ALGEBRAICITY OF RATIOS OF SPECIAL VALUES OF RANKIN–SELBERG L-FUNCTIONS AND APPLICATIONS TO DELIGNE’S CONJECTURE

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Abstract. In this paper, we prove new cases of Blasius’ and Deligne’s conjectures on the algebraicity of critical values of tensor product L-functions and symmetric odd power L-functions associated to modular forms. We also prove an algebraicity result on critical values of Rankin–Selberg L-functions for $GL_n \times GL_2$. These results are applications of our main result on the algebraicity of ratios of special values of Rankin–Selberg L-functions. Let $\Sigma, \Sigma', \Pi, \Pi'$ be algebraic automorphic representations of general linear groups over $\mathbb{Q}$ such that $\Sigma \cong \Sigma'$ and $\Pi \cong \Pi'$. Based on conjectures of Clozel and Deligne, and Yoshida’s computation of motivic periods, we expect the ratio

$$\frac{L(s, \Sigma \times \Pi) \cdot L(s, \Sigma' \times \Pi')}{L(s, \Sigma' \times \Pi') \cdot L(s, \Sigma \times \Pi')}$$

to be algebraic and Galois-equivariant at critical points. We show that this assertion holds under certain parity and regularity assumptions on the archimedean components. Our second main result is to prove an automorphic analogue of Blasius’ conjecture on the behavior of critical values of motivic L-functions upon twisting by Artin motives. We consider Rankin–Selberg L-functions twisted by finite order Hecke characters.

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1. INTRODUCTION

In [Del79], Deligne proposed a remarkable conjecture on the algebraicity of critical values of L-functions of motives, in terms of the periods obtained by comparing the Betti and de Rham realizations of the motives. In the introduction, we recall two classes of motivic L-functions associated to elliptic modular forms, and state our results toward Deligne’s conjecture for these L-functions. We begin with the tensor product L-functions of modular forms. Let $n$ be a positive integer and $f_i \in S_{k_i}(N_i, \omega_i)$, $1 \leq i \leq n$. For each prime $p \nmid N_i$, denote by $\alpha_{i,p}, \beta_{i,p}$ the Satake parameters of $f_i$ at $p$ and put

$$A_{i,p} = \begin{pmatrix} \alpha_{i,p} & 0 \\ 0 & \beta_{i,p} \end{pmatrix}.$$ 

Recall that $\alpha_{i,p}, \beta_{i,p}$ are the roots of the Hecke polynomial $X^2 - a_{f_i}(p)X + p^{k_i-1} \omega_i(p)$. Define the tensor product L-function $L(s, f_1 \otimes \cdots \otimes f_n)$ by an Euler product

$$L(s, f_1 \otimes \cdots \otimes f_n) = \prod_p L_p(s, f_1 \otimes \cdots \otimes f_n), \quad \text{Re}(s) > 1 + \sum_{i=1}^n \frac{k_i-1}{2}.$$ 

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Here the Euler factors are given by
\[
L_p(s, f_1 \otimes \cdots \otimes f_n) = \det \left( 1_{2^n} - \otimes_{i=1}^n A_{i,p} \cdot p^{-s} \right)^{-1}
\]
for \( p \nmid N_1 \cdots N_n \). Conjecturally, it admits meromorphic continuation to the whole complex plane and satisfies a functional equation relating \( L(s, f_1 \otimes \cdots \otimes f_n) \) to \( L(1 + \sum_{i=1}^n (\kappa_i - 1) - s, f'_1 \otimes \cdots \otimes f'_n) \), where \( f'_i \in S_{\kappa_i}^\infty \) is the normalized newform dual to \( f_i \). In [Bla87], Blasius proposed the following conjecture on the algebraicity of special values of the tensor product \( L \)-function, which is a refinement of Deligne’s conjecture [Del70] for tensor product motives associated to \( f_1 \otimes \cdots \otimes f_n \). For \( 1 \leq i \leq n \), let \( G(\omega_i) \) be the Gauss sum of \( \omega_i \cdot \| f_i \| \) be the Petersson norm of \( f_i \) defined by
\[
\| f_i \| = \text{vol}(\Gamma_0(N_i))/\delta \int_{\Gamma_0(N_i)/\delta} \| f_i(\tau) \|^2 y^{\kappa_i-2} \, d\tau,
\]
and \( \sigma f_i \in S_{\kappa_i}^\infty \) be the normalized elliptic newform with \( a(\sigma f_i)(m) = \sigma(a_{f_i}(m)) \) for \( \sigma \in \text{Aut}(\mathbb{C}) \).

**Conjecture A (Blasius, Conjecture 5.7).** Assume \( n \geq 2 \). Let \( m \in \mathbb{Z} \) be a critical point for \( L(s, f_1 \otimes \cdots \otimes f_n) \). Then the tensor product \( L \)-function is holomorphic at \( s = m \) and we have
\[
\sigma \left( \frac{L(m, f_1 \otimes \cdots \otimes f_n)}{(2\pi \sqrt{-1})^{2^n-1} \cdot c(f_1 \otimes \cdots \otimes f_n)} \right) = \frac{L(m, \sigma f_1 \otimes \cdots \otimes f_n)}{(2\pi \sqrt{-1})^{2^n-1} \cdot c(\sigma f_1 \otimes \cdots \otimes f_n)}, \quad \sigma \in \text{Aut}(\mathbb{C}).
\]
Here
\[
c(f_1 \otimes \cdots \otimes f_n) = (2\pi \sqrt{-1})^{2^n-2} \prod_{i=1}^n (1-\kappa_i) \prod_{i=1}^n \prod_{j=1}^n G(\omega_i)^{2^n-2} \cdot (\pi \cdot \| f_i \|)^{2^n-2 - t_i},
\]
for \( 1 \leq i \leq n \).

For \( n = 2 \), we have the pioneering result of Shimura in [Shi76]. For \( n = 3 \), the conjecture was considered by various authors and is partially proved, notably we have the results of Harris–Kudla [HK91], Garrett–Harris [GH93], and Furusawa–Morimoto [FM04], [FM06]. We prove in this paper the following theorem regarding Blasius’ conjecture for arbitrary \( n \):  

**Theorem A (Theorem 5.8).** Conjecture A holds if the following conditions are satisfied:

1. \( \kappa_1 + \cdots + \kappa_n = n \) (mod 2).
2. \( |\sum_{i=1}^n (\varepsilon_i - \varepsilon_i')| (\kappa_i - 1) | \geq 6 \) for all \( (\varepsilon_1, \cdots, \varepsilon_n) \) and \( (\varepsilon_1', \cdots, \varepsilon_n') \) in \( \{ \pm 1 \}^n \).
3. If \( n \geq 5 \), then \( N_1 = \cdots = N_{n-2} = 1 \).

As another example of Deligne’s conjecture on critical values of motivic \( L \)-functions, we consider symmetric power \( L \)-functions of modular forms. Let \( f \in S_{\kappa}(N, \omega) \) be a normalized elliptic modular newform. For \( n \geq 1 \), the symmetric \( n \)-th power \( L \)-function \( L(s, \text{Sym}^n(f)) \) is defined by an Euler product
\[
L(s, \text{Sym}^n(f)) = \prod_p L_p(s, \text{Sym}^n(f)), \quad \text{Re}(s) > 1 + \frac{n(n-1)}{2}.
\]
Here the Euler factors are defined by
\[
L_p(s, \text{Sym}^n(f)) = \det \left( 1_{n+1} - \text{Sym}^n(A_p) \cdot p^{-s} \right)^{-1}
\]
for \( p \nmid N \), where \( \text{Sym}^n : \text{GL}_2(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C}) \) is the symmetric \( n \)-th power representation. By the result of Barnet-Lamb, Geraghty, Harris, and Taylor [BLGHT], the symmetric power \( L \)-functions admit meromorphic continuation to the whole complex plane and satisfy functional equations relating \( L(s, \text{Sym}^n(f)) \) to \( L(1+n(\kappa-1)-s, \text{Sym}^n(f^*)) \). Associated to \( f \), we have a pure motive \( M_f \) over \( \mathbb{Q} \) of rank 2 with coefficients in \( \mathbb{Q}(f) \), whose existence was established by Deligne [Del71] and Scholl [Sch90], such that
\[
L(M_f, s) = (L(s, f^*)|_{\text{Sym}(f)})_{\sigma, \mathbb{Q}(f) \to \mathbb{C}}.
\]
Denote by \( c^\pm(M_f) \in (\mathbb{Q}(f) \otimes \mathbb{Q} \mathbb{Q})^\times / \mathbb{Q}(f)^\times \) the Deligne’s periods of \( M_f \). Under the canonical isomorphism \( \mathbb{Q}(f) \otimes \mathbb{Q} \mathbb{Q} \cong \prod_{\mathfrak{p} : \mathbb{Q}(f) \to \mathbb{Q}} \mathbb{Q} \), we write
\[
c^\pm(M_f) = (c^\pm(f)|_{\sigma, \mathbb{Q}(f) \to \mathbb{C}}.
\]
In [Del79 §7], Deligne explicitly computed the periods $c^\pm(\text{Sym}^n M_f)$ in terms of $c^\pm(M_f)$. In particular, specializing to the case when $n$ is odd, we have the following conjecture on the algebraicity of critical values of symmetric odd power $L$-functions.

**Conjecture B** (Deligne, Conjecture 5.10). Let $r \geq 1$. For a critical point $m \in \mathbb{Z}$ for $L(s, \text{Sym}^{2r+1}(f))$, we have

$$
\sigma \left( \frac{L(m, \text{Sym}^{2r+1}(f))}{(2\pi \sqrt{-1})^{(r+1)m} \cdot e^{(-1)^m(\text{Sym}^n(f))}} \right) = \frac{L(m, \text{Sym}^{2r+1}(\sigma f))}{(2\pi \sqrt{-1})^{(r+1)m} \cdot e^{(-1)^m(\text{Sym}^n(\sigma f))}} \quad \sigma \in \text{Aut}(\mathbb{C}).
$$

Here $c^\pm(\text{Sym}^{2r+1}(f)) = (2\pi \sqrt{-1})^{(r+1)(1-\kappa)/2} \cdot G(\omega)^{(r+1)/2} \cdot c^\pm(f)^{(r+1)(r+2)/2} \cdot c^\tau(f)^{(r+1)/2}$.

For $r = 1$, the conjecture was proved by Garrett and Harris [GH93] assuming $\kappa \geq 5$. We extend the result of Garrett and Harris to $\kappa = 3, 4$ in [Che21a]. For $r = 2$, we prove the conjecture assuming $\kappa \geq 6$ in [Che22b].

We prove in this paper the following theorem regarding Deligne’s conjecture for arbitrary $r$:

**Theorem B** (Theorem 5.11). Conjecture B holds when $\kappa$ is odd and $\kappa \geq 5$.

Theorems A and B are proved by mathematical induction. In the induction step, a key ingredient is our main result Theorem 1.2 which we shall now introduce.

1.1. **Main results.** Denote by $\hat{A}$ the ring of adeles of $\mathbb{Q}$. For isobaric automorphic representations $\Sigma$ and $\Pi$ of $\text{GL}_n(\hat{A})$ and $\text{GL}_{n'}(\hat{A})$, respectively, let $L(s, \Sigma \times \Pi)$ be the Rankin–Selberg $L$-function of $\Sigma \times \Pi$. We denote by $L^\sigma(s, \Sigma \times \Pi)$ the $L$-function obtained by excluding the archimedean factor. These $L$-functions play important roles in number theory. The analytic properties were studied by various authors (cf. [JSS1a], [JSS1b], [JPSS83], [Sha81], [MW89], and [CPS04]). In particular, $L(s, \Sigma \times \Pi)$ admits meromorphic continuation to the whole complex plane and satisfies a functional equation relating $L(s, \Sigma \times \Pi)$ to $L(1-s, \Sigma^\vee \times \Pi^\vee)$. Besides its analytic properties, the algebraic aspect is also an important subject of study. In this paper, we study the algebraicity of ratios of special values of Rankin–Selberg $L$-functions. Following Clozel in [Clo90], we consider algebraic automorphic representations of general linear groups. For algebraic automorphic representation $\Pi$, we denote by $Q(\Pi)$ its rationality field and $^\sigma \Pi$ its $\sigma$-conjugate for $\sigma \in \text{Aut}(\mathbb{C})$. It is conjectured that the rationality field is a number field and the $\sigma$-conjugates are automorphic. The assertion was proved in loc. cit. for regular algebraic automorphic representations. In [HR20], Harder and Raghuram prove a remarkable result on the algebraicity of ratios of critical values of a fixed Rankin–Selberg $L$-function. More precisely, let $\Sigma$ and $\Pi$ be regular algebraic (cohomological) cuspidal automorphic representations of $\text{GL}_n(\hat{A})$ and $\text{GL}_{n'}(\hat{A})$, respectively. Assume $n$ and $n'$ are both even, they show that

$$
\sigma \left( \frac{L(m_0, \Sigma \times \Pi)}{L(m_1, \Sigma \times \Pi)} \right) = \frac{L(m_0, \Sigma \times \Pi)}{L(m_1, \Sigma \times \Pi)}
$$

for all $\sigma \in \text{Aut}(\mathbb{C})$ and all critical points $m_0, m_1$ for $L(s, \Sigma \times \Pi)$ with $L(m_1, \Sigma \times \Pi) \neq 0$. If $n$ is even and $n'$ is odd, then the ratios are expressed in terms of the relative periods of $\Sigma$ (cf. [HR20 Definition 5.3]). The result is compatible with Deligne’s conjecture [Del79] on critical values of motivic $L$-functions, and it reduces the study of algebraicity of critical values to that of the rightmost or leftmost critical values. As a different aspect of ratios of critical values, we consider ratios of product of different Rankin–Selberg $L$-functions at a fixed critical point. Under reasonable assumptions, we expect these ratios to be algebraic and propose the following conjecture:

**Conjecture 1.1.** Let $\Sigma, \Sigma'$ (resp. $\Pi, \Pi'$) be algebraic automorphic representations of $\text{GL}_n(\hat{A})$ (resp. $\text{GL}_{n'}(\hat{A})$) such that

$$
\Sigma_{\infty} = \Sigma'_{\infty}, \quad \Pi_{\infty} = \Pi'_{\infty}.
$$

Let $m_0 \in \mathbb{Z} + \frac{n+n'}{2}$ be a critical point for $L(s, \Sigma \times \Pi)$ such that $L(m_0, \Sigma \times \Pi) \cdot L(m_0, \Sigma' \times \Pi') \neq 0$. Then, for $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\sigma \left( \frac{L(m_0, \Sigma \times \Pi) \cdot L(m_0, \Sigma' \times \Pi')}{L(m_0, \Sigma \times \Pi') \cdot L(m_0, \Sigma' \times \Pi)} \right) = \frac{L(m_0, \Sigma \times \Pi) \cdot L(m_0, \Sigma' \times \Pi')}{L(m_0, \Sigma \times \Pi') \cdot L(m_0, \Sigma' \times \Pi)}.
$$

In particular, we have

$$
\frac{L(m_0, \Sigma \times \Pi) \cdot L(m_0, \Sigma' \times \Pi')}{L(m_0, \Sigma \times \Pi') \cdot L(m_0, \Sigma' \times \Pi)} \in \mathbb{Q}(\Sigma) \cdot \mathbb{Q}(\Sigma') \cdot \mathbb{Q}(\Pi) \cdot \mathbb{Q}(\Pi').
$$
The conjecture is also compatible with Deligne’s conjecture, as we will explain in §1.2 below. The conjecture can be verified in some special cases when $\Sigma$ and $\Pi$ are regular algebraic and algebraicity results for $GL_n \times GL_{n'}$ are known. For instance, it holds for $n = n' = 2$ by the result of Shimura [Shi76]. It also holds for $n' = n - 1$ by the result of Raghuram [Rag10] (in fact, a stronger version where $\Pi, \Pi'$ can be isobaric, see Theorem 4.1) under the assumptions that $\Sigma, \Sigma'$ are cuspidal and $(\Sigma, \Pi, \Sigma_\infty)$ is balanced (cf. [Che22a, Corollary 2.6]). In practice, compared with known methods in the literature, Conjecture 1.1 provides another approach to prove algebraicity results for automorphic/motivic $L$-functions. For instance, we have apply Conjecture 1.1 for $GL_4 \times GL_3$ in [Che22a] to prove Deligne’s conjecture for symmetrical fifth $L$-functions of modular forms. Applications to Deligne’s conjecture for motivic $L$-functions of higher degree are given in §6 below. The following theorem is the main result of this paper. We prove Conjecture 1.1 for cohomologically tamely isobaric automorphic representations (i.e. regular algebraic automorphic representations with essentially tempered archimedean components) under some assumptions. Given a cohomologically tamely isobaric automorphic representation of $GL_n(\mathbb{A})$, we can associate a tuple of integers $(\kappa; \omega)$, called the infinity type of the representation, which is determined by its archimedean component (cf. §2.1). Note that $\kappa = (\kappa_1, \cdots, \kappa_r) \in \mathbb{Z}^r$ with $\kappa_1 \geq \cdots \geq \kappa_r \geq 2$ and $r = \lfloor \frac{n}{2} \rfloor$, and some parity condition is imposed on the infinity type (cf. (2.1)). We say $\kappa$ is $N$-regular if $\kappa_i - \kappa_j \geq N$ for all $1 \leq i < j \leq r$. The assumptions in our main result are some parity and regularity conditions on the infinity types. More precisely, our main result is as follows:

**Theorem 1.2.** Let $\Sigma$ and $\Pi$ be cohomological tamely isobaric automorphic representations of $GL_n(\mathbb{A})$ and $GL_{n'}(\mathbb{A})$ with infinity types $(\kappa; \omega)$ and $(\ell; u)$, respectively. Then Conjecture 1.1 holds if the following conditions are satisfied:

1. $nn'$ is even and $w + u = n + n'$ (mod 2).
2. $\min \{\kappa_i, \ell_j\} \geq 3$ if $n$ and $n'$ are even, $\min \{\kappa_i, \ell_j\} \geq 5$ if $n$ or $n'$ is odd, and $\kappa \sqcup \ell$ is $(4 + 2\delta)$-regular, where $\delta = 0$ if $w$ or $u$ is odd and $\delta = 1$ otherwise.

Here $\kappa \sqcup \ell$ is the $(\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n'}{2} \rfloor)$-tuple in descending order with entries from $\kappa$ and $\ell$.

We consider yet another aspect of ratios of critical values. We have the following conjecture on the algebraicity of ratios of critical values of Rankin–Selberg $L$-functions twisted by finite order Hecke characters. It is an automorphic analogue of Blasius’ conjecture [Blas97] on the behavior of critical values of motivic $L$-functions upon twisting by Artin motives.

**Conjecture 1.3.** Let $\Sigma$ and $\Pi$ be algebraic cuspidal automorphic representations of $GL_n(\mathbb{A})$ and $GL_{n'}(\mathbb{A})$, respectively. Assume $nn'$ is even. Let $m_0, m_1 \in \mathbb{Z} + \frac{n + n'}{2}$ be critical points for $L(s, \Sigma \times \Pi)$ and $\chi$ a finite order Hecke character of $\mathbb{A}^\times$ such that $L(m_1, \Sigma \times \Pi) \neq 0$ and $(-1)^{m_0 - m_1} \chi^s(-1)$ if $n$ or $n'$ is odd. Then, for $\sigma \in Aut(\mathbb{C})$, we have

$$\sigma \left( \frac{L(m_0, \Sigma \times \Pi \otimes \chi)}{(\sqrt{-1})^{(m_0 - m_1)mn/2} \cdot G(\chi)^{mn/2} \cdot L(m_1, \Sigma \times \Pi)} \right) = \frac{L(m_0, \sigma \Sigma \times \sigma \Pi \otimes \chi)}{(\sqrt{-1})^{(m_0 - m_1)mn/2} \cdot G(\sigma \chi)^{mn/2} \cdot L(m_1, \sigma \Sigma \times \sigma \Pi)}.$$

Here $G(\chi)$ is the Gauss sum of $\chi$.

Besides the results for $GL_2 \times GL_2$ and $GL_n \times GL_{n-1}$ mentioned above, the conjecture is also known when $n' = 1$, $m_0 = m_1$, and $\Sigma$ is regular algebraic and symplectic by the result of Grobner and Raghuram [GR14]. Following theorem is our second main result. We prove Conjecture 1.3 for regular algebraic (cohomological) cuspidal automorphic representations for arbitrary $n, n'$, without the essential self-duality assumption, under some parity and regularity conditions on the infinity types. More precisely:

**Theorem 1.4.** Let $\Sigma$ and $\Pi$ be cohomological cuspidal automorphic representations of $GL_n(\mathbb{A})$ and $GL_{n'}(\mathbb{A})$ with infinity types $(\kappa; \omega)$ and $(\ell; u)$, respectively. Write $r = \lfloor \frac{n}{2} \rfloor$ and $r' = \lfloor \frac{n'}{2} \rfloor$. Then Conjecture 1.3 holds if either the conditions in Theorem 1.2 are satisfied or the following conditions are satisfied:

1. $\kappa_1 > \ell_1$ and for each $1 \leq i \leq r - 1$ (resp. $i = r$), there exists at most one $1 \leq j \leq r'$ such that $\kappa_i \geq \ell_j > \kappa_{i+1}$ (resp. $\kappa_r > \ell_j$).
2. If $n$ is even, then $\ell_{r'} > \kappa_r \geq 3$ when $\min \{|\kappa_i - \ell_j|\}$ is even and $\ell_{r'} > \kappa_r \geq 6$ when $\min \{|\kappa_i - \ell_j|\}$ is odd.
3. If $n$ is odd and $\ell_{r'} > \kappa_r$ (resp. $\kappa_r > \ell_{r'}$), then $\kappa_r \geq 5$ (resp. $\ell_{r'} \geq 3$) when $\min \{|\kappa_i - \ell_j|\}$ is even and $\kappa_r \geq 11$ (resp. $\ell_{r'} \geq 6$) when $\min \{|\kappa_i - \ell_j|\}$ is odd.
(4) If \( \min \{|\kappa_i - \ell_j|\} \) is even (resp. \( \min \{|\kappa_i - \ell_j|\} \) is odd), then \( \min \{|\kappa_i - \ell_j|\} \geq 2 \) (resp. \( \min \{|\kappa_i - \ell_j|\} \geq 5 \)) and \( \kappa \) is 4-regular (resp. 10-regular).

(5) If \( n \) and \( n' \) are both even, then \( u \) is odd.

When \( n' = 1 \), we replace \( \min \{|\kappa_i - \ell_j|\} \) by \( \min \{|\kappa_i - 1|\} \) in the above conditions.

We have the following remark on the conditions in Theorems 1.2 and 1.4.

**Remark 1.5.** Condition (1) in Theorem 1.2 is equivalent to the existence of integers \( t_1, t_2 \) such that the isobaric sum \( (\Sigma \boxtimes | l_i^{1/2}) \boxplus (\Pi^\vee \boxtimes | l_i^{1/2}) \) is cohomological and tamely isobaric. Condition (2) in Theorem 1.2 is necessary for the existence of a cohomological cuspidal automorphic representation of \( \text{GL}_{n+n'+1}(A) \) such that Raghuram’s result [Rag10] is applicable to the corresponding Rankin–Selberg \( L \)-function for \( \text{GL}_{n+n'+1} \times \text{GL}_{n+n'} \). Conditions (1)-(5) in Theorem 1.2 are slightly artificial in nature. It guarantees the existence of cohomological tamely isobaric automorphic representation \( \Pi' \) of \( \text{GL}_{n-n'-1}(A) \) and integers \( t_1, t_2 \) such that \( \Psi = (\Pi \boxtimes | l_i^{1/2}) \boxplus (\Pi^\vee \boxtimes | l_i^{1/2}) \) is cohomological and tamely isobaric and Raghuram’s result [Rag10] is applicable to \( L(s, \Sigma \times \Psi) \). If we assume \( \Sigma \) and \( \Pi' \) are cuspidal, then Conjecture 1.1 also holds under conditions (1)-(5) in Theorem 1.3. The main difficulty to lift these conditions is to generalize Raghuram’s result to cohomological tamely isobaric automorphic representations. We refer to [GH10] § 6.6 for a discussion.

### 1.2. Outline of the proof.

The main ingredients in the proof of Theorem 1.2 are:

- The result of Raghuram [Rag10] on the algebraicity of critical values of Rankin–Selberg \( L \)-functions for \( \text{GL}_n \times \text{GL}_{n-1} \).
- An analogue of the result [GL21] Theorem 2.6] of Grobner and Lin on period relations for the Betti–Whittaker periods over CM-fields.

For the second ingredient, we follow the idea in [GL21] § 2 and carry out the details when the base field is \( \mathbb{Q} \) (actually, can be any totally real number fields). More precisely, in Theorem 3.2, we prove period relations for the Betti–Whittaker periods of a cohomological tamely isobaric automorphic representation in terms of the Betti–Whittaker periods of its summands and critical values of Rankin–Selberg \( L \)-functions. For instance, let \( \Sigma \) and \( \Pi \) be cohomological tamely isobaric automorphic representations of \( \text{GL}_n(A) \) and \( \text{GL}_{n'}(A) \) respectively with both \( n \) and \( n' \) even. The parity condition (1) in Theorem 1.2 implies that we may assume the isobaric sum \( \Sigma \boxplus \Pi^\vee \) is cohomological and tamely isobaric. Then Theorem 3.2 gives the period relation

\[
p(\Sigma \boxplus \Pi^\vee, \pm) \sim G(\omega_\Pi)^{-n} \cdot L^{(\infty)}(1, \Sigma \times \Pi) \cdot p(\Sigma, \pm(-1)^{n/2}) \cdot p(\Pi^\vee, \pm(-1)^{n/2}),
\]

where \( p(\cdot, \pm) \) refers to the Betti–Whittaker periods. By the regularity condition (2) in Theorem 1.2, we prove in Lemma 1.6 the existence of an auxiliary cohomological cuspidal automorphic representation \( \Psi \) of \( \text{GL}_{n+n'+1}(A) \) such that the result of Raghuram can be applied to the Rankin–Selberg \( L \)-function \( L(s, \Psi \times (\Sigma \boxplus \Pi^\vee)) \) which has at least two critical points. Then we will conclude that

\[
L^{(\infty)}(1, \Sigma \times \Pi) \sim \frac{L^{(\infty)}(m + \frac{1}{2}, \psi \times \Sigma) \cdot L^{(\infty)}(m + \frac{1}{2}, \psi \times \Pi^\vee)}{q_\infty(m) \cdot G(\omega_\Pi^{-m-n+1}) \cdot p(\psi, \varepsilon_m) \cdot p(\Sigma, \varepsilon_m') \cdot p(\Pi^\vee, \varepsilon_m'^\vee)}
\]

for any critical points \( m + \frac{1}{2} \) for \( L(s, \Psi \times (\Sigma \boxplus \Pi^\vee)) \). Here \( q_\infty(m) \in \mathbb{C}, \varepsilon_m, \varepsilon_m', \varepsilon_m'^\vee \in \{\pm 1\} \) depend only on \( m, \psi, \Sigma, \Pi \). It then follows that Conjecture 1.1 holds for the critical point \( m_0 = 1 \). The result for other critical points is a consequence of this special case together with the result of Harder and Raghuram [HR20]. For Theorem 1.4 the proof is similar but slightly more complicated. There is one more ingredient which is the result of Raghuram and Shahidi [RS08] on the period relations for the Betti–Whittaker periods upon twisting by algebraic Hecke characters.

This paper is organized as follows. In § 2 we recall the definition of the Betti–Whittaker periods of cohomological tamely isobaric automorphic representations. The definition depends on choice of generators in relative Lie algebraic cohomology groups. We specify a canonical choice in § 2.3. In § 3 we prove the period relations for the Betti–Whittaker periods in Theorem 3.2. In § 4 we prove our main results Theorems 1.2 and 1.4. In § 5 we give applications of Theorem 1.2 to Deligne’s conjecture for some motivic \( L \)-functions. This section is logically independent of § 2–4 except for some notation in § 2.1 and some preliminaries on Deligne’s conjecture in § 4.5. Readers interested in the applications can start with § 5.
1.3. Notations. Let \( \mathbb{A} \) be the ring of adeles of \( \mathbb{Q} \). Let \( \mathbb{A}_f \) be the finite part of \( \mathbb{A} \), and \( \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \) be its maximal compact subring. Let \( \psi = \otimes_v \psi_v \) be the standard additive character of \( \mathbb{Q} \backslash \mathbb{A} \) defined so that
\[
\psi_p(x) = e^{-2\pi i x/p} \quad \text{for} \quad x \in \mathbb{Z}/p^{-1} \mathbb{Z},
\]
\[
\psi_{\infty}(x) = e^{2\pi i x} \quad \text{for} \quad x \in \mathbb{R}.
\]
For each place \( v \) of \( \mathbb{Q} \), let \( | \cdot |_v \) be the absolute value on \( \mathbb{Q}_v \), normalized so that \( |p|_p = p^{-1} \) if \( p \) is a prime number and \( |x| = 1 \) is the ordinary absolute value on \( \mathbb{R} \) if \( v = \infty \). Let \( | \cdot |_A = \prod_v | \cdot |_v \) be the normalized absolute value on \( \mathbb{A} \).

Let \( \chi \) be an algebraic Hecke character of \( \mathbb{A}^\times \). We denote by \( G(\chi) \) the Gauss sum of \( \chi \) defined by
\[
G(\chi) = \prod_p \varepsilon(0, \chi_p, \psi_p),
\]
where \( \varepsilon(s, \chi_p, \psi_p) \) is the \( \varepsilon \)-factor of \( \chi_p \) with respect to \( \psi_p \) defined in \cite{Tat79}. For \( \sigma \in \text{Aut}(\mathbb{C}) \), let \( \sigma \chi \) the unique algebraic Hecke character of \( \mathbb{A}^\times \) such that \( \sigma \chi(a) = \sigma(\chi(a)) \) for \( a \in \mathbb{A}_f^\times \). It is easy to verify that
\[
\sigma(G(\chi)) = \sigma(u_\sigma)G(\sigma \chi),
\]
where \( u_\sigma \in \hat{\mathbb{Z}}^\times \) is the unique element such that \( \sigma(\psi(x)) = \psi(u_\sigma x) \) for \( x \in \mathbb{A}_f \).

We denote by \( W_n \) the Weyl group of \( \text{GL}_n \) with respect to the standard maximal torus consisting of diagonal matrices. We will take permutation matrices in \( \text{GL}_n \) as representatives of elements in \( W_n \). For an ordered partition \( (n_1, \ldots, n_k) \) of \( n \), let \( P_{(n_1, \ldots, n_k)} \) by the standard parabolic subgroup of \( \text{GL}_n \) defined by
\[
P_{(n_1, \ldots, n_k)} = \left\{ \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1k} \\ 0 & g_{22} & \cdots & g_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{kk} \end{pmatrix} : g_{ij} \in \text{GL}_{n_i}, g_{ij} \in M_{n_i, n_j} \text{ for } 1 \leq i < j \leq k \right\}.
\]
For a standard parabolic subgroup \( P \) of \( \text{GL}_n \), we denote by \( U_P \) and \( M_P \) the unipotent radical and standard Levi subgroup consisting of block-diagonal matrices of \( P \), respectively. Let \( W_n^P \) be the minimal coset representatives of \( W_{M_P} \backslash W_n \), where \( W_{M_P} \) is the Weyl group of \( M_P \).

The following notation will appear only in the proof of algebraicity results: Let \( (a_\sigma)_{\sigma \in \text{Aut}(\mathbb{C})} \) and \( (b_\sigma)_{\sigma \in \text{Aut}(\mathbb{C})} \) be sequences of complex numbers index by \( \text{Aut}(\mathbb{C}) \) such that \( b_\sigma \neq 0 \) for all \( \sigma \). We write \( a_{id} \sim b_{id} \) if
\[
\sigma \left( \frac{a_{id}}{b_{id}} \right) = \frac{a_\sigma}{b_\sigma}
\]
for all \( \sigma \in \text{Aut}(\mathbb{C}) \).

2. Betti–Whittaker periods for \( \text{GL}_n \)

2.1. Cohomological representations. We identify the set of dominant integral weights for \( \text{GL}_n \) (with respect to its standard maximal torus consisting of diagonal matrices and the standard Borel subgroup consisting of upper triangular matrices) with the set of \( n \)-tuples \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \) such that \( \lambda_1 \geq \cdots \geq \lambda_n \). For a dominant integral weight \( \mu \), we denote by \( M_\mu \) the irreducible algebraic representation of \( \text{GL}_n(\mathbb{Q}) \) with highest weight \( \mu \). Let \( M_{\mu, \mathbb{C}} = M_\mu \otimes \mathbb{C} \) be the base change to a representation of \( \text{GL}_n(\mathbb{C}) \).

Let \( K_n^\circ = \mathbb{R}_+ \cdot \text{SO}(n) \) and \( K_n = \mathbb{R}_+ \cdot \text{O}(n) \). Here we regard the set \( \mathbb{R}_+ \) of positive real numbers as the topological connected component of the center of \( \text{GL}_n(\mathbb{R}) \). We denote by \( \mathfrak{g}_n \) and \( \mathfrak{t}_n \) the Lie algebras of \( \text{GL}_n(\mathbb{R}) \) and \( K_n \), respectively. Let \( \Pi \) be an irreducible admissible \( (\mathfrak{g}_n, \text{O}_n(\mathbb{R})) \)-module. Recall \( \Pi \) is cohomological if there exists some irreducible algebraic representation \( M_\mathbb{C} \) of \( \text{GL}_n(\mathbb{C}) \) such that the \( (\mathfrak{g}_n, K_n^\circ) \)-cohomology is non-zero:
\[
H^*(\mathfrak{g}_n, K_n^\circ; \Pi \otimes M_\mathbb{C}) \neq 0.
\]
In this case, we say \( \Pi \) is cohomological with coefficients in \( M_\mathbb{C} \). Moreover, if we assume further that \( \Pi \) is essentially tempered, then \( M_\mathbb{C} \) is uniquely determined. Indeed, an irreducible admissible, cohomological, essentially tempered \( (\mathfrak{g}_n, \text{O}_n(\mathbb{R})) \)-module is realized as the space of \( \text{O}_n(\mathbb{R}) \)-finite vectors of an induced
representation of the form

\[
\begin{cases}
\text{Ind}_{P_{2,\ldots,2}(\mathbb{R})}^{GL_n(\mathbb{R})}(D_{\kappa_1} \otimes \cdots \otimes D_{\kappa_r}) \otimes |w/2| & \text{if } n = 2r, \\
\text{Ind}_{P_{2,\ldots,2,1}(\mathbb{R})}^{GL_n(\mathbb{R})}(D_{\kappa_1} \otimes \cdots \otimes D_{\kappa_r} \otimes \text{sgn}^\delta) \otimes |w/2| & \text{if } n = 2r + 1,
\end{cases}
\]

for some \(\kappa_1 > \cdots > \kappa_r \geq 2\) and \(\delta \in \{0, 1\}\) if \(n\) is odd, such that

\[
\begin{aligned}
\kappa_1 &\equiv \cdots \equiv \kappa_r \equiv w \pmod{2} & & \text{if } n = 2r, \\
\kappa_1 &\equiv \cdots \equiv \kappa_r \equiv w + 1 \equiv 1 \pmod{2} & & \text{if } n = 2r + 1.
\end{aligned}
\]

Here \(D_{\kappa}\) is the discrete series representation of \(\text{GL}_2(\mathbb{R})\) with weight \(\kappa \geq 2\). We write \(\kappa = (\kappa_1, \ldots, \kappa_r)\) and call \((\kappa; w)\) the infinity type of \(\Pi\). When \(n\) is odd, we also define the signature \(\varepsilon(\Pi)\) of \(\Pi\) by

\[
\varepsilon(\Pi) = (-1)^{r+i+w/2}.
\]

In this case, \(\Pi\) is cohomological with coefficients in the uniquely determined algebraic representation \(M_{\mu, \mathbb{C}}\) with

\[
\mu = \left(\frac{1-n}{2}, \frac{3-n}{2}, \cdots, \frac{n-1}{2} \right) - \left(\frac{w}{2}, \cdots, \frac{w}{2}\right) + \left\{\frac{\kappa-1}{2}, \cdots, \frac{\kappa_r-1}{2}, \frac{1-\kappa_r}{2}, \cdots, \frac{1-\kappa_1}{2}\right\} \quad \text{if } n = 2r,
\]

\[
\mu = \left\{\frac{\kappa-1}{2}, \cdots, \frac{\kappa_r-1}{2}, \frac{1-\kappa_r}{2}, \cdots, \frac{1-\kappa_1}{2}\right\} \quad \text{if } n = 2r + 1.
\]

Moreover, \(\bullet = b_n = \lceil \frac{n^2}{2} \rceil\) is the least non-vanishing degree of the \((\mathfrak{g}_n, K_n^\mathbb{C})\)-cohomology for \(\Pi \otimes M_{\mu, \mathbb{C}}\), and we have

\[
\dim_{\mathbb{C}} H^n_{\rho}(\mathfrak{g}_n, K_n^\mathbb{C}; \Pi \otimes M_{\mu, \mathbb{C}}) = \begin{cases} 
2 & \text{if } n \text{ is even,} \\
1 & \text{if } n \text{ is odd.}
\end{cases}
\]

Note that \(\pi_0(\text{GL}_n(\mathbb{R})) = \text{GL}_n(\mathbb{R})/\text{GL}_n(\mathbb{R})^0 \cong O_n(\mathbb{R})/SO_n(\mathbb{R})\) naturally acts on the \((\mathfrak{g}_n, K_n^\mathbb{C})\)-cohomology groups. For \(\varepsilon \in \{\pm 1\}\), we denote by \(H^n_{\rho}(\mathfrak{g}_n, K_n^\mathbb{C}; \Pi \otimes M_{\mu, \mathbb{C}}; \varepsilon)\) the \(\varepsilon\)-isotypic space under the action of \(\pi_0(\text{GL}_n(\mathbb{R}))\). The \(\varepsilon\)-isotypic space is one-dimensional when \(n\) is even, and is non-zero for \(\varepsilon = \varepsilon(\Pi)\) when \(n\) is odd. In §2.3 below, we will specify a canonical choice of generator of the \(\varepsilon\)-isotypic space under the Whittaker realization of \(\Pi\).

### 2.2. Tamely isobaric automorphic representations

Let \(\Pi\) be an isobaric automorphic representation of \(\text{GL}_n(\mathbb{A})\). We have

\[
\Pi = \Pi_{11} \oplus \cdots \oplus \Pi_k
\]

for some cuspidal automorphic representation \(\Pi_i\) of \(\text{GL}_{n_i}(\mathbb{A})\) for \(1 \leq i \leq k\). Following [LLS21], we say \(\Pi\) is tamely isobaric if there exists \(s \in \mathbb{R}\) such that \(\Pi_i \otimes |s\rangle\) is unitary for \(1 \leq i \leq k\). In this case, \(\Pi\) is fully induced (cf. [Ber84] and [Vog86]). We recall the realization of \(\Pi\) in the space \(\mathcal{A}(\text{GL}_n)\) of automorphic forms on \(\text{GL}_n(\mathbb{A})\) using cuspidal Eisenstein series as follows: Let \(P = P_{(n_1, \ldots, n_k)}\) be the standard parabolic subgroup of \(\text{GL}_n(\mathbb{R})\) of type \((n_1, \cdots, n_k)\). Let \(I_P^{\text{GL}_{n_k}}(\otimes_{i=1}^k \Pi_i)\) be the space of smooth and \(O_n(\mathbb{R})\)-finite functions \(h : U_P(\mathbb{A}) M_P(\mathbb{Q}) \backslash \text{GL}_n(\mathbb{A}) \to \mathbb{C}\) such that for all \(g \in \text{GL}_n(\mathbb{A})\), the function

\[
M_P(\mathbb{A}) \to \mathbb{C}, \quad m \mapsto \rho_P(m)^{-1} \cdot h(mg)
\]

is a cusp form in \(V_{\Pi_1} \otimes \cdots \otimes V_{\Pi_k}\). Here \(\rho_P\) is the square-root of the modulus character of \(P(\mathbb{A})\), and \(V_{\Pi_i}\) is the space of \(\Pi_i\) realized in the space of cusp forms on \(\text{GL}_{n_i}(\mathbb{A})\) for \(1 \leq i \leq k\). It is clear that \(I_P^{\text{GL}_{n_k}}(\otimes_{i=1}^k \Pi_i)\) can be identified with the space of \(O_n(\mathbb{R})\)-finite functions in the induced representation \(\text{Ind}_{(\otimes_{i=1}^k \Pi_i)}^{\text{GL}_{n_k}}(\mathbb{A})\), which we denote by \(\text{Ind}_{(\otimes_{i=1}^k \Pi_i)}^{\text{GL}_{n_k}}(\mathbb{A})|_{O_n(\mathbb{R})}\). For \(h \in I_P^{\text{GL}_{n_k}}(\otimes_{i=1}^k \Pi_i)\) and \(\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{C}^k\), let \(h_\lambda \in I_P^{\text{GL}_{n_k}}(\otimes_{i=1}^k \Pi_i |_{|\lambda_i|})\) defined by

\[
h_\lambda(g) = \prod_{i=1}^k |\det m_i|^{\lambda_i}_{\mathbb{A}} \cdot h(g)
\]

for \(g = \text{umk} \in U_P(\mathbb{A}) M_P(\mathbb{A}) (O_n(\mathbb{R}) \times \text{GL}_n(\mathbb{Z}))\) with \(m = \text{diag}(m_1, \cdots, m_k)\). We define the Eisenstein series

\[
E(g, h_\lambda) = \sum_{\gamma \in \mathcal{P}(\mathbb{Q}) \backslash \text{GL}_n(\mathbb{Q})} h_\lambda(\gamma g),
\]

which converges absolutely when \(\text{Re}(\lambda_i - \lambda_j)\) is sufficiently large for \(1 \leq i < j \leq k\). By Langlands’ work [Lan70], it admits meromorphic continuation to \(\lambda \in \mathbb{C}^k\) (cf. [MW99 §IV.1.8]). Moreover, since \(\Pi\) is tamely
isobaric, the Eisenstein series is holomorphic at \( \lambda = 0 \) (cf. [MW95, §IV.1.11]). We then realize \( II \) in \( \mathcal{A}(\text{GL}_n) \) as the image of the \( ((\mathfrak{g}_n, O_n(\mathbb{R})) \times \text{GL}_n(\mathbb{A}_f)) \)-equivariant injective homomorphism
\[
I^\text{GL}_n_P(\otimes_{i=1}^k \Pi_i) \to \mathcal{A}(\text{GL}_n), \quad h \mapsto E(h) = E(h_\lambda)|_{\lambda = 0}.
\]
(2.4)

For an automorphic form \( \varphi \) on \( \text{GL}_n(\mathbb{A}) \), recall the constant term of \( \varphi \) along \( P \) is defined by
\[
\varphi_P(g) = \int_{U_P(\mathbb{Q}) \backslash U_P(\mathbb{A})} \varphi(g) \, du^\text{Tam}, \quad g \in \text{GL}_n(\mathbb{A}).
\]
Here \( du^\text{Tam} \) is the Tamagawa measure on \( U_P(\mathbb{A}) \). For \( h \in I^\text{GL}_n_P(\otimes_{i=1}^k \Pi_i) \) and \( w \in W_n \) such that \( w M_P w^{-1} = M_P \), let
\[
M^P_w(h) : U_P(\mathbb{A}) M_P(\mathbb{Q}) \backslash \text{GL}_n(\mathbb{A}) \to \mathbb{C}
\]
be the function defined so that \( M^P_w(h)(g) \) is the evaluation at \( \lambda = 0 \) of the meromorphic function defined by the intertwining integral
\[
\int_{(U_P(\mathbb{A}) \cap w U_P(\mathbb{A}) w^{-1}) \backslash U_P(\mathbb{A})} h_\lambda(w^{-1}u) \, du^\text{Tam}
\]
when \( \text{Re}(\lambda_i - \lambda_j) \) is sufficiently large for \( 1 \leq i < j \leq k \) (cf. [MW95, §II.1.6]). Note that the meromorphic function is holomorphic at \( \lambda = 0 \), since \( II \) is tamely isobaric (cf. [MW95, §IV.1.11]). The constant term of the Eisenstein series \( E(h) \) along \( P \) is given by (cf. [MW95 §II.1.7])
\[
E_P(h) = \sum_{w \in W^P_n \cap (W^P_n)^{-1}} M^P_w(h).
\]
(2.5)

For \( w \in W^P_n \cap (W^P_n)^{-1} \) with \( w M_P w^{-1} = M_P \), there exists a unique permutation \( \tau_w \) on \( \{1, \ldots, k\} \) such that
\[
w \cdot \text{diag}(m_1, \ldots, m_k) \cdot w^{-1} = \text{diag}(m_{\tau_w(1)}, \ldots, m_{\tau_w(k)}), \quad \text{diag}(m_1, \ldots, m_k) \in M_P.
\]
(2.6)

One can easily verify that the association \( w \to \tau_w \) is injective. We have an \( ((\mathfrak{g}_n, O_n(\mathbb{R})) \times \text{GL}_n(\mathbb{A}_f)) \)-equivariant isomorphism
\[
M^P_w : I^\text{GL}_n_P(\otimes_{i=1}^k \Pi_i) \to I^\text{GL}_n_P(\otimes_{i=1}^k \Pi_{\tau_w(i)}).
\]

2.3. Whittaker functionals. Let \( U_n \) be the standard maximal unipotent subgroup of \( \text{GL}_n \) consisting of upper unipotent matrices. Let \( \psi_{U_n} : U_n(\mathbb{Q}) \backslash U_n(\mathbb{A}) \to \mathbb{C} \) be the additive character defined by
\[
\psi_{U_n}(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1,n}), \quad u = (u_{ij}) \in U_n(\mathbb{A}).
\]
For each place \( v \) of \( \mathbb{Q} \), let \( \psi_{U_n,v} \) be the local component of \( \psi_{U_n} \) at \( v \). Let \( \psi_{U_n,f} = \otimes_v \psi_{U_n,v} \).

Let \( II = \Pi_1 \otimes \cdots \otimes \Pi_k \) be a tamely isobaric automorphic representation of \( \text{GL}_n(\mathbb{A}) \) as in §2.2. For an automorphic form \( \varphi \in II \), the Whittaker function \( W(\varphi) \) with respect to \( \psi_{U_n} \) is defined by
\[
W(g, \varphi) = \int_{U_n(\mathbb{Q}) \backslash U_n(\mathbb{A})} \varphi(ug) \psi_{U_n}(u) \, du^\text{Tam}, \quad g \in \text{GL}_n(\mathbb{A}).
\]
(2.7)

Here \( du^\text{Tam} \) is the Tamagawa measure on \( U_n(\mathbb{A}) \). Let \( \mathcal{W}(II) \) be the space of Whittaker functions of \( II \). We say \( II \) is globally generic if \( \mathcal{W}(II) \) is non-zero. Since \( II \) is fully induced, \( II \) is locally generic everywhere. For each place \( v \) of \( \mathbb{Q} \), let \( \mathcal{W}(II_v) \) be the space of Whittaker functions of \( II_v \) with respect to \( \psi_{U_n,v} \). When \( v = p \) is a finite place such that \( II_p \) is unramified, let \( \mathcal{W}^\circ_{II_p} \in \mathcal{W}(II_p) \) be the \( \text{GL}_n(\mathbb{Z}_p) \)-invariant Whittaker function normalized so that \( \mathcal{W}^\circ_{II_p}(1_n) = 1 \). Let \( \mathcal{W}(II_f) = \prod_p \mathcal{W}(II_p) \) be the restricted product with respect to \( \mathcal{W}^\circ_{II_p} \) for unramified \( p \). Assume \( II \) is globally generic. Then \( \mathcal{W}(II) = \prod_v \mathcal{W}(II_v) \). For \( W_\infty \in \mathcal{W}(II_\infty) \) and \( W_f \in \mathcal{W}(II_f) \), there exists a unique automorphic form \( \varphi \in II \) such that
\[
W(\varphi) = W_\infty \cdot W_f.
\]
In this case, we obtain a \( ((\mathfrak{g}_n, O_n(\mathbb{R})) \times \text{GL}_n(\mathbb{A}_f)) \)-equivariant isomorphism
\[
\mathcal{W}(II_\infty) \otimes \mathcal{W}(II_f) \to II.
\]
(2.8)
Recall \( P = P_{(n_1, \ldots, n_k)}. \) For \( f \in \text{Ind}_{P_{(k)}}^{\GL_n}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}) \) and \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k, \) let \( f_\lambda \in \text{Ind}_{P_{(k)}}^{\GL_n}((\otimes_{i=1}^k \mathcal{W}(I_i \otimes | \lambda_i))_{O_n(\mathbb{R})}) \) be defined as in \([23, 24]\), and \( \mathcal{W}(f_\lambda) \) be the Whittaker function of the induced representation \( \text{Ind}_{P_{(k)}}^{\GL_n}((\otimes_{i=1}^k (I_i \otimes | \lambda_i))_{O_n(\mathbb{R})}) \) defined by

\[
\mathcal{W}(g, f_\lambda) = \int_{w_p \in P_{(k)}} f_\lambda(w_p^{-1}ug)(1_{n_1}, \ldots, 1_{n_k}) du, \quad g \in \GL_n(\mathbb{A}).
\]

Here \( U_p \) is the transpose of \( U_p, \) \( w_p \in W_n \) is the Weyl element given by

\[
w_p = \begin{pmatrix} & & 1_{n_k} \\ & \ddots & \\ 1_{n_1} & & \end{pmatrix},
\]

and \( du = \prod_v du_v \) is the Haar measure on \( w_p U_p(\mathbb{A})w_p^{-1} \) so that \( \text{vol}(w_p U_p(\mathbb{A})w_p^{-1}, du_p) = 1 \) if \( v = p \) is finite and \( du_v \) is the product measure of the Lebesgue measures on \( \mathbb{R} \) if \( v = \infty. \) Note that the integral converges absolutely when \( \text{Re}(\lambda_i - \lambda_j) \) is sufficiently large for \( 1 \leq i < j \leq k \) and admits holomorphic continuation to \( \lambda \in \mathbb{C}^k \) (cf. \([10, \S 3]\)). For each place \( v \) of \( \mathbb{Q} \) and \( f_v \in \text{Ind}_{P_{(k)}}^{\GL_n(\mathbb{Q}_v)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}), \) let \( \mathcal{W}(f_{v, \lambda}) \) be the Whittaker function of \( \text{Ind}_{P_{(k)}}^{\GL_n(\mathbb{Q}_v)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}) \) defined in a similar way as above. For \( h \in I_{P_{(k)}}^{\GL_n(\mathbb{Q}_v)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}), \) let \( f(h) \in \text{Ind}_{P_{(k)}}^{\GL_n(\mathbb{Q}_v)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}) \) defined by composing with the global Whittaker functional \([23, 24]\) with \( U_n \) replaced by \( U_n \cap M_P, \) that is, for \( g \in \GL_n(\mathbb{A}), \) \( f(h)(g) \in \otimes_{i=1}^k \mathcal{W}(I_i) \) is given by

\[
f(h)(g)(m) = \rho_{P_{(k)}}(m)^{1 - j} \int_{(U_n \cap M_P)_{(k)}} \hat{h}(umg) psi_{U_n}(u) d\mu_{\text{Tam}}, \quad m \in M_P(\mathbb{A}).
\]

We then have the following equality between Whittaker functions as holomorphic functions in \( \lambda \in \mathbb{C}^k \) outside the singularities of \( E(h_\lambda) \):

\[(2.9) \quad W(E(h_\lambda)) = \mathcal{W}(f(h_\lambda)).\]

**Lemma 2.1.** Assume \( I_i \neq I_j \) if \( i \neq j. \) Then \( \Pi \) is globally generic.

**Proof.** Let \( S \) be a finite set of places of \( \mathbb{Q} \) consisting of \( \infty \) such that \( I_p \) is unramified for all \( p \notin S. \) For \( p \notin S, \) let \( f_p^\circ \) be the \( \GL_n(\mathbb{Z}_p) \)-invariant section in \( \text{Ind}_{P_{(k)}}^{\GL_n(\mathbb{Q}_p)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}) \) normalized so that \( f_p^\circ(1_n) = \otimes_{i=1}^k \mathcal{W}_{I_i, p}^\circ. \) By \([10, \text{Proposition 7.1.4}]\), we have

\[
\mathcal{W}(1_n, f_p^\circ) = \prod_{1 \leq i < j \leq k} L(1 + \lambda_i - \lambda_j, I_{i, p} \times I_{j, p})^{-1}.
\]

For any place \( v, \) the equivariant map

\[
\mathcal{W} : \text{Ind}_{P_{(k)}}^{\GL_n(\mathbb{Q}_v)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}) \to \mathcal{W}(I_v), \quad \mathcal{W}(f_v) = \mathcal{W}(f_v, \lambda = 0)
\]

is an isomorphism by \([92, \text{Theorem 15.4.1}]\) and \([10, \text{Corollaries 3.4.9 and 3.6.11}]\) (for \( v = \infty, \) we take \( O_n(\mathbb{R}) \)-finite part on the left-hand side). For \( v \in S, \) fix \( f_v \in \text{Ind}_{P_{(k)}}^{\GL_n(\mathbb{Q}_v)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}) \) (\( O_n(\mathbb{R}) \)-finite if \( v = \infty \)) such that \( \mathcal{W}(1_n, f_v) \neq 0. \) We then define \( h \in I_{P_{(k)}}^{\GL_n(\mathbb{Q}_v)}((\otimes_{i=1}^k \mathcal{W}(I_i))_{O_n(\mathbb{R})}) \) to be the unique section such that \( f(h) \) is \( \mathcal{W}(1_n, f_v) \otimes (\otimes_{p \notin S} f_p^\circ). \) By \([23, 24]\), we deduce that

\[
W(1_n, E(h)) = \prod_{v \in S} W(1_n, f_v) \cdot \prod_{1 \leq i < j \leq k} L^S(1 + \lambda_i - \lambda_j, I_i \times I_j)^{-1}(\lambda = 0).
\]

By our assumption, \( L(s, I_i \times I_j) \) is holomorphic and non-vanishing at \( s = 1 \) for \( 1 \leq i < j \leq k \) (cf. \([33, \text{Propositions 3.3 and 3.6}]\) and \([10, \text{Theorem 5.2}]\)). We conclude that \( W(1_n, E(h)) \neq 0. \) This completes the proof. \( \square \)
2.4. Betti–Whittaker periods. Let $\Pi = \Pi_1 \oplus \cdots \oplus \Pi_k$ be a tamely isobaric automorphic representation of $\GL_n(\A_f)$ as in §2.2. Recall $p_\mu$ is the square-root of the modulus character of $P(\A_f)$ for $1 \leq i \leq k$, let $\rho_i$ be the restriction of $p_\mu$ to the factor $\GL_n$ of the Levi component $M_f \cong \GL_n \times \cdots \times \GL_n$, and we abbreviate $\Pi_i \rho_i$ for $\Pi_i \otimes \rho_i$. Assume further that $\Pi$ is cohomological, that is, $\Pi_{\infty}$ is cohomological. By [BW00, III, Theorem 3.3], $\Pi_i \rho_i$ is cohomological for $1 \leq i \leq k$. It then follows from Clozel’s purity lemma [Clo90, Lemme 4.9] that $\Pi_{i, \infty}$ is essentially tempered for $1 \leq i \leq k$. Therefore, $\Pi_{\infty}$ is essentially tempered.

As we recalled in §2.4.2, there exists a unique irreducible algebraic representation $M_\mu$ of $\GL_n(\Q)$ such that the $(\g_n, K_n^0)$-cohomology is non-zero:

$$H^\bullet(\g_n, K_n^0, \Pi_{\infty} \otimes M_\mu, \C) \neq 0.$$ 

By the description of $\Pi_{\infty}$ in §2.1 we also see that $\Pi_{i, \infty} \neq \Pi_{j, \infty}$ if $i \neq j$. In particular, $\Pi$ is globally generic by Lemma 2.1.

2.4.1. Rational structure via Whittaker model. Let $\sigma \in \Aut(\C)$. For $1 \leq i \leq k$, let $\sigma(\Pi_i \rho_i)$ be the irreducible admissible $((\g_n, \O_n(\R)) \times \GL_n(\A_f))$-module defined by

$$\sigma(\Pi_i \rho_i) = (\Pi_i \rho_i) \otimes \sigma(\Pi_i \rho_i, \infty) \otimes \sigma(\Pi_i \rho_i, j).$$

Since $\Pi_i \rho_i$ is cohomological, $\sigma(\Pi_i \rho_i)$ is cohomological and cuspidal by the result of Clozel [Clo90 Théorème 3.19]. Let $\sigma \Pi$ be the isobaric automorphic representation defined by

$$\sigma \Pi = \bigoplus_{i=1}^k \sigma(\Pi_i \rho_i) \rho_i^{-1}.$$ 

It is clear that $\sigma \Pi$ is cohomological and tamely isobaric. Moreover, we have $(\sigma \Pi)_f = \sigma(\Pi_f)$ (cf. [Gro18, Lemma 1.2]). We write $\sigma \Pi_f = (\sigma \Pi)_f$. Let $\Q(\Pi)$ be the rationality field of $\Pi$ which is the fixed field of $\{\sigma \in \Aut(\C)| \sigma \Pi = \Pi\}$. Note that $\Q(\Pi)$ is equal to the composite of the rationality fields of $\Q(\Pi_i)$ for $1 \leq i \leq k$. In particular, $\Q(\Pi)$ is a number field. Let $t_\sigma : \mathcal{W}(\Pi_f) \to \mathcal{W}(\sigma \Pi_f)$ be the $\sigma$-linear $\GL_n(\A_f)$-equivariant isomorphism defined by

$$t_\sigma W(g) = \sigma \left(W(\text{diag}(u_{\sigma}^{-n+1}, u_{\sigma}^{-n+2}, \cdots, 1)), g \in \GL_n(\A_f).$$

Here $u_{\sigma} \in \hat{\Z}^\times$ is the unique element such that $\sigma(x) = \psi(u_{\sigma} x)$ for all $x \in \A_f$. We thus obtain a $\Q(\Pi)$-rational structure on $\mathcal{W}(\Pi_f)$ given by taking the Galois invariants:

$$\mathcal{W}(\Pi_f)^{\Aut(\C/\Q(\Pi))} = \{W \in \mathcal{W}(\Pi_f) | t_\sigma W = W \text{ for } \sigma \in \Aut(\C/\Q(\Pi))\}.$$ 

Note that $\mathcal{W}(\Pi_f)^{\Aut(\C/\Q(\Pi))}$ is non-zero by the newform theory for $\GL_n(\A_f)$ [JPSSS1], the existence of the Kirillov model for $\Pi_f$ [BZ76, Theorem 5.20], and [Wal85, Lemme 1.1].

2.4.2. Rational structure via sheaf cohomology. Consider the orbifold

$$S_n = \GL_n(\Q) \backslash \GL_n(\A_f)/K_n^0.$$ 

The algebraic representation $M_\mu$ defines a sheaf $\mathcal{M}_\mu$ of $\Q$-vector spaces on $S_n$ (cf. [HR20 §2.8]). We denote by

$$H^\bullet(S_n, \mathcal{M}_\mu)$$

the sheaf cohomology groups of $\mathcal{M}_\mu$. It naturally endowed with an action of $\pi_0(\GL_n(\R)) \times \GL_n(\A_f)$ (cf. [HR20 §2.3]). When we go to the transcendental level, the base change $M_{\mu, \C} = M_\mu \otimes \C$ gives a sheaf $\mathcal{M}_{\mu, \C}$ of $\C$-vector spaces on $S_n$, and we have

$$H^\bullet(S_n, \mathcal{M}_{\mu, \C}) = H^\bullet(S_n, \mathcal{M}_\mu) \otimes \C.$$ 

At the transcendental level, via the de Rham complex, the cohomology groups $H^\bullet(S_n, \mathcal{M}_{\mu, \C})$ are isonically isomorphic to the $(\g_n, K_n^0)$-cohomology groups of $C^\bullet(\GL_n(\Q), \GL_n(\A_f)) \otimes M_{\mu, \C}$. Moreover, we have the celebrated result of Franke [Fra98, Theorem 18] which shows that we can replace the space of smooth functions on $\GL_n(\Q) \backslash \GL_n(\A_f)$ by the space $\mathcal{A}(\GL_n)$ of automorphic forms on $\GL_n(\A_f)$. The homomorphism (2.4.3) then induces a $\pi_0(\GL_n(\R)) \times \GL_n(\A_f)$-equivariant homomorphism

$$\Psi^\bullet_\Pi : H^\bullet(S_n, K_n^0, \Pi \otimes M_{\mu, \C}) \to H^\bullet(S_n, \mathcal{M}_{\mu, \C}).$$
We mention that the image of $\Psi_{II}$ lies in the Eisenstein cohomology (resp. cuspidal cohomology) if $k \geq 2$ (resp. $k = 1$). For $\sigma \in \text{Aut}(C)$, we have the $\sigma$-linear isomorphism $M_{\mu,C} \rightarrow M_{\mu,C}$ defined by

$$v \otimes z \mapsto v \otimes \sigma(z)$$

for $v \otimes z \in M_{\mu,C} = M_{\mu}\otimes_{\mathbb{Q}} C$. It naturally induces a $\sigma$-linear $\pi_0(\text{GL}_n(\mathbb{R})) \times \text{GL}_n(\mathbb{A}_f)$-equivariant isomorphism

$$\sigma^*: H^*(S_n, \mathcal{M}_{\mu,C}) \rightarrow H^*(S_n, \mathcal{M}_{\mu,C}).$$

Following the proof of [Gro18] Propositions 1.6 and 1.7 (based on the result of Grobner [Gro13] Theorem 4) which is a refinement of the result of Franke [Fra98] Theorem 14), we deduce that $\Psi_{II}^b$ is injective and the image of $\sigma^* \circ \Psi_{II}^b$ is equal to the image of $\Psi_{II}^b$. We will write $\Psi_{II}^b = \Psi_{II}$ for brevity. This induces a $\sigma$-linear $\pi_0(\text{GL}_n(\mathbb{R})) \times \text{GL}_n(\mathbb{A}_f)$-equivariant isomorphism

$$\sigma^*: H^b_{\text{b}}(g_n, K_n^\sigma; \Pi \otimes M_{\mu,C}) \rightarrow H^b_{\text{b}}(g_n, K_n^\sigma; \sigma \Pi \otimes M_{\mu,C}).$$

Let $\varepsilon \in \{\pm 1\}$ if $n$ is even, and $\varepsilon = \varepsilon(\Pi_\varepsilon)$ if $n$ is odd. We then obtain a $\mathbb{Q}(\Pi)$-rational structure on $H^b_{\text{b}}(g_n, K_n^\sigma; \Pi \otimes M_{\mu,C})[\varepsilon]$ given by taking the Galois invariants:

$$H^b_{\text{b}}(g_n, K_n^\sigma; \Pi \otimes M_{\mu,C})[\varepsilon]^{\text{Aut}(\mathbb{Q}(\Pi))}$$

$$= \{\omega \in H^b_{\text{b}}(g_n, K_n^\sigma; \Pi \otimes M_{\mu,C})[\varepsilon] \mid \sigma^* \omega = \omega \text{ for } \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))\}.$$ 

Now we fix a choice of a generator

$$[\Pi_\varepsilon]^\varepsilon \in H^b_{\text{b}}(g_n, K_n^\sigma; \mathcal{W}(\Pi_\varepsilon) \otimes M_{\mu,C})[\varepsilon].$$

Let

$$\Phi_{II}^\varepsilon: \mathcal{W}(\Pi_\varepsilon) \rightarrow H^b_{\text{b}}(g_n, K_n^\sigma; \Pi \otimes M_{\mu,C})[\varepsilon]$$

be the $\text{GL}_n(\mathbb{A}_f)$-equivariant isomorphism defined as follows: For $W \in \mathcal{W}(\Pi_\varepsilon)$, we have

$$[\Pi_\varepsilon]^\varepsilon \otimes W \in H^b_{\text{b}}(g_n, K_n^\sigma; \mathcal{W}(\Pi_\varepsilon) \otimes \mathcal{W}(\Pi_\varepsilon) \otimes M_{\mu,C})[\varepsilon].$$

Then $\Phi_{II}^\varepsilon(W)$ is the image of $[\Pi_\varepsilon]^\varepsilon \otimes W$ under the $\text{GL}_n(\mathbb{A}_f)$-equivariant isomorphism induced by the isomorphism (2.28). Comparing the $\mathbb{Q}(\Pi)$-rational structures given by (2.11) and (2.15), we have the following lemma/definition of the Betti–Whittaker periods of $\Pi$.

**Lemma 2.2.** Let $\varepsilon \in \{\pm 1\}$ if $n$ is even, and $\varepsilon = \varepsilon(\Pi_\varepsilon)$ if $n$ is odd. There exists $p(\Pi, \varepsilon) \in \mathbb{C}^\times$, unique up to $\mathbb{Q}(\Pi)^\times$, such that

$$\frac{\Phi_{II}^\varepsilon(\mathcal{W}(\Pi_\varepsilon))^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}}{p(\Pi, \varepsilon)} = H^b_{\text{b}}(g_n, K_n^\sigma; \Pi \otimes M_{\mu,C})[\varepsilon]^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}.$$ 

Moreover, we can normalize the periods so that

$$\sigma^*(\Phi_{II}^\varepsilon(W)) = \frac{\Phi_{II}^\varepsilon(t_\sigma W)}{p(\sigma \Pi, \varepsilon)}.$$

It is clear that the Betti–Whittaker period $p(\Pi, \varepsilon)$ of $\Pi$ depends on the choice of $[\Pi_\varepsilon]^\varepsilon$. In §2.6 below, we will specify a canonical choice of generator. When $n$ is odd, for brevity we will also write

$$p(\Pi, \varepsilon) = p(\Pi).$$

In the following theorem, we recall the result of Raghuram and Shahidi [RSh08] on the period relations for Betti–Whittaker periods twisting by algebraic Hecke characters.

**Theorem 2.3** (Raghuram–Shahidi). Let $\Pi$ be a cohomological tamely isobaric automorphic representation of $\text{GL}_n(\mathbb{A})$ and $\chi$ an algebraic Hecke character of $\mathbb{A}_f^\times$. Let $\varepsilon \in \{\pm 1\}$ if $n$ is even, and $\varepsilon = \varepsilon(\Pi_\varepsilon)$ if $n$ is odd. Assume the generators $[\Pi_\varepsilon]^\varepsilon$ and $[\Pi_\varepsilon \otimes \chi_\varepsilon]^\varepsilon \varepsilon(\chi_\varepsilon)$ satisfy the compatibility relation in Lemma 2.6 below. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\frac{p(\Pi \otimes \chi_\varepsilon \cdot \varepsilon(\chi_\varepsilon))}{G(\chi)^{n(n-1)/2} \cdot p(\Pi, \varepsilon)} = \frac{p(\sigma \Pi \otimes \chi_\varepsilon \cdot \varepsilon(\chi_\varepsilon))}{G(\sigma \chi)^{n(n-1)/2} \cdot p(\sigma \Pi, \varepsilon)}.$$ 

**Remark 2.4.** The result of Raghuram and Shahidi is stated for cuspidal $\Pi$. Nonetheless, the proof goes without change for general case. One can also deduce the general case from the cuspidal one by using the period relation in Theorem 3.2 below.
2.5. Choices of generators for relative Lie algebra cohomology. Let $\Pi$ be an irreducible admissible essentially tempered $(\mathfrak{g}_n, O_n(\mathbb{R}))$-module. Assume $\Pi$ is cohomological with coefficients in $\mathcal{M}_\mu$. Let $W(\Pi)$ be the space of Whittaker functions of $\Pi$ with respect to $\psi_{\nu_n, \infty}$. Let $\varepsilon \in \{\pm 1\}$ if $n$ is even, and $\varepsilon = \varepsilon(\Pi)$ if $n$ is odd. The aim of this section is to fix a canonical choice of generator of $H^0_n(\mathfrak{g}_n, K_n^\circ; W(\Pi) \otimes \mathcal{M}_\mu, \mathcal{C})[\varepsilon]$.

We begin with the case $n = 1$. Then $\Pi = \chi = \text{sgn}^\delta |w|$ for some $\delta \in \{0, 1\}$ and $w \in \mathbb{Z}$. In this case $\varepsilon = (-1)^{\delta + w}$, and $\mathcal{M}_\mu = \mathcal{Q}$ with $\text{GL}_1(\mathcal{Q}) = \mathcal{Q}^\times$ acts by $(-w)$-th power multiplication. Note that $W(\chi)$ consisting of functions $W : \mathbb{R}^\times \rightarrow \mathbb{C}$ such that $W(a) = \chi(a)W(1)$. Let $W_\chi \in W(\chi)$ be the Whittaker function normalized so that $W_\chi(1) = 1$. Then we define the class

$$[\chi]^\varepsilon \in H^0_n(\mathfrak{g}_1, K_1^\circ; W(\chi) \otimes \mathcal{M}_\mu, \mathcal{C})[\varepsilon]$$

by

$$[\chi]^\varepsilon = W_\chi \otimes 1 \otimes 1. \tag{2.17}$$

Here we identify $\bigwedge^0_0(\mathfrak{g}_1, \mathfrak{t}_1) \ast \mathcal{C}$.

Secondly, we consider the case $n = 2$. Then $\Pi = D_n \otimes |w|^{w/2}$ for some $\kappa \geq 2$ and $w \in \mathbb{Z}$ such that $\kappa \equiv w \pmod{2}$. In this case, $\mathcal{M}_\mu$ can be realized as the $\mathcal{Q}$-vector space consisting of homogeneous polynomials over $\mathcal{Q}$ of degree $\kappa - 2$ in variables $x, y$, and $\text{GL}_2(\mathcal{Q})$ acts on it by

$$(g \cdot P)(x, y) = (\text{det } g)^{-\kappa + 2 - w/2} \cdot P((x, y)g).$$

Let $Y_\pm \in \mathfrak{g}_2, \mathfrak{t}_2 \ast \mathcal{C}$ be defined by

$$Y_\pm = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \sqrt{-1} \pm \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and $\{Y_+, Y_-\} \subset (\mathfrak{g}_2, \mathfrak{t}_2) \ast \mathcal{C}$ be the corresponding dual basis. Let $W_\Pi^\pm \in W(\Pi)$ be the Whittaker function of weight $\pm \kappa$ under the action of $\text{SO}_2(\mathbb{R})$ and normalized so that

$$W_\Pi^\pm(\text{diag}(a, 1)) = |a|^{(\kappa + w)/2} e^{-2\pi|a|} \cdot 1_{\mathbb{R}_+}(\pm a).$$

We define the class

$$[\Pi]^\varepsilon \in H^1(\mathfrak{g}_2, K_2^\circ; W(\Pi) \otimes \mathcal{M}_\mu, \mathcal{C})[\varepsilon]$$

by

$$[\Pi]^\varepsilon = W_\Pi^+ \otimes Y_+^\ast \otimes (\sqrt{-1}x + y)^{\kappa - 2} + \varepsilon \cdot (\sqrt{-1})^w \cdot W_\Pi^- \otimes Y_-^\ast \otimes (x + \sqrt{-1}y)^{\kappa - 2}. \tag{2.18}$$

Note that the normalization here differs from the one in [RTT1 (3.6) and (3.7)] by a rational number.

Now we consider general $n$. Assume there exist a standard parabolic subgroup $P$ of $\text{GL}_n$ of type $(n_1, \cdots, n_k)$ and irreducible admissible essentially tempered $(\mathfrak{g}_{n_i}, O_{n_i}(\mathbb{R}))$-module $\Pi_i$ for $1 \leq i \leq k$ such that

$$\Pi \cong \text{Ind}^\text{GL}_n(\mathbb{R})_{O_n(\mathbb{R})}^{\text{GL}_n(\mathbb{R})} \left( \bigotimes_{i=1}^k \Pi_i \right). \tag{2.19}$$

Then $\Pi_i \rho_i$ is cohomological with coefficients in some $\mathcal{M}_{\mu_i}$ for $1 \leq i \leq k$. Also the essentially temperedness of $\Pi$ implies that $\Pi_1, \cdots, \Pi_k$ are simultaneously unitary up to a common twist. Moreover, by the result of Borel and Wallach [BW00, III, Theorem 3.3] (also known as Delorme’s lemma in [Har, Theorem 9.2.1]), there exists a unique Weyl element $w \in W_n$ such that $(\mu_1, \cdots, \mu_k) = w \cdot (\mu + \rho_n) - \rho_n$, and we have a canonical $\pi_0(\text{GL}_n(\mathbb{R}))$-equivariant isomorphism

$$I_{\bigotimes_{i=1}^k W(\Pi_i)} : H^0_n(\mathfrak{g}_n, K_n^\circ; \text{Ind}_P^\text{GL}_n(\mathbb{R})_{O_n(\mathbb{R})}^{\text{GL}_n(\mathbb{R})} \left( \bigotimes_{i=1}^k W(\Pi_i) \right) \otimes \mathcal{M}_\mu, \mathcal{C}) \rightarrow \left( \bigotimes_{i=1}^k H^0_n(\mathfrak{g}_n, K_n^\circ; W(\Pi_i \rho_i) \otimes \mathcal{M}_{\mu_i}, \mathcal{C}) \right)^{\pi_0(\text{GL}_n(\mathbb{R}))}. \tag{2.20}$$

Here $\rho_n$ is half the sum of positive roots for $\text{GL}_n$ and $K_n^{M_P}$ is the image of $K_n^\circ \cap P(\mathbb{R})$ under the canonical map $P \rightarrow M_P$. Note that the tensor product on the right-hand side is a $\pi_0(M_P(\mathbb{R}))$-module. After taking the $\pi_0(K_n^{M_P})$-invariant part, it is naturally a $\pi_0(\text{GL}_n(\mathbb{R}))$-module by the short exact sequence

$$1 \rightarrow \pi_0(K_n^{M_P}) \rightarrow \pi_0(M_P(\mathbb{R})) \rightarrow \pi_0(\text{GL}_n(\mathbb{R})) \rightarrow 1.$$
By taking the $\varepsilon$-isotypic component on both sides, we obtain the isomorphism

$$I_{\otimes \iota=1}^\varepsilon W(\Pi_i) : H^{bn}(g_n, K_n^\varepsilon; \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})}(\otimes_{j=1}^k W(\Pi_j))_{O_n(\mathbb{R})} \otimes M_{\mu, \varepsilon})[\varepsilon] \longrightarrow \bigotimes_{i=1}^k H^{bn}(g_n, K_n^\varepsilon; W(\Pi_i) \otimes M_{\mu, \varepsilon})[\varepsilon].$$

Recall the equivariant isomorphism $W : \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})}(\otimes_{i=1}^k W(\Pi_i))_{O_n(\mathbb{R})} \rightarrow W(\Pi)$ defined in §2.4. This in turn defines a $\pi_0(\text{GL}_n(\mathbb{R}))$-equivariant isomorphism

$$W^* : H^{bn}(g_n, K_n^\varepsilon; \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})}(\otimes_{i=1}^k W(\Pi_i))_{O_n(\mathbb{R})} \otimes M_{\mu, \varepsilon}) \longrightarrow H^{bn}(g_n, K_n^\varepsilon; W(\Pi) \otimes M_{\mu, \varepsilon}).$$

We define the class

$$[\Pi]^\varepsilon \in H^{bn}(g_n, K_n^\varepsilon; W(\Pi) \otimes M_{\mu, \varepsilon})$$

by

$$(2.21) [\Pi]^\varepsilon = W^* \circ \left( I_{\otimes \iota=1}^\varepsilon W(\Pi_i) \right)^{-1} \left( \otimes_{i=1}^k [\Pi_i \rho_i]^\varepsilon \right)$$

for some specific choices of $P$, $\Pi$, and $[\Pi_i \rho_i]^\varepsilon$ described as follows: Let $P = P_{2, \ldots, 2}$ if $n$ is even and $P = P_{2, \ldots, 2, 1}$ if $n$ is odd. Let $(\underline{\underline{\kappa}}, \underline{\underline{w}})$ be the infinity type of $\Pi$ with $\underline{\underline{k}} = (\kappa_1, \ldots, \kappa_r)$. Thus $k = 1$ if $n$ is even and $k = r + 1$ if $n$ is odd. We define $\Pi_i \rho_i = D_{\kappa_i} \otimes |\underline{\underline{w}}/2$ for $1 \leq i \leq r$. When $n$ is odd, let $\Pi_{r+1} \rho_{r+1}$ be the algebraic character of $\mathbb{R}^\times$ such that $\Pi_{r+1} \rho_{r+1}(-1) = (-1)^r \underline{\underline{w}}/2(-1)$ and $|\underline{\underline{w}}/2| = |\underline{\underline{w}}/2|$. The generators $[\Pi_1 \rho_1]^\varepsilon, \ldots, [\Pi_{k} \rho_k]^\varepsilon$ are the ones fixed in (2.17) and (2.18) for $n = 1, 2$.

For an algebraic character $\chi$ of $\mathbb{R}^\times$, let $A_\chi : W(\Pi) \rightarrow W(\Pi \otimes \chi)$ be the isomorphism defined by

$$A_\chi(W)(g) = \chi(\det g) \cdot W(g), \quad g \in \text{GL}_n(\mathbb{R}).$$

Let $u \in \mathbb{Z}$ be the integer such that $|\chi| = |\chi|^u$ and we identify the representation spaces of $M_\mu$ and $M_{\mu-(u, \ldots, u)}$. Then the isomorphism

$$A_\chi \otimes \text{id} \otimes \text{id} : W(\Pi) \otimes \bigotimes_{i=1}^n (g_n, C/t_n, C)^* \otimes M_{\mu, \varepsilon} \longrightarrow W(\Pi \otimes \chi) \otimes \bigotimes_{i=1}^n (g_n, C/t_n, C)^* \otimes M_{\mu-(u, \ldots, u), \varepsilon}$$

induces an isomorphism

$$A_\chi^* : H^{bn}(g_n, K_n^\varepsilon; W(\Pi) \otimes M_{\mu, \varepsilon}) \longrightarrow H^{bn}(g_n, K_n^\varepsilon; W(\Pi \otimes \chi) \otimes M_{\mu-(u, \ldots, u), \varepsilon}).$$

By construction, $A_\chi^*$ sends $\varepsilon$-isotypic component to $\varepsilon \cdot \varepsilon(\chi)$-isotypic component. Moreover, we have the following lemma.

**Lemma 2.5.** Let $\chi$ be an algebraic character of $\mathbb{R}^\times$. We have

$$A_\chi^*([\Pi]^\varepsilon) = [\Pi \otimes \chi]^\varepsilon(\chi).$$

**Proof.** Let $B_\chi : \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})}(\otimes_{i=1}^k W(\Pi_i)) \rightarrow \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})}(\otimes_{i=1}^k W(\Pi_i \otimes \chi))$ be the isomorphism defined by

$$B_\chi(f) = \chi(\det g) \cdot f(g).$$

Similarly it induces an isomorphism

$$B_\chi^* : H^{bn}(g_n, K_n^\varepsilon; \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})}(\otimes_{i=1}^k W(\Pi_i))_{O_n(\mathbb{R})} \otimes M_{\mu, \varepsilon}) \longrightarrow H^{bn}(g_n, K_n^\varepsilon; \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})}(\otimes_{i=1}^k W(\Pi_i \otimes \chi))_{O_n(\mathbb{R})} \otimes M_{\mu-(u, \ldots, u), \varepsilon}).$$

By the construction of the isomorphism $I_{\otimes \iota=1}^\varepsilon W(\Pi_i)$ (cf. [BW00] III, § 3.4)], we have

$$(\otimes_{i=1}^k A_\chi^*) \circ I_{\otimes \iota=1}^\varepsilon W(\Pi_i) = I_{\otimes \iota=1}^\varepsilon W(\Pi_i \otimes \chi) \circ B_\chi^*.$$
The following lemma is one of the crucial ingredients in the proof of the period relation in Theorem 3.2.

**Lemma 2.6.** Assume (2.1) holds. We have
\[ \mathbb{W}^* \circ \left( \bigotimes_{i=1}^k \mathbb{W}(\Pi_i) \right)^{-1} \left( \bigotimes_{i=1}^k [\Pi_i \rho_i]^\varepsilon \right) = C \cdot [\Pi]^\varepsilon \]
for some constant \( C \in \mathbb{Q}^\times \).

**Proof.** Note that the isomorphisms \( I_{\bigotimes_{i=1}^k \mathbb{W}(\Pi_i)} \) and \( \mathbb{W}^* \) commute with parabolic induction in stages. Therefore, by the definition of \([\Pi]^\varepsilon, [\Pi_1 \rho_1]^\varepsilon, \ldots, [\Pi_k \rho_k]^\varepsilon\), to prove the assertion of the lemma, we are reduced to prove the following assertion: Let \( \Sigma \) be an irreducible admissible essentially tempered \((\mathfrak{g}_N, \mathfrak{o}_N(\mathbb{R}))\)-module such that
\[ \Sigma \cong \text{Ind}_{P(n_1, n_2)(\mathbb{R})}^{GL_N(\mathbb{R})} (\Sigma_1 \otimes \Sigma_2)_{\mathfrak{o}_N(\mathbb{R})} \]
for some irreducible admissible essentially tempered \((\mathfrak{g}_N, \mathfrak{o}_n(\mathbb{R}))\)-module \( \Sigma_i \) for \( i = 1, 2 \) with \( n_1 + n_2 = N \). Then we have
\[ \mathbb{W}^* \circ \left( I_{W(\Sigma_1) \otimes W(\Sigma_2)} \right)^{-1} \left( [\Sigma_1] \otimes [\Sigma_2] \right) = C \cdot \mathbb{W}^* \circ \left( I_{W(\Sigma_2) \otimes W(\Sigma_1)} \right)^{-1} \left( [\Sigma_2] \otimes [\Sigma_1] \right) \]
for some \( C \in \mathbb{Q}^\times \). Here \( \varepsilon' \in \{ \pm 1 \} \) if \( N \) is even and \( \varepsilon' = \varepsilon(\Sigma) \) if \( N \) is odd. (It is clear that we can even assume \( N \leq 4 \).) Write \( P = P(n_1, n_2) \) and \( Q = P(n_2, n_1) \). To prove (2.22), we begin with the intertwining operator
\[ M^Q : \text{Ind}_{P(n_1, n_2)(\mathbb{R})}^{GL_N(\mathbb{R})} (W(\Sigma_1) \otimes W(\Sigma_2)) \rightarrow \text{Ind}_{Q(n_1, n_2)(\mathbb{R})}^{GL_N(\mathbb{R})} (W(\Sigma_2) \otimes W(\Sigma_1)), \quad f \mapsto M^Q(f) \]
so that \( M^Q(f)(g) \) is the evaluation at \( \lambda = 0 \) of the meromorphic function defined by the integral
\[ \int_{U_Q(\mathbb{R})} f_A \left( \begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} \right) u g \ du \]
when \( \text{Re}(\lambda_1 - \lambda_2) \) is sufficiently large. Let \( (M^Q)^* \) be the corresponding isomorphism for the relative Lie algebra cohomology group in bottom degree \( b_N \). Consider the following diagram of equivariant isomorphisms:
\[ \begin{array}{c}
\text{Ind}_{P(n_1, n_2)(\mathbb{R})}^{GL_N(\mathbb{R})} (W(\Sigma_1) \otimes W(\Sigma_2))_{\mathcal{O}_N(\mathbb{R})} \\
\downarrow M^Q \\
\text{Ind}_{Q(n_1, n_2)(\mathbb{R})}^{GL_N(\mathbb{R})} (W(\Sigma_2) \otimes W(\Sigma_1))_{\mathcal{O}_N(\mathbb{R})}
\end{array} \]
As a special case of the result of Shahidi on local coefficients in [Sha85, Theorem 3.1], we have
\[ \mathbb{W} = \varepsilon \cdot \gamma(0, \Sigma_1 \times \Sigma_2, \psi_\varepsilon) \cdot \mathbb{W} \circ M^Q \]
for some \( \varepsilon \in \{ \pm 1 \} \). Here \( \gamma(s, \Sigma_1 \times \Sigma_2, \psi_\varepsilon) \) is the \( \gamma \)-factor of \( \Sigma_1 \times \Sigma_2 \) with respect to \( \psi_\varepsilon \). We refer to [Sha10, Remark 5.1.3] for the appearance of \( \varepsilon \). This induces an isomorphism at the level of relative Lie algebra cohomology groups in degree \( b_N \):
\[ \mathbb{W}^* = \varepsilon \cdot \gamma(0, \Sigma_1 \times \Sigma_2, \psi_\varepsilon) \cdot \mathbb{W}^* \circ (M^Q)^* \]
Therefore, (2.22) holds if and only if
\[ \gamma(0, \Sigma_1 \times \Sigma_2, \psi_\varepsilon) \cdot (M^Q)^* \circ \left( I_{W(\Sigma_1) \otimes W(\Sigma_2)} \right)^{-1} \left( [\Sigma_1] \otimes [\Sigma_2] \right) = C \cdot \left( I_{W(\Sigma_2) \otimes W(\Sigma_1)} \right)^{-1} \left( [\Sigma_2] \otimes [\Sigma_1] \right) \]
for some \( C \in \mathbb{Q}^\times \). The above equality was proved independently by Harder [HR20, Theorem 8.7] and Weselmann [HR20, §9.6.12] with \( \gamma(0, \Sigma_1 \times \Sigma_2, \psi_\varepsilon) \) replaced by \( (2\pi \sqrt{-1})^{-n_1 n_2/2} \). Finally, an immediate computation, together with the condition that \( \Sigma, \Sigma_1, \Sigma_2 \) are essentially tempered, shows that
\[ \gamma(0, \Sigma_1 \times \Sigma_2, \psi_\varepsilon) \in (2\pi \sqrt{-1})^{-n_1 n_2/2} \cdot \mathbb{Q}^\times. \]
This completes the proof. \qed
3. Period relations

Let $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_k$ be a cohomologically tame isobaric automorphic representation of $GL_n(\mathbb{A})$, where $\Pi_i$ is a cuspidal automorphic representation of $GL_{n_i}(\mathbb{A})$ for $1 \leq i \leq k$. Let $P$ be the standard parabolic subgroup of $GL_n$ of type $(n_1, \ldots, n_k)$. We denote by $\text{Ind}_{P(\mathbb{A})}^{GL_n(\mathbb{A})}$ the algebraic induction (without the normalizing factor $\rho_P$). Then we have

$$\Pi \cong \text{Ind}_{P(\mathbb{A})}^{GL_n(\mathbb{A})}((\Pi_1 \rho_1) \otimes \cdots \otimes \Pi_k \rho_k)_{0_n(\mathbb{R})}.$$

Recall here the subscript refers to the $O_n(\mathbb{R})$-finite part. The main result of this section is Theorem 3.2, which is an analogue of the result [GL21, Theorem 2.6] of Grobner and Lin. We prove a period relation between the Betti–Whittaker period of $\Pi$ and the Betti–Whittaker periods of $\Pi_1 \rho_1, \ldots, \Pi_k \rho_k$ together with critical values of some Rankin–Selberg $L$-functions. The idea of the proof is to give a cohomological interpretation of the Langlands–Shahidi method for the Rankin–Selberg $L$-functions. We begin with the following lemma on the Galois-equivariance property of the local Whittaker integral $W$ defined in [2.10] at finite places.

**Lemma 3.1.** Let $p$ be a prime number. For $\sigma \in \text{Aut}(\mathbb{C})$, we have the equality of $\sigma$-linear isomorphisms from $\text{Ind}_{P(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)}(\bigotimes_{i=1}^k W((\Pi_i \rho_i)_p))$ to $W(\sigma \Pi_p)$ as follows:

$$t_\sigma \circ W = \prod_{1 \leq i < j \leq k} \omega(\Pi_i \rho_j)_p \cdot (u_{\sigma,p})^{-n_i} \cdot W \circ \text{Ind}_{P(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)}(\bigotimes_{i=1}^k t_\sigma).$$

Here $t_\sigma$ is the $\sigma$-linear isomorphism defined in (2.10), and $u_{\sigma,p} \in \mathbb{Z}_p^\times$ is the unique element such that $\sigma(x) = \psi_p(u_{\sigma,p} x)$ for all $x \in \mathbb{Q}_p$.

**Proof.** We drop the subscript $p$ for brevity. Since the Whittaker integrals $W$ commute with parabolic induction in stages, we may assume $k = 2$. For $\sigma \in \text{Aut}(\mathbb{C})$ and $N \geq 1$, let $d_{\sigma,N} = \text{diag}(u_{\sigma,N+1}^{-1}, u_{\sigma,N+2}^{-1}, \ldots, 1) \in GL_N(\mathbb{Z}_p)$. Let $f \in \text{Ind}_{P(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)}(W((\Pi_1 \rho_1) \otimes W((\Pi_2 \rho_2))))$ and $g \in GL_n(\mathbb{Q}_p)$. By [Sha10, Theorem 3.4.7], the Whittaker integral $W(g, f)$ is a stable integral, that is, there exists a sufficiently small integer $m$ such that

$$W(g, f) = \int_{M_{n_2,n_1}(p^m \mathbb{Z}_p)} f \left( \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ n_1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & x \\ n_2 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ n_1 & 1 \end{pmatrix} \right) g \left( \begin{pmatrix} 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ n_2 & 1 \end{pmatrix} \right) \psi(x n_{2,1}) \, dx.$$ 

For $\sigma \in \text{Aut}(\mathbb{C})$, since $M_{n_2,n_1}(p^m \mathbb{Z}_p)$ is compact, we have

$$\begin{align*}
(t_\sigma \circ W)(g, f) &= \int_{M_{n_2,n_1}(p^m \mathbb{Z}_p)} \sigma \left( f \left( \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ n_1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1_n & 0 \\ n_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & x \\ n_1 & 1 \end{pmatrix} \right) \psi(u_{\sigma,x_{n_{2,1}}}) \, dx \\
&= \int_{M_{n_2,n_1}(p^m \mathbb{Z}_p)} \sigma \left( f \left( \begin{pmatrix} d_{\sigma,n_1} & 0 \\ 0 & u_{\sigma}^{-n_1} \cdot d_{\sigma,n_2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1_n & 1 \end{pmatrix} \begin{pmatrix} u_{\sigma,n_1} & d_{\sigma,n_2}^{-1} \\ 1_n & 0 \end{pmatrix} \right) \psi(u_{\sigma,x_{n_{2,1}}}) \, dx \\
&= \sigma(\Pi_2 \rho_2(u_{\sigma}^{-n_1}) \cdot \int_{M_{n_2,n_1}(p^m \mathbb{Z}_p)} \sigma \left( f \left( \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ n_2 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1_n & 0 \\ n_2 & 1 \end{pmatrix} \right) g \left( \begin{pmatrix} 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ n_2 & 1 \end{pmatrix} \right) \psi(y_{n_{2,1}}) \, dy \\
&= \sigma(\Pi_2 \rho_2(u_{\sigma}^{-n_1}) \cdot W \left( g \circ \text{Ind}_{P(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)}(t_\sigma \otimes t_\sigma)(f) \right). 
\end{align*}$$

Here in the third equality we make a change of variable from $u_{\sigma,n_2}^{-1} \cdot d_{\sigma,n_2}^{-1} \cdot d_{\sigma,n_1}$ to $y$. This completes the proof.
Theorem 3.2. Let $H = H_1 \oplus \cdots \oplus H_k$ be a cohomological tamely isobaric automorphic representation of $GL_n(\mathbb{A})$. Let $\varepsilon \in \{\pm 1\}$ if $n$ is even, and $\varepsilon = \varepsilon(H_\rho)$ if $n$ is odd. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\sigma \left( \frac{p(H_\rho, \varepsilon)}{\prod_{1 \leq i < j \leq k} G(\omega H_\rho)} \right) = \prod_{1 \leq i < j \leq k} G(\omega H_\rho) \cdot \prod_{1 \leq i \leq k} p(H_\rho, \varepsilon).$$

Proof. Recall the orbifold $\mathcal{S}_n$ in [2.12]. Let $\mathcal{S}_n$ be the Borel–Serre compactification of $\mathcal{S}_n$ (cf. [BS73] and [Roh96]) and $\partial_\mathcal{P}$ be the boundary component of $\mathcal{S}_n$ associated to a standard parabolic subgroup $P$. Let $M_\mu$ be an irreducible algebraic representation of $GL_n(\mathbb{Q})$ and $\mathcal{M}_\mu$ be the sheaf of $\mathbb{Q}$-vector spaces on $\mathcal{S}_n$ associated to $M_\mu$ (cf. §2.4.2). The closed embedding $\partial_\mathcal{P} \mathcal{S}_n \hookrightarrow \mathcal{S}_n$ defines the $\pi_0(GL_n(\mathbb{R})) \times GL_n(\mathbb{A})$-equivariant restriction map

$$\tau_\mathcal{P}: H^b_n(S_n, M_\mu) \longrightarrow H^b_n(\partial_\mathcal{P} \mathcal{S}_n, M_\mu).$$

We denote by $\tau_{\mathcal{P}, \mathcal{C}}$ the corresponding map at the transcendental level (cf. 2.13). Then we have

$$\sigma^* \circ \tau_{\mathcal{P}, \mathcal{C}} = \tau_{\mathcal{P}, \mathcal{C}} \circ \sigma^*$$

for $\sigma \in \text{Aut}(\mathbb{C})$. For the boundary cohomology, we have the following canonical $\pi_0(GL_n(\mathbb{R})) \times GL_n(\mathbb{A})$-equivariant isomorphism (cf. [HR20] Proposition 4.3):

$$I_{\partial_\mathcal{P}}: H^b_n(\partial_\mathcal{P} \mathcal{S}_n, M_\mu) \longrightarrow \bigoplus_{\omega \in W_n^P} \text{aInd}_{P(\mathbb{A})}^{GL_n(\mathbb{A})} \left[ \left( \bigotimes_{i=1}^k H^q_i(S_{\mu_i}, \mathcal{M}_{\mu_{\mu_i}}) \right) \right].$$

Here $(\mu_{w_1}, \cdots, \mu_{w_k}) = w \cdot (\mu + \rho_n) - \rho_n$ for $w \in W_n^P$. We denote by $I_{\partial_\mathcal{P}, \mathcal{C}}$ the corresponding map at the transcendental level, and it is clear that

$$\text{aInd}_{P(\mathbb{A})}^{GL_n(\mathbb{A})} \circ I_{\partial_\mathcal{P}, \mathcal{C}} = I_{\partial_\mathcal{P}, \mathcal{C}} \circ \sigma^*$$

for $\sigma \in \text{Aut}(\mathbb{C})$.

We now assume $P = P_{(n_1, \cdots, n_k)}$ and $M_\mu$ is the unique algebraic representation such that $H_\xi$ is cohomological with coefficients in $M_{\mu, \mathcal{C}}$. For $1 \leq i \leq k$, let $M_{\mu_i}$ be the irreducible algebraic representation of $GL_n(\mathbb{Q})$ such that $(H_\rho_i)_{\mathcal{C}}$ is cohomological with coefficients in $M_{\mu_i, \mathcal{C}}$. Let $\omega_0 \in W_n^P$ be the unique Kostant representative such that $(\mu_{w_0, 1}, \cdots, \mu_{w_0, k}) = (1, \cdots, 1)$. We denote by $H^b_n(S_n, M_{\mu, \mathcal{C}})| \varepsilon \times H_f$ the image of $H^b_n(\mathcal{S}_n, K_n^0; H \otimes \mathcal{M}_\mu)| \varepsilon$ under $\Psi_\mathcal{P}$ in [2.14]. We define the $GL_n(\mathbb{A})$-equivariant homomorphism

$$\tau_{\Psi_\mathcal{P}}: H^b_n(S_n, M_{\mu, \mathcal{C}})| \varepsilon \times H_f \longrightarrow \bigoplus_{\omega_0 \in W_n^P} \text{aInd}_{P(\mathbb{A})}^{GL_n(\mathbb{A})} \left[ \left( \bigotimes_{i=1}^k H^q_i(S_{\mu_i}, \mathcal{M}_{\mu_i}) \right) \right]$$

by restrict the homomorphism $I_{\partial_\mathcal{P}} \circ \tau_\mathcal{P}$ to $H^b_n(S_n, M_{\mu, \mathcal{C}})| \varepsilon \times H_f$, followed by the projection to the summand index by $w_0$. Then $\tau_{\Psi_\mathcal{P}}$ is injective with image equal to $\text{aInd}_{P(\mathbb{A})}^{GL_n(\mathbb{A})} \left( \bigotimes_{i=1}^k H^b_{\mu_i}(S_{\mu_i}, M_{\mu_i, \mathcal{C}})| \varepsilon \times (H_{\rho_i})_f \right)$. Indeed, given $\omega \in H^b_n(S_n, M_{\mu, \mathcal{C}})| \varepsilon \times H_f$, we have

$$(\Psi_\mathcal{P})^{-1}(\omega) = \sum_{\alpha \in I_\rho, \beta \in J} E(h_{\alpha, \beta}) \otimes X^*_\alpha \otimes v_\beta$$

for some index sets $I, J$, $h_{\alpha, \beta} \in I_{\mu, \mathcal{C}}(\otimes_{i=1}^k I_i)$, $X_{\alpha}^* \in \bigwedge^H_{\mathcal{C}}(\mathcal{S}_n, \mathcal{M}_{\mu, \mathcal{C}})^*$, and $v_\beta \in \mathcal{M}_{\mu, \mathcal{C}}$. By [Sch83] Satz 1.10 (see also [Har] §9.2.1) and the formula for constant term (2.5), the class $\tau_{\mathcal{P}, \mathcal{C}}(\omega)$ is then represented by

$$\sum_{w \in W_n^P \cap (W_n^P)^{-1}} \sum_{\omega \in W_n^P \cap (W_n^P)^{-1}} M_{\mu_w}(h_{\alpha, \beta}) \otimes X^*_{\alpha} \otimes v_\beta.$$

For $w \in W_n^P \cap (W_n^P)^{-1}$ with $wM_{\mu_w}w^{-1} = M_{\mu}$, there exists a unique $w' \in W_n^P$ such that $(H_{\rho_{\nu}(w)} \rho_{\nu})_\mu$ is cohomological with coefficients in $M_{\mu_{\mu_{\nu}}, \mathcal{C}}$ for $1 \leq i \leq k$. Here $\tau_{w}$ is the permutation defined in [2.6]. It is
clear that the association $w \mapsto w'$ is injective and we have $1 \mapsto w_0$ by definition. If $w \mapsto w'$, then we have the $\pi_0(\GL_n(\mathbb{R})) \times \GL_n(\mathbb{A}_f)$-equivariant isomorphism (cf. (2.20))

$$I_{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}} : H^{b_n}(g_n, K_n^\circ; I_P^{\GL_n}(\otimes_{i=1}^k \Pi_{\tau_{w(i)}}) \otimes M_{\mu, \mathbb{C}})$$

$$\xrightarrow{\text{aInd}_{\GL_n(\mathbb{A}_f)}^{\GL_n(\mathbb{A}_f)}} \left( \bigotimes_{i=1}^k H^{b_n}(g_n, K_n^\circ; \Pi_{\tau_{w(i)}}\rho_i \otimes M_{\mu, \mathbb{C}}) \right)$$

by [BW00] III, Theorem 3.3. We denote by $I_{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}}$ the isomorphism obtained by restriction to the $\varepsilon$-isotypic components on both sides. By (2.3) and the constructions of $I_{\varepsilon, \mathbb{C}}$ and $I_{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}}$, we see that the image of $I_{\varepsilon, \mathbb{C}} \circ \tau_{\mathbb{C}}(\omega)$ in the summand indexed by $w'$ is equal to

$$\text{aInd}_{\GL_n(\mathbb{A}_f)}^{\GL_n(\mathbb{A}_f)}(\bigotimes_{i=1}^k \Psi_{\Pi_{\tau_{w(i)}}(\rho_i)} \circ I_{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}}) \left( \sum_{\alpha \in I, \beta \in J} M_\alpha(\pi_{\alpha, \beta}) \otimes X_{\alpha} \otimes \nu_{\beta} \right).$$

On the other hand, if $w' \in W^P$ is not in the image of the above association, then $I_{\varepsilon, \mathbb{C}} \circ \tau_{\mathbb{C}}(\omega)$ has zero contribution to the summand indexed by $w'$. For our purpose here, we can consider the projection of $I_{\varepsilon, \mathbb{C}} \circ \tau_{\mathbb{C}}$ to an arbitrary summand indexed by some $w' \in W^P$ which lies in the image of the association $w \mapsto w'$. We choose $w' = w_0$ to ease the use of notation. In conclusion, we have the following commutative diagram of $\GL_n(\mathbb{A}_f)$-equivariant isomorphisms:

$$H^{b_n}(g_n, K_n^\circ; \Pi \otimes M_{\mu, \mathbb{C}})[\varepsilon] \xrightarrow{\Psi_{\Pi}} H^{b_n}(S_n, M_{\mu, \mathbb{C}})[\varepsilon \times \Pi_f]$$

$$\xrightarrow{(E^*)^{-1}} H^{b_n}(g_n, K_n^\circ; I_P^{\GL_n}(\otimes_{i=1}^k \Pi_{\tau_{w(i)}}) \otimes M_{\mu, \mathbb{C}})[\varepsilon]$$

$$\xrightarrow{\text{aInd}_{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}}^{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}}} \text{aInd}_{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}}^{\otimes_{i=1}^k \Pi_{\tau_{w(i)}}},$$

here in the diagram *Ind is the abbreviation for *Ind$^P_{\Pi(\mathbb{A}_f)}$, and $E^*$ is the isomorphism induced from the automorphic realization (2.3) of $\Pi$.

To prove the period relation, let $\sigma \in \text{Aut}(\mathbb{C})$ and consider the following diagram (non-commutative):

$$C \cdot \prod_{1 \leq i < j \leq k} G(\omega_{\Pi_i \rho_i})^{-n_i} L(\xi)(1, \Pi_i \times \Pi_j)^{-1} \cdot \prod_{i=1}^k p(\Pi_i \rho_i, \varepsilon)^{-1}$$

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and
\[ C \cdot \prod_{1 \leq i < j \leq k} G(\omega_{II_i\rho_i})^{-n_i} L^{(\times)}(1, \sigma(I_i\rho_i)\rho_i^{-1} \times \sigma(I_j^\vee \rho_j)\rho_j^{-1})^{-1} \cdot \prod_{i=1}^k p(\sigma(I_i\rho_i), \varepsilon)^{-1}, \]
respectively. We prove this later assertion by consider the remaining five faces of the diagram as follows:

**Right face:** The right face of the diagram is commutative by (3.1) and (3.2).

**Down face:** By definition and normalization of the Betti–Whittaker periods of \( I_i\rho_i \) and \( \sigma(I_i\rho_i) \) for \( 1 \leq i \leq k \), the down face of the diagram is commutative after we multiply \( a\text{Ind} \left( \bigotimes_{i=1}^k \left( \Psi_{I_i\rho_i} \circ \Phi_{I_i\rho_i}^\sigma \right) \right) \) and \( a\text{Ind} \left( \bigotimes_{i=1}^k \left( \Psi_{I_i\rho_i} \circ \Phi_{I_i\rho_i}^\sigma \right) \right) \) by \( \prod_{i=1}^k p(I_i\rho_i, \varepsilon)^{-1} \) and \( \prod_{i=1}^k p(I_i\rho_i, \varepsilon)^{-1} \), respectively.

**Left face:** We abbreviate \( a\text{Ind}_p \) for \( a\text{Ind}^{GL_n(Q_p)} \) for each prime number \( p \). When \( I_p \) is unramified, let \( f_{I_p}^o \in a\text{Ind}_p(\bigotimes_{i=1}^k W((I_i\rho_i)_p)) \) be the \( GL_n(Z_p) \)-invariant section normalized so that \( f_{I_p}^o(1_n) = \bigotimes_{i=1}^k W((I_i\rho_i)_p) \) (see the notation in §2.3). Then we have
\[ \mathbb{W}(1_n, f_{I_p}^o) = \prod_{1 \leq i < j \leq k} L(1, I_i \times I_j^\vee). \]
by [Sha10 Proposition 7.1.4]. Note that \( a\text{Ind} \left( \bigotimes_{i=1}^k W((I_i\rho_i)_f) \right) \) is generated by sections of the form
\[ f = \bigotimes_p f_p \]
such that \( f_p = f_{I_p}^o \) for almost all \( p \). We fix such section \( f \) and consider the effect of the homomorphisms in the left face of the diagram. Let \( g \in GL_n(A_f) \). There is a finite set \( S \) of finite places of \( Q \) such that \( I_p \) is unramified, \( f_p = f_{I_p}^o \), and \( g_p \in GL_n(Z_p) \) for all \( p \notin S \). Then we have
\[ \mathbb{W}(g, f) = \prod_{1 \leq i < j \leq k} L^{S,\times}(1, I_i \times I_j^\vee)^{-1} \cdot \prod_{p \notin S} \mathbb{W}(g_p, f_p). \]
It then follows from (1.1) and Lemma 3.1 that
\[ t_\sigma \circ \mathbb{W}(g, f) = \sigma(L^{S,\times}(1, I_i \times I_j^\vee)^{-1}) \cdot \prod_{1 \leq i < j \leq k} \frac{G(\omega_{II_i\rho_i})^{-n_i}}{\sigma(G(\omega_{II_i\rho_i}))^{n_i}} \cdot \prod_{p \notin S} \mathbb{W}(g_p, a\text{Ind}_p \left( \bigotimes_{i=1}^k t_\sigma \right)(f_p)). \]
On the other hand, we have
\[ a\text{Ind} \left( \bigotimes_{i=1}^k t_\sigma \right)(f) = \bigotimes_p a\text{Ind}_p \left( \bigotimes_{i=1}^k t_\sigma \right)(f_p) \]
and \( a\text{Ind}_p \left( \bigotimes_{i=1}^k t_\sigma \right)(f_p) = f_{I_p}^o \) for \( p \notin S \) by definition. Therefore, similarly we have
\[ \mathbb{W}(g, a\text{Ind} \left( \bigotimes_{i=1}^k t_\sigma \right)(f)) = \prod_{1 \leq i < j \leq k} L^{S,\times}(1, \sigma(I_i\rho_i)\rho_i^{-1} \times \sigma(I_j^\vee \rho_j)\rho_j^{-1})^{-1} \cdot \prod_{p \notin S} \mathbb{W}(g_p, a\text{Ind}_p \left( \bigotimes_{i=1}^k t_\sigma \right)(f_p)). \]
Note that for all prime numbers \( p \), we have (cf. [Rag10 Proposition 3.17] and [Mor14 Corollary 5.5])
\[ \sigma(L(1, I_i \times I_j^\vee, p)) = L(1, \sigma(I_i\rho_i)\rho_i^{-1} \times \sigma(I_j^\vee \rho_j)\rho_j^{-1}). \]
We thus conclude that
\[ t_\sigma \circ \left( \prod_{1 \leq i < j \leq k} G(\omega_{II_i\rho_i})^{-n_i} L^{(\times)}(1, I_i \times I_j^\vee) \right) \]
\[ = \left( \prod_{1 \leq i < j \leq k} G(\omega_{II_i\rho_i})^{-n_i} L^{(\times)}(1, \sigma(I_i\rho_i)\rho_i^{-1} \times \sigma(I_j^\vee \rho_j)\rho_j^{-1}) \right) \circ a\text{Ind}_p \left( \bigotimes_{i=1}^k t_\sigma \right). \]

**Front and back faces:** We show that there exists \( C \in Q^\times \) such that the front and back faces of the diagram
are commutative after we multiply $\Psi_H \circ \Phi_{II}$ and $\Psi_{II} \circ \Phi_{\varepsilon II}$ by $C$. For the back face of the diagram, by the commutative diagram in the previous paragraph, it suffices to consider the following diagram:

$$
\begin{array}{ccc}
W(II) & \xrightarrow{\Phi_{II}} & H^{bn}(g_n, K_n^c; II \otimes M_{\mu, C})[\varepsilon] \\
\downarrow W & & \downarrow (W*)^{-1} \\
H^{bn}(g_n, K_n^c; \bigotimes_{i=1}^{k} W(\Pi_i) \otimes M_{\mu, C})[\varepsilon] & \xrightarrow{\text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] & \xrightarrow{\text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] \\
\end{array}
$$

By the definition of $\Phi_{II}$ and $\Phi_{\varepsilon II}$, for $1 \leq i \leq k$, we can further separate the diagram into two parts. First consider the following diagram:

$$
\begin{array}{ccc}
H^{bn}(g_n, K_n^c; W(II) \otimes M_{\mu, C})[\varepsilon] & \xrightarrow{\text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] & H^{bn}(g_n, K_n^c; II \otimes M_{\mu, C})[\varepsilon] \\
H^{bn}(g_n, K_n^c; \text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] & \xrightarrow{\text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] & \xrightarrow{\text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] \\
\end{array}
$$

Here the horizontal isomorphisms are induced by (2.8), and $W^*$ is induced by the equivariant isomorphism

$$
W : \text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] \xrightarrow{(W*)^{-1}} \bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon]
$$

defined in §223. It is clear that the lower square of the diagram is commutative. By (2.9) (evaluating at $\lambda = 0$), the upper square of the diagram is also commutative. We are reduced to consider the following diagram:

$$
\begin{array}{ccc}
W(II) & \xrightarrow{\Phi_{II}} & H^{bn}(g_n, K_n^c; W(II) \otimes M_{\mu, C})[\varepsilon] \\
\downarrow W & & \downarrow (W*)^{-1} \\
H^{bn}(g_n, K_n^c; \bigotimes_{i=1}^{k} W(\Pi_i) \otimes M_{\mu, C})[\varepsilon] & \xrightarrow{\text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] & \xrightarrow{\text{Ind}_{\chi, 1}^{GL, \chi}(\bigotimes_{i=1}^{k} W(\Pi_i)) \otimes M_{\mu, C})[\varepsilon] \\
\end{array}
$$

Here the upper and lower horizontal isomorphisms are defined respectively by

$$
W \xrightarrow{\varepsilon} W, \quad f \mapsto (\bigotimes_{i=1}^{k} (\Pi_i)_{\chi})[\varepsilon] \otimes f.
$$

We then deduce from Lemma 2.7 that there exists $C \in \mathbb{Q}^\times$ such that the above diagram is commutative if we multiply the upper horizontal isomorphism by $C$. Similarly, the same constant $C$ also makes the front face of diagram 3.3 commutative.

Based on the above discussion, we conclude that the up face of diagram 3.3 is commutative after multiply $\Psi_H \circ \Phi_{II}$ and $\Psi_{II} \circ \Phi_{\varepsilon II}$ by the prescribed complex numbers. This completes the proof.

**Remark 3.3.** The factor $\prod_{1 \leq i < j \leq k} G(\omega_{\Pi_i \Pi_j})^{n_i}$ in the period relation is unfortunately missing in the statement of the analogous result [G.21] Theorem 2.6. This is because the equality in [G.21] p.894] (between (2.8) and (2.9) in that page) does not hold unless we multiply this factor, which is a consequence of Lemma 3.1.

The following corollary is actually equivalent to Theorem 3.2. We state it in a form which is more convenient to use in the proof of Theorem 3.2.
Corollary 3.4. Let \( \Sigma \) and \( \Pi \) be cohomological tamely isobaric automorphic representations of \( \text{GL}_n(\mathbb{A}) \) and \( \text{GL}_m(\mathbb{A}) \), respectively. Let \( \xi \), \( \eta \) be infinity types \((\xi, \eta)\) and \((\xi', \eta')\), respectively. We assume \( n \) is even and
\[
\begin{cases}
    w + u = 0 & \text{if } n' \text{ is even}, \\
    w + u = 1 & \text{if } n' \text{ is odd}.
\end{cases}
\]
Then we have the following period relations: Let \( \sigma \in \text{Aut}(\mathbb{C}) \). If \( n' \) is even, then
\[
\sigma \left( \frac{p(\xi \otimes \eta \otimes \Pi \otimes \Pi', \pm)}{G(\xi \otimes \eta \otimes \Pi \otimes \Pi', \pm)} \right) = \frac{p(\xi \otimes \eta \otimes \Pi \otimes \Pi', \pm)}{G(\xi \otimes \eta \otimes \Pi \otimes \Pi', \pm)}
\]
If \( n' \) is odd, then
\[
\sigma \left( \frac{p((\xi \otimes \eta) \otimes \Pi \otimes \Pi', \pm)}{G(\xi \otimes \eta \otimes \Pi \otimes \Pi', \pm)} \right) = \frac{p((\xi \otimes \eta) \otimes \Pi \otimes \Pi', \pm)}{G(\xi \otimes \eta \otimes \Pi \otimes \Pi', \pm)}
\]
Proof. Put \( \delta \in \{0, 1\} \) with \( \delta = n' \pmod{2} \). Let \( \varepsilon \in \{\pm 1\} \) if \( n' \) is even, and \( \varepsilon = (-1)^{n/2} \varepsilon(\Pi_0) \) if \( n' \) is odd. The assumptions on \( n, w, u \) imply that the isobaric sum \((\xi \otimes \eta) \otimes \Pi \otimes \Pi' \) is cohomological and tamely isobaric.

By Theorem 2.2 we have
\[
p((\xi \otimes \eta) \otimes \Pi \otimes \Pi', \varepsilon) \sim G(\xi \otimes \eta \otimes \Pi \otimes \Pi', \varepsilon)
\]
By Theorem 2.3 we have
\[
p((\xi \otimes \eta) \otimes \Pi \otimes \Pi', \varepsilon) \sim p((\xi \otimes \eta) \otimes \Pi \otimes \Pi, \varepsilon)
\]
Recall our convention (2.10). This completes the proof. 

4. PROOF OF MAIN RESULTS

The aim of this section is to prove our main results Theorems 1.2 and 1.4. In §4.5, we show that Conjecture 1.3 is compatible with Deligne’s conjecture.

4.1. Rankin–Selberg L-functions. Let \( \Sigma \) and \( \Pi \) be isobaric automorphic representations of \( \text{GL}_n(\mathbb{A}) \) and \( \text{GL}_m(\mathbb{A}) \), respectively. Denote by \( L(s, \Sigma \times \Pi) \) the Rankin–Selberg L-function for \( \Sigma \times \Pi \). Assume \( \Sigma \) and \( \Pi \) are algebraic. A critical point for \( L(s, \Sigma \times \Pi) \) is a half-integer \( m_0 \in \mathbb{Z} + \frac{n + m}{2} \) such that the archimedean local factors \( L(s, \Sigma \times \Pi) \) and \( L(1 - s, \Sigma \times \Pi) \) are holomorphic at \( s = m_0 \). For instance, if \( \Sigma \) and \( \Pi \) are cohomological and tamely isobaric with infinity types \((\xi, \eta)\) and \((\xi', \eta')\), respectively, and \( n \) is even, then the set of critical points are given by (cf. [HR20, Proposition 7.7])
\[
\left\{ m_0 \in \mathbb{Z} + \frac{n}{2} \left| \frac{2 - d(\xi, \xi') - w - u}{2} \leq m_0 \leq \frac{d(\xi, \xi') - w - u}{2} \right. \right\},
\]
where
\[
d(\xi, \xi') = \begin{cases}
    \min\{|\kappa_i - \ell_j|\} & \text{if } n' \text{ is even}, \\
    \min\{|\kappa_i - \ell_j|, |\kappa_i - 1|\} & \text{if } n' \text{ is odd}.
\end{cases}
\]
In this case, \( L(s, \Sigma \times \Pi) \) admits critical points if and only if \( d(\xi, \xi') \geq 1 \). Also, it admits a central critical point if and only if \( d(\xi, \xi') \geq 1 \) and \( w + u = n + n' + 1 \pmod{2} \). Moreover, when it admits critical points, the cuspidal summands of \( \Sigma \) and \( \Pi \) are non-isomorphic, therefore \( L(s, \Sigma \times \Pi) \) are holomorphic (and non-vanishing) at (non-central) critical points.
4.2. Some algebraicity results and an existence lemma. In this section, we recall some algebraicity results for Rankin–Selberg $L$-functions for $\text{GL}_n \times \text{GL}_{n-1}$, which is a key ingredient in our proof of main results. Then we prove a result on the existence of cohomological cuspidal automorphic representations, which will also be used in the proof. We begin with the following result of Raghuram [Rag10], where the algebraicity is expressed in terms of Betti–Whittaker periods.

**Theorem 4.1** (Raghuram). Let $\Sigma$ and $\Pi$ be cohomological tamely isobaric automorphic representations of $\text{GL}_n(\mathbb{A})$ and $\text{GL}_{n-1}(\mathbb{A})$ with infinity types $(\xi; w)$ and $(\xi'; u)$, respectively. Assume $\Sigma$ is cuspidal and $(\Sigma, \Pi)$ is balanced, that is,

$$\kappa_1 > \ell_1 > \kappa_2 > \ell_2 > \cdots.$$

Then there exist non-zero complex numbers $p(m, \Sigma \times \Pi)$ defined for critical points $m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ for $L(s, \Sigma \times \Pi)$ such that

$$\sigma \left( \frac{L^{(\xi)}(m + \frac{1}{2}, \Sigma \times \Pi)}{p(m, \Sigma \times \Pi) \cdot G(\omega \Pi) \cdot p(\Sigma, \varepsilon_m) \cdot p(\Pi, \varepsilon_m)} \right) = \frac{L^{(\xi)}(m + \frac{1}{2}, \sigma \Sigma \times \sigma \Pi)}{p(m, \Sigma \times \Pi) \cdot G(\omega \Pi) \cdot p(\sigma \Sigma, \varepsilon_m) \cdot p(\sigma \Pi, \varepsilon_m)}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Here $\varepsilon_m = \varepsilon(\Sigma)$ if $n$ is odd, $\varepsilon'_m = \varepsilon(\Pi)$ if $n$ is even, and $\varepsilon_m \varepsilon'_m = (-1)^{n+m}$.

**Remark 4.2.** The theorem was proved subject to a non-vanishing hypothesis [Rag10, Hypothesis 3.10] which was proved by Sun in [Sim17]. Also the theorem was stated for cuspidal $\Sigma$ and $\Pi$ in [BR17]. For tamely isobaric $\Pi$, the proof goes without change, except we consider the image of $H^{b_n-1}(\mathbb{C}, \mathbb{C}, n-1; \Pi \otimes M_{\mu, c})$ instead of the cohomology with compact support when $\Pi$ is cuspidal. We refer to [LLS21, §6] for a general base field (see also [GH16], [Gro18] when the base field is a CM-field).

**Remark 4.3.** As the notation suggests, the number $p(m, \Sigma \times \Pi)$ depends only on the archimedean components $\Sigma$ and $\Pi$. More precisely, it depends on the choice of generators $[\Sigma]\chi^m$ and $[\Pi]\chi'^m$. In [LLS21], Li, Liu, and Sun prove the following archimedean period relation: For critical points $m_1 + \frac{1}{2}, m_2 + \frac{1}{2}$, we have

$$\frac{p(m_1, \Sigma \times \Pi)}{p(m_2, \Sigma \times \Pi)} \in (2\pi \sqrt{-1})^{(m_1-m_2)n(n-1)/2} \cdot \mathbb{Q}^\times.$$

When the critical points are non-central, the above period relation also follows immediately from the result of Harder and Raghuram [HR20, Theorem 7.21].

In [RT11], the above theorem was further refined by Raghuram and Tanabe when $n = 2$. We recall it in the first assertion of the following theorem. The second assertion is a refinement of the main result in [BR17] for $\text{GL}_2$. It is a period relation between product of Betti–Whittaker periods and Petersson norm of normalized newforms on $\text{GL}_2$. We refer to [Che22a, Lemma 4.1] for a proof.

**Theorem 4.4.** Let $\Pi$ be a cohomological cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ with infinity type $(\kappa; w)$.

1. (Raghuram–Tanabe) Let $\chi$ be a finite order Hecke character of $\mathbb{A}^\times$ and $m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ a critical point for $L(s, \Pi \otimes \chi)$. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\sigma \left( \frac{L^{(\xi)}(m + \frac{1}{2}, \Pi \otimes \chi)}{(2\pi \sqrt{-1})^{m+(\kappa+\omega)/2} \cdot G(\chi) \cdot p(\Pi, (-1)^m \chi \zeta(-1))} \right) = \frac{L^{(\xi)}(m + \frac{1}{2}, \sigma \Pi \otimes \sigma \chi)}{(2\pi \sqrt{-1})^{m+(\kappa+\omega)/2} \cdot G(\chi) \cdot p(\sigma \Pi, (-1)^m \chi \zeta(-1))}.$$

2. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\sigma \left( \frac{p(\Pi, +) \cdot p(\Pi, -)}{(2\pi \sqrt{-1})^{\omega} \cdot G(\Pi) \cdot |f_H|} \right) = \frac{p(\sigma \Pi, +) \cdot p(\sigma \Pi, -)}{(2\pi \sqrt{-1})^{\omega} \cdot G(\sigma \Pi) \cdot |f_H|^\omega}.$$

Here $f_H$ is the normalized newform of $\Pi$ and $|f_H|$ is the Petersson norm of $f_H$ defined by

$$|f_H| = \int_{\mathbb{A}^\times} \int_{\text{GL}_2(\mathbb{Q})\backslash \text{GL}_2(\mathbb{A})} |f_H(g)|^2 \det g^{-w} \, dg^\text{Tam},$$

where $dg^\text{Tam}$ is the Tamagawa measure on $\mathbb{A}^\times \backslash \text{GL}_2(\mathbb{A})$.  

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Remark 4.5. We also have the pioneering result due to Shimura in \[\text{Shi77}\], where the Betti–Whittaker periods are replaced by some rightmost critical values. We recall the comparison between the Betti–Whittaker periods of \(\Pi\) and the Deligne’s periods of the rank 2 motive associated to \(\Pi\) in \([\text{H20}]\).

The following lemma is on the existence of cohomological cuspidal automorphic representations of \(\text{GL}_n(\mathbb{A})\). When \(n\) is even, the existence was established by Ramakrishnan and Wang in \([\text{RW04}]\) Theorem A.1, using automorphic induction from Hecke characters over CM-fields. We give a proof for the existence when \(n\) is odd.

**Lemma 4.6.** Let \((\kappa; w)\) be the infinity type of an irreducible admissible, cohomological, essentially tempered \((\mathfrak{g}_n, \text{O}_n(\mathbb{R}))\)-module such that \(\kappa_1\) is 4-regular and \(\min\{\kappa_i\} \geq 5\) if \(n\) is odd. Then there exists a cohomological cuspidal automorphic representation of \(\text{GL}_n(\mathbb{A})\) with infinity type \((\kappa; w)\).

**Proof.** Assume \(n = 2r + 1\) is odd. Recall the parity condition \((\text{2.4})\) in this case. We prove the existence based on Arthur’s multiplicity formula \([\text{Art13}]\) and the existence theorem of Shin \([\text{Shi12}]\) applied to \(\text{Sp}_{2r}\). Let \(\pi\) and \(\tau\) be irreducible admissible representations of \(\text{Sp}_{2r}(\mathbb{Q}_2)\) and \(\text{Sp}_{2r}(\mathbb{Q}_3)\) satisfying the following conditions:

- The Langlands parameter \(\phi_{\pi} : W_{\mathbb{Q}_2} \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{2r+1}(\mathbb{C})\) of \(\pi\) factors through the Weil group \(W_{\mathbb{Q}_2}\) and is irreducible as a \((2r + 1)\)-dimensional representation of \(W_{\mathbb{Q}_2}\).
- \(\pi\) is tempered and unramified.

For the existence of \(\pi\), we refer to the construction of Bushnell and Henniart in \([\text{BH11}]\) Corollary 3.3. Let \(\lambda\) be a dominant integral weight for \(\text{Sp}_{2r}\), with respect to its standard maximal torus and Borel subgroup, given by \(\lambda = (\lambda_1, \cdots, \lambda_r)\) with \(\lambda_i = \frac{\text{dim} \mathfrak{g}}{\text{dim} \mathfrak{b}} + i - (r + 1)\). Then our assumption on \(\kappa\) is equivalent to the condition that \(\lambda\) is regular, that is, \(\lambda_1 > \cdots > \lambda_r > 0\). Denote by \(\xi_{\lambda}\) the irreducible algebraic representation of \(\text{Sp}_{2r}(\mathbb{C})\) with highest weight \(\lambda\). By the existence theorem \([\text{Shi12}]\) Theorem 5.7] with \(S = \{2, 3\}\) and \(\xi = \xi_{\lambda}\), there exists a cuspidal automorphic representation \(\Pi_{\xi}\) of \(\text{Sp}_{2r}(\mathbb{A})\) satisfying the following conditions:

- \(\Pi_2 = \pi\).
- \(\Pi_3\) is induced from the Borel subgroup with inducing data equivalent to that of \(\tau\) up to twist by unramified unitary characters of \((\mathbb{Q}_n)^\ast\).
- \(\Pi_{\xi}\) is cohomological with coefficients in \(\xi_{\lambda}\).

Since \(\lambda\) is regular, the set of irreducible admissible, cohomological representations of \(\text{Sp}_{2r}(\mathbb{R})\) with coefficients in \(\xi_{\lambda}\) is a \(L\)-packet for \(\text{Sp}_{2r}(\mathbb{R})\) consisting of discrete series representations. Therefore, the local Langlands transfer of \(\Pi_{\xi}\) to \(\text{GL}_{2r+1}(\mathbb{R})\) must be cohomological and tempered, as verified in \([\text{RS13}]\). More explicitly, the local transfer has infinity type \((\kappa; 0)\) by \([\text{RS18}]\) Proposition 5.2. The global Arthur parameter of \(\Pi\) is of the form

\[
\Psi = \Psi_1[d_1] \boxplus \cdots \boxplus \Psi_k[d_k]
\]

for some self-dual cuspidal automorphic representation \(\Psi_i\) of \(\text{GL}_{n_i}(\mathbb{A})\) and some \(d_i \geq 1\) for \(1 \leq i \leq k\) satisfying certain conditions. By Arthur’s multiplicity formula \([\text{Art13}]\) Theorems 1.5.1 and 1.5.2], it suffices to show that \(d_1 = \cdots = d_k = 1\) and \(k = 1\). Indeed, this would imply that \(\Psi_1\) is the global functorial lift of \(\Pi\) to \(\text{GL}_{2r+1}(\mathbb{A})\) whose archimedean component is cohomological with infinity type \((\kappa; 0)\). Thus \(\Psi_1 \otimes |\cdot|^{w/2}|_{\mathbb{A}}\) is cohomological with infinity type \((\kappa; w)\). Since \(\Pi_3\) is tempered and unramified, we must have \(d_1 = \cdots = d_k = 1\) by \([\text{Moe09}]\) Proposition 4.4]. Therefore, for each place \(v\) of \(\mathbb{Q}\), the localization of \(\Psi\) at \(v\) is equal to the local \(L\)-parameter of \(\Pi\) at \(v\). In particular, we have \(\phi_{\Psi_1} = \phi_{\Psi_{1,2}} \oplus \cdots \oplus \phi_{\Psi_{k,2}}\), where \(\Psi_{1,2}\) refers to the local component of \(\Psi_1\) at \(2\). We thus conclude from the irreducibility of \(\phi_\pi\) that \(k = 1\). This completes the proof. \(\square\)

### 4.3. Proof of Theorem 1.2

Let \(\Sigma, \Sigma'\) (resp. \(\Pi, \Pi'\)) be algebraic automorphic representations of \(\text{GL}_n(\mathbb{A})\) (resp. \(\text{GL}_n(\mathbb{A})\)) such that

\[
\Sigma_{\infty} = \Sigma'_{\infty}, \quad \Pi_{\infty} = \Pi'_{\infty}.
\]

Let \((\kappa; w)\) and \((\lambda; u)\) be the infinity types of \(\Sigma\) and \(\Pi\), respectively. Consider the following assertion:

For all critical points \(m_0 \in \mathbb{Z} + \frac{n+n'}{2}\) for \(L(s, \Sigma \times \Pi)\) with \(L(m_0, \Sigma \times \Pi') \cdot L(m_0, \Sigma' \times \Pi) \neq 0\), we have

\[
\left| \frac{L(m_0, \Sigma \times \Pi) \cdot L(m_0, \Sigma' \times \Pi')}{L(m_0, \Sigma \times \Pi') \cdot L(m_0, \Sigma' \times \Pi)} \right| \approx 1.
\]
In this section, we prove assertion (4.4) under conditions (1) and (2) in Theorem 1.2. By condition (1), we may assume $n$ is even and, after twisting $\Sigma$ and $\Pi$ by integral powers of $|_h$, we have

$$
\begin{cases}
w + u = 0 & \text{if } n' \text{ is even}, \\
w + u = 1 & \text{if } n' \text{ is odd}.
\end{cases}
$$

Put $\delta \in \{0, 1\}$ with $\delta \equiv n' \pmod{2}$. Consider an isobaric automorphic representation of $\mathrm{GL}_{n+n'}(\mathbb{A})$ defined by

$$
\tau(\Sigma, \Pi) = (\Sigma \otimes |_h^{-\delta/2}) \oplus \Pi \vee.
$$

Note that it is cohomological and tamely isobaric by condition (1), and the archimedean component $\tau(\Sigma, \Pi)_\infty$ has infinity type $(\underline{\kappa} \sqcup \underline{\lambda}, w - \delta)$. We write

$$
\underline{\kappa} \sqcup \underline{\lambda} = (\kappa'_1, \ldots, \kappa'_{r'}) \\
\tau' = \frac{n + n'}{2},
$$

Let $r = \lceil \frac{n + n' + 1}{2} \rceil$ and define $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ by

$$
\begin{cases}
\lambda_i = \kappa'_i & \text{if } w \text{ or } u \text{ is odd}, \\
\lambda_i = \kappa'_i + 1 & \text{otherwise},
\end{cases}
$$

for $1 \leq i \leq r'$ and $\lambda_r = \kappa'_r - 2$ if $n'$ is odd. Note that $\lambda_r \geq 5$ when $n'$ is even by our assumption on $\kappa'_r$ in condition (2). Also the parity condition (1) implies that $(\underline{\lambda}; \delta)$ is the infinity type of an irreducible admissible, cohomological, essentially tempered $(\mathfrak{g}_{n+n'+1}, O_{n+n'+1}(\mathbb{R}))$-module. By Lemma 4.15 there exists a cohomological cuspidal automorphic representation $\Psi$ of $\mathrm{GL}_{n+n'+1}(\mathbb{A})$ with infinity type $(\underline{\lambda}; \delta)$. It is clear that the Rankin–Selberg $L$-function $L(s, \Psi \times \tau(\Sigma, \Pi))$ admits exactly $2$ non-central critical points (cf. (4.1)), and $(\Psi_\Sigma, \tau(\Sigma, \Pi)_\Sigma)$ is balanced by the definition of $\underline{\lambda}$. Fix a non-central critical point $m + \frac{\delta}{2} \in \mathbb{Z} + \frac{1}{2}$ for $L(s, \Psi \times \tau(\Sigma, \Pi))$. By Theorem 1.1, we have

$$
L^{(\infty)}(m + \frac{1}{2}; \Psi \times \tau(\Sigma, \Pi)) \sim p(m, \Psi \times \tau(\Sigma, \Pi)_\Sigma) \cdot G(\omega_\Sigma \omega_\Pi^{-1}) \cdot p(\Psi, \varepsilon_m) \cdot p(\tau(\Sigma, \Pi), \varepsilon_m'),
$$

where $\varepsilon_m = \varepsilon(\Psi_\Sigma)$ if $n'$ is even, $\varepsilon'_m = \varepsilon(\tau(\Sigma, \Pi)_\Sigma)$ if $n'$ is odd, and $\varepsilon_m \varepsilon'_m = (-1)^{n' + m + 1}$. By the period relation in Theorem 3.2 we have

$$
p(\tau(\Sigma, \Pi), \varepsilon'_m) \sim G(\omega_\Pi)^{-n} \cdot L^{(\infty)}(1 - \frac{\delta}{2}, \Sigma \times \Pi) \cdot p(\Sigma \otimes |_h^{(n' - \delta)/2} \varepsilon'_m) \cdot p(\Pi^{\vee} \otimes |_h^{-n/2} \varepsilon'_m).
$$

Note that

$$
L(s, \Psi \times \tau(\Sigma, \Pi)) = L(s - \frac{\delta}{2}, \Psi \times \Sigma) \cdot L(s, \Psi \times \Pi^{\vee}).
$$

Therefore, we conclude that

$$
L^{(\infty)}(1 - \frac{\delta}{2}; \Sigma \times \Pi) \sim \frac{L^{(\infty)}(m + \frac{1 - \delta}{2}; \Psi \times \Sigma) \cdot L^{(\infty)}(m + \frac{1}{2}; \Psi \times \Pi^{\vee})}{p(m, \Psi \times \tau(\Sigma, \Pi)_\Sigma) \cdot G(\omega_\Sigma \omega_\Pi^{-1}) \cdot p(\Psi, \varepsilon_m) \cdot p(\Sigma \otimes |_h^{(n' - \delta)/2} \varepsilon'_m) \cdot p(\Pi^{\vee} \otimes |_h^{-n/2} \varepsilon'_m)}.
$$

It then follows immediately from the assumptions $\Sigma_\infty = \Sigma'_\infty$ and $\Pi_\infty = \Pi'_\infty$ that assertion (4.4) holds for the critical point $m_0 = 1 - \frac{\delta}{2}$. Note that condition (1) implies that all critical points for $L(s, \Sigma \times \Pi)$ are non-central (cf. (4.1)). In particular, all critical values are non-zero. Therefore, by the result of Harder and Raghuram [HR20 Theorem 7.21], we then conclude that (4.4) holds for all critical points. This completes the proof of Theorem 1.2.

4.4. Proof of Theorem 1.4 Let $\Sigma$ and $\Pi$ be cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_{n'}(\mathbb{A})$ with infinity types $(\underline{\kappa}; w)$ and $(\underline{\lambda}; u)$, respectively. Consider the following assertion:

Assume $nn'$ is even. Let $\chi$ be a finite order Hecke character of $\mathbb{A}^\times$. Let $m_0, m_1 \in \mathbb{Z} + \frac{m + n'}{2}$ be critical points for $L(s, \Sigma \times \Pi)$ such that $\Lambda(m_1, \Sigma \times \Pi) \neq 0$ and $(-1)^{m_0 - m_1} = \chi_{\infty}(-1)$ if $n$ or $n'$ is odd. Then we have

$$
\frac{L^{(\infty)}(m_0, \Sigma \times \Pi \otimes \chi)}{L^{(\infty)}(m_1, \Sigma \times \Pi)} \sim (2\pi \sqrt{-1})^{(m_0 - m_1)n'n'}/2 \cdot G(\chi)^{n'n'/2}.
$$
In this section, we prove this assertion under two different sets of assumptions. Firstly we prove it under conditions (1)-(5) in Theorem 1.4. By conditions (2)-(4), $L(s, \Sigma \times \Pi)$ admits at least two (resp. four) non-central critical points if $\min\{|\kappa_i - \ell_j|\}$ is even (resp. odd). Therefore, to prove (1.5) holds for all $m_0, m_1$, it suffices to show that it holds for some non-central critical points $m_0, m_1$ with given parity. Indeed, let $m_0, m_1$ be non-central critical points such that (1.5) holds. Let $m_0' \in m_0 + 2\mathbb{Z}$ and $m_1' \in m_1 + 2\mathbb{Z}$ be other critical points such that $L(m'_1, \Sigma \times \Pi) \neq 0$. By the result of Harder and Raghuram [HR20] Theorem 7.21, we have

$$L(\chi)(m_0, \Sigma \times \Pi \otimes \chi) \sim (2\pi \sqrt{-1})^{(m_0-m_0')nn'/2}, \quad L(\chi)(m_1, \Sigma \times \Pi) \sim (2\pi \sqrt{-1})^{(m_1-m_1')nn'/2}.$$

Thus (1.5) also holds for the critical points $m_0', m_1'$.

Let $I$ be the subset of $\{1 \leq i \leq r\}$ consisting of $1 \leq i \leq r-1$ such that there is no $j$ with $\kappa_i > \ell_j > \kappa_{i+1}$, and also contains $r$ if $n$ is odd and $\ell_r > \kappa_r$. By conditions (1)-(3), it is clear that $n > n'$ and

$$\mathbb{Z}(I) = \begin{cases} n-n'-1 & \text{if } n \text{ or } n' \text{ is odd,} \\ n-n'-2 & \text{if } n \text{ is even and } n' \text{ is even.} \end{cases}$$

Let $(\ell'_i)_{i \in I}$ be a sequence of integers chosen such that:

- If $1 \leq i \leq r-1$ and $i \in I$, then $\kappa_i > \ell'_i > \kappa_{i+1}$ and $\ell'_i \equiv u + n'$ (mod 2).
- If $r \in I$, then $\ell'_r = \begin{cases} \kappa_r - 2 & \text{if } \min\{|\kappa_i - \ell_j|\} \text{ is even,} \\ \kappa_r - 5 & \text{if } \min\{|\kappa_i - \ell_j|\} \text{ is odd.} \end{cases}$
- $\min\{|\kappa_i - \ell'_i|\} = \begin{cases} 2 & \text{if } \min\{|\kappa_i - \ell_j|\} \text{ is even,} \\ 5 & \text{if } \min\{|\kappa_i - \ell_j|\} \text{ is odd.} \end{cases}$

The existence of $\ell'_i$ is guaranteed by conditions (3) and (4). For $i \in I$, let $\Pi'_i$ be a cohomological cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ with infinity type

$$\begin{cases} (\ell'_i; u) & \text{if } n' \text{ is even,} \\ (\ell'_i; u + 1) & \text{if } n' \text{ is odd.} \end{cases}$$

Let $\Pi'$ be a cohomological tamely isobaric automorphic representation of $\text{GL}_{n-n'-1}(\mathbb{A})$ defined by

$$\Pi' = \begin{cases} \prod_{i \in I} \Pi'_i & \text{if } n \text{ or } n' \text{ is odd,} \\ \prod_{i \in I} (\Pi'_i \otimes |_{\mathbb{A}}^{-1/2}) \boxtimes |_{\mathbb{A}}^{(u-1)/2} & \text{if } n \text{ is even and } n' \text{ is even.} \end{cases}$$

Here we have use condition (5) when $n$ and $n'$ are both even. For finite order Hecke characters $\chi, \eta$ of $\mathbb{A}^\times$, let $\Psi(\chi, \eta)$ be a cohomological tamely isobaric automorphic representation of $\text{GL}_{n-1}(\mathbb{A})$ defined by

$$\Psi(\chi, \eta) = \begin{cases} (\Pi' \otimes \chi) \boxtimes (\Pi' \otimes \eta) & \text{if } n \text{ is odd and } n' \text{ is even,} \\ (\Pi' \otimes \chi) \boxtimes (\Pi' \otimes \eta) |_{\mathbb{A}}^{-1/2} & \text{if } n \text{ is even and } n' \text{ is odd,} \\ (\Pi \otimes \chi) |_{\mathbb{A}}^{-1/2} \boxtimes (\Pi' \otimes \eta) & \text{if } n \text{ is even and } n' \text{ is even.} \end{cases}$$

When $\chi = \eta = 1$, we write $\Psi(1, 1) = \Psi$. Recall the convention (2.10). By the period relations in Theorem 2.3 and Corollary 3.3 we have the following period relations:

- If $n$ is odd and $n'$ is even, then

$$p(\Psi(\chi, \eta), \pm) \sim G(\chi)^{\frac{1}{2}(n'-1)/2} \cdot G(\eta)^{\frac{1}{2}(n-1)/2} \cdot L(\chi, (1, \Pi) \times (\Pi')^\vee \otimes \chi_{\mathbb{A}} \eta_{\mathbb{A}}(-1)) \cdot p(\Pi, \pm(-1)^{(n'-1)/2} \chi_{\mathbb{A}}(-1)) \cdot p(\Pi', \pm(-1)^{n'/2} \eta_{\mathbb{A}}(-1)).$$

- If $n$ is even and $n'$ is odd, then

$$p(\Psi(\chi, \eta), \pm) \sim G(\chi)^{\frac{1}{2}(2n-n'-3)/2} \cdot G(\eta)^{\frac{1}{2}(n'-1)/2} \cdot G(\omega_{\Pi'}) \cdot L(\chi, (\frac{1}{2}, \Pi' \times \Pi^\vee \otimes \chi_{\mathbb{A}} \eta_{\mathbb{A}}\chi_{\mathbb{A}} \eta_{\mathbb{A}}(-1)) \cdot p(\Pi, \Pi', \pm(-1)^{(n'-2)/2} \chi_{\mathbb{A}}\eta_{\mathbb{A}}(-1)).$$

- If $n$ is even and $n'$ is even, then

$$p(\Psi(\chi, \eta), \pm) \sim G(\chi)^{\frac{1}{2}(n'-1)/2} \cdot G(\eta)^{\frac{1}{2}(n'-1)/2} \cdot G(\omega_{\Pi'}) \cdot L(\chi, (\frac{1}{2}, \Pi' \times (\Pi')^\vee \otimes \chi_{\mathbb{A}} \eta_{\mathbb{A}}(-1)) \cdot p(\Pi', \Pi').$$
On the other hand, by Theorems 4.1 and 2.3 for critical point \( m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2} \) for \( L(s, \Sigma \times \Psi) \), we have

\[
L^{(x)}(m + \frac{1}{2}, \Sigma \times \Psi(\chi, \eta))
\sim p(m, \Sigma \times \Psi) \cdot G(\omega H \omega H')
\times \begin{cases}
G(\chi)^{n'} \cdot G(\eta)^{n-n'-1} \cdot p(\Sigma) \cdot p(\Psi(\chi, \eta), (\chi^{-1})^{1+m+n}(\Sigma \eta)) & \text{if } n \text{ is odd and } n' \text{ is even,} \\
G(\chi^{-1})^{n-n'-1} \cdot G(\chi)^{n(n-1)/2} \cdot p(\Sigma, (\chi^{-1})^{m+n}(\Psi \chi) \chi \chi \eta) \cdot p(\Psi(\chi, \chi^{-1})) & \text{if } n \text{ is even and } n' \text{ is odd,} \\
G(\chi^{-1})^{n'} \cdot G(\eta)^{n(n-1)/2} \cdot p(\Sigma, (\chi^{-1})^{m+n}(\Psi \chi) \chi \chi \eta) \cdot p(\Psi(\chi^{-1})) & \text{if } n \text{ is even and } n' \text{ is even.}
\end{cases}
\]

Fix a finite order Hecke character \( \chi \) of \( \mathbb{A}^\times \). Let \( m_1 + \frac{1}{2}, m_2 + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2} \) be non-central critical points for \( L(s, \Sigma \times \Psi) \) such that \( (\chi, \chi^{-1})^{n+m} = \chi_{\Sigma}(1) \) if \( n \) or \( n' \) is odd. We consider the ratio

\[
\frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Psi(\chi, \eta))}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Psi)}.
\]

- Assume \( n \) is odd and \( n' \) is even. By \( (4.2), (4.4), (4.9) \), and the condition \( (\chi^{-1})^{n+m} = \chi_{\Sigma}(1) \), we have

\[
\frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi \otimes \chi)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi)} \sim \frac{(2\sqrt{-1})^{(m_1-m_2)n(n-n'-1)/2} \cdot G(\chi)^{n(n-1)/2} \cdot p(\Pi, (\chi^{-1})^{1+m+n/2}(\Sigma \eta))}{p(\Pi, (\chi^{-1})^{1+m+n/2}(\Sigma \eta))}.
\]

(4.10)

In particular, by taking \( \eta = 1 \) in (4.10), we have

\[
\frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi \otimes \chi)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi)} \sim \frac{(2\sqrt{-1})^{(m_1-m_2)n(n-1)/2} \cdot G(\chi)^{n(n-1)/2} \cdot L^{(x)}(1, \Pi \times (\Pi')^\nu \otimes \chi)}{L^{(x)}(1, \Pi \times (\Pi')^\nu)}.
\]

When \( n' = 2 \), by the result of Shimura \( [Sh76] \) Theorem 3], we have

\[
\frac{L^{(x)}(1, \Pi \times (\Pi')^\nu \otimes \chi)}{L^{(x)}(1, \Pi \times (\Pi')^\nu)} \sim G(\chi)^{n-3}.
\]

Therefore, when \( n' = 2 \), we have

\[
\frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi \otimes \chi)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi)} \sim (2\sqrt{-1})^{(m_1-m_2)n} \cdot G(\chi)^n.
\]

(4.11)

On the other hand, by taking \( \eta = \chi \) in (4.10) and the condition \( (\chi^{-1})^{n+m} = \chi_{\Sigma}(1) \), we have

\[
\frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi \otimes \chi)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi)} \sim (2\sqrt{-1})^{(m_1-m_2)n(n-1)/2} \cdot G(\chi)^{n(n-1)/2}.
\]

(4.12)

- Assume \( n \) is even and \( n' \) is odd. By \( (4.2), (4.4), (4.9) \), and the condition \( (\chi^{-1})^{n+m} = \chi_{\Sigma}(1) \), we have

\[
\frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi \otimes \chi) \cdot L^{(x)}(m_1, \Sigma \times \Pi' \otimes \eta)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi) \cdot L^{(x)}(m_2, \Sigma \times \Pi')}
\sim (2\sqrt{-1})^{(m_1-m_2)n(n-1)/2} \cdot G(\chi^{-1})^{(n-n'-1)/2} \cdot G(\chi)^{n(n-1)/2}
\times \frac{L^{(x)}(1, \Pi' \times (\Pi')^\nu \otimes \chi^{-1}(\eta)) \cdot p(\Pi, (\chi^{-1})^{1+(n-2)/2}(\Sigma \eta))}{L^{(x)}(1, \Pi' \times (\Pi')^\nu)}.
\]

(4.13)
By [HR20, Theorem 7.21], we have
\[ \frac{L^{(x)}(m_1, \Sigma \times \Pi')}{L^{(x)}(m_2, \Sigma \times \Pi')} \sim (2\pi \sqrt{-1})^{(m_1-m_2)n(n-n'-1)/2}. \]

When \( n' = 1 \), by Theorems 4.4 and 3.2, we have
\[ \frac{L^{(x)}(\frac{1}{2}, \Pi' \times \Pi \otimes \chi^{-1}) \cdot p(\Pi', (-1)^{(n-2)/2} \varepsilon(\Pi_{\chi'}) \chi_{\Pi}(1))}{L^{(x)}(\frac{1}{2}, \Pi' \times \Pi \cdot p(\Pi', (-1)^{(n-2)/2} \varepsilon(\Pi_{\chi})))} \sim G(\chi^{-1})^{(n-2)/2}. \]

Therefore, when \( n' = 1 \), we have
\[ \frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi \otimes \chi)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi)} \sim (2\pi \sqrt{-1})^{(m_1-m_2)n/2} \cdot G(\chi)^{n/2}. \quad (4.14) \]

On the other hand, by taking \( \eta = \chi \) in (4.13), we have
\[ \frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi \otimes \chi) \cdot L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi' \otimes \eta)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi) \cdot L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi')} \sim (2\pi \sqrt{-1})^{(m_1-m_2)n(n-1)/2} \cdot G(\chi)^{(n-1)/2}. \quad (4.15) \]

\[ \begin{aligned} &\sim (2\pi \sqrt{-1})^{(m_1-m_2)n(n-1)/2} \cdot G(\chi)^{(n-1)/2} \cdot \frac{p(\Sigma, (-1)^{m_1} \varepsilon(\Psi_{\chi}) \eta_{\chi}(1))}{p(\Sigma, (-1)^{m_2} \varepsilon(\Psi_{\chi}))} \\ &\times \frac{L^{(x)}(\frac{1}{2}, \Pi \times (\Pi')^* \otimes \chi^{-1}) \cdot p(\Pi', (-1)^{(n-2)/2} \varepsilon(\Pi_{\chi'}') \chi_{\Pi}(1))}{L^{(x)}(\frac{1}{2}, \Pi \times (\Pi')^* \cdot p(\Pi', (-1)^{(n-2)/2} \varepsilon(\Pi_{\chi'})))}. \end{aligned} \]

By [HR20, Theorem 7.21] and Theorem 3.2, we have
\[ \frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi')}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi')} \sim (2\pi \sqrt{-1})^{(m_1-m_2)n(n-n'-1)/2} \cdot \frac{p(\Sigma, (-1)^{m_1} \varepsilon(\Pi_{\chi'}))}{p(\Sigma, (-1)^{m_2} \varepsilon(\Pi_{\chi'}))}. \]

When \( n' = 2 \), by [Shi76, Theorem 3] and Theorems 4.4 and 3.2, we have
\[ \frac{L^{(x)}(\frac{1}{2}, \Pi \times (\Pi')^* \otimes \chi) \cdot p(\Pi', (-1)^{(n-2)/2} \varepsilon(\Pi_{\chi'}) \chi_{\Pi}(1))}{L^{(x)}(\frac{1}{2}, \Pi \times (\Pi')^* \cdot p(\Pi', (-1)^{(n-2)/2} \varepsilon(\Pi_{\chi'})))} \sim G(\chi)^{n-3}. \]

Note that \( \varepsilon(\Psi_{\chi}) = (-1)^{n'/2} \varepsilon(\Pi_{\chi'}'). \) Therefore, when \( n' = 2 \), we have
\[ \frac{L^{(x)}(m_1, \Sigma \times \Pi \otimes \chi)}{L^{(x)}(m_2, \Sigma \times \Pi)} \sim (2\pi \sqrt{-1})^{(m_1-m_2)n} \cdot G(\chi)^{n}. \quad (4.17) \]

On the other hand, by taking \( \eta = \chi \) in (4.14), we have
\[ \frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi') \cdot L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi' \otimes \eta)}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi) \cdot L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi')} \sim (2\pi \sqrt{-1})^{(m_1-m_2)n(n-1)/2} \cdot \frac{p(\Sigma, (-1)^{m_1} \varepsilon(\Psi_{\chi}) \chi_{\Pi}(1))}{p(\Sigma, (-1)^{m_2} \varepsilon(\Psi_{\chi}))}. \quad (4.18) \]

By (4.11), (4.14), and (4.17), we see that (4.15) holds if \( n' = 1, 2 \). Note that conditions (1)-(5) are satisfied by the infinity types of \( \Sigma \) and each summand of \( \Pi' \). In particular, since \( \Pi' \) is an isobaric sum of cuspidal automorphic representations of \( \mathbb{A}^\times \) and \( \text{GL}_2(\mathbb{A}) \), we conclude that (4.3) also holds for \( L(s, \Sigma \times \Pi') \). Therefore, by (4.12) and (4.15), we see that (4.15) holds if \( n \) or \( n' \) is odd. Assume \( n \) and \( n' \) are both even. It follows from (4.18) that (4.15) holds if \( (-1)^{m_1-m_2} = \chi_{\Pi}(1) \). If \( (-1)^{m_1-m_2} = -\chi_{\Pi}(1) \), then
\[ \frac{L^{(x)}(m_2 + \varepsilon + \frac{1}{2}, \Sigma \times \Pi')}{L^{(x)}(m_2 + \frac{1}{2}, \Sigma \times \Pi')} \sim (2\pi \sqrt{-1})^{\varepsilon n(n-n'-1)/2} \cdot \frac{p(\Sigma, (-1)^{m_1} \varepsilon(\Pi_{\chi'}) \chi_{\Pi}(1))}{p(\Sigma, (-1)^{m_2} \varepsilon(\Pi_{\chi'}))}. \]

Here \( \varepsilon \in \{ \pm 1 \} \) is chosen so that \( m_2 + \varepsilon + \frac{1}{2} \) is a non-central critical point for \( L(s, \Sigma \times \Pi') \). Also we have
\[ \frac{L^{(x)}(m_1 + \frac{1}{2}, \Sigma \times \Pi' \otimes \chi)}{L^{(x)}(m_2 + \varepsilon + \frac{1}{2}, \Sigma \times \Pi')} \sim (2\pi \sqrt{-1})^{(m_1-m_2-\varepsilon)n(n-n'-1)/2} \cdot G(\chi)^{n(n-n'-1)/2}. \]
Therefore, in this case (4.15) also follows from (4.18). This completes the proof of Theorem 1.4 if conditions (1)-(5) are satisfied.

Now we show that (4.15) holds if conditions in Theorem 1.2 are satisfied. By condition (1) in Theorem 1.2 without lose of generality we may assume $n$ is even and

$$\begin{cases} w + u = 0 & \text{if } n' \text{ is even}, \\
 w + u = 1 & \text{if } n' \text{ is odd}.
\end{cases}$$

For a finite order Hecke character $\chi$ of $\mathbb{A}^\times$, let $\Pi'(\chi)$ be a cohomological tamely isobaric automorphic representation of $GL_{n+n'}(\mathbb{A})$ defined by

$$\Pi'(\chi) = \begin{cases} (\Sigma \otimes \chi) \otimes \Pi' & \text{if } n' \text{ is even}, \\
 (\Sigma \otimes \chi |_{\mathbb{A}^{1/2}}) \otimes \Pi' & \text{if } n' \text{ is odd}.
\end{cases}$$

When $\chi = 1$, we write $\Pi'(1) = \Pi'$. By condition (2) in Theorem 1.2 and Lemma 1.6 as in the proof of Theorem 1.2 in [1.3] there exists a cohomological cuspidal automorphic representation $\Psi$ of $GL_{n+n'}(\mathbb{A})$ such that $(\Psi, \Pi'_\chi)$ is balanced and $L(s, \Psi \times \Pi')$ admits exactly 2 non-central critical points. By Theorems 2.3 and 4.1 and Corollary 3.4 for critical point $m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ for $L(s, \Psi \times \Pi')$, we have

(4.19)

$$L^{(\infty)}(m + \frac{1}{2}, \Psi \times \Pi'(\chi)) \\ \sim p(m, \Psi \times \Pi'(\chi)) \cdot G(\omega) \cdot G(\omega)^{-n-1} \cdot G(\chi)^{n(n+1)/2}$$

\times \begin{cases} p(\Psi) \cdot L^{(\infty)}(1, \Sigma \otimes \Pi \otimes \chi) \cdot p(\Sigma, (-1)^{1+m+n'/2}(\varphi_{\varphi}(\chi_{\varphi}(-1))) \cdot p(\Sigma, (-1)^{1+m+n/2}(\varphi_{\varphi}_{\varphi}(\chi_{\varphi}(-1)))) & \text{if } n' \text{ is even}, \\
p(\Psi) \cdot L^{(\infty)}(1, \Sigma \otimes \Pi \otimes \chi) \cdot p(\Sigma, (-1)^{(n+n'-1)/2}(\varphi_{\varphi}(\chi_{\varphi}(-1))) \cdot p(\Sigma, (-1)^{(n+n'/2}(\varphi_{\varphi}(\chi_{\varphi}(-1)))) & \text{if } n' \text{ is odd}.
\end{cases}

Note that

$$L(s, \Psi \times \Pi'(\chi)) = \begin{cases} L(s, \Psi \times \Pi' \otimes \chi) \cdot L(s, \Psi \times \Pi') & \text{if } n' \text{ is even}, \\
L(s - \frac{1}{2}, \Psi \times \Pi' \otimes \chi) \cdot L(s, \Psi \times \Pi') & \text{if } n' \text{ is odd}.
\end{cases}$$

Assume $n'$ is even. Let $m_1 + \frac{1}{2}, m_2 + \frac{1}{2}$ be critical points for $L(s, \Psi \times \Pi')$ such that $(-1)^{m_1-m_2} = \chi_{\varphi}(1)$. By (4.19), we have

$$\frac{L^{(\infty)}(m_1 + \frac{1}{2}, \Psi \times \Sigma \otimes \chi) \cdot L^{(\infty)}(m_2 + \frac{1}{2}, \Psi \times \Pi')} {L^{(\infty)}(m_2 + \frac{1}{2}, \Psi \times \Sigma) \cdot L^{(\infty)}(m_1 + \frac{1}{2}, \Psi \times \Pi')}$$

$$\sim (2\pi)^{-1}(m_1-m_2)(n+n')^{n(n+1)/2} \cdot G(\chi)^{n(n+1)/2} \cdot \frac{L^{(\infty)}(1, \Sigma \otimes \Pi \otimes \chi) \cdot p(\Sigma, (-1)^{1+m+n/2}(\varphi_{\varphi}(\chi_{\varphi}(-1)))) \cdot p(\Sigma, (-1)^{1+m+n/2}(\varphi_{\varphi}(\chi_{\varphi}(-1)))} {L^{(\infty)}(1, \Sigma \otimes \Pi) \cdot p(\Pi, (-1)^{1+m+n/2}(\varphi_{\varphi}(\chi_{\varphi}(-1))))}.$$
for any critical point \( m + \frac{1}{2} \) for \((L(s, \Psi \times \Pi^*)\). Since the infinity types of \( \Psi \) and \( \Sigma \) satisfy conditions (1)-(5) in Theorem \[1.3\] we have
\[
\frac{L^{(\infty)}(m, \Psi \times \Sigma \otimes \chi)}{L^{(\infty)}(m, \Psi \times \Sigma)} \sim G(\chi)^{n(n+n'+1)/2}.
\]
When \( \chi_{\infty}(-1) = -1 \), by \[HR20, \text{Theorem 7.21}\], we have
\[
\frac{L^{(\infty)}(\frac{1}{2}, \Sigma \times \Pi)}{L^{(\infty)}(-\frac{1}{2}, \Sigma \times \Pi)} \sim (2\pi \sqrt{-1})^{nn'/2} \cdot \frac{p(\Sigma, (-1)^{(n+n'+1)/2} \varepsilon(\Pi_{\infty}))}{p(\Sigma, (-1)^{(n+n'-1)/2} \varepsilon(\Pi_{\infty}))}.
\]
We thus conclude that
\[
\frac{L^{(\infty)}(\frac{1}{2}, \Sigma \times \Pi \otimes \chi)}{L^{(\infty)}(\frac{1}{2}, \Sigma \times \Pi)} \sim G(\chi)^{nn'/2}
\]
if \( \chi_{\infty}(-1) = 1 \), and
\[
\frac{L^{(\infty)}(\frac{1}{2}, \Sigma \times \Pi \otimes \chi)}{L^{(\infty)}(-\frac{1}{2}, \Sigma \times \Pi)} \sim (2\pi \sqrt{-1})^{nn'/2} \cdot G(\chi)^{nn'/2}
\]
if \( \chi_{\infty}(-1) = -1 \). Hence \[1.3\] holds for the special cases \( m_0 = m_1 = \frac{1}{2} \) or \( m_0 = \frac{1}{2}, \ m_1 = -\frac{1}{2} \). The general cases then follow from these special cases together with \[HR20, \text{Theorem 7.21}\]. This completes the proof of Theorem \[1.4\].

4.5. Compatibility with Deligne’s conjecture. In this section, we verify in Lemma \[4.10\] that Conjecture \[1.1\] is compatible with Deligne’s conjecture \[De72\], based on Clozel’s conjecture \[Clo90\] and Yoshida’s computation of motivic periods \[Yos01\]. Let \( M \) be a motive over \( \mathbb{Q} \) with coefficients in a number field \( \mathbb{E} \). Associated to \( M \), we have the Betti realization \( H_B(M) \), the de Rham realization \( H_{dR}(M) \), and the l-adic realization \( H_l(M) \) for each finite places \( l \) of \( \mathbb{E} \). The dual motive \( M^\vee \) of \( M \) is also over \( \mathbb{Q} \) with coefficients in \( \mathbb{E} \) whose realizations are dual to that of \( M \). The Betti realization \( H_B(M) \) is a finite dimensional \( \mathbb{E} \)-vector space equipped with an action of archimedean Frobenius \( F_\infty \) and a Hodge decomposition:
\[
H_B(M) \otimes \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} H_B^{p,q}(M),
\]
where \( H_B^{p,q}(M) \) is a free \( \mathbb{E} \otimes \mathbb{Q} \) \( \mathbb{C} \)-module of rank \( h_B^{p,q}(M) \) such that \( F_\infty(H_B^{p,q}(M)) = H_B^{p,q}(M) \). Let \( H_B^{p,q}(M) \) be the \pm\-eigenspace of \( H_B(M) \) under the action of \( F_\infty \), and write \( d^\pm(M) = \dim_{\mathbb{E}} H_B^{p,q}(M) \). We say \( M \) is pure of weight \( w \) if \( H_B^{p,q}(M) \) is non-zero only when \( p+q = w \). We say \( M \) is regular if the non-zero Hodge numbers are all equal to \( 1 \). When \( M \) is pure of weight \( w \), the archimedean local factor of \( M \) is defined by
\[
L_\infty(M, s) = \prod_{p \in \mathbb{Z}} \Gamma_C(s-p)^{h_M^p,\pm} \cdot \left\{ \begin{array}{ll}
1 & \text{if } w \text{ is not even}, \\
\Gamma_R(s - \frac{w}{2})^{h_M^{w/2,\pm}} \cdot \Gamma_R(s - \frac{w}{2} + 1)^{h_M^{w/2,-}} & \text{if } w \text{ is even}.
\end{array} \right.
\]
Here \( h_M^{w/2,\pm} \) and \( h_M^{w/2,-} \) are the rank of the eigenspaces of \( H_B^{w/2,\pm}(M) \) under the action of \( F_\infty \) with eigenvalues \(-1)^{w/2}\) and \(-1)^{w/2+1}\), respectively, and
\[
\Gamma_R(s) = \pi^{-s/2} \Gamma(\frac{s}{2}), \quad \Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s).
\]
A critical point for \( M \) is an integer \( m \in \mathbb{Z} \) such that \( L_\infty(M, s) \) and \( L_\infty(M^\vee, 1-s) \) are holomorphic at \( s = m \). Note that \( h_M^{0,q} = h_M^{q,0} \) and \( h_M^{w/2,\pm} = h_M^{w/2,\pm} \) if \( w \) is even. Therefore, \( M \) admits critical points if and only if \( h_M^{w/2,\pm} = 0 \) or \( h_M^{w/2,-} = 0 \) when \( w \) is even. The de Rham realization \( H_{dR}(M) \) is a finite dimensional \( \mathbb{E} \)-vector space equipped with a decreasing Hodge filtration \( \{F_i^{dR}(M)\}_{i \in \mathbb{Z}} \). We have the comparison isomorphism
\[
I_\infty : H_B(M) \otimes \mathbb{Q} \mathbb{C} \to H_{dR}(M) \otimes \mathbb{Q} \mathbb{C}
\]
such that \( I_\infty(F_\infty \otimes c) = id \otimes c \) and \( I_\infty(F_i^{dR}(M)) = F_i^{dR}(M) \) for all \( i \in \mathbb{Z} \). Assume \( M \) is pure and admits critical points. Then there are subspaces \( F^\pm(M) \) of \( H_{dR}(M) \) from the Hodge filtration such that \( \dim_{\mathbb{E}} F^\pm(M) = d^\pm(M) \). In this case, \( I_\infty \) induces isomorphisms
\[
I_\infty^\pm : H_B^\pm(M) \otimes \mathbb{Q} \mathbb{C} \to H_B^\pm(M) \otimes \mathbb{Q} \mathbb{C} \to H_{dR}(M) \otimes \mathbb{Q} \mathbb{C} \to H_{dR}(M)/F^\pm(M) \otimes \mathbb{Q} \mathbb{C}.
\]
With respect to fixed \( \mathbb{E} \)-rational bases on both sides, the Deligne’s periods \( c^\pm(M) \) are defined by
\[
c^\pm(M) := \det(I_\infty^\pm) \in (\mathbb{E} \otimes \mathbb{Q} \mathbb{C})^*.
\]
The l-adic realizations \( \{ H_t(M) \}_t \) forms a strictly compatible system of Galois representations of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on \( \mathbb{E}_t \)-vector spaces \( H_t(M) \). Under assumptions [Del79 §1.2.1] on rationality and independency of local factors, we can define the \( \mathbb{E} \otimes \mathbb{C} \)-valued L-function \( L(M,s) \) of \( M \). Note that

\[
dim_{\mathbb{E}} H_B(M) = \dim_{\mathbb{E}} H_{DR}(M) = \dim_{\mathbb{E}} H_1(M).
\]

This common dimension is denoted by \( d(M) \), called the rank of \( M \). In [Del79, Conjecture 2.8], Deligne proposed the following:

**Conjecture 4.7** (Deligne). Assume \( M \) is pure and admits critical points. For a critical point \( m \in \mathbb{Z} \) for \( L(M,s) \), we have

\[
L(M,m) \left( \frac{2\pi \sqrt{-1}}{d(-1)^m(M)m \cdot c(-1)^m(M)} \right) \in \mathbb{E}.
\]

An automorphic representation \( \Pi \cong \bigotimes_v \Pi_v \) of \( \text{GL}_n(\mathbb{A}) \) is called algebraic if it is isobaric and the infinitesimal character of \( \Pi_v \) belongs to \( (\mathbb{Z} + \frac{\mathbb{A}^*}{\mathbb{A}^*})^n \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), let \( \sigma \Pi \) be the irreducible admissible representation of \( \text{GL}_n(\mathbb{A}) \) so that \( \sigma \Pi_\mathfrak{p} = \Pi_\mathfrak{p} \) and \( \sigma \Pi_p \) is the \( \sigma \)-conjugate of \( \Pi_p \) for each prime number \( p \). The rationality field \( Q(\Pi) \) of \( \Pi \) is defined to be the fixed field of \( \{ \sigma \in \text{Aut}(\mathbb{C}) | \sigma \Pi = \Pi \} \). Clozel conjectured in [Clo90] that if \( \Pi \) is algebraic, then \( \sigma \Pi \) is automorphic for all \( \sigma \in \text{Aut}(\mathbb{C}) \) and \( Q(\Pi) \) is a number field. The assertion was proved in *loc. cit.* when \( \Pi \) is regular algebraic (cohomological) and cuspidal, based on the study of cuspidal cohomology of the locally symmetric spaces associated to \( \text{GL}_n \). Moreover, we have the following conjecture proposed by Clozel in [Clo90 Conjecture 4.5]:

**Conjecture 4.8** (Clozel). Let \( \Pi \) be an algebraic cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}) \). There exists an absolutely irreducible, pure motive \( M_{\Pi} \) over \( Q \) of rank \( n \) with coefficients in a number field \( \mathbb{E} \) containing \( Q(\Pi) \) satisfying the following properties:

1. Let \( \phi_{\Pi_v} \) be the Langlands parameter associated to \( \Pi_v \) by the local Langlands correspondence. Write

\[
\phi_{\Pi_v} = (h_1 \cdot \phi_{\kappa_1} \oplus \cdots \oplus h_r \cdot \phi_{\kappa_r} \oplus h_+ \cdot 1 \oplus h_- \cdot \text{sgn}) \otimes | \cdot |^{w/2}
\]

for some \( \kappa_1, \ldots, \kappa_r \geq 2, w \in \mathbb{Z} \), and some multiplicities \( h_1, \ldots, h_r, h_{\pm} \), where \( \phi_{\kappa} \) corresponds to the discrete series representation of \( \text{GL}_2(\mathbb{R}) \) with weight \( \kappa \geq 2 \). Then \( M_{\Pi} \) is pure of weight \( -w - n + 1 \) and the possible non-zero hodge numbers are given by

\[
h_{M_{\Pi}}(-w-n+2)/2,(-w-n)/2 = h_i, \quad h_{M_{\Pi}}(-w-n+1)/2,(-w-n)/2 = h_{\pm}
\]

for \( 1 \leq i \leq r \).

2. Let \( p \neq l \) be prime numbers and \( l \) be a finite place of \( \mathbb{E} \) lying over \( l \). Then we have

\[
WD \left( \left. H_t(M) \otimes_{\mathbb{E}_l} \overline{\mathbb{Q}}_l \right|_{\text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)} \right)_{F-ss} \cong \iota_l^{-1} \text{rec}_{\mathbb{Q}_l} \left( \Pi_p \otimes | \cdot |^{(p-1)/2} \right).
\]

Here \( WD(\cdot) \) denotes the associated Weil–Deligne representation, \( F-ss \) denotes the Frobenius semisimplification, \( \text{rec}_{\mathbb{Q}_l} \) denotes the local Langlands correspondence, and \( \iota_l : \overline{\mathbb{Q}}_l \to \mathbb{C} \) is an isomorphism.

In particular, we have \( L_{\nu_X}(M_{\Pi},s) = L(s + \frac{n-1}{2}, \Pi_{\mathfrak{X}}) \) and

\[
L(M,s) = \left( L^{(\infty)}(s + \frac{n-1}{2}, \sigma \Pi) \right)_{\sigma : \mathbb{E} \to \mathbb{C}}.
\]

**Remark 4.9.** The conjectural motive \( M_{\Pi} \) was constructed by Deligne [Del79] and Scholl [Sch90] when \( n = 2 \) and \( \Pi \) is regular algebraic. Under the canonical isomorphism \( \mathbb{E} \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\sigma : \mathbb{E} \to \mathbb{C}} \mathbb{C} \), we write

\[
c^{\pm}(M_{\Pi}) = (c^{\pm}('\Pi))_{\sigma : \mathbb{E} \to \mathbb{C}}.
\]

In this case, Conjecture 4.7 is also known (cf. [Del79 §7]). In particular, we have the following period relation between the Deligne’s periods and the Betti–Whittaker periods of \( \Pi \):

\[
\sigma \left( \frac{c^{\pm}(\Pi)}{(2\pi \sqrt{-1})^{(\kappa + w)/2} \cdot p(\Pi, \pm)} \right) = \frac{c^{\pm}(\Pi)}{(2\pi \sqrt{-1})^{(\kappa + w)/2} \cdot p(\sigma \Pi, \pm), \sigma \in \text{Aut}(\mathbb{C})}
\]

where \( (\kappa; w) \) is the infinity type of \( \Pi \). Here we need to use the existence of non-zero twisted central critical values when there is only one critical value which is zero (cf. [Rob99]).
Let $\Pi$ be an algebraic automorphic representation of $GL_n(\mathbb{A})$ with isobaric summands $\Pi_1, \ldots, \Pi_k$. Then $\Pi_i \otimes |_{\mathbb{A}_k}^{(n-n_i)/2}$ is algebraic cuspidal by definition for $1 \leq i \leq k$, where $n_i$ is the degree of $\Pi_i$. It is clear that $\Pi_k$ is essentially tempered if and only if $\Pi$ is tamely isobaric. Let $M_H$ be the conjectural motive over $\mathbb{Q}$ of rank $n$ with coefficients in some number field containing $\prod_{i=1}^k \mathbb{Q}(\Pi_i \otimes |_{\mathbb{A}_k}^{(n-n_i)/2})$ defined by

$$M_H = M_{H_1 \otimes |_{\mathbb{A}_k}^{(n-n_1)/2}} \otimes \cdots \otimes M_{H_k \otimes |_{\mathbb{A}_k}^{(n-n_k)/2}}.$$  

Then $M_H$ is pure if and only if $\Pi$ is tamely isobaric.

**Lemma 4.10.** If Conjectures 4.7 and 4.8 hold, then Conjecture 4.7 holds.

**Proof.** The assertion is a direct consequence of a result of Yoshida in [Yos01]. We use freely the terminology therein. Let $M$ and $N$ be pure motives over $\mathbb{Q}$ with coefficients in a number field $E$. Denote by $X_M \in M_{d(M),d(M)}(\mathbb{E}_E \otimes \mathbb{C})$ and $X_N \in M_{d(N),d(N)}(\mathbb{E}_E \otimes \mathbb{C})$ the period matrices of $M$ and $N$, respectively (well-defined up to $E^\times$). In [Yos01] Proposition 12, Yoshida explicitly computed the Deligne’s periods $c^{\pm}(M \otimes N)$ of the tensor motive $M \otimes N$, and show that there exist polynomial functions $f^\pm$ and $g^\pm$ on $M_{d(M),d(M)}$ and $M_{d(N),d(N)}$ respectively whose types are determined by the Hodge numbers of $M$ and $N$ (including the ranks of $\pm$-eigenspaces of the middle Hodge types), such that

$$c^{\pm}(M \otimes N) \in f^\pm(X_M)g^\pm(X_N) \cdot \mathbb{E}^\times.$$

Therefore, if $M'$ and $N'$ are another pure motives over $\mathbb{Q}$ with coefficients in $E$ such that the Hodge numbers of $M'$ and $N'$ are equal to that of $M$ and $N$, respectively, then we have

$$c^{\pm}(M \otimes N) \cdot c^{\pm}(M' \otimes N') \in f^\pm(X_M)g^\pm(X_N) \cdot f^\pm(X_{M'})g^\pm(X_{N'}) \cdot \mathbb{E}^\times = \mathbb{E}^\times.$$

Let $\Sigma, \Sigma'$ (resp. $\Pi, \Pi'$) be algebraic automorphic representations of $GL_n(\mathbb{A})$ (resp. $GL_m(\mathbb{A})$) such that

$$\Sigma_k = \Sigma'_k, \quad \Pi_k = \Pi'_k.$$

When $\Sigma$ and $\Pi$ are tamely isobaric (then so are $\Sigma'$ and $\Pi'$), we see that Conjecture 4.7 follows from Conjecture 4.7 and (4.21) by taking

$$M = M_\Sigma, \quad M' = M_{\Sigma'}, \quad N = M_\Pi, \quad N' = M_{\Pi'}.$$

The general case follows from this special case by decomposing $\Sigma, \Sigma', \Pi, \Pi'$ into sums of algebraic tamely isobaric automorphic representations according to the exponents of the summands.

**5. Applications to Deligne’s conjecture**

This section is devoted to the applications of Theorem 4.2 to Conjecture 4.7. In general, Deligne’s conjecture is widely open. It is known for rank 1 motives over totally real or CM fields (cf. [Bla88], and for motives associated to cohomological cuspidal automorphic representations of $GL_2(\mathbb{A})$ (cf. [Del79], [Sch90]). Recently, Grobner, Harris, and Lin [GHL21] proved the conjecture for tensor product of two motives associated to cohomological conjugate self-dual cuspidal automorphic representations over CM-fields under various assumptions (see also [GHL21]). For some other known cases in the literature, we have results for Rankin–Selberg $L$-functions for $GL_n \times GL_2$ in the unbalanced case, tensor product $L$-functions for $GL_2$, and symmetric odd power $L$-functions for $GL_2$ recalled in §§5.3, 5.4 and §§5.5 respectively. In these known cases, the conjecture is proven based on integral representation of the $L$-function in question, and a cohomological interpretation of the integral representation (or algebraicity of the Eisenstein series involved in the construction). As applications of our main result Theorem 4.2 we provide another approach to prove Deligne’s conjecture for these motives in higher rank cases, where integral representation of the $L$-functions are not available. The main results of this section are Theorems 5.5, 5.8 and 5.11 which are proved based on mathematical induction. The strategy is as follows:

- Show that the automorphic/motivic $L$-function in question is equal to (a factor of) a Rankin–Selberg $L$-function $L(s, \Sigma \times \Pi)$ for some cohomological tamely isobaric automorphic representations $\Sigma$ and $\Pi$.
- Choose auxiliary tamely isobaric automorphic representations $\Sigma'$ and $\Pi'$ with $\Sigma_k = \Sigma'_k$ and $\Pi_k = \Pi'_k$, such that the Rankin–Selberg $L$-functions $L(s, \Sigma \times \Pi'), L(s, \Sigma' \times \Pi)$, and $L(s, \Sigma' \times \Pi')$ decompose into product of automorphic/motivic $L$-functions in question with smaller degree.
• By Theorem 5.2 and the induction hypothesis, we are reduced to the cases for \(GL_2 \times GL_1, GL_2 \times GL_2,\) and \(GL_2 \times GL_2 \times GL_2\) which are already known thanks to [Shi 76], [Shi 77], and [GH 93]. In the proof of Theorem 5.5 we also need the results [Har 97], [Har 21], and [GL 16] for \(GL_n \times GL_1\) over CM-fields.

We recall in §5.1 and §5.2 some notation and algebraicity results.

5.1. Automorphic tensor products. For automorphic representations \(H_1\) and \(H_2\) of \(GL_n(A)\) and \(GL_n'(A)\), we denote by \(H_1 \boxtimes H_2\) the automorphic induction of \(H_1\) and \(H_2\). It is an irreducible admissible ((\(g_{nn'}, O_{nn'}(R)\)) \(\times GL_{nn'}(A))\)-module defined by the restricted tensor product

\[
H_1 \boxtimes H_2 = \bigotimes_v H_{1,v} \boxtimes H_{2,v}.
\]

Here \(H_{1,v} \boxtimes H_{2,v}\) is the irreducible admissible representation of \(GL_{nn'}(Q_v)\) (resp. \((g_{nn'}, O_{nn'}(R))\)-module) if \(v\) is finite (resp. archimedean) defined by the local Langlands correspondence with respect to the tensor representation

\[
GL_n(C) \times GL_{nn'}(C) \longrightarrow GL_{nn'}(C).
\]

As special cases of the Langlands functoriality conjecture, automorphic tensor products are expect to be automorphic. Thanks to the results of Ramakrishnan [Ram 00], Dieulefait [Die 20], and Newton–Thorne [NT 21a, NT 21b], all the automorphic tensor products appear in this section will be automorphic and tamely isobaric.

5.2. Cohomological cuspidal automorphic representations of \(GL_2(A)\) of CM-type. Let \(k\) be an imaginary quadratic extension of \(Q\). Let \(c \in \text{Gal}(k/Q)\) be the non-trivial automorphism. We identify \(k_{\infty}\) with \(C\) by the natural inclusion \(k \subset C\). Let \(\chi\) be a Hecke character of \(A_k^\times\). Assume \(\chi\) is algebraic, that is, there exist \(a, b \in Z\) such that

\[
\chi_{\infty}(z) = z^a z^b.
\]

In this case, let \(k(\chi)\) be the number field generated over \(k\) by the values of \(\chi\) on the finite ideles \(A_k^\times\). As in [HK 91 Appendix] and [Har 93 §1], we denote by

\[
p(\chi, \tau) \in C^\times/k(\chi)^\times
\]

the CM-period of \(\chi\) defined with respect to the CM-type \(\{\tau\} \subset \text{Gal}(k/Q)\). Moreover, we normalize the sequence of periods \(\{p(\chi, \sigma \circ \tau)\}_{\sigma \in \text{Aut}(C/k)}\) in a compatible way as in [Har 93 Lemma 1.3 and Remark 1.3.1]. Then we have

\[
\sigma \left( \frac{L(\chi)(m, \chi)}{(2\pi \sqrt{-1})^m \cdot p(\chi, \tau \chi)} \right) = \frac{L(\chi)(m, \sigma \chi)}{(2\pi \sqrt{-1})^m \cdot p(\sigma \chi, \tau \chi)}
\]

for all critical points \(m \in Z\) for \(L(s, \chi)\) and \(\sigma \in \text{Aut}(C/k)\). Here \(\tau \chi = \text{id}\) (resp. \(\tau \chi = c\)) if \(a < b\) (resp. \(a > b\)). More generally, we have the following result on the algebraicity of twisted standard \(L\)-functions for \(GL_n(A_k)\) due to Harris and Guerberoff–Lin. We only state the result in the unbalanced case.

Theorem 5.1 (Harris [Har 97, Har 21], Guerberoff–Lin [GL 16]). Let \(\Psi\) be a cohomological conjugate self-dual cuspidal automorphic representation of \(GL_n(A_k)\) with central character \(\omega \Psi\). Let \(\{z^a \bar{z}^{-a} \}_{1 \leq i \leq n}\) be the infinitesimal type of \(\Psi\). We assume the following assumption is satisfied:

• There exists a \(n\)-dimensional non-degenerate definite Hermitian space \(V\) over \(k\) such that \(\Psi\) descends to an automorphic representation of \(U(V)(A)\).

Let \(\chi\) be an algebraic Hecke character of \(A_k^\times\) with infinitesimal type \(z^a \bar{z}^b\) such that \(a \neq b\) and \(2a_i + a - b > 0\) for all \(1 \leq i \leq n\). For a critical point \(m_0 \in Z + \frac{a - b}{2}\) and \(\sigma \in \text{Aut}(C/k)\), we have

\[
\sigma \left( \frac{L(\chi)(m_0, \Psi \otimes \chi)}{(2\pi \sqrt{-1})^{nm_0} \cdot p(\omega \Psi, \epsilon) \cdot p(\chi, \epsilon)^n} \right) = \frac{L(\chi)(m_0, \sigma \Psi \otimes \sigma \chi)}{(2\pi \sqrt{-1})^{nm_0} \cdot p(\omega \Psi, \epsilon) \cdot p(\sigma \chi, \epsilon)^n}.
\]

Here \(L(s, \Psi \otimes \chi)\) is the standard \(L\)-function of \(\Psi \otimes \chi\) and \(\bar{\chi} = (\chi^c)^{-1}\).

Remark 5.2. Let \(V\) be a definite Hermitian space over \(k\). By Arthur’s multiplicity formula for \(U(V)(A)\) announced by Kaletha–Mínguez–Shin–White in [KMSW Theorem* 1.7.1], the obstruction to descending \(\Psi\) to \(U(V)(A)\) is purely local. In particular, if either \(n\) is odd or \(n \equiv 0 \mod 4\), then the assumption always holds. When \(n \equiv 2 \mod 4\), the assumption holds if we assume further that there exists a prime \(p\) such that \(\Psi\) is a discrete series representation and \(p\) is inert in \(k\).
For an algebraic Hecke character \( \chi \) of \( \mathbb{A}^\times_\mathbb{K} \) with infinity type \( z^\omega \), we denote by \( I_\mathbb{K}(\chi|_{1/2}) \) the automorphic induction of \( \chi|_{1/2} \) to \( GL_2(\mathbb{A}) \). Recall that the central character of \( I_\mathbb{K}(\chi|_{1/2}) \) is equal to \( \omega_{\mathbb{K}/\mathbb{Q}}\chi|_{1/2} \), where \( \omega_{\mathbb{K}/\mathbb{Q}} \) is the quadratic Hecke character of \( \mathbb{A}^\times_\mathbb{K} \) associated to \( \mathbb{K}/\mathbb{Q} \) by class field theory. If \( a \neq b \), then \( I_\mathbb{K}(\chi|_{1/2}) \) is cohomological cuspidal with infinity type
\[
(a-b+1; \ a+b+1).
\]
In the following lemma, we recall a period relation for cohomological cuspidal automorphic representations of \( GL_2(\mathbb{A}) \) of CM-type.

**Lemma 5.3.** Let \( \chi \) be an algebraic Hecke character of \( \mathbb{A}^\times_\mathbb{K} \) with infinity types \( z^\omega \). For \( \sigma \in Aut(\mathbb{C}/\mathbb{K}) \), we have
\[
\sigma \left( \frac{p(I_\mathbb{K}(\chi|_{1/2}), \pm)}{(2\pi \sqrt{-1})^{-\max(a,b)} \cdot p(\chi, \tau_\chi)} \right) = \frac{p(I_\mathbb{K}(\tau\chi|_{1/2}), \pm)}{(2\pi \sqrt{-1})^{-\max(a,b)} \cdot p(\sigma\tau_\chi, \tau_\chi)}.
\]
Here \( \tau_\chi = \text{id} \) (resp. \( \tau_\chi = c \)) if \( a < b \) (resp. \( a > b \)).

**Proof.** Note that for any finite order Hecke character \( \xi \) of \( \mathbb{A}^\times_\mathbb{K} \), we have
\[
L(s, \chi \cdot (\xi \circ N_{\mathbb{K}/\mathbb{Q}})) = L(s - \frac{1}{2}, I_\mathbb{K}(\chi|_{1/2}) \otimes \xi).
\]
On the other hand, by [Har93, Proposition 1.4] and [Har97, (1.10.10)], we have
\[
\sigma \left( \frac{p(\chi \cdot (\xi \circ N_{\mathbb{K}/\mathbb{Q}}), \tau)}{G(\xi) \cdot p(\chi, \tau)} \right) = \frac{p(\tau\chi \cdot (\xi \circ N_{\mathbb{K}/\mathbb{Q}}), \sigma \cdot \tau)}{G(\xi) \cdot p(\tau\chi, \sigma \cdot \tau)}
\]
for all \( \sigma \in Aut(\mathbb{C}) \) and \( \tau \in Gal(\mathbb{K}/\mathbb{Q}) \). The assertion then follows immediately from Theorem 4.4-(1) and (5.1) by choosing \( \xi \) so that all the critical values of \( L(s, \chi \cdot (\xi \circ N_{\mathbb{K}/\mathbb{Q}})) \) are non-zero (cf. [Roh89]). \( \square \)

### 5.3. Rankin–Selberg L-functions for \( GL_n \times GL_2 \)
Let \( \Sigma \) and \( \Pi \) be cohomological cuspidal automorphic representations of \( GL_n(\mathbb{A}) \) and \( GL_2(\mathbb{A}) \) with infinity types \( (\kappa; \omega) \) and \( (\kappa'; \omega') \), respectively. Let \( f_\Pi \) be the normalized newform of \( \Pi \) in the sense of Casselman [Cas73], and \( \|f_\Pi\| \) be its Petersson norm defined as in (1.3). In [FM10, Appendix A], Morimoto explicitly computed the (conjectural) Deligne’s periods of the tensor product motive \( M_\Sigma \otimes M_\Pi \) when \( n \) is even and \( \kappa' > \kappa_1 \). Similar computation can be carried out when \( n \) is odd (cf. [Bha14]). The resulting formula, under the unbalanced condition \( \kappa' > \kappa_1 \), is expressed in terms of \( G(\omega_\Sigma) \) and motivic periods \( c^\pm(\Pi) \) associated to \( \Pi \) (cf. Remark 4.9), which is irrelevant to the existence of \( M_\Sigma \). We have the following refinement of Conjecture 4.7 for \( M_\Sigma \otimes M_\Pi \):

**Conjecture 5.4.** Assume \( \kappa' > \kappa_1 \). Put \( \delta \in \{0, 1\} \) with \( \delta \equiv n \pmod{2} \). For a critical point \( m + \frac{\delta}{2} \in \mathbb{Z} + \frac{\omega}{2} \) and \( \sigma \in Aut(\mathbb{C}) \), we have
\[
\sigma \left( \frac{L(\kappa, (m + \frac{\delta}{2}, \Sigma \times \Pi))}{(2\pi \sqrt{-1})^{nm} \cdot q^{(-1)^m(\Sigma \times \Pi)}} \right) = \frac{L(\kappa, (m + \frac{\delta}{2}, \sigma \Sigma \times \sigma \Pi))}{(2\pi \sqrt{-1})^{nm} \cdot q^{(-1)^m(\sigma \Sigma \times \sigma \Pi)}}.
\]
Here
\[
q^\pm(\Sigma \times \Pi) = (2\pi \sqrt{-1})^{\frac{n\delta}{2} + (nw + (n-1)(\kappa' + \omega'))/2} \cdot (\sqrt{-1})^{\left(\frac{n\delta}{2} + \frac{n}{2}\right) |\omega'| \cdot G(\omega_\Sigma) \cdot G(\omega_\Pi) \cdot f_\Pi \|f_\Pi\| \cdot \varepsilon}
\]
and \( \varepsilon = (-1)^{\frac{n\delta}{2} + \omega_\Sigma \cdot \omega_{\Sigma} \cdot \varepsilon} \) if \( n \) is odd.

In [FM13] and [FM10], Furusawa and Morimoto proved algebraicity results for \( SO(V) \times GL_2 \), which imply Conjecture 5.4 under the following global and regularity assumptions:
- A twist of \( \Sigma \) by some integral power of \( |_{1/2} \) is self-dual and descends to an automorphic representation of \( SO(V)(\mathbb{A}) \) for some anisotropic quadratic space \( V \) over \( \mathbb{Q} \). In particular, \( n \) must be even and \( \Sigma \) is essentially self-dual.
- \( \kappa' \geq \kappa_1 + \dim_{\mathbb{Q}} V + 2 \).

We prove the conjecture for essentially self-dual \( \Sigma \) under some parity and regularity conditions on the infinity types, and a local condition when \( n \equiv 2 \pmod{4} \). More precisely:
Theorem 5.5. Conjecture \[5.4\] holds if the following conditions are satisfied:

1. \(\Sigma\) is essentially self-dual.
2. If \(n \equiv 2 \pmod 4\), then \(\Sigma_p\) is an essentially discrete series representation for some prime \(p\).
3. \(w + w' \equiv n \pmod 2\).
4. \(\min\{\kappa_i\} \geq 3\) if \(n\) is even, \(\min\{\kappa_i\} \geq 5\) if \(n\) is odd, and \(\kappa = (4 + 2\delta)\)-regular, where \(\delta = 0\) if \(w\) or \(w'\) is odd and \(\delta = 1\) otherwise.

Proof. Put \(\delta \equiv n \pmod 2\) and \(r = \lfloor \frac{n}{2} \rfloor\). We rewrite \(q^\pm(\Sigma \times \Pi)\) in terms of the Betti–Whittaker periods of \(\Pi\) (cf. (1.20)):

\[
q^\pm(\Sigma \times \Pi) = (2\pi\sqrt{-1})^{\delta+n(w+w'+w')/2} \cdot (\sqrt{-1})^{w'} \cdot G(\omega_\Sigma) \cdot G(\omega_{\Pi})^r \cdot \|f_H\|^{r'}
\]

\[
\times \begin{cases} 1 & \text{if } n \text{ is even,} \\
 p(\Pi, \pm(-1)^{r+w/2}\omega_{\Sigma,\chi}(-1)) & \text{if } n \text{ is odd.}
\end{cases}
\]

We choose auxiliary cohomological tamely isobaric automorphic representations \(\Pi'\) and \(\Pi''\) of \(\text{GL}_n(\mathbb{A})\) and \(\text{GL}_2(\mathbb{A})\) respectively as follows:

- For \(1 \leq i \leq r\), let \(\Pi_i\) be a cohomological cuspidal automorphic representation of \(\text{GL}_2(\mathbb{A})\) with infinity type \((\kappa_i; w + \delta)\). If \(n\) is odd, let \(\xi\) be a finite order Hecke character of \(\mathbb{A}^\times\) such that \(\xi_{\kappa}(-1) = (-1)^r\omega_{\Sigma,\chi}(-1)\). We define \(\Sigma'\) by

\[
\Sigma' = (\Pi_1 \otimes |_{\mathbb{A}}^{\delta/2}) \oplus \cdots \oplus (\Pi_r \otimes |_{\mathbb{A}}^{-\delta/2}) \oplus \begin{cases} 0 & \text{if } n \text{ is even,} \\
 \xi |_{\mathbb{A}}^{w/2} & \text{if } n \text{ is odd.}
\end{cases}
\]

- Let \(K\) be an imaginary quadratic extension of \(\mathbb{Q}\) such that \(n \equiv 2 \pmod 4\), there exists a prime number \(p\) which is inert in \(K\) and \(\Sigma_p\) is an essentially discrete series representation. We then choose \(\Pi' = I_K(\chi')|_{\mathbb{A}^\times}^{1/2}\) for some algebraic Hecke character \(\chi'\) of \(\mathbb{A}_K^\times\) with infinity type \(z(\kappa' + w' - 2)/2\).

Let \(m + \frac{\delta}{2} \in \mathbb{Z} + \frac{\delta}{2}\) be a critical point for \(L(s, \Sigma \times \Pi)\). By Theorem \[3.4\] and the result of Shimura \[Shi76\], Conjecture \[5.4\] is known when \(n = 1, 2\). Therefore, a direct computation shows that

\[
L^{(\kappa)}(m + \frac{\delta}{2}, \Sigma' \times \Pi') \sim (2\pi\sqrt{-1})^{nm} \cdot q^{-(1)^n} (\Sigma' \times \Pi'),
\]

\[
L^{(\kappa)}(m + \frac{\delta}{2}, \Sigma' \times \Pi') \sim (2\pi\sqrt{-1})^{nm} \cdot q^{-(1)^n} (\Sigma' \times \Pi').
\]

By condition (1) in Theorem \[5.5\] we have \(\Sigma' = \Sigma \otimes \chi\) for some algebraic Hecke character \(\chi\) of \(\mathbb{A}_K^\times\). Fix an algebraic Hecke character \(\eta\) of \(\mathbb{A}_K^\times\) such that \(\eta_{\kappa}(z) = z^{-w}\) and \(\eta|_{\mathbb{A}^\times} = \chi\) (cf. \[CHT08\], Lemma 4.1.4]). Let \(BC_k(\Sigma)\) be the base change of \(\Sigma\) to \(\text{GL}_n(\mathbb{A}_K)\). Put \(\Psi = BC_k(\Sigma) \otimes \eta\). Then \(\Psi\) is cohomological, conjugate self-dual, and tamely isobaric. Moreover, the infinity type of \(\Psi\) is given by \(\{z^a\}_{1 \leq a \leq n}\) with

\[
a_i = \kappa_i - 1 - w,
\]

\[
a_i + a_{n-i+1} = -w\]

for \(1 \leq i \leq r\) and \(a_{r+1} = -w\) if \(n\) is odd. By the adjointness between automorphic induction and base change, we have

\[
L(s, \Sigma \times \Pi') = L(s + \frac{1}{2}, BC_k(\Sigma) \otimes \chi'),
\]

\[
L(s + \frac{1}{2}, \Psi \otimes \eta^{-1} \chi').
\]

We assume further that \(K\) is chosen so that \(BC_k(\Sigma)\) is cuspidal. Note that the assumption in Theorem \[5.1\] holds for \(BC_k(\Sigma)\) (cf. Remark \[5.2\]) It then follows from the assumption \(\kappa' > \kappa_1\) and Theorem \[5.1\] that

\[
L^{(\kappa)}(m + \frac{\delta}{2}, \Sigma \times \Pi') \sim_k (2\pi\sqrt{-1})^{(m + (\delta + 1)/2)} \cdot p(\omega, c) \cdot p(\eta^{-1} \chi', c)\]

Here \((\sim_k)\) means the ratio is equivariant under \(\text{Aut}(\mathbb{C}/\mathbb{K})\) (recall the notation in \[11.3\], here we replace \(\text{Aut}(\mathbb{C})\) by \(\text{Aut}(\mathbb{C}/\mathbb{K})\)). By Theorem \[4.3\] (2) and Lemma \[5.3\] we have

\[
p(\chi', c)^n \sim_k (2\pi\sqrt{-1})^{r+n(\kappa' + w' - 2)/2} \cdot (\sqrt{-1})^{r'} \cdot G(\omega_{\Pi'})^r \cdot \|f_{\Pi'}\|^{r'}
\]

\[
\times \begin{cases} 1 & \text{if } n \text{ is even,} \\
 p(\Pi', \pm) & \text{if } n \text{ is odd.}
\end{cases}
\]

By \[Har93\] Proposition 1.4-(b)], for algebraic Hecke characters \(\chi_1, \chi_2\) of \(\mathbb{A}_K^\times\), we have

\[
p(\chi_1 \chi_2, \tau) \sim_k p(\chi_1, \tau) \cdot p(\chi_2, \tau), \quad \tau \in \text{Gal}(\mathbb{K}/\mathbb{Q}).
\]
In particular, since $\omega_\Psi = \langle \omega_{\Sigma} \circ \mathbb{N}_{K/Q} \rangle \cdot \eta^n$, we have
\[
P(\tilde{\omega}_\Psi, c) \cdot p(\tilde{\eta}^{-1}, c)^n \sim_K p((\omega_{\Sigma} \circ \mathbb{N}_{K/Q}), c)
\sim_K (2\pi \sqrt{-1})^{nw/2} \cdot G(\omega_{\Sigma}).
\]
Here we have used [Har97, (1.10.9) and (1.10.10)]. We thus conclude that
\[
L^{(\infty)}(m + \frac{\delta}{2}, \Sigma \times \Pi') \sim_K (2\pi \sqrt{-1})^{nm} \cdot q^{(-1)m}(\Sigma \times \Pi').
\]
(5.3)

Note that conditions (3) and (4) imply that the conditions in Theorem 1.2 are satisfied. Therefore, by (5.2),
\[
L^{(\infty)}(m + \frac{\delta}{2}, \Sigma \times \Pi') \sim_K (2\pi \sqrt{-1})^{nm} \cdot q^{(-1)m}(\Sigma \times \Pi')
\sim_K (2\pi \sqrt{-1})^{nm} \cdot q^{(-1)m}(\Sigma' \times \Pi').
\]
(5.4)

Here the first and second equalities follow from (5.5) and (5.4), respectively. This completes the proof.

Remark 5.6. If we assume the validity of Conjecture [1.1], then we can prove Conjecture [5.4] under assumptions (1) and (2), and the regularity condition that $K \cap \kappa'$ is 2-regular.

5.4. Tensor product $L$-functions for $GL_2$. Let $\Pi_1, \cdots, \Pi_n$ be cohomological cuspidal automorphic representations of $GL_2(\mathbb{A})$. For $1 \leq i \leq n$, let $(\kappa_i; w_i)$ be the infinity type of $\Pi_i$. Let $f_{\Pi_i}$ be the normalized newform of $\Pi_i$, and $\|f_{\Pi_i}\|$ be its Petersson norm defined as in [13]. We consider the automorphic tensor product $\Pi_1 \boxtimes \cdots \boxtimes \Pi_n$ of $\Pi_1, \cdots, \Pi_n$. As a special instance of Langlands functoriality conjecture, we expect $\Pi_1 \boxtimes \cdots \boxtimes \Pi_n$ to be automorphic and tamely isobaric. For $n = 2$, the functoriality was proved by Ramakrishnan [Ram99]. For $n \geq 3$, Dieulefait [Die20] proved the functoriality under the assumptions that $\Pi_i$ has level 1 for all $1 \leq i \leq n$ and the infinitesimal character of $\Pi_1, \cdots, \Pi_n$ is regular. Consider the tensor product $L$-function defined by the Euler product
\[
L(s, \Pi_1 \times \cdots \times \Pi_n) = \prod_v L(s, \Pi_{1,v} \boxtimes \cdots \boxtimes \Pi_{n,v}),
\]
which converges absolutely for Re$(s) > 1 - \frac{1}{2} \sum_{i=1}^n w_i$. A critical point for $L(s, \Pi_1 \times \cdots \times \Pi_n)$ is a half-integer in $\mathbb{Z} + \frac{\delta}{2}$ which is not a pole of the archimedean local factors $L(s, \Pi_{1,\infty} \boxtimes \cdots \boxtimes \Pi_{n,\infty})$ and $L(1-s, \Pi_{1,\infty} \boxtimes \cdots \boxtimes \Pi_{n,\infty})$. The tensor product $L$-function admits critical points if and only if
\[
\sum_{i=1}^n \varepsilon_i (\kappa_i - 1) \neq 0
\]
for all $(\varepsilon_1, \cdots, \varepsilon_n) \in \{\pm 1\}^n$. In [Bla87], when $n \geq 2$, Blasius explicitly computed Deligne’s period of the tensor product motive $M_{\Pi_1} \otimes \cdots \otimes M_{\Pi_n}$ in terms of the Petersson norms $\|f_{\Pi_1}\|, \cdots, \|f_{\Pi_n}\|$. More precisely, we have the following refinement of Conjecture [1.7] for $M_{\Pi_1} \otimes \cdots \otimes M_{\Pi_n}$:

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Conjecture 5.7 (Blasius). Assume $n \geq 2$ and \eqref{5.6} holds. Put $\delta \in \{0, 1\}$ with $\delta \equiv n \pmod{2}$. The tensor product $L$-function admits meromorphic continuation to $s \in \mathbb{C}$ and is entire if it admits critical points. For a critical point $m + \frac{\delta}{2} \in \mathbb{Z} + \frac{\delta}{2}$ and $\sigma \in \text{Aut}(\mathbb{C})$, we have

\begin{equation}
(5.7) \quad \sigma \left( \frac{L^{(\infty)}(m + \frac{\delta}{2}, \Pi_1 \times \cdots \times \Pi_n)}{(2\pi \sqrt{-1})^{2n-1}m \cdot q(\Pi_1 \times \cdots \times \Pi_n)} \right) = \frac{L^{(\infty)}(m + \frac{\delta}{2}, \sigma \Pi_1 \times \cdots \times \sigma \Pi_n)}{(2\pi \sqrt{-1})^{2n-1}m \cdot q(\sigma \Pi_1 \times \cdots \times \sigma \Pi_n)}.
\end{equation}

Here

\[ q(\Pi_1 \times \cdots \times \Pi_n) = (2\pi \sqrt{-1})^{2n-2}(\delta + \sum_{i=1}^{n} w_i) \cdot \prod_{i=1}^{n} G(\omega \Pi_i)^{2n-2} \cdot (\pi^{\kappa_i} \cdot ||f_{\Pi_i}||)^{2n-2-t_i}. \]

with $t_i$ equal to the cardinality of the set

\begin{equation}
(5.8) \quad \left\{ (\varepsilon_1, \ldots, \varepsilon_n) \in \{ \pm 1 \}^n \mid 2(\kappa_i - 1) < \sum_{j=1}^{n} \varepsilon_j (\kappa_j - 1), \quad \varepsilon_i = 1 \right\}
\end{equation}

for $1 \leq i \leq n$.

For $n = 2$, the conjecture was proved by Shimura in \cite[Theorem 3]{Shimura}. When $n = 3$, there are two cases according to $\kappa_1 + \kappa_2 + \kappa_3 \geq 2 \max\{\kappa_1, \kappa_2, \kappa_3\}$ or $2 \max\{\kappa_1, \kappa_2, \kappa_3\} > \kappa_1 + \kappa_2 + \kappa_3$. The former (resp. later) case is called the balanced (resp. unbalanced) case. In the balanced case, the conjecture was proved by Garrett and Harris in \cite{GH93} under some conditions which were lifted by the author in \cite{Che21a}. We refer to the introduction of \cite{Che21a} for a survey of known results in this case. In the unbalanced case, the conjecture is partially proved. In \cite{HK91}, Harris and Kudla prove the case for central critical values when $\omega_{\Pi_1, \omega_{\Pi_2}, \omega_{\Pi_3}} = 1$. In \cite{FM14} and \cite{FM16}, as an application to their main result, Furusawa and Morimoto proved the conjecture under some local and global assumptions. Assume $\kappa_1 > \kappa_2 + \kappa_3$, by applying Theorem \ref{5.5} to $\Pi = (\Pi_2 \boxtimes \Pi_3) \boxtimes | \frac{A}{A} |^{t_i}$ and $\Pi' = \Pi_1$, the conjecture holds under the following conditions:

\begin{equation}
(5.9) \quad w_1 + w_2 + w_3 \equiv 1 \pmod{2}, \quad \kappa_1 \geq \kappa_2 + \kappa_3 + 3 \geq |\kappa_2 - \kappa_3| + 9 \geq 11.
\end{equation}

For higher $n$, we prove the conjecture under some parity and regularity conditions on the infinity types, together with an assumption on functoriality if $n \geq 5$. More precisely:

Theorem 5.8. Assume $n \geq 2$ and \eqref{5.6} holds. Conjecture \ref{5.7} holds if the following conditions are satisfied:

1. $w_1 + \cdots + w_n \equiv n \pmod{2}$.
2. $| \sum_{i=1}^{n} (\varepsilon_i - \varepsilon'_i)(\kappa_i - 1) | \geq 6$ for all $(\varepsilon_1, \ldots, \varepsilon_n)$ and $(\varepsilon'_1, \ldots, \varepsilon'_n)$ in $\{ \pm 1 \}^n$.
3. $\Pi_1 \boxtimes \cdots \boxtimes \Pi_k$ is automorphic and tamely isobaric for all $1 \leq k \leq n - 2$.

In particular, condition (3) holds if either $n \leq 4$ or $\Pi_1, \ldots, \Pi_{n-2}$ have level 1.

Proof. First note that condition (3) implies that $L(s, \Pi_1 \times \cdots \times \Pi_n)$ is entire. Indeed, by \eqref{5.6}, the isobaric summands of $\Pi_1 \boxtimes \cdots \boxtimes \Pi_{n-2}$ and $\Pi_{n-1}' \boxtimes \Pi_{n-1}''$ must be non-isomorphic. For the algebraicity assertion \eqref{5.7}, we will prove it by induction on $n$. As we mentioned in \S 5.4, it holds unconditionally when $n = 2$ and holds for $n = 3$ under conditions (1) and (2) in the unbalanced case. Now let $n \geq 4$ and assume \ref{5.7} holds for $n - 1$. We choose some auxiliary automorphic representations $\Pi(\varepsilon), \Pi_{n-1}', \Pi_{n-1}''$ as follows:

- For $\varepsilon \in \{ \pm 1 \}^{n-2}/\{ \pm 1 \}$, let $\Pi(\varepsilon)$ be a cohomological cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ with infinity type

\[ \left( \sum_{i=1}^{n-2} \varepsilon_i (\kappa_i - 1) + 1; \sum_{i=1}^{n-2} w_i + 1 + \delta \right). \]

- Let $\Pi_{n-1}'$ and $\Pi_{n-1}''$ be cohomological cuspidal automorphic representations of $\text{GL}_2(\mathbb{A})$ with infinity types

\[ (\kappa_{n-1} + \kappa_n - 1; w_{n-1} + w_n - 1), \quad (|\kappa_{n-1} - \kappa_n| + 1; w_{n-1} + w_n - 1) \]

respectively.

Let $\Sigma$ and $\Sigma'$ be cohomological tamely isobaric automorphic representations of $\text{GL}_{2n-2}(\mathbb{A})$ defined by

\[ \Sigma = (\Pi_1 \boxtimes \cdots \boxtimes \Pi_{n-2}) \boxtimes | \frac{A}{A} |^{(1+\delta)/2}, \quad \Sigma' = \bigoplus_{\varepsilon \in \{ \pm 1 \}^{n-2}/\{ \pm 1 \}} \Pi(\varepsilon). \]
Let $\Pi$ and $\Pi'$ be cohomologically tamely isobaric automorphic representations of $\text{GL}_4(\mathbb{A})$ defined by

$$\Pi = (\Pi_{n-1} \boxtimes \Pi_n) \otimes |_\lambda^{-1/2}; \quad \Pi' = \Pi'_{n-1} \boxtimes \Pi''_{n-1}.$$  

Then we have $\Sigma_\lambda = \Sigma'_\lambda$ and $\Pi_\lambda = \Pi'_\lambda$. Put $\delta \in \{0,1\}$ with $\delta \equiv n \pmod{2}$. Let $m + \frac{\delta}{2} \in \mathbb{Z} + \frac{\delta}{2}$ be a critical point for $L(s, \Pi_1 \times \cdots \times \Pi_n)$. It is clear that we have the following equality and factorization:

$$L(s, \Sigma \times \Pi) = L(s + \frac{\delta}{2}, \Pi_1 \times \cdots \times \Pi_n),$$

$$L(s, \Sigma \times \Pi') = L(s + \frac{\delta + \delta}{2}, \Pi_1 \times \cdots \times \Pi_{n-2} \times \Pi'_{n-1}) \cdot L(s + \frac{\delta + \delta}{2}, \Pi_1 \times \cdots \times \Pi_{n-2} \times \Pi''_{n-1}).$$

Recall the sequence $(t_1, \ldots, t_n) \in \mathbb{Z}^n$ defined in (5.8) with respect to $(\kappa_1, \ldots, \kappa_n)$. Similarly, let $(t_1', \ldots, t'_{n-1})$ and $(t_{n}', \ldots, t''_{n-1})$ be sequences of integers in $\mathbb{Z}^{n-1}$ defined with respect to $(\kappa_1, \ldots, \kappa_{n-2}, \kappa_{n-1} + \kappa_n - 1)$ and $(\kappa_1, \ldots, \kappa_{n-2}, |\kappa_{n-1} - \kappa_n| + 1)$, respectively. It is clear that conditions (1)-(3) in Theorem 5.8 are satisfied for the tensor product $L$-functions $L(s, \Pi_1 \times \cdots \times \Pi_{n-2} \times \Pi'_{n-1})$ and $L(s, \Pi_1 \times \cdots \times \Pi_{n-2} \times \Pi''_{n-1})$. Therefore, by induction hypothesis, we have

$$(5.10) \quad L^{(\infty)}(m, \Sigma \times \Pi')$$

$$\sim (2\pi \sqrt{-1})^{2-1} \cdot m + 2^{n-2}(\delta + \sum_{i=1}^{n-1} w_i) \cdot \prod_{i=1}^{n-2} G(\omega_{n_{i-1}})^{2-2} \cdot (\pi_{\Sigma_i} \cdot \|f_{n_{i-1}}\|)^{2-2} - t'_{n-1}$$

$$\times G(\omega_{n_{n-1}})^{2-2} \cdot G(\omega_{n_{n-1}}')^{2-2} \cdot (\pi_{\Sigma_i} \cdot \|f_{n_{n-1}}\|)^{2-2} - t'_{n-1} \cdot (\pi_{\Sigma_i} \cdot \|f_{n_{n-1}}\|)^{2-2} - t''_{n-1}.$$  

On the other hand, we have the following factorizations:

$$L(s, \Sigma' \times \Pi') = \prod_{\xi \in \{\pm 1\}^{n-2}/(\pm 1)} L(s, \Pi(\xi) \times \Pi'_{n-1}) \cdot L(s, \Pi(\xi) \times \Pi''_{n-1}),$$

$$L(s, \Sigma' \times \Pi) = \prod_{\xi \in \{\pm 1\}^{n-2}/(\pm 1)} L(s - \frac{1}{2} \xi, \Pi \times \Pi_{n-1} \times \Pi_n).$$

For each $\xi$, it is clear that conditions (1) and (2) hold for the triple product $L$-function $L(s, \Pi(\xi) \times \Pi_{n-1} \times \Pi_n)$. Therefore, apply (5.7) to $n = 2, 3$, we have

$$(5.11) \quad \frac{L^{(\infty)}(m, \Sigma' \times \Pi')}{L^{(\infty)}(m, \Sigma' \times \Pi)} \sim \frac{G(\omega_{n_{n-1}}')^{2-3} \cdot G(\omega_{n_{n-1}})^{2-3}}{G(\omega_{n_{n-1}}) \cdot G(\omega_{n_{n-1}}')^{2-2} \cdot G(\omega_{n_{n-1}}')^{2-2}} \cdot \frac{1}{\prod_{\xi \in \{\pm 1\}^{n-2}/(\pm 1)} (\pi_{\Sigma_i} \cdot \|f_{n_{n-1}}\|)^{2-2} \cdot (\pi_{\Sigma_i} \cdot \|f_{n_{n-1}}\|)^{2-2} - t'_{n-1} \cdot (\pi_{\Sigma_i} \cdot \|f_{n_{n-1}}\|)^{2-2} - t''_{n-1}) \cdot (\pi_{\Sigma_i} \cdot \|f_{n_{n-1}}\|)^{2-2} - t'_{n-1} \cdot (\pi_{\Sigma_i} \cdot \|f_{n_{n-1}}\|)^{2-2} - t''_{n-1}).$$

Here

$$t_{n-1}(\xi) = \begin{cases} (\xi', \xi_{n-1}) \in \{\pm 1\}^2 & \kappa_{n-1} - 1 < \xi' \sum_{i=1}^{n-2} \epsilon_i(\kappa_i - 1) + \epsilon_{n-1}(\kappa_{n-1} - 1) \end{cases},$$

$$t_n(\xi) = \begin{cases} (\xi', \xi_{n-1}) \in \{\pm 1\}^2 & \kappa_{n-1} - 1 < \xi' \sum_{i=1}^{n-2} \epsilon_i(\kappa_i - 1) + \epsilon_{n-1}(\kappa_{n-1} - 1) \end{cases},$$

$$t'_{n-1}(\xi) = \begin{cases} \xi' \in \{\pm 1\} & \kappa_{n-1} + \kappa_n - 2 < \xi' \sum_{i=1}^{n-2} \epsilon_i(\kappa_i - 1) \end{cases},$$

$$t''_{n-1}(\xi) = \begin{cases} \xi' \in \{\pm 1\} & |\kappa_{n-1} - \kappa_n| < \xi' \sum_{i=1}^{n-2} \epsilon_i(\kappa_i - 1) \end{cases}.$$  

Finally, it is clear from definition that

$$\sum_{\xi \in \{\pm 1\}^{n-2}/(\pm 1)} t_{n-1}(\xi) = t_{n-1}, \quad \sum_{\xi \in \{\pm 1\}^{n-2}/(\pm 1)} t_n(\xi) = t_n,$$

$$\sum_{\xi \in \{\pm 1\}^{n-2}/(\pm 1)} t'_{n-1}(\xi) = t'_{n-1}, \quad \sum_{\xi \in \{\pm 1\}^{n-2}/(\pm 1)} t''_{n-1}(\xi) = t''_{n-1}.$$
We thus conclude from (5.10) and (5.11) that (5.7) holds for $L^{(x)}(m + \frac{r}{2}, \Pi_1 \times \cdots \times \Pi_n)$. This completes the proof. 

\[\Box\]

**Remark 5.9.** If we assume the validity of Conjecture [14] then we can prove that Conjecture 5.7 holds for all non-central critical points under assumption (3).

5.5. **Symmetric odd power $L$-functions for $GL_2$.** Let $\Pi$ be a cohomological cuspidal automorphic representation of $GL_2(\mathbb{A})$ with infinity type $(\kappa; \chi)$. For $n \geq 1$, let $\text{Sym}^n \Pi$ be the functorial lift of $\Pi$ to $GL_{n+1}(\mathbb{A})$ with respect to the symmetric $n$-th power representation of $GL_2(\mathbb{C})$. The existence of the lifts was proved recently by Newton and Thorne in [NT21a] and [NT21b] (we refer to loc. cit. for a survey of known cases for $n \leq 8$). Note that $\text{Sym}^n \Pi$ is cohomological (cf. [Rag10, Theorem 5.3]) and tamely isobaric with infinity type 

$$
((n-2)(n-1) + 1)_{0 \leq i \leq \frac{n-1}{2}} ; \text{nw}
$$

Moreover, it is cuspidal if and only if $\Pi$ is not of CM-type. In [Del79, Proposition 7.7], Deligne explicitly computed the motivic periods of $\text{Sym}^n \Pi$ in terms of $c^\pm(\Pi) = (c^\pm(\Pi))_{\sigma: \mathcal{Q}(\Pi) \to \mathcal{C}}$. In particular, when $n$ is odd, we have the following refinement of Conjecture [17] for $\text{Sym}^n \Pi$:

**Conjecture 5.10** (Deligne). Let $r \geq 1$ and $\chi$ be a finite order Hecke character of $\mathbb{A}$. For a critical point $m + \frac{r}{2} \in \mathbb{Z} + \frac{1}{2}$ for the twisted symmetric $(2r+1)$-th power $L$-function $L(s, \text{Sym}^{2r+1} \Pi \otimes \chi)$ and $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\sigma \left( \frac{L^{(x)}(m + \frac{r}{2}, \text{Sym}^{2r+1} \Pi \otimes \chi)}{(2\pi \sqrt{-1})^{(r+1)m} \cdot G(\chi)^{r+1} \cdot q(-1)^{m \chi_{\sigma}(-1)}(\text{Sym}^{2r+1} \Pi)} \right) = \frac{L^{(x)}(m + \frac{r}{2}, \text{Sym}^{2r+1} \Pi \otimes \sigma \chi)}{(2\pi \sqrt{-1})^{(r+1)m} \cdot G(\sigma \chi)^{r+1} \cdot q(-1)^{m \chi_{\sigma}(-1)}(\text{Sym}^{2r+1} \Pi \otimes \sigma \chi)}.
$$

Here

$$q^\pm(\text{Sym}^{2r+1} \Pi) = (2\pi \sqrt{-1})^{r(r+1)(\omega r - 1)/2} \cdot G(\omega_{\Pi})^{r(r+1)/2} \cdot c^\pm(-1)^r(\Pi)^{(r+1)(r+2)/2} \cdot c^\mp(-1)^r(\Pi)^{(r+1)/2}.
$$

For $r = 1$, the conjecture was proved by Garrett and Harris [GH93] assuming $\kappa \geq 5$. We extend the result of Garrett and Harris to $\kappa = 3, 4$ in [Che21a]. For $r = 2$, we prove the conjecture assuming $\kappa \geq 6$ in [Che21b]. For symmetric even power $L$-functions, we have the results [Stu89] and [Stu89] of Sturm for $\text{Sym}^2$ and [Mor21], [Che21b], [Che21c] for $\text{Sym}^4$ and $\text{Sym}^6$ due to Morimoto and the author. For higher $r$, our result is as follows:

**Theorem 5.11.** Conjecture 5.10 holds when $\kappa$ is odd and $\kappa \geq 5$.

**Remark 5.12.** If we assume the validity of Conjecture [14] then we can prove Conjecture 5.10 for all $\kappa \geq 3$.

As in the previous sections, we prove Theorem 5.11 by induction on $r$ using Theorem 1.2. In the induction step, we will encounter critical values of Rankin–Selberg $L$-functions for $\text{Sym}^{\prime} \Pi \times \Pi' \times \Pi''$ for $\kappa = r, r \pm 1, r - 2$ and some cohomological cuspidal automorphic representation $\Pi''$ of $GL_2(\mathbb{A})$. Based on the same strategy, the algebraicity of these critical values can be obtained. The precise statement is as follows:

**Theorem 5.13.** Let $\Pi''$ be a cohomological cuspidal automorphic representation of $GL_2(\mathbb{A})$ with infinity type $(\kappa'; \chi')$. Let $n \geq 0$. Put $\delta \in \{0, 1\}$ with $\delta \equiv n + 1 \pmod{2}$. Assume the following conditions are satisfied:

1. $\omega \equiv \omega' \equiv 1 \pmod{2}$.
2. $\kappa' - 1 > \frac{(n-2)(\kappa - 1)}{2}$ and $\min\{|n(\kappa - 1) - (\kappa' - 1)|, |(n-2)(\kappa - 1) - (\kappa' - 1)|\} \geq 4$.

Then, for a critical point $m + \frac{r}{2} \in \mathbb{Z} + \frac{n+1}{2}$ for $L(s, \text{Sym}^n \Pi \times \Pi')$ and $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$
\sigma \left( \frac{L^{(x)}(m + \frac{r}{2}, \text{Sym}^n \Pi \times \Pi')}{(2\pi \sqrt{-1})^{(n+1)m} \cdot q(-1)^m(\text{Sym}^n \Pi \times \Pi')} \right) = \frac{L^{(x)}(m + \frac{r}{2}, \text{Sym}^{n-\sigma} \Pi \times \sigma \Pi')}{(2\pi \sqrt{-1})^{(n+1)m} \cdot q(-1)^m(\text{Sym}^{n-\sigma} \Pi \times \sigma \Pi')},
$$

Here $q^\pm(\text{Sym}^n \Pi \times \Pi')$ are defined as follows:
(i) If \( \kappa' - 1 \geq n(\kappa - 1) + 4 \), then
\[
q^\pm (\text{Sym}^n II \times II') = (2\pi \sqrt{-1})^{ \frac{n+1}{2} |\delta+(n+1)(n\kappa'+w')/2} \cdot (\sqrt{-1})^{\frac{n+1}{2} |\omega'} \\
\times G(\omega_{II})^{n(n+1)/2} \cdot G(\omega_{II'})^{\frac{n+1}{2} |f_{II'}}^{\frac{n+1}{2} |f_{II}} \cdot p(\Pi', \pm(-1)^{n/2}) \cdot \begin{cases} 1 & \text{if } n \text{ is even,} \\ n(\kappa - 1) - 4 \geq \kappa' - 1 \geq (n - 2)(\kappa - 1) + 4, \end{cases}
\]

(ii) If \( n(\kappa - 1) - 4 \geq \kappa' - 1 \geq (n - 2)(\kappa - 1) + 4 \), then
\[
q^\pm (\text{Sym}^n II \times II') = (2\pi \sqrt{-1})^{ \frac{n+1}{2} |\delta+n\kappa+n(n+1)w/2+(n-1)(\kappa'+(n+1)w')/2} \cdot (\sqrt{-1})^{nw+\frac{n+1}{2} |\omega'} \\
\times G(\omega_{II})^{n(n+1)/2} \cdot G(\omega_{II'})^{\frac{n+1}{2} |f_{II'}}^{\frac{n+1}{2} |f_{II}} \cdot p(\Pi', \pm(-1)^{n/2}) \cdot \begin{cases} 1 & \text{if } n \text{ is even,} \\ n(\kappa - 1) - 4 \geq \kappa' - 1 \geq (n - 2)(\kappa - 1) + 4. \end{cases}
\]

Proof. The assertion will be proved by induction on \( n \). Therefore, \( II \) will be fixed and \( II' \) varies with \( n \). When \( n = 1 \), Cases (i) and (ii) are special cases of Conjecture \ref{Conjecture:GarrettHarris} and are proved by Shimura \cite[Theorem 3]{Shimura:1976}. When \( n = 2 \), Case (i) is a special case of Theorem \ref{Theorem:GarrettHarris} \cite{GarrettHarris:1993}. When \( n = 2 \), Case (ii) is a direct consequence of the result of Garrett and Harris \cite{GarrettHarris:1993} on triple product \( L \)-functions for \( GL_2 \) in the balanced case (cf. \cite[Theorem 6]{Morimoto:2021}). Note that assumptions therein can be further relaxed due to our main result in \cite{Chen:2021}. In fact, we only need to assume \( \kappa' - 1 \geq 2(\kappa - 1) + 2 \) in this case. In the induction step, the arguments for Cases (i) and (ii) are similar, we only give details for Case (ii) and leave it to the readers to verify Case (i). We stress that in the induction step, we only use induction hypothesis and Theorem \ref{Theorem:Induction} in Case (i). Whereas in Case (ii), we also need the result for Case (i). Let \( n \geq 3 \) and assume the assertion for Case (ii) holds for all \( n - 1 \geq n' \geq 1 \). Let \( II' \) and \( \delta \) be as in the assertion with \( w' \) odd and
\[
n(\kappa - 1) - 4 \geq \kappa' - 1 \geq (n - 2)(\kappa - 1) + 4.
\]

We choose some auxiliary automorphic representations \( II_j, II'_j, II''_j \), and \( \chi \) as follows:

- For \( 0 \leq j \leq \lfloor \frac{n-2}{2} \rfloor \), let \( II_j \) be a cohomological cuspidal automorphic representation of \( GL_2(\mathbb{A}) \) with infinity type
\[
((-n - 1)j)(\kappa - 1) + 1; (n - 1)w + 1 - \delta).
\]
- If \( n \) is even and \( j = \frac{n-2}{2} \), let
\[
II_j = II \otimes | \mathbb{A}_{\kappa - 1/2} \).
\]
- Let \( II'_j \) and \( II''_j \) be cohomological cuspidal automorphic representations of \( GL_2(\mathbb{A}) \) with infinity types
\[
(\kappa + \kappa' - 1; w + w' - 1), \quad (|\kappa - \kappa'| + 1; w + w' - 1)
\]
respectively.

- When \( n \) is odd, let \( \chi \) be a finite order Hecke character of \( \mathbb{A}^\times \) such that \( \chi_{\kappa' - 1}(-1) = (-1)^{(n-1)(1+w')/2} \).

Let \( \Sigma \) and \( \Sigma' \) be cohomological tamely isobaric automorphic representations of \( GL_n(\mathbb{A}) \) defined by
\[
\Sigma = \text{Sym}^{n-1} II, \\
\Sigma' = \begin{cases} \bigoplus_{j=0}^{n-2} II_j & \text{if } n \text{ is even,} \\
\bigoplus_{j=0}^{n+1} (II_j \otimes | \mathbb{A}^{(-1)/2} \) \boxtimes | \mathbb{A}^{(n-1)/2} & \text{if } n \text{ is odd.}
\end{cases}
\]

Let \( \Psi \) and \( \Psi' \) be cohomological tamely isobaric automorphic representations of \( GL_4(\mathbb{A}) \) defined by
\[
\Psi = (II \boxtimes II') \otimes | \mathbb{A}^{-1/2}, \quad \Psi' = II'_1 \oplus II'_2.
\]

Then we have \( \Sigma_{\kappa'} = \Sigma'_{\kappa'} \) and \( \Psi_{\kappa'} = \Psi'_{\kappa'} \). Let \( m + \frac{\delta}{2} \in \mathbb{Z} + \frac{n+1}{2} \) be a critical point for \( L(s, \text{Sym}^n II \times II') \). Then \( m + \frac{\delta+1}{2} \) is critical for \( L(s, \Sigma \times \Psi) \). Indeed, we have
\[
\text{Sym}^{n-1} II \boxtimes II = \text{Sym}^n II \boxplus (\text{Sym}^{n-2} II \otimes \omega_{II}).
\]

In particular, we have the factorization of \( L \)-function:
\[
L(s, \Sigma \times \Psi) = L(s, \text{Sym}^{n-1} II \times II') \cdot L(s, \text{Sym}^{n-2} II \times II' \otimes \omega_{II}).
\]

Also it is clear that
\[
L(s, \Sigma \times \Psi') = L(s, \text{Sym}^{n-1} II \times II') \cdot L(s, \text{Sym}^{n-1} II \times II'_2).
\]
Applying result for Case (i) to \(L(m + \frac{\delta}{2}, \Sym^{n-2} \Pi \times \Pi')\) and \(L(m + \frac{\delta+1}{2}, \Sym^{n-1} \Pi \times \Pi'_1)\), we obtain
\[
\frac{L(m + \frac{\delta+1}{2}, \Sigma \times \Psi)}{L(m + \frac{\delta}{2}, \Sigma \times \Psi')} \sim \frac{L^{(\omega)}(m + \frac{\delta}{2}, \Sym^n \Pi \times \Pi')}{L^{(\omega)}(m + \frac{\delta+1}{2}, \Sym^{n-1} \Pi \times \Pi'_1)}
\times (2\pi\sqrt{-1})^{-m+\frac{\delta-1}{2}-(n(w+\kappa'+w')/2)} \cdot (\sqrt{-1})^{\frac{\delta}{2}(1+w)}
\times G(\omega_H \omega_{H'})^{-n} \cdot G(\omega_{H'_1} \omega_{H'_2})^{-\frac{1}{2}} \cdot \|f_H\|^{-n+1} \cdot \|f_{H'}\|^{-\frac{n-1}{2}} \cdot \|f_{H'_1}\|^{-\frac{n-1}{2}} \cdot \|f_{H'_2}\|^{-\frac{n-1}{2}}
\times \begin{cases} p(H', (-1)^{m+n/2})^{-1} & \text{if } n \text{ is even}, \\ p(H'_1, (-1)^{m+(n-1)/2})^{-1} & \text{if } n \text{ is odd}. \end{cases}
\tag{5.12}
\]
Here, when \(n\) is even, we also use Theorem 4.4 (2) to replace \(p(H', (-1)^{m+(n-2)/2})\) by
\[(2\pi\sqrt{-1}) \cdot (\sqrt{-1})^{w}, G(\omega_H) \cdot p(H', (-1)^{m+n/2})^{-1} \cdot \|f_H\|.
\]
On the other hand, we have the following factorizations:
\[
L(s, \Sigma' \times \Psi') = \prod_{j=0}^{n-2} L(s + \frac{\delta+1}{2}, \Pi_j \times (\Pi'_1 \oplus \Pi'_2)) \cdot \begin{cases} 1 & \text{if } n \text{ is even}, \\ L(s + \frac{(n-1)w}{2}, (\Pi'_1 \oplus \Pi'_2) \otimes \chi) & \text{if } n \text{ is odd}. \end{cases}
\]
\[
L(s, \Sigma' \times \Psi) = \prod_{j=0}^{n-2} L(s + \frac{\delta-1}{2}, \Pi_j \times \Pi') \cdot \begin{cases} 1 & \text{if } n \text{ is even}, \\ L(s + \frac{(n-1)w-1}{2}, \Pi \times \Pi' \otimes \chi) & \text{if } n \text{ is odd}. \end{cases}
\]
When \(n\) is even and \(j = \frac{n-2}{2}\), we have
\[
L(s, \Pi_j \times \Pi' \otimes \chi) = L(s + \frac{(n-2)w}{2}, \Sym^2 \Pi \times \Pi') \cdot L(s + \frac{(n-2)w}{2}, \Pi' \otimes \omega_H).
\]
Therefore, by Theorem 5.5 for \(L(s + \frac{(n-2)w}{2}, \Sym^2 \Pi \times \Pi')\) (Case (i) with \(n = 2\)) and Theorem 4.4 for \(L(s + \frac{(n-2)w}{2}, \Pi' \otimes \omega_H)\), we see that Conjecture 5.7 holds for \(L(s, \Pi_j \times \Pi' \otimes \Pi')\). For \(0 \leq j \leq \frac{n-1}{2}\), the conditions in 5.9 are satisfied, thus Conjecture 5.7 also holds for \(L(s, \Pi_j \times \Pi' \otimes \Pi')\). We also have algebraic results for
\[
L(m + \delta, \Pi_j \times (\Pi'_1 \oplus \Pi'_2)), \quad L(m + \frac{(n-1)w}{2}, \Pi \times \Pi' \otimes \chi) \quad (\text{when } n \text{ is odd})
\]
and
\[
L(m + \frac{(n-1)w+1}{2}, (\Pi'_1 \oplus \Pi'_2) \otimes \chi) \quad (\text{when } n \text{ is odd}),
\]
by [Shi76 Theorem 3] and Theorem 4.4 respectively. A tedious computation then shows that
\[
\frac{L(m + \frac{\delta+1}{2}, \Sigma \times \Psi)}{L(m + \frac{\delta}{2}, \Sigma \times \Psi')} \sim (2\pi\sqrt{-1})^\delta (\sqrt{-1})^{nw'+1}
\times G(\omega_H \omega_{H'})^{-n} \cdot G(\omega_{H'_1} \omega_{H'_2})^{-\frac{1}{2}} \cdot \|f_H\|^{-n+1} \cdot \|f_{H'}\|^{-\frac{n-1}{2}} \cdot \|f_{H'_1}\|^{-\frac{n-1}{2}} \cdot \|f_{H'_2}\|^{-\frac{n-1}{2}}
\times \begin{cases} 1 & \text{if } n \text{ is even}, \\ p(H'_1, (-1)^{m+n/2}) \cdot p(H'_2, (-1)^{m+(n-1)/2}) & \text{if } n \text{ is odd}. \end{cases}
\tag{5.13}
\]
Note that our assumptions (1) and (2) imply that the conditions in Theorem 1.2 on the infinity types of \(\Sigma\) and \(\Psi\) are satisfied. Therefore, by Theorem 1.2 we have
\[
\frac{L(m + \frac{\delta}{2}, \Sigma \times \Psi)}{L(m + \frac{\delta+1}{2}, \Sigma \times \Psi')} \cdot \frac{L(m + \frac{\delta}{2}, \Sigma \times \Psi)}{L(m + \frac{\delta+1}{2}, \Sigma \times \Psi')} \sim 1.
\tag{5.14}
\]
We then deduce from (5.12) and (5.14) that
\[
\frac{L^{(\omega)}(m + \frac{\delta}{2}, \Sym^n \Pi \times \Pi')}{L^{(\omega)}(m + \frac{\delta+1}{2}, \Sym^{n-1} \Pi \times \Pi'_1)} \sim (2\pi\sqrt{-1})^m \cdot \|f_H\|^{-\frac{m-1}{2}-(n(w+\kappa'+w')/2)} \cdot (\sqrt{-1})^{\frac{m}{2}(1+w)+nw'+1}
\times G(\omega_H)^{-n} \cdot G(\omega_{H'_1} \omega_{H'_2})^{-\frac{1}{2}} \cdot \|f_H\|^{-\frac{m-1}{2}} \cdot \|f_{H'_1}\|^{-\frac{m-1}{2}} \cdot \|f_{H'_2}\|^{-\frac{m-1}{2}}
\times \begin{cases} p(H', (-1)^{m+n/2})^{-1} & \text{if } n \text{ is even}, \\ p(H'_1, (-1)^{m+(n-1)/2})^{-1} & \text{if } n \text{ is odd}. \end{cases}
\]
Then we have
\[ \Sigma \]
periods of \( \Psi \). Let
\[ \Sigma \]
and
\[ \Pi' \]
representations of \( \text{GL} \).

Remark 5.15. Contrary to condition (2) in Theorem 5.5, here we do not need to assume the local condition on finite place.

Remark 5.14. Since \( \text{Sym}^n \Pi \) is essentially self-dual, the result for Case (i) is a special case of Conjecture 5.4. Contrary to condition (2) in Theorem 5.5, here we do not need to assume the local condition on finite place.

Now we begin the proof of Theorem 5.11. We rewrite \( q^\pm(\text{Sym}^{2r+1} \Pi) \) in terms of the Betti–Whittaker periods of \( \Pi \) (cf. [Che21a, Theorem 1.6]) under the assumption that \( \kappa \geq 3 \). Let \( r \geq 2 \) and assume the conjecture holds for all \( r - 1 \geq r' \geq 1 \). Put \( \delta \in \{0, 1\} \) with \( \delta \equiv r \pmod{2} \). Let \( \Pi' \) and \( \Pi'' \) be cohomological cuspidal automorphic representations of \( \text{GL}_2(\mathbb{A}) \) with infinity types
\[ (\kappa', \mathbf{w}') = ((r + 1)(\kappa - 1) + 1; (r + 1)\mathbf{w} + \delta), \]
\[ (\kappa'', \mathbf{w}'') = (r(\kappa - 1) + 1; r\mathbf{w} + 1 - \delta). \]

Let \( \Sigma \) and \( \Sigma' \) be cohomological tamely isobaric automorphic representations of \( \text{GL}_{r+2}(\mathbb{A}) \) defined by
\[ \Sigma = \text{Sym}^{r+1} \Pi, \quad \Sigma' = (\text{Sym}^{r-1} \Pi \otimes | \mathbf{w}'') \oplus (\Pi' \otimes | \mathbf{w}'^\delta). \]

Let \( \Psi \) and \( \Psi' \) be cohomological tamely isobaric automorphic representations of \( \text{GL}_2(\mathbb{A}) \) defined by
\[ \Psi = \text{Sym}^r \Pi \otimes \chi, \quad \Psi' = (\text{Sym}^{r-2} \Pi \otimes \chi | | \mathbf{w}'') \oplus (\Pi'' \otimes | | \mathbf{w}'^{\delta-1} \otimes \chi). \]

Then we have \( \Sigma = \Sigma' \) and \( \Psi = \Psi' \). Let \( m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2} \) be a critical point for \( L(s, \text{Sym}^{2r+1} \Pi \otimes \chi) \). Then it is also a critical point for \( L(s, \Sigma \times \Psi) \). Indeed, we have
\[ \text{Sym}^{r+1} \Pi \otimes \text{Sym}^r \Pi = \bigoplus_{j=0}^r \text{Sym}^{2(r-j)+1} \Pi \otimes \omega^j. \]

In particular, we have
\[ L(s, \Sigma \times \Psi) = \prod_{j=0}^r L(s, \text{Sym}^{2(r-j)+1} \Pi \otimes \omega^j). \]

Similarly, we have
\[ L(s, \Sigma' \times \Psi) = \prod_{j=0}^{r-1} L(s + w, \text{Sym}^{2(r-j)-1} \Pi \otimes \omega^j \chi) \cdot L(s - \frac{1}{2}, \text{Sym}^r \Pi \times \Pi' \otimes \chi). \]

By Theorem 6.13 we have
\[ L^{(\infty)}(m + \frac{1}{2}, \text{Sym}^r \Pi \times \Pi' \otimes \chi) \sim (2\pi)^{(r+1)m} \cdot q^{-m} \cdot (\text{Sym}^r \Pi \times \Pi' \otimes \chi). \]
By induction hypothesis, we have
\[
H(x) = \sum_{j=0}^{r-1} \frac{L(x) (m + \frac{1}{2}, \Sigma \times \Psi)}{L(x) (m + w + \frac{1}{2}, \Sigma' \times \Psi)} \sim G(\omega_H)^{r(\gamma+1)/2} \prod_{j=0}^{r-1} \frac{q^\gamma (\text{Sym}^{2(r-j)-1} H \otimes \omega^j_H \chi)}{q^{-\gamma} (\text{Sym}^{2(r-j)-1} H \otimes \omega^j_H \chi)}.
\]

Therefore, we have
\[
L(m + \frac{1}{2}, \Sigma \times \Psi) \sim \frac{L(x) (m + \frac{1}{2}, \Sigma' \times \Psi)}{L(x) (m + w + \frac{1}{2}, \Sigma' \times \Psi)} \sim (2\pi\sqrt{-1})^{(r-1)m+2w} \cdot \frac{q^{(1-\gamma)m} (\text{Sym}^{r-1} H \otimes H') \cdot q^{(1-\gamma)m+1} (\text{Sym}^{r-2} H \otimes H') \cdot q^{(1-\gamma)m} (\text{Sym}^{r+1} H \otimes H')}{q^{(1-\gamma)m} (\text{Sym}^{r+1} H \otimes H')}.
\]

By Theorem 5.13 we have
\[
L(x) (m, H' \otimes H') \cdot L(x) (m + w + \frac{1}{2}, \text{Sym}^{r-1} H \otimes H') \cdot L(x) (m + w + \frac{1}{2}, \text{Sym}^{r-1} H \otimes H') \sim (2\pi\sqrt{-1})^{(r-1)m+2w} \cdot \frac{q^{(1-\gamma)m} (\text{Sym}^{r-1} H \otimes H') \cdot q^{(1-\gamma)m+1} (\text{Sym}^{r-2} H \otimes H') \cdot q^{(1-\gamma)m} (\text{Sym}^{r+1} H \otimes H')}{q^{(1-\gamma)m} (\text{Sym}^{r+1} H \otimes H')}
\]

By induction hypothesis, we have
\[
\prod_{j=0}^{r-1} \frac{L(x) (m + 2w + \frac{1}{2}, \text{Sym}^{2(r-j)-1} H \otimes \omega_H^j \chi)}{L(x) (m + w + \frac{1}{2}, \text{Sym}^{2(r-j)-1} H \otimes \omega_H^j \chi)} = \frac{L(x) (m + 2w + \frac{1}{2}, \text{Sym}^{2(r-j)-1} H \otimes \omega_H^j \chi)}{L(x) (m + w + \frac{1}{2}, \text{Sym}^{2(r-j)-1} H \otimes \omega_H^j \chi)} \cdot \prod_{j=0}^{r-3} \frac{L(x) (m + 2w + \frac{1}{2}, \text{Sym}^{2(r-j)-1} H \otimes \omega_H^j \chi)}{L(x) (m + w + \frac{1}{2}, \text{Sym}^{2(r-j)-1} H \otimes \omega_H^j \chi)}
\]

\[
\sim (2\pi\sqrt{-1})^{(1-r)m+(s+w)/2} \cdot G(\omega_H)^{-(r-1)(r-2)/2} \cdot G(\chi)^{1-r} \cdot \frac{p(\Pi, \varepsilon)}{q^{-\varepsilon} (\text{Sym}^{2(r-2)-3} H \otimes H')} \cdot \prod_{j=0}^{r-3} \frac{q^{(1-\gamma)m} (\text{Sym}^{2(r-j)-3} H \otimes H')}{q^{-\gamma} (\text{Sym}^{2(r-j)-3} H \otimes H')}.
\]

Therefore, we have
\[
\frac{L(m + \frac{1}{2}, \Sigma' \times \Psi)}{L(m + \frac{1}{2}, \Sigma \times \Psi)} \sim (2\pi\sqrt{-1})^{(2r-1)w+(s+w)/2} \cdot G(\omega_H)^{-(r-1)(r-2)/2} \cdot G(\chi)^{1-r} \cdot \frac{q^{(1-\gamma)m} (\text{Sym}^{r-1} H \otimes H') \cdot q^{(1-\gamma)m+1} (\text{Sym}^{r-2} H \otimes H')}{q^{(1-\gamma)m} (\text{Sym}^{r+1} H \otimes H')} \cdot \prod_{j=0}^{r-3} \frac{q^{(1-\gamma)m} (\text{Sym}^{2(r-j)-3} H \otimes H')}{q^{-\gamma} (\text{Sym}^{2(r-j)-3} H \otimes H')}.
\]
Since $w$ is odd and $\kappa \geq 5$, the conditions in Theorem 1.2 on the infinity types of $\Sigma$ and $\Psi$ are satisfied. By Theorem 1.2, we have

$$L(m + \frac{1}{2}, \Sigma \times \Psi) \cdot L(m + \frac{1}{2}, \Sigma' \times \Psi') \sim 1.$$  

(5.17)

We thus deduce from (5.15)-(5.17) that

$$L(x)(m + \frac{1}{2}, \text{Sym}^{2r+1} \Pi \otimes \chi) \sim (2\pi \sqrt{-1})^{(r+1)m-(2r+1)w-(\kappa+w)/2} \cdot G(\omega_{\Pi})^{-2r+1} \cdot G(\chi)^{r-1} \times \frac{q^{-1}(\Pi') \cdot q^{-1}(\Pi') \cdot q^{-1}(\Pi') \cdot q^{-1}(\Pi') \cdot \cdots}{q^{-1}(\Pi') \cdot q^{-1}(\Pi') \cdot q^{-1}(\Pi') \cdot q^{-1}(\Pi') \cdot \cdots} \sim 1.$$  

By a direct computation, we have

$$\frac{q^x(\text{Sym}^{2r+1} \Pi \times \Pi' \otimes \chi)}{q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot \cdots} \sim (2\pi \sqrt{-1})^{r-1} \cdot (2\pi \sqrt{-1})^{r+1} \cdot G(\omega_{\Pi})^{3r-1} \cdot G(\chi)^2 \cdot p(\Pi, +)^{r+1} \cdot p(\Pi, -)^{r+1}.$$  

Note that

$$\frac{q^x(\text{Sym}^{2r-1} \Pi \times \Pi' \otimes \chi)}{q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot \cdots} = \frac{p(\Pi, \pm (1)^j)}{p(\Pi, \pm (1)^j)}.$$  

Hence we have

$$\prod_{j=0}^{r-3} \frac{q^x(\text{Sym}^{2(r-j)-3} \Pi \times \Pi' \otimes \chi)}{q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot \cdots} \prod_{j=0}^{r-3} \frac{q^x(\text{Sym}^{2(r-j)-3} \Pi \times \Pi' \otimes \chi)}{q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot \cdots} = \prod_{j=1}^{r-3} \frac{q^x(\text{Sym}^{2(r-j)-3} \Pi \times \Pi' \otimes \chi)}{q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot q^x(\Pi') \cdot \cdots} = \begin{cases} p(\Pi, \pm (1)^j) \\ p(\Pi, \pm (1)^j) \end{cases}$$  

if $r$ is even,  

1 if $r$ is odd.

We conclude that

$$L(x)(m + \frac{1}{2}, \text{Sym}^{2r+1} \Pi \otimes \chi) \sim (2\pi \sqrt{-1})^{(r+1)m-(2r+1)w-(\kappa+w)/2} \cdot G(\omega_{\Pi})^{r-1} \cdot G(\chi)^{r+1} \times q^x(\text{Sym}^{2r-1} \Pi \times \Pi' \otimes \chi)^{r+1} \cdot p(\Pi, \pm (1)^j)^{r+1} \cdot p(\Pi, \pm (1)^j)^{r+1} \cdot G(\chi)^r \cdot \text{Sym}^{2r+1} \Pi \otimes \chi.$$  

This completes the proof of Theorem 5.11.

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