REPRESENTATIONS AND DEFORMATIONS OF 3-HOM-\(\rho\)-LIE ALGEBRAS

E. PEYGHAN \(^1\), Z. BAGHERI \(^2\), I. GULTEKIN \(^3\) AND A. GEZER \(^4\)

Abstract. The aim of this paper is to introduce 3-Hom-\(\rho\)-Lie algebra structures generalizing the algebras of 3-Hom-Lie algebra. Also, we investigate the representations and deformations theory of this type of Hom-Lie algebras. Moreover, we introduce the definition of extensions and abelian extensions of 3-hom-\(\rho\)-Lie algebras and show that associated to any abelian extension, there is a representation and a 2-cocycle.

1. Introduction

The structure of Hom-Lie algebra appeared first as a generalization of Lie algebra by Hartwig, Larsson and Silvestrov in \(^{[12]}\). In 1994, the concept of \(\rho\)-Lie algebra or Lie color algebra introduced by Bongaarts \(^{[0]}\) and then in 1998, Scheunert and Zhang introduced the cohomology theory of Lie color algebras in \(^{[10]}\). Also, in 2012, Yuan \(^{[23]}\) introduced the notion of a hom-Lie color algebra which can be viewed as an extension of Hom-Lie superalgebras to \(G\)-graded algebras, where \(G\) is any abelian group. In 2015, Abdaoui, Ammarto and Makhlouf defined representations and a cohomology of the Hom-Lie color algebra in \(^{[3]}\). After two years, in 2017, \(T^*\)-extensions and abelian extensions of the Hom-Lie color algebras are studied by Bing Sun, Liangyun Chen and Yan Liu in \(^{[20]}\).

Filippov, in 1985 introduced a concept that is called \(n\)-Lie algebra. These Lie algebras are represented with various names such as Filippov algebra, Nambu-Lie algebra, Lie \(n\)-algebra. The notion of \(n\)-Lie algebra has close relationships with many fields in mathematics and mathematical physics, for their applications refer to \(^{[21][8][17][4][5]}\). The cohomology theory and deformation theory for \(n\)-Lie algebras was introduced respectively by Takhtajan and Gautheron in \(^{[22][7][9]}\). H. Ataguema, A. Makhlouf and S. Silvestrov in \(^{[2]}\) have introduced the notion of 3-hom-Lie algebras and representations and module-extensions of 3-hom-Lie algebras have investigated by Y. Liu, L. Chen and Y. Ma in \(^{[13]}\). The notion of 3-Lie colour algebras have introduced by T. Zhang and have studied Cohomology and deformations of 3-Lie colour algebras by him (see \(^{[24]}\), for more details).

In this paper, we introduce the notion of 3-Hom-Lie colour algebras or 3-Hom-\(\rho\)-Lie algebras and study the representation and deformation theory of this kind of Hom-Lie algebras.

This paper is arranged as follows: In Section 2, we recall some necessary background knowledge including \(\rho\)-commutative and Hom-\(\rho\)-Lie algebras. In the next, we discuss about the 3-Hom-\(\rho\)-Lie algebras and define representations, modules, \(\phi^k\) derivations of it and show that representations and modules of 3-Hom-\(\rho\)-Lie algebras are equivalent. This section also contains the \(T^*\)-extension of 3-Hom-\(\rho\)-Lie algebras. Section 3 is contained abelian extensions of 3-Hom-\(\rho\)-Lie algebras and the reader will get some results in this case. In this section we show that associated to any abelian extension, there is a representation and a 2-cocycle. Section 4 is devoted to discuss about deformations and the Hom Nijenhuis operator of

2010 Mathematics Subject Classification. 11R20, 17B10, 16W25, 17B56, 17B70, 17B75.

Key words and phrases. 3-Hom-\(\rho\)-Lie algebras, Abelian extensions, Deformations, Representations.
3-Hom-$\rho$-Lie algebras. Furthermore, we show that $\omega$ generates a $t$-parameter infinitesimal deformation of the 3-Hom-$\rho$-Lie algebra $A$ is equivalent to 3-Hom-$\rho$-Lie algebra which is 1-cocycle of $A$ with coefficients in the adjoint representation.

2. 3-HOM-$\rho$-LIE ALGEBRAS

In this section, we summarize some definitions concerning $\rho$-commutative algebras and Hom-$\rho$-Lie algebras. We also introduce the notion of 3-Hom-$\rho$-Lie algebras. Representations, modules, $\phi^k$-derivations and some results about them are studied in this section.

Let $A$ be an associative and unital algebra over a field $k$ ($k = \mathbb{R}$ or $k = \mathbb{C}$), grading by an abelian group $(G, +)$ that is the vector space $A$ has a $G$-grading $A = \oplus_{a \in G} A_a$ such that $A_a A_b \subset A_{a+b}$. A map $\rho : G \times G \rightarrow k^*$ is called a two-cycle if the following conditions hold

$$\rho(a, b) = \rho(b, a)^{-1}, \quad a, b \in G,$$

(2.1)

$$\rho(a + b, c) = \rho(a, c)\rho(b, c), \quad a, b, c \in G.$$

(2.2)

The above conditions say that $\rho(a, b) \neq 0$, $\rho(0, b) = 1$ and $\rho(c, c) = \pm 1$ for all $a, b, c \in A$, $c \neq 0$.

Let us denote by $Hg(A)$ the set of homogeneous elements in $A$. The $\rho$-commutator of two homogeneous elements $f, g$ is

$$[f, g]_\rho = fg - \rho(|f|, |g|)gf,$$

where $|f|$ denotes the $G$-degree of a (non-zero) homogeneous element $f \in A$.

A $\rho$-commutative algebra is a $G$-graded algebra $A$ with a given two-cycle $\rho$ such that $fg = \rho(|f|, |g|)gf$ for all homogeneous elements $f$ and $g$ in $A$ (i.e., $[f, g]_\rho = 0$).

**Definition 2.1.** A 2-Hom-$\rho$-Lie algebra or for simply a Hom-$\rho$-Lie algebra is a $G$-graded vector space $A$ together with a bilinear map $[,]_\rho : A \times A \rightarrow A$, a two-cycle $\rho$ and a linear map $\phi : A \rightarrow A$ satisfying the following conditions

- $|[f, g]_\rho| = |f| + |g|,
- [f, g]_\rho = -\rho(f, g)[g, f]_\rho,
- \rho(h, f)[\phi(f), [g, h]_\rho]_\rho + \rho(g, h)[\phi(h), [f, g]_\rho]_\rho + \rho(f, g)[\phi(g), [h, f]_\rho]_\rho = 0.$

The third condition is equivalent to

$$[[f, g]_\rho, \phi(h)]_\rho = [[f, g]_\rho, \phi(h)]_\rho + \rho(f, g)[\phi(g), [h, f]_\rho]_\rho.$$

**Definition 2.2.** A quadruple $(A, [, [, ]], \rho, \phi)$ consisting of a $G$-graded vector space $A = \bigoplus_{a \in G} A_a$, a trilinear map $[, [, ]] : A \times A \times A \rightarrow A$, a two-cycle $\rho : G \times G \rightarrow k^*$ and an even linear map $\phi : A \rightarrow A$ is called a 3-Hom-$\rho$-Lie algebra if the following condition are satisfied

1. $|[f_1, f_2, f_3]| = |f_1| + |f_2| + |f_3|,
2. $[\phi(f_1), \phi(f_2), [g_1, g_2, g_3]] = [[f_1, f_2, g_1], \phi(g_2), \phi(g_3)] - \rho(f_1 + f_2, g_1)[\phi(g_1), [f_1, f_2, g_2], \phi(g_3)]$

$$+ \rho(f_1 + f_2, g_1 + g_2)[\phi(g_1), \phi(g_2), [f_1, f_2, g_3]].$$

The second property is called $\rho$-fundamental identity.
Note that, the bracket introduced in the above definition has the \(\rho\)-skew symmetry property with respect to the displacement of every two elements of itself.

**Definition 2.3.** A 3-Hom-\(\rho\)-Lie algebra \((A,\ldots,\rho,\phi)\) is said to be multiplicative if \(\phi\) is a Lie algebra morphism, i.e. for \(f,g,h \in A\), \(\phi[f,g,h] = [\phi(f),\phi(g),\phi(h)]\), regular if \(\phi\) is an automorphism for \([\ldots,\ldots]\), and involutive if \(\phi^2 = \text{Id}_A\).

**Definition 2.4.** Let \((A,\ldots,\rho,\phi)\) and \((B,\ldots,\rho,\psi)\) be two 3-Hom-\(\rho\)-Lie algebra. A linear map \(\alpha: A \rightarrow B\) is said to be a morphism of 3-Hom-\(\rho\)-Lie algebras if

\[
\alpha[f,g,h]_A = [\alpha(f),\alpha(g),\alpha(h)]_B,
\]

for all \(f,g,h \in A\) and

\[
\alpha \circ \phi = \psi \circ \alpha.
\]

Let us denote by \(\vartheta = \{(f,\alpha(f))|f \in A\} \subseteq A \oplus B\) the group of linear maps \(\alpha: A \rightarrow B\).

Let \((A,\ldots,\rho,\phi)\) be a multiplicative 3-Hom-\(\rho\)-Lie algebra. We define the following operation on the fundamental set \(\mathcal{L} = \wedge^2 A\) by

\[
[(f_1,f_2),(g_1,g_2)]_{\mathcal{L}} = ([f_1,f_2,g_1],\phi(g_2)) + \rho(f_1 + f_2,g_1)(\phi(g_1),[f_1,f_2,g_2]).
\]

If we define the even linear map \(\phi_1: \mathcal{L} \rightarrow \mathcal{L}\) by \(\phi_1(f_1,f_2) = (\phi(f_1),\phi(f_2))\), then we have the multiplicative Hom-\(\rho\)-Lie algebra \((\mathcal{L},\ldots,\rho,\phi_1)\).

Let \(A\) be a 3-Hom-\(\rho\)-Lie algebra and \(V\) be a \(G\)-graded vector space. Recall that End\(_G\)(\(V\),\(V\)) and Hom\(_G\)(\(A\),\(V\)) are \(G\)-graded vector spaces.

**Definition 2.5.** Let \(A\) be a 3-Hom-\(\rho\)-Lie algebra, \(V\) be a \(G\)-graded vector space and \(\mu\) be a linear map from \(\mathcal{L} = \wedge^2 A\) to End\(_G\)(\(V\)). Then \((V,\mu)\) is called a representation of \(A\) with respect to \(\beta \in \text{End}_G(V)\) if the following conditions are satisfied

\[
\mu([f_1,f_2),(g_1,g_2)]_{\mathcal{L}} \circ \beta = \mu(\phi_1(f_1,f_2))\mu(g_1,g_2) - \rho(f_1 + f_2,g_1 + g_2)\mu(\phi_1(g_1,g_2))\mu(f_1,f_2),
\]

\[
\mu([g_1,g_2,g_3],\phi(f)) \circ \beta = \mu(\phi_1(g_1,g_2))\mu(g_3,f) + \rho(g_1 + g_2 + g_3)\mu(\phi_1(g_2,g_3))\mu(g_1,f)
\]
\[
+ \rho(g_1 + g_2 + g_3)\mu(\phi_1(g_3,g_1))\mu(g_2,f),
\]

\[
\mu(\phi(g),[f_1,f_2,f_3]) \circ \beta = \rho(g,f_1 + f_2)\mu(\phi_1(f_1,f_2))\mu(f_1,f_3) + \rho(g,f_2 + f_3)\rho(f_1,f_2 + f_3)\mu(\phi_1(f_2,f_3))\mu(g,f_1)
\]
\[
+ \rho(g,f_1 + f_3)\rho(f_1 + f_2,f_3)\mu(\phi_1(f_3,f_1))\mu(g,f_2),
\]

\[
\mu(\phi_1(f_1,f_2))\mu(g_1,g_2) = \rho(f_1 + f_2,g_1 + g_2)\mu(\phi_1(g_1,g_2))\mu(f_1,f_2)
\]
\[
+ \rho(f_1 + f_2,g_1)\mu(\phi(g_1),[f_1,f_2,g_2]) \circ \beta + \mu([f_1,f_2,g_1],\phi(g_2)) \circ \beta.
\]

**Example 2.6.** Let \(A\) be a 3-Hom-\(\rho\)-Lie algebra, \(V = A\) and \(\phi = \beta \in \text{End}_G(A)\). Then the linear map \(\text{ad}: A \times A \rightarrow \text{End}_G(A)\) defined by \(\text{ad}(f_1,f_2)(f_3) = [f_1,f_2,f_3]\) is a representation of \(A\) with respect to \(\beta = \phi\).
It is enough to check the conditions \(2.5\) and \(2.6\). For the condition \(2.5\), by \(2.4\) and the \(\rho\)-fundamental identity of 3-Hom-\(\rho\)-Lie algebra \(A\), we have

\[
\text{ad}([f_1, f_2], (g_1, g_2)]_c \phi(g_3) = \text{ad}([f_1, f_2, g_1], \phi(g_2))\phi(g_3) + \rho(f_1 + f_2, g_1)\text{ad}(\phi(g_1), [f_1, f_2, g_2])\phi(g_3)
\]

\[
= [[f_1, f_2, g_1], \phi(g_2), \phi(g_3)] + \rho(f_1 + f_2, g_1)[\phi(g_1), [f_1, f_2, g_2], \phi(g_3)]
\]

\[
= [\phi(f_1), \phi(f_2), [g_1, g_2, g_3]] - \rho(f_1 + f_2, g_1 + g_2)[\phi(g_1), \phi(g_2), [f_1, f_2, g_3]]
\]

\[
= \text{ad}(\phi(f_1), \phi(f_2))\text{ad}(g_1, g_2, g_3) - \rho(f_1 + f_2, g_1 + g_2)\text{ad}(\phi(g_1), \phi(g_2))\text{ad}(f_1, f_2, g_3).
\]

The other conditions prove similarly.

**Definition 2.7.** Let \((A, [[\ldots, \cdot]], \rho, \phi)\) be a 3-Hom-\(\rho\)-Lie algebra. Consider the triple \((V, \beta, \cdot)\) consisting of a \(G\)-graded vector space \(V\), an even homomorphism \(\beta\) of vector spaces and a linear operation \(\cdot : \mathcal{L} \times V \rightarrow V\) such that \(\mathcal{L}^g \cdot V_h \subseteq \mathcal{L}^{g+h}\) for all \(g, h \in G\). Then \((V, \beta, \cdot)\) is called an \(A\)-module if

\[
([f_1, f_2], (g_1, g_2)]_c \beta(m) = \phi_1(f_1, f_2) \cdot ((g_1, g_2) \cdot m) - \rho(f_1 + f_2, g_1 + g_2)\phi_1(g_1, g_2) \cdot ((f_1, f_2) \cdot m),
\]

\[
([f_1, f_2, f_3], \phi(g)) \cdot \beta(m) = \phi_1(f_1, f_2) \cdot ([f_3, g] \cdot m) + \rho(f_1 + f_2, f_3 + f_3)\phi_1(f_1, f_2) \cdot ((f_3, g) \cdot m)
\]

\[
+ \rho(f_1 + f_2, f_3)\phi_1(f_3, f_1) \cdot ((f_1, g) \cdot m),
\]

\[
\phi_1(f_1, f_2) \cdot (g_1, g_2) \cdot \beta(m) = \rho(f_1 + f_2, g_1)\phi_1(g_1, g_2) \cdot ((f_1, f_2) \cdot m)
\]

\[
+ \rho(f_1 + f_2, g_1 + g_2)\phi_1(g_1, g_2) \cdot ((f_1, f_2) \cdot m).
\]

**Lemma 2.8.** Let \((A, [[\ldots, \cdot]], \rho, \phi)\) be a 3-Hom-\(\rho\)-Lie algebra. The modules and the representations of \(A\) are equivalent.

**Proof.** Let \((V, \mu, \beta)\) be a representation of \(A\). We define the linear operation \(\cdot : \mathcal{L} \times V \rightarrow V\) by \((f, g, m) \mapsto (f, g) \cdot m = \mu(f, g)(m)\) and show that \((V, \beta, \cdot)\) is an \(A\)-module. For this purpose, it is sufficient to check two relations \(2.9\) and \(2.10\). Then we have

\[
([f_1, f_2], (g_1, g_2)]_c \beta(m) = \mu([[f_1, f_2], (g_1, g_2)]_c)\beta(m) = \mu(\phi_1(f_1, f_2))\mu(g_1, g_2)m
\]

\[
- \rho(f_1 + f_2, g_1 + g_2)\mu(\phi_1(g_1, g_2))\mu(f_1, f_2)m
\]

\[
= \phi_1(f_1, f_2) \cdot ((g_1, g_2) \cdot m) - \rho(f_1 + f_2, g_1 + g_2)\phi_1(g_1, g_2) \cdot ((f_1, f_2) \cdot m).
\]

For the next condition, we have

\[
([f_1, f_2, f_3], \phi(g)) \cdot \beta(m) = \mu([[f_1, f_2, f_3], \phi(g)])\beta(m) = \mu(\phi_1(f_1, f_2))\mu(f_3, g)m
\]

\[
+ \rho(f_1 + f_2, f_3)\mu(\phi_1(f_3, f_1))\mu(f_1, g)m + \rho(f_1 + f_2, f_3)\mu(\phi_1(f_3, f_1))\mu(f_1, g)m
\]

\[
= \phi_1(f_1, f_2) \cdot ((f_3, g) \cdot m) + \rho(f_1 + f_2, f_3)\phi_1(f_2, f_3) \cdot ((f_1, g) \cdot m)
\]

In the same method, we can check the last two conditions.

For the converse, consider the linear map \(\mu : A \times A \rightarrow \text{End}_G(V)\) by \(\mu(f, g)m = (f, g) \cdot m\). It is easy to see that for an even homomorphism \(\beta : V \rightarrow V\), the triple \((V, \beta, \cdot)\) is an \(A\)-module. \(\square\)
Definition 2.9. Let $A$ be a $3$-Hom-$\rho$-Lie algebra and $(V, \mu)$ be an $A$-module. An $n$-cochain on $A$ is a $\rho$-skew symmetric morphism $\omega$ from $\wedge^{2n+1}(A)$ into $V$ of degree $|\omega|$. Let us denote by $C^n(A, V)$ the set of all $n$-cochains on $A$, in the sense of

$$C^n(A, V) = \text{Hom}(\wedge^{2n+1}(A), V), \quad \omega(f_1, \ldots, f_{2n+1}) \in V_{|f_1|+\cdots+|f_{2n+1}|+|\omega|},$$

where $f_1, \ldots, f_{2n+1} \in Hg(A)$. $\omega \in C^n(A, V)$ is called an $n$-Hom-cochain on $A$ if for $f_1, \ldots, f_{2n+1} \in Hg(A)$ and $\beta \in \text{End}_G(V)$, the following relation holds

$$\beta(\omega(f_1, \ldots, f_{2n+1})) = \omega(\phi(f_1), \ldots, \phi(f_{2n+1})).$$

We denote by $C^n_\phi(A, V)$ the set of all $n$-Hom-cochains on $A$.

Let $A$ be a multiplication $3$-Hom-$\rho$-Lie algebra and $\beta = 1d_V$. Define the coboundary operator $d_{n-1} : C^{n-1}_\phi(A, V) \longrightarrow C^n_\phi(A, V)$ by

$$d_{n-1}\omega(f_1, \ldots, f_{2n+1}) = (-1)^{n+1}\rho(f_1 + \cdots + f_{2n-2}, f_{2n-1} + f_{2n+1})\rho(f_{2n-1} + f_{2n}, f_{2n+1})$$

$$\times \mu(\phi^{n-1}(f_{2n+1}), \phi^{n-1}(f_{2n-1}))\omega(f_1, \ldots, f_{2n-2}, f_{2n})$$

$$+ (-1)^{n+1}\rho(f_1 + \cdots + f_{2n-2}, f_{2n} + f_{2n+1})\rho(f_{2n-1}, f_{2n} + f_{2n+1})$$

$$\times \mu(\phi^{n-1}(f_{2n}), \phi^{n-1}(f_{2n+1}))\omega(f_1, \ldots, f_{2n-1})$$

$$+ \sum_{k=1}^n (-1)^{k+1}\rho(f_1 + \cdots + f_{2k-2}, f_{2k-1} + f_{2k})$$

$$\times \mu(\phi^{n-1}(f_{2k-1}), \phi^{n-1}(f_{2k}))\omega(f_1, \ldots, f_{2k-1}, f_{2k}, \ldots, f_{2n+1})$$

$$+ \sum_{k=1}^{2n+1} \sum_{j=2k+1}^{2n+1} (-1)^k\rho(f_1 + \cdots + f_{2k}, f_{2k+1} + \cdots + f_{j-1})$$

$$\times \omega(\phi(f_1), \ldots, \phi(\overbrace{f_{2k-1}}^{f_1}, \phi(\overbrace{f_{2k}}^{f_2}), \ldots, [f_{2k-1}, f_{2k}, f_j], \ldots, \phi(f_{2n+1}))),$$

for $n \geq 1$, and $\omega \in C^n_\phi(A, V)$, where $\phi(\overbrace{f_{2k-1}}^{f_1})$ means that $\phi(f_{2k-1})$ is omitted. Note that $|d\omega| = |\omega|$ and $d_n \circ d_{n-1} = 0$ (the condition $d_n \circ d_{n-1} = 0$ does not follow if the condition $\omega \circ \phi = \omega$ is omitted, so it is necessary to define the differential operator on $n$-Hom-cochains).

For $n = 1$:

$$d_0\omega(f_1, f_2, f_3) = \mu(f_1, f_2)\omega(f_3) + \rho(f_1 + f_2, f_3)\mu(f_3, f_1)\omega(f_2)$$

$$+ \rho(f_1, f_2 + f_3)\mu(f_2, f_3)\omega(f_1) - \omega([f_1, f_2, f_3]).$$

For $n = 2$:

$$d_1\omega(f_1, f_2, g_1, g_2, g_3) = \omega(\phi(f_1), \phi(f_2), [g_1, g_2, g_3]) + \mu(\phi(f_1), \phi(f_2))\omega(g_1, g_2, g_3)$$

$$- \omega([f_1, f_2, g_1], \phi(g_2), \phi(g_3)) + \rho(f_1 + f_2, g_1)\omega(\phi(g_1), [f_1, f_2, g_2], \phi(g_3))$$

$$- \rho(f_1 + f_2, g_1 + g_2)\omega(\phi(g_1), \phi(g_2), [f_1, f_2, g_3])$$

$$- \rho(f_1 + f_2 + g_3)\rho(g_1, g_2 + g_3)\mu(\phi(g_2), \phi(g_3))\omega(f_1, f_2, g_1)$$

$$- \rho(f_1 + f_2, g_1 + g_3)\rho(g_1 + g_2, g_3)\mu(\phi(g_3), \phi(g_1))\omega(f_1, f_2, g_2)$$

$$- \rho(f_1 + f_2, g_1 + g_2)\mu(\phi(g_1), \phi(g_2))\omega(f_1, f_2, g_3).$$
Definition 2.10. Let $A$ be a 3-Hom-$\rho$-Lie algebra and $(V, \mu)$ be an $A$-module. Then a morphism $\nu \in \text{Hom}_G(A, V)$ is called 0-Hom-cocycle if and only if $d_0\nu = 0$, in the other word
\[
\mu(f_1, f_2)\nu(f_3) + \rho(f_1 + f_2, f_3)\mu(f_3, f_1)\nu(f_2) + \rho(f_1, f_2 + f_3)\mu(f_2, f_3)\nu(f_1) = \nu([f_1, f_2, f_3]).
\]
Also, $\omega \in \text{Hom}(\wedge^3 A, V)$ is called 1-Hom-cocycle with respect to $\mu$ if and only if $d_1\omega = 0$.

Definition 2.11. Let $(A, [\cdot, \cdot, \cdot], \rho, \phi)$ be a multiplicative 3-Hom-$\rho$-Lie algebra. For any non-negative integer $k$, denote the $k$ times composition of $\phi$ by $\phi^k$ ($\phi^k = \phi \circ \cdots \circ \phi$ ($k$ times)) such that $\phi^0 = \text{id}$ and $\phi^1 = \phi$. A $\phi^k$-derivation of degree $|X|$ on $A$ is a linear map $X : A \to A$ such that
\[
X \circ \phi = \phi \circ X \quad \text{i.e.,} \quad [X, \phi]_{\mu} = 0,
\]
and
\[
X[f, g, h] = [X(f), \phi^k(g), \phi^k(h)] + \rho(X, f)[\phi^k(f), X(g), \phi^k(h)] + \rho(X, f + g)[\phi^k(f), \phi^k(g), X(h)],
\]
for all $f, g, h \in A$. Let us denote by $\rho\text{-Der}_{\phi^k} A$ the set of all $\phi^k$-derivations of $A$.

Example 2.12. We define the even linear map $ad_k(f_1, f_2) : A \to A$ by $ad_k(f_1, f_2)(g) = [f_1, f_2, \phi^k(g)]$ for $g \in A$. If we assume that for any $f_1, f_2 \in A$, $\phi(f_1) = f_1$ and $\phi(f_2) = f_2$, then $ad_k(f_1, f_2)$ is a $\phi^{k+1}$-derivation. If we check the accuracy of the equality (2.13), the assertion follows. Thus, we have
\[
ad_k(f_1, f_2)[f, g, h] = [f_1, f_2, \phi^k[f, g, h]] = [\phi(f_1), \phi(f_2), \phi^k[f, g, h]]
\]
\[
= [[f_1, f_2, \phi^k(f)], \phi^{k+1}(g), \phi^{k+1}(h)]
\]
\[
+ \rho(f_1 + f_2, f)[\phi^{k+1}(f), [f_1, f_2, \phi^k(g)], \phi^{k+1}(h)]
\]
\[
+ \rho(f_1 + f_2, f + g)[\phi^{k+1}(f), \phi^{k+1}(g), [f_1, f_2, \phi^k(h)]]
\]
\[
= [ad_k(f_1, f_2)(f), \phi^{k+1}(g), \phi^{k+1}(h)]
\]
\[
+ \rho(f_1 + f_2, f)[\phi^{k+1}(f), ad_k(f_1, f_2)(g), \phi^{k+1}(h)]
\]
\[
+ \rho(f_1 + f_2, f + g)[\phi^{k+1}(f), \phi^{k+1}(g), ad_k(f_1, f_2)(h)].
\]

$ad_k(f_1, f_2)$ is called an inner $\phi^{k+1}$-derivation. Denote by $\text{Inn}_{\phi^k}(A)$ the set of inner $\phi^k$-derivation, i.e.,
\[
\text{Inn}_{\phi^k}(A) = \{[f_1, f_2, \phi^{k-1}(\cdot)] f_1, f_2 \in A, \phi(f_i) = f_i \ i = 1, 2\}.
\]

Definition 2.13. Let $A$ be a 3-Hom-$\rho$-Lie algebra.

1: A linear map $X : A \to A$ is said to be a homogeneous generalized $\phi^k$-derivation of degree $|X|$ of $A$, if there exist three linear maps $Y, Z, W : A \to A$ such that
\[
[X, \phi] = 0, \ [Y, \phi] = 0, \ [Z, \phi] = 0, \ [W, \phi] = 0,
\]
and
\[
W[f, g, h] = [X(f), \phi^k(g), \phi^k(h)] + \rho(X, f)[\phi^k(f), Y(g), \phi^k(h)]
\]
\[
+ \rho(X, f + g)[\phi^k(f), \phi^k(g), Z(h)],
\]
for all \( f, g, h \in A \). We denote the set of all homogeneous generalized \( \phi^k \)-derivation of degree \( |X| \) of \( A \) by \( \text{GDer}_{\phi^k}(A) \).

2: We call \( X : A \rightarrow A \) a homogeneous \( \phi^k \)-quasi derivation of degree \( |X| \) of \( A \), if there exists a linear map \( Y : A \rightarrow A \) such that

\[
[X, \phi] = 0, \ [Y, \phi] = 0,
\]

and

\[
Y[f, g, h] = [X(f), \phi^k(g), \phi^k(h)] + \rho(X, f)[\phi^k(f), X(g), \phi^k(h)]
\]

\[
+ \rho(X, f + g)[\phi^k(f), \phi^k(g), X(h)],
\]

for all \( f, g, h \in A \). We denote the set of all homogeneous \( \phi^k \)-quasi derivation of degree \( |X| \) of \( A \) by \( \text{QDer}_{\phi^k}(A) \).

3: We call \( X : A \rightarrow A \) a homogeneous \( \phi^k \)-centroid element of degree \( |X| \) of \( A \), if it satisfies for all \( f, g, h \in Hg(A) \)

\[
X[f, g, h] = [X(f), \phi^k(g), \phi^k(h)] + \rho(X, f)[\phi^k(f), X(g), \phi^k(h)]
\]

\[
= \rho(X, f + g)[\phi^k(f), \phi^k(g), X(h)].
\]

We denote the set of all homogeneous \( \phi^k \)-centroid elements of degree \( |X| \) of \( A \) by \( C_{\phi^k}(A) \).

4: \( X : A \rightarrow A \) is said to be a homogeneous \( \phi^k \)-quasi centroid element of degree \( |X| \) of \( A \), if it satisfies for all \( f, g, h \in Hg(A) \)

\[
X[f, g, h] = [X(f), \phi^k(g), \phi^k(h)] = \rho(X, f)[\phi^k(f), X(g), \phi^k(h)]
\]

\[
= \rho(X, f + g)[\phi^k(f), \phi^k(g), X(h)].
\]

We denote the set of all homogeneous \( \phi^k \)-quasi centroid elements of degree \( |X| \) of \( A \) by \( \text{QCent}_{\phi^k}(A) \).

**Proposition 2.14.** Let \((A, [\ldots,], \rho, \phi)\) be a multiplication 3-Hom-\(\rho\)-Lie algebra. If \( X \in \text{GDer}_{\phi^k}(A) \) and \( X' \in C_{\phi^k}(A) \), then \( X'X \in \text{GDer}_{\phi^{k+\ast}}(A) \).

**Proof.** Since \( X \in \text{GDer}_{\phi^k}(A) \), then there exist \( Y, Z, W : A \rightarrow A \) such that

\[
W[f, g, h] = [X(f), \phi^k(g), \phi^k(h)] + \rho(X, f)[\phi^k(f), Y(g), \phi^k(h)]
\]

\[
+ \rho(X, f + g)[\phi^k(f), \phi^k(g), Z(h)].
\]

On the other hand, since \( X' \in C_{\phi^k}(A) \) we have

\[
X'W[f, g, h] = X'[X(f), \phi^k(g), \phi^k(h)] + \rho(X, f)X'[\phi^k(f), Y(g), \phi^k(h)]
\]

\[
+ \rho(X, f + g)X'[\phi^{k+\ast}(f), \phi^{k+\ast}(g), Z(h)]
\]

\[
= [X'X(f), \phi^{k+\ast}(g), \phi^{k+\ast}(h)] + \rho(X, f)[\phi^{k+\ast}(f), X'Y(g), \phi^{k+\ast}(h)]
\]

\[
+ \rho(X, f + g)[\phi^{k+\ast}(f), \phi^{k+\ast}(g), X'Z(h)].
\]

Therefore, \( X'X \in \text{GDer}_{\phi^{k+\ast}}(A) \). \qed

**Proposition 2.15.** Let \((A, [\ldots,], \rho, \phi)\) be a multiplication 3-Hom-\(\rho\)-Lie algebra and \( X \in C_{\phi^k}(A) \). Then \( X \) is a \( \phi^k \)-quasi derivation of \( A \).
Proof. Assuming that \( f, g, h \in Hg(A) \). So, we have
\[
[X(f), \phi^k(g), \phi^k(h)] + \rho(X, f)[\phi^k(f), X(g), \phi^k(h)]
+ \rho(X, f + g)[\phi^k(f), \phi^k(g), X(h)]
= 3[X(f), \phi^k(g), \phi^k(h)] = 3X[f, g, h]
= X'[f, g, h].
\]

\( \square \)

2.1. The Coadjoint Representation. We consider \( A^* \) as a dual space of \( A \), \( A^* \) is a \( G \)-graded space, where \( A^n = \{ \alpha \in A^* | \alpha(f) = 0, \forall f : |f| \neq -n \} \). Moreover, \( A^* \) is a graded \( A \)-module. Also, since \( A = \oplus_{a \in G} A_a \) and \( A^* = \oplus_{a \in G} A^*_a \) are \( G \)-graded spaces, the direct sum
\[
A \oplus A^* = \oplus_{a \in G} (A \oplus A^*)_a = \oplus_{a \in G} (A_a \oplus A^*_a),
\]
is \( G \)-graded. Consider a homogeneous element of \( A \oplus A^* \) as \( f + \alpha \) such that \( f \in A \) and \( \alpha \in A^* \), with \(|f + \alpha| = |f| = |\alpha| \).

Let \( \langle A, [\ldots]_A, \rho, \phi \rangle \) be a 3-Hom-\( \rho \)-Lie algebra and \( (V, \mu, \beta) \) be a representation of \( A \). Let \( V^* \) be the dual vector space of \( V \). We define the linear map \( \tilde{\mu} : A \times A \rightarrow \text{End}(V^*) \) by \( \tilde{\mu}(f_1, f_2)(g) = -\rho(f_1 + f_2, g)\circ \mu(f_1, f_2) \), where \( f_1, f_2 \in A \), \( g \in V^* \) and set \( \tilde{\beta}(g) = g \circ \beta \).

Proposition 2.16. Let \( \langle A, [\ldots]_A, \rho, \phi \rangle \) be a 3-Hom-\( \rho \)-Lie algebra and \( (V, \mu, \beta) \) be a representation of \( A \). Then the triple \( (V^*, \tilde{\mu}, \tilde{\beta}) \) defines a representation of \( A \) if and only if

1. \( \beta(\mu([f_1, f_2], (g_1, g_2)]_A) = \mu(f_1, f_2)\mu(\phi(g_1), \phi(g_2)) - \mu(f_1 + f_2, g_1 + g_2)\mu(\phi(f_1), \phi(f_2)) \)
2. \( \beta \mu([g_1, g_2, g_3], \phi(f)) = -\rho(g_1 + g_2, g_3 + f)\mu(g_3, f)\mu(\phi(g_1), \phi(g_2)) - \rho(g_2 + g_3, g_1 + f)\mu(g_1, f)\mu(\phi(g_2), \phi(g_3)) - \rho(g_3 + g_1, g_2 + f)\mu(g_2, f)\mu(\phi(g_3), \phi(g_1)) \)
3. \( \beta \mu(\phi(g), [f_1, f_2, f_3]) = -\rho(f_1 + f_2, f_3)\mu(g, f_3)\mu(\phi(f_1), \phi(f_2)) - \mu(g, f_1)\mu(\phi(f_2), \phi(f_3)) - \rho(f_1 + f_2, f_3)\mu(g, f_2)\mu(\phi(f_3), \phi(f_1)) \)
4. \( \rho(f_1 + f_2, g_1)\beta \mu(\phi(g_1), [f_1, f_2, g_2]) + \beta \mu([f_1, f_2, g_1], \phi(g_2)) = \mu(f_1, f_2)\mu(\phi(g_1), \phi(g_2)) - \rho(f_1 + f_2, g_1 + g_2)\mu(g_1, g_2)\mu(\phi(f_1), \phi(f_2)) \)

Proof. Assuming that the conditions (1)-(4) hold. We prove that \( \tilde{\mu} \) is a representation of \( A \), thus we must check the properties (2.5)-(2.8) for \( (\tilde{\mu}, \tilde{\beta}, V^*) \). For this, we check the property (2.5) and the others will prove similarly. Using (1), we start with the computation of the left-hand side of (2.5):
\[
\tilde{\mu}((f_1, f_2), (g_1, g_2)]_A \tilde{\beta}(g)(m) = -\rho(f_1 + f_2 + g_1 + g_2, g_2)(\rho(\phi(g_2))((\mu((f_1, f_2), (g_1, g_2)]_A)
= \rho(f_1 + f_2 + g_1 + g_2, g_2)\rho(f_1 + f_2, g_1 + g_2)\mu(g_1, g_2)\mu(\phi(f_1), \phi(f_2))(m)
- \rho(f_1 + f_2 + g_1 + g_2, g_2)\mu(f_1, f_2)\mu(\phi(g_1), \phi(g_2))(m)
= \tilde{\mu}(\phi(f_1), \phi(f_2))\tilde{\mu}(g_1, g_2)\phi(m)
- \rho(f_1 + f_2 + g_1 + g_2)\mu(\phi(g_1), \phi(g_2))\tilde{\mu}(f_1, f_2)\phi(m).
\]
Therefore
\[ \tilde{\mu}((f_1, f_2), (g_1, g_2)) \circ \tilde{\beta} = \tilde{\mu}(\phi(f_1), \phi(f_2))\tilde{\mu}(g_1, g_2) - \rho(f_1 + f_2, g_1 + g_2)\tilde{\mu}(\phi(g_1), \phi(g_2))\tilde{\mu}(f_1, f_2). \]

The proof of the converse is also straightforward. □

**Corollary 2.17.** Let \((A, [\ldots, ]_A, \rho, \phi)\) be a 3-Hom-\(\rho\)-Lie algebra with the adjoint representation and \(A^*\) be the dual of \(A\). The linear map \(ad^*: A \times A \rightarrow \text{End}(A^*)\) defined by \(ad^*(f_1, f_2)(\varphi) = -\rho(f_1 + f_2, \varphi)\varphi(f_1, f_2)\varphi(f_3)\) for \(f_1, f_2, f_3 \in A\) and \(\varphi \in A^*\) is a representation of \(A\) that is called coadjoint representation.

**Theorem 2.18.** Let \((A, [\ldots, ]_A, \rho, \phi)\) be a 3-Hom-\(\rho\)-Lie algebra with the adjoint representation and \(A^*\) be the dual of \(A\). Consider the linear map \(\omega: A \times A \times A \rightarrow A^*\) and let \(\tilde{\mu} = ad^*\). The \(G\)-graded space \(A \oplus A^*\) together with the bracket and linear map
\[
[f_1 + \alpha_1, f_2 + \alpha_2, f_3 + \alpha_3]_{A \oplus A^*} = [f_1, f_2, f_3]_A + \omega(f_1, f_2, f_3) + \tilde{\mu}(f_1, f_2)(\alpha_3) + \rho(f_1 + f_2, f_3)\tilde{\mu}(f_3, f_2)(\alpha_2) + \rho(f_1, f_2 + f_3)\tilde{\mu}(f_2, f_3)(\alpha_1),
\]
\[
\phi'(f + \alpha) = \phi(f) + \alpha \circ \phi,
\]
for all \(f_i \in A\) and \(\alpha_i \in A^*\), \(i = 1, 2, 3\), is a 3-Hom-\(\rho\)-Lie algebra if and only if \(\omega\) is a 1-Hom-cocycle with respect to \(\tilde{\mu}\).

**Proof.** Using (2.10) and Definition 2.10 the result easily follows.

**Definition 2.19.** The 3-hom-\(\rho\)-Lie algebra \((A \oplus A^*, [\ldots, ]_{A \oplus A^*}, \phi')\) is called the \(T^*_\omega\)-extension of \((A, [\ldots, ]_A, \phi)\).

\[ \square \]

### 3. Abelian Extension of 3-Hom-\(\rho\)-Lie Algebra

In this section, we discuss about extensions and abelian extensions of 3-Hom-\(\rho\)-Lie algebra \(A\) and show that associated to any abelian extension, there is a representation and a 2-cocycle. We assume that the 3-Hom-\(\rho\)-Lie algebra \(A\) is multiplicative.

**Definition 3.1.** A sub-vector space \(I \subseteq A\) is called a Hom subalgebra of \((A, [\ldots, ]_A, \phi)\) if \([I, I, I]_A \subseteq I\) and \(\phi(I) \subseteq I\), \(I\) also is called a Hom ideal of \(A\) if \(\phi(I) \subseteq I\) and \([I, A, A]_A \subseteq I\). \(I\) is said to be a Hom abelian ideal of \(A\) if \([A, I, I]_A = 0\).

**Definition 3.2.** Let \((A, [\ldots, ]_A, \rho, \phi), (V, [\ldots, ]_V, \psi_V)\) and \((B, [\ldots, ]_B, \psi)\) be three 3-Hom-\(\rho\)-Lie algebras and \(i: V \rightarrow B, p: B \rightarrow A\) be homomorphisms. The sequence
\[
0 \longrightarrow V \overset{i}{\longrightarrow} B \overset{p}{\longrightarrow} A \longrightarrow 0,
\]
of 3-Hom-\(\rho\)-Lie algebras is a short exact sequence if \(\text{Im}(i) = \text{Ker}(p), \text{Ker}(i) = 0, \text{Im}(p) = A\) and \(\psi_V(V) = \psi(V)\). In this case, \(B\) is called an extension of \(A\) by \(V\) and denote it by \(E_B\). Also, we call \(B\) an abelian extension of \(A\) if \(V\) is a Hom abelian ideal of \(B\) i.e., \([, u, v]_B = 0\) for all \(u, v \in V\). A linear map \(\delta: A \rightarrow B\) is called a section of \(p: B \rightarrow A\) if \(p \circ \delta = \text{id}_A\) and \(\delta \circ \phi = \psi \circ \delta\).
Definition 3.3. Two extensions $0 \rightarrow V \rightarrow B \rightarrow A \rightarrow 0$ and $0 \rightarrow V \rightarrow B \rightarrow A \rightarrow 0$ of $3$-Hom-$\rho$-Lie algebra $A$ are equivalent if there exists a morphism $F : B \rightarrow \tilde{B}$ of $3$-Hom-$\rho$-Lie algebras such that the following diagram commutes:

$$
\begin{array}{ccc}
0 & \rightarrow & V \\
\downarrow{\text{id}} & & \downarrow{f} \\
0 & \rightarrow & V \\
\end{array}
\begin{array}{ccc}
\rightarrow & \rightarrow & A \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
\rightarrow & \rightarrow & A \\
\end{array}
$$

Let $B$ be an abelian extension of $A$ by $V$ and $\delta : A \rightarrow B$ be a section. Define $\mu : \wedge^2 A \rightarrow \text{End}(V)$ by

$$
\mu(f)(u) = \mu(f_1, f_2)(u) = [\delta(f_1), \delta(f_2), u]_B = \text{ad}(\delta(f))u,
$$

for all $f = (f_1, f_2) \in \wedge^2 A, u \in V$.

Proposition 3.4. Let $(V, \phi_V)$, $(A, \phi)$ and $(B, \psi)$ be multiplication $3$-Hom-$\rho$-Lie algebras and $B$ be an abelian extension of $A$ by $V$. Consider $\mu$ with (3.1). Then, $(V, \phi_V, \mu)$ is a representation of $(A, \phi)$ and does not depend on the choice of the section $\delta$. Moreover, equivalent abelian extensions give the same representation.

Proof. At first, we show that $\mu$ is independent of the choice of $\delta$. If $\delta' : A \rightarrow B$ is another section, then

$$
p(\delta(f_i) - \delta'(f_i)) = f_i - f_i = 0,
$$

thus, $\delta(f_i) - \delta'(f_i) \in V$. So $\delta'(f_i) = \delta(f_i) + u$ for some $u \in V$. Since $[., u, v]_B = 0$ for all $u, v \in V$, we deduce that

$$
[\delta'(f_1), \delta'(f_2), w]_B = [\delta(f_1) + u, \delta(f_2) + v, w]_B
$$

$$
= [\delta(f_1), \delta(f_2) + v, w]_B + [u, \delta(f_2) + v, w]_B
$$

$$
= [\delta(f_1), \delta(f_2), w]_B + [\delta(f_1), v, w]_B + [u, \delta(f_2), w]_B + [u, v, w]_B
$$

$$
= [\delta(f_1), \delta(f_2), w]_B.
$$

So, $\mu$ is independent of the choice of $\delta$. In the next, we show that $(V, \phi_V, \mu)$ is a representation of $(A, \phi)$. For this it is enough to check the conditions (2.5) and (2.6). Let $\beta = \phi_V \in \text{End}(V)$. By the third property of $3$-Hom-$\rho$-Lie algebras, we have

$$
[\psi(\delta(f_1)), \psi(\delta(f_2)), \delta(g_1), \delta(g_2), u]_B = [[[\delta(f_1), \delta(f_2), \delta(g_1)]_B, \psi(\delta(g_2))], \psi(u)]_B
$$

$$
+ \rho(f_1 + f_2, g_1)[\psi(\delta(g_1))], [\delta(f_1), \delta(f_2), \delta(g_2)]_B, \psi(u)]_B
$$

$$
+ \rho(f_1 + f_2, g_1 + g_2)[\psi(\delta(g_1)), \psi(\delta(g_2)), [\delta(f_1), \delta(f_2), u]_B].
$$

Using (3.1) and this fact that $\psi \circ \delta = \delta \circ \phi$, we have

$$
\mu(\phi(f_1), \phi(f_2))\mu(g_1, g_2)u = \mu((f_1, f_2), (g_1, g_2))\phi_V(u)
$$

$$
+ \rho(f_1 + f_2, g_1 + g_2)\mu(\phi(g_1), \phi(g_2))\mu(f_1, f_2)u.
$$

Therefore

$$
\mu((f_1, f_2), (g_1, g_2))\phi_V(u) = \mu(\phi(f_1), \phi(f_2))\mu(g_1, g_2)u
$$

$$
- \rho(f_1 + f_2, g_1 + g_2)\mu(\phi(g_1), \phi(g_2))\mu(f_1, f_2)u.
$$
This gives us the condition (2.5). In the continues, we try to prove the correctness of the condition (2.6). Since $\phi_V(V) = \psi(V)$, $\delta \circ \phi = \psi \circ \delta$, $[\delta(f_1), \delta(f_2), \delta(g_1)|_B - \delta(f_1, f_2, g_1)_A \in V$ and $V$ is abelian ideal then

$$
\mu([f_1, f_2, g_1], \phi(g_2))\phi_V(u) = [\delta(f_1, f_2, g_1), \delta(g_2)], \phi_V(u)
$$

(3.2)

On the other hand, we have

$$
[\psi(\delta(f_1)), \psi(u), [\delta(g_1), \delta(g_2), \delta(g_3)]_B] = [\delta(f_1), u, \delta(g_1)]_B, [\psi(\delta(g_2)), \delta(g_3)]_B
$$

By invoking (3.2) and this fact that $\delta \circ \phi = \psi \circ \delta$ and $\beta = \phi_V$, we conclude that

$$
\rho(f_1 + u, g_1 + g_2 + g_3)\mu([g_1, g_2, g_3], \phi(f_1))\phi_V(u) = \rho(f_1 + u, g_1 + g_2 + g_3)\rho(f_1 + u, g_1)\mu(\phi(g_1), \phi(g_2))\mu(g_3, f_1)u
$$

$$
+ \rho(f_1 + u, g_1 + g_2 + g_3)\rho(f_1 + u, g_1)\mu(\phi(g_2), \phi(g_3))\mu(g_1, f_1)u
$$

$$
+ \rho(f_1 + u, g_2 + g_3)\rho(f_1 + u, g_1 + g_2 + g_3)\mu(\phi(g_3), \phi(g_1))\mu(g_2, f_1)u.
$$

So, this statements lead us to

$$
\mu([g_1, g_2, g_3], \phi(f_1)) \circ \phi_V = \mu(\phi(g_1), \phi(g_2))\mu(g_3, f_1)
$$

$$
+ \rho(g_1, g_2 + g_3)\mu(\phi(g_2), \phi(g_3))\mu(g_3), f_1)
$$

$$
+ \rho(g_1 + g_2, g_3)\mu(\phi(g_3), \phi(g_1))\mu(g_2, f_1).
$$

Therefore, the result holds. At last, we investigate that equivalent abelian extension give the same representation. For this, suppose that $E_B$ and $\tilde{E}_B$ are equivalent abelian extensions presented by

$$
0 \longrightarrow V \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0
$$

$$
0 \longrightarrow V \xrightarrow{j} \tilde{B} \xrightarrow{q} A \longrightarrow 0,
$$

and $F : B \longrightarrow \tilde{B}$ is the 3-Hom-$\rho$-Lie algebra homomorphism, satisfying $F \circ i = j$, $q \circ F = p$. Choose linear sections $\delta$ and $\delta'$ of $p$ and $q$. So, we obtain $q \circ F(\delta(f_1)) = p \circ \delta(f_1) = f_1 = q \circ \delta'(f_1)$. Then,

$$
\rho(\delta(f_1) - \delta'(f_1)) \in \text{Ker}(q) \cong V. \text{ Thus, we have}
$$

$$
[\delta(f_1), \delta(f_2), u]_B = [F \circ \delta(f_1), F \circ \delta(f_2), u]_{\tilde{B}} = [\delta'(f_1), \delta'(f_2), u]_{\tilde{B}}.
$$

Therefore, we get the result.

\[\Box\]

**Proposition 3.5.** Let $\delta : A \longrightarrow B$ be a section of the abelian extension of $A$ by $V$. Define the map

$$
\omega(f_1, f_2, f_3) = [\delta(f_1), \delta(f_2), \delta(f_3)]_B - \delta([f_1, f_2, f_3], A),
$$

for all $f_1, f_2, f_3 \in A$. Then $\omega$ is a 1-cocycle, where the representation $\mu$ is given by (3.1).

\[\begin{array}{c}
\text{Proof.}\end{array}\]

Since $B$ is a 3-Hom-$\rho$-Lie algebra, we have

$$
[\psi(\delta(f_1)), \psi(\delta(f_2)), [\delta(g_1), \delta(g_2), \delta(g_3)]_B] = [[[\delta(f_1), \delta(f_2), \delta(g_1)]_B, \psi(\delta(g_2)), \psi(\delta(g_3))]_B
$$

$$
+ \rho(f_1 + f_2, g_1)[\psi(\delta(g_1)), [\delta(f_1), \delta(f_2), \delta(g_2)]_B, \psi(\delta(g_3))]
$$

$$
+ \rho(f_1 + f_2 + g_1 + g_2)[\psi(\delta(g_1)), \psi(\delta(g_2)), [\delta(f_1), \delta(f_2), \delta(g_3)]_B]_B.
$$

(3.3)
On the other hand, we have
\[
[\psi(\delta(f_1)), \psi(\delta(f_2)), [\delta(g_1), \delta(g_2), \delta(g_3)]_B]_B = [\psi(\delta(f_1)), \psi(\delta(f_2)), \omega(g_1, g_2, g_3) + \delta[g_1, g_2, g_3]_A]_B \\
= \mu(\phi(f_1), \phi(f_2))\omega(g_1, g_2, g_3) + [\psi(\delta(f_1)), \psi(\delta(f_2)), \delta[g_1, g_2, g_3]_A]_B \\
= \mu(\phi(f_1), \phi(f_2))\omega(g_1, g_2, g_3) + \omega(\phi(f_1), \phi(f_2), [g_1, g_2, g_3]_A) \\
+ \delta[\psi(\delta(f_1)), \psi(\delta(f_2)), [g_1, g_2, g_3]_A]_B.
\]

Similarly, the right hand side of \([3.3]\) is equal to
\[
\rho(f_1 + f_2 + g_1, g_2 + g_3)\mu(\phi(g_2), \phi(g_3))\omega(f_1, f_2, g_1) + \omega([f_1, f_2, g_1]_A, \phi(g_2), \phi(g_3)) \\
+ \delta[[f_1, f_2, g_1]_A, \phi(g_2), \phi(g_3)]_A + \rho(f_1 + f_2, g_1)\rho(f_1 + f_2 + g_2, g_3)\rho(g_1, g_2, g_3)\mu(\phi(g_3), \phi(g_1))\omega(f_1, f_2, g_2) \\
+ \rho(f_1 + f_2, g_1)\delta[\phi(g_1), [f_1, f_2, g_2]_A, \phi(g_3)]_A + \rho(f_1 + f_2, g_1 + g_2)\mu(\phi(g_1), \phi(g_2))\omega(f_1, f_2, g_3) \\
+ \rho(f_1 + f_2, g_1 + g_2)\omega(\phi(g_1), \phi(g_2), [f_1, f_2, g_3], A) + \rho(f_1 + f_2, g_1 + g_2)\delta[\phi(g_1), \phi(g_2), [f_1, f_2, g_3], A].
\]

So, we get
\[
\omega(\phi(f_1), \phi(f_2), [g_1, g_2, g_3]_A) + \mu(\phi(f_1), \phi(f_2))\omega(g_1, g_2, g_3) = \omega([f_1, f_2, g_1]_A, \phi(g_2), \phi(g_3)) \\
+ \rho(f_1 + f_2, g_1)\omega(\phi(g_1), [f_1, f_2, g_2]_A, \phi(g_3)) \\
+ \rho(f_1 + f_2, g_1 + g_2)\omega(\phi(g_1), \phi(g_3), [f_1, f_2, g_3]_A) \\
+ \rho(f_1 + f_2, g_1)\rho(f_1 + f_2 + g_2, g_3)\rho(g_1, g_2, g_3)\mu(\phi(g_3), \phi(g_1))\omega(f_1, f_2, g_2) \\
+ \rho(f_1 + f_2, g_1 + g_2)\mu(\phi(g_1), \phi(g_2))\omega(f_1, f_2, g_3) \\
+ \rho(f_1 + f_2 + g_1 + g_2)\mu(\phi(g_2), \phi(g_3))\omega(f_1, f_2, g_1).
\]

Therefore, \(\omega\) is a 1-cocycle. \(\square\)

4. Infinitesimal deformations of 3-Hom-\(\rho\)-Lie algebras

In this section, we introduce infinitesimal deformations of 3-Hom-\(\rho\)-Lie algebras and define Hom-Nijenhuis operator of it.

Let \(A\) be a 3-Hom-\(\rho\)-Lie algebra and \(\omega : \wedge^3A \rightarrow A\) be a morphism. Consider a \(t\)-parametrized family of linear operations
\[
[f, g, h]_t = [f, g, h]_A + t\omega(f, g, h).
\]

If \(A\) with all the brackets \([\ldots, \ldots]_t\) endow regular 3-Hom-\(\rho\)-Lie algebra structure \((A, [\ldots, \ldots]_t, \rho, \phi)\) which is denoted by \(A_t\), we say that \(\omega\) generates a \(t\)-parameter infinitesimal deformation of the 3-Lie colour algebra \(A\).

**Theorem 4.1.** \(\omega\) generates a \(t\)-parameter infinitesimal deformation of the 3-Hom-\(\rho\)-Lie algebra \(A\) is equivalent to

1. \(\omega\) itself defines a 3-Hom-\(\rho\)-Lie algebra structure on \(A\).
2. \(\omega\) is a 1-cocycle of \(A\) with coefficients in the adjoint representation.

**Proof.** Since \([\ldots, \ldots]_t\) endow \((A, [\ldots, \ldots]_t, \rho, \phi)\) a regular 3-Hom-\(\rho\)-Lie algebra structure, then,
\[
[\phi(f_1), \phi(f_2), [g_1, g_2, g_3]_t]_t = [[f_1, f_2, g_1]_t, \phi(g_2), \phi(g_3)]_t - \rho(f_1 + f_2, g_1)\phi(g_1), [f_1, f_2, g_2]_t, \phi(g_3)]_t \\
+ \rho(f_1 + f_2, g_1 + g_2)\phi(g_1), \phi(g_2), [f_1, f_2, g_3]_t].
\]
The left-hand side is equal to
\[
[\omega(f_1), \phi(f_2), [g_1, g_2, g_3]]_A + t\omega([\omega(f_1), \phi(f_2), [g_1, g_2, g_3]]_A) + \omega(g_1, g_2, g_3)]_A
\]
\[+ t^2 \omega([\omega(f_1), \phi(f_2), \omega(g_1, g_2, g_3)]_A).
\]

The right-hand side too is equal to
\[
[[f_1, f_2, g_1], \phi(g_1), \phi(g_3)] + t\omega([[f_1, f_2, g_1], \phi(g_2), \phi(g_3)] + [t\omega(f_1, f_2, g_1), \phi(g_2), \phi(g_3)]_A
\]
\[+ t^2 \omega(f_1, f_2, g_1), \phi(g_2), \phi(g_3)] + \rho(f_1 + f_2, g_1)\phi(\phi(\phi(g_1)), [f_1, f_2, g_2], A, \phi(g_3))_A
\]
\[+ \rho(f_1 + f_2, g_1)\omega(\phi(g_1), [f_1, f_2, g_2], A, \phi(g_3))_A
\]
\[+ \rho(f_1 + f_2, g_1)\phi(\phi(g_1), \phi(g_2), [f_1, f_2, g_3], A) + \rho(f_1 + f_2, g_1 + g_2)t\omega(\phi(g_1), \phi(g_2), [f_1, f_2, g_3], A)
\]
\[+ \rho(f_1 + f_2, g_1 + g_2)\phi(\phi(g_1), \phi(g_2), t\omega(f_1, f_2, g_3)] + \rho(f_1 + f_2, g_1 + g_2)t^2 \omega(\phi(g_1), \phi(g_2), f_1, f_2, g_3)]_A.
\]

Thus, we have
\[
\omega(\phi(f_1), \phi(f_2), [g_1, g_2, g_3]]_A + [f_1, f_2, \omega(g_1, g_2, g_3)]_A = \omega([[f_1, f_2, g_1], \phi(g_2), \phi(g_3))
\]
\[+ \rho(f_1 + f_2, g_1)\omega(\phi(g_1), [f_1, f_2, g_2], A, \phi(g_3))_A
\]
\[+ \rho(f_1 + f_2, g_1 + g_2)\omega(\phi(g_1), \phi(g_2), [f_1, f_2, g_3], A)
\]
\[+ \omega(\omega(f_1, f_2, g_1), \phi(g_2), \phi(g_3)]_A
\]
\[+ \rho(f_1 + f_2, g_1)\phi(\phi(g_1), \phi(g_2), \omega(f_1, f_2, g_3)]_A,
\]
and
\[
\omega(\phi(f_1), \phi(f_2), \omega(g_1, g_2, g_3) = \omega(\omega(f_1, f_2, g_1), \phi(g_2), \phi(g_3)) + \rho(f_1 + f_2, g_1)\omega(\phi(g_1), \omega(f_1, f_2, g_2), \phi(g_3))
\]
\[+ \rho(f_1 + f_2, g_1 + g_2)\phi(\phi(g_1), \phi(g_2), \omega(f_1, f_2, g_3))_A.
\]

Therefore, \(\omega\) defines a 3-Hom-\(\rho\)-Lie algebra structure on \(A\) and \(\omega\) is a 1-cocycle of \(A\) with the coefficient in the adjoint representation. \(\square\)

An infinitesimal deformation is said to be trivial if there exists a grade-preserving map \(N : A \rightarrow A\) such that for \(T_t = id + tN : A_t \rightarrow A\) the following relation holds
\[
T_t[f_1, f_2, f_3] = [T_t f_1, T_t f_2, T_t f_3].
\]

**Definition 4.2.** A linear operator \(N : A \rightarrow A\) is called a Hom Nijenhuis operator if
\[
N^2[f_1, f_2, f_3] = N[N f_1, f_2, f_3] + N[f_1, N f_2, f_3] + N[f_1, f_2, N f_3]
\]
\[+ \omega(f_1, f_2, f_3) - ([N f_1, f_2, f_3] + [N f_1, f_2, N f_3] + [f_1, N f_2, N f_3]).
\]

If we define \([\, , \, , \, ]_N\) by
\[
[f_1, f_2, f_3]_N = [N f_1, f_2, f_3] + [f_1, N f_2, f_3] + [f_1, f_2, N f_3] - N[f_1, f_2, f_3],
\]
then (4.1) is equivalent to
\[
N[f_1, f_2, f_3] = [N f_1, N f_2, f_3] + [N f_1, f_2, N f_3] + [f_1, N f_2, N f_3].
\]
Theorem 4.3. Let $N$ be a Nijenhuis operator for $A$. Setting 
$$\omega(f_1, f_2, f_3) = [f_1, f_2, f_3]_N,$$
then $\omega$ is an infinitesimal deformation of $A$. Furthermore, this deformation is a trivial one.

Proof. By a direct calculation, we can see $d_i\omega = 0$, therefore $\omega$ is a 1-cocycle of $A$ with the coefficients in the adjoint representation. We must check the $\rho$-fundamental identity for $\omega$, that is 
$$\omega(\phi(f_1), \phi(f_2), \omega(g_1, g_2, g_3)) = \omega(\omega(f_1, f_2, g_1), \phi(g_2), \phi(g_3)) + \rho(f_1 + f_2, g_1)\omega(\phi(g_1), \omega(f_1, f_2, g_2), \phi(g_3))$$
$$+ \rho(f_1 + f_2, g_1 + g_2)\omega(\phi(g_1), 
\phi(g_2), \omega(f_1, f_2, g_3)).$$
This identity follows easily by a direct calculation, using the $\rho$-fundamental identity for $A$ and this fact that $N$ is a Hom Nijenhuis operator for $A$. Suppose that $T_i = id + tN$, then 
$$T_i[f_1, f_2, f_3]_t = id + tN([f_1, f_2, f_3]_t) = id + tN([f_1, f_2, f_3]_4 + t\omega(f_1, f_2, f_3))$$
$$= [f_1, f_2, f_3] + t\omega(f_1, f_2, f_3) + tN([f_1, f_2, f_3] + t\omega(f_1, f_2, f_3))$$
$$= [f_1, f_2, f_3] + t(\omega(f_1, f_2, f_3) + N[f_1, f_2, f_3]) + t2N\omega(f_1, f_2, f_3),$$
and
$$[T_if_1, T_if_2, T_if_3] = [f_1 + tNf_1, f_2 + tNf_2, f_3 + tNf_3]$$
$$= [f_1, f_2, f_3] + t([Nf_1, f_2, f_3] + [f_1, Nf_2, f_3] + [f_1, f_2, Nf_3])$$
$$+ t2([Nf_1, Nf_2, f_3] + [Nf_1, f_2, Nf_3] + [f_1, Nf_2, Nf_3]) + t3[Nf_1, Nf_2, Nf_3].$$
Then, we have 
$$T_i[f_1, f_2, f_3]_t = [T_if_1, T_if_2, T_if_3],$$
which implies that the infinitesimal deformation is trivial. \qed

References

1. A. R. Armakan, S. Silvestrov and M. R. Farhangdoost, *Extensions of hom-Lie color algebras*, Georgian Mathematical Journal, (2019), preprint.
2. H. Ataguema, A. Makhlouf and S. Silvestrov, Generalization of $n$-ary Nambu algebras and beyond, J. Math. Phys., 50(8) (2009), 15 pages.
3. K. Abdaoui, F. Ammar and A. Makhlouf, *Construction and cohomology of color Hom-Lie algebras*, J. Commu. Alg., 43(11) (2015), 4581–4612.
4. J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, Phys. Rev. D, 77(6) (2008), 6 pages.
5. J. Bagger and N. Lambert, *Comments on multiple M2-branes*, J. High Energy Phys., 2 (2008), 15 pages.
6. P. J. Bongaarts and H. G. J. Pijs, *Almost commutative algebra and differential calculus on the quantum hyperplane*, J. Math. Phys., 35(2) (1994), 959–970.
7. Y. Daletskii and L. Takhtajan, *Leibniz and Lie algebra structures for Nambu algebra*, Lett. Math. Phys., 39 (1997), 127–141.
8. A. Gustavsson, *Algebraic structure on parallel M2-branes*, Nuclear Phys. B, 811 (1-2) (2009) 66–76.
9. P. Gautheron, *Some remarks concerning Nambu mechanics*, Lett. Math. Phys., 37 (1996), 103–116.
10. M. Scheunert, R. Zhang, *Cohomology of Lie superalgebras and their generalizations*, J. Math. Phys., 9 (1998), 5024–5061.
11. Y. Fregier, A. Gohr, *On Hom-type algebras*, J. Gen. Lie Theory. Appl., 4 (2010), 16 pages.
12. J. T. Hartwig, D. Larsson and S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Alg., **295** (2006), 314–361.

13. Y. Liu, L. Chen and Y. Ma, *Representations and module-extensions of 3-hom-Lie algebras*, J. Geom. Phy., **98** (2015), 376–383.

14. A. Makhlouf and S. D. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory. Appl., **2** (2008), 51–64.

15. A. Makhlouf, *Hom-alternative algebras and Hom-Jordan algebras*, Int. Electron. J. Alg., **8** (2010), 177–190.

16. A. Makhlouf and S. D. Silvestrov, *Notes on Formal Deformations of Hom-associative algebras and Hom-Lie algebras*, Forum Math., **22**(4) (2010), 715–739.

17. P. Ho, R. Hou and Y. Matsuo, *Lie 3-algebra and multiple M2-branes*, J. High. Energy Phys., (2008).

18. Y. Sheng and D. Chen, *Hom-Lie 2 algebras*, J. Alg., **376** (2013), 174–195.

19. Y. Sheng, *Representations of Hom-Lie algebras*, J. Alg. Rep. Theo., **15** (2012), 1081–1098.

20. B. Sun, L. Chen and Y. Liu, *T*-extensions and abelian extensions of Hom-Lie color algebras, Revista de la unión matemática Argentina, **59**(1) (2018), 123–142.

21. L. Takhtajan, On foundation of the generalized Nambu mechanics, Commun. Math. Phys., (1994), 295–316.

22. L. Takhtajan, *A higher order analog of Chevally-Eilenberg complex and deformation theory of n-algebras*, St. Petersburg Math. J., **6** (1995), 429–438.

23. L. Yuan, *Hom-Lie color algebra structures*, J. Comm. Alg., **40**(2) (2012), 575–592.

24. T. Zhang, *Cohomology and deformations of 3-Lie colour algebras*, Lin. multi. alg., **63**(4) (2015), 651–671.

---

1, 2 Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran.

*Email address: e-peyghan@araku.ac.ir* 1, *z-bagheri@phd.araku.ac.ir* 2

3, 4 Ataturk University, Faculty of Science, Department of Mathematics, 25240, Erzurum-Turkey.

*Email address: igultekin@atauni.edu.tr* 3, *agezer@atauni.edu.tr* 4