C\(^1\)-UMBILICS WITH ARBITRARILY HIGH INDICES

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ABSTRACT. In this paper, the existence of C\(^1\)-umbilics with arbitrarily high indices is shown. This implies that more than C\(^1\)-regularity is required to prove Loewner’s conjecture.

1. INTRODUCTION

The index of an isolated umbilic on a given regular surface is the index of the curvature line flow of the surface at that point, which takes values in the set of half-integers. Loewner’s conjecture asserts that any isolated umbilic on an immersed surface must have index at most 1. Carathéodory’s conjecture asserts the existence of at least two umbilics on an immersed sphere in \(\mathbb{R}^3\), which follows immediately from Loewner’s conjecture. Although this problem was investigated mainly on real-analytic surfaces, several geometers recently became interested in non-analytic cases (cf. \([A, B, GH, GMS, SX]\)). In particular, Smyth-Xavier \([SX]\) observed that Enneper’s minimal surface is inverted to a branched sphere such that the index of the curvature line flow at the branch point is equal to two. Bates \([B]\) found that the graph of the function

\[
B(x, y) := 2 + \frac{xy}{\sqrt{1 + x^2} \sqrt{1 + y^2}}
\]

has no umbilics on \(\mathbb{R}^2\) and inversion of it gives a genus zero surface without self-intersections, which is differentiable at the image of infinity under that inversion. Ghomi-Howard \([GH]\) gave similar examples of genus zero surfaces using inversion. Moreover, they showed that Carathéodory’s conjecture for closed convex surfaces can be reduced to the problem of existence of umbilics of certain entire graphs over \(\mathbb{R}^2\). A brief history of Carathéodory’s conjecture and recent developments are written also in \([GH]\). Recently, Guilfoyle-Klingenberg \([GK1]\) and \([GK2]\) gave an approach to proving the Caratheodory and Loewner conjecture in the smooth case.

Let \(P : U \to \mathbb{R}^4\) be a \(C^1\)-immersion defined on an open subset \(U\) of \(\mathbb{R}^2\) such that \(P\) is \(C^\infty\)-differentiable on \(U \setminus \{q\}\) and not \(C^2\)-differentiable at \(q\). Then the point \(q \in U\) is called a \(C^1\)-umbilic if the umbilics of \(P\) on \(U \setminus \{q\}\) do not accumulate to...
Let \( U_1 (\subset \mathbb{R}^2) \) be the unit disk centered at the origin. For each positive integer \( m \), there exists a \( C^1 \)-function \( f : U_1 \to \mathbb{R} \) satisfying the following properties:

1. \( f \) is real-analytic on \( U_1^* := U_1 \setminus \{(0, 0)\} \).
2. \((0, 0, f(0, 0))\) is a \( C^1 \)-umbilic of the graph of \( f \) with index \( 1 + (m/2) \).

It should be remarked that the inversion of the graph of Bates’ function \( B(x, y) \) has a differentiable umbilic of index 2 although not of class \( C^1 \) (see Example 2.3). It was classically known that curvature line flows are closely related to the eigen-flows of the Hessian matrices of functions (see Appendix A). As an application of the above result, we can show the following:

**Corollary 1.2.** For each \( m (\geq 1) \), there exists a \( C^1 \)-function \( \lambda : U_1 \to \mathbb{R} \) satisfying

1. \( \lambda \) is real-analytic on \( U_1^* \), and
2. the eigen-flow of the Hessian matrix of \( \lambda \) has an isolated singular point \((0, 0)\) with index \( 1 + (m/2) \).

When we consider the eigen-flow of the Hessian matrix of \( f \), it is well-known that the index of the flow at an isolated singular point is equal to half of the index of the vector field

\[
d_f := 2f_{xy} \frac{\partial}{\partial x} + (f_{yy} - f_{xx}) \frac{\partial}{\partial y}.
\]

In addition, if \( o := (0, 0) \) is an isolated singular point of the eigen-flow of the Hessian matrix of \( f \), then its index is equal to \( 1 + \text{ind}_o(\delta_f)/2 \) (see Appendix B), where \( \text{ind}_o(\delta_f) \) is the index of the vector field

\[
\delta_f := 2(rf_r \theta - f_\theta) \frac{\partial}{\partial x} + (-r^2 f_{rr} + rf_r + f_{\theta\theta}) \frac{\partial}{\partial y},
\]

at \( o \), and \( x = r \cos \theta, y = r \sin \theta \). In order to prove the above theorem, we introduce vector fields \( D_f \) and \( \Delta_f \) analogous to \( d_f \) and \( \delta_f \), respectively (cf. Propositions \( \ref{proposition:1} \) and \( \ref{proposition:2} \)), and prove the theorem by computing the index of \( \Delta_f \) at infinity for each of the functions (cf. Section 5)

\[
(f =) f_m(r, \theta) := 1 + \tanh (r^a \cos m \theta) \quad (0 < a < 1/4, \ m = 1, 2, \ldots).
\]

In addition, we give an alternative proof of Theorem 1.1 without use of inversion, by an explicit example of \( \lambda \) (cf. (6.1)) satisfying (1) and (2) of Corollary 1.2 (see Section 6).

2. **The Regularity of the Inversion**

Let \( R \) be a positive number. Consider a function \( f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R} \), where

\[
\Omega_R := \{(x, y) \in \mathbb{R}^2 ; \sqrt{x^2 + y^2} \leq R\}.
\]
Then $F = (x, y, f(x, y))$ gives a parametrization of the graph of $f$. The inversion of $F$ is given by $F/(F \cdot F)$, where the dot denotes the inner product on $\mathbb{R}^3$. We consider the following coordinate change

\[(2.2)\]

\[x = \frac{u}{u^2 + v^2}, \quad y = \frac{v}{u^2 + v^2}.\]

Then

\[(2.3)\]

\[\Psi_f := \frac{1}{\rho^2 f^2 + 1} (u, v, \rho^2 \hat{f}), \quad \hat{f}(u, v) := f \left( \frac{u}{\rho^2}, \frac{v}{\rho^2} \right)\]

gives a parametrization of the inversion, where $\rho := \sqrt{u^2 + v^2}$. The map $\Psi_f$ is defined on the domain

\[(2.4)\]

\[U^*_1/R := U_1/R \setminus \{0\}, \quad \left( U_1/R := \{(u, v) \in \mathbb{R}^2; \sqrt{u^2 + v^2} < \frac{1}{R} \} \right),\]

where $o := (0, 0)$. If we set

\[(2.5)\]

\[x = r \cos \theta, \quad y = r \sin \theta,\]

where $r > 0$, then \[(2.2)\] yields

\[(2.6)\]

\[\rho = \frac{1}{r}, \quad u = \rho \cos \theta, \quad v = \rho \sin \theta.\]

In particular, the angular parameter is common in the $xy$-plane and the $uv$-plane.

**Proposition 2.1.** Let $f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}$ be a $C^\infty$-function such that $f/r$ is bounded. Then the inversion $\Psi_f : U^*_1/R \to \mathbb{R}^3$ can be continuously extended to $(0, 0)$. Moreover, if

\[(2.7)\]

\[\left| \frac{f^2 - 2rf f_r}{r^2} \right| < 1 \quad (r > R),\]

then the image of $\Psi_f = (X, Y, Z)$ can be locally expressed as the graph of a function $Z = Z_f(X, Y)$ on a neighborhood of $(0, 0)$ in the $XY$-plane. Under the assumption \[(2.7)\], the function $Z_f(X, Y)$ is differentiable if and only if

\[\lim_{r \to \infty} \frac{f}{r} = 0.\]

**Proof.** We can write

\[(2.8)\]

\[\Psi_f(u, v) = \frac{1}{1 + \varphi(u, v)^2} \left( u, v, \varphi(u, v) \sqrt{u^2 + v^2} \right),\]

where

\[(2.9)\]

\[\varphi(u, v) = \sqrt{u^2 + v^2} \hat{f}(u, v) = \frac{f(x, y)}{r}.\]

Since $f/r$ is bounded, the function $\varphi$ is bounded on $U^*_1/R$. Thus, one can prove $\lim_{r \to 0} \Psi_f = (0, 0, 0)$ using \[(2.8)\], that is, $\Psi_f(u, v)$ can be continuously extended to $(0, 0)$. We denote by $\Pi : \mathbb{R}^3 \ni (x, y, z) \mapsto (x, y) \in \mathbb{R}^2$ the orthogonal projection. By setting

\[\psi(\rho, \theta) := \frac{\rho}{1 + \varphi(\rho \cos \theta, \rho \sin \theta)^2},\]
it holds that
\begin{equation}
\Pi \circ \Psi_f(u, v) = \left( \psi(\rho, \theta) \cos \theta, \psi(\rho, \theta) \sin \theta \right).
\end{equation}

Since \( \hat{f}(\rho \cos \theta, \rho \sin \theta) = f(\cos \theta / \rho, \sin \theta / \rho) \), we have
\[ \varphi_\rho = f - r f_r. \]
In particular, it holds that
\[ \varphi_\rho = \frac{1 - (f^2 - 2rf f_r) / r^2}{(1 + f^2 / r^2)^2}. \]

By (2.7), there exists \( \varepsilon > 0 \) such that \( \rho \mapsto \psi(\rho, \theta) \) \((|\rho| \leq \varepsilon)\) is a monotone increasing function for each \( \theta \). Thus, by (2.10), we can conclude that \( \Pi \circ \Psi_f : U_\varepsilon \to \mathbb{R}^2 \) is an injection. Since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, the inverse map \( G : \Omega \to U_\varepsilon \) of \( \Pi \circ \Psi_f |_{U_\varepsilon} \) is continuous, where \( \Omega \) is a neighborhood of the origin of the \( XY \)-plane in \( \mathbb{R}^3 \). Then the graph of
\begin{equation}
Z_f = \frac{\rho \varphi}{1 + \varphi^2} = \frac{\varphi(G(X, Y)) \rho(G(X, Y))}{1 + \varphi(G(X, Y))^2}
\end{equation}
coincides with the image of \( \Psi_f = (X, Y, Z) \) around \((0, 0, 0)\). Then
\[ X = \frac{u}{1 + \varphi^2}, \quad Y = \frac{v}{1 + \varphi^2}, \quad Z = \frac{\rho \varphi}{1 + \varphi^2}. \]

Since \( \rho \to 0 \) as \((X, Y) \to (0, 0)\), we obtain
\begin{equation}
\lim_{(X,Y) \to (0,0)} \frac{Z_f(X, Y)}{\sqrt{X^2 + Y^2}} = \lim_{(X,Y) \to (0,0)} \frac{\varphi_\rho}{\sqrt{u^2 + v^2}} = \lim_{\rho \to 0} \varphi = \lim_{r \to \infty} \frac{f}{r},
\end{equation}
proving the last assertion. \( \square \)

**Corollary 2.2.** Suppose that \( f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R} \) is a bounded \( C^\infty \)-function satisfying
\begin{equation}
\lim_{r \to \infty} \frac{f_r}{r} = 0.
\end{equation}
Then the inversion \( \Psi_f : U_{1/R}^* \to \mathbb{R}^3 \) can be continuously extended to \((0, 0, 0)\). Moreover, the image of \( \Psi_f \) is locally a graph which is differentiable at \((0,0)\).

**Example 2.3.** Bates’ example (cf. (1.1)) mentioned in the introduction is differentiable. In fact, \( B(x, y) \) is bounded and \( B_r / r \) converges to zero as \( r \to \infty \). However, the inversion of \((x, y, B(x, y))\) is not \( C^1 \). In fact, the unit normal vector field of the graph of \( B \) is not continuously extended to the point at infinity. Since the inversion preserves the angle, the unit normal vector field of its inversion cannot be continuously extended to \((0, 0, 0)\).

**Example 2.4.** Ghomi-Howard \([\text{GH}]\) gave an example
\begin{equation}
f_{GH} = 1 + \lambda \frac{1 + x + y^2}{\sqrt{1 + (x + y^2)^2}} \quad (\lambda > 0).
\end{equation}
The graph of \( f_{GH} \) is an umbilic free (see Example 3.5 in Section 3). The function \( f_{GH} \) is bounded. In addition, since \((f_{GH})_r \) is bounded, (2.13) is obvious. Therefore,
as pointed out in [GH], the inversion of \((x, y, f_{GH}(x, y))\) is differentiable. However, it is not a \(C^1\)-map. In fact, the limit of the unit normal vector field along \(y = 0\) of the graph of \(f_{GH}\) is not equal to that along \(x + y^2 = 0\) at the point at infinity.

Next, we give a condition for \(\Psi_f\) to be extendable as a \(C^1\)-map to \((0, 0)\).

**Proposition 2.5.** Suppose that \(f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}\) is a bounded \(C^\infty\)-function satisfying

(a) \(\lim_{r \to \infty} f_r = 0\), \hspace{1cm} (b) \(\lim_{r \to \infty} f_\theta/r = 0\).

Then \(\Psi_f = (X, Y, Z)\) can be extended to \((0, 0)\) as a \(C^1\)-map. Moreover, the map \((u, v) \mapsto (X(u, v), Y(u, v))\) is a \(C^1\)-diffeomorphism from a neighborhood of the origin in the \(uv\)-plane onto a neighborhood of the origin in the \(XY\)-plane.

To prove this, we prepare the following lemma.

**Lemma 2.6.** The conditions \((a)\) and \((b)\) in Proposition 2.5 are equivalent to the following two conditions, respectively:

(1) \(\lim_{\rho \to 0} \rho^2 \hat{f}_\rho = 0\), \hspace{1cm} (2) \(\lim_{\rho \to 0} \rho \hat{f}_\theta = 0\).

**Proof.** The equivalency of \((2)\) and \((b)\) is obvious. The equivalency of \((1)\) and \((a)\) follows from the identity \(\hat{f}_\rho = -f_\rho/\rho^2\). \(\square\)

**Proof of Proposition 2.5.** We see by Corollary 2.2 that \(\Psi_f\) can be extended to \((0, 0)\) as a differentiable map and the map \((u, v) \mapsto (X(u, v), Y(u, v))\) is a homeomorphism from a neighborhood of \((0, 0)\) onto a neighborhood of \((0, 0)\). We set

\[(2.15) \hspace{1cm} h := \rho^2 \hat{f}(= \rho \varphi), \hspace{1cm} k := (\rho \hat{f})^2(= \varphi^2).\]

By (2.15), we can write

\[(2.16) \hspace{1cm} \Psi_f = (X, Y, Z) = \frac{1}{k + 1}(u, v, h).\]

To show that \(\Psi_f\) is a \(C^1\)-map at \((0, 0)\), it is sufficient to show that \(h, k\) are \(C^1\)-functions. Since \(h\) and \(k\) are \(C^\infty\)-functions on \(U_{1/R}^*\), they satisfy

\[(2.17) \hspace{1cm} h_u = \rho \left(2 \hat{f} + \rho \hat{f}_\rho\right) \cos \theta - \hat{f}_\theta \sin \theta, \hspace{1cm} h_v = \rho \left(2 \hat{f} + \rho \hat{f}_\rho\right) \sin \theta + \hat{f}_\theta \cos \theta,\]

\[(2.18) \hspace{1cm} k_u = 2 \hat{f} \rho \left(\cos \theta (\hat{f} + \rho \hat{f}_\rho) - \hat{f}_\theta \sin \theta\right), \hspace{1cm} k_v = 2 \hat{f} \rho \left(\sin \theta (\hat{f} + \rho \hat{f}_\rho) + \hat{f}_\theta \cos \theta\right)\]

on \(U_{1/R}^*\). Using (1), (2) in Lemma 2.6, (2.17) and (2.18), one can easily see that

\[(2.19) \hspace{1cm} \lim_{\rho \to 0} h_u = \lim_{\rho \to 0} h_v = \lim_{\rho \to 0} k_u = \lim_{\rho \to 0} k_v = 0,\]

which shows that \(\Psi_f\) extends to \((0, 0)\) as a \(C^1\)-map. By (2.16) and (2.19), we have

\[X_u(0, 0) = 1, \hspace{1cm} X_v(0, 0) = 0, \hspace{1cm} Y_u(0, 0) = 0, \hspace{1cm} Y_v(0, 0) = 1.\]

Thus the second assertion follows from the inverse mapping theorem, because the Jacobi matrix of the map \((u, v) \mapsto (X(u, v), Y(u, v))\) is regular at \((0, 0)\). \(\square\)
In Section 5, we need the following:

**Proposition 2.7.** Let \( f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R} \) be a bounded \( C^\infty \) function satisfying the conditions (a) and (b) of Proposition 2.5. If there exists a constant \( 0 \leq c < 1/2 \) such that
\[
 r^{1-c/2} f_r, \quad r^{-c/2} f_\theta, \quad r^{2-c} f_{rr}, \quad r^{1-c} f_r \theta, \quad r^{-c} f_{\theta \theta}
\]
are bounded on \( \mathbb{R}^2 \setminus \Omega_R \), then the map \((u, v) \mapsto (X(u, v), Y(u, v))\) is a \( C^2 \)-map at \((0, 0)\), where \( \Psi_f = (X, Y, Z) \).

We prepare the following lemmas:

**Lemma 2.8.** The boundedness of the five functions in Proposition 2.7 is equivalent to the boundedness of the functions
\[
 \rho^{1+c/2} \hat{f}_\rho, \quad \rho^{c/2} \hat{f}_\theta, \quad \rho^{2+c} \hat{f}_{\rho \rho}, \quad \rho^{1+c} \hat{f}_{\rho \theta}, \quad \rho^c \hat{f}_{\theta \theta}
\]
on \( \{0, 0\} \setminus \{0\} \), where \( \rho \) is sufficiently small neighborhood of \((0, 0)\).

**Proof.** Differentiating \( \hat{f} = \hat{f}(\rho \cos \theta, \rho \sin \theta) \) by \( \rho \), we have \( \rho \hat{f}_\rho = -r \hat{f}_r \) and \( \rho^2 \hat{f}_{\rho \rho} = 2r \hat{f}_r + r^2 \hat{f}_{rr} \). Using these relations, the assertion can be easily checked. \( \square \)

**Lemma 2.9.** Suppose that the five functions in (2.20) are bounded on \( \{0, 0\} \setminus \{0\} \). Then \( \rho^{2c} k_{uu}, \rho^{2c} k_{uv} \) and \( \rho^{2c} k_{vv} \) are also bounded on \( \{0, 0\} \setminus \{0\} \), where \( k \) is the function given in (2.15).

**Proof.** In fact, each of \( k_{uu}, k_{uv}, k_{vv} \) is written as a linear combination of
\[
1, \quad \rho \hat{f}_\rho, \quad \hat{f}_\theta, \quad (\rho \hat{f}_\rho)^2, \quad \rho \hat{f}_\rho \hat{f}_\theta, \quad \hat{f}_\theta^2, \quad \rho \hat{f}_{\rho \rho}, \quad \rho \hat{f}_{\rho \theta}, \quad \hat{f}_{\theta \theta}
\]
with coefficients that are bounded functions. For example,
\[
k_{uv} = \sin 2\theta \left( \rho^2 \hat{f}_\rho^2 + \hat{f} (\rho^2 \hat{f}_{\rho \rho} + 3\rho \hat{f}_\rho - \hat{f}_{\theta \theta}) - \hat{f}_\theta^2 \right)
\]
\[
+ 2 \cos 2\theta \left( \hat{f}_\theta (\rho \hat{f}_\rho + \hat{f}) + \rho \hat{f} \hat{f}_{\theta \theta} \right).
\]
Thus, we get the assertion. \( \square \)

**Proof of Proposition 2.7** By Lemmas 2.8 and 2.9, the fact that \( 2c < 1 \) yields that
\[
\lim_{\rho \to 0} \rho k_{uu} = \lim_{\rho \to 0} \rho k_{uv} = \lim_{\rho \to 0} \rho k_{vv} = 0.
\]

Since
\[
X_{uu} = \frac{2u k_{uu}^2 - 2(k + 1) k_u - u(k + 1) k_{uu}}{(k + 1)^3}, \quad X_{uv} = \frac{k_u (-2u k_u + k + 1) + u(k + 1) k_{uv}}{(k + 1)^3}, \quad X_{vv} = \frac{-u ((k + 1) k_{uv} - 2k_v^2)}{(k + 1)^3},
\]
we have that \( X_{uu}, X_{uv}, X_{vv} \) tend to 0 as \( \rho \to 0 \). This implies that \( X_u, X_v \) are \( C^1 \)-functions. Similarly, \( Y_u, Y_v \) are also \( C^1 \)-functions. \( \square \)
3. The pair of identifiers for umbilics

Let $U$ be a domain on $\mathbb{R}^2$. Consider a flow (i.e. a 1-dimensional foliation) $\mathcal{F}$ defined on $U \setminus \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ are distinct points in $U$. We are interested in the case that $\mathcal{F}$ is

- the curvature line flow of an immersion $P : U \to \mathbb{R}^3$,
- the eigen-flow of a matrix-valued function on $U$, or
- the flow induced by a vector field on $U$.

We fix a simple closed smooth curve $\gamma : T^1 \to U \setminus \{p_1, \ldots, p_n\}$, where $T^1 := \mathbb{R}/2\pi \mathbb{Z}$. We set

$$\partial_x := \frac{\partial}{\partial x}, \quad \partial_y := \frac{\partial}{\partial y}.$$ 

Then one can take a smooth vector field

$$V(t) := a(t)\partial_x + b(t)\partial_y$$

along the curve $\gamma(t)$ such that $V(t)$ is a non-zero tangent vector of $\mathbb{R}^2$ at $\gamma(t)$ which points in the direction of the flow $\mathcal{F}$. Then the map

$$\tilde{V} : T^1 \ni t \mapsto \frac{(a(t), b(t))}{\sqrt{a(t)^2 + b(t)^2}} \in S^1 := \{x \in \mathbb{R}^2 ; |x| = 1\}$$

is called the Gauss map of $\mathcal{F}$ with respect to the curve $\gamma$. The mapping degree of the map $\tilde{V}$ is called the rotation index of $\mathcal{F}$ with respect to $\gamma$ and denoted by $\text{ind}(\mathcal{F}, \gamma)$, which is a half-integer, in general. If $\gamma$ surrounds only $p_j$, then $\text{ind}(\mathcal{F}, \gamma)$ is independent of the choice of such a curve $\gamma$. So we call it the (rotation) index of the flow $\mathcal{F}$ at $p_j$, and it is denoted by $\text{ind}_{p_j}(\mathcal{F})$. If the flow $\mathcal{F}$ is generated by a vector field $V$ defined on $U \setminus \{p_1, \ldots, p_n\}$, then $\text{ind}_{p_j}(\mathcal{F})$ is an integer, and we denote it by $\text{ind}_{p_j}(V)$.

We denote by $S_2(\mathbb{R})$ the set of real symmetric 2-matrices. Let $U$ be a domain in $\mathbb{R}^2$, and

$$A = \begin{pmatrix} a_{11}(x,y) & a_{12}(x,y) \\ a_{12}(x,y) & a_{22}(x,y) \end{pmatrix} : U \to S_2(\mathbb{R})$$

a $C^\infty$-map. A point $p \in U$ is called an equi-diagonal point of $A$ if $a_{11} = a_{22}$ and $a_{12} = 0$ at $p$. We now suppose that $p$ is an isolated equi-diagonal point. Without loss of generality, we may assume that $A$ has no equi-diagonal points on $U \setminus \{p\}$. Since two eigen-flows of $A$ are mutually orthogonal, the indices of the two eigen-flows of the $S_2(\mathbb{R})$-valued function $A$ are the same half-integer at $p$. We denote it by $\text{ind}_p(A)$.

It is well-known that for an $S_2(\mathbb{R})$-valued function $A$, the formula

$$\text{ind}_p(A) = \frac{1}{2} \text{ind}_p(v_A)$$

holds, where $v_A$ is the vector field on $U$ given by

$$v_A := (a_{11} - a_{22})\partial_x + a_{12}\partial_y.$$

We shall apply these facts to the computation of the indices of isolated umbilics on regular surfaces in $\mathbb{R}^3$ as follows. Let $f : U \to \mathbb{R}$ be a $C^\infty$-function. The symmetric
matrices associated with the first and the second fundamental forms of the graph of \( f \) are given by

\[
I := \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}, \quad II := \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.
\]

We consider a \( GL(2, \mathbb{R}) \)-valued function

\[
P := \begin{pmatrix} 0 & \sqrt{1 + f_x^2} \\ -\sqrt{(1 + f_x^2 + f_y^2)/(1 + f_x^2)} & f_x f_y/\sqrt{1 + f_x^2} \end{pmatrix},
\]

which satisfies the identity \( P^T \mathcal{P} P = \mathcal{I} \), where \( P^T \) is the transpose of \( P \). Then

\[
A_f := P^{-1} II (P^T)^{-1} = P^T (I^{-1} II) (P^T)^{-1}
\]
is an \( S_2(\mathbb{R}) \)-valued function. The umbilics of the graph of \( f \) correspond to the equi-diagonal points of \( A_f \). We show the following:

**Proposition 3.1.** The symmetric matrix \( A_f(p) \) is proportional to the identity matrix at \( p \in U \) if and only if \( p \) gives an umbilic of the graph of \( f \). Moreover, if \( p \) is an isolated umbilic, then \( \text{ind}_p(A_f) \) coincides with the index of the umbilic \( p \).

**Proof.** The first assertion follows from the definition of \( A_f \). Without loss of generality, we may assume that \( p \) coincides with the origin \( o := (0, 0) \), and the graph of \( f \) has no umbilics other than \( o \) on \( U \). Take a sufficiently small positive number \( \varepsilon > 0 \) so that the circle \( \gamma(t) = \varepsilon (\cos t, \sin t) \) \((0 \leq t \leq 2\pi)\) is null-homotopic in \( U \).

We denote by \((a_1(t), b_1(t))^T \) and \((a_2(t), b_2(t))^T \) eigenvectors of \( I^{-1} II \) and \( A_f \) at \( \gamma(t) \), respectively. We may suppose

\[
(a_1(t), b_1(t)) P(\gamma(t)) = (a_2(t), b_2(t)) \quad (0 \leq t \leq 2\pi).
\]

We set

\[
w_i(t) := a_i(t) \partial_x + b_i(t) \partial_y \quad (i = 1, 2).
\]

Then \( w_1 \) points in one of the principal directions of the graph of \( f \). The matrix \( P(\gamma(t)) \) takes values in the set

\[
\mathcal{T} := \left\{ \begin{pmatrix} 0 & x \\ -y & z \end{pmatrix} : x, y > 0, z \in \mathbb{R} \right\}.
\]

Since the set \( \mathcal{T} \) is null-homotopic, the mapping degree of \( w_1(t) \) with respect to the origin is equal to that of \( w_2(t) \). Since the degree of \( w_2(t) \) with respect to \( o \) coincides with \( \text{ind}_o(A_f) \), we get the second assertion. \( \square \)

By a straightforward calculation, one can get the following identity:

\[
\tilde{A}_f := h k^3 A_f = \begin{pmatrix} f_x f_y (f_x f_y f_{xx} - 2 h f_{xy}) + h^2 f_{yy} \\ l k & k^2 f_{xx} \end{pmatrix},
\]

where

\[
h := 1 + f_x^2, \quad k := \sqrt{1 + f_x^2 + f_y^2}, \quad l := -h f_{xy} + f_x f_y f_{xx}.
\]

Then the coefficients of the vector field

\[
v_{\tilde{A}_f} = v_1 \partial_x + v_2 \partial_y
\]
defined as in (3.3) for \( A = \tilde{A}_f \) are given by

\[
v_1 = \tilde{a}_{11} - \tilde{a}_{22} = (1 + f_x^2)f_y^2f_{xx} - hf_{xx} - 2h f_x f_y f_y + h^2 f_{yy},
\]

\[
v_2 = \tilde{a}_{12} = -k(hf_{xy} - f_x f_y f_{xx}),
\]

where \( \tilde{A}_f = (\tilde{a}_{ij})_{i,j=1,2} \). Hence, we get the following identity

\[
v_1 = \frac{2f_x f_y}{k} v_2 + k \left( -f_{xx}(1 + f_y^2) + (1 + f_x^2)f_{yy} \right).
\]

Consequently, we get the following fact (cf. Ghomi-Howard [GH, (10)]): 

**Fact 3.2.** The graph of the function \( z = f(x, y) \) defined on \( U \) has an umbilic at \( p \in U \) if and only if the functions

\[
d_1(x, y) := (1 + f_x^2)f_{xy} - f_x f_y f_{xx}, \quad d_2(x, y) := (1 + f_y^2)f_{yy} - f_x f_y f_{xx}
\]

both vanish at \( p \).

We consider the vector field

\[
D_f := d_1 \partial_x + d_2 \partial_y
\]

defined on the domain \( U \) in the \( xy \)-plane. Suppose that \( p \) is a zero of \( D_f \). The following assertion holds:

**Proposition 3.3.** If \( p \) gives an isolated umbilic of the graph of \( f \), then half of the index of the vector field \( D_f \) at \( p \) coincides with the index of the umbilic \( p \).

**Proof.** The half of the index of the vector field

\[
X := -v A_f = (2f_x f_y d_1 - hd_2) \partial_x + kd_1 \partial_y
\]

at \( p \) is equal to \( \text{ind}_p(\tilde{A}_f) \). We now set

\[
X_s := (\partial_x, \partial_y) \begin{pmatrix}
\frac{2sf_x f_y}{1 + s(f_x^2 + f_y^2)} & -1 - sf_y^2 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2
\end{pmatrix}
(0 \leq s \leq 1).
\]

Then \( X = X_1 \) and \( X_0 = -d_2 \partial_x + d_1 \partial_y \), and the rotation index of \( X_s \) at \( p \) does not depend on \( s \in [0, 1] \). Since the rotation index of \( D_f = (d_1, d_2) \) at \( p \) coincides with that of \( X_0 \), we can conclude that \( X \) has the same rotation index as \( D_f \) at \( p \). \( \square \)

We call \( d_1, d_2 \) the Cartesian umbilic identifiers of the function \( f \).

**Example 3.4.** For a function \( f(x, y) := \text{Re}(z^3) = x^3 - 3xy^2 \) \( (z = x + iy) \), the Cartesian umbilic identifiers are given by \( d_1 = -6y \varphi_1, \ d_2 = -6x \varphi_2 \), where

\[
\varphi_1 := -9x^4 + 9y^4 + 1, \quad \varphi_2 := 9x^4 + 18x^2 y^2 + 9y^4 + 2.
\]

Since \( \varphi_i \) \( (i = 1, 2) \) are positive at the origin \( (0, 0) \), the vector field \( D_f \) can be continuously deformed into the vector field \( -y \partial_x - x \partial_y \) preserving the property that the origin is an isolated zero. Thus \( D_f \) is of index \(-1\), and the graph of the function \( f \) has an isolated umbilic of index \(-1/2\) at the origin.
Example 3.5. Bates’ function $B(x, y)$ has no umbilics since $d_1 > 0$ on $R^2$. On the other hand, the identifier $d_1$ with respect to Ghomi-Howard’s function $f_{GH}(x, y)$ in (2.14) vanishes if and only if $y = 0$ or $x = -y^2$. Since $d_2$ never vanishes on these two sets, the graph of $f_{GH}$ also has no umbilics on $R^2$.

4. THE PAIR OF POLAR IDENTIFIERS FOR UMBILICS

Let $U$ be a domain in the $xy$-plane, and $f : U \rightarrow R$ a $C^\infty$-function. Let $(r, \theta)$ be the polar coordinate system associated to $(x, y)$ as in (2.5). Then

$$F(r, \theta) := (r \cos \theta, r \sin \theta, f(r \cos \theta, r \sin \theta))$$

gives a parametrization of the graph of $f$ with the unit normal vector

$$\nu := \frac{1}{\sqrt{f_\theta^2 + r^2 (1 + f_r^2)}} \left( f_\theta \sin \theta - rf_r \cos \theta, -rf_r \sin \theta - f_\theta \cos \theta, r \right).$$

Then $\hat{I} := \begin{pmatrix} 1 + f_r^2 & f_r f_\theta \\ f_r f_\theta & r^2 + f_\theta^2 \end{pmatrix}$ is the symmetric matrix consisting of the coefficients of the first fundamental form of $F$. If we set

$$Q = \begin{pmatrix} 0 & \sqrt{1 + f_r^2} \\ -\sqrt{f_\theta^2 + r^2 (1 + f_r^2)} / \sqrt{1 + f_r^2} & f_r f_\theta / \sqrt{1 + f_r^2} \end{pmatrix},$$

then $QQ^T = \hat{I}$. The symmetric matrix consisting of the coefficients of the second fundamental form is given by

$$\hat{H} := \frac{1}{\sqrt{f_\theta^2 + r^2 (1 + f_r^2)}} \begin{pmatrix} r f_{rr} & rf_{r\theta} - f_\theta \\ rf_{r\theta} - f_\theta & r (f_{\theta\theta} + rf_{rr}) \end{pmatrix}.$$

Then the symmetric matrix

$$B_f = Q^{-1} \hat{H} (Q^{-1})^T = Q^T (\hat{I}^{-1} \hat{H}) (Q^T)^{-1}$$

satisfies

$$\hat{B}_f = \hat{h} \hat{k}^3 B_f = \begin{pmatrix} r f_{rr}^2 f_\theta^2 f_{rr} + \hat{h} f_r & rf_{r\theta} + 2 f_\theta \hat{h} + r^2 \hat{h} f_{\theta\theta} + \hat{h} \hat{k} \\ \hat{k} & r^2 f_{rr} \end{pmatrix},$$

where

$$\hat{h} := 1 + f_r^2, \quad \hat{k} := \sqrt{f_\theta^2 + r^2 (1 + f_r^2)}, \quad \hat{l} := f_\theta \left( \hat{h} + r f_r f_{rr} \right) - rf_{r\theta}.$$

The following holds.

**Proposition 4.1.** The symmetric matrix $\hat{B}_f(p)$ is proportional to the identity matrix at $p \in U \setminus \{o\}$ if and only if $p$ gives an umbilic of the graph of $f$. Moreover, if $o$ is an isolated umbilic of the graph of $f$, then the index of the umbilic at $o$ is equal to $1 + \text{ind}_o(\hat{B}_f)$.

**Proof.** The first assertion follows from the above discussions. So we now prove the second assertion. Suppose $o$ is an isolated umbilic. We take a simple closed smooth curve $\gamma(t) (0 < t < 2\pi)$ in the $xy$-plane which surrounds the origin $o$ anticlockwise, and does not surround any other umbilics. Let $w_1 : [0, 2\pi] \rightarrow R^2$ be a
vector field along $\gamma$ such that $w_1(t)$ is an eigen-vector of the matrix $I^{-1}H$ at $\gamma(t)$ for each $t \in [0, 2\pi]$. Since
\[
\partial_r = \cos \theta \partial_x + \sin \theta \partial_y, \quad \partial_\theta = -r \sin \theta \partial_x + r \cos \theta \partial_y,
\]
we have that
\[
(\partial_r, \partial_\theta) = (\partial_x, \partial_y)T_0, \quad T_0 := \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}.
\]
Then, it holds that
\[
\hat{I}^{-1}H = (T_0)^{-1}(I^{-1}H)T_0.
\]
In particular,
\[
w_2(t) := T_0(\gamma(t))^{-1}w_1(t) \quad (0 \leq t \leq 2\pi)
\]
gives an eigen-vector of the matrix $\hat{I}^{-1}H$ at $\gamma(t)$. Let $T_s : U \to GL(2, \mathbb{R}) \quad (0 \leq s \leq 1)$ be a map defined by
\[
T_s := \begin{pmatrix} \cos \theta & -(r(1-s)+s) \sin \theta \\ \sin \theta & (r(1-s)+s) \cos \theta \end{pmatrix} \quad (0 \leq s \leq 1).
\]
Then it gives a continuous deformation of $T_0$ to the rotation matrix $T_1$. Since the winding number of the curve $\gamma(t)$ with respect to the origin $o$ is equal to 1, the difference between the rotation indices of $w_1$ and $w_2$ is equal to 1. Since the eigen-flow of the symmetric matrix $\hat{B}_f$ is associated with that of the matrix $\hat{I}^{-1}H$ by $Q$, the fact that $Q$ takes values in the set $\mathcal{T}$ in Section 3 yields that the index of the umbilic $o$ is equal to $1 + \text{ind}_o(\hat{B}_f)$. \hfill $\square$

We now set
\[
\delta_1 := \frac{\hat{b}_{12}}{k} = -f_\theta \left(1 + f_r^2 + rf_r f_{rr}\right) + r \left(1 + f_r^2\right) f_{r\theta},
\]
where $\hat{B}_f = (\hat{b}_{ij})_{i,j=1,2}$. Then we have
\[
\hat{b}_{11} - \hat{b}_{22} = -2 f_r f_\theta \delta_1 + r \left(1 + f_r^2\right) \delta_2,
\]
where
\[
\delta_2 := (1 + f_r^2) \left(1 + f_r f_{\theta\theta} - f_{rr} \left(1 + f_\theta^2\right)\right).
\]
Thus, as in the proof of Proposition 3.3 we get the following assertion.

**Proposition 4.2.** Let $U$ be a neighborhood of the origin $o := (0,0)$. Let $f : U \to \mathbb{R}$ be a $C^\infty$-function. Then the graph of $f$ has an umbilic at $p \in U \setminus \{o\}$ if and only if the two functions $\delta_1(r, \theta)$, $\delta_2(r, \theta)$ both vanish at $p$, where $x = r \cos \theta$ and $y = r \sin \theta$. Moreover, if $o$ is an isolated umbilic, then half of the index of the vector field
\[
\Delta_f := \delta_1 \partial_x + \delta_2 \partial_y
\]
at $o$ equals $-1 + I_f(o)$, where $I_f(o)$ is the index of the umbilic $o$.

We call $\delta_1, \delta_2$ the polar umbilic identifiers of the function $f$.  

Example 4.3. Consider the function \((z = x + iy)\)
\[
f(x, y) := \text{Re}(z^2) = x^3 + xy^2 = r^3 \cos \theta.
\]
By straightforward calculations, we have
\[
\delta_1 = -2r^3 \sin \theta, \quad \delta_2 = -2r^3 \left(2 - 3r^4 - 6r^4 \cos 2\theta \right) \cos \theta.
\]
Since \(2 - 3r^4 - 6r^4 \cos 2\theta \) is positive for sufficiently small \(r > 0\), the vector field \(\Delta_f\) can be continuously deformed into the vector field \(- \sin \theta \partial_r - \cos \theta \partial_\theta\) preserving the property that the origin is an isolated zero. Thus the rotation index of \(\Delta_f\) at \(o\) is equal to \(-1\), and \(I_f(o) = 1 - 1/2 = 1/2\).

We give a modification of Proposition 4.2 for the computation of index of the curvature line flow of a surface along an arbitrarily given simple closed curve surrounding the origin as follows. Let \(z = f(x, y)\) be a \(C^\infty\)-function defined on \(\mathbb{R}^2\) admitting only isolated umbilics. Suppose that \(\gamma : \mathbb{R} \to \mathbb{R}^3\) be a \(C^\infty\)-map satisfying \(\gamma(t + 2\pi) = \gamma(t)\) which gives a simple closed curve in the \(xy\)-plane such that it surrounds a bounded domain containing the origin \(o\) anti-clockwisely. Moreover, we assume that \(\gamma(t)\) does not pass through any points corresponding to umbilics of the graph of \(f\). We denote by \(I_f(\gamma)\) (resp. ind\(_r\)(\(\Delta_f\))) the rotation index of the curvature line flow (resp. of the vector field \(\Delta_f\)) along the simple closed curve \(\gamma\). Then the formula
\[
I_f(\gamma) = 1 + \frac{\text{ind}_r(\Delta_f)}{2}
\]
can be proved by modifying the proof of Proposition 4.2 Suppose that there exist at most finitely many points \(t = t_1, \ldots, t_k \in [0, 2\pi]\) such that \(\delta_1(\gamma(t))\) vanishes at \(t = t_j\). We now assume that \(\delta_1(\gamma(t)) := d\delta_1(\gamma(t))/dt\) does not vanish at \(t = t_j\) \((j = 1, \ldots, k)\). We set
\[
\varepsilon(t_j) = \begin{cases} 
0 & (\delta_2(\gamma(t_j)) < 0), \\
1 & (\delta'_1(\gamma(t_j)) > 0 \text{ and } \delta_2(\gamma(t_j)) > 0), \\
-1 & (\delta'_1(\gamma(t_j)) < 0 \text{ and } \delta_2(\gamma(t_j)) > 0).
\end{cases}
\]
Then, it holds that
\[
\text{ind}_r(\Delta_f) = - \sum_{j=1}^k \varepsilon(t_j).
\]

5. Proof of the main theorem

In this section, using the function \(f = f_m\) \((m = 1, 2, 3, \ldots)\) given in (1.2), we prove Theorem 1.1 and Corollary 1.2 in the introduction. More generally, we consider the function
\[
(g :=) g_m(r, \theta) := 1 + F(r^a \cos m\theta) \quad (0 < a < 1/4, \ m = 1, 2, 3, \ldots),
\]
which is defined on \(\{(r, \theta) : r > R\}\), where \(R\) is an arbitrarily fixed positive number, and \(F : \mathbb{R} \to \mathbb{R}\) is a bounded \(C^\infty\)-function satisfying the following conditions:

(i) \(F(x)\) is an odd function, that is, it satisfies \(F(-x) = -F(x)\),
Figure 1. The inversion of the graph $f_5$ for $a = 1/5$ (left) and its enlarged view (right). In these two figures, the $z$-axis points toward the downward direction.

(ii) the derivative $F'(x)$ of $F$ is a positive-valued bounded function on $\mathbb{R}$,
(iii) the second derivative $F''(x)$ is a bounded function on $\mathbb{R}$ such that $F''(x) < 0$ for $x > 0$,
(iv) there exist three constants $\alpha, \beta$ and $\gamma$ ($\beta \neq 0, \gamma > 0$) such that

$$\lim_{x \to \infty} e^{\gamma x} F'(x) = \alpha, \quad \lim_{x \to \infty} e^{\gamma x} F''(x) = \beta.$$ 

One can easily construct a bounded $C^\infty$-function $F(x)$ satisfying the properties (i-iv). For example, one can construct an odd $C^\infty$-function satisfying (ii) and (iii) so that

$$F(x) = 1 - e^{-x} \quad (x \in [M, \infty)).$$

Then it satisfies also (iv). However, to prove Theorem 1.1 we must choose the function $F(x)$ to be real-analytic, and

$$F(x) := \tanh x$$

satisfies all of the properties required. From now on, we shall prove Theorem 1.1 and Corollary 1.2 using only the above four properties of $F(x)$.

The function $g$ can be considered as a $C^\infty$-function on $\mathbb{R}^2 \setminus \Omega_R$ in the $xy$-plane for any $R > 0$. The graph of $g$ lies between two parallel planes orthogonal to the $z$-axis, and is symmetric under rotation by the angle $2\pi/m$ with respect to the $z$-axis (the entire figure of the inversion of the graph of $f_5$ is given in the left-hand side of
Figure 1. The partial derivatives of the function \( g \) are given by

\[
\begin{align*}
g_r &= ar^{a-1}c_m F'(r^a c_m), \\
g_{rr} &= ar^{a-2}c_m \left( ar^a c_m F''(r^a c_m) + (a - 1)F'(r^a c_m) \right), \\
g_{r\theta} &= -ar^{a-1}s_m \left( F''(r^a c_m) + F'(r^a c_m) \right), \\
g_{\theta\theta} &= m^2 r^a \left( r^a s_m F''(r^a c_m) - c_m F'(r^a c_m) \right),
\end{align*}
\]

where

\[
c_m := \cos m\theta, \quad s_m := \sin m\theta.
\]

Since \( F(x) \) is a bounded function, \( g \) is bounded and satisfies (2.13), since \( a < 2 \). Therefore, the inversion \( \Psi_g \) can be expressed as a graph near \((0, 0, 0)\). Since \( 0 < a < 1 \), the function \( g \) satisfies (a) and (b) of Proposition 2.5. Then \( Z = Z_f(X, Y) \) as in (2.11) with \( f := g \) is a \( C^1 \)-function at \((0, 0)\). The graph of \( Z_g \) for \( g = f_3 \) near \((0, 0, 0)\) is indicated in the right-hand side of Figure 1. To prove Theorem 1.1 it is sufficient to show that there exists a positive number \( R \) such that the graph of \( g \) has no umbilics if \( r > R \). We then compute the index \( I_g(\Gamma) \) with respect to the circle

\[
\Gamma(\theta) := (r \cos \theta, r \sin \theta) \quad (0 \leq \theta \leq 2\pi, \ r > R),
\]

using (4.1) and (4.2), which does not depend on the choice of \( r > R \), as follows. We set

\[
\delta_j(\theta) := \delta_j(\Gamma(\theta)) \quad (j = 1, 2).
\]

The first polar identifier is given by

\[
\delta_1 = -m r^a s_m \left( ar^a c_m F''(r^a c_m) + (a - 1)F'(r^a c_m) \right).
\]

Since \( 0 < a < 1 \), the condition (ii) yields that

\[
(a - 1)F'(r^a c_m) < 0.
\]

On the other hand, by (i) and (iii), it holds that

\[
x F''(x) \leq 0 \quad (x := r^a c_m).
\]

By (5.7) and (5.8), we can conclude that \( \delta_1(\theta) \) changes sign only at the zeros of the function \( \sin m\theta \). Since the function \( g \) is symmetric with respect to rotation by angle \( 2\pi/m \), to compute the rotation index of \( \Delta_g \) along \( \Gamma \), it is sufficient to check the sign changes of \( \delta_i(\theta) \) for \( \theta = 0 \) and \( \theta = \pi/m \). By (5.6), (5.7) and (5.8), we get the following:

\[
\frac{d\delta_1}{d\theta} \bigg|_{\theta=0} > 0, \quad \frac{d\delta_1}{d\theta} \bigg|_{\theta=\pi/m} < 0.
\]
Theorem 1.1. the index of the curvature line flow along symmetry of \( g \).

If we choose (resp. an odd function), we have

\[ r^2 \delta_2 = r^2 a^2 F''(r^2) \]

where

\[ F := \frac{Z}{1 + Z_X^2 + Z_Y^2} \]

where \( Z := Z_g \) is the function given in (5.11). Suppose that \( \lambda \) and \( \lambda \nu \) are a \( C^1 \)-function and a \( C^1 \)-vector field defined on a sufficiently small neighborhood of \((X, Y) = (0, 0)\), respectively, where \( \nu \) is a unit normal vector field of the graph of \( Z_g \). Then the map

\[ \Phi : (X, Y) \mapsto (\xi(X, Y), \eta(X, Y)) \]

given by (A.4) for \( \bar{f} = Z_{f_m} \) is a local \( C^1 \)-diffeomorphism, and is real-analytic on \( U \setminus \{(0,0)\} \). Then the proof of Fact A.1 in the appendix is valid in our situation, and we can conclude that the eigen-flow of the Hessian matrix of \( \lambda(\xi, \eta) \) is equal to the curvature line flow of the map \( P(\xi, \eta) \) given by (A.8). Since the image of \( P(\xi, \eta) \) coincides with that of \( \Psi_{f_m}(u, v) \), we get the proof of the corollary in the introduction.

Thus, it is sufficient to show that \( \lambda \) and \( \lambda \nu \) are \( C^1 \) at \((X, Y) = (0, 0)\). By (5.11), we have the following expression

\[ \lambda \nu = \frac{ZZ_X Z_Z_Y - Z}{1 + \sqrt{1 + Z_X^2 + Z_Y^2}} \]
By (5.11) and (5.12), we can conclude that $\lambda(X, Y)$ and $\lambda(X, Y) v(X, Y)$ are $C^1$ at $(0, 0)$ if

$$
\lim_{(X, Y) \to (0, 0)} ZZ_{XX} = \lim_{(X, Y) \to (0, 0)} ZZ_{XY} = \lim_{(X, Y) \to (0, 0)} ZZ_{YY} = 0
$$

hold. So to prove the corollary, it is sufficient to show (5.13). It can be easily seen that all of $r^{1-a} g_r, r^{-a} g_\theta, r^{2-2a} g_{rr}, r^{1-2a} g_{r\theta}$ and $r^{-2a} g_{\theta\theta}$ are bounded functions on $\mathbb{R}^2 \setminus \Omega_R$. Since $0 < a < 1/4$, Proposition 2.7 yields that the map $(u, v) \mapsto (X, Y) = \Pi \circ \Psi_g(u, v)$ is a $C^\infty$-map. Then (5.13) is equivalent to

$$
\lim_{(u, v) \to (0, 0)} ZZ_{uu} = \lim_{(u, v) \to (0, 0)} ZZ_{uv} = \lim_{(u, v) \to (0, 0)} ZZ_{vv} = 0.
$$

Since $Z = h/(k + 1)$, (5.14) follows from (2.19), (2.21) and the fact that

$$
\lim_{\rho \to 0} \rho h_{uu} = \lim_{\rho \to 0} \rho h_{uv} = \lim_{\rho \to 0} \rho h_{vv} = 0.
$$

6. AN ALTERNATIVE PROOF OF THE MAIN THEOREM

In the previous section, we have proved Corollary 1.2. However, it is natural to expect that one can give an explicit description of the function with the desired properties. The function $\lambda$ given in (5.11) does not have a simple expression. On the other hand, we will see that functions

$$
(\Lambda =) \Lambda_m := r^2 \tanh(r^{-a} \cos m \theta) \quad (m = 1, 2, 3, \ldots)
$$

satisfy (1) and (2) of Corollary 1.2 if $0 < a < 1$. We set

$$
(\lambda =) \lambda_m := r^2 F(r^{-a} \cos m \theta),
$$

where $\xi = r \cos \theta$, $\eta = r \sin \theta$, and $F : \mathbb{R} \to \mathbb{R}$ is a function satisfying the properties (i–iv) given in the beginning of Section 5. Then $\Lambda_m$ is a special case of $\lambda_m$ for $F(x) := \tanh x$. It holds that

$$
\lambda_r = r \left(2F(r^{-a} c_m) - ac_m r^{-a} F'(r^{-a} c_m)\right), \quad \lambda_\theta := -mr^{2-a} s_m F''(r^{-a} c_m),
$$

$$
\lambda_{rr} = 2F(r^{-a} c_m) + ar^{-2a} c_m \left((a-3) r^a F'(r^{-a} c_m) + ac_m F''(r^{-a} c_m)\right),
$$

$$
\lambda_{r\theta} = ms_m r^{-2a} \left((a-2) r^a F'(r^{-a} c_m) + ac_m F''(r^{-a} c_m)\right),
$$

$$
\lambda_{\theta\theta} = -m^2 r^{2-2a} \left(r^a c_m F''(r^{-a} c_m) - s_m^2 F''(r^{-a} c_m)\right),
$$

where $c_m$ and $s_m$ are defined in (5.3). We set

$$
\zeta_1 := 2(r \lambda_{r\theta} - \lambda_\theta), \quad \zeta_2 := -r^2 \lambda_{rr} + r \lambda_r + \lambda_{\theta\theta}.
$$

Then each component of the vector field $\delta_\lambda := \zeta_1 \partial_x + \zeta_2 \partial_y$ is an identifier for the eigen-flow of the Hessian matrix of $\lambda$ at the origin given in the introduction (cf. (1.2)).
By a direct calculation, we have
\[
\zeta_1 = 2mr^2 - 2a s_m F''(r - a c_m) + (a - 1)r a F'(r - a c_m),
\]
\[
\zeta_2 = -r^2(2a^2 - m^2 s_m) F''(r - a c_m) - \left(a^2 - 2a + m^2 \right) r^2 s_m F'(r - a c_m).
\]
By the property (ii) of \( F \), \((a - 1)r a F'(r - a c_m) \) is negative, and by (ii) and (iii), \( c_m F''(r - a c_m) \) is also negative. So \( \zeta_1 \) is positively proportional to \(-s_m (= - \sin m \theta)\).

In particular, \( \zeta_1 \) vanishes only when \( s_m = 0 \). Moreover, for fixed \( r \), it holds that \( d\zeta_1/d\theta < 0 \) (resp. \( d\zeta_1/d\theta > 0 \)) if \( c_m = 1 \) (resp. \( c_m = -1 \)).

On the other hand, if \( s_m = 0 \) and \( r \) tends to zero, then \( c_m = \pm 1 \) and \( F''(\pm r - a) \) and \( F''(\pm r - a) \) tend to zero with exponential order (cf. the condition (iv) for \( F(x) \)). Therefore, the leading term of \( \zeta_2 \) for small \( r \) is \(-r^2(2a^2 - m^2 s_m) F''(r - a c_m)\). Hence, for a fixed sufficiently small \( r \), the function \( \zeta_2 \) is positive (resp. negative) if \( c_m = 1 \) (resp. \( c_m = -1 \)). Summarizing these facts, one can easily show that the index of the vector field \( \delta_\lambda \) at \( o := (0, 0) \) is equal to \( m \). So the index of the eigen-flow of the Hessian matrix of \( \Lambda \) at \( o \) is equal to \( 1 + m/2 \) (cf. Appendix B). One can easily check that \( \lambda \) is a \( C^1 \)-function at \( o \) and the function \( \lambda \) satisfies (1) and (2) of Corollary 1.2. Since \( \Lambda \) is a special case of \( \lambda \), we proved that \( \Lambda \) satisfies the desired properties.

**Figure 2.** The image of \( P (r \leq 1/2) \) for \( m = 2 \) and \( a = 1/2 \).

To give an alternative proof of Theorem 1.1, we consider the real analytic map \( P : \mathbb{R}^2 \setminus \{o\} \to \mathbb{R} \) defined by (cf. (6.8))
\[
P(\xi, \eta) := (\xi, \eta, \Lambda(\xi, \eta)) - \Lambda(\xi, \eta) \nu(\xi, \eta),
\]
where
\[
\nu := \frac{1}{\Lambda_\xi^2 + \Lambda_\eta^2 + 1} (2\Lambda_\xi, 2\Lambda_\eta, \Lambda_\xi^2 + \Lambda_\eta^2 - 1).
\]
One can easily verify that
\[
\Lambda_\xi = r^{1 - a} \left( m s_m c_m - ac_1 c_m \right) \mathrm{sech}^2 \left( r^{-a} c_m \right) + 2r^a c_1 \tanh \left( r^{-a} c_m \right),
\]
\[
\Lambda_\eta = r^{1 - a} \left( 2r^a c_1 \tanh \left( r^{-a} c_m \right) - (as_1 c_m + mc_1 s_m) \mathrm{sech}^2 \left( r^{-a} c_m \right) \right),
\]
where \( c_1 = \cos \theta \) and \( s_1 = \sin \theta \). Using them, one can get the following expressions

\[
\Lambda_{\xi \xi} = \frac{1}{r^{2a}} h_1(r, \theta), \quad \Lambda_{\xi \eta} = \frac{1}{r^{2a}} h_2(r, \theta), \quad \Lambda_{\eta \eta} = \frac{1}{r^{2a}} h_3(r, \theta),
\]

where \( h_i(r, \theta) (i = 1, 2, 3) \) are continuous functions defined on \( \mathbb{R}^2 \). Using (6.2), (6.3) and the fact \( \lim_{r \to 0} \Lambda / r^{2a} = 0 \), we have

\[
\lim_{r \to 0} \Lambda_{\nu \xi} = \lim_{r \to 0} \Lambda_{\nu \eta} = 0,
\]

and also

\[
\lim_{r \to 0} \Lambda_{\nu \eta} = 0.
\]

Using (6.4), (6.5) and the fact

\[
d(\Lambda_{\nu}) = (d\Lambda)_{\nu} + \Lambda_{\nu} d\nu,
\]

we can conclude that \( \Lambda_{\nu} \) can be extended as a \( C^1 \)-function at \( o \) and \( \Phi : (\xi, \eta) \mapsto (X(\xi, \eta), Y(\xi, \eta)) \) is a local \( C^1 \)-diffeomorphism, where \( P = (X, Y, Z) \). In particular,

\[
Z_{\Lambda} := Z(\Phi^{-1}(X, Y))
\]
gives a function defined on a neighborhood of \( (X, Y) = (0, 0) \). By Fact A.1 in the appendix, the index of the curvature line flow at \( (0, 0) \) of the graph of \( Z_{\Lambda} \) is equal to the index of the eigen-flow of the Hessian matrix of \( \Lambda \), which implies Theorem 1.1.

The image of \( P \) for \( m = 3 \) and \( a = 1/2 \) is given in Figure 2.

7. The Duality of Indices

At the end of this paper, we consider the index at infinity for eigen-flows of Hessian matrices. Let

\[
f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R}, \quad g : U_{1/R} \setminus \{o\} \to \mathbb{R}
\]

be \( C^2 \)-functions, where \( \Omega_R \) and \( U_{1/R} \) are disks defined in Section 2. Let \( \mathcal{H}_f \) (resp. \( \mathcal{H}_g \)) be the eigen-flow of the Hessian matrix of \( f \) (resp. \( g \)). If the Hessian matrix of \( f \) has no equi-diagonal points, then we can consider the index \( \text{ind}(\mathcal{H}_f, \Gamma) \) with respect to the circle \( \Gamma \) given in (5.4) and it is independent of the choice of \( r > R \). So we denote it by \( \text{ind}_\infty(\mathcal{H}_f) \). Similarly, if the Hessian matrix of \( g \) has no equi-diagonal points, then we can consider the index \( \text{ind}(\mathcal{H}_g, \Gamma') \) with respect to the circle \( \Gamma'(\theta) := (\rho \cos \theta, \rho \sin \theta) \) \( (0 \leq \theta \leq 2\pi, \rho < 1/R) \). Since it is independent of the choice of \( \rho < 1/R \), we denote it by \( \text{ind}_\infty(\mathcal{H}_g) \). Consider the plane-inversion

\[
\iota : \mathbb{R}^2 \ni (u, v) \mapsto \frac{1}{u^2 + v^2}(u, v) \in \mathbb{R}^2.
\]

Then the following assertion holds.
Proposition 7.1 (The duality of indices). Let \( f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R} \) be a \( C^2 \)-function whose Hessian matrix has no equi-diagonal points. Then the function \( g : \Omega_R \to \mathbb{R} \) defined by

\[
g(x, y) := (u^2 + v^2)f \circ \iota(u, v)
\]

(called the dual of \( f \)) satisfies

\[
\text{ind}_o(H_g) + \text{ind}_\infty(H_f) = 2.
\]

Proof. Using the identification of \( (u, v) \) and \( z = \frac{u + \bar{v}}{2} \), it holds that \( u = \frac{z + \bar{z}}{2} \) and \( v = \frac{z - \bar{z}}{2i} \). In particular, \( f \) can be considered as a function of variables \( z \) and \( \bar{z} \), and can be denoted by \( f = f(z, \bar{z}) \). Since \( \iota(z) = 1/\bar{z} \), we can write

\[
g(z, \bar{z}) := \frac{z\bar{z}f(1/\bar{z}, 1/z)}{z^3}.
\]

holds, where

\[
\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).
\]

Since \( \Gamma(\theta) = re^{i\theta} \), we have that

\[
g_{zz}(\Gamma(\theta)) = \frac{\bar{z}f_{\bar{z}\bar{z}}(\iota \circ \Gamma(\theta))}{r^2e^{4i\theta}}.
\]

Thus, it holds that

\[
\text{ind}_o(g_{zz}, \Gamma) = -4 + \text{ind}_o(f_{\bar{z}\bar{z}}, \iota \circ \Gamma).
\]

By (B.1), we have

\[
\text{ind}_o(g_{zz}, \Gamma) = -2 \text{ind}_o(H_g), \quad \text{ind}_o(f_{\bar{z}\bar{z}}, \iota \circ \Gamma) = - \text{ind}_o(f_{zz}, \iota \circ \Gamma) = 2 \text{ind}_\infty(H_f).
\]

Thus we get the assertion. \( \square \)

Applying Proposition 7.1 for the function \( g = \Lambda_m \) (cf. (6.1)), we get the following:

Corollary 7.2. For each \( m(\geq 1) \), there exists a \( C^1 \)-function \( f : \mathbb{R}^2 \setminus \Omega_R \to \mathbb{R} \) satisfying

1. \( f \) is real-analytic on \( \mathbb{R}^2 \setminus \Omega_R \),
2. the eigen-flow of the Hessian matrix of \( f \) has no singular points, and
3. the index at infinity of the eigen-flow of \( H_f \) is equal to \( 1 - m/2 \).

The function \( \Lambda_m \) used in the second proof of Theorem 1.1 coincides with the dual of the function \( f_m - 1 \) given in (1.5).
APPENDIX A. THE CLASSICAL REDUCTION

In this appendix we show the existence of a special coordinate system \((\xi, \eta)\) of the graph of a function \(f(x, y)\) which reduces the curvature line flow to the Hessian of a certain function, called Ribaucour’s parametrization (the third author learned this from Konrad Voss at the conference of Thessaloniki 1997). Although, the existence of such a coordinate system was classically known, and a proof is in the appendix of [5], the authors will give the proof here for the sake of convenience. We set \(P = (x, y, f(x, y))\), and suppose that \(f(0, 0) = f_x(0,0) = f_y(0,0) = 0\). Consider a sphere which is tangent to the graph of \(f\) at \(P\) and also tangent to the \(xy\)-plane at a point \(Q\). Then, it holds that

\[
Q + \lambda e_3 = P + \lambda \nu,
\]

where \(e_3 = (0, 0, 1)\) and \(\nu = (f_x, f_y, -1)/\sqrt{1 + f_x^2 + f_y^2}\). Taking the third component of \((A.1)\), we get

\[
\lambda = \frac{f \sqrt{1 + f_x^2 + f_y^2}}{1 + \sqrt{1 + f_x^2 + f_y^2}}
\]

In particular, \(\lambda(0,0) = 0\). Since \(f_x(0,0) = f_y(0,0) = 0\), we have that

\[
d\lambda(0,0) = df(0,0) = 0.
\]

Taking the exterior derivative of \((A.1)\), and using \((A.3)\) and \(\lambda(0,0) = 0\), we have \(dP(0,0) = dQ(0,0)\). So, if we set \(Q = (\xi(x, y), \eta(x, y), 0)\), then it holds that

\[
\begin{align*}
(\xi_x(0,0)dx + \xi_y(0,0)dy, \eta_x(0,0)dx + \eta_y(0,0)dy, 0) &= dQ \\
= dP &= (dx, dy, f_x(0,0)dx + f_y(0,0)dy) = (dx, dy, 0),
\end{align*}
\]

which implies that the Jacobi matrix of the map

\[
\Phi : (x, y) \mapsto (\xi(x, y), \eta(x, y))
\]

is the identity matrix at \((0,0)\). So we can take \((\xi, \eta)\) as a new local coordinate system. Differentiating \((A.1)\) by \(\xi\) and \(\eta\), we get the following two identities:

\[
Q_{\xi} + \lambda_{\xi} e_3 = P_{\xi} + \lambda\xi \nu + \lambda \nu_{\xi}, \quad Q_{\eta} + \lambda_{\eta} e_3 = P_{\eta} + \lambda\eta \nu + \lambda \nu_{\eta}.
\]

Taking the inner products of them and \(\nu\), these two equations yield

\[
Q_{\xi} \cdot \nu + \lambda_{\xi} \nu_3 = \lambda_{\xi}, \quad Q_{\eta} \cdot \nu + \lambda_{\eta} \nu_3 = \lambda_{\eta},
\]

where we set \(\nu = (\nu_1, \nu_2, \nu_3)\). Since \(Q = (\xi, \eta, 0)\), we have that \(Q_{\xi} = (1, 0, 0)\) and \(Q_{\eta} = (0, 1, 0)\). So it holds that \(Q_{\xi} \cdot \nu = \nu_1\) and \(Q_{\eta} \cdot \nu = \nu_2\). Substituting this into \((A.5)\), we have

\[
\lambda_{\xi} = \frac{\nu_1}{1 - \nu_3}, \quad \lambda_{\eta} = \frac{\nu_2}{1 - \nu_3}.
\]

This implies that \((\lambda_{\xi}, \lambda_{\eta})\) is the image of \(\nu\) via the stereographic projection, and we can write

\[
\nu = \frac{1}{1 + \lambda_{\xi}^2 + \lambda_{\eta}^2}(2\lambda_{\xi}, 2\lambda_{\eta}, \lambda_{\xi}^2 + \lambda_{\eta}^2 - 1).
\]
By (A.1), we have

\[(A.8) \quad P = (\xi, \eta, 0) - \lambda \nu + (0, 0, \lambda).\]

We prove the following

**Fact A.1.** The curvature line flow of the graph \( z = f(x, y) \) coincides with the eigen-flow of the Hessian of the function \( \lambda(\xi, \eta) \) given by (A.2).

**Proof.** Noticing (A.8), we set

\[
\Delta(\xi, \eta) := \det \begin{pmatrix} \nu \\ dP \\ d\nu \end{pmatrix} = \det \begin{pmatrix} \nu \\ d\xi, d\eta, d\lambda \end{pmatrix}.
\]

Then this gives a map \( \Delta(\xi, \eta) : T(\xi, \eta) \mathbb{R}^2 \to \mathbb{R} \) such that

\[
\Delta(\xi, \eta) \left( a \frac{\partial}{\partial \xi} + b \frac{\partial}{\partial \eta} \right) = \det \begin{pmatrix} \nu, aP_\xi(\xi, \eta) + bP_\eta(\xi, \eta), a\nu_\xi(\xi, \eta) + b\nu_\eta(\xi, \eta) \end{pmatrix} \in \mathbb{R}.
\]

It is well-known that \( \nu \in T(\xi, \eta) \mathbb{R}^2 \) points in a principal direction of \( P \) at \( (\xi, \eta) \) if and only if \( \Delta(\xi, \eta)(\nu) = 0 \). Since \( (\nu_1)^2 + (\nu_2)^2 + (\nu_3)^2 = 1 \), (A.6) yields that

\[
\lambda_\xi \nu_1 + \lambda_\eta \nu_2 = \frac{(\nu_1)^2 + (\nu_2)^2}{1 - (\nu_3)^2} = \frac{1 - (\nu_3)^2}{1 - (\nu_3)^2} = 1 + \nu_3,
\]

which implies \( \nu_3 = \lambda_\xi \nu_1 + \lambda_\eta \nu_2 - 1 \). We now set \( \mu = 2/(1 + \lambda_\xi^2 + \lambda_\eta^2) \). Differentiating (A.7), we have

\[
d\nu = \frac{d\mu}{\mu} \nu + \mu (d\lambda_\xi, d\lambda_\eta, \lambda_\xi d\lambda_\xi + \lambda_\eta d\lambda_\eta).
\]

The first term of the right hand-side of the above equation is proportional to \( \nu \) and does not affect the computation of \( \Delta(\xi, \eta) \). So we have that

\[
\Delta(\xi, \eta) = \mu \begin{vmatrix} \nu_1 & \nu_2 & \lambda_\xi \nu_1 + \lambda_\eta \nu_2 - 1 \\ d\xi & d\eta & \lambda_\xi d\xi + \lambda_\eta d\eta \\ d\lambda_\xi & d\lambda_\eta & \lambda_\xi d\lambda_\xi + \lambda_\eta d\lambda_\eta \end{vmatrix} = -\mu \begin{vmatrix} \nu_1 & \nu_2 & -1 \\ d\xi & d\eta & 0 \\ d\lambda_\xi & d\lambda_\eta & 0 \end{vmatrix} = \mu \begin{vmatrix} (\lambda_\xi \xi - \lambda_\eta \eta) d\xi d\eta - \lambda_\xi \eta d\xi^2 - \lambda_\xi \eta d\eta^2 \end{vmatrix}.
\]

Fact A.1 follows from this representation of \( \Delta(\xi, \eta) \). \qed

**APPENDIX B. INDICES OF EIGEN-FLOWS OF HESSIAN MATRICES**

Let \( g : \Omega_R \setminus \{o\} \to \mathbb{R} \) be a \( C^2 \)-function, where \( \Omega_R \) is the closed disk of radius \( R \) centered at the origin \( o := (0, 0) \) (cf. (2.1)). The Hessian matrix of \( g \) is given by

\[
H_g := \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}.
\]
We denote by $\mathcal{H}_g$ the eigen-flow of $H_g$. A point $p \in \Omega_R \setminus \{o\}$ is called an equi-diagonal point of $\mathcal{H}_g$ if $H_g(p)$ is proportional to the identity matrix. Consider the circle

$$\Gamma(\theta) := r(\cos \theta, \sin \theta) \quad (0 \leq \theta < 2\pi, r < R).$$

If there are no equi-diagonal points on $\Omega_R \setminus \{o\}$, then we can define the index $\text{ind}(\mathcal{H}_g, \Gamma)$ of the eigen-flow $\mathcal{H}_g$ with respect to $\Gamma$, which does not depend on the choice of $r$. We call it the index of $\mathcal{H}_g$ at the origin and denote it by $\text{ind}_o(\mathcal{H}_g)$.

Consider the vector field
d\!g := 2g_{xy} \frac{\partial}{\partial x} + (g_{yy} - g_{xx}) \frac{\partial}{\partial y}.

It is well-known that the mapping degree of the Gauss map (cf. (3.1))

$$d_g : T^1 := \mathbb{R}/2\pi \mathbb{Z} \ni \theta \mapsto \frac{d_g(\Gamma(\theta))}{|d_g(\Gamma(\theta))|} \in S^1 := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$$

is equal to $2 \text{ind}_o(\mathcal{H}_g)$. Using the correspondence $(x, y) \mapsto x + iy$, we identify $\mathbb{R}^2$ with $\mathbb{C}$, where $i = \sqrt{-1}$. Then

$$g_z = \frac{1}{2}(g_x - ig_y), \quad g_{zz} = \frac{1}{4}((g_{xx} - g_{yy}) - 2ig_{xy}),$$

where $g_z := \partial g/\partial z, g_{zz} := \partial^2 g/\partial z^2$ and

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

Thus, $d_g$ can be identified with the right-angle rotation of $g_{zz}$. In particular, we have

(B.1) \quad \text{ind}_o(\mathcal{H}_g) = -\frac{1}{2} \text{ind}_o(g_{zz}).

Here $g_{zz}$ is considered as a vector field and $\text{ind}_o(g_{zz})$ is its index at the origin. Let $(r, \theta)$ be as in (2.3). Then $z = re^{i\theta}$ and

$$g_z = \frac{e^{-i\theta}}{2r}(rg_r - ig_\theta), \quad g_{zz} = \frac{e^{-2i\theta}}{4r^2} \left( (r^2 g_{rr} - rg_r - g_\theta) + 2i(g_\theta - rg_\theta) \right).$$

We consider the vector field defined by

(B.2) \quad \delta_g := 2(rg_r - g_\theta) \frac{\partial}{\partial x} + (-r^2 g_{rr} + rg_r - g_\theta) \frac{\partial}{\partial y}.

Since

$$\text{ind}_o(g_{zz}) = 2 + \text{ind}_o(\delta_g),$$

we obtain the following:

Lemma B.1. The identity $\text{ind}_o(\mathcal{H}_g) = 1 + \text{ind}_o(\delta_g)/2$ holds.

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