Linking electroweak and gravitational generators

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Using complexified quaternions, an intriguing link between generators of two different and surprisingly commuting four-dimensional representations of the SU(2) × U(1) Lie group, and generators of two four-dimensional spin \( \frac{1}{2} \) representations of the Spin(3, 1) Lie group is established: the former generators completely determine the latter ones, and cross-combined they constitute two different, but closely related, four-dimensional representations of Spin(3, 1) × SU(2) × U(1). These representations are used to construct a Spin(3, 1) × SU(2) × U(1) gauge invariant Lagrangian, containing two four-spinors consisting not as usual of Weyl two-spinors of opposite helicity and equal weak isospin, but instead of Weyl two-spinors of opposite weak isospin and equal helicity, a construction which arises naturally from the mathematical formalism itself. A possible future generalization, using complexified octonions, is discussed.

I. INTRODUCTION AND MAIN RESULT

The quest for unification of the fundamental forces of Nature began with the unification of electricity and magnetism, by Maxwell, resulting in the electromagnetic force \( [1] \). A century later this force was united with the weak nuclear force, resulting in the electroweak force \( [2] \). Even though there have been promising theoretical propositions of unification of this force with the strong nuclear force, none of these have been experimentally verified. Gravity \( [3] \) is in its own category, stubbornly refusing to join the quantum-party of unification, the currently most promising, though highly speculative, theoretical proposition being string/M-theory \([4, 5]\).

This article makes no claim of any unification. More modestly, it is the purpose of this article to point out, and utilize, an intriguing link between generators of two different and surprisingly commuting four-dimensional representations of the SU(2) × U(1) Lie group, relevant for the weak nuclear force, of course. The complex numbers \( \mathbb{C} \) and the quaternions \( \mathbb{H} \) belong (note that they are not division algebras), see Ref. \([6, 7]\). For a comprehensive review of the octonions, see Ref. \([11]\). For a monograph on octonions and other nonassociative algebras, see Ref. \([13]\).

The paper is organized as follows: Sec. II introduces the necessary notation and conventions used. Sec. III sets up the main machinery needed. Sec. IV contains the main result of the paper; the Spin(3, 1) × SU(2) × U(1) gauge invariant Lagrangian, Eq. (14). Sec. V discusses various notable features of this Lagrangian, and points to a future generalization, using complexified octonions. There are two appendices: Appendix A contains various useful identities valid for any composition algebra, a class to which both the complex quaternions and complex octonions belong. Appendix B contains the proofs of most of the assertions of Sec. III.

II. NOTATION AND CONVENTIONS

The set of complexified quaternions is denoted \( \mathbb{C} \otimes \mathbb{H} \), equal to \( \mathbb{H} \otimes \mathbb{C} \) because the complex numbers \( \mathbb{C} \) and the quaternions \( \mathbb{H} \) are assumed to commute. The imaginary unit of \( \mathbb{C} \) is denoted \( i \), obeying \( i^2 = -1 \), of course. The imaginary units of \( \mathbb{H} \) are denoted \( e_i = (e_1, e_2, e_3) \), obeying \( e_ie_j = -\delta_{ij} + \epsilon_{ijk}e_k \), where \( \epsilon_{ijk} \) is the Levi-Civita symbol with \( \epsilon_{123} = +1 \). The basis for \( \mathbb{C} \otimes \mathbb{H} \) (over \( \mathbb{C} \)) is taken as \( e_0 = (e_0, e_1) = (1, e_1) \).

Latin indices from the beginning of the alphabet run from 0 to 3, and are raised and lowered with \( \eta^{ab} \) and \( \eta_{ab} \), respectively, \( \eta_{ab} \) being the Minkowski metric. Latin
indices from the middle of the alphabet, beginning at \(i\), run from 1 to 3, and are raised and lowered with \(\delta^{ij}\) and \(\delta_{ij}\), respectively. Greek indices run from 0 to 3, and are raised and lowered with \(\eta^{\mu\nu}\) and \(g_{\mu\nu}\), respectively, \(g_{\mu\nu}\) being the metric of curved spacetime. The Einstein summation convention is adhered to throughout.

Let \(c^a \in \mathbb{C}\). Complex conjugation is the involution \(\cdot^* : \mathbb{C} \otimes \mathbb{H} \to \mathbb{C}^* \otimes \mathbb{H}\) defined by \((c^a e^a)^* = -(c^b)^* e_0 + (c^i)^* e_i\), and quaternionic conjugation is the involution \(\overline{c} : \mathbb{C} \otimes \mathbb{H} \to \mathbb{C} \otimes \mathbb{H}\) defined by \(\overline{c}^* e_a = c^0 e_0 - c^i e_i\). Note that \(\overline{c} = -c^*\).

The bilinear inner product \(\langle \cdot, \cdot \rangle : (\mathbb{C} \otimes \mathbb{H})^2 \to \mathbb{C}\) is defined by
\[
2 \langle x, y \rangle = x\overline{y} + y\overline{x} = \overline{x}y + \overline{y}x.
\]
Note that \(\langle e_a, e_b \rangle = \eta_{ab}\).

The set of \(n\)-dimensional square matrices over some field \(\mathbb{F}\) is denoted \(M(n, \mathbb{F})\). The \(n\)-dimensional identity matrix is denoted \(1_n\), and the \(n\)-dimensional matrix with zero entries only is denoted \(0_n\). The so-called 'eta-transpose' \(\cdot^\eta : M(4, \mathbb{C}) \to M(4, \mathbb{C})\) is defined by
\[
A^\eta = \eta A^T \eta,
\]
where \(\eta\) is the Minkowski metric as a matrix; \((\eta)^a_b = \eta_{ab}\). In terms of this 'eta-transpose', the two (anti)commutator-like (even though they do not have all the properties of the usual (anti)commutator) brackets \([\cdot, \cdot]_{\eta\pm} : M(4, \mathbb{C}) \to M(4, \mathbb{C})\), are defined by
\[
[A, B]_{\eta\pm} = A^\eta B \pm B^\eta A.
\]

**III. SETUP**

For analytical proofs of various assertions of this section, see Appendix D.

**A. Generators of SU(2) \(\times\) U(1)**

Define the matrices \(\Gamma_{L|a}, \Gamma_{R|a} \in M(4, \mathbb{C})\) by
\[
(\Gamma_{L|a})_{cd} = \langle e_c, e_a e_d \rangle, \quad (\Gamma_{R|a})_{cd} = \langle e_c, e_d e_a \rangle,
\]
where \(L\) and \(R\) refer to whether \(e_a\) is multiplied from the left or the right. They obey (note the mixed positions of \(L\) and \(R\))
\[
\Gamma^\dagger_{L|a} = -\Gamma_{R|a}, \quad \Gamma^\dagger_{R|a} = -\Gamma_{L|a},
\]
Also, they obey the Lie algebra (where \(X\) denotes either \(L\) or \(R\), a shorthand notation frequently used below)
\[
-2\varepsilon_{ij}^k \Gamma_{L|b} = \left[\Gamma_{L|i}, \Gamma_{L|j}\right], \quad +2\varepsilon_{ij}^k \Gamma_{R|b} = \left[\Gamma_{R|i}, \Gamma_{R|j}\right],
\]
so that \(\Gamma^\dagger_{L|a}\) and \(\Gamma^\dagger_{R|a}\) (note the i's) each constitute hermitian generators of two different, but closely related (by complex conjugation), four-dimensional representations of SU(2) \(\times\) U(1). In fact, due to the surprising relation (which originally prompted this research)
\[
0_4 = \left[\Gamma_{L|a}, \Gamma_{R|b}\right],
\]
they constitute two commuting representations.

Furthermore, they obey the anticommutator-like relation
\[
2\eta_{ab} 1_4 = \left[\Gamma_{X|a}, \Gamma_{X|b}\right]_{\eta} = \Gamma^\eta_{X|a} \Gamma_{X|b} + \Gamma^\eta_{X|b} \Gamma_{X|a},
\]
which is the reason for the choice of '\(T\)' as the designating letter, \(\Gamma_{L|a}\) and \(\Gamma_{R|a}\) being reminiscent of the usual Clifford gamma matrices.

**B. Generators of Spin(3,1)**

Define the matrices \(\Sigma_{L|ab}, \Sigma_{R|ab} \in M(4, \mathbb{C})\) by
\[
4i (\Sigma_{L|ab})_{cd} = \langle e_a e_c, e_b e_d \rangle - \langle e_a e_d, e_b e_c \rangle, \quad 4i (\Sigma_{R|ab})_{cd} = \langle e_c e_a, e_d e_b \rangle - \langle e_d e_a, e_c e_b \rangle,
\]
where \(L\) and \(R\) refer to whether \(e_a\) and \(e_b\) are multiplied from the left or the right. They are related to \(\Gamma_{L|a}\) and \(\Gamma_{R|b}\) by the commutator-like relations
\[
4i \Sigma_{X|ab} = \left[\Gamma_{X|a}, \Gamma_{X|b}\right]_{\eta} = \Gamma^\eta_{X|a} \Gamma_{X|b} - \Gamma^\eta_{X|b} \Gamma_{X|a}.
\]
They obey (note the mixed positions of \(L\) and \(R\))
\[
\Sigma^\dagger_{L|ab} = -\Sigma_{R|ab}, \quad \Sigma^\dagger_{R|ab} = -\Sigma_{L|ab},
\]
Also, they obey the Lie algebra
\[
i \left[\Sigma_{X|ab}, \Sigma_{X|cd}\right] = \eta_{ac} \Sigma_{X|bd} - \eta_{ad} \Sigma_{X|bc} - \eta_{bc} \Sigma_{X|ad} + \eta_{bd} \Sigma_{X|ac},
\]
so that \(\Sigma_{L|ab}\) and \(\Sigma_{R|ab}\) constitute generators of two different, but closely related (by complex conjugation), four-dimensional spin \(\frac{3}{2}\) representations of Spin(3,1), because
\[
\frac{1}{2} \Sigma^2_{L|ij} \Sigma_{L|i} = \frac{1}{2} \Sigma^2_{R|ij} \Sigma_{R|i} = \frac{3}{4} 1_4.
\]
C. Generators of Spin (3, 1) \times SU (2) \times U (1)

Eq. (5) implies that
\[
\begin{align*}
0_4 &= [\Gamma_L |a, \Sigma_{R|cd}], \\
0_4 &= [\Gamma_R |a, \Sigma_{L|cd}].
\end{align*}
\]

In conjunction with Eq. (5), these relations imply that (note the crossing of the

D. Generators of SO (3, 1)

Define the matrices \( \Sigma_{V|ab} \in M (4, \mathbb{R}) \) by
\[
i (\Sigma_{V|ab})^c_{\mu} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}.
\]

They obey the exact same Lie algebra as do \( \Sigma_{L|ab} \) and \( \Sigma_{R|ab} \), i.e., the Lie algebra given by Eq. (10) with \( \Sigma_{X|ab} \) replaced by \( \Sigma_{V|ab} \), so they constitute generators of the vector representation of SO (3, 1), because
\[
\frac{1}{2} (\Sigma_{V|ij} \Sigma_{V|j})_{kl} = 2 (\mathbb{I}_3)_{kl}.
\]

Remark Note that under transposition, \( \Sigma_{V|ab} \) transforms as \( \Sigma_{V|a'b'} \) due to the results of Sec. III, the Lagrangian is invariants.

E. Transformations and invariants

Define the matrices \( \Lambda_L, \Lambda_R \in M (4, \mathbb{C}) \) and \( \Lambda_V \in M (4, \mathbb{R}) \) by
\[
\begin{align*}
\Lambda_L &= \exp \left( -i \frac{1}{2} g_{ab} \Sigma_{L|ab} \right), \\
\Lambda_R &= \exp \left( -i \frac{1}{2} g_{ab} \Sigma_{R|ab} \right), \\
\Lambda_V &= \exp \left( -i \frac{1}{2} g_{ab} \Sigma_{V|ab} \right),
\end{align*}
\]

where \( \theta_{ab} = -\theta_{ba} \in \mathbb{R} \). From Eq. (9) it straightforwardly follows that
\[
\begin{align*}
\Lambda_L^T &= \Lambda_R, \\
\Lambda_R^T &= \Lambda_L, \\
\Lambda_L = \eta \Lambda_L^{-1} \eta, \\
\Lambda_R = \eta \Lambda_R^{-1} \eta, \\
\Lambda_V^T &= \eta \Lambda_V^{-1} \eta, \\
\Lambda_V = \eta \Lambda_V^{-1} \eta.
\end{align*}
\]

Remark Note that under transposition, \( \Lambda_L \) and \( \Lambda_R \) surprisingly behave exactly as does \( \Lambda_V \), which obeys \( \Lambda_V^T = \eta \Lambda_V^{-1} \eta \), the relation responsible for the invariance of the line element in the special theory of relativity.

From Eqs. (13) it follows that
\[
\begin{align*}
(\Lambda_V)^a_{\ b} \Gamma^b_{\ c} &= \Lambda_L^a_{\ b} \Gamma^b_{\ c} \Lambda_X, \\
(\Lambda_V^{-1})^a_{\ b} \Gamma^b_{\ c} &= \Lambda_L^a_{\ b} \Gamma^b_{\ c} \Lambda_X.
\end{align*}
\]

These relations imply, that if \( \psi_X \) are two four-spinors transforming as \( \psi'_X = \Lambda_X \psi_X \) under a Lorentz transformation, then \( \psi'_X \Gamma^X_{\ a} \partial_a \psi_X \) are invariants, because \( \partial_a \) transforms as \( \partial_a = (\Lambda_V^{-1})^a_{\ b} \partial_b \), and \( \psi'_X \eta \psi_R \) and \( \psi'_X \eta \psi_L \) (note the surprising appearance of \( \eta \)) are invariants.

IV. LAGRANGIAN

Consider the Lagrangian (note the explicit appearance of \( \eta \) in the mass terms)
\[
\mathcal{L} = \Psi^\dagger \left( i e^\mu_a \Gamma^L_{\ a|\mu} D \psi + m^a_{\ \eta} e^\mu_a \Gamma^R_{\ a|\mu} D \psi \right) \Psi + \text{h.c.,}
\]

where (note for the inner interactions \( G^{\text{inner}}_{X|\mu} \) the crossing of the \( L \) and \( R \) sectors)
\[
\begin{align*}
D_{X|\mu} &= 1_d \partial_\mu + G^{\text{outer}}_{X|\mu} + G^{\text{inner}}_{X|\mu}, \\
G^{\text{outer}}_{X|\mu} &= \frac{1}{2} \omega_{ab} \Sigma_{X|ab}, \\
G^{\text{inner}}_{X|\mu} &= \frac{i}{\hbar} g t \Lambda_{L|\mu} (i \Gamma_{R|\mu}) + \frac{i}{\hbar} g' y L B_\mu (i \Gamma_{R|\mu}), \\
G^{\text{inner}}_{R|\mu} &= \frac{i}{\hbar} g t \Lambda_{R|\mu} (i \Gamma_{L|\mu}) + \frac{i}{\hbar} g' y R B_\mu (i \Gamma_{L|\mu}).
\end{align*}
\]

The fields are: An eight-spinor field \( \Psi^T = (\psi^T_L, \psi^T_R) \), a vierbein field \( e^a_\mu \) and its associated minimal spin connection \( \omega_{ab} = g_{ab} \partial_\mu e^{\mu}_{\ \sigma} \) \( \nabla^e \sigma \), see Ref. [2 Sec. 31A], and \( SU (2) \) and \( U (1) \) gauge fields \( W^i_\mu \) and \( B_\mu \), respectively. The constants are: Two real masses \( m_1, m_2 \in \mathbb{R} \) combined into a complex mass \( m = m_1 + i m_2 \), coupling constants \( g \) and \( g' \) for \( SU (2) \) and \( U (1) \), respectively, and charges \( t_X \) and \( y_L \) for \( SU (2) \) and \( U (1) \), respectively, where \( t_X = 0 \) corresponds to an \( SU (2) \) singlet, and \( t_X = \frac{1}{2} \) corresponds to an \( SU (2) \) doublet. The factor \( \frac{1}{2} \) in connection with \( g' \) is present to be consistent with conventions [13 p. 428].

Due to the results of Sec. III, the Lagrangian is Spin (3, 1) \( \times SU (2) \times U (1) \) gauge invariant, and it describes an eight-spinor field \( \Psi \) coupled to the external fields \( \omega_{ab} \), and \( W^i_\mu \) and \( B_\mu \).

Remark In Eq. (14), hermitian conjugation h.c. effectively applies to only the terms of the Lagrangian arising from \( 1_d \partial_\mu \) and \( G^{\text{outer}}_{X|\mu} \), because the terms arising from \( G^{\text{inner}}_{X|\mu} \) are hermitian due to Eqs. (13) and (5), and the mass terms that couple \( \psi_L \) and \( \psi_R \) are the others hermitian conjugate.
V. DISCUSSION

A notable feature of the Lagrangian, Eq. (14), is that the $\Gamma_{X|\nu}$’s appearing in front of the (covariant) derivatives, as do the usual Dirac gamma matrices, also appear, although crossed in the $L$ and $R$ sense, in $G_{X|^a}$ as SU (2) and $G_{X|A}$ as $SU (2) \times U (1)$ generators. Is that profound?

Define the matrices $P_{X|e}, P_{X|\nu} \in M (4, \mathbb{C})$ by (note the sign difference between the $L$ and $R$ sectors)

$$P_{L|e} = - \frac{i}{2} (\Gamma_{R|0} - \Gamma_{R|3}) ,$$
$$P_{L|\nu} = - \frac{i}{2} (\Gamma_{R|0} + \Gamma_{R|3}) ,$$
$$P_{R|e} = - \frac{i}{2} (\Gamma_{L|0} + \Gamma_{L|3}) ,$$
$$P_{R|\nu} = - \frac{i}{2} (\Gamma_{L|0} - \Gamma_{L|3}) .$$

They obey $P_{X|e}^2 = P_{X|e}$ and $P_{X|\nu}^2 = P_{X|\nu}$, and $1_4 = P_{X|e} + P_{X|\nu}$ and $0_4 = P_{X|e}P_{X|\nu}P_{X|e}$, so they are projection operators in the $L$ and $R$ sector, respectively. Because of Eqs. (13) and (5), $P_{L|e}$ and $P_{L|\nu}$ commute with all terms in $D_{L|\mu}$ except the terms arising from $\Sigma_{L|ab}$ and $\Gamma_{R|1}$ and $\Gamma_{R|2}$. Analogously for $P_{R|e}$ and $P_{R|\nu}$. So, in the light of Eqs. (14a)-14b the matrices $P_{X|e}$ and $P_{X|\nu}$ may be considered weak isospin projection operators, a fact from which their subscripts $e$ and $\nu$, referring to the electron and neutrino, respectively, are derived from. Furthermore, because of the sign difference between the $L$ and $R$ sectors, most importantly (otherwise the mass terms would couple the different isospin components) they obey

$$0_4 = P_{L|e} \eta P_{R|\nu} = P_{L|\nu} \eta P_{R|e} ,$$
$$0_4 = P_{R|e} \eta P_{L|\nu} = P_{R|\nu} \eta P_{L|e} .$$

Therefore, defining the four four-spinors $\psi_{X|e} = P_{X|e}\psi_X$ and $\psi_{X|\nu} = P_{X|\nu}\psi_X$, the mass terms of the Lagrangian may be written as

$$\text{Re} \left( m^* \psi_{L|e}^\dagger \eta \psi_{R|e} \right) + \text{Re} \left( m^* \psi_{L|\nu}^\dagger \eta \psi_{R|\nu} \right) .$$

What significance, if any, is there to the explicit appearance of $\eta$, comparing it with the usual 2D-block diagonal $\gamma^0$ in the mass term of the Dirac Lagrangian? Could the non-2D-block diagonal form of $\eta$, singling out one of four components, be connected with the missing component of the neutrino? And generally, is it any improvement that the usual Dirac projection operators $\frac{i}{2} (1_4 \pm \gamma_5)$ are not present?

On a more speculative note, what happens when the complexified quaternions, here considered, is (almost irresistibly) generalized to the complexified octonions? Mathematically, there are some very compelling reasons for such a generalization:

1. The set of complexified quaternions is a natural subset of the set of complexified octonions, as the former can be embedded into the latter in numerous ways.

2. The proofs of Appendix B with the sole exception being the proof of Eq. (9), which relies on associativity, a property the octonions does not have, carry over without any change for matrices $\Gamma_{X|a}$ and $\Sigma_{X|AB}$ (replacing $\Gamma_{X|a}$ and $\Sigma_{X|ab}$ considered in this article) defined by

$$\langle \Gamma_{L|a} \rangle_{CD} = \langle ec, e AeD \rangle ,$$
$$\langle \Gamma_{R|a} \rangle_{CD} = \langle ec, e eA \rangle ,$$

and [generators of the spinor representations of Spin (7,1)]

$$4i \langle \Sigma_{L|aB} \rangle_{CD} = \langle e AeC, e BeD \rangle - \langle e AeD, e BeC \rangle ,$$
$$4i \langle \Sigma_{R|aB} \rangle_{CD} = \langle e eAe, e eB \rangle - \langle e eD, e eC eB \rangle ,$$

where $e_A = (i, e_I)$ is a basis for the complexified octonions: $e_I$ are the seven imaginary units of $\mathbb{O}$, obeying $e_I e_J = - \varepsilon_{IJK} e_K$, where $\varepsilon_{IJK}$ is the octonionic structure constants, see for instance Refs. [8, 9, 10, 11]. Of course, various other replacements must be made, for instance replacing $\eta \in M (4, \mathbb{R})$ by the eight-dimensional Minkowski metric $\eta_8 \in M (8, \mathbb{R})$. Might the requirement of associativity in the proof of Eq. (9), which holds for the complexified quaternions, but not for the complexified octonions, be the explanation for the four-dimensionality of spacetime, somehow forcing a $\mathbb{C} \otimes \mathbb{H}$-fibration of $\mathbb{C} \otimes \mathbb{O}$?

3. The quaternions and octonions share a unique property, although not utilized in this article: They allow the definition of triple cross products $X_L, X_R : (\mathbb{C} \otimes \mathbb{D})^3 \rightarrow \mathbb{C} \otimes \mathbb{D}$ (where $\mathbb{D}$ denotes either $\mathbb{H}$ or $\mathbb{O}$) by

$$3! X_L (x, y, z) = x (y \bar{z} - \bar{y} z) + \text{cyclic perm} ,$$
$$3! X_R (x, y, z) = (x \bar{y} - y \bar{x}) z + \text{cyclic perm} .$$

The cross products $X_L$ and $X_R$ possess both the orthogonality property and the (generalized) Pythagorean property [6],

$$0 = \langle X (x_1, x_2, x_3), x_i \rangle ,$$
$$\det ((x_i, x_j)) = \langle X (x_1, x_2, x_3), X (x_1, x_2, x_3) \rangle ,$$

where the suppressed subscript means that the relations apply to both $L$ and $R$. Trilinear cross products possessing both these properties exist only over algebras of real (or complex) dimension 4 or 8, see Refs. [6, 10, 11], the underlying reason being the existence of precisely the division algebras $\mathbb{H}$ and $\mathbb{O}$.

4. The seemingly insignificant relation

$$\varepsilon_{abcd} = 1 \langle X (e_a, e_b, e_c), e_d \rangle ,$$
links duality in four-dimensional spacetime, as controlled by \( \varepsilon_{abcd} \), with two natural structures of the (complex) quaternions, the inner product and the cross product, as defined above. This relation may be straightforwardly generalized to

\[
\chi_{L|ABCD} = i \langle X_L (e_A, e_B, e_C) , e_D \rangle ,
\]

\[
\chi_{R|ABCD} = i \langle X_R (e_A, e_B, e_C) , e_D \rangle ,
\]

where \( \chi_{L|ABCD} \) and \( \chi_{R|ABCD} \) are nonequal because of the nonassociativity of the (complexified) octonions. These structure constants \( \chi_{L|ABCD} \) and \( \chi_{R|ABCD} \) allow for the definition of self-duality in eight-dimensional spacetime of rank two tensors:

\[
T_{AB} = \frac{i}{2} \lambda X \chi_{X|ABCD} T^{CD} .
\]

It can be shown that the eigenvalues are \( \lambda_L \in \{ +1, -1/3 \} \) and \( \lambda_R \in \{ -1, +1/3 \} \). In the quaternionic case the eigenvalues are \( \pm 1 \), as is well-known. Is the appearance of \( \pm 1/3 \) in the octonionic case somehow related to fractional (hyper)charges of the quarks?

It is the hope that some or all of these issues will be resolved in the near future.

**APPENDIX A: IDENTITIES**

The following Lemma lists some useful identities for composition algebras, a class to which the complexified quaternions belong, see for instance [13] or [14]. Note, though, that the normalization of the inner product in [13] and [14] differ by a factor of 2. The normalization used in Eq. (1) is the normalization used in [13]. However, [14] is mentioned because its overall presentation is clearer than that of [13], and as such may be valuable to the reader.

**Lemma (See [13] or [14])** The following identities hold for any composition algebra:

\[
\langle x, y \rangle \equiv \langle y, x \rangle , \quad (A1)
\]

\[
\langle x, y \rangle \equiv \langle \overline{y}, x \rangle , \quad (A2)
\]

and

\[
\langle x, y z \rangle \equiv \langle y , x z \rangle , \quad (A3)
\]

\[
\langle x y , z \rangle \equiv \langle x , z y \rangle , \quad (A4)
\]

and

\[
x (\overline{y} z) + (y \overline{z}) \equiv 2 \langle x , y \rangle z , \quad (A5)
\]

\[
(x \overline{y}) z + (x \overline{z}) y \equiv 2 \langle y , z \rangle x . \quad (A6)
\]

**APPENDIX B: PROOFS**

Throughout this section the identities of the Lemma of Appendix A will be used without being explicitly referred to. Although the equations being proved below could reasonably simply be checked by explicit numerical calculation, by first calculating explicitly the four-dimensional matrices \( \Gamma_{L|a} \) and \( \Gamma_{R|a} \), and \( \Sigma_{L|ab} \) and \( \Sigma_{R|ab} \) using Eqs. (2) and (7), the main purpose of presenting analytical proofs is that the majority of these as stated, the sole exception being the proof of Eq. (3), apply to any composition algebra, and therefore in particular to both the complexified quaternions and the complexified octonions.

1. **Proof of Eq. (3)**

By direct calculation, using \( e_a^* = -\overline{e}_a \) and \( \overline{e}_a = -\delta_{ab} e^b \), respectively:

\[
\left( \Sigma_{L|a}^* \right)_{cd} = \left( \langle \Sigma_{L|a} \rangle_{cd} \right)^* = \langle e_c, e_a e_d \rangle^* = \langle e^c, e^a e^d \rangle
\]

\[
= - \langle \overline{e}_c, e_a e_d \rangle = - \langle e_c, e_a e_d \rangle = - \langle \Sigma_{R|a} \rangle_{cd} ,
\]

and

\[
\left( \Sigma_{L|a}^T \right)^c_d = \delta^{ce} \langle \Sigma_{L|a} \rangle^f_d e_f e_d = \delta^{ce} \langle e^f, e_a e_d \rangle \delta_{fd}
\]

\[
= \langle (\delta^{de} e_f), e_a (\delta^{ce}) \rangle = \langle \overline{e}_d e^c, e_a \rangle = \langle e^c, e_a e_d \rangle = \langle \Sigma_{R|a} \rangle^c_d .
\]

The remaining assertion, Eq. (3c), readily follows from the matrix identity \( M^T \equiv (M^*)^T \equiv (M^T)^\dagger \).

2. **Proof of Eq. (5)**

Using the completeness relation \( (\langle x, e_a \rangle , \langle x, e^a \rangle) = \langle x, y \rangle \):

\[
\left[ \Sigma_{L|a} \right]^c_e \left[ \Sigma_{R|b} \right]^e_d = \langle e^c, e_a e_c \rangle \langle e^c, e_a e_d \rangle = \langle \overline{e}_a e^c, e_a e_d \rangle ,
\]

\[
\left[ \Sigma_{R|b} \right]^c_e \left[ \Sigma_{L|a} \right]^e_d = \langle e^c, e_b e_a \rangle \langle e^c, e_a e_d \rangle = \langle \overline{e}_b e^c, e_a e_d \rangle .
\]

These two expressions are equal because the (complexified) quaternions are associative (a property which breaks down when generalizing to complexified octonions) so that

\[
\langle \overline{e}_a e^c, e_a e_d \rangle = \langle e^c, (e_a e_d) \rangle = \langle e^c, (e_a e_d) \rangle = \langle e^c, e_a e_d \rangle = \langle \overline{e}_b e^c, e_a e_d \rangle .
\]
3. Proof of Eqs. (6) and (8)

Only the proof for \( L \) will be given, as the proof for \( R \) is completely analogous. Eq. (8) is proved as follows:

\[
\left( \Sigma^\eta_{L[a]} \right)^c_d = (\eta \Sigma^T_{L[a]} \eta)^c_d = (\eta)^c e \left( \Sigma^T_{L[a]} \right)^c\left( \eta \right)^f d \\
= \eta^{ce} \left( \Sigma^e_{L[a]} \right)^c_f \varepsilon \eta fd = \eta^{ce} \langle e^f, e_a e_c \rangle \eta fd \\
= \langle e_d, e_a e_c \rangle ,
\]

which, using the completeness relation \( \langle x, e_a \rangle \langle e^a, y \rangle \equiv \langle x, y \rangle \), implies that

\[
\left( \Sigma^\eta_{L[a]} \Sigma^{}_{L[b]} \right)^c_d \equiv \left( \Sigma^\eta_{L[a]} \right)^c e \left( \Sigma^{}_{L[b]} \right)^c d \\
= (e_c, e_a e^c) \langle e^c, e_b e_d \rangle \\
= \langle e_a e^c, e_b e_d \rangle ,
\]

which implies that

\[
\left( \Gamma^\eta_{L[a]} \Gamma^{}_{L[b]} \right)^c d = \left( \Sigma^\eta_{L[a]} \Sigma^{}_{L[b]} + \Sigma^\eta_{L[b]} \Sigma^{}_{L[a]} \right)^c d \\
= (e_c, e_a e^c) \langle e^c, e_b e_d \rangle + (e_d, e_a e^c) \langle e^c, e_b e_d \rangle \\
= (e_c, e_a e^c) \langle e^c, e_b e_d \rangle + (e_d, e_a e^c) \langle e^c, e_b e_d \rangle \\
= 2 (e_a, e_b \langle e^c, e_d \rangle) = 2 \eta_{ab} (1_4)^c d .
\]

Eq. (8) follows directly from the second equation in the proof above for Eq. (6), and the defining equation of \( \Sigma^{}_{L[a]} \) and \( \Sigma^{}_{R[a]} \), Eq. (7).

4. Proof of Eq. (9)

Using Eqs. (5a) and (8):

\[
-4i \Sigma^*_{L[a]} = \left( \Gamma^\eta_{L[a]} \Gamma^{}_{L[b]} - \Gamma^\eta_{L[b]} \Gamma^{}_{L[a]} \right)^* \\
= \eta \left( \Gamma^T_{L[a]} \eta \right) \Gamma^*_{L[b]} - \eta \left( \Gamma^*_{L[a]} \eta \right) \Gamma^{}_{L[b]} \\
= \eta \Gamma^T_{R[a]} \eta \Gamma^{}_{R[b]} - \eta \Gamma^*_{R[b]} \eta \Gamma^{}_{R[a]} \\
= 4i \Sigma^{}_{R[a]} .
\]

Using Eq. (8):

\[
4i \Sigma^T_{X[a]} = \left( \Gamma^\eta_{X[a]} \Gamma^{}_{X[b]} - \Gamma^\eta_{X[b]} \Gamma^{}_{X[a]} \right)^T \\
= \Gamma^T_{X[b]} \eta \Gamma^*_{X[a]} - \eta \Gamma^*_{X[a]} \eta \Gamma^{}_{X[b]} \\
= \eta \left( \Gamma^T_{X[b]} \eta \right) \Gamma^{}_{X[a]} - \eta \left( \Gamma^*_{X[a]} \eta \right) \Gamma^{}_{X[b]} \\
= -4i \eta \Sigma^{}_{X[a]} .
\]

The remaining assertion, Eq. (9c), readily follows from the matrix identity \( M^U = (M^*)^T = (M^T)^* \).

5. Proof of Eq. (10)

To compactify the calculations, the subscript \( X \) has been dropped throughout. Consider the expression \( \Gamma^\eta_{a} \Gamma^{}_{b} \Gamma^\gamma_{d} \Gamma^{}_{c} \). Using fourfoldly Eq. (8) to move \( \Gamma^\eta_{b} \Gamma^{}_{d} \) through \( \Gamma^\gamma_{c} \Gamma^{}_{d} \), it follows that

\[
\Gamma^\eta_{a} \Gamma^{}_{b} \Gamma^\gamma_{d} \Gamma^{}_{c} = 2 \eta_{ac} \eta^\gamma_{d} \Gamma^{}_{b} - 2 \eta_{ad} \eta^\gamma_{b} \Gamma^{}_{c} \\
+ 2 \eta_{bd} \eta^\gamma_{c} \Gamma^{}_{d} .
\]

Using fourfoldly this result in the expression

\[
-16 [\Sigma^{}_{ab}, \Sigma^{}_{cd}] = \left[ \Gamma^\eta_{a} \Gamma^{}_{b}, \Gamma^\gamma_{d} \Gamma^{}_{c} \right] - \left[ \Gamma^\gamma_{a} \Gamma^{}_{b}, \Gamma^\eta_{d} \Gamma^{}_{c} \right] \\
+ \left[ \Gamma^\gamma_{a} \Gamma^{}_{b}, \Gamma^\eta_{d} \Gamma^{}_{c} \right] - \left[ \Gamma^\eta_{a} \Gamma^{}_{b}, \Gamma^\gamma_{d} \Gamma^{}_{c} \right] ,
\]

collecting identical terms, and using again Eq. (8), yields

\[
-16 [\Sigma^{}_{ab}, \Sigma^{}_{cd}] = -4 \eta_{bc} [\Gamma^{}_{a}, \Gamma^{}_{d}]_{\eta} - 4 \eta_{ad} [\Gamma^{}_{a}, \Gamma^{}_{c}]_{\eta} \\
+ 4 \eta_{bd} [\Gamma^{}_{a}, \Gamma^{}_{c}]_{\eta} - 4 \eta_{db} [\Gamma^{}_{a}, \Gamma^{}_{c}]_{\eta} ,
\]

from which the result follows.

Remark The proof is completely analogous to the proof of the assertion that \( -\frac{1}{2} [\gamma_a, \gamma_b] \), where \( \gamma_a \) are the Dirac matrices obeying \( 2 \eta_{ab} 1_4 = \{ \gamma_a, \gamma_b \} \), are generators of Spin (3, 1). That is the main reason for introducing above the (anti)commutator-like brackets \([\cdot, \cdot]_{\eta\pm}\).

6. Proof of Eq. (11)

Only the proof of Eq. (11a) will be given, as the proof of Eq. (11b) is completely analogous. Using \( \eta_a = -\delta_{ab} \varepsilon^b \):

\[
\left( \Sigma^\eta_{L[a]} \right)^c_d = (\eta \Sigma^T_{L[a]} \eta)^c_d = (\eta)^c e \left( \Sigma^T_{L[a]} \right)^c\left( \eta \right)^f d \\
= \eta^{ce} \left( \Sigma^e_{L[a]} \right)^c_f \varepsilon \eta fd = \eta^{ce} \langle e^f, e_a e_c \rangle \eta fd \\
= \langle e_d, e_a e_c \rangle ,
\]

which, using Eq. (5a), implies that

\[
4i \left[ \Gamma^{}_{L[a]}, \Sigma^{}_{R[cd]} \right] = \left[ \Gamma^{}_{L[a]}, \Gamma^{}_{R[c]}, \Gamma^{}_{R[d]} \right]_{\eta} \\
= \Gamma^{}_{L[a]} \Gamma^{}_{R[c]}, \Gamma^{}_{R[d]} - \Gamma^{}_{L[a]} \Gamma^{}_{R[d]}, \Gamma^{}_{R[c]} \\
- \Gamma^{}_{L[a]} \Gamma^{}_{R[d]}, \Gamma^{}_{R[c]} + \Gamma^{}_{L[a]} \Gamma^{}_{R[c]}, \Gamma^{}_{R[d]} \\
= \frac{1}{2} \left[ \Gamma^{}_{L[a]}, \Gamma^{}_{R[c]}, \Gamma^{}_{R[d]} \right] \\
- \frac{1}{2} \left[ \Gamma^{}_{L[a]}, \Gamma^{}_{R[d]}, \Gamma^{}_{R[c]} \right] \\
- \frac{1}{2} \left[ \Gamma^{}_{L[a]}, \Gamma^{}_{R[c]}, \Gamma^{}_{R[d]} \right] \\
+ \delta_{de} \left[ \Gamma^{}_{L[a]}, \Gamma^{}_{R[e]} \right] \Gamma^{}_{R[d]} \\
+ \delta_{de} \left[ \Gamma^{}_{L[a]}, \Gamma^{}_{R[e]} \right] \Gamma^{}_{R[d]} = 0_4 .
\]
7. Proof of Eq. (13)

Only the proof of Eq. (13a) will be given, as the proof of Eq. (13b) is analogous. Using Eq. (3c):

\[
(\Gamma^\eta_{X|a} \Gamma_{X|b})^\dagger = \Gamma^\dagger_{X|b} \left( \Gamma^\dagger_{X|a} \right)^\eta = \Gamma_{X|b} \Gamma^\eta_{X|a},
\]

which, using Eqs. (6) and (8), implies that (where to compactify the calculations, the subscript \(X|\) has been dropped throughout)

\[
4i \left( \Sigma^\dagger_{ab} \Gamma^c - \Gamma^c \Sigma_{ab} \right) = \Gamma_a \left[ 2\delta^c_a \Gamma_b - (\Gamma^c)^\eta \Gamma_b \right] - \Gamma^c \Gamma^\eta_{ab} \Gamma_b
- \Gamma_b \left[ 2\delta^c_b \Gamma_a - (\Gamma^c)^\eta \Gamma_a \right] + \Gamma^c \Gamma^\eta_{ab} \Gamma_a
= 2\delta^c_a \Gamma_b - \left( \Gamma_c \Gamma_a \right)^\eta \Gamma_b
- 2\delta^c_b \Gamma_a + \left( \Gamma_c \Gamma_b \right)^\eta \Gamma_a
= 2\delta^c_a \Gamma_b - \left( \Gamma^c \Gamma_a \right)^\eta \Gamma_b
- 2\delta^c_b \Gamma_a + \left( \Gamma^c \Gamma_b \right)^\eta \Gamma_a
= 4 \left( \delta^c_b \eta_{ad} - \delta^c_a \eta_{bd} \right) \Gamma^d
= -4i \left( \Sigma_{V|ab} \right)^c_d \Gamma^d.
\]

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