WE PROPOSEG THROUGH SPARSE POTENTIAL BARRIERS

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ABSTRACT. We prove that 3-dimensional Schrödinger operator with slowly decaying sparse potential has an a.c. spectrum that fills $\mathbb{R}^+$. A new kind of WKB asymptotics for Green’s function is obtained. The absence of positive eigenvalues is established as well.

Consider the Schrödinger operator

$$H = -\Delta + V, x \in \mathbb{R}^d$$

We are interested in studying the scattering properties of $H$ for the slowly decaying potential $V$. The following conjecture is due to Barry Simon

Conjecture. If $V(x)$ is such that

$$\int_{\mathbb{R}^d} V^2(x) \frac{1}{1+|x|^d} dx < \infty$$

then $\sigma_{ac}(-\Delta + V) = \mathbb{R}^+$.

Some progress was recently made for slowly decaying oscillating potentials and potentials asymptotically close to the spherically symmetric. But even for $V$ satisfying the bound $|V(x)| < C < x >^{-\gamma}$, $(\gamma < 1, < x > = (1+|x|^2)^{0.5})$, there are no results. We also want to mention that for the problem on the Bethe lattice (Caley tree), we have relatively good understanding: consider a rooted Caley tree and denote the root by $O$. Assume that each point has three neighbors and $O$ has only two. Consider the discrete Schrödinger operator with potential $V$ and denote by $d\sigma_O$ the spectral measure corresponding to the discrete delta function at $O$. Let $w(\lambda) = (4\pi)^{-1}(8 - \lambda^2)^{1/2}$ on $[-2\sqrt{2}, 2\sqrt{2}]$, and $\rho_O = \sigma_O(\lambda)|w(\lambda)|^{-1}$, a relative density of the spectral measure at the point $O$. Consider all paths that go from $O$ to infinity without self-intersections and define the probability space on the set of these paths by assigning to each of them the same weight (i.e. as we go from $O$ to infinity, we toss the coin at any vertex and move to one of the neighbors further from $O$ depending on the result). In [5], we proved the following

Theorem 0.1. For any bounded $V$, define

$$s_O = \int_{-2\sqrt{2}}^{2\sqrt{2}} \ln \rho_O(\lambda) w(\lambda) d\lambda$$
Then the following inequality is true
\[
\exp s_O \geq \mathbb{E} \left\{ \exp \left[ -\frac{1}{4} \sum_{n=1}^{\infty} V^2(x_n) \right] \right\}
\]  
where the expectation is taken with respect to all paths \( \{x_n\} \) going from \( O \) to infinity without self-intersections. In particular, if the r.h.s. of (3) is positive, then \( [-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H) \).

Notice that for the r.h.s. to be positive we just need to make sure that there are “enough” paths over which the potential is square summable. This is much weaker than (2).

In the current paper, we consider the following model. Let \( R_n \) be a sparse sequence of real numbers, i.e. \( R_n \to +\infty, R_{n+1}/R_n \to \infty \) as \( n \to \infty \). Consider concentric spherical layers \( \Sigma_n = \{x : R_n < |x| < R_{n+1}\} \) and assume that the measurable functions \( v_n(x) = 0 \) outside these layers and \( |v_n(x)| < v_n, x \in \Sigma_n, v_n \to 0 \) as \( n \to \infty \). Take
\[
V_n(x) = \sum_{j=1}^{n} v_j(x), V(x) = \sum_{j=1}^{\infty} v_j(x)
\]
We will study the scattering properties of the corresponding \( H \). In particular, we want to study the a.c. spectrum and the spatial asymptotics of the Green function. The sparse potentials of general form were studied earlier (see, e.g. [13, 14, 10] and references there). It is also necessary to mention the one-dimensional result first (see [15, 9]).

**Theorem 0.2.** (see [9]) Assume that \( d = 1, \frac{R_n}{R_{n+1}} \to 0 \) and \( v_n \to 0 \). Then for
\[
V(x) = \sum_{n=1}^{\infty} v_n \phi(x - R_n)
\]
(\( \phi(x) \) - a nonzero, nonnegative bump function), we have the following: if \( v_n \in \ell^p(\mathbb{Z}^+), p \leq 2 \), the spectrum of \( H \) is purely a.c. on \( \mathbb{R}^+ \). For \( p > 2 \), it is singular continuous.

We are interested in studying the same phenomena in the multidimensional case. Clearly, by taking the potential spherically symmetric, one can show that the spectrum can be singular continuous for \( v_n \in \ell^p(\mathbb{Z}^+), p > 2 \) even in multidimensional case. We will address the following question: assume that the sequence \( \{R_n\} \) is as sparse as we like and \( v_n \) is an arbitrary sequence from \( \ell^2(\mathbb{Z}^+) \). Is it true that the a.c. spectrum exists?

From Weyl’s theorem, we know that \( \sigma_{ess}(H) = [0, \infty) \). We need to introduce certain quantities. Let \( z = k^2 \) and \( \Pi = \{k \in \mathbb{C}^+, 0 < \text{Im} k < 1\} \). If \( f(x) \) is nonzero \( L^2 \) function with compact support, say, within the unit ball, then \( u_n(x, k) = (-\Delta + V_n - k^2)^{-1} f \) has the following asymptotics at infinity
\[
u_n(x, k) = \exp(ikr) \left( A_n(f, k, \theta) + \bar{o}(1) \right),
\]
\[
\frac{\partial u_n(x, k)}{\partial r} = \frac{ik}{r} \exp(ikr) \left( A_n(f, k, \theta) + \bar{o}(1) \right), \quad (\text{Sommerfeld’s radiation conditions})
\]
\[
\theta = \frac{\theta}{|x|}, |x| \to \infty
\]
(4)
as long as \( k \in \mathbb{C}^+ \) and \( k^2 \notin \sigma(H_n) \). Since \( V_n \) has compact support, one can show that \( A_n \) is continuous in \( k \) up to the real line. The physical meaning of \( A_n(f, k, \theta) \) is the amplitude of the outgoing spherical wave in the direction \( \theta \) after propagation through \( n \) concentric barriers. For the free case (i.e. \( V = 0 \)) the amplitude of function \( f \) is given by the formula

\[
A^0(f, k, \theta) = (4\pi)^{-1} \int_{\mathbb{R}^3} \exp(-ik < \theta, x>) f(x) dx
\]

Notice that for the fixed \( \theta \) this function is entire in \( k \) (nonzero \( f \) has compact support) and so has finite number of zeroes inside any compact.

Instead of proving the asymptotics of the Green function of \( H \), we will study an asymptotical behavior of the sequence \( A_n(f, k, \theta) \), as \( k \in \Pi \), and \( n \to \infty \). That will contain all relevant information on the scattering mechanism. The following is the main result of the paper

**Theorem 0.3.** If the sequence \( R_n \) is sparse enough\(^1\), then for any \( v_n \in \ell^2(\mathbb{Z}^+) \), we have

\[
A_n(f, k, \theta) = WKB_n(k, \theta) \tilde{A}_n(f, k, \theta)
\]

where

\[
WKB_n(k, \theta) = \exp \left[ -(4\pi)^{-1} \int_{\mathbb{R}^3} \exp(ik(|t| < \theta, t>)|t| V_n(t) dt \right], |\tilde{A}_n(f, k, \theta)| < C(k)
\]

uniformly in \( n \) for \( k = \tau + i\epsilon, \tau > 0, 0 < \epsilon < 1 \). If \( k \) is fixed, then

\[
\tilde{A}_n(f, k, \theta) = A^0(f, k, \theta) + \delta(\|v\|_2, k)
\]

where \( \delta(\|v\|_2, k) \to 0 \) as \( \|v\|_2 \to 0 \) uniformly in \( n \). Moreover, \( \sigma_{ac}(H) = \mathbb{R}^+ \).

**Proof.** For simplicity, assume \( \|v\|_2 < 1 \). From now on, we will reserve the symbol \( C \) for the constant whose value can change from one formula to another. We need two lemmas first.

**Lemma 0.1.** Consider potential \( V(x) : V(x) = 0 \) for \( |x| > R \) and \( |V(x)| < 1 \). Denote the Green function by \( G(x, y, k) \). Then, for \( k = \tau + i\epsilon, \tau, \epsilon > 0 \), we have

\[
G(x, y, k) = \frac{e^{ik|x-y|}}{4\pi |x-y|} + \delta(x, y, k)
\]

where

\[
|\delta(x, y, k)| \leq C \frac{R^3 e^{2\epsilon R-|x|}}{\tau \epsilon(|x| - R)(|y| - R)} |x|, |y| >> R
\]

Moreover, if

\[
B(\hat{x}, y, k) = \lim_{|x| \to \infty} \left( |x| e^{-ik|x|} G(x, y, k) \right)
\]

then

\[
|\partial B(\hat{x}, y, k)| < (4\pi)^{-1} \epsilon |y| \exp(\epsilon < \hat{x}, y>) + CRk \left( \frac{R^3 e^{2\epsilon R-|y|}}{\tau \epsilon(|y| - R)} \right), \hat{x} = x/|x|
\]

\(^1\)For example, an estimate \( R_{n+1} > e^{\alpha R_n}, \alpha > 1, n \in \mathbb{Z}^+ \) will be sufficient.
and the derivative is taken with respect to \( \hat{x} \in \Sigma \) (a unit sphere).

**Proof.** By the second resolvent identity,
\[
G(x, y, k) = (4\pi)^{-1} \frac{e^{ik|x|}}{|x|} - (4\pi)^{-2} \int \frac{e^{ik|x-u|}}{|x-u|} V(u) e^{ik|u-y|} \, du + (4\pi)^{-2} \int \frac{e^{ik|x-u|}}{|x-u|} V(u) \int G(u, t, k) V(t) \frac{e^{ik|t-y|}}{|t-y|} \, dt du.
\]
For the second term,
\[
\left| \int \frac{e^{ik|x-u|}}{|x-u|} V(u) \int G(u, t, k) V(t) \frac{e^{ik|t-y|}}{|t-y|} \, dt du \right| < C R^3 e^{|x| (|y| - |y|)}
\]
To estimate the third term, we notice that \( \text{Im} k^2 = 2\tau \) and therefore
\[
\left| \int \frac{e^{ik|x-u|}}{|x-u|} V(u) \int G(u, t, k) V(t) \frac{e^{ik|t-y|}}{|t-y|} \, dt du \right| < (2\tau \epsilon)^{-1} \left( \int_{|u| < R} \frac{e^{-2\epsilon|x-u|}}{|x-u|^2} \, du \right)^{1/2} \left( \int_{|t| < R} \frac{e^{-2\epsilon|t-y|}}{|t-y|^2} \, dt \right)^{1/2}
\]
Now (9) is straightforward.
To obtain (10), we write the following bounds for any unit vector \( \nu \in \mathbb{T}_{\hat{x}} \),
\[
|\partial_\nu B(\hat{x}, y, k)| < (4\pi)^{-1} |k < \nu, y > e^{-ik<\hat{x}, y>}| + (4\pi)^{-2} \left| \int k < \nu, u > e^{-ik<\hat{x}, u>} V(u) \frac{e^{ik|u-y|}}{|u-y|} \, du \right|
\]
\[
+ (4\pi)^{-2} \left| \int k < \nu, u > e^{-ik<\hat{x}, u>} V(u) \int G(u, t, k) V(t) \frac{e^{ik|t-y|}}{|t-y|} \, dt du \right|
\]
Then, we estimate the second and the third terms as before. \( \square \)

**Lemma 0.2.** Under the conditions of lemma 0.1, consider nonzero \( f(x) \in L^2 \) with compact support (say, inside the unit ball). Define the corresponding amplitude \( A(f, k, \hat{x}) \). Then,
\[
u(x, k) = (H - k^2)^{-1} f = \frac{\exp(ik|x|)}{|x|} (A(f, k, \hat{x}) + \rho(x, k))
\]
where
\[
|\rho(x, k)| < C(\tau \epsilon)^{-1} R^3 e^{|x| (|x| - |x|)} |x| >> R
\]
**Proof.** We have \( -\Delta u + Vu = f + k^2 u \) which can be rewritten \( u = (\Delta - k^2)^{-1} \mu \), where \( \mu = f - Vu \) has compact support. So,
\[
u(x, k) = (4\pi)^{-1} \int \frac{e^{ik|x-t|}}{|x-t|} \mu(t) dt
\]
Clearly,
\[A(f, k, \hat{x}) = (4\pi)^{-1} \int e^{-ik<\hat{x}, t>} \mu(t) dt\]
and
\[
\int \left( \frac{|x|}{|x-t|} e^{ik|x-t| - |x|} - e^{-ik<\hat{x}, t>} \right) \mu(t) dt
\]
that the wave, propagated through the level of physical intuition, by requiring the sparseness of barriers, we make sure representation for the amplitude, so well-known in the one-dimensional case. On which basically looks like this:

We will see that

Introducing the spherical variables, we see that the second term is not greater than

The difficulty of the problem comes from noncommutativity of \( q \) and \( \hat{q} \).

Then, the obvious estimates lead to (12).

Now, let us proceed to the proof of the theorem. The strategy is rather simple. We want to obtain recursion for \( A_n \). To do that, we will consecutively perturb \( H_n \) by \( v_{n+1}(x) \), \( H_{n+1} \) by \( v_{n+2}(x) \), etc. That will allow us to obtain almost multiplicative representation for the amplitude, so well-known in the one-dimensional case. On the level of physical intuition, by requiring the sparseness of barriers, we make sure that the wave, propagated through \( n \) barriers, hits the \( n+1 \) barrier almost like an outgoing spherical wave. That makes an analysis doable.

Let us write the second resolvent identity for \( H_{n+1} = -\Delta + V_n(x) + v_{n+1}(x) \):

\[
G_{n+1}(x,y,k) = G_n(x,y,k) - \int G_n(x,u,k)v_{n+1}(u)G_n(u,y,k)\,du + \int G_n(x,u,k)v_{n+1}(u)\int G_{n+1}(u,s,k)v_{n+1}(s)G_n(s,y,k)\,ds
\]

(13)

Therefore,

\[
\begin{align*}
\quad u_{n+1}(x) &= \int G_{n+1}(x,y,k)f(y)\,dy = u_n(x) - \int G_n(x,y,k)v_{n+1}(y)u_n(y)\,dy \\
&\quad + \int G_n(x,y,k)v_{n+1}(y)\int G_{n+1}(y,s,k)v_{n+1}(s)u_n(s)\,ds\,dy
\end{align*}
\]

Taking \(|x|\) to infinity, we get

\[
A_{n+1}(k,\hat{x}) = A_n(k,\hat{x}) - (4\pi)^{-1} \int e^{-ik<\hat{x},u>}v_{n+1}(u)\frac{e^{ik|u|}}{|u|^4}A_n(k,\hat{u})\,du + r_n(k,\hat{x})
\]

(14)

Introducing the spherical variables, we see that the second term is not greater than

\[
C\|A_n\|_\infty v_{n+1}e^{-1}
\]

(15)

We will see that \( r_n \) can be regarded as a small correction to the recurrence relation which basically looks like this:

\[
l_{n+1} = \left[ I - \int_{R_{n+1}} O_t q_t dt \right] l_n
\]

(16)

where \( l_n(\theta) \) are the functions on \( \Sigma \),

\[
O_t f(\theta) = (4\pi)^{-1}t \int_{\Sigma} e^{ikt(1-<\theta,s>)} f(s)\,ds
\]

(17)

and \( q_t \) is just an operator of multiplication by the function \( q_t(\theta) \) given on the unit sphere. The difficulty of the problem comes from noncommutativity of \( O_n \) and \( q_n \). If not the correction \( r_n \) we would just have the product of operators. But to get the
needed asymptotics for this product, we will have to essentially use the sparseness condition again.

The rest of the proof goes as follows: we first obtain rough apriori estimates on $A_n(k, \theta)$ and $\partial A_n(k, \theta)$. Then, we will use them to obtain an accurate asymptotics for $A_n(k, \theta)$. In the last part, this asymptotics will be used to show the presence of a.c. component of the spectral measure.

For $r_n$, we have $r_n = I_1 + \ldots + I_7$. Applying lemma 0.1 and lemma 0.2, we get the following estimates for $I_j$:

\[
I_1 = - \lim_{|x| \to \infty} |x|e^{-ik|x|} \int \frac{e^{ik|x-u|}}{4\pi|x-u|} v_{n+1}(u) \frac{e^{ik|u|}}{|u|} \rho_n(u, k) du
\]

\[
|I_1| < C \left| \int e^{-ik\langle \hat{x}, u \rangle} v_{n+1}(u) \frac{e^{ik|u|}}{|u|} \rho_n(u, k) du \right|
\]

\[
< \sigma_n(\tau \epsilon)^{-1} v_{n+1} \int_{R_{n+1} < |u| < R_{n+1}+1} e^{\epsilon \langle \hat{x}, u \rangle - |u|} \frac{\rho_n(u, k)}{|u|} du
\]

where

\[
\sigma_n = R_n^{3.5} e^{R_n(R_{n+1} - R_n)^{-1}}
\]

By introducing the spherical coordinates, we estimate the last integral

\[
\int_{R_{n+1} < |u| < R_{n+1}+1} e^{\epsilon \langle \hat{x}, u \rangle - |u|} \frac{\rho_n(u, k)}{|u|} du = C \int_{R_{n+1}}^{R_{n+1}+1} \rho \frac{1 - e^{-2\epsilon \rho}}{\epsilon \rho} d\rho < C \epsilon^{-1}
\]

Thus,

\[
|I_1| < C(\tau \epsilon^2)^{-1} \sigma_n v_{n+1}
\]

For $I_2$:

\[
I_2 = - \lim_{|x| \to \infty} |x|e^{-ik|x|} \int \delta_n(x, t, k) v_{n+1}(t) \frac{e^{ik|t|}}{|t|} A_n(\hat{t}, k) dt
\]

\[
|I_2| < C \int \frac{R_n^3 e^{2\epsilon R_n - \epsilon|t|}}{\tau \epsilon(t - R_n)} v_{n+1}(t) \frac{e^{-\epsilon|t|}}{|t|} |A_n(\hat{t}, k)| dt
\]

\[
< C(\tau \epsilon)^{-1} R_n^3 v_{n+1} e^{2\epsilon (R_n - R_{n+1})} \| A_n(\hat{t}, k) \|_{L^\infty(\Sigma)}
\]

as long as

\[
R_{n+1} > 2R_n
\]

Define $I_3$ as

\[
I_3 = - \lim_{|x| \to \infty} |x|e^{-ik|x|} \int \delta_n(x, t, k) v_{n+1}(t) \frac{e^{ik|t|}}{|t|} \rho_n(t, k) dt
\]

\[
|I_3| < C(\tau \epsilon)^{-1} R_n^3 \sigma_n \int \frac{e^{2\epsilon R_n - 2\epsilon|t|}}{\tau \epsilon(|t| - R_n)|t|} v_{n+1}(t) dt < C \tau^{-2} \epsilon^{-2} v_{n+1} \sigma_n R_n^3 e^{2\epsilon (R_n - R_{n+1})}
\]

(22)
For the other terms, we will be using the so-called Combes-Thomas inequality \[Q\], which says the following. Assume that potential \( Q \) is bounded. Then, for the operator kernel, we have \(^2\)

\[ \|\chi(x)(-\Delta + Q - z)^{-1}\chi(x)\| < C(\text{Im } z)^{-1} e^{-\gamma \text{Im } |x-y|} \]

where \( \chi(x) \) is characteristic function of the unit cube centered at \( x \), \( \gamma \) is fixed positive parameter and the norm is understood as the norm of operator acting in \( L^2(\mathbb{R}^3) \). Although an estimate on the operator kernel is not the same as pointwise estimate on the Green function, it is almost the same in our case. Let us accurately show that for \( I_4 \), for the other terms, we will be skipping details. We have

\[
I_4 = \lim_{|x| \to \infty} |x|e^{-ik|x|} \int \frac{e^{ik|x-u|}}{4\pi|x-u|} G_{n+1}(u, s, k) e^{i\gamma|s|} A_n(s, k) dy
\]

\[
|I_4| < C \int e^{<\hat{x}, u> \mid G_{n+1}(u, s, k) e^{i\gamma|s|} A_n(s, k) ds} du
\]

\[
< \sum_{u_i, s_j} e^{<\hat{x}, u> \mid G_{n+1}(u, s, k) e^{i\gamma|s|} A_n(s, k) ds} du
\]

\[
(23)
\]

where \( C(u_i) \) are all unit cubes from the \( \mathbb{Z}^3 \) partition of \( \mathbb{R}^3 \) that intersect \( \Sigma_{n+1} \). Points \( u_i \) (same as \( s_j \)) are the centers of these cubes. Therefore, we have

\[
|I_4| < Cv_{n+1}^2 \| A_n \| \int_{R_{n+1} + 2 < |u| < R_{n+1} + 3} e^{-\gamma \tau \tau \epsilon < u, s_j>} e^{-\epsilon |s_j|} dsdu
\]

\[
< Cv_{n+1}^2 \| A_n \| \int_{R_{n+1} - 2 < |u| < R_{n+1} + 3} e^{-\gamma \tau \tau \epsilon < u, s_j>} e^{-\epsilon |s_j|} dsdu
\]

So,

\[
|I_4| < C \epsilon^4 \| A_n \| \int_{R_{n+1}}^\infty \tau^{-4} \epsilon^{-5}
\]

(24)

In the same way, we have

\[
I_5 = \lim_{|x| \to \infty} |x|e^{-ik|x|} \int \delta_n(x, y, k) G_{n+1}(y, s, k) e^{i\gamma|s|} A_n(s, k) dy
\]

\[
|I_5| < C \int \frac{R_n^3 e^{2\tau \epsilon < y, R_{n+1}}} {\tau (|y| - R_n)} |G_{n+1}(y, s, k) e^{i\gamma|s|} A_n(s, k) ds| dy
\]

\[
< C(\tau \epsilon)^{-5} R_n^3 \| A_n \| \int_{R_n}^{R_{n+1}} \epsilon^4 R_n e^{2\tau \epsilon < y, R_{n+1}>
\]

(25)

For \( I_6 \):

\[
I_6 = \lim_{|x| \to \infty} |x|e^{-ik|x|} \int \delta_n(x, y, k) G_{n+1}(y, s, k) e^{i\gamma|s|} \rho_n(s, k) dy
\]

\[
|I_6| < C(\tau \epsilon)^{-6} R_n^3 \| A_n \| \int_{R_n}^{R_{n+1}} \epsilon^4 R_n e^{2\tau \epsilon < y, R_{n+1}>
\]

(26)

\(^2\)The actual estimate obtained in \[Q\] is stronger.
Define $I_7$:

$$I_7 = \lim_{|x| \to \infty} |x|e^{-ik|x|} \int \frac{e^{ik|x-y|}}{4\pi|x-y|} v_{n+1}(y) \int G_{n+1}(y, s, k)v_{n+1}(s) \frac{e^{ik|s|}}{|s|} \rho_n(s, k) dy$$

$$|I_7| < C(\tau\varepsilon)^{-6}\sigma_n v_{n+1}^2$$

(27)

From now on we assume that $\tau$ changes within the interval $I = [a, b], a > 0$. Then, we can disregard dependence on $\tau$ and will keep track on $\epsilon$ only. The estimates on $I_j$ and amount to

$$\|A_{n+1}\|_\infty < \|A_n\| \left( 1 + C v_{n+1} \epsilon^{-5} + C v_{n+1} \epsilon^{-5} R_n^3 \epsilon^{2\epsilon(R_n-R_{n+1})} \right)$$

$$+ C \epsilon^{-6} \sigma_n v_{n+1} + C \epsilon^{-6} \sigma_n R_n^3 v_{n+1} \epsilon^{2\epsilon(R_n-R_{n+1})}$$

(28)

The following lemma is trivial

**Lemma 0.3.** If $x_n, a_n, b_n \geq 0$ and $x_{n+1} \leq a_n x_n + b_n$, then

$$x_{n+1} \leq (x_0 + \sum_{j=0}^{n} b_j) \max \{1, a_j \cdot a_{j+1} \cdot \ldots \cdot a_{n-1} \cdot a_n\}$$

**Proof.** The proof follows from the iteration of the given inequality.

**Lemma 0.4.** The following estimates hold

$$x^j e^{-\epsilon x} \leq (j/\epsilon)^j \epsilon^{-j}$$

for any $x > 0, j > 0, \epsilon > 0$.

**Proof.** The function $f(x) = x^j e^{-\epsilon x}$ has maximum at the point $x^* = j\epsilon^{-1}$.

By using lemma 0.3 and estimate (21), we get

$$\|A_{n+1}\|_\infty < \|A_n\| \left( 1 + C v_{n+1} \epsilon^{-8} \right) + C \epsilon^{-9} \sigma_n v_{n+1}$$

(29)

From lemma 0.3 it follows

$$\|A_n\|_\infty < (\|A_0\|_\infty + \epsilon^{-9} \sum_{j=0}^{n-1} v_{j+1} \sigma_j) \exp(C \epsilon^{-8} \|v\|_2 n^{0.5})$$

(30)

Let us make another, rather strong, assumption on sparseness of $R_n$:

$$\sigma_n < \epsilon^{-n}$$

(31)

Then we have the following apriori bound on $\|A_n\|_\infty$:

$$\|A_n\|_\infty < g_n = (\|A_0\|_\infty + C \|v\|_2 \epsilon^{-9}) \exp(C \epsilon^{-8} \|v\|_2 n^{0.5})$$

(32)

We will need analogous estimate for the derivatives of $A_n$. Let us use formula (14) and the same bounds as before together with estimate (10). From (10), we see that we only pick up extra $R_{n+1}$ in the inequalities. So,

$$|\partial A_{n+1}| < |\partial A_n| + C \|A_n\| R_{n+1} v_{n+1} \epsilon^{-8} + C \epsilon^{-9} R_{n+1} \sigma_n v_{n+1}$$

and from (22) we get

$$||\partial A_{n+1}\|_\infty < g'_{n+1} = ||\partial A_0|| + R_{n+1} \epsilon^{-9} ||v||_2$$
+nR_{n+1}^2\|v\|_{2\epsilon^{-8}\exp(C\epsilon^{-8}\|v\|_{2\epsilon^{-8}}^{0.5})} \left(\|A_0\|_{\infty} + C\|v\|_{2\epsilon^{-9}}\right) \tag{33}
}

Now we are going to use these apriori estimates to obtain asymptotics of the sequence $A_n$. By considering $R_1$ big enough, we may assume that $A_0$ is the amplitude for the unperturbed operator. Consider the second term in (14). It can be written as

$$(4\pi)^{-1}A_n(k, \hat{x}) \int \frac{v_{n+1}(u)}{|u|} e^{-ik\langle \hat{x}, u \rangle + ik|u|} du$$

$$+(4\pi)^{-1} \int \frac{v_{n+1}(u)}{|u|} (A_n(k, \hat{u}) - A_n(k, \hat{x})) e^{-ik\langle \hat{x}, u \rangle + ik|u|} du$$

Denote

$$\kappa_n = (4\pi)^{-1} \int \frac{v_{n+1}(u)}{|u|} e^{-ik\langle \hat{x}, u \rangle + ik|u|} du$$

Notice that

$$|\kappa_n| < Ce^{-5}v_{n+1} \in \ell^2(\mathbb{Z}^+) \tag{34}$$

The second term can be bounded using an apriori bound on the derivative. It is not greater than

$$C\|\nabla A_n\|_{\infty} \int e^{-\epsilon(|u| - \langle \hat{x}, u \rangle)} v_{n+1}(u) \frac{|\hat{u} - \hat{x}|}{|u|} du$$

$$< Cg'_{n+1}R_{n+1} \left[ \epsilon^{-1.5}R_{n+1}^{-1.5} + \epsilon^{-1}R_{n+1}^{-1} e^{-2\epsilon R_{n+1}} \right] \tag{35}$$

and used lemma [13] in the last inequality. Now, we will pay special attention to $I_4$. The other terms will be of little importance. We have

$$I_4 = \beta_n A_n(k, \hat{x})$$

$$+(4\pi)^{-1} \int e^{-ik\langle \hat{x}, u \rangle} v_{n+1}(u) \int G_{n+1}(u, s, k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} (A_n(k, \hat{s}) - A_n(k, \hat{x})) ds du \tag{36}$$

where

$$\beta_n = (4\pi)^{-1} \int e^{-ik\langle \hat{x}, u \rangle} v_{n+1}(u) \int G_{n+1}(u, s, k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} ds du$$

and upon using Combes-Thomas estimate, we have

$$|\beta_n| < Ce^{-5}v_{n+1}^2 \tag{37}$$

and it is very important that $\beta_n \in \ell^1(\mathbb{Z}^+)$. That is the only place where we essentially use $\ell^2$ condition on $v$. The other term can be estimated by

$$C\epsilon^{-1}v_{n+1}^2 \|\nabla A_n\|_{\infty} \int_{R_{n+1} - 2|u| < R_{n+1} + 3} e^{\epsilon\langle \hat{x}, u \rangle}$$
The second term in the last expression is bounded by

$$\sum_{n=1}^{N} e^{-\gamma|\alpha_n|} \left| e^{-\epsilon |\alpha_n|} \frac{\epsilon^{-\epsilon |\alpha_n|}}{|s|} (|\hat{s} - \hat{u}| + |\hat{u} - \hat{x}|) ds \right|$$

The first term in the last expression is bounded by

$$C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

$$< C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

$$< C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

The second term in (38) is estimated by

$$C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

$$< C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

$$< C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

The second term in (38) is estimated by

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$$< C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

$$< C v_{n+1} \epsilon^{-2} g_n R_n^2 \int_{\Sigma} e^{-\gamma R_n+1 \|\hat{u} - \hat{x}\|} ds$$

For the other $I_j$, we are using estimates obtained before and get the following recursion

$$A_{n+1}(k, \theta) = A_n(k, \theta) (1 - \kappa_n + \beta_n) + \eta_n$$

For $\eta_n$, we apply estimates (15), (20), (22), (24), (27), (32), (33), (38), (39), (40), (41) to get

$$|\eta_n| < C \epsilon^{-d} v_{n+1} \left\{ g_{n+1} R_{n+1}^{0.5} + \sigma_n + R_n g_n e^{2\epsilon (R_n - R_{n+1})} \right\}$$

with some $d$ that will be reserved for the positive constant that might change from formula to formula, we will not care for its particular value. From the estimates on $g_n$ and $g_{n+1}$, we get

$$|\eta_n| < C \epsilon^{-d} v_{n+1} \left\{ e^{2\epsilon (R_n - R_{n+1})} \left( n R_n R_{n+1}^{0.5} + R_n^{3} \right) \right\}$$

Assuming

$$n R_n R_{n+1}^{0.5} < e^{-2n}, 2R_n < R_{n+1}, \sigma_n < e^{-n}$$

we have

$$|\eta_n| < C \epsilon^{-d} v_{n+1} \exp(C \epsilon^{8} 0.5 - 1.5\alpha n) < \exp(C \epsilon^{-d} v_{n+1} \exp(-\epsilon n)$$

Now, we are going to use the following lemma

**Lemma 0.5.** Assume that $x_{n+1} = x_n (1 + q_n) + d_n$ where $q_n \in \ell^2(\mathbb{Z}^+)$, $|d_n| < \omega e^{-\alpha n}, \alpha > 0$, and $n = 0, 1, \ldots$. Then, we have

$$x_n = \mathcal{Q} \left[ \exp(C \|q\|_2^2) \right] \exp \left[ \sum_{j=0}^{n} q_j \right] \left( x_0 + \omega \sum_{j=1}^{n} \exp(-\alpha j + \|q\|_2^2) \right)$$

(45)
Proof. Iterating, we get
\[ x_{n+1} = (1 + q_n) \ldots (1 + q_0)x_0 + (1 + q_n) \ldots (1 + q_1)d_0 + \ldots + (1 + q_n)d_{n-1} + d_n \]
Clearly,
\[ (1 + q_n) \ldots (1 + q_k) = \exp \left[ \sum_{j=0}^{n} q_j \right] r_{k,n} \]
where
\[ r_{k,n} = \exp \left[ - \sum_{j=k}^{n-1} q_j \right] \prod_{j=k}^{n} l(q_j) \]
and
\[ l(z) = (1 + z)e^{-z} \]
It is obvious that \(|l(z)| < \exp[C|z|^2]|. Then,
\[ \exp \left[ - \sum_{j=0}^{k-1} q_j \right] < \exp \left[ \sqrt{k}||q||_2 \right] \]
and
\[ \prod_{j=k}^{n} |l(q_j)| < \exp[C||q||_2^2] \]
Now, (45) easily follows. \[\square\]

Let us now apply this lemma to (41) bearing in mind (44), (34), (37). Then, for \(R_0\) large enough, we get
\[ A_n(k,\theta) = O \left( \exp \left[ - \sum_{j=0}^{n-1} \kappa_j \right] \right) \exp \left[ A^0(\theta,k) + \nu_n \right] \] \quad (46)
with
\[ |\nu_n| < ||v||_2 \exp(C\epsilon^{-d}) \sum_{j=1}^{n} \exp(-\epsilon j + C\epsilon^{-d} j^{0.5}) < ||v||_2 \exp(C\epsilon^{-d}) \] \quad (47)
and
\[ A_n(k,\theta) = \exp \left[ - \sum_{j=0}^{n-1} \kappa_j \right] \exp \left[ \exp(C\epsilon^{-d}) \right] \] \quad (48)
which proves (5), (6). Now, we need to show (7), otherwise that would not be an asymptotical result. Indeed, fix \(k\). Then, from (46) and (47), we easily get (7) as \(||v||_2 \to 0\).

Let us show that the a.c. spectrum of \(H\) fills \(\mathbb{R}^+\). We will prove that the interval \(I^2 = \{k^2, k \in I\}\) supports the a.c. component of the spectrum. Following [7, 9], consider an isosceles triangle \(T\) in \(\Pi\) with the base equal to \(I\) and the adjacent angles both equal to \(\pi/\gamma_1\), \(\gamma_1 > d\) with \(d\) from (48). Then, for simplicity, fix \(f\)–nonzero \(L^2\) spherically-symmetric function with compact support within, say, a unit ball centered at origin. Clearly, we can find some point \(k_0\) within this triangle \(T\) at which \(|A^0(f,k_0,\theta)| > C > 0\) for \(\theta \in \Sigma\). It follows just from the analyticity of \(A^0(f,k,\theta)\) in \(k\) for fixed \(\theta\) and spherical symmetry in \(\theta\) for fixed \(k\). Let us fix this \(k_0\). Then, consider a new potential \(\hat{V} = \chi_{|x|>R}V(x)\) with \(R\) large enough. By
Rozenblum-Kato theorem \cite{17}, the a.c. spectrum is not changed. Now, by \cite{8}, we can choose $R$ large enough (that means $\|\hat{v}_n\|_2$ is small) to guarantee that
\[ |\hat{A}_n(f, k_0, \theta)| > C > 0 \] (49)
uniformly in $n$ and all $\theta \in \Sigma$. Then, we will prove that $\sigma'_f(k^2) > 0$ for a.e. $k \in I$, where $d\sigma_f(E)$ is the spectral measure of $f$ with respect to operator $\hat{H} = -\Delta + \hat{V}$. That will show that $I^2$ is in the support of the a.c. spectrum of $H$. Since $I$ is arbitrary, that will mean $\sigma'_{ac}(H) = \mathbb{R}^+.$

To implement this strategy, we use the following factorization identity (\cite{20}, pages 40-42)
\[ \sigma'_{n,f}(E) = k\pi^{-1}\|A_n(f, k, \theta)\|_{L^2(\Sigma)}^2, E = k^2 \] (50)
where $\sigma_{n,f}(E)$ is the spectral measure of $f$ with respect to the operator with potential $\hat{V}_n = \chi_{|x| > R}V_n$ and $A_n$ is an amplitude with respect to the same potential. Let $\omega(k_0, s), s \in \partial_T$ denote the value at $k_0$ of the Poisson kernel associated to $T$. One can easily show that
\[ 0 \leq \omega(k_0, s) < C|s - s_{1(2)}|^{\gamma_1 - 1}, s \in \partial T \] (51)
where $s_{1(2)}$ are endpoints of $I$. It is also nonnegative function. Let us write the following inequalities
\[ \int_I \omega(k_0, s) \ln \|A_n(f, s, \theta)\|_{L^2(\Sigma)}^2 ds = \int_I \omega(k_0, s) \ln \|WKB_n(s, \theta)\hat{A}_n(f, s, \theta)\|^2 d\theta ds > 2(J_1 + J_2) \]
by Jensen's inequality, where
\[ J_1 = \int_I \omega(k_0, s) \int_\Sigma \ln |WKB_n(s, \theta)| d\theta ds \]
\[ J_2 = \int_I \omega(k_0, s) \int_\Sigma \ln |\hat{A}_n(f, s, \theta)| ds d\theta = \int_\Sigma \int_I \omega(k_0, s) \ln |\hat{A}_n(f, s, \theta)| ds d\theta \]
We will estimate from below each of these terms. The function $\ln |\hat{A}_n(f, k, \theta)|$ is subharmonic in $k \in T$ for fixed $\theta$, so we have a mean-value inequality
\[ \int_I \omega(k_0, s) \ln |\hat{A}_n(f, s, \theta)| ds \geq \ln |\hat{A}_n(f, k_0, \theta)| - \int_{I_1 \cup I_2} \omega(k_0, s) \ln |\hat{A}_n(f, s, \theta)| ds \]
where $I_{1(2)}$ are the sides of triangle $T$. From \cite{18}, \cite{19}, and \cite{61}, we obtain the bound
\[ \int_I \omega(k_0, s) \ln |\hat{A}_n(f, s, \theta)| ds > C > -\infty \]
uniformly in $n$ and $\theta \in \Sigma$. Here, the possible growth of $\ln |\hat{A}_n(f, s, \theta)|$ near the real line is compensated by the zero of the kernel $\omega(k_0, s)$. Therefore, $J_2 > C > -\infty$.
uniformly in \( n \). Thus, we are left to show that the same is true for \( J_1 \).

\[
J_1 = -(4\pi)^{-1} \int_I \omega(k_0, s) \int \sum \frac{\cos(s(|u| - \theta, u >))}{|u|} V_n(u) du
\]

\[
= -(4\pi)^{-1} \int_I \omega(k_0, s) \int \sin(2s|u|) \frac{V_n(u)}{|u|^2} duds
\]

Since \( \omega(k_0, s) \) is smooth on \( I \) and equals to 0 at the endpoints, we have

\[
|J_1| < C \int |V_n(u)| (1 + |u|^2)^{-1.5} du < C
\]

upon integration by parts in \( s \).

Thus, from (50), we get

\[
\int I \ln \sigma'_{n,f}(k^2) dk > C
\]

uniformly in \( n \). But then the standard argument on the semicontinuity of the entropy (see [8], Section 5) shows that

\[
\int I \ln \sigma'_{f}(k^2) dk > -\infty
\]

It follows from the fact that \( d\sigma_{n,f}(E) \) converges weakly to \( d\sigma_f(E) \) as \( n \to \infty \). Let us collect the conditions on sparseness that we used:

\[
2R_n < R_{n+1}, \sigma_n = R_n^{3.5} e^{R_0} (R_{n+1}-R_n)^{-1} < e^{-n}, nR_n R_{n+1}^{0.5} < e^{-2n}, n < R_{n+1}-R_n
\]

Obviously, the condition on \( \sigma_n \) is the strongest one and we can satisfy all of them by requiring

\[
R_{n+1} > e^{\alpha R_n}, \alpha > 1
\]

and \( R_0 \) is big enough. In particular, \( R_n = g^{(n)}(R_0) \) will work for \( g(x) = e^{2x} \).

We believe that the restrictions on sparseness can be relaxed by more detailed, rather straightforward analysis. It is also likely that by controlling the oscillatory integrals for real \( k \) one can show that the spectrum is purely a.c. on \( \mathbb{R}^+ \). We do not want to pursue that technically difficult problem in this paper. It also might be that formula (5.4) from [8] can be used to obtain the asymptotics of Green’s function in a simpler way. Notice also that condition (2) is satisfied under the assumptions of the theorem.

The WKB asymptotics we proved in the theorem is quite new to the best of our knowledge. It is different from the correction obtained in [15] and in earlier papers. Consider the randomized model: e.g., \( v_n(x) = \sum_{k \in \Delta_n} \omega_k^2 \nu_k^n(x) \) where \( \{\omega_k^2\} \) are independent random variables with mean zero and uniformly bounded dispersion, functions \( \nu_k^n(x) \), \( (k \in \Delta_n - \text{set of indexes}) \)— bump functions living within the small nonintersecting balls all lying inside the \( n \)-th spherical layer, which also satisfy the bound \( \max_{k \in \Delta_n} \|\nu_k^n\|_\infty \in l^2(\mathbb{Z}^+) \) in \( n \). Then,

\[
E_\omega \left| \int_{\mathbb{R}^3} \frac{\exp(ik(|t| - \theta, t >))}{|t|} V_n(t) dt \right|^2 < C
\]
uniformly in $n \in \mathbb{Z}^+$, $k \in \mathbb{C}^+$, and $\theta \in \Sigma$. So, one does not have any modification to the asymptotics really. That is due to oscillations and was observed before \cite{3}.

Now, we want to discuss the following issue. In the last theorem, we established the WKB asymptotics away from the real line. Recall that in the one-dimensional situation, the corresponding WKB correction was given by

$$WKB(k) = \exp \left( \frac{-i}{2k} \int_0^\infty V(r)dr \right)$$

and its absolute value is equal to one for real $k$. Clearly, this is not the case for the multidimensional WKB that we have got. We had an estimate \cite{52} that was sufficient to conclude the presence of a.c. spectrum but rather than that this WKB can exhibit quite a bad behavior. Apparently, the actual WKB asymptotics should be understood differently. The level sets of the function $|t| - <t, \theta>$ from the formula \cite{9} are paraboloids and it suggests that some evolution equation of the heat-transfer type might be involved. Consider operators $O_t$ given by the formula \cite{17}.

**Lemma 0.6.** The following parametrix representation is true for any $k \in \mathbb{C}^+$

$$-2ikO_tf = \exp \left[ -\frac{B}{2ikt} f + O(t^{-1})\|f\|_2, \quad t > 1 \right]$$

where $B$ is the Laplace-Beltrami operator on the unit sphere.

**Proof.** It is easy to show that $-2ikO_t \to I$ in the strong sense as $t \to \infty$. Consider $g_t = \exp \left[ -\frac{B}{2ikt} \right] f$. It solves the following problem

$$g'_t = \frac{B}{2ikt^2} g, \quad g(\infty) = f$$

Take $\psi(t) = -2ikO_t f - g(t)$. Then,

$$\psi' - \frac{B}{2ikt^2} \psi = C \left[ \int_\Sigma e^{ikt(1-<x,y>)}(1- <x,y>)f(y)dy + 
+ ikt \int_\Sigma e^{ikt(1-<x,y>)}[1- <x,y> + \frac{1}{2}(<x,y>^2 - 1)]f(y)dy \right]$$

We estimate the integral operators in $L^{1,1}$, $L^{\infty,\infty}$ norms first and then interpolate by Riesz-Thorin theorem to get

$$\psi' - \frac{B}{2ikt^2} \psi = O(t^{-2})\|f\|_2, \quad \psi(\infty) = 0$$

Since $B$ is nonpositive, we get \cite{53} by integration. \hfill \Box

Consider the following evolution equations

$$\frac{d}{dt} U_0(\tau, t, k) = -(2ik)^{-1} \frac{B}{t^2} U_0(\tau, t, k), U_0(\tau, \tau, k) = I \quad (54)$$

$$\frac{d}{dt} U(\tau, t, k) = -(2ik)^{-1} \left[ \frac{B}{t^2} - V(t) \right] U(\tau, t, k), U(\tau, \tau, k) = I \quad (55)$$
Then,
\[ \exp \left[ -\frac{B}{2ikt} \right] = U_0(t, \infty, k) \]

Since \( A_{R_n} \) oscillates relatively slow with the respect to the \((n+1)\)-th layer, an expression
\[ \int_{R_{n+1}}^{R_{n+1}} O_t q_t dt \]
from (16) basically coincides with the linear in \( V \) term for the Duhamel expansion of \( U(R_{n+1}, R_{n+2}, k) \).

An interesting open question is: what is the WKB correction for the real \( k \)? The likely candidate might be a solution to the following evolution problem
\[ 2iku_r = -\frac{B}{r^2}u(r, k) + V(r)u(r, k) \]  
(56)

Unfortunately, we cannot control the terms corresponding to the multiple collisions within the same layer (e.g. higher order in \( v_n \) terms) for the real \( k \). But if one considers only those that are linear in \( v_n \), then the conjecture seems to be reasonable. The same evolution equation can be obtained via the formal asymptotical expansion for the 3-dim Schrödinger operator written as the one-dimensional operator with operator-valued potential. This new candidate for the correct WKB and modification of wave operators preserves the \( L^2(\Sigma) \) norm. Notice that for \( k \) real, we cannot reduce the asymptotics to the scalar version because \( v_{n+1} \) can be rough. So there is no any contradiction really with what we proved in theorem 0.3.

Although we can prove asymptotics of Green’s function for \( k \in \mathbb{C}^+ \) only, some analysis is possible on the real line too. The following simple result on the absence of embedded eigenvalues holds

**Theorem 0.4.** If \( R_n \) is sparse enough, then there are no positive eigenvalues for any bounded \( v_n \).

**Proof.** The idea is quite simple and was used before to treat the one-dimensional problem: we show that the solutions can not decay too fast between the layers where the potential is zero. Then, since the solution and its gradient are square summable in \( \mathbb{R}^3 \), we get the contradictions for \( R_n \) large enough. To make the argument work, we also need some apriori bounds from below that we borrow from [1].

Assume that \( \psi(x) \) is a real-valued eigenfunction corresponding to an eigenvalue \( E > 0 \). Since \( \psi(x) \to 0 \) as \( |x| \to \infty \) (see Chapter 2, [2]), there is a point \( x_0 \) such that \( \psi(x_0) = \max_{x \in \mathbb{R}^3} |\psi(x)| = 1 \), this is our normalization of \( \psi \). We also know that \( \|\psi\|_2 \) is finite but we have no control over this quantity.

Introduce the spherical change of variables and consider \( \omega(r) = r\psi(r\sigma), \sigma \in \Sigma, x = r\sigma \). Now, \( \omega(r) \in L^2(\mathbb{R}^+, L^2(\Sigma)) \). Expanding in the spherical harmonics,
\[ \omega(r) = \sum_{m=0}^{\infty} \sum_{l_m=-m}^{m} Y_{m,l_m}(\sigma)f_{m,l_m}(r) \]
and we have for any \( l_m \)
\[ -f'''_{m,l_m}(r) - m(m+1)r^{-2}f_{m,l_m}(r) = Ef_{m,l_m}(r), R_n + 1 < r < R_{n+1} \]  
(57)
From [1], lemma 3.10, we infer the following bound
\[ \int_{R_n+1 < |x| < R_n+2} \psi^2(x) dx > CR_n^2 e^{-\gamma R_n^{4/3} \ln R_n}, n >> 1 \] (58)
with \( \gamma(E) > 0 \) which is an independent constant (at this point we used the normalization of \( \psi \) at \( x_0 \)). Since the potential \( V \) is bounded, we have
\[ \|\Delta \psi\|_\infty < C, \|\nabla \psi\|_\infty < C \] (59)
The bounds (58), (59) lead to the existence of \( r_n \in [R_n + 1, R_n + 2] \) such that
\[ \sum_{m=0}^{\infty} \sum_{l_m = -m}^m |f_{m,l_m}(r_n)|^2 > CR_n^2 e^{-\gamma R_n^{4/3} \ln R_n} \] (60)
\[ \sum_{m=0}^{\infty} (m+1)^2 \sum_{l_m = -m}^m |f_{m,l_m}(r_n)|^2 < CR_n^2 \] (61)
Consider
\[ p_m = \sum_{k=m}^{\infty} \sum_{l_k = -k}^k |f_{k,l_k}(r_n)|^2 \]
Then, (61) leads to
\[ \sum_{m=1}^{\infty} mp_m < CR_n^2 \]
and
\[ p_m < CR_n^2 m^{-1} \]
Take integer \( k_n \) so large that \( R_n^2 k_n^{-1} < R_n \exp(-\gamma R_n^{4/3} \ln R_n) \). From (60), we have an estimate
\[ \sum_{m=0}^{k_n} \sum_{l_m = -m}^m |f_{m,l_m}(r_n)|^2 > (C_1 R_n^2 - C_2 R_n) \exp(-\gamma R_n^{4/3} \ln R_n) \]
with \( C_1 > 0 \). The following estimate easily follows from (57) (e.g., by introducing the Prüfer transform [9], formula (2.4))
\[ |f_{m,l_m}(r)|^2 + E^{-1} |f'_{m,l_m}(r)|^2 \]
\[ > \left[ |f_{m,l_m}(r_n)|^2 + E^{-1} |f'_{m,l_m}(r_n)|^2 \right] \exp\left[ -(2\sqrt{E})^{-1} m(m+1)(r-r_n)r^{-1}r_n^{-1} \right] \]
\[ > |f_{m,l_m}(r_n)|^2 \exp\left[ -(2\sqrt{E})^{-1} (m+1)^2 r^{-1} \right] \]
Integrating the last estimate in \( r \) and summing over the indices, we have
\[ C > \int_{r_n < |x| < R_n+1} \left[ \psi^2(x) + E^{-1}(\psi_r(x) + r^{-1}\psi(x))^2 \right] dx \] (62)
\[ > (R_n+1 - r_n) \sum_{m=0}^{k_n} \sum_{l_m = -m}^m |f_{m,l_m}(r_n)|^2 \exp\left[ -(2\sqrt{E})^{-1} (m+1)^2 r^{-1} \right] \]
\[ > (R_n+1 - R_n - 2)(C_1 R_n^2 - C_2 R_n) \exp(-\gamma R_n^{4/3} \ln R_n) \exp\left[ -(2\sqrt{E})^{-1} (k_n + 1)^2 (R_n + 2)^{-1} \right] \] (63)
Now, choose \( \{ R_n \} \) so sparse that for any \( \gamma > 0 \) and \( E > 0 \)
\[
(R_{n+1} - R_n - 2) R_n^2 \exp \left[ -\gamma R_n^{4/3} \ln R_n - (2 \sqrt{E})^{-1} (k_n + 1)^{2} (R_n + 2)^{-1} \right] \to \infty, \text{ as } n \to \infty
\]
Then, for \( n \) large enough, we get the contradiction in (63) because the constant in the left hand side of (62) is independent of \( n \). Notice that the sparseness conditions were chosen independent of the eigenfunction, eigenvalue \( E \), and even of \( \|V\|_{\infty} \). It is satisfied, for instance, if \( R_{n+1} > \exp(\exp(R_n^3)) \), \( \beta > 4/3 \) and \( R_0 \) is large enough. \( \square \)

Remark. Apparently, the estimates from [1] that we used can be improved in our case. Thus, the sparseness conditions can also be relaxed.

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