Abstract

Recent literature has made much progress in understanding online LQR: a modern learning-theoretic take on the classical control problem where a learner attempts to optimally control an unknown linear dynamical system with fully observed state, perturbed by i.i.d. Gaussian noise. The optimal regret over time horizon $T$ against the optimal control law scales as $\tilde{\Theta}(\sqrt{T})$. In this paper, we show that the same regret rate (against a suitable benchmark) is attainable even in the considerably more general non-stochastic control model, where the system is driven by arbitrary adversarial noise [3]. We attain the optimal $\tilde{O}(\sqrt{T})$ regret when the dynamics are unknown to the learner, and $\text{poly}(\log T)$ regret when known, provided that the cost functions are strongly convex (as in LQR). Our algorithm is based on a novel variant of online Newton step [19], which adapts to the geometry induced by adversarial disturbances, and our analysis hinges on generic regret bounds for certain structured losses in the OCO-with-memory framework [6].

1 Introduction

In control tasks, a learning agent seeks to minimize cumulative loss in a dynamic environment which responds to its actions. While dynamics make control problems immensely expressive, they also pose a significant challenge: the learner’s past decisions affect future losses incurred.

This paper focuses on the widely-studied setting of linear control, where the learner’s environment is described by a continuous state, and evolves according to a linear system of equations, perturbed by process noise, and guided by inputs chosen by the learner. Many of the first learning-theoretic results for linear control focused on online LQR [1, 13, 12, 25], an online variant of the classical Linear Quadratic Regulator (LQR) [21]. In online LQR, the agent aims to control an unknown linear dynamical system driven by independent, identically distributed Gaussian process noise. Performance is measured by regret against the optimal LQR control law on a time horizon $T$, for which the optimal regret rate is $\Theta(\sqrt{T})$ [12, 25, 26, 9]. Theoretical guarantees for LQR rely heavily on the strong stochastic modeling assumptions for the noise, and may be far-from-optimal if these assumptions break. A complementary line of work considers non-stochastic control, replacing stochastic process noise with adversarial disturbances to the dynamics [3, 28]. Here, performance is measured by regret: performance relative to the best (dynamic) linear control policy in hindsight, given full knowledge of the adversarial perturbations.

Though many works have proposed efficient algorithms which attain sublinear regret for non-stochastic control, they either lag behind optimal guarantees for the stochastic LQR problem, or require partial stochasticity assumptions to ensure their regret. And while there is a host of literature demonstrating that, in many online learning problems without dynamics, the worst-case rates of regret for the adversarial and stochastic settings are the same [8, 31, 19], whether this is true
in control is far from clear. Past decisions affect future losses in control settings, and this may be fundamentally more challenging when perturbations are adversarial and unpredictable. Despite this challenge, we propose an efficient algorithm that matches the optimal $\sqrt{T}$ regret bound attainable the stochastic LQR problem, but under arbitrary, non-stochastic disturbance sequences and arbitrary strongly convex costs. Thus, from the perspective of regret with respect to a benchmark of linear controllers, we show that the optimal rate for non-stochastic control matches the stochastic setting.

Our Setting Generalizing LQR, we consider partially-observed linear dynamics:

$$x_{t+1} = A_s x_t + B_s u_t + w_t, \quad y_t = C_s x_t + e_t$$  \hspace{1cm} (1.1)

Here, the state $x_t$ and process noise $w_t$ lie in $\mathbb{R}^{d_x}$, the observation $y_t$ and observation noise $e_t$ lie in $\mathbb{R}^{d_y}$, and the input $u_t \in \mathbb{R}^{d_u}$ is elected by the learner, and $A_s, B_s, C_s$ are matrices of appropriate dimensions. We call the $(w_t, e_t)$ the disturbances, and let $(w, e)$ denote the entire disturbance sequence. Unlike LQR, we assume that the disturbances are selected by an oblivious adversary, rather than from a mean zero stochastic process, and the learner observes the outputs $y_t$, but not the full state $x_t$. Appendix C describes how our setting strictly generalizes the online LQR problem, and relates to its partially observed analogue LQG. A policy $\pi$ is a (possibly randomized) sequence of mappings $u_t := \pi_t(y_{1:t}, u_{1:t-1})$. We denote by $y_t^\pi$ and $u_t^\pi$ sequence the realized sequence of outputs and inputs produced by policy $\pi$ and the noise sequence $(w, e)$. At each time $t$, a convex cost $\ell_t : \mathbb{R}^{d_x \times d_u} \rightarrow \mathbb{R}$ is revealed, and the learner observes the current $y_t$, and suffers loss $\ell_t(y_t, u_t)$. The cost functional of a policy $\pi$ is

$$J_T(\pi) := \sum_{t=1}^{T} \ell_t(y_t^\pi, u_t^\pi),$$

measuring the cumulative losses evaluated on the outputs and inputs induced by the realization of the disturbances $(w, e)$. The learner’s policy $\text{alg}$, is chosen to attain low control regret with respect to a pre-specified benchmark class $\Pi$ of reference policies,

$$\text{ControlReg}_{T}(\text{alg}; \Pi) := J_T(\text{alg}) - \inf_{\pi \in \Pi} J_T(\pi),$$  \hspace{1cm} (1.2)

which measures the performance of $\text{alg}$ (on the realized losses/disturbances) compared to the best policy $\pi \in \Pi$ in hindsight (chosen with knowledge of losses and disturbances). We consider a restricted a benchmark class $\Pi$ consisting of linear, dynamic controllers, formalized in Definition 3.1. While this class encompasses optimal control laws for many classical settings [28], in general it does not include the optimal control law for a given realization of noise. This is unavoidable: even in the simplest settings, it is impossible to attain sublinear regret with respect to the optimal control law [24]. We assume that the losses $\ell_t(\cdot)$ are $\alpha$-strongly convex, and grow at most quadratically:

**Assumption 1.** We suppose that all $\ell_t : \mathbb{R}^{d_x + d_u} \rightarrow \mathbb{R}$ are L-subquadratic: $0 \leq \ell(v) \leq L \max \{1, \|v\|_2^2\}$, and $\|\nabla \ell_t(v)\|_2 \leq L \max \{1, \|v\|\}$. We also assume that $\ell_t$ are twice-continuously differentiable, and $\alpha$-strong convex ($\nabla^2 \ell_t \succeq \alpha I$). For simplicity, we assume $L \geq \max \{1, \alpha\}$.

This assumption is motivated by classical LQR/LQG, where the loss is a strongly convex quadratic of the form $\ell(y, u) = y^\top R y + u^\top Q u$ for $R, Q > 0$. The central technical challenge of this work is that, unlike standard online learning settings, the strong convexity of the losses does not directly yield fast rates [4, 16].

1.1 Our Contributions

For the above setting, we propose Disturbance Reponse Control via Online Newton Step, or DRC-ONS - an adaptive control policy which attains fast rates previously only known for settings with stochastic or semi-stochastic noise [25, 28, 12, 4]. Our algorithm combines the DRC controller parametrization [28] with Semi-ONS, a novel second-order online learning algorithm tailored to our setting. We show that DRC-ONS achieves logarithmic regret when the learner knows the dynamics:

**Theorem 3.1** (informal) When the agent knows the dynamics (1.1) (but does not have foreknowledge of disturbances nor the costs $\ell_t$), DRC-ONS has ControlReg$_T = O(\frac{L^2}{\alpha} \cdot \text{poly}(\log T))$.  

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$^1$The oblivious assumption is only necessary if the dynamics are unknown to the learner; if the dynamics are known, our guarantees hold against adaptive adversaries as well.
This is the first bound to guarantee logarithmic regret with general strongly convex losses and non-stochastic noise. Past work required stochastic or semi-stochastic noise [4, 28], or was limited to fixed quadratic costs [16]. For unknown dynamics, we find:

**Theorem 3.2** (informal) *When the dynamics are unknown, DRC-ONS with an initial estimation phase attains ControlReg\(_T\) = \(O\left(\frac{L^2}{\alpha} \sqrt{T}\right)\).*

This bound matches the optimal \(\sqrt{T}\)-scaling for stochastic online LQR [26]. Thus, from the perspective of regret minimization with respect to the benchmark II, non-stochastic control is *almost* as easy as stochastic. This is not without many caveats, which are left to the discussion in Appendix B.1.

**Technical Contributions** While our main results are control theoretic, our major technical insights pertain to online convex optimization (OCO). Our control algorithm leverages a known reduction [3] to the online convex optimization with memory (OCOM) framework [6], which modifies OCO by allowing losses to depend on past iterates. Past OCOM analyses required bounds on both the standard OCO regret and total Euclidean variation of the iterates produced (Section 2.4). But for the the losses that arise in our setting, Theorem 2.3 shows that there is a significant tradeoff between the two, obviating sharp upper bounds. To overcome this, we show that online control enjoys additional structure we call OCO with affine memory, or OCOAM. We propose a novel second order method, Semi-ONS, based on online Newton step (ONS, [19]), tailored to this structure. Under a key technical condition satisfied by online control, we establish logarithmic regret.

**Theorem 2.1** (informal) *Under the aforementioned assumption (Definition 2.2), the Semi-ONS algorithm attains \(O\left(\frac{1}{\alpha} \log T\right)\) regret in the OCOAM setting.*

The above bound directly translates to logarithmic control regret for known systems, via the control-to-OCOAM reduction spelled out in Section 3. For control of unknown systems, the undergirding OCOAM bound is quadratic sensitivity to \(\epsilon\)-approximate losses:

**Theorem 2.2** (informal) *Consider the OCOAM setting with \(\epsilon\)-approximate losses (in the sense of Assumption 2). Then, Semi-ONS has regret \(O\left(\frac{1}{\alpha} \log T \cdot T\epsilon^2\right)\).*

Quadratic sensitivity to errors in the gradients was previously demonstrated for strongly convex stochastic optimization [15], and subsequently for strongly convex OCO [28]. Extending this guarantee to Semi-ONS is the most intricate technical undertaking of this paper.

### 1.2 Prior Work

In the interest of brevity, we restrict our attention to previous works regarding online control with a regret benchmark; for a survey of the decades old field of adaptive control, see e.g. [29]. Much work has focused on obtaining low regret in online LQR with unknown dynamics [1, 13, 25, 12], a setting we formally detail in Appendix C.1. Recent algorithms [25, 12] attain \(\sqrt{T}\) regret for this setting, with polynomial runtime and polynomial regret dependence on relevant problem parameters. This was recently demonstrated to be optimal [26, 9], with Cassel et al. [9] showing that logarithmic regret is possible the partial system knowledge. In the related LQG setting (partial-observation, stochastic process and observation noise, Appendix C.2), Mania et al. [25] present perturbation bounds which suggest \(T^{2/3}\) regret, improve to \(\sqrt{T}\) by Lale et al. [23], matching the optimal rate for LQG. For LQG with both non-denegerate process and observation noise, Lale et al. [22] attain \(\text{poly}(\log T)\) regret, demonstrating that in the presence of observation, LQG is in fact *easier* than LQR (with no observation noise) in terms of regret; see Appendix B.1 for further discussion.

Recent work first departed from online LQR by considering adversarially chosen costs under known stochastic or noiseless dynamics [2, 11]. Agarwal et al. [4] obtain logarithmic regret for fully observed systems, stochastic noise and adversarially chosen, strongly convex costs. The non-stochastic control setting we consider in this paper was established in Agarwal et al. [3], who obtain \(\sqrt{T}\)-regret for convex, Lipschitz (not strongly convex) cost functions and known dynamics. Hazan et al. [20] attains \(T^{2/3}\) regret for the same setting with unknown dynamics. Simchowitz et al. [28] generalizes both guarantees to partial observation, and generalize the optimal rate of logarithmic and \(\sqrt{T}\) for known and unknown systems, respectively to strongly convex losses and a “semi-stochastic” noise model. This assumption requires the noise to have a well-conditioned, stochastic component; in contrast, our methods allow *truly adversarial* noise sequences. Lastly, for the known system setting, Foster and Simchowitz [16] propose a different paradigm which yields logarithmic regret with
truly adversarial noise, but fixed quadratic cost functions and with full observation. In contrast, our algorithm accommodates both partial observation and arbitrary, changing costs, and its analysis and presentation are considerably simpler. Our work also pertains to the broader literature of online optimization with policy regret and loss functions with memory [7, 6], and our lower bound (Theorem 2.3) draws on the learning-with-switching-costs literature [5, 10, 14].

1.3 Organization and Notation

Section 2 formulates the general OCOAM setting, describes our Semi-ONS algorithm, and states its guarantees (Theorems 2.1 and 2.2), and the regret-movement tradeoff that hindered past approaches (Theorem 2.3). Section 3 turns to the control setting, describing the reduction to OCOAM, the DRC-ONS algorithm, and stating our main results (Theorems 3.1 and 3.2). Discussion of our results is deferred to Appendix B.1. All proofs are deferred to our appendix, whose organization of the appendix is detailed in Appendix A. Throughout, let \( a \lesssim b \) denote that \( a \leq Cb \), where \( C \) is a universal constant independent of problem parameters. We use \( \Omega(\cdot), O(\cdot) \) as informal asymptotic notation. We let \( a \vee b \) denote \( \max\{a, b\} \), and \( a \wedge b \) to denote \( \min\{a, b\} \). For vectors \( x \) and \( \Lambda \geq 0 \), we denote \( \|x\|_\Lambda := \sqrt{x^\top \Lambda x} \), and use \( \|x\| \) and \( \|x\|_2 \) interchangeably for Euclidean norm. We let \( \|A\|_{op} \) denote the operator norm, and given a sequence of matrices \( G = (G_i)_{i \geq 0} \), we define \( \|G\|_{\ell, \text{op}} := \sum_{i \geq 0} \|G_i\|_{op} \). We use \( (\cdot; \cdot) \) to denote vertical concatenation of vectors and matrices. Finally, non-bold arguments (e.g. \( z \)) denote function arguments, and bold (e.g. \( z_t \)) denote online iterates.

2 Fast Rates for OCO with Affine Memory

Building on past work [28, 3], our results for control proceed via a reduction to online convex optimization (OCO) with memory, proposed by Anava et al. [6], and denoted by OCO\( M \) in this work. Our lower bound in Section 2.5 explains why this past strategy is insufficient. Thus, we consider a structured special case, OCOAM, which arises in control, present a second-order algorithm for this setting, Semi-ONS, and state its main guarantees.

OCOM preliminaries Let \( C \subset \mathbb{R}^d \) be a convex constraint set. OCOM is an online learning game where, at each time \( t \), the learner plays an input \( z_t \in C \), nature reveals an \( h + 1 \)-argument loss \( F_t : C^{h+1} \to \mathbb{R} \), and the learner suffers loss \( F_t(z_t, z_{t-1}, \ldots, z_{t-h}) \), abbreviated as \( F_t(z_{t:t-h}) \). For each \( F_t \), we define its unary specialization \( f_t(z) := F_t(z, \ldots, z) \). The learner’s performance is measured by what we term memory-regret:

\[
\text{MemoryReg}_T := \sum_{t=1}^T F_t(z_{t:t-h}) - \inf_{z \in C} \sum_{t=1}^T f_t(z).
\] (2.1)

Because the learner’s loss is evaluated on a history of past actions, OCOM encodes learning problems with dynamics, such as our control setting. This is in contrast to the standard OCO setting, which measures regret evaluated on the unary \( f_t \): \( \text{OCOReg}_T := \sum_{t=1}^T f_t(z_t) - \inf_{z \in C} \sum_{t=1}^T f_t(z) \).

Our goal is to attain logarithmic memory-regret, and quadratic sensitivity to structured errors (in a sense formalized below).

2.1 OCO with Affine Memory

While we desire logarithmic memory regret, Theorem 2.3 shows that existing analyses cannot yield better rates than \( \Omega(T^{1/3}) \). Luckily, the control setting gives us more structure. Let us sketch this with a toy setting, and defer the full reduction to Section 3. Consider a nilpotent, fully observed system:

\[ y_t \equiv x_t, \quad A^p_t = 0. \]

Defining \( G[i] := [A^{i-1}_t B_t; I : i=0] \), the linear dynamics give \( x_t; u_t[i] := \sum_{i=0}^h G[i] u_{t-i} + [x_t; 0] \), where \( x_t[0] = \sum_{i=0}^h A_t w_{t-i} \). For simple policies parametrized by \( u^x_t := z \cdot w_t, z \in \mathbb{R} \), the loss incurred under iterates \( z_{t:t-h}, \ell_t([x_t;0]; : i=0) + \sum_{i=0}^h G[i] w_{t-i} z_{t-i} := F_t(z_{t:t-h}), \) exhibits affine dependence on the past. Generalizing the above, the OCO with affine memory (OCO\( AM \)) setting is as follows. Fix \( G = (G[i])_{i \geq 0} \in (\mathbb{R}^{p \times d_m})^B \) across rounds. At each \( t \geq 1 \), the learner selects \( z_t \in C \subset \mathbb{R}^d \), and the adversary reveals a convex cost \( \ell_t : \mathbb{R}^p \to \mathbb{R} \),

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2Throughout the initial iterates \( (z_t)_{t \leq 5} \) are arbitrary elements of \( C \). We note that Anava et al. [6] referred to MemoryReg_T as “policy regret”, but this differs slightly from the policy regret proposed by Arora et al. [7]. To avoid confusion, we use “memory regret”.

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an offset vector $\mathbf{v}_t \in \mathbb{R}^p$, and a matrix $\mathbf{Y}_t \in \mathbb{R}^{d_{in} \times d}$. The learner suffers loss with-memory loss $F_t(\mathbf{z}_{t-1}, h)$, given by $F_t(\mathbf{z}_{t-1}) := \ell_t(\mathbf{v}_t + \sum_{i=0}^{h'} G[i] \mathbf{Y}_{t-i} \mathbf{z}_{t-i})$. The induced unary losses are

$$f_t(z) := \ell_t(\mathbf{v}_t + \mathbf{H}_t z), \quad \text{where } \mathbf{H}_t := \sum_{i=0}^{h} G[i] \mathbf{Y}_{t-i}. \quad (2.2)$$

We consider two settings for OCOAM. In the exact setting, $G$ is known to the learner, and $\ell_t, \mathbf{v}_t, \mathbf{Y}_t$ are revealed at each $t$. Thus $f_t$ and $\mathbf{H}_t$ can be computed after each round. The approximate setting, the learner knows only an approximation $\hat{G}$ of $G$, and receives an estimate $\hat{\mathbf{v}}_t$ of $\mathbf{v}_t$ ($\mathbf{Y}_t$ and $\ell_t$ remain exact). Our algorithm uses approximate unary losses:

$$\hat{f}_t(z) := \ell_t(\hat{\mathbf{v}}_t + \hat{\mathbf{H}}_t z), \quad \text{where } \hat{\mathbf{H}}_t := \sum_{i=0}^{h} \hat{G}[i] \mathbf{Y}_{t-i}. \quad (2.3)$$

We desire low sensitivity to the approximation errors of $\hat{G}$ and $\hat{\mathbf{v}}$, translating to low estimation error sensitivity for control of an unknown system. For both exact and approximate losses, memory regret is evaluated on the exact losses $F_t, f_t$, consistent with OCOAM.

### 2.2 The Semi-ONS Algorithm

The standard algorithmic template for OCOAM is to run an online optimization procedure on the unary losses $f_t$, otherwise disregarding $F_t$ (but accounting for the discrepancy between the two in the analysis) [6]. We take this approach here, but with a tailored second order method. Let $\mathbf{z}_{t-1}, \ldots, \mathbf{z}_0 \in \mathcal{C}$ be arbitrary initial parameters. For step size and regularization parameters $\eta > 0$ and $\lambda > 0$, and setting $\mathbf{V}_t := \nabla f_t(\mathbf{z}_t)$, the Semi-ONS (Algorithm 1) iterates:

$$\zeta_{t+1} := \zeta_t - \eta \Lambda_t^{-1} \nabla \zeta_t, \quad \zeta_{t+1} := \arg \min_{\zeta \in \mathcal{C}} \| \Lambda_t^{1/2} (\zeta_{t+1} - z) \|, \quad \Lambda_t := \lambda I + \sum_{s=1}^{t} \mathbf{H}_s^\top \mathbf{H}_s, \quad (2.4)$$

The updates are nearly identical to online Newton step (ONS) [19], but whereas the ONS uses pre-conditioner $\Lambda_{t, ONS} := \lambda I + \sum_{s=1}^{t} \nabla f_s(\mathbf{z}_s) (\nabla f_s(\mathbf{z}_s))^\top$, Semi-ONS uses outer products of $\mathbf{H}_s$. This decision is explained in the paragraph concluding Section 2.4. In the approximate setting Semi-ONS proceeds using the following approximations, with $\hat{\mathbf{V}}_t := \nabla \hat{f}_t(\mathbf{z}_t)$

$$\zeta_{t+1} := \zeta_t - \eta \Lambda_t^{-1} \nabla \zeta_t, \quad \zeta_{t+1} := \arg \min_{\zeta \in \mathcal{C}} \| \Lambda_t^{1/2} (\zeta_{t+1} - z) \|, \quad \hat{\Lambda}_t := \lambda I + \sum_{s=1}^{t} \hat{\mathbf{H}}_s^\top \hat{\mathbf{H}}_s, \quad (2.5)$$

defined using the quantities in Eq. (2.3). In other words, approximate Semi-ONS is equivalent to exact Semi-ONS, treating $(\hat{f}_t, \hat{H}_t)$ like the true $(f_t, H_t)$.

### Algorithm 1: Online Semi-Newton Step - Semi-ONS($\lambda, \eta, \mathcal{C}$)

**parameters:** Learning rate $\eta > 0$, regularization parameter $\lambda > 0$, convex domain $\mathcal{C} \subset \mathbb{R}^d$.
**initialize:** $\Lambda_0 := \lambda I, \mathbf{z}_1 := 0_d$
**for** $t = 1, 2, \ldots$ **do**

- **receive triple** $(\ell_t, \mathbf{v}_t, \mathbf{H}_t)$ . % For approximate setting, replace $(\mathbf{v}_t, \mathbf{H}_t) \leftarrow (\hat{\mathbf{v}}_t, \hat{\mathbf{H}}_t)$
- $\nabla \zeta_t := \nabla f_t(\zeta_t)$, where $f_t(\zeta) = \ell_t(\mathbf{v}_t + \mathbf{H}_t \zeta)$
- $\Lambda_t := \Lambda_{t-1} + \mathbf{H}_t^\top \mathbf{H}_t$
- $\zeta_{t+1} := \zeta_t - \eta \Lambda_t^{-1} \nabla \zeta_t$
- $\zeta_{t+1} := \arg \min_{z \in \mathcal{C}} \| \Lambda_t^{1/2} (z - \zeta_{t+1}) \|_2$

### 2.3 Guarantees for Semi-ONS

To state our guarantees, we assume the $\alpha$-strong convexity and $L$-subquadratic assumption of Assumption 1. We assume various upper bounds on relevant quantities:

**Definition 1** (Bounds on Relevant Parameters). We assume $\mathcal{C}$ contains the origin. Further, we define the diameter $D := \max \{ \| \mathbf{z} - \mathbf{z}' \| : \mathbf{z}, \mathbf{z}' \in \mathcal{C} \}$, Y-radius $R_Y := \max_t \| \mathbf{Y}_t \|_{op}$, and $R_{\mathcal{C}} := \max_{\mathbf{z} \in \mathcal{C}} \max_{\mathbf{z}' \in \mathcal{C}} \| \mathbf{Y}_t \mathbf{z} \|$; In the exact setting, we define the radii $R_v := \max_t \max_{\mathbf{v}} \| \mathbf{v} \|_2$ and $R_G := \max_t \| G \|_{\ell_1,op}$; In the approximate setting, $R_v := \max_t \max_{\mathbf{v}} \| \mathbf{v} \|_2, \| \mathbf{\hat{v}} \|_2$, $R_G := \max_t \{ \| G \|_{\ell_1,op}, \| \mathbf{\hat{G}} \|_{\ell_1,op} \}$; For settings, we define the $H$-radius $R_H = R_GR_Y$, and define the effective Lipschitz constant $L_{a,t} := L \max \{ 1, R_v + R_H \}$. 

Lastly, our analysis requires that the smallest singular value of $G$, viewed as linear operator acting by convolution with sequences $(u_1, u_2, \ldots) \in (\mathbb{R}^d_{\text{un}})^N$, is bounded below:

**Definition 2.2.** We define the convolution invertibility-modulus as $\kappa(G) := 1 \land \inf_{(u_0, u_1, \ldots)} \{ \sum_{i=0}^n \|G[i]u_{n-i}\|_2^2 : \sum_i \|u_i\|_2^2 = 1 \}$, and the decay function $\psi_G(n) := \sum_{i \geq n} \|G[i]\|_{\text{op}}$.

A Fourier-analytic argument (Lemma 3.1) demonstrates that $\kappa(G) > 0$ when expressing reducing our control setting to OCOAM (Section 3), and stability of our control parametrization ensures $\psi_G(n)$ decays exponentially; the reader should have in mind the scalings $\kappa(G) = \Omega(1)$ and $\psi_G(n) = \exp(-\Omega(n))$. For the exact setting, we have the following guarantee:

**Theorem 2.1** (Semi-ONS regret, exact case). Suppose $\kappa = \kappa(G) > 0$, Assumption 1 holds, and consider the update rule Eq. (2.4) with parameters $\eta = \frac{3}{\alpha}$, $\lambda := 6hR^2_GR^2_T$. Suppose in addition that $h$ is large enough to satisfy $\psi_G(h + 1)^2 \leq R^2_G/T$. Then, we have $\text{MemoryReg}_T \leq 3chD^2R^2_H + \frac{3dh^2L^2_GR_T}{\alpha \kappa \gamma^2} \log(1 + T)$.

The above regret mirrors fast rates for strongly convex rates OCOM and exp-concave standard OCO. Its proof departs significantly from those of existing OCOM bounds, and is sketched in Section 2.4, and formalized in Appendix F. For the approximate setting, we assume

**Assumption 2** (Approximate Semi-ONS assumptions). We assume that $\|\hat{G} - G_\ast\|_{\ell_1, \text{op}} \leq \epsilon_G$, $\max_{i \geq 1} \|v_t - \hat{v}_t\|_2 \leq c_v \epsilon_G$ for some $c_v > 0$, and that $\hat{G}[i] = 0$ for all $i > h$.

For simplicity, the following theorem considers $\epsilon_G \geq 1/\sqrt{T}$, which arises in our estimation-exploitation tradeoff for control of unknown linear systems. It shows that Semi-ONS exhibits a quadratic sensitivity to the estimation error $\epsilon_G$, with MemoryReg$_T$ scaling as $\frac{1}{\alpha} \log T \cdot T\epsilon^2_G$.

**Theorem 2.2** (Semi-ONS regret, approximate case). Suppose Assumptions 1 and 2 hold, and in addition $\nabla^2 \ell_i \preceq L I$ uniformly, and $\epsilon_G \geq 1/\sqrt{T}$. Consider the update rule Eq. (2.5) with parameters $\eta = \frac{3}{\alpha}$ and $\lambda = (T\epsilon^2_G + hR^2_G)$. Then MemoryReg$_T \leq \log T \left( \frac{c_1^2}{\alpha \epsilon_G^2} + C_2 \right)$.

The above mirrors the strongly convex setting, where online gradient descent with $\epsilon$-approximate gradients attains $\frac{1}{\alpha} T\epsilon^2$ regret [28]. In Appendix G we provide two stronger versions: The first (Theorem 2.2a) includes a certain negative regret term which is indispensible for the control setting, and accommodates misspecified $\lambda$. The second (Theorem G.1) allows for $\epsilon_G \ll 1/\sqrt{T}$, establishing $(T\epsilon_G)^{2/3}$ regret for small $\epsilon_G$. Appendix G also details the proof of Theorem 2.2, which constitutes the main technical undertaking of the paper. The proof draws heavily on ideas from the proof of Theorem 2.1, which we presently sketch.

### 2.4 Proof Sketch for Exact Semi-ONS (Theorem 2.1)

Recall the with-memory and unary regret defined at the start of Section 2, and set $\nabla_t := \nabla f_t(z_t)$. Following [6], our analysis begins with the following identity:

$$\text{MemoryReg}_T = \text{OCOReg}_T + \text{MoveDiff}_T,$$

where $\text{MoveDiff}_T := \sum_{t=1}^T F(z_{t:t-1}) - f(z_t)$.

That is, $\text{MemoryReg}_T$ equals the standard regret on the $f_t$ sequence, plus the cumulative difference between $F_t$ (with memory) and $f_t$ (unary). The bound on $\text{OCOReg}_T$ for Semi-ONS mirrors the analysis of standard ONS, using that $\nabla^2 f_t(z_t) \succeq \mathbf{H}_t^T \mathbf{H}_t \succeq \nabla \nabla^T$ (Lemma F.2). To bound $\text{MoveDiff}_T$, past work on OCOM applies the triangle inequality and an $L$-Lipschitz condition on $F$ to bound the movement difference by movement in the Euclidean norm:

$$\text{MoveDiff}_T \leq \text{poly}(L, h) \cdot \text{EucCost}_T,$$

where $\text{EucCost}_T := \sum_{t=1}^T \|z_t - z_{t-1}\|$.

The standard approach is to run OGD on the unary losses [6] When doing so, the differences $\|z_t - z_{t-1}\|$ scale with Lipschitz constant $L$ and step sizes $\eta_t$. In particular, for the standard $\eta_t \propto \frac{1}{\alpha}$ step size, the strongly convex case, we obtain $O(\frac{\text{poly}(L, h)}{\alpha} \log T)$ memory regret. However, when $\ell_t$ are strongly convex,
the induced OCOAM losses $f_t$ need not be [16], and Theorem 2.3 shows that it is impossible to attain both logarithmic regret and logarithmic movement cost simultaneously. As a work around, we establish a refined movement bound in terms of $Y_t$-sequence (see Lemma F.6):

$$\text{MoveDiff}_T \leq \text{poly}(L, h) \cdot \text{AdapCost}_T, \quad \text{AdapCost}_T := \sum_{i=1}^{h} \sum_{t=1}^{T} ||Y_t(zt_{t-i} - zt_{t-i-1})||_2,$$

Via Lemma F.7, the Semi-ONS updates and an application of Cauchy-Schwartz yields:

$$\text{AdapCost}_T \leq O \left( \text{poly}(L, h) \right) \cdot \left( \sum_{t=1}^{T} \nabla_i^\top \Lambda_t^{-1} \nabla_t \right)^{1/2} \cdot \left( \sum_{t=1}^{T} Y_t^\top \Lambda_t^{-1} Y_t \right)^{1/2}. \quad (2.7)$$

Readers familiar with the analysis of ONS will recognize the \(\nabla\)-movement as the dominant term in its regret bound, and can be bounded in a similar fashion. To address the \(Y\)-movement, we use the convolution-invertibility assumption (Definition 2.2). This assumption implies that convolution with \(G = (G[i])_{i \geq 0}\) is invertible, meaning that we can essentially invert the sequence \((H_1, H_2, \ldots)\) defined by \(H_t := \sum_{i=0}^{h} G[i] Y_{t-i}\), so as to back out \((Y_1, Y_2, \ldots)\). Linear algebraically, this implies (see Proposition F.8) \(\Lambda_t - \lambda I = \sum_{t=1}^{T} H_s^\top H_s \geq \kappa(G) \lambda I \sum_{s=1}^{T} Y_s^\top Y_s - O(1)\). In other words, up to an additive remainder term and multiplicative factor of \(\kappa(G)\), the \(\Lambda_s\)-covariance dominates that \(Y_s\)-covariance. Hence, \(\Lambda_t\) roughly dominates \(\sum_{t=1}^{T} Y_s^\top Y_s + \lambda I\). Hence, \(Y\)-movement is also \(O(d \log T)\) by an application of the log-determinant lemma (Lemma F.5). This yields a logarithmic upper bound on MoveDiff, and thus logarithmic memory regret.

**Semi-ONS v.s. ONS** Standard ONS uses a preconditioner based on outer products of \(\nabla_i\). However, the movement difference depends on gradients of the with-memory loss \(F_t(\cdot, \cdot, \ldots)\), which may not be aligned with direction of \(\nabla_i\). Indeed, \(\nabla_i \in \text{RowSpace}(Y_t)\), but this is in general a strict inclusion; that is, \(Y_t\) accounts for more possible directions of movement that \(\nabla_i\). Thus, Semi-ONS forms its preconditioner to ensure slower movement in all \(Y_t\)-directions, using \(H_t\) as a proxy via the convolution-invertibility analysis.

### 2.5 The Regret-Movement Tradeoff

As described above, the standard analysis of OCOAM bounds the sum of the unary regret and Euclidean total variation of the iterates. While this permits logarithmic regret when \(f_t\) are strongly convex, OCOAM losses \(f_t\) are not strongly convex even if \(\ell_t\) are (see e.g. below). We now show that for a simple class of quadratic OCOAM losses, there is a nontrivial trade-off between the two terms. We lower bound \(\mu\)-Reg$_T := \text{OCOReg}_T + \mu\text{EucCost}_T = \sum_{t=1}^{T} f_t(z_t) + \mu||z_t - z_{t-1}|| - \inf_{z \in C} \sum_{t=1}^{T} f_t(z_t)\), which characterizes the Pareto curve between unary regret and Euclidean movement. We consider \(d = 1, \ C = [-1, 1], \ \ell(u) = u^2\), and the memory-1 OCOAM losses \(f_t = \ell(v_t - \varepsilon)\), where \(\varepsilon \in (0, 1]\) is fixed and \(v_t \in \{-1, 1\}\) are chosen by an adversary. On \(C\), \(f_t\) are \(O(\varepsilon)\)-Lipschitz, and have Hessian \(\varepsilon^2\) (thus arbitrarily small strong convexity). Still, \(\ell\) satisfies Assumption 1 with \(\alpha = L = 1\). We prove the following in Appendix J.1:

**Theorem 2.3.** Let \(c_1, \ldots, c_4\) be constants. For \(T \geq 1\) and \(\mu \leq c_1 T\), there exists \(c_2 \geq c_2 T^{1/3}\) such that any proper (i.e. \(z_t \in C\) for all \(t\)) possibly randomized algorithm alg suffers \(\mu\text{-Reg}_T \geq c_2 (T \mu^2)^{1/3}\). In particular, \(\mathbb{E}[\text{1-Reg}_T] \geq c_2 T^{1/3}\) and if \(\mathbb{E}[\text{OCOReg}_T] \leq R \leq c_3 T\), then \(\mathbb{E}[\text{EucCost}_T] \geq c_4 \sqrt{T/R}\).

Hence, existing analyses based on Euclidean movement cannot ensure better than \(T^{1/3}\) regret. Moreover, to ensure \(\text{OCOReg}_T = O(\log T)\), then one must suffer \(\sqrt{T/\log T}\) movement. In Theorem J.1 in Appendix J.2, we show that standard ONS with an appropriately tuned regularization parameter attains this optimal tradeoff (up to logarithmic and dimension factors), even in the more general case of arbitrary exp-concave losses.

### 3 From OCOAM to Online Control

This section proposes and analyzes the DRC-ONS algorithm via OCOAM. Recall the control setting with dynamics described by Eq. (1.1), and regret defined by Eq. (1.2). Throughout, we assume
that the losses satisfy the strong convexity and quadratic growth assumption of Assumption 1. Outputs $y$ lie in $\mathbb{R}^{d_y}$, inputs $u$ lie in $\mathbb{R}^{d_u}$. For the main text of this paper, we assume knowledge of a stabilizing, static feedback policy: that is a matrix $K \in \mathbb{R}^{d_y \times d_u}$ such that the policy $u_t = Ky_t$ is stabilizing ($\rho(A_* + B_* K C_*) < 1$, where $\rho$ denotes the spectral radius). For this stabilizing $K$, we select inputs $u_t^{alg} := Ky_t^{alg} + u_t^{ex, alg}$, where $u_t^{ex, alg}$ is the exogenous output dictated by an online learning procedure. We let the nominal iterates $y_t^K, u_t^K$ denote the sequence of outputs and inputs that would occur by selecting $u_t^{alg} = Ky_t^{alg}$, with no exogenous inputs. We exploit the superposition identity (using $[\cdot; \cdot]$ to denote vertical concatenation)

$$\begin{bmatrix} y_t^{alg} ; u_t^{alg} \end{bmatrix} = [y_t^K ; u_t^K] + \sum_{i=1}^{t-1} G_i^K u_{i-1}^{alg},$$  

(3.1)

where $G_i^K = [0; I_{d_u}]$ and $G_i^{alg} = [C_1; KC_*] (A_* + B_* K C_*)^{i-1} B_*$ for $i \geq 1$. We call $G^K$ the nominal Markov operator. Since $K$ is stabilizing, we will assume that $G_i^K$ decays geometrically, and that the nominal iterates are bounded. For simplicity, we take $x_1 = 0$.

Assumption 3. For some $c_K > 0$ and $\rho_K \in (0, 1)$ and all $n \geq 0$, $\|G_i^K\|_{\text{op}} \leq c_K \rho_K^n$.

Assumption 4. We assume that $(w_t, e_t)$ are bounded such that, for all $t \geq 1$, $\|\langle y_t^K, u_t^K \rangle\|_2 \leq R_{\text{nat}}$

Assumption 3 is analogous to “strong stability” [11], and holds for any stabilizing $K$. Assumption 4 is analogous to the bounded assumption in Simchowitz et al. [28]: since $K$ is stabilizing, any bounded sequence of disturbances implies a uniform upper bound on $\|\langle y_t^K, u_t^K \rangle\|_2$.

Benchmark Class We compete with linear dynamical controllers (LDCs) $\pi \in \Pi_{\text{ide}}$ whose closed loop iterates are denoted $(y_t^?, u_t^?, \hat{x}_t^?)$ (see Definition E.2 for further details). These policies include static feedback laws $u_t^? = Ky_t^?$, but are considerably more general due to the internal state. We consider stabilizing $\pi$: for all bounded disturbance sequences $\max_{t \geq 1} \|w_t\|, \|e_t\| < \infty$, it holds that $\max_{t \geq 1} \|y_t^\pi\|, \|u_t^\pi\| < \infty$. These policies enjoy geometric decay, motivating the following parametrization of our benchmark class.

Definition 3.1 (Policy Benchmark) Fix parameters $\rho_* \in (0, 1)$ and $c_* > 0$. Our regret benchmark competes LDC’s $\pi \in \Pi_* := \Pi_{\text{stab}}(c_*, \rho_*)$, where we define $\Pi_{\text{stab}}(c, \rho) := \{\pi \in \Pi_{\text{ide}} : (\|G_i^K\|_{\text{op}} \leq c \rho^n, \forall n \geq 0\}$. where the Markov operator $G_{\pi, c1}$ is in Definition E.3.

Known vs. Unknown Dynamics We refer to the known dynamics setting as the setting where the learner knows the matrices $A_*, B_*, C_*$ defining the dynamics in Eq. (1.1). In the unknown dynamics setting, the learner does not know these matrices (but knows $K$).

The DRC parametrization Given radius $R_M > 0$ and memory $m \in \mathbb{N}$, we adopt the DRC parametrization of memory-$m$ controllers $M \in \mathcal{M}$ [28]:

$$\mathcal{M} = M_{\text{drc}}(m, R_M) := \{M = (M_i^{[i]} )_{i=0}^{m-1} \in (\mathbb{R}^{d_y d_u})^m : \sum_{i=0}^{m-1} \|M_i\|_{\text{op}} \leq R_M\}.$$  

(3.2)

Controllers $M \in \mathcal{M}$ are then applied to estimates of the nominal outputs $y_t^K$. When the dynamics are known, $y_t^K$ and $u_t^K$ are recovered exactly via Eq. (3.1). If $A_*, B_*, C_*$ are not known, we use an estimate $\hat{G}$ of $G_K$ to construct estimates $\hat{y}_{t}^{K}$, $\hat{u}_{t}^{K}$:

$$\begin{bmatrix} \hat{y}_t^K ; \hat{u}_t^K \end{bmatrix} = \begin{bmatrix} y_t^{alg} ; Ky_t^{alg} \end{bmatrix} - \sum_{i=1}^{t-1} \hat{G}_i^{alg} u_{i-1}^{alg}.$$

(3.3)

Going forward, we use the more general $\hat{y}_t^K$ notation, noting that it specializes to $y_{t}^{K}$ for known systems (i.e. when $\hat{G} = G_K$). The DRC parametrization selects exogenous inputs as linear combinations of $\hat{y}_{t}^{K}$ under $M \in \mathcal{M}$: via $u_t^{ex}(M \mid \hat{y}_{t}^{K}) := \sum_{i=0}^{m-1} M_i^{[i]} \hat{y}_{t-i}^{K}$.

3.1 Reducing DRC to OCMO Fixing the DRC length $m \geq 1$, let $d = d_y d_u d_m$, and $p = d_y + d_u$. Further, let $(\hat{y}_{t}^{K}, \hat{u}_{t}^{K})_{t \geq 1}$ and $\hat{G}_t$ denote estimates of $(y_t^K, u_t^K)_{t \geq 1}$ and $G_K$, respectively

---

1 This may be restrictive for partially observed systems [17], see Appendix D for generalizations.

2 The assumed bound can be stated in terms of $\max_{t} \|w_t, e_t\|_2$. One may allow $R_{\text{nat}}$ to grow logarithmically (e.g. $R_{\text{nat}} = \mathcal{O}(\log^{1/2} T)$ for subgaussian noise), by inflating logarithmic factors in the final bounds.
We now define the relevant OCOAM quantities for control. Let $e[\cdot]$ denote the natural embedding of $M \in \mathcal{M}$ into $\mathbb{R}^d$, and let $e^{-1}[\cdot]$ denote its inverse; Define the OCOAM matrices $Y_t := e_y[\hat{y}^K_{t,m+1}]$, where $e_y$ is embedding satisfying $Y_t z = e^{x}(M \mid \hat{y}^K_t) \forall z \in e[M]$; Define the offset $v^K_t = (\hat{y}^K_t, \hat{u}^K_t) \in \mathbb{R}^p$, and its approximation $\hat{v}^K_t = (\hat{y}^K_t, \hat{u}^K_t) \in \mathbb{R}^p$; Define the constraint set $C := e(M) \subset \mathbb{R}^d$ (that is, embed the DRC set into $\mathbb{R}^d$).

We now define the relevant OCOAM losses as those consistent with the above notation.

**Definition 3.3 (OCOAM losses for control).** Let $Y_t, v^K_t, \hat{v}^K_t$ be as above. For $h \in \mathbb{N}$, define the exact losses $F_t(\hat{v}^K_{t-h}) = \ell_t(v^K + t_{i-t-h}, i) + f_t(z) = \ell_t(v^K + H_t z)$, where $H_t := \sum_{i=0}^{h} G^i [Y_{i-t-1}]$. Given an estimate $\hat{G}$ of $G$, the approximate unary loss is $\hat{F}_t(z) = \ell_t(\hat{G} + H_t z)$ with $H_t := \sum_{i=0}^{h} \hat{G} [Y_{i-t-1}]$.

We take $h = \Theta(\log T)$, since the exponential decay assumption (Assumption 3) ensures $G^i_t = \exp(-t h) \approx 0$ for $i > h$. The resulting OCOAM problem is to produce a sequence of iterates $z_t$ minimizing MemoryReg$_T$ on the sequence $(F_t, f_t)$. Since $z_t$ are embeddings of controllers, this gives rise to a natural control algorithm: for each iterate $z_t$, back out a DRC controller $M_t = e^{-1}(z_t)$, and applies exogenous input $u^x_{t+1} := u^x(M_t \mid y^K_t)$. In Appendix E, we streamline past work [28] by providing black-box reductions bounding the control regret (Eq. (1.2)) of such an algorithm by its memory regret. Proposition E.5 addresses the known system case, and Proposition E.8 the unknown case. Because the latter is more intricate, we conclude the present discussion with an informal statement of the known system reduction:

**Proposition E.5 (informal).** Let algorithm alg which produces iterates $z_t \in \mathbb{R}^d$. Let alg' denote the control algorithm which selects $u^x_{t+1} := u^x(M_t \mid y^K_t)$, where $M_t = e^{-1}(z_t)$. Then, for $m = O(1)$, we have ControlReg$_T$(alg) $\leq$ MemoryReg$_T$(alg) + $O(1)$.

**Remark 3.1 (Hat-accent notation).** We use $Y_t$ even when defined using the approximate $\hat{y}^K_t$. However, $G$ and $v^K$ do receive hat-acents when estimates are used. This is because, while OCOAM can account for the approximation error on $G$ and $v^K$ (Theorem 2.2), the approximation error introduced by setting $Y_t := e_y[\hat{y}^K_{t,m+1}]$ requires control specific arguments.

### 3.2 The DRC-ONS algorithm and guarantees

Stating the DRC-ONS algorithm is now a matter of putting the pieces together. For known systems, the learner constructs the losses in Definition 3.3 with $\hat{G} = G_K$, and runs Semi-ONS on $f_t$, and uses these to prescribe a DRC controller in accordance with the above discussion. For unknown systems, one constructs the estimate $\hat{G}$ via least squares, and then runs Semi-ONS on $\hat{f}_t$; formal pseudocode is given in Algorithms 2 and 3 in Appendix D.1. Our formal guarantees are

**Theorem 3.1 (Guarantee for Known System).** Suppose Assumptions 1, 3 and 4 hold, and for given $\rho_K L_1 \leq 1$ for simplicity, also assume $c_5 \geq c_5$, $\rho_K \geq \rho_K$. Then, for a suitable choice of parameters, DRC-ONS (Algorithm 2) achieves the bound ControlReg$_T$(alg; $\Pi_t) \leq 2\min(1, \|K\|_{op})^2 \cdot d_u d_y R^2_{nat} \cdot \frac{L^2}{\alpha}$.

**Theorem 3.2 (Guarantee for Unknown System).** Suppose Assumptions 1, 3 and 4 hold, and for given $\rho_K L_1 \leq 1$ for simplicity, also assume $c_5 \geq c_5$, $\rho_K \geq \rho_K$. In addition, assume $\nabla^2 2 \leq 1$ uniformly. Then, for any $\delta \in (0, 1/T)$. DRC-ONS with an initial estimation phase (Algorithm 3) for an appropriate choice of parameters has the following regret with probability $1 - \delta$: ControlReg$_T$(alg; $\Pi_t) \leq \sqrt{T} \log^3 (1 + T) \log^3 \left(\frac{1}{\delta} \cdot \frac{d_u d_y R^2_{nat} \cdot L^2}{\alpha} \cdot \min(1, \|K\|_{op})^2 \cdot d_u d_y \cdot \frac{R_0}{\alpha} \cdot \frac{L^2}{\alpha} \right)$

Together, these bounds match the optimal regret bounds for known and unknown control, up to logarithmic factors [4, 26]. The above theorems are proven Appendix E, which also gives complete statements which specify the parameter choices Theorems 3.1a and 3.2a. In addition, Appendix D generalizes the algorithm by replacing static $K$ in the DRC algorithm with a dynamic nominal controller $\Pi_0$, for which analogous guarantees are stated in Appendix E. Importantly, Appendix E.3 verifies that convolution-invertibility holds:

**Lemma 3.1.** For $\kappa$ as in Definition 2.2, we have $\kappa(G_K) \geq \frac{1}{4} \min(1, \|K\|_{op}^2)$. 

9
Broader Impact

Though this paper is primarily theoretical in nature, we believe that the non-stochastic control setting is an important one. Historically, one of the greatest strengths of control theory is its ability to provide robust, mathematical guarantees on performance quality. As control theory merges with recent developments in reinforcement learning, we see novel applications in domains with little room for error: control algorithms in automated transportation, server cooling, and industrial robotics can wreak havoc when gone awry. These tasks may range from easy-to-model to wildly unpredictable, and purely stochastic models may not suffice to capture the full extent of the uncertainty in the task. On the other hand, traditional techniques from robust control may be overly conservative, and deem certain tasks infeasible from the outset.

While far from perfect, we believe that the non-stochastic control model inches us closer towards robustness to modeling assumptions, without succumbing to excessive pessimism. As such, we find it important to understand what, if any, challenges this more accommodating model poses to data-driven control. We hope that our central theoretical contribution - demonstrating that the uncertainty in the noise model is in fact not a significant barrier to achieving near optimal performance - may encourage practitioners not to abandon considerations of robustness for fear of sacrificing performance. But there is still a long road ahead, and we recognize that non-stochastic control does not capture many important senses of robustness in the decades-old control literature. We also recognize that there are, and will continue to be, instances when performance must be sacrificed for robustness, and hope our work will contribute a small but helpful part in a broader dialogue about the tensions between safety and performance in data-driven control.

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References

[1] Yasin Abbasi-Yadkori and Csaba Szepesvári. Regret bounds for the adaptive control of linear quadratic systems. In *Proceedings of the 24th Annual Conference on Learning Theory*, pages 1–26, 2011.

[2] Yasin Abbasi-Yadkori, Peter Bartlett, and Varun Kanade. Tracking adversarial targets. In *International Conference on Machine Learning*, pages 369–377, 2014.

[3] Naman Agarwal, Brian Bullins, Elad Hazan, Sham Kakade, and Karan Singh. Online control with adversarial disturbances. In *International Conference on Machine Learning*, pages 111–119, 2019.

[4] Naman Agarwal, Elad Hazan, and Karan Singh. Logarithmic regret for online control. In *Advances in Neural Information Processing Systems*, pages 10175–10184, 2019.

[5] Jason Altschuler and Kunal Talwar. Online learning over a finite action set with limited switching. *arXiv preprint arXiv:1803.01548*, 2018.

[6] Oren Anava, Elad Hazan, and Shie Mannor. Online learning for adversaries with memory: price of past mistakes. In *Advances in Neural Information Processing Systems*, pages 785–792, 2015.

[7] Raman Arora, Ofer Dekel, and Ambuj Tewari. Online bandit learning against an adaptive adversary: from regret to policy regret. In *International Conference on Machine Learning (ICML)*, pages 1747–1754, 2012.

[8] Peter Auer, Nicolo Cesa-Bianchi, Yoav Freund, and Robert E Schapire. The nonstochastic multiarmed bandit problem. *SIAM journal on computing*, 32(1):48–77, 2002.

[9] Asaf Cassel, Alon Cohen, and Tomer Koren. Logarithmic regret for learning linear quadratic regulators efficiently. *arXiv preprint arXiv:2002.08095*, 2020.

[10] Lin Chen, Qian Yu, Hannah Lawrence, and Amin Karbasi. Minimax regret of switching-constrained online convex optimization: No phase transition. *arXiv preprint arXiv:1910.10873*, 2019.

[11] Alon Cohen, Avinatan Hasidim, Tomer Koren, Nevena Lazic, Yishay Mansour, and Kunal Talwar. Online linear quadratic control. In *International Conference on Machine Learning*, pages 1028–1037, 2018.

[12] Alon Cohen, Tomer Koren, and Yishay Mansour. Learning linear-quadratic regulators efficiently with only $\sqrt{T}$ regret. In *International Conference on Machine Learning*, pages 1300–1309, 2019.

[13] Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. Regret bounds for robust adaptive control of the linear quadratic regulator. In *Advances in Neural Information Processing Systems*, pages 4188–4197, 2018.

[14] Ofer Dekel, Jian Ding, Tomer Koren, and Yuval Peres. Bandits with switching costs: $T^{2/3}$ regret. In *Proceedings of the forty-sixth annual ACM symposium on Theory of computing*, pages 459–467, 2014.

[15] Olivier Devolder, François Glineur, and Yurii Nesterov. First-order methods of smooth convex optimization with inexact oracle. *Mathematical Programming*, 146(1-2):37–75, 2014.

[16] Dylan J Foster and Max Simchowitz. Logarithmic regret for adversarial online control. *arXiv preprint arXiv:2003.00189*, 2020.

[17] Yoram Halevi. Stable lqg controllers. *IEEE Transactions on Automatic Control*, 39(10):2104–2106, 1994.

[18] Elad Hazan. Introduction to online convex optimization. *arXiv preprint arXiv:1909.05207*, 2019.

[19] Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2-3):169–192, 2007.

[20] Elad Hazan, Sham Kakade, and Karan Singh. The nonstochastic control problem. In *Proceedings of the 31st International Conference on Algorithmic Learning Theory*, pages 408–421. PMLR, 2020.
[21] Rudolph Emil Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82.1:35–45, 1960.

[22] Sahin Lale, Kamyar Azizzadenesheli, Babak Hassibi, and Anima Anandkumar. Logarithmic regret bound in partially observable linear dynamical systems. *arXiv preprint arXiv:2003.11227*, 2020.

[23] Sahin Lale, Kamyar Azizzadenesheli, Babak Hassibi, and Anima Anandkumar. Regret minimization in partially observable linear quadratic control. *arXiv preprint arXiv:2002.00082*, 2020.

[24] Yingying Li, Xin Chen, and Na Li. Online optimal control with linear dynamics and predictions: Algorithms and regret analysis. In *Advances in Neural Information Processing Systems*, pages 14887–14899, 2019.

[25] Horia Mania, Stephen Tu, and Benjamin Recht. Certainty equivalence is efficient for linear quadratic control. In *Advances in Neural Information Processing Systems*, pages 10154–10164, 2019.

[26] Max Simchowitz and Dylan J Foster. Naive exploration is optimal for online LQR. *arXiv preprint arXiv:2001.09576*, 2020.

[27] Max Simchowitz, Ross Boczar, and Benjamin Recht. Learning linear dynamical systems with semi-parametric least squares. *arXiv preprint arXiv:1902.00768*, 2019.

[28] Max Simchowitz, Karan Singh, and Elad Hazan. Improper learning for non-stochastic control. *arXiv preprint arXiv:2001.09254*, 2020.

[29] Robert F Stengel. *Optimal control and estimation*. Courier Corporation, 1994.

[30] Dante Youla, Hamid Jabr, and Jr Bongiorno. Modern wiener-hopf design of optimal controllers—part ii: The multivariable case. *IEEE Transactions on Automatic Control*, 21(3): 319–338, 1976.

[31] Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th international conference on machine learning (icml-03)*, pages 928–936, 2003.
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A Organization of the Appendix and Notation

The appendix is organized as follows:

- **Appendix B** provides further discussion, describing how our work serves to characterize the relative difficulty of adversarial noise in online control settings when compared to stochastic.
- **Appendix C** provides an in-depth comparison with the classic LQR and LQG settings, together with an in-depth discussion in Appendix B.1 about the extent to which stochasticity affects the optimal regret rates in online control.
- **Appendix D** provides the full statement of the algorithm DRC-ONS algorithm for the known and unknown settings, and describes the more general DRC-ONS-DYN algorithm for use with a non-static internal controller.
- **Appendix E** provides full statements and proofs of our main regret bounds for the control setting, Theorems 3.1 and 3.2. In particular, we provide the full analogues with the full parameter settings required for the regret bounds, Theorems 3.2b and G.1. We also provide generalizations of our DRC-ONS-DYN algorithm, Theorems 3.1b and 3.2b.
- **Appendix F** gives the full proof of the logarithmic regret bound for Semi-O NS, Theorem 2.1, and Appendix H provides the omitted proofs.
- **Appendix G** gives the full proof of the quadratic error sensitivity of Semi-O NS, Theorem 2.2, and Appendix I provides the omitted proofs.
- **Appendix J** gives the proof of Theorem 2.3, and then demonstrates the standard online Newton step matches the tradeoff (Theorem J.1)

Notation: We use $a = O(b)$ and $a \lesssim b$ interchangably to denote that $a \leq Cb$, where $C$ is a universal constant independent of problem parameters. We also use $a \lor b$ to denote $\max\{a, b\}$, and $a \land b$ to denote $\min\{a, b\}$. Notation relevant to the control problem is reviewed where-necessary in Appendices C and D. In what follows, we review notation relevant to the generic analyses of Semi-O NS.

In Semi-O NS, we have the with-memory loss functions

$$ F_t(z_t, \ldots, z_{t-h}) := \ell_t(v_t + \sum_{i=0}^{h} G[i] Y_{t-i} z_{t-i}), $$

and their unary specializations

$$ f_t(z) := F_t(z, \ldots, z) = \ell_t(v_t + H_t z), \quad H_t := \sum_{i=0}^{h} G[i] Y_{t-i}. $$

Here the losses $\ell_t, v_t, Y_t$ change at each round, and $G = (G[i])_{i \geq 0}$ is regarded as part of an infinite-length Markov operator which is fixed throughout.

For unknown systems, we are use approximate losses, where $\tilde{v}_t \approx v_t, \tilde{G} \approx G$,

$$ \tilde{f}_t(z) := \tilde{F}_t(z, \ldots, z) = \ell_t(\tilde{v}_t + \tilde{H}_t z), \quad \tilde{H}_t := \sum_{i=0}^{h} \tilde{G}[i] Y_{t-i}. $$

Throughout, we use bold $z_t$ to refer to the iterates of the algorithm.

B Further Discussion

B.1 Discussion of Results

In this work, we demonstrate that fast rates for online control, and in particular, the optimal $\sqrt{T}$ regret rate [26] for the online LQR setting, are achievable with non-stochastic noise. Interestingly, simultaneous work by Lale et al. [22] shows that the presence of observation noise implies that the
optimal regret for purely stochastic LQG is in fact polylogarithmic. At first this seems puzzling because, on face, LQG appears to be a strict generalization of LQR. However, poly(log $T$) regret occurs when LQG has a strictly non-degenerate stochastic observation noise $e_t$, which is not the case in LQR. This faster rate is achievable because the noise on the observation provides continuous exploration, allowing the learner to continue to learn with dynamics while simultaneously exploiting near-optimal policies. Alternatively, this observation noise can be understood as making the baseline comparator easier (i.e. min$_{\pi \in \Pi}$ $K_T(\pi)$ is larger), because the underlying control problem is more difficult.

Since we are not guaranteed this observation noise in purely non-stochastic control (indeed, there may be no observation noise at all), $\sqrt{T}$ is still the optimal rate in our setting. Thus, our regret guarantees contribute to the following surprising characterization of regret (with respect to linear dynamic policies) in linear control:

- For known system dynamics, non-stochastic control is just easy as stochastic (Theorem 3.1). There is no substantial price to pay for past mistakes, even under potentially unpredictable, non-stochastic disturbances.
- For unknown system dynamics, stochastic process noise confers little advantage over adversarial noise; both have quadratic sensitivity to error (Theorem 3.2).
- However, there is an advantage to having non-degenerate observation noise. But this is due to continual exploration induced by stochastic noise, and not because stochastic reduces sensitivity to error.

As mentioned in the introduction, competing with arbitrary policies (e.g. the optimal control law given the noise) requires regret which is linear in $T$ [24]. Understanding the optimal competitive ratio, or further assumptions which allow sublinear regret with respect to the optimal control law, remain an interesting direction for future work.

**B.2 Conclusion**

In this work, we demonstrate that fast rates for online control, and in particular, the optimal $\sqrt{T}$ regret rate [26] for the online LQR setting, are achievable with non-stochastic noise.

**Future Work** It is an interesting direction for future research to determine if non-degenerate observation noise can be used to attain polylogarithmic regret for unknown systems in the semi-stochastic regime considered by Simchowitz et al. [28]. This regime interpolates between purely stochastic non-degenerate noise, and arbitrary adversarial noise considered in this setting.

Furthermore, it may be possible that $\sqrt{T}$ regret for unknown systems is attainable even without strongly convex cost function; currently, the state of the art in this setting is $T^{2/3}$ [28, 20].

Finally, we hope future work will take up a more ambitious direction of inquiry, investigating whether these techniques can be applied beyond linear time invariant systems with bound noise. Such directions understanding slowly-varying dynamics, robustness to non-linearities, and model-predictive control.

**Open Question: System Stability and Fast Rates** Lastly, an open question that remains is the extent to which stability of the dynamics affects the extent to which stochastic control is easier than non-stochastic. For example, the guarantees in Lale et al. [22] assume that the dynamics of the system are internally stable, which presumambly simplifies the system identification procedure. On the other hand, our work assumes only that our system can be stabilized by a static feedback controller, which holds without loss of generality for fully observed systems.

As discussed in Appendix D, there are many partially observed systems which cannot be stabilized even by static feedback, but can be stabilized by more general linear control laws. For such systems, our guarantees do extend, but under the opaque technical assumption on the dynamics induced by this more general stabilizing controller have the invertibility property of Definition 2.2. Recall that for the simple case of static feedback, this invertible property is proven to hold in Lemma 3.1.

On the other hand, Simchowitz et al. [28] show that for semi-stochastic disturbances (disturbances with a non-degenerate stochastic component), one can still achieve fast rates for any any linear sta-
bilizing scheme.\textsuperscript{5} This seems to suggest that for controller parametrizations based on more powerful stabilizing controllers, stochasticity may in fact be beneficial. It is an interesting direction for future work to understand whether these more general stabilizing controllers admit fast regret rates for non-stochastic control.

Part I

Appendices for Control

C Past Work and Classical Settings

In this section, we describe in detail how our non-stochastic control setting compares with other control settings considered in the literature. At the end of the section, we conclude with a more thorough discussion of the separations (and lack thereof) between stochastic and non-stochastic control. Recall that our linear system is described by the dynamic equations

\[ x_{t+1} = A_* x_t + B_* u_t + w_t, \quad y_t = C_* x_t + e_t, \]  

(C.1)

Of special interest are the fully observed settings, where \( y_t = x_t \). We may also imagine an intermediate, full-rank observation setting, where \( d_y = d_x \), and \( \sigma_{\text{min}}(C_*) > 0 \). Note that this latter setting allows for observation noise \( e_t \), while the former does not. Finally, in full generality \( C_* \in \mathbb{R}^{d_y \times d_x} \) may have rank \( \text{rank}(C_*) < d_x \), and thus states cannot in general be recovered from observations.

C.1 Online LQR

The linear quadratic regularity, or LQR, corresponds to the setting where the state is fully observed \( x_t = y_t \), and the noise \( w_t \) is selected from a mean-zero, light-tailed stochastic process - typically i.i.d. Gaussian. Crucially, the noise \( w_t \) is assumed to have some non-degenerate covariance: e.g., \( w_t \sim \mathcal{N}(0, \Sigma) \) for some \( \Sigma \succ 0 \). One then considers quadratic cost functions which do not vary with time:

\[ \ell_t(x, u) = \ell(x, u) = x^\top Rx + u^\top Qu, \]

where \( R \) and \( Q \) are positive definite matrices. In particular, \( \ell(x, u) \) is a strong-convex function, and thus the LQR setting is subsumed by our present work.

For the above setting, the optimal control policy (in the limit as \( T \to \infty \)) is described by a static feedback law \( u_t = K_* x_t \), where \( K_* \) solves the Discrete Algebraic Riccati Equation, or DARE; we denote the corresponding control policy \( \pi_{K_*} \). Note that this is in fact the optimal unrestricted control policy (say, over any policy which executes inputs as functions of present and past observations), despite having the simple static feedback form.

Results for online LQR consider a regret benchmark typically considered performance with respect to this benchmark (see e.g. [1, 13, 25, 12])

\[ R_T(\text{alg}) := J_T(\text{alg}) - T \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_w[J_n(\pi_{K_*})] \]

where the righthand term is the infinite horizon average cost induced by placing the optimal control law \( K_* \). One can show (e.g. [26]) \( \mathbb{E}_w[J_n(\pi_{K_*})] \) is increasing in \( n \). Thus, by Jensen’s inequality, it

\textsuperscript{5}Intuitively, this is because with (semi-stochastic) noise, one can replace the infinite-horizon invertibility condition \( \kappa(G) \) of Definition 2.2 with a finite-horizon analogue, \( \kappa_{m,h}(G) \). It is shown that this analogue decays at most polynomially in \( m, h \), even though \( \kappa(G) \) may be zero. This translates into a polynomial dependence on \( m, h \) in the final bound, which contributes only logarithmic factors for the typical choice \( m, h = \mathcal{O}(\log T) \).
holds that for any $\Pi \subset \Pi_{\text{ide}}$ containing $\pi^{K}$,
\[
\mathbb{E}_w[\mathcal{R}_T(\text{alg})] = \mathbb{E}_w[J_T(\text{alg})] - T \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_w[J_n(\pi^{K})] \\
\leq J_T(\text{alg}) - \mathbb{E}_w[J_T(\pi^{K})] \\
= \mathbb{E}_w[J_T(\text{alg})] - \inf_{\pi \in \Pi} \mathbb{E}_w[J_T(\pi)] \\
\leq \mathbb{E}_w[J_T(\text{alg})] - \inf_{\pi \in \Pi} J_T(\pi) \\
\leq \mathbb{E}_w[J_T(\text{alg})] - \inf_{\pi \in \Pi} J_T(\pi) := \mathbb{E}_w[\text{Regret}_T(\text{alg}; \Pi)],
\]
where $\text{Regret}_T$ is our non-stochastic benchmark. Hence, we find that, in expectation, the standard benchmark for online LQR is weaker than ours. Nevertheless, the two benchmark typically coincide up to lower order terms due to martingale concentration. Observe however a key conceptual difference: the LQR regret $\mathcal{R}_T$ can be defined with an \textit{a priori} benchmark, because the dynamics are stochastic. On the other hand, the non-stochastic benchmark is defined \textit{a posteriori}, after because the noises are selected by an adversary.

## C.2 Online LQG

In the LQG, or linear quadratic gaussian control, one typically assumes a partially observed dynamical system, inheriting the full generality of Eq. (C.1). Again, the cost function is typically taken to be quadratic function of input and output:
\[
\ell_t(y, u) = \ell(y, u) = y^\top R y + y^\top Q y,
\]
Again, $R, Q$ are assumed to be positive defined, and thus our assumption that $\ell_t$ are strongly convex subsumes the LQG setting. Typically, online LQG assumes that both the process noise $w_t$ and the observation noise $e_t$ are not only mean zero and stochastic, but also well conditioned. For example, $w_t \overset{i.i.d}{\sim} \mathcal{N}(0, \Sigma_w)$ and $e_t \overset{i.i.d}{\sim} \mathcal{N}(0, \Sigma_e)$, where $\Sigma_w, \Sigma_e > 0$.

Whereas the unconstrained optimal policy in LQG is an \textit{static} feedback law, the optimal LQG policy is \textit{dynamic} linear controller of the form considered in this work. This is true even if $C_\star = I$ but there is non-zero process noise $e_t$; that is, $y_t = x_t + e_t$.

## D Pseudocode, and Dynamic Feedback Generalization

### D.1 Full Pseudocode for Static Feedback Parametrization

parameters:
- Newton parameters $\eta, \lambda > 0$
- DRC parameters radius $R_M > 0$, DRC length $m \geq 1$, memory length $h \geq 0$

initialize:
- Constraint set $\mathcal{M} \leftarrow \mathcal{M}_{\text{dir}}(h, R_M)$ (Eq. (3.2))
- Semi-ONS subroutine $\mathcal{A} \leftarrow \text{Semi-ONS}(\eta, \lambda, \epsilon(\mathcal{M}))$ (Algorithm 1)
- initial values $\tilde{y}_t^K, \hat{y}_t^K, \ldots, \hat{y}_{t+m+h}^K \leftarrow 0$ for $t = 1, 2, \ldots$
- receive $y_t^{\text{alg}}$ from environment, iterate $z_t$ from $\mathcal{A}$, and set DRC parameter $\mathcal{M}_t \leftarrow e^{-1}[z_t]$
- Construct estimate $\hat{v}_t^K = (\hat{y}_t^K, \hat{u}_t^K)$ via Eq. (3.3)
- play input $u_t^{\text{alg}} \leftarrow K \hat{y}_t^{\text{alg}} + u_t^{\text{ex}}(\mathcal{M}_t | \hat{y}_{t+1}^K)$
- suffer loss $\ell_t(y_t^{\text{alg}}, u_t^{\text{alg}})$, and observe $\ell_t(z_t)$

Algorithm 2: Disturbance Response Control via Online Newton Step (DRC-ONS).

### D.2 Stabilizing with dynamic feedback

In general, a partially observed system can not be able to be stabilized by static feedback. To circumvent this, we describe stabilizing the system with a \textit{dynamic feedback controller}, a parameterization
Input:
- Newton parameters $\eta, \lambda > 0$
- DRC parameters radius $R_M > 0$, DRC length $m \geq 1$, memory length $\mathit{h} \geq 0$
- Estimation Length $N \geq 0$, $\% \; N \propto \sqrt{T}$

Initialize $\hat{G}^{[0]} = \begin{bmatrix} 0_{d_w \times d_y} & I_{d_w} \end{bmatrix}$, and $\hat{G}^{[i]} = 0$ for $i > \mathit{h}$.

for $t = 1, 2, \ldots, N$ do
  receive $\hat{y}^\alg_t$
  play $\hat{u}^\alg_t = \hat{u}^\text{ex,alg}_t + K \hat{y}^\alg_t$, where $\hat{u}^\text{ex,alg}_t \sim \mathcal{N}(0, I_{d_w})$.
  estimate $\hat{G}^{[1:h]} = (\hat{G}^{[i]})_{i \in [h]} \leftarrow \arg \min_{G^{[1:h]}} \sum_{i=0}^{N} \| \hat{v}^\alg_t - \sum_{i=1}^{\mathit{h}} G^{[i]} \hat{u}^\text{ex,alg}_{t-i} \|^2$.
run Algorithm 2 for times $t = N + 1, N + 2, \ldots, T$, using $\hat{G}$ as the Markov parameter estimate, and parameters $m, h, \lambda, \eta$.

Algorithm 3: Full DRC-ONS for Unknown System, with estimation

we refer to as DRC-DYN. The following exposition mirrors Simchowitz et al. [28], but is abridged considerably. Specifically, we assume that our algorithm maintains an internal state $s^\alg_t$, which evolves according to the dynamical equations

$$s^{\alg}_{t+1} = A \pi_0 s^\alg_t + B \pi_0 y^\alg_t + B \pi_0 \hat{w}^\alg_t,$$  \hspace{1cm} (D.1)

and selects inputs as a combination of an exogenous input $\hat{u}^\text{ex}_t$, and an endogenous input determined by the system:

$$\hat{u}^\alg_t = \hat{u}^\text{ex,alg}_t + (C_{\pi_0} s^\alg_t + D_{\pi_0} y^\alg_t).$$  \hspace{1cm} (D.2)

Lastly, the algorithmic prescribes a control output, denoted by $\omega_t$, given by

$$\omega^\alg_{t+1} = C_{\pi_0, \omega} s^\alg_t + D_{\pi_0, \omega} y^\alg_t \in \mathbb{R}^{d_w},$$

which we use to parameterize the controller. In the special case of static feedback, we take $C_{\pi_0, \omega} = 0$ and $D_{\pi_0, \omega} = I$, so that $\omega^\alg_t = y^\alg_t$. We assume that $\pi_0$ is stabilizing, meaning that, if we have $\max_t \| e_t \|, \| w_t \|, \| u^\text{ex,alg}_t \| < \infty$ are bounded, then with $\max_t \| u^\alg_t \|, \| y^\alg_t \|, \| \omega^\alg_t \| < \infty$. As a consequence of the Youla parametrization [30], one can always construct a controller $\pi_0$ which has this property for sufficiently non-pathological systems.

Analogous to the sequence $y^K_t, \hat{u}^K_t$, we consider a sequence that arises under no exogenous inputs:

**Definition D.1.** We define the ‘Nature’ sequence $y^\nat_t, \hat{u}^\nat_t, \omega^\nat_t$ as the sequence obtained by executing the stabilizing policy $\pi_0$ in the absence of $u^\text{ex}_t = 0$; we see $y^\nat_t = (y^\nat_t, \hat{u}^\nat_t) \in \mathbb{R}^{d_y + d_w}$.

Each such sequence is determined uniquely by the disturbances $w_t, e_t$.

Moreover, the ‘Nature’ sequences can be related to the sequences visited by the algorithm via linear Markov operators

**Definition D.2.** We define the linear Markov operators $G^{\text{ex} \rightarrow u}, G^{\text{ex} \rightarrow \omega}$ as the operators for which

$$\omega^\alg_t = \omega^\nat_t + \sum_{i=1}^{t} G^{[t-i]} \hat{u}^\text{ex}_i, \quad y^\alg_t = y^\nat_t + \sum_{i=1}^{t} G^{[t-i]} \hat{u}^\text{ex}_i.$$

We note that $G^{[0]}_{\text{ex} \rightarrow \omega} = 0_{d_w \times d_w}$ by construction.

Finally, we describe our controller parametrization:

**Definition D.3** (DRC with dynamic stabilizing controller), Generalizing Eq. (3.2), let $\mathcal{M}_{\text{drc}}(m, R_M)$ denote $M \in \mathbb{G}^{d_w \times d_w}$ for which $\| M \|_{t, \text{op}} \leq R_M$, and $M^{[i]} = 0$ for all $i \geq m$. Given estimates $\hat{\omega}^\nat_{t-m+1}, \ldots, \hat{\omega}^\nat_t$, we select

$$\hat{u}^\text{ex}_i (M \mid \hat{\omega}^\nat_{1:t}) := \sum_{i=0}^{m-1} M^{[i]} \hat{\omega}^\nat_{t-i}.$$
We recover the static feedback setting in the following example:

**Example D.1 (Static Feedback).** To recover the special case of static feedback, we make the following substitutions

- We set $s_t^{\text{alg}} = 0$ for all $t$, $C_{\pi_0} = 0$ and $D_{\pi_0} = K$.
- We set $C_{\pi_0,\omega} = 0$ and $D_{\pi_0,\omega} = I$, so that $u_t^{\text{alg}} = u_t^{\text{ex,alg}} + Ky_t^{\text{alg}}$
- We set we set $C_{\pi_0,\omega} = 0$ and $D_{\pi_0,\omega} = I$, so that $\omega_t^{\text{alg}} = y_t^{\text{alg}}$ for all $t$.
- The quantities $y_t^{\text{nat}}$ and $\omega_t^{\text{nat}}$ both correspond to $y^K_t$, and $u_t^{\text{nat}} = u^K_t$, the operator $G_{ex \rightarrow v}$ becomes the Markov operator $G_K$, and $G_{ex \rightarrow \omega}$ becomes the top $d_y \times d_u$ block of $G_K$, capturing the response from $u_t^{\text{ex}} \rightarrow y_t$.
- Thus, $u_t^{\text{ex}}(M \mid \omega_t^{\text{nat}})$ corresponds to $u_t^{\text{ex}}(M \mid \hat{y}^K_t)$.

**D.3 Full Algorithm under Dynamic Feedback**

Let us now turn to the specific of the main algorithm with dynamic feedback, DRC-ONS-DYN. Throughout the algorithm, we maintain an internal state updated according to the nominal controller $\pi_0$ via Eq. (D.1). Moreover, all inputs are selected as $u_t^{\text{alg}} = u_t^{\text{ex,alg}} + (C_{\pi_0} s_t^{\text{alg}} + D_{\pi_0} y_t^{\text{alg}})$ in accordance with Eq. (D.2).

Next, we specify how we recover $v_t^{\text{alg}}$ and $\omega_t^{\text{alg}}$. Given estimates $\hat{G}_{ex \rightarrow (y,u)}, \hat{G}_{ex \rightarrow \omega}$, we parallel Eq. (3.3) in defining

$$\hat{u}_t^{\text{alg}} := \begin{bmatrix} y_t^{\text{nat}} \\ \hat{y}_t^{\text{nat}} \end{bmatrix} = \begin{bmatrix} y_t^{\text{alg}} \\ C_{\pi_0} s_t^{\text{alg}} + D_{\pi_0} y_t^{\text{alg}} \end{bmatrix} - \sum_{i=1}^{t-1} \hat{G}_{ex \rightarrow (y,u)}^{[i]} u_{t-i}^{\text{ex,alg}},$$

$$\omega_t^{\text{nat}} := \omega_t^{\text{alg}} - \sum_{i=1}^{t-1} \hat{G}_{ex \rightarrow \omega}^{[i]} u_{t-i}^{\text{ex,alg}}.$$  \hfill (D.3)

As in the static feedback case, the above exactly $v_t^{\text{nat}}, \omega_t^{\text{nat}}$ for exact estimates $\hat{G}_{ex \rightarrow (y,u)} = G_{ex \rightarrow v}$ and $\hat{G}_{ex \rightarrow \omega} = G_{ex \rightarrow \omega}$. We then contract optimization losses as follows, mirroring Eq. (2.3):

$$\hat{f}_t(z) := \ell_t(\hat{y}_t^{K} + \hat{H}_t z), \text{ where } \hat{H}_t := \sum_{i=0}^{h} \hat{G}_{ex \rightarrow (y,u)}^{[i]} Y_{t-i}, \text{ and } Y_s = c_\omega[\omega_s^{\text{nat}} m] - m,$$  \hfill (D.4)

where $c_\omega$ is an embedding map analogues to $c_y$.

With these estimates and definitions, Algorithms 4 and 5 provides the pseudocode generalizing Algorithms 2 and 3 to our setting. The main differences are

- Using $\omega_t^{\text{nat}}$ for the controller parameterization, rather than $y^K_t$.
- Maintaining the internal state $s_t^{\text{alg}}$
- Estimating two sets of Markov parameters, $\hat{G}_{ex \rightarrow \omega}$ and $\hat{G}_{ex \rightarrow (y,u)}$.

**E Full Control Regret Bounds and Proofs**

This section states and proves our main results for the control setting. We state and prove Theorems 3.1b and 3.2b for the general, dynamic-internal controllers described in Appendix D. We then derive the regret bounds Theorems 3.1 and 3.2 in the main text as consequences of the above theorems. In addition, we state variations of the main-text bounds which make explicit the parameter settings which attain the desired regret (Theorems 3.1a and 3.2a). The section is organized as follows:

- **Appendix E.1** gives the requisite assumptions and conditions for the general setup of Appendix D, which replaces the static $K$ controller with dynamics internal controller.
parameters: Newton parameters $\eta, \lambda$, radius $R_M$, DRC length $m$, memory $h$, closed-loop Markov operator estimate $\hat{G}_{ex \rightarrow \omega}, \hat{G}_{ex \rightarrow (y, u)}$, initial internal state $s^{alg}_1$

initialize:
- constraint set $\mathcal{M} \leftarrow \mathcal{M}_{thr}(h, R_M)$ (Eq. (3.2)), with $\mathcal{C} \leftarrow \epsilon(\mathcal{M})$.
- optimization subroutine $\mathcal{A} \leftarrow \text{Semi-ONS}(\eta, \lambda, \mathcal{C})$ (Algorithm 1), with iterates $z_k$
- initial values $\hat{\omega}^{nat}_0, \hat{\omega}^{nat}_1, \ldots, \hat{\omega}^{nat}_{(m+h)} \leftarrow 0$

for $t = 1, 2, \ldots$
- receive $y_{t}^{alg}$ from environment
- Construct estimate $\hat{\mu}^{nat}_t = (\hat{y}^{nat}_t, \hat{u}^{nat}_t)$ and $\hat{\omega}^{nat}_t$ via Eq. (D.3)
- Recieve iterate $z_t$ from $\mathcal{A}$, and back out DRC parameter $M_t \leftarrow \epsilon^{-1}[z_t]$.
- play input $u_t^{alg} \leftarrow D_{\pi_0}y_t^{alg} + C_{\pi_0}s_t^{alg} + u_t^{ex}(M_t | \hat{\omega}^{nat}_t)$.
- suffer loss $\ell_t(y_t^{alg}, u_t^{alg})$, and observe $\ell_t(\cdot)$
- feed $\mathcal{A}$ the pair $(\hat{f}_t, \hat{H}_t)$, defined in Eq. (D.4), and update $\mathcal{A}$
- update internal state $s^{alg}_{t+1}$ according to Eq. (D.1).

**Algorithm 4:** DRC-ONS-DYN from Markov Parameter Estimates

Input: Number of samples $N$, system length $h$, DRC length $m$, learning parameters $\eta, \lambda$.

Initialize $\hat{G}_{ex \rightarrow (y, u)}^{[0]} = \begin{bmatrix} 0_{d_x \times d_y} & I_{d_u} \end{bmatrix}$, and $\hat{G}_{ex \rightarrow (y, u)}^{[i]} = 0$ for $i > h$, and $\hat{G}_{ex \rightarrow \omega}^{[i]} = 0$ for $i = 0$ and for $i > h, v_1^{alg} = 0$ for $t = 1, 2, \ldots, N$

draw $u_t^{ex, alg} \sim N(0, I_{d_u})$
receive $v_t^{alg} = (y_t^{alg}, u_t^{alg})$ and $\omega_t^{alg}$.
play $u_t^{alg} = u^{ex, alg} + (C_{\pi_0}s_t^{alg} + D_{\pi_0}y_t^{alg})$
update internal state $s^{alg}_{t+1}$ according to Eq. (D.1).

estimate $\hat{G}_{ex \rightarrow (y, u)}^{[1:h]}$ via

$$\hat{G}_{ex \rightarrow (y, u)}^{[1:h]} \leftarrow \arg \min_{G^{[1:h]}} \sum_{t=h+1}^N \|v^{alg}_t - \sum_{i=1}^{h} G[i]u^{ex,alg}_{t-i}\|_2$$

$$\hat{G}_{ex \rightarrow \omega}^{[1:h]} \leftarrow \arg \min_{G^{[1:h]}} \sum_{t=h+1}^N \|\omega^{alg}_t - \sum_{i=1}^{h} G[i]u^{ex,alg}_{t-i}\|_2$$

run Algorithm 4 for times $t = N + 1, N + 2, \ldots, T$, using $\hat{G}_{ex \rightarrow \omega}, \hat{G}_{ex \rightarrow (y, u)}$ as the Markov parameter estimates, and parameters $m, h, \lambda, \eta$, and state $s^{alg}_{t+1}$.

**Algorithm 5:** Full DRC-ONS-DYN for Unknown System (with estimation)

- Appendix E.2 states the general regret guarantees Theorems 3.1b and 3.2b for the dynamic-internal-controller setup. It also states Theorems 3.1a and 3.2a - the complete regret bounds for static feedback with parameter settings made explicit. The static regret bounds are derived in Appendix E.2.1.

- Appendix E.3 proves the bound on the invertibility modulus $\kappa(G_K)$, Lemma 3.1. It also provides discussion regarding the invertibility modulus in the dynamically-stabilized setting (see Remark E.2).

- Appendix E.4 proves the dynamically-stabilized setting guarantee for the known system, Theorem 3.1b. The proof combines the regret decomposition from Simchowitz et al. [28] with our policy regret bound, Theorem 2.1.

- Appendix E.5 proves the dynamically-stabilized setting guarantee for the unknown system, Theorem 3.1b. Again, we combine the existing regret decompositions with the policy regret bound Theorem 2.2.
The arguments that follow essentially reuse lemmas from [28] to port over our policy regret bounds for Semi-ONs to the control setting. We state formal reductions for the known and unknown system settings in Propositions E.5 and E.8, which may be useful in future works applying the DRC parameterization.

The only significant technical difference from [28] is in the analysis of the unknown system, where we use an intermediate step in their handling of one of the approximation errors. This yields an offset in the $Y_r$-geometry (see Proposition E.8), which is explained further in Appendix E.5.

**Asymptotic Notation:** Throughout, we will use $O_{\text{const}(b)}$ to denote a quantity $a$ which is at most $Cb$, where $C$ is a universal constant independent of problem parameters. Equivalently, $a = O_{\text{const}(b)}$ if and only if $a \lesssim b$. We use both notations interchangeably, and $O_{\text{const}(\cdot)}$ affords convenience.

### E.1 Preliminaries and Assumptions for Dynamic Feedback

While the main theorems in the main body of the main text assume explicitly geometric decay, the results in this result will be established with a more abstract, yet theoretically more streamlined construction called a decay function:

**Definition E.1** (Decay Function). For a Markov operator $G = (G[i])_{i \geq 0}$, we define the decay function as $\psi_G(n) := \sum_{i \geq n} \|G[i]\|_{\text{op}}$. We say that $G$ is stable if $\psi_G(0) < \infty$, which implies that $\lim_{n \to \infty} \psi_G(n) = 0$. In general, we say that $\psi$ is a proper, stable decay function if $\psi(n)$ is non-negative, non-increasing, and $\psi(0) < \infty$.

**Assumption 3b** (Stability). We assume that $R_{\pi_0} := \max\{\|G_{\text{ex} \to v}\|_{\ell_1, \text{op}}, \|G_{\text{ex} \to w}\|_{\ell_1, \text{op}}\} < \infty$. We further assume that the decay function of $G_{\text{ex} \to v}$ and $G_{\text{ex} \to w}$ are upper bounded by a proper, stable decay function $\psi_{\pi_0}$. Note that, when the static analogue Assumption 3b holds, we can take

$$R_{\pi_0} = \frac{c_K}{1 - \rho_K}, \quad \psi_{\pi_0}(n) = R_{\pi_0} \rho_K^n.$$ 

For any stabilizing $\pi_0$, Assumption 3b always holds, and in fact $\psi_{\pi_0}$ will have geometric decay. In the special case of static feedback $K$, Assumption 3 implies that

$$\psi_K(n) \leq \frac{c_K \rho_K^n}{1 - \rho_K}. \quad \text{(E.1)}$$

Again, since $\pi_0$ is stabilizing, we also may also assume that the iterates $y_t^K$, $e_t^K$ are bounded for all $t$.

**Assumption 4b** (Bounded Nature’s-iterates). We assume that $(w_t, e_t)$ are bounded such that, for all $t \geq 1$, $\|v^{\text{nat}}\|, \|\omega^{\text{nat}}\| \leq R_{\text{nat}}$. This is equivalent to Assumption 4 in when $\pi_0$ corresponds to static feedback $K$.

### E.1.1 Policy Benchmarks

**Definition E.2** (Linear Dynamic Controller). An LDC is specified by a linear dynamical system $(A_{\pi}, B_{\pi}, C_{\pi}, D_{\pi})$, with internal state $s_t^{\pi} \in \mathbb{R}^{d_{\pi}}$, equipped with the internal dynamical equations $s_{t+1}^{\pi} = A_{\pi} s_t^{\pi} + B_{\pi} y_t^{\pi}$ and $\hat{u}_t^{\pi} := C_{\pi} s_t^{\pi} + D_{\pi} y_t^{\pi}$. We let $\Pi_{\text{ldc}}$ denote the set of all LDC’s $\pi$.

These policies include static feedback laws $\hat{u}_t^{\pi} = K y_t^{\pi}$, but are considerably more general due to the internal state. The closed loop iterates $(y_t^{\pi}, u_t^{\pi}, x_t^{\pi}, s_t^{\pi})$ denotes the unique sequence consistent with Eq. (1.1), the above internal dynamics, and the equalities $\hat{u}_t^{\pi} = u_t$, $y_t^{\pi} = y_t$. The sequence $(y_t^K, u_t^K)$ is a special case with $D_{\pi} = K$ and $C_{\pi} = 0$.

**Dynamic Policy Benchmark** Lastly, let us quantitatively define our policy benchmark, from [28].

**Definition 3.1b** (Policy Benchmark). We define a $\pi_0 \to \pi$ as a Markov operator $G_{\pi_0 \to \pi}$ such that the inputs $u_t^{\pi_0 \to \pi} := \sum_{t=1}^{t} G_{\pi_0 \to \pi}^{[t-1]} w_t^{\pi_0}$ satisfies the following for all $t$:

$$\begin{bmatrix} y_t^{\pi_0} \\ u_t^{\pi_0} \end{bmatrix} = \begin{bmatrix} y_t^{\text{nat}} \\ u_t^{\text{nat}} \end{bmatrix} + \sum_{i=1}^{t} G_{\text{ex} \to \pi}^{[t-i]} u_t^{\pi_0 \to \pi},$$

where $(y_t^{\pi_0}, u_t^{\pi_0})$ is the sequence obtained by executed LDC $\pi$. We define the comparator class $\Pi_{\delta} := \Pi_{\text{stab}, \pi_0}(R, \psi_{\pi_0})$, where $\Pi_{\text{stab}, \pi_0}(R, \psi_{\pi_0}) := \{ \pi \in \Pi_{\text{ldc}} : \|G_{\pi_0 \to \pi}\|_{\ell_1, \text{op}} \leq R, \psi_{G_{\pi_0 \to \pi}}(n) \leq \psi(n), \forall n \}$.

Exact expressions for conversion operators are detailed in Simchowitz et al. [28, Appendix C].

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Static Policy Benchmark

**Definition E.3 (Static Feedback Operator).** Let \( G_{\pi,cl} \) denote the Markov operator \( G_{\pi,cl}^{[i]} = D_{\pi,cl}I_{i=0} + C_{\pi,cl}A_{\pi,cl}^{i-1}B_{\pi,cl}I_{i>0} \), where we define

\[
A_{\pi,cl} := \begin{bmatrix} A_{\pi} + B_{\pi}D_{\pi}C_{\pi} & B_{\pi}C_{\pi} \\ B_{\pi}C_{\pi} & A_{\pi} \end{bmatrix}, \quad B_{\pi,cl} := \begin{bmatrix} B_{\pi}D_{\pi} & -B_{\pi} \\ B_{\pi} & 0 \end{bmatrix}
\]

\( C_{\pi,cl} := \begin{bmatrix} (D_{\pi} - D_{\pi_0})C_{\pi} & C_{\pi} \end{bmatrix}, \quad D_{\pi,cl} = \begin{bmatrix} D_{\pi} & 0 \end{bmatrix} \)

To specialize to the static-feedback setting described in the main text of the paper, we develop the following concrete expression:

**Lemma E.1 (Conversion operators for static feedback).** Consider the special case of the above, where \( \pi_0 \) corresponds to static feedback with matrix \( G \). Then, the following is a \( K \to \pi \) conversion operator:

\[
G_{K \to \pi}^{[i]} = D_{\pi}I_{i=0} + I_{i>0}C_{\pi,cl}A_{\pi,cl}^{i-1}B_{\pi,cl} \quad \begin{bmatrix} I \\ K \end{bmatrix},
\]

Next, fix \( c_\pi > 0, \rho_\pi \in (0,1) \), and recall the set \( \Pi_{\text{st}}(c_\pi, \rho_\pi) := \{ \pi : \forall n, \|G_{\pi,cl}^{[n]}\|_{\text{op}} \leq c_\pi \rho_\pi^n \} \). Then defining

\[
\psi_\pi(n) := \frac{(1 + \|K\|_{\text{op}})(c_\pi^n \rho_\pi^n)}{1 - \rho_\pi^n}, \quad R_\pi := \frac{(1 + \|K\|_{\text{op}})}{1 - \rho_\pi^n}
\]

we have that \( \pi \in \Pi_\pi, \) where \( \Pi_\pi := \Pi_{\pi_{\text{st}}} \quad \begin{bmatrix} R_\pi, \psi_\pi \end{bmatrix} \) as defined in Definition 3.1b. Lastly, in the special case where the target policy \( \pi \) corresponds to another static feedback law \( u_t = K_{\pi}y_t \), then

\[
G_{K \to \pi}^{[i]} = I_{i=0}K_{\pi} + (K_{\pi} - K_{\pi}C_{\pi}A_{\pi} + B_{\pi}K_{\pi}C_{\pi})^{i-1}B_{\pi}(K_{\pi} - K)
\]

**Proof.** The first and third statements are a special case of Simchowitz et al. [28, Proposition 1], taking \( D_{\pi_0} = K_{\pi} \), and \( A_{\pi_0}, B_{\pi_0}, C_{\pi_0} \) identically zero. For the second statement follows from the fact that \( \|G_{K \to \pi}^{[i]}\|_{\text{op}} \leq (1 + \|K\|_{\text{op}})\|G_{\pi,cl}^{[i]}\|_{\text{op}} \). \( \square \)

**E.2 Complete Statement of Regret Bounds for control setting**

Here, we state our main regret bounds for both general dynamical internal controllers (Theorems 3.1b and 3.2b), and specialization for static controllers, Theorems 3.1a and 3.2a. The main theorems in the text Theorems 3.1 and 3.2 are special cases of the latter. Proofs of specialization to static controllers are provided in Appendix E.2.1 below.

**Assumption 5 (Invertibility Modulus).** For the setting setting, where the system is stabilized by a possibility non-static nominal controller \( \pi_0 \), we assuch that the Markov operator \( G_{\text{ex} \to \pi} \) satisfies \( \kappa(G_{\text{ex} \to \pi}) > 0 \).

**Remark E.1 (Conditions under which Assumption 5 holds).** From Lemma 3.1, we note that Assumption 5 holds whenever \( \pi_0 \) corresponds to stabilizing the system with a static controller. In general, it is more opaque when Assumption 5 assumption holds. We discuss this in more detail in the Appendix E.3.

With our general setting and notation in place, we are ready to state our general bound. Throughout, we consider a comparator class

\[
\Pi_{\text{cl}} := \Pi_{\text{cl}}(R, \psi), \quad \text{where}
\]

\[
\Pi_{\text{cl}}(R, \psi) := \{ \pi \in \Pi_{\text{de}} : \|G_{\pi_{\text{de}}}\|_{\text{op}} \leq R, \psi_{G_{\pi_{\text{de}}} \to \pi}(n) \leq \psi(n), \forall n \},
\]

as defined in Definition 3.1b.

**Theorem 3.1b (Main Regret Guarantee of DRC-ONS-DYN: Known System).** Suppose that 1.3b, 4b, 5 hold. Moreover, choose \( \lambda = 6hR_{\text{ns}}^2R_{\eta}^2/\eta = 1/\alpha \), and suppose that \( m, h \) are selected so that that \( \psi_{\pi_0}(h+1) \leq R_{\pi_0}/T, \psi_{\pi}(m) \leq cR_{\pi}/T, \) and \( R_{\pi} \geq R_{\pi} \). Then, the DRC-ONS-DYN algorithm (Algorithm 4) enjoys the following regret bound:

\[
\text{ControlReg}_{\text{T}}(\text{alg}; \Pi_{\text{cl}}) \lesssim (\alpha \sqrt{\eta})^{-1} mh^2 d_{\text{u}} d_{\omega} R_{\text{ns}}^4 R_{\eta}^2 L^2 \log (1 + T),
\]

The above guarantee is also inherited by DRC-ONS (Algorithm 2) as a special case.
Suppose Assumptions 1, 3 and 4 holds, and for given system variant of $D_{\ell}$. Suppose that Assumptions 1, 3b, 4b, 5 hold, and that the following specializes to static control:

Then, the following regret bound holds with probability $\delta$:

$$\text{ControlReg}_{T}(\text{alg}; \Pi_{*}) \lesssim \frac{\lambda R_{\text{nat}}^{2} \log(1/\delta) \sqrt{T}}{\alpha K^{2}} + \frac{hR_{\text{nat}}^{2}}{\alpha \lambda} \cdot d_u d_y R_{\text{nat}}^{2} \cdot \frac{L^{2}}{\alpha} \log^{4}(1 + T)$$

For unknown systems, the following guarantees $\tilde{O} \left( \sqrt{T} \right)$ regret:

**Theorem 3.2b** (Main Regret Guarantee of DRC-ONS-DYN: Unknown System). Suppose that Assumptions 1, 3, 4, 5 hold, and that $\ell_{t}$ are L-smooth ($\nabla^{2} \ell_{t} \preceq L$). Lastly, fix $\delta \in (0, 1/T)$. Then, when the unknown-system variant of DRC-ONS-DYN with estimation (Algorithm 5) is run with the following choice of parameters:

- $\lambda = R_{\text{nat}}^{2} \log(1/\delta) \sqrt{T} + hR_{\text{nat}}^{2}$ and $\eta = 3/\alpha$
- $N = h^{2} \sqrt{T} \max\{d_{w}, d_{y} + d_{u}\}$
- $\sqrt{T} \geq 4 \cdot 1.764 h^{2} R_{\text{nat}}^{2} \tau_{\text{opt}}^{2} + c_{0} h^{2} d_{u}^{2}$, where $c_{0}$ is a universal constant arising from conditioning of the least squares problem\(^6\).
- $m \geq m_{*} + 2 h$ and $R_{\text{M}} \geq 2 R_{\text{*}}$
- $\psi_{\tau_{0}}(h + 1) \leq R_{\tau_{0}}/T$, $\psi_{\star}(m) \leq R_{\star}/T$

Then, the following regret bound holds with probability $1 - \delta$:

$$\text{ControlReg}_{T}(\text{alg; } \Pi_{*}) \lesssim \log(1 + T) \frac{(d_{w} + d_{y})(d_{y} + d_{u})m h L^{2} R_{\text{nat}}^{4} R_{\text{M}}^{5} \sqrt{T} \log(1/\delta)}{\alpha K^{2}}$$

The same guarantee also holds for the static analogue Algorithm 3.\(^a\)

The following specializes to static control:

**Theorem 3.2a** (Main Regret Guarantee of DRC-ONS: Unknown System, with Explicit Parameters). Suppose that Assumptions 1, 3, 4, 5 hold, and that $\ell_{t}$ are L-smooth ($\nabla^{2} \ell_{t} \preceq L$). For simplicity, further select comparator parameters $\rho_{*} \geq \rho_{K}$, $c_{*} \geq c_{K}$. Finally, fix $\delta \in (0, 1/T)$. Then, when the unknown-system variant of DRC-ONS-DYN with estimation (Algorithm 5) is run with the following choice of parameters:

- $h = \lceil \log(1/\delta) \log T \rceil$, $m = 3 h$, $R_{\text{M}} = 2 \frac{\lambda}{\alpha} \frac{h c_{K}}{1 - \rho_{*}}$
- $\lambda = R_{\text{nat}}^{2} \log(1/\delta) \sqrt{T} + h c_{K}^{2} / (1 - \rho_{K})^{2}$ and $\eta = 3/\alpha$
- $N = h^{2} \sqrt{T} (d_{y} + d_{u})$
- $\sqrt{T} \geq c \log^{2} T (1 - \rho_{*})^{-6} c_{K}^{2} (1 + \|K\|_{\text{op}})^{2} + (1 - \rho_{*})^{-2} d_{u}^{2}$ for some universal constant $c$ (satisfied for $T = \tilde{O}(1)$).

\(^a\)Empirically, one can just verify whether the LS problem is well conditioned.
Then, the following regret bound holds with probability $1 - \delta$:

$$\text{ControlReg}_{\text{alg}}(\Pi, n) \lesssim \sqrt{T} \cdot \frac{c_K^4 c_1^4 (1 + \|K\|_{\text{op}})^5}{\left(1 - \rho_K\right)^4 (1 - \rho_N)^6} \cdot \frac{L^2 R_{\text{nat}}^5}{\alpha} \cdot \log^3 (1 + T) \log (1/\delta)$$

The same guarantee also holds for the static analogue Algorithm 3).

### E.2.1 Specializing Dynamic Stabilizing Controller to Static

**Proof of Theorems 3.1 and 3.1a.** For the static case, as noted in Assumptions 3b and 4b, Assumption 4 implies Assumption 4b, and Assumption 3 implies Assumption 3b with

$$R_{\pi_0} = \frac{c_K}{1 - \rho_N}, \quad \psi_{\pi_0}(n) = R_{\pi_0} n.$$ 

Moreover, recall that our benchmark is $\pi \in \Pi_{\text{stab}}(c_*, \rho_*)$, as defined in Definition 3.1. from Lemma E.1, this benchmark is subsumed by the benchmark $\Pi_*$ for the choice of $\psi_*, R_*$, as in Eq. (E.2):

$$R_* := \frac{(1 + \|K\|_{\text{op}}) c_*}{1 - \rho_*}, \quad \psi_*(n) \leq R_* n.$$ 

Let us now use the following technical claim:

**Fact E.2.** Let $\rho \in (0, 1)$. Then $\rho^n \leq 1/T$ for $n \geq \frac{\log T}{1 - \rho}$.

**Proof of Fact E.2.** We have $\rho^n \leq 1/T$ for $n \geq \log(T) / \log(1/\rho)$. But $\log(1/\rho) \leq 1 - \rho = 1 - \frac{1 - \rho}{\rho}$, so it suffices to select $n \geq \log(T)(\rho/1 - \rho) \geq \log(T)/(1 - \rho)$. □

Thus, our conditions $\psi_{\pi_0}(h + 1) \leq R_{\pi_0}/T$, $\psi_*(m) \leq c R_* / T$, and $R_M \geq R_*$ hold as soon as

$$h \geq \frac{\log T}{1 - \rho_*} \geq \frac{\log T}{1 - \rho_N}.$$ 

Thus, setting $h = \lceil \frac{\log T}{1 - \rho_N} \rceil$, $m = \lceil \frac{\log T}{1 - \rho_*} \rceil$, and $R_M = R_* = (1 + \|K\|_{\text{op}}) \bar{c}_*, \kappa(G_K) \geq 1 \min\{1, \|K\|_{\text{op}}^{-2}\} \gtrsim (1 + \|K\|_{\text{op}})^{-2}$, we obtain

$$\text{ControlReg}_{\text{alg}}(\Pi, n) \lesssim \frac{c_K^2 c_1^2 (1 + \|K\|_{\text{op}})^3}{(1 - \rho_K)^5 (1 - \rho_N)^2} \cdot d_u d_y R_{\text{nat}}^2 \cdot \frac{L^2}{\alpha} \log^3 (1 + T).$$

This requires the step size choice of $\eta = 1/\alpha$ and $\lambda = 6 h R_{\text{nat}}^2 c_K^2 (1 - \rho_K)^2$. □

**Theorem 3.2a.** For static feedback, we have $d_w = d_y$. Thus, $(d_w + d_y)(d_y + d_u) = d_y(d_y + d_u)$. Next, we have $R^4_{\pi_0} R^4_M = (1 - \rho_K)^{-4} c_K^4 \cdot (1 + \|K\|_{\text{op}})^4 (1 - \rho_N)^{-2} c^4_1$, and $h \leq m \lesssim (1 - \rho_N)^{-1} \log(1 + T)$. This gives

$$(d_w + d_y)(d_y + d_u) m h R^4_{\pi_0} R^4_M \lesssim d_y(d_y + d_u) \frac{c_K^4 c_1^4 (1 + \|K\|_{\text{op}})^4}{(1 - \rho_K)^4 (1 - \rho_N)^6} \log^2 (1 + T).$$

Using $1/\sqrt{\pi} \lesssim (1 + \|K\|_{\text{op}}), we then get

$$\text{ControlReg}_{\text{alg}}(\Pi, n) \lesssim \sqrt{T} \cdot \frac{c_K^4 c_1^4 (1 + \|K\|_{\text{op}})^5}{(1 - \rho_K)^4 (1 - \rho_N)^6} \cdot \frac{R_{\text{nat}}^5 L^2}{\alpha} \cdot \log^3 (1 + T) \log (1/\delta)$$

The correctness of the various parameter settings can be checked analogously. □
E.3 Invertibility-Modulus and Proof of Lemma 3.1

In this section, we bound the condition-modulus $\kappa(G_K)$ defined in Definition 2.2, and generalize the notion to DRC-DYN parametrizations. To begin, we recall our desired bound:

**Lemma 3.1.** For $\kappa$ as in Definition 2.2, we have $\kappa(G_K) \geq \frac{1}{2} \min \{ 1, \| K \|_\text{op} \}$.

For general DRC-DYN parameters, the Z-transform yields a clean lower bound for the condition-modulus of $\hat{G}_{ex\rightarrow v}$ from Definition 2.2:

**Proposition E.3.** Define the Z-transform

$$\hat{G}_{ex\rightarrow v}(z) = \sum_{i=0}^{\infty} G^{[i]}_{ex\rightarrow v} z^{-i}$$

Then, we have the lower bound:

$$\kappa(G_{ex\rightarrow v}) \geq \min_{z \in \mathbb{T}} \sigma_{\text{min}}(\hat{G}_{ex\rightarrow v}(z))^2,$$

where $\kappa(G_{ex\rightarrow v})$ is the condition-modulus of $G_{ex\rightarrow v}$, as defined in Definition 2.2. In particular, if $G_{ex\rightarrow v}$ takes the form

$$G^{[i]}_{ex\rightarrow v} = \mathbb{I}_{i=0} D_{ex\rightarrow v} + \mathbb{I}_{i>0} C_{ex\rightarrow v} A^{i-1}_{ex\rightarrow v} B_{ex\rightarrow v},$$

then

$$\kappa_{\sigma_0} \geq \min_{z \in \mathbb{T}} \sigma_{\text{min}}(D_{ex\rightarrow v} + C_{ex\rightarrow v}(zI - A_{ex\rightarrow v})^{-1} B_{ex\rightarrow v})^2.$$

**Proof of Proposition E.3.** Part 2 applies the well-known formula that the Z-transform of an LTI system with operator $G^{[i]} = D\mathbb{I}_{i=0} + CA^{-1}B\mathbb{I}_{i>0}$, which can be computed via

$$\hat{G}(z) = D + C \left( \sum_{i \geq 1} A^{i-1} z^{-i} \right) B$$

$$= D + C \left( z^{-1} \sum_{i \geq 0} (A/z)^i \right) B$$

$$= D + C \left( z^{-1}(I - A/z)^{-1} \right) B$$

$$= D + C (zI - A)^{-1} B,$$

where we use formal identity identity $\sum_{i \geq 0} X^i = (I - X)^{-1}$.

Let us turn to the first part of the proof. We adopt the argument from [28, Appendix F]. Fix $u_0, u_1, \ldots$ with $\sum n = 0^\infty \| u_n \|^2 = 1$, and define a Markov-shaped vector $U = (U^{[i]})$, with $U^{[i]}$, and its Z-transform $\hat{U}(z) := \sum_{i=0}^{n} U^{[i]} z^{-i}$. We have that

$$\sum_{n \geq 0} \left\| \sum_{i=0}^{n} G^{[i]} u_{n-i} \right\|_2^2 = \sum_{n \geq 0} \| (G * U)^{[n]} \|_2^2$$

where $*$ denotes the convolution operator. By Parseval’s identity, we have that

$$\sum_{n \geq 0} \left\| (G * U)^{[n]} \right\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \| \hat{G} \hat{U}(e^{i\theta}) \|_2^2 d\theta,$$

where $(\hat{G} \hat{U})(z) = \sum_{i \geq 0} (G \hat{U})^{[i]} z^{-i}$ is the Z-transform of $G \hat{U}$. Because convolutions become multiplications under the Z-transform, we have that for the Z-transform of $U$,

$$\frac{1}{2\pi} \int_0^{2\pi} \| (G \hat{U})(e^{i\theta}) \|_2^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} \| \hat{G} \hat{U}(e^{i\theta}) \|_2^2 d\theta.$$
This establishes the first equality of the claim. For the inequality, we have
\[
\frac{1}{2\pi} \int_0^{2\pi} \| \hat{G}(e^{i\theta}) \hat{U}(e^{i\theta}) \|^2_2 d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} \sigma_{\min}(\hat{G}(e^{i\theta}))^2 \| \hat{U}(e^{i\theta}) \|^2_2 d\theta
\]
\[
\geq \min_{z \in \mathbb{T}} \sigma_{\min}(z)^2 - \frac{1}{2\pi} \int_0^{2\pi} \| \hat{U}(e^{i\theta}) \|^2_2 d\theta.
\]
To conclude, we note that by Parseval’s identity, \[
\frac{1}{2\pi} \int_0^{2\pi} \| \hat{U}(e^{i\theta}) \|^2_2 d\theta = \sum_{n \geq 0} \| U[n] \|^2 = \sum_{n \geq 0} \| G[i] u_{n-1} \|^2_2 = \frac{1}{2\pi} \int_0^{2\pi} \| \hat{G}(e^{i\theta}) \hat{U}(e^{i\theta}) \|^2_2 d\theta \geq \min_{z \in \mathbb{T}} \sigma_{\min}(z)^2,
\]
as needed.

We now turn to giving an explicit lower for the static-feedback stabilized setting:

**Proof of Lemma 3.1.** For the special case of static feedback, we recall from Eq. (3.1) that
\[
G_{ex \rightarrow v}^[[i]] = G_K^[[i]] = \mathbb{I}_{i=0}^{[0]} + \mathbb{I}_{i>0}^{[1]} C_* K C_* \left( A_* + B_* K C_* \right)^{-1} B_*, \quad i \geq 1.
\]
Thus, defining \( \hat{A}(z) := (zI - A_* + B_* K C_*)^{-1} \), we have from Proposition E.3 that
\[
G_{ex \rightarrow v}(z) = \left[ C_* \hat{A}(z) B_* \right]
\]
\[
I + K C_* \hat{A}(z) B_*
\]
where the above holds for all \( z \in \mathbb{T} \) since \( K \) is stabilizing. We now invoke a simple linear algebraic fact:

**Claim E.4** (Lemma F.2 in [28]). *Consider a matrix of the form*
\[
W = \begin{bmatrix} YZ \\ I + XZ \end{bmatrix} \in \mathbb{R}^{(d_1 + d) \times d},
\]
*with \( Y \in \mathbb{R}^{d_1 \times d_1}, X, Z \in \mathbb{R}^{d \times d_1}. \) Then, \( \sigma_{\min}(W) \geq \frac{1}{2} \min\{1, \sigma_{\min}(Y)\}. \)

Applying the above claim with \( Y = I, X = K, \) and \( W = C_* \hat{A}(z) B_* \), we conclude that
\[
\sigma_{\min}(G_{ex \rightarrow v}(z)) \geq \frac{1}{2} \min\{1, \| K \|_{op}^{-1}\}
\]
for all \( z \in \mathbb{C} \). Thus, by Proposition E.3, \( \kappa(G_K) \geq \left( \frac{1}{2} \min\{1, \| K \|_{op}^{-1}\} \right)^2 = \frac{1}{2} \min\{1, \| K \|_{op}^{-2}\} \), as needed. \( \square \)

**Remark E.2** (Generic Bounds on Invertibility). In general, we do not have a generic lower bound on the invertibility modulus which is verifiably no-negative for all choices of stabilizing controllers. For one, it is not clear that our lower bound in Proposition E.3 is sharp, in part because we are working with real operators. However, there are certain conditions (e.g. Youla parametrization, where \( A_* \) has no eigenvalues \( z \in \mathbb{T} \), Simchowitz et al. [28, F.2.3]) where we have \( \min_{z \in \mathbb{T}} \sigma_{\min}(G_{ex \rightarrow v}(z))^2 \) is strictly positive.

### E.4 Control Proofs for Known System

We focus on the dynamic version of our algorithm, DRC-ONS-DYN, with stabilizing controller \( \pi_0 \). For known Markov operator, this algorithm specializes to DRC-ONS in the case of static feedback. The following theorem reduces to bounding the policy regret:

**Proposition E.5** (Reduction to policy regret for known dynamics). *Consider the DRC-ONS-DYN algorithm (Algorithm 4) initialized with the exact Markov operators \( G_{ex \rightarrow (y,u)} = G_{ex \rightarrow v}, G_{ex \rightarrow \omega} = G_{ex \rightarrow \omega}, \) and iterates \( M_t \) produced by an arbitrary black-box optimization procedure \( A \). Further, suppose that \( \psi_* (m) \leq c R_* / T, \psi_{\pi_0} (h+1) \leq c \psi_{\pi_0} (h+1) / R_{\pi_0} \) for some \( c > 0 \). Then,
\[
\text{ControlReg}_T(\text{alg}) \leq \text{MemoryReg}_T(\text{alg}) + 12 L c R_M^2 R_{\pi_0} \mathcal{R}_{\text{nat}}^2.
\]
where, for the \( F_t, f_t \) losses in Definition 3.3b, we define
\[
\text{MemoryReg}_T(\text{alg}) := \sum_{t=1}^T F_t(z_{t:t-h} \mid \omega_{1:t}^\text{nat}) - \inf_{z \in M_t} \sum_{t=1}^T f_t(z \mid \omega_{1:t}^\text{nat})
\]
The same is true for Algorithm 2 (for static feedback).
Remark E.3. In the above, we allow a slack parameter $c$ on the choice of $m, h$. This means that our main theorems can be generalized slightly to accommodate when $m, h$ are chosen larger-than-needed.

Next, we bound the relevant parameters required:

**Lemma E.6 (Parameter Bounds).** Assume $R_{\text{nat}}, R_M \geq 1$. The following bounds hold

(a) We have $D = \max \{\|z - z'\| : z, z' \in M_\mathbb{C}\} \leq 2\sqrt{m}R_M$.

(b) We have $R_Y := \max_t \|Y_t\|_{\text{op}} = \max_t \|\epsilon_\omega(\psi_{1:t}^\text{nat})\|_{\text{op}} \leq R_{\text{nat}}$.

(c) We have $R_Y \mathcal{L} = \max_z \max_{z' \in \mathcal{C}} \|Y_t z\| \leq R_M R_{\text{nat}}$.

(d) For $G = G_{\text{ex}} \mathcal{v}$, we have $R_G = \|G_{\text{ex}} \mathcal{v}\|_{\text{op}} \leq R_{\psi_0}, \psi_G \leq \psi_{\psi_0}$, and $R_H \leq R_{\psi_0} R_{\text{nat}}$.

(e) We have $R_V \leq R_{\text{nat}}$, and $L_{\text{eff}} \leq 2LR_{\psi_0} R_M R_{\text{nat}}$.

Moreover, $d = m\eta \log (1 + \sqrt{\lambda})$.

We are now ready to prove our general regret bound for the known system case, encompassing

**Proof of Theorem 3.1b.** From Theorem 2.1, we have the bound:

$\text{MemoryReg}_{\text{alg}}(\text{alg}) = \sum_{i=1}^{T} F_i(z_{t:i-h}) - \min_{z' \in \mathbb{C}} \sum_{i=1}^{T} f_i(z) \leq 3\alpha hD^2 R_M^2 + \frac{3dh^2 L_{\text{eff}}^2 R_G}{\alpha \kappa^{3/2}} \log (1 + T)$,

Let us now specify the above constants using Lemma E.6. From this lemma, we have that $\alpha hD^2 R_M^2 = \alpha h m R_{\psi_0}^2 R_{\text{nat}}^2 R_M$. Moreover, $dh^2 L_{\text{eff}}^2 R_G = 4m h^2 d_i d_a L^2 R_{\psi_0}^2 R_{\text{nat}}^2 R_M$. Thus, with $\lambda := 6h R_{\text{nat}}^2 R_{\psi_0}^2$ and $\eta = \log (1/\alpha)$, we get

$\text{MemoryReg}_{\text{alg}}(\text{alg}) \leq m^2 h^2 R_{\psi_0}^2 R_{\text{nat}}^2 R_M^2 (\alpha + (\alpha \sqrt{\overline{\kappa}})^{-1} L^2 R_{\psi_0} d_i d_a \log (1 + T) ) \leq (\alpha \sqrt{\overline{\kappa}})^{-1} m^2 h^2 d_i d_a R_{\psi_0}^3 R_{\text{nat}}^2 R_M^2 L^2 \log (1 + T)$,

where we used that $L^2/\alpha \sqrt{\overline{\kappa}} \geq L^2/\alpha \geq \alpha$ by the assumption $\alpha \leq L$. Combining with Proposition E.5 and again using $L \leq L^2/\alpha \sqrt{\overline{\kappa}}$ ensures that the total control regret $\text{ControlReg}_{\text{alg}}$ suffers an additional constant $L$ in the bound, yielding at most

$\text{ControlReg}_{\text{alg}}(\text{alg}) \leq (\alpha \sqrt{\overline{\kappa}})^{-1} m^2 h^2 d_i d_a R_{\psi_0}^3 R_{\text{nat}}^2 R_M^2 L^2 \log (1 + T)$,

as needed.

\[\square\]

**E.4.1 Proof of Proposition E.5**

We follow the regret decomposition from [28], noting that our assumptions on the dynamics, magnitude bounds, and costs $c_i$ all align. To facilitate reuse of the technical material from [28], we introduce the following loss notation in the $M$-domain:

**Definition 3.8 (Losses for the analysis).** Generalizing Definition 3.3, we introduce the $z$-space losses,

$F_i(z_{t:i-h} | \hat{\psi}_{1:t}^{\text{nat}}) := \ell_i(v_t^{\text{nat}} + \sum_{i=0}^{h} G_{\text{ex}}^{[i] \rightarrow v} Y_{t-i} z_t-i), \text{ where } Y_s = \epsilon_\omega(\hat{\psi}_{1:t}^{\text{nat}}),$

with unary specialization $F_i(z_{t:t-h} | \hat{\psi}_{1:t}^{\text{nat}}) := f_i(z, \ldots, z | \hat{\psi}_{1:t}^{\text{nat}})$, and their analogues in $M$-space

$\tilde{F}_i(M_{t:t-h} | \hat{\psi}_{1:t}^{\text{nat}}) := \ell_i(v_t^{\text{nat}} + \sum_{i=0}^{h} G_{\text{ex}}^{[i] \rightarrow v} u_{\text{alg}}(M| \hat{\psi}_{1:t}^{\text{nat}}))$,

and unary specialization $\tilde{f}_i(M | \hat{\psi}_{1:t}^{\text{nat}}) := \tilde{F}_i(M, \ldots, M | \hat{\psi}_{1:t}^{\text{nat}})$. Observe that, for $z_s = \epsilon(M_s)$ for $s \in [T]$, and $z = \epsilon(M)$, then

$F_i(z_{t:t-h} | \hat{\psi}_{1:t}^{\text{nat}}) = F_i(M_{t:t-h} | \hat{\psi}_{1:t}^{\text{nat}}), \text{ and } f_i(z | \hat{\psi}_{1:t}^{\text{nat}}) = \tilde{f}_i(M | \hat{\psi}_{1:t}^{\text{nat}})$. (E.4)
Moving forward, let \((y^M, u^M)\) denote the sequence produced by selecting input \(u^\pi_t \in (M \mid \omega^\pi_t)\) at each \(t\). We then have

\[
\text{ControlReg}_T(\text{alg}; \Pi_*) \\
= \sum_{t=1}^T \ell_t(y^\text{alg}_t, u^\text{alg}_t) - \inf_{\pi \in \Pi_*} \sum_{t=1}^T \ell_t(y^\pi_t, u^\pi_t) \\
\leq \sum_{t=1}^T \left[ \ell_t(y^\text{alg}_t, u^\text{alg}_t) - \bar{F}_t(M_{t:t-h} \mid \omega^\text{nat}_t) \right] + \sum_{t=1}^T \bar{F}_t(M_{t:t-h} \mid \omega^\text{nat}_t) - \inf_{\pi \in \Pi_*} \sum_{t=1}^T \ell_t(y^\pi_t, u^\pi_t) \\
+ \max_{M \in \mathcal{M}} \sum_{t=1}^T \left[ \bar{f}_t(M \mid \omega^\text{nat}_t) - \ell_t(y^M_t, u^M_t) \right] + \inf_{M \in \mathcal{M}} \sum_{t=1}^T \ell_t(y^M_t, u^M_t) - \inf_{\pi \in \Pi_*} \sum_{t=1}^T \ell_t(y^\pi_t, u^\pi_t).
\]

Let’s proceed term by term. From Simchowitz et al. [28, Lemma 5.3] (replacing their notation \(R_G, \psi_G\), with our notation \(R_{\pi_0}, \psi_{\pi_0}\),)

\[(i.a) + (i.b) \leq 4LTR_{\pi_0}R^2_M R^2_{\text{nat}} \psi_{\pi_0}(h + 1) \tag{E.5}\]

Secondly, from Eq. (E.4), we have

\[(ii) = \sum_{t=1}^T \bar{F}_t(a_{t:t-h} \mid \omega^\text{nat}_t) - \inf_{\pi \in \Pi_*} \sum_{t=1}^T f_t(z \mid \omega^\text{nat}_t) := \text{MemoryReg}_T(\text{alg}). \tag{E.6}\]

Finally, from Simchowitz et al. [28, Theorem 1b], we have that for \(R_M \geq R_*\),

\[(iii) \leq 2LTR_* R^2_{\pi_0} R^2_{\text{nat}} \psi(m) \tag{E.7}\]

Thus, we obtain

\[
\text{ControlReg}_T(\text{alg}; \Pi_*) \leq (i.a) + (i.b) + (ii) + (iii) \\
\leq \text{MemoryReg}_T(\text{alg}) + 4LTR^2_M R^2_{\pi_0} R^2_{\text{nat}} \left( \psi_* \frac{(m)}{R_*} + \frac{2\psi_{\pi_0}(h + 1)}{R_{\pi_0}} \right).
\]

Finally, bound \(\psi_*(m) \leq cR_*/T\) and \(2\psi_{\pi_0}(h + 1) \leq cR_{\pi_0}/T\) concludes. \(\square\)

E.4.2  Proof of Lemma E.6

We go term by term:

(a) We have \(D \leq 2\max\{\|z\| : z \in \mathcal{C}\}\). For \(z = \epsilon(M)\), have that \(\|z\| = \|M\|_F \leq \sqrt{m}\|M\|_{\ell_1, \text{op}} \leq \sqrt{m}R_M\) by Simchowitz et al. [28, D.1]

(b) Each matrix \(Y_t\) can be represented as a block diagonal, with blocks as rows corresponding to \(\omega^\text{nat}\) for \(s \in \{t, t - 1, \ldots, t - m + 1\}\). This matrix has operator norm as most \(\max\{\|\omega^\text{nat}_s\| : s \in \{t, t - 1, \ldots, t - m + 1\}\} \leq R_\text{nat}\).

(c) We have that \(Y_t \in u^\text{alg}_t(M \mid \omega^\text{nat}_t) \leq \sum_{i=0}^{m-1} \|M[i]\|_{\ell_1, \text{op}} \|\omega^\text{nat}_{i-t}\|_{\ell_1, \text{op}} \leq R_M R_\text{nat}\) by Holder’s inequality.

(d) These bounds follow directly from our definitions.

(e) We have \(R_v \leq R_\text{nat}\) by assumption, and \(L_{\text{eff}} := 2LR_{\pi_0} R_M R_\text{nat}\) follows from the definition \(L_{\text{eff}} = L \max\{R_v + R_G R_Y C\}\), and the assumption \(s R_M, R_\text{nat} \geq 1\), and \(R_{\pi_0} \geq 1\) by definition \((R_{\pi_0} = \|G^\text{ex} - v\|_{\ell_1, \text{op}}\) and \(G^\text{ex} = 0\).

\(\square\)
E.5 Unknown System

We begin by stating guarantees for the estimation procedures Algorithm 3 and Algorithm 5, which follow directly past work:

**Lemma E.7** (Theorem 6b in Simchowitz et al. [28]). Let \( \delta \in (e^{-T}, T^{-1}) \), \( N, d_u \leq T \), and \( \psi_{G_{\epsilon}}(h + 1) \leq \frac{1}{\sqrt{N}} \). Define \( d_{\text{max}} = \max\{d_y, d_u, d_w\} \), and set

\[
\epsilon_G(N, \delta) = \frac{h^2 R_{\text{nat}}}{\sqrt{N}} C_\delta, \quad \text{where} \quad C_\delta := 14 \sqrt{d_u + d_{\text{max}} + \log \frac{1}{\delta}}, \quad \text{and} \quad R_{u,\text{est}} := 3 \sqrt{d_u + \log(1/\delta)}.
\]

and suppose that \( N \geq h^4 C_2^2 R_{u,\text{est}}^2 R_M^2 + c_0 h^2 d_u^2 \) for an appropriately large \( c_0 \), which can be satisfied by taking

\[
N \geq 1764(d_{\text{max}} + d_u + \log(1/\delta))^2 h^4 R_M^2 R_{\pi_0}^2 + c_0 h^2 d_u^2.
\]

Then with probability \( 1 - \delta - N^{-2} \), Algorithm 5 satisfies the following bounds

1. \( \epsilon_G \leq 1/\max\{R_{u,\text{est}}, R_M R_{\pi_0}\} \).
2. For all \( t \in [N] \), \( \|u_t\| \leq R_{u,\text{est}} := 3 \sqrt{d_u + \log(1/\delta)} \).
3. For estimation error is bounded as

\[
\|\hat{G}_{\text{ex}}(\omega) - G_{\text{ex}}(\omega)\|_{l_1,\text{op}} \leq \|\hat{G}_{\text{ex}}^{[0:h]} - G_{\text{ex}}^{[0:h]}\|_{l_1,\text{op}} + R_{u,\text{est}} \psi_{G_{\epsilon}}(h + 1) \leq \epsilon_G.
\]

Moreover, Algorithm 3 also satisfies the above for \( \hat{G}_{\text{ex}}(\omega) = G \) and \( G_{\text{ex}}(\omega) = G_K \).

The above bounds are in turn a consequence of Simchowitz et al. [27]. We denote the event of Lemma E.7 as \( \mathcal{E}_{\text{est}} \), and the following exposition assumpt it holds.

Next, we state a blackbox reduction to the DRC online controller framework. This reduction crucially uses the fact that we have over-parameterized the set \( \mathcal{M} \). Specifically, over comparator set is

\[
\mathcal{M}_* := M_{\text{drc}}(m_*, R_*)
\]

whereas the algorithm uses the over-parametrized set

\[
\mathcal{M} := M_{\text{drc}}(m, R_M), \text{ with } R_M \geq 2R_* \text{ and } m \geq 2m_* + h. \quad (E.8)
\]

By over-parametrizing the controller set as above, we obtain the following guarantee:

**Proposition E.8** (Reduction to policy regret for known dynamics). Suppose that Eq. (E.8) holds, and that \( \psi_{\gamma_0}(h + 1) \leq c R_{\gamma_0}/T \) and \( \psi_{\nu}(m) \leq c R_*/T \) for some \( c > 1 \), and that \( N \geq m + h \).

Consider the DRC-ONS-DYN algorithm with estimation (Algorithm 5) initialized with the exact Markov operators \( \hat{G}_{\text{ex}}(y,u) = G_{\gamma_0}^{(v)}, G_{\text{ex}}(\omega) = G_{\text{ex}}(\omega) \), and iterates \( \mathcal{M}_t \) produced by an arbitrary black-box optimization procedure \( \mathcal{A} \).

\[
\text{ControlReg}_{T}(\text{alg}; \Pi_*) \leq \text{MemoryReg}_{T}(z_*) + \nu \sum_{t=N+m+2h+1}^{T} \|Y_t(z_t - z_*)\|^2_2 + \mathcal{O}(\text{const}(LR_{\gamma_0}^3(N+c\nu))) (d_u + \log(1/\delta) + R_{\text{nat}}^2) + \mathcal{O}(\text{const}(LR_{M}^3 R_{\nu}^2 R_{\gamma_0}^2 T_{\gamma_0})) \left( 1 + \frac{LmR_{\gamma_0}^2}{\nu} \right)
\]

where \( \mathcal{O}(\text{const}(1)) \) hides a universal numerical constants. Here, for the \( F_t, f_t \) losses in Definition 3.3b, we define the term:

\[
\text{MemoryReg}_{T}(z_*) := \sum_{t=N+m+2h+1}^{T} F_t(z_{t:t-h} | \hat{\omega}_{1:t}^{\text{nat}}) + \inf_{z \in \mathcal{M}_t} \sum_{t=N+m+2h+1}^{T} f_t(z | \hat{\omega}_{1:t}^{\text{nat}}).
\]

Moreover, the same guarantee is also true of Algorithm 3.
Again, we allow a slack parameter \( c \) to allow for over-specifying \( m, h \), demonstrating low sensitivity to imperfectly tuned algorithm parameters. Next, we translate the parameter bounds from the control setting to the ones required for the policy regret analysis of Semi-O:

**Lemma E.9** (Parameter Bounds for Unknown Setting). Assume \( R_{\text{nat}} \geq 1 \), and that \( \cdot \). Then, for \( t_0 := N + m + h + 1 \), the following hold:

(a) We have \( D = \max\{ \| z - z' \| : z, z' \in \mathcal{M}_t \} \leq \sqrt{m}R_M \).

(b) We have \( R_Y := \max_{t \geq t_0} \| Y_t \| \leq 2R_{\text{nat}} \).

(c) We have \( R_{Y,c} = \max_{t \geq t_0} \max_{z \in \mathcal{C}} \| Y_t z \| \leq 2R_MR_{\text{nat}} \).

(d) For \( G = G_{\text{ex}} \), we have \( R_G = \| G_{\text{ex}} \| \leq 2R_{\text{nat}} \).

(e) We have \( R_v := \max_{t \geq t_0} \| v^K_t \| \leq 2R_{\text{nat}} \).

(f) We can take \( c_v \) to be \( 3R_MR_{\text{nat}} \).

Moreover, \( d = d_d d_y m \).

Finally, we are in place to prove our main theorem:

**Proof of Lemma E.9.** The bounds follow analogously to those in **Lemma E.6**, with the modification that, for \( t \geq N + h \), we have \( \| \hat{w}_t \| \leq 2R_{\text{nat}} \) (by Simchowitz et al. [28, Lemma 6.1]), and that \( \| G_{\text{ex}} \| \leq 2R_{\text{nat}} \) under \( \mathcal{E}_{\text{ext}} \). Moreover, we can take the constant \( c_v \) which bounds \( \| \hat{u}_t - v_t \| \leq c_v c_G \) to be \( 3R_MR_{\text{nat}} \) by Simchowitz et al. [28, Lemma 6.4b].

**Proof of Theorem 3.2b.** Let us prove the bound for the dynamic-controller variant Algorithm 5; the static-controller variant works similarly. Recall that we assume the following:

- \( \lambda = R^2_{\text{nat}} \log(1/\delta) \sqrt{T} + hR^2_{\pi_0}, \eta = 3/\alpha \)
- \( N = h^2 \sqrt{T} d_{\text{max}} \)
- \( \sqrt{T} \geq 4 \cdot 176h^2 R^2_M R^2_{\pi_0} + c_h h^2 d_u \)
- \( m \geq m_* + 2h, R_M \geq 2R_* \)
- \( \psi_t(h + 1) \leq \psi_t / T, \psi_t(m) \leq R_* / T. \)

Let \( \epsilon_G \) be an upper bound on the estimation error, which we will set to be greater than \( \sqrt{T} \). By taking \( \lambda \in [c_\lambda, 1] \), and applying **Theorem 2.2a**, we can bound

\[
\text{MemoryReg}^f(z_t) + \nu \sum_{t = N + m + 2h + 1}^T \| Y_t (z_t - z_t) \|_2^2 \lesssim C_1^{-1} \log(1 + \frac{T}{c_\lambda}) \left( \frac{C_1}{\alpha \kappa^{1/2}} + C_2 \right) \left( T c^2_G + h^2 (R^2_G + R_Y) \right),
\]

where \( C_1 := (1 + R_Y) R_G (h + d) L_{\text{eff}}, C_2 := (L^2 c^2_G + \alpha D^2), \) and \( \nu = \frac{\alpha \sqrt{T}}{48(1 + R_Y)} \) are constants which we must bound presently. Since \( d = d_d d_y m \geq h, L \geq \alpha, \) and \( \kappa \leq 1 \)

\[
C_1 \lesssim d_d d_y m R_{\text{nat}} R_{\pi_0} L_{\text{eff}} \lesssim d_d d_y m L^2 R^3_{\text{nat}} R^3_{\pi_0} R^2_M
\]

\[
C_2 \lesssim L^2 / \alpha R^2_{\text{nat}} R^2_M + m R^2_M \lesssim L^2 / \alpha (m R^2_{\text{nat}} R^2_M) \lesssim \frac{L^2}{\alpha \sqrt{\kappa}} (m R^2_{\text{nat}} R^2_M).
\]

Thus, we can bound

\[
\left( \frac{C_1}{\alpha \kappa^{1/2}} + C_2 \right) \lesssim \frac{d_d d_y m L^2 R^3_{\text{nat}} R^3_{\pi_0} R^2_M}{\alpha \kappa^{1/2}}.
\]
Thus, from Proposition E.8 with $\nu = \nu_*$, taking $c = 1$, and bounding $R_G \lesssim R_{\pi_0}$, $R_Y \lesssim R_{\text{nat}}$ from Lemma E.9

\[
\text{ControlReg}_{T}(\text{alg}; \Pi_*) \lesssim c_{\lambda}^{-1} \log(1 + \frac{T}{c_{\lambda}}) \frac{d_w d_y mL^2 R_{\pi_0}^3 R_{\text{nat}}^3 R_{M}^3}{\alpha k^{1/2}} \left( T c_G^2 + h^2(R_{\pi_0}^2 + R_{\text{nat}}^2) \right).
\]

Using the above bounds we have $\nu_* = \frac{\alpha \sqrt{\pi}}{4 \sqrt{1+R_Y}} \geq \alpha \sqrt{\pi} / R_{\text{nat}}$. Thus, for $L \geq \alpha$ and $\kappa \leq 1$, the term $\frac{Lm R_{\pi_0}^2}{\nu_*}$ dominates 1, and we have

\[
LR_{\pi_0}^2 R_{\text{nat}}^2 \left( 1 + \frac{Lm R_{\pi_0}^2}{\nu_*} \right) \lesssim \frac{L^2 R_{\pi_0}^2 R_{\text{nat}}^2 R_{M}^3}{\alpha \sqrt{\pi}}.
\]

Moreover, using $N \geq m$ by assumption and aggregating terms and simplifying

\[
\text{ControlReg}_{T}(\text{alg}; \Pi_*) \lesssim c_{\lambda}^{-1} \log(1 + \frac{T}{c_{\lambda}}) \frac{d_w d_y mL^2 R_{\pi_0}^4 R_{\text{nat}}^3 R_{M}^3}{\alpha k^{1/2}} \left( T c_G^2 + h^2(R_{\pi_0}^2 + R_Y) \right),
\]

Next, recall $d_{\max} := \max\{d_u + d_y, d_w\}$, let us take $N = h^2 \sqrt{T} d_{\max}$. From Lemma E.7, this yields $c_G^2 = \frac{h^4 R_{\pi_0}^2 R_{\text{nat}}^2 \kappa^2}{N} \approx \frac{h^4 R_{\pi_0}^2 (d_{\max} + \log(1/\delta))}{N} = R_{\text{nat}}^2 \log(1/\delta) / \sqrt{T}$ and that $c_G^2 \geq \sqrt{T}$. This yields

\[
\text{ControlReg}_{T}(\text{alg}; \Pi_*) \lesssim c_{\lambda}^{-1} \log(1 + \frac{T}{c_{\lambda}}) \frac{d_w d_y mL^2 R_{\pi_0}^4 R_{\text{nat}}^3 R_{M}^3}{\alpha k^{1/2}} \left( \sqrt{T} R_{\text{nat}}^2 \log(1/\delta) + h^2(R_{\pi_0}^2 + R_{\text{nat}}^2) \right),
\]

Finally, we use bound $LR_{\pi_0}^2 h^2 \sqrt{T} d_{\max} \left( d_u + \log(1/\delta) + R_{\pi_0}^2 R_{\text{nat}}^2 \right) \leq LR_{\pi_0}^2 R_{\text{nat}}^2 h^2 \log(1/\delta) d_u$, and take $L \leq L^2 / \alpha \leq L^2 / \sqrt{\pi}$. Thus, we can bound the above by

\[
\text{ControlReg}_{T}(\text{alg}; \Pi_*) \lesssim c_{\lambda}^{-1} \log(1 + \frac{T}{c_{\lambda}}) \frac{d_w d_y (m + h^2) L^2 R_{\pi_0}^4 R_{\text{nat}}^3 R_{M}^3}{\alpha k^{1/2}} \left( \sqrt{T} R_{\text{nat}}^2 \log(1/\delta) + R_{\pi_0}^2 \right).
\]

Finally, for $\lambda = R_{\text{nat}}^2 \log(1/\delta) \sqrt{T} + h R_{\pi_0}^2$, we can take $c_{\lambda} \approx 1$. Together with $m + h^2 \leq m h$, under the present assumption, we conclude

\[
\text{ControlReg}_{T}(\text{alg}; \Pi_*) \lesssim \log(1 + T) \frac{(d_w d_y + d_{\max} d_u) m h L^2 R_{\pi_0}^4 R_{\text{nat}} R_{M}^3}{\alpha k^{1/2}} \left( \sqrt{T} R_{\text{nat}}^2 \log(1/\delta) + R_{\pi_0}^2 \right).
\]

Finally, we require $N \geq 1764(d_{\max} + d_u + \log(1/\delta))^2 h^2 R_{\pi_0}^2 R_{M}^2 + c_0 h^2 d_u$, which means for our choice of $N = h^2 \sqrt{T} d_{\max}$ and $d_u \geq d_u$, our stipulation that $\sqrt{T} \geq 4 \cdot 1764 h^2 R_{\pi_0}^2 R_{M}^2 + c_0 h^2 d_u$ suffices. This ensures in turn that $\sqrt{T} R_{\text{nat}}^2 \log(1/\delta)$ dominates $R_{\pi_0}^2$, allowing us to drop the term from the final bound, ultimately yields

\[
\text{ControlReg}_{T}(\text{alg}; \Pi_*) \lesssim \log(1 + T) \frac{(d_w d_y + d_{\max} d_u) m h L^2 R_{\pi_0}^4 R_{\text{nat}} R_{M} \sqrt{T} \log(1/\delta)}{\alpha k^{1/2}}.
\]

Finally, using $d_{\max} = \max\{d_w, d_y + d_u\}$, we have $(d_w d_y + d_{\max} d_u) \leq d_w (d_y + d_u) + d_u (d_y + d_u) = (d_w + d_y)(d_y + d_u)$, concluding the bound.

\[\square\]

### E.5.1 Proof of Proposition E.5

Recall that $\tilde{f}_i, \tilde{F}_i$ losses from Definition 3.3b. In a fixed a comparator matrix $\tilde{M} \in \mathcal{M}$, where we recall $\mathcal{M} = M_{\text{dir}}(m, R_{M})$, where $R_{M} \geq 2 R_{\Pi}$ and $m \geq 2 m_* - 1 + h$. $\tilde{M}$ will be chosen towards
the proof in a careful way, and is not necessarily the best-in-hindsight parameter on the $M$ sequence. Our regret decomposition is as follows:

$$\text{ControlReg}_{T}(\text{alg; } \Pi_*) = \sum_{t=1}^{T} \ell_t(y_t, u_t) - \inf_{\pi \in \Pi_*} \sum_{t=1}^{T} \ell_t(y_t, u_t^*)$$

$$\leq \sum_{t=1}^{T} \ell_t(y_t, u_t) + \sum_{t=N+m+2h+1}^{T} \ell_t(y_t, u_t^*) - \sum_{t=N+m+2h+1}^{T} \ell_t(M | \omega_{1:t}^{\text{nat}})$$

$$+ \sum_{t=N+m+2h+1}^{T} \ell_t(M_t | \omega_{1:t}^{\text{nat}}) - \sum_{t=N+m+2h+1}^{T} \ell_t(M_t | \omega_{1:t}^{\text{nat}})$$

$$+ \max_{M \in M_*} \left| \sum_{t=1}^{T} \ell_t(M | \omega_{1:t}^{\text{nat}}) - \ell_t(y_t, u_t^M) \right| + \inf_{M \in M_*} \sum_{t=1}^{T} \ell_t(y_t, u_t^M) - \inf_{\pi \in \Pi_*} \sum_{t=1}^{T} \ell_t(y_t, u_t^*)$$

Again, let us work term-by-term, starting with the terms which are most similar to the terms that arise in the known system. Together with $R_M \geq R_*$, the last two terms can be bounded via Eq. (E.5) and Eq. (E.7)

$$(ii,b) + (v) \lesssim LTR^2 R^2 \psi_0 R^2 \left( \frac{\psi_0(m_*)}{R_*} + \frac{2\psi_0(h + 1)}{R_0} \right).$$

Moreover, similar arguments can be used to bound $(ii,a) \lesssim \text{RHS of Eq. (E.5)}$ (specifically, one replaces the appearance of $\omega_{1:t}^{\text{nat}}$ in the proof Simchowitz et al. [28, Lemma 5.3] with $\omega_t^{\text{nat}}$, and uses the bound $\|\omega_{1:t}^{\text{nat}}\| \leq 2T$ by Simchowitz et al. [28, Lemma 6.1]). Thus, we have so far

$$(ii,a) + (ii,b) + (v) \lesssim LTR^2 R^2 \psi_0 R^2 \left( \frac{\psi_0(m_*)}{R_*} + \frac{2\psi_0(h + 1)}{R_0} \right).$$

Next, analogously to Eq. (E.6), we recognize that

$$(iii) = \text{MemoryReg}_{T}(z_*)$$

for $z_* := c(M)$. Furthermore, from Simchowitz et al. [28, Lemma 6.3] and the definition of the term $\overline{R}_u$ in Simchowitz et al. [28, Lemma 6.1b], and with $N \geq m + 2h$, we have $(i) \lesssim LNR^2 R^2 \left( \frac{\psi_0(m_*)}{R_*} + \frac{2\psi_0(h + 1)}{R_0} \right),$ Thus, collecting what we have thus far, we obtain

$$\text{ControlReg}_{T}(\text{alg; } \Pi_*) \leq \text{MemoryReg}_{T}(z_*) + (iv)$$

$$+ O_{\text{const}}(N)(N + c)(R^2 \psi_0 + R^2 \psi_0 R^2),$$

where $O_{\text{const}}(1)$ supersedes a universal constant. It remains to account for the term $(iv)$. In particular, for $\psi_0(h + 1) \leq cR_0/T$ and $\psi_0(m_*) \leq cR_*/T$, the above simplifies to

$$\text{ControlReg}_{T}(\text{alg; } \Pi_*) \leq \text{MemoryReg}_{T}(z_*) + (iv)$$

$$+ O_{\text{const}}(L)(N + c)(R^2 \psi_0 + R^2 \psi_0 R^2),$$

(E.9)

Lemma E.10 (Slight Modification of Equation E.6 in Simchowitz et al. [28], altering numerical constants and allowing $c$ dependence). Suppose that $E_{\text{est}}$ holds, and that $\psi_0(h + 1) \leq cR_0/T.$
Further, assume $R_M \geq 2R_\ast$ and $m \geq 2m_\ast + h$. Then, there exists an $\overline{M} \in M$ such that, for all $\nu > 0$, we have

\[
\text{Term (iv)} \leq \mathcal{O}_{\text{const}(1)} \cdot LR_M^2 R_{\pi_0}^2 R_{\text{nat}}^2 T_{2, G}^2 \left( 1 + \frac{LmR_{\pi_0}^2}{\nu} \right) + \mathcal{O}_{\text{const}(c)} LR_M^2 R_{\pi_0}^2 R_{\text{nat}}((R_{\text{u,est}} + R_M R_{\text{nat}})R_{\pi_0} + m) + \nu \sum_{t=N+m+2h+1}^{T} \|u_j^*(M_j | \widehat{\omega}_{1,j}) - u_j^*(\overline{M} | \widehat{\omega}_{1,j}^\text{nat})\|^2_2.
\]

(E.10)

Absorbing the first $h$ terms in the sum into the term on the first line (using arguments as in Lemma E.6, this contributes $\mathcal{O}_{\text{const}}(R_M^2 R_{\text{nat}}^2 h) \leq \mathcal{O}_{\text{const}}(R_M^2 R_{\text{nat}}^2 m)$), and translating back to our $Y, z$-notation, we have that there exists a $z_\ast \in \mathcal{M}_\ast$ such that

\[
\text{Term (iv)} \leq \mathcal{O}_{\text{const}(1)} \cdot LR_M^2 R_{\pi_0}^2 R_{\text{nat}}^2 T_{2, G}^2 \left( 1 + \frac{LmR_{\pi_0}^2}{\nu} \right) + \mathcal{O}_{\text{const}(c)} R_M^2 R_{\pi_0}^2 R_{\text{nat}}((R_{\text{u,est}} + R_M R_{\text{nat}})R_{\pi_0} + m) + \nu \sum_{t=N+m+2h+1}^{T} \|Y_t(z_t - z_\ast)\|^2_2.
\]

Putting things together with Eq. (E.9), we have the bound that for $\psi_{\pi_0}(h+1) \leq R_{\pi_0}/T$ and $\psi_\ast \leq R_\ast/T$, we find

\[
\text{ControlReg}_T(\text{alg; } \Pi_\ast) \leq \text{MemoryReg}_T(z_\ast) + \nu \sum_{t=N+m+2h+1}^{T} \|Y_t(z_t - z_\ast)\|^2_2 + \mathcal{O}_{\text{const}(1)} \cdot LR_M^2 R_{\pi_0}^2 R_{\text{nat}}^2 T_{2, G}^2 \left( 1 + \frac{LmR_{\pi_0}^2}{\nu} \right) + \mathcal{O}_{\text{const}(c)} R_M^2 R_{\pi_0}^2 R_{\text{nat}}((R_{\text{u,est}} + R_M R_{\text{nat}})R_{\pi_0} + m) + c R_m^2 R_{\text{nat}}((R_{\text{u,est}} + R_M R_{\text{nat}})R_{\pi_0} + m)\]

Finally, since $N \geq m$, we bound

\[
LR_{\pi_0}^2 \cdot (N(R_{\text{u,est}}^2 + c R_M^2 R_{\text{nat}}^2) + R_M^2 R_{\text{nat}}((R_{\text{u,est}} + R_M R_{\text{nat}})R_{\pi_0} + m)) \leq \mathcal{O}_{\text{const}(L)} R_{\pi_0}^2 ((N + cm)(R_{\text{u,est}}^2 + R_M^2 R_{\text{nat}}^2) + cm R_{\text{u,est}} R_M^2 R_{\text{nat}}) \leq \mathcal{O}_{\text{const}(L)} R_{\pi_0}^2 ((N + cm)(R_{\text{u,est}}^2 + c R_M^2 R_{\text{nat}}^2),
\]

where the last step is by AM-GM. Thus,

\[
\text{ControlReg}_T(\text{alg; } \Pi_\ast) \leq \text{MemoryReg}_T(z_\ast) + \nu \sum_{t=N+m+2h+1}^{T} \|Y_t(z_t - z_\ast)\|^2_2 + \mathcal{O}_{\text{const}(L)} R_{\pi_0}^2 ((N + cm)(R_{\text{u,est}}^2 + c R_M^2 R_{\text{nat}}^2) \left( 1 + \frac{LmR_{\pi_0}^2}{\nu} \right),
\]

which after substituting in $R_{\text{u,est}}^2 \lesssim d_u + \log(1/\delta)$ (Lemma E.7), concludes the bound.

Part II

Appendices for OCOM

F Proof of Logarithmic Memory Regret (Theorem 2.1)

This section proves Theorem 2.1. We begin by bounding the standard (no-memory) regret in Appendix F.1, and then turn to agressing the contribution of memory in Appendix F.2. All omitted proofs, as well as the proof of Proposition F.8, are given in Appendix I in numerical order.

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F.1 Bounding the (unary) OCO Regret

As a warmup, we establish a bound on the no-memory regret for Semi-ONS. Throughout, recall the parameters from Definition 2.1, which we assume to be finite.

Proposition F.1. Suppose the the losses satisfy Assumption 1, and \( \kappa_h := \kappa(G) > 0 \). Then, for \( \eta \geq \frac{1}{\alpha} \), Semi-ONS(\( \lambda, \eta, C \)) fed pairs \((f_t, H_t)\) satisfies the following:

\[
\text{OCOReg}_{\eta} := \sum_{t=1}^{T} f_t(z_t) - \min_{z \in \mathcal{C}} \sum_{t=1}^{T} f_t(z) \leq \frac{\eta d L_{\text{eff}}^2}{2} \log \left( 1 + \frac{T R_{\text{eff}}^2}{\lambda} \right) + \frac{\lambda D^2}{2\eta}.
\]

This section proves the above proposition, and all omitted proofs in the proofs in this section are deferred to Appendix H.2. First, let us establish two simple structural properties of \( f_t \):

Lemma F.2. For all \( z \in \mathcal{C} \)

1. \( \nabla^2 f_t(z) \succeq \alpha H_t^\top H_t \)

2. There exists a function \( g_t(z) \in \mathbb{R}^{d_c} \) such that \( \nabla f_t(z) = H_t^\top g_t(z) \), and \( \|g_t(z)\| \leq L_{\text{eff}} \). In particular, \( \nabla f_t(z) \nabla f_t(z)^\top \succeq L_{\text{eff}}^2 H_t^\top H_t \).

Proof. Point (1): By the chain rule and the fact that \( \nabla^2 (z \mapsto v_t + H_t z) = 0 \), we have \( \nabla^2 f_t(z) = H_t^\top \nabla^2 \ell(v_t + H_t z) H_t \). Since \( \ell_t \) is strongly convex, \( \nabla^2 \ell(v_t + H_t z) \succeq \alpha I \). Point (2): Again invoking the chain rule, \( \nabla f_t(z) = H_t^\top g_t(z) \), where \( G_t(z) = \nabla \ell_t(v_t + H_t z) \). Since \( \ell_t \) is \( L \)-subquadratic, \( \|g_t(z)\| \leq L \max\{1, \|v_t + H_t z\|_2\} \leq L \max\{1, R_v + R_G \max_{t,z \in \mathcal{C}} \|Y_t z\|_2\} = L \max\{1, R_v + R_G R_{Y_t,C}\} = L_{\text{eff}} \). \( \square \)

Next, we establish a simple quadratic lower bound, which mirrors the basic inequality in analysis of standard ONS:

Lemma F.3 (Quadratic Lower Bound). For all \( z_1, z_2 \in \mathcal{C} \), we have

\[
f_t(z_1) \geq f_t(z_2) + \nabla f_t(z_2)^\top (z_1 - z_2) + \frac{\alpha}{2} \|H_t(z_1 - z_2)\|^2.
\]

Proof of Lemma F.3. By Taylor’s theorem, there exists a \( z_3 \) on the segment joining \( z_1 \) and \( z_2 \) for which \( f_t(z_1) \geq f_t(z_2) + \nabla f_t(z_2)^\top (z_1 - z_2) + \frac{1}{2} \|H_t(z_1 - z_2)\|^2 \). By Lemma F.2, \( \nabla^2 f_t(z_3) \succeq \alpha H_t^\top H_t \). \( \square \)

Remark F.1. Observe that Lemma F.3 uses the fact that \( \nabla^2 f_t(z) \succeq \alpha H_t^\top H_t \) globally. Lemma F.3 may be false if instead one replaces \( H_t \), \( H_t \) in the definition with \( \nabla^2 f_t(z_t) \), because the latter may be very large at a given point. This is why we use \( H_t^\top H_t \) in the definition of \( \Lambda_t \), as opposed to the full-Hessian. This is no longer an issue if one assumes that \( \nabla^2 f_t(z) \preceq \beta I \) globally, in which case one pays for the conditioning \( \beta/\alpha \).

Remark F.2 (Comparison to Canonical Online Newton). Let us compare the above to the canonical Online Newton Step algorithm [19]. This algorithm applies to exp-concave functions, which satisfy the bound \( \nabla^2 f \succeq \alpha \nabla f(\nabla f)^\top \) globally. For these functions, the analogue of Lemma F.3, with \( f_t(z_1) \geq f_t(z_2) + \nabla f_t(z_2)^\top (z_1 - z_2) + \frac{\alpha}{2} \|\nabla f_t(z_1 - z_2)\|^2 \) does in fact hold, albeit due to a somewhat trickier argument [18, Lemma 4.3]. This enables the algorithm to use the preconditioner \( \Lambda_t = \lambda I + \sum_{s=1}^{t} \nabla f(\nabla f)^\top \). Note however that this yields a smaller pre-conditioner \( \Lambda_t \), for which Proposition F.8 may fail.

As a consequence, we obtain intermediate regret bound for Semi-ONS, which mirrors the standard analysis of online Newton step (e.g. Hazan [18, Chapter 4]).

Lemma F.4 (Online Semi-Newton Step Regret). Suppose that \( \eta \geq \frac{1}{\alpha} \). Then,

\[
\sum_{t=1}^{T} f_t(z_t) - \inf_{z \in \mathcal{C}} \sum_{t=1}^{T} f_t(z) \leq \frac{\lambda D^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \nabla f_t^\top \Lambda_t^{-1} \nabla f_t,
\]

Lastly, we recall a standard log-det potential lemma. To facilitate reuse, the lemma is stated for a slightly more general sequence of matrices \( \Lambda_t \):
Lemma F.5 (Log-det potential). Suppose that \( \bar{\Lambda}_t \geq c \sum_{t=1}^{T} H_t^T H_t + \lambda_0 \). Then,
\[
\sum_{t=1}^{T} \text{tr}(H_t \bar{\Lambda}_t^{-1} H_t^T) \leq \frac{d}{c} \log \left( 1 + \frac{cTR_H^2}{\lambda_0} \right)
\]

Proof. Define \( \bar{\Lambda}_t = \sum_{t=1}^{T} H_t^T H_t + \frac{\lambda_0}{c} \). Then, \( \sum_{t=1}^{T} \text{tr}(H_t \bar{\Lambda}_t^{-1} H_t^T) \leq \frac{1}{c} \sum_{t=1}^{T} \text{tr}(H_t \bar{\Lambda}_t^{-1} H_t^T) \). The result now follows from the standard log-det potential lemma (see e.g. Hazan [18, Proof of Theorem 4.4]).

Proof of Proposition F.1. Begin with the unary bound:
\[ \text{OCoReg}_T := \sum_{t=1}^{T} f_t(z_t) - \inf_{z \in \mathcal{C}} \sum_{t=1}^{T} f_t(z) \leq \frac{\lambda D^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \nabla_t^T \Lambda_t^{-1} \nabla_t. \]

From Lemma F.2, we have \( \nabla_t \nabla_t^T \leq L_{\text{eff}}^2 H_t^T H_t \). Since \( \Lambda_t > 0 \), this implies that \( \nabla_t^T \Lambda_t^{-1} \nabla_t = (\nabla_t, \Lambda_t^{-1}) \leq L_{\text{eff}}^2 (H_t^T H_t, \Lambda_t^{-1}) = L_{\text{eff}}^2 \text{tr}(H_t \Lambda_t^{-1} H_t^T) \). Thus, by Lemma F.5,
\[
\frac{\eta}{2} \sum_{t=1}^{T} \nabla_t^T \Lambda_t^{-1} \nabla_t \leq \frac{\eta L_{\text{eff}}^2}{2} \sum_{t=1}^{T} \text{tr}(H_t \Lambda_t^{-1} H_t^T) \leq \frac{d\eta L_{\text{eff}}^2}{2} \log \left( 1 + \frac{TR_H^2}{\lambda} \right). \quad (F.1)
\]

F.2 Memory Regret for Known System

In this section, we address movement costs, thereby proving Theorem 2.1. In what follows, we make the simplifying assumption that \( z_s = z_1 \) for \( s \leq 1 \). We will remove this assumption at the end of the proof. Our goal is to bound:
\[
\text{MemoryReg}_T := \sum_{t=1}^{T} F_t(z_t, \ldots, z_{t-h}) - \min_{z \in \mathcal{C}} f_t(z)
\]
\[
= \sum_{t=1}^{T} F_t(z_t, \ldots, z_{t-h}) - f_t(z_t) + \sum_{t=1}^{T} f_t(z_t) - \min_{z \in \mathcal{C}} \sum_{t=1}^{T} f_t(z). \quad (\text{MoveDiff}_T)
\]
\[
\text{OCoReg}_T \quad (\text{MoveDiff}_T)
\]

The second term is bounded by direct application of Proposition F.1. For the first term, we begin with the following lemma, which shows that the relevant movement cost is only along the \( Y_{t-i} \) directions:

Lemma F.6 (Movement Cost). For all \( t \geq 1 \), we have
\[
|F_t(z_t, \ldots, z_{t-h}) - f_t(z_t)| \leq L_{\text{eff}} R_G \sum_{i=1}^{h} \| Y_{t-i}(z_t - z_{t-i}) \|_2.
\]

Therefore, by the triangle inequality, rearranging summations, and the assumption \( z_s = z_1 \) for \( s \leq 1 \),
\[
\text{MoveDiff}_T \leq hL_{\text{eff}} R_G \sum_{s=1-h}^{h-1} \sum_{t=1}^{T} \| Y_s(z_{s+i+1} - z_{s+i}) \|_2 \cdot \| Y_{t-i} \|_2.
\]

Next, let us develop a bound on \( \| Y_s(z_{t+1} - z_t) \|_2 \):

Lemma F.7. Adopt the convention \( \Lambda_s = \Lambda_1 \) for \( s \leq 0 \). Further, consider \( s \leq t \), with \( t \geq 1 \) and \( s \) possibly negative. Then, \( \| Y_s(z_{t+1} - z_t) \|_2 \leq \eta L_{\text{eff}} \text{tr}(Y_s \Lambda_s^{-1} Y_s)^{1/2} \text{tr}(H_t^T \Lambda_t^{-1} H_t)^{1/2} \). Therefore,
\[
\text{MoveDiff}_T \leq \eta h L_{\text{eff}} R_G \cdot \sum_{t=1-h}^{T} \text{tr}(Y_t \Lambda_t^{-1} Y_t) \cdot \sum_{t=1}^{T} \text{tr}(\nabla_t^T \Lambda_t^{-1} \nabla_t).
\]
Now, we already bounded the sum of the terms $\text{tr}(\nabla_t^\top A_t^{-1} \nabla_t)$ in Eq. (F.1):
\[
\sum_{s=1}^{T} \text{tr}(\nabla_t A_t^{-1} \nabla_t) \leq d \mathcal{L}_{\text{eff}}^2 \log(1 + \frac{TR_H^2}{\lambda}).
\] (F.2)

The main technical challenge is to reason about the sum $\text{tr}(Y_t A_t^{-1} Y_t)$. We bound this quantity using the following proposition:

**Proposition F.8.** Suppose that $\kappa(G) > 0$, and define $c_{\psi;t} := 1 + \frac{t \psi_G(h+1)^2}{\kappa R_H^2}$. Then, for any $Y_{1-h}, Y_{2-h}, \ldots, Y_t$, the matrices $H_s = \sum_{i=0}^{[h]} G[i] Y_{s-i}$ satisfy
\[
\sum_{s=1}^{t} H_s^\top H_s \geq \frac{\kappa(G)}{2} \cdot \left( \sum_{s=1-h}^{t} Y_s^\top Y_s \right) - 5h R_H^2 c_{\psi;t} I.
\]

The above proposition is proved in Appendix H.1. Under the assumption of the theorem, we have $c_{\psi;t} \leq 1$, so $5h R_H^2 c_{\psi;t} \leq 5h R_H^2$. Thus, for $t = 6h R_H^2$, we have $\Lambda_t \geq \frac{h}{2} I + \kappa \sum_{s=1-h}^{t} Y_s^\top Y_s$. Note that this holds even for $t \leq 0$, with the above convention $\Lambda_t = \Lambda_1$ for negative $t$. Thus, Lemma F.5 and the simplifications $R_Y \leq R_H, \kappa \leq 1$ gives
\[
\sum_{s=1}^{T} \text{tr}(Y_t A_t^{-1} Y_t) \leq \frac{2d}{\kappa} \log \left( 1 + \frac{6\kappa R_H^2}{2\lambda} \right) \leq \frac{2d}{\kappa} \log \left( 1 + \frac{3R_H^2}{\lambda} \right).
\]

(F.3)

We can now complete the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Combining Lemma F.7, Eqs. (F.2) and (F.3), and finally the unary regret bound from Proposition F.1:

\[
\text{MoveDiff}_T + \text{OcoReg}_T \\
\leq \text{OcoReg}_T + \frac{\eta h^2 \mathcal{L}_{\text{eff}}^2 R_G}{\kappa} \cdot \sqrt{\sum_{t=1}^{T} \text{tr}(Y_t A_t^{-1} Y_t)} - \sqrt{\sum_{t=1}^{T} \text{tr}(H_t^\top \Lambda_t^{-1} H_t)}.
\]

\[
\leq \text{OcoReg}_T + \frac{\sqrt{2}}{\kappa} d h^2 \mathcal{L}_{\text{eff}}^2 R_G \log(1 + \frac{3R_H^2}{\lambda})/d.
\]

Finally, since $\lambda = 6h R_H^2$, $\log(1 + \frac{3R_H^2}{\lambda}) \leq \log(1 + T)$. Thus, combining with the unary regret bound from Proposition F.1,

\[
\text{MoveDiff}_T + \text{OcoReg}_T \leq \frac{\lambda D^2}{\eta} + \frac{\eta h^2 \mathcal{L}_{\text{eff}}^2 d}{2} + h^2 R_G \sqrt{\frac{2}{\kappa}} \log(1 + T).
\]

To conclude, we use $\eta = \frac{1}{\alpha}$ so that with $\lambda = 6h R_H^2$, yields $\frac{\lambda D^2 R_G^2}{\eta} = 3\alpha R_G^2 D^2$. Moreover, noting $h^2 R_G \sqrt{\frac{1}{\kappa}} \geq 17$, we arrive at

\[
\text{MemoryReg}_T = \text{MoveDiff}_T + \text{OcoReg}_T \leq 3\alpha D R_G^2 + \frac{2dh^2 \mathcal{L}_{\text{eff}}^2 R_G}{\alpha R_G^2} \log(1 + T).
\]

(F.4)

Recall that the above bound follows under the assumption that $z_s = z_1$ for $s \leq 1$. Let us remove this assumption presently. Observe that the iterates $z_s$ for $s < 1$ do not alter the trajectory of future iterates $z_t$ for $t \geq 1$; they only appear in the memory regret bound via the with memory loss $L_1(z_{t-1}, z_s)$. Thus, introducing $\tilde{z}_t := \mathbb{1}(t \geq 1)z_t + \mathbb{1}(t < 1)z_1$, imposing the above assumption ($z_s = z_1$ for $s \leq 1$) comes at the expense of regret at most
\[
\sum_{t=1}^{h} |F_t(\tilde{z}_{t-1}) - F(\tilde{z}_{t-1})| = \sum_{t=1}^{h} |F_t(z_{t-1}) - F(z_{t-1})|.
\]

With routine computations and the assumption that $L \geq 1$, each term in the above can be bounded by $L_{\text{eff}} \sum_{s=0}^{[h]} G[s] \|Y_{s-1} (z_{s-1} - z_1)\|_2 \leq \mathcal{L}_{\text{eff}} R_G R_Y \leq \mathcal{L}_{\text{eff}}^2 R_G$. This contributes a total addition of cost of $h L_{\text{eff}}^2$, which can be absorbed into the right-most term on Eq. (F.4) at the expense of replacing the constant 2 with a factor of 3.\[\square\]
G Regret with Quadratic Error Sensitivity (Theorem 2.2)

This section proves Theorem 2.2 and its generalizations. It is organized as follows:

- In Appendix G.1, we two bounds which make explicit a certain negative regret term. Theorem 2.2a gives the generalization of Theorem 2.2 in the $c_G^2 \geq \sqrt{T}$ regime (and allows for slight mis-specification of $\lambda$), and Theorem G.1 proves a guarantee that degrades as $(Tc_G)^{2/3}$ for small $c_G$. We prove Theorem 2.2a from Theorem G.1 in Appendix G.1.1.
- The remainder of the section is dedicated to the proof of Theorem G.1. This begins with Appendix G.2, which introduces relevant preliminaries.
- Appendix G.3 provides a careful analysis of initial regret terms, and controlling the contribution of errors introduced by using the $f_t$ sequence rather than $f_t$.
- Appendix G.4 details our careful “blocking argument”, which we use to offset the errors the terms $\sum_{t=1}^T \|Y_t(z_t - z_*)\|$ from the gradients by a negative terms $\sum_{t=1}^T \|X_t(z_t - z_*)\|^2$ that arise in the regret analysis.
- Appendix G.5 concludes the proof of Theorem G.1, bounding first the movement cost and then tuning relevant parameters in the analysis.

All ommitted proofs are provided in Appendix I, organized into subsections and presented in numerical order.

G.1 Bounds for Unknown Systems with Negative Regret

Here, we provide bounds which explicitly account for an appropriate negative regret term, scaling with $\sum_{t=1}^T \|Y_t(z_t - z_*)\|^2$. Specifically, for any fixed comparator $z_* \in C$, our goal is to bound

$$\text{MemoryReg}_T(\nu; z_*) := \sum_{t=1}^T F_t(z_t; t - h) - f_t(z_*) + \nu \sum_{t=1}^T \|Y_t(z_t - z_*)\|^2,$$

(G.1)

which gives a negative regret term by re-arranging $\nu \sum_{t=1}^T \|Y_t(z_t - z_*)\|^2$ to the right-hand side of the above display. Note that we prove this bound for any fixed comparator $z_*$, not just the “best-in-hindsight” comparator. Moreover, proving this bound for the best-in-hindsight comparator does not imply the bound for all $z_* \in C$, because the terms $\delta_t$ in the negative-regret term differ as a function of $z_*$.

To state our bound on $\text{MemoryReg}_T$, we recall the relevant parameter bounds:

Definition 2.1 (Bounds on Relevant Parameters). We assume $C$ contains the origin. Further, we define the diameter $D := \max\{\|z - z'\| : z, z' \in C\}$, $Y$-radius $R_Y := \max_{z \in C} \|Y_t\|_{\text{op}}$, and $R_Y,C := \max_{z \in C} \max_{\lambda \in \mathbb{R}} \|Y_t z\|$; in the exact setting, we define the radii $R_v := \max_{z \in C} \max_{\lambda \in \mathbb{R}} \|Y_t\|_{\text{op}}$ and $R_G := \max_{z \in C} \max_{\lambda \in \mathbb{R}} \|Y_t\|_{\text{op}}$. In the approximate setting, $R_v := \max_{z \in C} \max_{\lambda \in \mathbb{R}} \|Y_t\|_{\text{op}}$, $R_G := \max_{z \in C} \max_{\lambda \in \mathbb{R}} \|Y_t\|_{\text{op}}$. For settings, we define the $H$-radius $R_H := R_G R_Y$, and define the effective Lipschitz constant $L_{\text{eff}} := L \max\{1, R_v + R_G R_Y\}$.

Our main result in this section is as follows. We also allow $\lambda$ to be slightly under-specified. This show’s relative insensitivity to the selection of $\lambda$, and is also useful when porting the bound over to the control setting:

Theorem 2.2a. Consider the setting of Theorem 2.2, but where instead $\lambda \in (c_\lambda, 1] \cdot (Tc_G^2 + hR_G^2)$ for $c_\lambda \in (0, 1]$. Equivalently, consider the setting of Theorem G.1 below, but with the additional conditions $c_G \geq \sqrt{T}$ and $\beta = L$. Then for any $z_* \in C$,

$$c_\lambda \text{MemoryReg}_T(\nu_*; z_*) \leq \log(1 + \frac{T}{c_\lambda}) \left( \frac{C_1}{\alpha R^2} + C_2 \right) \left( Tc_G^2 + h^2 (R_G^2 + R_Y) \right),$$

where $C_1 := (1 + R_Y) R_G (h + d) L_{\text{eff}}^2$, $C_2 := (L^2 c_G^2 / \alpha + \alpha D^2)$, and $\nu_* := \frac{\alpha \sqrt{T}}{48 (1 + R_Y)}$.

Theorem 2.2 is an immediate consequence of Theorem 2.2a. We prove the above guarantee from a more statement, which allows for $c_G^2 \leq \sqrt{T}$ as well.
Granular Guarantee for Semi-ONS with errors

To state our generic guarantee, we specify the following constants:

**Definition G.1** (Constants for Unknown $G$ Regret Analysis). We define the constants

\[ C_{\text{mid}} := (1 + \frac{\alpha^2}{T}) (1 + R_Y) h L_{\text{eff}}^2 + \beta^2 \sqrt{\kappa} c_v^2 + \alpha^2 \sqrt{\kappa} D^2 \]  
\[ C_{\text{hi}} := (1 + R_Y) L_{\text{eff}} R_G^2 R_Y c (h + d) + \alpha D^2 \]  
\[ C_{\text{low}} := (1 + R_Y)^2 R_G h^2 \cdot d L_{\text{eff}}^2 . \]

Finally, we define a logarithmic factor

\[ \mathcal{Q} := \log(1 + R_H^2 T/L), \quad \text{with } \mathcal{Q} \leq \log(1 + T) \text{ for } \lambda \geq R_H^2. \]  

Our more granular result is the following:

**Theorem G.1** (Granular Regret Guarantee for Semi-ONS on an unknown system). Consider running Semi-ONS on the empirical loss sequence \((\tilde{f}_t, \tilde{H}_t)\). Suppose that

- The losses \(\ell_t\) are $L$-subquadratic and \(\alpha\)-strongly convex for \(L \geq 1 \lor \alpha\) (Assumption 1), and are \(\beta\) smooth (\(\nabla^2 \ell_t \leq \beta I\)).
- Suppose that \(\|G - G_\ast\|_{\ell_{i,\text{op}}} \leq \epsilon_G, \tilde{G}^{[i]} = 0\) for \(i > h\), and \(\max_{t \geq 1} \|v_t - \tilde{v}_t\|_2 \leq c_v \epsilon_G\) for some constant \(c_v \geq 0\).
- The step size is \(\eta = 3/\alpha\), and \(\lambda\) lies in \(\lambda \in [c_\lambda, 1]\) \((T \epsilon_G^2 + (T \epsilon_G)^{2/3} + h R_G^2)\) for some \(c_\lambda \in (0, 1]\).
- All relevant quantities are bounded as in Definition 2.1.

Then, the memory regret on the true loss sequence \((f_t, H_t)\) is bounded by

\[ c_\lambda \text{MemoryReg}_T (\nu_\ast; z_\ast) \lesssim C_{\text{hi}} (T \epsilon_G) \frac{2/3}{\alpha \sqrt{\kappa}} + \frac{C_{\text{mid}} T \epsilon_G^2}{\alpha \sqrt{\kappa}} + \frac{C_{\text{low}} \mathcal{Q}}{\alpha \sqrt{\kappa}} + \alpha h R_G^2 D^2. \]

Observe that, when \(\epsilon_G^2 \geq \sqrt{T}\), the dominating term is \(T \epsilon_G^2\). However, for \(\epsilon \leq \sqrt{T}\), the term \((T \epsilon_G)^{2/3}\) dominates.

**G.1.1 Proof of Theorem 2.2a from Theorem G.1**

Theorem 2.2a follows from the granular Theorem G.1 as a consequence of the following tedious simplifications. Recall that Theorem 2.2 adds the assumptions that \(\epsilon_G^2 \geq \sqrt{T}\), and \(\beta = L\). This enables the following simplifications. First, since \((T \epsilon_G^{2/3}) \geq 1\) we can take \(\nu_\ast = \frac{\alpha \sqrt{\kappa}}{48(1 + R_Y)}\), which is precisely the value of \(\nu\) used in the theorem. Second, we have \((T \epsilon_G)^{2/3} / T \epsilon_G^2 = 1 / (T \epsilon_G^{1/3}) \leq 1\). This means that the choice of \(\lambda = c_\lambda (T \epsilon_G^2 + h R_G^2)\) is valid for Theorem G.1, up to rescaling \(c_\lambda\) by a factor of 2. Thus, we have

\[ c_\lambda \text{MemoryReg}_T \left( \frac{\alpha \sqrt{\kappa}}{48(1 + R_Y)}; z_\ast \right) \lesssim C_{\text{hi}} (T \epsilon_G) \frac{2/3}{\alpha \sqrt{\kappa}} + \frac{C_{\text{mid}} (T \epsilon_G^2)}{\alpha \sqrt{\kappa}} + \frac{C_{\text{low}} \mathcal{Q}}{\alpha \sqrt{\kappa}} + \alpha h R_G^2 D^2 \]

\[ \lesssim \frac{\mathcal{Q}}{\alpha \sqrt{\kappa}} \left( (T \epsilon_G^2) (C_{\text{hi}} \alpha \sqrt{\kappa} + C_{\text{mid}}) + C_{\text{low}} + \alpha^2 \sqrt{\kappa} h R_G^2 D^2 \right). \]

First, let us simplify \(C_{\text{hi}} \alpha \sqrt{\kappa} + C_{\text{mid}}\). Using the simplifying condition \(\beta = L\), and using \(R_G R_{Y,c} \leq L_{\text{eff}}\) (again, \(L \geq 1\)), we have

\[ C_{\text{hi}} \alpha \sqrt{\kappa} + C_{\text{mid}} \lesssim (1 + R_Y) (h L_{\text{eff}}^2 + L_{\text{eff}} R_G R_{Y,c} (h + d) + L^2 \sqrt{\kappa} c_v^2 + \alpha^2 \sqrt{\kappa} D^2 \]

\[ \lesssim (1 + R_Y) R_G (h + d) L_{\text{eff}}^2 + L^2 \sqrt{\kappa} c_v^2 + \alpha^2 \sqrt{\kappa} D^2. \]
Hence,

\[(C_{hi}\alpha\sqrt{\kappa} + C_{mid})Tc_G^2 + C_{low} + \alpha^2 \sqrt{\kappa} h R_G^2 D^2 \leq (1 + R_Y) R_G (h + d) L^2_{\text{eff}} (Tc_G + (1 + R_Y) h^2) + (L^2 \sqrt{\kappa} c_G^2 + \alpha^2 \sqrt{\kappa} D^2) (Tc_G^2 + h R_G^2) \leq C_1 (Tc_G^2 + (1 + R_Y) h^2) + \alpha \sqrt{\kappa} C_2 (Tc_G^2 + h R_G^2) \leq (C_1 \alpha \sqrt{\kappa} C_2) (Tc_G^2 + (1 + R_Y) h^2 + h R_G^2) \leq (C_1 \alpha \sqrt{\kappa} C_2) (Tc_G^2 + h^2 (R_G^2 + R_Y))\]

for \( C_1 := (1 + R_Y) R_G (h + d) L^2_{\text{eff}} \) and \( C_2 := (L^2 \alpha^{-1} c_G^2 + \alpha D^2) \). Thus we conclude that

\[\text{MemoryReg}_T \left( \frac{\alpha \sqrt{\kappa}}{48(1 + R_Y)} ; z_* \right) \leq c_\lambda^{-1} \log(1 + T) \left( \frac{C_1}{\alpha \kappa^{1/2}} + C_2 \right) \left( Tc_G^2 + h^2 (R_G^2 + R_Y) \right),\]

as needed.

G.2 Preliminaries for Proof of Theorem G.1

**Notation:** Let us begin by introducing relevant notation. Set \( \nabla_t = \nabla f_t(z_t) \) to denote the gradients of the true counterfactual stationary counterfactual costs \( f_t \), and let \( \hat{\nabla}_t := \nabla f_t(z_t) \) denote the gradient of their approximations. Analogously, define the matrices

\[\hat{\Lambda}_t = \lambda I + \sum_{t=1}^{T} \hat{H}^\top_t \hat{H}_t, \quad \Lambda_t = \lambda I + \sum_{t=1}^{T} H^\top_t H_t\]

For \( t \leq 1 \), we will use the conventions \( \Lambda_t = \Lambda_1 \) and \( \hat{\Lambda}_t = \hat{\Lambda}_1 \). Throughout, we fix an arbitrary comparator \( z_* \in \mathcal{C} \), and further introduce the notation

\[\delta_t := z_t - z_*, \quad \text{err}_t = \hat{\nabla}_t - \nabla_t\]

to denote the difference of \( z_t \) from the comparator, and difference between gradients, respectively.

We recall that \( \lambda, \eta \) are the algorithm parameters dictating the magnitude of the regularizer in \( \Lambda_t \), and step size, respectively. We will also introduce a “blocking parameter” \( \tau \), whose purposes is described at length in Appendix G.4. For simplicity, most of the proof will focuses on the unary regret analogue of MemoryReg$_T$, defined as follows:

\[\text{OcoReg}_T(\nu; z_*) := \sum_{t=1}^{T} f_t(z_t) - f_t(z_*) + \nu \sum_{t=1}^{T} \| Y_t \delta_t \|^2, \quad \delta_t := z_t - z_*, \quad (G.7)\]

We extend to memory regret in Appendix G.5. denote a logarithmic factor that will appear throughout.

**Reduction \( z_s = z_1 \) for \( s \leq 1 \):** As in the proof of Theorem 2.1 in Appendix F.2, we can assume that \( z_s = z_1 \), at the expense of an additional factor of \( h L^2_{\text{eff}} \) in the regret. This term is dominated by the factor of \( C_{low} \mathcal{L} \) in Theorem G.1, and can thus be disregarded in the following argument.

G.3 Bounding Regret in Terms of Error

We begin with the following basic regret bound, controls the excess regret of using inexact gradients compared to standard bounds from online Newton.

**Lemma G.1.** Let \( \lambda \geq 1 \). Then regret on measured on the \( f_{t}^{\text{pred}} \) sequence is bounded by

\[\sum_{t=1}^{T} f_t(z_t) - f_t(z_*) \leq \sum_{t=1}^{T} \text{err}_t^\top \delta_t + \frac{1}{2\eta} \sum_{t=1}^{T} (\| \hat{H}_t \delta_t \|^2 - \eta \alpha \| H_t \delta_t \|^2) + \hat{\text{Reg}}_T, \]

where \( \hat{\text{Reg}}_T := \frac{\nu \delta_1^\top \mathcal{L}^2}{2} + \frac{\lambda D^2}{2\eta} \) arises from the regret bound in Proposition F.1, and we recall \( \mathcal{L} := \log(e + TR^2_H) \).
Next, let us turn to bounding the mismatch arising from the terms $\sum_{t=1}^T (\|\hat{H}_t \delta_t\|^2 - \eta \alpha \|H_t \delta_t\|^2)$:

**Lemma G.2.** For $\eta \geq \frac{3}{\alpha}$, we have $\|\hat{H}_t \delta_t\|^2 - \eta \alpha \|H_t \delta_t\|^2 \leq -\|H_t \delta_t\|^2 + 8R_{Y,C}^2c_G^2$. Hence, we have the regret bound:

$$\sum_{t=1}^T f_t(z_t) - f_t(z_*) \leq \sum_{t=1}^T \text{err}_t \delta_t - \frac{1}{2\eta} \sum_{t=1}^T \|H_t \delta_t\|^2 + \frac{4}{\eta} T R_{Y,C}^2c_G^2 + \text{Reg}_T.$$  

**G.3.1 Controlling the error contributions**

Next, we turn to bounding the contribution of the error in estimating the gradient:

**Lemma G.3.** There exists $g_{1,t}$ and $g_{2,t}$ with $\|g_{1,t}\|_2 \leq L_{\text{eff}}$ and $\|g_{2,t}\| \leq \beta \epsilon_G(c_v + 2R_{Y,C})$ such that

$$\text{err}_t = (\hat{H}_t - H_t)^T g_{1,t} + H_t^T g_{2,t}.$$  

By levering the specific structure of $\text{err}_t$, we obtain:

**Lemma G.4.** For $\eta \geq \frac{3}{\alpha}$, the following regret bound holds for all $z_* \in \mathcal{C}$ and all $\nu > 0$:

$$\sum_{t=1}^T f_t(z_t) - f_t(z_*) \leq \frac{\nu}{4\eta} \sum_{t=1}^T \left( \frac{\nu}{h+1} \sum_{i=0}^h \|\hat{Y}_{t-i} \delta_t\|^2 - \|H_t \delta_t\|^2 \right) + T \epsilon_G \cdot \text{ERR}(\nu) + \text{Reg}_T,$$  

where $\text{ERR}(\nu) := \left( \frac{\eta(h+1)L_{\text{eff}}^2}{\nu} + \eta \beta^2 \left( c_v + 2R_{Y,C} \right)^2 + \frac{4R_{Y,C}}{\eta} \right)$.  

As a consequence, we have

$$\text{O(c\text{Reg}_T)} \left( \frac{\nu}{4\eta} z_* \right) \leq \frac{\nu}{4\eta} \sum_{t=1}^T \left( \frac{\nu}{h+1} \sum_{i=0}^h \|\hat{Y}_{t-i} \delta_t\|^2 - \|H_t \delta_t\|^2 \right) + T \epsilon_G^2 \text{ERR}(\nu) + \text{Reg}_T,$$  

**G.4 The ‘blocking argument’**

A this stage of the proof, the main challenge is to show that for some small constant $\nu$, the terms $\|\hat{Y}_{t-i} \delta_t\|^2$ in Eq. (G.9) are offset by $\|H_t \delta_t\|^2$ on aggregate. We do this by dividing times into “blocks” of size $\tau = \Theta(\sqrt{T})$, centering at the terms $\delta_t$ at times $t = k_j + 1$, for indices $j$ defined below. We define $j_{\text{max}} := \lceil T/\tau \rceil$ as the number of blocks. We then argue that, within any block

$$\sum_{t \in \text{block } j} \|H_t \delta_t\|^2 \gtrsim \sum_{t \in \text{block } j} \frac{1}{\nu} \|\hat{Y}_{t-i} \delta_t\|^2 + \mathcal{O}(1)$$  

for appropriate $\nu$ and block size $\tau$. The reason we should expect an inequality of the above form to holds is that, from adapting Proposition F.8, we have the inequality that

$$\sum_{t \in \text{block } j} H_t H_t^\top \gtrsim \sum_{t \in \text{block } j} Y_t Y_t^\top - \mathcal{O}(1) \cdot I,$$  

However, Eq. (G.11) does not directly imply a bound of the form Eq. (G.10), because the vectors $\delta_t$ differ for each $t$. Instead, we ‘re-center’ the $\delta_t$ terms in the sum $\delta_t = \delta_{k_j + 1}$, and argue

$$\sum_{t \in \text{block } j} \|H_t \delta_{k_j + 1}\|^2 \approx \sum_{i=0}^h \|\hat{Y}_{t-i} \delta_{k_j + 1}\|^2 - \mathcal{O}(1).$$  

The above bound can be established from an estimate of the form Eq. (G.11). Summing this up across all $j_{\text{max}}$ blocks, we see that the negative regret from the terms $\|H_t \delta_{k_j + 1}\|^2$ cancels the regret from the terms $\|\hat{Y}_{t-i} \delta_{k_j + 1}\|^2$. Accounting for all $j_{\text{max}} = \Theta(T/\tau)$ blocksgives

$$\sum_{j=1}^{j_{\text{max}}} \sum_{t \in \text{block } j} \|H_t \delta_{k_j + 1}\|^2 \approx \sum_{j=1}^{j_{\text{max}}} \sum_{i=0}^h \|\hat{Y}_{t-i} \delta_{k_j + 1}\|^2 - \mathcal{O}(T/\tau).$$  

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incurs an additive factor of $T/\tau$, favoring larger block sizes $\tau$.

But, we must also argue that not too much is lost by approximating the statement Eq. (G.10) with the centered analogue Eq. (G.13). The cost of recentering will ultimatex as $O(\tau)$, so trading off $\tau$ with the bound of yields $\sqrt{T}$ regret in the final bound.

Interestingly, the cost of recentering is intimately tied to bounding the movement of the iterates $z_t$. Thus, we find that the same properties that allow Semi-ONS to attain logarithmic regret for the known system case are also indispensable in achieving low sensitivity to error in the unknown system case.

### G.4.1 Formalizing the blocking argument

Formally, the cost of the above re-centering argument is captured by the following lemma:

**Lemma G.5 (Blocking Argument).** Given parameter $\tau \in \mathbb{N}$, and introduce the $k_j = \tau (j - 1)$, and $j_{\max} := \lceil T/\tau \rceil$. Then, with the understanding that $z_s = 0$ for $s \leq 1$, the following holds for all $i \in [h]$,

$$
\begin{align*}
&\sum_{t=1}^{T} \| Y_{t-1} \delta_t \|_2^2 \leq 4\tau R_{Y,C} + \sum_{j=1}^{j_{\max}} \sum_{s=1}^{\tau} \| Y_{k_j+s-1} \delta_{k_j+1} \|_2^2 + 4R_{Y,C} \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| Y_{t-s} (z_{t-s} - z_{t-s-1}) \|_2, \\
&\sum_{t=1}^{T} \| h_{t-h} \delta_t \|_2^2 \geq \sum_{j=1}^{j_{\max}} \sum_{s=1}^{\tau} \| h_{k_j+s} \delta_{k_j+1} \|_2^2 - 4R_{Y,C} R_G \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| h_t (z_{t-s} - z_{t-s-1}) \|_2.
\end{align*}
$$

Notice that, while the left-hand side depends on $\delta_t$, the right hand side is ‘centered’ at $\delta_{k_j+1}$ for $j \in [j_{\max}]$, at the expense of movement penalties on $z_{t-s} - z_{t-s-1}$. Let us re-write the above bound to give a useful regret decomposition. We introduce bounding terms $\text{Reg}_{Y\text{-move},i}$ and $\text{Reg}_{H\text{-move}}$ for the movement costs above associated with the centering argument, and $\text{Reg}_{\text{cancel}}$ associated with the offsetting argument described above. Formally,

$$
\begin{align*}
\text{Reg}_{Y\text{-move},i} &:= \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| Y_{t-s} (z_{t-s} - z_{t-s-1}) \|_2, \\
\text{Reg}_{H\text{-move}} &:= \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| H_t (z_{t-s} - z_{t-s-1}) \|_2, \\
\text{Reg}_{\text{cancel}} &:= \sum_{j=1}^{j_{\max}} \sum_{s=1}^{\tau} \left( \sum_{i=0}^{h} \left( \frac{1}{h+1} + I_{i=0} \right) \| Y_{k_j+s-i} \delta_{k_j+1} \|_2^2 - \| H_{k_j+s} \delta_{k_j+1} \|_2^2 \right).
\end{align*}
$$

Then, from Lemma G.5, the upper bound on $\overline{O\text{COReg}}_T$ in Eq. (G.9) can be expressed as

$$
\overline{O\text{COReg}}_T \left( \frac{\nu}{4\eta}; z^* \right) \leq \frac{1}{4\eta} \text{Reg}_{\text{block}} + T \epsilon_G^2 \text{ERR}(\nu) + \overline{\text{Reg}}_T,
$$

where we define and bound

$$
\text{Reg}_{\text{block}} := \sum_{t=1}^{T} \left( \nu \| Y_t \delta_t \| + \frac{\nu}{h+1} \sum_{i=0}^{h} \| Y_{t-i} \delta_t \|^2 - \| H_t \delta_t \|^2 \right) \leq 8\tau \cdot \nu R_{Y,C} + 8\nu R_{Y,C} \left( \max_{i \in [h]} \text{Reg}_{Y\text{-move},i} \right) + 4R_{Y,C} R_G \cdot \text{Reg}_{H\text{-move}} + \text{Reg}_{\text{cancel}}.
$$

Thus, we shall conclude our argument by developing bounds on $\text{Reg}_{Y\text{-move},i}$, $\text{Reg}_{H\text{-move}}$ and $\text{Reg}_{\text{cancel}}$.

**Movement Costs** Via Eq. (G.15) and the definitions of $\text{Reg}_{Y\text{-move},i}$ and $\text{Reg}_{H\text{-move}}$, the cost of the re-centering argument is given by a movement costs, which we bound presently. Since the
movement of the algorithm are small in the norms induced by the preconditioning matrices \( \hat{\Lambda} \), our main argument invokes steps of the form

\[
\| H_t(z_{t-s} - z_{t-s-1}) \|_2^2 \leq \frac{\| H_t^T \hat{\Lambda}_{t-s-1}^{-1} H_t \|_{op}^2}{2} + \frac{\| (z_{t-s} - z_{t-s-1}) \|_{\hat{\Lambda}_{t-s-1}}^2}{2},
\]

much like the regret analysis in the known system case. Moreover, the contributions of the \( \| (z_{t-s} - z_{t-s-1}) \|_{\hat{\Lambda}_{t-s-1}}^2 \) can be bounded via an application of the log-det potential argument, as in Proposition F.1.

However, we observe that the conditioning of the relevant movement costs is in terms of the \( \hat{\Lambda} \) matrix. To bound terms \( \| H_t^T \hat{\Lambda}_{t-s-1}^{-1} H_t \|_{op}^2 \), we will need to relate the matrices \( \hat{\Lambda}_{t-s-1} \), constructed based on the estimated sequence \( \{\hat{H}_t\} \), and with delays up to \((s + 1) = \tau\), to the matrixes \( \hat{\Lambda}_t \), based on \( \{H_t\} \) and current time \( t \). This is accomplished by the following lemma:

**Lemma G.6.** For \( c_\lambda \in (0, 1] \), set \( c_\lambda(\tau) := 2(1 + R_Y) + 2 c_\lambda^2 \sqrt{\tau R_2} \). Then, for \( \lambda \geq c_\lambda T \varepsilon_G^2 \), we have that for all \( t \in [T] \),

\[
\hat{\Lambda}^{-1}_{t-\tau} \leq c_\lambda(\tau)^2 \Lambda_{t-1},
\]

where we adopt the convention \( \hat{\Lambda}_s = \hat{\Lambda}_1 \) and \( \Lambda_s = \Lambda_1 \) for \( s \leq 1 \).

For our scalings of \( \tau \) and \( \lambda, c_\lambda \) will be roughly constant in magnitude. With the above lemma in hand, we show that the movement terms from the blocking argument scale proportionally to \( \tau \).

**Lemma G.7.** Recall the logarithmic factor \( \mathcal{L} := \log(e + TR^2_H) \). If \( \lambda \) is chosen such that \( \lambda \geq \frac{h^2}{n} T \varepsilon_G^2 + c_\lambda h R_2^2 \), then the movement terms admit the following bounds for \( i \in \{0, \ldots, h\} \):

\[
\text{Reg}_{Y,move,i} := \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| Y_{t-i}(z_{t-s} - z_{t-s-1}) \|_2 \leq \tau c_\lambda \varepsilon_G^{1/2} \cdot d_L \sqrt{T(1 + 10 R^2_2)} \mathcal{L},
\]

\[
\text{Reg}_{H,move} := \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| H_t(z_{t-s} - z_{t-s-1}) \|_2 \leq \tau c_\lambda \varepsilon_G^{1/2} \cdot d_L \mathcal{L}.
\]

**Cancellation within blocks** Next, let us argue that the term \( \text{Reg}_{\text{cancel}} \) is small, which leverages cancellation within blocks. As per the proof sketch at the beginning of the section, we show that the terms \( \| Y_{k_i+s-\delta_{k_i+1}} \|_2^2 \) offset the terms \( \| H_{k_i+s} \delta_{k_i+1} \|_2^2 \) up to an \( O(1) \) factor for each \( j \), incurring an error scaling as \( j_{max} \approx T/\tau \) (thereby inducing a trade-off on the parameter \( \tau \)).

**Lemma G.8.** For \( \nu \leq \frac{\sqrt{\varepsilon}}{4} \), we have

\[
\text{Reg}_{\text{cancel}} \leq \frac{20T}{\tau} \cdot \nu h R_2^2 R_Y \varepsilon_G^2 + 5T \varepsilon_G^2 \cdot \kappa R^2_{Y,C}.
\]

**G.4.2 Summarizing the blocking argument**

Grouping all the terms that have emerged thus far, we summarize the current state of our argument in the following lemma:

**Lemma G.9.** Assuming \( L_{\text{eff}} \geq 1, \nu \leq \frac{\sqrt{\varepsilon}}{4(1+R_Y)}, \text{ and } \lambda \geq c_\lambda(\frac{1}{\nu} T \varepsilon_G^2 + h R_2^2 + \tau), \) we have that for all \( z_{*} \in C \),

\[
c_\lambda \text{OCoReg}_T \left( \frac{\nu}{4\eta}; z_{*} \right) \lesssim \frac{T \varepsilon_G^2}{\alpha} \cdot \left( \frac{h L_{\text{eff}}}{\nu} + \varepsilon_L^2(\varepsilon_G + R_Y) \right) + \widehat{\text{Reg}}_T.
\]

\[
+ \frac{\nu}{\tau} \cdot (\alpha h R_2^2 R_Y + \tau) \cdot (\alpha(1 + R_Y)R_Y \cdot R_G^2 \cdot d_L \mathcal{L}),
\]

Let us take stock of what we have so far. The bound \( \text{OCoReg}_T(\nu/4\eta; z_{*}) \) has four components:

- \( \widehat{\text{Reg}}_T \), which accounts for the regret on the \( \hat{f}_t \) sequence.
• A term scaling with $T c_{G}^{2}$, which accounts for the sensitivity to error. This term also involves the offset $\nu$.
• A term scaling as $\frac{L^2}{\nu}$, yielding a penalty for the number of blocks in the blocking argument.
• A term scaling as linearly in $\tau$, arising from the movement costs from the recentering argument.

The final regret bound will follow from carefully trading off the parameters $\nu$ and $\tau$ in the analysis, and from setting $\lambda$ appropriately. Before continuing, we first address with “with-memory” portion of the bound, passing from unary regret to memory regret.

### G.5 Concluding the Bound

Before concluding the bound, we need to bound the movement cost that appears:

**Lemma G.10 (Movement Cost: Unknown System).** Under the conditions of Lemma G.9,

$$\text{MoveDiff}_{T} := \sum_{i=1}^{T} F_{i}(x_{i}; -h) - f_{i}(z_{i}) \leq 9\eta\kappa^{-\frac{1}{2}}(1 + R_{Y})^{2}R_{G}h^{2} \cdot dL_{\text{eff}}^{2} \cdot \mathcal{L}$$

We are now ready to prove our main theorem:

**Proof of Theorem G.1.** Let us begin by unpacking

$$\overline{\text{Reg}}_{T} + \text{MoveDiff}_{T} \leq 9\eta\kappa^{-\frac{1}{2}}(1 + R_{Y})^{2}R_{G}h^{2} \cdot dL_{\text{eff}}^{2} \cdot \mathcal{L} + \frac{\eta dL_{\text{eff}}^{2} \cdot \mathcal{L}}{2} + \frac{\lambda D^{2}}{2\eta},$$

where we use $\eta = \frac{3}{\alpha}$. Thus, from Lemma G.9, the term $\text{MemoryReg}_{T}$ defined in Eq. (G.1) satisfies the following for any $z_{*} \in C$, provided that the conditions of Lemma G.9 hold:

$$c_{\lambda}\text{MemoryReg}_{T}(\frac{\nu}{4\eta}; z_{*}) \leq c_{\lambda}\text{CorrectReg}_{T}(\frac{\nu}{4\eta}; z_{*}) + \text{MoveDiff}_{T}$$

where above we use $c_{\lambda} \leq 1$. Let us now specialize parameters. As per our theorem, we take

$$\lambda = c_{\lambda}\left(T c_{G}^{2} + c(T \epsilon_{G})^{2/3} + hR_{G}^{2}\right), \quad \tau = (T \epsilon_{G})^{2/3}, \quad c_{\lambda} \in (0, 1)$$

which we verify satisfies the condition on $\lambda$ placed by Lemma G.9. For this choice of parameters, we have

$$\text{MemoryReg}_{T}(\frac{\nu}{4\eta}; z_{*}) \leq \frac{1}{\alpha\sqrt{R}}(1 + R_{Y})^{2}R_{G}h^{2} \cdot dL_{\text{eff}}^{2} \cdot \mathcal{L} + \alpha hR_{G}^{2} \cdot D^{2}$$

$$+ \frac{T c_{G}^{2}}{\alpha} \cdot \left(\beta^{2}(c_{\lambda}^{2} + R_{Y,C}^{2}) + \alpha^{2} \cdot D^{2}\right)$$

$$+ \alpha(T \epsilon_{G})^{2/3} \cdot (D^{2} + (1 + R_{Y})R_{Y,C}^{2} \cdot dL_{\text{eff}}^{2})$$

$$+ \frac{T c_{G}^{2}}{\alpha} \cdot hL_{\text{eff}}^{2} + \frac{T \nu}{\tau} \cdot (\alpha hR_{G}^{2} R_{Y,C})$$.

Next, let’s tune $\nu$. Define $\nu_{0} := \frac{\sqrt{T}}{4(1 + R_{Y})}$ to denote the upper bound on $\nu$ imposed by Lemma G.9. Moreover, let $\nu_{1}$ denote the value of $\nu$ that minimizes the upper bound above, namely

$$\nu_{1} = \left(\frac{T c_{G}^{2}}{\alpha} \cdot hL_{\text{eff}}^{2}\right)^{1/2} \cdot \left(\frac{T}{\tau} \cdot \alpha hR_{G}^{2} R_{Y,C}\right)^{-1/2}.$$
We set $\bar{\nu} = \min\{\nu_0, \nu_1\}$. For this value, we have that
\[
\frac{T \epsilon_G^2}{\alpha} \cdot \frac{h L_{\text{eff}}^2}{\nu} + \frac{T \bar{\nu}}{\tau} \cdot (\alpha h R_g^2 R_{Y, c}) \leq \frac{T \epsilon_G^2}{\alpha} \cdot \frac{h L_{\text{eff}}^2}{\nu_0} + \frac{T \epsilon_G^2}{\alpha} \cdot \frac{h L_{\text{eff}}^2}{\nu_1} + \frac{T \nu_1}{\tau} \cdot (\alpha h R_g^2 R_{Y, c}) \\
\leq \frac{T \epsilon_G^2}{\alpha} \cdot \frac{h L_{\text{eff}}^2}{\nu_0} + 2\sqrt{\frac{T^2 \epsilon_G^2 h^2 L_{\text{eff}}^2 R_g^2 R_{Y, c}}{\tau}} \\
\leq \frac{T \epsilon_G^2}{\alpha} \cdot \frac{h L_{\text{eff}}^2}{\nu_0} + 2(T \epsilon_G)^{2/3} \sqrt{h L_{\text{eff}} R_g R_{Y, c}} \\
\leq \frac{T \epsilon_G^2}{\alpha \sqrt{\kappa}} \cdot (1 + R_Y) h L_{\text{eff}}^2 + (T \epsilon_G)^{2/3} h L_{\text{eff}} R_g R_{Y, c}.
\]

Combining with the above,
\[
\text{MemoryReg}_{G,T}(\frac{\bar{\nu}}{4\alpha}; z_*) \lesssim \frac{T \epsilon_G^2}{\alpha \sqrt{\kappa}} \cdot \left( (1 + R_Y) h L_{\text{eff}}^2 + \beta^2 \sqrt{\kappa} (\nu^2 + R_{Y, c} + R_{Y, c}^2) + \alpha^2 \sqrt{\kappa} D^2 \right) := C_{\text{mid}}' \\
+ (T \epsilon_G)^{2/3} L \cdot \left( h L_{\text{eff}} R_g R_{Y, c} + \alpha D^2 + \alpha (1 + R_Y) R_{Y, c} R_{G}^2 \cdot d_{\text{eff}} \right) := C_{\text{hi}}' \\
+ \frac{\nu}{\alpha \sqrt{\kappa}} \cdot (1 + R_Y)^2 R_g h^2 \cdot d_{\text{eff}} + \alpha \lambda D^2 := C_{\text{low}}
\]

where we use $C_{\text{hi}}'$, $C_{\text{mid}}'$ as intermediate constants that we simplify as follows. Recalling the
\[
C_{\text{mid}}' = (1 + R_Y) h L_{\text{eff}}^2 + \beta^2 \sqrt{\kappa} (\nu^2 + R_{Y, c} + R_{Y, c}^2) + \alpha^2 \sqrt{\kappa} D^2 \\
\leq (1 + \frac{\nu}{2})^2 (1 + R_Y) h L_{\text{eff}}^2 + \beta^2 \sqrt{\kappa} \nu^2 + \alpha^2 \sqrt{\kappa} D^2 := C_{\text{mid}} \\
C_{\text{hi}}' = h L_{\text{eff}} R_g R_{Y, c} + \alpha D^2 + \alpha (1 + R_Y) R_{Y, c} R_{G}^2 \cdot d_{\text{eff}} \\
\leq (1 + R_Y) L_{\text{eff}} R_g^2 R_{Y, c}^2 (h + d) + \alpha D^2 := C_{\text{hi}} 
\]

Note that the constant $C_{\text{hi}}'$, $C_{\text{low}}$, $C_{\text{mid}}'$ coincided with those in Definition G.1. Thus, writing our regret bound compactly, we have
\[
\text{MemoryReg}_{G,T}(\frac{\bar{\nu}}{4\alpha}; z_*) \lesssim C_{\text{hi}} (T \epsilon_G)^{2/3} L + \frac{C_{\text{mid}}(T \epsilon_G^2)}{\alpha \sqrt{\kappa}} + C_{\text{low}} \frac{\nu}{\alpha \sqrt{\kappa}} + \alpha \lambda D^2.
\]

Finally, let us expose $\bar{\nu}$. Recall we set $\bar{\nu} = \min\{\nu_0, \nu_1\}$, with $\nu_0 = \frac{\sqrt{\kappa}}{4(1 + R_Y)}$, and
\[
\nu_1 = \left( \frac{T \epsilon_G^2}{\alpha} \cdot h L_{\text{eff}}^2 \right)^{1/2} \cdot \left( \frac{T}{\tau} \cdot \alpha h R_g^2 R_{Y, c} \right)^{-1/2} \\
= \frac{L_{\text{eff}} \epsilon_G \sqrt{\kappa}}{\alpha R_g R_{Y, c}} = \frac{L_{\text{eff}}(T \epsilon_G^4)^{1/3}}{\alpha R_g R_{Y, c}},
\]

finally yielding
\[
\bar{\nu} = \min \left\{ \frac{L_{\text{eff}}(T \epsilon_G^4)^{1/3}}{\alpha R_g R_{Y, c}}, \frac{\sqrt{\kappa}}{4(1 + R_Y)} \right\} .
\]

To conclude, we parametrize $\bar{\nu}' = \frac{\bar{\nu}}{4\eta}$. Since $\eta = \frac{3}{\alpha}$, we take
\[
\bar{\nu}' = \frac{\alpha \sqrt{\kappa}}{48(1 + R_Y)} \min \left\{ \frac{4(1 + R_Y) L_{\text{eff}}(T \epsilon_G^4)^{1/3}}{\alpha \sqrt{\kappa} R_g R_{Y, c}}, 1 \right\} \\
\geq \frac{\alpha \sqrt{\kappa}}{48(1 + R_Y)} \min \left\{ \frac{4(1 + R_Y) L_{\text{eff}}(T \epsilon_G^4)^{1/3}}{R_g R_{Y, c}}, 1 \right\} \\
\geq \frac{\alpha \sqrt{\kappa}}{48(1 + R_Y)} \min \left\{ \frac{4(1 + R_Y)(T \epsilon_G^4)^{1/3}}{1}, 1 \right\} := \nu_*
\]

where in the last line we use $L \geq 1$ to bound $L_{\text{eff}} \geq R_g R_{Y, c}$. Thus, taking $\nu_*$ to be the above lower bound on $\bar{\nu}'$ concludes. 

\[\Box\]
H Omitted Proofs from Appendix F

H.1 Proof of Proposition F.8

Proof of Proposition F.8. Let \( v \in \mathbb{R}^{d_w} \), with \( \|v\| = 1 \), and let \( u_s = Y_s v \) for \( s \in \{1 - h, 2 - h, \ldots, t\} \), and set \( u_s = 0 \) for \( s \leq t - h \) and \( s > t \). From Fact I.2, which shows that \( \|v + w\|_2^2 \geq \frac{1}{2} \|v\|^2 - \|w\|^2 \), we have

\[
v^T \sum_{s=1}^{t} H_s^T H_s v := \sum_{s=1}^{t} \left\| H_s v \right\|_2^2 = \sum_{s=1}^{h} \left\| G[i] Y_{s-t} v \right\|_2^2
\]

\[
\geq \sum_{s=1}^{t-h} \sum_{i=0}^{h} \left\| G[i] u_{s-i} \right\|_2^2 - 2h R_G^2 R_Y^2.
\]

\[
\geq \frac{1}{2} \sum_{s=1-h}^{t+h} \sum_{i=0}^{\infty} \left\| G[i] u_{s-i} \right\|_2^2 - \sum_{s=1-h}^{t+h} \sum_{i>h}^{\infty} \left\| G[i] u_{s-i} \right\|_2^2 - 2h R_G^2 R_Y^2
\]

\[
= \frac{1}{2} \sum_{s=1-h}^{t+h} \sum_{i=0}^{\infty} G[i] u_{s-i} \|_2^2 - (t \psi_G(h+1)^2 + 4h R_G^2) R_Y^2,
\]

where we use \( \psi_G(h+1) \leq \psi_G(0) = R_G^2 \) in the last line. Moreover, setting \( \tilde{u}_s = u_{s-h} \),

\[
\sum_{s=1-h}^{t+h} \sum_{i=0}^{\infty} G[i] u_{s-i} \|_2^2 = \sum_{s=1}^{t+2h} \sum_{i=0}^{\infty} G[i] \tilde{u}_{s-i} \|_2^2
\]

\[
\geq \sum_{s=1-h}^{t+h} \sum_{i=0}^{\infty} G[i] \tilde{u}_{s-i} \|_2^2
\]

\[
\geq \kappa_0 \sum_{s=1}^{\infty} \| \tilde{u}_s \|_2^2
\]

\[
= \kappa_0 \sum_{s=1-h}^{t+2h} \| u_{s-h} \|_2^2
\]

where (i) uses that we have \( \tilde{u}_s = 0 \) for \( s \leq 0 \) and for \( s \geq t + 2h \), and (ii) invokes Definition 2.2. Combining the two displays, we have

\[
v^T \sum_{s=1}^{t} H_s^T H_s v \geq \frac{\kappa_0}{2} \sum_{s=1}^{\infty} \| u_{s-h} \|_2^2 - \gamma_{t,h}
\]

\[
\geq \frac{\kappa_0}{2} \sum_{s=1}^{t+h} \| Y_{s-h} v \|_2^2 - \gamma_{t,h}
\]

\[
= v^T \left( \frac{\kappa_0}{2} \sum_{s=1-h}^{t} Y_s^T Y_s - \gamma_{t,h} I \right) v,
\]

where the last line uses \( \|v\| = 1 \). Finally, defining \( c_{\psi_G} := \max \left\{ 1, \frac{\psi_G(h+1)^2}{h R_G^2} \right\} \), we have \( \gamma_{t,h} = R_G^2 (t \psi_G(h+1)^2 + 4h R_G^2) \leq R_G^2 (h c_{\psi_G} R_G^2 + 4h R_G^2) \leq 5h R_G^2 c_{\psi_G} \), yielding the desired bound. \( \square \)
H.2 Proof of Lemma F.4

Let \( z_* \in \arg\min_{z \in C} \sum_{t=1}^{T} f_t(z) \). Following the standard analysis of Online Newton Step (e.g. Hazan [18, Chapter 4] with \( \gamma \leftarrow 1/\eta \)), one has

\[
\sum_{t=1}^{T} \nabla_i (z_t - z_*) \leq \eta \frac{1}{2} \sum_{t=1}^{T} \nabla_i^T \Lambda_t^{-1} \nabla_i + \frac{1}{2\eta} \sum_{t=1}^{T} (z_t - z_*)^T (\Lambda_t - \Lambda_{t-1}) (z_t - z_*) + \frac{1}{2\eta} (z_1 - z_*)^T \Lambda_0 (z_1 - z_*)
\]

The last term is at most \( \frac{1}{2\eta} D^2 \). Moreover, since \( \Lambda_t - \Lambda_{t-1} = H_t H_t^T \),

\[
\sum_{t=1}^{T} \nabla_i (z_t - z_*) - \frac{1}{2\eta} \|H_t(z_t - z_*)\|^2 \leq \lambda D^2 + \frac{1}{2\eta} \sum_{t=1}^{T} \nabla_i^T \Lambda_t^{-1} \nabla_i.
\]

Finally, for \( \eta \geq \frac{1}{2\lambda} \), we recognize that \( \nabla_i (z_t - z_*) - \frac{1}{2\eta} \|H_t(z_t - z_*)\|^2 \geq \nabla_i (z_t - z_*) - \frac{\eta}{2\lambda} \|H_t(z_t - z_*)\|^2 \geq f_t(z_t) - f_t(z_*) \) by Lemma F.3. Thus,

\[
\sum_{t=1}^{T} f_t(z_t) - f_t(z_*) \leq \lambda D^2 + \frac{\eta}{2} \sum_{t=1}^{T} \nabla_i^T \Lambda_t^{-1} \nabla_i,
\]

as needed.

H.3 Proof of Lemma F.6

We have \( F_t(z_1, \ldots, z_{t-h}) - f_t(z_t) = F_t(z_1, \ldots, z_{t-h}) - F_t(z_t, \ldots, z_t) \). Therefore Taylor’s theorem, there exists some \( \mu \in [0, 1] \) such that, for \( z_t = \mu z_{t-i} + (1 - \mu) z_t \),

\[
F_t(z_1, \ldots, z_{t-h}) - f_t(z_t) = (\nabla F_t(z_1, \ldots, z_{t-h}))^T (0, z_{t-1} - z_t, z_{t-2} - z_t, \ldots, z_{t-h} - z_t).
\]

By the Chain Rule, we then have

\[
|F_t(z_1, \ldots, z_{t-h}) - f_t(z_t)| = \left| \nabla \ell_t (v_t + \sum_{i=0}^{h} G^{[i]} Y_{t-i} z_t) \right| \left( \sum_{i=1}^{h} G^{[i]} Y_{t-i} \right) \left( \sum_{i=0}^{h} G^{[i]} Y_{t-i} \right) \| Y_{t-i} (z_t - z_{t-i}) \|_2.
\]

Analogous to the Lemma F.2, we have \( \| \nabla \ell_t (v_t + \sum G^{[i]} Y_{t-i} z_t) \|_2 \leq L_{eff} \), concluding the first part of the proof. For the second display, we have

\[
\sum_{t=1}^{T} F_t(z_1, \ldots, z_{t-h}) - f_t(z_t) \leq L_{eff} R_G \sum_{i=1}^{T} \max_{i \in \{1, \ldots, h\}} \| Y_{t-i} (z_t - z_{t-i}) \|_2
\]

\[
\leq L_{eff} R_G \sum_{t=1}^{T} \sum_{i=1}^{h} \| Y_{t-i} (z_t - z_{t-i}) \|_2
\]

\[
\leq L_{eff} R_G \sum_{t=1}^{T} \sum_{i=1}^{h} \sum_{j=1}^{i-1} \| Y_{i-j} (z_{t-j+1} - z_{t-j}) \|_2
\]

\[
= L_{eff} R_G \sum_{i=1}^{T} \sum_{s=1}^{h-i} \| Y_{s} (z_{s+i} - z_{s+i-1}) \|_2
\]

\[
\leq h L_{eff} R_G \sum_{s=1}^{T} \sum_{i=1}^{h-s} \| Y_{s} (z_{s+i} - z_{s+i}) \|_2 : I_{s+i+1 \leq t}.
\]

Finally, since \( z_t - z_{t-1} = 0 \) for \( t \leq 1 \), the above indicator \( I_{s+i+1 \leq t} \) can be replaced with \( I_{2 \leq s+i+1 \leq t} = I_{1 \leq s+i \leq t-1} \), completing the proof.
H.4 Proof of Lemma F.7

For $t \leq 0$, $\|Y_s(z_{t+1} - z_t)\|_2 = 0$. Otherwise, we have

$$\|Y_s(z_t - z_{t-1})\|_2 = \|Y_s\Lambda_t^{-1/2}\Lambda_t^{1/2}(z_{t+1} - z_t)\|_2$$

$$\leq \|Y_s\Lambda_t^{-1/2}\|_{op} \cdot \|\Lambda_t^{1/2}(z_{t+1} - z_t)\|_2$$

$$\leq \|Y_s\Lambda_t^{-1/2}\|_{op} \cdot \|\Lambda_t^{1/2}(z_{t+1} - z_t)\|_2$$

$$= \|Y_s\Lambda_t^{-1/2}\|_{op} \|\Lambda_t^{1/2}\cdot \eta\Lambda_t^{-1}\nabla_t\|_2$$

$$= \eta\sqrt{\|Y_s\Lambda_t^{-1/2}\|_2^2 \|\Lambda_t^{-1/2}\nabla_t\|_2^2}, \quad (H.1)$$

where $(i)$ follows from the Pythagorean theorem, using that $z_{t+1}$ is projected in the $\Lambda_t$-norm. Finally, we can crudely bound $\|Y_s\Lambda_t^{-1}Y_s\|_{op} \leq \text{tr}(Y_s\Lambda_t^{-1}Y_s)$. Since we consider indices $t \geq s$, we have $\text{tr}(Y_s\Lambda_t^{-1}Y_s) \leq \text{tr}(Y_s\Lambda_s^{-1}Y_s)$, where we have the understanding that $\Lambda_s = \Lambda_1$ for $s \leq 0$. Thus, we see that for $t > 0$,

$$\|Y_s(z_{t+1} - z_t)\|_2 \leq \eta\text{tr}(Y_s\Lambda_s^{-1}Y_s)^{1/2}\text{tr}(\nabla_t^\top\Lambda_t^{-1}\nabla_t)^{1/2}$$

Thus, from Lemma F.6 and by Cauchy Schwarz,

$$\text{MoveDiff}_T \leq hL_{eff}R_G \sum_{i=1}^{h-1} \sum_{s=1-h}^T \|Y_s(z_{s+i+1} - z_{s+i})\|_2 I_{1 \leq s+i \leq t-1}$$

$$\leq \eta hL_{eff}R_G \cdot \sum_{i=1}^{h-1} \sum_{s=1-h}^T \|Y_s(z_{s+i+1} - z_{s+i})\|_2 I_{1 \leq s+i \leq t-1} \cdot \text{tr}(Y_s\Lambda_s^{-1}Y_s) \leq \eta h^2 L_{eff} R_G \cdot \sqrt{\sum_{s=1-h}^T \text{tr}(Y_s\Lambda_s^{-1}Y_s)} \sqrt{\sum_{s=1}^T \text{tr}(\nabla_t^\top\Lambda_t^{-1}\nabla_t)}$$

as needed. 

I Ommited Proofs from Appendix G

I.1 Useful Facts for Analysis

We begin by listing some useful elementary facts:

Fact I.1. For all $t \geq 1$ and all $z \in C$, we have $\|H_t - \hat{H}_t\|_\infty \leq \epsilon_G R_Y$ and $\|(H_t - \hat{H}_t)z\|_\infty \leq \epsilon_G R_Y$.

Proof. $\|H_t - \hat{H}_t\|_\infty = \|\sum_{i=0}^{h}(G_t^{[i]} - \hat{G}_t^{[i]}Y)\|_\infty \leq \|R_Y\|_\infty \sum_{i=0}^{h} \|G_t^{[i]} - \hat{G}_t^{[i]}\|_\infty \leq \epsilon_G R_Y$. The second bound is similar. 

Fact I.2. Given two vectors $v, w \in \mathbb{R}^m$, $\|v + w\|_2^2 \geq \frac{1}{2} \|v\|^2 - \|w\|^2$.

Proof. $\|v + w\|_2^2 = \|v\|^2 + \|w\|^2 + 2(v, w) \geq \|v\|^2 + \|w\|^2 - 2\|v\|\|w\| \geq \|v\|^2 + \|w\|^2 - \frac{1}{2} \|v\|^2 - 2\|w\|^2 = \frac{\|v\|^2}{2} - \|w\|^2$, as needed.

Fact I.3. $\|a\|^2 \leq \|b\|^2 + (\|a\| + \|b\|)\|b - a\|$

Proof. $\|a\|_2^2 = \langle a, a \rangle = \langle b - a, a \rangle + \langle b, a \rangle = \langle b - a, a \rangle + \langle b, a - b \rangle + \|b\|^2$. The bound now follows form Cauchy-Schwartz.
1.2 Proof of Lemma G.1

Let \( z_* \in C \) be an arbitrary comparator point. Analogous to the proof of Lemma F.4,

\[
\sum_{t=1}^{T} \widehat{f}_t(z_t) - \hat{f}_t(z_*) \leq \sum_{t=1}^{T} \nabla_t^T (z_t - z_*) - \frac{\alpha}{2} \|\widehat{H}_t(z_t - z_*)\|^2. \tag{1.1}
\]

One the other hand, the standard inequality obtained from applying Semi-ONS to the \((\hat{f}_t)\)-sequence (see, for analogy, page 58 of [18]), we obtain

\[
\nabla_t^T (z_t - z_*) \leq \frac{\eta}{2} \|\nabla\|_{\Lambda_t}^2 + \frac{2}{\eta} \|z_t - z_*\|^2 - \frac{2}{\eta} \|z_{t+1} - z_*\|^2. \nabla_t^T (z_t - z_*) \leq \frac{\eta}{2} \|\nabla\|_{\Lambda_t}^2 + \frac{2}{\eta} \|z_t - z_*\|^2 - \frac{2}{\eta} \|z_{t+1} - z_*\|^2.
\]

Summing up over \( t \) and telescoping

\[
\sum_{t=1}^{T} \nabla_t^T (z_t - z_*) \leq \frac{\eta}{2} \sum_{t=1}^{T} \|\nabla\|_{\Lambda_t}^2 + \frac{1}{2\eta} \sum_{t=1}^{T} \|\widehat{H}_t(z_t - z_*)\|^2 \leq \frac{\lambda D^2}{2\eta}, \tag{1.2}
\]

where we use \( \Lambda_t - \Lambda_{t-1} = \widehat{H}_t \widehat{H}_t^\top \), and \( \Lambda_0 = \lambda I \). Thus, introducing \( \text{err}_t := \nabla \hat{f}_t(z) - \nabla f(z_t) \) and combining (1.1) and (1.2),

\[
\sum_{t=1}^{T} f_t(z_t) - f_t(z_*) \leq \sum_{t=1}^{T} \text{err}_t^T (z_t - z_*) + \frac{1}{2\eta} \sum_{t=1}^{T} \|\widehat{H}_t(z_t - z_*)\|^2 - \eta \alpha \|\widehat{H}_t(z_t - z_*)\|^2
\]

Plugging in \( \delta_t = z_t - z_* \) concludes the proof, and re-iterating the proof of Proposition F.1 concludes the proof.

1.3 Proof of Lemma G.2

First, we can bound \( \|\widehat{H}_t \delta_t\|^2 \leq 2\|\widehat{H}_t \delta_t\|^2 + 2\| (\widehat{H}_t - \widehat{H}_t) \delta_t\|^2 \), and

\[
\|(\widehat{H}_t - \widehat{H}_t) \delta_t\| \leq \| (\widehat{H}_t - \widehat{H}_t) z_t\| + \|(\widehat{H}_t - \widehat{H}_t) z_*\| \leq 2R_{Y,C} \epsilon_G
\]

by Fact I.1. Taking \( \eta \geq \frac{3}{\lambda} \), we find then that

\[
\|\widehat{H}_t \delta_t\|^2 - \eta \alpha \|\widehat{H}_t \delta_t\|^2 \leq 2\|\widehat{H}_t \delta_t\|^2 + 8 R_{Y,C}^2 \epsilon_G^2\|H_t \delta_t\|^2 \leq -\|\widehat{H}_t \delta_t\|^2 + 8 R_{Y,C}^2 \epsilon_G^2.
\]

The second statement of the lemma follows by substitution into Lemma G.1.

1.4 Proof of Lemma G.3

We have the bound

\[
\text{err}_t := \nabla \hat{f}_t(z) - \nabla f(z_t)
\]

\[
\begin{aligned}
&= \widehat{H}_t \nabla \ell_t (\widehat{v}_t + H_t z_t) - H_t \nabla \ell_t (v_t^* + H_t z_t) \\
&= (\widehat{H}_t - H_t) \nabla \ell_t (\widehat{v}_t + \widehat{H}_t z_t) + H_t \left( \nabla \ell_t (\widehat{v}_t + \widehat{H}_t z_t) - \nabla \ell_t (v_t^* + H_t z_t) \right).
\end{aligned}
\]

Defining

\[
g_{t,1} := \nabla \ell_t (\widehat{v}_t + \widehat{H}_t z_t)
\]

\[
g_{t,2} := \left( \nabla \ell_t (\widehat{v}_t + \widehat{H}_t z_t) - \nabla \ell_t (v_t^* + H_t z_t) \right)
\]

We have that \( \|g_{t,1}\|^2 \leq \epsilon_{off} \) by analogy to Lemma F.2. Moreover, since \( \beta \)-smoothness implies that the gradients are \( \beta \)-Lipschitz, and by invoking Fact I.1, we have

\[
\left( \nabla \ell_t (\widehat{v}_t + \widehat{H}_t z_t) - \nabla \ell_t (v_t^* + H_t z_t) \right) \leq \beta \| (\widehat{v}_t + \widehat{H}_t z_t) - (v_t^* + H_t z_t) \| \leq \beta (c_\epsilon + 2\epsilon G R_{Y,C}).
\]

\( \square \)
I.5 Proof of Lemma G.4

Recall that from Lemma G.2, we have the bound

\[
\sum_{t=1}^{T} f_t(z_t) - f_t(z^*_t) \leq \sum_{t=1}^{T} \text{err}_t^\top \delta_t - \frac{1}{2\eta} \sum_{t=1}^{T} \|H_t \delta_t\|^2 + \frac{4}{\eta} T R_{\eta,c}^2 + \text{Reg}_T. \tag{I.3}
\]

Let us now bound the sum \(\sum_{t=1}^{T} \text{err}_t^\top \delta_t\) via Lemma G.3. The lemma ensures \(\text{err}_t = (\hat{H}_t - H_t)^\top g_{1,t} + H_t^\top g_{2,t}\), where \(\|g_{1,t}\|_2 \leq L_{\text{eff}}\) and \(\|g_{2,t}\|_2 \leq \beta \epsilon_G(c_v + 2R_{Y,c})\). The contribution of the term including \(g_{2,t}\) is easily addressed:

\[
(H_t^\top g_{2,t})^\top \delta_t \leq \|g_{2,t}\|_2 \|H_t \delta_t\|_2 \leq \beta \epsilon_G(c_v + 2R_{Y,c}) \|H_t \delta_t\|_2 \leq \eta \beta^2 \epsilon_G^2(c_v + 2R_{Y,c})^2 + \frac{1}{4\eta} \|H_t \delta_t\|_2,
\]

by the AM-GM inequality. Next, we handle the term \((\hat{H}_t - H_t)^\top g_{1,t}\). First we bound

\[
((\hat{H}_t - H_t)^\top g_{1,t})^\top \delta_t \leq \|g_{1,t}\|_2 \|H_t \delta_t\|_2 \leq L_{\text{eff}} \|((\hat{H}_t - H_t) \delta_t\|_2.
\]

Plugging into Eq. (I.3) gives

\[
\sum_{t=1}^{T} f_t(z_t) - f_t(z^*_t) \leq \sum_{t=1}^{T} L_{\text{eff}} \|((\hat{H}_t - H_t) \delta_t\| - \frac{1}{4\eta} \sum_{t=1}^{T} \|H_t \delta_t\|^2 + T \left( \eta \beta^2 (c_v + 2\epsilon_G R_{Y,c})^2 + \frac{4R_{\eta,c}^2}{\eta} \right) \epsilon_G^2 + \text{Reg}_T. \tag{I.4}
\]

For arbitrary sequences \(H_t, \hat{H}_t\), there is no obvious way to cancel the terms \(L_{\text{eff}} \|((\hat{H}_t - H_t) \delta_t\|\) and \(-\|H_t \delta_t\|^2\) to achieve a \(O(T \epsilon_G^2)\)-error dependence. However, there is additional structure we can leverage. We can observe that

\[
\|(\hat{H}_t - H_t) \delta_t\|_2^2 = \sum_{i=0}^{h} (\hat{G}^{[i]} - G^*_t)^{[i]} Y_{t-i} \delta_t \|_2^2 \leq \epsilon_G \max_{i\in[0:h]} \|Y_{t-i} \delta_t\|_2.
\]

Hence, by AMG-GM, we have that for any \(\nu > 0\),

\[
L_{\text{eff}} \|(\hat{H}_t - H_t) \delta_t\| \leq \nu^{-1} (h + 1) \eta \epsilon_G^2 + \frac{\nu}{4(h + 1) \eta} \max_{i\in[0:h]} \|Y_{t-i} \delta_t\|_2.
\]

Together with Eq. (I.4), the above display implies

\[
\sum_{t=1}^{T} f_t(z_t) - f_t(z^*_t) \leq \frac{1}{4\eta} \sum_{t=1}^{T} \left( \frac{\nu}{h + 1} \sum_{i=0}^{h} \|Y_{t-i} \delta_t\|_2^2 - \|H_t \delta_t\|_2^2 \right) + T \epsilon_G^2 \cdot \text{ERR}(\nu) + \text{Reg}_T,
\]

where \(\text{ERR}(\nu) := \left( \frac{(\eta(h+1)\epsilon_G^2)}{\nu} + \eta \beta^2 (c_v + 2R_{Y,c})^2 + \frac{4R_{\eta,c}^2}{\eta} \right).
\]
I.6 Proof of Lemma G.5

Fix a block length \( \tau \in \mathbb{N} \), and recall the index \( k_j = (j - 1)\tau \), and \( j_{\text{max}} \) as the largest \( j \) such that \( j_{\text{max}}\tau \leq T \). We bound

\[
\sum_{t=1}^{T} \| Y_{t-i} \delta_t \|_2^2 \\
= \sum_{j=1}^{j_{\text{max}}} \sum_{s=1}^{\tau} \| Y_{k_j+s-i} \delta_{k_j+s} \|_2^2 + \sum_{s=1+\tau(j_{\text{max}}-1)}^{T} \| Y_{i-h} \delta_i \|_2^2 \\
\leq 4\tau R_{Y,C} + \sum_{j=1}^{j_{\text{max}}} \sum_{s=1}^{\tau} \| Y_{k_j+s-i} \delta_{k_j+s} \|_2^2 \\
\leq 4\tau R_{Y,C} + \sum_{j=1}^{j_{\text{max}}} \sum_{s=1}^{\tau} \| Y_{k_j+s-i} \|_2^2 + 4 R_{Y,C} \sum_{j=1}^{j_{\text{max}}} \sum_{s=1}^{\tau} \| Y_{k_j+s-i} \|_2^2, \quad \text{(1.5)}
\]

Where \( (i) \) uses the inequality \( \| a \|^2 \leq \| b \|^2 + (\| a \| \| b \|) \| b - a \| \) from Fact I.3, and where \( (ii) \) uses the \( \| Y_{i}(\delta_i) \| \leq \| Y_{i}z_i \| + \| Y_{i}z_i \| \leq 2R_{Y,C} \).

Next, recalling \( \delta_t := z_t - z_\star \), we develop

\[
\sum_{j=1}^{j_{\text{max}}} \sum_{s=1}^{\tau} \| Y_{k_j+s-i}(\delta_{k_j+s} - \delta_{k_j+1}) \|_2^2 = \sum_{j=1}^{j_{\text{max}}} \sum_{s=2}^{\tau} \| Y_{k_j+s-i}(z_{k_j+s} - z_{k_j+1}) \|_2^2 \\
\leq \sum_{j=1}^{j_{\text{max}}} \sum_{s=0}^{\tau-1} \sum_{s'=0}^{s-1} \| Y_{k_j+s-i}(z_{k_j+s-s'} - z_{k_j+s'-1}) \|_2^2 \\
\leq \sum_{j=1}^{j_{\text{max}}} \sum_{s=0}^{\tau-1} \sum_{s'=0}^{s-1} \| Y_{k_j+s-i}(z_{k_j+s-s'} - z_{k_j+s'-1}) \|_2^2 \\
\leq \sum_{t=1}^{T} \| Y_{t-i}(z_{t-s'} - z_{t-s'-1}) \|_2^2,
\]

where above we use the convention \( z_t = 0 \) for \( t \leq 1 \), and that the induces \( k_j + s \) range over a subset of \( t \in [T] \). Relabeling \( s' \) with \( s \), and combining with Eq. (1.5) this finally yields

\[
\sum_{t=1}^{T} \| Y_{t-i} \delta_t \|_2^2 \geq 4\tau R_{Y,C} + \sum_{j=1}^{j_{\text{max}}} \sum_{s=1}^{\tau} \| Y_{k_j+s-i} \delta_{k_j} \|_2^2 + 4 R_{Y,C} \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| Y_{t-i}(z_{t-s} - z_{t-s-1}) \|_2^2.
\]

Following similar steps (but using Fact I.1 to bound \( \| H_z \| \leq R_{G,R_{Y,C}} \)), we obtain

\[
\sum_{t=1}^{T} \| H_t \delta_t \|_2^2 \geq \sum_{j=1}^{j_{\text{max}}} \sum_{s=1}^{\tau} \| H_{k_j+s} \delta_{k_j} \|_2^2 - 4R_{Y,C}R_G \sum_{t=1}^{T} \sum_{s=0}^{\tau-1} \| H_t(z_{t-s} - z_{t-s-1}) \|_2,
\]

\[\square\]
I.7 Proof of Lemma G.6

Recall our convention \( \hat{\Lambda}_s = \hat{\Lambda}_1 \) and \( \Lambda_s = \Lambda_1 \) for \( s \leq 1 \). For any \( \mu \in (0, 1] \), we have the bound

\[
\hat{\Lambda}_{t-\tau} = \lambda I + \sum_{s=1}^{t-\tau} \hat{H}_s^\top \hat{H}_s \geq \lambda I + \mu \sum_{s=1}^{t-\tau} \hat{H}_s^\top \hat{H}_s \\
\geq (\lambda - \mu \tau R_H^2) I + \mu \sum_{s=1}^{t} \hat{H}_s^\top \hat{H}_s \\
\geq (\lambda - \mu \tau R_H^2) I + \frac{\mu}{2} \sum_{s=1}^{t} H_s^\top H_s - \mu (\hat{H}_s - H_s)^\top (\hat{H}_s - H_s),
\]

where the last step follows from Fact I.2. We can crudely bound \( (\hat{H}_s - H_s)^\top (\hat{H}_s - H_s) \leq \| \hat{H}_s - H_s \|^2 I \leq R_Y^2 c_G^2 I \) via Fact I.1, giving

\[
\hat{\Lambda}_{t-\tau} \geq (\lambda - \mu \tau R_H^2 - \mu R_Y^2 c_G^2) I + \frac{\mu}{2} \sum_{s=1}^{t} H_s^\top H_s.
\]

Bounding \( t \leq T \), and taking \( \mu = \min\{1, \frac{\lambda}{2(\tau R_H^2 + R_Y^2 c_G^2)}\} \), we obtain

\[
\hat{\Lambda}_{t-\tau} \geq \frac{\lambda}{2} + \frac{\mu}{2} \sum_{s=1}^{t} H_s^\top H_s \geq \frac{\mu}{2} \Lambda_t.
\]

Thus, for any upper bound \( c_\Lambda \geq \sqrt{\frac{2}{\mu}} \),

\[
\hat{\Lambda}^{-1}_{t-\tau} \leq \frac{2}{\mu} \Lambda_t^{-1} \leq c_\Lambda^2 \Lambda^{-1}_t. \tag{I.6}
\]

Finally, we can bound

\[
\sqrt{\frac{T}{\mu}} = \sqrt{\max\{2, \frac{4(\tau R_H^2 + R_Y^2 c_G^2 T)}{\lambda}\}} \\
= \sqrt{\max\{2, 4 R_Y^2 \frac{\tau R_H^2 + c_G^2 T}{\lambda}\}} \\
\geq (i) \sqrt{\max\{2, 4 c_\Lambda^{-1} R_Y^2 (1 + \frac{\tau R_H^2}{\lambda})\}} \\
\leq 2(1 + R_Y) + 2c_\Lambda^{-1} R_Y \sqrt{\frac{\tau R_H^2}{\lambda}} := c_\Lambda,
\]

where we use that \( \lambda \geq c_\Lambda T c_G^2 \) in (i). This verifies that \( c_\Lambda \) in the lemma is an upper bound on \( \sqrt{2/\mu} \), and the lemma now follows from Eq. (I.6).

I.8 Proof of Lemma G.7

Let \( \tau \in \mathbb{N} \) denote our blocking parameter. Again, adopt the convention \( \hat{\Lambda}_s = \hat{\Lambda}_1 \) and \( \Lambda_s = \Lambda_1 \) for \( s \leq 0 \), and let \( c_\Lambda \) be such from Lemma G.5, which ensures that, for all \( t \),

\[
\hat{\Lambda}^{-1}_{t-\tau} \leq c_\Lambda^2 \Lambda^{-1}_t. \tag{I.7}
\]

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Then, any for \( s \in \{0, \ldots, \tau - 1\} \) such that \( s \leq t - 1 \) any \( \mu > 0 \), we have

\[
\|Y_{t-i}(z_{t-s} - z_{t-s-1})\|_2 \leq \|Y_{t-i} \tilde{\Lambda}^{-\frac{1}{2}}_{t-s-1}\|_{\text{op}} \|\tilde{\Lambda}^{\frac{1}{2}}_{t-s-1}(z_{t-s} - z_{t-s-1})\|_2
\]
\[
\leq \|Y_{t-i} \tilde{\Lambda}^{-\frac{1}{2}}_{t-s-1}\|_{\text{op}} \|\tilde{\Lambda}^{\frac{1}{2}}_{t-s-1} \hat{\nabla}t-s-1\|_2
\]
\[
\leq \sqrt{\text{tr}(Y_{t-i} \tilde{\Lambda}^{-1}_{t-s}Y_{t-i}) \cdot \|\hat{\nabla}t-s-1\|^2_{\tilde{\Lambda}^{-1}_{t-s}}}
\]
\[
\leq c\sqrt{\text{tr}(Y_{t-i} \tilde{\Lambda}^{-1}_{t-s}Y_{t-i}) \cdot \|\hat{\nabla}t-s-1\|^2_{\tilde{\Lambda}^{-1}_{t-s}}}. \quad (\text{Eq. (1.7)})
\]

Note that the above expression does not depend on \( \tau \). Thus, since \( z_{t-s} - z_{t-s-1} = 0 \) for \( s > t - 1 \) (recall here we assume \( z_i = z_1 \) for \( i \leq 1 \)), an application of Cauchy Schwartz yields

\[
\sum_{t=1}^{T} \sum_{s=0}^{t-1} \|Y_{t-i}(z_{t-s} - z_{t-s-1})\|_2 \leq c\sqrt{\sum_{t=1}^{T} \text{tr}(Y_{t-i} \tilde{\Lambda}^{-1}_{t-s}Y_{t-i}) \cdot \sum_{t=1}^{T} \|\hat{\nabla}t-s-1\|^2_{\tilde{\Lambda}^{-1}_{t-s}}}
\]

\[
\leq c\sqrt{\sum_{t=1}^{T} \text{tr}(Y_{t-i} \tilde{\Lambda}^{-1}_{t-s}Y_{t-i}) \cdot \sum_{t=1}^{T} \|\hat{\nabla}t-s-1\|^2_{\tilde{\Lambda}^{-1}_{t-s}}}. \quad (1.8)
\]

Arguing as in the proof of Proposition F.1, and using \( \lambda \geq h R_G^2 \geq 1 \),

\[
\sum_{t=1}^{T} \|\hat{\nabla}t\|^2_{\tilde{\Lambda}^{-1}_t} \leq \sum_{t=1}^{T} \text{tr}(H_t \tilde{\Lambda}^{-1}_{t}H_t) \leq dL^2 \log(1 + \frac{T R_G^2}{\lambda}) \leq dL^2 \log \frac{2d}{\mu_0 \kappa}. \quad (1.9)
\]

We now develop a simple claim, which is a consequence of Proposition F.8:

**Claim I.4.** Recall \( c_{\psi,t} := \max\{1, \frac{1}{\kappa h R_G^2} \} \), and set \( \mu_0 = \min\{1, \frac{\lambda}{10h R_H^2 c_{\psi,t}}\} \). We have

\[
\sum_{t=1-h}^{T} \text{tr}(Y_{t-i} \tilde{\Lambda}^{-1}_{t-i}Y_{t-i}) \leq \frac{2d}{\mu_0 \kappa}. \]

**Proof of Claim I.4.** From Proposition F.8, we have the bound

\[
\sum_{s=1}^{t} H_s^\top H_s \geq \sum_{s=1-h}^{t} Y_s^\top Y_s - 5h R_H^2 c_{\psi,t} \quad \text{I}.
\]

Thus, for any \( \mu_0 = \min\{1, (10h R_H^2 c_{\psi,t})^{-1}\} \leq 1 \),

\[
\Lambda_t = \lambda I + \sum_{s=1}^{t} H_s^\top H_s \geq \lambda I + \mu_0 \sum_{s=1}^{t} H_s^\top H_s
\]
\[
= \lambda I + \mu_0 \left( \frac{\kappa}{2} \sum_{s=1-h}^{t} Y_s^\top Y_s - 5h R_H^2 c_{\psi,t} \right) \geq \lambda I + \frac{\mu_0 \kappa}{2} \sum_{s=1-h}^{t} Y_s^\top Y_s
\]

Hence, from the log-det potential bound of Lemma F.5, the bounds \( \mu_0, \kappa \leq 1 \) and \( R_H = R_G R_Y \)

\[
\sum_{s=1-h}^{T} \text{tr}(Y_s^\top \Lambda_s^{-1} Y_s) \leq \frac{2d}{\mu_0 \kappa} \log(1 + \frac{\mu_0 \kappa T R_H^2}{\lambda}) \leq \frac{2d}{\mu_0 \kappa} \log(1 + \frac{T R_H^2}{\lambda}) = \frac{2d}{\mu_0 \kappa} \log. \]

\[\square\]
To apply the above, let us simplify our expression for \( \mu_0 \). Recall that

\[
\mu_0 = \min \left\{ 1, \frac{\lambda}{10 h R^2_G c_{\psi ; T}} \right\}, \quad c_{\psi ; T} := \max \left\{ 1, \frac{T \psi_G (h + 1)^2}{h R^2_G} \right\} \leq (1 + T c^2_G / h R^2_G),
\]

where we note that \( c_G = \| \hat{G} - G \|_{\text{op}} \geq \sum_{i>h} \| G[i] \|_{\text{op}} \geq \psi_G (h + 1) \), since \( G[i] = 0 \) for \( i > h \).

Using the bounds \( R_H / R_G = R_Y \) and \( \lambda \geq c_\lambda (T c^2_G + h R^2_G) \) for \( c_\lambda \in (0, 1] \),

\[
\mu_0^{-1} \leq 1 + \frac{10 h R^2_H c_{\psi ; T}}{\lambda} \leq 1 + \frac{10 h R^2_H (1 + T c^2_G / h R^2_G)}{\lambda} = 1 + \frac{10 R^2_H (h R^2_G + R^2_T c^2_G / h)}{\lambda} \leq 1 + c^{-1} R^2_G.
\]

Together with Claim I.4, we obtain

\[
\sum_{t=1}^T \text{tr}(Y_t \Lambda_t^{-1} Y_t) \leq \frac{2 d}{\mu_0 \kappa} \cdot \mathcal{L} \leq \frac{2 d (1 + 10 R^2_Y)}{\kappa} \cdot \mathcal{L}, \tag{I.10}
\]

Thus, putting together Equations (I.8), (I.9), and (I.10),

\[
\sum_{t=1}^T \sum_{s=0}^{\tau-1} \| Y_{t-s} (z_{t-s} - z_{t-s-1}) \|_2 \leq \tau c_\Lambda \frac{\kappa}{2} \cdot L_{\text{eff}} \mathcal{d} \sqrt{\frac{2 (1 + 10 R^2_Y)}{\kappa} \mathcal{L}},
\]

which is the first inequality of the lemma. For the second inequality, we establish the following analogue of Eq. (I.8):

\[
\sum_{t=1}^T \sum_{s=0}^{\tau-1} \| H_t (z_{t-s} - z_{t-s-1}) \|_2 \leq \tau c_\Lambda \left( \sum_{t=1}^T \text{tr}(H_t \Lambda_t^{-1} H_t) \right)^{\frac{1}{2}} \left( \sum_{t=1}^T \| \nabla_t \|_2 \right)^{\frac{1}{2}}.
\]

Again, we bound \( \sum_{t=1}^T \| \nabla_t \|_2 \leq d L_{\text{eff}} \cdot \mathcal{L} \) as in Eq. (I.9). Moreover, from Eq. (F.2), we can bound \( \sum_{t=1}^T \text{tr}(H_t \Lambda_t^{-1} H_t) \leq d \mathcal{L} \). Thus,

\[
\sum_{t=1}^T \sum_{s=0}^{\tau-1} \| H_t (z_{t-s} - z_{t-s-1}) \|_2 \leq \tau d L_{\text{eff}} c_\Lambda \frac{1}{2} \mathcal{L},
\]

which is precisely the second inequality of the lemma.

\[ \square \]

I.9 Proof of Lemma G.8

We state a slightly sharper variant of Proposition F.8, which considers directions limited to \( \delta \in \mathcal{C} - \mathcal{C} \):

**Claim I.5.** Set \( c_{\psi ; t} := \max \{ 1, \frac{\psi_G (h + 1)^2}{h R^2_G} \} \). Let \( \delta = z - z' \) for some \( z, z' \in \mathcal{C} \). Then,

\[
\delta^T \left( \sum_{s=1}^T H_s H_s \right) \delta \geq \frac{K}{2} \delta^T \left( \sum_{s=1-h}^T H_s H_s \right) \delta - 20 h R^2_Y \cdot R^2_G c_{\psi ; t}.
\]

**Proof.** The proof is analogous to Proposition F.8, but instead, the remainder term need only account for directions \( z - z' \) for \( z, z' \in \mathcal{C} \). This replaces the factor of \( R_Y \) one would otherwise obtain with a factor of \( \max_{s \neq t} \| Y \delta_s \| \leq 2 R_Y \), yielding a remainder term of \( 20 h R^2_Y \cdot R^2_G c_{\psi ; t} \) instead of \( 5 h R^2_Y \cdot R^2_G c_{\psi ; t} \) in the original proposition. \[ \square \]
Let us now turn to the proof of our lemma. From Claim I.5, we have
\[
\sum_{s=1}^{\tau} \|H_{kj} \delta_{kj+s}\|^2_2 = \delta_{kj+1}^T \left( \sum_{s=1}^{\tau} H_{kj+s}^T H_{kj+s} \right) \delta_{kj+1} \\
\geq \frac{\kappa}{2} \delta_{kj+1}^T \left( \sum_{s=1-h}^{\tau} Y_{kj+s}^T Y_{kj+s} \right) \delta_{kj+1} - 20hc_{\psi,T} R_G^2 R_{Y,C}^2
\]
Moreover, for any \(i \in [h]\), we have
\[
\sum_{s=1}^{\tau} \sum_{i=0}^{h} \|Y_{kj+s-i} \delta_{kj}\|^2_2 = \delta_{kj+1}^T \left( \sum_{s=1}^{\tau} Y_{kj+s-i}^T Y_{kj+s-i} \right) \delta_{kj+1} \\
\leq \delta_{kj+1}^T \left( \sum_{s=1-h}^{\tau} Y_{kj+s}^T Y_{kj+s} \right) \delta_{kj+1}
\]
Thus, for \(\nu \leq \frac{\kappa}{4}\), we have
\[
\sum_{s=1}^{\tau} \sum_{i=0}^{h} \nu(h^{-1} + \mathbb{I}_{i=0}) \|Y_{kj+s-i} \delta_{kj}\|^2_2 \leq 2\nu \delta_{kj+1}^T \left( \sum_{s=1-h}^{\tau} Y_{kj+s}^T Y_{kj+s} \right) \delta_{kj+1} \\
\leq \frac{\kappa}{2} \delta_{kj+1}^T \left( \sum_{s=1-h}^{\tau} Y_{kj+s}^T Y_{kj+s} \right) \delta_{kj+1} \\
\leq \sum_{s=1}^{\tau} \|H_{kj} \delta_{kj}\|^2_2 + 20hc_{\psi,T} R_G^2 R_{Y,C}^2
\]
Hence, rearranging, we have
\[
\text{Reg}_{\text{cancel}} := \sum_{j=1}^{\max} \sum_{s=1}^{\tau} \left( \sum_{i=0}^{h} \nu(h^{-1} + \mathbb{I}_{i=0}) \|Y_{kj+s-i} \delta_{kj}\|^2_2 - \|H_{kj+s} \delta_{kj}\|^2_2 \right) \\
\leq \sum_{j=1}^{\max} 20hc_{\psi,T} R_G^2 R_{Y,C}^2 \\
\leq \frac{T}{\tau} 20hc_{\psi,T} R_G^2 R_{Y,C}^2
\]
Finally, let us simplify the dependence on \(c_{\psi,T}\). We have
\[
c_{\psi,T} = \max \left\{ \frac{1}{\tau}, \frac{c_{\psi,T}^2 (h+1)^2}{h R_G^2} \right\} \leq \frac{1}{\tau} + \frac{c_{\psi,T}^2}{h R_G^2}
\]
Together with \(\nu \leq \frac{\kappa}{4}\), this gives
\[
\text{Reg}_{\text{cancel}} \leq \frac{20\nu h}{\tau} T c_{\psi,T} R_G^2 R_{Y,C}^2 \leq \frac{20\nu h}{\tau} T R_G^2 R_{Y,C}^2 + 20\nu T c_{\psi,T}^2 R_{Y,C}^2 \\
\leq \frac{20T}{\tau} \cdot \nu h R_G^2 R_{Y,C}^2 + 5T c_{\psi,T}^2 \cdot \kappa R_{Y,C}^2.
\]
\[
\square
\]
### I.10 Proof of Lemma G.9

From Eq. (G.15), we bound
\[
\text{OCOReg}_T \left( \frac{\nu}{4\eta}; z* \right) \leq \frac{1}{4\eta} \text{Reg}_{\text{block}} + T c_{G}^2 \text{ERR}(\nu) + \hat{\text{Reg}}_T,
\]
where from Eq. (G.14) we have
\[
\text{Reg}_{\text{block}} \leq 8\tau \cdot \nu R_{Y,C} + 8\nu R_{Y,C} \left( \max_{i \in [h]} \text{Reg}_{Y,\text{move},i} \right) + 4R_{Y,C} R_{\pi_0} \cdot \text{Reg}_{H,\text{move}} + \text{Reg}_{\text{cancel}}.
\]
\[
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\]
Let us develop the above bound on $\text{Reg}_{\text{block}}$. From Lemma G.7, we have

$$\text{Reg}_{Y,\text{move},i} \leq \tau c_\alpha c_\lambda^{\frac{1}{2}} \cdot d_{\text{eff}} \frac{2 \left( 1 + 10 R_G^2 \right)}{\kappa} \| \mathcal{L} \|, \quad \text{and} \quad \text{Reg}_{H,\text{move}} \leq \tau c_\lambda c_\lambda^{\frac{1}{2}} \cdot d_{\text{eff}} \| \mathcal{L} \|,$$

and from Lemma G.8, we have $\text{Reg}_{\text{cancel}} \leq \frac{20 T}{\tau} \cdot \nu h R_G^2 R_{\lambda,\mathcal{C}}^2 + 5 T \epsilon_\mathcal{G}^2 \cdot \kappa R_{\lambda,\mathcal{C}}^2$. Thus, using followed by

$$\text{Reg}_{\text{block}} \leq 8 \tau \cdot \nu Y, + 8 \nu Y, \left( \max_{t \in [h]} \text{Reg}_{Y,\text{move},i} \right) + 4 R_Y, \cdot R_{\pi_0} \cdot \text{Reg}_{H,\text{move}} + \text{Reg}_{\text{cancel}}$$

\begin{align*}
\overset{(i)}{\leq} & 8 \tau c_\alpha c_\lambda^{\frac{1}{2}} Y, \left( \nu + \nu d_{\text{eff}} \frac{2 \left( 1 + 10 R_G^2 \right)}{\kappa} \| \mathcal{L} \| + d_R G \cdot L_{\text{eff}} \| \mathcal{L} \| \right) + \frac{20 T}{\tau} \cdot \nu h R_G^2 R_{\lambda,\mathcal{C}}^2 + 5 T \epsilon_\mathcal{G}^2 \cdot \kappa R_{\lambda,\mathcal{C}}^2 \\
\overset{(ii)}{\leq} & 8 \tau c_\alpha c_\lambda^{\frac{1}{2}} Y, \cdot d_{\text{eff}} \| \mathcal{L} \| \left( \nu \sqrt{\frac{2 \left( 1 + 10 R_G^2 \right)}{\kappa} + 2 R_G} \right) + \frac{20 T}{\tau} \cdot \nu h R_G^2 R_{\lambda,\mathcal{C}}^2 + 5 T \epsilon_\mathcal{G}^2 \cdot \kappa R_{\lambda,\mathcal{C}}^2
\end{align*}

where $(i)$ uses the above bounds together with $c_\alpha c_\lambda^{\frac{1}{2}} \geq 1$ (see Lemma G.6), and $(ii)$ uses $\nu \leq 1 \leq L_{\text{eff}}$ and $d_R G \cdot \| \mathcal{L} \| \geq 1$, and where the last line disposal of constants. Using $R_G \geq 1$, and the assumption $\nu \leq \frac{\sqrt{2}}{4(1 + R_Y)}$, the above is at most

$$\text{Reg}_{\text{block}} \lesssim \tau c_\alpha^{-\frac{1}{2}} Y, \cdot R_G d_{\text{eff}} \cdot \| \mathcal{L} \| + \frac{T}{\tau} \cdot \nu h R_G^2 R_{\lambda,\mathcal{C}}^2 + T \epsilon_\mathcal{G}^2 \cdot \kappa R_{\lambda,\mathcal{C}}^2,$$

Next, using $\lambda \geq c_\lambda \tau$, we have from Lemma G.6,

$$c_\lambda = 2(1 + R_Y) + 2 R_Y \sqrt{\frac{\tau R_G^2}{\lambda}} \lesssim c_\lambda^{\frac{1}{2}} \left( 1 + R_Y \right) R_G.$$

Thus, we obtain

$$\text{Reg}_{\text{block}} \lesssim c_\lambda^{-\frac{1}{2}} \tau \left( 1 + R_Y \right) \cdot R_Y, \cdot R_G^2 \cdot d_{\text{eff}} \cdot \| \mathcal{L} \| + \frac{T}{\tau} \left( \alpha \nu h R_G^2 R_{\lambda,\mathcal{C}}^2 \right) + \epsilon_\mathcal{G}^2 \left( \alpha R_{\lambda,\mathcal{C}}^2 + \text{Err} \left( \nu \right) \right) + \text{Reg}_T.$$

Combining with $\eta = \frac{3}{\tau}$, we have

$$\text{OCOR}_T \left( \frac{\nu}{4 \eta} ; z_* \right)$$

\begin{align*}
\leq & \frac{1}{4 \eta} \text{Reg}_{\text{block}} + T \cdot \epsilon_\mathcal{G}^2 \cdot \text{Err} \left( \nu \right) + \text{Reg}_T \\
\lesssim & c_\lambda^{-\frac{1}{2}} \left( \alpha \left( 1 + R_Y \right) R_Y, R_G^2 \cdot d_{\text{eff}} \cdot \| \mathcal{L} \| + \frac{T}{\tau} \left( \alpha h R_G^2 R_{\lambda,\mathcal{C}}^2 \right) + T \epsilon_\mathcal{G}^2 \left( \alpha R_{\lambda,\mathcal{C}}^2 + \text{Err} \left( \nu \right) \right) + \text{Reg}_T. \\
\end{align*}

Finally, let us substitute in

$$\text{Err} \left( \nu \right) := \frac{\eta (h + 1) L_{\text{eff}}^2}{\nu} + \eta \beta^2 (c_v + 2 R_Y, \mathcal{C})^2 + \frac{4 R_Y, \mathcal{C}}{\eta},$$

$$\lesssim \frac{h L_{\text{eff}}^2}{\alpha \nu} + \frac{1}{\alpha} \beta^2 (c_v^2 + R_{\lambda,\mathcal{C}}^2) + \alpha R_{\lambda,\mathcal{C}}^2.$$

Since $\alpha \leq \beta$ by necessity and $\kappa \leq 1$, we have $\alpha \leq \frac{\beta^2}{\alpha}$, so that

$$\text{Err} \left( \nu \right) + \alpha \kappa R_{\lambda,\mathcal{C}}^2 \lesssim \frac{h L_{\text{eff}}^2}{\alpha \nu} + \beta^2 (c_v^2 + R_{\lambda,\mathcal{C}}^2 + R_{\lambda,\mathcal{C}}^2).$$

Altogether, combined with the bound $c_\lambda \leq 1$, this yields

$$c_\lambda \text{OCOR}_T \left( \frac{\nu}{4 \eta} ; z_* \right) \lesssim \frac{T \epsilon_\mathcal{G}^2}{\alpha} \left( \frac{h L_{\text{eff}}^2}{\nu} + \beta^2 (c_v^2 + R_{\lambda,\mathcal{C}}^2 + R_{\lambda,\mathcal{C}}^2) \right) + \text{Reg}_T.$$

as needed. \qed
I.11 Proof of Lemma G.10

Consider \( \text{MoveDiff}_T := \sum_{t=1}^{T} F_t(z_t) - f_t(z_t) \). The decomposition Lemma F.6 holds verbatim, and by appropriately modifying Lemma F.7 to use the fact that the iterates are based on \( \hat{H}_t, \hat{\Lambda}_t \), we arrive at

\[
\text{MoveDiff}_T \leq \eta h^2 L_{eff}^2 R_G \cdot \sqrt{\sum_{t=1}^{T} \text{tr}(Y_t \hat{\Lambda}_t^{-1} Y_t)} \cdot \sqrt{\sum_{t=1}^{T} \text{tr}(\hat{H}_t^\top \hat{\Lambda}_t^{-1} \hat{H}_t)}.
\]

As in Eq. (F.2), we bound

\[
\sum_{t=1}^{T} \text{tr}(\hat{H}_t^\top \hat{\Lambda}_t^{-1} \hat{H}_t) \leq d \log(1 + T R_H^2 / \lambda) \leq d \mathcal{L},
\]

where we take \( \lambda \geq 1 \) and use \( \mathcal{L} = \log(1 + T R_H^2 / \lambda) \) from Eq. (G.6). Moreover, applying Lemma G.6 with \( \tau = 0 \), we have that \( \hat{\Lambda}_t^{-1} \preceq 4(1 + R_Y)^2 \Lambda_t^{-1} \), giving

\[
\sum_{t=1}^{T} \text{tr}(Y_t \hat{\Lambda}_t^{-1} Y_t) \leq 4(1 + R_Y)^2 \sum_{t=1}^{T} \text{tr}(Y_t \Lambda_t^{-1} Y_t) \leq 4(1 + R_Y)^2 2d(1 + 10 R_Y^2) \mathcal{L}.
\]

where the last inequality uses Eq. (I.10). Thus,

\[
\text{MoveDiff}_T \leq 9 \eta (1 + R_Y) h^2 L_{eff}^2 \mathcal{L} R_G \cdot \sqrt{(1 + R_Y^2) / \kappa}.
\]

\[
\leq 9 \eta \kappa^{1/2} (1 + R_Y)^2 R_G h^2 \cdot d L_{eff}^2 \mathcal{L}
\]

\[\square\]

J Lower and Upper Bounds on Euclidean Movement

J.1 Proof of Theorem 2.3

Our construction is loosely based of of [5, Theorem 13].

Recall the lower bound set up \( \mathcal{C} = [-1, 1], f_t(z) = (v_t - \epsilon z)^2, \) and \( \epsilon \leq 1 \). Let \( E \) be an epoch length to be selected, and suppose for simplicity that \( k = T/E \) is an integer. Let \( T_i := 1 + E \cdot (i - 1) \) denote the start of each epoch for \( i \geq 1 \). Let us define the distribution \( \mathcal{D} \) over \( v_1, \ldots, v_T \) via:

\[
v_t := \begin{cases} \mathcal{I}_{i,d} \sim \text{Unif}([-1, 1]) & t = T_i \\ \mathcal{V}_{T_i} & t \in \{T_i + 1, \ldots, T_{i+1} - 1\} \end{cases}
\]

Lastly, recall the definition:

\[
\mu-\text{Reg}_T := \sum_{t=1}^{T} f_t(z_t) - \inf_{z \in \mathcal{C}} \sum_{t=1}^{T} f_t(z) + \mu \sum_{t=1}^{T} |z_{t-1} - z_t|
\]

Our key technical ingredient is the following lemma, which shows that if the regularizer is large enough, the optimal strategy is essentially to select \( z_t = z_{T_i} \) within any given epoch \( i \):

Lemma J.1. For \( \mu \geq 4E\epsilon \),

\[
\sum_{t=T_{i+1}}^{T_{i+1}} f_t(z_t) + \mu |z_t - z_{t-1}| \geq (E - 1) f_t(v_{T_i} - z_{T_i}).
\]
Proof. We can write
\[
\sum_{t=T_i+1}^{T_{i+1}-1} f_t(z_t) + \mu|z_t - z_{t-1}| = \sum_{t=T_i+1}^{T_{i+1}-1} f_{T_i}(z_t) + \mu|z_t - z_{t-1}|
\]
\[
\geq \sum_{s=1}^{E-1} f_{T_i}(z_t) + \mu \cdot \max_{t=T_{i+1}, \ldots, T_{i+1}-1} |z_{T_i} - z_t|
\]
\[
\geq \sum_{s=1}^{E-1} f_{T_i}(z_t) + \frac{\mu}{E - 1} |z_{T_i} - z_t|, \quad \text{where} \quad z := g(z)
\]
where the first inequality uses the triangle inequality, and the second replaces the maximum by the average. Define \( \mu_0 = \frac{\mu}{(E-1)T_i} \), and set \( g(z) := f_{T_i}(z_t) + \frac{\mu}{E-1} |z_{T_i} - z_t| = (v_{T_i} - \epsilon z)^2 + 2\epsilon \mu_0 |z_{T_i} - z_t| \). Then,
\[
\partial g(z) = 2\epsilon (\epsilon z - v_{T_i} + \mu_0 \sigma(z))
\]
where \( \sigma(z) = 1 \) if \( z_{T_i} > z \), \(-1 \) if \( z_{T_i} < z \), and is in interval \([-1, 1]\) if \( z = z_{T_i} \). Now, if \( \mu_0 \geq 2 \), then, \( |\epsilon z - v_{T_i}| \leq \mu_0 \), so that the first order optimality conditions are met by selecting \( z^* = z_{T_i} \). This yields
\[
g(z^*) = (v_{T_i} - \epsilon z_{T_i})^2.
\]
The bound follows. \( \square \)

By summing within different epochs, the above lemma implies a simple lower bound on \( \mu \cdot \text{Reg}_T \):
\[
\mu \cdot \text{Reg}_T = \sum_{i=1}^{k} \sum_{t=T_i}^{T_{i+1}-1} f_i(z_t) - f_i(z) + \mu \|z_t - z_{t-1}\|
\]
\[
\overset{(i)}{=} \sum_{i=1}^{k} f_{T_i}(z_{T_i}) - E f_{T_i}(z) + \mu \|z_{T_i} - z_{T_i-1}\| + \left( \sum_{i=T_i}^{T_{i+1}-1} f_i(z_t) + \mu \|z_t - z_{t-1}\| \right)
\]
\[
\overset{(ii)}{\geq} \sum_{i=1}^{k} f_{T_i}(z_{T_i}) - E f_{T_i}(z) + \mu \|z_{T_i} - z_{T_i-1}\| + (E - 1)f_{T_i}(z_{T_i})
\]
\[
\geq \sum_{i=1}^{k} f_{T_i}(z_{T_i}) - E f_{T_i}(z) + (E - 1)f_{T_i}(z_{T_i})
\]
\[
= \sup_{z \in \mathcal{C}} E \left( \sum_{i=1}^{K} f_{T_i}(z_{T_i}) - f_{T_i}(z) \right),
\]
where (i) uses that \( f_i = f_{T_i} \) in epoch \( i \) and (ii) uses Lemma J.1. Crucially, the above quantity is scaled up by a factor of \( E \), and the learner is forced to commit to a single iterate per epoch. Continuing with \( f_{T_i}(z) = (v_{T_i} - \epsilon z)^2 \),
\[
\mu \cdot \text{Reg}_T \geq \sup_{z \in \mathcal{C}} E \left( \sum_{i=1}^{k} (v_{T_i} - \epsilon z_{T_i})^2 - (v_{T_i} - \epsilon z)^2 \right)
\]
\[
= \sup_{z \in \mathcal{C}} E \left( \sum_{i=1}^{k} -2\epsilon v_{T_i} z_{T_i} + \epsilon^2 z_{T_i}^2 + 2\epsilon \epsilon v_{T_i} - \epsilon^2 \cdot \epsilon^2 \cdot \epsilon^2 \leq 1 \right)
\]
\[
\geq \sup_{z \in \mathcal{C}} E \left( \sum_{i=1}^{k} -2\epsilon v_{T_i} z_{T_i} + 2\epsilon \epsilon v_{T_i} - k\epsilon^2 \right).
\]
Taking an expectation, and noting that \( E[v_T, z_T] = 0 \) by construction, we have that

\[
E[\mu - \text{Reg}_T] \geq 2\epsilon E \left(2\epsilon E \left[ \sup_{z \in C} \sum_{i=1}^{k} v_i \right] - k\epsilon^2 \right)
\]

\[
= 2\epsilon E \left( \sum_{i=1}^{k} v_i \right) - \frac{k\epsilon}{2}
\]

\[
\geq 2\epsilon \left( c\sqrt{k} - \frac{k\epsilon}{2} \right),
\]

where \( c \leq 1 \) is a universal constant. \(^8\)

Let us now tune the above bound. Select

- \( k = (8Tc/\mu)^{2/3} \)
- \( \epsilon = \mu/4E \).

We first check that these parameters are valid:

**Claim J.2.** For a universal constant \( c_1 \), it holds that if \( \mu \leq c_1 T \), then \( k \geq 1 \) and \( \epsilon \leq 1 \).

**Proof.** For \( \mu \leq 8cT \), \( k \geq 1 \) Moreover,

\[
\epsilon = \frac{\mu}{4E} = \frac{\mu}{4T} \leq (8Tc/\mu)^{2/3} \frac{\mu}{T} = 4\epsilon^{2/3} (\mu/T).
\]

Hence, for \( \mu \leq T/4\epsilon^{2/3} \), the above is at most 1. Setting \( c_1 = \min\{8c, 1/4\epsilon^{2/3}\} \) concludes. \( \square \)

For the above choices, we have

\[
E[\mu - \text{Reg}_T] \geq 2\epsilon E \left( c\sqrt{k} - \frac{k\epsilon}{2} \right)
\]

\[
= \frac{\mu}{2} \left( c\sqrt{k} - \frac{k^2\mu}{8T} \right) = \frac{c\sqrt{k}\mu}{4} \left( 2 - \frac{k^{3/2}}{8Tc/\mu} \right)
\]

\[
\geq \frac{c\sqrt{k}\mu}{4} \geq \frac{c\epsilon(8Tc/\mu)^{2/3}}{4}
\]

\[
\geq c_{2}\mu(T/\mu)^{1/3} = c_{2}(\mu^{2}T)^{1/3},
\]

for some universal constant \( c_2 \). Moreover, suppose that that \( E[\text{OcoReg}_T] \leq R \). Then, for \( \mu \geq c_1 T \)

\[
c_{2}(\mu^{2}T)^{1/3} \leq E[\mu - \text{Reg}_T] \leq R + \mu E[\text{EucCost}_T].
\]

Rearranging, we have that if \( c_{2}(\mu^{2}T)^{1/3} \geq 2R \), \( E[\text{EucCost}_T] \geq \frac{c_{2}}{2}(T/\mu)^{1/3} \). For this to hold, we take \( \mu = \sqrt{(2R/c_{2})^{3}/T} \), yielding

\[
E[\text{EucCost}_T] \geq \frac{c_{2}}{2}(T/\sqrt{(2R/c_{2})^{3}/T})^{1/3} = \frac{c_{2}}{2}(c_{2}T/2R)^{1/2} \geq c_{3}\sqrt{T/R}.
\]

(J.1)

Finally, we need to ensure that \( \mu \leq c_1 T \), which hold for \( (2R/c_{2})^{3}/T \leq c_{1}^{3}T^{2} \), i.e. for \( R \leq c_{4}T \) for a universal \( c_4 \).

### J.2 Matching Tradeoff via ONS

We now show that ONS matches the tradeoff in **Theorem 2.3** up to logarithmic factors, problem constants and dimension. To show this, we first check that OCOAM losses satisfy the general ONS regularity conditions. We say \( f \) is \( \tau \)-exp concave if \( \nabla^2 f \geq \tau \cdot \nabla f \nabla f \)^\(1\) [18]. The following is a direct consequence of **Lemma J.2**

**Lemma J.3.** Let \( f_\ell \) be an OCOAM loss with parameters bounded as in **Definition 2.1**, where \( \ell \) satisfies **Assumption 1**. Then \( f_\ell \) is \( \frac{\ell}{L^{\text{eff}}_{\text{eff}}} \)-exp concave, and \( R_{H}L_{\text{eff}}\text{-Lipschitz on } C \).

\(^8\)Note the folklore results that the expectation average of \( k \) Rademacher random variables scales as \( \sqrt{k} \)
We now show that ONS matches the optimal \((\mu^2 T)^{1/3}\) scaling up to dimension and logarithmic factors:

**Theorem J.1.** Consider ONS on a sequence family of \(G\)-Lipschitz, \(\tau\)-exp concave functions on a convex set \(C\) of diameter \(D\). Let define \(R_0 = (GD + \tau^{-1}) \cdot d \log T\) be the standard upper bound (up-to-constants) on the regret of ONS [18]. Then, for any \(\mu \in \mathbb{R}\), there exists a choice of regularization parameter \(\lambda\) such that ONS with \(\eta = 2 \max\{4GD, 1/\tau\}\) has:

\[
\mu\text{-Reg}_T \lesssim (R_0 D^2 \cdot T \mu^2)^{1/3} + R_0.
\]

For the special case of OCOAM, the above guarantee can also be satisfied for by Semi-ONS (albeit with modified dependence on problem parameters).

Consider the ONS algorithm, with updates

\[
z_{t+1} = z_t - \eta A_t^{-1} \nabla f_t, \quad z_{t+1} = \arg\min_{z \in \mathcal{C}} \|z_{t+1} - z\|^2_{\Lambda_t}, \quad \Lambda_t := \lambda I + \sum_{s=1}^{t-1} \nabla f_s \nabla f_s^T, \quad \nabla f_t := \nabla f_t(z_t)
\]

(\text{J.2})

Set \(\eta = 2 \max\{4GD, 1/\tau\}, \lambda \geq G^2\). From Hazan [18, Section 4.3], with the notation change \(\eta \leftarrow 1/\gamma, \tau \leftarrow \alpha \Lambda_t \leftarrow A_t\), and \(\lambda \leftarrow \epsilon\), ONS has unary regret bouned by

\[
\text{OcoReg}_T \leq \eta \frac{T}{2} \sum_{t=1}^{T} \|\nabla f_t\| + \frac{D^2 \lambda}{2\eta} \leq \frac{d\eta}{2} \log(1 + T) + \frac{D^2 \lambda}{2\eta}.
\]

Moreover, we can bound

\[
\text{EucCost}_T = \sum_{t=1}^{T} \|z_t - z_{t-1}\| \leq \frac{1}{\sqrt{\lambda}} \sum_{t=1}^{T} \|\Lambda_{t}^{1/2}(z_t - z_{t-1})\|
\]

\[
\leq \frac{1}{\sqrt{\lambda}} \sum_{t=1}^{T} \|\Lambda_{t}^{1/2}(z_t - z_t)\| = \frac{1}{\sqrt{\lambda}} \sum_{t=1}^{T} \|\Lambda_{t}^{-1/2} \nabla f_t\|
\]

\[
\leq \frac{\eta}{\sqrt{\lambda}} \sqrt{T \sum_{t=1}^{T} \|\nabla f_t\|^2_{\Lambda_t^{-1}}} \leq \frac{\eta}{\sqrt{\lambda}} \sqrt{T d \log(1 + T)},
\]

where (i) uses \(\Lambda_t \succeq \lambda\), (ii) uses the Pythagorean theorem, (iii) uses Cauchy-Schwartz, and (iv) applies the log-determinant lemma as in Hazan [18, Section 4.3] with \(\lambda \geq G^2\). Hence,

\[
\mu\text{-Reg}_T \leq \frac{d\eta}{2} \log(1 + T) + \frac{D^2 \lambda}{2\eta} + \frac{\eta \mu}{\sqrt{\lambda}} \sqrt{T d \log(1 + T)}.
\]

Set \(\lambda_0\) to satisfy

\[
\frac{D^2 \lambda_0}{2\eta} + \frac{\mu}{\sqrt{\lambda_0}} \sqrt{T d \log(1 + T)} = \frac{D^2 \lambda_0}{\eta}
\]

\[
= \frac{D^2}{\eta} \cdot \left(\frac{2\eta^2}{D^2} \mu \sqrt{T d \log(1 + T)}\right)^{2/3}
\]

\[
= \frac{D^2}{\eta} \cdot \left(\frac{2\eta^{3/2} \mu^2 T d \log(1 + T)}{D^4}\right)^{1/3}
\]

\[
= (2D^2 \cdot \mu^2 T \cdot \eta d \log(1 + T))^{1/3}.
\]

Setting \(\lambda = G^2 \vee \lambda_0\) yields

\[
\mu\text{-Reg}_T \leq \frac{d\eta}{2} \log(1 + T) + \frac{G^2 D^2}{2\eta} + \frac{d\eta}{2} \log(1 + T) + \frac{D^2 \lambda_0}{2\eta} + \frac{\mu}{\sqrt{\lambda_0}} \sqrt{T d \log(1 + T)}
\]

\[
\leq \frac{d\eta}{2} \log(1 + T) + \frac{G^2 D^2}{2\eta} + \left(2D^2 \cdot \mu^2 T \cdot \eta d \log(1 + T)\right)^{1/3}.
\]

Substituting in \(\eta = 2 \max\{4GD, 1/\tau\}\), and defining \(R_0 = \max\{GD, 1/\tau\} \cdot d \log(1 + T)\) gives that the above is at most

\[
\mu\text{-Reg}_T \lesssim (R_0 D^2 \cdot T \mu^2)^{1/3} + R_0.
\]