On vertex decomposable simplicial complexes and their Alexander duals

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History

- Pure $k$-decomposable simplicial complexes, first were defined by Provan and Billera for pure simplicial complexes, in connection with their study of diameter problems for pure complexes.

- For 0-decomposable simplicial complexes (known as vertex decomposable), the definition was extended to non-pure complexes for by Björner and Wachs to obtain a new class of shellable simplicial complexes.
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- Extension of $k$-decomposability to non-pure complexes was introduced by Woodroofe in order to study the independence complex of a chordal clutter.
History

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Extension of $k$-decomposability to non-pure complexes was introduced by Woodroofe in order to study the independence complex of a chordal clutter.
Defined in a recursive manner, vertex decomposable simplicial complexes form a well-behaved class of simplicial complexes. In many research papers vertex decomposability was used in an interesting way to study the algebraic properties of monomial ideals and nice results on edge ideals were obtained by combinatorial topological techniques.
1. Preliminares

2. Vertex decomposable simplicial complexes and vertex splittable ideals
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3. Vertex cover ideals of graphs
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Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ with $\deg(x_i) = 1$.

Any finitely generated graded $R$-module $M$ (such as a homogenous ideal $I$) has a minimal graded free resolution of length at most $n$, which can be presented as follows:

$$0 \to \bigoplus_{j \geq 0} R(-j)^{\beta_{p,j}} \to \cdots \to \bigoplus_{j \geq 0} R(-j)^{\beta_{1,j}} \to \bigoplus_{j \geq 0} R(-j)^{\beta_{0,j}} \to M \to 0$$
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Basic facts

*The numbers* $\beta_{i,j}$ *are uniquely determined by* $M$:

$$\beta_{i,j}(M) = \dim_k \text{Tor}^R_i(M, k)_j$$

*and are called the* graded Betti numbers *of* $M$. 
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A particularly simple sort of graded module is provided by the ideals generated by monomials, which are called monomial ideals. However, despite the misleading appearance of simplicity, it is still an open problem to describe explicitly the graded Betti numbers even in this case.

An explicit minimal resolution for a family of monomial ideals, which are called stable ideals, has been given by Eliahou and Kervaire.
A particularly simple sort of graded module is provided by the ideals generated by monomials, which are called **monomial ideals**. However, despite the misleading appearance of simplicity, it is still an **open problem** to describe explicitly the graded Betti numbers even in this case.

An explicit minimal resolution for a family of monomial ideals, which are called **stable ideals**, has been given by **Eliahou and Kervaire**.
Betti splitting

**Eliahou-Kervaire Splitting:** Let $I$, $J$ and $K$ be monomial ideals such that $G(I)$, the unique set of minimal generators of $I$, is the disjoint union of $G(J)$ and $G(K)$. Then $I = J + K$ is an **Eliahou-Kervaire splitting** if there exists a splitting function $G(J \cap K) \to G(J) \times G(K)$ sending $w \mapsto (\varphi(w), \phi(w))$ such that

1. $w = \text{lcm}(\varphi(w), \phi(w))$ for all $w \in G(J \cap K)$, and
2. for every subset $S \subseteq G(J \cap K)$, $\text{lcm}(\varphi(S))$ and $\text{lcm}(\phi(S))$ strictly divide $\text{lcm}(S)$. 
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Betti splitting

[Fatabbi]. When \( I = J + K \) is an Eliahou-Kervaire splitting,

\[
\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)
\]

for all \( i \in \mathbb{N} \) and (multi)degrees \( j \).
There are other conditions on $I$, $J$ and $K$, beyond the criterion of Eliahou and Kervaire, that imply that formula for Betti numbers holds. Consider the ideal

$I = (x_1 x_2 x_3, x_1 x_3 x_5, x_1 x_4 x_5, x_2 x_3 x_4, x_2 x_4 x_5)$. There is no Eliahou and Kervaire splitting of $I$, but there are many ways to partition the minimal generators of $I$ to form smaller ideals $J$ and $K$ so that the formula for Betti numbers still holds. Set

$I = (x_1 x_2 x_3, x_1 x_3 x_5, x_1 x_4 x_5) + (x_2 x_3 x_4, x_2 x_4 x_5)$. 
Definition [Francisco, Ha, Van Tuyl]. Let $I$, $J$ and $K$ be monomial ideals such that $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Then $I = J + K$ is a Betti splitting if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

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[Francisco, Ha, Van Tuyl]. Let $I$, $J$ and $K$ be monomial ideals such that $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Considering

$$0 \rightarrow J \cap K \rightarrow J \oplus K \rightarrow J + K = I \rightarrow 0 \quad (1)$$

the following are equivalent:

1. $I = J + K$ is a Betti splitting.
2. the map $\text{Tor}_i(k, J \cap K)_j \rightarrow \text{Tor}_i(k, J)_j \oplus \text{Tor}_i(k, K)_j$ in the long exact sequence in Tor induced from (1) is the zero map.
3. applying the mapping cone construction to (1) gives a minimal free resolution of $I$. 

Betti splitting
Betti splitting

In general the mapping cone construction applied to (1) produces a free resolution of \( I \) that is \textit{not} necessarily minimal. In particular, the mapping cone construction implies that

\[
\beta_{i,j}(I) \leq \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)
\]

Betti splitting

If $I = J + K$ is a Betti splitting, then

\[
\text{reg}(I) = \max\{\text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1\}
\]
If $I = J + K$ is a Betti splitting, then

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- $\text{pd}(I) = \max\{\text{pd}(J), \text{pd}(K), \text{pd}(J \cap K) + 1\}$
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Alexander dual of a simplicial complex and an ideal

For a simplicial complex $\Delta$ with the vertex set $X$, the Alexander dual simplicial complex $\Delta^\vee$ of $\Delta$ is defined as follows:

$$\Delta^\vee = \{ F \subseteq X; X \setminus F \notin \Delta \}$$

For a squarefree monomial ideal $I = (x_{1,1} x_{1,2} \cdots x_{1,k_1}, \ldots, x_{n,1} x_{n,2} \cdots x_{n,k_n})$, Alexander dual ideal of $I$ is defined as:

$$I^\vee = (x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \cap \cdots \cap (x_{n,1}, x_{n,2}, \ldots, x_{n,k_n})$$
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$$I_{\Delta}^\vee = (I_{\Delta})^\vee = (x_F^c : F \text{ is a facet of } \Delta)$$
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For a squarefree monomial ideal

$I = (x_1, x_1, x_1, \ldots, x_n, x_n, x_n, \ldots, x_{n,k_n})$, Alexander dual ideal of $I$ is defined as:

$$I^\vee = (x_1, x_1, x_1, \ldots, x_1, x_1) \cap \cdots \cap (x_n, x_n, x_n, \ldots, x_n, x_n)$$

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Alexander dual ideal

\[ \Delta = \langle \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 5\}, \{2, 5\} \rangle \]

\[ \mathcal{N}(\Delta) = \{\{1, 2, 5\}, \{1, 3, 5\}, \{2, 4, 5\}, \{2, 3, 5\}, \{1, 4\}\} \]

\[ I_{\Delta} = (x_1x_2x_5, x_1x_3x_5, x_2x_4x_5, x_2x_3x_5, x_1x_4) \]

\[ I_{\Delta^\vee} = (x_4x_5, x_1x_5, x_1x_2, x_2x_3x_4, x_1x_3x_4) = (x_1, x_2, x_5) \cap (x_1, x_3, x_5) \cap (x_2, x_4, x_5) \cap (x_2, x_3, x_5) \cap (x_1, x_4) \]
Ideals with linear quotients were defined by Herzog and Takayama in connection to their work on minimal free resolution of monomial ideals.

A monomial ideal $I = (f_1, \ldots, f_m)$ has linear quotients, if there exists an order $f_1 < \cdots < f_m$ on the minimal generators of $I$ such that the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by a subset of variables for all $2 \leq i \leq m$. 
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[Herzog-Hibi-Zheng]. Let $\Delta$ be a simplicial complex and $I = I_\Delta$. Then

$\Delta$ is shellable $\iff$ $I^\wedge$ has linear quotients
Alexander dual concepts

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[Herzog-Hibi-Zheng]. Let \( \Delta \) be a simplicial complex and \( I = I_\Delta \). Then

\[ \Delta \text{ is shellable } \iff I^\vee \text{ has linear quotients} \]
Alexander dual concepts

[Eagon-Reiner]. Let $\Delta$ be a simplicial complex. Then

$$k[\Delta] \text{ is Cohen–Macaulay } \iff I_{\Delta^\vee} \text{ has linear resolution}$$

[Herzog-Hibi]. $k[\Delta]$ is sequentially Cohen-Macaulay if and only if $I_{\Delta^\vee}$ is componentwise linear.
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[Terai]. For a simplicial complex $\Delta$, $\text{pd}(I_{\Delta}) = \text{reg}(R/I_{\Delta^\vee})$. 
Alexander dual concepts

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[Terai]. For a simplicial complex \( \Delta \), \( \text{pd}(I_\Delta) = \text{reg}(R/I_{\Delta^\vee}) \).
Vertex decomposable simplicial complex

Let \( \Delta \) be a simplicial complex and \( F \in \Delta \). The deletion of \( F \) is defined as:

\[
\text{del}_{\Delta}(F) = \{ G \in \Delta : G \cap F = \emptyset \}
\]

and the link of \( F \)

\[
\text{lk}_{\Delta}(F) = \{ G \in \Delta : G \cap F = \emptyset, \ G \cup F \in \Delta \}
\]

Let \( \Delta \) be a simplicial complex on the vertex set \( V = \{ x_1, \ldots, x_n \} \). Then \( \Delta \) is vertex decomposable if either:

1. The only facet of \( \Delta \) is \( \{ x_1, \ldots, x_n \} \), or \( \Delta = \emptyset \).
2. There exists a vertex \( x \in V \) such that \( \text{del}_{\Delta}(x) \) and \( \text{lk}_{\Delta}(x) \) are vertex decomposable, and such that every facet of \( \text{del}_{\Delta}(x) \) is a facet of \( \Delta \).
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The vertex $x$ is called a shedding vertex for $\Delta$. 
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The vertex $x$ is called a shedding vertex for $\Delta$. 
If $\Delta$ is vertex decomposable with shedding vertex $x$, then

$$\Delta = del_{\Delta}(x) \cup (lk_{\Delta}(x) \ast \{x\}).$$

What is the properties of Alexander dual ideal of a vertex decomposable simplicial complex?
If $\Delta$ is vertex decomposable with shedding vertex $x$, then

$$\Delta = \text{del}_\Delta(x) \cup (\text{lk}_\Delta(x) * \{x\}).$$

What is the properties of Alexander dual ideal of a vertex decomposable simplicial complex?
Vertex splittable ideal

A monomial ideal $I$ of $R$ is called vertex splittable if it can be obtained by the following recursive procedure:

(i) If $u$ is a monomial and $I = (u)$, $I = (0)$ or $I = R$, then $I$ is a vertex splittable ideal.

(ii) If there is a variable $x \in X$ and vertex splittable ideals $I_1$ and $I_2 \subseteq k[X \setminus \{x\}]$ so that $I = xl_1 + I_2$, $I_2 \subseteq I_1$ and $G(I)$ is the disjoint union of $G(xl_1)$ and $G(I_2)$, then $I$ is a vertex splittable ideal.

With the above notations if $I = xl_1 + I_2$ is a vertex splittable ideal, then $xl_1 + I_2$ is called a vertex splitting for $I$. 
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(ii) If there is a variable $x \in X$ and vertex splittable ideals $I_1$ and $I_2 \subseteq k[X \setminus \{x\}]$ so that $I = xl_1 + I_2$, $I_2 \subseteq I_1$ and $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(xl_1)$ and $\mathcal{G}(I_2)$, then $I$ is a vertex splittable ideal.

With the above notations if $I = xl_1 + I_2$ is a vertex splittable ideal, then $xl_1 + I_2$ is called a **vertex splitting** for $I$. 
Vertex splittable ideal

Example.
▶ Let $I = (x_1^2 x_2^3, x_3 x_5^2, x_1 x_2 x_5, x_1 x_3 x_5, x_1 x_4 x_5, x_2 x_5^3 x_6)$. Then

$$I = x_1(x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5) + (x_3 x_5^2, x_2 x_5^3 x_6)$$

and

$$(x_3 x_5^2, x_2 x_5^3 x_6) \subseteq (x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5)$$

$$I_1 = (x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5) = x_5(x_2, x_3, x_4) + (x_1 x_2^3)$$

$$(x_2, x_3, x_4) = x_2(1) + (x_3, x_4)$$

$$I_2 = (x_3 x_5^2, x_2 x_5^3 x_6) = x_3(x_5^2) + (x_2 x_5^3 x_6)$$

▶ $I = (x_1 x_3, x_2 x_4)$ is not vertex splittable.
Example.

Let \( I = (x_1^2 x_2^3, x_3 x_5^2, x_1 x_2 x_5, x_1 x_3 x_5, x_1 x_4 x_5, x_2 x_5^3 x_6) \). Then

\[
I = x_1(x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5) + (x_3 x_5^2, x_2 x_5^3 x_6)
\]

and

\[
(x_3 x_5^2, x_2 x_5^3 x_6) \subseteq (x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5)
\]

\[
l_1 = (x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5) = x_5(x_2, x_3, x_4) + (x_1 x_2^3)
\]

\[
(x_2, x_3, x_4) = x_2(1) + (x_3, x_4)
\]

\[
l_2 = (x_3 x_5^2, x_2 x_5^3 x_6) = x_3(x_5^2) + (x_2 x_5^3 x_6)
\]

\[
I = (x_1 x_3, x_2 x_4)
\]

is not vertex splittable.
Properties of vertex splittable ideals

**Theorem.** A simplicial complex $\Delta$ is vertex decomposable if and only if $I_{\Delta^\vee}$ is a vertex splittable ideal.

**Theorem.** Any vertex splittable ideal has linear quotients.
Properties of vertex splittable ideals

**Theorem.** A simplicial complex $\Delta$ is vertex decomposable if and only if $I_\Delta$ is a vertex splittable ideal.

**Theorem.** Any vertex splittable ideal has linear quotients.

**Outline of proof:** Let $I = xI_1 + I_2$, where $I_1$ and $I_2$ are vertex splittable. By induction, let $f_1 < \cdots < f_r$ and $g_1 < \cdots < g_s$ be the order of linear quotients on the minimal generators of $I_1$ and $I_2$, respectively. The ordering

$$xf_1 < \cdots < xf_r < g_1 < \cdots < g_s$$

is an order of linear quotients on the minimal generators of $I$. 

Properties of vertex splittable ideals

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Properties of vertex splittable ideals

Example. \( I = x_1(x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) + (x_3x_5^2, x_2x_5^3x_6) \)
\( x_2x_5 < x_3x_5 < x_4x_5 < x_1x_2^3 \) and \( x_3x_5^2 < x_2x_5^3x_6 \) are order of linear quotients.

\[ \downarrow \]

\( x_1x_2x_5 < x_1x_3x_5 < x_1x_4x_5 < x_1^2x_2^3 < x_3x_5^2 < x_2x_5^3x_6 \)

is an order of linear quotients for \( I \).
Properties of vertex splittable ideals

Example. \( I = x_1(x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) + (x_3x_5^2, x_2x_5^3x_6) \)
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is an order of linear quotients for \( I \).

Corollary. Let \( I \) be a vertex splittable ideal generated by monomials in the same degrees. Then \( I \) has a linear resolution. (\( \Delta \) pure + vertex decomposable \( \Rightarrow \) \( \Delta \) is Cohen-Macaulay)
Properties of vertex splittable ideals

Example. \( I = x_1(x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) + (x_3x_5^2, x_2x_5^3x_6) \)
\( x_2x_5 < x_3x_5 < x_4x_5 < x_1x_2^3 \) and \( x_3x_5^2 < x_2x_5^3x_6 \) are order of linear quotients.

\( \Downarrow \)

\( x_1x_2x_5 < x_1x_3x_5 < x_1x_4x_5 < x_1^2x_2^3 < x_3x_5^2 < x_2x_5^3x_6 \)

is an order of linear quotients for \( I \).

Corollary. Let \( I \) be a vertex splittable ideal generated by monomials in the same degrees. Then \( I \) has a linear resolution.
\((\Delta \text{ pure } + \text{ vertex decomposable } \Rightarrow \Delta \text{ is Cohen-Macaulay})\)
Betti splitting

Resolution by mapping cone for ideals with linear quotients:
[Herzog, Takayama].
I : a monomial ideal with linear quotients with the ordering
f_1 < \cdots < f_m on its minimal generators
I_j = (f_1, \ldots, f_j)
L_j = (u_1, \ldots, u_j) : u_{j+1}
I_{j+1}/I_j \cong R/L_j \Rightarrow we get the exact sequences

0 \to R/L_j \xrightarrow{u_{j+1}} R/I_j \to R/I_{j+1} \to 0
Betti splitting

Resolution by mapping cone for ideals with linear quotients: [Herzog, Takayama].

$I$: a monomial ideal with linear quotients with the ordering $f_1 < \cdots < f_m$ on its minimal generators

$l_j = (f_1, \ldots, f_j)$

$L_j = (u_1, \ldots, u_j): u_{j+1}$

$l_{j+1}/l_j \cong R/L_j \Rightarrow$ we get the exact sequences

$$0 \rightarrow R/L_j \xrightarrow{u_{j+1}} R/l_j \rightarrow R/l_{j+1} \rightarrow 0$$

$F(j)$: minimal graded free resolution of $R/l_j$

$K(j)$: the Koszul complex for the regular sequence $x_{k_1}, \ldots, x_{k_l}$, where $\text{set}(u_{j+1}) = \{x_{k_1}, \ldots, x_{k_l}\}$. 
Betti splitting

Resolution by mapping cone for ideals with linear quotients:
[Herzog, Takayama].
$I$ : a monomial ideal with linear quotients with the ordering
$f_1 < \cdots < f_m$ on its minimal generators
$I_j = (f_1, \ldots, f_j)$
$L_j = (u_1, \ldots, u_j) : u_{j+1}$
$I_{j+1}/I_j \cong R/L_j \Rightarrow$ we get the exact sequences

$$0 \to R/L_j \overset{u_{j+1}}{\to} R/I_j \to R/I_{j+1} \to 0$$

$F(j) : \text{minimal graded free resolution of } R/I_j$
$K(j) : \text{the Koszul complex for the regular sequence } x_{k_1}, \ldots, x_{k_l}$,
where set$(u_{j+1}) = \{x_{k_1}, \ldots, x_{k_l}\}$. 
Betti splitting

\[ \psi(j) : K(j) \to F(j) \text{ a graded complex homomorphism lifting} \]
\[ R/L_j \to R/I_j \]
The mapping cone \( C(\psi(j)) \) of \( \psi(j) \) yields a minimal graded free resolution of \( R/I_{j+1} \).

\[ \beta_{i,j}(I) = \sum_{\deg(f_t) = j-i} \binom{|\text{set}_I(f_t)|}{i} \cdot \]

By iterated mapping cones one obtains step by step a minimal graded free resolution of \( R/I \).
Betti splitting

$I$: a vertex splittable ideal with vertex splitting $I = xl_1 + l_2$

$$\beta_{i,j}(I) = \sum_{\deg(f_t) = j-i} \binom{|\text{set}_I(f_t)|}{i}.$$ 

$f_1 < \cdots < f_r$: the order of linear quotients on the minimal generators of $l_1$

g_1 < \cdots < g_s$: the order of linear quotients on the minimal generators of $l_2$

$$xf_1 < \cdots < xf_r < g_1 < \cdots < g_s$$

is an order of linear quotients for $l$. 
Betti splitting

I : a vertex splittable ideal with vertex splitting \( I = xI_1 + I_2 \)

\[
\beta_{i,j}(I) = \sum_{\text{deg}(f_t)=j-i} \binom{|\text{set}_l(f_t)|}{i}.
\]

\( f_1 < \cdots < f_r : \) the order of linear quotients on the minimal generators of \( I_1 \)

\( g_1 < \cdots < g_s : \) the order of linear quotients on the minimal generators of \( I_2 \)

\( xf_1 < \cdots < xf_r < g_1 < \cdots < g_s \)

is an order of linear quotients for \( I \),

\[
\text{set}_l(xf_t) = \text{set}_{l_1}(f_t) \quad (1 \leq t \leq r)
\]

and

\[
\text{set}_l(g_k) = \{x\} \cup \text{set}_{l_2}(g_k) \quad (1 \leq k \leq s)
\]
Betti splitting

$I : a vertex splittable ideal with vertex splitting $I = xI_1 + I_2$

\[
\beta_{i,j}(I) = \sum_{\deg(f_t) = j-i} \binom{\left| \text{set}_I(f_t) \right|}{i}.
\]

$f_1 < \cdots < f_r :$ the order of linear quotients on the minimal generators of $I_1$

g_1 < \cdots < g_s :$ the order of linear quotients on the minimal generators of $I_2$

$xf_1 < \cdots < xf_r < g_1 < \cdots < g_s$

is an order of linear quotients for $I$,

\[
\text{set}_I(xf_t) = \text{set}_{I_1}(f_t) \quad (1 \leq t \leq r)
\]

and

\[
\text{set}_I(g_k) = \{x\} \cup \text{set}_{I_2}(g_k) \quad (1 \leq k \leq s)
\]
Betti splitting

\[ \beta_{i,j}(I) = \sum_{\text{deg}(f_t)=j-i-1} \binom{|\text{set}_I(xf_t)|}{i} + \sum_{\text{deg}(g_k)=j-i} \binom{|\text{set}_I(g_k)|}{i} \]

\[ \downarrow \]

\[ \beta_{i,j}(I) = \sum_{\text{deg}(f_t)=j-i-1} \binom{|\text{set}_{I_1}(f_t)|}{i} + \sum_{\text{deg}(g_k)=j-i} \binom{|\text{set}_{I_2}(g_k)| + 1}{i} \]

Applying the equality

\[ \binom{|\text{set}_{I_2}(g_k)| + 1}{i} = \binom{|\text{set}_{I_2}(g_k)|}{i} + \binom{|\text{set}_{I_2}(g_k)|}{i-1} \]
Betti splitting

\[
\beta_{i,j}(I) = \sum_{\deg(f_t) = j - i - 1} \left( \left| \text{set}_1(xf_t) \right| \right) + \sum_{\deg(g_k) = j - i} \left( \left| \text{set}_1(g_k) \right| \right)
\]

\[
\downarrow
\]

\[
\beta_{i,j}(I) = \sum_{\deg(f_t) = j - i - 1} \left( \left| \text{set}_1(f_t) \right| \right) + \sum_{\deg(g_k) = j - i} \left( \left| \text{set}_2(g_k) \right| + 1 \right)
\]

Applying the equality

\[
\left( \left| \text{set}_2(g_k) \right| + 1 \right) = \left( \left| \text{set}_2(g_k) \right| \right) + \left( \left| \text{set}_2(g_k) \right| \right)
\]
Betti splitting

\[
\sum_{\deg(g_k) = j-i} \binom{|\text{set}_{l_2}(g_k)| + 1}{i} = \\
\sum_{\deg(g_k) = j-i} \binom{|\text{set}_{l_2}(g_k)|}{i} + \sum_{\deg(g_k) = j-i} \binom{|\text{set}_{l_2}(g_k)|}{i-1} \\
= \beta_{i,j}(l_2) + \beta_{i-1,j-1}(l_2)
\]

Also

\[
\sum_{\deg(f_t) = j-i-1} \binom{|\text{set}_{l_1}(f_t)|}{i} = \beta_{i,j-1}(l_1)
\]
Betti splitting

$$\beta_{i,j}(l) = \beta_{i,j-1}(l_1) + \beta_{i,j}(l_2) + \beta_{i-1,j-1}(l_2).$$

$l_2 \subseteq l_1$ implies that $xl_1 \cap l_2 = xl_2$. Also $\beta_{i,j-1}(l_1) = \beta_{i,j}(xl_1)$ and $\beta_{i-1,j-1}(l_2) = \beta_{i-1,j}(xl_2)$.

$$\Rightarrow$$

$$\beta_{i,j}(l) = \beta_{i,j}(xl_1) + \beta_{i,j}(l_2) + \beta_{i-1,j}(xl_1 \cap l_2).$$
Betti splitting

\[ \beta_{i,j}(l) = \beta_{i,j-1}(l_1) + \beta_{i,j}(l_2) + \beta_{i-1,j-1}(l_2). \]

\( l_2 \subseteq l_1 \) implies that \( x l_1 \cap l_2 = x l_2 \). Also \( \beta_{i,j-1}(l_1) = \beta_{i,j}(x l_1) \) and \( \beta_{i-1,j-1}(l_2) = \beta_{i-1,j}(x l_2) \).

\[ \downarrow \]

\[ \beta_{i,j}(l) = \beta_{i,j}(x l_1) + \beta_{i,j}(l_2) + \beta_{i-1,j}(x l_1 \cap l_2). \]
**Theorem.** Let \( I = xl_1 + l_2 \) be a vertex splitting for the monomial ideal \( I \). Then \( I = xl_1 + l_2 \) is a Betti splitting.

\[
\begin{align*}
0 &\to xl_2 \to xl_1 \oplus l_2 \to I \to 0
\end{align*}
\]

Applying the mapping cone to gives a minimal free resolution of \( I \)
Betti splitting

**Theorem.** Let $I = xl_1 + l_2$ be a vertex splitting for the monomial ideal $I$. Then $I = xl_1 + l_2$ is a Betti splitting.

\[\downarrow\]

**Applying the mapping cone to**

\[0 \rightarrow xl_2 \rightarrow xl_1 \oplus l_2 \rightarrow I \rightarrow 0\]

**gives a minimal free resolution of $I$**
Corollary. For a vertex splittable ideal \( I \) with vertex splitting \( I = xI_1 + I_2 \), the graded Betti numbers of \( I \) can be computed by the following recursive formula

\[
\beta_{i,j}(I) = \beta_{i,j-1}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j-1}(I_2).
\]

Corollary. Let \( \Delta \) be a vertex decomposable simplicial complex, \( x \) a shedding vertex of \( \Delta \), \( \Delta_1 = \text{del}_\Delta(x) \) and \( \Delta_2 = \text{lk}_\Delta(x) \). Then

\[
\beta_{i,j}(I_{\Delta^\vee}) = \beta_{i,j-1}(I_{\Delta_1^\vee}) + \beta_{i,j}(I_{\Delta_2^\vee}) + \beta_{i-1,j-1}(I_{\Delta_2^\vee}).
\]
Corollary. For a vertex splittable ideal $I$ with vertex splitting $I = xI_1 + I_2$, the graded Betti numbers of $I$ can be computed by the following recursive formula

$$\beta_{i,j}(I) = \beta_{i,j-1}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j-1}(I_2).$$

Corollary. Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$, $\Delta_1 = \text{del}_\Delta(x)$ and $\Delta_2 = \text{lk}_\Delta(x)$. Then

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Corollary. For a vertex splittable ideal $I$ with vertex splitting $I = xI_1 + I_2$, the graded Betti numbers of $I$ can be computed by the following recursive formula

$$\beta_{i,j}(I) = \beta_{i,j-1}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j-1}(I_2).$$

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$$\beta_{i,j}(I_{\Delta \vee}) = \beta_{i,j-1}(I_{\Delta_1 \vee}) + \beta_{i,j}(I_{\Delta_2 \vee}) + \beta_{i-1,j-1}(I_{\Delta_2 \vee}).$$

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Corollary. Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$ and $\Delta_1 = \text{del}_\Delta(x)$ and $\Delta_2 = \text{lk}_\Delta(x)$. Then

$$\text{pd}(R/I_\Delta) = \max\{\text{pd}(R/I_{\Delta_1}) + 1, \text{pd}(R/I_{\Delta_2})\},$$

$$\text{reg}(R/I_\Delta) = \max\{\text{reg}(R/I_{\Delta_1}), \text{reg}(R/I_{\Delta_2}) + 1\}.$$
Vertex cover ideal of a graph

Let $G = (V(G), E(G))$ be a graph. A subset $C \subseteq V(G)$ is called a vertex cover of $G$ if it intersects all the edges of $G$.

A vertex cover $C$ is a minimal vertex cover if no proper subset of $C$ is a vertex cover.
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Let $C_1, \ldots, C_k$ be the minimal vertex covers of $G$. Then

$$J(G) = (x^{C_i} : 1 \leq i \leq k)$$

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$$J(G) = I(G)^\vee = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j)$$
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A graph $G$ is called vertex decomposable if $\Delta_G$ is vertex decomposable.

If $\Delta = \Delta_G$ and $v \in V(G)$, then $\text{del}_{\Delta}(v) = \Delta_G \{v\}$ and $\text{lk}_{\Delta}(v) = \Delta_G \setminus N_G[v]$. 
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**Theorem.** Let $G$ be a vertex decomposable graph, $v \in V(G)$ be a shedding vertex of $G$, $G' = G \setminus \{v\}$, $G'' = G \setminus N_G[v]$ and $\deg_G(v) = t$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G''')).$$
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$$
Vertex cover ideal of a vertex decomposable graph

[Francisco, Ha, Van Tuyl]. Let $G$ be a Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are Cohen-Macaulay and $\deg_G(y) = t$. Then

$$\beta_i(J(G)) = \beta_i(J(G')) + \beta_i(J(G'')) + \beta_{i-1}(J(G''))$$
Theorem. Let $G$ be a sequentially Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are sequentially Cohen-Macaulay and $\deg_G(y) = t$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G''))$$

Also $y$ is a shedding vertex of $G$. 

$$\{\text{sequentially Cohen-Macaulay bipartite graph}\} \subseteq \{\text{vertex decomposable graphs}\}$$
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**Theorem.** Let $G$ be a sequentially Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are sequentially Cohen-Macaulay and $\deg_G(y) = t$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G''))$$

\[\text{\{sequentially Cohen-Macaulay bipartite graph\}} \subseteq \text{\{vertex decomposable graphs\}}\]

Also $y$ is a shedding vertex of $G$. 
Vertex cover ideal of a vertex decomposable graph

A graph $G$ is called **chordal**, if it contains no induced cycle of length 4 or greater.

In a graph $G$, a vertex $x$ is called a **simplicial vertex** if the induced subgraph on the set $N_G[x]$ is a complete graph.
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Vertex cover ideal of a vertex decomposable graph

[Dirac]. Any chordal graph has a simplicial vertex.

Theorem. Let $G$ be a chordal graph with simplicial vertex $x$ and $y \in N_G(x)$ with $\deg_G(y) = t$. Let $G' = G \setminus \{y\}$ and $G'' = G \setminus N_G[y]$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G''))$$
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**Chordal $\Rightarrow$ vertex decomposable**

A neighbour of a simplicial vertex is a shedding vertex.
Vertex cover ideal of a vertex decomposable graph

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**Theorem.** Let $G$ be a chordal graph with simplicial vertex $x$ and $y \in N_G(x)$ with $\deg_G(y) = t$. Let $G' = G \setminus \{y\}$ and $G'' = G \setminus N_G[y]$. Then

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**Chordal $\Rightarrow$ vertex decomposable**

A neighbour of a simplicial vertex is a shedding vertex.
Vertex splittable edge ideals

**Theorem.** Let $G$ be a chordal graph. Then $I(G^c)$ is a vertex splittable ideal.

$x \in V(G):$ a simplicial vertex

$N_G(x) = \{x_1, \ldots, x_k\}$



$I(G^c) = x(x_{k+1}, \ldots, x_n) + I(G_0^c)$

, where $G_0 = G \setminus \{x\}$

$I(G_0^c) \subseteq (x_{k+1}, \ldots, x_n)$

$G_0$ is chordal $\Rightarrow$ $I(G_0^c)$ is vertex splittable

$(x_{k+1}, \ldots, x_n)$ is vertex splittable
Vertex splittable edge ideals

**Theorem.** Let $G$ be a chordal graph. Then $I(G^c)$ is a vertex splittable ideal.

Let $x \in V(G)$: a simplicial vertex $N_G(x) = \{x_1, \ldots, x_k\}$

$$I(G^c) = x(x_{k+1}, \ldots, x_n) + I(G^c_0)$$

where $G_0 = G \setminus \{x\}$

$$I(G^c_0) \subseteq (x_{k+1}, \ldots, x_n)$$

$G_0$ is chordal $\Rightarrow$ $I(G^c_0)$ is vertex splittable

$(x_{k+1}, \ldots, x_n)$ is vertex splittable
[Fröberg]. For a graph $G$, the edge ideal $I(G)$ has linear resolution if and only if $G^c$ is a chordal graph.

\[\downarrow\]

**Corollary.** For a graph $G$, the edge ideal $I(G)$ is vertex splittable if and only if $I(G)$ has linear resolution.
For a graph $G$, the edge ideal $I(G)$ has linear resolution if and only if $G^c$ is a chordal graph.

\[\Downarrow\]

**Corollary.** For a graph $G$, the edge ideal $I(G)$ is vertex splittable if and only if $I(G)$ has linear resolution.
REFERENCES

[1] G.A. Dirac, *On rigid circuit graphs*. Abh. Math. Sem. Univ. Hamburg 24 (1961), 71-76.
[2] J. A. Eagon; V. Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*. J. Pure Appl. Algebra 130 (1998), no. 3, 265–275.
[3] S. Eliahou; M. Kervaire, *Minimal resolution of some monomial ideals*. J. of Algebra 129, (1990), 1-25.
[4] C. A. Francisco; H. T. Hà; A. Van Tuyl, *Splittings of monomial ideals*. Proc. Amer. Math. Soc. 137 (2009), no. 10, 3271–3282.
[5] R. Fröberg, *On Stanley-Reisner rings*. Topics in Algebra, Banach Center Publications, 26 (1990), 57–70.
[6] H. T. Hà; R. Woodroofe, *Results on the regularity of square-free monomial ideals*. Preprint, arXiv:math.AC/1301.6779v1.
[7] J. Herzog; Y. Takayama, *Resolutions by mapping cones*. The Roos Festschrift volume, 2. Homology Homotopy Appl. 4, no. 2, part 2, (2002), 277-294.
[8] S. Moradi; F. Khosh-Ahang *On vertex decomposable simplicial complexes and their Alexander duals*. To appear in Math. Scand.
[9] J. S. Provan; L. J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*. Math. Oper. Res. 5 (1980), no. 4, 576–594.
Thanks!