Superconformal Symmetry
and
Correlation Functions

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Declaration

This dissertation is my own original work, except where explicitly stated in the text or the acknowledgements. Chapters 2-4, parts of chapter 5 and appendices A-D are based on published work [1] with my supervisor, Prof. Hugh Osborn. Parts of chapter 5 and appendices E-G are unpublished work in progress with my supervisor.

No part of this dissertation has been submitted for a degree, diploma, or other qualification at any other university, or is concurrently submitted for any degree, diploma or qualification.
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Michael Nirschl

Superconformal Symmetry and Correlation Functions

Summary

The constraints that $\mathcal{N} = 2, 4$ superconformal symmetry imposes in $d = 4$ for four point functions of chiral primary $\frac{1}{2}$-BPS operators are derived. The operators are described by symmetric traceless tensors of the internal $R$-symmetry group. A substantial simplification compared to earlier work is achieved by introduction of null vectors. They reduce complicated tensorial expressions to polynomials of one or two invariant cross ratios for $\mathcal{N} = 2, 4$ respectively, similar to the 2 conformally invariant cross ratios of 4 points. Two variable polynomials corresponding to the different $R$-symmetry representations are constructed. The Ward identities for superconformal symmetry are then obtained as simple differential equations. The general solution of these identities is presented in terms of a constant, a single variable function and a two variable function. In the extremal case it is shown that the amplitude has to be a constant. In the next-to-extremal case the amplitude contains a constant and a single variable function only. An interpretation in terms of the operator product expansion is given for the case of fields of equal dimension and for the so called (next-to)extremal cases. The result is shown to accommodate long multiplets as well as semishort and short multiplets with protected dimension. Generically also non-unitary multiplets can appear. It is shown how to remove them using appropriate semishort and long multiplets to obtain a unitary theory. Where possible, positivity of OPE coefficients required by unitarity is confirmed. Implications of crossing symmetry for the four point functions studied are derived and discussed. It is shown that crossing symmetry fixes the single variable function in the general solution to be of free field form using singularity arguments. For a restricted set of next-to-extremal correlation functions with $S_3$ symmetry amongst the first three fields it is shown that the amplitude is fixed up to normalization to be of free field form. Starting from a known expression for the large $N$ amplitude of $[0, 4, 0]$ operators we simplify it further and present it in a manifestly crossing symmetric form. We compute the coefficients of the conformal partial wave expansion of all representations in this amplitude and use them to compute an averaged value of the anomalous dimensions for long multiplets given spin and twist in each relevant representation at first order in $1/N$. Finally assuming the previously observed universal singularity structure in the large $N$ amplitude we derive the general large $N$ amplitude of four identical $\frac{1}{2}$-BPS operators in the $[0, p, 0]$ representation in terms of $\overline{D}$ functions. Explicit expressions for all coefficients are given.
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1. Introduction

1.1. Motivation

One of the most important and powerful concepts in the development of physics in the 20th century was the concept of symmetry. Einstein discovered the Poincaré symmetry of spacetime which was the integral part of his special theory of relativity in 1905. Later in 1915 he developed the general theory of relativity which forms our basis of understanding of gravity. The key concept is again considering the general symmetry of change of coordinates of an observer.

Later in the 20th century gauge symmetry became the central tool to construct quantum theories which describe all the forces we observe in nature apart from gravity. Using the same formalism with different symmetries we can accurately describe the electromagnetic force as well as the weak and the nuclear forces which forms the basis of the Standard Model. Therefore gauge field theories, also called Yang Mills theories (YM), have been and are in the very centre of research of quantum field theories.

Generally, all particles we observe fall into two classes characterized by their spin. Particles with integer spin are called bosons, and those are the particles which transmit all the forces we observe in nature. Particles with half-integer spin are called fermions and constitute all the matter in the universe. Bosons and fermions show very different behaviour. Fermions obey Fermi-Dirac statistics, i.e. they can never be in the same state as another one, which is a generalization of the Pauli principle and is responsible for the stability of atoms and thus the world surrounding us. Bosons on the other hand obey Bose-Einstein statistics, i.e. they tend to clump together, an effect which is used in lasers to generate light with very long coherence length as well as in Bose-Einstein condensates to generate a fourth phase of matter.

In 1971 a new symmetry was proposed, supersymmetry. It unifies the two different classes of particles, bosons and fermions, by allowing them to transform into each other. This was first studied by Ramond, Neveu and Schwarz in the context of string theory. 1974 Wess and Zumino wrote down the first example of a 4-dimensional QFT with su-
persymmetry. Haag, Sohniues and Lopuszanski showed in 1975 that the unification of
supersymmetry and conformal symmetry, called superconformal symmetry, is actually the
maximal spacetime symmetry group possible for a QFT permitting a non-trivial $S$-matrix.
This was an extension of the Coleman-Mandula theorem that had established that the
maximal symmetry Lie group for a non-trivial $S$-matrix is conformal symmetry.

It turns out that supersymmetry has several very desirable effects, which is why many
theories studied until today are supersymmetric although we still wait for experimental
confirmation of its existence. First of all, supersymmetry cures some of the divergences
which appear in quantum field theories and in string theory. This makes the theories look
mathematically more consistent and in the case of superstring theory or the quantum field
theory we will study they actually become finite. Another technical advantage of super-
symmetry is that it gives more control over computations and therefore more quantities
are accessible in supersymmetric theories, generically because of so called nonrenormalization
theorems. Most quantities in a QFT depend strongly on the coupling constant which
measures the strength of the forces. The two extreme regimes are on the one hand the
perturbative regime where the interactions are very weak, and the strong coupling regime
with very intense forces on the other hand. Supersymmetry guarantees that certain quan-

tities are not affected by the renormalization flow which describes the transition between
the different regimes of the theory. Thus they can be computed in the perturbative regime
and then be extrapolated to the strong coupling regime. This is why supersymmetric
theories are also often used as models for their non-supersymmetric counterparts. Finally
from a phenomenological point of view supersymmetry is very desirable if one wants to
unify electroweak and color forces. Without supersymmetry the three coupling constants
never quite meet. Supersymmetry achieves exactly that. Also the hierarchy problem, or
the question why the Higgs mass is so small compared to the Planck scale, can be solved
with supersymmetry.

In 1997 Maldacena [2] found that there exists a very deep connection between string
theory and quantum field theories. The AdS/CFT correspondence conjectures that string
theory in the geometric background of $AdS_5 \times S^5$ with 5-form flux of $N$ units is equivalent
to $\mathcal{N} = 4$ Supersymmetric Yang Mills (SYM) on $S^4$, the boundary of $AdS_5$, with gauge group $SU(N)$. The isometry group of $S^5$ is $SO(6) \simeq SU(4)$ which is the internal $R$-symmetry group. $\mathcal{N} = 4$ SYM is the unique unitary quantum field theory (QFT) in 4 dimensions with $\mathcal{N} = 4$ superconformal symmetry apart from the choice of the gauge group. Remarkably the AdS/CFT correspondence relates the weak coupling limit of one theory to the strong coupling region of the other. This allows obtaining nonperturbative information about string and quantum field theory from perturbative calculations in the respective other theory. For a review of this and a complete list of references check [3].

The maximal amount of symmetry in $d = 4$ apart from internal symmetries is $\mathcal{N} = 4$ extended superconformal symmetry \footnote{More supersymmetry is excluded because the theory would include a graviton and since gravity is not renormalizable this theory cannot easily make sense as a QFT.}. This is the symmetry we will study in this thesis along with its sibling $\mathcal{N} = 2$ superconformal symmetry, both in the context of SYM theories.

One interesting question to ask is what constraints superconformal symmetry imposes on correlation functions. That is we will purely focus on the symmetry aspect of the theory. Therefore all our results are independent of the interactions of the theory and apply to all possible dynamics. Conformal symmetry fixes the two and three point functions up to normalization. Therefore it will be natural to study four-point functions which from a conformal symmetry point of view may contain arbitrary functions of two conformal invariants that one can construct out of four points. We will look at a special set of operators, $\frac{1}{2}$-BPS operators which in the AdS/CFT correspondence correspond to fundamental fields and Kaluza Klein modes. They are described by shortening conditions, i.e. some supercharges annihilate them. From this it follows that their dimension is not renormalized and thus their anomalous dimensions vanish.

Also one can use four point functions to analyse the operator product expansion (OPE) of the theory. By taking a limit where two fields approach the other two fields, the four point function reduces to a sum of two point functions multiplied by OPE coefficients. Finally we use conformal partial wave expansions to obtain further information about the
four point functions we study. Conformal partial waves correspond to intermediate states when interpreting the correlation function as describing the scattering of 2 incoming states into 2 outgoing states. An expansion of the amplitude in a basis of partial waves describing different intermediate states can reveal information about the intermediate states appearing.

Also in the light of the AdS/CFT correspondence we will examine the four point functions of four identical operators in the strong coupling limit more closely. We will study the amplitudes obtained using supergravity techniques as well as from a more field theoretic point of view by considering their singularity structure. We will see how the singularity structure actually uniquely determines these amplitudes together with crossing symmetry.

1.2. Superconformal Symmetry in \( d = 4 \)

The conformal group in \( d = 4 \) is \( SO(2, 4) \simeq SU(2, 2) \). It contains translations generated by \( P_\mu \), Lorentz transformations generated by \( M_{\mu\nu} \), dilations generated by \( D \) and special conformal transformations generated by \( K_\mu \). The scale dimensions of the generators are

\[
[M] = [D] = 0, \quad [P] = 1, \quad [K] = -1. \tag{1.1}
\]

In any theory there might be internal symmetries generated by \( T^A \).

The superconformal group in \( d = 4 \) is given by \( SU(2, 2|\mathcal{N}) \). The bosonic part of \( SU(2, 2|\mathcal{N}) \) consists of the conformal group and the \( R \)-symmetry \( U(\mathcal{N}) \) generated by \( R^i_j \). The \( R \)-symmetry corresponds to automorphisms of the SUSY generators. For the case of \( \mathcal{N} = 4 \) supersymmetry one can read off from the commutation relations [4] of the \( R \)-symmetry that it is consistent to remove the traces and thus restrict the \( R \)-symmetry to \( SU(4) \). The fermionic part contains Poincaré supersymmetries generated by \( Q^i_\alpha \) and \( \bar{Q}_{i\dot{\alpha}} \), also called SUSY generators in the following. Since their action does not commute with the action of the special conformal transformations, the algebra is closed by generators of conformal supersymmetries \( S_{i\alpha} \) and \( \bar{S}_{\dot{\alpha}} \). The scale dimensions of these additional generators
are

$$[T^A] = 0, \quad [Q] = [\bar{Q}] = \frac{1}{2}, \quad [S] = [\bar{S}] = -\frac{1}{2}. \quad (1.2)$$

For the complete commutation relations check [4].

1.3. Physical Fields in $\mathcal{N} = 4$ SYM

The $\mathcal{N} = 4$ Super Yang Mills theory contains 6 scalars $X_r$, 4 gauginos $\lambda_i$ and the gauge field $A_\mu$ with field strength $F_{\mu\nu}$. The scalars transform in the adjoint representation of $SU(4)$, the gauginos in the fundamental representation. The scalars and the gauge field have mass dimension 1, the gauginos $\frac{3}{2}$. The gauge group $G$ can be arbitrary. The moduli of the theory are given by the gauge coupling $g$, the instanton angle $\theta$ and the expectation values of the scalars $\langle X_r \rangle$. We will only be interested in the superconformal phase of the theory when all the expectations values of the scalars vanish $\langle X_r \rangle = 0$. Then the gauge group $G$ is unbroken and the theory possesses full superconformal symmetry given by the supergroup $PSU(2,2|4)$. This symmetry holds classically and also at the quantum level. The operators we consider are gauge invariant combinations of the physical fields and transform under unitary representations of $PSU(2,2|4)$.

1.4. Conformal Scalar 2, 3, 4-point Functions in $d = 4$

As mentioned before, conformal symmetry restricts two and three point functions completely up to normalization. This is due to the fact that the conformal group can be used to map any set of two or three points into any other set of two or three points. Or in other words, there are no invariants under conformal symmetry that can be constructed out of two or three points. The spacetime dependence is completely fixed by the scaling behaviour.

First we define for convenience

$$x_{ij} = x_i - x_j, \quad r_{ij} = x_{ij}^2 \quad (1.3)$$

and

$$\Delta_{ij} = \Delta_i - \Delta_j. \quad (1.4)$$
For a scalar field $\varphi_i(x)$ with scale dimension $\Delta_i$ the two point function reads

$$\langle \varphi_i(x_1)\varphi_j(x_2) \rangle = \delta_{ij} \frac{N}{r_{12}^{\Delta_i}} ,$$

(1.5)

where $N$ is an arbitrary normalization which we choose to be $N = 1$ in the following. The three point functions for scalar fields are also determined by conformal symmetry

$$\langle \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3) \rangle = C_{ijk} \frac{1}{r_{12}^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_k)}r_{13}^{\frac{1}{2}(\Delta_i+\Delta_k-\Delta_j)}r_{23}^{\frac{1}{2}(\Delta_j+\Delta_k-\Delta_i)}} ,$$

(1.6)

The constants $C_{ijk}$ only depend on which fields are present in the correlation function and will find an interpretation in the next section in the context of the operator product expansion.

The four point function may be expressed as

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4) \rangle = \frac{r_{23}^{\Sigma-\Delta_2-\Delta_3}r_{34}^{\Sigma-\Delta_3-\Delta_4}}{r_{13}^{\Delta_3}r_{24}^{\Sigma-\Delta_3}} F(u,v)$$

$$= \frac{1}{r_{12}^{\frac{1}{2}(\Delta_1+\Delta_2)}r_{24}^{\frac{1}{2}(\Delta_3+\Delta_4)}} \left( \frac{r_{24}}{r_{14}} \right)^{\frac{1}{2}\Delta_{12}} \left( \frac{r_{14}}{r_{13}} \right)^{\frac{1}{2}\Delta_{34}} G(u,v) ,$$

(1.7)

where $2 \Sigma = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ and $u, v$ are the two independent conformal invariants of four points

$$u = \frac{r_{12}r_{34}}{r_{13}r_{24}} , \quad v = \frac{r_{14}r_{23}}{r_{13}r_{24}} .$$

(1.8)

$F(u,v)$ or $G(u,v)$ are arbitrary functions of $u,v$ within the context of conformal symmetry and can only be constrained by the dynamics of a particular theory considered. $F(u,v), G(u,v)$ are obviously related by

$$F(u, v) = u^{-\frac{1}{2}(\Delta_1+\Delta_2)}v^{-\frac{1}{2}(\Delta_{12}+\Delta_{34})} G(u, v) .$$

(1.9)

As was shown in [5] the superconformal Ward identities are greatly simplified if they are expressed in terms of new variables $x, \bar{x}$ rather than the usual conformal invariants $u, v$. In terms of the standard correspondence for the space-time coordinates $x^\mu \to x = x^\mu \sigma_\mu$  

---

1 Thus 4-vectors are identified with $2 \times 2$ matrices using the hermitian $\sigma$-matrices $\sigma_\mu, \bar{\sigma}_\mu, \sigma_\mu \bar{\sigma}_\nu = -\eta_{\mu\nu}1$, $x^\mu \to x_{\alpha\dot{\alpha}} = x^\mu (\sigma_\mu)_{\alpha\dot{\alpha}}$, $\bar{x}^{\dot{\alpha}\alpha} = x^\mu (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}$, with inverse $x^{\mu} = -\frac{1}{2} tr(\sigma^{\mu}\bar{x})$. We have $x \cdot y = x^\mu y_\mu = -\frac{1}{2} tr(\bar{x}y)$, det $x = -x^2$, $x^{-1} = -\bar{x}/x^2$. 

8
for four points \(x_1, x_2, x_3, x_4\), \(x, \bar{x}\) may be defined, as shown in [6], as the eigenvalues of \(x_{12}^{-1} x_{43} x_{13}^{-1}\). By conformal transformations we may choose a frame such that \(x_2 = 0, x_3 = \infty, x_4 = 1\) and \(x_1 = \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}\). The two conformal invariants in \((1.8)\) are then given in terms of \(x, \bar{x}\) by

\[
 u = \det(x_{12}^{-1} x_{43} x_{13}^{-1}) = x \bar{x}, \\
 v = \det(1 - x_{12} x_{42}^{-1} x_{43} x_{13}^{-1}) = \det(x_{14} x_{24}^{-1} x_{23} x_{13}^{-1}) = (1 - x)(1 - \bar{x}).
\]

(1.10)

For a Euclidean metric on space-time \(x, \bar{x}\) are complex conjugates. We will see how this change of variables strongly simplifies not only the identities but also the derivation of those. They are also essential in writing down expressions for conformal partial waves.

1.5. Operator Product Expansion

Four point functions are also very useful to study the properties of the operator product expansion OPE of a QFT. In a conformal field theory a complete set of operators locally satisfies an OPE

\[
 \varphi_i(x) \varphi_j(0) \sim \sum_k C_{ij}^{kl} \frac{1}{(x^2)^{\frac{1}{2}(\Delta_i + \Delta_j - \Delta + \ell)}} x_{\mu_1} \ldots x_{\mu_\ell} O_{\mu_1 \ldots \mu_\ell}(0) + \ldots
\]

(1.11)

for \(x \sim 0\). Here \(O_{\mu_1 \ldots \mu_\ell}(0)\) is a symmetric traceless tensor operator of scale dimension \(\Delta\) and spin \(\ell\).

To derive a short distance limit for \(x_1 \to x_2\) of the scalar four point function \((1.7)\) we first write down the following expression for the three point function of a tensor operator of spin \(\ell\) and two scalar fields

\[
 \langle O_{\mu_1 \ldots \mu_\ell}(x_2) \varphi_3(x_3) \varphi_4(x_4) \rangle = C_{34O} \frac{1}{r_{23}^{\frac{1}{2}(\Delta_3 + \Delta_4 - \Delta - \ell)}} \frac{1}{r_{24}^{\frac{1}{2}(\Delta_2 + \Delta_4 - \Delta - \ell)}} \frac{1}{r_{34}^{\frac{1}{2}(\Delta_3 + \Delta_4 - \Delta - \ell)}} X_{\mu_1} \ldots X_{\mu_\ell};
\]

(1.12)

where

\[
 X_{\mu} = \frac{x_{24\mu}}{r_{24}} - \frac{x_{23\mu}}{r_{23}}, \quad X^2 = \frac{r_{34}}{r_{23} r_{24}}
\]

(1.13)

Since \(1 + u - v = x + \bar{x}\) and \(1 + u^2 + v^2 - 2uv - 2u - 2v = (x - \bar{x})^2\) it is easy to invert these results to obtain \(x, \bar{x}\) in terms of \(u, v\) up to the arbitrary sign of the square root \(\sqrt{(x - \bar{x})^2}\). For any \(f(u, v)\) there is a corresponding symmetric function \(\hat{f}(x, \bar{x}) = \hat{f}(\bar{x}, x)\) such that \(\hat{f}(x, \bar{x}) = f(u, v)\).
Using (1.11) and (1.12) we can derive the following short distance limit for $x_1 \to x_2$ for
the four point function of four scalar fields

$$
\left\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4) \right\rangle
\sim_{x_{12} \to 0} \frac{C_{12\Diamond} C_{34\Diamond}}{(r_{12})^{\frac{1}{2}(\Delta_1+\Delta_2)}(r_{34})^{\frac{1}{2}(\Delta_3+\Delta_4)}} \left(\frac{r_{24}}{r_{23}}\right)^{\frac{1}{2}\Delta_{34}} \left(\frac{r_{12} r_{34}}{r_{23} r_{24}}\right)^{\frac{1}{2}\Delta} C_{\ell}^{\frac{d}{2}-1}(t) \frac{\ell!}{2^\ell(\frac{1}{2}d-1)^\ell},
$$

(1.14)

where

$$
t = \frac{x_{12} \cdot X}{(r_{12}X^2)^{\frac{1}{2}}} \sim_{x_{12} \to 0} \frac{1-v}{2\sqrt{u}} + O(\sqrt{u}, 1-v) = -\frac{x + \overline{x}}{2(x\overline{x})^{\frac{1}{2}}} + O(\sqrt{x\overline{x}}, x + \overline{x}).
$$

(1.15)

The function $C_{\ell}^{\frac{d}{2}-1}(t)$ is a Gegenbauer polynomial which arises from

$$
a_{\mu_1} \ldots a_{\mu_\ell} b_{(\mu_1} \ldots b_{\mu_\ell)} = (a^2)^{\frac{1}{2}\ell} (b^2)^{\frac{1}{2}\ell} C_{\ell}^{\frac{d}{2}-1}(t) \frac{\ell!}{2^\ell(\frac{1}{2}d-1)^\ell},
$$

(1.16)

for $a, b$ $d$-dimensional vectors. It follows that the amplitude $G(u, v)$ in the four point function (1.7) has the following form

$$
G(u, v) \sim_{x_{12} \to 0} C_{12\Diamond} C_{34\Diamond} u^{\frac{1}{2}\Delta} C_{\ell}^{\frac{d}{2}-1}(t) \frac{\ell!}{2^\ell(\frac{1}{2}d-1)^\ell}.
$$

(1.17)

Now for $d = 4$ the Gegenbauer polynomial reduces to the following form

$$
C_{\ell}^{1}(\frac{x + \overline{x}}{2(x\overline{x})^{\frac{1}{2}}}) = (-1)^\ell (x\overline{x})^{-\frac{1}{2}\ell} \frac{x^{\ell+1} - \overline{x}^{\ell+1}}{x - \overline{x}},
$$

(1.18)

and thus $G(u, v)$ simplifies to

$$
G(u, v) \sim_{x_{12} \to 0} C_{12\Diamond} C_{34\Diamond} (x\overline{x})^{\frac{1}{2}(\Delta-\ell)} (-\frac{1}{2})^\ell x^{\ell+1} - \overline{x}^{\ell+1}
$$

(1.19)

a simple expression for the leading term contribution in the short distance limit of the four point function for an operator of scale dimension $\Delta$ and spin $\ell$.

The full analysis of the OPE is facilitated by a simple expression for the contribution of the conformal block of a quasi-primary operator of dimension $\Delta$ and spin $\ell$ to the four point function. The conformal block of a quasi-primary operator includes the operator itself, and all its descendants formed by the actions of derivatives. The explicit expression
was obtained in [7] and gives a concrete expression for the contribution in the function $G(u, v)$ in (1.7)

$$G(u, v) = G^{(\ell)}_\Delta(u, v; \Delta_{21}, \Delta_{43}) = u^{\frac{1}{2}(\Delta - \ell)} G^{(\ell)}_\Delta(u, v).$$  \hspace{1cm} (1.20)

The functions $G^{(\ell)}_\Delta(u, v)$ are given in terms of hypergeometric functions of type $2 F_1(a, b; c; x)$ by

$$G^{(\ell)}_\Delta(u, v) = G^{(\ell)}(\frac{1}{2}(\Delta - \Delta_{12} - \ell), \frac{1}{2}(\Delta + \Delta_{34} - \ell), \Delta; u, v),$$

$$G^{(\ell)}(a, b, c; u, v) = \frac{(-\frac{1}{\pi} x)^{\ell} F(a + \ell, b + \ell; c + \ell; x) F(a - 1, b - 1; c - 2 - \ell; \frac{\pi}{x}) - x \leftrightarrow \frac{\pi}{x}}{x - \frac{\pi}{x}}.$$  \hspace{1cm} (1.21)

Note that the leading terms in (1.21) agrees with (1.19) as required. The conformal partial wave functions satisfy the following relations

$$G^{(\ell)}_\Delta(u, v; \Delta_{21}, \Delta_{43}) = (-1)^{\ell} v^{\frac{1}{2}(\Delta_{43} - \Delta_{21})} G^{(\ell)}_\Delta(u/v, 1/v; -\Delta_{21}, \Delta_{43})$$

$$= v^{\frac{1}{2}(\Delta_{43} - \Delta_{21})} G^{(\ell)}_\Delta(u, v; -\Delta_{21}, -\Delta_{43}).$$  \hspace{1cm} (1.22)

Finally the OPE requires an expansion of $G(u, v)$ in terms of conformal partial waves

$$G(u, v) = \sum_{\Delta, \ell} a_{\Delta, \ell} G^{(\ell)}_\Delta(u, v),$$  \hspace{1cm} (1.23)

which determines the spectrum of possible operators in the OPE. The conformal partial waves coefficients are given by the OPE coefficients as

$$a_{\Delta, \ell} = C_{12O} C_{34O}.$$  \hspace{1cm} (1.24)

\textbf{1.6. $N = 4$ Superconformal Multiplets}

The unitary irreducible representations of $SU(2, 2|4)$ with positive energy are labelled by the quantum numbers of the maximal bosonic subgroup which is $SO(2, 4) \times SU(4) \cong SO(2) \times SO(4) \times SU(4)$. There is a mapping of representations of this group to representations of $SO(1, 1) \times SO(1, 3) \times SU(4)$. The quantum number of $SO(1, 1)$ is the scale dimension $\Delta$ which is non-negative for positive energy representations. The quantum
numbers of $SO(1, 3) \simeq SU(2) \times SU(2)$ are the ‘spins’ $(j_1, j_2)$ where $j_1, j_2$ are non-negative half-integers. The representations of $SU(4)$ are labelled by their Dynkin labels $[p_1, p_2, p_3]$ where $p_i$ are non-negative integers. Complex conjugation of a representation $[p_1, p_2, p_3]$ yields $[p_3, p_2, p_1]$.

In any superconformal multiplet there is one unique operator $\mathcal{O}$, called the superconformal primary operator, which commutes with the generators of the conformal supersymmetries. Notice that because of $\{S, S\} \sim K$ this is a stricter condition than commuting with special conformal generators, i.e. being a conformal primary operator. Looking at the dimensions of the generators, $[S, \mathcal{O}] = 0$ implies that $\mathcal{O}$ is the operator of lowest dimension in the multiplet. The rest of the multiplet is generated by the action of the Poincaré supersymmetries and the operators are therefore called descendants of $\mathcal{O}$. Their dimension is related to the one of $\mathcal{O}$ since acting with Poincaré supersymmetries raises the dimension by $\frac{1}{2}$.

There exists a complete classification of unitary representations for primary operators with vanishing ‘spin’ $j_1, j_2$ of $SU(2, 2|4)$[8]:

1.) $\Delta = 2p_1 + p_2$ where $p_1 = p_3$

2.) $\Delta = \frac{3}{2}p_1 + p_2 + \frac{1}{2}p_3$ where $p_1 \geq p_3 + 2$

3.) $\Delta = \frac{1}{2}p_1 + p_2 + \frac{3}{2}p_3$ where $p_3 \geq p_1 + 2$

4.) $\Delta \geq \text{Max}[2 + \frac{3}{2}p_1 + p_2 + \frac{1}{2}p_3; 2 + \frac{1}{2}p_1 + p_2 + \frac{3}{2}p_3]$.

The first three classes are discrete. They correspond to BPS operators, also called short or chiral since they are chiral under a $\mathcal{N} = 1$ subalgebra of $SU(2, 2|4)$. The shortness implies that the operators do not receive perturbative corrections to their dimension, i.e. their anomalous dimensions vanish. This originates from a shortening condition that states that some of the supercharges annihilate the chiral primary operator (CPO).

The fourth class has continuous scale dimension. Primary operators in this class do not satisfy any shortening condition and thus the multiplets are long multiplets with dimension proportional to $2^{16}$.

Primary operators in the fourth class, satisfying the threshold for which equality holds
in (1.25) are called semi-short. They transform in the \([q, p, q]\) representation and thus are self-conjugate. Their dimension is \(\Delta = 2q + p + 2\).

Notice that there is a gap of dimension 2 between the discrete classes and the continuous one. Therefore a long multiplet cannot be made short by going to the free field limit. We will actually study in chapter 3 how a long multiplet at the threshold decomposes into 2 semi-short multiplets. Conversely this also implies that semi-short multiplets are protected against anomalous dimensions unless there exists a second multiplet to pair up with to form a long multiplet.

We will mostly be interested in \(\frac{1}{2}\)-BPS operators belonging to the first class of representations. They transform in the \([0, p, 0], p \geq 2\) representation and thus have dimension \(p\). These operators are self-conjugate. The simplest case is a single-trace operator containing \(p\) scalars. The single trace operators we consider are of the form, \(r = 1, \ldots, 6\)

\[
\varphi_{r_1 \ldots r_p} = \text{tr}(X_{\{r_1 \cdots X_{r_p}\}}). \tag{1.26}
\]

Curly brackets always denote the completely symmetric and traceless part. These \(\frac{1}{2}\)-BPS trace operators are local gauge invariant operators of the theory with protected dimension. The fact that they are superconformal primaries can be read off the schematic action of the Poincaré supersymmetries on the fields

\[
\{Q, \lambda\} = F^+ + [X, X], \\
\{Q, \bar{\lambda}\} = DX, \\
\{X, F\} = \lambda, \\
\{Q, F\} = D\lambda. \tag{1.27}
\]

Trace operators containing only products of scalars in a totally symmetrized way do not contain any of the terms on the right hand side. Thus they cannot be generated by the action of \(Q\) on another operator. Therefore they are superconformal primaries. Since we are interested in operators in irreducible representations we will in addition impose that the products of scalars are traceless in their \(R\)-symmetry indices. The simplest operators are given by single-trace operators like \(\text{tr}(X_r X_r)\), the Konishi operator, which is a singlet, or
\( \text{tr}(X_{t}X_{s}) \) which transforms in the \([0, 2, 0]\) representation of the \(R\)-symmetry and contains the energy-momentum tensor. There are of course multi-trace operators given by products of single trace operators projected onto a \([0, p, 0]\) \(R\)-symmetry representation. Since the \(X_{r}\) are traceless the multi trace operators have dimension \(\Delta = p \geq 4\).

The first three levels of the short \([0, p, 0]\) \(\mathcal{N} = 4\) multiplets we will use are given by

\[
\begin{align*}
\psi_{r_{1}\ldots r_{p-1}\alpha} & \quad \psi_{r_{1}\ldots r_{p-1}\dot{\alpha}} \\
\varphi_{r_{1}\ldots r_{p}} & \quad J_{r_{1}\ldots r_{p-1}\alpha\dot{\alpha}} \\
\ldots & \quad \ldots
\end{align*}
\]

All fields are traceless and symmetric in the \(r\)-indices of the \(R\)-symmetry \(SU(4)\). The \(\alpha, \dot{\alpha}\) indices are spacetime spinor indices. In addition the fields at level 2 satisfy an irreducibility constraint

\[
\gamma \cdot \psi = \bar{\psi} \cdot \bar{\gamma} = 0.
\] (1.29)

where \(\gamma, \bar{\gamma}\) are the \(SU(4)\) gamma matrices, \(\gamma_{r}\bar{\gamma}_{s} + \gamma_{s}\bar{\gamma}_{r} = -2\delta_{rs}1\).

Tensorial complications in the analysis of superconformal Ward identities and also in applying the operator product expansion are avoided here by taking

\[
\varphi_{r_{1}\ldots r_{p}}(x) \to \varphi^{(p)}(x, t) = \varphi_{r_{1}\ldots r_{p}}(x) t_{r_{1}} \ldots t_{r_{p}},
\] (1.30)

where \(t\) is an arbitrary complex vector satisfying

\[
t^{2} = 0.
\] (1.31)

(For a more mathematical discussion of using such vectors for the treatment of representations of \(SO(n)\) see [11], see also appendix A in [12]). Clearly \(\varphi_{r_{1}\ldots r_{p}}\) can be recovered from

\footnote{Such null vectors may also be motivated by considering the harmonic superspace approach and were used similarly for instance in [9,10]. Our application is independent of the harmonic superspace formalism and is essentially motivated just by the requirement of simplifying the treatment of arbitrary rank symmetric traceless tensors, we do not anywhere consider the conjugate of \(t\).}
\( \varphi(p) \). The four point function then becomes a homogeneous polynomial in \( t_1, t_2, t_3, t_4 \), of respective degree \( p_1, p_2, p_3, p_4 \), invariant under simultaneous rotations on all \( t_i \)'s.

The four point correlation functions of interest then have the form

\[
\langle \varphi^{(p_1)}(x_1, t_1) \varphi^{(p_2)}(x_2, t_2) \varphi^{(p_3)}(x_3, t_3) \varphi^{(p_4)}(x_4, t_4) \rangle = r_{\Sigma - p_2 - p_3}^{\Sigma - p_3 - p_4} F(u, v; t) , \quad 2 \Sigma = p_1 + p_2 + p_3 + p_4 ,
\]

Due to the condition (1.31) for each \( t_i \) the conformally covariant four point function is reducible to an invariant function \( F(u, v; \sigma, \tau) \) with \( \sigma, \tau \) the two independent invariants, homogeneous of degree zero in each \( t_i \), which are analogous to the conformal invariants \( u, v, \)

\[
\sigma = \frac{t_1 \cdot t_3 t_2 \cdot t_4}{t_1 \cdot t_2 t_3 \cdot t_4} , \quad \tau = \frac{t_1 \cdot t_4 t_2 \cdot t_3}{t_1 \cdot t_2 t_3 \cdot t_4} .
\]

We make the following choice for the definition of \( F(u, v; \sigma, \tau) \)

\[
F(u, v, t) = (t_1 \cdot t_4)^{p_1 - E} (t_2 \cdot t_4)^{p_2 - E} (t_1 \cdot t_2)^{E} (t_3 \cdot t_4)^{p_3} F(u, v; \sigma, \tau) ,
\]

where we ordered \( p_1 \leq p_2 \leq p_3 \leq p_4 \) and

\[
2E = p_1 + p_2 + p_3 - p_4 .
\]

In general \( F(u, v; \sigma, \tau) \) is a polynomial in \( \sigma, \tau \), with degree determined by the \( p_i \), where the number of independent terms match exactly the number of tensorial invariants necessary for the general decomposition of the four point function for the corresponding symmetric traceless tensorial fields, for \( p_i = p \) there are \( \frac{1}{2} (p + 1)(p + 2) \) terms.

1.7. \( \mathcal{N} = 2 \) Superconformal Multiplets

Analogously to the \( \mathcal{N} = 4 \) case the unitary irreducible representations of \( SU(2, 2|2) \) with positive energy are labelled by the quantum numbers of \( SO(2, 4) \times U(2) = SO(2) \times SO(4) \times SU(2)_R \times U_R(1) \). There is a mapping of representations of this group to representations of \( SO(1, 1) \times SO(1, 3) \times SU(2)_R \times U_R(1) \). The quantum number of \( SO(1, 1) \) is the
scale dimension $\Delta$ which is non-negative for positive energy representations. The quantum numbers of $SO(1,3) \sim SU(2) \times SU(2)$ are the ‘spins’ $(j_1, j_2)$ where $j_1, j_2$ are non-negative half-integers. The representations of $SU(2)_R$ are labelled by the $R$-symmetry spin $R$, the representations of $U_R(1)$ by the $R$-symmetry charge $r$. We will encounter short or $\frac{1}{2}$-BPS, semi-short and long multiplets. Short multiplets are characterized by the condition $\Delta = 2R + r$. We will actually consider operators analogous to the $\frac{1}{2}$-BPS CPO’s in $\mathcal{N} = 4$ which will have $j_1 = j_2 = r = 0$. Therefore the shortening condition in our case will always be $\Delta = 2R$. Long representations are characterized by the bound $\Delta \geq 2R + r + 2$. If the dimension actually satisfies the bound, the multiplet is called semi-short with $\Delta = 2R + r + 2$.

Notice the gap in dimension between short and semi-short multiplets with the same quantum numbers. The first three levels of the short $\mathcal{N} = 2$ multiplets with $R$-spin $n$ are given by, $r_i = 1, 2, 3$

\[
\begin{align*}
\varphi_{r_1 \ldots r_n} & \
\psi_{r_1 \ldots r_{n-1} \alpha} & \quad \overline{\chi}_{r_1 \ldots r_{n-1} \dot{\alpha}} \\
\ldots & \quad J_{r_1 \ldots r_{n-1} \alpha \dot{\alpha}} & \quad \ldots
\end{align*}
\]

(1.36)

All fields are symmetric and traceless in the $r$-indices and the fermions at level 2 satisfy the irreducibility constraint

\[
\tau \cdot \psi = \overline{\psi} \cdot \tau = 0. \tag{1.37}
\]

where $\tau$ are the $SU(2)$ Pauli matrices with the usual algebra $\tau_r \tau_s + \tau_s \tau_r = 2\delta_{rs}$.

We will make the same definitions for the vectors $t_i$ as for the $\mathcal{N} = 4$ case. For $\mathcal{N} = 2$ there will be a slight difference since in 3 dimensions four vectors are not linearly independent. This implies a relation between the four $t_i$ of a four point function. It turns out that this relation translates into the following relation for $\sigma, \tau$ in the $\mathcal{N} = 2$ case

$$\sigma^2 + \tau^2 + 1 = 2(\sigma + \tau + \sigma \tau).$$

In appendix E we present an alternative treatment of the $\mathcal{N} = 2$ case. Basically we consider fields with symmetric fundamental $SU(2)$ indices instead of symmetric traceless adjoint $SU(2)$ indices. There is a one to one mapping between the two pictures for an even number of fundamental $SU(2)$ indices. The fact that there is no trace condition on
the fundamental $SU(2)$ indices will simplify the treatment considerably since this renders derivatives with respect to $u$ ordinary derivatives and all the complications discussed in appendix A will be unnecessary to consider.

1.8. Large $N$ Amplitudes

In the large $N$ limit four point functions of four identical operators have been computed using supergravity techniques and the AdS/CFT correspondence [13,9,14]. They can be expressed in terms of $\mathcal{D}$-functions $\mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v)$ which are defined by conformal 4-point integrals and only depend on the two conformal cross-ratios of two points $u,v$. We will take the results available for the case of $[0,4,0]$ operators [14] and perform a conformal partial wave expansion to ultimately obtain averaged first order anomalous dimensions. A conformal partial wave expansion has an analogue from studying scattering amplitudes. Four point functions can be interpreted as two incoming fields scattering and thus producing two outgoing fields. The amplitude can now be expanded in a basis of functions corresponding to intermediate fields. In our context these are called conformal partial waves characterized by the dimension $\Delta$ and the spin $\ell$. The crucial ingredients will be the explicit expression for the contribution of a conformal block (1.7) and (1.20), and a power series expansion of the part of the $\mathcal{D}$ functions containing a factor of $\log u$ [9]. The aim of the partial wave expansion is to find the coefficients of the expansion of the amplitude consisting of $\mathcal{D}$ functions in terms of the $G^{(\ell)}_\Delta$ describing a conformal block.

The $\mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v)$ may be expressed in three parts. First we define

$$\Sigma = \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4), \quad s = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4),$$

(1.38)

and, restricting $\Sigma, s$ to be integer, for $s = 1, 2, \ldots$ we have [9,15]

$$\mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = \sum_{m=0}^{s-1} u^{-s+m} (-1)^m m! (s-m-1)! f_{\Delta_1 \Delta_2 \Delta_3 \Delta_4+m}(u,v)$$

$$+ \log u a_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) + b_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v),$$

(1.39)

where $a_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v), b_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v)$ are both given by power series in $u, 1 - v$. For $s = 0$ the first term in (1.39) is omitted. If $s < 0$ we can use (G.2b) to invert the sign of
In (1.39) the relevant terms here are given by [9,15]

\[ f_{\Delta_1\Delta_2\Delta_3\Delta_4}(v) = \frac{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)\Gamma(\Delta_4)}{\Gamma(\Delta_3 + \Delta_4)} \, 2F_1(\Delta_2, \Delta_3; \Delta_3 + \Delta_4; 1 - v), \]

\[ a_{\Delta_1\Delta_2\Delta_3\Delta_4}(u, v) \]

\[ = -\log u \left( -1 \right)^s \frac{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Sigma - \Delta_3)\Gamma(\Sigma - \Delta_4)}{m! n! \Gamma(\Delta_1 + \Delta_2)} \times \sum_{m,n=0}^{\infty} \frac{(\Delta_1)_m (\Sigma - \Delta_3)_m (\Delta_2)_{m+n} (\Sigma - \Delta_4)_{m+n}}{m! n! (s+1)_m (\Delta_1 + \Delta_2)_{2m+n}} u^m (1 - v)^n. \]  

(1.40)

The part singular in \( u \) given by \( f_{\delta_1,\delta_2,\delta_3,\delta_4}(v) \) will be used to restrict the possible form of general \( p \) large \( N \) amplitudes in the final part of chapter 5. Note that in (1.39) we only use \( f_{\delta_1,\delta_2,\delta_3,\delta_4} \) with \( \delta_1 + \delta_2 = \delta_3 + \delta_4 \) so we could eliminate \( \delta_1 \).

1.9. Known Results about Superconformal Ward Identities

It was found that the two and three point functions of CPO’s in \( \mathcal{N} = 2, 4 \) SYM are not renormalized. This can be shown using Intriligator’s reduction formula [16]

\[ \frac{\partial}{\partial g} \langle \varphi_1(x_1) \cdots \varphi_n(x_n) \rangle = \int d^4 y \langle \mathcal{L}(y) \varphi_1(x_1) \cdots \varphi_n(x_n) \rangle. \]

The \( \mathcal{N} = 4 \) SYM on-shell Lagrangian \( \mathcal{L} \) is part of the stress tensor multiplet with a numerical coefficient of \( g^2 \). Therefore in the path integral formalism the derivative with respect to the coupling constant brings the Lagrangian down which is equivalent to inserting the stress tensor multiplet into the correlator and picking the Lagrangian term with a suitable superspace integration. It was shown that only nilpotent invariants can contribute to this and since there are no nilpotent invariants for 4 or less points [17] it follows that 2 and 3 point functions are not renormalized.

Also, for \( \mathcal{N} = 4 \) surprising nonrenormalization theorems for extremal and next-to-extremal correlation functions were found [18]. The fields in the correlator are chiral primary \( \frac{1}{2} \)-BPS operators transforming in the \( R \)-symmetry representation with Dynkin labels \([0, p_i, 0]\) and having dimension \( p_i \). The condition of extremality for an \( N \)-point correlation function is \( \sum_{i=1}^{N-1} p_i = p_N \) or \( E = 0 \) in (1.35). Later this was generalized
to a similar case, the next-to-extremal case, where \( \sum_{i=1}^{N-1} p_i = p_N + 2 \). Initially it was observed that extremal three point functions in \( \mathcal{N} = 4 \) SYM coincide in the zero and the strong coupling limit [19]. The nonrenormalization of extremal \( n \)-point functions was conjectured from supergravity considerations [18]. It was shown that these correlators receive no \( g^2 \)-corrections [18] and no instanton corrections in the field theory [18]. The non-renormalization was also shown using \( \mathcal{N} = 2 \) harmonic superspace techniques [18].

It is unknown if more nonrenormalization theorems exist for SYM theories. A well known fact is that conformal symmetry fixes two and three point functions in their functional form. Four point functions may contain an arbitrary function of the two conformal invariants of four points. It is not fully known how much further superconformal symmetry restricts correlation functions. The conformal invariants extend to superconformal invariants and additionally one can construct nilpotent superconformal invariants for \( n \) points with \( n \geq 5 \) [17]. This does not lead to any further constraints. But the fields are actually constrained fields in irreducible representations of the \( R \)-symmetry. This together with the symmetry can impose further restrictions on the correlation functions. What is needed is a computation of the superconformal Ward identities. Once the superconformal Ward identities are solved, results obtained will automatically be compatible with superconformal symmetry.

We will consider here four point functions of \( \frac{1}{2} \)-BPS chiral primary operators (CPO’s). They are short and therefore cannot receive anomalous dimensions. For an analysis of their three point functions refer to [19], the vanishing of perturbative corrections was shown in [20,21] and also in the harmonic superspace approach in [22]. The first four point function studied were the ones for \( \frac{1}{2} \)-BPS CPO’s with dimension 2 which contains the energy momentum tensor. Perturbative results can be found in [23] and in the supergravity regime in [13]. In [5] the same approach as in this thesis was taken. For dimensions \( \Delta = 3, 4 \) the results can be found in [10,9,14]. Using these results the analysis of the operator product expansion (OPE) was performed in [24,25,26,27,28,29,30]. Similar results to ours were previously obtained by Heslop and Howe [6] based on expansions in terms of Schur
polynomials for $SU(2,2|2)$ and $PSU(2,2|4)^4$. Using the formalism of harmonic superspace [31] also provides a method for deriving superconformal identities which are equivalent to those obtained here.

The analysis we perform in general in this thesis was first undertaken in [5] for the simplest case of $[0,2,0]$-CPO’s and then carried on for the case of $p = 3$ in [9]. Unfortunately it turned out to be not easy to generalize to CPO’s of higher dimension. The tensors involved for CPO’s with higher dimension become increasingly complicated. For each correlation function in the Ward identities appropriate invariant tensors have to be constructed and it is not obvious how to perform an analysis for CPO’s of arbitrary dimension. We will solve this issue by the introduction of null vectors as explained earlier.

1.10. Known large $N$ results

In this thesis we will endeavour to work out the consequences of superconformal symmetry for the four point correlation functions of BPS operators. As a result our considerations are lacking in dynamical input since we do not consider any details of $\mathcal{N} = 2$ or $\mathcal{N} = 4$ superconformal theories. The results of our analysis will demonstrate that the details of the dynamics resides in the function $\mathcal{K}$ as in (2.60) or (3.51). In particular cases results have been obtained using perturbation theory [23] or with the AdS/CFT correspondence [13,9,14]. We here summarise some of the results obtained in [13,9,14] in the context of this paper.

The large $N$ results obtained through the AdS/CFT correspondence are expressible in terms of functions $\overline{D}_{n_1 n_2 n_3 n_4}(u,v)$ which satisfy various identities listed in [5],[9] and appendix G. When $p = 2$

$$\mathcal{H}^{(2)}(u,v;\sigma,\tau) = -\frac{4}{N^2} u^2 \overline{\mathcal{D}}_{2422}(u,v), \quad (1.41)$$

and for $p = 3$,

$$\mathcal{H}^{(3)}(u,v;\sigma,\tau) = -\frac{9}{N^2} u^3 \left( (1 + \sigma + \tau) \overline{\mathcal{D}}_{3533} + \overline{\mathcal{D}}_{3522} + \sigma \overline{\mathcal{D}}_{2523} + \tau \overline{\mathcal{D}}_{2532} \right). \quad (1.42)$$

\footnote{In eq. (49) of [6] $S_{020}(Z) = (X_1 - Y_1)(X_1 - Y_2)(X_2 - Y_1)(X_2 - Y_2)$ which appears as an overall factor in the Schur polynomial for long representations.}
The expression for \( p = 4 \) can be found in chapter 5 where we present a new simplified form obtained starting from the result in [14].

1.11. Outline

In detail the structure of this thesis is then as follows. In chapter 2 we derive the superconformal Ward identities for \( \mathcal{N} = 2 \) superconformal symmetry, discuss the solution and apply it by analysing the contributions of different supermultiplets in the OPE. The discussion is extended to the \( \mathcal{N} = 4 \) case in chapter 3. For the operator product expansion it is shown how there are potential contributions from non-unitary semi-short supermultiplets. We explicitly show how they may be cancelled in such a way that only unitary multiplets remain given an appropriate operator content. In chapter 4 we take into account the restrictions imposed by crossing symmetry making use of \( S_3 \) representations. In chapter 5 we present a simplified expression for the \( p = 4 \) amplitude and perform a conformal partial wave expansion to compute the averaged first order anomalous dimensions for the different representations. Following up on an observation made in [15] we present work in progress showing how the amplitudes for higher \( p \) might be constrained by crossing symmetry and the structure of singularities in \( u \). By generalizing a universal structure in the singularities as observed for \( p = 2, 3, 4 \) [13,9,14,15] and using the crossing symmetry present we will be able to first reproduce the results computed before and finally also work out the amplitude for general \( p \). Some final comments on the work done and future investigations are made in the conclusion.

Various technical issues are addressed in seven appendices. In appendix A we discuss how derivatives involving the null vector \( t_r \) are compatible with \( t^2 = 0 \). In appendix B we consider two variable harmonic polynomials, depending on \( \sigma, \tau \) given by the two cross ratios of four points, which are used in the expansion of general four point correlation functions. Appendix C describes some differential operators which play an essential role in our analysis whereas in appendix D we consider non unitary semi-short representations for \( PSU(2,2|4) \) which are important in our operator product analysis. Appendix E contains a simpler derivation of the Ward identities for the \( \mathcal{N} = 2 \) case than presented in chapter
2. We pursue the more complicated route in chapter 2 since only this easily generalizes to the $\mathcal{N} = 4$ case in chapter 3. In appendix F we present a Mathematica program used for computation of conformal partial wave expansions in the large $N$ limit of the $p = 4$ case. Finally, in appendix G we list a number of standard identities for $\overline{D}$ functions used in chapter 5.
2. \( \mathcal{N} = 2 \)

2.1. Superconformal Ward Identities

The algebraic complications involved in the analysis of Ward identities are much simpler for \( \mathcal{N} = 2 \) superconformal symmetry. In this case the \( R \)-symmetry group is just \( U(2) \) and discussion of the representations is much easier. In order to facilitate the comparison with the \( \mathcal{N} = 4 \) case later we consider BPS chiral primary operators which belong to representations of \( SU(2)_R \) symmetry for \( R = n \), an integer. The BPS condition requires that the scale dimension \( \Delta = 2n \). Such fields form superconformal primary states for a short supermultiplet with necessarily unrenormalised scale dimensions. The fields in this case are represented by symmetric traceless tensors \( \varphi_{r_1...r_n} \) with \( r_i = 1, 2, 3 \). To derive the Ward identities we need to consider just the superconformal transformations at the lowest levels of the multiplet. First

\[
\delta \varphi_{r_1...r_n} = \hat{\epsilon} \tau_{r_n} \psi_{r_1...r_{n-1}} + \bar{\psi}_{r_1...r_{n-1}} \tau_{r_n} \hat{\epsilon}, \tag{2.1}
\]

where \( \psi_{r_1...r_{n-1}} \), \( \bar{\psi}_{r_1...r_{n-1}} \) are spinor fields, traceless and symmetric on the indices \( r_1...r_{n-1} \), satisfying irreducibility constraints, with \( i = 1, 2 \) and \( \tau_r \) the usual Pauli matrices,

\[
\tau_r \psi_{r_1...r_{n-2}} = 0, \quad \bar{\psi}_{r_1...r_{n-2}} \tau_r = 0. \tag{2.2}
\]

Thus both \( \psi \) and \( \bar{\psi} \) belong to \( SU(2)_R \) representations with \( R = n - \frac{1}{2} \). In (2.1) we have

\[
\hat{\epsilon}^i(x) = \epsilon_i^\alpha - i \bar{\eta}_{i\dot{\alpha}} \bar{x}^{\dot{\alpha} \alpha}, \quad \hat{\epsilon}^{i\dot{\alpha}}(x) = \bar{\epsilon}^{i\dot{\alpha}} + i \bar{x}^{\dot{\alpha} \alpha} \eta_i^\alpha. \tag{2.3}
\]

where \( \epsilon_i^\alpha, \bar{\eta}_{i\dot{\alpha}}, \bar{\epsilon}^{i\dot{\alpha}}, \eta_i^\alpha \) are the \( R = \frac{1}{2} \) anticommuting parameters for an \( \mathcal{N} = 2 \) superconformal transformation. In addition to (2.1) we use

\[
\delta \psi_{r_1...r_{n-1}} = i \partial_\alpha \bar{\varphi}_{r_1...r_{n-1}} \tau_s \hat{\epsilon}^{\dot{\alpha}} - 4n \varphi_{r_1...r_{n-1}} \tau_s \bar{\eta}_\alpha + \frac{n - 1}{2n - 1} \tau_{(r_1} \bar{J}_{r_2...r_{n-1})s} \bar{\eta}_\alpha \tau_s \hat{\epsilon}^{\dot{\alpha}}. \tag{2.4}
\]

where \( \bar{J}_{r_1...r_{n-1}} \), a symmetric traceless rank \( n - 1 \) tensor, is a \( R = n - 1 \) current. Using (2.4) together with its conjugate we may verify closure of the superconformal algebra acting on \( \varphi_{r_1...r_n} \),

\[
[\delta_2, \delta_1] \varphi_{r_1...r_n} = -v \cdot \partial \varphi_{r_1...r_n} - n(\sigma + \bar{\sigma}) \varphi_{r_1...r_n} + n t_{(r_n|s} \varphi_{r_1...r_{n-1})s}, \tag{2.5}
\]
where $v^a$, which is quadratic in $x$, and $\sigma, \bar{\sigma}, t_{rs} = -t_{sr}$, which are linear in $x$, are constructed from $\hat{\epsilon}_1, \hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_2$.

Similarly to the definition of $\varphi$ in (1.30) we define

$$\psi_{\alpha}^{(n-1)}(x, t) = \psi_{r_1...r_{n-1}\alpha}(x) t_{r_1} \cdots t_{r_{n-1}}$$

and

$$\bar{\psi}_{\dot{\alpha}}^{(n-1)}(x, t) = \bar{\psi}_{r_1...r_{n-1}\dot{\alpha}}(x) t_{r_1} \cdots t_{r_{n-1}}$$

while

$$J_{\alpha\dot{\alpha}}^{(n-1)}(x, t) = J_{r_1...r_{n-1}\alpha\dot{\alpha}}(x) t_{r_1} \cdots t_{r_{n-1}}.$$  

With this notation (2.1) may be rewritten as

$$\delta \varphi^{(n)}(t) = \hat{\epsilon} \tau \cdot t \psi^{(n-1)}(t) + \bar{\psi}^{(n-1)}(t) \tau \cdot t \hat{\epsilon}, \quad (2.6)$$

and (2.4) becomes

$$\delta \psi_{\alpha}^{(n-1)}(t) = \frac{1}{n} \tau \cdot t \frac{\partial}{\partial t} \bar{i} \partial_{\alpha\dot{\alpha}} \varphi^{(n)}(t) \hat{\epsilon} \hat{\epsilon} + 4 \tau \cdot t \frac{\partial}{\partial t} \varphi^{(n)}(t) \eta_{\alpha}$$

$$+ \left( 1 - \frac{1}{2n-1} \tau \cdot t \tau \cdot t \frac{\partial}{\partial t} \right) J_{\alpha\dot{\alpha}}^{(n-1)}(t) \hat{\epsilon} \hat{\epsilon}. \quad (2.7)$$

A precise form for differentiation with respect to $t_r$ satisfying (1.31) is given in appendix A. The conditions (2.2) are now

$$\tau \frac{\partial}{\partial t} \psi_{\alpha}^{(n-1)}(t) = 0, \quad \bar{\psi}^{(n-1)}(t) \tau \cdot \frac{\bar{i}}{\partial t} = 0. \quad (2.8)$$

$F(u, v; t)$ is also a $SU(2)_R$ scalar which was specified in (1.32), clearly there is a freedom to modify it by suitable powers of $u$ or $v$ at the expense of changing the terms involving $r_{ij}$ in (1.32). The choice made on (1.32) has some convenience in the later discussion.

The fundamental superconformal Ward identities arise from expanding

$$\delta \langle \psi_{\alpha}^{(n_1-1)}(x_1, t_1) \varphi^{(n_2)}(x_2, t_2) \varphi^{(n_3)}(x_3, t_3) \varphi^{(n_4)}(x_4, t_4) \rangle = 0, \quad (2.9)$$
using (2.6) and (2.7). This gives, suppressing the arguments $t_i$ for the time being,

$$
\frac{1}{n_1} i \partial_{1\alpha} \tau_i \frac{\partial}{\partial t_1} \left\langle \varphi^{(n_1)}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \right\rangle \hat{e}^\alpha(x_1)
$$

$$
+ 4 \tau_i \frac{\partial}{\partial t_1} \left\langle \varphi^{(n_1)}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \right\rangle \eta_\alpha
$$

$$
+ \left( 1 - \frac{1}{2n_1 - 1} \tau_i t_1 \tau_j \frac{\partial}{\partial t_1} \right) \left\langle J^{(n_1-1)}_{\alpha\alpha}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \right\rangle \hat{e}^\alpha(x_1)
$$

$$
+ \left\langle \psi^{(n_1-1)}_{\alpha}(x_1) \overline{\psi}^{(n_2-1)}_{\alpha}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \right\rangle \tau_i \tau_j \hat{e}^\alpha(x_2)
$$

$$
+ \left\langle \psi^{(n_1-1)}_{\alpha}(x_1) \varphi^{(n_2)}(x_2) \overline{\psi}^{(n_3-1)}_{\alpha}(x_3) \varphi^{(n_4)}(x_4) \right\rangle \tau_i \tau_j \hat{e}^\alpha(x_3)
$$

$$
+ \left\langle \psi^{(n_1-1)}_{\alpha}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \overline{\psi}^{(n_4-1)}_{\alpha}(x_4) \right\rangle \tau_i \tau_j \hat{e}^\alpha(x_4) = 0.
$$

To apply this we make use of general expressions compatible with conformal invariance for each four point function which appears. Thus

$$
\left\langle \psi^{(n_1-1)}_{\alpha}(x_1) \overline{\psi}^{(n_2-1)}_{\alpha}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \right\rangle
$$

$$
= 2i \frac{r_{34}^\Sigma - 2n_2 - 2n_3}{r_{13}^\Sigma - 2n_2 - 2n_3} \frac{r_{24}^\Sigma - 2n_3 - 2n_4}{r_{14}^\Sigma - 2n_3 - 2n_4} \left( \frac{1}{r_{12}^\Sigma \Sigma_{12}} R_2 + \frac{1}{r_{13}^\Sigma \Sigma_{13}} (x_{13} \bar{x}_{34} x_{42})_{\alpha\alpha} S_2 \right),
$$

$$
\left\langle \psi^{(n_1-1)}_{\alpha}(x_1) \varphi^{(n_2)}(x_2) \overline{\psi}^{(n_3-1)}_{\alpha}(x_3) \varphi^{(n_4)}(x_4) \right\rangle
$$

$$
= 2i \frac{r_{34}^\Sigma - 2n_2 - 2n_3}{r_{13}^\Sigma - 2n_2 - 2n_3} \frac{r_{24}^\Sigma - 2n_3 - 2n_4}{r_{14}^\Sigma - 2n_3 - 2n_4} \left( \frac{1}{r_{12}^\Sigma \Sigma_{12}} R_3 + \frac{1}{r_{13}^\Sigma \Sigma_{13}} (x_{13} \bar{x}_{42} x_{23})_{\alpha\alpha} S_3 \right),
$$

$$
\left\langle \psi^{(n_1-1)}_{\alpha}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \overline{\psi}^{(n_4-1)}_{\alpha}(x_4) \right\rangle
$$

$$
= 2i \frac{r_{34}^\Sigma - 2n_2 - 2n_3}{r_{13}^\Sigma - 2n_2 - 2n_3} \frac{r_{24}^\Sigma - 2n_3 - 2n_4}{r_{14}^\Sigma - 2n_3 - 2n_4} \left( \frac{1}{r_{12}^\Sigma \Sigma_{12}} R_4 + \frac{1}{r_{13}^\Sigma \Sigma_{13}} (x_{13} \bar{x}_{32} x_{24})_{\alpha\alpha} S_4 \right),
$$

where $R_n, S_n$ are functions of $u, v$ and also scalars formed from $t_i$ (to verify completeness of the basis chosen in (2.11) we use relations such as $x_{13} \bar{x}_{34} x_{42} + x_{14} \bar{x}_{43} x_{32} = r_{34} x_{12}$). In addition we have

$$
\left\langle J^{(n_1-1)}_{\alpha\alpha}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \right\rangle
$$

$$
= 2i \frac{r_{34}^\Sigma - 2n_2 - 2n_3}{r_{13}^\Sigma - 2n_2 - 2n_3} \frac{r_{24}^\Sigma - 2n_3 - 2n_4}{r_{14}^\Sigma - 2n_3 - 2n_4} \left( X_{1[23]} \alpha\alpha I + X_{1[43]} \alpha\alpha J \right),
$$

for

$$
X_{i[jk]} = \frac{x_{ij} \bar{x}_{jk} x_{ki}}{r_{ij} r_{ik}} = \frac{1}{r_{ij}} x_{ij} - \frac{1}{r_{ik}} x_{ik},
$$

which transforms under conformal transformations as a vector at $x_i$ and is antisymmetric in $jk$.  

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Using (2.11) and (2.12) in (2.10), noting that
\[ i \partial_{1\alpha\dot{\alpha}} \frac{1}{r_{13}^{2n_1}} \hat{\xi}^\alpha(x_1) + 4n_1 \frac{1}{r_{13}^{2n_1}} \eta_\alpha = -4n_1 i \frac{1}{r_{13}^{2n_1+1}} \hat{\xi}^\alpha(x_3), \] (2.14)
and \( \partial_{1\alpha\dot{\alpha}} u = 2u X_1[23]_{\alpha\dot{\alpha}}, \) \( \partial_{1\alpha\dot{\alpha}} v = 2v X_1[43]_{\alpha\dot{\alpha}} \), we may decompose (2.10) into independent contributions involving \( \hat{\xi}(x_1) \) and \( \hat{\xi}(x_3) \) (note that \( x_{12} \hat{\xi}(x_2)/r_{12} = x_{13} \hat{\xi}(x_2)/r_{13} + X_1[23] \hat{\xi}(x_1) \)) and also for \( x_2 \rightarrow x_4 \) giving two linear relations,
\[ \frac{1}{n_1} \tau \frac{\partial}{\partial t_1} (X_1[23] u \partial_u + X_1[43] v \partial_v) F + \left( 1 - \frac{1}{2n_1 - 1} \tau \cdot t_1 \tau \cdot \frac{\partial}{\partial t_1} \right) (X_1[13] I + X_1[43] J) + X_1[23] R_2 \tau \cdot t_2 + X_1[43] R_4 \tau \cdot t_4 + \left( u X_1[23] - v X_1[43] \right) (S_2 \tau \cdot t_2 - S_4 \tau \cdot t_4) = 0, \] (2.15a)
\[ \left( -2 \tau \cdot \frac{\partial}{\partial t_1} F + (R_2 + (u - v) S_2) \tau \cdot t_2 + R_3 \tau \cdot t_3 + (R_4 + (1 - u + v) S_4) \tau \cdot t_4 \right) \frac{1}{r_{13}} x_{13} + \left( v S_2 \tau \cdot t_2 + S_3 \tau \cdot t_3 - v S_4 \tau \cdot t_4 \right) \frac{1}{r_{14} r_{23}} x_{14} \bar{x}_{42} x_{23} = 0. \] (2.15b)
It is easy to decompose \((2.15a, b)\) into independent equations but crucial simplifications are obtained essentially by diagonalising each \( 2 \times 2 \) spinorial equation in terms of new variables \( x, \bar{x} \) which, as mentioned in the introduction, are the eigenvalues of \( x_{12} x_{42}^{-1} x_{43} x_{13}^{-1} \).
These are related to the conformal invariants \( u, v \) defined in (1.8) by (1.10). In (2.15b) the spinorial matrix \( \bar{x}_{41} x_{42} \bar{x}_{32} x_{31} \) may be replaced by \( 1/(1 - x) \) and in (2.15a) we may effectively replace \( X_1[23] \rightarrow 1/x \) and \( X_1[43] \rightarrow -1/(1 - x) \) and in each case also for \( x \rightarrow \bar{x} \).
Using
\[ \frac{\partial}{\partial x} = \bar{x} \frac{\partial}{\partial u} - (1 - \bar{x}) \frac{\partial}{\partial v}, \] (2.16)
and the definitions
\[ T_2 = R_2 + x S_2, \quad T_3 = R_3 + \frac{1}{1 - x} S_3, \quad T_4 = R_4 + (1 - x) S_4, \quad K = \frac{1}{x} I - \frac{1}{1 - x} J, \] (2.17)
we may then obtain from \((2.15a, b)\)
\[ \frac{1}{n_1} \tau \cdot \frac{\partial}{\partial t_1} \frac{\partial}{\partial x} F = - \left( 1 - \frac{1}{2n_1 - 1} \tau \cdot t_1 \tau \cdot \frac{\partial}{\partial t_1} \right) K - \frac{1}{x} T_2 \tau \cdot t_2 + \frac{1}{1 - x} T_4 \tau \cdot t_4, \] (2.18a)
\[ 2 \tau \cdot \frac{\partial}{\partial t_1} F = T_2 \tau \cdot t_2 + T_3 \tau \cdot t_3 + T_4 \tau \cdot t_4. \] (2.18b)
Together with the corresponding equations obtained by \( x \rightarrow \bar{x} \) in \((2.18a, b)\) with also \( T_i \rightarrow \bar{T}_i, K \rightarrow \bar{K} \), which are defined just as in (2.17) for \( x \rightarrow \bar{x} \).
The equations in (2.18a,b) are equations for $\partial F/\partial t_1$. The integrability conditions, which are required by virtue of $(\tau \cdot \partial t_1)^2 = 0$, are satisfied since we have, for $i = 2, 3, 4$,

$$
\tau \cdot \frac{\partial}{\partial t_1} T_i = 0, \quad T_i \tau \cdot \frac{\partial}{\partial t_i} = 0, \quad (2.19)
$$
as a consequence of (2.8). To reduce (2.18a,b) into equations which ultimately allow $T_i$ and $K$ to be eliminated we first write, since $T_i$ is a $2 \times 2$ matrix,

$$
T_i \tau \cdot t_i = \tau \cdot V_i + W_i, \quad (2.20)
$$
where $W_i$ and $V_{i,r}$ are respectively a scalar and a vector. From the results of appendix A we further decompose $V_i$ uniquely in the form

$$
V_i = \frac{1}{n_1} \frac{\partial}{\partial t_1} U_i + \hat{V}_i, \quad t_1 \cdot V_i = U_i, \quad t_1 \cdot \hat{V}_i = 0. \quad (2.21)
$$
The first equation in (2.19) then separates into $SU(2)$ scalar and vector equations,

$$
\frac{\partial}{\partial t_1} V_i = 0, \quad i \frac{\partial}{\partial t_1} \times V_i + \frac{\partial}{\partial t_1} W_i = 0, \quad (2.22)
$$
where we may let $V_i \rightarrow \hat{V}_i$ without change since both $\partial_1^2 U_i$ and $\partial_1 \times \partial_1 U_i$ are zero. From (2.22) we may then find

$$
L_1 W_i = i n_1 \hat{V}_i, \quad (2.23)
$$
where we define the $SU(2)_R$ generators by

$$
L_i = t_i \times \frac{\partial}{\partial t_i}. \quad (2.24)
$$
Substituting (2.20) into (2.18a) gives

$$
\frac{\partial}{\partial x} F = -\frac{1}{x} U_2 + \frac{1}{1-x} U_4, \quad (2.25)
$$
and

$$
\frac{n_1}{2n_1 - 1} K = -\frac{1}{x} W_2 + \frac{1}{1-x} W_4, \quad (2.26)
$$
which is just an equation giving $K$, and also

$$
-\frac{1}{2n_1 - 1} iL_1 K = -\frac{1}{x} \hat{V}_2 + \frac{1}{1-x} \hat{V}_4. \quad (2.27)
$$
It is easy to see that this follows from (2.26) as a consequence of (2.23). Similarly substituting (2.20) into (2.18b) gives three equations
\[ 2n_1 F = \sum_{i=2}^{4} U_i, \tag{2.28} \]
and
\[ \sum_{i=2}^{4} \dot{V}_i = 0, \tag{2.29} \]
as well as
\[ \sum_{i=2}^{4} W_i = 0. \tag{2.30} \]
Clearly (2.29) follows from (2.30).

An essential constraint may also be obtained from the second equation in (2.19) which gives
\[ (n_i + 1)T_i \tau \cdot t_i = -(T_i \tau \cdot t_i) i\tau \cdot L_i. \tag{2.31} \]
With the decomposition (2.20) this leads to
\[ (n_i + 1)W_i = -iL_i V_i, \tag{2.32a} \]
\[ (n_i + 1)V_i = -L_i \times V_i - iL_i W_i. \tag{2.32b} \]
Contracting (2.32b) with \( L_i \), and using \( L_i \times L_i = -L_i, \ L_i^2 W_i = n_i(n_i+1)W_i \), gives (2.32a). In addition we have from \( T_i(\tau \cdot t_i)^2 = 0 \)
\[ t_i \cdot V_i = 0, \quad it_i \times V_i = t_i W_i. \tag{2.33} \]
With the aid of the results in appendix A we may obtain \( (2n_i + 1) \partial_i \cdot (t_i W_i - i t_i \times V_i) = (2n_i + 3)( (n_i + 1)W_i + iL_i \cdot V_i) \) so that (2.33) implies (2.32a). Similarly, since \( \partial_i \times (t_i \times V_i) = (\partial_i \times t_i) \times V_i + \partial_i (t_i \cdot V_i) - \partial_i \cdot (t_i V_i), \) we have from appendix A \( (2n_i + 1) \partial_i \times (t_i \times V_i) = -(2n_i + 3)(L_i \times V_i + (n_i + 4)V_i) \) and \( (2n_i + 1) \partial_i \times (t_i W_i) = -(2n_i + 3) L_i W_i \). Hence it is clear that (2.33) also implies (2.32b)\(^5\).

\(^5\) The converse follows using \( t_i \cdot L_i = 0, \ t_i \times L_i = t_i t_i \cdot \partial_i \) and \( t_i \times (L_i \times V_i) = -t_i \times V_i. \)
Using (2.21) and (2.23) for \( \hat{V}_i \) in (2.32) we obtain

\[
(L_1 \cdot L_i + n_i(n_i + 1))W_i = \frac{i}{2}((L_1 + L_i)^2 + (n_1 + n_i)(n_1 + n_i + 1))W_i = -i \frac{\partial}{\partial t_1} \cdot L_i U_i. \tag{2.34}
\]

\( U_i(u, v; t) \), which is defined by (2.20), is a homogeneous polynomial in \( t_1, t_i \) of \( O(t_1^{n_1}, t_i^{n_i}) \) such that the \( SU(2)_R \) representation with \( R_{(1i)} = n_1 + n_i \) is absent. In consequence the operator \( (L_1 + L_i)^2 + (n_1 + n_i)(n_1 + n_i + 1) \), which commutes with \( \partial_1 \cdot L_i \), in (2.34) may be inverted to give \( W_i \) in terms of \( U_i \). Alternatively we may obtain from (2.33)

\[
i t_i \times \partial_1 U_i = -t_1 \times L_1 W_i + n_1 t_i W_i. \tag{2.35}
\]

To analyse these equations further we now consider the decomposition of \( F \) and also \( U_i \) in terms of \( SU(2)_R \) scalars. We first assume the \( n_i \) are ordered so that

\[
n_1 \leq n_2 \leq n_3 \leq n_4, \tag{2.36}
\]

and further assume

\[
n_4 = n_1 + n_2 + n_3 - 2E, \tag{2.37}
\]

for integer \( E = 0, 1, 2, \ldots \), where \( E \) is a measure of how close the correlation function is to the extremal case. With (2.36) and (2.37) \( F \), which is \( O(t_1^{n_1}, t_2^{n_2}, t_3^{n_3}, t_4^{n_4}) \), can in general be written in the form

\[
F(u, v; t) = (t_1 \cdot t_4)^{n_1-E}(t_2 \cdot t_4)^{n_2-E}(t_1 \cdot t_2)^E(t_3 \cdot t_4)^{n_3} F(u, v; \sigma, \tau), \tag{2.38}
\]

where \( F \) is a polynomial in \( \sigma, \tau \), defined in (1.33), with all terms \( \sigma^p \tau^q \) satisfying \( p + q \leq E \).

If \( E > n_1 \) then all terms in \( F \) must contain a factor \( \mu^{E-n_1} \) to cancel negative powers of \( t_1 \cdot t_4 \) in (2.38). Since \( t_i \) are three dimensional vectors \( t_1[t_2, t_3, t_4, u] = 0 \) so that \( \sigma, \tau \) are not independent but obey the relation

\[
\Lambda \equiv \sigma^2 + \tau^2 + 1 - 2\sigma\tau - 2\sigma - 2\tau = 0. \tag{2.39}
\]

This may be solved in terms of a single variable \( \alpha \) by

\[
\sigma = \alpha^2, \quad \tau = (1 - \alpha)^2, \tag{2.40}
\]

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so that
\[ \hat{F}(u,v;\sigma,\tau) = \hat{F}(x,\bar{x};\alpha). \tag{2.41} \]

\( \hat{F}(x,\bar{x};\alpha) \) is symmetric in \( x,\bar{x} \) and, for \( E \leq n_1 \), is a polynomial in \( \alpha \) of degree \( 2E \), so that there are \( 2E + 1 \) independent coefficients, while if \( E > n_1 \) then it must be of the form \((1 - \alpha)^{2(E-n_1)}p(\alpha)\) with \( p \) a polynomial of degree \( n_1 \), so that the number of coefficients is \( 2n_1 + 1 \). These results correspond exactly of course to the number of \( SU(2)_R \) invariants which can be formed in the four point function, subject to (2.37), together with (2.36), that can be found using standard \( SU(2) \) representation multiplication rules.

A similar expansion to (2.38) can be given for each \( U_i \)
\[ U_i(x,\bar{x};t) = (t_1 \cdot t_4)^{n_1 - E} (t_2 \cdot t_4)^{n_2 - E} (t_1 \cdot t_2)^E (t_3 \cdot t_4)^{n_3} U_i(x,\bar{x};\sigma,\tau), \tag{2.42} \]
\[ \hat{U}_i(x,\bar{x};\sigma,\tau) = \hat{U}_i(x,\bar{x};\alpha). \]

The analysis of (2.26) and (2.30) depends on using (2.34), or (2.35), as shown in appendix C, to relate \( W_i \) and \( U_i \). Defining
\[ W_i = i t_2 \cdot (t_3 \times t_4) (t_1 \cdot t_4)^{n_1 - E} (t_2 \cdot t_4)^{n_2 - E} (t_1 \cdot t_2)^{E-1} (t_3 \cdot t_4)^{n_3-1} W_i, \tag{2.43} \]
then we obtain
\[ 2(2n_1 - 1) W_i = \hat{D}_i \hat{U}_i, \tag{2.44} \]
where \( \hat{D}_i \) are linear operators given by
\[ \hat{D}_2 = \frac{d}{d\alpha} + \frac{2(E-n_1)}{1-\alpha}, \quad \hat{D}_3 = \frac{d}{d\alpha} - \frac{2n_1}{\alpha} + \frac{2(E-n_1)}{1-\alpha}, \quad \hat{D}_4 = \frac{d}{d\alpha} + \frac{2E}{1-\alpha}. \tag{2.45} \]

The superconformal identities (2.25), (2.28) and (2.30) then become
\[ \frac{\partial}{\partial x} \hat{F} = -\frac{1}{x} \hat{U}_2 + \frac{1}{1-x} \hat{U}_4, \tag{2.46a} \]
\[ 2n_1 \hat{F} = \hat{U}_2 + \hat{U}_3 + \hat{U}_4, \tag{2.46b} \]
\[ \hat{D}_2 \hat{U}_2 + \hat{D}_3 \hat{U}_3 + \hat{D}_4 \hat{U}_4 = 0. \tag{2.46c} \]

By acting on (2.46b) with \( \hat{D}_3 \) and using (2.46c) we may obtain
\[ \hat{D}_3 \hat{F} = -\frac{1}{\alpha} \hat{U}_2 - \frac{1}{\alpha(1-\alpha)} \hat{U}_4, \tag{2.47} \]
and substituting in (2.46a) gives
\[
\left( x \frac{\partial}{\partial x} - \alpha \hat{D}_3 \right) \hat{F} = \left( \frac{x}{1 - x} + \frac{1}{1 - \alpha} \right) \hat{U}_4. \tag{2.48}
\]
The right hand side of (2.48) vanishes when $\alpha = 1/x$ leaving an equation for $\hat{F}$ alone. With the explicit form for $\hat{D}_3$ in (2.45) we have
\[
\frac{\partial}{\partial x} \left( x^{2n_1} \left( 1 - \frac{1}{x} \right)^{2(n_1 - E)} \hat{F} \left( x, \bar{x}; \frac{1}{x} \right) \right) = 0. \tag{2.49}
\]
Together with its partner or conjugate equation involving $\partial/\partial \bar{x}$ (2.49) provides the final result for the constraints due to superconformal identities for the four point function when $\mathcal{N} = 2$.

For the $\mathcal{N} = 2$ case we may also require instead of (2.37)
\[
n_4 = n_1 + n_2 + n_3 - 2E - 1, \tag{2.50}
\]
since $F$ can then be written as
\[
F(u, v; t) = (t_1 \cdot t_4)^{n_1 - E} (t_2 \cdot t_4)^{n_2 - E - 1} (t_1 \cdot t_2)^E (t_3 \cdot t_4)^{n_3 - 1} t_2 \cdot t_3 \times t_4 \, \hat{F}(x, \bar{x}; \alpha). \tag{2.51}
\]
There is an essentially unique expression in (2.51), with a single function $\hat{F}$ as a consequence of identities for the various possible vector cross products for null vectors which take the form
\[
\begin{align*}
t_1 \cdot t_2 \times t_3 & \cdot t_4 = \frac{1}{2} (\sigma - \tau + 1) t_2 \cdot t_3 \times t_4 \cdot t_1 \cdot t_2, \\
t_1 \cdot t_2 \times t_4 & \cdot t_3 \cdot t_4 = \frac{1}{2} (\sigma - \tau - 1) t_2 \cdot t_3 \times t_4 \cdot t_1 \cdot t_2, \\
t_1 \cdot t_3 \times t_4 & \cdot t_2 \cdot t_4 \tau = \frac{1}{2} (\sigma + \tau - 1) t_2 \cdot t_3 \times t_4 \cdot t_1 \cdot t_4.
\end{align*} \tag{2.52}
\]
Since, as shown in appendix B, effectively $t_1 \cdot t_4 \cdot t_2 \cdot t_3 \times t_4 = O(1 - \alpha)$ we can take in (2.51), if $n_1 - E \geq 1$, $(1 - \alpha) \hat{F}(x, \bar{x}; \alpha)$ to be a polynomial of degree $2E + 1$. If $n_1 - E < 1$ then $\hat{F}(x, \bar{x}; \alpha)$ must contain a factor $(1 - \alpha)^{2(E - n_1) - 1}$. It is easy to see that the number of independent coefficients matches with the number of independent terms in the four point function obtained by counting possible representations in each case.

There is a similar expansion as (2.51) for $U_i$. Instead of (2.43) and (2.44) we now have
\[
W_i = \frac{i}{2n_1 - 1} \left( t_1 \cdot t_4 \right)^{n_1 - E - 1} \left( t_2 \cdot t_4 \right)^{n_2 - E} \left( t_1 \cdot t_2 \right)^E \left( t_3 \cdot t_4 \right)^{n_3} \tau \hat{D}_i \hat{U}_i, \tag{2.53}
\]
with $\hat{D}_i$ exactly as in (2.45). In consequence the superconformal identities reduce to (2.46a, b, c) and we may derive the final result (2.49), albeit with $E$ given by (2.50).
2.2. Solution of Identities

Although in the $\mathcal{N} = 2$ case the identities can be solved rather trivially we show here how they may be put in a form which makes the connection with the operator product expansion, and the possible supermultiplets which may contribute to it, rather obvious.

For the purposes of analysing the operator product expansion for $x_1 \sim x_2$ we use the expression of the four point function in terms of the function $G(u,v; t)$ in (1.32), so we write

$$\langle \varphi^{(n_1)}(x_1, t_1) \varphi^{(n_2)}(x_2, t_2) \varphi^{(n_3)}(x_3, t_3) \varphi^{(n_4)}(x_4, t_4) \rangle = \frac{1}{r_{12}^{n_1+n_2} r_{14}^{n_3+n_4}} \left( \begin{array}{c} r_{24} \cr r_{14} \end{array} \right)^{n_1-n_2} \left( \begin{array}{c} r_{14} \cr r_{13} \end{array} \right)^{n_3-n_4} G(u,v; t),$$

where

$$G(u,v; t) = u^{n_1+n_2} v^{n_1+n_4-n_2-n_3} F(u,v; t).$$  \hspace{1cm} (2.54)

For application of the superconformal Ward identities here it is convenient here to replace the variable $\alpha$ by $y$ where

$$y = 2\alpha - 1,$$

and $x, \bar{x}$ by $z, \bar{z}$ given by

$$z = \frac{2}{x} - 1, \quad \bar{z} = \frac{2}{\bar{x}} - 1.$$  \hspace{1cm} (2.56)

Assuming now

$$G(u,v; t) = (t_1\cdot t_4)^{n_1-E} (t_2\cdot t_4)^{n_2-E} (t_1\cdot t_2)^E (t_3\cdot t_4)^{n_3} G(u,v; y),$$

the solution of (2.49) and its conjugate equation, maintaining the symmetry under $z \leftrightarrow \bar{z}$, becomes

$$G(u,v; z) = u^{n_1+n_2-2E} f(z), \quad G(u,v; \bar{z}) = u^{n_1+n_2-2E} f(z),$$

where $f$ is an unknown single variable function. Since $G(u,v; y)$ is just a polynomial in $y$ (2.59) requires

$$G(u,v; y) = u^{n_1+n_2-2E} \frac{(y-\bar{z})f(z) - (y-z)f(z)}{z-\bar{z}} + (y-z)(y-\bar{z}) K(u,v; y),$$

where $K(u,v; y)$ is undetermined and contains the dynamical part of the correlation function. If $G(u,v; y)$ is a polynomial of degree $2E$ in $y$ then clearly $K$ is a polynomial of degree $2E - 2$. 

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2.3. OPE Analysis

The operator product expansion applied to this correlation function is realised by expanding it in terms of conformal partial waves \( G_\Delta^{(\ell)}(u, v; \Delta_{21}, \Delta_{43}) \), which represent the contribution to a four point function for four scalar fields, with scale dimensions \( \Delta_i \), from an operator of scale dimension \( \Delta \) and spin \( \ell \), and all its conformal descendants. Explicit expressions, in four dimensions, were found in [7] which are simple in terms of the variables \( x, \bar{x} \) defined in (1.10). For the definition refer to (1.20).

For this case the expansion is also over the contributions for differing \( SU(2)_R \) representations and has the form, if \( n_1 \geq E \),

\[
\mathcal{G}(u, v; y) = \sum_{R=n_4-n_3}^{n_1+n_2} \sum_{\Delta, \ell} a_{R, \Delta, \ell} P_{R+n_3-n_4}^{(2n_1-2E, 2n_2-2E)}(y) \mathcal{G}_\Delta^{(\ell)}(u, v; 2(n_2-n_1), 2(n_4-n_3)),
\]

(2.61)

with \( P_n^{(a, b)} \) a Jacobi polynomial. For \( a \) a negative integer \( P_n^{(a, b)}(y) \propto (1 - y)^{-a} \) and \( n+a \geq 0 \). Hence when \( n_1 < E \) we require a similar expansion to (2.61) but with \( R = n_2 - n_1, \ldots, n_1 + n_2 \) and then \( \mathcal{G}(u, v; y) \propto \tau^{E-n_1} \) as required in (2.58) to avoid negative powers of \( t_1 t_4 \). The different terms appearing in the sum in (2.61) then determine the necessary spectrum of operators required by this correlation function. The symmetry properties of this operator product expansion follow from (2.61) and \( P_n^{(a, b)}(y) = (-1)^n P_n^{(b, a)}(-y) \).

We first consider the case when \( n_1 = n_2 = n_3 = n_4 = n \), so that \( E = n \). To apply (2.59) we first consider the expansion in terms of Legendre polynomials (to which the Jacobi polynomial reduce in this case),

\[
\mathcal{G}(u, v; y) = \sum_{R=0}^{2n} a_R(u, v) P_R(y), \quad \mathcal{K}(u, v; y) = \sum_{R=0}^{2n-2} A_R(u, v) P_R(y).
\]

(2.62)

The \( P_R(y) \) in (2.62) correspond to the \( 2n+1 \) possible \( SU(2)_R \) invariants for the four point function (2.54) and, as a consequence of results in appendix B, the coefficients \( a_R \) represent the contribution to the correlation function from operators belonging just to the \( SU(2)_R \) \( R \)-representation in the operator product expansion for \( \varphi^{(n)}(x_1, t_1) \varphi^{(n)}(x_2, t_2) \).

From (2.60) it is easy to see that the single variable function \( f \) involves terms linear
in $y$ and so contributes only for $R = 0, 1$ giving
\[ a_0^f = \frac{zf(z) - \bar{z}f(\bar{z})}{z - \bar{z}}, \quad a_1^f = -\frac{f(z) - f(\bar{z})}{z - \bar{z}}. \] (2.63)

Using the expansion in (2.62) for $\mathcal{K}$ in (2.60) and standard recurrence relations for Legendre polynomials gives corresponding expressions for $a_R$. For the terms involving $A_R$ we have
\[ a_{R+2}^A = \frac{(R + 1)(R + 2)}{(2R + 1)(2R + 3)} A_R, \quad a_{R-2}^A = \frac{(R - 1)R}{(2R - 1)(2R + 1)} A_R, \]
\[ a_{R+1}^A = -\frac{2(R + 1)}{2R + 1} \frac{1 - v}{u} A_R, \quad a_{R-1}^A = -\frac{2R}{2R + 1} \frac{1 - v}{u} A_R, \] (2.64)
\[ a_R^A = \left(2 \frac{1 + v}{u} - \frac{1}{2} + \frac{1}{2(2R - 1)(2R + 3)}\right) A_R. \]

For $R \geq 2$ $a_R$ is therefore given in terms $A_{R+2}, A_{R+1}, A_R$ while for $R = 0, 1$, with (2.63), we have
\[ a_0 = a_0^f + \left(2 \frac{1 + v}{u} - \frac{2}{3}\right) A_0 - \frac{2}{3} \frac{1 - v}{u} A_1 + \frac{2}{15} A_2, \]
\[ a_1 = a_1^f - \frac{1 - v}{u} A_0 + \left(2 \frac{1 + v}{u} - \frac{2}{5}\right) A_1 - \frac{4}{5} \frac{1 - v}{u} A_2 + \frac{6}{35} A_3. \] (2.65)

In (2.64) and (2.65) any contributions involving $A_R$ for $R > 2n - 2$ should be dropped.

The significance of the results given by (2.64) and (2.65) is that they correspond exactly to the $\mathcal{N} = 2$ supermultiplet structure of operators appearing in the operator product expansion. Each $a_R(u, v)$ may then be expanded in terms of $G^{(\ell)}_\Delta(u, v) = G^{(\ell)}_\Delta(u, v; 0, 0)$
\[ a_R(u, v) = \sum_{\Delta, \ell} b_{R, \Delta, \ell} G^{(\ell)}_\Delta(u, v). \] (2.66)

The conformal partial waves $G^{(\ell)}_\Delta(u, v)$ satisfy crucial recurrence relations [5],
\[ -2 \frac{1 - v}{u} G^{(\ell)}_\Delta(u, v) = 4 G^{(\ell+1)}_{\Delta-1}(u, v) + G^{(\ell-1)}_{\Delta-1}(u, v) + a_s G^{(\ell+1)}_{\Delta+1}(u, v) + \frac{1}{4} a_{\ell-1} G^{(\ell+1)}_{\Delta-1}(u, v), \]
\[ \left(2 \frac{1 + v}{u} - 1\right) G^{(\ell)}_\Delta(u, v) = 4 G^{(\ell+2)}_{\Delta-2}(u, v) + 4 a_s G^{(\ell+2)}_{\Delta}(u, v) + \frac{1}{4} a_{\ell-1} G^{(\ell-2)}_{\Delta}(u, v) \]
\[ + \frac{1}{4} a_s a_{\ell-1} G^{(\ell)}_{\Delta+2}(u, v), \] (2.67)
where
\[ s = \frac{1}{2}(\Delta + \ell), \quad t = \frac{1}{2}(\Delta - \ell), \quad a_s = \frac{s^2}{(2s - 1)(2s + 1)}. \] (2.68)

In (2.67) $a_{\ell-1} > 0$ if $\Delta > \ell + 3$. 34
2.4. Long Operators

If $A_R$ is restricted to a single partial wave so that

$$A_R \to \mathcal{G}^{(\ell)}_{\Delta+2}.$$ (2.69)

then, using (2.67) with (2.64),

$$a_{A_R} \to a_{A_{R'}}(A_{R,\ell}) = \sum_{(\Delta',\ell')} b_{(\Delta',\ell')} \mathcal{G}^{(\ell')}_{\Delta'},$$

$$|R' - R| = 2, \ (\Delta'; \ell') = (\Delta + 2; \ell), \ |R' - R| = 1, \ (\Delta'; \ell') = (\Delta + 3, \Delta + 1; \ell \pm 1),$$

$$R' - R = 0, \ (\Delta'; \ell') = (\Delta + 4, \Delta; \ell), \ (\Delta'; \ell') = (\Delta + 2; \ell \pm 2, \ell).$$ (2.70)

This gives exactly the expected contributions corresponding to those operators present in a long $\mathcal{N} = 2$ supermultiplet, which we may denote $A_{R,\ell}$, whose lowest dimension operator has dimension $\Delta$, spin $\ell$ belonging to the $SU(2)_R$ $R$-representation. From (2.64) and the positivity constraints for (2.67) we may then easily see that in (2.66) $b_{(\Delta',\ell')} > 0$ for $\Delta > \ell + 1$. For a unitary representation, so that all states in $A_{R,\ell}$ have positive norm, (we consider here multiplets whose $U(1)_R$ charge is zero) the requirement is

$$\Delta \geq 2R + \ell + 2.$$ (2.71)

Since $\mathcal{G}^{(\ell)}_{\Delta}(u, v) = u^{\frac{1}{2}(\Delta - \ell)} F(u, v)$ with $F(u, v)$ expressible as a power series in $u, 1 - v$ we must have from (2.69) for $u \sim 0$,

$$A_R(u, v) \sim u^{R + 2 + \epsilon}, \quad \epsilon \geq 0.$$ (2.72)

2.5. Semishort Operators

The contribution of the single variable function $f$ (2.63) represents operators just with twist $\Delta - \ell = 2$. From the results in [7] we have

$$\mathcal{G}^{(\ell)}_{\ell+2}(u, v) = u \frac{g_{\ell+1}(x) - g_{\ell+1}(\bar{x})}{x - \bar{x}} = -2 \frac{g_{\ell+1}(x) - g_{\ell+1}(\bar{x})}{z - \bar{z}},$$ (2.73)

for

$$g_\ell(x) = (-\frac{1}{2} x)^{\ell-1} x F(\ell, \ell; 2\ell; x) = -\frac{2}{z^2} F\left(\frac{1}{2} \ell, \frac{1}{2} \ell + \frac{1}{2}; \ell + \frac{1}{2}; \frac{1}{z^2}\right),$$ (2.74)
where $F$ is just an ordinary hypergeometric function$^6$. As shown in [5] $g_\ell$ satisfies

$$z g_\ell(x) = -g_{\ell-1}(x) - a_\ell g_{\ell+1}(x).$$

(2.75)

In general we therefore expand the single variable function $f$ in (2.63) in the form

$$f(z) = \sum_{\ell=0}^{\infty} b_\ell g_\ell(x).$$

(2.76)

For this to be possible $f(z)$ must be analytic in $1/z$, or equivalently in $x$. If we consider just $f \to 2g_{\ell+2}$ and use (2.75) in (2.63) then $a_R^I \to a_R(C_{0,\ell})$ where

$$a_1(C_{0,\ell}) = G^{(\ell+1)}_{\ell+3}, \quad a_0(C_{0,\ell}) = G^{(\ell)}_{\ell+2} + a_{\ell+2} G^{(\ell+2)}_{\ell+4}.$$ (2.77)

These results for $a_0, a_1$ then correspond to the contributions of operators belonging to a semi-short $\mathcal{N} = 2$ supermultiplet $C_{0,\ell}$ whose lowest dimension operator is a $SU(2)_R$ singlet with spin $\ell$ and $\Delta = \ell + 2$, i.e. at the unitarity threshold (2.71).

In general we denote by $C_{R,\ell}$ the semi-short multiplet whose lowest dimension operator has spin $\ell$, belongs to the representation $R$, and has $\Delta = 2R + \ell$, so that the bound (2.71) is saturated. At the unitarity threshold given by (2.71) a long multiplet $\mathcal{A}_{R,\ell}^\Delta$ may be decomposed into two semi-short supermultiplets $C_{R,\ell}$ and $C_{R+1,\ell-1}$, [4]. This is reflected in the contributions to the four point function since, with $a_R'(\mathcal{A}_{R,\ell}^\Delta)$ defined by (2.69) and (2.70),

$$a_R'(\mathcal{A}_{R,\ell}^{2R+\ell+2}) = 4 a_R'(C_{R,\ell}) + \frac{R+1}{2R+1} a_R'(C_{R+1,\ell-1})$$

(2.78)

where we take

$$a_R(C_{R,\ell}) = G^{(\ell)}_{2R+\ell+2} + \frac{1}{4} a_R G^{(\ell)}_{2R+\ell+4} + a_{R+\ell+2} G^{(\ell+2)}_{2R+\ell+4},$$

$$a_{R-1}(C_{R,\ell}) = \frac{R}{2R+1} \left\{ G^{(\ell+1)}_{2R+\ell+3} + \frac{1}{4} G^{(\ell-1)}_{2R+\ell+3} + \frac{1}{4} a_{R+\ell+2} G^{(\ell+1)}_{2R+\ell+5} \right\},$$

$$a_{R-2}(C_{R,\ell}) = \frac{(R-1)R}{4(2R-1)(2R+1)} G^{(\ell)}_{2R+\ell+4},$$

$$a_{R+1}(C_{R,\ell}) = \frac{R+1}{2R+1} G^{(\ell+1)}_{2R+\ell+3}. \quad (2.79)$$

$^6 \ g_\ell(x) \propto Q_{\ell-1}(z)$ with $Q_\nu$ an associated Legendre function.
For $R = 0$ (2.79) coincides with (2.77). Thus the contribution of any semi-short supermultiplet $C_{R,\ell}$, $R = 0, 1, \ldots, 2n - 1$, to the four point function may be obtained by combining the results for long supermultiplets at unitarity threshold with (2.77). There is no reason why any particular $C_{R,\ell}$, except $C_{0,0}$ which contains the energy momentum tensor and the conserved $SU(2)_R$ current, should be present but if $f(z)$ is non zero it is necessary for there to be at least one semi-short contribution involving operators with protected dimensions.

2.6. Short Operators

A special case arises if we set $\ell = -1$ in the result for semishort operators. Formally, as shown in [4], $C_{R,-1} \simeq B_{R+1}$ where $B_R$ denotes the short supermultiplet whose lowest dimension operator belongs to the $R$-representation with $\Delta = 2R$, $\ell = 0$, obeying the full $\mathcal{N} = 2$ shortening conditions. The conformal partial waves as shown in [5] satisfy

$$ (\frac{1}{4})^{\ell-1} \mathcal{G}_{\Delta}^{(-\ell)} = -\mathcal{G}_{\Delta}^{(\ell-2)}, \quad \mathcal{G}_{\Delta}^{(-1)} = 0, \quad (2.80) $$

and hence from (2.79) we have

$$ a_{R'}(C_{R,-1}) = \frac{R + 1}{2R + 1} a_{R'}(B_{R+1}), \quad (2.81) $$

where

$$ a_R(B_R) = \mathcal{G}_{2R}^{(0)}, \quad a_{R-1}(B_R) = \frac{R}{2R + 1} \mathcal{G}_{2R+1}^{(1)}, $$

$$ a_{R-2}(B_R) = \frac{(R - 1)R}{4(2R - 1)(2R + 1)} \mathcal{G}_{2R+2}^{(0)}. \quad (2.82) $$

The operators whose contributions appear in (2.82) are just those expected for the short supermultiplet $B_R$ and there are possible contributions to the four point function for $R = 1, 2, \ldots, 2n$. Since

$$ \mathcal{G}_{0}^{(0)}(u,v) = 1, \quad (2.83) $$

then it is easy to see from (2.82) that

$$ a_R(B_0) = a_R(I) = \delta_{R0}, \quad (2.84) $$
where \( a_R(I) \) denotes the contribution of the identity in the operator product expansion. Besides (2.81) we may also note that

\[
a_{R'}(C_{R,-2}) = -4 a_{R'}(B_R). \tag{2.85}
\]

For \( R = 0 \) this is in accord with (2.84) since \( G_0^{(-2)} = -4 \).

2.7. (Next-to-) Extremal Correlators

Apart from the case of the correlation function for four identical operators as considered above there are other solutions of the superconformal Ward identities which are of interest corresponding to extremal and next-to-extremal correlation functions [18]. The extremal case corresponds to taking \( E = 0 \) in (2.37). There is then a unique \( SU(2)_R \) invariant coupling which also follows from the requirement that \( F \) in (2.42), or \( G \) where in (2.55) \( G(u, v; t) = (t_1 \cdot t_4)^{n_1} (t_2 \cdot t_4)^{n_2} (t_3 \cdot t_4)^{n_3} G(u, v) \), must be independent of \( \sigma, \tau \) and hence equivalently also of \( \alpha \). In this case the result (2.49) for \( \partial_x \) and its conjugate for \( \partial_{\bar{x}} \) simply imply

\[
G(u, v) = C u^{n_1 + n_2}, \tag{2.86}
\]

where \( C \) is independent of both \( x, \bar{x} \) and thus a constant. To interpret this in terms of the operator product expansion for \( \varphi^{(n_1)}(x_1, t_1) \varphi^{(n_2)}(x_2, t_2) \) we may use the result from [7],

\[
G_{\Delta_1 + \Delta_2}^{(0)}(u, v; \Delta_2, \Delta_1 + \Delta_2) = u^{\frac{1}{2} (\Delta_1 + \Delta_2)}. \tag{2.87}
\]

The result (2.86) then shows that the only operators contributing to the operator product expansion in the extremal case have \( \Delta = 2(n_1 + n_2), \ell = 0 \) and necessarily \( R = n_1 + n_2 \). Such operators can only be found as the lowest dimension operator in the short supermultiplet \( B_{n_1 + n_2} \).

For the next-to-extremal case we set \( E = 0 \) in (2.50). The solution of (2.49) can be conveniently expressed as

\[
(1 - z) G(u, v; z) = u^{n_1 + n_2 - 1} f(\bar{z}), \tag{2.88}
\]
where in (2.55) we have $G(u, v; t) = (t_1 \cdot t_4)^{n_1} (t_2 \cdot t_4)^{n_2-1} (t_3 \cdot t_4)^{n_3-1} t_2 \cdot t_3 \times t_4 \ G(u, v; y)$. For $E = 0, (1 - y)G(u, v; y)$ is linear in $y$ and from (2.88) and its conjugate we may find
\[
(1 - y)G(u, v; y) = u^{n_1 + n_2 - 1} \frac{(y - \bar{z})f(\bar{z}) - (y - z)f(z)}{z - \bar{z}}, \tag{2.89}
\]
so that $G$ is determined just by the single variable function $g$ in this case. Since $K$ does not appear this implies that no dynamical information is contained in the correlation function confirming the nonrenormalization property.

For the next-to-extremal correlation function there are just two independent $SU(2)_R$ invariant couplings. In a similar fashion to (2.62), we have an expansion, from appendix B, in terms of two Jacobi polynomials
\[
(1 - y) \ G(u, v; y) = a_{n_1 + n_2 - 1}(u, v) P_0^{(2n_1 - 1, 2n_2 - 1)}(y) + a_{n_1 + n_2}(u, v) P_1^{(2n_1 - 1, 2n_2 - 1)}(y), \tag{2.90}
\]
where $a_R, R = n_1 + n_2 - 1, n_1 + n_2$ represent the contribution of the two possible $R$-representations of $SU(2)_R$ in this case. From (2.89) we obtain
\[
a_{n_1 + n_2 - 1} = - \frac{1}{n_1 + n_2} u^{n_1 + n_2} \frac{f(z) - f(\bar{z})}{z - \bar{z}}, \tag{2.91}
a_{n_1 + n_2} = u^{n_1 + n_2} \left( \frac{zf(z) - \bar{z}f(\bar{z})}{z - \bar{z}} + \frac{n_1 - n_2}{n_1 + n_2} \frac{f(z) - f(\bar{z})}{z - \bar{z}} \right).
\]
To interpret this in terms of the operator product expansion we may use, extending (2.73),
\[
G_{\Delta_1 + \Delta_2 + \ell}^{(\langle \ell \rangle)}(u, v; \Delta_2, \Delta_1, \Delta_2 + 2) = u^{\frac{1}{2}(\Delta_1 + \Delta_2)} \ \frac{g_{\ell+1}^{(\langle \ell \rangle)}(x; \Delta_1, \Delta_2) - g_{\ell+1}^{(\langle \ell \rangle)}(\bar{x}; \Delta_1, \Delta_2)}{x - \bar{x}},
\]
\[
ge_{\ell}(x; \Delta_1, \Delta_2) = (-\frac{1}{2} x)^{\ell-1} x F(\ell + \Delta_2 - 1, \ell; 2\ell + \Delta_1 + \Delta_2 - 2; x). \tag{2.92}
\]
In consequence only operators with twist $\Delta_1 + \Delta_2$ can contribute for the solution for $a_R$ given by (2.91). If in (2.91) let we $f(z) \rightarrow 2g_{\ell+2}(x; 2n_1, 2n_2)$ then $a_R \rightarrow a_R(C_{n_1 + n_2 - 1, \ell})$ where
\[
a_{n_1 + n_2}(C_{n_1 + n_2 - 1, \ell}) = \frac{1}{n_1 + n_2} G_{2n_1 + 2n_2 + \ell + 1}^{(\ell+1)},
\]
\[
a_{n_1 + n_2 - 1}(C_{n_1 + n_2 - 1, \ell}) = G_{2n_1 + 2n_2 + \ell}^{(\ell)} + (n_2 - n_1) \frac{(\ell + 1)(2n_1 + 2n_2 + \ell)}{(n_1 + n_2 + \ell)(n_1 + n_2 + \ell + 1)(n_1 + n_2)} G_{2n_1 + 2n_2 + \ell + 1}^{(\ell+1)}
\]
\[
+ \frac{(\ell + 2)(2n_1 + \ell + 1)(2n_2 + \ell + 1)(2n_1 + 2n_2 + \ell)}{(n_1 + n_2 + \ell + 1)^2(2n_1 + 2n_2 + 2\ell + 1)(2n_1 + 2n_2 + 2\ell + 3)} G_{2n_1 + 2n_2 + \ell + 2}^{(\ell+2)}, \tag{2.93}
\]
where $G_{2n_1+2n_2+\ell}^{(\ell)}$ are as in (2.92) with $\Delta_i \to 2n_i$. The contributions appearing in (2.93) correspond to those expected from the semi-short supermultiplet $C_{n_1+n_2-1,\ell}$. Using $C_{R,-1} \simeq B_{R+1}$ again we may obtain the contribution for the short multiplet $B_{n_1+n_2}$,

$$a_R(C_{n_1+n_2-1,-1}) = \frac{1}{n_1+n_2} a_R(B_{n_1+n_2}), \quad (2.94)$$

giving

$$a_{n_1+n_2}(B_{n_1+n_2}) = G_{2n_1+2n_2}^{(0)},$$

$$a_{n_1+n_2-1}(B_{n_1+n_2}) = \frac{4n_1n_2}{(n_1+n_2)(2n_1+2n_2+1)} G_{2n_1+2n_2+1}^{(1)}. \quad (2.95)$$

For the next-to-extremal correlation function therefore only the protected short and semi-short supermultiplets $B_{R}$ and $C_{R-1,\ell}$ can contribute to the operator product expansion.

2.8. Summary

By analysis [32,33] of three point functions the possible $N=2$ supermultiplets which may appear in the operator product expansion of two $N=2$ short supermultiplets is determined by the decomposition, for $n_2 \geq n_1$,

$$B_{n_1} \otimes B_{n_2} \simeq \bigoplus_{n=n_2-n_1}^{n_2+n_1} B_{n} \oplus \bigoplus_{\ell \geq 0} \left( \bigoplus_{n=n_2-n_1}^{n_2+n_1-1} C_{n,\ell} \oplus \bigoplus_{n=n_2-n_1}^{n_2+n_1-2} A_{n,\ell}^{\Delta} \right), \quad (2.96)$$

where for $A_{n,\ell}^{\Delta}$ all $\Delta > 2n + \ell + 2$ is allowed. By considering also the corresponding result for $B_{n_3} \otimes B_{n_4}$ in all cases discussed above the general solution of the $N=2$ superconformal identities accommodates all possible $N=2$ supermultiplets which may contribute to the four point function in the operator product expansion according to (2.96). In the extremal case it is clear that only $B_{n_1+n_2}$ contributes while for the next-to-extremal case long multiplets which undergo renormalisation are also excluded.
3. $\mathcal{N} = 4$

3.1. Superconformal Ward Identities

We here describe an analysis of the superconformal Ward identities for the four point function of $\mathcal{N} = 4$ chiral primary operators belonging to the $SU(4)_R$ representation with $\Delta = p$ represented by symmetric traceless fields $\varphi_{r_1,\ldots,r_p}(x)$, $r_i = 1,\ldots,6$. As in (1.30) we define $\varphi^{(p)}(x,t)$, homogeneous of degree $p$ in $t$, in terms of a six dimensional null vector $t_r$. The superconformal transformation of $\varphi^{(p)}(x,t)$ is then expressible in the form

$$
\delta \varphi^{(p)}(x,t) = -\hat{\epsilon}(x) \gamma \cdot t \psi^{(p-1)}(x,t) + \bar{\psi}^{(p-1)}(x,t) \gamma \cdot t \hat{\epsilon}(x),
$$

(3.1)

where the conformal Killing spinors $\hat{\epsilon}^\alpha_i(x)$, $\hat{\epsilon}^{i\dot{\alpha}}(x)$ are as in (2.3), with $i = 1, 2, 3, 4$ and $\gamma_r^{ij} = -\gamma_{rji}$, $\bar{\gamma}_{rji} = \frac{1}{2} \varepsilon_{ijkl} \gamma_r^{kl}$ are $SU(4)$ gamma matrices, $\gamma_r \bar{\gamma}_s + \gamma_s \bar{\gamma}_r = -2\delta_{rs}1$, $\gamma_r^\dagger = -\bar{\gamma}_r$. In (3.1) $\psi^{(p-1)}_{i\alpha}(x,t)$, $\bar{\psi}^{(p-1)i\dot{\alpha}}(x,t)$ are homogeneous spinor fields of degree $p - 1$ in $t$ and satisfy constraints similar to (2.8)

$$
\gamma_r \frac{\partial}{\partial t} \psi^{(p-1)}_{i\alpha}(x,t) = 0, \quad \bar{\psi}^{(p-1)}(x,t) \gamma^r \frac{\partial}{\partial t} = 0,
$$

(3.2)

which are necessary for them to belong to $SU(4)_R$ representations $[0,p-1,1]$, $[1,p-1,0]$. At the next level the superconformal transformations involve a current belonging to the $[1,p-1,1]$ representation which corresponds to a homogeneous field of degree $p - 1$ with one $SU(4)_R$ vector index $J^{(p-1)}_{r\alpha\dot{\alpha}}(x,t)$ satisfying

$$
t_r J^{(p-1)}_{r\alpha\dot{\alpha}}(x,t) = 0, \quad \frac{\partial}{\partial t_r} J^{(p-1)}_{r\alpha\dot{\alpha}}(x,t) = 0.
$$

(3.3)

The superconformal transformation of $\psi^{(p-1)}(x,t)$, neglecting $\hat{\epsilon}$ terms, is then

$$
\delta \psi^{(p-1)}_{\alpha}(x,t) = \frac{1}{p} \bar{\gamma} \cdot \frac{\partial}{\partial t} i \partial_{\alpha\dot{\alpha}} \varphi^{(p)}(x,t) \hat{\epsilon}^{\dot{\alpha}}(x) + 2 \bar{\gamma} \cdot \frac{\partial}{\partial t} \varphi^{(p)}(x,t) \eta_\alpha \\
+ \left(1 + \frac{1}{2p} \bar{\gamma} \cdot t \gamma \cdot \frac{\partial}{\partial t}\right) J^{(p-1)}_{r\alpha\dot{\alpha}}(x,t) \bar{\gamma}_r \hat{\epsilon}^{\dot{\alpha}}(x).
$$

(3.4)

Superconformal transformations which generate the full BPS multiplet listed in [34] can be obtained similarly to [5] but the superconformal Ward identities depend only on (3.1) and (3.4).
The general four point function of chiral primary operators can be written in an identical form to (1.32),

\[
\langle \varphi^{(p_1)}(x_1, t_1) \varphi^{(p_2)}(x_2, t_2) \varphi^{(p_3)}(x_3, t_3) \varphi^{(p_4)}(x_4, t_4) \rangle = r_{23}^{\Sigma-p_2-p_3} r_{34}^{\Sigma-p_3-p_4} F(u, v; t), \quad \Sigma = \frac{1}{2}(p_1 + p_2 + p_3 + p_4). \tag{3.5}
\]

The derivation of superconformal Ward identities initially follows an almost identical path as that in chapter 2 leading to (2.18a, b). With similar definitions to (2.11), (2.12), taking \(2n_i \to p_i\), and (2.17) we find

\[
\frac{1}{p_1} \bar{\gamma} \frac{\partial}{\partial t_1} \frac{\partial}{\partial x} F = -\left(1 + \frac{1}{2p_1 + 2} \bar{\gamma} \cdot t_1 \gamma \cdot \frac{\partial}{\partial t_1}\right) \bar{\gamma} \cdot K - \frac{1}{x} T_2 \bar{\gamma} \cdot t_2 + \frac{1}{1 - x} T_4 \bar{\gamma} \cdot t_4, \tag{3.6a}
\]

\[
\bar{\gamma} \cdot \frac{\partial}{\partial t_1} F = T_2 \bar{\gamma} \cdot t_2 + T_3 \bar{\gamma} \cdot t_3 + T_4 \bar{\gamma} \cdot t_4. \tag{3.6b}
\]

Instead of (2.19) we have the constraints, which follow from (3.2) and (3.3),

\[
\gamma \cdot \frac{\partial}{\partial t_1} T_i = 0, \quad T_i \frac{\partial}{\partial t_i} \bar{\gamma} \cdot \frac{\partial}{\partial t_i} = 0, \quad t_1 \cdot K = \frac{\partial}{\partial t_1} \cdot K = 0. \tag{3.7}
\]

As with (2.20) we exhibit the dependence on \(SU(4)\) gamma matrices by writing

\[
T_i \bar{\gamma} \cdot t_i = \bar{\gamma} \cdot V_i + \frac{1}{6} \bar{\gamma}[r \gamma_s \bar{\gamma}_u] W_{i, rsu}. \tag{3.8}
\]

Since we take\(^7\) \(\gamma[r \gamma_s \gamma_u \gamma_v \gamma_w \gamma_z] = i\varepsilon_{rsuvwz}\),

\[
\bar{\gamma}[r \gamma_s \bar{\gamma}_u] = -\frac{1}{6} i\varepsilon_{rsuvwz} \bar{\gamma}_v \gamma_w \bar{\gamma}_z, \tag{3.9}
\]

so that we must require the self-duality condition

\[
W_{i, rsu} = \frac{1}{6} i\varepsilon_{rsuvwz} W_{i, uvw}. \tag{3.10}
\]

Imposing the first equation in (3.7) we have

\[
\frac{\partial}{\partial t_1} V_i = 0, \quad \partial_{1[r} V_{i,s]} = \partial_{1u} W_{i, rsu}. \tag{3.11}
\]

\(^7\) Note that \((\gamma_1 \bar{\gamma}_2 \gamma_3 \bar{\gamma}_4 \gamma_5 \bar{\gamma}_6)^t = -\gamma_1 \bar{\gamma}_2 \gamma_3 \bar{\gamma}_4 \gamma_5 \bar{\gamma}_6\).
Just as in (2.21) we write,

\begin{equation}
V_i = \frac{1}{p_1} \frac{\partial}{\partial t_1} U_i + \dot{V}_i, \quad t_1 \cdot V_i = U_i, \quad t_1 \cdot \dot{V}_i = 0. \tag{3.12}
\end{equation}

so that in (3.11) we may let \( V_i \to \dot{V}_i \).

Using (3.8) with (3.12), and \( \hat{\gamma} \cdot t_1 \gamma \cdot \partial_1 \hat{\gamma} \cdot K = -p_1 \hat{\gamma} \cdot K + \frac{1}{2} \hat{\gamma} \cdot s \hat{\gamma} \cdot K_{1,rs} K_u \), (3.6a, b) may be decomposed into three pairs of equations,

\begin{equation}
\frac{\partial}{\partial x} F = -\frac{1}{x} U_2 + \frac{1}{1 - x} U_4, \quad p_1 F = \sum_{i=2}^{4} U_i, \tag{3.13}
\end{equation}

and

\begin{equation}
\frac{p_1 + 2}{2p_1 + 2} K_r = -\frac{1}{x} \dot{V}_{2,r} + \frac{1}{1 - x} \dot{V}_{4,r}, \quad \sum_{i=2}^{4} \dot{V}_{i,r} = 0, \tag{3.14}
\end{equation}

and

\begin{equation}
\frac{3}{2p_1 + 2} (L_{1,[rs]} K_{u})_{sd} = -\frac{1}{x} W_{2,rsu} + \frac{1}{1 - x} W_{4,rsu}, \quad \sum_{i=2}^{4} W_{i,rsu} = 0, \tag{3.15}
\end{equation}

where we define for \( i = 1, 2, 3, 4 \)

\begin{equation}
L_{i,rs} = t_{ir} \partial_{is} - t_{is} \partial_{ir}, \tag{3.16}
\end{equation}

and for any \( X_{rsu} = X_{[rsu]} \) the self dual part, satisfying (3.10), is given by

\begin{equation}
(X_{rsu})_{sd} = \frac{1}{2} X_{rsu} + \frac{1}{12} i \varepsilon_{rsuvwz} X_{uwz}. \tag{3.17}
\end{equation}

Since \( 2t_1 s \partial_1 [r \dot{V}_{i,s}] = -p_1 \dot{V}_{i,r} \) we may obtain from (3.11)

\begin{equation}
p_1 \dot{V}_{i,r} = -L_{1,rs} W_{i,rsu}, \tag{3.18}
\end{equation}

which gives \( \dot{V}_{i,r} \) in terms of \( W_{i,rsu} \). Furthermore from (3.7) \( L_{1,rs} K_s = K_r \) and using also, as a consequence of the commutation relations for \( L_1, [L_{1,rs}, L_{1,ru}] = -4L_{1,sw} \) we have \( L_{1,rs} L_{1,ru} K_s = 3K_u \). With, in addition, \( \frac{1}{2} L_{1,rs} L_{1,rs} K_u = -(p_1 - 1)(p_1 + 3)K_u \) we may then obtain

\begin{equation}
3L_{1,rs} L_{1,[rs]} K_u = -2p_1 (p_1 + 2) K_u. \tag{3.19}
\end{equation}
Since also $\varepsilon_{rsuvwz}L_{1, su}L_{1, vu} K_z = 0$ it is clear from (3.18) and (3.19) that eqs. (3.15) imply (3.14). However, if we define

$$ \bar{W}_{i, rsu} = 3 (L_{1, [rsu]} \hat{V}_{i, u})_{sd} - (p_1 + 2) W_{i, rsu}, $$

(3.20)

with $\hat{V}_{i, u}$ determined by (3.18), then as a consequence of (3.14) and (3.15) we must also require

$$ \frac{1}{x} W_{2, rsu} = \frac{1}{1 - x} W_{4, rsu}. $$

(3.21)

From the second equation in (3.7) we may obtain $\frac{1}{2} L_{i, rs}(T_i \bar{\gamma} \cdot t_i) \gamma_r \bar{\gamma}_s = (p_i + 4) T_i \bar{\gamma} \cdot t_i$ which leads to the relations

$$ L_{i, rs} V_{i, s} - L_{i, su} W_{i, rsu} = (p_i + 4) V_{i, r}, $$

(3.22a)

$$ 3 (L_{i, rs} V_{i, u})_{sd} + 3 L_{i, [r]u} W_{i, su} u = (p_i + 4) W_{i, rsu}, $$

(3.22b)

where $L_{i, [u]v} W_{i, rs} u$ is self dual as a consequence of (3.10). We also have from $T_i \bar{\gamma} \cdot t_i \gamma \cdot t_i = 0$

$$ t_i \cdot V_i = 0, \quad t_{i[r} V_{i, s]} + W_{i, rsu} t_{iu} = 0. $$

(3.23)

For consistency we note that $\partial t_s(t_{i[r} V_{i, s]} + W_{i, rsu} t_{iu}) = 0$ is identical with (3.22a). Furthermore using (3.10) we have $(\partial t_{i[r} W_{i, su} v t_{iv})_{sd} = \frac{1}{2} (\partial t_{i[r} W_{i, su} v t_{iv} - \partial t_{i[r} W_{i, su} v}) + \frac{1}{3} h_{t v} \partial t_{i[r} W_{i, rsu} t_{iu})$ and, from appendix A, $(p_i + 2) h_{t v} W_{i, rsu} t_{iv} = (p_i + 3) (p_i + 4) W_{i, rsu}$ while acting on $W_{i, su} v$ similarly $(p_i + 2) h_{t v} W_{i, su} v t_{iv} = (p_i + 3) (p_i + 4) W_{i, rsu}$ while acting on $W_{i, su} v$. Hence we have demonstrated that $(\partial t_{i[r} (t_{is} V_{i, u]} + W_{i, su} v t_{iv})_{sd} = 0$ is identical to (3.22b) so that this equation is also implied by (3.23).

Combining (3.23) with (3.12) gives the essential equation

$$ t_{i[r} \partial t_{1s]} U_i + p_1 t_{i[r} \hat{V}_{i, s]} + p_1 W_{i, rsu} t_{iu} = 0, $$

(3.24)

where $p_1 \hat{V}_{i, s}$ is determined by (3.18).

As in (2.42) we may expand the correlation function $F$, as defined in (3.5), in terms of $SU(4)$ invariants

$$ F(u, v; t) = (t_1 \cdot t_4)^{p_1 - E} (t_2 \cdot t_4)^{p_2 - E} (t_1 \cdot t_2)^E (t_3 \cdot t_4)^{p_3} F(u, v; \sigma, \tau), $$

(3.25)
where we assume
\[ p_1 \leq p_2 \leq p_3 \leq p_4, \quad 2E = p_1 + p_2 + p_3 - p_4. \] (3.26)

In (3.25) \( F(u, v; \sigma, \tau) \) is a polynomial in \( \sigma, \tau \) consistent with \( F(u, v; t) = O(t^{p_1}, t^{p_2}, t^{p_3}, t^{p_4}) \) and hence \( E \geq 0 \) is an integer. For \( p_1 \geq E \) then \( F \) is expressible as a polynomial of degree \( E \) in \( \sigma, \tau \), i.e. a linear expansion in the \( \frac{1}{2}(E + 1)(E + 2) \) independent monomials \( \sigma^p \tau^q \) with \( p + q \leq E \). For \( p_1 < E \) it is necessary also that \( q \geq E - p_1 \) giving only \( \frac{1}{2}(p_1 + 1)(p_1 + 2) \) independent terms. It is easy to see that this matches the number of invariants that may be constructed by finding common representations in \([0, p_1, 0] \otimes [0, p_2, 0]\) and \([0, p_3, 0] \otimes [0, p_4, 0]\) using the tensor product result
\[
[0, p_1, 0] \otimes [0, p_2, 0] \simeq \bigoplus_{r=0}^{p_1} \bigoplus_{s=0}^{p_1 - r} [r, p_2 - p_1 + 2s, r].
\] (3.27)

Hence representations \([r, p_2 - p_1 + 2s, r]\) may contribute for \( s = 0, \ldots, n - r, \ r = 0, \ldots, n \) with \( n = E \) if \( p_1 \geq E \), otherwise \( n = p_1 \).

In an exactly similar fashion to (3.25) we may express \( U_i(x, \bar{x}; t) \) in terms of \( U_i(x, \bar{x}; \sigma, \tau) \) so that (3.13) becomes
\[
\frac{\partial}{\partial x} F = -\frac{1}{x} U_2 + \frac{1}{1 - x} U_4, \quad p_1 F = U_2 + U_3 + U_4. \] (3.28)

Furthermore we may also decompose \( W_{i,rsu}(x, \bar{x}; t) \) for \( i = 2, 3, 4 \) in terms of four independent self dual tensors,
\[
W_{i,rsu} = -(t_1 \cdot t_4)^{p_1 - E} (t_2 \cdot t_4)^{p_2 - E} (t_1 \cdot t_2)^{E - 2} (t_3 \cdot t_4)^{p_3 - 1}
\times \left( (t_1[r, t_2, t_3, t_4])_{sd} t_2 \cdot t_4 A_i + (t_1[r, t_4, t_2, t_3])_{sd} t_2 \cdot t_3 B_i \right.
+ (t_1[r, t_3, t_4, t_4])_{sd} t_2 \cdot t_3 t_2 \cdot t_4 C_i + (t_2[r, t_3, t_4, t_4])_{sd} t_1 \cdot t_2 W_i \bigg), \] (3.29)

with \( A_i, B_i, C_i \) and \( W_i \) polynomials in \( \sigma, \tau \) of degree \( E - 2 \) and \( E - 1 \), if \( p_1 \geq E \). From its definition in (3.8) we must have
\[
C_2 = B_3 = A_4 = 0. \] (3.30)
The result (3.15) then requires
\[ A_2 + A_3 = 0, \quad B_2 + B_4 = 0, \quad C_3 + C_4 = 0, \quad W_2 + W_3 + W_4 = 0. \tag{3.31} \]

We may similarly decompose \( \hat{V}_{i,r} \) in the form
\[
\hat{V}_{i,r} = (t_1 t_4)^{p_1 - E - 1} (t_2 t_4)^{p_2 - E} (t_1 t_2)^{E - 1} (t_3 t_4)^{p_3 - 1}
\times ((t_2 r t_1 t_4 - t_4 r t_1 t_2) t_3 t_4 \mathcal{I}_i + (t_3 r t_1 t_4 - t_4 r t_1 t_3) t_2 t_4 \mathcal{J}_i + t_1 r t_2 t_4 t_3 t_4 \mathcal{V}_i),
\tag{3.32}
\]
where we impose \( t_1 \cdot \hat{V}_i = 0 \). The coefficient of \( t_{1r} \) is determined by the requirement \( \partial_1 \hat{V}_i = 0 \),
\[
(p_1 + 2) \mathcal{V}_i = -\mathcal{O}_\sigma \mathcal{I}_i + \mathcal{I}_i - (\sigma \mathcal{O}_\sigma - \tau \mathcal{O}_r) \mathcal{J}_i + \mathcal{J}_i,
\tag{3.33}
\]
with differential operators
\[
\mathcal{O}_\sigma = (\sigma + \tau - 1) \frac{\partial}{\partial \sigma} + 2\tau \frac{\partial}{\partial \tau} + p_1 - 2E + 1,
\]
\[
\mathcal{O}_r = 2\sigma \frac{\partial}{\partial \sigma} + (\sigma + \tau - 1) \left( \frac{\partial}{\partial \tau} + \frac{p_1 - E}{\tau} \right) - p_1 + 1. \tag{3.34}
\]

Using (3.18) we get
\[
6p_1 \mathcal{I}_i = (p_1 + 2) (\sigma A_i - \tau B_i) - (\sigma \mathcal{O}_\sigma - \tau \mathcal{O}_r) \mathcal{W}_i,
\]
\[
6p_1 \mathcal{J}_i = - (p_1 + 2) (A_i - \tau C_i) + \mathcal{O}_\sigma \mathcal{W}_i.
\tag{3.35}
\]
From (3.33) we then obtain
\[
6p_1 \mathcal{V}_i = \tau \left( (\mathcal{O}_\sigma + 1) B_i - (\mathcal{O}_r + 1) A_i - (\sigma (\mathcal{O}_\sigma + 1) - \tau (\mathcal{O}_\tau + 1)) C_i \right). \tag{3.36}
\]

As a consequence of (3.24) the coefficients in (3.25) are not independent but we have relations which determine \( A_i, B_i, C_i \) for each \( i \),
\[
(p_1 + 1) A_2 = 3 \frac{\partial}{\partial \sigma} \mathcal{U}_2 + \frac{1}{2}(\mathcal{O}_\sigma - p_1) \mathcal{W}_2,
\]
\[
(p_1 + 1) B_2 = -3 \left( \frac{\partial}{\partial \tau} + \frac{p_1 - E}{\tau} \right) \mathcal{U}_2 + \frac{1}{2}(\mathcal{O}_r - p_1) \mathcal{W}_2,
\]
\[
(p_1 + 1) \sigma A_3 = 3 \left( \sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} - E \right) \mathcal{U}_3 + \frac{1}{2}(\sigma \mathcal{O}_\sigma - \tau \mathcal{O}_r - p_1) \mathcal{W}_3,
\]
\[
(p_1 + 1) \sigma C_3 = 3 \left( \frac{\partial}{\partial \tau} + \frac{p_1 - E}{\tau} \right) \mathcal{U}_3 - \frac{1}{2}(\mathcal{O}_r + p_1) \mathcal{W}_3,
\]
\[
(p_1 + 1) \tau B_4 = -3 \left( \sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} - E \right) \mathcal{U}_4 - \frac{1}{2}(\sigma \mathcal{O}_\sigma - \tau \mathcal{O}_r + p_1) \mathcal{W}_4,
\]
\[
(p_1 + 1) \tau C_4 = -3 \frac{\partial}{\partial \sigma} \mathcal{U}_4 - \frac{1}{2}(\mathcal{O}_\sigma + p_1) \mathcal{W}_4.
\tag{3.37}
\]
Combining this with (3.31) and also the result in (3.28) for \( p_1 \mathcal{F} \) leads to
\[
\left( \frac{\tau}{\partial \tau} + p_1 - E \right) \mathcal{F} = U_4 + \frac{1}{6}(\sigma - \tau - 1)W_4 - \frac{1}{3} \tau W_2, \tag{3.38}
\]
\[
\left( \frac{\sigma}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} - E \right) \mathcal{F} = U_2 + \frac{1}{6}(\sigma - \tau - 1)W_2 - \frac{1}{3} W_4.
\]

If \( \bar{W}_{i,rsu} \) given by (3.20) is defined in terms of \( \bar{A}_i, \bar{B}_i, \bar{C}_i \) and \( \bar{W}_i \) as in (3.29) then it is easy to see that \( \bar{W}_i = -(p_1 + 2)W_i \) and, as a consequence of (3.35) and (3.37),
\[
2(p_1 + 1)\bar{A}_i = \left( \frac{\partial}{\partial \sigma} (\sigma \mathcal{O}_\sigma - \tau \mathcal{O}_\tau + p_1 + 2) + \left( \frac{\sigma}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} - E + 1 \right) (\mathcal{O}_\sigma - p_1 - 2) \right) W_i, \tag{3.39}
\]
\[
\tau \bar{C}_i - \bar{A}_i = \mathcal{O}_\sigma W_i, \quad \tau \bar{B}_i - \sigma \bar{A}_i = (\sigma \mathcal{O}_\sigma - \tau \mathcal{O}_\tau) W_i.
\]
Hence (3.21) reduces to just
\[
\frac{1}{x} W_2 = \frac{1}{1 - x} W_4. \tag{3.40}
\]

Just as the invariants \( u, v \) are expressed in terms of \( x, \bar{x} \) it is convenient to write \( \sigma, \tau \)
in a similar form involving new variables \( \alpha, \bar{\alpha} \),
\[
\sigma = \alpha \bar{\alpha}, \quad \tau = (1 - \alpha)(1 - \bar{\alpha}). \tag{3.41}
\]
In terms of these variables \( \alpha, \bar{\alpha} \), (3.38) becomes
\[
\alpha(1 - \alpha) \frac{\partial}{\partial \alpha} \mathcal{F} + E \alpha \mathcal{F} - p_1 \mathcal{F} = -(1 - \alpha)U_2 - U_4 + \frac{1}{6}(\alpha - \bar{\alpha})((1 - \alpha) W_2 + W_4), \tag{3.42}
\]
together with the conjugate equation obtained for \( \alpha \leftrightarrow \bar{\alpha} \). If this is used together with (3.28) for \( \partial_x \mathcal{F} \) we may eliminate \( U_2 \) to obtain
\[
\left( x \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial \alpha} - E - \frac{\alpha}{1 - \alpha} + p_1 \frac{1}{1 - \alpha} \right) \mathcal{F}
\]
\[
= \left( \frac{x}{1 - x} + \frac{1}{1 - \alpha} \right) U_4 - \frac{1}{6}(\alpha - \bar{\alpha}) \left( W_2 + \frac{1}{1 - \alpha} W_4 \right) \tag{3.43}
\]
\[
= \frac{1 - \alpha x}{(1 - \alpha)(1 - x)} \left( U_4 - \frac{1}{6}(\alpha - \bar{\alpha}) W_4 \right),
\]
where we have used (3.40). Writing \( \mathcal{F}(u,v;\sigma,\tau) = \hat{\mathcal{F}}(x,\bar{x};\alpha,\bar{\alpha}) \) evidently
\[
\left( x \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial \alpha} + E + (p_1 - E) \frac{1}{1 - \alpha} \right) \hat{\mathcal{F}}(x,\bar{x};\alpha,\bar{\alpha}) \bigg|_{\alpha = \frac{1}{x}} = 0, \tag{3.44}
\]
which is solved by writing
\[
u^E (1 - \nu)^{p_1 - E} \hat{\mathcal{F}}(x,\bar{x};\alpha,\bar{\alpha}) = f(\bar{x}, \bar{\alpha}). \tag{3.45}
\]
Together with the conjugate equation in which \( \alpha \to \bar{\alpha} \) (3.45) is the basic solution of the superconformal Ward identities in this context.
3.2. Solution of Identities

As previously it is more convenient for consideration of the operator product expansion to change from \( F(u, v; t) \) to \( G(u, v; t) \), defined in (1.32). Writing \( G(u, v; t) \) in a similar fashion to (3.25) then the corresponding function \( G \) is given in terms of \( F(u, v; \sigma, \tau) \) by

\[
G(u, v; \sigma, \tau) = u^{\frac{1}{2}(p_1+p_2)} v^{p_1-E} F(u, v; \sigma, \tau). \tag{3.46}
\]

For the applications in this section it is convenient to write

\[
G(u, v; \sigma, \tau) = \hat{G}(u, v; y, \bar{y}) = \hat{G}(u, v; \bar{y}, y), \tag{3.47}
\]

where \( \hat{G} \) depends on the variables

\[
y = 2\alpha - 1, \quad \bar{y} = 2\bar{\alpha} - 1. \tag{3.48}
\]

The solution (3.45) then gives, with \( z, \bar{z} \) defined in (2.57),

\[
\hat{G}(u, v; z, \bar{y}) = u^{\frac{1}{2}(p_1+p_2)-E} f(\bar{z}, \bar{y}), \quad \hat{G}(u, v; \bar{z}, y) = u^{\frac{1}{2}(p_1+p_2)-E} f(z, y). \tag{3.49}
\]

For consistency, since \( f(z, z) = f(\bar{z}, \bar{z}) \), we must have

\[
f(z, z) = k. \tag{3.50}
\]

\( G(u, v; y, \bar{y}) \) is a symmetric polynomial in \( y, \bar{y} \) with degree \( p \). Since it must also be symmetric in \( z, \bar{z} \) (3.49) implies

\[
\hat{G}(u, v; y, \bar{y}) = -k + \frac{(y - z)(\bar{y} - \bar{z})(f(z, \bar{y}) + f(\bar{z}, y) - (y - \bar{z})(y - z)(f(z, y) + f(\bar{z}, \bar{y}))}{(z - \bar{z})(y - \bar{y})} + (y - z)(y - \bar{z})(\bar{y} - z)(\bar{y} - \bar{z}) K(u, v; \sigma, \tau), \tag{3.51}
\]

with \( K(u, v; \sigma, \tau) = \hat{K}(u, v; y, \bar{y}) \) defining an undetermined symmetric polynomial in \( y, \bar{y} \) of degree \( p - 2 \). Since this is the part of the correlation function not constrained by the Ward identities, it must contain the full dynamical information.
3.3. OPE Analysis

In general the conformal partial wave expansion and the decomposition into contributions for differing \( SU(4)_R \) representations further into conformal partial waves is realised by writing for \( p_1 \geq E \).

\[
\hat{G}(u, v; y, \bar{y}) = \sum_{0 \leq m \leq n \leq E} a_{nm}(u, v) P_{nm}^{(p_1 - E, p_2 - E)}(y, \bar{y}) \\
= \sum_{0 \leq m \leq n \leq E} \sum_{\Delta, \ell} a_{nm, \Delta, \ell} P_{nm}^{(p_1 - E, p_2 - E)}(y, \bar{y}) G_{\Delta}^{(\ell)}(u, v; p_2 - p_1, p_4 - p_3),
\]

(3.52)

where \( G_{\Delta}^{(\ell)} \) are described in the introduction in (2.55) and \( P_{nm}^{(a, b)}(y, \bar{y}) \) are symmetric polynomials of degree \( n \) (i.e. for an expansion in terms of the form \( (y\bar{y})^s(y^t + \bar{y}^t), s + t \leq n \)) which are discussed in appendix B and which are given in terms of Jacobi polynomials

\[
P_{nm}^{(a, b)}(y, \bar{y}) = \frac{P_{n+1}^{(a, b)}(y)P_{m}^{(a, b)}(\bar{y}) - P_{m}^{(a, b)}(y)P_{n+1}^{(a, b)}(\bar{y})}{y - \bar{y}} = -P_{m-1, n+1}^{(a, b)}(y, \bar{y}),
\]

(3.53)

In (3.52) \( a_{nm, \Delta, \ell} \) then corresponds to the presence of an operator in the operator product expansion if \( \varphi^{(p_1)} \) and \( \varphi^{(p_2)} \) belonging to the \( SU(4)_R \) representation with Dynkin labels \( [n - m, p_1 + p_2 - 2E + 2m, n - m] \) and with scale dimension \( \Delta \), spin \( \ell \). The expansion (3.52) also extends to \( p_1 < E \) save that then \( m \geq E - p_1 \) and \( P_{nm}^{(a, b)}(y, \bar{y}) \propto \tau^{E - p_1} \).

To take account of the constraint (3.50) we write

\[
f(z, y) = k + (y - z)\hat{f}(z, y),
\]

(3.54)

with \( \hat{f}(z, y) \) a free function, polynomial in \( y \) of degree \( p - 1 \). In terms of this new function \( \hat{f}(z, y) \) we can write \( \hat{G}(u, v; y, \bar{y}) \) as

\[
\hat{G}(u, v; y, \bar{y}) = k + \frac{(y - z)(\bar{y} - \bar{z})(\bar{y} - \bar{z})f(z, \bar{y}) + (y - z)(\bar{y} - \bar{z})(y - \bar{z})f(\bar{z}, y)}{(z - \bar{z})(y - \bar{y})}
\]

\[
- \frac{(y - \bar{z})(\bar{y} - z)(y - z)\hat{f}(z, y) + (y - \bar{z})(\bar{y} - z)(\bar{y} - \bar{z})f(\bar{z}, \bar{y})}{(z - \bar{z})(y - \bar{y})}
\]

\[
+ (y - \bar{z})(y - \bar{z})(\bar{y} - z)(\bar{y} - \bar{z})K(u, v; \sigma, \tau).
\]

(3.55)
The decomposition of $\hat{G}(u,v; y, \bar{y})$ into the contributions for different possible $SU(4)_R$ representations is given by (3.52) where $a_{nm}$ are for this case the coefficients corresponding to the representation with Dynkin labels $[n - m, 2m, n - m]$. For this case in (3.53) $P^{(0,0)}_n(y) = P_n(y)$, conventional Legendre polynomials.

We consider the contribution of the two variable function $\mathcal{K}$ in (3.51) which is expanded, for $P_{nm}(y, \bar{y}) \equiv P^{(0,0)}_{nm}(y, \bar{y})$, as

$$\mathcal{K}(u,v; y, \bar{y}) = \sum_{0 \leq m \leq n \leq p - 2} A_{nm}(u, v) P_{nm}(y, \bar{y}),$$

(3.56)

with $\frac{1}{2}(p - 1)p$ terms. In this case the Legendre recurrence relations give

$$a_{n-2m-2}^A = \frac{(m - 1)mn(n + 1)}{(2m - 1)(2m + 1)(2n + 1)(2n + 3)} A_{nm},$$

$$a_{n-2m+2}^A = \frac{(m + 1)(m + 2)n(n + 1)}{(2m + 1)(2m + 3)(2n + 1)(2n + 3)} A_{nm},$$

$$a_{n+2m-2}^A = \frac{(m - 1)m(n + 2)(n + 3)}{(2m - 1)(2m + 1)(2n + 3)(2n + 5)} A_{nm},$$

$$a_{n+2m+2}^A = \frac{(m + 1)(m + 2)(n + 2)(n + 3)}{(2m + 1)(2m + 3)(2n + 3)(2n + 5)} A_{nm},$$

$$a_{n-2m-1}^A = -\frac{2mn(n + 1)}{(2m + 1)(2n + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n-2m+1}^A = -\frac{2(m + 1)n(n + 1)}{(2m + 1)(2n + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n-1m-2}^A = -\frac{2(m - 1)m(n + 1)}{(2m - 1)(2m + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n-1m+2}^A = -\frac{2(m + 1)(m + 2)(n + 1)}{(2m + 1)(2m + 3)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n+2m-1}^A = -\frac{2m(n + 2)(n + 3)}{(2m + 1)(2n + 3)(2n + 5)} \frac{1 - v}{u} A_{nm},$$

$$a_{n+2m+1}^A = -\frac{2(m + 1)(n + 2)(n + 3)}{(2m + 1)(2n + 3)(2n + 5)} \frac{1 - v}{u} A_{nm},$$

$$a_{n+1m-2}^A = -\frac{2(m - 1)m(n + 2)}{(2m - 1)(2m + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n+1m+2}^A = -\frac{2(m + 1)(m + 2)(n + 2)}{(2m + 1)(2m + 3)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n-1m-1}^A = \frac{4m(n + 1)}{(2m + 1)(2n + 3)} \frac{(1 - v)^2}{u^2} A_{nm},$$

50
\begin{align*}
a^A_{n-1m+1} &= \frac{4(m+1)(n+1)}{(2m+1)(2n+3)} \frac{(1-v)^2}{u^2} A_{nm}, \\
a^A_{n+1m-1} &= \frac{4m(n+2)}{(2m+1)(2n+3)} \frac{(1-v)^2}{u^2} A_{nm}, \\
a^A_{n+1m+1} &= \frac{4(m+1)(n+2)}{(2m+1)(2n+3)} \frac{(1-v)^2}{u^2} A_{nm}, \\
a^A_{n-m-2} &= \frac{2n(n+1)}{(2n+1)(2n+3)} B_m A_{nm}, \\
a^A_{n-m-2} &= \frac{2(m-1)m}{(2m-1)(2m+1)} B_{n+1} A_{nm}, \\
a^A_{n-1m} &= -\frac{4(n+1)}{2n+3} B_m \frac{1-v}{u} A_{nm}, \\
a^A_{n-1m} &= -\frac{4m}{2m+1} B_{n+1} \frac{1-v}{u} A_{nm}, \\
a^A_{nm} &= 4B_m B_{n+1} A_{nm},
\end{align*}

where

\begin{equation}
B_m = \frac{1+v}{u} - \frac{m^2 + m - 1}{(2m-1)(2m+3)}.
\end{equation}

For \( m = n, n-1, n-2, n-3 \), and also if \( n = 0, 1, 2 \), (3.53) may be used to combine terms to ensure that we only have \( a^A_{nm} \) for \( 0 \leq m' \leq n' \). For \( m = n = 0 \) this prescription gives

\begin{align*}
a_{22} &= \frac{4}{15} A_{00}, \\
a_{21} &= -\frac{4}{5} \frac{1-v}{u} A_{00}, \\
a_{20} &= \frac{4}{15} \left( 3 \frac{1+v}{u} - 1 \right) A_{00}, \\
a_{11} &= \frac{4}{15} \left( 10 \frac{(1-v)^2}{u^2} - 5 \frac{1+v}{u} + 1 \right) A_{00}, \\
a_{10} &= -\frac{4}{3} \left( 2 \frac{1+v}{u} - 1 \right) \frac{1-v}{u} A_{00}, \\
a_{00} &= \frac{4}{15} \left( 15 \frac{(1+v)^2}{u^2} - 5 \frac{(1-v)^2}{u^2} - 8 \frac{1+v}{u} + 1 \right) A_{00},
\end{align*}

which is equivalent to the results in [5]. Similarly for \( n = 1, m = 0, 1 \) the resulting \( a^A_{nm} \) correspond to those in [9].

To analyse the contributions arising from the function \( \hat{f}(z,y) \) this may be expanded as

\begin{equation}
\hat{f}(z,y) = \sum_{n=0}^{p-1} f_n(z) P_n(y).
\end{equation}

Using this in (3.54) and (3.51) then \( f_n \) gives rise to the following contributions to \( a_{nm} \) just
for \( m = 0, 1, \)

\[
a_{n+1}^n = \frac{(n+1)(n+2)}{(2n+1)(2n+3)} F_{nm}(z, \bar{z}), \quad a_{n-3}^n = \frac{(n-1)n}{(2n-1)(2n+1)} F_{nm}(z, \bar{z}),
\]

\[
a_{nm}^n = -\frac{n+1}{2n+1} (z + \bar{z}) F_{nm}(z, \bar{z}), \quad a_{n-2}^n = -\frac{n}{2n+1} (z + \bar{z}) F_{nm}(z, \bar{z}), \quad (3.61)
\]

\[
a_{n-1}^n = \left( z \bar{z} + \frac{1}{2} + \frac{1}{2(2n-1)(2n+3)} \right) F_{nm}(z, \bar{z}),
\]

where

\[
F_{n1}(z, \bar{z}) = -\frac{f_n(z) - f_n(\bar{z})}{z - \bar{z}}, \quad F_{n0}(z, \bar{x}) = \frac{zf_n(z) - \bar{z}f_n(\bar{z})}{z - \bar{z}}.
\]

(3.62)

For low \( n \) the results need to be modified but these can be obtained from (3.61) by taking into account the symmetry relation in (3.53). For \( n = 0 \), \( a_{11}^0, a_{10}^0 \) are as in (3.61) but for \( a_{00}^0 \) we need to take

\[
a_{00}^0 - a_{-11}^0 \to a_{00}^0 = -\frac{(z^2 - \frac{1}{3})f_0(z) - (\bar{z}^2 - \frac{1}{3})f_0(\bar{z})}{z - \bar{z}},
\]

(3.63)

while for \( n = 1 \), \( a_{21}^1, a_{20}^1, a_{11}^1, a_{10}^1 \) are given by (3.61) but

\[
a_{00}^1 = \frac{\bar{z}(z^2 - \frac{1}{3})f_1(z) - z(\bar{z}^2 - \frac{1}{3})f_1(\bar{z})}{z - \bar{z}} + \frac{4}{15} F_{11}.
\]

(3.64)

3.4. Long Operators

The solution of the superconformal identities given by (3.57) may now be naturally interpreted in terms of the operator product expansion. If in (3.57) we consider a single conformal partial wave for \( A_{nm} \) by letting

\[
A_{nm} \to G_{\Delta+4}^{(\ell)},
\]

then, if \( A_{[q,p,q],\ell}^{\Delta} \) denotes a long superconformal multiplet whose lowest state has spin \( \ell \), scale dimension \( \Delta \) and which belongs to a \( SU(4)_R \) representation with Dynkin labels \( [q, p, q] \), we obtain

\[
a_{n'm'}^{A_{nm}} \to a_{n'm'}^{A_{nm}^{\Delta}} \equiv A_{n',m'}^{\Delta}(A_{nm}^{\Delta}, \ell), \quad A_{nm,\ell}^{\Delta} \equiv A_{[n-m,2m,n-m],\ell}^{\Delta}.
\]

(3.66)
The non zero results obtained from (3.57) with (3.65) may be conveniently expressed in the form

$$a_{n+i m+j} \left( A_{nm,\ell}^\Delta \right) = N_{n+1,j} N_{m,j} A_{|i| |j|}^{nm}, \quad i, j = \pm 2, \pm 1, 0,$$

(3.67)

for

$$N_{m,2} = \frac{(m + 1)(m + 2)}{(2m + 1)(2m + 3)}, \quad N_{m,1} = \frac{m + 1}{2m + 1}, \quad N_{m,0} = 1,$$

(3.68)

and using (2.67) we have

$$a_{n+i m+j} \left( A_{nm,\ell}^\Delta \right) = \sum_{(\Delta'; \ell')} b(\Delta'; \ell') G_{\Delta'}^\ell \, \ell' \Delta,$$

(3.69)

In consequence $a_{n'm'}(A_{nm,\ell}^\Delta)$ corresponds to the contribution in the operator product expansion applied to the correlation function for all expected operators belonging to $A_{nm,\ell}^\Delta$.

In (3.69) $b(\Delta'; \ell') > 0$ if $\Delta > \ell + 1$. If $m \leq n \leq m + 3$ the results are modified since we then obtain from (3.57) contributions with $m' > n'$. In this case, for $n' \geq m'$ and $a_{m'−1 n'+1}(A_{nm,\ell}^\Delta)$ non zero, we should take

$$a_{n'm'}(A_{nm,\ell}^\Delta) - a_{m'−1 n'+1}(A_{nm,\ell}^\Delta) \rightarrow a_{n'm'}(A_{nm,\ell}^\Delta).$$

(3.70)

Furthermore any contribution with $m' = n' + 1$ should be dropped. Using this result and (3.70) we may then easily show that

$$a_{n'm'}(A_{nm,\ell}^\Delta) = 0,$$

(3.71)

and for later reference we also note the symmetry relation

$$a_{n'm'}(A_{nm,\ell}^\Delta) = a_{m'−1 n'+1}(A_{m-1 n+1,\ell}^\Delta).$$

(3.72)
The unitarity condition for a long multiplet \( A_{nm,\ell}^\Delta \) requires
\[
\Delta \geq 2n + \ell + 2,
\] (3.73)
and so, as in (2.72), using (3.65) we must have for \( u \sim 0 \),
\[
A_{nm}(u, v) \sim u^{n+3+\epsilon}, \quad \epsilon \geq 0.
\] (3.74)

3.5. Semi-Short and Non-Unitary Operators

To analyse the contribution of the single variable functions \( f_n \) in (3.60) we use the result (2.73) for the conformal partial wave for twist two operators as well as
\[
\mathcal{G}_\ell^{(\ell)}(u, v)\big|_{\Delta_1=\Delta_2=\Delta_3=\Delta_4} = \frac{1}{2} \frac{\bar{z} g_\ell(x) - z g_\ell(\bar{x})}{z - \bar{z}},
\] (3.75)
with \( g_\ell \) as in (2.74), for twist zero. Taking
\[
f_{n+1}(z) \to \frac{1}{2} g_{\ell+2}(x), \quad n = 0, 1, 2, \ldots,
\] (3.76)
in (3.62) and (3.60) then leads to results corresponding to only twist zero and twist two operators. These operators can be interpreted as belonging to a multiplet \( D_{n0,\ell} \), where in general we denote by \( D_{n,\ell} \equiv D_{[n-m,2m,n-m],\ell} \) the semi-short supermultiplet in which the lowest dimension operator has \( \Delta = 2m+\ell \), or twist \( 2m \), and belongs to the \([n-m,2m,n-m] \) \( SU(4)_R \) representation. These non unitary super multiplets are discussed in appendix D.

For \( D_{n,\ell} \) the conformal partial waves may be expressed in general in the form
\[
a_{n+i m+j}(D_{n,\ell}) = N_{n+1,i} N_{m,j} D_{n+i m+j}^{nm}, \quad D_{[i]2}^{nm} = 0.
\] (3.77)

Corresponding to (3.76) we then have
\[
D_{21}^{n0} = \frac{1}{4} \mathcal{G}_{\ell+3}^{(\ell+1)}, \quad D_{20}^{n0} = \frac{1}{4} \left( \mathcal{G}_{\ell+2}^{(\ell)} + a_{\ell+2} \mathcal{G}_{\ell+4}^{(\ell+2)} \right),
\]
\[
D_{11}^{n0} = \mathcal{G}_{\ell+2}^{(\ell+2)} + \frac{1}{4} \left( \mathcal{G}_{\ell+2}^{(\ell)} + a_{\ell+2} \mathcal{G}_{\ell+4}^{(\ell+2)} \right),
\]
\[
D_{10}^{n0} = \mathcal{G}_{\ell+1}^{(\ell+1)} + a_{\ell+2} \mathcal{G}_{\ell+3}^{(\ell+3)} + \frac{1}{4} \left( \mathcal{G}_{\ell+1}^{(\ell-1)} + b_\ell \mathcal{G}_{\ell+3}^{(\ell+1)} + a_{\ell+2} a_{\ell+3} \mathcal{G}_{\ell+5}^{(\ell+3)} \right),
\] (3.78)
\[
D_{01}^{n0} = \mathcal{G}_{\ell+1}^{(\ell+1)} + a_{\ell+2} \mathcal{G}_{\ell+3}^{(\ell+3)} + \frac{1}{4} b_n \mathcal{G}_{\ell+3}^{(\ell+1)},
\]
\[
D_{00}^{n0} = \mathcal{G}_{\ell}^{(\ell)} + b_\ell \mathcal{G}_{\ell+2}^{(\ell+2)} + a_{\ell+2} a_{\ell+3} \mathcal{G}_{\ell+4}^{(\ell+4)} + \frac{1}{4} b_n \left( \mathcal{G}_{\ell+2}^{(\ell)} + a_{\ell+2} \mathcal{G}_{\ell+4}^{(\ell+2)} \right),
\]
whereas $a_\ell$ is as in (2.68) and
\[
    b_\ell = a_{\ell+2} + a_{\ell+1} = \frac{2\ell^2 + 6\ell + 3}{(2\ell + 1)(2\ell + 5)}.
\]

A list of relevant representations for differing dimensions contained in $D_{n,0,\ell} \equiv D_{[n,0,n],\ell}$ is listed in appendix D, the twist zero and twist two representations correspond with those necessary for (3.78). For $f_0$ these results are modified. From (3.63) only twist two contributions are required since, taking now $f_0(z) \to 2g_{\ell+3}(x)$,
\[
    a_{00}(C_{00,\ell}) = G^{(\ell+2)}_{\ell+2} + \frac{2(\ell + 2)(\ell + 3)}{3(2\ell + 3)(2\ell + 7)} G^{(\ell+2)}_{\ell+4} + a_{\ell+3}a_{\ell+4} G^{(\ell+4)}_{\ell+6},
\]
\[
    a_{10}(C_{00,\ell}) = \frac{2}{3} \left( G^{(\ell+1)}_{\ell+3} + a_{\ell+3} G^{(\ell+3)}_{\ell+5} \right),
\]
\[
    a_{11}(C_{00,\ell}) = \frac{2}{3} G^{(\ell+2)}_{\ell+4}.
\]

Here we denote by $C_{nm,\ell} \equiv C_{[n-m,2m,n-m],\ell}$ the semi-short supermultiplet in which the lowest dimension operator has $\Delta = 2n + \ell + 2$ and belongs to the $[n-m,2m,n-m]$ $SU(4)_R$ representation.

The multiplets $D_{[q,p,q],\ell}$ fail to satisfy the unitarity condition (3.73) on $\Delta$ and so their contributions as in (3.78) must be cancelled in a unitary theory. This may be achieved by a corresponding long multiplet contribution. When $\Delta = 2m + \ell$ or $\Delta = 2n + \ell + 2$ the long multiplet $A_{nm,\ell}^\Delta$ can be decomposed into semi-short multiplets resulting in
\[
    a_{n'm'}(A_{nm,\ell}^{2m+\ell}) = 16 a_{n'm'}(D_{nm,\ell}) + \frac{4(m + 1)}{2m + 1} a_{n'm'}(D_{nm+1,\ell-1}),
\]
and, at the unitarity threshold (3.73),
\[
    a_{n'm'}(A_{nm,\ell}^{2n+\ell+2}) = 16 a_{n'm'}(C_{nm,\ell}) + \frac{4(n + 2)}{2n + 3} a_{n'm'}(C_{n+1m,\ell-1}).
\]

When $n = m$ we have the special case
\[
    a_{n'm'}(A_{nn,\ell}^{2n+\ell}) = 16 a_{n'm'}(D_{nn,\ell}) + \frac{(n + 1)(n + 2)}{(2n + 1)(2n + 3)} a_{n'm'}(C_{n+1n+1,\ell-2}).
\]

The results (3.81), (3.82) and (3.83) reflect a decomposition of long multiplets at particular values of $\Delta$ as described in appendix D. From (3.81) we may obtain $a_{n'm'}(D_{nm,\ell})$
iteratively starting from (3.78). With the notation in (3.77) the results are

\[
D_{n,m}^{\ell} = \begin{cases}
\frac{1}{16} G_{2m+\ell+2}^{(\ell)}, & D_{21}^{n,m} = \frac{1}{4} G_{2m+\ell+3}^{(\ell+1)}, \\
\frac{1}{16} (G_{2m+\ell+3}^{(\ell-1)} + 4 G_{2m+\ell+3}^{(\ell+1)} + a_{m+\ell+2} G_{2m+\ell+5}^{(\ell+1)}), & \\
\frac{1}{16} (4 G_{2m+\ell+2}^{(\ell)} + a_m G_{2m+\ell+4}^{(\ell)} + 4a_m G_{2m+\ell+4}^{(\ell+1)}), & \\
\frac{1}{16} (G_{2m+\ell+3}^{(\ell-1)} + 4 G_{2m+\ell+3}^{(\ell+1)} + \frac{1}{4}a_m G_{2m+\ell+5}^{(\ell-1)} + a_{m+\ell+2} G_{2m+\ell+5}^{(\ell+1)}), & \\
\frac{1}{4} (G_{2m+\ell+2}^{(\ell+2)} + 4 G_{2m+\ell+2}^{(\ell+2)} + \frac{1}{4}a_m G_{2m+\ell+4}^{(\ell+2)} + a_{m+\ell+2} G_{2m+\ell+4}^{(\ell+2)}), & \\
\frac{1}{16} (G_{2m+\ell+2}^{(\ell-2)} + \frac{1}{4}a_m G_{2m+\ell+4}^{(\ell-2)} \\
+ 8 G_{2m+\ell+2}^{(\ell)} + (b_{m+\ell} + a_m) G_{2m+\ell+4}^{(\ell)} + \frac{1}{4}a_m G_{2m+\ell+6}^{(\ell)} \\
+ 16 G_{2m+\ell+2}^{(\ell+2)} + 8a_m G_{2m+\ell+4}^{(\ell+2)} + a_{m+\ell+2} G_{2m+\ell+6}^{(\ell+2)}), & \\
\frac{1}{16} (4 G_{2m+\ell+1}^{(\ell-1)} + 2a_m G_{2m+\ell+3}^{(\ell-1)} + \frac{1}{4}a_m a_m + G_{2m+\ell+5}^{(\ell-1)} \\
+ 16 G_{2m+\ell+1}^{(\ell+1)} + 4(b_{m+\ell} + a_m) G_{2m+\ell+3}^{(\ell+1)} + 2a_m G_{2m+\ell+5}^{(\ell+1)} \\
+ 16a_m G_{2m+\ell+3}^{(\ell+3)} + 4a_m G_{2m+\ell+3}^{(\ell+3)} G_{2m+\ell+5}), & \\
\frac{1}{16} (4 G_{2m+\ell+2}^{(\ell)} + \frac{1}{4}a_m G_{2m+\ell+4}^{(\ell)} + b_n G_{2m+\ell+4}^{(\ell)} \\
+ 4a_m G_{2m+\ell+4}^{(\ell+2)} + \frac{1}{4}a_m a_m + G_{2m+\ell+6}^{(\ell)}), & \\
\frac{1}{4} (G_{2m+\ell+1}^{(\ell+1)} + \frac{1}{4}a_m G_{2m+\ell+3}^{(\ell-1)} + b_n G_{2m+\ell+3}^{(\ell+1)} \\
+ 4a_m G_{2m+\ell+3}^{(\ell+3)} + \frac{1}{4}a_m a_m + G_{2m+\ell+6}^{(\ell+1)}), & \\
\frac{1}{16} (4a_m + 1 G_{2m+\ell+3}^{(\ell-3)} + 4 G_{2m+\ell+1}^{(\ell-1)} + (a_m + b_n) G_{2m+\ell+3}^{(\ell-1)} + \frac{1}{4}a_m + b_m G_{2m+\ell+5}^{(\ell-1)} \\
+ 16 G_{2m+\ell+1}^{(\ell+1)} + 4(b_m + b_n) G_{2m+\ell+3}^{(\ell+1)} + a_m + b_n G_{2m+\ell+5}^{(\ell+1)} \\
+ \frac{1}{4}a_m + a_m + 2a_m + G_{2m+\ell+7}^{(\ell+1)} \\
+ 16a_m + b_m G_{2m+\ell+3}^{(\ell+3)} + 4a_m G_{2m+\ell+3}^{(\ell+3)} G_{2m+\ell+5}), & \\
\frac{1}{16} a_m G_{2m+\ell+2}^{(\ell-2)} + \frac{1}{4}a_m a_m + G_{2m+\ell+4}^{(\ell-2)} + 16 G_{2m+\ell+2}^{(\ell)} + 4(a_m + b_n) G_{2m+\ell+2}^{(\ell)} \\
+ a_m (b_{m+\ell} + b_n) G_{2m+\ell+4}^{(\ell)} + \frac{1}{4}a_m + a_m + 2a_m + G_{2m+\ell+6}^{(\ell+6)} \\
+ 16b_m + b_n G_{2m+\ell+2}^{(\ell+2)} + 4a_m + b_n G_{2m+\ell+4}^{(\ell+2)} + a_m G_{2m+\ell+2} G_{2m+\ell+6} \\
+ 16a_m + b_n G_{2m+\ell+4}^{(\ell+4)}, &
\end{cases}
\]

The corresponding results for the semi-short multiplet $C_{n,m,\ell}$ may be obtained from

\[56\]
those for $\mathcal{D}_{n,m,\ell}$ given above by taking

$$a_{n,m'}(C_{n,m,\ell}) = a_{m'-1,n'+1}(\mathcal{D}_{m-1,n+1,\ell}).$$  \tag{3.85}

Using (3.72) then (3.82) easily follows from (3.81). We may also verify that (3.83) is satisfied. Combining (3.81) for $m = n + 1$ with (3.71) we may then obtain

$$a_{n,m'}(A_{n,n,\ell}) - a_{m'-1,n'+1}(A_{n,n,\ell}) = 16(a_{n,m'}(\mathcal{D}_{n,n,\ell}) - a_{m'-1,n'+1}(\mathcal{D}_{n,n,\ell}))$$

$$+ \frac{(n+1)(n+2)}{(2n+1)(2n+3)} \left( -a_{m'-1,n'+1}(C_{n+1,n+1,\ell-2}) + a_{n,m'}(C_{n+1,n+1,\ell-2}) \right),$$ \tag{3.86}

which for $n' \geq m'$, and noting the requirement (3.70), gives exactly (3.83).

In general the results from (3.85) can be expressed as

$$a_{n+m,j}(C_{n,m,\ell}) = N_{n+1,i}N_{m,j} C_{i,j}^{nm}, \quad C_{2,2}^{nm} = 0.$$ \tag{3.87}

For general $n, m$ the necessary operators are just those given in table 4 of [4]. For $m = n$ the relation (3.85) combined with (3.84) in this case and applying the corresponding results to (3.70) gives

$$C_{11}^{mn} = G_{2n+\ell+4}^{(\ell+2)},$$

$$C_{10}^{mn} = G_{2n+\ell+5}^{(\ell+1)} + \frac{1}{4} a_n G_{2n+\ell+5}^{(\ell+3)},$$

$$C_{00}^{mn} = G_{2n+\ell+2}^{(\ell)} + \frac{1}{4} a_n G_{2n+\ell+4}^{(\ell+2)} + (b_{n+\ell+1} - a_{n+1}) G_{2n+\ell+4}^{(\ell+2)},$$

$$+ \frac{1}{16} a_n a_{n+1} G_{2n+\ell+6}^{(\ell+2)} + \frac{1}{4} a_n a_{n+1} + 3 G_{2n+\ell+6}^{(\ell+2)} + a_{n+\ell+3} a_{n+\ell+4} G_{2n+\ell+6}^{(\ell+4)},$$

$$C_{m-1}^{mn} = \frac{1}{4} G_{2n+\ell+4}^{(\ell)} + \frac{1}{4} a_n G_{2n+\ell+4}^{(\ell+2)} + \frac{1}{4} a_{n+\ell+3} G_{2n+\ell+6}^{(\ell+2)},$$

$$C_{m-1}^{mn} = \frac{1}{4} G_{2n+\ell+3}^{(\ell-1)} + G_{2n+\ell+3}^{(\ell+1)} + \frac{1}{4} a_n a_{n+1} G_{2n+\ell+5}^{(\ell+1)}$$

$$+ \frac{1}{16} a_n a_{n+1} a_{n+\ell+3} G_{2n+\ell+7}^{(\ell+1)} + \frac{1}{4} a_n a_{n+\ell+3} G_{2n+\ell+4}^{(\ell+3)},$$

$$C_{-1}^{mn} = \frac{1}{16} G_{2n+\ell+4}^{(\ell+2)} + \frac{1}{4} G_{2n+\ell+4}^{(\ell+2)} + \frac{1}{16} (b_{n+\ell+1} - a_{n+2}) G_{2n+\ell+6}^{(\ell)}$$

$$+ \frac{1}{4} a_n a_{n+\ell+3} G_{2n+\ell+6}^{(\ell+2)} + \frac{1}{16} a_n a_{n+\ell+3} G_{2n+\ell+4}^{(\ell+2)},$$

$$C_{1-2}^{mn} = \frac{1}{4} G_{2n+\ell+5}^{(\ell+1)},$$

$$C_{0-2}^{mn} = \frac{1}{4} G_{2n+\ell+4}^{(\ell)} + \frac{1}{16} a_n a_{n+1} G_{2n+\ell+6}^{(\ell+2)} + \frac{1}{4} a_n a_{n+\ell+3} G_{2n+\ell+6}^{(\ell+2)},$$

$$C_{-1}^{mn} = \frac{1}{16} G_{2n+\ell+5}^{(\ell+1)} + \frac{1}{4} G_{2n+\ell+5}^{(\ell+1)} + \frac{1}{16} a_n a_{n+\ell+3} G_{2n+\ell+7}^{(\ell+1)},$$

$$C_{-2}^{mn} = \frac{1}{16} G_{2n+\ell+6}^{(\ell+1)}.$$ \tag{3.88}
The necessary operators correspond exactly to those listed in [4] (see table 3) as present in the semi-short supermultiplet for this case. For \( n = 0 \) (3.88) reproduces (2.73). We may also note that, since for \( m \geq 1, \frac{1}{4} < a_m \leq \frac{1}{3} \) and \( b_n > \frac{1}{2} \), all coefficients in (3.88) are positive as required by unitarity.

3.6. Short Operators

As in the \( \mathcal{N} = 2 \) case the semi-short results also include the contributions for short BPS multiplets when extended to negative \( \ell \). Formally as shown in [4] \( C_{[q,p,q],-1} \simeq B_{[q+1,p,q+1]} \) where \( B_{[q,p,q]} \) denotes the BPS supermultiplet whose lowest state has spin zero, \( \Delta = 2q+p \), and belongs to the \( SU(4)_R \) \([q,p,q]\) representation. For \( q > 0 \) the lowest state is annihilated by \( \frac{1}{4} \) of the \( Q \) and also \( \bar{Q} \) supercharges whereas when \( q = 0 \) we have a \( \frac{1}{2} \)-BPS multiplet with \( \frac{1}{2} \) the \( Q \) and \( \bar{Q} \) supercharges annihilating the lowest state. As earlier we identify, for \( n \geq m, B_{n m} \equiv B_{[n-m,2m,n-m]} \) and we then have

\[
a_{n'm'}(C_{n m, -1}) = \frac{n+1}{2n+1} a_{n'm'}(B_{n+1 m}),
\]

(3.89)

where

\[
a_{n+i m+j}(B_{n m}) = N_{n+1,i} N_{m,j} B_{i j}^{n m}, \quad B_{2 j}^{n m} = B_{1 j}^{n m} = 0.
\]

(3.90)

For general \( n, m \) we have

\[
B_{i j}^{n m} = B_{1 j}^{n m},
\]

(3.91)

and

\[
B_{02}^{n m} = \frac{1}{4} G_{2n+2}^{(0)},
\]

\[
B_{-22}^{n m} = \frac{1}{16} G_{2n+4}^{(0)},
\]

\[
B_{-12}^{n m} = \frac{1}{4} G_{2n+3}^{(1)},
\]

\[
B_{01}^{n m} = G_{2n+1}^{(1)} + \frac{1}{4} a_{n+1} G_{2n+3}^{(1)},
\]

\[
B_{n 1}^{n m} = \frac{1}{4} G_{2n+3}^{(1)} + \frac{1}{16} a_{n+2} G_{2n+5}^{(1)},
\]

\[
B_{-11}^{n m} = \frac{1}{4} G_{2n+2}^{(2)} + \frac{1}{16} a_n G_{2n+4}^{(2)} + \frac{1}{4} a_{n+2} G_{2n+4}^{(2)},
\]

\[
B_{00}^{n m} = G_{2n}^{(0)} + \frac{1}{4} (b_{n-1} - a_n) G_{2n+2}^{(0)} + a_{n+1} G_{2n+2}^{(2)} + \frac{1}{16} a_n a_{n+1} G_{2n+4}^{(0)},
\]

\[
B_{-20}^{n m} = \frac{1}{4} G_{2n+2}^{(0)} + \frac{1}{16} (b_{n-1} - a_n+1) G_{2n+4}^{(0)} + \frac{1}{4} a_{n+2} G_{2n+4}^{(2)} + \frac{1}{64} a_n a_{n+1} a_{n+2} G_{2n+6}^{(0)},
\]

\[
B_{-10}^{n m} = G_{2n+1}^{(1)} + \frac{1}{4} b_{n-1} G_{2n+3}^{(1)} + a_{n+2} G_{2n+3}^{(3)} + \frac{1}{16} a_n a_{n+2} G_{2n+5}^{(1)}.
\]

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Again all coefficients are positive and the necessary operators are exactly as expected for this supermultiplet (see table 2 in [4]). For $n = m + 1$ the multiplet is truncated with, in (3.90), the following non zero,

\[
\begin{align*}
B_{01}^{m+1} &= \mathcal{G}_{2m+3}^{(1)}, \\
B_{00}^{m+1} &= \mathcal{G}_{2m+2}^{(0)} + \frac{1}{4}a_m \mathcal{G}_{2m+4}^{(0)} + a_{m+2} \mathcal{G}_{2m+4}^{(2)}, \\
B_{-10}^{m+1} &= \mathcal{G}_{2m+3}^{(1)} + \frac{1}{4}a_m \mathcal{G}_{2m+5}^{(1)} + a_{m+3} \mathcal{G}_{2m+5}^{(3)}, \\
B_{0-1}^{m+1} &= \mathcal{G}_{2m+3}^{(1)} + \frac{1}{4}a_m \mathcal{G}_{2m+5}^{(1)}, \\
B_{-1-1}^{m+1} &= \frac{1}{4} \mathcal{G}_{2m+4}^{(0)} + \mathcal{G}_{2m+4}^{(2)} + \frac{1}{16}a_{m+1} \mathcal{G}_{2m+6}^{(0)} + \frac{1}{4}a_{m+3} \mathcal{G}_{2m+4}^{(2)}, \\
B_{-2-2}^{m+1} &= \frac{1}{4} \mathcal{G}_{2m+5}^{(1)} + \frac{1}{16}a_{m+3} \mathcal{G}_{2m+7}^{(1)} \\
B_{0-2}^{m+1} &= \frac{1}{4} \mathcal{G}_{2m+4}^{(2)}, \\
B_{-1-2}^{m+1} &= \frac{1}{4} \mathcal{G}_{2m+5}^{(1)}, \\
B_{-2-2}^{m+1} &= \frac{1}{16} \mathcal{G}_{2m+6}^{(0)} . 
\end{align*}
\]

(3.93)

The necessary operators correlate again with those expected for this $\frac{1}{4}$-BPS multiplet (see table 5 in [4]).

If we consider the semi-short multiplet for $\ell = -2$ we get

\[
a_{n'm'}(C_{n,m-2}) = -4a_{n'm'}(B_{n,m}),
\]

(3.94)

which allows results for $a_{n'm'}(B_{n,m})$ to be derived for $m = n$ in addition to $m < n$ as given by (3.89). However in this case there is a further decomposition into contributions corresponding to $\frac{1}{2}$-BPS multiplets. Such $\frac{1}{2}$-BPS contributions are obtained in (3.90) by letting $B_{nm} \rightarrow \hat{B}_{nn}$ and $B_{nm}^{nn} \rightarrow \hat{B}_{ij}^{nn}$ where

\[
\begin{align*}
\hat{B}_{00}^{nn} &= \mathcal{G}_{2n}^{(0)}, \\
\hat{B}_{01}^{nn} &= \mathcal{G}_{2n+1}^{(1)}, \\
\hat{B}_{-10}^{nn} &= \mathcal{G}_{2n+2}^{(0)}, \\
\hat{B}_{-11}^{nn} &= \frac{1}{4} \mathcal{G}_{2n+3}^{(1)}, \\
\hat{B}_{-2-2}^{nn} &= \frac{1}{16} \mathcal{G}_{2n+4}^{(0)}, \\
\hat{B}_{-2-2}^{nn} &= \frac{1}{16} \mathcal{G}_{2n+4}^{(0)},
\end{align*}
\]

(3.95)

(the relevant operators here correspond to table 1 in [4]). With the result given in (3.95) we can then write in (3.94)

\[
a_{n'm'}(B_{n}n) = a_{n'm'}(\hat{B}_{n}n) - \frac{(n+1)(n+2)}{4(2n+1)(2n+3)} a_{n'm'}(\hat{B}_{n+1}n+1).
\]

(3.96)

From (2.83) and (3.99) it is also easy to see that

\[
a_{nm}(\hat{B}_{00}) = a_{nm}(I).
\]

(3.97)

Any $\frac{1}{2}$-BPS contribution $a_{n'm'}(\hat{B}_{n}n)$ may then be isolated by considering appropriate linear combinations of $a_{n'm'}(C_{n}n,-2)$ together with $a_{n'm'}(I)$.
3.7. Identity Operator

We consider the contribution resulting from the constant $k$ in (3.51) and (3.54). It is easy to see that this gives only

$$a_{00}^k = k. \quad (3.98)$$

The constant $k$ clearly corresponds to the identity operator,

$$a_{nm}(I) = \delta_{n0}\delta_{m0}. \quad (3.99)$$

3.8. (Next-to-) Extremal Case

We also consider the extremal and next-to-extremal cases. When $E = 0$ $G$ is independent of $y, \bar{y}$ and so must also be the function $f$ in (3.49). From (3.50) and (2.87) we then get the solution

$$G(u, v) = u^{\frac{1}{2}p^+} k, \quad (3.100)$$

where we define

$$p_\pm = p_2 \pm p_1. \quad (3.101)$$

Noting that

$$P_{00}^{(p_1, p_2)}(y, \bar{y}) = \frac{1}{2}(p_+ + 2), \quad G_{p_+}^{(0)}(u, v; p_-, p_+) = u^{\frac{1}{2}p^+}, \quad (3.102)$$

it is clear that the only operator which is necessary in the operator product expansion has $\Delta = p_+$ and is spinless belonging to the $[0, p_+, 0]$ representation. This is of course may be identified with the contribution of just the $\frac{1}{2}$-BPS operator belonging to the short $\mathcal{B}_{[0, p_+, 0]}$ supermultiplet so that for the extremal case, up to a constant factor,

$$a_{nm}(\mathcal{B}_{[0, p_+, 0]}) = \delta_{n0}\delta_{m0} G_{p_+}^{(0)}. \quad (3.103)$$

The correlation function in this case has the very simple form

$$\langle \phi^{(p_1)}(x_1, t_1) \phi^{(p_2)}(x_2, t_2) \phi^{(p_3)}(x_3, t_3) \phi^{(p_4)}(x_4, t_4) \rangle |_{p_4 = p_1 + p_2 + p_3} = \frac{(t_1 \cdot t_4)^{p_1}(t_2 \cdot t_4)^{p_2}(t_1 \cdot t_3)^{p_3}}{r_1^{p_1} r_2^{p_2} r_3^{p_3}} k. \quad (3.104)$$
For the next-to-extremal case, $E = 1$, we have a similar solution to that given by (3.51) and (3.54), but with no arbitrary $K$ term and $\hat{f}$ a single variable function of $z$,

$$\hat{G}(u, v; y, \bar{y}) = u^{\frac{1}{2}p} (k - \frac{1}{z - \bar{z}} ((y - z)(\bar{y} - z)\hat{f}(z) - (y - \bar{z})(\bar{y} - \bar{z})\hat{f}(\bar{z})))$$

$$= \sum_{0 \leq m \leq n \leq 1} a_{nm}(u, v) P^{(p_{n+1}, p_{2n-1})}_{nm}(y, \bar{y}),$$

(3.105)

where we have expanded in terms of the different possible $SU(4)_R$ representations. From this we obtain

$$\frac{1}{16} p_{+} (p_{+} + 1)(p_{+} + 2) a_{11} = \hat{a}_{11} = F_{0},$$

$$\frac{1}{8} (p_{+} + 1)(p_{+} + 2) a_{10} = \hat{a}_{10} = F_{1} + \frac{p_{-}}{p_{+}} F_{0},$$

$$\frac{1}{2} p_{+} a_{00} = \hat{a}_{00} = k u^{\frac{1}{2}p_{+} - 1} + F_{2} + \frac{2p_{-}}{p_{+} + 2} F_{1} + \frac{p_{-} - (p_{+} + 2)}{(p_{+} + 1)(p_{+} + 2)} F_{0},$$

(3.106)

for

$$F_{n}(z, \bar{z}) = (-1)^{n} u^{\frac{1}{2}p_{+} - 1} z^{n} \frac{\hat{f}(z) - \bar{z}^{n} \hat{f}(\bar{z})}{z - \bar{z}}.$$

(3.107)

Keeping only the term in (3.106) involving $k$ we may easily from (3.102) see that this represents the contribution of just the $\frac{1}{2}$-BPS chiral primary operator belonging to the $\mathcal{B}_{[0, p_{+} - 2, 0]}$ supermultiplet so that in the next-to-extremal case we have

$$\hat{a}_{nm} (\mathcal{B}_{[0, p_{+} - 2, 0]}) = \delta_{n0} \delta_{m0} g_{p_{+} - 2}^{(0)}.$$

(3.108)

If in (3.106) and (3.107) we let $\hat{f}(z) \rightarrow 2g_{\ell+3}(x; p_{1}, p_{2})$ and use the definitions in (2.92) we obtain the contributions for the semi-short supermultiplet $\mathcal{C}_{[0, p_{+} - 2, 0], \ell}$,

$$\hat{a}_{11} (\mathcal{C}_{[0, p_{+} - 2, 0], \ell}) = g_{p_{+} + \ell + 2}^{(\ell + 2)},$$

$$\hat{a}_{10} (\mathcal{C}_{[0, p_{+} - 2, 0], \ell}) = g_{p_{+} + \ell + 1}^{(\ell + 1)} + b_{\ell+2} g_{p_{+} + \ell + 3}^{(\ell + 3)} + \frac{4(\ell + 2)p_{-}(p_{+} + \ell + 1)}{p_{+}(p_{+} + 2\ell + 2)(p_{+} + 2\ell + 4)} g_{p_{+} + \ell + 2}^{(\ell + 2)},$$

$$\hat{a}_{00} (\mathcal{C}_{[0, p_{+} - 2, 0], \ell}) = g_{p_{+} + \ell}^{(\ell)} + b_{\ell+2} b_{\ell+3} g_{p_{+} + \ell + 4}^{(\ell + 4)} + c_{\ell+2} g_{p_{+} + \ell + 2}^{(\ell + 2)} + \frac{8(\ell + 1)p_{-}(p_{+} + \ell + 1)}{(p_{+} + 2)(p_{+} + 2\ell)(p_{+} + 2\ell + 4)} g_{p_{+} + \ell + 1}^{(\ell + 1)} + \frac{8(\ell + 2)p_{-}(p_{+} + \ell + 2) b_{\ell+2}}{(p_{+} + 2)(p_{+} + 2\ell + 2)(p_{+} + 2\ell + 6)} g_{p_{+} + \ell + 3}^{(\ell + 3)},$$

(3.109)
for
\[
b_\ell = \frac{4(\ell + 1)(p_1 + \ell)(p_2 + \ell)(p_+ + \ell - 1)}{(p_+ + 2\ell - 1)(p_+ + 2\ell)^2(p_+ + 2\ell + 1)},
\]
\[
c_\ell = \frac{2\ell(p_+ + \ell - 1)}{(p_+ + 1)(p_+ + 2\ell - 3)(p_+ + 2\ell + 1)} \left( p_+ - 1 + \frac{p^2(8(\ell - 1)(p_+ + \ell) - p_+(p_+ - 1))}{(p_+ + 2)(p_+ + 2\ell - 2)(p_+ + 2\ell)} \right).
\]

The necessary operators required for (3.109) correspond exactly with those in this semi-short supermultiplet (see table 3 in [4]).

Just as previously we may extend these (3.109) to \( \ell = -1, -2 \) to obtain results for short multiplets. Thus
\[
\hat{a}_{nm}(C_{[0,p_+ - 2,0],-1}) = \hat{a}_{nm}(B_{[1,p_+ - 2,1]}),
\]
\[
\hat{a}_{nm}(C_{[0,p_+ - 2,0],-2}) = \hat{a}_{nm}(B_{[0,p_+ ,0]}) - 4\hat{a}_{nm}(B_{[0,p_+ - 2,0]}),
\]
where, together with (3.108),
\[
\hat{a}_{11}(B_{[1,p_+ - 2,1]}) = G_{p_+ + 1}^{(1)},
\]
\[
\hat{a}_{10}(B_{[1,p_+ - 2,1]}) = G_{p_+}^{(0)} + \frac{4p_-(p_+ + 2)}{p_+(p_+ + 2)} G_{p_+ + 1}^{(1)} + b_1 G_{p_+ + 2}^{(2)},
\]
\[
\hat{a}_{00}(B_{[1,p_+ - 2,1]}) = b_1 \left( G_{p_+ + 1}^{(1)} + \frac{8p_-(p_+ + 2)}{p_+(p_+ + 2)(p_+ + 4)} G_{p_+ + 2}^{(2)} + b_2 G_{p_+ + 3}^{(3)} \right),
\]
and
\[
\hat{a}_{11}(B_{[0,p_+ ,0]}) = G_{p_+}^{(0)}, \quad \hat{a}_{10}(B_{[0,p_+ ,0]}) = b_0 G_{p_+ + 1}^{(1)}, \quad \hat{a}_{00}(B_{[0,p_+ ,0]}) = b_0 b_1 G_{p_+ + 2}^{(2)}. \quad (3.113)
\]
The necessary operators here correspond to table 5 and table 1 in [4].

3.9. Summary

The results obtained above show that the operator product expansion for \( \frac{1}{2} \)-BPS operators can be decomposed into short, semi-short and long supermultiplets. For \( p_- = p_2 - p_1 \geq 0, \)
\[
B_{[0,p_1,0]} \otimes B_{[0,p_2,0]} \simeq \bigoplus_{0 \leq m \leq n \leq p_1} B_{[n-m,p_- + 2m,n-m]} + \bigoplus_{\ell \geq 0} \bigoplus_{0 \leq m \leq n \leq p_1 - 1} C_{[n-m,p_- + 2m,n-m],\ell} \bigoplus_{\ell \geq 0} \bigoplus_{0 \leq m \leq n \leq p_1 - 2} A_{[n-m,p_- + 2m,n-m],\ell}, \quad (3.114)
\]

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in accordance with the results of Eden and Sokatchev [33]. In (3.114) we identify
$B_{[0,0,0]} \simeq I$, corresponding to the unit operator in the operator product expansion. It
immediately follows from (3.114) that long supermultiplets, with non zero anomalous di-
mensions, cannot contribute to extremal and next-to-extremal correlation functions.
4. Crossing Symmetry

The operator product expansion provides the strongest constraints when combined with crossing symmetry. For a correlation function for four identical chiral primary operators the correlation function is invariant under permutations of all \(x_i, t_i\) for all \(i = 1, 2, 3, 4\). Permutations of the form \((ij)(kl)\) act trivially so we may restrict to permutations leaving \(x_4, t_4\) invariant so that crossing symmetry transformations correspond to the permutation group \(S_3\), which is of order 6. The action of each permutation on the essential conformal invariants \(u, v\) or \(x, \bar{x}\) or \(y, \bar{y}\) and also on the \(R\)-symmetry invariants \(\sigma, \tau\) or \(\alpha, \bar{\alpha}\) or \(y, \bar{y}\) is given in table 1, where the transformations of \(\bar{x}\) are identical to those of \(x\), and similarly for \(\bar{\alpha}_i, \bar{\alpha}\).

| \(e\) | (12) | (13) | (23) | (123) | (132) | \(e\) | (12) | (13) | (23) | (123) | (132) |
|------|------|------|------|------|------|------|------|------|------|------|------|
| \(u\) | \(u\) | \(v\) | \(\frac{1}{u}\) | \(\frac{v}{u}\) | \(\frac{1}{v}\) | \(\sigma\) | \(\tau\) | \(\frac{\sigma}{\tau}\) | \(\frac{1}{\sigma}\) | \(\frac{1}{\tau}\) | \(\frac{\tau}{\sigma}\)
| \(v\) | \(\frac{1}{v}\) | \(u\) | \(\frac{v}{u}\) | \(\frac{1}{u}\) | \(\frac{u}{v}\) | \(\tau\) | \(\sigma\) | \(\frac{1}{\tau}\) | \(\frac{\tau}{\sigma}\) | \(\frac{\sigma}{\tau}\) | \(\frac{1}{\sigma}\)
| \(x\) | \(\frac{x}{x-1}\) | \(1-x\) | \(\frac{1}{x}\) | \(\frac{x-1}{x}\) | \(\frac{1}{1-x}\) | \(\alpha\) | \(1-\alpha\) | \(\frac{\alpha}{\alpha-1}\) | \(\frac{1}{\alpha}\) | \(\frac{1}{1-\alpha}\) | \(\frac{\alpha-1}{\alpha}\)
| \(z\) | \(-z\) | \(\frac{z+3}{z-1}\) | \(\frac{3-z}{1+z}\) | \(\frac{3+z}{1-z}\) | \(\frac{z-3}{z+1}\) | \(y\) | \(-y\) | \(\frac{y+3}{y-1}\) | \(\frac{3-y}{1+y}\) | \(\frac{y+3}{y-1}\) | \(\frac{y-3}{y+1}\)

Table 1. Symmetry transformations of variables under crossing.

4.1. \(\mathcal{N} = 4\)

For the \(\mathcal{N} = 4\) case with \(p_i = p\) the crossing symmetry conditions on the correlation function \(\mathcal{G}(u, v; \sigma, \tau)\) are generated by considering just (12) and (13) which give

\[
\mathcal{G}(u, v; \sigma, \tau) = \mathcal{G}(u/v, 1/v; \tau, \sigma) = \left(\frac{u}{v}\right)^p \mathcal{G}(v, u; \sigma/\tau, 1/\tau).
\] (4.1)

The general construction of such invariant correlation functions follows by determining polynomials in \(\sigma, \tau\) which transform according to the irreducible representations of \(S_3\). We first consider symmetric polynomials satisfying

\[
S_p(\sigma, \tau) = S_p(\tau, \sigma) = \tau^p S_p(\sigma/\tau, 1/\tau).
\] (4.2)
As described by Heslop and Howe [6], for any given \( p, S_3 \) acts on the \( \frac{1}{2}(p + 1)(p + 2) \) monomials \( \sigma^r \tau^s, r + s \leq p \), giving chains of length 6 or 3 or 1 which may be added to give minimal polynomial solutions of (4.2). If the chain contains a monomial \( (\sigma \tau)^r \), for \( 0 \leq r \leq \lfloor \frac{1}{2}p \rfloor \), where \( \lfloor x \rfloor \) denotes the integer part of \( x \), then this term is invariant under the action of the permutation (12) and the chain is of length 3, except if \( p \) is divisible by 3 then \( (\sigma \tau)^{p/3} \) satisfies (4.2) by itself and so forms a chain of length 1. All other chains are of length 6. With this counting the number of independent such minimal symmetric polynomials is,

\[
N_p = \begin{cases} 
(n + 1)3n + 1, & p = 6n; \\
(n + 1)(3n + q), & p = 6n + q, \; q = 1, 2, 3, 4, 5.
\end{cases}
\]

We list the first few non trivial cases in table 2, of course \( S_0(\sigma, \tau) = 1 \).

| \( p \) | polynomial | \( (i, j) \) |
| --- | --- | --- |
| 1 | \( \sigma + \tau + 1 \) | (0, 0) |
| 2 | \( \sigma^2 + \tau^2 + 1 \) | (0, 0), (1, 0) |
| | \( \sigma \tau + \sigma + \tau \) | (0, 1) |
| 3 | \( \sigma^3 + \tau^3 + 1 \) | (0, 0), (1, 0) |
| | \( \sigma \tau^2 + \sigma^2 \tau + \sigma^2 + \tau + \sigma + \tau \) | (0, 1) |
| 4 | \( \sigma^4 + \tau^4 + 1 \) | (0, 0), (1, 0), (2, 0) |
| | \( \sigma^3 \tau + \sigma^3 \tau^2 + \sigma^3 + \tau + \sigma + \tau \) | (0, 1) |
| | \( \sigma^2 \tau^2 + \sigma^2 + \tau^2 \) | |
| | \( \sigma^2 \tau + \sigma \tau^2 + \sigma + \tau \) | |
| 5 | \( \sigma^5 + \tau^5 + 1 \) | (0, 0), (1, 0), (2, 0) |
| | \( \sigma^4 \tau + \sigma^4 + \sigma^4 + \tau^4 + \sigma + \tau \) | (0, 1), (1, 1) |
| | \( \sigma^3 \tau^2 + \sigma^3 + \tau^3 + \sigma^3 + \tau^2 + \sigma + \tau \) | |
| | \( \sigma^3 \tau + \sigma \tau^3 + \sigma + \tau \) | |
| | \( \sigma^2 \tau^2 + \sigma^2 + \tau^2 \) | |

Table 2. Symmetric polynomials.

An alternative basis for \( S_p \), valid for general \( p \), may be obtained by constructing from \( \sigma, \tau \) two invariants \( I_1, I_2 \) under \( S_3 \) and then introducing for any \( p \) a factor to ensure that (4.2) holds. With suitable restrictions the result becomes a polynomial expressible in the
form

\[ S_{p,(i,j)}(\sigma, \tau) = (\sigma + \tau + 1)^p I_1(\sigma, \tau)^i I_2(\sigma, \tau)^j, \]

\[ I_1(\sigma, \tau) = \frac{\sigma \tau + \sigma + \tau}{(\sigma + \tau + 1)^2}, \quad I_2(\sigma, \tau) = \frac{\sigma \tau}{(\sigma + \tau + 1)^3}, \quad (4.4) \]

\[ i = 0, 1, \ldots, \left[ \frac{1}{2}p \right], \quad j = 0, 1, \ldots, \left[ \frac{1}{3}(p - 2i) \right]. \]

Lists of possible \((i, j)\) for \(p\) up to 7 are given in table 2. This result may also be easily expressed as symmetric polynomial in \(y, \bar{y}\) by using

\[ \sigma + \tau + 1 = \frac{1}{2}(y \bar{y} + 3), \quad \sigma \tau = \frac{1}{16}(1 - y^2)(1 - \bar{y}^2), \]

\[ \Lambda = (\sigma + \tau + 1)^2 - 4(\sigma \tau + \sigma + \tau) = \frac{1}{4}(y - \bar{y})^2, \quad (4.5) \]

where \(\Lambda\) is defined in (2.39). Completeness of the basis provided by (4.4) is straightforwardly demonstrated by showing that it gives the same number of independent polynomials \(N_p\) as given in (4.3).

For the antisymmetric representation of \(S_3\) we require,

\[ a \rightarrow_{(12)} -a, \quad a \rightarrow_{(132)} a. \quad (4.6) \]

while the two-dimensional mixed symmetry representation of \(S_3\) is defined on a basis \((b, c)\) where

\[ \begin{pmatrix} b \\ c \end{pmatrix} \rightarrow_{(12)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}, \quad \begin{pmatrix} b \\ c \end{pmatrix} \rightarrow_{(123)} \begin{pmatrix} -1 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}. \quad (4.7) \]

It is easy to see that the tensor products formed by \(aa'\) and \(bb' + cc'\) are symmetric while \((bb' + cc', bb' + cc')\) is a basis for a mixed symmetry representation and \(bc' - cb'\) is antisymmetric.

For functions of \(\sigma, \tau\) (4.6) is satisfied by

\[ a(\sigma, \tau) = \frac{(\sigma - \tau)(\sigma - 1)(\tau - 1)}{(\sigma + \tau + 1)^3}. \quad (4.8) \]

For \(p \geq 3\), \(a(\sigma, \tau)S_{p,(i,j)}(\sigma, \tau)\) is a polynomial if we allow \(i = 0, 1, \ldots, \left[ \frac{1}{2}(p - 3) \right]\) and \(j = 0, 1, \ldots, \left[ \frac{1}{3}(p - 2i - 3) \right]\) giving \(N_{p-3}\) antisymmetric polynomials. For the mixed symmetry transformations in (4.7) there essentially two independent possibilities

\[ b_1(\sigma, \tau) = \frac{\sigma - \tau}{\sigma + \tau + 1}, \quad c_1(\sigma, \tau) = \frac{\sigma + \tau - 2}{\sqrt{3}(\sigma + \tau + 1)}. \quad (4.9) \]
and
\[ b_2(\sigma, \tau) = \frac{\sigma - \tau}{(\sigma + \tau + 1)^2}, \quad c_2(\sigma, \tau) = \frac{-\sigma + \tau - 2\sigma \tau}{\sqrt{3}(\sigma + \tau + 1)^2}. \] (4.10)

By considering \((b_r(\sigma, \tau), c_r(\sigma, \tau))S_{p,(i,j)}(\sigma, \tau)\) for \(p \geq r, r = 1, 2\), for appropriate \(i, j\) we obtain \(N_{p-r}\) polynomial mixed symmetry representations of \(S_3\). Together with the symmetric polynomials \(S_{p,(i,j)}\) and \(aS_{p,(i,j)}\) these provide a complete basis for two variable polynomials in \(\sigma, \tau\) of order \(p\) since \(N_p + 2(N_{p-1} + N_{p-2}) + N_{p-3} = \frac{1}{2}(p + 1)(p + 2)\). We may also note that these polynomials form a closed set under multiplication since
\[
\begin{align*}
 b_1^2 + c_1^2 &= \frac{4}{3} - 4I_1, \quad \sqrt{3}(2b_1c_1, b_1^2 - c_1^2) = 2(b_1 - 3b_2, c_1 - 3c_2), \\
 b_1b_2 + c_1c_2 &= \frac{2}{3}I_1 - 6I_2, \quad \sqrt{3}(b_1c_2 + c_1b_2, b_1b_2 - c_1c_2) = 2(I_1b_1 - b_2, I_1c_1 - c_2), \\
 b_2^2 + c_2^2 &= \frac{4}{3}I_1^2 - 4I_2, \quad \sqrt{3}(b_2c_2, b_2^2 - c_2^2) = 2(3I_2b_1 - I_1b_2, 3I_2c_1 - I_1c_2), \\
 \sqrt{3}(b_1c_2 - c_1b_2) &= 2a, \quad a^2 = I_1^2 - 4I_1^3 + 18I_1I_2 - 4I_2 - 27I_2^2. \tag{4.11}
\end{align*}
\]

where \(I_1, I_2\) are the invariants defined in (4.4).

The superconformal Ward identities require
\[
G(u, v; \sigma, \tau)\big|_{\alpha = \frac{u}{v}} = f(x, \alpha), \tag{4.12}
\]
so that (4.1) gives
\[
f(x, \alpha) = f\left(\frac{x}{x-1}, 1 - \alpha\right) = \left(\frac{x(\alpha - 1)}{1-x}\right)^p f\left(1-x, \frac{\alpha}{\alpha-1}\right) = (x^p f\left(\frac{1}{x}, \frac{1}{\alpha}\right). \tag{4.13}
\]

To obtain an extension to a fully crossing symmetric correlation function we may consider for any \(S_p\) satisfying (4.2)
\[
G(u, v; \sigma, \tau) = S_p\left(u \sigma, \frac{u}{v} \tau\right), \tag{4.14}
\]
which obeys (4.1) as a consequence of (4.2). From (4.12) we obtain
\[
f(x, \alpha) = S_p\left(x \alpha, \frac{x(1 - \alpha)}{x-1}\right), \tag{4.15}
\]
which automatically satisfies (4.13).

The function \(f(x, \alpha)\) is required to be a general solution of the crossing symmetry conditions given by
\[
f(x, \alpha) = f\left(\frac{x}{x-1}, 1 - \alpha\right) = \left(\frac{x(\alpha - 1)}{1-x}\right)^p f\left(1-x, \frac{\alpha}{\alpha-1}\right) = (x^p f\left(\frac{1}{x}, \frac{1}{\alpha}\right). \tag{4.16}
\]

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which is also a polynomial of degree $p$ in $\alpha$. It is also analytic in $x$ in the neighbourhood of $x = 0$ with singularities only at $x = 1, \infty$. If we write

$$f(x, \alpha) = P(x, \alpha)^p g(x, \alpha), \quad P(x, \alpha) = \frac{x^2 \alpha - 2x\alpha + 2x - 1}{x - 1}, \quad (4.17)$$

then $g$ is an invariant under the action of $S_3$, as displayed in Table 1. Determining a general form for $g$ is then reducible to finding a basis for all possible independent invariants which may be formed from $x$ and $\alpha$. Crossing symmetry in one variable $\alpha$ will be studied in the case of $N = 2$. We will need the results for the different representations of $S_3$ here. Basically they are obtained from the two variable case by setting $\bar{\alpha} = \alpha$. The symmetric or invariant representation is

$$s(\alpha) = \frac{\alpha^2 (1 - \alpha)^2}{(\alpha^2 - \alpha + 1)^3}. \quad (4.18)$$

The antisymmetric representation is given by

$$a(\alpha) = (2\alpha - 1) \frac{(\alpha - 2)(\alpha - 1)\alpha(\alpha + 1)}{(\alpha^2 - \alpha + 1)^3}. \quad (4.19)$$

The 2 mixed symmetry representation solutions are

$$b_1(\alpha) = \frac{2\alpha - 1}{\alpha^2 - \alpha + 1},$$
$$c_1(\alpha) = \frac{1}{\sqrt{3}} \frac{2\alpha^2 - 2\alpha - 1}{\alpha^2 - \alpha + 1}, \quad (4.20)$$

and

$$b_2(\alpha) = (2\alpha - 1) \frac{\alpha(\alpha - 1)}{(\alpha^2 - \alpha + 1)^2},$$
$$c_2(\alpha) = \sqrt{3} \frac{\alpha(\alpha - 1)}{(\alpha^2 - \alpha + 1)^2}. \quad (4.21)$$

Since the action of $S_3$ on any polynomial in $\alpha$ may be decomposed, up to functions of the invariant $s(\alpha)$, into contributions linear in 1, $a(\alpha)$ and $(b_r(\alpha), c_r(\alpha)), r = 1, 2$, as given in (4.18), (4.19), (4.20), (4.21), then a basis for such invariants is obtained, in addition to the separate invariants $s(x), s(\alpha)$, by combining these non trivial irreducible representations with corresponding representations involving $x$ to give

$$A(x, \alpha) = a(x^{-1}) a(\alpha), \quad (4.22)$$
where $a(x^{-1}) = -a(x)$, and also

\[ S_1(x, \alpha) = b_1(x^{-1}) b_1(\alpha) + c_1(x^{-1}) c_1(\alpha) = \frac{4}{3} - \frac{2(x\alpha - 1)^2}{(\alpha^2 - \alpha + 1)(x^2 - x + 1)} , \]
\[ S_2(x, \alpha) = b_2(x^{-1}) b_2(\alpha) + c_2(x^{-1}) c_2(\alpha) = \frac{2\alpha(1 - \alpha) x(1 - x)}{(\alpha^2 - \alpha + 1)(x^2 - x + 1)^2} (x\alpha - 2\alpha - 2x + 1) , \]
\[ S_3(x, \alpha) = b_2(x^{-1}) b_1(\alpha) + c_2(x^{-1}) c_1(\alpha) = \frac{2x(1 - x)}{(\alpha^2 - \alpha + 1)(x^2 - x + 1)^2} (x\alpha^2 - 2x\alpha + 2\alpha - 1) , \]
\[ S_4(x, \alpha) = b_1(x^{-1}) b_2(\alpha) + c_1(x^{-1}) c_2(\alpha) = \frac{2\alpha(1 - \alpha)}{(\alpha^2 - \alpha + 1)^2 (x^2 - x + 1)} (x^2\alpha - 2x\alpha + 2x - 1) . \]

These are not independent since

\[ A(x, \alpha) = \frac{4}{3} (S_1(x, \alpha) S_2(x, \alpha) - S_3(x, \alpha) S_4(x, \alpha)) , \]
\[ S_2(x, \alpha) - \frac{1}{2} S_1(x, \alpha) S_3(x, \alpha) - \frac{1}{2} S_3(x, \alpha) = -2s(x) , \]
\[ S_2(x, \alpha) - \frac{1}{2} S_1(x, \alpha) S_4(x, \alpha) - \frac{1}{2} S_4(x, \alpha) = -2s(\alpha) , \]
\[ 2(S_3(x, \alpha) + S_4(x, \alpha)) - 6S_2(x, \alpha) + S_1(x, \alpha)^2 - \frac{2}{3} S_1(x, \alpha) = \frac{8}{9} . \]

A crucial further constraint arises from (3.50) which here requires that $f(x, x^{-1})$ is a constant. Since $P(x, x^{-1}) = 3$ we also require that $g$ depends on invariants $s_r(x, \alpha)$ such that $s_r(x, x^{-1})$ are constants. Taking account of the relations in (4.24) there are then two independent solutions which we take as

\[ s_1(x, \alpha) = 2 \frac{S_3(x, \alpha) s(\alpha)}{S_4(x, \alpha)^2} = \frac{R(x, \alpha)}{P(x, \alpha)^2} , \quad s_2(x, \alpha) = 8 \frac{s(x) s(\alpha)^2}{S_4(x, \alpha)^3} = \frac{Q(x, \alpha)}{P(x, \alpha)^3} , \]
\[ R(x, \alpha) = \frac{x(x\alpha^2 - 2x\alpha + 2\alpha - 1)}{1 - x} , \quad Q(x, \alpha) = \frac{x^2 \alpha(1 - \alpha)}{x - 1} . \]

where $R(x, x^{-1}) = 3, Q(x, x^{-1}) = 1$. It is then evident that $g$ in (4.17) must be of the form

\[ g = \sum_{i,j \geq 0, 2i + 3j \leq p} c_{ij} s_1^i s_2^j . \]
Noting that

\[ P(x, \alpha) = \left( u \sigma + \frac{u}{v} \tau + 1 \right) \bigg|_{\bar{\alpha} = \frac{1}{x}}, \quad Q(x, \alpha) = \frac{u^2}{v} \sigma \tau \bigg|_{\bar{\alpha} = \frac{1}{x}}, \]

\[ R(x, \alpha) = \left( \frac{u^2}{v} \sigma \tau + u \sigma + \frac{u}{v} \tau \right) \bigg|_{\bar{\alpha} = \frac{1}{x}}, \]

it is easy to see, as a consequence of (4.4), that \( I_r(u\sigma, u\tau/v)|_{\bar{\alpha} = 1/x} = s_r(x, \alpha) \) and hence the expression given by (4.17) and (4.26) for the function \( f \) may always be extended to a fully crossing symmetric result for the full correlation function \( G \) of the form (4.14) with \( S_p(\sigma, \tau) = (\sigma + \tau + 1)^p \sum_{i,j} c_{ij} I_1(\sigma, \tau)^i I_2(\sigma, \tau)^j \) and where \( f \) satisfies (4.15). With appropriate coefficients for the independent terms in \( S_p \) (4.14) corresponds to the results of free field theory. In general, using the formalism of harmonic superspace, the Intriligator insertion technique [16] demonstrates that only \( K \) as in (2.60) or (3.51), can depend on the coupling \( g \), and so are dynamical. The functions \( f(x) \) or \( f(x, \alpha) \) are then identical with the free theory, or \( g = 0 \), results.

The remaining part of the correlation function may also be expressed in terms of \( S_3 \) representations. It is convenient to define from (4.9) and (4.10) \( (b'_r(u, v), c'_r(u, v)) = (b_r(1/u, v/u), c_r(1/u, v/u)). \) We may then write

\[
(\alpha x - 1)(\alpha x - 1)(\bar{\alpha} x - 1)(\bar{\alpha} x - 1) = \frac{1}{16} u^2 (y - z)(y - \bar{z})(\bar{y} - z)(\bar{y} - \bar{z})
\]

\[
= v + \sigma^2 uv + \tau^2 u + \sigma v(v - 1 - u) + \tau(u + v - 1) + \sigma \tau u(u - 1 - v)
\]

\[
= (\sigma + \tau + 1)^2(u + v + 1)^2 \left( \frac{1}{3} I_1(u, v) + \frac{1}{3} I_1(\sigma, \tau) - 2I_1(u, v) \right)
\]

\[
- \frac{1}{2} \left( b'_1(u, v) b_2(\sigma, \tau) + c'_1(u, v) c_2(\sigma, \tau) \right) + b'_2(u, v) b_1(\sigma, \tau) + c'_2(u, v) c_1(\sigma, \tau)
\]

\[
- 3 b'_2(u, v) b_2(\sigma, \tau) - 3 c'_2(u, v) c_2(\sigma, \tau) \right). \quad (4.28)
\]

The function \( K \) in (3.51) must then satisfy the crossing symmetry relations

\[
K(u, v; \sigma, \tau) = K(u/v, 1/v; \tau, \sigma) = \left( \frac{u}{v} \right)^{p+2} \tau^{p-2} \tau(v, u; \sigma/\tau, 1/\tau). \quad (4.29)
\]

4.2. \( N = 2 \)

For \( N = 2 \) there are further restrictions as a consequence of (2.39). Taking \( p \to 2n \) we construct, instead of (4.2) since \( \sigma, \tau \) are expressible in terms of just \( \alpha \) by (2.40), the
single variable polynomials \( f_n \) of degree 2\( n \), satisfying under the action of \( S_3 \)

\[
f_n(\alpha) = f_n(1 - \alpha) = (\alpha - 1)^2n f_n\left(\frac{\alpha}{\alpha - 1}\right) = \alpha^{2n} f_n\left(\frac{1}{\alpha}\right) = (\alpha - 1)^2n f_n\left(\frac{1}{1 - \alpha}\right) = (\alpha - 1)^2n f_n\left(\frac{\alpha - 1}{\alpha}\right). \tag{4.30}
\]

As shown by Heslop and Howe, [6], the sum of terms produced by the action of \( S_3 \) as given by (4.30) starting from \( \alpha^r \) generates a linearly independent set of polynomials for \( r = 0, 1, \ldots, \lfloor \frac{1}{3}n \rfloor \), giving \( \lfloor \frac{1}{3}n \rfloor + 1 \) solutions for \( f_n \). Alternatively an equivalent basis is provided by

\[
S_{n,j}(\alpha) = (\alpha^2 - \alpha + 1)^n s(\alpha)^j, \quad j = 0, 1, \ldots, \lfloor \frac{1}{3}n \rfloor, \tag{4.31}
\]

where \( s(\alpha) \) is the \( S_3 \) invariant

\[
s(\alpha) = \frac{\alpha^2(1 - \alpha)^2}{(\alpha^2 - \alpha + 1)^3} = 4 \frac{(1 - y^2)^2}{(y^2 + 3)^3}. \tag{4.32}
\]

The solutions given by (4.31) correspond to (4.4) for \( i = 0 \) since \( \Lambda = 0 \) in this case. A general polynomial solution of (4.30) is then given by

\[
f_n(\alpha) = (\alpha^2 - \alpha + 1)^n P(s(\alpha)), \tag{4.33}
\]

with \( P(s) \) a polynomial of degree \( \lfloor \frac{1}{3}n \rfloor \).

We may also consider other representations of \( S_3 \). For the antisymmetric representation, as in (4.8), we may define

\[
a(\alpha) = (2\alpha - 1) \frac{(\alpha - 2)(\alpha - 1)\alpha(\alpha + 1)}{(\alpha^2 - \alpha + 1)^3} = 4 \frac{y(y^2 - 1)(y^2 - 9)}{(y^2 + 3)^3}, \tag{4.34}
\]

so that \( a(\alpha)S_{n,j}(\alpha) \) is then a polynomial for \( n \geq 3 \) and \( j = 0, 1, \ldots, \lfloor \frac{1}{3}n \rfloor - 1 \). For the mixed symmetry representation there are two essential solutions which can be written in the form

\[
b_1(\alpha) = \frac{2\alpha - 1}{\alpha^2 - \alpha + 1} = 4 \frac{y}{y^2 + 3},
\]

\[
c_1(\alpha) = \frac{1}{\sqrt{3}} \frac{2\alpha^2 - 2\alpha - 1}{\alpha^2 - \alpha + 1} = \frac{2}{\sqrt{3}} \frac{y^2 - 3}{y^2 + 3}, \tag{4.35}
\]

and

\[
b_2(\alpha) = (2\alpha - 1) \frac{\alpha(\alpha - 1)}{(\alpha^2 - \alpha + 1)^2} = 4 \frac{y(y^2 - 1)}{(y^2 + 3)^2},
\]

\[
c_2(\alpha) = \sqrt{3} \frac{\alpha(\alpha - 1)}{(\alpha^2 - \alpha + 1)^2} = 4\sqrt{3} \frac{y^2 - 1}{(y^2 + 3)^2}. \tag{4.36}
\]
It is easy to see that \((b_r(\alpha), c_r(\alpha))S_{n,j}(\alpha)\) are polynomials for \(j = 0, 1, \ldots, \lfloor \frac{1}{r}(n-r) \rfloor\) if \(n \geq r\) for \(r = 1, 2\). The basis provided by \(S_{n,j}(\alpha), (b_r(\alpha), c_r(\alpha))S_{n,j}(\alpha), r = 1, 2\) and \(a(\alpha)S_{n,j}(\alpha)\) is then complete in that it gives \(2n+1\) linearly independent polynomials, allowing for the expansion of any arbitrary polynomial of degree \(2n, 2\left(\lfloor \frac{1}{r}n \rfloor + \lfloor \frac{1}{r}(n-1) \rfloor + \lfloor \frac{1}{r}(n-2) \rfloor \right) + 5 = 2n + 1\).

For the \(\mathcal{N} = 2\) case instead of (4.1) we have

\[
\mathcal{G}(u, v; \sigma, \tau) = \mathcal{G}(u/v, 1/v; \tau, \sigma) = (\frac{u^2}{v^2} \tau)^n \mathcal{G}(v, u; \sigma/\tau, 1/\tau),
\]

(4.37)

where, with \(\sigma, \tau\) constrained as in (2.40), the superconformal Ward identities are

\[
\mathcal{G}(u, v; \sigma, \tau)\big|_{\alpha=\frac{1}{x}} = f(x) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^{2n} f(1-x) = x^{2n} f\left(\frac{1}{x}\right),
\]

(4.38)

where we also exhibit the crossing symmetry relations for the single variable function \(f\). The corresponding solution to (4.14) is given by

\[
\mathcal{G}(u, v; \sigma, \tau) = S_n\left(u^2 \sigma, \frac{u^2}{v^2} \tau\right),
\]

(4.39)

which implies

\[
f(x) = S_n\left(x^2, \frac{x^2}{(1-x)^2}\right).
\]

(4.40)

In this case if we consider the contribution of individual factors in the basis given by (4.4) to \(f(x)\) as expected from (4.39) and (4.40) we have

\[
P = \left(u^2 \sigma + \frac{u^2}{v^2} \tau + 1\right)\big|_{\alpha=\frac{1}{x}} = p^2, \quad Q = \left(\frac{u^4}{v^2} \sigma \tau\right)\big|_{\alpha=\frac{1}{x}} = q^2,
\]

(4.41)

\[
R = \left(\frac{u^4}{v^2} \sigma \tau + u^2 \sigma + \frac{u^2}{v^2} \tau\right)\big|_{\alpha=\frac{1}{x}} = 2pq,
\]

where

\[
p(x) = \frac{x^2 - x + 1}{1 - x}, \quad q(x) = \frac{x^2}{1 - x},
\]

(4.42)

so that we have the relation \(R^2 = 4PQ\). In consequence we may restrict in (4.4) to those polynomials with \(i = 0, 1\).

Conversely we may argue that for the \(\mathcal{N} = 2\) case all single variable functions \(f(x)\) may be expressible in terms of \(S_n\) as in (4.40) and therefore may be extended to a fully
crossing symmetric form for \( G(u, v; \sigma, \tau) \) as exhibited in (4.39). To demonstrate this we suppose all solutions of the crossing symmetry relations in (4.38) for \( f \) are solvable by writing

\[
f(x) = p(x)^{2n} g(s(x)), \quad s(x) = \frac{q(x)}{p(x)^3},
\]

for some function \( g \) of the crossing invariant \( s \) given by (4.32). Note that for \( x \to 0, s \sim x^2 \), \( x \to 1, s \sim (1 - x)^2 \) and for \( x \to \infty, s \sim 1/x^2 \). From the superconformal representation theory for the corresponding contributions to the operator product expansion \( f(x) \) should be analytic in the neighbourhood of \( x = 0 \) with singularities only at \( x = 1, \infty \). In consequence \( g(s) \) must be a polynomial which is then restricted to have maximal degree \([\frac{2}{3}n]\) to avoid singularities when \( x^2 - x + 1 = 0 \). It is then easy to see that \( f \) can be written as a polynomial in \( P, Q \) with terms also linear in \( R \), as defined in (4.41), which is consistent with (4.40) where \( S_n \) has an expansion in terms of \( S_{n,(i,j)} \) with \( i = 0, 1 \) and \( j \) restricted as in (4.4).

The remaining part of the correlation function may also be expressed in terms of \( S_3 \) representations. It is convenient as for \( \mathcal{N} = 4 \) to define from (4.9) and (4.10)

\[
(b_r'(u, v), c_r'(u, v)) = (b_r(1/u, v/u), c_r(1/u, v/u)).
\]

We may then write for the factor which appears in the solution of the superconformal identities in (2.60)

\[
(\alpha x - 1)(\bar{\alpha} \bar{x} - 1) = (\alpha^2 - \alpha + 1)(u + v + 1)(\frac{1}{x} - \frac{1}{\bar{x}}) (b_1'(u, v) b_1(\alpha) + c_1'(u, v) c_1(\alpha)) \quad (4.44)
\]

4.3. (Next-to-)Extremal Case

It is also of interest to extend the considerations of crossing symmetry to the next-to-extremal case when \( p_1 = p_2 = p_3 = p, p_4 = 3p - 2 \). In this case \( G \), defined by (3.46), must satisfy for the permutations (12) and (23)

\[
G(u, v; \sigma, \tau) = v^{p-1} G\left(\frac{u}{v}, \frac{1}{v}; \tau, \sigma\right), \quad G(u, v; \sigma, \tau) = u^{2p-1} \sigma G\left(\frac{1}{u}, \frac{v}{u}; \frac{1}{\sigma}, \frac{\tau}{\sigma}\right). \quad (4.45)
\]

The solution (3.105) can be rewritten as

\[
G(u, v; \sigma, \tau) = u^{p-1} \left( k + \frac{x \bar{x}}{x - \bar{x}} \left( (\alpha - 1/x)(\bar{\alpha} - 1/\bar{x})f(x) - (\alpha - 1/\bar{x})(\bar{\alpha} - 1/x)f(\bar{x}) \right) \right), \quad (4.46)
\]
and then (4.45) requires

\[ f(x) = -f\left(\frac{x}{x-1}\right), \quad f(x) = -x^2 f\left(\frac{1}{x}\right) + kx. \quad (4.47) \]

A particular solution of (4.47) is given by

\[ f(x) = \frac{k}{3} \left( x - \frac{x}{x-1} \right). \quad (4.48) \]

To obtain a general solution of (4.47) it is then sufficient to seek the general solution \( f_0(x) \) of (4.47) with \( k = 0 \). Using results obtained above this is

\[ f_0(x) = \frac{(x-2)x(x+1)(2x-1)}{(x-1)(x^2 - x + 1)} h(s(x)), \quad (4.49) \]

where \( s \) is the invariant defined by (4.43) and (4.42). This introduces unphysical singularities for \( x^2 - x + 1 = 0 \) unless cancelled by \( h \). However, for compatibility with semi-short representations, \( h(s) \) must be analytic in \( s \) for \( s \sim 0 \) (if \( h(s) = 1/s \), which cancels the singularity at \( x^2 - x + 1 = 0 \), then \( f_0(x) \sim 1/x \) for \( x \to 0 \)). Hence we conclude that there is no possible solution of the form (4.49) and hence we only have (4.48). In this case

\[ G(u, v; \sigma, \tau) = \frac{1}{3} k u^{p-1} \left( 1 + \sigma u + \tau \frac{u}{v} \right). \quad (4.50) \]
5. Large $N$ Results

5.1. Simplification of the Amplitude for $p = 4$

For $p = 4$ the results obtained in [14] are expressible in terms of two functions $\mathcal{F}(u,v), \tilde{\mathcal{F}}(u,v)$ in the following way

\[
\mathcal{H}^{(4)}(u, v; \sigma, \tau) = \frac{u}{v} \mathcal{F}(u,v) + \sigma^2 \frac{u^3}{v^2} \mathcal{F}(1/v, u/v) + \tau^2 \frac{u}{v^2} \mathcal{F}(v,u) \\
+ \sigma \tau \frac{u^3}{v^2} \tilde{\mathcal{F}}(u,v) + \sigma \frac{u^2}{v} \tilde{\mathcal{F}}(v,u) + \tau \frac{u^2}{v^3} \tilde{\mathcal{F}}(1/v, u/v). \tag{5.1}
\]

The functions $\mathcal{F}(u,v), \tilde{\mathcal{F}}(u,v)$ both satisfy the same crossing symmetry constraint

\[
\mathcal{F}(u,v) = \frac{1}{v} \mathcal{F}(u/v, 1/v), \quad \tilde{\mathcal{F}}(u,v) = \frac{1}{v} \tilde{\mathcal{F}}(u/v, 1/v). \tag{5.2}
\]

This ensures that $\mathcal{H}^{(4)}(u, v; \sigma, \tau)$ satisfies the crossing symmetry relations

\[
\mathcal{H}^{(4)}(u, v; \sigma \tau) = \frac{1}{v^2} \mathcal{H}^{(4)}(u/v, 1/v; \tau, \sigma) \\
= \left( \frac{u}{v} \right)^2 \tau^2 \mathcal{H}^{(4)}(v, u; \sigma/\tau, 1/\tau). \tag{5.3}
\]

The results from [14] give $\mathcal{F}(u,v), \tilde{\mathcal{F}}(u,v)$ in terms of $\overline{D}$ functions (1.39) (note $\mathcal{F}(u,v) = \alpha_1(u,v), \tilde{\mathcal{F}}(u,v) = \beta_3(v,u)$) which were discussed in the introduction and identities for which can be found in appendix G. Using identities (G.1), (G.2b) we directly obtain

\[
\mathcal{F}(u,v) = -\frac{4}{N^2} (u^2 \overline{D}_{4446} + 2u \overline{D}_{3346} + 2\overline{D}_{2246}) , \\
\tilde{\mathcal{F}}(u,v) = -\frac{16}{N^2} v(\overline{D}_{4446} - \overline{D}_{3355} + 5\overline{D}_{3344} + \overline{D}_{4244} + \overline{D}_{2444} + \overline{D}_{3243} + \overline{D}_{2343} + \overline{D}_{2242}). \tag{5.4}
\]

Using (G.2a) and (G.2b) we can rewrite

\[
\mathcal{F}(u,v) = -\frac{4}{N^2} u^3 v(\overline{D}_{4644} + 2\overline{D}_{4633} + 2\overline{D}_{4622}). \tag{5.5}
\]

To simplify $\tilde{\mathcal{F}}(u,v)$ we first use (G.2b) to pull a factor of $u$ out of $\tilde{\mathcal{F}}(u,v)$

\[
\tilde{\mathcal{F}}(u,v) = -\frac{16}{N^2} uv(\overline{D}_{6444} - u\overline{D}_{5533} + 5\overline{D}_{4433} + \overline{D}_{4424} + \overline{D}_{4442} + \overline{D}_{3423} + \overline{D}_{3432} + \overline{D}_{2422}). \tag{5.6}
\]
Now we use (G.4c) for $\overline{D}_{5533}$ to obtain

$$\tilde{F}(u, v) = -\frac{16}{N^2} uv (\overline{D}_{4444} - \overline{D}_{4444} + 4 \overline{D}_{4433} + \overline{D}_{4424} + \overline{D}_{4424} + \overline{D}_{3423} + \overline{D}_{3432} + \overline{D}_{2422}).$$  (5.7)

Using (G.2a) for $\overline{D}_{6444}$ and (G.4b) for $\overline{D}_{3432}, \overline{D}_{4442}, \overline{D}_{4424}, \overline{D}_{2422}$ we can simplify $\tilde{F}(u, v)$ to

$$\tilde{F}(u, v) = -\frac{16}{N^2} uv (\overline{D}_{4644} + \overline{D}_{4552} + \overline{D}_{4533} + \overline{D}_{3542} + \overline{D}_{4433} + \overline{D}_{3533} + \overline{D}_{3542} + \overline{D}_{2532}).$$  (5.8)

Now we use (G.5) for $3 \overline{D}_{4433}$ and (G.3) and (G.2a) for $\overline{D}_{5443}$ to obtain

$$\tilde{F}(u, v) = -\frac{16}{N^2} uv^2 (\overline{D}_{4644} + \overline{D}_{4552} + \overline{D}_{4543} + \overline{D}_{3533} + \overline{D}_{3542} + \overline{D}_{2532}).$$  (5.9)

We use (G.4a) for $\overline{D}_{2532}$ and obtain

$$\tilde{F}(u, v) = -\frac{16}{N^2} uv^2 (\overline{D}_{4644} + \overline{D}_{4552} + \overline{D}_{4543} + \overline{D}_{3533} + \overline{D}_{2633}).$$  (5.10)

From (G.3) for $\overline{D}_{4552}$ and (G.4a) for $\overline{D}_{4543}$ we get

$$\tilde{F}(u, v) = -\frac{16}{N^2} uv^2 (\overline{D}_{4644} + \overline{D}_{3643} + \overline{D}_{3634} + \overline{D}_{2633}).$$  (5.11)

Now we use (G.4a) for $\overline{D}_{3643}$ and obtain

$$\tilde{F}(u, v) = -\frac{16}{N^2} uv^2 (\overline{D}_{4644} - 2 \overline{D}_{2633} + \overline{D}_{2734} + \overline{D}_{3634} + \overline{D}_{2633}).$$  (5.12)

Finally, we use (G.5) for $\overline{D}_{2734} + \overline{D}_{3634}$. This gives us the final result

$$\tilde{F}(u, v) = -\frac{16}{N^2} uv^2 (\overline{D}_{4644} - 2 \overline{D}_{2644} + 2 \overline{D}_{2633}).$$  (5.13)

Using this result we can write down the full amplitude in a simplified and manifestly crossing symmetric form

$$\mathcal{H}^{(4)}(u, v; \sigma, \tau) = -\frac{4}{N^2} u^4 \left( (1 + \sigma^2 + \tau^2 + 4\sigma + 4\tau + 4\sigma\tau) \overline{D}_{4644} + 2(\overline{D}_{4633} + \overline{D}_{4622}) + 2\sigma^2(\overline{D}_{3643} + \overline{D}_{2642}) + 2\tau^2(\overline{D}_{3643} + \overline{D}_{2642}) - 4\sigma(\overline{D}_{4624} - 2\overline{D}_{3623}) - 4\tau(\overline{D}_{4642} - 2\overline{D}_{3632}) - 4\sigma\tau(\overline{D}_{2644} - 2\overline{D}_{2633}) \right).$$  (5.14)
Since $K(u, v; \sigma, \tau) = \frac{1}{16} u^2 H(u, v; \sigma, \tau)$ it is easy to verify both the crossing symmetry conditions (4.29) using $\tilde{D}$ identities. Furthermore the results given by (1.41), (1.42) and (5.14), in which overall factors of $u^p$ are present, are manifestly compatible with the unitarity conditions flowing from (3.56) and (3.74) since the leading log. term $\tilde{D}_{n_1 n_2 n_3 n_4}(u, v)$ is log $u$ itself. When expressed in terms of conformal partial waves $G^{(\ell)}_{\Delta+4}$ it is easy to see in each case that only contributions with minimum twist $\Delta - \ell = 2p$ are required. Hence (1.41), (1.42) and (5.14) require the presence of operators belonging to long multiplets which have anomalous dimensions with twist, at zeroth order in $1/N$, $\Delta - \ell = 2(p + t)$, $t = 0, 1, 2, \ldots$ for the lowest scale dimension operators in each multiplet. The condition $\Delta - \ell = 2p$ is stronger than that required by unitarity (3.74), with $n \leq p - 2$, which shows that for any representation some low twist multiplets decouple (thus for the singlet case twist 2 is absent as it disappears in the large $N$ limit but twist 4 multiplets, which are necessary in the $p = 2$ correlation function, decouple from the correlation functions for $p = 3, 4$).

5.2. Computation of First Order Anomalous Dimensions

For $p_i = p$ in the large $N$ limit the leading result for the free contribution of single trace operators may be, with a suitable normalisation, simply obtained from disconnected graphs in free field theory

$$G^{(p)}_0(u, v; \sigma, \tau) = 1 + (\sigma u)^p + \left(\tau \frac{u}{v}\right)^p. \quad (5.15)$$

The definitions (3.49) and (3.50) then give

$$f^{(p)}_0(z, y) = 1 + \left(\frac{1 + y}{1 + z}\right)^p + \left(\frac{1 - y}{1 - z}\right)^p, \quad k = 3. \quad (5.16)$$

Using (3.51) we can then determine, assuming $K^{(p)}(u, v; \sigma, \tau) = \frac{1}{16} u^2 H^{(p)}(u, v; \sigma, \tau)$, the free field expression

$$H^{(2)}_0(u, v; \sigma, \tau) = 1 + \frac{1}{v^2}, \quad (5.17)$$

$$H^{(3)}_0(u, v; \sigma, \tau) = \frac{1}{v^3} \left(\frac{1}{2}(\sigma + \tau) u(1 + v^3) + \frac{1}{2}(\sigma - \tau) \left(-3u(1 - v^3) + 2(1 - v)(1 + v^3)\right) + u(1 + v^3) - 1 + 2v + 2v^3 - v^4\right), \quad (5.18)$$

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\[
\mathcal{H}_0^{(4)}(u, v; \sigma, \tau) = \frac{1}{v^4} \left( \sigma\tau u^2(1 + v^4) + \frac{1}{2}(\sigma - \tau)^2(2(1 - v)^2(1 + v^4) - 5u(1 + v^5) + 3uv(1 + v^3) + 4u^2(1 + v^4)) + \frac{1}{2}(\sigma^2 - \tau^2)(u(1 - v)(1 + v^4) - 2u^2(1 - v^4)) + \frac{1}{2}(\sigma + \tau)(- (1 - v)^2 + u(1 + v))(1 + v^4) + \frac{1}{2}(\sigma - \tau)((1 - v)(-3(1 + v) + 7u)(1 + v^4) + 8v(1 - v)(1 + v^3) - 4u^2(1 - v^4)) + 1 + v^6 - 3v(1 - v)(1 - v^2) - 2u(1 - v)(1 - v^4) + u^2(1 + v^4) \right).
\]

(5.19)

In each case the crossing symmetry relation \( \mathcal{H}_0^{(p)}(u, v; \sigma, \tau) = \mathcal{H}_0^{(p)}(u/v, 1/v; \tau, \sigma)/v^2 \) is satisfied but the corresponding one for \( u \leftrightarrow v \) is not since it is necessary to take account of the function \( f_0^{(p)}(z, y) \) then as well.

To obtain the anomalous scale dimensions in detail it is necessary to decompose both (1.41), (1.42), (5.14) and (5.17), (5.18), (5.19) in terms of different representations, as in (3.56), and then to expand each term in conformal partial waves. The expressions (5.17), (5.18) and (5.19) require contributions with twist zero and above but the corresponding low twist operators in long supermultiplets, for which there are no anomalous dimensions, are cancelled by semi-short multiplets which are required by the expansion of \( f_0(z, y) \). For \( p = 2, 3 \) and 4 a detailed discussion is contained in [5],[9] and [15](although some details are different the analysis is equivalent to the the results that would be obtained by expanding \( \mathcal{H} \) as given by (5.17) and (5.18)). For \( p = 4 \) we first decompose \( \mathcal{H} \) into contributions for each \( SU(4) \) representation [15]

\[
A_{22} = - \frac{2}{5N^2} u^4 \left( D_{4644} + D_{2633} - D_{2644} - \frac{1}{3}(D_{1643} + D_{1634}) \right),
\]

\[
A_{21} = - \frac{2}{5N^2} u^4 \left( D_{3634} - D_{3643} - D_{1634} + D_{1643} \right),
\]

\[
A_{20} = - \frac{4}{5N^2} u^4 \left( -D_{2633} - \frac{1}{3}(D_{1643} + D_{1634}) \right),
\]

\[
A_{11} = - \frac{4}{5N^2} u^4 \left( 8D_{4644} + \frac{20}{3}D_{4633} - \frac{1}{3}D_{2633} - \frac{14}{3}D_{2644} - (D_{1643} + D_{1634}) \right),
\]

\[
A_{10} = - \frac{2}{3N^2} u^4 \left( 4(D_{3623} - D_{3632}) + 5(D_{3634} - D_{3643}) - D_{1634} + D_{1643} \right),
\]

\[
A_{00} = - \frac{4}{N^2} u^4 \left( \frac{5}{2}D_{4644} + \frac{10}{3}D_{4633} + 2D_{4622} + \frac{1}{30}D_{2633} - \frac{5}{6}D_{2644} \right).
\]

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\[ -\frac{1}{10}(\mathcal{D}_{1643} + \mathcal{D}_{1634}) \]. \tag{5.20} 

This simplified form is achieved by using the following identities for \(\mathcal{D}\)-functions \([15]\) which can be derived from the basic identities in appendix G:

\[
\begin{align*}
\mathcal{D}_{3634}(u, v) + \mathcal{D}_{3643}(u, v) &= -\mathcal{D}_{2644}(u, v) - \mathcal{D}_{2633}(u, v) + 2\frac{1 + v}{uv^3}, \\
\mathcal{D}_{2624}(u, v) + \mathcal{D}_{2642}(u, v) &= -2\mathcal{D}_{2633}(u, v) - \mathcal{D}_{1634}(u, v) + 2\frac{1 + v}{uv^3}, \\
\mathcal{D}_{3623}(u, v) + \mathcal{D}_{3632}(u, v) &= -\mathcal{D}_{2633}(u, v) + \frac{1}{u^2v^2}, \\
\mathcal{D}_{4624}(u, v) + \mathcal{D}_{4642}(u, v) &= -2\mathcal{D}_{4633}(u, v) + \mathcal{D}_{2644}(u, v) - 2\mathcal{D}_{2633}(u, v) + \frac{1}{u^2v^2}, \\
\mathcal{D}_{4624}(u, v) + \mathcal{D}_{4642}(u, v) &= -\mathcal{D}_{1634}(u, v) + \mathcal{D}_{1643}(u, v) - 2\frac{1 - v}{uv^3}, \\
\mathcal{D}_{4624}(u, v) - \mathcal{D}_{4642}(u, v) &= \mathcal{D}_{3632}(u, v) - \mathcal{D}_{3634}(u, v) + \mathcal{D}_{3643}(u, v). 
\end{align*} \tag{5.21} 
\]

Although \(\mathcal{D}_{1634} + \mathcal{D}_{1643} = \mathcal{D}_{4361} + \mathcal{D}_{3461}\) and we might use the standard identity (G.5) this would lead to \(\Delta_4 = 0\) and the appearance of \(\mathcal{D}_{3371}\). This will introduce two singularities. Still a similar relation taking account of the singularities in this case can be found in [15]. We have omitted terms in the amplitudes which do not contain any \(\log u\) factors and which thus do not contribute to anomalous dimensions. The \(L^2\)-eigenfunctions we used are consistent with the \(p_{nm}\)’s in appendix B, which differ from the \(Y_{nm}\)’s by a normalization factor of 3, 10 for \(n = 1, 2\) respectively. We split the amplitude into two parts, one regular in \(u\) and one containing a factor of \(\log u\)

\[
A_{nm}(u, v) = \frac{1}{2} \log u \hat{A}_{nm}(u, v) + \ldots \tag{5.22} 
\]

Now we perform a conformal partial wave expansion of \(\hat{A}_{nm}\) using the computer program in appendix F:

\[
\hat{A}_{nm}(u, v) = \sum_{\tau \geq 4, \ell} A_{nm, \tau \ell} u^\tau G^{(\ell)}_{\ell+2\tau+4}(u, v). \tag{5.23} 
\]

Here we set \(\tau = \frac{1}{2}(\Delta - \ell)\) since only even twists occur. We factor off a universal part

\[
A_{nm, \tau \ell} = (\tau - 3)(\ell + 1)!^2 \frac{(\ell + \tau + 1)!^2}{45(2\ell + 2\tau + 2)!(2\tau)!} a_{nm, \tau \ell}. \tag{5.24} 
\]
There is an arbitrary choice of normalization here which agrees with the one used in [15].

The results we obtain for the expansion coefficients are

\[
\begin{align*}
    a_{22,\tau \ell} &= - \frac{4}{75N^2} ((\ell + 1)(\ell + 2\tau + 2)(25(\ell + \tau + 1)(\ell + \tau + 2) + 57\tau(\tau + 1) - 90) \\
    & \quad + 60(\tau - 1)\tau(\tau + 1)(\tau + 2)), \\
    a_{21,\tau \ell} &= - \frac{4}{25N^2} (\ell + 1)(\ell + 2\tau + 2)(25(\ell + \tau + 1)(\ell + \tau + 2) + 9\tau(\tau + 1) - 30), \\
    a_{20,\tau \ell} &= - \frac{8}{15N^2} (\ell + 1)(\ell + 2\tau + 2)(5(\ell + \tau + 1)(\ell + \tau + 2) - 3\tau(\tau + 1)), \\
    a_{11,\tau \ell} &= - \frac{8}{25N^2} ((\ell + 1)(\ell + 2\tau + 2)(25(\ell + \tau + 1)(\ell + \tau + 2) + 97\tau(\tau + 1) - 1140) \\
    & \quad + 160(\tau - 3)\tau(\tau + 1)(\tau + 4)), \\
    a_{10,\tau \ell} &= - \frac{4}{3N^2} (\ell + 1)(\ell + 2\tau + 2)(5(\ell + \tau + 1)(\ell + \tau + 2) + 21\tau(\tau + 1) - 270), \\
    a_{00,\tau \ell} &= - \frac{4}{5N^2} ((\ell + 1)(\ell + 2\tau + 2)(5(\ell + \tau + 1)(\ell + \tau + 2) + 37\tau(\tau + 1) - 450) \\
    & \quad + 100(\tau - 3)(\tau - 2)(\tau + 3)(\tau + 4)).
\end{align*}
\]  

(5.25)

The final quantity we compute is \(\langle \eta_{nm,\tau \ell} \rangle = \frac{a_{nm,\tau \ell}}{a_{nm,\tau \ell}^{(0)}}\). \(\langle \eta \rangle\) is the expectation value of the first order anomalous dimension of all the operators in one particular representation of the amplitude. In particular for only one operator present this actually directly gives the first order anomalous dimension. For \(A^{(0)}\) we take the free amplitudes computed in [15](setting \(a = b = c = 0\) since these are proportional to \(1/N^2\))

\[
A_{nm,\tau \ell}^{(0)} = 2^{\ell-5} \frac{(\tau!)^2(\ell + \tau + 1)!^2}{45(2\ell + 2\tau + 2)!(2\tau)!} a_{nm,\tau \ell}^{(0)}.
\]  

(5.26)

\[
\begin{align*}
    a_{22,\tau \ell}^{(0)} &= \frac{1}{3} (\tau - 1)\tau(\tau + 1)(\tau + 2)(\ell + 1)(\ell + 2\tau + 2) \\
    & \quad \times (\ell + \tau)(\ell + \tau + 1)(\ell + \tau + 2)(\ell + \tau + 3), \\
    a_{21,\tau \ell}^{(0)} &= (\tau - 2)\tau(\tau + 1)(\tau + 3)(\ell + 1)(\ell + 2\tau + 2) \\
    & \quad \times (\ell + \tau - 1)(\ell + \tau + 1)(\ell + \tau + 2)(\ell + \tau + 4), \\
    a_{20,\tau \ell}^{(0)} &= \frac{2}{3} (\tau - 2)(\tau - 1)(\tau + 2)(\tau + 3)(\ell + 1)(\ell + 2\tau + 2) \\
    & \quad \times (\ell + \tau - 1)(\ell + \tau)(\ell + \tau + 3)(\ell + \tau + 4),
\end{align*}
\]
\[ a_{11, \tau \ell}^{(0)} = 2(\tau - 3)\tau(\tau + 1)(\tau + 4)(\ell + 1)(\ell + 2\tau + 2) \]
\[ \times (\ell + \tau - 2)(\ell + \tau + 1)(\ell + \tau + 2)(\ell + \tau + 5), \]
\[ a_{10, \tau \ell}^{(0)} = \frac{5}{3}(\tau - 3)(\tau - 1)(\tau + 2)(\tau + 4)(\ell + 1) \]
\[ \times (\ell + 2\tau + 2)(\ell + \tau - 2)(\ell + \tau)(\ell + \tau + 3)(\ell + \tau + 5), \]
\[ a_{00, \tau \ell}^{(0)} = (\tau - 3)(\tau - 2)(\tau + 3)(\tau + 4)(\ell + 1)(\ell + 2\tau + 2) \]
\[ \times (\ell + \tau - 2)(\ell + \tau - 1)(\ell + \tau + 4)(\ell + \tau + 5). \quad (5.27) \]

This together with the results of our expansion gives the following expressions for \( \langle \eta \rangle \)

\[ \langle \eta_{22, \tau \ell} \rangle = -\frac{4}{25N^2} \frac{(\tau - 3)(\tau - 2)(\tau + 3)(\tau + 4)}{\ell + \tau)(\ell + \tau + 1)(\ell + \tau + 2)(\ell + \tau + 3)(\ell + 1)(\ell + 2\tau + 2)} \]
\[ \times ((\ell + 1)(\ell + 2\tau + 2)(25(\ell + \tau + 1)(\ell + \tau + 2) + 57\tau(\tau + 1) - 90) \]
\[ + 60(\tau - 1)(\tau + 1)(\tau + 2)), \]
\[ \langle \eta_{21, \tau \ell} \rangle = -\frac{4}{25N^2} \frac{(\tau - 3)(\tau - 1)(\tau + 2)(\tau + 4)}{\ell + \tau - 1)(\ell + \tau + 1)(\ell + \tau + 2)(\ell + \tau + 4)} \]
\[ \times (25(\ell + \tau + 1)(\ell + \tau + 2) + 9\tau(\tau + 1) - 30), \]
\[ \langle \eta_{20, \tau \ell} \rangle = -\frac{4}{5N^2} \frac{(\tau - 3)(\tau + 1)(\tau + 4)}{\ell + \tau - 1)(\ell + \tau + 1)(\ell + \tau + 2)(\ell + \tau + 4)} \]
\[ \times (5(\ell + \tau + 1)(\ell + \tau + 2) - 3\tau(\tau + 1)), \]
\[ \langle \eta_{11, \tau \ell} \rangle = -\frac{4}{25N^2} \frac{(\tau - 2)(\tau - 1)(\tau + 2)(\tau + 3)}{\ell + \tau - 2)(\ell + \tau + 1)(\ell + \tau + 2)(\ell + \tau + 5)(\ell + 1)(\ell + 2\tau + 2)} \]
\[ \times ((\ell + 1)(\ell + 2\tau + 2)(25(\ell + \tau + 1)(\ell + \tau + 2) + 97\tau(\tau + 1) - 1140) \]
\[ + 160(\tau - 3)(\tau + 1)(\tau + 4)), \]
\[ \langle \eta_{10, \tau \ell} \rangle = -\frac{4}{5N^2} \frac{(\tau - 2)(\tau + 1)(\tau + 3)}{\ell + \tau - 2)(\ell + \tau + 1)(\ell + \tau + 3)(\ell + \tau + 5)} \]
\[ \times (5(\ell + \tau + 1)(\ell + \tau + 2) + 21\tau(\tau + 1) - 270), \]
\[ \langle \eta_{00, \tau \ell} \rangle = -\frac{4}{5N^2} \frac{(\tau - 1)(\tau + 1)(\tau + 2)}{\ell + \tau - 2)(\ell + \tau - 1)(\ell + \tau + 1)(\ell + \tau + 3)(\ell + 1)(\ell + 2\tau + 2)} \]
\[ \times ((\ell + 1)(\ell + 2\tau + 2)(5(\ell + \tau + 1)(\ell + \tau + 2) + 37\tau(\tau + 1) - 450) \]
\[ + 100(\tau - 3)(\tau - 2)(\tau + 3)(\tau + 4)). \quad (5.28) \]

Notice that the \( \tau \)-dependence in the numerator of the first factor can easily be seen to be

\[ u_{nm} = (\tau - n - 1)(\tau - m)(\tau + m + 1)(\tau + n + 2). \quad (5.29) \]
Now we define

\[ v_r = \frac{(\tau - r)(\tau + r + 1)}{(\ell + \tau - r + 1)(\ell + \tau + r + 2)}. \]  

(5.30)

Using this we can write the results in the following relatively simple way for \( \tau = 4, 5, \ldots \)

\[
\begin{align*}
\langle \eta_{22}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{22}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 + \frac{12}{25} v_0 + \frac{4}{5} v_1 + \frac{3}{25} v_0 v_1 \right), \\
\langle \eta_{21}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{21}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 + \frac{4}{25} v_0 - \frac{4}{5} v_2 - \frac{9}{25} v_0 v_2 \right), \\
\langle \eta_{20}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{20}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 - \frac{2}{5} v_1 - \frac{6}{5} v_2 + \frac{3}{5} v_1 v_2 \right), \\
\langle \eta_{11}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{11}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 + \frac{2}{25} v_0 + \frac{14}{5} v_3 + \frac{63}{25} v_0 v_3 \right), \\
\langle \eta_{10}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{10}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 - \frac{4}{25} v_1 + \frac{84}{25} v_3 - \frac{21}{5} v_1 v_3 \right), \\
\langle \eta_{00}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{00}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 + \frac{4}{5} v_2 + \frac{28}{5} v_3 + \frac{63}{5} v_2 v_3 \right).
\end{align*}
\]  

(5.31)

For comparison we can also rewrite the results for \( p = 2 \)

\[ \langle \eta_{00}, \tau \ell \rangle = -\frac{4}{N^2} \frac{u_{00}}{(\ell + 1)(\ell + 2\tau + 2)}, \]

and \( p = 3 \)

\[
\begin{align*}
\langle \eta_{11}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{11}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 + \frac{1}{2} v_0 \right), \\
\langle \eta_{10}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{10}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 - v_1 \right), \\
\langle \eta_{00}, \tau \ell \rangle &= -\frac{4}{N^2} \frac{u_{00}}{(\ell + 1)(\ell + 2\tau + 2)} \left( 1 + 5 v_2 \right),
\end{align*}
\]

in this form. This suggests a universal form for these anomalous dimensions for general \( p \).

In particular it seems to emerge a universal behaviour for large \( \ell \)

\[ \langle \eta_{nm, \tau \ell} \rangle = -\frac{4}{N^2} \frac{u_{nm}}{(\ell + 1)(\ell + 2\tau + 2)} (1 + \mathcal{O}(\ell^{-2})). \]

5.3. Conjectures for General Chiral Four Point Functions

It was observed in [15] that the structure of the singularities in \( u \) is universal in the sense, that up to order \( p - 1 \) in \( u \) the terms in the amplitude are given by a universal
function $\mathcal{F}$ and thus for increasing $p$ new singular terms appear, but the existing ones are not modified. Also, from this it follows that we expect the free field part of the amplitude to have the same form with opposite sign. The reason is that these singular contributions correspond to long multiplets which have no corresponding log $u$ terms and thus do not receive anomalous dimensions. Assuming that all long multiplets do receive anomalous dimensions we expect these protected contributions to be cancelled by the free field part of the amplitude. We will use this observation as an assumption together with crossing symmetry and will work out the constraints it imposes on the general $p$ amplitude. This is still work in progress since some of our assumptions about the singularity structure are not well justified yet.

We will attempt to write down expressions for the general large $N$ amplitude for a four point function of four identical single trace $\frac{1}{2}$-BPS operators belonging to the $SU(4)$ $[0, p, 0]$ representation. The following will be based on what has been observed [15] in the examples for $p = 2, 3, 4$ which have been explicitly calculated using the AdS/CFT correspondence [13,9,14]. We believe that after making use of the freedom of decomposing long multiplets into semi-short ones the free part $H_0$ of the amplitude $G$ defines a function identical to $F$ but with opposite sign

$$H_0^{(p)}(u, v; \sigma, \tau) = \frac{p^2}{N^2} F(u, v; \sigma, \tau), \quad (5.32)$$

such that the singular contributions of the free and the dynamical part cancel and all remaining contributions from long operators possess log $u$ terms generating anomalous dimensions as well. Given this, it would provide the opportunity to compute $F(u, v; \sigma, \tau)$ from the free field amplitude. One technical complication is the ambiguity of the distinction between $H, \hat{f}$ in (3.51), which is not really understood yet.

While developing these ideas, Mathematica has proven very useful as a tool to test our ideas and check what amplitudes would be the result for $p = 2, \ldots, 9$ long before we had explicit expressions for the coefficients.

Crossing symmetry plays an essential role in what follows. We first exhibit a basis for crossing symmetric polynomials in the variables $\sigma, \tau$ of degree $n$, i.e. they may be expanded
in monomials $\sigma^g \tau^h$ with $g + h \leq n$, which are defined by

$$S^{(n)}(\sigma, \tau) = S^{(n)}(\tau, \sigma) = \tau^n S^{(n)}(\sigma/\tau, 1/\tau).$$  \hfill (5.33)

A simple basis for these polynomials is given by

$$S^{(n)}_{ab}(\sigma, \tau) = \begin{cases} 
\sigma^a \tau^a + \sigma^a \tau^{n-2a} + \sigma^{n-2a} \tau^a, & a = b, \\
\sigma^a \tau^b + \sigma^b \tau^a + \sigma^a \tau^a, & 2a + b = n, \\
\sigma^n \tau^n, & \frac{1}{3} n \in \mathbb{N}, \\
\sigma^a \tau^b + \sigma^b \tau^a + \sigma^a \tau^{n-a-b} + \sigma^{n-a-b} \tau^a + \sigma^b \tau^{n-a-b} + \sigma^{n-a-b} \tau^b, & \text{otherwise},
\end{cases}$$  \hfill (5.34)

where $a, b$ are integers satisfying

$$0 \leq b \leq a, \quad 2a + b \leq n.$$  \hfill (5.35)

Notice that this basis is related to the one used in Table 2 in chapter 4 by $a = i + j, b = j$ and $n = p$ whereas here it will be $n = p - 2$. The first two cases in (5.34) are distinguished according to whether $n > 3a$ or $3a > n$ respectively. For any $n$ the set of possible $(a, b)$ are the points of an integer lattice inside or on a triangle with vertices $(0, 0), (\frac{1}{3} n, \frac{1}{3} n)$ and $(\frac{1}{2} n, 0)$. The number of independent crossing symmetric polynomials as in (4.3) is

$$(n - 3[\frac{1}{6} n])([\frac{1}{6} n] + 1) + \delta_{n, 6[\frac{1}{6} n]}.$$  \hfill (5.36)

From (5.34) it is trivial to verify that the number of terms $n^{(n)}_{ab}$ in the above symmetric polynomials is in each case

$$n^{(n)}_{aa} = n^{(n)}_{a n-2a} = 3, \quad n^{(n)}_{\frac{1}{3} n \frac{1}{3} n} = 1, \quad n^{(n)}_{ab} = 6 \quad \text{otherwise},$$  \hfill (5.37)

and then

$$\sum_{0 \leq b \leq a \leq n-a-b} n^{(n)}_{ab} = \frac{1}{2} (n + 1)(n + 2),$$  \hfill (5.38)

which is the number of independent polynomials in $\sigma, \tau$ of degree $n$.

The four point function for general $p$ can be reduced to an amplitude $\mathcal{H}^{(p)}$, polynomial of degree $p - 2$ in $\sigma, \tau$, satisfying

$$\mathcal{H}^{(p)}(u, v; \sigma, \tau) = \frac{1}{v^2} \mathcal{H}^{(p)}(u/v, 1/v; \tau, \sigma) = \left(\frac{u}{v}\right)^{p-2} \tau^{p-2} \mathcal{H}^{(p)}(v, u; \sigma/\tau, 1/\tau).$$  \hfill (5.39)
The results for $p = 2, 3, 4$ [13,9,14] can be reduced to the following form

$$
\mathcal{H}^{(p)}(u, v; \sigma, \tau) = -\frac{p^2}{N^2} u^p \sum_{0 \leq k \leq a} \sum_{2a + b \leq p - 2} c_{ij,k}^{(p)} T_{ijk,ab}^{(p)}(u, v; \sigma, \tau),
$$

(5.39)

where $T_{ijk,ab}^{(p)}$ are completely crossing symmetric combinations of $D \overline{D}$ functions which are related to the crossing symmetric polynomials in (5.34). For $p = 2, 3, \ldots$ and restricting $0 \leq b \leq a$, $2a + b \leq p - 2$, corresponding to (5.35), we define

$$
T_{ijk,ab}^{(p)}(u, v; \sigma, \tau)
= \frac{1}{6} n_{ab}^{(p-2)} \left(\sigma^a \tau^b \overline{D}_{ip+2,jk}(u, v) + \sigma^b \tau^a \overline{D}_{ik+2,j}(u, v)
+ \sigma^a \tau^{p-2-a-b} \overline{D}_{jp+2,ik}(u, v) + \sigma^{p-2-a-b} \tau^a \overline{D}_{j,kp+2,ii}(u, v) + \sigma^{p-2-a-b} \tau^b \overline{D}_{k,p+2,ji}(u, v)\right).
$$

(5.40)

The crossing identities for $\overline{D}$ functions (G.1) ensure that $u^p T_{ijk,ab}^{(p)}(u, v; \sigma, \tau)$ satisfies (5.38) without any additional factors of $u, v$ being necessary. The factors $\frac{1}{6} n_{ab}^{(p-2)}$ are introduced for later convenience, essentially since, for the boundary values of $a, b$, we have

$$
T_{ijk,aa}^{(p)} = T_{ikj,aa}^{(p)} = T_{ijk,ap-2-2a}^{(p)} = T_{kji,ap-2-2a}^{(p)} = T_{jki,aa}^{(p)}.
$$

(5.41)

When $a, b$ satisfy the conditions in (5.41) then (5.40) can be simplified in appropriate cases,

$$
T_{ij,jj,aa}^{(p)}(u, v; \sigma, \tau) = T_{jij,ap-2-2a}^{(p)}(u, v; \sigma, \tau)
= \sigma^a \tau^a \overline{D}_{ip+2,jj}(u, v) + \sigma^a \tau^{p-2-2a} \overline{D}_{jp+2,ij}(u, v) + \sigma^{p-2-2a} \tau^a \overline{D}_{j,p+2,ji}(u, v),
$$

(5.42)

$$
T_{iii,p+2,ii}^{(p)}(u, v; \sigma, \tau) = (\sigma \tau)^{\frac{1}{3}(p-2)} \overline{D}_{ip+2,ii}(u, v).
$$

To determine contributions in the operator product expansion from long multiplets corresponding to different $SU(4)$ representations for the lowest dimension operators we may expand in terms of $SU(4)$ harmonics

$$
\mathcal{H}^{(p)}(u, v; \sigma, \tau) = \sum_{0 \leq m \leq n \leq p-2} A_{nm}(u, v) Y_{nm}(\sigma, \tau),
$$

(5.43)

where $nm$ corresponds to a $SU(4)$ representation with Dynkin labels $[n - m, 2m, n - m]$. $Y_{nm}(\sigma, \tau)$ are polynomials of degree $n$ in $\sigma, \tau$. The condition that only long multiplets
may contribute to $A_{nm}(u, v)$ satisfying the unitarity condition requires that for $u \sim 0$ we must have

$$A_{nm}(u, v) = O(u^{n+1}). \quad (5.44)$$

The $\mathcal{D}$ functions that can appear in (5.40) are constrained by the unitarity conditions (5.44). To obtain these we first list a few more essential properties used here apart from the ones introduced in the introduction (1.39). From standard relations for hypergeometric functions we have

$$n_1 f_{n_1 n_2 n_3 n_4}(v) = f_{n_1+1 n_2 n_3+1 n_4}(v) + f_{n_1+1 n_2 n_3 n_4+1}(v), \quad (5.45)$$

and also

$$f_{n_1 n_2 n_3 n_4}(v) = v^{-n_2} f_{n_1 n_2 n_3 n_4}(1/v). \quad (5.46)$$

The log $u$ terms in (1.39) result in anomalous dimensions for operators belonging to long multiplets which have twist $\Delta - \ell \geq 2p$ at zeroth order in the $1/N$ expansion, where $\Delta$ is the scale dimension and $\ell$ the spin. The terms involving negative powers $u^{-s+m}$ have no corresponding log $u$ terms and would correspond in the operator product expansion to contributions from long multiplets which are unrenormalised. We assume that these are all cancelled although there remain contributions from various semi-short multiplets which cannot be combined to form a long multiplet, thus all multiplets not satisfying a shortening condition gain anomalous dimensions in the $1/N$ expansion as expected. This is the essential assumption that leads to strong constraints on the expansion (5.39).

Using the properties of the $SU(4)$ harmonics $Y_{nm}(\sigma, \tau)$ it is clear that the unitarity condition (5.44) is satisfied if for any contribution in (5.39) involving

$$\sigma^g \tau^h u^{p+2} \mathcal{D}_{n_1 p+n_3 n_4}(u, v), \quad (5.47)$$

we require

$$0 \leq s \leq p - g - h - 1. \quad (5.48)$$

With the assumed expansion given by (5.39) and (5.40) the condition $s \geq 0$ ensures from (1.39) that only long multiplets with twist $\Delta - \ell \geq 2p$ with non zero anomalous dimensions...
in the large $N$ limit can contribute to the operator product expansion of the chiral four point function.

Applying the condition (5.48) to (5.40) gives the following inequalities

$$p - 2a \leq i + j - k \leq p + 2,$$
$$p - 2b \leq i + k - j \leq p + 2,$$
$$2(a + b + 2) - p \leq j + k - i \leq p + 2.$$  \hspace{1cm} (5.49)

We also impose

$$i, j, k \leq p.$$  \hspace{1cm} (5.50)

It is clear that there is a finite number of possibilities for $i, j, k$, note that $p + 4 \leq i + j + k \leq 3p$. We should also note that the expansion (5.40) is not unique since

$$\frac{1}{2} (i + j + k - p - 2) D_{i j k} = D_{i j k} + D_{j i k} + D_{i k j}.$$  \hspace{1cm} (5.51)

This is the only relation for $D$ functions of the form appearing in (5.40). Correspondingly we may take

$$\frac{1}{2} (i + j + k - p - 2) T_{ijk, ab}^{(p)} = T_{ijk, ab}^{(p)} + T_{jik, ab}^{(p)} + T_{kij, ab}^{(p)}.$$  \hspace{1cm} (5.52)

which allows the expansion (5.39) to be simplified if each term in (5.52) satisfies the constraints (5.49) and (5.50). For this to be the case we must have

$$i + j - k, i + k - j, j + k - i \leq p, \hspace{0.5cm} i, j, k \leq p - 1.$$  \hspace{1cm} (5.53)

Whenever (5.53) is satisfied one of the terms appearing in (5.52) may be omitted in the general expansion.

If we use (5.53) to remove terms with the lowest value of $i + j + k$ whenever appropriate then for general $p$ the list of possible terms obtained from (5.49), (5.50) modulo (5.53) is

$$T_{p j, ab}^{(p)} \hspace{1cm} j = a + b + 2, \ldots, p,$$
$$T_{p i, ab}^{(p)} \hspace{1cm} i = p - a, \ldots, p - 1, \hspace{0.5cm} a \geq 1,$$
$$T_{p i, ab}^{(p)} \hspace{1cm} i = p - b, \ldots, p - 1, \hspace{0.5cm} b \geq 1,$$
$$T_{i p i, ab}^{(p)} \hspace{1cm} i = p - a + 1, \ldots, p, \hspace{0.5cm} k = a + b + 3, \ldots, p, \hspace{0.5cm} a \geq 1,$$
$$T_{i j i + j - p - 2 a, ab}^{(p)} \hspace{1cm} i = p - b + 1, \ldots, p, \hspace{0.5cm} j = a + b + 3, \ldots, p, \hspace{0.5cm} b \geq 1,$$
$$T_{j + k - p - 2 j, ab}^{(p)} \hspace{1cm} j = p - a + 1, \ldots, p, \hspace{0.5cm} k = p - b + 1, \ldots, p, \hspace{0.5cm} a, b \geq 1.$$  \hspace{1cm} (5.54)

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When \( a = b \) or \( b = p - 2 - 2a \) it is necessary also to take into account the symmetry conditions in (5.41) to obtain an independent basis. If \( N_{ab}^{(p)} \) are the number of possibilities for each \( a, b \) then we have

\[
N_{ab}^{(p)} = (a + b + 1)(p - a - b - 1) + ab, \quad 0 \leq b < a < \frac{1}{2}(p - 2 - b),
\]

\[
N_{aa}^{(p)} = (a + 1)(p - \frac{3}{2}a - 1), \quad 0 \leq a < \frac{1}{2}(p - 2),
\]

\[
N_{ap-2-2a}^{(p)} = (a + 1)(p - \frac{3}{2}a - 1), \quad \frac{1}{3}(p - 2) < a \leq \frac{1}{2}(p - 2),
\]

\[
N_{\frac{p}{3}(p-2)\frac{p}{3}(p-2)}^{(p)} = \frac{1}{18}(p + 1)(p + 4), \quad p = 2 \text{ mod } 3.
\]

The crucial assumption, in addition to (5.40), is that the leading terms in the expansion of \( H^{(p)} \) in powers of \( u \), which do not involve any \( \log u \) terms, are universal, i.e. we have for any \( p = 2, 3, \ldots \)

\[
H^{(p)}(u, v; \sigma, \tau) = -\frac{p^2}{N^2} F(u, v; \sigma, \tau) + O(u^p),
\]

where \( F \) is independent of \( p \). It satisfies the same crossing properties under \( u \to u/v, v \to 1/v \) and \( \sigma \leftrightarrow \tau \) as \( H^{(p)} \) in (5.38). In the expansion (5.39) for \( H^{(p)} \), with \( T_{ijk,ab}^{(p)} \) given by (5.40), then only the leading singular terms in the expansion of the \( D_j \) functions shown in (1.39) contribute to \( F \) in (5.56). This ensures that it is expressible as an expansion with the form

\[
F(u, v; \sigma, \tau) = \sum_{n \geq 1} u^n F_n(v; \sigma, \tau)
\]

\[
= \sum_{g, h \geq 0} \sum_{l, m} d_{lm, gh} \sigma^g \tau^h u^{g+h+1} f_{l+m-g-h-3g+h+3lm}(v),
\]

where (5.46) has been used to restrict to terms of the form \( u^n f_{l+m-n-2n+2lm}(v) \).

In general for each contribution in \( F \) proportional to \( \sigma^g \tau^h \) we may expect a series involving \( n = g + h + 1, g + h + 2, \ldots \), but as verified later it is possible to restrict to just the minimal form shown in (5.57) when only \( n = g + h + 1 \) for each \( g, h \). This is an essential assumption for determining the coefficients \( c_{ppp,ab}^{(p)} \) for which we currently have no justification other than we observe that it is possible to impose this restriction and obtain results consistent with known numerical results. Subject to using (5.45) we may also require

\[
l, m \leq g + h + 2, \quad l + m \geq g + h + 4, \quad 0 \leq m - l \leq g - h \text{ or } 0 \leq l - m \leq h - g.
\]
Apart from the relation (5.45) the functions \( u^n f_{l+m-n-2} n+2 l+m(v) \) appearing in (5.57) are assumed to be linearly independent. With the restricted form in (5.57) then, as shown later, we are both able to determine \( \mathcal{F} \) from (5.56) and using the explicit form of \( \mathcal{F} \), up to terms of \( \mathcal{O}(u^{p-1}) \), to find all the coefficients \( c^{(p)}_{ijk,ab} \) which appear in the expansion (5.39).

A simple consequence of (5.56) and (5.39), which does not require any restrictions of the form for \( \mathcal{F} \), is that for \( p = 3, 4, \ldots \)

\[
u^p \sum_{0 \leq b \leq a} \sum_{i,j,k} c^{(p)}_{ijk,ab} T^{(p)}_{ijk,ab}(u, v; \sigma, \tau) = u^{p-1} \sum_{0 \leq b \leq a} \sum_{i,j,k} c^{(p-1)}_{ijk,ab} T^{(p-1)}_{ijk,ab}(u, v; \sigma, \tau) + \mathcal{O}(u^{p-1}) ,
\]

where only the leading singular terms displayed explicitly in (1.39) need be considered. These equations are invariant under \( \sigma \leftrightarrow \tau \) and \( u \rightarrow u/v, v \rightarrow 1/v \). In (5.59) all \( \overline{D} \) functions with \( s = 1, 2, \ldots \) are relevant on the right hand side but only those with \( s = 2, 3, \ldots \) on the left hand side. It is important to note that (5.59) does not constrain, \( c^{(p)}_{ppp,ab} \), the coefficient for \( \overline{D}_{pp+2pp} \) which is present for any \( a, b \).

### 5.4. Applications for Low \( p \)

We now show how the above suggestions work out in practice for low \( p \), initially using only (5.59).

For \( p = 2 \) \( \mathcal{H}^{(2)} \) is independent of \( \sigma, \tau \) and there is just one possible \( \overline{D} \) function which is in accord with the simplification of results obtained from AdS/CFT,

\[
\mathcal{H}^{(2)}(u, v) = -\frac{4}{N^2} u^2 \overline{D}_{2422}(u, v).
\]

(5.60)

For \( p = 3 \) we must have again \( a = b = 0 \) and there are just two crossing symmetric forms

\[
\mathcal{H}^{(3)}(u, v; \sigma, \tau) = -\frac{9}{N^2} u^3 \left( c^{(3)}_{322,00} T^{(3)}_{322,00} + c^{(3)}_{333,00} T^{(3)}_{333,00} \right),
\]

\[
= -\frac{9}{N^2} u^3 \left( c^{(3)}_{333,00}(1 + \sigma + \tau) \overline{D}_{3533} + c^{(3)}_{322,00}(\overline{D}_{3522} + \sigma \overline{D}_{2523} + \tau \overline{D}_{2532}) \right).
\]

(5.61)

The equations (5.59) give just one condition arising from \( u^3 \overline{D}_{3522}(u, v) \sim uf_{1322}(v) \) and in (5.60) \( u^2 \overline{D}_{2422}(u, v) \sim uf_{1322}(v) \) so that we require

\[
c^{(3)}_{322,00} = 1.
\]

(5.62)
The known results also give

\[ c^{(3)}_{333,00} = 1. \] (5.63)

For \( p = 4 \), for comparison with previous results, we write

\[
\mathcal{H}^{(4)}(u, v; \sigma, \tau) = -\frac{16}{N^2} u^4 \left( c^{(4)}_{222,00} T^{(4)}_{222,00} + c^{(4)}_{333,00} T^{(4)}_{333,00} + c^{(4)}_{444,00} T^{(4)}_{444,00} \right.
\]

\[ + c^{(4)}_{323,10} T^{(4)}_{323,10} + c^{(4)}_{424,10} T^{(4)}_{424,10} + c^{(4)}_{444,10} T^{(4)}_{444,10} \left), \right. (5.64)\]

where from (5.41) and (5.52) \( T^{(4)}_{334,10} = T^{(4)}_{433,10} = \frac{1}{2}(T^{(4)}_{323,10} - T^{(4)}_{424,10}) \) so that such contributions are discarded. Expanding (5.64) then gives

\[
\mathcal{H}^{(4)}(u, v; \sigma, \tau) = -\frac{16}{N^2} u^4 \left( c^{(4)}_{444,00}(1 + \sigma^2 + \tau^2) + c^{(4)}_{444,10}(\sigma + \tau + \sigma \tau) \right) \bar{D}_{4644}
\]

\[ + c^{(4)}_{323,10}(\sigma \bar{D}_{3623} + \tau \bar{D}_{3632} + \sigma \tau \bar{D}_{2633})
\]

\[ + c^{(4)}_{424,10}(\sigma \bar{D}_{4624} + \tau \bar{D}_{4642} + \sigma \tau \bar{D}_{2642})
\]

\[ + c^{(4)}_{422,00}(\bar{D}_{4622} + \sigma^2 \bar{D}_{2624} + \tau^2 \bar{D}_{2642})
\]

\[ + c^{(4)}_{433,00}(\bar{D}_{4633} + \sigma^2 \bar{D}_{3634} + \tau^2 \bar{D}_{3643}) \right). (5.65)\]

Applying (5.59) then gives for the 1 terms

\[ c^{(4)}_{422,00} = \frac{1}{2}, \quad c^{(4)}_{422,00} - c^{(4)}_{433,00} = 0, \] (5.66)

and from the \( \sigma \) terms

\[ c^{(4)}_{323,10} f_{1423}(v) + c^{(4)}_{424,10} f_{2424}(v) = f_{2433}(v) + f_{1423}(v). \] (5.67)

Using (5.45) this is easily solved giving

\[ c^{(4)}_{323,10} = 2, \quad c^{(4)}_{424,10} = -1. \] (5.68)

The remaining coefficients which are undetermined by (5.59) are

\[ c^{(4)}_{444,10} = 1, \quad c^{(4)}_{444,00} = \frac{1}{4}. \] (5.69)

For the basis corresponding to (5.54) then, instead of (5.65), we should take

\[
\mathcal{H}^{(4)}(u, v; \sigma, \tau) = -\frac{16}{N^2} u^4 \left( \sum_{j=2}^{4} c^{(4)}_{4j,j,00} T^{(4)}_{4j,j,00} \right.
\]

\[ + 2c^{(4)}_{433,10} T^{(4)}_{433,10} + c^{(4)}_{424,10} T^{(4)}_{424,10} + c^{(4)}_{444,10} T^{(4)}_{444,10} \left). \right. (5.70)\]
Here we introduce a factor 2 for the $c_{433,10}^{(4)}$ terms to count equal contributions from $T_{433,10}^{(4)}$ and $T_{434,10}^{(4)}$. This ensures uniformity with later general results. In this case (5.66) and (5.68) are unchanged but instead of (5.68) we should take

$$c_{433,10}^{(4)} = 2, \quad c_{424,10}^{(4)} = 1.$$  (5.71)

On the basis of the results for $p = 2, 3, 4|13,9,14$ we determine the first few terms in the function $F$ introduced in (5.56),

$$\mathcal{F}(u,v;\sigma,\tau) = uf_{1322}(v) + \sigma u^2(f_{1423}(v) + f_{2433}(v)) + \tau u^2(f_{1432}(v) + f_{2433}(v)) + O(u^4).$$

(5.72)

This result is in accord with the assumed form in (5.57).

For $p = 5$ we have the general form

$$\mathcal{H}^{(5)}(u,v;\sigma,\tau) = -\frac{25}{N^2} u^5 \left( \sum_{j=2}^{5} c_{5jj,00}^{(5)} T_{5jj,00}^{(5)} + \frac{1}{2} \sum_{j=2}^{5} c_{5jj,10}^{(5)} T_{5jj,10}^{(5)} + c_{445,10}^{(5)} T_{445,10}^{(5)} + c_{544,10}^{(5)} T_{544,10}^{(5)} + c_{535,10}^{(5)} T_{535,10}^{(5)} + 3c_{544,11}^{(5)} T_{544,11}^{(5)} + 3c_{553,11}^{(5)} T_{553,11}^{(5)} + c_{555,11}^{(5)} T_{555,11}^{(5)} \right).$$

(5.73)

Here we note from (5.41) that $T_{i,j,k,11}^{(5)}$ is completely symmetric in $i,j,k$ and $\frac{1}{3} T_{333,11}^{(5)} = T_{443,11}^{(5)} = T_{544,11}^{(5)} + \frac{1}{2} T_{553,11}^{(5)}$, so neither of these terms are present in the expansion in (5.73). By using (5.52), we also eliminate terms involving $T_{434,10}^{(5)}$, $T_{423,10}^{(5)}$. As in (5.70) we introduce factors to take account of the sum over identical terms related by (5.41). With the basis in (5.73) and the results for $\mathcal{H}^{(4)}$ we may readily solve the equations (5.59) giving

$$c_{522,00}^{(5)} = c_{533,00}^{(5)} = \frac{1}{6}, \quad c_{544,00}^{(5)} = \frac{1}{12},$$

$$c_{533,10}^{(5)} = c_{544,10}^{(5)} = 1, \quad c_{524,10}^{(5)} = c_{535,10}^{(5)} = \frac{1}{2}, \quad c_{445,10}^{(5)} = \frac{3}{4},$$

$$c_{544,11}^{(5)} = 3, \quad c_{553,11}^{(5)} = 1.$$  (5.74)
Only $c^{(5)}_{555,00}, c^{(5)}_{555,10}, c^{(5)}_{555,11}$ are undetermined at this stage. For the expansion (5.57) we may now obtain

\[
\mathcal{F}_4(v; \sigma, \tau) = \left( c^{(5)}_{555,00} - \frac{1}{36} \right) f_{4655}(v) + (\sigma + \tau + \sigma^2 + \tau^2) \left( c^{(5)}_{555,10} - \frac{1}{4} \right) f_{4655}(v) + \sigma \tau \left( c^{(5)}_{555,11} - 1 \right) f_{4655}(v) \\
+ \sigma^3 \left( \frac{1}{6} f_{1625}(v) + \frac{1}{6} f_{2635}(v) + \frac{1}{12} f_{3645}(v) + 2c^{(5)}_{555,00} f_{4655}(v) \right) \\
+ \tau^3 \left( \frac{1}{6} f_{1652}(v) + \frac{1}{6} f_{2653}(v) + \frac{1}{12} f_{3654}(v) + 2c^{(5)}_{555,00} f_{4655}(v) \right) \\
+ \sigma^2 \tau \left( f_{1634}(v) + \frac{1}{2} f_{2644}(v) + \frac{1}{4} f_{3645}(v) + c^{(5)}_{555,10} f_{4655}(v) \right) \\
+ \sigma \tau^2 \left( f_{1643}(v) + \frac{1}{2} f_{2644}(v) + \frac{1}{4} f_{3654}(v) + c^{(5)}_{555,10} f_{4655}(v) \right) .
\]

(5.75)

The restrictions, imposed by the additional constraint that just the leading term for $n = g + h + 1$ appears in the expansion of $\mathcal{F}$ for each $g, h$ as assumed in (5.57), is easily achieved in (5.75) by setting

\[
c^{(5)}_{555,00} = \frac{1}{36}, \quad c^{(5)}_{555,10} = \frac{1}{4}, \quad c^{(5)}_{555,11} = 1.
\]

(5.76)

For $p = 6$ we have the following expansion in terms of independent crossing symmetric functions

\[
\mathcal{H}^{(6)}(u; \sigma, \tau) = -\frac{36}{N^2} u^6 \left( \sum_{j=2}^{6} c^{(6)}_{6jj,00} T^{(6)}_{6jj,00} + \sum_{j=3}^{6} c^{(6)}_{6jj,10} T^{(6)}_{6jj,10} + c^{(6)}_{556,10} T^{(6)}_{556,10} + c^{(6)}_{624,10} T^{(6)}_{624,10} + c^{(6)}_{635,10} T^{(6)}_{635,10} + c^{(6)}_{636,10} T^{(6)}_{636,10} + \right. \\
+ \sum_{j=4}^{6} c^{(6)}_{6jj,11} T^{(6)}_{6jj,11} + 2c^{(6)}_{556,11} T^{(6)}_{556,11} + 2c^{(6)}_{635,11} T^{(6)}_{635,11} + 2c^{(6)}_{646,11} T^{(6)}_{646,11} + 2c^{(6)}_{466,11} T^{(6)}_{466,11} + 2c^{(6)}_{525,20} T^{(6)}_{525,20} \left. \right) .
\]

(5.77)

Here for $a = b = 1$ and $a = 2, b = 0$ we have used (5.41) to reduce the number of necessary
terms. In this case the equations give

\[ c_{622,00}^{(6)} = c_{633,00}^{(6)} = \frac{1}{24}, \quad c_{644,00}^{(6)} = \frac{1}{48}, \quad c_{655,00}^{(6)} = -\frac{1}{48} + c_{555,00}^{(5)}, \]
\[ c_{633,10}^{(6)} = c_{644,10}^{(6)} = \frac{1}{3}, \quad c_{655,10}^{(6)} = -\frac{1}{12} + c_{555,10}^{(5)}, \]
\[ c_{624,10}^{(6)} = c_{635,10}^{(6)} = \frac{1}{6}, \quad c_{646,10}^{(6)} = \frac{1}{12}, \quad c_{656,10}^{(6)} = \frac{1}{12} + c_{556,10}^{(5)}, \]
\[ c_{644,11}^{(6)} = \frac{3}{2}, \quad c_{655,11}^{(6)} = \frac{1}{2} + c_{555,11}^{(5)}, \quad c_{466,11}^{(6)} = \frac{1}{4}, \]
\[ c_{635,11}^{(6)} = c_{646,11}^{(6)} = \frac{1}{2}, \quad c_{555,11}^{(6)} = \frac{3}{4} + c_{555,10}^{(5)}, \]
\[ c_{644,20}^{(6)} = \frac{3}{8}, \quad c_{655,20}^{(6)} = \frac{1}{8} + c_{555,10}^{(5)} \quad c_{525,20}^{(6)} = c_{646,20}^{(6)} = \frac{1}{4}, \quad c_{635,20}^{(6)} = \frac{1}{4}. \]

(5.78)

We may now extend the results given by (5.72) and (5.75), assuming (5.76), to obtain

\[
\mathcal{F}_5(v; \sigma, \tau) = (c_{666,00}^{(6)} - \frac{1}{576}) f_{576}(v) \\
+ (\sigma + \tau + \sigma^3 + \tau^3)(c_{666,10}^{(6)} - \frac{1}{36}) f_{576}(v) \\
+ \sigma \tau (1 + \sigma + \tau)(c_{666,11}^{(6)} - \frac{1}{4}) f_{576}(v) \\
+ (\sigma^2 + \tau^2)(c_{666,20}^{(6)} - \frac{1}{16}) f_{576}(v) \\
+ \sigma^4 \left( \frac{1}{24} f_{1726}(v) + \frac{1}{24} f_{2736}(v) + \frac{1}{48} f_{3746}(v) + \frac{1}{144} f_{4756}(v) + c_{666,00}^{(6)} f_{576}(v) \right) \\
+ \tau^4 \left( \frac{1}{24} f_{1762}(v) + \frac{1}{24} f_{2763}(v) + \frac{1}{48} f_{3764}(v) + \frac{1}{144} f_{4765}(v) + c_{666,00}^{(6)} f_{576}(v) \right) \\
+ \sigma^3 \tau \left( \frac{1}{3} f_{1735}(v) + \frac{1}{3} f_{2745}(v) + \frac{1}{18} f_{4765}(v) + c_{666,10}^{(6)} f_{576}(v) \right) \\
+ \sigma \tau^3 \left( \frac{1}{3} f_{1753}(v) + \frac{1}{3} f_{2754}(v) + \frac{1}{18} f_{4765}(v) + c_{666,10}^{(6)} f_{576}(v) \right) \\
+ \sigma^2 \tau^2 \left( \frac{3}{4} f_{1744}(v) + \frac{3}{8} f_{3755}(v) + c_{666,20}^{(6)} f_{576}(v) \right).
\]

(5.79)

Again the same fashion as (5.76) and in accord with (5.57) we also obtain

\[ c_{666,00}^{(6)} = \frac{1}{576}, \quad c_{666,10}^{(6)} = \frac{1}{36}, \quad c_{666,11}^{(6)} = \frac{1}{4}, \quad c_{666,20}^{(6)} = \frac{1}{16}. \]

(5.80)

5.5. General Solutions

We here discuss how the equations which follow from (5.56), assuming (5.39) with \( T_{ijk,ab}^{(p)} \) restricted as in (5.54), can be solved if we also suppose that the only contributions in the expansion for \( \mathcal{F} \) are restricted to the form shown in (5.57). The general expansion
has the form

\[ \mathcal{H}^{(p)} = -\frac{p^2}{N^2} u^p \sum_{2a+b \leq p-2} \left( \sum_{j=a+b}^{p} c_{p,j}^{(p)} T_{p,j}^{(p)} \right) + \sum_{i=p-a}^{p-1} c_{i,p}^{(p)} T_{i,p}^{(p)} + \sum_{i=p-b}^{p-1} c_{i,p-1}^{(p)} T_{i,p-1}^{(p)} \]

\[ + \sum_{i=p-a+1}^{p} \sum_{k=a+b+3}^{p} c_{i+k-p-2}^{(p)} T_{i+k-p-2}^{(p)} \]

\[ + \sum_{i=p-b+1}^{p} \sum_{j=a+b+3}^{p} c_{i+j-p-2}^{(p)} T_{i+j-p-2}^{(p)} \]

\[ + \sum_{j=p-a+1}^{p} \sum_{k=p-b+1}^{p} c_{j+k-p-2}^{(p)} T_{j+k-p-2}^{(p)} \]. \quad (5.81)\]

We first consider terms independent of \( \sigma, \tau \) which arise only from \( \sum_{j=2}^{p} c_{p,j}^{(p)} T_{p,j}^{(p)} \) where \( T_{p,j}^{(p)} \to D_{p+2,j} \). Requiring \( F(u,v;\sigma,\tau) = uf_{1322}(v) + O(\sigma,\tau) \) as in (5.72) we obtain

\[ \sum_{j=2}^{p} \sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p-j-1)! c_{p,j}^{(p)} u^{j+m-1} f_{j-1+m,j+1+m,j+m,m}(v) = uf_{1322}(v), \]

(5.82)

which requires

\[ (p-2)! c_{p22,00} = 1, \quad \sum_{m=0}^{k-2} \frac{(-1)^m}{m!} c_{p,k-m-k-m,00} = 0, \quad k = 3, \ldots, p. \]

(5.83)

This is easily solved giving

\[ c_{p,j}^{(p)} = \frac{1}{(p-2)!(j-2)!}, \quad j = 2, \ldots, p. \]

(5.84)

We next consider the calculation of the coefficients \( c_{ij}^{(p),ab} \) for \( a \geq 1, b = 0 \). These are determined in (5.56) by the terms in the expansion (5.57) with \( g = a, h = 0 \). Contributions proportional to \( \sigma^g \) first arise in an expansion in powers of \( u \) of \( \mathcal{H}^{(p)} \) from \( \sum_{j=2}^{p} c_{p,j}^{(p)} T_{p,j}^{(p)} \), with \( T_{p,j}^{(p)}(u,v;\sigma,\tau) \to \sigma^p \mathcal{D}_{p+2,jp}(u,v) \), for \( p = g+2 \). Using (5.56) with (5.84) this gives the relevant contribution to \( F \) for this case,

\[ F(u,v;\sigma,\tau)|_{\sigma^g} = \sigma^g u^{g+1} \frac{1}{g!} \sum_{j=2}^{g+2} \frac{1}{(j-2)!} f_{j-1+g+3,jg+2}(v). \]

(5.85)
Assuming this form for $\mathcal{F}$ in general then for $p \geq a + 3$ (5.56) requires, for the contributions which involve powers $u^n$ with $n < p$ arising only from the $T_{i p j j, a 0}^{(p)}$, $T_{i i + k - p - 2, a 0}^{(p)}$ and $T_{i i p, a 0}^{(p)}$ terms in (5.81), keeping just the first term in (5.40),

$$
\sum_{j=a+2}^{p} c_{p j j, a 0}^{(p)} \sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p - j - m)! u^{j+m-1} f_{j-1+m-j+1+m+j+m}(v) 
+ \sum_{i=p-a+1}^{p} \sum_{k=a+3}^{p} c_{i i + k - p - 2, a 0}^{(p)} \sum_{m=0}^{p-k+1} \frac{(-1)^m}{m!} (p - k + 1 - m)! \times u^{k+m-2} f_{i+k-p-2+m+k-m+i+k-p-2+m+k+m}(v) 
+ \sum_{i=p-a}^{p-1} c_{i i p, a 0}^{(p)} u^{p-1} f_{i-1+p+1 i p}(v) 
= u^{a+1} \frac{1}{a!} \sum_{j=2}^{a+2} \frac{1}{(j-2)!} f_{j-1+a+3 \cdot j+a+2}(v).
$$

(5.86)

To analyse (5.86) we consider first all terms proportional to $u^{a+1}$ when we obtain

$$
(p - a - 2)! c_{p a+2 a+2, a 0}^{(p)} f_{a+1 a+3 a+2 a+2}(v) 
+ (p - a - 2)! \sum_{i=p-a+1}^{p} c_{i i+a-p+1 a+3 a+2 a+2}(v) 
= \frac{1}{a!} \sum_{j=2}^{a+2} \frac{1}{(j-2)!} f_{j-1+a+3 \cdot j+a+2}(v).
$$

(5.87)

Applying (5.45) repeatedly for $f_{j-1+a+3 \cdot j+a+2}(v)$ we then get

$$
c_{i i+a-p+1 a+3 a+2, a 0}^{(p)} = \frac{1}{(p - a - 2)! a! (i + a - p - 1)!}, \quad i = p - a + 1, \ldots, p,
$$

$$
c_{p a+2 a+2, a 0}^{(p)} = \frac{a+1}{(p - a - 2)! a!^2}.
$$

(5.88)

From contributions in (5.86) proportional to $u^{k-1} f_{k-1+k+1 k k}(v)$, for $k = a + 3, \ldots, p - 1$, and $u^{k-2} f_{i+k-p-2 k i+k-p-2 k}(v)$, for $k = a + 4, \ldots, p$, $i = p - a + 1, \ldots, p$, we get

$$
\sum_{m=0}^{k-a-2} \frac{(-1)^m}{m!} c_{p k-m-k-m, a 0}^{(p)} = 0, \quad \sum_{m=0}^{k-a-3} \frac{(-1)^m}{m!} c_{i i+k-p-2-m k-m, a 0}^{(p)} = 0,
$$

(5.89)

which in conjunction with (5.88) may be solved giving

$$
c_{p j j, a 0}^{(p)} = \frac{a+1}{(p - a - 2)! a!^2 (j-a-2)!}, \quad j = p + 2, \ldots, p - 1,
$$

$$
c_{i i+k-p-2 k, a 0}^{(p)} = \frac{1}{(p - a - 2)! a! (i + a - p - 1)! (k - a - 3)!}, \quad k = a + 3, \ldots, p, \quad i = p - a + 1, \ldots, p.
$$

(5.90)
Using (5.90) the remaining part of (5.86) becomes

\[
\begin{align*}
&\left(\left(\frac{c^{(p)}_{ppp,a_0}}{a + 1}\right) - \frac{a + 1}{(p - a - 2)!^2 a!^2}\right) f_{p - 1 + p + 1}(v) \\
&- \frac{1}{(p - a - 2)!^2 a!} \sum_{j=p-a+1}^{p} \frac{1}{(j + a - p - 1)!} f_{j - 1 + j - 1 + 1}(v) \\
&+ \sum_{i=p-a}^{p-1} c^{(p)}_{ii p, a_0} f_{i - 1 + i + p}(v) = 0.
\end{align*}
\]

(5.91)

With the aid of (5.45) again we finally obtain for this case

\[
\begin{align*}
&\frac{c^{(p)}_{ppp,a_0}}{a!^2} = \frac{1}{(p - a - 2)!^2 a!^2}, \\
&\frac{c^{(p)}_{ii p, a_0}}{a!} = \frac{a + 1}{(p - a - 2)!^2 a! (i + a - p)!}, \quad i = p - a, \ldots, p - 1.
\end{align*}
\]

(5.92)

For \( p = 2a + 2 \) the symmetry conditions (5.41) ensure that in the expansion (5.81) we may require \( c^{(p)}_{p j j, a_0} = c^{(p)}_{j j p, a_0} \) and \( c^{(p)}_{i i + k - p - 2 k, a_0} \) is symmetric in \( i, k \). With these constraints instead of (5.86) we have now

\[
\begin{align*}
&\sum_{j=a+2}^{p} c^{(p)}_{p j j, a_0} \left( \sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p - j - m)! u^{j+m-1} f_{j-1+m j+1+m j+m j+m}(v) \\
&+ u^{p-1} f_{j-1+p+1 j p}(v) \right) \\
&+ \sum_{i,k=a+3}^{p} c^{(p)}_{i i + k - p - 2 k, a_0} \sum_{m=0}^{p-k+1} \frac{(-1)^m}{m!} (p - k + 1 - m)! \times u^{k+m-2} f_{i+k-p-2+m k+m i+k-p-2+m k+m}(v) \\
&= u^{a+1} \sum_{j=2}^{a+2} \frac{1}{a!} \sum_{j=2}^{j-2} \frac{1}{(j-2)!} f_{j-1+a+3 j a+2}(v).
\end{align*}
\]

(5.93)

The solution of (5.93) is essentially as before giving in this case

\[
\begin{align*}
&c^{(p)}_{ppp,a_0} = \frac{1}{a!}, \quad c^{(p)}_{p j j, a_0} = \frac{a + 1}{a!^3 (j - a - 2)!}, \quad j = a + 2, \ldots, p - 1. \\
&c^{(p)}_{i i + k - p - 2 k, a_0} = \frac{1}{a!^2 (i - a - 3)! (k - a - 3)!}, \quad i, k = a + 3, \ldots, p.
\end{align*}
\]

(5.94)

These results are just as expected from (5.90) and (5.92) after substituting \( p = 2a + 2 \). The results manifestly satisfy the necessary symmetry conditions.
For completeness it is also necessary to analyse the other contributions which are present in crossing symmetric expressions exhibited in (5.40) for $T_{pjj,a0}^{(p)}$, $T_{i+i+k−p−2,k,a0}^{(p)}$ and $T_{iip,a0}^{(p)}$. The six terms in (5.40) form three pairs related by $\sigma \leftrightarrow \tau$ under which our equations are invariant. For those terms proportional to $\sigma^{p-2-a}$ we obtain

$$
\sigma^{p-2-a} \left( \sum_{j=a+2}^{p} c_{pjj,a0}^{(p)} u^{p-1} f_{j-1} p_{j} p(v) \right.
\left. + \sum_{i=p-a+1}^{p} \sum_{k=a+3}^{p} c_{i+k−p−2,k,a0}^{(p)} \frac{(-1)^{m}}{m!} (p−i+1−m)! \times u^{i+m−2} f_{i+k−p−2+m i+m i+k−p−2+m i+m(v)} \right)
\left. + \sum_{i=p-a}^{p-1} \sum_{m=0}^{p-i} \frac{(-1)^{m}}{m!} (p+1−i−m)! u^{i+m−1} f_{i+1+m i+1+m i+m i+m(v)} \right)
\left. = \sigma^{p-2-a} u^{p-1-a} \frac{1}{(p-2-a)!} \sum_{j=2}^{p-a} \frac{1}{(j-2)!} f_{j−1} p−a+1 j p−a(v) \right)
\left. = F(u, v; \sigma, \tau)|_{\sigma^{p-2-a}}, \quad (5.95) \right.
$$

using (5.85). The identity shown in (5.95) is obtained by following the identical procedure as in calculations described above after using

$$
c_{i+k−p−2,k,a0}^{(p)} \leftrightarrow c_{k+i+k−p−2 i,a0}^{(p)}, \quad c_{pjj,a0}^{(p)} \leftrightarrow c_{jjp,a0}^{(p)}, \quad \text{for} \hspace{0.2cm} a \rightarrow p − a − 2, \hspace{0.2cm} (5.96)
$$

which are easily seen to be a property of the solutions (5.90) and (5.92). The result (5.95) is then in accord with expectation from (5.56).

For the remaining terms we consider those proportional to $\sigma^{p−2−a} \tau^{a}$ for which, up to terms of $O(u^p)$, we have just

$$
\sigma^{p−2−a} \tau^{a} u^{p−1} \left( \sum_{j=a+2}^{p} c_{pjj,a0}^{(p)} f_{j−1} p_{j} p(v) + \sum_{i=p−a}^{p−1} c_{iip,a0}^{(p)} f_{i−1} p_{i} p(v) \right). \quad (5.97)
$$

These are identified as required by (5.56) with the following term in $F$, after using the
expressions (5.90) and (5.92),

\[ F(u, v; \sigma, \tau)\big|_{\sigma^g\tau^h} = \sigma^g \tau^h u^{g+h+1} \left( \sum_{j=h+2}^{g+h+1} \frac{h+1}{g! \cdot (j-h-2)!} f_{j-1} g+h+3 j g+h+2(v) \right) \]

\[ + \frac{g+1}{g! \cdot (j-h-2)!} f_{j-1} g+h+3 g+h+2 j(v) \]

\[ + \frac{1}{g! \cdot (j-h-2)!} f_{g+h+1} g+h+3 g+h+2 g+h+2(v) \] \hspace{1cm} (5.98)

This result is obtained from (5.97) for \( g > h \), the corresponding result for \( g < h \) is obtained by using the symmetry under \( \sigma \leftrightarrow \tau \), for \( g = h \) it is necessary to use (5.96). For \( h = 0 \) (5.98) coincides with (5.85). Although (5.98) is not immediately of the form expected from (5.57) and (5.58), it can be reduced to it by application of (5.45).

With the determination of \( F \) in general in (5.98) we can now determine the remaining coefficients in (5.81). For terms proportional to \( \sigma^a \tau^b \) in (5.56) we have, corresponding to (5.86),

\[ \sum_{j=a+b+2}^{p} c_{ppj,ab}^{(p)} \sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p-j-m)! u^{j+m-1} f_{j-1+m} j+1+m j+m+1(v) \]

\[ + \sum_{i=p-a+1}^{p} \sum_{k=a+b+3}^{p} c_{i+k-p-2,ab}^{(p)} \sum_{m=0}^{p-k+1} \frac{(-1)^m}{m!} (p-k+1-m)! \]

\[ \times u^{k+m-2} f_{i+k-p-2+m} k+m i+k-p-2+m k+m(v) \]

\[ + \sum_{i=p-b+1}^{p} \sum_{j=a+b+3}^{p} c_{i+j-p-2,ab}^{(p)} \sum_{m=0}^{p-j+1} \frac{(-1)^m}{m!} (p-j+1-m)! \]

\[ \times u^{j+m-2} f_{i+j-p-2+m} j+m i+j-p-2+m j+m(v) \]

\[ + \sum_{i=p-a}^{p-1} c_{ip,ab}^{(p)} u^{p-1} f_{i-1+p+1} i p(v) + \sum_{i=p-b}^{p-1} c_{ip,ab}^{(p)} u^{p-1} f_{i-1+p+1} i p(v) \] \hspace{1cm} (5.99)

\[ = u^{a+b+1} \left( \sum_{j=b+2}^{a+b+1} \frac{b+1}{a! b! (j-b-2)!} f_{j-1} a+b+3 j a+b+2(v) \right) \]

\[ + \sum_{j=a+2}^{a+b+1} \frac{a+1}{a! b! (j-a-2)!} f_{j-1} a+b+3 a+b+2 j(v) \]

\[ + \frac{1}{a! b!} f_{a+b+1} a+b+3 a+b+2 a+b+2(v) \]
This may be analysed in a similar fashion to previously. For terms proportional to \( u^{a+b+1} \),

\[
(p-a-b-2)! \sum_{i=p-a+1}^{p} c_{i \cdot a+b+2}^{(p)} \sum_{a+b+2,ab} f_{a+b+1} \sum_{a+b+3}^{a+b+2} a+b+2(v) \\
+ (p-a-b-2)! \sum_{i=p-b+1}^{p} c_{i \cdot a+b+2}^{(p)} \sum_{a+b+2}^{a+b+1} f_{i+a+b-p+1} \sum_{a+b+3}^{a+b+2} a+b+2(v) \\
+ (p-a-b-2)! \sum_{i=p-a+1}^{p} c_{i \cdot a+b+2}^{(p)} \sum_{a+b+2}^{a+b+1} f_{i+a+b-p+1} \sum_{a+b+3}^{a+b+2} a+b+2(v) \\
= \sum_{j=a+2}^{a+b+1} \frac{a+1}{a! b! ((j-a-2))!} f_{j-1} a+b+3 a+b+2 j(v) + 1 \frac{1}{a^{2} b!} f_{a+b+1} a+b+3 a+b+2 a+b+2(v). \tag{5.100}
\]

Using (5.45) this may be decomposed to give

\[
c_{i \cdot a+b+i-p+1}^{(p)} a+b+3,ab = \frac{1}{(p-a-b-2)! a! b! (i+a-p-1)!}, \quad i = p-a+1, \ldots, p, \\
c_{i \cdot a+b+3}^{(p)} a+b+i-p+1,ab = \frac{1}{(p-a-b-2)! a! b! (i+b-p-1)!}, \quad i = p-b+1, \ldots, p, \\
c_{p \cdot a+b+2}^{(p)} a+b+2,ab = \frac{a+b+1}{(p-a-b-2)! a! b!}. \tag{5.101}
\]

To obtain (5.101) we have used the identity \( \sum_{m=0}^{k}(n-1+m)!/m! = (n+k)!/n k! \). For terms proportional to \( u^{n} \) in (5.99) for \( n = a+b+2, \ldots, p-2 \) we get

\[
k-a-b-2 \sum_{m=0}^{k} \frac{(-1)^{m}}{m!} c_{p \cdot k-m-k-m,ab}^{(p)} = 0, \quad k = a+b+3, \ldots, p-1, \\
k-a-b-3 \sum_{m=0}^{k} \frac{(-1)^{m}}{m!} c_{i \cdot k-p-2-m-k-m,ab}^{(p)} = 0, \quad \left\{ \begin{array}{l}
i = p-a+1, \ldots, p \\\nk = a+b+4, \ldots, p \end{array} \right., \tag{5.102} \\
j-a-b-3 \sum_{m=0}^{j} \frac{(-1)^{m}}{m!} c_{i \cdot j-m+i-p-2-m,ab}^{(p)} = 0, \quad \left\{ \begin{array}{l}
i = p-b+1, \ldots, p \\\nj = a+b+4, \ldots, p \end{array} \right.
\]
Combining with (5.101) then gives

\[
\begin{align*}
C_{pjj,ab}^{(p)} &= \frac{a + b + 1}{(p - a - b - 2)! a!^2 b!^2 (j - a - b - 2)!}, \quad j = a + b + 2, \ldots, p - 1, \\
C_{ki+k-p-2,k,ab}^{(p)} &= \frac{1}{(p - a - b - 2)! a! b! (i + a - p - 1)! (k - a - b - 3)!}, \quad k = a + b + 3, \ldots, p, i = p - a + 1, \ldots, p, \\
C_{ij+i+j-p-2,ab}^{(p)} &= \frac{1}{(p - a - b - 2)! a!^2 b! (i + b - p - 1)! (j - a - b - 3)!}, \quad j = a + b + 3, \ldots, p, i = p - b + 1, \ldots, p.
\end{align*}
\]

For the remaining terms in (5.99) proportional to \( u^{p-1} \) after using the result (5.103) we have

\[
\left( C_{ppp,ab}^{(p)} - \frac{a + b + 1}{(p - a - b - 2)! a!^2 b!^2} \right) f_{p-1p+1pp}(v) - \frac{1}{(p - a - b - 2)!^2 a! b!^2} \sum_{j=p-a+1}^{p} \frac{1}{(j + a - p - 1)!} f_{j-1p+1j-1p+1}(v)
\]

\[
- \frac{1}{(p - a - b - 2)! a! b!} \sum_{j=p-b+1}^{p} \frac{1}{(j + b - p - 1)!} f_{j-1p+1p+1j-1}(v)
\]

\[
+ \sum_{i=p-a}^{p-1} c_{iip,ab}^{(p)} f_{i-1p+1i+1p}(v) + \sum_{i=p-b}^{p-1} c_{ipi,ab}^{(p)} f_{i-1p+1i+1p}(v) = 0.
\]

This may be solved giving

\[
C_{ppp,ab}^{(p)} = \frac{1}{(p - a - b - 2)!^2 a! b!^2},
\]

\[
C_{iip,ab}^{(p)} = \frac{p - a - 1}{(p - a - b - 2)!^2 a! b!^2 (i + a - p)!}, \quad i = p - a, \ldots, p - 1,
\]

\[
C_{ipi,ab}^{(p)} = \frac{p - b - 1}{(p - a - b - 2)!^2 a! b!^2 (i + b - p)!}, \quad i = p - b, \ldots, p - 1.
\]

The coefficients satisfy the crucial relations

\[
C_{i+j+p-2,j,ab}^{(p)} \leftrightarrow C_{j+i+j+p-2,i,ab}^{(p)}, \quad C_{i+j+p-2,a,ab}^{(p)} \leftrightarrow C_{i+j+p-2,i,ab}^{(p)},
\]

\[
C_{iip,ab}^{(p)} \leftrightarrow C_{ipi,ab}^{(p)}, \quad C_{ipi,ab}^{(p)} \leftrightarrow C_{ipi,ab}^{(p)}, \quad \text{for } a \to p - 2 - a - b,
\]

and

\[
C_{i+j+p-2,j,ab}^{(p)} \leftrightarrow C_{i+j+p-2,i,ab}^{(p)}, \quad C_{i+j+p-2,a,ab}^{(p)} \leftrightarrow C_{i+j+p-2,i,ab}^{(p)},
\]

\[
C_{iip,ab}^{(p)} \leftrightarrow C_{ipi,ab}^{(p)}, \quad C_{ipi,ab}^{(p)} \leftrightarrow C_{ipi,ab}^{(p)}, \quad \text{for } b \to p - 2 - a - b.
\]
There is also a similar relation for $a \leftrightarrow b$ which can be obtained by combining (5.106) and (5.107). These relations require for the undetermined coefficient so far

$$c_{j+k-p-2,j,k,ab}^{(p)} = \frac{1}{(p-a-b-2)!a!b!(j+a-p-1)!(k+b-p-1)!},$$  

(5.108)

$$j = p-a+1, \ldots, p, k = p-b+1, \ldots, p.$$  

The results (5.103), (5.105) and (5.108) hence determine the expansion of $\mathcal{H}^{(p)}$ for general $p$. The symmetry conditions (5.106) and (5.107) are necessary to ensure that the other terms in the expression for $T_{ijk,ab}^{(p)}$, defined in (5.40), contribute as required to the terms in $\mathcal{F}$ proportional to $\sigma^g \tau^h$ with $g = p-2-a-b$, $h = b$ and $g = a, h = p-2-a-b$. The results given by (5.103), (5.105) and (5.108) also satisfy

$$c_{ij i+j-p-2,aa}^{(p)} = c_{i i+j-p-2,ja}^{(p)}, \quad c_{ij+k-p-2,jk,aa}^{(p)} = c_{j+k-p-2,jk,aa}^{(p)},$$

$$c_{ii,p,aa}^{(p)} = c_{ii,p,aa}^{(p)}, \quad c_{ii,p,aa}^{(p)} = c_{ii,p,aa}^{(p)},$$

$$c_{ij i+j-p-2,ap-2-2a}^{(p)} = c_{i+j-p-2,ij,a p-2-2a}^{(p)}, \quad c_{ij+k-p-2,ap-2-2a}^{(p)} = c_{i+k-p-2,ap-2-2a}^{(p)},$$

$$c_{ii,p,ap-2-2a}^{(p)} = c_{ii,p,ap-2-2a}^{(p)}, \quad c_{ii,p,ap-2-2a}^{(p)} = c_{ii,p,ap-2-2a}^{(p)},$$  

(5.109)

which shows that they remain valid in these cases when the symmetry requirements in (5.41) hold.
6. Conclusion and Future Investigations

In this thesis we have derived $\mathcal{N} = 2, 4$ superconformal Ward identities for four point functions of $\frac{1}{2}$-BPS operators. They were obtained by considering invariance of a correlation function obtained from the pure CPO correlation function by action of one supercharge. In that sense the Ward identities are a manifestation of the first order action of the supercharges.

There could possibly exist further identities which require consideration of higher order actions of supercharges, although we have no reason to believe so. A related question is whether all four point functions of all members of the multiplet generated from the CPOs we considered can be uniquely obtained by action of a differential operator acting on the pure CPO four point function. For the case of three point functions for the case $p = 2$ this was done by hand in [5]. Ideally we would like to show that there is a set of differential operators which together with the superconformal constraints derived here uniquely generate all four point functions for the short multiplets we considered.

Another area of possible future investigation is whether further constraints from the operator product expansion can be developed to extend the requirements superconformal symmetry imposes. For example crossing symmetry allowed us to fix the form of the one variable function completely to free field form. This argument is restricted to the case of three identical fields in the four point function.

Our results for the four point functions of $[0, p, 0] \frac{1}{2}$-BPS operators in the large $N$ limit suggest a form for the averaged first order anomalous dimension for general $p$ at order $1/N$. To fix the concrete expression we would have to find a way to compute the constants present in the results for $p = 2, 3, 4$ for general $p$. It would be very interesting to compute the large $N$ amplitude for $p = 5$ which could confirm this suggestion and could possibly shed some light on the structure of the coefficients determining the averaged first order anomalous dimensions.

Finally, we followed up on an observation made in [15] about the universal singularity structure in $u$ of the large $N$ amplitudes and showed how this might be exploited together
with crossing symmetry to restrict the possible expressions for amplitudes with higher $p$. By making one additional technical assumption about the details of the singularity structure we managed to derive explicit expressions for all coefficients of $\bar{D}$ functions appearing in a general $p$ large $N$ four point amplitude of four identical $\frac{1}{2}$-BPS single trace operators.

Our assumptions about the singularity structure are attempts to generalize from the cases explicitly known for $p = 2, 3, 4$ [13,9,14,15]. They are based on the expectation that all long operators should receive anomalous dimensions. Therefore we assume the cancellation of all terms with lower order than $p$ in $u$. An important step further in this direction would be the computation of the free field amplitude which would provide an independent derivation of the function $F$ and a check on the first terms we computed for $F$ from the dynamical part of the amplitude. To do this, also the ambiguity in the split of the amplitude into $\mathcal{H}, \hat{f}$ has to be understood better. Also, once these results were established one could attempt to use them to derive the structure of the general averaged first order anomalous dimension as mentioned above.
Appendix A. Results for Null Vectors

We discuss here some results for null vectors \( t \), which are useful in the text. For generality we allow \( t \) to be \( d \)-dimensional. As a consequence of (1.31) differentiation requires some care but for any null vector \( a \) we may define as usual

\[
\frac{\partial}{\partial t} (a \cdot t)^n = n (a \cdot t)^{n-1} a .
\]  

(A.1)

More generally for a set of null vectors \( a_1, a_2, \ldots, a_p \) we have

\[
\frac{\partial}{\partial t} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = \sum_{i=1}^{p} n_i (a_1 \cdot t)^{n_1} \ldots (a_i \cdot t)^{n_i-1} \ldots (a_p \cdot t)^{n_p} a_i \\
- R \sum_{1 \leq i < j \leq p} n_i n_j a_i \cdot a_j (a_1 \cdot t)^{n_1} \ldots (a_i \cdot t)^{n_i-1} \ldots (a_j \cdot t)^{n_j-1} \ldots (a_p \cdot t)^{n_p} t ,
\]

(A.2)

where

\[
R = \frac{2}{2N + d - 4} , \quad N = \sum_{i=1}^{p} n_i .
\]  

(A.3)

The right hand side of (A.2), with (A.3), may be represented in the form

\[
\left( \frac{\partial}{\partial t} - t \frac{1}{2t \cdot \partial + d} \partial^2 \right) \prod_{i=1}^{p} (a_i \cdot t)^{n_i} ,
\]  

(A.4)

where the action of the derivatives is as usual, without regard to the constraint \( t^2 = 0 \). The resulting operator is equivalent to a definition given in [11] for an interior differential operator on the complex null cone.

From (A.2) we may readily find

\[
\left[ \frac{\partial}{\partial t_r}, \frac{\partial}{\partial t_s} \right] \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = 0 , \quad \frac{\partial}{\partial t} \frac{\partial}{\partial t} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = 0 , \quad t \cdot \frac{\partial}{\partial t} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = N \prod_{i=1}^{p} (a_i \cdot t)^{n_i} .
\]

(A.5)

We also have

\[
\left[ \frac{\partial}{\partial t_r}, t_s \right] \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = \left( \delta_{rs} - \frac{2}{2N + d - 2} t_r \frac{\partial}{\partial t_s} \right) \prod_{i=1}^{p} (a_i \cdot t)^{n_i} ,
\]

(A.6)
which implies
\[ \frac{\partial}{\partial t_r} \left( t_r \prod_{i=1}^{p} (a_i \cdot t)^{n_i} \right) = \frac{(2N + d)(N + d - 2)}{2N + d - 2} \prod_{i=1}^{p} (a_i \cdot t)^{n_i}. \] (A.7)

Defining the generators of $SO(d)$ by
\[ L_{rs} = t_r \partial_s - t_s \partial_r, \] (A.8)
then the above results give
\[ L_{rs} L_{su} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = -((N + d - 3) t_r \partial_s + (N - 1) t_s \partial_r + N \delta_{rs}) \prod_{i=1}^{p} (a_i \cdot t)^{n_i}, \] (A.9)
and
\[ \frac{1}{2} L_{rs} L_{su} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = -N(N + d - 2) \prod_{i=1}^{p} (a_i \cdot t)^{n_i}, \] (A.10)
which reproduces the appropriate eigenvalue of the Casimir operator for the representation formed by traceless rank $N$ tensors.

If $V_r(t)$ is homogeneous of degree $N$ then in general
\[ V_r = \hat{V}_r + \frac{1}{N + 1} \frac{\partial}{\partial t_r} (t_s V_s), \quad t_r \hat{V}_r = 0, \] (A.11)
as used in (2.21) and (3.12). If $V_r$ also satisfies
\[ \frac{\partial}{\partial t_r} V_s - \frac{\partial}{\partial t_s} V_r = 0, \quad \frac{\partial}{\partial t_r} V_r = 0, \] (A.12)
then, by contracting with $t_s$ and using (A.5), (A.6), we easily see that $\hat{V}_r = 0$. As a further corollary if $V_r = \partial_s U_{rs}$, $U_{rs} = -U_{sr}$, $\partial_{[r} U_{su]} = 0$ then $(N + 1)V_r = \partial_r (\frac{1}{2} L_{su} U_{su})$ with $L_{su}$ as in (A.8). In general we have the decomposition
\[ V_r = \frac{2N + d - 4}{(2N + d - 2)(N + d - 3)} t_r \partial V \]
\[- \frac{1}{(2N + d)(N + d - 3)} ((2N + d - 2) \partial_s (t_r V_s - t_s V_r) + 2 \partial_r (t \cdot V)). \] (A.13)

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Appendix B. Two Variable Harmonic Polynomials

For the expansion of four point functions in terms of $R$-symmetry representations we consider here the eigenfunctions of the $SO(d)$ Casimir operator

\[ L^2 = \frac{1}{2} L_{rs} L_{rs}, \quad (B.1) \]

where the generators are

\[ L_{rs} = t_1 r_1 s - t_1 s_1 r + t_2 r_2 s - t_2 s_2 r, \quad (B.2) \]

formed by homogeneous functions of the null vectors $t_1, t_2, t_3, t_4$. Obviously $L_{rs} t_1 t_2 = 0$ and hence $L^2 (t_1 t_2)^k (t_3 t_4)^l f(\sigma, \tau) = (t_1 t_2)^k (t_3 t_4)^l L^2 f(\sigma, \tau)$, where $\sigma, \tau$ are given by (1.33). We therefore first consider eigenfunctions which are polynomials in $\sigma, \tau$

\[ Y(\sigma, \tau) = \sum_{t \geq 0} \sum_{q = 0}^t c_{t,q} \sigma^{t-q} \tau^q, \quad (B.3) \]

satisfying

\[ L^2 Y(\sigma, \tau) = -2 CY(\sigma, \tau). \quad (B.4) \]

With the aid of the given in appendix A we may easily calculate the action of $L^2$ on a monomial formed from $\sigma, \tau$,

\[ L^2 (\sigma^p \tau^q) = -2((d-2)(p+q) + 4pq)\sigma^p \tau^q + 2(1-\sigma-\tau)(p^2 \sigma^{p-1} \tau^q + q^2 \sigma^p \tau^{q-1}), \quad (B.5) \]

or

\[ \frac{1}{2} L^2 \rightarrow D_d = (1-\sigma-\tau) \left( \frac{\partial}{\partial \sigma} \sigma \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau} \tau \frac{\partial}{\partial \tau} \right) - 4\sigma \tau \frac{\partial^2}{\partial \sigma \partial \tau} - (d-2) \left( \sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} \right). \quad (B.6) \]

Alternatively $D_d$ may be written in the form

\[ D_d = \frac{1}{w} \partial^T w G \partial, \quad G = \begin{pmatrix} \sigma(1-\sigma-\tau) & -2\sigma \tau \\ -2\sigma \tau & \tau(1-\sigma-\tau) \end{pmatrix}, \quad \partial = \begin{pmatrix} \partial_\sigma \\ \partial_\tau \end{pmatrix}, \quad (B.7) \]

where, with $\Lambda = (\sqrt{\sigma} + \sqrt{\tau} + 1)(\sqrt{\sigma} + \sqrt{\tau} - 1)(\sqrt{\sigma} - \sqrt{\tau} + 1)(\sqrt{\sigma} - \sqrt{\tau} - 1)$ as in (2.39),

\[ w = \Lambda^{\frac{d}{2}}(d-5). \quad (B.8) \]
In general, for a polynomial as in (B.3) with $t_{\text{max}} = n$, we must have that $c_{n,q}$ forms an eigenvector for an $(n+1) \times (n+1)$ matrix $M_n$,

$$M_{n,pq}c_{n,q} = Cc_{n,p}, \quad M_{n,pq} = \delta_{pq}(n(n+d-2)+2p(n-p))+\delta_{p,q-1}q^2+\delta_{p,q+1}(n-q)^2. \quad (B.9)$$

The coefficients $c_{t,q}$ with $t < n$ may then be obtained by solving recurrence relations. For given $n$ there are $n+1$ eigenvectors solving (B.9) and the corresponding eigenfuctions are

$$Y_{nm}(\sigma,\tau), \quad C_{nm} = n(n+d-3) + m(m+1), \quad n = 0,1,2,\ldots, m = 0,\ldots n. \quad (B.10)$$

As a consequence of (B.7) and (B.8) the polynomials are orthogonal for $d > 5$ with respect to integration over $\sigma, \tau \geq 0, \sqrt{\sigma} + \sqrt{\tau} \leq 1$ with weight $w$ (for a general discussion of such two variable orthogonal polynomials see [35,36]).

The polynomials $Y_{nm}$ are also eigenfunctions for higher order Casimir invariants. Letting

$$\frac{1}{4} L_{rs}L_{st}L_{tu}L_{ur} - \frac{1}{2} L^2 L^2 + \frac{1}{4}(d-2)(d-3)L^2 \to Q$$

then acting on any $Y(\sigma,\tau)$ we may express $Q$ in a form similar to (B.7),

$$Q = \frac{1}{\Lambda^{\frac{1}{2}}(d-5)} (\partial_\sigma^2 \partial_\tau^2) \Lambda^{\frac{1}{2}}(d-3) \left( \begin{array}{cc} \sigma^2 & -\sigma \tau \\ -\sigma \tau & \tau^2 \end{array} \right) \left( \begin{array}{c} \partial_\sigma^2 \\ \partial_\tau^2 \end{array} \right)$$

$$+ (d-3) \frac{1}{\Lambda^{\frac{1}{2}}(d-5)} (\partial_\sigma \partial_\tau) \Lambda^{\frac{1}{2}}(d-5) \left( \begin{array}{cc} 2\sigma & \sigma + \tau - 1 \\ \sigma + \tau - 1 & 2\tau \end{array} \right) \left( \begin{array}{c} \partial_\sigma \\ \partial_\tau \end{array} \right)$$

$$+ (d-2) \frac{1}{\Lambda^{\frac{1}{2}}(d-5)} (\partial_\sigma \Lambda^{\frac{1}{2}}(d-3) \partial_\tau + \partial_\tau \Lambda^{\frac{1}{2}}(d-3) \partial_\sigma). \quad (B.12)$$

The harmonic polynomials then satisfy

$$Q Y_{nm} = -(n - m)(n + m + 1)(n + m + d - 3)(n - m + d - 4) Y_{nm}. \quad (B.13)$$

Using (B.5) it is straightforward to construct the first few eigenfunctions satisfying
(B.4) by hand. With an arbitrary normalisation, we find for \( n = 0, 1, 2, 3, \)

\[
Y_{00}(\sigma, \tau) = 1,
Y_{10}(\sigma, \tau) = \sigma - \tau,
Y_{11}(\sigma, \tau) = \sigma + \tau - \frac{2}{d},
Y_{20}(\sigma, \tau) = \sigma^2 + \tau^2 - 2\sigma\tau - \frac{2}{d-2} (\sigma + \tau) + \frac{2}{(d-2)(d-1)},
Y_{21}(\sigma, \tau) = \sigma^2 - \tau^2 - \frac{4}{d+2} (\sigma - \tau),
Y_{22}(\sigma, \tau) = \sigma^2 + \tau^2 + 4\sigma\tau - \frac{8}{d+4} (\sigma + \tau) + \frac{8}{(d+2)(d+4)},
Y_{30}(\sigma, \tau) = \sigma^3 - 3\sigma^2\tau + 3\sigma\tau^2 - \tau^3 - \frac{6}{d} (\sigma^2 - \tau^2) + \frac{12}{d(d+1)} (\sigma - \tau),
Y_{31}(\sigma, \tau) = \sigma^3 - \sigma^2\tau - \sigma\tau^2 + \tau^3 - \frac{8(d-1)}{(d+4)(d-2)} (\sigma^2 + \tau^2) + \frac{8(d-2)}{(d+4)(d-2)} \sigma \tau + \frac{4(3d+2)}{(d+1)(d+4)(d-2)} (\sigma + \tau) - \frac{8}{(d+1)(d+4)(d-2)},
Y_{32}(\sigma, \tau) = \sigma^3 + 3\sigma^2\tau - 3\sigma\tau^2 - \tau^3 - \frac{12}{d+6} (\sigma^2 - \tau^2) + \frac{24}{(d+4)(d+6)} (\sigma - \tau),
Y_{33}(\sigma, \tau) = \sigma^3 + 9\sigma^2\tau + 9\sigma\tau^2 + \tau^3 - \frac{18}{d+8} (\sigma^2 + \tau^2) - \frac{72}{d+8} \sigma \tau + \frac{72}{(d+6)(d+8)} (\sigma + \tau) - \frac{48}{(d+4)(d+6)(d+8)}.
\]

Up to an overall normalisation for \( d = 6 \) each term may be identified with terms in the projection operators constructed in [9] where \( Y_{nm} \) corresponds to the \( SU(4) \simeq SO(6) \) representation with Dynkin labels \([n-m, 2m, n-m]\). For \( m = n \) in (B.10) we have \( c_{n,q} = \binom{n}{q}^2 \) and the recurrence relations may be easily solved giving

\[
Y_{nn}(\sigma, \tau) = A_n F_4(-n, n + \frac{1}{2} d - 1; 1, 1; \sigma, \tau),
\]

where \( F_4 \) is one of Appell’s generalised hypergeometric functions\(^8\) and \( A_n \) is some overall constant.

To obtain more general forms (see [37]) we used the variables \( \alpha, \bar{\alpha} \) defined in (3.41). Acting on \( Y(\sigma, \tau) = \mathcal{P}(\alpha, \bar{\alpha}) = \mathcal{P}(\bar{\alpha}, \alpha) \)

\[
\frac{1}{2} L^2 \mathcal{P}(\alpha, \bar{\alpha}) = \hat{D}_d \mathcal{P}(\alpha, \bar{\alpha}),
\]

\[\]

\[
F_4(a, b; c, c'; x, y) = \sum_{m,n} \binom{a}{m+n} \binom{b}{m+n} (c) (c')_{n! m!} x^m y^n.
\]

\(^8\)
where, using (B.5) or (B.6), we now have

\[ \hat{D}_d = \frac{\partial}{\partial \alpha} \alpha (1 - \alpha) \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha} \bar{\alpha} (1 - \bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}} + (d - 4) \frac{1}{\alpha - \bar{\alpha}} \left( \alpha (1 - \alpha) \frac{\partial}{\partial \alpha} - \bar{\alpha} (1 - \bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}} \right). \]  

(B.17)

Corresponding to (B.4) and (B.10) we have

\[ \hat{D}_d P_{nm}(\alpha, \bar{\alpha}) = -\left( n(n + d - 3) + m(m + 1) \right) P_{nm}(\alpha, \bar{\alpha}), \]  

(B.18)

where \( P_{nm}(\alpha, \bar{\alpha}) \) are generalised symmetric Jacobi polynomials. For particular \( d \) simplified formulae may be found in terms of well known single variable Legendre polynomials \( P_n \).

When \( d = 4 \) it is clear from (B.17) that \( \hat{D}_4 \) is just the sum of two independent Legendre differential operators so that

\[ P_{nm}(\alpha, \bar{\alpha}) = \frac{1}{2} \left( P_n(y) P_m(\bar{y}) + P_m(y) P_n(\bar{y}) \right) \]  

for \( n \geq m \),

(B.19)

with \( y, \bar{y} \) defined in (3.48). For \( d = 6 \) we may use the result

\[ \hat{D}_6 \frac{1}{\alpha - \bar{\alpha}} = \frac{1}{\alpha - \bar{\alpha}} (\hat{D}_4 + 2), \]  

(B.20)

to see that we can take the eigenfunctions to be of the form

\[ P_{nm}(\alpha, \bar{\alpha}) = p_{n+1m}(y, \bar{y}), \quad n \geq m, \]  

(B.21)

where

\[ p_{nm}(y, \bar{y}) = -p_{mn}(y, \bar{y}) = \frac{P_n(y) P_m(\bar{y}) - P_m(y) P_n(\bar{y})}{y - \bar{y}}. \]  

(B.22)

It is also of interest to consider \( d = 8 \) when we take

\[ P(\alpha, \bar{\alpha}) = \frac{F(\alpha, \bar{\alpha})}{(\alpha - \bar{\alpha})^2}, \]  

(B.23)

and the eigenvalue equation becomes

\[ \hat{D}_6 F(\alpha, \bar{\alpha}) = -\frac{2}{(\alpha - \bar{\alpha})^2} \left( \alpha (1 - \alpha) \frac{\partial}{\partial \alpha} ((\alpha - \bar{\alpha}) F(\alpha, \bar{\alpha})) - \bar{\alpha} (1 - \bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}} ((\alpha - \bar{\alpha}) F(\alpha, \bar{\alpha})) \right) \]  

\[ = -(C + 4) F(\alpha, \bar{\alpha}). \]  

(B.24)
If we assume

\[ F(\alpha, \bar{\alpha}) = \sum_{n,m} a_{nm} p_{nm}(y, \bar{y}), \quad (B.25) \]

and use, from standard identities for Legendre polynomials,

\[
\frac{1}{y - \bar{y}} \left( (1 - y^2) \frac{\partial}{\partial y} - (1 - \bar{y}^2) \frac{\partial}{\partial \bar{y}} \right) ((y - \bar{y}) p_{nm}(y, \bar{y})) = \frac{m(m + 1)}{2m + 1} (p_{nm+1}(y, \bar{y}) - p_{nm-1}(y, \bar{y})) - \frac{n(n + 1)}{2n + 1} (p_{n+1m}(y, \bar{y}) - p_{n-1m}(y, \bar{y}))
\]

\[
(y - \bar{y}) p_{nm}(y, \bar{y}) = \frac{1}{2n + 1} ((n + 1) p_{n+1m}(y, \bar{y}) + n p_{n-1m}(y, \bar{y}))
\]

\[
- \frac{1}{2m + 1} ((m + 1) p_{nm+1}(y, \bar{y}) + m p_{nm-1}(y, \bar{y})) , \quad (B.26)
\]

then we may set up recurrence relations for \( a_{nm} \) which for the appropriate value of \( C \) have just four terms. For \( C = n(n + 1) + m(m + 1) - 6 \) (B.25) gives a solution

\[
q_{nm}(y, \bar{y}) = \frac{1}{(y - \bar{y})^2} \left\{ \frac{n + 1}{2n + 1} (n + m)(n - m - 1) p_{n+1m}(y, \bar{y}) \right. \\
+ \frac{n}{2n + 1} (n + m + 2)(n - m + 1) p_{n-1m}(y, \bar{y}) \\
- \frac{m + 1}{2m + 1} (n + m)(n - m + 1) p_{nm+1}(y, \bar{y}) \\
- \frac{m}{2m + 1} (n + m + 2)(n - m - 1) p_{nm-1}(y, \bar{y}) \right\}, \quad (B.27)
\]

where \( q_{nm}(y, \bar{y}) = -q_{mn}(y, \bar{y}), \ q_{nn}(y, \bar{y}) = q_{n+1n}(y, \bar{y}) = 0. \) Hence we can take

\[
P_{nm}(\alpha, \bar{\alpha}) = q_{n+2m}(y, \bar{y}). \quad (B.28)
\]

The above results for harmonic polynomials in \( \sigma, \tau \) are relevant for discussing four point functions when each field belongs to the same \( SO(d) \) representation. For the more general case we also consider instead of (B.4),

\[
L^2((t_1 \cdot t_4)^a (t_2 \cdot t_4)^b Y^{(a,b)}(\sigma, \tau)) = -2C((t_1 \cdot t_4)^a (t_2 \cdot t_4)^b Y^{(a,b)}(\sigma, \tau)), \quad (B.29)
\]

where now the action of \( L^2 \) is determined by

\[
L^2((t_1 \cdot t_4)^a (t_2 \cdot t_4)^b \sigma^p \tau^q)
\]

\[
= (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b \left( (2D_d - (a + b)(a + b + d - 2) - 4ap - 4bq)(\sigma^p \tau^q) + 2(1 - \sigma - \tau)(bp \sigma^{p-1} \tau^q + aq \sigma^p \tau^{q-1}) \right), \quad (B.30)
\]
\[ \frac{1}{2} L^2 ((t_1 \cdot t_4)^a (t_2 \cdot t_4)^b f(\sigma, \tau)) = (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b (D_{d}^{(a,b)} - \frac{1}{2}(a+b)(a+b+d-2)) f(\sigma, \tau), \]  

(B.31)

where

\[ D_d^{(a,b)} = D_d + (1 - \sigma - \tau) \left( a \frac{\partial}{\partial \tau} + b \frac{\partial}{\partial \sigma} \right) - 2a \sigma \frac{\partial}{\partial \sigma} - 2b \tau \frac{\partial}{\partial \tau}. \]  

This may also be written in the form (B.7) with \( w = \sigma^b \tau^a \Lambda^{\frac{1}{2}(d-5)}. \) The possible eigenvalues for polynomial eigenfunctions with maximum power \( p + q = n \) are then determined by the matrix

\[ M_{n,pq} = \delta_{pq} (n(n + d - 2 + a + b) + \frac{1}{2}(a + b)(a + b + d - 2) + 2p(n - p) + a(n - p) + bp) \]
\[ + \delta_{p,q-1} q(q + a) + \delta_{p,q+1} (n - q)(n - q + b). \]  

(B.33)

The eigenfunctions \( Y_{nm}^{(a,b)}(\sigma, \tau) \) for \( m = 0, 1, \ldots, n \) then have eigenvalues

\[ C_{nm} = \left( n + \frac{1}{2}(a + b) \right) \left( n + \frac{1}{2}(a + b) + d - 3 \right) + (m + \frac{1}{2}(a + b)) \left( m + \frac{1}{2}(a + b) + 1 \right). \]  

(B.34)

For \( d = 6 \) \( Y_{nm}^{(a,b)} \) corresponds to the representation \([n - m, a + b + 2m, n - m] \). The simplest non trivial examples are

\[ Y_{10}^{(a,b)}(\sigma, \tau) = \sigma - \tau + \frac{a - b}{a + b + d - 2}, \]
\[ Y_{11}^{(a,b)}(\sigma, \tau) = \frac{1}{b + 1} \sigma + \frac{1}{a + 1} \tau - \frac{1}{a + b + \frac{1}{2}d}. \]  

(B.35)

Corresponding to (B.15) we have in general

\[ Y_{nn}^{(a,b)}(\sigma, \tau) = A_n F_4(-n, n + a + b + \frac{1}{2}d - 1; b + 1, a + 1; \sigma, \tau). \]  

(B.36)

Again more explicit results can be obtained by using the variables \( \alpha, \bar{\alpha} \). In (B.32) the differential operator now becomes

\[ \hat{D}_d^{(a,b)} = \hat{D}_d - (a \alpha - b(1 - \alpha)) \frac{\partial}{\partial \alpha} - (a \bar{\alpha} - b(1 - \bar{\alpha})) \frac{\partial}{\partial \bar{\alpha}}, \]  

(B.37)

with \( \hat{D}_d \) given in (B.17). Denoting the eigenfunctions of \( \hat{D}_d^{(a,b)} \) by \( P_{nm}^{(a,b)}(\alpha, \bar{\alpha}) \) then previous results for \( d = 4, 6 \) for the eigenfunctions can be extended by using Jacobi polynomials.
For \( d = 4 \) \( \hat{D}_{4}^{(a,b)} = D_{\alpha}^{(a,b)} + D_{\bar{\alpha}}^{(a,b)} \) where \( D^{(a,b)} \) is the ordinary differential operator defined by

\[
D_{\alpha}^{(a,b)} = \frac{d}{d\alpha} \alpha(1 - \alpha) \frac{d}{d\alpha} - a \alpha \frac{d}{d\alpha} + b(1 - \alpha) \frac{d}{d\alpha}.
\]  

(B.38)

The eigenfunctions of \( D_{\alpha}^{(a,b)} \) are just \( P_{n}^{(a,b)}(y) \), where \( y = 2\alpha - 1 \) and the eigenvalues are \(-n(n + a + b + 1)\). For \( d = 6 \) the generalisation of (B.21) and (B.22) is then

\[
\mathcal{P}_{nm}^{(a,b)}(\alpha, \bar{\alpha}) = \frac{P_{n+1}^{(a,b)}(y)F_{m}^{(a,b)}(\bar{y}) - P_{m}^{(a,b)}(y)F_{n+1}^{(a,b)}(\bar{y})}{y - \bar{y}}.
\]  

(B.39)

When \( d = 3 \) the above results need to be considered separately since \( \sigma, \tau \) are not independent and satisfy the constraint (2.39). The eigenfunctions \( Y_{nm}^{(a,b)}(\sigma, \tau) \) are also restricted since \( Y_{nm}^{(a,b)}(\sigma, \tau) = 0 \) for \( m < n - 1 \) as a consequence of (2.39). To obtain eigenfunctions of \( L^2 \) in general we make use of the solution (2.40) which amounts to setting \( \alpha = \bar{\alpha} \) in the above, so that we are restricted just to single variable functions. Instead of (B.31) we have

\[
L^2((t_1 \cdot t_4)^a(t_2 \cdot t_4)^b f(\alpha)) = (t_1 \cdot t_4)^a(t_2 \cdot t_4)^b(D_{\alpha}^{(2a,2b)} - (a + b)(a + b + 1)) f(\alpha),
\]  

(B.40)

using the definition (B.38). In consequence

\[
L^2((t_1 \cdot t_4)^a(t_2 \cdot t_4)^b P_{n}^{(2a,2b)}(y)) = -(n + a + b)(n + a + b + 1) (t_1 \cdot t_4)^a(t_2 \cdot t_4)^b P_{n}^{(2a,2b)}(y),
\]  

(B.41)

corresponding to the \((n+a+b)\)-representation for \( SU(2) \simeq SO(3) \). Hence for \( d = 3 \) we may then take

\[
\mathcal{P}_{nn}^{(a,b)}(\alpha, \alpha) = P_{2n}^{(2a,2b)}(y), \quad \mathcal{P}_{n n-1}^{(a,b)}(\alpha, \alpha) = P_{2n-1}^{(2a,2b)}(y),
\]  

(B.42)

with \( \mathcal{P}_{nm}^{(a,b)}(\alpha, \alpha) = 0 \) for \( m < n - 1 \).

For \( d = 3 \) there are also eigenfunctions involving cross products. To consider these we first define

\[
T_1 = t_1 \cdot t_3 \times t_4 (t_1 \cdot t_4)^{a-1}(t_2 \cdot t_4)^b, \quad T_2 = t_2 \cdot t_3 \times t_4 (t_1 \cdot t_4)^a(t_2 \cdot t_4)^{b-1},
\]  

(B.43)

and consider eigenfunctions of the form \( T_1 f_1(\alpha) + T_2 f_2(\alpha) \). The action of \( L^2 \) on such functions is given by

\[
(L^2 + (a + b - 1)(a + b)) (T_1 f_1 + T_2 f_2)
\]

\[
= (T_1 \quad T_2) \begin{pmatrix} D^{(2a-1,2b+1)} - 2a & -2a \\ -2b & D^{(2a+1,2b-1)} - 2b \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.
\]  

(B.44)
However for \( t_i \) three dimensional null vectors the basis given by (B.43) is not independent since we have from (2.52)
\[
T_1(1 - \alpha) + T_2\alpha = 0, \quad \text{(B.45)}
\]
so that \( f_1, f_2 \) are not unique. If we use this freedom to set \( f_2 = 0 \) the eigenvalue equation for \( L^2 \) reduces to
\[
\left(D^{(2a-1,2b+1)} - 2a + 2b - \frac{1 - \alpha}{\alpha} - (a + b - 1)(a + b)\right)f_1
= \frac{1}{\alpha}\left(D^{(2a-1,2b-1)} - (a + b - 1)(a + b)\right)(\alpha f_1) = -Cf_1, \quad \text{(B.46)}
\]
which has solutions proportional to Jacobi polynomials,
\[
f_1(\alpha) = \frac{1}{\alpha}P_n^{(2a-1,2b-1)}(y), \quad C = (n + a + b - 1)(n + a + b). \quad \text{(B.47)}
\]
For \( n \geq 1 \) the apparent singularity for \( \alpha \to 0 \) may be removed by using (B.45) to give an appropriate non zero \( f_2 \). The eigenfunctions for the solution in (B.47) correspond to the \( SU(2) \ (n+a+b-1)\)-representation. Alternatively we may set \( f_1 = 0 \) and obtain the corresponding equation
\[
\frac{1}{1 - \alpha}\left(D^{(2a-1,2b-1)} - (a + b - 1)(a + b)\right)((1 - \alpha)f_2) = -Cf_2. \quad \text{(B.48)}
\]
Appendix C. Calculation of Differential Operators

A non trivial aspect in the derivation of the superconformal identities is the determination of the differential operators (5.51) which appear in (2.44). To sketch how these were obtained we first obtain, for any dimension \(d\) and arbitrary \(f(\sigma, \tau)\),

\[
\frac{1}{2}(k+a + \frac{1}{2}d - 2) L_{2[r_s \partial t_1 u]} ((t_1 \cdot t_2)^k (t_3 \cdot t_4)^l (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b f)
= -(t_1 \cdot t_2)^{k-2} (t_3 \cdot t_4)^{l-1} (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b (t_1[r_t t_2 s t_3 u] t_2 \cdot t_4 \mathcal{D}_1 + t_1[r_t t_4 s t_2 u] t_2 \cdot t_3 \mathcal{D}_2
+ (k + a + \frac{1}{2}d - 2) t_2[r_t t_3 s t_4 u] t_1 \cdot t_2(\mathcal{D}_\sigma - \mathcal{D}_\tau)) f, 
\]

(C.1)

where

\[
\mathcal{D}_1 = \frac{\partial}{\partial \sigma} \mathcal{D}_\sigma + (\sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} + 1 - k)(\mathcal{D}_\sigma - \mathcal{D}_\tau),
\]

\[
\mathcal{D}_2 = -\left(\frac{\partial}{\partial \tau} + \frac{a}{\tau}\right) \mathcal{D}_\sigma + (\sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} + 1 - k)(\mathcal{D}_\sigma - \mathcal{D}_\tau), 
\]

(C.2)

for

\[
\mathcal{D}_\sigma = \sigma(1 - \sigma) \frac{\partial^2}{\partial \sigma^2} - \tau^2 \frac{\partial^2}{\partial \tau^2} - 2\sigma\tau \frac{\partial^2}{\partial \sigma \partial \tau}
+ (b + 1) \frac{\partial}{\partial \sigma} - (a + b + \frac{1}{2}d)(\sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau}) + k(k + a + b + \frac{1}{2}d - 1),
\]

\[
\mathcal{D}_\tau = \tau(1 - \tau) \frac{\partial^2}{\partial \tau^2} - \sigma^2 \frac{\partial^2}{\partial \sigma^2} - 2\sigma\tau \frac{\partial^2}{\partial \sigma \partial \tau}
+ (a + 1) \frac{\partial}{\partial \tau} - (a + b + \frac{1}{2}d)(\sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau}) + k(k + a + b + \frac{1}{2}d - 1).
\]

(C.3)

In terms of (B.32) and (B.6) we have

\[
\Delta_{d}^{(a,b)} \equiv \mathcal{D}_\sigma + \mathcal{D}_\tau + (\sigma - \tau)(\mathcal{D}_\sigma - \mathcal{D}_\tau) = \mathcal{D}_{d}^{(a,b)} + 2k(k + a + b + \frac{1}{2}d - 1).
\]

(C.4)

The operators in (C.2) satisfy the identity

\[
2(\sigma - \tau + 1) \mathcal{D}_1 + 2(\tau - \sigma + 1) \mathcal{D}_2
= \mathcal{D}_2 \Delta_{d}^{(a,b)} - \Lambda \left(\frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau} + \frac{a}{\tau}\right)(\mathcal{D}_\sigma - \mathcal{D}_\tau) - (2k + 2a - 1)(\mathcal{D}_\sigma - \mathcal{D}_\tau),
\]

(C.5)

with \(\Lambda\) as in (2.39) and, as well as (C.4), defining

\[
\mathcal{D}_2 = (\sigma - \tau + 1) \frac{\partial}{\partial \sigma} + (\sigma - \tau - 1) \left(\frac{\partial}{\partial \tau} + \frac{a}{\tau}\right).
\]

(C.6)
When \( d = 3 \), and \( \Lambda = 0 \), this result with (2.52) leads to the simplified form for (C.1)

\[
4(k + a - \frac{1}{2}) \frac{\partial}{\partial t_2} \cdot L_2 ((t_1 \cdot t_2)^k (t_3 \cdot t_4)^l (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b \hat{f})
\]

\[
= -t_2 \cdot t_3 \times t_4 (t_1 \cdot t_2)^{k-1} (t_3 \cdot t_4)^{l-1} (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b \hat{D}_2 (D^{(2a,2b)} + 2k(2k + 2a + 2b + 1)) \hat{f},
\]

(C.7)

letting \( f(\sigma, \tau) = \hat{f}(\alpha) \) and \( D_2 \rightarrow \hat{D}_2 \) for

\[
\hat{D}_2 = \frac{d}{d\alpha} - \frac{2a}{1 - \alpha}.
\]

(C.8)

From (B.40) the operator \( D^{(2a,2b)} + 2k(2k + 2a + 2b + 1) \) acting on \( \hat{f}(\alpha) \) corresponds to \( L^2 + (2k + a + b)(2k + a + b + 1) \). We may also note that

\[
\hat{D}_2 D^{(2a,2b)} = \frac{1}{1 - \alpha} D^{(2a - 1,2b + 1)}(1 - \alpha) \hat{D}_2,
\]

(C.9)

and in (C.7) from (B.44)

\[
t_2 \cdot t_3 \times t_4 (t_1 \cdot t_2)^{k-1} (t_3 \cdot t_4)^{l-1} (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b
\]

\[
\times \frac{1}{1 - \alpha} (D^{(2a-1,2b+1)} + 2k(2k + 2a + 2b + 1))(1 - \alpha) f
\]

\[
= (L^2 + (2k + a + b)(2k + a + b + 1)) t_2 \cdot t_3 \times t_4 (t_1 \cdot t_2)^{k-1} (t_3 \cdot t_4)^{l-1} (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b f.
\]

(C.10)

The equivalent results to (C.7) for \( L_2 \rightarrow L_3 \) and \( L_2 \rightarrow L_4 \) can be found by using the permutations \( 2 \rightarrow 3 \rightarrow 4 \rightarrow 2 \), along with \( a \rightarrow a' = k - l, \ b \rightarrow -a, \ k \rightarrow a + l, \ l \rightarrow a + b + l \) and \( \alpha \rightarrow \alpha' = -(1 - \alpha)/\alpha \), and also \( 2 \rightarrow 4 \rightarrow 3 \rightarrow 2 \), along with in this case \( a \rightarrow a'' = -b, \ b \rightarrow l - k, \ k \rightarrow a + b + l, \ l \rightarrow b + k \) and \( \alpha \rightarrow \alpha'' = 1/(1 - \alpha) \). From (C.7) we then find

\[
\hat{D}_3 = \alpha^{2(a+l-1)}(1 - \alpha)^{-2a} \hat{D}_2 \alpha^{-2(a+l)}(1 - \alpha)^{2a} = \frac{d}{d\alpha} + \frac{2a}{1 - \alpha} - \frac{2(k + a)}{\alpha},
\]

\[
\hat{D}_4 = \alpha^{-2b}(1 - \alpha)^{2(b+k-1)} \hat{D}_2 \alpha^{2b}(1 - \alpha)^{-2(b+k)} = \frac{d}{d\alpha} + \frac{2k}{1 - \alpha}.
\]

(C.11)

Together with (C.8), (C.11) is equivalent to (C.4).

For the analysis of the \( N = 4 \) superconformal identities a particular solution of the constraints (3.7) is obtained by expressing \( T_i \) in terms of scalar functions \( Y_i(u,v,t) \)

\[
T_i = -\tilde{\gamma} \frac{\partial}{\partial t_1} Y_i \gamma \frac{\partial}{\partial t_i}.
\]

(C.12)
(3.8) and (3.12) then give
\[ U_i = (L_{1,r}sL_{i,rs} + p_1 p_i) Y_i, \quad W_{i,rsu} = 3(\partial_{1,r}[L_{i,stu} Y_i]_{sd}, \quad (C.13) \]
\[ \dot{V}_{i,r} = \partial_{1,s} L_{i,rs} Y_i - \frac{1}{p_1} \partial_{1,r} (\frac{1}{2} L_{1,stu} L_{i,stu} Y_i). \]
Writing
\[ Y_i(u, v; t) = (t_1 \cdot t_4)^{p_1 - E}(t_2 \cdot t_4)^{p_2 - E}(t_1 \cdot t_2)^{E}(t_3 \cdot t_4)^{p_3} \mathcal{Y}_i(u, v; \sigma, \tau), \quad (C.14) \]
then for \( i = 2 \), using (C.1) with \( k = E, k + a = p_1, k + b = p_2 \), we find
\[ \mathcal{U}_2 = \Delta_0^{(p_1 - E, p_2 - E)} \mathcal{Y}_2, \quad \mathcal{W}_2 = 6(D_\sigma - D_\tau) \mathcal{Y}_2. \quad (C.15) \]
\( \mathcal{A}_2 \) and \( \mathcal{B}_2 \) are then given by (C.1) and (C.2) in terms of \( \mathcal{U}_2, \mathcal{W}_2 \) in accord with (3.37). The other results may be obtained by cyclic permutations. For \( 2 \to 3 \to 4 \to 2 \), when \( \sigma \to \tau/\sigma, \tau \to 1/\sigma \) and \( E \to E - p_4 - p_2 \), then \( \mathcal{U}_2 \to \tau^{p_1 - E} \sigma^{p_2 - p_4 - E} \mathcal{U}_3, \mathcal{W}_2 \to \tau^{p_1 - E} \sigma^{p_2 - p_4 - E + 1} \mathcal{W}_3, \mathcal{A}_2 \to \tau^{p_1 - E} \sigma^{p_2 - p_4 - E + 2} \mathcal{C}_3 \) and \( \mathcal{B}_2 \to \tau^{p_1 - E} \sigma^{p_2 - p_4 - E + 2} \mathcal{A}_3 \).
For \( 2 \to 4 \to 3 \to 2 \), so that \( \sigma \to 1/\tau, \tau \to \sigma/\tau \) and \( E \to -E + p_1 + p_2 \), in this case \( \mathcal{U}_2 \to \tau^{-p_2} \sigma^{-p_2 - E} \mathcal{U}_4, \mathcal{W}_2 \to \tau^{1 - p_2} \sigma^{-p_2 - E} \mathcal{W}_4, \mathcal{A}_2 \to \tau^{2 - p_2} \sigma^{-p_2 - E} \mathcal{B}_4 \) and \( \mathcal{B}_2 \to \tau^{2 - p_2} \sigma^{-p_2 - E} \mathcal{C}_4 \).

However the representation (C.12) is not valid in general since it excludes contributions involving the \( \varepsilon \)-tensor. Nevertheless equivalent results may be obtained by use of (3.24).

With the expansion
\[ t_{2[r} \partial_{1{s]} U_2 + p_1 t_{2[r} \dot{V}_{2,s]} + p_1 W_{2,rsu} t_{2u} \]
\[ = (t_1 \cdot t_4)^{p_1 - E - 1}(t_2 \cdot t_4)^{p_2 - E}(t_1 \cdot t_2)^{E - 1}(t_3 \cdot t_4)^{p_3 - 1} \]
\[ \times \left( t_{2[r} t_{3<s]} t_{1\cdot t_4 t_2\cdot t_4} \left( \partial_\sigma U_2 + p_1 J_2 - \frac{1}{p_1} \partial_1 p_1 (A_2 + W_2) \right) \right. \]
\[ + t_{2[r} t_{4<s]} t_{1\cdot t_4 t_2\cdot t_4} \left( \partial_{\tau'} U_2 - p_1 \frac{1}{\tau}(I_2 + \sigma J_2) + \frac{1}{p_1} p_1 (B_2 + W_2) \right) \]
\[ + t_{1[r} t_{2<s]} t_{2\cdot t_4 t_3\cdot t_4} \left( \frac{\tau}{p_1 + 1} \left( (\partial_\sigma + \partial_{\tau'})(E - \sigma \partial_\sigma - \tau \partial_\tau) + \partial_\sigma \partial_{\tau'} \right) U_2 \right. \]
\[ \left. - p_1 V_2 - \frac{1}{p_1} p_1 \tau (A_2 - B_2) \right), \quad (C.16) \]
where \( \partial_{\tau'} = \partial_{\tau} + (p_1 - E)/\tau \), then (3.24) requires
\[ 6 \partial_\sigma U_2 = -6 p_1 J_2 + p_1 (A_2 + W_2) = 2(p_1 + 1) A_2 - (O_\sigma - p_1) W_2, \quad (C.17) \]
\[ 6 \partial_{\tau'} U_2 = 6 p_1 \frac{1}{\tau} (I_2 + \sigma J_2) - p_1 (B_2 + W_2) = -2(p_1 + 1) B_2 + (O_\sigma - p_1) W_2, \]

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using (3.35) for $i = 2$, which gives the first two equations in (3.37). The remaining results in (3.37) can be obtained by using permutations. In addition with (3.36) we also obtain

$$(p_1 + 1)((\mathcal{O}_\sigma - p_1 + 1)B_2 - (\mathcal{O}_\tau - p_1 + 1)A_2)$$

$$= 6((E - 1 - \sigma \partial_\sigma - \tau \partial_\tau)(\partial_\sigma + \partial_\tau) + \partial_\sigma \partial_\tau')u_2$$  \hspace{1cm} (C.18)

$$= -3((\mathcal{O}_\sigma - p_1 + 1)\partial_\tau' + (\mathcal{O}_\tau - p_1 + 1)\partial_\sigma)u_2.$$

It is then straightforward to see that (C.18) follows from (C.17) using $[\mathcal{O}_\sigma, \mathcal{O}_\tau] = \mathcal{O}_\sigma - \mathcal{O}_\tau$. 

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Appendix D. Non Unitary Semi-short Representations

In chapter 3 the analysis of the operator product expansion in general potentially required contributions below the unitarity threshold on the scale dimension $\Delta$. We show here how such truncations of the full representation space arise for the superconformal algebra $PSU(2,2|4)$, following the approach in [4]\textsuperscript{9}

The essential results are found by considering the chiral subalgebra $SU(2|4)$ (although no hermeticity conditions are imposed) which has generators $Q^i_\alpha, S_i^\alpha, \alpha = 1, 2, i = 1, \ldots 4$, where
\[
\{Q^i_\alpha, S_j^\beta\} = 4 \{\delta^i_j (M_\alpha^\beta + \frac{1}{2} \delta_\alpha^\beta \hat{D}) - \delta_\alpha^\beta R^i_j\},
\]
(D.1)
as well as $\{Q^i_\alpha, Q^j_\beta\} = \{S_i^\alpha, S_j^\beta\} = 0$. In (D.1) $M_\alpha^\beta$ are generators of $SU(2)$ and $R^i_j$ are generators of $SU(4)$, $\sum_i R^i_i = 0$, with standard commutation relations. $\hat{D}$ is the dilation operator, with eigenvalues the scale dimension. The commutators with $Q^i_\alpha$ and $S_i^\alpha$ are then
\[
[M_\alpha^\beta, Q^i_\gamma] = \delta_\gamma^i Q^i_\alpha - \frac{1}{2} \delta_\alpha^\beta Q^i_\gamma, \quad [M_\alpha^\beta, S_i^\gamma] = -\delta_\alpha^\gamma S_i^\beta + \frac{1}{2} \delta_\alpha^\beta S_i^\gamma,
\]
\[
[R^i_j, Q^k_\alpha] = \delta^k_j Q^k_\alpha - \frac{1}{2} \delta^i_j Q^k_\alpha, \quad [R^i_j, S_k^\alpha] = -\delta^i_j S_k^\alpha + \frac{1}{4} \delta^i_j S_k^\alpha, \quad (D.2)
\]
\[
[\hat{D}, Q^i_\alpha] = \frac{1}{2} Q^i_\alpha, \quad [\hat{D}, S_i^\alpha] = -\frac{i}{2} S_i^\alpha.
\]

In terms of the usual $J_3, J_^ \pm$
\[
[M_\alpha^\beta] = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}, \quad (D.3)
\]
and it clear then that $(Q^1_1, Q^2_2)$ and $(S^2_i, -S^1_i)$ form $j = \frac{1}{2}$ doublets. In terms of a standard Chevalley basis $E_r^\pm, H_r, r = 1, 2, 3$, where $H_r^\pm = H_r, E_r^\pm = E_r^-$ with commutators $[H_r, H_s] = 0, [E_r^+, E_s^-] = \delta_{rs}H_s, [H_r, E_s^\pm] = \pm K_{rs} E_s^\pm$, for $[K_{rs}]$ the $SU(4)$ Cartan matrix,
\[
[K_{rs}] = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad (D.4)
\]
then we may take $R^{i_1}_1 = E_1^+, R^{i_2}_2 = E_2^+, R^{i_3}_3 = E_3^+$ and $R^{i_4}_i = \frac{1}{4} (3H_1 + 2H_2 + H_3) - \sum_{r=1}^{i-1} H_r$.

\textsuperscript{9} This reproduces an analysis in [38].
For $SU(4) \otimes SU(2)$ highest weight states $|p_1, p_2, p_3; j\rangle^{hw} \equiv |p; j\rangle^{hw}$ we have

$$H_r|p; j\rangle^{hw} = p_r|p; j\rangle^{hw}, \quad J_3|p; j\rangle^{hw} = j|p; j\rangle^{hw}, \quad J_+|p; j\rangle^{hw} = E_r^+|p; j\rangle^{hw} = 0, \quad (D.5)$$

from which states defining a representation with Dynkin labels $[p_1, p_2, p_3]j$ are constructed by the action of $E_r^-, J_-$. The representations of $SU(2|4)$ may then be formed from a highest weight state which is also superconformal primary,

$$\hat{D}|p; j\rangle^{hw} = \Delta|p; j\rangle^{hw}, \quad S_\alpha^\alpha|p; j\rangle^{hw} = 0. \quad (D.6)$$

The states of a generic supermultiplet, labelled by $a^\Delta_{[p_1, p_2, p_3]j}$, are obtained by the action of the supercharges, giving $\prod_{i,\alpha} (Q_\alpha^i)^{n_{i\alpha}}|p; j\rangle^{hw}$ with $n_{i\alpha} = 0, 1$, together with the lowering operators $E_r^-$. The possible $SU(4) \otimes SU(2)$ representations $[p'_1, p'_2, p'_3]j'$, with scale dimension $\Delta'$, forming the supermultiplet $a^\Delta_{[p_1, p_2, p_3]j}$ are obtained by adding the $SU(4), SU(2)$ weights with $n_{i\alpha} = 0, 1$ so that

$$p'_r = p_r + \sum_\alpha (n_{r\alpha} - n_{r+1\alpha}), \quad j' = j + \frac{1}{2} \sum_i (n_{i1} - n_{i2}), \quad \Delta' = \Delta + \frac{1}{2} \sum_{i,\alpha} n_{i\alpha}, \quad (D.7)$$

It is easy to see that $\dim a^\Delta_{[p_1, p_2, p_3]j} = 2^{s_d}(p_1, p_2, p_3)(2j + 1)$, where $d(p_1, p_2, p_3)$ is the dimension of the $SU(4)$ representation with Dynkin labels $[p_1, p_2, p_3]$. If in (D.7) any $p'_r$ or $j'$ are negative the Racah-Speiser algorithm, described in [4], provides a precise prescription for removing such $[p'_1, p'_2, p'_3]j'$.

Shortening conditions arise for suitable $\Delta$ when descendant representations satisfy the conditions (D.6) to be superconformal primary. Since all $S_\alpha^\alpha$ are obtained by commutators of $S_1^1$ with $E_r^+$ and $J_+$ it is sufficient to impose only that $S_1^1$ annihilates the highest weight state of the representation. In such cases we may impose that the appropriate combinations of $Q_\alpha^i$ annihilate $|p; j\rangle^{hw}$. For application here it is convenient to define, acting on states $|\psi\rangle$ such that $J_3|\psi\rangle = j|\psi\rangle$,

$$\hat{Q}_i^i = Q_2^i - \frac{1}{2j} Q_1^1 J_- \quad (D.8)$$

If $J_+|\psi\rangle = 0$ then $J_+\hat{Q}_i^i|\psi\rangle = 0$ and $J_3\hat{Q}_i^i|\psi\rangle = (j - \frac{1}{2})\hat{Q}_i^i|\psi\rangle$. From (D.1) we have

$$\frac{1}{2} j \{ S_1^1, \hat{Q}_1^1 \} = (2j - J_3 - \frac{1}{2} \hat{D} + \frac{1}{4} (3H_1 + 2H_2 + H_3)) J_- \quad \frac{1}{2} j \{ S_1^1, \hat{Q}_2^2 \} = E_1^- J_- \quad (D.9)$$

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It is straightforward to show that $\tilde{Q}^i|p; j\rangle^{\text{hw}} \sim |p_1 + 1, p_2, p_3; j - \frac{1}{2}\rangle$. The shortening conditions considered in [4] and previously are obtained by imposing

$$\tilde{Q}^i|p; j\rangle^{\text{hw}} = 0 \begin{cases} i = 1, & \text{if } p_1 = 0, \\ i = 1, 2, & \text{if } p_1 = p_2 = 0, \\ i = 1, 2, 3, 4, & \text{if } p_1 = p_2 = p_3 = 0. \end{cases} \quad (D.10)$$

In each case there is a consistency condition on $\Delta$ which can be found by using (D.9) and (D.6),

$$\Delta = 2 + 2j + \frac{1}{2}(3p_1 + 2p_2 + p_3). \quad (D.11)$$

The corresponding supermultiplet is here denoted by $c_{[p_1, p_2, p_3]}j$. Detailed results were given in [4], the $SU(4) \times SU(2)$ representations present may be calculated as in (D.7) with the restriction $n_{i2} = 0$ for those $i$ listed in (D.10) for each case.

There are also additional shortening conditions of the form

$$\left(\tilde{Q}^2 - \frac{1}{p_1} \tilde{Q}^1 E_1^-\right)|p; j\rangle^{\text{hw}} = 0, \quad p_1 > 0, \quad \tilde{Q}^1\tilde{Q}^2|0, p_2, p_3; j\rangle^{\text{hw}} = 0, \quad (D.12)$$

where the left hand sides correspond to highest weight states $|p_1 - 1, p_2 + 1, p_3; j - \frac{1}{2}\rangle$ and $|0, p_2 + 1, p_3; j - 1\rangle$ respectively. Using (D.9) these conditions then require

$$\Delta = 2j + \frac{1}{2}(-p_1 + 2p_2 + p_3). \quad (D.13)$$

For $p_2 = 0$ the condition (D.12) extends also to $\left(\tilde{Q}^3 - \frac{1}{p_1} \tilde{Q}^1[E_2^-, E_1^-]\right)|p; j\rangle^{\text{hw}} = 0$. The supermultiplet in each case is denoted by $d_{[p_1, p_2, p_3]}j$. The representations are obtained as in (D.7) with $n_{i2} = 0$, or if $p_2 = 0$ then $n_{22} = n_{32} = 0$. For $p_1 = 0$ it is sufficient to exclude $n_{12} = n_{22} = 1$.

These semi-short representations lead to decompositions of the generic multiplet,

$$a^{2 + 2j + \frac{1}{2}(3p_1 + 2p_2 + p_3)}_{[p_1, p_2, p_3]} \simeq c_{[p_1, p_2, p_3]} j \oplus c_{[p_1 + 1, p_2, p_3]} j - \frac{1}{2},$$

$$a^{2j + \frac{1}{2}(-p_1 + 2p_2 + p_3)}_{[p_1, p_2, p_3]} \simeq d_{[p_1, p_2, p_3]} j \oplus d_{[p_1 - 1, p_2 + 1, p_3]} j - \frac{1}{2}, \quad (D.14)$$

$$a^{2j + \frac{1}{2}(2p_2 + p_3)}_{[0, p_2, p_3]} j \simeq d_{[0, p_2, p_3]} j \oplus c_{[0, p_2 + 1, p_3]} j - 1.$$ 

Formally, as discussed in [4], we have

$$c_{[p_1, p_2, p_3]} j - \frac{1}{2} \simeq b_{[p_1 + 1, p_2, p_3]}, \quad c_{[p_1, p_2, p_3]} j - 1 \simeq -b_{[p_1, p_2, p_3]} \quad (D.15)$$
where \( b_{[p_1,p_2,p_3]} \) is the short supermultiplet formed by imposing \( Q_1^1|p;0\rangle_{hw} = 0 \) where we require \( \Delta = \frac{1}{2}(3p_1 + 2p_2 + p_3) \). Just as in (D.15) \( d_{[p_1,p_2,p_3]} \) may be identified with a multiplet obtained from the highest weight state \(|p_1 - 1, p_2 + 1, p_3;0\rangle_{hw} \), with \( \Delta = \frac{1}{2}(-p_1 + 2p_2 + p_3 + 1) \), annihilated by \( Q_1^1 - \frac{1}{p_1} Q_1^1 E_1^- \). Formally we have

\[
d_{[p_1,p_2,p_3]} \simeq c_{[-p_1 - 2, p_1 + 2p_2 + p_3 + 2, -p_3 - 2]} ,
\]

where, in accord with the Racah-Speiser algorithm described in [4], we may identify \( SU(4) \) representations \([p_1,p_2,p_3] \simeq [-p_1 - 2, p_1 + 2p_2 + p_3 + 2, -p_3 - 2] \) which are related by an element of the Weyl group. This allows the detailed representation content and dimension for \( d_{[p_1,p_2,p_3]} \) to be determined from the results given in [4].

The generators of the superconformal group \( PSU(2,2|4) \) are obtained by extending those of \( SU(2|4) \) to include the hermitian conjugates, \( \tilde{Q}_{i\dot{\alpha}} = Q^i_\alpha \), \( \tilde{S}^{i\dot{\alpha}} = S^i_\alpha \), \( \tilde{M}^\beta_{\dot{\alpha}} = (M^\alpha_\beta)^\dagger \), with an algebra obtained by conjugation of that for \( SU(2|4) \), assuming \( \hat{D}^1 = -\hat{D} \) and \( (R^i_j)^\dagger = R^j_i \). In addition \( \{S^i_\alpha, Q_j^\dot{\alpha}\} = \{Q^i_\alpha, \tilde{S}^{j\dot{\alpha}}\} = 0 \) and \( \{Q^i_\alpha, \tilde{Q}_{j\dot{\alpha}}\} = 2\delta^i_j (\sigma^a)^\alpha_\alpha P_a \), \( \{S^{i\dot{\alpha}}, S_j^\alpha\} = 2\delta^i_j (\tilde{\sigma}^a)^\dot{\alpha}\alpha K_a \) where \( P_a \) is the momentum operator and \( K_a \) the generator of special conformal transformations. The supermultiplets are generated from highest weight superconformal primary states \(|p_1,p_2,p_3;\bar{j},\bar{j}\rangle_{hw} \), where \( \bar{j} \) is the \( SU(2) \) quantum number for \( \bar{J}_\pm, \bar{J}_3 \) obtained from \( \tilde{M}^\beta_{\dot{\alpha}} \), which is annihilated by \( S^i_\alpha, \tilde{S}^{i\dot{\alpha}} \) and \( K_a \). The representations are of course infinite dimensional, since they are generated by arbitrary powers of \( P_a \), but they are formed by a finite set of conformal primary representations, annihilated by \( K_a \), which are straightforwardly constructed from \( SU(2|4) \) supermultiplet representations, as described above, combined with with their conjugates formed by the action of \( \tilde{Q}_{i\dot{\alpha}} \). For these \( j \rightarrow \bar{j} \) and \( p_1 \leftrightarrow p_3 \). The generic supermultiplet is denoted \( \mathcal{A}^\Delta_{[p_1,p_2,p_3]|(j,\bar{j})} \) where \([p_1,p_2,p_3]|(j,\bar{j})\) are the labels for the representation with lowest scale dimension \( \Delta \). The conformal primary states form representations labelled by \([p'_1,p'_2,p'_3]|(j',\bar{j}')\), with scale dimension \( \Delta' \), which are given by

\[
p'_{r'} = p_r + \sum_{\alpha} (n_{r\alpha} - n_{r+1\alpha}) + \sum_{\dot{\alpha}} (\bar{n}_{r+1\dot{\alpha}} - \bar{n}_{r\dot{\alpha}}),
\]

\[
j' = j + \frac{1}{2} \sum_i (n_{i1} - n_{i2}), \quad \bar{j}' = \bar{j} + \frac{1}{2} \sum_i (\bar{n}_{i2} - \bar{n}_{i1}),
\]

\[
\Delta' = \Delta + \frac{1}{2} \sum_{i,\alpha} n_{i\alpha} + \frac{1}{2} \sum_{i,\dot{\alpha}} \bar{n}_{i\dot{\alpha}}, \quad n_{i\alpha}, \bar{n}_{i\dot{\alpha}} = 0, 1.
\]
The total dimension is \(2^{16}d(p_1, p_2, p_3)(2j+1)(2j+1)\).

We here consider the case when shortening conditions are imposed for both the \(Q\) and \(\bar{Q}\) charges. Requiring (D.10) with (D.11) together with its conjugate we have the semi-short multiplets,

\[
C_{[p_1, p_2, p_3]}(j, \bar{j}), \quad p_1 - p_3 = 2(j - \bar{j}), \quad \Delta = 2 + j + \bar{j} + p_1 + p_2 + p_3 \tag{D.18}
\]

where we impose in (D.17) \(n_{12} = \bar{n}_{41} = 0\), with further restrictions if \(p_1\) or \(p_3\) are zero. Requiring (D.12) and (D.13) in both cases gives

\[
D_{[p_1, p_2, p_3]}(j, \bar{j}), \quad p_1 - p_3 = 2(j - \bar{j}), \quad \Delta = j + \bar{j} + p_2 \tag{D.19}
\]

and we require in (D.17) \(n_{22} = \bar{n}_{31} = 0\). If \(p_2 = 1\) then we exclude \(n_{32} = \bar{n}_{21} = 1\) while if \(p_2 = 0\) then we require \(n_{32} = \bar{n}_{21} = 0\) as well. Corresponding to (D.16) we have

\[
D_{[p_1, p_2, p_3]}(j, \bar{j}) \simeq C_{[-p_1-2, p_1+p_2+p_3+2,-p_3-2]}(j, \bar{j}) \tag{D.20}
\]

This result has essentially been used in (3.85). We may also impose (D.10) with (D.11) for the \(Q\) charges and (D.12) and (D.13) for \(\bar{Q}\) giving

\[
E_{[p_1, p_2, p_3]}(j, \bar{j}), \quad p_1 + p_3 = 2(j - \bar{j} - 1), \quad \Delta = 2 + j + \bar{j} + p_1 + p_2 \tag{D.21}
\]

and we may also obtain a conjugate \(E_{[p_1, p_2, p_3]}(\bar{j}, j)\). Only for (D.18), where \(\Delta\) is at the unitarity threshold, is there a unitary representation.

For relevance in chapter 3 we list the self-conjugate representations, when \(p_1 = p_2, j = \bar{j}\), arising in \(D_{[q, 0, q]}(j, \bar{j})\), obtained by the action of equal powers of the \(Q\) and \(\bar{Q}\) supercharges for each \(\Delta\)

| \(\ell\) | \(\ell + 1\) |
|---------|-------------|
| \([q, 0, q]_{\ell}\) | \([q-1, 0, q-1]_{\ell+1}, [q-1, 2, q-1]_{\ell+1}, [q, 0, q]_{\ell+1}, [q+1, 0, q+1]_{\ell+1}\
| & \([q-1, 0, q-1]_{\ell-1}, 2[q, 0, q]_{\ell-1}, [q+1, 0, q+1]_{\ell-1}\) |
| \(\ell + 2\) | 
| \([q-2, 2, q-2]_{\ell+2}, [q-1, 0, q-1]_{\ell+2}, 2[q-1, 2, q-1]_{\ell+2}, 4[q, 0, q]_{\ell+2}, [q, 2, q]_{\ell+2}, [q+1, 0, q+1]_{\ell+2}\
| & \[q-2, 0, q-2]_{\ell}, [q-2, 2, q-2]_{\ell}, 5[q-1, 0, q-1]_{\ell}, 2[q-1, 2, q-1]_{\ell}, 8[q, 0, q]_{\ell}, [q, 2, q]_{\ell}, 5[q+1, 0, q+1]_{\ell}, [q+2, 0, q+2]_{\ell}\
| & \[q, 0, q]_{\ell-2}\) |
Table 3. Diagonal representations for each $\Delta$ in $\mathcal{D}_{[q,0,q]}(\frac{1}{2},\frac{1}{2},\ell)$.

For application in the text we have the decompositions of self conjugate multiplets

\[
\mathcal{A}_{[q,p,q]}^{2j+p+2q}(j,j) \cong \mathcal{C}_{[q,p,q]}(j,j) \oplus \mathcal{C}_{[q+1,p,q]}(j-\frac{1}{2},j) \oplus \mathcal{C}_{[q,p,q+1]}(j,j-\frac{1}{2}) \oplus \mathcal{C}_{[q+1,p,q+1]}(j-\frac{1}{2},j-\frac{1}{2})
\]

\[
\mathcal{A}_{[q,p,q]}^{2j+p}(j,j) \cong \mathcal{D}_{[q,p,q]}(j,j) \oplus \mathcal{D}_{[q-1,p+1,q]}(j-\frac{1}{2},j) \oplus \mathcal{D}_{[q,p+1,q-1]}(j,j-\frac{1}{2}) \oplus \mathcal{D}_{[q-1,p+2,q-1]}(j-\frac{1}{2},j-\frac{1}{2})
\]

\[
\mathcal{A}_{[0,p,0]}^{2j+p}(j,j) \cong \mathcal{E}_{[0,p,0]}(j,j) \oplus \mathcal{E}_{[0,p+1,0]}(j-1,j) \oplus \mathcal{E}_{[0,p+1,0]}(j,j-1) \oplus \mathcal{C}_{[0,p+2,0]}(j-1,j-1)
\]

The first case represents the decomposition of a long multiplet into semi-short multiplets at the unitarity threshold, the second plays a crucial role in chapter 3 in relating the solution of the superconformal Ward identities to the operator product expansion.
Appendix E. Alternative Derivation of Ward identities for $N = 2$

The fields we consider have $N$ symmetric fundamental $SU(2)$ indices $\varphi_{i_1 \ldots i_N}(x)$. In the case of even $N = 2n$ these can be mapped to $n$ symmetric traceless adjoint $SU(2)$ indices $\varphi^{(2n)}_{r_1 \ldots r_n}(x)$. As in chapter 2 we can contract the $SO(3)$ indices with null vectors $t$. Because of the mapping between $SO(3)$ and $SU(2)$ indices we can correspondingly map the $SO(3)$ vector $t$ to a $SU(2)$ spinor $\tilde{u}$ as follows. In three dimensions a null vector $t$ may be represented [11] in terms of two component spinors $u = (u^1, u^2)$ by $t_a = u^\tau_a \tilde{u}$ for $\tilde{u} = \epsilon u^T$ for $\epsilon$ the $2 \times 2$ antisymmetric matrix. Then $t_1 t_2 = -2(u_2 \tilde{u}_1)^2$. Since $u$ is in the fundamental representation of $SU(2)$ it squares to $0 = u^i u^j \epsilon_{ij}$. This ensures that the $t_i$ constructed from the $u$'s are in fact null vectors. Since there is no null condition on $u$ derivatives with respect to $u$ will be ordinary derivatives which is the main reason for the simplicity of this derivation. In this case the variable $\alpha$ defined in chapter 2 becomes

$$\alpha = \frac{u_3 \tilde{u}_1 u_2 \tilde{u}_4}{u_2 \tilde{u}_1 u_3 \tilde{u}_4} = 1 - \frac{u_2 \tilde{u}_3 u_1 \tilde{u}_4}{u_2 \tilde{u}_1 u_3 \tilde{u}_4}.$$  

Note that also $i t_1 \times t_2 \cdot t_3 = 4 u_1 \tilde{u}_2 u_2 \tilde{u}_3 u_3 \tilde{u}_1$. This will allow us to avoid the distinction between two cases as necessary in chapter 2.

So now we can define

$$\varphi^{(N)}(x, u) = \varphi_{i_1 \ldots i_N} u^{i_1} \ldots u^{i_N}$$

$$\psi^{(N-1)}_\alpha(x, u) = \psi_{\alpha i_1 \ldots i_{N-1}} u^{i_1} \ldots u^{i_{N-1}}$$

$$\overline{\psi}^{(N-1)}_\alpha(x, u) = \overline{\psi}_{\bar{\alpha} i_1 \ldots i_{N-1}} u^{i_1} \ldots u^{i_{N-1}}$$

$$J^{(N-2)}_{\alpha \bar{\alpha}}(x, u) = J_{\alpha \bar{\alpha} i_1 \ldots i_{N-2}} u^{i_1} \ldots u^{i_{N-2}}$$

(E.1)

In these new variables, the needed superconformal transformations are

$$\delta \varphi^{(N)} = \tilde{\epsilon}_i \varphi^{(N-1)} + \varphi^{(N)} \tilde{u}_i \tilde{\epsilon}$$

$$\delta \psi^{(N-1)}_\alpha = i \frac{\partial}{\partial u^i} \varphi^{(N)} \tilde{\epsilon} + 2 \frac{\partial}{\partial u^i} \psi^{(N-2)}_\alpha \tilde{u}_i \tilde{\epsilon}.$$  

(E.2)

One can check that these close acting on $\varphi$. Now we derive the Ward identity by starting
from
\[ 0 = \delta \langle \psi^{(N_1-1)}(x_1) \varphi^{(N_2)}(x_2) \varphi^{(N_3)}(x_3) \varphi^{(N_4)}(x_4) \rangle \]
\[ = \frac{i}{N_1} \frac{\partial}{\partial u_1^i} \partial_{\alpha \dot{\alpha}} \langle \varphi^{(N_1)}(x_1) \varphi^{(N_2)}(x_2) \varphi^{(N_3)}(x_3) \varphi^{(N_4)}(x_4) \rangle \hat{e}^{i \dot{\alpha}}(x_1) \]
\[ + 2 \frac{\partial}{\partial u_1^i} \langle \varphi^{(N_1)}(x_1) \varphi^{(N_2)}(x_2) \varphi^{(N_3)}(x_3) \varphi^{(N_4)}(x_4) \rangle \eta^i_{\alpha}(x_1) \]
\[ + \langle J^{(N_1-2)}(x_1) \varphi^{(N_2)}(x_2) \varphi^{(N_3)}(x_3) \varphi^{(N_4)}(x_4) \rangle \tilde{u}_{i1} \hat{e}^{i \dot{\alpha}}(x_1) \]
\[ + \langle \psi^{(N_1-1)}(x_1) \dot{\psi}^{(N_2-1)}(x_2) \varphi^{(N_3)}(x_3) \varphi^{(N_4)}(x_4) \rangle \tilde{u}_{i2} \hat{e}^{i \dot{\alpha}}(x_2) \]
\[ + \langle \psi^{(N_1-1)}(x_1) \varphi^{(N_2)}(x_2) \dot{\psi}^{(N_3-1)}(x_3) \varphi^{(N_4)}(x_4) \rangle \tilde{u}_{i3} \hat{e}^{i \dot{\alpha}}(x_3) \]
\[ + \langle \psi^{(N_1-1)}(x_1) \varphi^{(N_2)}(x_2) \varphi^{(N_3)}(x_3) \dot{\psi}^{(N_4-1)}(x_4) \rangle \tilde{u}_{i4} \hat{e}^{i \dot{\alpha}}(x_4), \tag{E.3} \]
where we suppress the \( u \) arguments of the fields. Following the same steps as in chapter 2 equations (2.14) to (2.17) we may decompose this equation into independent ones and separate into \( x, \overline{x} \) resulting in
\[ \frac{1}{N_1} \frac{\partial}{\partial u_1^i} \partial_x F + K \tilde{u}_{i1} + \frac{1}{x} T_2 \tilde{u}_{i2} - \frac{1}{1 - x} T_4 \tilde{u}_{i4} = 0, \tag{E.4} \]
\[ \frac{\partial}{\partial u_1^i} F = T_2 \tilde{u}_{i2} + T_3 \tilde{u}_{i3} + T_4 \tilde{u}_{i4}, \]
which is very similar of the result in chapter 2 in equation (2.18). Now we contract the first equation with \( u_1 \) to eliminate the derivative in \( u_1 \). Since \( F \) is homogeneous of degree \( N_1 \) in \( u_1 \) this simply leaves
\[ \partial_x F = -\frac{1}{x} T_2 u_1 \tilde{u}_2 + \frac{1}{1 - x} T_4 u_1 \tilde{u}_4. \tag{E.5} \]
The second equation we contract with \( u_3 \) to project out the term involving \( T_3 \)
\[ u_3^i \frac{\partial}{\partial u_1^i} F = T_2 u_3 \tilde{u}_2 + T_4 u_3 \tilde{u}_4. \tag{E.6} \]
We solve this for \( T_2 \) and insert it into the other equation
\[ \left( u_3^i \frac{\partial}{\partial u_1^i} + \frac{u_3 \tilde{u}_2}{u_1 \tilde{u}_2} x \partial_x \right) F = T_4 u_3 \tilde{u}_4 \frac{1 - \alpha x}{1 - x}. \tag{E.7} \]
We now compute the action of the differential operator on the left hand side. Since we know that \( F \) is homogeneous of degrees \( N_1, N_2, N_3, N_4 \) in \( u_1, u_2, u_3, u_4 \) we expand it in the form
\[ F(u, v; u_i) = (u_1 \tilde{u}_4)^{N_1} (u_2 \tilde{u}_4)^{N_2} (u_1 \tilde{u}_2)^{N_3} F(x, \overline{x}; \alpha). \tag{E.8} \]
where here $2E = N_1 + N_2 + N_3 - N_4$. The differential operator in (E.7) acting on this gives

$$
\left( u_3 \frac{\partial}{\partial u_1} + \frac{u_3 \bar{u}_2}{u_1 \bar{u}_2} x \partial_x \right) F(u, v; u_i) = u_3 \bar{u}_2 (u_1 \bar{u}_4)^{N_1 - E} (u_2 \bar{u}_4)^{N_2 - E} (u_1 \bar{u}_2)^{E - 1} (u_3 \bar{u}_4)^{N_3} \times \left( -\alpha \partial_\alpha + x \partial_x + \frac{N_1 - E}{1 - \alpha} + E \right) F(x, \bar{x}; \alpha).
$$

(E.9)

For $\alpha = \frac{1}{x}$ the right hand side of (E.7) vanishes. Therefore the Ward identity we obtain is

$$
\partial_x \left( x^E (x - 1)^{N_1 - E} F(x, \bar{x}; \frac{1}{x}) \right) = 0.
$$

(E.10)

You can easily check that this is identical to (2.49) we obtained before.

We may also define $O(3)$ generators by $L_a = \frac{1}{2} u \tau_a \frac{\partial}{\partial u}$. Since

$$(L_1 + L_2)^2 (u_1 \bar{u}_4)^a (u_2 \bar{u}_4)^b f(\alpha) = \left(-D_{\alpha}^{(a,b)} + \frac{1}{4} (a + b)(a + b + 2)\right) f(\alpha),$$

the relevant $O(3)$ eigenfunctions for each representation may be directly obtained and shown to be identical to the ones obtained before.
Appendix F. Mathematica Computation of Conformal Partial Wave Expansions

In this appendix we will describe how the Mathematica program attached can automate most of the steps in the computation of conformal partial wave expansions. The source code is divided into blocks preceded by “In[x] :=”, where we will refer to x as the number of the block of code.

First, to motivate the function of the program, we make the simple observation that the contributions of operators of twist $t$ always have a $u$-leading term of the form $cu^{\frac{t}{2}} = c(x\tau)^{\frac{t}{2}}$. Looking at the expression for $G_{\Delta}^{(\ell)}(x,\tau)$ we see that the contribution of an operator with twist $t$ and spin $\ell$ will have leading terms in $x$, $\tau$ of order $\frac{t}{2} + \ell + 1$, $\frac{t}{2}$ respectively. Therefore by expanding up to a finite order in $x$, $\tau$, we can simultaneously eliminate all twists and scale dimensions beyond a certain limit. We will actually use it to compute the coefficients first twist by twist to then later fit to a general ansatz. In our example $p = 4$ the overall factor to the amplitude is $u^4$. Thus by only making a power expansion in $\tau$ up to power 4, we eliminate all twist contributions for $t > 8$. It is also apparent that no operators with twist $t < 8$ can contribute. Therefore the leading expansion will automatically reduce to $t = 8$ contributions only. Then we can plug the coefficients back into the expansion which we perform then up to power 5 in $\tau$ to obtain the coefficients for twist 10. We keep iterating this until we have enough coefficients to read off the general twist dependence.

In the following we assume general familiarity with Mathematica. We need the basic commands for simplification of expressions, table management, Taylor series and obtaining a list of the coefficients. Also an understanding of the use of rules is required.

Block 1 switches off certain warnings which are not important.

Block 2 sets $p = 4$ which is the case for which we will illustrate the use of the program. It can also be used to confirm the $p = 2, 3$ results or to perform the expansion for higher $p$’s. In principle also other functions might be expanded.

Block 3 defines hypergeometric functions of type $2F_1$ as a power series.
Block 5 contains the definition of the basic $G^{(\ell)}_{\Delta}(x,\bar{x})$ function. Also we set up a function which already contains the universal factor of $u^{\frac{1}{2}}$. Notice that this function already has a cutoff for powers of $x, \bar{x}$. This will save time since we ultimately only want to compute up to a certain power of $x, \bar{x}$.

Block 8 contains the expressions for the different amplitudes in terms of $\mathcal{D}$-functions. For different values of $p$ we have to insert the appropriate expressions here. At the end of block 8 they are all summed up in one array for convenience.

Block 15 contains first the choice of amplitude to compute which is just the position of the amplitude in the array. Also a variable $\text{even}$ is set to 1=true or 0=false, signaling if even or odd coefficients are non-zero. This also has to be adjusted for different values of $p$.

Block 16 defines the appropriate expression in the array now as $H_0$ which is the function we will try to expand in partial waves. It has parameters $TT, DD, o$ which set limits for the maximum twist considered, maximum dimension considered and order to which to expand.

Block 17 sets up the expansion coefficients $c[a, \Delta = t + \ell, \ell]$ where $a$ is the index of the amplitude in the array, $\Delta$ is the dimension, $t$ is the twist and $\ell$ the spin. Immediately the constraint is implemented that only even or odd twist can contribute depending on which amplitude we deal with. This might be too restrictive for other expansions which might have less symmetries.

Block 18 already contains the results computed for the amplitude $A_{22}$ which has index 6 for the twist range $8-24$. Initially this would be empty and is built up successively twist by twist. Correspondingly Blocks 27,36,45,54,63 contain the already computed values for amplitudes $A_{21}, A_{20}, A_{11}, A_{10}, A_{00}$.

Block 72 contains a definition of the $\mathcal{D}$-functions as a power series in $x, \bar{x}$ (1.40).

Block 73 defines the general ansatz we make

$$HH2 = \sum_{t,\ell} c_{a,\Delta,\ell} u^{\frac{1}{2}} G^{(\ell)}_{\Delta},$$

(F.1)
where immediately a power expansion up to a fixed order is performed to truncate the
series and also limits for $t, \ell$ are chosen such that no terms with a too high power of $x, \overline{r}$
are considered. The maximum twist we want to include is specified by the parameter $TT$,
the maximum dimension by $DD$ and the order to which to expand by $o$.

Block 74 equates the function to expand with the general ansatz and performs a Taylor
expansion of the resulting equation. The expansion order for $\overline{r}$ is chosen such that only
terms at the first twist level not yet computed survive. The expansion order in $x$ is chosen
high enough to accomodate all the partial waves in the specified dimensional range.

Block 75 computes the resulting equations for the coefficients for a certain twist set
by the variable $twist$, here 26, which is the first level not yet computed. It produces a set
of rules which solve this system of equations.

Block 77 uses the solution of the last step to compute the partial wave coefficients
and stores them in a table.

Block 79 strips off universal factors of $\ell$-dependence such that the remainder is poly-
nomial in $\ell$ (schematically)

$$c_{2\tau,\ell} = \frac{2^\ell (\ell + \tau - 1)!(\ell + \tau)!(\tau)!^2}{(2\ell + 2\tau - 1)!(2\tau)!} c'_{2\tau,\ell}. \quad (F.2)$$

This is an ansatz which needs change if for example other functions are supposed to be
expanded. The table contains the coefficients for fixed twist $t$, but for a range of values of
the spin $\ell$

Block 82 takes these values and tries to fit them against a polynomial of order 4. This
is a crucial point of the procedure. Certainly there exists some sort of expression of the
table values in terms of $\ell$, but one has to ensure that through the removal of the universal
factor only a polynomial dependence is left over. If the guess in the factorization is roughly
correct, but for example one of the factorials has a wrong argument, one can increase the
order of the fitting polynomial. The problem is that eventually this will make the equations
used to fit underdetermined and thus requires computation of more coefficients to start
from. Therefore for the factorization the best known ansatz should be used.
Block 85 uses the polynomial fitted to the numerical values and puts back the universal factor. It will generate output which is the final expression of the partial wave expansion coefficients for this particular twist but for general $\ell$. Using copy & paste this now needs to be fed back into one of the Blocks above with the previously computed values and one can return to Block 75 to compute for the next higher twist. This is the way the Blocks 18-63 were obtained.

To summarize, so far the program computed the general $\ell$-dependence of the expansion coefficients. Now the program has a second stage where we will now try to determine the general dependence on twist. The coefficients we determined so far are general expressions for $\ell$ but only valid for the specific values of $t$ they were computed for. The final result we want is a general expression valid for all values of $t, \ell$.

Block 89 contains the definition of an ansatz to take out a universal factor containing factorial and exponential (basically any non-polynomial) dependence on $t$ similar to the case of $\ell$ (schematically)

\[
\frac{c_{t,\ell}}{c'_{t,\ell}} = \frac{2^{\ell - 1}(\ell + \tau + 1)!\ell + 2\tau + 2^{1-\text{event}}((\ell + 2\tau + 2))^{1-\text{event}}}{(2\ell + 2\tau + 1)!}.
\]

We use the variable $\tau = \frac{t}{2}$ since we already know that only even twists can appear. The variable $\text{event}$ takes care of the fact that for odd $\ell$ certain additional factors appear. Here the same applies as for block 79/82. This ansatz needs to be close enough to the expected result for the program to work.

Block 90 defines coefficients $d$ which are the expansion coefficients $c$ with the ansatz just made factored off.

Block 91 defines a general ansatz for the $d$ coefficients which is a 4-th order polynomial in $\ell$ and 8-th order in $\tau$. The orders have to be high enough to capture the complexity of the expansion coefficients, but also not too large, since otherwise with a limited number of coefficients computed in the first part, a system of equations to fit the polynomial to the numerical values would be underdetermined.

Block 92 tries to simplify the $d$-coefficients to speed up the computation.
Block 93 fits the $d$ coefficients twist by twist against the corresponding polynomial ansatzes we just made. The result is then returned by the function ploy.

Blocks 94-99 show the output, the function ploy will generate for the $p = 4$ case, the individual amplitude expansion coefficients for $A_{00}, A_{10}, A_{11}, A_{20}, A_{21}, A_{22}$. The polynomials obtained still look quite complicated but should describe the general $\tau, \ell$ dependence of the expansion coefficients. Further simplifications have to be done by hand.
In[1]:= Off[General::spell1]; Off[Solve::svars];

In[2]:= p := 4;

In[3]:= HG[a_, b_, c_, x_] := 
  Sum[Pochhammer[a, n] Pochhammer[b, n] Pochhammer[c, n] x^n/n!, {n, 0, o}]
HG /: HG[a_, b_, c_, x_] /: (c == 0) && (a == 0) && (b == 0)

In[5]:= g /: g[dd_, ll_, o_, xx_, xxb_] := 
  ((-1/2 xx)^11 xx HG[1/2 (dd - 11) - 1, 1/2 (dd - 11) - 1, dd - 2 - 11, o, xxb]
  HG[1/2 (dd + 11), 1/2 (dd + 11), dd + 11, o, xx] -
  (-1/2 xxb)^11 xxb HG[1/2 (dd - 11) - 1, 1/2 (dd - 11) - 1, dd - 2 - 11, o, xx]
HG[1/2 (dd + 11), 1/2 (dd + 11), dd + 11, o, xxb])
g /: g[dd_, ll_, o_, xx_, xxb_] := 1 /; dd = 11 + 2;
gg[dd_, ll_, o_, xx_, xxb_] := (xx xxb)^1/2 (dd - 11) - p g[dd + 4, 11, o, xx, xxb]

In[8]:= A22[TT_, DD_, o_] := 
  (x - xb) (DDD[4, 6, 4, 4, o, x, xb] + DDD[2, 6, 3, 3, o, x, xb])
A21[TT_, DD_, o_] := 
  (x - xb) (DDD[3, 6, 4, 4, o, x, xb] - DDD[1, 6, 3, 4, o, x, xb])
A20[TT_, DD_, o_] := 
  (x - xb) (-DDD[2, 6, 3, 3, o, x, xb] -
  (DDD[3, 4, 6, 1, o, x, xb] + DDD[4, 3, 6, 1, o, x, xb]))
A11[TT_, DD_, o_] := 
  (x - xb) (8 DDD[4, 6, 4, 4, o, x, xb] + 20/3 DDD[4, 6, 3, 3, o, x, xb] -
  1/3 DDD[2, 6, 3, 3, o, x, xb] - 14/3 DDD[2, 6, 4, 4, o, x, xb])
A10[TT_, DD_, o_] := 
  (x - xb) (4 DDD[3, 6, 2, 3, o, x, xb] - DDD[3, 6, 3, 2, o, x, xb]) +
  5 DDD[3, 6, 3, 4, o, x, xb] - DDD[1, 6, 3, 4, o, x, xb])
A00[TT_, DD_, o_] := 
  (x - xb) (5/2 DDD[4, 6, 4, 4, o, x, xb] +
  10/3 DDD[4, 6, 3, 3, o, x, xb] + 2 DDD[4, 6, 2, 2, o, x, xb]) +
  1/10 (DDD[3, 6, 3, 1, o, x, xb] + DDD[4, 3, 6, 1, o, x, xb])

Amps[TT_, DD_, o_] := 
  {A00[TT, DD, o], A10[TT, DD, o], A11[TT, DD, o],
  A20[TT, DD, o], A21[TT, DD, o], A22[TT, DD, o]}

In[15]:= amp := 2; If[(amp = 2) || (amp = 5), even := 0, even := 1];

In[16]:= H0[TT_, DD_, o_] = Amps[TT, DD, o][[amp]];

In[17]:= Clear[c]; If[even = 1, c[amp, a_, b_] := 0 /; OddQ[a], c[amp, a_, b_] := 0 /; EvenQ[a]]
\[\begin{align*}
\text{In}[18]:= \quad & c[6, a_, i_] := -\frac{2^i (1584 + 902 i + 203 i^2 + 22 i^3 + i^4) (4 + i) (5 + i)!}{3 (9 + 2 i)!}; (a - i) = 8; \\
c[6, a_, i_] := \frac{2^{-1+i} (1488 + 7722 i + 1439 i^2 + 130 i^3 + 5 i^4) (5 + i)! (6 + i)!}{3 (11 + 2 i)!}; (a - i) = 10; \\
c[6, a_, i_] := \frac{2^{-1+i} (152656 + 60810 i + 9679 i^2 + 750 i^3 + 25 i^4) (6 + i)! (7 + i)!}{11 (13 + 2 i)!}; (a - i) = 12; \\
c[6, a_, i_] := \frac{-72^{2+i} (259872 + 90134 i + 12527 i^2 + 850 i^3 + 25 i^4) (7 + i)! (8 + i)!}{39 (15 + 2 i)!}; (a - i) = 14; \\
c[6, a_, i_] := \frac{-72^i (415152 + 127566 i + 15739 i^2 + 950 i^3 + 25 i^4) (8 + i)! (9 + i)!}{195 (17 + 2 i)!}; (a - i) = 16; \\
c[6, a_, i_] := \frac{-212^{-1+i} (126200 + 34818 i + 3863 i^2 + 210 i^3 + 5 i^4) (9 + i)! (10 + i)!}{85 (19 + 2 i)!}; (a - i) = 18; \\
c[6, a_, i_] := \frac{-492^{1+i} (184272 + 46138 i + 4651 i^2 + 230 i^3 + 5 i^4) (10 + i)! (11 + i)!}{323 (21 + 2 i)!}; (a - i) = 20; \\
c[6, a_, i_] := \frac{-112^{-2+i} (1301616 + 298350 i + 27559 i^2 + 1250 i^3 + 25 i^4) (11 + i)! (12 + i)!}{323 (23 + 2 i)!}; (a - i) = 22; \\
c[6, a_, i_] := \frac{-332^{1+i} (1788592 + 378054 i + 32227 i^2 + 1350 i^3 + 25 i^4) (12 + i)! (13 + i)!}{7429 (25 + 2 i)!}; (a - i) = 24; \\
\end{align*}\]
In[45]:= \[\begin{align*}
c[4, a_, i_] & := -\frac{2^i (1 + i) (2 + i) (9 + i) (10 + i) (4 + i)! (5 + i)!}{3 (9 + 2 i)!} /; (a - i) = 8; \\
c[4, a_, i_] & := -\frac{5^{2 - i} (1 + i) (12 + i) (24 + 13 i + i^2) (5 + i)! (6 + i)!}{3 (11 + 2 i)!} /; (a - i) = 10; \\
c[4, a_, i_] & := -\frac{5^{2 - i} (1 + i) (14 + i) (154 + 75 i + 5 i^2) (6 + i)! (7 + i)!}{11 (13 + 2 i)!} /; (a - i) = 12; \\
c[4, a_, i_] & := -\frac{35^{2 - i} (1 + i) (16 + i) (192 + 85 i + 5 i^2) (7 + i)! (8 + i)!}{39 (15 + 2 i)!} /; (a - i) = 14; \\
c[4, a_, i_] & := -\frac{72^i (1 + i) (18 + i) (234 + 95 i + 5 i^2) (8 + i)! (9 + i)!}{39 (17 + 2 i)!} /; (a - i) = 16; \\
c[4, a_, i_] & := -\frac{21^2 (1 + i) (20 + i) (56 + 21 i + i^2) (9 + i)! (10 + i)!}{17 (19 + 2 i)!} /; (a - i) = 18; \\
c[4, a_, i_] & := -\frac{245^2 (1 + i) (22 + i) (66 + 23 i + i^2) (10 + i)! (11 + i)!}{323 (21 + 2 i)!} /; (a - i) = 20; \\
c[4, a_, i_] & := -\frac{55^2 (1 + i) (24 + i) (384 + 125 i + 5 i^2) (11 + i)! (12 + i)!}{323 (23 + 2 i)!} /; (a - i) = 22; \\
c[4, a_, i_] & := -\frac{165^2 (1 + i) (26 + i) (442 + 135 i + 5 i^2) (12 + i)! (13 + i)!}{7429 (25 + 2 i)!} /; (a - i) = 24;
\end{align*}\]
In[54]:= \[c[2, a_, i_] := \frac{2^i (-1 + i) (1 + i) (10 + i) (12 + i) (4 + i)! (5 + i)!}{(9 + 2 i)!}; \quad (a - i) = 8;\]
\[c[2, a_, i_] := \frac{5 \cdot 2^{-1+i} (1 + i) (12 + i) (-54 + 13 i + i^2) (5 + i)! (6 + i)!}{(11 + 2 i)!}; \quad (a - i) = 10;\]
\[c[2, a_, i_] := \frac{15 \cdot 2^{-1+i} (1 + i) (14 + i) (-524 + 75 i + 5 i^2) (6 + i)! (7 + i)!}{(13 + 2 i)!}; \quad (a - i) = 12;\]
\[c[2, a_, i_] := \frac{35 \cdot 2^{-2+i} (1 + i) (16 + i) (-822 + 85 i + 5 i^2) (7 + i)! (8 + i)!}{(15 + 2 i)!}; \quad (a - i) = 14;\]
\[c[2, a_, i_] := \frac{7 \cdot 2^i (1 + i) (18 + i) (-1164 + 95 i + 5 i^2) (8 + i)! (9 + i)!}{(17 + 2 i)!}; \quad (a - i) = 16;\]
\[c[2, a_, i_] := \frac{63 \cdot 2^{-1+i} (-10 + i) (1 + i) (20 + i) (31 + i) (9 + i)! (10 + i)!}{(19 + 2 i)!}; \quad (a - i) = 18;\]
\[c[2, a_, i_] := \frac{735 \cdot 2^{-1+i} (1 + i) (22 + i) (-396 + 23 i + i^2) (10 + i)! (11 + i)!}{323 (21 + 2 i)!}; \quad (a - i) = 20;\]
\[c[2, a_, i_] := \frac{165 \cdot 2^{-2+i} (1 + i) (24 + i) (-2454 + 125 i + 5 i^2) (11 + i)! (12 + i)!}{323 (23 + 2 i)!}; \quad (a - i) = 22;\]
\[c[2, a_, i_] := \frac{495 \cdot 2^i (1 + i) (26 + i) (-2972 + 135 i + 5 i^2) (12 + i)! (13 + i)!}{7429 (25 + 2 i)!}; \quad (a - i) = 24;\]

In[63]:= \[c[1, a_, i_] := \frac{2^{-1+i} (3120 + 1078 i + 219 i^2 + 22 i^3 + i^4) (4 + i)! (5 + i)!}{5 (9 + 2 i)!}; \quad (a - i) = 8;\]
\[c[1, a_, i_] := \frac{2^{-2+i} (10728 + 2418 i + 355 i^2 + 26 i^3 + i^4) (5 + i)! (6 + i)!}{(11 + 2 i)!}; \quad (a - i) = 10;\]
\[c[1, a_, i_] := \frac{3 \cdot 2^{-2+i} (127376 + 21810 i + 2579 i^2 + 150 i^3 + 5 i^4) (6 + i)! (7 + i)!}{11 (13 + 2 i)!}; \quad (a - i) = 12;\]
\[c[1, a_, i_] := \frac{7 \cdot 2^{-2+i} (251712 + 35054 i + 3507 i^2 + 170 i^3 + 5 i^4) (7 + i)! (8 + i)!}{13 (15 + 2 i)!}; \quad (a - i) = 14;\]
\[c[1, a_, i_] := \frac{7 \cdot 2^{-1+i} (443952 + 52326 i + 4559 i^2 + 190 i^3 + 5 i^4) (8 + i)! (9 + i)!}{65 (17 + 2 i)!}; \quad (a - i) = 16;\]
\[c[1, a_, i_] := \frac{63 \cdot 2^{-2+i} (144760 + 14826 i + 1147 i^2 + 42 i^3 + i^4) (9 + i)! (10 + i)!}{85 (19 + 2 i)!}; \quad (a - i) = 18;\]
\[c[1, a_, i_] := \frac{147 \cdot 2^{-2+i} (222672 + 20194 i + 1407 i^2 + 46 i^3 + i^4) (10 + i)! (11 + i)!}{323 (21 + 2 i)!}; \quad (a - i) = 20;\]
\[c[1, a_, i_] := \frac{33 \cdot 2^{-2+i} (1637136 + 133350 i + 8459 i^2 + 250 i^3 + 5 i^4) (11 + i)! (12 + i)!}{323 (23 + 2 i)!}; \quad (a - i) = 22;\]
\[c[1, a_, i_] := \frac{99 \cdot 2^{-1+i} (2322032 + 171774 i + 10007 i^2 + 270 i^3 + 5 i^4) (12 + i)! (13 + i)!}{7429 (25 + 2 i)!}; \quad (a - i) = 24;\]
In[72]:= DDD[a1_, a2_, a3_, a4_, o_, x, xb] := Module[{s, S = 1/2 (a1 + a2 + a3 + a4);
    s - 1/2 (a1 + a2 - a3 - a4); Sum[(-1)^s Gamma[a1 + m] Gamma[a2 + m + n]
    Gamma[a3 + s + m + n] Gamma[a4 + s + m] / m! / n! / Gamma[s + m + 1] / Gamma[a1 + a2 + 2 m + n]
    (x xb)^m (1 - (1 - x) (1 - xb))^n, {m, 0, o}, {n, 0, o - m}]]

In[73]:= HH2[TT_, DD_, o_] := Expand[Sum[Sum[c[amp, 1 + t, l] gg[1 + t, 1, o - t / 2 + 4, x, xb], {1, 1 - even, DD - t}], {t, 2 p, TT, 2}];

In[74]:= equns3[TT_, DD_, DD_] := CoefficientList[Normal[Series[H0[TT, DD, DD - TT / 2 + 2] - HH2[TT, DD, DD - TT / 2 + 2], {x b, 0, TT / 2 - p}, {x, 0, DD - TT - 1}], {x, xb}]]

In[75]:= twist := 26;
   tmp = equns3[twist, twist + 30];

In[77]:= tmp2 = Flatten[Solve[Simplify[Take[tmp, Length[tmp] - 2]] == 0]];
   tmp3 = Table[c[amp, i - twist + i], {i, 0, 27, 1}] /. tmp2;

In[79]:= ansatz[i_] :=
   2 (i + twist - 1)! / (i + IntegerPart[twist / 2])! / (i + IntegerPart[twist / 2] - 1)! / 2^(i);
   norm := (twist)! / ((twist / 2))^2;
   tmp4 = Table[norm ansatz[i] tmp3[[i]], {i, 2 - even, Length[tmp3], 2}];

In[82]:= tmp5 = Table[c4 i^4 + c0 i^3 + c1 i^2 + c2 i + c3, {i, 1, Length[tmp4]}];
   tmp6 = Flatten[Solve[(tmp4 - tmp5) == 0]];
   tmp7[i_] = Factor[c4 i^4 + c0 i^3 + c1 i^2 + c2 i + c3] /. tmp6;

In[85]:= refinedansatz[i_] = Factor[1 / norm / ansatz[i] tmp7[(i - even) / 2 + even]];
   final = Factor[refinedansatz[i]]; 
   ffinal = Simplify[final //. (i - i + 1)];
   Factor[ffinal]

Out[88]= \frac{143 2^{1-i} (1+i) (28+i) (-3534+145 i+5 i^2) (13+i) (14+i)}{2185 (27+2 i)!}

In[89]:= prefac[ampl_, a_, b_] :=
   Module[(evenl), If[(ampl == 2) || (ampl == 5), evenl := 0, evenl := 1];
   2^(a - 1) (a + b / 2 + 1) ! (a + b / 2) ! / (2 a + b + 1) ! (a + 1)^(1 - evenl) (a + b + 2)^(1 - evenl)]

In[90]:= d[ampl_, a_, b_] := c[ampl, a + b, a] / prefac[ampl, a, b]

In[91]:= ananz[t_, a_] := Sum[a^kk Pochhammer[t - p + 1, p]
   (t) ! (t + p)! / (2 t)! (Sum[c[o, k] t^o, {o, 0, 8}]), {kk, 0, 4}];

In[92]:= pss[ampl_, ii_] := Factor[Expand[FullSimplify[d[ampl, L, 2 ii]]]]

In[93]:= plo[y[ampl_, ii_] :=
   Module[{ii, ps}, For[ii = p, ii <= 12, ii++, ps = Table[pss[ampl, ii], {ii, p, 12}]];
   Factor[prefac[ampl, L, 2 t] ananz[t, L] /.
   Flatten[Solve[(Table[anz[t, L], {t, p, 12}] - ps) == 0]]]
In[94]:= \[2\] \[1\] (+A00\[2\])

Out[94]= \[\frac{1}{225 (2 \tau) ! (1 + 2 L + 2 \tau)!} (2^{-6} L (-3 + \tau) (-2 + \tau) (-1 + \tau) \tau

\[\frac{1}{675 (2 \tau) ! (1 + 2 L + 2 \tau)!} \]

In[95]:= \[\frac{1}{plo[2] (+A10\[2\])}

Out[95]= (2^{-6} L (1 + L) (-3 + \tau) (-2 + \tau) (-1 + \tau) \tau (2 + L + 2 \tau) (340 + 15 L + 5 L^2 - 12 \tau + 10 L \tau - 22 \tau^2) \tau ! (4 + \tau) ! (L + \tau) ! (1 + L + \tau) !) / (135 (2 \tau) ! (1 + 2 L + 2 \tau) !)

In[96]:= \[\frac{1}{plo[3] (+A11\[2\])}

Out[96]= \[\frac{1}{225 (2 \tau) ! (1 + 2 L + 2 \tau)!} (2^{-6} L (-3 + \tau) (-2 + \tau) (-1 + \tau) \tau

\[\frac{1}{675 (2 \tau) ! (1 + 2 L + 2 \tau)!} \]

In[97]:= \[\frac{1}{plo[4] (+A20\[2\])}

Out[97]= - (2^{-6} L (1 + L) (-3 + \tau) (-2 + \tau) (-1 + \tau) \tau (2 + L + 2 \tau) (10 + 15 L + 5 L^2 + 12 \tau + 10 L \tau + 2 \tau^2) \tau ! (4 + \tau) ! (L + \tau) ! (1 + L + \tau) !) / (225 (2 \tau) ! (1 + 2 L + 2 \tau) !)

In[98]:= \[\frac{1}{plo[5] (+A21\[2\])}

Out[98]= (2^{-6} L (1 + L) (-3 + \tau) (-2 + \tau) (-1 + \tau) \tau (2 + L + 2 \tau) (380 + 75 L + 25 L^2 + 36 \tau + 50 L \tau - 14 \tau^2) \tau ! (4 + \tau) ! (L + \tau) ! (1 + L + \tau) !) / (225 (2 \tau) ! (1 + 2 L + 2 \tau) !)

In[99]:= \[\frac{1}{plo[6] (+A22\[2\])}

Out[99]= \[\frac{1}{675 (2 \tau) ! (1 + 2 L + 2 \tau)!} (2^{-6} L (-3 + \tau) (-2 + \tau) (-1 + \tau) \tau

\[\frac{1}{675 (2 \tau) ! (1 + 2 L + 2 \tau)!} \]

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Appendix G. $\mathcal{D}$ Identities

We give a list of standard relations\cite{9} for $\mathcal{D}$ functions here since they are used at various points in chapter 5. First there are identities which relate $\mathcal{D}$ functions with crossing transformed arguments (in the following we use $\Sigma = \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)$)

$$
\mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = v^{-\Delta_4} \mathcal{D}_{\Delta_1 \Delta_2 \Delta_4 \Delta_3}(u/v, 1/v),
$$

$$
= v^{\Delta_1 - \Sigma} \mathcal{D}_{\Delta_2 \Delta_1 \Delta_4 \Delta_3}(u/v, 1/v),
$$

$$
= \mathcal{D}_{\Delta_3 \Delta_2 \Delta_1 \Delta_4}(v, u),
$$

$$
= u^{-\Delta_4} \mathcal{D}_{\Delta_4 \Delta_2 \Delta_3 \Delta_1}(1/u, v/u).
$$

Then there is a set of relations between individual functions which leave the arguments invariant and permute the indices

$$
\mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = u^{\Delta_3 + \Delta_4 - \Sigma} \mathcal{D}_{\Delta_1 \Delta_2 \Delta_4 \Delta_3}(u,v),
$$

$$
= u^{\Delta_3 + \Delta_4 - \Sigma} \mathcal{D}_{\Delta_4 \Delta_3 \Delta_2 \Delta_1}(u,v).
$$

Also there exists a reflection property of the indices

$$
\mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = \mathcal{D}_{\Sigma - \Delta_3 \Sigma - \Delta_4 \Sigma - \Delta_1 \Sigma - \Delta_2}(u,v).
$$

Now we list identities relating $\mathcal{D}$ functions with different values of $\Sigma$. First there are the two up, two down relations

$$
(\Delta_2 + \Delta_4 - \Sigma) \mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = \mathcal{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u,v) - \mathcal{D}_{\Delta_1 + 1 \Delta_2 + 1 \Delta_3 + 1 \Delta_4}(u,v),
$$

$$
(\Delta_1 + \Delta_4 - \Sigma) \mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = \mathcal{D}_{\Delta_1 + 1 \Delta_2 \Delta_3 + 1 \Delta_4 + 1}(u,v) - v \mathcal{D}_{\Delta_1 + 1 \Delta_2 + 1 \Delta_3 + 1 \Delta_4}(u,v),
$$

$$
(\Delta_3 + \Delta_4 - \Sigma) \mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = \mathcal{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 + 1 \Delta_4 + 1}(u,v) - u \mathcal{D}_{\Delta_1 + 1 \Delta_2 + 1 \Delta_3 + 1 \Delta_4}(u,v).
$$

Finally there is a formula for the sum of 3 $\mathcal{D}$ functions with the same value of $\Sigma$

$$
\Delta_4 \mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u,v) = \mathcal{D}_{\Delta_1 + 1 \Delta_2 \Delta_3 \Delta_4 + 1}(u,v) + \mathcal{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u,v)
$$

$$
+ \mathcal{D}_{\Delta_1 \Delta_2 \Delta_3 + 1 \Delta_4 + 1}(u,v).
$$

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Superconformal Symmetry and Correlation Functions

Summary

The constraints that $\mathcal{N} = 2, 4$ superconformal symmetry imposes in $d = 4$ for four point functions of chiral primary $\frac{1}{2}$-BPS operators are derived. The operators are described by symmetric traceless tensors of the internal $R$-symmetry group. A substantial simplification compared to earlier work is achieved by introduction of null vectors. They reduce complicated tensorial expressions to polynomials of one or two invariant cross ratios for $\mathcal{N} = 2, 4$ respectively, similar to the 2 conformally invariant cross ratios of 4 points. Two variable polynomials corresponding to the different $R$-symmetry representations are constructed. The Ward identities for superconformal symmetry are then obtained as simple differential equations. The general solution of these identities is presented in terms of a constant, a single variable function and a two variable function. In the extremal case it is shown that the amplitude has to be a constant. In the next-to-extremal case the amplitude contains a constant and a single variable function only. An interpretation in terms of the operator product expansion is given for the case of fields of equal dimension and for the so called (next-to)extremal cases. The result is shown to accommodate long multiplets as well as semishort and short multiplets with protected dimension. Generically also non-unitary multiplets can appear. It is shown how to remove them using appropriate semishort and long multiplets to obtain a unitary theory. Where possible, positivity of OPE coefficients required by unitarity is confirmed. Implications of crossing symmetry for the four point functions studied are derived and discussed. It is shown that crossing symmetry fixes the single variable function in the general solution to be of free field form using singularity arguments. For a restricted set of next-to-extremal correlation functions with $\mathcal{S}_3$ symmetry amongst the first three fields it is shown that the amplitude is fixed up to normalization to be of free field form. Starting from a known expression for the large $\mathcal{N}$ amplitude of $[0, 4, 0]$ operators we simplify it further and present it in a manifestly crossing symmetric form. We compute the coefficients of the conformal partial wave expansion of all representations in this amplitude and use them to compute an averaged value of the anomalous dimensions for long multiplets given spin and twist in each relevant representation at first order in $1/\mathcal{N}$. 