A survey of known results on the \( m \)-step solvable anabelian geometry for hyperbolic curves

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Abstract

In this survey, we introduce the three theorems about the \( m \)-step solvable Grothendieck conjecture in anabelian geometry of hyperbolic curves by H. Nakamura, S. Mochizuki, and the author. We also give sketches of the proofs of these theorems.

Introduction

From now on, we fix the following notation. Let \( m \) be an integer greater than or equal to 1, \( k \) a field of characteristic \( p \geq 0 \), \( X \) (resp. \( X_1, X_2 \)) a proper, smooth curve over \( k \) (in other words, \( X \) (resp. \( X_1, X_2 \)) is geometrically connected, proper, smooth of dimension one over \( k \)), and \( E \) (resp. \( E_1, E_2 \)) a closed subscheme of \( X \) (resp. \( X_1, X_2 \)) which is finite, étale over \( k \). We set \( U := X - E \) (resp. \( U_1 := X_1 - E_1 \), \( U_2 := X_2 - E_2 \)) and call \( U \) (resp. \( U_1 \), \( U_2 \)) a smooth curve over \( k \). We write \( g \) (resp. \( g_1 \), \( g_2 \)) for a geometric genus of \( X_k \) (resp. \( X_1^k \), \( X_2^k \)), and set \( r := |E(k)| \) (resp. \( r_1 := |E_1(k)| \), \( r_2 := |E_2(k)| \)). We say that the smooth curve \( U \) is hyperbolic if \( 2 - 2g - r < 0 \).

Let \( K(U_{k_{sep}}) \) be the function field of \( U_{k_{sep}} \), \( \Omega \) an algebraically closed field containing \( K(U_{k_{sep}}) \), \( \varpi : \text{Spec}(\Omega) \to U_{k_{sep}} \) the corresponding geometric point, \( \Sigma \) a set of primes that contains at least one prime different from \( p \), and \( \Sigma^1 := \Sigma - \{ p \} \). We set

\[ \Pi_U := \pi_1^{tame}(U, \varpi), \quad \text{and} \quad \Pi_U^{(m)} := \pi_1^{tame}(U_{k_{sep}}, \varpi). \]

For a profinite group \( G \), we write \( G^\Sigma \) for the maximal pro-\( \Sigma \) quotient of \( G \), and \( [G, G] \) for the closed subgroup of \( G \) which is (topologically) generated by the commutator subgroup of \( G \). We set \( G^{[0]} := G \), \( G^{[m]} := [G^{[m-1]}, G^{[m-1]}] \), \( G^{0} := G/G^{[0]} \), and \( G^{m} := G/G^{[m]} \). We set \( G^{m, \Sigma} := (G^\Sigma)^m \) for simplicity. We define

\[ \Pi_U^{(m)} := \Pi_U / \Pi_U^{(m)}, \quad \Pi_U^{(m, \Sigma)} := \Pi_U / \text{Ker}(\Pi_U \to \Pi_U^{(m, \Sigma)}). \]

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Π_U^{(0)} := Π_U/Π_U^{(0)}(≃ G_k), and Π_U^{(0,Σ)} := Π_U/Ker(Π_U → Π_U^{(0,Σ)}(≃ G_k)). When Σ = {ℓ}, we write Π_U^{pro-ℓ}, Π_U^{m,pro-ℓ}, Π_U^{(m,pro-ℓ)} instead of Π_U^{(0)}, Π_U^{(m,Σ)}, Π_U^{(m,Σ)}, respectively. When Σ is the set of all primes different from p, we write Π_U^{(p)}, Π_U^{(m,p)}, Π_U^{(m,p)} instead of Π_U^{(0)}, Π_U^{(m,Σ)}, Π_U^{(m,Σ)}, respectively. For each quotient Π_U → Q which satisfies Π_U ⊂ Ker(Π_U → Q), we write pr : Q → G_k for the natural projection.

We write “FF”, “NF”, “FGF”, “FGF∞” for “finite field”, “number field”, “field finitely generated over the prime field”, “finite field finitely generated over the prime field”, respectively. Moreover, we write “SSF” for “sub ℓ-adic field” (i.e., a subfield of a finitely generated field extension of Q_ℓ).

In anabelian geometry, we have a fundamental conjecture called the (weak) Grothendieck conjecture, which predicts: if a G_k-isomorphism Π_U₁ ≃ Π_U₂ exists, a k-isomorphism U₁ ≃ U₂ exists (we only consider the case of p = 0 for simplicity). About the Grothendieck conjecture, we already have many results, e.g., [5], [8], [2], [7]. We can consider a variant of the Grothendieck conjecture by replacing Π_U with Π_U and Π_U with Π_U instead of Π_U and Π_U, respectively. Moreover, we write Π_U for “finite field” (i.e., a subfield of a finitely generated field extension of Q_ℓ).

When m ≥ 2, and k is an NF that satisfies one of the following (a)-(c).

(a) k is the rational number field Q.

(b) k is a quadratic field F ≠ Q(√2).

(c) There exists a prime ideal p of O_k unramified in k/Q such that |O_k/p| = 2.

Then the m-step solvable Grothendieck conjecture for 4-projected projective lines over k holds.

Theorem 0.1 [4] Theorem A, see Theorem 1.1. Assume that m ≥ 2, and k is an NF that satisfies one of the following (a)-(c).

1. When m ≥ 2, (g, r) = (0, 4) and k is an NF with certain conditions

Through this section, we assume that k is an NF, and write O_k for the ring of integers of k. In this section, we introduce the paper [4]. In [4], Nakamura exploited the theory of weights and proved the following theorem.

Theorem 1.1 [4] Theorem A. Let λ_i ∈ k \{0, 1\} and set Λ_i = {0, 1, ∞, λ_i}, for each i = 1, 2. Assume that m ≥ 2, and k satisfies one of the following (a)-(c).
(a) \( k \) is the rational number field \( \mathbb{Q} \).
(b) \( k \) is a quadratic field \( \neq \mathbb{Q}(\sqrt{2}) \).
(c) There exists a prime ideal \( \mathfrak{p} \) of \( O_k \) unramified in \( k/\mathbb{Q} \) such that \( |O_k/\mathfrak{p}| = 2 \).

Then the following holds.

\[
\Pi_{\mathfrak{p}^m}^{(m)} \simeq \Pi_{\mathfrak{p}^{m-1}}^{(m-1)} \quad \text{if} \quad \mathfrak{p} \mathfrak{A} \cap \Pi_{\mathfrak{p}^{m-1}}^{(m-1)} = \Pi_{\mathfrak{p}^{m-1}}^{(m-1)} \Rightarrow \mathfrak{p}_{\mathfrak{p}^{m-1}}^{(m-1)} - \mathfrak{A} \simeq \Pi_{\mathfrak{p}^{m-1}}^{(m-1)} - \mathfrak{A}.
\]

In other words, the \( m \)-step solvable Grothendieck conjecture for 4-punctured projective lines over \( k \) holds.

We will see the proof of Theorem [1.1] in this section. We divide the proof into the following three steps.

Step 1: Let \( \Lambda \) be a finite subset of \( k \cup \{ \infty \} \) which satisfies \( \{0, 1, \infty \} \subset \Lambda \) and \( |\Lambda| \geq 4 \). We define \( \Gamma(\Lambda) \) as the subgroup of \( k^\times \) generated by \( \{ \lambda - \lambda' \mid \lambda, \lambda' \in \Lambda \} \). We will show that, for all \( N \geq 1 \), the field \( k(\Gamma(\Lambda)^{\text{sep}}) \) is reconstructed group-theoretically from \( \Pi_{\mathfrak{p}^{2}}^{(2)} \) (see Proposition [1.3]), where \( \Gamma(\Lambda)^{\text{sep}} \) stands for the set of all elements of \( k^{\times} \), whose \( N \)-th power is contained in \( \Gamma(\Lambda) \). This step corresponds to [4] section 2.

Step 2: For each \( i = 1, 2 \), let \( \Lambda_i \) be a finite subset of \( k \cup \{ \infty \} \) which satisfies \( \{0, 1, \infty \} \subset \Lambda_i \) and \( |\Lambda_i| \geq 4 \). In this step, we will show that, if \( k(\Gamma(\Lambda_1)^{\text{sep}}) = k(\Gamma(\Lambda_2)^{\text{sep}}) \) holds for all primes \( \ell \) and all integers \( n \geq 0 \), then \( \Gamma(\Lambda_1) = \Gamma(\Lambda_2) \) holds (see Proposition [1.4]). This step corresponds to [4] section 3.

Step 3: For each \( \lambda \in k - \{0, 1\} \), we define the set:

\[
J(\lambda) = \left\{ \lambda, 1/\lambda, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda - 1}{\lambda} \right\}.
\]

Let \( \lambda_i \in k - \{0, 1\} \). The curves \( \mathbb{P}_k^1 - \{0, 1, \infty, \lambda_i\} \) and \( \mathbb{P}_k^1 - \{0, 1, \infty, \lambda_j\} \) are isomorphic over \( k \) if and only if \( J(\lambda_i) = J(\lambda_j) \). In this step, we find out a certain relation between the equalities \( J(\lambda_1) = J(\lambda_2) \) and \( \Gamma(\{0, 1, \infty, \lambda_1\}) = \Gamma(\{0, 1, \infty, \lambda_2\}) \). In particular, we will prove the main theorem of this section. This step corresponds to [4] section 4.

**Step 1**

Let \( \ell \) be a prime. First, we introduce the weight filtration of \( \prod_{U}^{\text{pro-}\ell} \). Let \( \mathbb{Z}[E(k)] \) be the free \( \mathbb{Z} \)-module generated by \( E(k) \) and regard it as a \( G_k \)-module via the \( G_k \)-action on \( E(k) \). We have the following isomorphism and exact sequence of \( G_k \)-modules.

\[
\begin{align*}
\prod_{U}^{\text{pro-}\ell} &\simeq T_{\ell}(J_X) \\
0 &\rightarrow \mathbb{Z}(1) \rightarrow \mathbb{Z}[E(k)] \otimes_{\mathbb{Z}} \mathbb{Z}(1) \rightarrow \prod_{U}^{\text{pro-}\ell} \rightarrow T_{\ell}(J_X) \rightarrow 0 \\
&\left( r = 0 \right) \quad \left( r \neq 0 \right).
\end{align*}
\]

Here \( T_{\ell}(J_X) \) stands for the \( \ell \)-adic Tate module of the Jacobian variety \( J_X \) of \( X \). The \( G_k \)-actions on \( \mathbb{Z}[E(k_{\text{sep}})] \otimes_{\mathbb{Z}} \mathbb{Z}(1) \) and \( T_{\ell}(J_X) \) have weights \(-2\) and \(-1\) (see [4] section 2). The same assertion is true if \( k \) is an \( FGF \) and \( \ell \neq p \), see [9] subsection 1.3.

For an open subgroup \( H \subset \Pi_{U}^{(n)} \) containing \( \Pi_{U}^{(n-1)} \cap \Pi_{U}^{(n)} \), let \( \text{W}_{-2}(\Pi_{U}^{(1)}) \) be the unique maximal \( \text{pr}(H) \) submodule of \( \Pi_{U}^{(1)} \) of weight \(-2\) (see [4] (2.1)Proposition). We have the Galois representation \( \phi_{H}^{(\ell)} : \text{pr}(H) \rightarrow \text{Aut}(\Pi_{U}^{(1)} \otimes_{\mathbb{Z}} \mathbb{Z}(1)) \) for all primes \( \ell \). Using these Galois representations, we define the strong rigidity invariant. (In [4], the strong rigidity invariant is defined for an arbitrary \( G_k \)-augmented profinite group (see [4] section 2), but we only consider the strong rigidity invariant for \( \Pi_{U}^{(2)} \).

**Definition 1.2** (cf. [4] (2.2)Definition). \( \Pi_{U}^{(n)} \) is a Galois \( G \)-model of \( H \) if \( H \cap \Pi_{U}^{(n)} = H \), \( \text{pr}(H) = G \), and \( \text{pr}^{-1}(G) \supset H \).
(2) Let \( N \in \mathbb{Z}_{\geq 1} \), and \( \prod_U^{N,\text{-th}} \) the set of the \( N \)-th powers of all elements of \( \prod_U^{1} \). We define the \emph{strong rigidity invariant} \( \kappa_N = \kappa_N(\Pi_U^{(2)}) \) to be the subfield of \( \mathbb{F} \) consisting of the elements fixed by all the automorphisms of \( \mathbb{F} \) belonging to

\[
\bigcup_{\ell \text{ prime}} \bigcup_{H \in \mathcal{H}_N^\ell} \text{Ker}(\phi_H^{(\ell)})
\]

where \( \mathcal{H}_N \) is the set of all open subgroups \( H \) of \( \Pi_U^{(2)} \) containing \( \prod_U^{[1]} / \prod_U^{[2]} \) such that the image of \( H \) in \( \Pi_U^{(1)} \) is a Galois \( G_{k(\mu_n)} \)-model of \( \prod_U^{1, N, \text{-th}} \).

**Proposition 1.3** (cf. [1] (2.9) Theorem, [1] (2.10) Corollary). Let \( \Lambda \) be a finite subset of \( k \cup \{\infty\} \) which satisfies \( \{0, 1, \infty\} \subset \Lambda \) and \( |\Lambda| \geq 4 \). For each \( N \in \mathbb{Z}_{\geq 1} \), the following holds:

\[
\kappa_N(\Pi_U^{(2)}_{p^k - \Lambda}) = k(\Gamma(\Lambda)^{\wedge})
\]

**Sketch of Proof.** Set \( \tilde{k} = k(\mu_N) \). We fix \( H \in \mathcal{H}_N \). For a closed subgroup \( W \) of \( \Pi_U^{(2)}_{p^k - \Lambda} \), let \( U_W \) be the cover of \( \mathbb{P}^1_k - \Lambda \) corresponding to \( W \subset \Pi_U^{(2)}_{p^k - \Lambda} \), and \( E_W \) the inverse image of \( E \) by the natural projection \( U_W^{\text{pt}} \to X \). By [1] (2.7) Proposition, the fixed field of \( \text{Ker}(\phi_H^{(\ell)}) \) coincides with \( \mathbb{Q}(\mu_\ell) \cdot \tilde{k}(E_H) \). Thus, first, we consider \( \tilde{k}(E_H) \). We have that \( K(U_{\prod_U^{[1, N, \text{-th}}}}) \) coincides with \( \mathbb{F}(t - \lambda)^{\wedge} \mid \lambda \in \Lambda - \{\infty\} \) (= the composite field of \( \{\tilde{k}(t - \lambda)^{\wedge}\}_{\lambda \in \Lambda - \{\infty\}} \)). Then we obtain that \( K(U_H) \) coincides with the composite field of \( \{\tilde{k}(t - \lambda)^{\wedge}\} \cap K(U_H) \}_{\lambda \in \Lambda - \{\infty\}} \). Since \( \tilde{k}(t - \lambda)^{\wedge} \cap K(U_H) \) is a \( \mathbb{Z}/N\mathbb{Z} \)-cover of \( \mathbb{P}^1_k - \{\lambda, \infty\} \), there exists \( \lambda_\ell \in \mathbb{F}^\times \) such that:

\[
\mathbb{F}(t - \lambda)^{\wedge} \cap K(U_H) = \tilde{k}(\epsilon_{\lambda}(t - \lambda)^{\wedge})
\]

by Kummer theory. Hence we obtain that \( \tilde{k}(E_H) = \tilde{k}(\epsilon_{\lambda}(\lambda - \lambda'))^{\wedge} \mid \lambda, \lambda' \in \Lambda \). Since \( \tilde{k}(E_H) \supseteq (\epsilon_1(0 - 1))^{\wedge}, (\epsilon_0(1 - 0))^{\wedge}, (\epsilon_0(\lambda - \lambda))^{\wedge} = (-\epsilon_0)^{\wedge}, (\epsilon_1(\lambda - \lambda))^{\wedge} = (-\epsilon_1)^{\wedge} \), we get \( \tilde{k}(E_H) \cap k(\Gamma(\lambda)^{\wedge}) \). There exists an open subgroup \( H_0 \) of \( \Pi_U^{(2)} \) containing \( \prod_U^{[1]} / \prod_U^{[2]} \) such that the image of \( H_0 \) in \( \Pi_U^{(1)} \) is a Galois \( G_{k(\mu_n)} \)-model of \( \prod_U^{1, N, \text{-th}} \) and that \( \epsilon_{\lambda} = 1 \) for all \( \lambda \in \Lambda - \{\infty\} \). We have \( \tilde{k}(E_{H_0}) = k(\Gamma(\Lambda)^{\wedge}) \).

Thus, we obtain

\[
\kappa_N(\Pi_U^{(2)}_{p^k - \Lambda}) = \bigcap_{\ell}(\mathbb{Q}(\mu_\ell) \cdot k(\Gamma(\Lambda)^{\wedge})) = k(\Gamma(\Lambda)^{\wedge}).
\]

\[\square\]

**Step 2**

**Proposition 1.4** (cf. [1] (3.1) Lemma). Let \( k' \) be a field finitely generated over \( \mathbb{Q} \), and \( \Gamma_1, \Gamma_2 \) finitely generated subgroups of \( k'^{\times} \). If \( k'(\Gamma_1^{\wedge}) = k'(\Gamma_2^{\wedge}) \) for all \( n \in \mathbb{Z}_{\geq 0} \) and all primes \( \ell \), then \( \Gamma_1 = \Gamma_2 \).

**Sketch of Proof.** We may replace \( k' \) with a field finitely generated over \( k' \), hence we may assume that \( k' \supseteq \mu_4 \). Replacing \( \Gamma_2 \) with \( \Gamma_1 \Gamma_2 \) if necessary, we may assume \( \Gamma_1 \subset \Gamma_2 \). Let \( \gamma \in \Gamma_2 \). We have \( k'(\mu_\ell \cdot \gamma^{\wedge}) \subset k'(\Gamma_2^{\wedge}) = k'(\Gamma_1^{\wedge}) \) and then \( \gamma \in \Gamma_1 \cdot k'(\mu_\ell \cdot \gamma^{\wedge}) \) by Kummer theory. By [1] Lemma 2.1.3 (where we need the assumption \( k' \supseteq \mu_4 \)), we obtain \( \gamma \in (\Gamma_1 \cdot k'(\mu_\ell \cdot \gamma^{\wedge}) \cap k') = \Gamma_1 \cdot k'^{\times} \). Let \( R \) be the integral closure of \( \mathbb{Z}[\Gamma_2] \) in \( k' \). We have that \( R^{\times} \) is a finitely generated \( \mathbb{Z} \)-module because \( R \) is finitely generated over \( \mathbb{Z} \). We have \( \gamma \in \Gamma_1 \cdot R^{\times} \) for all \( n \in \mathbb{Z}_{\geq 0} \) and all primes \( \ell \). Thus, we obtain \( \gamma \in \Gamma_1 \).

**Step 3**

In this step, we find out a relation between the equalities \( J(\lambda_1) = J(\lambda_2) \) and \( \Gamma(\{0, 1, \infty, \lambda_1\}) = \Gamma(\{0, 1, \infty, \lambda_2\}) \), and show Theorem 1.1. We write \( \mathcal{E} \) for the set of all elements \( e \in \mathbb{Z}^\times \) which satisfies \( (1 - e) \mid 2 \).

**Proposition 1.5** (cf. [1] (4.7) Theorem). Let \( \lambda_1, \lambda_2 \in k - \{0, 1\} \). Suppose that:

(i) \( \Gamma(\{0, 1, \infty, \lambda_1\}) = \Gamma(\{0, 1, \infty, \lambda_2\}) \),
(ii) $J(\lambda_1)$ contains no elements of $\mathcal{E}$.

Then $J(\lambda_1) = J(\lambda_2)$.

**Sketch of Proof.** Since $\Gamma(\{0, 1, \infty, \lambda_1\}) = -1, \lambda_1, 1 - \lambda_1\}$, we obtain

(a) $(-1, \lambda_1, 1 - \lambda_1) = (-1, \lambda_2, 1 - \lambda_2)$

by (i). When $J(\lambda_1) \cap O_k^\times \neq \emptyset$, we may assume that $\lambda_1 \in O_k^\times$ because the conditions (i)(ii) and the conclusion are preserved if $\lambda_1$ is replaced with another element of $J(\lambda_1)$. Then we get $(1 - \lambda_1) \in O_k - O_k^\times$ by (ii). Hence there exists an additive discrete valuation $v$ of $k$ such that $v(1 - \lambda_1) > 0$. When $J(\lambda_1) \cap O_k^\times = \emptyset$, there exists an additive discrete valuation $v$ of $k$ such that $v(1 - \lambda_1) \neq 0$. Replacing $\lambda_1$ by $\frac{\lambda_1}{\lambda_1 - 1}$ if necessary, we obtain that $v(1 - \lambda_1) > 0$. Thus, the following holds by replacing $\lambda_1$ with another element of $J(\lambda_1)$.

(b) There exists an additive discrete valuation $v$ of $k$ such that $v(1 - \lambda_1) > 0$.

If $(1 - \lambda_1)(1 + \lambda_1^n) = 2$ for some $n \in \mathbb{Z}$, then $\lambda_1 \in O_k^\times$ and we get $\lambda_1 \in \mathcal{E}$. Thus, the following holds by (ii)

(c) $(1 - \lambda_1)(1 + \lambda_1^n) \neq 2$ for all $n \in \mathbb{Z}$.

By [4] (4.8) Lemma and (a)(b)(c), we get $J(\lambda_1) = J(\lambda_2)$.

**Sketch of Proof of Theorem [17].** Already we have $\Gamma(\{0, 1, \infty, \lambda_1\}) = \Gamma(\{0, 1, \infty, \lambda_2\})$ by Proposition [13] and Proposition [14]. If $J(\lambda_1) \cap \mathcal{E} = \emptyset$, then the assertion follows by Proposition [13]. Thus, we may assume that $J(\lambda_i) \cap \mathcal{E} \neq \emptyset$ for $i = 1, 2$. When $k = \mathbb{Q}$, $J(\lambda_1) = J(\lambda_2) = \{-1, 2, \frac{1}{2}\}$ by $\mathcal{Q} \cap \mathcal{E} = \{1\}$ and $\mathcal{E}$ holds.

When $k$ is a quadratic field $\neq \mathbb{Q}(\sqrt{5})$, we can directly show that $e \in k \cap \mathcal{E}$ is either

$$-1, \pm \sqrt{-1}, \frac{1 + \sqrt{-3}}{2}, 2 \pm \sqrt{3}, \frac{3 \pm \sqrt{5}}{2}, \frac{1 \pm \sqrt{5}}{2}, \text{ or } \pm 2 \pm \sqrt{5}.$$  

Hence we may only consider the cases where $k = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$. In these cases, we can show that $\Gamma(\{0, 1, \infty, \lambda_1\}) \neq \Gamma(\{0, 1, \infty, \lambda_2\})$ for all $\lambda_1, \lambda_2 \in k \cap \mathcal{E}$ with $\lambda_1 \neq \lambda_2$. Thus, the assertion follows.

When there exists a prime ideal $p$ of $O_k$ unramified in $k/\mathbb{Q}$ such that $|O_k/p| = 2$, we show that every element $\lambda_1 \in k \cap \mathcal{E}$ satisfies the conditions (b) and (c) in the proof of Proposition [13]. Let $v$ denote an additive discrete valuation of $k$ corresponding to $p$. By $|O_k/p| = 2$, we have $\lambda_1 - 1 \in p$. Then $v(1 + \lambda_1^n) > 0$ for all $n \in \mathbb{Z}$. In particular, the condition (b) in the proof of Proposition [13] follows. If $(1 - \lambda_1)(1 + \lambda_1^n) = 2$ for some $n \in \mathbb{Z}$, then $2 \in p^2$ because $1 - \lambda_1, 1 + \lambda_1^n \in p$. However, this contradicts the assumption. Thus, the condition (c) in the proof of Proposition [13] follows. Now, the assertion follows by [4] (4.8) Lemma.

2 When $m \geq 5$ and $k$ is an $S\ell F$

Through this section, we fix a prime $\ell$ and assume that $k$ is an $S\ell F$. In the paper [2], Mochizuki proved the Grothendieck conjecture for hyperbolic curves over $k$ ([2] Theorem A) in a much wider form than the one defined in the Introduction. Moreover, Mochizuki also showed the $m$-step solvable Grothendieck conjecture for hyperbolic curves over $k$ ([2] Theorem A') as one of the $m$-step solvable versions of the main theorem. In this section, we introduce the following theorem.

**Theorem 2.1** ([2] Theorem A'). Assume that $m \geq 5$, and $U_2$ is a hyperbolic curve over $k$. Let $\Phi_m : \Pi_{U_1}^{(m, \text{pro-} \ell)} \to \Pi_{U_2}^{(m, \text{pro-} \ell)}$ be a continuous open $G_k$-homomorphism, and $\Phi_{m-3} : \Pi_{U_1}^{n-3, \text{pro-} \ell} \to \Pi_{U_2}^{n-3, \text{pro-} \ell}$ the homomorphism defined by $\Phi_m$. Then $\Phi_m$ induces a unique dominant $k$-morphism $\mu : U_1 \to U_2$ whose induced homomorphism on geometric fundamental groups coincides (up to composition with an inner automorphism arising from $\Pi_{U_2}^{n-3, \text{pro-} \ell}$) with $\Phi_{m-3}$. In particular, the (pro-\(\ell\)) $m$-step solvable Grothendieck conjecture for hyperbolic curves over $k$ holds.

**Remark.** 1. In [2] Theorem A', it is not assumed that $U_1$ is a curve over $k$, but instead, it is assumed that $U_1$ is a (smooth) variety over $k$. 


2. The uniqueness assertion for \( \mu \) in Theorem 2.1 is not explicitly stated in [2] Theorem A'. However, the uniqueness follows from the fact that a dominant \( k \)-morphism \( X_1 \to X_2 \) is determined by the open homomorphism \( \prod_{X_1}^{\text{pro-}\ell} \to \prod_{X_2}^{\text{pro-}\ell} \) induced by \( X_1 \to X_2 \) when \( g_2 \geq 2 \). See the first paragraph of the proof of [2] Theorem 14.1.

3. Mochizuki also proved the \( m \)-step solvable Grothendieck conjecture when \( U_1 \) is a pro-variety (i.e., a \( k \)-scheme which can be written as a projective limit of smooth varieties over \( k \) such that the transition morphisms are all birational) and \( U_2 \) is a hyperbolic pro-curve (i.e., a \( k \)-scheme which can be written as a projective limit of smooth hyperbolic curves over \( k \) such that the transition morphisms are all birational) and \( m \geq 3m_1 + 6tr_k(K(U_1)) + 2 \), where \( m_1 \) stands for the minimal transcendence degree over \( Q_k \) of all finitely generated field extensions of \( Q_k \) that contain \( k \). See [2] Theorem A''.

**Outline of Proof.** The second assertion follows from the first assertion (we remark that this proof needs the uniqueness of \( \mu \)). We consider the first assertion. In the next paragraph, we assume that: (i) \( k \) is a finite extension of \( Q_k \); (ii) \( U_2 \) is a non-hyperelliptic proper hyperbolic curve; (iii) \( m = 3 \).

By [2] Lemma 0.4, we reconstruct group-theoretically the \( \ell \)-adic cohomology group \( H^1(U_1, Q_\ell) \) from \( \Pi_1^{(1, \text{pro-}\ell)} \). Using the Hodge-Tate decomposition for \( \ell \)-adic cohomology, we obtain \( H^0(U_1, \omega_{U_1/k}) \) as the \( G_k \)-invariant part of \( H^1(U_1, Q_\ell) \otimes Q_\ell \mathcal{C}_\ell(1) \), where \( \mathcal{C}_\ell(1) \) is the Tate-twist of \( \mathcal{C}_\ell \). Hence, we get a (injective) morphism \( \theta : H^0(U_2, \omega_{U_2/k}) \to H^0(U_1, \omega_{U_1/k}) \) from \( \Phi_\mathfrak{m} \), which induces a morphism \( \varphi(\theta) : \varphi(H^0(U_1, \omega_{U_1/k})) \to \varphi(H^0(U_2, \omega_{U_2/k})) \). We have a canonical morphism \( \chi : \varphi(H^0(X_1, \omega_{U_1/k})) \to \varphi(H^0(U_2, \omega_{U_2/k})) \) for \( i = 1, 2 \). Moreover, since \( U_2 \) is non-hyperelliptic proper hyperbolic, \( X_2 = U_2 \) and \( U_2 \to \varphi(H^0(U_2, \omega_{U_2/k})) \) is an embedding. For each positive integer \( i \), we set:

\[
\mathcal{R}^i := \operatorname{Ker}(\bigotimes_i \mathcal{H}^0(U_2, \omega_{U_2/k}) \to \mathcal{H}^0(U_2, \omega_{U_2/k}^{\otimes i})),
\]

which is the set of relations of degree \( i \) defining \( U_2 \to \varphi(H^0(U_2, \omega_{U_2/k})) \). If the map:

\[
\kappa^i : \bigotimes_i \mathcal{H}^0(U_2, \omega_{U_2/k}) \otimes^{\theta} \bigotimes_i \mathcal{H}^0(U_1, \omega_{U_1/k}) \to \mathcal{H}^0(U_1, \omega_{U_1/k}^{\otimes i})
\]

satisfies \( \kappa^i(\mathcal{R}^i) = 0 \) for all \( i \), we say that \( \theta \) preserves relations. When \( \theta \) preserves relations, \( \varphi(\theta) \) induces a morphism \( U_1 \to U_2 \). Thus, we have only to show that the morphism \( \theta \) (induced by \( \Phi_3 : \Pi_1^{(3, \text{pro-}\ell)} \to \Pi_2^{(3, \text{pro-}\ell)} \)) preserves relations. Let \( X_1 \) be a (proper) model of \( U_1 \) over \( O_k \), and \( \mathfrak{p} \) an irreducible component of a special fiber of \( X_1 \). We write \( L \) for the quotient field of the completion of the local ring \( O_{X_1, \mathfrak{p}} \). From the definition of \( L \), we have a natural \( L \)-valued point \( \operatorname{Spec}(L) \to U_1 \). In particular, we get \( \alpha^L : G_L \to \Pi_2^{(1, \text{pro-}\ell)} \). If the morphism \( \alpha^L \) is geometric (i.e., comes from some \( L \)-valued point \( \operatorname{Spec}(L) \to U_2 \)), then \( \alpha^L \) preserves relations. Thus, we have only to show that \( \alpha^L \) is geometric. This step is the main part of the first half of [2]. To construct a line bundle of degree prime to \( \ell \) in this step, we need \( \Pi_2^{(1, \text{pro-}\ell)} \). For more detail, see [2] sections 1-7, 18.

For the assumption (i), see [2] section 15. For arbitrary hyperbolic curve \( U_2 \), there always exists a covering \( V_2 \to U_2 \) corresponding to an open subgroup of \( \Pi_2^{(2, \text{pro-}\ell)} \) such that \( g(V_2) \geq 2 \), that the compactification \( V_2^{\text{cpt}} \) is non-hyperelliptic, and that \( V_2^{\text{cpt}} \to U_2^{\text{cpt}} \) has arbitrarily large (specified) ramification over all points of \( U_2^{\text{cpt}} \to U_2 \). By using this cover, we can lift the assertion (ii) if \( m \geq 5 \).

\begin{proof}

\end{proof}

\section{When \( m \geq 3, g = 0 \) and \( k \) is an FGF.}

Through this section, we assume that \( k \) is an FGF. In the paper [3], Nakamura proved the Grothendieck conjecture for genus 0 hyperbolic curves over fields finitely generated over \( Q \). A key step of the proof in [3] is to reconstruct group-theoretically the inertia groups of \( \Pi_1 \) from \( \Pi_2 \). In the paper [3], the author proved the following theorem by giving the \( m \)-step solvable version of the method of the proof in [5].

**Theorem 3.1** ([3] Theorem 2.4.1). Assume that \( m \geq 3 \), and \( U_1 \) is a genus 0 hyperbolic curve over \( k \). If, moreover, \( p > 0 \), we assume that: (i) For each \( S' \subset E_{1, \overline{k}} \) with \( |S'| = 4 \), the curve \( X_{1, \overline{k}} - S' \) does not descend
to a curve over $\mathbb{F}_p$. Then the following holds.

$$I_{U_1} = U_2 \iff \left\{ \begin{array}{rl} U_1 & \cong \Pi^{(m,p')}_{k,\Sigma} \\
 & \text{for some } n_1, n_2 \in \mathbb{Z}_{\geq 0} \end{array} \right. \quad (p = 0)$$

In other words, the (pro-prime to $p$) $m$-step solvable Grothendieck conjecture for genus 0 hyperbolic curves over $k$ under $(\dagger)$ holds.

In this section, we introduce the paper [9] and a sketch of the proof of Theorem 3.1. In subsection 3.1, we explain the group-theoretical reconstruction of inertia groups. In subsection 3.2, we explain the outline of the proof of Theorem 3.1 in detail. In subsection 3.3, we discuss extensions of Theorem 3.1.

### 3.1 The group-theoretical reconstruction of inertia groups

For a scheme $S$, denote by $S^{\text{cl}}$ the set of all closed points of $S$. We define $\hat{X}^{m,\Sigma}$ as the cover of $X$ corresponding to $\{1\} \subset \Pi_U^{(m,\Sigma)}$, and write $\hat{U}^{m,\Sigma}$ and $\hat{E}^{m,\Sigma}$ for the inverse image of $U$ and $E$ by $\hat{X}^{m,\Sigma} \to X$, respectively.

Let $I_{\hat{e},\Pi_U^{(m,\Sigma)}}$ (resp. $D_{\hat{e},\Pi_U^{(m,\Sigma)}}$) be the stabilizer of $\hat{e} \in (\hat{X}^{m,\Sigma})^{\text{cl}}$ in $\Pi_U^{(m,\Sigma)}$ (resp. $\Pi_U^{(m,\Sigma)}$) with respect to the natural action $\Pi_U^{(m,\Sigma)} \curvearrowright (\hat{X}^{m,\Sigma})^{\text{cl}}$ (resp. $\Pi_U^{(m,\Sigma)} \curvearrowright (\hat{X}^{m,\Sigma})^{\text{cl}}$). We call it the inertia group (resp. the decomposition group) at $\hat{e}$. For each $x \in E$, we set

$$I_{x,\Pi_U^{(m,\Sigma)}} := \{ I_{\hat{e},\Pi_U^{(m,\Sigma)}} \mid \hat{e} \in (\hat{X}^{m,\Sigma})^{\text{cl}}, \hat{e} \text{ is above } x \},$$

and set $I_{\Pi_U^{(m,\Sigma)}} := \{ I_{x,\Pi_U^{(m,\Sigma)}} \mid x \in (\hat{X}^{m,\Sigma})^{\text{cl}} \}$. In this subsection, we show the following proposition.

**Proposition 3.2** (cf. [5] (3.5)Corollary, [9] Corollary 1.4.8). Assume either “$m \geq 2$ and $r \geq 2$” or “$m = 1$ and $r \geq 3$”. Let $\Phi_{m+2}: \Pi_U^{(m+2,\Sigma)} \overset{\sim}{\longrightarrow} \Pi_U^{(m,\Sigma)}$ be a $G_k$-isomorphism, and $\Phi_m : \Pi_U^{(m,\Sigma)} \overset{\sim}{\longrightarrow} \Pi_U^{(m,\Sigma)}$ the isomorphism induced by $\Phi_{m+2}$. Then $\Phi_m$ preserves the decomposition groups at cusps.

We divide the proof of Proposition 3.2 into the following two steps.

**Step 1**: Let $F$ be a free pro-$\Sigma$ group, $X \subset F$ a set of free generators of $F$, and $x \in X$. For each element \( f \in F \), we write $Z_{F^n}(f)$ for the centralizer of $f$ in $F^m$. First, we show that $Z_{F^n}(x^n) = \langle x \rangle$ if $m \geq 2$ and $n \in \mathbb{Z} - \{ 0 \}$ (see Lemma 3.3). By using this result, we get $D_{y,\Pi_U^{(m,\Sigma)}} = N_{\Pi_U^{(m,\Sigma)}}(I_{y,\Pi_U^{(m,\Sigma)}})$ for all $y \in \hat{E}^{m,\Sigma}$.

**Step 2**: We consider the group-theoretical reconstruction of inertia groups of $\Pi_U^{(m,\Sigma)}$ from $\Pi_U^{(m+2,\Sigma)}$. For this, we use the maximal cyclic subgroups of cyclotomic type (see Definition 3.3). Since $D_{y,\Pi_U^{(m,\Sigma)}} = N_{\Pi_U^{(m,\Sigma)}}(I_{y,\Pi_U^{(m,\Sigma)}})$, we obtain the group-theoretical reconstruction of decomposition groups at cusps of $\Pi_U^{(m,\Sigma)}$ from $\Pi_U^{(m+2,\Sigma)}$.

**Step 1**

Let $F$ be a free pro-$\Sigma$ group and $X \subset F$ a set of free generators of $F$. We write $Z_{F^n}(y)$ for the centralizer of $y$ in $F^m$ ($y \in F$).

**Lemma 3.3** (cf. [9] Proposition 1.1.10). Let $n \in \mathbb{Z} - \{ 0 \}$ and $x \in X$. If $m \geq 2$, then $Z_{F^n}(x^n) = \langle x \rangle$. In particular, $F^m$ is either abelian or center-free.

**Sketch of Proof.** Let $\alpha \in \hat{Z}^{\Sigma} - \{ 0 \}$. By [9] Proposition 1.1.6, we obtain that $Z_{F^n}(x^n) \subset \langle x \rangle \cdot F^{m-1}/F^m$. (The proof of [9] Proposition 1.1.6 is essentially the same as [6] Lemma 2.1.2.) For simplicity, we only consider the case of $|X| < \infty$. A calculation shows that $\hat{Z}^{\Sigma}[\langle F^1 \rangle] \ni x^n - 1$ is a non-zero-divisor ([9] Lemma 1.1.7(1)). In [9] Appendix, we establish the Blanchfield-Lyndon theory for $F$. By using this theory, we obtain that $\hat{Z}^{\Sigma}[\langle F^1 \rangle] \ni x^n - 1$ is a non-zero-divisor if and only if $Z_{F^n}(x^n) = \langle x \rangle$. Hence we get $Z_{F^n}(x^n) = \langle x \rangle$. Finally, the assertion follows by the induction on $m$. \( \square \)
Let \( y, y' \in \hat{E}^{m, \Sigma^1} \) with \( y \neq y' \), and assume that \( r \geq 2 \) and \( m \geq 2 \). Since \( \Pi_U^{\Sigma^1} \) is a free pro-\( \Sigma \) group and each inertia group of \( \Pi_U^{\Sigma^1} \) is generated by a free generator of \( \Pi_U^{\Sigma^1} \), we obtain that \( I_{y, \Pi}^{\Sigma^1} \) and \( I_{y', \Pi}^{\Sigma^1} \) are not commensurable by Lemma 3.3 if \( y(x_{\Gamma}) = y'(x_{\Gamma}) \). When \( y(x_{\Gamma}) \neq y'(x_{\Gamma}) \), we also have that \( I_{y, \Pi}^{\Sigma^1} \) and \( I_{y', \Pi}^{\Sigma^1} \) are not commensurable by [9] Lemma 1.2.1. In particular, \( I_{y, \Pi}^{\Sigma^1} = N_{\Pi_U^{\Sigma^1}}(I_{y, \Pi}^{\Sigma^1}) \) and \( D_{y, \Pi}^{\Sigma^1} = N_{\Pi_U^{\Sigma^1}}(I_{y, \Pi}^{\Sigma^1}) \) hold ([9] Proposition 1.2.3).

**Step 2**

We consider the group-theoretical reconstruction of inertia groups of \( \Pi_U^{m, \Sigma^1} \) from \( \Pi_U^{m+2, \Sigma^1} \). For this, we use the maximal cyclic subgroups of cyclotomic type. They are first defined in [5] (3.3) Definition in the case of the full fundamental group. The following definition differs from that of [5] for the following two points; we weaken the self-normalizing property in [5], and we generalize the definition from NFs to FGFs.

**Definition 3.4** (cf. [5] (3.3) Definition, [9] Definition 1.4.3). Let \( J \) be a closed subgroup of \( \Pi_U^{m, \Sigma^1} \). If \( J \) satisfies the following conditions, then \( J \) is called a maximal cyclic subgroup of cyclotomic type.

(i) \( J \cong \hat{Z}^{\Sigma^1} \)

(ii) Write \( \mathcal{J} \) for the image of \( J \) by \( \Pi_U^{m, \Sigma^1} \rightarrow \Pi_U^{\Sigma^1} \). Then \( \mathcal{J} \trianglelefteq \mathcal{J} \) and \( \mathcal{J} / \mathcal{J} \) is torsion-free.

(iii) \( \text{pr}(N_{\Pi_U^{m, \Sigma^1}}(J))^{\text{op}} \subset G_k \)

(iv) Let \( \chi_{\text{cycl}} : G_k \rightarrow (\hat{Z}^{\Sigma^1})^\times \) be the cyclotomic character and \( N_{\Pi_U^{m, \Sigma^1}}(J) \rightarrow \text{Aut}(J) = (\hat{Z}^{\Sigma^1})^\times \) the character obtained from the conjugate action. Then the following diagram is commutative.

\[
\begin{array}{ccc}
N_{\Pi_U^{m, \Sigma^1}}(J) & \longrightarrow & \text{Aut}(J) \\
\downarrow & & \parallel \\
G_k & \xrightarrow{\chi_{\text{cycl}}} & (\hat{Z}^{\Sigma^1})^\times 
\end{array}
\]

**Lemma 3.5** (cf. [9] Lemma 1.4.1). Let \( Q \) be a finite group, \( G \) a profinite group, \( J \) a closed subgroup of \( G \), and \( \mathcal{I} \) a closed subset of \( G \). If there exists \( \rho_J : J \rightarrow Q \) such that \( \mathcal{I} \cap J \subset \ker(\rho_J) \), then there exist an open subgroup \( H \subset G \) with \( J \subset H \) and a surjection \( \rho_H : H \rightarrow Q \) such that \( \mathcal{I} \cap H \subset \ker(\rho_H) \) and that \( J \rightarrow H \rightarrow Q \) coincides with \( \rho_J \).

**Proof.** By [1] Lemma 1.1.6(b), there exists an open subgroup \( H_0 \subset G \) with \( J \subset H_0 \) and a morphism \( \rho_{H_0} : H_0 \rightarrow Q \) such that the morphism \( J \rightarrow H_0 \rightarrow Q \) coincides with \( \rho_J \). Let \( \mathcal{H} := \{ H \subset G \mid H \subset H_0 \} \), and \( \rho_{\mathcal{H}} : H \rightarrow H_0 \rightarrow Q \) for \( H \in \mathcal{H} \). Since \( (\mathcal{I} \cap H) = \mathcal{I} \cap H \subset \ker(\rho_{\mathcal{H}}) = (\mathcal{I} \cap H) = \ker(\rho_{\mathcal{H}}) \) is closed in \( G \) and \( \cap_{H \in \mathcal{H}} (\mathcal{I} \cap H) = \mathcal{I} \cap H \cap \ker(\rho_{\mathcal{H}}) = \mathcal{I} \cap H \cap \ker(\rho_{\mathcal{H}}) \) by the compactness argument.

**Lemma 3.6** (cf. [5] (3.4) Theorem, [9] Proposition 1.4.5). Assume that \( r \geq 2 \). Then, for any subgroup \( I \) of \( \Pi_U^{m, \Sigma^1} \), the following conditions are equivalent.

(a) \( I \) is the inertia group at a point in \( \hat{E}^{m, \Sigma^1} \).

(b) There exists a maximal cyclic subgroup of cyclotomic type \( J \) of \( \Pi_U^{m, \Sigma^1} \) whose image by \( \Pi_U^{m+2, \Sigma^1} \rightarrow \Pi_U^{m, \Sigma^1} \) coincides with \( I \).

**Sketch of Proof.** For simplicity, we set \( \Delta := (\Pi_U^{\Sigma^1})^{m+1}/(\Pi_U^{\Sigma^1})^{m+2} \). We set \( \mathcal{I} := \bigcup_{\gamma \in \hat{X}^{m+2, \Sigma^1}} I_{\gamma, \Pi_U^{m+2, \Sigma^1}} \). For all open subgroups \( H \) of \( \Pi_U^{\Sigma^1} \) containing \( J \cdot \Delta \), we obtain that \( J \subset H^{[1]} \cdot \langle \mathcal{I} \cap H \rangle \) by Definition 3.3 (iii) (iv) and \( I_{H^1} = W_{-2}(H^1) := \prod_{\ell \in \Sigma^1} W_{-2}(H^{1, \pro^\ell}) \).
We fix a prime $\ell$ with $J^\ell \subseteq J$. Suppose that $I \cap (J \cdot \Delta) \subset J^\ell \cdot \Delta$. Then there exists an open subgroup $H$ of $(\prod_U^{\Sigma^1})^{m+2}$ containing $J \cdot \Delta$ and $\rho_H : H \to \mathbb{Z}/\ell \mathbb{Z}$ such that $\rho_H(I \cap H) = \{1\}$ and $\rho_H(J) \neq \{1\}$ by Lemma 3.5. This is absurd. Thus, we get $I \cap (J \cdot \Delta) \not\subset J^\ell \cdot \Delta$, and hence $I \cap (J \cdot \Delta) \neq \{1\}$. Therefore, we obtain that $J \subset I \cdot ((\prod_U^{\Sigma^1})^{m+1}/(\prod_U^{\Sigma^1})^{m+2})$ by Lemma 1.4.2.

Let $z \in J$ be a generator, and $\bar{z}$ the image of $z$ by $\prod_U^{m+2, \Sigma^1} \to \prod_U^{n, \Sigma^1}$. Since there exists an inertia group $I$ that contains $\bar{z}$, we have $\langle \bar{z} \rangle \equiv I$ mod $(\prod_U^{\Sigma^1})^{\{1\}/(\prod_U^{\Sigma^1})^{\{m\}}}$ by Definition 3.4(i)(ii). Since $I$ mapped injectively by $\prod_U^{\Sigma^1} \to \prod_U^{n, \Sigma^1}$, we obtain $\langle z \rangle = \bar{I}$.

Sketch of Proof of Proposition 3.2. When $m \geq 2$ and $r \geq 2$, the assertion follows from Lemma 3.9 and $D^r_{\bar{y}, \prod_U^{m, \Sigma^1}} = N_{\prod_U^{m, \Sigma^1}}(I^r_{\bar{y}, \prod_U^{m, \Sigma^1}})$ for $\bar{y} \in \hat{E}^{n, \Sigma^1}$. When $m = 1$, the above method cannot be used because $D^r_{\bar{y}, \prod_U^{m, \Sigma^1}} \subset N_{\prod_U^{m, \Sigma^1}}(I^r_{\bar{y}, \prod_U^{m, \Sigma^1}})$ in general. In this case, we need to think about the maximal nilpotent quotient of $\prod_U^{\Sigma^1}$. For this, see the proof of [9] Proposition 1.4.6.

3.2 The proof of the main theorem

The important difference between the proofs of Theorem 1.1 and Theorem 3.1 is Proposition 3.2. In Theorem 1.1, we do not reconstruct group-theoretically inertia groups. Thus, we could only consider the case of $\mathbb{P}^1_k$ minus 4 points, but we get the results for the case of $m = 2$. In Theorem 3.1, we have the group-theoretical reconstruction of inertia groups. Thus, we can consider the case of all genus 0 curves, but we should assume that $m \geq 3$.

In this subsection, we show Theorem 3.1. As in section 1, we divide the proof into the following three steps.

Step 1: Let $\Lambda$ be a finite subset of $k \cup \{\infty\}$ with $|\Lambda| \geq 4$. Let $x_1, x_2, x_3, x_4$ be distinct elements of $\Lambda$ and $\varepsilon = \{x_1, x_2\}, \delta = \{x_3, x_4\}$. Set the following notation for $\varepsilon$ and $\delta$.

$$\lambda(\varepsilon, \delta) := \frac{x_4 - x_1}{x_4 - x_2} \frac{x_3 - x_2}{x_3 - x_1}$$

(The isomorphism $\mathbb{P}^1_k \cong \mathbb{P}^1_k$ satisfying $x_1 \mapsto 0$, $x_2 \mapsto \infty$, $x_3 \mapsto 1$ maps $x_4$ to $\lambda(x_1, x_2, x_3, x_4)$.)

We will show that, for all $n \in \mathbb{Z}_{\geq 0}$ and for all primes $\ell$ that differ from $p$, the field $k(\mu_{pn}, \lambda(\varepsilon, \delta)^{\frac{1}{\ell^n}})$ is reconstructed group-theoretically from $\mathbb{P}^1_{\mathbb{P}^1_k - \Lambda}$ (see Proposition 3.8). This step corresponds to [9] subsection 2.1.

Step 2: Assume that $p = 0$ (resp. $p > 0$). For each $i = 1, 2$, let $\Gamma_i$ be a finitely generated subgroup of $k^\times$. In this step, we will show that, if $k(\Gamma_1^{\frac{1}{\ell^n}}) = k(\Gamma_2^{\frac{1}{\ell^n}})$ for all $n \in \mathbb{Z}_{\geq 0}$ and for all primes $\ell$ that differ from $p$, then $\Gamma_1 = \Gamma_2$ (resp. $\bigcup_{n \in \mathbb{Z}_{\geq 0}} \Gamma_1^{\frac{1}{\ell^n}} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \Gamma_2^{\frac{1}{\ell^n}}$) holds (see Lemma 3.9). In particular, we get the reconstruction of $\lambda \in k^\times - \{1\}$ (resp. a non-torsion element $\lambda$ of $k^\times$) from $\langle \lambda \rangle$ and $\langle 1 - \lambda \rangle$ in $k^\times$. This step corresponds to [9] subsection 2.1.

Step 3: In this step, we first show Theorem 3.1 for punctured projective lines. If $p = 0$, then the result easily follows from Step 1 and Step 2. However, if $p > 0$, then we have to consider the gluing of the Frobenius twists. Finally, we show Theorem 3.1 in general by genus 0 descent. This step corresponds to [9] subsections 2.2, 2.3 and 2.4.

Step 1

First, we define the weak rigidity invariant of $\mathbb{P}^1_k - \Lambda$ (where $\Lambda$ stands for a finite set of $k$-rational points of $\mathbb{P}^1_k$ with $|\Lambda| \geq 4$). The weak rigidity invariant was first defined in [5] (4.2) for the full fundamental group $\Pi_{\mathbb{P}^1_k - \Lambda}$ when $p = 0$. However, this definition is applicable to arbitrary $p \geq 0$ and essentially uses only $\Pi_{\mathbb{P}^1_k - \Lambda}^{(1, p')}$.

Thus, in [9] subsection 1.4, the author redefined the weak rigidity invariant as follows.
Definition 3.7 (cf. [5] (4.2), [9] Definition 2.1.1). Let \( n \in \mathbb{Z}_{\geq 0} \), \( \ell \) a prime different from \( p \), and \( \Lambda \) a finite set of \( k \)-rational points of \( \mathbb{P}^1_k \) with \( |\Lambda| \geq 4 \). Let \( x_1, x_2, x_3, x_4 \) be distinct elements of \( \Lambda \) and \( \varepsilon = \{ x_1, x_2 \}, \delta = \{ x_3, x_4 \} \). We define \( \kappa_{\ell^n}(\varepsilon, \delta) \) to be the subfield of \( k_{\text{sep}} \) consisting of the elements fixed by all the automorphisms of \( k_{\text{sep}} \) belonging to \( \bigcup_{\mathcal{H}} (\mathcal{H} \cap D_{\mathcal{H}}) \).

Here \( \mathcal{H} \) run over all closed points of \( \tilde{\Lambda}^{\ell^p} \) above \( \delta \), and \( \mathcal{H} \) run over all open subgroups of \( \Pi_{\mathbb{P}^1_k - \Lambda}^{(1, p')} \) satisfying the following conditions.

(i) \( \mathcal{H} := H \cap \Pi_{\mathbb{P}^1_k - \Lambda}^{(1, p')} \) contains \( I_{x, p} \Pi_{\mathbb{P}^1_k - \Lambda}^{(1, p')} \) for all \( x \in E - \varepsilon \),

(ii) \( \Pi_{\mathbb{P}^1_k - \Lambda}^{(1, p')}/\mathcal{H} \cong \mathbb{Z}/\ell^n\mathbb{Z} \)

(iii) \( \text{pr}(\mathcal{H}) = \text{G}^{(k(\mu_{\ell^n}))} \)

(iv) \( \text{pr}^{-1}(\text{G}(\mu_{\ell^n})) \supset \mathcal{H} \)

We call \( \kappa_{\ell^n}(\varepsilon, \delta) \) the weak rigidity invariant for \( \varepsilon, \delta \) of \( U \).

In [9], the weak rigidity invariant \( \kappa_N(\varepsilon, \delta) \) is defined for all integers \( N \in \mathbb{Z}_{\geq 1} \), whose all prime factors are contained in \( \Sigma^1 \). In the following proofs, we only need the weak rigidity invariant in the case of \( N = \ell^n \).

**Proposition 3.8** (cf. [5] (4.3)Proposition, [9] Proposition 2.1.2). Under the notation of Definition 3.7, the following holds.

\[
\kappa_{\ell^n}(\varepsilon, \delta) = k(\mu_{\ell^n}, \lambda(\varepsilon, \delta)^{\mathbb{P}})
\]

**Sketch of Proof.** Let \( t : \mathbb{P}^1_k \to \mathbb{P}^1_k \) be the isomorphism that satisfies \( t(x_1) = 0, t(x_2) = \infty \) and \( t(x_3) = 1 \). (In particular, \( t(x_4) = \lambda(\varepsilon, \delta) \).) Let \( H \) be an open subgroup of \( \Pi_{\mathbb{P}^1_k - \Lambda}^{(1, p')} \) satisfying the conditions (i)(ii)(iii)(iv) in Definition 3.7. Then \( (U_{\mathcal{H}})^{\text{cpt}} \to X_{k_{\text{sep}}} \) is identified with \( \mathbb{P}^1_{k_{\text{sep}}} \to \mathbb{P}^1_{k_{\text{sep}}}, x \mapsto x^{\ell^n} \), and the corresponding extension of function fields is \( k_{\text{sep}}(t^{\mathbb{P}})/k_{\text{sep}}(t) \). Let \( U_{\mathcal{H}} \to U(k(\mu_{\ell^n})) \) be the cover corresponding to \( H <_p U_{k(\mu_{\ell^n})} \). Then there exists \( \omega_H \in k(\mu_{\ell^n})^\times \) such that \( K(U_H) = k(\mu_{\ell^n}, \omega_H t^{\mathbb{P}}) \) by the conditions (i)(ii)(iii)(iv) in Definition 3.7 and Kummer theory. Since the fixed field \( \kappa_H \) by \( \bigcap_{\mathcal{H}} \text{pr}(\mathcal{H} \cap D_{\mathcal{H}}) \) is the composite field of residue fields of all points of \( X_H := (U_{\mathcal{H}})^{\text{cpt}} \) above \( x_3 \) and \( x_4 \), we get \( \kappa_H = k(\mu_{\ell^n}, \{ \omega_H \}^{\mathbb{P}}, \{ \omega_H \lambda(\varepsilon, \delta) \}^{\mathbb{P}}) \). There exists an open subgroup \( H_0 \) of \( \Pi_{\mathbb{P}^1_k - \Lambda}^{(1, p')} \) with \( \omega_{H_0} = 1 \) that satisfies the conditions (i)(ii)(iii)(iv) in Definition 3.7. Note that we have \( \kappa_H \supset \kappa_{H_0} \). Hence, the assertion follows.

**Step 2**
First, we consider a generalization of Proposition 1.4 to arbitrary \( p \geq 0 \). When \( p > 0 \), for a finitely generated subgroups \( \Gamma \) of \( k^\times \), we write \( \Gamma_{\text{perf}} \subset k^\times \) for the set of all elements whose \( p^n \)-th power is in \( \Gamma \) for some \( n \in \mathbb{Z}_{\geq 0} \). In other words, \( \Gamma_{\text{perf}} := \bigcup_{n \in \mathbb{Z}_{\geq 0}} \Gamma_{p^n} \).

**Lemma 3.9** (cf. [9] (3.1)Lemma, [9] Proposition 2.1.4.). Let \( \Gamma_1, \Gamma_2 \) be finitely generated subgroups of \( k^\times \). If \( k(\Gamma_1^{\mathbb{P}}) = k(\Gamma_2^{\mathbb{P}}) \) for all \( n \in \mathbb{Z}_{\geq 0} \) and for all primes \( \ell \) that differ from \( p \), then the following hold.

(1) If \( p = 0 \), then \( \Gamma_1 = \Gamma_2 \).

(2) If \( p > 0 \), then \( \Gamma_1^{\text{perf}} = \Gamma_2^{\text{perf}} \).

**Sketch of Proof.** (1) is the same as Proposition 1.4. We consider (2). Assume that \( p > 0 \). As in the proof of Proposition 1.4, we define \( R \) as the integral closure of \( \mathbb{F}_p[\Gamma_2] \) in \( k(\mu_4) \). We have that \( R^\times \) is a finitely generated \( \mathbb{Z} \)-module because \( R \) is finitely generated over \( \mathbb{F}_p \) by assumption. In the notation of Proposition 1.4, we have \( \gamma \in \Gamma_1 \cdot R^{\ell^n} \) for all \( n \in \mathbb{Z}_{\geq 0} \) and for all primes \( \ell \) that differ from \( p \). Thus, we obtain \( \gamma \in \Gamma_1^{\text{perf}} \).
Lemma 3.10 (cf. [9] Lemma 2.2.1, [9] Lemma 2.3.4). Let \( \lambda_1, \lambda_2 \in k^\times - \{1\} \), and let \( \rho \in \overline{\mathbb{F}}_p \) be a primitive 6-th root of unity.

(1) If \( p = 0 \), \( \langle \lambda_1 \rangle = \langle \lambda_2 \rangle \) and \( \{1 - \lambda_1\} = \{1 - \lambda_2\} \) in \( k^\times \), then \( \lambda_1 = \lambda_2 \) or \( \{\lambda_1, \lambda_2\} = \{\rho, \rho^{-1}\} \).

(2) If \( p > 0 \), \( \lambda_1 \not\in k \cap \overline{\mathbb{F}}_p \), \( \langle \lambda_2 \rangle^{perf} = \langle \lambda_2 \rangle^{perf} \) and \( \{1 - \lambda_1\}^{perf} = \{1 - \lambda_2\}^{perf} \) in \( k^\times \), then there exists unique \( n \in \mathbb{Z} \) such that \( \lambda_2 = \lambda_1^{n} \).

Sketch of Proof. (1) We fix an embedding \( k \hookrightarrow \mathbb{C} \) and regard \( k \) as a subfield of \( \mathbb{C} \). In particular, we may assume that \( \rho = e^{i \pi \sqrt{-1}} \). If \( |\lambda_1| \neq 1 \), then \( \lambda_1 \) is a non-torsion element and \( \mathbb{Z} \cong \langle \lambda_1 \rangle = \langle \lambda_2 \rangle \), hence \( \lambda_1 \in \{\lambda_2, \lambda_2^{*}\} \). Similarly, if \( |1 - \lambda_1| \neq 1 \), then \( 1 - \lambda_1 \in \{1 - \lambda_2, (1 - \lambda_2)^{-1}\} \). Then we obtain either \( \lambda_1 = \lambda_2 \) or \( |\lambda_1| = |\lambda_2| = |1 - \lambda_1| = |1 - \lambda_2| = 1 \) by an easy calculation. The second case coincides with \( \{\lambda_1, \lambda_2\} = \{e^{i \pi \sqrt{-1}}, e^{-i \pi \sqrt{-1}}\} \).

(2) By assumption, \( \lambda_1, 1 - \lambda_1, \lambda_2, 1 - \lambda_2 \) are non-torsion elements of \( k^\times \). Since \( \langle \lambda_1 \rangle^{perf} = \langle \lambda_2 \rangle^{perf} \) and \( \{1 - \lambda_1\}^{perf} = \{1 - \lambda_2\}^{perf} \), there exist a unique pair \( u, v \in \mathbb{Z} \) such that \( \lambda_1^{u} \in \{\lambda_2, \lambda_2^{-1}\} \) and \( \lambda_1^{v} \in \{1 - \lambda_2, (1 - \lambda_2)^{-1}\} \). We may assume that \( \lambda_1^{u} = \lambda_2^{-1} \) and \( 1 - \lambda_1^{v} = (1 - \lambda_2)^{-1} \). Hence \( \lambda_1 \) satisfies \( \lambda_1^{u+v} - \lambda_2^{u+v} + 1 = 0 \). Take \( W \in \mathbb{N} \) that satisfies \( u + v + W \geq 0 \) and \( v + W \geq 0 \). Then \( \lambda_1 \) is a root of the polynomial \( t_0^u + vt_0^w + 1 \in \mathbb{F}_p[t] \). Hence we get \( \lambda_1 \in k \cap \overline{\mathbb{F}}_p \). This is absurd.

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Step 3

First, we consider the case of \( p = 0 \). In this case, we obtain the \( m \)-step solvable Grothendieck conjecture for hyperbolic punctured projective lines by Step 1 and Step 2.

Proposition 3.11 (cf. [3] (4.4)Theorem, [9] Proposition 2.2.4). Let \( \Lambda_1, \Lambda_2 \) be finite sets of \( k \)-rational points of \( \mathbb{P}^1_k \) with \( |\Lambda_1| \geq 3 \). Assume that \( p = 0 \) and \( m \geq 3 \). Let \( \Phi_m : \Pi^{(m)}_{k - \Lambda_1} \overset{\sim}{\longrightarrow} \Pi^{(m)}_{k - \Lambda_2} \) be a \( G_k \)-isomorphism, and \( \Phi : \Pi^{(1)}_{k - \Lambda_1} \overset{\sim}{\longrightarrow} \Pi^{(1)}_{k - \Lambda_2} \) the isomorphism induced by \( \Phi_m \). Then there exists \( f \in \text{Aut}_k(\mathbb{P}^1_k) \) such that

(a) \( f(\Lambda_1) = \Lambda_2 \), and

(b) for each pair \( x_1 \in \Lambda_1 \) and \( x_2 \in \Lambda_2 \), \( f(x_1) = x_2 \) if and only if \( \Phi_1(I_{x_1, k - \Lambda_1}^{r, s}) = I_{x_2, k - \Lambda_2}^{r, s} \).

Sketch of Proof. We may assume that \( |\Lambda_1| \geq 4 \). Let \( x_1, x_2, x_3, x_4 \) be distinct elements of \( \Lambda_1 \) and \( \varepsilon = \{x_1, x_2\}, \delta = \{x_3, x_4\} \). Since \( m \geq 3 \), \( \Phi_1 \) preserves decomposition groups at cusps (Proposition 3.2). Let \( \alpha^* : \Lambda_1 \overset{\sim}{\longrightarrow} \Lambda_2 \) be the unique bijection satisfying \( \alpha^*(x_1) = x_2 \Leftrightarrow \Phi_1(I_{x_1, k - \Lambda_1}^{r, s}) = I_{x_2, k - \Lambda_2}^{r, s} \). By Proposition 3.8 and Lemma 3.9 we get \( \lambda(\varepsilon, \delta) = \lambda(\alpha^* \varepsilon, \alpha^* \delta) \). Set \( \varepsilon' = \{x_3, x_2\}, \delta' = \{x_1, x_2\} \). Then \( \lambda(\varepsilon', \delta') = 1 - \lambda(\varepsilon, \delta) \). Hence we get \( (1 - \lambda(\varepsilon, \delta)) = (1 - \lambda(\alpha^* \varepsilon, \alpha^* \delta)) \). Thus, by Lemma 3.10, we obtain either \( \lambda(\varepsilon, \delta) = \lambda(\alpha^* \varepsilon, \alpha^* \delta) \) or \( \lambda(\varepsilon, \delta) = \lambda(\alpha^* \varepsilon, \alpha^* \delta) = \{\rho, \rho^{-1}\} \). There is no isomorphism \( \Pi^{(2, \text{pro-2})}_{k - (0, 1, \rho, \rho^{-1})} \overset{\sim}{\longrightarrow} \Pi^{(2, \text{pro-2})}_{k - (0, 1, \rho, \rho^{-1})} \) which maps \( I_{0, \Pi^{(2, \text{pro-2})}_{k - (0, 1, \rho, \rho^{-1})}} \) to \( I_{0, \Pi^{(2, \text{pro-2})}_{k - (0, 1, \rho, \rho^{-1})}}, I_{\Pi^{(2, \text{pro-2})}_{k - (0, 1, \rho, \rho^{-1})}}, I_{\Pi^{(2, \text{pro-2})}_{k - (0, 1, \rho, \rho^{-1})}} \), respectively (see [9] Lemma 2.2.3(2)). Together with a certain improvement ([9] Corollary 1.4.8(2)) of Proposition 3.2 in the pro-\( \ell \) setting, we get \( \lambda(\varepsilon, \delta) = \lambda(\alpha^* \varepsilon, \alpha^* \delta) \) for an arbitrary pair \( \varepsilon, \delta \). Therefore, the assertion follows.

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Next, we consider the case of \( p > 0 \). This case has difficulties arising from the existence of Frobenius twists.

Proposition 3.12 (cf. [9] Proposition 2.3.7). Let \( \Lambda_1, \Lambda_2 \) be finite sets of \( k \)-rational points of \( \mathbb{P}^1_k \) with \( |\Lambda_1| \geq 3 \). Assume that \( p > 0 \) and \( m \geq 3 \). Let \( \Phi_m : \Pi^{(m, \rho')}_{k - \Lambda_1} \overset{\sim}{\longrightarrow} \Pi^{(m, \rho')}_{k - \Lambda_2} \) be a \( G_k \)-isomorphism, and \( \Phi_{1(v_1, v_2)} : \Pi^{(1, \rho')}_{k - \Lambda_1(v_1)} \overset{\sim}{\longrightarrow} \Pi^{(1, \rho')}_{k - \Lambda_2(v_2)} \) the isomorphism induced by \( \Phi_m \) for each pair \( v_1, v_2 \in \mathbb{Z}_{\geq 0} \). We assume that the following condition: (f) For each \( S' \subset \Lambda_1 \) with \( |S'| = 4 \), the curve \( \mathbb{P}^1_k - S' \) does not descend to a curve over \( \overline{\mathbb{F}}_p \). Then there exist \( w_1, w_2 \in \mathbb{Z}_{\geq 0} \) and \( f : \mathbb{P}^1_k \rightarrow \mathbb{P}^1_k \) such that.
(a) $f(A_1(w_1)) = A_2(w_2)$, and

(b) for each pair $x_1 \in A_1(w_1)$ and $x_2 \in A_2(w_2)$, $f(x_1) = x_2$ if and only if

$$
\Phi_{1, \cdot}^{w_1, w_2}(I_{x_1, \Pi_{e}^{p'}}) = I_{x_2, \Pi_{e}^{p'}}.
$$

**Sketch of Proof.** Let $v_1, v_2 \in \mathbb{Z}_{\geq 0}$. Since $\Phi_{1}^{w_1, w_2}$ preserves decomposition groups at cusps by Corollary 3.2.1, there exists a bijection $\alpha_{1}^{s_1, s_2} : A_1(v_1) \to A_2(v_2)$ such that $\alpha_{1}^{s_1, s_2}(x_1) = x_2 \iff \Phi_{1, \cdot}^{s_1, s_2}(I_{x_1, \Pi_{e}^{p'}}) = I_{x_2, \Pi_{e}^{p'}}$.

For simplicity, we only consider the cases that $|A_1| = 4$ and $|A_1| = 5$.

Let $x_1, x_2, x_3, x_4$ be distinct elements of $A_1$ and set $\varepsilon = \{x_1, x_2\}$, $\delta = \{x_3, x_4\}$. Using Proposition 3.8, Lemma 3.9 and Lemma 3.10(2), there exists a unique $n \in \mathbb{Z}$ such that $\lambda(\alpha_{1}^{s_1, s_2}(0), \alpha_{1}^{s_1, s_2}(0)) = (\lambda(\varepsilon, \delta))$ by the same way as the proof of Proposition 3.11 (Remark: $\lambda(\varepsilon, \delta)$ is not contained in $k \cap \mathbb{P}_p$ by the condition ($\dagger$)). Thus, the assertion follows when $|A_1| = 4$.

We consider the case that $|A_1| = 5$. Assume that $A_1 = \{0, \infty, 1, \lambda_1, 1_2\}$, $A_2 = \{0, \infty, 1, \lambda_2, 1_2\}$ and $\alpha_{1}^{s_1, s_2}(0) = 0$, $\alpha_{1}^{s_1, s_2}(\infty) = \infty$, $\alpha_{1}^{s_1, s_2}(1) = 1$, $\alpha_{1}^{s_1, s_2}(\lambda_1, i) = \lambda_2, i$ for $i = 1, 2$. Since $\lambda(\{0, \infty\}, \{1, \lambda_1, i\}) = \lambda_{1, 1}$, there exists a unique $n \in \mathbb{Z}$ such that $\lambda_1 = \lambda_{1, 1}^n$ ($i = 1, 2$). Similarly, there exists a unique $\sigma, \tau, \zeta \in \mathbb{Z}$ such that $\lambda_{1, 2} = (\lambda_{1, 1}^n \lambda_{2, 1}), \lambda_{2, 2} = (\lambda_{2, 1}^n \lambda_{1, 1})$.

Thus, by Lemma 2.3.5 and Lemma 2.3.6, we obtain either $n_1 = n_2$ or $\sigma = \tau = \zeta$. When $n_1 = n_2$, the assertion follows. Let $T, T' \subset A_1$ with $|T \cap T'| \geq 3$. If there exist $w_1, w_2 \in \mathbb{Z}$ such that the conditions (a)(b) for $(w_1, w_2, \mathbb{P}_k^1 - T)$ and $(w_1, w_2, \mathbb{P}_k^1 - T')$ hold, then the conditions (a)(b) for $(w_1, w_2, \mathbb{P}_k^1 - T - T')$ also follows. Thus, the assertion follows when $\sigma = \tau = \zeta$.

Proposition 3.11 and Proposition 3.12 are stronger than the (weak) m-step solvable Grothendieck conjecture. In [5], Nakamura uses such functoriality and Galois descent to show the Grothendieck conjecture for genus 0 hyperelliptic curves. In [9], the author followed this method to show Theorem 3.1.

**Sketch of Proof of Theorem 3.1.** Let $\Phi_{m} : \Pi_{U_1}^{(m, p')} \simeq_{G_k} \Pi_{U_2}^{(m, p')}$, and let $\Phi_{1} : \Pi_{U_1}^{(1, p')} \simeq_{G_k} \Pi_{U_2}^{(1, p')}$ the isomorphism induced by $\Phi_{m}$. We may assume that $g_2 = 0$ by Corollary 1.3.5. When $p = 0$ (resp. $p > 0$), there exists (resp. exist) a finite Galois extension $K$ of $k$ (resp. and $M \in \mathbb{Z}_{\geq 0}$) such that $U_{1, K} \cong \mathbb{P}_k^1 - \Lambda_1$ and $U_{2, K} \cong \mathbb{P}_k^1 - \Lambda_2$ (resp. $U_{1, K}(M) \cong \mathbb{P}_k^1 - \Lambda_1$ and $U_{2, K}(M) \cong \mathbb{P}_k^1 - \Lambda_2$) for some $\Lambda_1, \Lambda_2 \subset \mathbb{P}_k^1(K)$. By Proposition 3.11 (resp. Proposition 3.12), there exists (resp. exist) $f : \Pi_{K}^{1} \simeq_{G_k} \Pi_{K}^{1}$ (resp. and $w_1, w_2 \in \mathbb{Z}_{\geq 0}$) such that the conditions (a)(b) in Proposition 3.11 (resp. Proposition 3.12) hold. We define $U_1^f, U_2^f, E_1^f, E_2^f, \Phi_{1, K}^f$ as $U_1, U_2, E_1, E_2, \Phi_{1, K}$ (resp. $U_1(M + w_1), U_2(M + w_2), E_1(M + w_1), E_2(M + w_2)$), $\Pi_{K}^{1, p'} \simeq_{G_k} \Pi_{K}^{1, p'}(M + w_1)$, respectively. Hence there exists $f : \Pi_{K}^{1} \simeq_{G_k} \Pi_{K}^{1}$ such that $f(K) = \Pi_{K}^{1}$, and $f(x_1) = x_2 \iff \Phi_{1, K}^f(I_{x_1, \Pi_{e}^{p'}}) = I_{x_2, \Pi_{e}^{p'}}$ by above.

Let $\rho(U_1^f)$ be the image of $\rho(\Pi_{K}^{1})$ by $Gal(K/k) \to Aut_{U_1^f}(U_1^f)$ (i = 1, 2). Let $\rho(U_1^f)$ be the inverse image of $\rho(\Pi_{K}^{1})$ by $p_{U_1/K} : \Pi_{U_1}^{(1, p')} \to G_k$ and $\rho(U_1^f)$ the image of $\rho(U_1^f)$ by $\Pi_{U_1}^{(1, p')} \simeq_{G_k} \Pi_{U_1}^{(1, p')}$. We have $\Phi_{1, K}^f(I_{\rho(U_1)(x_1), \Pi_{e}^{p'}}) = \rho(U_1^f) \cdot \Phi_{1, K}^f(I_{x_1, \Pi_{e}^{p'}}) \cdot \rho(U_1^f)^{-1}$ for all $x_1 \in E_1^f, K$. Hence we get $\rho(U_1^f)(x_1) = \rho(U_1^f)(f(x_1))$. As $|E_1^f| \geq 3$, it follows that $f \circ \rho(U_1^f) = \rho(U_1^f) \circ f$ for all $\rho$. By Galois descent, we obtain $U_1^f \cong U_2^f$.

### 3.3 Work in Progress

We notice that there are several possible extensions to Theorem 3.1. For example, can $m \geq 3$ in Theorem 3.1 be replaced with $m \geq 2$? This question is open for now. However, the author thinks that the answer to
this question would be affirmative because we have Theorem 1.1 (see the beginning of subsection 3.2). As other examples, we consider the following two extensions.

(i) The Grothendieck conjecture is also proved when \( k \) is a finite field. Hence we can expect that the \( m \)-step solvable Grothendieck conjecture is also true when \( k \) is a finite field. Unfortunately, by the condition (†), Theorem 3.1 does not imply the result of the case that \( k \) is a finite field.

(ii) We can also expect that the \( m \)-step solvable Grothendieck conjecture for (general, in other words, genus \( \geq 0 \)) hyperbolic curves is true. To consider this problem, we need an approach different from the method in section 3.

For the present, the author is trying to show the above two extensions (i)(ii) by constructing the \( m \)-step solvable version of the methods in [8, 3].

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