Motivated Planning for Kinematic systems
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Abstract—In this paper, we present a general theory of motion planning for kinematic systems. This theory has been developed for long by one of the authors in a previous series of papers. It is mostly based upon concepts from subriemannian geometry. Here, we summarize the results of the theory, and we improve on, by developing in details an intricate case: the ball with a trailer, which corresponds to a distribution with flag of type 2,3,5,6.

This paper is dedicated to Bernard Bonnard for his 60th birthday.

Index Terms—Optimal control, Subriemannian geometry, robotics, motion planning

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I. Introduction

Here we present the main lines of a theory of motion planning for kinematic systems, which is developed for about ten years in the papers \[8, 9, 10, 11, 12, 13, 14\]. One of the purposes of the paper is to survey the whole theory disseminated in these papers. But also we improve on the theory, by treating one more case, in which "the fourth order brackets are involved". We also improve on several previous results (periodicity of our optimal trajectories for instance).

Potential application of this theory is motion planning for kinematic robots. We will show several basic examples here.

The theory starts from the seminal work of F. Jean, in the papers \[15, 16, 17\]. At the root of this point of view in robotics, there are also more applied authors like J.P. Laumond \[20\]. See also \[25\].

We consider kinematic systems that are given under the guise of a vector-distribution \( \Delta \) over a \( n \)-dimensional manifold \( M \). The rank of the distribution is \( p \), and the corank \( k = n - p \). Motion planning problems will always be local problems in an open neighborhood of a given finite path \( \Gamma \) in \( M \). Then we may always consider that \( M = \mathbb{R}^n \). From a control point of view, a kinematic system can be specified by a control system, linear in the controls, typically denoted by \( \Sigma \):

\[
(\Sigma)\dot{x} = \sum_{i=1}^{p} F_i(x)u_i,
\]

where the \( F_i \)'s are smooth \((C^\infty)\) vector fields that span the distribution \( \Delta \). The standard controllability assumption is always assumed, i.e. the Lie algebra generated by the \( F_i \)'s is transitive on \( M \). Consequently, the distribution \( \Delta \) is completely nonintegrable, and any smooth path \( \Gamma : [0, T] \to M \) can be uniformly approximated by an admissible path \( \gamma : [0, \theta] \to M \), i.e. a Lipschitz path, which is almost everywhere tangent to \( \Delta \), i.e., a trajectory of \( \Gamma \).

This is precisely the abstract answer to the kinematic motion planning problem: it is possible to approximate uniformly nonadmissible paths by admissible ones. The purpose of this paper is to present a general constructive theory that solves this problem in a certain optimal way.
More precisely, in this class of problems, it is natural to try to minimize a cost of the following form:

$$J(u) = \int_0^\theta \sqrt{\sum_{i=1}^p (u_i)^2} dt,$$

for several reasons: 1. the optimal curves do not depend on their parametrization, 2. the minimization of such a cost produces a metric space (the associated distance is called the subriemannian distance, or the Carnot-Caratheodory distance), 3. Minimizing such a cost is equivalent to minimize the following (called the energy of the path) quadratic cost $J_E(u)$, in fixed time $\theta$:

$$J_E(u) = \int_0^\theta \sum_{i=1}^p (u_i)^2 dt.$$

The distance is defined as the minimum length of admissible curves connecting two points, and the length of the admissible curve corresponding to the control $u : [0, \theta] \to M$ is just $J(u)$.

In this presentation, another way to interpret the problem is as follows: the dynamics is specified by the distribution $\Delta$ (i.e. not by the vector fields $F_i$, but their span only). The cost is then determined by an Euclidean metric $g$ over $\Delta$, specified here by the fact that the $F_i$'s form an orthonormal frame field for the metric.

At this point we would like to make a more or less philosophical comment: there is, in the world of nonlinear control theory, a permanent twofold critic against the optimal control approach: 1. the choice of the cost to be minimized is in general rather arbitrary, and 2. optimal control solutions may be non robust.

Some remarkable conclusions of our theory show the following: in reasonable dimensions and codimensions, the optimal trajectories are extremely robust, and in particular, do not depend at all (modulo certain natural transformations) on the choice of the metric, but on the distribution $\Delta$ only. Even stronger: they depend only on the nilpotent approximation along $\Gamma$ (a concept that will be defined later on, which is a good local approximation of the problem). For a lot of low values of the rank $p$ and corank $k$, these nilpotent approximations have no parameter (hence they are in a sense universal). The asymptotic optimal syntheses (i.e. the phase portraits of the admissible trajectories that approximate up to a small $\varepsilon$) are also universal.

Given a motion planning problem, specified by a (non-admissible) curve $\Gamma$, and a Subriemannian structure $(\Pi)$, we will consider two distinct concepts, namely: 1. The metric complexity $MC(\varepsilon)$ that measures asymptotically the length of the best $\varepsilon$-approximating admissible trajectories, and 2. The interpolation entropy $E(\varepsilon)$, that measures the length of the best admissible curves that interpolate $\Gamma$ with pieces of length $\varepsilon$.

The first concept was introduced by F. Jean in his basic paper [16]. The second concept is closely related with the entropy of F. Jean in [17], which is more or less the same as the Kolmogorov’s entropy of the path $\Gamma$, for the metric structure induced by the Carnot-Caratheodory metric of the ambient space.

Also, along the paper, we will deal with generic problems only (but generic in the global sense, i.e. stable singularities are considered). That is, the set of motion planning problems on $\mathbb{R}^n$ is the set of couples $(\Gamma, \Sigma)$, embedded with the $C^\infty$ topology of uniform convergence over compact sets, and generic problems (or problems in general position) form an open-dense set in this topology. For instance, it means that the curve $\Gamma$ is always transversal to $\Delta$ (except maybe at isolated points, in the cases $k = 1$ only). Another example is the case of a surface of degeneracy of the Lie bracket distribution $[\Delta, \Delta]$ in the $n = 3, k = 1$ case. Generically, this surface (the Martinet surface) is smooth, and $\Gamma$ intersects it transversally at a finite number of points only.

Also, along the paper, we will illustrate our results with one of the following well known academic examples:

**Example 1**: the unicycle:

$$\dot{x} = \cos(\theta)u_1, \quad \dot{y} = \sin(\theta)u_1, \quad \dot{\theta} = u_2 \quad (2)$$

**Example 2**: the car with a trailer:

$$\dot{x} = \cos(\theta)u_1, \quad \dot{y} = \sin(\theta)u_1, \quad \dot{\theta} = u_2, \quad \phi = u_1 - \sin(\varphi)u_2 \quad (3)$$

**Example 3**: the ball rolling on a plane:

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{\theta} = \frac{1}{L}(\cos(\theta)u_1 + \sin(\theta)u_2), R \quad (4)$$

where $(x, y)$ are the coordinates of the contact point between the ball and the plane, $R \in SO(3, \mathbb{R})$ is the right orthogonal matrix representing an othonormal frame attached to the ball.

**Example 4**: the ball with a trailer

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{\theta} = \frac{1}{L}(\cos(\theta)u_1 + \sin(\theta)u_2), R \quad (5)$$

Typical motion planning problems are: 1. for example (2), the parking problem: the non admissible curve $\Gamma$ is $s \to (x(s), y(s), \theta(s), \varphi(s)) = (s, 0, \frac{\pi}{2}, 0)$, 2. for example (3), the full rolling with slipping, $\Gamma : s \to (x(s), y(s), \theta(s)) = (s, 0, Id)$, where $Id$ is the identity matrix. On figures [2] we show our approximating trajectories for both problems, that are in a sense universal. In figure [1] of course, the $x$-scale is much larger than the $y$-scale.

Up to now, our theory covers the following cases:

(C1) The distribution $\Delta$ is one-step bracket generating (i.e. dim($[\Delta, \Delta]$) $= n$) except maybe at generic singularities,

(C2) The number of controls (the dimension of $\Delta$) is $p = 2$, and $n \leq 6$.

The paper is organized as follows: In the next section [1] we introduce the basic concepts, namely the metric complexity,
the interpolation entropy, the nilpotent approximation along \( \Gamma \), and the normal coordinates, that will be our basic tools.

Section [III] summarizes the main results of our theory, disseminated in our previous papers, with some complements and details. Section [IV] is the detailed study of the case \( n = 6, k = 4 \), which corresponds in particular to example 4, the ball with a trailer. In Section [V] we state a certain number of remarks, expectations and conclusions.

II. BASIC CONCEPTS

In this section, we fix a generic motion planning problem \( \mathcal{P} = (\Gamma, \Sigma) \). Also, along the paper there is a small parameter \( \varepsilon \) (we want to approximate up to \( \varepsilon \)), and certain quantities \( f(\varepsilon), g(\varepsilon) \) go to \(+\infty\) when \( \varepsilon \) tends to zero. We say that such quantities are equivalent \( (f \simeq g) \) if \( \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1 \). Also, \( d \) denotes the subriemannian distance, and we consider the \( \varepsilon \)-subriemannian tube \( T_\varepsilon \) and cylinder \( C_\varepsilon \) around \( \Gamma \):

\[
T_\varepsilon = \{ x \in M \mid d(x, \Gamma) \leq \varepsilon \}, \\
C_\varepsilon = \{ x \in M \mid d(x, \Gamma) = \varepsilon \}.
\]

A. Entropy versus metric complexity

\textbf{Definition 5:} The metric complexity \( MC(\varepsilon) \) of \( \mathcal{P} \) is \( \frac{1}{\varepsilon} \) times the minimum length of an admissible curve \( \gamma_\varepsilon \) connecting the endpoints \( \Gamma(0), \Gamma(T) \) of \( \Gamma \), and remaining in the tube \( T_\varepsilon \).

\textbf{Definition 6:} The interpolation entropy \( E(\varepsilon) \) of \( \mathcal{P} \) is \( \frac{1}{\varepsilon} \) times the minimum length of an admissible curve \( \gamma_\varepsilon \) connecting the endpoints \( \Gamma(0), \Gamma(T) \) of \( \Gamma \), and \( \varepsilon \)-interpolating \( \Gamma \), that is, in any segment of \( \gamma_\varepsilon \) of length \( \geq \varepsilon \), there is a point of \( \Gamma \).

These quantities \( MC(\varepsilon), E(\varepsilon) \) are functions of \( \varepsilon \) which tends to \(+\infty\) as \( \varepsilon \) tends to zero. They are considered \textit{up to equivalence}. The reason to divide by \( \varepsilon \) is that the second quantity counts the number of \( \varepsilon \)-balls to cover \( \Gamma \), or the number of pieces of length \( \varepsilon \) to interpolate the full path. This is also the reason for the name "entropy".

\textbf{Definition 7:} An asymptotic optimal synthesis is a one-parameter family \( \gamma_\varepsilon \) of admissible curves, that realizes the metric complexity or the entropy.

Our main purpose in the paper is twofold:

1. We want to estimate the metric complexity and the entropy, in terms of certain invariants of the problem. Actually, in all the cases treated in this paper, we will give explicit formulas.
2. We shall exhibit explicit asymptotic optimal syntheses realizing the metric complexity or/and the entropy.

B. Normal coordinates

Take a \textbf{parametrized} \( p \)-dimensional surface \( S \), transversal to \( \Delta \) (maybe defined in a neighborhood of \( \Gamma \) only),

\[ S = \{ q(s_1, \ldots, s_{p-1}, t) \in \mathbb{R}^n \}, \text{ with } q(0, \ldots, 0, t) = \Gamma(t). \]

Such a \textit{germ} exists if \( \Gamma \) is not tangent to \( \Delta \). The exclusion of a neighborhood of an isolated point where \( \Gamma \) is tangent to \( \Delta \) (that is \( \Gamma \) becomes "almost admissible"), will not affect our estimates presented later on (it will provide a term of higher order in \( \varepsilon \)).

In the following, \( C_\varepsilon^S \) will denote the cylinder \( \{ \xi \mid d(S, \xi) = \varepsilon \} \).

\textbf{Lemma 8:} (Normal coordinates with respect to \( S \)). There are mappings \( x : \mathbb{R}^n \to \mathbb{R}^p, y : \mathbb{R}^n \to \mathbb{R}^{k-1}, w : \mathbb{R}^n \to \mathbb{R} \), such that \( \xi = (x, y, w) \) is a coordinate system on some neighborhood of \( S \) in \( \mathbb{R}^n \), such that:

1. \( S(y, w) = (0, y, w), \Gamma = \{ (0, 0, w) \} \)
2. The restriction \( \Delta_S = \ker dw \cap_{i=1}^{k-1} \ker dy_i \), the metric \( g_S = \sum_{i=1}^{p-1} (dx_i)^2 \)
3. \( C_\varepsilon^S = \{ \xi \mid \sum_{i=1}^{p-1} x_i^2 = \varepsilon^2 \} \), the geodesics of the Pontryagin’s maximum principle meeting the transversality conditions w.r.t. \( S \) are the straight lines through \( S \), contained in the planes \( P_{y_0, w_0} = \{ \xi | (y, w) = (y_0, w_0) \} \). Hence, they are orthogonal to \( S \).

These normal coordinates are unique up to changes of coordinates of the form

\[ \dot{x} = T(y, w)x, (\dot{y}, \dot{w}) = (y, w), \]

where \( T(y, w) \in O(p) \), the \( p \)-orthogonal group.
C. Normal forms. Nilpotent approximation along $\Gamma$

1) Frames: Let us denote by $F = (F_1,...,F_p)$ the orthonormal frame of vector fields generating $\Delta$. Hence, we will also write $P = (\Gamma, F)$. If a global coordinate system $(x, y, w)$, not necessarily normal, is given on a neighborhood of $\Gamma$ in $\mathbb{R}^n$, with $x \in \mathbb{R}^p$, $y \in \mathbb{R}^{k-1}$, $w \in \mathbb{R}$, then we write:

$$F_j = \sum_{i=1}^{k-1} Q_{i,j}(x, y, w) \frac{\partial}{\partial x_i} + \sum_{i=1}^{k-1} L_{i,j}(x, y, w) \frac{\partial}{\partial y_i}$$

$$+ M_j(x, y, w) \frac{\partial}{\partial w},$$

$j = 1,...,p$.

Hence, the SR metric is specified by the triple $(Q, L, M)$ of smooth $x, y, w$-dependent matrices.

2) The general normal form: Fix a surface $S$ as in Section ?? and a normal coordinate system $\xi = (x, y, w)$ for a problem $P$.

Theorem 9: (Normal form, [2]) There is a unique orthonormal frame $F = (Q, L, M)$ for $(\Delta, y)$ with the following properties:

1. $Q(x, y, w)$ is symmetric, $Q(0, y, w) = Id$ (the identity matrix),
2. $Q(x, y, w)x = x$,
3. $L(x, y, w)x = 0, M(x, y, w)x = 0$.

4. Conversely if $\xi = (x, y, w)$ is a coordinate system satisfying conditions 1, 2, 3 above, then $\xi$ is a normal coordinate system for the SR metric defined by the orthonormal frame $F$ with respect to the parametrized surface $\{(0, y, w)\}$.

Clearly, this normal form is invariant under the changes of normal coordinates $\xi$.

Let us write:

$$Q(x, y, w) = Id + Q_1(x, y, w) + Q_2(x, y, w) + ... ,$$

$$L(x, y, w) = 0 + L_1(x, y, w) + L_2(x, y, w) + ... ,$$

$$M(x, y, w) = 0 + M_1(x, y, w) + M_2(x, y, w) + ... ,$$

where $Q_r, L_r, M_r$ are matrices depending on $\xi = (x, y, w)$, the coefficients of which have order $k$ w.r.t. $x$ (i.e. they are in the $r^{th}$ power of the ideal of $C^\infty(x, y, w)$ generated by the functions $x_r, r = 1, ..., n-p$). In particular, $Q_1$ is linear in $x, Q_2$ is quadratic, etc... Set $u = (u_1, ..., u_p) \in \mathbb{R}^p$. Then $\sum_{j=1}^{k-1} L_1(x, y, w)u_j = L_{1,y,w}(x, u)$ is quadratic in $(x, u)$, and $\mathbb{R}^{k-1}$-valued. Its $i^{th}$ component is the quadratic expression denoted by $L_{1,y,w}(x, u)$. Similarly $\sum_{j=1}^{k-1} M_1(x, y, w)u_j = M_{1,y,w}(x, u)$ is a quadratic form in $(x, u)$. The corresponding matrices are denoted by $L_{1,y,w}, i = 1, ..., k-1,$ and $M_{1,y,w}$.

The following was proved in [2, 5] for corank 1:

Proposition 10: 1. $Q_1 = 0$.
2. $L_{1,y,w}, i = 1, ..., p-1,$ and $M_{1,y,w}$ are skew symmetric matrices.

A first useful very rough estimate in normal coordinates is the following:

Proposition 11: If $\xi = (x, y, w) \in T_\varepsilon$, then:

$$||x||_2 \leq \varepsilon,$$

$$||y||_2 \leq k\varepsilon^2,$$

for some $k > 0$.

At this point, we shall split the problems under consideration into two distinct cases: first the 2-step bracket-generating case, and second, the 2-control case.

3) Two-step bracket-generating case: In that case, we set, in accordance to Proposition [11] that $x$ has weight 1, and the $y_i$’s and $w$ have weight 2. Then, the vector fields $\frac{\partial}{\partial x_i}$ have weight -1, and $\frac{\partial}{\partial y_i}, \frac{\partial}{\partial w}$ have weight -2.

Inside a tube $T_\varepsilon$, we write our control system as a term of order -1, plus a residue, that has a certain order w.r.t. $\varepsilon$. Here, $O(\varepsilon^k)$ means a smooth term bounded by $c\varepsilon^k$. We have, for a trajectory remaining inside $T_\varepsilon$:

\begin{align}
\dot{x} &= u + O(\varepsilon^2); \\
\dot{y}_i &= \frac{1}{2} x' L^i(w) u + O(\varepsilon^2); \quad i = 1,...,p-1; \\
\dot{w} &= \frac{1}{2} x' M(w) u + O(\varepsilon^2),
\end{align}

where $L^i(w), M(w)$ are skew-symmetric matrices depending smoothly on $w$.

Remark 12: In [3, 1] (1), the term $O(\varepsilon^2)$ can seem surprising. One should wait for $O(\varepsilon)$. It is due to (1) in Proposition [10].

In that case, we define the Nilpotent Approximation $\hat{P}$ along $\Gamma$ of the problem $P$ by keeping only the term of order -1:

\begin{align}
\hat{P}: \quad \dot{x} &= u; \\
\dot{y}_i &= \frac{1}{2} x' L^i(w) u; \quad i = 1,...,p-1; \\
\dot{w} &= \frac{1}{2} x' M(w) u.
\end{align}

Consider two trajectories $\xi(t), \hat{\xi}(t)$ of $P$ and $\hat{P}$ corresponding to the same control $u(t)$, issued from the same point on $\Gamma$, and both arclength-parametrized (which is equivalent to $||u(t)|| = 1$). For $t \leq \varepsilon$, we have the following estimates:

\begin{align}
||x(t)-\hat{x}(t)|| &\leq c\varepsilon^3, \quad ||y(t)-\hat{y}(t)|| \leq c\varepsilon^3, \quad ||w(t)-\hat{w}(t)|| \leq c\varepsilon^3,
\end{align}

for a suitable constant $c$.

Remark 13: It follows that the distance (either $d$ or $\hat{d}$-the distance associated with the nilpotent approximation) between $\xi(t), \hat{\xi}(t)$ is smaller than $\varepsilon^{1+\alpha}$ for some $\alpha > 0$.

This fact comes from the estimate just given, and the standard ball-box Theorem (15). It will be the key point to reduce the motion planning problem to the one of its nilpotent approximation along $\Gamma$.

4) The 2-control case:
5) Normal forms: In that case, we have the following general normal form, in normal coordinates. It was proven first in [11], in the corank 1 case. The proof holds in any corank, without modification.

Consider Normal coordinates with respect to any surface $S$. There are smooth functions, $\beta(x,y,w), \gamma_i(x,y,w), \delta(x,y,w)$, such that $P$ can be written as (on a neighborhood of $\Gamma$):

\[
\begin{align*}
\dot{x}_1 &= (1 + (x_2)^2 \beta)u_1 - x_1x_2\beta u_2, \\
\dot{x}_2 &= (1 + (x_2)^2 \beta)u_2 - x_1x_2u_1, \\
\dot{y}_i &= \gamma_i(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{w} &= \delta(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2),
\end{align*}
\]

where moreover $\beta$ vanishes on the surface $S$.

The following normal forms can be obtained, on the tube $T_\varepsilon$, by just changing coordinates in $S$ in certain appropriate way. It means that a trajectory $\xi(t)$ of $P$ remaining in $T_\varepsilon$ satisfies:

**Generic 4 - 2 case (see [12]):**

\[
\begin{align*}
\dot{x}_1 &= u_1 + 0(\varepsilon^3), \dot{x}_2 = u_2 + 0(\varepsilon^3), \\
\dot{y}_i &= \gamma_i(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{w} &= \delta(u_1)\dot{w}_1 + \delta(u_2)\dot{w}_2 + O(\varepsilon^3).
\end{align*}
\]

We define the nilpotent approximation as:

\[
\begin{align*}
\dot{\tilde{x}}_1 &= u_1, \dot{\tilde{x}}_2 = u_2, \dot{\tilde{y}} = \frac{x_2}{2}u_1 - \frac{x_1}{2}u_2, \\
\dot{\tilde{w}} &= \delta(\tilde{w}_1)\dot{\tilde{w}}_1 + \delta(\tilde{w}_2)\dot{\tilde{w}}_2 + O(\varepsilon^3).
\end{align*}
\]

Again, we consider two trajectories $\xi(t), \tilde{\xi}(t)$ of $P$ and $\tilde{P}$ corresponding to the same control $w(t)$, issued from the same point on $\Gamma$, and both arclength-parametrized (which is equivalent to $||u(t)|| = 1$). For $t \leq \varepsilon$, we have the following estimates:

\[
||x(t) - \tilde{x}(t)|| \leq c\varepsilon^4, \ ||y(t) - \tilde{y}(t)|| \leq c\varepsilon^3, \ ||w(t) - \tilde{w}(t)|| \leq c\varepsilon^4.
\]

Which implies that, for $t \leq \varepsilon$, the distance (d or $\tilde{d}$) between $\xi(t)$ and $\tilde{\xi}(t)$ is less than $\varepsilon^{1+\alpha}$ for some $\alpha > 0$, and this will be the keypoint to reduce our problem to the Nilpotent approximation.

**Generic 5 - 2 case (see [13]):**

\[
\begin{align*}
\dot{x}_1 &= u_1 + 0(\varepsilon^3), \dot{x}_2 = u_2 + 0(\varepsilon^3), \\
\dot{y} &= \gamma_i(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{z} &= \frac{x_2}{2}u_1 - \frac{x_1}{2}u_2, \\
\dot{w} &= \delta(u_1)\dot{w}_1 + \delta(u_2)\dot{w}_2 + O(\varepsilon^3).
\end{align*}
\]

We define the nilpotent approximation as:

\[
\begin{align*}
\dot{\tilde{x}}_1 &= u_1, \dot{\tilde{x}}_2 = u_2, \dot{\tilde{y}} = \frac{x_2}{2}u_1 - \frac{x_1}{2}u_2, \\
\dot{\tilde{z}} &= \frac{x_2}{2}u_1 - \frac{x_1}{2}u_2, \\
\dot{\tilde{w}} &= \delta(\tilde{w}_1)\dot{\tilde{w}}_1 + \delta(\tilde{w}_2)\dot{\tilde{w}}_2 + O(\varepsilon^3).
\end{align*}
\]

The estimates necessary to reduce to Nilpotent approximation are:

\[
\begin{align*}
||x(t) - \tilde{x}(t)|| \leq c\varepsilon^4, && ||y(t) - \tilde{y}(t)|| \leq c\varepsilon^3, \\
||z(t) - \tilde{z}(t)|| \leq c\varepsilon^4, && ||w(t) - \tilde{w}(t)|| \leq c\varepsilon^4.
\end{align*}
\]

**Generic 6 - 2 case (proven in Appendix):**

\[
\begin{align*}
\dot{x}_1 &= u_1 + 0(\varepsilon^3), \dot{x}_2 = u_2 + 0(\varepsilon^3), \\
\dot{y} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^3), \\
\dot{z} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^3), \\
\dot{w} &= Q_w(x_1, x_2)(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^4),
\end{align*}
\]

where $Q_w(x_1, x_2)$ is a quadratic form in $x$ depending smoothly on $w$.

We define the nilpotent approximation as:

\[
\begin{align*}
\dot{\tilde{x}}_1 &= u_1, \dot{\tilde{x}}_2 = u_2, \\
\dot{\tilde{y}} &= (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{\tilde{z}} &= \frac{x_2}{2}u_1 - \frac{x_1}{2}u_2, \\
\dot{\tilde{w}} &= Q_w(x_1, x_2)(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2).
\end{align*}
\]

The estimates necessary to reduce to nilpotent approximation are:

\[
\begin{align*}
||x(t) - \tilde{x}(t)|| \leq c\varepsilon^4, && ||y(t) - \tilde{y}(t)|| \leq c\varepsilon^3, \\
||z(t) - \tilde{z}(t)|| \leq c\varepsilon^4, && ||w(t) - \tilde{w}(t)|| \leq c\varepsilon^5.
\end{align*}
\]

In fact, the proof given in Appendix, of the reduction to this normal form, contains the other cases 4-2 and 5-2.

6) Invariants in the 6-2 case, and the ball with a trailer:

Let us consider a one form $\omega$ that vanishes on $\Delta'' = [\Delta, [\Delta, \Delta]]$. Set $\alpha = d\omega|_\Delta$, the restriction of $d\omega$ to $\Delta$. Set $H = [F_1, F_2]$, $I = [F_1, H]$, $J = [F_2, H]$, and consider the $2 \times 2$ matrix $A(\xi) = \begin{pmatrix} d\omega(F_1, I) & d\omega(F_2, I) \\ d\omega(F_1, J) & d\omega(F_2, J) \end{pmatrix}$.

Due to Jacobi Identity, $A(\xi)$ is a symmetric matrix. It is also equal to $\left( \begin{pmatrix} \omega([F_1, I]) & \omega([F_2, I]) \\ \omega([F_1, J]) & \omega([F_2, J]) \end{pmatrix} \right)$, using the fact that $\omega([X, Y]) = d\omega(X, Y)$ in restriction to $\Delta''$.

Let us consider a gauge transformation, i.e. a feedback that preserves the metric (i.e. a change of orthonormal frame $(F_1, F_2)$ obtained by setting $F_1 = \cos(\theta(\xi)) F_1 + \sin(\theta(\xi)) F_2$, $F_2 = -\sin(\theta(\xi)) F_1 + \cos(\theta(\xi)) F_2$).

It is just a matter of tedious computations to check that the matrix $A(\xi)$ is changed for $A(\xi) = R_\theta A(\xi) R_{-\theta}$. On the other hand, the form $\omega$ is defined modulo multiplicity by a nonzero function $f(\xi)$, and the same holds for $\alpha$, since $d(f \omega) = f d\omega + df \land \omega$, and $\omega$ vanishes over $\Delta''$. The following lemma follows:

**Lemma 14:** The ratio $r(\xi)$ of the (real) eigenvalues of $A(\xi)$ is an invariant of the structure.

Let us now consider the normal form [14], and compute the form $\omega = \omega_1 dx_1 + \ldots + \omega_6 du$ along $\Gamma$ (that is, where $x, y, z = 0$). Computing all the brackets show that $\omega_1 = \omega_2 =$
... = \omega_5 = 0$. This shows also that in fact, along $\Gamma$, $A(\xi)$ is just the matrix of the quadratic form $Q_\omega$. We get the following:

**Lemma 15:** The invariant $r(\Gamma(t))$ of the problem $\mathcal{P}$ is the same as the invariant $r(\Gamma(t))$ of the nilpotent approximation along $\Gamma$.

Let us compute the ratio $r$ for the ball with a trailer, Equation (5). We denote by $A_1, A_2$ the two right-invariant vector fields over $SO(3, \mathbb{R})$ appearing in (5). We have:

$$F_1 = \frac{\partial}{\partial x_1} + A_1 - \frac{1}{L} \cos(\theta) \frac{\partial}{\partial \theta},$$

$$F_2 = \frac{\partial}{\partial x_2} + A_2 - \frac{1}{L} \sin(\theta) \frac{\partial}{\partial \theta}.$$  

Then, we compute the brackets:

$$[A_1, A_2] = A_3, [A_1, A_3] = -A_2, [A_2, A_3] = A_1.$$  

Then, for each couple of an interval $I_1 = [\tilde{\varepsilon}, \varepsilon_1 + \varepsilon], [\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + \varepsilon_3], \ldots$ with $\varepsilon \leq \varepsilon_1 < \varepsilon(1 + \varepsilon^\alpha)$, $i = 1, 2, \ldots$.

**Remark 18:** This concept of an $\varepsilon$-modification is for the following use: we will construct asymptotic optimal syntheses for the nilpotent approximation $\hat{\mathcal{P}}$ of problem $\mathcal{P}$. Then, the asymptotic optimal syntheses have to be slightly modified in order to realize the interpolation constraints for the original (non-modified) problem. This has to be done "slightly" for the length of paths remaining equivalent.

In this section it is always assumed but not stated that **we consider generic problems only**. One first result is the following:

**Theorem 19:** In the cases 2-step bracket generating, 4-2, 5-2, 6-2, (without singularities), an asymptotic optimal synthesis [relative to the entropy] for $\mathcal{P}$ is obtained as an $\varepsilon$-modification of an asymptotic optimal synthesis for the nilpotent approximation $\hat{\mathcal{P}}$. As a consequence, the entropy $E(\varepsilon)$ of $\mathcal{P}$ is equal to the entropy $E(\varepsilon)$ of $\hat{\mathcal{P}}$.

This theorem is proven in [12]. However, we can easily get an idea of the proof, using the estimates of formulas [10, 12, 13, 16].

All these estimates show that, if we apply an $\varepsilon$-interpolating strategy to $\hat{\mathcal{P}}$, and the same controls to $\mathcal{P}$, at time $\varepsilon$ (or length $\varepsilon$-since it is always possible to consider arclength-parametrized trajectories), the endpoints of the two trajectories are at subriemannian distance (either $d$ or $d$) of order $\varepsilon^{1+\alpha}$, for some $\alpha > 0$. Then the contribution to the entropy of $\mathcal{P}$, due to the correction necessary to interpolate $\Gamma$ will have higher order.

Also, in the one-step bracket-generating case, we have the following equality:

**Theorem 20:** (one step bracket-generating case, corank $k \leq 3$) The entropy is equal to $2\pi$ times the metric complexity: $E(\varepsilon) = 2\pi MC(\varepsilon)$.

The reason for this distinction between corank less or more than 3 is very important, and will be explained in the section III-C.

Another very important result is the following **logarithmic lemma**, that describes what happens in the case of a (generic) singularity of $\Delta$. In the absence of such singularities, as we shall see, we shall always have formulas of the following type, for the entropy (the same for the metric complexity):

$$E(\varepsilon) \simeq \frac{1}{\varepsilon^p} \int_\Gamma dt \chi(t), \quad (17)$$

where $\chi(t)$ is a certain invariant along $\Gamma$. When the curve $\Gamma(t)$ crosses transversally a codimension-$1$ singularity of $\Delta'$, $\Delta''$, the invariant $\chi(t)$ vanishes. This may happen at isolated points $t_i$, $i = 1, \ldots, \ldots$. In that case, we always have the following:

**Theorem 21:** (logarithmic lemma). The entropy (resp. the metric complexity) satisfies:

$$E(\varepsilon) \simeq -2 \lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta} \sum_{i=1}^n \rho(t_i)^{-1},$$

where $\rho(t) = \left| \frac{d\chi(t)}{dt} \right|$. On the contrary, there are also generic codimension $1$ singularities where the curve $\Gamma$, at isolated points, becomes tangent to $\Delta$, or $\Delta'$, ... At these isolated points, the invariant $\chi(t)$ of Formula [17] tends to infinity. In that case, the formula [17] remains valid (the integral converges).

**B. Generic distribution in $\mathbb{R}^3$**

This is the simplest case, and it is important, since many cases just reduce to it. Let us describe it in details.

Generically, the 3-dimensional space $M$ contains a 2-dimensional singularity (called the Martinet surface, denoted by $\mathcal{M}$). This singularity is a smooth surface, and (except at...
isolated points on $\mathcal{M}$), the distribution $\Delta$ is not tangent to $\mathcal{M}$. Generically, the curve $\Gamma$ crosses $\mathcal{M}$ transversally at a finite number of isolated points $t_i$, $i = 1, ..., r$. These points are not the special isolated points where $\Delta$ is tangent to $\mathcal{M}$ (this would be not generic). They are called Martinet points. This number $r$ can be zero. Also, there are other isolated points $\tau_j$, $j = 1, ..., l$, at which $\Gamma$ is tangent to $\Delta$ (which means that $\Gamma$ is almost admissible in a neighborhood of $\tau_j$). Out of $\mathcal{M}$, the distribution $\Delta$ is a contact distribution (a generic property).

Let $\omega$ be a one-form that vanishes on $\Delta$ and that is 1 on $\Gamma$, defined up to multiplication by a function which is 1 along $\Gamma$. Along $\Gamma$, the restriction 2-form $d\omega|_\Delta$ can be made into a skew-symmetric endomorphism $A(\Gamma(t))$ of $\Delta$ (skew symmetric with respect to the scalar product over $\Delta$), by duality: $< A(\Gamma(t))X,Y >= d\omega(X,Y)$. Let $\chi(t)$ denote the moduli of the eigenvalues of $A(\Gamma(t))$. We have the following:

**Theorem 22:** 1. If $r = 0$, $MC(\varepsilon) \simeq \frac{\varepsilon^2}{2} \int_\Gamma \frac{dt}{\rho(t)}$. At points where $\chi(t) \to +\infty$, the formula is convergent.

2. If $r \neq 0$, $MC(\varepsilon) \simeq -\frac{2\ln(\varepsilon)}{\pi} \sum_{i=1}^r \frac{1}{\rho(t_i)}$, where $\rho(t) = \frac{|dx(t)|}{|d\omega(t)|}$.

3. $E(\varepsilon) = 2\pi MC(\varepsilon)$.

Let us describe the asymptotic optimal syntheses. They are shown on Figures 3, 4.

Figure 3 concerns the case $r = 0$ (everywhere contact type). The points where the distribution $\Delta$ is not transversal to $\Gamma$ are omitted (they again do not change anything). Hence $\Delta$ is also transversal to the cylinders $C_\varepsilon$, for $\varepsilon$ small. Therefore, $\Delta$ defines (up to sign) a vector field $X_\varepsilon$ on $C_\varepsilon$, tangent to $\Delta$, that can be chosen of length 1. The asymptotic optimal synthesis consists of: 1. Reaching $C_\varepsilon$ from $\Gamma(0)$. 2. Follow a trajectory of $X_\varepsilon$. 3. Join $\Gamma(t)$. The steps 1 and 3 cost $2\varepsilon$, which is negligible w.r.t. the full metric complexity. To get the optimal synthesis for the interpolation entropy, one has to make the same construction, but starting from a subriemannian cylinder $C'_\varepsilon$ tangent to $\Gamma$.

In normal coordinates, in that case, the $x$-trajectories are just circles, and the corresponding optimal controls are just trigonometric functions, with period $\frac{2\pi}{\varepsilon}$.

Figure 4 concerns the case $r \neq 0$ (crossing Martinet surface). At a Martinet point, the vector-field $X_\varepsilon$ has a limit cycle, which is not tangent to the distribution. The asymptotic optimal strategy consists of: a. following a trajectory of $X_\varepsilon$ till reaching the height of the center of the limit cycle, b. crossing the cylinder, with a negligible cost $2\varepsilon$, c. Following a trajectory of the opposite vector field $-X_\varepsilon$. The strategy for entropy is similar, but using the tangent cylinder $C'_\varepsilon$.

**C. The one-step bracket-generating case**

For the corank $k \leq 3$, the situation is very similar to the 3-dimensional case. It can be completely reduced to it. For details, see [10].

At this point, this strange fact appears: there is the limit corank $k = 3$. If $k > 3$ only, new phenomena appear. Let us explain now why the reason for this.

Let us consider the following mapping $B_\varepsilon : \Delta \times \Delta \to T_x M/\Delta$, $(X,Y) \to [X,Y] + \Delta$. It is a well defined tensor mapping, which means that it actually applies to vectors (and not to vector fields, as expected from the definition). This is due to the following formula, for a one-form $\omega : d\omega(X,Y) = \omega([X,Y]) + \omega(Y)X - \omega(X)Y$. Let us call $L$ the image by $B_\varepsilon$ of the product of two unit balls in $\Delta$. The following holds:

**Theorem 23:** For a generic $P$, for $k \leq 3$, the sets $I_{\Gamma(t)}$ are convex.

This theorem is shown in [10], with the consequences that we will state just below.

This is no more true for $k > 3$, the first catastrophic case being the case 10-4 (a $p = 4$ distribution in $\mathbb{R}^{10}$). The intermediate cases $k = 4, 5$ in dimension 10 are interesting, since on some open subsets of $\Gamma$, the convexity property may hold or not. These cases are studied in the paper [13].

The main consequence of this convexity property is that everything reduces (out of singularities where the logarithmic lemma applies) to the 3-dimensional contact case, as is shown in the paper [10]. We briefly summarize the results.

Consider the one forms $\omega$ that vanish on $\Delta$ and that are 1 on $\Gamma$, and again, by the duality w.r.t. the metric over $\Delta$, define $d\omega|_\Delta(X,Y) = < AX, Y >$, for vector fields $X, Y$ in $\Delta$. Now, we have along $\gamma$, a $(k-1)$-parameter affine family of skew symmetric endomorphisms $A(\Gamma(t))$ of.
\[ \Delta_{\Gamma(t)} = \lambda_0 \sum_{j=1}^{k-1} \lambda_j A_{\gamma(t)}^j. \] Set \( \chi(t) = \inf \lambda ||A_{\gamma(t)}(\lambda)|| = ||A_{\gamma(t)}(\lambda^*(t))||. \)

Out of isolated points of \( \Gamma \) (that count for nothing in the metric complexity or in the entropy), the \( t \)-one parameter family \( A_{\gamma(t)}(\lambda^*(t)) \) can be smoothly block-diagonalized (with \( 2 \times 2 \) blocks), using a gauge transformation along \( \Gamma \). After this gauge transformation, the 2-dimensional eigenspace corresponding to the largest (in moduli) eigenvalue of \( A_{\gamma(t)}(\lambda^*(t)) \), corresponds to the two first coordinates in the distribution, and to the 2 first controls. In the asymptotic optimal synthesis, all other controls are put to zero (here the convexity property is to the 2 first controls. In the asymptotic optimal synthesis, it is exactly that of the 3-dimensional contact case. We still have the formulas:

\[ MC(\epsilon) \approx \frac{2}{\epsilon^2} \int_0^T dt \sum_{j=1}^{r} j \lambda_j^2, \] the optimal controls being of the form:

\[ u_{2j-1}(t) = - \frac{j \lambda_j^2}{\sum_{j=1}^{r} j \lambda_j^2} \sin(\frac{2 \pi j t}{\epsilon}), \] \[ u_{2j}(t) = \frac{j \lambda_j^2}{\sum_{j=1}^{r} j \lambda_j^2} \cos(\frac{2 \pi j t}{\epsilon}), \quad j = 1, ..., r, \] \[ u_{2r+1}(t) = 0 \] if \( p \) is odd.

These last formulas hold in the free case only (i.e. the case where the corank \( k = \frac{p(p-1)}{2} \), the dimension of the second homogeneous component of the free Lie-algebra with \( p \) generators). The non free case is more complicated (see [14]).

To prove all the results in this section, one has to proceed as follows: 1. use the theorem of reduction to nilpotent approximation (19), and 2. use the Pontriaguin's maximum principle on the normal form of the nilpotent approximation, in normal coordinates

\[ D. \ The \ 2\text{-}control \ case, \ in \ \mathbb{R}^4 \ and \ \mathbb{R}^5. \]

These cases correspond respectively to the car with a trailer (Example [2] and the ball on a plate (Example [3]).

We use also the theorem [19] of reduction to Nilpotent approximation, and we consider the normal forms \( \mathcal{P}_{4,2}, \mathcal{P}_{5,2} \) of Section II-C. In both cases, we change the variable \( \bar{w} \) for \( w \) such that \( dw = \frac{du}{\Delta u} \). We look for arclength-parametrized trajectories of the nilpotent approximation (i.e. \( u_1^2 + u_2^2 = 1 \)), that start from \( \Gamma(0) \), and reach \( \Gamma \) in fixed time \( \epsilon \), maximizing \( \int_0^{\epsilon} \bar{w}(\tau) d\tau \). Abnormal extremals do no come in the picture, and optimal curves correspond to the hamiltonian

\[ H = \sqrt{(PF_1^2 + PF_2^2)}, \] where \( P \) is the adjoint vector. It turns out that, in our normal coordinates, the same trajectories are optimal for both the 4-2 and the 5-2 case (one has just to notice that the solution of the 4-2 case meets the extra interpolation condition corresponding to the 5-2 case).

Setting as usual \( u_1 = \cos(\varphi) = PF_1, u_2 = \sin(\varphi) = PF_2 \), we get \( \varphi = P[F_1, F_2], \dot{\varphi} = -P[F_1, [F_1, F_2]]PF_1 - P[F_2, [F_1, F_2]]PF_2 \).

At this point, we have to notice that only the components \( F_1, F_2 \) of the adjoin vector \( P \) are not constant (the hamiltonian in the nilpotent approximation depends only on the \( x \)-variables), therefore, \( P[F_1, [F_1, F_2]] \) and \( P[F_2, [F_1, F_2]] \) are constant (the third brackets are also constant vector fields). Hence, \( \varphi = \alpha \cos(\varphi) + \beta \sin(\varphi) = \alpha \dot{x}_1 + \beta \dot{x}_2 \) for appropriate constants \( \alpha, \beta \). It follows that, for another constant \( k \), we have, for the optimal curves of the nilpotent approximation, in normal coordinates \( x_1, x_2 \):

\[ \dot{x}_1 = \cos(\varphi), \dot{x}_2 = \sin(\varphi), \] \[ \dot{\varphi} = k + \lambda x_1 + \mu x_2. \]

Remark 24: 1. It means that we are looking for curves in the \( x_1, x_2 \) plane, whose curvature is an affine function of the position,

2. In the two-step bracket generating case (contact case), optimal curves were circles, i.e. curves of constant curvature,

3. the conditions of \( \epsilon \)-interpolation of \( \Gamma \) say that these curves must be periodic (there will be more details on this point in the next section), that the area of a loop must be zero \( (\mu(\epsilon) = 0) \), and finally (in the 5-2 case) that another moment must be zero.

It is easily seen that such a curve, meeting these interpolation conditions, must be an elliptic curve of elastica-type. The periodicity and vanishing surface requirements imply that it is the only periodic elastic curve shown on Figure 5 parametrized in a certain way.
The formulas are, in terms of the standard Jacobi elliptic functions:
\[
\begin{align*}
u_1(t) &= 1 - 2dn(K(1 + \frac{4t}{\varepsilon}))^2, \\
u_2(t) &= -2dn(K(1 + \frac{4t}{\varepsilon})sn(K(1 + \frac{4t}{\varepsilon}))sn(\frac{\varphi_0}{2}),
\end{align*}
\]
where \(\varphi_0 = 130^\circ\) (following \([22]\), p. 403) and \(\varphi_0 = 130,692^\circ\) following Mathematica\(^\text{®}\), with \(k = \sin(\frac{\varphi_0}{2})\) and \(K(k)\) is the quarter period of the Jacobi elliptic functions. The trajectory on the \(x_1, x_2\) plane, shown on Figure 5 has equations:
\[
\begin{align*}x_1(t) &= -\frac{\varepsilon}{4K}[\frac{-4Kt}{\varepsilon} + 2(Eam(\frac{4Kt}{\varepsilon} + K) - Eam(K))], \\
x_2(t) &= k\frac{\varepsilon}{2K}cn(\frac{\varepsilon}{2} + K).
\end{align*}
\]

On the figure 2 one can clearly see, at the contact point of the ball with the plane, a trajectory which is a “repeated small perturbation”. In fact, it is the only abnormal extremal on the level set. It holds only for the ball with a trailer.

Furthermore, we get
\[
\dot{\varphi} = P[F,G], \quad \dot{\varphi} = -PFFG.PF - PGFG.PG,
\]
where \(FFG = [F,[F,G]]\) and \(GFG = [G,[F,G]]\). We set \(\lambda = -PFFG, \mu = -PGFG\). We get that
\[
\dot{\varphi} = \lambda \sin(\varphi) + \mu \cos(\varphi).
\]

Now, we compute \(\lambda\) and \(\mu\). We get, with similar notations as above for the brackets (we bracket from the left):
\[
\lambda = PFFFG.PF + PGFFG.PG,
\]
and computing the brackets, we see that \(GFG = FGFG = 0\). Also, since the hamiltonian does not depend on \(y, z, w\), we get that \(p_3, p_4, p_5, p_6\) are constants. Computing the brackets \(FFG\) and \(GFG\), we get that
\[
\lambda = \frac{3}{2}p_4 + p_6x_1, \quad \mu = \frac{3}{2}p_5 + p_6x_2,
\]
and then, \(\dot{\lambda} = p_6 \sin(\varphi)\) and \(\dot{\mu} = p_6 \cos(\varphi)\). Then, by (21),
\[
\dot{\lambda} = p_6 \frac{\lambda}{p_6} + \frac{\mu}{p_6}, \quad \text{and finally:}
\]
\[
\dot{x}_1 = \sin(\varphi), \quad \dot{x}_2 = \cos(\varphi),
\]
\[
\dot{\varphi} = K + \frac{1}{2p_6}(\lambda^2 + \mu^2),
\]
\[
\lambda = p_6 \sin(\varphi), \quad \mu = p_6 \cos(\varphi).
\]

Setting \(\omega = \frac{\lambda}{p_6}, \delta = \frac{\mu}{p_6}\), we obtain:
\[
\dot{\omega} = \sin(\varphi), \quad \dot{\delta} = \cos(\varphi),
\]
\[
\dot{\varphi} = K + \frac{1}{2}(\omega^2 + \delta^2).
\]

It means that the plane curve \((\omega(t), \delta(t))\) has a curvature which is a quadratic function of the distance to the origin. Then, the optimal curve \((x_1(t), x_2(t))\) projected to the horizontal plane of the normal coordinates has a curvature which is a quadratic function of the distance to some point. Following the lemma (23) in the appendix, this system of equations is integrable.

Summarizing all the results, we get the following theorem.

Theorem 26: (asymptotic optimal synthesis for the ball with a trailer) The asymptotic optimal synthesis is an \(\varepsilon\)-modification of the one of the nilpotent approximation, which has the following properties, in projection to the horizontal plane \((x_1, x_2)\) in normal coordinates:

1. It is a closed smooth periodic curve, whose curvature is a quadratic function of the position, and a function of the square distance to some point.
2. The area and the \(2\text{nd}\) order moments \(\int_\Gamma x_1(x_2dx_1 - x_1dx_2)\) and \(\int_\Gamma x_2(x_2dx_1 - x_1dx_2)\) are zero.
3. The entropy is given by the formula: 

\[ E(\varepsilon) = \frac{\sigma}{\pi} \int_0^\infty \frac{dw}{\delta(w)} \]

where \( \delta(w) \) is the main invariant from (20), and \( \sigma \) is a universal constant.

In fact we can go a little bit further to integrate explicitly the system (22). Set 

\[ \bar{\lambda} = \cos(\varphi)\lambda - \sin(\varphi)\mu, \quad \bar{\mu} = \sin(\varphi)\lambda + \cos(\varphi)\mu. \]

we get:

\[ \frac{d\bar{\lambda}}{dt} = -\bar{\mu}(K + \frac{1}{2p_0}(\bar{\lambda}^2 + \bar{\mu}^2)), \]
\[ \frac{d\bar{\mu}}{dt} = p_0 + \bar{\lambda}(K + \frac{1}{2p_0}(\bar{\lambda}^2 + \bar{\mu}^2)). \]

This is a 2 dimensional (integrable) hamiltonian system. The hamiltonian is:

\[ H_1 = -p_0\bar{\lambda} - \frac{2p_0}{4}(K + \frac{1}{2p_0}(\bar{\lambda}^2 + \bar{\mu}^2))^2. \]

This hamiltonian system is therefore integrable, and solutions can be expressed in terms of hyperelliptic functions. A little numerics now allows to show, on figure 7 the optimal x-trajectory in the horizontal plane of the normal coordinates.

On the figure 8 we show the motion of the ball with a trailer on the plane (motion of the contact point between the ball and the plane). Here, the problem is to move along the x-axis, keeping constant the frame attached to the ball and the angle of the trailer.

V. EXPECTATIONS AND CONCLUSIONS

Some movies of minimum entropy for the ball rolling on a plane and the ball with a trailer are visible on the website.

A. Universality of some pictures in normal coordinates

Our first conclusion is the following: there are certain universal pictures for the motion planning problem, in corank less or equal to 3, and in rank 2, with 4 brackets at most (could be 5 brackets at a singularity, with the logarithmic lemma).

These figures are, in the two-step bracket generating case: a circle, for the third bracket, the periodic elastica, for the 4th bracket, the plane curve of the figure 6.

They are periodic plane curves whose curvature is respectively: a constant, a linear function of of the position, a quadratic function of the position.

This is, as shown on Figure 8, the clear beginning of a series.

B. Robustness

As one can see, in many cases (2 controls, or corank \( k \leq 3 \)), our strategy is extremely robust in the following sense: the asymptotic optimal syntheses do not depend, from the qualitative point of view, of the metric chosen. They depend only on the number of brackets needed to generate the space.

C. The practical importance of normal coordinates

The main practical problem of implementation of our strategy comes with the \( \varepsilon \)-modifications. How to compute them,
how to implement? In fact, the $\varepsilon$-modifications count at higher order in the entropy. But, if not applied, they may cause deviations that are not neglectable. The high order w.r.t. $\varepsilon$ in the estimates of the error between the original system and its nilpotent approximation (Formulas [10] [12] [13]? ) make these deviations very small. It is why the use of our concept of a nilpotent approximation along $\Gamma$, based upon normal coordinates is very efficient in practice.

On the other hand, when a correction appears to be needed (after a noneglectable deviation), it corresponds to brackets of lower order. For example, in the case of the ball with a trailer (4th bracket), the $\varepsilon$-modification corresponds to brackets of order 2 or 3. The optimal pictures corresponding to these orders can still be used to perform the $\varepsilon$-modifications.

D. Final conclusion

This approach, to approximate optimally nonadmissible paths of nonholonomic systems, looks very efficient, and in a sense, universal. Of course, the theory is not complete, but the cases under consideration (first, 2-step bracket-generating, and second, two controls) correspond to many practical situations. But there is still a lot of work to do in order to cover all interesting cases. However, the methodology to go ahead is rather clear.

VI. APPENDIX

A. Appendix 1: Normal form in the 6-2 case

We start from the general normal form [11] in normal coordinates:

$$\dot{x}_1 = (1 + (x_2)^2)u_1 - x_1x_2u_2,$$

$$\dot{x}_2 = (1 + (x_1)^2)u_2 - x_1x_2u_1,$$

$$\dot{y}_1 = (x_2 - x_1)\gamma_1(y, w),$$

$$\dot{w} = (x_2 - x_1)\delta(y, w).$$

We will make a succession of changes of parametrisation of the surface $S$ (w.r.t. which normal coordinates were constructed). These coordinate changes will always preserve the fact that $\Gamma(t)$ is the point $x = 0, y = 0, w = t$.

Remind that $\beta$ vanishes on $S$, and since $x$ has order 1, we can already write on $T_x: \dot{x} = u + O(\varepsilon^2)$. One of the $\gamma_i$’s (say $\gamma_1$) has to be nonzero (if not, $\Gamma$ is tangent to $\Delta$). Then, $y_1$ has order 2 on $T_x$. Set for $i > 1, \dot{y}_i = y_i - \frac{\gamma_i}{\gamma_1}$. Differentiating, we get that $\frac{d\dot{y}_i}{dt} = \dot{y}_i - \frac{\gamma_i}{\gamma_1}\dot{y}_1 + O(\varepsilon^2)$, and $z_1 = \dot{y}_2, z_2 = \dot{y}_3$ have order 3. We set also $w := w - \frac{1}{\gamma_1}$, and we are at the following point:

$$\dot{x} = u + O(\varepsilon^3), \quad \dot{y} = (x_2u_1 - x_1u_2)\gamma_1(w) + O(\varepsilon^2),$$

$$\dot{z}_i = (x_2u_1 - x_1u_2)\gamma_i(w) + O(\varepsilon^3),$$

$$\dot{w} = (x_2u_1 - x_1u_2)\delta(w) + O(\varepsilon^3),$$

where $L_i(w), x, \delta(w), x$ are liner in $x$. The function $\gamma_1(w)$ can be put to 1 in the same way by setting $y := \frac{y}{\gamma_1(w)}$. Now let $T(w)$ be an invertible 2\times2 matrix. Set $\tilde{z} = T(w)z$. It is easy to see that we can chose $T(w)$ for we get:

$$\dot{x} = u + O(\varepsilon^3), \quad \dot{y} = (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2) + O(\varepsilon^2),$$

$$\dot{z}_i = (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)\gamma_i(w) + O(\varepsilon^3),$$

$$\dot{w} = (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)\delta(w) + O(\varepsilon^3),$$

Another change of the form: $w := w + L(w)x$, where $L(w), x$ is linear in $x$ kills $\delta(w)$ and brings us to $\dot{w} = (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)O(\varepsilon^2)$. This $O(\varepsilon^2)$ can be of the form $Q_w(x) + h(w)y + O(\varepsilon^3)$ where $Q_w(x)$ is quadratic in $x$. If we kill $h(w)$, we get the expected result. This is done with a change of coordinates of the form: $w := w + \varphi(w)\frac{\varepsilon}{2}$.

B. Appendix 2: Plane curves whose curvature is a function of the distance to the origin

This result was known already, see [24]. However we provide here a very simple proof.

Consider a plane curve $(x(t), y(t))$, whose curvature is a function of the distance from the origin, i.e.:

$$\dot{x} = \cos(\varphi), \dot{y} = \sin(\varphi), \varphi = k(x^2 + y^2). \quad (23)$$

Equation [23] is integrable.

Proof: Set $\tilde{x} = x \cos(\varphi) + y \sin(\varphi), \dot{y} = -x \sin(\varphi) + y \cos(\varphi)$. Then $k(\tilde{x}^2 + \tilde{y}^2) = k(x^2 + y^2)$. Just computing, one gets:

$$\frac{d\tilde{x}}{dt} = 1 + \tilde{y}k(\tilde{x}^2 + \tilde{y}^2),$$

$$\frac{d\tilde{y}}{dt} = -\tilde{x}k(\tilde{x}^2 + \tilde{y}^2).$$

We just show that [24] is a hamiltonian system. Since we are in dimension 2, it is always Liouville-integrable. Then, we are looking for solutions of the system of PDE’s:

$$\frac{\partial H}{\partial \tilde{x}} = 1 + \tilde{y}k(\tilde{x}^2 + \tilde{y}^2),$$

$$\frac{\partial H}{\partial \tilde{y}} = -\tilde{x}k(\tilde{x}^2 + \tilde{y}^2).$$

But the Schwartz integrability conditions are satisfied: $\frac{\partial^2 H}{\partial \tilde{y} \partial \tilde{x}} = 2\tilde{x}\tilde{y}k^3$.

C. Appendix 3: periodicity of the optimal curves in the 6-2 case

Proof: We consider the nilpotent approximation $\tilde{P}_{6,2}$ given in formula [15]

$$\tilde{P}_{6,2} \quad \tilde{x}_1 = u_1, \tilde{x}_2 = u_2, \dot{y} = \frac{x_2}{2}u_1 - \frac{x_1}{2}u_2, \quad (25)$$

$$\dot{z}_1 = x_2(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \dot{z}_2 = x_1(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2),$$

$$\dot{w} = Q_w(x_1, x_2)(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2).$$
We consider the particular case of the ball with a trailer. Then, according to Lemma 16 the ratio $r(\xi) = 1$.

It follows that the last equation can be rewritten $\dot{w} = \delta(w)((x_1)^2 + (x_2)^2)((\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)$ for some non-vanishing function $\delta(w)$ (vanishing would contradict the full rank of $\Delta(\xi)$). We can change the coordinate $w$ for $\tilde{w}$ such that $d\tilde{w} = dw$.

We get finally:

$$
\dot{\tilde{p}}_{6,2} = \begin{cases} 
\dot{x}_1 = u_1, \\
\dot{x}_2 = u_2, \\
\dot{y} = (\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{z}_1 = x_2(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{z}_2 = x_1(\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2), \\
\dot{w} = ((x_1)^2 + (x_2)^2)((\frac{x_2}{2}u_1 - \frac{x_1}{2}u_2)
\end{cases}
$$

This is a right invariant system on $\mathbb{R}^6$ with cocoordinates $\xi = (\zeta, \omega) = (x, y, z, w)$, for a certain Nilpotent Lie group structure over $\mathbb{R}^6$ (denoted by $G$). It is easily seen (just expressing right invariance) that the group law is of the form $(\zeta_2, \omega_2)(\zeta_1, \omega_1) = (\zeta_1 * \zeta_2, \omega_1 + \omega_2 + \Phi(\zeta_1, \zeta_2))$, where $*$ is the multiplication of another Lie group structure on $\mathbb{R}^6$, with coordinates $\zeta$ (denoted by $G_0$). In fact, $G$ is a central extension of $\mathbb{R}$ by $G_0$.

Lemma 27: The trajectories of (26) that maximize $\int \tilde{w} dt$ in fixed time $\varepsilon$, with interpolating conditions $\zeta(0) = \zeta(\varepsilon) = 0$, have a periodic projection on $\zeta$ (i.e. $\zeta(t)$ is smooth and periodic of period $\varepsilon$).

Remark 28: 1. Due to the invariance with respect to the $w$ coordinate of (26), it is equivalent to consider the problem with the more restrictive terminal conditions $\zeta(0) = \zeta(\varepsilon) = 0$, $w(0) = 0$.

2. The scheme of this proof works also to show periodicity in the 4-2 and 2-2 cases.

The idea for the proof was given to us by A. Agrachev.

Proof: Let $(\zeta, \omega_1), (\zeta, \omega_2)$ be initial and terminal points of an optimal solution of our problem. By right translation by $(\zeta, \omega_0)$, this trajectory is mapped into another trajectory of the system, with initial and terminal points $(0, w_1 + \Phi(\zeta, \omega_1))$ and $(0, w_1 + \Phi(\zeta, \omega_2))$. Hence, this trajectory has the same value of the cost $\int \tilde{w} dt$. We see that the optimal cost is in fact independent of the $\zeta$-coordinate of the initial and terminal condition.

Therefore, the problem is the same as maximizing $\int \tilde{w} dt$ but with the (larger) endpoint condition $\zeta(0) = \zeta(\varepsilon)$ (free). Now, we can apply the general transversality conditions of Theorem 12.15 page 188 of [4]. It says that the initial and terminal covectors $(p^1, p^1_w)$ and $(p^2, p^2_w)$ are such that $p^1 = p^2$. This is enough to show periodicity.

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