Supplementary Information On

**Stochastic dynamics of gene switching and energy dissipation for gene expression**

Quan Liu¹, FengZhen Yu², Liang Yi³, Yijun Gao¹, Rong Gui¹, Ming Yi⁴, Jianqiang Sun⁵,*

¹Department of Physics, College of Science, Huazhong Agricultural University, Wuhan, China

²Department of Chemistry, College of Science, Huazhong Agricultural University, Wuhan, China

³College of Animal Science and Technology and College of Veterinary Medicine, Huazhong Agricultural University, Wuhan, China

⁴School of Mathematics and Physics, China University of Geosciences, Wuhan, China

⁵School of Automation and Electrical Engineering, Linyi University, Linyi, China

1 DETAILED DERIVATION FROM CHEMICAL MASTER EQUATIONS TO STEADY-STATE DISTRIBUTIONS

Since it is difficult to obtain the theoretical results of chemical master equations, the method of probability-generating functions is used to change chemical master equations into differential equations which may be solved with mathematical skill. Based on this method, two factorial probability-generating functions $G_i(z) = \sum_{m=0}^{\infty} P_i(m) z^m$ with $i = 0, 1$ are introduced. Substituting these into Eq. 2:

$$k_{\text{off}} G_1(z) - k_{\text{on}} G_0(z) + k_2 z G_1'(z) - a k_1 G_0(z) + a k_3 (z - 1) G_0(z) - (z - 1) G_0'(z) = 0,$$

$$-k_{\text{off}} G_1(z) + k_{\text{on}} G_0(z) - k_2 z G_1'(z) + a k_1 G_0(z) + a k_3 (z - 1) G_1(z) - (z - 1) G_1'(z) = 0.$$ (S1)

Here, all parameters have been normalized by $k_4$ as mentioned in Sec.II. It is found from Eq. S1 that

$$e^{-ak_3 z} G_0 = -e^{ak_3 z} G_1'.$$ (S2)

$e^{-ak_3 z} G_1$ is set as $H_1$ for simplicity. From Eq. S2, we can determine that $H_0' = -H_1'$. This allows us to obtain $H_0$ from $H_1$. In the following, we will only focus on the expression of $H_1$, which can be deduced from Eq. S1 as

$$A(z) H_1'' + B(z) H_1' + C H_1 = 0$$ (S3)

in which $A(z) = (k_2 + 1) z - 1$, $B(z) = a k_2 k_3 z + k_{\text{off}} + k_{\text{on}} + a k_1 + k_2 + 1$, and $C = a k_2 k_3$. Until now, it seems difficult to obtain the analytical distribution because the coefficients of Eq. S3 are dependent on the variable $z$. Another transformation of the form $H_1(z) = e^{m z} W(nz + p)$ (here, $m$, $n$ and $p$ are undetermined constants) is introduced in Eq. S3 and hence a solvable differential equation about $W$ is obtained as

$$\omega W''(\omega) - (\beta - \omega) W'(\omega) - \alpha W(\omega) = 0$$ (S4)
where

\[
\begin{align*}
\omega &= \frac{a k_2 k_3}{k_2 + 1} z - \frac{a k_2 k_3}{(k_2 + 1)^2}, \\
\alpha &= \frac{a k_2 k_3}{(k_2 + 1)^2} + \frac{k_{\text{on}} + k_{\text{off}} + ak_1}{k_2 + 1}, \\
\beta &= \frac{a k_2 k_3}{(k_2 + 1)^2} + \frac{k_{\text{on}} + k_{\text{off}} + ak_1}{k_2 + 1} + 1.
\end{align*}
\]

(S5)

There are two independent solutions of Eq. S4. One is expressed by the Tricomi function of the form \(U(\alpha, \beta; \omega)\). This result is inadmissible because it is required that \(P_i(m) \to 0\) for \(m \to \infty\). The other solution is expressed by the confluent hypergeometric function of the form \(1 F_1(\alpha, \beta; \omega)\). The analytical expressions for the probability-generating functions are presented as below

\[
\begin{align*}
G_0(z) &= A_0\left[e^{a k_3(z-1)} - e^{ak_3 z/(k_2+1)} 1 F_1(\alpha - 1, \beta - 1; \omega)\right], \\
G_1(z) &= A_0 e^{ak_3 z/(k_2+1)} 1 F_1(\alpha, \beta; \omega)
\end{align*}
\]

(S6)

Furthermore, based on the relationship between probability distribution and generating function:

\[
P_i(m) = \frac{1}{m!} \frac{d^m}{dz^m} G_0(z) \big|_{z=0}.
\]

(S7)

The analytic results of probability distributions are obtained and shown in the text (i.e., Eq.3 in Sec.II).

2 THE RESULTS OF NUMERICAL SIMULATIONS

The direct method to verify our analytical results is numerical simulations. The approximate probability distributions of \(P_i(m)\) obtained by Monte Carlo simulation are given in Fig[S1]

![Figure S1](image_url)

Figure S1. The numerical simulated results about probability distribution functions of the gene’s states. The parameters are the same with Fig.3(a) and (b): \(k_1 = 20, k_2 = 10, k_3 = 1.5, k_{\text{on}} = k_{\text{off}} = 2\). \(P_0(m)\) is the probability of the gene’s “off” state indicated with orange curve, \(P_1(m)\) is the probability of gene’s “on” state indicated with blue curve.
Figure S2. The numerical simulated results about gene’s state dominance factor $\delta$ which is consistent with that in Fig.4(c). $\delta$ will increase with the increasing of the strength of external signal in Mode I. The values of other parameters are listed in Table I.

Moreover, the curve between $\delta$ and $a$ which is obtained by Monte Carlo simulation with the same set of parameters in Fig.4(c) is shown in Fig. S2. It is obviously that the trend of $\delta$ is similar with the curve in Fig.4(c).

3 THE COMPARISONS BETWEEN THE TOTAL ENERGY DISSIPATION AND THE ENERGY DISSIPATION IN THE SYNTHESIS-DEGRADATION PROCESS OF ROCK IN MODE I AND II

The Fig. S3 shows the trends of the total energy dissipation (i.e., $EP$) and the energy dissipation in the synthesis-degradation process of ROCK (i.e., $EP_m$) with the increase of the strengthen of external stimulations (i.e., $a$) in Mode I and II. It is obviously that their trends are consistent as $a$ increases in its respective modes. Specifically, $EP$ and $EP_m$ increase simultaneously in Mode I and decrease simultaneously in Mode II with the enhancing of the strengthen of external stimulations (i.e., $a$). Moreover, the difference between $EP$ and $EP_m$ diminishes both in Mode I and Mode II when $a$ increases.

Figure S3. The comparisons between the total energy dissipation and the energy dissipation in the synthesis-degradation process of ROCK in Mode I and II. The values of all parameters are listed in Table I.
4 PARAMETERS TABLE

The values of the main parameters used in calculations are given in Table I. All these parameters have been normalized by the degradation rate of ROCK (i.e., $k_4$).

Table I: The values of parameters used in calculations

| Parameters                                         | Value                |
|----------------------------------------------------|----------------------|
| The reaction rate of positive control ($k_1$)      | 20                   |
| The reaction rate of negative control ($k_2$)      | 10                   |
| The synthetic rate of ROCK ($k_3$)                 | 1.5 (Mode I) or 5 (Mode II) |
| The basic switch rate of gene to “on” state ($k_{on}$) | 2                   |
| The basic switching rate of gene to “off” state ($k_{off}$) | 2                   |