HOMOMORPHISMS OF DISTRIBUTIVE LATTICES
AS RESTRICTIONS OF CONGRUENCES.
III. RECTANGULAR LATTICES AND
TWO CONVEX SUBLATTICES

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To the memory of E. T. (Tomi) Schmidt,
whose ideas still inspire us

Abstract. Let \( L \) be a finite lattice and let \( I \) be an ideal of \( L \). Then the
restriction map is a bounded lattice homomorphism of the congruence lattice
of \( L \) into the congruence lattice of \( I \). In a 2009 paper, the authors proved the
converse.

In a 2012 paper, G. Czédli proved an analogous result for rectangular lattices.
In this paper, we prove a stronger form of Czédli’s result and provide a short,
elementary, and direct proof.

1. Introduction

Let \( L \) be a finite lattice and let \( K \) be a convex sublattice of \( L \). Then the
restriction map, defined as \( \text{re} : \alpha \mapsto \alpha \upharpoonright K \) (the congruence \( \alpha \) of \( L \) is mapped to its restriction
to \( K \)), is a bounded (that is, \( \{0,1\} \)-) lattice homomorphism of \( \text{Con} L \) into \( \text{Con} K \).

G. Grätzer and H. Lakser [17] proved the converse.

Theorem 1. Let \( D \) and \( E \) be finite distributive lattices and let \( \varphi : D \to E \) be
a bounded lattice homomorphism. Then there exist a finite lattice \( L \), a convex
sublattice \( G \) of \( L \) that can be chosen to be either an ideal or a filter of \( L \), and
isomorphisms \( \alpha : D \cong \text{Con} L \) and \( \beta : E \cong \text{Con} G \), making the following diagram
commutative:

\[
\begin{array}{ccc}
D & \xrightarrow{\alpha \cong} & \text{Con} L \\
\varphi \downarrow & & \downarrow \text{re} \\
E & \xrightarrow{\beta \cong} & \text{Con} G
\end{array}
\]

where \( \text{re} \) is the restriction map: for a congruence \( \alpha \) of \( L \), \( \text{re}(\alpha) \) is \( \alpha \) restricted to \( G \).

See E. T. Schmidt [24] for an alternative proof.

Theorem 1 is an abstract/abstract result. The congruence lattices are given as
abstract finite distributive lattices, the finite lattices \( L \) and \( G \) are constructed.

We can improve on Theorem 1 in two ways.

First, we can construct the finite lattices \( L \) and \( G \) in smaller classes of lattices.
G. Grätzer and H. Lakser [18] constructed them as planar lattices, while G. Grätzer

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and H. Lakser [19] represented them as isoform lattices. 16 years later, G. Grätzer and E. Knapp [15] represented finite distributive lattices as congruence lattices of rectangular lattices. Theorem 1 for rectangular lattices was done in G. Czédli [1]. (Using filters instead of ideals, see Section 5.)

Second, we can try to loosen the abstract/abstract representation. This was first done in G. Czédli [1].

**Theorem 2.** Let $G$ be a rectangular lattice, and let $\varphi$ be a $\{0,1\}$-lattice homomorphism of a finite distributive lattice $D$ to $\text{Con} G$. Then there is a rectangular lattice $L$ containing $G$ as a convex sublattice, which can be chosen as a filter of $L$, and a lattice isomorphism $\alpha: D \to \text{Con} L$ such that $\varphi = \text{re} \circ \alpha$, where $\text{re}$ is the restriction map of congruences from $L$ to $G$. That is, the diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\alpha} & \text{Con} L \\
\parallel & \downarrow \text{re} & \parallel \\
D & \xrightarrow{\varphi} & \text{Con} G
\end{array}
$$

is commutative.

Thus, $G$ is made into a filter and $\varphi$ is represented as restriction of congruences to $G$. This is a concrete/abstract result: instead of the finite distributive lattice $E$, we are given the rectangular lattice $G$ and $\text{Con} G$ plays the role of $E$.

E. T. Schmidt [25] proved this result for the special case when $\varphi$ is injective.

In this paper, we present the following result.

**Theorem 3.** Let $F$ and $G$ be rectangular lattices and let $\varphi: \text{Con} F \to \text{Con} G$ be a bounded lattice homomorphism. Then there is a rectangular lattice $L$ containing $F$ and $G$ as convex sublattices, where $G$ can be chosen as a filter of $L$, satisfying the following two conditions:

(i) the lattice $L$ is a congruence-preserving extension of $F$;
(ii) for $\alpha$ of $\text{Con} L$, the map $\varphi$ is $\alpha|F \to \alpha|G$.

This is a concrete/concrete result: instead of the finite distributive lattice $D$, we are given the rectangular lattice $F$ and $\text{Con} F$ plays the role of $D$; instead of the finite distributive lattice $E$, we are given the rectangular lattice $G$ and $\text{Con} G$ plays the role of $E$. The lattices constructed are rectangular and $G$ is constructed as a filter, so Theorem 3 is stronger form of Czédli’s result. Our approach also provides a short, elementary, and direct proof based on the construction in G. Grätzer and E. T. Schmidt [22].

2. **Preliminaries**

2.1. **Notation.** We use the notation as in [8]. You can find the complete

*Part I. A Brief Introduction to Lattices and Glossary of Notation* of [8] at

[ tinyurl.com/lattices101 ]

2.2. **BRT.** The Birkhoff Representation Theorem (BRT, for short) relates bounded homomorphisms of finite distributive lattices and isotone maps of finite ordered sets.

For a finite distributive lattice $D$, let $J(D)$ denote the ordered sets of join-irreducible elements of $D$; for $a \in D$, let $\text{spec}(a)$ denote the set of join-irreducible elements $\leq a$. 
Theorem 4 (BRT). Let $D$ and $E$ be finite distributive lattices and set $P = J(D)$ and $Q = J(E)$. Then the following five statements hold.

(i) With every bounded homomorphism $\varphi: D \rightarrow E$, we can associate an isotone map $J(\varphi): Q \rightarrow P$ defined by

$$J(\varphi)x = \bigwedge \{ e \in D \mid x \leq \varphi e \}$$

for $x \in Q$.

(ii) With every isotone map $\psi: Q \rightarrow P$, we can associate a bounded homomorphism $\text{Down}(\psi): D \rightarrow E$ defined by

$$\text{Down}(\psi)(e) = \bigvee \psi^{-1}(\text{spec}(e))$$

for $e \in D$.

(iii) The constructions of (i) and (ii) are inverses to one another, and so yield together a bijection between bounded homomorphisms $\varphi: D \rightarrow E$ and isotone maps $J(\varphi): Q \rightarrow P$.

(iv) $\varphi$ is one-to-one iff $J(\varphi)$ is onto.

(v) $\varphi$ is onto iff $J(\varphi)$ is an order-embedding.

The two main formulas in this result are easy to visualize. For the formula in (i), for $x \in Q$, take the set $X$ of all elements of $D$ mapped by $\varphi$ onto $x$ or above. Since $\varphi$ is a homomorphism, it follows that the meet of all these elements, let us denote it by $x^\dagger$, is still mapped by $\varphi$ onto $x$ or above. The map $J(\varphi)$ maps $x$ into $x^\dagger$.

With every isotone map $\psi: Q \rightarrow P$, we can associate a bounded homomorphism $\text{Down}(\psi): D \rightarrow E$ defined by

$$\text{Down}(\psi)(e) = \bigvee \psi^{-1}(\text{spec}(e))$$

for $e \in D$.

Now for the formula in (ii), take an element $e \in D$. We form the set $Y$ of join-irreducible elements of $E$ mapped by $\psi$ to an element $\leq e$. The join of $Y$ denoted by $e^\ddagger$, is an element of $E$. The map $\text{Down}(\psi)$ maps $e$ into $e^\ddagger$.

The constructions of (i) and (ii) are inverses to one another, and so yield together a bijection between bounded homomorphisms $\varphi: D \rightarrow E$ and isotone maps $\psi: Q \rightarrow P$.

2.3. Triple gluing. For rectangular lattices, we need the following variant of gluing.

Let $G, Y, Z, U$ be rectangular lattices, arranged in the plane as in Figure 1 such that the facing boundary chains have the same number of elements. First, we glue $U$ and $Y$ together over the boundaries, to obtain the rectangular lattice $X$, then we glue $Z$ and $G$ together over the boundaries, to obtain the rectangular lattice $W$. Finally, we glue $X$ and $W$ together over the boundaries, to obtain the rectangular lattice $V$, which we call the triple gluing of $G, Y, Z, U$.

Lemma 5. A congruence $\alpha$ of $V$ is uniquely associated with the four congruences $\alpha_G$ of $G$, $\alpha_Y$ of $Y$, $\alpha_Z$ of $Z$, and $\alpha_U$ of $U$, satisfying the condition that $\alpha_G$ and $\alpha_Y$ agree on the facing boundaries, and the same for $\{G, Z\}$, $\{Y, U\}$, and $\{Z, U\}$.

This is a very easy lemma. For a formal proof of a more general statement, see the Appendix.
3. Proof by Picture

For the “Proof by Picture”, we choose

\[ F = G = S_7. \]

Let

\[ D = \text{Con } F, \quad E = \text{Con } G, \quad P = \text{J}(D), \quad Q = \text{J}(E), \]

and let \( \varphi \) the bounded homomorphism \( D \to E \), as in Figure 2(ii). Note that \( S_7 \)

is “square”, as a result all the rectangular lattices look “square” in this and the

subsequent diagrams of this chapter.

We construct the rectangular lattice \( L \) of Theorem 3 in a few steps, as illustrated

in Figure 3.

Step 1. We construct a congruence-preserving extension \( B \) of \( F \) in which all

join-irreducible congruences of \( F \) appear on both upper boundaries of \( B \).

The distributive lattice \( E = \text{Con } G \) has 3 join-irreducible congruences, so we

start with the glued sum of 3 copies of \( M_3 \) and \( F \) and extend this to a rectangular

lattice \( A \), see Figure 3.

We add three more eyes to \( A \) to form \( M_3 \)-s, to make sure that any two edges

of the same color generate the same congruence; thus we obtain the rectangular

lattice \( B \) of Figure 3.

In Figures 3–5 an \( M_3 \), all whose edges are colored by \( x \) are pictured as \( x \).

Step 2. We form a rectangular lattice \( C \) containing all the congruences of \( F \) and \( G \).

To do this, we form the glued sum of \( B \) and \( G \) and extend this to a rectangular

lattice \( C \) as in Figure 3.

Step 3. We go back to Figure 2 to the element \( v \) of \( Q \). The element \( b \) of \( D \)

is the smallest element mapped by \( \varphi \) to an element of \( E \) with \( \geq v \); the element \( b \)
is join-irreducible, so it is in \( P \). We identify \( b \) and \( v \) (as per the discussion of the

Birkhoff Representation Theorem), by finding a cover preserving \( C_2 \) colored by \( b \)

and \( v \) and adding an eye (colored gray in Figure 3). We proceed the same way with

the element \( w \in Q \), again finding \( b \in E \), and identify \( b \) and \( w \). Finally, we do the

same for the element \( u \in Q \) and identify the congruences represented by \( u \) and \( a \).
Thereby, $L$ is a congruence-preserving extension of $F$, since each join-irreducible congruence of $L$ is one of $a, b, c$. This finishes the construction of the rectangular lattice $L$, as in Figure 3.

Now we infer from BRT, that the map $\varphi$ is represented by $\alpha|F \to \alpha|G$, for $\alpha \in \text{Con } F$, as required in Theorem 3.

4. Proof of Theorem 3

We use the notation (3.1).

The first triple gluing. For the rectangular lattice $F$ of Theorem 3, let $\text{bl}_F$, $\text{br}_F$, $\text{tl}_F$, and $\text{tr}_F$ denote the number of elements of the bottom left and right, and top left and right boundaries, respectively, and let $j$ denote the number of join-irreducible congruences. If the lattice $F$ is understood, the subscripts may be omitted.

We prove the following statement.

Lemma 6. The rectangular lattice $F$ has a congruence-preserving rectangular extension $R$ such that all join-irreducible congruences of $R$ appear on both upper boundaries of $R$. 
Figure 3. The rectangular lattices $A$, $B$, $C$, and $L$. 
Proof. Let us define the rectangular lattices, $Y$ and $Z$ as follows.

$$Y = \mathbb{C}_{bl} \times \mathbb{C}_{j+1}, \quad Z = \mathbb{C}_{j+1} \times \mathbb{C}_{br}. $$

We also need the rectangular lattice $U$ that we obtain from $\mathbb{C}_{2j+1}$ by adding eyes $j$-times to the covering $C_{2j}$-s on the main diagonal.

We label $F$; there are $j$ labels. We use these to label the $M_3$-s on the main diagonal of $U$, bijectively.

Now we form the triple gluing of $F, Y, Z, U$ to form the rectangular lattice $V$, as illustrated in Figure 4. The labeling of $V$ is not a coloring, because two edges of the same label do not necessarily generate the same congruence. Let $x$ be the label of a join-irreducible congruence of $F$. There is an edge $A_x$ of label $x$ in the lower boundary of $F$, say, in the lower left boundary. Let $A'_x$ and $B'_x$ be the edges of $Y$ that are identified with $A_x$ and $B_x$, respectively, in the triple gluing. Let $\alpha_x$ and $\beta_x$ denote the congruences of $Y$ generated by $A'_x$ and $B'_x$, respectfully.

We claim that for the color $x$, there is a covering square $S_x = C_2^2$ in $Y$, that is colored by both $\alpha_x$ and $\beta_x$. Indeed, take the trajectory $r$ of $Y$ containing $A'_x$; it is a normal-up trajectory. Take the trajectory $t$ of $Y$ containing $B'_x$; it is a normal-down trajectory. Therefore, $r$ and $t$ intersect in a covering square in $Y$ colored by both $\alpha_x$ and $\beta_x$, as claimed.

We add an eye to $S_x$, as in Figure 4. We do it for all colors $x$ in $F$, to obtain the rectangular lattice $R$, illustrated in Figure 4. Now the labelling is a coloring, the color $x$ of $F$ is the same as the color $x$ of $U$.

Let $\gamma_l$ be the restriction of $\alpha_x$ to $C_{bly}$, the lower left boundary of $Y$. Similarly, associate with the congruence $x$ of $U$ the congruence of $Y$ generated by $C'_x$ that is identified with $C_x$, and restrict it to the lower right boundary of $Y$; call it $\gamma_r$.
Finally, \( R \) is a congruence-preserving extension of \( F \), as claimed in this lemma. To accomplish this, we define the congruences \( \alpha_F \) of \( F \), \( \alpha_Y \) of \( Y \), \( \alpha_Z \) of \( Z \), and \( \alpha_U \) of \( U \), as follows:

(i) The congruence \( \alpha_F \) of \( F \) is \( x \), equivalently, \( \alpha_F \) is generated in \( F \) by the edge \( A_x \).

(ii) We define \( \alpha_Y = \gamma_l \times \gamma_r \).

(iii) The congruence \( \alpha_Z \) of \( Z \) is defined symmetrically.

(iv) The congruence \( \alpha_U \) of \( U \) is generated by the edge \( B_x \subseteq U \).

By construction, these congruences satisfy the conditions of Lemma 5, so there is a congruence \( \alpha_R \) extending all four, and this is the extension of the color \( x \) to \( R \). This gives us that there is at least one such extension. The uniqueness follows from the fact the every edge of \( R \) is perspective to an edge of \( F \) or \( U \).

We conclude by observing that the congruence extension property for join-irreducible congruences is equivalent to the congruence extension property (for all congruences). \( \square \)

The second triple gluing. We prove the following statement in this section.

**Lemma 7.** The rectangular lattice \( G \) has a congruence-preserving extension \( L \) such that \( \downarrow_{0G} = R \) (the rectangular lattice of Lemma 6).

**Proof.** The technical aspects of the proof are very similar to the proof of Lemma 6, *mutatis mutandis*.

We use three auxiliary rectangular lattices, defined as follows: the rectangular lattice \( R \) constructed in the previous section, and \( Y = C_{\text{bl}} \times C_{\text{tl}} \), \( Z = C_{\text{tr}} \times C_{\text{br}} \).

Then we form the triple glued sum of \( G, Y, Z, R \) to form the rectangular lattice \( U \), as illustrated in Figure 5(ii).

Recall that we use the notation \((3.1)\).

Let \( x \) be the color of a join-irreducible congruence of \( G \), that is, \( x \in Q \). As in Section 3 there is a smallest element \( y \in D \) for which \( \varphi y \geq x \) holds, namely, \( J(\varphi)x \in P \). There is an edge \( A_x \) of color \( x \) in the lower boundary of \( G \), say, in the lower left boundary. There is also an edge \( B_y \) of color \( y \) in the upper left boundary of \( R \).

Let \( A'_x \) and \( B'_y \) be the edges of \( Y \) that are identified with \( A_x \) and \( B_y \), respectfully, in the second riple gluing. Let \( \alpha_x \) and \( \beta_y \) denote the congruences of \( Y \) generated by \( A'_x \) and \( B'_y \), respectfully.

As the proof of Lemma 6 we identify \( x \) and \( y \) by finding a cover preserving \( C_2 \) in \( Y \) colored by \( x \) and \( y \) and adding an eye (colored gray in Figure 3). We do it for all colors \( x \) in \( G \), to obtain the rectangular lattice \( L \), illustrated in Figure 5.

We verify the properties of \( L \) as we verified the properties of \( R \) in the previous section. \( \square \)

**Completing the proof of Theorem 3.** In the previous section, we constructed a rectangular lattice \( L \) containing the filter \( G \) and the convex sublattice \( F \) such that \( 3(i) \) holds. To complete the proof of Theorem 3 we have to verify that \( 3(ii) \) also holds for \( L \). By BRT, it is equivalent to verify that for \( \alpha \in Q \), the map \( J(\varphi) \) is represented by the construction, which is evident.
5. Ideal Embeddability

First some terminology. Let $G$ be a rectangular lattice. We say $G$ is *simple-embeddable* if it is embeddable as an ideal in a simple rectangular lattice.
We say $G$ is \textit{abstractly ideal-representable} if, for any finite distributive lattice $D$ and any bounded lattice homomorphism $\varphi: D \to \text{Con} G$, there is a rectangular lattice $L$ containing $G$ as an ideal, and an isomorphism $\eta: D \to \text{Con} L$ such that $\varphi = \text{re} \circ \eta$.

We say $G$ is \textit{concretely ideal-representable} if, for any rectangular lattice $F$ and any bounded lattice homomorphism $\varphi: \text{Con} F \to \text{Con} G$, there is a rectangular congruence-preserving extension $L$ of $F$ that contains $G$ as an ideal such that for any $\alpha$ of $\text{Con} L$, the map $\varphi$ is $\alpha | F \mapsto \alpha | G$.

\textbf{Lemma 8.} Let $L$ be a lattice, let $I$ be an ideal in $L$, and let $\alpha$ be a meet-congruence on $I$. Extend $\alpha$ to an equivalence relation $\beta$ on $L$ by setting the equivalence class of any $x \in L \setminus I$ to be the singleton $\{x\}$. Then $\beta$ is a meet-congruence on $L$.

\textit{Proof.} We need only show that for $x, y, z \in L$, if $x \neq y$ and $x \equiv y (\beta)$, then $x \land z \equiv y \land z (\beta)$. But then $x, y \in I$, and $x \equiv y (\alpha)$. Now,

$$z' = (x \lor y) \land z \in I$$

and

$$x \land z = x \land z', \quad y \land z = y \land z'.$$

Thus $x \land z \equiv y \land z (\alpha)$, that is, $x \land z \equiv y \land z (\beta)$. \hfill $\square$

\textbf{Lemma 9.} Let $L$ be a rectangular lattice, and let the ideal $I$ of $L$ also be a rectangular lattice. Then $\text{lc}(I)$ is in the left lower chain of $L$ and $\text{rc}(I)$ is in the right lower chain of $L$.

\textit{Proof.} The corners of $I$, the elements $\text{lc}(I)$ and $\text{rc}(I)$, are join-irreducible in the ideal $I$, and so are join-irreducible elements of $L$. But any join-irreducible element of $L$ is either an eye or an element of one of the two lower chains of $L$. Being incomparable, they cannot both lie in the same chain.

On the other hand, we claim that neither can be an eye in $L$. An eye $e$ in $L$ has a unique upper cover in $L$ which has two additional lower covers, one to the left of $e$ and one to the right of $e$. But each of $\text{lc}(I)$ and $\text{rc}(I)$ has an upper cover in one of the upper chains of the ideal $I$, and so, by the uniqueness of the upper cover, if $\text{lc}(I)$ were an eye, we would get the contradiction that $I$ contains an element to the left of $\text{lc}(I)$, and, similarly for $\text{rc}(I)$, an element to the right of $\text{rc}(I)$.

Thus $\text{lc}(I)$ is in the left lower chain of $L$ and $\text{rc}(I)$ is in the right lower chain of $L$. \hfill $\square$

We note the following.

\textbf{Lemma 10.} Let $L$ be a rectangular lattice and let $x \in L$ not be an eye. Then $x = (x \land \text{lc}(L)) \lor (x \land \text{rc}(L))$.

\textbf{Lemma 11.} Let $L$ be a rectangular lattice, and let the ideal $I$ of $L$ also be rectangular. Then for each $x \in L \setminus I$, either $x > \text{lc}(I)$ or $x > \text{rc}(I)$.

\textit{Proof.} We first consider the case where $x$ is not an eye. Then, by Lemma 10, $x = (x \land \text{lc}(L)) \lor (x \land \text{rc}(L))$. Now, $C_{\text{ul}}(I) L$ is a chain containing both $x \land \text{lc}(L)$ and, by Lemma 9, $\text{lc}(I)$. Thus

either $x \land \text{lc}(L) \leq \text{lc}(I)$ \quad or \quad $x \land \text{lc} L > \text{lc}(I)$.

Similarly,

either $x \land \text{rc}(L) \leq \text{rc}(I)$ \quad or \quad $x \land \text{rc} L > \text{rc}(I)$. 

But, if \( x \wedge \text{lc}(L) \leq \text{lc}(I) \) and \( x \wedge \text{rc}(L) \leq \text{rc}(I) \), we get the contradiction \( x \leq \text{lc}(I) \lor \text{rc}(I) \in I \). Thus, either \( x \wedge \text{lc} L > \text{lc}(I) \), and so \( x > \text{lc}(I) \), or \( x \wedge \text{rc} L > \text{rc}(I) \), and so \( x > \text{rc}(I) \), establishing our claim if \( x \) is not an eye.

On the other hand, if \( x \) is an eye, then \( x \) has a unique lower cover \( x' \) in \( L \), and \( x' \) is not an eye. If \( x' \in \text{ul}(I) \cup \text{ur}(I) \), then

\[
\text{either } x > x' \geq \text{lc}(I) \quad \text{or} \quad x > x' \geq \text{rc}(I).
\]

Otherwise, \( x' \in L \setminus I \), and so, by the first case of the proof, either \( x' > \text{lc}(I) \) or \( x' > \text{rc}(I) \), and thus either \( x > \text{lc}(I) \) or \( x > \text{rc}(I) \), establishing our claim also when \( x \) is an eye.

\[\square\]

**Lemma 12.** Let \( L \) be a rectangular lattice, let \( I \) be an ideal in \( L \) which is also rectangular and let \( \alpha \) be a congruence on \( I \) that collapses no interval in either upper chain of \( I \). Extend \( \alpha \) to an equivalence relation \( \beta \) on \( L \) by setting the equivalence class of any \( x \in L \setminus I \) to be the singleton \( \{x\} \). Then \( \beta \) is a congruence on \( L \).

**Proof.** By Lemma 8, the relation \( \beta \) is preserved by meet. So, we need only show that it is preserved by join. We need only consider elements \( x, y, z \in L \) with \( x < y \) and \( x \equiv y(\beta) \). Being distinct, \( x, y \in I \), by definition of \( \beta \), and so \( x \equiv y(\alpha) \). If \( z \in I \) then \( x \lor z \equiv y \lor z(\alpha) \) and we are done.

Otherwise, by Lemma 11, we may assume that \( z > \text{lc}(I) \). But then

\[
x \lor z = x \lor \text{lc}(I) \lor z \quad \text{and} \quad y \lor z = y \lor \text{lc}(I) \lor z.
\]

Now \( x \lor \text{lc}(I) \) and \( y \lor \text{lc}(I) \) are in the left upper chain of \( I \) and \( x \lor \text{lc}(I) \equiv y \lor \text{lc}(I) \)(\( \alpha \)). By our condition on \( \alpha \) we get \( x \lor \text{lc}(I) = y \lor \text{lc}(I) \), and so \( x \lor z = y \lor z \). Thus \( \beta \) is also preserved by join. \[\square\]

We have an immediate corollary:

**Lemma 13.** Let the rectangular lattice \( G \) have a congruence that collapses no interval in either upper chain. Then \( G \) cannot be embedded as an ideal in any simple rectangular lattice.

**Lemma 14.** Let \( G \) be a rectangular lattice such that each non-trivial congruence collapses some edge in one of its upper chains.

Let \( F \) also be a rectangular lattice and let \( \varphi : \text{Con } F \rightarrow \text{Con } G \) be a bounded lattice homomorphism. Then there is a rectangular lattice \( L \) with \( G \) as an ideal and \( F \) as a filter satisfying the following two conditions:

(i) the lattice \( L \) is a congruence-preserving extension of \( F \);

(ii) for \( \alpha \) of \( \text{Con } L \), the map \( \varphi \) is \( \alpha[F] \rightarrow \alpha[G] \).

**Proof.** (Outline) We proceed as in the proof of Theorem 8, mutatis mutandis. We have \( J(\varphi) : J(G) \rightarrow J(F) \). Embed \( F \) as a filter in a preserving rectangular extension \( F' \) where each non-trivial congruence appears as a color on each lower chain. We then glue \( F' \) to the top of \( G \) and add the two “flaps”. Finally, insert eyes in the flaps so that, for each join-irreducible congruence \( \alpha \) of \( G \), each edge on either upper chain of \( G \) that is colored by \( \alpha \) is connected to an edge on the corresponding lower chain of \( F' \) that is colored by \( J(\varphi)(\alpha) \). The lattice \( L \) is the required rectangular lattice. \[\square\]

By Lemmas 13 and 14, we have our main theorem.
Theorem 15. Let $G$ be a rectangular lattice. The following conditions are equivalent:

(i) each non-trivial congruence of $G$ collapses some edge in one of its upper chains;
(ii) $G$ is concretely ideal-representable;
(iii) $G$ is abstractly ideal-representable;
(iv) $G$ is simple-embeddable.

6. Discussion

6.1. Rectangular extensions. In Section 3, Proof by Picture, we write: “extend this to a rectangular lattice” in Step 1. The is unambiguous but not very precise. The formal way is to use triple gluing.

G. Grätzer and E. Knapp [14] prove the existence of a rectangular congruence-preserving extension of an SPS lattice. The uniqueness of a rectangular extension is discussed in G. Czédli [2].

6.2. $C_1$-diagrams. G. Czédli [2] introduced $C_1$-diagrams for slim rectangular lattices. Informally, a diagram is a $C_1$-diagram, if all edges are drawn at directions of $(1,1)$ and $(1,-1)$, normal edges, except the middle edges of $S_7$-s are drawn at directions $(\cos \alpha, \sin \alpha)$ with $\pi/2 < \alpha < 3\pi/2$, steep edges. G. Czédli proves that all slim rectangular lattices have $C_1$-diagrams, see [2]. See also G. Grätzer [10].

All the diagrams of rectangular lattices of this paper are $C_1$-diagrams.

6.3. Automorphisms. It is easy to see that the lattice $L$ of Theorem 3 can be constructed with a given automorphism group.

Appendix

Any gluing can be described as a lattice $L$ with an ideal $A$ and a filter $B$ such that $A \cap B \neq \emptyset$; the lattice is the gluing of any lattice $A'$ isomorphic to $A$ and any lattice $B'$ isomorphic to $B$ over the filter of $A'$ corresponding to $A \cap B$ and ideal of $B'$ isomorphic to $A \cap B$. All that is required is that $A$ and $B$ have a nonempty intersection.

We recall the following characterization of how congruences on $A$, $B$, and $L$ relate.

Lemma 16. Let the lattice $L$ be a gluing of an ideal $A$ and a filter $B$. Let $\alpha_A$ be a congruence of $A$ and $\alpha_B$ a congruence of $B$. Then there is a congruence $\alpha$ of $L$ with $\alpha_A = \alpha \uparrow A$ and $\alpha_B = \alpha \downarrow B$ iff

$$\alpha \uparrow (A \cap B) = \alpha_B \downarrow (A \cap B),$$

and, in that event,

$$\alpha = \alpha_A \cup \alpha_B \cup (\alpha_A \circ \alpha_B) \cup (\alpha_B \circ \alpha_A),$$

uniquely determined by $\alpha_A$ and $\alpha_B$.

We now introduce triple gluing.

Let $L$ be a lattice and let $c \in L$. We assume that $L$ is the union of four convex sublattices, $U$, $V$, the ideal $X = \downarrow c$, and the filter $Y = \uparrow c$.

We assume that the following properties hold:

(Pi) $A = U \cup X$ is an ideal of $L$.
(Pii) $B = V \cup Y$ is a filter of $L$. 

(Piii) \( U \) is a filter of \( A \) and \( X \) is an ideal of \( A \).
(Piv) \( V \) is an ideal of \( B \) and \( Y \) is a filter of \( B \).
(Pv) \( U \cap V = \{ c \} \).

By (Pv), \( c \) is an element of each of \( X \), \( Y \), \( U \), and \( V \); thus no pairwise intersection is empty. Then (Pi) and (Pii) state that \( L \) is a gluing of \( A \) and \( B \) over \( A \cap B \). (Piii) states that \( A \) is a gluing of \( X \) and \( U \) over \( X \cap U \), and (Piv) states that \( B \) is a gluing of \( V \) and \( Y \) over \( V \cap Y \). We call such a configuration a *triple gluing*; see Figure 6.

Property (Pv) will be utilized in the following results.

**Lemma 17.** \( U \cup Y \) is a sublattice of \( L \), with \( U \) an ideal and \( Y \) a filter, so \( U \cup Y \) is a gluing of \( U \) and \( Y \) over \( U \cap Y \).

Dually, \( X \cup V \) is a sublattice of \( L \) with \( V \) a filter and \( X \) an ideal, so \( X \cup V \) is a gluing of \( V \) and \( X \) over \( V \cap X \).

**Proof.** By duality, we need only prove the first statement.

We first show that \( U \cup Y \) is a sublattice of \( L \). Let \( x, y \in U \cup Y \). If \( x, y \in U \) (respectively, \( x, y \in Y \)), then \( x \wedge y, x \vee y \in U \) (respectively \( \in Y \)), since \( U \) and \( V \) are sublattices of \( L \).

Otherwise, we may assume that \( x \in U \) and \( y \in Y \). Then \( x \vee y \in Y \), since \( Y \) is a filter of \( L \). So, we need only show that \( x \wedge y \in U \). We have \( y \geq c \), by definition of \( Y \). Thus,

\[
x \wedge c \leq x \wedge y \leq x.
\]

Then \( c \in U \) follows from (Pv). Thus \( x \wedge c, x \in U \), and, since \( U \) is a convex sublattice of \( L \), we conclude that \( x \wedge y \in U \). Thus \( U \cup Y \) is a sublattice of \( L \).

Now \( Y \) was defined as a filter of \( L \), and so is clearly a filter of \( U \cup Y \).

We then need only show that \( U \) is an ideal of \( U \cup Y \). By (Pi), \((U \cup Y) \cap A \) is an ideal of \( U \cup Y \). But

\[
(U \cup Y) \cap A = U \cup (Y \cap X) = U \cup \{ c \} = U,
\]

![Figure 6. Triple gluing](image-url)
Then there exists a unique congruence \( \alpha \) on \( L \) such that \( \alpha \big|_X = \alpha_X \), \( \alpha \big|_Y = \alpha_Y \), \( \alpha \big|_U = \alpha_U \), and \( \alpha \big| V = \alpha_V \).

*Proof.* We apply Lemma 16 successively to the lattice \( A = U \cup X \), the lattice \( B = V \cup Y \), and the lattice \( L = A \cup B \).

By (6.1), \( \alpha_X \), and \( \alpha_U \) extend uniquely to the congruence
\[
\beta = \alpha_X \cup \alpha_U \cup (\alpha_X \circ \alpha_U) \cup (\alpha_U \circ \alpha_X)
\]
on \( A \). Similarly, by (6.3), the congruences \( \alpha_Y \) and \( \alpha_V \) extend uniquely to the congruence
\[
\gamma = \alpha_Y \cup \alpha_V \cup (\alpha_Y \circ \alpha_V) \cup (\alpha_V \circ \alpha_Y)
\]
on \( B \).

To conclude the proof, we have to show that \( \beta \) and \( \gamma \) extend uniquely to a congruence \( \alpha \) of \( L \), that is, that
\[
\beta \big| (A \cap B) = \gamma \big| (A \cap B).
\]

By duality, it suffices to show that
\[
\beta \big| (A \cap B) \leq \gamma \big| (A \cap B),
\]
that is, that for any pair \( x < y \) of elements of \( A \cap B \) where \( x \equiv y \) (\( \beta \)), it follows that \( x \equiv y \) (\( \gamma \)). We first determine \( A \cap B \). By definition of \( X \) and \( Y \), \( X \cap Y = \{c\} \).

Then, by (Pv), we get
\[
A \cap B = (V \cap X) \cup (U \cap Y).
\]

If \( x, y \in V \cap X \), then \( x \equiv y \) (\( \alpha_X \)), and by (6.2), \( x \equiv y \) (\( \alpha_V \)) and so \( x \equiv y \) (\( \gamma \)).

Similarly, if \( x, y \in U \cap Y \), we conclude, by (6.4), that \( x \equiv y \) (\( \gamma \)).

Otherwise, since \( X = c \) and \( Y = c \), we conclude that \( x \in V \cap X \) and \( y \in U \cap Y \), and so that \( x < c < y \). Then, \( c \equiv y \) (\( \beta \)). But, \( c, y \in U \cap Y \) and
\[
\beta \big| (U \cap Y) = \alpha_U \big| (U \cap Y) = \alpha_Y \big| (U \cap Y) \subseteq \gamma.
\]

Thus \( c \equiv y \) (\( \gamma \)).

Furthermore, by (6.5), \( x \equiv y \) (\( \alpha_X \circ \alpha_U \)), that is, there is \( z \in X \cup U \) with \( x \equiv z \) (\( \alpha_X \)) and \( z \equiv y \) (\( \alpha_U \)).

Now \( x \in V \cap X \), a filter in \( X \), and \( z \in U \cap X \), another filter in \( X \), Thus \( x \lor z \in (U \cap X) \cap (V \cap X) = \{c\} \), that is \( x \lor z = c \). Then \( x \equiv c \) (\( \alpha_X \)). Since \( x, c \in V \cap X \), \( x \equiv c \) (\( \alpha_U \)) by (6.2). Thus \( x \equiv c \) (\( \gamma \)). Therefore \( x \equiv y \) (\( \gamma \)), thereby concluding the proof of (6.6), and so concluding the proof of the lemma. \( \square \)
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