Remarks about an “exact” RG theory of Goldstone modes

J. Kaupužs *
Institute of Mathematics and Computer Science, University of Latvia
29 Rainja Boulevard, LV–1459 Riga, Latvia

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Abstract
A renormalization group (RG) theory of Goldstone mode singularities in the \(O(n > 1)\)-symmetric \(\varphi^4\) model is discussed. This perturbative RG theory is claimed to be asymptotically exact, as regards the long–wave limit of the correlation functions, where it predicts a purely Gaussian behavior of the transverse correlation function. However, we show that the results of this theory are incorrect, and the Gaussian behavior originates from a rough error in mathematics. Other relevant perturbative theories are discussed, as well.

1 Introduction

Recently, our theory of grouping of Feynman diagrams \[1\] has been extended to include the region below the critical point \[2\]. The two–point correlation function in Fourier representation has been studied, which depends on the wave vector \(\mathbf{k}\). Based on our method of systematic consideration of all diagrams, we have shown in \[2\] that the behavior of the longitudinal \((G_\parallel(\mathbf{k}))\) and transverse \((G_\perp(\mathbf{k}))\) correlation functions in \(O(n > 1)\)–symmetric \(\varphi^4\) model below the critical temperature \(T_c\) is non–Gaussian. It means that \(G_\perp(\mathbf{k}) \simeq a k^{-\lambda_\perp}\) and \(G_\parallel(\mathbf{k}) \simeq b k^{-\lambda_\parallel}\) with exponents \(d/2 < \lambda_\perp < 2\) and \(\lambda_\parallel = 2\lambda_\perp - d\) is the long–wave (at \(k \to 0\)) solution of our equations at the spatial dimensionality \(2 < d < 4\). These results coincide also with a non–perturbative renormalization group (RG) analysis provided in our paper.

It turns out, however, that some perturbative RG theory \[12\] \[13\], which is claimed to be exact by its founders, contradicts our results – it predicts a purely Gaussian behavior with \(\lambda_\perp = 2\). We have put some effort to clarify this question and have found that the Gaussian behavior in this theory originates from a rough error in mathematics.

2 The critical analysis

We consider a \(\varphi^4\) model with the Hamiltonian

\[
H/T = \int \left( r_0 \varphi^2(x) + c(\nabla \varphi(x))^2 + u \varphi^4(x) - h \varphi(x) \right) dx ,
\]

\[1\] (E-mail: kaupuzs@latnet.lv)
where the order parameter $\varphi(x)$ is an $n$–component vector with components $\varphi_i(x)$, depending on the coordinate $x$. $T$ is the temperature, $h$ is an external field. The $\varphi^4$ model exhibits a nontrivial behavior in close vicinity, as well as below the critical temperature $T_c$, if the order parameter is an $n$–component vector with $n > 1$. The related long–wave divergence of the longitudinal and transverse correlation functions (in Fourier representation) at $T < T_c$ has been studied in [3] based on the hydrodynamical (Gaussian) approximation. Essentially the same problem has been studied before in [4] in terms of the Gaussian spin–wave theory [5]. Later perturbative renormalization group (RG) studies [6, 7, 8, 9, 10, 11, 12, 13, 14] support the Gaussian approximation. The RG method is claimed to be asymptotically exact. Hence, we disprove this statement. Our analysis in [2] predicts a non–Gaussian behavior, and we show by general physical arguments that it must be the true behavior to coincide with the known rigorous results for the classical $XY$ model.

According to the conventional belief [6, 7, 8, 9, 10, 11, 12, 13, 14], the transverse correlation function $G_{\perp}(k)$ diverges like $k^{-\lambda_{\perp}}$ with $\lambda_{\perp} = 2$ at $k \to 0$ below $T_c$ for the systems with $O(n > 2)$ rotational symmetry. It corresponds to the $G_{\parallel}(k) \sim k^{d-4}$ divergence of the longitudinal correlation function. Besides, the singular structure of the correlation functions is represented by an expansion in powers of $k^{4-d}$ and $k^{d-2}$ in [11, 12]. Formally, our analysis in [2] is consistent with these results at $\lambda_{\parallel} = 2$, although in reality $\lambda_{\parallel} < 2$ holds. Below we will show that $\lambda_{\perp} < 2$ holds near two dimensions at $n = 2$.

As usually accepted in lattice models, here we define that all the parameters of the normalized Hamiltonian $H/T$ are proportional to the inverse temperature $1/T$. In this case $r_0$ is negative. The assumption that $G_{\perp}(k) \simeq a(T) k^{-2}$ holds with some temperature–dependent amplitude $a(T)$) in the stable region below the critical point, i. e., at $T \leq T_c/C$, where $C$ is an arbitrarily large constant, leads to a conclusion that the critical temperature $T_c$ continuously tends to zero at $d \to 2$ (supposed $d > 2$). Really, at $\lambda_{\parallel} = 2$ we have

$$\langle \varphi^2(x) \rangle = \frac{1}{V} \sum_{i,k} G_i(k) \simeq M^2$$

$$+(2\pi)^{-d} \left[ \int G'_{\parallel}(k)dk + \frac{(n-1)S(d)\Lambda^{d-2}a(T)}{d-2} \right],$$

where $G_i(k) = \langle |\varphi_i(k)|^2 \rangle$ is the $i$–th component (one of which is longitudinal, other ones – transverse) of the correlation function, $M$ is the magnetization, $\Lambda$ is the upper cutoff of the wave vector magnitude, and $S(d)$ is the area of unit sphere in $d$ dimensions. Here $G_{\parallel}(k) = G'_{\parallel}(k) + \delta_{k,0}M^2V$ is the Fourier transform of the real–space correlation function $\langle \varphi_{\parallel}(0)\varphi_{\parallel}(x) \rangle$, whereas $G'_{\parallel}(k)$ denotes the Fourier transform of $\langle \varphi_{\parallel}(0)\varphi_{\parallel}(x) \rangle - M^2$, where $\varphi_{\parallel}(x)$ is the longitudinal component of the order–parameter field. Since the amplitude of the transverse fluctuations never can vanish at a finite temperature, Eq. (2) implies that the average $\langle \varphi^2(x) \rangle$ diverges at $T = T_c/C$ when $d \to 2$, if $T_c$ remains finite. Thus, we obtain an unphysical result unless the critical temperature $T_c$ and, therefore, $a(T_c/C)$ tend to zero at $d \to 2$.

On the other hand, it is a rigorously stated fact [15, 16] that the classical 2D $XY$ model undergoes the Kosterlitz–Thouless phase transition at a finite temperature $T_{KT}$. It means that a certain structural order without the spontaneous magnetization exists within the temperature region $T < T_{KT}$. There is a general tendency of disordering with decreasing the spatial dimensionality $d$, and not vice versa. Thus, since the structural order exists at $T < T_{KT}$ and $d = 2$, some kind of order necessarily exists also at $T < T_{KT}$ and $d > 2$. Since
the classical XY model undergoes the disorder → long–range order phase transition at 
\( d > 2 \), this obviously is the long–range order. Thus, the critical temperature at 
\( d = 2 + \varepsilon \) is \( T_c \geq T_{kT} \neq 0 \) for an infinitesimal and positive \( \varepsilon \), and it drops to zero by a jump at 
\( d = 2 - \varepsilon \), as consistent with the rigorous consideration in \([15]\).

The classical XY model belongs to the same universality class as the actual \( \varphi^4 \) model
at \( n = 2 \), which means that both models become fully equivalent after a renormalization
(a suitable renormalization has been discussed in \([2]\)). Thus, \( T_c \) does not vanish at \( d \rightarrow 2 \)
(for \( d > 2 \)) also in the \( \varphi^4 \) model. In such a way, the assumption \( G_\perp(k) \simeq a(T)k^{-2} \)
leads to a contradiction. In the stable region \( T < T_c/C \), the Gaussian approximation
\( G_\perp(k) \simeq 1/(2ck^2) \) makes sense at finite not too small values of \( k \). The above contradiction
means that the Gaussian approximation with \( \lambda_\perp = 2 \) cannot be extended to \( k \rightarrow 0 \) in
vicinity of \( d = 2 \). The contradiction is removed only if \( \lambda_\perp < 2 \) holds at \( d \rightarrow 2 \) in the actual
case of \( n = 2 \).

It has been stated in \([12,14]\) that the essentially Gaussian result \( \lambda_\perp = 2 \) of the
perturbative RG theory should be exact. However, the underlying method is not rigorous:
it is assumed without proof that the renormalized Hamiltonian has the form of the Landau–
Ginzburg–Wilson expansion with few terms included, as in the original Hamiltonian \([11]\).
Besides the “exact” result is obtained by cutting (at one loop order) a purely formal
divergent perturbation series, where the expansion parameters are in no sense small. One
claims \([12,13,14]\) that the one–loop–diagram approximation provides asymptotically exact
results at \( r_0 \rightarrow -\infty \) (which corresponds to \( m_0 \rightarrow \infty \) or \( m \rightarrow \infty \) in \([12,13]\)) and, according
to the provided there renormalization group arguments, also in the long–wavelength limit
\( k \rightarrow 0 \). However, some of these “exact” results are rather unphysical. In particular, we
find from Eq. (3.6) in \([13]\) and from the formula \( \langle \pi^2 \rangle = -6A/u_0 \) given in the line just
below that
\[
\langle \pi^2(x) \rangle = (N - 1) \int \frac{d^dq}{q^2}
\]
holds, where \( \pi(x) \) is the transverse \((N - 1)\)–component field. Eq. (3) represents a sense-
less result, since \( \langle \pi^2(x) \rangle \) given by this equation diverges at \( d \rightarrow 2 \). It is clear that \( \langle \pi^2(x) \rangle \)
cannot diverge in reality, as it follows from the Hamiltonian density (2.1) in \([13]\) (Hamiltonian
\([11]\) in our paper): any field configuration with diverging \( \pi^2(x) \) provides a divergent
\( \pi(x) \)–dependent term
\[
\sim \frac{1}{2} \left| \nabla \pi(x) \right|^2 + \frac{u_0}{4!}(\pi^2(x))^2
\]
in the Hamiltonian density and, therefore, gives no essential contribution to the statistical
averages. The result (3) corresponds to a poor approximation where the second term in (4)
is neglected. In a surprising way, based on Ward identities, authors of \([13]\) and related
papers have lost all the purely transverse diagrams, generated by the term \( \frac{u_0}{4!}(\pi^2(x))^2 \),
and stated that this is the exact result at \( r_0 \rightarrow -\infty \). These diagrams, evidently, cannot
vanish due to \( r_0 \rightarrow -\infty \), simply, because they are independent of \( r_0 \). According to \([12,13]\),
the actual transverse term appears to be hidden in a shifted longitudinal field \( \bar{s} \), which is
considered as an independent Gaussian variable (cf. Eqs. (3.5) and (3.6) in \([13]\)). Obviously,
this is the fatal trivial error which leads to the above discussed unphysical result (3), since
\( \bar{s} \) is not an independent field in reality – the determinant of the transformation Jacobian
(from \( \pi, s \) to \( \pi, \bar{s} \)) is omitted in the relevant functional integrals! Such an introduction
of the shifted field to obtain a formally Gaussian Hamiltonian does not make any sense,
because the transformation Jacobian then must be included and the resulting model all
the same is not Gaussian. Since (3) comes from

\[ \langle \pi^2(x) \rangle = (N - 1) (2\pi)^{-d} \int G_\perp(k) \, d^d k , \]  

(5)

the unphysical divergence of \( \langle \pi^2(x) \rangle \) means that the predicted Gaussian form of the transverse correlation function \( G_\perp(k) \) is incorrect. Another aspect is that the method used in [12, 13] gives \( \lambda_\perp \equiv 2 \) also at \( n = 2 \) in contradiction with our previous discussion concerning the known rigorous results for the XY model.

Our consideration does not contradict the conventional statement (see [14] and references therein) that the Gaussian spin–wave theory [5] becomes exact at low temperatures, but only in the sense that it holds for any given nonzero \( k \) at \( T \to 0 \), and in the limit \( \lim_{k \to 0} \lim_{T \to 0} \). Therefore, it is impossible to make any rigorous conclusion regarding the exponent \( \lambda_\perp \) (or any related exponent) based on the fact that the Gaussian spin–wave theory becomes exact at \( T \to 0 \). One has to prove that the limits can be exchanged! This problem persists also in the classical treatment of the many–particle systems in [4] which, in essence, is based on the Gaussian spin wave theory at \( T \to 0 \). This treatment, evidently, is not exact, since it breaks down at \( d \to 2 \) for the two–component (\( n = 2 \)) vector model (where \( T_c \) remains finite and we fix the temperature \( 0 < T < T_c \)) just like we have discussed already – the average \( \langle \pi^2(x) \rangle \) is given by the integral (6.8) in [4] which diverges in this case (supposed \( (2\pi)^{-3} d^d k \to (2\pi)^{-d} d^d k \).)

A slightly different perturbative RG approach has been developed in [10] to analyze the nonlinear \( \sigma \) model. In this case the modulus of \( \varphi(x) \) is fixed which automatically removes the divergence of \( \langle \pi^2(x) \rangle \). A finite external field \( h \) has been introduced there to make an expansion. The correlation functions have the power–like singularities of interest only at \( h = +0 \), which means that in this case we have to consider the limit \( \lim_{k \to 0} \lim_{h \to 0} \). The results in [10] are not rigorous since the expansions used there are purely formal, i. e., they break down in this limit. Besides, the renormalization of [10] is no more than an uncontrolled approximation scheme: contrary to the assumptions in [10], it should be clear that the exact renormalization is a fairly nontrivial problem which cannot be reduced to a finding of only two renormalization constants. If, e. g., we make a real–space renormalization of the Heisenberg model, say, with the scaling factor \( s = 2 \), then the statistically averaged block–spins of the Kadanoff transformation (composed of \( s^d \) original spins) do not have a fixed modulus – simply, the original model does not include the constraint \( |\varphi(x)| = \text{const} \) for the block averages. It means that the transformation with any finite \( s \) yields a Hamiltonian of form different from the original one, i. e., the original Hamiltonian with merely renormalized coupling constant can never be the fixed–point Hamiltonian. This is the common problem in the perturbative RG theory that the renormalized Hamiltonian is assumed (or even “proved” like in [10]) to have the same form as the original one. This, however, appears to be wrong in view of the exact renormalization.

In absence of rigorous proofs, theoretical predictions can be supported by appropriate empirical data. However, the experimental measurements of susceptibility \( \chi \) depending on field \( h \) in isotropic ferromagnets like high–purity polycrystalline Ni [17] are incompatible with the prediction \( \chi \sim h^{(d-4)/2} \) of the “exact” RG theory. An explanation has been given in [9] that the divergence of the susceptibility is not observed in real crystals due to the symmetry breaking perturbations causing easy axes. However, this argument is not valid
for polycrystalline materials, since they are practically isotropous in a length scale which
exceeds remarkably the size of grains. Thus, the theory should work at $h \rightarrow 0$, where the
relevant length measure, i.e. the correlation length, diverges.

3 Conclusions

The “exact” RG theory of Goldstone modes developed in \[12\] and related papers is
not exact, it is unphysical and mathematically erroneous. The essential claims about the
Gaussian character of the $O(n > 1)$–symmetric $\phi^4$ model below $T_c$ are based in \[12\] on the
fact that the Hamiltonian can be formally written in an apparently Gaussian form. The
author, however, forgot to include the determinant of the transformation Jacobian in the
relevant functional integrals, according to which the resulting model all the same is not
Gaussian. Surprisingly, the experts in the perturbative RG theory have not mentioned
this evidently rough error in their theory till now. Other versions of the perturbative RG
theory do not have any rigorous mathematical basis, as well. Besides, the prediction of the
“exact” RG theory that the susceptibility below $T_c$ diverges as $h^{-1/2}$ in three dimensions
evidently disagree with the experimental measurements in isotropous ferromagnets.

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