A quantum measure of the multiverse

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Abstract. It has been recently suggested that probabilities of different events in the multiverse are given by the frequencies at which these events are encountered along the worldline of a geodesic observer (the “watcher”). Here I discuss an extension of this probability measure to quantum theory. The proposed extension is gauge-invariant, as is the classical version of this measure. Observations of the watcher are described by a reduced density matrix, and the frequencies of events can be found using the decoherent histories formalism of Quantum Mechanics (adapted to open systems). The quantum watcher measure makes predictions in agreement with the standard Born rule of QM.

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1 Introduction

A long-standing problem of inflationary cosmology is the so-called measure problem. In an eternally inflating universe, any event having a non-vanishing probability will occur an infinite number of times, and in order to assign probabilities to different events these infinities must be regulated. The problem is that the results turn out to be highly sensitive to the choice of the regulator. For example, if one counts only events prior to some global time $t$, the resulting probabilities are sensitive to the choice of the time variable. (For a review of the measure problem, see, e.g., [1].) An additional puzzle arises in the context of quantum measurements. Even if the regulator is specified, the probabilities of different measurement results cannot be expressed as expectation values of projection operators, as required by the Born rule [2]. The reason is that when the universe is so large that it contains multiple copies of the experiment, one needs an additional rule that would select a specific observer among the identical copies (or assign probabilities to different copies [3]). Moreover, any acceptable probability measure should satisfy the ‘correspondence principle’, that is, its predictions should agree with those of regular Quantum Mechanics in the range of applicability of the latter. Some of the currently popular measures may satisfy this principle, at least under certain conditions, but this has not been explicitly demonstrated.

It has been recently suggested [4] that the classical measure problem can be naturally resolved by using a measure based on a probe geodesic traversing the multiverse and encountering each type of event an infinite number of times. This geodesic can be thought of as the worldline of an eternal observer, the “watcher”. (It should be remembered though...
that the watcher is not a physical entity and has no degrees of freedom of its own.) Probabilities of different events are then identified with the frequencies at which these events are encountered by the watcher. ¹ It was shown in [4] that the same probability distribution is obtained for all geodesics, except for a set of measure zero. A crucial assumption underlying this prescription is that the geodesics do not terminate. In particular, it is assumed that big crunches in the interiors of negative-energy (Anti-de Sitter) bubbles are non-singular and are followed by bounces with subsequent expansion, so that geodesics can be continued through the crunches. ²

The purpose of this paper is to extend the watcher measure to include quantum measurements. I will argue that quantum probabilities in this measure can be calculated using the formalism of decoherent histories in Quantum Mechanics, adapted to open systems. Furthermore, I will show that the Born rule is consistent with the watcher measure and can even be derived from it (modulo certain caveats).

The paper is organized as follows. In the next section, I argue that the horizon region of a geodesic observer in the multiverse should be regarded as an open system interacting with its environment, in contrast with a widely accepted view. In section 3, I review the classical watcher measure and propose its quantum extension. A special case where the effect of environment can be described by a Markov stochastic process is considered in section 4. Consistency with the Born rule is demonstrated in section 5. Finally, the conclusions of the paper are briefly summarized in section 6.

2 Causal patch as an open system

Observations of the watcher are confined to its causal patch, that is, to the spacetime region inside its future horizon. In practice, a physical observer can monitor only a small fraction of the degrees of freedom in her causal patch. The remaining degrees of freedom, both inside and outside the horizon, should be regarded as “the environment” and should be traced over. The dynamics of the causal patch will then be described by a reduced density matrix. An important point, however, is that a density matrix description appears to be unavoidable, even if the observer monitors all the accessible degrees of freedom. The reason is that there is a constant flow of information through the horizon to the exterior space. As a result, the quantum state of the horizon region gets entangled with that of the exterior. Since the part of spacetime outside the horizon cannot be observed, even in principle, the corresponding degrees of freedom should be traced over. The horizon region should therefore be described by a reduced density matrix, even if the entire multiverse is in a pure quantum state [7].

It has often been argued [5, 8, 9] that a causal patch in a purely de Sitter landscape evolves like a finite closed system, with the horizon acting as an impenetrable membrane, allowing no loss of information. It has also been suggested [10] that a similar picture should apply even in the presence of Anti-de Sitter vacua. The causal patch would then reach a state of thermal equilibrium and would remain in that state forever. Apart from very rare large thermal fluctuations, this state would not exhibit any arrow of time. I will now briefly summarize some of the arguments for and against this equilibrium picture of de Sitter space,

¹This measure prescription has close similarity to Nomura’s single observer measure [5] and to the so-called “fat geodesic” measure [6], but there are also important differences. For a detailed discussion see [4].
²If the vacuum landscape includes stable Minkowski vacua, then a generic geodesic eventually enters such a vacuum and stays there forever. Watcher’s worldlines should then be selected from a measure-zero class of geodesics which do not get stuck in stable Minkowski vacua [4].
but I will say at the outset that in the present paper I will not subscribe to this view and assume instead that the causal patch interacts with the environment and evolves as an open system.

In classical GR, a causal patch in de Sitter space can be described using a static coordinate system,
\[
  ds^2 = (1 - H^2 r^2) dt^2 - (1 - H^2 r^2)^{-1} dr^2 - r^2 d\Omega^2.
\] (2.1)

Timelike and null geodesics approach the horizon \( r = H^{-1} \) in the asymptotic future, but do not leave the causal patch in a finite time \( t \). Moreover, no timelike or null geodesic can enter the causal patch from outside. This seems to suggest that the causal patch is indeed a closed system. However, things may be different in quantum theory. A detector at \( r = 0 \) detects a thermal spectrum of particles emanating from the horizon. These particles can be thought of as being produced in pairs, with the two members of the pair being on opposite sides of the horizon. As a result the particles within the causal patch are entangled with the particles outside.

It has been recently suggested [11–13] that the semiclassical spacetime geometry may be drastically modified in the vicinity of black hole horizons, resulting in “fuzzballs” or “firewalls”. The firewalls absorb and re-emit all incoming information, so the evolution of the region outside the horizon is unitary. One might expect that a similar picture could apply to de Sitter space, but the problem is that de Sitter horizons are observer-dependent. Spacetime distortion and firewalls near the horizon appear to be inconsistent with the fact that all points in de Sitter space are equivalent.

As pointed out in [4], an interesting situation arises in multiverse models with a purely de Sitter landscape, where all vacua have positive energy density. The semiclassical transition rate from vacuum \( j \) to vacuum \( i \), detected by an inertial observer, is then given by [14]
\[
  \kappa_{ij} = (4\pi/3) H_j^{-3} \Gamma_{ij},
\] (2.2)
where \( H_j = (8\pi \rho_j / 3)^{1/2} \) is the de Sitter expansion rate in vacuum \( j \) (in Planck units), \( \rho_j \) is the corresponding vacuum energy density, and \( \Gamma_{ij} \) is the nucleation rate per unit spacetime volume for bubbles of vacuum \( i \) in parent vacuum \( j \),
\[
  \Gamma_{ij} = A_{ij} e^{-I_{ij} - S_j}.
\] (2.3)
Here, \( I_{ij} \) is the Euclidean action of the tunneling instanton, \( A_{ij} \) is a prefactor arising from integration of small perturbations around the instanton, and \( S_j \) is the entropy of vacuum \( j \). The instanton action and the prefactor \( A_{ij} \) are symmetric with respect to interchange of \( i \) and \( j \) [15]. Hence, we can write
\[
  \frac{\kappa_{ij}}{\kappa_{ji}} = (H_i / H_j)^3 \exp(S_i - S_j).
\] (2.4)

The quantity \( e^{S_j} \) has the interpretation of the number of (accessible) microstates in a horizon region of vacuum \( j \), and the relation
\[
  \frac{\kappa_{ij}}{\kappa_{ji}} \propto \exp(S_i - S_j)
\] (2.5)
can be thought of as expressing the detailed balance condition, which is necessary for equilibrium (microcanonical) distribution to establish [16, 17]. However, the prefactor in (2.4)
violates the detailed balance. Even if this violation is small, it indicates that the equilibrium picture of de Sitter space can only be approximate.\(^3\)

Furthermore, in the presence of Anti-de Sitter vacua, it was argued in ref. \([4]\) that strong violations of detailed balance are likely to occur at bounces replacing the big crunch singularities in Anti-de Sitter bubbles. Because of the high energy densities reached near the bounce, the crunch regions are likely to be excited above the energy barriers between different vacua, so transitions to other vacua are likely to occur \([18–20]\). The duration of the bounce, however, is very short, so there is no time for the region to explore its available phase space and reach thermal equilibrium. In particular, there seems to be no reason for transition probabilities from the bounce region to different de Sitter vacua to be related to the entropies of those vacua. Such violations of ergodicity may be responsible for the observed arrow of time.

Realizing that these issues are far from being settled, here we shall adopt the following assumptions and explore heir consequences for the measure problem. (1) Anti-de Sitter bounces do occur and are accompanied by strong violations of ergodicity. The bounces allow a semiclassical description, so the watcher’s geodesic can be continued through the bounces.\(^4\) (2) The causal patch of the watcher interacts with its environment; its evolution can be described by a reduced density matrix.

3 The watcher measure and its quantum extension

3.1 The classical watcher measure

In the classical version of the watcher measure \([4]\), an event \(A\) is counted if the watcher’s geodesic crosses the spacetime domain \(D_A\) of the event. The domain \(D_A\) is defined as the minimal spacetime region necessary to distinguish this type of event from others. In order to account for the different sizes of the domains, the number of encountered events is then renormalized by a factor \(\sigma_{D_A}^{-1}\), where \(\sigma_A\) is the cross-section (having the dimension of 3-volume) that the domain \(D_A\) presents to the watcher’s geodesic.

Here I am going to use a slightly different, but essentially equivalent prescription. For each domain \(D_A\) we shall define a point that we shall call its center. If \(D_A\) is small enough, so that spacetime curvature in \(D_A\) can be neglected, then we can define its center by analogy with the center of mass: in the standard Minkowski coordinates, the center is a point \(x_{\mu_A}^\mu\), such that

\[\int_{D_A} d^4x (x^\mu - x_{\mu_A}^\mu) = 0. \tag{3.1}\]

This definition can be generalized to curved spacetime as

\[\int_{D_A} d^4x \sqrt{-g} u^\mu(x, x_A) = 0, \tag{3.2}\]

\(^3\)One might think that quantum gravity corrections to the entropy might compensate for the prefactor \(H^{-3}\), thus restoring detailed balance. However, the factor \(e^{S_j}\) in (2.3) is shorthand for the semiclassical path integral around the Euclidean de Sitter saddle point corresponding to the parent vacuum and already includes quantum corrections \([4]\).

\(^4\)A “geodesic” is a classical concept that can only be defined in a semiclassical spacetime background. Some approaches to quantum gravity, in particular the holographic ideas, suggest that a full quantum description should be in terms of the wave function (or density matrix) of a region encompassed by an apparent horizon surface (e.g., \([5, 7, 21]\)). Then geodesics representing possible trajectories of a watcher may play no fundamental role, except perhaps in some appropriate limit.
where \(v^\mu(x, x_A)\) is a vector at point \(x_A\) pointing in the direction of the (shortest) geodesic connecting the points \(x_A\) and \(x\) and having magnitude proportional to the length of that geodesic. In fact, the precise definition of the center is unimportant, as long as it is a well specified point in the domain \(D_A\).

We shall adopt the prescription that an event of type \(A\) is counted whenever the center of its domain lies within a specified small distance range \(\epsilon\) from the watcher’s geodesic. If this condition is satisfied, we shall say that the event has been encountered by the watcher. The precise value of \(\epsilon\) is also unimportant, as long as it is sufficiently small. This prescription is essentially the same as that in the fat geodesic measure \(\mu\), except that we assume that \(\epsilon\) is smaller than the domain \(D_A\), so no more than one event can be encountered at a time.

Let \(t = 0\) be an arbitrary point on the watcher’s worldline and \(N_A(T)\) the number of events of type \(A\) encountered during the time interval \(0 < t < T\). The relative probability of events \(A\) and \(B\) is then identified with the relative frequency of these events as they are encountered by the watcher,

\[
p_A = \lim_{T \to \infty} \frac{N_A(T)}{N_B(T)}.
\] (3.3)

In this formulation, no renormalization of the numbers of events is required. It is also clear that the resulting probabilities are independent of the choice of the time variable \(t\), as long as it is monotonic along the watcher’s geodesic.

### 3.2 Decoherent histories for an open system

An extension of the watcher measure to quantum theory is most naturally obtained using the decoherent histories formulation of Quantum Mechanics [23–25]. Possible histories of the watcher can be represented by chains of projection operators at a sequence of times, \(0 < t_1 < t_2 < \ldots < T\),

\[
h : P_{i_1}, \ldots P_{i_n},
\] (3.4)

where \(t_n = T\) and the subscript \(i_k\) indicates the alternative that has been chosen at time \(t_k\) in the particular history \(h\). The projectors \(P_{i_k}\) act in the Hilbert space of the watcher’s causal patch; the watcher itself is completely specified by its geodesic and has no independent degrees of freedom of its own.

We assume that the projectors \(P_{i_k}\) belong to an exhaustive and mutually exclusive set,

\[
\sum_i P_i = 1,
\] (3.5)

\[
P_i P_j = P_i \delta_{ij}.
\] (3.6)

We shall think of these projectors as representing records of events that happened in the time interval \(t_{k-1} < t < t_k\), rather than the events that occurred at the moment \(t_k\). For example, if some measurement was made in this time interval, then \(P_{i_k}\) are the projectors on the possible outcomes of the measurement.

For any two histories \(h\) and \(h'\), we can define the decoherence functional [25]

\[
D(h', h) = \text{Tr} \left( P_{i_n} K_{i_n-1}^{t_n} \cdots P_{i_1} K_{i_1}^{t_1} [P_{i_n} \rho(0) P_{i_n}] \cdots P_{i_1} \rho(0) P_{i_1} \right).
\] (3.7)

\(^5\)The choice of a center may be important in cases where the event \(A\) represents a “story” whose domain extends over more than a Hubble time in the time direction. This leads to the Guth-Vanchurin paradox [22], and the probabilities will depend on whether we choose the center at the beginning or at the end of the story.
Here, the ‘super-operators’ $K_{t_i}^{t_{i+1}}$ evolve the density matrix from $t_i$ to $t_{i+1}$,

$$K_{t_i}^{t_{i+1}}[\rho(t_i)] = e^{-i\mathcal{H}(t_{i+1}-t_i)} \rho(t_i) e^{i\mathcal{H}(t_{i+1}-t_i)} = \rho(t_{i+1}), \quad (3.8)$$

$\rho(0)$ is the initial density matrix at $t = 0$ and $\mathcal{H}$ is the Hamiltonian.

We say that histories $h$ and $h'$ decohere if $D(h', h) \approx 0$. If all histories in the set (3.4) decohere, then each history can be assigned a probability,

$$p(h) = D(h, h). \quad (3.9)$$

Histories generally decohere when the time intervals $\Delta t_k = t_k - t_{k-1}$ are increased and when the history is coarse grained (that is, when the projectors are bunched together into a smaller set of projectors). With a suitable choice of basis (the so-called ‘pointer basis’), decoherence can be achieved even with minimal coarse-graining and for a very small time separation between successive events [26].

In the case of an open system, the universe is divided into the ‘system of interest’ $S$ and the ‘environment’ $E$. The histories of the system are then specified by projectors of the form $P = I_E \otimes P_S$, where $I_E$ is the identity in the Hilbert space of the environment and $P_S$ is a projector in the Hilbert space of the system. However, the trace in eq. (3.7) for the decoherence functional should still be taken over the full Hilbert space, including both the system and the environment, and the Hamiltonian $\mathcal{H}$ in the evolution operators in (3.8) is the full Hamiltonian. It is not generally possible to express the decoherence functional only in terms of the reduced density matrix of the system,

$$\tilde{\rho}(t) = \text{Tr}_E \rho(t). \quad (3.10)$$

The reason is that the evolution of the system is generally influenced by the correlations that develop between the system and the environment. We expect, however, that such correlations should be unimportant for the causal patch of the watcher. For example, the members of particle-antiparticle pairs outside the de Sitter horizon are quickly driven away by the de Sitter expansion, and it seems reasonable to assume that they have little effect on the subsequent evolution of the horizon interior.$^6$ Assuming this to be the case, it should be possible to define the evolution operator $\tilde{K}_{t_i}^{t_{i+1}}$ for the reduced density matrix [26]

$$\tilde{K}_{t_i}^{t_{i+1}}[\tilde{\rho}(t_i)] = \tilde{\rho}(t_{i+1}). \quad (3.11)$$

The decoherence functional is then given by

$$D(h', h) = \text{Tr}_S \left( P_{t_n} \tilde{K}_{t_{n-1}}^{t_n} \cdots P_{t_1} \tilde{K}_{t_0}^{t_1} [\tilde{\rho}(0)] P_{t'_1} \cdots P_{t'_{n-1}} \right) P_{t'_n}. \quad (3.12)$$

It should be noted that the evolution operator $\tilde{K}_{t_i}^{t_{i+1}}$ for an open system is not generally given by eq. (3.8).

A potential problem with the decoherent histories formalism is that one can generally consider a number of different consistent sets of projection operators $P_k$. The formalism works fine as long as we work within a single consistent set, but once we try to compare the probabilities of histories belonging to different sets, we get into trouble [27, 28]. As a simple

$^6$In models of inflation, regions outside the apparent horizon can later become observable. In such cases super-horizon correlations can be significant, but it appears that correlations beyond the true causal horizon can still be neglected.
example, suppose the projector at time \( t_1 \) represents a measurement of the spin projection of a spin\(-\frac{1}{2}\) particle on some axis. However, the axis is not fixed and the observer can choose it to be either \( x \) or \( z \)-axis. For the first choice, the corresponding projection operators are \( P_x^+ \) and \( P_x^- \), and for the second choice they are \( P_z^+ \) and \( P_z^- \), where + and − refer to the spin projection along and opposite to the chosen axis, respectively. Either of these operator pairs can be incorporated into a consistent set of histories, but we cannot incorporate both of them at once.

This issue has been debated for some time, and a number of possible solutions have been suggested [29–31]. Perhaps the simplest one is to allow the sets of operators defining the consistent histories to be branch-dependent [31]. That is, the set of alternatives at a given time \( t_k \) generally depends on the specific history at \( t < t_k \). In the above example, one introduces an additional set of projection operators, \( P_{\text{choose}}^\pm \) and \( P_{\text{choose}}^\mp \), acting at some \( t_0 < t_1 \). The subsequent projectors are \( P_x^\pm \) or \( P_z^\pm \), depending on whether the \( x \) or \( z \) direction is chosen at this step. Our analysis can be straightforwardly extended to this modified consistent histories formalism.

### 3.3 The quantum watcher measure

In order to extend the watcher measure to quantum theory, we shall assume that the set of projection operators \( P_{ik} \) includes projectors on the states corresponding to all types of events of interest. The number of events of type \( A \) in a history \( h \) defined by the chain of projectors (3.4) is then

\[
N_A(h; T) = \delta_{i_1,A} + \delta_{i_2,A} + \ldots , \tag{3.13}
\]

and the average number of such events in the time interval \( 0 < t < T \) is given by the sum over histories

\[
N_A(T) = \sum_h p(h)N_A(h; T) . \tag{3.14}
\]

As in the classical version of the measure, we shall identify the relative probability of events \( A \) and \( B \) with their relative frequency,

\[
\frac{p_A}{p_B} = \lim_{T \to \infty} \frac{N_A(T)}{N_B(T)} , \tag{3.15}
\]

where \( N_A \) and \( N_B \) are now given by (3.14). We assume that the time intervals \( \Delta t_i = t_i - t_{i-1} \) are sufficiently small, so that no relevant events are missed between the sampling times \( t_i \). For example, one can choose \( \Delta t_i \) to be somewhat larger than the characteristic time of decoherence. We also assume that the typical time separation \( \Delta t_i \) is kept fixed in the limit \( T \to \infty \), so the number of sampling moments \( n \) becomes infinite in the limit.

Assuming that the vacuum landscape is irreducible, that is, that any vacuum can be reached from any other vacuum in a finite number of transitions, the initial state at \( t = 0 \) will eventually be completely forgotten, and the asymptotic frequencies of events will be determined entirely by the properties of the landscape. The relative probability (3.15) should therefore be independent of the initial density matrix \( \tilde{\rho}(0) \) (which appears in the definition of \( p(h) \); see eqs. (3.9), (3.12)). One can, for example use some pure state \( |\psi\rangle\langle\psi| \) or the asymptotic density matrix \( \tilde{\rho}(\infty) \).
3.4 Timestep evolution

Suppose we want to find relative probabilities for some set of events, labeled by index $J = 1, 2, \ldots$. To distinguish these events of interest from other, irrelevant events, we shall refer to them as “marked events”. We shall consider a set of alternative histories specified by different sequences of marked events, $J_1, J_2, \ldots$, without specifying the times at which the events occurred. These histories can be thought of as representing branching Everett’s worlds, with the branching points corresponding to marked events.

Next, we define a branching ratio $T_{IJ}$ as the probability to observe event $I$ after observing event $J$ (without any marked events in between). This can be found as

$$T_{IJ} = \sum_J p(h),$$

(3.16)

where the summation is over all histories $h$ starting at $J$ and encountering $I$ before any other marked event.\(^7\) (Note that we do not assume that all histories in the sum (3.16) encounter event $I$ at the same time.) From the definition of the branching ratios it is clear that they satisfy

$$\sum_I T_{IJ} = 1$$

(3.17)

and that they do not depend on the choice of the time variable $t$.

We now introduce a discrete timestep variable $n$, which takes integer values, $n = 0, 1, 2, \ldots$, and which is incremented by one at every branching transition. Given a probability distribution $p_J(n)$ at step $n$, the distribution at step $(n + 1)$ can be found from the equation

$$p_I(n + 1) = \sum_J T_{IJ} p_J(n).$$

(3.18)

The distribution $p_I(n)$ can be thought of as describing an ensemble of branching histories, with each history including $n$ events. In the limit $n \to \infty$, $p_I(n)$ approaches a stationary distribution $p_I^{(\infty)}$ satisfying

$$p_I^{(\infty)} = \sum_J T_{IJ} p_J^{(\infty)}.$$

(3.19)

This has a unique solution, assuming that the marked set of events is irreducible, that is, that it does not split into subsets which cannot be accessed from one another [4]. Since $T_{IJ}$ do not depend on the choice of $t$, the solution $p_I^{(\infty)}$ should also be gauge-independent. The standard relation between the ensemble and time averaging implies that $p_I^{(\infty)}$ should be equal to the frequency at which event $I$ is encountered by the watcher. We therefore expect the frequencies of events found from eq. (3.19) to agree with those found from (3.15).

\(^7\)We are usually interested in macroscopic events, represented by a large number of microscopically indistinguishable states. Such states are characterized by density matrices which may depend on prior events. Here I ignore this complication and assume that each marked event corresponds to a pure state.
4 Markovian evolution

4.1 The Lindblad equation

As an illustration, we shall now consider a simplified model, where the reduced density matrix \( \tilde{\rho}(t) \) can be represented as

\[
\tilde{\rho}(t) = \sum_j p_j(t) |j\rangle \langle j |. \tag{4.1}
\]

Here, \(|j\rangle\) are Schrodinger state vectors,

\[
i \frac{\partial}{\partial t} |j\rangle = \tilde{H} |j\rangle, \tag{4.2}
\]

which are assumed to form an orthonormal basis in the Hilbert space, and \( \tilde{H} \) is the Hamiltonian of the watcher’s system (the causal patch). The quantity \( p_j(t) \) has the meaning of the probability to find the system in state \( j \), and

\[
\sum_j p_j(t) = 1. \tag{4.3}
\]

For a closed system we would have \( p_j = \text{const} \), but we shall assume that the Hamiltonian evolution (4.2) is punctuated by transitions caused by interaction with the environment, resulting in time variation of \( p_j \). We shall also assume that the evolution of \( p_j(t) \) is Markovian\(^8\) and is described by the rate equation

\[
\dot{p}_i = \sum_j (\kappa_{ij} p_j - \kappa_{ji} p_i), \tag{4.4}
\]

where \( \kappa_{ij} \) are the corresponding transition rates. These assumptions should apply if the transitions are due to bubble nucleation in the multiverse; then eq. (4.4) is the standard rate equation [32] with \( \kappa_{ij} \) given by (2.2) (assuming that the time variable \( t \) is the proper time along the watcher’s geodesic). More generally, the assumptions may give a reasonable approximation if the basis \(|i\rangle\) is chosen to be the “pointer basis”, in which decoherence occurs on a very short timescale [26]. I will later indicate how the assumptions can be relaxed.

Differentiating eq. (4.1) with respect to \( t \) and using eqs. (4.2), (4.4), we obtain

\[
\dot{\tilde{\rho}}(t) = -i[\tilde{H}, \tilde{\rho}] + \sum_{i,j} \kappa_{ij} p_j (|i\rangle \langle i| - |j\rangle \langle j|). \tag{4.5}
\]

Introducing the operators

\[
Q_{ij} = |i\rangle \langle j|, \tag{4.6}
\]

\[
Q_{ij}^\dagger = |j\rangle \langle i| = Q_{ji}, \tag{4.7}
\]

we have

\[
Q_{ij}^\dagger \dot{\tilde{\rho}}(t) Q_{ij} = p_i(t) |j\rangle \langle j|, \tag{4.8}
\]

\[
Q_{ij} Q_{ij}^\dagger = |i\rangle \langle i|, \tag{4.9}
\]

\(^8\)Markovian means that the evolution has no memory, so that \( \dot{\tilde{\rho}}(t) \) depends on \( \tilde{\rho} \) at time \( t \), but not at earlier times.
and
\[ Q_{ij} Q_{ij}^\dagger \dot{\rho}(t) = p_i(t) |i\rangle \langle i| = \dot{\rho}(t) Q_{ij} Q_{ij}^\dagger. \] (4.10)

With the aid of these relations, we can rewrite eq. (4.5) as
\[
\dot{\rho} = -i[\hat{\mathcal{H}}, \rho] - \frac{1}{2} \sum_{i,j} \kappa_{ij} [Q_{ij}^\dagger Q_{ij} \rho + Q_{ij}^\dagger \rho Q_{ij}^\dagger - 2 Q_{ij} \rho Q_{ij}^\dagger] \equiv -i[\hat{\mathcal{H}}, \rho] + \mathcal{L} \rho.
\] (4.11)

An equation of the form (4.11) is known as the Lindblad equation [33]. This is the general form of a linear, Markovian evolution equation for the density matrix that preserves its unit trace and positivity. These properties hold for arbitrary operators $Q_{ij}$. With $Q_{ij}$ of the form (4.6), the Lindblad equation (4.11) has no more content than the rate equation (4.4), but it may be useful in a more general context, when the ansatz (4.1) for the density matrix is not imposed. Nonlocal generalizations of eq. (4.11), where the evolution is non-Markovian, have also been discussed (e.g., [34]). A method for solving the Lindblad equation (4.11) in an operator form has been given in [35].

The microcanonical distribution,
\[
\tilde{\rho} \propto I,
\] (4.12)
where $I$ is a unit operator, is a solution of eq. (4.11), provided that
\[
\sum_{i,j} \kappa_{ij} [Q_{ij}, Q_{ij}^\dagger] = 0.
\] (4.13)

The latter condition is satisfied, for example, when $Q_{ij}^\dagger = Q_{ji}$ and $\kappa_{ij} = \kappa_{ji}$. If (4.12) is a solution, then this solution is approached asymptotically at late times. According to our discussion in section 2, we expect (4.13) to be weakly violated in a purely de Sitter landscape and to be strongly violated in the presence of Anti-de Sitter bounces. In the general case, since $\kappa_{ij}$ and $Q_{ij}$ do not have explicit time dependence, we expect $\tilde{\rho}(t)$ to approach asymptotically a stationary distribution with
\[
[H, \tilde{\rho}(t \to \infty)] = 0, \quad \mathcal{L} \tilde{\rho}(t \to \infty) = 0.
\] (4.14)

### 4.2 Gauge-independence

The quantities $\kappa_{ij}$ in the rate equation (4.4) are transition rates per unit time, and their magnitude depends on one’s choice of the time variable. If $\tau$ is the proper time along the watcher’s geodesic, we can introduce a new variable $t$ as
\[
dt = H^\beta d\tau,
\] (4.15)
where $H$ is the Hubble expansion rate. (For $\beta = 1$, $t$ is the scale factor time.) The transition rates in the new time variable are then related to the proper time rates by
\[
\kappa_{ij}^{(\beta)} = H^{-\beta}_{j} \kappa_{ij}^{(0)},
\] (4.16)
so the rate equation can be rewritten as
\[
\dot{p}_i = \sum_j \left( \kappa_{ij}^{(0)} H^{-\beta}_j p_j - \kappa_{ji}^{(0)} H^{-\beta}_i p_i \right),
\] (4.17)
Solutions of this equation for $\beta \neq 0$ will clearly be different from the $\beta = 0$ proper time solutions. This applies in particular to the stationary distribution, which is approached in the limit $t \to \infty$,

$$p_j^{(\beta)}(t \to \infty) = H_j^{\beta} p_j^{(0)}(t \to \infty). \quad (4.18)$$

The density matrices for different values of $\beta$ will therefore also be different.

This gauge-dependence is not surprising. The density matrix $\tilde{\rho}(t)$ describes an ensemble of watchers at a given value of the global time $t$, and the gauge-dependence has the same origin as in a global time cutoff. An important point, however, is that the branching ratios $(3.16)$ which appear in eq. $(3.19)$ for the probabilities are gauge-independent. The gauge-independence of $T_{ij}$ can be verified explicitly in the case where the marked set of ‘events’ coincides with the complete set of states $|i\rangle$ in the pointer basis. Then we have

$$T_{ij} = \frac{\kappa_{ij}}{\sum_k \kappa_{kj}}, \quad (4.19)$$

and the gauge-dependent factor in eq. $(4.16)$ for $\kappa^{(\beta)}_{ij}$ cancels out.

5 The Born rule

5.1 Consistency with quantum mechanics

We shall now verify that our measure prescription agrees with the usual rules of quantum mechanics. That is, that the probabilities of different measurement results for a quantum system are given by the standard Born rule. The following argument is essentially the same as in refs. [36–40], except that these papers consider an ensemble of identical experiments, while we are interested in experiments encountered at different times along the watcher’s worldline.

As a simple example, we shall consider an experiment measuring the spin projection of a spin-1/2 particle on a given axis. Disregarding for a moment the environment degrees of freedom, the quantum state prior to the measurement is

$$|\psi\rangle_{\text{before}} = (\alpha |\uparrow\rangle + \beta |\downarrow\rangle) |A_r\rangle, \quad (5.1)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are respectively the states with spin in the “up” and “down” directions along the axis, $\alpha$ and $\beta$ are complex coefficients satisfying $|\alpha|^2 + |\beta|^2 = 1$, and $|A_r\rangle$ is the “ready” quantum state of the measuring apparatus. After the spin and the apparatus are allowed to interact, the spin state is entangled with that of the apparatus,

$$|\psi\rangle_{\text{after}} = \alpha |\uparrow\rangle |A_+\rangle + \beta |\downarrow\rangle |A_-\rangle, \quad (5.2)$$

where $|A_+\rangle$ and $|A_-\rangle$ are the states of the apparatus corresponding to “spin up” and “spin down” measurements, respectively. Shortly afterwards, interaction with the environment causes the superposition in $(5.2)$ to decohere. At that point the measurement is over, and its result is represented by the density matrix, obtained by tracing over all degrees of freedom except those of the apparatus,

$$\rho_A = |\alpha|^2 |A_+\rangle \langle A_+| + |\beta|^2 |A_-\rangle \langle A_-|. \quad (5.3)$$

It has been noted in [37] that the argument can also be applied to repeated measurements on a single system.
Like any other process, the above described measurement will be observed by the watcher an infinite number of times. The successive observations will be separated by enormous time intervals. If the watcher’s world line passes near the center of a spin measurement experiment in some habitable bubble, it is rather unlikely to hit an identical experiment in the same bubble. The world line will move on into the multiverse, crossing a number of different bubbles. It will eventually return to the quantum state (5.1) on a timescale which is set by bubble nucleation rates (and is typically very large). Let us consider histories including, apart from other irrelevant events, a sequence of \( N \) spin measurements at times \( t_1, t_2, t_3, \ldots, t_N \). There are \( 2^N \) distinct histories, represented by chains of projection operators of the form

\[
h : P_↑ \ldots P_↑ \ldots P_↓ \ldots ,
\]

where \( P_↑ \) and \( P_↓ \) are projectors onto the states \( |A_↑⟩ \) and \( |A_↓⟩ \) of the measuring device. The probability of a given history is

\[
p(h) = (|α|^2)^{N_↑} (|β|^2)^{N_↓},
\]

where \( N_↑ \) and \( N_↓ \) are respectively the numbers of up and down spin measurements in that history. Here I assume that all measurements are uncorrelated, which is justified, considering that the system completely forgets its initial state on the recurrence timescale.

The probabilities in eq. (5.5) are normalized so that

\[
\sum_h p(h) = 1,
\]

where the summation is over the histories (5.4). (Note that the measurement times \( t_1, t_2, \ldots \) are the same for all histories.) There is a factor \( |α|^2 \) for each spin up and a factor \( |β|^2 \) for each spin down measurement in (5.5). It follows that the probability of having a given number \( N_↑ \) of spin up measurements is given by a binomial distribution [38],

\[
p_N(N_↑) = \binom{N}{N_↑} (|α|^2)^{N_↑} (|β|^2)^{N-N_↑}.
\]

Let \( f_↑ = N_↑/N \) be the fraction of measurements that gave spin up. Its mean value is then

\[
⟨ f_↑ ⟩ = |α|^2,
\]

and its variance is

\[
δf_↑ ≡ (⟨ f_↑^2 ⟩ − ⟨ f_↑ ⟩^2)^{1/2} = \frac{|αβ|}{\sqrt{N}}.
\]

In the limit \( N → ∞ \), we have \( δf_↑ \to 0 \), and the distribution approaches a delta-function,

\[
p_N(f_↑) \to δ(f_↑ − |α|^2).
\]

Applied to the watcher measure, this means that the frequencies of spin up and spin down measurements are precisely given by the Born rule for all watcher histories, except a set of measure zero.\(^{10}\) It can be shown that the same conclusion applies to measurements that can have more than two different outcomes [36, 37, 39, 40] and to situations where the states of the macroscopic measuring device corresponding to different outcomes include large groups of macroscopically indistinguishable microstates [40].

\(^{10}\)It should be emphasized that this argument cannot be regarded as a derivation of the Born rule. In fact, we used the Born rule to find the probabilities (5.5) for different histories. We showed that these probabilities are consistent with the standard Born-rule probabilities for individual measurement outcomes, when the latter are identified with the frequencies of occurrence of different outcomes in the watcher’s history.
5.2 ‘Derivation’ of the Born rule

It has been suggested in refs. [36–40] (see also [41]) that a slight modification of the above analysis can be regarded as a derivation of the Born rule. Here is a rough sketch of the argument, adopted to our case.

A sequence of spin measurements observed by the watcher can be represented by a density matrix

\[
\rho = \rho_A^{(1)} \otimes \rho_A^{(2)} \otimes \cdots \otimes \rho_A^{(N)},
\]

with \( \rho_A^{(k)} \) corresponding to the measurement at time \( t_k \). All of \( \rho_A^{(k)} \) have the form (5.3). We can now rewrite eq. (5.11) as

\[
\rho = \sum_h p(h) |A_1^{(N)}| \cdots |A_j^{(1)}| \cdots |A_j^{(N)}| = \sum f \rho_N(f) \rho_N(f).
\]

Here, the summation in the first step of eq. (5.12) is over histories \( h = \{A_1^{(1)}, \ldots, A_j^{(N)}\} \) with \( i, j = \uparrow, \downarrow \), the sum in the second step is over the values \( f = 0, 1/N, 2/N, \ldots, 1 \), \( \rho_N(f) \) is the binomial distribution (5.7) expressed in terms of \( f \), and \( \rho_N(f) \) is the symmetrized and normalized density matrix for states with a given value of \( f \). For example, for \( f = M/N \), where \( M < N \) is an integer,

\[
\rho_N(f) = \left( \frac{N}{M} \right)^{-1} \left( |A_1^{(N)}| \cdots |A_1^{(M+1)}| |A_1^{(M)}| \cdots |A_1^{(1)}| \cdots |A_1^{(N)}| \middle| A_1^{(1)} \cdots A_1^{(N)} \right) + \text{permutations},
\]

\[
\text{Tr} \rho_N(f) = 1.
\]

Now, in the limit \( N \to \infty \), \( \rho_N(f) \) becomes a delta-function (5.10), so terms with \( f \neq |\alpha|^2 \) drop out of the sum (5.12), and only histories satisfying the Born rule, \( f = |\alpha|^2 \), are represented in the density matrix. The final step of the argument, leading to the conclusion that quantum probabilities are given by the Born rule, is nontrivial and requires additional assumptions. For example, one may have to assume [37] that a measurement of an observable \( \mathcal{O} \) in a quantum state which is an eigenstate of \( \mathcal{O} \) gives the corresponding eigenvalue with a 100% probability. It should also be noted that a careful definition of the limit \( N \to \infty \) is a delicate issue, which is still being debated [39, 42–44]. Here, I will not discuss these issues any further, since they are peripheral to the main subject of the present paper.

6 Conclusions

We discussed a possible extension of the watcher measure, introduced in [4], to quantum theory. This measure identifies probabilities of different events in the multiverse with frequencies at which these events are encountered along the watcher’s geodesic. We have adopted a picture where the observable region of the watcher undergoes a stochastic evolution, due to its interaction with the environment. The quantum state of this region is described by a reduced density matrix \( \tilde{\rho}(t) \), where \( t \) is a monotonic time variable along the watcher’s worldline.

A quantum extension of the watcher measure is most naturally obtained using the decoherent histories formulation of Quantum Mechanics. In this formulation, histories are specified by chains of projection operators, and the average number of events in a given time interval \( T \) can be expressed in terms of a sum over histories, eq. (3.14). The relative
probability of events is then given by the ratio of the corresponding occurrence numbers in
the limit $T \to \infty$, eq. (3.15). As in the classical version of the watcher measure, the resulting
probabilities are independent of the choice of time variable $t$. Note also that the ambiguity
related to choosing between identical observers in an ensemble, pointed out in refs. [2, 3],
does not arise in this measure, since there is no more than one observation that needs to be
considered at any time.

Any acceptable measure should be in agreement with the standard predictions of Quantum
Mechanics. In particular, the probabilities of possible outcomes of any quantum measure-
ment should be given by the expectation values of the corresponding projection operators
(the Born rule). We have verified that this is indeed the case for the watcher measure. Fur-
thermore, modulo the caveats associated with the $N \to \infty$ limit, the Born rule can be derived
from the watcher measure in a manner similar to refs. [36, 37, 40].

The watcher measure prescription gives a specific implementation of the idea that many
worlds of the multiverse are the same as Everett’s branching worlds [5, 7]. Here, Everett’s
worlds are represented by the decoherent histories of the watcher, and the probabilities of
different measurements in the multiverse are obtained from the watcher’s density matrix. As
it now stands, this measure prescription is not fully quantum, since it relies on a semiclassical
picture of spacetime and on classical concepts like the watcher’s geodesic. According to
some approaches to quantum gravity, a full quantum description should be in terms of the
wave function (or density matrix) of the causal patch of an observer (e.g., [5, 7, 21]). The
background spacetime and geodesics representing possible trajectories of a watcher would
then play no fundamental role, except perhaps in some appropriate limit. Implementation
of this approach, however, would require a better understanding of quantum gravity.

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