R)}) & \text{if } |s| \in \text{dom}(d(s)) \\
p(d(s), A v, x . p([d(s)](|s|)) * x @ v, 0_{X^* \rightarrow (X \rightarrow R)}) & \text{otherwise.}
\end{cases}
$$

As will become clear below, $p$ plays the role of a continuation such that if $d(s)$ is not defined at $|s|$, we make a bar recursive call to all ‘updates’ of $d(s)$ of the form $[d(s)](|s|) * x @ v$. For the parameter $\tilde{b}$, define

$$
\tilde{b}(s^{Y \rightarrow R}) := \begin{cases} 
b(d(s)) & \text{if } \varphi(\tilde{d}(s)) \in \text{dom}(d(s)) \\
\tilde{\phi}(d(s), A x^X. (\pi_1 \tilde{\phi}(\tilde{d}(s)))(d(s)))(x) & \text{otherwise.}
\end{cases}
$$

Finally, let

$$
\tilde{\varphi}(\alpha^{Y^{\infty}}) := \varphi(\tilde{d}^{\infty}(\alpha)).
$$

The basic idea behind our construction is as follows: a finite partial state $u$ in the computation of $\Psi$ is represented by a finite sequence $s_u$ in our instance of $\text{BR}$. If $n \in \text{dom}(u)$ then in the first component $s_u(n)$ contains the state $u' \subseteq u$ that was present when point $n$ was updated, and if $n \notin \text{dom}(u)$ then the second component forms a continuation that allows us to make bar recursive calls on updates of the state at any point in the future.

We now define formally define the finite sequence $s_u$, primitive recursively in $\text{BR}^{\tilde{\phi}, \tilde{b}, \tilde{\varphi}}$, as

$$
s_{u,0} := \langle \rangle
$$

$$
s_{u,j+1} := \begin{cases} 
[s_{u,j}](n_i) * \langle u_{i+1}, 0_{X^* \rightarrow (X \rightarrow R)} \rangle & \text{if } n_i < |s_{u,j}| \\
s_{u,j} * \langle v_i, f_{s_{u,j}} \rangle * \ldots * \langle v_1, f_{s_{u,1}} \rangle * \langle u_{i+1}, 0 \rangle & \text{otherwise}
\end{cases}
$$

where be use the abbreviations $u_i := [u]_{\varphi}(i), n_i := \varphi(\tilde{u}_i), v_i := d(s_{u,i})$ and

$$
f_n := X^* \rightarrow (X \rightarrow R) \begin{cases} 
\lambda w, x . \text{BR}^{\tilde{\phi}, \tilde{b}, \tilde{\varphi}}([s_{u,j+1}](n) * \langle [v_j](n) * x @ w, 0 \rangle) & \text{if } n \in \text{dom}(v) \\
0 & \text{otherwise.}
\end{cases}
$$

Let $s_u := s_{u,|\text{dom}(u)|}$. We claim that

$$
\Phi(u) := \begin{cases} 
0 & \text{if } \neg S_{\varphi}(u) \\
\text{BR}^{\tilde{\phi}, \tilde{b}, \tilde{\varphi}}(s_u) & \text{otherwise}
\end{cases}
$$
satisfies the defining equations for $\Psi^{\varphi,b,\varphi}$. This is clearly the case for $\neg S_\varphi(u)$, so from now on we assume that $S_\varphi(u)$ is true. By induction on $i$ we can show that $d(s_{u,i}) = u_i$ for all $i$, and therefore by Lemma 2.5 we have $d(s_u) = u$ and, assuming in addition that the canonical element $\Theta_\varphi \equiv \emptyset$ we have $d^\infty(\tilde{s}) = \tilde{d}(\tilde{s})$ for all $s$: $Y^*$ and so in particular $d^\infty(\tilde{s}_u) = \tilde{u}$. Therefore, we have

$$\tilde{\varphi}(\tilde{s}_u) = \varphi(d^\infty(\tilde{s}_u)) = \varphi(\tilde{u}).$$

There are now two cases to deal with.

(I) If $\tilde{\varphi}(\tilde{s}_u) < |s_u|$ then $\Phi(u) = \tilde{b}(s_u)$. There are now two subcases: either $\varphi(\tilde{u}) \in \text{dom}(u)$ and then $\tilde{b}(s_u) = b(d(s_u)) = b(u)$, or otherwise

$$\tilde{b}(s_u) = \phi_u(\lambda x \cdot (\pi_1(s_u),(\varphi(\tilde{u}))(u)(x)))$$

where for the last part we have $\phi_u(\lambda x \cdot \Phi(u \oplus (n,x)))$ where $i$ is the least index such that $j \in \text{dom}(u) \iff j \in \text{dom}(u_i)$ for all $j < n$. But then $[v_i](n) = [d(s_{u,i})](n) = [u_i](n) = [u](n)$ and hence $[v_i](n) \ast x \oplus u = u \oplus (n,x)$. In addition, since $n_i > n$ for all $i > i'$ we have $[s_{u,i+1}](n) = [s_u](n)$ by construction, and therefore

$$(\pi_1(s_u),(\varphi(\tilde{u}))(u)(x)) = \text{BR}([s_{u,i+1}](n) \ast ([v_i](n) \ast x \oplus u, 0))$$

where for the last part we have $S_\varphi(u) = S_\varphi(u \oplus (n,x))$ for $n = \varphi(\tilde{u}) \notin \text{dom}(u)$.

(II) If $\tilde{\varphi}(\tilde{s}_u) \geq |s_u|$ then $\Phi(u) = \tilde{f}_{\varphi_u}(\lambda z \cdot \text{BR}(s_u \ast z))$. Let

$$t_m := \forall n' : s_u \ast \langle u, f_{s_u} \rangle \ast \ldots \ast \langle u, f_{m-1} \rangle$$

where $f_{m'}$ is defined analogously to above:

$$f_{m'} := \begin{cases} A w, x \cdot \text{BR}(t_{m'} \ast ([u](n') \ast x \oplus w, 0)) & \text{if } n' \notin \text{dom}(u) \\ 0 & \text{otherwise.} \end{cases}$$

We prove by induction that $\text{BR}(s_u) = \text{BR}(t_m)$ for $|s_u| \leq m \leq n+1$ where as before $n := \tilde{\varphi}(\tilde{s}_u) = \varphi(\tilde{u})$. This is true by definition for $m = |s_u|$, so assuming it is also true for arbitrary $m < n+1$ we have

$$\text{BR}(s_u) \overset{IH}{=} \text{BR}(t_m)$$

$$= \tilde{\Phi}_u(\lambda z \cdot \text{BR}(t_m \ast z))$$

$$= \begin{cases} \text{BR}(t_m \ast \langle u, 0 \rangle) & \text{if } m \in \text{dom}(u) \\ \text{BR}(t_m \ast \langle u, A w, x \cdot \text{BR}(t_m \ast ([u](m) \ast x \oplus w, 0)) \rangle) & \text{otherwise} \end{cases}$$

$$= \text{BR}(t_m \ast \langle u, f_m \rangle)$$

$$= \text{BR}(t_{m+1})$$
where (\ast) follows from \( \check{\psi}(t_m) = \check{\psi}(d(t_m)) = \varphi(\hat{u}) = n \geq m \). In particular, we have proved that \( \text{BR}(s_u) = \text{BR}(t_{n+1}) \).

But \( \check{\psi}(t_{n+1}) = \varphi(\hat{u}) = n < n + 1 \) and therefore

\[
\text{BR}(t_{n+1}) = \hat{b}(t_{n+1}) = \begin{cases} 
  b(u) & \text{if } \varphi(\hat{u}) \in \text{dom}(u) \\
  \phi_{\varphi}(\lambda x . (\pi_1(t_{n+1})_n)(u)(x)) & \text{otherwise}
\end{cases}
\]

All that remains to show is that \( (\pi_1(t_{n+1})_n)(u)(x) = \Phi(s_{\alpha(n,x)}) \) for \( n = \varphi(\hat{u}) \notin \text{dom}(u) \). In this case

\[
(\pi_1(t_{n+1})_n)(u)(x) = f_u(u)(x)
\]

\[
= \text{BR}(t_n * ([u](n) \oplus u, 0))
\]

\[
= \text{BR}(t_n * (u \oplus (n, x), 0))
\]

\[
= \text{BR}(s_{\alpha(n,x)})
\]

where for the last equality we have \( s_{\alpha(n,x)} \subset \text{dom}(u) \) and \( s_u \) and since \( \varphi(\hat{u}) \geq |s_u| \) we have \( s_{\alpha(n,x)} \subset \text{dom}(u) + 1 \) for all \( m < n \), and therefore \( s_{\alpha(n,x)} = t_m \) for all \( m < n \), and therefore \( s_{\alpha(n,x)} = t_m \).

All that remains to show is that \( \text{BR} \) defines \( s\text{BR} \) is to prove that the restricted version \( \Psi \) is equivalent to the full version.

**Lemma 3.4.** \( s\text{BR} \) is primitive recursively definable from \( \Psi \), provably in \( \text{E-HA}^\omega + (\Psi) + s\text{Bl} \).

**Proof.** For some arbitrary input \( v \) (not necessarily satisfying \( S_{\varphi}(v) \)) we define \( s\text{BR}^{b,\bar{b},\check{\psi}, \varphi}(v) := \Psi^{\theta, \check{\psi}, \varphi}(\emptyset) \) where

\[
\phi^v_{\check{\psi}}(\lambda x . R) =_R \begin{cases} 
  b(v \oplus u) & \text{if } \varphi(\hat{u}) \in \text{dom}(v) \\
  \phi_{\varphi}(\lambda u . (x, \Phi^\varphi(u)(x))) & \text{otherwise}
\end{cases}
\]

We prove that \( \Psi^{\theta, \check{\psi}, \varphi}(\emptyset) = s\text{BR}^{b,\bar{b},\check{\psi}, \varphi}(v) \) by \( s\text{Bl} \) on the predicate \( P(u) \equiv \Psi(u) = s\text{BR}(v \oplus u) \). First observe that \( \Psi \) is sufficient to define the function \( \theta \) of Lemma 2.8 so provably in \( \text{E-HA}^\omega + (\Psi) \), for all \( \alpha : X^\omega \) there exists a least \( n \) such that \( \check{\psi}(\langle \alpha \rangle(\varphi, n)) = \check{\psi}(v \oplus \langle \alpha \rangle(\varphi, n)) \subset \text{dom}(\langle \alpha \rangle(\varphi, n)) \subset \text{dom}(v \oplus \langle \alpha \rangle(\varphi, n)) \).

In this case \( \langle \alpha \rangle(\varphi, n) \in S_{\varphi} \) and therefore

\[
\Psi(\langle \alpha \rangle(\varphi, n)) = b^\varphi(\langle \alpha \rangle(\varphi, n)) = s\text{BR}(\langle \alpha \rangle(\varphi, n)).
\]

For the induction step, for any \( u \in S_{\varphi} \) we have

\[
\Psi^\varphi(u) = \begin{cases} 
  b(v \oplus u) & \text{if } \varphi(\hat{u}) \in \text{dom}(v \oplus u) \\
  \phi_{\varphi}(\lambda u . (x, \Phi^\varphi(u)(x))) & \text{otherwise}
\end{cases}
\]

for \( m := \varphi(\hat{u}) \). In the second case, since \( u \oplus (m, x) \in S_{\varphi} \) and \( \varphi(\hat{u}) \notin \text{dom}(v \oplus u) \Rightarrow \varphi(\hat{u}) \notin \text{dom}(u) \) we have

\[
\Psi(u \oplus (m, x)) = s\text{Bl}(v \oplus [u \oplus (m, x)])
\]

the first equality following from the bar induction hypothesis, and the last from \( m \notin \text{dom}(v) \), and hence \( \Psi^\varphi(u) = s\text{Bl}(v \oplus u) \). This establishes the premise of \( s\text{Bl} \), from which we obtain \( P(\emptyset) \), which completes the proof.

To summarise, in the previous two sections we introduced a new variant of Spector’s bar recursion, equivalent up to primitive recursive definability, but in which we relax the restriction on recursive calls being carried out in a strictly sequential manner, rather letting the control functional dictate exactly where to update next. Our ultimate aim in doing this, of course, is to give an alternative computational interpretation to the axiom of countable choice that has greater potential for extracting intuitive and efficient programs from proofs in classical analysis, and this will be the focus of the rest of the paper.
4 The Dialectica interpretation of countable choice

In the following sections we assume that the reader is broadly familiar with Gödel’s Dialectica interpretation and its role in the extraction of computational content from proofs - details of which can be found in e.g. [3] [12] - although we make an effort to keep the main flow of ideas as self-contained as possible. The Dialectica interpretation translates each formula \( A \) in the language of some theory \( T \) to a quantifier-free formula \( \|A\|_l \) in some functional verifying theory \( S \), where \( x \) and \( y \) are (possibly empty) tuples of variables of some finite type. The idea is that \( A \) is (classically) equivalent to \( \exists x \forall y |A|_l \), and that the interpretation \( T \to S \) is sound if whenever \( T \vdash A \) we can extract a realizer \( f \) for \( \exists x \) so that \( S \vdash \forall y |A|_l \). When \( T \) is a classical theory, one typically precomposes the Dialectica interpretation with a negative translation in order to obtain soundness, a combination normally referred to as the \( \text{ND interpretation} \).

The Dialectica interpretation was conceived by Gödel in the 1930s, and published much later in a seminal paper of 1958 [9] in which it was shown that Peano arithmetic can be ND interpreted into the system \( T \) of primitive recursive functionals in all finite types. In fact it is not too difficult to lift Gödel’s soundness proof to the higher-type theory \( \text{WE-PA}^\omega + \text{QF-AC} \) of \textit{weakly-extensional} Peano arithmetic with the quantifier-free axiom of choice (see [12] for details). On the other hand, for the addition of computationally non-trivial choice principles such as the axiom of countable choice

\[
\text{AC}^{\omega, X} : \forall n^X \exists x A_n(x) \to \exists f^{\omega \to X} \forall n A_n(f(n))
\]

where \( A \) is \textit{arbitrary}, the primitive recursive functionals no longer suffice for soundness of the interpretation. In fact, over \( \text{WE-PA}^\omega \) countable choice is strong enough to derive the full comprehension schema

\[
\text{CA} : \exists f^{\omega \to X} \forall n(f(n) = 0 \leftrightarrow A(n))
\]

and so the theory \( \text{WE-PA}^\omega + \text{QF-AC} + \text{AC}^{\omega} \) is already capable of formalising a large portion of mathematical analysis, and is thus considerably stronger than Peano arithmetic. Nevertheless, just a few years after Gödel’s paper, C. Spector [17] proved that one could indeed extend the Dialectica interpretation to full classical analysis provided we add bar recursion to system \( T \).

4.1 The countable choice problem

Spector’s main idea can be appreciated from a completely abstract perspective, independent of the world of proof interpretations. Spector observed that in order to extend the ND interpretation to \( \text{WE-PA}^\omega + \text{QF-AC} + \text{AC}^{\omega} \), it suffices to find some way of realizing the Dialectica interpretation of the double negation shift, a non-constructive principle given by

\[
\text{DNS} : \forall n \neg \neg B(n) \to \neg \neg \forall n B(n).
\]

Now, suppose that the Dialectica interpretation of \( B(n) \) is \( |B(n)|_l \) where \( x : X \) and \( y : Y \) are sequences of variables of the appropriate type. Then the Dialectica interpretation of \( \text{DNS} \) is given by

\[
|\text{DNS}|_{f,p,n}^{\phi,q} = |B(n)|_{f,p,n}^{\phi,q} \to |B(\phi f)|_{q_f}^{l(\phi f)}
\]

In other words, to solve the Dialectica interpretation of \( \text{DNS} \), for each given formula \( B \) one must produce realizers \( f : X^\omega, p : X \to Y \) and \( n : \mathbb{N} \) in terms of the parameters \( e : \mathbb{N} \to ((X \to Y) \to X) \), \( q : X^\omega \to Y \) and \( \phi : X^\omega \to \mathbb{N} \) satisfying \( |\text{DNS}|_{f,p,n}^{\phi,q} \). Spector approached this by tackling a stronger problem, namely to solve the underlying system of equations

\[
\begin{align*}
\varphi f &= n \\
f(\varphi f) &= e_n p \\
q f &= p(e_n p)
\end{align*}
\]
in \( f, p \) and \( n \). We call the equations Spector’s equations, and the issue of solving them the countable choice problem. It is clear that a solution Spector’s equations is also a realizer for DNS, even independent of the formula \( B \), and thus to extend the ND interpretation to classical analysis it suffices to find a general solution to the countable choice problem.

### 4.2 Spector’s bar recursive solution

We now present Spector’s remarkable solution to the countable choice problem. Spector first formulated the well-known general form of bar recursion \( \text{BR} \) defined in Section 2 although the countable choice problem in fact only requires a ‘special’, auxiliary mode of bar recursion \( \Phi_X \), which is easily definable from the general bar recursor. Here we formulate Spector’s special form of bar recursion in a variant which outputs a finite as opposed to an infinite sequence, akin to the \( \text{fBR} \) of [15] its defining equation is

\[
\Phi^{\varphi}(s) = \begin{cases} 
\langle \rangle & \text{if } \varphi(s) < |s| \\
\Phi^{\varphi}(\hat{s} \ast a_s) & \text{otherwise},
\end{cases}
\]

where \( a_s = \varepsilon_{|s|}(\lambda x \cdot \hat{q}(\Phi^{\varphi}(\hat{s} \ast x))) \). Here, the types of the parameters \( \varepsilon, q \) and \( \varphi \) are the same as those for the countable choice problem. The key step in the construction of a solution to (6) is Theorem 4.2 below.

**Lemma 4.1.** Let \( t = \Phi^{\varphi}(s) \). Then

\[
t = \Phi^{\varphi}(\langle \rangle)
\]

for all \(|s| \leq i \leq |t|\).

**Proof.** Induction on \( i \). For the base case \( i = |s| \) we have \(|t|(|s|) = s \) from the defining equation of \( \Phi \) and hence \( t = \Phi([t]([s])) \) by definition. Now, assume that (1) holds for some \(|s| \leq i < |t|\). First, we claim that \( \hat{q}([t](i)) \geq i \). If this were not the case then by definition we would have \( \Phi([t](i)) = [t](i) \ast \langle \rangle = [t](i) \) and so by the induction hypothesis \(|t| = i < |t|\), a contradiction. Therefore, we obtain

\[
t \downarrow \Phi([t](i)) = [t](i) \ast \Phi([t](i) \ast a_{[t](i)}) = \Phi([t](i) \ast a_{[t](i)}),
\]

the last equality holding because \([t](i)\) is a prefix of \( \Phi([t](i) \ast a_{[t](i)}) \). But then \( t_i = a_{[t](i)} \) and hence \( t = \Phi([t](i) \ast t_i) = \Phi([t](i + 1)) \), which completes the induction. \( \square \)

**Theorem 4.2.** Define,

\[
t = \lambda x \cdot \Phi^{\varphi}(\langle \rangle)
\]

\[
p_i = \lambda x . \hat{q}(\Phi^{\varphi}(\langle t \ast x \rangle))
\]

where \( i < |t| \) in the second equation. Then for all \( 0 \leq i < |t| \) we have

\[
t_i = \varepsilon_i p_i
\]

\[
\hat{q}(t) = p(\varepsilon; p_i).
\]

**Proof.** As in the proof of Lemma 4.1 we must have \( \hat{q}([t](i)) \geq i \) for all \( 0 \leq i < |t| \). Thus

\[
t_i \downarrow \Phi([t](i)) = \Phi([t](i) \ast a_{[t](i)}) = a_{[t](i)} = \varepsilon(\lambda x . \hat{q}(\Phi([t](i) \ast x))) = \varepsilon_i p_i,
\]

\[\text{In fact, the authors show in [15] that the theory } T \vdash (\Phi_X) \text{ is equivalent to } T \vdash (\text{BR}_{X,R}) \text{ when the recursors to range over all types, although } \Phi \text{ is in general weaker than } \text{BR} \text{ in a pointwise sense (see [15] Remark 5.4).}\]
which establishes the first equation. For the second, note that \( i+1 \leq \left| t \right| \) and so we can appeal once more to Lemma \[4.1\]

\[
\hat{q}(t) \overset{\text{Def.}}{=} \hat{q}(\Phi([t](i+1))) = p_i(t_i) = p_i(e; p_i).
\]

\[\square\]

**Corollary 4.3.** Define \( t \) and \( p_i \) in the parameters \( \varepsilon, q \) and \( \varphi \) as in Theorem \[4.2\]. Then \( f := \hat{t}, p := p_{\hat{t}(i)} \) and \( n := \hat{q}(t) \) solve Spector’s equations.

**Proof.** It suffices to show that \( n = \hat{q}(t) < \left| t \right| \) (this is also necessary for ensuring that our definition of \( p \) makes sense). If not, then by Lemma \[4.1\] we would have \( t = \Phi(t) = t \circ \Phi(t + a_i) \), a contradiction. But now, the first line of \[\text{(6)}\] follows by definition, and the remaining two directly from \[\text{(3)}\], where we have \( f(\varphi f) = \hat{t}_n = t_n = e_n(p_n) = e_n(p) \) and \( q(f) = \hat{q}(t) = p_n(e_n p_n) = p(e_n p) \).

\[\square\]

### 4.3 A symmetric solution

Postponing any further comment or motivation for the time being, we present our alternative solution to the countable choice problem which is based on our symmetric bar recursor \( \text{BR} \) instead of the usual Spector recursor \( \text{BR} \). Our first step is to define a symmetric version of the special recursor \( \Phi \), which takes parameters \( \varepsilon, q \) and \( \varphi \) of the same type as \( \Phi \), but whose input and output are now finite partial sequence:

\[
\Psi_{\varphi,q}(u^i) = \begin{cases} 
\emptyset & \text{if } \varphi(\hat{u}) \in \text{dom}(u) \\
\Psi(u \oplus (n_u, au)) & \text{otherwise}
\end{cases}
\]

where \( n_u = \varphi(\hat{u}) \) and \( a_u = e_{\varphi(\hat{u})}(\lambda x. \hat{q}(\Psi(u \oplus (n_u, x))) \). We note without proof the following fact.

**Proposition 4.4.** The functional \( \text{BR}_{XX'}^{\varphi,q,\text{id},\varphi} \), where

\[
\Phi_{\varphi,q}^{\text{id}}(p^{X \rightarrow X'}) = X, u \circ p(e_{\varphi(\hat{u})}(\lambda x. \hat{q}(p(x)))),
\]

satisfies the defining equation of \( \Psi_{\varphi,q} \), provably in \( \text{E-HA}^0 \).

Our construction and verification of a solution to Spector’s equations now broadly follows \[4.1\] of the previous section.

**Lemma 4.5.** Assume \( u \) is such that \( S_{\varphi}(u) \) holds (cf. Definition \[2.4\]). Let \( v = \Psi_{\varphi,q}(u) \). Then

\[
v = \Psi_{\varphi,q}(\{v\}_{\varphi}(i)) \quad (3)
\]

for all \( |\text{dom}(u)| \leq i \leq |\text{dom}(v)| \).

**Remark 4.6.** Here the restriction on \( u \) is merely a convenience as opposed to a necessity, allowing us to smoothly import the notation from Definition \[2.2\].

**Proof.** Again, we use induction. For \( i = |\text{dom}(u)| \) we have \( \{v\}_{\varphi}(i) = u \) since \( u \subseteq v \) by definition and \( S_{\varphi}(u) \) holds, and therefore \( v = \Psi(\{v\}_u(\text{dom}(u))) \). Assuming that \[3\] is true for \( |\text{dom}(u)| \leq i < |\text{dom}(v)| \), we can see that \( \hat{q}(\{v\}_{\varphi}(i)) \notin \text{dom}(\{v\}_{\varphi}(i)) \). If this were the case, then we would have \( \Psi(\{v\}_{\varphi}(i)) = \{v\}_{\varphi}(i) \circ \emptyset = \{v\}_{\varphi}(i) \), and hence \( |\text{dom}(v)| = i < |\text{dom}(v)| \). Therefore we obtain

\[
v \overset{\text{Def.}}{=} \Psi(\{v\}_{\varphi}(i)) = \{v\}_{\varphi}(i) \circ \Psi(\{v\}_{\varphi}(i) \oplus (n_i, a_i)) = \Psi(\{v\}_{\varphi}(i) \oplus (n_i, a_{\varphi(\hat{v})}))
\]

where \( n_i = \varphi(\{v\}_{\varphi}(i)) \), the last equality holding because \( \{v\}_{\varphi}(i) \) is contained in \( \Psi(\{v\}_{\varphi}(i) \oplus (n_i, a_{\varphi(\hat{v})})) \). But \( \{v\}_{\varphi}(i+1) = \{v\}_{\varphi}(i) \oplus (n_i, v(i)) = \{v\}_{\varphi}(i) \oplus (n_i, a_{\varphi(\hat{v})}) \) and therefore \( v = \Psi(\{v\}_{\varphi}(i+1)) \), which completes the induction step. \[\square\]
Theorem 4.7. Define
\[
v = \xi \cdot \phi^{\widehat{n}}(\emptyset) \\
p_i = \chi \cdot \phi^{\widehat{n}}(\Psi_{k}(\bigcup (n, i) \oplus (n, x))
\]
where in the second equation \( i < |\text{dom}(v)| \) and \( n = \phi^{\widehat{n}}(\phi(v)(i)) \). Then for all \( 0 \leq i < |\text{dom}(v)| \) we have
\[
v(n_i) = e_n p_i \\
\hat{q}(v) = p_i(e_n p_i).
\]

Proof. It is easy to show, as before, that \( \phi(v)(i) \notin \text{dom}(v(i)) \) for all \( 0 \leq i < |\text{dom}(v)| \). Therefore, analogously to the proof of Theorem 4.2 we have
\[
v(n_i) \stackrel{\text{Lem 4.5}}{=} \phi(\Psi_{(v)}(i))(n_i) = \phi(\Psi_{(v)}(i) \oplus (n, n_i))(n_i) = a_{(v)}(i \oplus (n, x)) = e_n p_i(n_i),
\]
while for the second, we obtain
\[
\hat{q}(v) \stackrel{\text{Lem 4.5}}{=} \phi(\Psi_{(v)}(i+1)) = p_i(v(n_i)) = p_i(e_n p_i).
\]

\[\Box\]

Corollary 4.8. Define \( v \) and \( p_i \) in the parameters \( \epsilon, q \) and \( \phi \) as in Theorem 4.7. Then \( f = \phi(v) \) and \( n = n_k = \phi(v(k)) \) solve Spector’s equations, where \( k < |\text{dom}(v)| \) is the unique index satisfying \( n = \phi(v) \).

Proof. This is a little more intricate than the corresponding proof of Corollary 4.3. First we show that \( \phi(v) \in \text{dom}(v) \). If this were not the case, then since \( S_{\phi}(v) \) holds by definition and \( v = \phi(v)(i) \) for \( i = |\text{dom}(v)| \) by Lemma 4.5 we would have \( v = \phi(v) = \phi(v \oplus (n, i)) \) for some \( n \notin \text{dom}(v) \), which is a contradiction. As a result, and again using the fact that \( S_{\phi}(v) \) holds, we must have \( \phi(v) = n_k \) for some \( k < |\text{dom}(v)| \), and so our proposed solution is well-defined.

The solution is now easily verified: \( n = n_k = \phi(v) \) is true by definition, and by Theorem 4.7 we have \( f(\phi) = \phi(n_k) = \phi(n_i) = e_n(p_k) = e_n(p) \) and \( q(f) = \hat{q}(v) = p_i(e_n p_k) = p(e_n p) \).

Let us now reflect on what we have done. In this section we recounted Spector’s well-known reduction of the problem of realizing the extension of the ND interpretation of classical analysis to that of solving a simple set of equations \([6]\). We presented, as well as his original solution using bar recursion, a novel solution to these equations using a new, symmetric variant of bar recursion. So what is the essential difference between these two approaches?

Spector’s solution to the countable choice works by computing finite sequences \( s \) of approximate solutions to the last pair of equations in \([6]\), backtracking every time \( \phi(\delta) < |s| \) to ensure that we can always set \( \phi(\delta) = n \) for some \( n \) in the approximation. This method eventually succeeds by well-foundedness of the underlying tree. While the solution given by bar recursion is elegant in its simplicity, from a computational perspective it is potentially inefficient, as solutions are always computed for the last two of Spector’s equations for all \( i < n \) where \( n \) is the eventual solution of the first.

Our new method of constructing solutions to Spector’s equations uses a new algorithm which constructs finite partial functions of approximate solutions to the last two equations whose domain is exactly determined by the parameter \( \phi \). This means that, in stark constrast to Spector’s method, we do not necessarily need to have computed solutions for this pair for every \( i \) in some initial segment of \( \mathbb{N} \) containing \( n \), but only for certain values. While our solution was a somewhat more complicated to verify, and in particular is based on a form of recursion for which it is seemingly more difficult to prove termination (cf Section 2), from a purely practical perspective it is possible that it gives rise to a much more efficient method of extracting computational content from proofs.

We now present a short and informal case study in order to provide a concrete illustration of the differences between the two methods of program extraction. Our aim is to highlight that in practise the realizers based on symmetric bar recursion compare favourably to the traditional Spector bar recursion.
5 Case study: No injection from \(\mathbb{N} \rightarrow \mathbb{N}\)

It is a basic mathematical fact that there is no injection from the real numbers (or equivalently the set off all functions \(\mathbb{N} \rightarrow \mathbb{N}\)) to the natural numbers \(\mathbb{N}\). This fact can be formalized as a \(\Pi_2\)-statement in the language of finite-types as follows, and moreover the standard proof by a diagonal argument can be formalized using an instance of \(\text{AC}^{\mathbb{N},\mathbb{N} \rightarrow \mathbb{N}}\).

**Theorem 5.1.** \(\forall H: \mathbb{N}^N \rightarrow \mathbb{N} \exists \alpha, \beta: \mathbb{N} \rightarrow \mathbb{N} \exists i : \mathbb{N}(\alpha i \neq \beta i \land H \alpha = H \beta)\).

**Classical Proof.** As a simple case of the law of excluded middle (also known as the drinker’s paradox) we have

\[
\forall n \mathbb{N}^N \exists \beta(N \beta = n) \rightarrow H \alpha = n.
\]

Applying \(\text{AC}^{\mathbb{N},\mathbb{N} \rightarrow \mathbb{N}}\) to (5) yields a functional \(f: \mathbb{N} \rightarrow \mathbb{N}^N\) satisfying

\[
\forall n(\exists \beta(N \beta = n) \rightarrow H(f(n)) = n).
\]

The map \(f\) produces for each \(n\) a function \(f(n): \mathbb{N} \rightarrow \mathbb{N}\) such that whenever \(n\) is in the range of \(H\), \(f(n)\) maps to \(n\). Now, define \(\alpha_H := \lambda n.f(n)(n) + 1\) and let \(i_H := H \alpha_H\). Then since \(i_H\) is in the range of \(H\), by (6) we must have \(H(f(i_H)) = i_H\). But \(\alpha_H \neq f(i_H)\) since \(\alpha_H(i_H) = f(i_H)(i_H) + 1 \neq f(i_H)(i_H)\), and therefore \(\alpha_H\) and \(\beta_H := f(i_H)\) witness the fact that \(H\) is not an injection.

It is an intriguing consequence of Spector’s ND interpretation of classical analysis that we are *a priori* guaranteed to be able to convert the classical diagonal argument above into an intuitionistic proof and directly construct using bar recursion explicit witnesses for \(\alpha_H, \beta_H\) and \(i_H\) whenever \(H\) exists in a model of bar recursion. An explicit bar recursive witness was given by the first author in (14).

However, Spector’s reduction of the ND interpretation of analysis to the countable choice problem demonstrates that, more generally, it is true that a direct realizer of Theorem 5.1 can be constructed primitive recursively in an arbitrary solution to the equations (5). In particular, we can replace Spector’s bar recursion with an instance of symmetric bar recursion to give an alternative procedure for refuting injectiveness for the same class of functionals \(H\) (virtue of the fact that sBR is primitive recursively equivalent to BR).

**Proposition 5.2.** Any computable solution to Spector’s equations allows us to effectively extract witnesses for \(f, g\) and \(i\) in Theorem 5.1.

**Proof.** The solution to Spector’s equations acts as a computational interpretation of the instance of \(\text{AC}^{\mathbb{N},\mathbb{N} \rightarrow \mathbb{N}}\) used in the classical proof. The rest of the proof is interpreted by the parameters we construct for the equations. First, the initial instance of law of excluded middle is interpreted by the term \(e: \mathbb{N} \rightarrow (\mathbb{N}^N \rightarrow \mathbb{N}^N) \rightarrow \mathbb{N}^N\) given by

\[
e_n(p^\mathbb{N}^N := \begin{cases} 0 & \text{if } H(p\epsilon)p \neq n \\ p & \text{if } H(p\epsilon)p = n. \end{cases}
\]

It is easy to verify that \(e\) satisfies

\[
\forall n, p(H(p(\epsilon_n p)) = n \rightarrow H(e_n p) = n),
\]

which is just the ND interpretation of (5). A computable solution to Spector’s equations allows us to effectively construct an approximation to a choice sequence \(f\) in the variables \(q: (\mathbb{N} \rightarrow \mathbb{N}^N) \rightarrow \mathbb{N}^N\) and \(\varphi: (\mathbb{N} \rightarrow \mathbb{N}^N) \rightarrow \mathbb{N}\) that satisfies

\[
H(q(f)) = \varphi(f) \rightarrow H(f(\varphi f)) = \varphi(f).
\]

Now, simply defining \(q(f) = \lambda n.f(n)(n) + 1\) and \(\varphi(f) = H(q(f))\) the premise of (8) holds by definition and hence we obtain \(H(f(\varphi f)) = \varphi(f)\). Finally, setting \(\alpha_H := q(f)\) and \(\beta_H := f(\varphi f)\) we have \(H\beta_H = H\alpha_H\) but \(\alpha_H\) and \(\beta_H\) differ at \(i_H := \varphi(f)\).

\[\square\]
In a very general manner of speaking, the reason one is able to convert the classical proof of Theorem 5.1 into a construction which computes \( \alpha \) and \( \beta \) for any given \( H \) is that the solution of Spector’s equations will typically only work in a subset of the full set-theoretic type structure. Solutions can be obtained for instance of either continuity or majorizability is assumed (cf. models of bar recursion).

However, the exact nature of the procedure which computes these counterexamples will depend on our chosen solution of Spector’s equations, and we now briefly analyse the program which arises from choosing our symmetric bar solution in place of Spector’s bar recursion.

### 5.1 Numerical performance on sample input

Procedures based on both forms of bar recursion were implemented in Haskell, and a series of tests run for a range of primitive recursive choices for \( H \). In almost all cases the procedure based on symmetric bar recursion outperformed \(^3\) that based on Spector’s bar recursion by a considerable margin. As an typical example, take the family of functionals \( H_n \) defined by

\[
H_n(f) = \prod_{i=0}^{n}(1 + fi).
\]

The table below indicates the number of recursive calls and the domain size of the output made when computing the first ten positions of \( \Phi(\langle \rangle) \) and \( \Psi(\emptyset) \) respectively for \( n = 3, 4, 5 \),

| \( n \)  | Spectator (calls/size) | Symmetric (calls/size) |
|---------|-------------------------|------------------------|
| 3       | 331 / 17                | 4 / 1                  |
| 4       | 1366 / 33               | 4 / 1                  |
| 5       | 5674 / 65               | 4 / 1                  |

This disparity in performance can be explained in quite intuitive terms by computing by hand what each method produces. For each \( n \) the symmetric bar recursor \( \Psi(\emptyset) \) calculates the singleton finite partial sequence \( \{ (2^n, 1) \} \) (where \( 1 : \mathbb{N}^\mathbb{N} \) indicates the constant function returning value 1), leading to the pair of counterexamples

\[
f_{H_n} = \lambda i. \begin{cases} 
2 & \text{if } i = 2^n \\
1 & \text{otherwise}
\end{cases} \quad \text{and} \quad g_{H_n} = 1.
\]

The recursor \( \Phi(\langle \rangle) \), on the other hand, returns the same counterexamples but computes the entire finite sequence \( \langle 0, 0, \ldots, 0, 1 \rangle \) of length \( 2^{n+1} \) in order to do so. Unsurprisingly, if we adjust \( H_n \) to try a bit harder to find an injection, for example

\[
H_n(f) = \prod_{i=0}^{n}(1 + 1^{f(i)}),
\]

the disparity is even more extreme:

| \( n \)  | Spectator | Symmetric |
|---------|-----------|-----------|
| 2       | 883 / 37  | 4 / 1     |
| 3       | 173651 / 577 | 4 / 1    |

Of course, it is not the case that \( \Psi \) is more efficient than \( \Phi \) for all potential inputs \( H \). As a somewhat contrived example, take the family of functionals \( H_n \) given by

\[
H_n(f) = \begin{cases} 
greatest i \leq n(f(i)=1) \text{ if it exists, else } n \\
0 & \text{if } f(0) = f(1) = 2 \\
1 & \text{if } f(0) = 1 \land f(1) = 2 \text{ or } f(0) = 2 \land f(1) = 1 \\
\end{cases}
\]

\(^3\)The interested reader is invited to try their own examples using our source code [http://www.eecs.qmul.ac.uk/~pbo/code/symmetric-br/] to witness this for themselves.
Here the linear backtracking associated with $\Phi$ means that the first clause in the case distinction is never triggered, and that $\Phi$ always returns a sequence of length 2. On the other hand, $\Psi$ ends up with a finite partial function of size $n$, and so its runtime complexity is proportional to $n$.

Nevertheless, the fact that such $H$ exist does not in any way detract from our symmetric bar recursion being a useful alternative to Spector’s bar recursion in a large number of situations. In particular, when using our realizer as, for instance, a building block for a more complex realizing term arising from a classical proofs that uses Theorem 5.1 as a Lemma, it is at least reasonable to assume that $H$ will take the form of a fairly natural recursive function, and the authors conjecture that in many such cases symmetric bar recursion will drastically outperform Spector’s bar recursion.

6 Discrete choice

We conclude this paper by discussing, on an informal level, how our move from $\text{BR}$ to $\text{sBR}$ opens the door to extending bar recursion to more complex types, thus leading to a computational interpretation of a wider range of choice principles. Up until now, we have considered bar recursion over either finite sequences or finite partial functions: in other words over objects of type $\mathbb{N} \to X + 1$ with finite support. It is natural ask whether we can further generalise bar recursion to take as input objects of type $D \to X + 1$ with finite support, for some suitable class of indexing types $D$.

It is clear that for such a bar recursor to be well-defined we require equality on $D$ to be decidable, and moreover for well-foundedness we require that the stopping condition $\varphi(\hat{u}) \in \text{dom}(u)$ is eventually reached for $u$ with sufficiently large domain. This first condition is already highly restrictive: in $\text{PA}^\omega$ decidability of equality is only guaranteed for types of lowest level. However, it has been shown by Escardó [8] that in the Kleene-Kreisel continuous functionals, decidability of equality can be established for a somewhat wider range of types.

Definition 6.1 (Escardó [8]). Inductively define the discrete and compact types by

\[
\begin{align*}
\text{discrete} & := \mathbb{B} | \mathbb{N} | \text{discrete} \times \text{discrete} | \text{compact} \to \text{discrete} \\
\text{compact} & := \mathbb{B} | \text{compact} \times \text{compact} | \text{discrete} \to \text{compact}.
\end{align*}
\]

This terminology is based on the fact that the space $C_X$ of Kleene-Kreisel continuous functionals of type $X$ is discrete under the usual (quotient of the) Scott topology whenever $X$ is discrete, and is compact whenever $X$ is compact. Several properties of discrete and compact types are established in [8], including the fact that for arbitrary discrete $X$ the space $C_X$ is both computably enumerable and has decidable equality\footnote{However, equality may not be \textit{primitive recursively} decidable as in $\text{PA}^\omega$: for non-trivial discrete types one must appeal to the so-called \textit{infinite product of selection functions} (see [3] for details).} (this is striking given that the discrete types contain genuine higher-types such as $\mathbb{B}^\omega \to \mathbb{N}$). Moreover, by a standard topological argument one can extend the usual sequential continuity property for functionals on infinite sequences to the following principle:

$$\text{Cont}[D] : \forall \varphi^{X \to D}, \alpha^{X^0} \exists S \forall \beta(\forall s \in S (\alpha(s) =_X \beta(s)) \to \varphi(\alpha) =_D \varphi(\beta)),$$

where $S_{\text{fin}}$ is a finite subset of $D$.

Let $X^{(D)}$ denote the type of finite partial functions from $D$ to $X$, i.e. partial functions $u : D \to X$ with finite domain. We define $\text{sBR}[D]$ where $D$ ranges over discrete types by

\[
\text{sBR}[D]_{x^D}^{\varphi, b, \varrho} (u^{x^0}) =^R \begin{cases} b(u) & \text{if } \varphi(\hat{u}) \in \text{dom}(u) \\
\varrho b((\lambda x^{x^{\text{fin}}} \cdot \text{sBR}[D]_{x^D}^{\varphi, b, \varrho} (u \oplus (\varphi(\hat{u}), x)))) & \text{otherwise}
\end{cases}
\]
where \( u : X^D \) and \( \varphi : (D \to X) \to D \). Note that \( sBR \) as defined in Section\[\text{2}\] is just \( sBR[\mathbb{N}] \). This generalised form of bar recursion is well-defined in the continuous functionals since equality on \( D \) is decidable, and therefore the constructions (\( \otimes \)) \( : X^D \to X^D \) and (\( \oplus \)) \( : X^D \to (D \times X) \to X^D \) are still continuous (which would not be the case for e.g. \( D = \mathbb{N} \to \mathbb{N} \)).

Moreover, the recursor \( sBR[D] \) is well founded by \( \text{Cont}[D] \) - here we give a direct argument using classical logic as opposed to the more delicate treatment of \( sBR[\mathbb{N}] \) given earlier: Suppose that \( sBR(u) \) is not total for some total input \( u \). Then by classical dependent choice we can construct a sequence recursively by \( u_0 := u \) and \( u_{i+1} := u_i \oplus (n_i, x_i) \), where for each \( i \) we have

\[
(i) \; n_i = \hat{\varphi}(u_i) \notin \text{dom}(u_i) \quad (ii) \; sBR(u_i) \text{ is not total.}
\]

By classical countable choice define \( \alpha : D \to X + 1 \) by \( \alpha(d) := x_i \) if \( d = n_i \) for some \( i \), and undefined otherwise. Then by \( \text{Cont}[D] \) there exists a finite subset \( S \subseteq D \) such that \( \varphi(\alpha) = \varphi(\beta) \) whenever \( \alpha(s) = \beta(s) \) for all \( s \in S \). Now since \( \alpha \) is the domain-theoretic limit of the \( u_i \), there is some index \( I \) such that \( u_I = \alpha \) on \( S \), and therefore \( n_I = \hat{\varphi}(u_I) = \varphi(\alpha) \). Now by definition we have \( n_I \in \text{dom}(u_{I+1}) \), and so in particular \( n_I \in \text{dom}(\alpha) \). It is clear that \( n_I \notin S \), since by (i) we have \( n_I \notin \text{dom}(u_I) \), but \( n_I \in \text{dom}(\alpha) \), contradicting the definition of \( I \). But then \( u_{I+1} = u_I \) on \( S \), and therefore \( n_{I+1} = n_I = \varphi(\alpha) \) and hence \( u_{I+1} \in \text{dom}(u_{I+1}) \), a contradiction.

Therefore \( sBR[D] \) is a well-defined, total continuous functional, and so by an entirely analogous argument as in the case of \( sBR[\mathbb{N}] \), one can construct \( f \) and \( p \) in \( sBR[D] \) which satisfy the appropriate generalisation of Spector’s equations:

\[
\varphi f =_{D} d
\]

\[
f(\varphi f) =_{X} e \; d \; p
\]

\[
a f =_{Y} p(\epsilon _dpi).
\]

As a result, we gain a computation interpretation of the following axiom of discrete choice:

\[
\text{AC}^{D,X} : \forall d^D \exists x^X A_d(x) \to \exists f^{D\to X} \forall d A_d(f d).
\]

We remark that, as shown in \( \text{[8]} \), the set \( C_D \) is recursively enumerable for any discrete type \( D \), and so can be encoded in the usual type of natural numbers \( \mathbb{N} \). Therefore in theory we could have defined both \( sBR[D] \) and the analogous generalisation \( BR[D] \) of Spector’s bar recursion in terms of \( sBR[\mathbb{N}] \) and \( BR[\mathbb{N}] \) respectively. However, this reduction to the base level would of course rely explicitly on the encoding of \( C_D \) into \( \mathbb{N} \) on the meta-level. To avoid this and define the generalised recursor directly seems only possible for the symmetric recursor \( sBR[D] \), as Spector’s bar recursion relies inherently on the underlying ordering of the natural numbers, and is therefore prime-facie undefined for higher level discrete types on which no natural total ordering exists.

We consider this to be another key advantage of our symmetric recursor over Spector’s bar recursion.

7 Final remarks

We introduced a variant of bar recursion that, unlike Spector’s bar recursion, carries out recursive calls in symmetric manner, as dictated by the control parameter. We proved that at the level of definability our general symmetric recursor is entirely equivalent to Spector’s bar recursor, and therefore exists in both the total continuous functionals and the majorizable functionals.

We then isolated the particular set of equations sufficient to give a Dialectica interpretation to the axiom of countable choice, and demonstrated that these can be solved with a special form of symmetric bar recursion, analogously to Spector’s original bar recursive solution. Finally, we compared concrete realizers obtained from the classical proof that there is no injection from \( \mathbb{N}^\mathbb{N} \to \mathbb{N} \) using both variants of bar recursion, and demonstrated that our new method of extracting programs from proofs in classical analysis performs drastically more efficiently in many cases.
Our work fits in to the much broader program of adapting and refining traditional proof theoretic techniques so that they are better suited to their role in modern proof theory - in our case taking a well-known method of proving the consistency of classical analysis and improving it so that it becomes better suited as a tool for extracting programs from proofs.

However, our analysis of the performance of symmetric bar recursion in practise is restricted to a single case study, and we believe that there is much further potential for exploiting this variant of recursion in proof theory. In particular, it is natural to wonder whether one could obtain new bounds on the complexity of symmetric bar recursive programs in certain restricted contexts which would in turn lead to new results in proof mining. Finally, it would be interesting to study the procedural behaviour of symmetric bar recursion, which also seems much more natural than Spector’s bar recursion and has close links to both the update procedures of Avigad [2] and the learning-based realizers of Aschieri et al. [1]. Making this relationship precise could lead to a much better understanding of the relationship between proof interpretations like the Dialectica interpretation and more direct learning-based interpretations of classical logic.

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