The intrinsic square function characterizations of weighted Hardy spaces

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Abstract

In this paper, we will study the boundedness of intrinsic square functions on the weighted Hardy spaces $H^p(w)$ for $0 < p < 1$, where $w$ is a Muckenhoupt’s weight function. We will also give some intrinsic square function characterizations of weighted Hardy spaces $H^p(w)$ for $0 < p < 1$.

Keywords: Intrinsic square function; weighted Hardy spaces; $A_p$ weights; atomic decomposition

1. Introduction and preliminaries

First, let’s recall some standard definitions and notations. The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [8]. Let $w$ be a nonnegative, locally integrable function defined on $\mathbb{R}^n$, all cubes are assumed to have their sides parallel to the coordinate axes. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{p}{p-1}} \, dx \right)^{p-1} \leq C \quad \text{for every cube } Q \subseteq \mathbb{R}^n,$$

where $C$ is a positive constant which is independent of the choice of $Q$.

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \inf_{x \in Q} w(x) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

For the case $p = \infty$, $w \in A_{\infty}$, if for any given $\varepsilon > 0$, we can find a positive number $\delta > 0$ such that if $Q$ is a cube, $E$ is a measurable subset of $Q$ with $|E| < \delta |Q|$, then $\int_E w(x) \, dx < \varepsilon \int_Q w(x) \, dx$.

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It is well known that $A_\infty = \bigcup_{1 < p < \infty} A_p$, namely, a nonnegative, locally integrable function $w(x)$ satisfies the condition $A_\infty$ if and only if it satisfies the condition $A_p$ for some $1 < p < \infty$. We also know that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. Therefore, we will use the notation $q_w = \inf\{q > 1 : w \in A_q\}$ to denote the critical index of $w$. Obviously, if $w \in A_q$, $q > 1$, then we have $1 \leq q_w < q$.

Given a cube $Q$ and $\lambda > 0$, $\lambda Q$ denotes the cube with the same center as $Q$ whose side length is $\lambda$ times that of $Q$. $Q = Q(x_0, r)$ denotes the cube centered at $x_0$ with side length $r$. For a weight function $w$ and a measurable set $E$, we set the weighted measure $w(E) = \int_E w(x) \, dx$, and we denote the characteristic function of $E$ by $\chi_E$.

We shall need the following lemmas. For the proofs of these results, please refer to [3, Chap IV] and [4, Chap 9].

**Lemma A.** Let $w \in A_p$, $p \geq 1$. Then, for any cube $Q$, there exists an absolute constant $C > 0$ such that

$$w(2Q) \leq Cw(Q).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda Q) \leq C\lambda^{np}w(Q),$$

where $C$ does not depend on $Q$ nor on $\lambda$.

**Lemma B.** Let $w \in A_q$, $q > 1$. Then, for all $r > 0$, there exists a constant $C$ independent of $r$ such that

$$\int_{|x| \geq r} \frac{w(x)}{|x|^{nq}} \, dx \leq Cr^{-nq}w(Q(0, 2r)).$$

**Lemma C.** Let $w \in A_\infty$. For any $0 < \varepsilon < 1$, there exists a positive number $0 < \delta < 1$ such that if $E$ is a measurable subset of a cube $Q$ with $|E|/|Q| > \varepsilon$, then we have $w(E)/w(Q) > \delta$.

**Lemma D.** Let $w \in A_p$, $p \geq 1$. Then there exists an absolute constant $C > 0$ such that

$$C\left(\frac{|E|}{|Q|}\right)^p \leq \frac{w(E)}{w(Q)},$$

for any measurable subset $E$ of a cube $Q$. 

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Given a Muckenhoupt’s weight function $w$ on $\mathbb{R}^n$, for $0 < q < \infty$, we denote by $L^q_w(\mathbb{R}^n)$ the space of all functions satisfying
\[\|f\|_{L^q_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^q w(x) \, dx\right)^{1/q} < \infty.\]
When $q = \infty$, $L^\infty_w$ will be taken to mean $L^\infty$, and we set $\|f\|_{L^\infty_w} = \|f\|_{L^\infty}$.

As we all know, for any $0 < p < \infty$, the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ can be defined in terms of maximal functions. Let $\varphi$ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \varphi(x) \, dx = 1$. Set
\[\varphi_t(x) = t^{-n} \varphi(x/t), \quad t > 0, \ x \in \mathbb{R}^n.\]

We will define the maximal function $M_\varphi f(x)$ by
\[M_\varphi f(x) = \sup_{t > 0} |f * \varphi_t(x)|.\]

Then $H^p_w(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $M_\varphi f \in L^p_w(\mathbb{R}^n)$ with $\|f\|_{H^p_w} = \|M_\varphi f\|_{L^p_w}$. For every $1 < p < \infty$, as in the unweighted case, we have $L^p_w(\mathbb{R}^n) = H^p_w(\mathbb{R}^n)$.

The real-variable theory of weighted Hardy spaces have been studied by many authors. In 1979, Garcia-Cuerva studied the atomic decomposition and the dual spaces of $H^p_w$ for $0 < p \leq 1$. In 2002, Lee and Lin gave the molecular characterization of $H^p_w$ for $0 < p \leq 1$, they also obtained the $H^p_w(\mathbb{R})$, $\frac{1}{2} < p \leq 1$ boundedness of the Hilbert transform and the $H^p_w(\mathbb{R}^n)$, $\frac{n}{n+1} < p \leq 1$ boundedness of the Riesz transforms. For the results mentioned above, we refer the readers to [2,6,9] for further details.

In this article, we will use Garcia-Cuerva’s atomic decomposition theory for weighted Hardy spaces in [2,9]. We characterize weighted Hardy spaces in terms of atoms in the following way.

Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index $q_w$. Set $[\cdot]$ the greatest integer function. For $s \in \mathbb{Z}_+$ satisfying $s \geq \lfloor n(q_w/p - 1)\rfloor$, a real-valued function $a(x)$ is called $(p,q,s)$-atom centered at $x_0$ with respect to $w$(or $w$-$(p,q,s)$-atom centered at $x_0$) if the following conditions are satisfied:

(a) $a \in L^q_w(\mathbb{R}^n)$ and is supported in a cube $Q$ centered at $x_0$,
(b) $\|a\|_{L^q_w} \leq w(Q)^{1/q-1/p}$,
(c) $\int_{\mathbb{R}^n} a(x)x^\alpha \, dx = 0$ for every multi-index $\alpha$ with $|\alpha| \leq s$.

**Theorem E.** Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index $q_w$. For each $f \in H^p_w(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$...
of \( w-(p,q,\lfloor n(q/p -1)\rfloor) \)-atoms and a sequence \( \{\lambda_j\} \) of real numbers with 
\[ \sum_j |\lambda_j|^p \leq C \|f\|_{H_w^p}^p \] 
such that \( f = \sum_j \lambda_j a_j \) both in the sense of distributions and in the \( H_w^p \) norm.

2. The intrinsic square functions and our main results

The intrinsic square functions were first introduced by Wilson in [10] and [11], the so-called intrinsic square functions are defined as follows. For \( 0 < \alpha \leq 1 \), let \( C_\alpha \) be the family of functions \( \varphi \) defined on \( \mathbb{R}^n \) such that \( \varphi \) has support containing in \( \{x \in \mathbb{R}^n : |x| \leq 1\} \), \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 0 \) and for all \( x, x' \in \mathbb{R}^n \),
\[ |\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha. \]

For \( (y,t) \in \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty) \) and \( f \in L_{loc}^1(\mathbb{R}^n) \), we set
\[ A_\alpha(f)(y,t) = \sup_{\varphi \in C_\alpha} |f * \varphi(y)|. \]

Then we define the intrinsic square function of \( f \) (of order \( \alpha \)) by the formula
\[ S_\alpha(f)(x) = \left( \iint_{\Gamma(x)} \left( A_\alpha(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \]
where \( \Gamma(x) \) denotes the usual cone of aperture one:
\[ \Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}. \]

We can also define varying-aperture versions of \( S_\alpha(f) \) by the formula
\[ S_{\alpha,\beta}(f)(x) = \left( \iint_{\Gamma_\beta(x)} \left( A_\alpha(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \]
where \( \Gamma_\beta(x) \) is the usual cone of aperture \( \beta > 0 \):
\[ \Gamma_\beta(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < \beta t\}. \]

The intrinsic Littlewood-Paley \( g\)-function (could be viewed as “zero-aperture” version of \( S_\alpha(f) \)) and the intrinsic \( g_\lambda^\ast \)-function (could be viewed as “infinite aperture” version of \( S_\alpha(f) \)) will be defined respectively by
\[ g_\alpha(f)(x) = \left( \int_0^\infty \left( A_\alpha(f)(x,t) \right)^2 \frac{dt}{t} \right)^{1/2}. \]
and
\[
g_{\lambda,\alpha}(f)(x) = \left( \int_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^\lambda \left( A_{\alpha}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]

Similarly, we can also introduce the so-called similar-looking square functions \( \tilde{S}_{(\alpha,\varepsilon)}(f)(x) \), which are defined via convolutions with kernels that have unbounded supports, more precisely, for \( 0 < \alpha \leq 1 \) and \( \varepsilon > 0 \), let \( C_{(\alpha,\varepsilon)} \) be the family of functions \( \varphi \) defined on \( \mathbb{R}^n \) such that for all \( x \in \mathbb{R}^n \),
\[
|\varphi(x)| \leq (1 + |x|)^{-n-\varepsilon},
\]
and for all \( x, x' \in \mathbb{R}^n \),
\[
|\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha ((1 + |x|)^{-n-\varepsilon} + (1 + |x'|)^{-n-\varepsilon}),
\]
and also satisfy \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 0 \).

Let \( f \) be such that \( |f(x)|(1 + |x|)^{-n-\varepsilon} \in L^1(\mathbb{R}^n) \). For any \( (y,t) \in \mathbb{R}_+^{n+1} \), set
\[
\tilde{A}_{(\alpha,\varepsilon)}(f)(y,t) = \sup_{\varphi \in C_{(\alpha,\varepsilon)}} |f * \varphi_t(y)|.
\]
We define
\[
\tilde{S}_{(\alpha,\varepsilon)}(f)(x) = \left( \int_{\Gamma(x)} \left( \tilde{A}_{(\alpha,\varepsilon)}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]
\[
\tilde{g}_{(\alpha,\varepsilon)}(f)(x) = \left( \int_{0}^{\infty} \left( \tilde{A}_{(\alpha,\varepsilon)}(f)(x,t) \right)^2 \frac{dt}{t} \right)^{1/2},
\]
and
\[
\tilde{g}_{(\alpha,\varepsilon)}^*(f)(x) = \left( \int_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^\lambda \left( \tilde{A}_{(\alpha,\varepsilon)}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.
\]

In [11], Wilson proved that the intrinsic square functions are bounded operators on the weighted Lebesgue spaces \( L^p_w(\mathbb{R}^n) \) for \( 1 < p < \infty \), namely, he showed the following result.

**Theorem F.** Let \( w \in A_p \), \( 1 < p < \infty \) and \( 0 < \alpha \leq 1 \). Then there exists a positive constant \( C > 0 \) such that
\[
\|S_{\alpha}(f)\|_{L^p_w} \leq C \|f\|_{L^p_w}.
\]

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Recently, Huang and Liu [5] studied the boundedness of intrinsic square functions on the weighted Hardy spaces $H^1_w(\mathbb{R}^n)$. Moreover, they obtained the intrinsic square function characterizations of $H^1_w(\mathbb{R}^n)$.

As a continuation of their work, the purpose of this paper is to investigate the boundedness of intrinsic square functions on the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $0 < p < 1$. Furthermore, we will characterize the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $0 < p < 1$ by the intrinsic square functions including the Lusin area function, Littlewood-Paley $g$-function and $g^*_\lambda$-function.

In order to state our theorems, we need to introduce the Lipschitz space $\text{Lip}(\alpha, 1, 0)$ for $0 < \alpha \leq 1$. Set $b_Q = \frac{1}{|Q|} \int_Q b(x) \, dx$.

$$\text{Lip}(\alpha, 1, 0) = \{ b \in L_{\text{loc}}(\mathbb{R}^n) : \| b \|_{\text{Lip}(\alpha, 1, 0)} < \infty \},$$

where

$$\| b \|_{\text{Lip}(\alpha, 1, 0)} = \sup_Q \frac{1}{|Q|^{1+\alpha/n}} \int_Q |b(y) - b_Q| \, dy$$

and the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$.

Our main results are stated as follows.

**Theorem 1.** Let $0 < \alpha \leq 1$, $\frac{n}{n+\alpha} < p < 1$, $w \in A_{p(1+\frac{2}{\alpha})}$ and $\varepsilon > \alpha$. Suppose that $f \in (\text{Lip}(\alpha, 1, 0))^*$, then a tempered distribution $f \in H^p_w(\mathbb{R}^n)$ if and only if $g_\alpha(f) \in L^p_w(\mathbb{R}^n)$ or $\tilde{g}_{(\alpha,\varepsilon)}(f) \in L^p_w(\mathbb{R}^n)$ and $f$ vanishes weakly at infinity.

**Theorem 2.** Let $0 < \alpha \leq 1$, $\frac{n}{n+\alpha} < p < 1$, $w \in A_{p(1+\frac{2}{\alpha})}$ and $\varepsilon > \alpha$. Suppose that $f \in (\text{Lip}(\alpha, 1, 0))^*$, then a tempered distribution $f \in H^p_w(\mathbb{R}^n)$ if and only if $S_\alpha(f) \in L^p_w(\mathbb{R}^n)$ or $\tilde{S}_{(\alpha,\varepsilon)}(f) \in L^p_w(\mathbb{R}^n)$ and $f$ vanishes weakly at infinity.

**Theorem 3.** Let $0 < \alpha \leq 1$, $\frac{n}{n+\alpha} < p < 1$, $w \in A_{p(1+\frac{2}{\alpha})}$, $\varepsilon > \alpha$ and $\lambda > \frac{3n+2\alpha}{n}$. Suppose that $f \in (\text{Lip}(\alpha, 1, 0))^*$, then a tempered distribution $f \in H^p_w(\mathbb{R}^n)$ if and only if $g^*_{\lambda,\alpha}(f) \in L^p_w(\mathbb{R}^n)$ or $\tilde{g}^*_{\lambda,\alpha,\varepsilon}(f) \in L^p_w(\mathbb{R}^n)$ and $f$ vanishes weakly at infinity.

**Remark 1.** Clearly, if for every $t > 0$, $\varphi_t \in C_\alpha$, then we have $\varphi_t \in \text{Lip}(\alpha, 1, 0)$. Thus the intrinsic square functions are well defined for tempered distributions in $(\text{Lip}(\alpha, 1, 0))^*$.

**Remark 2.** We say that a tempered distribution $f$ vanishes weakly at infinity, if for any $\varphi \in \mathcal{S}$, we have $f \ast \varphi_t(x) \to 0$ as $t \to \infty$ in the sense of distributions.
Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{A}{B} \leq C$.

3. The necessity of our theorems

We shall first prove the following lemma.

**Lemma 3.1.** Let $0 < p < 1$ and $w \in A_\infty$. Then for every $f \in H^p_w(\mathbb{R}^n)$, we have that $f$ vanishes weakly at infinity.

**Proof.** For any given $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \varphi(x) \, dx = 1$, we denote the nontangential maximal function of $f$ by

$$M^*_\varphi(f)(x) = \sup_{|y-x|<t} |f \ast \varphi_t(y)|.$$  

Then we have $|f \ast \varphi_t(x)| \leq M^*_\varphi(f)(y)$ whenever $|x-y| < t$. As a consequence, we have the following inequality

$$\int_{|x-y|<t} |f \ast \varphi_t(x)|^p w(y) \, dy \leq \int_{|x-y|<t} (M^*_\varphi(f)(y))^p w(y) \, dy.$$  

Hence

$$|f \ast \varphi_t(x)|^p \leq \frac{1}{w(Q(x,\sqrt{2t}))} \|M^*_\varphi(f)\|_{L^p_w}^p \leq C \frac{1}{w(Q(x,\sqrt{2t}))} \|M_\varphi(f)\|_{L^p_w}^p.$$  

It is well known that for given $w \in A_\infty$, $w$ satisfies the doubling condition (Lemma A). Furthermore, we can easily prove that $w$ also satisfies the reverse doubling condition; that is, for any cube $Q$, there exists a constant $C_1 > 1$ such that $w(2Q) \geq C_1 w(Q)$. From this property, we can deduce $w(2^k Q) \geq C_1^k w(Q)$ by induction. Set $Q = Q(x, \sqrt{2})$. So we can get

$$\lim_{k \to \infty} \frac{1}{w(2^k Q)} = 0,$$

which implies

$$\lim_{t \to \infty} \frac{1}{w(Q(x,\sqrt{2t}))} = 0.$$  

This completes the proof of the lemma. \hfill \Box

From the definitions of intrinsic square functions, we know that when $\varphi \in \mathcal{C}_\alpha$, $0 < \alpha \leq 1$, then there exists a positive constant $c$ depending only on $\alpha, \varepsilon$, and $n$, such that $c \varphi \in \mathcal{C}_{(\alpha,\varepsilon)}$. Thus we can get the pointwise inequality

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Proposition 3.2. Let $0 < \alpha \leq 1$, $\frac{n}{n+\alpha} < p < 1$ and $w \in \mathcal{A}_{p(1+\frac{n}{n})}$. Then for every $f \in H_{p}^{\alpha}(\mathbb{R}^{n})$, we have

$$\|g_{\alpha}(f)\|_{L_{w}^{p}} \leq C\|f\|_{H_{p}^{\alpha}}.$$  

Proof. Set $q = p(1 + \frac{n}{n})$. Then for $w \in \mathcal{A}_{q}$, we have $\lfloor n(q_{w}/p - 1) \rfloor = 0$. By Theorem E, it suffices to show that for any $w$-$(p, q, 0)$-atom $a$, there exists a constant $C > 0$ independent of $a$ such that $\|g_{\alpha}(a)\|_{L_{w}^{p}} \leq C$.

Let $a$ be a $w$-$(p, q, 0)$-atom with supp $a \subset Q = Q(x_{0}, r)$, and let $Q^{*} = 2\sqrt{n}Q$. By using Hölder inequality, Lemma A and Theorem F, we have

$$\int_{Q^{*}} |g_{\alpha}(a)(x)|^{p}w(x)\,dx \leq \left( \int_{Q^{*}} |g_{\alpha}(a)(x)|^{q}w(x)\,dx \right)^{p/q} \left( \int_{Q^{*}} w(x)\,dx \right)^{1-p/q} \leq \|g_{\alpha}(a)\|_{L_{w}^{p}}^{p} \cdot w(Q^{*})^{1-p/q} \leq C\|a\|_{L_{w}^{p}}^{p} \cdot w(Q)^{1-p/q} \leq C\|a\|_{L_{w}^{p}}^{p} \cdot w(Q) \leq C.$$  

Below we give the estimate of the integral $I = \int_{Q^{*}} |g_{\alpha}(a)(x)|^{p}w(x)\,dx$.

For any $\varphi \in C_{\alpha}$, by the vanishing moment condition of atom $a$, we have

$$|a \ast \varphi_{\lambda}(x)| = \left| \int_{Q} \varphi_{\lambda}(y-x)\,dy \right| \leq \int_{Q} \frac{|y-x_{0}|^{\alpha}}{t^{n+\alpha}}|a(y)|\,dy \leq C \cdot \frac{t^{\alpha}}{t^{n+\alpha}} \int_{Q} |a(y)|\,dy.$$  

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Denote the conjugate exponent of \( q > 1 \) by \( q' = q/(q - 1) \). Hölder’s inequality and the \( A_q \) condition yield

\[
\int_Q |a(y)| dy \leq \left( \int_Q |a(y)|^q w(y) dy \right)^{1/q} \left( \int_Q w(y)^{-1/(q-1)} dy \right)^{1/q'}
\]

\[
\leq C \|a\|_{L^q_w} \left( \frac{|Q|^q}{w(Q)} \right)^{1/q}
\]

\[
\leq C \frac{|Q|}{w(Q)^{1/p}}. \quad (3)
\]

We note that \( \text{supp } \varphi \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\} \), then for any \( y \in Q, x \in (Q^*)^c \), we have \( t \geq |x - y| \geq |x - x_0| - |y - x_0| \geq \frac{|x - x_0|}{2} \).

Substituting the above inequality (3) into (2), we thus obtain

\[
|g_\alpha(a)(x)|^2 = \int_0^\infty \left( \sup_{\varphi \in \mathcal{C}_\alpha} |a \ast \varphi_t(x)| \right)^2 \frac{dt}{t} \leq C \left( \frac{|Q|}{w(Q)^{1/p}} \right)^2 \int_0^\infty \frac{dt}{t^{2(n+\alpha)+1}} \quad (4)
\]

It follows from (4), Lemma A and Lemma B that

\[
I = \int_{(Q^*)^c} |g_\alpha(a)(x)|^p w(x) dx
\]

\[
\leq C \left( \frac{r^{n+\alpha}}{w(Q)^{1/p}} \right)^p \int_{|x-x_0| \geq \sqrt{r}} \frac{w(x)}{|x-x_0|^q} dx
\]

\[
= C \left( \frac{r^{n+\alpha}}{w(Q)^{1/p}} \right)^p \int_{|y| \geq \sqrt{r}} \frac{w_1(y)}{|y|^q} dy \quad (5)
\]

\[
\leq C \left( \frac{r^{n+\alpha}}{w(Q)^{1/p}} \right)^p r^{-nq} w_1(Q_1)
\]

\[
= C \left( \frac{r^{n+\alpha}}{w(Q)^{1/p}} \right)^p r^{-nq} w(Q)
\]

\[
\leq C,
\]

where \( w_1(x) = w(x + x_0) \) is the translation of \( w(x) \), \( Q_1 \) is a cube which is the translation of \( Q \). It is obvious that \( w_1 \in A_q \) for \( w \in A_q, q > 1 \), and \( q w_1 = q w \).

Therefore, Proposition 3.2 is proved by combining (1) and (5). \( \square \)
Proposition 3.3. Let \(0 < \alpha \leq 1\), \(\frac{n}{n+\alpha} < p < 1\), \(w \in A_{p(1+\alpha/n)}\) and \(\lambda > \frac{3n+2\alpha}{n}\). Then for every \(f \in H^p_w(\mathbb{R}^n)\), we have

\[
\|g^{*}_{\lambda,\alpha}(f)\|_{L^p_w} \leq C\|f\|_{H^p_w}.
\]

Proof. Let \(q = p(1+\frac{\alpha}{n})\). As in the proof of Proposition 3.2, we only need to show that for any \(w-(p,q,0)\)-atom \(a\), there exists a constant \(C > 0\) independent of \(a\) such that \(\|g^{*}_{\lambda,\alpha}(a)\|_{L^p_w} \leq C\).

Let \(a\) be a \(w-(p,q,0)\)-atom with supp \(a \subset Q = Q(x_0, r)\), and let \(Q_k^* = 2\sqrt{n}(2^kQ)\). From the definition, we readily see that

\[
g^{*}_{\lambda,\alpha}(a)(x)^2 = \int_{\mathbb{R}^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}}
\]

\[
= \int_0^\infty \int_{|x-y|<t} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}}
\]

\[
+ \sum_{k=1}^\infty \int_0^\infty \int_{2^{k-1}t \leq |x-y| < 2^k t} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}}
\]

\[
\leq C \left[ S_\alpha(a)(x)^2 + \sum_{k=1}^\infty 2^{-k\lambda n} S_{\alpha,2^k}(a)(x)^2 \right].
\]

Since \(0 < p < 1\), we thus get

\[
\|g^{*}_{\lambda,\alpha}(a)\|_{L^p_w}^p \leq C \left[ \|S_\alpha(a)\|_{L^p_w}^p + \sum_{k=1}^\infty 2^{-k\lambda n} \|S_{\alpha,2^k}(a)\|_{L^p_w}^p \right].
\]

By Proposition 3.2, we can obtain \(\|S_\alpha(a)\|_{L^p_w} \leq C\). It remains to estimate \(\|S_{\alpha,2^k}(a)\|_{L^p_w}\) for \(k = 1, 2, \ldots\).

First we claim that the following inequality holds.

\[
\|S_{\alpha,2^k}(a)\|_{L^p_w} \leq C \cdot 2^{knq} \|S_\alpha(a)\|_{L^p_w}^2 \quad k = 1, 2, \ldots
\]

(6)

In fact, by the Fubini theorem and Lemma A, we can get

\[
\|S_{\alpha,2^k}(a)\|_{L^p_w}^2 = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n+1}} (A_\alpha(a)(y,t))^2 \chi_{|x-y|<2^k t} \frac{dydt}{t^{n+1}} \right) w(x) dx
\]

\[
= \int_{\mathbb{R}^{n+1}} \left( \int_{|x-y|<t} w(x) dx \right) (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}}
\]

\[
\leq C \cdot 2^{knq} \int_{\mathbb{R}^{n+1}} \left( \int_{|x-y|<t} w(x) dx \right) (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}}
\]

\[
= C \cdot 2^{knq} \|S_\alpha(a)\|_{L^p_w}^2.
\]
Using Hölder’s inequality, Lemma A, Theorem F and (6), we obtain
\[
\left( \int_{Q_k^+} |S_{\alpha,2^k}(a)(x)|^p w(x) \, dx \right)^{1/p} \leq \|S_{\alpha,2^k}(a)\|_{L_p^w} w(Q_k^+) \cdot \left( \frac{1}{p} \right) \leq C \cdot 2^{\frac{knq}{2}} \|S_{\alpha}(a)\|_{L_p^w} \left( 2^{knq} w(Q) \right)^{\frac{1}{p} - \frac{1}{2}} \leq C \cdot 2^{\frac{knq}{2}} \|a\|_{L_p^w} \left( w(Q) \right)^{\frac{1}{p} - \frac{1}{2}} \leq C \cdot 2^{\frac{knq}{2}},
\]
where we have used the fact that \( w \in A_\gamma, 1 < q < 1 + \frac{\alpha}{n} \leq 2 \), then \( w \in A_2 \).

Below we give the estimate of the integral \( J = \int_{(Q_k^+)^c} |S_{\alpha,2^k}(a)(x)|^p w(x) \, dx \).

Note that \( \text{supp } \varphi' \subset \{ x \in \mathbb{R}^n : |x| \leq 1 \} \), by a simple calculation, we know that for any \((y, t) \in \Gamma_{2^k}(x), x \in (Q_k^+)^c\), then \( t \geq \frac{|x - x_0|}{2^k} \). It follows from (2) and (3) that
\[
|S_{\alpha,2^k}(a)(x)|^2 \leq C \left( \frac{|Q|}{w(Q)^{1/p}} \right)^2 r^{2\alpha} \int_{\Gamma_{2^k}(x)} \frac{dydt}{t^{2(n+\alpha)} \cdot \lambda^{n+1}} \leq C \left( \frac{|Q|}{w(Q)^{1/p}} \right)^2 r^{2\alpha} 2^{kn} \int_{|x - x_0| > 2^k} \frac{dt}{t^{2(n+\alpha)+1}} \leq C \cdot 2^{3kn+2\alpha} \left( \frac{r^{n+\alpha}}{w(Q)^{1/p}} \right)^2 \frac{1}{|x - x_0|^{2(n+\alpha)}}.
\]

Using Lemma A, Lemma B and (8), we have
\[
J = \int_{(Q_k^+)^c} |S_{\alpha,2^k}(a)(x)|^p w(x) \, dx \leq C \cdot 2^{k(p(n+\alpha) - 2)} \int_{|x - x_0| > 2^k} \frac{w(x)}{|x - x_0|^{2q}} \, dx \leq C \cdot 2^{k(p(n+\alpha) - 2)} \int_{|x - x_0| > 2^k} \frac{w(x)}{2^{knq} w(Q)} \, dx \leq C \cdot 2^{k(p(n+\alpha) - 2)} \int_{|x - x_0| > 2^k} \frac{w(x)}{2^{knq} w(Q)} \, dx \leq C \cdot 2^{k(p(n+\alpha))}.
\]

where the notations \( w_1 \) and \( Q_1 \) are the same as Proposition 3.2, we have \( w_1(Q_1) = w(Q) \). Hence, by the above estimates (7) and (9), we obtain
\[
\|S_{\alpha,2^k}(a)\|_{L_p^w} \leq C \cdot \left( 2^{k(p(n+\alpha))} + 2^{k(p(3n+2\alpha)/2)} \right) \leq C \cdot 2^{k(p(3n+2\alpha)/2)}.
\]

Therefore
\[
\|\tilde{S}_{\lambda,\alpha}(a)\|_{L_p^w} \leq C \sum_{k=1}^{\infty} 2^{k\lambda n^q} \cdot 2^{k(p(3n+2\alpha)/2)} \leq C,
\]

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where the last inequality holds since \( \lambda > 3 + 2\alpha/n \). The proof of Proposition 3.3 is complete.

Using the same arguments as above, we can also show the \( H^p_w-L^p_w \) boundedness of \( \tilde{g}^*_{\lambda,(\alpha,\varepsilon)} \); that is,

\[
\|\tilde{g}^*_{\lambda,(\alpha,\varepsilon)}(f)\|_{L^p_w} \leq C\|f\|_{H^p_w},
\]

Therefore, by Lemma 3.1, Proposition 3.2, Proposition 3.3 and (10), we have proved the necessity of Theorems 1, 2 and 3.

4. The sufficiency of our theorems

We shall need the following Calderón reproducing formula given in [1].

**Lemma 4.1.** Let \( \psi \in \mathcal{S}(\mathbb{R}^n) \), supp \( \psi \subset \{ x \in \mathbb{R}^n : |x| \leq 1 \} \), \( \int_{\mathbb{R}^n} \psi(x) \, dx = 0 \) and

\[
\int_0^\infty |\hat{\psi}(\xi t)|^2 \frac{dt}{t} = 1 \quad \text{whenever} \quad \xi \neq 0.
\]

Then for any \( f \in \mathcal{S}'(\mathbb{R}^n) \), \( f \) vanishes weakly at infinity, we have

\[
f(x) = \int_0^\infty \int_{\mathbb{R}^n} f * \psi_t(y) \psi_t(x-y) \frac{dy \, dt}{t},
\]

where the equality holds in the sense of distribution.

Suppose that \( \psi \) satisfies the conditions of Lemma 4.1. For every \( f \in \mathcal{S}'(\mathbb{R}^n) \), we define the area integral of \( f \) by

\[
S_{\psi}(f)(x) = \left( \int_{|x-y|\leq t} |f * \psi_t(y)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}.
\]

We now prove the following result.

**Proposition 4.2.** Let \( 0 < \alpha \leq 1 \), \( \frac{n}{m+\alpha} < p < 1 \) and \( w \in A_{p(1+\frac{n}{m})} \). Then for any \( f \in \mathcal{S}'(\mathbb{R}^n) \), \( f \) vanishes weakly at infinity, we have

\[
\|f\|_{H^p_w} \leq C\|S_{\psi}(f)\|_{L^p_w}.
\]

**Proof.** We follow the same constructions as in [7]. For any \( k \in \mathbb{Z} \), set

\[
\Omega_k = \{ x \in \mathbb{R}^n : S_{\psi}(f)(x) > 2^k \}.
\]

Let \( \mathcal{D} \) denote the set formed by all dyadic cubes in \( \mathbb{R}^n \) and let

\[
\mathcal{D}_k = \left\{ Q \in \mathcal{D} : |Q \cap \Omega_k| > \frac{|Q|}{2} , |Q \cap \Omega_{k+1}| \leq \frac{|Q|}{2} \right\}.
\]
Obviously, for any \( Q \in \mathbb{D} \), there exists a unique \( k \in \mathbb{Z} \) such that \( Q \in \mathbb{D}_k \). We also denote the maximal dyadic cubes in \( \mathbb{D}_k \) by \( Q^l_k \). Set

\[
\tilde{Q} = \{(y,t) \in \mathbb{R}^{n+1}_+: y \in Q, l(Q) < t \leq 2l(Q)\},
\]

where \( l(Q) \) denotes the side length of \( Q \).

If we set \( Q^l_k = \bigcup_{Q \supseteq \tilde{Q}} Q^l_k \), then we have \( \mathbb{R}^{n+1} = \bigcup_{k,l} Q^l_k \). Hence, by (11), we obtain

\[
f(x) = \sum_k \sum_l \int_{Q^l_k} f \ast \psi_t(y) \psi_t(x-y) \frac{dydt}{t} = \sum_k \sum_l \lambda_{kl} \alpha^l_k(x),
\]

where

\[
\alpha^l_k(x) = \lambda_{kl}^{-1} \int_{Q^l_k} f \ast \psi_t(y) \psi_t(x-y) \frac{dydt}{t}
\]

and

\[
\lambda_{kl} = w(Q^l_k)^{1/p}(\int_{Q^l_k} \frac{2w(Q^l_k)}{|Q^l_k|} \frac{dydt}{t})^{1/2}.
\]

By the properties of \( \psi \), we can easily get \( \text{supp} \alpha^l_k \subseteq 5Q^l_k \). Let \( q = p(1 + \frac{\alpha}{n}) \), \( w \in A_q \). Since

\[
\|a^l_k\|_{L^q_w} = \sup_{\|b\|_{L^p_{q'}} \leq 1} \left| \int_{\mathbb{R}^n} a^l_k(x)b(x)w(x) \, dx \right|
\]

Then Hölder’s inequality and the definition of \( \lambda_{kl} \) imply

\[
\left| \int_{\mathbb{R}^n} a^l_k(x)b(x)w(x) \, dx \right|
\leq \lambda_{kl}^{-1} \int_{Q^l_k} |f \ast \psi_t(y)||g \ast \psi_t(y)| \frac{dydt}{t}
\leq \lambda_{kl}^{-1} \left( \int_{Q^l_k} \frac{2dydt}{t} \right)^{1/2} \left( \int_{Q^l_k} |g \ast \psi_t(y)|^2 \frac{dydt}{t} \right)^{1/2}
\leq \frac{|Q^l_k|^{1/2}}{w(Q^l_k)^{1/p}} \left( \int_{Q^l_k} |g \ast \psi_t(y)|^2 \frac{dydt}{t} \right)^{1/2},
\]

where \( g(x) = \chi_{5Q^l_k}(x)b(x)w(x) \). A simple calculation shows that

\[
|g \ast \psi_t(y)| \leq C \cdot t^{-n}\|b\|_{L^p_{q'}} w(Q^l_k)^{1/q}.
\]
Hence we have
\[
\|a_k^t\|_{L^p_w} \leq C \cdot \frac{|Q_k^l|^{1/2}}{w(Q_k^l)^{1/p}} w(Q_k^l)^{1/q} \left( \int_{\Omega_k^l} \frac{dydt}{t^{2n+1}} \right)^{1/2} \\
\leq C \cdot w(Q_k^l)^{1/q-1/p},
\]
where in the last inequality we have used the fact that for any \((y, t) \in Q_k^l\), we have \(t^n \sim |Q_k^l|\). Therefore these functions \(a_k^t\) defined above are all \(w\)-(\(p, q, 0\))-atoms.

Set \(\Omega_k^* = \left\{ x \in \mathbb{R}^n : M_w(x, \Omega_k)(x) > \frac{C_0}{2} \right\}\), where \(C_0\) is a constant to be determined later and \(M_w(f)(x) = \sup_{x \in Q} \frac{1}{w(Q)} \int_Q |f(y)| w(y) dy\). Using the weighted weak type estimate of weighted maximal operator \(M_w\), we have \(w(\Omega_k^*) \leq C w(\Omega_k)\). Consequently
\[
\int_{\Omega_k^* \setminus \Omega_{k+1}} S_w(f)(x)^2 w(x) \, dx \leq (2^{k+1})^2 w(\Omega_k^*) \leq C \cdot 2^{2k} w(\Omega_k).
\]

We set \(E = \{ x \in \Omega_k^* \setminus \Omega_{k+1} : |x - y| < t \}\), then we have
\[
\int_{\Omega_k^* \setminus \Omega_{k+1}} S_w(f)(x)^2 w(x) \, dx = \int_{\mathbb{R}^n+1} \left\{ \int_{\mathbb{R}^n} \chi_E(x) w(x) \, dx \right\} |f * \psi_t(y)|^2 dydt \left( \frac{t^{n+1}}{t^{n+1}} \right) w(E).
\]

For any \(x \in Q, Q \in \mathbb{D}_k\), we have \(|Q \cap \Omega_k| > \frac{1}{2} |Q|\), which is equivalent to
\[
\frac{1}{|Q|} \int_Q \chi_{\Omega_k}(y) \, dy > \frac{1}{2}. \tag{12}
\]

Hölder’s inequality and the \(A_q\) condition give
\[
\frac{1}{|Q|} \int_Q \chi_{\Omega_k}(y) \, dy \leq \frac{1}{|Q|} \left( \int_Q |\chi_{\Omega_k}(y)|^q w(y) \, dy \right)^{1/q} \left( \int_Q w^{-1/(q-1)} \, dy \right)^{(q-1)/q} \\
\leq [w]_{A_q}^{1/q} \left( \frac{1}{w(Q)} \right)^{1/q} \int_Q \chi_{\Omega_k}(y) w(y) \, dy. \tag{13}
\]

It follows immediately from (12) and (13) that \(M_w(\chi_{\Omega_k})(x) > (\frac{1}{2})^q [w]_{A_q}^{-1}\).

So if we choose \(C_0 = (\frac{1}{2})^{q-1} [w]_{A_q}^{-1}\), we have \(x \in \Omega_k^*\), which implies \(Q \subseteq \Omega_k^*\).
Hence $w(Q \cap \Omega^*_k) = w(Q)$. Since $|Q \cap \Omega_{k+1}| \leq \frac{1}{2}|Q|$, $w \in A_{\infty}$, then there exists a constant $0 < C' < 1$ such that $w(Q \cap \Omega_{k+1}) \leq C'w(Q)$. Consequently
\begin{align*}
 w(E) & \geq w(Q \cap (\Omega^*_k \setminus \Omega_{k+1})) \\
 & \geq w(Q) - w(Q \cap \Omega_{k+1}) \\
 & \geq (1 - C')w(Q). \tag{14}
\end{align*}

Suppose that $Q^l_k$ is the maximal dyadic cubes containing $Q$ which belong to $D_k$. Then by Lemma D and the inequality (14), we can get
\begin{align*}
 2^{2k}w(\Omega_k) & \geq C \sum_{Q \in D_k} \int_{\tilde{Q}} |f * \psi_l(y)|^2 w(Q) \frac{dydt}{t^{n+1}} \\
 & \geq C \sum_{Q \in D_k} \int_{\tilde{Q}} |f * \psi_l(y)|^2 w(Q^l_k) \left( \frac{|Q|}{|Q_k|} \right)^q \frac{dydt}{t^{n+1}} \\
 & \geq C \sum_{l} \int_{\tilde{Q}^l_k} |f * \psi_l(y)|^2 \frac{w(Q^l_k)}{|Q^l_k|} \frac{1}{|Q^l_k|^\alpha} \frac{dydt}{t^{1-\alpha}} \\
 & \geq C \sum_{l} \int_{\tilde{Q}^l_k} |f * \psi_l(y)|^2 \frac{w(Q^l_k)}{|Q^l_k|} \frac{dydt}{t}, \tag{15}
\end{align*}

where the last inequality holds since $t \sim l(Q^l_k)$. For any $l \in \mathbb{Z}_+$, since $|Q^l_k \cap \Omega_k| > \frac{1}{2}|Q^l_k|$, $w \in A_{\infty}$, then by Lemma C, we have that there exists a constant $0 < C'' < 1$ such that $w(Q^l_k \cap \Omega_k) > C''w(Q^l_k)$. Note that the maximal dyadic cubes $Q^l_k$ are pairwise disjoint, we thus obtain
\begin{align*}
 w(\Omega_k) & \geq w\left( \bigcup_l Q^l_k \right) \cap \Omega_k \\
 & = \sum_l w(Q^l_k \cap \Omega_k) \\
 & > C'' \sum_l w(Q^l_k). \tag{16}
\end{align*}

Then it follows from Hölder’s inequality, (15) and (16) that
\begin{align*}
 \sum_k \sum_l |\lambda_{kl}|^p & = \sum_k \sum_l \left( w(Q^l_k) \right)^{1-p/2} \left( \int_{\tilde{Q}^l_k} |f * \psi_l(y)| \left( \frac{2w(Q^l_k)}{|Q^l_k|} \frac{dydt}{t} \right)^{p/2} \\
 & \leq \sum_k \left( \sum_l w(Q^l_k) \right)^{1-p/2} \left( \sum_l \int_{\tilde{Q}^l_k} |f * \psi_l(y)| \left( \frac{2w(Q^l_k)}{|Q^l_k|} \frac{dydt}{t} \right)^{p/2} \right)^{p/2} \\
 & \leq \sum_k \left( \sum_l w(Q^l_k) \right)^{1-p/2} \left( \sum_l \int_{\tilde{Q}^l_k} |f * \psi_l(y)| \right)^{p/2}.
\end{align*}
\[
\leq C \sum_{k} \left( w(\Omega_k) \right)^{1-p/2} \left( 2^{2k} w(\Omega_k) \right)^{p/2} \\
\leq C \|S_\psi(f)\|_{L^p_w}^p.
\]

Therefore, by using the atomic decomposition of weighted Hardy spaces, we get the desired result. \( \square \)

Finally, we choose a function \( \psi \) satisfying the conditions of Lemma 4.1. Obviously, we have \( \psi \in C_\alpha \) for \( 0 < \alpha \leq 1 \), which implies

\[
S_\psi(f)(x) \leq S_\alpha(f)(x) \leq C S(f)(x) \leq C g^*_\lambda(x), \quad (17)
\]

Combining the above inequality (17) and Proposition 4.2, we have proved the sufficiency of Theorems 1, 2 and 3.

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