The Bohr–Pál Theorem and the Sobolev Space $W^{1/2}_2$

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Abstract. The well-known Bohr–Pál theorem asserts that for every continuous real-valued function $f$ on the circle $\mathbb{T}$ there exists a change of variable, i.e., a homeomorphism $h$ of $\mathbb{T}$ onto itself, such that the Fourier series of the superposition $f \circ h$ converges uniformly. Subsequent improvements of this result imply that actually there exists a homeomorphism that brings $f$ into the Sobolev space $W^{1/2}_2(\mathbb{T})$. This refined version of the Bohr–Pál theorem does not extend to complex-valued functions. We show that if $\alpha < 1/2$, then there exists a complex-valued $f$ that satisfies the Lipschitz condition of order $\alpha$ and at the same time has the property that $f \circ h \not\in W^{1/2}_2(\mathbb{T})$ for every homeomorphism $h$ of $\mathbb{T}$.

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1. Introduction

For an arbitrary integrable function $f$ on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (where $\mathbb{R}$ is the real line and $\mathbb{Z}$ is the group of integers) consider its Fourier series:

$$f(t) \sim \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikt}, \quad t \in \mathbb{T}.$$ 

Recall that the Sobolev space $W^{1/2}_2(\mathbb{T})$ is the space of all (integrable) functions $f$ with

$$\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 |k| < \infty.$$ 

Let $C(\mathbb{T})$ be the space of all continuous functions on $\mathbb{T}$.

The well-known Bohr–Pál theorem states that for every real-valued function $f \in C(\mathbb{T})$ there exists a homeomorphism $h$ of the circle $\mathbb{T}$ onto itself, such that the superposition $f \circ h$ belongs to the space $U(\mathbb{T})$ of uniformly convergent Fourier series. (The theorem was obtained in a somewhat weaker form by J. Pál in [11], and in the final form by H. Bohr in [2].) The original
method of proof of this result uses conformal mappings and in fact allows (see [9, Sec. 3]) to obtain the following representation:

\[ f \circ h = g + \psi, \quad g \in W^{1/2}_2 \cap C(T), \quad \psi \in V \cap C(T), \quad (1) \]

where \( V(T) \) is the space of functions of bounded variation on \( T \). It is well-known that both \( W^{1/2}_2 \cap C(T) \) and \( V \cap C(T) \) are subsets of \( U(T) \), thus (1) implies \( f \circ h \in U(T) \).

A substantial improvement of the Bohr–Pál theorem was obtained by A. Sahakian [12, Corollary 1], who showed that if \( a(n), n = 0, 1, 2, \ldots \), is a given positive sequence satisfying the condition \( \sum_n a(n) = \infty \) and a certain condition of regularity, then for every real-valued \( f \in C(T) \) there is a homeomorphism \( h \) such that \( \widehat{f \circ h}(k) = O(a(|k|)) \). An immediate consequence of Sahakian’s result is that the term \( \psi \) in (1) can be omitted, i.e., the following refined version of the Bohr–Pál theorem holds: for every real-valued \( f \in C(T) \) there exists a homeomorphism \( h \) of \( T \) onto itself, such that \( f \circ h \in W^{1/2}_2(T) \). This refined version also follows from a result on conjugate functions, obtained by W. Jurkat and D. Waterman in [4] (see also [3, Theorem 9.5]). We note that Sahakian’s result is obtained by purely real analysis technique whereas Jurkat and Waterman use an approach similar to the one used by Bohr and Pál. A very short proof of the refined version of the Bohr–Pál theorem was communicated to the author by A. Olevskii, see [7, Sec. 3].

Another improvement of the Bohr–Pál theorem was obtained by J.-P. Kahane and Y. Katznelson [6] (see also [9], [5]). These authors showed that if \( K \) is a compact family of functions in \( C(T) \), then there exists a homeomorphism \( h \) of \( T \) such that \( f \circ h \in U(T) \) for all \( f \in K \). This result naturally leads to a question if it is possible to attain the condition \( f \circ h \in W^{1/2}_2(T) \) for all \( f \in K \). This question was posed by A. Olevskii in [10]. A negative answer was obtained by the author of this work in [7, Theorem 4], it turns out that, given a real-valued \( u \in C(T) \), the property that for every real-valued \( v \in C(T) \) there is a homeomorphism \( h \) such that both \( u \circ h \) and \( v \circ h \) are in \( W^{1/2}_2(T) \) is equivalent to the boundness of variation of \( u \). Thus, in general, there is no single change of variable which will bring two real-valued functions in \( C(T) \) into \( W^{1/2}_2(T) \). Certainly this amounts to the existence of a complex-valued \( f \in C(T) \) such that \( f \circ h \notin W^{1/2}_2(T) \) for every homeomorphism \( h \) of \( T \).
The purpose of this work is to show that there exists a complex-valued function \( f \) that is very smooth but at the same time has the property that \( f \circ h \notin W^{1/2}_2(\mathbb{T}) \) for every homeomorphism \( h \) of \( \mathbb{T} \).

Note that, as one can easily verify (see, e.g., [7], Sec. 3), the following two semi-norms

\[
\|f\|_{W^{1/2}_2(\mathbb{T})} = \left( \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 |k| \right)^{1/2}
\]

\[
\|f\|_{W^{1/2}_2(\mathbb{T})} = \left( \int_0^{2\pi} \frac{1}{\theta^2} \int_0^{2\pi} |f(t + \theta) - f(t)|^2 d\theta d\theta \right)^{1/2}
\]

(2)

are equivalent semi-norms on \( W^{1/2}_2(\mathbb{T}) \), i.e., \( f \) is in \( W^{1/2}_2(\mathbb{T}) \) if and only if \( \|f\|_{W^{1/2}_2(\mathbb{T})} < \infty \), and \( c_1\|f\|_{W^{1/2}_2(\mathbb{T})} \leq \|f\|_{W^{1/2}_2(\mathbb{T})} \leq c_2\|f\|_{W^{1/2}_2(\mathbb{T})} \) for all \( f \in W^{1/2}_2(\mathbb{T}) \), where \( c_1, c_2 > 0 \) do not depend on \( f \). Thus, we see that every function that satisfies the Lipschitz condition of order greater than 1/2 belongs to \( W^{1/2}_2(\mathbb{T}) \). We shall show that, in general, there is no change of variable which will bring a complex-valued function that satisfies the Lipschitz condition of order less than 1/2 into \( W^{1/2}_2(\mathbb{T}) \). The author does not know if the same holds for the functions satisfying the Lipschitz condition of order 1/2 (see Remarks at the end of the paper).

2. Result

Let \( \omega \) be a modulus of continuity, i.e., a nondecreasing continuous function on \( [0, +\infty) \) such that \( \omega(0) = 0 \) and \( \omega(x + y) \leq \omega(x) + \omega(y) \). By \( \text{Lip}_\omega(\mathbb{T}) \) we denote the class of all complex-valued functions \( f \) on \( \mathbb{T} \) with \( \omega(f, \delta) = O(\omega(\delta)), \ \delta \to +0 \), where

\[
\omega(f, \delta) = \sup_{|t_1 - t_2| \leq \delta} |f(t_1) - f(t_2)|, \quad \delta \geq 0,
\]

is the modulus of continuity of \( f \). For \( 0 < \alpha \leq 1 \) we just write \( \text{Lip}_\alpha \) instead of \( \text{Lip}_{\omega} \).

**Theorem.** Suppose that \( \lim \sup_{\delta \to +0} \omega(\delta)/\sqrt{\delta} = \infty \). Then there exists a complex-valued function \( f \in \text{Lip}_\omega(\mathbb{T}) \) such that \( f \circ h \notin W^{1/2}_2(\mathbb{T}) \) for every homeomorphism \( h \) of the circle \( \mathbb{T} \) onto itself. In particular, if \( \alpha < 1/2 \), then there exists a function of class \( \text{Lip}_\alpha(\mathbb{T}) \) with this property.
Ideologically the method of the proof of this theorem is close to the one used by the author to prove Theorem 4 in [7].

We shall need certain preliminary constructions and lemmas. Simple Lemma 1 below is purely technical.

**Lemma 1.** *Under the assumption of the theorem on* $\omega$ *there exists a sequence* $\delta_k > 0$, $k = 1, 2, \ldots$, *such that*

\[
\sum_{k=1}^{\infty} \delta_k < 2\pi/6, \quad (3)
\]

\[
\sum_{k=1}^{\infty} (\omega(\delta_k))^2 = \infty. \quad (4)
\]

*Proof.* For each $j = 1, 2, \ldots$ we can find $\varepsilon_j$ so that $0 < \varepsilon_j < 2^{-(j+1)}$ and

\[
\frac{(\omega(\varepsilon_j))^2}{\varepsilon_j} \geq 2^j.
\]

Chose positive integers $n_j$ satisfying

\[
\frac{1}{2^{j+1}\varepsilon_j} \leq n_j < \frac{1}{2j\varepsilon_j}, \quad j = 1, 2, \ldots.
\]

Let $N_0 = 1$ and let $N_j = N_{j-1} + n_j$ for $j = 1, 2, \ldots$. We define the sequence $\delta_k$, $k = 1, 2, \ldots$, by setting $\delta_k = \varepsilon_j$ if $N_{j-1} \leq k < N_j$, $j = 1, 2, \ldots$. This yields

\[
\sum_{k=1}^{\infty} \delta_k = \sum_{j=1}^{\infty} \sum_{N_{j-1} \leq k < N_j} \delta_k = \sum_{j=1}^{\infty} n_j \varepsilon_j \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1,
\]

and at the same time

\[
\sum_{N_{j-1} \leq k < N_j} (\omega(\delta_k))^2 = n_j (\omega(\varepsilon_j))^2 \geq n_j \varepsilon_j 2^j \geq \frac{1}{2}.
\]

The lemma is proved.
For a closed interval $I = [a, b] \subseteq (0, 2\pi)$ let $\Delta_I$ denote the “triangle” function supported on $I$, i.e., a continuous function on the interval $[0, 2\pi]$ such that $\Delta_I(t) = 0$ for all $t \in [0, a] \cup [b, 2\pi]$, $\Delta_I(c) = 1$, where $c = (a + b)/2$ is the center of $I$, and $\Delta_I$ is linear on $[a, c]$ and on $[c, b]$.

Let $\delta_k, k = 1, 2, \ldots$, be the sequence from Lemma 1. Consider intervals $I_k = [a_k, b_k] \subseteq (0, 2\pi)$ of length $b_k - a_k = 6\delta_k$, where $a_k < b_k < a_{k+1}$, $k = 1, 2, \ldots$ (see (3)). For each $k$ let $J_k$ denote the left half of $I_k$, i.e., $J_k = [a_k, (a_k + b_k)/2]$, $k = 1, 2, \ldots$.

Everywhere below we use $u$ and $v$ to denote two real-valued functions on $T$ defined by

$$u(t) = \sum_{k=1}^{\infty} \omega(\delta_k) \Delta_{I_k}(t), \quad v(t) = \sum_{k=1}^{\infty} \omega(\delta_k) \Delta_{J_k}(t), \quad t \in [0, 2\pi].$$

We shall show that the function $f = u + iv$ satisfies the assertion of the theorem.

**Lemma 2.** The functions $u$ and $v$ are of class $\text{Lip}_\omega(T)$.

**Proof.** It is clear that for an arbitrary (closed) interval $I \subseteq (0, 2\pi)$, the function $\Delta_I$ satisfies

$$|\Delta_I(t_1) - \Delta_I(t_2)| \leq \frac{2}{|I|} |t_1 - t_2|, \quad \text{for all } t_1, t_2 \in [0, 2\pi], \quad (5)$$

where $|I|$ is the length of $I$.

Note also that if $0 < x \leq y$, then $\omega(y)/y \leq 2\omega(x)/x$. Indeed, let $n = \lfloor y/x \rfloor + 1$, where $\lfloor \alpha \rfloor$ denotes the integer part of a number $\alpha$, then we have $y \leq nx \leq 2y$, so

$$\frac{\omega(y)}{y} \leq \frac{\omega(nx)}{y} \leq \frac{n\omega(x)}{y} \leq 2\frac{\omega(x)}{x}.\$$

Let us show that $u \in \text{Lip}_\omega(T)$; for $v$ the proof is similar. It is easy to see that to prove the inclusion $u \in \text{Lip}_\omega(T)$ it suffices to verify that for all $t_1, t_2 \in \bigcup_k I_k$ we have

$$|u(t_1) - u(t_2)| \leq c\omega(|t_1 - t_2|),$$

where $c > 0$ does not depend on $t_1$ and $t_2$. 


First we consider the case when \( t_1 \) and \( t_2 \) belong to the same interval \( I_k \). If that is the case, then, since \(|t_1 - t_2| \leq |I_k| = 6\delta_k\), we have

\[
\frac{\omega(6\delta_k)}{6\delta_k} \leq 2 \frac{\omega(|t_1 - t_2|)}{|t_1 - t_2|},
\]

so (see (5)),

\[
|u(t_1) - u(t_2)| = \omega(\delta_k)|\Delta_{I_k}(t_1) - \Delta_{I_k}(t_2)| \leq \omega(\delta_k) \frac{2}{6\delta_k}|t_1 - t_2| \leq 2 \omega(6\delta_k) \frac{|t_1 - t_2|}{6\delta_k} \leq 4 \omega(|t_1 - t_2|).
\]

Consider now the case when \( t_1 \in I_{k_1}, t_2 \in I_{k_2}, k_1 \neq k_2 \). We can assume that \( t_1 < t_2 \), and hence \( 0 < t_1 < b_{k_1} < a_{k_2} < t_2 < 2\pi \). Using the previous estimate, we obtain

\[
|u(t_1) - u(t_2)| \leq |u(t_1)| + |u(t_2)| = |u(t_1) - u(b_{k_1})| + |u(t_2) - u(a_{k_2})| \leq 8\omega(|t_1 - t_2|).
\]

The lemma is proved.

For \( n = 1, 2, \ldots \) we define functions \( u_n \) by

\[
u_n(t) = \max\{u(t), 1/n\}, \quad t \in \mathbb{T}.
\]

As above, \( V(\mathbb{T}) \) stands for the class of functions of bounded variation on \( \mathbb{T} \).

**Lemma 3.** The functions \( u_n, n = 1, 2, \ldots \), have the following properties:

\[
|u_n(t_1) - u_n(t_2)| \leq |u(t_1) - u(t_2)| \quad \text{for all} \quad t_1, t_2 \in \mathbb{T} \quad \text{and all} \quad n; \quad (6)
\]

\[
u_n \in V(\mathbb{T}) \quad \text{for all} \quad n; \quad (7)
\]

\[
\sup_n \left| \int_{\mathbb{T}} v(t)du_n(t) \right| = \infty. \quad (8)
\]

**Proof.** Properties (6) and (7) are obvious. Let us verify (8). To this end consider the middle thirds of the intervals \( J_k \), namely, the intervals \( J_k^* = [a_k + \delta_k, a_k + 2\delta_k], k = 1, 2, \ldots \). Note that if

\[
\frac{\omega(\delta_k)}{3} \geq \frac{1}{n}, \quad (9)
\]
then the function \( u_n \) coincides with \( u \) on \( J_k^* \). So, if (9) holds, then \( u_n \) is monotonically increasing on \( J_k^* \), and for its values at the endpoints of \( J_k^* \) we have
\[
u_n(a_k + \delta_k) = \omega(\delta_k)/3, \quad u_n(a_k + 2\delta_k) = 2\omega(\delta_k)/3.
\]
It is easily seen, that for each \( k \)
\[
\min_{J_k^*} v = 2\omega(\delta_k)/3.
\]
Taking into account that \( u \), and hence \( u_n \), is non-decreasing on each interval \( J_k \), we see that for all \( n \) and \( k \) satisfying condition (9)
\[
\int_{J_k} vdu_n \geq \frac{2}{3}\omega(\delta_k) \int_{a_k + \delta_k}^{a_k + 2\delta_k} du_n = \frac{2}{3}\omega(\delta_k) \frac{1}{3}\omega(\delta_k) = \frac{2}{9}(\omega(\delta_k))^2.
\]
In addition (since \( u_n \) is non-decreasing on each \( J_k \)) we have
\[
\int_{J_k} vdu_n \geq 0
\]
for all \( n \) and \( k \). Thus, taking into account that \( v \) vanishes outside \( \bigcup_{k=1}^{\infty} J_k \), we obtain
\[
\int_{T} vdu_n = \sum_{k=1}^{\infty} \int_{J_k} vdu_n \geq \sum_{k: \omega(\delta_k) \geq 3/n} \int_{J_k} vdu_n \geq \sum_{k: \omega(\delta_k) \geq 3/n} \frac{2}{9}(\omega(\delta_k))^2.
\]
Applying (4) we see that (8) holds. The lemma is proved.

We shall also need the following auxiliary lemma.

**Lemma 4.** If \( x, y \in W_2^{1/2} \cap C(\mathbb{T}) \) and \( y \in V(\mathbb{T}) \), then
\[
\left| \frac{1}{2\pi} \int_{\mathbb{T}} x(t)dy(t) \right| \leq \|x\|_{W_2^{1/2}(\mathbb{T})}\|y\|_{W_2^{1/2}(\mathbb{T})}.
\]

**Proof.** Integration by parts yields
\[
\frac{1}{2\pi} \int_{0}^{2\pi} e^{ikt}dy(t) = -\frac{1}{2\pi} \int_{0}^{2\pi} y(t)de^{ikt} = -ik\hat{y}(-k).
\]
So, if $x$ is a trigonometric polynomial, then, using Cauchy inequality, we obtain
\[
\left| \frac{1}{2\pi} \int_{\mathbb{T}} x(t)dy(t) \right| = \left| \sum_{k} \hat{x}(k) \frac{1}{2\pi} \int_{\mathbb{T}} e^{ikt}dy(t) \right| = \left| \sum_{k} \hat{x}(k)(-ik)\hat{y}(-k) \right| \leq \|x\|_{W^{1/2}_2(\mathbb{T})} \|y\|_{W^{1/2}_2(\mathbb{T})}.
\]

To see that the assertion of the lemma holds in the general case, consider the Fejér sums $\sigma_N(x)$ of the function $x$:
\[
\sigma_N(x)(t) = \sum_{|k| \leq N} \left(1 - \frac{|k|}{N}\right) \hat{x}(k)e^{ikt}.
\]

Since $|\sigma_N(x)(k)| \leq |\hat{x}(k)|$ for all $k \in \mathbb{Z}$, we have $\|\sigma_N(x)\|_{W^{1/2}_2(\mathbb{T})} \leq \|x\|_{W^{1/2}_2(\mathbb{T})}$.

Hence,
\[
\left| \frac{1}{2\pi} \int_{\mathbb{T}} \sigma_N(x)(t)dy(t) \right| \leq \|\sigma_N(x)\|_{W^{1/2}_2(\mathbb{T})} \|y\|_{W^{1/2}_2(\mathbb{T})} \leq \|x\|_{W^{1/2}_2(\mathbb{T})} \|y\|_{W^{1/2}_2(\mathbb{T})}.
\]

At the same time, since $y$ is of bounded variation and $\sigma_N(x)$ converges uniformly to $x$ it is clear that
\[
\frac{1}{2\pi} \int_{\mathbb{T}} \sigma_N(x)(t)dy(t) \to \frac{1}{2\pi} \int_{\mathbb{T}} x(t)dy(t)
\]
as $N \to \infty$. The lemma is proved.

Now we proceed directly to the proof of the theorem. Let $f = u + iv$. Lemma 2 yields $f \in \text{Lip}_\omega(\mathbb{T})$, so it remains to show that $f \circ h \notin W^{1/2}_2(\mathbb{T})$ for every homeomorphism $h$ of $\mathbb{T}$. It is obvious that if a function is in $W^{1/2}_2(\mathbb{T})$, then both its real and imaginary parts are in $W^{1/2}_2(\mathbb{T})$ as well. Assume that, contrary to the assertion of the theorem, $f \circ h \in W^{1/2}_2(\mathbb{T})$ for a certain homeomorphism $h$. Then we have $u \circ h \in W^{1/2}_2(\mathbb{T})$ and $v \circ h \in W^{1/2}_2(\mathbb{T})$.

Note that (6) implies $|u_n \circ h(t_1) - u_n \circ h(t_2)| \leq |u \circ h(t_1) - u \circ h(t_2)|$ for all $t_1, t_2 \in \mathbb{T}$. Using the equivalence of the semi-norms $\| \cdot \|_{W^{1/2}_2(\mathbb{T})}$ and $\| \cdot \|_{W^{1/2}_2(\mathbb{T})}$ (see (2)), we infer that $u_n \circ h \in W^{1/2}_2(\mathbb{T})$ for all $n = 1, 2, \ldots$, and
\[
\|u_n \circ h\|_{W^{1/2}_2(\mathbb{T})} \leq c\|u \circ h\|_{W^{1/2}_2(\mathbb{T})}, \quad n = 1, 2, \ldots, \quad (10)
\]
where \(c > 0\) does not depend on \(n\).

The property of a function to be of bounded variation is invariant under homeomorphic changes of variable, hence from (7) it follows that \(u_n \circ h \in V(\mathbb{T})\) for all \(n\). Certainly we also have \(u_n \circ h \in C(\mathbb{T})\). Applying Lemma 4, and taking (10) into account, we obtain

\[
\left| \frac{1}{2\pi} \int_{\mathbb{T}} v(t) du_n(t) \right| = \left| \frac{1}{2\pi} \int_{\mathbb{T}} v \circ h(t) du_n \circ h(t) \right| \leq \|v \circ h\|_{W_{s/2}^1(\mathbb{T})} \|u_n \circ h\|_{W_{s/2}^1(\mathbb{T})} \leq c \|v \circ h\|_{W_{s/2}^1(\mathbb{T})} \|u \circ h\|_{W_{s/2}^1(\mathbb{T})},
\]

which contradicts (8). The theorem is proved.

**Remarks.**

1. For \(s > 0\) consider the Sobolev space \(W_s^s(\mathbb{T})\) i.e., the space of all (integrable) functions \(f\) with

\[
\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 |k|^{2s} < \infty.
\]

As the author of this paper showed in [7, Corollary 3], for each compact family \(K\) in \(C(\mathbb{T})\) (or equivalently for each class \(\text{Lip}_\omega(\mathbb{T})\)) there exists a homeomorphism \(h\) of \(\mathbb{T}\) such that \(f \circ h \in \bigcap_{s<1/2} W_s^s(\mathbb{T})\) for all \(f \in K\) (for all \(f \in \text{Lip}_\omega(\mathbb{T})\)).

2. There exists a real-valued \(f \in C(\mathbb{T})\) such that \(f \circ h \notin \bigcup_{s>1/2} W_s^s(\mathbb{T})\) for every homeomorphism \(h\) of \(\mathbb{T}\). This is a simple consequence of the inclusion \(\bigcup_{s>1/2} W_s^s \cap C(\mathbb{T}) \subseteq A(\mathbb{T})\), where \(A(\mathbb{T})\) is the Wiener algebra of absolutely convergent Fourier series, and a well-known result of Olevskiĭ, that provides a negative answer to Lusin’s rearrangement problem: there exists a real-valued \(f \in C(\mathbb{T})\) such that \(f \circ h \notin A(\mathbb{T})\) for every homeomorphism \(h\) ([8], see also [9]).

3. The function \(f(t) = \sum_{k \geq 0} 2^{-k/2} e^{2k t}\) is in \(\text{Lip}_{1/2}(\mathbb{T})\), (see, e.g., [1, Ch. XI, Sec. 6]), but it is obvious, that \(f \notin W_{1/2}(\mathbb{T})\); thus \(\text{Lip}_{1/2}(\mathbb{T}) \not\subseteq W_{1/2}^{1/2}(\mathbb{T})\). The author does not know if the assertion of the theorem proved in this paper holds for \(\omega(\delta) = \delta^{1/2}\). At the same time there is no change of variable which will bring the whole class \(\text{Lip}_{1/2}(\mathbb{T})\) into \(W_{1/2}^{1/2}(\mathbb{T})\); a proof will be presented in another paper.
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