Ising spin glass models versus Ising models: an effective mapping at high temperature: II. Applications to graphs and networks

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Received 21 July 2006
Accepted 20 September 2006
Published 13 October 2006

Online at stacks.iop.org/JSTAT/2006/P10005
doi:10.1088/1742-5468/2006/10/P10005

Abstract. By applying a recently proposed mapping, we derive exactly the upper phase boundary of several Ising spin glass models defined over quenched graphs and random graphs, generalizing some known results and providing new ones.

Keywords: phase diagrams (theory), disordered systems (theory), exact results, random graphs, networks

ArXiv ePrint: cond-mat/0607518
1. Introduction

The science of graphs and networks is nowadays a well studied topic receiving more and more attention due to its large number of applications in the study of complex systems in technological, biological and social phenomena [1]–[3]. As a basic starting point in the analysis of the cooperative behaviour typical of the complex systems, the study of the simplest dichotomy-interaction models, such as the Ising model, plays a crucial role. In recent years, progress has been made toward the solution of this model [4,5], which on a network, especially on the so-called ‘scale free networks’, may manifest critical phenomena very different from those exhibited over regular lattice systems [6,7]. However, the Ising model with purely ferromagnetic couplings only allows for a ‘friendly’ interaction between the elements of the network, \( J_b = +J \), so that only a two-state ‘friendly’ collective behaviour is possible. To take into account an unfriendly interaction between the elements of the network, \( J_b = \pm J \), the Ising spin glass model should instead be considered [8].

In part I of this work [9], we have introduced a general mapping between a given Ising spin glass model and a related Ising model in which, roughly speaking, the couplings are uniform. The mapping turns out to be of immediate application whenever the solution of this related Ising model is known.

The above mentioned mapping becomes exact when the dimensionality \( D \) of the system goes to infinity. More precisely, we have the following results. For a system having a number of first neighbours per vertex proportional to \( D \), the mapping applies exactly as \( D \to \infty \) throughout the paramagnetic region (\textit{infinite dimensional in a strict sense}). In general, the mapping becomes an exact method for finding the upper critical surface when, in the thermodynamic limit, the total number of infinitely long paths per vertex

doi:10.1088/1742-5468/2006/10/P10005
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goes to infinity and the probability that any two of them, chosen randomly, overlap for an infinite number of bonds goes to zero (infinite dimensional in a broad sense).

In [9] we have applied the mapping to several models in which the above conditions are satisfied including the generalized Sherrington–Kirkpatrick models, which are infinite dimensional in a strict sense, and the spin glass model defined on the Bethe lattice, which is infinite dimensional in the broad sense.

In this paper we will apply the mapping to general graphs and networks which are infinite dimensional in the broad sense. First, we will consider generalized quenched tree-like structures also allowing for a finite number of loops per vertex. Second, we will analyse the case of random graphs, namely an ensemble $G$ of graphs $g$, infinite dimensional in the broad sense, with a given probability $P(g)$, $g \in G$. The latter class includes, in particular, the so-called ‘uncorrelated equilibrium random graphs’ [2]. In both cases, we find a general formula for the exact upper critical surface: paramagnetic–spin glass and paramagnetic–disordered ferromagnetic/antiferromagnetic transitions. Whereas for the generalized tree-like structures these critical surfaces represent a completely novel result (except for the ferromagnetic disordered transition with positive couplings [10]), for the random graphs critical surfaces have been already derived in [8] relatively to a particular subclass of the uncorrelated equilibrium random graphs. Beyond the uncorrelated equilibrium random graphs, we provide a formula which applies exactly to the most general case of networks whose set of bonds turns out to be infinite dimensional in the broad sense. This formula gives us the upper critical surface of the Ising spin glass model in terms of the critical temperature of the mapped Ising model over the network. In general, the critical temperature of the mapped model has to be found numerically. However, since this model is a non-random one, the measurement of its critical temperature turns out to be hugely easier than a direct measurement of the critical temperature in the spin glass model.

The paper is organized as follows. In sections 2 and 3, we recall briefly the models we are considering and the main result of [9]. In section 4, we apply the mapping to quenched generalized tree-like structures. In section 5, the random graphs are considered. In section 5.1, we recall the critical temperature of an Ising model over an ensemble of random graphs. Thanks to the mapping, this result is used in section 5.2 to find the upper critical surface of a generic Ising spin glass over an ensemble of random graphs. Finally, some conclusions are drawn in section 6.

2. Models

In this paper we will apply the mapping to systems defined over a graph $g$ of $N$ vertices whose set of links $\Gamma_g$ is defined through the adjacency matrix of the graph, $g_{i,j} = 0,1$:

$$\Gamma_g \equiv \{ b = (i_b, i_b) : i_b, j_b \in g, g_{i_b,j_b} = 1, i_b < j_b \}. \quad (1)$$

The fully connected graph will be indicated with $\Gamma_f$:

$$\Gamma_f \equiv \{ b = (i_b, i_b) : i_b, j_b = 1, \ldots, N, i_b < j_b \}. \quad (2)$$

The Hamiltonian of these spin glass models can be written as

$$H (\{ \sigma_b \}; \{ J_b \}) \equiv - \sum_{b \in \Gamma_g} J_b \sigma_b \quad (3)$$
where the $J_b$'s are quenched couplings, and $\tilde{\sigma}_b$ stands for the product of two Ising variables, $\tilde{\sigma}_b = \sigma_{i_b} \sigma_{j_b}$, with $i_b$ and $j_b$ such that $b = (i_b, j_b)$.

The free energy $F$ is defined by

$$-\beta F \equiv \int d\mathcal{P} \{ J_b \} \log(Z(\{ J_b \})),$$

(4)

where $Z(\{ J_b \})$ is the partition function of the quenched system,

$$Z(\{ J_b \}) = \sum_{\{ \sigma \}} e^{-\beta H(\{ \sigma \}; \{ J_b \})},$$

(5)

and $d\mathcal{P} \{ J_b \}$ is a product measure given in terms of normalized measures $d\mu_b \geq 0$ (we are considering a general measure $d\mu_b$ also allowing for a possible dependence on the bonds),

$$d\mathcal{P} \{ J_b \} \equiv \prod_{b \in \Gamma_g} d\mu_b (J_b), \quad \int d\mu_b (J_b) = 1.$$  

(6)

We take the Boltzmann constant $k_B = 1$. A generic inverse critical temperature of the spin glass model, if any, will be indicated with $\beta_c$.

3. Mapping

In part I of this work we have provided a mapping between an Ising spin glass model and a related Ising model. The mapping is exact at and above the critical temperature when, in the thermodynamic limit, the dimensionality is infinite. For the aims of the present work, we will need to recall only the following definitions and rules of the mapping.

Given a spin glass model through equations (1)–(6), we define, on the same set of links $\Gamma_g$, its related Ising model through the following Ising Hamiltonian:

$$H_I(\{ \sigma_i \}; \{ J_b \}) \equiv -\sum_{b \in \Gamma_g} J_b^{(I)} \tilde{\sigma}_b$$

(7)

where the Ising couplings $J_b^{(I)}$ have non-random values such that $\forall b, b' \in \Gamma_g$

$$J_b^{(I)} = J_b^{(I)} \quad \text{if} \quad d\mu' \equiv d\mu_b,$$

(8)

$$J_b^{(I)} \neq 0 \quad \text{if} \quad \int d\mu_b (J_b) J_b \neq 0 \quad \text{or} \quad \int d\mu_b (J_b) J_b^2 > 0.$$  

(9)

In the following, a suffix $I$ will be used to indicate quantities which refer to the related Ising system with Hamiltonian (7). Finally, with $z_b$ ($z_b^{(I)}$) we will indicate the universal parameters $z_b \equiv \tanh(\beta J_b)$ ($z_b^{(I)} \equiv \tanh(\beta J_b^{(I)})$), and with $w_b$ ($w_b^{(I)}$) their critical value at $\beta_c$ ($\beta_c^{(I)}$).
3.1. Case of a uniform measure (same disorder for any bond)

If \(d\mu_b \equiv d\mu\) for any bond \(b\) of \(\Gamma\), the related Ising model corresponds to a uniform Ising model having a single coupling \(J_b^{(I)} \equiv J^{(I)}\) and its critical behaviour will be characterized by, at most, two points \(w_F^{(I)} = \tanh(J^{(I)}F) > 0\) and \(w_{AF}^{(I)} = \tanh(J^{(I)}AF) < 0\), if any, where \(\beta_F^{(I)}\) and \(\beta_{AF}^{(I)}\) are the critical ferromagnetic and antiferromagnetic temperatures of the related Ising model, respectively. If \(\Gamma\) is infinite dimensional in the broad sense, the critical inverse temperature of the spin glass model, \(\beta_c\), is given by

\[
\beta_c = \min\{\beta_c^{(SG)}, \beta_c^{(F/AF)}\},
\]

where \(\beta_c^{(SG)}\) and \(\beta_c^{(F/AF)}\) are, respectively, the solutions of the equations, if any,

\[
\int d\mu \tanh^2(\beta_c^{(SG)} J_b) = w_F^{(I)},
\]

\[
\int d\mu \tanh(\beta_c^{(F/AF)} J_b) = w_{F/AF}^{(I)}.
\]

In equation (12) \(F\) or \(AF\) are used in both the lhs and rhs.

3.2. Generalization

In the case of an arbitrary (not uniform) measure \(d\mu_b\), useful for example for anisotropic models, in which we have to consider even a given number of bond dependences, the related Ising model is defined by a set of, usually a few, independent couplings \(\{J_b^{(I)}\}\), through equations (8) and (9) and its critical behaviour will be fully characterized by the solutions of a vectorial equation of the type \(G_I(\{z_b\}) = 0\). If \(\Gamma\) is infinite dimensional in the broad sense, equations (10)–(12) are generalized as follows. The critical inverse temperature of the spin glass model \(\beta_c\) is given by

\[
\beta_c = \min\{\beta_c^{(SG)}, \beta_c^{(F/AF)}\},
\]

where \(\beta_c^{(SG)}\) and \(\beta_c^{(F/AF)}\) are, respectively, solutions of the two sets of equations

\[
G_I\left(\int d\mu_b \tanh^2(\beta_c^{(SG)} J_b)\right) = 0,
\]

\[
G_I\left(\int d\mu_b \tanh(\beta_c^{(F/AF)} J_b)\right) = 0.
\]

3.3. Phase diagram

Equations (11) and (12) and their generalizations (14) and (15) give the exact critical paramagnetic–spin glass ((P–SG), \(\beta_c^{(SG)}\)) and paramagnetic–ferro/antiferromagnetic ((P–F/AF), \(\beta_c^{(F/AF)}\)) surfaces. According to equations (10) or (13), the stability of these surfaces depends on which of them is the minimum. In the case of a uniform measure, the suffix \(F/AF\) stands for ferromagnetic or antiferromagnetic, respectively. In the general case, such a distinction is not possible and the symbol \(F/AF\) stresses only that the transition is not P–SG.

doi:10.1088/1742-5468/2006/10/P10005
4. Spin glass over generalized tree-like graphs (quenched graphs)

In part I of this work [9], as an application of the mapping (11) and (12), starting from the critical value of the universal parameter of the Ising model over a Bethe lattice of coordination number $q$,

$$w_F^{(I)} = \tanh \left( \beta_c^{(I)} J^{(I)} \right) = \frac{1}{q - 1},$$  \hspace{1cm} (16)

we have recovered the critical surface of the corresponding Ising spin glass model. We have found that a spin glass (SG) transition and a disordered ferromagnetic (F) transition take place at $\beta_c^{(SG)}$ and $\beta_c^{(F)}$, respectively solutions of the following two equations:

$$\int d\mu \, \tanh^2 \left( \beta_c^{(SG)} J_b \right) = \frac{1}{q - 1},$$  \hspace{1cm} (17)

$$\int d\mu \, \tanh \left( \beta_c^{(F)} J_b \right) = \frac{1}{q - 1},$$  \hspace{1cm} (18)

which, by using equation (10), give the upper phase boundary.

We now generalize equations (17) and (18) to a spin glass defined over a given generalized tree-like graph $g$. Here $g$ is said to be a generalized tree-like graph if for any vertex there is at most a finite number of loops (see figure 1). This application provides a novel non-trivial result.

Figure 1. An example of a generalized tree-like graph with several loops.
Following [10], let us choose a vertex as the root 0. If \( v \) is a vertex, \(|v|\) will indicate the number of links of the shortest path connecting 0 to \( v \). A cutset \( \Pi \) will indicate a finite set of vertices having the property that every infinite path starting from 0 intersects \( \Pi \). In other words a cutset generalizes the concept of shell used in the Bethe lattice. Finally, the branching number of \( \Gamma_g \), denoted by \( \text{br } \Gamma_g \), is introduced to generalize the concept of branching:

\[
\text{br } \Gamma_g \equiv \inf \left\{ \lambda > 0 : \inf_{\Pi} \sum_{v \in \Pi} \lambda^{-|v|} = 0 \right\} .
\] (19)

It is easy to check that in the case of a regular lattice of coordination number \( q \), one recovers \( \text{br } \Gamma_g = q - 1 \). Note however that, in general, \( \text{br } \Gamma_g \) does not coincide with the average of the number of branches per vertex.

In [10] it is proved that in the uniform Ising model with positive coupling \( J(I) \) defined over a generalized tree-like graph, the ferromagnetic phase transition occurs at

\[
w_F(I) = \tanh \left( \beta_c(I) J(I) \right) = \frac{1}{\text{br } \Gamma_g} .
\] (20)

which generalizes equation (16). Correspondingly, by observing that a generalized tree-like graph is infinite dimensional in the broad sense (see section 3 of [9]), and by using the general rule (11) and (12) holding for an Ising spin glass with a uniform measure, \( d\mu_b \equiv d\mu \), we immediately find that a spin glass transition and a disordered ferromagnetic transition take place at \( \beta_c^{(SG)} \) and \( \beta_c^{(F)} \), respectively solutions of the following two equations:

\[
\int d\mu \ \tanh^2 \left( \beta_c^{(SG)} J_b \right) = \frac{1}{\text{br } \Gamma_g} ,
\] (21)

\[
\int d\mu \ \tanh \left( \beta_c^{(F)} J_b \right) = \frac{1}{\text{br } \Gamma_g} .
\] (22)

Equations (21) and (22) generalize (17) and (18) to tree-like structures.

We can further generalize the above result as follows. In [10] it is proved that for an arbitrary, i.e. non-uniform, Ising model with positive couplings \( \{J_b(I)\} \) defined over a generalized tree-like graph, the ferromagnetic phase transition occurs at a critical temperature \( T_c^{(I)} \) given by

\[
T_c^{(I)} = \inf \left\{ T : \inf_{\Pi} \sum_{v \in \Pi} \prod_{\tau \leq v} \tanh(J_b(\tau)/T) = 0 \right\} .
\] (23)

For any vertex \( v \in \Pi \), the product in the rhs of the above expression is extended to all the vertices \( \tau \) belonging to the shortest path from 0 to \( v \) (\( \tau \leq v \)) and \( b(\tau) \) denotes the link connecting \( \tau \) to the preceding point along the shortest path for \( v \). Formally, in terms of the universal quantities \( \{z_b(I)\} = \{\tanh(\beta J_b)\} \), equation (23) says that the critical point \( \{w_b^{(I)}\} \) of this non-uniform Ising model is located at

\[
\{w_b^{(I)}\} = \sup \left\{ \{z_b\} : \inf_{\Pi} \sum_{v \in \Pi} \prod_{\tau \leq v} z_b(\tau) = 0 \right\} .
\] (24)
Correspondingly, by applying the rules (14) and (15) to a spin glass with a measure $d\mathcal{P} = \prod_{b \in \Gamma} d\mu_b$, we find that a spin glass transition and a disordered ferromagnetic transition take place at $\beta_c^{(SG)}$ and $\beta_c^{(F)}$, respectively solutions of the following two equations:

\[
\beta_c^{(SG)} = \sup \left\{ \beta : \inf \sum_{v \in \Pi} \prod_{\tau \leq v} \int d\mu_{b(\tau)} \tanh^2(\beta J_{b(\tau)}) = 0 \right\}, \tag{25}
\]

\[
\beta_c^{(F)} = \sup \left\{ \beta : \inf \sum_{v \in \Pi} \prod_{\tau \leq v} \int d\mu_{b(\tau)} \tanh(\beta J_{b(\tau)}) = 0 \right\}. \tag{26}
\]

Finally, the upper critical surface follows by applying equation (13), whereas the multicritical point $P$–$F$–$SG$ is the simultaneous solution of the system of equations (25) and (26).

Note that equations (22) and (26) for a disordered ferromagnetic Ising model were already found in [10] but limited to measures $d\mu_b$ with positive support of the couplings, $\{J_b \geq 0\}$, whereas equations (21) and (25) for the spin glass boundary constitute a completely novel result.

5. Random graphs

In the case of random graphs, an ensemble of graphs $G$, together with a probability distribution $P(g)$ for the elements $g$ of $G$, are given. The free energy of a spin glass model defined over the ensemble $G$ is then evaluated as

\[
-\beta F \equiv \sum_{g \in G} P(g) \int d\mathcal{P} (\{J_b\}) \log (Z_g (\{J_b\})) , \tag{27}
\]

where $Z_g (\{J_b\})$ is the partition function of the quenched system in the graph $g$ with couplings $\{J_b\}$.

In the following, we will apply the mapping to the so-called ‘uncorrelated equilibrium random graphs’, which are graphs maximally random with the constraint that the degree distribution of a vertex is a given one, $p(k)$, $k$ being the number of connections of the vertex. For this family of graphs the corresponding sets of links $\Gamma_g$ are infinite dimensional in the broad sense. In fact, even if the number of loops with infinite length per vertex is infinite (see e.g. [12] and [13]), due to the fact that the bonds are randomly distributed with the only constraint of the given degree distribution $p(k)$, the probability that two of these loops overlap for an infinite number of bonds is zero. The fact that the number of finite loops is infinite in certain scale free networks has been instead recently raised as a possible doubt about the validity of the random tree approximation for the solution of the Ising model on the equilibrium random graphs [12]. Note, however, that the mapping would not be affected by such short loops; we recall in fact that the validity of the mapping relies only on the infinitely long loops.

5.1. Ising model over equilibrium random graphs

Before considering the Ising spin glass over a random graph, we need to recall the general solution for the critical temperature of the Ising model over equilibrium random
graphs \([4, 5]\). Let us indicate with \(\langle f(k)\rangle_p \equiv \sum_{k=0}^{\infty} p(k)f(k)\) the average of the degree function \(f(k)\) with respect to the degree distribution \(p(k)\). The critical inverse temperature \(\beta_c\) of the uniform Ising model with a coupling \(J\) is given by the following relation:

\[
\tanh(\beta_c J) = \frac{\langle k \rangle_p}{\langle k^2 \rangle_p - \langle k \rangle_p}.
\]  

It is interesting to see that this result can be approximately derived by directly using our mapping. Let us consider a random graph with \(N\) vertices and \(L\) bonds so that the average degree is \(\langle k \rangle_p = 2L/N\). A given couple of vertices among the total of \(N(N-1)/2\) couples will be connected by one of the \(L\) bonds with a probability \(p = 2L/(N(N-1)) = \langle k \rangle_p/(N - 1)\). Therefore, if we neglect the correlations among the bonds, an Ising model with coupling \(J\) defined over the graph will have an effective distribution of couplings described by

\[
dP\{\{J_b\}\} \equiv \prod_{b \in \Gamma_f} d\mu_b(J_b),
\]  

where

\[
d\mu(J_b) \equiv \delta(J_b - J)p + \delta(J_b)(1 - p).
\]

According to equations (7)–(9), the related Ising model of this disordered model, whose disorder is given by equations (29) and (30), is the Ising model defined over the fully connected graph \(\Gamma_f\) for which a ferromagnetic or antiferromagnetic phase transition occurs at \(\beta_c^{(I)}J^{(I)}N = 1\), \(J^{(I)}\) being a generic coupling, positive or negative, respectively. Therefore, if we now use the measure (30) in equation (12), and \(v_{F/AF}^{(I)} = \tanh(\beta_c^{(I)}J^{(I)}) \to \pm 1/N\) for \(N\) large, we find that the critical inverse temperature \(\beta_c\) of the Ising system over the random graph, is given by \(\tanh(\beta_cJ) = \pm 1/\langle k \rangle_p\). This is equal to equation (28) only in the case of the classical random graph, for which \(\langle k^2 \rangle_p - \langle k \rangle_p = \langle k \rangle_p^2\) \([11]\). The error is due to the fact that the correlations among the bonds have been neglected or, in other words, that we have assumed for \(dP\{\{J_b\}\}\) a product measure. On the other hand, in a random graph the distribution of the bonds does not follow the degree distribution of the vertices. In fact, as is well known, the probability that a bond points to a node of degree \(k\) is not \(p(k)\), but \(p(k)/\langle k \rangle_p\). Correspondingly, one expects that in equation (30) the degree average of a vertex \(\langle k \rangle_p\) should be replaced with the degree average of the end of a bond: \(\langle k^2 \rangle_p/\langle k \rangle_p\). By implementing this substitution one arrives at \(\tanh(\beta_c^{(I)}J^{(I)}) = \langle k \rangle_p/\langle k^2 \rangle_p\), which constitutes a good approximation to the exact result (28)\(^1\).

5.2. Spin glass over random graphs

As pointed out above, in a random graph it is not possible to use an effective product measure \(dP\{\{J_b\}\}\) to rigorously take into account the degree distribution as well. However, for an exact treatment we can proceed as follows. Let some disorder \(d\mu(J_b)\) be given (we consider here only the case of a disorder identical for any bond). We observe that in equation (27) the averages with respect to \(dP\{\{J_b\}\}\) and \(P(g)\) are interchangeable, so

\(^1\) Such an intuitive argument has already been used in [4] and [5], though within different procedures.
that, if for any fixed graph \( g \) we apply our mapping (see equations (67) of [9]), for the free energy \( F \) we find

\[
-\beta F = N \log(2) + \sum_{g \in G} P(g) \sum_{b \in \Gamma_g} \int d\mu_b(J_b) \log(\cosh(\beta J_b)) + N \sum_{g \in G} P(g) \varphi(g),
\]

(31)

where \( \varphi(g) \) is proportional to the non-trivial part of the high temperature expansion of the free energy density over the graph \( g \) and, at least in the limit \( \beta \to \beta_c(g) \), it can be evaluated by means of \( \varphi_{\text{eff}}(g) \) through the mapping equations (24) and (25) of [9], \( \beta_c(g) \) being the critical inverse temperature of the spin glass model defined over the (quenched) graph \( g \) to be determined according to equations (10)–(12). In other words, for any fixed graph \( g \), the mapping says that the critical behaviour of the system is encoded in the effective density \( \varphi_{\text{eff}}(g) \) which is nothing else that the non-trivial part of the high temperature expansion of the uniform Ising model defined over the graph \( g \) and in which, for any \( b \in \Gamma_g \), the bond parameter \( \tanh(\beta J_b) \) is simply replaced by \( \int d\mu(J_b) \tanh(\beta J_b) \) or \( \int d\mu(J_b) \tanh^2(\beta J_b) \), depending on which transition we are interested in: disordered ferromagnetic or spin glass, respectively. Now, since in the thermodynamic limit the relative fluctuations of the graphs vanish [15, 16], we have that \( \beta_c(g) \) becomes the same for almost any \( g \in G \) (with respect to the measure \( P(g) \)) so that in the limit \( \beta \to \beta_c(g) \), the ensemble \( G \) of the Ising spin glass models can be effectively read as an ensemble \( G \) of uniform Ising models in which the parameters of the high temperature expansion are given by \( \int d\mu(J_b) \tanh(\beta J_b) \) or \( \int d\mu(J_b) \tanh^2(\beta J_b) \), depending on which transition we are interested in. Due to this Ising representation, by using equation (28) (which holds for an ensemble of uniform Ising models and where now \( \beta_c \) and \( J \) must be meant as quantities of the related Ising model, i.e., \( \beta_c^{(I)} \) and \( J^{(I)} \), respectively) in equations (11) and (12) applied to an ensemble of random graphs with weights \( P(g) \) and bond parameters \( \int d\mu(J_b) \tanh(\beta J_b) \) or \( \int d\mu(J_b) \tanh^2(\beta J_b) \), we immediately find the following disordered ferromagnetic or spin glass critical temperatures, respectively:

\[
\int d\mu(J_b) \tanh(\beta_c^{(F)} J_b) = \frac{\langle k \rangle_p}{\langle k^2 \rangle_p - \langle k \rangle_p},
\]

(32)

\[
\int d\mu(J_b) \tanh^2(\beta_c^{(SG)} J_b) = \frac{\langle k \rangle_p}{\langle k^2 \rangle_p - \langle k \rangle_p}.
\]

(33)

Finally, by applying equation (10) the upper critical surface of the phase diagram follows, whereas the multicritical P–F–SG point is the simultaneous solution of the system of equations (32) and (33).

Equations (32) and (33), together with their consequences, have been recently derived in the framework of the replica approach in [8], where an analysis of the order parameters has also been carried out. We stress however that, unlike that of [8], where the random graphs considered were confined to the particular class generated by the so-called ‘static model’ [14], our derivation shows that equations (32) and (33) hold for the most general uncorrelated equilibrium random graphs. More generally, by using the same above scheme, given an ensemble of random graphs \( G \) distributed with some distribution \( P(g) \), if the set \( \Gamma_g \) is infinite dimensional in the broad sense for almost any \( g \in G \) with respect to \( P(g) \), the critical disordered ferromagnetic and spin glass critical temperatures are, respectively,
the solutions of the following two equations:

\[ \int d\mu (J_b) \tanh(\beta_c^{(F)} J_b) = w^{(I)}(G; P) \tag{34} \]
\[ \int d\mu (J_b) \tanh^2(\beta_c^{(SG)} J_b) = w^{(I)}(G; P), \tag{35} \]

where \( w^{(I)}(G; P) \) is the (universal) Ising critical parameter in the ensemble \( G \) with distribution \( P \) of the uniform Ising model having coupling \( J^{(I)} \); \( w^{(I)}(G; P) = \tanh(\beta_c^{(I)}(G; P) J^{(I)}) \), \( \beta_c^{(I)}(G; P) \) being the inverse critical temperature. In the case of equilibrium random graphs, equations (34) and (35) reduce to equations (32) and (33), respectively.

As anticipated in the introduction, we stress that, in general, a numerical experiment to measure the Ising critical parameter \( w^{(I)}(G; P) \) turns out to be hugely easier than a direct measurement of the critical temperature \( \beta_c \) in the spin glass model.

6. Conclusions

A general mapping presented in part I of this work [9], allows one to derive the upper critical surface of an Ising spin glass model starting from the critical temperature of a related Ising model. When the dimensionality of the set of links \( \Gamma \) is infinite, the mapping is exact. In this paper, we have applied such a mapping to the case of quenched generalized tree-like structures and networks.

Since in a generalized tree-like structure the number of loops per vertex is finite, these structures turn out to be infinite dimensional. Therefore, by using the known solution for the Ising model over a generalized tree-like structure, we have derived the upper critical surface. The result found for the ferromagnetic disordered transition represents a generalization to arbitrary measures \( d\mu_b \) of a result previously established, holding for measures with positive support (\( \{ J_b \geq 0 \} \)), whereas those found for the spin glass transition are completely novel results.

As regards networks, which are ensembles of random graphs characterized by some distribution \( P(g) \), the dimensionality is infinite as long as \( P(g) \) does not generate infinite long paths with an infinite overlapping of bonds, as happens in the so-called equilibrium random graphs. In the case of equilibrium random graphs, whose \( P(g) \) can be completely described in terms of the only degree distribution of the vertices \( p(k) \), the known solution for the Ising model allowed us to find the exact upper critical surface of the Ising spin glass model over the network. In this case, as in the Ising model, the critical surface is expressed in terms of the first and second moments of \( p(k) \). We stress that this result is very general, holding for any kind of uncorrelated equilibrium random graph and not only for the so-called ‘static model’ [8]. Yet, we have provided a more general formula which, in terms of the critical temperature of the Ising model, applies exactly to any network as long as almost (with respect to the measure \( P(g) \)) any graph \( g \) is infinite dimensional. Unlike in the case of the equilibrium random graphs, however, now the critical temperature of the Ising model and, therefore, of the Ising spin glass model are expressed in terms of averages more complicated than the simple moments of the degree distribution \( p(k) \). We point out, however, that, although in these cases the analytical knowledge of the critical temperature of the Ising model is lacking, a numerical measurement of this quantity turns
out to be hugely easier than a direct measurement of the critical temperature of the spin glass model.

By looking back at: equations (17) and (18) for the Bethe lattice case; equations (21) and (22) or (25) and (26) for the generalized tree-like graphs; equations (32) and (33) for the equilibrium random graphs; and equations (34) and (35) for more general random graphs, we recognize a sort of common structure for the critical surface and the critical behaviour. This is not surprising considering that the related Ising models of these models are essentially reduced to an Ising model over some tree-like graph distributed, if necessary, according to some density whose fluctuations are to be taken into account. In this sense, we can say that the mapping establishes a sort of universality among different Ising spin glass models.

We conclude by observing that, as already mentioned in part I of this work, our mapping does not seem to be peculiar to the Ising spin glass models. For more general models, a mapping at high temperature, ‘random system’ \( \rightarrow \) ‘non-random system’, seems possible; so a known solution for a given non-random model can be used to find the upper critical surface of the corresponding random model over infinite dimensional graphs or infinite dimensional networks.

Acknowledgments

This work was supported by DYSONET under NEST/Pathfinder initiative FP6, and by the FCT (Portugal) grant SFRH/BPD/24214/2005. I am grateful to A V Goltsev for useful discussions and to F Mukhamedov for bringing reference [10] to my attention. I also thank C Presilla for a critical reading of the manuscript.

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doi:10.1088/1742-5468/2006/10/P10005