Some Aspects of the AdS/CFT Correspondence

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Abstract

This is a very brief review of some aspects of the AdS/CFT correspondence with an emphasis on the role of the topology of the boundary and the meaning of the sum over bulk geometries. To appear in the proceedings of the 73rd Meeting between Physicists and Mathematicians “(A)dS/CFT correspondence,” Strasbourg, September 11-13, 2003.

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1 Introduction

Since its incarnation in 1997 [1, 2, 3] (for a review see [4]), the AdS/CFT correspondence has been one of the prime subjects of interest in string theory. It provides a duality between a theory with quantum gravity in $d$ dimensions and a field theory in $d−1$ dimensions. This is a rare example where we have a complete non-perturbative definition of string theory in a certain background, and quite amazingly it is equivalent to just an ordinary field theory. Strongly coupled string theory is equivalent to weakly coupled field theory and vice versa, and therefore the AdS/CFT correspondence can help understand the physics of strongly coupled gauge theories. At the same time, although Anti-de Sitter space has different asymptotics than Minkowski space or our universe (as far as we know), the properties of gravity at short distances should be somewhat independent of the asymptotic behavior of the space. Therefore the AdS/CFT correspondence should also be useful in understanding the puzzles associated to quantum gravity, in particular those associated with black hole creation and evaporation and information loss in black holes.

Though it is easy to say these words, to actually implement them in practice is not quite so straightforward. Given a certain manifold $M$ on which the CFT lives, which can be either Lorentzian or Euclidean, the dual gravitational description involves a sum over all geometries whose conformal boundaries are equal to $M$. This by itself is a mathematical question, namely the classification of all solutions to the Einstein equations with a negative cosmological constant with given conformal boundary. However, this is not the full story, since we should really sum over all solutions of the string theory equations of motion with the right asymptotic behavior. String theory has various other fields in addition to the metric, and therefore this set can be larger than the set of purely gravitational solutions. Furthermore, the string theory configurations that have to be summed over do not need to be weakly coupled everywhere and can in principle include stringy objects such as branes in the interior. The full classification of all such solutions is still in its infancy and a further understanding seems crucial in order to make progress in our understanding of quantum gravity.

With these motivations, we will here briefly summarize some of the known problems and solutions associated to finding the bulk solutions with a priori given asymptotia. We have no pretense of being complete and/or exhaustive, the main purpose of these notes is to provide
some food for further thought.

There are in principle many cases to consider, and we organized them as follows. In section 2 we consider solutions with Euclidean signature and a single connected boundary. In section 3 we consider static solutions with Lorentzian signature and a single connected boundary. In section 4 we discuss solutions with more than one boundary, and in section 5 we briefly comment on the relation of all this with the puzzles associated to singularities and black holes in quantum gravity. In section 6 we comment on the role of Chern-Simons theory in 2 + 1 dimensions; this is an especially interesting case since gravity has no propagating degrees of freedom in 2 + 1 dimensions. Finally, in section 7 we mention some examples that involve time-dependent geometries, and in section 8 some of the problems associated to extending all this to zero and positive cosmological constant are summarized.

2 Euclidean, single boundary setups

AdS/CFT – the statement

The original statement of AdS/CFT is a relationship between the partition function of string theory on AdS×X geometry and that of a CFT living on the boundary of AdS:

\[ Z_{\text{CFT}}(\partial M; \gamma) = \int D\Phi_{\text{string}} e^{-S_{\text{SFT}} \text{ saddle point}} \approx \sum_i Z_{\text{String}}(M_i). \] (1)

The CFT lives on \( \partial M \) which carries a metric in a fixed conformal class indicated by \( \gamma \), and the LHS of (1) is the CFT partition function. We have schematically written the string path integral as an integral over string fields approaching \( \partial M \) on the boundary, and \( S_{\text{SFT}} \) represents the string field theory action. In going from this somewhat schematic expression to a more useful expression \( \sum_i Z_{\text{String}}(M_i) \), we have made a saddle point approximation. \( M_i \) are backgrounds which satisfy the string equations of motion and have \( (\partial M; \gamma) \) as their conformal boundary. In most cases, the string coupling constant and the inverse curvature radius of the AdS space are free parameters and we can take them arbitrarily small. In this limit\(^1\) we can replace the string theory partition function with the classical supergravity

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\(^1\)On the dual CFT side, the interpretation of this limit depends on the model, but generically involves taking a strong coupling limit, and also often involves sending the rank of a gauge group to infinity.
contribution, which is simply \( Z_{\text{Sugra}} = e^{-S(M)} \) where \( S(M) \) is the classical supergravity action evaluated on the solution \( M \).

Notice the sum over different manifolds with the same conformal boundary. This plays a crucial role in the study of phase transitions in the boundary CFT. In a certain regime of parameters, generically one of the spaces \( M_i \) will dominate the sum on the RHS of (1). However, by varying the parameters, which of these \( M_i \)'s dominates may change, leading to phase transitions. The existence of a phase transition is perhaps surprising if the boundary \( \partial M \) is compact (as will mostly be the case in what we study below). However, in the large \( N \) limit, we can still have sharp phase transitions, even on a compact volume.

An important ill-understood feature of (1) is the precise relative normalization of the contributions on the right hand side. In the absence of a background independent formulation of string field theory is is not obvious how to compute these from first principles. The naive guess to take just the supergravity actions is incorrect in the example discussed in [5], but the microscopic origin of the relative normalization found in that paper is not clear.

In this section, we will discuss Euclidean setups, with the bulk space \( M \) being a solution to Einstein’s equations with \( \Lambda < 0 \), which has a single boundary \( \partial M \). We restrict ourselves to the case of a four dimensional bulk which is one of the best studied cases [6].

2.1 \( \partial M = S^3, \quad M = B^4 \)

This is the best understood example. Euclidean AdS\(_4\) can be described as the open unit ball \( B^4 \), with coordinates \( x_i \) such that \( \sum_{i=1}^{4} x_i^2 < 1 \) and the metric

\[
ds^2 = \frac{4dx^2}{(1 - |x|^2)^2}.
\]

This metric does not extend to the boundary at \( |x|^2 = 1 \). However, the metric can be extended to the boundary by defining a function \( f \) on the closure of \( M \) such that it has a simple zero at the boundary and is positive in the interior. Then, the metric \( d\tilde{s}^2 = f^2 ds^2 \) extends to a metric on the boundary, but given a bulk metric, only the conformal structure of the boundary metric can be uniquely determined. Following a theorem of Graham and Lee, for every conformal structure on \( S^3 \) sufficiently close to the standard one, there exists a metric on \( B^4 \) with that \( S^3 \) as a conformal boundary. Analogous statements hold for the
scalar fields as well as the gauge fields—bulk fields are uniquely specified by their behavior on the boundary. This means that when we apply the AdS/CFT correspondence in this background, we can compute the correlation functions in the boundary theory by evaluating the bulk action for field configurations which asymptotically approach a given boundary data. On the RHS of (1), there is only one term in the summation.

### 2.2 \( \partial M = S^1 \times S^2, \quad M = S^1 \times R^3 \) and \( M = R^2 \times S^2 \)

With the boundary \( S^1_\beta \times S^2 \) (where \( S^1_\beta \) is a circle of radius \( \beta \)), there are two known asymptotically AdS bulk solutions with this boundary. One is AdS itself (with topology \( S^1 \times R^3 \)), with the Euclidean time direction being a circle. This background is appropriate for the finite temperature bulk physics at temperature \( 1/\beta \). Another solution with the same boundary behavior is Euclidean AdS-Schwarzschild (this has topology \( R^2 \times S^2 \)). The boundary theory is a CFT at finite temperature.

It was shown by Witten [7] that the first solution dominates the partition function computation at low temperatures while the latter becomes dominant at high temperatures. This difference in behavior corresponds to the confinement-deconfinement phase transition in the field theory. This phase transition was first studied by Hawking and Page [8] who showed that above a critical temperature, thermal radiation is unstable to the formation of an AdS Schwarzschild black hole. There are, in fact, two black hole solutions, with different masses for a given value of \( \beta \) (the temperature). The smaller value of the masses leads to a black hole with a negative specific heat, which means that the black hole is unstable to decay as Hawking radiation. For the higher value of the masses, the Hawking radiation is in thermal equilibrium with the thermal radiation in the background.

### 2.3 \( \partial M = T^3, \quad M = R^2 \times T^2 \)

These are the AdS toroidal black hole metrics, which are of the form

\[
ds^2 = U^{-2}dr^2 + U^2d\theta^2 + r^2ds_{T^2}^2,
\]

where \( U^2 = r^2 - \frac{2m}{\rho} \). The conformal boundary of this space is \( S^1_\theta \times T^2 = T^3 \). Given a boundary metric, there are actually infinitely many different ways of filling in the bulk...
metric, each corresponding to a choice of one cycle in $T^3$ which is ‘filled in’ to obtain a bulk solution. For details, see [6]. The boundary theory is a finite temperature CFT on $T^2$. The multiple classical solutions perhaps correspond to different phases in this theory. Heuristically, the CFT partition function can be written as

$$Z_{\text{CFT}} \sim \sum_{g \in SL(3,\mathbb{Z})/H} \exp\left(-I(M_g)\right).$$

(4)

This expression should be taken with a large grain of salt. We do not really understand how to perform this sum here. In one lower dimension, for a two dimensional boundary, a similar summation was performed in [5] where the elliptic genus of the conformal field theory was computed by writing it as a sum over different asymptotically AdS$_3 \times S^3$ bulk geometries (also see section 5.3). The AdS$_3$ string theory should reduce to a Chern Simons theory at large distances. The calculation of the bulk partition function in this Chern-Simons theory is a state in the space of conformal blocks of the boundary theory and therefore transforms non-trivially under the modular group [9]. On the other hand, the string theory computation in [5] gives a modular invariant partition function. This apparent paradox is resolved [9] by the special appearance of a modular invariance restoring chiral “spectator boson” on the boundary.

### 2.4 AdS Taub-bolt metrics

The AdS Taub-bolt metrics are locally asymptotically AdS. The conformal boundary is an $S^1$ bundle over $S^2$, with non-zero first Chern number. For vanishing first Chern number, the boundary is the product space $S^1 \times S^2$ and the space is asymptotically AdS. This is one of the cases we discussed above. However, for non-vanishing first Chern number, $k$, the conformal boundary is a squashed $S^3$ with $|k|$ points identified along the $S^1$. These metrics have a U(1) isometry which acts on the $S^1$ fiber in the natural way. For the AdS Taub-Bolt metric, the fixed point set of this isometry is two-dimensional (called a bolt). The line element is given by

$$ds^2 = -\frac{3}{4\Lambda}E\left[\frac{F(r)}{E(r^2 - 1)}(d\tau + E^{1/2}\cos \theta d\phi)^2 + \frac{4(r^2 - 1)}{F(r)}dr^2 + (r^2 - 1)(d\theta^2 + \sin^2 \theta d\phi^2)\right],$$

(5)
with
\[ F_{\text{Bolt}}(r) = Er^4 + (4 - 6E)r^2 + \left( -Es^3 + (6E - 4)s + \frac{3E - 4}{s} \right)r + 4 - 3E, \] (6)
and
\[ E = \frac{2ks - 4}{3(s^2 - 1)}. \] (7)

Here \( \Lambda < 0 \) is the cosmological constant, \( \tau \) has period \( \beta = \frac{4\pi E^{1/2}}{k} \), \( s \) is an arbitrary parameter (the bolt is at \( r = s \)) and \( k \) is the Chern number of the \( S^1 \) bundle over \( S^2 \), which is the conformal boundary of this solution. \(|k| \) points on the \( S^1 \) fiber are identified.

There is another class of closely related metrics for which the fixed point set of \( \frac{\partial}{\partial \tau} \) is just a point. These are the AdS Taub-Nut metrics. The line element for these metrics has the same form as (5) but the function \( F(r) \) is now given by
\[ F_{\text{NUT}}(r) = Er^2 + (4 - 6E)r^2 + (8E - 8)r + 4 - 3E. \] (8)

Now, \( E \) is an arbitrary parameter which parameterizes the squashing of the \( S^3 \) which is the conformal boundary.

The AdS TN and AdS TB have the same asymptotic behavior for \( k = 1 \). For \(|k| > 1 \), if we identify \(|k| \) points on the \( S^1 \) fiber of the AdS TN solution, we obtain a space which has the same boundary structure as the AdS TB solution with parameter \( k \). This identified AdS TN solution, however, has a conical singularity at the origin, which can be smoothed out.

For computation for the analogue of the ADM mass, and action for these solutions, we need to compare it to some reference metric which has the same boundary behavior. The reference metric is taken to be the AdS Taub-Nut metrics discussed above, with appropriate identifications along the \( S^1 \) fiber to get the same asymptotic structure. Then, the Hamiltonian calculation reveals that there are two AdS Taub-Bolt metrics with the same temperature, but different masses. The one with the lower masses is thermodynamically unstable, since it has a negative specific heat. In addition, as in the AdS case, there is a phase transition in the system (for \( k = 1 \)). The AdS Taub-Nut solution exists for all temperatures. However, the AdS Taub-Bolt solution can only exist for temperatures above a minimum value \( T_0 \).

Furthermore, since we have multiple bulk solutions with the same boundary behavior, the partition function of the boundary CFT will receive contributions from the different bulk
spaces with the same boundary behavior. To determine which one dominates, we need to evaluate the action for these solutions. It can be shown \cite{10} that for temperatures below $T_1$ ($> T_0$), the AdS Taub-Nut background is favored, whereas for temperatures above $T_1$, the AdS Taub-Bolt solution dominates, and the AdS Taub-Nut background will decay into it. This presumably corresponds to a confinement/deconfinement phase transition for the boundary theory living on the squashed $S^3$.

3 Static Lorentzian spacetimes, $\Lambda < 0$

In the previous section, we discussed Euclidean situations, where specifying the boundary values of the various fields at the conformal boundary determines the bulk configurations, in some cases uniquely, and in others, up to a few discrete choices. The situation in Lorentzian signature is more subtle. The normalizable mode solutions to the equations of motion, which exist in Lorentzian signature, can be arbitrarily added to a bulk solution with a given boundary behavior without affecting the boundary behavior. The choice of the normalizable part of the solution corresponds to the choice of state in the conformal field theory in which the partition function and hence the correlation functions are computed. Here again we will only deal with the case of four dimensional bulk, as it is one of the most studied cases \cite{11}.

3.1 $\partial M = \mathbb{R} \times S^2$, $M = \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$

This is the usual Lorentzian AdS/CFT setup. The boundary CFT lives on $\mathbb{R} \times S^2$. Given a certain boundary metric (with non-negative Ricci scalar), a bulk metric always exists with that boundary behavior \cite{11}. The uniqueness of such a metric is not guaranteed in general. However for the boundary metric $ds^2 = -dt^2 + ds_{S^2}^2$, there is a unique globally static bulk metric with conformal compactifiable smooth acausal equal time slices. This is just the standard metric of AdS$_4$ \cite{11}.

3.2 $\partial M = \mathbb{R} \times S^2$, $M = \mathbb{R} \times (\mathbb{R}^+ \times S^2)$

Looking now at the same boundary manifold but at $M = \mathbb{R} \times (\mathbb{R}^+ \times S^2)$ as a bulk manifold, we find a rather different situation. Taking the boundary metric to be again $ds^2 = -dt^2 + ds_{S^2}^2$,
one can check that the following family of 1-parameter bulk metrics all have the required asymptotics - these are the Lorentzian AdS Schwarzschild black holes with the metric

\[ ds^2 = -U^2 dt^2 + U^{-2} dr^2 + r^2 (d\theta^2 + \cos^2 \theta d\phi^2), \tag{9} \]

where

\[ U^2 = 1 + r^2 - \frac{2m}{r}, \quad m > 0. \tag{10} \]

This background corresponds to a thermal state in the boundary CFT. It differs from the usual setup in (3.1) by the choice of the state on the boundary.

One can also show that these AdS Schwarzschild black hole metrics are the unique globally static metrics smooth up to the horizon with conformal compactifiable smooth acausal equal time slices \[11\).

The dual CFT corresponding to the boundary conditions of both the global \( AdS_4 \) metric described in section 3.1 and these AdS Schwarzschild black holes is a CFT defined on a spatial manifold \( S^2 \). To discuss this theory at finite temperature, one effectively needs to calculate the partition function on \( S^2 \times S^1 \), where the radius of the extra \( S^1 \) factor is related to the inverse of the temperature, and it can be thought of as the time direction, Wick rotated to Euclidean signature. The calculation of the partition function then follows the one we had in section 2, for Euclidean spacetimes. Therefore the Hawking-Page phase transition occurs here and is seen in the Field theory as a confinement - deconfinement transition. Of course in cases where the field theory is not in a finite temperature, it is hard to tell which geometry would dominate the partition function.

### 3.3 \( \partial M = \mathbb{R} \times T^2 \), \( M = \mathbb{R} \times (D^2 \times S^1) \)

Let us look at the boundary metric \( ds^2 = -dt^2 + ds^2_{T^2} \). Then one can show \[11\ \[12\] that all the globally static metrics on \( M \) with such asymptotics and with conformal compactifiable smooth acausal equal time slices are of the “AdS soliton” type discussed by Horowitz and Myers \[13\]:

\[ ds^2 = -r^2 dt^2 + U^{-2} dr^2 + U^2 d\phi^2 + r^2 d\theta^2, \tag{11} \]

where

\[ U^2 = r^2 - \frac{2m}{r}, \quad m > 0. \tag{12} \]
\( \phi \) is a periodic angle of period \( \beta = \frac{4\pi}{3(2m)^{1/3}} \), and \( \theta \) is of arbitrary period.

One can also show that for any given boundary metric on \( T^2 \) there are countably many such filling metrics parameterized by the choice of an \( S^1 = \partial D^2 \).

These geometries have the interesting property that their mass is negative (relative to the choice where conformal flatness means zero energy). This fact has a natural interpretation on the CFT side - it was shown in [13] that the corresponding CFT has a negative Casimir energy, related to the breaking of supersymmetry on the CFT by the boundary conditions on the fermions. In fact it was conjectured in [13] that the “AdS soliton” metrics are the lowest energy solutions with these given boundary conditions.

3.4 \( \partial M = \mathbb{R} \times T^2, \quad M = \mathbb{R} \times (\mathbb{R}^+ \times T^2) \)

In this case, taking again the boundary metric to be \( ds^2 = -dt^2 + ds^2_{T^2} \), one can show that there is a 1-parameter family, this time of toroidal black holes with these asymptotics. This family of toroidal Kottler metrics is given by

\[
    ds^2 = -U^2 dt^2 + U^{-2} dr^2 + r^2 d\phi^2 + r^2 d\theta^2,
\]

where as before

\[
    U^2 = r^2 - \frac{2m}{r}, \quad m > 0,
\]

and where both \( \phi, \theta \) are periodic of arbitrary period. These metrics have a horizon at \( r^4 = 2m \) which is \( \mathbb{R} \times T^2 \). As before these filling metrics are the unique ones which are globally static and with conformal compactifiable smooth acausal equal time slices [11].

The energy of these black holes is greater than that of the AdS solitons of the same boundary structure, in accordance with the conjecture made by Horowitz and Myers [13]. However, a thermodynamical analysis [14, 15, 16], shows that the free energy of these black holes can be greater or smaller than that of the solitons, leading to a phase transition, somewhat similar to the Hawking-Page transition we mentioned earlier. It has been shown that small, hot black holes are unstable and decay to small, hot solitons. Large cold black holes are stable. The order parameter for the transition depends both on the horizon area and on the temperature of the black hole (which are two independent parameters for these black holes). On the side of the CFT, this phase transition can be related to a confinement - deconfinement transition [14, 17].
4 Multiple boundary configurations

In cases where the boundary of spacetime has multiple disconnected components, the issue of a dual holographic description is more involved. On the one hand the holographic theory is defined on a union of disjoint manifolds. There is no obvious way in which the theories on the different manifolds are coupled, and a priori it seems natural to expect that the holographic theory would just be the product of the theories on each one of the boundary components. On the other hand the bulk theory seems to induce correlations between the different boundary regions.

This seeming puzzle bears a somewhat different nature depending on whether one is discussing Euclidean or Lorentzian settings.

In the Lorentzian case, for asymptotically AdS spacetimes (i.e. $\Lambda < 0$), a topological censorship theorem was proved [18], which basically states that under certain conditions, the presence of multiple boundaries forces the bulk to be separated by horizons, in such a way that different boundary components are not causally connected through the bulk. This implies that the different holographic theories living on the different boundary components would indeed be uncorrelated and will not interact dynamically. The only correlations could be ones in initial states of the theory \(^2\). Let us state the topological censorship theorem more precisely now: Let $M'$ be a globally hyperbolic spacetime with boundary, with timelike boundary $\mathcal{I}$ that satisfies the average null energy condition \(^3\). Let $\mathcal{I}_0$ be a connected component of $\mathcal{I}$ of $M'$. Furthermore assume that either (i) $\mathcal{I}_0$ admits a compact spacelike cut or (ii) $M'$ satisfies the generic condition \(^4\). Then $\mathcal{I}_0$ cannot communicate with any other component of $\mathcal{I}$, i.e. $J^+(\mathcal{I}_0) \cap (\mathcal{I} \setminus \mathcal{I}_0) = \emptyset$.

In the Euclidean case, for asymptotically AdS Einstein spacetimes, the puzzle is avoided due to a theorem by Witten and Yau [20], basically stating that if one of the boundary components has $R > 0$, then the boundary is connected. This theorem was later generalized

\(^2\)One example for this is the case of Schwarzschild AdS black holes, and in particular the BTZ black hole. These were studied in [19] and we would make a few comments about them below.

\(^3\)The average null energy condition states that for each point $p$ in $\mathcal{M}$ near $\mathcal{I}$ and any future complete null geodesic $s \to \eta(s)$ in $\mathcal{M}$ starting at $p$ with tangent $X$, $\int_0^\infty \text{Ric}(X, X) \, ds \geq 0$ ($\text{Ric}(X, X)$ denotes $R_{ab}X^aX^b$). This condition is satisfied by spacetimes created from physically reasonable matter sources.

\(^4\)A spacetime satisfies the generic condition if every timelike or null geodesic with tangent vector $X$ contains a point at which $X^aX^bX^c[R_{ab}X^dX^f]$ is nonzero.
by Cai and Galloway [21] to cases where the boundary has zero scalar curvature. Let us state the general theorem: Let $M^{n+1}$ be a complete Riemannian manifold which admits a conformal compactification, with conformal boundary $N^n$, and with the Ricci tensor of $M$ satisfying $\text{Ric} \geq -ng$ such that $\text{Ric} \to -ng$ sufficiently fast on approach to conformal infinity $^5$. If $N$ has a component of nonnegative curvature, then the following holds: (i) $N$ is connected (ii) If $M$ is orientable, then $H_n(M, Z) = 0$ (iii) The map $i_* : \Pi_1(N) \to \Pi_1(M)$ ($i=$inclusion) is onto.

This theorem therefore implies that the puzzle we described does not arise in asymptotically AdS Einstein spacetimes with nonnegative boundary curvature. One might wonder then about the case where the boundary has negative curvature. In such cases, it can be shown that the holographic theory living on the negative curvature boundary would be unstable for any boundary dimension $n \geq 3$. However, the Witten Yau theorem can be avoided if we turn on extra supergravity fields, and thus look not at Einstein spacetimes, but rather at spacetimes obeying the more general supergravity equations. The instability related to negative curvature boundary can also be avoided if we look at specific settings of 3-dimensional bulk (i.e. $n = 2$). Such examples will be presented in section 4.3.

Let us now discuss a few examples of multi boundary situations where one can say something about the AdS/CFT correspondence. We’ll focus in the case where the bulk is 3-dimensional, where things are better known. In fact in the case of 3-dimensions, the only solution to Einstein’s equations with a negative cosmological constant is locally $AdS_3$. Different spacetimes can only differ from each other by global identifications.

Starting from Lorentzian configurations, one can build configurations with multiple boundary components by taking a 2-dimensional slice of global $AdS_3$, cutting and gluing it along geodesics and then letting it evolve in time $^\text{23}$. Such constructions lead, in agreement with the topological censorship theorem, to spacetimes with the same number $h$ of boundaries and horizons, and with any number $g$ of handles behind the horizons.

In the special case where there are only two boundaries ($h = 2$, $g = 0$), the spacetime describes the eternal BTZ black hole.

$^5$ i.e. $r^{-2}(\text{Ric} + ng) \to 0$ as $r \to 0$ where the bulk metric is expanded in a neighborhood of the boundary as $g = \frac{1}{r^2}(dr^2 + g_\gamma)$, and the conformal boundary is at $r \to 0$. For some discussion on a physical interpretation of these conditions, see $^{22}$.
4.1 Eternal BTZ

An eternal BTZ black-hole has two boundaries. More precisely, it has two asymptotically AdS$_3$ regions each of which is separated from the other by a horizon (see the Penrose diagram in fig 1). Locally, it is isometric to AdS$_3$ but differs from it in its global properties. Three dimensional anti-de Sitter space is a maximally symmetric space of constant negative curvature. It is the hyperboloid

$$\text{AdS}_3 \leftrightarrow \mathbb{R}^{2,2}$$

$$-u^2 - v^2 + x^2 + y^2 = -l^2,$$  \(15\)

in flat $\mathbb{R}^{2,2}$. By construction, the isometry group is SO(2, 2). The Killing vectors of the metric generate the Lie algebra $\mathfrak{so}(2, 2)$ of the isometry group, and are described in terms of the embedding space $\mathbb{R}^{2,2}$ as

$$J_{ab} = x_b \partial_a - x_a \partial_b,$$  \(16\)

with $x^a \equiv (u, v, x, y)$ and $x_a = \eta_{ab} x^b$, with $\eta_{ab} = (-, -, +, +)$.

The BTZ black holes are obtained by identifying AdS$_3$ by the discrete action generated by the Killing vector $[24]

$$\xi_{\text{BTZ}} = \frac{r_+}{l} J_{12} - \frac{r_-}{l} J_{03} - J_{13} + J_{23}. $$  \(17\)

In the non-extremal case, $r_+^2 - r_-^2 > 0$ and by a SO(2, 2) transformation, $\xi_{\text{BTZ}}$ can be brought into the form:

$$\xi'_{\text{BTZ}} = \frac{r_+}{l} J_{12} - \frac{r_-}{l} J_{03}. $$  \(18\)
The mass and angular momentum of the black hole are given by

\[ M = \frac{1}{l^2} (r_+^2 + r_-^2) \quad , \quad J = \frac{2}{l} r_+ r_- . \] (19)

The extremal black hole is obtained by taking the limit \( r_+ \to r_- \) in (17), so that the generator becomes

\[ \xi_{BTZ} \to \frac{r_+}{l} (J_{12} - J_{03}) - J_{13} + J_{23} . \] (20)

The holographic description of the space-time is given in terms of two non-interacting identical CFTs. In Lorentzian AdS/CFT, a holographic description includes the specification of a state in the CFT. The relevant state in two CFTs which describes the BTZ background is a particular entangled state given by

\[ |\Psi\rangle = \sum_n e^{-\frac{2}{l} E_n} |E_n\rangle_1 \times |E_n\rangle_2 , \] (21)

where \( |E_n\rangle_{1,2} \) denotes an energy eigenstate in the two CFTs and \( \beta = \frac{\pi l}{r_+} \).

Computation of the correlation functions involving only operators in one of the two CFTs in this entangled state lead to thermal correlation functions:

\[ \langle \Psi | \mathcal{O}_1 | \Psi \rangle = \sum_n e^{-\beta E_n} \langle E_n | \mathcal{O}_1 | E_n \rangle_1 = \text{Tr} (\rho_\beta \mathcal{O}_1) , \] (22)

where \( \rho_\beta \) is the thermal density matrix.

We will come back to the eternal BTZ in Section 5.

4.2 Two null cylinder boundaries

This space has two boundaries, each of which are null cylinders, i.e. flat space with a compact null direction. The metric is given by

\[ ds^2 = l^2 \left( -(dt)^2 + (d\phi)^2 + 2 \sinh(2z) dtd\phi + (dz)^2 \right) , \] (23)

with \( \phi \) being an angular coordinate taking values in \([0, 2\pi)\), and the coordinates \((z, \phi, t)\) give a global parameterization of the space. The two boundaries are at \( z \to \infty \). Unlike the BTZ,
the boundaries are not separated by a horizon. The space is stationary but not static. It is a quotient of AdS$_3$ by action of a subgroup of SO(2, 2) isomorphic to $\mathbb{Z}$.

\[ P \rightarrow e^{i\xi} P \quad , \quad t = 0, \pm 2\pi, \pm 4\pi, \cdots \quad \forall P \in \text{AdS}_3 \]  

(24)

where

\[ \xi = \frac{1}{2} (J_{02} + J_{13}) \ . \]  

(25)

This generator is a linear combination of a boost in the ux-plane and vy-plane in the embedding space $\mathbb{R}^{2,2}$.

The metric (25) is an $S^1$ fibration over AdS$_2$. Indeed, we can rewrite this metric as

\[ g = l^2 \left( - \cosh^2(2z) \, dt^2 + dz^2 + (d\phi + \sinh(2z) \, dt)^2 \right) \ . \]  

(26)

Compactifying on $\phi$ now gives the metric

\[ g_2 = - \cosh^2 2z \, dt^2 + dz^2 , \]

\[ A_1 = \sinh 2z \, dt . \]

The metric is precisely that of AdS$_2$, but there is also a constant electric field.$^6$

The fact the the boundaries are null cylinders implies that the boundary theory is defined through a discrete light cone quantization procedure. The exact map between the bulk and the boundary theory is still somewhat mysterious. For more details, see [25].

4.3 Wormholes

Now looking at Euclidean setups, in order to describe multiple boundary spaces (i.e. wormholes), we must look at configurations which do not satisfy the conditions of the Witten-Yau theorem. One possibility is to look at spaces which have two boundaries, each being a Riemann surface of genus $g \geq 2$, $\Sigma_g$ [26]:

\[ ds^2 = d\rho^2 + \cosh^2 \rho \, ds_{\Sigma_g}^2 , \]  

(27)

$^6$The field is constant in the sense that the field strength of the U(1) connection is proportional to the AdS$_2$ volume form.
The boundaries have constant negative curvature, so the Witten-Yau theorem does not apply here, but unlike in higher dimensions, the two-dimensional field theories on these Riemann surfaces are well-defined and stable. Such spaces can be created by a quotient of $H_3$ by a discrete subgroup $\Gamma \in SL(2, \mathbb{C})$. The two Riemann surfaces have the same genus, but may differ in their moduli $t^a$. Performing a quotient of $SL(2, \mathbb{C})$ by a Fuchsian subgroup $\Gamma$ results in two Riemann surfaces which have the same moduli. Performing a quotient by a quasi-Fuchsian subgroup results in two Riemann surfaces of different moduli. In fact, according to the Bers simultaneous uniformization theorem [27] the quasi-fuchsian space of a Riemann surface $\Sigma_{g,n}$ of genus $g$ and $n$ punctures: $QF(\Sigma_{g,n})$ is homeomorphic to pairs of points in the Teichmuller space of $\Sigma_{g,n}$: $Teich(\Sigma_{g,n}) \times Teich(\Sigma_{g,n})$.

In such cases, the puzzle we described previously for multiple boundary spaces is apparent, and it is not clear whether correlations would or would not exist between the two boundary holographic theories [26]. Actually it is interesting to note that it is also possible to construct by discrete quotients of $H_3$, spaces with a single boundary which could be any Riemann surface $\Sigma_{n,g}$ (this is guaranteed by the classical retrosection theorem). In this case the discrete subgroup is a Schottky group, and such constructions were described in [28]. It would be interesting to see if the geometry with disconnected boundaries (i.e. a union of two such spaces, each with a single Riemann surface as a boundary) has larger or smaller action than the geometry that connects the two boundaries. If the disconnected geometry is the dominant one, then it is possible that correlations between the two boundaries will indeed be suppressed.

Another way to avoid the Witten-Yau theorem and still build a Euclidean spacetime with multiple boundaries of positive curvature, is to add to the pure Einstein gravity some gauge fields. An example in 4 dimensions was built in [26], where an $SU(2)$ gauge field was introduced, and the 4-dimensional action is

$S \sim \int d^4x \sqrt{g}[-R + \Lambda + F_{\mu\nu}^a F^{a\mu\nu}]$. \hspace{1cm} (28)

Here the field strength is given in terms of the gauge field by $F = dA + A^2$ and the cosmological constant is normalized to be $\Lambda = -6$.  \hspace{1cm} $^7$ The following two-boundary solution for this

\hspace{1cm} $^7$It was shown in [26] that this action is a consistent reduction of 11-dimensional supergravity.
action was constructed:

\[
    ds^2 = d\rho^2 + e^{2w} ds_{S^3}^2 \quad ; \quad e^{2w} = \frac{1}{2}(\sqrt{5} \cosh 2\rho - 1)
\]
\[
    A = \frac{1}{2} \omega^a,
\]

(29)

where \( \omega^a \) are the left-invariant, \( su(2) \)-valued one-forms on \( S^3 \), such that \( ds_{S^3}^2 = \frac{1}{4} \omega^a \omega^a \). For large \( \rho \) the geometry becomes that of \( H_4 \) and the boundaries are 3-spheres, and for \( \rho = 0 \) we have a finite throat size. The precise way in which the AdS/CFT correspondence would work here is quite mysterious. It would be very interesting to understand its exact meaning and interpretation for such configurations. For more details, other examples of wormhole setups, and some speculations regarding possible resolutions of the AdS/CFT puzzle for such configurations see [26].

5 Some interesting questions

5.1 The information paradox

Hawking showed that black holes are not really black but in fact emit a thermal radiation. This follows from a semi-classical analysis. We now imagine matter in a pure state falling into the black hole. Hawking’s semi-classical analysis reveals that it will eventually be radiated out as thermal radiation, which is a mixed state. This exposes a paradox: in quantum mechanics, pure states evolve to pure states. How then can a pure state evolve into a mixed state in a black hole background. The information about the initial pure state seems to have been lost inside the black hole. This has been called the information loss paradox [29] – after matter falls into the black hole, the correlators with infalling matter decay exponentially, so if we wait a long enough time, the correlation functions will eventually vanish. This violates unitarity. In AdS, this poses a particularly sharp paradox since asymptotically AdS black holes can live for ever.

In section 4.1 we described the eternal BTZ black hole and its description in the two boundary CFTs in terms of an entangled state. In [19], Maldacena considered a deformation of the thermal Schwarzchild AdS state by adding an operator to the second boundary and showed that the correlations indeed die off exponentially. This change in the black hole
state, although minor, is still detectable: the one point function of the same operator in
the first CFT which was previously zero is now non-zero, but dies exponentially fast at a
rate $e^{-\beta}$ where $c$ is a numerical constant. The puzzle is that correlation functions of the
boundary CFT cannot decay at late times since it signals a loss of unitarity (one can in fact
show that to be consistent with unitarity, the correlations could be as small as $e^{-cS}$ where
$S$ is the entropy of the ensemble and $c$ is a numerical factor). The resolution comes from
the fact that in AdS/CFT, we need to sum over all geometries with a prescribed boundary
behavior. In fact, there are other geometries than just the Schwarzschild geometry that we
have so far considered. The additional geometry that provides the relevant effect consists of
two separate global AdS spaces with a gas of particles on them in an entangled state. This
geometry contributes with a small weight because it has a very small free energy compared
to the Schwarzschild geometry, but it indeed gives a non-decaying answer of order $e^{-cS}$ as
expected from constraints of unitarity; see also [30], [31], [32], [33] and [34] for further
discussion. In the last paper it is shown that although the sum over geometries does yield a
non-decaying answer, it seems extremely difficult to obtain the required quasi-periodic answer
for correlation functions this way, unless one manages to perform the sum over geometries
in closed form.

There is another picture of black holes due to Mathur et al that has got some attention
recently which we describe in more detail in section 5.3. According to this picture, the
black hole should be thought of as an ensemble of classical geometries, each of which has no
horizon. The absence of a horizon in each of the different geometries evades the information
loss problem.

5.2 Singularity beyond the horizon

In the AdS/CFT correspondence, the region outside the horizon of a black hole is represented
holographically by a boundary CFT at finite temperature. Since the black hole singularity
is inside the horizon, at first sight it seems that AdS/CFT cannot be used to gain insight
into the nature of the singularity. However, the situation is more subtle for the eternal black
holes, which have multiple asymptotic regions. For example, the eternal BTZ black hole
has two asymptotically AdS regions. The holographic dual is given by two decoupled CFTs
living on the two boundaries, living in an entangled state as described in Section 4.1. For
correlations functions where all the operators belong to one of the two CFTs, we can trace over the states of the other CFT leading to correlation functions in a thermal state. Such correlation functions will not contain any non-trivial information about the physics beyond the event horizon. However, correlation functions of operators in each of the two CFTs will contain information about the region beyond the horizon. This is most easily seen by using the geodesic approximation to compute the correlators. For example, for a two point function of operators inserted on the two boundaries, the WKB approximation is good for bulk fields of large mass. In this approximation, space-like geodesics dominate the contribution to the 2-point function, and these geodesics traverse the region behind the horizon. In the case of the BTZ black hole, these two point computations can be carried out exactly, by using appropriate bulk to boundary propagators, and then moving the bulk point to the boundary and removing an overall rescaling. In [35], Kraus et al defined these amplitudes by an analytic continuation procedure from Euclidean signature. This analytic continuation can be done in different ways. In one way of performing the analytic continuation (I), in Lorentzian signature, the contribution comes from only the region behind the horizon whereas in the other procedure (II), the contribution comes from both the region outside and behind the horizon. Since the two analytic continuation procedures are equivalent and finite, (I) manifestly so, while in (II), the singularity can be regulated by an $i\epsilon$ prescription inherent in the analytic continuation procedure, and the contribution from the past and future singularity can be shown to cancel. Its not entirely clear how much information behind the horizon can really be inferred from this procedure: the fact that we can obtain the same correlation function by integrating in the region outside the horizon seems to suggest that no real information behind the horizon can really be contained in these correlatons functions. Similar analysis can be carried out for rotating BTZ’s [36, 37].

The situation for AdS Schwarzchild black holes in higher dimensions is more involved. As was shown in [38], the Penrose diagram is not a square, which results in a contribution from an almost null geodesic in real coordinates which bounces off the singularity at a fixed boundary time $t_c$. This would imply that in the CFT correlation function, there is a light cone singularity at $t = t_c$. This is problematic because such a singularity is ruled out in the CFT on very generic grounds. However, it was shown in [38] that the CFT correlation function is in fact dominated by a complexified geodesic. There is a branch cut in the
CFT correlation function at $t = t_c$, and the information about the black hole singularity is contained in the analytic structure near $t = t_c$.

5.3 Where are the microstates of the black hole?

An important breakthrough in our understanding of black holes was the realization that the horizon area of black holes has all the properties of thermodynamical entropy \cite{[39]}. This seemed to suggest that there exists a large number of microstates (of the order of the exponential of the horizon area in appropriate units) building up the black hole. In some settings, this large number of states can be reproduced from the dual holographic field theory. However, the question remains how all these different states are manifested in terms of the actual gravity description of the black hole.

In a recent series of papers \cite{[40]}, it was suggested that in fact the black hole is not one classical solution having a singularity and a horizon, but rather is a "coarse grained" description of an ensemble of different geometries, each being completely regular, and each corresponding to a microstate in the dual field theory. These geometries are all very similar to each other and to the 'naive' black hole geometry, when probed with particles of large wavelength, but differ in a small region which defines the location of a 'horizon'. One must note that this horizon has nothing to do with the classical horizon we are used to. it is not a special surface and there is no singularity inside it. It is just the characteristic location where all the different geometries start to differ from each other.

Such geometries were actually built for a specific system of branes - the two charge rotating D1-D5 system (characterized by the charges of the branes: $Q_1$ and $Q_5$ respectively and by an angular momentum related parameter $a$), which is supersymmetric (1/4 BPS) \cite{[42]}. At some scaling limit, it describes $AdS_3 \times S^3$, whose holographic dual is a 1+1 dimensional CFT with $SO(4)$ symmetry. For this system the geometries built are asymptotically flat (at $r >> (Q_1 Q_5)^{1/4}$), and have a finite throat, which at some length scales ($(Q_1 Q_5)^{1/4} >> r >> a$) describes an $AdS_3 \times S^3$ geometry, and deeper ($a > r$) describes a different geometry, whose details depend on the specific state associated. These systems are also related by U-duality to the 1/4 BPS supertubes \cite{[41]} and for critical values of the angular momentum have a similar 'blow-up' mechanism \cite{[42]}.

One problematic aspect of these systems is that their macroscopic entropy really vanishes,
as the number of microscopic states, although finite, is too small to give any macroscopic entropy. It would be nice if these ideas could be also shown for the 3-charge D1-D5-momentum system (which is 1/8 BPS), where the macroscopic entropy is nonzero. The main problem is that it is not known how to build a general family of geometries dual to all these microstates. First attempt in this direction have recently been made in [43, 61].

Another approach to counting the geometrical microstates of this system is to try and manipulate the expression for the elliptic genus of the conformal field theory and rewrite it as a sum over different geometries with $\text{AdS}_3 \times S^3$ asymptotics [44, 5].

6 Chern-Simons theory

Pure Gravity in three dimensions has many special and interesting features, making it, on the one hand a convenient laboratory for studying gravity and holography, but on the other hand less generic and harder to generalize to a different number of dimensions. Many of these features are related to the fact that three dimensional gravity can be rewritten as a topological Chern-Simons theory [45].

Starting with the regular Einstein action

$$ S = \frac{1}{16\pi G} \int d^3 x \sqrt{|g|} [R - 2\Lambda], \quad (30) $$

one can change variables from the metric $g_{\mu\nu}$ to the first order forms - the dreibeins $e_\mu^a$ (such that $g_{\mu\nu} = e_\mu^a e_\nu^a$) and the spin connections $\omega_\mu^a \equiv \frac{1}{2} \epsilon_{abc} \omega^b \mu^c$, where the action is re-written as

$$ S = \frac{2}{16\pi G} \int d^3 x [e^a \wedge (d\omega^a + \frac{1}{2} \epsilon_{abc} \omega^b \mu^c) + \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c]. \quad (31) $$

Then changing variables from $e^a, \omega^a$ to $A_{L,R}^a = \omega^a \pm \frac{1}{\sqrt{-\Lambda}} e^a$, the action becomes

$$ S = k_L S_{CS}[A_L] - k_R S_{CS}[A_R], $$

$$ S_{CS}[A] \equiv \frac{1}{2} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (32) $$

and $k_L = \frac{\sqrt{-\Lambda}}{8\pi G}$, $k_R = k_L^*$. In [32] we regard $A_{L,R}$ as taking value in a Lie algebra. In case we are discussing Lorentzian gravity they are two independent 1-forms, each taking values

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8This is also true for three dimensional supergravity [46].
in \( sl(2, R) \), and in case we are discussing Euclidean gravity, they are complex conjugates of each other, taking values in \( sl(2, C) \).

The fact that one can recast pure gravity as a gauge theory which is topological is very appealing. However, the change of variables we presented here between the second and first order formalisms is naive and ignores an important subtlety. Namely, the mapping is not one-to-one, and it is not clear whether one should include also degenerate \( A_{L,R} \) or not. Therefore there might be problems defining the measure in the path integrals for the gravity action and for the Chern-Simons theory. Another possibly related problem is that it seems the Chern-Simons theory cannot account for the black hole entropy (as calculated from its horizon area), and predicts a much smaller number of states \([17]\). One therefore might question whether the theories actually exist. There are also other interesting subtleties related to the AdS/CFT correspondence in this setting, which we would not get into here (see for example \([9]\)).

One interesting aside is that in fact many 3-manifolds admit a hyperbolic structure (i.e. admit metrics of constant negative curvature). For such compact manifolds, there exists a natural complexification of the volume of the manifold, which involves the Chern-Simons topological invariant, and has good analytic properties \([18]\) :

\[
Z(M) \sim \exp[\frac{2}{\pi} \text{Vol}(M) + 4\pi i \text{CS}(M)],
\]

(where above we set \( \Lambda = -1 \)). It is not clear if and what would be the importance and interpretation of this invariant in the context of the AdS/CFT correspondence. For some discussion of this invariant and its possible applications see \([62]\).

7 Time dependence

There are many interesting issues that arise in time dependent or cosmological space times. For example, there are generically multiple natural vacua that we can choose in such backgrounds. Also, most such backgrounds exhibit cosmological particle production. The holographic theory reflects these phenomena in an interesting way. We now discuss an explicit example of such a background.
7.1 AdS bubbles of nothing

By starting with the AdS Schwarzschild solution and performing a double analytic continuation, we obtain interesting time dependent backgrounds which are called AdS bubbles of nothing [49, 50, 51]. The AdS Schwarzschild metric is given in (9). The analytic continuation $t \rightarrow i\chi$ and $\theta \rightarrow i\tau$ yields a time dependent space time which is a vacuum solution to five dimensional gravity with a negative cosmological constant:

$$ds^2 = (1 + \frac{r^2}{l^2} - \frac{2m}{r})d\chi^2 + (1 + \frac{r^2}{l^2} - \frac{2m}{r})^{-1}dr^2 + r^2(-d\tau^2 + \cosh^2 \tau d\phi^2).$$ (34)

This is a smooth spacetime if $\chi$ is periodic with period $\frac{4\pi r_+ l^2}{3r_+^3 + l^2}$. Here $r_+$ is the minimum value of the coordinate $r$ and is the largest positive root of the equation $r_+^3 + l^2 r_+ - 2ml^2 = 0$. For a fixed $\tau$ and for $r > r_+$, the space is basically $S^1_\chi \times S^1_\phi$. As $r \rightarrow r_+$, the circle parameterized by $\chi$ collapses and the circle parameterized by $\phi$ approaches a finite size $r_+^2 \cosh^2 \tau$. This circle is the boundary of a bubble of nothing. The metric on the boundary of the bubble is 2d de Sitter space. This space is asymptotically AdS with the conformal boundary being two dimensional de Sitter space times a circle. So the holographic dual to this bubble lives on dS$_2 \times S^1$ [50].

Similar time dependent spacetimes can be constructed by performing double analytic continuations of AdS-Kerr and Reissner-Nordstrom AdS black holes.

8 Zero and positive cosmological constant

In the previous sections solutions of the Einstein equations with a negative cosmological constant and in particular Anti-de Sitter space played a prominent role. It is an obvious question to what extent similar results can be obtained for spaces with zero or a positive cosmological constant. Much less is known in these cases, and it is in fact not clear to what extent a meaningful holographic duality can be formulated.

8.1 Positive cosmological constant

The maximally symmetric solution of the Einstein equations with a positive cosmological constant is de Sitter space. In Euclidean signature it is simply a $d$-sphere, whereas in
Lorentzian signature it is the time-dependent geometry

\[ ds^2 = -dt^2 + \cosh^2 t \Omega_{d-1}^2, \]

with \( d\Omega_{d-1}^2 \) the metric on a round \((d-1)\)-sphere. The metric (35) describes a sphere that contracts exponentially in the past until it reaches a fixed size, and then expands exponentially again. According to recent experimental data, the present day expanding universe is well described by de Sitter space.

Whether or not quantum gravity (or rather, string theory) on a space like (35) is dual to a field theory of some sort is unclear. Attempts to find such field theories run into various kinds of problems (see e.g. [52]). Unfortunately, despite a lot of recent work, a clean explicit example of a solution to the string theory equations of motion of the form (35) is still lacking. Such a solution would obviously be very helpful in exploring the physics of quantum gravity in de Sitter space.

The space (35) has two boundaries and a cosmological horizon. Associated to the cosmological horizon is a finite Hawking temperature, and in addition it has a finite area, similar to what one has for a black hole horizon. This suggests that there might exist some version of holography which applies to cosmological horizons and associates a finite entropy to them. If one additionally believes that a version of black hole complementarity applies to cosmological horizons - i.e. the Hilbert space of a single observer is sufficient to describe both sides of the horizon - then a dual description of de Sitter space might involve a theory with a finite dimensional Hilbert space. In such a theory there is not enough resolving power to measure arbitrary small distances, and therefore it can at best yield a dual of (35) which is a good description for a finite amount of time but not asymptotically as \( t \to \pm\infty \).

If we try to apply the AdS/CFT philosophy more directly to (35), we should first find all possible solution of the Einstein equations that are asymptotically identical to (35). It is known [53] that for sufficiently small deformations of the boundary metrics a smooth solution with the same asymptotic behavior still exists. However, the situation for large deformations has not been resolved. Under significantly large perturbations de Sitter space can break into pieces and in particular disconnected geometries (for example so-called big bang/crunch geometries) will start to contribute [54, 55]. Therefore it is quite possible that the sum over geometries involved in a putative dS/CFT correspondence will involve a much larger set of metrics and geometries than just (35).
Perhaps a better strategy is to first study some simpler aspects of de Sitter space before engaging in a full-fledged holographic correspondence. For example, whether or not a positive mass theorem for de Sitter space exists and if so what its precise formulation is, is still an open problem. A preliminary positive mass theorem was described in [55], and it was verified in many different examples, but a closer look [56] suggests that its formulation is not quite complete as it stands.

### 8.2 Zero cosmological constant

The case of zero cosmological constant is at least as problematic as the case of a positive cosmological constant. The maximally symmetric solution of the Einstein equations is Minkowski space (or smooth quotients thereof). There have been a few attempts at finding a dual description of quantum gravity in Minkowski space. First, one can try to find a dual description of a subset of Minkowski space by putting suitable “holographic screens” in it, see [57]. This has not led to a concrete dual description however. Another approach involves taking a decompactification limit of AdS/CFT [58]. This turns out to be quite difficult and has not led to a precise dual description either.

A third approach involves the conformal boundary of Minkowski space, in particular past and future null infinity. It has been known for a long time that the asymptotic symmetry group of this boundary is the so-called BMS group, an infinite dimensional group. In the spirit of AdS/CFT, one might try to look for a theory that carries representations of this large group. This is also quite problematic, see [59] for a recent discussion.

A final approach involves slicing of Minkowski space in Anti-de Sitter and de Sitter slices. These slices are given by the equation $\eta_{\mu\nu}x^\mu x^\nu = r$, where $r < 0$ gives rise to AdS slices and $r > 0$ gives rise to dS slices. The case $r = 0$ corresponds to the light-cone. The idea is now to apply holography to each slice separately and then to combine the results. In this way one obtains a holographic dual theory that lives on the boundary of the light-cone, i.e. in two dimensions less, and which has infinitely many degrees of freedom. Although various

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9One problem is that the dS/AdS slicing of Minkowski space doesn’t really survive small deformations. In 3+1 dimensions, there are no deformations which give self-similar slices (i.e slices which only differ from each other by rescaling). Such deformations may exist in higher dimensions, however the global spaces always have a singularity. We thank M. Anderson for his comment regarding this. See [1] for more details.
miracles happen it remains to be seen whether these have essentially a kinematic origin, or whether they reveal some true holographic nature of Minkowski space.

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