THE COMPLEX SHADE OF A REAL SPACE AND ITS APPLICATIONS

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Abstract. A natural oriented \((2k+2)\)-chain in \(\mathbb{C}P^{2k+1}\) with boundary twice \(\mathbb{R}P^{2k+1}\), its complex shade, is constructed. Via intersection numbers with the shade, a new invariant, the shade number, of \(k\)-dimensional subvarieties with normal vector fields along their real part, is introduced. For an even-dimensional real variety, the shade number and the Euler number of the complement of the normal vector field in the real normal bundle of its real part agree. For an odd-dimensional orientable real variety, a linear combination of the shade number and the wrapping number (self-linking number) of its real part is independent of the normal vector field and equals the encomplexed writhe as defined by Viro \cite{Viro}. Shade numbers of varieties without real points and encomplexed writhes of odd-dimensional real varieties are, in a sense, Vassiliev invariants of degree 1.

Complex shades of odd-dimensional spheres are constructed. Shade numbers of real subvarieties in spheres have properties analogous to those of their projective counterparts.

1. Introduction

The manifold \(\mathbb{R}P^n\), \(n > 0\), is orientable if and only if \(n\) is odd. With its standard orientation, \(\mathbb{R}P^{2k+1} \subset \mathbb{C}P^{2k+1}\) is homologous to zero, since \(H_{2k+1}(\mathbb{C}P^{2k+1}; \mathbb{Z}) = 0\). (Notice the difference between even and odd dimensions: \(\mathbb{R}P^{2k}\) represents a generator of \(H_{2k}(\mathbb{C}P^{2k}; \mathbb{Z}_2)\). Our main concern are subvarieties in “double dimension + 1” and so we consider only the odd-dimensional case.) The complex shade of \(\mathbb{R}P^{2k+1}\) consists of the complexifications of all real lines through a point \(p\) in \(\mathbb{R}P^{2k+1}\). It is an oriented \((2k+2)\)-chain \(\Gamma_p\) in \(\mathbb{C}P^{2k+1}\) with boundary twice \(\mathbb{R}P^{2k+1}\), see Definition \[\text{[4]}\]. Geometrically distinct shades are in 1-1 correspondence with points in \(\mathbb{R}P^{2k+1}\). They all represent the same homology class, the shade class \([\Gamma]\) \(\in H_{2k}(\mathbb{C}P^{2k+1}, \mathbb{R}P^{2k+1})\). This homology class \([\Gamma]\) has three characterizing properties: it is invariant under complex conjugation, it has boundary twice \(\mathbb{R}P^{2k+1}\), and its intersection number with the class represented by a complex \(k\)-dimensional linear space without real points has absolute value equal to 1, see Proposition \[\text{[8]}\].

The shade class will be used to measure certain linking phenomena.

Recall that if \(M\) is an \(m\)-dimensional oriented manifold, and \(B\) and \(C\) are disjoint oriented cycles of dimensions \(k\) and \(m-k-1\), respectively, which are weakly zero-homologous (i.e. represent torsion classes in homology) then the linking number

\[\text{linking number} = \frac{1}{2} \left( \frac{\int_B \int_C \alpha \wedge \beta}{\int_B \alpha} \right)\]

\[\text{where } \alpha \text{ and } \beta \text{ are cohomology classes representing } B \text{ and } C, \text{ respectively.} \]
\( \text{lk}(B, C) \) is defined as \( \frac{1}{2} \) times the intersection number of an oriented \((k + 1)\)-chain \( A \) with boundary \( n \cdot B \) and \( C \) in \( M \).

An oriented \( 2k \)-cycle \( C \) in the complement of \( \mathbb{R}P^{2k+1} \) in \( \mathbb{C}P^{2k+1} \) need not be weakly zero-homologous and the linking number of \( \mathbb{R}P^{2k+1} \) and \( C \) is in general not defined. In fact, there are many possible choices of a counterpart of linking number in this case: the intersection number of any oriented \( 2k \)-chain with boundary \( \mathbb{R}P^{2k+1} \) and \( C \) would do. The shade class provides a choice: we measure the linking of \( C \) and \( \mathbb{R}P^{2k+1} \) in \( \mathbb{C}P^{2k+1} \) as \( \frac{1}{2} \) times the intersection number of the shade class and \( C \). This construction has straightforward applications to \( k \)-dimensional complex projective varieties without real points and give rise to what will be called shade numbers, see \( \S 2.A \).

Most \( k \)-dimensional complex varieties in projective \((2k + 1)\)-space (varieties in open dense subset of the Chow-variety) are without real points. But the definition of shade number can be extended to arbitrary \( k \)-dimensional projective subvarieties provided they have been equipped with additional structure. (The additional structure is a certain kind of vector field along the real part in \( \mathbb{R}P^{2k+1} \), on which the shade number will depend. Any \( k \)-dimensional variety admits such additional structure and in special cases it appears naturally, see \( \S 7.A \).)

The most interesting applications of shade numbers arise in the study of the interplay between the real and the complex geometry/topology of generic real projective varieties (see \( \S 2.B \)): there are connections between shade numbers and topological invariants of the real part equipped with the above mentioned vector field. The nature of these connections depends on the parity of the dimension of the variety and the differences between even and odd dimensions are striking, see \( \S 2.D \) and \( \S 2.E \). Since spheres frequently occurs as real algebraic ambient spaces, we define complex shades of spheres as well, see \( \S 2.C \).

1.A. Applications to real algebraic links. The writhe of a generic projection of a knot in \( \mathbb{R}^3 \) to a plane (a knot diagram) is the signed sum of double points of its image. This notion has a straightforward generalization to plane projections of knots in \( \mathbb{R}P^3 \). In general the writhe changes as the projection varies.

It was observed by Viro [7] that the writhe of a knot may be enhanced if it is real algebraic (a real algebraic knot in projective 3-space is a smooth 1-dimensional projective variety with connected real part). In [9], Viro introduced the encomplexed writhe which shows that the classification of real algebraic knots up to rigid isotopy (a smooth isotopy which is also a continuous family of real algebraic curves, see [1] and [3]) is more refined than the corresponding classification up to smooth isotopy.

Viro’s definition is diagrammatic: the encomplexed writhe of a real algebraic knot \( V \) is expressed as the signed sum of real double points in a generic real projection of \( V \) to a plane. Note that the preimage of a double point in such a projection may be either two distinct points in the real part of \( V \), or two complex conjugate non-real points of \( V \). Double points of the later kind are called solitary and they enhance the writhe to the encomplexed writhe, which is independent of generic projection and invariant under rigid isotopy.

The initial motivation for the study undertaken in this paper was to give an intrinsic 3-dimensional explanation of Viro’s invariant. Such an explanation was indeed found. In Theorem 2.11 we show how to express the encomplexed writhe in terms of the shade number and the wrapping (self-linking) number.
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2. Statements of main results

In this section, some notation is introduced and the main results are stated.

2.A. Linking of complex varieties with real projective space. If $W$ is a projective $k$-dimensional variety in $(2k+1)$-dimensional complex projective space then its set of closed points $CW \subset CP^{2k+1}$, with the subspace topology and the induced complex orientation, is an oriented $2k$-cycle in $CP^{2k+1}$. Assume that $RW = CW \cap RP^{2k+1} = \emptyset$ and let $[CW]$ denote the homology class of $CW$ in $H_{2k}(CP^{2k+1} - RP^{2k+1})$.

Definition 2.1. The number $\text{sh}(W) = \frac{1}{2} ([\Gamma] \cdot [CW]) \in \frac{1}{2}\mathbb{Z}$, where $[\Gamma]$ is the shade class and $\cdot$ denotes the intersection product, is called the shade number of $W$.

The range of the shade number can be expressed in terms of degrees:

Theorem 2.2. For projective $k$-dimensional varieties in complex projective $(2k+1)$-space of degree $d$ without real points, the range of the shade number consists of all half-integers between $-\frac{1}{4}d^2$ and $\frac{1}{4}d^2$ which are congruent to $\frac{1}{2}d$ modulo 1.

Theorem 2.2 is proved in § 4.B.

Proposition 2.3. If $W_0$ and $W_1$ are projective $k$-dimensional varieties in complex projective $(2k+1)$-space without real points and if $A$ is a $(2k+1)$-chain in $CP^{2k+1}$ such that $\partial A = CW_1 - CW_0$ then

$$\text{sh}(W_1) - \text{sh}(W_0) = RP^{2k+1} \cdot A,$$

where $\cdot$ denotes intersection number.

Proposition 2.3 is proved in § 4.A.

Remark 2.4. Proposition 2.3 gives information about how the shade number changes under deformations of a variety. If there is an ambient isotopy $\phi_t$ of $CP^{2k+1}$ carrying $CW_0$ to $CW_1$ in $CP^{2k+1}$ (e.g. $\phi_t$ could be a 1-parameter family of complex projective transformations, or induced by a rigid isotopy) then $A$ can be taken as $A = \bigcup_t \phi_t(CW_0)$. In particular, the shade number changes by $\pm 1$ when a deformation crosses $RP^{2k+1}$ transversely. So, in a sense, it is a first order Vassiliev invariant (this notion is borrowed from knot theory, see [8]) of $k$-dimensional varieties in $CP^{2k+1}$ without real points.

2.B. Shade numbers of armed real varieties. If $V$ is a real projective subvariety of real projective $n$-space, then after base extension, it is a projective subvariety of complex projective $n$-space. We shall use the notions $CV \subset CP^n$ for its set of closed points with the topology induced from the complex manifold $CP^n$, and $RV$ to denote $CV \cap RP^n$ (often without further mentioning of the base extension).

Following [1], a submanifold $M$ of a manifold $Y$ which is equipped with a non-vanishing normal vector field $n$ will be called an armed submanifold. We denote it $(M, n)$. Analogously, we say that a real projective variety $V$ in real projective
$m$-space is \textit{armed} if its real part $\mathbb{R}V$ is an armed submanifold of $\mathbb{R}P^m$. We denote it $(V,n)$, where $n$ refers to the vector field. Note that, for dimensional reasons, the real part of any $k$-dimensional subvariety of $(2k + 1)$-dimensional projective space admits a non-vanishing normal vector field.

For real projective $k$-dimensional varieties $V$ in projective $(2k + 1)$-space, the intersection $\mathbb{R}V = CV \cap \mathbb{R}P^{2k+1}$ may be large and there are many ways of pushing $\mathbb{C}V$ off $\mathbb{R}P^{2k+1}$ in $\mathbb{C}P^{2k+1}$. For armed varieties $(V,n)$, the normal vector field $n$ determines such a push-off: in $(i = \sqrt{-1})$ is a normal vector field of both $\mathbb{C}V$ and of $\mathbb{R}P^{2k+1}$ along $\mathbb{R}V$ in $\mathbb{C}P^{2k+1}$. If $\mathbb{C}V$, with its complex orientation, is shifted slightly along a normal vector field $\nu$ extending $i\nu$, and with support in a small neighborhood of $\mathbb{R}V$ in $\mathbb{C}V$, then an oriented $2k$-cycle $\mathbb{C}V_n$ in $\mathbb{C}P^{2k+1} - \mathbb{R}P^{2k+1}$ is obtained. Let $[\mathbb{C}V_n]$ denote its homology class in $H_{2k}(\mathbb{C}P^{2k+1} - \mathbb{R}P^{2k+1})$.

\textbf{Definition 2.5.} The number
\[
\sh(V,n) = \frac{1}{2} (\langle [\Gamma] \bullet [\mathbb{C}V_n] \rangle) \in \frac{1}{2} \mathbb{Z},
\]
where $[\Gamma]$ is the shade class and $\bullet$ denotes the intersection product, is called the shade number of $(V,n)$.

It is easy to see that $\sh(V,n)$ is independent of the choice of $\nu$ (the extension of $i\nu$) as long as its support and the shifting distance are sufficiently small.

\textbf{2.C. Real algebraic spheres.} Spheres frequently occurs as real algebraic ambient spaces (e.g. the link of a complex plane curve singularity is a real subvariety of the sphere rather than of projective space) and so we define shade numbers of armed real varieties also in this setting: the $n$-dimensional sphere $S^n$ is the set of real points of a real quadric $Q^n$ in projective $(n + 1)$-space (see \S\ 3.C.1 for explicit equations), $S^n \approx \mathbb{R}Q^n$. This real variety $Q^n$ will be called simply the \textit{real $n$-sphere}. Its set of complex points is the complex manifold $\mathbb{C}Q^n \subset \mathbb{C}P^{n+1}$.

Using a projection $\mathbb{C}Q^{2k+1} \to \mathbb{C}P^{2k+1}$ which is a double cover branched over a purely imaginary quadric and which restricts to the standard double cover $\mathbb{R}Q^{2k+1} \to \mathbb{R}P^{2k+1}$, we construct the complex shade of $\mathbb{R}Q^{2k+1}$ in $\mathbb{C}Q^{2k+1}$ see Definition 2.5 and Remark 3.8. Definition 2.5 then applies also to armed $k$-dimensional varieties in $\mathbb{R}Q^{2k+1}$.

Our notation for subvarieties of the sphere is analogous to the one in the projective case: if $V$ is a subvariety of the real $n$-sphere then $\mathbb{C}V \subset \mathbb{C}Q^n$ denotes its set of closed points (after base extension) with topology induced from the complex manifold $\mathbb{C}Q^n$ and $\mathbb{R}V = \mathbb{C}V \cap \mathbb{R}Q^n$.

One may study also other real algebraic ambient spaces. In this paper we will however restrict attention to the two basic cases of projective spaces and spheres.

\textbf{2.D. Even-dimensional real varieties.} If $(M,n)$ is an armed 2$j$-dimensional submanifold of an oriented manifold $Y$ of dimension $4j + 1$ then let $\epsilon(M,n)$ denote the Euler number of the 2$j$-dimensional vector bundle $\xi(n)$ which is the normal bundle of $M$ in $Y$ divided by its 1-dimensional subbundle generated by $n$. (Note that the orientation of $Y$ together with $n$ induce an orientation on the total space of $\xi(n)$, hence the Euler number is defined, see \S\ 3.A.3)

\textbf{Theorem 2.6.} Let $(V,n)$ be an armed 2$j$-dimensional projective variety without real singularities in real projective $(4j + 1)$-space or in the real $(4j + 1)$-sphere.
Then

\[ \text{sh}(V, n) = (-1)^j \frac{1}{2} e(\mathbb{R}V, n) \]

Theorem 2.6 is proved in § 5.B.

2.E. Odd-dimensional real varieties. For even-dimensional armed varieties, the shade number equals a topological invariant. This is not the case in odd-dimensions. As we shall see, for orientable odd-dimensional armed varieties, a linear combination of the shade number and a topological invariant (the wrapping number) is independent of the normal vector field (which always exists, see § 2.B), and is invariant under a certain class of deformations (weak rigid isotopies).

The wrapping number of an armed orientable \((2j + 1)\)-dimensional submanifold \((M, n)\) of \(S^{4j+3}\) \((\mathbb{R}P^{4j+3})\) is the orientation independent part of its self-linking number. It will be denoted \(\text{wr}(M, n)\). In \(S^{4j+3}\), \(\text{wr}(M, n) \in \mathbb{Z}\), in \(\mathbb{R}P^{4j+3}\), \(\text{wr}(M, n) \in \frac{1}{2}\mathbb{Z}\), see § 5.B.13 With this term introduced we define the linear combination mentioned above:

**Definition 2.7.** Let \((V, n)\) be an armed projective \((2j + 1)\)-dimensional variety without real singularities in real projective \((4j + 3)\)-space or in the real \((4j + 3)\)-sphere, with \(\mathbb{R}V\) orientable. Define

\[ \mathbb{C}w(V) = \text{wr}(\mathbb{R}V, n) + (-1)^j \text{sh}(V, n) \in \frac{1}{2}\mathbb{Z}. \]

A weak rigid isotopy of a real subvariety \(V\) in a nonsingular projective real algebraic variety \(Y\) is a continuous 1-parameter family of real subvarieties \(V_t, 0 \leq t \leq 1\), such that \(V_0 = V\), and such that the induced 1-parameter family \(\mathbb{R}V_t \subset \mathbb{R}Y\) is given by \(\phi_t(\mathbb{R}V)\), where \(\phi_t : \mathbb{R}Y \rightarrow \mathbb{R}Y, 0 \leq t \leq 1\), is a 1-parameter family of diffeomorphisms starting at the identity.

For example, rigid isotopies are weak rigid isotopies but not conversely: the 1-parameter family \((\mathbb{C}V_t, \mathbb{R}V_t)\) induced by a rigid isotopy satisfies \((\mathbb{C}V_t, \mathbb{R}V_t) = \psi_t(\mathbb{C}V, \mathbb{R}V)\), where \(\psi_t\) is a continuous 1-parameter family of diffeomorphisms of the pair \((\mathbb{C}V, \mathbb{R}V)\). Also, continuous paths of real projective transformations starting at the identity induce weak rigid isotopies in projective space.

**Theorem 2.8.** Let \(V\) be a projective \((2j + 1)\)-dimensional variety without real singularities in real projective \((4j + 3)\)-space or in the real \((4j + 3)\)-sphere, with \(\mathbb{R}V\) orientable. Then \(\mathbb{C}w(V)\) is an integer, independent of the choice of normal vector field, independent of the choice of orientation, and invariant under weak rigid isotopy.

Theorem 2.8 is proved in § 5.B. For explicit computations of \(\mathbb{C}w\) of real algebraic representatives of the unknot and the trefoil knot in \(\mathbb{R}Q^3\), see § 8.C and § 8.D respectively.

The invariant \(\mathbb{C}w\) may change under deformations which are not weak rigid isotopies. For certain deformations it changes in a controlled manner. We begin by describing such deformations:

Let \(\epsilon > 0\) and let \(V_t, t \in (-\epsilon, \epsilon)\) be a continuous 1-parameter family of real projective \((2j + 1)\)-dimensional varieties in real projective \((4j + 3)\)-space or in the real \((4j + 3)\)-sphere such that \(V_t\) is without real singularities and \(\mathbb{R}V_t\) is connected and orientable, for \(t \neq 0\). Assume that \(V_0\) has exactly one real double point where either two branches of \(\mathbb{R}V_0\) intersect cleanly or where two complex conjugate branches of \(\mathbb{C}V_0\) intersect cleanly. In the former case assume that the intersection point of
the traces of the deformations of the two branches of $\mathbb{R}V_t$ in $\mathbb{R}P^{4j+3} \times (-\epsilon, \epsilon)$ ($\mathbb{R}Q^{4j+3} \times (-\epsilon, \epsilon)$) is transverse, in the later assume that the intersection of the trace of (either) one of the branches of $\mathbb{C}V_t$ meets $\mathbb{R}P^{4j+3} \times (-\epsilon, \epsilon)$ ($\mathbb{R}Q^{4j+3} \times (-\epsilon, \epsilon)$) transversely in $\mathbb{C}P^{4j+3} \times (-\epsilon, \epsilon)$ ($\mathbb{C}Q^{4j+3} \times (-\epsilon, \epsilon)$).

**Theorem 2.9.** For $\epsilon > t > 0$ and $V_t$ as described above

$$Cw(V_t) - Cw(V_{-t}) = \pm 2.$$

**Remark 2.10.** Theorem 2.9 implies in particular that $Cw$ is a first order Vassiliev invariant of real algebraic embeddings $\phi$ of a real projective $(2j + 1)$-dimensional variety $V$, with $\mathbb{R}V$ connected and orientable, into real projective $(4j + 3)$-space, where $\phi$ varies in a given linear system $\Lambda$ of dimension $N > 4j + 3$ on $V$ ($\Lambda$ presents $V$ as a real algebraic variety without real singularities in real projective $N$-space).

This can be seen as follows: in $\Lambda$ there is a discriminant hypersurface consisting of all maps $\phi$ with real singularities. The top-dimensional strata of this discriminant consists of maps with one real-real or one complex-complex-conjugate double point and 1-parameter families intersecting the discriminant transversely satisfies the assumptions on the deformations in Theorem 2.9. Therefore the first jump of $Cw$ is $\pm 2$, and the second jump is zero (see $\[2\]$ for the notion of jump). That is, $Cw$ is a non-trivial first order Vassiliev invariant.

2.F. Shade number and encomplexed writhe. Shade numbers provide an intrinsic 3-dimensional formula, homological in nature, for Viro’s invariant of real algebraic links mentioned in § 1.A.

**Theorem 2.11.** Let $V$ be a real algebraic link in real projective 3-space. Then $Cw(V)$ equals the encomplexed writhe of $V$ as defined in $\[8\]$.

**Remark 2.12.** There are different possible definitions of the encomplexed writhe of a real algebraic link $V$ such that $\mathbb{R}V$ has more than one connected component, see $\[8\]$, §1.4. We choose one such definition, see Remark 6.2.

In $\[8\]$, Section 3.3, it is mentioned that the diagrammatic approach to the encomplexed writhe can be generalized to give rigid isotopy invariants of non-singular orientable real projective $(2j + 1)$-dimensional varieties in real projective $(4j + 3)$-space. Carrying out the proposed generalization, invariants which agree with $Cw$ (Definition 2.7) are obtained. That is, the counterpart of Theorem 2.11 holds in the high-dimensional situation and two formulas, one diagrammatic and one intrinsic homological, for the same invariant are obtained.

3. Complex shades and their characteristics

In this section, orientation conventions are specified, complex shades of projective spaces and of spheres are formally defined, and an axiomatic characterization of the shade class is given.

3.A. Orientation conventions. Let $M$ be an oriented $k$-dimensional submanifold of an oriented manifold $Y$ of dimension $n$. Let $NM$ denote the normal bundle of $M \subset Y$, and let $x \in M$. A basis $(n_1, \ldots, n_{n-k})$ of $N_xM$ is positively oriented if
for any positively oriented basis \((t_1, \ldots, t_k)\) of the tangent space \(T_x M\) of \(M\) at \(x\), 
\((a_1, \ldots, a_{n-k}, t_1, \ldots, t_k)\) is a positively oriented basis of \(T_x Y\).

If \(M\) is an oriented manifold with boundary \(\partial M\), we induce an orientation on \(\partial M\) by requiring that the outward normal vector field of \(\partial M\) in \(M\) induces the positive orientation of the normal bundle of \(\partial M\) in \(M\).

3.B. Definition of the shade of projective space. The complex shade of \(\mathbb{R}P^{2k+1}\) is constructed as follows. Fix a point \(p \in \mathbb{R}P^{2k+1}\). Let \(L(p) \approx \mathbb{R}P^{2k}\) denote the set of all real lines through \(p\).

If \(l \in L(p)\) then \((\mathbb{C}l, \mathbb{R}l) \approx (\mathbb{C}P^1, \mathbb{R}P^1)\). Let \(X = \bigcup_{l \in L(p)} (\mathbb{C}l - \mathbb{R}l)\). Then \(X\) is a disk-bundle over the \(2k\)-sphere with fibers open \(2\)-disks which are naturally identified with connected components of \(C\).

The fibers of \(\bar{X}\) are then naturally identified with the closures of components of \(C - \mathbb{R}l\) and there is a canonical map

\[
\gamma_p: \bar{X} \rightarrow \mathbb{C}P^{2k+1}.
\]

The restriction of \(\gamma_p\) to \(\partial \bar{X} - \gamma_{p}^{-1}(p)\) is a double cover of \(\mathbb{R}P^{2k+1} - \{p\}\). The standard orientation of \(\mathbb{R}P^{2k+1}\) therefore induces an orientation on \(\partial \bar{X}\) which in turn induces an orientation on \(\bar{X}\).

**Definition 3.1.** The complex shade \(\Gamma_p\) of \(\mathbb{R}P^{2k+1}\), constructed using the point \(p\), is the relative \((2k+2)\)-cycle in \((\mathbb{C}P^{2k+1}, \mathbb{R}P^{2k+1})\) defined by \(\Gamma_p = \gamma_p(\bar{X})\), where \(\gamma_p\) and \(\bar{X}\) are as in (3.1).

**Remark 3.2.** It is straightforward to check that \(\Gamma_p\) in Definition 3.1 has the following properties.

- The homology class of \(\Gamma_p\) in \(H_{2k+2}(\mathbb{C}P^{2k+1}, \mathbb{R}P^{2k+1})\), independent of \(p\).
- The boundary \(\partial \Gamma_p\) equals \(2\mathbb{R}P^{2k+1}\).
- If \(\ast: \mathbb{C}P^{2k+1} \rightarrow \mathbb{C}P^{2k+1}\) denotes complex conjugation then \(\Gamma_p^\ast = \Gamma_p\).

3.C. Definition of the shade of the sphere. Before we construct complex shades of spheres we introduce some notation which will be used throughout the rest of the paper.

Let \(Q^n\) denote the quadric in real projective \((n+1)\)-space defined by the homogeneous equation \(-x_0^2 + x_1^2 + \cdots + x_{n+1}^2 = 0\) in projective coordinates \([x_0, \ldots, x_{n+1}]\). Then \(\mathbb{R}Q^n \approx S^n\).

Let \(\Pi: \mathbb{C}Q^n \rightarrow \mathbb{C}P^n\) denote the projection from the point \([1, 0, \ldots, 0]\) to the hyperplane in \(\mathbb{C}P^{n+1}\) given by the equation \(x_0 = 0\). Note that \(\Pi\) is a double cover branched over the purely imaginary quadric given by the equations \(x_1^2 + \cdots + x_{n+1}^2 = 0, x_0 = 0\). If \(\pi: \mathbb{R}Q^n \rightarrow \mathbb{R}P^n\) denotes the restriction of \(\Pi\) then \(\pi\) is the standard double cover identifying antipodal points on \(\mathbb{R}Q^n\). Thus, inverse images of lines in \(\mathbb{R}P^n\) are great circles in \(\mathbb{R}Q^n\).

We construct the shade of \(\mathbb{R}Q^{2k+1}\). Fix a point \(p \in \mathbb{R}P^{2k+1}\). Let \(\mathbb{R}^{-1}(p) = \overline{\{\bar{p}_0, \bar{p}_1\}} \subset \mathbb{R}Q^{2k+1}\). Then \(\bar{p}_0\) and \(\bar{p}_1\) are antipodal points. Let \(G(p) \approx \mathbb{R}P^{2k}\) denote the set of all great circles through \(\bar{p}_0\) and \(\bar{p}_1\).

If \(g \in G(p)\) then \(g\) is the intersection of \(\Delta\) and \(Q^{2k+1}\), where \(\Delta\) is a real projective \(2\)-plane in \(\mathbb{R}P^{2k+2}\) through \(\bar{p}_0, \bar{p}_1\), and one other point in \(\mathbb{R}Q^{2k+1}\). Thus, \(g\) is a quadric in \(\Delta\) and \((\mathbb{C} \mathbb{P}^1, \mathbb{R}P^1)\).

Let \(Y = \bigcup_{g \in G(p)} (\mathbb{C}g - \mathbb{R}g)\). Then \(Y\) is a disk-bundle over the \(2k\)-sphere with fibers open \(2\)-disks which are naturally identified with connected components of
The fibers of $\bar{Y}$ are then naturally identified with the closures of components of $Cg - Rg$ and there is a canonical map
\[
\gamma_p : \bar{Y} \to CQ^{2k+1}.
\] (3.2)

The restriction of $\gamma_p$ to $\partial\bar{Y} - \gamma_p^{-1}(\{\bar{p}_0, \bar{p}_1\})$ is a (trivial) double cover of of $RQ^{2k+1} - \{\bar{p}_0, \bar{p}_1\}$. The standard orientation of $RQ^{2k+1}$ therefore induces an orientation on $\partial\bar{Y}$ which in turn induces an orientation on $\bar{Y}$.

**Definition 3.3.** The complex shade $\Gamma_p$ of $RQ^{2k+1}$, constructed using $p \in R\mathbb{P}^{2k+1}$, is the relative $(2k+2)$-cycle in $(CQ^{2k+1}, RQ^{2k+1})$ defined by $\Gamma_p = \gamma_p(\bar{Y})$, where $\gamma_p$ and $\bar{Y}$ are as in (3.2).

**Remark 3.4.** Remark 3.3 carries over word by word from the projective to the spherical case if “$R\mathbb{P}^{2k+1}$” and “$CQ^{2k+1}$” are replaced by “$RQ^{2k+1}$” and “$CQ^{2k+1}$”, respectively.

**Remark 3.5.** Let $p \in R\mathbb{P}^{2k+1}$ and let $\Gamma_p^P$ and $\Gamma_p^Q$ denote the shades of $R\mathbb{P}^{2k+1}$ and $RQ^{2k+1}$, respectively, constructed using $p$. Then $\Gamma_p^Q = \Pi(\Gamma_p^P)$. Moreover, the map $\Pi$ restricted to any $Cg \subset \Gamma_p^P$ is the standard double cover of a $\mathbb{C}l \subset \Gamma_p^P$, branched at two distinct complex conjugate points.

3.D. **Homology characterization of the shade.** Let $L^P$ denote the complex $k$-dimensional subspace in complex projective $(2k+1)$-space given by the equations
\[
i_z 0 - z_1 = i_z 2 - z_3 = \cdots = i_z 2k - z_{2k+1} = 0,
\] (3.3)
in projective coordinates $[z_0, \ldots, z_{2k+1}]$. Then $RL^Q = \emptyset$.

Let $L^Q$ denote the complex $k$-dimensional subspace in complex projective $(2k+2)$-space given by the equations
\[
i_z 1 - z_2 = i_z 3 - z_4 = \cdots = i_z 2k+1 - z_{2k+2} = 0, \quad z_0 = 0.
\] (3.4)
in projective coordinates $[z_0, \ldots, z_{2k+2}]$. Then $CL^Q \subset CQ^{2k+1}$, see §3.C, and $RL^Q = \emptyset$.

Let $[L^P]$ and $[L^Q]$ denote the homology classes in $H_{2k}(CP^{2k+1} - R\mathbb{P}^{2k+1})$ and $H_{2k}(CQ^{2k+1} - RQ^{2k+1})$ of $CL^P$ and $CL^Q$, with their complex orientations, respectively. Let $[F^P] \in H_{2k}(CP^{2k+1} - R\mathbb{P}^{2k+1})$ and $[F^Q] \in H_{2k}(CQ^{2k+1} - RQ^{2k+1})$ denote the homology classes corresponding to the fiber classes in the normal bundles of $R\mathbb{P}^{2k+1}$ and $RQ^{2k+1}$ in $CP^{2k+1}$ and $CQ^{2k+1}$, respectively.

To prove the next lemma we need the homology of $CQ^{2k+1}$. A straightforward calculation (the adjunction formula gives the total Chern class of $CQ^{2k+1}$ and, in particular, its Euler characteristic $2k + 2$, apply Lefschetz hyperplane theorem, Poincaré duality and the universal coefficient theorem) shows that
\[
H_r(CQ^{2k+1}) = \begin{cases} 
\mathbb{Z} & \text{if } r \text{ is even,} \\
0 & \text{if } r \text{ is odd.}
\end{cases}
\] (3.5)
Lemma 3.6. Let \( n = 2k + 1 \). The homology groups \( H_{n-1}(\mathbb{C}P^n - \mathbb{R}P^n) \) and \( H_{n-1}(\mathbb{C}Q^n - \mathbb{R}Q^n) \) are isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \), are generated by \([L^Q]\) and \([F^Q]\) respectively \([L^Q]\) and \([F^Q]\), and the homomorphisms

\[
\alpha^P : H_{n+1}(\mathbb{C}P^n, \mathbb{R}P^n) \to \text{Hom}(H_{n-1}(\mathbb{C}P^n - \mathbb{R}P^n); \mathbb{Z}); \quad \alpha^P(\Sigma)(\Delta) = \Sigma \cdot \Delta,
\]

\[
\alpha^Q : H_{n+1}(\mathbb{C}Q^n, \mathbb{R}Q^n) \to \text{Hom}(H_{n-1}(\mathbb{C}Q^n - \mathbb{R}Q^n); \mathbb{Z}); \quad \alpha^Q(\Sigma)(\Delta) = \Sigma \cdot \Delta,
\]

where \( \cdot \) denotes the intersection product, are isomorphisms.

Proof. The two cases will be treated simultaneously. Therefore, we let \((\mathbb{C}P^n, \mathbb{R}P^n)\) denote either \((\mathbb{C}P^n, \mathbb{R}P^n)\) or \((\mathbb{C}Q^n, \mathbb{R}Q^n)\), and we drop the superscripts on \( F, L, \) and \( \alpha \).

Poincaré duality implies that the following diagram with exact rows commutes and that all vertical arrows (which are cap-products with the orientation class of \( \mathbb{C}Y^n \)) are isomorphisms.

\[
\begin{array}{ccc}
\cdots H^{2n-r-1}(\mathbb{R}Y^n) & \xrightarrow{\delta} & H^{2n-r}(\mathbb{C}Y^n, \mathbb{R}Y^n) \xrightarrow{} H^{2n-r}(\mathbb{C}Y^n) \\
\downarrow & & \downarrow \\
\cdots H_{r+1}(\mathbb{C}Y^n, \mathbb{C}Y^n - \mathbb{R}Y^n) & \xrightarrow{\partial} & H_r(\mathbb{C}Y^n - \mathbb{R}Y^n) \xrightarrow{} H_r(\mathbb{C}Y^n) \\
\end{array}
\]

It follows that \( H_{n-1}(\mathbb{C}Y^n - \mathbb{R}Y^n) \approx \mathbb{Z} \oplus \mathbb{Z} \), generated as claimed, and that \( H_{n-2}(\mathbb{C}Y^n - \mathbb{R}Y^n) = 0 \), see Equation (3.5). The universal coefficient theorem then implies that

\[
\text{Hom}(H_{n-1}(\mathbb{C}Y^n - \mathbb{R}Y^n); \mathbb{Z}) \approx H^{n-1}(\mathbb{C}Y^n - \mathbb{R}Y^n).
\]

Let \( M^{2n} \) be the compact manifold which is the complement of an open tubular neighborhood of \( \mathbb{R}Y^n \) in \( \mathbb{C}Y^n \). By homotopy,

\[
H^r(M^{2n}) \approx H^r(\mathbb{C}Y^n - \mathbb{R}Y^n).
\]

Poincaré duality gives \( H^r(M^{2n}) \approx H_{2n-r}(M^{2n}, \partial M^{2n}) \). The inclusion \( M^{2n} \to \mathbb{C}Y^n \) induces a map \( (M^{2n}, \partial M^{2n}) \to (\mathbb{C}Y^n, T) \), where \( T \) is a closed tubular neighborhood of \( \mathbb{R}Y^n \) in \( \mathbb{C}Y^n \). By excision,

\[
H_r(M^{2n}, \partial M^{2n}) \approx H_r(\mathbb{C}Y^n, T).
\]

It now follows from homotopy and the 5-lemma that

\[
H_r(\mathbb{C}Y^n, T) \approx H_r(\mathbb{C}Y^n, \mathbb{R}Y^n).
\]

Hence,

\[
H_{n+1}(\mathbb{C}Y^n, \mathbb{R}Y^n) \approx \text{Hom}(H_{n-1}(\mathbb{C}Y^n - \mathbb{R}Y^n); \mathbb{Z}).
\]

Remark 3.2 (Remark 3.4) implies that the homology class of the shade \( \Gamma_p \) is independent of \( p \). We denote this homology class \([\Gamma]\). If \([\mathbb{R}P^{2k+1}] \in H_{2k+1}(\mathbb{R}P^{2k+1}) \) \(([\mathbb{R}Q^{2k+1}] \in H_{2k+1}(\mathbb{R}Q^{2k+1}))\) denotes the orientation class and \( ^* \) denotes complex conjugation then this class can be characterized as follows.

Proposition 3.7. The shade class \([\Gamma]\) is the unique class in \( H_{2k+2}(\mathbb{C}P^{2k+1}, \mathbb{R}P^{2k+1}) \) \((H_{2k+2}(\mathbb{C}Q^{2k+1}, \mathbb{R}Q^{2k+1}))\) which satisfies the following conditions

\begin{enumerate}
\item \([\Gamma] = [\Gamma]^*\),
\item \([\partial[\Gamma]] = 2[\mathbb{R}P^{2k+1}] \quad ([\partial[\Gamma]] = 2[\mathbb{R}Q^{2k+1}]),
\end{enumerate}
(c) \(|[\Gamma] \bullet [L^P]| = 1\) \((|[\Gamma] \bullet [L^Q]| = 1)\),

where \(\bullet\) denotes the intersection product.

**Remark 3.8.** If \(k\) is even then (c) is a consequence of (a) and (b). Note that Proposition 3.7 could be taken as definition of the shade class.

**Proof.** We use notation as in the proof of Lemma 3.4. Also, let \(J^P\) be a real \((k+1)\)-dimensional linear subspace of real projective \((2k+1)\)-space, and \(J^Q\) be the intersection of \(Q^{2k+1}\) and the \((k+2)\)-dimensional linear subspace, of real projective \((2k+2)\)-space, given by the equations \(x_2 = x_4 = \cdots = x_{2k} = 0\) in projective coordinates \([x_0, \ldots, x_{2k+2}]\). We write \([J]\) for the homology class represented by \(\mathbb{C}J^P (\mathbb{C}J^Q)\) with its complex orientation in \(H_{n+1}(\mathbb{C}Y^n, \mathbb{R}Y^n)\).

By Remarks 3.2 and \(2.3, \partial[\Gamma] = 2[\mathbb{R}Y^{2k+1}].\) Hence \([\Gamma]\) satisfies (a).

Let \(n = 2k + 1\) and consider the exact sequence

\[
0 \to H_{n+1}(\mathbb{C}Y^n) \to H_{n+1}(\mathbb{C}Y^n, \mathbb{R}Y^n) \xrightarrow{\partial} H_n(\mathbb{R}Y^n) \to 0.
\]

Since \(\partial[\Gamma] = 2[\mathbb{R}Y^n]\), any homology class \(\xi\) such that \(\partial \xi = 2[\mathbb{R}Y^n]\) can be written as

\[
[\Gamma] + a[J],
\]

where \(a \in \mathbb{Z}\). (In the spherical case, note that \(\mathbb{C}J^Q \bullet \mathbb{C}L^Q = 1\), thus \([J]\) generates the image of \(H_{n+1}(\mathbb{C}Q^n)\).)

By Proposition 3.4, the group \(H_{n-1}(\mathbb{C}Y^n - \mathbb{R}Y^n)\) dual to \(H_{n+1}(\mathbb{C}Y^n, \mathbb{R}Y^n)\) is generated by \([F]\) and \([L]\). It is straightforward to check that \([\Gamma] \bullet [F] = [\Gamma]^* \bullet [F]\) and that \([\Gamma] \bullet [L] = [\Gamma]^* \bullet [L]\). It follows from this that \([\Gamma] = [\Gamma]^*\). Hence \([\Gamma]\) satisfies condition (b). Moreover, \([J]^* = (-1)^{k+1}[J]\). Thus, Equation (3.6) implies that the homology class \([\Gamma]\) is uniquely determined by (a) and (b) if \(k\) is even.

Finally, \(|[\Gamma] \bullet [L]| = 1\). Hence \([\Gamma]\) satisfies condition (c). Equation (3.6), together with the fact that \([J] \bullet [L] = 1\) imply that \([\Gamma]\) is uniquely determined by (a)-(c), if \(k\) is odd. \(\Box\)

**Remark 3.9.** The proof of Proposition 3.7 gives a partial picture of the relation between conjugation and \(2k\)-dimensional homology. The following two equations complete it:

\[
[F]^* = -[F],
\]

\[
[L]^* = \begin{cases} 
- [L] & \text{if } k \text{ is odd,} \\
[L] - ([\Gamma] \bullet [L])[F] & \text{if } k \text{ is even,}
\end{cases}
\]

The first equation is obvious. To see that the second equation holds we consider first projective space and use a \((2k+1)\)-cycle in \(\mathbb{C}P^{2k+1}\) interpolating between \(\mathbb{C}L^P\) and \(\mathbb{C}(L^P)^*\), where \(\mathbb{C}(L^P)^*\) denotes the complex linear subspace conjugate to \(L\), with its complex orientation.

More precisely, consider the sequence of maps \(\phi_j : \mathbb{C}P^k \times [-1, 1] \to \mathbb{C}P^{2k+1}\) defined as follows. For \(0 \leq j \leq k\), let \(p_j\) denote the coordinate vector in coordinates \([z_0, \ldots, z_{2k+1}]\) on \(\mathbb{C}P^{2k+1}\) which has \(z_{2j} = 1, z_{2j+1} = i,\) and \(z_m = 0\) for \(m \neq 2j, 2j + 1\), let \(p_j^*\) be the complex conjugate coordinate vector, and let \(q_j\) denote
the coordinate vector which has $z_{2j+1} = i$ and $z_m = 0$ for $m \neq 2j + 1$. Let 
$[u] = [u_0, \ldots, u_k]$ be homogeneous coordinates on $\mathbb{CP}^k$ and define

$$\phi_j([u], t) = u_0p_0^* + \cdots + u_j-1p_{j-1}^* + u_j(p_j - (1 + t)q_j) + u_{j+1}p_{j+1} + \cdots + u_kp_k,$$

where the left-hand side is to be interpreted as a coordinate vector.

Note that $\phi_j([u], 1) = \phi_{j+1}([u], -1)$. The cycle $\phi_j(\mathbb{CP}^k \times [-1, 1])$ intersects $\mathbb{RP}^{2k+1}$ transversely in exactly one point and the sign of the intersection points of $\phi_j$ and that of $\phi_{j+1}$ are opposite. Finally, $\phi_0(\mathbb{CP}^k, -1)$ is $\mathbb{CL}^m$ and $\phi_k(\mathbb{CP}^k, 1)$ is $\mathbb{CL}^m$. Since $[L^p]^* = (-1)^k$ times the homology class of $\mathbb{CL}^m$ it follows that $[L^p]^* = -[L^p]$ if $k$ is odd and, since $[[L^p] \bullet [\Gamma]] = 1$ and $[\Gamma] \bullet [F^p] = 2$, $[L^p]^* = [L^p] - ([\Gamma] \bullet [L^p])[F^p]$ if $k$ is even, as claimed.

The argument in $\mathbb{C}Q^{2k+1}$ is similar: write the coordinates on $\mathbb{CP}^{2k+2}$ as $[u, z]$ where $z = [z_0, \ldots, z_{2k+1}]$ is as above, and use the maps

$$\psi_j([u], t) = \sqrt{1 - t^2}, \phi_j([u], t),$$

where $\phi_j([u], t)$ is as above, to interpolate between $\mathbb{CL}^q$ and $\mathbb{CL}^q$.

4. Varieties without real points

In this section, Proposition 2.3 and Theorem 2.2 are proved.

4.A. Proof of Proposition 2.3. We use the notation $[F^p]$ as in Lemma 3.4. If the intersection number of the $(2k+1)$-chain $A$ with $\mathbb{RP}^{2k+1}$ is $m$ then $\mathbb{CP}([-1, 1]) = m[F^p]$ in $H_{2k}(\mathbb{CP}^{2k+1} - \mathbb{RP}^{2k+1})$. Since $[\Gamma] \bullet [F^p] = 2$ the statement follows.

4.B. Proof of Theorem 2.2. Let $L^p$ be as in Lemma 3.4. Then $\mathbb{sh}(L^p) = \pm \frac{1}{2}$. Let $W$ be a $k$-dimensional variety of degree $d$ with $\mathbb{RW} = 0$. By definition of degree, $\mathbb{CW}$ is homologous to $d \cdot \mathbb{CL}^p$ in $\mathbb{CP}^{2k+1}$. Proposition 2.3 then implies that $\mathbb{sh}(W)$ is congruent to $\frac{d^2}{4}$ modulo 1.

Let $p \in \mathbb{RP}^{2k+1}$ be a point and $H$ a hyperplane in real projective $(2k+1)$-space such that $p \notin \mathbb{RH}$. Let $pr : (\mathbb{CP}^{2k+1} - \{p\}) \to \mathbb{CH}$ denote linear projection and let $W^*$ denote the complex conjugate variety of $W$.

Possibly after a small algebraic perturbation, we may assume that $pr(\mathbb{CW})$ meets $pr(\mathbb{CW}^*)$ transversely. (Proposition 2.3 implies that small perturbations do not affect $\mathbb{sh}(W)$.)

If $x$ is an intersection point of $\Gamma_p$ and $\mathbb{CW}$ then $x$ lies on $\mathbb{Cl}$ for some real line $l$ through $p$. The complex conjugate $x^*$ of $x$ satisfies $x^* \in \mathbb{Cl}$ and $x^* \in \mathbb{CW}^*$. Now, $pr(\mathbb{Cl})$ is a point and hence $pr(x) = pr(x^*)$. Thus, to each intersection point of $\Gamma_p$ and $\mathbb{CW}$ there corresponds an intersection point of $pr(\mathbb{CW})$ and $pr(\mathbb{CW}^*)$ in $\mathbb{CH}$. The $k$-dimensional varieties $pr(W)$ and $pr(W^*)$ both have degree $d$. They meet transversely, and hence intersect in $d^2$ points. It follows that $|\mathbb{sh}(W)| \leq \frac{1}{2}d^2$.

To see that the shade number takes all possible values between $-\frac{1}{2}d^2$ and $\frac{1}{2}d^2$ we construct explicit varieties:

Let $K$ be a large positive real number. Define

$$P(u, v) = K(u - v)(u - 2v) \ldots (u - dv).$$

For $t \in \mathbb{R}$, define

$$Q_t(u, v) = K \left( u - (t + \frac{1}{3t^2})v \right) \left( u - (t + \frac{2}{3t^2})v \right) \ldots \left( u - (t + \frac{d}{3t^2})v \right).$$
Then the equations $P(u, 1) = 1$ and $Q_t(u, 1) = 1$ has $d$ real solutions $\theta_1 < \cdots < \theta_d$ and $\phi_1(t), \ldots, \phi_d(t)$, respectively. The distances $|\theta_j - j|$ and $|\phi_j(t) - (t + \frac{j}{d+1})|$, $j = 1, \ldots, d$, can be made arbitrarily small by choosing $K$ sufficiently large.

For $t \in \mathbb{R}$, let $W_t$ be the variety in complex projective $(2k + 1)$-space defined by the equations

\[
P(z_1, z_0) + iQ_t(z_2, z_0) - (1 + i)z_0^d = 0,
\]

\[
z_1 + z_2 - iz_3 = 0,
\]

\[
z_{2j} - iz_{2j+1} = 0, \quad \text{for } j = 2, \ldots, k.
\]

in projective coordinates $[z_0, \ldots, z_{2k+1}]$ (with complex conjugation given by conjugation on all coordinates).

Consider the shade $\Gamma_p$ constructed using the point $p = [0, 0, 0, 1, 0, \ldots, 0]$ in $\mathbb{R}P^{2k+1}$. A straightforward check shows that $\Gamma_p \cap C W_t$ consists of the points $[1, \theta_j, \phi_k(t), i(\theta_j + \phi_k(t)), 0, \ldots, 0]$, $1 \leq j, k \leq d$. For large negative $t$, all intersection points contribute with the same sign to $\Gamma_p \cdot C W_t$ and $\text{sh}(W_t) = \pm \frac{1}{2}d^2$. As $t$ increases, the roots $\phi_1(t), \ldots, \phi_d(t)$ approaches $-\theta_d$. At some time $t'$, $\phi_d(t') = -\theta_d$ and $W_{t'}$ has a real point $[1, \theta_d, \phi_d(t'), 0, 0, \ldots, 0]$. As $t$ passes $t'$ the sign of the intersection point $[1, \theta_d, \phi_d(t), i(\theta_d + \phi_d(t)), 0, \ldots, 0]$ changes. This changes the intersection number $\Gamma_p \cdot C W_t$ by $\mp 2$ and hence $\text{sh}(W_t)$ by $\mp 1$. As $t$ increases further the roots $\phi_{d-1}(t), \ldots, \phi_1(t)$ passes $-\theta_d$ and at each passage $\text{sh}(W_t)$ changes by $\mp 1$. As $t$ increases even further the same things happen at $-\theta_{d-1}, \ldots, -\theta_1$ until $\phi_1(t)$ has passed $-\theta_1$ and we have $\text{sh}(W_t) = \mp \frac{1}{2}d^2$.

5. Topological invariants, shade numbers, twists, and generic shades

In this section, two topological invariants of armed submanifolds are described. For later use, the behavior of these invariants and of shade numbers under local deformations of normal vector fields are studied.

The existence of shades that have good properties with respect to a given real variety is established and used to demonstrate a certain symmetry property of shade numbers in even dimensions.

5A. Euler numbers. For the readers convenience we recall the definition of Euler number: let $M$ be a $k$-dimensional manifold and let $\xi$ be $k$-dimensional vector bundle over $M$. Assume that the total space $E(\xi)$ of $\xi$ is orientable.

Let $M$ and $F$ denote the integer local coefficient systems over $M$ associated to the orientation bundle of $M$ and the fiber orientation bundle of $\xi$, respectively. Then the orientation of $E(\xi)$ gives

\[
M \otimes F \approx \mathbb{Z}, \tag{5.1}
\]

where $\mathbb{Z}$ denotes the (trivialized) local coefficient system associated to the the orientation bundle of $E(\xi)$, restricted to the zero-section $M$. An isomorphism $M \approx F$, is specified by requiring that, at each point, (5.1) is given by ordinary multiplication of integers.

This specified isomorphism in turn gives a well-defined pairing

\[
\langle , \rangle : H^k(M; F) \otimes H_k(M; M) \to \mathbb{Z}.
\]

The Euler number $e(\xi)$ of $\xi$ is defined as $\langle e, [M] \rangle$, where $e$ is the Euler class of the bundle (the obstruction to finding a non-vanishing section), and $[M]$ is the orientation class of $M$. 

One can compute the Euler number of $\xi$ by choosing any section $s$ of $\xi$ transverse to the zero-section and sum up the local intersection numbers at zeros of $s$. The local intersection number at a zero $p$ of $s$ is the intersection number in $E(\xi)$ of some neighborhood $U \subset M$ of $p$, with any chosen orientation, and $s(U)$, with the orientation induced from the chosen orientation on $U$.

It is straightforward to check that if $k$ is odd then $e(\xi) = 0$.

5.B. Wrapping numbers. Let $(M, n)$ be an armed $k$-dimensional orientable smooth submanifold of $\mathbb{R}P^{2k+1}$ or $S^{2k+1}$. Let $K(1), \ldots, K(m)$ denote the connected components of $M$ and let $K(j)_n$ denote a copy of $K(j)$ shifted slightly along $n$. Fix some orientation of $M$ and note that it induces an orientation on each $K(j)_n$.

**Definition 5.1.** The number

$$\text{wr}(M, n) = \sum_{j=1}^{m} \text{lk}(K(j), K(j)_n)$$

is called the wrapping number of $(M, n)$.

**Remark 5.2.** Note that $\text{wr}(M, n)$ is independent of the choice of orientation on $M$. If $M$ is endowed with an orientation then other self-linking numbers can be associated to it. They are expressible in terms of $\text{wr}(M, n)$ and the linking numbers $\text{lk}(K(j), K(l))$, $j \neq l$.

**Remark 5.3.** An argument similar to the proof of Theorem 2.6 (§ 6.A) shows that if $\dim(M)$ is even then $\text{wr}(M, n) = -\frac{1}{2} e(M, n)$. Note that the definition of the wrapping number applies only if $M$ is orientable whereas the Euler number is defined also for non-orientable manifolds.

5.C. Twists. Let $M$ be a $k$-dimensional submanifold of an oriented manifold $Y$ of dimension $2k + 1$. Let $NM$ denote the normal bundle of $M \subset Y$, and let $N^0M$ denote the bundle of non-zero vectors in $NM$.

If $n$ and $m$ are sections in $N^0M$ then a standard obstruction theory argument shows that $n$ can be made homotopic to $m$ by local changes which we call local twists:

Pick a Riemannian metric in the normal bundle $NM$ and let $\text{UNM}$ denote the corresponding bundle of unit normal vectors. Let $v$ be a section of $\text{UNM}$ and let $p \in M$. Choose a local coordinate neighborhood $X \subset M$ around $p$ and a local trivialization $\beta: X \times S^k \to \text{UNM}|X$ such that $\beta^{-1}(v(x)) = (x, a) \in X \times S^k$ for $x$ varying in $X$ and $a$ fixed in $S^k$. Fix an orientation on $X$. This orientation together with the orientation on $Y$ induce an orientation on the fibers of $NM|X$, which in turn induces an orientation on the fibers $S^k$ of $\text{UNM}|X$. Let $D^k \subset X$ be a disk centered at $p$ and let $r: D^k \to S^k$ be a map of degree $\pm 1$ such that $r(\partial D^k) = a$. Define $v'$ as

$$v'(x) = \begin{cases} v(x), & x \in M - D^k \\ \beta(x, r(x)), & x \in D^k \end{cases}.$$  

We say that $v'$ is obtained from $v$ by adding a local twist at $p$ and that the degree of $r$ is the sign of the local twist.

We shall now study how Euler numbers, wrapping numbers and shade numbers are affected by local twists.
Lemma 5.4. Let $M$ be a $2j$-dimensional submanifold of an oriented manifold $Y$ of dimension $4j + 1$. Let $m$ and $n$ be non-vanishing normal vector fields of $M$.

(a) If $m$ and $n$ are homotopic as sections in $N^0 M$ then $e(M, m) = e(M, n)$.
(b) If $m$ is obtained from $n$ by adding a local twist of sign $\sigma = \pm 1$ then $e(M, m) = e(M, n) + 2\sigma$.
(c) $e(M, n) = -e(M, -n)$.

Proof. Assertion (a) is immediate. To prove (b), let $p$ be the point where the twist is added and fix local coordinates $x = (x', x'', x_{4j+1}) \in \mathbb{R}^{4j+3}$, $x' = (x_1, \ldots, x_{2j})$, $x'' = (x_{2j+1}, \ldots, x_{4j})$, on an open set $X \subseteq Y$ such that $M \cap X = \{ x : x'' = 0, x_{4j+1} = 0 \}$, and $p$ corresponds to $x = 0$.

Write $\partial_i = \frac{\partial}{\partial x_i}$. We may assume that $n$ is given by $x' \mapsto \frac{1}{\sqrt{2j}} \sum_{k=2j+1}^{4j} \partial_k$ in coordinates $x'$ on $M \cap X$. Identify the bundle $\xi(n)$ with the orthogonal complement of $n(x')$ in the normal bundle and choose a section $s$ of $\xi(n)$ which is given by $x' \mapsto \partial_{4j+1}$ in $M \cap X$.

Let $S^{2j} = \{ x : x' = 0, (x'')^2 + x_{4j+1}^2 = 1 \}$ and let $D^{2j}$ be a small disk where the twist, $x' \mapsto r(x') \in S^{2j}$ is added to $n$ to obtain $m$. The bundle $\xi(m)$ is then identical to $\xi(n)$ outside $D^{2j}$ and over $D^{2j}$, $\xi(m)$ can be identified with the bundle with fiber over points $x'$ equal to $T_{r(x')} S^{2j}$. Let $s'$ be the section of $\xi(m)$ which agrees with $s$ outside $D^{2j}$, and inside $D^{2j}$ is given by orthogonal projection of $s(x')$ into $T_{r(x')} S^{2j}$. That is,

$$s'(x') = \partial_{4j+1} - \langle \partial_{4j+1}, r(x') \rangle r(x'),$$

where $\langle , \rangle$ denotes the Euclidean metric on $X$. Then $s'(x') = 0$ if and only if $r(x') = \pm \partial_{4j+1}$. At such an $x'$,

$$ds'(x') = \mp dr(x').$$

Since changing the sign of each vector in an even dimensional frame does not change its orientation, it follows that the difference of the algebraic number of zeroes of $s'$ and $s$ equals twice the degree of $r$. This implies (b).

Assertion (c) follows from the fact that the bundles $\xi(n)$ and $\xi(-n)$ are canonically isomorphic, but the orientations of the total space $E(\xi(n)) = E(\xi(-n))$ induced by $n$ and $-n$, respectively, are opposite. \qed

Lemma 5.5. Let $M$ be an orientable $(2j + 1)$-dimensional submanifold of $S^{4j + 3}$ or $\mathbb{R}P^{4j + 3}$. Let $m$ and $n$ be non-vanishing normal vector fields of $M$.

(a) If $m$ and $n$ are homotopic as sections in $N^0 M$ then $\text{wr}(M, m) = \text{wr}(M, n)$.
(b) If $m$ is obtained from $n$ by adding a local twist of sign $\sigma = \pm 1$ then $\text{wr}(M, m) = \text{wr}(M, n) + \sigma$.

Proof. Assertion (a) is immediate. To prove (b), let $p$ be the point where the twist is added and fix local coordinates $x = (x', x'', x_{4j+3}) \in \mathbb{R}^{4j+3}$, $x' = (x_1, \ldots, x_{2j+1})$, $x'' = (x_{2j+2}, \ldots, x_{4j+2})$, on an open set $X$ in the ambient space such that $M \cap X = \{ x : x' = 0, x_{4j+3} = 0 \}$, and $p$ corresponds to $x = 0$.

Let the orientation of the ambient space along $X$ be given by the frame $(\partial', \partial'', \partial_{4j+3})$, where $\partial' = (\partial_1, \ldots, \partial_{2j+1})$ and $\partial'' = (\partial_{2j+2}, \ldots, \partial_{4j+2})$. We may assume that $n$ is given by $x' \mapsto \frac{1}{\sqrt{2j+1}} \sum_{k=2j+2}^{4j+2} \partial_k$ in coordinates $x'$ on $M \cap X$.

Choose the orientation on $M$ given by the frame $\partial'$. \qed
Let $K$ denote the component of $M$ which intersects $X$. To find $\text{lk}(K^{2j+1}, K^{2j+1}_n)$ we need a $(2j+2)$-chain $A$ with boundary $2K$. We may choose $A$ so that $A \cap X = a_1 \cup a_2$, where

$$a_1 = \{x: x'' = 0, x_{4j+3} \geq 0\},$$

$$a_2 = \{x: x'' = 0, x_{4j+3} \leq 0\}.$$  

Then, by our orientation conventions (see § 5.4), $(\partial', \partial_{4j+3})$ and $(\partial', -\partial_{4j+3}$ are positively oriented frames along $a_1$ and $a_2$, respectively.

Let $S^{2j+1} = \{x: x' = 0, (x'')^2 + x_{4j+3}^2 = 1\}$ and let $D^{2j+1}$ be a small disk where the twist $x' \mapsto r(x') \in S^{2j+1}$ is added to $n$ to obtain $m$. If $U$ is a small neighborhood of $D^{2j+1}$ in $X$ then outside of $U$, $K_n = K_m$, and inside $U$,

$$K_n = \{x: x'' = \delta'', x_{2k+1} = 0\},$$

where $\delta'' = (\delta, \ldots, \delta)$ for some small $\delta > 0$, and

$$K_m = \{x: (x'', x_{2k+1}) = \delta r(x')\}.$$  

Thus $U \cap K_n \cap A = \emptyset$, and $U \cap K_m \cap A$ consists of points $(x', \delta r(x'))$ such that $r(x') = \pm \partial_{4j+3}$. Assume that $\pm \partial_{4j+3}$ are regular values of $r$.

Consider a point $x'$ such that $r(x') = \partial_{4j+3}$. The corresponding intersection point $(x', \delta r(x'))$ lies in $a_1$ and the local intersection number of $A$ and $K_m$ is given by the sign of the orientation of the frame

$$(\partial', \partial_{4j+3}, \partial' + dr\partial'),$$

which is just the local degree of $r$ at $x'$. At a point $(x', \delta r(x'))$ such that $r(x') = -\partial_{4j+3}$ the sign of $\partial_{4j+3}$ in Formula (5.2) must be changed since the intersection point lies in $a_2$ and not $a_1$ and again the local intersection number equals the local degree of $r$.

It follows that the intersection numbers $A \bullet K_m$ and $A \bullet K_n$ differs by twice the degree of $r$. Since there is a factor $\frac{1}{2}$ in the definition of $\text{lk}$ (see Section 5.4), (b) follows.

**Lemma 5.6.** Let $(V, n)$ be a $k$-dimensional projective variety without real singularities in real projective $(2k+1)$-space or in the real $(2k+1)$-sphere. Write $k = 2j+1$ if $k$ is odd and $k = 2j$ if $k$ is even. If $m$ is obtained by adding a local twist of sign $\sigma = \pm 1$ to $n$ then

$$\text{sh}(V, m) = \text{sh}(V, n) + (-1)^{k+j+1}\sigma.$$  

**Proof.** Let $\mathbb{R}Y^{2k+1}$ and $\mathbb{C}Y^{2k+1}$ denote the real- and the complex points of the ambient space, respectively. Let $p \in \mathbb{R}V$ be the point where the twist is added and let $K$ be the connected component of $\mathbb{R}V$ which contains $p$.

Introduce (real analytic) local coordinates $x = (x', x'', x_{2k+1}) \in \mathbb{R}^{2k+1}$, where $x' = (x_1, \ldots, x_k)$ and $x'' = (x_{k+1}, \ldots, x_{2k})$, on an open set $X \subset \mathbb{R}^{2k+1}$ so that $\mathbb{R}V \cap X = \{x: x'' = 0, x_{2k+1} = 0\}$, and so that $p$ corresponds to $x = 0$ (and hence $\mathbb{R}V \cap X = K \cap X$). Then there exist holomorphic local coordinates $z = (z', z'', z_{2k+1}) \in \mathbb{C}^{2k+1}$ on a neighborhood $Z \subset \mathbb{C}Y^{2k+1}$ of $p$ such that $Z \cap \mathbb{R}Y^{2k+1} = X$, such that $z_j = x_j + iy_j$, $j = 1, \ldots, 2k + 1$, such that the complex conjugation on $\mathbb{C}Y^{2k+1}$ is given by conjugation of coordinates in $Z$, and such that $\mathbb{C}V$ is locally given by $\mathbb{C}V \cap Z = \{z: z'' = 0, z_{2k+1} = 0\}$. We use the notions $\partial'$ and $\partial''$ as in the proof of Lemma 5.4. Let $(\partial', \partial'', \partial_{2k+1})$ be a positively oriented frame of $T\mathbb{R}Y^{2k+1}|X$. 


The shade $\Gamma = \Gamma_q$ ($q \in \mathbb{R}P^{2k+1}$) of $\mathbb{R}Y^{2k+1}$ is a relative $(2k+2)$-cycle in $(\mathbb{C}Y^{2k+1}, \mathbb{R}Y^{2k+1})$. After a homotopy, supported in a small neighborhood of $Z$, we may assume that $\Gamma \cap Z$ equals $\gamma_1 \cup \gamma_2$, where

$$\gamma_1 = \{z = x + iy: y'' = 0, y_{2k+1} \geq 0\},$$

$$\gamma_2 = \{z = x + iy: y'' = 0, y_{2k+1} \leq 0\}.$$

Since $\partial\Gamma = 2\mathbb{R}Y^{2k+1}$, our orientation conventions (§ 3.4) imply that

$$(\partial', \partial'', \partial_{2k+1}, i\partial_{2k+1})$$

are positively oriented frames of $\gamma_1$ and $\gamma_2$, respectively.

We may assume that the normal vector field $n$ is given by $x' \mapsto \frac{1}{\sqrt{k}} \sum_{j=k+1}^{2k} \partial_j$ in coordinates $x'$ on $K \cap X$. Let $S^k = \{x: x'' = 0, (x'')^2 + x_{2k+1}^2 = 1\}$ and let $D^k$ be a small disk where the twist $x' \mapsto r(x') \in S^{2j+1}$ is added to $n$ to obtain $m$.

If $E$ is a small neighborhood of $D^{2j+1}$ in $Z$ then outside of $E$, $\mathbb{C}V_n = \mathbb{C}V_m$, and inside $E$,

$$\mathbb{C}V_n = \{z: x'' = z_{2k+1} = 0, y'' = \delta''\}$$

and

$$\mathbb{C}V_m = \{z: x'' = x_{2k+1} = 0, (y'')^2 = \delta'\}$$

where $\phi$ is a positive function such that $\phi(y') = \delta$ for $(y')^2 < \epsilon$ and $\phi(y') = 0$ for $(y')^2 > 2\epsilon$ for some very small $\epsilon > 0$. Thus, $E \cap \mathbb{C}V_n \cap \Gamma = \emptyset$, and $E \cap \mathbb{C}V_m \cap \Gamma$ consists of points $(x' + 0i, 0 + i\delta r(x'))$, where $r(x') = \pm \partial_{2k+1}$. Hence, the difference between $\text{sh}(V, m)$ and $\text{sh}(V, n)$ equals one half times the sum of local intersection numbers of $\mathbb{C}V_m$ and $\Gamma$ in $E$. We calculate this difference:

Let $dx' \wedge dy' = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_k \wedge dy_k$, interpret $dx'' \wedge dy''$ and $dx \wedge dy$ similarly. Then the complex orientation of $\mathbb{C}Y^{2k+1}$ is given by the $(4k+2)$-form $dx \wedge dy$ and the complex orientation of $\mathbb{C}V$ is given by $dx' \wedge dy'$.

Assume that $r(x') = \partial_{2k+1}$. Then the local intersection number of $\Gamma$ and $\mathbb{C}V_m$ at the corresponding intersection point is given by the sign of

$$(-1)^j dx \wedge dy (\partial', \partial'', \partial_{2k+1}, i\partial_{2k+1}, i\partial r(x')) \wedge \partial'$$

and let $\langle -i \partial r(x') \rangle$.

It follows that the local intersection number of $\Gamma$ and $\mathbb{C}V_m$ at the intersection point corresponding to $x'$ equals $(-1)^{j+k}$ times the local degree of $r$ at $x'$.

At an intersection point with $r(x') = -\partial_{2k+1}$ a similar calculation shows that local intersection number is again $(-1)^{j+k}$ times the local degree of $r$ at $x'$. (Both the sign of $i\partial_{2k+1}$ in Formula (5.3) and that of $\partial_{2k+1}$ in Formula (5.4) change.)

5.D. **Generic shades.** We shall treat projective spaces and spheres simultaneously. The notation introduced in § 3.4 will be used repeatedly.

Let $V$ be a real projective $k$-dimensional variety without real singularities in real projective $(2k+1)$-space (in the real $(2k+1)$-sphere). We define three subsets of $\mathbb{R}P^{2k+1}$, which we want to avoid.
First, consider the variety \( \Sigma \) of singular points of \( V \). By assumption \( \mathbb{R}\Sigma = \emptyset \). Moreover, the dimension of \( \Sigma \) is at most \( k-1 \). Let \( A \) be the chordal variety (see (3)) of \( \Sigma \) \( (\Pi(\Sigma)) \). Then the complex dimension of \( CA \) is at most \( 2k-1 \) and \( RA \) is a stratiﬁed set of real dimension at most \( 2k-1 \).

Second, let \( B \) be the tangential variety (see (3)) of \( V \) \( (\Pi(V)) \). Then the complex dimension of \( CB \) is at most \( 2k \) and \( RB \) is a stratiﬁed set of real dimension at most \( 2k \).

Third, let \( D \) be the variety which is the closure of all points on chords through two distinct smooth points \( p,q \in CV \) \( (p,q \in \Pi(CV)) \) such that the projective tangent spaces of \( CV \) \( (\Pi(CV)) \) at \( p \) and \( q \) intersect. Then the complex dimension of \( CD \) is at most \( 2k \) and \( RD \) is a stratiﬁed set of real dimension at most \( 2k \).

A point \( p \in \mathbb{R}P^{2k+1} \) will be called generic (with respect to \( V \)) if \( q \notin RA \cup RB \cup RD \). Note that the set of generic points form a dense open set in \( \mathbb{R}P^{2k+1} \).

If \( p \in \mathbb{R}P^{2k+1} \) is a generic point then the shade \( \Gamma_p \) of \( \mathbb{R}P^{2k+1} \) \( (\mathbb{R}Q^{2k+1}) \), constructed using \( p \), does not intersect \( \mathbb{C}\Sigma \) (since \( p \notin RA \)), and at any point \( x \in \Gamma_p \cap (CV - RV) \) \( (x \in \Gamma_p \cap (CV - RV)) \) the intersection is transverse (since \( p \notin RD \)).

Assume that \( RV \) is orientable and oriented, and let \( K \) be a component of \( RV \). To describe linking properties of \( K \) we need a chain with boundary \( 2K \), see Section 1.

Such chains can be constructed using generic points.

First consider \( \mathbb{R}P^{2k+1} \): If \( y,z \in \mathbb{R}P^{2k+1} \) are distinct then let \( l(y,z) \) denote the line containing them. Let \( C = \bigcup_{x \in K} \mathbb{R}l(p,x) \). If \( p \) is a generic point then \( C \) is an immersed submanifold, except possibly at \( p \) (since \( p \notin RB \)). Clearly, we can orient the chain \( C \) so that \( \partial C = 2K \).

Second consider \( \mathbb{R}Q^{2k+1} \): If \( y \in \mathbb{R}P^{2k+1} \), \( z \in \mathbb{R}Q^{2k+1} \), and \( z \notin \pi^{-1}(y) \), then let \( g(y,z) \) denote the great circle through \( \pi^{-1}(y) \) and \( z \). Let \( p \in \mathbb{R}P^{2k+1} \) and let \( C = \bigcup_{x \in K} \mathbb{R}g(p,x) \). If \( p \) is generic then \( C \) is an immersed submanifold, except possibly at \( p \) (since \( p \notin RB \)). Clearly, we can orient the chain \( C \) so that \( \partial C = 2K \).

Remark 5.7. If \( (V,n) \) is an armed real projective \( k \)-dimensional variety without real singularities in \( (2k+1) \)-dimensional real projective space or in the real \( (2k+1) \)-sphere and if \( p \in \mathbb{R}P^{2k+1} \) is a generic point, then there are two types of intersection points in \( \Gamma_p \cap CV \) and in \( C \cap RV \): points near \( RV \) and points far from \( RV \). More precisely, since \( p \) is generic there is a small neighborhood \( E \) of \( RV \) in \( \mathbb{C}P^{2k+1} \) \( (\mathbb{C}Q^{2k+1}) \) such that \( \Gamma_p \cap CV \cap E = RV \). We write \( \Gamma_p(E) = \Gamma_p \cap E \) and \( \Gamma_p'(E) = \Gamma_p - \Gamma_p(E) \) and say that a point \( x \in \Gamma_p \cap CV \) is of type

- \( \text{(C)} \) if \( x \in \Gamma'(E) \cap CV \), and of type \( \text{(CN)} \) if \( x \in \Gamma(E) \cap CV \).

We use similar notation for the chain \( C \): Let \( U = E \cap \mathbb{R}P^{2k+1} \) \( (U = E \cap \mathbb{R}Q^{2k+1}) \) and write \( C(U) = C \cap U \) and \( C'(U) = C - C(U) \) and say that a point \( y \in C \cap K_n \) is of type

- \( \text{(RF)} \) if \( x \in C'(U) \cap K_n \), and of type \( \text{(RN)} \) if \( x \in C(U) \cap K_n \).
5.E. Symmetry in even dimensions.

**Lemma 5.8.** Let $(V,n)$ be an armed projective $2j$-dimensional variety without real singularities in real projective $(4j + 1)$-space or in the real $(4j + 1)$-sphere. Then

$$\text{sh}(V,n) = -\text{sh}(V,-n).$$

**Proof.** Pick a generic point $p$ in $\mathbb{R}P^{4j+1}$ and let $\Gamma = \Gamma_p$ be the shade constructed using $p$. Let $\mathbb{R}Y^{4j+1}$ and $\mathbb{C}Y^{4j+1}$ denote the real- and complex parts of the ambient space, respectively. Let $E$ be a neighborhood of $\mathbb{R}V$ in $\mathbb{C}Y^{4j+1}$ with properties as in Remark 5.7.

Assume that the vector field $\nu$ (which extends in) used to shift $\mathbb{C}V$ has support inside $\mathbb{C}V \cap E$. As in Remark 5.7, we distinguish two types (CF) and (CN) of intersection points in $\mathbb{C}V \cap \Gamma$:

If $x$ is an intersection point of type (CF) then, since both $\Gamma$ and $\mathbb{C}V$ are invariant under conjugation, also the complex conjugate point $x^*$ of $x$ is an intersection point of type (CF). We compare their signs: let $o(\Gamma, x)$ and $o(\mathbb{C}V, x)$ denote positively oriented frames of $T_x\Gamma$ and $T_x\mathbb{C}V$, respectively. The intersection number at $x$ is then given by the sign $\sigma$ of the frame $(o(\Gamma, x), o(\mathbb{C}V, x))$. Since the complex dimension of $\mathbb{C}Y^{4j+1}$ is odd, complex conjugation $*$ reverses orientation and hence the sign of the frame $(o(\Gamma_p, x)^*, o(\mathbb{C}V, x)^*)$ at $x^*$ is $-\sigma$. The orientations of $o(\Gamma, x^*)$ and $o(\Gamma, x)^*$ agree (recall, $[\Gamma]$ is invariant under conjugation). Since $\mathbb{C}V$ is complex even-dimensional the orientations $o(\mathbb{C}V, x)^*$ and $o(\mathbb{C}V, x^*)$ agree as well. It follows that points of type (CF) do not contribute at all to $\text{sh}(V,n)$.

Consider an intersection point $q$ of type (CN). Let $Z$ be a small neighborhood of $q$ and let $X = \mathbb{R}Y^{4j+1} \cap Z$. We use coordinates on $X$ and $Z$ as in the proof of Lemma 5.6, more precisely:

Let $\mathbb{R}V \cap X = \{x : x'' = 0, x_{4j+3} = 0\}$ in coordinates $x = (x', x'', x_{4j+1})$ on $X$, and let be $(z', z'', z_{4j+1})$, where $z' = x' + iy'$, $z'' = x''$, and $z_{4j+1} = x_{4j+1} + iy_{4j+1}$ be holomorphic coordinates on $Z$. Also, $\Gamma \cap Z = \gamma_1 \cup \gamma_2$, where

$$\gamma_1 = \{z = x + iy; y' = y'' = 0, y_{4j+1} \geq 0\},$$

$$\gamma_2 = \{z = x + iy; y' = y'' = 0, y_{4j+1} \leq 0\}.$$

We view the normal vector field $n$ as a map $x' \mapsto n(x') \in \{x : x' = 0, (x'')^2 + x_{4j+1}^2 = 1\}$. It is then clear that the intersection point $q$ corresponds to a point $x'$ such that $n(x') = \pm \partial_{4j+1}$. Assume that $n(x') = \partial_{4j+1}$ and that the intersection is transverse. The contribution from $q$ to $\Gamma \bullet \mathbb{C}V_n$ is then given by the sign of

$$(-1)^j dx \land dy(\partial', \partial'', \partial_{4j+1}, i\partial_{4j+1}, idn\partial', i\partial').$$

Now consider the vector field $-n$. It is easy to see that to each intersection point in $\mathbb{C}V_n \cap \Gamma$ there is a corresponding intersection point of the same type, (CF) or (CN), in $\mathbb{C}V_{-n} \cap \Gamma$. As we have seen above, only the points of type (CN) contribute.

Consider the intersection point $q'$ corresponding to the point $q$ (of type (CN) considered above). The contribution from $q'$ to $\Gamma \bullet \mathbb{C}V_{-n}$ is given by the sign of

$$(-1)^j dx \land dy(\partial', \partial'', \partial_{4j+1}, -i\partial_{4j+1}, -idn\partial', i\partial'),$$

where the sign in front of $i\partial_{4j+1}$ appears since $q' \in \gamma_2$ (in contrast to $q$, which lies in $\gamma_1$). Since multiplying all vectors in an even-dimensional frame by $-1$ does not change its orientation, the contributions from $q$ and $q'$ have opposite signs. The lemma follows. 

\[\square\]
6. Real varieties

In this section, Theorems 2.6, 2.8, and 2.9 are proved. The diagrammatic definition of the encomplexed writhe is presented, following \([8]\) and, with this definition at hand, Theorem 2.11 is proved.

6.A. Proof of Theorem 2.6. Let \(N\) denote the set of homotopy classes of sections in the bundle of non-zero normal vectors of \(\mathbb{R}V\).

Let \(\Phi, \Psi : N \to \frac{1}{2}\mathbb{Z}\) be defined as

\[
\Phi(n) = \frac{1}{2} e(\mathbb{R}V, n), \\
\Psi(n) = \text{sh}(V, n),
\]

where \(n\) is a representative of the homotopy class \(n\). Then it follows from Lemma 5.4 (a) that \(\Phi\) is well-defined and from Lemma 5.5 (a) that \(\Psi\) is well-defined. Moreover, Lemma 5.4 (b) and Lemma 5.5 (b) imply that \(\Phi - (-1)^j \Psi\) is a constant function equal to \(c\) say. But then Lemma 5.4 (c) and Lemma 5.8 implies that

\[
c = \frac{1}{2} e(\mathbb{R}V, n) - (-1)^j \text{sh}(V, n) = \frac{1}{2} e(\mathbb{R}V, -n) - (-1)^j \text{sh}(V, -n) = -(-\frac{1}{2} e(\mathbb{R}V, n) - (-1)^j \text{sh}(V, n)) = -c.
\]

Hence, \(c = 0\) and the theorem follows.

6.B. Proof of Theorem 2.8. It follows from Lemmas 5.5 and 5.6 that \(C_w\) is independent of the choice of normal vector field of \(\mathbb{R}V\). Also, \(C_w\) is easily seen to be independent of orientation on \(\mathbb{R}V\), and to be invariant under weak rigid isotopy.

It remains to show that \(C_w\) is integer-valued: fix a generic point \(p \in \mathbb{R}P^{4j+3}\) and let \(K\) be a component of \(\mathbb{R}V\). To calculate \(\text{lk}(K, K_n)\) we consider \(C \bullet K_n\), where \(C\) is constructed using the generic point \(p\), see § 5.D. Let \(U = E \cap \mathbb{R}Y^{4j+3}\). As there, two types (CF) and (CN) of points in \(\Gamma \cap CV_n\) will be distinguished. As in the proof of Lemma 5.8, the points of type (CF) come in pairs, the same argument as there shows that, in the odd-dimensional case under consideration, the two points in a pair of type (CF) contribute with the same sign to \(\Gamma \bullet CV_n\).

We consider first points of type (RF). Let \(x'\) be such a point.

- If ambient space is projective space then \(x'\) lies very close to a chord \(l\) of \(K\), which passes through \(p\).
- If the ambient space is the sphere then \(x'\) lies very close to a great circle \(g\) which passes through two points on \(K\) and through \(\tilde{p}_0\) and \(\tilde{p}_1\) (see § 3.C for notation).

Let \(x\) and \(y\) be the points in \(K\) on \(\mathbb{R}l(\mathbb{R}g)\) and let \(x\) be the one close to \(x'\). Then there is another intersection point \(y'\) of type (RF) close to \(y\). Thus, points of type (RF) come in pairs.

We now consider points of types (CN) and (RN). Using the local models of Lemmas 5.5 and 5.6, we observe that to each intersection point \(z\) of type (RN) in \(C(U) \cap K_n\) there corresponds exactly one point \(r\) of type (CN) in \(\Gamma(E) \cap CV_n\). It follows that \(C_w\) is integer-valued.
Remark 6.1. For future reference we establish the relation between the local intersection numbers in the pairs considered in the above proof:

Let \( x' \) and \( y' \) be a pair of intersection points of type (RF), as in the proof above. Then the local intersection numbers of \( C' \) and \( K_n \) at \( x' \) and \( y' \) equal the local intersection numbers of \( C' \) and \( K \) at \( x \) and \( y \), respectively. These agree and therefore the intersection points contribute with the same sign to the intersection number \( C \cdot K_n \).

To see that this is the case, pick a sub-arc \( a \) of \( \mathbb{R} (Rg) \) between \( x \) and \( y \), which does not contain \( p (\tilde{p}_0, \text{see } \S 5.1.1) \). Let \( X \) and \( Y \) be positively oriented frames of \( T_xK \) and \( T_yK \), respectively.

- If the ambient space is projective space, \( a \) lies in an affine part of \( \mathbb{R}P^{4j+1} \) which does not contain \( p \). If \( u \) is a non-vanishing tangent vector field of \( a \) then the local intersection number of \( C \) and \( K \) at \( x \) is given by the orientation of the frame \( (\pm u, Y, X) \) and the local intersection number at \( y \) is given by the frame \( (\mp u, X, Y) \). Since \( K \) is odd-dimensional, these frames have the same orientation.

- If the ambient space is the sphere, we use stereographic projection \( s: \mathbb{R}Q^{2k+1} \to \mathbb{R}^{2k+1} \) from \( \tilde{p}_0 \) such that \( s(\tilde{p}_1) = 0 \). Two separate cases must be considered.

  First, assume that \( \tilde{p}_1 \notin a \). If \( u \) is a non-vanishing tangent vector field of \( s(a) \) then the orientation of the frame \( (\pm u, ds(Y), ds(X)) \) gives the local intersection number of \( C \) and \( K \) at \( x \) and the orientation of the frame \( (\mp u, ds(X), ds(Y)) \) gives the local intersection number at \( y \).

  Second, assume that \( \tilde{p}_1 \in a \). If \( u \) is a non-vanishing tangent vector field of \( s(a) \) then the orientation of the frame \( (\pm u, -ds(Y), ds(X)) \), where \( -ds(Y) \) indicates the frame which is obtained if all vectors in \( ds(Y) \) are multiplied by \( -1 \), gives the local intersection number of \( C \) and \( K \) at \( x \) and the orientation of the frame \( (\mp u, -ds(X), ds(Y)) \) gives the local intersection number at \( y \).

In any case, since \( K \) is odd-dimensional, the local intersection numbers are of the same sign.

Consider intersection points \( z \) of type (CN) and \( r \) of type (RN), as in the end of the above proof. Using the local models of Lemmas 3.3 and 3.6 (cf. also the Proof of Theorem 2.11) one calculates that the local intersection number of \( C(U) \) and \( RV_n \) at \( r \) equals \( \lambda \) if and only if the local intersection number of \( \Gamma \) and \( CV_n \) at \( z \) equals \((-1)^{j+2} + 1 \lambda = (-1)^{j+1} \lambda \). Thus, the total contribution of a pair of a point of type (CN) and a corresponding point of type (RN) to \( \mathbb{C}w \) equals 0.

6.C. Proof of Theorem 2.9. We must check how the invariant \( \mathbb{C}w(V_t) \) changes at the double point instance. We separate the cases:

(RR) \( V_0 \) has a real-real double point, where two real branches pass through each other, or

(CC) \( V_0 \) has a complex-complex-conjugate double point, where two non-singular complex conjugate branches pass through each other at a point in \( \mathbb{R}P^{4j+3} \).

We use the coordinate notation as in Lemma 5.3. There exist (analytic) local coordinates \( x + iy = z = (z_1, \ldots, z_{4j+3}) = (z', z'', z_{4j+3}) \in C^{4j+3} \) in a neighborhood \( Z \subset \mathbb{C}P^{4j+3} \) of the singular point \( s \in RV_0 \) with the following properties in cases (a) and (b) above.
The two local branches of \( RV_i \) near \( s \) are given by the equations \( x'' = 0, x_{4j+3} = t \) and \( x' = 0, x_{4j+3} = -t \), respectively.
(cc) The two local complex conjugate branches \( B_t \) and \( B^*_t \) of \( CV_t \) near \( s \) are given by the equations \( z' + iz'' = 0, z_{4j+3} = it \) and \( z' - iz'' = 0, z_{4j+3} = -it \), respectively.

Moreover, after changing the relative cycle \( \Gamma \) by homotopy we may assume that \( \Gamma \cap Z = \gamma_1 \cup \gamma_2 \), where
\[
\gamma_1 = \{ z = x + iy: y' = y'' = 0, y_{4j+3} \geq 0 \},
\gamma_2 = \{ z = x + iy: y' = y'' = 0, y_{4j+3} \leq 0 \}
\]

In case (a), choose the normal vector field \( n \) so that \( n = \partial_{2j+2} \) along the first local branch and \( n = \partial_1 \) along the second one. At \( t = 0 \), the two local branches experiences a crossing change. Since \( RV_t \) is connected it follows that \( wr(RV_t, n) = \text{lk}(RV_t, (RV_t)_n) \) changes by \( \pm 2 \). Clearly, \( sh(V_t, n) \) is constant in \( t \) and thus, \( Cw(V_t) \) changes by \( \pm 2 \) at \( t = 0 \) in case (RR).

In case (cc) we may assume that the normal vector field \( \nu \) of \( CV_t \), which is the extension of the vector field in along \( RV_t \), vanishes along \( B_t \) and \( B^*_t \). Note that, for \( t \neq 0 \), there are two complex conjugate points \( b_t \) and \( b^*_t \) in \( \Gamma \cap (CV_t)_n \cap Z = \Gamma \cap CV_t \cap Z \), \( b_t \in \gamma_1 \) and \( b^*_t \in \gamma_2 \).

The orientations of \( \gamma_1 \) and \( \gamma_2 \) are “opposite” (see the proof of Lemma 5.6) and since the dimension of \( CV_t \) is odd, complex conjugation changes orientation. Therefore, the local intersection numbers at \( b_t \) and \( b^*_t \) agree.

As \( t \to 0+ \), \( b_t \) and \( b^*_t \) come closer together. At \( t = 0 \), the two points collide at \( s \), and as \( t \) becomes negative \( b_t \) has moved to \( \gamma_2 \) and \( b^*_t \) to \( \gamma_1 \). It follows that \( 2 \text{sh}(V_t, n) = [\Gamma] \bullet ([CV_t]_n) \) changes by \( \pm 4 \) at \( t = 0 \). Clearly, \( wr(RV_t, n) \) is constant in \( t \). Thus, \( Cw \) changes by \( \pm 2 \) at \( t = 0 \) also in case (cc).

6.D. The diagrammatic definition of encomplexed writhe. We present the definition of the local writhe at a double point in a projection of a real algebraic knot, see \([3]\), Sections 2.1 and 2.2.

Let \( K \) be a variety in real projective 3-space such that \( RRK \) is a knot and let \( c \) be a generic point in \( RP^3 \). Fix some hyperplane \( H \) in \( RP^3 \), \( c \notin RH \). Let \( p_c: (RP^3 - \{c\}) \to RH \) and \( p_c: (CP^3 - \{c\}) \to CH \) denote the projections. The image of \( K \) under projection may have two kinds of double points:

- (RDP) double points \( x \in RH \) which are transverse intersection of two branches of \( p_c(RK) \), and
- (SDP) double points \( s \in RH \) which are intersections of two non-singular complex conjugate branches of \( p_c(CK) \), both meeting \( RH \) transversely in \( CH \) (such double points are called solitary).

Let \( x \) be a double point of type (RDP) and let \( l \) be the line which is the preimage of \( x \) under the projection. Denote by \( a \) and \( b \) the points of \( Rl \cap RK \). The points \( a \) and \( b \) divide \( Rl \) into two segments. Choose one of these and denote it \( S \). Choose an orientation of \( RRK \). Let \( v \) be \( w \) be nonzero tangent vectors of \( RRK \) at \( a \) and \( b \) respectively, which are directed along the selected orientation of \( RRK \). Let \( u \) be the tangent vector of \( Rl \) at \( a \) directed inside \( S \). Let \( w' \) be a vector at \( a \) which is tangent to the plane containing \( l \) and \( w \) and directed to the same side of \( S \) as \( w \) (in the affine part of the plane containing \( S \) and \( w \)). The local writhe at \( x \) is the sign of the orientation of the frame \( (v, u, w') \) in \( T_xRP^3 \). It is independent of the choices of \( a \) and \( S \), and of the choice of orientation of the knot.
Let $s \in RH$ be a double point of type (SDP) and let $l$ be the preimage of $s$ under the projection. Then $Cl$ is the preimage of $s$ under $P_c$ and $Cl \cap CK$ consists of two imaginary complex conjugate points. Call them $a$ and $b$. Since $a$ and $b$ are conjugate they belong to different components of $Cl - RH$. Let $Dl(a)$ and $Dl(b)$ denote the closures of the components of $Cl - Rh$ containing $a$ and $b$ respectively. The complex orientations of $Dl(a)$ and $Dl(b)$ induce orientations $u_a$ and $u_b$ respectively on $l$ (their common boundary). Let $CK(a)$ and $CK(b)$ denote neighborhoods in $Ck$ of $a$ and $b$, respectively. The images $P_c(CK(a))$ and $P_c(CK(b))$ intersects $RH$ transversely in $CH$ at $s$. Let $o_a$ be the local orientation of $RH$ at $s$ which gives a positive local intersection number of $RH$ and $P_c(CK(a))$ in $CH$ (with its complex orientation). Define $o_b$ similarly using $P_c(CK(b))$ instead. The signs of the orientations $(o_a, u_a)$ and $(o_b, u_b)$ of $RP^3$ at $s$ agree. This sign is the local writhes at $s$.

It turns out that the sum of local writhes of all crossing points in a generic projection is independent of the projection chosen, see [8], Sections 2.3 and 3.1. This sum is the encomplexed writhe of the knot.

**Remark 6.2.** For $V$ such that $RV$ is a many component link, the local writhes at a double point of type (SDP) is still well-defined. The local writhes at a double point of type (RDP) is well-defined only if the preimages of the double point belong to the same component of $RV$. To incorporate other double points in the definition of encomplexed writhe one must fix a semi-orientation (an orientation up to sign) on $RV$, see [8], Section 1.4. However, the local writhes then depend on the semi-orientation.

Another possibility is to neglect these other double points and define the encomplexed writhe of a real algebraic link as the sum of local writhes at double points of type (SDP) and at those double points of type (RDP) for which the preimages belong to the same component of $RV$. *This is the definition we use in the present paper.* It has the advantage of depending only on the algebraic link itself, not on chosen orientations.

**6.E. Proof of Theorem 2.11.** Fix an orientation of $RV$ and endow $RP^3$ with the metric coming from the standard metric on $S^3 \approx \mathbb{R}Q^3$. Choose a generic point $c \in RP^3$ and a hyperplane $H$, $c \notin RH$. As in § 6.D, the notions $p_c$ and $P_c$ denote the real and complex projections from $c$ to $RH$ and $CH$, respectively.

Let $K$ be a connected component of $RV$. We shall define a normal vector field $n$ along $K$. Consider $C = \bigcup_{k \in K} l(c, k)$ as in § 5.D. Let $NK$ denote the bundle of vectors in $TRP^3|K$ orthogonal to the tangent vector of $K$. Let $\nu K$ denote the bundle of vectors in $TRP^3|K$ tangent to $C$ and orthogonal to the tangent vector of $K$. Since $c$ is generic, $\nu K$ is a 1-dimensional sub-bundle of the 2-dimensional bundle $NK$. Let $\xi K$ denote the orthogonal complement of $\nu K$ in $NK$. It is easy to see that the bundles $\xi K$ and $\nu K$ are trivial if and only if $K$ is contractible in $RP^3$.

We make separate choices of $n$ for contractible and non-contractible components $K$ of $RV$.

- If $K$ is contractible then let $n$ be a non-zero section in $\xi K$.
- If $K$ is not contractible then fix a small sub-arc $J$ of $K$ between $k_1$ and $k_2$, say, and an orientation preserving diffeomorphism $[0, \pi] \to J$, $0 \mapsto k_1$. The restriction of $n$ to $K - J$ is defined to be the (unique up to sign) non-zero section of $\xi K|(K - J)$. Choose a trivialization $NK|J \approx [0, \pi] \times \mathbb{R}^2$ such that if $(e_1, e_2)$ is the standard basis of $\mathbb{R}^2$ then $Re_1$ corresponds to $\xi K$ and...
\( \mathbb{R}^2 \) to \( \nu K \). Then \( n(k_1) = n(0) = e_1 \) and \( n(k_2) = n(\pi) = -e_1 \). Define \( n(\theta) = \cos(\theta)e_1 + \sin(\theta)e_2 \), \( 0 \leq \theta \leq \pi \).

We use \( C \) to compute the linking number \( \text{lk}(K, K_n) \). We first prove the theorem under the assumption that all components \( K \) of \( V \) are contractible:

Let \( W \) be a sufficiently small neighborhood of \( \mathbb{R}V \) in \( \mathbb{C}P^3 \) and let \( U = \mathbb{R}P^3 \cap W \), as in Remark 5.7 (we use notions \( C(U), C'(U), \Gamma(W) \), and \( \Gamma'(W) \), as there). First, we relate \( \text{wr}(\mathbb{R}V, n) \) to the sum of local writhes at double points \( x \) of \( p_c(\mathbb{R}V) \) (the double points labeled (rdp) in § 6.D). Since the vector field \( n \) is everywhere orthogonal to \( TC, C(U) \cap K_n = \emptyset \).

To each double point \( x \) of type (rdp) there corresponds exactly two points in \( C'(U) \cap K \), which are the preimages of \( x \) under \( p_c \). Denote these two points \( a \) and \( b \), respectively. We use the notions \( v, w, w', S \), and \( u \) as in § 6.D. Extend \( u \) to a nonzero tangent vector field of \( S \). Then, according to our orientation convention and since \( \partial C = 2K \), \( (u, w') \) is a positively oriented basis of \( TC'(U) \) at \( a \) and the local intersection number equals the sign of the orientation \( (v, u, w') \), which is just the local writh. Interchanging \( a \) and \( b \) in the above argument, we see that the local intersection number of \( C'(U) \) and \( K \) at \( b \) also equals the local writh. Hence, the sum of local writhes at double points of type (rdp) the preimages of which lies on \( K \), equals \( \text{lk}(K, K_n) \). It follows that the sum of local writhes at double points of type (rdp), the preimages of which lies on the same component, equals \( \text{wr}(\mathbb{R}V, n) \).

Second, we relate \( sh(V, n) \) to the sum of local writhes at double points of type (sdp). Use \( c \) to construct the shade \( \Gamma = \Gamma_c \). Since \( n \) is orthogonal to \( TC \), it follows that \( in \) is transverse to \( TT_c \) along \( \mathbb{R}V \). Hence, \( \Gamma(W) \cap CV_n = \emptyset \). Then

\[
\Gamma \bullet CV_n = \Gamma'(W) \bullet CV_n = \Gamma'(W) \bullet CV.
\]

To each double point \( s \in \mathbb{R}H \) of \( P_c(CV) \) (of type (sdp)) there corresponds exactly two complex conjugate points in \( \Gamma'(W) \cap CV \), which are the preimages of \( s \) under \( p_c \). Denote these two points \( a \) and \( b \), respectively. Note that \( a \) and \( b \) lie in \( CI \subset \Gamma \) where \( l \) is the line through \( s \) and \( c \).

Consider the local intersection number of \( CV \) and \( \Gamma'(W) \) at \( a \). We use the notions \( Dl(a), Dl(b), CV(a) \), and \( CV(b) \) as in § 6.D. Let \( \alpha \subset Dl(a) \) be an arc connecting \( a \) to \( s \). Push \( CV(a) \) along \( \alpha \) to \( s \), keeping it transverse to \( \Gamma \), and so that at the end of the push \( CV(a) \) agrees (locally around \( s \)) with \( P_c(CV(a)) \). From this construction it follows that the local intersection number of \( \Gamma'(W) \) and \( CV(a) \) at \( a \) equals the local intersection number of \( \Gamma \), with orientation coming from the part of \( \Gamma \) containing \( Dl(a) \), and \( P_c(CV(a)) \) at \( s \).

The orientation of \( T_s \Gamma \) coming from the part of \( \Gamma \) containing \( Dl(a) \) (see § 6.B and 6.C) is given by the orientation of the frame \( (v, iv, u, w) \), where \( v \) is a real tangent vector along \( RL \) pointing in the direction of the orientation of \( l \) induced from \( Dl(a) \), and \( (u, w) \) is a frame in \( TH \) such that \( (u, w, v) \) is a positive basis of \( T_s \mathbb{R}P^3 \).

Let \( (f, if) \) be a complex frame of \( T_s P_c(CV(a)) \subset T_s \mathbb{C}H \) (neither \( f \) nor \( if \) lies in \( T_s \mathbb{R}H \)). Then the local intersection number of \( \Gamma \) and \( P_c(CV(a)) \) is given by the sign of the orientation of the frame

\[
(v, iv, u, w, f, if),
\]

which is positive if and only if the frame \( (u, w, f, if) \) gives the complex orientation of \( \mathbb{C}H \).
On the other hand, the local writhe of $s$ is the sign of the orientation of the frame $(v, u', w')$ of $T_z\mathbb{R}P^k$, where $(u', w')$ is a frame of $T_z\mathbb{C}H$ such that $(u', w', f, i)$ gives the complex orientation of $\mathbb{C}H$. So, the local writhe is positive if and only if $(u, w, f, i)$ gives the complex orientation of $\mathbb{C}H$.

It follows that the sum of local writhes at double points of type (SDP) equals $\text{sh}(V, n)$.

The above together with Theorem 2.8 proves the theorem for real algebraic links without non-contractible components.

Now assume that $\mathbb{R}V$ has non-contractible components. The only difference from the case considered above is that, for each non-contractible component $K$, there will be one intersection point of $C(U)$ and $K_n$ (a point of type (RN)) and a corresponding intersection point of $\Gamma(W)$ and $\mathbb{C}V_n$ (of type (CN)). These intersection points appear near the sub-arc $J$ of $K$. We compare their signs (cf. the end of Remark 6.1).

We use local coordinates in a neighborhood of $J \subset K$ as in the proof of Lemma 5.6 (the case $k = 1$): $K$ is given by the equation $x_2 = x_3 = 0$, $C \simeq A = a_1 \cup a_2$, and $\Gamma \simeq \gamma_1 \cup \gamma_2$.

We may assume that $0 \leq x_1 \leq \pi$ corresponds to $J$, that the orientation of $K$ is the one from 0 to $\pi$, and that $n(0) = \partial_2$. The intersection point of $A$ and $K_n$ (type (RN)) is the point $(\vec{\tau}, 0, \delta)$ and the sign of the intersection is the sign of the frame $(\partial_1, \partial_3, -\partial_2)$, which is $+1$. The intersection point of $\Gamma$ and $\mathbb{C}V_n$ (type (CN)) is $(\vec{\tau}, 0, i\delta)$ and the sign of the intersection is the sign of the frame

$$(\partial_1, \partial_2, \partial_3, i\partial_3, -i\partial_2, i\partial_1),$$

which is $-1$. Hence, the contributions to $\mathbb{C}w$ from the two extra intersection points for each non-contractible component cancel and the theorem follows in the general case.

### 7. Generalizations

In this section, it is shown that any $k$-dimensional subvariety of $(2k + 1)$-dimensional complex projective space or of the complexification of the real $(2k + 1)$-sphere can be equipped with additional structure which allows for shade numbers to be defined. Throughout this section, we call the complexification of the real $(2k + 1)$-sphere the complex $(2k + 1)$-sphere and we let $\mathbb{R}Y^{2k+1}$ denote $\mathbb{R}P^{2k+1}$ or $\mathbb{R}Q^{2k+1}$ and $\mathbb{C}Y^{2k+1}$ be the associated complex manifold.

**7.A. Additional structure.** Let $W$ be a projective $k$-dimensional variety in complex projective $(2k + 1)$-space or in the complex $(2k + 1)$-sphere. A vector field $n$ in $\mathbb{R}Y^{2k+1}$ along $\mathbb{R}W$ (i.e. a section of the bundle $T\mathbb{R}Y^{2k+1}$ restricted to $\mathbb{R}W$) is *admissible* if it has the following properties.

- $n$ is the restriction of a smooth vector field defined in some neighborhood of $\mathbb{R}W$ in $\mathbb{R}Y^{2k+1}$.
- For all smooth extensions $\nu$ of $nW$ supported in sufficiently small neighborhoods of $\mathbb{R}W$ in $\mathbb{C}Y^{2k+1}$, the flow $\Phi_t$ of $\nu$ satisfies $\Phi_t(CW) \cap \mathbb{R}Y^{2k+1} = \emptyset$, for all sufficiently small $t > 0$.

If $(W, n)$ is a variety with an admissible vector field $n$ then its shade number can be defined as follows.

$$\text{sh}(W, n) = \frac{1}{2}(\{C\} \bullet [CW_n]) \in \frac{1}{2}\mathbb{Z}, \quad (7.1)$$
where $[CW_n]$ denotes the homology class in $H_{2k}(CY^{2k+1} \ominus RW^{2k+1})$ of the cycle $CW_n$ obtained by shifting $CW$ slightly by the flow of $\nu$. The shade number is independent of the extension $\nu$ of $in$ as long as its support is sufficiently small.

A general position argument shows that any $k$-dimensional variety has an admissible vector field: $CW$ is a stratified set of real dimension $2k$ and standard transversality arguments applied to a tubular neighborhood of $RY^{2k+1}$ in $CY^{2k+1}$ show that there exists a smooth purely imaginary vector field $v$ along $RY^{2k+1}$ in $CY^{2k+1}$ supported in an arbitrarily small neighborhood of $RW$, such that if $\Phi_t$ is any 1-parameter deformation of $RY^{2k+1}$ with $\frac{d}{dt}\Phi_t|_{t=0} = v$ then $\Phi_t(RY^{2k+1}) \cap CW = \emptyset$, for all sufficiently small $t > 0$. Let $n(x) = iv(x)$ for $x \in RW$. Then $in = -v$ and $n$ is admissible.

The cases considered earlier in this paper, varieties with empty real set ($\S$ 2.A) and armed real varieties without real singularities ($\S$ 2.B) are special cases of the definition given in Equation (7.1). (In fact, they are the generic cases for complex respectively real varieties.) In particular, the case of armed real varieties shows that admissible vector fields are not unique and that different choices may give different shade numbers.

We now turn our attention to some particular non-generic cases which are similar to the case of real varieties studied earlier in that admissible vector fields are easy to find and their non-uniqueness can be controlled.

7.B. Varieties with manifolds as real sets. Let $W$ be a projective $k$-dimensional variety in complex projective $(2k+1)$-space or in the complex $(2k+1)$-sphere. Assume that $RW$ is submanifold of $RY^{2k+1}$ and that the points in $RW$ are smooth points of $W$. (The connected components of $RW$ may have different dimensions, the maximal possible dimension of a component is $k$.) Assume also that if $x \in RW$ is any point in an $r$-dimensional component $K$ of $RW$ then $T_x CW \cap T_x RW^{2k+1} = T_x K$. (In other words, $CW$ and $RY^{2k+1}$ intersect cleanly along the manifold $RW$).

Under these conditions, an admissible vector field along $RW$ can be constructed as follows.

Let $K$ be an $r$-dimensional component of $RW$. The normal bundle $NK$ of $K$ in $RY^{2k+1}$ is $(2k+1 - r)$-dimensional. It has a natural $(r + 1)$-dimensional subbundle $N'K$: if $x \in K$ then $N_x'K = T_x CY^{2k+1} / \text{pr}(T_x CW)$, where $\text{pr}: T_x CY^{2k+1} \to T_x RW^{2k+1}$ is the projection. For dimensional reasons there exists non-zero sections of $N'K$. If $n$ is a vector field along $RW$ which projects to a non-zero section of $N'K$ for each component $K$ of $RW$ then $n$ is admissible.

Conversely, any admissible vector field gives rise to such sections. Moreover, two admissible vector fields which give rise to homotopic sections in $N'K$ for every $K$ clearly give the same shade number. It is straightforward to describe all possible homotopy classes of non-zero sections in these bundles.

If $K = \{x\}$ is a 0-dimensional component of $RW$, then the bundle $N'K$ is 1-dimensional. The tangent space $T_x RW^{2k+1}$ has a preferred orientation from a fixed orientation of $RY^{2k+1}$ and the image $\text{pr}(T_x CW)$ has an orientation induced by the complex orientation of $T_x CW$. This allows us to define $n(x)$ in a canonical way: let $n(x)$ be the positive normal of $\text{pr}(T_x W)$ in $T_x RW^{2k+1}$. It is easy to check that each 0-dimensional component of $RW$ endowed with the canonical vector field contributes $(-1)^{k+1}$ to $\text{sh}(W, n)$. This observation leads to the following analogue of Theorem 2.2 for varieties $W$ in $CP^{2k+1}$ with $RW$ 0-dimensional:
Theorem 7.1. For varieties $W$ as above of degree $d$ and with $\mathbb{R}W = \{p_1, \ldots, p_m\}$ (note that $m \leq d^2$), the range of the shade number consists of all half-integers between $(-1)^{k}\frac{1}{2}(m - d^2)$ and $(-1)^{k}\frac{3}{2}d^2$ congruent to $\frac{1}{2}d$ modulo 1.

7.C. Real varieties with singularities. Let $V$ be a real projective $k$-dimensional variety in real projective $(2k+1)$-space or in the real $(2k+1)$-sphere and let $n$ be an admissible vector field along $\mathbb{R}V$. When $\operatorname{dim}(V)$ was odd, $V$ was without real singularities, and $\mathbb{R}V$ orientable, we compared the shade number to the wrapping number. One may study related issues in more general situations there are however differences between the singular and the non-singular cases:

The shade number is defined once an admissible vector field has been picked. To have a counterpart of the wrapping number defined, there are two obvious requirements which must be met: that all small shifts of $\mathbb{R}V$ along $n$ shifts $\mathbb{R}V$ off itself, and that $\mathbb{R}V$ is an orientable $k$-cycle in $\mathbb{R}P^{2k+1}$. Even with these two conditions met, it is not clear how to define the counterpart of the wrapping number of the $k$-cycle. One reason is that an orientable connected $k$-cycle, in contrast to an orientable connected manifold, may have more than two orientations. To get a reasonable counterpart of the wrapping number one must fix a semi-orientation (an orientation up to sign) on the $k$-cycle $\mathbb{R}V$.

We next study properties of shade numbers for real varieties with the simplest singularities: varieties with double points as that of $V_0$ in Theorem 2.9 (this refers to the double points, $\mathbb{R}V$ need not have global properties as there, i.e. it need neither be connected nor orientable).

In this case, admissible vector fields are easy to find: pick any normal vector field along $\mathbb{R}V$ which is transverse to the (real) $2k$-dimensional intersection of the Zariski tangent space of $V$ (i.e. of $V$ after base extension) and $T_x\mathbb{R}Y^{2k+1}$ at the double points $x$.

In the odd-dimensional case ($\operatorname{dim}(V)$ is odd), the shade number of a variety with one double point is the mean value of the shade numbers of its (two) resolutions, with the induced admissible vector field, (if the double point is real-real then the shade numbers of the resolutions are the same, if it is complex-complex-conjugate then the shade numbers of the resolutions differ by $\pm 2$).

In the even-dimensional case, the shade number of a variety with one complex-complex-conjugate double point differs from the common value of the shade numbers of its resolutions by $\pm 1$, depending on the direction of the shift at the double point, and the shade number of a variety with a real-real double point agree with the common value of the shade numbers of its resolutions.

8. Examples

In this section, several examples are presented.

8.A. Two families of real algebraic knots. Let $[x_0, x_1, x_3, x_4]$ be projective coordinates on real projective 3-space and let $a \in \mathbb{R}$. Let $\epsilon = \pm 1$ and let $K_a(\epsilon)$ be the variety defined by the equations

$$ax_0x_2 - x_1x_3 = 0,$$

$$a^2x_1x_0 - \epsilon a^2x_0^2 - x_3^2 = 0,$$

$$x_3x_2^2 + \epsilon x_1^2x_3 - ax_1^2x_2 = 0,$$

$$x_2^2x_0 - x_1^3 + \epsilon x_1^2x_0 = 0.$$
It is easy to check that for \( a \neq 0 \), \( K_a(\epsilon) \) are smooth curves. The curve \( K_0(\epsilon) \) has one double point. For \( \epsilon = -1 \) it is a real-real double point and for \( \epsilon = +1 \) it is a complex-complex-conjugate double point. In fact, \( K_a(\epsilon) \) is the rational curve given, in projective coordinates, by

\[
[s, t] \mapsto [s^3, st^2 + \epsilon s^3, t^3 + \epsilon s^2 t, ats^2].
\]

It follows from Theorem 2.8 (ii) that \( \mathbb{C}w(K_a(\epsilon)) \) changes by \( \pm 2 \) at \( a = 0 \). Hence, \( \mathbb{C}w(K_a(\epsilon)) \neq \mathbb{C}w(K_{-a}(\epsilon)) \), \( a \neq 0 \) and thus \( K_a(\epsilon) \) is not weak rigid isotopic to \( K_{-a}(\epsilon) \). However, it is clear that \( \mathbb{R}K_a(\epsilon) \) and \( \mathbb{R}K_{-a}(\epsilon) \) are topologically isotopic. Also the projective isomorphism \([x_0, x_1, x_2, x_3] \mapsto [x_0, x_1, -x_2, x_3] \) takes \( K_a(\epsilon) \) to \( K_{-a}(\epsilon) \).

8.B. An armed real projective plane. Let \([x_0, x_1, x_2] \) and \([y_0, \ldots, y_5] \) be projective coordinates on real projective 2- and 5-space, respectively. Consider the map \( \phi_0 : \mathbb{R}P^2 \to \mathbb{R}P^5 \)

\[
\phi_0([x_0, x_1, x_2]) = [x_0, x_1, x_2, 0, 0, 0] \]

Then \( \phi_0 \) gives a parameterization of the variety \( V \) defined by the equations \( y_3 = y_4 = y_5 = 0 \). Let \( \phi_t \) be the 1-parameter variation of \( \phi_0 \) given by

\[
\phi_t([x_0, x_1, x_2]) = [x_0, x_1, x_2, tx_0, tx_1, tx_2],
\]

and let \( n = \frac{4}{\delta}t \). Then \( n \) is a normal vector field along \( \mathbb{R}V^2 \) in \( \mathbb{R}P^5 \).

Claim. If the armed variety \((V, n)\) is as above then \( \text{sh}(V, n) = -\frac{1}{2} \).

Proof. Consider \( \Gamma_p \), where \( p = [0, 0, 0, 0, 0, 1] \). It is straightforward to check that \( \mathbb{C}V_n \cap \Gamma_p = \{0, 0, 1, 0, 0, i\delta\} \), where \( \delta > 0 \) is small. In coordinates

\[
(x_1 + iy_1, \ldots, x_5 + iy_5) \mapsto [1, x_1 + iy_1, \ldots, x_5 + iy_5],
\]

the sign of the intersection point equals the sign of the orientation of the frame

\[
(\partial_1, \ldots, \partial_5, i\partial_5, \partial_1 + i\partial_3, i\partial_1, \partial_2 + i\partial_4, i\partial_2),
\]

which is \(-1\).

8.C. An unknot. Let \([x_0, \ldots, x_4] \) be homogeneous coordinates on real projective 4-space. Let \( Q^3 \) be as in § 5.D. If \( O \) is the intersection of \( Q^3 \) and the 2-plane given by \( x_3 = x_4 = 0 \) then \( \mathbb{R}O \subset \mathbb{R}Q^3 \) is a representative of the unknot.

Claim. If \( O \) is as described above then \( \mathbb{C}w(O) = 0 \).

Proof. Let \([s, t] \) be projective coordinates on the real projective line. The variety \( O \) admits the rational parameterization

\[
[s, t] \mapsto [s^2 + t^2, 2st, s^2 - t^2, 0, 0].
\]

Let \( p = [0, 0, 0, 0, 1] \in \mathbb{R}P^3 \) (as in § 5.D we think of \( \mathbb{R}P^3 \subset \mathbb{R}P^4 \) given by the equation \( x_0 = 0 \)). Consider \( \Gamma = \Gamma_p \) and \( C \), where \( C \subset \mathbb{R}Q^3 \), \( \partial C = 2O \), is the chain constructed using \( p = [1, 0, 0, 0, 1] \), see § 5.D.

Let \( W \) be a small neighborhood of \( \mathbb{R}O \) in \( \mathbb{C}Q^3 \) and let \( U = W \cap \mathbb{R}Q^3 \), as in Remark 5.3 (and using the same notation as there), \( \mathbb{C}w(O) \) equals half of the sum of the intersection numbers \( \Gamma'(W) \bullet \mathbb{C}O \) and \( C'(U) \bullet O \) (see the proof of Theorem 2.8 and Remark 5.1).

It is straightforward to check that \( \mathbb{R}O \cap C'(U) = \emptyset \). A point

\[
[x_0, x_1, x_2, 0, 0] = [s^2 + t^2, 2st, s^2 - t^2, 0, 0] \in \mathbb{C}O
\]
(here we think of \([s, t]\) as homogeneous coordinates on \(\mathbb{C}P^1\)) lies in \(\Gamma'(W)\) if and only if it is not a real point and all the products \(x_i x_j^*, \ i \neq j\) are real numbers \((x^*\) denotes the complex conjugate of \(x\)). Since \([0, 1] \subset \mathbb{C}P^1\) maps to a real point we can restrict attention to the coordinate chart \(t \in \mathbb{C} \mapsto [1, t] \in \mathbb{C}P^1\). Then \(x_1 x_3^* \in \mathbb{R}\) only if \(\rho^2 t = 1 - t^2\) for some real number \(\rho\). However, this equation has only real solutions for every \(\rho \in \mathbb{R}\). Hence also \(\Gamma'(W) \cap \mathbb{C}O = \emptyset\), and we conclude \(\mathbb{C}w(O) = 0\).

\[\square\]

8.D. A trefoil knot. Let \([x_0, \ldots, x_4]\) be coordinates on real projective 4-space and let \((z, w)\) be coordinates on \(\mathbb{C}^2\). The trefoil knot is the link of the singularity \(z^2 = w^3\). We think of \(\mathbb{C}^2 \approx \mathbb{R}^4\) as the affine part of \(\mathbb{R}P^4\) given by the condition \(x_0 \neq 0\). For simpler formulas below we violate our conventions slightly and represent \(\mathbb{R}Q^3\) as the quadric \(-2x_0^2 + x_1^2 + \cdots + x_4^2 = 0\). The intersection of the subset \(z^2 = w^3\) of \(\mathbb{R}^4 \subset \mathbb{R}P^4\) and \(Q^3\) is a variety \(K\) such that \(\mathbb{R}K \subset \mathbb{R}Q^3\) is a representative of the trefoil knot.

**Claim.** If \(K\) is as above then \(\mathbb{C}w(K) = 4\).

**Proof.** The knot \(\mathbb{R}K\) can be parameterized by \(\theta \in [0, 2\pi] \subset \mathbb{R}\) as

\[
\theta \mapsto [1, \cos 3\theta, \sin 3\theta, \cos 2\theta, \sin 2\theta].
\]

Using the rational parameterization \([s, t] \mapsto \left(\frac{2st}{1+2t^2}, \frac{s^2-t^2}{1+2t^2}\right)\) of \(\mathbb{S}^1\) we get the rational parameterization \(\phi([s, t]) = [x_0([s, t])], \ldots, x_4([s, t])]\) of \(K\), where

\[
x_0([s, t]) = (s^2 + t^2)^3,
\]

\[
x_1([s, t]) = -2st \left(3(s^2 + t^2)^2 - 16s^2 t^2\right),
\]

\[
x_2([s, t]) = -(s^2 - t^2) \left((s^2 + t^2)^2 - 16s^2 t^2\right),
\]

\[
x_3([s, t]) = -(s^2 + t^2) \left((s^2 + t^2)^2 - 8s^2 t^2\right),
\]

\[
x_4([s, t]) = 4st(s^2 + t^2)(s^2 - t^2).
\]

(\text{It is easy to check that the same formulas over complex numbers give a holomorphic immersion of } \phi: \mathbb{C}P^1 \rightarrow \mathbb{C}Q^3 \text{ and thus } \phi \text{ parameterizes } K.\)

We use the points \(\tilde{p}_0 = [1, 0, \sqrt{2}, 0, 0]\) and \(\tilde{p}_1 = [1, 0, -\sqrt{2}, 0, 0]\) to construct the shade \(\Gamma \subset \mathbb{C}Q^3\), and the chain \(C \subset \mathbb{R}Q^3\), \(\partial C = 2K\), see §8.D. The sphere \(\mathbb{R}Q^3\) has the standard orientation coming from \(\mathbb{C}^2 \subset \mathbb{R}P^4\). As in §8.C, we must calculate \(C'(U) \bullet \mathbb{R}K\) and \(\Gamma'(W) \bullet \mathbb{C}K\), where \(W\) is a small neighborhood of \(\mathbb{R}K\) in \(\mathbb{C}Q^3\) and \(U = \mathbb{R}Q^3 \cap W\).

We start with \(C'(U) \bullet \mathbb{R}K\). Using the parameterization of \(\mathbb{R}K\) by \(\theta \in [0, 2\pi] \subset \mathbb{R}\) as given above, there are nine point pairs in \(C'(U) \cap \mathbb{R}K\). These nine pairs \((\alpha, \beta)\) are of two types:

\[
(\alpha, \beta) = \begin{cases} 
\text{type 1: } (\frac{\alpha}{4} + n\frac{\pi}{3}, \frac{3\alpha}{4} + n\frac{\pi}{3}) & n = 0, 1, 2, 3, 4, 5, \\
\text{type 2: } (\frac{\alpha}{6} + n\frac{\pi}{4}, \frac{3\alpha}{6} + n\frac{\pi}{4}) & n = 0, 1, 2.
\end{cases}
\]

Each point pair contributes \(\pm 2\) to \(C'(U) \bullet \mathbb{R}K\). Using stereographic projection \(s: \mathbb{R}Q^3 - \{\tilde{p}_0\} \rightarrow \mathbb{R}^3, s(\tilde{p}_1) = 0 \in \mathbb{R}^3\) as in §6.B, one finds that a point pair of type 1 contributes \((-1)^n 2\) to \(C'(U) \bullet \mathbb{R}K\) and that the point pair of type 2 contributes \(+2\). Hence, \(\frac{1}{2} \text{degree}(C'(U) \bullet \mathbb{R}K) = 3\).

We proceed to calculate \(\Gamma'(W) \bullet \mathbb{C}K\). Note that \(\phi([s, t])\) lies in \(\Gamma\) if and only if \(x_i([s, t]) x_j^*([s, t]) \in \mathbb{R}, i, j \in \{1, 3, 4\}\) \((x^*\) denotes the complex conjugate of \(x\)). We
are looking for points in $\Gamma'(W) \cap \mathbb{C}K$, and so we are interested in non-real points $[s, t] \in \mathbb{C}P^1$ for which this condition holds. Since $[1, 0]$ and $[0, 1]$ are real points in $\mathbb{C}P^1$ we may work in the coordinate chart $t \in \mathbb{C} \to [1, t] \in \mathbb{C}P^1 - \{[1, 0], [0, 1]\}$. The condition $x_1([1, t])x_1^*(([1, t]) \in \mathbb{R}$ is then equivalent to

$$(3(1 + t^2)^2 - 16t^2)(1 - (t^*)^4) \in \mathbb{R},$$

which holds if and only if $t = \pm 1, \pm i$ or

$$(3(1 + t^2)^2 - 16t^2) = \rho(1 - t^4), \text{ for } \rho \in \mathbb{R}.$$ 

This equation has solutions $t^2 = \frac{5\pm\sqrt{16+\rho^2}}{3+\rho}$ if $\rho \neq -3$ and $t = \pm \frac{3}{\sqrt{10}}$ if $\rho = -3$. We conclude that $x_1([1, t])x_1^*(([1, t]) \in \mathbb{R}$ only if $t \in \mathbb{R}$ or $t \in i\mathbb{R}$. The latter case is the one of interest. For $t = ia$, $a \in \mathbb{R} - \{0\}$ the condition $x_4([1, t])x_4^*(([1, t]) \in \mathbb{R}$ reads

$$4ia(1 + a^2)(1 - a^2)^2((1 - a^2)^2 + 8a^2) \in \mathbb{R}.$$ 

This condition is met for $a = \pm 1, \pm i$ or $a^2 = -3 \pm \sqrt{3} \mathbb{R}$ and hence the only points in $\mathbb{R} - \{0\}$ which meet the condition is $a = \pm 1$. We conclude that the only points in $\Gamma'(W) \cap \mathbb{C}K$ are $\phi([1, \pm i]) = [0, \pm i, 1, 0, 0]$. We must calculate the intersection number. Around $[0, -i, 1, 0, 0]$, the shade $\Gamma$ can be parameterized by $(z, h, k) \in U \times I \times I$, where $U \subset \mathbb{C}$ is a small disk around $i$ and $I \subset \mathbb{R}$ denotes a small open interval around 0, as follows

$$(z, h, k) \mapsto \left[ \frac{z^2 + 1}{\sqrt{2(z^2 - 1)}}, \frac{2z\sqrt{1 - h^2 - k^2}}{z^2 - 1}, 1, \frac{2hz}{z^2 - 1}, \frac{2kz}{z^2 - 1} \right],$$

and the orientation of $\Gamma$ is the one induced by the complex orientation of $U$ followed by the orientation of the frame $(\partial_h, \partial_k)$ on $I \times I$. A straightforward calculation shows that the intersection number of $\Gamma'(W)$ and $\mathbb{C}K$ at $[0, 1, -i, 0, 0]$ is $+1$. By the proof of Theorem 2.8 the intersection number at $[0, 1, i, 0, 0]$ is the same. Thus $\frac{1}{2}(\Gamma'(W) \bullet \mathbb{C}K) = 1$. Adding the terms, $\mathbb{C}w(K) = 3 + 1 = 4$. 

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