THE GROUP OF UNITAL $C^*$-EXTENSIONS

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Abstract. Let $A$ and $B$ be separable $C^*$-algebras, $A$ unital and $B$ stable. It is shown that there is a natural six-terms exact sequence which relates the group which arises by considering all semi-split extensions of $A$ by $B$ to the group which arises by restricting the attention to unital semi-split extensions of $A$ by $B$. The six-terms exact sequence is an unpublished result of G. Skandalis.

Let $A, B$ be separable $C^*$-algebras, $B$ stable. As is well-known the $C^*$-algebra extensions of $A$ by $B$ can be identified with $\Hom(A, Q(B))$, the set of $\ast$-homomorphisms $A \rightarrow Q(B)$ where $Q(B) = M(B)/B$ is the generalized Calkin algebra. Two extensions $\varphi, \psi : A \rightarrow Q(B)$ are unitarily equivalent when there is a unitary $u \in M(B)$ such that $\Ad q(u) \circ \psi = \varphi$, where $q : M(B) \rightarrow Q(B)$ is the quotient map. The unitary equivalence classes of extensions of $A$ by $B$ have an abelian semi-group structure thanks to the stability of $B$: Choose isometries $V_1, V_2 \in M(B)$ such that $V_1 V_1^* + V_2 V_2^* = 1$, and define the sum $\varphi \oplus \psi : A \rightarrow Q(B)$ of $\varphi, \psi \in \Hom(A, Q(B))$ by

\[(\psi \oplus \varphi)(a) = \Ad q(V_1) \circ \psi(a) + \Ad q(V_2) \circ \varphi(a).\] (1)

The isometries, $V_1$ and $V_2$, are fixed in the following. An extension $\varphi : A \rightarrow Q(B)$ is split when there is a $\ast$-homomorphisms $\pi : A \rightarrow M(B)$ such that $\varphi = q \circ \pi$. To trivialize the split extensions we declare two extensions $\varphi, \psi : A \rightarrow Q(B)$ to be stably equivalent when there is a split extension $\pi$ such that $\psi \oplus \pi$ and $\varphi \oplus \pi$ are unitarily equivalent. This is an equivalence relation because the sum (1) of two split extensions is itself split. We denote by $\Ext(A, B)$ the semigroup of stable equivalence classes of extensions of $A$ by $B$. It was proved in $[\text{Th}]$, as a generalization of results of Kasparov, that there exists an absorbing split extension $\pi_0 : A \rightarrow Q(B)$, i.e. a split extension with the property that $\pi_0 \oplus \pi$ is unitarily equivalent to $\pi_0$ for every split extension $\pi$. Thus two extensions $\varphi, \psi$ are stably equivalent if and only if $\varphi \oplus \pi_0$ and $\psi \oplus \pi_0$ are unitarily equivalent. The classes of stably equivalent extensions of $A$ by $B$ is an abelian semigroup $\Ext(A, B)$ in which any split extension (like 0) represents the neutral element. As is well-documented the semi-group is generally not a group, and we denote by

$\Ext^{-1}(A, B)$

the abelian group of invertible elements in $\Ext(A, B)$. It is also well-known that this group is one way of describing the $KK$-groups of Kasparov. Specifically, $\Ext^{-1}(A, B) = KK(SA, B) = KK(A, SB)$.

Assume now that $A$ is unital. It is then possible, and sometimes even advantageous, to restrict attention to unital extensions of $A$ by $B$, i.e. to short exact sequences

\[0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0\] (2)
of C*-algebras with E is unital, or equivalently to *-homomorphisms $A \to Q(B)$ that are unital. The preceding definitions are all amenable to such a restriction, if done consistently. Specifically, we say that a unital extension $\varphi : A \to Q(B)$ is unitally split when there is a unital *-homomorphism $\pi : A \to M(B)$ such that $\varphi = q \circ \pi$. The sum $\oplus$ of two unital extensions is again unital, and we say that two unital extensions are unitally stably equivalent when there is a unital split extension $\varphi, \psi : A \to Q(B)$ such that $\psi \oplus \pi$ and $\varphi \oplus \pi$ are unitarily equivalent. It was proved in [Th] that there always exist a unitally absorbing split extension $\pi_0 : A \to Q(B)$, i.e. a unitally split extension with the property that $\pi_0 \oplus \pi$ is unitarily equivalent to $\pi_0$ for every unitally split extension $\pi$. Thus two unital extensions $\varphi, \psi$ are unitally stably equivalent if and only if $\varphi \oplus \pi_0$ and $\psi \oplus \pi_0$ are unitarily equivalent. The classes of unitally stably equivalent extensions of $A$ by $B$ is an abelian semi-group which we denote by $\text{Ext}_{\text{unital}}(A, B)$. The unitally absorbing split extension $\pi_0$, or any other unitally split extension, represents the neutral element of $\text{Ext}_{\text{unital}}(A, B)$, and we denote by

$$\text{Ext}_{\text{unital}}^{-1}(A, B)$$

the abelian group of invertible elements in $\text{Ext}_{\text{unital}}(A, B)$. As we shall see there is a difference between $\text{Ext}_{\text{unital}}^{-1}(A, B)$ and $\text{Ext}^{-1}(A, B)$ arising from the fact that while the class in $\text{Ext}^{-1}(A, B)$ of a unital extension $A \to Q(B)$ can not be changed by conjugating it with a unitary from $Q(B)$, its class in $\text{Ext}_{\text{unital}}^{-1}(A, B)$ can. In a sense the main result of this note is that this is the only way in which the two groups differ.

Note that there is a group homomorphism

$$\text{Ext}_{\text{unital}}^{-1}(A, B) \to \text{Ext}^{-1}(A, B),$$

obtained by forgetting the word ‘unital’. It will be shown that this forgetful map fits into a six-terms exact sequence

$$
\begin{array}{ccc}
K_0(B) & \xrightarrow{u_0} & \text{Ext}_{\text{unital}}^{-1}(A, B) \\
\downarrow{i_5^*} & & \downarrow{i_1^*} \\
\text{Ext}^{-1}(A, SB) & \xrightarrow{u_1} & K_1(B)
\end{array}
$$

(3)

where $SB$ is the suspension of $B$, i.e. $SB = C_0(0, 1) \otimes B$, and the maps $u_k$ and $i_k^*, k = 0, 1,$ will be defined shortly. This six-terms exact sequence is mentioned in 10.11 of [S], but the proof was never published.

Fix a unitally absorbing *-homomorphism $\alpha_0 : A \to M(B)$, which exists by Theorem 2.4 of [Th]. It follows then from Theorem 2.1 of [Th] that $\alpha = q \circ \alpha_0$ is a unitally absorbing split extension as defined above.

**Lemma 1.** The *-homomorphisms $\text{Ad} V_1 \circ \alpha_0 : A \to M(B)$ and $(\alpha_0 \circ A) : A \to M(M_2(B))$ are both absorbing.

**Proof.** There is a *-isomorphism $M(M_2(B)) = M_2(M(B)) \to M(B)$ given by

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \mapsto V_1 m_{11} V_1^* + V_1 m_{12} V_2^* + V_2 m_{21} V_1^* + V_2 m_{22} V_2^*,$$

which sends $M_2(B)$ to $B$ and $(\alpha_0 \circ A)$ to $\text{Ad} V_1 \circ \alpha_0$, so it suffices to show that the latter is an absorbing *-homomorphism. By definition, cf. Definition 2.6 of [Th], we must show that the unital *-homomorphism $A \oplus C \ni (a, \lambda) \mapsto V_1 \alpha_0(a) V_1^* + \lambda V_2 V_2^*$ is unitally absorbing.
For this we check that it has property 1) of Theorem 2.1 of [Th]. So let \( \varphi : A \oplus \mathbb{C} \to B \) be a completely positive contraction. Since \( \alpha_0 \) has property 1), there is a sequence \( \{W_n^*\} \) in \( M(B) \) such that \( \lim_{n \to \infty} W_n^*b = 0 \) for all \( b \in B \) and \( \lim_{n \to \infty} W_n^*\alpha_0(a)W_n = \varphi(a) \) for all \( a \in A \). Since \( B \) is stable there is a sequence \( \{S_n\} \) of isometries in \( M(B) \) such that \( \lim_{n \to \infty} S_n^*b = 0 \) for all \( b \in B \). Set

\[
T_n = V_1W_n + V_2S_n\varphi(0,1)^{\frac{1}{2}}.
\]

Then \( \lim_{n \to \infty} T_nb = 0 \) for all \( b \in B \), and

\[
T_n^* (V_1\alpha_0(a)V_1^* + \lambda V_2V_2^*) T_n = W_n^*\alpha_0(a)W_n + \varphi(0,\lambda)
\]

for all \( n \). Since the last expression converges to \( \varphi(a,\lambda) \) as \( n \) tends to infinity, the proof is complete.

Set

\[
C_\alpha = \{ m \in M_2(M(B)) : m (\alpha_0(a)_0) - (\alpha_0(a)_0) m \in M_2(B) \forall a \in A \}
\]

and

\[
A_\alpha = \{ m \in C_\alpha : m (\alpha_0(a)_0) \in M_2(B) \forall a \in A \}.
\]

We can define a \( \ast \)-homomorphism \( C_\alpha \to \alpha(A)' \cap Q(B) \) such that

\[
(m_{11} m_{12} m_{21} m_{22}) \mapsto q (m_{11}).
\]

Then kernel is then \( A_\alpha \), so we have a \( \ast \)-isomorphism \( C_\alpha/A_\alpha \simeq \alpha(A)' \cap Q(B) \). By Lemma [4] \((\alpha_0)_0\) is an absorbing \( \ast \)-homomorphism, so we conclude from Theorem 3.2 of [Th] that there is an isomorphism

\[
K_1 (\alpha(A)' \cap Q(B)) \simeq KK(A,B).
\]

(4)

Since the unital \( \ast \)-homomorphism \( \mathbb{C} \to M(B) \) is unitally absorbing, this gives us also the well-known isomorphism

\[
K_1 (Q(B)) \simeq KK(\mathbb{C}, B).
\]

(5)

Let \( i : \mathbb{C} \to A \) be the unital \( \ast \)-homomorphism. For convenience we denote the map \( K_1 (\alpha(A)' \cap Q(B)) \to K_1(Q(B)) \) induced by the inclusion \( \alpha(A)' \cap Q(B) \subseteq Q(B) \) by \( i^* \). It is then easy to check that the isomorphisms [4] and [5] match up to make the diagram

\[
\begin{array}{ccc}
K_1 (\alpha(A)' \cap Q(B)) & \xrightarrow{i^*} & K_1(Q(B)) \\
\downarrow & & \downarrow \\
KK(A,B) & \xrightarrow{i^*} & KK(\mathbb{C}, B)
\end{array}
\]

commute.

Let \( v \) be a unitary in \( M_n(Q(B)) \). By composing the \( \ast \)-homomorphism \( \text{Ad} v \circ (1_n \otimes \alpha) : A \to M_n(Q(B)) \) with an isomorphism \( M_n(Q(B)) \simeq Q(B) \) which is canonical in the sense that it arises from an isomorphism \( M_n(B) \simeq B \), we obtain a unital extension \( e(v) : A \to Q(B) \) of \( A \) by \( B \). By use of a unitary lift of \((v,v^*)\) one sees that \( e(v) \oplus e(v^*) \) is split, proving that \( e(v) \) represents an element in \( \text{Ext}^{-1}_{\text{unital}}(A,B) \). If \( v_t, t \in [0,1] \), is a norm-continuous path of unitaries in \( M_n(Q(B)) \) there is a partition \( 0 = t_0 < t_1 < t_2 < \cdots < t_N = 1 \) of \([0,1]\) such that \( v_t, v_{t+1}^* \) is in the connected component of 1 in the unitary group of \( M_n(Q(B)) \)
and hence has a unitary lift to \( M_n(M(B)) \). It follows that \( e(v_0) = e(v_1) \), and it is then clear that the construction gives us a group homomorphism
\[
u : K_1(Q(B)) \to \text{Ext}^{-1}_{\text{unital}}(A, B).
\]

**Lemma 2.** The sequence
\[
\begin{array}{ccc}
K_1(Q(B)) & \xrightarrow{u} & \text{Ext}^{-1}_{\text{unital}}(A, B) \xrightarrow{i^*} \text{Ext}^{-1}(A, B) \\
\uparrow & & \uparrow \\
K_1(\alpha(A)' \cap Q(B)) & \xrightarrow{i^*} & \text{Ext}^{-1}(\mathbb{C}, B)
\end{array}
\]
is exact.

**Proof.** Exactness at \( K_1(Q(B)) \): If \( v \) is a unitary in \( M_n(\alpha(A)' \cap Q(B)) \), the extension \( \text{Ad} v \circ (1_n \otimes \alpha) = 1_n \otimes \alpha \) (of \( A \) by \( M_n(B) \)) is split, proving that \( u \circ i^* = 0 \). To show that \( \ker u \subseteq \text{im} i^* \), let \( v \in Q(B) \) be a unitary such that \( u[v] = 0 \). Then \( \text{Ad} v \circ \alpha \oplus \alpha \) is unitarily equivalent to \( \alpha \oplus \alpha \), which means that there is a unitary \( S \in M(M_2(B)) \) such that
\[
\text{Ad} \left( \left( \text{id}_{M_2(\mathbb{C})} \otimes q \right)(S) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \left( \begin{smallmatrix} a(o) \\ \alpha(a) \end{smallmatrix} \right) = \left( \begin{smallmatrix} a(o) \\ \alpha(a) \end{smallmatrix} \right)
\]
for all \( a \in A \). Since the unitary group of \( M(M_2(B)) \) is normconnected by \([\mathbb{M}]\) or \([\mathbb{CH}]\), the unitary \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) is homotopic to \( (\text{id}_{M_2(\mathbb{C})} \otimes q)(S) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) which is in \( M_2(\alpha(A)' \cap Q(B)) \) by \([\mathbb{S}]\). This implies that \( [v] \in \text{im} i^* \). The same argument works when \( v \) is a unitary \( M_n(Q(B)) \) for some \( n \geq 2 \).

Exactness at \( \text{Ext}^{-1}_{\text{unital}}(A, B) \): For any unitary \( v \in Q(B) \),
\[
(\text{Ad} v \circ \alpha) \oplus 0 = \text{Ad} \left( \text{id}_{M_2(\mathbb{C})} \otimes q \right)(T) \circ (\alpha \oplus 0),
\]
where \( T \in M_2(M(B)) \) is a unitary lift of \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \). Hence \( [\text{Ad} v \circ \alpha] = 0 \) in \( \text{Ext}^{-1}(A, B) \). The same argument works when \( v \) is a unitary \( M_n(Q(B)) \) for some \( n \geq 2 \), and we conclude that the composition \( K_1(Q(B)) \to \text{Ext}^{-1}_{\text{unital}}(A, B) \to \text{Ext}^{-1}(A, B) \) is zero. Let \( \varphi : A \to Q(B) \) be a unital extension such that \( [\varphi] = 0 \) in \( \text{Ext}^{-1}(A, B) \). By Lemma 1 this means that there is a unitary \( T \in M(M_3(B)) \) such that
\[
\text{Ad} \left( \text{id}_{M_3(\mathbb{C})} \otimes q \right)(T) \circ \left( \begin{smallmatrix} \varphi & \alpha \\ \alpha & \alpha \end{smallmatrix} \right) = \left( \begin{smallmatrix} \alpha & \alpha \\ \alpha & \alpha \end{smallmatrix} \right).
\]
It follows that \( (\text{id}_{M_3(\mathbb{C})} \otimes q)(T) = (V_r) \) for some unitaries \( V \in M_2(Q(B)) \) and \( r \in Q(B) \). Hence
\[
(\varphi |_{A}) = \text{Ad} V^* \circ (\alpha |_{A}).
\]
Thus \( [\varphi] = u[V^*] \).

Exactness at \( \text{Ext}^{-1}(A, B) \): It is obvious that \( i^* \) kills the image of \( \text{Ext}^{-1}_{\text{unital}}(A, B) \), so consider an invertible extension \( \varphi : A \to Q(B) \) such that \( [\varphi \circ i] = 0 \) in \( \text{Ext}^{-1}(\mathbb{C}, B) \). By Lemma 1 applied with \( A = \mathbb{C} \), this means that there is a unitary \( T \in M_3(M(B)) \) such that
\[
(\text{id}_{M_3(\mathbb{C})} \otimes q)(T) \left( \begin{smallmatrix} \varphi(1) & 0 \\ 0 & \alpha \end{smallmatrix} \right) (\text{id}_{M_3(\mathbb{C})} \otimes q)(T^*) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix} \right).
\]
Set \( \psi = \varphi \oplus \alpha \oplus 0 \). It follows from \([\mathbb{S}]\) that there are isometries \( W_1, W_2, W_3 \in M(B) \) and a unitary \( u \in M(B) \) such that \( W_i^* W_j = 0, i \neq j, W_1 W_1^* + W_2 W_2^* + W_3 W_3^* = 1 \) and \( \text{Ad} q(u) \circ \psi(1) = q(W_2 W_2^*) \). Then \( \text{Ad} q(u) \circ \psi + \text{Ad} q(W_1) \circ \alpha + \text{Ad} q(W_3) \circ \alpha \) is a unital extension which is invertible because it admits a completely positive contractive lifting.
to $M(B)$ since $\psi$ does, cf. [A]. As it represents the same class in $\text{Ext}^{-1}(A, B)$ as $\varphi$, the proof is complete.

In order to complete the sequence of Lemma 2 let $i_1^*: \text{Ext}^{-1}(A, B) \to K_1(B)$ be the composition

$$\text{Ext}^{-1}(A, B) \xrightarrow{i^*} K_1(\alpha(A) \cap Q(B)) \xrightarrow{i^*} K_1(Q(SB))) \xrightarrow{} K_1(B),$$

where the first map is the isomorphism (11) and the last is the well-known isomorphism. Let $u_1: K_1(B) \to \text{Ext}^{-1}_{\text{unital}}(A, SB)$ be the composition

$$K_1(B) \xrightarrow{} K_1(Q(SB)) \xrightarrow{u} \text{Ext}^{-1}_{\text{unital}}(A, SB),$$

where the first map is the well-known isomorphism (the inverse of the one used in (10)) and second is the $u$-map as defined above, but with $SB$ in place of $B$. Let $i_0^*: \text{Ext}^{-1}(A, SB) \to K_0(B)$ be the composition

$$\text{Ext}^{-1}(A, SB) \xrightarrow{i^*} \text{Ext}^{-1}(C, SB) \xrightarrow{} K_0(B),$$

where the second map is the well-known isomorphism. Finally, let $u_0: K_0(B) \to \text{Ext}^{-1}_{\text{unital}}(A, B)$ be the composition

$$K_0(B) \xrightarrow{} K_1(Q(B)) \xrightarrow{u} \text{Ext}^{-1}_{\text{unital}}(A, B),$$

where the first map is the well-known isomorphism. We have now all the ingredients to prove

**Theorem 3.** The sequence

$$\begin{array}{cccccc}
K_0(B) & \xrightarrow{u_0} & \text{Ext}^{-1}_{\text{unital}}(A, B) & \xrightarrow{i_0^*} & \text{Ext}^{-1}(A, B) & \\
\downarrow{i_0^*} & & \downarrow{i^*} & & \downarrow{i_1^*} \\
\text{Ext}^{-1}(A, SB) & \xleftarrow{i^*} & \text{Ext}^{-1}_{\text{unital}}(A, SB) & \xleftarrow{u_1} & K_1(B) \\
\end{array}$$

is exact.

**Proof.** If we apply Lemma 2 with $B$ replaced by $SB$ we find that the sequence

$$\begin{array}{ccc}
\text{Ext}^{-1}(C, SB) & K_1(\alpha(A) \cap Q(SB)) & \\
\downarrow{i^*} & \downarrow{i^*} & \\
\text{Ext}^{-1}(A, SB) & \text{Ext}^{-1}_{\text{unital}}(A, SB) & \xleftarrow{u} K_1(Q(SB)) \\
\end{array}$$

is exact. Thanks to the commuting diagram (3) we can patch this sequence together with the sequence from Lemma 2 with the stated result. \qed
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