ON THE SPECTRAL THEORY OF POSITIVE OPERATORS
AND PDE APPLICATIONS

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Dedicated to the 70th birthday of Professor Wei-Ming Ni

Abstract. The strong version of the Krein-Rutman theorem requires that the positive cone of the Banach space has nonempty interior and the compact map is strongly positive, mapping nonzero points in the cone into its interior. In this paper, we first generalize this version of the Krein-Rutman theorem to the case of “semi-strongly positive” operators; and we prove it in a totally elementary fashion. We then prove the equivalence of semi-strong positivity and irreducibility in a Banach lattice, linking the afore-mentioned result with the Krein-Rutman theorem for irreducible operators. One of the things we emphasize is to use “upper and lower spectral radii” to characterize, in the fashion of Collatz-Wielandt formula for nonnegative irreducible matrices, the principal eigenvalue of these operators. For reducible operators, we prove that the lower spectral radius always serves as the least upper bound of the set of eigenvalues pertaining to positive eigenvectors, and the upper spectral radius the greatest lower bound of the set. Finally, we demonstrate the full power of these Krein-Rutman theorems on some PDE examples such as elliptic eigenvalue problems on non-smooth domains, and cooperative systems which may or may not be fully coupled, by using as few PDE tools as possible.

1. Introduction. The Perron–Frobenius theorem states that if a real \( n \times n \) matrix \( T \) has nonnegative entries and is irreducible, then its spectral radius \( r(T) \) is positive and is an algebraically simple eigenvalue with a corresponding eigenvector whose components are positive. Collatz [11] and Wielandt [39] characterized this
eigenvalue, which is called the principal eigenvalue of $T$, in the following variational fashions:

$$\inf_{x \geq 0} \sup_{1 \leq i \leq n} \frac{(Tx)_i}{x_i} = r(T) = \sup_{x \geq 0} \inf_{1 \leq i \leq n} \frac{(Tx)_i}{x_i},$$  \hspace{1cm} (1)

where $x = (x_1, \cdots, x_n) \gg 0$ means every component $x_i$ is positive. The usefulness of these characterizations is obvious: one can choose test vectors to get upper and lower bounds of the principal eigenvalue. Inspired by Collatz-Wielandt formula (1), the notion of upper and lower spectral radii (which are the generalizations of the min-max and the max-min characterizations in (1)) was introduced in the study of Krein-Rutmann theory for linear positive operators by Marek [24] and for increasing, positively 1-homogeneous nonlinear mappings by Chang [9] and [10]; and for these mappings in the last two papers, the notion of semi-strong positivity, which is equivalent to irreducibility in the case of matrices, was introduced to generalize the notion of strong positivity. In this paper, we shall give an elementary proof of the strong version of the Krein-Rutmann Theorem for semi-strongly positive operators (including strongly positive operators), and study the relationship between the semi-strong positivity and the ideal-irreducibility in a Banach lattice, as well as the upper and lower spectral radii for reducible linear positive operators. To show the full strength of these abstract results, we apply them on some PDE examples.

Let $X$ be a real Banach space and let $P$ be a cone in $X$ (i.e., closed and convex subset of $X$ satisfying $xP \subset P$, $\forall \lambda \geq 0$, and $P \cap (-P) = \{0\}$). The cone induces an ordering “$\leq$” in $X$, i.e.,

$$\forall x, y \in X, x \leq y \iff y - x \in P.$$  

Let $\hat{P} = P \setminus \{0\}$, we denote $x < y$ if $y - x \in \hat{P}$, and $x \gg 0$ if $x \in \text{int}(P)$ (the interior of $P$, if it exists).

In this paper, the cone $P$ is always assumed to be total: $X = P - P$.

Let $P^*$ be the dual cone of $P$, i.e., $P^* = \{x^* \in X^* \mid \langle x^*, x \rangle \geq 0, \forall x \in P\}$. Let $\hat{P}^* = P^* \setminus \{0\}$. $\forall x \in \hat{P}$, let $P^*(x) = \{x^* \in P^* \mid \langle x^*, x \rangle > 0\}$. It is well-known that $P^*(x)$ is non-empty and even dense in $P^*$.

A linear operator $T \in L(X)$ is called positive (strongly positive), if $0 \leq x \Rightarrow 0 \leq Tx$ ($0 < x \Rightarrow 0 < Tx$).

A pair $(\lambda, x) \in [0, \infty) \times \hat{P}^*$ is called a positive eigen-pair if

$$Tx = \lambda x.$$  

In their paper [20], Krein and Rutman gave two versions, of what we now call “Krein-Rutman Theorem”, with the “weak” one reading as follows (see also [12][40]).

**Theorem 1.1** (Linear Krein-Rutman (the weak version)). Let $X$ be a Banach space, $P \subset X$ be a total cone and $T \in L(X)$ be compact, positive with its spectral radius $r(T) > 0$. Then $r(T)$ is an eigenvalue of $T$ pertaining to a positive eigenvector; it is also an eigenvalue of $T^*$ with an eigenvector in $P^*$.

In the strong version of the Krein Rutmann theorem, the cone $P$ is assumed to have nonempty interior, and $T$ is assumed to be strongly positive (see also [12][40]).

**Theorem 1.2** (Linear Krein-Rutman (the strong version)). Let $X$ and $P$ be given as in the weak version, with $\text{int}(P) \neq \emptyset$. Suppose $T \in L(X)$ is compact and strongly positive. Then

1. $r(T) > 0$ is a geometrically simple eigenvalue of $T$ pertaining to an eigenvector $x_0 \in \text{int}(P)$.
2. \( r(T) \) is algebraically simple.
3. \( \forall \) eigenvalue \( \lambda \neq r(T) \), we have \( |\lambda| < r(T) \).
4. Any real eigenvalue of \( T \) with positive eigenvector must be \( r(T) \).
5. \( r(T) \) is an algebraically simple eigenvalue of \( T^* \), with an eigenvector \( x_0^* \) which is a strictly positive functional: \( \langle x_0^*, x \rangle > 0, \forall x > 0 \); moreover, any eigenvector of \( T^* \) in \( \hat{P}^* \) corresponds to \( r(T) \).

In [9, 10], again by assuming \( \text{int}(P) \neq \emptyset \), the strong positivity was relaxed to semi-strong positivity (s.s.p.), i.e., \( \forall x \in \hat{P} \setminus \text{int}(P), \exists x^* \in P^* \) such that 

\[ \langle x^*, Tx \rangle > 0 = \langle x^*, x \rangle. \]

The notion of s.s.p. was introduced in [9], where it was proved that in the case of matrices, s.s.p. is equivalent to irreducibility. In there, two numbers \( r^*(T) \) and \( r_*^*(T) \) were defined for a positive operators \( T \). They are intended as a replacement of the spectral radius \( r(T) \) for a linear operator \( T \).

**Definition 1.3.** \( \forall x \in \hat{P} \), we define

\[ \mu_*(x) = \inf_{x^* \in P^*(x)} \frac{\langle x^*, Tx \rangle}{\langle x^*, x \rangle}, \]

\[ \mu^*(x) = \sup_{x^* \in P^*(x)} \frac{\langle x^*, Tx \rangle}{\langle x^*, x \rangle}, \]

and 

\[ r_*(T) = \sup_{x \in P} \mu_*(x), \quad r^*(T) = \inf_{x \in P^*} \mu^*(x). \]

\( r_*(T) \) and \( r^*(T) \) are called lower and upper spectral radii of \( T \), respectively.

It can be easily proved ([10]) that for positive operators, linear or nonlinear, that for \( x > 0 \),

\[ \mu_*(x) = \sup\{ \mu \geq 0 | \mu x \leq Tx \}, \]

\[ \mu^*(x) = \inf\{ \mu \geq 0 | \mu x \geq Tx \}. \]

Theorem 1.2 was extended to s.s.p. nonlinear operators, but when specializing in the linear case, the result reads as follows

**Theorem 1.4 (Linear Krein-Rutman (refined strong version 1)[10]).** Let \( X \) and \( P \) be given as in the weak version, with \( \text{int}(P) \neq \emptyset \). Suppose \( T \in L(X) \) is compact and semi-strongly positive. Then

1. \( r(T) > 0 \) is a geometrically simple eigenvalue with an eigenvector \( x_0 \in \text{int}(P) \).
2. \( r_*(T) = r(T) = r^*(T) \).
3. Any real eigenvalue of \( T \) with positive eigenvector must be \( r(T) \).

According to the Extension Theorem for Positive Functionals (see for instance, [12], Proposition 19.3(a)), a strongly positive linear mapping is semi-strongly positive. And it is easy to see “strong positivity \( \neq \) semi-strong positivity”by examples in matrices. One of the main features of the above theorem is part (2), which, in the cases of matrices, covers the Collatz-Wielandt formula, and provides two characterizations of the “principal eigenvalue” \( r(T) \), that will in turn yield a min-max as well as a min-max characterizations of the principal eigenvalue for partial differential operators (see Section 5).

In this paper, we shall present a refined version of the above theorem, see Theorem 2.1 below, which not only restores almost the full strength of Theorem 1.2 (short
of part (3), which is natural because s.s.p. matrices are just irreducible matrices and for these matrices, the eigenvalues on the spectral circle are in general cyclic) but also adds to it part (2) of Theorem 1.4. We should mention that our proof of Theorem 2.1 is elementary and hence also has pedagogical value. We repeatedly use what we call “Touching Lemma”, making the proof rather transparent. Recall Krein and Rutman’s original proof of the strong K-R Theorem 1.2 is based on the weak K-R Theorem 1.1, whose proof involves heavy usage of resolvent analysis. P. Rabinowitz came up with a shorter and direct proof of the strong K-R Theorem 1.2, using the Leray-Schauder Theorem, as recorded in the book [34]. At the writing of the last drafts of this paper, we became aware of another elementary proof of the strong K-R Theorem 1.2 (without part (5)), due to Takáč [36]. Takáč’s proof and our proof of Theorem 2.1 are very different, and both do not use anything about compact operators beyond the fact that any non-zero spectral point must be an eigenvalue of the operator, except that when we prove the conclusion about $T^*$, we use the facts on algebraic simple eigenvalues and their associated projection operators; moreover, our proof of the fact that $r_s(T) = r(T) = r^*(T)$ (which Takáč did not mention) is cohesively mixed with the proof of other parts. Through [36], we became aware of yet another elementary proof of the strong K-R Theorem 1.2 (short of part (5)), due to N. Alikakos and G. Fusco [2]. Their proof is based on the concept of $\omega$—limit set of a dynamical system generated by $T$; the proof does not even use any spectral theory of compact operators, but is technical.

In many Banach spaces arising in applications, such as $L^p$ spaces, the positive cone does not have interior points. In this case, it is also not possible to define “strong positivity” of mappings. It turns out that the notions of “quasi interior” and “ideal-irreducibility” are the right substitutes, if the ordered Banach space is a “Banach lattice”.

A Banach lattice is an ordered Banach space $X$ (with norm $\|\cdot\|$), satisfying that (i) $\forall x, y \in X$, the least upper bound $\max\{x, y\}$ and the greatest lower bound $\min\{x, y\}$, both defined in the natural way, exist in $X$, and hence the absolute value operation $|\cdot|$ is also defined (again in the natural way); (ii) this lattice operation is consistent with the norm in the sense of $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$.

An ideal of $X$ is a linear subspace $I$ of it, such that $y \in I, x \in X$ and $|x| \leq |y| \Rightarrow x \in I$. Note that if $I$ is an ideal, then so is $\bar{I}$.

\[ I_a = \{ x \in X | \exists \lambda > 0 \text{ such that } |x| \leq \lambda a \}. \]

(We see easily that $I_a$ is an ideal, which is called the principal ideal generated by $a$.) A point $a \in \bar{P}$ is called a quasi-interior point of the cone $P$, if $I_a = X$. The set of quasi-interior points of $P$ is denoted as $qit(P)$. It is known [30] that (i) if $\text{int}(P) \neq \emptyset$, then $qit(P) = \text{int}(P)$; (ii) $a \in qit(P) \Leftrightarrow P^*(a) = P^* \setminus \{0\}$.

A positive operator $T \in L(X)$ is called ideal-irreducible, or simply, irreducible, if it does not have a $T$-invariant closed ideal, which is neither $\{0\}$ nor $X$.

The crown jewel of the Krein-Rutman theory for irreducible operators is the following result (see for instance [26], [30], [25])

**Theorem 1.5.** Suppose that $T \in L(X)$ is a positive, ideal-irreducible, compact operator in a Banach lattice $X$ whose dimension is bigger than 1, then we have

1. $r(T) > 0$.
2. $r(T)$ is an algebraically simple eigenvalue of $T$, with an eigenvector $x_0 \in qit(P)$. 
3. \( r(T) \) is also an algebraically simple eigenvalue of \( T^* \), with a strictly positive eigenvector \( x_0^* : \langle x_0^*, x \rangle > 0, \forall x \in \bar{P} \).

4. \( T \) and \( T^* \) have no (real) eigenvalues other than \( r(T) \) associated with eigenvectors in \( \bar{P} \) and \( \bar{P}^* \), respectively.

In Section 4.1, we will add an observation that

\[ r(T) = r^*(T) = r_+(T). \]  

This is almost what we can achieve in the case of strong positivity (Theorem 1.2, with the lacking of part (3)), and exactly the same as in the case of semi-strong positivity (Theorem 2.1). The lacking of part (3) of Theorem 1.2 is again expected, just as explained before for the case of semi-strong positivity.

So far we have introduced three notions that are stronger than “positivity”: “strong positivity”, “semi-strong positivity”, and “irreducibility”. It is easy to see that the first implies the other two in a Banach lattice whose positive cone has interior. It is natural to study the relationship between the last two notions. In doing so we compare them again in a Banach lattice, whose positive cone may not have interior. In this scenario, we modify the previous definition of semi-positivity by replacing “interior” by “quasi interior”. In Theorem 3.2, we will show that s.s.p. is equivalent to irreducibility.

With the success of using \( r_+(T) \) and \( r^*(T) \) to characterize \( r(T) \) for s.s.p. operators and irreducible positive operators, we naturally want to know what would happen if the operator is reducible. Before addressing this case, we pause to discuss more general cases. It should be known (but we cannot find it explicitly stated in the literature) that in a general ordered Banach space, under the conditions in the weak K-R Theorem 1.1, we always have

\[ r^*(T) \leq r(T) = r_+(T). \]  

The proof of this is rather straightforward: by the definition of \( r_+(T) \), \( \forall 0 < \lambda < r_+(T) \), \( \exists x > 0 \) such that \( T x \geq \lambda x \), from which we have \( T^n x \geq \lambda^n x, \forall n \geq 1 \).

Picking a \( x^* \in P^* \) (and letting it act on both sides of the above inequality, we have \( |x^*||T^n x|||x|| \geq |x^*|T^n x \geq \lambda^n < x^*, x > \Rightarrow \limsup_{n \to \infty} \|T^n\|^{\frac{1}{n}} \geq \lambda \). Thus \( r(T) \geq \lambda \) and hence

\[ r(T) \geq r_+(T). \]  

On the other hand, from the definition of \( r_+(T) \) and \( r^*(T) \), it follows that any eigenvalue \( \lambda \) of \( T \) with a positive eigenvector satisfies \( r^*(T) \leq \lambda \leq r_+(T) \). By the weak Krein-Rutman Theorem 1.1, \( r(T) > 0 \) is such an eigenvalue so we have \( r^*(T) \leq r(T) \leq r_+(T) \). Combining this with (4), we have (3).

Notice that the simple proof of (4) only requires that \( T \) is positive and bounded. In fact, the same proof works for increasing, positively 1-homogeneous nonlinear mappings. In this connection, we mention [37] in which Thieme surveys and discusses various notions of spectral radii for these nonlinear mappings. One of the spectral radii discussed there is the cone spectral radius \( r_+(T) \), defined as \( \lim_{n \to \infty} \|T^n\|^{1/n} \) where \( |T|_+ = \sup\{\|Tx\| \mid x \geq 0, \|x\| \leq 1\} \). Obviously, for linear \( T \), \( r_+(T) \leq r(T) \); and according to Mallet-Paret and Nussbaum [23] the equality holds if the cone \( P \) is “generating”, i.e., \( X = P - P \) (in particular, if the interior of \( P \) is non-empty or if \( X \) is a lattice). Among many other things, the following are proved in [37] (when specialized for linear \( T \)): Suppose \( P \) has non-empty interior, and \( T \) is linear, bounded and positive (so \( r_+(T) = r(T) \)). (i) if \( \exists x > 0 \) such that \( T x \geq r(T)x \), then \( r_+(T) = r_+(T) \leq r^*(T) \), where \( r^*(T) \) is defined slightly
differently from $r^*(T)$: $\hat{r}^*(T) = \inf_{x \geq 0} \inf \{ \lambda \geq 0 \mid Tx \leq \lambda x \}$ (so $\hat{r}^*(T) \geq r^*(T)$).

Now we come back to the case of reducible $T$. Our new discovery is Theorem 4.5, which says that in a Banach lattice with “order continuous norm” (it is known that all reflexive Banach lattices have order continuous norm), if $T$ is reducible and compact, then $r^*(T)$ is either 0 or the infimum of the set of all eigenvalues $\lambda$ with a positive eigenvector of $T$. The proof of this theorem involves applying Frobenius Decomposition Theorem for positive reducible operators i.e., decomposing a reducible positive operator in a Banach lattice into a direct sum of irreducible positive operators and a quasi-nilpotent operator, as developed in [18] and [16].

The last part of this paper is devoted to PDE examples. The central concern is the existence, uniqueness, multiplicity and variational characterizations of the principal eigenvalues of PDE operators, such as second order elliptic operators. These tasks can be finished without using the Krein-Rutman theory at all, see for example [4] and [22] for the case of bounded spatial domains, and [5, 6] for the case of unbounded spatial domains. Here we adopt the functional analytic approach, because we intend to demonstrate the full power of various Krein-Rutman theorems mentioned or proved in this paper; to this end, we purposefully use as few PDE tools as possible. The four examples are taken from the papers of Berestycki, Nirenberg and Varadhan [4], Birindelli [7], Li, Coville and Wang [22], Sweers [35], Birindelli, Mitidieri and Sweers [8] and Lam and Lou [21]. Here we emphasize characterizations, in the min-max and max-min fashions, of the principal eigenvalue (we call any eigenvalue of a PDE operator principal eigenvalue if it pertains to a positive eigenfunction), though along the way for the sake of completeness and the readability we also discuss, in the more functional analytic style, the principal eigenvalue’s existence, uniqueness and multiplicity which may have been proved in the afore-mentioned papers via “pure PDE” method or via Krein-Rutman (in the latter case our presentation differs in places from those in previous works).

On the min-max and max-min characterizations of the principal eigenvalue of non-symmetric elliptic operators, we mention two pioneering works. In 1937, Barta [3] observed that the principal Dirichlet eigenvalue $\lambda_1$ of $-\Delta$ on a bounded smooth domain $\Omega$ satisfies

$$\lambda_1 \geq \inf_{\Omega} \frac{-\Delta \phi(x)}{\phi(x)},$$

for any $\phi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying $\phi > 0$ in $\Omega$. Since the positive principal eigenfunction satisfies the condition on $\phi$, we have the following max-min characterization

$$\lambda_1 = \sup_{\phi \in C^2(\Omega) \cap C^1(\bar{\Omega})} \inf_{\Omega > 0 \text{ in } \Omega} -\frac{\Delta \phi(x)}{\phi(x)}.$$

On the hand, in 1966, Protter and Weinberger [27] studied a general second order elliptic, nonsymmetric operator $L$ with general boundary operator $B$ (covering the cases of Dirichlet, Neumann and Robin), whose Theorem 1 implies a generalization of (5). Combining this with Remark 2 of [27], one has two characterizations of the principal eigenvalue of $L$ and $B$:

$$\lambda_1 = \sup_{\phi \in C^2(\Omega) \cap C^1(\bar{\Omega})} \inf_{\phi > 0 \text{ on } \partial \Omega \phi > 0 \text{ in } \Omega} L \phi(x) \phi(x).$$
These characterizations, which have analogs (Collatz-Wielandt formula) for non-negative irreducible square matrices, are useful because by choosing test functions we can obtain lower and upper bounds for $\lambda_1$. In all but Example 2, the characterization of the second kind was not proved in the related papers mentioned above. Moreover, in Example 3 for a cooperative, but not fully coupled elliptic system, even though the principal eigenvalues are not unique, we are able to use our new theorem on reducible operators (Theorem 4.5) to obtain a characterization of the largest principal eigenvalue, as well as the one for the smallest principal eigenvalue, also in the fashion of (7).

For issues related to (7) in unbounded domains, see [5, 6].

The Krein-Rutman theory presented here can of course be applied to the principal eigenvalues of the so called periodic parabolic operators, but we choose not to touch this topic here.

2. Touching lemma and elementary proof of Krein-Rutman theorem.

The purpose of this section is to give an elementary proof of

**Theorem 2.1** (Linear Krein-Rutman (refined strong version 2)). Let $X$ be a Banach space ordered by a cone $P$, with $\text{int}(P) \neq \emptyset$. Suppose $T \in L(X)$ is compact and semi-strongly positive. Then

1. $r(T) > 0$ is a geometrically simple eigenvalue of $T$ with an eigenvector $x_0 \in \text{int}(P)$.
2. $r(T)$ is algebraically simple.
3. Any real eigenvalue of $T$ with a positive eigenvector must be $r(T)$.
4. $r(T)$ is an algebraically simple eigenvalue of $T^*$, with an eigenvector $x_0^*$ which is a strictly positive functional: $\langle x_0^*, x \rangle > 0, \forall x > 0$; moreover, any eigenvector of $T^*$ in $\dot{P}^*$ corresponds to $r(T)$.
5. $r_+(T) = r(T) = r^+(T)$.

The proof involves repeatedly using the following simple fact.

**Lemma 2.2** (Touching Lemma). Let $X$ be a Banach space ordered by a cone $P$, with $\text{int}(P) \neq \emptyset$. Suppose $T \in L(X)$ is s.s.p. in $X$. Assume $\exists x > 0, y \in X$, and $\lambda \geq \mu \geq 0$, such that

$$\mu x \geq Tx \quad \text{and} \quad \lambda y \geq Ty.$$  \hspace{1cm} (8)

Then either $y \geq 0$ or $x = -\delta y$ for a positive real number $\delta$, $\lambda = \mu$ and $\lambda x = Tx$ and $\lambda y = Ty$.

This result is inspired by its PDE analog (see, e.g., Lemma 5.4 in Section 5), which can be traced back to Walter [38]. The proof of the PDE analog involves adding to a positive super-solution (of a second order elliptic operator) a positive constant multiple of another super-solution which takes negative values, then gradually increasing the constant multiplier until the sum touches 0 at a point. This is why we name this result “Touching Lemma”.

To prepare the proofs of Theorem 2.1 and Touching Lemma, we recall some basic facts. Suppose $\text{int}(P) \neq \emptyset$, $x_0 \in \text{int}(P)$ and $y \notin P$. Define

$$\delta_{x_0}(y) = \sup \{ \delta > 0 \mid x_0 + ty \in \text{int}(P), \forall 0 \leq t \leq \delta \}. \hspace{1cm} (9)$$

Then (see [40]) $0 < \delta_{x_0}(y) < \infty$; moreover,
1. $x_0 + \delta y \in \text{int}(P)$ if $0 \leq \delta < \delta_{x_0}(y)$,
2. $x_0 + \delta_{x_0}(y)y \in P \setminus \text{int}(P) = \partial P$,
3. $x_0 + \delta y \notin P$ if $\delta > \delta_{x_0}(y)$.

**Lemma 2.3.** If $a \in \text{int}(P)$, then $\forall x \in X, \exists r > 0$ such that $a \gg rx$.

*Proof.* We assume that $x \neq 0$. By definition, $\exists \delta > 0$ such that $B_\delta(a) \subset \text{int}(P)$, which implies that $a - \delta \frac{x}{\|x\|} \gg 0$. Let $r = \frac{\delta}{\|x\|}$, the conclusion is proved.

**Lemma 2.4.** If $T$ is s.s.p. and if $\exists (\lambda, x) \in [0, \infty) \times \hat{P}$, such that $\lambda x \geq Tx$, then $x \in \text{int}(P)$.

*Proof.* If not, then $\exists x_0 \in \hat{P} \setminus \text{int}(P)$ satisfying $\lambda_0 x_0 \geq Tx_0$ for some $\lambda_0 \geq 0$. Then by s.s.p., $\exists x^* \in P^*$ such that

$$0 = \langle x^*, x_0 \rangle < \langle x^*, Tx_0 \rangle \leq \lambda_0 \langle x^*, x_0 \rangle = 0.$$  

This is impossible.

**Lemma 2.5.** If $T$ is s.s.p. and if $x \in \text{int}(P)$, then $Tx \in \text{int}(P)$.

*Proof.* First we claim that $Tx \in \hat{P}$. For otherwise, $Tx = 0$, but $\forall y \in \hat{P}, \exists r > 0$ such that $0 \leq ry \leq x$. Thus $Ty = 0$. In particular, we choose $y \in \hat{P} \setminus \text{int}(P)$, by the semi strong positivity of $T$, one should have $x^* \in P^*$ such that $\langle x^*, y \rangle = 0 < \langle x^*, Ty \rangle$. This is a contradiction.

Then we claim that $Tx \in \text{int}(P)$. If not, then $Tx \in \hat{P} \setminus \text{int}(P)$. By the semi-strong positivity of $T$, one should have $x^* \in P^*$ such that

$$\langle x^*, Tx \rangle = 0 < \langle x^*, T^2 x \rangle.$$  

(10)

But by Lemma 2.3, we have $r > 0$ such that $x \geq r Tx$, which implies that $Tx \geq rT^2 x$. This contradicts (10).

*Proof of Touching Lemma.* From Lemma 2.4, it follows $x \in \text{int}(P)$. If $y \geq 0$, then we have nothing to do. Otherwise, $y \notin P$ and so $\delta_{x}(y) > 0$. Let $z = x + \delta_{x}(y)y$. Then by the definition of $\delta_{x}(y)$, $z \in \partial P$: either $z = 0$ or $z \in \partial P \setminus \{0\}$. But since

$$Tz = Tx + \delta_{x}(y)Ty = \mu x + \lambda \delta_{x}(y)y \leq \lambda z,$$

by Lemma 2.4, $z \neq 0$ is impossible. Therefore $z = 0$, that is, $x = -\delta_{x}(y)y$. Combining this with (8), we have

$$\mu x \geq Tx = -\delta_{x}(y)Ty \geq -\delta_{x}(y)\lambda y = \lambda x,$$

from which it follows that $\lambda = \mu$, $\lambda x = Tx$ and $\lambda y = Ty$.

*Proof.* (Elementary proof of the strong version of linear Krien-Rutman (refined strong version 2).)

**Step 1. Prove** $r^*(T) > 0$. As mentioned before,

$$r^*(T) = \inf_{x \in P} \inf \{\mu \geq 0 | \mu x \geq Tx\}.$$  

We only need to show that $\exists r > 0$ such that if $(\mu, x) \in [0, \infty) \times \hat{P}$ satisfies $\mu x \geq Tx$ then $\mu \geq r$. To this end, fix $x_0 \in \text{int}(P)$. By Lemma 2.5, $Tx_0 \in \text{int}(P)$, and hence by Lemma 2.3, $\exists r > 0$ such that $Tx_0 > r x_0$, which is equivalent to $r(-x_0) > T(-x_0)$. 

If $\mu \leq r$, by Touching Lemma we have $T(-x_0) = r(-x_0)$, which is impossible. Step 1 is complete.

**Step 2. Prove** $r_*(T) \leq r^*(T)$. Recall $r_*(T) = \sup_{x \in P} \mu_*(x)$ where $\mu_*(x) = \sup \{ \mu \geq 0 \mid \mu x \leq Tx \}$. For any $x_0 \in P$, if $\mu_*(x_0) = \infty$, then $\forall \mu > 0, \mu x_0 \leq Tx_0 \Rightarrow x_0 \leq \frac{Tx_0}{\mu} \to 0$ as $\mu \to \infty$, which implies $x_0 \leq 0$. But this is impossible. Thus $\forall x_0 \in P, \mu_*(x_0) < \infty$ which implies $\mu_*(x_0)x_0 \leq Tx_0$, i.e., $\mu_*(x_0)(-x_0) \geq T(-x_0)$. By the proof of Step 1, we have $r^*(T) \geq \mu_*(x_0)$ and hence $r^*(T) \geq r_*(T)$.

**Step 3. Prove** $\forall \lambda > r^*(T), \exists x \in \sigma(T)$ (resolvent set of $T$). If not, $\exists \lambda \in \sigma(T)$ (spectrum set of $T$). By Riesz-Schauder, $\lambda$ is an eigenvalue of $T$ with eigenvector $y \in X$: $Ty = \lambda y, y \neq 0$. Without loss of generality, assume $y \notin P$ (otherwise, consider $-y$). On the other hand, by the definition of $r^*(T), \forall \mu > r^*(T), \exists x \in P$, such that $Tx \leq \mu x$. Choosing $\mu < \lambda$, we have by Touching Lemma, $\mu \lambda = \lambda$. That is a contradiction. Thus $\lambda \in \rho(T)$.

**Step 4. Searching for positive eigenpair by approximation.** Now $\forall \lambda \in (r^*(T), r^*(T) + 1)$, $\forall \epsilon \in (0, 1), \exists x_\lambda^\epsilon \in X$, such that $(\lambda I - T)x_\lambda^\epsilon = \epsilon Tx_\lambda^\epsilon$ ($x_\lambda^\epsilon \geq 0$ is fixed). We claim that $x_\lambda^\epsilon \in \text{int}(P)$. In fact, since $\lambda x_\lambda^\epsilon \geq Tx_\lambda^\epsilon$, by Lemma 2.4, it is sufficient to verify $x_\lambda^\epsilon \in P$. By the definition of $r^*(T), \exists \mu \in (r^*(T), \lambda)$ and $\bar{x} \in P$, such that $\mu \bar{x} \geq T\bar{x}$. If $x_\lambda^\epsilon \notin P$, then by Touching Lemma, we would have $\lambda x_\lambda^\epsilon = T\bar{x}$. Impossible. Thus we must have $x_\lambda^\epsilon \in P$.

Now observe that there are only two cases:

**Case 1.** $\forall \epsilon \in (0, 1), \exists \lambda \in (r^*(T), r^*(T) + 1)$, such that $x_\epsilon = x_\lambda^\epsilon$ satisfies $\|x_\epsilon\| > 1$.

**Case 2.** $\exists \epsilon \in (0, 1)$, such that $\forall \lambda \in (r^*(T), r^*(T) + 1), \|x_\lambda^\epsilon\| \leq 1$.

**Step 5. Prove that Case 2 does not occur.** If Case 2 occurs, then after passing to a subsequence of $\lambda \nearrow r^*(T)$, $Tx_\lambda^\epsilon$ converges in $X$. Then one has $x_\lambda^\epsilon = (\epsilon Tx_\lambda^\epsilon + Tx_\lambda^\epsilon) / \lambda$ converges in $X$ to some $\bar{x} \in P$ (because $x_\lambda^\epsilon \in P$ and $P$ is closed), and so $(r^*(T)I - T)\bar{x} = \epsilon Tx_\lambda^\epsilon \neq 0$, from which it also follows $\bar{x} \in P$. By Lemma 2.5, $Tx_\lambda^\epsilon \in \text{int}(P)$, hence by Lemma 2.3, $\exists \delta > 0$, such that $\epsilon Tx_\lambda^\epsilon \geq \delta \bar{x}$, so is $(r^*(T) - \delta)\bar{x} \geq T\bar{x}$. But $r^*(T) \leq \inf \{ \mu \geq 0 \mid \mu \bar{x} \geq T\bar{x} \} \leq r^*(T) - \delta$. That is a contradiction.

**Step 6. Prove the existence of a positive eigen-pair $(\bar{\lambda}, \bar{y})$.** Thus only Case 1 occurs. After passing to a subsequence of $\epsilon \to 0, \lambda_\epsilon \to \lambda$ such that $\lambda_\epsilon \in [r^*(T), r^*(T) + 1]$. Let $y_\epsilon = \frac{x_\epsilon}{\|x_\epsilon\|}$. Then $\|y_\epsilon\| = 1$ and $(\lambda_\epsilon I - T)y_\epsilon = \frac{\epsilon}{\|x_\epsilon\|}Tx_\epsilon$. By the compactness of $T$, it follows $y_\epsilon \to \bar{y}$ some $\bar{y} \in P$, which satisfies $\|\bar{y}\| = 1, (\lambda I - T)\bar{y} = 0$. By Step 3, there exists no eigenvalue of $T$ bigger than $r^*(T)$, thus $\bar{\lambda} = r^*(T)$. Now $\bar{y} \in P$, so $r_*(T) = r^*(T)$, which is an eigenvalue with an eigenvector $\bar{y}$. This eigenvector must be in $\text{int}(P)$, according to Lemma 2.4.

**Step 7. Prove that the algebraic multiplicity of $\bar{\lambda}$ is 1.** We first prove that the geometric multiplicity is 1. Suppose $\exists x \in X \setminus \{0\}$, such that $Tx = \bar{\lambda}x$. Without loss of generality, assume $x \notin P$. By Touching Lemma, $\bar{y} = -\delta\bar{y}(x)$. Done.

Recall the algebraic multiplicity of $\bar{\lambda}$ is $\dim \bigcap_{k=1}^\infty \ker(\lambda I - T)^k$. It suffices to show $\ker(\lambda I - T)^2 = \text{span}\{\bar{y}\}$. Let $x \in X \setminus \{0\}$, such that $(\lambda I - T)^2x = 0$. Then $(\lambda I - T)x \in \ker(\lambda I - T) = \text{span}\{\bar{y}\}$, and so $\exists c \in \mathbb{R}, \text{such that } (\lambda I - T)x = cy$. If $c = 0$, then $x \in \ker(\lambda I - T) = \text{span}\{\bar{y}\}$, we are done.
We now show that $c \neq 0$ is impossible. In fact, if $c \neq 0$, then without loss of generality, we assume $c = 1$. Then $(\lambda I - T)x = \bar{y} \in \text{int}(P)$, hence $\lambda x \gg Tx$. Combining this with $\lambda \bar{y} = T\bar{y}$ and applying Touching Lemma, we conclude either $x \geq 0$ or $\lambda x = Tx$. But the latter is impossible. Hence $x \geq 0$ and thus $x \in P$. Now
\[
\lambda x \gg Tx, \quad x > 0
\]
\[
\lambda(-\bar{y}) = T(-\bar{y}), \quad -\bar{y} \notin P,
\]
so applying Touching Lemma again, we obtain $\lambda x = Tx$, which is impossible.

\textbf{Step 8.} \textbf{Proof that if $\lambda$ is a real eigenvalue of $T$ with a positive eigenvector $x$, then $\lambda = r^*(T) = r_*(T)$.} Since $T$ is positive, $\lambda \geq 0$. Thus $r^*(T) \leq \lambda \leq r_*(T)$. Recall the upper and lower spectral radii are equal, as proved in Step 6.

\textbf{Step 9.} \textbf{Prove parts (4) and (5) of the theorem.} Since $r(T)$ is an algebraically simple eigenvalue of $T$, it is so for $T^*$; moreover, $X = \text{Ker}(r(T)I - T) \oplus R(r(T)I - T)$, and $r(T)$ is a simple pole of the resolvent operator $R(\lambda) := (\lambda I - T)^{-1}$.

Let $P = \lim_{\lambda \downarrow r(T)}(\lambda - r(T))R(\lambda)$. Then $P$ is the projection operator of $X$ onto $\text{Ker}(r(T)I - T)$ along $R(r(T)I - T)$; furthermore, $P$ is positive.

So $\forall x > 0$, $\exists$ real number $c \geq 0$ and $y \in X$ such that $x = cx_0 + (r(T)I - T)y$, where $x_0$ is the $\bar{y}$ obtained in Step 6. (This is still true $\forall x \in X$, but not necessarily $c \geq 0$.) We claim that $\forall x > 0, c \neq 0$. Otherwise, $r(T)y > Ty$. Since $r(T)x_0 = Tx_0$, by Touching Lemma, we must have $y > 0$. By Touching Lemma again, we must have $r(T)y = Ty$, impossible.

Let $x_0^*$ be an eigenvector of $T^*$ corresponding to $r(T)$, such that $\langle x_0^*, x_0 \rangle \geq 0$. Since $x_0^*$ is “orthogonal ” to $R(r(T)I - T)$, $\langle x_0^*, x_0 \rangle > 0$ (otherwise $x_0^* = 0$). Now $\forall x > 0, \langle x_0^*, x \rangle = c\langle x_0^*, x_0 \rangle > 0$.

Finally, let $x^* \in P^*$ be an eigenvector of $T^*$ with eigenvalue $\lambda$. Then
\[
r(T)\langle x^*, x_0 \rangle = \langle x^*, Tx_0 \rangle = \langle T^*x^*, x_0 \rangle = \lambda \langle x^*, x_0 \rangle.
\]

Since $x_0 \gg 0$, $\langle x^*, x_0 \rangle > 0$. Now $r(T) = \lambda$. \hfill \Box

3. \textbf{Semi-strongly positive operators.} Now, we study the equivalence of the ideal-irreducibility and the semi-strong positivity of a positive operator. These two notions have to be compared in a Banach lattice whose positive cone may not have interior, so we modify the definition of s.s.p. defined in Section 1 as follows.

\textbf{Definition 3.1.} Let $X$ be a Banach lattice, a positive operator $T \in L(X)$ is called \textit{semi-strongly positive} if $\forall x \in \hat{P} \setminus \text{qit}(P), \exists x^* \in P^*$ such that
\[
\langle x^*, Tx \rangle > 0 = \langle x^*, x \rangle.
\]

If $\text{Int}(P) \neq 0$, then $\text{qit}(P) = \text{Int}(P)$, so we recover the previous definition as a special case.

\textbf{Theorem 3.2.} Suppose $X$ is a Banach lattice and $T \in L(X)$ is positive. If $T$ is ideal-irreducible, then it is semi-strongly positive.

If further, $X$ is separable, then the converse is true, i.e., if $T$ is semi-strongly positive, then $T$ is ideal-irreducible.

\textbf{Proof.} $\Rightarrow$: The proof is by contradiction. Suppose $\exists x_0 \in \hat{P} \setminus \text{qit}(P)$ satisfying
\[
\forall x^* \in P^*, \langle x^*, x_0 \rangle = 0 \Rightarrow \langle x^*, Tx_0 \rangle = 0. \tag{11}
\]
Let us denote by $I_{x_0}$ the principal ideal generated by $x_0$. We shall prove that $I_{x_0}$ is $T$-invariant. If this is true, then by the irreducibility of $T$, $I_{x_0} = X$ and then 

$$
\langle x^*, x_0 \rangle = 0 \Rightarrow \langle x^*, x \rangle = 0, \forall x \in X,
$$

and then $x^* = 0$. This is impossible, because $x_0 \in \hat{P} \setminus \text{qit}(P)$ implies that $\exists x_0^* \in \hat{P}^*$ such that $\langle x_0^*, x_0 \rangle = 0$.

Now we return to verify the closed ideal $I_{x_0}$ is $T$-invariant. We proceed in two steps.

**Step 1.** $Tx_0 \in \bar{I}_{x_0}$.

If not, then by Hahn-Banach theorem, we have a bounded linear functional $x_0^*$, such that $x_0^* = 0$ on $\bar{I}_{x_0}$ and $(x_0^*, Tx_0) > 0$. By Proposition 4.2 of [30], $\exists$ positive, bounded linear functional $x_0^{*+} (= x_0^* \lor 0)$ such that $x_0^{*+}$ and $x_0^*$ are identical on $P$. In particular, $(x_0^{*+}, x_0^*) = 0 < (x_0^{*+}, Tx_0)$, contradicting (11).

**Step 2.** $\bar{I}_{x_0}$ is a $T$-invariant ideal.

$\forall y \in \bar{I}_{x_0}$, $\exists \{y_n\} \subseteq I_{x_0}$ and $\lambda_n$, such that $|y_n| \leq \lambda_n x_0$ and $y_n \to y$ as $n \to \infty$. We have

$$
\text{dist}(Ty, y) \leq \lambda_n \text{dist}(Ty, \bar{I}_{x_0}).
$$

Noticing that $Tx_0 \in \bar{I}_{x_0}$ and $\bar{I}_{x_0}$ is an ideal, we have $Ty_n \in \bar{I}_{x_0}$. Since $y_n \to y$, we have $Ty_n \to Ty$ as $n \to \infty$. Therefore $Ty \in \bar{I}_{x_0}$.

We prove it by contradiction. Suppose $T$ is reducible, then $\exists$ some closed $T$-invariant ideal $I \neq X$, $\{0\}$. Note that $P \nsubseteq I$. Since $X$ is separable and complete, $I$ is also separable and complete. Thus $\text{qit}(P \cap I) \neq \emptyset$ by [30] (Proposition 6.2, p.97). Take $x_0 \in \text{qit}(P \cap I)$ and $y_0 \in P$ but not in $I$. Using Hahn-Banach, there exists a linear and bounded functional $x_0^*$ on $X$ such that it is identically equal to $0$ on $I$ and $1$ at $y_0$. Define $x_0^{*+} \in \hat{P}^*$ as above; we still have $\langle x_0^{*+}, x_0^* \rangle = 0$, which implies $x_0 \in \hat{P} \setminus \text{qit}(P)$. On the other hand, since $x_0 \in \text{qit}(P \cap I)$ and $I$ is $T$-invariant, we have $I = I_{x_0} \cap I$ and $Tx_0 \in \bar{I}_{x_0}$. Therefore $\forall x^* \in \hat{P}$, if $\langle x^*, x_0 \rangle = 0$, then $\langle x^*, Tx_0 \rangle = 0$. This contradicts the definition of the semi-strong positivity of $T$. \hfill \Box

4. $r_*(T)$, $r^*(T)$, $r(T)$ and positive eigenpairs.

4.1. Ideal-irreducible operators.

**Theorem 4.1.** Assume that $X$ is a Banach lattice, and that $T \in L(X)$ is a positive irreducible compact operator, then

$$
r_*(T) = r(T) = r^*(T).
$$

**Proof.** Let $\lambda = r(T)$, then by Theorem 1.5, $\lambda > 0$, corresponding to which there exists strictly positive eigenvector $x_0^* \in \hat{P}^*$ of $T^*$. By (3), we only need to show $r^*(T) \geq \lambda$. Since $x_0^* \in \hat{P}^*$ is strictly positive, we have $\langle x_0^*, x \rangle > 0, \forall x \in \hat{P}$, and now

$$
\mu^*(x) \langle x_0^*, x \rangle \geq \langle x_0^*, Tx \rangle = \langle T^* x_0^*, x \rangle = \lambda \langle x_0^*, x \rangle,
$$

from which it follows that

$$
\mu^*(x) \geq \lambda, \ \forall x \in \hat{P},
$$

and $r^*(T) \geq \lambda$. \hfill \Box

**Remark 1.** We were informed by Lei Zhang that this theorem follows from a general result of Marek [24](Theorem 3.1). However, it seems that the above theorem was not explicitly written down in the literature in a context like ours, and our proof is rather straightforward, so we decide to present the result and proof here.
The compactness in the assumption of Theorem 4.1 is to ensure the strict positivity of \( r(T) \). In fact, before the work of de Pagter [26], Schaefer [30, 31], Niro-Sawashima [25] and Sawashima [29] proved the following (see also [13]).

**Theorem 4.2.** Assume that \( X \) is a Banach lattice and \( T \in L(X) \) is a positive irreducible operator with \( r(T) > 0 \). If the resolvent \( (\lambda I - T)^{-1} \) of \( T \) has a pole at \( \lambda = r(T) \), then

1. \( r(T) \) is a simple pole of the resolvent operator \( R(\lambda) \).
2. \( r(T) \) is an algebraically simple eigenvalue of \( T \) with an eigenvector \( x_0 \in \text{qit}(P) \).
3. \( r(T) \) is an algebraically simple eigenvalue of \( T^* \) with an eigenvector \( x_0^* \) which is a strictly positive functional.

By the same proof as Theorem 4.1, under the assumptions of Theorem 4.2, again we have the same conclusion (12).

4.2. Quasi-nilpotent operators. A linear bounded operator \( T \in L(X) \) is called quasi-nilpotent if

\[
r(T) = 0,
\]

i.e. if \( \lim_{n \to \infty} \| T^n \|^{1/n} = 0 \).

**Theorem 4.3.** Assume that \( X \) is a Banach lattice, and \( T \in L(X) \) is a positive quasi-nilpotent operator, then

\[
r_*(T) = r^*(T) = 0.
\]

**Proof.**

**Step 1.** First, we prove \( r_*(T) = 0 \). Observe that

\[
\mu_*(x)x \leq Tx, \quad \forall x \in \hat{P};
\]

\[
\mu_*(x)^2x \leq \mu_*(x)Tx = T(\mu_*(x)x) \leq T^2x, \quad \forall x \in \hat{P};
\]

\[
\mu_*(x)^nx \leq T^nx, \quad n = 1, 2, \ldots, \forall x \in \hat{P}.
\]

Since in a Banach lattice, the norm preserves the order of nonnegative elements, we have

\[
\mu_*(x)^n\|x\| \leq \|T^n\|\|x\|, \quad n = 1, 2, \ldots, \forall x \in \hat{P};
\]

\[
\mu_*(x) \leq \|T^n\|^{1/n}, \quad n = 1, 2, \ldots, \forall x \in \hat{P}.
\]

So

\[
r_*(T) = \sup_{x \in \hat{P}} \mu_*(x) \leq \lim_{n \to \infty} \|T^n\|^{1/n} = 0.
\]

**Step 2.** Next, we prove \( r^*(T) = 0 \). To this end, \( \forall \epsilon > 0 \), we are going to construct \( x_\epsilon \in \hat{P} \) such that \( \epsilon x_\epsilon \geq Tx_\epsilon \). Once such an \( x_\epsilon \) is constructed, we have

\[
r^*(T) = \inf_{x \in \hat{P}} \inf \{ \mu \geq 0 | \mu x \geq T x \} = 0.
\]

To construct \( x_\epsilon \), observe that since \( r(T) = 0 \), \( \forall \epsilon > 0 \), the resolvent

\[
(\epsilon I - T)^{-1} = \frac{1}{\epsilon} \sum_{k=0}^\infty \left( \frac{T}{\epsilon} \right)^k
\]

is positive. Pick any \( a \in \hat{P} \), let \( x_\epsilon = (\epsilon I - T)^{-1}a \), then \( x_\epsilon \in \hat{P} \), which is what we look for, because \( \epsilon x_\epsilon - Tx_\epsilon = a > 0 \).
4.3. **Reducible operators.** When the positive operator \( T \in L(X) \) is reducible, the Frobenius decomposition, i.e., decomposing a reducible operator into direct sum of ideal-irreducible operators, is needed. Jang and Victory, Jr [18], and Grobler and Reinecke [16] have studied the Frobenius decomposition for positive compact operators on Banach lattices. Let us review a few terminologies:

In a Banach lattice \( X \), an ideal \( I \) is called a band if for any \( A \subseteq I \), \( \sup A \in I \) whenever \( \sup A \) exists in \( X \). If the Banach lattice \( X \) has an order continuous norm, i.e., every order-monotone and order-bounded sequence is norm-convergent in \( X \), then every closed ideal \( I \) of \( X \) is a band, and hence a projection band: according to Riesz Decomposition Theorem, one has \( X = I \oplus I^\perp \). (This means that \( \forall x \in X \), there exists a unique decomposition: \( x = y + z \) with \( y \in I, z \in I^\perp \), and \( |y| \wedge |z| = 0 \) [30].)

A band \( J \) such that \( P_J TP_J : J \to J \) is ideal-irreducible and \( r(P_J TP_J) > 0 \) is called a significant \( T \)-band, where \( P_J \) is the “band projection” from \( X \) onto \( J \) along the “orthogonal complement” \( J^\perp \). (Recall the projections \( P_J \) and \( P_{J^\perp} \) both are positive (30), page 61.)

A significant \( T \)-band \( J \), which is not properly contained in any significant \( T \)-band, is called a principal \( T \)-band.

A principal \( T \)-band \( J \) with \( r(P_J TP_J) = r(T) \) is called a basic \( T \)-band.

The following was proved in [18] and [16]:

**Proposition 1.** Let \( X \) be a Banach lattice with order continuous norm and \( T \in L(X) \) be positive, reducible, eventually compact (meaning \( \exists \) integer \( k \) such that \( T^k \) is compact) with spectral radius equal to unity. If \( \lambda = 1 \) is a simple pole of the resolvent \( R(\lambda, T) = (\lambda I - T)^{-1} \), then we can decompose \( X \) into a direct sum of bands of \( X \) as \( J_1^\perp \oplus J_1 \oplus \cdots \oplus J_m \oplus J \), such that \( J_i \), \( 1 \leq i \leq m \), are basic \( T \)-bands, \( J \) is \( T \)-invariant, \( r(P_J TP_J) < 1 \) and \( r(P_J TP_J^\perp) < 1 \). With respect to this decomposition, \( T \) can be represented in lower triangular form

\[
T = \begin{bmatrix}
T_{1,0} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & T_{1,1} & 0 & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & T_{1,m} & \cdots \\n\hline
T_{2,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & T_2
\end{bmatrix}
\]

where \( T_{1,0} = P_{J_0^\perp} TP_{J_0^\perp}, T_{1,i} = P_{J_i} TP_{J_i}, T_2 = P_{J_0} TP_J, T_{2,1} = P_{J_0} TP_{J_0^\perp}. \)

In [18] and [16] a more general case was studied: suppose \( r(T) = 1 \) is a pole of order \( k > 1 \) of the resolvent \( R(\lambda, T) \) possessing a residuum of finite rank:

\[
Q^{(1)}_{-k} = \lim_{\lambda \to 1} (\lambda - 1)^k R(\lambda, T).
\]

Define

\[
G_k = \{ x \in X | Q^{(1)}_{-k} |x| = 0 \}.
\]
Then $G_k$ is a $T$-invariant band. We have a direct sum decomposition:

$$X = G_k^\perp \oplus G_k,$$

and let $J_k = G_k^\perp$, $T_1^{(k)} = P_{J_k}TP_{J_k}$, $T_2^{(k)} = T|_G$. Then $\lambda = 1$ is a simple pole of $R(\lambda, T_1^{(k)})$ of order 1 and that of $R(\lambda, T_2^{(k)})$ of order $k - 1$.

Inductively, one produces a sequence of closed ideals $\{J_1, \ldots, J_k, G_1\}$, satisfying

1. $X = J_k \oplus \cdots \oplus J_1 \oplus G_1$,
2. $T^{(1)}_2 := T|_{G_1}$ and $r(T|_{G_1}) < 1$,
3. $T^{(i)}_1 := P_{J_i}TP_{J_i}$, $r(T^{(i)}_1) = 1$ and $R(\lambda, T^{(i)}_1)$ has simple pole at $\lambda = 1$,
4. With the decomposition in (1), $T$ can be represented as

$$T = \begin{bmatrix}
T^{(k)}_1 \\
\vdots \\
\vdots \\
\vdots \\
T^{(1)}_1 \\
\vdots \\
\vdots \\
T^{(1)}_2
\end{bmatrix}$$

Thus $G_i := J_{i-1} \oplus \cdots \oplus J_1 \oplus G_1$ is $T$-invariant for any $1 \leq i \leq k$.

We can further decompose $J_i$ by property (3) and Proposition 1; we can also decompose $G_1$ if $r(T|_{G_1}) > 0$. It’s a natural question that how many bands we will get if we decompose $X$ thoroughly. So we need the following from [18] (Proposition II.2).

**Proposition 2.** For arbitrary $\lambda \in (0, 1]$, there are at most finite number of principal $T$-bands $J_\alpha$ for which $r(P_{J_\alpha}TP_{J_\alpha}) > \lambda$.

**Theorem 4.4** (Frobenius Decomposition [18] [16]). Let $X$ be a Banach lattice with order continuous norm, and let $T$ be a positive, reducible, eventually compact linear operator from $X$ to itself, with positive $r(T)$. Then $X$ can be decomposed into a direct sum of at most countably many principal $T$-bands and a band $\gamma$ on which $P_\gamma TP_\gamma$ is quasi-nilpotent:

$$X = \bigoplus_{i=1}^{\infty} J_i \oplus \gamma.$$

We are now ready to state our result on the upper and lower spectral radii of reducible operators.

**Theorem 4.5.** Let $X$ be a Banach lattice with order continuous norm, and let $T$ be a positive, reducible, compact linear operator mapping $X$ into itself with $r(T) > 0$. Let $\sigma_+(T)$ be the set of eigenvalues of $T$ pertaining to positive eigenvectors. Then

$$r_+(T) = \sup \sigma_+(T) = r(T)$$

and

$$r^*(T) = \begin{cases} 
0 & \text{if there exists a nontrivial } T\text{-invariant band } B \subset \gamma, \\
\inf \sigma_+(T) & \text{otherwise.}
\end{cases}$$

(15)
where $\gamma$ is the remainder of the decomposition in the Frobenius Decomposition Theorem.

Proof. By the weak version of Krein-Rutman Theorem, $r(T) \in \sigma_+(T)$; hence $r(T) = \text{sup} \sigma_+(T)$. Now (14) follows from (3).

Now we focus on $r^*(T)$. Suppose there exists a nontrivial band $B \subset \gamma$, which is invariant under $T$, then by Theorem 4.4, $r^*(T) \leq r^*(T|_B) = 0$. Thus $r^*(T) = 0$.

In the following we assume that there exists no such nontrivial $T$-invariant band $B \subset \gamma$. We already have $r^*(T) \leq \text{inf} \sigma_+(T)$, so we only need to show that $r^*(T) \geq \text{inf} \sigma_+$, or equivalently that $\forall \epsilon > 0$, there exists a positive eigen-pair $(\lambda_1, z_1)$ of $T$ with $0 < \lambda_1 < r^*(T) + \epsilon$.

By the definition of $r^*(T)$, $\forall \epsilon > 0$, there exist $z_0 \in \hat{P}$ and $\lambda_0 \in [r^*(T), r^*(T) + \epsilon)$, such that

$$Tz_0 \leq \lambda_0 z_0.$$  \hfill (16)

First, we observe that if there exists a nontrivial $T$-invariant band $J$ (indeed, such $J$ exists because $T$ is reducible), then $X$ can be decomposed as $X = J^{-1} \oplus J$. Let $z_0 = z_0^1 + z_0^2$ where $z_0^1 \in J^{-1}$ and $z_0^2 \in J$. Let $T_1 = PJ_\perp TP_{J_\perp}$, $T_{21} = PJP_{J_\perp}$, $T_2 = PJPJ$. Then $T$ can be represented as

$$\begin{pmatrix} T_1 & 0 \\ T_{21} & T_2 \end{pmatrix}.$$

From $Tz_0 \leq \lambda_0 z_0$, it follows that

$$\begin{cases} T_1 z_0^1 \leq \lambda_0 z_0^1 \\ T_{21} z_0^1 + T_2 z_0^2 \leq \lambda_0 z_0^2. \end{cases}$$

Since the projections $P_J$ and $P_{J_\perp}$ both are positive, $z_0^1$, $z_0^2$ are nonnegative and $T_1, T_{21}, T_2$ are positive.

**Case 1*. $z_0^2 = 0$.

Then $z_0^1 \neq 0$ satisfies

$$\begin{cases} T_1 z_0^1 \leq \lambda_0 z_0^1 \\ T_{21} z_0^1 = 0. \end{cases}$$

Let $J' = \hat{I}_{z_0^1}$ (the principal ideal generated by $z_0^1$). We will show that $J' \subset J_\perp$ is a $T$-invariant band. In fact, $\forall x \in J'$, $\exists$ a sequence $\{x_n\}$, such that $x_n$ converges to $x$ as $n$ goes to $\infty$ and $|x_n| \leq \lambda_n z_0^1$, where $\lambda_n > 0$ depends on $x_n$. Thus

$$|Tx_n| \leq |x_n| \leq \lambda_n Tz_0^1 = \lambda_n T_1 z_0^1 \leq \lambda_n \lambda_0 z_0^1,$$

from which it follows $Tx_n \in I_{z_0^1}$. By the continuity of $T$, $Tx \in J'$. Therefore $J'$ is invariant under $T$. Since we are assuming that there are no nontrivial bands in $\gamma$, $r(T|_{J'}) > 0$.

**Case 2*. $z_0^2 \neq 0$.

We have $T_2 z_0^2 \leq \lambda_0 z_0^2$, $z_0^2 \in \hat{P}$. Again, since we are assuming that there are no nontrivial bands in $\gamma$, $r(T|_{J'}) > 0$.

**Summary**: in either case, we can always find a smaller $T$-invariant band in which there exists $z_0 > 0$ satisfying (16) and the spectral radius of $T$ restricted in the band is positive.

Next, we apply the decomposition $X = J_k \oplus G_k$ (see the discussion before Proposition 4.6). Notice that $G_k$ is $T$-invariant, so we can think of it as the $J$ in the above discussion. Let $z_0 = z_0^{(k)} + z_2^{(k)}$. If $z_2^{(k)} = 0$, as discussed in case 2*, we can find a nontrivial $T$-invariant band $J' \subset J_k$ with $z_1^{(k)} \in J'$ and $Tz_1^{(k)} \leq \lambda_0 z_1^{(k)}$. If $z_2^{(k)} > 0$,
then as discussed in case 1*, we can focus on $G_k$ and $z^{(k)}_{\overline{2}}$. Similarly, since $G_k$ can be decomposed as $G_k = J_{k-1} \oplus G_{k-1}$, by the similar argument and induction, we have two cases: (Let $x_0 = z^{(k)}_{1} + z^{(k-1)}_{1} + \cdots + z^{(1)}_{1} + z^{(1)}_{2}$, $z^{(i)}_{1} \in J_i$ and $z^{(1)}_{2} \in G_1$.)

**Case 1.** We can find some nontrivial $T$-invariant $J' = \overline{I}_{z^{(i)}_{1}} \subset J_i$, where $z^{(i)}_{1} \neq 0$ satisfies

$$Tz^{(i)}_{1} \leq \lambda_0 z^{(i)}_{1}.$$  

**Case 2.** We focus on $G_1$ and nonzero $z^{(1)}_{2} \in G_1$ which satisfies

$$Tz^{(1)}_{2} \leq \lambda_0 z^{(1)}_{2}.$$  

For Case 2, we will deal with it together with the Subcase 2 of Case 1. Here we treat Case 1 first.

By Property (3) in the discussion before Proposition 4.6, we can apply Proposition 4.5 to decompose $J_i$ further:

$$J_i = \bigoplus_{j=0}^{n_i} J^{(i)}_{1,j} \oplus J^{(i)}_2$$

where $J^{(i)}_{1,j} (j = 1, \cdots n_i)$ are basic $T$-bands, and $r(P^{(i)}_{j_{1,0}} TP^{(i)}_{j_{1,0}}) < r(T)$, $r(P^{(i)}_{j_{2}} TP^{(i)}_{j_{2}}) < r(T)$.

To decompose $J'$, we write $z^{(i)}_{1} \in J_i$ as $z^{(i)}_{1} = z^{(i)}_{1,0} + \cdots + z^{(i)}_{1,n_i} + z^{(i)}_{2}$ where $z^{(i)}_{1,j} \in J^{(i)}_{1,j}$ and $z^{(i)}_{2} \in J^{(i)}_2$. Notice that the decomposition is orthogonal and by the construction of $J'$, we have $z^{(i)}_{1,j}, z^{(i)}_{2} \in J'$ as well.

**Claim.** If $J^{(i)}_{1,j} \cap J' \neq \emptyset$ ($j \geq 1$), then $J^{(i)}_{1,j} \subset J'$.

To prove this we observe that $J^{(i)}_{1,j} \cap J'$ is also $P^{(i)}_{j_{1,0}} TP^{(i)}_{j_{1,0}}$-invariant, and $P^{(i)}_{j_{1,j}} TP^{(i)}_{j_{1,j}}$ is irreducible on $J^{(i)}_{1,j}, J^{(i)}_{1,j} \cap J'$ is $J^{(i)}_{1,j}$ or $\emptyset$. This proves the claim.

By the claim, we have

$$J' = (J^{(i)}_{1,0} \cap J') \bigoplus \bigoplus_k J^{(i)}_{1,j_k} \oplus (J^{(i)}_2 \cap J')$$

where $j_k$'s are the integers, such that $z^{(i)}_{1,j_k} \neq 0$. We can further assume $\{j_k\}$ is increasing. With this decomposition, $T|_{J''}$ can be represented as lower triangular block matrix, i.e. the sum of last finitely many bands is $T$-invariant. As before, we can break down the problem to a nontrivial $T$-invariant band $J''$ which is included either in $J^{(i)}_{1,j_k}$, or $J^{(i)}_{1,0} \cap J'$, or $J^{(i)}_2 \cap J'$.

**Subcase 1.** $J'' \subset J^{(i)}_{1,j_k}$ for some $k$.

Since $J''$ is $T$-invariant, $J''$ is also $P^{(i)}_{j_{1,j_k}} TP^{(i)}_{j_{1,j_k}}$-invariant. By the same argument as the proof of the above claim, $J'' = J^{(i)}_{1,j_k}$. It implies that $T$ is irreducible on $J''$. Then one may apply Theorem 4.1 to ensure $r^*(T|_{J''}) = r(T|_{J''})$. By the weak Krein-Rutmann theorem, $\exists$ a positive eigen-pair $(\lambda_1, z_1)$ such that

$$Tz_1 = \lambda_1 z_1, \quad \lambda_1 = r(T|_{J''})$$

and then by the definition of $r^*$,

$$r^*(T|_{J''}) \leq \lambda_0 < r^*(T) + \epsilon.$$
We are done.

**Subcase 2.** $J'' \subset J^{(1)}_{1,0}$ or $J'' \subset J^{(2)}_{2} \cap J'$. The same argument for this subcase will apply to Case 2. For the notational simplicity, we don’t distinguish $J^{(1)}_{1,0}$, $J^{(2)}_{2}$ and $G_{1}$. We will call the nontrivial $T$-invariant band we focus on $J''$ and the band containing $J''$ which we can decompose further $J^1$. Here $J^1$ can be $J^{(1)}_{1,0}$, $J^{(2)}_{2}$ or $G_{1}$.

Note that we have $r(T|_{J^1}) < r(T)$.

We continue to decompose $J^1$ and argue as before as in Case 1, Subcase 1.

If after finitely many steps, we reach Subcase 1, then we are done. So we may assume that the process never stops and in the $m$-th step we get a pair of bands $J'' \subset J^m$, satisfying $r(T) > r(P_{J''} TP_{J''}) > \cdots > r(P_{J^m} TP_{J^m})$. According to Proposition 4.6, $\forall 0 < \lambda \leq r(T)$, $\exists$ at most finitely many principal ideals $J_\alpha$ such that $r(P_{J_\alpha} TP_{J_\alpha}) > \lambda$. This means that after finitely many steps, we arrive at $r(P_{J_m} TP_{J_m}) < r^*(T) + \epsilon$.

Once again, since we are assuming that there are no nontrivial bands in $\gamma$, $r(T|_{J''}) > 0$, we can apply the weak Krein-Rutman theorem to $T|_{J''}$ to get $z_1 \in J''$ satisfying $Tz_1 = \lambda_1 z_1$, $\lambda_1 = r(T|_{J''})$. Since $J'' \subset J^m$, $P_{J''} \leq P_{J^m}$, from which it follows that

$$ P_{J^m} TP_{J''} \leq P_{J^m} TP_{J^m}, $$

and hence

$$(P_{J''} TP_{J''})^n \leq (P_{J^m} TP_{J^m})^n, \; \forall n.$$  

Since $E$ is a Banach lattice,

$$ \| (P_{J''} TP_{J''})^n \| \leq \| (P_{J^m} TP_{J^m})^n \|, \; \forall n, $$

and so

$$ r(T|_{J''}) \leq r(P_{J^m} TP_{J^m}) < r^*(T) + \epsilon.$$  

Therefore

$$ \lambda_1 \leq r^*(T) + \epsilon.$$

The proof is complete. \hfill $\square$

5. **PDE examples.** The main concern of this section is the application of the abstract Krein-Rutman theory, in particular the theory associated with $r^*(T)$ and $r_*(T)$ (which is the aspect of Krein-Rutman theory that has not been used in PDE applications, as far as we know), on some PDE eigenvalue problems. In order to see the full potential of the abstract theory, we refrain from using PDE tools unnecessarily.

5.1. **Example 1.** Second order elliptic operators with Dirichlet boundary condition in non-smooth bounded domains.

The first paper that studies the principal eigenvalue of such operators in arbitrary bounded domains (without smoothness assumption on its boundary) is that of Berestycki, Nirenberg and Varadhan[4]. The operator has the form

$$ Lu = -a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + c(x) u $$

on an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$, where $a_{ij} \in C(\Omega)$, $b_i, c \in L^\infty(\Omega)$, and for some constant $C_0 > 0$,

$$ \frac{1}{C_0} |\eta|^2 \leq a_{ij}(x) \eta_i \eta_j \leq C_0 |\eta|^2, \quad \forall \eta = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n, \quad \forall x \in \Omega.$$
In the standard literature (see, e.g. Theorem 9.30 of [15]), the Dirichlet boundary value problem

\[
\begin{align*}
Lu &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega 
\end{align*}
\]  

(18)
is uniquely solvable in the class of \(W^{2,n}_{loc}(\Omega) \cap C(\bar{\Omega})\) if \(f \in L^n(\Omega), c \geq 0\) and \(\partial \Omega\) satisfies the exterior cone condition. When \(\partial \Omega\) does not satisfy any smoothness condition, [4] introduces a sense in which the Dirichlet boundary condition is satisfied: First, they prove the existence and uniqueness of a solution \(u_0 \in W^{2,n}_{loc}(\Omega) \cap L^\infty(\Omega)\) of

\[
Mu = 1 \quad \text{in } \Omega
\]

where \(Mu = Lu - c(x)u\), satisfying \(u_0 = 0\) on \(\partial \Omega\) in the sense that “\(u_0\) can be extended as a continuous function to every point \(y \in \partial \Omega\) ‘admitting a strong barrier’ by setting \(u_0(y) = 0\)” (for example, the exterior cone condition at \(y\)). Next, they introduce the notation \(\|x_j u_0\|_{\partial \Omega} \rightarrow 0\) if \(x_j \in \Omega \rightarrow \partial \Omega\) and \(u_0(x_j) \rightarrow 0\), and the notation \(u u_0 = 0\) on \(\partial \Omega\) if \(x_j u_0 \rightarrow \partial \Omega\) implies \(u(x_j) \rightarrow 0\).

Let

\[
\lambda_1 = \sup \{ \lambda \in \mathbb{R} \mid \exists \phi \in W^{2,n}_{loc}(\Omega) \text{satisfying } \phi > 0, L\phi \geq \lambda \phi \text{ in } \Omega \}
\]

(19)

**Theorem 5.1** (Theorems 1.1 and 1.2 of [4]). (i) \(\lambda_1 > 0 \Leftrightarrow \) the refined maximum principle holds, i.e. if \(w \in W^{2,n}_{loc}(\Omega)\) is bounded above, and

\[
\begin{align*}
\{ Lu &\leq 0 \quad \text{in } \Omega \\
\limsup_{j \rightarrow \infty} w(x_j) &\leq 0 \quad \text{for any } x_j \xrightarrow{u_0} \partial \Omega,
\end{align*}
\]

(20)

then \(w \leq 0\) in \(\Omega\).

(ii) Suppose \(\lambda_1 > 0\). Then \(\forall f \in L^n(\Omega), \exists!\) solution \(u \in W^{2,n}_{loc}(\Omega) \cap L^n(\Omega)\) of

\[
\begin{align*}
Lu &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(21)
satisfying \(\|u\|_{L^\infty(\Omega)} \leq A \|f\|_{L^n(\Omega)}, \) where \(A\) is a constant depending on \(\Omega, C_0, \lambda_1, \|b\|_{L^\infty(\Omega)}\) and \(\|c\|_{L^\infty(\Omega)}\).

Concerning the principal eigenvalue of \(L\), among other interesting results, the following are also proved in [4].

**Theorem 5.2.** (i) \(\exists \phi_1 \in W^{2,n}_{loc}(\Omega) \cap L^\infty(\Omega)\) such that \(0 < \phi_1 \leq Cu_0\) in \(\Omega\) for a constant \(C\), and

\[
\begin{align*}
L\phi_1 &= \lambda_1 \phi_1 \quad \text{in } \Omega, \\
\phi_1 &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(22)

(ii) \(\lambda_1\) is algebraically simple in the sense that it is geometrically simple, and that \(\nexists \psi \in W^{2,n}_{loc}(\Omega)\) and bounded above in \(\Omega\), such that

\[
\begin{align*}
(L - \lambda_1)\psi &= \phi_1 \quad \text{in } \Omega, \\
\psi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
(iii) Let \( \psi \in W^{2,n}_{\text{loc}}(\Omega) \cap L^n(\Omega) \) be a real eigenfunction with the corresponding eigenvalue \( \lambda \neq \lambda_1 \). Then \( \psi \) changes sign in \( \Omega \).

In proving Theorem 5.1 (i.e. Theorems 1.1 and 1.2 of [4]), the existence of \( \phi_1 \) in Theorem 5.2(i) is used in [4] (see §6 there). But one can replace \( \phi_1 \) by an appropriate test function \( \phi \) in the definition (19) of \( \lambda_1 \). Moreover, in proving Theorem 5.1, one does not need to know \( \lambda_1 \) is a principal eigenvalue. In a word, Theorem 5.1 does not logically rely on Theorem 5.2. Thus it makes sense to ask if it is possible to use the Krein-Rutman theory and, of course, the basic existence and the maximum principle Theorem 5.1 to prove the existence of the principal eigen-pair, and perhaps more. Using the weak Krein-Rutman theorem (Theorem 1.1), Birindelli [7] proves the existence of an eigenvalue \( \tilde{\lambda}_1 \) with a positive eigenfunction \( \tilde{\phi}_1 \); then relying on Theorem 5.2 (i) and Corollary 2.2 of [4], she concludes that \( \tilde{\lambda}_1 = \lambda_1 \) and \( \tilde{\phi}_1 \) is a constant multiple of \( \phi_1 \).

Here we want to push the Krein-Rutman approach further by using Theorems 1.5 and 4.1, obtaining not only the max-min characterization (19) and another min-max one for the principal eigenvalue, as well as (iii) and a slightly weaker version of (ii) of Theorem 5.2. We do so by using Theorem 5.1 only (and of course the usual strong maximum principle - see, e.g. Theorem 9.6 of [15]).

We start with the framework built by Birindelli, and then go further so we can use the Krein-Rutman theory for irreducible operators. Take a large constant \( k \) such that \( c(x) + k > 1 \) in \( \Omega \). Then \( \lambda_1 \) defined in (19) with \( L \) replaced by \( L + k \) is \( \geq 1 \) (take \( \phi \equiv 1 \)), and hence Theorem 5.1 applies, which leads us to define a linear bounded operator

\[
T : v \in L^n(\Omega) \mapsto u \in W^{2,n}_{\text{loc}}(\Omega) \cap L^\infty(\Omega) \subset L^n(\Omega)
\]

where \( u \) is the unique solution of

\[
\begin{align*}
(L + k)u &= v \quad \text{in } \Omega, \\
u &\equiv 0 \quad \text{on } \partial \Omega. 
\end{align*}
\]

Combining the refined maximum principle and the strong maximum principle, we also have that \( v \geq 0 \Rightarrow u \geq 0 \); \( v \geq, \neq 0 \Rightarrow u > 0 \) in \( \Omega \); in particular, \( T \) is positive. It is also proved in [7] that \( T \) is compact.

We now prove that \( T \) is ideal-irreducible. Let \( I \) be a \( T \)-invariant, closed lattice ideal in \( L^n(\Omega) \). By [30] (pp.157-158), \( \exists \) measurable subset \( S \subset \Omega \) such that \( I = \{ f \in L^n(\Omega) \mid f(x) = 0 \text{ a.e. on } \Omega \setminus S \} \). So \( T\chi_S \in I \) where \( \chi_S \) is the characteristic function of \( S \). If \( S \) has positive measure, then \( T\chi_S > 0 \) in \( \Omega \Rightarrow \Omega \setminus S = \emptyset \). Thus \( I \) has to be a trivial ideal in \( L^n(\Omega) \), and \( T \) is irreducible.

Now by Theorems 1.5 and 4.1, we have that \( \mu_0 := r(T) = r_\ast(T) = r^\ast(T) > 0 \), and is an eigenvalue with an eigenvector \( v_0 \in qit(P) \) (\( P \) set of non-negative functions in \( L^n(\Omega) \)); and that the algebraic multiplicity of \( \mu_0 \) is 1. Letting \( \lambda_1 = \frac{1}{\mu_0} - k \), \( \tilde{\phi}_1 = Tv_0 (> 0 \text{ in } \Omega) \), we see that \( (\lambda_1, \tilde{\phi}_1) \) is an eigen-pair of the eigenvalue problem

\[
\begin{align*}
L\phi &= \lambda\phi \quad \text{in } \Omega, \\
\phi &\equiv 0 \quad \text{on } \partial \Omega, \\
\phi &\in W^{2,n}_{\text{loc}}(\Omega) \cap L^n(\Omega). 
\end{align*}
\]
Let
\[
\lambda_1^* = \sup_{\phi \in W^{2,n}_{\text{loc}}(\Omega), \phi > 0 \text{ in } \Omega} \inf_{\Omega} \frac{L\phi(x)}{\phi(x)},
\]
\[
\lambda_{1*} = \inf_{\phi \in W^{2,n}_{\text{loc}}(\Omega), \phi > 0 \text{ in } \Omega} \sup_{\Omega} \frac{L\phi(x)}{\phi(x)}.
\] (25)

**Theorem 5.3.** \(\tilde{\lambda}_1 = \lambda_1^* = \lambda_{1*} = \lambda_1\).

**Remark 2.** The characterization of the principal eigenvalue by \(\lambda_1^*\) was not proved in [4] and [7].

**Proof.** Taking \(\phi = \tilde{\phi}_1\), we see \(\tilde{\lambda}_1 \leq \lambda_1^*\). If \(\tilde{\lambda}_1 < \lambda_1^*\), then \(\exists \phi \in W^{2,n}(\Omega) \cap L^n(\Omega)\) and \(\lambda_1 < \lambda := \inf_{\Omega} \frac{L\phi(x)}{\phi(x)}\), from which we have
\[
(L + k)\phi \geq (\lambda + k)\phi \text{ in } \Omega,
\]
and hence
\[
T(L + k)\phi \geq (\lambda + k)T\phi \text{ in } \Omega.
\]

Letting \(\psi = T(L + k)\phi\), we are led to
\[
\begin{cases}
(L + k)\psi = (L + k)\phi \text{ in } \Omega, \\
\psi \equiv 0 \text{ on } \partial\Omega \text{ and } 0 \leq \liminf_{x_j \to \partial\Omega} \phi(x_j).
\end{cases}
\]

From the refined maximum principle, we have
\[
\psi \leq \phi \text{ in } \Omega.
\]

Now
\[
\psi \geq (\lambda + k)T\phi \geq (\lambda + k)T\psi \text{ in } \Omega,
\]
and therefore
\[
\frac{1}{\lambda_1 + k} = \mu_0 = r^*(T) = \inf_{\mu \geq 0} \inf_{v \in \mathcal{P}} \{\mu \geq 0 \mid \mu v \geq Tv\} \leq \frac{1}{\lambda + k}.
\]

We reach \(\tilde{\lambda}_1 \geq \lambda\), which is impossible. We have shown \(\tilde{\lambda}_1 = \lambda_1^*\); similarly, we can show \(\tilde{\lambda}_1 = \lambda_{1*}\).

Now we show \(\lambda_1\) in (19) and \(\lambda_1^*\) in (25) are the same. Obviously \(\lambda_1^* \leq \lambda_1\). On the other hand, \(\forall \lambda < \lambda_1\), consider
\[
\begin{cases}
(L - \lambda)\phi = 1 \text{ in } \Omega, \\
\phi \equiv 0 \text{ on } \partial\Omega.
\end{cases}
\] (26)

Since \(\lambda_1^{L-\lambda}\), which is defined by (19) with “L” replaced by \(L - \lambda\), satisfies \(\lambda_1^{L-\lambda} > 0\), we can use Theorem 5.1 to obtain the existence and uniqueness of a solution \(\phi \in W^{2,n}_{\text{loc}}(\Omega) \cap L^n(\Omega)\), with \(\phi > 0\) in \(\Omega\). Using this \(\phi\) as a test function we have \(\lambda_1^* \geq \lambda\). Letting \(\lambda \nearrow \lambda_1\), we have \(\lambda_1^* \geq \lambda_1\).

**Proof of (ii) of Theorem 5.2 modified as:** \(\tilde{\lambda}_1\) is algebraically simple in the sense that it is geometrically simple, and that \(\tilde{\psi} \in W^{2,n}_{\text{loc}}(\Omega) \cap L^n(\Omega)\) (so by Theorem 5.1 (ii), \(\psi\) is bounded), satisfying (22) ( (ii) of Theorem 5.2 only requires that \(\psi\) be bounded above).
The geometric simplicity of \( \tilde{\lambda}_1 \) follows from that of \( \mu_0 \). And if \( \psi \) exists, then

\[
(L + k - (\tilde{\lambda}_1 + k))\psi = \tilde{\phi}_1,
\]

\[
(I - (\tilde{\lambda}_1 + k)T)\psi = T\tilde{\phi}_1 = \mu_0\tilde{\phi}_1,
\]

\[
(\mu_0 - T)\psi = \mu_0^2\tilde{\phi}_1.
\]

This contradicts the algebraic simplicity of \( \mu_0 \).

**Proof of (iii) of Theorem 5.2.** If \( (\lambda, \phi) \) is an eigen-pair of (24) with \( \phi > 0 \) in \( \Omega \), then \( \lambda = \tilde{\lambda}_1 \).

This follows immediately from the corresponding statement in Theorem 1.5.

5.2. **Example 2.** Second order elliptic operators with a nonlocal term and general boundary conditions in smooth domains.

This is an example that was studied in [22]. The operators have the form

\[
Lu = -a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + c(x)u + a(x) \int_{\Omega} b(y)u(y)dy
\]

in a \( C^2 \)-smooth bounded domain in \( \mathbb{R}^n \), and the boundary operators have the form

\[
Bu = u \quad \text{or} \quad Bu = \alpha_i(x) \frac{\partial u}{\partial x_i} + \beta(x)u, \quad x \in \Omega,
\]

where \( a_{ij} \in C(\Omega), \ b_i, c, a \in L^\infty(\Omega), \ b \in L^1(\Omega); \ a_{ij} = a_{ji}, \ (a_{ij})_{n \times n} \geq c_0 I_{n \times n} \) in \( \Omega \) for a constant \( c_0 > 0; \ \alpha_i, \beta \in C^1(\partial \Omega), \ (\alpha_1, \ldots, \alpha_n) \cdot \nu > 0 \) on \( \partial \Omega \), \( \nu \) is the unit outer normed vector on \( \partial \Omega \); \( a(x) \leq 0 \) and \( b(x) \geq 0 \) on \( \Omega \), or \( a(x) \geq 0 \) and \( b(x) \leq 0 \) on \( \Omega \).

**Without resorting to Krein-Rutman theory, [22] proves the existence, the algebraic simplicity and two min-max characteristics (in the fashion of Theorem 5.3) of the principal eigenvalue of \( (L, B) \).** Here, we use the strong Krein-Rutman theorem (the refined version, i.e. Theorem 5.3) to regain the results of [22] mentioned above.

It is well-known (see, for example, Theorem 2.7 of Shi-Wang [33]) that for any \( n < p < \infty \),

\[
\tilde{L} : W_B^{2,p}(\Omega) \to L^p(\Omega)
\]

is Fredholm of index 0, where \( W_B^{2,p}(\Omega) = \{ u \in W^{2,p}(\Omega) \mid Bu = 0 \} \), \( \tilde{L} \) is \( L \) with the integral term removed. Since the integral part is a compact operator, \( L \) is also Fredholm of index 0. Being such, for the boundary value problem for \( L \) and \( B \), the existence is equivalent to the uniqueness.

Take \( u_0 \in C^2(\Omega) \), such that \( u_0 \geq 1 \) in \( \Omega \) and \( Bu_0 \geq 1 \) on \( \partial \Omega \) (such \( u_0 \) can be constructed in the form \( 1 + g(x) \), where \( 1 \geq g \geq 0 \) in \( \Omega \) and \( \alpha \cdot \nabla g |_{\partial \Omega} \) is large).

Let

\[
\lambda_0 = \inf_{\Omega} \frac{Lu_0(x)}{u_0(x)}.
\]

We will apply

**Lemma 5.4** (Touching Lemma, Lemma 3.1 of [22]). Let \( \lambda \in \mathbb{R} \). Assume \( \exists u \in W^{2,p}(\Omega), \) such that \( u > 0, (L - \lambda)u \geq 0 \) in \( \Omega \), and \( Bu \geq 0 \) on \( \partial \Omega \). If \( \exists v \in W^{2,p}(\Omega) \) such that \( (L - \lambda)v \geq 0 \) in \( \Omega \) and \( Bv \geq 0 \) on \( \partial \Omega \), then \( v \) does not change sign on \( \Omega \).

Furthermore, if \( v < 0 \) somewhere in \( \Omega \), then \( (L - \lambda)u = 0 = (L - \lambda)v, \) \( Bu = 0 = Bv \), and \( v \) is a negative constant multiple of \( u \).

**Remark 3.** Results of this sort can be traced back at least to [38], where Walter studied operators without integral terms.
Let \( w \in W^{2,p}(\Omega) \) be a solution of
\[
\begin{align*}
(L - \lambda_0)w &= 0 \quad \text{in } \Omega, \\
Bw &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (27)

Applying Touching lemma with \( u = u_0, v = w \) and \( \lambda = \lambda_0 \), we see \( w \equiv 0 \). So the Fredholm Alternatives implies that \( \forall v \in L^p(\Omega), \exists ! \) solution \( u \in W^{2,p}(\Omega) \) of
\[
\begin{align*}
(L - \lambda_0)u &= v \quad \text{in } \Omega, \\
Bu &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (28)

Moreover, by Touching lemma, strong Maximum principle and Hopf boundary point lemma, if \( v \not\equiv 0 \) in \( \Omega \), then \( u > 0 \) in \( \Omega \) (in \( \bar{\Omega} \) in the Neumann/Robin B.C. case).

This leads us to define
\[
T : v \in X \mapsto u \in X,
\]
where
\[
X = \begin{cases} 
C_0^1(\bar{\Omega}) & \text{in Dirichlet B.C. case}, \\
C^1(\bar{\Omega}) & \text{in Neumann/Robin B.C. case}.
\end{cases}
\]

The positive cone \( P \) in \( X \) is defined to be
\[
P = \{ u \in X \mid u \geq 0 \text{ in } \bar{\Omega} \},
\]
with
\[
\text{int}(P) = \begin{cases} 
\{ u \in X \mid u > 0 \text{ in } \Omega, \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial \Omega \} & \text{in Dirichlet B.C. case}, \\
\{ u \in X \mid u > 0 \text{ in } \bar{\Omega} \} & \text{in Neumann/Robin B.C. case}.
\end{cases}
\]

Now \( T : X \to X \) is linear, compact and strongly positive, hence Theorem 1.2 and Theorem 2.1 apply. In the same way as in Example 1, we can proceed to reprove the following results in [22].

**Theorem 5.5.** (i) \( L \) with boundary operator \( B \) has an eigen-pair \( (\lambda_1, \phi_1) \), with \( \phi_1 \in W^{2,p}(\Omega), \phi_1 > 0 \text{ in } \Omega \) (in \( \bar{\Omega} \) in case of Neumann/Robin B.C.).
(ii) \( \lambda_1 \) can be characterized in two ways:
\[
\begin{align*}
\lambda_1 &= \sup_{u \in W^{2,p}(\Omega) \atop Bu \geq 0 \text{ on } \partial \Omega} \inf_{u > 0 \text{ in } \Omega} \frac{Lu(x)}{u(x)}, \\
\lambda_1 &= \inf_{u \in W^{2,p}(\Omega) \atop Bu \leq 0 \text{ on } \partial \Omega} \sup_{u > 0 \text{ in } \bar{\Omega}} \frac{Lu(x)}{u(x)}.
\end{align*}
\] (28)

(iii) The algebraic multiplicity of \( \lambda_1 \) is 1, when thinking of \( L \) as an operator from \( W^{2,p}_{B}(\Omega) \) to \( L^p(\Omega) \).
(iv) \( \lambda_1 \) is the only eigenvalue associated to a positive eigenfunction in \( \Omega \).
5.3. Example 3. Cooperative systems with Dirichlet boundary condition in bounded domains satisfying exterior cone condition.

In [35], Sweers studied the following eigenvalue problem for an elliptic system

\[
\begin{cases}
(L - H)u = \lambda u \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where

\[
L = \begin{bmatrix}
L_1 \\
\vdots \\
L_k
\end{bmatrix}, \quad L_\mu u_\mu = -a^\mu_{ij}(x) \frac{\partial^2 u_\mu}{\partial x_i \partial x_j} + b^\mu_i(x) \frac{\partial u_\mu}{\partial x_i} + c^\mu(x) u_\mu,
\]

\(\Omega\) is a bounded domain in \(\mathbb{R}^n\) satisfying the exterior cone condition; \(a^\mu_{ij} \in C(\Omega) \cap L^\infty(\Omega), \ b^\mu_i, c^\mu \in L^\infty(\Omega), \) and \(\exists\) constant \(c_0 > 0, \) such that \(a^\mu_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2, \) \(\forall x \in \Omega, \forall \xi \in \mathbb{R}^n; \ H = (h_{ij})\) is a \(k \times k\) matrix of functions, all in \(L^\infty(\Omega).\)

We shall always assume \(H\) is cooperative: the off-diagonal elements are \(\geq 0,\) and the diagonal elements \(\equiv 0\) in \(\Omega\) (the latter is assumed without loss of any generality).

Only at the end of this subsection, we shall also assume that \(H\) is “fully coupled”, i.e. \(H(x)\) is an irreducible matrix, which means that \(\nexists\) proper subsets \(I, J\) of \(\{1, 2, \cdots, k\}, \) such that \(I \cap J = \emptyset, I \cup J = \{1, 2, \cdots, k\}, \)

\[
h_{ij} \equiv 0 \quad \text{in } \Omega, \quad \forall i \in I, \forall j \in J.
\]

5.3.1. The case of not necessarily fully coupled \(H.\) Without assuming fully coupled \(H\) condition, we first establish the existence and uniqueness of

\[
\begin{cases}
(L + \beta - H)u = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where \(\beta\) is a large constant, \(f \in (L^n(\Omega))^k.\) (This follows from Remark 1.2 of [35] but our proof will be somewhat different.) We then use the inverse of \(L + \beta - H\) as the operator “T” to apply the abstract Theorem 4.5.

We shall use the Fredholm Alternatives to prove the existence of (31) by first proving the uniqueness. To that end, we remark the following version of the weak maximum principle holds. Suppose \(u \in (W^{2,n}_{\text{loc}}(\Omega))^k \cap (C(\Omega))^k\) satisfies

\[
\begin{cases}
(L + \beta - H)u \geq 0 \text{ in } \Omega, \\
u \geq 0 \text{ on } \partial \Omega,
\end{cases}
\]

where \(\beta\) is large enough such that

\[
\beta + c^\mu(x) - \sum_{i=1}^k h_{\mu i}(x) \geq 0 \quad \text{in } \Omega, \quad \forall \mu = 1, \cdots, k.
\]

Then \(u \geq 0 \) in \(\Omega.\)

In the classical setting when \(u\) is \(C^2\) smooth in \(\Omega,\) this was proved in [28]. In the current situation, we define \(M = \min\{u_\mu(x) \mid \mu = 1, \cdots, k; \ x \in \Omega\},\) and to argue by contradiction we assume \(M < 0.\) Then \(\exists \mu \) and \(x_0 \in \Omega\) such that \(u_\mu(x_0) = M.\)
Observe that
\[
(L + \beta - H) \begin{pmatrix}
  u_1 - M \\
  \vdots \\
  u_k - M
\end{pmatrix} \geq 0 \quad \text{in } \Omega,
\]
\[
(L + \beta) \begin{pmatrix}
  u_1 - M \\
  \vdots \\
  u_k - M
\end{pmatrix} \geq H \begin{pmatrix}
  u_1 - M \\
  \vdots \\
  u_k - M
\end{pmatrix} \geq 0 \quad \text{in } \Omega,
\]
hence for every \(\mu\), we have
\[
(L \mu + \beta)(u_\mu - M) \geq 0 \quad \text{in } \Omega.
\]

By the strong maximum principle for \(W^{2,n}_{loc}(\Omega)\) functions (Theorem 9.6 of [15]), we have \(u_\mu \equiv M < 0\), contradicting the boundary condition.

We now start to build a framework in which \(L + \beta - H\) is a Fredholm operator of index 0. First, using the existence theory for the scalar case (Theorem 9.30 of [15] or Theorem 1.2 of [4]), we see
\[
\begin{aligned}
(L + \beta)^{-1} : (L^n(\Omega))^k &\to (W^{2,n}_{loc}(\Omega) \cap C_0(\bar{\Omega}))^k \\
X &\mapsto (L + \beta)^{-1}X,
\end{aligned}
\]
is well-defined. Let \(X = (L + \beta)^{-1}(L^n(\Omega))^k\). Equip \(X\) with norm \(\|u\|_X = \|v\|_{(L^n(\Omega))^k}\), where \(u = (L + \beta)^{-1}v\) is the unique \((W^{2,n}_{loc}(\Omega) \cap C_0(\bar{\Omega}))^k\) solution of
\[
\begin{aligned}
(L + \beta)u &= v \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

By Theorem 1.2 of [4] which is included in Theorem 5.1 of this paper,
\[
\|u\|_{(L^{n}(\Omega))^k} \leq A \|v\|_{(L^{n}(\Omega))^k}
\]
for a constant \(A > 0\).

We claim that \(X\) is a Banach space. Let \(\{u_i\}_{i=1}^\infty\) be Cauchy in \(X\), i.e. \(\{v_i\}\) be Cauchy in \((L^n(\Omega))^k\), where \(v_i = (L + \beta)u_i\). Then \(\{u_i\}\) is Cauchy in \((C_0(\bar{\Omega}))^k\), and in \((W^{2,n}(K))^k\) for any sub-domain \(K\) compactly contained in \(\Omega\) (by the standard interior \(L^p\) estimates). So \(u_i \to u\) in \((C_0(\bar{\Omega}))^k\) and \((W^{2,n}_{loc}(\Omega))^k\); \(v_i \to v\) in \((L^n(\Omega))^k\). Now \(u_i \to u = (L + \beta)^{-1}v\) in \(X\), and hence \(X\) is Banach.

In the similar way, we show that \(X\) is compactly embedded into \((L^n(\Omega))^k\): Let \(\{u_i\}_{i=1}^\infty\) be bounded in \(X\), i.e. \(\{v_i\}\) be bounded in \((L^n(\Omega))^k\), where \(v_i = (L + \beta)u_i\). By the afore-mentioned \(L^p\) estimates, the continuous imbedding \(W^{2,n}(K)\) into \(C_0(K)\) \((0 < \alpha < 1)\) and Arzela-Ascoli Theorem, we have that, after passing to a subsequence, \(u_i \to u\) locally uniformly in \(\Omega\). This and (33) imply that \(u_i \to u\) in \((L^n(\Omega))^k\).

We think of \(L + \beta - H\) as an operator from \(X\) to \((L^n(\Omega))^k\); we write \(L + \beta - H = (L + \beta)(I_X - (L + \beta)^{-1}H)\). \(L + \beta : X \to (L^n(\Omega))^k\) is linear, bounded, 1-1 and onto. We claim that
\[
(L + \beta)^{-1}H : X \to X \text{ is compact}.
\]

Let \(\{u_i\}\) be bounded in \(X\), i.e. \(v_i = (L + \beta)u_i\) is bounded in \((L^n(\Omega))^k\). By (33), the interior \(L^p\) estimates and the compact imbedding \(W^{2,n}_{loc}(\Omega) \to C_{loc}(\Omega)\), we have that after passing to a subsequence, \(\forall \text{ subdomain } K\) compactly contained in \(\Omega\), \(u_i \to u\) in \((W^{2,n}(K))^k\) and \((C(K))^k\). Note that \(\{u_i\}\) is bounded in \((L^\infty(\Omega))^k\). Now \(u_i \to u\) in \((L^n(\Omega))^k\) \(\Rightarrow Hu_i \to Hu\) in \((L^n(\Omega))^k\), which is equivalent to \((L + \beta)^{-1}Hu_i \to (L + \beta)^{-1}Hu\) in \(X\), completing the proof of the compactness of
(L + \beta)^{-1}H. Therefore I_X - (L + \beta)^{-1}H is Fredholm of index 0, and consequently so is L + \beta - H : X \to (L^n(\Omega))^k. It follows that the latter is 1-1 and onto, by the uniqueness of (31).

Define T := (L + \beta - H)^{-1} : (L^n(\Omega))^k \to (L^n(\Omega))^k. It is positive by the weak maximum principle; it is compact because the imbedding X \hookrightarrow (L^n(\Omega))^k is compact. Moreover, by [30](p.92), L^n(\Omega) is a Banach lattice with order continuous norm. We are almost set to apply Theorem 4.5, except that we have to prove that the spectral radius r(T) > 0.

In fact, we shall prove more: \# non-zero band B which is T-invariant and T\mid_B is quasi-nilpotent. Suppose such a B exists. We use the notation: 0 < z \in (L^n(\Omega))^k if its components are nonnegative in \Omega and at least one of them is not identically equal to 0. Since B is non-zero, \exists 0 < z^0 = (z^0_1, \ldots, z^0_k) \in B. Without loss of generality, assume z^0_1 \geq 0 in \Omega. Define

\[ B_1 = \{ z_1 \in L^n(\Omega) \mid (z_1, 0, \ldots, 0) \in B \}. \]

Since z^0 \in B, which is an ideal, z^0_1 \in B_1, and hence B_1 \neq \emptyset; since B is a closed ideal, B_1 is also a closed ideal in L^n(\Omega). Being such, \exists measurable set S \subseteq \Omega such that B_1 = \{ z_1 \in L^n(\Omega) \mid z_1(x) = 0 \text{ a.e. on } \Omega \setminus S \} (see [30]). In particular, the characteristic function \chi_S \in B_1. Since B is T-invariant, u := T(\chi_S, 0, \ldots, 0) \in B. Observe that u \in (W^2_{loc}(\Omega) \cap C_0(\Omega))^k, and u \geq 0 in \Omega (because T is positive). So

\[
\begin{cases}
(L_1 + \beta)u_1 \geq \chi_S & \text{in } \Omega, \\
u_1 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By the maximum principle, u_1 > 0 in \Omega. Since u \in B, (u_1, 0, \ldots, 0) \in B \Rightarrow u_1 \in B_1 \Rightarrow \Omega \setminus S is of measure 0. Thus B_1 = L^n(\Omega) and so \phi_1^1 \in B_1, where \phi_1^1 is a positive eigenfunction of L_1 + \beta with Dirichlet boundary condition, with the corresponding eigenvalue \lambda_1^1 > 0. Observe (\phi_1^1, 0, \ldots, 0) \in B and

\[
(L + \beta - H) \begin{pmatrix} \phi_1^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1^1 \phi_1^1 \\ -h_{21} \phi_1^1 \\ \vdots \\ -h_{k1} \phi_1^1 \end{pmatrix} \leq \lambda_1^1 \begin{pmatrix} \phi_1^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ in } \Omega.
\]

\[
\frac{1}{\lambda_1} \begin{pmatrix} \phi_1^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leq T \begin{pmatrix} \phi_1^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ in } \Omega.
\]

Thus r(T\mid_B) \geq \frac{1}{\lambda_1} > 0. But T\mid_B is supposed to be quasi-nilpotent, a contradiction.

Now by Theorem 4.5,

\[ r_+(T) = \sup \sigma_+(T) = r(T) > 0, r_-(T) = \inf \sigma_+(T). \]

Slightly modifying the argument in Example 1, we obtain

**Theorem 5.6.** (i) L - H with Dirichlet boundary condition has an eigenvalue \lambda_1, pertaining to an eigenfunction \phi_1 \in (W^2_{loc}(\Omega) \cap C_0(\Omega))^k, whose components are nonnegative in \Omega.
Let \( \sigma_+ (L - H) \) be the set of eigenvalues of \( L - H \) (with Dirichlet boundary condition) associated with nonnegative eigenfunctions. Then

\[
\sup \sigma_+ (L - H) = \sup_{\phi \in U} \inf_{\Omega} \frac{(L - H)\phi(x)}{\phi(x)},
\]

\[
\inf \sigma_+ (L - H) = \inf_{\phi \in V} \sup_{\Omega} \frac{(L - H)\phi(x)}{\phi(x)},
\]

where \( U = \{ \phi \in (W_{loc}^{2,n}(\Omega))^k \mid \phi \geq 0 \text{ in } \Omega \}, V = \{ \phi \in (W_{loc}^{2,n}(\Omega))^k \mid \phi \geq 0 \text{ in } \Omega \}, \phi, L\phi \in (L^n(\Omega))^k, \phi|_{\partial \Omega} = 0 \}, \inf_{\Omega} \frac{(L - H)\phi(x)}{\phi(x)} \text{ is understood as } \sup_{\lambda \in \mathbb{R}} (L - H)\phi \geq \lambda \phi \text{ in } \Omega \}
\]

and \( \sup_{\Omega} \frac{(L - H)\phi(x)}{\sigma(\phi)} \text{ as } \inf_{\lambda \in \mathbb{R}} \{ \lambda \in \mathbb{R} \mid (L - H)\phi \leq \lambda \phi \text{ in } \Omega \}. \)

**Remark 4.** Note here we do not assume the fully coupledness of \( H \). To our knowledge, (ii) is new. (i) was proved by Sweers [35].

**Remark 5.** We claim that \( \sigma_+ (L - H) \) is a finite set and that we have upper bound estimate

\[
\inf \sigma_+ (L - H) \leq \max \{ \lambda_1(L_1), \cdots, \lambda_1(L_k) \},
\]

where \( \lambda_1(L_\mu) \) is the principal eigenvalue of \( L_\mu \) under Dirichlet boundary condition. To prove this claim, observe that

\[
\sigma_+ (L - H) = \frac{1}{\sigma_+(T)} - \beta.
\]

Since the nonzero eigenvalues of the compact operator \( T \) are isolated, so are the eigenvalues in \( \sigma_+ (L - H) \); on the other hand, by taking test functions in (34) we have upper and lower bounds for \( \sigma_+ (L - H) \). In particular, taking the test function \( \phi = (0, \cdots, 0, \phi_i, 0, \cdots, 0) \), where \( \phi_i \) is the positive principal eigenfunction pertaining to \( \lambda_1(L_\mu) \), we have (35).

**5.3.2. The case of coupled \( H \).** From now on in this example, we assume that \( H(x) \) is fully coupled. Then by Remark 1.1 of [35] (involving the strong maximum principle and the irreducibility nature of \( H \)), if \( f \geq 0 \text{ in } \Omega \), then every component of the unique solution of (31) is positive in \( \Omega \). This implies that \( T : (L^n(\Omega))^k \to (L^n(\Omega))^k \) is, in addition to being positive and compact, irreducible. Consequently, Theorem 1.5 applies and we have, after slightly modifying the argument in Example 1, the following

**Theorem 5.7.** Let \( H \) be cooperative and fully coupled. Then (i) \( L - H \) with Dirichlet boundary condition has an algebraically simple eigenvalue \( \lambda_1 \) pertaining to an eigenfunction \( \phi_1 \in (W_{loc}^{2,n}(\Omega) \cap C_0(\Omega))^k \), whose components are positive in \( \Omega \); (ii) any eigenvalue of \( L - H \) with Dirichlet boundary condition pertaining to a nonnegative eigenfunction is equal to \( \lambda_1 \) (i.e., \( \sigma_+ (L - H) = \{ \lambda_1 \} \)); (iii) \( \lambda_1 \) is characterized by (34).

**Remark 6.** In [35], Proposition 3.1, the geometric simplicity but not the algebraic simplicity of \( \lambda_1 \) was mentioned; a characterization of \( \lambda_1 \) similar to the first one in (34) was also proved, but the second one was not. In [8], a weighted eigenvalue problem for cooperative elliptic systems was studied on a general bounded domain, which covers our eigenvalue problem, the existence of a positive strong supersolution was assumed but that is not needed for our eigenvalue problem, their results cover
(i) and (ii), and a characterization of the principal eigenvalue similar to the first one in (34), but not the second one.

5.4. Example 4. Cooperative systems with mixed boundary condition in bounded smooth domains.

Let \( L \) and \( H \) be given as in Example 3, but now \( a_{ij} \in C(\bar{\Omega}) \), and \( \Omega \) is a bounded domain with \( C^2 \)-smooth boundary (so Neumann/Robin boundary conditions can be defined). The boundary operator is given as

\[
Bu = (B_1 u_1, \cdots, B_k u_k),
\]

where either \( B_\mu u_\mu = u_\mu \) or

\[
B_\mu u_\mu = \alpha_\mu^\nu(x) \frac{\partial u}{\partial x_i} + \beta_\mu^\nu(x) u
\]

with \( \alpha_\mu^\nu, \beta_\mu^\nu \in C^1(\partial \Omega) \), \( (\alpha_1^\nu, \cdots, \alpha_k^\nu) \cdot \nu > 0 \) on \( \partial \Omega \) (\( \nu \) is the unit outer normed vector of \( \partial \Omega \)).

We modify the arguments in Example 2 as follows.

(a) Fix \( p > n \). It is well-known that for large constant \( \beta \)

\[
L + \beta - H : (W^{2,p}_B(\Omega))^k \to (L^p(\Omega))^k
\]

is an isomorphism (see [1], [14] and [33]).

(b) The Touching Lemma in Example 2 can be modified as follows.

**Lemma 5.8** (Touching Lemma). Let \( \lambda \in \mathbb{R} \). Suppose \( \exists u \in (W^{2,p}(\Omega))^k \) such that \( u > 0 \), \( (L - \lambda - H)u \geq 0 \) in \( \Omega \) and \( Bu \geq 0 \) on \( \partial \Omega \). If \( \exists v \in (W^{2,p}(\Omega))^k \) such that \( (L - \lambda - H)v \geq 0 \) in \( \Omega \) and \( Bv \geq 0 \) on \( \partial \Omega \), then, without assuming the fully coupled condition on \( H \), we have that either \( v \geq 0 \) in \( \Omega \) or \( \exists \mu \in \{1, \cdots, k\} \) and a \( \bar{\delta} > 0 \) such that \( u_\mu + \bar{\delta} v_\mu \equiv 0 \) in \( \Omega \) and consequently

\[
(L_\mu - \lambda)u_\mu - \sum_{i=1}^k h_{\mu i}(x)u_i(x) \equiv 0 \equiv (L_\mu - \lambda)v_\mu - \sum_{i=1}^k h_{\mu i}(x)v_i(x) \quad \text{in } \Omega,
\]

\[
B_\mu u_\mu \equiv 0 \equiv B_\mu v_\mu \quad \text{on } \partial \Omega.
\]

If we impose the fully coupled condition on \( H \), we have that either \( v \geq 0 \) in \( \Omega \) or \( u + \bar{\delta} v \equiv 0 \) in \( \Omega \).

\[
(L - \lambda - H)u \equiv 0 \equiv (L - \lambda - H)v \quad \text{in } \Omega, \quad Bu \equiv 0 \equiv Bv \quad \text{on } \partial \Omega.
\]

This can be proved by slightly modifying the proof Lemma 3.1 of [22]. As in there, if \( v \not\equiv 0 \) in \( \Omega \), then define \( \delta = \sup \{ \delta > 0 \mid u + \delta v > 0 \text{ in } \Omega \} \). Then \( 0 < \delta < \infty \) and \( \exists \mu \in \{1, \cdots, k\} \) such that \( u_\mu + \delta v_\mu = 0 \) somewhere in \( \Omega \), and other components of \( u + \delta v \geq 0 \) in \( \Omega \). Then

\[
(L_\mu - \lambda)(u_\mu + \delta v_\mu) \geq \sum_{j \neq \mu} h_{\mu j}(x)(u_j + \delta v_j) \geq 0 \quad \text{in } \Omega, \quad B_\mu(u_\mu + \delta v_\mu) \geq 0 \quad \text{on } \partial \Omega.
\]

So the strong maximum principle and the Hopf boundary point lemma leads to \( u_\mu + \delta v_\mu \equiv 0 \) in \( \Omega \). If we also have the fully coupledness of \( H \), by Remark 1.1 in [35], all the other components must also satisfy this equation and hence \( u + \delta v \equiv 0 \) in \( \Omega \).

(c) Like in Example 2, we can construct \( u_0 \in (C^2(\bar{\Omega}))^k \) such that each component of \( u_0 \) is \( \geq 1 \) in \( \Omega \), \( Bu_0 \geq (1, \cdots, 1) \) on \( \partial \Omega \). Define

\[
\lambda_0 = \inf_{\Omega} \frac{(L - H)u_0(x)}{u_0(x)}.
\]
Then using Touching Lemma, we see
\[ L - \lambda_0 - H : (W_B^{2,p}(\Omega))^k \to (L^p(\Omega))^k \]
is 1-1 and hence by (a) it is onto.

(d) Now we are ready to apply Theorem 4.5 in the case of not fully coupled \( H \) as in Example 3, and Theorem 2.1 in the case of fully coupled \( H \) as in Example 2, we obtain the following theorems.

**Theorem 5.9.** Suppose \( H \) is cooperative. Then

(i) \( L - H \) with boundary operator \( B \) has an eigenvalue \( \lambda_1 \), pertaining to an eigenfunction \( \phi_1 \in (W_B^{2,p}(\Omega))^k \) \((\forall n < p < \infty)\), whose components are nonnegative in \( \Omega \).

(ii) Let \( \sigma_+(L - H) \) be the set of eigenvalues of \( L - H \) (with boundary operator \( B \)) associated with nonnegative eigenfunctions in \( (W_B^{2,p}(\Omega))^k \). Then

\[
\sup_{\phi \in (W_B^{2,p}(\Omega))^k} \inf_{B\phi \geq 0 \text{ on } \partial \Omega} \frac{(L - H)\phi(x)}{\phi(x)} = \sup_{\phi \in (W_B^{2,p}(\Omega))^k} \inf_{B\phi \leq 0 \text{ on } \partial \Omega} \frac{(L - H)\phi(x)}{\phi(x)},
\]

where \( \inf_{\Omega} \frac{(L - H)\phi(x)}{\phi(x)} \) is understood as \( \sup \{ \lambda \in \mathbb{R} \mid (L - H)\phi \geq \lambda \phi \text{ in } \Omega \} \), and \( \sup_{\Omega} \frac{(L - H)\phi(x)}{\phi(x)} \) as \( \inf \{ \lambda \in \mathbb{R} \mid (L - H)\phi \leq \lambda \phi \text{ in } \Omega \} \).

**Remark 7.** To our knowledge, (ii) is new; but (i) was proved by Lam and Lou [21], where they used the weak Krein-Rutman theorem in an indirect way.

**Theorem 5.10.** Suppose \( H \) is cooperative and fully coupled. Then

(i) \( L - H \) with boundary operator \( B \) has an algebraically simple eigen-pair \( (\lambda_1, \phi_1) \), with \( \phi_1 \in (W_B^{2,p}(\Omega))^k \), and all components of \( \phi_1 > 0 \) in \( \Omega \) (in \( \bar{\Omega} \) in case of Neumann/Robin B.C.).

(ii) \( \lambda_1 \) can be characterized in two ways by (36).

(iii) \( \lambda_1 \) is the only eigenvalue associated to a nonnegative eigenfunction in \( \Omega \).

**Remark 8.** As recorded in Sweers[35], Bandle had remarked privately that Sweers’s results also hold for the case of Neumann/Robin boundary conditions; so in that sense (i)(short of the algebraic simplicity) and (iii) were known, at the time of publication of [35]; but (ii) is new.

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