Non-analytic magnetic field dependence of quasi-particle properties of two-dimensional metals

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We show that in a weak external magnetic field $H$ the quasi-particle residue and the renormalized electron Landé factor of two-dimensional Fermi liquids exhibit a non-analytic magnetic field dependence proportional to $|H|$ which is due to electron-electron interactions. We explicitly calculate the corresponding prefactors to second order in the interaction and show that they are determined by low-energy scattering processes involving only momenta close to the Fermi surface. Experimentally, these non-analytic terms can be detected from measurements of the magnetic field dependence of the density of states and the magnetococonductivity.

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I. INTRODUCTION

The non-analytic dependence of thermodynamic susceptibilities of Fermi liquids on relevant control parameters such as the temperature $T$ or a magnetic field $H$ has recently received a lot of attention\cite{1,2}. The non-analytic corrections are due to electron-electron interactions and exist for dimensions $D$ in the range $1 < D < 3$. While in three dimensions the leading corrections to the dominant analytic terms predicted by the Sommerfeld expansion are only logarithmic in $T$ and $H$, for $D < 3$ the non-analytic corrections to the free energy and its derivatives scale as $T^{D-1}$ and $|H|^{D-1}$. Recently Belitz and Kirkpatrick\cite{3} have shown that within the framework of the renormalization group these corrections can be related to the existence of a certain leading irrelevant coupling constant with scaling dimension $-(D-1)/2$; a scaling argument then gives a simple explanation for the non-analyticities of thermodynamic quantities as a function of $T$ and $H$ for $1 < D < 3$.

The non-analytic magnetic field dependence of the free energy and the resulting spin susceptibility has been obtained by Maslov and Chubukov\cite{4}. In two dimensions they found that the magnetic field dependence of the grand canonical potential per unit volume can be written as

$$ f(h) = f(0) - \frac{\chi(0)}{2} h^2 - \frac{\chi_1}{6} |h|^3 + \mathcal{O}(h^4). \quad (1) $$

Here $h = g\mu_B H/2$ is the Zeeman energy of an electron in a magnetic field $H$, where $g \approx 2$ is the electron Landé factor and $\mu_B$ is the Bohr magneton. To second order in the relevant dimensionless interaction $u = vU$ (where $U$ is a short-range bare interaction between two electrons with antiparallel spin) the coefficient $\chi_1$ is at zero temperature given by\cite{4}

$$ \chi_1 = 2\nu u^2 / E_F + \mathcal{O}(u^3). \quad (2) $$

Here $\nu = m/(2\pi)$ is the two-dimensional density of states (per spin projection) at the Fermi energy $E_F$, where $m$ is the electronic mass. For convenience we use units where $\hbar = k_B = 1$ and normalize the zero-field spin-susceptibility such that in the non-interacting limit $\chi(0) = 2\nu$. The expansion (1) of the thermodynamic potential implies that the field-dependent spin-susceptibility

$$ \chi(h) = -\partial^2 f(h) / \partial h^2 = \chi(0) + \chi_1 |h| + \mathcal{O}(h^2) \quad (3) $$

exhibits a non-analytic magnetic field dependence proportional to $|h|$. While in second order perturbation theory the coefficient $\chi_1$ is positive, to third order in the interaction one obtains a negative correction to $\chi_1$, so that within perturbation theory one cannot exclude the possibility that the sign of $\chi_1$ becomes negative for sufficiently strong interaction\cite{1,2}. Although it has been argued\cite{3} that this does not happen, non-perturbative calculations of $\chi_1$ retaining all scattering channels are not available. One can perform a partial resummation to all orders by collecting the dominant logarithmic corrections in the Cooper channel, which occur starting at the third order. In this way the non-analytic dependence of the spin-susceptibility on temperature $\theta$, magnetic field $H$, or an external momentum $\mathbf{q}$ have been calculated. In all three cases the coefficient of the non-analytic correction changes sign if the corresponding parameter is sufficiently small. The crossover happens at the energy scale associated with the Kohn-Luttinger instability, but is numerically larger. It is still likely, though, that the characteristic energy scale is unmeasurably small.

In this work we show that some quasi-particle properties such as the quasi-particle residue and the renormalized Landé factor also exhibit a non-analytic magnetic field dependence which in two dimensions is proportional to $|H|$. We explicitly calculate the corresponding prefactors to leading order in the interaction. We also discuss consequences for experimentally accessible quantities; specifically, we show that the tunneling density of states and the magnetococonductivity both have corrections linear in $|H|$ which arise from the momentum-dependence of the self-energy.
II. NON-ANALYTIC MAGNETIC FIELD
DEPENDENCE OF QUASI-PARTICLE
PROPERTIES

Since the non-analytic corrections are due to electron-electron interactions we ignore the lattice and consider the following Euclidean action of spin-dependent Grassmann-fields \( c^\sigma_K \) describing electrons with mass \( m \) interacting with a momentum-independent bare interaction \( U \) acting between different spin projections \( \sigma_1 \neq \sigma_2 \),

\[
S[c] = -\int_K \sum_{\sigma} [G_0^\sigma(K)]^{-1} c^\sigma_K c^\sigma_K + \frac{U}{2} \int_K \int_K \int_{Q} \sum_{\sigma_1 \neq \sigma_2} c^\sigma_{K+Q} c^\sigma_{K+Q} c^\sigma_{K} c^\sigma_{K} . \tag{4}
\]

The Gaussian part of the action depends on the inverse non-interacting Matsubara Green function

\[
[G_0^\sigma(K)]^{-1} = i\omega - k^2/(2m) + \mu + \sigma h , \tag{5}
\]

where \( \mu \) is the chemical potential, \( k \) is the electronic momentum and \( i\omega \) is a fermionic Matsubara frequency. For convenience we have introduced collective labels \( K = (k, i\omega) \) and \( Q = (q, i\omega) \), where \( i\omega \) is a bosonic Matsubara frequency. We focus on the zero temperature and infinite volume limit in two dimensions where the integration symbol reduces to \( \int_K = \int \frac{d^2k}{(2\pi)^2} \int \frac{d\omega}{2\pi} \).

To second order in the interaction, the momentum- and frequency-dependent part of the self-energy of electrons with spin-projection \( \sigma \) can be written in the following three equivalent ways (which can be generated from each other by relabeling and changing the order of integrations)

\[
\Sigma^\sigma(K) = -U^2 \int_Q \Pi_0^{-\sigma,-\sigma}(Q) G_0^\sigma(K - Q) \tag{6a}
\]

\[
= -U^2 \int_Q \Pi_0^{↑↓}(Q) G_0^\sigma(K - Q) \tag{6b}
\]

\[
= -U^2 \int_Q \Phi_0^{↑↓}(Q) G_0^\sigma(-K + Q) \tag{6c}
\]

where the particle-hole and particle-particle bubbles are

\[
\Pi_0^{-\sigma,-\sigma}(Q) = \int_K G_0^\sigma(K) G_0^\sigma(K - Q) , \tag{7a}
\]

\[
\Phi_0^{↑↓}(Q) = \int_K G_0^\sigma(K) G_0^\sigma(-K + Q) . \tag{7b}
\]

The three different ways of writing the self-energy in Eqs. \(6a - 6c\) are depicted in Fig. 1 (a-c). In addition, there is a second order self-energy diagram of the Hartree type shown in Fig. 1 (d) which we ignore because it is independent of momentum and frequency. A priori it is not clear which of the three expressions in Eqs. \(6a - 6c\) is most convenient for the explicit evaluation of the self-energy. However, given the fact that the non-analytic magnetic field dependence of the free energy can be expressed in terms of the small-momentum part of the antiparallel-spin particle-hole bubble \( \Pi_0^{↑↓}(Q) \), let us start from the representation \(6b\). In two dimensions the relevant particle-hole bubble can be calculated exactly at zero temperature. The result can be written as

\[
P(q, i\omega; h) = 1 - \text{sgn} \omega \frac{k_F}{q} \left[ \sqrt{1 - \left( \frac{q}{2k_F} \right)^2 + \left( \frac{\omega - 2ih}{v_F q} \right)^2 - \frac{i\omega}{2E_F}} - \sqrt{1 - \left( \frac{q}{2k_F} \right)^2 + \left( \frac{\omega - 2ih}{v_F q} \right)^2 + \frac{i\omega}{2E_F}} \right] , \tag{8}
\]
where \( v_F \) is the Fermi velocity and the Fermi momentum \( k_F = mv_F = \sqrt{2\pi n} \) in the absence of a magnetic field is fixed by the total density \( n \). Here and below the root symbol denotes the principal branch of the complex root, defined by \( \text{Re}\sqrt{z} \geq 0 \). Note that for large \( q \) the function \( P(q, i\omega; h) \) vanishes as
\[
P(q, i\omega; h) \sim 2(k_F/q)^2, \quad (9)
\]
and for large \(|\omega|\),
\[
P(q, i\omega; h) \sim \frac{2h}{i\omega} - \frac{2h^2}{\omega^2} + \frac{(v_F q)^2}{2\omega^2} \left[ 1 - \left( \frac{q}{2k_F} \right)^2 \right], \quad (10)
\]
implies that the integral (6b) is convergent.

We are interested in the quasi-particle properties, which are encoded in the expansion of the self-energy for small frequencies and for wave-vectors close to the Fermi surface. Note that for finite \( h \) the Fermi momentum \( k_F^h \) of spin-up electrons has a different value than the Fermi momentum \( k_F \) of spin-down electrons. Given the self-energy \( \Sigma^\sigma(k, i\omega) \) (note that the \( k \)-dependence of the self-energy appears only via \(|k| = k_F \)), the true Fermi surface of electrons with spin-projection \( \sigma \) is defined via the solution of
\[
\frac{(k_F^\sigma)^2}{2m} + \Sigma^\sigma(k_F^\sigma, 0) = \mu + \sigma h. \quad (11)
\]
If we neglect the self-energy correction, this yields \( (k_F^\sigma)^2 \approx k_F^2 \left( \frac{1 + \sigma h/E_F}{1} \right) \) and to leading order \( k_F^\sigma \approx k_F + \sigma h/v_F \). The quasi-particle properties are encoded in the expansion of the self-energies around the true Fermi surface,
\[
\Sigma^\sigma(K) \approx \Sigma^\sigma(k_F^\sigma, 0) - (1 - Y^{-1})\xi_k^\sigma + (1 - Z^{-1})i\omega, \quad (12)
\]
where \( \xi_k^\sigma = |k^2 - (k_F^\sigma)^2|/(2m) \). The fermionic Green function assumes then the quasi-particle form
\[
G^\sigma(K) = \frac{1}{Z^{-1}i\omega - Y^{-1}\xi_k^\sigma} = \frac{Z}{i\omega - k^2 - \xi_k^\sigma + \sigma g_h h}, \quad (13)
\]
where the effective mass renormalization factor is given by
\[
m_* = \frac{m}{Y/Z}, \quad (14)
\]
while the average Fermi momentum \( \tilde{k}_F \) and the renormalization factor \( g_* \) of the electron Landé factor are defined by
\[
\frac{\tilde{k}_F^2}{2m} = \mu - \Sigma^\uparrow(k_F^\uparrow, 0) + \Sigma^\downarrow(k_F^\downarrow, 0), \quad (15)
\]
\[
g_* = \frac{Z}{Y} \left[ 1 - \frac{\Sigma^\uparrow(k_F^\uparrow, 0) - \Sigma^\downarrow(k_F^\downarrow, 0)}{2h} \right] = \frac{Z}{Y} X. \quad (16)
\]
To calculate the weak-field expansions of the interaction- and magnetic field dependent renormalization factors \( X, \ Y, \) and \( Z \), we substitute the exact expression (8) for the particle-hole bubble into Eq. (6b) and perform the angular integration using
\[
\int_0^{2\pi} \frac{d\varphi}{2\pi} = \frac{1}{\sqrt{(a + ib)^2 - 1}} = \frac{1}{\text{sgn}a\sqrt{\frac{R + 1}{2}} + \text{isgnb}\sqrt{\frac{R - 1}{2}}}, \quad (17)
\]
where \( A = a^2 - b^2 - 1, \ B = 2ab, \) and \( R = \sqrt{A^2 + B^2} \). Note that for \(|a| < 1\) the integral (17) has a discontinuity at \( b = 0 \), so that it is not allowed to commute the partial derivatives of the integral with respect to \( a \) and \( b \) with the angular integration. In fact, writing \( \text{sgn}b = 2\Theta(b) - 1 \) and using the identity(13)
\[
\delta(x)f(\Theta(x)) = \delta(x) \int_0^1 dt f(t), \quad (18)
\]
where \( f(\Theta(x)) \) is an arbitrary function of the step function \( \Theta(x) \), it is easy to show that(13)
\[
\frac{\partial}{\partial a} \int_0^{2\pi} \frac{d\varphi}{2\pi} = \frac{1}{2\pi a + ib + \cos \varphi} = \frac{1}{\text{sgn}(\xi_k^\sigma + \frac{q^2}{2m} + \sigma h)} \frac{1}{\sqrt{(i\omega - \xi_k^\sigma - \frac{q^2}{2m} - \sigma h)^2 - (kq/m)^2}}, \quad (19)
\]
where \( \xi_k^\sigma = (k^2 - k_F^2)/(2m) \). Setting in Eq. (20) \( k = k_F^\sigma \) and \( \omega = 0 \) we obtain \( \Sigma^\sigma(k_F^\sigma, 0) \) and hence the factor \( X \) defined Eq. (16) which gives the renormalization of the Landé factor,
\[
X = 1 + \frac{\nu U^2}{2h} \int \frac{d\omega}{2\pi} \int_0^\infty \frac{dq}{2\pi} \left[ P(q, i\omega; \sigma h) \times \frac{\text{sgn}(\xi_k^\sigma + \frac{q^2}{2m} + 2h)}{(i\omega + \frac{q^2}{2m} + 2h)^2 - (v_F q)^2} \right]^{1/2} - (h \to -h), \quad (21)
\]
while \( v_F^\sigma = k_F^2/m \). To calculate the quantities \( Y \) and \( Z \) defined via Eq. (12), we need the derivatives of the self-energy (20) with respect to \( \xi_k^\sigma \) and \( \omega \). Carefully keeping track of the term arising from the discontinuity in Eq. (19), we obtain
\[
\frac{1}{Y} = 1 + \frac{\nu U^2}{2\pi} \int \frac{d\omega}{2\pi} \int_0^\infty \frac{dq}{2\pi} P(q, i\omega; \sigma h) \times \frac{(i\omega - \frac{q^2}{2m} + 2h)\text{sgn}(\frac{q^2}{2m} + 2\sigma h)}{[(i\omega + \frac{q^2}{2m} + 2h)^2 - (v_F q)^2]^{3/2}}, \quad (22)
\]
\[
\frac{1}{Z} = 1 + \nu U^2 \int \frac{d\omega}{2\pi} \int_0^\infty \frac{dq}{2\pi} P(q, \omega; \sigma h)
\]
\[
\times \left[ (i\omega + \frac{q^2}{2m} + 2\sigma h) \text{sgn}(\frac{q^2}{2m} + 2\sigma h) \right]
\left[ (i\omega + \frac{q^2}{2m} + 2\sigma h)^2 - (\nu q^2)^2 \right]^{3/2}
+ 2\delta(\omega)
\right]
\]
\[
\Theta((\nu q^2)^2 - (\frac{q^2}{2m} + 2\sigma h)^2) \left[ (\nu q^2)^2 - (\frac{q^2}{2m} + 2\sigma h)^2 \right]^{1/2}.
\]
(23)

The remaining two integrations over \( q \) and \( \omega \) can be done numerically. By plotting the results versus \( h \) and fitting the \( h \)-dependence to straight lines we find that for small \( h \) all three coefficients \( X, Y, \) and \( Z \) have a non-analytic term linear in \( |h| \),
\[
X(h) = X(0) + X_1|h| + O(h^2),
\]
(24a)
\[
Y(h) = Y(0) + Y_1|h| + O(h^2),
\]
(24b)
\[
Z(h) = Z(0) + Z_1|h| + O(h^2),
\]
(24c)

where, with a numerical accuracy of the order of one percent, all prefactors of the non-analytic terms have the same numerical value,
\[
X_1 \approx Y_1 \approx Z_1 \approx 0.50 \frac{u^2}{E_F} + O(u^3).
\]
(25)

Here \( u = \nu U \) is the relevant dimensionless interaction. Moreover, from our numerical evaluation of the integrals in Eqs. (21,23) we find that the numerical values of \( Y_1 \) and \( Z_1 \) are determined by exchange-momenta \( q \) in the entire interval \( q \in [0,2k_F] \), indicating that other scattering processes than exchange scattering contribute. On the other hand, the value of \( X_1 \) is completely determined by the small-\( q \) part of \( \Sigma_0^{1+}(Q) \), which is also the case for the susceptibility coefficient \( \chi_1 \) in Eq. (2).

The identity \( Y_1 = Z_1 \) implies that, at least to this order in the interaction, the effective mass \( m_\sigma/m = Y/Z \) does not exhibit a linear dependence on the magnetic field. In order to prove this and to classify the various contributions to the non-analytic terms, let us anticipate that the numerical values of the coefficients \( X_1, Y_1, \) and \( Z_1 \) are completely determined by low-energy scattering processes involving momenta in the vicinity of the Fermi surface. This implies that these coefficients can be calculated from an effective low-energy model containing only states with momenta in a thin shell around the Fermi surface. It turns out that in two dimensions only the three types of low-energy scattering processes shown in Fig. 2 are possible. Let us therefore replace the bare interaction in our original model by an effective low-energy interaction containing only scattering processes with momenta in the vicinity of the Fermi surface. In two dimensions the corresponding Euclidean action can be written as
\[
S_{\text{int}}[\sigma] \approx \frac{1}{2} \int Q \Theta(\Lambda_0 - |q|) \sum_{\sigma_1 \neq \sigma_2} \left[ U_f D_Q^{\sigma_1 \sigma_1} D_Q^{\sigma_2 \sigma_2} - U_x D_Q^{\sigma_1 \sigma_2} D_Q^{\sigma_1 \sigma_2} + U_c C_Q^{\sigma_1 \sigma_3} C_Q^{\sigma_1 \sigma_2} \right],
\]
(26)

where \( \Lambda_0 \ll k_F \) is an ultraviolet cutoff and we have introduced the bilinear composite fermion fields \( D_Q^{\sigma_1 \sigma_2} = \int K \tilde{c}_K^{\sigma_1} c_{K+Q}^{\sigma_2} \) and \( C_Q^{\sigma_1 \sigma_2} = \int K c_{-K}^{\sigma_1} c_{K+Q}^{\sigma_2} \). Our original interaction in Eq. (4) corresponds to \( U_f = U_x = U_c = U \), but it is instructive to introduce separate interaction constants \( U_f \) (forward scattering), \( U_x \) (exchange scattering), and \( U_c \) (Cooper scattering). For our low-energy model the three contributions to the self-energy shown in Fig. 1 are additive, so that to second order in the interaction we may approximate
\[
\Sigma^\sigma(K) \approx \Sigma_f^\sigma(K) + \Sigma_x^\sigma(K) + \Sigma_c^\sigma(K),
\]
(27)
Then the relevant integrals can be performed analytically and we finally obtain for the universal coefficients of the terms linear in $|h|$,

$$Y_1 = Z_1 = \frac{u_x^2 + u_y^2}{4E_F}, \quad (31)$$

where $u_i = \nu U_i$. Note that forward scattering processes do not contribute to the above non-analytic corrections. Setting $u_y = u_c = u$ in Eq. (31), we obtain agreement with our direct numerical evaluation in Eq. (25), so that for this type of interaction the identity $Y_1 = Z_1$ is established exactly.

The evaluation of the coefficient $X_1$ is more tricky, because the integral (21) decays more slowly than the integrals defining $Y_1$ and $Z_1$. In practice, we first calculate $\partial \Sigma^\sigma(k^2, 0)/\partial h$ and integrate the resulting expression over $h$ to recover the difference in Eq. (21). Unfortunately, our low-energy approximation (29) for the particle-particle bubble with magnetic-field independent Fermi surface cutoff $\Omega_0$ seems not to be sufficient to extract the complete non-analytic $h$-dependence of the coefficient $X(h)$. However, we know from our direct numerical evaluation of Eq. (21) that the relevant integral is completely determined by small exchange momenta $q \ll k_F$, as discussed in the text after Eq. (25). To evaluate $X_1$ within our low-energy model, it is therefore sufficient to take only the contribution from the exchange scattering channel into account. After a straightforward calculation we obtain

$$X_1 = \frac{u_x^2}{2E_F}, \quad (32)$$

which agrees with Eq. (25) if we set $u_x = u$.

We conclude that the coefficients $X_1$, $Y_1$ and $Z_1$ of the terms linear in $|h|$ are indeed determined by low-energy processes involving momenta close to the Fermi surface, as anticipated. Using the methods developed in Ref. [11], we can show analytically that the numerical values of these coefficients are determined by scattering processes involving momentum transfers in the entire range $[0, 2k_F]$. As a consequence, there are no correlations between the momenta $k$ and $k'$ of the two incoming electrons shown in Fig. 2 so that for a momentum-dependent interaction $X_1$, $Y_1$, and $Z_1$ will depend on all angular harmonics of the interaction on the Fermi surface. This result should be contrasted with the well-known non-analytic behavior of the real part of the self-energy on the real frequency axis on resonance ($\omega = \xi_k$) for vanishing magnetic field, which is proportional to $\omega |\omega|$, with a prefactor that depends exclusively on scattering processes involving momentum transfers $q = 0$ and $q = 2k_F$. Hence, for real frequencies and finite magnetic field the non-analytic dependence of $\Sigma(k, \omega)$ on $\omega$ and $h$ must be described by a non-trivial cross-over function depending on the ratio $|\omega|/|h|$ which takes the different nature of the dominant scattering processes in the two limits $|\omega|/|h| \ll 1$ and $|\omega|/|h| \gg 1$ into account.
III. EXPERIMENTAL DETECTION OF THE LINEAR MAGNETIC FIELD DEPENDENCE

We now discuss experimental observables which are sensitive to the non-analytic magnetic field dependence of the quasi-particle properties. The quasi-particle residue and the renormalized single-particle dispersion can be determined via tunneling experiments, which would be the most direct method to reveal the non-analytic magnetic field dependence of $Z$ predicted in this work. Due to the identity $Y_1 = Z_1$, the effective mass $m_e/m = Y/Z$ does not exhibit any non-analytic magnetic field dependence to second order in the interaction. Whether this remains true also to higher order in the interaction remains an open problem.

A. Tunneling density of states

The non-analytic magnetic field dependence of the renormalization factor $Y(h)$ associated with the momentum-derivative of the self-energy determines the renormalized density of states $\nu_\star(h)$, which in the quasi-particle approximation is given by

$$\nu_\star(h) = Z \int \frac{d^2 k}{(2\pi)^2} \delta \left( k^2 - k_F^2 - g_\star h \right) = Y(h) \frac{m_\star}{2\pi}. \tag{33}$$

Hence, the magnetic field dependence of the renormalized density of states is given by

$$\frac{\nu_\star(h) - \nu_\star(0)}{\nu} = Y_1 |h| + O(h^2) \tag{34}$$

$$(u_x^2 + u_y^2)|h| + O(h^2, u^3|h|). \tag{35}$$

Note that the renormalized density of states can be measured via tunneling experiments.

B. Magnetoconductivity

It turns out that the factor $Y(h)$ appears also in the magnetoconductivity $\sigma(h)$ via the renormalization of the current vertices in the Kubo formula. In order to obtain a finite conductivity, we add elastic impurity scattering to our model. In the quasi-particle approximation the disorder averaged fermionic Green function is then

$$G^\sigma(K) = \frac{Z}{i\omega - (m/m_\star)\xi_1^0 + isgn\omega/(2\tau_\star)}. \tag{36}$$

To determine the inverse scattering time $1/\tau_\star$, we calculate the contribution $\Sigma_{imp}(i\omega)$ to the self-energy due to impurity scattering for short-range disorder within the Born approximation using the quasi-particle approximation for the propagator in the loop integral. Evaluating the corresponding Feynman diagram shown in Fig. 3 (a) we obtain

$$\Sigma_{imp}(i\omega) = -isgn\omega \frac{Y}{2\tau}. \tag{37}$$

where $\tau$ is the elastic lifetime in the absence of interactions. In this approximation, the renormalized inverse scattering time is given by

$$\frac{1}{\tau_\star} = -2Z \text{Im} \Sigma_{imp}(i\eta) = ZY/\tau, \tag{38}$$

where $\eta > 0$ is infinitesimal. Using the Drude formula for the conductivity of a Fermi liquid with effective mass $m_\star$ and scattering time $\tau$, we obtain

$$\sigma(h) = \frac{ne^2\tau_\star}{m_\star} = \frac{\sigma_0}{Y^2}. \tag{39}$$

Here $\sigma_0 = ne^2\tau/m$ is the conductivity without interactions and $n = k_F^2/(2\pi)$ is the total electronic density. Alternatively, Eq. (38) can be obtained from the Kubo formula for the conductivity of a $D$-dimensional Fermi liquid

$$\sigma = \frac{e^2}{2\pi D} \sum_\sigma \left[ 1 + \frac{\partial \Sigma^\sigma(k,0)}{\partial k^\sigma} \right]_{\xi^0=0}^2$$

$$
\times \int \frac{d^D k}{(2\pi)^D} G^\sigma(k, i\eta) G^\sigma(k, -i\eta). \tag{40}$$

For vanishing magnetic field, this formula has first been derived by Lange, and has been justified by several other authors. A diagrammatic representation of Eq. (40) is shown in the first line of Fig. 3. The factor containing the momentum-derivative of the self-energy in the Langer-formula takes the low-energy part of vertex corrections due to interactions into account, as explained in the caption of Fig. 3. The fact that vertex corrections in the Kubo formula can be expressed in terms of the
We use the so-called Aslamazov-Larkin corrections to the conductivities up to second order in the interaction. When inserted into the conductivity bubble, the diagrams (4) and (5) generate the so-called Aslamazov-Larkin corrections to the conductivity (39). In Appendix A we show explicitly how to recover the interaction processes of the type shown in Fig. 3 (b) and (c). In Appendix A we show explicitly how to recover the leading weak coupling behavior of the ZNA-formula (11) from the disorder averaged Kubo formula; we also show that the Langer-formula (39) does not take the leading vertex corrections due to impurity scattering in the Kubo formula into account.

Although the weak-coupling expansion of the ZNA-formula (11) generates a term of order $(F^2_0)$, it is not obvious whether this term includes the nonanalytic corrections to the current vertices which can be related to the momentum-derivative of the self-energy. In any case, the ZNA-formula (11) is incomplete because ZNA have ignored the Cooper channel, which according to Eq. (40) contributes to the weak-coupling expansion of the magnetoconductivity on equal footing with the exchange channel. On the other hand, our use of the Langer-formula (39), which is by itself on solid grounds, in a situation where quantum-interference effects produce non-analytic terms in $H$ and $T$ is not completely justified, as discussed in Appendix A.

Interaction-induced magnetoresistance of a two-dimensional electron gas in a transverse magnetic field has been calculated by Gornyi and Mirlin (28). They have taken into account the same interaction processes as ZNA but have considered also the case of long-range disorder, where their interaction correction is exponentially suppressed by the disorder correlation function. On the other hand, our interaction correction to the conductivity, which we have related via the Langer-formula to the momentum-dependence of the self-energy in the clean limit, is not exponentially suppressed for long-range dis-
order, indicating that also Gornyi and Mirlin did not take the non-analytic corrections discussed in our work into account.

IV. SUMMARY AND CONCLUSIONS

In summary, we have shown that in a two-dimensional Fermi liquid the quasi-particle residue and the renormalized electronic Landé factor exhibit a non-analytic magnetic field dependence proportional to $|H|$ at zero temperature. We have explicitly calculated the corresponding prefactors to second order in the interaction. To this order, the terms linear in $|H|$ cancel in the renormalized effective mass. We have also shown that the magnetic field dependence generated by the momentum-derivative of the electronic self-energy gives rise to a non-analytic correction to the density of states which is linear in $|H|$ and proportional to the square of the interaction in the weak coupling regime.

Using the Langer-formula\cite{21} for the conductivity of a Fermi liquid, we have also shown that the momentum-derivative of the self-energy generates a correction to the magnetoconductivity which is linear in $|H|$ and quadratic in the coupling constants in both the exchange and the Cooper channel. We have compared our result with previous work by Zala, Narozhny and Aleiner\cite{26} who have focused on the contribution from the exchange channel. Whether the ZNA-formula implicitly takes into account the non-analytic corrections due to the renormalization of the current vertices requires further investigations which are beyond the scope of this work.

While in this work we have focused on the non-analytic magnetic field dependence of the self-energy at vanishing temperature, in two-dimensional Fermi liquids similar non-analyticities appear also as a function of temperature $T$. In particular, we show in Appendix B that for $H = 0$ and $T \ll E_F$ the factor $Y$ defined in Eq. (12) has for a constant bare interaction $U$ acting between electrons with opposite spin the low-temperature expansion

$$Y(T) = Y(0) + \tilde{Y}_1 T + O(T^2), \quad \tilde{Y}_1 = \frac{u^2}{4E_F}.$$  (42)

A linear temperature dependence of quasi-particle properties of two-dimensional Fermi liquids has been discussed previously in Refs. [10] and [29] and is closely related to terms of order $\omega T$ in the real part of the self-energy on the mass shell discussed in Ref. [2]. Using again the Langer-formula\cite{33}, we find that the conductivity of two-dimensional Fermi liquids in the ballistic regime exhibits a linear temperature dependence, in agreement with ZNA\cite{26}.

Finally, let us point out two interesting extensions of our calculations. First of all, it would be interesting to calculate the coefficients $X_1$, $Y_1$, and $Z_1$ of the terms linear in the magnetic field in the low-energy expansion of the electronic self-energy beyond the leading order in the interaction. Such a calculation should take vertex corrections and the interference of the different low-energy scattering channels into account. We believe that the low-energy model with interaction (26) is a good starting point for such a calculation. It would also be interesting to revisit earlier calculations of the magnetoconductivity in the presence of disorder\cite{26,28}, consistently taking all sources of non-analytic behavior by all low-energy scattering channels into account. In this context a perturbative calculation of the disorder averaged Kubo formula in the ballistic regime which systematically includes all diagrams up to second order in the interaction would be very instructive.

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APPENDIX A: DERIVATION OF THE ZNA-FORMULA TO FIRST ORDER IN THE INTERACTION

In this appendix we re-derive the ZNA-formula (11) for the magnetoconductivity in the ballistic regime to leading (linear) order in the interaction. For simplicity, we assume that the bare interaction $U$ is momentum independent and acts only between electrons with antiparallel spin projection, see Eq. (1). In the ballistic regime and to first order in the interaction the magnetoconductivity is then determined by the three diagrams shown in Fig. 5. The diagrams (b) and (c) depend on the first order self-energy diagrams shown in Fig. 3 (b) and (c). These diagrams give rise to non-analytic corrections to the renormalized scattering time. To determine the corresponding contribution to the conductivity, we can simply insert the relevant scattering time in the Drude formula. Assuming $\delta$-function correlated disorder, the sum of the two diagrams (b) and (c) in Fig. 3 yields the following first order contribution to the self-energy of electrons with spin-projection $\sigma$,

$$\Sigma^{\sigma}(k, i\omega) = \frac{2U}{2\pi\nu \tau} \int \frac{d^2q}{(2\pi)^2} \Pi_0^{\sigma, -\sigma}(q, 0) G_0^\sigma(k - q, i\omega).$$  (A1)

The parallel-spin particle-hole bubble in a magnetic field is for vanishing frequency given by $\Pi_0^{\sigma, \sigma}(q, 0) = -\nu P^\sigma(q)$.
we obtain for the spin-projection $\sigma = \uparrow$ to leading order in $h > 0$,

$$\text{Im} \Sigma^\uparrow_1(k_F^+, i\eta) - \text{Im} \Sigma^\downarrow_1(k_F^+, i\eta) \bigg|_{h=0} = -\frac{2u}{2\pi \nu \tau v_F} \int_{-2h/\nu v_F}^{2h/\nu v_F} \left. \frac{dp \text{e}^{i\eta p + 2h}}{2h - \nu v_F p} \right. = -\frac{uh}{E_F \tau}. \quad (A7)$$

For the other spin projection we obtain for $h > 0$,

$$\text{Im} \Sigma^\downarrow_1(k_F^+, i\eta) - \text{Im} \Sigma^\uparrow_1(k_F^+, i\eta) \bigg|_{h=0} = 0, \quad (A8)$$

so that the renormalized scattering rates are for $h > 0$,

$$\frac{1}{\tau_{s\uparrow}} = \frac{1}{\tau} - 2\text{Im} \Sigma^\uparrow_1(k_F^+, i\eta) = \frac{1}{\tau} \left[ 1 + \frac{2uh}{E_F} \right], \quad (A9)$$

and $1/\tau_{s\downarrow} = 1/\tau + O(h^2)$. The correction to the conductivity due to the diagrams shown in Fig. 5 (b) and (c) is therefore

$$\sigma(h)_{b+c} = \frac{ne^2}{m} \frac{\tau_{s\uparrow} + \tau_{s\downarrow}}{2} = \sigma_0 \left[ 1 - u \frac{|h|}{E_F} + O(u^2, h^2) \right]. \quad (A10)$$

Keeping in mind that to leading order in the bare interaction we may identify $F_0^\sigma = -u$ and approximate $g(F_0^\sigma) \approx 1$ in Eq. (41), the correction in Eq. (A10) is a factor of two smaller than the corresponding correction obtained from the weak-coupling expansion of the ZNA-formula (41). The missing contribution is due to the vertex correction diagram shown in Fig. 5 (a), which yields the following correction to the conductivity in the ballistic limit,

$$\sigma(h)_a = \frac{e^2}{2\pi m^2 \nu \tau} \sum \int \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k'}{(2\pi)^2} \int \frac{d^2 k''}{(2\pi)^2} \frac{1}{2}$$

$$\times \Pi_0^{\sigma,-\sigma}(\mathbf{k} - \mathbf{k}', \mathbf{k}'') G_0^\sigma(\mathbf{k}, \mathbf{i}\eta) G_0^{\sigma}(\mathbf{k}, -\mathbf{i}\eta) \times G_0^{\sigma}(\mathbf{k}', \mathbf{i}\eta) G_0^{\sigma}(\mathbf{k}'', -\mathbf{i}\eta). \quad (A11)$$

The correction linear in $|h|$ can be extracted analytically and we obtain

$$\sigma(h)_a - \sigma(0)_a = -\sigma_0 u |h| \frac{1}{E_F}. \quad (A12)$$

Combining this with Eq. (A10) we finally obtain

$$\frac{\sigma(h) - \sigma(0)}{\sigma_0} = -2u \frac{|h|}{E_F} + O(u^2, h^2). \quad (A13)$$

Keeping in mind that to leading order $F_0^0 = -u$ for our model, Eq. (A13) agrees with the leading weak coupling behavior of the ZNA-formula (41) in the ballistic regime.

Finally, let us show that the vertex correction (A11) is not taken into account in the Langer-formula (39). As discussed in Sec. III B, the momentum-derivative of the self-energy in the Langer-formula implicitly takes vertex corrections due to interactions in the Kubo formula into account. However, the vertex correction in Eq. (A11),

FIG. 5. (Color online) Corrections to the disorder averaged Kubo formula for the conductivity in the ballistic regime to first order in a momentum-independent interaction which acts only between electrons with opposite spin-projection. The black triangles are the bare current vertices. The other symbols are the same as in Fig. 3. The vertex correction (a) does not transfer any energy between the two branches of the bubble and is not taken into account in the Langer-formula (39). The diagrams (b) and (c) generate a non-analytic contribution to the renormalized scattering time.

with

$$P^\sigma(q) = 1 - \frac{2k_F^2}{q} \Theta(q - 2k_F^2) \sqrt{\frac{q}{2k_F^2}} - 1, \quad (A2)$$

where $k_F^2 = k_F \sqrt{\frac{1 + \sigma h}{E_F}} \approx k_F [1 + \sigma h/(2E_F)]$. After carrying out the angular integration in Eq. (A1), we obtain

$$\Sigma^\sigma(k, i\omega) = \frac{2u}{2\pi \nu \tau} \int_0^{\infty} dq \frac{P^\sigma(q)}{2\pi}$$

$$\times \frac{\text{sgn}(\xi_k + \frac{q^2}{2m} - \sigma h)}{[i\omega - \xi_k - \frac{q^2}{2m} + \sigma h]^2 - (\nu k q)^2}. \quad (A3)$$

It turns out that the non-analytic magnetic-field dependence of the self-energy (A3) is due to the non-analyticity of the polarization bubble (A2) at $q = 2k_F^2$. It is therefore sufficient to replace Eq. (A2) by its approximation for $|q - 2k_F^2| \ll 2k_F^2$,

$$P^\sigma(q) \approx 1 - \Theta(q - 2k_F^2) \sqrt{\frac{q - 2k_F^2}{k_F^2}}. \quad (A4)$$

To obtain the interaction correction to the scattering rate we need

$$\text{Im} \Sigma^\sigma_1(k_F^+, i\eta) = \frac{2u}{2\pi \nu \tau} \int_0^{\infty} dq \frac{P^\sigma(q)}{2\pi}$$

$$\times \frac{\text{sgn}(\frac{q^2}{2m} - 2\sigma h)}{\sqrt{\left[ \frac{q^2}{2m} - 2\sigma h - i\eta \right]^2 - (\nu k q)^2}}. \quad (A5)$$

Setting $q = 2k_F + p$ and approximating

$$\left[ \frac{q^2}{2m} - 2\sigma h - i\eta \right]^2 - (\nu k q)^2 \approx 8E_F |v_F p - 2\sigma h - i\eta|, \quad (A6)$$

we obtain for the spin-projection $\sigma = \uparrow$ to leading order in $h > 0$,
which does not transfer any energy and can be viewed as an interference correction involving an effective renormalized impurity line, is not included in the Langer-formula. To see this, note that to first order in the interaction the velocity renormalization factor in the Langer-formula can only be due to the momentum-derivative of the self-energy \( \Sigma'(k, i\omega) \) given in Eq. (A1). However, \( \Sigma'(k, i\omega) \) vanishes for \( 1/\tau \to 0 \), so that in the ballistic limit the velocity renormalization factor associated with the self-energy \( \Sigma'(k, i\omega) \) vanishes and therefore does not contribute to the conductivity, in contrast to Eq. (A1).

APPENDIX B: NON-ANALYTIC TEMPERATURE DEPENDENCE

In this appendix we briefly describe the derivation of the leading temperature dependence of the renormalization factor \( Y \) for vanishing magnetic field given in Eq. (42). The derivation will be done along the lines of Ref. 2. We begin by considering the exchange channel and express the antiparallel-spin particle-hole bubble through its spectral representation

\[
\Pi_{0}^{\uparrow\downarrow}(q, i\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\Omega \frac{\text{Im}\Pi_{0}^{\uparrow\downarrow}(q, \Omega + i\eta)}{\Omega - i\omega}, \tag{B1}
\]

where \( \eta > 0 \) is infinitesimal and the integral is now over real frequencies \( \Omega \). The Matsubara sum in Eq. (6b) can then be performed. Using \( \text{Im}G_{0}(k, \omega + i\eta) = -\pi \delta(\omega - \xi_{k}) \) we find for the imaginary part of the self-energy after analytic continuation to real frequencies,

\[
\text{Im}\Sigma_{x}(k, \omega + i\eta) = \frac{U_{x}^{2}}{2} \int \frac{d^{2}q}{(2\pi)^{2}} \Theta(\Lambda_{0} - |q|) \int_{-\infty}^{\infty} d\Omega \text{Im}\Pi_{0}^{\uparrow\downarrow}(q, \Omega + i\eta) \delta(\xi_{k} - q - \omega + \Omega) \left[ \coth\left(\frac{\Omega}{2T}\right) - \tanh\left(\frac{\omega - \Omega}{2T}\right) \right].
\]

The angular integration is now trivial due to the \( \delta \)-function while the remaining momentum integration can also be done. After performing a Kramers-Kronig transform we obtain for the singular contribution to the real part of the self-energy,

\[
\text{Re}\Sigma_{x}(k, \omega + i\eta) = \frac{\nu U_{x}^{2}}{(2\pi)^{3}v_{F}} P \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} \int_{-\infty}^{\infty} d\Omega \Omega \ln \left| \frac{v_{F} \Lambda_{0}}{2\Omega + \omega' - \xi_{k}} \right| \left[ \coth\left(\frac{\Omega}{2T}\right) - \tanh\left(\frac{\omega' + \Omega}{2T}\right) \right], \tag{B3}
\]

where \( P \) denotes the principal part. Finally taking the derivative with respect to \( \xi_{k} \) and setting \( \omega = \xi_{k} = 0 \) we can scale out the temperature dependence. By symmetrizing the integrand with respect to \( \omega' \), the remaining frequency integrals can then be done. We find that the exchange channel yields a contribution \( u_{x}^{2}/(8E_{F}) \) to the coefficient \( \tilde{Y}_{1} \) in Eq. (42).

The contributions from the two other channels can be found in a similar way. In the Cooper channel the same steps lead to the following expression for the singular contribution to the imaginary part of the self-energy,

\[
\text{Im}\Sigma_{c}(k, \omega + i\eta) = -\frac{\nu U_{c}^{2}}{(2\pi)^{2}} \int_{0}^{\Lambda_{0}} dq q \int_{-\infty}^{\infty} d\Omega \ln \frac{2\Omega}{\sqrt{\Omega^{2} + (v_{F}q)^{2}}} \left[ \coth\left(\frac{\Omega}{2T}\right) - \tanh\left(\frac{\omega' + \Omega}{2T}\right) \right]. \tag{B4}
\]

Differentiating with respect to \( \xi_{k} \), setting then \( \omega = \xi_{k} = 0 \), and using the Kramers-Kronig transform we find for the real part,

\[
\frac{\partial \text{Re}\Sigma_{c}(k, i\eta)}{\partial \xi_{k}} \bigg|_{\xi_{k}=0} = \frac{\nu U_{c}^{2}}{(2\pi)^{2}v_{F}^{3}} \int_{0}^{\nu v_{F}^{2}/2} dx \int_{-\infty}^{\infty} d\Omega \ln \frac{2\Omega}{\sqrt{\Omega^{2} + x^{2}}} \times \left| \frac{2\Lambda_{0}}{\Omega + \sqrt{\Omega^{2} + x^{2}}} \right| \coth\left(\frac{\Omega}{2T}\right) - \tanh\left(\frac{\omega' + \Omega}{2T}\right), \tag{B5}
\]

with \( x = v_{F}^{2}q^{2} \). Since the integrand is odd under the simultaneous shift of \( \Omega \to -\Omega \) and \( \omega' \to -\omega' \) the integral vanishes. Finally for vanishing magnetic field the contribution from the forward scattering channel to \( Y \) can be
taken into account by simply replacing \( u_x \rightarrow u_f \) in the result from the exchange scattering channel, so that for \( u_x = u_f \) forward and exchange scattering channels yield identical contributions to \( Y \). Collecting all terms and setting \( u_x = u_f = u_c = u \) we finally arrive at Eq. (42).

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