Effective Medium Theory for Elastic Metamaterials in Thin Elastic Plates

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An effective medium theory for resonant and non-resonant metamaterials for flexural waves in thin plates is presented. The theory provides closed-form expressions for the effective parameters of arrangement of inclusions or resonators in thin plates as a function of the filling fraction of the inclusions, their physical properties and the frequency. It is shown that positive or negative effective elastic parameters are possible depending on the symmetry of the resonance, being the negative Young’s modulus and Poisson’s ratio due to the combination of monopolar and quadrupolar resonances, as it happens for bulk elastic wave but the responsible for the negative mass density behaviour is due to the monopolar term only, not to a dipolar one as in bulk elastic waves, showing also that, at least for the first order in the scattering coefficients, the dipolar resonance plays no role in the description of the effective medium. Several examples are given for both non-resonant and resonant effective parameters and the results are verified by multiple scattering theory.

I. INTRODUCTION

In the last few years the field of metamaterials has received increasing attention, due to the extraordinary properties of these structures to control the propagation characteristics of electromagnetic, acoustic or elastic waves. Consisting essentially in periodic arrangements of wave interacting units or scatterers, these advanced structures behave as materials with extraordinary constitutive parameters, like negative permittivity or negative elastic modulus. The wide variety of phenomena and applications found for these structures has motivated the research in this field not only for bulk waves but also for confined or surface waves.

More specifically, the study of the propagation of flexural waves in thin elastic plates with internal microstructure has been widely studied. For instance, the dispersion relation of plates with periodic arrangements of pillars or point-like spring-mass resonators attached to them has been investigated by several groups, as well as the resonant properties of complex inclusions.

However, most of these works are mainly numerical or experimental, and a theory for metamaterials for flexural waves, essentially relating the the effective parameters with the symmetry of the resonances, has not been considered so far.

In this work we present a theory for elastic metamaterials for flexural waves based on the scattering properties of arrangements of scatterers or resonators. This homogenization method has been previously employed for either electromagnetic or acoustic and elastic waves and the resonant effective parameters have been properly assigned to different symmetries of the fields; in this work we obtain the same relationships for flexural waves.

Thus, the presented theory obtains the effective parameters as a function of the filling fraction of the inclusion, their physical properties and the frequency. The theory is valid for low and mid filling fractions, being necessary the inclusion of the multiple scattering terms in order to cover the full range of inclusions’ radii, however it still provides a good description of the effective materials’ properties for both frequency dependent and non-dependent structures.

The paper is organized as follows: After this introduction, in Section II the general procedure for obtaining the effective parameters from the scattering properties of clusters of inclusions or resonators is explained. Following, Section III describes the scattering properties of a circular inclusion in a thin plate, and the low frequency behaviour of these scattering properties are analysed by means of the so called $T$ matrix. This low-frequency behaviour is used in Section IV to obtain closed form expressions for the effective parameters of arrangements of scatterers and resonators in thin plates. Finally, in Section VI multiple scattering simulations are performed to verify the theory and results are summarized in Section V. Appendix A contains some mathematical details about the analytical derivations.

II. HOMOGENIZATION FROM THE SCATTERING PROPERTIES

In this section the general procedure to obtain the effective properties of a given collection of scatterers will be explained. Later on, the theory will be applied to the case of scatterers for flexural waves in thin plates.

Figure III shows the schematic view of a general homogenization procedure based on multiple scattering: An incident field $\psi_0$ impinges a circular cluster of scatterers, and it excites a scattered field $\psi_{sc} = T_{cls} \psi_0$, being $T_{cls}$ the $T$ matrix of the cluster, which is a function of the frequency and the physical properties of the scatterers. We expect that, in the low frequency limit, this cluster behaves like an effective circular scatterer of radius $R_{eff}$ and some effective parameters which will depend on the nature of the wave under study. In the present case, as will be shown later, the parameters describing any scat-
Poisson’s ratio $\nu$ and $\rho$ behaviour of the these effective parameters from the low frequency be-

The left hand side of the above equation is a known quantity, since we decide the nature, size and position of the scatterers, while the right hand side of the equation contains the parameters to be determined, therefore the above equation provides a solution for the effective parameters of the medium.

In general, in either two and three dimensions, both the incident field and the scattered field are expanded in multipolar fields, for instance, in two dimensions we have that the fields

$$\psi_0 = \sum_q A_q \psi_q^0(k_b r) e^{i q \theta} \quad (2a)$$

$$\psi_{sc} = \sum_q B_q \psi_q^{sc}(k_b r) e^{i q \theta}, \quad (2b)$$

being $r$ and $\theta$ the polar coordinates and $q$ an integer number representing the mode’s symmetry. Here, $k_b$ is the backgrounds wave number and $\psi_q^0$ and $\psi_q^{sc}$ are usually Bessel and Hankel (or related) functions representing the incident and scattered fields, respectively. The $T$ matrix relates the $B_q$ coefficients with the $A_q$, that is

$$B_q = \sum_s T_{qs} A_s. \quad (3)$$

In the case of a circular homogeneous scatterer, the $T$ matrix becomes diagonal, then $T_{qs} = T_q \delta_{qs}$, however the $T$ matrix of a cluster of scatterers is not diagonal in general, since it is obvious that the structure is not invariant under rotations. However, in the low frequency limit, this effective $T$ matrix becomes diagonal $26$ since in this limit it is behaving like a homogeneous scatterer (actually this is only true for isotropic arrangements of scatterers, the anisotropic case is different and beyond the scope of the present work).

In the low frequency limit, all the $T_q$ elements goes to zero, however the trend is different for each multipolar order $q$. For instance, in the case of electromagnetic or acoustic scatterers, the dominant terms are the $T_0$ and $T_1$ elements, which goes to zero as $\omega^2$, while for elastic waves in solids the dominant terms are the $T_0$, $T_1$ and $T_2$ elements. It is from the dominant terms from which we obtain the effective parameters, and the number of dominant terms is consistent with the number of parameters to determine. Thus, for electromagnetic waves we need two dominant terms, since we have to obtain the effective permeability $\mu$ and permittivity $\varepsilon$. Similarly, in acoustics we need to obtain the effective bulk modulus $B$ and mass density $\rho$ and again we have two dominant terms. Finally, for elastic waves in solids, we need three parameters, the two Lamé coefficients $\lambda$ and $\mu$ and the mass density, and we have three dominant terms in the low frequency limit.

It can be shown$^{26,27}$ that the general structure of the dominant term of the $T$ matrix of a homogeneous scatterer of radius $R_a$ is, in two dimensions,

$$T_q \approx \frac{i \pi k_b^2 R_a^2}{4} \Gamma_q^a, \quad (4)$$

being $\Gamma_q^a$ a function of the scatterer’s parameters, whose explicit expression depends on the nature of the field considered. The expression is valid as well for the effective scatterer, therefore in this limit the dominant terms of the effective $T$ matrix are

$$(T_q)_{eff} \approx \frac{i \pi k_b^2 R_{eff}^2}{4} \Gamma_q^{eff}, \quad (5)$$

also, it can be shown that the $T$ matrix of the cluster is, as a first approximation, i.e., neglecting the multiple scattering terms, the addition of the $T$ matrices of all the scatterers$^{28}$ if all them are identical and we have $N$ scatterers in the cluster we get

$$(T_q)_{cls} \approx N \frac{i \pi k_b^2 R_a^2}{4} \Gamma_q^a. \quad (6)$$
The homogenization procedure implies equating the last two equations, thus we get, defining the filling fraction as \( f = N R_a^2/R_{eff}^2 \),

\[
\Gamma_{q}^{eff} = f \Gamma_{q}^{a},
\]

(7)

from which we can obtain the effective parameters. If instead of having \( N \) identical scatterers we have \( N_1 \) scatterers of type 1, \( N_2 \) scatterers of type 2 and so on, and they form an isotropic medium, we have that

\[
\Gamma_{q}^{eff} = \sum_i f_i \Gamma_{q}^{a},
\]

(8)

where \( f_i \) is the partial filling fraction of the scatterer of type \( i \).

For instance, for acoustic waves, we have that \( \Gamma_{q}^{a} = 1 - B_b/B_a \), being \( B_b \) and \( B_a \) the bulk modulus of the background and the scatterer, respectively. Using the above equation we obtain the well know expression for the effective bulk modulus of a composite

\[
1 \over B_{eff} = 1 - f B_b + f B_a,
\]

(9)

or, in the most general case,

\[
1 \over B_{eff} = 1 - f \over B_b + \sum_i f_i \over B_i.
\]

(10)

In the next section the scattering of flexural waves in thin plates is described in terms of the \( T \) matrix, and the dominant terms are derived for later use in the extraction of the effective parameters of an ensemble of scatterers.

### III. SCATTERING OF FLEXURAL WAVES BY A CIRCULAR INCLUSION

Flexural waves in thin plates are described by the fourth order differential equation \[28,29\]

\[
- \frac{\partial^2}{\partial x^2} \left( D_b \left[ \frac{\partial^2 W}{\partial x^2} + \nu_b \frac{\partial^2 W}{\partial y^2} \right] \right) - \frac{\partial^2}{\partial y^2} \left( D_b \left[ \frac{\partial^2 W}{\partial y^2} + \nu_b \frac{\partial^2 W}{\partial x^2} \right] \right) - 2 \frac{\partial^2}{\partial x \partial y} \left( D_b (1 - \nu_b) \frac{\partial^2 W}{\partial x \partial y} \right) = \rho_b \frac{\partial^2 W}{\partial t^2},
\]

(11)

being \( W(x,y) \) the vertical displacement of the plate, \( \nabla = \partial_x \partial_x + \partial_y \partial_y \), \( \rho_b \) the mass density of the plate, \( h_b \) its thickness and \( D_b = E_b h_b^3/12(1-\nu_b^2) \) the rigidity of the plate, with \( E_b \) and \( \nu_b \) the Young’s modulus and Poisson’s ratio, respectively. When the background’s parameters are constant and we assume harmonic time dependence of the field \( W \), the above equation reduces to

\[
(D_b \nabla^4 - \rho_b h_b \omega^2) W(x,y) = 0,
\]

(12)

whose solution in cylindrical coordinates is given by a linear combination of Bessel and modified Bessel functions \[20\] of argument \( k_b \), such that

\[
k_b^4 = \frac{\rho_b h_b}{D_b} \omega^2.
\]

(13)

For a scattering problem, the incident field is expressed as

\[
\Psi_0 = \sum_q \left[ A_q^i J_q(k_b r) + A_q^i I_q(k_b r) \right] e^{iq\theta},
\]

(14)

while the scattered field is given by

\[
\Psi_{sc} = \sum_q \left[ B_q^H H_q(k_b r) + B_q^K K_q(k_b r) \right] e^{iq\theta}.
\]

(15)

If the scatterer is a circular inhomogeneity of radius \( R_a \) we have that, inside the scatterer \((r < R_a)\), since there are no sources, the field is expressed as

\[
\Psi_i = \sum_q \left[ C_q^i J_q(k_a r) + C_q^i I_q(k_a r) \right] e^{iq\theta}.
\]

(16)

Boundary conditions are explained for instance in \[30\] and they provide a system of four equations which solves for the fourth unknowns: two scattering coefficients \( B_q^H, B_q^K \) and the two internal coefficients \( C_q^i, C_q^i \). The system of equations can be expressed as (see Appendix A for details)

\[
X_0^a A_q + X_0^{sc} B_q = X_0^q C_q,
\]

(17a)

\[
Y_0^a A_q + Y_0^{sc} B_q = Y_0^q C_q,
\]

(17b)

where the matrices \( X_q^i \) and \( Y_q^i \), with \( i = 0, sc, a \), are \( 2 \times 2 \) matrices given in Appendix A and the coefficient vectors are \( A_q = (A_q^i, A_q^a) \) and \( B_q = (B_q^H, B_q^K) \). Solving for \( C_q^i \) from equation (17a) and inserting into equation (17b) gives

\[
Y_0^q A_q + Y_0^{sc} B_q = Y_0^q a (X_0^q)^{-1} (X_0^q a + X_0^{sc} B_q),
\]

(18)

from which we can solve for the \( B_q \) as a function of \( A_q \)

\[
B_q = - (Y_0^{sc} - Y_0^q a (X_0^q)^{-1} X_0^{sc} a)^{-1} \left( Y_0^q a - Y_0^q a (X_0^q)^{-1} X_0^q a \right) A_q.
\]

(19)

The above equation defines the \( T \) matrix of the scatterer, and gives the scattering coefficients \( B_q \) as a function of \( A_q \). It is a \( 2 \times 2 \) matrix and each element of the matrix relates the excitation of a different mode, that is, in full matrix form we have

\[
\begin{pmatrix}
B_1^H \\
B_1^K
\end{pmatrix} =
\begin{pmatrix}
T_1^{HJ} & T_1^{HI} \\
T_1^{JH} & T_1^{JI}
\end{pmatrix}
\begin{pmatrix}
A_1^J \\
A_1^K
\end{pmatrix}.
\]

(20)

If the scatterer is a hole, the clamped free boundary conditions gives simply

\[
B_q = -(Y_q^{sc})^{-1} Y_q^0 A_q
\]

(21)
The effective medium defined by a cluster of scatterers will be described by means of the three parameters appearing in the wave equation (11), which are the mass density \( \rho_{eff} \) (actually the surface mass density \( \rho_{eff} h_{eff} \)), the effective rigidity \( D_{eff} \) and the effective Poisson’s ratio \( \nu_{eff} \). Thus, in principle we would expect that the dominant terms of the \( T \) matrix be three, as it happens for bulk elasticity, however it will be shown that the case of flexural waves is different.

As shown in [31], in the low frequency (wavenumber) limit, the elements of the \( T \) matrix of a hole depend on each multipolar order \( q \), as expected. Thus, the \( q = 0 \) element is

\[
T_0 \approx \frac{i \pi (k_b R_a)^2}{4} \frac{1}{1 - \nu_b} \left( \frac{-\nu_b}{-2i/\pi} \frac{-1}{2i\nu_b/\pi} \right),
\]

while the \( q = 1 \) is

\[
T_1 \approx \frac{i \pi (k_b R_a)^4}{32} \frac{1}{1 - \nu_b} \left( \frac{1 + \nu_b}{-4i/\pi} \frac{-2}{2i(1 + \nu_b)/\pi} \right),
\]

finally, for \( q \geq 2 \) we have

\[
T_q \approx \frac{i \pi (k_b R_a)^2 q^{-2}}{2^{2q-1}(q-1)![(q-2)]^3 + \nu_b} \left( \frac{1}{2i/\pi} \frac{1}{2i/\pi} \right).
\]

The above expressions show that in the low frequency limit the dominant terms of the \( T \) matrix are the \( q = 0 \) and the \( q = 2 \), unlike in other waves like acoustic or electromagnetic where the dominant terms are the \( q = 0 \) and the \( q = 1 \). Only in the elastic case we find the \( q = 2 \) as a dominant term, but it also includes the \( q = 1 \). However, although in the case of flexural waves we have found only two dominant terms, and we still have three effective parameters to obtain, the \( T \) matrix elements are actually \( 2 \times 2 \) matrices, which moreover have the following form

\[
T_0 \approx \frac{i \pi (k_b R_a)^2}{4} \left( \frac{\Gamma_0^{11}}{2i/\pi \Gamma_0^{12}} \frac{\Gamma_0^{12}}{2i/\pi \Gamma_0^{11}} \right)
\]

and

\[
T_2 \approx \frac{i \pi (k_b R_a)^2}{4} \left( \frac{\Gamma_2}{2i/\pi \Gamma_2} \frac{\Gamma_2}{2i/\pi \Gamma_2} \right)
\]

with

\[
\Gamma_0^{11} = \frac{\nu_b}{1 - \nu_b},
\]
\[
\Gamma_0^{12} = -\frac{1}{1 - \nu_b},
\]
\[
\Gamma_2 = \frac{11 - \nu_b}{23 + \nu_b}
\]

which shows that indeed we have only three independent terms. This structure of the \( T \) matrix should be maintained for a general inhomogeneity, since it is from that expression from which we will obtain the effective parameters. The demonstration is tedious and long, and some details are given in Appendix A but as expected the dominant terms of the \( T \) matrix of an elastic inhomogeneity have the same behaviour and form, and it is found that

\[
\Gamma_0^{11} = \frac{1}{2} \left( \frac{D_b}{D_b(1 - \nu_b) + D_a(1 + \nu_a)} \right) - 1
\]
\[
\Gamma_0^{12} = \frac{1}{2} \left( \frac{D_b}{D_b(1 - \nu_b) + D_a(1 + \nu_a)} \right)
\]
\[
\Gamma_2 = \frac{D_b(1 - \nu_b) - D_a(1 - \nu_a)}{2D_b(3 + \nu_b) + D_a(1 - \nu_a)}
\]

Notice that we recover the expressions for the holes by setting \( D_a = 0 \) and \( \rho_a = 0 \). From the above expressions it is now possible to obtain the effective parameters for an ensemble of scatterers in a thin plate. These parameters are derived in next section, first for the low frequency limit and, later on, the generalization for a resonant medium.

\section{Effective Parameters}

The low frequency limit of the \( T \) matrix of a inhomogeneity in a thin plate responds to the general behaviour explained before, therefore application of equation (7) is straightforward, giving

\[
(G_0^{11})_{eff} = f(G_0^{11})
\]
\[
(G_0^{12})_{eff} = f(G_0^{12})
\]
\[
(G_2)_{eff} = f(G_2)
\]

From the above equations we can solve for the effective parameters as a function of the filling fraction, giving

\[
\rho_{eff} = (1 + f(G_0^{11} + G_0^{12})) \rho_b
\]
\[
D_{eff}(1 + \nu_{eff}) = \frac{1 + \nu_b - f(G_0^{11} - G_0^{12})(1 - \nu_b)}{1 + f(G_0^{11} - G_0^{12})} D_b
\]
\[
D_{eff}(1 - \nu_{eff}) = \frac{1 - \nu_b - 2fG_2(3 + \nu_b)}{1 + 2G_2} D_b
\]

The above equations shows that the effective mass density is obtained from the monopolar scattering terms. When these terms be resonant, as will be shown later, we will have an effective medium with a dispersive effective mass density, reaching positive and negative values. It is clear as well that the effective rigidity and Poisson’s ratio are defined from the monopolar and quadrupolar scattering elements, which also happens for bulk elasticity.

\subsection{Non-Resonant Effective Parameters}

The effective parameters for a non-resonant medium are obtained by inserting the low frequency parameters \( G_i \) into equations (32), from which we obtain the following expressions
Notice that the equation for the effective mass density is identical than that found for bulk elastic waves, however here has been obtained from the monopolar term while in elasticity is obtained from the dipolar one. The relevance of this result will be clarified in next section.

From the above equations it is possible to obtain the effective phase velocity as

\[ c_{\text{eff}} = (D_{\text{eff}}/\rho_{\text{eff}})^{1/4}. \]  

Special mention deserves the system of holes in an elastic plate. As said before, the expressions for the holes are obtained by setting \( \rho_a = 0 \) and \( D_a = 0 \). It is easy to see that then the the effective phase velocity \( c_{\text{eff}} \) related with the phase velocity of the background \( c_b \) depends only on the plate’s Poisson’s ratio. Effectively, in this particular case, equations (33) become

\[ \rho_{\text{eff}} = (1 - f)\rho_b, \]

\[ D_{\text{eff}}(1 + \nu_{\text{eff}}) = \frac{(1 + \nu_b)(1 - f)}{D_b} (1 - \nu_b) + f((1 + \nu_b) D_b/(1 - \nu_b) - D_a(1 + \nu_a)) D_b \]

(33b)

\[ D_{\text{eff}}(1 - \nu_{\text{eff}}) = \frac{(1 - \nu_b)(3 + \nu_b)(1 - f)}{D_b} (3 + \nu_b) + f(3 + \nu_b)D_b(1 - \nu_b) - D_a(1 - \nu_a) D_b \]

(33c)

from which it is clear that the ratio \( D_{\text{eff}}/\rho_{\text{eff}} \) depends only on the filling fraction and the background’s Poisson’s ratio.

Figure 2 shows the effective phase velocity computed using equations (33) for a triangular arrangement of holes in a plate for different Poisson’s ratio (continuous lines) compared with the second order approximation for the phase velocity given by equation (35) of reference 31 (dashed lines). Notice that the the two expressions gives the same velocity for low filling fractions but they split for filling fractions above 0.2, where the expression given in the present work is more accurate. It must be also remarked than equations (33) give also an approximated expression for high filling fractions, where the multiple scattering terms should be included in the theory. However, these expressions are accurate for low and mid filling fractions.

Figure 3 shows the effective mass density \( \rho_{\text{eff}} \) (panel (a)), rigidity \( D_{\text{eff}} \) (panel (b)), Poisson’s ratio \( \nu_{\text{eff}} \) (panel (c)) and phase velocity \( c_{\text{eff}} \) (panel (d)) as a function of the filling fraction for triangular arrangements of inclusions of holes, rubber and lead in an aluminium plate, with the elastic parameters given in table I. The effective mass density follows the well known linear relationship with the filling fraction, since it is simply the volume average, as for bulk elastic waves. The effective rigidity \( D_{\text{eff}} \) has a decreasing trend, being indistinguishable the case of holes to that of rubber due to the low Young’s modulus ratio between the rubber and the Aluminium. The same phenomenon is observed for the Poisson’s ratio but not for the phase velocity, where the differences between the rubber and the holes are more evident.

It must be pointed out that the values of the effective Poisson’s ratio for the case of holes and the rubber inclusions reach zero at filling fractions close to 0.6. This is due to the fact that the scatterers are very strong, and the multiple scattering corrections must be added for high filling fractions, as explained above. The case of lead inclusions is well-behaved in this sense, it does not means that these multiple scattering terms are not necessary for a proper description, nevertheless their in-

![FIG. 2. Continuous lines: Effective phase velocity for a triangular array of holes embedded in an elastic plate of different Poisson’s ratio. Dashed lines: Results from an approximated model (see text for further details).](image-url)
fluence will be weaker. A full analysis of these multiple scattering terms is beyond the scope of the present work, since they are important for strong scatterers and high filling fractions only.

**B. Resonant Effective Parameters**

The effective parameters given by equations (33) are obtained in the low frequency limit, which implies that the wavenumber in the background, scatterer and effective medium are negligible. However, as long as the wavenumber in the background be small (that is, the wavelength larger than the typical distance between scatterers), the description of the system as an effective medium makes sense. It means that we can allow the wavelength inside the scatterer be short and then have complex field oscillations while the field in the background sees an average medium. If this is the case, we will have a frequency-dependent effective medium. In this case, the asymptotic forms of Bessel functions cannot be employed thus the quantities $\Gamma_i$ have to be obtained directly from the $T$ matrix, similarly as was done in references [23] and [24]. Thus, if we assume that equations (25) and (26) holds as long as $\lambda/R_a << 1$, being $a$ the typical distance between scatterers, we can obtain the frequency-dependent $\Gamma_i$ parameters as

\[
\Gamma_{11}^0(\omega) = -\frac{4i}{\pi(k_0R_a)^2} T_{01}^{11}(\omega) \quad (36a)
\]
\[
\Gamma_{12}^0(\omega) = -\frac{4i}{\pi(k_0R_a)^2} T_{01}^{12}(\omega) \quad (36b)
\]
\[
\Gamma_2(\omega) = -\frac{4i}{\pi(k_0R_a)^2} T_{02}^{22}(\omega) \quad (36c)
\]

which inserted into equations (32) gives the frequency-dependent parameters $\rho_{eff}(\omega), D_{eff}(\omega)$ and $\nu_{eff}(\omega)$.

Figure 4 shows the three frequency dependent $\Gamma_i$ parameters as a function of frequency for the case of Rubber inclusions in Aluminium, since the holes and the Lead inclusions does not present low frequency resonances. It can be seen that only the $\Gamma_{11}^0$ and $\Gamma_{12}^0$ elements present a resonance in a low enough frequency, that is, for wavelengths in the background such that $\lambda > 4a$, where the medium could be described as an effective medium. The resonance for the $\Gamma_2$ element could be found at higher frequencies, but there the description as an effective medium has no sense.

Given that the $\Gamma_{11}^0$ and $\Gamma_{12}^0$ elements appear in the constitutive equations for the effective mass density, rigidity and Poisson’s ratio (see equations (32)), the resonance of these elements will affect all the effective parameters, as will be seen below.

Figure 5 panel (a) shows the effective mass density for the mentioned system of rubber inclusions. It is clear that the monopolar resonance produces a resonance in the mass density, which allow it to have positively and negatively divergent values. The monopolar resonance also affects the effective rigidity $D_{eff}$ and Poisson’s ratio $\nu_{eff}$, as can be seen from equations (32) and panels (b) and (c) of Figure 5 however the resonance does not

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**FIG. 3.** Effective mass density (a), rigidity (b), Poisson’s ratio (c) and phase velocity (d) as a function of the filling fraction for an hexagonal arrangement of circular inclusions in Aluminium. Results are shown for holes (blue continuous line), lead inclusions (green dashed line) and rubber inclusions (red dash-dotted line).

**FIG. 4.** Real part of the frequency-dependent parameters $\Gamma_0, \Gamma_1$ and $\Gamma_2$ for a triangular arrangement of rubber inclusions in an aluminium matrix. Only the $\Gamma_0$ and $\Gamma_1$ parameters present a resonance, while the $\Gamma_2$ parameter is nearly constant along all the frequency range.
produce any divergent value, since the role in which the 
\( \Gamma_i \) quantities appear is in the form of \( \Gamma_i^{11} - \Gamma_i^{02} \), which cancels near the resonance. Therefore, although the system of soft inclusions in an elastic matrix can easily produce a resonant negative mass density, in order to obtain the same thing for the other two parameters it will be required a more complex scatterer or other type of resonator capable to produce a low frequency quadrupolar resonance.

\[ \frac{\rho_{\text{eff}}}{\rho_b} \]

\[ \frac{D_{\text{eff}}}{D_b} \]

\[ \nu_{\text{eff}} \]

\[ \frac{a}{\lambda} \]

\[ \frac{x}{a} \]

\[ \frac{y}{a} \]

FIG. 5. Frequency-dependent effective mass density (a), rigidity (b) and Poisson’s ratio (c) for the system described in Fig. 4. The monopolar resonance creates a resonant behaviour in the three parameters, although only for the mass density is divergent.

V. MULTIPLE SCATTERING SIMULATIONS

Multiple scattering theory (MST) allows for the computation of the total field scattered by arbitrarily located inclusions. This is described for instance in reference [33] for flexural waves. In this section the MST method is employed to compare the scattered field by a cluster of the different inclusions considered in the preceding sections with the scattered field by a single inclusion with the same effective parameters than those given by equations (33) for the case of holes and Lead inclusions but with those given by equations (32) and (36) for the case of rubber inclusions.

Figure 6, left panel, shows a multiple scattering simulation of a circular cluster of 151 holes of radius \( R_a = 0.3 a \) arranged in a triangular lattice of parameter \( a \) and in an Aluminium plate. A plane wave of wavelength \( \lambda = 15 a \) comes from the left and impinges the cluster, producing the multiple scattering process illustrated. The right panel shows the same wave arriving at a homogeneous inclusion with the parameters \( \rho_{\text{eff}} = 0.67 \rho_b, D_{\text{eff}} = 0.48 D_b \) and \( \nu_{\text{eff}} = 0.14 \), as given by equations (33) for the corresponding filling fraction of \( f = 2\pi/\sqrt{3}R_a^2 = 0.3265 \). Notice that the field distributions are very similar in the two figures, both in the far and near fields, which shows that the cluster and the effective scatterer behaves in the same way. Figure 7 shows the same situation but for a cluster of lead inclusions of the same radius as the holes. The corresponding effective parameters are \( \rho_{\text{eff}} = 2.04 \rho_b, D_{\text{eff}} = 0.868 D_b \) and \( \nu_{\text{eff}} = 0.37 \) and, as before, the field patterns are very similar.

Finally, Figure 8 shows the same cluster but made of rubber inclusions, where a frequency-dependent behaviour is expected. The corresponding wavelength is \( \lambda = 6.165 a \), where a negative mass density is expected. We see from the left panel that there is no propagation
inside the cluster, since the wavenumber is complex and no propagation is possible inside the medium. In this case, the effective parameters couldn’t be obtained by means of equations (32) and (36), since they would give us positive parameters and we would observe propagation inside the effective inclusion. Instead, by means of equations (32) and (36), the corresponding effective parameters are \( \rho_{\text{eff}} = -1.831 \rho_0, D_{\text{eff}} = 0.7436 D_0 \) and \( \nu_{\text{eff}} = 0.4411 \), which give us a negative mass density and, as we see from the right panel, the field patterns are identical for both the cluster and the effective medium.

**VI. SUMMARY**

In summary, an effective medium description for the propagation of flexural waves in thin elastic plates with isotropic arrangements of inclusions or resonators have been presented. The theory is based on the scattering properties of these inclusions, which are compared with that of an effective scatterer and, from the low frequency limit of the expressions found for the scattering coefficients, closed form expressions for the effective elastic parameters of the arrangement are obtained.

The theory is valid not only in the quasi-static limit, but also for frequency-dependent effective parameters, situation that happens in the case of having a long wavelength in the background but not inside the scatterers.

For the resonant medium it is found that the negative mass density is obtained from a monopolar resonance, unlike acoustic or bulk elastic waves where this extraordinary behaviour is due to a dipolar resonance. Additionally, the resonant behaviour of both the rigidity and the Poisson’s ratio is given to both the monopolar and the quadrupolar resonance, being the latter the most important contribution, although for the systems studied in the present work this resonance has not been found.

It has been found that, surprisingly, the dipolar term plays no role in the effective parameters in the present approximation, valid for low and mid filling fractions, however it is expected that including the full multiple-scattering terms in the equations the dipolar resonance plays a secondary role, being therefore the challenge for this type of metamaterials the finding of a good quadrupolar resonator.

Multiple scattering simulations in which the scattered field by a cluster of inclusions is compared with the scattered field by an effective homogeneous scatterer are presented, supporting in this way the presented theory.

This work opens the door for the efficient design of simple, doubly and triply resonant metamaterials for flexural waves in thin plates, and can be the basis for its generalization of all the type of Lamb waves, with promising applications in the control of vibrations in all scales.

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**Appendix A: Closed form expression for the \( T \) matrix**

The expressions for the \( X^i_q \) and \( Y^i_q \) matrices are required to obtain the \( T \) matrix of a circular inhomogeneity, and they are obtained after applying boundary conditions at \( r = R_a \), being

\[
\begin{align*}
X_0 &= \begin{pmatrix} J_q(k_b R_a) & I_q(k_b R_a) \\ k_b J'_q(k_b R_a) & k_b I'_q(k_b R_a) \end{pmatrix} \\
Y_0 &= \begin{pmatrix} S^I_q(k_b R_a) & S^I_q(k_b R_a) \\ T^I_q(k_b R_a) & T^I_q(k_b R_a) \end{pmatrix} \\
X_a &= \begin{pmatrix} J_q(k_a R_a) & I_q(k_a R_a) \\ k_a J'_q(k_a R_a) & k_a I'_q(k_a R_a) \end{pmatrix} \\
Y_a &= \begin{pmatrix} S^I_q(k_a R_a) & S^I_q(k_a R_a) \\ T^I_q(k_a R_a) & T^I_q(k_a R_a) \end{pmatrix} \\
X_{sc} &= \begin{pmatrix} H_q(k_b R_a) & K_q(k_b R_a) \\ k_b H'_q(k_b R_a) & k_b K'_q(k_b R_a) \end{pmatrix} \\
Y_{sc} &= \begin{pmatrix} S^H_q(k_b R_a) & S^H_q(k_b R_a) \\ T^H_q(k_b R_a) & T^H_q(k_b R_a) \end{pmatrix}
\end{align*}
\]

where
where the upper sign applies when $X$ is $J_q, H_q$ and the lower sign when $X$ is $I_q, K_q$.

The $T$ matrix is defined by means of equation (19) as

\[
T_q = - (Y_q^{sc} - Y_q^{a}(X_q^a)^{-1}X_q^{sc})^{-1} \times (Y_q^0 - Y_q^{a}(X_q^a)^{-1}X_q^0). \quad (A5)
\]

The low frequency terms of the above expression can be obtained after some manipulation. First, the inclusion’s terms can be developed, being

\[
Y_q^a(X_q^a)^{-1} = \frac{1}{\Delta} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (A6)
\]

where, with $x = k_a R_a$,

\[
\begin{align*}
a_{11} &= k_a (I_q^I(x) S_q^I(x) - J_q^I(x) S_q^I(x)) \\
a_{12} &= J_q(x) S_q^I(x) - I_q(x) S_q^I(x) \\
a_{21} &= k_a (I_q^I(x) T_q^I(x) - J_q^I(x) T_q^I(x)) \\
a_{22} &= J_q(x) T_q^I(x) - I_q(x) T_q^I(x) \end{align*} \quad (A7)
\]

Then we have

\[
\Delta = k_a (J_q(x) I_q^I(x) - J_q^I(x) I_q(x)) \quad (A7e)
\]

The above expressions are now suitable for their analysis in the low frequency limit, since essentially they contain linear combinations of Bessel, Hankel and modified Bessel functions. The details are tough, but the results given by Eqs. (27) have been verified both analytically and with the help of a symbolic math computer software. Then, it has been found that, in the low frequency limit, the dominant terms of the $T$ matrix are the $q = 0$ and the $q = 2$, since they go to zero as $k_b^2$, while the $q = 1$ goes as $k_b^2$. The other elements goes as $k_b^{2q-2}$.

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