Precise calculation of the threshold of various directed percolation models on a square lattice

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Using Monte Carlo simulations on different system sizes we determine with high precision the critical thresholds of two families of directed percolation models on a square lattice. The thresholds decrease exponentially with the degree of connectivity. We conjecture that $p_c$ decays exactly as the inverse of the coordination number.

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Directed percolation (DP) describes generically the dynamics of adsorbing processes and has been applied to epidemics, forest fires, surface catalysis, etc [1-6]. DP displays a phase transition as function of the propagation probability $p$ between an “adsorbing” state and a percolating (or active) state at a critical threshold $p_c$. The scaling properties of DP have been known since over two decades.

The value of $p_c$ depends on the lattice and the rule of connectivity. In 1+1 dimension the most studied case is the tilted square lattice with nearest neighbor connectivity (see Fig. 1) giving $p_c = 0.6447 \pm 0.0001$ [7]. For many applications it is of interest to consider a longer range of connectivities. In particular also an infinite range model with a probability decaying as a powerlaw with distance has been studied [8]. However, what is astonishingly missing in the literature are studies of propagation probabilities of intermediate range although these are in practice the most common situations. It is therefore the aim of the present paper to calculate the percolation thresholds of two families of models in 1+1 dimensions having finite, varying range of interactions.

Model I is defined on a tilted square lattice as shown in Fig. 1a. The connectivity $k$ is given as the number of neighbors to which a site $i$ can be connected in the row below. In model I the $k$ is always even. Fig. 1a also shows the cases $k = 2$, 4 and 6. Model II is defined on the standard square lattice and in Fig. 1b we can see the cases $k = 1$, 3 and 5. For this case, $k$ is always odd.

We studied the two models up to $k = 15$ on lattices of sizes ranging from $256 \times 20000$ till $2048 \times 20000$. We simulated for each value of $k$ and $L$ the model for different values of $p$ averaging over 2000 initial configurations and monitored if the system percolated or not. From the inflection point of the histogram as function of $p$ we determined $p_c(L)$. Finally we extrapolated to the thermodynamic limit $p_c$ through the expression

$$p_c(L) = p_c(1 + a L^{-1/\nu})$$ (1)

In table 1 we show the obtained thresholds with their estimated error bars and in Fig. 2 we see these values in a double-logarithmic plot. Since the slope is $-1 \pm 0.05$,
we conclude that

\[ p_c = \frac{\eta}{k} \]  \hspace{1cm} (2a)

with \( \eta = 1.3 \pm 0.1 \) for model I and

\[ p_c = \frac{\eta}{(k - 1)} \]  \hspace{1cm} (2b)

with \( \eta = 0.9 \pm 0.1 \) for model II.

In summary, we have described in this short paper numerical evidence that the percolation threshold of directed percolation decays like \( \frac{\eta}{k} \) as function of the coordination \( k \). This result is consistent with the dependence of the percolation threshold on the coordination number on the Cayley tree \( \mathbb{K}^\infty \) and we conjecture it to be exact.

As a future work, it would be interesting to calculate the thresholds for higher dimensional lattices.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{k} & \textbf{I} & \textbf{II} \\
\hline
2 & 0.6447 \pm 0.0002 & 3 \text{ } 0.4395 \pm 0.0003 \\
4 & 0.3272 \pm 0.0002 & 5 \text{ } 0.2249 \pm 0.0003 \\
6 & 0.2121 \pm 0.0003 & 7 \text{ } 0.1470 \pm 0.0002 \\
8 & 0.1553 \pm 0.0003 & 9 \text{ } 0.1081 \pm 0.0002 \\
10 & 0.1220 \pm 0.0002 & 11 \text{ } 0.0851 \pm 0.0002 \\
12 & 0.0999 \pm 0.0002 & 13 \text{ } 0.0701 \pm 0.0002 \\
14 & 0.0846 \pm 0.0003 & 15 \text{ } 0.0549 \pm 0.0002 \\
\hline
\end{tabular}
\caption{Percolation threshold of directed percolation to model I and model II.}
\end{table}

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