Cyclic unparticle physics

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We explore phenomenological consequences of coupling a non-conformal scale-invariant theory to the standard model. We point out that, under certain circumstances, non-conformal scale-invariant theories have oscillating correlation functions which can dramatically modify standard model processes. We dub this scenario cyclic unparticle physics, or simply cyclunparticle physics. We compute phase spaces and amplitudes associated with final state cyclunparticle and cyclunparticle exchange, respectively. We show detailed formulae in a simple example.

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1. Introduction

The physical consequences of a quantum field theory can all be obtained from its $S$-matrix, provided the $S$-matrix is well defined. Of course, the $S$-matrix directly encodes information on scattering cross sections and hence on widths and masses of resonances too. The computation of some other quantities, such as static properties of stable particles like the gyromagnetic ratio of the electron or the neutron, is only slightly less direct. But even quantities such as the thermodynamic potential of a system in a thermal bath, something that would appear quite unrelated to the $S$-matrix, can be obtained from it \[1\]. However, the $S$-matrix does not exist in theories with scale invariance.

In the absence of an $S$-matrix one is left to study correlation functions. While these are not physical, they surely encode measurable information. To address the question of how to access this information, Georgi proposed a model with two sectors that are very weakly coupled through irrelevant operators \[2\]. The first sector, call it the ‘SM’ sector, is by itself non-scale-invariant. An example of such a sector is the standard model of electroweak and strong interactions. The second sector, call it the ‘SI’ sector, is, considered in its own right, a scale-invariant theory (SIT). In this construction an observer made up of SM stuff can conduct experiments that excite and probe the SI sector. For one example, scattering of SM particles into SI stuff appears like the scattering into invisible particles with possibly fractional dimensions of phase space. And for another, exclusive scattering processes of SM to SM particles can present unusual patterns of SM-SI interference \[3\]. Georgi dubbed this construction ‘unparticle physics’ because of the apparent fractional dimension of the phase space in ‘unparticle’ production.

In this letter we distinguish an SI sector that is merely scale-invariant from one that displays full fledged conformal invariance. Virtually all of the (too numerous to cite) literature on phenomenology of unparticles assumes the SI sector is a conformal field theory (CFT). It was pointed out in Ref. \[4\] that unitarity imposes constraints on the dimensions of non-scalar unparticle operators that render them phenomenologically less accessible. In avoiding these constraints, work on unparticle phenomenology that appeared after Ref. \[4\] often appeals to an SI sector that is not a CFT \[1\]. But until recently the structure of non-CFT SI systems was largely unknown. Our work has uncovered a remarkable feature of non-CFT SI systems: rather than corresponding to fixed points of an RG flow, they are found as limit cycles (or ergodic trajectories) in the flows \[5\]. This most remarkable feature

\[1\] Even if only scale invariance is assumed, there are still constraints on dimensions of operators; see footnote 9 in Ref. \[4\].
of non-CFT scale-invariant theories has been missed in all of the unparticle literature. We propose to investigate it.

This serves another purpose. One may wonder if these cycles, being unfamiliar, have any physical consequences at all. One may suspect that they don’t, because on the SI cycle one may recast the RGE as one with a vanishing beta function but with a universal shift of anomalous dimensions. We will lay this question to rest once and for all: probing a cycle of the SI system through the unparticle setup results in observable oscillations as a function of energy in, for example, cross sections. In our mind, settling this issue, which has been raised informally by many, is no less important than phenomenological implications.

We hasten to indicate that cyclic unparticles, or ‘cyclunparticles,’ are more conjectural than plain unparticles. For both, one conjectures the existence of an extension to the SM Lagrangian consisting of an SI sector and of irrelevant operators coupling the SM to the SI. But while the existence of CFTs that can make up the SI sector in plain vanilla unparticle scenarios is well established, at present we have only established non-CFT scale-invariant flows in $D = 4 - \epsilon$ spacetime dimensions. We must note that we have found exact solutions to the two-loop beta functions that correspond to flows with limit cycles in $D = 4$ spacetime dimensions. These solutions, however, could be destabilized by three-loop effects which have not been studied. Our approach here is to assume the existence of SITs even at strong coupling and use the properties that follow from general considerations.

2. The main ingredient

The SI sector consists of $N \geq 2$ real scalars, $n \geq 1$ Weyl spinors and non-Abelian gauge interactions. The kinetic energy terms for the real scalars and Weyl fermions display $SO(N)$ and $U(n)$ symmetries, respectively, which are broken by the scalars’ quartic self-interactions with coupling constants $\lambda_{abcd}$, the Yukawa interactions with couplings $y_{ai|ij}$, and by gauge interactions. Here $a, b, c, d$ run from 1 through $N$, while $i, j$ run from 1 through $n$. At a scale but not conformal invariant point there are solutions to

$$\beta_{abcd} - Q_{abcd} = 0, \quad (2.1a)$$

$$\beta_{a|ij} - P_{a|ij} = 0. \quad (2.1b)$$

where

$$Q_{abcd} = -Q_{a'a'}\lambda_{a'bc'd} + 3 \text{ permutations}, \quad (2.2a)$$

$$P_{a|ij} = -Q_{a'a'y_{a'|ij}} - P_{i'j'y_{a'|ij}} - P_{j'j'y_{a'|ij'}}. \quad (2.2b)$$
such that the beta functions themselves do not all vanish \[6\]. That is, one can find a value of the coupling constants and a real anti-symmetric matrix \(Q\) and an anti-Hermitian matrix \(P\) such that Eqs. \((2.1)\) are satisfied without setting all terms to zero.

Once a solution has been found, the SI cycle is given by constant gauge couplings and, with \(t = \ln(\mu_0/\mu)\),

\[
\lambda_{abcd}(t) = \hat{Z}_{a'|a}(t)\hat{Z}_{b'|b}(t)\hat{Z}_{c'|c}(t)\hat{Z}_{d'|d}(t)\lambda_{a'b'c'd'},
\]

\[
y_{a|ij}(t) = \hat{Z}_{a|a}(t)\hat{Z}_{b|j}(t)y_{a|b'j'},
\]

where \(\hat{Z}_{ab}(t) = (e^{Qt})_{ab}\) and \(\hat{Z}_{ij}(t) = (e^{Pt})_{ij}\) are clearly elements of \(SO(N)\) and \(U(n)\), respectively.

Let us remark here that scale-invariance requires that \(Q\) and \(P\) be constant. To see that, notice that once a solution to Eqs. \((2.1)\) is found, then Eqs. \((2.1)\) remain satisfied even if the running couplings \((2.3)\) are used. Therefore, if one finds a scale-invariant point, one can be sure that there is a trajectory going through that point with the same \(Q\) and \(P\). Trajectories with \(Q\) and/or \(P\) that are functions of RG time are not possible, for then those, at every value of \(Q\) and \(P\), would intersect trajectories with constant \(Q\) and \(P\).

For simplicity we will assume that \(P = 0\) in what follows. The interesting effects arise from orbits in either of the groups \(SO(N)\) and \(U(n)\), so considering only one will suffice to illustrate the main features of this novel physics.

We assume the reader is familiar with Georgi’s work. In order to study the cyclun-particle analogue all we need is the general form of two-point functions of cyclun-particle operators. For this, it is necessary to specify the transformation properties under \(SO(N)\) and the Lorentz group of the operators in the two-point function. For example, if operators \(O_1\) and \(O_2\) have scaling dimensions \(\Delta_1\) and \(\Delta_2\), respectively, and are scalars under \(SO(N)\), then\(^2\)

\[
\langle O_1(p)O_2(-p) \rangle = C(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta_1 + \Delta_2 - 4)},
\]

where \(C\) is a constant. Here and below we write correlation functions in Minkowski space. By contrast, for an \(SO(N)\)-vector of scalar operators, \(O_a\), with matrix of scaling dimensions \(\Delta_{ab}\), one has

\[
\langle O_a(p)O_b(-p) \rangle = (-p^2 - i\epsilon)^{-2}[(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta + Q)}C(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta - Q)}]_{ab},
\]

where now \(C\) is an \(N \times N\) matrix of constants. Similarly, for an \(SO(N)\)-vector of vector operators \(O^\mu_a\),

\[
\langle O^\mu_a(p)O^\nu_b(-p) \rangle = (-p^2 - i\epsilon)^{-3}[(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta + Q)}(p^2 g^{\mu\nu} C_1 + p^\mu p^\nu C_2)(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta - Q)}]_{ab},
\]

\(^2\)Here and after we define two-point functions in momentum space without the usual \((2\pi)^4 \delta^{(4)}(0)\) factor.
where now both $C_1$ and $C_2$ are $N \times N$ matrices. Note that, as opposed to the case of CFTs, $C_1$ and $C_2$ are independent. (In CFTs one may simultaneously diagonalize $\Delta$, $C_1$ and $C_2$, and then give the entries of $C_2$ in terms of the corresponding diagonal entries in $C_1$ and $\Delta$.) Similar expressions can be immediately written for other $SO(N)$ representations and for correlators with mixed representations, e.g.,

$$\langle O_a(p)O_b(-p) \rangle = (-p^2 - i\epsilon)^{\frac{1}{2}(\Delta'-4)}[(-p^2 - i\epsilon)^{\frac{1}{2}(\Delta+Q)}]_{ab}C_b.$$

(2.6)

3. Cyclunparticle phase space

In his first paper Georgi considers rates $r(X \rightarrow YU)$ for a state $X$ with one or more standard particles to a state $Y$ of one or more standard particles and one unparticle $U$. If the state $X$ consists of a single particle, $r$ stands for a decay rate, else it stands for a cross section. In either

$$dr = \kappa(2\pi)^4\delta(4)(P_X - p_Y - p_U) d\Phi(Y) d\Phi(U) |M|^2,$$

where $P_X$ is the total 4-momentum of the initial state $X$, $M(X \rightarrow YU)$ is the transition amplitude and $\kappa^{-1} = 2\sqrt{P_X^2}$ or $4\sqrt{(P_{X+} \cdot P_{X-})^2 - P_{X+}^2 P_{X-}^2}$ depending on whether $r$ is a decay rate or a two-particle cross section (here $X_+$ and $X_-$). The phase space factor $d\Phi(Y)$ is the product of single (normal) particle ones, $(2\pi)^4 \delta(p^2 - m^2)\theta(p^0) \frac{d^4p}{(2\pi)^4}$ for a particle of mass $m$. The novelty in unparticle physics is the phase space factors for the unparticle. This can be obtained from the absorptive part of the two-point function. For example, for a scalar operator (see Eq. (4.4) in Ref. [4]):

$$d\Phi(U) = C_O(p^2)^{\Delta_O - 2}\theta(p^2)\theta(p^0) \frac{d^4p}{(2\pi)^4},$$

where $\Delta_O$ is the scaling dimension of the CFT operator $O$. The constant $C_O$ depends on the normalization of this operator. While this does not properly belong in the phase space, it does belong in the product of phase space and amplitude-squared, so nothing is lost by including it here rather than in $|M|^2$. In any case, the important point that Georgi makes is that the phase space for the invisible ‘unparticle’ is unusual in that it behaves much like a multi-particle state with fractional particle number $\Delta_O$. This conclusion is independent of the specific $X \rightarrow YU$ process considered.

The extension to cyclunparticles is immediate. One need only replace the unparticle phase space, $d\Phi(U)$, with the cyclunparticle one, $d\Phi(C)$. This can be determined from the absorptive part of the two-point function, obtained by taking the imaginary part of, for
example, Eqs. (2.4)–(2.6). The phase space for the cyclunparticle associated with some operator \( \mathcal{O} \) is

\[
d\Phi(C_\mathcal{O}) = F(p^2)\theta(p^2)\theta(p^0)\frac{d^4p}{(2\pi)^4}
\]

(3.1)

where \( F(p^2) \) is the coefficient of \( \theta(p^2) \) in the imaginary part of the two-point function of \( \mathcal{O} \).

Consider, for example, the case that \( Y \) consists of a single particle of mass \( m \). Then the particle + cyclunparticle production differential rate is

\[
\frac{dr}{dE} = \frac{\kappa}{4\pi^2} |\mathcal{M}|^2 \sqrt{E^2 - m^2} F(p^2),
\]

where \( E = P_X \cdot p_Y / \sqrt{P_X^2} \) is the observable energy in the CM frame and \( p^2 = P_X^2 + m^2 - 2\sqrt{P_X^2}E \). At large CM energy (negligible mass) this simplifies. In terms of the fraction of the total energy in the cyclunparticle, \( \xi = 1 - 2E/\sqrt{P_X^2} \), we have

\[
\frac{dr}{d\xi} = \frac{\kappa}{16\pi^2} |\mathcal{M}|^2 P_X^2 (1 - \xi) F(P_X^2 \xi).
\]

Our task is then to compute the function \( F(p^2) \) for specific cyclunparticles, at least for some specific cases.

Consider for definiteness an \( SO(N) \)-vector, scalar operator in the SIT, \( \mathcal{O}_a(x) \), coupled to an external source \( \chi \) through \( L \supset g_a \chi \mathcal{O}_a + h.c. \). This leads to the following tree-level \( \chi \rightarrow \chi \) forward scattering amplitude:

\[
\mathcal{M}^{\text{fwd}} = g_a g_b |\chi|^2 \left[ (-p^2 - i\epsilon) \frac{1}{2}(\Delta + Q)^{-1} C(-p^2 - i\epsilon) \frac{1}{2}(\Delta - Q)^{-1} \right]_{ab}.
\]

Taking its imaginary part we obtain

\[
F(p^2) = -g_a g_b \left[ (p^2) \frac{1}{2}(\Delta + Q)^{-1} \left\{ \cos \left[ \frac{(\Delta + Q)}{2} \pi \right] C \sin \left[ \frac{(\Delta - Q)}{2} \pi \right] \right. \right.
\]
\[
+ \left. \sin \left[ \frac{(\Delta + Q)}{2} \pi \right] C \cos \left[ \frac{(\Delta - Q)}{2} \pi \right] \} (p^2) \frac{1}{2}(\Delta - Q)^{-1} \right]_{ab}. \tag{3.2}
\]

This, with Eq. (3.1), gives the phase space for the cyclunparticle corresponding to the linear combination of operators \( g_a \mathcal{O}_a \).

Similarly, for an \( SO(N) \)-vector, vector operator \( \mathcal{O}_a^\mu(x) \) in the SIT, coupled to an external source \( \chi_\mu \) through \( L \supset g_a \chi_\mu \mathcal{O}_a^\mu + h.c. \), the forward scattering \( \chi \rightarrow \chi \) amplitude is

\[
\mathcal{M}^{\text{fwd}} = -g_a g_b \left[ (-p^2 - i\epsilon) \frac{1}{2}(\Delta + Q)^{-1} \left( \chi \cdot \chi^\dagger C_1 + \frac{\chi \cdot p}{p^2} C_2 \right) (p^2) \frac{1}{2}(\Delta - Q)^{-1} \right]_{ab}.
\]

We obtain the phase space factor \( F(p^2) \) for the cyclunparticle corresponding to the linear combination of operators \( g_a \epsilon_\mu \mathcal{O}_a^\mu \) by taking the imaginary part of the forward scattering.
amplitude:

\[ F(p^2) = g_ag_b \left( (p^2)^{1/2} \right)^{(\Delta+Q)-1} \left\{ \cos \left[ \left( \frac{\Delta+Q}{2} \right) \pi \right] \tilde{C} \sin \left[ \left( \frac{\Delta-Q}{2} \right) \pi \right] + \sin \left[ \left( \frac{\Delta+Q}{2} \right) \pi \right] \tilde{C} \cos \left[ \left( \frac{\Delta-Q}{2} \right) \pi \right] \right\} (p^2)^{1/2(\Delta-Q)-1} \right]_{ab}, \quad (3.3) \]

where \( \tilde{C} = \epsilon \cdot \epsilon^\dagger C_1 + \frac{|\epsilon \cdot p|^2}{p^2} C_2. \)

4. Cyclunparticle exchange

The second class of examples studied by Georgi corresponds to processes in which both initial and final states contain only standard particles, but there are virtual unparticle contributions to the amplitude.

Consider for definiteness an \( SO(N) \)-vector, scalar operator in the SIT, \( O_a(x) \), coupled to external sources \( \chi^{(1)} \) and \( \chi^{(2)} \) through \( L \supset g_a^{(1)} \chi^{(1)} O_a + g_a^{(2)} \chi^{(2)} O_a + \text{h.c.} \). The sources \( \chi^{(1)} \) and \( \chi^{(2)} \) model the standard particle initial and final states and the coupling constants \( g_a \) characterize the strength of the interaction. This leads to the following \( s \)-channel cyclunparticle exchange contribution to the \( \chi^{(1)} \rightarrow \chi^{(2)} \) scattering amplitude,

\[ M^{\text{cyc}} = g_a^{(1)} g_b^{(2)} \chi^{(1)} \chi^{(2)\dagger} \left[ (-p^2 - i\epsilon)^{1/2(\Delta+Q)-1} C(-p^2 - i\epsilon)^{1/2(\Delta-Q)-1} \right]_{ab}. \]

Similarly, for an \( SO(N) \)-vector, vector operator \( O^\mu_a(x) \) in the SIT, coupled to external sources \( \chi^{(1)}_\mu \) and \( \chi^{(2)}_\mu \) through \( L \supset g_a^{(1)} \chi^{(1)}_\mu O^\mu_a + g_a^{(2)} \chi^{(2)}_\mu O^\mu_a + \text{h.c.} \), the cyclunparticle contribution to the \( s \)-channel scattering amplitude for \( \chi^{(1)} \rightarrow \chi^{(2)} \) amplitude is

\[ M^{\text{cyc}} = -g_a^{(1)} g_b^{(2)} \left[ (-p^2 - i\epsilon)^{1/2(\Delta+Q)-1} \left( \chi^{(1)} \cdot \chi^{(2)\dagger} C_1 \right. \right.

\[ + \left. \left. \chi^{(1)} \cdot \frac{p \chi^{(2)\dagger} \cdot p}{p^2} C_2 \right) (-p^2 - i\epsilon)^{1/2(\Delta-Q)-1} \right]_{ab}. \]

Consider, for example, an amplitude for which the SM gives an \( s \)-channel contribution mediated by a photon, say, \( e^+e^- \rightarrow \mu^+\mu^- \). If a vector cyclunparticle couples to the same currents as the photon, then the interference between the photon and cyclunparticle exchange gives a fractional deviation of the cross section,

\[ \frac{\sigma - \sigma_{\text{SM}}}{\sigma_{\text{SM}}} = 2R + R^2 + I^2, \quad (4.1) \]
where

\[
R = p^2 \frac{g_a^{(1)} g_b^{(2)}}{e^2} \left[ (p^2)^{1/2} (\Delta+Q)^{-1} \left\{ \cos \left[ \left( \frac{\Delta+Q}{2} \right) \pi \right] C_1 \cos \left[ \left( \frac{\Delta-Q}{2} \right) \pi \right] - \sin \left[ \left( \frac{\Delta+Q}{2} \right) \pi \right] C_1 \sin \left[ \left( \frac{\Delta-Q}{2} \right) \pi \right] \right\} (p^2)^{1/2} (\Delta-Q)^{-1} \right]_{ab} ,
\]

\[
I = -p^2 \frac{g_a^{(1)} g_b^{(2)}}{e^2} \left[ (p^2)^{1/2} (\Delta+Q)^{-1} \left\{ \cos \left[ \left( \frac{\Delta+Q}{2} \right) \pi \right] C_1 \sin \left[ \left( \frac{\Delta-Q}{2} \right) \pi \right] + \sin \left[ \left( \frac{\Delta+Q}{2} \right) \pi \right] C_1 \cos \left[ \left( \frac{\Delta-Q}{2} \right) \pi \right] \right\} (p^2)^{1/2} (\Delta-Q)^{-1} \right]_{ab} .
\]

It is straightforward to exhibit examples displaying corrections to processes where other particles are exchanged. To better understand the physical content of these expressions we turn to a more specific case.

5. An SO(2) example

The formal expressions of the previous sections belie their complexity. To better understand them we look at their explicit form in the simplest case, that of \( N = 2 \). Consider first the matrix of vector correlators, Eq. (2.4). Using \( Q_{12} = q \) and \( \Delta = d \mathbb{1} + \gamma \), where \( d \) is the naive dimension of the scalar operators\(^3\) and \( \gamma \) the matrix of anomalous dimensions, we have

\[
\langle O_a(p) O_b(-p) \rangle = (-p^2)^{d-2+1/2 \text{Tr} \gamma}
\times \left( c + \frac{1}{2 \omega} (\gamma_{11} - \gamma_{22}) s \quad \frac{1}{2 \omega} (\gamma_{12} + q) s \quad c - \frac{1}{2 \omega} (\gamma_{11} - \gamma_{22}) s \right)
\times C \left( c + \frac{1}{2 \omega} (\gamma_{11} - \gamma_{22}) s \quad \frac{1}{2 \omega} (\gamma_{12} - q) s \quad c - \frac{1}{2 \omega} (\gamma_{11} - \gamma_{22}) s \right),
\]

where \( c = \cos(\frac{1}{2} \omega \ln(-p^2/\mu^2)) \), \( s = \sin(\frac{1}{2} \omega \ln(-p^2/\mu^2)) \) and \( \omega = \sqrt{q^2 - \frac{1}{4} (\gamma_{11} - \gamma_{22})^2 - \gamma_{12}^2} \). Notice that the two-point function displays oscillations in momentum provided \( \omega \) is real, which in turn requires that \( q^2 > \frac{1}{4} (\gamma_{11} - \gamma_{22})^2 + \gamma_{12}^2 \). Much as in the case of unparticle physics, where large anomalous dimensions are assumed in the strongly interacting CFT, for cyclunparticle physics we assume large \( q \) and \( \gamma \) with real \( \omega \) in the strongly interacting SI model.

\(^3\) We have taken the two scalar operators to have the same naive dimensions. In mass independent subtraction schemes operators with different naive dimension do not mix with each other. While the cross correlators are generically non-vanishing, one is not required to consider them, as would be the case if they mixed under renormalization.
Taking the imaginary part of (5.1) we obtain the function $F(p^2)$ in the phase space (3.1). For example, for the cyclunparticle associated with the operator $g_1 O_1$ the phase space function $F$ is given by

$$F(p^2) = g_1^2 |p^2|^{\frac{1}{2}} \frac{1}{\text{Tr } \Delta} \left\{ 2c_d c_h s_h (c + as)(s - ac) - s_d \left( c_h^2 (c + as)^2 - s_h^2 (s - ac)^2 \right) \right\} C_{11}$$

$$+ 2 \frac{\gamma_{12} + q}{\omega} \left[ c_d c_h s_h (s(s - ac) - c(c + as)) - s_d \left( c_h^2 s(c + as) + s_h^2 c(s - ac) \right) \right] C_{12}$$

$$- \frac{(\gamma_{12} + q)^2}{\omega^2} \left[ 2c_d c_h s_h c s_d c_h^2 s^2 - s_d^2 c^2 \right] C_{22} \right\}.$$  

(5.2)

Here $a = (\gamma_{11} - \gamma_{22})/2\omega$, $c_d = \cos(\frac{1}{2}\pi \text{Tr } \Delta)$, $s_d = \sin(\frac{1}{2}\pi \text{Tr } \Delta)$, $c_h = \cosh(\frac{1}{2}\pi \omega)$, $s_h = \sinh(\frac{1}{2}\pi \omega)$ and $c$ and $s$ are as before (modulo a minus sign), $c = \cos(\frac{1}{2}\omega \ln(p^2/\mu^2))$ and $s = \sin(\frac{1}{2}\omega \ln(p^2/\mu^2))$.

This function displays very unusual behavior. Beyond the scaling corresponding to fractional particle number, familiar from unparticle physics, the phase space exhibits oscillations with angular frequency $\omega$. We show in Fig. 1 the plot of $F$ for a specific choice of parameters (as given in the figure caption).

$$F(p^2)$$

\[ \text{ln } p^2 \]

Fig. 1: Plot of the function $F(p^2)$ determining the phase space for cyclunparticle production per Eq. (3.1), for the case of a cyclunparticle that transforms as the first component of a vector of $SO(2)$. The parameters used in this example are $d = 2$, $\gamma_{11} = 0.6$, $\gamma_{22} = 0.4$, $\gamma_{12} = 0.2$, $q = 1.4$, $C_{11} = -2.9$, $C_{22} = -2.1$ and $C_{12} = 0.2$; see Eq. (5.2). The frequency is $\omega = 1.38$ and two periods are shown in the figure.

A couple of important remarks: first, positivity of $F(p^2)$ is required by unitarity. This,
However, is not guaranteed by the form of the absorptive part of the two-point function. Rather, positivity restricts the parameters of the theory, namely the matrix of dimensions $\Delta$, the coefficient of the virial operator $Q$ and the two-point function normalization matrix $C$. This is not unlike the situation in CFT—unitarity restricts the representations of the conformal group precisely in that the dimensions of operators are restricted \[7\]. The solution to the unitarity problem in SIT, namely, the conditions that $\Delta$, $Q$ and $C$ must satisfy to insure the positivity of the absorptive part of the two-point function at all momenta, is not known even for the simple $N = 2$ example of this section, let alone the general case\[\text{\footnote{Except for $SO(N) \times U(n)$ singlet operators, for which the condition on the dimensions is given in Ref. 4.}}\] That the function $F$ in the example of Fig. 1 is positive only shows a judicious choice of parameters, but the reader can easily find examples for which $F$ fails to remain positive for all momenta.

Second, as pointed out above, oscillatory behavior in the two-point function requires real $\omega$. This imposes conditions on the parameters of the SIT that, however, are a priori independent from the unitarity conditions. But in a Lagrangian formulation of the theory the parameters of the two-point function are derived from the coupling constants and hence are not mutually independent. In that case the unitarity conditions must be automatically satisfied (provided the Hamiltonian is Hermitian), but not so the reality of $\omega$. This begs a question: is $\omega$ real for the few known examples of perturbative SI RG-flows in $D = 4 - \epsilon$? In the (few) known examples the anomalous dimensions for the scalar fields already receive contributions at one-loop order, while the equation for $Q$ requires that one goes to at least two-loops. Hence, $\omega$ is purely imaginary in those cases. While there is no reason to forbid real $\omega$ in a strongly coupled SIT, this possibility is an assumption in this work.

We turn now to an example of a cyclunparticle exchange. Consider the fractional deviation of the cross section for an $s$-channel photon exchange, as in, say, $e^+e^- \rightarrow \mu^+\mu^-$. We have displayed the general expression in Eq. (4.1). For $N = 2$ taking into account only the interference term we can write more explicitly,

$$\frac{\sigma - \sigma_{\text{SM}}}{\sigma_{\text{SM}}} = \frac{2g_1^{(1)}g_1^{(2)}}{\epsilon^2}(p^2)^{\frac{1}{2}}\text{Tr}^{-1}G(p^2),$$

where for a cyclunparticle associated with the first component of the $SO(N)$ vector only
(that is, $g^{(i)}_a = 0$ for $a \neq 1$) we have

$$G(p^2) = \left[ 2s_d c_h s_h (c + a s)(s - a c) + c_d \left( c_h^2 (c + a s)^2 - s_h^2 (s - a c)^2 \right) \right] C_{11}$$

$$+ 2 \frac{\gamma_{12} + q}{\omega} \left[ s_d c_h s_h (s - a c) - c (c + a s) \right] + c_d \left( c_h^2 s (c + a s) + s_h^2 c (s - a c) \right) \right] C_{12}$$

$$- \left( \frac{\gamma_{12} + q}{\omega^2} \right)^2 \left[ 2 s_d c_h s_h c s + c_d \left( c_h^2 s^2 - s_h^2 c^2 \right) \right] C_{22}. \quad (5.4)$$

We show in Fig. 2 a plot of the fractional correction to the cross section (assumed small so that the interference term is dominant) for the same parameters as the example of Fig. 1. The units are arbitrary since the normalization of the cyclunparticle operator is free. Only

\begin{center}
\includegraphics[width=\textwidth]{fig2.png}
\end{center}

**Fig. 2:** Plot of the fractional correction to the cross section, in arbitrary units, for $e^+ e^- \rightarrow \mu^+ \mu^-$ resulting from the exchange of a vector cyclunparticle transforming as the first component of a vector of $SO(2)$, as given per Eqs. (5.3) and (5.4). The parameters used in this example are the same as in Fig. 1 except that $d = 3$. Two periods are shown in the figure.

a range of center of mass energy $\sqrt{p^2/2}$ corresponding to two cycles is shown. Oscillations are apparent. The envelope of the correction grows with energy because the cyclunparticle two-point function scales more slowly than the photon’s.

6. Conclusions

We examined some consequences of coupling a non-conformal scale-invariant sector to the standard model. Our most important result is that the oscillating behavior of non-
conformal scale-invariant correlation functions is physical—it appears both in phase spaces and in amplitudes and cross sections. This leads to novel effects in standard model processes.

A simple example exhibiting oscillations has been provided. Possibly more interesting effects could be achieved by coupling $SO(3)$ cyclunparticles to standard model flavor currents. Clearly, the potential model building applications of non-conformal scale-invariant theories are largely unknown.

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