Abstract. A Boolean algebra $A$ equipped with a (finitely-additive) positive probability measure $m$ can be turned into a metric space $(A, d_m)$, where $d_m(a, b) = m((a \land \neg b) \lor (\neg a \land b))$, for any $a, b \in A$, sometimes referred to as metric Boolean algebra. In this paper, we study under which conditions the space of atoms of a finite metric Boolean algebra can be isometrically embedded in $\mathbb{R}^N$ (for a certain $N$) equipped with the Euclidean metric. In particular, we characterize the topology of the positive measures over a finite algebra $A$ such that the metric space $(\text{At}(A), d_m)$ embeds isometrically in $\mathbb{R}^N$.

1. Introduction

Every Boolean algebra $A$ equipped with a positive probability measure $m$ is a metric space, where the Kolmogorov distance between two elements $a, b \in A$ is given by the value (in $[0, 1]$) assigned by $m$ to the symmetric difference between $a$ and $b$, namely the element $(a \land \neg b) \lor (\neg a \land b)$. Algebras “metrized” by positive measures are called metric Boolean algebras, a nomenclature introduced by Kolmogorov [15], or normed Boolean algebras [21].

The algebraic and topological interest around metric (or normed) Boolean algebras mainly arises from probability theory and its subfield of stochastic geometry [8]. Moreover, these structures have recently found interesting applications in the theory of random sets (see e.g. [18]), which can be represented as a random element taking values in a normed Boolean algebra [7]. Applications of the theory of metric Boolean algebras mostly rely on the idea that they are structures well suited to treat the probability of (two-valued) events (see e.g. [21]). However, the observation that positive probability measures turn an algebra into a metric space adds an importance going far beyond logic and probability. Indeed, in the last few years, the interest around the geometry of discrete metric spaces has relevantly increased. Due to innovative ideas of Gromov [9] and others [1, 3, 13], the study of traditionally relevant geometrical properties usually conceived for Riemannian manifolds, such as length of curves, measures, sectional and Ricci curvatures, has been fruitfully extended to discrete spaces. In this context, the present work focuses on expanding the geometry of metric Boolean algebras and, in particular, to understand for which choices of measures $m$ these metric spaces resemble Euclidean spaces, in the sense that the subspace of atoms of a metric Boolean algebra can be isometrically embedded in the prototypical example of metric space, $\mathbb{R}^N$ (for some $N$) equipped with the Euclidean metric. The choice of the space of the atoms (forcing to confine our attention to atomic algebras only) is driven, on the one side, by its relevance in probability theory, as, in an atomic $\sigma$-algebra of events, the probability of any event depends on the...
probability distribution on the set of atoms and, on the other, by the impossibility of getting
an isometric embedding of the entire algebra (see Remark 5).

The paper is structured as follows: in Section 2 are introduced all the preliminary notions
necessary to go through the reading of the whole paper. Section 3 discusses the details
of isometric embeddings of generic metric spaces into Euclidean spaces (ruled by Morgan’s
theorem) and shows the existence of a probability measure allowing the embedding of the
atoms of a metric Boolean algebra in \( \mathbb{R}^N \), for an appropriate \( N \) (Corollary 12). Section
4 contains the main findings of the paper, namely the study of the topology of the space
of the measures for which the metric space of the atoms of a finite Boolean algebra can be
isometrically embedded in \( \mathbb{R}^N \), for some \( N \): the main finding is that this space is contractible,
while its complement is simply connected but not contractible. Finally, Section 6 contains
the proof of the very useful technical Lemma 10, which is applied throughout the whole
paper, for establishing the existence/non-existence of isometric embeddings in \( \mathbb{R}^N \).

2. Preliminaries: metric Boolean algebras

Let \( A \) be a Boolean algebra equipped with a \textit{strictly positive} (finitely additive) probability
measure, i.e. a map \( m: A \to [0,1] \) such that:

1. \( m(\top) = 1 \),
2. \( m(a \lor b) = m(a) + m(b) \), for every \( a, b \in A \) such that \( a \land b = \bot \),
3. \( m(a) > \bot \), for every \( a \in A, a \neq \bot \).

We indicate with \( \top \) and \( \bot \) the top and bottom element, respectively, of a Boolean algebra
(to avoid confusion with the numbers 0, 1).

The following recalls the well-known properties of probability measures.

**Proposition 1.** Let \( m \) be a finitely additive probability measure over a Boolean algebra \( A \),
then the following hold, for every \( a, b \in A \):

1. \( m \) is monotone, i.e. if \( a \leq b \) then \( m(a) \leq m(b) \);
2. \( m(a) + m(b) = m(a \lor b) + m(a \land b) \);
3. \( m(a \lor b) \leq m(a) + m(b) \);
4. \( m(a') = 1 - m(a) \);
5. if \( m \) is strictly positive, \( a < b \) implies \( m(a) < m(b) \);
6. If \( A \) is finite then \( \sum_{i=1}^{k} m(a_i) = 1 \), where \( \{a_1, \ldots, a_k\} \) is the set of atoms of \( A \).

**Remark 2.** Let \( A \) be a Boolean algebra equipped with a strictly positive (finitely additive)
probability measure \( m \). Then \( (A, d_m) \) is a metric space, where
\[
d_m(a, b) := m(\Delta a b) = m((a \land b') \lor (a' \land b)),
\]
for every \( a, b \in A \).

1) \( d_m(a, b) = d_m(b, a) \) is immediate by commutativity of \( \land \) and \( \lor \).
2) for triangular inequality, first recall that, in any Boolean algebra \( A \), it holds \( a \land c' \leq \)
\( (a \land b') \lor (b \land c') \), for any \( a, b, c \in A \). We have to show that \( m(\Delta a c) \leq m(\Delta a b) + m(b \land c) \).
Observe that \( \Delta a c = (a \land c') \lor (a' \land c) \leq (a \land b') \lor (b \land c') \lor (a' \land b) \lor (b' \land c) = ((a \land b') \lor (a' \land b)) \lor ((b \land c') \lor (b' \land c)) = \Delta a b \lor \Delta b c \). It then follows, from Proposition 1, that
m(a \triangle c) \leq m((a \triangle b) \lor (b \triangle c)) \leq m(a \triangle b) + m(b \triangle c).

3) \ d_m(a, b) = 0 \text{ if and only if } a = b. \text{ Right to left is obvious. For the other direction, suppose } m(a \triangle b) = 0, \text{ then, since } m \text{ is strictly positive, } (a \land b') \lor (a' \land b) = \bot, \text{ i.e. } a \land b' = \bot \text{ and } b \land a' = \bot, \text{ and this means that } b' = a', \text{ thus } a = b. \ \blacksquare

From now on, we always intend a Boolean algebra \( A \) equipped with a strictly positive (finitely additive) probability measure \( m \) and refer to it as metric Boolean algebra, namely as the metric space \((A, d_m)\) (we will not make explicit reference to the metric \( d_m \)). Observe that the assumption that \( m \) is strictly positive is crucial to have a metric space: in case \( m \) is not strictly positive then \((A, d_m)\) is a pseudo-metric space.

In general, not every Boolean algebra can be equipped with a strictly positive (finitely additive) probability measure: a characterization of those Boolean algebras for which such measures exist is due to Kelley [14, Theorem 4] (see also [12]). For the purpose of the present paper, we observe that every atomic Boolean algebra whose sets of atoms is enumerable admits a strictly positive measure [11, Theorem 2.5].

The basic properties of the term-operation \( \triangle \) are recalled in the following.

**Proposition 3.** For any \( a, b \in A \), the following hold:

1) \( a \triangle \bot = a \),

2) \( a \triangle \top = a' \),

3) \( a' \triangle b' = a \triangle b \),

4) \( a \triangle a' = \top \).

We now prove an easy fact about atoms of (atomic) Boolean algebras that will be used in the next section. We indicate by \( \text{At}(A) \) the set of atoms of an atomic Boolean algebra \( A \).

**Lemma 4.** Let \( A \) be an atomic metric Boolean algebra and let \( a, b \in \text{At}(A) \) (with \( a \neq b \)). Then \( d_m(a, b) = m(a) + m(b) \).

**Proof.** Let \( a, b \in \text{At}(A) \), with \( a \neq b \). Since \( a \land b = \bot \), then \( a \leq b' \) and \( b \leq a' \); thus \( d_m(a, b) = m((a \land b') \lor (a' \land b)) = m(a \land b') + m(a' \land b) = m(a) + m(b). \) \ \blacksquare

3. **Isometric embeddings in \( \mathbb{R}^N \)**

It is natural to wonder whether the metric space \((A, d_m)\) embeds isometrically into \( \mathbb{R}^N \) (for some \( N \)) with the Euclidean metric. Unfortunately this is never the case.

**Remark 5.** A Boolean algebra \( A \) (with \(|A| > 2\)) can not be isometrically embedded in \( \mathbb{R}^N \), for any \( N \in \mathbb{N} \), equipped with the Euclidean distance. Indeed, let \( a \in A \) be any element different from the constants. Then \( d_m(a, 0) = m(a), \ d_m(a, 1) = m(a') = 1 - m(a) \) and \( d_m(0, 1) = 1 \), which implies that any isometric embedding maps \( a, 0, 1 \) on the same line, where (for analogous reasoning) should lie also \( a' \). Since \( d_m(a, a') = 1 \), the only possibility is setting \( \iota: A \to \mathbb{R}^N, \ i(a) = i(0) \) and \( i(a') = i(1) \) (or, viceversa), but such an embedding can not be isometric, as \(|\iota(a) - \iota(0)| = 0 \) while \( d_m(a, 0) \neq 0 \). \ \blacksquare

We now turn our attention to a relevant subspace of a metric atomic Boolean algebra, namely the space of \( \text{At}(A) \) of its atoms (with the metric \( d_m \)).
In the proof of Lemma 10, it is actually enough to assume that Remark 11.

\[\] Proof. See Appendix 6.

\[\hat{e}\] where

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Let

\[x\]

A metric space \((X,d)\) is flat if the determinant of the \(n \times n\) matrix \(M(x_0, \ldots, x_n)\), whose generic entry is \(\langle x_i, x_j, x_0 \rangle = \frac{1}{2}(d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2)\), is non-negative for every \(n\)-simplex, namely every choice of \(n + 1\) points \(\{x_0, \ldots, x_n\}\) in \(X\).

Definition 7. The dimension of a space \((X,d)\) is the greatest \(N\) (if exists) such that there exists a \(N\)-simplex with positive determinant.

Theorem 8 (Morgan [19]). A metric space \((X,d)\) embeds in \(\mathbb{R}^N\) if and only if it is flat and has dimension less or equal to \(N\).

Morgan’s theorem is constructive. Indeed, given \((X,d)\) flat and with dimension \(N\), then the embedding into \(\mathbb{R}^N\) is given by:

\[f : X \to \mathbb{R}^N\]

\[x \mapsto (\langle x, x_1, x_0 \rangle, \ldots, \langle x, x_N, x_0 \rangle)\]

for a generic \(n\)-simplex \(\{x_0, x_1, \ldots, x_N\}\).

From now on, our analysis will be confined to finite metric Boolean algebras.

Remark 9. In order to simplify notation, given a (finite) metric Boolean algebra \(A\) with \(k+1\) atoms \(\text{At}(A) = \{a_0, a_1, \ldots, a_k\}\), we set \(x_\alpha = m(a_\alpha)\) (thus \(x_\alpha > 0\), for every \(\alpha \in \{0, 1, \ldots, k\}\)). The matrix \(M(x_0, \ldots, x_n), 2 \leq n \leq k\), introduced in Definition 6, associated to \(A\) has generic entry

\[\langle x_i, x_j, x_0 \rangle = (x_0 + x_i)^2 \delta_{ij} + (x_0^2 + x_0 x_i + x_0 x_j - x_i x_j)(1 - \delta_{ij}).\] (1)

Indeed, \(\langle x_i, x_j, x_0 \rangle = \frac{1}{2}(d_m(a_\alpha, a_i)^2 + d_m(a_\alpha, a_j)^2 - d_m(a_i, a_j)^2)\), and, for \(i = j\), \(d_m(a_i, a_j) = 0\), hence, by Lemma 4, \(\langle x_i, x_j, x_0 \rangle = \frac{1}{2}(2(x_0 + x_i)^2) = (x_0 + x_i)^2\). Else, for \(i \neq j\), \(\langle x_i, x_j, x_0 \rangle = \frac{1}{2}((x_0 + x_i)^2 + (x_0 + x_j)^2 - (x_i + x_j)^2) = \frac{1}{2}(x_0^2 + x_i^2 + 2x_0 x_i + x_0^2 + x_j^2 + 2x_0 x_j - x_i^2 - x_j^2 - 2x_i x_j) = x_0^2 + x_0 x_i + x_0 x_j - x_i x_j.\)

Lemma 10. Let \(A\) be a finite metric atomic Boolean algebra with \(k+1\) atoms and \(M(x_0, \ldots, x_n), 2 \leq n \leq k\) be the matrix defined above. Then

\[\det(M(x_0, \ldots, x_n)) = 2^{n-1} \left[ \left( \sum_{\alpha=0}^{n} x_\alpha \cdots \hat{x}_\alpha \cdots x_n \right)^2 - (n-1) \left( \sum_{\alpha=0}^{n} x_\alpha^2 \cdots \hat{x}_\alpha^2 \cdots x_n^2 \right) \right],\] (2)

where \(\hat{x}_i\) means that \(x_i\) has to be omitted.

Proof. See Appendix 6.

Remark 11. In the proof of Lemma 10, it is actually enough to assume that \(x_0, \ldots, x_n\) satisfies the property introduced in (1); thus from now on, we will weaken the assumption that \(x_0, \ldots, x_n\) are probabilities: we will just assume that they are elements in \((0, 1)\) satisfying equation (1).
Lemma 10 allows to provide a positive answer to the above state question. Indeed, as a corollary we get the embedding of the atoms from a finite metric Boolean algebra which are assigned with the same probability, according to the principle of indifference.

**Corollary 12.** Let \( \mathbf{A} \) be a finite metric Boolean algebra with \( k + 1 \) atoms (\( |\mathbf{A}| = 2^{k+1} \)) and \( m \) a finitely additive probability measure such that \( m(a_i) = \frac{1}{k+1} \), for every \( a_i \in \text{At}(\mathbf{A}) \). Then \( \text{At}(\mathbf{A}) \) embeds in \( \mathbb{R}^k \) with the Euclidean metric.

**Proof.** In virtue of Theorem 8, it is enough to show that \( \text{At}(\mathbf{A}) \) is flat and has dimension \( k \).

By assumption \( x_0 = x_1 = \cdots = x_k = \frac{1}{n+1} > 0 \). Hence, applying Lemma 10, we get

\[
\det(M(x_0, \ldots, x_n)) = 2^{n-1} \left[ ((n+1)x_0^n)^2 - (n-1)(n+1)x_0^{2n} \right] \\
= 2^{n-1} \left[ (n+1)^2 x_0^{2n} - (n^2 - 1)x_0^{2n} \right] \\
= 2^{n-1} 2n x_0^{2n} > 0,
\]

for every \( 2 \leq n \leq k \). \( \square \)

It follows from Theorem 8 and Lemma 10 that the space \( \text{At}(\mathbf{A}) \) of a Boolean algebra \( \mathbf{A} \) with 3 atoms (\( |\mathbf{A}| = 8 \)) embeds in \( \mathbb{R}^2 \) (it always true that a metric space of cardinality 3 embeds isometrically in \( \mathbb{R}^2 \)).

**Remark 13.** It is easy to check that \( \det(M(x_0, x_1, x_2)) > 0 \), a property that we will apply insofar with no explicit mention.

**Remark 14.** Observe that for any \( \lambda \in \mathbb{R} \), \( \det(M(\lambda x_0, \ldots, \lambda x_n)) = \lambda^{2n} \det(M(x_0, \ldots, x_n)) \). Moreover, \( \forall n \ (2 \leq n \leq k) \), \( M(x_0, \ldots, x_n) \geq 0 \) if and only if \( \left( \sum_{\alpha=0}^{n} \frac{1}{x_{\alpha}} \right)^2 - (n-1) \sum_{\alpha=0}^{n} \frac{1}{x_{\alpha}^2} \geq 0 \).

It is not always the case that the space of atoms of a (finite) metric Boolean algebra embeds in \( \mathbb{R}^N \). In the following we consider the probability assignment in accordance with the binomial distribution.

**Example 15.** The binomial distribution (with parameters \( n \) and \( p \)) is the probability distribution of the number of successes in a sequence of \( n \) independent experiments (Bernoulli process), each asking a “yes-no” question, and each with a two-valued outcome: success (with probability \( p \)) or failure (with probability \( q = 1 - p \)). Thus the Boolean algebra of events is \( \mathcal{P}(\Omega) \), where \( \Omega = \{1, \ldots, n\} \) and atoms consists of all the sequences (regardless of the order) of successes and failures. The probability \( x_\alpha = m(a_\alpha) \) of an atom \( a_\alpha \) of \( \mathcal{P}(\Omega) \) is:

\[
x_\alpha = \binom{n}{\alpha} p^\alpha q^{n-\alpha},
\]

with \( p \in (0, 1) \) and \( q = 1 - p \). For the sake of simplicity, we set \( p = q = \frac{1}{2} \). Relying on Remark 14, it is easy to check that \( M(x_0, \ldots, x_3), M(x_0, \ldots, x_4) > 0 \). On the other hand, for \( n = 5 \), we have \( (1 + \frac{1}{5} + \frac{1}{10} + \frac{1}{10} + \frac{1}{5} + 1)^2 - 4(1 + \frac{1}{25} + \frac{1}{100} + \frac{1}{100} + \frac{1}{25} + 1) = -\frac{41}{25} \).
4. Main results

Let $\mathcal{M}(A)$ be the space of probability measures over a finite Boolean algebra $A$, with $|A| = 2^{k+1}$ and $\mathcal{M}(\text{At}(A))$ the space of probability measures over the atoms of $A$ ($|\text{At}(A)| = k+1$). $\mathcal{M}(\text{At}(A)) \subset (0,1)^{k+1}$ is a convex (open) subspace of $(0,1)^{k+1}$ (we are considering only the positive measures). $\mathcal{M}(\text{At}(A))$ can be naturally identified with the space $(0,1)^{k+1}$ by identifying a measure $m \in \mathcal{M}(\text{At}(A))$ with its values $\vec{x} = (x_0, x_1, \ldots, x_k) \in \mathbb{R}_{+}^{k+1}$, $x_\alpha = m(a_\alpha)$, $\alpha = 0, \ldots, k$.

The space $\mathcal{M}(\text{At}(A))$ can be described as:

$$\mathcal{M}(\text{At}(A)) = (0,1)^{k+1} \cap \Pi_k,$$

where $\Pi_k$ is the standard $k$-simplex (or probability simplex) of $\mathbb{R}^{k+1}$, namely

$$\Pi_k = \{ \vec{x} \in (0,1)^{k+1} \mid \sum_{\alpha=0}^{k} x_\alpha = 1 \}.$$

At the light of the above discussion, we define the space of measures $\mathcal{M}_{\text{ind}}(\text{At}(A))$ induced by the flat metric of $\mathbb{R}^N$, namely those measures $m$ such that $(\text{At}(A), d_m)$ admits an isometric embedding into some Euclidean space $\mathbb{R}^N$. By Morgan’s theorem, $m \in \mathcal{M}_{\text{ind}}(\text{At}(A))$ if and only if the metric space $(\text{At}(A), d_m)$ is flat, hence:

$$\mathcal{M}_{\text{ind}}(\text{At}(A)) = \bigcap_{n=3}^{k} C_n \cap \Pi_k,$$

where

$$C_n = \{ \vec{x} \in \mathbb{R}_{+}^{k+1} \mid \det(M(x_0, \ldots, x_n)) \geq 0 \}, \text{ with } 3 \leq n \leq k. \quad (4)$$

Notice that we are taking $\vec{x} \in \mathbb{R}_{+}^{k+1}$ and not $\vec{x} \in (0,1)^{k+1}$. We are interested in the solution of the following.

**Problem.** Study the topology of $\mathcal{M}_{\text{ind}}(\text{At}(A))$, and of its complement $\mathcal{M}(\text{At}(A)) \setminus \mathcal{M}_{\text{ind}}(\text{At}(A))$, with the topology induced by $(0,1)^{k+1} \subset \mathbb{R}^{k+1}$.

The main contribution of the present paper is the solution to the above mentioned problem (see Theorem 20). In order to tackle it, we begin by analyzing the topology of $C_n$.

**Lemma 16.** For each $3 \leq n \leq k$, the space $C_n \cong H_n \times \mathbb{R}_{+}^{k-n}$ where $H_n$ is a solid half-hypercone in $\mathbb{R}_{+}^{n+1}$.

**Proof.** Consider the involutive homeomorphism

$$\Phi : \mathbb{R}_{+}^{k+1} \to \mathbb{R}_{+}^{k+1}, \vec{x} = (x_0, \ldots, x_k) \mapsto \left( \frac{1}{x_0}, \ldots, \frac{1}{x_k} \right). \quad (5)$$

In view of Lemma 10 and Remarks 13 and 14 the image of $C_n$ via $\Phi$ is given by

$$\Phi(C_n) = \{ (z_0, \ldots, z_k) \in \mathbb{R}_{+}^{k+1} \mid (z_0 + \cdots + z_n)^2 - (n-1)(z_0^2 + \cdots + z_n^2) \geq 0, \; 3 \leq n \leq k \},$$
Equivalently,
\[ \Phi(C_n) = \{(z_0, \ldots, z_k) \in \mathbb{R}^{k+1}_+ | (n-2) \sum_{\alpha=0}^n z_\alpha^2 - \sum_{\alpha,\beta=0}^n z_\alpha z_\beta \leq 0\} \]

We claim that, for every \(3 \leq n \leq k\), \(\Phi(C_n)\) is affinely homeomorphic the product of a solid half-hypercone \(H_n \subset \mathbb{R}^{n+1}_+\) with \(\mathbb{R}^{k-n}_+\), i.e. \(\Phi(C_n) \cong H_n \times \mathbb{R}^{k-n} \subset \mathbb{R}^{k+1}_+\). In order to show the claim, observe that

\[ \Phi(C_n) = \{(z_0, \ldots, z_k) \in \mathbb{R}^{k+1}_+ | \tilde{z}^t A \tilde{z} \leq 0\}, \]

where \(A\) is the matrix (of order \(n+1\))

\[ A = \begin{pmatrix}
  n-2 & -1 & \cdots & -1 \\
  -1 & n-2 & \cdots & -1 \\
  \vdots & \vdots & \ddots & \vdots \\
  -1 & -1 & \cdots & n-2 \\
\end{pmatrix}. \]

It is not difficult to check that the eigenvalues of \(A\) are \(\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = n-1\) and \(\lambda_n = -2\) and an orthonormal basis of eigenvectors

\[ v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad v_i = c_i \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad v_n = \frac{1}{\sqrt{n+1}} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \]

with \(c_i = \frac{1}{\sqrt{(i+1)(i+2)}}\), for \(i = \{1, \ldots, n-1\}\), and 1 in the first \(i+1\) entries. Thus if \(P = (v_0 \ldots v_n)\) is the associated orthogonal matrix and \(D\) the diagonal matrix of eigenvalues we can write

\[ \Phi(C_n) = \{(z_0, \ldots, z_k) \in \mathbb{R}^{k+1}_+ | \tilde{z}^t D P^t \tilde{y} \leq 0\} = \{(y_0, \ldots, y_k) \in \mathbb{R}^{k+1} | \tilde{y}^t D \tilde{y} \leq 0\} \cap \{P \tilde{y} > 0\} \]

\[ = \{(y_0, \ldots, y_k) \in \mathbb{R}^{k+1} | (n-1)^2(y_0^2 + \cdots + y_{n-1}^2) - 2y_n^2 \leq 0\} \cap \{P \tilde{y} > 0\}, \]

with \(\tilde{y} = P^t \tilde{z}\). The affine transformation \(P^{-1} = P^t: \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}\) shows that, for every \(3 \leq n \leq k\), \(\Phi(C_n)\) is affinely homeomorphic to

\[ (H_n \cap \mathbb{R}^{n+1}_+) \times \mathbb{R}^{k-n}_+, \]

where \(H_n = \{(y_0, \ldots, y_n) \in \mathbb{R}^{n+1} | (n-1)^2(y_0^2 + \cdots + y_{n-1}^2) - 2y_n^2 \leq 0\}\) is a solid hypercone in \(\mathbb{R}^{n+1}\). In order to prove the claim we have to show that \(H_n \subset \mathbb{R}^{n+1}_+\).

Observe that \(H_n\) is indeed obtained by a rotation around the line generated by \((0, \ldots, 0, 1)\) of angle \(\alpha\), with \(\cos(\alpha) = \sqrt{\frac{n-2}{n}}\). Hence \(P(H_n)\) is a solid hypercone obtained by a rotation around the line generated by \((1, \ldots, 1, 1)\) in \(\mathbb{R}^{n+1}\) of the same angle. On the other hand,
the angle $\alpha_i$ between the vector $(1, \ldots, 1)$ and each vectors of the canonical basis $e_i$ of $\mathbb{R}^{n+1}$ satisfies $\cos(\alpha_i) = \frac{1}{\sqrt{n+1}}$, for every $i \in \{0, \ldots, n\}$. Thus, since $n \geq 3$,

$$\sqrt{\frac{n-2}{n}} = \cos(\alpha) > \cos(\alpha_i) = \frac{1}{\sqrt{n+1}}, \quad i = 0, \ldots, n.$$ 

It follows that $\alpha < \alpha_i$ therefore $P(H_n) \subset \mathbb{R}^{n+1}_+$ proving our claim and concluding the proof of the lemma. ■

In order to provide an answer to the above stated problem, we need to prove some rather technical lemmas, whose proofs consist of adaptations of the proof techniques used in a standard results for compact convex subsets (with non-empty interior) in $\mathbb{R}^k$ (see e.g. [16, Proposition 5.1]).

Recall that a set $X \subseteq S^k$ in the unit $k$-dimensional sphere $S^k \subset \mathbb{R}^{k+1}$ is geodesically convex if, for every $x_1, x_2 \in X$ there exits a (unique) minimal geodesic connecting them in $X$.

**Lemma 17.** Let $X \subseteq S^k_+ = S^k \cap \mathbb{R}^{k+1}_+$ be a compact, geodesically convex set with $\text{Int}(X) \neq \emptyset$. Then any geodesic starting in $p \in \text{Int}(X)$ intersects $X$ in exactly one point.

**Proof.** Let $g$ be a geodesic starting in $p \in \text{Int}(X)$. Since $X$ is compact, then $X \cap g$ is compact and thus bounded. So let $x_0$ be the point having the maximal (spherical) distance from $p$, $x_0 = \max\{d(x, p) \mid x \in X\}$. Let $B_r(p)$ an open (spherical) ball contained in $X$ and $I_c = \{y \in B_r(p) \mid y \in \text{Int}(X)\}$ the geodesic segment connecting $y$ to $x_0$ (it is the “ice-cream cone” formed by the geodesic from $B_r(p)$ to $x_0$). Since $X$ is geodesically convex, then, for every $x \in g \setminus \{x_0\}$, there is an open ball $B_{r_2}(x) \subseteq X$. This holds for every $x \in g \setminus \{x_0\}$ and thus shows that $X \cap g = \{x_0\}$. ■

**Lemma 18.** Let $K \subset \mathbb{R}^{k+1}$ be a compact, star-shaped space with respect to 0, with $\text{Int}(K) \neq \emptyset$ and such that any ray from 0 intersects $K$ in exactly one point. Then there exists a homeomorphism $F : \overline{B_1(0)} \to K$ such that $F(S^k) = \partial K$, where $B_1(0) \subset \mathbb{R}^{k+1}$ denotes the open unit ball centered at the origin.

**Proof.** Define $f : \partial K \to S^k$, $x \mapsto f(x) = \frac{x}{\|x\|}$, i.e. $f(x)$ is the intersection of the ray (from $O$) with the sphere $S^k$. By definition, $f$ is continuous and, since any ray from 0 intersects $S$ in exactly one point, is also bijective. Since $\partial K$ is compact, $f$ is a homeomorphism by the closed map lemma. We then define $F : \overline{B_1(0)} \to K$ as $x \mapsto \begin{cases} |x| f^{-1}(\frac{x}{\|x\|}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ radial segment $\overline{0x}$ with $x \in S^{k-1}$ in the radial segment $\overline{0f^{-1}(x)}$ in $K$, with $f^{-1}(x) \in \partial K$ (it is well defined since $S$ is star-shaped in 0). $F$ is continuous as $f^{-1}$ is and $\lim_{x \to 0} f(x) = 0$; it is injective as any ray from 0 intersects $K$ in exactly one point and surjective as any $y \in K$ belongs to some ray. Finally, since $K$ is compact, $F$ is a homeomorphism by the closed map lemma. ■

**Lemma 19.** Let $K_1, K_2 \subseteq \mathbb{R}^{k+1}$ be compact, star-shaped spaces with respect to 0, with $\text{Int}(K_i) \neq \emptyset$ (for $i = 1, 2$), $K_1 \subset \text{Int}(K_2)$ and such that any ray from 0 intersects $\partial K_i$ (for $i = 1, 2$) in exactly one point. Then there exists a homeomorphism $F : \overline{B_2(0)} \setminus B_1(0) \to \overline{B_1(0)} \setminus B_0(0)$.
Define $S^k$ be compact and contractible being a strong deformation retract of $X$. Moreover, $\partial K_i$ is contractible, observe that, for points $x_2 = \lambda x_1$ (w.l.o.g. $\lambda > 0$) and $F(x_1) = F(x_2)$. Upon setting $F_1(\frac{\bar{x}}{|\bar{x}|}) = a \in \partial K_1$ and $F_2(\frac{\bar{x}}{|\bar{x}|}) = b \in \partial K_1$, we have $(2 - |x_1|)a + (|x_1| - 1)b = (2 - |x_2|)a + (|x_2| - 1)b$, thus $(|x_2| - |x_1|)(a - b) = 0$ and, since $a \neq b$, $|x_1| = |x_2|$, i.e. $x_1 = x_2$ (as $x_2 = \lambda x_1$). Finally, $F$ is a homeomorphism by the closed map lemma. The last part follows by restricting $F$ to $\text{Int}K_2 \setminus K_1$ and by the fact that the annulus $B_2(0) \setminus \overline{B_1(0)}$ is homeomorphic to $S^k \times (0,1)$. 

The solution to the above presented problem is given by the following.

**Theorem 20.** Let $k \geq 3$. Then:

1. $\mathcal{M}_{\text{ind}}(\text{At}(A))$ is contractible.
2. $\mathcal{M}(\text{At}(A)) \setminus \mathcal{M}_{\text{ind}}(\text{At}(A))$ is simply-connected (not contractible).

**Proof.** (1) Consider the (open) retraction
\[
s : \mathbb{R}^{k+1}_+ \to \Pi_k \subset (0,1)^{k+1}, \quad \bar{x} = (x_0, \ldots, x_k) \mapsto \frac{\bar{x}}{s(\bar{x})},
\]
where $s(\bar{x}) := \sum_{\alpha=0}^k x_\alpha$. Then
\[
\mathcal{M}_{\text{ind}}(\text{At}(A)) = \bigcap_{n=3}^k C_n \cap \Pi_k = s\left(\bigcap_{n=3}^k C_n\right)
\]
is contractible being a strong deformation retract of $\bigcap_{n=3}^k C_n$ which is contractible (it is homeomorphic to $\bigcap_{n=3}^k H_n \times \mathbb{R}^{k-n}_+$ by Lemma 16).

(2) Let $C = \bigcap_{n=3}^k C_n$. Then $\mathcal{M}(\text{At}(A)) \setminus \mathcal{M}_{\text{ind}}(\text{At}(A))$ is a strong deformation retract of $\mathbb{R}^{k+1}_+ \setminus C$ using (6). Let $X = C \cap \overline{S^k_+}$. Then $X$ is compact (the intersection of closed sets in the compact $S^k_+$), $C \cong X \times (0, +\infty)$ (by Remark 14) and $\mathbb{R}^{k+1}_+ \setminus C = (S^k_+ \setminus X) \times (0, +\infty)$. Moreover, $X$ has non-empty interior (it follows from Corollary 12 and the proof of Lemma 16 that, for instance, $p = (1, \ldots, 1) \in \text{Int}(X)$) and it is geodesically convex as a subset of $S^k_+$. To see this, consider $x_1, x_2 \in X \subset C$. Let $\frac{\bar{x}_1 - \bar{x}_2}{|\bar{x}_1 - \bar{x}_2|} \in C$ be the segment connecting $x_1, x_2$.
(C is convex) and \( H \subset C \) the hypercone in \( \mathbb{R}^{k+1} \) generated by \( x_1 x_2 \) (the inclusion \( H \subset C \) follows by Remark 14). Then \( H \cap S^k_+ \subset X \) is the minimal geodesic segment in \( S^k \), since it is the intersection of \( S^k_+ \) with an hyperplane containing 0, \( x_1 \) and \( x_2 \). Thus, by Lemma 17, the geodesic from \( p = (1, \ldots, 1) \) intersects \( X \) exactly in one point. Consider the stereographic projection \( \pi \) from the \( S^k_+ \setminus \{-p\} \) to the tangent space \( T_p S^k \) of the sphere \( S^k \) at the point \( p \), namely the homeomorphism which to a point \( x \) of \( S^k_+ \setminus \{-p\} \) associates the intersection of the line joining \( x \) to \(-p\) with \( T_p S^k \). Thus \( K_1 := \pi(X) \) and \( K_2 := \pi(S^k_+) \) are subsets in \( \mathbb{R}^k \cong T_p S^k \) satisfying the assumptions of Lemma 19. It follows that
\[
\text{Int} K_2 \setminus K_1 \cong S^k_+ \setminus X \cong S^{k-1} \times (0,1)
\]
and therefore \( \mathbb{R}^k_+ \setminus C = S^k_+ \setminus X \times (0, +\infty) \cong S^{k-1} \times (0,1) \times (0, +\infty) \) is homotopically equivalent to \( S^{k-1} \), thus simply connected (\( k \geq 3 \)) and not contractible.

**Remark 21.** Consider the set of measure which can be induced in \( \mathbb{R}^N \) with a fixed \( N \). Then one can show that this set is open in \( \Pi_k \) for \( N \geq k \) since each \( C_n \) with \( n \leq k \) is open.

5. Conclusion and future work

The space of **strictly positive** probability measures \( \mathcal{M}(A) \) of a finite Boolean algebra \( A \) is an open convex sets (the boundary being given by all probability measures) in \( [0,1]^n \), with \( n = |A| \). The main contribution of the present work consists in “splitting” such convex space into the (disjoint) union of the contractible set \( \mathcal{M}_{ind}(\text{At}(A)) \), corresponding to those measures \( m \) for which the space (\( \text{At}(A), d_m \)) embeds isometrically in \( \mathbb{R}^N \), and its complement. We have given examples of measures belonging to each of the two components: the measure corresponding to the principle of indifference (see Corollary 12) and the binomial distribution (see Example 15), respectively. As a remarkable consequence of our topological characterization, we get that \( m \in \mathcal{M}_{ind}(\text{At}(A)) \) if and only if \( m \) is homeotopically equivalent to the measure of indifference; on the other hand, all measures \( m \) for which (\( \text{At}(A), d_m \)) can not be isometrically embedded in \( \mathbb{R}^N \) are path connected (though not homeotopically equivalent) to the binomial distribution.

It is natural to wonder in which component of \( \mathcal{M}(\text{At}(A)) \) can be located well-known distributions finding applications in probability (e.g. the hypergeometrical distribution) and exploits the potential applications of the fact that measures belonging to the component \( \mathcal{M}_{ind}(\text{At}(A)) \) have the same homotopy. In order to tackle the former problem, refinements of the useful Lemma 10 shall be found to ease calculations.

Our main result (Theorem 20) relies on the topological characterization (see Lemma 16) of the objects \( C_n = \{ \bar{x} \in \mathbb{R}^k_+ | \det(M(x_0, \ldots, x_n)) \geq 0 \} \) (\( 3 \leq n \leq k \)), for which the assumption about the finiteness of the Boolean algebra considered is crucial (the existence of atoms is only a byproduct). We leave to future work the extension of the present setting to infinite (atomic) Boolean algebras.

Finally, we confined our attention to metric Boolean algebras. However, in the last decades the theory of probability has been extended to algebraic semantics of several non-classical logics, via the development of the so-called theory of states. We will dedicate future work to the study of the metric properties, for instance, of MV-algebras (a study initiated in \( [20, 17] \)) equipped with a faithful state (see e.g. \( [6, 5, 4] \)) or involutive bisemilattices (see \( [2] \)) and their isometric embeddability into Euclidean spaces.
6. Appendix

This section is dedicated to the proof of Lemma 10, whose technicalities are not so important (we believe) for the reading of the whole message of the paper.

Recall that, given a (square) matrix \( A \) of order \( n \), the adjugate \( \text{Adj}(A) \) of \( A \) is the transpose of the cofactor matrix of \( A \). Equivalently, \( \text{Adj}(A) \) is the matrix of order \( n \) such that \( A \cdot \text{Adj}(A) = \text{det}(A) \cdot I_n \), where \( I_n \) is the identity matrix of order \( n \). We recall here a result from linear algebra (see e.g. [10]) that we will use in the proof of Lemma 10.

**Lemma 22** (Matrix determinant lemma). Let \( A \) be a matrix of order \( n \) and \( u,v \) column vectors in \( \mathbb{R}^n \). Then

\[
\text{det}(A + uv^t) = \text{det}(A) + v^t \text{Adj}(A)u.
\]

**Lemma 10.** Let \( A \) be a finite metric atomic Boolean algebra with \( k+1 \) atoms and \( M(x_0, \ldots, x_n) \), \( 2 \leq n \leq k \) be the matrix defined above. Then

\[
\text{det}(M(x_0, \ldots, x_n)) = 2^{n-1} \left[ \left( \sum_{\alpha=0}^{n} x_0 \cdots \hat{x}_\alpha \cdots x_n \right)^2 - (n-1) \left( \sum_{\alpha=0}^{n} x_0^2 \cdots \hat{x}_\alpha^2 \cdots x_n^2 \right) \right],
\]

where \( \hat{x}_i \) means that \( x_i \) has to be omitted.

**Proof.** Preliminarily observe that \( M(x_0, \ldots, x_n) = A + vv^t \), where \( A \) is the matrix whose generic entry is \( A_{ij} = -2(1 - \delta_{ij})x_ix_j \) and \( v = (x_0 + x_1, \ldots, x_0 + x_n) \in \mathbb{R}^n \).

**Claim 1.** \( \text{det}(A) = -2^n x_1^2 \cdots x_n^2(n-1) \).

Observe that \( \text{det}(A) = (-2)^n x_1^2 \cdots x_n^2 \text{det}(B) \), where \( B = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \), i.e. \( B_{ij} = 1 - \delta_{ij} \).

Moreover, \( B = cc^t - I_n \), for \( c = (1, \ldots, 1) \). Thus, by Lemma 22, \( \text{det}(B) = \text{det}(-I_n) + c\text{Adj}(-I_n)c^t \). Observe that \( \text{Adj}(-I_n) = (1)^{n-1}I_n \) (indeed \( -I_n(1)^{n-1}I_n = (1)^nI_n = \text{det}(-I_n) \cdot I_n \)), hence

\[
\text{det}(B) = \text{det}(-I_n) + c^t \text{Adj}(-I_n)c = (1)^n + c^t(-1)^{n-1}I_n c = (1)^n + (-1)^{n-1}c^tc = (1)^n + (1)^{-1}n = (1)^n(n-1).
\]

It then follows that \( \text{det}(A) = (-2)^n x_1^2 \cdots x_n^2 \text{det}(B) = (-2)^n x_1^2 \cdots x_n^2(1)^{-1}(n-1) = -2^n x_1^2 \cdots x_n^2(n-1) \), showing Claim 1.

**Claim 2.** \( \text{Adj}(A) = D \), with generic entry

\[
D_{ij} = 2^{n-1} x_1^2 \cdots x_n^2 \left( \frac{1}{x_i x_j} - \frac{\delta_{ij}}{x_i^2(n-1)} \right).
\]
By definition of adjugate, $A \cdot D = D \cdot A = \det(A) \cdot \mathbb{I}_n$, equivalently $
sum_{k=1}^{n} A_{ik} D_{kj} = \det(A) \cdot \delta_{ij}.$

$$
\sum_{k=1}^{n} A_{ik} D_{kj} = \sum_{k=1}^{n} -2(1 - \delta_{ik}) x_i x_k \cdot \left( 2^{n-1} x_1^2 \ldots x_n^2 \left( \frac{1}{x_i x_j} - \frac{\delta_{kj}}{x_k^2} (n-1) \right) \right)
$$

$$
= -2^n x_1^2 \ldots x_n^2 \sum_{k=1}^{n} (1 - \delta_{ik}) \left( \frac{x_i}{x_j} - \frac{x_i \delta_{kj}}{x_k} (n-1) \right)
$$

$$
= -2^n x_1^2 \ldots x_n^2 \left( \frac{x_i}{x_j} \sum_{k=1}^{n} (1 - \delta_{ik}) - x_i (n-1) \sum_{k=1}^{n} \frac{(1 - \delta_{ik}) \delta_{kj}}{x_k} \right)
$$

$$
= -2^n x_1^2 \ldots x_n^2 \left( \frac{x_i}{x_j} (n-1) - \frac{x_i}{x_j} (n-1) + x_i (n-1) \sum_{k=1}^{n} \frac{\delta_{ik} \delta_{kj}}{x_k} \right)
$$

$$
= -2^n x_1^2 \ldots x_n^2 (n-1) x_i \sum_{k=1}^{n} \frac{\delta_{ik} \delta_{kj}}{x_k} = \det(A) \cdot \delta_{ij}, \text{ which shows Claim 2.}
$$

To simplify notation, let us fix $E_{ij} = \frac{1}{x_i x_j} - \frac{\delta_{ij}}{x_i^2} (n-1)$ (a part of the generic entry of $D$). In order to conclude, observe that, by Lemma 22, $\det(M(x_0, \ldots, x_n)) = \det(A) + v D v^t$

$$
= -2^n x_1^2 \ldots x_n^2 (n-1) + 2^{n-1} x_1^2 \ldots x_n^2 v E v^t \quad \text{(Claims 1, 2)}
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 (-2(n-1) + v E v^t)
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 \left( -2(n-1) + \sum_{i,j=1}^{n} (x_0 + x_i) E_{ij} (x_0 + x_j) \right)
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 \left( -2(n-1) + \sum_{i,j=1}^{n} (x_0 + x_i) \left( \frac{1}{x_i x_j} - \frac{\delta_{ij}}{x_i^2} (n-1) \right) (x_0 + x_j) \right)
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 \left( -2(n-1) + \sum_{i,j=1}^{n} \left( \frac{x_0}{x_i} + 1 \right) \left( \frac{x_0}{x_j} + 1 \right) - \sum_{i,j=1}^{n} \left( \frac{x_0}{x_i} + 1 \right) (x_0 + x_j) \frac{\delta_{ij} (n-1)}{x_i} \right)
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 \left( -2(n-1) + \left[ \sum_{i=1}^{n} \left( \frac{x_0}{x_i} + 1 \right) \right]^2 - (n-1) \sum_{i=1}^{n} \left( \frac{x_0}{x_i} + 1 \right)^2 \right)
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 \left( -2(n-1) + \left[ n - 1 + \sum_{i=1}^{n} \frac{x_0}{x_i} \right]^2 - (n-1) \left( \sum_{i=1}^{n} \frac{x_0^2}{x_i^2} + 2 \sum_{i=1}^{n} \frac{x_0}{x_i} + n + 1 - 1 \right) \right)
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 (-2(n-1) + \left( 1 + \sum_{i=1}^{n} \frac{x_0}{x_i} \right)^2 + (n-1)^2 + 2(n-1) \sum_{i=1}^{n} \frac{x_0}{x_i} + 2(n-1) -
$$

$$
- (n-1) \sum_{i=1}^{n} \frac{x_0^2}{x_i^2} - 2(n-1) \sum_{i=1}^{n} \frac{x_0}{x_i} - (n-1)^2 - (n-1))
$$

$$
= -2^{n-1} x_1^2 \ldots x_n^2 \left[ \left( 1 + \sum_{i=1}^{n} \frac{x_0}{x_i} \right)^2 - (n-1) \left( 1 + \sum_{i=1}^{n} \frac{x_0^2}{x_i^2} \right) \right]$$
\begin{align*}
&= -2^{n-1} \left[ \left( x_1 \ldots x_n + \sum_{i=1}^{n} \frac{x_0 x_1 \ldots x_n}{x_i} \right)^2 - (n-1) \left( x_1^2 \ldots x_n^2 + \sum_{i=1}^{n} \frac{x_0^2 x_1^2 \ldots x_n^2}{x_i^2} \right) \right] \\
&= 2^{n-1} \left[ \left( \sum_{\alpha=0}^{n} x_0 \ldots \hat{x}_j \ldots x_n \right)^2 - (n-1) \left( \sum_{\alpha=0}^{n} x_0^2 \ldots \hat{x}_j^2 \ldots x_n^2 \right) \right].
\end{align*}

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