Robust Output Feedback Consensus for Networked Identical Nonlinear Negative-Imaginary Systems

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Abstract: A robust output feedback consensus problem for networked identical nonlinear negative-imaginary (NI) systems is investigated in this paper. Output consensus is achieved by applying identical linear output strictly negative-imaginary (OSNI) controllers to all the nonlinear NI plants in positive feedback through the network topology. First, we extend the definition of nonlinear NI systems from single-input single-output (SISO) systems to multiple-input multiple-output (MIMO) systems and also extend the definition of OSNI systems to nonlinear scenarios. Asymptotic stability is proved for the closed-loop interconnection of a nonlinear NI system and a nonlinear OSNI system under reasonable assumptions. Then, an NI property and an OSNI-like property are proved for networked identical nonlinear NI systems and networked identical linear OSNI systems, respectively. Output feedback consensus is proved for a network of identical nonlinear NI plants by investigating the stability of its closed-loop interconnection with a network of linear OSNI controllers. This closed-loop interconnection is proposed as a protocol to deal with the output consensus problem for networked identical nonlinear NI systems and is robust against uncertainty in the individual system’s model.

Keywords: nonlinear negative-imaginary systems, output strictly negative-imaginary systems, consensus, robust control.

1. INTRODUCTION

Negative-imaginary (NI) systems theory was introduced in Lanzon and Petersen (2008) and has rapidly attracted interest among control theory researchers (see Petersen and Lanzon (2010), Xiong et al. (2010), Bhilkaji et al. (2011), Patra and Lanzon (2011)). NI systems theory can be regarded as a complementary theory to positive-real (PR) systems theory. NI systems theory can deal with systems with relative degree from zero to two, while PR systems theory is only applicable to systems with relative degree zero or one. NI systems theory is typically applied to deal with systems with co-located position sensors and force actuators. NI system theory has achieved success in the control of flexible structures with highly-resonant dynamics (see Cai and Hagen (2010), Mabrok et al. (2014)); e.g., nano-positioning control (see Das et al. (2014b), Das et al. (2014a), Das et al. (2015)).

A square, real-rational, proper transfer function $G(s)$ is said to be NI if the following conditions are satisfied (Mabrok et al. (2014)): (1) $G(s)$ has no pole in $Re[s] > 0$; (2) $\forall \omega > 0$ such that $\omega^2$ is not a pole of $G(s)$, $\Im [G(j\omega) - G(j\omega^*)] \leq 0$; (3) If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, then it is a simple pole and the residue matrix $K = \lim_{s \to j\omega_0} (s - j\omega_0) G(s)$ is Hermitian and positive semi-definite; (4) If $s = 0$ is a pole of $G(s)$, then $\lim_{s \to 0} s^k G(s) = 0$ for all $k \geq 3$ and $\lim_{s \to 0} s^2 G(s)$ is Hermitian and positive semi-definite. This definition includes NI systems with free body motion. For a SISO NI system, its phase-lag is in the range $(0, \pi)$. Hence the Nyquist plot of the system’s frequency response $G(j\omega)$ is restricted to the lower half of the complex plane for all positive frequencies.

NI systems theory has recently been extended to nonlinear systems (see Gallab et al. (2018)). Generally speaking, a system is said to be nonlinear NI if there exists a positive definite storage function such that its time derivative is not greater than the product of the system’s input and
the time derivative of the system’s output. For a nonlinear system with input $u$, state $x$ and output $y$, where $y$ is only a function of $x$, the nonlinear NI property is equivalent to the dissipativity property with a supply rate $w(u, y) = u y$.

Ghallab et al. (2018) extends several existing results in linear NI systems theory to a class of nonlinear systems with the dissipativity property of NI systems. Asymptotic stability is proved for the closed-loop interconnection of a nonlinear NI system and a so-called weak strict nonlinear NI system under several assumptions.

A subclass of NI systems called output strictly negative-imaginary (OSNI) systems was motivated by the notion of output strictly passive (OSP) systems (see Bhowmick and Patra (2017) and Bhownick and Lanzon (2019)). A square real-rational, proper transfer function $M(s)$ is said to be OSNI if it is stable and there exists a scalar $\delta > 0$ such that

$$
\begin{align*}
j \omega [M(j \omega) - M(j \omega)^*] - \delta \omega^2 M(j \omega)^* M(j \omega) &\geq 0, \\
\forall \omega \in \mathbb{R} \cup \{\infty\} &\text{ where } M(j \omega) = M(j \omega) - M(\infty).
\end{align*}
$$

The index $\delta$ describes the level of output strictness. The OSNI property of a system with input $u$, output $y$ and a minimal realisation $(A, B, C, D)$ is equivalent to the dissipativity property with a supply rate of $w(u, y) = u \hat{y} - \delta \| \hat{y} \|^2$, where $\hat{y} = y - Du$.

A robust output feedback consensus problem for networked NI systems was investigated in Wang et al. (2015). Output feedback consensus is considered as the natural convergence of outputs to a common trajectory (not necessarily constant) by the subsystems themselves under the effect of a network connection. Identical strictly negative-imaginary (SNI) controllers are applied to provide positive feedback control to identical NI plants through a network topology. The output consensus problem is solved as an asymptotic stability problem of the networked system. The asymptotic stability of the interconnection of a single NI system and a single SNI system is guaranteed if the DC-gain of these two cascaded systems has its largest eigenvalue being less than unity. In contrast, the asymptotic stability of the networked control system under consideration is achieved when the cascaded DC-gain $P(0)P_r(0)$ of the NI plant and the SNI controller satisfies a condition involving the Laplacian matrix $\mathcal{L}_n$ corresponding to the network: $\lambda_{\max}(P(0)P_r(0)) < \frac{1}{\lambda_{\max}(\mathcal{L}_n)}$, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix.

This paper is motivated by the output feedback consensus problem of networked identical nonlinear NI systems. With the development of nonlinear NI systems theory, the output feedback consensus problem in Wang et al. (2015) can be investigated directly in nonlinear scenarios. This work differs from the previous work Ghallab et al. (2018), Bhownick and Lanzon (2019) and Wang et al. (2015) in the following aspects: (a) nonlinear NI systems are now defined for MIMO systems instead of SISO systems only; (b) Bhownick and Lanzon (2019) only considered linear OSNI systems while this work provides a definition for general nonlinear OSNI systems in terms of their dissipativity properties; (c) instead of a weak strict nonlinear NI system, a nonlinear OSNI system is now considered in closed-loop interconnection with a nonlinear NI system, and asymptotic stability is proved for the closed-loop interconnection; (d) Wang et al. (2015) provides a protocol for robust output feedback consensus of networked identical linear NI systems while this work provides a protocol to achieve robust output feedback consensus for networked identical nonlinear NI systems.

Notation: The notation in this paper is standard. $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively, and $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ denote the sets of real and complex matrices of dimension $m \times n$ respectively. $A^T$ and $A^*$ denote the transpose and the complex conjugate transpose of matrix $A$. $Re(s)$ and $Im(s)$ denote the real and imaginary parts of $s \in \mathbb{C}$ respectively. $\mathbb{R} \mathcal{h}_\infty^{m \times n}$ denotes the set of real-rational, stable transfer function matrices. $\mathfrak{M}$ denotes a constant vector or scalar. Given a matrix $A \in \mathbb{R}^{m \times n}$, $A > 0$ ($0 < A$) means $A$ is positive (negative) definite and $A \geq 0$ ($0 \leq A$) means $A$ is positive (negative) semi-definite. $N(A)$ is the null space of $A$. $I_n$ is the $n \times n$ identity matrix and $I_1$ is the $n \times 1$ vector with all elements being 1. $\text{diag}(a_1, a_2, \ldots, a_n)$ is a diagonal matrix with $a_1, a_2, \ldots, a_n$ on its diagonal. $A \otimes B$ denotes the Kronecker product of matrices $A$ and $B$.

Graph Theory Preliminaries: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{E} = \{e_1, e_2, \ldots, e_l\} \subseteq \mathcal{V} \times \mathcal{V}$ describes a graph with $n$ nodes and $l$ edges. The adjacency matrix denoted by $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is defined so that $a_{ii} = 0$, and $\forall i \neq j$, $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. $D = \text{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n \times n}$ denotes the degree matrix where $d_i = \sum_{j=1}^{n} a_{ij}$ denotes the degree of node $i$. The Laplacian matrix of a graph $\mathcal{G}$ is given by $\mathcal{L}_n = D - A$. A sequence of unreported edges in $\mathcal{E}$ that joins a sequence of nodes in $\mathcal{V}$ defines a path. An undirected graph is connected if there is a path between every pair of nodes.

2. PRELIMINARIES

Here, we recall the definitions of nonlinear negative-imaginary systems, output strictly negative-imaginary systems and a theorem that relates the output strictly negative-imaginary property to the dissipativity property. Consider the following general nonlinear system

$$
\begin{align*}
\dot{x} &= f(x, u); \\
y &= h(x)
\end{align*}
$$

where $f : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ is a Lipschitz continuous function and $h : \mathbb{R}^p \to \mathbb{R}$ is a class $C^1$ function.

Definition 1. (Ghallab et al. (2018)) The system described by (1) and (2) is said to be a nonlinear negative-imaginary system if there exists a positive definite storage function $V : \mathbb{R}^p \to \mathbb{R}$ of class $C^1$ such that

$$
\dot{V}(x(t)) \leq u(t) \dot{y}(t)
$$

for all $t \geq 0$.

Definition 2. (See also Bhownick and Lanzon (2019)) Let $M(s) \in \mathbb{R} \mathcal{h}_\infty^{m \times n}$. Then, $M(s)$ is said to be output strictly negative-imaginary (OSNI) if there exists a scalar $\delta > 0$ such that

$$
\begin{align*}
\text{Re} [M(j \omega) - M(j \omega)^*] - \delta \omega^2 M(j \omega)^* M(j \omega) &\geq 0, \\
\forall \omega \in \mathbb{R} \cup \{\infty\} &\text{ where } M(j \omega) = M(j \omega) - M(\infty).
\end{align*}
$$

The parameter $\delta > 0$ is an index which quantifies the level of output strictness of a given OSNI system.

Theorem 1. (Bhownick and Lanzon (2019)) Let $M(s)$ be a causal, square, LTI system described by the state-space equations $\dot{x} = Ax + Bu$, $x(0) = 0$ and $y = Cx + Du$, $\mathcal{L}_n$. \end{document}
where $A$ is Hurwitz, $D = D^T$ and $(A, B, C, D)$ is a minimal realisation. Let the associated transfer function matrix be $M(s)$ and define $\tilde{y} = y - Du$. Let $\delta > 0$ be a given scalar. Then, the following statements are equivalent:

i) $M(s)$ is OSNI with a level of output strictness $\delta$;

ii) There exists a real matrix $Y = Y^T > 0$ such that $AY + YA^T + 2\delta(CAY)^T(CAY) \leq 0$ and $B = -AYC^T$;

iii) The realisation $(A, B, C, D)$ is dissipative with respect to the supply rate $w(u, \tilde{y}) = u^T\tilde{y} - \delta|\tilde{y}|^2$.

3. AN INITIAL NI STABILITY RESULT

In this section, we show that under suitable assumptions, the closed-loop interconnection of a nonlinear NI system and a nonlinear OSNI system is asymptotically stable. First, we give a definition of nonlinear negative-imaginary systems for MIMO systems and a definition of nonlinear output strictly negative-imaginary systems.

3.1 Definitions of nonlinear MIMO NI systems and nonlinear OSNI systems

Consider the following general nonlinear system

\[ \dot{x}(t) = f(x(t), u(t)); \quad (3) \]

\[ y(t) = h(x(t)); \quad (4) \]

where $f : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a Lipschitz continuous function and $h : \mathbb{R}^p \to \mathbb{R}^m$ is a class $C^1$ function. Now we extend the nonlinear NI systems definition in Ghallab et al. (2018) from SISO systems to MIMO systems.

Definition 3. The system (3), (4) is said to be a nonlinear negative-imaginary system if there exists a positive definite storage function $V : \mathbb{R}^p \to \mathbb{R}$ of class $C^1$ such that

\[ \dot{V}(x(t)) \leq u(t)^T \dot{y}(t) \quad (5) \]

for all $t \geq 0$.

Let us give a definition for nonlinear OSNI systems based on the dissipativity property.

Definition 4. The system (3), (4) is said to be a nonlinear output strictly negative-imaginary system if there exists a positive definite storage function $V : \mathbb{R}^p \to \mathbb{R}$ of class $C^1$ and a constant $\delta > 0$ such that

\[ \dot{V}(x(t)) \leq u(t)^T \dot{y}(t) - \delta|\dot{y}(t)|^2 \quad (6) \]

for all $t \geq 0$. The quantity $\delta$ quantifies the level of output strictness of the nonlinear OSNI system.

Remark 1. According to Theorem 1, all linear OSNI systems with a minimal realisation in the form $\dot{x} = Ax + Bu; \quad y = Cx$ have a dissipativity property and hence also satisfy the nonlinear OSNI definition. Therefore, all the results in this paper relating to nonlinear OSNI systems are also true for linear OSNI systems with minimal realisations.

3.2 Asymptotic stability of the closed-loop interconnection of a nonlinear NI system and a nonlinear OSNI system

In this section, we use the dissipativity property to prove the asymptotic stability of the closed-loop interconnection of a nonlinear NI system and a nonlinear OSNI system, as shown in Fig. 1.
\[ W(x_1, x_2) = \dot{V}_1(x_1) + \dot{V}_2(x_2) - u_1^T \ddot{y}_1 - u_2^T \ddot{y}_2 \leq -\delta |\dot{y}_2|^2 \leq 0. \] (10)

This inequality shows that the system is Lyapunov stable. We will prove in the following that the equilibrium point at \((x_1, x_2) = (0, 0)\) is asymptotically stable.

According to (10), \(W(x_1, x_2) = 0\) can only hold when \(\dot{y}_2 = 0\). In other words, \(W(x_1, x_2)\) cannot remain at zero unless \(\dot{y}_2\) remains at zero. According to Assumptions I and II, \(\dot{y}_2(t) \equiv 0 \Rightarrow \dot{x}_j(t) \equiv 0 \Rightarrow \dot{u}_j(t) \equiv 0\). Due to the feedback interconnection of the systems \(H_1\) and \(H_2\), it follows that \(u_2(t) \equiv y_1(t)\). Hence, we obtain \(\ddot{y}_1(t) \equiv 0 \Rightarrow \dot{x}_1(t) \equiv 0 \Rightarrow \dot{u}_1(t) \equiv 0\). Therefore, the closed-loop system is in steady-state. We let the constant inputs, outputs and states of the systems \(H_1\) and \(H_2\) be denoted as \(\bar{u}_1, \bar{u}_2, \bar{y}_1, \bar{y}_2, \bar{x}_1\) and \(\bar{x}_2\), respectively. Considering the feedback interconnection, we must have \(\ddot{y}_2 = \ddot{u}_1\) and therefore inequality (9) implies

\[ \ddot{u}_1^T \ddot{u}_1 = |\ddot{u}_1|^2 \leq \gamma |\dot{u}_1|^2. \]

This can only hold when \(\dot{y}_2 = \ddot{u}_1 = 0\). According to Assumptions I and II, \(\dot{u}_2 \equiv 0 \Rightarrow \ddot{x}_2 \equiv 0 \Rightarrow \dot{u}_2 = 0 \Rightarrow \ddot{y}_1 \equiv 0 \Rightarrow \ddot{x}_1 = 0\). Therefore, \(W(x_1, x_2)\) will keep decreasing until \(x_1 = x_2 = 0\). According to LaSalle’s invariance principle, it follows that the equilibrium point of the closed-loop interconnection of \(H_1\) and \(H_2\) at \((x_1, x_2) = (0, 0)\) is asymptotically stable. This completes the proof.

4. STABILITY OF NETWORKED NI SYSTEMS

In this section, we extend Theorem 2 from the closed-loop interconnection of a nonlinear NI system and a nonlinear OSNI system to the closed-loop interconnection of networked nonlinear NI systems connected by linear OSNI systems. This system setting is also proposed as a protocol for the robust output feedback consensus of networked identical nonlinear NI systems. To extend Theorem 2 to networked systems, we first prove the nonlinear property for networked identical nonlinear NI systems, and an OSNI-like property for networked identical linear OSNI controllers. First, we investigate the OSNI property of networked identical linear OSNI controllers by decomposing the entire network into edge-linked pairs of nodes. The following subsection establishes the OSNI property of two connected linear OSNI systems.

4.1 OSNI property of two connected OSNI systems

Fig. 3. An undirected and connected graph consisting of two nodes.

**Lemma 3.** Let \(M(s)\) be a square transfer function matrix described by the state-space representation \(\dot{x} = Ax + Bu\), \(y = Cx\), where \(u \in \mathbb{R}^m\), \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}^p\); \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), \(C \in \mathbb{R}^{p \times n}\), and \((A, B, C)\) is a minimal realisation of \(M(s)\). Consider two identical systems with transfer function \(M(s)\) simply connected by the graph described by the Laplacian matrix \(L_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\), as shown in Fig. 3.

Suppose \(M(s)\) is an OSNI transfer function matrix with a level of output strictness \(\delta > 0\). Then the networked system with transfer function matrix \(L_2 \otimes M(s)\) is also OSNI with a level of output strictness \(\frac{\delta}{2}\).

**Proof.** According to Theorem 1, the realisation \((A, B, C)\) is dissipative with respect to the supply rate \(\omega(u, y) = u^T \ddot{y} - \delta |\dot{y}|^2\). Hence, there exists a positive storage function \(\dot{V}_2(x)\) that satisfies \(\dot{V}_2(x) \leq u^T \ddot{y} - \delta |\dot{y}|^2\). We show in the following that there is also a similar dissipativity property for system corresponding to the transfer function matrix \(L_2 \otimes M(s)\).

Let the states, inputs and outputs of the two identical systems with transfer function matrix \(M(s)\) be denoted as \(x_1, x_2; u_1, u_2\) and \(y_1, y_2\), respectively. Then the following state equations will be satisfied:

System \(i\) \((i = 1, 2)\):
\[ \begin{align*}
\dot{x}_i &= Ax_i + Bu_i; \\
y_i &= Cx_i,
\end{align*} \]

where \((A, B, C)\) defines a minimal realisation for \(M(s)\). Let us define the following quantities: \(\Delta x_{ij} = x_i - x_j\), \(\Delta u_{ij} = u_i - u_j\), \(\Delta \dot{y}_{ij} = \ddot{y}_i - \ddot{y}_j\). For the networked system with transfer function matrix \(L_2 \otimes M(s)\), we have

\[ L_2 \otimes M(s) = \begin{bmatrix} M(s) & -M(s) \\ -M(s) & M(s) \end{bmatrix}. \]

Also, we define the states of the networked system as \(\ddot{x} = \begin{bmatrix} \Delta x_{12} \\ \Delta x_{21} \end{bmatrix}\). Hence, a state-space realisation of the transfer function matrix \(L_2 \otimes M(s)\) can be written as follows.

\[ \begin{align*}
\dot{x} &= \begin{bmatrix} \Delta x_{12} \\ \Delta x_{21} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \Delta x_{12} \\ \Delta x_{21} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \Delta u_{12} \\ \Delta u_{21} \end{bmatrix}; \\
\ddot{y} &= \begin{bmatrix} \Delta \dot{y}_{12} \\ \Delta \dot{y}_{21} \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \Delta x_{12} \\ \Delta x_{21} \end{bmatrix}.
\end{align*} \] (11) (12)

Here, the input \(\ddot{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\) and the output \(\ddot{y} = \begin{bmatrix} y_1 - y_2 \\ y_2 - y_1 \end{bmatrix}\). By introducing this transfer function matrix, we arrange the system’s state equations as two independent state equations. The first row of the state equation (11) can be considered as a new system together with the first row of the output equation (12). This system has the same state equations as Systems 1 and 2, which is a minimal realisation of \(M(s)\). The state of the second row in the state equations (11), (12) is unobservable. Since we apply \(M(s)\) later as a control law to eliminate the difference between the outputs of edge-linked plants, we are only interested in the difference between the two controllers. Therefore, we split off the first row of the state-space model (11), (12) as a new system, which has the state-space model

\[ \begin{align*}
\Delta \dot{x}_{12} &= A \Delta x_{12} + B \Delta u_{12}; \\
\Delta \dot{y}_{12} &= C \Delta x_{12}.
\end{align*} \] (13) (14)

This system describes the difference of the two networked subsystems and is an OSNI system. However, the input
and output of this system are modified versions of the inputs and outputs of the original system with transfer function matrix $L_2 \otimes M(s)$. The OSNI property of transfer function matrix $L_2 \otimes M(s)$ is established in the following by using the equivalence of the OSNI property and the corresponding dissipativity property.

We take $V_2(\Delta x_{12})$ as the storage function for system (13), (14) with transfer function matrix $L_2 \otimes M(s)$. By the virtue of the transfer function matrix $M(s)$, for the system (13), (14), we have

$$V_2(\Delta x_{12}) \leq \Delta u_{12}^T \Delta y_{12} - \frac{1}{2} |\delta| \Delta y_{12}^2.$$  (15)

Though the input $\tilde{u}$ and output $\tilde{y}$ of system with transfer function matrix $L_2 \otimes M(s)$ do not appear in (15), we can replace the terms $\Delta u_{12}^T \Delta y_{12}$ and $|\Delta y_{12}|^2$ with the terms $\tilde{u}^T \tilde{y}$ and $\frac{1}{2}|\tilde{y}|^2$, respectively, according to the following calculation:

$$\tilde{u}^T \tilde{y} = [u_1^T \ u_2^T] \begin{bmatrix} \dot{y}_1 - \dot{y}_2 \\ \dot{y}_2 - \dot{y}_1 \end{bmatrix} = u_1^T (\dot{y}_1 - \dot{y}_2) + u_2^T (\dot{y}_2 - \dot{y}_1)$$  (16)

$$= (u_1 - u_2)^T (\dot{y}_1 - \dot{y}_2) = \Delta u_{12}^T \Delta y_{12};$$

$$|\tilde{y}|^2 = [\dot{y}_1 - \dot{y}_2 \dot{y}_2 - \dot{y}_1]^T [\dot{y}_1 - \dot{y}_2] = 2|\dot{y}_1 - \dot{y}_2|^2 = 2|\Delta y_{12}|^2.$$  (17)

Substituting (16) and (17) into (15), we have

$$\dot{V}_2(\Delta x_{12}) \leq \tilde{u}^T \tilde{y} - \frac{1}{2} \delta |\tilde{y}|^2.$$  

Therefore, the system (13), (14) with transfer function matrix $L_2 \otimes M(s)$ is dissipative with a supply rate $\omega(u, \tilde{y}) = \tilde{u}^T \tilde{y} - \frac{1}{2} \delta |\tilde{y}|^2$, which implies $L_2 \otimes M(s)$ is OSNI with a level of output strictness $\frac{1}{2}$.

4.2 An OSNI-like property for a network of OSNI systems

It is shown above that a system consisting of two identical linear OSNI systems connected by an undirected and connected graph is a linear OSNI system. We show in the following that if an undirected and connected graph connects more than two identical linear OSNI systems, then the networked system also has an OSNI-like property.

We construct a storage function for an $n$-node networked system based on the storage functions of all pairs of edge-linked nodes. Consider $n$ identical systems with transfer function matrix $M(s)$ connected according to an undirected and connected graph $\mathcal{G} = (V, E)$. Let this networked system be defined by the transfer function matrix $M_n(s) = L_n \otimes M(s)$. We define the storage function for the system which is a state-space realization of $M_n(s)$ as the sum of storage functions for all pairs of edge-linked nodes:

$$\dot{V}_2 = \frac{1}{2} \sum_{(v_i, v_j) \in E} V_2(\Delta x_{ij}) = \frac{1}{2} \sum_{(i, j) \in A} a_{ij} V_2(\Delta x_{ij}),$$

where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the adjacency matrix of the graph $\mathcal{G}$. The factor of $\frac{1}{2}$ is included because the summation counts the storage function of each pair of linked nodes $i$ and $j$ twice. According to (15), the time derivative of this storage function is

$$\dot{V}_2 = \frac{1}{2} \sum_{(v_i, v_j) \in E} V_2(\Delta x_{ij}) \leq \frac{1}{2} \sum_{(v_i, v_j) \in E} (\Delta u_{ij}^T \Delta \dot{y}_{ij} - \delta |\Delta \dot{y}_{ij}|^2)$$  (18)

where $U_2$ and $Y_2$ are the input and output vectors of the system defined by $M_n(s)$, respectively. Inequality (18) implies the system corresponding to $M_n(s)$ satisfies the nonlinear NI definition. Moreover, there is an additional term $\sum_{(v_i, v_j) \in E} |\Delta \dot{y}_{ij}|^2$ which describes the output strictness of the system. Comparing this term to $|\dot{V}_2|^2$, which is expected to replace $\sum_{(v_i, v_j) \in E} |\Delta \dot{y}_{ij}|^2$ in (18) if $M_n(s)$ corresponds to a standard OSNI system, the term $\sum_{(v_i, v_j) \in E} |\Delta \dot{y}_{ij}|^2$ gives a better measurement of the differences between the subsystem outputs in the networked systems. The term $\sum_{(v_i, v_j) \in E} |\Delta \dot{y}_{ij}|^2$ only involves the output differences between pairs of nodes that are directly connected by an edge, while $|\dot{V}_2|^2$ also involves the output differences of indirectly connected nodes, which is unnecessary. We say the system defined by $M_n(s)$ is an OSNI-like system.

4.3 Robust output feedback consensus of networked identical nonlinear NI systems

Consider $n$ identical nonlinear systems $H_i$ described as in the state-space model (3), (4) which operate independently in parallel and each of them has its own input $u_i \in \mathbb{R}^m$ and output $y_i \in \mathbb{R}^m$, ($i = 1, 2, \ldots, n$), as shown in Fig. 4. We combine their inputs and outputs respectively as the vectors

$$U_1 = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^{nm \times 1}, \quad \text{and} \quad Y_1 = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^{nm \times 1}.$$  

We have the following lemma.

Lemma 4. If $H_1$ is a nonlinear NI system, then $H_1$ is also a nonlinear NI system.

Proof. Since $H_1$ is a nonlinear NI system, there must exists a positive definite storage function $V_1(x)$ such that

Fig. 4. System $H_1$: a nonlinear system consisting of $n$ independent and identical nonlinear systems $H_1$, with independent inputs and outputs combined as the input and output of the networked system $H_1$.  

We have the following lemma.

Lemma 4. If $H_1$ is a nonlinear NI system, then $H_1$ is also a nonlinear NI system.

Proof. Since $H_1$ is a nonlinear NI system, there must exists a positive definite storage function $V_1(x)$ such that
\[ \dot{V}(x) \leq u_1^T \dot{y}_i. \] For the system \( \mathcal{H}_1 \), we define its storage function as \( \dot{V}_1 = \sum_{i=1}^n V_i(x_i) > 0. \) Then

\[ \dot{V}_1 = \sum_{i=1}^n \dot{V}_i(x_i) \leq \sum_{i=1}^n u_1^T \dot{y}_i = U_1^T \dot{Y}_1, \] (19)

which implies the nonlinear NI inequality (5). Therefore, \( \mathcal{H}_1 \) is a nonlinear NI system.

For a networked system \( \mathcal{H}_1 \) consisting of \( n \) identical nonlinear NI subsystems \( \mathcal{H}_i \) with each subsystem described by the state-space model (3), (4), as shown in Fig. 4, output feedback consensus is defined as follows.

**Definition 5.** A distributed output feedback control law achieves output feedback consensus for a network of systems if \( |y_i(t) - y_j(t)| \to 0 \) as \( t \to +\infty, \forall i, j \in \{1, 2, \ldots, n\} \).

Consider a network consisting of \( n \) identical nonlinear NI systems \( \mathcal{H}_1 \) with identical linear OSNI controllers defined by a transfer function matrix \( M(s) \); e.g., see Fig. 7 in the example. The entire system can be regarded as a closed-loop interconnection of two networked systems \( \mathcal{H}_1 \) and \( \mathcal{M}_n(s) \), as shown in Fig. 5. According to Lemma 4, the system \( \mathcal{H}_1 \) is a nonlinear NI system. Also, the system defined by \( \mathcal{M}_n(s) \) has an OSNI-like property described in (18). The stability of the closed-loop interconnection shown in Fig. 5 is investigated in the sequel to prove the output feedback consensus of the networked nonlinear NI systems \( \mathcal{H}_1 \) in \( \mathcal{H}_1 \). First, we assume the following assumption is satisfied for the open-loop consensus of the systems \( \mathcal{H}_1 \) and \( \mathcal{M}_n(s) \) as shown in Fig. 6.

**Assumption IV:** Given any constant input \( U_1(t) \equiv \bar{U}_1 \) for the system \( \mathcal{H}_1 \) and given its corresponding output \( Y_1(t) \) (not necessarily constant) as input \( U_2(t) \) to the system defined by \( \mathcal{M}_n(s) \). If the corresponding output of the system defined by \( \mathcal{M}_n(s) \) is constant; i.e., \( Y_2(t) \equiv Y_2 \), then there exists a constant \( \gamma \in (0, 1) \) such that \( \bar{U}_1 \) and \( Y_2 \) satisfy

\[ U_1^T Y_2 \leq \gamma |\bar{U}_1|^2. \] (20)

Note that in most cases, a constant output \( Y_2 \) can only be achieved when the system is in steady-state; i.e., when \( U_2(t) \equiv \bar{U}_2 \) and \( Y_1(t) \equiv Y_1 \) are also constant. However, a special situation is also considered when a constant input \( \bar{U}_1 \) in the open-loop interconnection leads to a constant output \( Y_2 \), while the system is not in steady-state, and \( Y_1(t) \) and \( U_2(t) \) still oscillate.

**Theorem 5.** Given an undirected and connected graph \( \mathcal{G} \) that models the communication links for networked identical nonlinear NI systems \( \mathcal{H}_1 \), and given OSNI control law \( M(s) \), robust output feedback consensus is achieved via the protocol

\[ U_1 = [\mathcal{L}_n \odot M(s)]Y_1 \]

as shown in Fig. 5, if \( \mathcal{H}_1 \) and \( M(s) \) satisfy Assumptions I and II, \( \mathcal{H}_1 \) and \( \mathcal{M}_n(s) \) satisfy Assumption IV, and the storage function defined as

\[ W := \dot{V}_1 + \dot{V}_2 - Y_1^T Y_2 \]

is positive definite, where \( \dot{V}_1 \) and \( \dot{V}_2 \) are positive definite storage functions that satisfy the nonlinear NI property (19) for system \( \mathcal{H}_1 \) and the OSNI-like property (18) for the system defined by \( \mathcal{M}_n(s) \), respectively. Here \( Y_1 \) and \( Y_2 \) are outputs of the systems \( \mathcal{H}_1 \) and \( \mathcal{M}_n(s) \), respectively.

**Proof.** We apply Lyapunov’s direct method and take the time derivative of the storage function \( \dot{W} \). According to (19) and (18), we have

\[ \dot{W} = \dot{V}_1 + \dot{V}_2 - Y_1^T Y_2 = \dot{V}_1 + \dot{V}_2 - U_1^T \dot{Y}_1 - U_2^T \dot{Y}_2 \leq -\frac{1}{2} \sum_{(v, v_j) \in \mathcal{E}} |\Delta \dot{y}_{ij}|^2 \leq 0 \]

This establishes the Lyapunov stability of this system. We now prove that output feedback consensus is achieved.

The Lyapunov derivative \( \dot{W} \) can only be zero when \( \sum_{(v, v_j) \in \mathcal{E}} |\Delta \dot{y}_{ij}|^2 = 0 \). This is equivalent to \( \Delta \dot{y}_{ij}(t) = \dot{y}_i(t) - \dot{y}_j(t) = 0 \) for all \( (v, v_j) \in \mathcal{E} \). In other words, \( W(X_1, X_2) \) cannot remain at zero unless \( \dot{y}_i(t) \equiv \dot{y}_j(t) \) for all \( (v, v_j) \in \mathcal{E} \). This means that there are always constant differences between the controller outputs \( y_i(t) \) and \( y_j(t) \) for all \((v, v_j) \in \mathcal{E}\); i.e., \( \Delta \dot{y}_{ij}(t) = \Delta \dot{x}_{ij} \).

Recall the state-space model (13), (14) corresponds to a minimal realisation \((A, B, C)\) of \( M(s) \). According to Assumptions I and II, \( \Delta \dot{y}_{ij}(t) \equiv 0 \implies \Delta \dot{x}_{ij}(t) \equiv 0 \implies \Delta u_{ij}(t) \equiv 0 \). This implies constant differences between both the controllers’ states \( x_i(t), x_j(t) \) and the controllers’ inputs \( u_i(t), u_j(t) \), for all \((v, v_j) \in \mathcal{E}\); i.e., \( \Delta x_{ij}(t) \equiv \Delta x_{ij}, \Delta u_{ij}(t) \equiv \Delta u_{ij}\). The state-space model (13), (14) can be modified to represent the constant differences between any pair of edge-connected controllers \( i \) and \( j \); i.e., \((v, v_j) \in \mathcal{E}\) implies

\[ 0 = \Delta \dot{x}_{ij} = A \Delta x_{ij} + B \Delta u_{ij}; \]

\[ \Delta y_{ij} = C \Delta x_{ij}. \]

According to Definition 2, \( A \) is Hurwitz and we can write

\[ \Delta x_{ij} = -A^{-1}B \Delta u_{ij}, \]

\[ \Delta y_{ij} = C \Delta x_{ij} = -CA^{-1}B \Delta u_{ij} = M(0) \Delta u_{ij}. \]
The $i$-th $m \times 1$ vector in $Y_2$, which is also the distributed input to the $i$-th plant, can be expressed as

$$\begin{align*}
(Y_2)_i = (\dot{Y}_2)_i = \sum_{j=1}^{n} a_{ij} \Delta y_{ij} = \sum_{j=1}^{n} a_{ij} M(0) \Delta u_{ij} \\
= M(0) \sum_{j=1}^{n} a_{ij} \Delta u_{ij} = M(0)[(\mathcal{L}_n \otimes I_m)U_2],
\end{align*}$$

(21)

Therefore, the output $Y_2$ of the system defined by $\mathcal{M}_n(s)$ is

$$Y_2 = \hat{Y}_2 = [I_n \otimes M(0)][(\mathcal{L}_n \otimes I_m)U_2] = [I_n \otimes M(0)]U_2.$$  

(22)

We now consider the closed-loop setting $\hat{Y}_2 = \hat{U}_1$. Inequality (20) implies

$$\dot{V}_1 = \dot{V}_2 = |\hat{U}_1|^2 \leq \gamma |U_1|^2,$$

which can only hold when $\hat{Y}_2 = \hat{U}_1 = 0$. According to (21), $\hat{Y}_2 \equiv 0 \implies \Delta y_{ij} \equiv 0 \forall (v_i, v_j) \in \mathcal{E}$. This implies $\Delta x_{ij} \equiv 0 \forall (v_i, v_j) \in \mathcal{E}$ according to Assumptions I and II. $\Delta u_{ij} \equiv 0 \forall (v_i, v_j) \in \mathcal{E}$ means the inputs of the controllers of any two edge-linked plants always have the same value, which means the outputs of the corresponding plants always have the same trajectory. Since the graph $\mathcal{G}$ is connected, this implies all plants have the same output trajectory. Hence output consensus is achieved. Otherwise, $\dot{W}$ cannot remain at zero, and according to LaSalle’s invariance principle, $\dot{W}$ will keep decreasing until either (i). output consensus is achieved; (ii). the states of all plants $H_1$ converge to zero, which also implies output consensus.

Remark 2. The protocol in Theorem 5 is robust against uncertainty in the system model for the subsystems connected in the network. For any network of identical nonlinear NI systems regardless of their model, we can apply this protocol to find a control law that enables the networked systems to achieve output feedback consensus.

5. EXAMPLE

In this section, we apply the output feedback consensus protocol developed in Section IV-C to a network of pendulum systems.

Consider a simple networked system consisting of four identical pendulum systems connected by a graph $\mathcal{G}$ as shown in Fig. 7. The Laplacian matrix of the graph $\mathcal{G}$ is

$$\mathcal{L}_4 = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 2 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}.$$  

Fig. 7. An undirected and connected graph consisting of four nodes.

The pendulum has a state-space model

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} = \begin{bmatrix}
1 \\
-\kappa x_1 - mg \sin(x_1 + u_1) \\
0 \\
0
\end{bmatrix};$$

where $m = 1kg$ is the mass of the bob, $l = 0.5m$ is the length of the rod, $\kappa = 5N \cdot m/rad$ is the spring constant of a torsional spring installed in the pivot, and $g \approx 9.8m/s^2$ is the gravitational acceleration. The system states are $x_1$ the counterclockwise angular displacement from the vertically downward position and $x_2$ is the system angular velocity. The system’s input $u$ is an external torsional force to the counterclockwise direction, and $y$ is the system’s output. The pendulum plant is not passive systems hence the existing passivity-based consensus methods is not applicable. In contrast, the consensus result proposed in this paper is effective because the pendulum plant is a nonlinear NI system with the storage function $V_1(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}m_l^2x_2^2 + mg(l \cos x_1)$. We apply a networked OSNI controller defined by $M(s) = \frac{\kappa}{s + \kappa}$ according to Fig. 5. A minimal realisation of $M(s)$ is

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} = \begin{bmatrix}
-bx_3 + au_2 \\
0 \\
x_3 \\
x_4
\end{bmatrix};$$

Fig. 8. Robust output feedback consensus for networked pendulum systems.

It is demonstrated in this example that the protocol presented in this paper is an alternative approach to achieve output feedback consensus when passivity-based consensus approaches are not applicable.

6. CONCLUSION

The robust output feedback consensus problem is investigated in this paper for networked identical nonlinear NI systems. To obtain more generality, the definition of nonlinear NI systems is extended to MIMO systems and the definition of OSNI systems is extended to include nonlinear systems. The closed-loop interconnection of a
nonlinear NI and a nonlinear OSNI system is proved to be asymptotically stable. The nonlinear NI property and an OSNI-like property is proved for networked identical nonlinear NI systems and networked identical linear OSNI systems, respectively. These properties are then applied to analyse the stability of the closed-loop interconnection of networked nonlinear NI systems and networked linear OSNI systems, which proves robust output feedback consensus for networked identical nonlinear NI plants.

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