Absolutely continuous copulas obtained by regularization of the Frechét–Hoeffding bounds

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Abstract

We show that the lower and upper Frechét-Hoeffding copulas, which are singular, can be regularized to absolutely continuous copulas. The method, which is constructive and explicit, states sufficient conditions for when an absolutely continuous copula can be achieved by averaging. A higher degree of regularisation cannot be achieved with the proposed method.

1 Introduction

A copula is a mathematical tool designed to join marginal probability distributions forming a joint multidimensional probability distribution. The advantage with copulas is that they reduce the problem of studying multidimensional probability distributions to studying a function, copula, defined on the unit hypercube. For a full introduction to copulas we refer to Nelsen [4]. As pointed out in e.g. [2],[6],[5] copulas have recently gained a lot in popularity. This is in part driven by their usefulness in applied mathematics, especially mathematical finance. This paper as well is motivated by a problem originating from applied mathematics but the application is to be found in toxicology.

Let $I = [0, 1]$ be the unit interval, and let $I^2 = I \times I$. A function $C(u, v)$ with piecewise continuous second derivatives is a cumulative distribution function on $I^2$ if and only if

\[
\begin{align*}
C_{uv}(u, v) &\geq 0, \ u, v \in (0, 1), \\
C(u, 0) &= 0, \ u \in [0, 1], \text{ and} \\
C(0, v) &= 0, \ v \in [0, 1]
\end{align*}
\]

The lower boundary values $C(u, 0), C(0, v)$ are zero; the upper boundary values $C(u, 1), C(1, v)$ are the marginal distributions. The function $C$ is said to be a copula if the marginals are uniform which is summarised in the following definition.

Definition 1 A two-dimensional copula is a function $C$ from $I^2$ to $I$ with the following properties:

\[
\begin{align*}
C_{uv}(u, v) &\geq 0, \ u, v \in (0, 1), \\
C(u, 0) &= 0, \ u \in [0, 1], \text{ and} \\
C(0, v) &= 0, \ v \in [0, 1]
\end{align*}
\]
1. For every \( u, v \) in \( I \),
\[
C(u, 0) = 0 = C(0, v) \tag{4}
\]
and
\[
C(u, 1) = u \quad \text{and} \quad C(1, v) = v; \tag{5}
\]
2. For every \( u_1, u_2, v_1, v_2 \) in \( I \) such that \( u_1 \leq u_2 \) and \( v_1 \leq v_2 \),
\[
C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0. \tag{6}
\]

Let us consider an example.

Example 2 The function \( C(u, v) = uv \) is a copula since \( C(u, 1) = u, C(1, v) = v, C(u, 0) = 0 = C(0, v) \) and
\[
C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) = (u_2 - u_1)(v_2 - v_1) \geq 0.
\]

It turns out that any copula \( C(u, v) \) is bounded from above by the Frechét-Hoeffding upper bound, and from below by the Frechét-Hoeffding lower bound. Indeed the Frechét-Hoeffding bounds are extremal copulas as they are copulas themselves, which is stated in the theorem below.

Definition 3 The Frechét-Hoeffding upper bound is given by \( M = \min(u, v) \) and the Frechét-Hoeffding lower bound is given by \( W = \max(u + v - 1) \).

Theorem 4 The upper and lower Frechét-Hoeffding bounds \( M \) and \( W \) are copulas. Moreover, for any copula \( C \) and all \( (u, v) \in I \),
\[
W(u, v) \leq C(u, v) \leq M(u, v). \tag{7}
\]

While generalising a model, the probit model \([1]\), for estimating the health impact on people exposed to hazardous chemicals we realised that the problem can be cast as a copula problem. The standard probit model is used to find a statistical description of the injury outcome of a population exposed to toxic substances. The model can be utilized to describe this outcome at several discrete injury states, such as light injuries, severe injuries, and death. The joint probability distribution of two succeeding injury states may be expressed as a multivariate normal distribution with normal marginal distributions. By Sklar’s theorem, see e.g. \([4]\), there exists a copula that prescribes how to joint, couple, the marginal distributions to achieve the probit model. As it happens the corresponding copula is the upper Frechét-Hoeffding copula \( M \). Our generalisation of the probit model aims at treating the population as an aggregate of individuals rather than as one entity. In this approach individuals are allocated one threshold value per injury state which determines at which exposure they will reach every state. The thresholds are distributed among the population according to the standard normal distribution. However, individuals must be able to possess different sensibility for the different injury states which means that they will not be located on the diagonal axis on the multivariate probability distribution which otherwise would be the trivial solution. As the thresholds are distributed, the overall statistics of the population must be kept intact. In copula terms this equates to rearranging the mass of the joint probability distribution without changing the marginal distributions. Adding the criteria that the probit model should readily support numerical investigation we arrived at the condition that the generalised probit model is feasible if the upper Frechét-Hoeffding copula \( M \) can be regularized and still remain a copula. In this paper we will focus on this mathematical problem while the full toxicological motivation and the generalised probit model will be published elsewhere in the toxicology literature.

The regularity that we are aiming for is absolute continuity, which is defined in the following way:
Definition 5 A two-dimensional copula $C$ is said to be absolutely continuous if $C(u, v)$ has a density $c(u, v)$ and

$$C(u, v) = \int_0^u \int_0^v c(s, t)dt ds \text{ for all } (u, v) \in I^2.$$  

(8)

We note that $\partial^2 C/\partial u \partial v$ exists almost everywhere in $I^2$, see Theorem 2.2.7 in [4], and is equal to the density $c(u, v)$. The example copula $C(u, v) = uv$ that we considered earlier is absolutely continuous.

Example 6 (cont.) The copula $C(u, v) = uv$ is absolutely continuous since

$$\int_0^u \int_0^v 1ds dt = uv \text{ for all } (u, v) \in I^2.$$  

The Frechét-Hoeffding copulas on the other hand are not absolutely continuous, indeed they are not absolutely continuous on any subdomain $\subset I^2$, and are thus singular. Geometrically the Frechét-Hoeffding upper and lower copula are $C^0$ surfaces each made up of two planar surfaces intersecting transversally along a line segment, see figure 1. Regularising these functions, $W$ and $M$, is, of course, trivial: any type of averaging around and along the transversal intersection will smooth out the $C^1$-discontinuity. The point here is that the regularized function should still be a copula. For our purposes, the probit model, it would suffice to work with only the upper Frechét-Hoeffding $M$ copula, but the regularization method works equally well for the lower one $W$. Thus the main contribution in this paper is that we prove under which conditions the upper and lower Frechét-Hoeffding copula can be regularised to absolutely continuous copulas, see Theorem 7. We then show that this class of copulas is non-empty by providing an explicit example. We remark that the proposed averaging method can not be used to prove regularity beyond, at most, $C^2$ due to unbounded third derivatives.

The question of regularity of copulas and the associated differentiability of copulas has been studied before and there are at least two well-known classes of differentiable copulas: Archimedean and Farlie-Gumbel-Morgenstern copulas. Recently a preprint claiming to characterize all twice differentiable copulas was presented [3]. In that paper also a new class, Fourier copulas, of twice differentiable copulas is introduced.

Figure 1: Visualisation of the graph of the Frechét-Hoeffding copulas $M$ and $W$. 

3
2 Setting of the problem

As was hinted at in the introduction, regularization of the upper and lower Frechét-Hoeffding copulas $M$ and $W$ will be achieved by averaging these copulas at, and in the vicinity of, their respective singularities: for $M$ the singularity is located along the diagonal while for $W$ it is the anti-diagonal (see figure 1). As averaging method we choose a family of discs and in each such disc we replace the value of the copula with the average value of the copula in that disc. Due to the simple geometry it turns out that these averages can be expressed explicitly, however the expressions become somewhat involved due to evaluating integrals on circle segments.

It is convenient to make the following change of coordinates

$$w = (v + u - 1)/\sqrt{2}$$
$$z = (v - u)/\sqrt{2}$$

(9) (10)

and study the problem in $(w, z)$-coordinates. Let $C(u, v)$ be a copula defined on $I^2$. With a slight abuse of notation, we write $C(w, z) = C(u, v)$. $C(w, z)$ is defined on

$$U = \{ (w, z) \in \mathbb{R}^2 : \|w\| + \|z\| \leq 1/\sqrt{2} \}.$$  (11)

The copula conditions, stated in Definition 1, can be summarized in the following way

$$C''_{ww} (u, v) \geq 0, \quad (u, v) \in \text{int}(I^2)$$
$$C = \max (0, u + v - 1) = \min (u, v), \quad (u, v) \in \partial I^2,$$

(12) (13)

and may be checked in $(w, z)$-coordinates instead. The inverse coordinate transformation is given by

$$u = \frac{w - z}{\sqrt{2}} + \frac{1}{2}$$
$$v = \frac{w + z}{\sqrt{2}} + \frac{1}{2}.$$

(14) (15)

which implies that the partial derivatives are transformed according to

$$\frac{\partial}{\partial u} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial w} - \frac{\partial}{\partial z} \right), \quad \frac{\partial}{\partial v} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial w} + \frac{\partial}{\partial z} \right),$$

(16)

$$\frac{\partial^2}{\partial u \partial v} = \frac{1}{2} \left( \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial z^2} \right).$$

(17)

and thus the copula conditions (12) and (13) becomes

$$C''_{ww} - C''_{zz} \geq 0, \quad (w, z) \in \text{int}(U)$$
$$C = \max \left( 0, \sqrt{2}w \right) = \frac{w - |z|}{\sqrt{2}} + \frac{1}{2}, \quad (w, z) \in \partial U$$

(18) (19)

in $(w, z)$-coordinates and where the interior $\text{int}(U)$ and the boundary $\partial U$ are given in the $(w, z)$-coordinates by:

$$\text{int}(U) = \left\{ (w, z) \in \mathbb{R}^2 : \|w\| + \|z\| < 1/\sqrt{2} \right\}$$
$$\partial U = \left\{ w, z \in \mathbb{R}^2 : \|w\| + \|z\| = 1/\sqrt{2} \right\}.$$  

(20) (21)
The expression for the boundary values occurring in the copula conditions is defined for all \((w, z) \in U\):

\[
W = \frac{w + |w|}{\sqrt{2}} = \max \left(0, \sqrt{2}w\right) = \max (0, v + u - 1).
\] (22)

This is a (non–smooth) copula, the lower Fréchet-Hoeffding bound, where all the probability mass is concentrated on the antidiagonal \(\{u + v = 1\} = \{w = 0\}\). There is also an upper bound

\[
M = \frac{w - |z|}{\sqrt{2}} + \frac{1}{2} = \min \left(\frac{w - z}{\sqrt{2}} + \frac{1}{2}, \frac{w + z}{\sqrt{2}} + \frac{1}{2}\right) = \min (u, v)
\] (23)

which is a non–smooth copula, the upper Fréchet-Hoeffding bound, where all the probability mass is concentrated on the diagonal \(\{u = v\} = \{z = 0\}\). Now we are in a position to state the main theorem describing under which conditions the Frechet-Hoeffding bounds can be regularised using the proposed averaging method.

**Theorem 7** Let \(r(z, w)\) be a twice continuously differentiable and strictly positive function defined for \(|w| + |z| < 1/\sqrt{2}\). Let

\[
w = (v + u + 1)/\sqrt{2}, z = (v - u)/\sqrt{2},
\] (24)

and let \(D_r(u, v)\) be the circular disk with radius \(r\) centered at \((u, v)\), and let

\[
g(\rho) = \begin{cases} 
2 \left(\rho \arcsin (\rho) + \sqrt{1 - \rho^2} / 3\right) / \pi & \text{if } |\rho| < 1 \\
|\rho| & \text{if } |\rho| \geq 1.
\end{cases}
\] (25)

Then

1. If \((r'_w)^2 \leq (1/2 - |r'_w|)^2 + 3/4\) and \(r''_{ww} \leq r''_{ww}\) then

\[
\bar{W}(u, v) \equiv \frac{1}{\pi r^2(w, z)} \iint_{D_r(u', v')} W(u', v') \, du' \, dv'
\] (26)

is an absolutely continuous copula with probability mass supported on \(|w| \leq r(w, z)\), and

\[
W(u, v) = \frac{1}{\sqrt{2}} \left(w + r(w, z) g \left(\frac{w}{r(w, z)}\right)\right)
\] (27)

2. If \((r'_w)^2 \leq (1/2 - |r'_w|)^2 + 3/4\) and \(r''_{ww} \leq r''_{ww}\) then

\[
\bar{M}(u, v) \equiv \frac{1}{\pi r^2(w, z)} \iint_{D_r(u', v')} M(u', v') \, du' \, dv'
\] (28)

is an absolutely continuous copula with probability mass supported on \(|z| \leq r(w, z)\) and

\[
M(u, v) = \frac{1}{\sqrt{2}} \left(w + \sqrt{2} - r(w) g \left(\frac{z}{r(w, z)}\right)\right)
\] (29)

We remark that the conditions stated in the theorem are not sharp, and this is pointed out in the proof which we postpone to the next section. Let us first consider an example showing that the class of copulas generated by the theorem is non-empty.
Example 8 Symmetric Gaussian copula. Let \( \Phi \) be the cumulative distribution function of a standard normal random variable, \( \varphi(x) = \Phi'(x) = \exp(-x^2/2) / \sqrt{2\pi} \) its probability density function, and consider a radius function \( r(w) \) implicitly defined by

\[
\Phi^{-1} \left( \frac{w + r(w)}{\sqrt{2}} + \frac{1}{2} \right) - \Phi^{-1} \left( \frac{w - r(w)}{\sqrt{2}} + \frac{1}{2} \right) = d
\]

where \( d > 0 \) is a given constant. Denote

\[
u = \frac{w - r(w)}{\sqrt{2}} + \frac{1}{2}, \quad x = \Phi^{-1}(u),
\]

\[
v = \frac{w + r(w)}{\sqrt{2}} + \frac{1}{2}, \quad y = \Phi^{-1}(v).
\]

Implicit differentiation gives

\[
\frac{1}{\varphi(y) \sqrt{2}} \frac{1 + r'}{\varphi(x) \sqrt{2}} - \frac{1}{\varphi(x) \sqrt{2}} = 0
\]

which gives

\[
r' = \frac{\varphi(y) - \varphi(x)}{\varphi(y) + \varphi(x)} = \tanh(s),
\]

where

\[
s = (x^2 - y^2) / 4 = -d(x + y) / 4
\]

by the definitions above. Hence \( |r'(w)| < 1 \). Furthermore,

\[
r'' = \tanh'(s) s' = -d \left( 1 - \tanh^2(s) \right) (x' + y') / 4
\]

hence it suffices to show that \( x'(w) + y'(w) \geq 0 \) for \( r''_{ww} \leq r''_{zz} = 0 \). But

\[
x' + y' = \frac{1}{\varphi(x) \sqrt{2}} \frac{1 - r'}{\varphi(y) \sqrt{2}} + \frac{1}{\varphi(y) \sqrt{2}} \frac{1 + r'}{\varphi(x) \sqrt{2}} \geq 0
\]

since \( |r'(w)| < 1 \) and \( \varphi > 0 \). Hence \( r(w) \) satisfies the assumptions in the theorem, so \( \bar{M}(u,v) \) is an absolutely continuous copula with probability mass supported on \( |z| \leq r(w) \).

It may be interesting to note that upon transforming the copula random variables \( (U, V) \) back to the original \( X = \Phi^{-1}(U), Y = \Phi^{-1}(V) \) we obtain a pair of Gaussian random variables \( (X, Y) \) with standard normal marginal distributions \( p_X(x) = \varphi(x), p_Y(y) = \varphi(y) \) and joint pdf \( p_{X,Y}(x, y) \) supported on \( |y - x| \leq d \).

As the example shows, verifying the conditions in Theorem 7 may require some work. In the example the radius function \( r(w, z) \) did not have a general dependence on both variables, indeed \( r(w, z) = r(w) \) in the example, and this simplified the verification of the conditions. In general the conditions are easier to verify if the radius function has a product-structure \( r(w, z) = p(w)q(z) \) which we state in the following corollary of Theorem 7.

Corollary 9 If \( r(w, z) = p(w)q(z) \) and \( p, q > 0 \) then the conclusions for \( \bar{M} \) are true if

\[
q(z)^2 \left( p'(w) \right)^2 \leq (1/2 - |q'(z)| p(w))^2 + 3/4
\]

\[
\frac{p''(w)}{p(w)} \leq \frac{q''(z)}{q(z)}
\]
for all $|w| + |z| < 1/\sqrt{2}$. In particular, it holds true if $\varepsilon \in [-1, 1]$, $p(w) > 0$ and $q(z) = (1 + \sqrt{2}\varepsilon)z$ and

$$\begin{align*}
(p'(w))^2 &\leq \frac{(1/2 - \sqrt{2}|\varepsilon| p(w))^2 + 3/4}{(1 + \sqrt{2}\varepsilon z)^2} \quad (40) \\
(p''(w)) &\leq 0
\end{align*}$$

for all $|w| + |z| < 1/\sqrt{2}$. In the symmetric case $\varepsilon = 0$, the condition reduces to

$$\begin{align*}
(p'(w))^2 &\leq 1 \\
p''(w) &\leq 0.
\end{align*}$$

**Remark 10** This allows for skew distributions, since the probability mass of $\bar{M}$ is supported on $|z| \leq (1 + \sqrt{2}\varepsilon) p(w)$, i.e.,

$$-\frac{p(w)}{1 + \sqrt{2}\varepsilon p(w)} \leq z \leq \frac{p(w)}{1 - \sqrt{2}\varepsilon p(w)}. \quad (44)$$

The ratio $\kappa$ between modulus of the upper and lower limit is

$$\kappa = \frac{1 + \sqrt{2}\varepsilon p(w)}{1 - \sqrt{2}\varepsilon p(w)} \leq \frac{1 + \sqrt{2}p(w)}{1 - \sqrt{2}p(w)}. \quad (45)$$

The corollary and the remark is stated for $\bar{M}(u,v)$, but the analogous result holds for $\bar{W}(u,v)$ by interchanging the roles of $p(w)$ and $q(z)$, and $w$ and $z$.

### 3 Proof of Theorem 7

The proof of Theorem 7 relies on being able to construct radius functions $r = r(w, z)$ such that the circular disc averages of the functions $|w|$ and $|z|$, $F(w, z) = \frac{1}{\pi r^2} \iint_{D_r(w, z)} |w'| dw' dz'$, $G(w, z) = \frac{1}{\pi r^2} \iint_{D_r(w, z)} |z'| dw' dz'$, satisfy

$$\begin{align*}
F''_{ww} - F''_{zz} &\geq 0 \quad (48) \\
G''_{zz} - G''_{ww} &\geq 0
\end{align*}$$

where $D_r(w, z)$ is the circular disc with radius $r$ centred at $(w, z)$. Once we have established what conditions the radius functions $r = r(w, z)$ have to satisfy (the averaging Lemma below) we pick those radius functions and apply the disc averaging method to the upper and the lower Frechét-Hoeffding copulas $\bar{M}$ and $\bar{W}$ and show that it results in absolutely continuous copulas: regularised Frechét-Hoeffding bounds $\bar{M}$ and $\bar{W}$.

Evaluating the disc averages of the functions $|w|$ and $|z|$ and exploring the differentiability properties of these averages with respect to the choice of radius function $r(w, z)$ and the part of the domain that is considered is summarised in the following technical averaging Lemma.
Lemma 11 Let \( r(w, z) \) be twice continuously differentiable and strictly positive, and let \( D_r(w, z) \) be the circular disc with radius \( r \) centred at \((w, z)\), and let

\[
g(\rho) = \begin{cases} 
2 \left( \rho \arcsin(\rho) + \sqrt{1 - \rho^2} \left( 2 + \rho^2 \right) / 3 \right) / \pi & \text{if } |\rho| < 1 \\
|\rho| & \text{if } |\rho| \geq 1.
\end{cases}
\]  

(50)

Then

\[
F(w, z) \equiv \frac{1}{\pi r(w, z)^2} \int\int_{\tilde{D}_r(w, z)} |w'| dw' dv' = r(w, z) g\left( \frac{w}{r(w, z)} \right)
\]

(51)

\[
G(w, z) \equiv \frac{1}{\pi r(w, z)^2} \int\int_{\tilde{D}_r(w, z)} |z'| dw' dz' = r(w, z) g\left( \frac{z}{r(w, z)} \right)
\]

(52)

and \( F, G \) are twice continuously differentiable. Moreover,

1. If \(|w| \geq r(w, z)\) then \( F(w, z) = |w| \) and \( F''_{uw} - F''_{zz} = 0 \),

2. If \(|z| \geq r(w, z)\) then \( G(w, z) = |z| \) and \( G''_{zz} - G''_{ww} = 0 \),

3. If \(|w| < r(w, z)\), \((r')^2 \leq (1/2 - |r'|^2)^2 + 3/4\) and \( r''_{zz} \leq r''_{ww} \), then \( F''_{ww} - F''_{zz} \geq 0 \), and

4. If \(|z| < r(w, z)\), \((r')^2 \leq (1/2 - |r'|^2)^2 + 3/4\) and \( r''_{ww} \leq r''_{zz} \), then \( G''_{zz} - G''_{ww} \geq 0 \).

Proof. Let us begin by showing the equality in equations (51) and (52), that is, evaluating the disc averages. Let \( \rho(w, z) = z/r(w, z) \). The case \(|\rho| \geq 1\) is immediate, since then the integrand is linear. Thereby statement 1. and 2. follow directly.

For the case \(|\rho| < 1\) we introduce the nondimensional coordinates

\[
\begin{align*}
\omega &= \frac{w - w'}{r}, \quad \zeta = \frac{z' - z}{r} \\
w' &= w + r\omega, \quad z' = z + r\zeta \\
dw' &= rd\omega, \quad dz' = rd\zeta
\end{align*}
\]

(53)

(54)

(55)

whereby

\[
G(w, z) = \frac{1}{\pi r^2} \int\int_{\tilde{D}_r(w, z)} |z'| dw' dz' = \frac{r}{\pi} \int\int_{\omega^2 + \zeta^2 \leq 1} \left| \frac{z'}{r} + \zeta \right| d\zeta d\omega
\]

\[
= -\frac{r}{\pi} \int_{-1}^{1} \int_{-z/r}^{z/r} \left( \frac{z'}{r} + \zeta \right) 2\sqrt{1 - \zeta^2} d\zeta = \frac{2z}{\pi} \int_{-1}^{1} \sqrt{1 - \zeta^2} d\zeta - \frac{z}{\pi} \int_{-z/r}^{z/r} \sqrt{1 - \zeta^2} d\zeta
\]

\[
= \frac{2z}{\pi} \int_{-1}^{1} \sqrt{1 - \zeta^2} d\zeta - \frac{z}{\pi} \int_{-z/r}^{z/r} \sqrt{1 - \zeta^2} d\zeta + \frac{r}{\pi} \int_{-1}^{1} 2\zeta \sqrt{1 - \zeta^2} d\zeta - \frac{r}{\pi} \int_{-z/r}^{z/r} 2\zeta \sqrt{1 - \zeta^2} d\zeta.
\]

Applying the standard integrals

\[
\begin{align*}
\int 2\zeta \sqrt{1 - \zeta^2} d\zeta &= -\frac{2}{3} (1 - \zeta^2)^{3/2} + C \\
\int \sqrt{1 - \zeta^2} d\zeta &= \frac{1}{2} \arcsin(\zeta) + \frac{1}{2} \zeta \sqrt{1 - \zeta^2}
\end{align*}
\]

(56)

(57)
gives us
\[
G(w, z) = \frac{1}{\pi r^2} \int_{D_r(w, z)} |z'| dw' dz'
\]
\[
= \frac{z}{\pi} \left[ \arcsin (\zeta) + \zeta \sqrt{1 - \zeta^2} \right]_{-z/r}^{z/r} - \frac{z}{\pi} \left[ \arcsin (\zeta) + \zeta \sqrt{1 - \zeta^2} \right]_{-1}^{1}
\]
\[
+ \frac{r}{\pi} \left[ -\frac{2}{3} (1 - \zeta^2)^{3/2} \right]_{-z/r}^{z/r} - \frac{r}{\pi} \left[ -\frac{2}{3} (1 - \zeta^2)^{3/2} \right]_{-1}^{1}
\]
\[
= 2z \pi \arcsin \left( \frac{z}{r} \right) + \frac{2z}{\pi} \sqrt{1 - \frac{z^2}{r^2}} + \frac{4r}{3\pi} \left( 1 - \frac{z^2}{r^2} \right)
\]
\[
= r \cdot \frac{2}{\pi} \left( \frac{z}{r} \arcsin \left( \frac{z}{r} \right) + \frac{1}{3} \sqrt{1 - \frac{z^2}{r^2}} \left( 2 + \frac{z^2}{r^2} \right) \right) = rg \left( \frac{z}{r} \right).
\]

which proves the equality in (52). Proving the disc average \( F(w, z) \), (51), is analogous.

In a preamble to studying how the derivatives of \( F(w, z) \) and \( G(w, z) \) behaves, we note that \( g(\rho) \geq 0 \) and
\[
g'(\rho) = \begin{cases} 
2 \left( \arcsin \rho + \rho \sqrt{1 - \rho^2} \right) / \pi & \text{if } |\rho| < 1 \\
\rho / |\rho| & \text{if } |\rho| > 1 
\end{cases}
\]
and
\[
0 \leq g''(\rho) = \begin{cases} 
4 \sqrt{1 - \rho^2} / \pi & \text{if } |\rho| < 1 \\
0 & \text{if } |\rho| > 1 
\end{cases}
\]
and \( g(\rho) \to 1, g'(\rho) \to \pm 1 \) and \( g''(\rho) \to 0 \) as \( \rho \to \pm 1 \), hence \( g \in C^2(\mathbb{R}) \) and convex. However, \( g \) is not \( C^3 \) since
\[
g'''(\rho) = -\frac{4}{\pi} \rho (1 - \rho^2)^{-1/2} \to \pm \infty \text{ as } \rho \to \pm 1, \quad |\rho| < 1.
\]

To simplify calculations, we define the auxiliary function
\[
h(\rho) = \begin{cases} 
4 (1 - \rho^2)^{3/2} / (3\pi) & \text{if } |\rho| < 1 \\
0 & \text{if } |\rho| \geq 1
\end{cases}
\]
and note that
\[
h(\rho) = g(\rho) - \rho g'(\rho) = (1 - \rho^2) g''(\rho) / 3, \quad \rho \neq 1
\]
\[
h'(\rho) = -g''(\rho), \quad \rho \neq 1.
\]

And, in addition, we observe that since
\[
\rho(w, z) = \frac{z}{r(w, z)}
\]
its first partial derivatives are given by
\[
\rho'_w = -\frac{\rho'_w}{r}, \quad \rho'_z = \frac{1 - \rho'_z}{r}.
\]
Now we are ready to study the properties of the second derivatives of $G$, with the aim of proving points 2. and 4. in the Lemma. Let us remind ourselves that $G$ is defined as, compare equation (52).

$$G(w, z) = r(w, z)g(\rho(w, z)).$$

Taking derivatives, first once and then twice, with respect to $w$ and $z$ respectively gives

$$G'_w = r'_w g + rg' \rho'_w = r'_w (g - \rho g') = hr'_w$$

and

$$G'_z = r'_z g + rg' \rho'_z = r'_z (g - \rho g') + g' = hr'_z + g'.$$

The last condition is only needed when the endpoints are nonnegative, i.e., $p_0$ and $p$.

A quadratic polynomial $p(\rho) = \alpha \rho^2 + \beta \rho + c$ is $\geq 0$ for $-1 \leq \rho \leq 1$ if and only if 1) The values at the endpoints are nonnegative, i.e., $p(-1) \geq 0$ and $p(1) \geq 0$, which is equivalent to $|b| \leq a + c$, and 2) if there is local minimum between the endpoints, i.e., if $|b| \leq 2|a|$, then this minimum is nonnegative, i.e., $c \geq b^2/(4a)$. If $|b| \leq 2|a|$ then $b^2/(4a) \leq 1$ so a simpler, sufficient condition would be $|b| \leq a + c$ and $c \geq 1$. Applying this condition to (70) we see that

$$r (G''_{zz} - G''_{ww}) = \rho^2 (p'_z)^2 - (p'_w)^2 + 1 - p'_z + \rho (rr''_{zz} - rr''_{ww})$$

$$= \rho^2 ((p'_z)^2 - (p'_w)^2 + 1 - p'_z + (1 - \rho^2) (rr''_{zz} - rr''_{ww}) / 3).$$

Dividing by $g''$ we denote the resulting quadratic polynomial in $p$ by $p(\rho)$

$$r (G''_{zz} - G''_{ww}) = \rho^2 ((p'_z)^2 - (p'_w)^2 - (rr''_{zz} - rr''_{ww}) / 3) - p'_z + 1 + (rr''_{zz} - rr''_{ww}) / 3 \equiv p(\rho).$$

(70)

A quadratic polynomial $p(\rho) = \alpha \rho^2 + \beta \rho + c$ is $\geq 0$ for $-1 \leq \rho \leq 1$ if and only if 1) The values at the endpoints are nonnegative, i.e., $p(-1) \geq 0$ and $p(1) \geq 0$, which is equivalent to $|b| \leq a + c$, and 2) if there is local minimum between the endpoints, i.e., if $|b| \leq 2|a|$, then this minimum is nonnegative, i.e., $c \geq b^2/(4a)$. If $|b| \leq 2|a|$ then $b^2/(4a) \leq 1$ so a simpler, sufficient condition would be $|b| \leq a + c$ and $c \geq 1$. Applying this condition to (70) we see that

$$r (G''_{zz} - G''_{ww}) \geq 0$$

(71)

if $|p'_z| \leq 1 + (p'_z)^2 - (p'_w)^2$ and $1 + (rr''_{zz} - rr''_{ww}) / 3 \geq 1$, which is equivalent to

$$(p'_z)^2 \leq (1/2 - |p'_z|)^2 + 3/4$$

and

$$rr''_{ww} \leq p''_{zz}. $$

(72)

(73)

The last condition is only needed when

$$|p'_z| < 2 \left| (p'_z)^2 - (p'_w)^2 - (rr''_{zz} - rr''_{ww}) / 3 \right|. $$

(74)

The radius function $r$ is always positive, and by (59) we have $g'' > 0$ for $|\rho| < 1$, hence equation (71) implies that $G''_{zz} - G''_{ww} > 0$ if conditions (72) and (73) are satisfied. Remember that $\rho = z/r(w, z)$, hence $|\rho| < 1$ is equivalent to $|z| < r(w, z)$. Thus we have proved statement 4. in the lemma.

We remark that the conditions stated in the lemma (and Theorem 7) are not sharp since we
chose a simplified condition in the quadratic polynomial, if more precision is needed we the exact condition
\[ 1 + \left( rr''_{zz} - rr''_{ww} \right)/3 \geq \frac{(r'_{z})^2}{4 \left( (r'_{z})^2 - (r'_{w})^2 - (rr''_{zz} - rr''_{ww})/3 \right)} \] (75)
which is equivalent to
\[ \frac{(r'_{z})^2 - 4 \left( (r'_{z})^2 - (rr''_{zz} - rr''_{ww})/3 \right) (rr''_{zz} - rr''_{ww})/3}{4 \left( (r'_{z})^2 - (r'_{w})^2 - (rr''_{zz} - rr''_{ww})/3 \right)} \leq 1 \] (76)
can be employed.

Next, let us consider the remaining unproved statement, statement 3. It follows analogously by the arguments above by interchanging the roles of \( w \) and \( z \). That is, let us consider
\[ F(w,z) = \int_{D_{r}(w,z)} \frac{\sqrt{2}}{\pi r^2} |w| \, dw' \, dz'. \] (77)
Interchanging the roles of \( w \) and \( z \), and applying the the same arguments as above to
\[ F(w,z) = r(w,z) \frac{g(w)}{r(w,z)} \] (78)
we obtain
\[ \frac{r \left( F''_{ww} - F''_{zz} \right)}{g'} \geq 0 \] (79)
if
\[ (r'_{z})^2 \leq (1/2 - |r'_{w}|)^2 + 3/4 \] (80)
and
\[ r''_{zz} \leq r''_{ww}. \] (81)
Thus we have proved the lemma. ■

We are now in a position to prove Theorem 7.

**Proof.** Let \( r(w,z) \) be a twice continously differentiable and strictly positive function such that \( (r'_{z})^2 \leq (1/2 - |r'_{w}|)^2 + 3/4 \) and \( r''_{zz} \leq r''_{ww} \). Considering the definition of \( \bar{W}(u,v) \) in (26) and inserting the expression for the lower Frechét-Hoeffding copula \( W \) stated in (22) gives
\[ \bar{W}(u,v) = \int_{D_{r}(w,z)} \frac{\sqrt{2}}{\pi r^2} |w| \, dw' \, dz'. \]
Applying Lemma 11 we obtain
\[ \bar{W}(u,v) = \frac{w + F(w,z)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left( w + r(z) \frac{g(w)}{r(w,z)} \right). \]
Likewise, letting \( r(w,z) \) be a twice continously differentiable and strictly positive function such that \( (r'_{w})^2 \leq (1/2 - |r'_{z}|)^2 + 3/4 \) and \( r''_{ww} \leq r''_{zz} \), and inserting the expression for the upper
Frechét-Hoeffding copula $M$, stated in (23), into the definition of $\bar{M}(u,v)$ given in (28) yields

$$\bar{M}(u,v) \equiv \frac{1}{\pi r^2(w, z)} \int \frac{w' - |z'|}{\sqrt{2}} + \frac{1}{2} dw' dz'$$

$$= \frac{1}{\sqrt{2} \pi r^2(w, z)} \int \left( w' + \frac{1}{\sqrt{2}} \right) dw' dz' - \frac{1}{\sqrt{2} \pi r^2(w, z)} \int |z'| dw' dz'$$

Applying Lemma 11 gives

$$\bar{M}(u, v) = \frac{w + 1}{\sqrt{2}} - G(w, z) \sqrt{2} = 1 + \frac{1}{\sqrt{2}} \left( w - r(w, z) g \left( \frac{z}{r(w, z)} \right) \right).$$

Since $r, g \in C^2$ it follows that $\bar{W}$ and $\bar{M}$ are $C^2$; however they are not $C^3$ since $g'''(\rho)$ is unbounded as $\rho \nearrow 1$ or $\rho \searrow -1$. The sufficient conditions on $r(w, z)$ in the lemma are fulfilled in the respective cases. Therefore $F''_{ww} - F''_{zz} \geq 0$ and $G''_{ww} - G''_{zz} \geq 0$, and hence it follows, see the copula definition (18), that $\bar{W}''_{ww} = W''_{ww} - W''_{zz} \geq 0$ and $\bar{M}''_{ww} = M''_{ww} - M''_{zz} \geq 0$. Furthermore, the averaging method employed does not alter the boundary values of the copula, hence also the copula condition (19) is satisfied since $W$ and $M$ are copulas. Thus we have proved that $\bar{W}$ and $\bar{M}$ are absolutely continuous copulas.

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