Estimation of residuals for the homogenized solution of quasi-periodic media

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Abstract

The convergence of the elastic coefficients residuals compared to the homogeneous solution, at varying size of the representative volume sample, is analysed in this work in the case of random two-dimensional media. In particular, it is considered the case of the masonry material which can be considered a heterogeneous solid with two phases (stones or bricks and mortar). Moreover, in relation to the peculiarities related to its effective realization, the masonry presents a quasi-periodic micro-structure. A procedure of numerical generation of wall portions has been developed by varying not only the scale ratio but also the mechanical ratio, between the characteristics of the stone and mortar, and the geometrical ratio, relative to the stones dimensions. Comparing to the homogeneous solution, the convergence of residuals has been highlighted in terms of probability density function and static moments, up to the second order, of the stiffness coefficients, log-Euclidean distance between matrices and closet isotropic material.

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1. Introduction

Several materials are characterized by a non homogeneous micro-structure and anisotropic mechanical behaviour. The differential equations that rule the problem related to mechanical response of these materials can be expressed using a two scales approach; the first scale can be associated to the microscopic characteristics while the second is associated with the macroscopic ones. The homogenization is a very useful tools to describe the macroscopic behaviour. In case of material with random micro-structure, the characteristics of the equivalent homogeneous material can be found by means of the Representative Volume Element (RVE). A special case is the masonry that can be regarded as an anisotropic heterogeneous material (composed by two phase: stone or bricks and mortar) with a quasi-periodic micro-structure. The homogenization approaches proposed in [1, 2, 3] permits to individuate the RVE and obtain the mechanical characteristics of the equivalent homogeneous material. An alternative method, which takes into account the quasi periodic micro-structure has been proposed in [4] by the Statistically Equivalent Periodic Unit Cell (SEPUC). In the context of the RVE approach a question of interest is the relation between the dimension of the selected RVE and the errors in estimation of equivalent mechanical characteristics. This aspect has received a very limited attention in literature. Let us consider to describe the elastic behaviour of heterogeneous media with random texture by an elliptic differential operators with random coefficient depend-
ing on small parameter. Under adequate condition [5, 6, 7], it is known that, as the parameter scale $\varepsilon \to 0$, the operator converge to an *averaged operator* with non random coefficients; this type of convergence is known as *G-convergence*. The accuracy of the approximation has been studied [8, 9, 10] but it is very difficult to adapt the proposed methods in case of special material as masonry which is characterized by a quasi-periodic texture. In order to investigate this problem, the authors, in previous papers [11, 12], dealt with the case of the beam with Young’s modulus randomly varying along the axis. In the present work, the approach is extended to the case of a region of the plane with randomly varying mechanical characteristics.

The convergence of the homogenized solution is analysed in terms of the parameters which characterize the microscopic scale ($\varepsilon$, ratio of elastic moduli, concentration ratio), by means of numerical simulations. The obtained results permits a preliminary estimation of the size of the RVE that assures a prefixed value of the error.

2. The quasi-periodic micro-structure of the historical masonry

When we consider the masonries of historical building, such as the two examples found in central Italy shown in Fig. 1, it appears evident that the micro-structure of this material is not completely random but some amount of regularity can be observed. In fact, the constituents phases, brick/stones and mortar, are assembled in a quasi-periodic texture. The stones have different dimensions (length and height), but those of roughly the same height are laid out to form horizontal rows. As a consequence, different rows have different height, nevertheless, the height of the stones of the same row have
small oscillation around the mean for that specific row. It should be noted that the length of the stones is also variable and the length-to-height ratio can assumes very different values (see Fig. 1); taking into account this observation, it is assumed, in what follows, that the length of the stones are uncorrelated with their height. Therefore, a quasi-periodic masonry arrangement of bricks/stones and mortar is obtained in such a way that horizontal stones of roughly the same height can be identified in the same row of the wall. Regarding the mortar joints, they have roughly the same thickness and the vertical ones are not aligned when two adjacent rows are considered.

[Figure 1 about here.]

3. Numerical generation of quasi-periodic samples

To assess the influence of both the mechanical and geometrical parameters of the constituent phases on the response of the masonry and homogenization residuals, a parametric analysis has been performed.

The fundamental characteristics of the quasi-periodic masonry are:

(i) the masonry wall is made by courses (rows) of stones;

(ii) it is possible to individuate continuous “bed joints” (or horizontal joints) of mortar, that is the length of every horizontal joints is equal to the width of the wall;

(iii) the “head joints” (or vertical joints) of two adjacent courses are not aligned.

[Figure 2 about here.]
The samples of quasi-periodic masonry are characterized by \( N \) rows with \( N \) stone (or \( N + 1 \) stones, in this latter case the first and the last are really half stones) and have a \( L_W \) overall length. In order to generate the samples, the following parameters have been chosen:

- the length scale ratio \( \varepsilon = \bar{L}_b/L_W \), where \( \bar{L}_b \) is the mean length of the stones;
- the geometrical ratio \( g_r = \Delta L_b/\bar{L}_b = \Delta H_{br}/\bar{H}_b \), where \( 2\Delta L_b \) is the variation interval of the length of the stones, \( \bar{H}_b \) and \( 2\Delta H_{br} \) are the mean and variation interval of the height of the stones;
- the mechanical ratio \( m_r = E_b/E_m \) between the elastic moduli of stones and mortar (the Poisson’s coefficient, both for stone and mortar, has been assumed equal to 0.2).

Given the previous conditions, the length ratio is \( \varepsilon = 1/N \).

Given \( \Delta L_b \) and \( \Delta H_{br} \), the length and eight of the stones have been assumed uncorrelated and vary randomly according a uniform law around the mean values. The same hypothesis have been assumed to generated the thickness of head \( t_h \) and bed mortar joints \( t_b \), around the means \( \bar{t}_h \) and \( \bar{t}_d \).

Considering the \( i \)-th course, the length of the \( j \)-th stone of the course is generated by

\[
L_{b,ij} = \bar{L}_b + U_j \Delta L_b
\]

where \( U_j \) indicates the \( j \)-th number generated from the uniform law in the interval \([-1/2, 1/2]\). The characteristic height of \( i \)-th the course is given by
\[ H^i_{br} = \bar{H}_b + U_i \Delta H_{br} \]

while the height of the \( j \)-th stone is obtained by

\[ H_{b,ij} = H^i_{br} + U_j \bar{t}_b \]

and its position is determined by translating its base by the quantity

\[ d_{ij} = U_j \frac{\bar{t}_b}{3} \]

Always considering the \( i \)-th course, the characteristic thickness of the bed mortar joint is give by

\[ t^i_b = \bar{t}_b + U_i \frac{\bar{t}_b}{3} \]

while \( j \)-th head mortar joint thickness thickness is

\[ t_{h,ij} = \bar{t}_h + U_j \frac{\bar{t}_h}{3} \]

The previous geometric quantities are shown in the Fig. 2(b).

For the numerical simulation the following mean values have been adopted: \( \bar{L}_b = 300 \text{ mm}, \bar{H}_b = 150 \text{ mm} \) and \( \bar{t}_b = \bar{t}_h = 10 \text{ mm} \). Moreover the length scale ratio, the mechanical ratio and the geometrical ratio used for the generations of the different sets are reported in Tab. 1.

[Table 1 about here.]

A total of 200 samples for each length scale ratio have been generated. Some examples of the obtained masonries are shown in Fig. 3.
Introducing the stress and strain mean values over the domain $V$

$$
\langle \sigma \rangle = \frac{1}{V} \int_{V} \sigma dv, \langle \epsilon \rangle = \frac{1}{V} \int_{V} \epsilon dv,
$$

with

$$
\sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix}
$$

the stiffness matrix $C$ has been determined by

$$
\langle \sigma \rangle = C \langle \epsilon \rangle \quad \text{with} \quad C = \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{33}
\end{bmatrix}
$$

The values of the stress and the strain have been evaluated by means of a finite element analysis, where four node plane stress elements have been used. In order to make comparisons, the stiffness tensor has been also evaluated using a sample with much larger dimension ($N = 12$), which can be assumed as the representative volume element: this tensor is referred to as the “homogenized” one.

4. Parametric analysis and stiffness coefficients behaviour

In the following the results obtained using the samples generated as described in the preceding section are presented. In particular, the estimates of the component $C_{11}$ of the stiffness tensor have been taken into account, nevertheless similar results hold also for others components. The values have been normalized to the corresponding component of the homogenized masonry, $C_{11}^h$. With reference to Tab. 1, it should be noted that the sets “set
are those with greater mechanical and geometrical ratios while those “set have smaller ratios. Moreover, $a$ denotes the “smallest” samples and $c$ the “greatest” samples in terms of dimension (or length scale ratio).

Indicating $c_1$ the concentration ratio, which is defined as the percentage of the volume sample occupied by the stones, the results for the same mechanical and geometrical ratio and different length scale ratio are shown in Fig. 4 where the dashed line denotes the value for the homogenized masonry as defined previously, whose concentration ratio was $c^h_1 = 0.845$. As can be observed, the estimates are less scattered as the dimension of the sample increases and as the mechanical characteristics ratio is smaller.

[Figure 4 about here.]

The estimations of the probability density functions are shown in Fig. 5. Again, it can be observed that the estimates are less scattered as the dimension of the sample increases and as the mechanical characteristics ratio are smaller.

[Figure 5 about here.]

Finally, the behaviour of the ratio between the mean value $\mu(C_{11})$ and the homogenized value $C^h_{11}$ in shown in Fig. 6(a), while the behaviour of the coefficient of variation $s(C_{11})/\mu(C_{11})$ is reported in Fig. 6(b). As can be observed, as the length scale ratio decreases (and therefore the dimension of the sample increases) the mean value tends to the homogenized value, while the coefficient of variation decreases. A similar behaviour can be also found for the others coefficients of the stiffness matrix.

[Figure 6 about here.]
5. Convergence measurement by distant between stiffness matrices

The previous analysis, that consider any single element of the stiffness matrix do not appear adequate, vice versa it is necessary to introduce a comprehensive measure of the convergence.

The stiffness matrix belong to $S(n)$ vector space of positive symmetric matrices in $M^{nxn}$, the space of $n \times n$ real matrices (in the present case $n = 3$). Given $S \in S(n)$ and introducing the spectral decomposition

$$S = \sum_{i=1}^{n} \lambda_i s_i s_i^T$$

where $\lambda_i$ are the eigenvalues, with $\lambda_i > 0$, and $s_i$ are the eigenvectors with $s_i^T s_j = \delta_{ij}$, the logarithm of the matrix is defined as

$$\text{Log } S = \sum_{i=1}^{n} \ln \lambda_i s_i s_i^T$$

For any pair $A, B \in S(n)$, the distance between these matrices can be quantified by three distinct metrics [13]:

- the conventional Euclidean or Frobenius metric $d_F$

$$d_F (A, B) = \| A - B \|$$

- the log-Euclidean distance $d_L$ [14]

$$d_L (A, B) = \| \text{Log } A - \text{Log } B \|$$

- and the Riemannian distance $d_R$ [15]

$$d_R (A, B) = \| \text{Log } (A^{-1/2}BA^{-1/2}) \|$$
where \( \| M \| = \left[ \text{tr} \left( M^T M \right) \right]^{1/2} \) for any \( M \in M(n) \).

Using the previously introduced distances, it is possible to compare the stiffness matrix of generated samples to the stiffness matrix of the homogenized material (relative to the wall with larger dimension, \( N = 12 \)). In the present paper the results relative to the log-Euclidean distance has been reported because this is invariant to respect the inversion operator, so that the distance in term of stiffness is equal to the one in terms of compliance [13].

In particular the distance \( d_L \) between the stiffness matrices of the samples obtained for different parameter and the homogenized one has been computed and the results are shown in Fig. 7

[Figure 7 about here.]

As can be observed, the minimum distance is for the samples which have a concentration ratio, \( c_1 \), closest to the homogenized one, \( c_1^h \) and larger dimension.

6. Closest isotropic moduli

For applications (e.g. numerical analysis of masonry building) assume a relevant interest to found the isotropic material closest to a given anisotropic material. This aspect has been considered in order to introduce a further convergence analysis.

It is worth noting that the stiffness matrix of an isotropic material depends only on the Young’s modulus, \( E \), and the Poisson’s ratio, \( \nu \), and in plane stress is given by
\[
C_{iso} = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\]

Therefore, it is possible to define the isotropic homogeneous material closest to an assigned orthotropic material as the one for which the distance between the corresponding stiffness matrices is minimum \[13\]; this allows to found the Young’s modulus and the Poisson’s ratio. Obviously, the results depend on the assumed distance: in what follows, the log-Euclidean distance \(d_L\) has been used.

The procedure is applied to the homogenized material, denoting the estimated values as \(E^h\) and \(\nu^h\), and to each generated sample.

In order to analyse the residual of homogenization, varying the length ratio and mechanical and geometrical characteristics of the sample, the results in terms of the Young’s modulus of the isotropic material closest to sample and the log-Euclidean distance \(d_L\) between the stiffness matrix of the isotropic material closest to sample and the (orthotropic) stiffness matrix of the sample are reported if Fig. 8 and Fig. 9 respectively, where the values have been normalized with respect to the values obtained for the homogenized material (relative to wall with larger dimension).

[Figure 8 about here.]

[Figure 9 about here.]

As can be observed, the values of the Young’s modulus follows the same trend of the values of \(C_{11}\) shown previously, whereas the distance tends to
decrease as the concentration ratio increases: this was expected, since as the $c_1$ increases the material tends to the homogenized material made of the stiffer phase only, which is obviously isotropic.

The estimation of the probability density function is shown in Fig. 10. Again, it can be observed that the estimates are less scattered as the dimension of the sample increases and as the mechanical characteristics ratio are smaller.

[Figure 10 about here.]

Finally, the behaviour of the ratio between the mean value $\mu(E)$ and the homogenized value $E_{11}^h$ in shown in Fig. 11(a), while the behaviour of the coefficient of variation $s(E)/\mu(E)$ is reported in Fig. 11(b). As can be observed, as the length scale ratio decreases (and therefore the dimension of the sample increases) the mean value tends to the homogenized value, while the coefficient of variation decreases.

[Figure 11 about here.]

These results allow to appreciate the rate of convergence and can be used to estimate the dimension of the RVE which permits to have an error under a prescribed value, in this contest the procedure proposed in a previous paper can be used.

7. Conclusions

The present paper dealt with the analysis of the elastic coefficients residuals compared to the homogeneous solution in the case of heterogeneous
two-dimensional media with random structure. Special attention is devoted to masonry material which can be considered as a solid composed by two phases (stones or bricks and mortar) with quasi-periodic micro-structure. A specific numerical procedure has been developed to generate numerically masonry wall samples by varying the scale ratio which has been defined as the ratio between the mean length of the stones and the length of the wall. At first the convergence of the elastic stiffness coefficients to the ones of the equivalent homogeneous continuum has been analysed. The effects of the geometrical ratio, among the stones sizes, and mechanical ratio, between Young modulus of phases, have been taken into account. The convergence to the homogeneous solution has been described in terms of probability density function and statistical moments up to the second order. Introducing several metrics, an overall criterium of convergence has been introduced in term of distance between the elastic stiffness matrices. In particular the log-Euclidean distance has been used because it is invariant to respect the inverse operator. Eventually, the effect of the length ratio, mechanical characteristic ratio and concentration on convergence and residuals behaviour have been analysed considering the closest isotropic elastic concept that can be very useful tool in the numerical analysis of large structure. Even if further studies are necessary, the obtained results, considering the convergence of the residuals, permit to introduce criteria to minimize errors in the detection of the representative volume element.
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[1] F. Cluni, V. Gusella, Homogenization of non-periodic masonry structures, International Journal of Solids and Structures 41 (2004) 1911–1923.

[2] V. Gusella, F. Cluni, Random field and homogenization for masonry with nonperiodic microstructure, Journal of Mechanics of Materials and Structures 1 (2) (2006) 357–386.

[3] N. Cavalagli, F. Cluni, V. Gusella, Strength domain of non-periodic masonry by homogenization in generalized plane state, European journal of mechanics A-solids 30 (2) (2011) 113–126.

[4] N. Cavalagli, F. Cluni, V. Gusella, Evaluation of a statistically equivalent periodic unit cell for a quasi-periodic masonry, International Journal of Solids and Structures 50 (25-26) (2013) 4226–4240.

[5] S. M. Kozlov, Averaging of random operators, Mathematics of the USSR-Sbornik 37 (2) (1980) 167–180.

[6] G. C. Papanicolaou, S. R. S. Varadhan, Boundary value problems with rapidly oscillating random coefficients, in: J. Fritz, J. L. Lebowitz, D. Szász (Eds.), Random fields, Vol. II of Colloquia mathematica Societatis János Bolyai, North-Holland, Amsterdam, 1981, pp. 835–873, (Esztergom, Hungary, 1979). MR 712714. Zbl 0499.60059.
[7] V. V. Yurinskii, Averaging of symmetric diffusion in random medium, Siberian Mathematical Journal 27 (4) (1986) 603–613.

[8] A. V. Pozhidaev, V. V. Yurinskiǐ, On the error of averaging symmetric elliptic systems, Mathematics of the USSR-Izvestiya 35 (1) (1990) 183–201.

[9] A. Bourgeat, A. Piatnitski, Estimates in probability of the residual between the random and the homogenized solutions of one-dimensional second-order operator, Asymptotic Analysis 21 (1999) 303–315.

[10] G. Bal, J. Garnier, S. Motsch, V. Perrier, Random integrals and correctors in homogenization, Asymptotic Analysis 58 (1-2) (2008) 1–26.

[11] F. Cluni, V. Gusella, Estimation of residuals for the homogenized solution of beam with random mechanical characteristics, Meccanica dei Materiali e delle Strutture 3 (2012) 49–56.

[12] F. Cluni, V. Gusella, Estimation of residuals for the homogenized solution: The case of the beam with random young’s modulus, Probabilistic Engineering Mechanics 35 (2014) 22–28.

[13] A. Norris, The isotropic material closest to a given anisotropic material, Journal of Mechanics of Materials and Structures 1 (2) (2006) 223–238.

[14] V. Arsigny, P. Fillard, X. Pennec, N. Ayache, Fast and simple calculus on tensors in the log-euclidean framework, in: J. Duncan, G. Gerig (Eds.), 8th Int. Conf. on Medical Image Computing and Computer-Assisted Intervention (MICCAI), no. 3749 in Lecture Notes in Computer Science, Springer, 2005, pp. 115–122.
[15] A. Norris, On the averaging of symmetric positive-definite tensors, Journal of Elasticity 82 (3) (2006) 273–296.
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