GEOMETRY AND TOPOLOGY OF GEOMETRIC LIMITS I

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ABSTRACT. In this paper, we are concerned with hyperbolic 3-manifolds $\mathbb{H}^3/G$ such that $G$ are geometric limits of Kleinian surface groups isomorphic to $\pi_1(S)$ for a finite-type hyperbolic surface $S$. In the first of the three main theorems, we shall show that such a hyperbolic 3-manifold is uniformly bi-Lipschitz homeomorphic to a model manifold which has a structure called brick decomposition and is embedded in $S \times (0,1)$. Conversely, any such manifold admitting a brick decomposition with reasonable conditions is bi-Lipschitz homeomorphic to a hyperbolic manifold corresponding to some geometric limit of quasi-Fuchsian groups. Finally, it will be shown that we can define end invariants for hyperbolic 3-manifolds appearing as geometric limits of Kleinian surface groups, and that the homeomorphism type and the end invariants determine the isometric type of a manifold, which is analogous to the ending lamination theorem for the case of finitely generated Kleinian groups.

0. Introduction

There are two notions of convergence in the theory of Kleinian group. One is algebraic convergence and the other is geometric convergence. Algebraic convergence is a convergence with respect to the topology induced from the natural topology on the space of representations of a group into $\text{PSL}_2\mathbb{C}$. On the other hand, geometric convergence corresponds to the convergence of the quotient hyperbolic 3-manifolds with respect to the pointed Gromov-Hausdorff topology. One of the main topics in the theory of Kleinian groups is studying the topological structure of deformation spaces. Deformation spaces have topology induced from the algebraic convergence. Still, their singularities, for instance those which are called self-bumping points, are caused by difference between the algebraic and the geometric convergences as was shown by work of Anderson-Canary [AC1] and McMullen [Mc2]. This suggests that studying geometric limits is important for understanding the deformation spaces.

For an algebraically convergent sequence of Kleinian groups, its geometric limit, which always exists passing to a subsequence, contains the algebraic limit but may be larger than that in general. The difference between algebraic limits and geometric ones is first observed by Jørgensen and Marden. In [JM], they gave an example of algebraically convergent sequence of infinite cyclic groups in $\text{PSL}_2\mathbb{C}$ which converges to a rank-2 parabolic group. This is a typical phenomenon for geometric limits, and is a cause of geometric limits larger than algebraic ones in more complicated situations such as an example of Kerckhoff-Thurston [KT].

Date: February 23, 2010.

2000 Mathematics Subject Classification. Primary 57M50; Secondary 30F40.

Key words and phrases. Kleinian groups, geometric limits, hyperbolic 3-manifolds, Ending Lamination Conjecture.
Kerckhoff and Thurston considered a sequence in the Bers slice of a quasi-Fuchsian space of a surface $S$, parametrised as $(m_0, \tau^n n_0) \in \mathcal{T}(S) \times \mathcal{T}(S)$ for a Dehn twist $\tau$ along an essential simple closed curve $c$ on $S$, where $m_0$ and $n_0$ are arbitrary points in $\mathcal{T}(S)$. They took a sequence of quasi-Fuchsian groups representing $(m_0, \tau^n n_0)$ so that it converges algebraically, which can always be done by Bers’s compactness theorem, and showed that such a sequence converges geometrically to a group $G$ such that $\mathbb{H}^3/G$ is homeomorphic to $S \times (0, 1) \setminus c \times \{\frac{1}{2}\}$. Here a tubular neighbourhood of $c \times \{\frac{1}{2}\}$ in $S \times (0, 1)$ corresponds to a $\mathbb{Z} \times \mathbb{Z}$-cusp of $\mathbb{H}^3/G$ where a phenomenon as in the case of Jørgensen-Marden occurs. By iterating this kind of procedure, it is also possible to construct an example of a geometric limit $G'$ of quasi-Fuchsian groups such that $\mathbb{H}^3/G'$ has infinitely many $\mathbb{Z} \times \mathbb{Z}$-cusps as was shown by Bonahon-Otal [BO], (see also Ohshika [Oh1]). In particular, this shows that the geometric limit of quasi-Fuchsian groups of isomorphic to $\pi_1(S)$ with a finite type surface $S$ can be infinitely generated.

Another important example of geometric limits of quasi-Fuchsian groups is given by Brock [Br]. He considered a homeomorphism $\phi: S \to S$ which is pseudo-Anosov on some essential subsurface $H$ of $S$ and is the identity outside, and a sequence parametrised as $(m_0, \phi^n n_0)$ in the Bers slice as in the case of Kerckhoff-Thurston. He showed that the geometric limit of such a sequence is a Kleinian group $G''$ such that $\mathbb{H}^3/G''$ is homeomorphic to $S \times (0, 1) \setminus H \times \{\frac{1}{2}\}$.

A natural problem arising from these examples is to determine what kind of Kleinian groups can appear as geometric limits of quasi-Fuchsian group, or more in general as a geometric limit of a sequence in the deformation space of a Kleinian group. The purpose of this paper is to answer this question. In this first part, we shall consider only geometric limits of Kleinian groups isomorphic to surface groups preserving the parabolicity, which are sometimes called Kleinian surface groups. In Theorems A, which is the first of the three main theorems of this paper, we shall give (bi-Lipschitz) model manifolds for geometric limits of Kleinian surface groups and determine the conditions which the model manifolds satisfy. In Theorem C we shall show that these conditions are really sufficient, i.e. that any model manifold satisfying the conditions in Theorem A is homeomorphic to a geometric limit of quasi-Fuchsian groups. Combining these two theorems, we have characterised completely Kleinian group which can appear as geometric limits of surface Kleinian groups. We plan to deal with more general Kleinian groups and give similar theorems in the second part.

Another problem is to classify hyperbolic manifolds corresponding to geometric limits up to isometries completely, which is the subject of Theorem D. The classification problem of finitely generated Kleinian groups, which is called the ending lamination conjecture, was solved by Minsky, collaborating with Brock, Canary and Masur ([MM1], [MM2], [Mi2], [BCM]). Since geometric limits of isomorphic non-elementary finitely generated Kleinian groups can be infinitely generated in general as explained above, the ending lamination conjecture is not sufficient for our situation. Still, we shall prove that the homeomorphism type and (generalised) end invariants are enough to determine the isometry type of geometric limits. This is what Theorem C claims for geometric limits of Kleinian surface groups. The general case is planned to be dealt with in the second part.
1. Main results

In this section, we shall state main results of this paper. We shall also give definitions of terms which are necessary for stating the main results, and a short outline of their proofs.

For a hyperbolic 3-manifold $N$, we denote by $N_0$ the complement of the $\varepsilon$-cusp neighbourhoods in $N$ for $\varepsilon$ less than the three-dimensional Margulis constant. Its homeomorphism type does not depend on the choice of a constant $\varepsilon$. By the solution of Marden’s tameness conjecture by Agol \cite{Ag} and Calegari and Gabai \cite{CG}, any relative end $e$ of a hyperbolic 3-manifold with finitely generated fundamental group is topologically tame, i.e. it has a neighbourhood homeomorphic to $F \times (0, \infty)$, where $F = F \times \{0\}$ is the frontier component of a relative compact core of $N_0$ facing $e$. It follows then from the results of Bonahon \cite{Bon} and Canary \cite{Ca1}, that $e$ is either geometrically finite or simply degenerate: the latter means that there is a sequence of closed geodesics tending to the end which are projected in $F \times \mathbb{R}$ to simple closed curves on $F$ whose projective classes converge in the projectivised Masur domain. However, in general, a hyperbolic 3-manifold $N$ with infinitely generated fundamental group may have infinitely many relative ends which are neither geometrically finite nor simply degenerate. We call such ends wild. To our knowledge, suitable invariants of wild ends which play the role of end invariants for tame ends have not been known up to now. Still, we shall show for hyperbolic 3-manifolds corresponding to geometric limits of surface Kleinian groups, wild ends are controlled in some way and they are determined only by the homeomorphism types, as we shall see in Theorem C.

Now, we turn to stating our main results. The first theorem, Theorem A, says that every geometric limit of Kleinian surface groups isomorphic to $\pi_1(S)$ has a bi-Lipschitz model which admits decomposition into standard blocks, and can be embedded into $S \times \mathbb{R}$. This gives also necessary conditions which hyperbolic 3-manifolds corresponding to geometric limits of Kleinian surface groups must satisfy. Before stating the theorem, we shall explain terms which will be used in the statement. A detailed account of these notions can be found in Section \ref{section:terminology}. A brick $B$ is a 3-manifold homeomorphic to $F \times J$ for a compact connected surface $F$ and an interval $J$ which is either closed or half-open. A brick manifold is a union of countably many bricks $F_n \times J_n$ which are glued each other along essential subsurfaces on their fronts $F_n \times \partial J_n$.

In a brick manifold, we consider to give the end of a half-open brick either a conformal structure at infinity or an ending lamination (i.e. a filling geodesic lamination). We call the brick geometrically finite in the former case and simply degenerate in the latter. Each half-open end of a brick constitutes an end of $M$, and the end is called geometrically finite or simply degenerate accordingly. The union of ideal boundaries on which conformal structures are given is called the boundary at infinity of $M$, and is denoted by $\partial_\infty M$. A brick manifold endowed with these end invariants is called a labelled brick manifold.

We say that a labelled brick manifold admits block decomposition when the manifold is decomposed into blocks in the sense of Minsky and solid tori in such a way that each block has horizontal and vertical directions coinciding with those of bricks. We also require the block decomposition for a half-open brick to accord with its end invariant. The blocks have standard metrics and we can choose gluing maps to be isometries. By identifying a solid torus with a Margulis tube which
is determined by information coming from the block decomposition, we can put a metric on the labelled brick manifold. We call such a metric a model metric. (See §§3.4 and 3.5 for details.)

**Theorem A.** Let \( \{ G_n \} \) be a sequence of Kleinian surface groups isomorphic to \( \pi_1(S) \) preserving the parabolicity, converging geometrically to a non-elementary Kleinian group \( G \). Then there are a labelled brick manifold \( M \) with the following properties, which admits block decomposition, and a bi-Lipschitz map from \( M \) with the model metric to the non-cuspidal part \( N_0 \) of the hyperbolic 3-manifold \( N = \mathbb{H}^3/G \).

(i) Each component of \( \partial M \) is either a torus or an open annulus.

(ii) There is no properly embedded annulus in \( M \) whose boundary components lie on distinct boundary components of \( M \).

(iii) If there is an embedded, incompressible half-open annulus \( S^1 \times [0, \infty) \) in \( M \) such that \( S^1 \times \{ t \} \) tends to a wild end \( e \) of \( M \) as \( t \to \infty \), then its core curve is homotopic into an annulus component of \( \partial M \) tending to \( e \).

(iv) The manifold \( M \) is (not necessarily properly) embedded in \( S \times (0, 1) \) in such a way that each brick has a form \( F \times J \) with an interval \( J \) and an essential subsurface \( F \) of \( S \) with respect to the product structure of \( S \times (0, 1) \) and the ends of geometrically finite bricks lie \( S \times \{ 0, 1 \} \). (We shall say that geometrically finite ends are peripheral to refer to the last condition.)

It should be noted that a result similar to this was announced in the introduction of Brock-Canary-Minsky [BCM]. By (iv), we see that the geometric limit manifold \( N_0 \) has at most \(-2\chi(S)\) geometrically finite ends.

The following corollary is easily deduced from Theorem A. It guarantees that we can make use of a generalised version of Sullivan’s Rigidity by McMullen [Mc1], which is crucial in the proof of Theorem D.

**Corollary B.** Let \( G \) be a non-elementary geometric limit of quasi-Fuchsian groups isomorphic to \( \pi_1(S) \) preserving the parabolicity. Then \( N = \mathbb{H}^3/G \) has at most countably many relative ends. Furthermore, there is an upper bound depending only on \( \chi(S) \) for the injectivity radii at points in the convex core of \( N \).

The next theorem claims the existence of a geometric limit which is bi-Lipschitz equivalent to a brick manifold with the properties in Theorem A provided there are no two simply degenerate ends with homotopic ending laminations.

**Theorem C.** Suppose that \( M \) is a labelled brick manifold satisfying the conditions (i)-(iv) in Theorem A such that the ending laminations of two simply degenerate ends of \( M \) are not homotopic each other in \( M \). (This condition is necessary only when \( M \) is homeomorphic to \( F \times (0, 1) \), for a compact essential subsurface \( F \) of \( S \) since ending laminations are filling.) Then \( M \) has block decomposition, and if we put \( M \) the model metric associated with the decomposition, then there exists a non-elementary geometric limit \( G \) of quasi-Fuchsian groups isomorphic to \( \pi_1(S) \) such that \( N = \mathbb{H}^3/G \) admits a \( K \)-bi-Lipschitz homeomorphism \( f : M \to N_0 \) which can be extended to a conformal map \( \partial_\infty M \to \partial_\infty N \) between the boundaries at infinity for a uniform constant \( K \geq 1 \).

Here we say that a constant is uniform if it depends only on \( \chi(S) \), and hence is independent of the end invariants of \( M \) and \( N \).
By applying Theorem C, we can construct various examples of geometric limits \( G \) of quasi-Fuchsian groups isomorphic to \( \pi_1(S) \) preserving the parabolicity such that \( N_0 \) has infinitely many simply degenerate ends and infinitely many wild ends simultaneously.

The last theorem is a classification theorem which is analogous to the ending lamination theorem for finitely generated case.

**Theorem D.** Suppose that \( G_1 \) and \( G_2 \) are non-elementary geometric limits of Kleinian surface groups isomorphic to \( \pi_1(S) \). If \( f : \mathbb{H}^3/G_1 \to \mathbb{H}^3/G_2 \) is a homeomorphism preserving their end invariants, then \( f \) is properly homotopic to an isometry.

**Remark 1.1.** In the beginning of the present work, we tried to use more classical topological approach involving only hyperbolic geometry to study topological properties of geometric limits of quasi-Fuchsian groups. Subsequently we found that, by invoking the bi-Lipschitz model theorem by Brock-Canary-Minsky, it is possible to simplify proofs of some results and moreover to obtain a deeper result on geometric properties of geometric limits. Therefore, we have changed our original plan and adopted the method relying upon work of [Mi2] and [BCM]. On the other hand, we have noticed that our original approach on geometric limits gives rise to a rather short proof of the bi-Lipschitz model theorem. See Soma [So].

Now, we shall outline the proofs of the main theorems. To prove Theorem A, we shall first apply Minsky’s bi-Lipschitz model theorem to each \( \mathbb{H}^3/G_n \) and get a model manifold \( M_n \) which can be decomposed into blocks with a bi-Lipschitz homeomorphism \( g_n \) from \( M_n \) to \( (\mathbb{H}^3/G_n)_0 \). We define \( M \) and a bi-Lipschitz homeomorphism from \( M \) to \( N_0 \) as the geometric limits of \( M_n \) and \( g_n \). We shall check these satisfy the required conditions (i)-(iv) one by one, among which the most difficult is (iv). Since \( M \) is the geometric limit of \( \{ M_n \} \), each finite sub-bricks can be proved to be embedded in \( S \times (0,1) \) preserving the product structures, but this does not imply immediately that the entire \( M \) can also be embedded. We shall need to rearrange the embeddings of sub-bricks by twisting them as will be shown in Lemma 3.1.

Next we turn to Theorem C. We shall first consider an ascending exhausting sequence of sub-brick manifolds \( W_n \) consisting of finite bricks within the given labelled brick manifold \( M \). These \( W_n \) may have very complicated homeomorphism types, but we shall construct from the \( W_n \) brick manifolds \( Z_n \) corresponding to geometrically finite Kleinian surface groups by applying Thurston’s uniformisation theorem, whose geometric limit is also \( M \). We shall approximate these Kleinian groups by quasi-Fuchsian groups, which are the groups as we wanted.

Finally, we shall outline the proof of Theorem D. We are given two geometric limits \( G_1 \) and \( G_2 \) such that \( N_1 = \mathbb{H}^3/G_1 \) and \( N_2 = \mathbb{H}^3/G_2 \) share the same topological type and end invariants. By Theorem A, we can construct a labelled model manifold \( M \) of \( (N_1)_0 \). By our assumption, there is a homeomorphism from \( M \) to \( (N_2)_0 \) preserving the end invariants. In Theorem 4.1 which is a generalisation of the bi-Lipschitz model theorem by Brock-Canary-Minsky [BCM], we shall prove that such a homeomorphism can be homotoped to a uniform bi-Lipschitz homeomorphism. This shows that \( G_1 \) and \( G_2 \) are quasi-conformally conjugate by a quasi-conformal homeomorphism which is conformal on the domain of discontinuity. The second statement of Corollary B makes it possible to apply McMullen’s
generalisation of Sullivan’s rigidity theorem and we shall be able to show that $G_1$ and $G_2$ are conformally conjugate.

2. Preliminaries

We refer the reader to Thurston [Th1], Benedetti and Petronio [BP], Matsuzaki and Taniguchi [MT], and Marden [Ma2] for the general theory on hyperbolic manifolds and Kleinian groups, and to Hempel [He] for the 3-manifold topology.

Throughout this paper, all manifolds are assumed to be oriented, and all homeomorphisms between manifolds are assumed to be orientation-preserving. When we talk about a surface $S$, we always assume that it is a connected surface of finite type possibly with punctures and $\chi(S) < 0$. Sometimes, we fix a hyperbolic structure of finite area on it for convenience. We denote by $\Sigma_{0,3}$, $\Sigma_{0,4}$, $\Sigma_{1,1}$ compact surfaces homeomorphic respectively to a three-holed sphere, a four-holed sphere and a one-holed torus.

2.1. The curve graph and tight geodesics. A subsurface $\Sigma$ of $S$ is called essential if no component of the frontier of $\Sigma$ is null-homotopic in $S$. We regard $S$ itself also as an essential subsurface of $S$. When $\Sigma$ is an open annulus we further assume that the frontier of $\Sigma$ is not homotopic to a puncture of $S$. We consider both closed essential subsurfaces and open ones. When we consider two essential subsurfaces, we assume that they do not have inessential intersection. If two essential subsurfaces are isotopic, they are assumed to coincide.

Let $\Sigma$ be a connected surface of finite type, possibly with punctures. In this paper, when we talk about curve graphs, we only consider the situation where $\Sigma$ is an open essential subsurface of some fixed surface $S$, including the case when $\Sigma = S$. The complexity of $\Sigma$ is defined by $\xi(\Sigma) = 3g + p$, where $g$ is the genus of $\Sigma$ and $p$ is the number of punctures of $\Sigma$. (This is more convenient than the Euler characteristic $\chi(S)$ for our purpose.) A surface $\Sigma$ with $\xi(\Sigma) = 3$ (resp. $\xi(\Sigma) = 4$) is homeomorphic to the interior of $\Sigma_{0,3}$ (resp. the interior of either $\Sigma_{0,4}$ or $\Sigma_{1,1}$).

When $\xi(\Sigma) > 4$, we define the curve graph $C(\Sigma)$ of $\Sigma$ to be a simplicial graph whose vertices are homotopy classes of non-contractible simple closed curves on $\Sigma$ which are not homotopic to punctures such that two vertices are connected by an edge if and only if they have disjoint representatives. We call a vertex of $C(\Sigma)$ or its representative a curve on $\Sigma$. For our convenience, we fix a complete hyperbolic structure on $\Sigma$ of finite area and take a uniquely determined geodesic as a representative for any curve in $C(\Sigma)$. The notion of curve graphs was first introduced by Harvey [Har] and extended and modified in [MM1, MM2, Mi1]. In the case when $\xi(\Sigma) = 4$, the curve graph $C(\Sigma)$ is defined so that the vertices are curves on $\Sigma$ and two curves $v, w$ are joined by an edge if and only if they have the minimum geometric intersection, i.e. $i(v, w) = 1$ when $\Sigma$ is $\text{Int}\Sigma_{1,1}$ and $i(v, w) = 2$ when $\Sigma$ is $\text{Int}\Sigma_{0,4}$. When $\Sigma$ is an open annulus embedded in $S$, we consider the covering $\tilde{\Sigma}$ of $S$ (with a fixed hyperbolic structure) associated to $\pi_1(\Sigma)$ and compactify $\tilde{\Sigma}$ to $\Sigma$ by attaching the ideal boundary. The vertices of curve complex $C(\Sigma)$ are homotopy classes of essential arcs on $\Sigma$ fixing the endpoints. Two vertices are connected by an edge if and only if they can be homotoped fixing the endpoints to arcs whose interiors are disjoint.

We put the path metric $d = d_{C(\Sigma)}$ on $C(\Sigma)$ setting the length of each edge to be 1. In the case when $\xi(\Sigma) > 4$, a finite subset $v$ of vertices in $C(\Sigma)$ is said to constitute
Definition 2.1. A sequence $\{v_i\}_{i \in J}$ of simplices in $C(\Sigma)$ is called a tight sequence if it satisfies one of the following conditions depending on whether $\xi(\Sigma)$ is greater than 4 or not, where $J$ is a finite or an infinite interval of $\mathbb{Z}$.

(i) When $\xi(\Sigma) > 4$, for any vertices $w_i$ of $v_i$ and $w_j$ of $v_j$ with $i \neq j$, it holds that $d(w_i, w_j) = |i - j|$. Moreover, if $\{i - 1, i, i + 1\} \subseteq J$, then $v_i$ is represented by the union of components of $\partial \Sigma^i_{i+1}$ which are non-peripheral in $\Sigma$, where $\Sigma^i_{i+1}$ is an isotopically minimal subsurface in $\Sigma$ with essential boundary containing the geodesic representatives of all the vertices of $v_{i-1}$ and $v_{i+1}$.

(ii) When $\xi(\Sigma) = 4$, each $v_i$ is a vertex in $C(\Sigma)$ and $d(v_i, v_j) = |i - j|$.

The surface $\Sigma$ is called the support of $g$ and is denoted by $D(g)$.

We regard a single vertex as a tight sequence of length 0. It follows from the definition that for any tight sequence $\{v_i\}$, if a vertex $w$ of $C(\Sigma)$ meets $v_i$ transversely, then $w$ meets at least one of $v_{i-1}$ and $v_{i+1}$ transversely.

For an open essential surface $F$ of $\Sigma$ and a tight geodesic $g$ in $C(\Sigma)$, we denote by $\phi_g(F)$ the union of simplices on $g$ which are disjoint from $F$. Here being disjoint means that they can be made disjoint by an isotopy. For a curve $c$ on $F$, we use the symbol $\phi_g(c)$ to denote $\phi_g(A(c))$, where $A(c)$ is an annular neighbourhood of $c$.

The following theorem is Lemma 5.14 in [MM1] (see also Theorem 1.2 in [Bow2]), which was crucial in the proof of the ending lamination conjecture.

Theorem 2.2. Let $u, w$ be distinct vertices or laminations in $C(\Sigma) \cup EL(\Sigma)$. Then there exists a tight sequence connecting $u$ with $w$.

A marking on $\Sigma$ is a simplex in $C(\Sigma)$ some of its vertices (possibly none) have transversals. Suppose that each of $I, T$ is either a marking on $\Sigma$ or a lamination in $EL(\Sigma)$. Then a tight sequence $g = \{v_i\}_{i \in I}$ on $\Sigma$ is said to be a tight geodesic with the initial marking $I(g) = I$ and the terminal marking $T(g) = T$ if it satisfies the following conditions.

- If $i_0 = \inf J > -\infty$, then $v_{i_0}$ is a vertex of $C(\Sigma)$ contained in $I$. Otherwise, $I = \lim_{i \to -\infty} v_i \in EL(\Sigma)$.
- If $j_0 = \sup J < \infty$, then $v_{j_0}$ is a vertex of $C(\Sigma)$ contained in $T$. Otherwise $T = \lim_{j \to \infty} v_j \in EL(\Sigma)$. 


For a simplex $v$ of a geodesic $g$ supported on $\Sigma$, a component of $\Sigma \setminus v$ and an annulus with core curve in $v$ is called a \textit{component domain} of $v$, and also a component domain of $g$. For a simplex $v_j$ of $g = \{v_j\}$, we define its predecessor $\text{pred}(v_j)$ to be $v_{j-1}$ if $j \neq 1$, and $I(g)$ if $j = 1$. Similarly we define the successor $\text{succ}(v_j)$. For a component domain $Y$ of $v_j$, we denote $\text{pred}(v_j)$ by $I(Y, g)$ and $\text{succ}(v_j)$ by $T(Y, g)$. Here in the case when $Y$ is an annulus a vertex in $C(Y)$ which $\text{pred}(v_j)$ determines when $j \neq 1$ and the transversal of the vertex $v_j$ determines if $j = 1$. The same definition applies for $\text{succ}(v_j)$. If $T(Y, g) \neq \emptyset$, then we write $Y \lesssim g$ and says that $Y$ is forward subordinate to $g$ at $v_j$. Similarly we define $g \gtrsim Y$ and says that $Y$ is backward subordinate to $g$ at $v_j$ if $I(Y, g) \neq \emptyset$. If a tight geodesic $k$ is supported on $Y$, the domain $Y$ is forward subordinate to $g$ at $v_j$, and $T(k) = T(Y, g)$, we say that $k$ is forward subordinate to $g$ at $v_j$ and denote by $k \lesssim g$. Similarly, we define $g \gtrsim k$.

**Definition 2.3.** A hierarchy $H$ of geodesics on $S$ is a family of tight geodesics on essential open subsurfaces of $S$ with the following properties.

1. There is a unique geodesic $g_H$ in $H$ with $D(g_H) = S$, which we call the \textit{main geodesic}.
2. Let $Y$ be a component domain of both a simplex $v$ of $g \in H$ and $w$ of $g' \in H$ such that $g \lesssim Y \lesssim g'$. (The geodesic $g$ and $g'$ may be the same.) Then there exists a unique geodesic $h$ in $H$ such that $D(h) = Y$ and $g \lesssim h \lesssim g'$.
3. For any geodesic $g$ in $H$ other than $g_H$, there exist geodesics $h, k \in H$ such that $h \lesssim g \lesssim k$.

A hierarchy $H$ is said to be \textit{complete} if for each component domain $X$ of $\xi(X) \neq 3$, there is a geodesic in $H$ supported on $X$. A geodesic $g$ in a hierarchy in $H$ whose domain $D(g)$ satisfies $\xi(D(g)) = 4$ is called a \textit{4-geodesic}. A sub-hierarchy of a complete hierarchy $H$ consisting of all the geodesics in $H$ supported on domains with $\xi \geq 4$ is called the $4$-sub-hierarchy.

**Definition 2.4.** Let $H$ be a hierarchy of geodesics on $S$. A \textit{slice} of $H$ is a set of pairs $\sigma = \{(g, v)\}$ of a geodesic $g \in H$ and a simplex $v$ on $g$ which has the following properties.

1. If $(g, v_1)$ and $(g, v_2)$ are contained in $\sigma$, then $v_1 = v_2$.
2. There is a pair $(g_\sigma, v_\sigma)$ called the bottom pair, and except for the bottom pair every pair $(h, u) \in \sigma$ is supported in a component domain of some other $(k, u) \in \sigma$.

We also call $g_\sigma$ the bottom geodesic and $v_\sigma$ the bottom simplex of $\sigma$.

A slice $\sigma$ is said to be \textit{saturated} when for any $(g, v) \in \sigma$ and its component domain $D$, if there is a geodesic $h$ in $H$ supported on $D$, there must be some simplex $w$ of $h$ such that $(h, w) \in \sigma$. We say that $\sigma$ is \textit{non-annular saturated} if the above holds provided that $D$ is not an annulus. For a slice $\sigma$, base($\sigma$) denotes the union of all vertices contained in simplices which appear in $\sigma$, which forms a simplex of $C(D(g_\sigma))$.

### 2.2. Hyperbolic 3-manifolds and geometric limits

A \textit{Kleinian group} $\Gamma$ is a discrete subgroup of $\text{PSL}_2 \mathbb{C}$. When $\Gamma$ contains an abelian subgroup of finite index,
it is called elementary. In this paper, we always assume that all Kleinian groups are torsion-free, or equivalently that they contain no elliptic elements. Under this assumption, a Kleinian group is elementary if and only if it is isomorphic to a free abelian group of rank at most two. For a Kleinian group \( \Gamma \), the quotient space \( N = \mathbb{H}^3/\Gamma \) is called the hyperbolic 3-manifold corresponding to \( \Gamma \).

The limit set \( \Lambda_\Gamma \) of \( \Gamma \) is the set of accumulation points of the orbit space \( \Gamma x_0 \) in the closed 3-ball \( \mathbb{H}^3 \cup \hat{\mathbb{C}} \) for a fixed point \( x_0 \in \mathbb{H}^3 \). It should be noted that \( \Lambda_\Gamma \) is contained in \( \hat{\mathbb{C}} \). The complement of \( \Lambda_\Gamma \) in \( \hat{\mathbb{C}} \) is called the region of discontinuity of \( \Gamma \). We can regard \( N \) as the interior of the manifold \( (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma \), which is called the Kleinian manifold corresponding to \( \Gamma \). The boundary \( \Omega_\Gamma/\Gamma \) at infinity is denoted by \( \partial_\infty N \). The Nielsen convex hull \( H_\Gamma \) of \( \Lambda_\Gamma \) in \( \mathbb{H}^3 \) is \( \Gamma \)-invariant. Its quotient \( C_\Gamma = H_\Gamma/\Gamma \) is called the convex core of \( N \). The Kleinian group \( \Gamma \) is said to be geometrically finite if the volume of the \( \delta \)-neighbourhood of \( C_\Gamma \) in \( N \) is finite for some \( \delta > 0 \).

For a positive number \( \varepsilon \), the \( \varepsilon \)-thin part \( N_{(0,\varepsilon]} \) of \( N \) is the set consisting of all points \( x \in N \) such that there exists a non-contractible loop \( \gamma \) of length \( \leq \varepsilon \) based at \( x \). The complement \( N_{[\varepsilon,\infty)} = N \setminus \text{Int}\,N_{(0,\varepsilon]} \) is called the \( \varepsilon \)-thick part of \( N \). A Margulis tube is an embedded, equidistant, tubular neighbourhood of a simple closed geodesic in \( N \). A \( \mathbb{Z} \) or a \( \mathbb{Z} \times \mathbb{Z} \)-cusp neighbourhood \( P \) is a subset of \( N \) such that each component of \( p^{-1}(P) \) is a horoball whose stabiliser in \( \Gamma \) is isomorphic to either \( \mathbb{Z} \) or \( \mathbb{Z} \times \mathbb{Z} \), where \( p : \mathbb{H}^3 \to N \) is the universal covering. By Margulis’s lemma [Th1 Corollary 5.10.2], there exists a constant \( \epsilon_0 > 0 \) independent of \( \Gamma \), called the Margulis constant, such that, for any \( 0 < \varepsilon < \epsilon_0 \), each component of \( N_{(0,\varepsilon]} \) is either a Margulis tube or a \( \mathbb{Z} \) or a \( \mathbb{Z} \times \mathbb{Z} \)-cusp neighbourhood. Let \( N_0 = N_{[\varepsilon,\infty)} \) be the union of \( N_{[\varepsilon,\infty)} \) and all the Margulis tube components of \( N_{(0,\varepsilon]} \), which we call the non-cuspidal part of \( N \). For any \( \epsilon_1 < \varepsilon_2 < \epsilon_0 \), there exists a \( K \)-bi-Lipschitz deformation retraction \( N_0^{\epsilon_2} \to N_0^{\epsilon_1} \) for some constant \( K \geq 1 \) depending only on \( \epsilon_1 \) and \( \varepsilon_2 \). It should also be noted that that \( N_0 \) is a deformation retract of \( N \). The end of \( N_0 \) is called the relative end of \( N \). Each component of the boundary \( \partial N_0 \) is either a Euclidean torus or a Euclidean open annulus. Since any parabolic cusp neighbourhood of \( N \) is covered by a horoball in \( \mathbb{H}^3 \) based at a single point of \( \hat{\mathbb{C}} \), the boundary at infinity \( \partial_\infty N_0 \) of \( N_0 \) is equal to \( \partial_\infty N \).

A sequence \( \{ (X_n, x_n) \} \) of complete metric spaces with base points converges geometrically (in the sense of Gromov) to a complete metric space \((Y, y)\) if there exist \((K_n, L_n)\)-quasi-isometric, \( L_n \)-dense map \( g_n : B_{R_n}(X_n, x_n) \to B_{K_n R_n}(Y, y) \) with \( K_n \searrow 1, L_n \searrow 0 \) and \( R_n \to \infty \), where \( B_R(X, x) \) denotes the \( R \)-metric ball in \( X \) centred at \( x \). A sequence of Kleinian groups \( \{ G_n \} \) is said to converge to a Kleinian group \( G \) if (i) each \( \gamma \in G \) is the limit of a sequence \( \{ \gamma_n \} \) with \( \gamma_n \in \Gamma_n \) and (ii) the limit of any convergent sequence \( \{ \gamma_n \} \) with \( \gamma_n \in \Gamma_n \) is an element of \( G \). It is well known that \( \{ \mathbb{H}^3/\Gamma_n \} \) converges geometrically to \( \mathbb{H}^3/G \) if we set basepoints to be the projections of a common basepoint point \( x_0 \) in \( \mathbb{H}^3 \) if and only if \( \{ G_n \} \) converges to \( G \) geometrically. Refer to [JM], [BP Chapter E] for more details on properties of geometric limits.

Suppose that \( \Sigma \) is an open essential subsurface of \( S \), possibly \( S \) itself. The Teichmüller space of \( \Sigma \) is denoted by \( \mathcal{T}(\Sigma) \). For a point \( \sigma \in \mathcal{T}(\Sigma) \), the surface \( \Sigma \) with a hyperbolic metric representing \( \sigma \) is denoted by \( \Sigma(\sigma) \). A proper map \( f \) from \( F(\sigma) \) to a hyperbolic 3-manifold \( N \) with \( \sigma \in \mathcal{T}(F) \) is called a pleated surface realising a geodesic lamination \( \lambda \) in \( F(\sigma) \) if \( f \) satisfies the following conditions.
(i) \( f \) maps each parabolic cusp of \( F(\sigma) \) to a parabolic cusp in \( N \).

(ii) The path-metric induced from \( N \) by \( f \) coincides with \( \sigma \), that is, for any rectifiable path \( \alpha \) in \( \Sigma(\sigma) \), its image \( f(\alpha) \) is also a rectifiable path in \( N \) with \( \text{length}_{F(\sigma)}(\alpha) = \text{length}_N(f(\alpha)) \).

(iii) \( f(l) \) is a geodesic in \( N \) for each leaf \( l \) of \( \lambda \).

(iv) For each component \( \Delta \) of \( \Sigma \setminus \lambda \), the restriction \( f|\Delta \) is a totally geodesic immersion into \( N \).

A relative end \( e \) of a hyperbolic 3-manifold \( N \) is said to be \textit{topologically tame} if there is a properly embedded compact surface \( F \) in \( N_0 \) which separates a submanifold containing \( e \) which is homeomorphic to \( F \times [0, \infty) \). All topologically tame ends of hyperbolic 3-manifolds considered in this paper are assumed to be \textit{incompressible}, i.e. the inclusion \( F \subset N \) is \( \pi_1 \)-injective. A topologically tame relative end \( e \) is called \textit{geometrically finite} if \( e \) has a neighbourhood which is disjoint from any closed geodesic. (Here we need to assume \( e \) to be topologically tame since we are considering also the case when \( \pi_1(N) \) is infinitely generated.) For a geometrically finite end, the conformal structure \( \nu(e) \) on the component of \( \partial_\infty N \) corresponding to \( e \) is defined to be the end invariant of \( e \). If \( \Gamma \) itself is geometrically finite, then every relative end of \( N \) is geometrically finite.

As was shown by Bonahon \cite{Bon}, if \( e \) is topologically tame and incompressible but not geometrically finite, then there exists a sequence of closed geodesics tending to \( e \) in a neighbourhood \( E \cong F \times [0, \infty) \) of \( e \) which are homotopic in \( E \) to essential simple closed curves \( c_n \) on \( F \). Moreover, it is shown in \cite{H1} that \( \{c_n\} \) converges in \( \UML(\text{Int} F) \) to a lamination \( \nu(e) \) contained in \( \EL(\text{Int} F) \) which is determined uniquely, independently of the choice of closed geodesics tending to \( e \). Then \( \nu(e) \) is called the \textit{ending lamination} of \( e \). In this situation, we say that the relative end \( e \) is \textit{simply degenerate} and define the end invariant of \( e \) to be the ending lamination \( \nu(e) \). An end which is not topologically tame is called \textit{wild}. Any reasonable invariant for a wild end is not know up to now. This forces us to define the \textit{end invariants} of \( N \) to be only those of topologically tame relative ends of \( N \).

3. Brick manifolds

3.1. Embeddings of brick manifolds with infinite bricks. We first introduce some notation for denoting the union of sets in a family which is convenient in the discussion on brick manifolds. Let \( \mathcal{Y} = \{Y_\alpha\}_{\alpha \in A} \) be a family of subsets of some set \( X \). We denote by \( \bigvee \mathcal{Y} \) the subset \( \bigcup_{\alpha \in A} Y_\alpha \) of \( X \). It should be noted that even when we are considering a sequence of families \( \{\mathcal{Y}_n\} \) of subsets of \( X \), the union \( \bigvee \mathcal{Y}_n \) is taken for each \( n \).

Now we shall give a precise definition of brick manifolds, upon which we have touched lightly before stating the main results.

Throughout the definition, \( S \) denotes some fixed surface. A brick is a 3-manifold homeomorphic to \( F \times J \) for a compact essential subsurface \( F \) of \( S \) with \( \xi(F) \geq 3 \) and \( J \) is either \([0, 1]\) or \([0, 1)\) or \((0, 1]\). In the latter two cases of \( J \), the brick is said to be \textit{half open}. We define \( \xi(B) \) to be \( \xi(F) \). For a brick \( B \), we set \( \partial_- B = F \times \{0\} \) and \( \partial_+ B = F \times \{1\} \) and called the \textit{upper front} and the \textit{lower front} respectively, even when \( B \) is half open. When \( B \) is half open, a front which is not contained in \( B \) is called the \textit{ideal front} of \( B \). On the other hand, \( \partial F \times J \) is called the vertical boundary of \( B \), and is denoted by \( \partial_v B \). A brick \( B = F \times J \) has two foliations: the horizontal foliation whose leaves consist of \( F \times \{t\} \) and vertical foliation whose
leaves consist of \( \{ x \} \times J \). A map from a brick to \( S \times I \) (where \( I \) is an interval in \( \mathbb{R} \)) is said to be leaf-preserving when leaves of the horizontal and the vertical foliations are mapped to leaves of the corresponding foliation of the range. (For \( S \times I \), the horizontal foliation consists of \( S \times \{ t \} \) whereas the vertical foliation consists of \( \{ x \} \times I \).)

A finite brick complex is a family of finitely many bricks \( \mathcal{K} = \{ B_1, \ldots, B_m \} \) realised as subsets of a 3-manifold with pairwise disjoint interiors satisfying the following two conditions:

1. \( \bigcup_{i=1}^{m} B_i \) is connected.
2. For any two bricks \( B_i, B_j \) in \( \mathcal{K} \) with \( F_{ij} = B_i \cap B_j \neq \emptyset \), there exists a leaf-preserving embedding \( \eta : B_i \cup B_j \to S \times [-1, 1] \) with \( \eta(B_i) \subset S \times [-1, 0] \), \( \eta(B_j) \subset S \times [0, 1] \) such that \( \eta(F_{ij}) \) is an essential subsurface of \( S \times \{ 0 \} \).

The union \( \bigvee \mathcal{K} \) is called a finite brick manifold with brick decomposition \( \mathcal{K} \). We call \( F_{ij} \) in the second condition above the joint of \( B_i \) and \( B_j \). A joint \( F_{ij} \) is said to be inessential if \( \partial_i B_i = F_{ij} = \partial_j B_j \).

Let \( \{ \mathcal{K}_n \}_{n=1}^{\infty} \) be an ascending sequence of finite brick complexes. Then the union \( \mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n \) is called a brick complex, and \( \bigvee \mathcal{K} \) is said to be a brick manifold with brick decomposition \( \mathcal{K} \). When a leaf-preserving embedding \( \eta : M \to S \times (0, 1) \) of a brick manifold is given, a half-open brick \( B \) in \( \mathcal{K} \) is said to be peripheral with respect to \( \eta \) if the ideal front of \( \eta(B) \) is contained in \( S \times \{ 0 \} \cup S \times \{ 1 \} \).

The following lemma is a key step in the proof of Theorem A.

**Lemma 3.1.** Let \( \{ M_n \} \) be a sequence of finite brick manifolds with brick decompositions \( \mathcal{K}_n \) such that \( \mathcal{K}_n \not\subseteq \mathcal{K}_{n+1} \). If there exists a leaf-preserving embedding \( \eta_n : M_n \to S \times (0, 1) \) for each \( n \in \mathbb{N} \), then the brick manifold \( M = \bigcup_{n=1}^{\infty} M_n \) has the following properties.

(i) There exists a leaf-preserving embedding \( \eta_{\infty} : M \to S \times (0, 1) \).
(ii) The number of ends of \( M \) is countable.
(iii) If \( B \in \mathcal{K}_n \) is peripheral with respect to \( \eta_n \) for all \( n \in \mathbb{N} \), then \( B \) is also peripheral with respect to \( \eta_{\infty} \).

We use the symbols \( \text{pr}_h : S \times [0, 1] \to [0, 1] \) to denote the projection to the second factor, and \( \text{pr}_v : S \times [0, 1] \to S \) to denote that to the first factor. For any brick \( B_i \in \mathcal{K}_n \), set \( \text{pr}_h \circ \eta_n(\partial_i B_i) = \alpha_{i,n} \) and \( \text{pr}_v \circ \eta_n(\partial_i B_i) = \beta_{i,n} \). (Here we regard \( \eta_n \) as being extended to ideal fronts continuously.) A half-open brick \( B_i \) is peripheral with respect to \( \eta_n \) if and only if either \( \alpha_{i,n} = 0 \) or \( \beta_{i,n} = 1 \). For integers \( n, m \) with \( 1 \leq n \leq m \), let \( T_{n,m} \) be the subset of \( [0, 1] \) consisting of the \( \alpha_{i,m}, \beta_{i,m} \) for \( B_i \in \mathcal{K}_n \), and set \( T_n = T_{n,n} \). Consider the correspondence \( \tau_{n,m} : T_n \to T_m \) which transfers \( \alpha_{i,n}, \beta_{i,n} \) respectively to \( \alpha_{i,m}, \beta_{i,m} \). Note that \( \tau_{n,m} \) is not necessarily a map. In fact, it may occur that \( \alpha_{i,n} = \alpha_{j,n} \) (resp. \( \beta_{i,n} = \beta_{j,n} \)) but \( \alpha_{i,m} \neq \alpha_{j,m} \) (resp. \( \beta_{i,m} \neq \beta_{j,m} \)) etc.

To prove Lemma 3.1 we shall make use of the following two kinds of rearrangement for \( \{ \mathcal{K}_n \} \).

**Rearrangement I.** Fix \( n \in \mathbb{N} \). Since there are only finitely many bricks in \( \mathcal{K}_n \), there are only finitely many ways to give them an order. Therefore, we can take a subsequence \( \{ \mathcal{K}_{n_k} \} \) of \( \{ \mathcal{K}_n \}_{n>n} \) so that the restriction \( \tau_{n_k,n_l} : T_{n_k,n_l} \to T_{n,n_l} \) is an order-preserving bijection whenever \( n_k \leq n_l \). For any \( k \geq n \), we define a new embedding \( \eta_k \) to be the old \( \eta_{n_k} \mid M_k \). Repeating the same argument, we can assume that \( \tau_{m_1,m_2} : T_{n,m_1} \to T_{n,m_2} \) is an order-preserving bijection for any
Deforming the new \( \eta_n \) by ambient isotopies of \( S \times I \), we may assume that \( \alpha_{i,n} = \alpha_{i,m} \) and \( \beta_{i,n} = \beta_{i,m} \) for any \( n \leq m \) and any \( i \) with \( B_i \in K_n \), and in particular, that \( T_n \) is a subset of \( T_m \).

**Rearrangement II.** Set \( T_n = \{a_0, a_1, \ldots, a_t\} \), where elements are arrayed in the increasing order, and \( R^n_j = \eta^{-1}_n(S \times [a_{j-1}, a_j]) \). See Figure 3.1. Passing again to a subsequence of \( \{\eta_n\} \) if necessary, we may assume that, for any \( j = 1, \ldots, t \), all \( \eta_m|R^n_j \ (m \geq n) \) define the same embedding up to isotopies and changes of the markings of \( S \times [a_{j-1}, a_j] \), i.e. there exists an orientation-preserving homeomorphism \( \gamma_{m,n} : S \times [a_{j-1}, a_j] \to S \times [a_{j-1}, a_j] \) with \( \gamma_{m,n} \circ \eta_m|R^n_j = \eta_n|R^n_j \). For, if we fix a topological type of a compact essential subsurface \( F \) of \( S \), there are only finitely many embeddings, up to isotopies and changes of markings, of \( F \) into \( S \) as an essential subsurface.

![Figure 3.1](image-url). The union of the shaded regions in the lower (resp. higher) level is \( \eta_n(R^n_4) \) (resp. \( \eta_n(R^n_{11}) \)).

Denote the bricks of \( K_n \) by \( B^{(n)}_j \). We call \( \Sigma_c^{(n)} := (S \times \{c\}) \setminus \text{Int}(\eta_n(M_n)) \) with \( c \in I \) the **slit** for \( \eta_n(M_n) \) at \( c \). By Rearrangement I and II, \( \chi(\Sigma_c^{(n)}) \) is monotone increasing; hence for all sufficiently large \( n \), the topological type of \( \Sigma_c^{(n)} \) does not vary with \( n \). The slit \( \Sigma_c^{(n)} \) is said to be **stable** if all the \( \Sigma_c^{(m)} \ (m \geq n) \) are homeomorphic.

Let \( T'_\infty \) be the set of accumulation points of \( T_\infty = \bigcup_{n \geq 1} T_n \). For \( c \in T'_\infty \), suppose that \( B^{(n)}_1, \ldots, B^{(n)}_k \) are the bricks in \( K_n \) with \( \eta_n(B^{(n)}_i) \cap S \times \{c\} \neq \emptyset \ (i = 1, \ldots, k) \). Take a sufficiently small \( \delta > 0 \) so that \( S \times ([c - \delta, c] \cup (c, c + \delta]) \) meets none of the
images under $\eta_n$ of the fronts of $B_i^{(n)}$ ($i = 1, \ldots, k$). Then we call the set

$$Q_\delta(S_{c}^{(n)}) = (S \times ([c - \delta, c) \cup (c, c + \delta]) \setminus \eta_n(B_i^{(n)}) \cup \cdots \cup \eta_n(B_k^{(n)}) \cup \Sigma_c^{(n)})$$

the $\delta$-region of the slit $\Sigma_c^{(n)}$ for $\eta_n(M_n)$. See Figure 3.2.

![Figure 3.2. The union of the bold horizontal segments represents $\Sigma_c^{(n)}$. The union of $\Sigma_c^{(n)}$ and the shaded regions is the $\delta$-region $Q_\delta(S_{c}^{(n)})$.](image)

For any integer $s \geq 1$, we define $T'_{\infty,s}$ to be the subset of $T'_{\infty}$ consisting of elements $c \in T'_{\infty}$ for which $-\chi_{\min}(\Sigma_c) = s$, where $-\chi_{\min}(\Sigma_c)$ is defined to be $\min \{-\chi(\Sigma_c^{(m)})\}$. For $c \in T'_{\infty,s}$, consider a sufficiently large $n$ such that $\Sigma_c^{(n)}$ is stable. For $m \geq n$, if $\Sigma_d^{(m)}$ ($d \in T_{\infty}$) is contained in $Q_\delta(S_{c}^{(m)}) \setminus \Sigma_c^{(m)}$ then $-\chi(\Sigma_d^{(m)}) \leq s$, and if further $-\chi_{\min}(\Sigma_d) = s$, then $\Sigma_d^{(m)}$ is parallel to $\Sigma_c^{(m)}$ in $S \times [0, 1] \setminus \eta_n(M_n)$, which implies that $J \cap T'_{\infty,s} = \emptyset$, where $J$ is an open interval between $c$ and $d$. (The last implication uses the assumption that $M_n$ is connected.) It follows that for $c \in T'_{\infty,s}$, there exists $\delta(c) > 0$ independent of $n$ such that $-\chi_{\min}(\Sigma_d) < s$ if $\Sigma_d^{(n)} \subset Q_{\delta(c)}(\Sigma_c^{(n)}) \setminus \Sigma_c^{(n)}$ and $d$ lies on the side of $c$ from which $T_{\infty}$ accumulates to $c$. Taking into account also the side from which $T_{\infty}$ does not accumulate to $c$, we can take possibly smaller $\delta(c)$ such that for any $\Sigma_d^{(n)}$ with $d \in T_{\infty}$ contained in $Q_{\delta(c)}(\Sigma_c^{(n)}) \setminus \Sigma_c^{(n)}$, we have $-\chi_{\min}(\Sigma_d) < s$. In particular, this implies that $T'_{\infty,s}$ is a countable subset of $[0, 1]$ for every $s$, and hence so is $T'_{\infty}$.

Let $F$ be a compact essential subsurface of $S \times \{a\}$ with $0 < a < 1$ and $\varphi : F \rightarrow F$ an orientation-preserving homeomorphism such that $\varphi|\partial F$ is the identity. Consider a 3-manifold $N_\varphi$ obtained from $S \times [0, 1] \setminus \text{Int}F$ by identifying the $(\pm)$-sides $F^{(\pm)}$ of $F$ by $\varphi : F^{(\rightarrow)} \rightarrow F^{(\leftarrow)}$ instead of the identity. The original $S \times [0, 1] \setminus \text{Int}F$ is naturally regarded as a subset of $N_\varphi$. We say that $N_\varphi$ is the manifold obtained from $S \times [0, 1] \setminus \text{Int}F$ by the $\varphi$-twist along $F$. Let $C_0$ be either $F \times [0, a]$ or $F \times (a, 1]$. Then there exists a homeomorphism $\xi_0 : N_\varphi \rightarrow S \times [0, 1]$ such that $\xi_0|\{(N_\varphi \setminus C_0)\}$ is the identity, whereas $\xi_0|C_0$ is $\varphi^{-1} \times \text{id}_{[0,a]}$ if $C_0$ is $F \times [0, a)$, and $\varphi \times \text{id}_{(a,1]}$ if $C_0$ is $F \times (a, 1]$. The submanifold $C_0$ is called the affected region of the twist. See Figure 3.3(a).

In the proof of Lemma 3.1, we shall use the following trick to reduce the affected region. We shall explain here only the case when $C_0$ is $F \times [0, a)$. The same trick
works also for the case when \( C_0 \) is \( F \times \{a, 1\} \). Let \( H \) be a non-peripheral horizontal essential subsurface in \( S \times [0, 1] \) with \( \text{pr}_c(H) \supset \text{pr}_c(F) \) which does not lie on \( S \times \{a\} \), say \( H \subset S \times \{b\} \) for some \( 0 < b < a \). Then there exists a homeomorphism \( \xi_1 : N_c \setminus H \to S \times [0, 1] \setminus H \) whose affected region is \( C_1 = F \times \{b, a\} \), i.e., \( \xi_1|_{N_c \setminus C_1} \) is the identity. See Figure 3.3(b).

We note that even if there exists another essential subsurface \( J \) of \( S \times \{c\} \) for some \( b < c < a \) satisfying \( \text{pr}_c(\text{Int} F) \cap \text{pr}_c(\text{Int} J) \neq \emptyset \) and \( R = \text{pr}_c(F) \setminus \text{pr}_c(J) \neq \emptyset \), we do not take into account a partial contribution of \( J \) for reducing the affected region of the \( \varphi \)-twist. To be more precise, the homeomorphism \( \xi_2 : N_c \setminus H \cup J \to S \times [0, 1] \setminus H \cup J' \) is the restriction of \( \xi_1 \), where \( J' = \xi_1(J) \). In particular, the affected region of \( \xi_2 \) is \( C_1 \setminus J \), not \( C_1 \setminus R \times \{b, c\} \). If the affected region is contained in a subset \( Q \) of \( S \times [0, 1] \), then we say that the effect of the twist is absorbed in \( Q \) under \( \xi_1 \).

Now we are ready to start the proof of Lemma 3.1.

**Proof of Lemma 3.1.** First we shall show the part (i). We shall define inductively a leaf-preserving embedding \( h_n : M_n \to S \times [0, 1] \) with \( h_n^{-1}(S \times [a_j, a_j]) = t_{a_j}^{-1}(S \times [a_j, a_j]) = \{a_0, a_1, \ldots, a_t\} \) for \( T_n = \{0, 0, \ldots, a_t\} \). Here \( t_{a_j} \) denotes the one which we obtained after applying the two kinds of rearrangement for the original \( t_{a_j} \). We set \( h_1 = t_{a_1} \). We assume that \( h_{n-1} \) has already been defined, and shall define \( h_n \).

Recall that we defined \( R_n \) to be \( t_{a_j}^{-1}(S \times [a_j, a_j]) \). By Rearrangement I, we have \( R_n \cap M_{n-1} = R_{n-1} \) for any \( j = 1, \ldots, t \). By Rearrangement II, there exists an embedding \( h_n^j : R_n \to S \times [a_j, a_j] \) such that \( h_n^j|_{R_n \cap M_{n-1} = h_{n-1}^{-1}(R_{n-1} \cap M_{n-1})} \). We note that the union of \( h_n^j \) does not necessarily match up on the boundaries of the \( R_n \) to define an entire embedding from \( M_n \) to \( S \times [0, 1] \). Let \( T_n \) be the subset of \( T_n \), consisting of elements \( \{a_j, \} \) for which \( -\chi(S_{\{a_j, \}}^{-1}) > -\chi(S_{\{a_j, \}}) \), where \( S_{\{a_j, \}}^{-1} = S \times \{a_j, \} \setminus \text{Int}(h_{n-1}(M_{n-1})) \) and \( S_{\{a_j, \}} = S \times \{a_j, \} \setminus \hat{h}_n^j(M_n) \) are slits for \( h_{n-1} \) and \( h^j_n \). In particular, \( c \in T_n \) implies that \( \hat{S}_{\{a_j, \}} \) is unstable.

For a brick \( B \) of \( M_n \) which is not contained in \( M_{n-1} \), the embeddings which \( h_{n-1} \) induces on \( \partial - B \) and \( \partial \) \( B \) may not be isotopic even if they are defined. Therefore, to define an embedding on the entire \( M_n \), we need to perform twist as defined before. For each \( a_j \in T_n \), we choose an orientation-preserving homeomorphism \( \varphi_{a_j} : S_{\{a_j, \}}^{-1} \to S_{\{a_j, \}}^{-1} \) with \( \varphi_{a_j}|_{\partial \hat{S}_{\{a_j, \}}} \) being the identity so that \( \bigcup_{j=1}^t \hat{h}_n^j \) extends...
to an embedding \( \hat{h}_n : M_n \to N_n \) with \( \hat{h}_n|_{M_{n-1}} = h_{n-1} \), where \( N_n \) is the manifold obtained from \( S \times [0, 1] \setminus \bigcup_{a_j \in \mathcal{T}_n} S_{\delta_j}^{(n-1)} \) by the composition of the \( \varphi_{a_j} \)-twists.

As was explained before starting the proof, for any \( c \in T'_\infty \), we can take sufficiently small \( \delta(c) > 0 \) so that for any \( d \in [c-\delta(c), c) \cup (c, c+\delta(c)] \) with \( d \in T_\infty \), if the surface \( \Sigma_d^{(n)} \) intersects \( Q_{\delta(c)}(\Sigma_c^{(n)}) \setminus \Sigma_d^{(n)} \), then \( -\chi_{\min}(\Sigma_d) < -\chi_{\min}(\Sigma_c) \). By making \( \delta(c) \) smaller if necessary, we can assume that \( Q_{\delta(c)}(\Sigma_c^{(n)}) \) is disjoint from \( \eta_c(B) \) for all \( w \geq n \) and for any \( c, c' \in T'_\infty \), either \([c-\delta(c), c+\delta(c)]\) and \([c'-\delta(c'), c'+\delta(c')]\) are disjoint or one of them contains the other. Since \( T_\infty \cup T'_\infty \) is compact, there exists a finite subset \( \{c_1, \ldots, c_k\} \) of \( T'_\infty \) such that \( T_\infty \setminus \bigcup_{i=1}^k [c_i - \delta(c_i), c_i + \delta(c_i)] \) contains only finitely many elements \( b_1, \ldots, b_n \). See Figure 3.4.

Let \( \Sigma_\infty^{(n)} \) be the union of stable slits \( \Sigma_c^{(n)} \) with \( c_1 \in \{c_1, \ldots, c_k\} \), and regard \( \Sigma_\infty^{(n)} \subset S \times [0, 1] \setminus \bigcup_{a_j \in \mathcal{T}_n} S_{\delta_j}^{(n-1)} \) naturally as a subset of \( N_n \). As was shown before we began the proof, for each \( \Sigma_{a_j} \) \( (a_j \in T_n) \) with \( a_j \in [c_i - \delta(c_i), c_i + \delta(c_i)] \), the \( \varphi_{a_j} \)-twist can be absorbed in either \( \Sigma_{c_i}^{(n)} \times [c_i, a_j] \) or \( \Sigma_{c_i}^{(n)} \times [a_j, c_i] \). Therefore, there is a homeomorphism \( \xi_n : N_n \setminus \Sigma_\infty^{(n)} \to S \times [0, 1] \setminus \Sigma_\infty^{(n)'} \) such that if \( a_j \) lies in some \([c_j - \delta(c_j), c_j + \delta(c_j)]\) then the affected regions of the \( \varphi_{a_j} \)-twist is absorbed in \( S \times [c_j - \delta(c_j), c_j + \delta(c_j)] \) by composing \( \xi_n \), where \( \Sigma_\infty^{(n)'} \) is the union of horizontal essential subsurfaces in \( S \times [0, 1] \) corresponding to the slits in \( \Sigma_\infty^{(n)} \). Then \( h_n = \xi_n \circ \hat{h}_n : M_n \to S \times [0, 1] \) is a leaf-preserving embedding, but \( h_n \) may no longer be an extension of \( h_{n-1} \). For thus defined sequence of embeddings \( h_n \), we shall show that for each brick \( B \) of \( \mathcal{K} \), the restriction \( h_n|B \) is eventually the same map.

**Figure 3.4.**
Let $B$ be a brick of $\mathcal{K}$. Then, there is $m$ such that $M_m$ contains $B$. Take a sufficiently large $w_0 \in \mathbb{N}$ so that $w_0 > m$, and all the $\Sigma_j^{(w_0)}$ are stable for all $j \in \{b_1, \ldots, b_u, c_1, \ldots, c_k\}$. This also means that all the twists along slits at $b_1, \ldots, b_u, c_1, \ldots, c_k$ are already finished by the $w_0$-th step. By our choice of $\delta(c_j)$, we have $\eta_n(B) \cap Q_{\delta(c_j)}(\Sigma_{c_i}^{(n)}) = \emptyset$ for all $n \geq w_0$. Since the effect of any twist along the slit $\Sigma_d^{(n)} (d \in T_\infty)$ with $d \in [c_i - \delta(c_i), c_i + \delta(c_i)]$ is absorbed in $Q_{\delta(c_j)}(\Sigma_{c_i}^{(n)})$ and $\eta_n(B) \cap Q_{\delta(c_j)}(\Sigma_{c_i}^{(n)}) = \emptyset$ for $i = 1, \ldots, k$ and $n \geq w_0$, it follows that all $h_n|B$ with $n \geq w_0$ are the same map. Thus a leaf-preserving embedding $\eta_\infty : M \to S \times [0, 1]$ is well defined by setting $\eta_\infty|B = h_{w_0}|B$ for such $w_0$. Thus we have completed the proof of the part (i).

If $B_j^{(1)} \in \mathcal{K}_1$ is peripheral with respect to $\eta_n$ for all $n \in \mathbb{N}$, then either $\alpha_{j,n} = 0$ or $\beta_{j,n} = 1$ for all $n$ by Rearrangement I. It follows from our definition of $\eta_\infty$ that either $\alpha_{j,\infty} = 0$ or $\beta_{j,\infty} = 1$ holds. This shows the part (iii).

Finally, we turn to the part (ii). We consider the ends of the embedded image $\eta_\infty(M)$ instead of $M$ itself. Fix a basepoint $x_0$ in $\eta_\infty(M)$. For an end $e$ of $\eta_\infty(M)$, consider an arc $\alpha_e$ in $\eta_\infty(M)$ emanating from $x_0$ and tending to $e$ which meets each horizontal leaf of all bricks $\eta_\infty(B_j) (B_j \in \mathcal{K})$ with $\alpha_e \cap \eta_\infty(B_j) \neq \emptyset$ transversely in a single point except for the one containing $x_0$. This implies that $\alpha_e$ meets each $S \times \{c\}$ at most at $-\chi(S)$ points. It follows that $\text{pr}_h(\alpha_e)$ converges to a point $b(e)$ of $T'_\infty$.

Now, for $c \in T'_\infty$, let $e_1, \ldots, e_m$ be $m$ ends of $\eta_\infty(M)$ with $b(e_1) = \cdots = b(e_m) = c$. For a sufficiently large $n$, these ends are contained in distinct components of $\eta_\infty(M \setminus M_n)$. Therefore, for each $j = 1, \ldots, m$, we can choose a subarc $\beta_{e_j}$ of $\alpha_{e_j}$ tending to $e_j$ in such a way that $\beta_{e_j}$ and $\beta_{e_j'}$ do not pass through the same bricks of $\eta_\infty(M)$ if $j \neq j'$. If we take a sufficiently small $\delta > 0$, then each $\beta_{e_j}$ passes through the $\delta$-region $S \times [c - \delta, c) \cup S \times (c, c + \delta]$ transversely to the horizontal foliation. It follows that $m \leq -2\chi(S)$ since there are at most $-\chi(S)$ ends in each of $S \times [c - \delta, c)$ and $S \times (c, c + \delta]$. Since $T'_\infty$ is a countable set, this implies that the number of ends of $\eta_\infty(M)$ is countable. This completes the proof of the part (ii).

\[\square\]

### 3.2. Conditions on labelled brick manifolds

A labelled brick manifold is a brick manifold $M$ in which every half-open brick has either a point in the Teichmüller space or an ending lamination attached to it as follows. Let $B$ be a half-open brick in $M$ which is homeomorphic to $F \times J$, where $J$ is either $[0, 1)$ or $(0, 1]$. Half-open bricks are divided into two categories: geometrically finite bricks and simply degenerate bricks. If $B$ is geometrically finite, then a point in $\mathcal{T}(\text{Int}F)$ is given to $B$, otherwise an ending lamination of $B$, which is contained in $\mathcal{EL}(F)$ is given.

For a geometrically finite brick $B$, the interior of the ideal front of $B$ is denoted by $\partial_\infty B$, and the point in $\mathcal{T}(\text{Int}F)$ is regarded as a marked conformal structure on $\partial_\infty B$. Also for a simply degenerate brick, the given ending lamination is regarded as attached to the end corresponding to its ideal front.

As in Theorem A, we shall consider connected labelled brick manifolds $M$ satisfying the following conditions.

A-(1) Every component of $\partial M$ is either a torus or an open annulus.

A-(2) There is no properly embedded essential annulus whose boundary components lie in distinct boundary components of $M$. 

16 KEN’ICHI OHSHIKA AND TERUHIKO SOMA
A-(3) If there is an embedded, incompressible half-open annulus $S^1 \times [0, \infty)$ in $M$ such that $S^1 \times \{t\}$ tends to a wild end $e$, then its core curve is homotopic into an open-annulus component of $\partial M$ tending to $e$.

A-(4) $M$ is embedded into $S \times (0, 1)$ preserving the horizontal and the vertical leaves in such a way that the ends of geometrically finite bricks are peripheral.

A-(5) Every geometrically finite half-open brick has real front which is an inessential joint.

We shall explain briefly why these conditions are necessary for a labelled brick manifold to be a model manifold. We consider a model manifold $M$ of geometric limit of Kleinian surface groups, whose corresponding hyperbolic 3-manifold we denote by $N$. The boundary of $M$ corresponds to the frontier of the non-cuspidal part $N_0$. This shows that the condition A-(1) must be satisfied. Moreover by Margulis’s lemma, no essential loops on two distinct components of $\text{Fr} N$ are an element of $M$.

We call the union $S$ of the complementary components of $\text{Fr} N$ is an element of $M$.

A-(3) If there is an embedded, incompressible half-open annulus $A-(3)$ lying in such a way that no closed curves can be homotoped to $e$ without obstructed by the complementary components except those lying on an annulus boundary component tending to $e$. We note that model manifolds of Kleinian surface groups (isomorphic to $\pi_1(S)$) constructed by Minsky can be regarded as labelled brick manifolds as will be explained later. Such brick manifolds can be embedded in $S \times (0, 1)$ preserving the horizontal and the vertical leaves. Lemma 3.1 implies that model manifolds of geometric limits can also be embedded in $S \times (0, 1)$ preserving the horizontal and the vertical leaves in such a way the geometrically finite ends are peripheral, which implies the condition A-(4).

The last condition A-(5) is just for convenience in defining a metric on a brick manifold later.

3.3. Tight tube unions. To construct model manifolds of Kleinian surface groups, Minsky considered a hierarchy of tight geodesics. In his construction, a tight geodesic is realised in the model manifold as a sequence of Margulis tubes. We shall consider a similar realisation of a tight geodesic in the model manifold, which we call a tight tube union.

Consider a brick $B = F \times [0, 1]$ with $\xi(F) > 4$. Suppose that we are given a pair of multi-curves $I \times \{0\}$ and $T \times \{1\}$ lying on $\text{Int} \partial_- B$ and $\text{Int} \partial_+ B$, which represent simplices in $C(\text{Int} F)$ by identifying $\partial_- B$ and $\partial_+ B$ with $F$ naturally. Let $g = \{v_i\}_{i=0}^n$ be a tight geodesic in $C(\text{Int} F)$ with $I(g) = I$ and $T(g) = T$. Then $\bigcup_{i=0}^n v_i \times [i/(n + 1), (i + 1)/(n + 1)]$ is a disjoint union $A_B$ of vertical annuli in $B$.

We call the union $A_B$ a tight annulus union in $B$ connecting $I \times \{0\}$ with $T \times \{1\}$.

Next we consider the case when $B$ is a half-open brick $F \times [0, 1)$ with $\xi(F) > 4$.

Suppose then that $I \times \{0\}$ is a multi-curve on $\text{Int} \partial_- B = \text{Int} F$, and that $T \times \{1\}$ is an element of $\mathcal{E}(\text{Int} \partial_+ B) = \mathcal{E}(\text{Int} F)$. Let $g = \{v_i\}_{i=0}^\infty$ be a tight geodesic ray in $C(\text{Int} F)$ with $I(g) = I$ and $T(g) = T$. Then the union $A_B = \bigcup_{i=0}^\infty v_i \times [1 - 1/2^i, 1 - 1/2^{i+1}]$ of vertical annuli in $B$ is called a tight annulus union in $B$ connecting $I \times \{0\}$ with $T \times \{1\}$. We can consider a similar construction for a half-open brick $F \times (0, 1]$ when an ending lamination on $\text{Int} \partial_+ B$ and a multi-curve on $\text{Int} \partial_- B$ are given, and define $A_B = \bigcup_{i=0}^\infty v_i \times [1/2^{i+1}, 1/2^i]$. 


When $\xi(F) = 4$, we need to modify our definition above to make annuli pairwise disjoint. In this case, we define a tight annulus union $A_B$ by $\bigcup_{i=0}^{n} v_i \times [i/(n+1), (2i+1)/(2n+2)]$ if $B = F \times [0, 1]$, by $\bigcup_{i=0}^{\infty} v_i \times [1 - 1/2^i, 1 - 3/2^{i+2}]$ if $B = F \times [0, 1)$, and $A_B = \bigcup_{i=0}^{\infty} v_i \times [3/2^{i+2}, 1/2]$ if $B = F \times (0, 1]$.

Let $A_B = \bigcup_i v_i \times J_i$ be a tight annulus union in a brick $B$. Take a sufficiently thin annular neighbourhood $R_i$ of $v_i$ on $F$ so that $R_i \times J_i$ are pairwise disjoint in $B$. Then $V_B = \bigcup_i R_i \times J_i$ is called a tight tube union in $B$ connecting $I \times \{0\}$ with $T \times \{1\}$.

### 3.4. Block decompositions of labelled brick manifolds.

In this subsection, we shall show that a labelled brick manifold $M$ admits a decomposition into blocks in the sense of Minsky provided that its brick decomposition $K$ satisfies the conditions A-(1)-(5) and the following additional condition (EL), which corresponds to the assumption on ending laminations of simply degenerate ends of $M$ given in Theorem C.

(EL) For any two simply degenerate bricks $B, B'$ in $K$, their ending laminations $\mu(B)$ and $\mu(B')$ are not homotopic in $M$.

Under the conditions A-(1)-(5), this condition is automatically satisfied unless $M$ is homeomorphic to $F \times (0, 1)$ for a compact essential subsurface $F$ of $S$ as we can see in the following way. Let $B_1$ and $B_2$ be two simply degenerate bricks with $B_1 = F_1 \times J_1$ and $B_2 = F_2 \times J_2$, where $J_1$ and $J_2$ are half-open intervals. Note that each component of $\partial S_{B_1}$ and $\partial S_{B_2}$ lies in $\partial M$. The condition A-(2) shows that $F_1 \times \{t\}$ and $F_2 \times \{t'\}$ cannot be homotopic in $M$ unless $M$ is homeomorphic to $F_1 \times (0, 1)$. Since $\mu(B_1)$ is contained in $E\mathcal{L}((\text{Int}F_1)$ whereas $\mu(B_2)$ lies in $E\mathcal{L}(\text{Int}F_2)$, which means that they are filling on non-homotopic surfaces, they cannot be homotopic in $M$.

Let $K_{\text{sf}}$ be the subset of $K$ consisting of geometrically finite bricks, and set $K_{\text{int}} = K \setminus K_{\text{sf}}$. The union $\partial_{\infty} M = \bigcup_{B \in K_{\text{sf}}} \partial_{\infty} B$ is called the boundary at infinity of $M$. Bricks contained in $K_{\text{int}}$ are called internal bricks.

We modify a brick decomposition by performing the following two operations.

1. **Removing inessential joints**: Suppose that there is an inessential joint $F$ of two bricks $B, B'$ in $K_{\text{int}}$. Then we replace $B, B'$ with the single brick $B \cup B'$. In the exceptional case when $M$ is homeomorphic to $F \times (0, 1)$ and has two simply degenerate bricks, this may generate a “brick” homeomorphic to $F \times (0, 1)$, which was not allowed in our definition. We still allow this operation and call thus obtained brick an open brick.

2. **Splitting bricks with non-overlapping annuli on the boundary**: Suppose that there is a brick $B = F \times [0, 1]$ in $K_{\text{int}}$ with a component $A$ of $\partial M \cap \partial B$ which does not overlap $\partial M \cap \partial_s B$. Here an annulus $A_1$ in $B$ is said to overlap a union of annuli $A$ in $B$ when the vertical projections of $A_1$ and $A$ to $F$ intersect essentially. Then we remove $A \times [0, 1]$ from $B$ and split $B$ into two bricks $B_1, B_2$. We can naturally identify $M$ with $M' \setminus A \times [0, 1]$ and regard $(K \setminus \{B\}) \cup \{B_1, B_2\}$ as a new brick decomposition of $M$. We can perform the same operation also when there is an annulus in $\partial M \cap \partial_s B$ which does not overlap $\partial M \cap \partial_s B$.

By repeating these two kinds of operation, we can assume
Assumption 3.2.  

(1) that there is no inessential joint for two bricks in $\mathcal{K}$,  
(2) and that for any brick $B$ both of whose fronts $\partial_- B$ and $\partial_+ B$ are real,  
each component of $\partial_- B \cap \partial M$ overlaps $\partial_+ B \cap \partial M$ and each component of  
$\partial_+ B \cap \partial M$ overlaps $\partial_- B \cap \partial M$.

By the condition A-(1), $\partial M$ is a union of tori and open annuli. Since $M$ is a brick manifold, each of such tori and annuli consists of horizontal annuli and vertical annuli whose interiors are pairwise disjoint, and contains at least one horizontal annulus. Let $H_A$ be the union of core curves of the horizontal annuli constituting the boundary components of $M$. (We take one core curve from each horizontal annulus.) For each geometrically finite brick $B_i$, we fix a multi-curve $s(B_i)$ on its real front $F_i$ which is the shortest pants decomposition of $F_i$ with respect to the hyperbolic structure given to $B_i$. Note that although we gave a conformal structure on the ideal front, we put the pants decomposition on the real front. Let $I(K)$ be the union of $H_A$, the $s(B_i)$ for the geometrically finite bricks $B_i$, and the ending laminations $\mu(B_j)$ for all simply degenerate brick $B_j$ in $\mathcal{K}_{\text{int}}$, which we regard as lying on the ideal fronts. See Figure 3.5(a).

![Figure 3.5](a) $B_1, B_{10}$ is geometrically finite and $B_3, B_8$ are simply degenerate. The real fronts of $B_1$ and $B_{10}$ are inessential joints. $B_2, B_3, B_7$ are connectable. (b) The union of shaded rectangles represents $\mathcal{V}^{(1)}$. The white rectangles are bricks in $\mathcal{K}_{\text{int}}^{(1)} \cup \mathcal{K}_{\text{gf}}$.

We set $M_{\text{int}} = \bigvee \mathcal{K}_{\text{int}}$. A brick $B$ in $\mathcal{K}_{\text{int}}$ is said to be connectable if neither $I(B) = \partial_- B \cap I(K)$ nor $T(B) = \partial_+ B \cap I(K)$ is empty. Notice that if $B$ is a simply degenerate brick, although $\mu(B)$ does not lie inside $M$, either $\partial_- B$ or $\partial_+ B$ intersects $\mu(B)$. It should be also noted that any brick $B$ in $\mathcal{K}_{\text{int}}$ that has greatest
\( \xi(B) \) among the bricks in \( K_{\text{int}} \) is connectable unless \( \xi(B) = 3 \). We denote by \( \xi_0 \) the greatest \( \xi(B) \).

For any connectable brick \( B \) of \( K_{\text{int}} \) with \( \xi(B) \geq 5 \), we take a tight tube union in \( B \) connecting \( I(B) \) with \( T(B) \), and denote it by \( V_B \). In the case when \( B \) is an open brick, the condition (EL) guarantees that there is a tight tube union connecting \( I(B) \) and \( T(B) \). We set \( V_B = \emptyset \) if either \( B \) is not connectable or \( \xi(B) \leq 4 \), and define \( V^{(1)} = \bigcup_{B \in K_{\text{int}}} V_B \). See Figure 3.5(b). Now, if there are two tubes \( T_1, T_2 \) in \( V^{(1)} \) which are homotopic in \( M \setminus (V^{(1)} \setminus (T_1 \cup T_2)) \) we merge them into one tube. Repeating this operation, we get a union of tubes \( V^{(1)} \) in which no two tori are homotopic in the complement of the rest of tubes.

Let \( M^{(1)}_{\text{int}} \) be the closure of \( M_{\text{int}} \setminus V^{(1)} \) in \( M_{\text{int}} \). Since \( V^{(1)} \) consists of tubes which are thicken vertical annuli in \( M_{\text{int}} \), the 3-manifold \( M^{(1)}_{\text{int}} \) has the local product structure induced from that on \( M_{\text{int}} \). Thus \( M^{(1)}_{\text{int}} \) has a brick decomposition \( K^{(1)}_{\text{int}} \) allowing a brick to be an open one having a form \( F \times (0,1) \) such that each brick is a maximal union of vertically parallel horizontal leaves in \( M^{(1)}_{\text{int}} \). By our operation modifying \( V^{(1)} \) to \( V^{(1)} \) and the condition A-(2) for \( M_{\text{int}} \), the same condition A-(2) holds also for \( M^{(1)}_{\text{int}} \).

Let \( B \) be a half-open or open brick in \( M^{(1)}_{\text{int}} \). Suppose that \( B \) meets infinitely many original internal bricks of \( K_{\text{int}} \). Then take an essential simple closed curve on the horizontal surface of \( B \) which is not homotopic into an annulus component of \( \partial M \). This gives rise to an incompressible half-open annulus with core curve not homotopic into an annulus component of \( \partial M \), which tends to a wild end of \( M_{\text{int}} \), contradicting the condition A-(3) for \( M_{\text{int}} \). (This end cannot be simply degenerate since each simply degenerate end is contained in one brick of \( K_{\text{int}} \).) Therefore, any brick in \( K^{(1)}_{\text{int}} \) meets only finitely many bricks of \( K_{\text{int}} \). Also, an ideal front \( F \) of \( B \) cannot be contained in the ideal front \( F' \) of some simply degenerate brick \( B' = F' \times J \) of \( K_{\text{int}} \) since \( \mu(B') \) is contained in \( EL(F') \), and hence there is no open annulus in \( B' \) disjoint from the tight union of tubes which we extracted. Thus \( M^{(1)}_{\text{int}} \) contains no half-open or open bricks in fact. Also, we should note that the greatest \( \xi(B) \) for the bricks \( B \) in \( M^{(1)}_{\text{int}} \), which we denote by \( \xi_1 \), is less than \( \xi_0 \) since bricks in \( M_{\text{int}} \) with \( \xi = \xi_0 \) are all connectable.

Next we consider the union \( V^{(2)} \) of tubes which we obtained by modifying the union of all tight tube unions \( V_B \) for all \( B \in K^{(1)}_{\text{int}} \) in the same way as we defined \( V^{(1)} \) in \( K_{\text{int}} \), and the closure \( M^{(2)}_{\text{int}} \) of \( M^{(1)}_{\text{int}} \setminus V^{(2)} \) in \( M^{(1)}_{\text{int}} \). By the same reason as before, the greatest \( \xi(B) \) for the bricks \( B \) in \( M^{(2)}_{\text{int}} \) is less than \( \xi_1 \). Therefore, repeating the same procedure at most \( \xi(S) - 4 \) times, we reach a brick decomposition \( K^{(k)}_{\text{int}} \) on \( M^{(k)}_{\text{int}} \) such that \( \xi(B) \) is either 3 or 4 for every brick \( B \in K^{(k)}_{\text{int}} \).

Let \( V^{(k+1)} \) be the union of tubes obtained by modifying in the same way as before the union of tight tube unions \( V_B \) for bricks \( B \in K^{(k)}_{\text{int}} \) with \( \xi(B) = 4 \), and let \( K^{(k+1)}_{\text{int}} \) be the brick decomposition on the closure \( M^{(k+1)}_{\text{int}} \) of \( M^{(k)}_{\text{int}} \setminus V^{(k+1)} \) such that each brick is a maximal union of parallel leaves with respect to the horizontal foliation on \( M^{(k+1)}_{\text{int}} \). Moving components of \( V^{(k+1)} \) vertically by an ambient isotopy of \( M^{(k)}_{\text{int}} \) if necessary, we can assume that \( \partial \pm B \) for a brick \( B \) does not go through the gaps of tubes, i.e. the following holds.
For any $B \in K_{\int}$ and $B' \in K_{\int}^{(k)}$ with $H = (\partial_+ B \cup \partial_- B) \cap B' \neq \emptyset$, each component of $H \setminus \text{Int} V_B$ is homeomorphic to $\Sigma_{0,3}$. We set $B_{\int}^{(k+1)} = K_{\int}(k+1)$, $B = K_{\int}^{(k+1)} \cup K_{gf}$, $M_0^{(k+1)} = M_{\int}^{(k+1)}$, $M[0] = M[0]_{\int} \cup (\bigvee K_{gf})$, and $V = \bigcup_{n=1}^{k+1} V(m)$. We call $B$ a block decomposition of $M[0]$ and each element of $B$ a block. Note that each block in $B_{\int}$ is homeomorphic to either $\Sigma_{0,3} \times J$ or $\Sigma_{1,1} \times J$ or $\Sigma_{0,4} \times J$, where $J$ is a closed or half-open or open interval, since all the bricks in $M_{\int}^{(k+1)}$ have $\xi$ at most 4. Also by our definition of bricks for $M_{\int}^{(k+1)}$ no two blocks meet at inessential joint.

Remark 3.3. It may appear our definition of blocks is slightly different from that of Minsky in [Mi2] as we allow blocks homeomorphic to $\Sigma_{0,3} \times J$. Still the difference is just a minor point since we can convert our block decomposition into that à la Minsky just by cutting a block of the form $\Sigma_{0,3} \times J$ into halves and paste each of them to the blocks above and below.

Each component of $V$ is a solid torus which is foliated by vertically parallel horizontal annuli. For each solid torus $V$ in $V$, its boundary $\partial V$ is contained in $\partial M[0] \cup \partial M$. If $M[0] \cap V$ consists of two vertical annuli $A_1, A_2$ for some $V \in V$, then $\partial V \setminus \text{Int}(A_1 \cup A_2)$ is a union of two horizontal annuli contained in $\partial M$, and hence each of $A_1, A_2$ is a properly embedded essential annulus in $M$. (These annuli cannot be boundary-parallel since a brick is not allowed to be a solid torus by definition.) This contradicts the condition A-(2) saying that $M$ must be acylindrical. Therefore, for any component $V$ of $V$, the intersection $M[0] \cap V$ is either a torus or an annulus. See Figure 3.6.

Figure 3.6. A local picture of $M$ in the case of $k = 0$. The white region is $M[0]$. $V_1 \cup V_2 \cup V_3 \cup V_6 \subset V[0]$ and $V_2 \cup V_3 \cup V_4 \cup V_7 \cup V_8 \subset V \setminus V[0]$.

Let $V[0]$ be the union of all components $V$ of $V$ such that $M[0] \cap V$ is a torus, and set $M^0 = M[0] \cup V[0]$. Then $M^0$ is obviously a deformation retract of $M$ and there exists a homeomorphism $\eta_M : M^0 \to M$ homotopic to the inclusion such that the restriction $\eta_M|_{V[0]}$ is the identity. We often identify the original brick manifold $M$ with $M^0$ via the map $\eta_M$. 
3.5. Model metrics on brick manifolds. Now we shall define a metric on a
brick manifold induced from its decomposition into blocks. We shall put a standard
metric on each block as was done in Minsky [Mi2], which is slightly different from his
for our convenience. Fix \( \varepsilon_1 > 0 \) less than the three-dimensional Margulis constant,
and a hyperbolic metric on the three-holed sphere \( \Sigma_{0,3} \) with respect to which each
component of \( \partial \Sigma_{0,3} \) is a closed geodesic of length \( \varepsilon_1 \). Let \( B_{0,3} \) be \( \Sigma_{0,3} \times [0,1] \)
endowed with the product metric of the hyperbolic metric on \( \Sigma_{0,3} \) and the standard
metric on \([0,1]\).

Consider two essential simple closed curves \( l_0, l_1 \) on \( \Sigma_{0,4} \) (resp. \( \Sigma_{1,1} \)) with the
geometric intersection number \( i(l_0, l_1) = 2 \) (resp. \( i(l_0, l_1) = 1 \)) and set \( B_\alpha \) to be a brick
in the form \( \Sigma_\alpha \times [0,1] \) for \( \alpha \in \{(0,4), (1,1)\} \). Let \( A_- \) and \( A_+ \) be annular
neighbourhoods of \( l_0 \times \{0\} \) and \( l_1 \times \{1\} \) in \( \partial_- B_\alpha \) and \( \partial_+ B_\alpha \) respectively. We define
a piecewise Riemannian metric on \( B_\alpha \) such that each component of \( \partial_- B_\alpha \setminus \text{Int} A_- \)
and \( \partial_+ B_\alpha \setminus \text{Int} A_+ \) is isometric to \( \Sigma_{0,3} \) with the hyperbolic metric given above,
all of \( A_- \), \( A_+ \) and \( \partial B_\alpha \) are isometric to the product annulus \( S^1(\varepsilon_1) \times [0,1] \) and
\( \text{dist}_{B_\alpha}(\partial_- B_\alpha, \partial_+ B_\alpha) = 1 \), where \( S^1(\varepsilon_1) \) is a round circle in the Euclidean plane of
circumference \( \varepsilon_1 \).

For any brick \( B \in B_{\text{int}} \) of type \( \beta \in \{(0,3), (0,4), (1,1)\} \), consider a diffeo-
morphism \( h_B : B_\beta \to B \) such that \( h_B(\partial_\beta B_\beta) = \partial_\beta B \) and moreover \( h_B(A_\pm) =
\partial_\pm B \setminus \mathcal{V}[0] \) when \( \xi(B) = 4 \). We can choose these homeomorphisms so that for
any \( B, B' \) in \( B_{\text{int}} \) with \( F = \partial_+ B \cap \partial_- B' \neq \emptyset \), \((h_{B'}|F)^{-1} \circ h_B|_{h_{B'}^{-1}(F)} \) is an isometry
with respect to the metric on \( B_\beta \) defined above. Then \( M[0]\text{int} \) has a piecewise
Riemannian metric induced from those on \( B_{0,3}, B_{0,4}, B_{1,1} \) via embeddings
\( h_B : B_\beta \to M[0]\text{int} \).

We shall next define metrics on geometrically finite bricks. Each geometrically
finite brick \( B \) of \( B \) is identified with \( F \times [-1, \infty) \) preserving the horizontal and the
vertical leaves for a compact core \( F \) of some open essential subsurface \( \tilde{F} \) of \( S \) with
\( \xi(F) \geq 0 \). Since \( \tilde{F} \) can be identified with \( \text{Int} F \), by our definition of geometrically
finite bricks, \( \tilde{F} = F \times \{\infty\} \) is given a conformal structure. Let \( \sigma(B) \) a complete
hyperbolic metric on \( \tilde{F} \) which is compatible with the given conformal structure. We
regard \( F \) as obtained from \( \tilde{F}(\sigma(B)) \) by deleting the cusp neighbourhoods which are
components of \( \tilde{F}(\sigma)(0, \varepsilon_1) \). Consider a piecewise Riemannian metric \( \tau(B) \) on \( \tilde{F} \)
obtained by rescaling \( \sigma(B) \) on the points of \( \tilde{F} \) in such a way that \( \tau(B)/\sigma(B) \) is
continuous and is equal to 1 on \( \tilde{F}(\sigma(B))|_{\varepsilon_1, \infty} \), and each component of \( \tilde{F}(\sigma(B))|_{0, \varepsilon_1} \)
is a Euclidean cylinder with respect to the \( \tau(B) \)-metric. On the other hand, we put a
piecewise Riemannian metric \( \psi(B) \) on \( F \) such that each component of \( F(\psi(B))|_{0, \varepsilon_1} \)
is a Euclidean cylinder, \( F(\psi(B))|_{0, \varepsilon_1} \times \{-1\} \) coincides with \( \partial M[0]\text{int} \cap B \), and each
each component of \( F(\psi(B))|_{\varepsilon_1, \infty} \) is isometric to \( \Sigma_{0,3} \). We choose such a metric so that the
identity \( F(\tau(B)) \to F(\psi(B)) \) is uniformly bi-Lipschitz (i.e. the bi-Lipschitz constant is bounded by a constant independent of \( B \) and \( F \)). We call a metric
constructed as the latter metric \( \psi(B) \) a cylinder-\( \Sigma_{0,3} \) metric on \( F \times \{-1\} \). We note
that our \( \psi(B) \) corresponds to the metric \( \sigma^{\text{in}} \) given in [Mi2] Subsection 8.3.

We put a piecewise Riemannian metric on \( F \times [-1, 0] \) such that its restriction to
\( F \times \{-1\} \) is equal to \( F(\psi(B)) \), its restriction to \( F \times \{0\} \) is equal to \( F(\tau(B)) \), and the
induced metric on \( F \times \{t\} \) is uniformly bi-Lipschitz to \( \tau(B) \) via the identification of \( F \times \{t\} \) with \( F \). Recall that \( F \) is a compact core of an open surface \( \tilde{F} \). We take a
diffeomorphism \( \eta : F \times [0, \infty) \to \tilde{F} \times [0, \infty) \) such that the restriction \( \eta|F \times \{0\} \) is
the identity and \( \eta(\partial F \times [0, \infty)) \) lies on \( \hat{F} \times \{0\} \). We put on \( F \times [0, \infty) \) the induce metric \( \eta^*(ds^2) \), where \( ds^2 \) is a piecewise Riemannian metric on \( F \times [0, \infty) \) defined by
\[
ds^2 = \tau(B)e^{2r} + dr^2 \quad (r \in [0, \infty)).
\]
We define a piecewise Riemannian metric on \( B \) by pasting the metrics on \( F \times [-1, 0] \) and \( F \times [0, \infty) \) along \( F \times \{0\} \). We may assume that the metric on \( M[0]_{\text{int}} \) and that on \( B \) are equal on \( M[0]_{\text{int}} \cap B = F \times \{-1\} \) deforming the map attaching \( B \) to \( M[0]_{\text{int}} \) by an ambient isotopy if necessary. Thus we have obtained a piecewise Riemannian metric on \( M[0] \), which we call the model metric on \( M[0] \). By our construction, each component \( C \) of \( \partial M[0] \) is either a Euclidean torus or a Euclidean cylinder which has a foliation \( \mathcal{F}_C \) whose leaves consist of closed geodesics of length \( \varepsilon_1 \).

3.6. Meridian coefficients. For a complex number \( z \) with \( \text{Im}(z) > 0 \) and a real number \( \eta > 0 \), we denote the covering map \( C \to C/\eta(\mathbb{Z} + z\mathbb{Z}) \) by \( \pi_{z,\eta} \). For any component \( V \) of \( \mathcal{V}[0] \), its boundary \( \partial V \) has a metric induced from the model metric on \( M[0] \) as above. Then there is a unique \( \omega \in C \) with \( \text{Im}(\omega) > 0 \) for which we have an orientation-preserving isometry from the quotient space \( C/\varepsilon_1(\mathbb{Z} + \omega\mathbb{Z}) \) to \( \partial V \) taking \( \pi_{\omega,\varepsilon_1}(\mathbb{R}) \) (resp. \( \pi_{\omega,\varepsilon_1}(\omega\mathbb{R}) \)) to a longitude (resp. a meridian) of \( V \). (Here a longitude of \( V \) is defined to be a horizontal essential simple closed curve on \( \partial V \).)
We define this \( \omega \) by \( \omega_M(V) \) and call it the meridian coefficient of \( \partial V \).

For any integer \( k > 0 \), consider the union \( \mathcal{V}[k] \) of components \( V \) of \( \mathcal{V}[0] \) with \( |\omega_M(V)| \geq k \) and set
\[
M[k] = M[0] \cup (\mathcal{V}[0] \setminus \mathcal{V}[k]).
\]
By definition, we have \( M[0] = M[k] \cup \mathcal{V}[k] \). We put each component \( V \) of \( \mathcal{V}[0] \) a hyperbolic metric induced from the Margulis tube whose boundary has exactly the Euclidean metric induced from the model metric on \( M[0] \). See Lemma 8.5 in [BCM] or Lemma 5.8 in [Bow3]. In this way, we extend the model metric on \( M[0] \) to a metric on \( M[0] \) whose restriction on \( \mathcal{V}[0] \) is hyperbolic. The brick manifold \( M \) has the metric induced from that on \( M[0] \) via \( \eta \). We also call these metrics on \( M[0] \) and \( M \) the model metrics.

4. The bi-Lipschitz model theorem for brick manifolds

Minsky constructed in [Mi2] model manifolds for hyperbolic 3-manifolds homeomorphic to \( S \times (0, 1) \) and proved that for any such hyperbolic manifold, there is a Lipschitz map, called a model map, from its model manifold, whose Lipschitz constant is uniformly bounded. Furthermore, in Brock-Canary-Minsky [BCM], it was shown that such a model map can be taken to be a bi-Lipschitz homeomorphism with uniformly bounded bi-Lipschitz constant. Using and generalising these results, we shall show that a homeomorphism from a labelled brick manifold satisfying the conditions A-(1)-(5) and (EL) to a hyperbolic 3-manifold preserving end invariants can be homotoped to a bi-Lipschitz homeomorphism with uniformly bounded bi-Lipschitz constant. Let us recall that for any hyperbolic 3-manifold \( N \) and a constant \( \varepsilon_1 > 0 \) less than the Margulis constant, \( N_0 = N_{\varepsilon_1}^0 \) denotes the \( \varepsilon_1 \)-non-cuspidal part, i.e. the union of \( N_{[\varepsilon_1, \infty)} \) and all Margulis tube components of \( N_{(0, \varepsilon_1]} \) as defined in Subsection 2.2.

**Theorem 4.1** (Bi-Lipschitz Model Theorem). Let \( M \) be a labelled brick manifold satisfying the conditions A-(1)-(5) and (EL), and \( N \) a hyperbolic 3-manifold with a homeomorphism \( f : M \to N_0 \) preserving the end invariants. Then \( f \) is properly
homotopic to a homeomorphism $g : M \to N_0 = N_0^1$ satisfying the following conditions, where $k \in \mathbb{N}$, $K \geq 1$ and $\varepsilon_1$ less than the Margulis constant depend only on $\xi(S)$.

(i) The image $g(\mathcal{V}[k]) = T[k]$ is a union of $\varepsilon_1$-Margulis tubes of $N_0$.
(ii) $g(M[k]) = N_0 \setminus \text{Int}T[k]$.
(iii) The restriction $g|_{M[k]} : M[k] \to N_0 \setminus \text{Int}T[k]$ is a $K$-bi-Lipschitz map.
(iv) The homeomorphism $g$ extends continuously to a conformal map from $\partial_\infty M$ to $\partial_\infty N$.

The whole of the present section is devoted to the proof of Theorem 4.1. We should note that by Lemma 3.4 there is a proper embedding $\iota_M$ of our model manifold into $S \times (0, 1)$. Accordingly, we have an embedding $\iota_N : N_0 \to S \times (0, 1)$ such that $\iota_N \circ f = \iota_M$. As in the previous section, we can modify the brick decomposition of $M$ so that Assumption 3.2 holds.

Applying the argument in Minsky [Mi2 Subsections 3.4 and 8.3], we can deform $f$ to a map $f_1$ by a proper homotopy so that for any geometrically finite half-open brick $B' \in K_{gf}$, the restriction $f_1|_{B'} : B' \to f_1(B')$ is a uniformly bi-Lipschitz homeomorphism which extends continuously to a conformal map from $\partial_\infty B'$ to $\partial_\infty f_1(B')$ and its real front is mapped into the boundary of the convex core of $N$.

We shall first show that $f_1$ can be properly homotoped to a $K$-Lipschitz map with a constant $K$ depending only on $\xi(S)$. We shall follow the line of Minsky’s argument in [Mi2]. Recall that we have a union of tubes $\mathcal{V}$ in $M$ which we constructed in 3.3 inducing a decomposition of $M$ into blocks, and that for each tube $V$ in $\mathcal{V}$, its meridian coefficient $\omega_M(V)$ is defined. The first step is to prove the following lemma.

**Lemma 4.2.** There is a universal constant $L$ depending only on $\xi(S)$ such that for the core curve $v$ of each tube $V$ in $\mathcal{V}[0]$, the length of the closed geodesic in $N$ homotopic to $f(v)$ is less than $L$.

**Proof.** This lemma corresponds to Lemma 7.9 in Minsky [Mi2]. We shall use its generalisation by Bowditch, Theorem 1.3 in [Bow2].

Recall that we constructed a block decomposition of $M$ repeating the process of putting tight tube unions in bricks, starting from the decomposition of $M$ into bricks of $\mathcal{K}$. At the first stage, for each connectable internal brick $B = F \times J$, we connected a component $\partial_- B \cap \mathcal{I}(K)$ with a component of $\partial_+ B \cap \mathcal{I}(K)$ by a tight geodesic. Since $f$ takes $\mathcal{I}(K)$ to either an ending lamination or a parabolic element in $N$, by applying Lemma 7.9 in Minsky [Mi2] or Theorem 1.3 in Bowditch [Bow2] to the covering of $N$ associated to $f_\# \pi_1(B)$, we see that there is $L_0$ depending only on $\xi(S)$ such that each curve in the simplices constituting the tight geodesic has length in $N$ bounded by $L_0$.

At the $n$-th stage, we have bricks $K_{int}^{(n)}$ constituting $M_{int}^{(n)}$ which is the complement of $\mathcal{V}_n = \bigcup_{m=1}^{n} \mathcal{V}^{(m)}$ in $M_{int}$. Let $L_n$ be the union of $\mathcal{I}(K)$ and core curves of $\mathcal{V}_n$ which are not homotopic to a simple closed curve in $\mathcal{I}(K)$. In each brick $B^{(n)}$ of $K_{int}^{(n)}$, we constructed a tight tube union connecting $\partial_- B^{(n)} \cap L_n$ and $\partial_+ B \cap L_n$. Therefore using Bowditch’s Theorem 1.3 inductively, we see that if the geodesic lengths in $N$ of curves in $L_n$ are bounded by $L_n$, then there is $L_{n+1}$ depending only on $L_n$ bounding the lengths in $N$ of $L_{n+1}$. Since we reached the block decomposition within $\xi(S) - 3$ steps, we see that there is a constant $L$ depending only on $\xi(S)$ which bounds the lengths of the geodesics corresponding to core curves of $\mathcal{V}$. □
4.1. Homotoping $f$ to a Lipschitz map preserving the thin part. Moving $V$ by an ambient isotopy of $M_{\text{int}}$ without changing the structure of block decomposition, we may assume that for any $B \in K_{\text{int}}$, every component of $\partial_+ B \setminus V$ and $\partial_- B \setminus V$ is homeomorphic to a thrice-punctured sphere. Let $F$ be a compact essential subsurface of $S$ such that $B$ is homeomorphic to $F \times J$ for an interval $J$. If $\partial_+ B$ is a real front, then $\partial_+ B \cap V$ determines a simplex in $C(F)$ inducing a pants decomposition of $F$. We now homotope $f_1$ so that each core curve of $V$ is mapped to a closed geodesic. By Lemma 4.2 all of such closed geodesics have length bounded by $L$. In this situation, we can apply Minsky’s construction in Steps 0-6 of [Mi2, Section 10] to get a map $f_2 : M \to N$ for which the following hold. Recall that we have fixed a constant $\varepsilon_1$ less than the three-dimensional Margulis constant.

1. We have $f_2[B] = f_1[B]$ for every $B' \in K_{\text{ext}}$.
2. For each block $B$ of $M[0]_{\text{int}}$, the $f_2|\partial_\pm B$ lies on a pleated surface with totally geodesic boundary each of whose component is a closed geodesic homotopic to $f_2(v)$ for a core curve $v$ of some $V \in \mathcal{V}$.
3. There exists a constant $\varepsilon_0 > 0$ depending only on $\xi(S)$ such that for a core curve $v$ of a solid torus component $V$ of $\mathcal{V}$, if the geodesic length of $f_2(v)$ is less than $\varepsilon_0$, then $f_2(V)$ is contained in the $\varepsilon_1$-Margulis tube with core curve $f_2(v)$.
4. The image of $f_2$ is contained in the union of the 1-neighbourhood of the convex core of $N$ and the $\varepsilon_1$-Margulis tubes of $N$.
5. For any $k$, there exists a positive number $\epsilon(k) < \varepsilon_1$ such that $f_2(M[k])$ is disjoint from the $\varepsilon_1$-Margulis tubes of $N$ whose core curves have length less than $\epsilon(k)$.
6. For any $k$, there exists a constant $L(k)$ such that $f_2(M[k])$ is $L(k)$-Lipschitz.

To modify $f_2$ further to get a Lipschitz map, we need the following lemma.

Lemma 4.3. Let $V$ be a tube in $\mathcal{V}[0]$, and $v$ its core curve. For any $\delta > 0$, there exists $k$ which depends on $\delta$ but is independent of $M$ and $N$ such that if $|\omega_M(V)| > k$ then the closed geodesic homotopic to $f_2(v)$ has length smaller than $\delta$.

Proof. This lemma corresponds to Lemma 10.1 in Minsky [Mi2]. In our situation, $V$ may be shared by blocks contained in distinct bricks. Therefore, we cannot apply Minsky’s result directly. Instead, we use an argument which can also be found in Soma [So]. Our argument is by contradiction. Suppose that there exist $\delta > 0$ and tubes $V_j$ with core curves $v_j$ such that $|\omega_M(V_j)| \to \infty$ whereas the closed geodesics homotopic to $f_2(v_j)$ have length greater than $\delta$.

Since $|\omega_M(V_j)| \to \infty$, by passing to a subsequence, we can assume that either $\Im \omega_M(V_j) \to \infty$ or $\Re \omega_M(V_j) \to \infty$ holds. We shall first consider the case when $\Im \omega_M(V_j) \to \infty$. By the definition of $\omega_M(V_j)$, there are $(3 \omega_M(V_j) - 2)$ blocks which intersect $\partial V_j$ along their vertical sides. This implies that there are at least $3 \omega_M(V_j)$ gluing surfaces, which are homeomorphic to $\Sigma_{(0,3)}$, having boundary components lying on $\partial V_j$. We should also note that no two distinct gluing surfaces are homotopic in $M$. Since we assumed that $\Im \omega_M(V_j)$ goes to $\infty$, there are $k_j$ pairwise non-homotopic gluing surfaces with boundary components on $\partial V_j$ with $k_j \to \infty$. The image of each gluing surface lies on a pleated surface with totally geodesic boundary one of whose components is the closed geodesic $\gamma_j$ homotopic to $f(v_j)$. Therefore, there are $k_j$ pairwise non-homotopic pleated surfaces from $\Sigma_{(0,3)}$ which have $\gamma_j$ as a boundary component.
Now, we put a basepoint \( x_j \) on \( \gamma_j \), and consider the geometric limit \((N_\infty, x_\infty)\) of \((N, x_j)\), passing to a subsequence if necessary. Since the length of \( \gamma_j \) is bounded from above by Lemma 4.3 and from below away from 0 by our assumption, the geometric limit exists (as a hyperbolic 3-manifold) if we take a subsequence, and does not depend on the choice of \( x_j \) as long as it lies on \( \gamma_j \) once we fix some geometrically convergent subsequence. Let \( \rho_i : B_{K_i}(N, x_j) \to B_{K_i}(N_\infty, x_\infty) \) be an approximate isometry associated to the geometric convergence with \( K_i \to \infty \) and \( K_i \to 1 \). In the geometric limit \( M_\infty \), we have the limit \( \gamma_\infty \) of \( \gamma_i \), which is a closed geodesic since the lengths of the \( \gamma_j \) are bounded away from 0. The geometric limit of pleated surfaces with boundary components on \( \gamma_j \) are pleated surfaces with a boundary component on \( \gamma_\infty \). We should also note that since all the pleated surfaces intersect the \( \epsilon \)-thin part of \( N \) only at near their boundary components other than \( \gamma_j \) for some fixed \( \epsilon > 0 \), the limit pleated surfaces can intersect the \( \epsilon \)-thin part only near their boundary components other than \( \gamma_\infty \). Since \( k_j \to \infty \), we can find among the limit pleated surfaces, two limit pleated surfaces \( F_1, F_2 \) such that \( F_2 \) is homotopic to \( F_1 \) in a small regular neighbourhood \( F_1 \) whereas \( \rho_1^{-1}(F_1) \) and \( \rho_1^{-1}(F_2) \) are not homotopic. This is a contradiction.

It remains to deal with the case when \( \Re \omega_M(V_j) \to \infty \) whereas \( \Im \omega_M(V_j) \) is bounded. Fix a horizontal simple closed curve \( c_i \) on \( \partial V_j \). We let \( d_i \) be a simple closed curve on \( \partial V_j \) intersecting \( c_i \) at one point and having shortest length among all simple closed curves intersecting \( c_i \) at one point. Let \( m_j \) be a meridian of \( V_j \). Since \( d_i \) intersects \( c_j \) at one point, as elements of the first homology group of \( \partial V_j \), we have \([d_j] = [m_j] + \alpha_j[c_j] \) with \( \alpha_j \in \mathbb{Z} \) if we fix orientations on \( c_j, m_j \) and \( d_j \). Since we assumed that \( \Re \omega_M(V_j) \to \infty \), we have \([\alpha_j] \to \infty \), and in particular, we can assume that \( \alpha_j \neq 0 \) by taking a subsequence. Since the length of \( d_j \) is bounded by that of \( m_j \), we have \( \text{length}_{\partial V_j}(d_j) \leq (\Im \omega_M(V_j) + 1)\epsilon_1 \). Now, since \( \partial V_j \) is contained in \( M[0] \), by the condition (6) above, we have \( \text{length}(f_2(d_j)) \leq L(0)(\Im \omega_M(V_j) + 1)\epsilon_1 \). The right hand side is bounded above since we have already proved \( \Im \omega_M(V_j) \) is bounded as \( j \to \infty \). Since \([d_j] = [m_j] + \alpha_j[c_j] \), the curve \( f_2(d_j) \) with an appropriate orientation is homotopic to the \( [\alpha_j] \)-time iteration of \( \gamma_j \). This implies \( \text{length}(f_2(d_j)) \geq |\alpha_j| \text{length}(\gamma_j) \). The right hand side goes to \( \infty \), whereas the left hand side is bounded as we have seen above. This is a contradiction. \( \square \)

Having proved Lemma 4.3, the rest of modification as in Minsky [Mi2] to get a proper, degree-1 map \( f_3 : M \to N_0 \) such that \( f_3[M[k]] \) is \( K_3 \)-Lipschitz with \( K_3 \) depending only on \( \xi(S) \) works without changes.

We state one more property of \( f_3 \).

(7) Since \( f_3 \) has degree 1, there exist constant \( k_2 \) and \( \epsilon(k_2) \) as in the condition (5) depending only on \( \xi(S) \) such that any \( \epsilon_1 \)-Margulis tube in \( N \) whose core curve has length less than \( \epsilon(k_2) \) is contained in the image of a component of \( \mathcal{V}[k_2] \).

4.2. Preliminary steps to homotope \( f_3 \) to a bi-Lipschitz map. We now turn to modify \( f_3 \) to a bi-Lipschitz homeomorphism, which was done in Brock-Canary-Minsky [BCM] for the case of surface Kleinian groups. Recall that we moved \( \mathcal{V} \) so that for each brick \( B \) in \( K_{\text{int}} \), if its upper or lower boundary \( \partial_{\pm} B \) is non-empty, then every component of \( \partial_{\pm} B \setminus \mathcal{V} \) is a thrice-punctured sphere. We parametrise \( B \) as \( F \times J \) with a closed or half-open interval \( J \). We define \( \hat{i}(B) \) to be a simplex in \( \mathcal{C}(F) \) with empty transversals such that \( \hat{i}(B) \times \{\min J\} \) is homotopic to \( \partial_{-} B \cap \mathcal{V} \).
if $\partial_- B$ is non-empty, and to be the ending lamination of the end corresponding to $F \times \{ \inf J \}$ if $\partial_- B$ is empty. Similarly we define $t(B)$ for the upper boundary of $B$. We shall first show that in this setting, the block decomposition of $B$ induced by $\mathcal{V}$ corresponds to a hierarchy in the sense of Masur-Minsky [MM2].

**Lemma 4.4.** Let $B$ be a brick in $\mathbb{K}_{int}$, which is homeomorphic to $F \times J$ with a closed or half-open interval $J$. Then there is a 4-complete hierarchy $h$ of tight geodesics on $F$ with $I(h) = i(B)$ and $T(h) = t(B)$. The 4-sub-hierarchy of $h$ gives rise to the same block decomposition of $B$ as the one induced by $\mathcal{V}$ converted as in Remark 3.3 to Minsky’s decomposition.

**Proof.** In the construction of $\mathcal{V}$ in the previous section, we began with putting tight tube unions in all connectable bricks in $M_{int}$ whose initial and terminal vertices are in $\mathcal{I}(K)$. After that, we merged homotopic tubes into one and let the obtained tube union be $\mathcal{V}^{(1)}$. Then we considered the brick manifold $M^{(1)}$ which is the complement of $\mathcal{V}^{(1)}$ and repeated the same procedure until we got a block decomposition. Now, we shall look more closely how tubes are put (and merged) in $B$ during this construction and define tight geodesics which constitute $h$. We define $I(B) = i(B) \times \inf J$ and $T(B) = t(B) \times \sup J$. (These may be larger than $I(B)$ and $T(B)$ defined in the previous section.)

If $B$ is connectable in the first step of the construction of $\mathcal{V}$, then we get a tube union $\mathcal{V}_B$ on $B$ in the first step, which corresponds to a tight geodesic $g_B$ in $\mathcal{G}(F)$ connecting a component of $\mathcal{I}(K) \cap \partial_- B$ with a component of $\mathcal{I}(K) \cap \partial_+ B$. (If one of them is an ending lamination, the geodesic $g_B$ refers to a tight geodesic ray tending to it.) Since $\mathcal{I}(K) \cap \partial_- B \subset i(B)$ and $\mathcal{I}(K) \cap \partial_+ B \subset t(B)$, we can assume that $g_B$ has $i(B)$ as the initial marking, and $t(B)$ as the terminal marking.

We next consider how the merging of tubes is reflected in the construction of geodesics in the hierarchy still under the assumption that $B$ is connectable at the first step. If there is a tube $V$ in $B$ which is merged with another homotopic tube $V'$ in another brick $B'$, then a core curve $v$ of $V$ must be in either $i(B)$ or $t(B)$ since $\partial_- B \setminus \mathcal{V}$ consists of thrice-punctured spheres. This can occur only when the core curve is contained in the first step, or the second, or the second but last, or the last simplex of the geodesic $g_B$. If $v$ is contained in the first or the last vertex of $g_B$, this procedure of merging does not affect tubes in $B$. Otherwise, $v$ is contained in either the second or the second to the last simplex of $g_B$. In this case, we regard the procedure as corresponding to putting a geodesic consisting of only one vertex which is subordinate to $g_B$ at the first or the last vertex.

Next, we shall consider the case when $B$ is not connectable at the first step. In the second step, either $B$ is contained in another brick $B$ constituting $M^{(1)}$ or $B$ is split into two (or more) in the process of merging two homotopic tubes of $\mathcal{V}^{(1)}$, one lying above $B$ and the other below $B$. In the latter case, let $V_1, \ldots, V_p$ be tubes in $\mathcal{V}^{(1)}$ which split $B$. We should note that these tubes have core curves which are homotopic to curves both in $I(B)$ and $T(B)$ then. Let $v_1, \ldots, v_p$ be curves on $F$ corresponding to their core curves. Then we define geodesics $g_1, \ldots, g_p$, each of which consists of only one vertex, such that $D(g_1) = F$, $D(g_j)$ is a component of $F \setminus \bigcup_{s=1}^{j-1} V_s$ for $j = 2, \ldots, p$, $I(g_j) = i(B) \cap D(g_j)$, $T(g_j) = t(B) \cap D(g_j)$, and $g_{j-1} \sqcup g_j \sqcup g_{j-1}$, and let them be geodesics in $h$.

In the former case, if $\bar{B}$ is connectable in $M^{(1)}$, then we consider $\mathcal{V}_{\bar{B}} \cap B$, where as explained above $B$ is assumed to be in a position such that $\mathcal{V} \cap \partial_- B$ is a regular
neighbourhood of $I(B)$ and $V \cap \partial_+ B$ is that of $T(B)$, and define $g_B$ to be the tight geodesic in $C(F)$ corresponding to $V_B \cap B$. As before, we define $I(g_B) = i(B)$ and $T(g_B) = t(B)$. If $B$ is not connectable, we proceed to the following step and repeat the same procedure depending on either there is a brick containing $B$ or $B$ is split by merging of homotopic tubes. Thus we have defined $g_B$, together with some more geodesics in $h$ in the case when $B$ is split. We shall now turn to subsequent steps.

In subsequent steps, we put a tight tube union $V_B$ into a brick $B'$ constituting a brick decomposition of $M \setminus \nu^{(k)}$. We shall show that the intersection with $B$ of each tube union in a connectable brick $B'$ in the $(k + 1)$-th step gives rise to a tight geodesic on $F$ which is subordinate to the ones obtained up to the $k$-th step. This implies that at the final step, we shall get a hierarchy on $F$ connecting $i(B)$ and $t(B)$. To show that, we shall analyse what a tube union in $B'$ brings about to $B$ dividing the situation into subcases depending on the location of $B'$ with regard to $B$. (Again, $B$ is in a position where $\partial_- B \cap V$ is a regular neighbourhood of $I(B)$ and $\partial_+ B \cap V$ is that of $T(B)$.) We parametrise $B'$ as $F' \times J'$ with $F' \subset F$, in such a way that horizontal leaves and vertical leaves are contained in those of bricks in $K_{\text{int}}$. Since $F' \times \{x\}$ for $x \in \text{Int} J'$ is a horizontal leaf whose boundary lies on $\partial \nu_k$, the surface $F'$ is a component domain of a geodesic corresponding to a tube union which was already put into $M$ up to the $k$-th step. Now we divide our argument into three, depending on the inclusive relation between $J$ and $J'$.

First, suppose that $B'$ is contained in $B$, which means that both $\partial_- B'$ and $\partial_+ B'$ lie in $B$ and $J'$ is contained in $J$. We define $I(B') = \partial_- B' \cap (\nu^{(k)} \cup (i(B) \times \text{inf} J))$ and $T(B') = \partial_+ B' \cap (\nu^{(k)} \cup (t(B) \times \text{sup} J))$, which we can assume to define pants decompositions on $\partial_- B'$ and $\partial_+ B'$. Note that $I(B')$ is the union of core curves in $\nu^{(k)} \cap \partial_- B'$ and $T(B')$ is that of $\nu^{(k)} \cap \partial_+ B'$, which are contained in $I(B')$ and $T(B')$ respectively. By our construction of $\nu^{(k + 1)}$, the tube union $V_B'$ in $B'$ connects a component of $I(B')$ to that of $T(B')$. We define $g_{B'}$ to be the tight geodesic corresponding to $V_B'$ whose initial and terminal markings are simplices corresponding to $I(B')$ and $T(B')$ respectively. The geodesic $g_{B'}$ corresponds to a tube union connecting a tube in $I(B')$ to that in $T(B')$, which are contained in $(\nu^{(k)} \cap B) \cup (i(B) \times \text{inf} J)$ and $(\nu^{(k)} \cap B) \cup (t(B) \times \text{sup} J)$ respectively. This shows that the tight geodesic $g_{B'}$ is both forward and backward subordinate to a geodesic in $h$ which was obtained up to the $k$-th step.

Next suppose that, one of $\partial_- B'$ and $\partial_+ B'$ is contained in $B$ whereas the other is not. This means that one of the endpoints of $J'$ lies in $J$ whereas the other does not. Now, we assume that $\partial_- B'$ is the one contained in $B$: for the other case, we can argue in the same way, just interchanging the directions. In this situation, $I(B')$ consists of core curves of $\partial_- B \cap \nu^{(k)}$ which is contained in $\nu^{(k)} \cap B$ as before. On the other hand, $T(B')$ may not lie in $B$. Now, by our definition of $V$, the tube union $V_B'$ is contained in $V$. Therefore, $V_{B'}$ intersects $\partial_+ B$ by a component of $\partial_+ B \cap V$ since we moved $V$ so that every component of $\partial_+ B \setminus V$ is a thrice-punctured sphere. (Recall that unless $\xi(B') = 4$, the upper front of each tube of $V_B'$ lies on the same horizontal level as the lower front of the subsequent tube. In the case when $\xi(B') = 4$, there is a gap between them, but we moved $V$ so that $\partial_+ B$ avoid such gaps.) Therefore, if we consider a sub-tube union $\nu_{B'}$ of $V_{B'}$ starting from the first tube and ending at a tube in $\partial_+ B \cap V$, then it is exactly what $V_{B'}$ brings about to $B$. Let $g_{B'}$ be the tight geodesic corresponding to $\nu_{B'}$ defining $I(g_{B'})$ to be a simplex consisting of curves corresponding to core curves of
Proof. Suppose, seeking a contradiction, that there are tubes hierarchy) we need to show the same kind of property for does not correspond to a hierarchy, (only its restriction to a brick corresponds to a Lemma 4.5. There are no two distinct tubes in \( V \) which are homotopic in \( M \).

Proof. Suppose, seeking a contradiction, that there are tubes \( V_1, V_2 \) in \( V \) which are homotopic each other in \( M \). Let \( k_1, k_2 \) be the numbers such that \( V_1 \in \gamma^{(k_1)} \) and \( V_2 \in \gamma^{(k_2)} \), and set \( k = \max\{k_1, k_2\} \). (When we say \( V_l \in \gamma^{(k_l)} \) for \( l = 1, 2 \), we
take the smallest $k_1$ such that $V^{(k_1)}$ contains $V_i$. We follow the same convention throughout the proof.) Then the longitudes of $V_1$ and $V_2$ pushed out to their boundaries are not homotopic in $M^{(k)}$, for otherwise they should have been merged into one in our construction. Let $U$ be the union of tubes in $U_{j=1}^{k} V_k$ which intersects essentially an embedded annulus $A$ bounded by the longitudes of $V_1$ and $V_2$ in $M$. (These are determined independently of the choice of an annulus since $M$ is toroidal.)

Let $U \in \mathcal{U}$ be a tube which appears in the earliest step among the tubes in $\mathcal{U}$, and suppose that $U \in \mathcal{V}^{(i)}$. Let $B \cong F \times J$ be a brick in $\mathcal{K}^{(l-1)}$ where $U$ appears as a tube in the tight tube union. If either $V_1$ or $V_2$, say $V_1$, intersects a front of $B$, then by replacing $V_1 \cap B$ with $V_1$, we can assume that both $V_1$ and $V_2$ are disjoint from the fronts of $B$. First suppose that both $V_1$ and $V_2$ are contained in $B$. In the following argument, for two tubes $U, V \in \mathcal{V}$, we write $U \approx V$ if $U = A_1 \times J_1$, $V = A_2 \times J_2$ and Int$J_1 \cap$ Int$J_2 \neq \emptyset$ for the parametrisation on $S \times (0, 1)$ in which $M$ is embedded by $\iota_M$. In the $k$-th step, a tight tube union $\mathcal{V}_B$ corresponding to a tight geodesic $g_B$ on $\mathcal{C}(F)$ is given. Then there are tubes $U_1, U_2$ in the tight tube union of $B$ such that $V_1 \approx U_1$ and $V_2 \approx U_2$. Since $U$ intersects $A$, we have $U_1 < U < U_2$ or $U_2 < U < U_1$ with respect to the ordering on the simplices of $g_B$. Let $u_1, u_2, u, v$ be vertices of $\mathcal{C}(F)$, which correspond to $U_1, U_2, U, V_1$. Then, we have $u_1, u_2 \in \phi_{g_B}(v)$ whereas $u \not\in \phi_{g_B}(v)$. This contradicts the fact that $\phi_{g_B}(u)$ consists of contiguous, which was proved in Lemma 4.10 in Masur-Minsky \cite{MM2}.

Next suppose that one of $V_1, V_2$, say $V_1$, lies outside $B$ whereas $V_2$ is contained in $B$. In this case, $A$ passes through a joint contained in the upper or the lower front of $B$. We only consider the case when $A$ passes through a joint in the upper front. The other case can be dealt with in the same way just by turning the figure upside down. Since $A$ passes through a joint in the upper front, we have $T(B) \cap V_1 = \emptyset$, which implies that the last vertex $u_\infty$ of $g_B$ is contained in $\phi_{g_B}(v)$. As in the previous paragraph, we have $u_1 < u < u_\infty$ and $u_1, u_2 \in \phi_{g_B}(v)$ whereas $u \not\in \phi_{g_B}(v)$, which contradicts Lemma 4.10 of \cite{MM2} as before. Also in the case when both $V_1$ and $V_2$ lie outside $B$, we can argue in the same way considering joints which $A$ passes contained in the upper and the lower fronts. Then we see that the first and the last vertices are contained in $\phi_{g_B}(v)$ whereas $u$ is not. This again contradicts Lemma 4.10 in \cite{MM2}. This completes the proof. \hfill \Box

The next lemma is obtained from Otal \cite{Ot} for hyperbolic 3-manifolds homeomorphic to $S \times \mathbb{R}$ for a hyperbolic surface $S$. Since the only condition that is necessary for the proof is the fact that the manifold can be filled up by incompressible pleated surfaces (with bounded genus), his argument also works in our settings.

**Lemma 4.6.** There is a constant $k_0$ depending only on $\chi(S)$ such that for any $k \geq k_0$ and tubes $V \in \mathcal{V}^{[k]}$, the core curves $c$ of the $V$ are mapped by $f_3$ to unknotted and unlinked closed geodesics, i.e. there is an isotopy of $S \times (0, 1)$ which takes $\iota_N(c)$ to disjoint collection of simple closed curves lying on horizontal surfaces.

Take $k_2$ in the condition (7) so that $\epsilon(k_2) < \rho(\epsilon(z))$ less than our fixed $\epsilon_1$ (less than the Margulis constant). By Lemma 4.3 there exists $k_1$ such that if $|\omega(V)| \geq k_1$, then $f(v)$ has length less than $\epsilon(k_2)$. We define $k_u = \max\{k_0, k_1, k_2\}$ for $k_0$ in the above lemma, and let $\epsilon_u$ be $\epsilon(k_u)$. 

Now, we recall the following definition of topological order introduced in Brock-Canary-Minsky [BCM], which we shall apply for surfaces in $M$ or $N_0$.

**Definition 4.7** (Brock-Canary-Minsky [BCM]). Let $f_1 : F_1 \to M$ and $f_2 : F_2 \to M$ be maps from essential subsurfaces $F_1, F_2 \subset S$ such that $\iota_M \circ f_1$ is homotopic to the inclusion $F_1 \to F_3 \times \{ t \}$. We say $f_1 \preceq_{\text{top}} f_2$ if and only if $\iota_M \circ f_1$ can be homotoped to $S \times \{ 0 \}$ without touching $\iota_M \circ f_2(F_2)$ and $\iota_M \circ f_2$ can be homotoped to $S \times \{ 1 \}$ without touching $\iota_M \circ f_1(F_1)$. We define the topological order on maps from surfaces to $N_0$ in the same way just replacing $M$ with $N_0$ and $\iota_M$ with $\iota_N$.

We should also recall that two embedded surfaces $F_1, F_2$ in $S \times (0, 1)$ are said to overlap if their projections to $S$ have essential intersection. We use this term also for surfaces in $M$ or $N_0$, for they can be embedded in $S \times (0, 1)$ by $\iota_M$ and $\iota_N$.

### 4.3. Homotoping $f_3$ to a homeomorphism.

We shall next consider to homotope $f_3$ so that its restriction to the union of the joints of the bricks is an embedding. Let $F$ be a joint of a brick with another brick. Recall that $F$ intersects $V$ in such a way that each component of $F \setminus V$ is a thrice-punctured sphere. We define $\hat{F}[k]$ to be an embedded surface in $B \cap M[k]$ obtained from $F$ by isotoping annuli in $F \cap V[k]$ to those on $\partial V[k]$. There are two choices for an annulus for each component of $F \cap V[k]$. We take annuli on $\partial V[k]$ which are situated lower than the others with respect to the embedding $\iota_M$.

Recall that the images of $V[k_u]$ are unknotted and unlinked $\varepsilon_1$-Margulis tubes whose core curves have lengths less than $\varepsilon(k_2)$. Conversely, every $\varepsilon_1$-Margulis tube whose core curve has length less than $\varepsilon(k_2)$ is the image of a component of $V[k_2]$ by $f_3$. Recall that we denote the union of the Margulis tubes which are images of tubes in $V[k_u]$ by $T[k_u]$. We denote $N_0 \setminus \text{Int}T[k_u]$ by $N[k_u]$. By Lemma 4.5, $f_3$ induces a bijection between the components of $V[k_u]$ and those of $T[k_u]$. Moreover, the image of $M[k_2]$ is disjoint from $T[k_u]$ by the condition (5). Again by Lemma 4.5 no tubes in $V[k_2] \setminus V[k_u]$ are mapped to $T[k_u]$. Therefore $f_3$ induces a Lipschitz map $f_3 : M[k_u] \to N[k_u]$.

**Proposition 4.8.** The Lipschitz map $f_3 : M[k_u] \to N[k_u]$ can be properly homotoped to a homeomorphism $f_4 : M[k_u] \to N[k_u]$, which extends to a homeomorphism between $M$ and $N_0$. This map $f_4$ may not be Lipschitz.

**Proof.** Let $B$ be a brick of $M_{\text{int}}$. We denote by $F_1^+, \ldots, F_\mu^+$ its joints contained in the upper front, and by $F_1^-, \ldots, F_\mu^-$ those contained in the lower front. (One of the fronts may be ideal; hence one of these families may be empty.) We consider $F_1^+[k_u], \ldots, F_\mu^+[k_u]$ and $F_1^-[k_u], \ldots, F_\mu^-[k_u]$ as defined above, and denote their unions by $\bar{F}^+$ and $\bar{F}^-$. Note that both $\bar{F}^+$ and $\bar{F}^-$ are incompressible in $M$. By changing each joint $F$ to $\bar{F}$, we get a brick decomposition of $M$ which is isotopic to the original one. From now on until the end of the proof of this proposition, when we refer to a brick $B$, we mean the one in this new decomposition, which is isotopic to the original $B$. Let $p_B : M_B \to M$ be the covering associated to the image of $\pi_1(B)$ in $\pi_1(M)$. Similarly, we consider the covering $N_B$ of $N_0$ associated to $(f_3)_*\pi_1(B)$. Let $\hat{f}_3 : M_B \to N_B$ be the lift of $f_3$ which is uniformly Lipschitz, and $\check{f} : M_B \to N_B$ that of $f$, which is a homeomorphism. The surfaces $\bar{F}_+, \bar{F}_-$ lift homeomorphically to surfaces $\check{F}_+, \check{F}_-$ lying on the boundary of a submanifold $\hat{B}$ homeomorphic to $B$ under $p_B$. Let $\hat{V}[k_2]$ and $\check{T}[k_u]$ be the preimages of $V[k_2]$ and
T[ku] respectively. We denote by MB[ku] the complement of Int ˜V[ku] in MB, and by NB[ku] the complement of Int ˜T[ku] in NB.

Note that f3( ˜F+ ∪ ˜F−) is properly homotopic to f(( ˜F+ ∪ ˜F−)) which is an embedding. We can assume that f3(( ˜F+ ∪ ˜F−)) is an immersion from the start by perturbing it. Then, as was shown in Freedman-Hass-Scott [FHS], f3( ˜F+ ∪ ˜F−) can be properly homotoped to an embedding by a homotopy which passes through only relatively compact components of NB \ f3( ˜F+ ∪ ˜F−). We note that each of such relatively compact components is homeomorphic to a trivial interval bundle over an essential compact subsurface of F+ ∪ F−.

Suppose that a component W of NB \ f3( ˜F+ ∪ ˜F−) intersects a component T of T[ku]. This means that W contains T since f3( ˜F) is disjoint from T[ku]. We should first observe that at most one component of T[ku] can be contained in W. We shall now prove the following claim.

Claim 4.9. The surfaces f3( ˜F+ ) and f3( ˜F− ) are homotopic to disjoint embeddings by proper homotopies which do not touch T.

Proof. Because f3 satisfies the conditions (3), (5) and (7), there is a unique component V of ˜V[ku], which is a solid torus, such that f3(V) = T. Since MB[ku] is mapped to NB[ku] and V[ku] bijects to T[ku], we see that f3(MB \ V) ⊂ NB \ T. Since every surface Kleinian group is tame, the interior of NB is homeomorphic to ∂B × (0, 1), and hence so is MB. Let V1, . . . , Vp be all components of ∂M, which are open annuli or tori, whose longitudes or core curves are homotopic into ˆF+ j ⊔ · · · ⊔ ˆF+w in M \ IntB. We renumber them in such a way that V1, . . . , Vr are disjoint from B whereas Vr+1, . . . , Vp intersect B along annuli. By the annulus theorem and standard cut-and-paste technique, we see that there are disjoint embedded annuli α1, . . . , αr realizing homotopies between longitudes or core curves on V1, . . . , Vp and simple closed curves on ˆF+ 1 ⊔ · · · ⊔ ˆF+w with ∂αj ⊂ Vj ⊔ ˆF+ j ⊔ · · · ⊔ ˆF+w. We can lift V1, . . . , Vp and α1, . . . , αr to open annuli A1, . . . , Ap on ∂MB and annuli ˜α1, . . . , ˜αr such that ˜Aj and ˜αj intersect at a core curve of Aj for j = 1, . . . , r. Similarly, we consider components V ′1, . . . , V ′q of ∂M whose longitudes or core curves are homotopic into ˆF− 1 ⊔ · · · ⊔ ˆF−w in M \ IntB, among which V ′1, . . . , V ′r lie outside B, and take annuli α ′1, . . . , α ′r realizing homotopies between longitudes or core curves to ˆF− 1 ⊔ · · · ⊔ ˆF−w. We lift V ′1, . . . , V ′q to open annuli A ′1, . . . , A ′q and α ′1, . . . , α ′r to ˜α ′1, . . . , ˜α ′r in the same way as before.

Let ˜A1, . . . , ˜Ap and ˜A ′1, . . . , ˜A ′q be core annuli of A1, . . . , Ap and A ′1, . . . , A ′q such that ˜Aj contains αj ∩ Aj for j ≤ r whereas ˜Aj = B ∩ Aj for j > r, and ˜A ′j contains α ′j ∩ A ′j for j ≤ s whereas ˜A ′j = B ∩ A ′j for j > s. By identifying ∂−B and ∂+B by vertical translation, we regard ˜A1, . . . , ˜Ap as also lying on ∂−B. To construct parts corresponding to the Z-cusps in ∂−B × (0, 1), we set U+ = ( ˜A1 ∪ · · · ∪ ˜Ar) × (7/8, 1), U− = ( ˜A ′r+1 ∪ · · · ∪ ˜Ap) × (3/4, 1), U+ = ( ˜A ′r+1 ∪ · · · ∪ ˜Ap) × (0, 1/4) and denote the union of these four parts by U. We parametrise MB by a proper homeomorphism IM : MB → ∂−B × R \ U, in such a way that ˜F− is identified with ∂−B × {1/4} \ IntU− whereas ˜F+ is identified with ∂+B × {3/4} \ IntU+.

Similarly, we parametrise NB by a homeomorphism IN : NB → ∂−B × (0, 1) \ U in such a way that IN( ˜f3(B)) lies in ∂−B × [1/4, 3/4] and IN(W) lies in ∂−B × [1/8, 7/8]. Note that each component of ∂U corresponds to the boundary of a
\textit{Z}-cusp neighbourhood of \(N_B\). Since \(N_B\) is the covering of the non-cuspidal part \(N_0\), we can extend \(N_B\) to a hyperbolic 3-manifold \(N_B\) which is the covering of \(N\) associated to \(\pi_1(B)\) by attaching cusp neighbourhoods. The parametrisation \(I_N\) extends to a homeomorphism \(\hat{I}_N : N_B \to \partial_-B \times (0, 1)\).

Since both \(\hat{F}_-\) and \(\hat{F}_+\) are disjoint from \(\hat{V}[k_u]\), the solid torus \(I_M(V)\) is contained in either \(\partial_-B \times (0, 1/4)\) or \(\partial_-B \times (1/4, 3/4)\) or \(\partial_-B \times (3/4, 1)\). We shall first consider the case when \(I_M(V)\) is contained in \(\partial_-B \times (0, 1)\). Take a sufficiently small number \(t_0\) so that both \(\hat{I}_N^{-1}(\partial_-B \times (1 - s_0, 1))\) and \(\hat{I}_N^{-1}(\partial_-B \times (0, s_0))\) are disjoint from the 1-neighbourhood of \(W\). Since \(f_3\) is proper and has degree 1, for sufficiently small \(t_0 > 0\), the surfaces \(I_N \circ \hat{f}_3 \circ I_M^{-1}(\partial_-B \times \{t_0\} \setminus U)\) and \(I_N \circ \hat{f}_3 \circ I_M^{-1}(\partial_-B \times \{1 - t_0\} \setminus U)\) are contained in \(\partial_-B \times (0, s_0)\) and \(\partial_-B \times (1 - s_0, 1)\) respectively. Denote \(I_M^{-1}(F_+ \cup \{1 - t_0\} \setminus U)\) by \(F'_+\), and \(I_M^{-1}(S \times \{t_0\} \setminus U)\) by \(F'_-\).

We can enlarge \(F'_-\) and \(F'_+\) to surfaces \(\tilde{F}_-\) and \(\tilde{F}_+\) homeomorphic to \(\tilde{F}_-\) and \(\tilde{F}_+\) respectively by joining pairs of parallel boundary components of \(F'_-\) lying on \(\partial U_-\) by annuli on \(\partial U_-\) bounded by them, and those of \(F'_+\) lying on \(\partial U_+\) bounded by them. On the other hand, since \(\hat{f}_3(F'_-)\) and \(\hat{f}_3(F'_+)\) are disjoint from the 1-neighbourhood of \(W\), we can enlarge \(f_3(F_-)\) and \(f_3(F_+)\) by joining each pair of parallel boundary component on \(I_N \circ \hat{f}_3(\partial U_- \cup \partial U_+) \subseteq N_0\) by an annulus embedded in the closure of an \(\varepsilon\)-cusp neighbourhood which is a component of \(N \setminus \text{Int}N_0\) so that their images under \(\hat{I}_N\) are contained in \(\partial_-B \times (0, s_0)\) and \(\partial_-B \times (1 - s_0, 1)\) respectively. These surfaces, which are homeomorphic to \(\tilde{F}_-\) and \(\tilde{F}_+\), are homotopic to embeddings \(\tilde{F}_-\) and \(\tilde{F}_+\) respectively outside the 1-neighbourhood of \(W\) by our choice of \(s_0\). Then by our choice of \(t_0\), we see that \(\hat{f}_3(F_-)\) and \(\hat{f}_3(F_+)\) are homotopic to \(\tilde{F}_-\) and \(\tilde{F}_+\) respectively by homotopies disjoint from \(W \supset T\).

Since \(\hat{f}_3(F_-)\) is homotopic to \(\tilde{f}_3(F'_-)\) outside \(T\) and \(\hat{f}_3(F_+)\) is homotopic to \(\tilde{f}_3(F'_+)\) outside \(T\) (for \(\hat{f}_3(MB \setminus V) \subset N_B \setminus T\)), the surfaces \(\tilde{f}_3(F_-)\) and \(\tilde{f}_3(F_+)\) are homotopic to disjoint embeddings by a homotopy disjoint from \(T\).

Next suppose that \(I_M(V)\) is contained in \(\partial_-B \times (0, 1/4)\). In this case, we shall consider to move both \(\tilde{F}_-\) and \(\tilde{F}_+\) in the + -direction. As in the previous case, there are sufficiently small \(s_0, t_0 > 0\) such that \(\hat{I}_N^{-1}(\partial_-B \times (1 - s_0, 1))\) is disjoint from the 1-neighbourhood of \(W\), and such that \(I_N \circ \hat{f}_3 \circ I_M^{-1}(\partial_-B \times \{1 - t_0\})\) is contained in \(\partial_-B \times (1 - s_0, 1)\). Then, by the same argument as in the previous case, we can see that both \(\tilde{f}_3(\tilde{F}_-)\) and \(\tilde{f}_3(\tilde{F}_+)\) are homotopic to an embedding contained in \(\hat{I}_N^{-1}(\partial_-B \times (1 - s_0, 1))\) by a homotopy outside \(T\). They can be homotoped to disjoint embeddings just by considering parallel copies of the embedding. Thus we are done also in this case. The argument for the case when \(I_M(V)\) is contained in \(\partial_-B \times (3/4, 1)\) is the same way just by changing the + -direction to the −-direction.

The above claim says that a homotopy from \(f_3(\tilde{F}_- \cup \tilde{F}_-)\) can be taken to be disjoint from \(W\) since any homotopy passing through \(W\) must intersect \(T\). We can repeat the same argument for every relatively compact component of \(N_0 \setminus \hat{f}_3(\tilde{F}_+ \cup \tilde{F}_-)\) containing a component of \(\hat{T}[k_u]\) and show that \(\tilde{f}_3(\tilde{F}_+ \cup \tilde{F}_-)\) can be homotoped to an embedding by a homotopy within \(N_B[k_u]\).

Now, we consider a new hyperbolic metric \(m_N\) on \(\text{Int}N[k_u]\) which makes every component of \(T[k_u]\) a torus cusp preserving the original cusps of \(N_0\). Pull back this metric to \(\text{Int}N_B[k_u]\) and denote it by \(m_B\). We consider a least area map
$h_3 : \tilde{F}_- \sqcup \tilde{F}_+ \to (\text{Int} N_B[k_u], m_B)$ homotopic to $\tilde{f}_3|\tilde{F}_- \sqcup \tilde{F}_+$. By the main result of Freedman-Hass-Scott [FHS], $h_3$ is an embedding.

In the following argument, we shall use the notion of topological order due to Brock-Minsky [BCM] which we explained in Definition 4.7.

Claim 4.10. Let $B$ be a brick in $M_{int}$ neither of whose fronts lies on the boundary of $M_{int}$. Then the embedding $h_3$ can be extended to an orientation-preserving embedding of $\tilde{B} \cap \text{Int} M_B[k_u]$ to $(\text{Int} N_B[k_u], m_B)$ taking $\tilde{B} \cap \tilde{V}[k_u]$ to cusps corresponding to $\partial_{\text{top}} \tilde{V}[k_u]$ and the homotopy classes of meridians of tube components of $\tilde{B} \cap \tilde{V}[k_u]$ to those of $\tilde{T}[k_u]$.

Proof. Recall that there is a homeomorphism $I_N : N_B \to \partial_{\text{top}} B \times (0,1) \setminus U$. By our definition of $k_u$, the images of $\tilde{T}[k_u]$ under $I_N$ are unknotted and unlinked in $\partial_{\text{top}} B \times (0,1)$. Since the ends of $h_3(\tilde{F}_- \sqcup \tilde{F}_+)$ other than those tending to cusps of $N_B$ tend to $I_N(\partial T[k_u])$, the surfaces $I_N \circ h_3(\tilde{F}_- \sqcup \tilde{F}_+)$ together with annuli on $I_N(\partial T[k_u])$ bound a submanifold homeomorphic to $\partial_{\text{top}} B \times [1/4, 3/4] \cong B$. We shall first prove that $I_N \circ h_3(\tilde{F}_-)$ is situated above $I_N \circ h_3(\tilde{F}_+)$. This is trivial when one of $\tilde{F}_+$ and $\tilde{F}_-$ is empty. Since we assumed that neither $\partial_{\text{top}} B$ nor $\partial_{\text{top}} B$ lie on the boundary of $M_{int}$, both $\partial_{\text{top}} B \cap \partial M_B$ and $\partial_{\text{top}} B \cap \partial M_B$ are non-empty.

By Assumption 5.2, every component of $\partial_{\text{top}} B \cap \partial M_B$ overlaps some component of $\partial_{\text{top}} B \cap \partial M_B$. Therefore, we can take components $X$ and $X'$ of $\tilde{V}[k_u]$ on which boundary components of $\tilde{F}_+$ and of $\tilde{F}_-$ lie respectively such that $X \cap \tilde{B}$ and $X' \cap \tilde{B}$ overlap. It follows that we have $X \cap \tilde{B} \prec_{\text{top}} X' \cap \tilde{B}$. Since $\tilde{f}_3$ is a proper degree 1-map and $\tilde{f}_3|\tilde{V}[k_u]$ is a homeomorphism to its image, this implies that $\tilde{f}_3(X \cap \tilde{B}) \prec_{\text{top}} \tilde{f}_3(X \cap \tilde{B})$. On the other hand, if $I_N \circ h_3(\tilde{F}_+)$ is situated under $I_N \circ h_3(\tilde{F}_-)$, then we should have $\tilde{f}_3(X' \cap \tilde{B}) \prec_{\text{top}} \tilde{f}_3(X \cap \tilde{B})$, which is a contradiction. Thus we have proved that $I_N \circ h_3(\tilde{F}_+)$ is situated above $I_N \circ h_3(\tilde{F}_-)$ and $h_3$ extends to an orientation-preserving homeomorphism from $\tilde{B}$ to a submanifold $B_N$ bounded by $h_3(\tilde{F}_- \sqcup \tilde{F}_+)$. We shall next show that this homeomorphism induces that between $\tilde{B} \cap M_B[k_u]$ to $B_N \cap N_B[k_u]$. For that, it suffices to show that for the components of $\tilde{V}[k_u]$ in $B$, the corresponding components of $\tilde{T}[k_u]$ are contained in $B_N$ preserving the topological order since all such components in $B_N$ are unknotted and unlinked. Let $V$ be a component of $\tilde{V}[k_u]$ contained in $\tilde{B}$. Then we have $\tilde{F}_- \prec_{\text{top}} V \prec_{\text{top}} \tilde{F}_+$. Let $T$ be a component of $\tilde{T}[k_u]$ with $T = \tilde{f}_3(V)$. Since $\tilde{f}_3$ is a proper degree 1-map and takes $M_B \setminus V$ to $N_B \setminus T$, we see that $\tilde{f}_3(\tilde{F}_-) \prec_{\text{top}} T \prec_{\text{top}} \tilde{f}_3(\tilde{F}_+)$. Since $h_3$ is homotopic to $f_3|\tilde{F}_- \cup \tilde{F}_+$ in $N_B[k_u]$, we also have $h_3(\tilde{F}_-) \prec_{\text{top}} T \prec_{\text{top}} h_3(\tilde{F}_+)$. Therefore any tube component of $\tilde{V}[k_u]$ in $\tilde{B}$ has its corresponding Margulis tube in $B_N$. Now suppose we have two such tube components $V_1, V_2$ with $V_1 \prec_{\text{top}} V_2$. Let $T_1, T_2$ be the components of $\tilde{T}[k_u]$ with $\tilde{f}_3(V_1) = T_1$ and $\tilde{f}_3(V_2) = T_2$. Then by the same argument as above using the bijective correspondence between the components of $\tilde{V}[k_u]$ and $\tilde{T}[k_u]$, we have $T_1 \prec_{\text{top}} T_2$. Thus we have shown that we have a homeomorphism $h_3$ from $\tilde{B} \cap M_B[k_u]$ to $B_N \cap N_B[k_u]$ which is an extension of $h_3$.

It remains to show that a meridian of a solid torus component of $\tilde{V}[k_u]$ contained in $\tilde{B}$ is taken to a meridian of $\tilde{T}[k_u]$. This is rather obvious from our construction: for $\tilde{f}_3$ takes meridians of solid torus components of $\tilde{V}[k_u]$ to those of $\tilde{T}[k_u]$. □
Now, for each brick $B$ of $K_{\text{int}}$ neither of whose fronts lies on the boundary of $M_{\text{int}}$, we consider $B \cap \text{Int} M[k_u]$, its lift $\tilde{B}$ in $\text{Int} M_B[k_u]$, and its embedding of $\text{Int} N_B[k_u]$ by an extension of $h_3$ as above, which we denote by $B_N$ as above. We denote the map taking $B \cap \text{Int} M[k_u]$ to $B_N$ in this way by $f_B$. We regard $B_N$ as a hyperbolic 3-manifold with boundary by restricting the metric $m_B$, and call $B_N$ with this metric the least-area realisation of $B$. When $B$ is a brick one of whose front lies on the boundary of $M_{\text{int}}$, we define $B_N$ to be a submanifold of $\text{Int} N_B[k_u]$ homeomorphic to $\partial_- B \times (0,1)$ obtained by cutting $\text{Int} N_B[k_u]$ along the embedding of one of the boundary components of $\tilde{B}$ whose projection in $M$ does not lie on the boundary of $M_{\text{int}}$, which is defined using the least area map in the same way as above.

Suppose that two bricks $B^1$ and $B^2$ share a joint $F$. We can assume $F$ is a component of $\partial_+ B^1$ and $\partial_- B^2$ by interchanging $B^1$ and $B^2$ if necessary. Construct the least-area realisations $B_N^1$ and $B_N^2$ as above. Then both of their boundaries contain a least area surface corresponding to $F$ as components. We denote by $B_j$ the one contained in $\partial B_N^j$ for $j = 1, 2$. Since the projections of $F^1$ and $F^2$ in $(\text{Int} N[k_u], m_N)$ are least-area surfaces homotopic to $f_3(F)$ (which might not be embeddings), they must coincide. Therefore, $F^1$ is isometric to $F^2$. Then we can consider the hyperbolic 3-manifold homeomorphic to $(\text{Int} B^1 \cup \text{Int} B^2 \cup F) \cap \text{Int} M[k_u]$ by pasting $B_N^1$ and $B_N^2$ along $F^1$ and $F^2$ by an isometry.

Repeating this procedure for every joint on $B^1$ and $B^2$, then again for all the bricks, we get a hyperbolic 3-manifold $N'[k_u]$ homeomorphic to $M[k_u]$. We denote the homeomorphism obtained by identifying $B \cap M[k_u]$ with $B_N$ in $N'[k_u]$ by $h : M[k_u] \to N'[k_u]$. We shall show that this manifold is isometric to $N[k_u]$.

**Claim 4.11.** There is an isometry $f' : N'[k_u] \to N[k_u]$, whose restriction to $B_N$ for each brick $B$ is an isometric embedding homotopic to $f_3 \circ f_B^{-1}$ in $N_0$.

**Proof.** For each brick $B$, by Claim 4.10 there is an embedding $h_3 : \tilde{B} \cap \text{Int} M_B[k_u] \to N_B[k_u]$ homotopic to $f_B(B \cap M_B[k_u])$. If we lift $f_B^{-1}(B_N)$ to $M_B[k_u]$, and embed it by $h_3$ into $N_B[k_u]$, then the map is isometric by our definition of the metric on $N'[k_u]$. By projecting it to $N[k_u]$, we get a locally isometric map from $B_N$ into $N[k_u]$. Since for two bricks sharing a joint, such maps induce the same map on the joint, we can glue this map at joints and get a local isometry $f' : N'[k_u] \to N[k_u]$.

(Note that if two bricks share a joint, then their images by $h_3$ lie on the opposite sides of the image of the joint by our way of extending $h_3$ in Claim 4.10 which guarantees that the map is also local isometry at joints.) Since $h_3$ is homotopic to $f_3|B$, we see that $f' \circ h$ is homotopic to $f_3$.

Since $f'$ induces an isomorphism between the fundamental group, to show that it is an isometry, it is sufficient to show that $f'$ is proper. Suppose, seeking a contradiction, that $f'$ is not proper. Then, there exists a sequence of distinct bricks $B^i$ and points $x_i \in B_N^i$ such that $f'(x_i)$ converges in $N[k_u]$. Since $f'(x_i)$ converges, the injectivity radius at $f'(x_i)$ is bounded below by a positive constant independent of $i$, hence so is the injectivity radius at $x_i$. We divide our argument depending on the distance between $x_i$ and $\partial B_N^i$ is bounded or not.

First we consider the case when the distance from $x_i$ and $\partial B_N^i$ is bounded as $i \to \infty$. Let $F^i$ be a least-area boundary component of $B_N^i$ from which $x_i$ is within uniformly bounded distance. Since $\xi(F^i) \leq \xi(S)$, the diameter of the thick part of $F^i$ is uniformly bounded. Since $x_i$ lies in the thick part, it is within uniformly
bounded distance from either an \( \varepsilon \)-Margulis tube or an \( \varepsilon \)-cusp neighbourhood touching \( F^i \) which corresponds to a component \( \tilde{V}^i \) of \( \tilde{T}[k_u] \). We denote by \( V^i \) a component of \( T[k_u] \) which is the projection of \( \tilde{V}^i \).

We can show that in \((\text{Int} N[k_u], m_N)\), for each component \( V \) of \( T[k_u] \) there are only finitely many images of joints by \( f \) touching \( V \) as follows. For any \( R > 0 \), there is a finitely many Margulis tubes and cusp neighbourhoods which can be reached from \( V \) within the distance \( R \) modulo the \( \varepsilon_0 \)-thin part. Since joints are subsurfaces of \( S \), there are only finitely many possibilities for the boundaries of their images in \( N[k_u] \). This implies there are only finitely many joints up to homotopy whose images can touch \( V \) since there are at most two kinds of homotopy classes of horizontal surfaces if we fix a boundary. Since no two distinct joints are homotopic as we removed inessential joints, it follows that there are only finitely many joints whose images touch \( V \).

Since our joints \( F^i \) are all distinct, we can assume that all the \( V^i \) are distinct by taking a subsequence. Since \( f_3 \) takes the components of \( V[k_u] \) to those of \( T[k_u] \) one-to-one, and no other part is mapped to \( T[k_u] \), we see that \( f' \) takes the \( V^i \) to distinct components of \( T[k_u] \). Therefore \( f'(x_i) \) is within bounded distance from infinitely many distinct components of \( T[k_u] \). Since the \( f'(x_i) \) are assumed to converge, this contradicts the fact that there are only finitely many components of \( T[k_u] \) within a bounded distance.

Thus, it only remains to consider the case when the distance from \( x_i \) to the boundary of \( B'_N \) goes to \( \infty \) as \( i \to \infty \). Recall that \( B'_N \) was originally a submanifold in \( N_B'[k_u] \). Therefore, we can regard \( x_i \) also as a point in \( N_B'[k_u] \). Since \( B'_N \) is bounded by least-area surfaces, it is contained in the convex core of \((\text{Int} N_B'[k_u], m_{B'})\). Therefore, there is a pleated surface \( k_i : \partial_- B^i \to N_{B'}[k_u] \) which is within bounded distance from \( x_i \) and is homotopic to the inclusion of \( \partial_- B^i \) as \( \partial_- B^i \times \{t\} \) with respect the parametrisation \( N_{B'} \cong \partial_- B^i \times (0, 1) \). Since the distance from \( x_i \) to \( \partial B'_N \) goes to \( \infty \), we can assume that the image of \( k_i \) is contained in \( B'_N \). Hence we can regard \( k_i \) as a pleated surface in \( N'[k_u] \). Also since the cuspidal part of \( N'[k_u] \) consists of those of \( N_0 \) and rank-2 cusps corresponding to \( T[k_u] \), we can take cusp neighbourhoods which are disjoint from all the images of \( k_i \).

We consider the pleated surfaces \( f' \circ k_i \). Since \( f'(x_i) \) converges and \( f' \circ k_i \) is disjoint from the cusp neighbourhoods which are images of those taken above, the pleated surface \( f' \circ k_i \) converges geometrically, passing to a subsequence. This implies in particular that there are distinct \( i_1, i_2 \) such that \( f' \circ k_{i_1} \) and \( f' \circ k_{i_2} \) are properly homotopic. Since \( f' \) induces an isomorphism between the fundamental groups, it follows that \( k_{i_1} \) and \( k_{i_2} \) are properly homotopic. This is a contradiction since no two horizontal surfaces of distinct bricks are properly homotopic. (Recall that \( N'[k_u] \) and \( M[k_u] \) are homeomorphic.) Thus we have established that \( f' \) is an isometry. By our construction, it is evident that \( f'|B_N \) is homotopic to \( f_3 \circ f_B^{-1} \) in \( N_0 \).

Thus \( N[k_u] \) is isometric to \( N'[k_u] \) which is the union of \( B_N \) each of which is homeomorphic to \( B \cap M[k_u] \) in \( M[k_u] \). This shows that there is a homeomorphism \( h : M[k_u] \to N'[k_u] \) such that \( f'^{\circ} \circ h \) is homotopic to \( f_3|M[k_u] \). By setting \( f_4 = f'^{\circ} \circ h \), we get a homeomorphism as we wanted.

It only remains to show that \( f_4 \) extends to a homeomorphism between \( M \) and \( N_0 \). To show this, it suffices to show that for each component \( V \) of \( V[k_u] \), its meridian is
sent to a meridian of a component of $T[k_n]$. If $V$ is contained in some brick $B$, then this follows from Claim 4.10. Since we isotoped the original brick decomposition to a new one by moving each joint $F$ to $\tilde{F}$, we see that every component of $V[k_n]$ is contained in some brick.

This completes the proof of Proposition 4.8. \qed

Having proved that $M[k_n]$ and $N[k_n]$ are homeomorphic, we shall next show that the Lipschitz map $f_3$ be homotoped to embed the joints preserving the Lipschitzity. For that, it is more convenient to consider a brick decomposition of $M[k_n]$ rather than that of $M$. As in [3.4] we define a brick of $M[k_n]$ to be a maximal union of vertically parallel horizontal leaves which are inherited from the horizontal foliation of $M$. By the same argument as in [3.4] we can check the conditions A-(1)-(5) are satisfied. (In reality, only A-(2) and A-(3) need to be checked.) We denote this brick decomposition of $M[k_n]$ by $K[k_n]$.

Before that, we shall first move $f_3$ so that it preserves the order of joints on the boundary except for parallel ones. Let $F$ be the union of joints of pairs of bricks in $\mathcal{K}[k_n]$. We introduce an equivalence relation $\sim$ in the set of components of $F$ such that $F_1 \sim F_2$ if they are parallel. By our definition of brick decomposition, there are not three distinct joints in $F$ which are all parallel. Therefore each equivalence class consists of at most two joints. We define the reduced union of joints to be the union of joints taken one from each equivalence class, and denote it by $\hat{F}$.

**Lemma 4.12.** There is a uniform constant $K'_3$ as follows. We can homotope $f_3$ to a proper, degree-1 map $f'_3 : M[k_n] \to N[k_n]$ with the following properties.

(i) $f'_3$ coincides with $f_3$ outside small pairwise disjoint neighbourhoods of $\partial M[k_n]$.

(ii) $f'_3$ is $K'_3$-Lipschitz.

(iii) On each component $T$ of $\partial M[k_n]$, distinct components of $F \cap T$ have disjoint images under $f'_3$.

(iv) On each component $T$ of $\partial M[k_n]$, the restriction $f'_3|T$ maps the components of $\hat{F} \cap T$ disjointly preserving the orientation of $\hat{F} \cap T$ and the order of $\{F \cap T \mid F$ is a component of $\hat{F}\}$. (When $T$ is a torus the order means the cyclic order.)

(v) For a component $F$ of $\hat{F}$, let $\hat{F}$ be a component of $\hat{F}$ equivalent to $F$. Then $f'_3$ also preserves the order of $((\hat{F} \setminus \hat{F}) \cup F) \cap T$ for any component $F$ of $\hat{F} \setminus \hat{F}$.

(vi) The order of $F \cap T$ and $\hat{F} \cap T$ may be reversed only when $f'_3(F) \cap f'_3(\hat{F}) = \emptyset$.

(vii) For each small $\delta > 0$, there is an universal number $n_0$ such that for any component $F$ of $\hat{F}$, there are at most $n_0$ joints $F_i$ such that $f'_3(F_i \cap T)$ are within the distance $\delta$ from $f'_3(F \cap T)$.

**Proof.** Let $T$ be a component of $\partial M[k_n]$, which is either a torus or an open annulus. As was shown before, $T$ consists of horizontal annuli and vertical annuli, and the joints intersect only vertical annuli. We shall show that we can homotope $f_3|T$ to a uniformly Lipschitz map by a homotopy moving each point at a uniformly bounded distance. We should note that $f_3|T$ is a degree-1 map to a boundary component $T'$ of $N[k_n]$. The foliation of $M$ by horizontal leaves induces a foliation on $T$ whose leaves are parallel horizontal circles. By our definition of the model metric, each leaf has length $\epsilon_1$. We can extend this foliation also to horizontal annuli so that they are also foliated by parallel circles with length $\epsilon_1$. We let $\gamma$ be a simple closed
geodesic with respect to the induced metric intersecting each leaf at one point when $T$ is a torus, and a geodesic ray intersecting each leaf at one point when $T$ is an open annulus.

Since $f_3$ is $K_3$-Lipschitz, the homotopy classes in $T'$ of the images of the leaves have (Euclidean) geodesic lengths bounded by $K_3\varepsilon_1$. We also note the their lengths are also bounded by $\varepsilon_1$ since $T'$ lies on the boundary of an $\varepsilon_1$-Margulis tube. We first homotope $f_3|A$ to $\tilde{f}_3|\gamma$ so that for each leaf $l$ of the foliation on $A$, the simple closed curve $\tilde{f}_3(l)$ is a closed geodesic with respect to the Euclidean metric on $T'$. If there are distinct components of $\mathcal{F} \cap \partial T$ which have the same images, we can perturb them to be disjoint moving them within the very small distances. We can take a homotopy $H_3 : A \times [0,1] \to T'$ from $f_3$ to $\tilde{f}_3$ as a $K_3$-Lipschitz map, where $K_3$ depends only on $\varepsilon_1$ and $K\varepsilon_1$, since the length of each closed curve $f_3(l)$ is between $\varepsilon_1$ and $K\varepsilon_1$ and the perturbation moves the images within uniformly bounded distances.

Now, the map from $\gamma$ to $f_3(\gamma) = \tilde{f}_3(\gamma)$ may not proceed in the positive direction monotonously. (As we shall see below, since $f_3|T$ has degree 1, the orientations of $T$ and $T'$ determine the positive direction to which $f_3(\gamma)$ should proceed.) This may cause a permutation of the order of $T \cap T'$ by $\tilde{f}_3$. We fix an orientation of the foliation on $T$, which, together with the orientation of $T$, induces a transverse orientation of the leaves and an orientation of $\gamma$. This also defines a transverse orientation of the foliation on $T'$ induced by the closed geodesics which are images of the leaves on $T$, since $f_3|T'$ has degree 1. We number the simple closed curves in $\mathcal{F}\cap T$ as $F_1, F_2, \ldots$ in accordance with the order determined by the orientation of $\gamma$. In the case when $T$ is a torus, we fix a leaf on the lower horizontal annulus, and let its intersection with $\gamma$, which we denote by $a_0$, be the starting point. The transverse orientation of the foliation on $T'$ gives an order on the images $\tilde{f}_3(F_1 \cap T), \ldots$, which may be different from the order on $T$. (We allow some of them go beyond $\tilde{f}_3(\{a_0\})$ in the negative direction. As long as $\tilde{f}_3(\gamma)$ moves in the negative direction, we regard it as receding with respect to the order on $T'$.)

Let $\sigma$ be a permutation such that $\tilde{f}_3(F_\sigma(1)), \ldots$ is the right order on $T'$, in other words $F_i$ is mapped to the $\sigma^{-1}(i)$-th curve in the order on $T'$. Now, we first look at $\tilde{f}_3(F_i \cap T)$ which is the $\sigma^{-1}(i)$-th curve on $T'$, and consider the curves $\tilde{f}_3(F_\sigma(1) \cap T), \ldots, \tilde{f}_3(F_\sigma(\sigma^{-1}(1)-1) \cap T)$ which are those situated before $\tilde{f}_3(F_i \cap T)$ on $T'$. Set $j = \max\{\sigma(1), \ldots, \sigma^{-1}(1)-1\}$. We shall consider to move $\tilde{f}_3(F_1 \cap T), \ldots, \tilde{f}_3(F_j \cap T)$ to correct their order. The point is that this can be done by a homotopy with bounded displacements.

Using the theory of Freedman-Hass-Scott [FHS], we shall bound uniformly the distance between any two of $\tilde{f}_3(F_1 \cap T), \ldots, \tilde{f}_3(F_j \cap T)$. Let $k$ be a number among $2, \ldots, j$. First consider the case when $\tilde{f}_3(F_k \cap T)$ comes before $\tilde{f}_3(F_i \cap T)$ on $T'$. Recall that $f_3$ is homotopic in $M[ku]$ to a homeomorphism $f_4 : M[ku] \to N[ku]$. By the same procedure as we used to construct $f_3$ from $f$, we can assume that $f_4$ also maps each leaf on $T$ to a closed geodesic with respect to the induced Euclidean metric on $T'$. Then, since both $F_1$ and $F_k$ are incompressible, by the theory of Freedman-Hass-Scott, we can homotope $\tilde{f}_3|F_1$ and $\tilde{f}_3|F_k$ fixing the boundary to embeddings $g_3^1$ and $g_3^k$ in $N[ku]$ which are contained in small regular neighbourhoods of $\tilde{f}_3(F_1)$ and $\tilde{f}_3(F_k)$ respectively. By perturbing $g_3^1$ and $g_3^k$, we can assume that they are transverse to $f_3(F_1)$ and $f_3(F_k)$ at their interiors. Then $g_3^1(F_1) \cup F_3(F_k) \cap T'$ bounds an annulus $A'_1$ which may degenerate to a circle. When $T$ is a torus, there are two choices for $A'_1$. We choose one which bounds a compact region.
with $g^4_1(F_1)$ and $f_4(F_1)$ which is disjoint from $\partial N[k_a] \setminus T'$. Similarly, we define an annulus $A'_k$ for $g^k_3(F_k)$ and $f_3(F_k)$. Since $g^4_1(F_1 \cap T)$ comes after $g^k_3(F_k \cap T)$ whereas $f_4(F_1 \cap T)$ is situated before $f_4(F_k \cap T)$, we see that $A'_1$ and $A'_k$ must intersect. Since $f_4(F_1) \cap f_4(F_k) = \emptyset$, both $F_1$ and $F_k$ are connected, and $F_1$ and $F_k$ are not parallel by our definition of $\hat{F}$, we see that $g^4_1(F_1)$ and $g^k_3(F_k)$ must intersect at their interiors. By our construction of $g^4_1$ and $g^k_3$, this implies that $f_3(F_1)$ and $f_3(F_k)$ also intersect at their interiors. Next suppose that $f_3(F_k \cap T)$ comes after $f_3(F_1 \cap T)$. By our definition of $j$, we see that $f_3(F_j \cap T)$ comes before $f_3(F_1 \cap T)$, hence also before $f_3(F_k \cap T)$. Since $k < j$, the order of $F_j \cap T$ and $F_k \cap T$ is reversed under $f_3$, and we can argue in the same way as above to conclude that $f_3(F_j)$ and $f_3(F_k)$ intersect at their interiors.

Recall that the diameters of $F_1$, ..., are uniformly bounded from above by a constant depending only on $\xi(S)$. Since $f_3$ is uniformly Lipschitz, their images $f_3(F_1)$, ..., also have diameters bounded from above by a constant $\lambda$ depending only on $\xi(S)$. This implies that for any $k = 2, \ldots, j$, the distance between $f_3(F_k \cap T)$ and either $f_3(F_1 \cap T)$ or $f_3(F_1 \cap T)$ is bounded by $2\lambda$. Therefore the distance between any two of $f_3(F_1 \cap T)$, ..., $f_3(F_j \cap T)$ is bounded by $4\lambda$. Recall that $f_3(F_1 \cap T)$, ..., $f_3(F_j \cap T)$ are parallel closed geodesics on $T'$ with respect to the induced Euclidean metric. By the uniform quasi-convexity of horoballs, we see that there is a number $\lambda_0$ depending only on $\chi(S)$ which bounds the distance between any two of $f_3(F_1 \cap T)$, ..., $f_3(F_j \cap T)$ on $T'$. Then we can homotope $f_3|T$ so that $f_3(F_1 \cap T)$, ..., $f_3(F_j \cap T)$ lie in the right order on $T'$ and near the original position of $f_3(F_{\sigma(1)} \cap T)$ so that all $f_3(F_i \cap T)$ with $i > j$ come after them, without changing the condition that every leaf is mapped to a closed geodesic preserved, by moving the image by $f_3$ of thin neighbourhoods of $F_1 \cap T$, ..., $F_j \cap T$ only at the distance at most $\lambda_0 + 1$. The map which we get after this homotopy is also uniformly Lipschitz since the displacements of points by the homotopy are uniformly bounded.

We now forget $F_1$, ..., $F_j$ and only consider $F_{j+1}$, ..., $F_{j_0}$. If $\sigma(j_0 + 1) = j_0 + 1$, we also forget $F_{j_0 + 1}$ and proceed to the first $j_0 > j$ with $\sigma(j_0) \neq j_0$. Regarding $f_3(F_{j_0} \cap T)$ instead of $f_3(F_1 \cap T)$ as the first one, we repeat the same argument. Then we can correct the order of $f_3(F_{j_0} \cap T)$, ..., $f_3(F_{j_1} \cap T)$ for some $j_1 > j_0$ and make them come after $f_3(F_{j_0 + 1})$ by moving $f_3$ in thin neighbourhoods of $F_{j_0} \cap T$, ..., $F_{j_1} \cap T$ only at the distance less than $\lambda_0 + 1$. We note that we do not touch the components $F_k \cap T$ with $k < j_0$ at this stage. We repeat the same process, and eventually we can homotope $f_3|T$ to a uniformly Lipschitz map $f^T_3 : T \to T'$ which preserves the order of $F_1 \cap T$, ..., by a homotopy moving every point within the distance $\lambda_0 + 1$.

(To be more precise, we need to define the homotopy inductively in the case when there are infinitely many components of $\hat{F} \cap T$.)

Having moved $f_3|T$ to $f^T_3$ which preserves the order of $\hat{F} \cap T$, we now turn to consider a component $F$ of $\hat{F} \setminus \hat{F}$. Suppose that $f^T_3$ does not preserve the order of $\hat{F} \cap T$ with $\hat{F}$ replaced with $F$. Then for each component $F'$ of $\hat{F} \setminus \hat{F}$ such that the order between $F$ and $F'$ is reversed by $f^T_3$, we see that $F$ must intersect $F'$ by the same argument as above, and we can move $f_3$ in a thin neighbourhood of $F' \cap T$ within the distance $\lambda_0 + 1$ to correct the order. Moreover, in the same way as above, we can correct the order of the images $F \cap T$ and $\hat{F} \cap T$ under $f^T_3$ by moving $f^T_3$ within the distance $\lambda_0 + 1$ if $f_3(F)$ and $\hat{f}_3(\hat{F})$ intersect. Thus we have shown that if we construct a uniform Lipschitz map whose restriction to $T$ is $f^T_3$, then the conditions (iii), (iv) and (v) in the statement are satisfied. We denote a homotopy
on $T$ by $H'_3$. This homotopy $H'_3$ is uniformly Lipschitz since the homotopy only passes through uniformly Lipschitz maps and its displacement function is uniformly bounded.

We shall next show that thus homotoped $f'_3$ satisfies the condition (iv) (with $f'_3$ in the statement replaced by $f'_3$). Fix some component $F$ of $\mathcal{F}$ and we shall bound the number of components $F'$ such that $f'_3(F \cap T)$ and $f'_3(F' \cap T)$ are within the distance $\delta$. By our construction of brick decomposition of $M[k_n]$, there are at most two joints whose boundary lie on exactly the same components. Therefore, for any natural number $\nu$ there exists $n$ such that if there are $n$ distinct joints, then there are at least $\nu$ boundary components of $M[k_n]$ which these joints intersect. As was shown above, the image of each component of $\mathcal{F}$ under $f'_3$ is bounded by $\lambda$. Now, since there is a bound for the number of components of $\partial N[k_n]$ which can be reached from $F$ within the distance $\delta + 2\lambda$, we get $n_0$ bounding the number of components of $F \cap T$ whose images by $f'_3$ are within the distance $\delta$ from $f'_3(F \cap T)$.

We shall finally show that $f'_3$ as defined above can be extended to a uniform Lipschitz map $f'_3$. We can take $r > 0$ depending only on $\varepsilon_1$ and $\chi(S)$ such that the boundary components of $M[k_n]$ have product $r$-neighbourhoods in $M[k_n]$ which are pairwise disjoint. Let $\mathcal{N}_r(T)$ denote the $r$-neighbourhood in $M[k_n]$ of a boundary component $T$ of $M[k_n]$, and we parametrise $\mathcal{N}_r(T)$ by $T \times [0, r]$ so that $T \times \{t\}$ is at the distance $t$ from $T$. We modify $f_3$ only inside $\cup \mathcal{N}_r(T)$ to get $f'_3$. We first define $f'_3|T \times [2r/3, r]$ to be rescaled $f_3|\mathcal{N}_r(T)$ so that $f'_3|\partial V \times \{2r/3\}$ is naturally identified with $f_3|T$. Next we define $f'_3|T \times [r/3, 2r/3]$ to realise the homotopy $H_3$ so that $f'_3|T \times \{t\}$ corresponds to $H_3(\cdot, 2 - 3t/r)$. Finally we define $f'_3|T \times [0, r/3]$ to realise the homotopy $H'_3$, so that $f'_3|T \times \{t\}$ corresponds to $H'_3(\cdot, 1 - 3t/r)$. Since $H_3$ and $H'_3$ are uniformly Lipschitz, we see that there is a uniform constant $K'_3$ such that $f'_3$ is $K'_3$-Lipschitz.

**Lemma 4.13.** There exists a constant $K'$ depending only on $\varepsilon(S)$ as follows. Let $\mathcal{F}$ be the family of joints of pairs of bricks in $\mathcal{K}[k_n]$. Then, there exists a $K'$-Lipschitz homotopy $H : \mathcal{F} \times [0, 1] \to N[k_n]$ fixing the boundary of $\mathcal{F}$ as follows.

1. $H|\mathcal{F} \times \{0\}$ coincides with $f'_3|\mathcal{F}$.
2. $H(x, t) = f'_3(x)$ for every $x \in \partial M[k_n] \cap \mathcal{F}$.
3. $H|\mathcal{F} \times [1/2, 1]$ is a $C^2$-embedding.
4. For each component $F$ of $\mathcal{F}$, the restriction $H|F \times [1/2, 1]$ is $K'$-bi-Lipschitz.

**Proof.** Let $F$ be a component of $\mathcal{F}$. Since the lengths of core curves in $V[0] \setminus V[k_n]$ are bounded below by $\epsilon_u$ by the condition (5) in [11] and $F \setminus V$ consists of thrice-punctured spheres, the modulus of $F$ is uniformly bounded. By the condition (7) in [11] and our choice of $k_n$, we see that there is no essential closed curve with length less than $\epsilon_u$ in $N[k_n]$. This shows that the map $f'_3|F$ is a uniformly bi-Lipschitz map to its image. (We should note that $f'_3|F$ may not be injective. The bi-Lipschitzity here means that the metric on $F$ induced from $M[k_n]$ and the one induced from $N[k_n]$ by $f'_3$ are bi-Lipschitz equivalent.) We can approximate $f'_3|F$ by an immersion fixing the boundary and preserving the uniform bi-Lipschitzity. Now, by Proposition [18], $f'_3|\mathcal{F}$ is properly homotopic to an embedding in $N[k_n]$ (without fixing the boundary).

We shall first show that each component $F$ of $\mathcal{F}$ can be homotoped fixing the boundary to a uniformly bi-Lipschitz embedding. Suppose, seeking a contradiction, that this is not the case. Then there exist sequences of labelled brick manifolds $M'$,
homeomorphisms $f^i : M^i \to N^i$, Lipschitz maps $f_3^i : M^i[k_u] \to N^i[k_u]$ corresponding to $f^i_3$ constructed above, and joints $F^i$ in $M^i[k_u]$ such that an embedding $g_3^i$ as above within the $\delta$-neighbourhood of $f_3^i(F^i)$ cannot be made $K_i$-bi-Lipschitz, with $K_i \to \infty$. We put the superscript $i$ for all the symbols related to $M^i$ and $N^i$. By taking a subsequence we can assume that all the $\hat{F}^i$ are homeomorphic to some fixed surface $F$. As was shown before, by our definition of the model metric, the moduli of $F^i$ are bounded. Therefore, we can choose a homeomorphism $\kappa_i : F \to F^i$ so that the pullback of the metric on $F^i$ by $\kappa_i$ converges as $i \to \infty$. Take a base-point $x$ on $F$, and consider geometric limits of $(\hat{F}_i, \kappa_i(x))$, $(M^i[k_u], \kappa_i(x))$, and $(N^i[k_u], f_3^i \circ \kappa_i(x))$. Since $f_3^i$ is uniformly Lipschitz, it converges to a Lipschitz map $f_3^\infty : M^\infty[k_u] \to N^\infty[k_u]$, where $M^\infty[k_u]$ and $N^\infty[k_u]$ are the geometric limits of $(M^i[k_u], \kappa_i(x))$ and $(N^i[k_u], f_3 \circ \kappa_i(x))$ respectively. Since the metrics induced from the $F^i$ on $F$ are bounded, the homeomorphism $\kappa_i$ converges to a homeomorphism $\kappa_\infty : F \to F^\infty$, where $F^\infty$ is embedded in $M^\infty[k_u]$. 

As before, we can assume that $f_3^i \circ \kappa_i$ and $f_3^\infty \circ \kappa_\infty$ are immersions. By a result of Freedman-Hass-Scott as was in the proof of Proposition 4.8, $f_3^i \circ \kappa_i$ is homotopic to $g_3^i$ relative to the boundary by a homotopy passing through only relatively compact components of $N^\infty[k_u] \setminus f_3^i \circ \kappa_i(F)$. Since $N^\infty[k_u]$ contains no Margulis tubes whose core curves have lengths less than $\epsilon_u$, these components have uniformly bounded diameters and converge geometrically to relatively compact components of $N^\infty[k_u] \setminus f_3^\infty \circ \kappa_\infty(F)$ through which $f_3^\infty \circ \kappa_\infty$ can be homotoped to an embedding (after a perturbation if necessary). Therefore, the geometric limit $f_3^\infty \circ \kappa_\infty$ can be homotoped to a bi-Lipschitz embedding in $N^\infty[k_u]$. By pulling back this embedding and a homotopy, we can homotope $f_3^i \circ \kappa_i$ to a uniformly bi-Lipschitz embedding, contradicting our assumption. Thus we have shown that $f_3^i[F]$ can be homotoped to a uniformly bi-Lipschitz embedding, which we shall let be $H(\cdot, 3/4)$.

Since $f_3^i$ preserves the order of $\hat{F} \cap T$ as was shown in Lemma 4.12(iii), $f_3^i$ is homotopic to a homeomorphism fixing $\hat{F} \cap T$. Therefore the least area surfaces homotopic to the restrictions of $f_3^i$ to the components of $\hat{F}$ fixing the boundary must be pairwise disjoint. The same holds even if we put $F$ of $F \setminus \hat{F}$ into $\hat{F}$ removing its counterpart $\hat{F}$ instead. Therefore, to show the disjointness of the least-area images of the components of $\hat{F}$, it suffices to show that the least area surfaces homotopic to $f_3^i(F)$ and $f_3^i(F')$ are disjoint for each component $F$ of $F \setminus \hat{F}$. This follows immediately from Freedman-Hass-Scott when $f_3^i(F)$ and $f_3^i(\hat{F})$ are already disjoint. If $f_3^i(F)$ and $f_3^i(\hat{F})$ intersect, then the condition (vi) of Lemma 4.12 implies that the order of $F \cap T$ and $\hat{F} \cap T$ is preserved under $f_3^i[T]$. Therefore, by considering $\hat{F} \cup F$ instead of $\hat{F}$, we see that the least area surfaces are disjoint.

It remains to show that we can take disjoint regular neighbourhoods of the components. (Since the restriction of $H(\cdot, 3/4)$ to each component of $\hat{F}$ is uniformly bi-Lipschitz, the uniform bi-Lipschitzity on $\hat{F} \times [1/2, 1]$ follows immediately once we prove that we can take regular neighbourhoods to be disjoint.) Recall that by Lemma 4.12(iv), there is a uniform positive lower bound for the distances between the images of distinct boundary components of $\hat{F}$ under $f_3^i$, hence also under $H(\cdot, 3/4)$. To get disjoint regular neighbourhoods, what we need is a lower bound for the distances between the images of distinct components of $\hat{F}$ under $H(\cdot, 3/4)$, not only for the boundary but for the entire surface. Suppose that
such a lower bound does not exist. Then by taking a geometric limit, we get two minimal surfaces which are tangent each other at their interiors. This contradicts the maximal principle of minimal surfaces. Thus, we have shown that there is a lower bound, and that we can take disjoint regular neighbourhoods.

4.4. **Topological ordering of joints.** Next we shall show thus obtained embedding $H(,1): \mathcal{F} \to N[k_u]$ preserves the topological order of joints.

**Lemma 4.14.** Let $F_1$ and $F_2$ be joints in $\mathcal{F}$ such that $\iota_M(F_1)$ and $\iota_M(F_2)$ are not homotopic in $S \times (0,1)$. If $F_1 \prec_{\text{top}} F_2$, then $H(F_1,1) \prec_{\text{top}} H(F_2,1)$.

**Proof.** Suppose that $F_1 \prec_{\text{top}} F_2$ whereas $\iota_M(F_1)$ is not homotopic to $\iota_M(F_2)$. Let $c$ be a boundary component of $F_2$ which overlaps $F_1$ if there are any. There is a component $T$ of $\partial M[k_u]$ on which $c$ lies. Then, Lemma 3.3 in Brock-Canary-Minsky [BCM] implies that $F_1 \prec_{\text{top}} c$. Recall from Proposition 4.8 that $\iota_M \circ f$ extends to a homeomorphism $f_4 : M \to N_0$ properly homotopic to $f$. Since $F_1 \prec_{\text{top}} c$, the surface $\iota_M(F_1)$ can be homotoped to $S \times \{0\}$ without touching $\iota_M(c)$. Since $\iota_M = \iota_N \circ f$, we see that $\iota_N \circ f(F_1)$ can be homotoped to $S \times \{1\}$ without touching $\iota_N \circ f(c)$, which implies $\iota_N \circ f(F_1) \prec_{\text{top}} \iota_N \circ f(c)$ by Lemma 3.16 in [BCM]. Because $f$ is properly homotopic to $f_4$, we also have $f_4(F_1) \prec_{\text{top}} f_4(c)$. Since $c$ lies on a component $T$ of $\partial M[k_u]$, the homeomorphism $f_4$ is homotopic to $f'_4$ in $N[k_u]$, and $H$ is a proper homotopy in $N[k_u]$, this topological order is preserved by $H(,1)$, and we have $H(F_1,1) \prec_{\text{top}} H(c,1)$. By the same argument by changing the direction of order, we see that for any boundary component $c'$ of $F_1$ that overlaps $F_2$, we have $H(c',1) \prec_{\text{top}} H(F_2,1)$. Since $F_1$ and $F_2$ are assumed not to be homotopic, by Lemma 3.15 in [BCM], this implies that $H(F_1,1) \prec_{\text{top}} H(F_2,1)$.

We next consider the case when two joints $F_1$ and $F_2$ are properly homotopic.

**Lemma 4.15.** Suppose that $F_1$ and $F_2$ are joints in $\mathcal{F}$ such that $\iota_M(F_1)$ is homotopic to $\iota_M(F_2)$. We further assume that $F_1 \cup F_2$ does not bound a brick in $M[k_u]$. If $F_1 \prec_{\text{top}} F_2$, then we have $H(F_1,1) \prec_{\text{top}} H(F_2,1)$.

**Proof.** Since $\iota_M(F_1)$ is homotopic to $\iota_M(F_2)$, for each component $c$ of $\partial F_1$, there is a unique component of $c'$ of $\partial F_2$ such that $\iota_M(c) \simeq \iota_M(c')$ in $S \times (0,1)$. Suppose first that $c$ and $c'$ are homotopic in $M[k_u]$ for all components $c$ of $\partial F_1$. Then $c$ and $c'$ lie on the same boundary component of $\partial M[k_u]$ by the condition A-(2) and the definition of $\mathcal{V}[k_u]$. Since this holds for every boundary component of $F_1$, we see that $F_1 \cup F_2$ bounds a submanifold $W$ in $M[k_u]$. If $W$ is homeomorphic to $F_1 \times [0,1]$, then by our definition of the brick decomposition of $M[k_u]$, we see that $W$ consists of only one brick. This contradicts our assumption that $F_1 \cup F_2$ does not bound a brick.

Therefore, $F_1 \cup F_2$ bounds a submanifold $W$ in $M[k_u]$, which is not homeomorphic to $F_1 \times [0,1]$. Then there is a component $T$ of $\partial M[k_u]$ which is contained in $W$. We take a horizontal curve $c$ contained in $T$. Then $c$ overlaps both $F_1$ and $F_2$ and $F_1 \prec_{\text{top}} c \prec_{\text{top}} F_2$. This implies that $f_4(F_1) \prec_{\text{top}} f_4(c) \prec_{\text{top}} f_4(F_2)$. Since $\iota_N \circ f_4$ is homotopic to $\iota_M$, we see that $\iota_N \circ f_4(F_1)$ is homotopic to $\iota_N \circ f_4(F_2)$ and the boundary of $\iota_N \circ (F_1)$ is unknotted and unlinked. Therefore, applying Lemma 3.14 in [BCM], we have $f_4(F_1) \prec_{\text{top}} f_4(F_2)$, which implies that $H(F_1,1) \prec_{\text{top}} H(F_2,1)$ as before. Thus it only remains to consider the case when there is a component $c$ of $\partial F_1$ which is not homotopic to $c'$ in $M[k_u]$. 

□
Since $\iota_M(c)$ and $\iota_M(c')$ are homotopic, and $\iota_M(c)$ and $\iota_M(c')$ are horizontal, there is an embedded annulus $A$ bounded by $\iota(c) \cup \iota(c')$ in $S \times (0, 1)$. It follows that there is a boundary component $T$ of $M[k_u]$ such that $\iota_M(T)$ intersects $A$ essentially. Take a longitude or a core curve $c$ of $T$. Then we have $F_1 \prec_{\text{top}} c \prec_{\text{top}} F_2$. Now since $f_4$ is a homeomorphism from $M$ to $N_0$, we see that $f_4(F_1) \prec_{\text{top}} f_4(c) \prec_{\text{top}} f_4(F_2)$, and as before, we have $H(F_1, 1) \prec_{\text{top}} H(c, 1) \prec_{\text{top}} H(F_2, 1)$. Since $\iota_N \circ H(F_1, 1)$ and $\iota_N \circ H(F_2, 1)$ are homotopic, Lemma 3.14 in [BCM] again implies that $H(F_1, 1) \prec_{\text{top}} H(F_2, 1)$. Thus we have completed the proof.

The remaining case is when $F_1$ and $F_2$ are homotopic in $M[k_u]$ and cobound a brick in $M[k_u]$. Let $B$ be a brick bounded by $F_1 \cup F_2$, and $h(B)$ the hierarchy on $B$ which we obtain by applying Lemma 4.14 to $M[k_u]$. We say that a tight geodesic $g \in h(B)$ is deep-seated if there is a component of $\text{Fr}D(g)$ whose corresponding tube in $V$ is disjoint from either $\partial_+ B$ or $\partial_- B$. In the case when $D(g)$ is an annulus, we regard a core curve of $D(g)$ as a component of $\text{Fr}D(g)$.

We shall first show that $h(B)$ cannot have a long deep-seated geodesic.

**Lemma 4.16.** There exists a constant $A$ depending only on $\xi(S)$ as follows. Let $B$ be a brick in $M[k_u]|_{\text{int}}$. Then every deep-seated geodesic in $h(B)$ has length less than $A$.

**Proof.** By Theorem 9.1 in [M2], we can take a constant $A$ such that if $g \in h(B)$ has length at least $A$, then for every component $c$ of $\text{Fr}D(g)$, either $c$ lies on $\partial M$ or a boundary component $\partial V$ for $V \in V$ such that $|\omega_M(V)| > k_u$. (Since we are considering geodesics in $h(B)$ whose geodesics consist of simplices on the curve complex of $C(\partial_- B)$, we can apply Minsky’s result on Kleinian surface groups.) Therefore every component of $\text{Fr}D(g)$ lies on $\partial M[k_u]$ in this situation. If $g$ is deep-seated, then some component $c$ of $\text{Fr}D(g)$ must lie on $\partial V$ which is disjoint either from $\partial_+ B$ or $\partial_- B$. Thus we see that if $h(B)$ has deep-seated geodesic with length at least $A$, then there is a component of $\partial M[k_u]$ which intersects $B$ but not at least one of $\partial_+ B$ and $\partial_- B$. This contradicts the assumption that $\partial_- B$ and $\partial_+ B$ are homotopic and bound $B$ in $M[k_u]$. □

Suppose that all the deep-seated geodesics in $h(B)$ have length less than $A$. We further divide our argument into two cases: the first is when the number of blocks constituting $B$ is large and the other is when it is small. It will turn out later that we do not need to show that the topological order is preserved in the latter case for the proof of Theorem 1.1.

**Lemma 4.17.** Fix a constant $A$ as in Lemma 4.16. There exists a constant $C$ depending only on $\xi(S)$ (and $A$) as follows. If $|h(B)| > C$, then we have $H(\partial_- B, 1) \prec_{\text{top}} H(\partial_+ B, 1)$.

**Proof.** We can take a constant $C$ so that if $h(B) > C$, then there must be a geodesic $g$ in $h(B)$ whose length is greater than $A$. By Lemma 4.16, then $g$ cannot be deep-seated. If $g$ is not deep-seated, then since every frontier component of $D(g)$ lies in $\partial M[k_u]$, the only possibility is $g$ is the main geodesic of $h(B)$. Then we can apply Theorem 7.1 in [BCM] to our hierarchy $h(B)$. The same argument as in the case 1b of the proof of Lemma 8.4 in [BCM] implies that $H(\partial_- B, 1) \prec_{\text{top}} H(\partial_+ B, 1)$. □

For the remaining case, we make the following definition. We say that a brick $B$ in $M[k_u]|_{\text{int}}$ is short in the remaining case: i.e. if $B$ satisfies the following:
(1) Their fronts $\partial_- B$ and $\partial_+ B$ are homeomorphic.
(2) $|h(B)| < C$.

4.5. **Deformation to a bi-Lipschitz map.** Having obtained the results in the previous subsection, we are now in a position to show that we can further homotope $H(\cdot)$ to make it bi-Lipschitz on the region between joints, applying arguments of §§8.2-8.4 in Brock-Canary-Minsky [BCM].

For a brick $B$ of $M[k_\omega]_{\text{int}}$ which is not short, we shall construct a cut system, following §4 and §8.2 in Brock-Canary-Minsky [BCM]. Our cut system $C_B$ is a set of slices of $h(B)$ having the following properties with a constant $d_1 > 5$ which will be specified later.

(1) For a geodesic $g \in h(B)$, let $C_B|g$ denote the subset of $C_B$ consisting of slices with bottom geodesic $g$. Then, for any geodesic $g \in h(B)$, the bottom simplices $\{v_\tau \mid \tau \in C_B|g\}$ cut $g$ into intervals whose lengths are between $d_1$ and $3d_1$.
(2) Two distinct slices in $C_B|g$ cannot have the same bottom simplex.
(3) For each $\tau \in C_B$ and any $(k, v)$ in $\tau$ other than the bottom one, $v$ is the first vertex of $k$.
(4) For non-annular $g$, any slice in $C_B|g$ is a non-annular saturated slice.
(5) For annular $g$, there is at most one slice in $C_B|g$.

We take a constant $d_1$ so that for any geodesic $g$ in the hierarchy $h(B)$, if $g$ has length at least $d_1$, then the geodesic length of each component of $f_3(\partial D(g))$ is less than $\varepsilon_u$. (Such $d_1$ exists by Lemma 4.17) Furthermore, we consider the constant $C$ which appeared in Lemma 4.17 depending only on $S$. We can take $d_1$ which is greater than $C$.

For each slice $\tau \in C_B$, we define *extended split level surfaces* as follows. Suppose that the bottom pair $(g_\tau, v_\tau)$ of $\tau$ is not supported in an annulus. Since $\tau$ is a non-annular saturated slice and $h(B)$ is 4-complete, base($\tau$) defines a pants decomposition of $D(g_\tau)$. For each pair of pants $Y$ constituting the pants decomposition, there is a horizontal boundary of two adjacent blocks à la Minsky in the form of $Y \times \{t\}$ with respect to the parametrisation of $S \times (0,1)$, along which the two are glued. (This lies at the middle of a block of the form $\Sigma_{0,3} \times J$ in our block decomposition in §3.) This horizontal surface is denoted by $F_Y$. We consider the union $F_\tau = \cup F_Y$ for all $Y$ constituting the pants decomposition, and call it the split level surface corresponding to $\tau$. For a cut system $C_B$ as above, the split level surface $F_\tau$ for $\tau \in C_B$ is called a *cut* in $B$. Let $F_B$ be the union of $F_\tau$ for all $\tau \in C_B$ and $F_b$ the union of $F_B$ for all bricks $B \in K[k_\omega]_{\text{int}}$. Let $V$ be a component of $\mathcal{V}$ on which a boundary component of $F_\tau$ lies. By the condition (1) of the definitions of $C_B$ and $d_1$, we see that $\omega(V) > k_\omega$, hence $V \in \mathcal{V}[k_\omega]$. Therefore, by adding $F_b$ to the joints of $M[k_\omega]$, we get a subdivision of $M[k_\omega]$ into smaller bricks, which may have inessential joints. We denote this refined brick manifold by $M'[k_\omega]$. (Note $M[k_\omega]$ and $M'[k_\omega]$ are the same as manifolds, only the brick structure differs.)

We shall show that $H(\cdot)$ can be homotoped so that each $F_\tau$ for $\tau \in C_B$ is a $K'$-bi-Lipschitz embedding.

**Lemma 4.18.** There exists a constant $K''$ depending only on $\xi(S)$ as follows. Then, there exists a $K''$-Lipschitz homotopy $H' : (F_b \cup F) \times [0,1] \to N[k_\omega]$, such that
(i) $H'(\mathcal{F}_b \cup \mathcal{F}) \times \{0\}$ coincides with $H(\mathcal{F}_b \cup \mathcal{F}, 1)$.
(ii) $H'(\mathcal{F}_b \cup \mathcal{F}) \times [1/2, 1]$ is a $K''$-bi-Lipschitz $C^2$-embedding.

**Proof.** Our argument is similar to the proof of Lemma 4.13. Let $T$ be a component of $\partial M[k_u]$ intersecting $B$ and $T'$ its image in $N[k_u]$ under $f_\lambda$. We first need to show that $H(\cdot, 1)$ can be moved to a uniformly Lipschitz map which preserves the order of $T \cap (\mathcal{F}_b \cup \mathcal{F})$ except for the fronts of short bricks by a homotopy whose displacement function is bounded from above by a uniform constant. Our situation is a little different from that of Lemma 4.12 since among our surfaces in $\mathcal{F}_b \cup \mathcal{F}$, there might be more than two components which are all homotopic to each other. Still, we can argue as in the proof of (vi) in Lemma 4.12 and see that there might be more than two components which are all homotopic to each other.

Next we shall show that the property as (vii) in Lemma 4.12 holds for $\mathcal{F}_B$ and $f_3''$; that is, for any $\delta$, there is a number $n_0$ bounding the number of components of $f_3''(\mathcal{F}_B \cap T)$ which are within distance $\delta$ from $f_3''(F \cap T)$ for any component $F$ of $\mathcal{F}_B$. Let $F_1, \ldots, F_n$ be distinct components of $\mathcal{F}$ such that $f_3''(F_1 \cap T), \ldots, f_3''(F_n \cap T)$ are within the distance $\delta$ from $f_3''(F \cap T)$. Then $H(F_1, 1), \ldots, H(F_n, 1)$ are within distance $3\lambda_0 + \delta$ from $H(F, 1)$, where $\lambda_0$ is the constant which we defined in the proof of Lemma 4.12. Recall that for each slice $\tau$ of $C_B$, each component of $F_{\tau} \setminus \mathcal{V}$ is a thrice-punctured sphere. By Lemma 4.14 for distinct slices $F_{\tau_1}, \ldots, F_{\tau_n}$, there are at least $n$ non-homotopic tubes in $\mathcal{V}$ which at least one of $F_{\tau_1}, \ldots, F_{\tau_n}$ intersects. By Lemma 4.12 each tube has a core curve with length less than $L$. Since $H(\cdot, 1)$ is uniformly Lipschitz, the lengths of the images of the core curves are universally bounded. Suppose that there is not a universal bound for $n$. By the usual argument using a geometric limit of model maps for which there are at least $i$ slices as above, we are lead to a contradiction to the fact that for a hyperbolic 3-manifold, a constant $R$ and a base point $x$, there are only finitely many homotopy classes which are represented by a closed curve of length less than $L$ contained in the $R$-ball centred at $x$. Thus we have shown that $f_3''|_{\mathcal{F}_B}$ has the same property as (vii) in Lemma 4.12. Combining this with Lemma 4.12 we see that for any $\delta$, there exists $n_0$ bounding the number of components of $f_3''((\mathcal{F}_b \cup \mathcal{F}) \cap T)$ within the distance $\delta$ from $f_3''(F \cap T)$ for any component $F$ of $\mathcal{F}_b \cup \mathcal{F}$.

Since the only short bricks of $M'[k_u]$ are those already contained in $M[k_u]$, by the same argument as Lemma 4.13, we see that $H(\cdot, 1)$ can be homotoped to a uniform Lipschitz map which embeds $\mathcal{F}_b \cup \mathcal{F}$ in such a way two distinct components have disjoint $\delta$-regular neighbourhoods.

By this homotopy $H'$, we can homotope $f_3$ to $f_5$ which is a uniform bi-Lipschitz map on each component of $\mathcal{F}_b \cup \mathcal{F}$ and embeds its regular neighbourhood. Recall that $f_5$ preserves the topological order of $\mathcal{F}$ except for the fronts of short brick by the results in (4.4) and Lemmata 4.16 4.17 If $B$ is short, then $B$ consists of
less than $D$ blocks, hence the diameter of $B$, which can be bounded by a linear function of the number of blocks, is bounded a constant depending only on $\xi(S)$. Therefore, we can isotope $f_5(\partial_-B)$ into a regular neighbourhood of $f_5(\partial_+B)$ so that $f_5(\partial_-B) \prec_{\text{top}} f_5(\partial_+B)$ preserving the condition on the bi-Lipschitzity. We should note that by Assumption 5.2 two short bricks cannot be adjacent each other. Therefore, we can perform this deformation for all short bricks so that $f_5(\partial_-B)$ and $f_5(\partial_+B)$ have regular neighbourhoods with uniform width. Since the embedding of each cut by $f_5$ has a regular neighbourhood with uniform width, $f_5$ is bi-Lipschitz not only on each cut or joint but also with respect to the induced metric on the entire $\hat{F} \cup \hat{F}_b$.

To complete the proof of Theorem 4.1 it remains to deform $f_5$ in the complement of $\hat{F}_b \cup \hat{F}$ in $M[k_u]_{\text{int}}$ to make it bi-Lipschitz without changing the map on the geometrically finite bricks. This can be done by the same argument as §8.4 in [BCM] without any modification. Thus we have completed the proof of Theorem 4.1 by setting $k$ to be $k_u$.

5. Proofs of Theorems

5.1. Geometric limits of geometrically finite bricks. Let $G_n$ be a surface Kleinian group, and set $N_n$ to be $\mathbb{H}^3/G_n$. Let $g_n : M_n \to (N_n)_0$ be a model map constructed in [BCM] which induces a bi-Lipschitz homeomorphism $g_n[k_u] : M_n[k_u] \to N_n[k_u]$. Suppose that $M_n$ has a geometrically finite brick $B_n \cong F_n \times [0, 1)$ or $F_n \times (0, 1]$. We shall consider only the case when $B_n \cong F_n \times [0, 1)$, for the other case can be dealt with in the same way just by turning everything upside down.

Lemma 5.1. Let $x_n$ be a point in $B_n$ in the above situation. Suppose that with respect to the metric $d_{M_n}$ on $B_n$ defined in [BCM] we have $d_{M_n}(F_n \times \{0\}, x_n) \to \infty$. Then the geometric limit of $(a subsequence of) \{(M_n, g_n(x_n))\}$ is elementary: i.e. isomorphic to $\mathbb{H}^3/\Gamma$ for an elementary Kleinian group $\Gamma$.

Proof. Let $C(N_n)$ be the convex core of $N_n = \mathbb{H}^3/G_n$. By the definition of the model map, we see that $d_{M_n}(C(N_n), g_n(x_n)) \to \infty$. Let $\Gamma$ be a Kleinian group such that $(\mathbb{H}^3/\Gamma, x_{\infty})$ is the Gromov limit of $(\{M_n, g_n(x_n)\})$ after passing to a subsequence. Suppose, seeking a contradiction, that there are non-commuting elements $g, h$ in $\Gamma$. Then, there exist elements $g_n, h_n$ in $G_n$ such that $\lim g_n = g$ and $\lim h_n = h$. Consider the action of $G_n$ on $\mathbb{H}^3$. Then $g_n$ and $h_n$ act on $\mathbb{H}^3$ as loxodromic or parabolic transformations. Let $l_n$ be a geodesic in $\mathbb{H}^3$ which is a common perpendicular of the axes of $g_n$ and $h_n$ if they are loxodromic, or a geodesic ray perpendicular to the axis of the loxodromic one to tending to the fixed point at infinity of the parabolic one when only one of them is loxodromic, or a geodesic connecting the fixed points at infinity of the two elements if both of them are parabolic.

We claim that the function $t(g_n, h_n)(x) = \max\{d(x, g_n(x)), d(x, h_n(x))\}$ takes minimum at a point $c_n$ on $l_n$. This can be seen by considering sets $V_{g_n}(r)$ and $V_{h_n}(r)$ consisting of points whose translation distances are less than or equal to $r$ under $g_n$ and $h_n$ respectively. The smallest $r$ for which $V_{g_n}(r) \cap V_{h_n}(r) \neq \emptyset$ realises the minimum of $t(g_n, h_n)$. (If $V_{g_n}(r)$ (resp. $V_{h_n}(r)$) reaches the axis of $h_n$ (resp. $g_n$) while $V_{h_n}(r)$ (resp. $V_{g_n}(r)$) is empty, we take such $r$ as the smallest.) By the convexity of these sets, we see that the intersection consists of one point $c_n$, and that it lies on $l_n$. Since $\{g_n\}$ and $\{h_n\}$ converge, the smallest $r$ is bounded from
above independently of $n$. Since the configurations of $V_{g_n}(r)$, $V_{h_n}(r)$ are compact, we see that $|t(g_n, h_n)(y) - 2d(y, c_n)|$ is bounded from above independently of $n$. (This follows from the fact that the displacement of a point can be approximated by twice the distance from the point to the axis if the translation length on the axis is bounded above.)

Obviously, $l_n$ is contained in the Nielsen convex hull of $G_n$. Take a lift $\tilde{x}_n$ of $x_n$ which converges to a lift $\tilde{x}_\infty$ of $x_\infty$. Since $d_{M_n}(C(M_n), g_n(x_n)) \to \infty$, the distance of $l_n$ from a lift $\tilde{x}_n$ of $x_n$ in $\mathbb{H}^3$ goes to $\infty$ as $n \to \infty$; hence $d(\tilde{x}_n, c_n) \to \infty$. From the above observation, this implies that $t(g_n, h_n)(\tilde{x}_n) \to \infty$. This contradicts the facts that $g = \lim g_n$ and $h = \lim h_n$ translate $\tilde{x}_\infty$ within a finite distance. \hfill \square

We next consider the geometric limit of geometrically finite bricks. By virtue of the previous lemma, we have only to consider the case when the basepoint lies on the real front along which the brick is pasted to other bricks. Let $x_m$ be a point in $B_m$ lying on $F_m \times \{0\}$. Since each $F_m = F_m \times \{0\}$ has the cylinder-$\Sigma_{0,3}$ metric $\tau_m$, if $\{(\tilde{F}_m, \tilde{x}_m)\}$ converges geometrically to $(\tilde{F}_\infty, \tilde{x}_\infty)$ passing to a subsequence, then $\tilde{F}_\infty$ also have such a metric $\tau_\infty$. Moreover, since $B_m$ is uniformly bi-Lipschitz to the brick $\tilde{F}_m \times [0, \infty)$ with metric $e^{2r}\tau_m + dr^2$ ($r \in [0, \infty)$), $\{(\tilde{F}_m, \tilde{x}_m)\}$ converges geometrically to a brick $B_\infty$ uniformly bi-Lipschitz to $\tilde{F}_\infty \times [0, \infty)$ with metric of the form $e^{2r}\tau_\infty + dr^2$ ($r \in [0, \infty)$) passing to a subsequence. In particular, $B_\infty$ is also a brick homeomorphic to $\tilde{F}_\infty \times [0, 1)$.

5.2. Proofs of Theorem A and Corollary B

Proof of Theorem A. Let $\{G_n\}$ be a sequence of Kleinian surface groups which converges geometrically to a non-elementary Kleinian group $G$. By the original bi-Lipschitz model theorem [BCM], for each $n \in \mathbb{N}$, there exist a model manifold $M_n$ and a model map $g_n : M_n \to (N_n)_0$ inducing a $K$-bi-Lipschitz homeomorphism $g_n : M_n[k_n] \to N_n[k_n]$, where $N_n = \mathbb{H}^3/G_n$.

The model manifold $M_n$ consists of $M_n[0]$, which is decomposed into internal blocks and boundary blocks, and Margulis tubes. Since $G_n$ is a Kleinian surface group, $M_n$ is embedded in $S_0 \times (0, 1)$ for a compact core $S_0$ of $S$ so that the boundary of a cusp neighbourhood which does not correspond to a boundary component of $S_0$ is a properly embedded open annulus both of whose ends go to the same end of $S_0 \times (0, 1)$, either to the $+$-direction or the $-$-direction. We put $M_n$ the structure of a brick manifold compatible with the block decomposition as follows. We first consider a proper embedding $\eta_n : M_n \to S \times (0, 1) with the following properties, which is obtained by isotoping blocks within $S \times (0, 1)$.

1. The embedding $\eta_n$ preserves the horizontal and the vertical leaves of each block. (Here for a block with the form $\Sigma \times J$, each $\Sigma \times \{t\}$ is a horizontal leaf and $\{x\} \times J$ is a vertical leaf.)
2. Each tube in $M_n$ is mapped to $A \times [t_1, t_2]$ for some essential annulus $A$ on $S$ and $t_1 < t_2$, and each torus boundary of $M_n$ is mapped to the boundary of $A \times [t_1, t_2]$.
3. Each open annulus boundary component of $M_n$ is mapped to the boundary of either $A \times \{t, 1\}$ or $A \times \{0, t\}$ for an essential annulus $A$ on $S$.
4. The geometrically finite ends of $M_n$ are peripheral, i.e. lie on $S \times \{0, 1\}$. 

GEOMETRY AND TOPOLOGY OF GEOMETRIC LIMITS I 47
This is exactly the situation as in the construction of brick decomposition for $M^{(1)}_{\text{int}}$ in §3.3. Therefore, we can endow a brick decomposition with $M_n$ by defining each to be a maximal family of parallel leaves.

Note that any internal block of $M_n[0]$ is isometric to either $\Sigma_{(0,4)} \times [0,1]$ or $\Sigma_{(1,1)} \times [0,1]$, or $\Sigma_{(0,3)} \times [0,1]$, each with a standard metric. (We can consider block decomposition in our sense or Minsky’s. Either will do.) Moreover, as was seen in Subsection 5.1 any sequence of geometrically finite bricks in $M_n[0]$ converges geometrically to a geometrically finite brick in $M[0]$ after taking a subsequence if we put a basepoint on the real front. This implies that the geometric limit $M[0]$ of $M_n[0]$ with basepoints $x_n$ either in an internal brick or within a bounded distance from the real front in a geometrically finite brick consists of geometrically finite bricks and the remaining part admitting a block decomposition. We denote by $M[0]|_{\text{int}}$ the part of $M[0]$ consisting of the limits of internal bricks. The complement of $M[0]|_{\text{int}}$ in $M[0]$ consists of geometrically finite bricks as was seen above. As before, we denote by $V_n$ the union of tubes in the tight tube unions giving a block decomposition of $M^{(n)}_n$. (Recall that $M^{(n)}_n$ is the complement of tubes in $V_n$ intersecting $M_n$ along annuli and is naturally identified with $M_n$.) For any $k$, we denote by $V_n[k]$ the subset of $V_n$ consisting of tubes $V$ with $[\omega_{M_n}(V)] \geq k$. Recall that $M_n[k] = (M_n)^0 \setminus V_n[k]$. We denote by $T_n[k]$ the union of Margulis tubes which is the image of $V_n[k]$ by $\eta_n$.

Each torus component $T$ of $\partial M[0]$ is a geometric limit of torus components $T_n$ of $\partial M_n[0]$. Since $T_n$ converges geometrically, either $\{\omega_{M_n}(T_n)\}$ converges or goes to $\infty$. If it converges, then $T_n$ bounds a hyperbolic tube $V_n$ converging geometrically to a hyperbolic tube $V$ bounded by $T$. We denote by $V_\infty$ the union of such tubes $V$. The gluing map of $V_n$ to $M_n[0]$ converges to a gluing map of $V$ to $M[0]$. We define the union of $M_n[0]$ with such tubes glued by the limit gluing maps to be $M$. We denote by $M[k]$ the union of $M[0]$ and tubes in $V_\infty$ for which $\lim_{n \to \infty} |\omega_{M_n}(T_n)| \leq k$. The argument above also implies in particular that $g_n$ with base point $x_n$ converges to a $K$-bi-Lipschitz homeomorphism $g : M[k_n] \to N[k_n]$. Since we put the metric on $V_n$ inherited from a Margulis tube determined by $\omega_{M_n}(V_n)$, each $g_n$ is extended to a $K$-bi-Lipschitz map in $V_n$. Therefore $g$ is also extended to a $K$-bi-Lipschitz homeomorphism from $M$ to $N_0$. We use the symbol $M_{\text{int}}$ to denote the union of $M[0]|_{\text{int}}$ and $V_\infty$.

If $\lim_{n \to \infty} \omega_{M_n}(V_n) = \infty$, then $g(T)$ is the boundary of a torus cusp neighbourhood of $N$, which is not contained in $N_0$. If we put a basepoint on $\partial V_n$, then the geometric limit of $V_n$ is also a $\mathbb{Z} \times \mathbb{Z}$-cusp which is $K$-bi-Lipschitz to the cusp neighbourhood bounded by $g(T)$ since $\omega_{M_n}(V_n)$ controls the modulus of the Margulis tube bounded by $g_n(T_n)$. (See §8.5 in [BCM].) Note that by our definition of $M$, the limit cusp neighbourhood is not contained in our model manifold $M$.

The properties (ii) that $M$ is acylindrical and (i) that $\partial M$ consists of tori and annuli in the statement of Theorem A are derived from the same properties for $N_0$. We shall show that $M$ is a brick manifold. Recall that $M[0]|_{\text{int}}$ admits a decomposition into blocks. Let $\rho_n : B^{(n)}_n(M_n, x_n) \to B^{(n)}_n(M, x_\infty)$ be a $(K_n, r_n)$-approximate isometry associated to the geometric convergence of $\{(M_n, x_n)\}$ to $(M, x_\infty)$. We can arrange $\rho_n$ so that for each block $b$ of $M[0]|_{\text{int}}$, its pull-back $\rho_n^{-1}(b)$ is also a block with respect to the block decomposition of the brick manifold $M_n$, and $\rho_n^{-1}|b$ preserves the vertical and horizontal leaves of $b$. Recall that the embedding $\eta_n$ of $M_n$ into $S \times (0, 1)$ preserves the vertical and the horizontal leaves.
of blocks. Therefore, at each point of $M$ the (two-dimensional) horizontal directions and the vertical direction are well defined. The horizontal directions in $M$ constitute a foliation whose leaves are incompressible in $M$ and homeomorphic to essential subsurface of $S$ (including $S$ itself) as we can see by considering their image under $\rho_n^{-1}$ for large $n$. We define a leaf of this foliation to be a horizontal leaf of $M$. Horizontal leaves are transversely oriented, by defining the $+$-direction of the second factor of $S \times (0, 1)$ to be the positive direction.

Now, we define a brick in $M$ to be a closed submanifold which is the closure of a maximal union of parallel horizontal leaves in $M$ if it has non-empty interior. It is evident that the bricks defined in this way are pairwise disjoint. We can further show the following, which implies that $M$ is a brick manifold.

**Lemma 5.2.** Every point in $M$ is contained in a brick. The bricks are locally finite.

**Proof.** Let $x$ be a point in $M$, and $F$ a horizontal leaf of $M$ on which $x$ lies. We say that a boundary component $T$ of $M$ touches $F$ from above if $T \cap F \neq \emptyset$ and if any leaf near $F$ and above $F$ intersects $T$ whereas any leaf below $F$ is disjoint from $T$. Similarly, we define touching from below. Every component of $\partial M$ is either a torus or an open annulus which can intersect a horizontal leaf along a single essential annulus. Since $M$ is acylindrical, there are no two annuli on $\partial M \cap F$ which are parallel on $F$. Therefore, the number of the components of $F \cap \partial M$ is uniformly bounded by a constant depending only on $\xi(S)$.

Let $h_1$ be the minimum of the heights above $F$ (with respect to the metric on $M$) of components of $\partial M$ intersecting $F$ but not touching from below, which we allow to be $\infty$. Then, if we move $F$ in the vertical direction to the positive side within the distance $h_1$, then the surface $F$ loses the interior of annuli which are intersection with components of $\partial M$ touching from above and all the horizontal leaves above $F$ within distance $h_1$ are parallel each other. Therefore if $x$ lies outside the intersection with components of $\partial M$ touching $F$ from above, then $x$ is contained in a brick which passes through $F$ or is situated above $F$ and touches $F$ at the boundary. Similarly there is $h_2$ such that all the horizontal surface below $F$ within distance $h_2$ are parallel each other. Also, if $x$ lies outside the intersection with components of $\partial M$ touching $F$ from below, then $x$ is contained in a brick which passes through $F$ or is situated below $F$ and touches $F$ at the boundary. Therefore, $F$ is contained in the closure of the (non-empty) union of finitely many bricks, one of which must contain $x$, and has a product neighbourhood intersecting only finitely many bricks.

By our definition of bricks in $M$ and that for $M_n$, for any brick $B$ in $M$ its pull-back $\rho_n^{-1}(B)$ is contained in one brick in $M_n$ for large $n$. Now, we are in a position to use Lemma 5.1 to verify the condition (iv) in our theorem. For any $r \in \mathbb{N}$, let $M(r)$ be the submanifold of $M$ consisting of bricks intersecting the $r$-ball centred at $x_\infty$ with respect to the metric induced from those on blocks. Then $M(r)$ contains only finitely many bricks by Lemma 5.2. If we take a sufficiently large $n$, then we can pull back $M(r)$ to $M_n$ by $\rho_n^{-1}$. Since the pull-back of each brick is contained in a brick of $M_n$, we can embed $M(r)$ to $S \times (0, 1)$ by $\eta_n \circ \rho_n^{-1}$ preserving the vertical and the horizontal leaves. Since $M = \cup_{r=1}^\infty M(r)$, by Lemma 5.1, we can embed $M$ into $S \times (0, 1)$ in such a way that every brick is mapped to a submanifold of the
form $F \times J$. Since the geometrically finite ends of $M_n$ are peripheral, we see that the same holds for $M$ by Lemma 3.1. This completes the proof of (iv).

Finally, we shall show (iii), that there is no incompressible half-open annulus tending to a wild end $e$ with core curve not homotopic to an annulus component of $\partial M$ tending to $e$. Suppose, seeking a contradiction, that $M$ has such an end $e$ to which an incompressible half-open annulus $A$ tends, and that the core curve of $A$ is not homotopic into an annulus component of $\partial M$ tending to $e$. Let $\{H_n\}$ be a sequence of properly embedded connected horizontal surfaces in $M$ meeting $A$ transversely and tending to $e$. (Since every point lies on some horizontal leaf, such a sequence of horizontal surfaces exist.) See Fig. 5.1 (a). For each $n$, the intersection $A \cap H_n$ is an essential simple closed curve, which we denote by $l_n$. By our assumption, $l_n$ is not homotopic into an annulus component of $\partial M$ tending to $e$. Therefore, $g(l_n)$ either represents a loxodromic element or is homotopic into a cusp which is disjoint from a small neighbourhood of $e$.

We first assume that $g(l_n)$ represents a loxodromic element. Let $h_n : H_n \to N_0$ be a pleated surface homotopic to $g|H_n$ realising $l_n$ as a closed geodesic, which we denote by $l^*$. We should note that $H_n$ is homeomorphic to an essential subsurface of $S$. For any $\delta > 0$, the pleated surfaces $h_n$ have an upper bound depending only on $\chi(S)$ and $\delta$ for the diameters modulo their $\delta$-thin parts. Since there are only finitely many $\varepsilon_1$-cusp neighbourhood within a bounded distance modulo the $\delta$-thin part of $N$ from $l^*$ and the images of $h_n$ contain $l^*$, by taking a subsequence we can assume that the homotopy class of $\partial H_n$ does not depend on $n$. By the condition (ii), this implies that the boundary components of $M$ on which $H_n$ lies does not depend on $n$. It follows that there is an essential subsurface $R$ of $S$ such that all the $H_n$ are vertically parallel to $R \times \{1/2\}$ in $S \times (0, 1)$. (Notice that they may not be parallel in $M$. To be more precise, we are claiming that the $i_M(H_n)$ are vertically parallel to $R \times \{1/2\}$ for the embedding $i_M$ of $M$ into $S \times (0, 1)$ obtained above. We omit to write $i_M$ here.)

Let $i_n : R \to H_n$ be a homeomorphism compatible with a homotopy from $R \times \{1/2\}$ within $S \times \{1/2\}$ to $H_n$ in $S \times (0, 1)$. Since the $l_n$ are homotopic each other in $S \times (0, 1)$, we can arrange the $i_n$ so that there is a simple closed curve $l$ on $R$ such that $i_n(l) = l_n$ for all $n$. Recall that there are only finitely many $\varepsilon_1$-cusp

![Figure 5.1](attachment:image.png)

**Figure 5.1.**
neighbourhoods which \( h_n(H_n) \) can touch. We extend \( l \) to a pants decomposition \( P \) of \( R \) so that no curve is mapped to a parabolic class representing a cusp which \( h_n(H_n) \) can touch. We now consider the sequence of pleated surfaces \( h_n \circ i_n \), which realises \( P \) as closed geodesics. Since there are only finitely many cusps which we must take into account, by applying the compactness of marked pleated surfaces without accidental parabolics (5.2.18 in Canary-Epstein-Green [CEG]), we see that passing to a subsequence, \( h_n \circ i_n \) converges to a pleated surface from a component \( R' \) of \( R \) containing \( l \), where \( \alpha \) is a possibly empty union of disjoint non-parallel essential annuli in \( R \), uniformly on every compact subset of \( R' \). It follows that there exists \( n_0 \in \mathbb{N} \), such that all \( h_n \circ i_n | R' \) \( (n \geq n_0) \) are properly homotopic in \( N_0 \). Pulling back this to \( M \), we see that there is no component of \( S \times (0,1) \setminus M \) which obstructs homotopies between the \( i_n(R') \) which are vertically parallel in \( M \) for all large \( n \). Therefore, there exists a submanifold \( R' \times [0,1] \) embedded in \( M \) preserving the horizontal and vertical leaves, which contains a neighbourhood of the end of \( A \) such that \( R' \times \{t\} \) tends to \( e \) as \( t \to 1 \). See Fig. 5.7(b).

We shall next show that we have the same kind product region even when \( \delta \)-cusp neighbourhood \( U_c \) of \( c \) within a bounded distance modulo the thin part. Therefore, as before, we can show that the \( H_n \) are parallel in \( S \times (0,1) \) after taking a subsequence.

As before, we can consider a homeomorphism \( i_n : R \rightarrow H_n \) compatible with the inclusion of \( R \) to \( S \), and a sequence of pleated surfaces \( h_n \circ i_n : R \setminus \text{Int}(l) \rightarrow N \) realising a pants decomposition \( P \) extending \( l \) none of whose curves is mapped to a cusp which can be reached by \( h_n(H_n) \). Then as in the previous case, there is a possibly empty union \( \alpha \) of non-separating disjoint essential annuli on \( R \), and for components \( R_1, R_2 \) of \( R \setminus (N(l) \cup \alpha) \) adjacent to \( N(l) \), which may coincide if \( l \) is non-separating, the pleated surfaces \( h_n \circ i_n | R_1 \cup R_2 \) converge uniformly on every compact set of \( R_1 \cup R_2 \). Let \( R'' \) be \( R_1 \cup R_2 \cup N(l) \). Since the \( h_n \circ i_n | R'' \) are homotopic each other for large \( n \), we see that the subsurfaces \( i_n(R'') \) on \( H_n \) are vertically parallel each other. This shows there is a region \( R'' \times [0,1] \) embedded in \( M \) preserving the horizontal and vertical leaves which contains a neighbourhood of the end of \( A \) such that \( R'' \times \{t\} \) tends to \( e \) as \( t \to 1 \).

In both cases, if every sequence of properly embedded connected horizontal surfaces tending to \( e \) is eventually contained in \( R' \times [0,1] \), then \( R'' \times [0,1] \) constitutes a neighbourhood of \( e \), contradicting the assumption that \( e \) is wild. Suppose that this is not the case. Then some component \( c \) of \( \text{Fr}R'' \) is not homotopic to a core curve of an annulus component of \( \partial M \) tending to \( e \). Therefore, we can repeat the above argument replacing \( A \) with \( c \times [0,1] \subset R'' \times [0,1] \) and get a larger subsurface \( R'' \) properly containing \( R'' \) and an leaf-preserving embedding \( R'' \times [0,1] \) such that \( R'' \times \{t\} \) tends to \( e \) as \( t \to 1 \). Since the topological type of \( S \) is fixed, in finite steps, this process terminates, and we get a neighbourhood of \( e \) in the form \( R_0 \times [0,1] \) for some essential subsurface \( R_0 \) of \( S \) (which might be \( S \) itself) such that \( \text{Fr}R_0 \times [0,1] \) lies on \( \partial M \). By our definition of brick decomposition of \( M \).
this \( R_0 \times [0, 1) \) is contained in one brick and \( e \) must be simply degenerate. This contradicts the assumption that \( e \) is wild. \( \square \)

**Proof of Corollary** \[ \] By Theorem \[A\] there is a brick manifold \( M \) having the properties listed in the theorem with a bi-Lipschitz homeomorphism \( g : M \to N_0 \). By Lemma \[3.1\] \( M \) has at most countably many ends, so does \( N_0 \).

Now, we turn to the second statement of our corollary. Let \( x \) be a point in the convex core of \( N_0 \) and \( y \) a point in \( M \) with \( g(y) = x \). We can assume that \( y \) does not lie in a geometrically finite brick since \( x \) is contained in the convex core. Let \( H_x \) be a properly embedded connected horizontal surface containing \( x \). If \( y \) is contained in \( V \) for a constant \( u \) as in the proof of Theorem \[A\] then \( x \) is contained in the \( \epsilon \)-Margulis tube and we are done. Otherwise take a shortest loop \( c_x \) on \( H_x \setminus V \) passing through \( x \). Recall that the modulus of horizontal surfaces outside \( V \) are bounded. Therefore, the length of \( c_x \) is bounded uniformly from above. Since \( g \) is a \( K \)-Lipschitz map, the length of \( g(c_x) \) is also bounded uniformly from above. This shows the injectivity radius at \( x \) is uniformly bounded from above. \( \square \)

**5.3. Proof of Theorem** \[C\]

**Proof of Theorem** \[C\]: Let \( M \) be a labelled brick manifold satisfying the conditions (i)-(iv) in Theorem \[A\] with end invariants given so that the condition (EL) is satisfied. Let \( K \) be a brick complex with \( \sqrt{K} = M \). By Subsections \[3.2\] and \[3.3\] we know that \( M \) admits a decomposition into blocks. We use the symbols \( V \) and \( V[k] \) etc. to denote the unions of tubes inducing the decomposition into blocks as before. This implies that the condition (BB) also holds. Since \( M \) is assumed to be embedded in \( S \times (0, 1) \), we often identify \( M \) and its image in \( S \times (0, 1) \).

For a simply degenerate brick \( B = F \times [0, 1) \) in \( K \), we consider a monotone increasing sequence \( \{p_n\} \) of positive numbers tending to 1 such that, for any \( n \in \mathbb{N} \), every component of \( F \times \{p_n\} \setminus \text{Int} V \) is homeomorphic to \( \Sigma_{0,1} \) and \( B(p_n) = F \times [0, p_n] \) contains at least \( n \) components of \( V[0] \). Let \( \{K_n\} \) be an ascending sequence of finite brick complexes with \( \bigcup_n K_n = K \). We may choose such \( K_n \) so that \( M_n = \sqrt{K_n} \) is connected for any \( n \in \mathbb{N} \). Since all geometrically finite bricks in \( K \) are peripheral in \( S \times (0, 1) \), the number of them is at most \(-2\chi(S)\). Hence we can choose \( \{K_n\} \) so that \( K_1 \) contains \( K_{gf} \).

Consider a brick complex \( K^-_n \) obtained from \( K_n \) by replacing all simply degenerate bricks \( B \) in \( K_n \) with \( B(p_n) \), and set \( M^-_n = \sqrt{K^-_n} \). For a simply degenerate brick \( B \) in \( K_n \), for all \( i \geq n \), the brick \( B \) is contained in \( K_i \) since \( \{K_i\} \) is ascending. Since \( B = \bigcup_{i \geq n} B(p_i) \), we have \( B \subseteq \bigcup_i M^-_i \). Therefore we see that \( M = \bigcup_n M^-_n \).

We fix a base point \( x_0 \) in \( M^-_1 \cap M[0] \). Let \( W_n[0] \) be the component of \( M^-_n \cap M[0] \) containing \( x_0 \), and \( W_n \) the union of \( W_n[0] \) and the components \( V[0] \) whose boundaries lie on \( \partial W_n[0] \). By the definition of \( W_n \), we have \( W_n \subseteq M^-_n \cap M[0] \). For any \( n \in \mathbb{N} \), there exists \( m \geq n \) such that every component of \( V[0] \) intersecting \( M^-_n \) is contained in the component of \( M^-_m \) containing \( x_0 \) since there are only finitely many components of \( V[0] \) intersecting \( M^-_n \). This means in particular that \( M^-_n \cap M[0] \) is contained in \( W_n \), and hence that \( \bigcup_m W_m = \bigcup_n (M^-_n \cap M[0]) = M[0] \). Taking a subsequence if necessary, we may assume that \( W_1 \) contains all of the geometrically finite bricks in \( K \).

Let \( V^\text{ext}_n \) be the union of components of \( V \setminus \text{Int} W_n \) intersecting \( \partial W_n \). It should be noted that \( V^\text{ext}_n \) might contain a component of \( V \setminus V[0] \). By the definition of \( W_n \), each component of \( W_n \cap V^\text{ext}_n \) is an annulus. Since \( M[0] \) is acylindrical, there
is no essential annulus \( A \) in \( W_n \) with \( \partial A \subset W_n \cap V_n \). Still there might be an
annulus \( A \) in \( S \times (0, 1) \) with \( \partial A \subset V_n \). Figure 5.2 illustrates such a situation.
By the acylindricity of \( M[0] \), for such an annulus \( A \), either there is a tube \( V_A \) in \( V \)
obstructing \( A \), or \( A \) goes out of \( M \) (i.e. \( A \) cannot be homotoped into \( M \)). In the
latter case, \( A \) must go out from a simply degenerate end \( B \), hence must intersect the
boundary of \( B(p_n) \) which \( V \) decompose into thrice-punctured spheres. Therefore,
also in this case there is a tube \( V_A \) in \( V \) obstructing \( A \). Since there are only finitely
many homotopy classes of such annuli, we can choose finitely many pairwise disjoint
tubes \( V_1', \ldots, V_m' \) in \( S \times (0, 1) \setminus W_n \) which obstruct all of such annuli. Then setting
\( V_n' = V_n^{\text{ext}} \cup V_1' \cup \cdots \cup V_m' \) and \( Z_n = S \times (0, 1) \setminus V_n' \) in \( S \times (0, 1) \), we see that \( Z_n \) is an
acylindrical finite brick manifold with a brick decomposition \( \mathcal{L} \) extending \( K_n \mid_{W_n} \).
(Figure 5.2 is an example of \( Z_n \).) Note that \( Z_n \) is not necessarily a subset of \( Z_{n+1} \)
although \( W_n \subset W_{n+1} \).

By the condition (BB) in Subsection 3.4, the closure of each component of
\( \partial W_n \setminus V_n^{\text{ext}} \) is homeomorphic to \( \Sigma_{0,3} \). For any \( B \) in \( \mathcal{L}_n \) with \( \partial B \cap V_n^{\text{ext}} \neq \emptyset \),
each component of \( \partial B \setminus \text{Int} V_n^{\text{ext}} \) is homeomorphic to \( \Sigma_{0,3} \). Therefore, we can
define a block decomposition on \( Z_n[0] \) whose restriction to \( W_n \) is equal to the original
block decomposition on \( W_n[0] \). Since \( W_n \) contains all geometrically finite bricks of
\( K \) and they are peripheral, we have \( \partial_{\infty} M[0] \subset \partial_{\infty} Z_n[0] \). We can define a model
metric on \( Z_n[0] \) whose conformal structure on \( \partial_{\infty} M[0] \) is equal to the given one.
Moreover the model metric on \( Z_n[0] \) is extended to the one on \( Z_n \) so that each
component of \( Z_n \setminus Z_n[0] \) is a Margulis tube. Since \( d_{Z_n}(x_0, Z_n \setminus W_n) \) goes to \( \infty \) as
\( n \to \infty \) with respect to the model metric \( d_{Z_n} \) on \( Z_n \), the geometric limit of \( \{Z_n \} \)
is equal to the geometric limit \( M^0 \) of \( \{W_n \} \).

By Thurston’s uniformisation theorem for atoroidal Haken manifolds [112] (see
Morgan [106] and Kapovich [92] for the proof), there exists a geometrically finite
hyperbolic 3-manifold \( N_n \) with a homeomorphism \( f_n : Z_n \setminus (N_n)_0 \) which can be
extended to the conformal map from \( \partial_{\infty} Z_n \) to \( \partial_{\infty} N_n \). By Theorem 4.1 (or the
original bi-Lipschitz theorem by Brock-Canary-Minsky), we may assume that \( f_n \) is a
\( K \)-bi-Lipschitz map. Since the geometric limit of \( Z_n \) based at \( x_0 \) is \( M^0 \), by the
Ascoli-Arzelà Theorem, \( \{f_n \} \) converges uniformly on any compact set of \( M^0 \) to a
\( K \)-bi-Lipschitz map \( f : M^0 \to N_0 \), where \( N \) is a geometric limit of \( N_n \). By our
construction of block decomposition of \( M[0] \), each simply degenerate brick \( F \times J \) has a sequence of tubes whose longitudes \( l_n \) regarded as simple closed curves on
\( F \) converge to the ending lamination \( \nu(e) \) given on the end \( e \) contained in \( F \times J \).
By our definition of the metric on \( M^0 \), and the Lipschitzity of \( f \), the lengths of
the \( l_n \) with respect to the model metric on \( M^0 \) are uniformly bounded. Since \( f \) is bi-Lipschitz, the geodesic loops \( l_n^* \) in \( N \) homotopic to \( f(l_n) \) have also uniformly
bounded lengths. This shows that \( l_n^* \) must tend to the end \( f(e) \) by the argument of
\S\S 6.3-6.4 of Bonahon [107]. Therefore, the end \( f(e) \) of \( N_0 \) has the ending lamination
\( f_* (\nu(e)) \).

Let \( G_n \) be a Kleinian group with \( \mathbb{H}^3 / G_n = N_n \). By the main theorem of [109],
there is a sequence of geometrically finite hyperbolic 3-manifolds \( N_n^k = \mathbb{H}^3 / G_n^k \) without
\( Z \)-cusps such that \( G_n^k \) converging algebraically to \( G_n \). We can choose \( N_n^k \)
so that the domain of discontinuity of \( G_n^k \) converges to that of \( G_n \) by defining \( G_n^k \)

The end of the previous paragraph is split into two lines. The following lines continue this.
Figure 5.2. This figure illustrates $Z_n$. The shaded region represents $W_n$ and the union of black rectangles is $V'_n$. $B(p_n) \in K_n$ is contained in a simply degenerate brick $B$ in $K_n$ with $B = B(p_n) \cup X$. $V_1$ splits $M_n$ into $W_n$ and $Y = M_n \setminus W_n$. $V_2$ and $V_3$ (resp. $V_4$ and $V_5$) are components of $\mathcal{V}_{ext}'$ parallel to each other in $S \times (0, 1)$. $V'_1$ (resp. $V'_2$) obstructs an annulus between $V_2$ and $V_3$ (resp. $V_4$ and $V_5$).

$k \to \infty$. By performing hyperbolic Dehn surgeries along the parabolic cusps of $N^k_n$ of type $(1, u_n)$ with sufficiently large $u_n \in \mathbb{N}$, we have quasi-Fuchsian manifolds $N^k_n$ geometrically approximating $N_n$ closer and closer as $k \to \infty$ as was shown in Bonahon-Otal [BO] and Ohshika [Oh1]. This gives rise to a sequence of quasi-Fuchsian manifolds $N^k_n$ converging geometrically to $N_n$ as $k \to \infty$. By the diagonal argument, we have a sequence of quasi-Fuchsian manifolds $N'_n$ converging geometrically to $N$. This completes the proof of Theorem C. \hfill \Box

5.4. Proof of Theorem D

Proof of Theorem D Let $G_1$ and $G_2$ be non-elementary geometric limits of Kleinian surface groups isomorphic to $\pi_1(S)$ preserving the parabolicity, and $f : N_1 = \mathbb{H}^3 / G_1 \to N_2 = \mathbb{H}^3 / G_2$ a homeomorphism preserving their end invariants. We may assume that $f((N_1)_0) = (N_2)_0$. By Theorem A there exists a brick manifold $M$ and a homeomorphism $\eta_1 : M \to (N_1)_0$ preserving the end invariants. Then the composition $\eta_2 = f \circ \eta_1 : M \to (N_2)_0$ is also a homeomorphism preserving the end invariants. By Theorem B we can make $\eta_1$ and $\eta_2$ $K$-bi-Lipschitz homeomorphisms. Therefore $\eta_2 \circ \eta_1^{-1} : (N_1)_0 \to (N_2)_0$ is a $K$-bi-Lipschitz homeomorphism preserving the end invariants, which can be extended to a bi-Lipschitz map $\Phi : N_1 \to N_2$. This $\Phi$ can be lifted to a bi-Lipschitz homeomorphism $\Phi : \mathbb{H}^3 \to \mathbb{H}^3$ between the universal coverings, which is equivariant with respect to the covering translations. Furthermore $\Phi$ is extended to a quasi-conformal homeomorphism $\Phi_\partial$ on the Riemann sphere $\hat{\mathbb{C}}$ such that $\Phi_\partial|_{\Omega_2}$ is a conformal homeomorphism from
$\Omega_{G_1}$ to $\Omega_{G_2}$. On the other hand, by Corollary 13 the injectivity radii in the convex cores of our manifolds $N_1$ and $N_2$ are bounded above. This makes it possible to apply McMullen’s generalisation (Theorem 2.9 in [Mc1]) of Sullivan’s rigidity theorem, and we see that $G_1$ and $G_2$ must be conformally conjugate. □

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