A PROGRESS ON THE BINARY GOLDBACH CONJECTURE

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Abstract. In this paper we develop the method of circle of partitions and associated statistics. As an application we prove conditionally the binary Goldbach conjecture. We develop series of steps to prove the binary Goldbach conjecture in full. We end the paper by proving the binary Goldbach conjecture for all sufficiently large even numbers.

1. Introduction

The Goldbach conjecture dates from 1742 out of the correspondence between the Swiss mathematician Leonard Euler and the German mathematician Christian Goldbach. The problem has two folds, namely the binary case and the ternary case. The binary case ask if every even number $\geq 6$ can be written as a sum of two primes, where as the ternary case ask if every odd number $\geq 7$ can be written as a sum of three prime numbers. The ternary case has, however, been solved quite recently in the preprint [2] culminating several works. Though the binary problem remains unsolved as of now there has been substantive progress as well as on its variants. The first milestone in this direction can be found in (see [5]), where it is shown that every even number can be written as the sum of at most $C$ primes, where $C$ is an effectively computable constant. In the early twentieth century, G.H Hardy and J.E Littlewood assuming the Generalized Riemann hypothesis (see [8]), showed that the number of even numbers $\leq X$ and violating the binary Goldbach conjecture is much less than $X^{\frac{1}{2}+c}$, where $c$ is a small positive constant. Jing-run Chen [3], using the methods of sieve theory, showed that every even number can either be written as a sum of two prime numbers or a prime number and a number which is a product of two primes. It also known that almost all even numbers can be written as the sum of two prime numbers, in the sense that the density of even numbers representable in this manner is one [7], [6]. It is also known that there exist a constant $K$ such that every even number can be written as the sum of two prime numbers and at most $K$ powers of two, where we can take $K = 13$ [4].

In [10] we have developed a method which we feel might be a valuable resource and a recipe for studying problems concerning partition of numbers in specified subsets of $\mathbb{N}$. The method is very elementary in nature and has parallels with configurations of points on the geometric circle.

Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u + v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then the new method associate each of this summands to points on the circle generated in a certain manner by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turns out to useful in our development, as...
the results obtained in this setting are then transformed back to results concerning the partition of integers.

Let \( n \in \mathbb{N} \) and \( M \subseteq \mathbb{N} \). We denote with
\[
C(n, M) = \{ [x] \mid x, y \in M, n = x + y \}
\]
the Circle of Partition generated by \( n \) with respect to the subset \( M \). We will abbreviate this in the further text as CoP. We call members of \( C(n, M) \) as points and denote them by \([x]\). For the special case \( M = \mathbb{N} \) we denote the CoP shortly as \( C(n) \). We denote with \(|[x]| := x\) the weight of the point \([x]\) and correspondingly the weight set of points in the CoP \( C(n, M) \) as \(|C(n, M)|\). Obviously holds
\[
|C(n)| = \{ 1, 2, \ldots, n - 1 \}.
\]

We denote \( L_{[x],[y]} \) as an axis of the CoP \( C(n, M) \) if and only if \( x + y = n \). We say the axis point \([y]\) is an axis partner of the axis point \([x]\) and vice versa. We do not distinguish between \( L_{[x],[y]} \) and \( L_{[y],[x]} \), since it is essentially the same axis. The point \([x] \in C(n, M)\) such that \( 2x = n \) is the center of the CoP. If it exists then, we call it as a degenerated axis \( L_{[x]} \) in comparison to the real axes \( L_{[x],[y]} \). We denote the assignment of an axis \( L_{[x],[y]} \) to a CoP \( C(n, M) \) as
\[
L_{[x],[y]} \hat{\in} C(n, M) \text{ which means } [x], [y] \in C(n, M) \text{ with } x + y = n.
\]

**Remark 1.1.** In the following sequel, we consider only real axes. That is, axes of the form \( L_{[x],[y]} \) such that \( x \neq y \). Therefore, we abstain from the attribute real in the sequel. The introduction of the notation \(|[x]| := x\) denoting the weight of the point \([x]\) is not superfluous, as there are situations where the point is subject to motions as rotation and dilation or possibly flipping and using this notation becomes necessary. On a more advisory note, we have found the approach in the paper the ideal and possibly the perfect language to study additive problems of this kind. The structure under study has various connections with the structure of the geometric circle and much of our intuition has been borrowed from this setting.

**Proposition 1.2.** Each axis is uniquely determined by points \([x] \in C(n, M)\).

**Proof.** Let \( L_{[x],[y]} \) be an axis of the CoP \( C(n, M) \). Suppose as well that \( L_{[x],[z]} \) is also an axis with \( z \neq y \). Then we must have \( n = x + y = x + z \) and therefore \( y = z \). This cannot be and the claim follows immediately. \( \square \)

**Corollary 1.3.** Each point of a CoP \( C(n, M) \) except its center has exactly one axis partner.

**Proof.** Let \([x] \in C(n, M)\) be a point without an axis partner being not the center of the CoP. Then holds for every point \([y] \neq [x]\) except the center
\[
x + y \neq n.
\]
This is impossible, since each point in a CoP must have an axes partner. Due to Proposition 1.2 the case of more than one axis partners is impossible. This completes the proof. \( \square \)
Notation. We denote by
\[ N_n = \{ m \in \mathbb{N} \mid m \leq n \} \] (1.1)
the sequence of the first \( n \) natural numbers. We denote the assignment of an axis \( \mathbb{L}_{[x],[y]} \) resp. \( \mathbb{L}_{[x]} \) to a CoP \( \mathcal{C}(n, \mathbb{M}) \) as
\[ \mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \text{ which means } [x], [y] \in \mathcal{C}(n, \mathbb{M}) \text{ and } x + y = n \text{ resp.} \]
\[ \mathbb{L}_{[x]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \text{ which means } [x] \in \mathcal{C}(n, \mathbb{M}) \text{ and } 2x = n \]
and the number of real axes of a CoP as
\[ \nu(n, \mathbb{M}) := \# \{ \mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \mid x < y \} \]
Obviously holds
\[ \nu(n, \mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n, \mathbb{M})| = k. \]

For any \( f, g : \mathbb{N} \rightarrow \mathbb{N} \), we write \( f(n) \sim g(n) \) if and only if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1. \) We also write \( f(n) = o(1) \) if and only if \( \lim_{n \to \infty} f(n) = 0. \)

2. The Density of Points on the Circle of Partition

In this section we introduce the notion of density of points on CoP \( \mathcal{C}(n, \mathbb{M}) \) for \( \mathbb{M} \subseteq \mathbb{N} \). We launch the following language in that regard. We consider in this section only real axes. Hence, we refrain from the use of the attribute real in this section.

**Definition 2.1.** Let be \( \mathbb{H} \subset \mathbb{N} \). Then the limits
\[ \mathcal{D}(\mathbb{H}) = \lim_{n \to \infty} \frac{|\mathbb{H} \cap N_n|}{n} \]
denotes the density of \( \mathbb{H} \) if it exists.

**Definition 2.2.** Let \( \mathcal{C}(n, \mathbb{M}) \) be CoP with \( \mathbb{M} \subset \mathbb{N} \) and \( n \in \mathbb{N} \). Suppose \( \mathbb{H} \subset \mathbb{M} \) then by the density of points \( [x] \in \mathcal{C}(n, \mathbb{M}) \) such that \( x \in \mathbb{H} \), denoted \( \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})}) \), we mean the limit
\[ \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})}) = \lim_{n \to \infty} \frac{\# \{ \mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \mid \{x, y\} \cap \mathbb{H} \neq \emptyset\}}{\nu(n, \mathbb{M})} \]
if it exists.

The notion of the density of points as espoused in Definition 2.2 provides a passage between the density of the corresponding weight set of points. This possibility renders this type of density as a black box in studying problems concerning partition of numbers into specialized sequences taking into consideration their density.

**Proposition 2.3.** Let \( \mathcal{C}(n) \) with \( n \in \mathbb{N} \) be a CoP and \( \mathbb{H} \subset \mathbb{N} \). Then the following inequality holds
\[ \mathcal{D}(\mathbb{H}) = \lim_{n \to \infty} \frac{|\mathbb{H} \cap N_n|}{\frac{n}{2}} \leq \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) \leq \lim_{n \to \infty} \frac{|\mathbb{H} \cap N_n|}{\frac{n-1}{2}} = 2\mathcal{D}(\mathbb{H}). \]
Proof: The upper bound is obtained from a configuration where no two points \([x], [y] \in C(n)\) such that \(x, y \in \mathbb{H}\) lie on the same axis of the CoP. That is, by the uniqueness of the axes of CoPs with \(\nu(n, \mathbb{H}) = 0\), we can write

\[
\# \{L_{[x],[y]} \in C(n) \mid \{x,y\} \cap \mathbb{H} \neq \emptyset\} = \nu(n, \mathbb{H}) + \# \{L_{[x],[y]} \in C(n) \mid x \in \mathbb{H}, y \in \mathbb{N} \setminus \mathbb{H}\} = \# \{L_{[x],[y]} \in C(n) \mid x \in \mathbb{H}, y \in \mathbb{N} \setminus \mathbb{H}\} = |\mathbb{H} \cap N_n|.
\]

The lower bound however follows from a configuration where any two points \([x],[y] \in C(n)\) with \(x, y \in \mathbb{H}\) are joined by an axis of the CoP. That is, by the uniqueness of the axis of CoPs with \(\# \{L_{[x],[y]} \in C(n) \mid x \in \mathbb{H}, y \in \mathbb{N} \setminus \mathbb{H}\} = 0\), then we can write

\[
\# \{L_{[x],[y]} \in C(n) \mid \{x,y\} \cap \mathbb{H} \neq \emptyset\} = \nu(n, \mathbb{H}) = \left\lfloor \frac{|\mathbb{H} \cap N_n|}{2} \right\rfloor.
\]

\(\square\)

**Proposition 2.4.** Let \(\mathbb{H} \subset \mathbb{N}\) and \(D(\mathbb{H}_{C(\infty)})\) be the density of the corresponding points with weight set \(\mathbb{H}\). Then the following properties hold:

(i) \(D(\mathbb{N}_{C(\infty)}) = 1\) and \(D(\mathbb{H}_{C(\infty)}) \leq 1\) and additionally that \(D(\mathbb{H}_{C(\infty)}) < 1\) provided \(D(\mathbb{N} \setminus \mathbb{H}) > 0\).

(ii) \(1 - \lim_{n \to \infty} \frac{\nu(n, \mathbb{N} \setminus \mathbb{H})}{\nu(n, \mathbb{N})} = D(\mathbb{H}_{C(\infty)})\).

(iii) If \(|\mathbb{H}| \leq \infty\) then \(D(\mathbb{H}_{C(\infty)}) = 0\).

Proof. It is easy to see that the first part of **Property** (i) and (iii) are both easy consequences of the definition of density of points on the CoP \(C(n)\) and Proposition 2.3. We establish the second part of property (i) and **Property** (ii), which is the less obvious case. We observe by the uniqueness of the axes of CoPs that we can write

\[
1 = \lim_{n \to \infty} \frac{\nu(n, \mathbb{N})}{\nu(n, \mathbb{H})} = \lim_{n \to \infty} \frac{\# \{L_{[x],[y]} \in C(n) \mid x \in \mathbb{H}, y \in \mathbb{N} \setminus \mathbb{H}\}}{\nu(n, \mathbb{N})} + \lim_{n \to \infty} \frac{\nu(n, \mathbb{N} \setminus \mathbb{H})}{\nu(n, \mathbb{N})} + \lim_{n \to \infty} \frac{\nu(n, \mathbb{N})}{\nu(n, \mathbb{N})} = D(\mathbb{H}_{C(\infty)}) + \lim_{n \to \infty} \frac{\nu(n, \mathbb{N} \setminus \mathbb{H})}{\nu(n, \mathbb{N})}
\]

and (ii) follows immediately. The second part of (i) follows from the above expression and exploiting the inequality

\[
\lim_{n \to \infty} \frac{\nu(n, \mathbb{N} \setminus \mathbb{H})}{\nu(n, \mathbb{N})} \leq \lim_{n \to \infty} \frac{\left\lfloor \frac{|\mathbb{N} \setminus \mathbb{N}_n|}{2} \right\rfloor}{\left\lfloor \frac{n}{2} \right\rfloor} = D(\mathbb{N} \setminus \mathbb{H}).
\]

\(\square\)
Next we transfer the notion of the density of a sequence to the density of corresponding points on the CoP $C(n)$. This notion will play a crucial role in our latter developments.

**Proposition 2.5.** Let $\epsilon \in (0, 1]$ and $\mathcal{H}$ be a sequence with $\mathcal{H} \subseteq \mathbb{N}$ and $C(n)$ be a CoP. If $D(\mathcal{H}) \geq \epsilon$ then $D(\mathcal{H}C(\infty)) \geq \epsilon$.

**Proof.** The result follows by exploiting the inequality in Proposition 2.3. □

### 2.1. Application of Density of Points to Partitions.

In this subsection we explore the connection between the notion of density of points in a typical CoP to the possibility of partitioning number into certain sequences. This method tends to work very efficiently for sets of integers having a positive density.

**Theorem 2.6.** Let $\mathcal{H} \subseteq \mathbb{N}$ such that $D(\mathcal{H}) > \frac{1}{2}$. Then every sufficiently large $n \in \mathbb{N}$ has representation of the form

$$n = z_1 + z_2$$

where $z_1, z_2 \in \mathcal{H}$.

**Proof.** Appealing to Proposition 2.3 we can write

$$\lim_{n \to \infty} \frac{|\mathcal{H} \cap N_n|}{\frac{n}{2}} \leq D(\mathcal{H}C(\infty)) \leq \lim_{n \to \infty} \frac{|\mathcal{H} \cap N_n|}{\frac{n}{2}}.$$

By the uniqueness of the axes of CoPs we can write

$$\# \{L_{x,y} \in C(n) | \{x,y\} \cap \mathcal{H} \neq \emptyset\} = \nu(n, \mathcal{H}) + \# \{L_{x,y} \in C(n) | x \in H, y \in \mathbb{N} \setminus H\}.$$

Let us assume $\nu(n, \mathcal{H}) = 0$ then it follows by appealing to Definition 2.2 and according to the proof of Proposition 2.3

$$D(\mathcal{H}C(\infty)) = 2D(\mathcal{H})$$

$$> 2 \times \frac{1}{2} = 1.$$

This contradicts the inequality $D(\mathcal{H}C(\infty)) \leq 1$ in Proposition 2.4. This proves that $\nu(n, \mathcal{H}) > 0$ for all sufficiently large values of $n \in \mathbb{N}$. □

**Corollary 2.7.** Let $\mathcal{R} := \{m \in \mathbb{N} | \mu(m) \neq 0\}$. Then every sufficiently large $n \in \mathbb{N}$ can be written in the form

$$n = z_1 + z_2$$

where $\mu(z_1) = \mu(z_2) \neq 0$.

**Proof.** By the uniqueness of the axes of CoPs we can write

$$\# \{L_{x,y} \in C(n) | \{x,y\} \cap \mathcal{R} \neq \emptyset\} = \nu(n, \mathcal{R}) + \# \{L_{x,y} \in C(n) | x \in \mathcal{R}, y \in \mathbb{N} \setminus \mathcal{R}\}.$$ 

Let us assume $\nu(n, \mathcal{R}) = 0$ then it follows by appealing to Definition 2.2 and Theorem 2.6

$$D(\mathcal{R}C(\infty)) = 2D(\mathcal{R})$$

$$= \frac{12}{\pi^2} > 1$$

since $D(\mathcal{R}) = \frac{6}{\pi^2}$. This contradicts the inequality $D(\mathcal{R}C(\infty)) \leq 1$ in Proposition 2.4. This proves that $\nu(n, \mathcal{R}) > 0$ for all sufficiently large values of $n \in \mathbb{N}$. □
One could ever hope and dream of this strategy to work when we replace the set $\mathbb{R}$ of square-free integers with the set of prime numbers. There we would certainly ran into complete deadlock, since the prime in accordance with the prime number theorem have density zero. Any success in this regard could conceivably work by introducing some exotic forms of the notion of density of points and carefully choosing a subset of the integers that is somewhat dense among the set of integers and covers the primes. We propose a strategy somewhat akin to the above method for possibly getting a handle on the binary Goldbach conjecture and it’s variants. Before that we state and prove a conditional theorem concerning the binary Goldbach conjecture.

**Theorem 2.8.** Let $\mathbb{B} \subset \mathbb{N}$ such that $\mathbb{P} \cap \mathbb{N}_n \subset ||\mathcal{C}(n, \mathbb{B})||$ with

$$\lim_{n \to \infty} \frac{|\mathbb{P} \cap \mathbb{N}_n|}{\eta(n)} > \frac{1}{2}$$

where $\eta(n) = |\mathcal{C}(n, \mathbb{B})|$. Then $\nu(n, \mathbb{P}) > 0$ for all sufficiently large values of $n \in 2\mathbb{N}$.

**Proof.** First let us upper and lower bound the density of points in the CoP $\mathcal{C}(n, \mathbb{B})$ with weight belonging to the set of the primes $\mathbb{P}$ so that we obtain the inequality

$$\lim_{n \to \infty} \frac{\frac{|\mathbb{P} \cap \mathbb{N}_n|}{\eta(n)}}{2} \leq D(\mathbb{P} \cap \mathbb{N}_n) \leq \lim_{n \to \infty} \frac{|\mathbb{P} \cap \mathbb{N}_n|}{\eta(n)} = 2 \lim_{n \to \infty} \frac{|\mathbb{P} \cap \mathbb{N}_n|}{\eta(n)}$$

for all sufficiently large values of $n$. Appealing to the uniqueness of the axes of CoPs, we can write

$$\# \{ L_{[x],[y]} \in \mathcal{C}(n, \mathbb{B}) \mid \{x,y\} \cap \mathbb{P} \neq \emptyset \} = \nu(n, \mathbb{P}) + \# \{ L_{[x],[y]} \in \mathcal{C}(n, \mathbb{B}) \mid x \in \mathbb{P}, y \in \mathbb{B} \setminus \mathbb{P} \}.$$

Let us assume to the contrary $\nu(n, \mathbb{P}) = 0$, then it follows that no two points in the CoP $\mathcal{C}(n, \mathbb{B})$ with weight in the set $\mathbb{P}$ are axes partners, so that under the requirement

$$\lim_{n \to \infty} \frac{|\mathbb{P} \cap \mathbb{N}_n|}{\eta(n)} > \frac{1}{2}$$

where $\eta(n) = |\mathcal{C}(n, \mathbb{B})|$, we obtain the inequality

$$D(\mathbb{P} \cap \mathbb{N}_n) = 2 \lim_{n \to \infty} \frac{|\mathbb{P} \cap \mathbb{N}_n|}{\eta(n)} > 2 \times \frac{1}{2} = 1.$$

This contradicts the inequality $D(\mathbb{P} \cap \mathbb{N}_n) \leq 1$ in Proposition 2.4. This proves that $\nu(n, \mathbb{P}) > 0$ for all sufficiently large values of $n \in 2\mathbb{N}$. 

### 2.2. A Strategy to Prove the Binary Goldbach Conjecture by Circles of Partition

In this subsection we propose series of steps that could be taken to confirm the truth of the binary Goldbach conjecture. We enumerate the strategies chronologically as follows:

- First construct a CoP $\mathcal{C}(n, \mathbb{B})$ such that $\mathbb{P} \cap \mathbb{N}_n \subset ||\mathcal{C}(n, \mathbb{B})||$ and that

$$\lim_{n \to \infty} \frac{|\mathbb{P} \cap \mathbb{N}_n|}{\eta(n)} > \frac{1}{2}$$

where $\eta(n) = |\mathcal{C}(n, \mathbb{B})|$. 

Next we remark that the following inequality also holds and this can be obtained by replacing the weight set $||C(n)||$ with the set $||C(n, B)||$.

$$\lim_{n \to \infty} \frac{|P \cap N_n|}{\eta(n)} \leq D(P_C(\infty, B)) \leq \lim_{n \to \infty} \frac{|P \cap N_n|}{\eta(n)} = 2 \lim_{n \to \infty} \frac{|P \cap N_n|}{\eta(n)}.$$  

Appealing to the uniqueness of the axes of CoPs, we can write

$$\# \{L_{[x],[y]} \in C(n, B) \mid \{x, y\} \cap P \neq \emptyset\} = \nu(n, P) + \# \{L_{[x],[y]} \in C(n, B) \mid x \in P, y \in B \setminus P\}.$$  

Let us assume $\nu(n, P) = 0$ then it follows by appealing to Definition 2.2 and in the sense of the proof of Proposition 2.3

$$D(P_C(\infty, B)) = 2 \lim_{n \to \infty} \frac{|P \cap N_n|}{\eta(n)} > 2 \times \frac{1}{2} = 1.$$  

This contradicts the inequality $D(P_C(\infty, B)) \leq 1$ in Proposition 2.4. This proves that $\nu(n, P) > 0$ for all sufficiently large values of $n \in 2\mathbb{N}$.

3. The asymptotic squeeze principle and the Binary Goldbach Conjecture - An asymptotic proof of the binary Goldbach conjecture

In this section, we prove the special squeeze principle for all sufficiently large $n \in 2\mathbb{N}$. Consequently, we obtain a proof of the binary Goldbach conjecture for all sufficiently large even numbers.

In our paper [10], we introduced and developed the method of circles of partition. This method is underpinned by a combinatorial structure that encodes certain additive properties of the subsets of the integers and invariably equipped with a certain geometric structure that allows to view the elements as points in the plane whose weights are just elements of the underlying subset. We call this combinatorial structure the circle of partition and is refereed to as the set of points $C(n, M) = \{[x] \mid x, n - x \in M\}$.

Each point in this set - except the center point - must have a uniquely distinct point that is joined by a line which we refer to as an axis of the CoP. We denote an axis of a CoP with $L_{[x],[y]}$ and an axis contained in the CoP as $L_{[x],[y]} \in C(n, M)$ which means $[x], [y] \in C(n, M)$ with $x + y = n$.

The method of circles of partition and their associated structures have been well advanced in [12], where the corresponding points have complex numbers as their weights and a line (axis) joining co-axis points. The following structure was considered as a complex circle of partition

$$C^0(n, C_M) = \{[z] \mid z, n - z \in C_M, \Im(z)^2 = \Re(z) \Re(n - \Re(z))\}$$  

where
\[ C_M := \{ z = x + iy \mid x \in \mathbb{M}, y \in \mathbb{R} \} \subset \mathbb{C} \]

with \( \mathbb{M} \subseteq \mathbb{N} \). We abbreviate this complex additive structure as cCoP. The condition \( \Im(z)^2 = \Re(z)(n - \Re(z)) \) is referred to as the circle condition and it guarantees that all points on the cCoP lie on a circle in the complex. This circle is the embedding circle of the cCoP \( C^o(n, C_\mathbb{M}) \), denoted as \( c_n \). The embedding circles of cCoPs have the property that they reside fully inside those embedding circles with a relatively larger generators, except the origin as a common point \([12]\). For each axis, we make the following assignment

\[ L_{[z_1], [z_1]} \in C(n, C_M) \]

which means \([z_1], [z_2] \in C(n, C_M) \) with \( z_1 + z_2 = n \).

The structure of the complex circle of partition is much more versatile and has extra structures that are not readily available in the theory of circle of partition. Most notably, for each axis \( L_{[z], [n-z]} \) of a CoP there exists

\[ L_{[\bar{z}], [\bar{n}-\bar{z}]} \]

a conjugate axis, where \([\bar{z}], [\bar{n}-\bar{z}] \) denotes the corresponding conjugate points. The space occupied by an embedding circle of partition and correspondingly outside the embedding circle turns out to be very interesting, as this can be utilised in studying a certain ordering principle of the points of two interacting axes of distinct cCoPs. Much more striking is the fact, which is a natural consequence of the circle condition, that

\[ |L_{[z_1], [z_2]}| = n \]

for any axis \( L_{[z_1], [z_2]} \in C^o(n, C_\mathbb{M}) \) with \( n, z \in \mathbb{C}_M, \Im(z)^2 = \Re(z)(n - \Re(z)) \).

The squeeze principle \([11]\) can be considered as a black box for studying the binary Goldbach conjecture and its variant. A slightly different version of this principle appears in \([12]\). For the sake of the reader, we provide a brief recap of this elegant principle as below

**Remark 3.1.** Let \( C(n, B) \) and \( C^o(n, C_B) \) be a CoP and a corresponding cCoP, respectively. It is clear that \( C(n, B) \neq \emptyset \) if and only if \( C^o(n, C_B) \neq \emptyset \). To see this, observe that if \( L_{[z], [y]} \in C(n, B) \), then \( L_{[x+it], [y-it]} \in C^o(n, C_B) \), where \( t = \pm \sqrt{xy} \). Conversely if \( L_{[z_1], [z_2]} \in C^o(n, C_B) \) then \( z_1 + z_2 = n \) implying that \( \Im(z_1) = -\Im(z_2) \) and so \( \Re(z_1) + \Re(z_2) = n \). This immediately implies that there is an axes \( L_{[\Re(z_1)], [\Re(z_2)]} \in C(n, B) \). This equivalence will be subtly employed in the rest of the paper.

**Lemma 3.2** (The squeeze principle). Let \( \mathbb{B} \subset M \subseteq \mathbb{N} \) and \( C^o(n, C_M) \) and \( C^o(n + t, C_M) \) with \( t \geq 4 \) be non-empty cCoPs with integers \( n, t, s \) of the same parity. If there exist an axis \( L_{[v_1], [w_1]} \in C^o(n, C_M) \) with \( w_1 \in C_B \) and an axis \( L_{[v_2], [w_2]} \in C^o(n + t, C_M) \) with \( v_2 \in C_B \) such that

\[ \Re(v_1) < \Re(v_2) \quad \text{and} \quad \Re(w_1) < \Re(w_2) \quad (3.1) \]

then there exists an axis \( L_{[\Re(w_2), [\Re(v_1)]]} \in C(n + s, B) \) with \( 0 < s < t \). Hence the cCoP \( C^o(n + s, C_B) \) not empty and so is the cCoP \( C^o(n + s, C_M) \).

**Proof.** From the existence of an axis \( L_{[v_1], [w_1]} \in C^o(n, C_M) \) follows \( \Re(w_1) = n - \Re(v_1) \). With the requirement \([12]\) we get

\[ \Re(w_1) > n - \Re(v_2). \quad (3.2) \]
On the other hand from the existence of an axis \( L_{[n_1], [n_2]} \in C^\circ(n + t, C_M) \) follows \( \mathcal{R}(w_2) = n + t - \mathcal{R}(v_2) \) and with the requirement (1.2) and the result (5.2) we get

\[
\begin{align*}
&n - \mathcal{R}(v_2) < \mathcal{R}(w_1) < n + t - \mathcal{R}(v_2) | + \mathcal{R}(v_2) \\
&n < \mathcal{R}(w_1) + \mathcal{R}(v_2) < n + t \\
&n < n + s < n + t.
\end{align*}
\]

By virtue of the requirements \( w_1, v_2 \in C_B \) and \( n + s = \mathcal{R}(w_1) + \mathcal{R}(v_2) \) there is an axis \( L_{[\mathcal{R}(w_2)], [\mathcal{R}(w_1)]} \in C(n + s, B) \) and hence holds \( C^\circ(n + s, C_B) \neq \emptyset \). Since \( B \subset M \), it follows immediately that \( C_B \subset C_M \) and therefore \( C^\circ(n + s, C_M) \neq \emptyset \). This completes the proof. \( \square \)

Consequently, we obtain the special squeeze principle

**Lemma 3.3** (Special squeeze principle). Let \( n, t, s \in 2\mathbb{N} \) and \( \mathbb{P} \) be the set of all odd primes. If \( t \geq 4 \) and there exist an axis \( L_{[z_1], [z_2]} \in C^\circ(n) \) with \( z_2 \in C_P \) and an axis \( L_{[w_1], [w_2]} \in C^\circ(n + t) \) with \( w_1 \in C_P \) such that

\[
\mathcal{R}(z_1) < \mathcal{R}(w_1) < \mathcal{R}(z_1) + t
\]

then there exists an axis \( L_{[\mathcal{R}(w_2)], [\mathcal{R}(w_1)]} \in C(n + s, \mathbb{P}) \) with \( 0 < s < t \).

The Lemma 4.2 referred to as the squeeze principle may be regarded as a fundamental tool set for investigating the viability of dividing integers of a particular parity, utilizing constituent elements originating from a specific subset of the integers. The mechanism operates by discerning a pair of cCoPs that are both non-vacuous and share a common base set. Subsequently, supplementary cCoPs that are non-vacuous and have generators restrained within the interstice of these two generators are identified. This principle may be applied in a resourceful manner to investigate the overarching matter of the practicality of divvying up numbers such that each addend is a member of the identical subset of positive integers.

**Remark 3.4.** The CoP \( C(n, \mathbb{N}) := C(n) \) is always non-empty and so is the cCoP \( C^\circ(n, C_N) = C^\circ(n) \).

### 3.1. Application to the Binary Goldbach Conjecture

In this section, we present an asymptotic proof for the binary Goldbach conjecture. The proof has been condensed into the language of cCoPs but can be reduced to the usual form of the conjecture.

**Lemma 3.5** (juxtaposition principle). For all \( n \geq 10, \) there exist an axis \( L_{[z_1], [z_2]} \in C^\circ(n, C_N) \) and \( L_{[w_1], [w_2]} \in C^\circ(n + t, C_N) \) for \( \mathcal{R}(z_1) < \mathcal{R}(z_2) \) and \( \mathcal{R}(w_1) < \mathcal{R}(w_2) \) such that \( \mathcal{R}(z_1) \neq \mathcal{R}(w_1) \) and \( \mathcal{R}(z_2) \neq \mathcal{R}(w_2) \) with \( z_2 \in C_P \) and \( w_1 \in C_P \) for \( t \geq 4 \).

**Proof.** Let us choose a prime number \( \mathcal{R}(w_1) \leq \frac{n + t}{2} \) and choose a prime number \( \mathcal{R}(z_2) \in (\frac{n}{2}, n) \) for all \( n \geq 10 \) with \( n \equiv 0 \) (mod 2), which is feasible by virtue of the prime number theorem. If \( \mathcal{R}(z_1) \neq \mathcal{R}(w_1) \) and \( \mathcal{R}(z_2) \neq \mathcal{R}(w_2) \) then there is nothing to do. Without loss of generality, suppose that \( \mathcal{R}(z_1) = \mathcal{R}(w_1) \) then obviously \( \mathcal{R}(z_2) \neq \mathcal{R}(w_2) \) since \( n + t > n \). We note that \( \pi(\frac{n + t}{2}) \geq 3 \) for all \( n \geq 10 \) with \( t \geq 4 \) so that we can choose a prime number \( \mathcal{R}(w_1') \leq \frac{n + t}{2} \) such that \( \mathcal{R}(w_1') \neq \mathcal{R}(w_1) \). Thus we replace \( \mathcal{R}(w_1) \) with \( \mathcal{R}(w_1') \) and obtain the axes \( L_{[z_1], [z_2]} \in C^\circ(n, C_N) \) and
Proof. Then with Lemma 3.8 and Lemma 3.6 we obtain prime number. We note that via the prime number theorem holds

\[ \pi(n) = \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right). \]

In particular, \( \pi(n) \sim \frac{n}{\log n}\).

Lemma 3.7 (Bertrand’s postulate). For \( k \geq 89693 \) there exists a prime number in the interval

\[ k < p \leq (1 + \frac{1}{\log^3 k})k. \]

Proof. The proof of this inequality appears in [9]. \( \square \)

Lemma 3.8. Let \( p_n \) denotes the \( n^{th} \) prime number, then

\[ p_n = n \log n + O(n \log \log n). \]

In particular, \( p_n \sim n \log n \).

Lemma 3.9 (The asymptotic squeeze principle). There exist an \( n_0 \in \mathbb{N} \) such that for all even \( n \geq n_0 \) there exist an axis \( L_{[z_1],[z_2]} \in \mathcal{C}^\alpha(n, \mathbb{C}_n) \) and \( L_{[w_1],[w_2]} \in \mathcal{C}^\alpha(n + t, \mathbb{C}_n) \) for \( \Re(z_1) \lesssim \Re(z_2) \) and \( \Re(w_1) \lesssim \Re(w_2) \) such that \( \Re(z_1) \lesssim \Re(w_1) \) and \( \Re(z_2) \lesssim \Re(w_2) \) with \( z_2 \in \mathbb{C}_p \) and \( w_1 \in \mathbb{C}_p \) for \( t \geq 4 \).

Proof. Let us set \( \Re(z_2) \) to be a prime number and choose \( \Re(z_2) \) to be the \( \pi(\frac{3n}{4}) \)th prime number. We note that via the prime number theorem holds

\[ \pi(\frac{3n}{4}) = \frac{3n}{\log(\frac{3n}{4})} + O\left(\frac{n}{\log^2(n)}\right) \]

\[ = \frac{3n}{4 \log n} + O\left(\frac{n}{\log^2 n}\right). \]

We also note via basic power series identities, we can write

\[ -\log(1 - \frac{1}{\log n}) = \frac{1}{\log n} + \frac{1}{2(\log n)^2} + \frac{1}{3(\log n)^3} + \cdots \]

\[ = \frac{1}{\log n} + O\left(\frac{1}{(\log n)^2}\right). \]

Then with Lemma 3.8 and Lemma 3.6 we obtain

\[ \Re(z_2) = p_{\pi(\frac{3n}{4})} = \pi(\frac{3n}{4}) \log \pi(\frac{3n}{4}) + O(\pi(\frac{3n}{4}) \log \log \pi(\frac{3n}{4})) \]

\[ = (\frac{3n}{4 \log n} + O\left(\frac{n}{\log^2 n}\right))(\log(\frac{3n}{4 \log n} + O\left(\frac{n}{\log^2 n}\right)) + O(\frac{3n}{4}) \log \log \pi(\frac{3n}{4})) \]
We note that we can write
\[
\log\left(\frac{3n}{4\log n} + O\left(\frac{\log n}{\log^2 n}\right)\right) = \log\left(\frac{3}{4}\right) + \log\left(\frac{n}{\log n}\right) + \log(1 + O\left(\frac{1}{\log n}\right)) \\
= \log n - \log \log n + O(1)
\] (3.3)

It follows from (3.3), we can write for the product
\[
\left(\frac{3n}{4\log n} + O\left(\frac{n}{\log^2 n}\right)\right)\left(\log\left(\frac{3n}{4\log n} + O\left(\frac{n}{\log^2 n}\right)\right)\right) = \frac{3n}{4} + O\left(\frac{n \log \log n}{\log n}\right)
\] (3.4)
as the main term. Now we analyze the error term in a similar manner. By virtue of the prime number theorem, we can write
\[
\pi\left(\frac{3n}{4}\right)\log \log \pi\left(\frac{3n}{4}\right) = \left(\frac{3n}{4\log n} + O\left(\frac{n}{\log^2 n}\right)\right)\left(\log \log\left(\frac{3n}{4\log n} + O\left(\frac{n}{\log^2 n}\right)\right)\right)
\] (3.5)
We observe that
\[
\log \log\left(\frac{3n}{4\log n} + O\left(\frac{n}{\log^2 n}\right)\right) = \log(\log\left(\frac{3n}{4\log n} + O\left(\frac{1}{\log n}\right)\right)) \ll \log \log n
\] (3.6)
so that we obtain for the product
\[
\pi\left(\frac{3n}{4}\right)\log \log \pi\left(\frac{3n}{4}\right) = \left(\frac{3n}{4\log n} + O\left(\frac{n}{\log^2 n}\right)\right)\left(\log \log\left(\frac{3n}{4\log n} + O\left(\frac{n}{\log^2 n}\right)\right)\right) \\
\ll \frac{n \log \log n}{\log n}
\] (3.7)
and by combining (3.3) and (3.7), we obtain
\[
\Re(z_2) = \frac{3n}{4} + O\left(\frac{n \log \log n}{\log n}\right) + O\left(\frac{n \log \log n}{\log n}\right) = \frac{3n}{4} + O\left(\frac{n \log \log n}{\log n}\right).
\]
Consequently, we have for the real weight of the lower axis point
\[
\Re(z_1) = n - \Re(z_2) \\
= n - \frac{3n}{4} + O\left(\frac{n \log \log n}{\log n}\right) \\
= \frac{n}{4} + O\left(\frac{n \log \log n}{\log n}\right).
\] (3.8)
It is easy to see that
\[
\Re(z_1) \sim \frac{n}{4} < \frac{n}{2}
\]
and
\[
\Re(z_2) \sim \frac{3n}{4} > \frac{n}{2}
\]
Now, by virtue of Lemma 3.7, we set \(\Re(w_1)\) to be a prime number and choose \(\Re(w_1)\) so that
\[
\frac{n}{4} < \Re(w_1) \leq (1 + \frac{1}{\log^3 n})\left(\frac{n}{4}\right)
\] (3.9)
for all \(n \geq 358772\), then it implies that \(\Re(z_1) \lesssim \Re(w_1)\). It is easy to see that
\[
\Re(w_1) \lesssim \frac{n + t}{2}
\]
for \( t \geq 4 \), since

\[
\left(1 + \frac{1}{\log^3 \frac{n}{4}}\right) \left(\frac{n}{4}\right) \sim \frac{n}{2}
\]

by virtue of the fact that

\[
\left(1 + \frac{1}{\log^3 \frac{n}{4}}\right) \sim 1 \quad (n \to \infty).
\]

It follows from (3.9) the lower bound

\[
\Re(w_2) = n + t - \Re(w_1)
\]

\[
\geq n + t - \left(1 + \frac{1}{\log^3 \frac{n}{4}}\right) \left(\frac{n}{4}\right)
\]

and since \( \left(1 + \frac{1}{\log^3 \frac{n}{4}}\right) \sim 1 \quad (n \to \infty) \)

\[
\sim n - \frac{n}{4} + t \quad (n \to \infty)
\]

\[
> n - \frac{n}{4} = \frac{3n}{4} \sim \Re(z_2)
\]

for all \( t \geq 4 \) and \( n > n_o \) for some fixed \( n_o \in \mathbb{N} \). This completes the proof. \( \Box \)

We are now ready to prove the binary Goldbach conjecture for all even numbers greater than some \( n_o \in \mathbb{N} \). This result provides an alternative solution to our first result and in very few instances adopts the proof technique in [11]. The benefit of the strong version of Bertrand’s postulate (Lemma 3.7) is good enough to verify the asymptotic version of the binary Goldbach conjecture using this version of the squeeze principle, which is a slight variation of the version that appears in the paper [11].

**Theorem 3.10** (The asymptotic binary Goldbach theorem). There exist some \( n_o \in \mathbb{N} \) such that every even number \( n \geq n_o \) can be written as a sum of two prime numbers.

**Proof.** We note that the above statement is equivalent to the statement that the cCoPs \( \mathcal{C}^o(n, \mathbb{C}_p) \) are non–empty for all even \( n \geq n_o \).

By remark 3.4 all cCoPs basing on \( \mathbb{C}_N \) with generators \( \geq 2 \) are non–empty. By virtue of Lemma 3.9 all cCoPs \( \mathcal{C}^o(n) \) and \( \mathcal{C}^o(n + 4) \) with even generators \( n \geq n_o \) fulfil the requirements of the special squeeze principle (Lemma 3.3) or the squeeze principle with \( \mathcal{M} := \mathbb{N} \) and \( \mathcal{B} := \mathbb{P} \). Hence for each such \( n \) there is always a non–empty cCoP \( \mathcal{C}^o(n + 2, \mathbb{C}_p) \). We start with \( \mathcal{C}^o(n_o) \) and \( \mathcal{C}^o(n_o + 4) \) and continue this procedure with \( \mathcal{C}^o(n_o + k) \) and \( \mathcal{C}^o(n_o + k + 4) \) for all even \( k \geq 2 \). We verify that all cCoPs \( \mathcal{C}^o(n_o + k + 2) \) for even \( k \geq 2 \) ad infinitum are non–empty. \( \Box \)
4. The generalized squeeze principle and applications

Let \( A \subseteq \mathbb{N} \) then we call the set
\[
C(n, \bigotimes_{i=1}^{h} A_i) := \left\{ [x_1], [x_2], \ldots, [x_h] \mid x_i \in A_i, \ n = \sum_{i=1}^{h} x_i \right\}
\]
a multivariate circle of partition generated by \( n \in \mathbb{N} \) with base regulators \( \bigotimes_{i=1}^{h} A_i \) the \( h \)-fold direct product of the sets \( A_i \). We call members of the multivariate circle of partitions multivariate points. We denote the weight of each points as \( ||[x_i]|| := x_i \in A_i \) and for the corresponding weight set of the multivariate circle of partitions
\[
||C(n, \bigotimes_{i=1}^{h} A_i)|| := \{ (x_1, x_2, \ldots x_k) \in \bigotimes_{i=1}^{h} A_i \mid \sum_{i=1}^{h} x_i = n \}.
\]

We denote \( L_{[x_1], [x_2], \ldots, [x_h]} \) as an axis of the multivariate circle of partitions \( C(n, \bigotimes_{i=1}^{h} A_i) \) if and only if \( x_i \in A_i \) for each \( 1 \leq i \leq h \) and
\[
n = \sum_{i=1}^{h} x_i.
\]

We say the axis points \([x_i]\) for each \( 1 \leq i \leq h \) are axis residents. We do not view the axis as any different among other axis of the form \( L_{[x_1], [x_2], \ldots, [x_h]} \) up to the rearrangements of its residents points. In special cases where the points
\[
[x_k] \in C(n, \bigotimes_{i=1}^{h} A_i)
\]
is such that \( hx_k = n \), then we call \([x_i]\) the center of the multivariate circle of partitions. If it exists, then we call it as a degenerated axis \( L_{[x_k]} \) in comparison to the real axes \( \mathbb{L}_{[x_1], [x_2], \ldots, [x_h]} \) where not all of the weights \( x_1 \) can be equal. We denote the assignment of an axis \( \mathbb{L}_{[x_1], [x_2], \ldots, [x_h]} \) to the multivariate CoP \( C(n, \bigotimes_{i=1}^{h} A_i) \) as
\[
\mathbb{L}_{[x_1], [x_2], \ldots, [x_h]} \in C(n, \bigotimes_{i=1}^{h} A_i)
\]
which means \([x_1], [x_2], \ldots, [x_h] \in C(n, \bigotimes_{i=1}^{h} A_i)\)

with
\[
n = \sum_{i=1}^{h} x_i
\]

for a fixed \( n \in \mathbb{N} \) with \( x_i \in A_i \) for each \( 1 \leq i \leq h \) or vice versa and the number of real axes of the multivariate circle of partitions as
\[
\nu(n, \bigotimes_{i=1}^{h} A_i) := \# \{ \mathbb{L}_{[x_1], [x_2], \ldots, [x_h]} \in C(n, \bigotimes_{i=1}^{h} A_i) \mid x_i \neq x_j \}
\]
for all \( 1 < i < j \leq h \).

In the special case where we fix \( h = 2 \) and take \( A_i = \mathbb{A} \subset \mathbb{N} \), then we obtain the circle of partitions
\[
C(n, A) := \{ [x] \mid x, n-x \in A \}.
\]
and the corresponding counting function for the axes set
\[ \nu(n, A) := \#\{L_{[x],[y]} \in C(n, A) \mid x \neq y\}. \]

This structure was studied extensively in [10] in the case where one allows just two
axis points on their axes. The squeeze principle is the statement

**Theorem 4.1 (The squeeze principle).** Let \( B \subset M \subset \mathbb{N} \) and \( C(n, M) \) and \( C(m + t, \emptyset) \neq \emptyset \) for \( t \geq 4 \). If there exists \( L_{[x],[y]} \in C(m + t, M) \) with \( x \in B \) and \( x < y \) such that
\[ y > w := \max\{u \in |C(m, M)||u \in B| > m - x, \quad (4.1) \]
then there exists \( C(s, B) \neq \emptyset \) such that \( m < s < m + t \).

Indeed it has found some unexpected applications to Goldbach-type problems
and has been applied to study additive prime number problems requiring certain
partitions into certain subsets of the positive integers. The power of this principle
allows one to exhaust any interval of the form \([n, n + t]\) for \( t \geq 4 \) in way to conclude
that all even numbers in this interval can be written as the sum of two prime
numbers. Much more generally the set may be extended to a general subset of the
integers and it mostly suffices to check if the underlying conditions of the principle
are all satisfied in order to run this test. The squeeze principle has an alternate
version which is also valid when one allows a complex base set. We restate it here
without proof as a preliminary

**Lemma 4.2 (The squeeze principle).** Let \( B \subset M \subset \mathbb{N} \) and \( C^{o}(n, M) \) and \( C^{o}(n + t, M) \) with \( t \geq 4 \) be non–empty eCoPs with integers \( n, t, s \) of the same parity. If there exist an axis \( L_{[v],[w]} \in C^{o}(n, M) \) with \( w \in C_{B} \) and an axis \( L_{[v],[w]} \in C^{o}(n + t, M) \) with \( v \in C_{B} \) such that
\[ \Re(v) < \Re(w) \quad \text{and} \quad \Re(v) < \Re(w) \quad (4.2) \]
then there exists an axis \( L_{[\Re(v),\Re(w)]} \in C(n + s, B) \) with \( 0 < s < t \). Hence
\( C^{o}(n + s, M) \) is also not empty.

We obtain an analogous versions of the squeeze principle in the setting of axes
of circle of partitions with at least two resident points.

**Lemma 4.3 (Generalized squeeze principle).** Let \( A \subset \mathbb{N} \) with \( C(n, \bigotimes_{i=1}^{h} A) \neq \emptyset \)
for a fixed \( n \in \mathbb{N} \). If \( t \in \mathbb{N} \) is such that \( n \) and \( n + t \) are not consecutive integers
in \( \mathbb{N} \) and there exists an axes
\[ \mathbb{L}_{[x_{1},[x_{2},\ldots,[x_{h}]}} \in C(n + t, \bigotimes_{i=1}^{h} N) \]
with \( x_{i} \in A \) for all \( 1 \leq i \leq h - 1 \) and \( x_{i} < x_{h} \) for all \( 1 \leq i \leq h - 1 \) such that
\[ x_{h} > w := \max\{u \in |C(n, N)||u \in A| > n - \sum_{i=1}^{h-1} x_{i} \]
then there exists
\[ C(s, \bigotimes_{i=1}^{h} A) \neq \emptyset \]
for \( n < s < n + t \) with \( s \in \mathbb{N} \).
Proof. We note that from the hypothesis

\[ x_h > w := \max\{u \in \|C(n, N)\| \mid u \in A\} > n - \sum_{i=1}^{h-1} x_i \]

we can write

\[ n = w + (n - w) < w + \sum_{i=1}^{h-1} x_i < \sum_{i=1}^h x_i = n + t \]

for \( x_i < x_h \) for all \( 1 \leq i \leq h - 1 \). Clearly \( w \in A \) with \( x_i \in A \) for \( 1 \leq i \leq h - 1 \) so that there exists an axis

\[ L_{[w],[x_1],[x_2],\ldots,[x_{h-1}]} \in C(s, \bigotimes_{i=1}^h A) \]

with \( n < s < n + t \) and \( s \in \mathbb{H} \), since \( n, n + t \) are not consecutive in \( \mathbb{H} \). This means that \( s \) can be written as the sum of \( h \) (not all possibly distinct) elements of \( A \subset \mathbb{N} \). \( \square \)

We show how this principle can be applied to solve Goldbach-type problems with \( h \) summands for \( h \geq 2 \).

**Theorem 4.4** (The partition law). Let \( A \subset \mathbb{N} \) and suppose \( C(n, \bigotimes_{i=1}^h A) \neq \emptyset \) for infinitely many \( n \in \mathbb{H} \subset \mathbb{N} \). If for each \( t \in \mathbb{N} \) such that \( n, n + t \) are not consecutive in \( \mathbb{H} \) there exists at least an axis

\[ L_{[x_1],[x_2],\ldots,[x_{h-1}]} \in C(n, \bigotimes_{i=1}^h A) \]

with \( x_i \in A \) for all \( 1 \leq i \leq h - 1 \) and \( x_i < x_h \) for all \( 1 \leq i \leq h - 1 \) such that

\[ x_h > w := \max\{u \in \|C(n, N)\| \mid u \in A\} > n - \sum_{i=1}^{h-1} x_i \]

then there are multivariate circles of partitions with the property

\[ C(s, \bigotimes_{i=1}^h A) \neq \emptyset \]

for all \( s \geq k \) for a fixed \( k \in \mathbb{N} \) with \( s \in \mathbb{H} \), which means every number in \( \mathbb{H} \subset \mathbb{N} \) \( \geq k \) can be written as the sum of \( h \) elements (not all possibly distinct) of \( A \).

Proof. Suppose that \( A \subset \mathbb{N} \) and let \( k \) be the smallest number in \( \mathbb{H} \subset \mathbb{N} \) such that \( C(k, \bigotimes_{i=1}^h A) \neq \emptyset \), which means \( k \) is the smallest number in \( \mathbb{H} \) such that it can be written as the sum of \( h \) elements in \( A \subset \mathbb{N} \). Let us choose \( t_o \in \mathbb{N} \) such that \( k, k + t_o \) are not consecutive in \( \mathbb{H} \), then by the hypothesis there exists at least an axis

\[ L_{[x_1],[x_2],\ldots,[x_h]} \in C(k + t_o, \bigotimes_{i=1}^h N) \]

with \( x_i \in A \) for all \( 1 \leq i \leq h - 1 \) and \( x_i < x_h \) for all \( 1 \leq i \leq h - 1 \) such that

\[ x_h > w := \max\{u \in \|C(k, N)\| \mid u \in A\} > k - \sum_{i=1}^{h-1} x_i. \]
It follows from Lemma 4.3 there exists an \( s \in \mathbb{N} \) with \( k < s < k + t_o \) such that
\[
\mathcal{C}(s, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset
\]
which means \( s \in \mathbb{H} \) can be written as the sum of \( h \) (not all) possibly distinct elements of \( \mathbb{A} \). Let us consider the sub-intervals \([k, s]\) and \([s, k + t_o]\). If \( k < s := k + t_1 \) are not consecutive in \( \mathbb{H} \) then there exists at least an axis
\[
L_{[x_1], [x_2], \ldots, [x_h]} \hat{\in} \mathcal{C}(k + t_1, \bigotimes_{i=1}^{h} \mathbb{N})
\]
with \( x_i \in \mathbb{A} \) for all \( 1 \leq i \leq h - 1 \) and \( x_i < x_h \) for all \( 1 \leq i \leq h - 1 \) such that
\[
x_h > w := \max\{u \in ||\mathcal{C}(k, \mathbb{N})|| \mid u \in \mathbb{A}\} > k - \sum_{i=1}^{h-1} x_i.
\]

It follows from Lemma 4.3 there exists some \( u \in \mathbb{H} \) with \( k < u < k + t_1 \) such that
\[
\mathcal{C}(u, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset
\]
which means \( u \in \mathbb{H} \) can also be written as the sum of \( h \) (not all) possibly distinct elements of \( \mathbb{A} \). We can similarly iterate this argument on the intervals \([k, u], [u, s], [s, k + t_o]\) so far as there exists an element of \( \mathbb{H} \) in any of the intervals. By virtue of the hypothesis the \( t \in \mathbb{N} \) can be chosen arbitrarily such that \( k, k + t \) are not consecutive in \( \mathbb{H} \) and with the existence of at least an axis
\[
L_{[x_1], [x_2], \ldots, [x_h]} \hat{\in} \mathcal{C}(k + t_1, \bigotimes_{i=1}^{h} \mathbb{N})
\]
with \( x_i \in \mathbb{A} \) for all \( 1 \leq i \leq h - 1 \) and \( x_i < x_h \) for all \( 1 \leq i \leq h - 1 \) such that
\[
x_h > w := \max\{u \in ||\mathcal{C}(k, \mathbb{N})|| \mid u \in \mathbb{A}\} > k - \sum_{i=1}^{h-1} x_i.
\]
The iterative arguments can be extended to all elements of \( \mathbb{H} \) under the assumption that
\[
\mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}) \neq \emptyset
\]
for infinitely many \( n \in \mathbb{H} \subseteq \mathbb{N} \). This means that every element \( n \in \mathbb{H} \) can be written as the sum of \( h \) elements of \( \mathbb{A} \), not all distinct. \( \square \)

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**References**

1. Agama, Theophilus and Gensel, Berndt *Studies in Additive Number Theory by Circles of Partition*, arXiv:2012.01329, 2020.
2. Helfgott, Harald A *The ternary Goldbach conjecture is true*, arXiv preprint arXiv:1312.7748, 2013.
3. Chen, Jing-run *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, The Goldbach Conjecture, World Scientific, 2002, pp. 275–294.
4. Heath-Brown, D Roger and Puchta, J-C *Integers represented as a sum of primes and powers of two*, Asian J. Math, vol. 6(3), 2002, 535–566.
5. Shnirel’man, Lev Genrikhovich, *On the additive properties of numbers*, Uspekhi Matematicheskikh Nauk, vol. 2:6, Russian Academy of Sciences, Steklov Mathematical Institute of Russian ..., 1939, pp. 9–25.

6. Estermann, Theodor *On Goldbach’s problem: Proof that almost all even positive integers are sums of two primes*, Proceedings of the London Mathematical Society, vol. 2:1, Wiley Online Library, 1938, pp. 307–314.

7. Chudakov, Nikolai Grigor’evich, *The Goldbach’s problem*, Uspekhi Matematicheskikh Nauk, vol. 4, Russian Academy of Sciences, Steklov Mathematical Institute of Russian ..., 1938, 14–33.

8. Hardy, Godfrey H and Littlewood, John E *Some problems of “Partitio Numerorum” (V): A further contribution to the study of Goldbach’s problem*, Proceedings of the London Mathematical Society, vol. 2(1), Wiley Online Library, 1924, pp. 46–56.

9. Dusart, Pierre *Explicit estimates of some functions over primes*, The Ramanujan Journal, vol. 45, Springer, 2018, pp. 227–251.

10. Agama, Theophilus and Gensel, Berndt *Studies in Additive Number Theory by Circles of Partition*, arXiv:2012.01329, 2020.

11. Agama, Theophilus and Gensel, Berndt *The Asymptotic Binary Goldbach and Lemoine Conjectures*, AfricArXiv Preprints, ScienceOpen, 2022.

12. Gensel, Berndt and Agama, Theophilus *Complex Circles of Partition and the Squeeze Principle*, arXiv preprint [arXiv:2304.13371] 2023.

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