ON KUROSH PROBLEM IN VARIETIES OF ALGEBRAS

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Abstract. We consider a couple of versions of classical Kurosh problem (whether there is an infinite-dimensional algebraic algebra?) for varieties of linear multioperator algebras over a field. We show that, given an arbitrary signature, there is a variety of algebras of this signature such that the free algebra of the variety contains multilinear elements of arbitrary large degree, while the clone of every such element satisfies some nontrivial identity. If, in addition, the number of binary operations is at least 2, then one can guarantee that each such clone is finitely-dimensional.

Our approach is the following: we translate the problem to the language of operads and then apply usual homological constructions in order to adopt Golod’s solution of the original Kurosh problem.

The paper is expository, so that some proofs are omitted. At the same time, the general relations of operads, algebras, and varieties are widely discussed.

1. Introduction

1.1. Kurosh problem and multioperator algebras. In his prognosis of the future of general algebra, A. G. Kurosh predicted in 1970 that the main interests of general algebra would moved, in the nearest decades, to the neutral territory between universal and general algebra [K, p. 9]. Following this tendency, in this note we try to extend some classical algebraic ideas of the time of Kurosh to a territory very close to universal algebra.

We start with a classical question of Kurosh: whether there can be a finitely-generated infinite-dimensional algebraic algebra? The first solution was found by E. S. Golod [G] in 1964, who constructed a number of analogues examples for different algebraic systems (such as associative algebras, Lie algebras and p–groups). Here we will discuss the Kurosh problem in the setting of linear universal algebra, for algebraic systems in place of algebras in the formulating. To construct suitable examples, we extend the original Golod technique, including the Golod–Shafarevich theorem [GS].

We consider the varieties of multioperator linear Ω–algebras over a field k (the term from [K § 13]). Every multilinear element of the countable generated free algebra F of a variety W may be identified with a family of multilinear operators (or operations) acting on the algebras of W. The linear combinations of finite compositions of a multioperator with itself form a linear subspace of F called
clone of the operator. We will see that the notion of clone of a single operator is analogous to the concept of one-generator subalgebra of an associative algebra. Therefore, one can consider two natural versions of the above Kurosh problem. The strong version (we will refer it as Burnside problem) is the following: does there exist a variety $W$ (of a given finite signature $\Omega$) such that it admits nonzero $n$-linear operations for all arbitrary large $n$ while every clone of a single operation is finite-dimensional? A weaker version of the problem is the following: does there exist a variety $W$, again with infinite-dimensional space of multi-linear operations, such that every such an operation satisfy nontrivial “identity”, that is, some nontrivial linear combination of composition of the operation with itself is zero?

One can consider also relative versions of the above problems, that is, with an additional restriction that the variety $W$ must be a subvariety of a given variety $\mathcal{V}$. In this paper, we focus on the non-relative versions, but our method (based on Theorem 4.1) gives also some examples to the relative case.

1.2. The language of operads. In linear universal algebra (when the algebras are modules over a ring or a field and the operations are multi-linear homomorphisms), there are two main languages for definitions and theorems which are used for different purposes.

First one is the classical language of varieties and identities: it is the most popular one in ring theory and in general algebra as well. The theory of algebras with polynomial identities is usually described in this language. The works of Kurosh are also written in it.

Another language is based on the concept of operad. It is used mostly in algebraic topology and mathematical physics. This language seems more appropriate for discussions on the homological properties of algebraic systems, including Massey operations and Kontsevich formality.

Our first step here is to give a brief dictionary of these two languages in Section 2. In this dictionary, we define operads and related notions in terms of varieties; dually, we define the varieties, identities etc in operadic terms. Therefore, it is assumed that the reader do understand at least one of these two languages. We hope that this brief dictionary will help the both kind of universal algebraists to understand each other, — or, at least, to recognize the well-known objects in foreign-language descriptions. We use this dictionary in order to get a more natural formulation of our versions of universal Kurosh problem.

For the reader convenience, let us give a rough and brief translation table in a phrase-book style.

**A phrase-book**

- variety $\rightarrow$ operad
- subvariety $\rightarrow$ quotient operad
- clone $\rightarrow$ suboperad
- signature $\rightarrow$ set of generators
- identities $\rightarrow$ relations
- free algebra $\rightarrow$ free algebra
- codimension series $\rightarrow$ generating function
- $T$-space $\rightarrow$ right ideal
- $T$-ideal $\rightarrow$ ideal
- Specht properties $\rightarrow$ Noether properties
Now, our two versions of Kurosh problem are translated as follows. Burnside problem (or strong Kurosh problem) for operads: *does there exist an infinite finitely generated operad P such that every its one-generated suboperad is finite?* Kurosh problem (in a weak version) for operads: *does there exist an infinite finitely generated operad P such that every its one-generated suboperad is not free?* The relative version sounds as follows: *given an operad S, can one choose the operad P above to be a quotient of S?*

In this paper, we give partial answers to both these problems. That is, we show that such varieties and operads do exists (in the stronger version, with a restriction on the signature). The relative versions are in general open.

Let us give a “bi-lingual” formulation (some details are given in Section 4 below).

**Theorem 1.1** (Corollaries 4.3, 4.4). 1) Let $\Omega$ be a finite signature. Then there exists a variety of algebras of signature $\Omega$ such that there are nonzero multilinear operations of arbitrary high order but the clone of each operation in a free algebra satisfies a non-trivial identity. If $\Omega(2)$ has at least 2 elements, then one can assert, in addition, that the clone of each operation in a free algebra is finite-dimensional.

2) Let $X = X(2), X(3), \ldots$ be an $S$-module, that is, a sequence of representation of symmetric groups $S_2, S_3, \ldots$ Then there exists an operad $P$ generated by $X$ such that each one-generated suboperad in it is not absolutely free. If $\dim X_2 \geq 3$, then we can assert, in addition, that every its one-generated suboperad is finite.

1.3. **Organization of the paper.** This paper is expository; most proofs are omitted and will be published in the subsequent paper [1].

The dictionary mentioned before is given in Section 2. In Section 3 we discuss the (well-known) analogy between operads and graded associative algebras. This leads to an analogy of linear universal algebra and graded ring theory. So, we describe the notions of ideals, modules, generators and other algebraic terms in operad theory.

This analogy allows to develop a version of classical homological algebra, including free resolutions and derived functor, in the category of modules of given operads. We use it in order to transfer the Golod–Shafarevich theorem to operads, see Section 4. This leads, by the Golod method, to a construction of infinite operad with finite one-generated sub-operads. This gives the solutions of weak and strong Kurosh problems for varieties (operads) for algebras of almost arbitrary signature, see Corollaries 4.3 and 4.4.

1.4. **Assumptions.** We consider varieties of multioperator linear algebras over a field $k$. To avoid technical details, we assume that the signature $\Omega$ of a variety is always locally finite and does not contain constants and unary operations, that is, $\Omega = \Omega_2 \cup \Omega_3 \cup \ldots$ there the subsets $\Omega_n$ of $n$-ary operations (i.e., $n$-linear operators) are finite. To simplify the notation, we assume that the identical operator (which does not belong to $\Omega$) is also applicable to any algebra of the variety. In Section 2 we assume, unless otherwise is stated, that the ground field $k$ has zero characteristic.

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2. Operads vs varieties: a dictionary

2.1. A definition of operad. Let $W$ be a variety of $k$–linear algebras (without constants, with identity and without other unary operations) of some signature $\Omega$. Consider the free algebra $F^W(X)$ on a countable set of indeterminates $X = \{x_1, x_2, \ldots \}$. Let $P(n) \subset F^W(X)$ be the subspace consisting of all multilinear generalized homogeneous polynomials on the variables $x_1, \ldots, x_n$.

Definition 2.1. Given such a variety $W$, the sequence $P_W = P := \{P(1), P(2), \ldots \}$ of the vector subspaces of $F^W(X)$ is called an operad. The signature $\Omega$ is called a generation set of the operad $P_W$.

The $n$-th component $P(n)$ may be identified with the set of all derived $n$-linear operations on the algebras of $W$: in particular, $P(n)$ carries a natural structure of a representation of the symmetric group $S_n$. Such a sequence $Q = \{Q(n)\}_{n \in \mathbb{Z}}$ of representations $Q(n)$ of the symmetric groups $S_n$ is called an $S$–module, so that an operad carries a structure of $S$-module. Also, the compositions of operations (that is, a substitution of an argument $x_i$ by a result of another polylinear operation, with a subsequent re-numerating the variables) gives natural equivariant maps of $S_n$-modules $\circ_i : P(n) \otimes P(m) \rightarrow P(n + m - 1)$. Note that the axiomatization of these operations gives an abstract definition of operads, see MSS for the discussion on different definitions.

A morphism of operads $f : P \rightarrow P'$ is a sequence of maps $f(n) : P(n) \rightarrow P'(n)$ of $S_n$-modules compatible with the compositions. For example, any inclusion $W \subset W'$ of varieties of the same signature gives a surjective operadic map $P_W \rightarrow P_{W'}$.

2.2. A definition of variety. Let $P$ be an operad (with $P(0)$ equal to zero and one-dimensional $P(1)$ spanned by the identity element) with some discrete generating set $\Omega$. Recall that an algebra over $P$ (= $P$-algebra) is a (non-graded) right $P$-module, that is, a vector space $V$ with $P$-action $P(n) : V \otimes^n V \rightarrow V$ compatible with compositions of operations and the $k[S_n]$-module structures on $P$. The class (or an additive category) of all algebras over $P$ is denoted by $P$-Alg.

Definition 2.2. Let $P$ be an operad with generating set $\Omega$. Then the category $P$-Alg is called a variety of algebras of signature $\Omega$.

From now, let us fix a variety $W$ and a corresponding operad $P$, so that $W = P_W$ and $P = P_W$. We fix also a minimal generating set (or signature) $\Omega$. The implications $(W = P_W) \iff (P = P_W)$ will be discussed later in Proposition 2.3.

2.3. (Co)dimensions. An $n$-th codimension of a variety $W$ is just the dimension of the respective operad component: $c_n(W) = \dim_k P_W(n)$. The codimension series of the variety $W$, or the generating function of the operad $P$, is a formal power series

$$P(z) := \sum_{n \geq 1} \frac{c_n(W)}{n!} z^n = \sum_{n \geq 1} \frac{\dim_k P_W(n)}{n!} z^n.$$ 

\footnote{More precisely, symmetric connected $k$–linear operad with identity.}
An analogous generation function $Q(z) = \sum_{n \geq 1} \frac{\dim_k Q(n)}{n^r} z^n$ is defined by for every $S$-module $Q$ with $\dim_k Q(n) < \infty$ for all $n \geq 1$. For a more general versions of this generating function (which involve, in particular, the characters of the representations $P(n)$ of groups $S_n$) we refer the reader to the well-known paper of Ginzburg and Kapranov [GK].

If the set $\Omega$ is finite, then the series $P(z)$ define an analytical function in a neighborhood of zero. For example, the operad $Ass$ of associative algebras has generating function $Ass(z) = \frac{1}{1 - z}$. For every proper quotient operad $P$ of $Ass$, we have $\lim_{n \to \infty} \frac{\ln \dim_k P(n)}{n} = \ln(c(P))$, where $c(P) \in \mathbb{Z}$ (due to Giambruno and Zaitev, see [GZ] and references therein): in particular, the function $P(z)$ is in this case analytical in the whole complex plane.

2.4. Free variety. Recall that a sequence $M = \{M(1), M(2), \ldots\}$, where each $M(i)$ is a $k[S_i]$-module, is called an $S$-module. Given a sequence $\Omega = \{\Omega(1), \Omega(2), \ldots\}$ of discrete sets, we naturally define a free $S$-module $\mathbb{S} \Omega = \{k[S_i]^{\Omega(1)}, k[S_i]^{\Omega(2)}, \ldots\}$. In particular, every operad is an $S$-module, and every subset of an $S$-module generate an $S$-submodule. If $\Omega$ is a minimal generating set of an operad $P$, then the $S$-submodule $S \Omega$ is also called a generating space of $P$.

Definition 2.3 (of free variety). The variety $W$ of signature $\Omega$ is called free, if the generating set $\Omega$ minimally generates a free $S$-submodule in the operad $P = P_W$ and the operad $P$ itself is free with generating $S$-submodule $S \Omega$.

A free algebra in a free variety is called absolutely free (of given signature).

We will call the operad of a free variety absolutely free.

2.5. Free operad. Let $P$ be an operad generated by a subset $\Omega$. The operad $P$ is called free (on the generating set $\Omega$), if the T-ideal $T$ of identities of the variety $W = W_P$ consists of the linear combinations of generators, that is, by elements of the $S$-submodule $X$ of $P$ generated by $\Omega$. Since the free operad $P$ is uniquely determined by the $S$-submodule $X$, it is denoted by $\Gamma(X)$ (notation from [AIS]).

For example, any absolutely free operad is free (since it has no relations). On the other hand, the operad $GenCom$ of general (nonassociative) commutative algebras is free with the multiplication $\mu$ as a generator ($\Omega = \{\mu\}$), but is not absolutely free because of the identity $[x_1, x_2] := \mu(x_1, x_2) - \mu \circ \tau(x_1, x_2) \in X$, where $\tau$ is the generator of the group $S_2$.

2.6. Relations of operads. For every two $S$-submodules $A$ and $B$ of an operad $P$, one can define a new $S$-submodule $A \circ B \subset P$ generated by all compositions $a(b_1, \ldots, b_n)$, where $a \in A \cap P(n)$ and $b_i \in B$. An $S$-submodule $I \subset P$ is called a left (respectively, right, two-sided) ideal, if $I = P \circ I$ (resp., $I = I \circ P$, $I = P \circ I \circ P$). The generating sets of ideals are defined in the obvious way.

It follows that the two-sided ideals are exactly the kernels of operadic morphisms. If an operad $P$ is represented as a quotient (=‘image of a surjective morphism’) of a free operad $P'$ by a two-sided ideal $I$, the elements of $I$ are called the relations of $P$. Given a generating set $\Omega$ of $P'$, all the relations becomes the identities of the variety $W_P$ in this signature.

For example, the operad $Com$ of commutative associative algebras, as a quotient of the free operad $GenCom$ described above, has the associativity relation $Ass(x_1, x_2, x_3) := (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3)$ (where $a \cdot b := \mu(a, b)$), and all other
relations belong to the two-sided ideal \( I \) in \( \text{GenCom} \) generated by this relation \( \text{Ass}(x_1, x_2, x_3) \).

2.4. Proposition. Let \( W \) be a field \( \text{Char}(k) > 0 \). Then for every \( p, q \in W \) one can define a composition \( p \circ q \in W \) (it is equal to \( p \) if \( p \) does not really depend on the variable \( x_i \)). Note that, in contrast to the operad composition, there is no re-numerating of variables after \( c \) composition, e. g., \( p(x_1, x_2) \circ_1 x_2 = p(x_2, x_2) \) etc. Then for every subset \( C \subseteq W \) one can define two composition subset \( F \circ C = \{ f \circ_i c \} \) and \( C \circ F = \{ c \circ_i f \} \) (where \( f, c, i \) runs through \( F, C \) and \( \mathbb{Z}_{\geq 1} \), respectively). A linear subspace \( C \subseteq F \) is called an ideal (respectively, \( T \)-space, subalgebra) if it is closed under composition of the first type (respectively, second type, both types).

Let \( W' \) be a free variety with signature \( \Omega \), and let \( P' = P_{W'} \) be the corresponding operad. The ideal \( I \) of relations of the operad \( P \) in \( P' \) generates a \( T \)-ideal \( T \) in the absolutely free algebra \( F = F_{W'}(X) \) with a countable generating set \( X \). The elements of \( T \) are called identities of the variety \( W \).

The standard linearization process gives a procedure to establish the following

**Proposition 2.4.** Let \( F \) be a free algebra with countable generating set of a variety \( W \) over field \( k \) of zero characteristic. Then every \( T \)-space or \( T \)-ideal \( Y \) in \( F \) is generated by the subset \( Y \cap P_W \) of multilinear elements. Moreover, this subset \( Y \cap P_W \) form an ideal (right- or two-sided, respectively) in the operad \( P_W \).

In particular, it follows that the \( T \)-ideal \( T \) of identities of an arbitrary variety \( W \) is generated by the relations of the operad \( P := P_W \), hence \( W = W_P \).

The proof of this proposition (essentially, it is a description of the linearization process mentioned above) is essentially the same for all types of linear algebras; see, e. g., [KBR] or [GZ] for the case of associative PI algebras. For example, the linearization of an identity \( x_1^2 \) in the free algebra of \( W_{\text{GenCom}} \) leads to the identity \( x_1 x_2 \) (because \( x_1 x_2 = \frac{1}{4}((x_1 + x_2)^2 - x_1^2 - x_2^2) \)), that is, the identity \( x_1^2 \) defines the variety of algebras with zero multiplication, that is, the category of vector spaces.

Note that the linearization is essentially depend on the assumption \( \text{Char}(k) = 0 \). If the characteristic of the field is positive, then only the implication \( (W = W_P) \implies (P = P_W) \) is valid, but the reverse implication fails.

## 3. Operads & Graded algebras: an Analogy

A homological theory of operads is similar to the homological theory of associative algebras. There is a number of homological construction, which are successively moved from rings to operads: (co)bar constructions, minimal models, (DG-)resolutions and (DG-)modules, Koszul duality etc. [F, GK, MSS]. Here we move to operads a part of classical homological algebra, namely, the theory of torsion functors. By a standard way, we will construct free resolutions of modules over operads and use them to define and calculate the derived functors of an operadic analogue of tensor product. This will be used later in our version of Golod–Shafarevich theorem.

Note that for every operad \( P \) one can define graded right and left modules over it: they are \( S \)-modules \( V \) with the structure of \( P \)-algebras (right modules) or with
the compositions $V(n) \circ P(m) \to V(n+m-1)$ (left modules), where in both cases the structure should be compatible with the operadic and $S$-module structures. The composition functor $- \circ P L$ (where $L$ is a graded left $P$-module) from the category $\text{mod} - P$ of graded right $P$-modules to the category $k - \text{mod}$ of graded vector spaces over $k$ is analogous to tensor product of modules over a graded algebra. It has left derived functors $\text{Tor}^P_i(R, L)$ which are analogous to usual $\text{Tor}$s of modules over graded algebras. These operadic torsion functors can be calculated using free resolutions (or cofibrant resolutions, in the DG case) of the first argument $[F]$. A formal explanation of these ideas can be given by the following chain of standard statements.

**Proposition 3.1.** Let $P$ be an operad. Then the category $\text{mod} - P$ of all graded right $P$-modules is abelian (hence, it is an abelian subcategory of the category of all $S$-modules).

Let $V$ be an arbitrary $S$-module. A composition right $P$-module $V \circ P$ is called free (and $V$ is called as its minimal $S$-module of generators). As an $S$-module, it is a composition product, so that its generation function is equal to $(V \circ P)(z) = V(P(z))$. For example, $P$ itself is a free right module generated by the trivial $S$-module $k id$.

Suppose that $M$ is a right graded $P$-module minimally generated by an $S$-module $V'$ isomorphic to $V$. Then there is a (unique up to an isomorphism $V \to V'$) surjective map of $P$-modules $p : V \circ P \to M$ which isomorphically maps $V$ to $V'$. The kernel $\ker p$ belongs to the ‘submodule of decomposables’ $V \otimes P_+ \subset V \circ P$, where $P_+ = P(2) \oplus P(3) \oplus \ldots$ is the maximal ideal of $P$. Iterating this construction, we get an exact sequence of right graded $P$-modules

$$\ldots F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0,$$

where all modules $F_i$ are free (that is why we refer the subsequence $F : \ldots F_2 \to F_1 \to F_0$ as free resolution of $M$). In addition, we have $\mathrm{im} d_i \subset F_{i-1} \circ P_+$ for all $i \geq 0$. A resolution with the last property will be called minimal.

The second part of the following standard proposition can proved by the same way as for graded modules over a connected graded associative algebra (where all graded projective modules are free).

**Proposition 3.2.** (i) Every right $P$-module $M$ admits a minimal free resolution $F$.

(ii) If $F'$ is another free resolution of $M$, then the complex $F$ of $P$-modules is a direct summand of $F'$.

(iii) The minimal resolution $F$ is unique up to isomorphism of complexes of right $P$-modules.

Let $R$ and $L$ be right and left graded $P$-modules. Then one can define a composition $k$-module $R \circ P L$. It is a quotient $k$-module of $R \circ L$ by the relations induced by the action of $P$.

**Proposition 3.3.** Let $L$ be a left graded $P$-module.

(i) The functor $C_L : X \mapsto X \circ P L$ is right exact on $\text{mod}$.

(ii) There exist a derived functors $L_* C_L(M)$ whose value $\text{Tor}^P_i(M, L) := L_i C_L(M)$ can be calculated, for each $i > 0$, as the $i$-th homology $S$-module of the complex

$$F \circ P L : \ldots F_2 \circ P L \to F_1 \circ P L \to F_0 \circ P L.$$
Note that the above derived functors has been introduced by Fresse \cite[2.2.4]{F} in
a more general context of DG-operad. The second part of the above Proposition 3.3
follows from \cite[Proposition 2.2.5]{F}.

For example, the minimal \( S \)-submodules that generate the modules \( F_i \) from the
minimal free resolution can be calculated as

\[ F_i / F_i \circ P_+ = \text{Tor}^P_1(M, P/\mathcal{P}_+) \]

4. A criterion for infinite operads and Kurosh problem

We give here an operadic version of famous Golod–Shafarevich theorem which
gives a criterion for an associative algebra to be infinite-dimensional. Despite the
original version of the proof of Golod and Shafarevich \cite{GS} can be almost directly
translated to the language of operads (with Shafarevich complex replaced by the
first step of the construction of a minimal model of the operad \( P \), see \cite{MSS}), we
prefer to adopt another approach (explained by Ufnarovski \cite{U}) based on a direct
construction of the minimal free resolution of the trivial module.

**Theorem 4.1.** Let \( P \) be an operad minimally generated by an \( S \)-module \( X \subset P \)
with a minimal \( S \)-module of relations \( R \subset \Gamma(X) \). We assume here that both these \( S \)-modules are locally finite, that is, all their graded
component are of finite dimension.

Suppose that the formal power series

\[ (1 - \frac{X(z)}{z} + \frac{R(z)}{z})^{-1} \]

has non-negative coefficients. Then the operad \( P \) is infinite.

**Sketch of proof.** Consider the trivial bimodule \( I = P/\mathcal{P}_+ \) (where \( \mathcal{P}_+ = P(2) \oplus
P(3) \oplus \ldots \) is the maximal ideal of \( P \), as before). For the generators of the begin-
ing of its minimal free resolution, we have \( \text{Tor}^P_0(I, I) \cong I \), \( \text{Tor}^P_1(I, I) \cong X \) and
\( \text{Tor}^P_2(I, I) \cong R \). This means that the beginning of the resolution looks as

\[ 0 \to \Omega^3 \to R \circ P \xrightarrow{d_3} X \circ P \to P \to I \to 0, \]

where \( \Omega^3 \) is the kernel of \( d_2 \).

Taking the Euler characteristics of the exact sequence \( (4.1) \), one can we get an
equality of formal power series

\[ \Omega^3(z) = (R \circ P)(z) - (X \circ P)(z) + P(z) - I(z). \]

Since the formal power series \( \Omega^3(z) \) has nonnegative coefficients, we obtain the
following coefficient-wise inequality

\[ R(P(z)) - X(P(z)) + P(z) - z \geq 0. \]

Manipulations with formal power series (including the Lagrangian inverse) complete
the proof. \( \square \)

For operads generated by binary operations, one can simplify the above condition.

**Corollary 4.2.** Let \( P \) be an operad and let \( X \) and \( R \) be as above. Suppose that \( P \)
is generated by binary operations (that is, \( X = X(2) \)). Suppose that the function

\[ \phi(z) = 1 - \frac{X(z)}{z} + \frac{R(z)}{z} \]
is analytical in a neighborhood of zero (it is always the case if $X$ is finitely generated) and has a positive real root $z_0$ in this neighborhood such that $\phi(z_0)' \neq 0$. Then the operad $\mathcal{P}$ is infinite.

**Corollary 4.3** (Weak Kurosh problem for multi-operational algebras). Suppose that the ground field $k$ is countable. Let $\Omega$ be an arbitrary nonempty countable signature. Then there is a variety $W$ of algebras of signature $\Omega$ such that the clone of every polilinear operation in this variety satisfies some non-trivial identity while there are multi-linear elements of the free algebra $F^W(x_1, x_2, \ldots)$ of arbitrary high degrees.

**Proof.** It is sufficient to prove that the suboperad in $\mathcal{P}_W$ generated by an arbitrary single multi-linear operation is not absolutely free. Let us enumerate by positive integers all elements (=operations) of degree (=arity) $\geq 2$ in the absolutely free operad $\mathcal{F}$ generated by $\Omega$. If the suboperad $\mathcal{P}$ in $\mathcal{F}$ generated by such an operation $p_i$ (where $i \in \mathbb{Z}_+$) is not absolutely free (e.g., if the $S$-submodule $X$ generated by $p_i$ in $\mathcal{F}$ is not free), then its image in $\mathcal{P}_W$ is not absolutely free as well.

Suppose that the operad $\mathcal{P}$ is absolutely free. Let us denote by $R_i$ the sum of all multi-linear compositions of $N = N_i$ copies of $p_i$, where the numbers $N_i$ are chosen so that the degrees $t_i$ of the elements $R_i$ increase: $t_1 < t_2 < \ldots$. Then every element $R_i$ is invariant under the action of the symmetric group $S_{t_i}$. Therefore, the generating function of the $S$-module $R$ generated by all these elements $R_i$ is $R(z) \leq \sum_{n \geq 1} z^{t_n}/t_n! \leq \sum_{n \geq t_1} z^n/n!$ (the last coefficient-wise inequality follows from the inequalities $\dim R(n) \leq 1$ for all $n \geq 1$). If the numbers $t_i$ are chosen so that they are sufficiently large ($0 << t_1 << t_2 << \ldots$), then Theorem 4.1 and the above estimate imply that the generating function of the operad $\mathcal{P}$ generated by $\Omega$ with the $S$-module of relations $R_i$s is infinite. $\square$

The next claim gives a stronger version of Kurosh problem, that is, the Burnside problem.

**Corollary 4.4** (Burnside problem for multi-operational algebras). Suppose that the ground field $k$ is countable.  

Let $X = X(2) \cup X(3) \cup \ldots$ be an $S$-module such that $\dim X(2) \geq 3$. Then there is an infinite operad $\mathcal{P}$ generated by $X$ such that every its element $x \in \mathcal{P}$ is strongly nilpotent, that is, the suboperad generated by $x$ is finite.

**Remark 4.5.** In the language of varieties, the above Corollary looks as follows: there exist a variety of algebras with two binary operations such that every operation in these algebras is nilpotent, that is, for every operation (that is, homogeneous element of free algebra) there is a number $N$ such that every composition of $N$ operations of this kind is zero for all possible substitutions of variables.

**Idea of proof.** For every multilinear operation $p$ (say, $n$-ary, where $n \geq 2$) and sufficiently large number $d$, we define a suitable set of relations $S(p, d)$ as a set of all possible compositions of the operation $p$ with itself of the following kind: in each of $d$ copies of $p$ in the compositions, at least $n-1$ of the inputs of $p$ are replaced by variables. That is, every element of $S(p, d)$ looks as a “branch” of length $d$ whose nodes are market by $p$. Then we choose $d$ sufficiently large for each $p$ and use Corollary 4.3. $\square$
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