ABSTRACT

Switched systems are known to exhibit subtle (in)stability behaviors requiring system designers to carefully analyze the stability of closed-loop systems that arise from their proposed switching control laws. This paper presents a formal approach for verifying switched system stability that blends classical ideas from the controls and verification literature using differential dynamic logic (dl), a logic for deductive verification of hybrid systems. From controls, we use standard stability notions for various classes of switching mechanisms and their corresponding Lyapunov function-based analysis techniques. From verification, we use dl’s ability to verify quantified properties of hybrid systems and dl models of switched systems as looping hybrid programs whose stability can be formally specified and proven by finding appropriate loop invariants, i.e., properties that are preserved across each loop iteration. This blend of ideas enables a trustworthy implementation of switched system stability verification in the Kελmaer X prover based on dl. For standard classes of switching mechanisms, the implementation provides fully automated stability proofs, including searching for suitable Lyapunov functions. Moreover, the generality of the deductive approach also enables verification of switching control laws that require non-standard stability arguments through the design of loop invariants that suitably express specific intuitions behind those control laws. This flexibility is demonstrated on three case studies: a model for longitudinal flight control by Branicky, an automatic cruise controller, and Brockett’s nonholonomic integrator.

CCS CONCEPTS

• Theory of computation → Logic and verification: Timed and hybrid models; • Computing methodologies → Computational control theory; • Computer systems organization → Embedded systems.

KEYWORDS

switched system stability, loop invariants, differential dynamic logic

1 INTRODUCTION

Switched systems provide a powerful mathematical paradigm for the design and analysis of discontinuous (or nondifferentiable) control mechanisms [10, 22, 27, 42]. Examples of such mechanisms include: bang-bang controllers that switch between on/off modes; gain schedulers that switch between a family of locally valid linear controllers; and supervisory control, where a supervisor switches between candidate controllers based on logical criteria [22, 27]. However, switched systems are known to exhibit subtle (in)stability behaviors, e.g., switching between stable subsystems can lead to instability [22], so it is important for system designers to adequately justify the stability of their proposed switching designs. Verification and validation are complementary approaches for such justifications: validation approaches, such as system simulations or lab experiments, allow designers to check that their models and controllers conform to real world behavior; verification approaches yield formal mathematical proofs that the stability properties hold for all possible switching decisions everywhere in the model’s infinite state space, not just for finitely-many simulated trajectories.

This paper presents a logic-based, deductive approach for verifying switched system stability under various classes of switching mechanisms. The key insight is that control-theoretic stability arguments for switching control can be formally justified by blending techniques from discrete program verification with continuous differential equations analysis using differential dynamic logic (dl), a logic for deductive verification of hybrid systems [32, 33]. Intuitively, switched systems are modeled in dl as looping hybrid programs [45], as in the following snippet (**^‘ denotes repetition):

\[
\begin{align*}
\{ & u := \text{ctrl}(x); \quad // \text{switching controller (discrete dynamics)} \\
& x' = f_u(x) \quad // \text{actuate decision (continuous dynamics)} \\
\}^*{}^{\text{invariant}}(\ldots) \quad // \text{switching loop with invariant annotation}
\end{align*}
\]

Accordingly, switched system stability is formally specified in dl as first-order quantified safety properties of switching loops (Section 2.2), and the resulting specifications can then be proved rigorously by combining fundamental ideas from verification and control, namely: i) identification of loop invariants (**^‘ invariants above), i.e., properties of the (discrete) loop that are preserved across all executions of the loop body, ii) compositional verification for separately analyzing the discrete and continuous dynamics of the loop body, and iii) Lyapunov functions, i.e., auxiliary energy functions that enable stability analysis for the continuous dynamics.

Section 3 identifies key loop invariants underlying stability arguments for various classes of switching mechanisms and derives sound stability proof rules for those mechanisms. Crucially, these **^‘ derivations are built from dl’s sound foundations for hybrid program reasoning [32, 33], without the need to introduce new...
mathematical concepts such as non-classical weak solutions or non-differentiable Lyapunov functions [9, 16]. The remaining practical challenge is how to (automatically) find suitable Lyapunov function candidates for a given switching mechanism; the correctness of any generated candidates can be soundly checked in dL. Section 4 adds support for switched systems in the KeYmaera X prover based on dL [12], including a modeling interface for switched systems, sum-of-squares search for Lyapunov function candidates [30, 36], and fully automatic verification of stability specifications for standard switching mechanisms. Notably, the implementation requires no extensions to KeYmaera X’s soundness-critical core and thereby directly inherits all of KeYmaera X’s correctness guarantees [12, 25]. This trustworthiness is necessary for computer-aided verification of any generated candidates can be soundly checked in dL. It then explains how stability for these models is formally directly inherits all of KeYmaera X’s correctness guarantees [12, 25].

This section recalls switched systems and their hybrid program models [45]. It then explains how stability for these models is formally directly inherits all of KeYmaera X’s correctness guarantees [12, 25].

2 BACKGROUND

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2.1 Switched Systems as Hybrid Programs

2.1.1 Hybrid Programs. The language of hybrid programs is generated by the following grammar, where \( x \) is a variable, \( e \) is a dL term, and \( Q \) is a formula of first-order real arithmetic [32, 33].

\[
\alpha, \beta ::= x' = f(x) \& Q \mid x := e \mid ?Q \mid \alpha; \beta \mid \alpha \cup \beta \mid \alpha^*
\]

Continuous dynamics are modeled using systems of ordinary differential equations (ODEs) \( x' = f(x) \& Q \) evolving within domain \( Q \); the ODE is written as \( x' = f(x) \) when there is no domain constraint, i.e., \( Q \equiv \top \). Discrete dynamics are modeled using assignments \( x := e \) assigns the value of term \( e \) to \( x \) and tests (?\( Q \) checks whether condition \( Q \) is true in the current state). The program combinators are used to piece together sub-programs to form programs with hybrid dynamics. The combinators are: sequential composition \( (\alpha; \beta) \), controls \( (\alpha \rightarrow \beta) \), nondeterministic choice \((\alpha \lor \beta) \), and nondeterministic repetition \((\alpha^*) \) repeats \( \alpha \) for any number of iterations.

Throughout this paper, \( x = (x_1, \ldots, x_n) \) denotes the vector of continuous state variables for the system under consideration. Other variables are used for program auxiliaries, e.g., to describe memory and timing components of switching controllers.

2.1.2 Switched systems. A switched system is described by a finite family \( P \) of ODEs \( x' = f_p(x) \) and \( p \in P \) and a set of switching signals \( \sigma : [0, \infty) \rightarrow P \) that prescribe the ODE \( x' = f_{\sigma(t)}(x) \) to follow at time \( t \) along the system’s evolution. Tan and Platzer [45] use hybrid programs as formal models for various classes of switching mechanisms; one example is arbitrary switching [22] where the system is allowed to follow any switching signal in order to model real world systems whose switching behavior is uncontrolled or a priori unknown. The hybrid program \( \sigma_{arb} \equiv \bigcup_{p \in P} x' = f_p(x) \) models arbitrary switching analogously to a computer simulation [45, Proposition 1]: on each loop iteration, the program makes a (discrete) nondeterministic choice of switching decision \( \bigcup_{p \in P} \cdot \) to select an ODE \( x' = f_p(x) \) which it then follows continuously for an arbitrarily chosen duration before repeating the simulation loop.

The hybrid programs language can be used to model various other classes of switching mechanisms [22, 45], including general controlled switching, as illustrated in Section 1, where a (discrete) control law \( u = \text{ctrl}(x) \) decides the ODE \( x' = f_u(x) \) to switch to on each loop iteration. Stability for these models is explained next.

2.2 Stability as Quantified Loop Safety

This paper studies uniform global pre-asymptotic stability (UGpAS) for switched systems [16, 17, 22], defined as follows:

Definition 1 (UGpAS [16, 17]). Let \( \Phi(x) \) denote the set of all (domain-obeying) solutions\footnote{A formal construction of the (right-maximal) solution \( \varphi \) for a given switching signal \( \sigma \) is available elsewhere [45, Appendix A].} \( \varphi : [0, T_{\varphi}) \rightarrow \mathbb{R}^n \) for a switched system from state \( x \in \mathbb{R}^n \). The origin \( 0 \in \mathbb{R}^n \) is:

- uniformly stable if, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that from all initial states \( x \in \mathbb{R}^n \) with \( ||x|| < \delta \), all solutions \( \varphi \in \Phi(x) \) satisfy \( ||\varphi(t)|| < \varepsilon \) for all times \( 0 \leq t \leq T_{\varphi} \);
- uniformly globally pre-attractive if, for all \( \varepsilon > 0 \), \( \delta > 0 \), there exists \( T \geq 0 \) such that from all initial states \( x \in \mathbb{R}^n \) with \( ||x|| < \delta \), all solutions \( \varphi \in \Phi(x) \) satisfy \( ||\varphi(t)|| < \varepsilon \) for all times \( T \leq t \leq T_{\varphi} \) and
- uniformly globally pre-asymptotically stable if the system is uniformly stable and uniformly globally pre-attractive.

The UGpAS definition can be understood intuitively for a system with a given switching control mechanism:
• stability means the mechanism keeps the system close to the origin if the system is initially perturbed close to the origin,
• global pre-attractivity means the mechanism drives the system to the origin asymptotically as \( t \to \infty \), and
• uniform means the stability and pre-attractivity properties are independent of both the nondeterminism in the switching mechanism (e.g., arbitrary switching) and the choice of initial states satisfying \( ||x|| < \delta \); for brevity in subsequent sections, “uniform” is elided when describing stability properties.

Remark 1. Switched systems whose solutions are all uniformly bounded in time, i.e., there exists \( T_o \) such that for all solutions \( \phi \), \( T_p \leq T_m \), are trivially pre-attractive. Goebel et al. [16, 17] introduce the notion of pre-attractivity as opposed to attractivity for hybrid systems because it separates considerations about whether a hybrid system’s solutions are complete, i.e., solutions exist for all (forward) time, from conditions for stability and attractivity. Pre-attractivity also sidesteps the difficult question of whether a switched system exhibits Zeno behavior, i.e., where infinitely many discrete switches occur in finite time [22, 48]. Indeed, it is common in the hybrid and switched systems literature to either ignore incomplete solutions or assume the models under consideration only have complete solutions [22, 26, 48]. Instead of predicting proofs on these hypotheses, this paper formalizes the (weaker) notion of UGpAS for switched systems, leaving proofs of completeness of solutions out of scope.

The definition of UGpAS nests alternating quantification over real numbers with temporal quantification over the solutions \( \phi \) of switched systems. This combination of quantifiers can be expressed formally using the formula language of \( dL \) [32, 33], whose grammar is shown below, \( \sim \in \{=, \neq, \geq, >, \leq, <\} \) is a comparison operator between \( dL \) terms \( e, e' \) and \( \alpha \) is a hybrid program:

\[
\phi, \psi ::= e \sim e \mid \phi \land \psi \mid \phi \lor \psi \mid \neg \phi \mid \forall \nu \phi \mid \exists \nu \phi \mid [\alpha]\phi \mid (\alpha)\phi
\]

This grammar extends the first-order language of real arithmetic (FOL\(_R\)) with the box \((\alpha)\phi\) and diamond \((\langle\alpha\rangle\phi)\) modality formulas which express that all or some runs of hybrid program \( \alpha \) satisfy postcondition \( \phi \), respectively. Real arithmetic FOL\(_R\) is decidable by quantifier elimination [46] and serves as a useful base specification language. Various specifications are equivalently definable in FOL\(_R\), e.g., Euclidean norm bounds \( ||x|| \sim e \equiv (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} \sim e^2 \) (for \( e \geq 0 \)) and topological operations such as the boundary \( \partial \phi \) and closure \( \overline{\phi} \) of the set characterized by formula \( \phi \) [3].

The box modality formula \([\alpha]\phi\) expresses safety properties \( \phi \) of program \( \alpha \) that must hold along all of its executions [33]. When \( \alpha \) models a switched system, the box modality quantifies (uniformly) over all times for all solutions arising from the switching mechanism. Accordingly, UGpAS for switched systems is formally specified by nesting the box modality with the first-order quantifiers.

Lemma 2 (UGpAS in differential dynamic logic). The origin \( 0 \in \mathbb{R}^n \) for a switched system modeled by program \( \alpha \) is UGpAS iff the \( dL \) formula UGpAS(\( \alpha \)) is valid. Variables \( e, \delta, T, t \) are fresh in \( \alpha \):

\[
\text{UGpAS}(\alpha) \equiv \text{UStab}(\alpha) \land \text{UGpAttr}(\alpha)
\]

Here, UStab(\( \alpha \)) and UGpAttr(\( \alpha \)) characterize stability and global pre-attractivity of \( \alpha \), respectively. In UGpAttr(\( \alpha \)), \( \alpha, t' = 1 \) denotes the hybrid program obtained from \( \alpha \) by augmenting its continuous dynamics so that variable \( t \) tracks the progression of time.

Formulas UStab(\( \alpha \)) and UGpAttr(\( \alpha \)) syntactically formalize in \( dL \) the corresponding quantifiers in Def. 1. In UGpAttr(\( \alpha \)), the fresh clock variable \( t \) is initialized to 0 and syntactically tracks the progression of time along switched system solutions. The program \( \alpha, t' = 1 \) can, e.g., be constructed by adding a clock ODE \( t' = 1 \) to all ODEs in the switched system model \( \alpha \). Accordingly, the post-condition \( t \geq T \to ||x|| < e \) expresses that the state system norm is bounded by \( e \) after \( T \) time units along any switching trajectory, as required in Def. 1. Various other stability notions are of interest in the continuous and hybrid systems literature [13, 17, 22, 28, 35, 42, 44]. These variations can also be formally specified in \( dL \) [44] but are left out of scope for this paper.

2.3 Proof Calculus

The \( dL \) proof calculus enables formal, deductive verification of UGpAS stability specifications through compositional reasoning principles for hybrid programs [32, 33] and a complete axiomatization for ODE invariants [34]. For example, an important syntactic tool for differential equations reasoning is the Lie derivative of term \( e \) along ODE \( x' = f(x) \), defined as \( \mathcal{L}_f(e) \equiv \nabla e \cdot f \). The sound calculation and manipulation of Lie derivatives is enabled in \( dL \) through the use of syntactic differentials [32].

All proofs are presented in a classical sequent calculus with the usual rules for manipulating logical connectives and sequents. The semantics of \( \text{sequent } \Gamma \vdash \phi \) is equivalent to the formula \( (\bigwedge \psi \in \Gamma \psi) \rightarrow \phi \) and a sequent is valid iff its corresponding formula is valid. The key (derived) \( dL \) proof rule used in this paper is:

\[
\begin{array}{c}
\text{loop} \\
\Gamma \vdash \text{Inv} \\
\text{Inv} \vdash [\alpha] \text{Inv} \\
\text{Inv} \vdash \phi \\
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash \phi^* \phi \\
\Gamma \vdash \phi^* \phi \\
\end{array}
\]

The loop rule says that, in order to prove validity of the conclusion (below the rule bar), it suffices to prove the three premises (above the rule bar), respectively from left to right: i) the initial assumptions \( \Gamma \vdash \text{Inv} \), ii) \( \text{Inv} \) is preserved across the loop body \( \alpha \), i.e., \( \text{Inv} \) is a loop invariant for \( \alpha^* \), and iii) \( \text{Inv} \) implies the postcondition \( \phi \). The identification of loop invariants \( \text{Inv} \) is crucial for formal proofs of UGpAS, as illustrated by the following deductive proof skeleton for stability (a similar skeleton is used for pre-attractivity):

**Deduction**

\[
\begin{array}{c}
\begin{array}{l}
\Gamma_1 \vdash \phi_1 \\
\vdots \\
\Gamma_k \vdash \phi_k \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{l}
\Gamma_1 \vdash \text{Inv} \\
\text{Inv} \vdash [\alpha] \text{Inv} \\
\text{Inv} \vdash ||x|| < e \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{l}
\text{loop} \\
\Gamma \vdash \phi^* \phi \\
\Gamma \vdash \phi^* \phi \\
\end{array}
\end{array}
\]

Proofs proceed upwards by deduction, where each reasoning step is justified by sound \( dL \) axioms and rules of inference, e.g., the loop rule. The proof skeleton above syntactically derives a proof rule that reduces a stability proof for \( \alpha^* \) to proofs of its top-most premises, \( \Gamma_1 \vdash \phi_1 \cdots \Gamma_k \vdash \phi_k \). These correspond to required logical
3 LOOP INVARIANTS FOR SWITCHED SYSTEM STABILITY

This section identifies loop invariants for proving UGpAS under various classes of switching mechanisms with Lyapunov functions [5, 21, 22]; relevant mathematical arguments are presented briefly (see supplement [43]). Throughout the section, loop invariants are progressively tweaked to account for new design insights behind increasingly complex switching mechanisms.

3.1 Arbitrary and State-Dependent Switching

3.1.1 Arbitrary Switching. Stability for the arbitrary switching model $\alpha_{arb}$ from Section 2 can be verified by finding a so-called common Lyapunov function $V$ for all of the ODEs $x' = f_p(x), p \in P$ satisfying the following arithmetical conditions [22, 42]:

i) $V(0) = 0$ and $V(x) > 0$ for all $\|x\| > 0$,

ii) $V$ is radially unbounded, i.e., for all $b$, there exists $\gamma > 0$ such that $\|x\| < \gamma$ for all $V(x) \leq b$, and

iii) for each ODE $x' = f_p(x), p \in P$, the Lie derivative $L_{f_p}(V)$ satisfies: $L_{f_p}(V)(0) = 0$ and $L_{f_p}(V)(x) < 0$ for all $\|x\| > 0$.

Conditions i)–iii) are generalizations of well-known conditions for stability of ODEs [8, 21] to arbitrary switching. Intuitively, conditions i) and iii) ensure that $V$ acts as an auxiliary energy function whose value decreases asymptotically to zero (at the origin) along all switching trajectories of the system; the radial unboundedness condition ii) ensures that this argument applies to all system states for global pre-attractivity [21]. Correctness of these conditions can be proved in dL using loop invariants, see Fig. 1 (explained below).

**Stability.** The specification $\text{UStab}(\alpha_{arb})$ requires that all trajectories of $\alpha_{arb}$ stay in the grey ball $\|x\| < \epsilon$, starting from a chosen ball $\|x\| < \delta$, see Fig. 1 (left). Condition i) guarantees that the ball $\|x\| < \epsilon$ contains (a connected component of) the sublevel set $V < W$ for some $W > 0$ (dashed blue curve) and this sublevel set contains a smaller ball $\|x\| < \delta$ [8, 21]. Condition iii) shows that this sublevel set is invariant for every ODE $x' = f_p(x), p \in P$ because $L_{f_p}(V)(x) \leq 0$, as illustrated by the dashed black and green arrows for two different switching choices $p \in P$ both locally pointing inwards on the boundary of the sublevel set. Thus, the formula $\text{Inv}_{V} \equiv \|x\| < \epsilon \land V < W$, which characterizes the blue sublevel set, is an invariant for all possible switching choices in the loop body of $\alpha_{arb}$, which makes $\text{Inv}_{V}$ a suitable loop invariant for $\text{UStab}(\alpha_{arb})$.

**Pre-attractivity.** The specification $\text{UGpAttr}(\alpha_{arb})$ requires that all trajectories of $\alpha_{arb}$ stay in the grey ball $\|x\| < \epsilon$ after a chosen time $T$, starting from the initial ball $\|x\| < \delta$, see Fig. 1 (right). The ball $\|x\| < \delta$ is bounded, so it is contained in a sublevel set satisfying $V < W$ for some $W > 0$ (outer dashed blue curve); this sublevel set is bounded by condition ii). Like the stability argument, condition i) guarantees that there is a sublevel set $V < U$ for some $U > 0$ (inner dashed blue curve) contained in the ball $\|x\| < \epsilon$, and condition iii) shows that the sublevel sets characterized by $V < W$ and $V < U$ are both invariants for every ODE in the loop body of $\alpha_{arb}$. The set characterized by formula $V \geq U \land V \leq W$ is compact and bounded away from the origin, which implies by condition iii) that there is a uniform bound $\delta > 0$ on this set, where for each ODE $x' = f_p(x), p \in P$, $L_{f_p}(V)(x) \leq k$. Thus, the value of Lyapunov function $V$ decreases at rate $k$, regardless of switching choices in the loop body of $\alpha_{arb}$, as long as it has not entered $V < U$. The loop invariant for $\text{UGpAttr}(\alpha_{arb})$ syntactically expresses this intuition: $\text{Inv}_{V} \equiv V < W \land (V \geq U \rightarrow V < W + kT)$. For a sufficiently large choice of $T$ with $W + kT \leq U$, trajectories at time $t \geq T$ satisfy $V < U$ so they are contained in the $\|x\| < \epsilon$ ball.

The loop invariants identified above enable derivation of a formal dL stability proof rule for $\alpha_{arb}$ (deferred to a more general version in Corollary 3 below). In fact, since arbitrary switching is the most permissive form of switching [22], UGpAS for any switching mechanism can be soundly justified using the loop invariants above in case a suitable common Lyapunov function can be found.

3.1.2 State-dependent Switching. The state-dependent switching mechanism [22] constrains arbitrary switching by allowing execution of (and switching to) an ODE $x' = f_p(x), p \in P$ only when the system state is in domain $Q_p$. This is modeled by the hybrid program $\alpha_{state} \equiv \bigcup_{p \in P} x' = f_p(x) & Q_p$ [45, Proposition 2], where arbitrary switching $\alpha_{arb}$ corresponds to the special case with $Q_p \equiv \text{true}$ for all $p \in P$.

The same loop invariants for $\alpha_{arb}$ are used for $\alpha_{state}$ to derive the following proof rule. For brevity, premises of all derived stability proof rules are implicitly conjunctively quantified over $p \in P$.

**Corollary 3 (UGpAS for state-dependent switching, CLF).** The following proof rule for common Lyapunov function $V$ with three stacked premises is syntactically derivable in dL.

- $V(0) = 0 \land \forall x (\|x\| > 0 \rightarrow V(x) > 0)$
- $\forall b \exists \gamma (\forall x (V(x) \leq b \rightarrow \|x\| \leq \gamma))$
- $L_{f_p}(V)(0) = 0 \land \forall x (\|x\| > 0 \land Q_p \rightarrow L_{f_p}(V)(x) < 0)$

$\vdash \text{UGpAS}(\alpha_{state})$

Corollary 3 syntactically derives a slight generalization of conditions i)–iii) from Section 3.1.1 for $\alpha_{state}$, where the Lie derivatives $L_{f_p}(V)(x)$ for each $p \in P$ are required to be negative on their respective domain closures$^2$. This generalization is justified by the

$^2$The topological closure $\overline{Q}$ of domain $Q$ is needed for soundness of a technical compactness argument used in the pre-attractivity proof (see supplement [43]).
Verifying Switched System Stability With Logic

with four stacked premises is syntactically derivable in $\text{MLF}$. The following proof rule for multiple Lyapunov functions $V_p, p \in \mathcal{P}$ with four stacked premises is syntactically derivable in $\text{DL}$.

\[ \vdash V_p(0) \iff 0 \land \forall x \left( \|x\| > 0 \rightarrow V_p(x) > 0 \right) \]

\[ \vdash \forall y \exists x \left( V_p(x) \leq y \rightarrow \|x\| \leq y \right) \]

\[ \vdash \mathcal{L}_f_p(V_p)(0) = 0 \land \forall x \left( \|x\| > 0 \land Q_p \rightarrow \mathcal{L}_f_p(V_p)(x) < 0 \right) \]

\[ \vdash \mathcal{L}_f_p(V_p)(0) = 0 \land \forall x \left( \|x\| > 0 \land Q_p \rightarrow \mathcal{L}_f_p(V_p)(x) < 0 \right) \]

\[ \vdash \text{UGpas}(\alpha_{c\text{state}}) \]

The top three premises of Corollary 4 are similar to those of Corollary 3, but are now required to hold for each Lyapunov function $V_p, p \in \mathcal{P}$ separately. The (new) bottom premise corresponds to a compatibility condition between the Lyapunov functions arising from the loop invariants. For example, the stability loop invariant (similarly for pre-attractivity) and suppose the system currently satisfies disjunct $Q_p \land V_p < W$ with $V_p$ justifying stability in domain $Q_p$. If the system switches to the ODE $x' = f_p(x)$ within domain $Q_p$, then Lyapunov function $V_p$ becomes the active Lyapunov function which must satisfy $V_p < W$ to preserve the stability loop invariant. The premise $Q_p \land Q_q \rightarrow V_p = V_q$ says that the Lyapunov functions $V_p, V_q$ are equal whenever such a switch is possible (in either direction), i.e., when their domains overlap.

3.2 Controlled Switching

This section turns to controlled switching models [45], where an explicit controller program is responsible for making logical switching decisions between the ODEs $x' = f_p(x), p \in \mathcal{P}$. This is in contrast to earlier models $\alpha_{\text{state}}, \alpha_{\text{state}}$ which exhibit autonomous switching, i.e., without an explicit control logic [6, 22]. General controlled switching is modeled by the hybrid program $\alpha_{c\text{trl}}$:

\[
\begin{align*}
\text{switching controller} & \quad \alpha_p (\text{plant, actuate decision}) \\
\alpha_{c\text{trl}} & \equiv \alpha_{i}(: \bigcup_{p \in \mathcal{P}} (\alpha_{i} : (\mathcal{L}(p = p, x' = f_p(x, y), y' = g_p(x, y) \\& Q_p)) \}
\end{align*}
\]

The model $\alpha_{c\text{trl}}$ uses three subprograms: $\alpha_i$ initializes the system, then $\alpha_p$ (modeling the switching controller) and $\alpha_d$ (modeling the continuous plant dynamics) are run in a switching loop. The discrete programs $\alpha_i, \alpha_d$ decide on values for the control output $u = p, p \in \mathcal{P}$ and the program $\alpha_p$ responds to this output by evolving the corresponding ODE $x' = f_p(x, y), y' = g_p(x, y) \land Q_p$. The programs $\alpha_i, \alpha_d$ must not modify the system state variables $x$, but they may modify other auxiliaries, including auxiliary continuous state variables $q$ used to model timers or integral terms used in controllers, see Section 5.2. This control-plant loop is a typical structure for hybrid systems modeled in $\text{DL}$ [31, 33], e.g., the controller $\alpha_u$ below models the discrete switching logic present in hybrid automata [6, 18, 31] (without jumps in the system state):

\[
\begin{align*}
\alpha_u & \equiv \bigcup_{p \in \mathcal{P}} \left( \mathcal{L}(p = p, q = q, q = q) ; R_p(q, u := q) \right) \\
R_p.q & \equiv y_1 := e_1 ; y_2 := e_2 ; \ldots ; y_k := e_k
\end{align*}
\]

For each mode $p \in \mathcal{P}$, the switching controller may nondeterministically switch to mode $q \in \mathcal{P}$ if the guard formula $G_p, q$ is true in the current state ($G_p, q \equiv \text{true for self-transitions}$); if the transition is taken, the reset map $R_p.q$ sets the values of auxiliary state variables $y_1, \ldots , y_k$ respectively to the value of terms $e_1, \ldots , e_k$.

Stability analysis for controlled switching proceeds by identifying suitable loop invariants $\text{Inv}$ for $\alpha_{c\text{trl}}$. A powerful proof technique applied here is compositional reasoning [31, 33] which separately analyses the discrete ($\alpha_i, \alpha_d$) and continuous ($\alpha_p$) dynamics, and then lifts those results to the full hybrid dynamics. This idea is exemplified by the following derived variation of the loop rule:

\[
\begin{align*}
\Gamma & \vdash \left[ \alpha_i \right] \text{Inv} \text{Inv} \vdash \left[ \alpha_d \right] \text{Inv} \text{Inv} \vdash \left[ \alpha_p \right] \text{Inv} \text{Inv} \vdash \phi \\
\Gamma & \vdash \left[ \alpha_i \alpha_d \alpha_p \right] \phi
\end{align*}
\]

The premises of rule $\text{loop}$ say that system initialization $\alpha_i$ puts the system in a state satisfying the invariant $\text{Inv}$, and that $\text{Inv}$ is...
compositional preservation by both the discrete switching logic $\alpha_d$ and the continuous dynamics $\sigma_n$. This rule is applied to analyze stability for two important special instances of $\alpha_{dtr}$ next.

### 3.2.1 Guarded State-dependent Switching

The instance $\alpha_{guard}$ corresponds to the automata controller from (1) with $\alpha_l \equiv \bigcup_{p \in \mathcal{P}} u := p$ and guard formulas $G_{p,q}$. It does not use auxiliary $y$ nor the reset map $R_{p,q}$. This model adds hysteresis [19] to the state-dependent switching model from Section 3.1.2, so that switching decisions at each $G_{p,q}$ depend explicitly on the current discrete mode $u$ in addition to the continuous state. This design change is reflected in the loop invariants and in the corresponding proof rule below.

**Stability.** The stability loop invariant is modified (cf. Section 3.1.2) to come split to possible discrete modes $u = p$ rather than the ODE domains: $Inv_a \equiv \|x\| < \varepsilon \land \forall \ p \in \mathcal{P} \ (u = p \land V_p < W)$.

**Pre-attractivity.** The pre-attractivity loop invariant is modified similarly: $Inv_a \equiv \forall \ p \in \mathcal{P} \ (u = p \land V_p < W \land (V_p \geq U \rightarrow V_p < W + k t))$.

**Corollary 5** (UGPAS for Guarded State-dependent Switching, MLF). The following proof rule for multiple Lyapunov functions $V_p, p \in \mathcal{P}$ with four stacked premises is syntactically derivable in $\mathcal{L}$. The proof skeleton shows the unfolding for the stability loop invariant $Inv_a$ corresponding to a switch from mode $p$ to mode $q$:

$\vdash V_p(0) = 0 \land \forall x (\|x\| > 0 \rightarrow V_p(x) > 0)$

$\vdash \forall \ p \in \mathcal{P} V_p(x) \leq b \rightarrow \|x\| \leq y$.

$\vdash \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

### 3.2.2 Time-dependent Switching

The instance $\alpha_{time}$ shown below models time-dependent switching, where the controller $\alpha_{tr}$ makes switching decisions based on the time $\tau$ elapsed in each mode.

**Stability.** The stability loop invariant expresses the required exponential bounds with a case split depending if $p \in S$ or $p \in \mathcal{U}$:

$Inv_{a} \equiv \forall \ p \in \mathcal{P} \ (u = p \land V_p < W e^{-\lambda_p \tau} \forall \ p \in S$

$\vdash \bigcup_{p \in \mathcal{P}} [\alpha_{tr}] Inv_a$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow V_p < W e^{-\lambda_p \tau}$

$\vdash \bigcup_{p \in \mathcal{P}} [\alpha_{tr}] Inv_a$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow V_p < W e^{-\lambda_p \tau}$

**Pre-attractivity.** The pre-attractivity loop invariant has similar exponential decay and growth bounds for each $p \in \mathcal{P}$ in the current mode. In addition, it has an overall exponential decay term $e^{-\sigma(t-t_r)}$ for some $\sigma > 0$, which ensures that the value of $V_p$ tends to 0 as $t \rightarrow \infty$ for all switching trajectories; recall $t$ is the global clock introduced in the specification of Pre-attractivity in Lemma 2.

$Inv_{a} \equiv \forall \ p \in \mathcal{P} \ (u = p \land V_p < W e^{-\sigma(t-t_r)} e^{-\lambda_p \tau})$

$\vdash \bigcup_{p \in \mathcal{P}} [\alpha_{tr}] Inv_a$

**Corollary 6** (UGPAS for Time-dependent Switching, MLF). The following proof rule for multiple Lyapunov functions $V_p, p \in \mathcal{P}$ with five stacked premises is syntactically derivable in $\mathcal{L}$.

$\vdash V_p(0) = 0 \land \forall x (\|x\| > 0 \rightarrow V_p(x) > 0)$

$\vdash \forall \ p \in \mathcal{P} G_{p,q} \rightarrow \forall \ x \in S \ (V_p(x) \leq b \rightarrow \|x\| \leq y)$

$\vdash \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

$\vdash \mathcal{L}_{dp}(V_p(x)) \rightarrow \bigcup_{p \in \mathcal{P}} G_{p,q} \rightarrow V_q \leq \lambda_{p/q}$

The two red premises on the bottom row are expanded to arithmetical conditions on $V_p$ by unfolding the program structure of $\alpha_n$ with $\mathcal{L}$ axioms in the supplement [43].
The bottom premises of MLF\(_R\) and MLF\(_G\) exemplify a key benefit of dl. stability reasoning: conditions on \(V_0\) that arise from Inv\(_X\), Inv\(_Y\) are derived by systematically unfolding the discrete dynamics of \(\alpha_s\) with sound dl axioms. This enables automatic, correct-by-construction derivation of those conditions, which is especially important for controlled switching because the number of possible transitions scales quadratically \(|P|^2\) with the number of modes \(|P|\).

4 KEYMAERA X IMPLEMENTATION

This section presents a prototype implementation of switched systems support in the KeYmaera X prover based on dl [12]. The implementation consists of \(\approx\)2700 lines and, crucially, does not require any extension to KeYmaera X’s existing soundness-critical core. Accordingly, verification results for switched systems obtained through this implementation directly inherit the strong correctness properties guaranteed by the design of KeYmaera X [12, 25].

4.1 Modeling and Proof Interface

The implementation builds on KeYmaera X’s proof IDE [24] to provide a convenient interface for modeling switching mechanisms, as shown in Fig. 3. The interface allows users to express switching mechanisms intuitively by rendering automaton plots while abstracting away the underlying hybrid programs. It provides templates for switched systems following the switching mechanisms of Section 3: state-dependent, guarded, timed, and general controlled switching (tabs “Autonomous”, “Guarded”, “Timed”, “Generic” in Fig. 3). From these templates, KeYmaera X automatically generates programs and stability specifications, ensuring that they have the correct dl hybrid program and formula structure.

Figure 3: Screenshot of the KeYmaera X switched systems modeling editor: automata input on top-left, rendered automaton top-right, generated hybrid program and specification(s) in dl at the bottom

Switched systems are represented internally with a common interface SwitchedSystem which is currently implemented by four classes: StateDependent \(\alpha_{state}\), Guarded \(\alpha_{guard}\), Timed \(\alpha_{time}\), and Controlled \(\alpha_{ctrl}\). The SwitchedSystem interface provides default stability and pre-attractivity specifications, which can be adapted by users on the UI if needed. Corollaries 3–6 are implemented as UG-pAS proof tactics in KeYmaera X’s Bellerophon tactic language [11]. These tactics automate all of the reasoning steps underlying stability proofs for their respective switching mechanisms, so that users only need to input candidate Lyapunov functions for KeYmaera X to (attempt to) complete their proofs. Additionally, when candidates are not provided by the user, the implementation uses sum-of-squares programming [30, 36] to automatically generate candidate Lyapunov functions for a subset of switching designs. The generated candidates are checked for correctness by KeYmaera X so the generator does not need to be trusted for correctness of the resulting proofs. Table 1 summarizes the available proof tactics and Lyapunov function generation for classes of switching mechanisms.

4.2 Examples

The implementation is tested on a suite of examples drawn from the literature [5, 19, 36, 42] featuring various switching mechanisms, with results summarized in Table 2. These examples have a 2 dimensional state space and switch between 2 modes except Example 6 (3 dimensions, 2 modes) and Example 4 (2 dimensions, 4 modes).

The proof tactics successfully prove most of the examples across various switching mechanisms. For Example 5, a suitable Lyapunov function (without numerical errors) could not be found. For the time-dependent switching models (Examples 8–10), KeYmaera X internally uses verified polynomial Taylor approximations to the exponential function for decidability of arithmetic [3, 46]. Example 10 needs a high degree approximation (15 terms in the polynomial) for

Table 1: Available tactics in KeYmaera X for switched systems stability proofs and Lyapunov function generation.

| SwitchedSystem       | Common Lyap. Proof | Gen. Lyap. Proof | Multiple Lyap. Proof | Gen. Lyap. Proof |
|----------------------|--------------------|-----------------|---------------------|-----------------|
| StateDependent \(\alpha_{state}\) | ✓                  | ✓               | ✓                   | ✓               |
| Guarded \(\alpha_{guard}\)    | ✓                  | ✓               | ✓                   | ✓               |
| Timed \(\alpha_{time}\)       | ✓                  | ✓               | ✓                   | ✓               |
| Controlled \(\alpha_{ctrl}\)  | ✓                  | ✓               | ✓                   | ✓               |

Table 2: Stability proofs for examples drawn from the literature. The “Time” columns indicate time (in seconds) to run the KeYmaera X proofs, ✓ indicates incomplete proof. A ✓ in the “Gen.” column indicates successful Lyapunov function generation, ? indicates that a candidate was generated but with numerical issues, and — indicates inapplicability. In the latter two cases (? —) known Lyapunov functions from the literature were used for the proofs (if available).

| Example | Model | Time (Stab.) | Time (Attr.) | Gen. |
|---------|-------|--------------|--------------|------|
| 1 [5, Ex. 2.1] | \(\alpha_{state}\) | 2.6 | 3.0 | ✓ |
| 2 [19, Motiv. ex.] | \(\alpha_{state}\) | 2.2 | 2.3 | ✓ |
| 3 [19, Ex. 1] | \(\alpha_{state}\) | 3.3 | 4.1 | ✓ |
| 4 [19, Ex. 2 & 3] | \(\alpha_{guard}\) | 2.8 | 3.8 | ? |
| 5 [36, Ex. 6] | \(\alpha_{guard}\) | × | × | ? |
| 6 [42, Ex. 2.45] | \(\alpha_{arb}\) | 19.4 | 11.1 | ✓ |
| 7 [42, Ex. 3.25] | \(\alpha_{state}\) | 2.4 | 2.9 | ✓ |
| 8 [42, Ex. 3.49] | \(\alpha_{time}\) | 4.4 | 5.6 | — |
| 9 [47, Ex. 1] | \(\alpha_{time}\) | 4.7 | 5.3 | — |
| 10 [47, Ex. 2] | \(\alpha_{time}\) | 256.9 | × | — |
sufficient accuracy and its attractivity proof could not be completed in reasonable time.

5 CASE STUDIES

This section presents three case studies applying the deductive verification approach to justify various non-standard stability arguments in KeYmaera X.³

5.1 Canonical Max System

Branicky [4] investigates the longitudinal dynamics of an aircraft with an elevator controller that mediates between two control objectives: i) tracking potentially unsafe pilot input and ii) respecting safety constraints on the aircraft’s angle of attack. Assuming a state feedback control law, the model is transformed to the following canonical max system [4, Remark 5], with state variables x, y and parameters a, b, f, g, y satisfying a, b, a − f, b − g > 0 and y ≤ 0.

\[ x' = y, \quad y' = -ax - by + \max(fx + gy + \gamma, 0) \] (2)

The right-hand side of system (2) is non-differentiable but the equations can be equivalently rewritten as a family of two ODEs corresponding to either possibility for the max function. Further insights from the controller design are used in the UGpAS proof in KeYmaera X.

Stability of this parametric system is not directly provable using standard techniques for state-dependent switching as presented in Section 3.1.2. For example, the ODE \( \overline{A} \) stabilizes the system to the origin but the ODE \( \overline{B} \) stabilizes to the point \((-\frac{\gamma}{f-a}, 0)\), away from the origin for \( \gamma < 0 \). Branicky proves global asymptotic stability (of (2)) with the following “noncustomary” [10] Lyapunov function involving a nondifferentiable integrand:

\[ V = \frac{1}{2}y^2 + \int_0^x a\xi - \max(f\xi + \gamma, 0)d\xi \] (3)

The key idea used to deductively prove stability here instead is ghost switching: analogous to ghost variables in program verification which are added for the sake of program proofs [29, 33, 34], ghost switching modes do not change the physical dynamics of the system but are introduced for the purposes of program proofs [29, 33, 34], which are added for the sake of program proofs [29, 33, 34]. Branicky proves global asymptotic stability of (2) with constraint \( f + gy + \gamma \geq 0 \), and ODE \( \overline{A} \) in domain \( fx + gy + \gamma \leq 0 \) and ODE \( \overline{B} \) in domain \( fx + gy + \gamma \geq 0 \).

To obtain closed form representations for the integral in (3), state variables x, \( a-x-b \) and ODE \( \overline{A} \) in domain \( fx + gy + \gamma \leq 0 \) and ODE \( \overline{B} \) in domain \( fx + gy + \gamma \geq 0 \).

\[ \alpha_m \equiv (\overline{A}_1 \cup \overline{A}_2 \cup \overline{B}_1 \cup \overline{B}_2) \quad \beta \equiv \int_0^x a\xi - \max(f\xi + \gamma, 0)d\xi \]

\[ P_m \equiv a > 0 \land b > 0 \land a-f \land b-g > 0 \land f \neq 0 \land \gamma \leq 0 \rightarrow UGpAS(\alpha_m) \]

The ghost switching modes enable a multiple Lyapunov function argument for stability using the following modified closed-form representations of Branicky’s Lyapunov function (3), with \( V_1 = \frac{1}{2}(bcx^2 + 2cxy + y^2) \) for \( \overline{A}_1, \overline{B}_1 \) and \( V_2 = \frac{1}{2}(bcx^2 + 2cxy + y^2 + (f+g)y^2) \) for \( \overline{A}_2, \overline{B}_2 \).

The sub-terms highlighted in red for \( V_1, V_2 \) are closed form expressions for \( \int_0^x a\xi - \max(f\xi + \gamma, 0)d\xi \) where \( f\xi + \gamma \leq 0 \) and \( f\xi + \gamma \geq 0 \) respectively. The Lyapunov functions \( V_1, V_2 \) are modified from (3) to use a quadratic form with an additional constant \( c \) satisfying constraints \( 0 < c < b, c < b-g, c < a-f(b-g) \) (a constant always exists under the assumptions on \( a, b, f, g \)). This technical modification is required to prove UGpAS for \( \alpha_m \) directly with the Lyapunov functions. Branicky’s earlier proof requires LaSalle’s principle [4].

Another challenging aspect of this case study is verification of the parametric arithmetical conditions for \( V_1, V_2 \), i.e., stability is verified for all possible parameter values \( a, b, f, g, y \) that satisfy the assumptions in \( P_m \). Such questions are decidable in theory [3, 46], but are difficult for automated solvers in practice (even out of reach of solvers that require numerically bounded parameters [14]). KeYmaera X enables a user-aided proof of the required arithmetic conditions. For example, the Lie derivative of the Lyapunov function \( V_1 \) for \( \overline{B}_2 \) is given by \( V_1' = -(b-c)y^2 - acc^2 + (cy + gy + \gamma) \), where \( V_1' \) is required to be strictly negative away from the origin for stability. The arithmetical argument is as follows: if \( cx + gy + \gamma \leq 0 \), then by constraint \( f + gy + \gamma \geq 0 \), \( V_1' \) satisfies \( V_1' \leq -(b-c)y^2 - acc^2 \). Otherwise, \( cx + gy + \gamma > 0 \), then by constraint \( f + gy + \gamma \geq 0 \), \( V_1' \) satisfies \( V_1' \leq -(b-g-c)y^2 - acc^2 + gcy \). In either case, the RHS bound is a negative definite quadratic form by the earlier choice of parameter \( c \) and therefore, \( V_1' \) is negative away from the origin.

5.2 Automated Cruise Control

Oehlerking [28, Sect. 4.6] verifies the stability of an automatic cruise controller modeled as a hybrid automaton with 6 operating modes and 11 transitions between them: normal proportional-integral (PI) control, acceleration, service braking (2 modes), and emergency braking (2 modes). Figure 4 shows an abridged version of the corresponding KeYmaera X model (using \( \alpha_{ctrl} \)) with the PI control mode, where \( v \) is the relative velocity to be controlled to \( u = 0 \) and \( x, t \) are auxiliary integral and timer variables used in the controller. Briefly, this controller is designed to use the PI controller near \( v = 0 \) for stability, while its other control modes drive the system toward \( v = 0 \) by accelerating or braking.

Lyapunov function candidates for this model can be successfully generated using the Stabhyli [26] stability tool for hybrid automata. However, Stabhyli (with default configurations) outputs a Lyapunov function candidate for the PI control mode that is numerically unsound, see the supplement [43]; this is a known issue with Stabhyli for control modes at the origin [26]. For this case study, the issue is manually resolved by truncating terms with very small magnitude coefficients in the generated output and then checking in KeYmaera X that the arithmetical conditions for the PI mode are satisfied exactly for the truncated candidate.

Further insights from the controller design are used in the UGpAS proof in KeYmaera X. Briefly, stability only concerns states and modes that are active near the origin so the stability argument and loop invariant only need to mention a single Lyapunov function for the PI control mode, while choosing \( \delta \) (in Def. 1) sufficiently small.

³See https://github.com/LS-Lab/KeYmaeraX-projects/blob/master/stability/UGpAS

⁴An important technical requirement for \( V_1 \) to be well-defined is \( f \neq 0 \). The case with \( f = 0 \) is also verified in KeYmaera X but the details are omitted here for brevity. It does not require ghost switching and uses only \( V_1 \) as its common Lyapunov function.
Verification of stabilizing control laws for Brockett’s nonholonomic integrator in the plane towards the origin. The closed-loop system is modeled for any given constant \( a \) asymptotically stabilize the integrator in the region \( x = 0 < \delta < y < \delta \) and \( z > 0 \). Notably, this is a classical example of a system that is not stabilizable by purely continuous feedback control. Intuitively, no choice of controls \( u, v \) can produce motion along the \( z \)-axis \( (x = y = 0) \). Thus, to stabilize the system to the origin, the controller must first drive the system away from the \( z \)-axis before switching to a control law that stabilizes the system from states away from the \( z \)-axis. This intuition can be realized using two different switching strategies that are analogous to the event-triggered and time-triggered CPS design paradigms respectively [33].

### 5.3 Brockett’s Nonholonomic Integrator

Verification of stabilizing control laws for Brockett’s nonholonomic integrator [7] is of significant interest because stabilizing for a large class of models can be reduced to that of the integrator via coordinate transformations, e.g., Liberzon [22] transforms a unicycle model to the integrator and provides a stabilizing switching control law corresponding to parking of the unicycle. The nonholonomic integrator is described by the system of differential equations \( x' = u, y' = u, z' = x, y, z \) and state feedback control inputs \( u = u(x, y, v), u = v(x, y, z) \) (to be determined below). Notably, this is a classical example of a system that is not stabilizable by purely continuous feedback control. Intuitively, no choice of controls \( u, v \) can produce motion along the \( z \)-axis \( (x = y = 0) \). Thus, to stabilize the system to the origin, the controller must first drive the system away from the \( z \)-axis before switching to a control law that stabilizes the system from states away from the \( z \)-axis. This intuition can be realized using two different switching strategies that are analogous to the event-triggered and time-triggered CPS design paradigms respectively [33].

#### 5.3.1 Event-triggered Controller

Bloch and Drakunov [2] use the switching controller \( u = -x + ay \text{sign}(z), v = -y - ax \text{sign}(z) \) to asymptotically stabilize the integrator in the region \( \frac{a}{2}(x^2 + y^2) \geq |z| \) for any given constant \( a > 0 \). This controller first drives the system towards the plane \( y = 0 \) and, once it reaches the plane, \( s \) slides along the plane towards the origin. The closed-loop system is modeled as an instance of state-dependent switching \( t_{\text{state}} \) with 3 modes depending on the sign of \( z \) and specification \( P_e \):

\[
\begin{align*}
\alpha &\equiv x' = -x + ay, y' = -y - ax, z' = a(x^2 + y^2) \geq |z| \\
\beta &\equiv x' = -x + ay, y' = -y + ax, z' = a(x^2 + y^2) \leq |z| \\
\gamma &\equiv x' = -x, y' = -y, z' = 0 \quad \text{and} \quad z = 0 \quad \alpha_e \equiv \left( \alpha \cup \beta \cup \gamma \right)^* \\
P_e &\equiv \exists a > 0 \Rightarrow \forall \varepsilon > 0 \exists T \geq 0 \forall x, y, z \left( \|x, y, z\| < \delta \land \frac{a}{2}(x^2 + y^2) \geq |z| \right) \rightarrow \\
&\left( t := 0; \alpha_e, t' = 1 \right) (t \geq T \Rightarrow \|x, y, z\| < \varepsilon)
\end{align*}
\]

The specification \( P_e \) is identical to UGpAS except it restricts pre-attractivity to the applicable region \( \frac{a}{2}(x^2 + y^2) \geq |z| \) for the controller. Its verification uses the squared norm \( V = x^2 + y^2 + z^2 \) as a common Lyapunov function. The key modification to the pre-attractivity proof, cf. Section 3.1, is to use (and verify) the fact that \( \frac{a}{2}(x^2 + y^2) \geq |z| \) is a loop invariant of \( \alpha_e \). This additional invariant corresponds to the fact that the controller keeps the system within its applicable region (if the system is initially within that region).

In fact, \( \alpha_e \) can be extended to a globally stabilizing controller, as modeled by \( \alpha_e \) below (if \( \text{else} \) branching is supported as an abbreviation in KeYmaera X [33]):

\[
\begin{align*}
\text{if}(x = 0) &\Rightarrow \exists u, v \in \mathbb{R} \quad \|x, y, z\| < \delta \land \frac{a}{2}(x^2 + y^2) \geq |z| \\
\text{else} &\Rightarrow \exists u, v \in \mathbb{R} \quad \|x, y, z\| < \delta
\end{align*}
\]

The applicable region is equivalently characterized by the real arithmetic formula \((z \geq 0 \rightarrow \frac{a}{2}(x^2 + y^2) \geq \delta) \land (z \leq 0 \rightarrow \frac{a}{2}(x^2 + y^2) \geq \delta)\), omitted for brevity.

![Figure 4](image-url)
If the system is in the applicable region (outer if branch), then the previous controller from $\alpha_t$ is used. Otherwise, outside the applicable region (outer else branch), the system applies a constant control $c > 0$ chosen to drive the system into the applicable region. The pair of ODEs $\mathbb{D}$ and $\mathbb{B}$ model an event-trigger in $\mathbb{D}$ [33], where the switching controller is triggered to make its next decision when the system reaches the switching surface $\frac{\partial}{\partial t} (x^2 + y^2) = |z|$.

The specification $P_e \equiv a \land c > 0 \land UGpAS(\alpha_t)$ is proved by modifying the loop invariants to account for an initial period where the system is outside the applicable region. For example, the stability loop invariant $\mathcal{In}_{\mathcal{V}_c} \equiv (\frac{\partial}{\partial t} (x^2 + y^2) \geq |z|) \land (\frac{\partial}{\partial t} (x^2 + y^2) \geq |z| < \delta) \land (\frac{\partial}{\partial t} (x^2 + y^2) \geq |z| < |e|) \land (\frac{\partial}{\partial t} (x^2 + y^2) \geq |z| < |e| - \delta)$ expresses that the controller keeps $|z|$ sufficiently small with $|z| < \delta$ to preserve stability outside the applicable region. The pre-attractivity loop invariant is similarly split between the two cases, with an explicit time estimate on the time it takes for the system to enter the applicable region.

### 5.3.2 Time-triggered Controller

The time-triggered switching strategy [33], modeled by $\alpha_t$ below, is similar to that proposed by Liberzon [22, Section 4.2]. If the system is on the $z$-axis and away from the origin (A), the controller sets an internal stopwatch $\tau$ and drives the system away from the axis for maximum duration $T_0 > 0$ with $u = z, v = z$. Otherwise (B), the controller drives the system towards the origin along a parabolic curve of the form $\frac{\partial}{\partial t} (x^2 + y^2) = z$.

$$\alpha_t \equiv \begin{cases} \{ & (x = 0 \land y = 0 \land z \neq 0) \\ & \tau := 0, x' = z, y' = z, z' = xz - yz \land \tau \leq T_0 \\ & \text{else} \{ a := \frac{2z}{x^2 + y^2}; \\ & x' = -x + ay, y' = y - ax, z' = -a(x^2 + y^2) \} \} \end{cases}$$

The specification $P_e \equiv T_0 > 0 \rightarrow UGpAS(\alpha_t)$ is again proved by analyzing both cases of the controller in the loop invariants, e.g., with the pre-attractivity invariant $\mathcal{In}_{\mathcal{V}_u}$:

$$\{ x = 0 \land y = 0 \land z \neq 0 \rightarrow |z| < \delta \land t = 0 \} \land$$

$$(\neg(x = 0 \land y = 0 \land z \neq 0) \rightarrow ||x, y, z|| > \epsilon \rightarrow ||x, y, z||^2 < \delta^2(2T_0^2 + 1) - \epsilon^2(t - T_0))$$

The top conjunct says the system may start transiently on the $z$-axis (away from $z = 0$) at time $t = 0$. The bottom conjunct gives explicit bounds on ||x, y, z||, which, for sufficiently large $t \geq T$, implies that the system enters $||x, y, z|| < \epsilon$ as required for pre-attractivity. The transient term $\delta^2(2T_0^2 + 1)$ upper bounds the (squared) norm of the system state after starting on the $z$-axis in ball $||x, y, z|| < \delta$ and following mode $\mathbb{A}$ for the maximum stopwatch duration $t = T_0$.

### 6 RELATED WORK

**Switched Systems**. Comprehensive introductions to the analysis and design of switching control can be found in the literature [10, 22, 42]. An important design consideration (which this paper sidesteps, cf. Remark 1) is whether a given switched or hybrid system has complete solutions [16, 17, 23, 48]. Justification of such design considerations, and other stability notions of interest for switching designs, e.g., quadratic, region, or set-based stability [16, 17, 22, 35, 42], can be done in $\mathbb{D}$ with appropriate formal specifications of the desired properties from the literature [31, 33, 44, 45]. Another complementary question is how to design a switching control law that stabilizes a given system. Switching design approaches are often guided by underlying stability arguments [22, 37, 42]; the loop invariants from Section 3 are expected to help guide correct-by-construction synthesis of such controllers.

**Stability Analysis and Verification**. Corollaries 3–6 formalize various Lyapunov function-based stability arguments from the literature [5, 47] using loop invariants, yielding trustworthy, computer-checked stability proofs in KeYmaera [11, 12]. Other computer-aided approaches for switched system stability analysis are based on finding Lyapunov functions that satisfy the requisite arithmetical conditions [20, 26, 28, 36, 39, 40]. Although the search for such functions can often be done efficiently with numerical techniques [26, 30, 36], various authors have emphasized the need to check that their outputs satisfy the arithmetical conditions exactly, i.e., without numerical errors compromising the resulting stability claims [1, 20, 38] (see, e.g., Section 5.2). This paper’s deductive approach goes further as it comprehensively verifies all steps of the stability argument down to its underlying discrete and continuous reasoning steps [32, 33]. The generality of this approach is precisely what enables verification of various classes of switching mechanisms all within a common logical framework (Section 3) and verification of non-standard stability arguments (Section 5). Alternative approaches to stability verification are based on abstraction [15, 41] and model checking [35].

### 7 CONCLUSION

This paper shows how to deductively verify switched system stability, using $\mathbb{D}$’s nested quantification over hybrid programs to specify stability, and $\mathbb{D}$’s axiomatics to prove those specifications. Loop invariants—a classical technique from verification—are used to succinctly capture the desired properties of a given switching design; through deductive proofs, these invariants yield systematic, correct-by-construction derivation of the requisite arithmetical conditions on Lyapunov functions for stability arguments in implementations. An interesting direction for future work is to use other Lyapunov function generation techniques [20, 26, 28, 40]—thanks to the presented approach—do not have to be trusted since their results can be checked independently by KeYmaera X. This would enable fully automated, yet sound and trustworthy verification of switched system stability based on $\mathbb{D}$’s parsimonious hybrid program reasoning principles.

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### REFERENCES

[1] Daniele Ahmed, Andrea Peruffo, and Alessandro Abate. 2020. Automated and Sound Synthesis of Lyapunov Functions with SMT Solvers. In TACAS (LNCS, Vol. 12078), Armin Biere and David Parker (Eds.). Springer, 97–114. https://doi.org/10.1007/978-3-030-45190-5_6
