A POSTERIORI ERROR ESTIMATES FOR A FINITE VOLUME SCHEME APPLIED TO A NONLINEAR REACTION-DIFFUSION EQUATION IN POPULATION DYNAMICS

ANOUAR EL HARRAK*, HATIM TAYEQ AND AMAL BERGAM

MAE2D laboratory, University Abdelmalek Essaadi
Polydisciplinary Faculty of Larache
Road of Rabat, Larache, Morocco

ABSTRACT. This work gives a posteriori error estimates for a finite volume implicit scheme, applied to a two-time nonlinear reaction-diffusion problem in population dynamics, whose evolution processes occur at two different time scales, represented by a parameter $\epsilon > 0$ small enough. This work consists of building error indicators concerning time and space approximations and using them as a tool of adaptive mesh refinement in order to find approximate solutions to such models, in population dynamics, that are often hard to be handled analytically and also to be approximated numerically using the classical approach.

An application of the theoretical results is provided to emphasize the efficiency of our approach compared to the classical one for a spatial inter-specific model with constant diffusivity and population growth given by a logistic law in population dynamics.

1. Introduction. During the last years, spatial dynamics of ecosystems has become of great interest (see [9, 11, 13], among others). Thanks to the fast development of computers, ecological modeling gives rise to a new generation of complex models, including huge dynamical systems. Such models are so hard to handle analytically. In population dynamics, we often look for the asymptotic dynamics of those models to describe population distribution in the long term. Thus, we need to present a practical algorithm in terms of computational cost for the numerical resolution of such model without losing the accuracy of simulations. Unfortunately, if we use a fixed mesh step algorithm, we will either do many useless calculations or lose the quality of simulations. Therefore, the use of an adaptive algorithm is highly required for mesh stepping.

Many theoretical results and numerical experiments provided in lots of works suggest its viability and efficiency for numerical computation. Self-adaptive mesh algorithm is often based on a posteriori error estimation techniques, and has become an indispensable tool in large scientific scale. Several works have been presented for different discretization techniques of many problems (see [4, 5, 10]). In addition, adaptive mesh stepping is an important tool to find an approximate solution with

---

2020 Mathematics Subject Classification. Primary: 65M08, 35K60, 92D25; Secondary: 65N08, 65M15, 35K57.

Key words and phrases. A posteriori analysis, Two time scales, Reaction-diffusion equations, Finite volume method, Self-adaptive algorithm, Upper bound for the error.

* Corresponding author: anouarelharrak1@gmail.com.
less computational cost, by controlling locally the error using quantities, called local error indicators.

In this work, we give a posteriori error estimates for a finite volume implicit scheme, applied to a two-time nonlinear reaction-diffusion problem in population dynamics, whose evolution processes occur at two different time scales represented by a parameter \( \varepsilon > 0 \) small enough. This model governs the evolution of a population whose density \( u(x,t) \) at position \( x \) and at time \( t \) is subjected to linear diffusion process and nonlinear local demography (growth, mortality, etc.). Moreover, the diffusion takes place at a faster time scale than local growth; it is frequent and widely used in ecology. Summarizing, this work consists of building error indicators concerning Spatio-temporal approximations and using them as a tool of adaptive mesh refinement.

We introduce in Section 2 the problem and its finite volume discretization. In section 3, we present a posteriori analysis of the problem by introducing error indicators with respect to time and space approximations in the adaptive mesh strategy. Finally, we provide numerical results to illustrate the applicability of such indicators in a Self-adaptive algorithm, in Section 4.

2. Problem statement and finite volume discretization. We are interested in deriving a numerical technique for solving a nonlinear diffusion-reaction equation which models the dynamics of population whose migrations process occurs at a fast time scale and the demography takes place at a slow one. Here, the population is living in a spatial region \( \Omega \subset \mathbb{R}^p \) \((p \geq 1)\), where \( \Omega \) is a non-empty bounded, open and connected set with smooth boundary \( \partial \Omega \in C^k \), \( k \geq 1 \). The problem is to find \( u, u(x,t) \) is the population density, such that

\[
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \frac{1}{\varepsilon} \text{div}(D(x)\nabla u(x,t)) + f(x,u(x,t)), \quad x \in \Omega, \ t \in ]0,T_f], \\
\frac{\partial u}{\partial \nu}(x,t) &= 0, \quad x \in \partial \Omega, \ t \in ]0,T_f], \\
u(x,0) &= u_0(x), \quad x \in \Omega.
\end{aligned}
\]  

We imposed the Neumann boundary condition to isolate the spatial domain from the external environment.

Let’s state the following hypotheses:

(H1) \( D \in C^2(\overline{\Omega}) \), \( D(x) \geq D_{\text{min}} > 0 \).

(H2) The nonlinear reaction term \( f(x,u) \) satisfies : \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is continuous and there exists a real-valued continuous positive function \( g \) defined on \( \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \) such that \( \forall x \in \overline{\Omega} \) and \( \forall u, v \in \mathbb{R} \):

\[
|f(x,u) - f(x,v)| \leq g(x,u,v) |u - v|.
\]

(H3) \( u_0 \in C(\overline{\Omega}) \) and \( u_0 \geq 0 \).

Under Hypotheses (H1), (H2), and (H3), Problem (1) admits a unique classical solution (see [12]). The existence of a global bounded solution requires the following additional smoothness assumption on the reaction term.

(H4) The function \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) satisfies the following:

1. \( f(x,0) = 0, \forall x \in \overline{\Omega} \).
2. There exists a constant \( C_f > 0 \) such that \( \forall x \in \overline{\Omega} \) and \( \forall u \in \mathbb{R} \) with \( |u| \geq C_f \), we have \( f(x,u) \leq 0 \).
If the initial data \( u_0 \) is non-negative, Hypothesis (H4) is sufficient to yield globally bounded non-negative solutions.

We equip the space \( L^2(0, T_T; H^1(\Omega)) \) with the norm, for all \( t \in [0, T_T] \):

\[
[V](t) = \left( \frac{1}{\varepsilon} \|v(t)\|^2 + \int_0^t \|v(s)\|^2\,ds \right)^{1/2},
\]

where we introduced the notation

\[
\|v\|^2 = \frac{1}{\varepsilon} \|D^{1/2}v\|^2, \quad \forall v \in H^1(\Omega),
\]

and \( \|\cdot\| \) stands for the \( L^2 \) norm over \( \Omega \).

In the following, we describe the finite volume discretization method applied to Problem (1). This discretization method is well suited for numerical simulation of various types of conservation laws. Let \( \Omega \) be an open polygonal bounded subset of \( \mathbb{R}^p \), with \( p = 2 \) or \( p = 3 \), \( T_T > 0 \) fixed. The time discretization may be performed by introducing a partition of the interval \( [0, T_T] \) into subintervals \( [t_{n-1}, t_n], \) \( 1 \leq n \leq N \) such that \( 0 = t_0 < t_1 < \cdots < t_N = T_T \), with a variable time step \( \tau_n = t_n - t_{n-1} \). We denote by \(|\tau|\) the maximum of the \( \tau_n, 1 \leq n \leq N \).

In order to perform a self-adaptive technique for the finite volume scheme, for each \( n, \) \( 0 \leq n \leq N \). To simplify we take \( p = 2 \). Let \( \mathcal{T}_h^n \) be a quasi-uniform triangulation of \( \Omega \) with \( h = \max h_T > 0 \), where \( h_T \) is the diameter of the triangle \( T \in \mathcal{T}_h^n \). We derive each triangulation \( \mathcal{T}_h^n \) from \( \mathcal{T}_h^{n-1} \) by refining or derefining some regions of the domain \( \Omega \).

To describe the finite volume method for solving Problem (1), we shall introduce a dual partition \( \mathcal{V}_h^n \) based upon the partition \( \mathcal{T}_h^n \), whose elements are called control volumes. Note that several triangulations can be used for mesh adaptivity at the same time \( t_n \) and we denote \( \mathcal{T}_h^n \) only the last one.

We consider the following implicit time discretization of Problem (1):

\[
\frac{u^n - u^{n-1}}{\tau_n} - \frac{1}{\varepsilon} \text{div}(D(x)\nabla u^n) = f(x, u^n), \tag{2}
\]

where \( u^n := u(t_n) \).

We construct the control volumes in the same way as in [7]. Let \( x_T \) be the barycenter of \( T \in \mathcal{T}_h^n \). We connect \( x_T \) with line segments to the midpoints of the edges of \( T \), thus partitioning \( T \) into three quadrilaterals \( Q_{x_i}, x_i \in X_h(T) \), where \( X_h(T) \) are the vertices of \( T \). Then, with each vertex \( x_i \in X_h^+ = \bigcup_{T \in \mathcal{T}_h^n} X_h(T) \), we associate a control volume \( V_{x_i} \), formed by the union of the subregions \( Q_{x_i} \) sharing the vertex \( x_i \). Thus, we finally obtain a group of control volumes covering the domain \( \Omega \), which is called the dual partition \( \mathcal{V}_h^n \) of the triangulation \( \mathcal{T}_h^n \), \( \mathcal{V}_h^n = (V_i)_{i=1,\ldots,M_h}; V_i := V_{x_i} \). We denote the set of edges \( E \) of \( \mathcal{T}_h^n \) by \( E_h^n \). Also, We define the set \( \mathcal{K}_h^i \) of the triangles \( K \) having \( x_i \) as a vertex and \( \gamma \) as an edge, \( \gamma \subset \partial V_i \).

Let us define the spaces \( \mathcal{P}_i(\mathcal{T}_h^n) \) and \( \mathcal{P}_0(\mathcal{V}_h^n) \) by

\[
\mathcal{P}_i(\mathcal{T}_h^n) = \{ v_h \in C^0(\bar{\Omega}) : v_h|_T \in \mathcal{P}_i; \forall T \in \mathcal{T}_h^n \},
\]

and

\[
\mathcal{P}_0(\mathcal{V}_h^n) = \{ w_h \in L^2(\bar{\Omega}) : w_h|_{V_i} \in \mathcal{P}_0; \forall i = 1, \ldots, M_h \},
\]

where \( \mathcal{P}_i \) is the set of polynomial functions of degree \( \leq l \), and set \( \mathcal{V}_h^n = \mathcal{P}_i(\mathcal{T}_h^n) \).

We formulate the finite volume discretization for Problem (1) as follows. Given a vertex \( x_i \in X_h^n \), integrating (2) over the associated control volume \( V_i \), we get
\[
\int_{V_i} \frac{u^n(x) - u^{n-1}(x)}{\tau_n} dx - \frac{1}{\varepsilon} \int_{V_i} \text{div}(D(x)\nabla u^n(x)) dx = \int_{V_i} f(x, u^n(x)) dx.
\]

Then, applying Green Formula and taking into account Neumann boundary condition, we find
\[
\int_{V_i} u^n(x) - u^{n-1}(x) \frac{dx}{\tau_n} - \frac{1}{\varepsilon} \sum_{\gamma \subset \partial V_i} \int_{\gamma} D(x) \nabla u^n(x) \cdot n_\gamma ds = \int_{V_i} f(x, u^n(x)) dx, \tag{3}
\]
where \(n_\gamma\) denotes the unit outer-normal vector on \(\gamma\).

Now, to define a numerical flux function for (3), we set
\[
u^n_h = \sum_{i=1}^{M_n} u^n_i \psi^n_i,
\]
where \(u^n_i := u^n(x_i)\) and the family \((\psi^n_i)_{i=1,\ldots,M_n}\) is the set of basis functions of \(V^n_h\).

Thus, the numerical flux could be defined as
\[
\int_{\gamma} D \nabla u^n \cdot n_\gamma ds \approx \int_{\gamma} D_h \nabla u^n_h \cdot n_\gamma ds,
\]
where \(D_h\) is a piecewise polynomial approximation of \(D\) such that there exists a constant \(c(D)\) only depending on \(D\) satisfying
\[
\|D - D_h\|_{L^\infty(\Omega)} \leq c(D) h^{l+1}. \tag{4}
\]

Hence, one could write:
\[
\int_{V_i} \frac{u^n_h - u^{n-1}_h}{\tau_n} dx - \frac{1}{\varepsilon} \sum_{\gamma \subset \partial V_i \setminus \partial \Omega} \int_{\gamma} D_h \nabla u^n_h \cdot n_\gamma ds = \int_{V_i} f(x, u^n_h) dx.
\]

The term \(\int_{V_i} f(x, u^n) dx\) could be approximated in (3) by
\[
\int_{V_i} \mathcal{F}_h(u^n_h) dx,
\]
where we use the notation
\[
\mathcal{F}_h(u^n_h) := \sum_{i=1}^{M_n} f(x_i, u^n_h) \psi^n_i.
\]

The obtained discrete problem associated to (1) is: Find the family \((u^n_h)_{0 \leq n \leq N}\), \(u^n_h \in V^n_h\), \(n = 0, \ldots, N\), satisfying
\[
\begin{cases}
  u^0_h = \Pi_h u_0 \text{ in } \Omega, \\
  \int_{V_i} \frac{u^n_h - u^{n-1}_h}{\tau_n} dx - \frac{1}{\varepsilon} \sum_{\gamma \subset \partial V_i \setminus \partial \Omega} \int_{\gamma} D_h \nabla u^n_h \cdot n_\gamma ds = \int_{V_i} \mathcal{F}_h(u^n_h) dx \\
  \text{for } i = 1, 2, \ldots, M_n, \ n = 1, \ldots, N,
\end{cases} \tag{5}
\]
where \(\Pi_h u_0\) is the projection of \(u_0\) on \(V^n_h\).

A priori analysis of Scheme (5) could be done and developed in similar way as in [6]. Indeed, under Hypotheses (H1)-(H4) and additional one on time step length \(\tau_n\), Scheme (5) admits a unique positive bounded solution. Even more, it is \(L^\infty\)-stable.
Theorem 3.1. Let \( u_{h\tau}(t) \) defined on \([0, T]\) which is affine on each interval \([t_{n-1}, t_n], n = 1, ..., N\). The function \( u_{h\tau} \) is defined by, for \( n = 1, ..., N\),

\[
u_{h\tau}(t) = u_{h\tau}^{n-1} + \frac{t - t_{n-1}}{\tau_n} (u_{h\tau}^n - u_{h\tau}^{n-1}), \quad \forall t \in [t_{n-1}, t_n].
\]

3. A posteriori analysis. In this section, we are interested in presenting error indicators and studying their equivalence with the error, using the finite volume method, those indicators are built as a tool of adaptive mesh refinement. For this, we will prove an upper bound for the error, \( ||u - u_{h\tau}||(t) \), as a function of the Hilbertian sum of the indicators.

We set respectively the residual and inter-element jump of \( (u_h^n)_n \) as:

\[
R_h^n|Q := \left( \mathcal{F}_h(u_h^n) - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \frac{1}{\varepsilon} \text{div}(D_h \nabla u_h^n) \right),
\]

and

\[
r_h^n|E := |D_h \nabla u_h^n.n_E|E,
\]

where \([\cdot]|_E\) denotes the jump across the edge \( E \).

The global spatial error indicator is given as

\[
(\eta_R^n)^2 := (\eta_R^n)^2 + (\eta^n)^2,
\]

where \((\eta_R^n)^2\) and \((\eta^n)^2\) are the local spatial indicators with

\[
(\eta_R^n)^2 := \sum_{V \in \mathcal{T}_h} \sum_{Q \subset V} \alpha_S^2 ||R_h^n||_Q^2,
\]

and

\[
(\eta^n)^2 := \varepsilon^{-3/2} D_{min}^{-1/2} \sum_{E \in \mathcal{E}_h} \alpha_S ||r_h^n||_E^2,
\]

such that \( \alpha_S := h_s \varepsilon^{1/2} D_{min}^{-1/2} \) for \( S = K, E; K \subset Q \) is a triangle and \( E \) is an edge.

We set the temporal error indicator as

\[
\Theta_h^n := \left\{ \frac{\tau_n}{3} \left( \varepsilon^{-1} ||(D_h)^{1/2} \nabla (u_h^n - u_h^{n-1})||^2 + ||u_h^n - u_h^{n-1}||^2 \right) \right\}^{1/2}.
\]

The error related to data as

\[
\mu^n(t) := ||f(\cdot, u_{h\tau}) - \mathcal{F}_h(u_{h\tau})|| + \varepsilon D_{min})^{-1/2} ||(D_h - D) \nabla u_{h\tau}||.
\]

In the following theorem, we give an upper bound for the error.

**Theorem 3.1.** Let \( u \) be the solution to (1) and \( (u_h^n)_{n \geq 1} \) the solution to (5), then

\[
||u - u_{h\tau}||(t_n) \leq C^* \left\{ ||u^0 - \Pi_h u^0||^2 + \sum_{n=1}^{N} (\eta_R^n)^2 \tau_n + (\Theta_R^n)^2 \right. \\
+ \left. \int_{t_{n-1}}^{t_n} ||\mu^n(t)||^2 \right\}^{1/2},
\]

where \( C^* \) is a constant independent of \( h \).
Proof. We have, for all \( v \in H^1(\Omega) \)
\[
\int_\Omega \frac{\partial}{\partial t}(u - u_{h\tau})vdx + \frac{1}{\varepsilon} \int_\Omega D\nabla(u - u_{h\tau}) \cdot \nabla vdx \\
= \int_\Omega (f(x, u) - f(x, u_{h\tau}))vdx \\
+ \int_\Omega \mathcal{F}_h(u_h^n)vdx - \int_\Omega \frac{u_h^n - u_h^{n-1}}{\tau_n}vdx - \frac{1}{\varepsilon} \int_\Omega D_h\nabla u_h^n \cdot \nabla vdx \\
+ \frac{1}{\varepsilon} \int_\Omega D_h\nabla(u_h^n - u_{h\tau}) \cdot \nabla vdx + \int_\Omega (\mathcal{F}_h(u_{h\tau}) - \mathcal{F}_h(u_h^n))vdx \\
+ \frac{1}{\varepsilon} \int_\Omega (D_h - D)\nabla u_{h\tau} \cdot \nabla vdx + \int_\Omega (f(x, u_{h\tau}) - \mathcal{F}_h(u_{h\tau}))vdx.
\]

For every \( v \in H^1(\Omega) \), we set:
\[
R_1(v) := \int_\Omega \mathcal{F}_h(u_h^n)vdx - \int_\Omega \frac{u_h^n - u_h^{n-1}}{\tau_n}vdx - \frac{1}{\varepsilon} \int_\Omega D_h\nabla u_h^n \cdot \nabla vdx,
\]
\[
R_2(v) := \int_\Omega \frac{1}{\varepsilon} D_h\nabla(u_h^n - u_{h\tau}) \cdot \nabla vdx + \int_\Omega (\mathcal{F}_h(u_{h\tau}) - \mathcal{F}_h(u_h^n))vdx,
\]
\[
R_3(v) := \int_\Omega (f(x, u_{h\tau}) - \mathcal{F}_h(u_{h\tau}))vdx + \frac{1}{\varepsilon} \int_\Omega (D_h - D)\nabla u_{h\tau} \cdot \nabla vdx,
\]
and
\[
R_4(v) := \int_\Omega (f(x, u) - f(x, u_{h\tau}))vdx.
\]

Afterwards we will bound each term: \( R_1(v) \), \( R_2(v) \), \( R_3(v) \), and \( R_4(v) \).

**Bound of \( R_1 \):**

We consider \( I_m : L^2(\Omega) \to \mathcal{P}_0(\mathcal{V}_h^n) \) the piecewise constant interpolation operator defined as:
\[
I_m v := \frac{1}{|V|} \int_V v dx, \text{ for } V \in \mathcal{V}_h.
\]

First, \( R_1 \) could be given under the following form:
\[
R_1(v) = \sum_{V \in \mathcal{V}_h^n} \int_V \left( \mathcal{F}_h(u_h^n)v - \frac{u_h^n - u_h^{n-1}}{\tau_n}v - \frac{1}{\varepsilon} D_h\nabla u_h^n \cdot \nabla (v - I_m v) \right) dx.
\]

Using integration by parts, we get:
\[
R_1(v) = \sum_{V \in \mathcal{V}_h^n} \sum_{Q \subset V} \left\{ \int_Q \left( \mathcal{F}_h(u_h^n)v - \frac{u_h^n - u_h^{n-1}}{\tau_n}v + \frac{1}{\varepsilon} \text{div} (D_h\nabla u_h^n)(v - I_m v) \right) dx \\
- \frac{1}{\varepsilon} \int_{\partial Q} D_h\nabla u_h^n \cdot n (v - I_m v) ds \right\},
\]
then we have:

$$R_1(v) = \sum_{v \in V_h} \sum_{Q \subset V} \int_Q \left( F_h(u_h^n) - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \frac{1}{\varepsilon} \text{div}(D_h \nabla u_h^n) \right) (v - I_m v) dx + \frac{1}{\varepsilon} \sum_{E \in \mathcal{E}_h} \int_E [D_h \nabla u_h^n, n E] E (v - I_m v) ds,$$

with

$$\tilde{R}_1 := \sum_{v \in V_h} \left\{ \sum_{Q \subset V} \int_Q \left( F_h(u_h^n) - \frac{u_h^n - u_h^{n-1}}{\tau_n} \right) I_m v dx + \frac{1}{\varepsilon} \int_{\partial V \setminus \partial \Omega} D_h \nabla u_h^n, n I_m v ds \right\}.$$

In addition, using the numerical scheme (5) we get:

$$\tilde{R}_1 = \sum_{v \in V_h} \left\{ \int_V \left( F_h(u_h^n) - \frac{u_h^n - u_h^{n-1}}{\tau_n} \right) I_m v dx + \frac{1}{\varepsilon} \int_{\partial V \setminus \partial \Omega} D_h \nabla u_h^n, n I_m v ds \right\}$$

$$= \sum_{v \in V_h} \left\{ \int_V \left( F_h(u_h^n) - \frac{u_h^n - u_h^{n-1}}{\tau_n} \right) dx + \frac{1}{\varepsilon} \int_{\partial V \setminus \partial \Omega} D_h \nabla u_h^n, ns \right\} I_m v$$

$$= 0.$$

So that:

$$R_1(v) = \sum_{v \in V_h} \sum_{Q \subset V} \int_Q \left( F_h(u_h^n) - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \frac{1}{\varepsilon} \text{div}(D_h \nabla u_h^n) \right) (v - I_m v) dx + \frac{1}{\varepsilon} \sum_{E \in \mathcal{E}_h} \int_E [D_h \nabla u_h^n, n E] E (v - I_m v) ds.$$

Finally, we get:

$$R_1(v) = \sum_{v \in V_h} \sum_{Q \subset V} \int_Q R^n_h (v - I_m v) dx + \frac{1}{\varepsilon} \sum_{E \in \mathcal{E}_h} \int_E r^n_h (v - I_m v) ds,$$

where $R^n_h$, the residual term, is given by:

$$R^n_h|Q := \left( F_h(u_h^n) - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \frac{1}{\varepsilon} \text{div}(D_h \nabla u_h^n) \right),$$

and $r^n_h$, the jump term, is presented as follows:

$$r^n_h|E := \left[ D_h \nabla u_h^n, n E \right] E.$$

The operator $I_m$ satisfies the estimations introduced in the following Lemma (See [1], among others).

**Lemma 3.2.** For $v \in H^1(\Omega)$, we have the following estimates:

1. $\|v - I_m v\|_K \leq c_1 \alpha K \|v\|_{W^K} \forall K \in \mathcal{K}; K \subset Q$,
2. $\|v - I_m v\|_E \leq c_2 \varepsilon^{1/4} D_{\min}^{-1/4} \frac{1}{\varepsilon} \|v\|_{W^K} \forall E \in \mathcal{E}_h,$

where $\alpha := h S D_{\min}^{-1/2} \varepsilon^{1/2}$ for $S = K, E$. The constants $c_1$ and $c_2$ are independent of $h$. 


Holder’s inequality together with the estimations given in Lemma 3.2, makes possible to establish:

$$R_1(v) \leq C_0 \left\{ \sum_{V \in \mathcal{V}_h \cap \mathcal{Q}} \sum_{Q \subset V} \alpha_Q \| R^n_h \|_Q \| v \|_{w_Q} + \sum_{E \in \mathcal{E}_h^s} \varepsilon^{-3/4} D_{\min}^{-1/4} \alpha_E^{1/2} \| r^n_h \|_E \| v \|_{w_E} \right\}.$$  

The domains $w_Q$ and $w_E$ consist of finite number of elements, so they are bounded by the minimal ratio of the diameter of its largest inscribed ball. Then, we have:

$$R_1(v) \leq C_0 \left( \sum_{V \in \mathcal{V}_h \cap \mathcal{Q}} \sum_{Q \subset V} \alpha_Q \| R^n_h \|_Q + \sum_{E \in \mathcal{E}_h^s} \varepsilon^{-3/4} D_{\min}^{-1/4} \alpha_E^{1/2} \| r^n_h \|_E \right) \| v \|,$$

then,

$$R_1(v) \leq C_0 \eta_h^{\alpha} \| v \|,$$

and

$$\sum_{m=1}^n \int_{t_{m-1}}^{t_m} R_1(v) dt \leq C_0 \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \eta_h^{\alpha} \| v \| dt \leq C_0 \sum_{m=1}^n \eta_h^{\alpha} \left( \int_0^{t_n} \| v \| dt \right)^{1/2}.$$  

**Bound of $R_2$:**

In (10), we have:

$$R_2(v) = R^2_D(v) + R^2_F(v),$$

where we set

$$R^2_D(v) := \frac{1}{\varepsilon} \int_\Omega D_\varepsilon \nabla (u_h^n - u_{h\tau}) \nabla v dx,$$

and

$$R^2_F(v) := \int_\Omega (\mathcal{F}_h(u_{h\tau}) - \mathcal{F}_h(u_h^n)) v dx.$$  

Using (4) and (6), we get for $t_{n-1} \leq t \leq t_n$:

$$R^2_D(v) = \frac{1}{\varepsilon} \int_\Omega D_\varepsilon \nabla (u_h^n - u_{h\tau}) \nabla v dx$$

$$= \frac{1}{\varepsilon} \int_\Omega D_\varepsilon \nabla \left( u_h^n - u_h^{n-1} - \frac{t - t_{n-1}}{\tau_n} (u_h^n - u_h^{n-1}) \right) \nabla v dx$$

$$= \frac{1}{\varepsilon} \left( \frac{t - t_{n-1}}{\tau_n} \right) \int_\Omega D_\varepsilon \nabla (u_h^n - u_h^{n-1}) \nabla v dx$$

$$\leq \frac{1}{\varepsilon} \left( \frac{t - t_{n-1}}{\tau_n} \right) \| D_\varepsilon^{1/2} \nabla (u_h^n - u_h^{n-1}) \| \| D_\varepsilon^{1/2} \nabla v \|$$

$$\leq \frac{1}{\varepsilon} \left( \frac{t - t_{n-1}}{\tau_n} \right) \left( 1 + \frac{\| D_\varepsilon - D \|_{L^\infty}^{1/2}}{D_{\min}} \right)^{1/2} \| D_\varepsilon^{1/2} \nabla (u_h^n - u_h^{n-1}) \| \| v \|$$

$$\leq C_1(D) \varepsilon^{-1/2} \left( \frac{t - t_{n-1}}{\tau_n} \right) \| D_\varepsilon^{1/2} \nabla (u_h^n - u_h^{n-1}) \| \| v \|,$$

where $C_1(D) = (1 + c(D) h^{l+1} / D_{\min})^{1/2}$.

Hence, we have

$$R^2_D(v) \leq C_1(D, \Omega) \varepsilon^{-1/2} \left( \frac{t - t_{n-1}}{\tau_n} \right) \| D_\varepsilon^{1/2} \nabla (u_h^n - u_h^{n-1}) \| \| v \|.$$
In addition, using the fact that $f$ is locally Lipschitz, we get for $t_{n-1} \leq t \leq t_n$:

$$R^2_F(v) = \int_\Omega \left( \mathcal{F}_h(u_{h\tau}) - \mathcal{F}_h(u^\tau_h) \right) v \, dx$$

$$= \int_\Omega \left( \sum_{i=1}^{M} (f(x_i, u_{h\tau}) - f(x_i, u^\tau_h)) \psi^n_i \right) v \, dx$$

$$\leq \int_\Omega \sum_{i=1}^{M} |f(x_i, u_{h\tau}) - f(x_i, u^\tau_h)||v|\psi^n_i \, dx$$

$$\leq \int_\Omega \sum_{i=1}^{M} g(x, u_n, u_{h\tau})|u_{h\tau} - u^n_h||v|\psi^n_i \, dx$$

$$\leq C_2 \int_\Omega \sum_{i=1}^{M} |u_{h\tau} - u^n_h||v|\psi^n_i \, dx,$$

where $C_2$ is due to the continuity of the function $g$ and boundedness of $u_h$ and $u_{h\tau}$ over $\Omega \times [0, T_f]$.

On the other hand, using the definition of $\psi^n_i$, we get:

$$R^2_F(v) \leq C_2 \int_\Omega \sum_{i=1}^{M} |u_{h\tau} - u^n_h||v|\psi^n_i \, dx$$

$$\leq C_2 \int_{\text{supp } \psi^n_i} |u_{h\tau} - u^n_h||v| \, dx$$

$$\leq 3C_2 \int_{T \in T^n_h} |u_{h\tau} - u^n_h||v| \, dx$$

$$\leq 3C_2 \int_\Omega |u_{h\tau} - u^n_h||v| \, dx.$$

Then, we find:

$$R^2_F(v) \leq 3C_2 \|u_{h\tau} - u^n_h\| \|v\|.$$

Using the definition of $u_{h\tau}$, (6), we get:

$$\int_\Omega \left( \mathcal{F}_h(u_{h\tau}) - \mathcal{F}_h(u^n_h) \right) v \, dx \leq 3C_2 \left( \frac{t_n - t}{\tau_n} \right) \|u^n_h - u^{n-1}_h\| \|v\|,$$

and

$$\sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} R_2(v) \, dt \leq C_3 \sum_{m=1}^{n} \Theta^n_h \left( \int_0^{t_n} (\|v\| + \|v\|)^2 \, dt \right)^{1/2},$$

where $C_3 = \max(C_1(D, \Omega), 3C_2)$.

**Bound of $R_3$:**

We have in (11),

$$R_3(v) = \int_\Omega (f(x, u_{h\tau}) - \mathcal{F}_h(u_{h\tau})) v \, dx + \frac{1}{\varepsilon} \int_\Omega (D_h - D) \nabla u_{h\tau} \cdot \nabla v \, dx.$$
Then, using (9), we get
\[
\sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} R_3(v)dt \leq \sum_{m=1}^{n} \left( \int_{t_{m-1}}^{t_m} |\mu_h^n(t)|^2 dt \right)^{1/2} \left( \int_{0}^{t_n} (\|v\| + \|v\|^2) dt \right)^{1/2}.
\]

**Bound of R₄:**

Bearing in mind the fact that \( f \) is locally Lipschitz, we have
\[
R₄(v) = \int_{\Omega} (f(x,u) - f(u_{h\tau})(u_{h\tau}))vdx
\leq \|f(u) - f(u_{h\tau})\|\|v\|
\leq C_4 \|u - u_{h\tau}\|\|v\|,
\]
where \( C_4 \) is due to the continuity of \( g \) and boundedness of \( u \) and \( u_{h\tau} \) over \( \Omega \times [0, T_f] \).

Using the first decomposition of the proof by taking \( v = u - u_{h\tau} \), we get:
\[
\int_{0}^{t_n} \int_{\Omega} \left( \frac{\partial (u-u_{h\tau})}{\partial t}(u-u_{h\tau}) + \frac{1}{\varepsilon}D(\nabla (u-u_{h\tau}))^2 \right) dxdt
\leq \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} R_1(u-u_{h\tau})dt + \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} R_2(u-u_{h\tau})dt
+ \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} R_3(u-u_{h\tau})dt + \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} R_4(u-u_{h\tau})dt.
\]

Let \( \Pi_h u_0 \) be the projection of \( u_0 \) onto \( V_h^0 \), then we have:
\[
\frac{1}{2} \|u-u_{h\tau}\|^2 + \int_{0}^{t_n} \|u - u_{h\tau}\|^2 dt
\leq \|u_0 - \Pi_h u_0\|^2 + C_0 \sum_{m=1}^{n} \tau^{1/2}_m n^m \left( \int_{0}^{t_n} \|u - u_{h\tau}\|^2 dt \right)^{1/2}
+ C_3 \sum_{m=1}^{n} \Theta_h^n \left( \int_{0}^{t_n} (\|u - u_{h\tau}\| + \|u - u_{h\tau}\|)^2 dt \right)^{1/2}
+ \sum_{m=1}^{n} \left( \int_{t_{m-1}}^{t_m} |\mu_h^n(t)|^2 dt \right)^{1/2} \left( \int_{0}^{t_n} (\|u - u_{h\tau}\| + \|u - u_{h\tau}\|)^2 dt \right)^{1/2}
+ C_4 \int_{0}^{t_n} \|u - u_{h\tau}\|^2 dt
\leq C_5 \left\{ \|u_0 - \Pi_h u_0\|^2 + \sum_{m=1}^{n} \left( \eta_h^m \right)^2 \tau_m + \left( \Theta_h^m \right)^2 + \int_{t_{m-1}}^{t_m} |\mu_h^n(t)|^2 dt \right\}
+ \frac{1}{2} \int_{0}^{t_n} \|u - u_{h\tau}\|^2 dt + C_6 \int_{0}^{t_n} \|u - u_{h\tau}\|^2 dt.
\]

Since:
\[
\frac{1}{2} \|u - u_{h\tau}\|^2 + \frac{1}{2} \int_{0}^{t_n} \|u - u_{h\tau}\|^2 dt - C_6 \int_{0}^{t_n} \|u - u_{h\tau}\|^2 dt
\leq C_5 \left\{ \|u_0 - \Pi_h u_0\|^2 + \sum_{m=1}^{n} \left( \eta_h^m \right)^2 \tau_m + \left( \Theta_h^m \right)^2 + \int_{t_{m-1}}^{t_m} |\mu_h^n(t)|^2 dt \right\}.
\]
Then, we have
\[
\|u - u_{h\tau}\|^2 + \int_0^{t_n} \|u - u_{h\tau}\|^2 dt - 2C_6 \int_0^{t_n} \|u - u_{h\tau}\|^2 dt \\
\leq 2C_5 \left\{ \|u_0 - \Pi_h u_0\|^2 + \sum_{m=1}^n \left( (\eta_{h}^m)^2 \tau_m + (\Theta_{h}^m)^2 + \int_{t_{m-1}}^{t_m} |\mu_{h}^m(t)|^2 dt \right) \right\}.
\]

Applying Gronwall’s Lemma, we obtain
\[
\|u - u_{h\tau}\|^2 + \int_0^{t_n} \|u - u_{h\tau}\|^2 dt \\
\leq 2C_5 e^{2C_6 T_f} \left\{ \|u_0 - \Pi_h u_0\|^2 + \sum_{m=1}^n \left( (\eta_{h}^m)^2 \tau_m + (\Theta_{h}^m)^2 + \int_{t_{m-1}}^{t_m} |\mu_{h}^m(t)|^2 dt \right) \right\}.
\]

Taking \((C^*)^2 = 2C_5 e^{2C_6 T_f}\), the proof is ended.

To ensure the total effectiveness of the indicators, a lower bound of the error is necessary for each indicator \(\Theta_{h}^m\) and \(\eta_{h}^m\) defined by (8) and (7) respectively.

The propositions of the lower bound are close to those developed by Verfürth in [14] using finite element method, because the numerical discretization is not introduced in the proof.

4. Numerical simulation. In this section, we give numerical results for the above studies to a spatial inter-specific model with constant diffusivity and population growth given by a logistic law, in population dynamics. To this end, we consider the model,
\[
\begin{aligned}
\frac{\partial u}{\partial t} & = \frac{1}{\varepsilon} \text{div}(D \nabla u) + ru \left( 1 - \frac{u}{K} \right), \quad x = (x_1, x_2) \in \Omega, \ t \in [0, T_f], \\
\frac{\partial u}{\partial \nu}(x, t) & = 0, \quad x \in \partial \Omega, \ t \in [0, T_f], \\
u(x, 0) & = u_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \(u(x, t)\) represents the population density. The function \(f\), representing the population growth, is defined by
\[
f(x, u) = r(x) u \left( 1 - \frac{u}{K(x)} \right),
\]

where \(r\) and \(K\) are real-valued continuous functions defined on \(\overline{\Omega}\), \(r(x) > 0\) and \(K(x) > 0\) for all \(x \in \overline{\Omega}\), which represent respectively the growth rate and the carrying capacity of the environment.

We give numerical results for Problem (12), by comparing number of triangles and CPU time before and after the use of mesh self-adaptation based on a posteriori error indicators developed in Section 3. The self-adaptive mesh algorithm used in this work is based on the h-adaptation technique, the later was subject to several works like [8, 14].

We have in Figure 1 the approximate solution of Problem (12) using the numerical scheme (5), with the following data: \(\varepsilon = 1, \ \Omega = (0, 1)^2, \ D(x) = 0.1(x_1 + x_2 + 1), \ K(x) = 5(\sin(\pi x_1^2/2) + \sin(\pi x_2^2/2) + 1)^{-1}, \ r(x) = 0.5 \exp(-50(x_1^2 + x_2^2)^2), \) and \(u_0(x) = 2(\cos(\pi x_1) + \cos(\pi x_2) + 2.2)^{-1} + 2.\)
In Figures 2 and 3, we give self-adaptive meshes at different instants and according to three levels, obtained by a self-adaptive algorithm of spatial mesh. Table 1 gives a comparison result between the number of triangles and CPU time before and after the use of the self-adaptive mesh algorithm. According to this numerical experiment, the local indicators allow us to reduce the number of triangles in each iteration, consequently CPU time has been significantly reduced as well.

Now, using adaptive algorithm, we could examine the asymptotic behavior of the dynamics of such model to describe population distribution over time, with less computation cost and without losing the accuracy of simulations. In Figure 1, we have found that the equilibrium (at $t = 10$) is given by a almost homogeneous distribution except a patch that is formed in the bottom left corner, due to the significant values of the growth rate $r$ and the carrying capacity of the environment $K$ in this region.
Figure 2. Self-adapted meshes of three levels at instant $t = 1$.

Table 1. Numerical tests for Problem (12)

| Instant | Mesh   | Level | Number of triangles | CPU time | Mean of $\eta^e_n$ |
|---------|--------|-------|---------------------|----------|-------------------|
| $t=1$   | Adaptive | 1     | 256                 | 1.957s   | $3.1229e - 2$     |
|         |         | 2     | 498                 | 2.321s   | $1.2628e - 2$     |
|         |         | 3     | 1071                | 5.492s   | $4.5722e - 3$     |
|         | Uniform | 1     | 6145                | 89.947s  | $3.3747e - 3$     |
| $t=10$  | Adaptive | 1     | 207                 | 9.861s   | $13053e - 2$      |
|         |         | 2     | 504                 | 16.087s  | $7.8965e - 3$     |
|         |         | 3     | 726                 | 23.411s  | $3.5019e - 3$     |
|         | Uniform | 1     | 6145                | 719.854s | $1.1385e - 3$     |
| $t=50$  | Adaptive | 1     | 356                 | 44.871s  | $1.1824e - 2$     |
|         |         | 2     | 603                 | 76.644s  | $7.5797e - 3$     |
|         |         | 3     | 879                 | 120.120s | $2.2937e - 3$     |
|         | Uniform | 1     | 6145                | 2686.813s| $1.1418e - 3$     |
5. Conclusion. The numerical simulation shows that the number of triangles has been significantly reduced, which reveals the efficiency of the local indicators developed in this work. Indeed, to obtain a good approximate solution to Problem 12, we used a uniform mesh of 6145 elements, at time $t = 50$, however this number is reduced to 879 elements by using the self-adaptive algorithm based on local error indicators. So that, a posteriori analysis of the error presents a powerful tool that allows to well manipulate available computer resources.

Models in Population dynamics require reliable and efficient mathematical tools to be treated. In future works, we develop a posteriori analysis of models applied to more realistic and complex phenomenon such as chemotaxis. Furthermore, new powerful and robust algorithms will also be developed and applied to this type of parabolic problem for which we will automatically adapt the spatial mesh and the temporal one for a better optimization of the computational cost.

The described algorithm is intended to be applied in many realistic applications in ecology; it is frequent and widely present in ecology to look for information on the asymptotic behavior of the population dynamics in order to know about the persistence of the population. As an extension, such studies can be applied and devoted to a new framework of mathematical models that involves fractional derivatives with respect to time and space.
Acknowledgments. The authors are grateful to the anonymous referees for their insightful comments and suggestions.

REFERENCES

[1] B. Amaziane, A. Bergam, M. El Ossmani and Z. Mghazli, A posteriori estimators for vertex centred finite volume discretization of a convection-diffusion-reaction equation arising in flow in porous media, J. Numer. Meth. Fluids, 59 (2009), 259–284.

[2] P. Auger and R. Bravo, Methods of aggregation of variables in population dynamics, Comptes Rendus de l’Académie des Sciences - Series III - Sciences de la Vie, 323 (2000), 665–674.

[3] P. Auger, J. C. Poggiale and E. Sánchez, A review on spatial aggregation methods involving several time scales, Ecological Complexity, 10 (2012), 12–25.

[4] A. Bergam, C. Bernardi and Z. Mghazli, A posteriori analysis of the finite element discretization of some parabolic equations, Math. Comp, 74 (2005), 1117–1138.

[5] A. Bergam, A. Chakib, A. Nachaoui and M. Nachaoui, Adaptive mesh techniques based on a posteriori error estimates for an inverse Cauchy problem, Applied Mathematics and Computation, 346 (2019), 865–878.

[6] A. El Harrak and A. Bergam, Preserving Finite-Volume Schemes for Two-Time Reaction-Diffusion Model, Applied Mathematics & Information Sciences, 14 (2020), 41–50.

[7] R. Eymard, T. Gallouët and R. Herbin, Finite volume methods, Hand-book of Numerical Analysis, P.G. Ciarlet and J.L. Lions eds, North-Holland, VII (2000), 713–1020.

[8] M. Fortin, Estimation a posteriori et adaptation de maillages, Revue Européenne des Eléments Finis, 9 (2000).

[9] W. Gurney and R. M. Nisbet, Ecological Dynamics, Oxford University Press, 1998.

[10] Z. Mghazli, R. Verfürth and A. Bergam, Estimateurs a posteriori d’un schéma de volumes finis pour un problème non linéaire, Numerische Mathematik, 95 (2003), 599–624.

[11] J. D. Murray, Mathematical Biology: I. an Introduction (Interdisciplinary Applied Mathematics), Springer-Verlag, New York, 2002.

[12] E. Sánchez, P. Auger and J. C. Poggiale, Two-time scales in spatially structured models of population dynamics: A semigroup approach, Journal of Mathematical Analysis and Applications, 375 (2011), 149–165.

[13] D. Tilman and P. Kareiva, Spatial ecology: The role of space in population dynamics and interspecific interactions (MPB-30), Princeton University Press, 30 (2018).

[14] R. Verfürth, A posteriori error estimates for non-stationary non-linear convection-diffusion equations, Calcolo, 55 (2018), Paper No. 20, 18 pp.

Received December 2019; revised June 2020.

E-mail address: anouarelharrak1@gmail.com
E-mail address: tayeq.hatim@gmail.com
E-mail address: bergamamali11@gmail.com