Article

An Averaging Principle for Stochastic Fractional Differential Equations Driven by fBm Involving Impulses

Jiankang Liu 1*, Wei Wei 2 and Wei Xu 2

1 School of Applied Science, Taiyuan University of Science and Technology, Taiyuan 030024, China
2 School of Mathematics and Statistics, Northwestern Polytechnical University, Xi’an 710129, China; wewei_nwpu@mail.nwpu.edu.cn (W.W); weixu@nwpu.edu.cn (W.X.)
* Correspondence: liujiankang@tyust.edu.cn

Abstract: In contrast to previous research on periodic averaging principles for various types of impulsive stochastic differential equations (ISDEs), we establish an averaging principle without periodic assumptions of coefficients and impulses for impulsive stochastic fractional differential equations (ISFDEs) excited by fractional Brownian motion (fBm). Under appropriate conditions, we demonstrate that the mild solution of the original equation is approximately equivalent to that of the reduced averaged equation without impulses. The obtained convergence result guarantees that one can study the complex system through the simplified system. Better yet, our techniques dealing with multi-time scales and impulsive terms can be applied to improve some existing results. As for application, three examples are worked out to explain the procedure and validity of the proposed averaging principles.

Keywords: stochastic fractional differential equations; impulsive dynamical systems; averaging principle; fractional Brownian motion; multi-time scale; convergence results

1. Introduction

In the past few decades, fractional calculus and fractional dynamical systems have gained widespread concern from scholars in various research fields. Different researchers explored fractional dynamical systems from both theoretical and practical aspects due to their applications and prospects in many scientific areas [1–3]. Specifically, taking into account the inevitable uncertainties and random factors, massive research achievements on stochastic fractional differential equations have emerged one after another [4–6]. Besides, some real-world systems are affected by instantaneous disturbances or undergo sudden changes, and impulsive differential equations are naturally used to characterize them. A wide range of mathematical models in the study of control theory, telecommunications, biology, ecology, epidemiology, finance and economics can be described by integer or fractional-order impulsive stochastic differential equations (ISDEs) [7,8].

As we know, random perturbations with LRD (long-range dependence) exist in various application fields; for instance, hydrology, meteorology, physical chemistry, bioengineering, mathematical finance, etc. [9]. Fractional Brownian motion has a long correlation time and can be used to model these stochastic excitations. Therefore, various types of stochastic fractional differential equations (SFDEs) excited by fBm are being emphasized and some preliminary study results [10–12] have been acquired. Meanwhile, some recent research [13,14] considered the well-posedness and controllability of different classes of ISFDEs. In particular, Pedjeu and Ladde [15] introduce a stochastic model under multi-time scales, which can model the complex dynamical processes in sciences, engineering, ecology and epidemiology. Correspondingly, considering that the correlated noises can not be ignored, Abouagwa, Cheng and Li [16] establish the existence and uniqueness results of SFDEs under fBm involving impulses based on Carathéodory successive approximation method.

On the other hand, the averaging method is an effective approach to explore different types of non-linear dynamical systems. This is because you can concentrate on the...
reduced autonomous averaged equation instead of the original complex time-varying one, providing an approximate way to remove some of the complexity. The core of demonstrating an averaging principle is examining the conditions when the averaged system is equivalent to the original system in some sense. Since Khasminskii’s contribution [17], stochastic averaging methods for SDEs have been widely developed and applied [18,19]. However, most of the noises they had considered are uncorrelated. Such results ruled out these stochastic differential systems under the random perturbations with long-term dependence. Fortunately, Pei, Xu and Guo [20,21] have recently obtained the averaging principles for several different kinds of SDEs under fBm with Hurst index $H \in (\frac{1}{2}, 1)$. With the development of research on ISDEs, in two recent years, a few study achievements on periodic averaging methods of ISDEs have appeared [22–25]. There are two disadvantages of this research. One is that the periodic assumptions of the coefficient functions and the impulses are strict; the other is the estimate of the difference between the impulsive term of the original equation and the impulsive integral term of the averaged equation is flawed. Here, it is noted that Liu and Xu [26] proved an averaging theorem for impulsive stochastic partial differential equations without periodic assumptions. In light of complex characteristics of stochastic fractional differential systems, Xu et al. [27] developed stochastic averaging for several classes of SFDEs driven by different noises. Abouagwa and Li [28] considered an averaging method for SFDEs under a non-Lipschitz condition. Luo and Zhu [29] discussed an averaging principle for SFDEs with delay. Shen et al. [30] showed an averaging theorem under a weaker condition for a kind of Caputo SFDEs. Guo et al. [31] examined an averaging theorem for SFDEs under a non-Lipschitz condition. Luo and Zhu [29] discussed an averaging principle for SFDEs with delay. Shen et al. [30] showed an averaging principle for SFDEs with Lévy noise and Markovian switching. Guo et al. [31] examined an averaging theorem under a weaker condition for a kind of Caputo SFDEs. Liu and Xu [32] proved an averaging principle for neutral ISFDEs. These works offer possible ways to reduce various types of SFDEs. However, these papers do not involve the partial differential equations without periodic assumptions. In light of complex characteristics of stochastic fractional differential systems, Xu et al. [27] developed stochastic averaging for several classes of SFDEs driven by different noises. Abouagwa and Li [28] considered an averaging method for SFDEs under a non-Lipschitz condition. Luo and Zhu [29] discussed an averaging principle for SFDEs with delay. Shen et al. [30] showed an averaging principle under a weaker condition for a kind of Caputo SFDEs. Guo et al. [31] examined an averaging theorem under a non-Lipschitz condition. Luo and Zhu [29] discussed an averaging principle for SFDEs with delay. Shen et al. [30] showed an averaging principle for SFDEs with Lévy noise and Markovian switching. Guo et al. [31] examined an averaging theorem under a weaker condition for a kind of Caputo SFDEs. Liu and Xu [32] proved an averaging principle for neutral ISFDEs. These works offer possible ways to reduce various types of SFDEs. However, these papers do not involve the impulsive effects and the random perturbations with long-range dependence with which we are concerned. Another gap is that they showed no uniformity of time scales. To be precise, the aforementioned results cannot solve the problem if the SFDEs are excited by fBm and involve impulses. In this work, our main objective is to fill these two gaps. We will establish an averaging principle for the SFDEs driven by fBm with impulses in the following form:

\[
\begin{cases}
  dx(t) = f(t, x(t))dt + g(t, x(t))(dt)^\beta + h(t, x(t))dH^\beta(t), & t \in [0, T], t \neq t_k, \\
  \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), & t = t_k, k \in \mathbb{N}, \\
  x(0) = x_0 \in \mathbb{R}^d,
\end{cases}
\]

where $f, g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, h : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ are measurable functions; the processes $B^H(t)$ stand for the independent fBms, in which the Hurst parameter $H \in (\frac{1}{2}, 1), 0 < \beta < 1, 0 < T < \infty, I_k : \mathbb{R}^d \to \mathbb{R}^d$ indicate the pulse size of $x$ at time $t_k$, with $x(t_k^+) = \lim_{\tau \to 0^+} x(t_k + \tau), x(t_k^-) = \lim_{\tau \to 0^-} x(t_k + \tau); x(t_k) = x(t_k^+)$; and the impulsive time sequence satisfies that $0 \leq t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ with $\lim_{k \to \infty} t_k = \infty$. The initial value $x_0 \in \mathbb{R}^d$ is a given random variable with $\mathbb{E}|x_0|^2 < \infty$.

Since Equation (1) is introduced under multi-time scales, the time scale of the standard form of system (1) should be uniform in the process of deriving the averaging principle. However, this point was overlooked in the present literature [28–30]. Additionally, the involvement of impulses makes the proof of the main results more technical. The techniques shown in the aforementioned research on periodic averaging methods for ISDEs are not applicable to our case. The innovation of this paper is not only that an averaging theorem for a new class of SFDEs is discussed, but also that some published results are improved. Naturally, we highlight the contributions of this article here.
• We propose an effective approximation for the original system (1); hence, the complexity of ISFDEs under fBm can be reduced.
• The problem of no match on each of the time scales of the standard stochastic fractional differential equations is pointed out and corrected.
• The obtained averaging principle is valid for stochastic fractional differential equations driven by fBm; that is, our results are new even for non-impulsive SFDEs with fBm.
• We present a method to estimate the impulsive terms, which is helpful to develop averaging principles for different types of ISDEs.

The rest of this work is as follows: Some essential notations and preliminaries are presented in Section 2. In Section 3, an averaging principle for system (1) is established under a Non-Lipschitz condition, and some useful remarks are also presented. In the end, Section 4 shows three examples to explain the procedure and validity of the averaging theorem for SFDEs and ISFDEs.

2. Preliminary

Let \( \phi(r, t) = H(2H - 1)|r - t|^{2H - 2} \), in which \( H \in (1/2, 1) \) is a constant. Defining

\[
L^2_\phi(\mathbb{R}^+) = \left\{ p : \|p\|^2_\phi = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} p(r)p(t)\phi(r, t)dtdr < \infty \right\},
\]

where \( p : \mathbb{R}^+ \to \mathbb{R} \) is a Borel measurable function; \( L^2_\phi(\mathbb{R}^+) \) comes to a separable Hilbert space when it is endowed with the following scalar product:

\[
\langle p_1, p_2 \rangle_\phi = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} p_1(r)p_2(t)\phi(r, t)dtdr, \quad p_1, p_2 \in L^2_\phi(\mathbb{R}^+).
\]

Denote \( S \) as the family of cylindrical and smooth random variables:

\[
F(\omega) = p \left( B^H(\psi_1), B^H(\psi_2), \ldots, B^H(\psi_n) \right),
\]

in which \( p \in \mathcal{C}_b^\infty(\mathbb{R}^n), n \geq 1 \) and \( \psi_i \in \mathcal{H} \), where \( \mathcal{H} \) represents the complete set of all measurable mappings with \( \|\psi\|^2_\phi < \infty \) with \( \langle \psi_i, \psi_j \rangle_\phi = \delta_{ij} \). Now, we can define the space \( |\mathcal{H}| \) of \( p \):

\[
\|p\|^2_{|\mathcal{H}|} = \int_0^T \int_0^T |p(t)||p(s)|\phi(t, s)dsdt < \infty.
\]

The Malliavin derivative \( D^H_t \) of \( F \in S \) is determined as follows:

\[
D^H_t F = \sum_{i=1}^n \frac{\partial p}{\partial \xi_i} \left( B^H(\psi_1), B^H(\psi_2), \ldots, B^H(\psi_n) \right) \psi_i(t).
\]

Moreover, for any positive integer \( k \), \( D^{H,k}_t \) symbolizes the iteration of the above formula. Then \( \mathcal{D}^{k,\phi} \) represents the closure of \( S \) and is normed by:

\[
\|F\|^q_{\mathcal{D}^{k,\phi}} = \mathbb{E}|F|^q + \mathbb{E} \sum_{j=1}^k \left\| D^{H,j}_t F \right\|^q_{\mathcal{H}^{\otimes j}}
\]

in which \( q \geq 1 \), \( \otimes \) stands for the tensor product. Likewise, let us introduce \( \mathcal{D}^{1,\phi}(|\mathcal{H}|) \) as the subspace of \( \mathcal{D}^{1,\phi}(\mathcal{H}) \). Then, the Malliavin \( \phi \)-derivative of \( F \) can be given:

\[
D^H_\phi F = \int_{\mathbb{R}^+} \phi(t, s) D^H_t Fds.
\]

There are three types of pathwise integrals definitions for fBm (i.e., forward pathwise integral, symmetric pathwise integral and backward pathwise integral) [33]. In this paper, we use the forward pathwise integral of fBm. For more details on fBm and stochastic integrals of fBm, one can refer [20,21,33,34].
Assume that $y(t)$ is a stochastic process on $[0, T]$ with integrable trajectories, $H \in (0, 1)$. If the limit of
\[
\lim_{\delta \to 0} \int_0^T y(t) \left[ \frac{B^H(t + \delta) - B^H(t)}{\delta} \right] dt
\]
exists in probability, then the above expression is called the forward integral of $y(t)$ for $B^H(t)$, denoted as $\int_0^T y(t) dB^H(t)$.

**Remark 1.** Defining $\mathcal{L}_\phi[0, T]$ as the space of stochastic processes $y(t)$ satisfying $\mathbb{E}||y(t)||^2 < \infty$, in which $y(t)$ is $\phi$-differentiable, and $B^H(y)$ meet that $\mathbb{E} \int_0^T \int_0^T |D^\phi_y(s)|^2 ds dt < \infty$. More details about $\mathcal{L}_\phi[0, T]$ are given in [21].

**Lemma 1 ([20]).** Let $y(t) \in \mathcal{L}_\phi[0, T] \cap D^{1,2}(|\mathcal{H}|)$, and $B^H(t)$ is a fBm with $H \in (\frac{1}{2}, 1)$, then for any $0 < T < \infty$, there exists a positive constant $C$, such that
\[
\mathbb{E} \left[ \int_0^T y(s) d-B^H(s) \right]^2 \leq 2HT^{2H-1} \mathbb{E} \left[ \int_0^T |y(s)|^2 ds \right] + 2TE^T_0 \int_0^T |D^\phi_y(s)|^2 ds \\
\leq 2HT^{2H-1} \mathbb{E} \left[ \int_0^T |y(s)|^2 ds \right] + 4CT^2.
\]

**Definition 1 ([33]).** A stochastic process $x(t)$ defined on $[0, T]$ is called a mild solution of (1) with initial value $x(0) = x_0$, if

(i) $x(t) \in \mathbb{R}^d$ is $\mathcal{F}_t$-adapted, and has càdlàg path a.e. on $[0, T]$;

(ii) for all $t \in [0, T]$, $x(t)$ meets the integral equation below:
\[
x(t) = x(0) + \int_0^t f(s, x(s)) ds + \beta \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\beta}} ds \\
+ \int_0^t h(s, x(s)) d-B^H(s) + \sum_{\delta < t_k < t} I_k(x(t_{\delta})). \tag{2}
\]

To achieve the approximate theoretical result, the following hypotheses are supposed throughout this work.

**Condition 1.** $f(t, \cdot), g(t, \cdot), h(t, \cdot) \in \mathcal{L}_\phi[0, T] \cap D^{1,2}(|\mathcal{H}|)$ satisfy that
\[
|f(t, x) - f(t, y)|^2 + |g(t, x) - g(t, y)|^2 + |h(t, x) - h(t, y)|^2 \\
+ |D^\phi_y(h(t, x) - h(t, y))|^2 \leq \mathcal{T}(t, |x - y|^2), \quad \forall t \in [0, T],
\]
where $\mathcal{T} : [0, +\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable with $t$, given any fixed $t \geq 0$, $\mathcal{T}$ is a continuous, non-decreasing, concave function with respect to $x$, $\mathcal{T}(t, 0) = 0$ and $\int_{0^+} \frac{1}{\mathcal{T}(t, x)} dx = +\infty$. Additionally, for any $t \in \mathbb{R}^+$ and positive constant $\kappa$, if the non-negative continuous function $\mathcal{Y}(t)$ fulfills that $\mathcal{Y}(t) \leq \kappa \int_0^t \mathcal{T}(s, \mathcal{Y}(s)) ds$, we have $\mathcal{Y}(t) \equiv 0$. 

Following Refs. [16,35], we show the definition of the mild solution for system (1).
Condition 2. \( f(t, \cdot), g(t, \cdot), h(t, \cdot) \in L_q[0, T] \cap D^{1,2}(|H|) \) satisfy that
\[
|f(t,x)|^2 + |g(t,x)|^2 + |h(t,x)|^2 + |D^\delta h(t,x)|^2 \leq \mathcal{M}(t, |x|^2), \quad \forall t \in [0, T],
\]
where \( \mathcal{M} : [0,\infty) \times \mathbb{R}^+ \to \mathbb{R}^+ \) is locally integrable for \( t \) and given every fixed \( t, \mathcal{M} \) is continuous, monotone non-decreasing, concave with \( x \). Moreover, given any positive constant \( \kappa \) and initial data \( x_0 \), the integral formula \( x(t) = x_0 + \kappa \int_0^t \mathcal{M}(s, x(s))\,ds \) admits a global solution.

Condition 3. The impulsive functions \( I_k \) satisfy the following Lipschitz condition and boundedness condition: there are positive constants \( \epsilon_k, d_k \) for all \( x, y \in \mathbb{R}^d \) such that
\[
|I_k(x) - I_k(y)|^2 \leq \epsilon_k |x - y|^2, \quad |I_k(x)|^2 \leq d_k.
\]

Remark 2. Following Theorem 3.1 in Ref. [16], the system (1) has a unique mild solution under Conditions 1–3. Additionally, the unique solution of (1) has the property that
\[
\mathbb{E} \sup_{0 \leq t \leq T} |x(t)|^2 \leq C,
\]
where \( C \) is a positive constant.

3. Main Results

In this part, we devote to develop an averaging principle for ISFDEs excited by fBm. That is, an approximate way of simplifying the system is presented.

The standard form of ISFDEs driven by fBm is defined as
\[
x_{\epsilon}(t) = x(0) + \epsilon \int_0^t f(s, x_{\epsilon}(s))\,ds + \epsilon^\beta \beta \int_0^t \frac{g(s, x_{\epsilon}(s))}{(t-s)^{1-\beta}}\,ds \\
+ \epsilon^H \int_0^t h(s, x_{\epsilon}(s))d^-B^H(s) + \epsilon \sum_{0 < k < t} I_k(x_\epsilon(t^-_k)),
\]
where \( t \in [0, T], \epsilon \in (0, \epsilon_1), (0 < \epsilon_1 \ll 1) \) represents a time-scale parameter; the coefficients \( f, g, h, I_k \) meet Conditions 1–3.

Remark 3. As discussed in Ref. [32], \( \beta \int_0^1 \frac{g(s, x_{\epsilon}(s))}{(t-s)^{1-\beta}}\,ds \) is gained from the integral of \( g(t, x_{\epsilon}(t)) \) with \( (dt)^\beta \), so the time scale of this term should be \( \epsilon^\beta \).

Next, we will demonstrate that when \( \epsilon \) tends to zero, the solution \( x_{\epsilon}(t) \) for Equation (1) converges to the solution \( z_{\epsilon}(t) \) of the following averaged equation:
\[
z_{\epsilon}(t) = x(0) + \epsilon \int_0^t \left( f(z_{\epsilon}(s)) + I(z_{\epsilon}(s)) \right)\,ds + \epsilon^\beta \beta \int_0^t \frac{g(z_{\epsilon}(s))}{(t-s)^{1-\beta}}\,ds \\
+ \epsilon^H \int_0^t h(z_{\epsilon}(s))d^-B^H(s),
\]
the coefficients \( f(\cdot), g(\cdot), h(\cdot) \in D^{1,2}(|H|) \cap L_q[0, T] \) and \( I : \mathbb{R}^d \to \mathbb{R}^d \) are all measurable, while meeting Conditions 1–3 and the hypotheses below:

**Hypothesis 1 (H1):**
\[
\frac{1}{S} \int_0^S \left| f(s,x) - \bar{f}(x) \right|^2 \,ds \leq \gamma_1(S)\theta(|x|^2),
\]

**Hypothesis 2 (H2):**
\[
\frac{1}{S} \int_0^S \left| g(s,x) - \bar{g}(x) \right|^2 \,ds \leq \gamma_2(S)\theta(|x|^2),
\]
Hypothesis 3 (H3):
\[
\frac{1}{S} \int_0^S \left| h(s, x) - \bar{h}(x) \right|^2 ds \leq \gamma_3(S) |x|^2,
\]

Hypothesis 4 (H4):
\[
\frac{1}{S} \sum_{0 < t_j < S} I_j(x) - ST(x) \leq \gamma_4(S)(1 + |x|),
\]

where \(S \in [0, T]\), \(\gamma_i(S)\) are positive bounded functions with \(\lim_{S \to \infty} \gamma_i(S) = 0, i = 1, 2, 3, 4\), and the function \(\theta : \mathbb{R}^+ \to \mathbb{R}^+\) is non-decreasing, continuous and concave.

Remark 4. As in Remark 2, under Hypotheses H1–H4, it is not difficult to derive that \(E \sup_{0 \leq t \leq T} |z_\varepsilon(t)|^2 < \infty\).

Now, the main theorem of this paper is presented as follows.

Theorem 1. Let Conditions 1–3 and Hypotheses H1–H4 hold. For any given small positive number \(\delta\), there are \(P > 0, \varepsilon_2 \in (0, \varepsilon_1]\) and \(\gamma \in (0, 1)\) such that, for all \(0 < \varepsilon \leq \varepsilon_2, 0 \leq t \leq P \epsilon^{-H_1}\), we have
\[
E |z_\varepsilon(t) - z_\varepsilon(t)|^2 \leq \delta.
\]

For the sake of showing Theorem 1, we state the following lemma in advance.

Lemma 2. Let the conditions of Theorem 1 hold; then, for any \(0 \leq s \leq t \leq T\), we have
\[
E |z_\varepsilon(t) - z_\varepsilon(s)|^2 \leq \varepsilon^2 Q_1(t-s)^2 + \varepsilon^2 Q_2(t-s)^{2H} + \varepsilon^2 Q_3(t-s)^{2H} + \varepsilon^2 Q_4(t-s)^2,
\]
in which \(Q_i(i = 1, 2, 3, 4)\) are four positive constants.

The proof of Lemma 2 is analogous to that of Theorem 3.1 in [16], so we just outline it here.

From Equation (4), with basic inequality, Cauchy–Schwarz inequality and Lemma 1, one can get
\[
E |z_\varepsilon(t) - z_\varepsilon(s)|^2 \leq 3\varepsilon^2 (t-s) \int_s^t \left| \bar{f}(z_\varepsilon(\theta)) + \bar{f}(z_\varepsilon(\theta)) \right|^2 d\theta
\]
\[
+ 3\varepsilon^2 \beta^2 \frac{(t-s)^{2\beta - 1}}{2\beta - 1} \int_s^t \left| \bar{f}(z_\varepsilon(\theta)) \right|^2 d\theta
\]
\[
+ 3\varepsilon^2 H \left( 2H(t-s)^{2H-1} E \int_s^t \left| \bar{f}(z_\varepsilon(\theta)) \right|^2 d\theta + 4C(t-s)^2 \right),
\]
then, with the aid of Condition 2, H1–H4 and Remark 4, one can conclude the statement of Lemma 2 without difficulty.

We are now turning back to the proof of Theorem 1.
Proof. Employing the elementary inequality, we obtain from (3) and (4):

\[
E|x(t) - z(t)|^2 \leq 4 \epsilon^2 E \left| \int_0^t (f(s, x(s)) - f(z(s))) ds \right|^2 + 4 \beta^2 E \left| \int_0^t (g(s, x(s)) - \bar{g}(z(s))) \frac{1}{(t-s)^{1-\beta}} ds \right|^2 + 4 \epsilon^{2H} E \left| \int_0^t h(s, x(s)) - \bar{h}(z(s)) d^H(s) \right|^2 + 4 \epsilon^2 \left| \sum_{0 < t_k < t} I_k(x(t_k^-)) - \int_0^t I(z(s)) ds \right|^2
\]

\[
= J_1 + J_2 + J_3 + J_4.
\]

Denote by \([0, \bar{a}] \subseteq [0, \bar{a}] \subseteq [0, T].\) Next, we estimate each term separately. For the first term, we have

\[
J_1 \leq 8 \epsilon^2 \bar{u} E \int_0^{\bar{a}} \mathcal{T}(s, |x(s) - z(s)|^2) ds.
\]

With the virtue of Cauchy–Schwarz inequality and Condition 1, it follows that

\[
J_{11} \leq 8 \epsilon^2 \bar{u} E \int_0^{\bar{a}} \mathcal{T}(s, |x(s) - z(s)|^2) ds.
\]

On the basis of Remark 4, the elementary inequality, Condition 1 and \(H1,\) one has

\[
J_{12} \leq 8 \epsilon^2 \bar{u} E \left[ \frac{1}{\bar{u}} \int_0^{\bar{a}} \left| f(s, z(s)) - \bar{f}(z(s)) \right|^2 ds \right] \leq 8 \epsilon^2 \bar{u}^2 C_{12},
\]

where \(C_{12}\) is a positive constant. Then, we arrive at

\[
J_1 \leq 8 \epsilon^2 \bar{u} E \int_0^{\bar{a}} \mathcal{T}(s, |x(s) - z(s)|^2) ds + 8 \epsilon^2 \bar{u}^2 C_{12}.
\]

Adopting Cauchy–Schwarz inequality to \(J_2,\) we get

\[
J_2 \leq 4 \beta^2 \epsilon^{2\beta} \int_0^t (t-s)^{2\beta-1} ds E \int_0^t \left| g(s, x(s)) - \bar{g}(z(s)) \right|^2 ds
\]

\[
\leq \frac{4 \beta^2 \epsilon^{2\beta} \bar{u}^{2\beta-1}}{2\beta - 1} E \int_0^t \left| g(s, x(s)) - \bar{g}(z(s)) \right|^2 ds,
\]

for \(E \int_0^t \left| g(s, x(s)) - \bar{g}(z(s)) \right|^2 ds,\) estimating analogously as above, one can easily obtain

\[
J_2 \leq \frac{8 \beta^2 \epsilon^{2\beta} \bar{u}^{2\beta-1}}{2\beta - 1} E \int_0^t \mathcal{T}(s, |x(s) - z(s)|^2) ds + \frac{8 \beta^2 \epsilon^{2\beta} \bar{u}^{2\beta}}{2\beta - 1} \sup_{0 \leq t \leq a} \gamma_2(t) E \left[ \phi \left( \sup_{0 \leq t \leq a} |z(t)|^2 \right) \right]
\]

\[
\leq \frac{8 \beta^2 \epsilon^{2\beta} \bar{u}^{2\beta-1}}{2\beta - 1} E \int_0^{\bar{a}} \mathcal{T}(s, |x(s) - z(s)|^2) ds + \frac{8 \beta^2 \epsilon^{2\beta} \bar{u}^{2\beta}}{2\beta - 1} C_{22},
\]

where \(\frac{1}{2} < \beta < 1, C_{22}\) is a positive constant.
To proceed, thanks to Lemma 1, the elementary inequality, Condition 1, Hypothesis H2 and Remark 4, one can derive that

\[
J_3 \leq 8 \epsilon^{2H} H t^{2H-1} \mathbb{E} \int_0^T (h(s, x(s)) - \overline{T}(z_s(s))^2 ds + 16 \epsilon^{2H} C t^2
\]

\[
\leq 16 \epsilon^{2H} H t^{2H-1} \mathbb{E} \int_0^T (h(s, x(s)) - h(s, z(s))^2 ds
\]

\[
+ 16 \epsilon^{2H} H t^{2H-1} \mathbb{E} \int_0^T (h(s, z(s)) - \overline{T}(z_s(s))^2 ds + 16 \epsilon^{2H} C t^2
\]

\[
\leq 16 \epsilon^{2H} H t^{2H-1} \mathbb{E} \int_0^T T(s, |x(s) - z(s)|^2) ds
\]

\[
+ 16 \epsilon^{2H} H \hat{u}^{2H} \sup_{0 \leq t \leq \hat{a}} \gamma_3(t) \mathbb{E} \left[ \theta \left( \sup_{0 \leq t \leq \hat{a}} |z_\epsilon(t)|^2 \right) \right] + 16 \epsilon^{2H} C \hat{u}^2
\]

\[
\leq 16 \epsilon^{2H} H \hat{u}^{2H-1} \mathbb{E} \int_0^T T(s, |x(s) - z(s)|^2) ds + 16 \epsilon^{2H} \hat{u}^{2H-1} HC_{32} + 16 \epsilon^{2H} C \hat{u}^2,
\]

where \( C_{32} \) is a positive constant.

For the last term \( J_4 \), we have

\[
J_4 \leq 16 \epsilon^{2H} \mathbb{E} \left| \sum_{0 < t_k < t} I_k(x(t_k^-)) - \int_0^t \overline{T}(x_k(t_k^-)) ds \right|^2
\]

\[
+ 16 \epsilon^{2H} \mathbb{E} \left| \int_0^t \overline{T}(x_k(t_k^-)) ds - \int_0^t \overline{T}(z_k(t_k^-)) ds \right|^2
\]

\[
+ 16 \epsilon^{2H} \mathbb{E} \left| \int_0^t \overline{T}(z_k(t_k^-)) ds - \int_0^t \overline{T}(z(s)) ds \right|^2
\]

\[
:= J_{41} + J_{42} + J_{43}.
\]

Notice that

\[
\sum_{0 < t_k < t} I_k(x(t_k^-)) = \sum_{i=1}^m \left( \sum_{0 < t_k < t_{i+1}} I_k(x(t_k^-)) - \sum_{0 < t_k < t_i} I_k(x(t_k^-)) \right)
\]

and

\[
\int_0^t \overline{T}(x_k(t_k^-)) ds = \sum_{i=0}^m (t_{i+1} - t_i) \overline{T}(x_k(t_k^-)).
\]

Estimating \( J_{41} \) in the following manner:

\[
J_{41} \leq 16 \epsilon^{2H} \mathbb{E} \left\{ \sum_{i=0}^m \left[ t_{i+1} \left( \frac{1}{t_{i+1}} \sum_{0 < t_k < t_{i+1}} I_k(x(t_k^-)) - \overline{T}(x_k(t_k^-)) \right) \right]^{2} \right. \]

\[
+ \left. t_i \left( \frac{1}{t_i} \sum_{0 < t_k < t_i} I_k(x(t_k^-)) - \overline{T}(x_k(t_k^-)) \right) \right\}^{2},
\]

then Hypothesis H4 yields that

\[
J_{41} \leq 32 \epsilon^2 (m + 1) \hat{u}^2 \left( \sup_{0 \leq t \leq \hat{a}} \gamma_4(t) \right)^2 \mathbb{E} \left( 1 + |x_\epsilon(t)|^2 \right)^2
\]

\[
\leq 64 \epsilon^2 (m + 1) \hat{u}^2 \left( \sup_{0 \leq t \leq \hat{a}} \gamma_4(t) \right)^2 \left( 1 + \mathbb{E} \sup_{0 \leq t \leq \hat{a}} |x_\epsilon(t)|^2 \right)
\]

\[
\leq 64 \epsilon^2 (m + 1) \hat{u}^2 C_{41},
\]
where $C_{41}$ is a positive constant.

As $T$ meets the Lipschitz condition, there is a positive constant $N$ involving with $c_k$ such that
\[ |\tilde{T}(x) - \tilde{T}(y)|^2 \leq N|x - y|^2, \quad \forall x, y \in \mathbb{R}^d. \]

Using Cauchy-Schwarz inequality, we have
\[
J_{42} \leq 16\varepsilon^2 uN \int_0^a E|\chi_x(t) - z_\varepsilon(t)|^2 \, ds \\
\leq 16\varepsilon^2 uN \int_0^a \sup_{0 \leq \theta \leq s} E|\chi_x(\theta) - z_\varepsilon(\theta)|^2 \, ds.
\]

With the aid of Lemma 2, one can reach
\[
J_{43} \leq 16\varepsilon^2 uN \int_0^a E|z_\varepsilon(t) - z_\varepsilon(s)|^2 \, ds \\
\leq 16\varepsilon^4 NQ_1\bar{u}^4 + 16\varepsilon^{2H+2}NQ_2\bar{u}^{2\beta+2} + 16\varepsilon^{2H+2}NQ_3\bar{u}^{2H+2} + 16\varepsilon^{2H+2}NQ_4\bar{u}^4. 
\]

Substituting (11), (12) and (13) together into (10), we get
\[
J_4 \leq 64\varepsilon^2 (m + 1)\bar{u}^2 C_{41} + 16\varepsilon^2 uN \int_0^a \sup_{0 \leq \theta \leq s} E|\chi_x(\theta) - z_\varepsilon(\theta)|^2 \, ds \\
+ 16\varepsilon^4 NQ_1\bar{u}^4 + 16\varepsilon^{2H+2}NQ_2\bar{u}^{2\beta+2} + 16\varepsilon^{2H+2}NQ_3\bar{u}^{2H+2} + 16\varepsilon^{2H+2}NQ_4\bar{u}^4. 
\]

Thanks to the fact that $T$ is concave in $x$, there is $w(t) > 0$ and $e(t) > 0$ such that
\[ T(t, x) \leq w(t) + e(t)x, \quad \int_0^T w(t) \, dt < \infty, \quad \int_0^T e(t) \, dt < \infty, \]
then, combining the estimates (7), (8), (9) and (14), we obtain
\[
\sup_{0 \leq t \leq a} E|\chi_x(t) - z_\varepsilon(t)|^2 \leq \left(8\varepsilon^2 \bar{u} + 16\varepsilon^{2H-H+1} \frac{8\varepsilon^{2\beta} \bar{u}^{2\beta-1}}{2\beta - 1} \right) \sup_{0 \leq t \leq s} E|\chi_x(\theta) - z_\varepsilon(\theta)|^2 \, ds \\
+ 16\varepsilon^2 uN \int_0^a \sup_{0 \leq \theta \leq s} E|\chi_x(\theta) - z_\varepsilon(\theta)|^2 \, ds \\
+ 80\varepsilon^2 \bar{u}^2 C_2 + 32\varepsilon^{2H} \bar{u}^{2H} C_3 + 64\varepsilon^{2H} C_4 \bar{u}^2 + \frac{16\varepsilon^{2\beta} \bar{u}^{2\beta} C_5}{2\beta - 1} \\
+ 16\varepsilon^4 C_6 \bar{u}^4 + 16\varepsilon^{2H+2} C_7 \bar{u}^{2\beta+2} + 16\varepsilon^{2H+2} C_8 \bar{u}^{2H+2} + 16\varepsilon^{2H+2} C_9 \bar{u}^4,
\]
where $C_{ij}, i = 1, 2, \cdots, 9$ are positive constants. Gronwall inequality shows that there are $P > 0$ and $\gamma \in (0, 1)$ fulfilling
\[
\sup_{t \in [0, Pe^{-H\gamma}]} E|\chi_x(t) - z_\varepsilon(t)|^2 \leq Qe^{(1-H\gamma)},
\]
where

\[
Q = \left[ 80c_2P^2e^{1-H\gamma} + 32c_3P^{2H}e^{(2H-1)(1-H\gamma)} + 64c_4P^2e^{(2H-1)(1-\gamma)} \right. \\
+ 16c_5P^{2\beta}e^{(2\beta-1)(1-H\gamma)} \left/ (2\beta - 1) \right. + 16c_6P^{4}(1-H\gamma) + 16c_7P^{2\beta+2}e^{(2\beta+1)(1-H\gamma)} \\
+ 16c_8P^{2H+2}e^{(2H+1)(1-H\gamma)} + 16c_9\exp \left( 8P^2c_1e^{2(1-H\gamma)} \right) \\
+ 16P^{2H}c_1e^{2H(1-H\gamma)} + 8P^{2H}c_1e^{2\beta(1-H\gamma)} \left/ (2\beta - 1) \right. + 16P^2e^{2(1-H\gamma)} \right]
\]

is a positive constant.

Therefore, for arbitrarily given small positive number \( \delta \), there is \( \varepsilon_2 \in (0, \varepsilon_1) \) such that

\[
\mathbb{E}[x_\varepsilon(t) - z_\varepsilon(t)]^2 \leq \delta,
\]

for \( \forall \varepsilon \in (0, \varepsilon_2], \forall t \in [0, Pe^{-H\gamma}] \subseteq [0, T] \). □

**Remark 5.** Noted that the order of convergence interval is \( e^{-H\gamma} \) in this study. It is not stronger as \( e^{-\gamma} \) of Bm case and Lévy case [27–30] because of the weakness that stochastic integral of fBm is not a martingale. However, the proposed result is still certainly a good theoretical method to simplify non-autonomous ISFDEs driven by fBm with long-term dependence.

**Remark 6.** When \( l_0 = 0 \), system (1) degenerates into stochastic fractional differential equation driven by fBm. Evidently, our obtained result is applicable for SFDEs with fBm.

4. Example

**Example 1.** Consider the SFDE under fBm below:

\[
dx_\varepsilon(t) = -\varepsilon \sin^2(t) x_\varepsilon(t) dt + e^\beta (\varepsilon dt)^\beta + e^H d^-B^H(t), t \in [0, T]
\]

\[
x_\varepsilon(0) = x_0,
\]

where \( \mathbb{E}|x_0|^2 < \infty \), \( f(t, x_\varepsilon(t)) = -\sin^2(t) x_\varepsilon(t), g(t, x_\varepsilon(t)) = 1, h(t, x_\varepsilon(t)) = 1 \). Let

\[
\tilde{f}(z_\varepsilon(t)) = -\frac{1}{2} \varepsilon z_\varepsilon(t), \tilde{g}(z_\varepsilon(t)) = 1, \tilde{h}(z_\varepsilon(t)) = 1.
\]

Obviously, Conditions 1, 2 and Hypotheses H1–H3 are satisfied. The simplified averaged SFDE is given by

\[
dz_\varepsilon(t) = -\frac{1}{2} \varepsilon z_\varepsilon(t) dt + e^\beta (\varepsilon dt)^\beta + e^H d^-B^H(t).
\]

Remark 6 illustrates that \( z_\varepsilon(t) \) is approximately equivalent to \( x_\varepsilon(t) \) on \([0, T]\). Then, a numerical comparison between the solution \( x_\varepsilon(t) \) of original Equation (15) and the solution \( z_\varepsilon(t) \) of the averaged Equation (16) is presented. Not surprisingly, good agreement can be observed in Figure 1.

![Figure 1](image)

**Figure 1.** Comparison of \( x_\varepsilon(t) \) and \( z_\varepsilon(t) \). (a) \( x_0 = 0, \varepsilon = 0.045, H = 0.65, \beta = 0.8 \), (b) \( x_0 = 0.5, \varepsilon = 0.01, H = 0.85, \beta = 0.6 \).
Example 2. Consider the SFDE with fBm below:

\[
\begin{align*}
dx_t(t) &= 2\sin^2(t)x_t(t)dt + 2e^\beta\cos^2(t)(dt)^\beta + e^H d^{-B^H}(t), t \in [0, T] \\
x_0 &= x_0,
\end{align*}
\]

where \(E|x_0|^2 < \infty, f(t, x_t(t)) = 2\sin^2(t)x_t(t), g(t, x_t(t)) = 2\cos^2(t), h(t, x_t(t)) = 1\). Let

\[
\tilde{f}(z_t(t)) = z_t(t), \quad \tilde{g}(z_t(t)) = 1, \quad \tilde{h}(z_t(t)) = 1.
\]

One can easily verify that Conditions 1, 2 and Hypotheses H1–H3 are satisfied. Meanwhile, a simplified averaged SFDE is presented as

\[
\begin{align*}
dz_t(t) &= \varepsilon \tilde{z}_t(t)dt + e^\beta(dt)^\beta + e^H d^{-B^H}(t), \\
\end{align*}
\]

Theorem 1 guarantees that \(z_t(t)\) can approximate \(x_t(t)\) on \([0, Pe^{-H\tau}] \subseteq [0, T]\). In succession, we show a numerical comparison between the solution \(x_t(t)\) to the original Equation (17) and the solution \(z_t(t)\) to the averaged Equation (18). It can be found that the two are consistent in a certain range in Figure 2.

![Figure 2](image)

Figure 2. Comparison of \(x_t(t)\) and \(z_t(t)\). (a) \(x_0 = 0, \varepsilon = 0.0045, H = 0.65, \beta = 0.7\), (b) \(x_0 = 1, \varepsilon = 0.01, H = 0.55, \beta = 0.65\).

Example 3. Consider the ISFDE with fBm below:

\[
\begin{align*}
dx_t(t) &= \varepsilon dt + 2e^\beta\sin^2(t)(dt)^\beta + e^H\cos^2(t)x_t(t)dt^{-B^H}(t), t \neq t_k, \\
\Delta x_{t_k}(t) &= x_{t_k^+}(t_k) - x_{t_k^-}(t_k) = \varepsilon \arctan(x_{t_k^-}(t_k)), t = t_k, \\
x_0 &= x_0,
\end{align*}
\]

where \(E|x_0|^2 < \infty, f(t, x_t(t)) = 1, g(t, x_t(t)) = 2\sin^2(t), h(t, x_t(t)) = \cos^2(t)x_t(t)\). Let

\[
\tilde{f}(z_t(t)) = 1, \quad \tilde{g}(z_t(t)) = 1, \quad \tilde{h}(z_t(t)) = \frac{1}{2}z_t(t), \quad \tilde{I}(z_t(t)) = \frac{1}{3} \arctan(z_t(t)),
\]

then, a new simplified averaged ISFDE without impulses is given as

\[
\begin{align*}
dz_t(t) &= \varepsilon \left(1 + \frac{1}{5} \arctan(z_t(t))\right)dt + e^\beta(dt)^\beta + \frac{1}{2}e^HDz_t(t)d^{-B^H}(t). \\
\end{align*}
\]

As we can see, the impulses are averaged out and the original system is simplified. Even better, only a small difference is generated in this proximate process. In the end, we carry out a numerical comparison between the solution process \(x_t(t)\) of original Equation (19) and the solution process \(z_t(t)\) of the simplified Equation (20). As expected, good agreement can be seen in Figure 3.
Figure 3. Comparison of $x_\epsilon(t)$ and $z_\epsilon(t)$ with $t_k = 4.8k$. (a) $x_0 = 0, \epsilon = 0.005, H = 0.65, \beta = 0.7$, (b) $x_0 = 1, \epsilon = 0.01, H = 0.55, \beta = 0.6$.

Author Contributions: Formal analysis, J.L.; methodology, J.L. and W.W.; writing—original draft preparation, J.L.; writing—review and editing, J.L. and W.W.; supervision, W.X. All authors have read and agreed to the published version of the manuscript.

Funding: We thank the support of the National Natural Science Foundation of China (Grant Nos. 12072261, 11872305) for our work, and J. Liu also thank the partial support of Fundamental Research Program of Shanxi Province (NO. 20210302122374) and TYUST SRIF (No. 20212074).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Ragusa, M.A. On weak solutions of ultraparabolic equations. *Nonlinear Anal. Theor.* 2001, 47, 503–511. [CrossRef]
2. Machado, J.T.; Mainardi, F.; Kiryakova, V. Fractional calculus: Quo vadimus? (Where are we going?) *Fract. Calc. Appl. Anal.* 2015, 18, 495–526. [CrossRef]
3. Khan, I.; Ullah, H.; AlSalman, H. Fractional analysis of MHD boundary layer flow over a stretching sheet in porous medium: A new stochastic Method. *J. Func. Spaces* 2021, 2021, 5844741. [CrossRef]
4. Yang, M. (Weighted pseudo) almost automorphic solutions in distribution for fractional stochastic differential equations driven by levy noise. *Filomat* 2021, 35, 2403–2424. [CrossRef]
5. Sakthivel, R.; Revathi, P.; Ren, Y. Existence of solutions for nonlinear fractional stochastic differential equations. *Nonlinear Anal. Theor.* 2013, 81, 70–86. [CrossRef]
6. Kamrani, M. Numerical solution of stochastic fractional differential equations. *Numer. Algor.* 2015, 68, 81–93. [CrossRef]
7. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific: Singapore, 1989.
8. Perestyuk, N.A.; Plotnikov, V.A.; Samoilenko, A.M.; Skripnik, N.V. *Differential Equations with Impulse Effects: Multivalued Right-Hand Sides with Discontinuities*; Walter de Gruyter: Berlin, Germany, 2011.
9. Hu, Y.; Øksendal, B. Fractional white noise calculus and applications to finance. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 2003, 6, 1–32. [CrossRef]
10. Liu, J.; Yan, L.; Cang, Y. On a jump-type stochastic fractional partial differential equation with fractional noises. *Nonlinear Anal. Theor.* 2012, 75, 6060–6070. [CrossRef]
11. Li, K. Stochastic delay fractional evolution equations driven by fractional Brownian motion. *Math. Methods Appl. Sci.* 2015, 38, 1582–1591. [CrossRef]
12. Xu, L.; Li, Z. Stochastic fractional evolution equations with fractional Brownian motion and infinite delay. *Appl. Math. Comput.* 2018, 336, 36–46. [CrossRef]
13. Chadha, A.; Pandey, D.N. Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay. *Nonlinear. Anal. Theor.* 2015, 128, 149–175. [CrossRef]
14. Dhayal, R.; Malik, M.; Abbas, S. Approximate controllability for a class of non-instantaneous impulsive stochastic fractional differential equation driven by fractional Brownian motion. *Differ. Equ. Dyn. Syst.* 2021, 29, 175–191. [CrossRef]
15. Pedjeu, J.C.; Ladde, G.S. Stochastic fractional differential equations: Modeling, method and analysis. *Chaos Solitons Fract.* 2012, 45, 279–293. [CrossRef]
16. Abouagwa, M.; Cheng, F.; Li, J. Impulsive stochastic fractional differential equations driven by fractional Brownian motion. *Adv. Differ. Equ.* 2020, 2020, 57. [CrossRef]
17. Khasminskii, R.Z. A limit theorem for the solution of differential equations with random right-hand sides. *Theory Probab. Appl.* 1963, 11, 390–405. [CrossRef]

18. Roberts, J.B.; Spanos, P.D. Stochastic averaging: An approximate method of solving random vibration problems. *Int. J. Nonlin. Mech.* 1986, 21, 111–134. [CrossRef]

19. Zhu, W.Q. Stochastic Averaging Methods in Random Vibration. *Appl. Mech. Rev.* 1988, 41, 189–199. [CrossRef]

20. Xu, Y.; Pei, B.; Guo, R. Stochastic averaging for slow-fast dynamical systems with fractional Brownian motion. *Discret. Contin. Dyn. Syst. B* 2015, 20, 2257–2267. [CrossRef]

21. Xu, Y.; Pei, B.; Wu, J.L. Stochastic averaging principle for differential equations with non-Lipschitz coefficients driven by fractional Brownian motion. *Stoch. Dyn.* 2017, 17, 1750013. [CrossRef]

22. Ma, S.; Kang, Y.M. Periodic averaging method for impulsive stochastic differential equations with Lévy noise. *Appl. Math. Lett.* 2019, 93, 91–97. [CrossRef]

23. Khalaf, A.D.; Abouagwa, M.; Wang, X. Periodic averaging method for impulsive stochastic dynamical systems driven by fractional Brownian motion under non-Lipschitz condition. *Adv. Differ. Equ.* 2019, 526. [CrossRef]

24. Cui, J.; Bi, N. Averaging principle for neutral stochastic functional differential equations with impulses and non-Lipschitz coefficients. *Stat. Probabil. Lett.* 2020, 163, 108775. [CrossRef]

25. Wang, P.; Xu, Y. Periodic averaging principle for neutral stochastic delay differential equations with impulses. *Complexity* 2020, 2020, 6731091. [CrossRef]

26. Liu, J.K.; Xu, W.; Guo, Q. Averaging principle for impulsive stochastic partial differential equations. *Stoch. Dynam.* 2021, 21, 2150014. [CrossRef]

27. Xu, W.J.; Duan, J.Q.; Xu, W. An averaging principle for fractional stochastic differential equations with lévy noise. *Chaos Interdiscip. J. Nonlinear Sci.* 2020, 30, 083126. [CrossRef] [PubMed]

28. Abouagwa, M.; Li, J. Approximation properties for solutions to Itô–Doob stochastic fractional differential equations with non-Lipschitz coefficients. *Stoch. Dynam.* 2019, 19, 1950029. [CrossRef]

29. Luo, D.F.; Zhu, Q.X.; Luo, Z.G. An averaging principle for stochastic fractional differential equations with time-delays. *Appl. Math. Lett.* 2020, 105, 106290. [CrossRef]

30. Shen, G.J.; Xiao, R.D.; Yin, X.W. Averaging principle and stability of hybrid stochastic fractional differential equations driven by Lévy noise. *Int. J. Syst. Sci.* 2020, 51, 2115–2133. [CrossRef]

31. Guo, Z.K.; Hu, J.H.; Yuan, C.G. Averaging principle for a type of Caputo fractional stochastic differential equations. *Chaos Interdiscip. J. Nonlinear Sci.* 2021, 31, 053123. [CrossRef]

32. Liu, J.K.; Xu, W. An averaging result for impulsive fractional neutral stochastic differential equations. *Appl. Math. Lett.* 2021, 114, 106892. [CrossRef]

33. Russo, F.; Vallois, P. Forward, backward and symmetric stochastic integration. *Probab. Theory Relat. Fields* 1993, 97, 403–421. [CrossRef]

34. Biagini, F.; Hu, Y.Z.; Øksendal, B.; Zhang, T.S. *Stochastic Calculus for Fractional Brownian Motion and Applications*; Springer: London, UK, 2008.

35. Shen, L.J.; Sun, J.T. Existence and uniqueness of mild solutions for nonlinear stochastic impulsive differential equation. *Abstr. Appl. Anal.* 2011, 2011, 439724. [CrossRef]