Counting characters in linear group actions

by

Thomas Michael Keller
Department of Mathematics
Texas State University
601 University Drive
San Marcos, TX 78666
USA
e–mail: keller@txstate.edu

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Abstract. Let $G$ be a finite group and $V$ be a finite $G$–module. We present upper bounds for the cardinalities of certain subsets of $	ext{Irr}(GV)$, such as the set of those $\chi \in \text{Irr}(GV)$ such that, for a fixed $v \in V$, the restriction of $\chi$ to $\langle v \rangle$ is not a multiple of the regular character of $\langle v \rangle$. These results might be useful in attacking the non–coprime $k(GV)$–problem.

1 Introduction

Let $G$ be a finite group and $V$ be a finite $G$–module of characteristic $p$. If $(|G|,|V|) = 1$, then in [3, Theorem 2.2] R. Knörr presented a beautiful argument showing how to obtain strong upper bounds for $k(GV)$ (the number of conjugacy classes of $GV$) by using only information on $C_G(v)$ for a fixed $v \in V$. Note that his result immediately implies the important special case that if $G$ has a regular orbit on $V$ (i.e., there is a $v \in V$ with $C_G(v) = 1$), then $k(GV) \leq |V|$, which was a crucial result in the solution of the $k(GV)$–problem. In this note we give a much shorter proof of this result (see Proposition 3.1 below).

The main objective of the paper, however, is to modify and generalize Knörr’s argument in various directions to include non–coprime situations. This way we obtain a number of bounds on certain subsets of $	ext{Irr}(GV)$, such as the following:

**Theorem A.** Let $G$ be a finite group and let $V$ be a finite $G$–module of characteristic $p$. Let $v \in V$ and $C = C_G(v)$ and suppose that $(|C|,|V|) = 1$. Then the number of irreducible characters whose restriction to $\langle v \rangle$ is not a multiple of the regular character of $\langle v \rangle$ is bounded above by

$$k(C) \sum_{i=1}^{k(C)} |C_V(c_i)|,$$

where the $c_i$ are representatives of the conjugacy classes of $C$.

**Theorem B.** Let $G$ be a finite group and $V$ be a finite $G$–module. Let $g \in G$ be of prime order not dividing $|V|$. Then the number of irreducible characters of $GV$ whose restriction to $A = \langle g \rangle$ is not a multiple of the regular character of $\langle g \rangle$ is bounded above by

$$|C_G(g)| \cdot n(C_G(g),C_V(g)),$$

where $n(C_G(g),C_V(g))$ denotes the number of orbits of $C_G(g)$ on $C_V(g)$.

Stronger versions and refinements of these results are proved in the paper. It is hoped that these results prove useful in solving the non–coprime $k(GV)$–problem, as discussed, for instance, in [2] and [1]. Theorem A and B will be proved in Sections 3 and 4 below respectively. In Section 2, we will generalize a recent result of P. Schmid [5, Theorem 2(a)] stating that in the situation
of the $k(GV)$–problem, if $G$ has a regular orbit on $V$, then $k(GV) = |V|$ can only hold if $G$ is abelian. We prove

**Theorem C.** Let $G$ be a finite group and $V$ a finite faithful $G$–module with $(|G|, |V|) = 1$. Suppose that $G$ has a regular orbit on $V$. Then

$$k(GV) \leq |V| - |G| + k(G).$$

Our proof is different from the approach taken in [5], and we actually will prove a slightly stronger result including some non–coprime actions.

Notation: If the group $A$ acts on the set $B$, we write $n(A, B)$ for the number of orbits of $A$ on $B$. All other notation is standard or explained along the way.

## 2 $k(GV) = |V|$ and regular orbits

In this paper we often work under the hypothesis of the $k(GV)$–problem which is the following.

**2.1 Hypothesis.** Let $G$ be a finite group and let $V$ be a finite faithful $G$–module such that $(|G|, |V|) = 1$. Write $p$ for the characteristic of $V$.

In [5, Theorem 2(a)] P. Schmid proved that under Hypothesis 2.1, if $G$ has a regular orbit on $V$, $V$ is irreducible, and $k(GV) = |V|$, then $G$ is abelian, and from this it follows easily that either $|G| = 1$ and $|V| = p$, or $G$ is cyclic of order $|V| - 1$. The proof in [5] is somewhat technical. The goal of this section is to give a short proof of a generalization of Schmid’s result based on a beautiful argument of Knörr [3]. We word it in such a way that we even do not need the coprime hypothesis, so that the result may even be useful to study the non–coprime $k(GV)$–problem. To do this, for any group $X$ and $x \in X$ we introduce the set

$$\text{Irr}(X, x) = \{ \chi \in \text{Irr}(G) | \chi|_{\langle x \rangle} \text{ is not an integer multiple of the regular character of } \langle x \rangle \}$$

and write

$$k(X, x) = |\text{Irr}(X, x)|.$$

**2.2 Theorem.** Let $G$ be a finite group and let $V$ be a finite $G$–module such that $G$ possesses a regular orbit on $V$. Let $v \in V$ be a representative of such an orbit. Then

$$k(GV, v) \leq |V| - |G| + k(G)$$
Proof. Let \( p \) be the characteristic if \( V \). We proceed exactly as in Case (ii) of the proof of [3, Theorem 2.2]. Write \( C = C_G(v) \). As \( C = 1 \), we see that for \( A = \langle v \rangle \) we trivially have that \( |C| \) and \( |A| \) are coprime, and so that proof yields
\[
(1) \quad (p - 1)|V| = \sum_{\tau \in \text{Irr}(GV)} (\tau \eta, \tau)_A
\]
where \( \eta \) is the character of \( A \) defined by \( \eta = p1_A - \rho_A \) with \( \rho_A \) being the regular character of \( A \). Now for any \( \tau \in \text{Irr}(GV) \) we have
\[
(2) \quad (\tau \eta, \tau)_A = \frac{1}{|A|} \sum_{a \in A} \tau(a)(p - \rho_A(a))\overline{\tau(a)}
\]
\[
= \sum_{1 \neq a \in A} |\tau(a)|^2 \begin{cases} 0 & \text{if } \tau|_A \text{ is an integer multiple of } \rho_A \\ \geq p - 1 & \text{otherwise} \end{cases}
\]
where the last step follows from [4, Corollary 4]. Next observe that if \( \tau \in \text{Irr}(GV) \) with \( V \leq \ker \tau \), then \( \tau \in \text{Irr}(G) \) and clearly \( \tau|_A \) is not a multiple of \( \rho_A \), and then clearly
\[
(3) \quad (\tau \eta, \tau)_A = \sum_{1 \neq a \in A} |\tau(a)|^2 = \sum_{1 \neq a \in A} \tau(1)^2 = (p - 1)\tau(1)^2.
\]
Thus with (1), (2), and (3) we get
\[
(p - 1)|V| = \sum_{\tau \in \text{Irr}(G)} (\tau \eta, \tau)_A + \sum_{\tau \in \text{Irr}(GV), \ V \not\leq \ker \tau} (\tau \eta, \tau)_A
\]
\[
\geq \sum_{\tau \in \text{Irr}(G)} (p - 1)\tau(1)^2 + (k(GV, v) - k(G))(p - 1)
\]
which yields
\[
|V| \geq \sum_{\tau \in \text{Irr}(G)} \tau(1)^2 + k(GV, v) - k(G) = |G| + k(GV, v) - k(G).
\]
This implies the assertion of the theorem, and we are done. □

The following consequence implies Schmid’s result [5, Theorem 2(a)].

2.3 Corollary. Assume Hypothesis 2.1 and that \( G \) has a regular orbit on \( V \). Then
\[
k(GV) \leq |V| - |G| + k(G).
\]
In particular, if $k(GV) = |V|$, then $G$ is abelian.

Proof. By Ito’s theorem and as $(|G|,|V|) = 1$, we know that $\chi(1)$ divides $|G|$ for every $\chi \in \text{Irr}(GV)$, so in particular $p$ does not divide $\chi(1)$. Thus for any $v \in V^\#$ we see that $\chi|_{\langle v \rangle}$ cannot be an integer multiple of $p_{\langle v \rangle}$. Therefore $k(GV, v) = k(GV)$. Now the assertion follows from Theorem 2.2. \hfill \diamond

3 Bounds for $k(GV)$

In this section we study more variations of Knörr’s argument in [3, Theorem 2.2] and generalize it to some non-coprime situations.

We begin, however, by looking at a classical application of it. An important and immediate consequence of Knörr’s result is that if under Hypothesis 2.1 $G$ has a regular orbit on $V$, then $k(GV) \leq |V|$. This important result can be obtained in the following shorter way.

3.1 Proposition. Let $G$ be a finite group and let $V$ be a finite faithful $G$–module. Let $v \in V$. Then

$$k(GV, v) \leq |C_{G}(v)||V|,$$

in particular, if $(|G|,|V|) = 1$ and $G$ has a regular orbit on $V$, then $k(GV) \leq |V|$.

Proof. Put $A = \langle v \rangle$. If $\tau \in \text{Irr}(GV, v)$, then by [4, Corollary 4] we know that $\sum_{1 \neq a \in A} |\tau(a)|^2 \geq p - 1$. With this and well–known character theory we get

$$(p - 1)k(GV, v) \leq k(GV, v) \min_{\tau \in \text{Irr}(GV, v)} \left( \sum_{1 \neq a \in A} |\tau(a)|^2 \right)$$

$$\leq \sum_{\tau \in \text{Irr}(GV)} \sum_{1 \neq a \in A} |\tau(a)|^2$$

$$= \sum_{1 \neq a \in A} \sum_{\tau \in \text{Irr}(GV)} \tau(a)\overline{\tau(a)}$$

$$= \sum_{1 \neq a \in A} |C_{GV}(a)|$$

$$= \sum_{1 \neq a \in A} |C_{G}(v)||V|$$

$$= (p - 1)|C_{G}(v)||V|$$

This implies the first result. If $(|G|,|V|) = 1$, then by Ito’s result $\tau(1)||G|$ for all $\tau \in \text{Irr}(GV)$, so $p$ cannot divide $\tau(1)$, and thus $k(GV, v) = k(GV)$, and the second result now follows by choosing
$v$ to be in a regular orbit of $G$ on $V$.  

Now we turn to generalizing Knörr’s argument. We discuss various ways to do so.

3.2 Remark. Let $G$ be a finite group and let $V$ be a finite faithful $G$–module of characteristic $p$. Let $v \in V$ and put $C = C_G(v)$ and $A = \langle v \rangle$. Let

$$\text{Irr}(GV, C, v) := \text{Irr}_0(GV) := \{\chi \in \text{Irr}(GV) \mid \chi|_{C \times \langle v \rangle} = \tau \times \rho_A \text{ for a character } \tau \text{ of } C\}$$

and

$$\text{Irr}_{p'}(GV) = \{\chi \in \text{Irr}(GV) \mid p \text{ does not divide } \chi(1)\},$$

so that clearly $\text{Irr}_{p'}(GV) \subseteq \text{Irr}_0(GV)$.

Note that if $(|G|, |V|) = 1$, then by Ito $\text{Irr}(GV) = \text{Irr}_{p'}(GV)$.

To work towards our next result, we again proceed somewhat similarly as in [3, Theorem 2.2]. In the following we work under the hypothesis that $(|C|, |V|) = 1$. Let $N = N_G(A)$. Then $|N : C|$ divides $p - 1$. Moreover, from Knörr’s proof we know that if $c_i$ ($i = 1, \ldots, k(C)$) with $c_1 = 1$ are representatives of the conjugacy classes of $C$ and $a_j$ ($j = 1, \ldots, \frac{p-1}{|N:C|}$) are representatives of the $N$–conjugacy classes of $A - 1$ then, the $c_ia_j$ are representatives of those conjugacy classes of $GV$ which intersect $C \times (A - 1)$ nontrivially.

Moreover recall from Knörr’s proof that for $c \in C, 1 \neq a \in A, g \in G, u \in V$ we know that

$$(ca)^{gu} \in C \times A \text{ if and only if } g \in N \text{ and } u \in C_V(c^g).$$

Now define a character $\eta$ on $C \times A$ by $\eta = 1_C \times (p1_A - \rho_A)$.

Then for $c \in C, a \in A$ we have

$$\eta(ca) = \begin{cases} p, & \text{if } a \neq 1 \\ 0, & \text{if } a = 1 \end{cases}$$

Therefore $\eta^{GV}$ vanishes on all conjugacy classes of $GV$ which intersect $C \times (A - 1)$ trivially, whereas for $c \in C, 1 \neq a \in A$ we have that

$$\eta^{GV}(ca) = \frac{1}{|C \times A|} \sum_{g \in G} \sum_{u \in V} \eta((ca)^{gu})$$

$$= \frac{1}{p|C|} \sum_{g \in N} \sum_{u \in C_V(c^g)} \eta(c^g a^g)$$

$$= \frac{1}{p|C|} \sum_{g \in N} |C_V(c^g)|p$$

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Thus if \( x_i \) \((i = 1, \ldots, k(GV))\) are representatives of the conjugacy classes of \( GV \), then we get

\[
\begin{align*}
\sum_{i=1}^{k(GV)} \eta^{GV}(x_i) &= \sum_{i=1}^{k(C)} \sum_{j=1}^{N/C} \eta^{GV}(c_i a_j) \\
&= \sum_{i=1}^{k(C)} \sum_{j=1}^{N/C} |N : C| |C V(c_i)| \\
&= \frac{p - 1}{|N : C|} |N : C| \sum_{i=1}^{k(C)} |C V(c_i)| \\
&= (p - 1) \sum_{i=1}^{k(C)} |C V(c_i)|,
\end{align*}
\]

and thus

\[
(p - 1) \sum_{i=1}^{k(C)} |C V(c_i)| = \sum_{i=1}^{k(GV)} \eta^{GV}(x_i) = \sum_{\tau \in \text{Irr}(GV)} (\tau \eta^{GV}, \tau)_{GV} \\
= \sum_{\tau \in \text{Irr}(GV)} (\tau \eta, \tau)_{C \times A} \quad (1).
\]

Now if \( \tau \in \text{Irr}(GV) \), as in [3] write

\[
\tau|_{C \times A} = \sum_{\lambda \in \text{Irr}(A)} \tau_\lambda \times \lambda \quad (2)
\]

where \( \tau_\lambda \) is a character of \( C \) or \( \tau_\lambda = 0 \).

Then as in [3] we see that

\[
(\tau \eta, \tau)_{C \times A} = \frac{1}{|C \times A|} \sum_{c \in C} \sum_{a \in A} \tau(ca) \eta(ca) \overline{\tau(ca)} \\
= \frac{1}{|C|} \sum_{\substack{c \in C \quad 1 \neq a \in A}} \tau(ca) \overline{\tau(ca)} \\
= \sum_{\lambda \neq \mu} (\tau_\lambda - \tau_\mu)(\tau_\lambda - \tau_\mu)|_C \quad (3)
\]
where ”≤” is some arbitrary ordering on \text{Irr}(A).

Now if \(\tau_\lambda - \tau_\mu\) is a nonzero multiple of \(\rho_{C}\), then
\[
(\tau_\lambda - \tau_\mu, \tau_\lambda - \tau_\mu)_{C} \geq |C| \tag{4}
\]
and thus
\[
(\tau_\eta, \tau)_{C \times A} \geq |C|.
\]

Moreover, note that if \(\tau \in \text{Irr}_0(GV)\), then not all \(\tau_\lambda - \tau_\mu\) can be equal to 0 as otherwise from (2) we see that \(\tau|_{C \times A}\) would be equal to \(\tau_\lambda \times \rho_{A}\) for any \(\lambda\). So we can partition the set \(\text{Irr}(A)\) into two disjoint nonempty subsets \(\Lambda_1 = \{\lambda \in \text{Irr}(A) \mid \tau_\lambda = \tau_1\}\) and \(\Lambda_2 = \{\lambda \in \text{Irr}(A) \mid \tau_\lambda \neq \tau_1\}\), and thus as in [3] we see that \(|\Lambda_1| |\Lambda_2| \geq p - 1\) so there are at least \(p - 1\) pairs \(\lambda, \mu \in \text{Irr}(A)\) such that \(\tau_\lambda - \tau_\mu \neq 0\). Thus
\[
(\tau_\eta, \tau)_{C \times A} \geq p - 1 \quad \text{for all} \quad \tau \in \text{Irr}_0(GV). \tag{5}
\]

Therefore by (1) and (5) we get that
\[
(p - 1) \sum_{i=1}^{k(C)} |C \times A| = \sum_{\tau \in \text{Irr}(GV)} (\tau_\eta, \tau)_{C \times A} \geq \sum_{\tau \in \text{Irr}_0(GV)} (\tau_\eta, \tau)_{C \times A} \geq (p - 1)|\text{Irr}_0(GV)|
\]
and thus
\[
|\text{Irr}_0(GV)| \leq \sum_{i=1}^{k(C)} |C \times A|. \tag{6}
\]

From now on we assume that \(C > 1\).

Now we repeat the arguments of this proof, but replace \(\eta\) by
\[
\eta_1 = (|C|1_\text{C} - \rho_{C}) \times (p1_\text{A} - \rho_{A}),
\]
so for \(c \in C\) and \(a \in A\) we have
\[
\eta_1(ca) = \begin{cases} |C|p & \text{if } c \neq 1 \text{ and } a \neq 1 \\ 0 & \text{if } c = 1 \text{ or } a = 1 \end{cases}
\]

Now from the above we know that the \(c_ia_j\) \((i = 2, \ldots, k(C), j = 1, \ldots, \frac{p-1}{|N: C|})\) are representatives of those conjugacy classes which intersect \((C - 1) \times (A - 1)\) nontrivially.

Clearly \(\eta_1^{GV}\) vanishes on all conjugacy classes of \(GV\) which intersect \((C - 1) \times (A - 1)\) trivially, whereas for \(1 \neq c \in C\), \(1 \neq a \in A\), if \((|C|, |V|) = 1\), we have that
\[
\eta_1^{GV}(ca) = \frac{1}{|C \times A|} \sum_{g \in G} \sum_{u \in V} \eta_1((ca)^{gu})
\]
\[
= \frac{1}{|C|} \sum_{\nu \in N} \sum_{u \in C \times (ca)_{\nu}} \eta_1(c^{\nu}a^{\nu})
\]
\[
= |N| |C \times V(c)|.
\]
Next we conclude that

\[
\sum_{i=1}^{k(GV)} \eta^G_i(x_i) = \sum_{i=2}^{k(G)} \sum_{j=1}^{p-1} \eta^G_j(c_ia_j) = (p-1)|C| \sum_{i=2}^{k(G)} |C_V(c_i)|,
\]

and so as in (1) we see that

\[
(p-1)|C| \sum_{i=2}^{k(C)} |C_V(c_i)| = \sum_{\tau \in Irr(GV)} (\tau \eta_1, \tau)_{C \times A} \quad (7).
\]

Now with (2) similarly as in [3] we see that

\[
(\tau \eta_1, \tau)_{C \times A} = \frac{1}{|C \times A|} \sum_{c \in C} \sum_{a \in A} \tau(ca) \eta_1(ca) \overline{\tau(ca)}
\]

\[
= \sum_{1 \neq c \in C} \sum_{1 \neq a \in A} \tau(ca) \overline{\tau(ca)}
\]

\[
= \sum_{1 \neq c \in C} \sum_{1 \neq a \in A} \tau_\lambda(c) \lambda(a) \sum_{\mu \in Irr(A)} \overline{\tau_\mu(c) \mu(a)}
\]

\[
= \sum_{\lambda, \mu \in Irr(A)} \sum_{1 \neq c \in C} \tau_\lambda(c) \overline{\tau_\mu(c)} \sum_{1 \neq a \in A} \lambda(a) \overline{\mu(a)}
\]

\[
= (p-1) \sum_{\lambda \in Irr(A)} \sum_{1 \neq c \in C} \tau_\lambda(c) \overline{\tau_\lambda(c)} - \sum_{\lambda, \mu \in Irr(A)} \sum_{1 \neq c \in C} \tau_\lambda(c) \overline{\tau_\mu(c)}
\]

\[
= p \sum_{\lambda \in Irr(A)} \sum_{1 \neq c \in C} \tau_\lambda(c) \overline{\tau_\lambda(c)} - \sum_{\lambda, \mu \in Irr(A)} \sum_{1 \neq c \in C} \tau_\lambda(c) \overline{\tau_\mu(c)}
\]

\[
= \sum_{\lambda < \mu} \sum_{1 \neq c \in C} (\tau_\lambda(c) - \tau_\mu(c))(\overline{\tau_\lambda(c)} - \overline{\tau_\mu(c)})
\]

\[
= \sum_{\lambda < \mu} \sum_{1 \neq c \in C} |\tau_\lambda(c) - \tau_\mu(c)|^2 \quad (8)
\]

for some arbitrary ordering \( \leq \) on \( \text{Irr}(A) \).

Now recall that if \( \tau \in \text{Irr}_0(GV) \), then not all of the \( \tau_\lambda - \tau_\mu \) can be 0. So choose \( \lambda, \mu \in \text{Irr}(C) \) such that \( \tau_\lambda - \tau_\mu \neq 0 \). If all the \( \tau_\mu \ (\mu \in \text{Irr}(A)) \) are integer multiples of \( \rho_C \) then put \( \Lambda_1 = \{ \phi \in \text{Irr}(A) \mid \tau_\phi = \tau_\lambda \} \) and \( \Lambda_2 = \{ \phi \in \text{Irr}(A) \mid \tau_\phi \neq \tau_\lambda \} \), so \( \Lambda_1 \neq \emptyset \) and \( \Lambda_2 \neq \emptyset \) and from

\[
0 \leq (|\Lambda_1| - 1)(|\Lambda_2| - 1)
\]

we clearly deduce that \( |\Lambda_1||\Lambda_2| \geq p-1 \), so there are at least \( p-1 \) pairs \( (\phi_1, \phi_2) \in \text{Irr}(A) \times \text{Irr}(A) \) such that \( \tau_{\phi_1} - \tau_{\phi_2} \) is a nonzero multiple of \( \rho_C \).

So next we assume that \( \tau_\lambda \) is not a multiple of \( \rho_C \).

Then put

\[
\Gamma_1 = \{ \phi \in \text{Irr}(A) \mid \tau_\lambda - \tau_\phi \text{ is a multiple of } \rho_C \}
\]
and
\[ \Gamma_2 = \{ \phi \in \text{Irr}(A) \mid \tau_\lambda - \tau_\phi \text{ is not a multiple of } \rho_C \}. \]

Clearly \( \lambda \in \Gamma_1 \), so \( \Gamma_1 \neq \emptyset \). If \( \Gamma_2 = \emptyset \), then \( \text{Irr}(A) = \Gamma_1 \), and if we define \( \Lambda_1, \Lambda_2 \) as in the previous argument, we see that there are at least \( (p - 1) \) pairs \((\phi_1, \phi_2) \in \text{Irr}(A) \times \text{Irr}(A)\) such that \( \tau_{\phi_1} - \tau_{\phi_2} \) is a nonzero multiple of \( \rho_C \).

So now suppose \( \Gamma_2 \neq \emptyset \). Then \( |\Gamma_1| + |\Gamma_2| = p \), and if \( \phi_1 \in \Gamma_1 \) and \( \phi_2 \in \Gamma_2 \), then \( \tau_{\phi_1} - \tau_{\phi_2} = (\tau_{\phi_1} - \tau_\lambda) + (\tau_\lambda - \tau_{\phi_2}) \) clearly is not a multiple of \( \rho_C \), and by the same argument as used before we see that \( |\Gamma_1||\Gamma_2| \geq p - 1 \), so there are at least \( (p - 1) \) pairs \((\phi_1, \phi_2) \in \text{Irr}(A) \times \text{Irr}(A)\) such that \( \tau_{\phi_1} - \tau_{\phi_2} \) is not a multiple of \( \rho_C \).

Altogether we thus have shown that for any \( \tau \in \text{Irr}_0(GV) \) one of the following holds:

(A) There are at least \( (p - 1) \) pairs \((\phi_1, \phi_2) \in \text{Irr}(A) \times \text{Irr}(A)\) such that
\[ \tau_{\phi_1} - \tau_{\phi_2} \text{ is a nonzero multiple of } \rho_C, \]

or

(B) there are at least \( (p - 1) \) pairs \((\phi_1, \phi_2) \in \text{Irr}(A) \times \text{Irr}(A)\) such that
\[ \tau_{\phi_1} - \tau_{\phi_2} \text{ is not a multiple of } \rho_C. \]

Now it remains to consider two cases:

Case 1: At least half of the \( \tau \in \text{Irr}_0(GV) \) satisfy (A).

Then for any of these \( \tau \) by (3) and (4) we have
\[ (\tau \eta, \tau)_{C \times A} = \sum_{\lambda < \mu} ((\tau_\lambda - \tau_\mu), (\tau_\lambda - \tau_\mu))_C \geq (p - 1)|C| \]

and so by (1) we see that
\[ (p - 1) \sum_{i=1}^{k(C)} |C_V(c_i)| \geq \sum_{\tau \in \text{Irr}_0(GV)} (\tau \eta, \tau)_{C \times A} \geq \frac{1}{2} |\text{Irr}_0(GV)| (p - 1)|C| \]

which implies
\[ |\text{Irr}_0(GV)| \leq \frac{2}{|C|} \sum_{i=1}^{k(C)} |C_V(c_i)| \]  \hspace{1cm} (9).

Case 2: At least half of the \( \tau \in \text{Irr}_0(GV) \) satisfy (B).

Then for any of these \( \tau \) by (8) and [4, Corollary 4] we have
\[ (\tau \eta_1, \tau)_{C \times A} \geq (p - 1)(k(C) - 1). \]

Thus by (7) we have that
\[ (p - 1)C \sum_{i=2}^{k(C)} |C_V(c_i)| \geq \sum_{\tau \in \text{Irr}_0(GV)} (\tau \eta_1, \tau)_{C \times A} \geq \frac{1}{2} |\text{Irr}_0(GV)| (p - 1) \cdot (k(C) - 1) \]
whence
\[
|\text{Irr}_0(GV)| \leq \frac{2|C|}{k(C) - 1} \sum_{i=2}^{k(C)} |C_V(c_i)| \tag{10}
\]
Now we drop the assumption \((|C|, |V|) = 1\) and work towards a general bound for \(|\text{Irr}_0(GV)|\).
For this, fix \(g_0 \in C\) such that \(g_0\) is of prime order \(q\) and put \(C_0 = \langle g_0 \rangle\) and \(N_0 = N_G(C_0)\). Trivially there are at most \(|C_0|(p - 1) = q(p - 1)\) conjugacy classes of \(GV\) that intersect \(C_0 \times (A - 1)\) nontrivially, and given \(1 \neq c \in C_0, 1 \neq a \in A\), we see that for \(g \in G, u \in V\)
\[
(ca)^g[a] = c^g[a]u^g \in C_0 \times A \text{ first implies } c^g \in C_0, \text{ i.e., } g \in N_0,
\]
and for each fixed \(g \in N_0\), the equation \([c^g, u]a^g \in A\) implies \([c^g, u] \in Aa^{-g}\) which has at most \(|C_V(c^g)| |Ag^{-1}| = p|C_V(g_0)|\) solutions \(u\).
Moreover, if \(c = 1\), then
\[
(ca)^g[a] = a^g \text{ implies } g \in N_G(A) = N \text{ and } u \in V.
\]
Now we define the character \(\eta_2\) on \(C_0 \times A\) by \(\eta_2 = 1_{C_0} \times (p1_A - pA)\). Thus \(\eta_2^{GV}\) vanishes on all conjugacy classes of \(GV\) which intersect \(C_0 \times (A - 1)\) trivially, whereas for \(1 \neq c \in C_0, 1 \neq a \in A\) we get
\[
\eta_2^{GV}(ca) = \frac{1}{|C_0 \times A|} \sum_{g \in G} \sum_{u \in V} \eta((ca)^gu)
\]
\[
\leq \frac{1}{qp} \sum_{g \in N_0} p|C_V(g_0)||p
\]
\[
= \frac{p}{q} |N_0||C_V(g_0)|,
\]
and for \(c = 1, 1 \neq a \in A\) we get
\[
\eta_2^{GV}(ca) = \eta_2^{GV}(a) = \frac{1}{qp} \sum_{q \in N} |V|p = \frac{1}{q}|N||V|.
\]
Thus if \(x_i (i = 1, \ldots, k(GV))\) are representatives of the conjugacy classes of \(GV\), then
\[
\sum_{i=1}^{k(GV)} \eta_2^{GV}(x_i) \leq (p - 1)\frac{1}{q}|N||V| + (q - 1)(p - 1)\frac{p}{q} |N_0||C_V(g_0)|
\]
and as in (1) we see that
\[
\sum_{i=1}^{k(GV)} \eta_2^{GV}(x_i) = \sum_{\tau \in \text{Irr}(GV)} (\tau \eta_2, \tau)_{C_0 \times A}.
\]

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Now arguing as in (2), (3), (5) and (6) above will yield

\[ |\text{Irr}_p'(GV)| \leq k(GV, v) \leq |\text{Irr}(GV, C_0, v)| \leq \frac{1}{q}(|N||V| + (q - 1)p|N_0||C_V(g_0)|), \]

where \( \text{Irr}(GV, C_0, v) \) is as defined at the beginning of Remark 3.2. Putting the main results together, altogether we have proved the following:

3.3 Theorem. Let \( G \) be a finite group and let \( V \) be a finite faithful \( G \)-module of characteristic \( p \). Let \( v \in V \) and put \( C = C_G(v) \). If \( c_i \ (i = 1, \ldots, k(C)) \) are representatives of the conjugacy classes of \( C \), then the following hold:

(a) If \((|C|, |V|) = 1\), then

\[ |\text{Irr}_0(GV)| \leq \sum_{i=1}^{k(C)} |C_V(c_i)| \]

and if \( C > 1 \), then

\[ |\text{Irr}_0(GV)| \leq \max \left\{ \frac{2}{|C|} \sum_{i=1}^{k(C)} |C_V(c_i)|, \frac{2|C|}{k(C) - 1} \sum_{i=2}^{k(C)} |C_V(c_i)| \right\} \]

(b) If \((|G|, |V|) = 1\), then

\[ \text{Irr}_0(GV) = \text{Irr}(G), \quad \text{so} \quad k(GV) = |\text{Irr}_0(GV)| \]

and the bounds in (a) hold true for \( k(GV) \) instead of \( |\text{Irr}_0(GV)| \).

(c) In general, if \( g \in C \) such that \( o(g) = q \) is a prime, then

\[ |\text{Irr}_p'(GV)| \leq k(GV, v) \leq \frac{1}{q} \left( |N_G(\langle v \rangle)||V| + (q - 1)p|N_G(\langle g \rangle)||C_V(g)| \right). \]

4 The dual approach

In the previous section, we always fixed \( v \in V \) and obtained bounds on the size of suitable subsets of \( \text{Irr}(GV) \) in terms of properties of the action of \( C_G(v) \) on \( V \). In this section we consider a "dual" approach:

We fix \( g \in G \) and find bounds in terms of the action of \( C_G(g) \) on \( C_V(g) \). For this, put

\[ \text{Irr}_g(GV) = \{ \chi \in \text{Irr}(G) \mid \chi|_{\langle g \rangle} \times C_V(g) \text{ cannot be written as } \rho_{\langle g \rangle} \times \psi \text{ for a character } \psi \text{ of } C_V(g) \}. \]
In particular, \( \text{Irr}(GV, g) \subseteq \text{Irr}_g(GV) \).

4.1 Theorem. Let \( G \) be a finite group and \( V \) be a finite \( G \)-module. Let \( g \in G \) such that \( (o(g), |V|) = 1 \). Write \( A = \langle g \rangle \), \( N = N_G(A) \) and \( C = C_V(g) \). Then

\[
(a) \ |\text{Irr}_g(GV)| \leq \frac{(n(N, A) - 1)n(C_G(A), C)}{(|A| - 1)|C|^2} \max_{1 \neq a \in A} (|N_G(\langle a \rangle)||C_V(a)|)
\]

(b) if \( g \) is of prime order, then

\[
|\text{Irr}_g(GV)| \leq |C_G(A)|n(C_G(A), C)
\]

(c) there are \( X, Y \subseteq \text{Irr}_g(GV) \) such that \( \text{Irr}_g(GV) \) is a disjoint union of \( X \) and \( Y \) and

\[
|X| \leq \frac{(n(N, A) - 1)n(C_G(A), C)}{(|A| - 1)|C|^2} \max_{1 \neq a \in A} (|N_G(\langle a \rangle)||C_V(a)|) \text{ and } |Y| \leq |C_G(A)|(n(C_G(A), C) - 1)
\]

(d) if \( g \) is of prime order and \( X, Y \) are as in (c), then

\[
|X| \leq \frac{|C_G(A)|n(C_G(A), C)}{|C|} \text{ and } |Y| \leq |C_G(A)|(n(C_G(A), C) - 1)
\]

Proof. If \( a_1, a_2 \in A \) and \( c_1, c_2 \in C - \{1\} \), then it is straightforward to see that \((a_1, c_1)^{GV} = (a_2, c_2)^{GV}\) implies that \(a_1^G = a_2^G\). Hence if \( T \) is a set of representatives of the orbits of \( N \) on \( A - \{1\} \), then every conjugacy class of \( GV \) that intersects nontrivially with \((A - \{1\}) \times C\) has a representative \(ac\) for some \(a \in T\) and some \(c \in C\). Moreover, for each \(a \in T\) we have that if \(c_3, c_4 \in C\) are \(C_G(A)\)-conjugate, then \(ac_3\) and \(ac_4\) are \(C_G(A)\)-conjugate and thus \((ac_3)^G = (ac_4)^G\). This shows that for each \(a \in T\) there are at most \(n(C_G(A), C)\) conjugacy classes of \(GV\) intersecting nontrivially with \(\{a\} \times C\). Hence altogether we see that there are at most

\[
|T|n(C_G(A), C) = (n(N, A) - 1)n(C_G(A), C) \quad (1)
\]

conjugacy classes of \(GV\) which intersect \((A - \{1\}) \times C\) nontrivially.

Moreover observe that for \(1 \neq a \in A, c \in C, h \in G\) and \(u \in V\) we have

\[
(ac)^{hu} \in A \times C \text{ if and only if } h \in N_G(\langle a \rangle), \ c^h \in C \text{ and } u \in C_V(a)
\]

because the condition \((ac)^{hu} = a^h[ac^h, u]^h \in A \times C\) first forces \(a^h \in A\) which implies (as \(A\) is cyclic) \(a^h \in \langle a \rangle\), so \(h \in N_G(\langle a \rangle)\), and then as \(c \in C \leq C_V(\langle a \rangle)\), it follows that \(c^h \in C_V(\langle a \rangle)\) and \([a^h, u] \in [\langle a \rangle, V]\). Now as by our hypothesis we have \(V = C_V(\langle a \rangle) \times [\langle a \rangle, V]\), we see that
\((ac)^{hu} \in A \times C\) now forces \([a^h, u] = 1\) and \(c^h \in C\). Hence \(u \in C_V(a^h) = C_V(a)\). Note that the direct product \(A \times C\) is a subgroup of \(GV\). We now define a generalized character \(\eta\) on \(A \times C\) by

\[
\eta = (|A| \cdot 1_A - \rho_A) \times 1_C
\]

where \(\rho_A\) is the regular character of \(A\). So for \(a \in A, c \in C\) we have

\[
\eta(ac) = \begin{cases} 
0, & a = 1 \\
|A|, & a \neq 1
\end{cases}
\]

Therefore \(\eta^{GV}\) vanishes on all conjugacy classes of \(GV\) which intersect \((A - \{1\}) \times C\) trivially, whereas for \(c \in C\) and \(1 \neq a \in A\) we have

\[
\eta^{GV}(ac) = \frac{1}{|A \times C|} \sum_{u \in V} \eta((ac)^{hu})
\]

\[
= \frac{1}{|A||C|} \sum_{h \in N_G((a)) \text{ with } c^h \in C} \sum_{u \in C_V(a)} \eta((ac)^{hu})
\]

\[
= \frac{1}{|A||C|} \sum_{h \in N_G((a)) \text{ with } c^h \in C} \sum_{u \in C_V(a)} \eta(a^hc^h)
\]

\[
= \frac{|C_V(a)|}{|A||C|} \sum_{h \in N_G((a)) \text{ with } c^h \in C} |A|
\]

\[
\leq \frac{|N_G((a))||C_V(a)|}{|C|} (2)
\]

Thus if \(\{x_i \mid i = 1, \ldots, k(GV)\}\) is a set of representatives for the conjugacy classes of \(GV\), then by (1) and (2) we see that

\[
(n(N, A) - 1)n(C_G(A), C) \cdot \frac{1}{|C|} \max_{1 \neq a \in A} (|N_G((a))||C_V(a)|) \geq \sum_{i=1}^{k(GV)} \eta^{GV}(x_i)
\]

\[
= \sum_{\tau \in \text{Irr}(GV)} (\tau \eta^{GV}, \tau)_{GV}
\]

\[
= \sum_{\tau \in \text{Irr}(GV)} (\tau \eta, \tau)_{A \times C} \quad (3).
\]

Observe that in case that \(A\) is of prime order, then

\[
n(N, A) - 1 = \frac{|A| - 1}{|N : C_G(A)|} = \frac{(|A| - 1)|C_G(A)|}{|N|}
\]
and \( \max_{1 \neq a \in A} (|N_G((a))| |C_V(a)|) = |N||C| \), so that (3) becomes

\[
|C_G(A)||(|A| - 1)n(C_G(A), C) \geq \sum_{\tau \in \text{Irr}(GV)} (\tau \eta, \tau)_{A \times C} \quad (3a)
\]

Since \( A \times C \) is a direct product, we can write

\[
\tau_{A \times C} = \sum_{\lambda \in \text{Irr}(C)} (\tau \times \lambda),
\]

where \( \tau \lambda \) is a character of \( A \) or \( \tau \lambda = 0 \). Then

\[
(\tau \eta, \tau)_{A \times C} = \frac{1}{|A \times C|} \sum_{a \in A} \sum_{c \in C} \tau(ac)\eta(ac)\overline{\tau(ac)}
= \frac{1}{|A||C|} \sum_{1 \neq a \in A} \sum_{c \in C} \tau(ac)\overline{\tau(ac)}
= \frac{1}{|C|} \sum_{1 \neq a \in A} \sum_{\lambda \in \text{Irr}(C)} \tau_\lambda(a)\lambda(c) \sum_{\mu \in \text{Irr}(C)} \overline{\tau_\mu(a)\mu(c)}
= \sum_{1 \neq a \in A} \sum_{\lambda, \mu \in \text{Irr}(C)} \tau_\lambda(a)\overline{\tau_\mu(a)}\lambda(c)\mu(c)
= \sum_{1 \neq a \in A} \sum_{\lambda, \mu \in \text{Irr}(C)} \tau_\lambda(a)\overline{\tau_\mu(a)}(\lambda, \mu)_C
\]

As \( (\lambda, \mu)_C = \begin{cases} 1, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases} \), we further obtain

\[
(\tau \eta, \tau)_{A \times C} = \sum_{1 \neq a \in A} \sum_{\lambda \in \text{Irr}(C)} \tau_\lambda(a)\overline{\tau_\lambda(a)}
= \sum_{\lambda \in \text{Irr}(C)} \sum_{1 \neq a \in A} |\tau_\lambda(a)|^2 \quad (4)
\]

Now observe that \( \tau(1) = \sum_{\lambda \in \text{Irr}(C)} \tau_\lambda(1) \).

If all the \( \tau_\lambda \) are multiples of \( \rho_A \), then clearly \( \tau_1 \notin \text{Irr}(GV) \), and so if \( \tau \in \text{Irr}(GV) \), then by \([4, \text{Corollary 4}] \) with (4) we see that

\[
(\tau \eta, \tau)_{A \times C} \geq |A| - 1 \quad (5)
\]

So (3) and (5) yield

\[
|\text{Irr}_g(GV)| \leq \frac{(n(N, A) - 1)n(C_G(A), C)}{|(A| - 1)|C|} \max_{1 \neq a \in A} (|N_G((a))| |C_V(a)|), \quad (6)
\]
and if \( g \) is of prime order, then (3a) and (5) yield

\[ |\text{Irr}_g(GV)| \leq |C_G(A)|n(C_G(A), C). \quad (6a) \]

Now as in Section 3, we now repeat the same arguments, but use

\[ \eta_1 = (|A|1_A - \rho_A) \times (|C|1_C - \rho_C) \]

instead of \( \eta \).

One can then easily check that

\[ (n(N, A) - 1)(n(C_G(A), C) - 1) \cdot \frac{1}{|C|} \max_{1 \neq a \in A} (|N_G(\langle a \rangle)||C_V(\langle a \rangle)) \geq \sum_{\tau \in \text{Irr}(GV)} (\tau \eta_1, \tau)_{A \times C} \quad (3b) \]

and if \( g \) is of prime order, then

\[ |C_G(A)|(\langle A \rangle - 1)(n(C_G(A), C) - 1) \geq \sum_{\tau \in \text{Irr}(GV)} (\tau \eta_1, \tau)_{A \times C} \quad (3c) \]

Moreover it is easily seen that

\[ (\tau \eta_1, \tau)_{A \times C} = \sum_{1 \neq a \in A} \tau(ac)\overline{\tau(ac)} \]

\[ = \sum_{1 \neq a \in A} \sum_{1 \neq c \in C} \sum_{\lambda, \mu \in \text{Irr}(C)} \tau_\lambda(a)\overline{\tau_\mu(a)} \sum_{1 \neq c \in C} \lambda(c)\mu(c), \]

and as \( \sum_{1 \neq c \in C} \lambda(c)\overline{\mu(c)} = \begin{cases} -1, & \text{if } \lambda \neq \mu \\ |C| - 1, & \text{if } \lambda = \mu \end{cases} \), it follows that

\[ (\tau \eta_1, \tau)_{A \times C} = \sum_{\lambda < \mu} \sum_{1 \neq a \in A} |\tau_\lambda(a) - \tau_\mu(a)|^2 \quad (7) \]

where ”\( \leq \)” is an arbitrary ordering on \( \text{Irr}(C) \).

Next suppose that there are exactly \( a \) characters \( \tau \in \text{Irr}_g(GV) \) such that there is a character \( \psi \) of \( A \) (depending on \( \tau \)) and there are \( a_\lambda \in \mathbb{Z} \) (\( \lambda \in \text{Irr}(C) \)) such that \( \tau_\lambda = \psi + a_\lambda \rho_A \) for all \( \lambda \in \text{Irr}(C) \) and \( \psi \) is not a multiple of \( \rho_A \). Then by (4) and [4, Corollary 4] we know that

\[ (\tau \eta, \tau)_{A \times C} = \sum_{\lambda \in \text{Irr}(C)} \sum_{1 \neq a \in A} |\psi(a)|^2 \geq |C|(\langle A \rangle - 1) \]

and hence by (3) we get

\[ a \leq \frac{(n(N, A) - 1)n(C_G(A), C)}{(\langle A \rangle - 1)|C|^2} \max_{1 \neq a \in A} (|N_G(\langle a \rangle)||C_V(\langle a \rangle))), \quad (8) \]
and if \( g \) is of prime order, then by (3a) even

\[
a \leq \frac{|C_G(A)| n(C_G(A), C)}{|C|} \quad (8a)
\]

Now let \( b \) be the number of \( \tau \in \text{Irr}_g(GV) \) such that there is no such \( \psi \).

Then there exist \( \lambda, \mu \in \text{Irr}(C) \) with

\[
\sum_{1 \neq a \in A} |\tau_\lambda(a) - \tau_\mu(a)|^2 \neq 0,
\]

and thus by [4, Corollary 4] we have

\[
(\tau_\eta_1, \tau) \geq |A| - 1 \quad (9)
\]

So (3b) and (9) yield

\[
b \leq \frac{(n(N, A) - 1)(n(C_G(A), C) - 1)}{|C|(|A| - 1)} \max_{1 \neq a \in A} (|N_G(\langle a \rangle)||C_V(a)|) \quad (10)
\]

and, if \( g \) is of prime order, then by (3c)

\[
b \leq |C_G(A)| (n(C_G(A), C) - 1), \quad (10b)
\]

and clearly \( a + b = |\text{Irr}_g(GV)| \), and hence all the assertions follow and we are done. \( \diamond \)

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