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Dynamics of fractional order COVID-19 model with a case study of Saudi Arabia

Yu-Ming Chu a, Aatif Ali b, Muhammad Altaf Khan c,d,e, Saeed Islam b, Saif Ullah e

a Department of Mathematics, Huzhou University, Huzhou 313000, PR China
b Department of Mathematics, Abdul Wali Khan University, Mardan, Khyber Pakhtunkhwa, Pakistan
c Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Viet Nam
d Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City Viet Nam
e Department of Mathematics, University of Peshawar, Khyber Pakhtunkhwa, Pakistan

ABSTRACT

The novel coronavirus disease or COVID-19 is still posing an alarming situation around the globe. The whole world is facing the second wave of this novel pandemic. Recently, the researchers are focused to study the complex dynamics and possible control of this global infection. Mathematical modeling is a useful tool and gains much interest in this regard. In this paper, a fractional-order transmission model is considered to study its dynamical behavior using the real cases reported in Saudi Arabia. The classical Caputo type derivative of fractional order is used in order to formulate the model. The transmission of the infection through the environment is taken into consideration. The documented data since March 02, 2020 up to July 31, 2020 are considered for estimation of parameters of system. We have the estimated basic reproduction number $R_0$ for the data is 1.2937. The Banach fixed point analysis has been used for the existence and uniqueness of the solution. The stability analysis at infection free equilibrium and at the endemic state are presented in details via a nonlinear Lyapunov function in conjunction with LaSalle Invariance Principle. An efficient numerical scheme of Adams-Molten type is implemented for the iterative solution of the model, which plays an important role in determining the impact of control measures and also sensitive parameters that can reduce the infection in the general public and thereby reduce the spread of pandemic as shown graphically. We present some graphical results for the model and the effect of the important sensitive parameters for possible infection minimization in the population.

Introduction

The fatal disease called a coronavirus (COVID-19) has grown beyond the expectation of everyone and seems to stay around for months or even years. The rapid spread of the virus with a high level of severity has become an unprecedented threat for the public health. To understand the mechanism of this virus and its possible eliminations, the researchers and scientists are looking to discover some treatments and vaccinations for its eliminations. The source of transmission is mainly through the droplets and person to person contact. To reduce the infection spread, the wearing of mask, stay at home, social distancing and wash the hands regularly are the highly recommended. Still the coronavirus infection spreading and according to the recent updates, over 20.2 million are infected with 787,909 deaths globally so far [20,13].

Like other countries, the rapid spreading of COVID-19 in Saudi Arabia; in which by the 30 of July 2020, an accounted 288609 confirmed cases of infected people, with 252035 recovered, and 3167 lost their lives as well as the severe economic hardship. The first case reported in Saudi Arabia was on second March 2020 while the second case has been reported on March 14 by the entry of the person from Iran via Bahrain [11]. After these cases, the government make plans and other preventive measure for the infection further spread. In this regard, a strict lock-down and curfew were announced, as a results the educations sector and the business markets were closed, flights suspended, Umrah and Hajj are restricted to a certain number of people. The development of the vaccine and the possible control of the infections, the researchers and the biologists around the globe started thinking to have some vaccine for the disease elimination. The virologists are focusing their attention on vaccine ongoing trials, many to developing the vaccine and also a specific treatment mechanism to prevent the
spread of this deadly virus.

The researchers around the world focus to study the coronavirus infection in different scenarios. Some were studying the statistical tools to understand the infected cases and to find some relationships that can be used further to help the infection minimization. Some studying the virus biologically and to think for some vaccine developments. Besides this, the others who developing mathematical models in terms of mathematics and to predict the infection eradication stage. The mathematical models are used in a lot for many infectious diseases and its outbreak. Like corona infection, the authors used the theory of differential equations to formulate mathematical models with different characteristics and provided useful results for its eliminations.

Mathematical modeling is a valuable tool to study very effectively the disease spread and its control. In this regard the researchers around the world presented many useful mathematical models in the last few decades to study the dynamics of infectious diseases and to determine some useful strategies for the better elimination of infection. The compartmental models along with real cases are more helpful to provide useful information about a particular disease outbreak. Several mathematical models are considered in literature to investigate and analyze the complex transmission pattern of the novel ongoing COVID-19 pandemic, see [18,19,28], using ordinary, stochastic and delay differential equations. More briefly, the COVID-19 dynamics through a mathematical with the help of generalized Mittag-Leffler function is defined as [5] : We give here in brief the essential definitions regarding fractional calculus and the model description in fractional derivative. The results of the system, categorized with the sections is as follows: In Section “Basics of fractional calculus and model descriptions”, we give the basics related to the fractional calculus and the model descriptions. The results of the system, existence and uniqueness shown in Section “Analysis of the fractional model” while the stability analysis of the model are discussed in Section “Iterative solution and stability analysis”. We presented the graphical results with discussion in Section “Results and discussion” and concluding the findings in Section “Conclusion”.

Basics of fractional calculus and model descriptions

We give here in brief the essential definitions regarding fractional calculus and the model description in fractional derivative. The following important definitions are considered that will be utilized as an application to our proposed study.

Definition 2.1. Consider \( y \in C^\alpha \) be function, then the well known classical Caputo derivative having fractional order \( a \) in \((n−1, n] \) where \( n \in \mathbb{N} \) is defined as:

\[
c^D_{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{y'(\tau)\, \tau^{n-\alpha-1}}{(t-\tau)^{\alpha}} \, d\tau.
\]

clearly \( c^D_{\alpha} y(t) \) tends to \( y'(t) \) as \( a \rightarrow 1 \).

Definition 2.2. The Corresponding integral with fractional order \( a > 0 \) of the function \( y : \mathbb{R} \rightarrow \mathbb{R} \) is described as follows:

\[
I^\alpha_{\alpha} y(t) = \frac{1}{\Gamma(a)} \int_0^t \frac{y(\tau)\, \tau^{a-1}}{(t-\tau)^{\alpha}} \, d\tau, \quad 0 < a < 1, \quad t > 0.
\]

Definition 2.3. The Atangana-Baleanu operator (ABC) developed with the help of generalized Mittag-Leffler function is defined as [5] :

\[
^\mathrm{ABC}_{\alpha} D^\alpha_{\alpha} y(t) = \frac{AB\mathrm{C}(\alpha)}{(1 - \alpha)} \int_0^t \frac{y'(\tau)E_{\alpha}[(-\alpha \frac{(t - \tau)}{1 - \alpha})^\alpha]}{\Gamma(\alpha)} \, d\tau, \quad \alpha \in [0, 1].
\]

Definition 2.4. The integral for the Definition 2.3 is given by:

\[
^\mathrm{ABC}_{\alpha} I^\alpha_{\alpha} y(t) = \frac{1 - \alpha}{AB\mathrm{C}(\alpha)} y(t) + \frac{\alpha}{AB\mathrm{C}(\alpha)\Gamma(\alpha)} \int_0^t y(\tau)(t - \tau)^{\alpha-1} \, d\tau,
\]

where \( a \in [0, 1] \).

Definition 2.5. Let \( y^* \) denotes the equilibrium point of the Caputo fractional model then:

\[
c^D_{\alpha} y(t) = h(t, y(t)), \quad \alpha \in (0, 1), \quad i f \quad h(t, y^*) = 0.
\]

Model descriptions

We consider here to generalize the COVID-19 model with the description of memory effects by using the Caputo derivative. For this purpose, we divide the total populations of humans at any time \( t \) in five different sub-classes. These sub-classes include, the susceptible \( S(t) \),
exposed $E(t)$, asymptomatic infected (no clinical symptoms but infect healthy people) $A(t)$, having disease symptoms and infected people or symptomatic infected $I(t)$ and the recovered individuals $R(t)$. The environmental class given by $B(t)$ shows the concentration of virus in the environment. Then, the following is the representations of the corona virus model in terms of fractional differential equations:

$$
\begin{aligned}
\mathcal{C}_D^\alpha S(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} S(\tau) d\tau, \\
\mathcal{C}_D^\alpha E(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} E(\tau) d\tau, \\
\mathcal{C}_D^\alpha I(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} I(\tau) d\tau, \\
\mathcal{C}_D^\alpha A(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} A(\tau) d\tau, \\
\mathcal{C}_D^\alpha R(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} R(\tau) d\tau.
\end{aligned}
$$

(1)

The COVID-19 model in Caputo operator form is:

$$
\begin{aligned}
\mathcal{C}_D^\alpha S(t) &= \Lambda - (\beta_1 E + \beta_2 I + \beta_3 A + \beta_4 B) \frac{S}{N} - \nu S, \\
\mathcal{C}_D^\alpha E(t) &= (\beta_1 E + \beta_2 I + \beta_3 A + \beta_4 B) \frac{S}{N} - (\mu + \nu) E, \\
\mathcal{C}_D^\alpha I(t) &= (1 - \omega) \mu E - (\nu + \psi_1 + \psi_2) I, \\
\mathcal{C}_D^\alpha A(t) &= \omega \mu E - (\nu + \phi_1) A, \\
\mathcal{C}_D^\alpha R(t) &= \phi_1 I + \phi_2 A - \nu R, \\
\mathcal{C}_D^\alpha B(t) &= \phi_3 E + \phi_4 I + \phi_5 A - \psi B,
\end{aligned}
$$

(2)

with the corresponding initial conditions (ICs) defined as:

$$
\begin{aligned}
S(0) &= S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, \\
A(0) &= A_0 \geq 0, R(0) = R_0 \geq 0, B(0) = B_0 \geq 0.
\end{aligned}
$$

(3)

The birth rate of human populations and the natural mortality rate is considered as $\Lambda$ and $\nu$ respectively. The parameters $\beta_i$ for $i = 1, 2, 3$ are used to describe the disease transmission rates through the direct transmission from the exposed, symptomatic and asymptomatic infected individual respectively. Whereas, $\beta_4$ shows the virus transmission rate from the environment or contaminated surfaces. The symptoms develop in exposed individuals at the rate $(1 - \omega) \mu$ they become infected and join $I(t)$ class while the remaining do not show any (or having mild) symptoms to asymptomatic class. The recovery rate in infected and asymptomatic class is $\psi_1$ and $\phi_2$ respectively. The people die at the rate $\nu_i$ from infected compartment. The concentration rate $\psi_i$ for $i = 1, 2, 3$ contributed by $E(t), I(t)$ and asymptomatic infected people to the environment and removal rate from the environment is $\psi$. In system (2), the net human population at time $t$ is described by the term $N(t)$. Moreover, $N(t) = S(t) + E(t) + I(t) + A(t) + R(t)$. By adding them, we observe

$$
\begin{aligned}
\mathcal{C}_D^\alpha N(t) &= \mathcal{C}_D^\alpha S(t) + \mathcal{C}_D^\alpha E(t) + \mathcal{C}_D^\alpha I(t) + \mathcal{C}_D^\alpha A(t) + \mathcal{C}_D^\alpha R(t), \\
&= \Lambda - \nu N - \nu_1 I \leq \Lambda - \nu N,
\end{aligned}
$$

we obtain

$$
N(t) \leq N(0) e^{\nu t} + \Lambda t e^{\nu t}.
$$

Therefore, we have

$$
\lim_{t \to \infty} N(t) \leq \frac{\Lambda}{\nu}. 
$$

The feasible region $\Omega$, shown by

$$\Omega = \left\{ \left( S(t), E(t), I(t), A(t), R(t) \right) \in \mathbb{R}_+^5 : N(t) \leq \frac{\Lambda}{\nu}, B(t) \in \mathbb{R}_+, \right\}.$$

Corollary 1. Suppose, $g(y) \in C[a_1, b_1]$ and $\mathcal{C}_D^\alpha g(y) \in (a_1, b_1)$, where $a \in (0, 1)$. Then if

(i) $\mathcal{C}_D^\alpha g(h) \geq 0$, $\forall h \in (a_1, b_1)$, then $y(h)$ is non-decreasing.

(ii) $\mathcal{C}_D^\alpha g(h) < 0$, $\forall h \in (a_1, b_1)$, then $y(h)$ is non-increasing.
Parameter estimation

In order to parameterize the biological parameters of the proposed model by considering the reported infected cases of corona virus in Saudi Arabia for the given period starting from March 02 till 31 July 2020 (the peak time of COVID-19), we make use of the well-known least square fitting technique. The birth rate denoted by $\Lambda$ and the natural mortality rate $\nu$ are estimated from literature [12] as can be found in the table given below. The total population of KSA is $N(0) = 34813871$ and the average life span is $1/74.87$ years. The other parameters of the model under consideration are estimated from the real data of the aforementioned period in KSA. Consequently, using the estimated and fitted parameters the reproduction number is evaluated &approx; 1.2937. The predicted curve is depicted in Fig. 1, which provides a reasonable fitting curve to the actual cumulative infected cases. The Table 1 describes the parameter values estimated from the real data.

Analysis of the fractional model

In this section, we investigate some basic and necessary mathematical features of the proposed model (2). To present the existence as well as the uniqueness (EU) of the problem solutions, we proceed as follow:

Existence and uniqueness (EU) of the model solution

This subsections explores the EU of the solution for Caputo operator with the help of fixed point theory. For this purpose, let $\mathcal{B}(\mathcal{Z})$ denoting a Banach space consists of real valued continuous functions over the interval $\mathcal{Z} = [0, a]$ with norm defined by $\|f\| = \sup_{t \in \mathcal{Z}} |f(t)|$, $|E| = \sup_{t \in \mathcal{Z}} |E(t)|$, $|I| = \sup_{t \in \mathcal{Z}} |I(t)|$, $|A| = \sup_{t \in \mathcal{Z}} |A(t)|$, $|R| = \sup_{t \in \mathcal{Z}} |R(t)|$ and $|B| = \sup_{t \in \mathcal{Z}} |B(t)|$, and the norm $\|(S, E, I, A, R, B)\| = \|S\| + \|E\| + \|I\| + \|A\| + \|R\| + \|B\|$.

Proof. System (2) gets the following for after utilizing the Caputo integral

$S(t) = S(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_S(t, S(\varsigma))d\varsigma$,

$E(t) = E(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_E(t, E(\varsigma))d\varsigma$,

$I(t) = I(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_I(t, I(\varsigma))d\varsigma$,

$A(t) = A(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_A(t, A(\varsigma))d\varsigma$,

$R(t) = R(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_R(t, R(\varsigma))d\varsigma$,

$B(t) = B(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_B(t, B(\varsigma))d\varsigma$,

Now by definition (2.2),

$$K_S(t, S(t)) = \Lambda - (\beta_1 E + \beta_2 I + \beta_3 A + \beta_4 B) \frac{S}{N} - \nu S,$$

$$K_E(t, E(t)) = (\beta_1 E + \beta_2 I + \beta_3 A + \beta_4 B) \frac{S}{N} - (\mu + \nu) E,$$

$$K_I(t, I(t)) = (1 - \omega) \mu E - (\upsilon + \pi_1 + \phi_1) I,$$

$$K_A(t, A(t)) = \omega \mu E - (\upsilon + \phi_2) A,$$

$$K_R(t, R(t)) = \phi_1 I + \phi_2 A - \nu R,$$

$$K_B(t, B(t)) = \phi_E E + \phi_I I + \phi_A A - \psi B.$$

The expressions $K_S(t, S(t))$, $K_E(t, E(t))$, $K_I(t, I(t))$, $K_A(t, A(t))$ and $K_R(t, R(t))$ and $K_B(t, B(t))$ fulfill the Lipschitz conditions with $S(t), E(t), I(t), A(t), R(t)$ and $B(t)$ having an upper bound. Let the two function $S(t)$ and $S'(t)$ into consideration, and in similar manner for other functions, we have

$$\|K_S(t, S(t)) - K_S(t, S'(t))\| = \frac{1}{\Gamma(\alpha)} \| (t - \varsigma)^{\alpha-1} (K_S(t, S(\varsigma)) - K_S(t, S'(\varsigma))) \|,$$

Now assuming $l_1 = \frac{1}{\Gamma(\alpha)} \| (t - \varsigma)^{\alpha-1} \|$, and continuing in the same way for the remaining equations, we get

$$\|K_S(t, S(t)) - K_S(t, S'(t))\| \leq l_1 \|S(t) - S'(t)\|,$$

$$\|K_E(t, E(t)) - K_E(t, E'(t))\| \leq l_2 \|E(t) - E'(t)\|,$$

$$\|K_I(t, I(t)) - K_I(t, I'(t))\| \leq l_3 \|I(t) - I'(t)\|,$$

$$\|K_A(t, A(t)) - K_A(t, A'(t))\| \leq l_4 \|A(t) - A'(t)\|,$$

$$\|K_R(t, R(t)) - K_R(t, R'(t))\| \leq l_5 \|R(t) - R'(t)\|,$$

$$\|K_B(t, B(t)) - K_B(t, B'(t))\| \leq l_6 \|B(t) - B'(t)\|,$$

where $l_1, l_2, l_3, l_4, l_5$ and $l_6$ denote the respective Lipschitz constants to the functions $K_S, K_E, K_I, K_A, K_R$ and $K_B$. Hence, the Lipschitz condition are satisfied. The equations in (4) can be shown recursively as:

$$S(t) = S(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_S(t, S(\varsigma))d\varsigma,$$

$$E(t) = E(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_E(t, E(\varsigma))d\varsigma,$$

$$I(t) = I(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_I(t, I(\varsigma))d\varsigma,$$

$$A(t) = A(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_A(t, A(\varsigma))d\varsigma,$$

$$R(t) = R(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_R(t, R(\varsigma))d\varsigma,$$

$$B(t) = B(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varsigma)^{\alpha-1} K_B(t, B(\varsigma))d\varsigma,$$

where the kernels are,
It is noted that, \( S_0(t) = \sum_{j=0}^{n} \Phi_j(t), E_0(t) = \sum_{j=0}^{n} \Phi_j(t), I_0(t) = \sum_{j=0}^{n} \Phi_j(t) \), \( A_0(t) = \sum_{j=0}^{n} \Phi_j(t), R_0(t) = \sum_{j=0}^{n} \Phi_j(t) \). Now, we consider that:

\[
\Phi_{S_{n-1}}(t) = S_{n-1}(t) - S_{n-2}(t), \quad \Phi_{E_{n-1}}(t) = E_{n-1}(t) - E_{n-2}(t), \\
\Phi_{I_{n-1}}(t) = I_{n-1}(t) - I_{n-2}(t), \quad \Phi_{A_{n-1}}(t) = A_{n-1}(t) - A_{n-2}(t), \\
\Phi_{R_{n-1}}(t) = R_{n-1}(t) - R_{n-2}(t), \quad \Phi_{B_{n-1}}(t) = B_{n-1}(t) - B_{n-2}(t),
\]

we get the following,

\[
\|\Phi_{S_{n}}(t)\| \leq \frac{1}{\Gamma(a)} \int_0^t (t - \xi)^{a-1} \|\Phi_{S_{n-1}}(\xi)\| d\xi,
\]

\[
\|\Phi_{E_{n}}(t)\| \leq \frac{1}{\Gamma(a)} \int_0^t (t - \xi)^{a-1} \|\Phi_{E_{n-1}}(\xi)\| d\xi,
\]

\[
\|\Phi_{I_{n}}(t)\| \leq \frac{1}{\Gamma(a)} \int_0^t (t - \xi)^{a-1} \|\Phi_{I_{n-1}}(\xi)\| d\xi,
\]

\[
\|\Phi_{A_{n}}(t)\| \leq \frac{1}{\Gamma(a)} \int_0^t (t - \xi)^{a-1} \|\Phi_{A_{n-1}}(\xi)\| d\xi,
\]

\[
\|\Phi_{R_{n}}(t)\| \leq \frac{1}{\Gamma(a)} \int_0^t (t - \xi)^{a-1} \|\Phi_{R_{n-1}}(\xi)\| d\xi,
\]

\[
\|\Phi_{B_{n}}(t)\| \leq \frac{1}{\Gamma(a)} \int_0^t (t - \xi)^{a-1} \|\Phi_{B_{n-1}}(\xi)\| d\xi.
\]

Hence, \( S(t), E(t), I(t), A(t), R(t) \) and \( B(t) \) describe the bounded functions and the expressions \( K_1, K_2, K_3, K_4, K_5 \) and \( K_6 \) satisfy the Lipschitz condition. \( \square \)

**Theorem 3.1.** The Caputo COVID-19 epidemic model has a unique solution for \( t \in [0, a] \) if

\[
\frac{1}{\Gamma(a)}< m < 1, \quad j = 1, ..., 6.
\]

**Proof.** It is observed that \( S(t), E(t), I(t), A(t), R(t) \) and \( B(t) \) are bounded and the expressions \( K_1, K_2, K_3, K_4, K_5 \) and \( K_6 \) fulfill the Lipschitz condition. By recursive principle the above equation implies:

\[
\|\Phi_{S_{n}}(t)\| \leq \|S_0(t)\| \left( \frac{m}{\Gamma(a)} \right)^n,
\]

\[
\|\Phi_{E_{n}}(t)\| \leq \|E_0(t)\| \left( \frac{m}{\Gamma(a)} \right)^n,
\]

\[
\|\Phi_{I_{n}}(t)\| \leq \|I_0(t)\| \left( \frac{m}{\Gamma(a)} \right)^n,
\]

\[
\|\Phi_{A_{n}}(t)\| \leq \|A_0(t)\| \left( \frac{m}{\Gamma(a)} \right)^n,
\]

\[
\|\Phi_{R_{n}}(t)\| \leq \|R_0(t)\| \left( \frac{m}{\Gamma(a)} \right)^n,
\]

\[
\|\Phi_{B_{n}}(t)\| \leq \|B_0(t)\| \left( \frac{m}{\Gamma(a)} \right)^n.
\]

Therefore the sequence exist and obey the conditions describes \( \|\Phi_{S_{n}}(t)\| \to 0, \|\Phi_{E_{n}}(t)\| \to 0, \|\Phi_{I_{n}}(t)\| \to 0, \|\Phi_{A_{n}}(t)\| \to 0, \) and \( \|\Phi_{B_{n}}(t)\| \to 0, \) as \( n \to \infty. \)

**Iterative solution and stability analysis**

To establish the results in details, we provide the following results:

**Theorem 4.1.** Let \( (\mathcal{B}, \|\cdot\|) \) denoting a Banach space and \( G' \) defines a self map on \( \mathcal{B} \). Further, \( z_{n+1} = h(G', z_n) \) shows the recursive expression while \( \mathcal{F}(G') \) denoting the fixed point set upon \( G' \). Further, by defining \( \|y_{n+1} - h(G', y_n)\| \) such that \( \{y_n\} \subseteq \mathcal{B} \). Then, the iterative approach, \( y_{n+1} = h(G', y_n) \) is stable if \( \lim_{n \to \infty} c_n = 0 \), that is \( \lim_{n \to \infty} c_n = p' \) for \( z_{n+1} = G' \), where, \( n \) is taken as the Picard’s iteration then \( G' \) iteration is stable. The theorem can be summarized as below:

\( (\mathcal{B}, \|\cdot\|) \) defines a Banach space and \( G' \) be a self-map upon \( \mathcal{B} \), then for all \( x, y \in \mathcal{B} \), we have

\[
S_{n+1}(t) = S_n(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \lambda (\beta_i E + \beta_j I + \beta_y A + \beta_B B) \frac{S}{N} - \lambda S \right\} \right\},
\]

\[
E_{n+1}(t) = E_n(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ (\beta_i E + \beta_j I + \beta_y A + \beta_B B) \frac{S}{N} - (\mu + \nu)E \right\} \right\},
\]

\[
I_{n+1}(t) = I_n(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ (1 - \omega)\mu E - (\nu + \phi_1)I \right\} \right\},
\]

\[
A_{n+1}(t) = A_n(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \alpha (\mu E - (\nu + \phi_1)A) \right\} \right\},
\]

\[
R_{n+1}(t) = R_n(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \phi_i I + \phi_2 A - \nu R \right\} \right\},
\]

\[
B_{n+1}(t) = B_n(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \phi_i E + \phi_3 I + \phi_4 A - \psi B \right\} \right\}.
\]

Let \( \mathcal{F} \) defines a self map, then we have

\[
\mathcal{F}[S_{n+1}(t)] = S_{n+1}(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \lambda (\beta_i E + \beta_j I + \beta_y A + \beta_B B) \frac{S}{N} - \lambda S \right\} \right\},
\]

\[
\mathcal{F}[E_{n+1}(t)] = E_{n+1}(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ (\beta_i E + \beta_j I + \beta_y A + \beta_B B) \frac{S}{N} - (\mu + \nu)E \right\} \right\},
\]

\[
\mathcal{F}[I_{n+1}(t)] = I_{n+1}(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ (1 - \omega)\mu E - (\nu + \phi_1)I \right\} \right\},
\]

\[
\mathcal{F}[A_{n+1}(t)] = A_{n+1}(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \alpha (\mu E - (\nu + \phi_1)A) \right\} \right\},
\]

\[
\mathcal{F}[R_{n+1}(t)] = R_{n+1}(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \phi_i I + \phi_2 A - \nu R \right\} \right\},
\]

\[
\mathcal{F}[B_{n+1}(t)] = B_{n+1}(t) + \mathcal{L}^{-1} \left\{ \frac{1}{\mathcal{Z}} \left\{ \phi_i E + \phi_3 I + \phi_4 A - \psi B \right\} \right\}.
\]
\( \mathcal{P} \) is stable if the following conditions are satisfied:
\[
\{ 1 - \beta_1 (Q + Q_0) f_j - \beta_2 (Q + Q_0) f_j - \beta_3 (Q + Q_0) f_j - \beta_4 (Q + Q_0) f_j = -K_1 \} \cap
\{ 1 - \beta_1 (Q + Q_0) f_i + \beta_2 (Q + Q_0) f_i + \beta_3 (Q + Q_0) f_i + \beta_4 (Q + Q_0) f_i = -K_2 \} \cap
\{ 1 - ((1 - \omega) \mu - (\omega + \nu + \phi_2)) f_j \} \cap \{ 1 - (\omega - (\omega + \phi_2)) f_i \} (1, \{ 1 - (\nu + \phi_2) f_i \} (1, \{ 1 - (\phi_1 + \phi_2 - \nu) K_3 \}) (1, \{ 1 - K_0 (\phi_1 + \phi_2 - \nu) = -K_3 \}.
\]

**Proof.** Since the map \( \mathcal{P} \) is a fixed point, therefore, we evaluate the following equation
\[
\begin{align*}
\mathcal{P}[S(t)] - \mathcal{P}[S_0(t)] &= S_0(t) - S_0(t), \\
\mathcal{P}[E(t)] - \mathcal{P}[E_0(t)] &= E_0(t) - E_0(t), \\
\mathcal{P}[I(t)] - \mathcal{P}[I_0(t)] &= I_0(t) - I_0(t), \\
\mathcal{P}[A(t)] - \mathcal{P}[A_0(t)] &= A_0(t) - A_0(t), \\
\mathcal{P}[R(t)] - \mathcal{P}[R_0(t)] &= R_0(t) - R_0(t), \\
\mathcal{P}[B(t)] - \mathcal{P}[B_0(t)] &= B_0(t) - B_0(t).
\end{align*}
\]
Taking the norm of both sides of above equation
\[
\| \mathcal{P}[S(t)] - \mathcal{P}[S_0(t)] \| = \| \mathcal{P}[E(t)] - \mathcal{P}[E_0(t)] \| = \| \mathcal{P}[I(t)] - \mathcal{P}[I_0(t)] \| = \| \mathcal{P}[A(t)] - \mathcal{P}[A_0(t)] \| = \| \mathcal{P}[R(t)] - \mathcal{P}[R_0(t)] \| = \| \mathcal{P}[B(t)] - \mathcal{P}[B_0(t)] \|.
\]
After simplification, we have
\[
\| \mathcal{P}[S(t)] - \mathcal{P}[S_0(t)] \| \leq \| S_0(t) - S_0(t) \| + \left\{ \frac{1}{\lambda} \left[ \| \beta_1 E_0 (t) - E_0(t) \| + \| \beta_2 I_0 (t) - I_0(t) \| + \| \beta_3 A_0 (t) - A_0(t) \| + \| \beta_4 B_0 (t) - B_0(t) \| \right] \right\}.
\]
Now it assumed that
\[
\begin{align*}
\| E_0(t) - E_0(t) \| &\leq \| S_0(t) - S_0(t) \|, \\
\| I_0(t) - I_0(t) \| &\leq \| S_0(t) - S_0(t) \|, \\
\| A_0(t) - A_0(t) \| &\leq \| S_0(t) - S_0(t) \|, \\
\| B_0(t) - B_0(t) \| &\leq \| S_0(t) - S_0(t) \|,
\end{align*}
\]
after substituting the above relation, we have
\[
\| \mathcal{P}[S(t)] - \mathcal{P}[S_0(t)] \| \leq \| S_0(t) - S_0(t) \| + \left\{ \frac{1}{\lambda} \left[ \| \beta_1 E_0 (t) \| + \| \beta_2 I_0 (t) \| + \| \beta_3 A_0 (t) \| + \| \beta_4 B_0 (t) \| \right] \right\}.
\]
Because the sequences \( S_0(t), E_0(t), I_0(t), A_0(t), \) and \( B_0(t) \) are convergent and bounded, their exist five different constant \( S_0 > 0, S_0 > 0, S_0 > 0, \) and \( S_0 > 0 \) for all \( t \). Hence we have \( \| S_0(t) \| < \| S_0 \|, \| E_0(t) \| < \| E_0 \|, \| I_0(t) \| < \| I_0 \|, \| A_0(t) \| < \| A_0 \|, \| B_0(t) \| < \| B_0 \|, (m, n) \in \mathbb{N} \times \mathbb{N} \).

Thus, the proof is complete. \( \square \)

**Disease free equilibrium (DFE) and the basic reproduction number**

We obtain the basic reproduction number of the system (2) and its equilibrium point at the disease free case. Let the disease free equilibrium DFE of the system (2) denoted by \( \mathcal{X}_0 = (S^0, E^0, I^0, A^0, R^0, B^0) \), and can be obtained by setting, \( C^D_1 S(t) = C^D_1 E(t) = C^D_1 I(t) = C^D_1 A(t) = C^D_1 R(t) = C^D_1 B(t) = 0 \). Solving these equations at the disease free state, we get
\[
\mathcal{X}_0 = \left( \frac{\lambda}{\nu}, 0, 0, 0, 0, 0 \right).
\]
In order to get the expression for the basic reproduction number \( \mathcal{R}_0 \) of the system (2), we apply the technique known as the next generation method [26]. The well-known next-generation procedure is utilized in order to derive the basic reproductive number \( \mathcal{R}_0 \) for the model (2). Let \( x = (E, I, A, B)^T \), then the necessary matrices are
\[
\mathcal{X} = \begin{pmatrix}
\beta_1 E + \beta_2 I + \beta_3 A + \beta_4 B \bar{S} \\
N \\
0 \\
0 \\
0 \\
0
\end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix}
\nu + \phi_1 \phi_2 + (1 - \omega) \mu E \\
(\nu + \phi_1) A - \alpha \nu E \\
\phi_1 \phi_2 \psi B - \phi_1 \phi_2 \psi I - \phi_1 \phi_2 \psi A
\end{pmatrix}.
\]
Now, for linearization the Jacobian of above matrices at disease free-state is:
\[
\mathcal{F} = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix}
\mu + \nu & 0 & 0 & 0 \\
-\nu \omega & 0 & \nu + \phi_1 & 0 \\
0 & -\nu \phi_1 & -\nu \phi_2 & -\nu \phi_3 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
We have
\[
\mathcal{F} \mathcal{V}^{-1} = \begin{pmatrix}
\beta_1 + \frac{k_1 \mu \nu (1 - \omega) k_2}{k_1 k_2 k_3} & \alpha \nu \mu \phi_1 k_1 k_3 & \beta_2 + \frac{k_2 \phi_1 k_2}{k_1 k_3} & \beta_3 + \frac{k_3 \phi_2}{k_1 k_3} & \beta_4 + \frac{k_3 \phi_3}{k_1 k_3} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Where, for simplicity we take
\[ k_1 = \mu + \nu, \quad k_2 = \nu + k_1 + \phi_1, \quad k_3 = \nu + \phi_2, \quad d_1 = \beta_2 k_1 k_2 \phi_1 + k_2 \mu \phi_2 + \phi_2 \mu \phi_3. \]

Thus, the expression of \( R_0 \) in term of model parameters is obtained as:

\[ R_0 = \frac{k_1 \mu \alpha (\beta_1 \phi_1 + \beta_2 \psi) + k_2 \beta_2 (\beta_1 \phi_1 + \beta_2 \psi) + \mu k_1 (1 - \alpha) (\beta_1 \phi_2 + \beta_2 \psi)}{k_1 k_2 \psi}. \]

where

\[ R_1 = \frac{\mu (1 - \alpha) (\beta_1 \phi_1 + \beta_2 \psi)}{k_1 k_2 \psi}, \quad R_2 = \frac{\mu \omega (\beta_1 \phi_1 + \beta_2 \psi)}{k_1 k_2 \psi}, \quad R_3 = \frac{\beta_1 \phi_1 + \beta_2 \psi}{k_1 \psi}. \]

Local stability of DFE

In order to study the local asymptotical stability of the system (2), we give the following result:

**Theorem 4.2.** For any two positive integers \( n_1, n_2 \) with \( \gcd(n_1, n_2) = 1 \) where \( \alpha = \frac{n_1}{n_2} \) and \( M = n_2 \). Then, the system (2) is GAS if \( \frac{\alpha}{M} > \frac{\beta}{2} \).

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\[ t \to \infty \text{ in } \Omega. \] So, the system (2) at \( \mathcal{R}_0 > 1 \) is GAS. \( \square \)

**The endemic equilibrium**

In order to find the endemic equilibrium (EE) of the model (2), we denote it by

\[ E^* = (S^*, E^*, I^*, A^*, R^*, B^*), \]

where, \( N^* = (S^* + E^* + I^* + A^* + R^*) \). The system (2) gives the following results at a steady state:

\[ \begin{align*}
S^* &= \frac{\Lambda}{\alpha + \nu} \\
E^* &= \frac{x^* \Lambda}{k_1(\alpha + \nu)} = \frac{x^* S^*}{k_1} \\
I^* &= \frac{\mu(s)(1-\alpha)\Lambda}{k_1 k_2 (\alpha + \nu)} = \frac{(1-\alpha)\mu E^*}{k_2}, \\
A^* &= \frac{\mu_0 \nu \Lambda}{k_3 k_4 (\alpha + \nu)} = \frac{\mu_0 \nu E^*}{k_3 k_4 (\alpha + \nu)}, \\
R^* &= \frac{\phi I^* + \phi_2 A^*}{\nu} = \frac{(\phi_1 (1-\alpha) + \phi_2 \mu \nu) s^* S^*}{\nu k_3 k_4 (\alpha + \nu)}, \\
B^* &= \frac{\phi_3 E^* + \phi_4 I^* + \phi_5 A^*}{\nu}.
\end{align*} \] (14)

Further, at endemic steady-state,
\[ \lambda^* = \frac{\beta_1 E^* + \beta_2 I^* + \beta_3 A^* + \beta_4 B^*}{N^*}. \] (15)

Substituting (14) in (15), and after simplification, we get

\[ F(\lambda^*) = d_1 \lambda^* + d_2, \]

where,
\[ \begin{align*}
d_1 &= \psi (\mu k_1 (1 - \alpha) (\phi_1 + \nu) + k_2 \mu_0 (\phi_1 + \nu) + \nu k_3)) \\
d_2 &= \nu k_3 k_4 (\alpha + \nu (1 - \phi_3)).
\end{align*} \]

If \( \mathcal{R}_0 > 1 \), then a unique EE exist.

**Lemma 4.2.** The system (2) has a unique EE \( (\mathcal{R}_0^*) \) whenever \( \mathcal{R}_0 > 1 \).

**Global asymptotical stability (GAS) of EE**

We wish to establish the global asymptotical stability of the system (2) for the special case when the contact rate \( (\beta_k) \) is 0, and induced mortality is negligible \( (\nu_1) \) is 0, then the following modal is presented:

\[
\begin{align*}
\frac{dI^*}{dt} &= \left( 1 - S^* \right) \frac{S^*}{S} D_I^* S(t) + \left( 1 - E^* \right) \frac{E^*}{E} D_I^* E(t) + \frac{k_1}{1 - \alpha \mu} \left( 1 - \frac{I^*}{I} \right) D_I^* I(t) \\
&\quad + \frac{k_1}{1 - \alpha \mu} \left( 1 - \frac{A^*}{A} \right) D_I^* A(t),
\end{align*} \]

where
\[ \begin{align*}
&\left( 1 - S^* \right) \frac{S^*}{S} (\Lambda - \lambda_i S - \kappa S) + \left( 1 - E^* \right) \frac{E^*}{E} (\lambda_i S - k_i E) \\
&\quad + \frac{k_1}{1 - \alpha \mu} \left( 1 - \frac{I^*}{I} \right) ((1 - \alpha) \mu E - k_3 I) + \frac{k_1}{1 - \alpha \mu} \left( 1 - \frac{A^*}{A} \right) (\omega E - k_4 A).
\end{align*} \]
Fig. 2. Simulation of the system (2) for various values of $\alpha$. 
From (19), we have

\[ (1 - S^{\ast\ast}) C D_{t}^{\alpha} S(t) = \left(1 - S^{\ast\ast}\right) \left( (\lambda_{i} + \nu) S^{\ast\ast} - (\lambda_{i} + \nu)S \right), \]

\[ = \beta_{1} S^{\ast\ast} E_{1} E_{1}^{\ast\ast} \left(1 - \frac{S^{\ast\ast}}{S} \frac{ES}{E^{\ast\ast}} + E^{\ast\ast} \right) \]

\[ + \beta_{2} S^{\ast\ast} I_{1}^{\ast\ast} \left(1 - \frac{S^{\ast\ast}}{S} I_{1} S^{\ast\ast} + I_{1} \right) \]

\[ + \beta_{3} S^{\ast\ast} A_{1}^{\ast\ast} \left(1 - \frac{S^{\ast\ast}}{S} \frac{AS}{A^{\ast\ast}} + A^{\ast\ast} \right) \]

\[ + \nu S^{\ast\ast} \left(2 - \frac{S^{\ast\ast}}{S} - \frac{S^{\ast\ast}}{S^{\ast\ast}} \right). \]

(20)

After substituting, we get

\[ C D_{t}^{\alpha} S(t) = \beta_{1} S^{\ast\ast} E_{1} E_{1}^{\ast\ast} \left(1 - \frac{S^{\ast\ast}}{S} \frac{ES}{E^{\ast\ast}} + E^{\ast\ast} \right) \]

\[ + \beta_{2} S^{\ast\ast} I_{1}^{\ast\ast} \left(1 - \frac{S^{\ast\ast}}{S} I_{1} S^{\ast\ast} + I_{1} \right) \]

\[ + \beta_{3} S^{\ast\ast} A_{1}^{\ast\ast} \left(1 - \frac{S^{\ast\ast}}{S} \frac{AS}{A^{\ast\ast}} + A^{\ast\ast} \right) \]

\[ + \nu S^{\ast\ast} \left(2 - \frac{S^{\ast\ast}}{S} - \frac{S^{\ast\ast}}{S^{\ast\ast}} \right). \]

(24)

We have the following fact,

\[ 2 - \frac{S^{\ast\ast}}{S} - \frac{S^{\ast\ast}}{S^{\ast\ast}} \leq 0. \]

Further, if

\[ 4 - \frac{S^{\ast\ast}}{S} A_{1}^{\ast\ast} E^{\ast\ast} IS^{\ast\ast} E_{1}^{\ast\ast} E_{2}^{\ast\ast} E_{3}^{\ast\ast} E_{4}^{\ast\ast} \leq 0, \]

\[ 4 - \frac{S^{\ast\ast}}{S} I_{1}^{\ast\ast} A_{1}^{\ast\ast} S^{\ast\ast} E_{1}^{\ast\ast} E_{2}^{\ast\ast} E_{3}^{\ast\ast} E_{4}^{\ast\ast} \leq 0, \]

\[ 4 - \frac{S^{\ast\ast}}{S} I_{1}^{\ast\ast} A_{1}^{\ast\ast} S^{\ast\ast} E_{1}^{\ast\ast} E_{2}^{\ast\ast} E_{3}^{\ast\ast} E_{4}^{\ast\ast} \leq 0. \]

(25)

Hence, \( C D_{t}^{\alpha} S(t) \) belongs to the its unique endemic equilibrium. So, whenever \( \mathcal{R}_{0} > 1 \), the system (16) at \( (S_{E_{1}}^{\ast\ast}) \) is globally asymptotically stable. \( \square \)

Conjecture 1. The unique EE of the system described in (2) is GAS in \( \Omega \), if \( \mathcal{R}_{0} > 1 \), in \( \Omega \).

Results and discussion

We present the simulation and discussion for the Caputo COVID-19 model (2) in the given section. We consider the numerical results given in Table 1 for the numerical results. The proposed fractional model is solved numerically using the method described in details in [10,14]. The biological parameters estimated from COVID-19 confirmed cases and tabulated in Table 1 to assess the dynamics of novel COVID-19 pandemic in KSA. In order to analyze the role of different parameters and memory index in the dynamical behavior of the disease incidence, we varied various values of the key parameters of the model and the \( \alpha \). The dynamics of proposed COVID-19 model (2) are shown graphically by taking the time unit in days. Fig. 2 describes the model differ component solution graphically for various values of \( \alpha \). It can be seen that the healthy individuators increases while the non-healthy individuals that shown by different compartments are decreasing with the decrease of the value of \( \alpha \). The influence of the contact rates \( \beta_{1} \) and \( \beta_{2} \) on the dynamics of only symptomatically-infected people are illustrated in Fig. 3 and 4 respectively. This behavior is interpreted for two values of the order of Caputo operator i.e., \( \alpha = 1, 0.90 \). It is found that by decreasing the contact rates to its estimated baseline for two distinct values of \( \alpha \), we can see a significant decrease. Although, a comparative faster decrease is observed in the infected population for \( \alpha = 0.90 \). But, the peaks of infection curves occur at the longer period of time in this case. The role of viral concentration on the COVID-19 incidence is depicted in Fig. 5. By decreasing the virus concentration rate, one can
Fig. 3. The impact of Contact rate $\beta_1$ on infected COVID-19 individuals for $\alpha = 1, \alpha = 0.90$.

Fig. 4. The impact of Contact rate $\beta_4$ on infected COVID-19 individuals for $\alpha = 1, \alpha = 0.90$.

Fig. 5. The impact of virus concentration ($\phi_1 = 0.2574, 0.2074, 0.1574, 0.1074$) on infected individuals for $\alpha = 1, \alpha = 0.90$. 
Fig. 6. The impact of $\psi$ on infected individuals for $\alpha = 1, \alpha = 0.85, \alpha = 0.75, \alpha = 0.65$.

Fig. 7. The impact of $\beta_1$ on cumulative symptomatic and asymptomatic COVID-19 individuals for $\alpha = 1, \alpha = 0.90$. 
Fig. 8. The impact of $\beta_4$ on cumulative symptomatic and asymptomatic individuals for $\alpha = 1, \alpha = 0.90$.

(a) \hspace{1cm} (b)

Fig. 9. The parameter $\phi_1 = 0.2574, 0.2074, 0.1574, 0.1074$ impact on cumulative infected and asymptomatic cases for $\alpha = 1, \alpha = 0.95, \alpha = 0.90$.

(a) \hspace{1cm} (b) \hspace{1cm} (c)
observe the decrease in the infected compartments as shown in Fig. 5. Fig. 6 shows that decay in the infected population due to decreasing the removal rate $\psi$. The behavior is even more prominent for a smaller value of fractional order $\alpha$ as can be seen in (b-d).

Moreover, we also investigated the impact of some parameters on the dynamical behavior of cumulative symptomatic and asymptomatic infected cases. The impact of contact rates $\beta_1$ and $\beta_4$ on the cumulative symptomatic and asymptomatic people is shown in Fig. 7 and Fig. 8 respectively. The reduction in these parameters can reduce the cumulative symptomatic and asymptomatic people significantly as can be seen in Fig. 7 and 8. This reveals that the disease incidence can be reduced following protective measures i.e., using a facemask, sanitizer, glues, etc. through disinfection spry in order to reduce the environmental viral load. Furthermore, the decrease in the cumulative symptomatic and asymptomatic population on the reduction in the virus concentration rate $\varphi_1$ due to exposed people is depicted in 9. This behavior becomes more prominent for smaller values of $\alpha$ as shown in 9 (b-c). From Fig. 10 the role of removal rate of the virus from the environmental (via disinfection spray etc.), is observed and seems to be more feasible for small values of the fractional operator $\alpha$.

**Conclusion**

We established a mathematical model for the understanding the corona virus infection in the Kingdom of Saudi Arabia through a fractional mathematical model in Caputo sense. We considered the real cases reported in the Kingdom have been analyzed and obtained a reasonable fit to the data. The environmental impact on the COVID-19 incidence is taken into consideration. Initially, the model in integer order were considered and then the fractional order operator with the power law kernel were applied for its generalization. We provided the related properties for the fractional model. The threshold quantity $R_0$ and parameters were estimated from the reported COVID-19 cases in Saudi Arabia with the help of nonlinear least square procedure and we found that $R_0 = 1.2937$. The stability analysis at the DFE and EEP are explored in detailed. We used a numerical scheme for the solution of the fractional order model and presented the graphical results. The predictor corrector scheme of Adams Molten type is utilized and studied the influence of various parameters by varying to its baseline value for distinct values of $\alpha$. The dramatic reduction in the disease burden was observed with an enhancement in contact tracing policy. The impact of variation in the environmental viral load due to symptomatic and asymptomatic COVID-19 infected individuals is analyzed graphically. It
is observed that by reducing the viral contribution in the environment by asymptomatic infected individuals (i.e., $\phi_3$) decrease the disease burden significantly. We also depicted the impact of variation in the removal rate of virus from the environment (or surfaces) graphically on the disease prevalence. In overall simulation results, that the sensitive parameters decrease the infection in the population very fast. Such important parameters that can be considered as a control for the infection eradication in the population. The present work can be extended by using the cases in the state of Saudi Arabia with the second wave in more recent introduced fractal-fractional operators.

CRediT authorship contribution statement

Yu-Ming Chu: Writing - review & editing, Conceptualization, Methodology, Visualization. Aatif Ali: Conceptualization, Methodology, Visualization, Investigation. Muhammad Altaf Khan: Conceptualization, Visualization, Writing - review & editing. Saeed Islam: Visualization, Writing - review & editing. Saif Ullah: Writing - original draft, Conceptualization, Methodology, Visualization, Investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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