Naturally Nonminimal Supersymmetry

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Abstract

We consider the bounds imposed by naturalness on the masses of superpartners for arbitrary points in nonminimal supersymmetric extensions of the standard model and for arbitrary messenger scales. We discuss appropriate measures of naturalness and the status of nonminimal supersymmetry in the light of recent experimental results.
1 Introduction

The problem of electroweak scale naturalness provides perhaps the most important motivation for the consideration of supersymmetric extensions of the standard model (SSMs). Softly broken supersymmetry stabilizes the electroweak scale against quadratically divergent radiative corrections, but not against logarithmic divergences and finite corrections on the order of the superpartner masses. It is thus necessary that supersymmetry (SUSY) be effectively restored at a scale not much higher than the electroweak scale in order to avoid fine-tuning.

Previous studies [1, 4] have derived bounds on superpartner masses by requiring that there be no fine-tuning among the GUT-scale parameters of the supergravity-inspired minimal model with universal scalar and gaugino masses. The authors of [2] relaxed this universality assumption slightly, allowing for nonuniversal scalar masses, but maintaining the assumption of gaugino mass universality and the assumption that soft SUSY-breaking parameters are set at the GUT scale. In this paper, we consider the degree of fine tuning present at arbitrarily nonminimal points in the SSM parameter space and make no assumption regarding the so-called messenger scale at which soft SUSY-breaking parameters are set.

1.1 Fine-tuning defined

In the context of the SSM, the electroweak scale is determined at tree level by three mass parameters \( (m^2_x, m^2_y, m^2_Z) \) according to the equations

\[
\frac{m^2_Z}{2} = \frac{m^2_x - m^2_y \tan^2 \beta}{\tan^2 \beta - 1} \tag{1.1}
\]

\[
\sin 2\beta = \frac{2m^2_z}{m^2_x + m^2_y} \tag{1.2}
\]

Expressions for these mass parameters in terms of SSM parameters will be given in section 2. Since the mass of the pseudoscalar Higgs boson \( m^2_A = m^2_x + m^2_y \), we can exchange the three mass parameters for \( m_Z, m_A, \) and \( \tan \beta \). Each point in \( (m_Z, m_A, \tan \beta) \) space represents a possible Higgs sector of the SSM, and the actual \( Z \) boson mass defines a surface in this space.

Qualitatively, the electroweak scale is fine-tuned if a fractionally small change in the value of a parameter results in a fractionally large change in
The degree of fine-tuning in a parameter $a$ can thus be quantified by a sensitivity parameter $\Delta$

$$\Delta = \frac{\delta m_Z^2 a}{m_Z^2 \delta a}$$  \hspace{1cm} (1.3)

which was first introduced by Barbieri and Giudice [1]. Bounding the allowed degree of fine-tuning, then, corresponds to requiring that the sensitivity $\Delta$ of $m_Z$ to any parameter under consideration be less than some specified value; traditionally, one has taken $\Delta \lesssim 10$, corresponding to 10% fine-tuning, as an acceptable level. Inserting equations (1.1-1.2) into (1.3), we obtain the expression

$$\Delta = \frac{2a}{(\tan^2 \beta - 1) m_Z^2} \left[ \frac{\delta m_x^2}{\delta a} - \tan^2 \beta \frac{\delta m_y^2}{\delta a} + \frac{\tan \beta}{\cos 2\beta} \left(1 + \frac{m_Z^2}{m_A^2} \right) \left( \sin 2\beta \left( \frac{\delta m_x^2}{\delta a} + \frac{\delta m_y^2}{\delta a} \right) - 2 \frac{\delta m_z^2}{\delta a} \right) \right]$$  \hspace{1cm} (1.4)

We will impose the requirement $\Delta < 10$ to place limits on the parameters of the SSM as functions of the messenger scale, $\tan \beta$, and $m_A$.

As pointed out by Dimopoulos and Giudice [2], the fact that the coefficients of $\delta m_x^2$ and $\delta m_z^2$ vanish in the $\tan \beta \to \infty$ limit implies that $m_y$ must exhibit a dependence on a parameter in order for the requirement of a particular degree of naturalness to impose a globally valid limit on the parameter. This situation arises because the decoupling limit

$$m_x \sim m_Z \tan \beta \quad m_y \sim m_Z \quad m_z \sim m_Z \tan^{1/2} \beta$$  \hspace{1cm} (1.5)

is a perfectly natural corner of parameter space with large $\tan \beta$. The form of the dependence of $m_x$ and $m_z$ on a parameter are nonetheless relevant in determining how a naturalness bound varies with $\tan \beta$ and $m_A$.

We should emphasize that the degree of fine-tuning present in a particular realization of the SSM depends upon the choice of a parameter set which we regard as specifying the theory. Previous studies have taken as a parameter set the DR soft-SUSY breaking parameters at the GUT scale. The messengers which communicate SUSY-breaking to the SM particles and their superpartners could in principle, however, exist at any scale. Whatever the scale at which the SUSY-breaking parameters are set, their effect on the electroweak scale is determined by integrating the renormalization group (RG)
equations which describe their flow from the messenger scale \( \tilde{m} \) down to the electroweak scale \( m_Z \).

In section 2, we integrate the relevant one-loop RG equations in order to find the dependence of the electroweak scale on the the DR parameters at the messenger scale, independent of any assumptions of universality. In section 3, we apply equation (1.4) to obtain limits these on parameters and, in turn, on sparticle masses as functions of the messenger scale. Finally, in section 4, we consider mass limits on sparticles which first contribute to \( m_Z \) at the two-loop level.

### 1.2 Naturalness vs. sensitivity

The equation of sensitivity with unnaturalness has been criticized by Anderson and Castaño [3], who rightly point out that it is inappropriate in some cases. For example, a small scale \( m = \Lambda e^{-\frac{4\pi}{g^2}} \) resulting from dynamical symmetry breaking exhibits a strong sensitivity to \( g \) even for perfectly natural values of \( g \sim 1 \). This motivates them to introduce a more refined measure of naturalness which, instead of simply reflecting the sensitivity of \( m_Z \) to an underlying parameter, compares that sensitivity to the average sensitivity over the entire allowed parameter space. We do not employ their refined definition for this study. We do, however, present in this section some considerations which are, to our knowledge, original and which we believe elucidate the relationship between naturalness and sensitivity and clarify the circumstances under which their equation is appropriate.

Suppose an underlying parameter set \( a \) determines the electroweak scale according to a function \( m(a) \) and we imagine that the underlying parameters are distributed according to a probability distribution \( p(a) \). Then the probability distribution of electroweak scales is given by

\[
p(m') = \int da \, p(a) \delta(m' - m(a))
\]

and the probability of the electroweak scale being as low or lower than the

\[ ^1 \text{In fact, this procedure only yields the contributions to } m_Z \text{ enhanced by factors of } \ln(\tilde{m}/m_Z). \text{ Unenhanced terms, while present, are not only presumably smaller, but also scheme-dependent. We therefore ignore the unenhanced terms, but note that the the limits derived here may therefore be considered valid only for messenger scales satisfying } \ln(\tilde{m}/m_Z) \gg 1. \]
measured value \( m \) is

\[
P(m) = \int_0^m dm' \, p(m') \tag{1.7}
\]

We typically imagine the probability density of parameters \( p(a) \) to be some flat function over an allowed range. Given some \( p(a) \), the probability \( P(m) \) would seem an appropriate measure of the naturalness of a low electroweak scale.

In a multidimensional parameter space like that of the SSM, the examination of the \( P(m) \) implied by equations (1.6-1.7) for various \( p(a) \) seems a daunting task. We can nevertheless glean some insight by considering the simplified case in which \( m \) depends only on a single parameter \( a \), distributed uniformly between between 0 and \( a_{\text{max}} \). Then

\[
p(m') = \frac{1}{a_{\text{max}}} \left( \frac{\delta m}{\delta a} \right)^{-1} \bigg|_{a=m^{-1}(m')}
\tag{1.8}
\]

and, assuming \( \delta m/\delta a \) to be relatively constant over the allowed values of \( a \), we obtain

\[
P(m) = \frac{m}{a_{\text{max}}} \frac{\delta a}{\delta m} \bigg|_{a=a_{\text{max}}}
\tag{1.9}
\]

which implies \( \Delta \sim 1/P \), i.e. sensitivity is the inverse of naturalness. In the example of Anderson and Castaño, it is the strong dependence of \( \delta m/\delta g \) on \( g \) which violates the assumption made for our toy example and belies the simple relationship between \( \Delta \) and \( P \). Moreover, the integrations described by equations (1.6,1.7) may be easily carried out for the Anderson and Castaño’s example to find that, for \( g \) distributed uniformly on the interval \( 0 \leq g \leq \mathcal{O}(1) \), a hierarchy of scales \( m/\Lambda \sim 10^{-10} \) is not terribly improbable, while hierarchy \( m/\Lambda \sim 10^{-100} \) is quite improbable.

We believe that equations (1.6,1.7) provide a quantitative definition of naturalness similar in spirit to that of Anderson and Castaño, but conceptually simpler, since it allows naturalness to be defined \( \text{a priori} \) without reference to sensitivity. Our experience with a toy example has suggested that the qualitative equation of sensitivity to unnaturalness is justified when the value of \( \delta m/\delta a \) is not strongly dependent on the location in parameter space \( a \). Since such an approximation is indeed valid for the parameters considered here, we believe our equation of sensitivity with naturalness to be qualitatively valid for the case under consideration.
1.3 The calculation sketched

Before embarking on this calculation, we outline a didactic method which yields back-of-the-envelope estimates of the limits set by naturalness and the form of their dependence on $\Delta$ and the messenger scale $\tilde{m}$. If a mass parameter $m$ contributes to a higgs mass at tree level, then we expect from naturalness that

$$\mathcal{O}(1) m^2 \lesssim m^2_Z \Delta \quad (1.10)$$

$$m \lesssim \mathcal{O}(300 \text{ GeV}) \left( \frac{\Delta}{10} \right)^{1/2} \quad (1.11)$$

If, on the other hand, a particle with mass $m$ contributes at one-loop via a coupling $g \sim 1$, we expect

$$\mathcal{O}(1) \frac{g^2}{(4\pi)^2} \ln \left( \frac{\tilde{m}}{m} \right) m^2 \lesssim m_Z^2 \Delta \quad (1.12)$$

$$m \lesssim \mathcal{O}(650 \text{ GeV}) \left( \frac{33 \Delta}{t/10} \right)^{1/2} \quad (1.13)$$

where $t = \ln(\tilde{m}/m_Z)$, so that $t \sim 33$ for a GUT messenger scale. As this calculation indicates, unlike the bounds on parameters entering at tree-level, bounds on the masses of particles contributing at one loop depend on the messenger scale.

Finally, if the particle contributes at two loops, we expect

$$\mathcal{O}(1) \frac{g^4}{(4\pi)^4} \ln \left( \frac{\tilde{m}}{m} \right) m^2 \lesssim m_Z^2 \Delta \quad (1.14)$$

$$m \lesssim (8000 \text{ GeV}) \left( \frac{33 \Delta}{t/10} \right)^{1/2} \quad (1.15)$$

which is, apparently, a significantly weaker bound.

From these considerations, we expect bounds on mass parameters which enter into higgs masses at tree level to be messenger scale independent while those which enter as radiative corrections to depend on the messenger scale approximately as $\ln^{-1/2}(\tilde{m}/m_Z)$. All mass bounds are expected to scale as $\Delta^{1/2}$. Our work in the following sections will bear out these qualitative expectations.
2 One-loop RG equations and their solutions

The tree-level relations between the parameters \(m_x^2, m_y^2, m_z^2\) and the soft supersymmetry breaking Higgs masses and higgsino mass parameter \(\mu\) are:

\[
\begin{align*}
    m_x^2 &= m_{h_d}^2 + \mu^2 \\
    m_y^2 &= m_{h_u}^2 + \mu^2 \\
    m_z^2 &= m_{ud}^2
\end{align*}
\]

We are thus required to integrate the RG equations \(^\text{[5]}\) for four mass parameters — \(m_{h_d}, m_{h_u}, \mu\) and \(m_{ud}\) — in order to determine their dependence on messenger scale parameters. The particles coupled to the higgses by gauge interactions or the top Yukawa coupling at one loop are: gauginos, higgsinos, left- and right-handed stops and left-handed sbottoms\(^\text{[2]}\). Since these couplings are large, we expect the masses of these particles to strongly renormalize the higgs masses; examination of the relevant one-loop RG equations bears out this expectation. Because of their strong effect on the higgs masses, naturalness places strong constraints on the masses of these particles; for this reason, the authors of \(^\text{[2]}\) referred to these particles as “brothers of the Higgs”.

We begin with the equations for \(m_{h_d}\) and \(m_{h_u}\). The RG equation governing the evolution of the down-type Higgs’ mass squared reads

\[
(4\pi)^2 \frac{d m_{h_d}^2}{dt} = -f_{h_d}
\]

while the up-type Higgs mass squared evolves according to the coupled equations

\[
(4\pi)^2 \frac{d}{dt} \begin{pmatrix} m_{h_d}^2 \\ m_{h_u}^2 \\ m_{ud}^2 \end{pmatrix} = \lambda^2 \begin{pmatrix} 6 & 6 & 6 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} m_{h_d}^2 \\ m_{h_u}^2 \\ m_{ud}^2 \end{pmatrix} - \begin{pmatrix} f_{h_u} \\ f_q \\ f_u \end{pmatrix}
\]

\(^\text{2}\)For \(\tan \beta \gtrsim m_t/m_b\), the bottom quark Yukawa coupling and potentially even the tau Yukawa coupling are large enough to require consideration. Since this represents an extreme corner of parameter space and would much complicate the RG equations, we do not consider this possibility here. We also ignore the effects of soft-SUSY breaking trilinear couplings, which may be present in principle, but are small in most models.
Here the inhomogeneous terms are

\[
\begin{align*}
  f_{hd} &= 6g_2^2m_2^2 + \frac{6}{5}g_1^2m_1^2 + \frac{3}{5}g_1^2M^2 \\
  f_{hu} &= 6g_2^2m_2^2 + \frac{6}{5}g_1^2m_1^2 - \frac{3}{5}g_1^2M^2 \\
  f_q &= \frac{32}{3}g_3^2m_3^2 + 6g_2^2m_2^2 + \frac{2}{15}g_1^2m_1^2 - \frac{1}{5}g_1^2M^2 \\
  f_u &= \frac{32}{3}g_3^2m_3^2 + \frac{32}{15}g_1^2m_1^2 + \frac{4}{5}g_1^2M^2
\end{align*}
\]  

(2.6) (2.7) (2.8) (2.9)

The gauge couplings and gaugino masses evolve according to

\[
\begin{align*}
  g_i^2(t) &= g_0^2Z_i(t) & m_i(t) &= (m_i)_0 Z_i(t)
\end{align*}
\]

(2.10)

where

\[
Z_i(t) = \left[1 - \frac{2B_i g_0^2}{(4\pi)^2}t\right]^{-1} \quad B_{1,2,3} = \left(\frac{33}{5}, 1, -3\right)
\]

(2.11)

The mass \(M\), defined by

\[
M^2 = 2\text{tr} \left[ Ym^2 \right] = n_{hu}^2 - m_{hd}^2 + \text{tr} \left[ m_q^2 - m_l^2 - 2m_u^2 + m_d^2 + m_e^2 \right]
\]

(2.12)

may be taken to be a RG invariant, since the terms proportional to gaugino masses vanish in its evolution equation, while the presumably dominant first and second generation scalar masses are only weakly renormalized by Yukawa couplings.

Before continuing, let us clarify our notation. Parameters without any additional annotation (e. g. \(g\) or \(m\)) refer to parameters evaluated at any scale \(t = \ln(m/m_0)\). Parameters annotated with a twiddle (e. g. \(\tilde{m}\)) refer to the value of the parameter at the messenger scale \(\tilde{t} = \ln(\tilde{m}/m_0)\), and thus constitute what we will regard as the SSM parameters \(a\) in the context of equations (1.3) and (1.4). Finally, parameters with a nought (e. g. \(g_0\)) refer to GUT-scale parameters evaluated at \(t_0 = 0\).

The equation for the evolution of \(m_{hd}^2\) (2.4) can be readily integrated

\[
\begin{align*}
  \frac{d}{d\tilde{t}} \tilde{m}_{hd}^2 &= \frac{1}{(4\pi)^2} \int_{\tilde{t}}^{\tilde{\tilde{t}}} dx f_{hd} \\
  \tilde{m}_{hd}^2 &= \tilde{m}_{hd}^2 - \frac{3}{2} \frac{Z_2^2 - \tilde{Z}_2^2}{Z_2^2} \tilde{m}_{hd}^2 - \frac{1}{22} \frac{Z_1^2 - \tilde{Z}_1^2}{Z_1^2} \tilde{m}_{hd}^2 - \frac{1}{22} \ln \left(\frac{Z_1}{\tilde{Z}_1}\right) M^2
\end{align*}
\]

(2.13) (2.14)
The solution of the coupled equations (2.5) governing the evolution of \(m_{h_u}\) is more complicated but can also be written in closed form in a useful approximation outlined in an appendix.

The RG equation governing the evolution of the \(\mu\)-parameter,

\[
(4\pi)^2 \frac{d\mu}{dt} = g(t)
\]

may be straightforwardly integrated to yield

\[
\mu = \tilde{\mu} e^{\int_t^\infty \frac{g}{(4\pi)^2} dt}
\]

Here \(g(t)\) is a function of gauge and Yukawa couplings whose specific form is irrelevant for our purposes.

Finally, the RG equation for the evolution of \(m_{ud}^2\),

\[
(4\pi)^2 \frac{dm_{ud}^2}{dt} = g(t) m_{ud}^2 + \left(6g_2^2 m_2 + \frac{6}{5}g_1^2 m_1\right) \mu
\]

may be likewise straightforwardly integrated

\[
m_{ud}^2 = \left[\tilde{m}_{us}^2 + \left(3 \frac{Z_2 - \tilde{Z}_2}{Z_2} \tilde{m}_2 + \frac{1}{11} \frac{Z_1 - \tilde{Z}_1}{Z_1} \tilde{m}_1\right) \tilde{\mu}\right] e^{\int_t^\infty \frac{g}{(4\pi)^2} dt}
\]

## 3 Limits on brothers’ masses

Having solved the RG equations for the relevant parameters, we can proceed to apply equation (1.4) to derive bounds from naturalness on the parameters which enter into their solutions.

### 3.1 Gluino masses

The algebra involved in limiting gluino mass parameter \(m_3\) is easiest, because it enters only into the expression (A.2) for \(m_{h_u}\). Applying the naturalness criterion (1.4) yields

\[
\frac{32 m_3^2}{9} \frac{1}{m_Z^2} \frac{1}{Z_3^2} \left[Z_3^2 - \tilde{Z}_3^2 + \left(\frac{m_Z}{m}\right)^\lambda \tilde{W}_3^{(2)} - W_3^{(2)}\right] < \Delta \left[1 - \frac{1}{\tan^2 \beta}\right] \left[1 + \frac{2c}{\tan^2 \beta - 1}\right]^{-1}
\]
Figure 1: $\Delta = 10$ limit on gluino mass as a function of the messenger scale in the large $\tan \beta$ limit. The scaling of this limit with $\tan \beta$ and $m_A$ is given by equation (3.1).

where $c = 1 + m_Z^2/m_A^2 > 1$, and $Z_i$, $\hat{\lambda}$ and $W_i^{(2)}$ are defined by equations (2.1), (A.6), and (A.7), respectively. We have used equation (2.10) to convert a limit on $\tilde{m}_3$ to one on $m_3$ itself. Note that the limit becomes significantly more stringent for smaller values of $\tan \beta$. Figure 1 shows the variation of the limit with the messenger scale $\tilde{m}$.

Because current experiments already require $m_3 \gtrsim 200$ GeV in the context of the minimal model with universal GUT-scale scalar and gaugino masses, our work allows us to deduce that the gaugino mass in the MSSM is fine-tuned to at least 11%; this required degree of fine-tuning increases by more than a factor of two as $\tan \beta$ is reduced to $\tan \beta \sim 2$.

3.2 Chargino and neutralino masses

The analysis of the limits on the bino, wino, and higgsino mass parameters $m_1$, $m_2$ and $\mu$ is considerably complicated by the fact that each of these parameters enters into more than one RG equation, and in such a way that
the application of equation (1.4) results in inequalities involving all three, so that we must solve three inequalities simultaneously for each value of \( \tan \beta \) and \( m_A \) in order to derive bounds on the parameters.

While this procedure can in principle be carried out numerically, the variation of the resulting bounds with \( \tan \beta \), \( m_A \), and the messenger scale \( \tilde{m} \) cannot be displayed in a compact form. We therefore content ourselves with the presentation of bounds in the large \( \tan \beta \) limit, in which the relevant inequalities decouple, and with the observation that, as was the case for gluino masses, mass bounds generically become more stringent as \( \tan \beta \) is lowered.

In the large \( \tan \beta \) limit, as noted previously, only the dependence of \( m_y^2 \) on the parameters matters, and the following inequalities are readily derived:

\[
6 \frac{m_2^2}{m_Z^2} \frac{1}{Z_2^2} \left[ \left( \frac{m_Z}{\tilde{m}} \right) \hat{\lambda} W_2^{(2)} - W_2^{(2)} \right] < \Delta \quad (3.2)
\]

\[
\frac{8}{99} \frac{m_1^2}{M_Z^2} \frac{1}{Z_1^2} \left[ Z_1^2 - \tilde{Z}_1^2 - \frac{13}{4} \left( W_1^{(2)} - \left( \frac{m_Z}{\tilde{m}} \right) \hat{\lambda} \tilde{W}_1^{(2)} \right) \right] < \Delta \quad (3.3)
\]

\[
\frac{4\mu^2}{m_Z^2} < \Delta \quad (3.4)
\]

These limits are displayed in figure 2 as a function of messenger scale. Most interesting is the limit on \( \mu \), first because it is the most stringent and second because, as we expected for parameters which enter at tree level, it is independent of the messenger scale.

Unlike the gluino mass \( m_3 \), the parameters \( m_1, m_2, \) and \( \mu \) are not themselves particle masses, but enter into mass matrices which must be diagonalized in order to determine particle masses. As pointed out by Barbieri and Giudice [4], however, from the structure of the relevant mass matrices it follows that the masses of the lightest chargino and neutralino are constrained by

\[
m_{\chi^\pm}^2 < m_W^2 + \min \left( m_2^2, \mu^2 \right) \quad (3.5)
\]

\[
m_{\chi^0}^2 < \min \left( m_1^2 + m_Z^2 \sin^2 \theta_W, m_2^2 + m_Z^2 \cos^2 \theta_W, \mu^2 + \frac{1}{2} m_Z^2 \right) \quad (3.6)
\]

Using these inequalities, our upper bounds on these parameters may be translated into upper bounds on the masses of the lightest chargino and neutralino.
Figure 2: $\Delta = 10$ limits on wino and higgsino mass parameters as a function of the messenger scale in the large $\tan \beta$ limit. The corresponding limit on the bino mass parameter is greater than 1 TeV.
In all cases, it is the bound on \( \mu \) which dominates, implying
\[
\begin{align*}
m_{\chi^\pm} &< 165 \text{ GeV} \quad (3.7) \\
m_{\chi^0} &< 160 \text{ GeV} \quad (3.8)
\end{align*}
\]
Because they arise from the bound on \( \mu \), these bounds are independent of the messenger scale. Although they do not scale exactly as \( \Delta^{1/2} \), they do so to a good approximation for \( \Delta \gtrsim 10 \).

### 3.3 Scalar masses

Now consider the limits on the scalar mass parameters \( \tilde{m}_{h_u}, \tilde{m}_q, \) and \( \tilde{m}_u \). From their contributions to expression (A.2) for \( m_{h_u} \) and the application of the naturalness criterion (1.4), we obtain
\[
\begin{align*}
\frac{\tilde{m}_{h_u}^2}{m_Z^2} \left[ 1 + \frac{(m_Z)}{\tilde{m}} \right] < \Delta \left[ 1 - \frac{1}{\tan^2 \beta} \right] \left[ 1 + \frac{2c}{\tan^2 \beta - 1} \right]^{-1} \quad (3.9) \\
\frac{\tilde{m}_{q,u}^2}{m_Z^2} \left[ 1 - \frac{(m_Z)}{\tilde{m}} \right] < \Delta \left[ 1 - \frac{1}{\tan^2 \beta} \right] \left[ 1 + \frac{2c}{\tan^2 \beta - 1} \right]^{-1} \quad (3.10)
\end{align*}
\]

The physical masses of the left- and right-handed stops, as determined by equations (A.3) and (A.4), receive contributions from both scalar and gaugino masses. In the case where the contributions from scalars dominate, it is easy to translate our bounds on scalar mass parameters at the messenger scale into bounds on the stop masses.

\[
\begin{align*}
m_q^2 < \frac{m_Z^2 \Delta}{6} \left[ 1 - \frac{1}{\tan^2 \beta} \right] \left[ 1 + \frac{2c}{\tan^2 \beta - 1} \right]^{-1} \\
	imes \left\{ 5 \left[ 1 - \frac{(m_Z)}{\tilde{m}} \right] \left[ 1 - \frac{(m_Z)}{\tilde{m}} \right]^{-1} + \left[ 1 - \frac{(m_Z)}{\tilde{m}} \right] \left[ 1 + \frac{(m_Z)}{\tilde{m}} \right]^{-1} \right\} \\
\quad (3.11)
\end{align*}
\]

\[
\begin{align*}
m_u^2 < \frac{m_Z^2 \Delta}{3} \left[ 1 - \frac{1}{\tan^2 \beta} \right] \left[ 1 + \frac{2c}{\tan^2 \beta - 1} \right]^{-1} \\
	imes \left\{ 2 \left[ 1 - \frac{(m_Z)}{\tilde{m}} \right] \left[ 1 - \frac{(m_Z)}{\tilde{m}} \right]^{-1} + \left[ 1 - \frac{(m_Z)}{\tilde{m}} \right] \left[ 1 + \frac{(m_Z)}{\tilde{m}} \right]^{-1} \right\} \\
\quad (3.12)
\end{align*}
\]
Figure 3: $\Delta = 10$ limits on left- and right-handed stop masses in the large \(\tan\beta\) region as a function of the messenger scale. The solid lines show the limits implied when scalar masses are large compared to gaugino masses; when gaugino masses are comparable, the dashed lines result. The scaling of the solid lines with $\tan\beta$ and $m_A$ is given by equations (3.11-3.12). The bump in the mass limit on right-handed stops near $10^{13}$ GeV is due to a positive contribution from the bino mass parameter, which is allowed to be large due to a zero in the coefficient with which it contributes to $m_{h_u}$ near this scale.

These bounds are shown as solid lines in figure 3. It is noteworthy that both of these bounds cannot be saturated simultaneously, since $\tilde{m}_u^2$ contributes negatively to $m_q^2$ and vice versa. The scaling of these limits with $\tan\beta$ makes them more stringent by more than a factor of two for low values of $\tan\beta \sim 2$.

When gaugino masses are comparable to stop masses, their possible contributions must also be considered. Inserting the gaugino mass limits (3.1) and (3.2-3.3) into equations (A.3) and (A.4) gives the stop mass limits indicated by the dashed lines in figure 3.

The scalar mass sum $M^2$ enters into the expressions (2.14, A.2) for both
Figure 4: $\Delta = 10$ limit on generic sparticle masses in $M \neq 0$ models in the large $\tan \beta$ regime as a function of the messenger scale. The scaling of this limit with $\tan \beta$ is determined by equation (3.13). The limit for right-handed up-type squarks is more stringent by a factor $\sqrt{2}$.

$m_{h_d}$ and $m_{h_u}$. Treating it as a parameter gives a naturalness limit

$$\frac{1}{11} \frac{M^2}{m_{Z_1}} \ln \left( \frac{Z_1}{\tilde{Z}_1} \right) < \Delta \frac{\tan^2 \beta - 1}{\tan^2 \beta + 1}$$

(3.13)

on $M$ and by implication on the masses of the scalars which enter into it. The variation of this limit with the messenger scale $\tilde{m}$ is shown in figure [4] in the large $\tan \beta$ limit; again, the limit becomes more stringent as $\tan \beta$ becomes smaller.

Of course, $M$ itself should not be regarded as a parameter but as a function of the scalar sparticle masses. As noted by Dimopoulos and Giudice [2], if some symmetry, e.g. scalar mass degeneracy or $SU(5)$, ensures $M = 0$ then equation (3.13) gives no limit on any scalar masses. If, on the other hand, scalar masses are not so constrained, this is an upper limit on the masses of the scalar partners of fermions of all three generations.
4 Cousins of the Higgs

If some symmetry principle does protect $M = 0$, the masses of the first and second generation sparticles enter the one-loop RG equations suppressed by tiny Yukawa couplings and their masses are thus essentially unlimited by one one-loop radiative corrections. On the other hand, at two loops the inhomogeneous terms in the RG equations described above get contributions

\begin{align}
(4\pi)^4 \frac{dm_{h_u}^2}{dt} &\supset \frac{3}{5} g_1^4 S_1^2 + 3 g_2^4 S_2^2 - \frac{6}{5} g_1^2 T^2 \quad (4.1) \\
(4\pi)^4 \frac{dm_{h_d}^2}{dt} &\supset \frac{3}{5} g_1^4 S_1^2 + 3 g_2^4 S_2^2 + \frac{6}{5} g_1^2 T^2 \quad (4.2) \\
(4\pi)^4 \frac{dm_{h_d}^2}{dt} &\supset \frac{1}{15} g_1^4 S_1^2 + 3 g_2^4 S_2^2 + \frac{16}{3} g_3^4 S_3^2 + \frac{2}{5} g_1^2 T^2 \quad (4.3) \\
(4\pi)^4 \frac{dm_{h_u}^2}{dt} &\supset \frac{4}{15} g_1^4 S_1^2 + \frac{16}{3} g_3^4 S_3^2 - \frac{8}{5} g_1^2 T^2 \quad (4.4)
\end{align}

where

\begin{align}
S_1^2 &= 2 \frac{3}{5} \text{tr} \left[ U(1)_Y^2 m^2 \right] \\
&= \frac{1}{5} \left( 3m_{h_u}^2 + 3m_{h_d}^2 + \text{tr} \left[ m_q^2 + 8m_u^2 + 2m_d^2 + 3m_l^2 + 6m_e^2 \right] \right) \quad (4.5) \\
S_2^2 &= 2 \text{tr} \left[ SU(2)^2 m^2 \right] = m_{h_u}^2 + m_{h_d}^2 + \text{tr} \left[ 3m_q^2 + m_l^2 \right] \quad (4.6) \\
S_3^2 &= 2 \text{tr} \left[ SU(3)^2 m^2 \right] = 2m_q^2 + m_u^2 + m_d^2 \quad (4.7)
\end{align}

and

\begin{align}
T^2 &= g_1^2 T_1^2 + g_2^2 T_2^2 + g_3^2 T_3^2 \quad (4.8) \\
T_1^2 &= \frac{3}{80} \left( m_{h_u}^2 - m_{h_d}^2 \right) + \text{tr} \left[ \frac{1}{30} m_q^2 - \frac{16}{15} m_u^2 + \frac{2}{15} m_d^2 - \frac{3}{10} m_l^2 + \frac{6}{5} m_e^2 \right] \quad (4.9) \\
T_2^2 &= \frac{3}{2} \left( m_{h_u}^2 - m_{h_d}^2 + \text{tr} \left[ m_q^2 - m_l^2 \right] \right) \quad (4.10) \\
T_3^2 &= \frac{8}{3} \text{tr} \left[ m_q^2 - 2m_u^2 + m_d^2 \right] \quad (4.11)
\end{align}

Including these mass sums gives a contribution to the up-type higgs mass

\begin{equation}
m_{h_u}^2 \supset \frac{\alpha_0}{8\pi} \left( C_1 S_1^2 + C_2 S_2^2 + C_3 S_3^2 + D_1 T_1^2 + D_2 T_2^2 + D_3 T_3^2 \right) \quad (4.12)
\end{equation}
where
\begin{align}
C_1 &= -\frac{2}{99} \left( \tilde{Z}_1 - Z_1 \right) - \frac{7}{99} \left( \frac{m_Z}{m} \right)^{\lambda} \tilde{W}_1^{(1)} - W_1^{(1)} \\
C_2 &= -3 \left[ \frac{m_Z}{m} \right]^{\lambda} \tilde{W}_2^{(1)} - W_2^{(1)} \\
C_3 &= \frac{8}{9} \left[ \frac{m_Z}{m} \right]^{\lambda} \tilde{W}_3^{(1)} - W_3^{(1)} - \tilde{Z}_3 + Z_3
\end{align}

(4.13)

(4.14)

(4.15)

and
\begin{align}
W_i^{(1)} &= Z_i - r_i e^{-r_i Z_i^{-1}} Ei (r_i Z_i^{-1}) \\
r_i &= \frac{6 \lambda^2}{B_i g_0^2}
\end{align}

(4.16)

Equation (4.12) implies naturalness limits for all sparticles of all generations given by
\begin{equation}
\frac{\alpha_0}{4\pi} \frac{m_s^2}{m_Z^2} |C_s| < \Delta
\end{equation}

(4.20)

where the coefficients $C_s$ may easily be read off as
\begin{align}
C_q &= \frac{1}{5} C_1 + 3C_2 + 2C_3 + \frac{1}{30} D_1 + \frac{3}{2} D_2 + \frac{8}{3} D_3 \\
C_u &= \frac{8}{5} C_1 + C_3 - \frac{16}{15} D_1 - \frac{16}{3} D_3 \\
C_d &= \frac{2}{5} C_1 + C_3 + \frac{2}{15} D_1 + \frac{8}{3} D_3 \\
C_l &= \frac{2}{5} C_1 + C_2 - \frac{3}{10} D_1 - \frac{3}{2} D_3 \\
C_e &= \frac{6}{5} C_1 + \frac{6}{5} D_1
\end{align}

(4.21)

(4.22)

(4.23)

(4.24)

(4.25)

The resulting limits are shown in figure [5]. This shows that it is possible to raise the masses of the $q$- and $d$-type first and second generation squarks to between 4 and 8 TeV, and those of the other squarks and sleptons even higher, without violating naturalness.
Figure 5: $\Delta = 10$ limits on sparticle masses from two-loop naturalness in the large $\tan \beta$ limit as a function of the messenger scale.
5 Conclusions

We have derived bounds on the masses of supersymmetric partners from the requirement of naturalness in arbitrarily nonminimal incarnations of the SSM in which SUSY-breaking is communicated by messengers at an arbitrary scale.

Examination of the variation of naturalness bounds with messenger scale reveals that these bounds may be increased from their GUT messenger values by between 15% and 100% by lowering the messenger scale for all superpartners except the lightest charginos and neutralinos, for which we obtain messenger scale independent mass bounds of $\sim 160$ TeV for 10% fine-tuning.

We have derived the variation of several of these several mass bounds with $\tan \beta$ and found that naturalness more significantly constrains sparticle masses in models with low $\tan \beta$ than in models with high $\tan \beta$. The mass constraints in the two regimes can differ by more than a factor of two.

The most problematic constraint for the traditional MSSM with GUT scale mediation of SUSY breaking is that placed on the gluino mass $m_3 \lesssim 260$ GeV. Present experiments thus require that the MSSM be at least 11% fine-tuned and more than 5% fine-tuned for $\tan \beta \lesssim 2$. This problem can be evaded by models which significantly lower the messenger scale or make the gluino the lightest supersymmetric particle.

As accelerator searches progress, this work will place increasingly stringent constraints on possible realizations of the SSM.

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A Appendix

Equations (2.5) represent an inhomogeneous linear differential system which may be solved formally in closed form. In order to make computational sense of this formal solution, we must examine the behavior of the top Yukawa
coupling $\lambda(t)$, which evolves according to the RG equation

$$\frac{d\lambda}{dt} = \frac{\lambda}{(4\pi)^2} \left[ 6\lambda^2 - \frac{16}{3} g_3^2 - 3g_2^2 - \frac{13}{15} g_1^2 \right]$$

(A.1)

This equation lacks a closed-form analytic solution, but nonetheless be integrated numerically. Numerical integration indicates that the top Yukawa coupling remains within ten percent of $\lambda \sim 1$ over nearly the entire interesting range of $t$ and $\tan \beta$. We will therefore perform the integrations considering $\lambda$ to be constant. This yields the results

$$m^2_{h_u} = \left[ 1 + \left( \frac{m}{\tilde{m}} \right)^\lambda \right] \frac{\tilde{m}_{h_u}^2}{2} - \left[ 1 - \left( \frac{m}{\tilde{m}} \right)^\lambda \right] \frac{\tilde{m}_{q}^2 + \tilde{m}_{u}^2}{2}$$

$$\begin{aligned}
&- \frac{8}{9} \frac{Z_3^2 - \tilde{Z}_3^2}{Z_3^2} \tilde{m}_3^2 - \frac{2}{99} \frac{Z_1^2 - \tilde{Z}_1^2}{Z_1^2} \tilde{m}_1^2 \\
&+ m_\lambda^2 + \frac{1}{22} \ln \left( \frac{Z_1}{\tilde{Z}_1} \right) M^2
\end{aligned}$$

(A.2)

$$m^2_q = \left[ 5 + \left( \frac{m}{\tilde{m}} \right)^\lambda \right] \frac{\tilde{m}_q^2}{6} - \left[ 1 - \left( \frac{m}{\tilde{m}} \right)^\lambda \right] \frac{\tilde{m}_{h_u}^2 + \tilde{m}_u^2}{6}$$

$$\begin{aligned}
&+ \frac{16}{27} \frac{Z_3^2 - \tilde{Z}_3^2}{Z_3^2} \tilde{m}_3^2 + \frac{2}{27} \frac{Z_2^2 - \tilde{Z}_2^2}{Z_2^2} \tilde{m}_2^2 - \frac{5}{297} \frac{Z_1^2 - \tilde{Z}_1^2}{Z_1^2} \tilde{m}_1^2 \\
&+ \frac{1}{3} m_\lambda^2 + \frac{1}{66} \ln \left( \frac{Z_1}{\tilde{Z}_1} \right) M^2
\end{aligned}$$

(A.3)

$$m^2_u = \left[ 2 + \left( \frac{m}{\tilde{m}} \right)^\lambda \right] \frac{\tilde{m}_u^2}{3} - \left[ 1 - \left( \frac{m}{\tilde{m}} \right)^\lambda \right] \frac{\tilde{m}_{h_u}^2 + \tilde{m}_u^2}{3}$$

$$\begin{aligned}
&+ \frac{8}{27} \frac{Z_3^2 - \tilde{Z}_3^2}{Z_3^2} \tilde{m}_3^2 - \frac{2}{27} \frac{Z_2^2 - \tilde{Z}_2^2}{Z_2^2} \tilde{m}_2^2 + \frac{11}{297} \frac{Z_1^2 - \tilde{Z}_1^2}{Z_1^2} \tilde{m}_1^2 \\
&+ \frac{2}{3} m_\lambda^2 - \frac{2}{33} \ln \left( \frac{Z_1}{\tilde{Z}_1} \right) M^2
\end{aligned}$$

(A.4)

\(^3\)The exception occurs for GUT scale values of $t$ when $\tan \beta \lesssim 1.5$. Even an exact treatment of the evolution of $\lambda$ would not be particularly helpful in this case, since its behavior in this region becomes highly sensitive to the exact value of $m_t$. 

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where

$$m_\lambda^2 = \frac{8}{9} \frac{1}{Z_3^2} \left[ W_3^{(2)} - \left( \frac{m}{\bar{m}} \right)^\lambda \tilde{W}_3^{(2)} \right] \bar{m}_3^2 + \frac{3}{2} \frac{1}{Z_2^2} \left[ \left( \frac{m}{\bar{m}} \right)^\lambda \tilde{W}_2^{(2)} - W_2^{(2)} \right] \bar{m}_2^2$$

$$+ \frac{13}{198} \frac{1}{Z_1^2} \left[ \left( \frac{m}{\bar{m}} \right)^\lambda \tilde{W}_1^{(2)} - W_1^{(2)} \right] \bar{m}_1^2$$

(A.5)

with

$$\dot{\lambda} = \frac{12\lambda^2}{(4\pi)^2}$$

(A.6)

and

$$W_i^{(2)} = Z_i^2 + r_i Z_i - r_i^2 e^{-r_i Z_i} Ei \left( r_i Z_i^{-1} \right) \quad r_i = \frac{6\lambda^2}{B_i g_0}$$

(A.7)

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