Non-interacting Cooper pairs inside a pseudogap.

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Abstract

I present a simple analytical model describing the normal state of a superconductor with a pseudogap in the density of states, such as in underdoped cuprates. In nearly two-dimensional systems, where the superconducting transition temperature is reduced from the mean-field BCS value, Cooper pairs may be present as slow fluctuations of the BCS pairing field. Using the self-consistent $T$-matrix (fluctuation exchange) approach I find that the fermion spectral weight exhibits two BCS-like peaks, broadened by fluctuations of the pairing field amplitude. The density of states becomes suppressed near the Fermi energy, which allows for long-lived low-energy Cooper pairs that propagate as a sound-like mode with a mass. A self-consistency requirement, linking the width of the pseudogap to the intensity of the pairing field, determines the pair condensation temperature. In nearly two-dimensional systems, it is proportional to the degeneracy temperature of the fermions, with a small prefactor that vanishes in two dimensions.

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I. INTRODUCTION

One of the strange features of high-temperature cuprate superconductors, a normal-state gap in the density of states, remains a subject of controversy. A great deal of experimental evidence (NMR, optical conductivity, specific heat, and, most recently, angle-resolved photoemission) seems to indicate that the superconducting energy gap survives the transition to the normal state and disappears only at a considerably higher temperature. Even the anisotropic character of the gap is preserved. This behavior is characteristic of “under-doped” compounds, in which doping of additional hole carriers increases the temperature of the superconducting transition $T_c$. There is no generally accepted theoretical model of this phenomenon, although arguably the most popular tentative explanation puts the blame on superconducting fluctuations near the transition point, which are expected to be enhanced in these highly anisotropic, almost two-dimensional, materials. While Monte-Carlo simulations of the attractive Hubbard model in two dimensions provide numerical evidence for the pairs-above-$T_c$ scenario, no analytical description of the pseudogap regime has been offered so far. The goal of this paper is to provide such a sketch.

Suppose that the pseudogap in cuprates does have the same origin as the superconducting gap — the scattering between fermion and hole states with the charge ±2 released in the form of a low-energy Cooper pair. This picture rests on certain assumptions. Simply saying that Cooper pairs are present above $T_c$ as a propagating mode is not enough. It is important that the fluctuating pairing field $\Delta$ look frozen on the time scale $1/|\Delta|$, otherwise the pseudogap will not be formed. Since the typical frequency of free propagating bosons in thermal equilibrium is given by their temperature, the pseudogap is expected to exist at temperatures $T \ll T_0$, where $T_0$ is loosely defined to be of order $|\Delta|/k_B$. In general, it may also be necessary to require the spatial coherence of the pairing field over a pair size $\xi_0 \approx v/|\Delta|$. In this particular model, however, the boson velocity is of order of the Fermi velocity $v$, so that as long as the pairing fluctuations are slow in time, they are also slow in space.

In a somewhat arbitrary way, $T_0$ can be identified with the transition temperature calcu-
lated in the BCS theory. As in any other mean-field theory, the onset of a long-range order coincides with the appearance of a short-range order (formation of pairs), the assumption being that the phase of the order parameter will lock up as soon as there is a non-zero amplitude. This, of course, breaks down in two spatial dimensions, as shown long ago by Hohenberg.\(^7\) In a highly anisotropic, almost two-dimensional system, long-range order sets in at a temperature \(T_c \ll T_0\). Above \(T_0\), Cooper pairs decay; between \(T_0\) and \(T_c\), they represent a propagating mode; finally, below \(T_c\), they form a Bose condensate. In the intermediate range, at least when \(T_c < T \ll T_0\), they represent a slowly fluctuating BCS pairing field and thus create a pseudogap. A well-defined pseudogap regime may be a peculiarity of low-dimensional systems.

A theoretical framework for treating the interplay between fermions and their boson-like bound states is thus needed. Following pioneering works by Eagles,\(^8\) Leggett,\(^9\) and Nozieres and Schmitt-Rink,\(^10\) several approximate methods have been suggested: the boson-fermion model,\(^11\)\(^12\) functional integration,\(^13\)\(^–\)\(^15\) and the self-consistent \(T\)-matrix approximation.\(^16\)\(^–\)\(^18\) In most cases, however, one has to resort to numerical computations. The present work is written with the purpose to provide an analytical sketch of a normal state with a pseudogap. As such, it inevitably contains further simplifications, which hopefully do not alter the nature of the problem: (1) The decay of low-energy bosons is neglected. (2) A clear separation of energy scales is assumed:

\[
k_B T_c \ll |\Delta| \ll \epsilon_F
\]

(\(\epsilon_F\) is the Fermi energy).

Using the self-consistent \(T\)-matrix approach, I derive approximate propagators for fermions and Cooper pairs. The width of the pseudogap is determined by the mean-square fluctuation of the pairing field \(\langle |\Delta(x, x')|^2 \rangle\) in a thermal ensemble. This expectation value depends, among other things, on the energy spectrum of Cooper pairs, which in turn is a function of the fermion energy spectrum. A closed set of coupled equations results, allowing one to determine the condensation temperature of pairs. The so determined \(T_c\) is propor-
tional to the fermion density and inverse mass, as seen on the Uemura plot. This is despite the fact that, as long as $|\Delta| \ll \epsilon_F$, the system is not in the limit of local (tightly bound) pairs. I also give a reason why quasiparticle peaks are quite broad near the Fermi surface, even when thermal fluctuations are slow ($k_B T_c \ll |\Delta|$). If correct, this sketch may provide a simple way to understand two puzzling features of underdoped cuprate superconductors – the pseudogap and the doping dependence of $T_c$ – from a unified standpoint.

The paper is organized as follows. The conserving $T$-matrix approximation is outlined in Section II; a bosonic propagator for a Cooper pair is introduced for the case of a “separable” interaction vertex. Section III contains a derivation of the fermion propagator in the presence of a slowly fluctuating pairing field. An exact model of Section IV reveals an important difference between a true superconducting gap and a pseudogap, stemming from the quantum nature of pairing fluctuations. Low-energy properties of Cooper pairs in the presence of a pseudogap and their condensation temperature are derived in Section V. The effect of fast fluctuations is estimated in the Appendix.

II. SELF-CONSISTENT $T$-MATRIX APPROXIMATION.

A good description of the self-consistent $T$-matrix approximation can be found in Ref. [16]. The relation to other approximate methods is outlined in Ref. [18].

The conserving $T$-matrix approximation is somewhat similar to the well-known Hartree-Fock principle [20]. The latter neglects any correlations between interacting particles (except for statistical ones) and describes the motion of independent entities in a self-consistent potential. The $T$-matrix approach includes pairwise correlations between colliding particles, which are particularly important when two-particle bound states are formed. The form of these correlations is inferred from the exact solution of a similar problem with two particles in vacuum, when the two-particle Green’s function

$$G_2(x, x'; y, y') \equiv -i\langle T[\psi(x)\psi(x')\psi(\dagger)(y')\psi(\dagger)(y)]\rangle$$

(2)

can be expressed in terms of the one-particle Green’s function $G(x; y) \equiv -i\langle T[\psi(x)\psi(\dagger)(y)]\rangle$. 

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and the two-body $T$ matrix [Fig. 1(a)]. In the many-body case, the latter is defined as a solution of the Bethe-Salpeter equation [Fig. 1(b)]:

$$T(P|k, k') = V(P|k, k') + \frac{i}{V} \sum_{k''} T(P|k, k'')G(P/2 + k'')G(P/2 - k'')V(P|k'', k'). \quad (3)$$

Here $P \equiv (\Omega, P)$ is the total 4-momentum of two fermions, $k$ and $k'$ are relative 4-momenta, and $V$ is the four-dimensional volume (with an imaginary time dimension). By reducing $G_2$ to a functional of $G$, one breaks the infinite hierarchy of equations for $n$-particle Green’s functions because the resulting approximate Dyson equation $G = G^{(0)} + G\Sigma G^{(0)}$ contains no higher-order Green’s functions [Fig. 1(c)]:

$$\Sigma(k) = -\frac{i}{V} \sum_{P} T(P|P/2 - k, P/2 - k)G(P - k), \quad (4)$$

Equations 3 and 4 form a closed set of equations that could be solved, at least in principle.

The use of full fermion propagators (instead of bare ones) in these equations is the main difference from the earlier approach of Thouless\cite{Thouless} equivalent to the BCS theory. In addition to having conservation laws built in\cite{Fetter}, the self-consistent approach is a step towards including the influence of Gaussian fluctuations on the phase transition. Near the transition temperature, fermions start to feel the presence of the emerging bosonic excitations, which in turn influences the process of pair formation. Dressing of the fermionic propagators in Eq. 3 allows one to account for this feedback, at least to some extent. The use of the dressed fermion propagator in the self-energy equation (4) increases the number of terms in the perturbation series for $G(k)$, which turns out to be important for treating properly the quantum nature of fluctuating Cooper pairs.

**A. Propagator of a Cooper pair.**

Although the $T$ matrix depends on three independent momenta, the non-trivial dependence should be associated with the conserved total momentum $P$. It is convenient to write the interaction vertex formally as [Fig. 1(d)]
$$V(P|k, k') = u^\dagger(k)D^{(0)}(P)u(k') \equiv \int u^*(k|x)D^{(0)}(P|x, x')u(k'|x')\, d^4x \, d^4x',$$  \hspace{1cm} (5)

where \( x \) and \( x' \) are some relative coordinates. One well-known example of such factorization is the case of a separable instantaneous potential \( V(P, k, k') = gv(k)v(k') \). This formalism is particularly convenient for lattice models with finite-range instantaneous interactions. In the case of the \( t-J \) model, for instance,

\[
V(P, k, k') = \begin{pmatrix}
+\infty & 0 & 0 \\
0 & -J & 0 \\
0 & 0 & -J
\end{pmatrix}
\begin{pmatrix}
1 \\
\cos k'_x \\
\cos k'_y
\end{pmatrix}
\begin{pmatrix}
1 \\
\cos k'_x \\
\cos k'_y
\end{pmatrix}^{\dagger}
\hspace{1cm} (6)
\]

in the singlet channel and zero in the triplet one. The infinite on-site repulsion imposes the constraint of no double occupancy. Phonon-mediated attraction can also be written in the form (5) with \( u(k|x) = e^{i(k\cdot x - i\omega t)} \).

The \( T \) matrix can now be found using the ansatz \( T(P|k, k') = u^\dagger(k)D(P)u(k') \) [Fig. 1(e)]. The new matrix \( D(P|x, x') \) satisfies the Dyson equation \( D = D^{(0)} + D\Pi D^{(0)} \) with the polarization matrix

\[
\Pi(P) = \frac{i}{V} \sum_k u(k)u^\dagger(k)G(P/2 + k)G(P/2 - k).
\hspace{1cm} (7)
\]

Clearly, \( D(P|x, x') \), which contains all the non-trivial information about the \( T \) matrix, can be regarded as the bosonic propagator of a Cooper pair whose internal structure is described by the dependence on the relative coordinates \( x \) and \( x' \).

The factorization procedure described above allows one to write the Bethe-Salpeter equation for the \( T \) matrix (3) in the form reminiscent of the Dyson equation for a doubly charged bosonic particle — see Fig. 1(f) — without introducing spurious degrees of freedom. The Thouless criterion for pair condensation, \( T(P) = \infty \) at \( P = 0 \), simply means that the quasiparticles with \( P = 0 \) have zero excitation energy.
III. FERMION PROPAGATOR IN THE T-MATRIX APPROXIMATION.

The fermion self-energy \( \Sigma \) can now be rewritten using the boson propagator \( D(P) \) [Fig. 1(g)]:

\[
\Sigma(k) = -\frac{i}{V} \sum_P u^\dagger(P/2 - k) D(P) u(P/2 - k) G(P - k).
\] (8)

This diagram illustrates two important points about the nature of the \( T \)-matrix approximation. (1) The irreversible decay of a fermionic quasiparticle is determined by the density of states “hole + Cooper pair” at a given 4-momentum. Note that both outgoing lines are dressed and thus represent actual excitations of the system. This should be contrasted to the functional-integration approach at the usual one-loop level\[13\]–\[15\] where the fermion self-energy is expressed in terms of the \emph{bare} fermion propagator. We will return to this point later in Sec. IV. (2) For a slowly fluctuating pairing field, the sum in Eq. 8 is dominated by the region near \( P = 0 \), so that a fermion with 4-momentum \( k \) is coupled mostly to the hole with the same 4-momentum. By replacing \( G(P - k) \) with \( G(-k) \), we obtain

\[
\Sigma(k) \approx -|\Delta(k)|^2 G(-k) \equiv \frac{|\Delta(k)|^2}{\Sigma(-k) - [G(0)(k)]^{-1}},
\] (9)

where

\[
|\Delta(k)|^2 \equiv \frac{i}{V} \sum_P u^\dagger(P/2 - k) D(P) u(P/2 - k)
\] (10)

The approximation made here may appear rather crude. Essentially, the scattering of a fermion into a \emph{continuum} of hole states is replaced by the coupling to a \emph{single} hole state with the same 4-momentum, a situation reminiscent of the BCS superconductor. If a bare propagator were used for the hole, the resulting Bogoliubov quasiparticles would be stable. However, the use of a full propagator \( G(-k) \) already makes the lifetime finite. In this case, neglecting temporal and spatial fluctuations of the pairing field does not look so bad, especially in a low-dimensional system.

The approximate equation for the self-energy (8) can be readily solved.\[22\] Iterating it once results in a quadratic equation for \( \Sigma(k) \). Since \( |\Delta(k)|^2 = |\Delta(-k)|^2 \), the dressed propagator is
The branch of the square root is fixed by the requirement that \( G(k) \to G^{(0)}(0) \) as \(|\Delta(k)| \to 0\).

In the simplest case of a short-range instantaneous attractive potential, \( \Delta(k) \) can be replaced by a constant and we have

\[
G(\omega, k) = \frac{1}{\omega - \epsilon_k} \left( \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{|\Delta|^2}{\omega^2 - \epsilon_k^2}} \right)^{-1},
\]

(12)

where \( \epsilon_k \) is the bare fermion energy. It is readily seen that the spectral weight of the dressed fermion is distributed over two finite regions \( \epsilon_k^2 < \omega^2 < \epsilon_k^2 + 4|\Delta|^2 \) as follows:

\[
\mathcal{A}(\omega, k) = \frac{|\omega + \epsilon_k|}{2\pi|\Delta|^2} \sqrt{\frac{\epsilon_k^2 + 4|\Delta|^2 - \omega^2}{\omega^2 - \epsilon_k^2}}.
\]

(13)

This distribution is reminiscent of the smeared BCS peaks at \( \omega^2 = \epsilon_k^2 + |\Delta|^2 \). In fact, the ratio of the spectral weights (13) at \( \omega = \pm \sqrt{\epsilon_k^2 + |\Delta|^2} \) is the same as in the BCS case.

Since the spectral weight is pushed away from the Fermi surface, a pseudogap opens up in the total density of states \( \mathcal{N}(\omega) \) (Fig. 4):

\[
\mathcal{N}(\omega) = \mathcal{N}_0 f(\omega/2|\Delta|),
\]

(14)

where \( \mathcal{N}_0 \) is the density of states in the free case and the function \( f(x) \) can be expressed in terms of complete elliptic integrals:

\[
f(x) = \begin{cases} 
(4/\pi)xE(x) & \text{for } 0 < x < 1, \\
(4/\pi)[K(1/x) - D(1/x)] & \text{for } x > 1.
\end{cases}
\]

(15)

It vanishes linearly for small values of \( x \) and approaches 1 as \( x \to \infty \).

IV. IDEAL FERMIONS INTERACTING WITH AN INDEPENDENT PAIRING FIELD.

The \( T \)-matrix approximation amounts to summing up an infinite number of terms of the perturbation series. Yet an even “greater” infinity of terms is left out. In such a case,
when approximations are not easily controlled, it is desirable to check the result using some exactly solvable model.

Consider a free fermion gas interacting with a classical external pairing field, which is represented by the action term

$$S_{\text{int}} = \int [\Delta(x, x') \psi^\dagger(x) \psi^\dagger(x') + \text{H.c.}] d^4x d^4x'. \quad (16)$$

The field $\Delta(x, x')$ can be interpreted as the wavefunction of a Cooper pair with its constituent fermions at $x$ and $x'$; the interaction term above removes two free fermions and forms a bound state.

The $U(1)$ symmetry, related to the conservation of charge, makes the phase of the complex field $\Delta$ unobservable. When pair formation lowers the energy of the system, the density of pairs $|\Delta|^2$ may become large. There are two distinct possibilities in this case. The first one occurs when the $U(1)$ symmetry is spontaneously broken and $\Delta$ chooses some direction in the complex plane with the well-known results (the BCS superconductor with a gap and a Bogoliubov sound mode). The second possibility is a symmetric phase with a fluctuating field $\Delta$. The quantum nature of these fluctuations leads to a non-trivial effect: a significant broadening of the fermion spectral weight.

Quantum fluctuations in this model can be implemented by using an ensemble of pairing fields $\Delta$ with a symmetric distribution. Although every single measurement of $\Delta$ produces a definite non-zero result, $\langle \Delta \rangle = 0$ upon averaging over a long series of such measurements. This procedure is Feynman’s way of quantizing a classical system. A closely related exactly solvable model for the Peierls gap in quasi-one-dimensional metals has been discussed by McKenzie following the analytic solution of Sadovskii.

A. Broken phase.

A generic diagram of the perturbation series for the fermion propagator contains $n$ incoming and $n$ outgoing dashed lines representing the pairing field [Fig. 3(a)]. If $\langle \Delta(x, x') \rangle \neq 0$, 

there is also the anomalous fermion propagator, which gives rise to the Meissner effect and superconductivity. Consider the extreme case when there are no fluctuations in the ensemble: \( \Delta(x, x') = \langle \Delta(x, x') \rangle \). The dressed fermion propagator is simply a geometrical series with a single term in each order. For an instantaneous short-range pairing field \( \langle \Delta(x, x') \rangle = \Delta \delta(x - x') \) (uniform for simplicity),

\[
G(k) = \frac{1}{\omega - \epsilon_k} \sum_{n=0}^{\infty} \left( \frac{\Delta^2}{\omega^2 - \epsilon_k^2} \right)^n.
\]

(17)

Although the geometrical series \( \sum_{n=0}^{\infty} z^n \) diverges when \( |z| > 1 \), its analytic continuation, the function \( (1 - z)^{-1} \), has only one singular point \( z = 1 \). The BCS result is recovered:

\[
G(k) = \frac{\omega + \epsilon_k}{\omega^2 - \epsilon_k^2 - \Delta^2}.
\]

(18)

The divergence of the perturbation series (17) in the range \( \epsilon_k^2 < \omega^2 < \epsilon_k^2 + |\Delta|^2 \) is associated with the transfer of the pole of the propagator from \( \omega = \epsilon_k \) to the new locations at \( \omega^2 = \epsilon_k^2 + |\Delta|^2 \).

B. Symmetric phase with quantum fluctuations.

The structure of the perturbation theory changes considerably in the symmetric phase. Assuming that the statistical distribution of the pairing fields is Gaussian, we can write any average that involves a product of \( n \) fields \( \Delta(x_m, x'_m) \equiv \Delta_m \) and \( n \) fields \( \Delta^*(y_m, y'_m) \equiv \Delta_m^* \) as a sum over all possible products of pairwise averages:

\[
\langle \Delta_1^* \Delta_1 \Delta_2^* \Delta_2 \ldots \Delta_n^* \Delta_n \rangle = \sum_P \langle \Delta_1^* \Delta_{P1} \rangle \langle \Delta_2^* \Delta_{P2} \rangle \cdots \langle \Delta_n^* \Delta_{Pn} \rangle,
\]

(19)

where \( P \) represents a permutation of the numbers 1...\( n \). Upon introducing a dashed line for each pair average \( \langle \Delta_i^* \Delta_j \rangle \), we find \( n! \) different diagrams with \( n \) such lines [Fig. 3(b) and (c)]. Again, take the example of a uniform instantaneous short-range pairing field:

\[
\langle \Delta^*(y, y') \Delta(x, x') \rangle = |\Delta|^2 \delta(x - x') \delta(y - y').
\]

(20)
Each single field breaks the phase symmetry and produces the effect described above. Averaging over the ensemble, however, restores this symmetry; the perturbation series now contains \( n! \) identical terms in the \( n \)-th order:

\[
G(k) = \frac{1}{\omega - \epsilon_k} \sum_{n=0}^{\infty} n! \left( \frac{|\Delta|^2}{\omega^2 - \epsilon_k^2} \right)^n.
\]

(21)

Clearly, the result is different from (17). Expression (21) was obtained in Ref. 26 in connection with the pseudogap in the boson-fermion model – as a leading term in two dimensions.

Although the series

\[
\sum_{n=0}^{\infty} n! z^n
\]

(22)

diverges for any \( z \neq 0 \), it represents an asymptotic Taylor expansion of the function

\[
f(z) = \int_0^\infty \frac{e^{-t}dt}{1 - vz}.
\]

(23)

The divergence of the series is related to the nonanalyticity of \( f(z) \) on the positive side of the real axis, where it has a cut. Rather then working with a series that converges only in the trivial case \( z = 0 \), we can sum up the diagrams for a single uniform field \( \Delta \) – as in Sec. IV A – and then average the result over a Gaussian ensemble with the weight \( \exp \left( -|\Delta|^2/|\Delta|^2 \right) \): \( d\Delta d\Delta^* \):

\[
G(k) = \frac{1}{\omega - \epsilon_k} \int \frac{e^{-t}dt}{1 - \frac{|\Delta|^2}{\omega^2 - \epsilon_k^2} t},
\]

(24)

which indeed has (21) as an asymptotic expansion. The quasiparticle spectral weight, concentrated at \( \omega = \epsilon_k \) in the free case, is now spilled into the region \( \omega^2 > \epsilon_k^2 \) (Fig. 4):

\[
\mathcal{A}(\omega, k) = \frac{\omega + \epsilon_k}{|\Delta|^2} \exp \left( -\frac{\omega^2 - \epsilon_k^2}{|\Delta|^2} \right), \quad \omega^2 > \epsilon_k^2.
\]

(25)

The density of states exhibits a pseudogap (Fig. 2), somewhat more pronounced than it was found in the \( T \)-matrix approximation: it now vanishes as \( \omega^2 \) near the Fermi level.

The results of this section can be understood in a simple intuitive way. Even though we have suppressed spatial and temporal fluctuations of the pairing field, its strength does
not yet have a definite value; the fermion pole, instead of being shifted to a definite new position, is smeared into a cut along some part of the real axis. Nevertheless, the existence of a scale $|\Delta|^2$ in the field distribution results in the fermion spectral function reminiscent of the BCS one.

C. Relation to the $T$-matrix approximation.

The $T$-matrix equation for the fermion self-energy $\Sigma(k)$, which includes a dressed fermion propagator, is equivalent to partial summation of the diagrams in Sec. IV B: only those without intersecting dashed lines are included. For instance, diagram (b) in Fig. 3 is present, while (c) is left out. The number of diagrams in the $n$-th order satisfies the recursion relation

$$C_{n+1} = \sum_{m=0}^{n} C_m C_{n-m} \tag{26}$$

($C_0 = 1$ by definition) with the solution

$$C_n = \prod_{m=1}^{n} \left(4 - \frac{6}{m+1}\right), \tag{27}$$

which grows as $4^n$ for a large $n$. Therefore, the series

$$\sum_{n=0}^{\infty} C_n z^n \tag{28}$$

has the radius of convergence equal to 1/4 and the Green’s function

$$G(k) = \frac{1}{\omega - \epsilon_k} \sum_{n=0}^{\infty} C_n \left(\frac{|\Delta|^2}{\omega^2 - \epsilon_k^2}\right)^n \tag{29}$$

is analytical (hence real) when $\omega^2 > \epsilon_k^2 + 4|\Delta|^2$. The spectral weight is contained in the finite region $\epsilon_k^2 < \omega^2 < \epsilon_k^2 + 4|\Delta|^2$.

The above discussion shows that the $T$-matrix approximation includes, at least partially, effects of quantum fluctuations of the pairing field on the fermion propagator. This is achieved through the use of a dressed fermion propagator in the equation for the fermion self-energy. The number of terms in the resulting perturbation series grows quickly enough to make the result non-analytical in a larger region than in the BCS broken phase. Inclusion
V. COOPER PAIRS IN A PSEUDOGAP.

A. The propagator of a Cooper pair.

With much of the spectral weight removed from the vicinity of the Fermi surface, low-energy bosonic excitations may represent a propagating mode. In this respect, such a normal state resembles a BCS superconductor just below $T_c$, where the opening of a true gap inhibits the decay of pairs. Therefore, a reasonable estimate of the boson energy spectrum can be obtained from Eq. 7 by using the dressed fermion propagator in the BCS form, which constitutes the two-pole ansatz of Ref. 18. Inclusion of fermion lifetime effects makes Cooper pairs unstable but is not expected to produce qualitative changes in the energy spectrum. For a short-range instantaneous attraction, $D^{(0)}(P) = g < 0$,

$$D^{-1}(P) = g^{-1} + \frac{i}{\mathcal{V}} \sum_k G(k)G(P - k),$$

with

$$G(k) \equiv G(\omega, \mathbf{k}) = \frac{\omega + \epsilon_k}{\omega^2 - \epsilon_k^2 - |\Delta|^2}. \quad (31)$$

At this level of approximation (no boson decay because of a full fermion gap), $D^{-1}(\Omega, \mathbf{P})$, analytically continued from Matsubara frequencies to the rest of the $\Omega$ plane, can be expanded in powers of $\Omega$ and $\mathbf{P}$:

$$D^{-1}(\Omega, \mathbf{P}) = D^{-1}(0, 0) + \frac{\partial D^{-1}(0, 0)}{\partial \Omega^2} \Omega^2 + \frac{\partial D^{-1}(0, 0)}{\partial \mathbf{P}^2} \mathbf{P}^2 + \ldots \quad (32)$$

or
\[ D(\Omega, P) \approx \frac{Z}{\Omega^2 - P^2 s^2 - M^2 s^4}. \] (33)

The mass term appears because the Thouless criterion no longer applies: \( D^{-1}(0, 0) \neq 0 \) in the normal state. The existence of two Cooper-pair poles at a given momentum \( P \), at the frequencies \( \Omega = \pm \sqrt{M^2 s^4 + P^2 s^2} \equiv \pm E_P \), reflects the fact that correlated propagation is possible not only for two fermionic particles, but also for two holes. If a pair is made out of fermionic excitations just around the Fermi surface, where the density of states is the same for particles and holes, the residues of the two poles are equal (up to a sign). A varying density of states will make the two poles less symmetric, both in terms of their residues and positions, but still low-energy Cooper pairs will have two poles.

This trend can be clearly seen in the numerically obtained two-particle density of states plotted in Fig. 4 of Ref. 18. Low-momentum pairs have two somewhat asymmetric peaks dispersing with momentum. The asymmetry is related to the instantaneous nature of the interaction in the attractive Hubbard model: the number of particle states involved in the formation of a pair greatly exceeds that of hole states at low fermion densities. No holes contribute to the formation of pairs with higher momenta, so that there is no pole at negative frequencies in this case. Such pairs, however, represent fast fluctuations of the pairing field and are irrelevant to the formation of the pseudogap. The numerical data also show an increasing lifetime of Cooper pairs as \( P \to 0 \), reflecting a stronger depletion of the fermion density of states near the Fermi energy.

When fermionic excitations across the gap are frozen out, the Cooper-pair propagator at zero momentum is

\[ D^{-1}(\Omega, 0) = g^{-1} - \int_{-\infty}^{\infty} N(\epsilon) \frac{\epsilon^2 + \bar{\epsilon}^2 + \epsilon \Omega}{\epsilon (\Omega^2 - 4 \bar{\epsilon}^2)} \] (34)

where \( \bar{\epsilon} \equiv \sqrt{\epsilon^2 + |\Delta|^2} \). With the exception of the zeroth-order term \( D^{-1}(0, 0) \equiv -Z^{-1} M^2 s^4 \), coefficients of the Taylor series (in powers of \( \Omega \)) are insensitive to large-\( \epsilon \) behavior of the density of states, and are thus determined by the lower energy scale, \textit{i.e.}, the gap width \( |\Delta| \), provided that variations of the bare density of states are small on that energy scale. In this
weak-coupling limit, the terms odd in $\Omega$ vanish because $\mathcal{N}(\epsilon)\epsilon$ is an odd function near $\epsilon = 0$. Quartic and higher-order terms can be dropped as long as $\Omega^2 \ll |\Delta|^2$. Then

$$D^{-1}(\Omega, 0) \approx Z^{-1}(\Omega^2 - M^2 s^4), \quad (35)$$

where $Z^{-1}$ is of order $\mathcal{N}_0/|\Delta|^2$. More exactly,

$$Z = \frac{12|\Delta|^2}{\mathcal{N}_0}. \quad (36)$$

Expansion in powers of $P$ allows one to determine the characteristic speed of bosons $s$, which is expected to be smaller than the Fermi velocity $v$ on physical grounds. In the weak-coupling limit, we recover the result of Bogoliubov, $s^2 = v^2/d$ in $d$ dimensions. This is roughly consistent with the dispersion of Cooper-pair poles at low momenta inferred from the data of Ref. 18.

At a given temperature, two as yet undetermined low-energy parameters of the boson field – the mass $M$ and the pole residue $Z$ – control the intensity of fermion scattering by the pairing field and thus determine the (also unknown) fermion energy gap, as discussed in some detail below. In addition, Eq. 36 relates $Z$ to the energy gap. Using this information, it should be possible to determine two out of the three unknown parameters. In the following section, the mass $M$ is found as a function of temperature.

**B. Evaluation of the fermion self-energy**

We now return to the analysis of the fermion self-energy $\Sigma$ using the approximate boson propagator $\mathcal{S}$. The following expression for the fermion self-energy results:

$$\Sigma(\omega, k) \approx \int \frac{d^dP}{(2\pi)^d} \frac{1}{2\pi i} \oint \frac{d\Omega}{e^{\beta\Omega} - 1} \frac{Z}{\Omega^2 - E_P^2} G(\Omega - \omega, P - k). \quad (37)$$

The sum over the bosonic Matsubara frequencies $\Omega$ has been converted into an integral around the poles of $(e^{\beta\Omega} - 1)^{-1}; E_P \equiv \sqrt{M^2 s^4 + P^2 s^2} > 0$ is the boson energy. It is convenient to write the fermion propagator in terms of its spectral weight:
\( G(\omega, k) = \int_{-\infty}^{+\infty} \frac{A(\epsilon', k) d\epsilon'}{\omega - \epsilon'} \). \hfill (38)

After deforming the integration contour in the standard way, one finds the self-energy as a sum of two terms, which correspond to emission (+) and absorption (−) of a Cooper pair:

\[
\Sigma^{(\pm)}(\omega, k) = Z \int \frac{d^4P}{(2\pi)^d} \frac{1}{2E_P} \int A(\epsilon', P - k) \, d\epsilon' \frac{N(E_P) + n(\pm \epsilon')}{\omega + \epsilon' \mp E_P}.
\] \hfill (39)

The part proportional to the boson occupation number \( N(E_P) \) can be regarded as induced emission or absorption with the rate proportional to the spectral intensity of the pairing field \( N(E_P)/E_P \). The fermionic term \( n(\pm \epsilon') \) is responsible for “spontaneous” processes.

If the pseudogap formation is governed by low-energy modes, the induced term is dominant since \( N(E_P) \gg n(\pm \epsilon') \) at low energies. The rate of spontaneous emission and absorption of pairs is estimated in the Appendix; it can be neglected in \( d = 2 \) dimensions. Upon shifting the integration variable \( \epsilon' \mp E_P \rightarrow \epsilon' \),

\[
\Sigma^{(\pm)}_{\text{ind}}(\omega, k) = Z \int \frac{d^4P}{(2\pi)^d} \frac{N(E_P)}{2E_P} \int \frac{A(\epsilon' \mp E_P, P - k) \, d\epsilon'}{\omega + \epsilon'}
\] \hfill (40)

Bosons are restricted to low-energy modes with \( E_P \) of order or less than \( k_B T \). In the case when the fermion spectral weight is distributed over a wider interval of energies, the variation of the fermion spectral weight on the energy scale of \( k_B T \) can be neglected. For the same reason, we can neglect the variation of the fermion momentum in the integrand, provided that fermion and boson velocities are of the same order (recall that \( s^2 = v^2/d \)). The integration over \( \epsilon' \) yields the fermion propagator \( G(-\omega, -k) \) and the self-energy has the form inferred previously (4):

\[
\Sigma^{(\pm)}_{\text{ind}}(\omega, k) + \Sigma^{(-)}_{\text{ind}}(\omega, k) \approx -|\Delta|^2 G(-\omega, -k),
\] \hfill (41)

where the average fluctuation (intensity) of the pairing field is

\[
|\Delta|^2 = Z \int \frac{d^4P}{(2\pi)^d} \frac{N(E_P)}{E_P}.
\] \hfill (42)

Essentially the same result can be obtained by considering a classical field \( \Delta(t, r) \) with the Lagrangian density determined by (33).
\[ \mathcal{L}(t, r) = Z^{-1} \left[ |\partial \Delta(t, r)/\partial t|^2 - |\nabla \Delta(t, r)|^2 - M s^2 |\Delta(t, r)|^2 \right], \] (43)

in thermal equilibrium. Application of the equipartition theorem yields

\[ |\Delta|^2 = Z \int \frac{d^d \mathbf{P}}{(2\pi)^d} \frac{k_B T}{E_{\mathbf{P}}}, \] (44)

which is the classical analogue of Eq. [42].

**C. Condensation temperature.**

By substituting the value of the boson residue (36) into Eq. [42], we obtain the self-consistency condition mentioned above. Remarkably, the width of the pseudogap cancels out and the resulting equation implicitly determines the boson mass as a function of temperature:

\[ \int \frac{d^d \mathbf{P}}{(2\pi)^d} \frac{N(E_{\mathbf{P}})}{E_{\mathbf{P}}} = \frac{N_0}{12}, \] (45)

where \( E_{\mathbf{P}} = \sqrt{M^2 s^4 + P^2 s^2} \), \( N(E) = (e^{\beta E} - 1)^{-1} \) is the boson occupation number, and \( N_0 \) is the density of bare fermion states. It is immediately obvious that there is no Bose condensation in \( d = 2 \) dimensions: for \( M = 0 \), the integral diverges at the lower limit. In \( d = 3 \) dimensions, this equation predicts a condensation temperature \( k_B T_c \) of order \( \epsilon_F \). This result should not be taken at face value because it was derived for a weakly attractive degenerate fermion gas. (Also, as discussed in the Appendix, the absence of a low energy scale implies that pairing fluctuations are not slow.) Nevertheless, the existence of such a high temperature scale is justified if one considers the limits of weak and strong attraction between fermions. In the weak-coupling limit, there are no pairs at the Fermi temperature, they form at a much lower temperature \( k_B T_0 \ll \epsilon_F \); therefore, the long-overdue condensation occurs immediately, which explains why the mean-field BCS approach works so well. In the opposite limit of local pairs (turned into hard-core bosons), condensation indeed occurs when the gas becomes degenerate.

Next, we estimate the condensation temperature in \( d = 2 + \varepsilon \) dimensions, which may be relevant to highly anisotropic cuprate superconductors. Recalling the asymptotic behavior of the Riemann zeta-function \( \zeta(1 + \varepsilon) \sim 1/\varepsilon \) we obtain
when $\varepsilon \ll 1$. For comparison, the condensation temperature of a Bose gas made of very small fermionic pairs is only higher by a factor $3/2$. (In both cases, $\epsilon_F$ is the Fermi energy of the ideal fermion gas with the same mass and density of particles.) It is thus clear that the condensation of Cooper pairs in the presence of a pseudogap occurs well below the fermion degeneracy temperature and a clear separation of energy scales is possible for moderately weak attraction strengths. If the attraction is strong, we end up with tightly bound pairs, in the local boson limit. In the case of a very weak attraction, the BCS pair formation temperature $T_0$ is lower than $T_c$ of Eq. (46); the BCS model takes over.

VI. CONCLUSION

I have considered the interaction between fermions and Gaussian pairing fluctuations (Cooper pairs without self-interaction). A well-defined pseudogap regime is found in $d = 2 + \varepsilon$ spatial dimensions, when the condensation temperature is much lower than the BCS mean-field one. The quantum character of Cooper pairs, related to the unobservable nature of the pair wavefunction $\Delta(x, x')$, leads to considerable broadening of the Bogoliubov quasiparticle peaks near the Fermi surface. Even when non-uniform configurations of the field $\Delta(x, x')$ are suppressed at low temperatures, fluctuations of its amplitude blur the strength of the pairing field, making the position of the quasiparticle poles at $\omega^2 = \epsilon_k^2 + |\Delta(k)|^2$ uncertain.

A well-pronounced suppression of the fermion density of states near zero energy makes room for long-lived pair states inside the gap. At low momenta and frequencies, they represent a Bogoliubov sound-like mode with a non-zero mass. The mass is determined from a self-consistency condition that links the width of the pseudogap to the mean fluctuation of the pairing amplitude. Zero mass signals the onset of Bose condensation. In $d = 2 + \varepsilon$ dimensions, the condensation temperature scales as $\varepsilon\epsilon_F/6$, which is $2/3$ times the condensation temperature for the corresponding ideal Bose gas. This may explain the doping dependence of $T_c$ in underdoped cuprates.
Approximations made in this work are twofold. First, an infinite hierarchy of equations for $n$-particle Green’s functions is replaced with the self-consistent $T$-matrix approach of Brueckner, which partially accounts for Gaussian pairing fluctuations. By solving an exact model with an independent pairing field, I show that the self-consistent $T$-matrix approximation takes into account the amplitude fluctuations of Cooper pairs in the normal state, a feature missing in other approximate models. Further approximations made in this paper are necessary to obtain an analytic solution. Firstly, thermally excited non-uniform configurations of the pairing field have little effect on the fermion propagator, provided that a clear separation of energy scales, $k_B T_c \ll |\Delta|$, exists. This appears to be the case in $d = 2+\varepsilon$ dimensions, i.e., for highly anisotropic, almost two-dimensional systems. Secondly, a residual density of fermionic states in the pseudogap, leading to the decay of Cooper pairs, is assumed to have negligible influence on the pair energy spectrum. A numerical work using the self-consistent $T$-matrix approach seems to indicate that this is a reasonable assumption: low-energy Cooper pairs represent a propagating mode with two Cooper-pair poles dispersing with the pair momentum. In order to ascertain that propagating Cooper pairs in the pseudogap are not an artifact of the self-consistent $T$-matrix approximation, it is necessary to look for them in quantum Monte-Carlo simulations on two-dimensional lattices.

There are many questions that remain open. For instance, in the case of a superconducting order parameter with nodes, the density of fermion states is less strongly suppressed. Will Cooper pairs be stable enough to represent a propagating mode? Also, what causes the strong broadening of fermion quasiparticle peaks above the temperature at which pairs are formed?

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APPENDIX:

Spontaneous emission or absorption of Cooper pairs is not restricted to low boson momenta and thus represents the effect of fast fluctuations. The emission or absorption rate depends on the phase space available to the products of a fermion decay (a pair and a hole), which tends to be smaller in a lower number of dimensions $d$. This indeed is confirmed by a simple calculation below. In $d = 3$ dimensions, the imaginary part of $\Sigma_{sp}(\omega, k)$ is linear in $\omega$ and thus leads to a renormalization of the quasiparticle spectral weight. In $d = 2$ dimensions, a constant imaginary part is introduced, which merely broadens the quasiparticle peak. Besides, the existence of a small parameter in the pseudogap regime (the ratio $|\Delta|/\epsilon_F$) makes this additional broadening insignificant in $d = 2 + \epsilon$ dimensions.

Let us estimate the impact of “vacuum” fluctuations on the self-energy of a fermion with momentum $k$ and bare energy $\epsilon_k = 0$. Inasmuch a hole created in the decay can be far away from the Fermi surface, we can neglect the influence of pair scattering on its spectrum and treat the hole as a stable particle with momentum $-k' = k - P$ and energy

$$-\epsilon_{k'} = -\epsilon_k + \frac{k \cdot P}{m} - \frac{|P|^2}{2m} \approx |P|v \cos \theta \equiv -\epsilon'(|P|, \theta),$$

(A1)

neglecting the curvature of the energy surface $\epsilon_k$; $\theta$ is the angle between $k$ and $P$. The boson energy is taken to be $E_P = s|P| \equiv E(|P|)$. Then

$$\text{Im} \Sigma_{sp}^{(\pm)}(\omega, k) = -Z\pi \int \frac{d^dP}{(2\pi)^d} \frac{n(\pm \epsilon') \delta(\omega + \epsilon' \mp E)}{2E}$$

(A2)

Energy conservation requires that

$$E = \frac{\omega s}{s + v \cos \theta}.$$  

(A3)

For definiteness, let us assume that the frequency of the incoming fermion $\omega > 0$. Then a pair can be emitted ($E > 0$) when $\cos \theta > -s/v$ and absorbed ($E < 0$) when $\cos \theta < -s/v$. 

20
Pair emission is accompanied by absorption of a second fermion with the energy in the range 
\[ \epsilon' > -v\omega/(v+s) \equiv \omega_+ . \] The second fermion can, thus, be taken from below the Fermi surface and the process will take place even at low temperatures! On the contrary, absorption of a pair requires emission of a second fermion with a negative energy \[ \epsilon' < -v\omega/(v-s) \equiv \omega_- . \] At low temperatures, such states are occupied and pair absorption is suppressed.

In \( d = 3 \) dimensions, the element of phase space reduces to \( E dE d\epsilon'/ (2\pi)^2 vs^2 \). The integration over the boson energy \( E \) is elementary because a \( \delta \)-function is present:

\[
\Im \Sigma_{sp}(\omega, k) = -\frac{Z}{8\pi vs^2} \left[ \int_{-\omega_+}^{\infty} n(\epsilon') d\epsilon' + \int_{-\omega_-}^{\omega_-} n(-\epsilon') d\epsilon' \right], \tag{A4}
\]

The first term in brackets (emission of a pair) grows linearly with \( \omega \). By inserting \( Z \) from (36), we obtain

\[
\Im \Sigma_{sp}(\omega, k) \sim -\frac{\pi |\Delta|^2 \omega}{\epsilon_F^2 (1 + 1/\sqrt{3})}, \tag{A5}
\]
as \( \omega \to +\infty \). Thus, spontaneous pair emission leads to a significant renormalization of the fermion propagator (the ratio \( |\Delta|^2/\epsilon_F^2 \) is not small for the pseudogap regime in \( d = 3 \)) and cannot be neglected.

In \( d = 2 \) dimensions, pair emission has a smaller available phase space:

\[
\Im \Sigma_{sp}^{(+)}(\omega, k) = -\frac{Z}{8\pi s\sqrt{v^2 - s^2}} \int_{-\omega_+}^{\infty} \frac{n(\epsilon') d\epsilon'}{\sqrt{(\epsilon' + \omega_+)(\epsilon' + \omega_-)}}. \tag{A6}
\]
The self-energy does not vary with \( \omega \) if \( n(\epsilon') \) is replaced with the step-function. This channel of decay is characterized by the rate

\[
-\Im \Sigma_{sp}(\omega, k) = \frac{3a|\Delta|^2}{\epsilon_F}, \tag{A7}
\]
where \( \sinh a = \sqrt{1/2} - 1/2 \). In \( d = 2 + \varepsilon \) dimensions, \( |\Delta|^2 \ll \epsilon_F^2 \) and this additional linewidth is small in comparison with the broadening caused by induced fluctuations, of order \( \sqrt{|\Delta|^2} \). Therefore, the effect of fast spontaneous fluctuations can be neglected in \( d = 2 + \varepsilon \) dimensions.
REFERENCES

1 W. W. Warren et al., Phys. Rev. Lett. 62, 1193 (1989).

2 J. Orenstein et al., Phys. Rev. B 42, 6342 (1990); C. C. Homes et al., Phys. Rev. Lett. 71, 1645 (1993).

3 J. W. Loram et al., Physica C 235–240, 134 (1994).

4 A. G. Loeser et al., Science 273, 325 (1996).

5 H. Ding et al., Nature 382, 51 (1996).

6 M. Randeria et al., Phys. Rev. Lett. 69, 2001 (1992); N. Trivedi and M. Randeria, Phys. Rev. Lett. 75, 312 (1995). See also A. Moreo and D. J. Scalapino, Phys. Rev. Lett. 66, 946 (1991).

7 P. C. Hohenberg, Phys. Rev. 158, 383 (1967).

8 D. M. Eagles, Phys. Rev. 186, 456 (1969).

9 A. J. Leggett, in *Modern Trends in the theory of condensed matter*, edited by A. Pekalski and J. Przystawa, Lecture Notes in Physics 115, Berlin (1980).

10 P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985).

11 R. Friedberg and T. D. Lee, Phys. Lett. A 138, 423 (1989).

12 J. Ranninger and J. M. Robin, Phys. Rev. B 53, 11961 (1996).

13 C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. 71, 3202 (1993). See also M. Drechsler and W. Zwerger, Ann. Phys. (Germany) 1, 15 (1992).

14 F. Pistolesi and G. C. Strinati, Phys. Rev. B 53, 15168 (1996).

15 V. P. Gusynin, V. M. Loktev, and I. A. Shovkovy, cond-mat/9508070.

16 R. Haussmann, Z. Phys. B 91 (1993).
17. N. E. Bickers, D. J. Scalapino, and S. R. White, Phys. Rev. Lett. 62, 961 (1989); N. E. Bickers and D. J. Scalapino, Ann. Phys. (NY) 193, 206 (1989).

18. R. Micnas et al., Phys. Rev. B 52, 16223 (1995).

19. Y. J. Uemura et al., Phys. Rev. Lett. 62, 2317 (1989). See also Y. J. Uemura, in Proceedings of the Workshop on High-Tc Superconductivity and the C_{60} family, Beijing, 1994, edited by S. Fung and H. C. Ren (Gordon and Breach, 1994).

20. G. Baym and L. P. Kadanoff, Phys. Rev. 124, 287 (1961); G. Baym, ibid., 127, 1391 (1962).

21. D. J. Thouless, Ann. Phys. 10, 553 (1960).

22. A. Schmid, Z. Phys., 231, 324 (1970). The main difference between the present work and Schmid’s lies in the properties of pairs: he considers a diffusing pairing field, as opposed to a propagating one. I thank R. Micnas for pointing out this reference to me.

23. R. P. Feynman and A. R. Hibbs, Quantum mechanics and path integrals (New York, 1965).

24. R. H. McKenzie, Phys. Rev. B 52, 16428 (1995); Phys. Rev. Lett. 74, 5140 (1995).

25. M. V. Sadovskii, Zh. Eksp. Teor. Fiz. 77, 2070 (1979) [Sov. Phys. JETP 50, 989 (1979)].

26. H. C. Ren, to be published.

27. R. V. Carlson and A. M. Goldman, Phys. Rev. Lett. 31, 880 (1973); ibid., 34, 11 (1975). These authors stress that, as soon as a superconducting gap opens up below $T_c$, pairing fluctuations become a propagating, sound-like mode. Well below $T_c$, this mode is pushed up to the plasmon frequency.

28. A lattice implementation of a $(2 + \varepsilon)$-dimensional Bose gas is discussed in R. Micnas, J. Ranninger, and S. Robaszkiewicz, Rev. Mod. Phys. 62, 113 (1990).

29. V. J. Emery and S. A. Kivelson, Nature 374, 434 (1995); Phys. Rev. Lett. 74, 3253 (1995).
FIGURES

FIG. 1. Diagrams of the self-consistent $T$-matrix approximation for the singlet channel.

FIG. 2. Pseudogap in the fermion density of states in the limit of slow pairing fluctuations. Dashed line: $T$-matrix approximation. Solid line: exact model with Gaussian fluctuations.

FIG. 3. Feynman diagrams for the exact model with a classical pairing field. Solid lines are bare fermion propagators. (a) A diagram of order $n = 3$. Dashed lines with circles are pairing fields $\Delta(x, x')$ and $\Delta^*(y, y')$. (b)-(c) Examples of diagrams generated from (a) after averaging over a Gaussian ensemble of pairing fields. Dashed lines are now pairwise averages $\langle \Delta(x, x')\Delta^*(y, y') \rangle$.

FIG. 4. Distribution of the fermion spectral weight $\mathcal{A}(\omega, k)$ in the model with Gaussian pairing fluctuations. $\epsilon_k$ is the bare energy of a fermion, $|\Delta|$ is the r.m.s. fluctuation of the pairing field. Also shown are positions of the fermion poles in a non-interacting fermion system (dotted line) and a BCS superconductor with the gap $|\Delta|$ (dashed line).
FIG. 1.
FIG. 2.
FIG. 3.
FIG. 4.