Gradient flow exact renormalization group— inclusion of fermion fields—

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The gradient flow exact renormalization group (GFERG) is an exact renormalization group motivated by the Yang–Mills gradient flow and its salient feature is a manifest gauge invariance. We generalize this GFERG, originally formulated for the pure Yang–Mills theory, to vector-like gauge theories containing fermion fields, keeping the manifest gauge invariance. For the chiral symmetry we have two options: one possible formulation preserves the conventional form of the chiral symmetry and the other simpler formulation realizes the chiral symmetry in a modified form à la Ginsparg–Wilson. We work out a gauge-invariant local Wilson action in quantum electrodynamics to the lowest nontrivial order of perturbation theory. This Wilson action reproduces the correct axial anomaly in $D = 2$. 

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1. Introduction

The Wilson exact renormalization group (ERG) \([1]\) (see also Refs. \([2, 3]\); for reviews, see Refs. \([4–8]\)) is important because, among many other things, it provides a unique framework to consider possible quantum field theories beyond perturbation theory. Given specific field contents, all possible quantum field theories are obtained by the continuum limit around each fixed point of the ERG equation with that field contents. In this sense, we may regard ERG as a theory of theories.

In particle physics, gauge symmetry is a fundamental principle and we are thus interested in ERG trajectories, i.e. solutions of the ERG equation, which preserve this symmetry. Traditional formulations, however, employ the momentum cutoff to define the ERG transformation, and this cutoff explicitly breaks the gauge symmetry in the conventional form. Although it is possible to define a modified gauge transformation which is consistent with the ERG evolution \([9]\) (see also Refs. \([6, 10]\) and references cited therein), and in principle one can maintain the gauge invariance in the modified form, since such a transformation depends on the Wilson action itself, it appears very hard to determine a nonperturbative truncation of the Wilson action being consistent with this exact symmetry of ERG. For non-perturbative applications of ERG in particle physics, therefore, a manifestly gauge-invariant ERG formulation is highly desirable. Such formulations have been developed, for instance in Refs. \([11–17]\).

The gradient flow exact renormalization group (GFERG) proposed in Ref. \([18]\) is one such manifestly gauge-invariant ERG formulation. This formulation is motivated by a similarity between the course graining process in ERG and the diffusion of a field configuration in spacetime. In particular, the diffusion defined by the Yang–Mills gradient flow \([19–21]\) has gauge-invariant meaning, and its renormalizability \([22]\) is also quite suggestive to ERG. Possible connections between ERG and the gradient flow or diffusion equations have been studied in Refs. \([23–33]\).

One direct connection between ERG and a diffusion equation may be observed as follows \([18, 34]\). The ERG evolution of the Wilson action \(S_\tau\) is described by the Wilson–Polchinski equation \([1, 35]\). For the scalar field theory in \(D\)-dimensional spacetime, in dimensionless variables, it reads

\[
\frac{\partial}{\partial \tau} e^{S_\tau[\phi]} = \int_p \left\{ \left[ \frac{\Delta(p)}{K(p)} + \frac{D + 2}{2} - \frac{\eta_p}{2} \right] \phi(p) + p \cdot \frac{\partial}{\partial p} \phi(p) \right\} \frac{\delta}{\delta \phi(p)} + \frac{1}{p^2} \left[ 2 \frac{\Delta(p)}{K(p)} k(p) + 2p^2 \frac{dk(p)}{dp^2} - \eta_p k(p) \right] \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{S_\tau[\phi]},
\]

where \(\tau\) parametrizes the ERG evolution and the functions \(K(p)\) and \(k(p)\) specify the ERG transformation; \(\Delta(p) \equiv -2p^2(\partial/\partial p^2)K(p)\).

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1 In momentum space, we adopt the convention

\[
\int_p \equiv \int \frac{d^Dp}{(2\pi)^D}, \quad \delta(p) \equiv (2\pi)^D \delta^{(D)}(p), \quad \frac{\delta \phi(p)}{\delta \phi(q)} \equiv \delta(p - q).
\]
As pointed out in Ref. [36], the ERG evolution of the Wilson action \( S_\tau \) under Eq. (1.2) can be neatly formulated as an equality,

\[
\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle^{K,k}_{S_\tau} = e^{-\tau n(D+2)/2} Z^{n/2}_{\tau}^{1/2} \langle\langle \phi(p_1 e^{-\tau}) \cdots \phi(p_n e^{-\tau}) \rangle\rangle^{K,k}_{S_\tau = 0} \tag{1.3}
\]

between the modified correlation functions defined by

\[
\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle^{K,k}_{S_\tau} = \prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle \exp \left[ - \int \frac{k(p)}{\pi^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] \phi(p_1) \cdots \phi(p_n) \right\rangle^S_{S_\tau = 0} \tag{1.4}
\]

where the correlation function on the right-hand side is the conventional one with respect to the action \( S \). The anomalous dimension \( \eta_{\tau} \) in Eq. (1.2) and the wave function renormalization factor \( Z_{\tau} \) in Eq. (1.3) are related by

\[
\eta_{\tau} = \frac{\partial}{\partial \tau} \ln Z_{\tau}. \tag{1.5}
\]

Equation (1.3) shows that the field variable is multiplicatively renormalized by \( Z_{\tau} \) under the ERG evolution, when it is viewed in terms of the modified correlation function. In this sense, what is suitable to characterize the scaling or critical behavior under the ERG transformation is the modified correlation function rather than the conventional correlation function. This fact explains why in Eq. (1.2) the anomalous dimension \( \eta_{\tau} \), which is related to a “rescaling” of the field variable, should appear not only in the coefficient of the first-order functional derivative, but also in the coefficient of the second-order functional derivative. On this issue, see Refs. [37, 38].

Now, it can be readily seen that Eq. (1.3) in coordinate space\(^2\) is represented in terms of a functional integral as

\[
e^{S_{\tau}^{'}[\phi]} = \exp \left[ \int dD x \frac{1}{2} \delta^2 \phi(x) \delta \phi(x) \right] \times \int [d\phi'] \prod_{x'} \delta \left( \phi(x) - e^{\tau(D-2)/2} Z^{1/2}_{\tau} \phi'(t, x' e^{\tau}) \right) \times \exp \left[ - \int d^D x'' \frac{1}{2} \delta^2 \phi'(x'') \delta \phi'(x'') \right] e^{S_{\tau = 0}[\phi']}. \tag{1.7}
\]

Here, we assume a particular form of \( K \) and \( k \)\(^3\)

\[
K(p) = e^{-p^2}, \quad k(p) = p^2. \tag{1.8}
\]

The point is that in Eq. (1.7), the field \( \phi'(t, x) \) inside the delta function is given by the solution of the diffusion equation

\[
\partial_t \phi'(t, x) = \partial^2 \phi'(t, x), \quad \phi'(t = 0, x) = \phi'(x), \tag{1.9}
\]

where the initial configuration for the diffusion is given by the integration variable \( \phi' \) in the functional integral in Eq. (1.7); the dimensionless diffusion or flow time \( t \) and the ERG

\[
\phi(x) = \int \int e^{ipx} \phi(p), \quad \phi(p) = \int d^D x e^{-ipx} \phi(x). \tag{1.6}
\]

\(^2\)We define

\(^3\)This particular choice is not essential. In fact, one can find the diffusion equation corresponding to Eq. (1.2) with general \( K \) and \( k \).
evolution parameter $\tau$ are related as

$$t \equiv e^{2\tau} - 1.$$  \hfill (1.10)\end{equation}

In this way, one can directly relate the ERG equation in Eq. (1.2) and the diffusion equation in Eq. (1.9). We note that the structure of Eq. (1.7) is very simple: it consists of exponential functions of the second-order functional derivative and the delta function which imposes the equality of the argument of the Wilson action and the diffused field. The diffused field $\varphi'$ is rescaled in the normalization by $e^{\tau(D-2)/2}Z^{1/2}_\tau$, where $(D - 2)/2$ is the canonical mass dimension of the field, and in the spacetime coordinate as $x \to x e^{\tau}$.

Considering the continuum limit around a fixed point of the ERG equation, the above connection relates the correlation function given by the functional integral with respect to the Wilson action with a finite momentum cutoff $\Lambda$ and the correlation function of the diffused field at the (dimensionful) diffused or flow time $t = 1/\Lambda^2$ with respect to the bare action (with the parameter renormalization, such as the one considered in Ref. [39]) [34]. This relation provides an intuitive understanding [34] of the fact that the renormalization of parameters and the wave function of the diffused elementary scalar field automatically make the equal-point product of diffused fields finite; the reason is that the functional integral with respect to the Wilson action possesses an ultraviolet (UV) cutoff $\Lambda$. This finiteness is analogous to a remarkable property [22] of the gauge field diffused by the Yang–Mills gradient flow. These observations motivated a proposal of GFERG in the pure Yang–Mills theory in Ref. [18].

In the present paper, we generalize the GFERG in Ref. [18] to vector-like gauge theories containing fermion fields. As a natural generalization, we can maintain the manifest gauge invariance. For the chiral symmetry, we have two options: one possible formulation (see Appendix A) preserves the conventional form of the chiral symmetry, while the other simpler formulation presented in Sect. 2 realizes the chiral symmetry in a modified form known as the Ginsparg–Wilson (GW) relation [40]. Our derivation of the GW relation in the present manifestly gauge-invariant ERG formulation is very simple. In Sect. 3 to have some idea how the GFERG equation works, we compute a gauge-invariant local Wilson action in quantum electrodynamics (QED) to the lowest nontrivial order of perturbation theory. Section 4 is devoted to our conclusion. In Appendix B we compute the axial anomaly in $D = 2$ by using our gauge-invariant local Wilson action obtained in Sect. 3.

2. GFERG for vector-like gauge theories

Our idea for the construction of a GFERG equation in vector-like gauge theories would be almost obvious from the elucidation in the previous section. Imitating the structure of Eq. (1.7), we define the Wilson action by

$$e^{S_G[A,\psi,\bar{\psi}]}$$

$$= \exp \left[ \int d^Dx \frac{1}{2} \delta A^a_\mu(x) \delta A^a_\mu(x) \right] \exp \left[ \int d^Dx' \frac{\delta}{\delta \psi(x')} \frac{\delta}{\delta \bar{\psi}(x')} \right]$$

$$\times \int [dA'd\psi'd\bar{\psi}'] \prod_{x''} \delta \left( A^b_\nu(x'') - e^\tau g_{\tau}^{-1} B^b_\nu(t,x'' e^\tau) \right)$$

$$\times \delta \left( \psi(x'') - e^\tau(D-1)/2 Z^{1/2}_\tau \chi'(t,x'' e^\tau) \right) \delta \left( \bar{\psi}(x'') - e^\tau(D-1)/2 Z^{1/2}_\tau \bar{\chi}'(t,x'' e^\tau) \right)$$

4
In this expression, the diffused gauge field $B'(t,x)$ is the solution to the Yang–Mills gradient flow equation \[4\]

\[\partial_t B'_\mu(t,x) = D_\mu G_{\nu\rho}^a(t,x) + \alpha_0 D_\mu \partial_\nu B'^a_\nu(t,x), \quad B'^a_\mu(t=0,x) = A'^a_\mu(x), \quad (2.2)\]

where $\alpha_0$ is a parameter and the initial configuration $A'$ is given by the integration variable in Eq. \[2.1\]. We have defined

\[C^a_{\mu\nu}(t,x) \equiv \partial_\mu B^a_\nu(t,x) - \partial_\nu B^a_\mu(t,x) + f^{abc} B^b_\mu(t,x) B^c_\nu(t,x), \]

\[D_\mu X^a(t,x) \equiv \partial_\mu X^a(t,x) + f^{abc} B^b_\mu(t,x) X^c(t,x) \quad (2.3)\]

from the structure constants of the gauge group $f^{abc}$ defined from anti-Hermitian generators $T^a$ by $[T^a, T^b] = f^{abc} T^c$. In Eq. \[2.1\] we have taken the canonical mass dimension of the gauge potential 1 and written the wave function renormalization factor of the gauge field as $g^2_\tau$; the reason for this convention will become clear later.\[4\] Similarly, for the fermion field, we use the diffusion equations in Ref. \[41\],

\[\partial_t \chi'(t,x) = \left[\Delta - \alpha_0 \partial_\mu B'_\mu(t,x) T^a\right] \chi'(t,x), \quad \chi'(t=0,x) = \psi(x), \quad (2.4)\]

\[\partial_t \bar{\chi}'(t,x) = \bar{\chi}'(t,x) \left[\bar{\Delta} + \alpha_0 \partial_\mu B'_\mu(t,x)\right], \quad \bar{\chi}'(t=0,x) = \bar{\psi}(x), \quad (2.4)\]

where

\[\Delta \equiv D_\mu D_\mu, \quad D_\mu \equiv \partial_\mu + B'^a_\mu T^a, \]

\[\bar{\Delta} \equiv \bar{D}_\mu \bar{D}_\mu, \quad \bar{D}_\mu \equiv \bar{\partial}_\mu - B'^a_\mu T^a. \quad (2.5)\]

As in Ref. \[18\], it is easy to see that the construction in Eq. \[2.1\] preserves the partition function:

\[Z = \int [dA d\psi d\bar{\psi}] e^{S_{\tau}[A,\psi,\bar{\psi}]} = \int [dA d\psi d\bar{\psi}] e^{S_{\tau=0}[A,\psi,\bar{\psi}]. \quad (2.6)\]

Let us examine other properties that follow from Eq. \[2.1\].

2.1. Gauge symmetry

In an almost identical way to Ref. \[18\], we can see that the Wilson action in Eq. \[2.1\] is invariant under the infinitesimal gauge transformation

\[A'^a_\mu(x) \rightarrow A'^a_\mu(x) + g_\tau^{-1} \partial_\mu \omega^a(x) + f^{abc} A'^b_\mu(x) \omega^c(x), \]

\[\psi(x) \rightarrow \psi(x) - \omega^a(x) T^a \psi(x), \]

\[\bar{\psi}(x) \rightarrow \bar{\psi}(x) + \bar{\psi}(x) \omega^a(x) T^a \quad (2.7)\]

\[\text{This convention and the convention in Ref. } \[18\] \text{(especially that in Appendix A) are related by } g_\tau = \lambda z(\tau).\]
if the initial action $S_{\tau=0}[A, \psi, \bar{\psi}]$ is invariant under the above transformation with $\tau = 0$: First, the exponential functions of the second-order functional derivatives
\[
\exp \left[ \int d^Dx \frac{1}{2} \frac{\delta^2}{\delta A^0_\mu(x) \delta A^0_\mu(x)} \right] \exp \left[ \int d^Dx' \frac{\delta}{\delta \psi'(x')} \frac{\delta}{\delta \psi'(x')} \right]
\] (2.8)
are manifestly invariant under the gauge transformation (see Ref. [18]). Then, the gauge transformation on the argument of the Wilson action in Eq. (2.1) is transmitted, through the delta functions, to the gauge transformation on the diffused fields $B', \chi'$, and $\bar{\chi}'$. This gauge transformation is then, through the gauge covariance of the diffusion equations, transmitted to that on the initial configurations $A', \psi'$, and $\bar{\psi}'$. Thus, the gauge invariance of the Wilson action finally depends on the gauge invariance of the initial action $S_{\tau=0}[A', \psi', \bar{\psi}']$ and of the integration measure $[dA'd\psi'd\bar{\psi}']$ (for which we assume its invariance).

In a similar manner, we can see the independence of $S_\tau[A, \psi, \bar{\psi}]$ from the parameter $\alpha_0$ in the diffusion equations, Eqs. (2.2) and (2.4). To see this, let us suppose that we make an infinitesimal change of the parameter, $\alpha_0 \rightarrow \alpha_0 + \delta \alpha_0$. For a fixed initial configuration $A', \psi'$, and $\bar{\psi}'$, the solution of Eqs. (2.2) and (2.4) will change under this. On the other hand, we see that if we make the following infinitesimal transformation in Eqs. (2.2) and (2.4),
\[
B^a_\mu(t, x) \rightarrow B^a_\mu(t, x) + D_\mu \omega^a(t, x),
\]
\[
\chi(t, x) \rightarrow \chi(t, x) - \omega^a(t, x) T^a \chi(t, x),
\]
\[
\bar{\chi}(t, x) \rightarrow \bar{\chi}(t, x) + \bar{\chi}(t, x) T^a \omega^a(t, x) T^a,
\]
(2.9)
where the function $\omega^a(t, x)$ is defined as the solution of
\[
(\partial_t - \alpha_0 D_\nu \partial_\nu) \omega^a(t, x) = \delta \alpha_0 \partial_\nu B^a_\nu(t, x),
\]
(2.10)
then the change of the parameter $\delta \alpha_0$ in the diffusion equations can be compensated. By integrating Eq. (2.10) “backward against time” from $\omega(t, x) = 0$ to $\omega(t = 0, x)$, we then have a gauge transformation $\omega(t = 0, x)$ on the initial configuration $A', \psi'$, and $\bar{\psi}'$ such that the solution, $B', \chi'$, and $\bar{\chi}'$, is identical to that before the change of $\alpha_0$. This shows that if the initial action $S_{\tau=0}[A', \psi', \bar{\psi}']$ and the integration measure in Eq. (2.11) are gauge invariant, then the Wilson action $S_\tau[A, \psi, \bar{\psi}]$ is independent of the parameter $\alpha_0$.

### 2.2. Modified chiral symmetry: GW relation

An important symmetry in a system containing the fermion field is the chiral symmetry. The Wilson action in Eq. (2.11) cannot be invariant under the conventional form of the chiral transformation, i.e.
\[
\psi(x) \rightarrow (1 + i\alpha \gamma_5) \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)(1 + i\alpha \gamma_5).
\]
(2.11)
This follows from the fact that, under Eq. (2.11),
\[
\frac{\delta}{\delta \psi(x)} \rightarrow \frac{\delta}{\delta \psi(x)}(1 + i\alpha \gamma_5), \quad \frac{\delta}{\delta \psi(x)} \rightarrow (1 + i\alpha \gamma_5) \frac{\delta}{\delta \psi(x)},
\]
(2.12)
and thus the exponential function in Eq. (2.11),
\[
\hat{s} \equiv \exp \left[ \int d^Dx \frac{\delta}{\delta \psi(x)} \frac{\delta}{\delta \psi(x)} \right]
\]
(2.13)
does not possess a simple transformation property under Eq. (2.11). One can avoid this drawback by putting an odd number of Dirac matrices in the expression such as
\[
\exp \left[ -i \int d^D x \frac{\delta}{\delta \psi(x)} \mathcal{D}_\tau \frac{\delta}{\delta \bar{\psi}(x)} \right],
\]
where we have to also put the covariant derivative
\[
\mathcal{D}_\tau \equiv \gamma_\mu \left( \partial_\mu + g_\tau A_\mu^a T^a \right)
\]
to preserve the gauge (and Lorentz) invariance. This “manifestly chiral-invariant formulation” is actually a possible option, and we write down the corresponding ERG equation in Appendix A.

Here, we pursue the simpler construction, Eq. (2.11). Quite interestingly, the Wilson action in Eq. (2.1) can be invariant under a modified chiral transformation; this is nothing but the chiral symmetry realized by the GW relation [40] in Eq. (2.1) can be invariant under a modified chiral transformation; this is nothing but lattice gauge theory, see Refs. [42–47]; for studies in the context of ERG, we may refer, for instance, to Refs. [48, 49].

To find the exact chiral symmetry in Eq. (2.11), we introduce differential operators,
\[
\hat{\gamma}_5 \equiv \int d^D x \left[ \gamma_5 \psi(x) \frac{\delta}{\delta \psi(x)} + \bar{\psi}(x) \gamma_5 \frac{\delta}{\delta \bar{\psi}(x)} \right],
\]
\[
\hat{\Gamma}_5 \equiv \hat{s} \hat{\gamma}_5 \hat{s}^{-1},
\]
where \(\hat{s}\) is given in Eq. (2.13). Then, from Eq. (2.1), we have
\[
\hat{\Gamma}_5 e^{S_{\tau}[A,\psi,\bar{\psi}]}
\]
\[
= \exp \left[ \int d^D x \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] \exp \left[ \int d^D x' \frac{\delta}{\delta \psi(x')} \frac{\delta}{\delta \bar{\psi}(x')} \right]
\]
\[
\times \int [dA' d\psi' d\bar{\psi}'] \prod_{x',\nu,b} \delta \left( A_\nu^b(x') - e^\tau g^{-1}_{\tau} B_\nu^b(t, x'' e^\tau) \right)
\]
\[
\times \hat{\gamma}_5 \delta \left( \psi(x'') - e^{(D-1)/2} Z^{1/2}_\tau \chi'(t, x'' e^\tau) \right) \hat{\gamma}_5 \delta \left( \bar{\psi}(x'') - e^{(D-1)/2} Z^{1/2}_\tau \bar{\chi}'(t, x'' e^\tau) \right)
\]
\[
\times \exp \left[ - \int d^D x'' \frac{\delta}{\delta \psi''} \frac{\delta}{\delta \bar{\psi}''} \right] \exp \left[ - \int d^D x''' \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x'') \delta A_\mu^a(x''')} \right]
\]
\[
\times e^{S_{\tau=0}[A',\psi',\bar{\psi}']}. \tag{2.17}
\]
In this expression, \(\hat{\gamma}_5\) acting on \(\psi\) and \(\bar{\psi}\) amounts, through the delta functions, to the chiral transformation on \(\chi'\) and \(\bar{\chi}'\) because, e.g., \(\delta(\gamma_5 \psi - e^{(D-1)/2} Z^{1/2}_\tau \chi') = \delta(\psi - e^{(D-1)/2} Z^{1/2}_\tau \gamma_5 \chi')\). Since the flow equations in Eq. (2.4) preserve the conventional chiral symmetry, the chiral transformation on \(\chi'\) and \(\bar{\chi}'\) induces the transformation on the initial configuration, \(\psi'\) and \(\bar{\psi}'\). Then, again using the definition of \(\hat{\Gamma}_5\), we see that
\[
\hat{\Gamma}_5 e^{S_{\tau}[A,\psi,\bar{\psi}]}
\]
\[\footnotetext{5}{We would like to thank Tetsuya Onogi for calling our attention to this point.}\]
\[\footnotetext{6}{Here, we assume that the functional measure \(|d\psi'/d\bar{\psi}'|\) is invariant under the conventional chiral transformation.}\]
transformation generated by $\hat{\Gamma}$

In this sense, our ERG evolution preserves the invariance under the modified chiral symmetry of the Wilson action $\hat{\Gamma}$.

We note that, from the definition,

$$
\hat{\Gamma} = \int d^D x \left\{ \gamma_5 \left[ \psi(x) - \frac{\delta}{\delta \psi(x)} \right] \frac{\delta}{\delta \bar{\psi}(x)} + \left[ \bar{\psi}(x) + \frac{\delta}{\delta \bar{\psi}(x)} \right] \gamma_5 \frac{\delta}{\delta \psi(x)} \right\}.
$$

From this, we have

$$
e^{-S_\tau} \hat{\Gamma} e^{S_\tau} = \int d^D x \left\{ S_\tau \frac{\delta}{\delta \psi(x)} \gamma_5 \bar{\psi}(x) + \frac{\bar{\psi}(x) \gamma_5 \delta}{\delta \bar{\psi}(x)} S_\tau - 2 S_\tau \frac{\gamma_5 \delta}{\delta \psi(x)} S_\tau \right\}.
$$

This shows that, if we assume that the action is bilinear in the fermion field, $S_\tau = -\int d^D x \bar{\psi}(x) D \psi(x) + \cdots$, the modified chiral symmetry of the Wilson action $\hat{\Gamma} e^{S_\tau} = 0$ implies the GW relation \cite{40},

$$
\gamma_5 D + D \gamma_5 + 2D \gamma_5 D = 0.
$$

Note that we have arrived at this relation by a very simple manipulation while maintaining a manifest gauge invariance; it would be interesting to see how this relation reproduces the axial anomaly in our GFERG formulation\footnote{In Appendix \[3\] we demonstrate that the gauge-invariant local Wilson action to $O(A)$ to be obtained in Sect.\[3\] actually reproduces the axial anomaly in $D = 2$ correctly.}.

### 2.3. GFERG equation

Let us derive an ERG equation that the Wilson action in Eq. \[2.21\] fulfills. This is readily obtained by taking the $\tau$ derivative of Eq. \[2.21\] in a way analogous to the derivation of the ERG equation in the Yang–Mills theory \cite{18}. The result is

$$
\frac{\partial}{\partial \tau} e^{S_\tau[A,\psi,\bar{\psi}]} = \int d^D x \frac{\delta}{g_\tau \delta A^\rho_\mu(x)}
$$
\[ \times \left[ -2D_{\nu}F_{\mu\nu}^a(x) - 2\alpha_0D_{\mu}\partial_{\nu}A_{\nu}^a(x) - \left( \frac{D - 2}{2} + \frac{\zeta_\tau}{2} + x \cdot \frac{\partial}{\partial x} \right) A_{\mu}^a(x) \right] \bigg|_{A \to g_\tau(A + \delta/\delta A)} \]
\[ \times e^{S_{\tau}[A,\psi,\bar{\psi}]} \]
\[ + \int d^Dx \left\{ \left[ 2\Delta - 2\alpha_0\partial_{\mu}A_{\mu}(x) + \left( \frac{D - 1}{2} + \frac{\eta_\tau}{2} + x \cdot \frac{\partial}{\partial x} \right) \right] \bigg|_{A \to g_\tau(A + \delta/\delta A)} \]
\[ \times \left[ \psi(x) - \frac{\delta}{\delta\psi(x)} e^{S_{\tau}[A,\psi,\bar{\psi}]} \right] \frac{\delta}{\delta\bar{\psi}(x)} \right\} \]
\[ + \int d^Dx \left\{ \frac{\delta}{\delta\bar{\psi}(x)} e^{S_{\tau}[A,\psi,\bar{\psi}]} \left[ \bar{\psi}(x) - \frac{\delta}{\delta\bar{\psi}(x)} \right] \right\} \]
\[ \times \left[ 2\Delta + 2\alpha_0\partial_{\mu}A_{\mu}(x) + \left( \frac{D - 1}{2} + \frac{\eta_\tau}{2} + \frac{\zeta_\tau}{2} \right) \cdot x \right] \bigg|_{A \to g_\tau(A + \delta/\delta A)} \right\}. \tag{2.23} \]

Here, we have defined the anomalous dimensions by (recall Eq. (1.5))
\[ \zeta_\tau \equiv 4 - D - \frac{d}{d\tau} \ln g_\tau^2, \]
\[ \eta_\tau \equiv \frac{d}{d\tau} \ln Z_\tau. \tag{2.24} \]

The ERG equation in Eq. (2.23) is the main result of the present paper. Once this GFERG equation has been obtained, we may forget about the underlying construction in Eq. (2.1). Possible requirements on the initial action \( S_{\tau=0} \) in Eq. (2.1) discussed so far, such as the gauge invariance and the chiral invariance, become implicit. If these properties of the Wilson action are considered to be desirable, we should simply pick up a solution or the initial condition of the ERG equation which fulfills these and other physical requirements (especially the locality and the Lorentz invariance). In this way, the issue of the existence of the UV regularization which makes the initial action finite becomes irrelevant. The renormalizability, i.e. whether we can tune parameters in the solution such that the correlation functions become finite in the continuum limit, is another issue, and we think that the results in Refs. [22, 41] become helpful in considering this question.

Since in Eq. (2.23) the power of the gauge potential always accompanies the power of \( g_\tau \), we see that \( g_\tau \) plays the role of the gauge coupling as the convention indicates. This parameter can thus be used as an expansion parameter which defines the perturbative expansion at the Gaussian fixed point [18].

3. Perturbative solution in QED to \( O(g_\tau^4) \)

To have some idea how the GFERG equation in Eq. (2.23) works, in this section we consider the \( U(1) \) gauge theory with a Dirac fermion with the charge \( e \), i.e.
\[ T^a \to -ie, \tag{3.1} \]
and solve the GFERG equation to the lowest nontrivial order of perturbation theory, \( O(g_\tau^4) \).
We first note that QED possesses charge conjugation symmetry, i.e. invariance under
\[
\psi(x) \to C\bar{\psi}^T(x), \quad \bar{\psi}(x) \to -\psi^T(x)C^{-1}, \quad A^a_\mu(x) \to -A^a_\mu(x), \quad (3.2)
\]
where the charge conjugation matrix satisfies \(C^{-1}\gamma_\mu C = -\gamma_\mu^T\). Since all elements in Eq. (2.1), especially the flow equations for QED (i.e. \(f^{abc} = 0\)), preserve the invariance under Eq. (3.2), if the initial action \(S_{\tau=0}[A', \psi', \bar{\psi}']\) is invariant under the charge conjugation, then \(S_{\tau}[A, \psi, \bar{\psi}]\) is too. In particular, we can forbid terms purely consisting of an odd number of gauge potentials; this is Furry’s theorem in the present ERG formulation. Taking this fact into account, we set the Wilson action as
\[
S_{\tau}[A, \psi, \bar{\psi}] = \frac{1}{2} \int p A^a_\mu(-p)T(\tau; p)(p^2\delta_{\mu\nu} - p_\mu p_\nu)A^a_\nu(p) - \int p \bar{\psi}(-p)G(\tau; p)\psi(p) + g_\tau \int_{p_1, p_2, p_3} \delta(p_1 + p_2 + p_3)\bar{\psi}(p_1)H^a_\mu(\tau; p_1, p_2, p_3)A^a_\mu(p_2)\psi(p_3) + O(g_\tau^2). \quad (3.3)
\]
For the first term, we already imposed the gauge invariance in \(O(g_\tau^{-1})\), i.e. the invariance under \(A^a_\mu(p) \to A^a_\mu(p) + g_\tau^{-1}p_\mu\omega^a(p)\). Note that the function \(G(\tau; p)\) is not necessarily invariant under \(p \to -p\), because it may contain the Dirac matrix such as \(\hat{p}\).

In momentum space, the ERG equation in Eq. (2.23) times \(e^{-S_{\tau}}\) reads, when \(a_0 = 1\),
\[
\frac{\partial}{\partial \tau} S_{\tau} = \int p \left(2p^2 + \frac{D}{2} + 1 - \frac{\zeta_\tau}{2} + p \cdot \frac{\partial}{\partial p} \right) A^a_\mu(p) \cdot \frac{\delta S_{\tau}}{\delta A^a_\mu(p)} + \int p \left(2p^2 + \frac{D}{2} + \frac{\zeta_\tau}{2} - \frac{\eta_\tau}{2} + p \cdot \frac{\partial}{\partial p} \right) \psi(p)
\]
\[
+ \int p \bar{\psi}(p) \left(2p^2 + \frac{D}{2} + \frac{1}{2} - \eta_\tau + \frac{\zeta_\tau}{2} + p \cdot \frac{\partial}{\partial p} \right) \frac{\delta}{\delta \bar{\psi}(p)} S_{\tau}
\]
\[
+ \int p (-1) \left(4p^2 + 1 - \eta_\tau\right) S_{\tau} \frac{\zeta}{\delta \bar{\psi}(p)} \frac{\delta}{\delta \bar{\psi}(p)} S_{\tau}
\]
\[
+ g_\tau \int_{p, p', p''} \delta(p + p' + p'') \times \left\{-4iS_{\tau} \frac{\zeta}{\delta \bar{\psi}(-p)} \left[A^a_\mu(p') + \frac{\delta S_{\tau}}{\delta A^a_\mu(-p')}\right] T^a_{\mu} p'' \psi(p'') + 4iv(p) p_\mu \left[A^a_\mu(p') + \frac{\delta S_{\tau}}{\delta A^a_\mu(-p')}\right] T^a_{\mu} \frac{\delta}{\delta \bar{\psi}(-p'')} S_{\tau}\right\}
\]
\[
+ g_\tau \int_{p, p', p''} \delta(p + p' + p'')
\]
\[\text{\footnote{Although for QED the index } a \text{ runs only over } a = 1, \text{ we keep this index for potential applications of the present lowest-order solution to non-Abelian theories.}}\]
\[
\times \text{tr} \left\{ -4i \frac{\delta}{\delta \bar{\psi}(-p)} S_\tau \cdot S_\tau \frac{\xi}{\delta \bar{\psi}(-p')} (p - p')_\mu \left[ A_\mu''(p'') + \frac{\delta S_\tau}{\delta A_\mu''(-p'')} \right] T^a \right\} + O(g_\tau^2),
\]
(3.4)

where we have retained only terms relevant to the ERG evolution of terms in Eq. (3.3).

3.1. \(O(g_\tau^0)\) terms
In the lowest order, \(O(g_\tau^0)\), the ERG equation in Eq. (3.4) requires, for the coefficient functions in Eq. (3.3),
\[
\frac{1}{2} \frac{\partial}{\partial \tau} T(\tau; p) = \left( -\frac{1}{2} p \cdot \frac{\partial}{\partial p} + 2p^2 - \frac{\zeta_\tau}{2} \right) T(\tau; p) + p^2 \left( 2p^2 + 1 - \frac{\zeta_\tau}{2} \right) T(\tau; p)^2,
\]
\[
\frac{\partial}{\partial \tau} G(\tau; p) = \left( -p \cdot \frac{\partial}{\partial p} + 4p^2 + 1 - \eta_\tau \right) G(\tau; p) + (4p^2 + 1 - \eta_\tau) G(\tau; p)^2.
\]
(3.5)

It can be seen that the general solutions to these are given by
\[
T(\tau; p) = -\frac{1}{e^{\tau(4-D)} g_\tau^2 C(e^{-\tau p}) e^{-2p^2 + p^2}}, \quad G(\tau; p) = -\frac{\dot{\rho}}{Z_\tau C(e^{-\tau p}) e^{-2p^2 + \rho}},
\]
(3.6)

where \(C(p)\) and \(\tilde{C}(p)\) are arbitrary functions of \(p^2\); for locality of the Wilson action, however, \(C(p)\) and \(\tilde{C}(p)\) must be analytic at \(p = 0\). In obtaining the above expression for \(G\), we have assumed parity symmetry and that \(\tilde{C}\) does not contain \(\gamma_5\), and thus \(\rho\) and \(\tilde{C}\) commute with each other.

3.2. GW relation in \(O(g_\tau^0)\)
Even in the the above lowest \(O(g_\tau^0)\) solution, it is interesting to see how the GW relation in Eq. (2.22) is realized. To this order, Eq. (2.22) implies
\[
\gamma_5 G + G\gamma_5 + 2G\gamma_5 G = 0 \iff G^{-1}\gamma_5 + \gamma_5 G^{-1} + 2\gamma_5 = 0.
\]
(3.7)

For Eq. (3.6), on the other hand, we have
\[
G^{-1}\gamma_5 + \gamma_5 G^{-1} + 2\gamma_5 = \frac{Z_\tau e^{-2p^2}}{\dot{\rho}} [\gamma_5, \tilde{C}(e^{-\tau p})].
\]
(3.8)

Therefore, if and only if \(\gamma_5\) and the function \(\tilde{C}\) in Eq. (3.6) commute, i.e. if and only if \(\tilde{C}\) does not contain \(\dot{\rho}\), the Wilson action satisfies the GW relation. Note that, since the \(\tau\) dependence of \(G\) arises only from the combination \(e^{-\tau p}\) in \(\tilde{C}\), the GW relation is preserved under the evolution of \(\tau\), as our general discussion shows.

---

\[1\] For this, we note that the differential equations in Eq. (3.5) become linear in terms of \(T^{-1}\) and \(G^{-1}\).
An interesting case in which the GW relation is not fulfilled is

\[ \tilde{C}(e^{-\tau}p) = \frac{e^{-\tau} \dot{p} - i m}{e^{-\tau} \dot{p} + i m} = \frac{\dot{p}}{\dot{p} + i e^\tau m}, \]

where \( m \) is a constant. In this case, the breaking of the GW relation in Eq. \([3.8]\) becomes

\[ G^{-1} \gamma_5 + \gamma_5 G^{-1} + 2 \gamma_5 = 2 i Z_T e^{-2p^2} \gamma_5 \frac{e^\tau m}{p^2 + e^2 m^2}. \]

The choice of \( \tilde{C} \) in Eq. \([3.9]\) actually realizes a massive fermion. The propagator of the fermion field with respect to the Wilson action to this order is given by

\[ \langle \psi(p) \bar{\psi}(q) \rangle_{S_r} = G(\tau; p)^{-1} \delta(p + q) \]

\[ = - \frac{Z_T e^{-2p^2}}{\dot{p} + i e^\tau m} \delta(p + q) - \delta(p + q). \]

This is not, however, the propagator that obeys the scaling law under the ERG evolution; recall the discussion at Eq. \([1.3]\). Such a propagator is given by the modified one \([36]\) defined by (see Eq. \([1.4]\) for the scalar field case)

\[ \langle \langle \psi(p) \bar{\psi}(q) \rangle \rangle_{S_r} \equiv e^{\tau^2} e^{\eta^2} \left\langle \exp \left[ \int \frac{\delta}{\delta \bar{\psi}(r)} \frac{\delta}{\delta \psi(-r)} \right] \psi(p) \bar{\psi}(q) \right\rangle_{S_r} \]

\[ = e^{\tau^2} e^{\eta^2} \left[ \langle \psi(p) \bar{\psi}(q) \rangle_{S_r} + \delta(p + q) \right] \]

\[ = - \frac{Z_T}{\dot{p} + i e^\tau m} \delta(p + q). \]

The correlation length in units of the UV cutoff is thus given by \( e^{-\tau} m^{-1} \), and \( m \) is the mass parameter; the critical surface is approached by \( m \to 0 \). As expected, the GW relation is broken by the amount of this mass parameter as Eq. \([3.10]\).

The GW relation in Eq. \([3.7]\) is satisfied if \( \tilde{C}(p) \) is a scalar function of \( p^2 \) so that \( \tilde{C} \) commutes with \( \gamma_5 \). In this case, the function \( G \) realizes a massless fermion.

### 3.3. \( O(g_\lambda^1) \) terms

Next, we consider the \( O(g_\lambda^1) \) terms. Equation \([3.4]\) requires, for the coefficient functions in Eq. \([3.4]\),

\[
\left[ \frac{\partial}{\partial \tau} + \sum_i p_i \frac{\partial}{\partial p_i} - 2 \sum_i p_i^2 + \eta_\tau \right] H_\mu^a(\tau; p_1, p_2, p_3)
- (4p_1^2 + 1 - \eta_\tau) G(\tau; -p_1) H_\mu^a(\tau; p_1, p_2, p_3) - (4p_3^2 + 1 - \eta_\tau) H_\mu^a(\tau; p_1, p_2, p_3) G(\tau; p_3)
- 2 \left( 2p_2^2 + 1 - \frac{\zeta_\tau}{2} \right) p_2^2 T(\tau; p_2) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) H_\mu^a(\tau; p_1, p_2, p_3)
= 4i T^a G(\tau; -p_3) p_{3\mu} - 4i T^a p_{3\mu} G(\tau; p_3) - 4i T^a G(\tau; -p_1) (p_1 - p_3)_\mu G(\tau; p_3)
+ 4i T^a p_2^2 T(\tau; p_2) G(\tau; -p_1) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{3\nu}
- 4i T^a p_2^2 T(\tau; p_2) G(\tau; p_3) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{1\nu}
\]
- 4iT^a p_2^2 T(\tau; p_2) G(\tau; -p_1) G(\tau; p_3) \left( \delta_{\mu\nu} - \frac{p_2^\mu p_2^\nu}{p_2^2} \right) (p_1 - p_3)_{\nu}.

To solve this differential equation we define the decomposition into transverse and longitudinal parts as

\[ H^a_\mu(\tau; p_1, p_2, p_3) = \left( \delta_{\mu\nu} - \frac{p_2^\mu p_2^\nu}{p_2^2} \right) H^a_\nu(\tau; p_1, p_2, p_3) + \frac{p_2^\mu p_2^\nu}{p_2^2} H^a_\nu(\tau; p_1, p_2, p_3) \]

\[ \equiv \ell^a_\mu(\tau; p_1, p_2, p_3) + \ell^a_\mu(\tau; p_1, p_2, p_3). \tag{3.14} \]

Then, the ERG equation is decomposed into

\[
\left[ \frac{\partial}{\partial \tau} + \sum_i p_i \cdot \frac{\partial}{\partial p_i} - 2 \sum_i p_i^2 + \eta \right] \ell^a_\mu(\tau; p_1, p_2, p_3)
\]

\[ - (4p_1^2 + 1 - \eta) G(\tau; -p_1) \ell^a_\mu(\tau; p_1, p_2, p_3) - (4p_3^2 + 1 - \eta) \ell^a_\mu(\tau; p_1, p_2, p_3) G(\tau; p_3)
\]

\[ = 4iT^a G(\tau; -p_1) \frac{p_2^\mu p_2^\nu}{p_2^2} p_{3\nu} - 4iT^a G(\tau; p_3) \frac{p_2^\mu p_2^\nu}{p_2^2} p_{1\nu}
\]

\[ - 4iT^a G(\tau; -p_1) G(\tau; p_3) \frac{p_2^\mu p_2^\nu}{p_2^2} (p_1 - p_3)_{\nu}. \tag{3.15} \]

and

\[
\left[ \frac{\partial}{\partial \tau} + \sum_i p_i \cdot \frac{\partial}{\partial p_i} - 2 \sum_i p_i^2 + \eta \right] \ell_\mu(\tau; p_1, p_2, p_3)
\]

\[ - (4p_1^2 + 1 - \eta) G(\tau; -p_1) \ell_\mu(\tau; p_1, p_2, p_3) - (4p_3^2 + 1 - \eta) \ell_\mu(\tau; p_1, p_2, p_3) G(\tau; p_3)
\]

\[ = 4iT^a [1 + p_2^2 T(\tau; p_2)] G(\tau; -p_1) \left( \delta_{\mu\nu} - \frac{p_2^\mu p_2^\nu}{p_2^2} \right) p_{3\nu}
\]

\[ - 4iT^a [1 + p_2^2 T(\tau; p_2)] G(\tau; p_3) \left( \delta_{\mu\nu} - \frac{p_2^\mu p_2^\nu}{p_2^2} \right) p_{1\nu}
\]

\[ - 4iT^a [1 + p_2^2 T(\tau; p_2)] G(\tau; -p_1) G(\tau; p_3) \left( \delta_{\mu\nu} - \frac{p_2^\mu p_2^\nu}{p_2^2} \right) (p_1 - p_3)_{\nu}. \tag{3.16} \]

3.3.1. Solution for \( \ell_\mu \). First, to solve Eq. (3.15), we set

\[ \ell^a_\mu(\tau; p_1, p_2, p_3) = e^{-2\tau} Z_\tau e^{-p_1} G(\tau; -p_1) e^{p_2^a} \ell^a_\mu(\tau; p_1, p_2, p_3) e^{-p_3^a} G(\tau; p_3). \]

Then, noting the relation

\[
\left[ \frac{\partial}{\partial \tau} + p \cdot \frac{\partial}{\partial p} - 2p_2^2 + \frac{\eta}{2} - (4p_1^2 + 1 - \eta) G(\tau; p) \right] e^{-\tau} Z_\tau^{1/2} e^{-p_1} G(\tau; p) = 0, \tag{3.18} \]

which follows from Eq. (3.5), Eq. (3.15) reduces to

\[
\left( \frac{\partial}{\partial \tau} + \sum_i p_i \cdot \frac{\partial}{\partial p_i} \right) \tilde{\ell}^a_\mu(\tau; p_1, p_2, p_3)
\]
Then Eq. (3.16) reduces to
\[ -4iT^a e^{2\tau} e^{-p_1^2 - p_2^2 + p_3^2} \tilde{C}(e^{-\tau} p_1) p_2 p_2 \tilde{C}(e^{-\tau} p_3) p_3 p_3 \]
where we have used Eq. (3.20). The general solution to this is given by
\[ \ell^a_{\mu}(\tau; p_1, p_2, p_3) = f^a_{\mu}(e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) \]
\[ - 4iT^a e^{2\tau} W(p_3; p_2, p_1) \tilde{C}(e^{-\tau} p_1) p_2 P_{2\mu} p_2 p_1 p_1 \]
\[ - 4iT^a e^{2\tau} W(p_1; p_2, p_3) \tilde{C}(e^{-\tau} p_3) p_2 P_{2\mu} p_2 p_3 p_3, \]
where \( f^a_{\mu}(p_1, p_2, p_3) \) is an arbitrary vector function of \( p_i \); in this expression, the function \( W \) is defined by
\[ W(p_1; p_2, p_3) \equiv 1 + \frac{1}{2} \frac{e^{p_1^2 - p_2^2 - p_3^2}}{p_1^2 - p_2^2 - p_3^2}, \]
which solves
\[ \left( \frac{\partial}{\partial \tau} + \sum_i p_i \cdot \frac{\partial}{\partial p_i} \right) e^{2\tau} W(p_1; p_2, p_3) = e^{2\tau} e^{p_1^2 - p_2^2 - p_3^2}. \]

Going back to Eq. (3.17), we have the general solution for \( \ell^a_{\mu} \),
\[ \ell^a_{\mu}(\tau; p_1, p_2, p_3) = e^{-2\tau} Z_\tau e^{-p_1^2 + p_2^2 - p_3^2} G(\tau; -p_1) f^a_{\mu}(e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) G(\tau; p_3) \]
\[ - 4iT^a Z_\tau e^{-p_1^2 + p_2^2 - p_3^2} W(p_3; p_2, p_1) G(\tau; -p_1) \tilde{C}(e^{-\tau} p_1) \tilde{C}(e^{-\tau} p_3) p_2 P_{2\mu} p_2 p_1 p_1 \]
\[ - 4iT^a Z_\tau e^{-p_1^2 + p_2^2 - p_3^2} W(p_1; p_2, p_3) G(\tau; -p_1) \tilde{C}(e^{-\tau} p_3) \tilde{C}(e^{-\tau} p_3) p_2 P_{2\mu} p_2 p_3 p_3. \]

3.3.2. Solution for \( t_\mu \). Next, to solve Eq. (3.16), we set
\[ t^a_{\mu}(\tau; p_1, p_2, p_3) = e^{-2\tau} Z_\tau e^{(4-D)} g\tau^2 e^{-p_1^2} G(\tau; -p_1) e^{-p_2^2} T(\tau; p_2) \tilde{t}^a_{\mu}(\tau; p_1, p_2, p_3) e^{-p_3^2} G(\tau; p_3), \]
in view of
\[ \left[ \frac{\partial}{\partial \tau} + p \cdot \frac{\partial}{\partial p} - 2p^2 - 2 \left( 2p^2 + 1 - \frac{\zeta_\tau}{2} \right) p^2 T(\tau; p) \right] e^{(4-D)} g\tau^2 e^{-p^2} T(\tau; p) = 0. \]

Then Eq. (3.16) reduces to
\[ \left( \frac{\partial}{\partial \tau} + \sum_i p_i \cdot \frac{\partial}{\partial p_i} \right) \tilde{t}^a_{\mu}(\tau; p_1, p_2, p_3) \]
\[ = 4iT^a e^{2\tau} e^{-p_1^2 - p_2^2 + p_3^2} \tilde{C}(e^{-\tau} p_1) \tilde{C}(e^{-\tau} p_2) \left( \delta_{\mu\nu} - \frac{p_2 P_{2\nu} p_2}{p_2} \right) p_{1\nu} \]
\[ + 4iT^a e^{2\tau} e^{-p_3^2 - p_2^2 - p_3^2} \tilde{C}(e^{-\tau} p_3) \tilde{C}(e^{-\tau} p_2) \left( \delta_{\mu\nu} - \frac{p_2 P_{2\nu} p_2}{p_2} \right) p_{3\nu}. \]

The general solution to this is given by
\[ \tilde{t}^a_{\mu}(\tau; p_1, p_2, p_3) = g^a_{\mu}(e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) \]
+ 4i T^a e^{2\tau} W(p_3; p_2, p_1) \frac{\tilde C(e^{-\tau} p_1)}{p_1} C(e^{-\tau} p_2) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{1\nu} \\
+ 4i T^a e^{2\tau} W(p_1; p_2, p_3) \frac{\tilde C(e^{-\tau} p_3)}{p_3} C(e^{-\tau} p_2) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{3\nu}, \quad (3.27)

where \( g_\mu^a(p_1, p_2, p_3) \) is an arbitrary vector function of \( p_i; \) \( g_\mu^a(p_1, p_2, p_3) = 0. \) Plugging this into Eq. (3.21), we have

\[
t^a_\mu(\tau; p_1, p_2, p_3) = e^{-2\tau} g_\tau^a - 2 e^{-p_1^2 - p_2^2} G(\tau; -p_1) T(\tau; p_2) g_\mu^a(e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) G(\tau; p_3) \\
+ 4i T^a e^{2\tau} e^{(4-D)} g_\tau^{-1} e^{p_1^2 + p_2^2} W(p_3; p_2, p_1) \frac{\bar C(e^{-\tau} p_1)}{p_1} \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{1\nu} \\
+ 4i T^a e^{2\tau} e^{(4-D)} g_\tau^{-1} e^{p_1^2 + p_2^2 - p_3^2} W(p_1; p_2, p_3) \frac{\bar C(e^{-\tau} p_3)}{p_3} \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{3\nu}.
\]

(3.28)

3.3.3. Gauge invariance. So far, we have obtained a most general form of the interaction vertex \( H_\mu^a \) in Eq. (3.3), which solves the ERG equation. We now impose the gauge invariance to the Wilson action and further restrict \( H_\mu^a. \) The gauge transformation in Eq. (2.7) reads, in momentum space,

\[
A_\mu^a(p) \rightarrow A_\mu^a(p) + g_\tau^{-1} i p_\mu \omega^a(p), \\
\psi(p) \rightarrow \psi(p) - \int q \omega^a(p - q) T^a \psi(q), \\
\bar \psi(p) \rightarrow \bar \psi(p) + \int q \bar \psi(q) \omega^a(p - q) T^a.
\]

(3.29)

The gauge invariance of the Wilson action in Eq. (3.3) to \( O(g_0^0) \) thus requires

\[
2 \mu p \mu H_\mu^a(\tau; p_1, p_2, p_3) = T^a G(\tau; p_3) - T^a G(\tau; -p_1). \quad (3.30)
\]

From Eqs. (3.14) and (3.23), we thus have (note that the transverse part in Eq. (3.28) does not contribute to this)

\[
i p_2 \mu H_\mu^a(\tau; p_1, p_2, p_3) \\
= e^{-2\tau} g_\tau^a(e^{-p_1^2 + p_2^2} G(\tau; -p_1) i p_2 \mu T^a \mu(e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) G(\tau; p_3) \\
+ T^a g_\tau(e^{-2p_1^2 - e^{-p_1^2 + p_2^2 - p_3^2}} G(\tau; -p_1) \frac{\bar C(e^{-\tau} p_1)}{p_1} \\
+ T^a g_\tau(e^{-2p_3^2 - e^{-p_1^2 + p_2^2 - p_3^2}} G(\tau; -p_1) \frac{\bar C(e^{-\tau} p_3)}{p_3} \\
= e^{-2\tau} g_\tau^a(e^{-p_1^2 + p_2^2 - p_3^2} G(\tau; -p_1) i p_2 \mu T^a \mu(e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) G(\tau; p_3) \\
- T^a g_\tau(e^{-p_1^2 + p_2^2 - p_3^2} G(\tau; -p_1) \frac{\bar C(e^{-\tau} p_1)}{p_1} G(\tau; p_3)
\]

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\begin{align*}
- T^a Z_\tau e^{-\nu^2 - \nu \cdot \tau} G(\tau; -p_1) \frac{\tilde{C}(e^{-\tau} p_3)}{p_3} G(\tau; p_3) \\
+ T^a G(\tau; p_3) - T^a G(\tau; -p_1),
\end{align*}

where we have used \( p_3^2 - p_2^2 - p_1^2 = 2p_1 \cdot p_2 \) and \( p_1^2 - p_2^2 - p_3^2 = 2p_2 \cdot p_3 \), which follow from the momentum conservation \( p_1 + p_2 + p_3 = 0 \) and the relation

\[
Z_\tau e^{-2\nu^2} G(\tau; p) \frac{\tilde{C}(e^{-\tau} p)}{p} = -1 - G(\tau; p),
\]

which follows from Eq. (3.6). Equations (3.30) and (3.31) show that we can achieve the gauge invariance by taking

\[
f^a_{\mu}(e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) = -i T^a \left[ \frac{\tilde{C}(e^{-\tau} p_1)}{e^{-\tau} p_1} + \frac{\tilde{C}(e^{-\tau} p_3)}{e^{-\tau} p_3} \right] e^{-\tau} p_{2\mu}.
\]

Note that the gauge invariance is preserved under the evolution of \( \tau \), reflecting the gauge invariance of the present ERG formulation.

Therefore, adding Eqs. (3.23) and (3.28), we have

\[
H^a_{\mu}(\tau; p_1, p_2, p_3) = e^{-2\tau} Z_\tau e^{\tau(1-D)} g_\tau e^{-\nu^2 - \nu \cdot \tau} G(\tau; -p_1) T(\tau; p_2) g_\mu(\tau; -p_2, e^{-\tau} p_3) G(\tau; p_3) \\
- i T^a Z_\tau e^{-\nu^2 - \nu \cdot \tau} G(\tau; -p_1) \left[ \frac{\tilde{C}(e^{-\tau} p_1)}{p_1} + \frac{\tilde{C}(e^{-\tau} p_3)}{p_3} \right] \frac{p_{2\mu}}{p_2} G(\tau; p_3) \\
- 4i T^a Z_\tau e^{-\nu^2 - \nu \cdot \tau} W(p_3; p_2, p_1) G(\tau; -p_1) \frac{\tilde{C}(e^{-\tau} p_1)}{p_1} G(\tau; p_3) \frac{p_{2\mu} p_{2\nu}}{p_2^2} p_{1\nu} \\
- 4i T^a Z_\tau e^{-\nu^2 - \nu \cdot \tau} W(p_1; p_2, p_3) G(\tau; -p_1) \frac{\tilde{C}(e^{-\tau} p_3)}{p_3} G(\tau; p_3) \frac{p_{2\mu} p_{2\nu}}{p_2^2} p_{3\nu}.
\]

\[
+ 4i T^a Z_\tau e^{\tau(1-D)} g_\tau e^{-\nu^2 - \nu \cdot \tau} W(p_3; p_2, p_1) G(\tau; -p_1) \frac{\tilde{C}(e^{-\tau} p_1)}{p_1} \\
\times T(\tau; p_2) C(e^{-\tau} p_2) G(\tau; p_3) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{1\nu} \\
+ 4i T^a Z_\tau e^{\tau(1-D)} g_\tau e^{-\nu^2 - \nu \cdot \tau} W(p_1; p_2, p_3) G(\tau; -p_1) \\
\times T(\tau; p_2) C(e^{-\tau} p_2) \frac{\tilde{C}(e^{-\tau} p_3)}{p_3} G(\tau; p_3) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) p_{3\nu} \\
= e^{-2\tau} Z_\tau e^{\tau(1-D)} g_\tau e^{-\nu^2 - \nu \cdot \tau} G(\tau; -p_1) T(\tau; p_2) g_\mu(\tau; -p_2, e^{-\tau} p_3) G(\tau; p_3) \\
- i T^a Z_\tau e^{-\nu^2 - \nu \cdot \tau} G(\tau; -p_1) \frac{1}{p_1} \left[ \tilde{C}(e^{-\tau} p_1) p_3 + \tilde{C}(e^{-\tau} p_3) p_1 \right] \frac{1}{p_3} G(\tau; p_3) \frac{p_{2\mu}}{p_2^2} \\
- 4i T^a e^{-\nu^2 - \nu \cdot \tau} W(p_3; p_2, p_1) [1 + G(\tau; -p_1)] G(\tau; p_3) p_{1\mu} \\
+ 4i T^a e^{-\nu^2 - \nu \cdot \tau} W(p_1; p_2, p_3) G(\tau; -p_1) [1 + G(\tau; p_3)] p_{3\mu} \\
- 4i T^a e^{-\nu^2 - \nu \cdot \tau} W(p_3; p_2, p_1) [1 + G(\tau; -p_1)] T(\tau; p_2) G(\tau; p_3) \\
\times (p_2^2 \delta_{\mu\nu} - p_{2\mu} p_{2\nu}) p_{1\nu} \\
+ 4i T^a e^{-\nu^2 - \nu \cdot \tau} W(p_1; p_2, p_3) G(\tau; -p_1) T(\tau; p_2) [1 + G(\tau; p_3)].
\]
where we have used
\[ e^{\tau (4-D)} g_\tau^{-2} e^{-2p^2} T(\tau; p) C(e^{-\tau} p) = -1 - p^2 T(\tau; p), \tag{3.35} \]
which follows from Eq. (3.6), and Eq. (3.32).

3.3.4. Locality. Finally, we impose the locality on the Wilson action and determine the so far arbitrary \( g_\mu^a \) in Eq. (3.34). For locality, the function \( H_\mu^a \) in Eq. (3.34) should be analytic at \( p_1 = 0 \). Since the functions \( T \) and \( G \) in Eq. (3.6) and \( W \) in Eq. (3.21) are analytic, the term in Eq. (3.34) that is non-analytic is
\[ -iT^a Z e^{-p_1^2 + p_3^2 - p_3^2} G(\tau; -p_1) \frac{1}{p_1} \left[ \tilde{C}(e^{-\tau} p_1) p_3 + \tilde{C}(e^{-\tau} p_3) \right] \frac{1}{p_3} G(\tau; p_3) \frac{p_{2\mu}}{p_2^2}, \tag{3.36} \]
Since \( G(\tau; -p_1) \propto \delta_1 \) and \( G(\tau; p_3) \propto \delta_3 \), as Eq. (3.6) shows, the only non-analyticity arises from the factor \( 1/p_2^2 \). We have to choose the as yet undetermined function \( g_\mu^a \) so that this singularity is cancelled.

Although it turns out that it is always possible to choose \( g_\mu^a \) so that the \( 1/p_2^2 \) singularity is cancelled, the expression of such a \( g_\mu^a \) for the general case is very complicated and not illuminating. Here, therefore, we are content with the expression for a particular case, i.e. the limit \( \tau \to \infty \). In this limit, all irrelevant operators die out and the expressions become much simpler. For \( \tau \to \infty \), because of the locality of \( T \) and \( G \), \( C(e^{-\tau} p) \to C_0 \), \( \tilde{C}(e^{-\tau} p) \to \tilde{C}_0 \), and
\[ T(\tau = \infty; p) \equiv T(p) = -\frac{1}{z C_0 e^{-2p_2^2 + p_2^2}}, \quad G(\tau = \infty; p) \equiv G(p) = -\frac{\delta}{Z C_0 e^{-2\delta^2 + \delta^2}}, \tag{3.37} \]
where
\[ z \equiv \lim_{\tau \to \infty} e^{\tau (4-D)} g_\tau^{-2}, \quad Z \equiv Z_{\tau=\infty}. \tag{3.38} \]
Equation (3.36) in this limit then becomes
\[ iT^a Z C_0 e^{-p_1^2 + p_3^2 - p_3^2} G(-p_1) \frac{1}{p_1} \gamma_\nu \frac{1}{p_3} G(p_3) \frac{p_{2\mu}}{p_2^2}, \tag{3.39} \]
under the momentum conservation \( p_1 + p_3 = -p_2 \). Then, the choice (recall that \( g_\mu^a \) must be transverse)
\[ e^{-2\tau} g_\mu^a (e^{-\tau} p_1, e^{-\tau} p_2, e^{-\tau} p_3) = -iT^a C_0 \frac{1}{p_1} \gamma_\nu \frac{1}{p_3} \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) \tag{3.40} \]
cancels the non-analyticity in Eq. (3.34) in the limit \( \tau \to \infty \). In fact, with this choice,
\[ H_\mu^a (\tau = \infty; p_1, p_2, p_3) \equiv H_\mu^a (p_1, p_2, p_3) = -iT^a Z C_0 e^{-p_1^2 + p_3^2 - p_3^2} G(-p_1) \frac{1}{p_1} \gamma_\nu \frac{1}{p_3} G(p_3) e^{-2\tau^2} T(p_2) \left( \delta_{\mu\nu} - \frac{p_{2\mu} p_{2\nu}}{p_2^2} \right) \]
\[ + iT^a Z C_0 e^{-p_1^2 + p_3^2 - p_3^2} G(-p_1) \frac{1}{p_1} \gamma_\nu \frac{1}{p_3} G(p_3) \frac{p_{2\mu}}{p_2^2} \]
\[ - 4iT^a e^{-p_1^2 + p_2^2 - p_3^2} W(p_3; p_2, p_1) [1 + G(-p_1)] G(p_3) \]
\[ + 4iT^a e^{-p_1^2 + p_2^2 - p_3^2} W(p_1; p_2, p_3) G(-p_1) [1 + G(p_3)] \]

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\[-4i T^a e^{p^1 + p^2 - \tau^2} W(p_3; p_2, p_1)[1 + G(-p_1)]G(p_3) T(p_2) \left(p_2^2 \delta_{\mu\nu} - p_2 \mu p_2 \nu\right) p_1 \nu \]
\[+ 4i T^a e^{-p^1 + p^2 + \tau^2} W(p_1; p_2, p_3) G(-p_1)[1 + G(p_3)] T(p_2) \left(p_2^2 \delta_{\mu\nu} - p_2 \mu p_2 \nu\right) p_3 \nu. \tag{3.41} \]

Since \( z C_0 e^{-2 \tau^2} T(p_2) \rightarrow -1 \) for \( p_2^2 \rightarrow 0 \), the singularity \( 1/p_2^2 \) is cancelled. Equation (3.41) provides the gauge-invariant local expression of the interaction vertex \( H^a_{\mu} \) at \( \tau \rightarrow \infty \)

The resulting Wilson action in Eq. (3.33) depends on the parameters \( Z C_0, z C_0, \) and \( g_F = e^{(4 - D)/2} \). In the present order of approximation, the first two are marginal and the last one is marginal for \( D = 4 \) and relevant for \( D < 4 \).

Interestingly, if we assume that the function \( G \) satisfies the GW relation in \( O(g_F^2) \), Eq. (3.37), then the solution in Eq. (3.41) also satisfies the GW relation in Eq. (2.22) in \( O(g_F^2) \), i.e.

\[ [1 + 2G(-p_1)] \gamma_5 H^a_{\mu}(p_1, p_2, p_3) + H^a_{\mu}(p_1, p_2, p_3) \gamma_5 [1 + 2G(p_3)] = 0. \tag{3.42} \]

This can be readily seen by noting Eq. (3.32) and

\[ [1 + 2G(-p_1)] \gamma_5 G(-p_1) = G(-p_1)(-\gamma_5), \]
\[ G(p_3) \gamma_5 [1 + 2G(p_3)] = (-\gamma_5)G(p_3), \tag{3.43} \]

which follow from Eq. (3.7). We do not have any understanding on whether this is accidental or inevitable. It would be troublesome, however, if Eq. (3.41) cannot fulfill the GW relation, because Eq. (3.41) provides essentially the unique gauge-invariant local solution of the GFERG equation to \( O(g_F^4) \).

As demonstrated in Appendix B, we can obtain the axial anomaly in \( D = 2 \) from the expression of the gauge-invariant local Wilson action to \( O(A) \) given by Eq. (3.33) with Eq. (3.41).

4. Conclusion

We have formulated an ERG equation (the GFERG equation) in vector-like gauge theories, Eq. (2.23), on the basis of the notion of the gradient flow and the fermion flow. The GFERG equation preserves the gauge invariance and the chiral symmetry in a modified (à la Ginsparg–Wilson) form. The formulation awaits applications such as the search for nontrivial fixed points in gauge theory on the basis of a certain gauge-invariant truncation of the Wilson action. Before going into such a nonperturbative study, however, we should better understand perturbative aspects of the GFERG equation. In this paper we obtained a gauge-invariant local Wilson action in QED by solving the GFERG equation to \( O(g_F^4) \) at the Gaussian fixed point. We should pursue this perturbative analysis at least to \( O(g_F^2) \), where we should be able to observe “quantum corrections.” Another important remaining issue is the finiteness of correlation functions in the continuum limit; we want to understand this question in GFERG, in a manner similar to the argument in Ref. [34], using the results of Refs. [22, 41] as a clue.

\[ \text{In the low-momentum limit } p_i \rightarrow 0, \text{H}_\mu^a \text{ reduces to the conventional expression of the vertex, } T^a/(iZ C_0) \gamma_\mu. \]

\[ \text{We assume that Dirac matrices and } \gamma_5 \text{ anti-commute.} \]

\[ \text{One may generalize the above gauge-invariant local solution by adding a local function of } e^{-\tau^2} \text{ that is proportional to } e^{-2\tau} (p_2^2 \delta_{\mu\nu} - p_2 \mu p_2 \nu) \text{ to } g_F^a(e^{-\tau^2} p_1, e^{-\tau^2} p_2, e^{-\tau^2} p_3) \text{ in Eq. (3.41). Such a generalization, however, introduces only irrelevant operators to the Wilson action and does not change } H_\mu^a(\tau = \infty; p_1, p_2, p_3). \]
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A. Manifestly chiral-invariant formulation

The GFERG formulation that is consistent with the conventional form of the chiral transformation in Eq. (2.11) can be obtained by setting

$$\exp\left[\int d^D x \frac{1}{2} \delta^2 \frac{\delta}{\delta A^a_{\mu}(x)} \frac{\delta}{\delta A^a_{\mu}(x)} \right]$$

for $\Delta_A$ and $\Delta_B$. We would like to thank Hidenori Sonoda for insightful remarks. This work was partially supported by Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research Grant Number JP20H01903.

\begin{align}
\frac{\partial}{\partial \tau} e^{S_{\tau}[A,\psi,\bar{\psi}]} &= \int d^D x \frac{1}{2} \delta^2 \frac{\delta}{\delta A^a_{\mu}(x)} \frac{\delta}{\delta A^a_{\mu}(x)} \left[ -i \int d^D x' \frac{\delta}{\delta \psi(x')} \frac{\delta}{\delta \bar{\psi}(x')} \right] \\
&\times \left[ \frac{\delta}{\delta \psi(x')} \right]_{\tau=0} \left[ \frac{\delta}{\delta \bar{\psi}(x')} \right]_{\tau=0} \\
&\times e^{S_{\tau=0}[A',\psi',\bar{\psi}']},
\end{align}

where $\bar{\psi}$ is given by Eq. (2.15).

Taking the $\tau$ derivative of this, we have the following GFERG equation:

$$\frac{\partial}{\partial \tau} e^{S_{\tau}[A,\psi,\bar{\psi}]}$$

\begin{align}
&\frac{\partial}{\partial \tau} e^{S_{\tau}[A,\psi,\bar{\psi}]} \\
&= \int d^D x \frac{1}{2} \delta^2 \frac{\delta}{\delta A^a_{\mu}(x)} \frac{\delta}{\delta A^a_{\mu}(x)} \left[ -i \int d^D x' \frac{\delta}{\delta \psi(x')} \frac{\delta}{\delta \bar{\psi}(x')} \right] \\
&\times \left[ \frac{\delta}{\delta \psi(x')} \right]_{\tau=0} \left[ \frac{\delta}{\delta \bar{\psi}(x')} \right]_{\tau=0} \\
&\times e^{S_{\tau=0}[A',\psi',\bar{\psi}']},
\end{align}

where $\bar{\psi}$ is given by Eq. (2.15).

Taking the $\tau$ derivative of this, we have the following GFERG equation:

$$\frac{\partial}{\partial \tau} e^{S_{\tau}[A,\psi,\bar{\psi}]}$$

\begin{align}
&\frac{\partial}{\partial \tau} e^{S_{\tau}[A,\psi,\bar{\psi}]} \\
&= \int d^D x \frac{1}{2} \delta^2 \frac{\delta}{\delta A^a_{\mu}(x)} \frac{\delta}{\delta A^a_{\mu}(x)} \left[ -i \int d^D x' \frac{\delta}{\delta \psi(x')} \frac{\delta}{\delta \bar{\psi}(x')} \right] \\
&\times \left[ \frac{\delta}{\delta \psi(x')} \right]_{\tau=0} \left[ \frac{\delta}{\delta \bar{\psi}(x')} \right]_{\tau=0} \\
&\times e^{S_{\tau=0}[A',\psi',\bar{\psi}']},
\end{align}

where $\bar{\psi}$ is given by Eq. (2.15).
\[ \times e^{S_{\tau}[A,\bar{\psi},\psi]} \frac{\delta}{\delta \psi(x)} \left\{ -2 \delta - 2 \alpha_0 \partial_\mu A_\mu(x) - \left( D \frac{1}{2} + \frac{\eta_\tau}{2} + \frac{\partial}{\partial x} \cdot \right) \right\} \bigg|_{A \rightarrow g_\tau(A + \frac{\delta}{\delta A})} \]

+ \int d^D x \text{tr} \left\{ i \mu^a \frac{\delta}{\delta \psi(x)} e^{S_{\tau}[A,\bar{\psi},\psi]} \frac{\delta}{\delta \psi(x)} \right\} \times \left[ -2 D_\nu F_{\nu\mu}^a(x) - 2 \alpha_0 D_\mu \partial_\nu A_\nu^a(x) - \left( 1 + x \cdot \frac{\partial}{\partial x} \right) A_\mu^a(x) \right] \bigg|_{A \rightarrow g_\tau(A + \frac{\delta}{\delta A})} \right\}.

B. Axial anomaly in \( D = 2 \)

We assume that the Wilson action is quadratic in the fermion field and set

\[ S_\tau = -\int d^D x d^D y \bar{\psi}(x) D(x,y) \psi(y) + \cdots. \quad (B1) \]

Then, from Eq. (2.21), the chiral invariance \( e^{-S_\tau} \hat{\Gamma}_5 e^{S_\tau} = 0 \) implies the GW relation,

\[ \gamma_5 D(x,y) + D(x,y) \gamma_5 + 2 \int d^D z D(x,z) \gamma_5 D(z,y) = 0. \quad (B2) \]

Under the infinitesimal chiral transformation in Eq. (2.11) with the localized parameter \( \alpha \rightarrow \alpha(x) \), the Wilson action in Eq. (B1) changes as

\[ S_\tau \rightarrow S_\tau - i \int d^D x d^D y [\alpha(y) - \alpha(x)] \bar{\psi}(x) D(x,y) \gamma_5 \psi(x) \]

\[ + 2i \int d^D x d^D y d^D z \alpha(x) \bar{\psi}(x) D(x,z) \gamma_5 D(z,y) \psi(x), \quad (B3) \]

where we have used the GW relation in Eq. (B2). Therefore, if the integration measure \([d\psi d\bar{\psi}]\) is invariant under the chiral transformation, we have the identity

\[ \int d^D x d^D y [\alpha(y) - \alpha(x)] \langle \bar{\psi}(x) D(x,y) \gamma_5 \psi(y) \rangle_{S_\tau} = 2 \int d^D x d^D y d^D z \alpha(x) \langle \bar{\psi}(x) D(x,z) \gamma_5 D(z,y) \psi(y) \rangle_{S_\tau}, \quad (B4) \]

and for a fixed gauge-field configuration, the right-hand side of this expression is computed as

\[ 2 \int d^D x d^D y d^D z \alpha(x) \langle \bar{\psi}(x) D(x,z) \gamma_5 D(z,y) \psi(y) \rangle_{S_\tau} \]

\[ = -2 \int d^D x d^D y d^D z \alpha(x) \text{tr} \left[ D(x,z) \gamma_5 D(z,y) \langle \psi(y) \bar{\psi}(x) \rangle_{S_\tau} \right] \]

\[ = -2 \int d^D x \alpha(x) \text{tr} [\gamma_5 D(x,x)], \quad (B5) \]
where we have used \( \int d^D y \, D(z, y) \langle \psi(y) \bar{\psi}(x) \rangle_{S_r} = \delta^{(D)}(z - x) \).

Now, for the parametrization of the Wilson action in Eq. (B35), we find

\[
D(x, x) = \int_\ell G(\tau; \ell) - g_\tau \int_p e^{ipx} A^a_\mu(p) \int_\ell \! H^a_\mu(\tau; -\ell - p, p, \ell) + \cdots \tag{B6}
\]

and thus the factor in Eq. (B35) is given by

\[
\text{tr} [\gamma_5 D(x, x)] = \text{tr} \left\{ \gamma_5 \left[ \int_\ell G(\tau; \ell) - g_\tau \int_p e^{ipx} A^a_\mu(p) \int_\ell \! H^a_\mu(\tau; -\ell - p, p, \ell) + \cdots \right] \right\}. \tag{B7}
\]

The term containing \( H^a_\mu \), being linear in the gauge potential, is relevant to the axial anomaly in \( D = 2 \).

So far, all elements in this paper have been dimensionless, i.e., everything is measured in units of a UV cutoff \( \Lambda_0 \). What we are eventually interested in is the continuum limit \( \Lambda_0 \to \infty \), in which the external momentum carried by the gauge potential in physical units, \( p\Lambda_0 \), is kept fixed. The axial anomaly in this “classical continuum limit” is given by a low-momentum limit of Eq. (B7). Then, in Eq. (B7), noting that \( \text{tr}(\gamma_5 F^a) = 0 \), we see that one can replace \( H^a_\mu \) given in Eq. (B31) (we consider the Wilson action with \( \tau \to \infty \) that is relevant in the continuum limit) by

\[
H^a_\mu(\tau = \infty; -\ell - p, p, \ell) = -iT^a Z\tilde{C}_0 e^{-(\ell+p)^2} G(\ell + p) \frac{1}{\ell + p} \gamma_\mu G(\ell)
\]

\[
+ 2iT^a \left[ 1 + G(\ell + p) \right] G(\ell) \epsilon_\mu + 2iT^a G(\ell + p) \left[ 1 + G(\ell) \right] \epsilon_\mu + O(p^2), \tag{B8}
\]

where we have noted that \( \lim_{p \to 0} W(\ell; p, -\ell - p) = \lim_{p \to 0} W(-\ell - p; p, \ell) = 1/2 \). Using this, after taking the trace over Dirac indices by \( \text{tr}(\gamma_5 \gamma_\mu \gamma_\nu) = 2i\epsilon_{\mu\nu} \) for \( D = 2 \) (we set \( \gamma_5 = -i\gamma_1 \) and \( \epsilon_{01} = 1 \)), we have

\[
\int_\ell \text{tr} [\gamma_5 H^a_\mu(\tau = \infty; -\ell - p, p, \ell)] = 2T^a Z^2 \tilde{C}_0^2 \int_\ell \left( \frac{e^{-4\ell^2} (1 + 4 \ell^2)}{(-Z^2 C_0^2 e^{-4\ell^2} + \ell^2)^2} \epsilon_{\mu\nu} p_\nu + O(p^2) \right)
\]

\[
= -\frac{1}{2\pi} T^a \mathcal{I}(Z^2 \tilde{C}_0^2) \epsilon_{\mu\nu} p_\nu + O(p^2), \tag{B9}
\]

where the integral

\[
\mathcal{I}(\xi) \equiv \int_0^\infty dx \frac{\xi e^{-4x}(1 + 4x)}{(\xi e^{-4x} + x)^2} = -\int_0^\infty dx \frac{d}{dx} \frac{\xi}{e^{4x} x + \xi} = 1, \tag{B10}
\]

is independent of \( \xi > 0 \).

When the external momentum is small, on the other hand, one sees that the left-hand side of Eq. (B14) reduces to

\[
\frac{1}{iZ\tilde{C}_0} \int d^D x \, \alpha(x) \partial_\mu \langle \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x) \rangle_{S_{r=\infty}} \equiv \int d^D x \, \alpha(x) \partial_\mu \langle \bar{\psi}(x) j_{5\mu}(x) \rangle_{S_{r=\infty}}. \tag{B11}
\]

Since the action is normalized as \( S_{r=\infty} = 1/(iZ\tilde{C}_0) \int d^D x \, \bar{\psi}(x) \gamma_5 \psi(x) + \cdots \) in the low-momentum limit, Eq. (B11) gives the total divergence of a correctly normalized axial-vector

\footnote{Here, we are assuming an Abelian gauge theory, for which the gauge-group generator is given by \( T^a = -ie \) as Eq. (B11).}
current $j_5\mu(x)$. Thus, finally, combining Eqs. (B11), (B4), (B5), (B7), and (B9), we have the axial anomaly in $D = 2$ as

$$\partial_\mu \langle j_5\mu(x) \rangle_{S_{\tau=\infty}} = -i \frac{\pi}{g_{\tau=\infty}} T^a \epsilon_{\mu\nu} \partial_\mu A_\nu^a(x). \quad (B12)$$

As anticipated from the gauge invariance and locality of our Wilson action, this reproduces the correct expression of the axial anomaly in $D = 2$.

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