ASSEMBLY MAPS FOR GROUP EXTENSIONS IN \( K \)-THEORY AND \( L \)-THEORY WITH TWISTED COEFFICIENTS

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Abstract. In this paper we show that the Farrell-Jones isomorphism conjectures are inherited in group extensions for assembly maps in \( K \)-theory and \( L \)-theory with twisted coefficients.

INTRODUCTION

Under what assumptions are the Farrell-Jones isomorphism conjectures inherited by group extensions or subgroups? We will formulate a version of the standard conjectures (see Farrell-Jones [10]) with twisted coefficients in an additive category, and then study these questions via the continuously controlled assembly maps of [11, §7]. A formulation using the Davis-Lück assembly maps [9] has already been given by Bartels and Reich [4], and applied there to show inheritance by subgroups. Recall that the Farrell-Jones conjecture in algebraic \( K \)-theory asserts that certain “assembly” maps

\[ H^n_G(E_{VC}G; \mathbb{K}_R) \to K_n(RG) \]

are isomorphisms, for a given ring \( R \), and all \( n \in \mathbb{Z} \). Here the space \( E_{VC}G \) is the universal \( G \)-CW-complex for \( G \)-actions with virtually cyclic isotropy, and the left-hand side denotes equivariant homology with coefficients in the non-connective \( K \)-theory spectrum for the ring \( R \).

Theorem A. Let \( N \to G \twoheadrightarrow K \) be a group extension, where \( N \triangleleft G \) is a normal subgroup, and \( K \) is the quotient group. Let \( \mathcal{A} \) be an additive category with \( G \)-action. Suppose that

(i) The group \( K \) satisfies the Farrell-Jones conjecture in algebraic \( K \)-theory, with twisted coefficients in any additive category with \( K \)-action.

(ii) Every subgroup of \( G \) containing \( N \) as a subgroup, with virtually cyclic quotient, satisfies the Farrell-Jones conjecture in algebraic \( K \)-theory, with twisted coefficients in \( \mathcal{A} \).

Then the group \( G \) satisfies the Farrell-Jones conjecture in algebraic \( K \)-theory, with twisted coefficients in \( \mathcal{A} \).

This is a special case of a more general result (see Theorem 4.7). The same statement holds for algebraic \( L \)-theory as well, where the coefficient categories are additive categories with involution. The corresponding result for the Baum-Connes conjecture was

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obtained by Oyono-Oyono [12], and our proof follows the outline given there. One of the 
main points is that the most effective methods known for proving the standard Farrell-
Jones conjectures (for particular groups $G$) also work for the twisted coefficient versions 
(compare [1], [3], [6], [7], [15], [16], and [17]). An immediate corollary to Theorem A is 
the following.

**Corollary** (Corollary 4.10). The Farrell-Jones conjecture with twisted coefficients is true 
for $G_1 \times G_2$ if and only if it is true for $G_1$, $G_2$, and every product $V_1 \times V_2$, where $V_1 \leq G_1$ 
and $V_2 \leq G_2$ are virtually cyclic subgroups.

The fibered isomorphism conjecture of Farrell and Jones [10] for a group $G$ and a 
ring $R$ asserts that for every group homomorphism, $\phi: H \to G$, the assembly map for 
$H$ relative to the family generated by the subgroups $\phi^{-1}(V)$, $V \subset G$ virtually cyclic, 
is an isomorphism. This conjecture implies the Farrell-Jones conjecture and has better 
inheritance properties. For example, the fibered version of our Theorem A is also true 
(see, for example, [2, Section 2.3]). The following result shows that the Farrell-Jones 
conjecture with twisted coefficients implies the Fibered Farrell-Jones conjecture.

**Theorem B.** Suppose that $\phi: H \to G$ is a group homomorphism. Then the Farrell-Jones 
conjecture for $G$, with twisted coefficients in any $G$-category, implies that the assembly map 
for $H$ relative to the family generated by the subgroups $\phi^{-1}(V)$, $V \subset G$ virtually cyclic, 
is an isomorphism with twisted coefficients in any $H$-category.

The corresponding result for the Davis-Lück assembly maps was obtained by Bartels-
Reich [4], who also pointed out a number of applications of the assembly map with twisted 
coefficients, including the study the $K$- and $L$-theory of twisted group rings (see also 
Example 4.8 and Example 4.9 below). One can check as in [11] that those assembly maps 
are equivalent to the continuously controlled assembly maps used in this paper.

### 1. Assembly via Controlled Categories

The controlled categories of Pedersen [13], Carlsson-Pedersen [6, 8] are our main tool 
for identifying various different assembly maps. We will recall the definition of these 
categories, and then the usual assembly maps are obtained by applying functors

$$H: G\text{-CW-Complexes} \to \text{Spectra}$$

as described in [11]. We will extend the earlier definitions in order to allow an additive 
category as coefficients, instead of just working with modules over a ring $R$. A formulation 
for assembly maps with coefficients in the setting of [9] has already been given in [4]. 
Following the method of [11], one can check that the two different descriptions give the 
same assembly maps.

Let $G$ be any discrete group, and let $X$ be a $G$-CW complex (we will use a left $G$-action). 
Subspaces of the form $G \cdot D \subset X$, with $D$ compact in $X$, are called $G$-compact 
subspaces of $X$. More generally, a subspace whose closure has this form is called relatively $G$-compact. 
A resolution of $X$ is a pair $(\overline{X}, p)$, where $\overline{X}$ is a free $G$-CW complex and $p: \overline{X} \to X$ is a 
continuous $G$-equivariant map, such that for every $G$-compact set $G \cdot D \subset X$ there exists
a $G$-compact set $G \cdot D \subset X$ such that $p(G \cdot D) = G \cdot D$. The notion of resolution comes from [13], and was developed further in [11 §3]. The original example was $X = G \times X$, with the diagonal $G$-action and first factor projection.

Let $A$ be an additive category with involution, and suppose that $A$ has a right $G$-action compatible with the involution. This is a collection of covariant functors $\{g^*: A \to A, \forall g \in G\}$, such that $(g \circ h)^* = h^* \circ g^*$ and $e^* = id$. We require that the functors $g^*$ commute with the involution $*: A \to A$ (an involution is a contravariant functor with square the identity).

**Definition 1.1.** Let $(Z, X)$ be a $G$-CW pair, where $X$ is a closed $G$-invariant subspace. Let $Y = Z - X$, and fix a resolution $p: Z \to Z$, whose restriction to $Y$ is denoted $\overline{Y}$. The category $\mathcal{D}(Z, X; A)$ has objects $A = (A_y)$ consisting of a collection of objects of $A$, indexed by $y \in \overline{Y}$, and morphisms $\phi: A \to B$ consisting of collections $\phi = (\phi_y^z)$ of morphisms $\phi_y^z: A_y \to B_z$ in $A$, indexed by $y, z \in \overline{Y}$, satisfying:

(i) the support $\{y \in \overline{Y} | A_y \neq 0\}$ is locally finite in $\overline{Y}$, and relatively $G$-compact in $\overline{Y}$.

(ii) for each morphism $\phi: A \to B$, and for each $y \in \overline{Y}$, the set $\{z | \phi_y^z \neq 0 \text{ or } \phi_y^z \neq 0\}$ is finite.

(iii) the morphisms $\phi: A \to B$ are continuously controlled at $X \subset Z$. For every $x \in X$, and for every $G_x$-invariant neighbourhood $U$ of $x$ in $Z$, there is a $G_x$-invariant neighbourhood $V$ of $x$ in $Z$ so that $\phi_y^z = 0$ and $\phi_y^z = 0$ whenever $p(y) \in (Y - U)$ and $p(z) \in (V \cap U \cap Y)$.

If $X = \emptyset$, we use the shorter notation $\mathcal{D}(Z; A) := \mathcal{D}(Z, \emptyset; A)$, and in this case the continuous control condition (iii) on morphisms is vacuous. If $S$ is a discrete left $G$-set, we denote by $\mathcal{D}^1(S \times Z, S \times X; A)$ the subcategory where the morphisms are $S$-level-preserving: $\phi_{(s, z)}^{(s', z)} = 0$ if $s \neq s' \in S$, for any $y, z \in Y$.

The category $\mathcal{D}(Z, X; A)$ is an additive category with involution, where the dual of $A$ is given by $(A^*)_y = A_y^*$ for all $y \in \overline{Y}$. It depends functorially on the pair $(Z, X)$ of $G$-CW complexes. The actions of $G$ on $A$ and $Z$ induce a right $G$-action on $\mathcal{D}(Z, X; A)$. For $g \in G$, we set $(gA)_y = g^* A_{gy}$ and $(g\phi)_y^z = g^*(\phi_{gy}^z)$. The fixed subcategory will be denoted $\mathcal{D}^G(Z, X; A)$. If $G = \{e\}$ is the trivial group, we use the notation $\mathcal{D}^0(Z, X; A)$. We have not included the resolution $(\overline{Z}, p)$ in the notation, because two different resolutions give $G$-equivalent categories (see [11 Prop. 3.5]). We can compare these fixed subcategories to the equivariant category $\mathcal{B}_G(Z, X; R)$ defined in [11 §7].

**Lemma 1.2.** There is an equivalence of categories $\mathcal{B}_G(Z, X; R) \simeq \mathcal{D}^G(Z, X; A)$, when $A$ is the category of finitely-generated free $R$-modules.

**Proof.** We define a functor $F: \mathcal{D}^G(Z, X; A) \to \mathcal{B}_G(Z, X; R)$ by sending an object $A$ to the free $R$-module $F(A)_y = \oplus_{g \in G_y} A_{(g,y)}$, for all $y \in Y$, with the obvious reference map to $Y$. Similarly, for a morphism $\phi: A \to B$, we define $F(\phi)_y^z = (\phi_{gy}^z)_{g, g' \in G}$, for all $y, z \in Y$. The verification that this definition makes sense will be left to the reader.

Conversely, we can define a functor $F^*: \mathcal{B}_G(Z, X; R) \to \mathcal{D}^G(Z, X; A)$ on objects by decomposing an object $A = (A_y)$ of $\mathcal{B}_G(Z, X; R)$ as $A_y = \oplus_{g \in G_y} (A_y)_g$, since $A_y$ is a
finitely-generated free $RG_y$-module. Now we let $F'(A)_{(g, y)} = (A_y)_{g}$, for all $y \in Y$, $g \in G$, and on morphisms by letting $F'(\phi)_{(g, y)} = \phi_{g, y}$. Again the verifications will be left to the reader (technically we should work with a category equivalent to $B_G(Z, X; R)$, in which the objects are based: each $A = R[T]$, where $T$ is a free $G$-set, and $T$ is equipped with a reference map to $X \times [0, 1]$).

For applications to assembly maps, we will let $X$ be a $G$-CW complex and $Z = X \times [0, 1]$ so that $Y = X \times [0, 1)$. The category just defined will be denoted

$$D^G(X \times [0, 1); A) := D^G(X \times [0, 1], X \times 1; A).$$

Let $D^G(X \times [0, 1); A)_\emptyset$ denote the full subcategory of $D^G(X \times [0, 1); A)$ with objects $A$ such that the intersection with the closure $\text{supp}(A) = \{(x, t) \in X \times [0, 1) \mid A_{(x, t)} \neq 0\} \cap (X \times 1)$ is the empty set.

**Example 1.3.** If $A$ is the additive category of finitely generated free $R$-modules, then $D^G(X \times [0, 1); A)_\emptyset$ is equivalent to the category of finitely generated free $RG$-modules, for any $G$-CW complex $X$.

The quotient category will be denoted $D^G(X \times [0, 1); A)^>0$, and we remark that this is a germ category (see [11, §7], [14], [6]). The objects are the same as in $D^G(X \times [0, 1); A)$ but morphisms are identified if they agree close to $\overline{X} = X \times 1$ (i.e. on the complement of a neighbourhood of $X \times 0$). Here is a useful remark.

**Lemma 1.4** ([11]). Let $S$ be a discrete left $G$-set. The forgetful functor

$$D^G_1(S \times X \times [0, 1); A)^>0 \rightarrow D^G(S \times X \times [0, 1); A)^>0$$

is an equivalence of categories.

**Proof.** In the germ category, every morphism has a representative which is level-preserving with respect to projection on $S$. □

The category $D^G(X \times [0, 1); A)^>0$ is an additive category with involution, and we obtain a functor $G$-CW-Complexes $\rightarrow$ AddCat$^-$. The results of [5, 1.28, 4.2] now show that the functors $F^\lambda : G$-CW-Complexes $\rightarrow$ Spectra defined by

$$F^\lambda_G(X; A) := \begin{cases} K^\infty(D^G(X \times [0, 1); A)^>0) \\ L^\infty(D^G(X \times [0, 1); A)^>0) \end{cases},$$

where $\lambda = K^\infty$ or $\lambda = L^\infty$ respectively, are $G$-homotopy invariant and $G$-excisive.

We can now extend the definition of the assembly maps to allow coefficients in any additive category with $G$-action.

**Definition 1.6.** We define the continuously controlled assembly map with coefficients in $A$ to be the map $F^\lambda_G(X; A) \rightarrow F^\lambda_G(\bullet; A)$. 
From the methods of [11], the continuously controlled assembly map with coefficients is homotopy equivalent to the assembly map with coefficients constructed in [4]. The most important example to consider is when $X = E_{VC}G$, in which case the Farrell-Jones conjecture with coefficients asserts that this assembly map is an equivalence. Given a discrete group $G$, a family of subgroups $\mathcal{F}$ of $G$, and coefficients $\mathcal{A}$, we will refer to

$$F^\lambda_G(\mathcal{E}_{\mathcal{F}G}; \mathcal{A}) \to F^\lambda_G(\bullet; \mathcal{A})$$

as the $(G, \mathcal{F}, \mathcal{A})$-assembly map.

By applying $K^{-\infty}$ or $L^{-\infty}$ to the sequence of additive categories (with involution):

$$D^G(X \times [0, 1]; \mathcal{A}) \to D^G(X \times [0, 1]; \mathcal{A}) \to D^G(X \times [0, 1]; \mathcal{A})^{>0}$$

we obtain a fibration of spectra [6]. As in [11], we have the following description for the assembly map.

**Theorem 1.7 ([11, §7]).** The continuously controlled assembly map

$$F^\lambda_G(X; \mathcal{A}) \to F^\lambda_G(\bullet; \mathcal{A})$$

is homotopy equivalent to the connecting map

$$\lambda(D^G(X \times [0, 1]; \mathcal{A})^{>0}) \to \Omega^{-1}\lambda(D^G(X \times [0, 1]; \mathcal{A})_0)$$

for $\lambda = K^{-\infty}$ or $\lambda = L^{-\infty}$.

See [11, §2] for the definition of homotopy equivalent functors from $G$-$CW$-Complexes $\to$ Spectra, and [9, 5.1] for the result that any functor $E : \text{Or}(G) \to \text{Spectra}$ out of the orbit category of $G$ may be extended uniquely (up to homotopy) to a functor $E^\% : G$-$CW$-Complexes $\to$ Spectra which is $G$-homotopy invariant and $G$-excisive. This will be our method for comparing functors. The orbit category $\text{Or}(G)$ is the category with objects $G/K$, for $K$ any subgroup of $G$, and the morphisms are $G$-maps.

## 2. Change of Coefficients

We will need some ‘change of coefficient’ properties for the categories defined in the last section. The first three properties are essentially just translations of [11 Proposition 2.8] into our language. The corresponding versions for additive categories with involution are needed to apply these change of coefficient functors to $L$-theory.

**Definition 2.1.** Let $K$ and $G$ be groups, $\mathcal{A}$ an additive category with commuting right $K$ and $G$-actions, and $S$ a $K$-$G$ biset. Then, the category $D^K(S; \mathcal{A})$ has a right $G$-action via $(g \cdot A)_y = g^*A_{yz}^{-1}$ and $(g \cdot \phi)_y^z = g^*\phi_{yz}^{-1}$, for all $y, z \in S$. We will mostly use the level-preserving subcategory $D^K(S; \mathcal{A})$.

If $T$ is a left $G$-set, and $S$ is a transitive $K$-$G$ biset (meaning that $K\backslash S/G$ is a point), we define a $K \times G$-action on $S \times T$ by the formula $(k, g) \cdot (s, t) := (ksg^{-1}, gt)$ for all $(k, g) \in K \times G$ and all $(s, t) \in S \times T$. This action is used in the statements below.
Lemma 2.2. Let $T$ be a left $G$-set, and $S$ be a transitive $K$-$G$ biset. Then there is an additive functor

$$F : \mathcal{D}_l^{K \times G}(S \times T \times [0,1]; \mathcal{A}) \to \mathcal{D}_l^G(T \times [0,1]; \mathcal{D}_l^K(S; \mathcal{A}))$$

which induces an equivalence of categories

$$\mathcal{D}_l^{K \times G}(S \times T; \mathcal{A}) \simeq \mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})).$$

Proof. We will take the standard resolutions $\overline{S} = K \times S$, with elements denoted $(k,s)$, for $k \in K$ and $s \in S$, and $\overline{T} = G \times T \times [0,1]$, with elements denoted $(g,t)$, for $g \in G$ and $t \in T \times [0,1]$. Therefore

$$\overline{S} \times \overline{T} = K \times G \times S \times T \times [0,1]$$

is a resolution for $S \times T \times [0,1]$. We define the functor

$$F : \mathcal{D}_l^{K \times G}(S \times T \times [0,1]; \mathcal{A}) \to \mathcal{D}_l^G(T \times [0,1]; \mathcal{D}_l^K(S; \mathcal{A}))$$

on objects by setting $B = F(A)(g,t)$ in $\mathcal{D}_l^K(S; \mathcal{A})$ as the object $B = (B_{(k,s)})$ with $B_{(k,s)} = A_{(k,g,s,t)}$ in $\mathcal{A}$. We use a similar formula for morphisms:

$$\left( F(\phi)^{(g',t')}_{(g,t)} \right)^{(k',s')}_{(k,s)} = \phi^{(k',g',s',t')}_{(k,g,s,t)}$$

The proof that this is a well-defined functor is given in Section 5 where step (5′) of the argument depends on the assumption that $S$ is a transitive $K$-$G$ biset.

Since $\mathcal{D}_l^{K \times G}(S \times T; \mathcal{A}) \simeq \mathcal{D}_l^{K \times G}(S \times T \times [0,1]; \mathcal{A})_0$ and $\mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})) \simeq \mathcal{D}_l^G(T \times [0,1]; \mathcal{D}_l^K(S; \mathcal{A}))_0$, the functor $F$ induces an additive functor

$$F : \mathcal{D}_l^{K \times G}(S \times T; \mathcal{A}) \to \mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})).$$

On this subcategory, we define an inverse additive functor

$$F' : \mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})) \to \mathcal{D}_l^{K \times G}(S \times T; \mathcal{A})$$

on objects by setting $F'(B)_{(k,g,s,t)} = (B_{(g,t)})_{(k,s)}$, and a similar formula for morphisms:

$$F'(\phi)^{(k',g',s',t')}_{(k,g,s,t)} = \phi^{(g',t')}_{(g,t)}$$

It is easy to check that $F'$ is a well-defined functor. The functors $F$ and $F'$ are inverses, so give an equivalence of categories. 

Corollary 2.3. Let $G$ and $K$ be groups, and $\mathcal{A}$ be an additive category with commuting right $K$ and $G$-actions,. Then

$$\mathcal{D}^{K \times G}(\bullet; \mathcal{A}) \simeq \mathcal{D}^G(\bullet; \mathcal{D}^{K}(\bullet; \mathcal{A})).$$

Proof. We substitute $S = \bullet$ and $T = \bullet$ in the statement above. Note that morphisms are automatically level-preserving in this case. 

□
**Lemma 2.4.** Let $K$ and $G$ be groups, $\mathcal{A}$ an additive category with commuting right $K$ and $G$-actions, and $S$ a transitive $K$-$G$ biset. Then, for any $G$-CW complex $X$, the functors

$$F^K_G(X; \mathcal{A})$$

and

$$F^K_{G,X}(S \times X; \mathcal{A})$$

are homotopy equivalent, where $\lambda = \mathbb{K}^\mathbb{C}$ or $\mathbb{L}^{-\infty}$. Here $K \times G$ acts on $S \times X$ by the formula $(k, g) \cdot (x, s) := (ksg^{-1}, gx)$.

**Proof.** By [9, 5.1] it is enough to show that the two functors are $G$-homotopy invariant, $G$-excisive, and homotopy equivalent when restricted to the orbit category $\text{Or}(G)$. For the first two properties, we apply [5, 1.28, 4.2]. For the last property, we follow the method of [11, §8]. Let $T = G/H$ and consider the following commutative diagram

$$
\begin{array}{c}
\mathcal{D}_1^T(S \times T \times [0, 1); \mathcal{A})_0 \longrightarrow \mathcal{D}_1^T(S \times T \times [0, 1); \mathcal{A}) \longrightarrow \mathcal{D}_1^T(S \times T \times [0, 1); \mathcal{A})^{>0} \\
\simeq \downarrow F \quad \downarrow F \quad \downarrow F \\
\mathcal{D}_1^T(T \times [0, 1); \mathcal{D}_1^T(S; \mathcal{A}))_0 \longrightarrow \mathcal{D}_1^T(T \times [0, 1); \mathcal{D}_1^T(S; \mathcal{A})) \longrightarrow \mathcal{D}_1^T(T \times [0, 1); \mathcal{D}_1^T(S; \mathcal{A})^{>0}
\end{array}
$$

where the vertical maps are induced by the additive functors of Lemma 2.2. We apply $\lambda = \mathbb{K}^\mathbb{C}$ or $\mathbb{L}^{-\infty}$ to obtain fibrations of spectra. Note that $\lambda$ applied to either of the middle two categories gives a spectrum with trivial homotopy groups (by an Eilenberg swindle). Therefore the first and third vertical maps induce a homotopy equivalence of spectra. Since the level-preserving condition is automatic on the germ categories, we are done.

The next property allows us to divide out a normal subgroup in suitable circumstances.

**Lemma 2.5.** Let $N$ be a normal subgroup of $G$, and $\mathcal{A}$ be an additive category with right $G$-action such that $N$ acts trivially. Let $X$ be a $G$-CW complex such that $N$ acts freely on $X$. Then there is an additive functor

$$\mathcal{D}^G(X \times [0, 1); \mathcal{A}) \rightarrow \mathcal{D}^{G/N}(N\backslash X \times [0, 1); \mathcal{A})$$

which induces an isomorphism on $K$-theory after taking germs away from the empty set.

**Proof.** We will construct a functor $F = F_2 \circ F_1$ inducing this isomorphism in two steps. First, we have a functor $F_1: \mathcal{D}^G(X \times [0, 1); \mathcal{A}) \rightarrow \mathcal{D}^G(N\backslash X \times [0, 1); \mathcal{A})$, which is the identity on objects and morphisms. The continuous control condition measured in $X$ is stronger than the continuous control condition measured in $N\backslash X$, so this is well-defined. This functor induces a homotopy invariant and $G$-excisive functor

$$F_1: \mathcal{D}^G(G/H \times [0, 1); \mathcal{A})^{>0} \rightarrow \mathcal{D}^G(N\backslash G/H \times [0, 1); \mathcal{A})^{>0}$$

for $X = G/H$, and an equivalence $\mathcal{D}^G(G/H; \mathcal{A}) \simeq \mathcal{D}^G(N\backslash G/H; \mathcal{A})$. Therefore $F_1$ induces isomorphisms on $K$-theory after taking germs away from the empty set (as in the proof of Lemma 2.1). Secondly, there is a functor

$$F_2: \mathcal{D}^G(N\backslash X \times [0, 1); \mathcal{A}) \rightarrow \mathcal{D}^{G/N}(N\backslash X \times [0, 1); \mathcal{A})$$
defined on objects by $F_2(A)_{(g,\bar{y})} = A_{(g,\bar{y})}$, where $\bar{y} \in N \setminus X \times [0,1)$. We define the functor on morphisms by $F_2(\phi)_{(g,\bar{y},g',\bar{y}')} = \phi_{(g,\bar{y},g',\bar{y}')}$. This is well-defined by $G$-invariance of the objects and morphisms in the domain, and the continuous control conditions on morphisms agree since both are measured in $N \setminus X$. It follows that $F_2$ is an equivalence of categories.

In the next statement, if $\mathcal{A}$ is an additive $G$-category, we denote by $\text{Res}_H \mathcal{A}$ the same category considered as an $H$-category under restriction to a subgroup $H$ of $G$. The following is “Shapiro’s Lemma” in our setting.

**Proposition 2.6.** Let $H$ be a subgroup of $G$, $\mathcal{A}$ be an additive category with $G$-action, and $X$ be an $H$-CW complex. There is an additive functor

$$\mathcal{D}^H(X \times [0,1); \text{Res}_H \mathcal{A}) \to \mathcal{D}^G(G \times H X \times [0,1); \mathcal{A})$$

which induces an equivalence of categories after taking germs.

**Proof.** This proposition is proven in [1, Proposition 8.3] in the case where $\mathcal{A}$ is the category of finitely generated free $R$-modules. The same proof works for any coefficient category once the functor $\text{Ind}: \mathcal{D}^H(X \times [0,1); \text{Res}_H \mathcal{A}) \to \mathcal{D}^G(G \times H X \times [0,1); \mathcal{A})$ is defined for general $\mathcal{A}$. Let $\phi: A \to B$ be a morphism in $\mathcal{D}^H(X \times [0,1); \text{Res}_H \mathcal{A})$. Then

$$\text{Ind}: \mathcal{D}^H(X \times [0,1); \text{Res}_H \mathcal{A}) \to \mathcal{D}^G(G \times H X \times [0,1); \mathcal{A})$$

is defined by $\text{Ind}(\mathcal{A})_{(g,\bar{y})} = (g^{-1})^*A_g$, and $\text{Ind}(\phi)_{(g',\bar{y}')} = (g^{-1})^*\phi_{g,\bar{y}}^{-1}g'\bar{y}'$ if $g^{-1}g' \in H$, and is zero otherwise. The inverse of this functor on the corresponding germ categories is induced by the inclusion $\iota: X \to G \times H X$. That is, $\text{Ind}^{-1}(M)_y = M_{\iota(y)}$ and $\text{Ind}^{-1}(\psi)_y' = \psi_{\iota(y)}$. □

**Remark 2.7.** The equivalences given in these three properties are natural with respect to equivariant maps $X \to X'$. If $\mathcal{A}$ is an additive category with involution, one can check that the above properties continue to hold in this context. This is needed for applications to the $L$-theory assembly maps.

3. Assembly and Subgroups

The properties of the continuously controlled categories given so far lead to a formal statement about assembly and subgroups. This is just our version of [4, Proposition 4.2]. If $H$ is a subgroup of $G$, and $\mathcal{A}$ is an additive $H$-category, we denote $\text{Ind}_H^G \mathcal{A} := \mathcal{D}^H(G; \mathcal{A})$ considered as a $G$-category by using the $H$-$G$ biset structure of $G$.

**Proposition 3.1.** Let $f: X \to X'$ be a $G$-equivariant map between $G$-CW complexes. Let $H$ be a subgroup of $G$, and let $\mathcal{A}$ be an additive category with $H$-action. Then there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}^H(\text{Res}_H X \times [0,1); \mathcal{A})^{>0} & \xrightarrow{f_*} & \mathcal{D}^H(\text{Res}_H X' \times [0,1); \mathcal{A})^{>0} \\
\simeq & & \simeq \\
\mathcal{D}^G(X \times [0,1); \text{Ind}_H^G \mathcal{A})^{>0} & \xrightarrow{f_*} & \mathcal{D}^G(X' \times [0,1); \text{Ind}_H^G \mathcal{A})^{>0}
\end{array}
$$
Proof. By Lemma 2.3 with $K = H$ and $S = G$, we have
\[ \mathcal{D}^G(X \times [0, 1); \text{Ind}_H^G(A)]^> \simeq \mathcal{D}^{H \times G}(G \times X \times [0, 1); A)]^> \]
where $1 \times G$ acts trivially on $A$ in the right-hand side. Finally,
\[ \mathcal{D}^{H \times G}(G \times X \times [0, 1); A)]^> \simeq \mathcal{D}^H(\text{Res}_H X \times [0, 1); A)]^> \]
by applying Lemma 2.5 to $H \times G$ with $N = G$. Note that $G$ acts freely on $G \times X$, with quotient isomorphic to $\text{Res}_H X$. □

**Corollary 3.2.** Let $H$ be a subgroup of $G$ and $\mathcal{F}$ be a family of subgroups of $G$. Suppose that the $K$-theory or $L$-theory $(G, \mathcal{F}, \mathcal{B})$-assembly map is an isomorphism (respectively injection or surjection) for every additive coefficient category $\mathcal{B}$ with $G$-action. Then the $(H, \mathcal{F}|_H, A)$-assembly map is an isomorphism (respectively injection or surjection) for any additive coefficient category $A$ with $H$-action.

Proof. Just substitute $X = E_{\mathcal{F}}G$ and $X' = \bullet$ in the diagram above. □

In particular, this says that the Farrell-Jones conjecture with coefficients is stable under taking subgroups. These ideas can be extended further to obtain a version of the fibered isomorphism conjecture.

**Proposition 3.3.** Let $\phi: H \to G$ be a group homomorphism, and let $\mathcal{F}$ be a family of subgroups of $G$. If the $K$-theory or $L$-theory assembly map for $G$ relative to the family $\mathcal{F}$ is an isomorphism (respectively injective or surjective), with twisted coefficients in any additive $G$-category, then the assembly map for $H$ relative to the pull-back family $\phi^* \mathcal{F} = \{ K \leq H \mid \phi(K) \in \mathcal{F} \}$ is an isomorphism (respectively injection or surjection), with twisted coefficients in any additive $H$-category.

Proof. The proof is the same as for Proposition 3.1 using $X = E_{\mathcal{F}}G$ and $X' = \bullet$, with the action of $H$ on $S = G$ and on $X$ defined via $\phi$, and $\text{Res}_\phi X = E_{\phi^* \mathcal{F}}G$. □

4. Assembly for Extensions

In [12] the Baum-Connes conjecture for topological $K$-theory is shown to pass to extensions. We show that there is a similar statement for algebraic $K$- and $L$-theory.

The proof outline used in [12] has two main steps, which we now translate into our setting. In the first step we use a discrete transitive right $G$-set $S$, which can be expressed as a single orbit $S = \{ s \} \cdot G$.

**Proposition 4.1.** Let $X$ be a $G$-CW complex, $S = \{ s \} \cdot G$, and $A$ be an additive $G$-category with involution. Then there is an additive functor
\[ \mathcal{D}^G_*(\text{Res}_{G*} X \times [0, 1); \text{Res}_{G*} A) \to \mathcal{D}^G_*(X \times [0, 1); \mathcal{D}^0_*(S; A)]^> \]
which induces a homotopy equivalence of spectra after applying $K^{-\infty}$ or $L^{-\infty}$. This equivalence is natural with respect to maps $X \to X'$ of $G$-CW complexes.
Proof. By Proposition 2.6

$$\mathbb{K}^\infty(\mathcal{D}^G((\text{Res}_{G_e} X \times [0, 1]; \text{Res}_{G_e} A))^{>0}) \simeq \mathbb{K}^\infty(\mathcal{D}^G(G \times_G, X \times [0, 1]; A)^{>0}).$$

Since $G \times_G X$ is $G$-equivariantly homeomorphic to $(G_s \backslash G) \times X = S \times X$, via the map $[g, x] \mapsto (H_{g^{-1}}, gx)$, and so

$$\mathcal{D}^G(G \times_G, X \times [0, 1]; A)^{>0} \simeq \mathcal{D}^G(S \times X \times [0, 1]; A)^{>0},$$

where $S \times X$ has the usual left $G$-action $g \cdot (s, x) = (sg^{-1}, gx)$. Finally, by Lemma 2.7.

$$\mathbb{K}^\infty(\mathcal{D}^G(S \times X \times [0, 1]; A)^{>0}) \simeq \mathbb{K}^\infty(\mathcal{D}^G(X \times [0, 1]; D^0(S; A))^{>0}).$$

The same proof works if we replace $\mathbb{K}$ by $\mathbb{L}$. □

Example 4.2. Let $\pi: G \to K$ be a surjection of groups, and $V \subset K$ be a subgroup. We consider $S = K$ as a right-$(G \times V)$-set via the transitive action $k \cdot (g, v) := \pi(g)^{-1}kv$, where $g \in G$, $v \in V$, and $k \in K$. Let $X$ be a $(G \times K)$-CW complex, and let $V' \subset G \times V$ denote the stabilizer subgroup of $e \in K$. Notice that $V' \cong \pi^{-1}(V)$, since $\pi(g)^{-1}v = e$ implies $g \in \pi^{-1}(v)$. By Proposition 4.1 we have a commutative diagram

$$
\begin{array}{ccc}
F^\lambda_{V'}(X; A) & \to & F^\lambda_{V'}(\bullet; A) \\
\downarrow \simeq & & \downarrow \simeq \\
F^\lambda_{G \times V}(X; \mathcal{D}^0_l(K; A)) & \to & F^\lambda_{G \times V}(\bullet; \mathcal{D}^0_l(K; A))
\end{array}
$$

for $\lambda = \mathbb{K}^\infty$ or $\lambda = \mathbb{L}^\infty$, which shows that the lower assembly map is a homotopy equivalence of spectra whenever the upper map is an equivalence.

Remark 4.3. In the proof of Theorem A, we will be using Example 4.2 with $X = E_{F_G} G \times E_{F_K} K$, where $F_G$ is a family of subgroups of $G$ and $F_K$ is a family of subgroups of $K$ such that $\pi(H) \in F_K$ for every $H \in F_G$. If $V \in F_K$, then the map $E_{F_G} G \times E_{F_K} K \to E_{F_G} G \times \bullet$ is a $G \times V$-equivariant homotopy equivalence. Therefore, it is a $V'$-equivariant homotopy equivalence. Since $V' \cong \pi^{-1}(V)$, we have the homotopy commutative diagram:

$$
\begin{array}{ccc}
F^\lambda_{\pi^{-1}(V)}(E_{F_G} G; A) & \to & F^\lambda_{\pi^{-1}(V)}(\bullet; A) \\
\downarrow \simeq & & \downarrow \simeq \\
F^\lambda_{G \times V}(X; \mathcal{D}^0_l(K; A)) & \to & F^\lambda_{G \times V}(\bullet; \mathcal{D}^0_l(K; A))
\end{array}
$$

where $X = E_{F_G} G \times E_{F_K} K$.

If $V = K$, then $G \cong V' \subset G \times K$ and $G$ acts on $X = E_{F_G} G \times E_{F_K} K$ via this isomorphism. Since we are assuming that $\pi(H) \in F_K$ for every $H \in F_G$, $X$ is a model for $E_{F_G} G$. Thus, we have the homotopy commutative diagram:

$$
\begin{array}{ccc}
F^\lambda_G(E_{F_G} G; A) & \to & F^\lambda_G(\bullet; A) \\
\downarrow \simeq & & \downarrow \simeq \\
F^\lambda_{G \times K}(X; \mathcal{D}^0_l(K; A)) & \to & F^\lambda_{G \times K}(\bullet; \mathcal{D}^0_l(K; A))
\end{array}
$$
**Definition 4.4.** Let $G_1$ and $G_2$ be discrete groups, and let $X_1$ and $X_2$ be $G_1$- and $G_2$-CW complexes, respectively. Let $A$ be a $G_1 \times G_2$-additive category with involution. The *partial assembly map*,

$$\mu_{G_1,G_2}^\lambda: F_{G_1 \times G_2}^\lambda(X_1 \times X_2; A) \to F_{G_2}^\lambda(X_2; D^{G_1}(\bullet; A)),$$

is the map induced by the second factor projection $X_1 \times X_2 \to \bullet \times X_2$, composed with the homotopy equivalence from Lemma 2.4 with $S = \bullet$.

**Lemma 4.5.** The partial assembly map is natural in the control spaces and involution. □

Now the second step of the proof outline gives a criterion for the partial assembly map to be an equivalence.

**Proposition 4.6.** Let $G$ and $K$ be groups, and let $B$ be an additive $G \times K$-category. Let $F_K$ be a family of subgroups of $K$. Let $X_1$ be a $G$-CW complex and $X_2$ be a $K$-CW complex with isotropy in $F_K$. Suppose that

$$F_{G \times V}^\lambda(X_1 \times \bullet; B) \to F_{G \times V}^\lambda(\bullet; B)$$

is a homotopy equivalence for all subgroups $V \in F_K$. Then the partial assembly map

$$\mu_{G,K}^\lambda: F_{G \times K}^\lambda(X_1 \times X_2; B) \to F_{K}^\lambda(X_2; D^{G}(\bullet; B))$$

is also an equivalence for $\lambda = K^-\infty$ or $\lambda = \mathbb{L}^-\infty$.

**Proof.** Suppose that $X_2 = K/V$ for some $V \in F_K$. Then, by Shapiro’s Lemma (Proposition 2.6),

$$F_{G \times V}^\lambda(X_1 \times \bullet; B) \xrightarrow{\mu_{G,V}^\lambda} F_{G}^\lambda(\bullet; D^{G}(\bullet; B))$$

and the upper map is an equivalence by assumption, since $F_{G}^\lambda(\bullet; D^{G}(\bullet; B)) \simeq F_{G \times V}^\lambda(\bullet; B)$. The functors $H(X_2) := F_{G \times K}^\lambda(X_1 \times X_2; B)$ and $H'(X_2) := F_{K}^\lambda(X_2; D^{G}(\bullet; B))$ are homotopy-invariant and $K$-excisive functors from $K$-CW complexes to spectra. Since $H(K/V) \simeq H'(K/V)$ for all $V \in F_K$, we conclude that $H(X_2) \simeq H'(X_2)$ for all $K$-CW complexes with isotropy in $F_K$. □

The following is our main result about extensions:

**Theorem 4.7.** Let $N \to G \xrightarrow{\pi} K$ be a group extension, where $N \triangleleft G$ is a normal subgroup, and $K$ is the quotient group. Let $F_G$ be a family of subgroups of $G$ and $A$ an additive category with right $G$-action. Let $F_K$ be a family of subgroups of $K$ such that $\pi(H) \in F_K$ for every $H \in F_G$. Suppose that for every $V \in F_K$ the $(\pi^{-1}(V), F_{G|\pi^{-1}(V)}, A)$-assembly map in algebraic $K$-theory is an isomorphism, and that for every additive category $B$ with right $K$-action the $(K,F_K,B)$-assembly map in algebraic $K$-theory is injective (resp. surjective). Then the $(G,F_G,A)$-assembly map in algebraic $K$-theory is injective (resp. surjective).

The same statement holds for algebraic $L$-theory as well.
Example 4.8. Suppose that $N$ is finite normal subgroup of $G$. Then the Farrell-Jones conjecture with twisted coefficients holds for $G$ if it holds for $K = G/N$.

Example 4.9. Suppose that $1 \to N \to G \to K \to 1$ is a group extension, and $\mathcal{F}_G$ and $\mathcal{F}_K$ both denote the family of finite subgroups of their respective groups. Then the conclusions of Theorem 4.7 hold provided that the assembly map is injective (resp. surjective) for $K$ and for every subgroup of $G$ containing $N$ as a subgroup of finite index.

The Proof of Theorem 4.7. Let $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$. Let $V \in \mathcal{F}_K$ be given. By Remark 4.3, we have a homotopy commutative diagram:

\[
\begin{array}{ccc}
F^\lambda_{\pi^{-1}(V)}(E_{\mathcal{F}_G} G; A) & \xrightarrow{a} & F^\lambda_{\pi^{-1}(V)}(\bullet; A) \\
\approx & \downarrow & \approx \\
F^\lambda_{G \times K}(X; D^0_G(K; A)) & \xrightarrow{b} & F^\lambda_{G \times V}(\bullet; D^0_G(K; A))
\end{array}
\]

Let $B = D^0_G(K; A)$, and note that the upper map $a$ is an equivalence by assumption, since $\text{Res}_{\pi^{-1}(V)} E_{\mathcal{F}_G} G$ is a universal space for the family $\mathcal{F}_G|_{\pi^{-1}(V)}$. Hence, the lower map $b$ is also an equivalence. By Proposition 4.6, we have the homotopy commutative diagram:

\[
\begin{array}{ccc}
F^\lambda_{G \times K}(X; B) & \xrightarrow{d} & F^\lambda_{G \times K}(\bullet; B) \\
\mu^{G, K} & \downarrow \approx & \downarrow \approx \\
F^\lambda_{K}(E_{\mathcal{F}_K} K; D^G(\bullet; B)) & \xrightarrow{e} & F^\lambda_{K}(\bullet; D^G(\bullet; B))
\end{array}
\]

By assumption, the map $e$ is injective (resp. surjective), which implies that $d$ is injective (resp. surjective).

Using Remark 4.3 again, we have the homotopy commutative diagram:

\[
\begin{array}{ccc}
F^\lambda_G(E_{\mathcal{F}_G} G; A) & \xrightarrow{c} & F^\lambda_G(\bullet; A) \\
\approx & \downarrow \approx \\
F^\lambda_{G \times K}(X; D^0_G(K; A)) & \xrightarrow{d} & F^\lambda_{G \times K}(\bullet; D^0_G(K; A))
\end{array}
\]

Therefore, the assembly map $c$ is injective (resp. surjective).

Corollary 4.10. The Farrell-Jones conjecture with twisted coefficients is true for $G_1 \times G_2$ if and only if it is true for $G_1$, $G_2$, and every product $V_1 \times V_2$, where $V_1 \leq G_1$ and $V_2 \leq G_2$ are virtually cyclic subgroups.

Proof. By our main result applied to the projection $G_1 \times G_2 \to G_2$, we may assume that $G_2$ is virtually cyclic. Similarly, we may assume that $G_1$ is virtually cyclic. Thus, we are reduced to knowing the conjecture for products $V_1 \times V_2$ of virtually cyclic subgroups of $G_1$ and $G_2$ respectively.

Remark 4.11. A product $V_1 \times V_2$ of virtually cyclic subgroups can be further reduced to the basic cases $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$ after quotients by finite normal subgroups.
5. The proof of Lemma 2.2

We will check the details of Lemma 2.2 which asserts that there is an additive functor

\[ F: \mathcal{D}_1^{K \times G}(S \times T \times [0, 1); \mathcal{A}) \to \mathcal{D}_1^G(T \times [0, 1); \mathcal{D}_1^K(S; \mathcal{A})) \]

defined by

\[
\begin{aligned}
(F(A)_{(g,t)})_{(k,s)} &= A_{(k,g,s,t)} \\
(F(\phi)_{(g',t')})_{(k',s')} &= \phi_{(k,g,s,t)}
\end{aligned}
\]

Here \( \mathcal{A} \) is an additive category with commuting right \( K \) and \( G \)-actions, \( T \) a left \( G \)-set and \( S \) a transitive \( K \)-\( G \) biset. The group \( K \times G \) acts on \( S \times T \) by the formula \( (k, g) \cdot (s, t) := (ksg^{-1}, gt) \). Recall the notation \( (k, s) \) for elements of \( K \times S \), and \( (g, t) \) for elements of \( G \times T \times [0, 1] \). We will let \( \epsilon: T \times [0, 1] \to T \) denote the projection map.

Notice that \( \phi^{(k',g',s',t')}_{(k,g,s,t)} = 0 \) unless \( s = s' \) and \( \epsilon(t) = \epsilon(t') \), since the morphisms \( \phi: A \to B \) in the domain category are assumed to be level-preserving. The free \((K \times G)\)-space

\[ \overline{S} \times \overline{T} = K \times G \times S \times T \times [0, 1] \]

is a resolution for \( S \times T \times [0, 1] \). The proof that \( F \) is a functor is done in the following steps.

(1. F(\phi \circ \psi) = F(\phi) \circ F(\psi). Since

\[
(F(\phi) \circ F(\psi))_{(g',t')}^{(g',t')} = \sum_{(g'',t'')} F(\phi)_{(g'',t'')} \circ F(\psi)_{(g,t)}
\]

we have that:

\[
\begin{aligned}
\left( (F(\phi) \circ F(\psi))_{(g',t')}^{(g',t')} \right)_{(k',s')} &= \left( \sum_{(g'',t'')} F(\phi)_{(g'',t'')} \circ F(\psi)_{(g,t)} \right)_{(k',s')}
\end{aligned}
\]

\[
= \sum_{(g'',t'')} \left( F(\phi)_{(g'',t'')} \circ F(\psi)_{(g,t)} \right)_{(k',s')}
\]

\[
= \sum_{(g'',t'')} \sum_{(s'',t'')} \phi_{(k',g',s'',t'')} \circ \psi_{(k,g,s,t)}
\]

\[
= (\phi \circ \psi)_{(k',g',s'',t'')}_{(k,g,s,t)}
\]

\[
= \left( F(\phi \circ \psi)_{(g',t')}^{(g',t')} \right)_{(k',s')}_{(k,s)}
\]

(2. \( F(A)_{(g,t)} \) is an object of \( \mathcal{D}_1^K(S; \mathcal{A}) \), for every \((g,t) \in G \times T \times [0, 1]) \).
(2’). \(F(A)_{(g,t)}\) is \(K\)-invariant. For each \(h \in K\),
\[
(h^*(F(A)_{(g,t)}))_{(k,s)} = h^*((F(A)_{(g,t)}))_{(hk,hs)} = h^*(A_{(hk,g,hs,t)}) = (h^*A)_{(k,g,s,t)} = A_{(k,g,s,t)} = (F(A)_{(g,t)})_{(k,s)}
\]

(2’’). The support of \(F(A)_{(g,t)}\) is \(K\)-compact in \(K \times S\).

Since a discrete \(K\)-set is \(K\)-compact if and only if its image under the quotient map is finite, we need to show that \(K \setminus \text{supp}(F(A)_{(g,t)})\) is finite. Let \(p\) be the projection map from \(K \times G \times S \times T \times [0,1]\) to \(K \times G \times S \times T\), \(M = p(\text{supp}(A))\), and \(N = p(\text{supp}(A) \cap K \times \{g\} \times S \times \{t\}) \subset M\). Consider the following commutative diagram, in which \(f(k',g',s',t') = (k',s'g')\), \(m_g(k,s) = (k,sg^{-1})\), and the vertical arrows are quotient maps.

\[
\begin{array}{ccc}
K \times G \times S \times T & \xrightarrow{f} & K \times S \\
\downarrow q_{K \times G} & & \downarrow q_K \\
K \setminus (K \times G \times S \times T) & \xrightarrow{\bar{f}} & K \setminus (K \times S) \\
\end{array}
\]

Since \(M\) is discrete and \((K \times G)\)-compact, \(q_{K \times G}(M)\) is finite. Since \(N \subset M\), \(q_{K \times G}(N)\) is also finite. Therefore, \((\bar{m}_g \circ \bar{f} \circ q_{K \times G})(N) = (q_K \circ m_g \circ f)(N) = q_K(\text{supp}(F(A)_{(g,t)}))\) is finite.

(3). \(F(\phi)_{(g',t')}^{(g,t)}\) is a morphism of \(\mathcal{D}^K_I(S;A)\), for every \((g,t),(g',t') \in G \times T \times [0,1]\).

(3’). \(F(\phi)_{(g',t')}^{(g,t)}\) is \(K\)-invariant. The proof is similar to the proof of (2’).

(3’’). Fix \((k,s) \in K \times S\). Then, the following set is finite:
\[
P = \left\{ (k',s') \in K \times S \left| \left( F(\phi)_{(g,t')}^{(g',t')} \right)_{(k,s)}^{(k',s')} \neq 0 \text{ or } \left( F(\phi)_{(g,t)}^{(g',t')} \right)_{(k',s')}^{(k,s)} \neq 0 \right\}
\]

The sets \(\left\{ (k',g',s',t') \in K \times G \times S \times T \times [0,1] \left| \phi_{(k,g,s,t)}^{(k',g',s',t')} \neq 0 \right\} \) and \(\left\{ (k',g',s',t') \in K \times G \times S \times T \times [0,1] \left| \phi_{(k',g,s',t')}^{(k,g,s,t)} \neq 0 \right\} \) are finite and their union projects onto \(P\).

(3’’’). \(F(\phi)_{(g,t')}^{(g',t)}\) is level preserving in \(S\). This is because \(\phi\) is level-preserving in \(S \times T\).

(4). \(F(A)\) is an object of \(\mathcal{D}^G_I(T \times [0,1]; \mathcal{D}^K_I(S;A))\).
(4'). \( F(A) \) is \( G \)-invariant. For each \( \gamma \in G \),
\[
(\gamma^*(F(A))(g,t))_{(k,s)} = (\gamma^*(F(A)(\gamma g,\gamma t)))_{(k,s)}
= \gamma^*((F(A)(\gamma g,\gamma t))_{(k,s^\gamma^{-1})})
= \gamma^*(A_{(k,\gamma g,s^\gamma^{-1},\gamma t)})
= (\gamma^*A)_{(k,g,s,t)}
= (F(A)(g,t))_{(k,s)}
\]

(4''). The support of \( F(A) \) is relatively \( G \)-compact in \( G \times T \times [0,1) \).

Let \( p: K \times G \times S \times T \times [0,1) \to G \times T \times [0,1) \) be the projection map. Since \( \text{supp}(A) \) is relatively \( (K \times G) \)-compact and \( p(\text{supp}(A)) = \text{supp}(F(A)) \), \( \text{supp}(F(A)) \) is relatively \( G \)-compact in \( G \times T \times [0,1) \).

(4''). The support of \( F(A) \) is locally finite in \( G \times T \times [0,1) \).

Let \( (g,t) \in \text{supp}(F(A)) \) be given. We must find an open neighborhood \( U \subset G \times T \times [0,1) \) of \( (g,t) \) such that \( U \cap \text{supp}(F(A)) = \{(g,t)\} \). Let
\[
L = \{(k,s) \in K \times S \mid (k,g,s,t) \in \text{supp}(A)\}.
\]
From (1''), we know that \( L \) is \( K \)-compact. That is, \( L = K \cdot (K_0 \times S_0) \), where \( K_0 \subset K \) and \( S_0 \subset S \) are finite sets. Since \( \text{supp}(A) \) is locally finite in \( K \times G \times S \times T \times [0,1) \), for each \( (k_i,s_i) \in K_0 \times S_0 \), there is a neighborhood \( U_i \subset T \times [0,1) \) of \( t \), such that
\[
\{(k_i) \times \{g\} \times \{s_i\} \times U_i\} \cap \text{supp}(A) = \{(k_i,g,s_i,t)\}.
\]
Thus, for each \( (k,s) \in L \), there is an \( i \), such that
\[
\{(k) \times \{g\} \times \{s\} \times U_i\} \cap \text{supp}(A) = \{(k,g,s,t)\}.
\]
Therefore, if we let \( U = \{g\} \times (\cap_i U_i) \), then \( U \cap \text{supp}(F(A)) = \{(g,t)\} \).

(5). \( F(\phi) \) is a morphism in \( D^G_\ast(T \times [0,1);D^K_\ast(S;A)) \).

(5'). \( F(\phi) \) is \( G \)-invariant. The proof is similar to the proof of (3').

(5''). Fix \( (g,t) \in G \times T \times [0,1) \). Then, the following set is finite
\[
\left\{(g',t') \in G \times T \times [0,1) \mid F(\phi)(g',t') \neq F(\phi)(g,t) \right\}
\]
As we saw in (2''), \( \text{supp}(A) \cap K \times \{g\} \times S \times \{t\} \) is \( K \)-compact. Therefore, it is contained in \( K \cdot (K_0 \times \{g\} \times S_0 \times \{t\}) \), for some finite subsets \( K_0 \subset K \) and \( S_0 \subset S \). Notice that by \( K \)-equivariance, \( F(\phi)(g',t') \neq F(\phi)(g,t) \) if and only if there exists an \( s_0 \in S_0 \), \( k_0 \in K_0 \) and \( k' \in K \) such that \( \phi^{(k',g',s_0,t')}_{(k_0,g,s_0,t)} \neq 0 \). But for each \( k_0 \in K_0 \) and each \( s_0 \in S_0 \), there are only finitely many \( k' \in K \), \( g' \in G \) and \( t' \in T \times [0,1) \) such that \( \phi^{(k',g',s_0,t')}_{(k_0,g,s_0,t)} \neq 0 \). Therefore, there are only finitely many \( g' \in G \) and \( t' \in T \times [0,1) \) such that \( F(\phi)(g',t') \neq 0 \). A similar argument shows that there are only finitely many \( g' \in G \) and \( t' \in T \times [0,1) \) such that \( F(\phi)(g,t) \neq 0 \).
(5’’). \(F(\phi)\) is continuously controlled in \(T \times [0,1]\).

Let \(\phi : A \to B\) be a morphism in \(D_1^{K \times G}(S \times T \times [0,1]; A)\). Let \((x_0, 1) \in T \times [0,1]\) and a \(G_{x_0}\)-invariant neighborhood \(U \subset T \times [0,1]\) of \((x_0, 1)\) be given. We must find a \(G_{x_0}\)-invariant neighborhood \(V \subset T \times [0,1]\) of \((x_0, 1)\), such that \(F(\phi)_{(g,t')} = 0 = F(\phi)_{(g,t)}\) whenever \((g,t) \in G \times V\) and \((g',t') \notin G \times U\).

By definition, \(\left(F(\phi)_{(g,t)}\right)_{(k,s)} = \phi_{(k',g',s,t')}\). Let \(s_0 \in S\) with \(K \cdot s_0 \cdot G = S\), and let \(H \subset K \times G\) be the stabilizer subgroup of \(s_0\). We will identify \(G \times T \times [0,1]\) with the level \(\{s_0\} \times G \times T \times [0,1]\). Notice that the intersection of \(\text{supp}(A)\) with \(K \times G \times \{s_0\} \times T \times [0,1]\) is contained in,

\[
\bigcup_{(a,b) \in H} a \cdot K_0 \times b \cdot G_0 \times \{s_0\} \times b \cdot T_0 \times [0,1],
\]

where \(K_0 \subset K\), \(G_0 \subset G\) and \(T_0 \subset T\) are finite sets. This holds since \(\text{supp}(A)\) is relatively \((K \times G)\)-compact and any element of \((K \times G) - H\) will move \(s_0\) to another level in \(S\).

Suppose that \(\phi_{(k',g',s,t')} \neq 0\) for some \(k' \in K\), \(g' \in G\) and \(t \in U\). Then we can write \(\tau s g^{-1} = s_0\), for some \(\tau \in K\) and some \(\gamma \in G\). By equivariance, \(\phi_{(\tau k, \gamma g, s_0, \gamma t)} \neq 0\). For this to happen, \((\tau k, \gamma g, s_0, \gamma t) \in \text{supp}(A)\). This implies that there exists \((a, b) \in H\) such that

\[
(\tau k, \gamma g, s_0, \gamma t) \in a \cdot K_0 \times b \cdot G_0 \times \{s_0\} \times b \cdot T_0 \times [0,1],
\]

which is equivalent to saying that

\[
(a^{-1} \tau k, b^{-1} \gamma g, s_0, b^{-1} \gamma t) \in K_0 \times G_0 \times \{s_0\} \times T_0 \times [0,1]
\]

In particular, \(b^{-1} \gamma t \in b^{-1} \gamma U \cap (T_0 \times [0,1])\).

Since \(T_0\) is finite, there are only finitely many elements of \(G\), say \(\{g_1, g_2, \ldots, g_r\}\), such that \(g_i U \cap (T_0 \times [0,1]) \neq \emptyset\). Therefore, \(\gamma = bg_i\) for some \((a, b) \in H\) that fixes \(s_0\) and some \(i\) with \(1 \leq i \leq r\).

Since \(\phi\) is continuously controlled at \(g_i \cdot (x_0, 1)\) along \(S \times T \times 1\), there is a neighborhood \(V_i \subset T \times [0,1]\) of \((x_0, 1)\) such that \(\phi_{(k',g',s_0,g_i t')} = 0\) if \(t \in V_i\) and \(t' \notin U\), for \(1 \leq i \leq r\).

Let \(V = \cap_i V_i\). Then, if \(t \in V\) and \(t' \notin U\), we have

\[
\phi_{(a^{-1} \tau k, g_i, g_i, g_i t')} = 0
\]

and hence

\[
0 = \phi_{(\tau k, bg_i g_i, s_0, bg_i t')} = \phi_{(\tau k, g_i, s_0, \gamma t)} = \phi_{(k', g', s, t')}
\]

by equivariance of the morphisms, and the relations \(\gamma = bg_i\), \(s_0 = \tau s g^{-1}\). A similar argument shows that \(F(\phi)_{(g',t')} = 0\) if \(t \in V\) and \(t' \notin U\). Therefore \(F(\phi)\) is continuously controlled along \(T \times 1\).

\[
\square
\]

References

[1] A. Bartels, F. T. Farrell, L. E. Jones, and H. Reich, On the isomorphism conjecture in algebraic K-theory. Topology 43 (2004), 157–213.

[2] A. Bartels, W. Lück, and H. Reich, On the Farrell-Jones conjecture and its applications, J. Topol. 1 (2008), 57–86.
[3] A. Bartels and H. Reich, *On the Farrell-Jones conjecture for higher algebraic K-theory*, J. Amer. Math. Soc. 18 (2005), 501–545 (electronic).

[4] ———, *Coefficients for the Farrell-Jones conjecture*, Adv. Math. 209 (2007), 337–362.

[5] M. Cárdenas and E. K. Pedersen, *On the Karoubi filtration of a category*, K-theory 12 (1997), 165–191.

[6] G. Carlsson and E. K. Pedersen, *Controlled algebra and the Novikov conjectures for K- and L-theory*, Topology 34 (1995), 731–758.

[7] ———, *Čech homology and the Novikov conjectures*, Math. Scand. 82 (1998), 5–47.

[8] G. Carlsson, E. K. Pedersen, and W. Vogell, *Continuously controlled algebraic K-theory of spaces and the Novikov conjecture*, Math. Ann. 310 (1998), 169–182.

[9] J. F. Davis and W. Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory*, K-Theory 15 (1998), 201–252.

[10] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. 6 (1993), 249–297.

[11] I. Hambleton and E. K. Pedersen, *Identifying assembly maps in K- and L-theory*, Math. Ann. 328 (2004), 27–57.

[12] H. Oyono-Oyono, *Baum-Connes conjecture and extensions*, J. Reine Angew. Math. 532 (2001), 133–149.

[13] E. K. Pedersen, *Continuously controlled surgery theory*, Surveys on surgery theory, Vol. 1, Princeton Univ. Press, Princeton, NJ, 2000, pp. 307–321.

[14] E. K. Pedersen and C. Weibel, *K-theory homology of spaces*, Algebraic Topology, (Arcata, 1986), Lecture Notes in Mathematics, vol. 1370, Springer, Berlin, 1989, pp. 346–361.

[15] D. Rosenthal, *Splitting with continuous control in algebraic K-theory*, K-Theory 32 (2004), 139–166.

[16] ———, *Continuous control and the algebraic L-theory assembly map*, Forum Math. 18 (2006), 193–209.

[17] D. Rosenthal and D. Schütz, *On the algebraic K- and L-theory of word hyperbolic groups*, Math. Ann. 332 (2005), 523–532.