ON THE PERIMETERS OF SIMPLE POLYGONS CONTAINED IN A PLANE CONVEX BODY

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Abstract. A simple $n$-gon is a polygon with $n$ edges such that each vertex belongs to exactly two edges and every other point belongs to at most one edge. Brass, Moser and Pach [2] asked the following question: For $n \geq 5$ odd, what is the maximum perimeter of a simple $n$-gon contained in a Euclidean unit disk? In 2009, Audet, Hansen and Messine [1] answered this question, and showed that the supremum is the perimeter of an isosceles triangle inscribed in the disk, with an edge of multiplicity $n - 2$. In [3], Lángi generalized their result for polygons contained in a hyperbolic disk. In this note we find the supremum of the perimeters of simple $n$-gons contained in an arbitrary plane convex body in the Euclidean or in the hyperbolic plane.

1. Introduction

A question regarding an isoperimetric problem about simple polygons was asked by Brass, Moser and Pach (see Problem 3 on p. 437 in [2]).

Problem 1 (Brass, Moser and Pach, 2005). For $n \geq 5$ odd, what is the maximum perimeter of a simple $n$-gon contained in a Euclidean unit disk?

The authors of [2] remarked that for $n$ even, the supremum of the perimeters is the trivial upper bound $2n$, as it can be approached by simple $n$-gons in which the vertices alternate between some small neighborhoods of two antipodal points of the disk. This argument cannot be applied if $n$ is odd. In 2009, Audet, Hansen and Messine [1] showed that for $n$ odd, the supremum is attained as the perimeter of an isosceles triangle inscribed in the disk, with an edge of multiplicity $n - 2$. The author of [3] gave a shorter proof of the same statement, and proved that for hyperbolic disks of any radius, the supremum is attained as the perimeter of an $n$-gon of the same kind; that is, that of an isosceles triangle with a multiple edge inscribed in the disk. He noted that for $n$ even and for any plane convex body $C$ in the Euclidean plane $\mathbb{E}^2$ or in the hyperbolic plane $\mathbb{H}^2$, the supremum of the perimeters of the simple $n$-gons contained in $C$ is the trivial bound $n \text{ diam} C$, where $\text{ diam} C$ is the diameter of $C$. He asked whether it is true that for $n$ odd, the supremum is the perimeter of a triangle with an edge of multiplicity $n - 2$, inscribed in $C$.

In this paper we answer this question. Our main result is the following.

\textit{1991 Mathematics Subject Classification.} 52B60, 52A40, 52A55.
\textit{Key words and phrases.} isoperimetric problem, simple polygon, perimeter, circumcircle.
Theorem. Let \( n \geq 3 \) be an odd integer, and let \( C \) be a plane convex body in \( \mathbb{E}^2 \) or in \( \mathbb{H}^2 \). For every simple \( n \)-gon \( P \) contained in \( C \) there is a triangle, inscribed in \( C \) and with side-lengths \( \alpha \geq \beta \geq \gamma \), such that \( \text{perim} \leq (n-2)\alpha + \beta + \gamma \).

In the proof we use the following notations. Let \( \mathbb{M} \in \{ \mathbb{E}^2, \mathbb{H}^2 \} \) and \( x, y \in \mathbb{M} \). The distance of \( x \) and \( y \) is denoted by \( \text{dist}(x,y) \). The closed (respectively, open) segment with endpoints \( x \) and \( y \) is denoted by \( [x,y] \) (respectively, \( (x,y) \)). If \( x \neq y \), \( L(x,y) \) denotes the straight line passing through \( x \) and \( y \), and \( R_t(x,y) \) denotes the closed ray in \( L(x,y) \) emanating from \( x \) and not containing \( y \).

For any set \( A \subseteq \mathbb{M} \), we use the standard notations \( \text{int} A, \text{bd} A, \text{diam} A, \text{perim} A, \text{area} A \) and \( \text{conv} A \) for the interior, the boundary, the diameter, the perimeter, the area, or the convex hull of \( A \). Points are denoted by small Latin letters, and sets of points by capital Latin letters.

In the proof we find a triangle \( T \), contained in \( C \), with side-lengths \( \alpha, \beta \) and \( \gamma \) such that \( \text{perim} \leq (n-2)\alpha + \beta + \gamma \), as in this case we can move the vertices of \( T \) to \( \text{bd} C \) in a way that no side-length of \( T \) decreases.

2. Proof of Theorem for the Euclidean plane

Let us consider a Descartes coordinate system. If \( z \in \mathbb{E}^2 \) is an arbitrary point, by \( z = (\mu, \nu) \) we mean that the \( x \)-coordinate of \( z \) is \( \mu \), and its \( y \)-coordinate is \( \nu \). Let \( [a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n] \) denote the edges of \( P \) such that \( a_0 = a_n \), and let \( a_i = (\omega_i, \theta_i) \) for \( i = 0, 1, 2, \ldots, n \). Without loss of generality, we may assume that \( [a_0, a_1] \) is a longest edge of \( P \), \( a_0 \) is the origin \((0,0)\), and that \( a_1 = (0,1) \).

For \( i = 0, 1, \ldots, n \), let \( \zeta_i = \theta_{i+1} - \theta_i \). Note that \( \zeta_0 = \zeta_n = 1 \). As \( n \) is odd, the sequence \( \{ \zeta_i \} \) consists of an even number of elements. Thus, it has two consecutive elements, say \( \zeta_{i-1} \) and \( \zeta_i \), that are both nonnegative or nonpositive. From this, we have that \( \theta_{j-1} \leq \theta_j \leq \theta_{j+1} \), or that \( \theta_{j-1} \geq \theta_j \geq \theta_{j+1} \), respectively. For simplicity, we denote \( a_0, a_1, a_{j-1}, a_j \) and \( a_{j+1} \) by \( p = (0,0), q = (0,1), a = (\omega_a, \theta_a), b = (\omega_b, \theta_b) \) and \( c = (\omega_c, \theta_c) \), respectively, and set \( p_a = (0, \theta_a) \) and \( p_c = (0, \theta_c) \). If \( \text{dist}(a,c) \geq 1 \), we have \( \text{perim} \leq (n-2)\text{dist}(a,c) + \text{dist}(a,b) + \text{dist}(a,c) \). Thus, in the following we may assume that \( \text{dist}(a,c) < 1 \) and also that \( \theta_a \leq \theta_b \leq \theta_c \).

![Figure 1](image-url)
Assume that \([a, b] \cap R_p(p, q) \neq \emptyset\). From this, it follows that \(\text{dist}(a, b) \leq 1 \leq \text{dist}(b, p)\) and that \(1 \leq \text{dist}(c, p)\) (cf. Figure 1). Thus, we have that \(\text{perim } P \leq n - 2 + \text{dist}(a, b) + \text{dist}(b, c) \leq (n - 2) \text{dist}(p, c) + \text{dist}(b, p) + \text{dist}(c, b)\). If \([b, c] \cap R_p[p, q] \neq \emptyset\), we may apply a similar argument.

Assume that \([a, b] \cap R_p(p, q) \neq \emptyset\), which yields that \(\theta_a \leq 0\). If \(\theta_b \leq 0\), then we may apply the argument in the previous paragraph, and thus, we may assume that \(0 < \theta_b\). From this and from \(\text{dist}(a, c) < 1\), we readily obtain that \(0 < \theta_b < \theta_c < 1\).

Let \(L\) denote the bisector of the segment \([c, q]\). Since \(\text{dist}(a, c) < 1 \leq \text{dist}(a, q)\), we have that \(L\) separates \(q\) from \(a\) and \(c\). Observe also that if \(\text{dist}(b, q) \geq \text{dist}(b, c)\), then \(\text{perim } P \leq (n - 2) \text{dist}(a, q) + \text{dist}(a, b) + \text{dist}(b, q)\), and hence, we may assume that \(L\) separates \(b\) and \(q\) from \(c\). Thus, \([a, b] \cap L \neq \emptyset \neq [b, c] \cap L\), which implies that \(b \in \text{conv} \{a, p_c, c\}\) and \(\text{dist}(a, b) + \text{dist}(b, c) \leq \text{dist}(a, p_c) + \text{dist}(p_c, c) \leq \text{dist}(a, c) + \text{dist}(c, q)\). From this, we obtain that \(\text{perim } P \leq (n - 2) \text{dist}(a, q) + \text{dist}(a, c) + \text{dist}(c, q)\). If \([b, c] \cap R_q(p, q) \neq \emptyset\), the assertion follows by a similar argument.

We are left with the case that \(a, b, c\) are in the same closed half-plane bounded by \(L(p, q)\). Let \(0 \leq \omega_a, \omega_b, \omega_c\). If \(\theta_a \leq 0\) and \(\theta_c \geq 1\), then \(\text{dist}(a, c) \geq 1\) and hence, we may assume that, say, \(\theta_a \geq 0\).

Assume that \(\theta_c \leq 1\) and that, say, \(\omega_a \leq \omega_c\). If \(\omega_b > \omega_c\), then \(\text{dist}(a, b) \leq \text{dist}(p_a, b) \leq \text{dist}(b, p)\) and \(\text{dist}(b, c) \leq \text{dist}(b, p_c) \leq \text{dist}(b, q)\), which yields that \(\text{perim } P \leq n - 2 + \text{dist}(p, b) + \text{dist}(b, q)\). If \(\omega_b \leq \omega_c\), then \(\text{dist}(a, b) + \text{dist}(b, c) \leq \text{dist}(a, p_c) + \text{dist}(p_c, c) \leq \text{dist}(a, c) + \text{dist}(c, c) \leq \text{dist}(a, c) + \text{dist}(c, q)\), from which it readily follows that \(\text{perim } P \leq n - 2 + \text{dist}(p, c) + \text{dist}(c, q)\).

Assume that \(\theta_c > 1\) and that \(b \notin \text{conv} \{p_a, p_c, a, c\}\). Then the three rays, emanating from \(a\), that pass through \(p, c\) and \(b\) are in this clockwise order around \(a\). Let \(L'\) denote the bisector of the segment \([p, a]\). Note that as \(\text{dist}(a, c) < 1 \leq \text{dist}(p, c)\), \(L'\) separates \([a, c]\) from \(p\). Hence, it follows from \(\theta_b \leq \theta_c\) that \(L'\) separates \([a, b]\) from \(p\) (cf. Figure 2). Then we obtain that \(\text{dist}(a, b) \leq \text{dist}(p, b)\), and hence, that \(\text{perim } P \leq (n - 2) \text{dist}(p, c) + \text{dist}(p, b) + \text{dist}(b, c)\).

![Figure 2](image_url)

Our last case is that \(\theta_c > 1\) and \(b \in \text{conv} \{p_a, p_c, a, c\}\). We may assume that \(\{p, q\} \cap \{a, b, c\} = \emptyset\), as otherwise the assertion clearly follows. If \(b \notin \text{conv} \{p, q, a, c\}\), then \(b \in \text{conv} \{q, p_c, c\}\), \(\text{dist}(b, c) \leq \text{dist}(c, q)\) and \(\text{perim } P \leq (n - 2) \text{dist}(p, c) + \text{dist}(p, q) + \text{dist}(q, c)\). Thus, we may assume that \(b \in \text{conv} \{p, q, a, c\}\), from which we obtain that \(\text{dist}(a, b) + \text{dist}(b, c) \leq \max \{\text{dist}(a, q) + \text{dist}(q, c), \text{dist}(a, p_a) + \text{dist}(p_a, c)\}\).
Assume that \( \text{dist}(a, p_a) + \text{dist}(p_a, c) \leq \text{dist}(a, q) + \text{dist}(q, c) \). Then \( \text{dist}(a, p_a) + \text{dist}(p_a, c) \leq \text{dist}(a, p_c) + \text{dist}(p_c, c) \), which yields that \( \omega_a \leq \omega_c \). Thus, the two legs of the right triangle \( \text{conv}\{p, c, p_a\} \) are pairwise greater than or equal to the two legs of \( \text{conv}\{q, a, p_a\} \), from which we obtain that \( \text{dist}(q, a) \leq \text{dist}(p, c) \) and that \( \text{perim} P \leq n - 2 + \text{dist}(p, c) + \text{dist}(q, c) \). Hence, we may assume that \( \text{dist}(a, b) + \text{dist}(b, c) \leq \text{dist}(a, p_a) + \text{dist}(p_a, c) \).

Assume that \( \theta_a \geq \frac{1}{2} \), or in other words, that \( \text{dist}(a, p) \geq \text{dist}(a, q) \). Then \( \text{dist}(a, b) + \text{dist}(b, c) \leq \text{dist}(u, v) + \text{dist}(u, c) \), where \( u = (0, \frac{1}{2}) \) and \( v = (\omega_a, \frac{1}{2}) \). Note that for any \( x, y, z \in \mathbb{E}^2 \), the function \( \tau \mapsto \text{dist}(x, y + \tau z) \) is a convex function on \( \mathbb{R} \). Thus,

\[
\text{dist}(u, v) + \text{dist}(u, c) \leq \frac{1}{2} (\text{dist}(a, p) + \text{dist}(a, q)) + \frac{1}{2} (\text{dist}(p, c) + \text{dist}(q, c)) \leq \max\{\text{dist}(a, p) + \text{dist}(a, q), \text{dist}(p, c) + \text{dist}(c, q)\},
\]

and the assertion readily follows.

Finally, assume that \( \theta_a \leq \frac{1}{2} \), which immediately yields that \( \theta_c - \theta_a \geq \frac{\theta_a}{\omega_a} \). If \( n = 5 \), then the remaining two edges of \( P \) are \([p, a]\) and \([q, c]\), and we obtain that \( \text{perim} P \leq 3 \text{dist}(p, c) + \text{dist}(p, q) + \text{dist}(q, c) \). Furthermore, if \( \text{dist}(a, c) \geq \text{dist}(p_a, c) \), then \( \text{perim} P \leq (n - 2) \text{dist}(p, c) + \text{dist}(p, a) + \text{dist}(a, c) \). Hence, we may assume that \( n \geq 7 \) and that \( \text{dist}(a, c) < \text{dist}(p_a, c) \), which yields that \( \omega_c \geq \frac{\theta_c}{\omega_a} \). Set \( w = (\frac{\theta_c}{\omega_a}, \theta_a) \).

Since \( n \geq 7 \), it suffices to prove that \( 5 + \text{dist}(a, b) + \text{dist}(b, c) \leq 5 \text{dist}(p, c) + \text{dist}(a, p) + \text{dist}(a, c) \); that is, that

(1) \( 5 + \text{dist}(a, p_a) + \text{dist}(p_a, c) \leq 5 \text{dist}(p, c) + \text{dist}(a, p) + \text{dist}(a, c) \).

Let \( M \) and \( N \) denote the left-hand side and the right-hand side of (1), respectively, and let us regard them as functions of \( c \). Consider the vector \( v = (1, 0) \). Note that \( 5 + \text{dist}(a, p_a) + \text{dist}(p_a, w) \leq 5 \text{dist}(p, w) + \text{dist}(p, a) + \text{dist}(a, w) \), and thus, that (1) holds for \( c = w \). Thus, we need only prove that in the direction of \( v \), the derivative of \( N \) is not smaller than that of \( M \); that is, using the standard differential geometric notation, that \( v(M) \leq v(N) \).

Let \( \phi, \chi \) and \( \psi \) denote the internal angles at \( c \) of \( \text{conv}\{p_a, p_c, c\} \), \( \text{conv}\{p, p_c, c\} \) and \( \text{conv}\{a, c, p_a, p_c\} \), respectively (cf. Figure 3). We observe that \( 0 < \phi \leq \pi - \psi < \pi \) and that \( \cos \phi \geq -\cos \psi \).
Note that $v(M) = \cos \phi$, and $v(N) = 5 \cos \chi + \cos \psi \geq 5 \cos \chi - \cos \phi$. We set $I = 5 \cos \chi - 2 \cos \phi \leq v(N) - v(M)$. Then an elementary calculation yields that

$$I = \frac{5\omega_c}{\sqrt{\omega_c^2 + \theta_c^2}} - \frac{2\omega_c}{\sqrt{\omega_c^2 + (\theta_c - \theta_a)^2}} \geq \frac{5\omega_c}{\sqrt{\omega_c^2 + \theta_c^2}} - \frac{2\omega_c}{\sqrt{\omega_c^2 + (\theta_c/2)^2}} = \frac{\omega_c (21\omega_c^2 + 4\theta_c^2)}{\sqrt{\omega_c^2 + \theta_c^2} \sqrt{\omega_c^2 + (\theta_c/2)^2}} \left(5\sqrt{\omega_c^2 + (\theta_c/2)^2} + 2\sqrt{\omega_c^2 + \theta_c^2}\right) \geq 0.$$

### 3. Proof of Theorem for the hyperbolic plane

In this section we show how to modify the argument in Section 2 to obtain the proof for polygons in the hyperbolic plane. Let $[a_0, a_1], \ldots, [a_{n-1}, a_n]$ be the edges of $P$, where $a_0 = a_n$. We use our notation in a way that $[a_0, a_1]$ is a longest edge of $P$ and, like in the previous section, we write $p = a_0$ and $q = a_1$.

Similarly like in Section 2 we obtain that $P$ has three consecutive vertices, which we denote by $a, b$ and $c$, with orthogonal projections $p_a, p_b$, and $p_c$ onto $L(p, q)$, respectively, such that $p_b \in [p_a, p_c]$. Without loss of generality, we may assume that, orienting the points of $L(p, q)$ in a way that $p$ precedes $q$, we have that $a$ precedes $c$. We set $\rho = \text{dist}(p, q)$, and denote the lines, orthogonal to $L(p, q)$, that pass through $p$ and $q$ by $L_p$ and $L_q$, respectively. Furthermore, we denote the closed infinite strip bounded by $L_p$ and $L_q$ by $S(p, q)$, and the one bounded by $L(a, p_a)$ and $L(c, p_c)$ by $S(a, c)$.

We may assume that $\text{dist}(a, c) < \rho$ and that $a, b$ and $c$ are in the same closed half-plane bounded by $L(p, q)$, since otherwise we may apply an argument similar to the one in Section 2. Thus, we obtain that one of the components of $\mathbb{H}^2 \setminus S(p, q)$, say the one containing $p$ in its boundary, is disjoint from $\{a, b, c\}$.

**Case 1, $c \in S(p, q)$.

Note that it yields that $\{a, b, c\} \subset S(p, q)$. First, assume that $b \in \text{conv}\{p_a, p_c, a, c\}$. Without loss of generality, let $\text{dist}(a, p_a) \leq \text{dist}(c, p_c)$. Then the assertion readily follows from $\text{dist}(a, b) + \text{dist}(b, c) \leq \max\{\text{dist}(a, p_a) + \text{dist}(p_a, c), \text{dist}(a, p_c) + \text{dist}(p_c, c)\} \leq \text{dist}(c, p_c) + \text{dist}(c, p_a) \leq \text{dist}(a, p) + \text{dist}(a, q)$. Thus, we may assume that $b \notin \text{conv}\{p_a, p_c, a, c\}$.
Let $D$ denote the closure of the component of $S(a, c) \setminus [a, c]$ that does not contain $[p_a, p_c]$, and observe that $b \in D$. We recall that for every hypercycle $H \subset \mathbb{H}^2$, there is a unique hyperbolic line such that the distances of the points of $H$ from $L$ are equal. We call this line the reference line of $H$. Let $H_a$ denote the hypercycle, orthogonal to $L(p, q)$, that passes through $p$ and $a$. In this case the reference line $L^*$ of $H_a$ is also orthogonal to $L(p, q)$. Let $H_c$ denote the hypercycle, with the reference line $L^*$, that passes through $c$. We observe that $H_c$ is also orthogonal to $L(p, q)$. Figure 4 shows these hypercycles in the conformal disk model such that $p$ is the centre of the model. We note that as $\text{dist}(a, c) < \rho$, we have that $H_c \cap [p, q] \neq \emptyset$.

Let $L_a$ and $L_c$ denote the line, orthogonal to both $H_a$ and $H_c$, that passes through $a$ and $c$, respectively. Observe that $L(p, q)$, $L_a$ and $L_c$ are pairwise disjoint. Since two hyperbolic lines intersect at most once, we obtain that the point $b \in D$ and the segment $[p_a, p_c]$ are separated by $L_a$ or by $L_c$. We show that in the first case we have $\text{dist}(a, b) \leq \text{dist}(p, b)$, and in the second case $\text{dist}(b, c) \leq \text{dist}(b, q)$.

First, assume that $b$ and $[p_a, p_c]$ are separated by $L_a$. Then the angle of $[a, b]$ and the arc of $H_a$ between $a$ and $p$ is not acute. Note that if $x \in \mathbb{H}^2$ is a point and $H \subset \mathbb{H}^2$ is a hypercycle, and we move a point $y \in H$ farther from the orthogonal projection of $x$ onto $H$, then the quantity $\text{dist}(x, y)$ does not decrease. Thus, it follows that $\text{dist}(a, b) \leq \text{dist}(p, b)$.

Now we assume that $b$ and $[p_a, p_c]$ are separated by $L_c$. Then, denoting the intersection point of $H_c$ and $[p, q]$ by $h_c$, we have that the angle of $[b, c]$ and the arc of $H_c$ between $c$ and $h_c$ is not acute, and that $\text{dist}(b, c) \leq \text{dist}(b, h_c)$. If $h_c \in [p, q]$, then $h_c \in [p_a, q]$, and clearly, $\text{dist}(b, c) \leq \text{dist}(b, q)$. If $h_c \in [p, p_c]$, then the angle between $[b, c]$ and $[c, p_c]$ is not acute, which yields that $\text{dist}(b, c) \leq \text{dist}(b, p_c) \leq \text{dist}(b, q)$ (cf. Figure 5).

![Figure 5](image)

We have obtained that $\text{dist}(a, b) \leq \text{dist}(p, b)$ or that $\text{dist}(b, c) \leq \text{dist}(b, q)$. Without loss of generality, let $\text{dist}(a, b) \leq \text{dist}(p, b)$. If $\text{dist}(p, c) \geq \rho$, then $\text{perim} P \leq (n - 2) \text{dist}(p, c) + \text{dist}(p, b) + \text{dist}(b, c)$, and hence, we may assume that $\text{dist}(p, c) < \rho$. Let $\hat{L}$ denote the line bisecting the segment $[q, c]$. If $b \in \text{conv}\{p, c, q\}$, then
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\[ \text{dist}(p, b) + \text{dist}(b, c) \leq \text{dist}(p, p_c) + \text{dist}(p_c, c) \leq \text{dist}(p, c) + \text{dist}(c, q), \] 
and the assertion readily follows. If \( b \notin \conv\{p, c, q\} \), then \( \bar{L} \) separates \( b \) and \( q \), and thus, the assertion follows from \( \text{dist}(b, c) \leq \text{dist}(b, q) \).

Case 2, \( c \notin S(p, q) \). Then \( \rho \leq \text{dist}(p, c) \).

We introduce \( M, N, \phi, \chi, \psi \) as in the Euclidean case. Note that the angle of \( \conv\{p, q, x, y\} \) at \( y \) is acute, and thus \( 0 < \phi \leq \pi - \psi < \pi \) and \( \cos \phi \geq -\cos \psi \).

Let \( v \) denote the unit vector at \( c \) (an element of the tangent plane \( T_{cH^2} \)) tangent to \( L(c, p) \) and pointing away from \( p_c \). Then, as in the Euclidean case, we need only prove that \( v(M) \leq v(N) \). Observe that \( v(M) = \cos \phi \), and \( v(N) = 5 \cos \chi + \cos \psi \geq 5 \cos \chi - \cos \phi \).

As in Section 2, we set \( I = 5 \cos \chi - 2 \cos \phi \leq v(N) - v(M) \). Set \( \theta_a = \text{dist}(p, a), \theta_c = \text{dist}(p, c) \) and \( \omega_c = \text{dist}(c, p_c) \).

By the hyperbolic Pythagorean and cosine theorems, we obtain that

\[ I = \frac{5 \cosh \theta_c \sinh \omega_c}{\cosh^2 \theta_c \cosh^2 \omega_c - 1} - \frac{2 \cosh(\theta_c - \theta_a) \sinh \omega_c}{\cosh^2(\theta_c - \theta_a) \cosh^2 \omega_c - 1} \]

Note that \( \theta_c - \theta_a \geq \frac{\theta_a}{2} \) and the function \( x \mapsto \frac{Ax}{\sqrt{Bx^2 - 1}} \), where \( A, B \in \mathbb{R} \) and \( A > 0 \), is strictly decreasing. Thus,

\[ I \geq \frac{5 \cosh \theta_c \sinh \omega_c}{\cosh^2 \theta_c \cosh^2 \omega_c - 1} - \frac{2 \cosh \frac{\theta_a}{2} \sinh \omega_c}{\cosh^2 \frac{\theta_a}{2} \cosh^2 \omega_c - 1} \]

Now the inequality \( I \geq 0 \) follows from the estimate \( \cosh \omega_c \geq 1 \) and by algebraic transformations similar to those used in the Euclidean case.

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