Abstract—We consider a dynamical linear network where nearest neighbours communicate via links whose states form binary (open/closed) valued independent and identically distributed Markov processes.

Our main result is the tight information-theoretic lower bound on the network traffic required by the connection state overhead, or the information required for all nodes to know their connected neighbourhood.

These results, and especially their possible generalisations to more realistic network models, could give us valuable understanding of the unavoidable protocol overheads in rapidly changing Ad hoc and sensor networks.

Index Terms—Connection state overhead, dynamic linear network, exact series solution, entropy rate of an infinite dimensional hidden Markov process.

I. INTRODUCTION

In a dynamical network it is essential to keep track of the connection state information in order to ensure efficient transmission of data. This requires additional information, in the form a connection state overhead, to be sent through the network. For networks with rapid dynamics (e.g. mobile networks) this overhead may be large, and it is therefore of relevance to find some quantitative measure of the required bandwidth.

In this paper we study a simple model of a one-dimensional network introduced by Dey [1], in which the links form identical, independent and time-homogeneous discrete-time Markov processes in an open/closed-binary space. In this case the required connectivity information at a given node is simply the length of the path of open links in either direction. The ensuing connection state overhead is then quantified using information-theoretic methods. The relevant quantity is the smallest possible number of bits per second required for the connectivity overhead. Our main result is a sequence of upper and lower bounds converging exponentially to this quantity, as well as a simple and efficient method for their computation.

To our knowledge [2] besides [1] is the only other work with the theme of quantifying the connection state overhead by information theory.

The outline of the paper is as follows. In Section II we introduce the network model and the connection state variables. The overhead is quantified in Section III we also introduce a sequence of bounds for this quantity, derive an algorithm for their computation and show their exponential convergence towards the exact optimal overhead cost.

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II. THE MODEL

The one-dimensional network is composed of nodes and links connecting neighbouring nodes. The nodes are labelled using the spatial variable $x \in \mathbb{Z}$. We choose $x$ to increase to the right (see Figure 1). The links are labelled using the index $x \in \mathbb{Z}$ such that the link $x$ connects the nodes $x$ and $x + 1$. The dynamics of the network is described using a discretised time variable $t \in \mathbb{N}$. The initial time is $t = 1$.

![Fig. 1. The linear network. Nodes and links are indexed as shown.](image-url)

The probability space $\Omega := \{0,1\}^{\mathbb{N} \times \mathbb{Z}}$ contains elements $\omega \in \Omega$ of the form $\omega = \{\omega_t : t \in \mathbb{N}, x \in \mathbb{Z}\}$. The state of a link $x$ at time $t$ is described by the random variable $X_t(x)$ which is by definition equal to $\omega_{tx}$; “1” stands for up or open, and “0” for down or closed. We shall introduce a probability measure $P$ of $\Omega$ on the $\sigma$-field generated by the finite-dimensional cylindrical subsets of $\Omega$.

All links $x$ are assumed to have identical and independent statistics: $P = \bigotimes_{x \in \mathbb{Z}} P$ is a product over each $x \in \mathbb{Z}$. We now consider $p$, i.e. the time evolution of a single link $x$. Since all links $x$ have identical statistics, we consider only the the link $x = 1$ and write $X_t := X_t(1)$. The time evolution of $X := \{X_t : t \in \mathbb{N}\}$ (and consequently of $X(x) := \{X_t(x) : t \in \mathbb{N}\}$) is given by an autonomous 1 Markov process. Using the abbreviation

$$p(b | a) := P[X_{t+1} = b | X_t = a],$$

where $a, b \in \{0,1\}$, the distribution of the Markov process $X$ is determined by the transition matrix

$$T = \begin{pmatrix} p(1 | 1) & p(1 | 0) \\ p(0 | 1) & p(0 | 0) \end{pmatrix} := \begin{pmatrix} \bar{\lambda} & \pi \\ \pi & \bar{\lambda} \end{pmatrix},$$

(1)

where $u, d \in (0,1)$ are the free parameters of the model, and $\bar{\lambda} := 1 - \lambda$ for any $\lambda \in [0,1]$. Thus $u$ (resp. $d$) is the probability that a closed (resp. open) link is opened (resp. closed) after one time step.

The above Markov chain has a steady state probability distribution on $\{0,1\}$. For $b \in \{0,1\}$ we have

$$p(b) := \lim_{t \to \infty} P[X_t = b | X_1 = a],$$

1We use the term ‘autonomous’ as a synonym for ‘time-homogeneous’.
regardless of the initial condition \( a \in \{0, 1\} \). From above we get

\[
U := p(1) = \frac{u}{u + d}, \quad (2a)
\]

\[
D := p(0) = \frac{d}{u + d}. \quad (2b)
\]

Thus \( U \) (resp. \( D \)) is the steady state probability of a link being up (resp. down).

For simplicity we assume that at time 1 all the link variables \( \{X_1(x), \ x \in \mathbb{Z}\} \) are distributed according to the stationary distribution. Autonomy of the links implies then that

\[
P[X_1(x) = 1] := U, \quad P[X_1(x) = 0] := D, \quad (3)
\]

for all \( x \in \mathbb{Z} \) and \( t \in \mathbb{N} \). Note that this restriction can always be lifted since all our results concern the limit \( t \to \infty \). For any given initial distribution, conditions (3) will hold with arbitrary accuracy for large enough times.

These remarks define \( P \) uniquely. Figure 2 shows a space-time diagram of a typical evolution of the link variables.

A glance at Figure 2 shows that the value of \( M_t(x) \) depends only on the link variables in the time-space-diagonal \( \Lambda(t) := \{ (s, x) : s = x + 1 \} \).

To simplify notation we re-index link states on the diagonals \( \Lambda(t) \),

\[
Z_t(x) := \begin{cases} 
X_{t+1-\epsilon}(x), & x \leq t, \\
0, & x > t, 
\end{cases}
\]

so that by (2) \( M_t \) becomes a deterministic function of the infinite dimensional random vector \( Z := (Z_t(1), Z_t(2), \ldots) \).

Similarly, we define re-indexed messages on \( \Lambda(t) \) by setting

\[
\tilde{M}_t(x) := M_{t+1-\epsilon}(x),
\]

A. Communications between the nodes

We make the following assumptions about the communication capabilities of the nodes.

(i) A node \( x \) is able to send a one message to its left neighbour \( x - 1 \) and another (independent) message to its right member \( x + 1 \) at each time \( t \) via links \( x - 1 \) and \( x \) respectively.

(ii) If the link \( x \) is up at time \( t \), i.e., \( X_t(x) = 1 \), then the nodes \( x \) and \( x + 1 \) can receive the messages they have (possibly) sent to each others at the previous time \( t - 1 \). If the link \( x \) is down at time \( t \) then these messages are lost. However, the nodes \( x \) and \( x + 1 \) are able to observe that \( X_t(x) = 0 \) in this case.

(iii) If a node \( x \) receives a message at time \( t \) it may resend it immediately, i.e., the destination neighbour is able to receive the message at the time \( t + 1 \) provided the link between it and \( x \) is up at \( t \).

Distant nodes are able to communicate by using the nodes between them as relays. We assume that when a link is open it forms a communication channel that has some finite transfer capacity. This last fact is not used for any calculations but is stated here to make the subsequent considerations meaningful.

B. The overhead messages

In order to use efficient routing schemes it is important that a fresh connectivity status of each node is known at all times. Since the network is linear the relevant information is, for each node \( x \), how far there exists an open path of links in both directions. Because of the finite data propagation speed this connection state information cannot be based on the current state of the network; rather, it is extracted from the newest available data at \( x \) on each link of the network. Since the network model is symmetric with respect to reflection about each node \( x \), how far there exists an open path of links in both directions.

Note that (3) implies that the equality in (6) holds even without the limit whenever \( t > m \).

Because of translation symmetry, we restrict ourselves to the studying of the node \( x = 1 \) and abbreviate \( M_t := M_t(1) \). Then (5) becomes

\[
M_t = \sum_{m=1}^{t} \prod_{k=1}^{m} X_{t+1-k}(x-1+k), \quad (5)
\]

As time advances nodes transmit information to their neighbours according to the recursive scheme

\[
M_t := X_t \left[ M_{t-1}(x + 1) + 1 \right], \quad (4)
\]

which, by the independence of the links, has a stationary distribution

\[
\lim_{t \to \infty} P[M_t = m] = D U^m. \quad (6)
\]

We denote positive integers by \( \mathbb{N} \) and write \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) for non-negative integers.
This may be easily evaluated to give
\[ C. \] Entropies related to the link variables
The entropy of a single link (say \( X_t(x) \)) is
\[ H(X_t) = h(U), \]
where \( h(\lambda) := -\lambda \log \lambda - (1 - \lambda) \log (1 - \lambda), \quad \lambda \in [0, 1]. \] (9)
The entropy rate of the process \( X \) is by definition given by
\[ \mathcal{H}(X) := \lim_{t \to \infty} \frac{1}{t} H(X_t). \]
Using the chain rule for entropy and Markovity (see [3] for details) we may write
\[ \mathcal{H}(X) = \lim_{t \to \infty} H(X_{t+1} \mid X_t). \]
Since we assumed that \( X_1 \) is distributed according to the stationary distribution, we get
\[ \mathcal{H}(X) = H(X_2 \mid X_1). \]
This may be easily evaluated to give
\[ \mathcal{H}(X) = U h(d) + D h(u). \]

In the following we shall also encounter Markov chains \( X^{(j)} = \{X_t^{(j)}, t \in \mathbb{N}\} \) defined by
\[ X_t^{(j)} := X_{jt}. \]
We therefore “skip” over \( j \) links at each time step. The corresponding transition probabilities are characterised by the two off-diagonal elements of \( T_j \), denoted by
\[ u_j := P[X_{t+1} | X_t = 0], \quad d_j := P[X_{t+1} = 0 | X_t = 1]. \]
Precisely as above, we find for the entropy rate of this process:
\[ \mathcal{H}(X^{(j)}) = U h(d_j) + D h(u_j), \]
where we used the fact that the stationary distribution of \( X^{(j)} \) is the same as that of \( X \).

III. OVERHEAD COST: ENTROPY RATE OF THE OVERHEAD MESSAGES
We now quantify the optimal (i.e. smallest possible) cost of the connection state information overhead by the entropy rate\(^3\) of the stochastic process \( M := \{M_t, t \in \mathbb{N}\} \). This corresponds to the minimum amount of bits that need to be used on average to keep up to date on the number of consecutive up-links in the right direction from a fixed node \( x \) (for more details see for instance [3], [4]). The rate is
\[ \mathcal{H}(M) := \lim_{t \to \infty} \frac{1}{t} H(M_1, \ldots, M_t) \]
\[ = \lim_{t \to \infty} H(M_t \mid M_{t-1}, \ldots, M_1), \] (12)
where the second equality follows by applying the chain rule of entropy (note that both limits exist since \( M \) is an autonomous ergodic aperiodic process; see [3] for details).

A. Bounds for the message entropy rate
The evaluation of (12) is tedious. A more practical approach is to compute lower and upper bounds that can be made as accurate as desired. Define for \( j \in \mathbb{N} \)
\[ \mathcal{B}_j := \lim_{t \to \infty} H(M_t \mid M_{t-1}, \ldots, M_{t-j+1}), \] (13)
\[ \mathcal{L}_j := \lim_{t \to \infty} H(M_t \mid M_{t-1}, \ldots, M_{t-j+1}, Z_{t-j}). \] (14)
It should not come as a surprise that \( \mathcal{B}_j \) (resp. \( \mathcal{L}_j \)) is an upper (resp. lower) bound for \( \mathcal{H}(M) \) that becomes arbitrarily accurate in the limit \( j \to \infty \). This is the content of the following.

**Lemma 3.1**: The sequence \( \{\mathcal{B}_j\}_{j \in \mathbb{N}} \) is non-increasing and \( \{\mathcal{L}_j\}_{j \in \mathbb{N}} \) is non-decreasing. Furthermore for all \( j \in \mathbb{N} \) we have
\[ \mathcal{L}_j \leq \mathcal{H}(M) \leq \mathcal{B}_j. \]
Finally,
\[ \mathcal{B}_j - \mathcal{L}_j \leq C |1 - u - d|^{j}, \]
for some constant \( C = C(u, v) \).

\(^3\)Note that this must still be multiplied by two to account for both right and left directions.
Proof: We omit the (easy) proof of monotonicity of the sequences as well as the fact that they are bounds for $\mathcal{H}(M)$ (see for instance Lemma 4.4.1 in [3]). The convergence of the bounds is postponed to Theorem 3.3 as it is easiest to prove using results from the following section.

B. A recursive scheme for the bounds

In this section we derive the main result: A recursive algorithm for computing the bounds $L_j$, $U_j$ and thus for approximating the exact entropy rate $\mathcal{H}(M)$ to an arbitrary accuracy.

For the proof it will be useful to rewrite the entropy by partitioning the probability space $\Omega$. Let $A \subset \Omega$ be an event. Define $H(\cdot : A)$ as the entropy functional computed using the conditional probability measure $P[\cdot | A]$. For two random variables $X, Y$ we have, for example,

$$H(X | Y : A) = \sum_{x,y} P[X = x, Y = y | A] \log P[X = x | Y = y, A].$$

Lemma 3.2: If $A$ lies in the $\sigma$-field $\sigma(X, Y)$ generated by $(X, Y)$, then

$$H(X | Y) = H(I_A | Y) + P[A] H(X | Y : A) + P[A'] H(X | Y : A'),$$

where $I_A$ is the indicator function of the event $A$, and $A'$ denotes the complement of the set $A$.

Note that if $A \in \sigma(Y)$ the first term of (15) vanishes.

Proof: Using the fact that $I_A$ is a deterministic function of $(X, Y)$ as well as the chain rule we have

$$H(X | Y) = H(X, Y | I_A) = H(X | I_A, Y).$$

The second term is equal to

$$\sum_{i \in \{0, 1\}} \sum_{x,y} P[X = x, Y = y, I_A = i] \log P[X = x | Y = y, I_A = i] = \sum_{i \in \{0, 1\}} P[I_A = i] \sum_{x,y} P[X = x, Y = y | I_A = i] \log P[X = x | Y = y, I_A = i]$$

$$= P[A] H(X | Y : A) + P[A'] H(X | Y : A').$$

We now introduce two sequences that will play a key role in the following. For $j \in \mathbb{N}$ define

$$p_j := \lim_{t \to \infty} P[M_t > \max\{M_{t-1}, \ldots, M_{t-j}\}],$$

we also set $p_0 := 1$. Define furthermore the differences

$$r_j := p_{j-1} - p_j,$$

for $j \in \mathbb{N}$.

In order to avoid writing explicit limits in the following we introduce the equivalence relation $\sim$ to denote asymptotic equality: $a(t) \sim b(t)$ means $\lim_{t \to \infty} a(t) = \lim_{t \to \infty} b(t)$.

Theorem 3.3: The sequence of bounds $L_j$, $U_j$ can be computed recursively from

$$L_{j+1} = L_j + \frac{p_j}{D} \left[ \mathcal{H}(X^{(j+1)}) - \mathcal{H}(X^{(j)}) \right],$$

$$U_{j+1} = L_j + \frac{p_j}{D} \left[ H(X_1) - \mathcal{H}(X^{(j)}) \right],$$

and

$$L_1 = \frac{1}{D} \mathcal{H}(X^{(1)}),$$

$$U_1 = \frac{1}{D} H(X_1) - \mathcal{H}(X^{(1)}).$$

Note that the probabilities $p_j$ (or, equivalently, the differences $r_j$) must still be computed; this is done in Appendix B. Everything else in the above expressions is known: $\mathcal{H}(X^{(j+1)})$ was computed in (13), and $H(X_1) = h(U)$.

A direct consequence of the theorem is an expression for the exact entropy rate: From (17a) and (16a) we get

$$\mathcal{H}(M) = \frac{1}{D} \sum_{j=1}^{\infty} r_j \mathcal{H}(X^{(j)}).$$

Proof: [Theorem 3.3] We first introduce some notation. Define the vector

$$M^{(j)} := (M_{t-1}, \ldots, M_{t-j})$$

and the $\infty$-norm $\cdot$ defined by

$$\| (m_1, \ldots, m_j) \| := \max\{m_1, \ldots, m_j\}.$$ A direct consequence of the theorem is an expression for the exact entropy rate: From (17a) and (16a) we get

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Proof: [Theorem 3.3] We first introduce some notation. Define the vector

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and the $\infty$-norm $\cdot$ defined by

$$\| (m_1, \ldots, m_j) \| := \max\{m_1, \ldots, m_j\}.$$
Let us consider the bounds from Lemma 3.5. (c) from Lemma 3.2; (d) from Lemma 3.4 (ii), (iii), (v); and (f) from Lemma 3.3, we have

\[ \mathcal{H}(Z_t) \]

Similarly,

\[ \mathcal{H}(M_t) \]

(a) from Lemma 3.2; (b) from Lemma 3.4; (c) from Lemma 3.5.

For any \( I \in \mathbb{N} \), we have \( \mathcal{H}(I) \) and the chain rule

\[ \mathcal{H}(M_t \mid Z_{t-j}) = \mathcal{H}(Z_t) + \mathcal{H}(M_t \mid Z_{t-j}) \]

where the last step follows from the fact that \( M_t \) is independent of \( Z_t \). Using translation invariance we therefore get

\[ \lim_{t \to \infty} \mathcal{H}(M_t \mid Z_{t-j}) = \mathcal{H}(X^{(j)}) + U \lim_{t \to \infty} \mathcal{H}(M_t \mid Z_{t-j}) \]

and \( (20) \) follows.

Furthermore from \( P[M_t = m] \sim U^m D \) we get

\[ \lim_{t \to \infty} \mathcal{H}(M_t) = - \sum_{m=0}^{\infty} U^m D \log(U^m D) \]

\[ = \frac{h(U)}{D} = \frac{H(X_1)}{D} \]

where the function \( h \) is defined in (2).

C. Convergence of the bounds

We now address the convergence of the bounds, thus completing the proof of Lemma 3.3.

**Theorem 3.6:** For any \( u, v \in (0,1) \) there exists a constant \( C = C(u, v) < \infty \) such that

\[ \mathcal{H}_t \leq C \left| 1 - u - d \right|^j. \]

**Proof:** We start with three auxiliary results.

First, we notice that the eigenvalues of the single link transition matrix \( T \) in (1) are \( 1, 1-u-d \), and since \( |1-u-d| < 1 \) the limit

\[ T_* = \lim_{j \to \infty} T^j = \begin{pmatrix} U & \ast \\ D & D \end{pmatrix} \]

exists. (This is just a restatement that the \( X \) has a unique stationary distribution.) The convergence is exponentially fast, i.e.,

\[ \|T^j - T_*\| \leq k_1 |1-u-d|^j, \]

where \( \| \cdot \| \) is a matrix norm and \( k_1 = k_1(u,v) \) is some finite constant.

Second, the smooth function \( g \) on \( 2 \times 2 \) matrices \((0,1)^2 \times 2\), defined by

\[ g(A) := -D A_{11} \log(A_{11}) - D A_{21} \log(A_{21}) - U A_{12} \log(A_{12}) - U A_{22} \log(A_{22}) \]

is Lipschitz continuous on closed subdomains. In particular, for all \( A, A' \in B_\varepsilon(T_*) \) holds

\[ |g(A) - g(A')| \leq k_2 \|A - A'\|, \]

1 ANJAIK AND KNOWLES: CONNECTION STATE OVERHEAD IN A DYNAMIC LINEAR NETWORK
provided that $\varepsilon > 0$ is small enough that the closure of the ball $B_\varepsilon(T_*=A \in \mathbb{R}^{2 \times 2}: \|A - T_*\| < \varepsilon)$ is contained in $(0, 1)^{2 \times 2}$, and the finite constant $k_2 = k_2(u, v, \varepsilon)$ is large enough.

Third, by a direct calculation we see that $g$ satisfies

$$\mathcal{H}(X^{(j)}) = g(T^j) \quad \text{and} \quad H(X_1) = g(T_*).$$

Therefore, by expressing the difference of the recursion relations $\mathcal{P}_j$ and $\mathcal{Q}_j$ with these identities and using the trivial bound $p_j \leq 1$, we get

$$\mathcal{H}_j - \mathcal{Q}_j = \frac{p_{j-1}}{D} \left[ \mathcal{H}(X^{(j)}) - H(X_1) \right] \leq \frac{1}{D} \left[ g(T^j) - g(T_*) \right].$$

If $j$ is large enough the estimates $\mathcal{P}_j$ and $\mathcal{Q}_j$ can be combined to yield

$$g(T^j) - g(T_*) \leq k_2 \left\| T^j - T_* \right\| \leq k_1 k_2 |1 - u - d|^j,$$

which together with $\mathcal{P}_j$ completes the proof.

Finally some remarks about convergence. From the theorem it is clear that if $u + d = 1$ the convergence is fast. Indeed, if $u + d = 1$ the first order terms $\mathcal{Q}_j$ are exact. This can also be seen directly: We have $u = U, d = D$, so that $T = T^2 = T_*$ and therefore $\mathcal{H}(X) = \mathcal{H}(X^{(j)}) = H(X_1)$. On the other hand, the convergence becomes slower if $u, d \approx 0$ or $u, d \approx 1$. The limiting case $u = d = 0$ corresponds to a static network and $u = d = 1$ is physically meaningless, which is also why we excluded both cases from our discussion.

IV. CONCLUSION

In a dynamic network information about connectivity must be sent through the network regularly. This connection state overhead consumes the available bandwidth of the network. It is therefore natural to ask what is the smallest possible (in the context of information theory) bandwidth required for the connection state overhead. In this work we provide the answer in the special case of a simple linear network model: As a main result we have presented an exact and rapidly converging series expression for the best achievable overhead data rate.

We have only considered a linear network model. However, the results derived here are also applicable to the case of a tree with the connectivity information at each node being whether or not it is connected to the root, since this model is fully equivalent to the one-dimensional network.

The generalisation of our results to linear networks with more general links that have a larger state space is probably possible by using the same or very similar techniques as here. However, the most interesting generalisations, such as more complex network topologies, seem to pose a far greater challenge.

APPENDIX I

AN EFFECTIVE ALGORITHM FOR COMPUTING $r_j$

A “brute force” computation of $p_j$ is too complex to be of any practical use if $j > 2$. We present here a more convenient method. The result is a simple recursive algorithm for calculating $r_j$. The probabilities $p_j$ can then be computed from

$$p_j = 1 - r_1 - \cdots - r_j.$$

For $j \in \mathbb{N}$ we have

$$r_j = p_{j-1} - p_j \sim \mathbb{P} \left[ M_t > \max\{M_{t-1}, \ldots, M_{t-j+1}\}, M_t \leq M_{t-j} \right] = \sum_{m=0}^{\infty} \mathbb{P}[M_t = m] \mathbb{P} \left[ M_{t-} \geq m, M_{t-j+1} < m, \ldots, M_{t-1} < m \mid M_t = m \right].$$

(25)

Define the new random variable

$$Z_i^{(m)} := \prod_{x=1}^m Z_i(x),$$

so that

$$\{Z_i^{(m)} = 0\} = \{M_t < m\}.$$

Then we get from above

$$r_j \sim \sum_{m=0}^{\infty} \mathbb{P} \left[ Z_i^{(m)} = 1, Z_i^{(m)} = \cdots = Z_i^{(m)} = 0 \right] \mathbb{P} \left[ Z_i^{(m)} = 1, Z_i(m+1) = 0 \right] \mathbb{P}[M_t = m],$$

$$\sim D \sum_{m=0}^{\infty} r^{(m)} U^m,$$

(26)

where we have used $\mathcal{A}$ and $r^{(m)}$ is the limit

$$\lim_{t \to \infty} \mathbb{P} \left[ Z_i^{(m)} = 1, Z_i^{(m)} = \cdots = Z_i^{(m)} = 0 \right] = 1.$$

The above discussion is meaningless if $m = 0$; from $\mathcal{A}$, however, we see that we must define

$$r_j^{(0)} := \begin{cases} 1, & j = 1; \\ 0, & j > 1; \end{cases}$$

for $\mathcal{A}$ to hold.

Define now for $j \in \mathbb{N}$

$$q_j^{(m)} := \lim_{t \to \infty} \mathbb{P} \left[ Z_i^{(m)} = 1 \mid Z_i^{(m)} = 1 \right].$$

For the following we note that the process obtained from $X$ by reversing the time is also a Markov process with transition probabilities identical to those of $X$; for example $\mathbb{P}[X_{t-1} = 1 \mid X_t = 0] = u$. Thus

$$q_j^{(m)} = r_j^{(m)}.$$

The recursion relation for $r_j^{(m)}$ arises as follows. We rewrite $q_j^{(m)}$ by decomposing the event $\{Z_i^{(m)} = 1, Z_i^{(m)} = 1\}$; By

$\text{We use here the fact that the links are distributed according to the stationary distribution at all times.}$
successively conditioning on the values of $Z_{i-1}^{(m)}$, $i = 1, \ldots, j$, we get

$$\left\{ Z_{i-j}^{(m)} = 1, Z_i^{(m)} = 1 \right\} = \sum_{i=1}^j \left\{ Z_{i-j}^{(m)} = 1, Z_i^{(m)} = 1 \right\},$$

where the sum means a union of disjoint events. Taking the probability measure of both sides and using Markovity of the time-reversed $X$ process we have

$$q_j^{(m)} = \sum_{i=1}^j q_{j-i}^{(m)} r_i^{(m)},$$

which gives

$$r_j^{(m)} = \frac{d_j^{(m)}}{d_j} - \sum_{i=1}^{j-1} d_{j-i}^{(m)} r_i^{(m)}. \tag{27}$$

This is the desired recursion relation expressing $r_j^{(m)}$ as a function of $r_1^{(m)}, \ldots, r_{j-1}^{(m)}$. Using $r_1^{(m)} = \frac{d_1^{(m)}}{d_1}$ we may therefore find $r_j^{(m)}$.

We summarise the results:

**Lemma 1.1:** The quantity $r_j$, $j \in \mathbb{N}$, may be computed from

$$r_j = D \sum_{m=0}^{\infty} r_j^{(m)} U^m,$$

where $r_j^{(m)}$, $m \in \mathbb{N}_0$, satisfies the recursion relation

$$r_j^{(m)} = D_{j}^{(m)} - \sum_{i=1}^{j-1} D_{j-i}^{(m)} r_i^{(m)},$$

$$r_1^{(m)} = D_{1}^{(m)}.$$

As an example, we compute $r_1$, $r_2$ and $r_3$:

$$r_1 = D \sum_{m=0}^{\infty} \frac{d_1^{(m)}}{d_1} U^m = D \frac{1}{1 - d_1},$$

$$r_2 = D \sum_{m=0}^{\infty} \left( \frac{d_2^{(m)}}{d_2} - \frac{d_1^{(2m)}}{d_1^2} \right) U^m = D \left[ \frac{1}{1 - d_2} - \frac{1}{1 - d_1^2} \right],$$

$$r_3 = D \sum_{m=0}^{\infty} \left( \frac{d_3^{(m)}}{d_3} - 2 \frac{d_2^{(m)}}{d_2} d_1^{(m)} + \frac{d_1^{(3m)}}{d_1^3} \right) U^m = D \left[ \frac{1}{1 - d_3} - \frac{2}{1 - d_2 d_1} + \frac{1}{1 - d_1^3} \right].$$

**APPENDIX II**

**Proof of Lemma 3.3**

The proof involves deriving equalities for conditional probabilities. These then induce equalities of the conditional entropies according to the following lemma.

Note that if $Z_i^{(m)} = 1$ then all of the relevant first $m$ links of $Z_i$ are known (to equal 1).

**Lemma 2.1:** Let $X, Y$ be random variables, $\phi$ a function on the range of $Y$, and suppose that, for all $x, y$,

$$P[X = x \mid Y = y] = P[X = x \mid Y \in \phi^{-1}(\phi(y))].$$

Then

$$H(X \mid Y) = H(X \mid \phi(Y)).$$

**Proof:** The proof is based on writing out the definition of the conditional entropy $H(X \mid Y)$, rewriting the sum $\sum_x \sum_y f(x, y)$ as $\sum_x \sum_{y: \phi(y) = s} f(x, y)$ and using the assumption. We omit further details. \[\square\]

**Proof:** [Lemma 3.3] Let us begin with (i). The conditioning event is

$$A^c = \{ M_t \leq |M_{t-1}| \}.$$ 

Let $m \in \mathbb{N}$ and define

$$i(m) := \min \{ k \in \{ 1, \ldots, j \} : m_{t-k} = |m| \}.$$ 

Let furthermore $z, z' \in \{ 0, 1 \}^N$ be chosen so that $\varphi(z) = m_{t-j}$ and $\varphi(z') = m_{t-i(m)}$, where $\varphi$ is a deterministic function that gives $M_t$ as a function of $Z_t$. Then we have, for $m \in \mathbb{N}$ and $z \in \{ 0, 1 \}^N$,

$$P \left[ M_t = m \mid M_{t-1}^{(j)} = m, Z_{t-j-1} = z, M_t \leq |m| \right]$$

(a) $$P \left[ Z_t(1) = \ldots = Z_t(m) = 1, Z_t(m + 1) = 0 \mid M_{t-1}^{(j)} = m, Z_{t-j-1} = z, M_t \leq |m|, Z_{t-i(m)}(1) = \ldots \right.$$

(b) $$P \left[ Z_t(1) = \ldots = Z_t(m) = 1, Z_t(m + 1) = 0 \mid M_{t-1}^{(j)} = m, M_t \leq |m|, Z_{t-i(m)}(1) = \ldots \right.$$

(c) $$P \left[ Z_t(1) = \ldots = Z_t(m) = 1, Z_t(m + 1) = 0 \mid M_{t-1}^{(j)} = m, M_t \leq |m|, Z_{t-i(m)}(1) = \ldots \right.$$

(d) $$P \left[ Z_t(1) = \ldots = Z_t(m) = 1, Z_t(m + 1) = 0 \mid M_{t-1}^{(j)} = m, M_t \leq |m|, Z_{t-i(m)}(1) = \ldots \right.$$

(e) $$P \left[ M_t = m \mid M_{t-1}^{(j-1)} = m, M_t \leq |m|, Z_{t-j-1} = z' \right].$$

where $m_{(j-1)}$ denotes the $j - 1$ first components of $m$: (a) follows from rewriting the conditions $M_t = m$, $M_{t-1}^{(m)} = |m|$; (b) from Markovity, independence and the fact that $m \leq |m|$; (c) from independence and $m \leq |m|$; (d) from Markovity; and (e) from independence and $m \leq |m|$. Then the assertion follows from Lemma 2.1 by choosing the functions $\phi_1(1^{(j-1)}, z', z) := (1^{(j-1)}, \varphi(z'), z)$, $\phi_2(1^{(j-1)}, z', z) := (1^{(j-1)}, \varphi(z'))$, and $\phi_3(1^{(j-1)}, z', z) := (1^{(j-1)}, 1^{(j-1)}).$
To prove (ii) choose \( m, z \) and \( z' \) as above and write:

\[
\begin{align*}
\Pr \left[ M_t > |m| \left| M_{t-1}^{(j)} = m, Z_{t-j-1} = z \right. \right] &
\end{align*}
\]

\[(a) \quad \Pr \left[ Z_t(1) = \cdots = Z_t(|m| + 1) = 1 \left| M_{t-1}^{(j)} = m \right. \right] = P
\]

\[(b) \quad \Pr \left[ Z_t(1) = \cdots = Z_t(|m| + 1) = 1 \left| M_{t-1}^{(j)} = m \right. \right] = P
\]

\[(c) \quad \Pr \left[ Z_t(1) = \cdots = Z_t(|m| + 1) = 1 \left| M_{t-1}^{(j)} = m \right. \right] = P
\]

where (a) follows from rewriting \( M_t > |m| \); (b) and (c) from independence and Markovity (the full details are exactly as above using the index variable \( i(m) \)).

The proofs of (iii), (iv) and (v) are almost identical; we only show (iv). Let \( m, m' \) and \( z \) be as above. First note that under the conditions \( M_{t-1}^{(j)} = m \) and \( M_t > |m| \) there is a bijective map between \( M_t \) and \( M_t(|m| + 2) \):

\[
M_t = \tilde{M}_t(|m| + 2) + |m| + 1,
\]

so that

\[
\begin{align*}
&H(M_t \mid Z_{t-j-1} : \{ M_{t-1}^{(j)} = m, M_t > |m| \}) \\
= &H(\tilde{M}_t(|m| + 2) \mid Z_{t-j-1} : \{ M_{t-1}^{(j)} = m, M_t > |m| \}) \quad \text{(29)}
\end{align*}
\]

Now

\[
\begin{align*}
\Pr \left[ \tilde{M}_t(|m| + 2) = m \left| M_{t-1}^{(j)} = m, Z_{t-j-1} = z, M_t > |m| \right. \right] &
\end{align*}
\]

\[(a) \quad \Pr \left[ \tilde{M}_t(|m| + 2) = m \left| M_{t-1}^{(j)} = m, Z_{t-j-1} = z, M_t > |m| \right. \right] = P
\]

\[(b) \quad \Pr \left[ \tilde{M}_t(|m| + 2) = m \left| Z_{t-j-1} = z \right. \right]
\]

where (a) follows from rewriting the condition \( M_t > |m| \), and (b) from independence. Now by Lemma \( 2.1 \) and \( 2.29 \) we get

\[
\begin{align*}
&H(M_t \mid Z_{t-j-1} : \{ M_{t-1}^{(j)} = m, M_t > |m| \}) \\
= &H(\tilde{M}_t(|m| + 2) \mid Z_{t-j-1}) \sim H(M_t \mid Z_{t-j-1})
\end{align*}
\]

where the last step follows from translation invariance. Therefore

\[
\begin{align*}
&H(M_t \mid M_{t-1}^{(j)}, Z_{t-j-1} : A) \\
= &\sum_{m} \Pr[M_{t-1}^{(j)} = m] \\
&\sum_{m} H(M_t \mid Z_{t-j-1} : \{ M_{t-1}^{(j)} = m, M_t > |m| \}) \\
&\sim H(M_t \mid Z_{t-j-1}) \sum_{m} \Pr[M_{t-1}^{(j)} = m] \\
= &H(M_t \mid Z_{t-j-1})
\end{align*}
\]

\[\blacksquare\]

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**REFERENCES**

[1] P. Dey, *Protocol Overheads in Networks*, Doctoral thesis, 2004 EPFL.
[2] Robert G. Gallager, “Basic Limits on Protocol Information in Data Communication Networks,” *Trans. Inform. Theory*, Vol. IT-22, No. 4, July 1976, pp. 385-398.
[3] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley & Sons, 1991 New York.
[4] A. I. Khinchin, *Mathematical Foundations of Information Theory*, Dover Publications, Inc., 1957 New York.