MATRIX MODELS FOR SPACETIME
TOPODYNAMICS

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Abstract
The machinery is suggested to describe the varying spacetime topology on the level of its substitutes by finite topological spaces.

Introduction
The approximations of (or substitutes for) continuous spacetime by finitary structures are studied in this paper. The results presented furnish a framework in which one might express such ideas as variable spacetime topology or, for instance, the topological fluctuations on small scales. The paper is organized as follows.

In Section 1 the coarse-graining procedure is described. Being applied to a continuous manifold, it yields the so-called pattern space \([12]\) which, being finite or at most countable set, may be thought of as topological space or, equivalently, as a directed graph. The kinematics of the spacetime topology is then addressed to that of the pattern space which substitutes its continuous predecessor.

In Section 2 the finitary counterpart of the superspace (in Wheeler’s sense) is introduced providing the arena for the variation of the topology of pattern spaces. To construe it we use the remarkable isomorphism between pattern spaces and finite quasiorders. The latter, being subject of combinatorial studies, are associated with certain finite-dimensional algebras \([9]\). Thus, the study of the variety of finite topological spaces (being, loosely speaking, discrete by its nature) is replaced by dealing with finite-dimensional algebras whose matrix representation is treated.

In Section 3 the main topological features of pattern spaces are formulated in algebraic terms.

In Section 4 the spatialization procedure is suggested restoring points of the pattern spaces by given finite-dimensional algebra. The ideas used in this procedure are Stanley’s techniques \([11]\) in algebraic combinatorics.

Now, possessing the algebraic means to capture the topological features, we are interested in introducing finitary substitutes for differential structures, to which the Section 5 is devoted. The elements of the tensor calculus needed to introduce the basic constituents of general relativity turn to be successfully transplanted to pattern spaces and their matrix representations. The Einstein-Hilbert variational principle is then rewritten in terms of matrix equations.

1 The coarse-graining procedure
1.1 From manifolds to pattern spaces

In the conventional general relativity, the spacetime manifold consists of events. Whereas, from the operationalistic perspective an individual event is an idealization of what can be directly measured. Such idealization is adequate within classical physics, but is unsatisfactory from the operationalistic point of view. In quantum theory the influence of a measuring apparatus on the object being observed can not in principle be removed. We could expect the metric of a quantized theory to be subject to fluctuations, whereas the primary tool to separate individual events is just the metric \[\text{3}\]. Thus a sort of smearing procedure for events is to be imposed into the quantized theory of spacetime.

To introduce the procedure, recall the definition of the topology \(\tau\) of a manifold \(M\). \(\tau\) is nothing but a family of subsets of \(M\) declared open and satisfying the following axioms:

- T1) \(\emptyset, X \in \tau\)
- T2) For any \(A, B \in \tau\) \(A \cap B \in \tau\)
- T3) For any collection \(\{A_j\}, A_j \in \tau\) \(\bigcup_{j \in J} A_j \in \tau\), where \(J\) is arbitrary index set, \(\cup, \cap\) are usual set union and intersection.

Thinking operationalistically, we can not have access to the infinite number of all open sets, thus to capture the topology of the manifold we consider its finite covering \(F\) by open subsets which we believe to be homeomorphic to open balls in \(\mathbb{R}^n\).

Supposed the covering \(F \subseteq \tau\) is closed under set intersections, the spacetime manifold acquires the cellular structure with respect to \(F\), so that the events belonging to one cell are thought of as operationally undistinguishable. Then, instead of considering the set \(M\) of all events we can focus on its finite subset \(X \subseteq M\) such that each cell contains at least one point of \(X\).

For each \(F \in F\) (that is, \(F \subseteq M\)) consider \(F' = F \cap X\) (which is not empty by the choice of \(X\)) and treat the collection \(F' = \{F' \mid F \in F\}\) as the base of a topology, denote it \(\tau_X\) on \(X\).

**Definition.** The finite topological space \((X, \tau_X)\) is called the pattern space for the manifold \(M\) with respect to the covering \(F\).

1.2 The graphs of pattern spaces

With each pattern space, being a finite topological space its Hasse graph can be associated in the following way \[\text{12}\]. The vertices of the graph are the points of \(X\). Two points \(x, y \in X\) are linked with the dart \(x \to y\) if and only if the following holds:

\[
\forall A \in \tau_X \quad A \ni x \Rightarrow A \ni y
\]

It can be verified directly that the obtained graph is reflexive and transitive. Note that in general there may exist points \(x, y \in X\) such that \(x \to y\) and \(y \to x\), (see section \[\text{13}\]).

**Lemma 1.** A subset \(A \subseteq X\) is open if and only if with each its point \(a \in A\) it contains all the points \(b \in X\) linked with \(a\):

\[
A \text{ is open } \iff \forall a \in A \quad (\forall b \in X a \to b \Rightarrow b \in A)
\]

**Proof** follows immediately from \[\text{1}\].
Corollary. A subset $B \subseteq X$ is closed if and only if with each its point $b \in B$ it contains all $c \in X$ such that $c \to b$:

$$B \text{ is closed } \iff \forall b \in B \quad (\forall c \in X c \to b \Rightarrow c \in B) \quad (3)$$

1.3 Example: a circle

Let $M = S^1$ be a circle: $M = \{e^{i\phi}\mod(2\pi)\}$. Consider its covering $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ defined as:

$$F_1 = (\pi/4, 3\pi/4) \quad F_2 = (-3\pi/4, -\pi/4)$$
$$F_3 = (-3\pi/4, 3\pi/4) \quad F_4 = (\pi/4, 7\pi/4)$$

Let $X = \{0, \pi/2, \pi, -\pi/2, \pi/6\}$. Then the graph of the appropriate pattern space is depicted on Fig. 1.

In the examples presented below the transitive darts will often be omitted.

1.4 Example: the real line

Let $M = \mathbb{R}^1$. Fix up a positive integer $N$ and consider the covering $\mathcal{F} = \{F_i \mid i = 0, \ldots, N\}$ (Fig. 2) with

$$F_0 = \mathbb{R}^1 \quad (4)$$
$$F_i = (i-1/2, +\infty) \quad (5)$$

Let $X = \{0, 1, \ldots, N\}$, then the Hasse graph of $X$ is presented on Fig. 3.
2 Algebraic superspace

2.1 Quasiorders and partial orders

As it was already established in 1.2, each pattern space can be associated with a reflexive and transitive directed graph. When such a graph is set up, we may consider its darts as pointing out a relation between the points of \( X \), denote it also \( \rightarrow \). This relation has the properties:

\[ \forall x \in X \quad x \rightarrow x \]
\[ \forall x, y \in X \quad x \rightarrow y \text{ and } y \rightarrow z \text{ imply } x \rightarrow z \]  

(6)

A relation on an arbitrary set having the properties (6) is called QUASIORDER. When a quasiorder is antisymmetric:

\[ \forall x, y \in X \quad x \rightarrow y \text{ and } y \rightarrow x \text{ imply } x = z \]  

(7)

the relation \( \rightarrow \) is called PARTIAL ORDER.

2.2 Incidence algebras

In the case when \( M \) is a compact manifold, there is the algebra \( A = C^\infty(M) \) of all smooth functions on \( M \). \( A \) can be treated as the algebraic substitute of \( M \) in that sense that, given \( A \) considered algebra (i.e. linear space with associative product operation), there exist the algebraic techniques (the Gel’fand procedure) which restore the points of \( M \) together with its topology. In the case when \( X \) is a finite topological space, the attempts to consider even a broader algebra of continuous functions \( X \rightarrow \mathbb{C} \) fails since the structure of such algebra captures only the number of connected components of \( X \) and nothing more [12]. Although, if we treat \( X \) as quasiordered set, we can broaden a well-known algebraic scheme from combinatorics, namely, that of incidence algebra [9], slightly generalized to pattern spaces (being quasiordered sets, in general).

**Definition.** For a quasiordered set \( X \) define its INCIDENCE ALGEBRA \( A_X \), or simply \( A \) if no ambiguity occurs, as the collection of all complex-valued functions of two arguments vanishing on non-comparable pairs:

\[ A = \{ a : X \times X \rightarrow \mathbb{C} \mid a(x, y) \neq 0 \Rightarrow x \rightarrow y \} \]  

(8)

To make the defined linear space \( A \) algebra we define the product of two elements \( a, b \in A \) as:

\[ ab(x, y) = \sum_{z : x \rightarrow z \rightarrow y} a(x, z)b(z, y) \]  

(9)

It can be proved that the so-defined product operation is associative [9]. Since the set \( X \) is finite, the algebra \( A \) is finite-dimensional associative (but not commutative, in general) algebra over \( \mathbb{C} \).

Now let us clear out the meaning of the elements of \( A \). Let \( a \in A \) and \( x, y \) be two points of \( X \). If they are not linked by a dart then, according to (8), the value \( a(x, y) \) always vanishes. So, \( a(x, y) \) can be thought of as an assignment of weights (or, in other
words, transition amplitudes) to the darts of the graph $X$. In these terms the product (9) has the following interpretation. Let $c = ab$, then $c(x, y)$ is the sum of the amplitudes of all allowed two-step transitions, the first step being ruled by $a$ and the second by $b$ (Fig. 4).

![Diagram](image)

Figure 4: Allowed transitions on pattern space.

while the element $c(x, y)$ of the multiple product $c = a_1 \ldots a_n$ looks similar to the Feynman sum over all paths from $x$ to $y$ allowed by the graph $X$ of the length $n$ and the closest physical counterpart of the elements of the incidence algebra are $S$-matrices.

So, the transition from pattern spaces to algebras is described. The inverse procedure of ”spatialization” will be described below in the Section 4.

### 2.3 The standard matrix representation of incidence algebras

Given the incidence algebra of a pattern space $X$, its standard matrix representation is obtained by choosing the basis of $\mathcal{A}$ consisting of the elements of the form $e_{ab}$, where $ab$ range over all ordered pairs $a \rightarrow b$ of elements of $X$, defined as:

$$e_{ab}(x, y) = \begin{cases} 1 & x = a \text{ and } y = b \text{ (provided } a \rightarrow b) \\ 0 & \text{otherwise} \end{cases}$$

We can also extend the ranging to all pairs of elements of $X$ by putting $e_{ab} \equiv 0$ for $a \nrightarrow b$. Then the product (9) reads:

$$e_{ab}e_{cd} = \delta_{bc}e_{ad}$$

With each $a \in \mathcal{A}$ the following $N \times N$-matrix ($N$ being the cardinality of $X$) is associated:

$$a \rightarrow a_{ik} = a(x_i, x_k)$$

Let $I$ be the incidence matrix of the graph $X$, that is

$$I_{ik} = \begin{cases} 1 & x_i \rightarrow x_k \\ 0 & \text{otherwise} \end{cases}$$

then the elements of $\mathcal{A}$ are represented as the matrices having the following property:

$$\forall i, k \ a_{ik}I_{ik} = a_{ik} \quad \text{no sum over } i, k$$

The product $c = ab$ of two elements is the usual matrix product:

$$c_{ik} = c(x_i, x_k) = \sum_{i \rightarrow l \rightarrow k} a(x_i, x_l)b(x_l, x_k) = \sum_{\text{forall} l} a_{il}b_{lk}$$

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That means, we have so embedded \( \mathcal{A} \) into the full matrix algebra \( \mathcal{M}_N(\mathbb{C}) \), that \( \mathcal{A} \) is represented by the set of all matrices satisfying (13). So, to specify an incidence algebra in the standard representation we have to fix the template matrix \( I_{ik} \). We can always re-enumerate the elements of \( X \) to make the template \( I_{ik} \) upper-block-triangular matrix with the blocks corresponding to cliques. In particular, when \( X \) is partially ordered, each clique contains exactly one element of \( X \), and the incidence matrix \( I \) is upper triangular.

### 2.4 Examples

Return to the examples of Section 1. The first example was the circesimulated by the pattern space \( X \) (Fig. 1) having the following incidence matrix:

\[
I_{\text{circle}} = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The standard matrix representation of the real line yields the algebra \( T_N \) of all upper triangular matrices with the incidence matrix

\[
I_{\text{line}} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

### 2.5 The algebraic superspace

Now we are in a position to introduce the arena for the future topodynamics of pattern spaces. If we fix up the cardinality \( N \), we already know that any pattern space of this cardinality can be isomorphically restored by its incidence algebra which, in turn, can be represented by \( N \times N \) matrices. So, the hazy question of what is the room for all topologies has the natural solution in finitary case: we can consider the space \( \mathcal{M} = \text{Mat}_N(\mathbb{C}) \) of all \( N \times N \) matrices. From now on this space will be referred to as FINITARY ALGEBRAIC SUPERSPACE.

As it will be shown in section 4, any subalgebra of \( \mathcal{M} \) gives rise to a finite topological space. So, if we treat \( \mathcal{M} \) as state space, the topologies are associated with the subspaces of \( \mathcal{M} \) with the only discrepancy with the conventional quantum mechanical approach that these subspaces are closed in a specific algebraic sense. Namely, for any subspace \( \mathcal{A} \subseteq \mathcal{M} \) its closure is build as the algebraic hull of \( \mathcal{A} \), that is, the intersection of all subalgebras of \( \mathcal{M} \) containing \( \mathcal{A} \).

### 3 Topological features and constructions in algebraic terms

#### 3.1 Connectedness

A topological space \( X \) is called CONNECTED if it contains no proper subsets being both closed and open, and LINEARLY CONNECTED if each pair \( x, y \) of its points can be connected by a path (a continuous mapping \( p : [0,1] \to X \) such that \( p(0) = x \) and \( p(1) = y \)). For finite topological spaces the notions of connectedness and linear connectedness coincide.
Definitions. Given a directed graph $X$, its underlying graph $UN\overline{X}$ is the undirected graph obtained from $X$ by forgetting the direction of all its darts. An undirected path in $X$ connecting $x, y \in X$ is the sequence $x = x_0, x_1, \ldots, x_n = y$ of vertices of $X$ such that each pair $(x_i, x_{i+1})$ is linked by an arc of the underlying graph $UN\overline{X}$.

Statement. Let $x, y \in X$. Then they can be connected by a path (in the topological sense) if and only if they can be connected by an undirected path defined above.

Proof. See \cite{12}.

3.2 Disjoint sums

Let $X_1, X_2$ be two pattern spaces such that $X_1 \cup X_2 = \emptyset$ and $X := X_1 \cap X_2$ be their disjoint sum. Denote by $A = A(X)$, $A_1 = A(X_1)$, $A_2 = A(X_2)$ the appropriate incidence algebras.

Statement. $A$ is the direct sum:

$$A = A_1 \bigoplus A_2$$  \hspace{1cm} (14)

Proof. Since $X_1 \cup X_2 = \emptyset$, both $X_1$ and $X_2$ are clopen subsets of $X$, hence, according to \cite{9}, no pair of points $x^1_i \in X_1$ and $x^2_k \in X_2$ can be connected by a dart of $X$. Therefore for any any $\alpha \in A$ we have $a(x^1_i, x^2_k) = a(x^1_i, x^2_i) = 0$. Consider the projections $\pi_\alpha : A \to A_\alpha$, $\alpha = 1, 2$ defined as follows:

$$(\pi_\alpha a)(x^\alpha_i, x^\alpha_k) = a(x^\alpha_i, x^\alpha_k)$$

and the injection $i_1 : A_1 \to A$:

$$(i_1 a_1)(x^1_i, x^1_k) = a_1(x^1_i, x^1_k)$$

$$(i_1 a_1)(x^2_i, y) = (i_1 a_1)(y, x^2_i) = 0 \quad \forall y \in X$$

Clearly $i_1 \pi_1 i_1 \pi_1 = i_1 \pi_1$ and $\pi_1 i_1 a_1 = a_1$ for all $a_1 \in A_1$, hence $A_1$ is the direct summand of $A$, as well as $A_2$ which is proved in the same way.

3.3 Boolean machinery

In \cite{12} the Boolean machinery to transform pattern spaces was suggested. A pattern space was treated as directed graph and two basic operations of stretching and cutting darts were introduced. This allowed to consider the stepwise changing of the topology of the pattern space. Let us translate these operations into the language of incidence algebras. Begin with the stretching operator.

Let $X$ be a pattern space and $a, b$ ($a \neq b$) be a pair of its vertices. The stretching operator $S^{ab}$ stretches the dart from $a$ to $b$ (in particular, does nothing if the dart already exists). The result of this only stretching may yield a non-transitive \cite{10} graph, so, to stay within pattern spaces we have to add the lacking darts to make it transitive. In the language of incidence algebras this procedure looks as follows. We have $A_X \subseteq \mathcal{M}$ in the standard representation \cite{10}. Applying $S^{ab}$ means adding $e_{ab}$ to the set of basis vectors and then we form the linear span of $(A_X \cup \{e_{ab}\})$. The result may not be the subalgebra of $\mathcal{M}$, so, to render it subalgebra we have to consider its algebraic span, which results in adding the basis elements $e_{a'b'}$ for each $e_{a'a} \neq 0$ and $e_{ab'}$ for each $e_{bb'} \neq 0$.

The cutting operator $C^{ab}$ removes the element $e_{ab}$ from the basis, and then the algebraic hull is formed. It may happen that the dart $a \to b$ being composite is unremovable and we return to the same algebra, and, hence, to the same pattern space.
4 The spatialization procedure

This section describes the procedure reverse to that described in the section 2. Namely, the suggested spatialization procedure having a finite-dimensional algebra on its input, manufactures a quasiordered set. Being applied to an incidence algebra $A_X$ of a quasiordered space $X$, it yields the initial space $X$ (up to a graph isomorphism).

It is assumed that the Reader of this section is familiar with the basic notions of the theory of associative algebras such as ideal, radical, semisimplicity and so on, and I will use these terms without defining them. Although, it seems appropriate to introduce the necessary definitions from the theory of partially ordered sets.

4.1 Interlude on partial and quasiorders

Let $(Y, \rightarrow)$ be a quasiordered set (6). Define the relation $\sim$ on $Y$

$$x \sim y \iff x \rightarrow y \text{ and } y \rightarrow x$$

being equivalence on $Y$, and consider the quotient set $X = Y/\sim$. Then $X$ is the partially ordered set (7).

When $Y$ is a pattern space, the transition from $Y$ to $X$ has the following meaning: $X$ is obtained from $Y$ by smashing cliques to points. Contemplating this procedure we see that $X$ may also be treated as the subgraph obtained from $Y$ by deleting all but one `redundant' vertices with adjacent (both incoming and outgoing) darts.

Now let us study how the relation between the quasiorders and associated partial orders looks in terms of incidence algebras. Let $\mathcal{A} = \mathcal{A}(Y)$ and $\mathcal{A}' = \mathcal{A}(Y/\sim)$. Then $\mathcal{A}'$ is the subalgebra of $\mathcal{A}$. In the standard matrix representation (10) $\mathcal{A}'$ is obtained as follows:

1. Select the set $X_R \subseteq X$ of redundant vertices (say, by checking out the identical rows of the incidence matrix $I$ (12))

2. Select the set $E_R$ of redundant basis elements (10):

$$e_{ab} \in E_R \iff a \in X_R \text{ or } b \in X_R$$

3. Delete the elements of $E_R$ from the basis, then

$$E_X = \{\text{the basis of } \mathcal{A}_Y\} \setminus E_R$$

is the basis of the incidence algebra $A_X$.

We shall also consider the inverse procedure of EXPANDING a partially set $X$ to a quasiorder $Y$. To each point of $x \in X$ a positive integer $n_x$ is assigned (which can be thought of as a sort of inner dimension — a room for gauge transformations). Then each $x$ is replaced by its $n_x$ copies linked between each other by two-sided darts and having all the incoming and outgoing darts the same as $x$.

So, given a quasiordered set $Y$, we can always represent it as the partially ordered set $x$ of its cliques equipped with the additional structure: to each $x \in X$ an integer $n_x \geq 1$ thought of as the cardinality of appropriate clique is assigned:

$$Y = (X, n_x)$$

(17)
4.2 The spatialization procedure.

Now let us explicitly describe the construction which will build pattern spaces by given finite-dimensional algebras. Let \( \mathcal{A} \) be a subalgebra of the full matrix algebra \( \text{Mat}(n, \mathbb{C}) \). Denote by \( \mathcal{R} \) the radical of the algebra \( \mathcal{A} \). To build the pattern space associated with \( \mathcal{A} \) the following is to be performed.

- **Step 1. Creating cliques.** Form the quotient \( \mathcal{A}' = \mathcal{A}/\mathcal{R} \) (being always semisimple since \( \mathcal{R} \) is the radical of \( \mathcal{A} \)). Denote by \( \mathcal{K} \) the center of \( \mathcal{A}' \):
  \[
  \mathcal{K} = \text{Center}(\mathcal{A}/\mathcal{R})
  \]
  Then define the set \( X \) of cliques as the set of all characters (i.e. linear multiplicative functionals) on \( \mathcal{K} \):
  \[
  X := \chi(\mathcal{K})
  \]

- **Step 2. Assigning cardinality to cliques.** Since the algebra \( \mathcal{A} \) is semisimple it is the direct sum of simple algebras and the set \( X \) labels its simple components \( \mathcal{A}'_x \):
  \[
  \mathcal{A}' = \bigoplus_{x \in X} \mathcal{A}'_x
  \]
  Each \( \mathcal{A}'_x \) being simple finite-dimensional algebra has the exact representation as the algebra of all \( n_x \times n_x \) matrices. Assign this number \( n_x \) to each \( x \in X \) and call it the cardinality of the clique \( x \).

- **Step 3. Stretching the darts.** For each character \( x \in X \) consider its annihilating subset \( \mathcal{K}_x \subseteq \mathcal{K} \subseteq \mathcal{A}' \) and span on it the two-sided ideal \( \mathcal{N}'_x \) in \( \mathcal{A}' \):
  \[
  \mathcal{N}'_x := \mathcal{A}'\mathcal{K}_x\mathcal{A}'
  \]
  and consider its preimage \( \mathcal{N}_x \) with respect to the canonical projection \( \pi : \mathcal{A} \to \mathcal{A}' = \mathcal{A}/\mathcal{R} \):
  \[
  \mathcal{N}_x := \pi^{-1}(\mathcal{N}'_x)
  \]
  being the ideal in \( \mathcal{A} \).
  For each pair \( x, y \in X \), \( x \neq y \) form two linear subspaces of \( \mathcal{A} \): \( \mathcal{N}_x \cap \mathcal{N}_y \) and \( \mathcal{N}_x \cdot \mathcal{N}_y \) and consider the quotient linear space:
  \[
  Q(x, y) := \mathcal{N}_x \cap \mathcal{N}_y / \mathcal{N}_x \cdot \mathcal{N}_y
  \]
  Then, if and only if \( Q(x, y) \neq 0 \) stretch the dart \( x \to y \).

**Remark.** The last step is based on the well known construction called scheme of a finite-dimensional algebra.

When (18) is checked for all \( x, y \), the non-transitive predecessor of the partially ordered set \( X \) is obtained. To have \( X \), form the transitive closure of \( X \):
  \[
  \text{darts}X := \{(x, x)\}_{x \in X} \cup \{(x, z) \mid \exists x = y_0, \ldots, y_n = z \in \mathcal{Q}(y_i, y_{i+1}) \neq 0\}
  \]

So, the pattern space \( Y = (X, n_x) \) (17) is completely built. In the sequel denote the quasiordered set furnished by the spatialization procedure applied to the algebra \( \mathcal{A} \) by \( \text{spat} \):
  \[
  Y = \text{spat}\mathcal{A}
  \]
Remark. Being applied to the incidence algebra of a quasiorder \( Y \), this procedure restores \( Y \) up to an isomorphism of quasiorders, as it follows from the Stanley’s theorem \([11]\).

5 Differential geometry in algebraic superspace

It was shown in the previous section how, starting from a particular algebra \( \mathcal{A} \) to extract from it something which may be interpreted as substitute of spacetime. Although, the question arises where this algebra \( \mathcal{A} \) can be taken from. As it was already outlined in section \( \ref{sec:algebra} \), the variety of this algebras can be placed into an algebraic superspace whose role in the case of fixed cardinality \( N \) of the pattern space we expect to be furnished, may be played by the full matrix algebra \( \text{Mat}(N, \mathbf{C}) \). In this section I am going to formulate the finitary counterparts of differential structures in which the basic notions of general relativity are formulated. For the overview of the mathematical problems related with such reinterpretation the Reader is referred to \([8]\).

5.1 Basic algebras.

Loosely speaking, the notion of basic algebra is the non-commutative generalization of Einstein algebras suggested by Geroch. His main observation was \([4]\) that, building general relativity, the notion of the spacetime manifold \( M \) is essentially used only once: to define the algebra \( \mathcal{A} = C^\infty(M) \) of all smooth functions on \( M \). All the forthcoming notions can be then reinterpreted in mere terms of \( \mathcal{A} \). For instance, vector fields are defined as derivations of \( \mathcal{A} \), that is linear mappings \( v : \mathcal{A} \to \mathcal{A} \) enjoying the Leibniz rule:

\[
v(ab) = va \cdot b + a \cdot vb
\]

and so on.

So, let \( \mathcal{S} \) be an algebraic superspace, that is, a finite dimensional algebra. Choose the basis \( \{E_0, E_1, \ldots\} \) of \( \mathcal{S} \):

\[
\mathcal{A} = \text{span}\{E_0, E_k \mid k = 1, \ldots\}
\]

so that

\[
E_0 = 1 \quad \text{Tr}E_k = 0
\]

where 1 is the unit element of \( \mathcal{A} \) (the meaning of this requirement will be clarified below).

The product in \( \mathcal{S} \) can be defined in terms of structural constants:

\[
E_0 E_i = E_i E_0 = E_i \quad ; \quad E_i E_k = P^i_{kl} E_l
\]

so that the associativity condition \((E_i E_k)E_l = E_i (E_k E_l)\) holds:

\[
P^m_{ik} P^n_{ml} = P^m_{im} P^n_{kl}
\]

Since \( \mathcal{S} \) is in general non-commutative, the commutators of basis elements do not vanish. Denote them \( L^l_{ik} \):

\[
E_i E_k = E_i E_k - E_k E_i = 2L^l_{ik} E_l
\]

\[
L^l_{ik} = \frac{1}{2}(P^l_{ik} - P^l_{ki}) = -L^l_{ik}
\]
5.2 Vectors.

The vectors are the derivations of the algebra $S$. In the case when $S = \text{Mat}(n, \mathbb{C})$ all derivatives are exhausted by inner ones. Moreover, they are in 1-1 correspondence with zero-trace matrices. So, if we choose the basis $E_1, \ldots$ elements of $S$ to be of zero trace, they can serve as the basis for the space $V$ of all vectors as well:

$$V = \text{span}\{E_k \mid k = 1, \ldots\}$$

(24)

The space $V$ is the Lie algebra with respect to matrix commutation, therefore the constants $L^l_{ik}$ are the Lie constants for $V$. Note that we always have the embedding $E_k \to E_k$ of $V$ into $S$ as that of linear spaces.

Being finite-dimensional Lie algebra, $V$ possesses two canonical forms: the trace of the Lie constants

$$L_i = L^m_{mi}$$

(25)

and the Killing form

$$K_{ij} = L^m_{ni}L^n_{mj}$$

(26)

Remark. In the case of the standard basis $\{e_{ab}\}$ of matrix units, the interpretation of the elements of $V$ as vector fields becomes very transparent. Namely, each derivation is associated with the assignment of a weight to each dart of the graph of pattern space.

5.3 Connection, curvature and all that.

Since the space $V$ plays the role of "tangent bundle", the connection can be defined as a linear mapping $D : V \times V \to V$ being derivation with respect to the second argument.

$$D(E_i, E_k E_l) = D(E_i, E_k E_l) + E_k D(E_i, E_l)$$

(27)

A particular choice of the connection $D$ can be set up by defining the set of appropriate structural constants

$$D(E_i, E_k) = D^m_{ik} E_m$$

(28)

then (27) reads:

$$D^m_{im} P^n_{kl} = D^m_{ik} P^n_{ml} + D^m_{il} P^n_{km}$$

Now, when the connection is defined, introduce the torsion associated with the connection $D$:

$$T(E_i, E_k) := D(E_i, E_k) - D(E_k, E_i) - [E_i, E_k]$$

In the sequel we shall be interested in torsion-free connections: $T(E_i, E_k) = 0$. It can be expressed as the following relations between the structural constants: $D_{ik}^l - D_{ki}^l - 2L_{ik}^l = 0$, or, in a more suitable form:

$$D_{ki}^l = D_{ik}^l - 2L_{ik}^l$$

(29)

In the non-commutative environment we can keep using the standard definition of Riemann and Ricci forms (see, e.g. [3]):

$$R(X, Y)Z = D_X (D_Y Z) - D_Y (D_X Z) - D_{[X,Y]} Z$$

(30)
Substitute basis vectors $E_i, E_k, E_l$ into (30) and decompose the left side over the basis denoting $R(E_i, E_k)E_l$ by $R^{n}_{ikl}E_n$, then these coefficients are expressed via Christoffel symbols $D^i_{jk}$ (28) as follows:

$$R^{n}_{ikl} = D^n_{im}D^m_{kl} - D^n_{km}D^m_{il} - 2D^n_{ml}L^m_{ik}$$  \hspace{1cm} (31)

The Ricci tensor is the trace of the Riemann one, and we can obtain its coefficients by contracting (31) over the pair of indices $n-k$:

$$R_{il} = R^k_{ikl} = D^k_{im}D^m_{kl} - D^k_{km}D^m_{il} - 2D^k_{ml}L^m_{ik}$$

which yields after replacing the summation indices in the first term and taking (29) into account:

$$R_{il} = D^k_{mi}D^m_{kl} - D^k_{km}D^m_{il}$$  \hspace{1cm} (32)

5.4 The metric and concertedness condition.

The metric tensor can be defined as a Hermitian form on $V$ taking the values in $S$. So, its most general form is:

$$g(E_i, E_k) = g_{ik}E_0 + g^m_{ik}E_m$$  \hspace{1cm} (33)

with $g_{ik} = \overline{g}_{ki}$ and $g^m_{ik} = \overline{g}^m_{ki}$, where the bar means usual complex conjugation. The requirement of nondegeneracy of the metric can be formulated here in different ways. We shall impose it in the following form:

$$\det g_{ik} \neq 0 \quad \text{and} \quad \det g^m_{ik} \neq 0 \quad \forall m$$  \hspace{1cm} (34)

By CONCERTEDNESS CONDITION I mean the analog of the well-known classical requirement $\nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ which can be rewritten as follows:

$$D(E_i, g(E_k, E_l)) = g(D(E_i, E_k), E_l) + g(E_k, D(E_i, E_l))$$  \hspace{1cm} (35)

Substituting (28, 33) to (35) and assuming the covariant derivative of $E_0$ to be zero, we obtain:

$$0 = g_{ml}D^m_{ik} + g_{km}D^m_{il}$$  \hspace{1cm} (36)

and consider (36) as equations with respect to the Christoffel symbols $D^m_{ik}$. From now on, to avoid considerable technical problems, restrict ourselves by such metrics that for any basis vectors $E_i, E_k$ the values of $g_{ik}$ are real, therefore

$$\text{Im} g_{ik} = 0 \Rightarrow g_{ik} = g_{ki}$$  \hspace{1cm} (37)

Now rewrite the first equation (36) using (29):

$$D^m_{ik}g_{ml} + D^m_{li}g_{km} = 2L^m_{li}g_{km}$$

and perform the cyclic permutation of the free indices $(ikl)$ taking (29) and (37) into account:

$$D^m_{ik}g_{lm} = L^m_{ik}g_{lm} + L^m_{il}g_{km} - L^m_{kl}g_{im}$$  \hspace{1cm} (38)

thus, by virtue of (34), the values of $D^m_{ik}$ are completely determined by those of $g_{ik}$. Therefore the remaining equations from (36) are to be considered as concertedness condition for $g^m_{ik}$ as well, completely determining the values of the "non-commutative part" of the metric when its commutative part $g_{ik}$ is given.
Remark. I do not consider here a possible reasonable weakening of the condition (34), namely, the requirement of the maximality of the rank of the matrix \((g_{ik}, g^{m}_{ik})\). In this case the role of \(g_{lm}\) in (38) can be played by a non-degenerate submatrix of \((g_{ik}, g^{m}_{ik})\).

Under the assumptions (34,37) the standard convention on raising and lowering tensor indices by means of \(g_{ik}\) can be used. Thus (38) can be rewritten as:

\[
D_{lik} = L_{lik} + L_{ki} - L_{ikl}
\]  

(39)

So, the formulas (38,39) allow us to calculate the coefficients of the connection concerted with the metric \(g\).

5.5 Einstein – Hilbert variational principle.

Having the nondegenerate metric tensor \(g_{ik}\) in our disposal, we can form the scalar curvature as follows:

\[
R = g^{ik} R_{ik} = g^{ik} D^{m}_{ni} D^{n}_{mk} - g^{ik} D^{n}_{nm} D^{m}_{ik}
\]  

(40)

where \(g^{ik}\) denotes, as usually, the matrix inverse to \(g_{ik}\). Begin the analysis of this formula from its second term \(g^{ik} D^{n}_{nm} D^{m}_{ik}\). It contains the factor \(D^{n}_{nm}\), which, according to

\[
D^{n}_{nm} = g^{ik} D_{ikm} = 2L^{k}_{km} = 2L_{m}
\]

does not depend on the metric. So, the second term of (40) reads:

\[
g^{ik} D^{n}_{nm} D^{m}_{ik} = 2g^{ik} L_{m} D^{m}_{ik} = g^{ik} D^{m}_{ik} = -4g^{ik} L_{i} L_{k}
\]  

(41)

The routine manipulations with indices turn the first term of (40) to:

\[
g^{ik} D^{m}_{ni} D^{n}_{mi} = D^{m}_{mi} D^{n}_{mi} = g^{ik} g^{jl} g^{mn} L^{m}_{ij} L^{n}_{kl} + 2g^{ik} K_{ik}
\]

So, (40) reads:

\[
R = g^{ik} (g^{jl} g^{mn} L^{m}_{ij} L^{n}_{kl} + 2K_{ik} + 4L_{i} L_{k})
\]  

(42)

Note that \(R\) has already numeric values rather than scalar ones, so we can consider its variation:

\[
\delta R = 2\delta g^{ik} (L_{min} L^{m}_{kn} + K_{ik} + 2L_{i} L_{k} - \frac{1}{2} L_{imn} L_{kmn}) = 2G_{ik} \delta g^{ik}
\]

where \(G_{ik}\) is the analog of the Einstein tensor:

\[
G_{ik} = L_{min} L^{m}_{kn} - \frac{1}{2} L_{imn} L_{kmn} + K_{ik} + 2L_{i} L_{k}
\]  

(43)

Note that the first two terms of (43) depend on the metric unlike the remaining two: \(K_{ik}\) and \(L_{i} L_{k}\).

5.6 Eigen-subalgebras and topologimeter.

Suppose we managed to solve the finitary matrix analog of the Einstein equation

\[
G = T
\]  

(44)

where \(G\) is the above defined Einstein tensor (43) and \(T\) is a finitary counterpart of the stress-energy tensor. And suppose that the resulting metric tensor \(g_{ik}\) splits the linear space \(V\) into a set of mutually orthogonal (with respect to \(g\)) subspaces:
\[ V = V_1 + V_2 + \ldots \]  
(45)

The crucial point of the techniques suggested is that we may consider the elements of \( V \) as those of the basic algebra \( S \). Therefore the decomposition (45) gives rise to the following decomposition of \( S \):

\[ S = E_0 + A_1 + A_2 + \ldots + \text{perhaps, some remainder} \]  
(46)

that is, with each \( V_i \) from (45) we can associate a subalgebra \( A_i \subseteq S \) spanned on the linear subspace \( V_i \) of \( S \). Consequently, we can apply the spatialization procedure described in section 4 to each \( A_i \), and then the solution \( g_{ik} \) of the Einstein-like equation (44) produces the family \( \{X_i\} \) of pattern spaces (47):

\[ X_i = \text{spat}(\text{span}_A(V_i)) \]  
(47)

In standard quantum mechanics an apparatus measuring an entity \( Q \) can be described as follows. We have the state space \( \mathcal{H} \) of a system and a family of mutually orthogonal (with respect to the inner product in \( \mathcal{H} \)) subspaces \( \{\mathcal{H}_i \subseteq \mathcal{H}\} \). With each subspace \( \mathcal{H}_i \) we associate a value of the measured entity \( Q \). In standard quantum mechanics these values are real numbers, though it is a mere matter of choice.

Now return to the suggested machinery. We can think of the pair \( (V, g) \) as state space, and consider the decomposition (45) as that associated with the measuring apparatus. But what should we assign to each \( V_i \)? The answer is given by (47): these are pattern spaces. So, we may conclude that the finite-topology-valued observable on the state space \( V \) is built, and the hypothetical device associated with the partition (45) may thus be called topologimeter.

6 Concluding remarks.

The machinery was suggested to draw the idea of description of varying spacetime topology to the level of calculations. It consists of the following:

- **The coarse-graining procedure** which replaces the continuous spacetime by finitary pattern space described in section 1.

- **The incidence algebras** associated with pattern spaces were introduced in section 2 to replace graphs being discrete object by their nature by linear spaces making it possible to embody them in a greater object (algebraic superspace) of the same type and enable the possibility to describe continuous evolution.

- **The spatialization procedure** suggested in section 4 is the inverse to the construction of incidence algebras and produces pattern spaces by given finite-dimensional algebras.

- **The Einstein – Hilbert variational principle** in finitary form was written down in section 5 for the algebraic superspace. It can give rise to a family of mutually orthogonal subspaces each of which is associated (via spatialization procedure) with certain topology. The construction is interpreted as the mathematical description of topologimeter.

So, what are the consequences of the suggested approach? We now have the machinery in our disposal which is able to describe the changing spacetime topology on the level of pattern spaces. It is seen within this approach that it is pointless to speak of both events forming the spacetime and its topology before a particular measurement is
performed. Moreover, the situation when we can speak of separation of events looks very special, namely, the solution of the Einstein equation in the superspace must support the decomposition (45) which may not take place.

I should also mention a crucial question remaining beyond the scope of the presented work. It is the correspondence principle: to what extent pattern spaces really substitute continuous spacetime. There are two modes of answering this question. That first is to claim that, as a matter of fact, nobody is able to prove that spacetime is really continuous: there is no operationalistically sound procedure checking the continuity of the spacetime since it would require to consider the infinite number of events. Moreover, the individual event itself is an idealization rather than a testable entity [3]. Another way to corroborate the correspondence principle is to use the techniques proposed in [10] where the inverse limits of pattern spaces converging to continuous manifolds are studied.

The finite-dimensional models I consider may be applied to other fundamental theories which are devoted to describe the structure of spacetime. I will dwell upon two such theories. The first is the histories approach to quantum mechanics suggested by Griffiths and Hartle [2]. In Isham’s [7] version of this approach each particular pattern space (substituting a spacetime) may serve as the counterpart of a particular history, and the application of topologimeter can be associated with the decoherence if we assign a numeric value to each subspace of the decomposition (45). The second theory where the machinery I suggest may be relevant is the construction of spacetime from elementary constituents called urs [5]. When we replace spacetime by a pattern space, we may consider the latter as the set of linked darts and then ask what is the law linking them. Each dart, in turn, can be described by the smallest pattern space consisting of two points: the appropriate algebraic superspace for each virtual dart is the algebra of $2 \times 2$ complex matrices, where the Einstein like equations (43) can be solved completely without any additional requirements.

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