ON AN ANALOGUE OF THE ICHINO–IKEDA CONJECTURE FOR WHITTAKER COEFFICIENTS ON THE METAPLECTIC GROUP

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Abstract. In previous papers we formulated an analogue of the Ichino–Ikeda conjectures for Whittaker–Fourier coefficients of automorphic forms on classical group and the metaplectic group. In the latter case we reduced the conjecture to a local identity. In this paper we will prove the local identity in the $p$-adic case, and hence the global conjecture under simplifying conditions at the archimedean places.

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1. Introduction

Let $G$ be a quasi-split group over a number field $F$ with ring of adeles $\mathbb{A}$. In a previous paper [LM14a] we formulated (under some hypotheses) a conjecture relating Whittaker–Fourier coefficients of cusp forms on $G(F) \backslash G(\mathbb{A})$ to the Petersson inner product. This conjecture is in the spirit of conjectures of Sakellaridis–Venkatesh [SV12] and Ichino–Ikeda [II10], which attempt to generalize the classical work of Waldspurger [Wal85, Wal81]. In the case of (quasi-split) classical groups, as well as the metaplectic group (i.e., the metaplectic double cover of the symplectic group) we explicated this conjecture using the descent construction of Ginzburg–Rallis–Soudry [GRS11] and the functorial transfer of generic...
representations of classical groups by Cogdell–Kim–Piatetski-Shapiro–Shahidi [CKPSS01, CKPPSS04, CPSS11].

In a follow-up paper [LM13b] we reduced the global conjecture in the metaplectic case to a local conjectural identity. We also gave a purely formal argument for the case of $\widetilde{\text{Sp}}_n$ (i.e., ignoring convergence issues). In this paper we will prove the local identity in the $p$-adic case, justifying the heuristic analysis (and extending it to the general case).

Let us recall the conjecture of [LM14a] in the case of the metaplectic group $\widetilde{\text{Sp}}_n(\mathbb{A})$, the double cover of $\text{Sp}_n(\mathbb{A})$ with the standard co-cycle. We view $\text{Sp}_n(F)$ as a subgroup of $\widetilde{\text{Sp}}_n(\mathbb{A})$. For any genuine function $\varphi$ on $\text{Sp}_n(F) \backslash \widetilde{\text{Sp}}_n(\mathbb{A})$ we consider the Whittaker coefficient

$$W(\hat{\varphi}) = W^{\psi_N}(\hat{\varphi}) := (\text{vol}(N'(F) \backslash N'(\mathbb{A})))^{-1} \int_{N'(F) \backslash N'(\mathbb{A})} \hat{\varphi}(u) \psi_N(u)^{-1} \, du.$$  

Here $\psi_N$ is a non-degenerate character on $N'(\mathbb{A})$, trivial on $N'(F)$ where $N'$ is the group of upper triangular matrices in $\text{Sp}_n$. (We view $N'(\mathbb{A})$ as a subgroup of $\widetilde{\text{Sp}}_n(\mathbb{A})$.) We also consider the inner product

$$(\tilde{\varphi}, \tilde{\varphi}^\vee)_{\text{Sp}_n(F) \backslash \text{Sp}_n(\mathbb{A})} = (\text{vol}(\text{Sp}_n(F) \backslash \text{Sp}_n(\mathbb{A})))^{-1} \int_{\text{Sp}_n(F) \backslash \text{Sp}_n(\mathbb{A})} \tilde{\varphi}(g) \tilde{\varphi}^\vee(g) \, dg.$$  

of two square-integrable genuine functions on $\text{Sp}_n(F) \backslash \widetilde{\text{Sp}}_n(\mathbb{A})$.

Another ingredient in the conjecture of [LM14a] is a regularized integral

$$\int_{N'(F_S)}^{st} f(u) \, du$$

for a finite set of places $S$ and for a suitable class of smooth functions $f$ on $N'(F_S)$. Suffice it to say that if $S$ consists only of non-archimedean places then

$$\int_{N'(F_S)}^{st} f(u) \, du = \int_{N'_1} f(u) \, du$$

for any sufficiently large compact open subgroup $N'_1$ of $N'(F_S)$. (The definition of the regularized integral is different in the archimedean case, however in this paper we do not use regularized integrals over archimedean fields.)

The conjecture of [LM14a] is applicable for $\psi_N$-generic representations which are not exceptional, in the sense that their theta $\psi$-lift to $\text{SO}(2n + 1)$ is cuspidal (or equivalently, their theta $\psi$-lift to $\text{SO}(2n - 1)$ vanishes). By [GRS11, Chapter 11] which is also based on [CKPSS01], there is a one-to-one correspondence between these representations and automorphic representations $\pi$ of $\text{GL}_{2n}(\mathbb{A})$ which are the isobaric sum $\pi_1 \boxplus \cdots \boxplus \pi_k$ of pairwise inequivalent irreducible cuspidal representations $\pi_i$ of $\text{GL}_{2n_i}(\mathbb{A})$, $i = 1, \ldots, k$ (with $n_1 + \cdots + n_k = n$) such that $L^S(\frac{1}{2}, \pi_i) \neq 0$ and $L^S(s, \pi_i, \lambda^2)$ has a pole (necessarily simple) at $s = 1$ for all $i$. Here $L^S(s, \pi_i)$ and $L^S(s, \pi_i, \lambda^2)$ are the standard and exterior square (partial) $L$-functions, respectively. More specifically, to any such $\pi$ one constructs a $\psi_N$-generic representation $\tilde{\pi}$ of $\widetilde{\text{Sp}}_n(\mathbb{A})$, which is called the $\psi_N$-descent of $\pi$. The theta $\psi$-lift
of $\tilde{\pi}$ is the unique irreducible generic cuspidal representation of $\text{SO}(2n+1)$ which lifts to $\pi$.

**Conjecture 1.1.** ([LM14a, Conjecture 1.3]) Assume that $\tilde{\pi}$ is the $\psi_{\tilde{N}}$-descent of $\pi$ as above. Then for any $\tilde{\varphi} \in \tilde{\pi}$ and $\tilde{\varphi}^\vee \in \tilde{\pi}^\vee$ and for any sufficiently large finite set $S$ of places of $F$ we have

\begin{equation}
\tilde{W}\psi_{\tilde{N}}(\tilde{\varphi})\tilde{W}\psi_{\tilde{N}}^{-1}(\tilde{\varphi}^\vee) = 2^{-k}(\prod_{i=1}^n \zeta_F(2i)) \frac{L^S(\frac{1}{2}, \pi)}{L^S(1, \pi, \text{sym}^2)} \times \\
(\text{vol}(N'(\mathcal{O}_S)\backslash N'(F_S)))^{-1} \int_{N'(F_S)}^{st} (\tilde{\pi}(u)\tilde{\varphi}, \tilde{\varphi}^\vee)_{\text{Sp}_n(F)\backslash\text{Sp}_n(\mathbb{A})}\psi_{\tilde{N}}(u)^{-1} \, du
\end{equation}

Here $\zeta_F(s)$ is the partial Dedekind zeta function and $\mathcal{O}_S$ is the ring of $S$-integers of $F$ and $L^S(s, \pi, \text{sym}^2)$ is the symmetric square partial $L$-function of $\pi$.

The main result in [LM13b] is

**Theorem 1.2.** ([LM13b, Theorem 6.2]) In the above setup we have

\begin{equation}
\tilde{W}\psi_{\tilde{N}}(\tilde{\varphi})\tilde{W}\psi_{\tilde{N}}^{-1}(\tilde{\varphi}^\vee) = 2^{-k}(\prod_{i=1}^n \zeta_F(2i)) \frac{L^S(\frac{1}{2}, \pi)}{L^S(1, \pi, \text{sym}^2)} \times \\
(\prod_{v \in S} c_{\pi_v}^{-1})(\text{vol}(N'(\mathcal{O}_S)\backslash N'(F_S)))^{-1} \int_{N'(F_S)}^{st} (\tilde{\pi}(u)\tilde{\varphi}, \tilde{\varphi}^\vee)_{\text{Sp}_n(F)\backslash\text{Sp}_n(\mathbb{A})}\psi_{\tilde{N}}(u)^{-1} \, du
\end{equation}

where $c_{\pi_v}, v \in S$ are certain non-zero constants which depend only on the local representations $\pi_v$.

The main result of this paper is

**Theorem 1.3.** In Theorem 1.2 we have $c_{\pi_v} = \epsilon(\frac{1}{2}, \pi_v, \psi_v)$ (the root number of $\pi_v$) for all finite places $v$.

We also show in Proposition 7.3 that the root number of $\pi_v$ equals the central sign of $\pi_v$. In [ILM14] it is shown that Theorem 1.3 implies the formal degree conjecture of Hiraga–Ichino–Ikeda [HII08] (or more precisely, its metaplectic analogue) for generic square-integrable representations of $\tilde{\text{Sp}}_n$. Conversely, in the real case (where the formal degree conjecture is a reformulation of classical results of Harish-Chandra) it is shown that $c_{\pi_v} = \epsilon(\frac{1}{2}, \pi_v, \psi_v)$ if $\pi_v$ is square-integrable. (Note that in the square-integrable case, matrix coefficients are integrable of $N'(F_v)$ and no regularization is necessary.) We conclude:

**Corollary 1.4.** Conjecture 1.3 of [LM14a] holds if $F$ is totally real and $\pi_\infty$ is discrete series.

Theorem 1.3 is the culmination of the series of papers [LM14c], [LM13b] and [LM14b]. More precisely, the theorem can be formulated as an identity – the Main Identity (MI) explicated in §3.5, (based on [LM13b]) between integrals of Whittaker functions in the induced spaces of $\pi$ (in a local setting). In principle, formal manipulations using the
functional equations of [LM14c] reduce the identity to the results of [LM14b]. Such an argument was described heuristically for the case \( n = 1 \) in [LM13b, §7]. However, making this rigorous (even in the case \( n = 1 \) and for \( \pi \) supercuspidal) seems non-trivial because the integrals only converge as iterated integrals. This is the main task of the present paper.

In §3.5 we reduce the theorem to the cases where \( \pi \) is tempered and satisfies some good properties. Here we rely on the classification result of Matringe [Mat] on generic representations admitting a non-trivial \( \text{GL}_n \times \text{GL}_n \)-invariant functionals. We also use a globalization result ([ILM14, Appendix A]) which is based on a result of Sakellaridis-Venkatesh [SV12].

The rest of the argument is purely local. In §5 (after the preparatory §4) we start the manipulation of the left-hand side of the Main Identity. It is technically important to restrict oneself to certain special sections in the induced space. This is possible by a nonvanishing result on the Bessel function of generic representations proved in Appendix A. Another useful idea is to write the left-hand side of the Main Identity (for \( W \) special) as \( B(W, M(\pi, \frac{1}{2})W, s) \) where \( B(W, W^\vee, s), s \in \mathbb{C} \) is an analytic family of bilinear forms on \( I(\pi, s) \times I(\pi^\vee, -s) \), and \( M(\pi, s) \) is the intertwining operator on \( I(\pi, s) \). This relies on results of Baruch [Bar05], generalized to the present context in [LM13a].

The reason for introducing this analytic family is that because of convergence issues, we can only apply the functional equations of [LM14c] for \( \Re s \ll 0 \). This is the most delicate step, which is described in §6. It entails a further restriction on \( W^\wedge \) (which is fortunately harmless for our purpose). The upshot is an expression (for special \( W \) and \( W^\wedge \) and for \( \Re s \ll 0 \))

\[
B(W, M(\pi, s)W^\wedge, s) = \int E^\psi(M(\pi, s)W, -s; t)E^\psi^{-1}(W^\wedge, s; t) \frac{dt}{|\det t|}
\]

where \( t \) is integrated over a certain \( n \)-dimensional torus and \( E^\psi \) is a certain integral of \( W \). The restriction on \( W^\wedge \) ensures that the integrand is compactly supported. Moreover, by [LM14b] after suitable averaging, \( E^\psi(W, s; t) \) is an entire function on \( s \). Therefore the above identity is meaningful for all \( s \) where \( M(\pi, s) \) is holomorphic. Speciliaing to \( s = \frac{1}{2} \) the remaining assertions are that \( E^\psi(M(\pi, \frac{1}{2})W, -\frac{1}{2}; t) \) (or rather, a suitable averaging thereof) is a constant function in \( t \) whose value can be explicited, while the integral of \( E^\psi^{-1}(W^\wedge, \frac{1}{2}; t) \) factors through \( M(\pi, \frac{1}{2})W^\wedge \), again in an explicit way. This is the content of Propositions 7.1 and 7.2 respectively, which follow from the representation-theoretic results established in [LM14b].

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2. Notation and preliminaries

For the convenience of the reader we introduce in this section the most common notation that will be used throughout.
We fix a positive integer \( n \). The letters \( i, j, k, l, m \) will denote auxiliary positive integers. Let \( F \) be a local field of characteristic 0.

2.1. Groups, homomorphisms and group elements. All algebraic groups are defined over \( F \). We typically denote algebraic varieties (or groups) over \( F \) by boldface letters (e.g., \( \mathbf{X} \)) and denote their set (or group) of \( F \)-points by the corresponding plain letter (e.g., \( X \)). (In most cases \( X \) will be clear from the context.)

- \([x, y] = xyx^{-1}y^{-1}\) denotes the commutator of \( x \) and \( y \).
- \( I_m \) is the identity matrix in \( \mathrm{GL}_m \), \( w_m \) is the \( m \times m \)-matrix with ones on the nonprincipal diagonal and zeros elsewhere; \( J_m := (-w_m, w_m) \in \mathrm{GL}_{2m} \).
- For any group \( Q \), \( Z_Q \) is the center of \( Q \); \( e \) is the identity element of \( Q \). We denote the module function of \( Q \) by \( \delta_Q \).
- \( \mathrm{Mat}_m \) is the vector space of \( l \times m \) matrices over \( F \).
- \( x \mapsto x^t \) is the transpose on \( \mathrm{Mat}_{m,m} \); \( x \mapsto \tilde{x} \) is the twisted transpose map on \( \mathrm{Mat}_{m,m} \) given by \( \tilde{x} = w_m x^t w_m \); \( g \mapsto g^* \) is the outer automorphism of \( \mathrm{GL}_m \) given by \( g^* = w_m^{-1} (g')^{-1} w_m \).
- \( \mathfrak{s}_m = \{ x \in \mathrm{Mat}_{m,m} : \tilde{x} = x \} \).
- \( \mathbb{M} = \mathrm{GL}_{2n}, \mathbb{M}' = \mathrm{GL}_n \).
- \( G = \mathrm{Sp}_{2n} = \{ g \in \mathrm{GL}_{4n} : g^t J_{2n} g = J_{2n} \} \).
- \( G' = \mathrm{Sp}_n = \{ g \in \mathrm{GL}_{2n} : g^t J_n g = J_n \} \).
- \( G' \) is embedded as a subgroup of \( G \) via \( g \mapsto \eta(g) = \mathrm{diag}(I_n, g, I_n) \).
- \( P = M \times U \) (resp., \( P' = M' \times U' \)) is the Siegel parabolic subgroup of \( G \) (resp., \( G' \)), with its standard Levi decomposition.
- \( \tilde{P} = P^t \) is the opposite parabolic of \( P \), with unipotent radical \( \tilde{U} = U^t \).
- We use the isomorphism \( \varrho(g) = \mathrm{diag}(g, g^*) \) to identify \( \mathbb{M} \) with \( M \subset G \). Similarly for \( \varrho' : \mathbb{M}' \to M' \subset G' \).
- We use the embeddings \( \eta_{\mathbb{M}}(g) = \mathrm{diag}(g, I_n) \) and \( \eta_{\mathbb{M}}^\vee(g) = \mathrm{diag}(I_n, g) \) to identify \( \mathbb{M}' \) with subgroups of \( \mathbb{M} \). We also set \( \eta_M = \varrho \circ \eta_{\mathbb{M}} \) and \( \eta_{\mathbb{M}}^\vee = \varrho \circ \eta_{\mathbb{M}}^\vee = \eta \circ \varrho' \).
- We denote by \( \star \) the involution on \( M \) defined by \( \varrho(m)^\star = \varrho(m^*) \).
- \( K \) is the standard maximal compact subgroup of \( G \).
- \( N \) is the standard maximal unipotent subgroup of \( G \) consisting of upper unitriangular matrices; \( T \) is the maximal torus of \( G \) consisting of diagonal matrices; \( B = T \times N \) is the Borel subgroup of \( G \).
- For any subgroup \( X \) of \( G \) we write \( X' = \eta^{-1}(X), X_M = X \cap M \) and \( X_M = g^{-1}(X_M) \); similarly \( X_{M'} = X' \cap M' \) and \( X_{M'} = g^{-1}(X_{M'}) \).
- \( \ell_M : \mathrm{Mat}_{n,n} \to N_{\mathbb{M}} \) is the homomorphism given by \( \ell_M(x) = (I_n x I_n) \). Similarly define \( \overline{\ell}_M(x) = (I_n x I_n) \).
- \( \ell : \mathfrak{s}_{2n} \to U \) is the isomorphism given by \( \ell(x) = (I_{2n} x I_{2n}) \). Similarly \( \overline{\ell}(x) = (I_{2n} x I_{2n}) \) is the isomorphism from \( \mathfrak{s}_{2n} \) to \( \tilde{U} \).
- \( \hat{G} = \hat{\mathrm{Sp}}_n \) is the metaplectic group, i.e., the two-fold cover of \( G' \). We write elements of \( \hat{G} \) as pairs \( (g, \epsilon) \), \( g \in G \), \( \epsilon = \pm 1 \) where the multiplication is given by Rao’s cocycle. (Cf. [GRS11].)
When \( g \in G' \), we write \( \tilde{g} = (g, 1) \in \tilde{G} \). (Of course, \( g \mapsto \tilde{g} \) is not a group homomorphism.)

\( \tilde{N} \) is the inverse image of \( N' \) under the canonical projection \( \tilde{G} \to G' \). We will identify \( N' \) with a subgroup of \( \tilde{N} \) via \( n \mapsto \tilde{n} \).

\( \xi_m = (0, \ldots, 0, 1) \in F^m \).

\( \mathcal{P} \) is the mirabolic subgroup of \( \tilde{M} \) consisting of the elements \( g \) such that \( \xi_{2n}g = \xi_{2n} \).

\( T'' = Z_M \times \eta_M(T''_{M'}) = \{ \text{diag}(t_1, \ldots, t_{2n}) : t_1 = \cdots = t_n \} \subset T_M \).

\( E = \text{diag}((-1)^{i-1}) \in \tilde{M} \). \( H \) is the centralizer of \( \varrho(E) \) in \( \tilde{G} \). It is isomorphic to \( \text{Sp}_n \times \text{Sp}_n \).

\( H_{\tilde{M}} \) is then the centralizer of \( E \) in \( \tilde{M} \). It is isomorphic to \( \text{GL}_n \times \text{GL}_n \).

\( R_{\tilde{m}} \) is the subspace of \( \text{Mat}_{m,m} \) consisting of the matrices whose first column is zero.

\( \tilde{R}_{\tilde{m}} \) is then the subspace of \( \text{Mat}_{m,m} \) consisting of matrices whose last row is zero.

\( U^\sim \) (resp. \( \widetilde{U}^\sim \)) is the image under \( \ell \) (resp. \( \widetilde{\ell} \)) of the vector subspace of \( \mathfrak{s}_{2n} \) consisting of \( (\frac{x}{y}) \) with \( x \in R_n \) and \( y \in \mathfrak{s}_n \).

\( \epsilon_{i,j} \in \text{Mat}_{m,m} \) is the matrix with one at the \((i,j)\)-entry and zeros elsewhere.

The one-parameter root subgroups \( N_{i,j}^M \), \( 1 \leq i < j \leq 2n \) of \( N_{\tilde{M}}^\sim \) are the groups \( \{I_{2n} + \lambda\epsilon_{i,j} : \lambda \in F\} \).

\( w_{0} = (w_{0_{2n}}, w_{0_{2n}}) = J_{2n} \in G \) represents the longest Weyl element of \( G \).

\( w_{0} = (-w_{0_{2n}}, w_{0_{2n}}) = J_{2n} \in G' \) represents the longest Weyl element of \( G' \).

\( w_{U} = (-I_{2n}, I_{2n}) \in G \) represents the longest \( M \)-reduced Weyl element of \( G \).

\( w_{U}' = (-I_{2n}, I_{2n}) \in G \) represents the longest \( M' \)-reduced Weyl element of \( G' \).

\( w_{0}^M = w_{2n} \in M \) represents the longest Weyl element of \( \tilde{M} \); \( w_{0}^M = \varrho(w_{0}^M) \).

\( w_{0}^{M'} = w_{n} \in M' \) represents the longest Weyl element of \( \tilde{M} \); \( w_{0}^{M'} = \varrho(w_{0}^{M'}) \).

\( w_{2n,n} := (I_{n}, I_{n}) \in \tilde{M} \).

\( \gamma = w_{U}\eta(w_{U}')^{-1} = \begin{pmatrix} I_{n} & I_{n} \\ -I_{n} & I_{n} \end{pmatrix} \in G \).

\( \epsilon_{1} = \ell_M(\epsilon_{1,n}) \left( \begin{array}{c} w_{0}^M \\ I_{n} \end{array} \right) \in \tilde{M} \), \( \epsilon_{2} = \ell_M(\text{diag}(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \ldots)w_{0}^{M'}) \in N_{\tilde{M}} \) and \( \epsilon_{3} = \varrho(\ell_{M}(-\epsilon_{n,n})) \in N_{\tilde{M}} \).

\( V_{R} \) is the unipotent subgroup \( \ell_M(R_n) \rtimes \eta_M((N_{m'}^M)') = \left( \begin{array}{c} I_{n} \\ x \end{array} \right) : n \in N_{M'}^M, x \in R_n \} \) of \( \tilde{M} \).

\( N_{0}^\sim_{\tilde{M}} \) is the subgroup of \( N_{\tilde{M}} \) consisting of matrices whose last column is \((0, \ldots, 0, 1)'\).

\( V \) (resp., \( V_{0} \)) is the unipotent radical in \( G \) of the standard parabolic subgroup with Levi \( \text{GL}_n^\sim \times \text{Sp}_n \) (resp., \( \text{GL}_n^{n-1} \times \text{Sp}_{n+1} \)). Thus \( N = V \rtimes \eta(N') \), \( V_{0} \) is normal in \( V \) and \( V/V_{0} \) is isomorphic to the Heisenberg group of dimension \( 2n + 1 \).

\( \gamma = V \cap \gamma^{-1}N\gamma = \eta(w_{U'})(N_{M'} \cap V)\eta(w_{U'})^{-1} = \eta_M(N_{M'}^M) \rtimes \{\ell((x, x)) : x \in \text{Mat}_{n,n}\} \).

2.2. Characters. We fix a non-trivial character \( \psi \) of \( F \).

\( \psi_{N_{\tilde{M}}} \) is a non-degenerate character of \( N_{\tilde{M}} \) given by

\( \psi_{N_{\tilde{M}}}(u) = \psi(u_{1,2} + \cdots + u_{2n-1,2n}). \)

\( \psi_{N_{M}} \) is the non-degenerate character of \( N_{M} \) such that \( \psi_{N_{M}}(\varrho(u)) = \psi_{N_{\tilde{M}}}(u) \).
• $\psi_{N'_{M'}}$ is the non-degenerate character of $N'_{M'}$ given by
  $$\psi_{N'_{M'}}(u') = \psi(u'_{1,2} + \cdots + u'_{n-1,n}).$$

• $\psi_{N'_{M'}}$ is the non-degenerate character of $N'_{M'}$ such that $\psi_{N'_{M'}}(g'(u)) = \psi_{N'_{M'}}(u)$.
  Thus, $\psi_{N'_{M'}}(n) = \psi_{N_{M}}(\gamma\eta(n)\gamma^{-1}).$

• $\psi_{U'}$ is the character on $U'$ given by $\psi_{U'}(u) = \psi(\frac{1}{2}u_{n,n+1})^{-1}.$

• $\psi_{\tilde{N}}$ is the genuine character of $\tilde{N}$ whose restriction to $N'$ is the non-degenerate character
  $$\psi_{\tilde{N}}(nu) = \psi_{N'_{M'}}(n)\psi_{U'}(u), \ q \in N'_{M'}, u \in U'.$$

• $\psi_{N}$ is the degenerate character on $N$ given by $\psi_{N}(nu) = \psi_{N_{M}}(n)$ for any $n \in N_{M}$ and $u \in U$.

• $\psi_{V}$ is the character of $V$ given by
  $$\psi_{V}(v) = \psi(v_{1,2} + \cdots + v_{n-1,n})^{-1}.$$

• $\psi_{V_{R}}$ is the character on $V_{R}$ with $\psi_{V_{R}}\left(\left(\begin{smallmatrix} I_{n} & \eta \end{smallmatrix}\right)\right) = \psi_{N'_{M'}}(n)$.

• $\hat{\psi}_{U}$ and $\hat{\psi}_{U}$ are the characters on $U$ and $\bar{U}$ respectively given by $\hat{\psi}_{U}(\ell(v)) = \hat{\psi}_{U}(\ell(v)) = \psi(\frac{1}{2}(v_{n,n+1} - v_{2n,1})).$

• $\hat{\psi}_{\bar{U}}$ is the character on $\bar{U}$ given by $\hat{\psi}_{\bar{U}}(\ell(v)) = \psi(v_{1,1}).$

2.3. Other notations.

• We use the notation $a \ll_{d} b$ to mean that $a \leq cb$ with $c > 0$ a constant depending on $d$.

• For any $g \in G$ define $\nu(g)$ by $\nu(ug(m)k) = \det m$ for any $u \in U$, $m \in M$, $k \in K$.
  Let $\nu'(g) = \nu(\eta(g))$ for $g \in G'$.

• $\text{CSGR}(Q)$ is the set of compact open subgroups of a topological group $Q$.

• When $K_{0} \in \text{CSGR}(G')$ is sufficiently small, $k \mapsto \tilde{k}$ gives a splitting of $K_{0}$ in the covering $\tilde{G}$. We let $\text{CSGR}^\times(G')$ be the set of those $K_{0} \in \text{CSGR}(G')$, and identify any $K_{0} \in \text{CSGR}^\times(G')$ with a subgroup of $\tilde{G}$ via $k \mapsto \tilde{k}$.

• For an $\ell$-group $Q$ let $C(Q)$ and $S(Q)$ be the space of continuous functions and Schwartz functions on $Q$ respectively.

• When $F$ is $p$-adic, if $Q'$ is a closed subgroup of $Q$ and $\chi$ is a character of $Q'$, we denote by $C(Q' \setminus Q, \chi)$ (resp., $C^{\text{sm}}(Q' \setminus Q, \chi)$, $C_{c}^{\infty}(Q' \setminus Q, \chi)$) the spaces of continuous (resp. $Q$-smooth,\(^1\), smooth and compactly supported modulo $Q'$) complex-valued left $(Q', \chi)$-equivariant functions on $Q$.

• For an $\ell$-group group $Q$ we write $\text{Irr} Q$ for the set of equivalence classes of irreducible representations of $Q$. If $Q$ is reductive we also write $\text{Irr}_{\text{sqr}} Q$ and $\text{Irr}_{\text{temp}} Q$ for the subsets of irreducible unitary square-integrable (modulo center) and tempered representations respectively. We write $\text{Irr}_{\text{gen}} M$ and $\text{Irr}_{\text{meta}} M$ for the subset of irreducible generic representations of $M$ and representations of metaplectic type.

\(^1\)i.e., right-invariant under an open subgroup of $Q$
Fix a polarization $W = W_+ \oplus W_-$. The group $\text{Sp}(W)$ acts on the right on $W$. We write a typical element of $\text{Sp}(W)$ as $(A, B)$ where $A \in \text{Hom}(W_+, W_+)$, $B \in \text{Hom}(W_+, W_-)$, $C \in \text{Hom}(W_-, W_+)$ and $D \in \text{Hom}(W_-, W_-)$. Let $\tilde{\text{Sp}}(W)$ be the metaplectic two-fold cover of $\text{Sp}(W)$ with respect to the Rao cocycle determined by the splitting. Consider the Weil representation $\omega_\psi$ of the group $\mathcal{H}_W \rtimes \tilde{\text{Sp}}(W)$ on $\mathcal{S}(W_+)$. Explicitly, for any $\Phi \in \mathcal{S}(W_+)$
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and $X \in W_+$ we have

\[(2.1a) \quad \omega_\psi(a, 0) \Phi(X) = \Phi(X + a), \quad a \in W_+,\]
\[(2.1b) \quad \omega_\psi(b, 0) \Phi(X) = \psi((X, b)) \Phi(X), \quad b \in W_-,
\[(2.1c) \quad \omega_\psi(0, t) \Phi(X) = \psi(t) \Phi(X), \quad t \in F,
\[(2.1d) \quad \omega_\psi((\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}), e) \Phi(X) = e \gamma_\psi(\det g) |\det g|^{\frac{1}{2}} \Phi(X^g), \quad g \in \text{GL}(W_+),
\[(2.1e) \quad \omega_\psi((\begin{smallmatrix} 1 & B \\ 0 & 1 \end{smallmatrix}), e) \Phi(X) = e \psi(\frac{1}{2} \langle X, XB \rangle) \Phi(X), \quad B \in \text{Hom}(W_+, W_-) \text{ self-dual},

where $\gamma_\psi$ is Weil’s factor.

We now take $W = F^{2n}$ with the standard symplectic form

\[\langle (x_1, \ldots, x_{2n}), (y_1, \ldots, y_{2n}) \rangle = \sum_{i=1}^{2n} x_i y_{2n+1-i} - \sum_{i=1}^{2n} y_i x_{2n+1-i}\]

and the standard polarization $W_+ = \{(x_1, \ldots, x_n, 0, \ldots, 0)\}$, $W_- = \{(0, \ldots, 0, y_1, \ldots, y_n)\}$. (We identify $W_+$ and $W_-$ with $F^n$.) The corresponding Heisenberg group is isomorphic to the quotient $V/V_0$ via $v \mapsto v_H := (v_{n+j})_{j=1, \ldots, 2n, \frac{1}{2}v_{n+3n+1}}$; (recall that $V$ and $V_0$ are the unipotent groups defined in §2.1.)

For $X = (x_1, \ldots, x_n), X' = (x'_1, \ldots, x'_{2n}) \in F^{2n}$ define

\[\langle X, X' \rangle' = x_1 x'_1 + \cdots + x_n x'_n.\]

For $\Phi \in \mathcal{S}(F^n)$ define

\[\hat{\Phi}(X) = \int_{F^n} \Phi(X') \psi(\langle X, X' \rangle') \, dX'.\]

Then, the Weil representation is realized on $\mathcal{S}(F^n)$ as follows.

\[(2.2a) \quad \omega_\psi((g', h)) \Phi(X) = |\det(h)|^{\frac{1}{2}} \beta_\psi(g') \Phi(X^h), \quad h \in \mathbb{H},
\[(2.2b) \quad \omega_\psi(w^{-1}_{\mathcal{U}}) \Phi(X) = \beta_\psi(w^{-1}_{\mathcal{U}}) \Phi(X),
\[(2.2c) \quad \omega_\psi((\begin{smallmatrix} 1 & B \\ 0 & 1 \end{smallmatrix})) \Phi(X) = \psi(\frac{1}{2} \langle X, XB \rangle) \Phi(X), \quad B \in s_n.

Here $\beta_\psi(g), g \in G'$ are certain roots of unity.

We extend $\omega_\psi$ to a representation $\omega_\psi$ of $V \rtimes \hat{G}$ by setting

\[(2.3) \quad \omega_\psi(v \tilde{g}) \Phi = \psi_V(v) \omega_\psi(v_H)(\omega_\psi(g) \Phi), \quad v \in V, \quad g \in G'.

Then for any $g \in G', v \in V$ we have

\[(2.4) \quad \omega_\psi((\eta(g)v \eta(g)^{-1}) \tilde{g}) \Phi = \omega_\psi(\tilde{g})(\omega_\psi(v) \Phi).\]
26. Stable integral. For the rest of the section, we assume \( F \) is \( p \)-adic.

Suppose that \( U_0 \) is a unipotent group over \( F \) with a fixed Haar measure \( du \). Recall that the group generated by a relatively compact subset of \( U_0 \) is relatively compact. In particular, the set \( \mathcal{CSGR}(U_0) \) is directed. Recall the following definition of stable integral in [LM14a]:

**Definition 2.1.** Let \( f \) be a smooth function on \( U_0 \). We say that \( f \) has a stable integral over \( U_0 \) if there exists \( U_1 \in \mathcal{CSGR}(U_0) \) such that for any \( U_2 \in \mathcal{CSGR}(U_0) \) containing \( U_1 \) we have

\[
(2.5) \quad \int_{U_2} f(u) \, du = \int_{U_1} f(u) \, du.
\]

In this case we write \( \int_{U_0}^{st} f(u) \, du \) for the common value (2.5) and say that \( \int_{U_0}^{st} f(u) \, du \) stabilizes at \( U_1 \). In other words, \( \int_{U_0}^{st} f(u) \, du \) is the limit of the net \( (\int_{U_1} f(u) \, du)_{U_1 \in \mathcal{CSGR}(U_0)} \) with respect to the discrete topology of \( \mathbb{C} \).

27. Remarks on convergence. Frequently, we will make use of the following elementary remark.

**Remark 2.2.** Let \( H \) be any algebraic group over \( F \) and \( H' \) a closed subgroup. Assume that \( \delta_H|_{H'} \equiv \delta_{H'} \). Suppose that \( f \in C^{sm}(H) \) and that the integral \( \int_{H} f(h) \, dh \) converges absolutely. Then the same is true for \( \int_{H'} f(h') \, dh' \).

We will also use the following elementary lemma.

**Lemma 2.3.** Let \( G_0 \) be a group and let \( C \) and \( D \) be subgroups which are isomorphic to \( F^m \) for some \( m \) and which are in duality with respect to a non-degenerate pairing \( \langle \cdot, \cdot \rangle' : C \times D \to F \). Let \( \chi \) be a continuous character of \( D \). Assume that \( f \in C^{sm}(G_0) \) is such that \( f(cd) = f(c)\chi(d)\psi((c,d)') \) for all \( c \in C, d \in D \). Then \( f|_C \) is compactly supported. Moreover, if \( f \in C(G_0)^{K_0} \) for some \( K_0 \in \mathcal{CSGR}(G_0) \) and \( \langle \cdot, \cdot \rangle' \) is restricted to a compact set \( \tilde{\Omega} \) of non-degenerate pairings then the support of \( f|_C \) is bounded in terms of \( K_0, \tilde{\Omega} \) and the conductor of \( \chi \).

3. Statement of main result

3.1. Local Fourier–Jacobi transform and explicit local descent. For any \( f \in C(G) \) and \( s \in \mathbb{C} \) define \( f_s(g) = f(g)\nu(g)^s, \, g \in G \). Let \( \pi \in \text{Irr}_{\text{gen}} M \) with Whittaker model \( \mathbb{W}^{\psi N M}(\pi) \). Let \( \text{Ind}(\mathbb{W}^{\psi N M}(\pi)) \) be the space of \( G \)-smooth left \( U \)-invariant functions \( W : G \to \mathbb{C} \) such that for all \( g \in G \), the function \( \delta_P(m)^{-\frac{s}{2}} W(mg) \) on \( M \) belongs to \( \mathbb{W}^{\psi N M}(\pi) \). For any \( s \in \mathbb{C} \) we have a representation \( \text{Ind}(\mathbb{W}^{\psi N M}(\pi), s) \) on the space \( \text{Ind}(\mathbb{W}^{\psi N M}(\pi)) \) given by \( (I(s, g)W)_x(x) = W_s(xg), \, x, g \in G \).

Following [GRS98], for any \( W \in \text{Ind}(\mathbb{W}^{\psi N M}(\pi)) \) and \( \Phi \in \mathcal{S}(F^n) \) define a genuine function on \( \tilde{G} \):

\[
(3.1) \quad A^\psi(W, \Phi, \tilde{g}, s) = \int_{V_\gamma \setminus V} W_s(\gamma \nu(\eta(g)))\omega_{\psi^{-1}(v\tilde{g})}\Phi(\xi_n) \, dv, \quad g \in G'.
\]
Lemma 3.1. Suppose that $F$ is $p$-adic. Then for any $K_0 \in C\mathcal{S}\mathcal{G}\mathcal{R}(G)$ there exists $\Omega \in C\mathcal{S}\mathcal{G}\mathcal{R}(\bar{U}^\gamma)$ such that for any $W \in C(N\backslash G, \psi_N)^{K_0}$ the support of $W|_{\bar{U}^\gamma}$ is contained in $\Omega$.

Define the intertwining operator $M(\pi, s) = M(s): \text{Ind}(\mathbb{W}^{\psi_N}(\pi), s) \to \text{Ind}(\mathbb{W}^{\psi_N}(\pi^\vee), -s)$ by (the analytic continuation of)

$$M(s)W(g) = \nu(g)^s \int_U W_\psi(\theta(t)w_u, ug) \, du$$

where $t = E = \text{diag}(1, -1, \ldots, 1, -1)$ is introduced in order to preserve the character $\psi_{NM}$. By abuse of notation we will also denote by $M(\pi, s)$ the intertwining operator $\text{Ind}(\mathbb{W}^{\psi_N}(\pi), s) \to \text{Ind}(\mathbb{W}^{\psi_N}(\pi^\vee), -s)$ defined in the same way.

Recall that $H_M$ is the centralizer of $E$, isomorphic to $\text{GL}_n \times \text{GL}_n$. The involution $w_0^M$ lies in the normalizer of $H_M$. We consider the class $\text{Irr}_{\text{meta}} M$ of irreducible representations of $M$ which admit a continuous non-zero $H_M$-invariant linear form $\ell$ on the space of $\pi$. It is known that any such $\pi$ is self-dual and $\ell$ is unique up to a scalar ([JR96, AG09]). Thus, $\ell \circ \pi(w_0^M) = \epsilon_\pi \ell$ where $\epsilon_\pi \in \{\pm 1\}$ does not depend on the choice of $\ell$. By [LM13b, Theorem 3.2], when $F$ is $p$-adic, for any $\pi \in \text{Irr}_{\text{meta,gen}} M$ we have

$$\epsilon_\pi = \epsilon\left(\frac{1}{2}, \pi, \psi\right)$$

where $\epsilon(s, \pi, \psi)$ is the standard $\epsilon$-factor attached to $\pi$.

Let $\pi \in \text{Irr}_{\text{gen,meta}} M$, considered also as a representation of $M$ via $\varrho$. By [LM13b, Proposition 4.1] $M(s)$ is holomorphic at $s = \frac{1}{2}$. Denote by $\mathcal{D}_\psi(\pi)$ the space of Whittaker functions on $\tilde{G}$ generated by $A^\psi(M(\frac{1}{2})W, \Phi, \cdot, -\frac{1}{2})$, $W \in \text{Ind}(\mathbb{W}^{\psi_N}(\pi))$, $\Phi \in \mathcal{S}(F^n)$. This defines an explicit descent map $\pi \mapsto \mathcal{D}_\psi(\pi)$ on $\text{Irr}_{\text{gen,meta}} M$. By [GRS99, Theorem in §1.3] $\mathcal{D}_\psi(\pi) \neq 0$.

3.2. **Good representations.** Let $\pi \in \text{Irr}_{\text{gen}} M$ and $\tilde{\sigma} \in \text{Irr}_{\text{gen,gen}}^{-1} \tilde{G}$ with Whittaker model $\mathbb{W}^{\psi_{\tilde{N}}}(\tilde{\sigma})$. Following Ginzburg–Rallis–Soudry [GRS98], for any $\tilde{W} \in \mathbb{W}^{\psi_{\tilde{N}}}(\tilde{\sigma})$, $W \in \text{Ind}(\mathbb{W}^{\psi_N}(\pi))$ and $\Phi \in \mathcal{S}(F^n)$ define the local Shimura type integral

$$\tilde{J}(\tilde{W}, W, \Phi, s) := \int_{N^\gamma \backslash \tilde{G}^\gamma} \tilde{W}(\tilde{g}) A^\psi(W, \Phi, \tilde{g}, s) \, d\tilde{g}.$$

By [GRS98, §6.3] and [GRS99] $\tilde{J}$ converges for $\Re s \gg \pi, \tilde{\sigma}$ and admits a meromorphic continuation in $s$. Moreover, for any $s \in \mathbb{C}$ we can choose $\tilde{W}$, $W$ and $\Phi$ such that $\tilde{J}(\tilde{W}, W, \Phi, s) \neq 0$.

Let $\pi \in \text{Irr}_{\text{gen,meta}} M$. We say that $\pi$ is **good** if the following conditions are satisfied for all $\psi$:

1. $\mathcal{D}_\psi(\pi)$ is irreducible.
Corollary 3.4. \( \tilde{J}(\tilde{W}, W, \Phi, s) \) is holomorphic at \( s = \frac{1}{2} \) for any \( \tilde{W} \in \mathcal{D}_{\psi^{-1}}(\pi), W \in \text{Ind}(\mathbb{W}_{NM}^{\psi}(\pi)) \) and \( \Phi \in S(F^n) \).

(3) There is a non-degenerate \( \tilde{G} \)-invariant pairing \([\cdot, \cdot]\) on \( \mathcal{D}_{\psi^{-1}}(\pi) \times \mathcal{D}_\psi(\pi) \) such that
\[
\tilde{J}(\tilde{W}, W, \Phi, \frac{1}{2}) = [\tilde{W}, A^\psi(M(\frac{1}{2})W, \Phi, \cdot, -\frac{1}{2})]
\]
for any \( \tilde{W} \in \mathcal{D}_{\psi^{-1}}(\pi) \), \( W \in \text{Ind}(\mathbb{W}_{NM}^{\psi}(\pi)) \) and \( \Phi \in C_c^\infty(F^n) \).

This property was introduced and discussed in [LM13b, §5]. In particular, if \( \pi \) is good and \( \tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi) \) then there exists a constant \( c_\pi \) such that for any \( \tilde{W} \in \mathbb{W}_{NM}^{\psi^{-1}}(\tilde{\pi}), W \in \text{Ind}(\mathbb{W}_{NM}^{\psi}(\pi)) \) and \( \Phi \in S(F^n) \) we have
\[
\int_{N'}^{st} \tilde{J}(\tilde{\pi}(n)\tilde{W}, W, \Phi, \frac{1}{2})\psi_N(n) \, dn = c_\pi \tilde{W}(e)A^\psi(M(\frac{1}{2})W, \Phi, e, -\frac{1}{2}).
\]

In other words, for any \( W \in \text{Ind}(\mathbb{W}_{NM}^{\psi}(\pi)), W^\wedge \in \text{Ind}(\mathbb{W}_{NM}^{\psi^{-1}}(\pi)), \Phi, \Phi^\vee \in S(F^n) \)
\[
\int_{N'}^{st} \tilde{J}(A^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, \Phi^\vee, \cdot, -\frac{1}{2}), W, \Phi, \frac{1}{2})\psi_N(n) \, dn
\]
\[
= c_\pi A^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, \Phi^\vee, e, -\frac{1}{2})A^\psi(M(\frac{1}{2})W, \Phi, e, -\frac{1}{2}).
\]

Let \( F \) be a \( p \)-adic field. In the rest of the paper we will prove the following statement:

**Theorem 3.2.** For any unitarizable \( \pi \in \text{Irr}_{gen, meta} \mathbb{M} \) which is good we have \( c_\pi = \epsilon_\pi \).

**Remark 3.3.** Of course, we expect Theorem 3.2 to hold in the archimedean case as well. However, we will not deal with the archimedean case in this paper.

Of particular importance will be the following special case (see Remark 3.7 below):

**Corollary 3.4.** Suppose \( \pi = \tau_1 \times \cdots \times \tau_l \) where \( \tau_j \in \text{Irr}_{sq, meta} \text{GL}_{2m_j}, \) \( j = 1, \ldots, l \) are distinct and \( n = m_1 + \cdots + m_l \). Then \( \pi \) is good and \( \tilde{\pi} := \mathcal{D}_{\psi^{-1}}(\pi) \) is square-integrable. Moreover,
\[
\int_{N'} \tilde{J}(\tilde{\pi}(n)\tilde{W}, W, \Phi, \frac{1}{2})\psi_N(n) \, dn = \epsilon_\pi \tilde{W}(e)A^\psi(M(\frac{1}{2})W, \Phi, e, -\frac{1}{2}),
\]
for any \( W \in \text{Ind}(\mathbb{W}_{NM}^{\psi}(\pi)), \tilde{W} \in \mathbb{W}_{NM}^{\psi^{-1}}(\tilde{\pi}), \Phi \in S(F^n) \).

3.3. **Relation to global statement.** Suppose now that \( F \) is a number field and \( \mathbb{A} \) is its ring of adeles. We say that an irreducible cuspidal representation \( \pi \) of \( \mathbb{M} \) is of *metaplectic type* if
\[
\int_{H_{sl}(F)\backslash H_{sl}(\mathbb{A}) \cap \mathbb{M}(\mathbb{A})^1} \varphi(h) \, dh \neq 0
\]
for some $\varphi$ in the space of $\pi$. Equivalently, $L^S(\frac{1}{2}, \pi) \res_{s=1} L^S(s, \pi, \lambda^2) \neq 0$ ([FJ93]). In particular, $\pi$ is self-dual and admits a trivial central character. We write $\text{Cusp}_\text{meta} M$ for the set of irreducible cuspidal representations of metaplectic type.

Consider the set $\text{MCusp} M$ of automorphic representations $\pi$ of $M(A)$ which are realized on Eisenstein series induced from $\pi_1 \otimes \cdots \otimes \pi_k$ where $\pi_i \in \text{Cusp}_\text{meta} \text{GL}_{2n_i}$, $i = 1, \ldots, k$ are distinct and $n = n_1 + \cdots + n_k$. The representation $\pi$ is irreducible: it is equivalent to the parabolic induction $\pi_1 \times \cdots \times \pi_k$. Moreover, $\pi$ determines $\pi_1, \ldots, \pi_k$ uniquely up to permutation [JS81b, JS81a].

Recall that Conjecture 1.1 is pertaining to the representations $\pi \in \text{MCusp} M$. The following fact will be crucial for us.

**Proposition 3.5.** ([LM13b, Theorem 6.2]) If $\pi \in \text{MCusp} M$ then its local components $\pi_v$ are good.

Thus Theorem 3.2 implies Theorem 1.3, our main result.

### 3.4. Relation to Bessel functions.

We record here a formal argument that relates Theorem 3.2 to an equation of Bessel functions defined in [LM13a]. We do not worry about convergence issues in this subsection.

Using the function equation (cf. [Kap14], note that the central character of $\pi$ is trivial)

$$
(3.7) \quad J(\tilde{W}, M(s)W, \Phi, -s) = |2|^{2ns} \frac{\gamma(\tilde{\pi} \otimes \pi, s + \frac{1}{2}, \psi)}{\gamma(\pi, s, \psi)\gamma(\pi, \lambda^2, 2s, \psi)} J(\tilde{W}, W, \Phi, s),
$$

(3.5) becomes:

$$
\int_{N'} J(\tilde{\pi}(\tilde{n})\tilde{W}, M(\frac{1}{2})W, \Phi, -\frac{1}{2})\psi_{\tilde{N}}(n) \, dn
= |2|^n \frac{\gamma(\tilde{\pi} \otimes \pi, 1, \psi)}{\gamma(\pi, 1, \psi)\gamma(\pi, \lambda^2, 1, \psi)} c_\pi \tilde{W}(\tilde{e}) A^\psi(M(\frac{1}{2})W, \Phi, e, -\frac{1}{2}).
$$

The left hand side is explicitly

$$
\int_{N'} \int_{N' \setminus G'} \tilde{W}(\tilde{g}\tilde{n}) A^\psi(M(\frac{1}{2})W, \Phi, \tilde{g}, -\frac{1}{2}) \, dg \psi_{\tilde{N}}(n) \, dn
= \int_{N'} \int_{N' \setminus G'} \tilde{W}(\tilde{g}\tilde{n}) \tilde{W}'(\tilde{g}) \, dg \psi_{\tilde{N}}(n) \, dn
$$

where $\tilde{W}' = A^\psi(M(\frac{1}{2})W, \Phi, \cdot, -\frac{1}{2})$. Using Bruhat decomposition this is

$$
\int_T \int_{N'} \int_{N'} \delta_B(t) \tilde{W}(\tilde{w}_0'\tilde{t}\tilde{n}') \tilde{W}'(\tilde{w}_0'\tilde{t}\tilde{n}') \psi_{\tilde{N}}(n) \, dn' \, dn \, dt
= \int_T \int_{N'} \int_{N'} \delta_B(t) \tilde{W}(\tilde{w}_0'\tilde{t}\tilde{n}) \tilde{W}'(\tilde{w}_0'\tilde{t}\tilde{n}) \psi_{\tilde{N}}(n) \psi_{\tilde{N}}^{-1}(n') \, dn' \, dn \, dt.
$$

\(^2\text{We can replace the partial } L\text{-function by the completed one since the local factors are holomorphic and non-zero.}\)
By definition of Bessel functions $\mathbb{B}_{\pi}^{\psi^{-1}}$ (see (5.1)), this is

$$\tilde{W}(e)\tilde{W}'(e) \int_{T'} \mathbb{B}_{\pi}^{\psi^{-1}}(\tilde{u}'_0 t) \mathbb{B}_{\tilde{\pi}}^{\psi}(\tilde{u}'_0 \tilde{t}) \delta_B(t) \, dt.$$ 

Here $\tilde{\pi}^\psi = D_\psi(\pi)$. Thus, on a formal level, Theorem 3.2 becomes the following inner product identity:

$$\int_{T'} \mathbb{B}_{\pi}^{\psi^{-1}}(\tilde{u}'_0 t) \mathbb{B}_{\tilde{\pi}}^{\psi}(\tilde{u}'_0 \tilde{t}) \delta_B(t) \, dt = |2|^n \frac{\gamma(\tilde{\pi} \otimes \pi, 1, \psi)}{\gamma(\pi, \Lambda^2, 1, \psi)}.$$ 

### 3.5. A reduction of the main theorem.

For the rest of the paper let $F$ be a $p$-adic field. The proof of Theorem 3.2 will be mostly local. However, the local proof will be under the additional assumption that $\pi \in \text{Irr}_{\text{temp}} \mathbb{M}$ and $\tilde{\pi} := D_{\psi^{-1}}(\pi) \in \text{Irr}_{\text{temp}} G$. We first show that it suffices to prove Theorem 3.2 under these additional assumptions. This will use a global argument as well as the following classification result due to Matringe. We denote by $\times$ parabolic induction for $\text{GL}_m$.

**Theorem 3.6 ([Mat]).** The set $\text{Irr}_{\text{gen,meta}} \mathbb{M}$ consists of the irreducible representations of the form

$$\pi = \sigma_1 \times \sigma_1^\psi \times \cdots \times \sigma_k \times \sigma_k^\psi \times \tau_1 \times \cdots \times \tau_l$$

where $\sigma_1, \ldots, \sigma_k$ are essentially square-integrable, $\tau_1, \ldots, \tau_l$ are square-integrable and $L(0, \tau_i, \Lambda^2) = \infty$ for all $i$.

(We expect the same result to hold in the archimedean case as well.)

Denote by $\omega_\sigma$ the central character of $\sigma \in \text{Irr} \text{GL}_m$.

Let $\tau_j \in \text{Irr}_{\text{sg,meta}} \text{GL}_{2n_j}$, $j = 1, \ldots, l$ be distinct and $\delta_i \in \text{Irr}_{\text{sgf}} \text{GL}_{n_i}$, $i = 1, \ldots, k$ with $n = n_1 + \cdots + n_k + m_1 + \cdots + m_l$. (Possibly $k = 0$ or $l = 0$.) For $\underline{s} = (s_1, \ldots, s_k) \in \mathbb{C}^k$ we consider the representation $\pi(\underline{s}) = \delta_1[s_1] \times \delta_1[-s_1] \times \cdots \times \delta_k[s_k] \times \delta_k[-s_k] \times \tau_1 \times \cdots \times \tau_l$. Suppose that our given p-adic field is the completion at a place $v$ of a number field. We claim that for a dense set of $\underline{s} \in i\mathbb{R}^k$, $\pi(\underline{s})$ is the local component at $v$ of an element of $\text{MCusp} \mathbb{M}$. This follows from [ILM14, Appendix A].\(^3\) Indeed, let $m = m_1 + \cdots + m_l$ and let $\rho \in \text{Irr}_{\text{sgf,gen}} \text{SO}(2m + 1)$ be the representation corresponding to $\tau_1 \times \cdots \times \tau_l$ under Jiang–Soudry [JS04] and let $\tilde{\rho} \in \text{Irr}_{\text{sgf,gen}} \hat{\text{Sp}}_m$ be the theta lift of $\rho$. Let $\sigma(\underline{s}) = \delta_1[s_1] \times \cdots \times \delta_k[s_k] \times \rho$ and $\tilde{\sigma}(\underline{s}) = \delta_1[s_1] \times \cdots \times \delta_k[s_k] \times \tilde{\rho}$. Then for $\underline{s}$ in a dense open subset of $i\mathbb{R}^k$ we have $\sigma(\underline{s}) \in \text{Irr} \text{SO}(2n + 1)$ and $\tilde{\sigma}(\underline{s}) \in \text{Irr} \hat{G}$ and moreover by [GS12] $\tilde{\sigma}(\underline{s})$ is the theta lift of $\sigma(\underline{s})$. By [ILM14, Corollary A.8], for a dense set of $\underline{s} \in i\mathbb{R}^k$, $\tilde{\sigma}(\underline{s})$ is the local component at $v$ of a generic cuspidal automorphic representation of $\hat{\text{Sp}}_n(\mathbb{A})$ whose theta lift to $\text{SO}(2n + 1)$ is cuspidal. The Cogdell–Kim–Piatetski-Shapiro–Shahidi lift of the latter to $\mathbb{M}$ is the required representation in $\text{MCusp} \mathbb{M}$.

It follows from Proposition 3.5 that for a dense set of $\underline{s} \in i\mathbb{R}^k$, $\pi(\underline{s}) \in \text{Irr}_{\text{temp,meta}} \mathbb{M}$ is good. Moreover, by [ILM14, Proposition 4.6] $D_{\psi^{-1}}(\pi(\underline{s})) = \tilde{\sigma}(\underline{s})$ and in particular it is tempered.

\(^3\)We remark that the appendices and §4 of [ILM14], and in particular the proof of [ILM14, Theorem 3.1], are independent of the results of current paper.
Suppose that $-\frac{1}{2} < \Re s_1, \ldots, \Re s_k < \frac{1}{2}$. We recall that by [LM13b, Lemma 4.12] the integral defining $\tilde{J}(A^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, \Phi^\vee, \cdot, -\frac{1}{2}), W, \Phi, s)$ converges and is holomorphic at $s = \frac{1}{2}$ for any $W \in \text{Ind}(\mathcal{W}^{\psi_{NM}}(\pi(\mathfrak{s}))), \ W^\wedge \in \text{Ind}(\mathcal{W}^{\psi_{\overline{NM}}}(-\mathfrak{s}(\mathfrak{r})))$. Moreover, by the properties of $A^\psi$ and $\tilde{J}$ (see [LM13b], (4.4) and (4.13)), the function $g \mapsto \tilde{J}(A^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, \Phi^\vee, \cdot, -\frac{1}{2}), W, \Phi, \frac{1}{2})$ is bi-$K_0$-invariant if $K_0 \in \mathcal{CSGR}^s(G^\prime)$ is such that $I(\frac{1}{2}, \eta(k))W = W, \ I(\frac{1}{2}, \eta(k))W^\wedge = W^\wedge, \ \omega_{\psi^{-1}}(\mathfrak{k})\Phi = \Phi, \ \omega_{\psi}(\mathfrak{k})\Phi^\vee = \Phi^\vee$ for all $k \in K_0$. It follows from [LM14a, Proposition 2.11] (applied to $\tilde{G}$) that the stable integral on the left-hand side of (3.6) can be written as an integral over a compact open subgroup of $N'$ depending only on $K_0$. Thus, taking $W \in \text{Ind}(\mathcal{W}^{\psi_{NM}}(\pi(\mathfrak{s}))), \ W^\wedge \in \text{Ind}(\mathcal{W}^{\psi_{\overline{NM}}}(-\mathfrak{s}(\mathfrak{r})))$ to be Jacquet integrals, both sides of (3.6) for $\pi(\mathfrak{s})$ are holomorphic functions of $\mathfrak{s}$ in the region $-\frac{1}{2} < \Re s_1, \ldots, \Re s_k < \frac{1}{2}$. Note that by [LM13b, Lemma 3.6] $\epsilon_{\pi(\mathfrak{s})} = \omega_{\delta_1}(-1) \cdots \omega_{\delta_k}(-1)\epsilon_{\tau_1} \cdots \epsilon_{\tau_l}$ is independent of $\mathfrak{s}$. Thus, in order to prove (3.6) in the region $-\frac{1}{2} < \Re s_1, \ldots, \Re s_k < \frac{1}{2}$ it is enough to show it for a dense set of $\mathfrak{s} \in i\mathbb{R}^k$. Since every unitarizable $\pi \in \text{Irr}_{\text{meta}} \mathbb{M}$ is of the form $\pi(\mathfrak{s})$ for some $\tau_1, \ldots, \tau_l, \delta_1, \ldots, \delta_k$ as above and $\mathfrak{s}$ with $-\frac{1}{2} < \Re s_1, \ldots, \Re s_k < \frac{1}{2}$, the reduction step follows.

**Remark 3.7.** In the case $k = 0$ we can globalize $\pi$ itself. Thus by Proposition 3.5 $\pi$ is good; $\mathcal{D}_{\psi^{-1}}(\pi)$ is irreducible and square integrable (see [ILM14, Theorem 3.1]). This yields Corollary 3.4 from Theorem 3.2.

In the remainder of the paper we will prove Theorem 3.2, i.e., the Main Identity

\[
\text{(MI)} \quad \int_{N^\prime}^{st} \tilde{J}(\tilde{\pi}(\tilde{n})\tilde{W}, \Phi, \frac{1}{2})\psi_{\mathcal{X}}(n) \ dn = \epsilon_{\pi} \tilde{W}(e)A^\psi(M(\frac{1}{2})W, \Phi, e, -\frac{1}{2}),
\]

under the assumptions that $\pi \in \text{Irr}_{\text{temp,meta}} \mathbb{M}$ is good and $\tilde{\pi} := \mathcal{D}_{\psi^{-1}}(\pi)$ is tempered. The strategy was described heuristically in [LM13b, §7] for the case $n = 1$ (which can serve as an introduction to this paper). The necessary ingredients to generalize this argument \textit{formally} for general $\mathfrak{n}$ were established in [LM14c, LM14b]. Our main task here is to carry out the argument (for general $\mathfrak{n}$) rigorously. Since $\pi$ is assumed to be good, we only need to establish (MI) for a single pair $(\tilde{W}, W)$ where at least one side of (MI) is nonzero. This will be a crucial fact in justifying the manipulations of various integrals for special pairs $(\tilde{W}, W)$. Grosso modo, we carry out here the steps outlined in §7.1–7.4 of [LM13b]. More precisely, we make a series of reductions culminating in Propositions 7.1 and 7.2 below which will be derived from the results of [LM14b].

### 4. The function $A^\psi(W, \Phi, \tilde{g}, s)$

In this section we use the formulas defining the Weil representation in §2.5 to calculate the function $A^\psi(W, \Phi, \tilde{g}, s)$ for $g = e$ and $g \in w_{\psi^\prime}^\prime F^\prime$.

Let $X_n$ be the image under $\varphi \circ \ell_M$ of the space of $\mathfrak{n} \times \mathfrak{n}$-matrices which are zero except possibly in the last row. For $c = \varphi(\ell_M(x)) \in X_n$, let $c \in F^n$ be the $\mathfrak{n}$-th (the only nontrivial) row of $x$. For $\Phi \in C_c(\mathbb{F}^n)$ and any function $f \in C^\infty(G)$ we set

\[
\Phi \ast f(g) = \int_{X_n} f(gc)\Phi(c) \ dc.
\]
For $g = e$ or $g \in w'_V P'$ we will express $A^\psi(W, \Phi, \tilde{g}, s)$ as $L_g(\Phi \ast (W_s))$ where $L_g$ is an explicit linear form on $C^{sm}(N \backslash G, \psi_N)$. The results are stated in Lemmas 4.1 and 4.3 below.

Throughout this section we fix $\pi \in \text{Irr}_{\text{gen}} M$. For simplicity we denote $M_s^*W := (M(s)W)_{-s}$ so that

$$M_s^*W = \int_U W_s(\tilde{g}(t)w_U u) \, du$$

for $\Re s \gg \pi$ 1. Set $M^*W := M_s^*W$. We also write $\Phi \ast_s W$ for the function $(\Phi \ast (W_s))_{-s}$. Thus,

$$(4.1) \quad M_s^*(\Phi \ast_s W) = \Phi \ast M_s^*W, \quad M(s)(\Phi \ast_s W) = \Phi \ast_{-s} M(s)W.$$ 

4.1. Let $\hat{\psi}_U$ and $\hat{\psi}_U$ be the characters on $U$ and $\bar{U}$ respectively given by

$$\hat{\psi}_U(\ell(v)) = \hat{\psi}_U(\ell(v)) = \psi(\frac{1}{2}(v_{n,n+1} - v_{2n,1})), \quad v \in \mathfrak{g}_{2n}.$$ 

Note that $\eta(U') \subset U$ and $\psi_U = \hat{\psi}_U \circ \eta$. Recall $U^\gamma$ and $\bar{U}^\gamma$ are defined in §2.1. The restriction $\hat{\psi}_{U^\gamma}$ of $\hat{\psi}_U$ to $U^\gamma$ is given by $\hat{\psi}_{U^\gamma}(\ell(v)) = \psi(\frac{1}{2}v_{n,n+1})$. Similarly define the restriction $\hat{\psi}_{\bar{U}^\gamma}$ of $\hat{\psi}_U$ to $\bar{U}^\gamma$.

Let $\epsilon_3 = \mathfrak{g}(\ell_M(-\epsilon_{n,n}))$, and define for any $W \in C^{sm}(N \backslash G, \psi_N)$

$$(4.2) \quad A^\psi_e(W) := \int_{\bar{U}^\gamma} W(v\gamma\epsilon_3)\hat{\psi}_{\bar{U}^\gamma}(v) \, dv.$$ 

**Lemma 4.1.** The integrand in (4.2) is compactly supported on $\bar{U}^\gamma$. For $W \in \text{Ind}(\mathbb{W}^\psi_{NM}(\pi))$ and $\Phi \in C^\infty_c(F^n)$ we have $A^\psi(W, \Phi, e, s) = A^\psi_e(\Phi \ast (W_s))$ for all $s \in \mathbb{C}$. In particular $A^\psi(M(\frac{1}{2})W, \Phi, e, -\frac{1}{2}) = A^\psi_e(\Phi \ast M^*W) = A^\psi_e(M^*(\Phi \ast_{\frac{1}{2}} W))$.

**Proof.** The support condition follows from Lemma 3.1. By (3.1) we can write

$$A^\psi(W, \Phi, e, s) = \int_{V_n \backslash V} W_s(\gamma v)\omega_{\psi^{-1}}(v) \Phi(\xi_n) \, dv.$$ 

Write $V = V_\gamma \times Y$ where $Y$ is the abelian group

$$Y = \{v = v(x, y) = \mathfrak{g}(\ell_M(x))(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}) | x \in \text{Mat}_{n,n}, y \in \mathfrak{g}_n\}.$$ 

Let $Y_0$ be the subgroup of $Y$ consisting of $v = v(x, y)$ with $x \in \bar{R}_n$ so that $Y = X_n \times Y_0$. From (2.3), (2.1a) and (2.1c) we have

$$\omega_{\psi^{-1}}(v(x, y)c)\Phi(\xi_n) = \Phi(\xi_n + c)\psi^{-1}(\frac{1}{2}y_{n,1}) \quad c \in X_n, \quad x \in \bar{R}_n, \quad y \in \mathfrak{g}_n.$$ 

Thus,

$$A^\psi(W, \Phi, e, s) = \int_{X_n} \int_{\bar{R}_n} \int_{\mathfrak{g}_n} W_s(\gamma v(x, y)c)\Phi(\xi_n + c)\psi^{-1}(\frac{1}{2}y_{n,1}) \, dy \, dx \, dc.$$
Changing $c \mapsto \epsilon_3 c$ we get
\[
\int_{X_n} \int_{\mathbb{R}_n} \int_{\mathbb{R}_n} W_s(\gamma(v(x,y)\epsilon_3 c) \Phi(\zeta) \psi^{-1}(\frac{1}{2}y_{n,1})) dy dx dc
= \int_{\mathbb{R}_n} \int_{\mathbb{R}_n} (\Phi \ast (W_s))(\gamma(v(x,y)\epsilon_3) \psi^{-1}(\frac{1}{2}y_{n,1})) dy dx
\]
which is equal to $A^\psi(\Phi \ast (W_s))$ since $\gamma(v(x,y)\gamma^{-1} = 7((x-y)) \in \bar{U}$ and $\psi \gamma^{-1}((x-y)) = \psi^{-1}(\frac{1}{2}y_{n,1})$.

### 4.2

Next we describe $A^\psi(W, \Phi, \tilde{g}, s)$ for $g$ in the big cell $P'w'_U U'$. Recall the group $V_0$ defined in §2.1.

#### Lemma 4.2

For $\Re s \gg \pi 1$ we have
\[
A^\psi(W, \Phi, \tilde{w}_U, \tilde{g}, s) = \beta_{\psi^{-1}}(w'_U) \int_{V_0 \cap M \setminus V} W_s(w_U v \eta(g)) \omega_{\psi^{-1}}(v \tilde{g}) \Phi(0) dv
\]
for all $g \in G'$.

**Proof.** From (3.1),
\[
A^\psi(W, \Phi, \tilde{w}_U, \tilde{g}, s) = \int_{V_0 \cap M \setminus V} W_s(\gamma(v \eta(w'_U g))) \omega_{\psi^{-1}}(v \tilde{w}_U, \tilde{g}) \Phi(\xi_n) dv.
\]
To prove (4.3), we may assume that $g = e$. Make a change of variable $v \mapsto \eta(w'_U) \eta(w'_U)^{-1}$. Since $V_0 = \eta(w'_U) (N_M \cap V) \eta(w'_U)^{-1}$, $\eta(w'_U)$ normalizes $V$, and $V = (N_M \cap V) \times (U \cap V)$, we have $V = \eta(w'_U)^{-1} V_0 \eta(w'_U) \times \eta(w'_U) \eta(w'_U)^{-1} (V \cap U) \eta(w'_U)$. By (2.4) we infer that
\[
A^\psi(W, \Phi, \tilde{w}_U, s) = \int_{V \cap U} W_s(w_U v) \omega_{\psi^{-1}}(w'_U, v) \Phi(\xi_n) dv
\]
and using (2.2b) we get
\[
\beta_{\psi^{-1}}(w'_U) \int_{V \cap U} \left( \int_{F^n} W_s(w_U v) \omega_{\psi^{-1}}(v) \Phi(Y) \psi(-Y_1) dY \right) dv.
\]
We claim that the double integral converges absolutely when $\Re s$ is large enough. Note that $V \cap U = \{ \ell((x \ y)) \}, x \in \text{Mat}_n, y \in s_n \}$ and hence $|\omega_{\psi^{-1}}(v) \Phi(Y)| = |\Phi(Y)|$ for $v \in V \cap U$. Thus,
\[
\int_{V \cap U} \int_{F^n} |W_s(w_U v) \omega_{\psi^{-1}}(v) \Phi(Y) \psi(-Y_1)| dY dv = |\Phi(0)| \int_{V \cap U} |W_s(w_U v)| dv.
\]
The integration on the right is absolutely convergent when $\Re s \gg \pi 1$. This follows from the convergence of $\int_U |W_s(w_U)| du$ and Remark 2.2.

For $c \in X_n$ such that $\zeta = Y \in F^n$, we have $W_s(w_U \epsilon g) = \psi(-Y_1) W_s(w_U g)$ by the equivariance of $W$ and $\omega_{\psi^{-1}}(c) \Phi(0) = \Phi(Y)$ by (2.1a). Thus
\[
A^\psi(W, \Phi, \tilde{w}_U, s) = \beta_{\psi^{-1}}(w'_U) \int_{V \cap U} \int_{X_n} W_s(w_U cv) \omega_{\psi^{-1}}(cv) \Phi(0) dc dv.
\]
It remains to note that the integrand on the right-hand side of (4.3) is left $V_0 \cap M$-invariant and that we can replace the integral over $V_0 \cap M \setminus V$ by integration over $X_n \times (V \cap U)$. □

4.3. When $\tilde{g} = \tilde{m}u$ with $m \in M'$ and $u \in U'$, we can get an expression of $A^\psi(W, \Phi, \tilde{w}_{U'} \tilde{g}, s)$ that holds for all $s$.

Let $X'_n \simeq F^n$ be the group $X'_n = \{ \ell((x, z)) : x \text{ supported in last row} \} \subset U$.

Lemma 4.3. For any $W \in C^\infty(N \setminus G, \psi_n)$ and $g \in P'$ define

\begin{equation}
A^\psi_{bc}(W, g) := \int_{X'_n} (\int_{U'} W(w_U \eta(g)vx)\hat{\psi}_{U^n}(v)^{-1} dv) dx.
\end{equation}

(1) The above iterated integrals are integrals of compactly supported functions, thus convergent.

(2) For any $W \in \text{Ind}(\mathbb{W}^{W \times M}(\pi))$, $m \in M'$, $u \in U'$ and $s \in C$ we have

\begin{equation}
A^\psi(W, \Phi, \tilde{w}_{U'} \tilde{m}u, s) = \nu'(m)^{n-\frac{1}{2}} \beta_{\psi^{-1}}(w_{U'}) \beta_{\psi^{-1}}(m) A^\psi_{bc}(\Phi * (W_s), m\mu).
\end{equation}

Proof. Observe that for any $g \in P'$, $\eta(g)$ normalizes $X'_n$ and $U^n$ and stabilizes the character $\hat{\psi}_{U^n}$. Therefore for the first part we can assume that $g = e$.

By Lemma 3.1, for fixed $x \in X'_n$, the inner integrand on the right-hand side of (4.4) is compactly supported. Let

\[ I_1(x) := \int_{U^n} W(w_U vx)\hat{\psi}_{U^n}(v)^{-1} dv. \]

We are left to show that $I_1$ is compactly supported on $X'_n$.

For $x \in X'_n \simeq F^n$, $c \in X_n \simeq F^n$, we have

\[ I_1(xc) = \int_{U^n} W(w_U vx)\hat{\psi}_{U^n}(v)^{-1} dv. \]

Note that $c$ normalizes $X'_n \times U^n$, with $[x, c] \in U^n$ and

\[ \hat{\psi}_{U^n}([x, c]) = \psi((x, c)'), \] where $(x, c)' := -\frac{1}{2} \sum_{i=1}^n c_i x_i. \]

Here $\underline{x} \in F^n$ is the $n$-th (the only nontrivial) row of $x'$ where $x = \ell((x', z'))$. Thus by a change of variable $v \mapsto v[x, c]^{-1}$:

\[ I_1(xc) = \int_{U^n} W(w_U vx)\hat{\psi}_{U^n}(v)^{-1} \psi((x, c)') dv. \]

Also $c$ normalizes $U^n$ while stabilizing the character $\hat{\psi}_{U^n}$. Using the equivariance of $W$ we get

\begin{align*}
I_1(xc) &= \int_{U^n} W(w_U vx)\psi((x, c)')\hat{\psi}_{U^n}(v)^{-1} dv \\
&= \int_{U^n} W(w_U vx)\psi((x, c)' - c, 1)\hat{\psi}_{U^n}(v)^{-1} dv = \psi((x, c)' - c, 1)I_1(x).
\end{align*}
From Lemma 2.3 we get that $I_1(x)$ is compactly supported. The first part follows.

We now show (4.5). Recall that $A^\psi(W, \Phi, \tilde{w}, \tilde{m}, s)$ is an entire function in $s$. As $A^\psi_{bc}(\Phi \ast (W_s), mu)$ is also entire by the first claim, we only need to show the identity when $\Re s \gg \pi 1$.

When $\Re s \gg \pi 1$, for $g = mu$ we get from (4.3)

$$A^\psi(W, \Phi, \tilde{w}, \tilde{m}, s) = \beta_{\psi^{-1}}(w') \int \mathcal{V}_{\tilde{w}, \tilde{m}} W_s(w_{U'} \eta(mu)) \omega_{\psi^{-1}}(m \tilde{m} \tilde{u}) \Phi(0) \, dv.$$ 

Making a change of variable $v \mapsto \eta(mu) \eta(mu)^{-1}$ this becomes (using (2.4))

$$\beta_{\psi^{-1}}(w') v(m)^{n-1} \int \mathcal{V}_{\tilde{w}, \tilde{m}} W_s(w_{U'} \eta(mu)v) \omega_{\psi^{-1}}(m \tilde{m} \tilde{u}v) \Phi(0) \, dv.$$ 

From (2.2a) and (2.2c), we get

$$\beta_{\psi^{-1}}(w') \beta_{\psi^{-1}}(m) v'(m)^{n-\frac{1}{2}} \int \mathcal{V}_{\tilde{w}, \tilde{m}} W_s(w_{U'} \eta(mu)v) \omega_{\psi^{-1}}(v) \Phi(0) \, dv.$$ 

As before, we can integrate over $X_n \times (V \cap U)$ instead. Note that $V \cap U = X_n' \times U'$. For $c \in X_n$, $x \in X_n'$ and $v \in U'$, by (2.1a)–(2.1c) and (2.3), we have

$$\omega_{\psi^{-1}}(vx) \Phi(0) = \Phi(\tilde{v}) \tilde{v}^{-1}.$$ 

Thus the above integral is

$$\int_{X_n} \int_{U'} \int_{X_n} W_s(w_{U'} \eta(mu)v) \Phi(\tilde{v}) \tilde{v}^{-1} \, dc \, dv \, dx.$$ 

The triple integral is simply $A^\psi_{bc}(\Phi \ast (W_s), mu)$. \qed

5. A second reduction step

In this section we will rephrase the Main Identity, eliminating the Schwartz function $\Phi$ from its formulation, and reduce it to Proposition 5.10 below. The latter will be eventually proved in §7. A key ingredient in the analysis of this section is the stability of the integral defining a Bessel function, which was proved in [LM13a] following ideas of Baruch [Bar05].

5.1. We use the main result of [LM13a] for the group $\tilde{G}$.

Lemma 5.1. [LM13a] Assume that $K_0 \in \mathcal{CSGR}^*(G')$, and $\tilde{W} \in C(\tilde{N} \backslash \tilde{G}, \psi_{\tilde{K}})^{K_0}$. Then for any $t \in T'$, the stable integral

$$\int_{\tilde{N}'} \tilde{W}(\tilde{w}_0 \tilde{t}, \tilde{n}) \psi_{\tilde{K}}(n)^{-1} \, dn$$

is well defined. More precisely, for any compact subset $C \subset T'$ there exists $\Omega \in \mathcal{CSGR}(N')$ depending only on $C$ and $K_0$ such that

$$\int_{\tilde{N}'} \tilde{W}(\tilde{w}_0 \tilde{t}, \tilde{n}) \psi_{\tilde{K}}(n)^{-1} \, dn = \int_{\Omega'} \tilde{W}(\tilde{w}_0 \tilde{t}, \tilde{n}) \psi_{\tilde{K}}(n)^{-1} \, dn$$. 

for any $t \in C$ and any $\Omega' \in \mathcal{CSGR}(N')$ containing $\Omega$. In particular, if $\tilde{W} \in \mathcal{W}^{\psi_N}(\tilde{\pi})$ with $\tilde{\pi} \in \text{Irr}_{\text{gen},\psi_N} \tilde{G}$ then

$$
(5.1) \quad \int_{N'}^{st} \tilde{W}(\tilde{w}_0'\tilde{t}\tilde{n})\psi_N(n)^{-1} \, dn = \mathbb{B}^{\psi_N}_{\tilde{\pi}}(\tilde{w}_0'\tilde{t})\tilde{W}(e)
$$

where $\mathbb{B}^{\psi_N}_{\tilde{\pi}}$ is the Bessel function of $\tilde{\pi}$.

We will apply the lemma to the function $\tilde{W}(g) = A^\psi(W, \Phi, \tilde{g}, s)$, noting that $K_0$ is independent of $s$. By (4.5), $A^\psi_{bc}(\Phi \ast (W_\lambda), w_0Mtn)$ is the product of a smooth function in $t$ and $A^\psi(W, \Phi, \tilde{w}_0'\tilde{t}\tilde{n}, s)$, (here we use the fact that $\beta_{\psi^{-1}}(gn) = \beta_{\psi^{-1}}(g)$ when $n \in N'$). Thus

**Corollary 5.2.** For any $t \in T'$ the stable integral

$$
Y^\psi(W, s; t) := \int_{N'}^{st} A^\psi_{bc}(W_s, w_0^{M'}tn)\psi_N(n)^{-1} \, dn
$$

is well defined. More precisely, for any compact subset $C \subset T'$ there exists $\Omega \in \mathcal{CSGR}(N')$ such that

$$
Y^\psi(W, s; t) = \int_{\Omega'}^{st} A^\psi_{bc}(W_s, w_0^{M'}tn)\psi_N(n)^{-1} \, dn
$$

for any $s \in C$, $t \in C$ and any $\Omega' \in \mathcal{CSGR}(N')$ containing $\Omega$. In particular, $Y^\psi(W, s; t)$ is entire in $s \in C$ and $Y^\psi(M(s)W, -s; t)$ is meromorphic in $s$; they are both locally constant in $t$, uniformly in $s \in C$.

Finally, assume that $\pi \in \text{Irr}_{\text{meta},\text{gen}} \mathbb{M}$ and that $\tilde{\pi} = D_{\psi^{-1}}(\pi)$ is irreducible. Then for any $W^\lambda \in \text{Ind}(\mathcal{W}^{\psi_NM}(\pi))$ we have

$$
(5.2) \quad Y^\psi_{\psi^{-1}}(M(\frac{1}{2})W^\lambda, -\frac{1}{2}; t) = \nu'(t)^{\frac{1}{2} - n} \kappa_{t,\psi^{-1}} \mathbb{B}^{\psi_N}_{\tilde{\pi}}(\tilde{w}_0'\tilde{t})A^{\psi^{-1}}_{\pi}(M^*W^\lambda)
$$

where $\kappa_{t,\psi^{-1}}$ is a certain root of unity.

**Proof.** It remains only to show (5.2). We will show this relation for $\Phi^\psi \ast_{\tilde{\pi}} W^\lambda$ in place of $W^\lambda$. We apply (5.1) to $\tilde{W} = A^{\psi^{-1}}(M(\frac{1}{2})W^\lambda, \Phi^\psi, \cdot, -\frac{1}{2}) \in \mathcal{W}^{\psi^{-1}_{\pi}}(\tilde{\pi})$ (and $\psi^{-1}$ instead of $\psi$).

By Lemma 4.1, the right-hand side of (5.1) is $\mathbb{B}^{\psi_N^{-1}}_{\tilde{\pi}}(\tilde{w}_0'\tilde{t})A^{\psi^{-1}}_{\pi}(\Phi^\psi \ast M^*W^\lambda)$. On the other hand, since $w'_0 = w'_0w_0^{M'}$ it follows from (4.5) that

$$
(5.3) \quad \int_{N'}^{st} A^\psi(W, \Phi, \tilde{w}'_0, w_0^{M'}tn, s)\psi_N(n)^{-1} \, dn = \nu'(t)^{\frac{1}{2} - n} \beta_{\psi^{-1}}(w'_0t)\beta_{\psi^{-1}}(w_0^{M'}t)Y^\psi(\Phi \ast s W, s; t).
$$

The corollary follows. \[ \square \]

5.2. We give an alternative expression for $Y^\psi(W, s; t)$ for $\Re s \gg 1$. First we need a convergence statement. Let $N_{M}^{b}$ be the subgroup of $N_M$ consisting of matrices whose last column is $(0, \ldots, 0, 1)^t$; let $N_{M}^{b} = o(N_{M}^{b})$. 
Lemma 5.3. Let $\pi \in \text{Irr}_\text{gen} M$. Then for $\Re s \gg \pi$ we have

\begin{align*}
(5.4a) \quad & \int_{\mathcal{N}_M^w} \int_U |W_s(w_0^M n v g)| \ dv \ dn < \infty, \\
(5.4b) \quad & \int_{(\mathcal{N}_M^w)\star} \int_G |W_s(w_0^M n v g)| \ dv \ dn < \infty,
\end{align*}

for any $W \in \text{Ind}(\mathcal{W}_w^{\text{Em}}(\pi)), g \in G$.

Proof. Let $\varrho(a(g))$ be the torus part in the Iwasawa decomposition of $g \in G$. Consider

$$a(w_0 \varrho(n)uw_0) = \text{diag}(a_1, \ldots, a_{2n}).$$

Notice that

\begin{equation}
(5.5) \quad \prod_{i=1}^{2n} |a_i| = |\det(a(w_0 \varrho(n)uw_0))| = |\det(a(w_0uw_0))|.
\end{equation}

We will show that for all $n \in \mathcal{N}_M^b \cup (\mathcal{N}_M^b)\star$ and $u \in U$ in the support of $W(w_0 \varrho(n)uw_0)$, for all $1 \leq i \leq 2n$,

\begin{equation}
(5.6) \quad |\det(a(w_0uw_0))| \ll_{W} |a_i| \ll_{W} |\det(a(w_0uw_0))|^{2^{-2n}}.
\end{equation}

We first show (5.4b) using (5.6). It suffices to take $g = w_0^M$, in which case the integral is

$$\int_{\mathcal{N}_M^w} \int_U |W_s(w_0 \varrho(n)uw_0)| \ dv \ dn.$$

It follows from (5.5) and (5.6) that there exists $\lambda \in \mathbb{R}$, depending only on $\pi$ such that

\begin{equation}
(5.7) \quad |W_s(w_0 \varrho(n)uw_0)| \ll_{W} |\det(a(w_0uw_0))|^{\Re s - \lambda}
\end{equation}

for any $u \in U$ and $n \in \mathcal{N}_M^b$. Meanwhile, each entry $n_{i,j}$ of $n$ is an $i \times i$ minor in the last $i$ rows of $w_0 \varrho(n)uw_0$, thus bounded by $\prod_{k=1}^{i} |a_k|^{-1}$. Thus over the support of $W_s(w_0 \varrho(n)uw_0)$, $|n_{i,j}| \ll_{W} |\det(a(w_0uw_0))|^{-i}$. Thus,

$$\int_{\mathcal{N}_M^w} \int_U |W_s(w_0 \varrho(n)uw_0)| \ dv \ dn \ll_{W} \int_U |\det(a(w_0uw_0))|^{\Re s - \lambda - (n-1)(2n-1)} \ dv$$

where the last integral is finite when $\Re s \gg \pi$ by a standard result on intertwining operators. The same argument also gives (5.4a).

We are left to prove (5.6). First consider the case when $n \in \mathcal{N}_M^b$. Then the $2n$-th row of $w_0 \varrho(n)uw_0$ is $(0, \ldots, 0, 1, 0, \ldots, 0)$. Thus, each nonzero $(2n+1) \times (2n+1)$ minor in the last $2n+1$ rows of $w_0 \varrho(n)uw_0$ is equal to a $2n \times 2n$ minor in the last $2n$ rows. We get

$$|a_{2n}| \prod_{i=1}^{2n} |a_i|^{-1} \leq \prod_{i=1}^{2n} |a_i|^{-1}$$

\[\text{Recall that } m^* = w_U m w_U^{-1}, m \in M \text{ i.e. } \varrho(x^*) = \varrho(x^*), x \in M.\]
as the two sides are the maxima of the norms of the above \((2n + 1) \times (2n + 1)\) minors and \(2n \times 2n\) minors respectively. Equivalently we have \(|a_{2n}| \leq 1\). Thus, over the support of \(W_s\), we have \(|a_1| \ll_W \cdots \ll_W |a_{2n}| \leq 1\). We get (5.6) since \(1 \leq |\det(a(w_0uw_0))|^{2-2n}\).

Next consider the case \(n \in (N_M^\ast)^\ast\). The last row of \(w_0\theta(n)uw_0\) is the same as the last row of \(w_0uw_0\). Thus \(|a_1|^{-1}\) is bounded above by the norm of the entries in \(u\), which is bounded above by \(|\det(a(w_0uw_0))|^{-1}\) (since each entry of \(u\) is a \(2n \times 2n\) minor in the last \(2n\) rows of \(w_0uw_0\)). Again over the support of \(W_s\), \(|a_1| \ll_W \cdots \ll_W |a_{2n}|\). From (5.5) we conclude that \(|\det(a(w_0uw_0))| \ll_W |a_1| \ll_W |\det(a(w_0uw_0))|^{2-2n}\).

\[\square\]

**Remark 5.4.** From the proof it is clear that we can take a uniform region of convergence \(\Re s \gg 1\) for all unitarizable \(\pi\).

**Lemma 5.5.** Let \(\pi \in \text{Irr}_{\text{gen}} M\). For any \(W \in \text{Ind}(\mathcal{W}^\psi M(\pi))\), \(t \in T'\) we have the identity

\[
Y^\psi(W, s; t) = \int_{N_M' \setminus U} W_s \left( \eta(w_0^{M'}tn)v w_U \right) \hat{\psi}_{U}(v) \psi_{N_M'}^{-1}(n) \, dv \, dn
\]

for \(\Re s \gg \pi 1\) where the right-hand side is absolutely convergent.

**Proof.** Observe that the double integral on the right-hand side of (5.8) can be viewed as a ‘partial integration’ of the double integral (5.4b) in Lemma 5.3. Thus by Remark 2.2 it is absolutely convergent for \(\Re s \gg \pi 1\). Note that \(w_U^{-1} \ell(v)w_U = \ell(-v)\) and \(w_U^{-1} mw_U = m^\ast\) for \(m \in M\). Thus, we can write the right-hand side of (5.8) as

\[
\int_{N_M' \setminus U} W_s \left( w_U \eta(w_0^{M'}tn)v \right) \hat{\psi}_{U}(v)^{-1} \psi_{N_M'}^{-1}(n) \, dv \, dn.
\]

For \(m \in M'\), let

\[
\varphi(m) = \int_{U} W_s (w_U \eta(m)v) \hat{\psi}_{U}(v)^{-1} \, dv
\]

so that the right-hand side of (5.8) equals \(\int_{N_M' \setminus U} \varphi(w_0^{M'}tn)\psi_{N_M'}^{-1}(n) \, dn\). Note that \(U = X_n' \times U^\gamma \times \eta(U')\). Moreover \(\hat{\psi}_{U}|_{X_n'} = 1\) and \(\hat{\psi}_{U}|_{U^\gamma} = \hat{\psi}_{U^\gamma}\). Thus,

\[
\varphi(m) = \int_{\eta(U')} \int_{X_n'} \int_{U^\gamma} W_s(w_U \eta(m)u'ux) \hat{\psi}_{U^\gamma}^{-1}(u)^{-1} \hat{\psi}_{U}(u')^{-1} \, du \, dx \, du'
\]

which by (4.4) and the fact that \(\hat{\psi}_{U} \circ \eta = \psi_{U'}\), is simply \(\int_{U} A_{bc}^\psi(W_s, mu') \psi_{U'}(u')^{-1} \, du'\). Thus, the right-hand side of (5.8) equals

\[
\int_{N_M' \setminus U} A_{bc}^\psi(W_s, w_0^{M'}tn'u') \psi_{U'}(u')^{-1} \psi_{N_M'}^{-1}(n')^{-1} \, du' \, dn' = \int_{N_M'} A_{bc}^\psi(W_s, w_0^{M'}tn') \psi_{N'(n')}^{-1} \, dn'
\]

which is equal to \(Y^\psi(W, s; t)\) as required. \(\square\)
5.3. To go further, we make a special choice of $W$. Consider the $P$-invariant subspace $\text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ$ of $\text{Ind}(\mathbb{W}^{\psi M}(\pi))$ consisting of functions supported in the big cell $Pw_UU$. Any element of $\text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ$ is a linear combination of functions of the form

\[(5.9) \quad W(u'mw_Uu) = \delta_P(m)^{\frac{1}{2}}W^M(m)\phi(u), \quad m \in M, u, u' \in U\]

with $W^M \in \mathbb{W}^{\psi M}(\pi)$ and $\phi \in C_c^\infty(U)$. Let $\eta_M$ be the embedding $\eta_M(g) = \left( \begin{smallmatrix} g & I_n \\ I_{2n} & g^* \end{smallmatrix} \right)$ of $M$ into $M'$. Also let $\eta_M = g \circ \eta_M$ so that $\eta_M(g) = \left( \begin{smallmatrix} g & I_{2n} \\ I_{2n} & g^* \end{smallmatrix} \right)$. Define $\text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$ to be the linear subspace of $\text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ$ generated by $W$'s as in (5.9) that satisfy the additional property that the function $(t, n) \mapsto W^M(\eta_M(tw_0^{M'} n))$ is compactly supported on $T_{M'}^r \times N_{M'}$, or equivalently, that the function $W^M \circ \eta_M$ on $M'$ is supported in the big cell $B_{M'}w_0^{M'}N_{M'}$ and its support is compact modulo $N_{M'}$. Note that if $W \in \text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$ then $\Phi \ast_s W \in \text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$ for any $\Phi$ and $s$.

Lemma 5.6. For $W \in \text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$ the identity (5.8) holds for all $s \in \mathbb{C}$ and the integrand on the right-hand side of (5.8) is compactly supported in $t, n, v$ uniformly in $s$ (i.e., the support in $(t, n, v)$ is contained in a compact set which is independent of $s$).

Moreover, for any $W \in \text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$ and $\Phi \in C_c^\infty(F^n)$, the function $A^{N}_{bc}(\Phi \ast(W_s), w_0^{M'}tn)$, and thus also $A^{N}(W, \Phi, \bar{\psi}_0, \bar{\ell}n, s)$, are compactly supported in $t \in T'$ and $n \in N_{M'}$, uniformly in $s \in \mathbb{C}$.

Proof. Suppose that $W \in \text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$ is of the form (5.9). Then for $n \in N_{M'}$, $\bar{\ell}(v) \in \bar{U}$ and $t \in T'$, we have $\eta(w_0^{M'}tn) \in M$ and

\[W \left( \eta(w_0^{M'}tn)^\ast \bar{\ell}(v)w_U \right) = \delta_P(\eta(t))^{-\frac{1}{2}}W^M(\eta(w_0^{M'}tn)^\ast)\phi(\ell(-v)).\]

Note that $\eta(g^\ast(m)) \ast = \eta_M(m^\ast)$ for any $m \in M'$. Also note that $W_s(g) = 0$ when $W(g) = 0$. The first part follows readily from the definition of $\text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$.

Similarly for the second part. □

5.4. For $W \in \text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2$ and $W^\psi \in \text{Ind}(\mathbb{W}^{\psi -1 M}(\pi^\psi))$ let

\[(5.10) \quad B(W, W^\psi, s) := \int_{T'} Y^\psi(W, s; t)Y^{\psi^{-1}}(W^\psi, -s; t)\delta_B(t)^\nu(t)^{2n-1}dt.\]

By Lemma 5.6 and Corollary 5.2 the integrand is smooth and compactly supported on $T'$ uniformly in $s \in \mathbb{C}$. Thus, $B(W, W^\psi, s)$ defines an entire function of $s$.

We will soon relate $B(W, W^\psi, s)$ to the left-hand side of the Main Identity (MI). But first we prove a non-vanishing result which follows from Theorem A.1 of Appendix A.

Lemma 5.7. Assume that $\pi \in \text{Irr}_{\text{gen.meta}} M$ and $\tilde{\pi} = D_{\psi^{-1}}(\pi)$ is irreducible and tempered. Then the bilinear form $B(W, M(\frac{1}{2})W^\psi, \frac{1}{2})$ does not vanish identically on $\text{Ind}(\mathbb{W}^{\psi M}(\pi))^\circ_2 \times \text{Ind}(\mathbb{W}^{\psi -1 M}(\pi^\psi))$. 
Proof. Since the image of the restriction map \( \mathbb{W}^{\psi N M}(\pi) \to C(N_M \setminus \mathcal{P}, \psi_{N_M}) \) contains \( C_c^\infty(N_M \setminus \mathcal{P}, \psi_{N_M}) \), it follows that for any \( \varphi \in C_c^\infty(T_{\pi'} \times N_M) \) and \( \phi \in C_c^\infty(U) \) there exists \( W \in \text{Ind}(\mathbb{W}^{\psi N M}(\pi)) \) (necessarily in \( \text{Ind}(\mathbb{W}^{\psi N M}(\pi))^0 \)) of the form (5.9) such that \( \varphi(t, n) = W^M(\pi_M((tw_M n)^*)) \). It follows from (5.8) that the linear map \( \text{Ind}(\mathbb{W}^{\psi N M}(\pi))^0 \to C_c^\infty(T') \) given by \( W \mapsto Y^\psi(W, \frac{1}{2} \cdot \cdot \cdot) \) is onto. Therefore, from the definition (5.10) of \( B \), the lemma amounts to the non-vanishing of the linear form \( W^\wedge \mapsto Y^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, -\frac{1}{2}; t) \) on \( \text{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi)) \) for some \( t \in T' \). This follows from Theorem A.1 and the relation (5.2). \( \square \)

**Proposition 5.8.** Assume that \( \pi \in \text{Irr}_{\text{gen,meta}} \mathbb{M} \) and \( \bar{\pi} = D_{\psi^{-1}}(\pi) \) is irreducible. Then for any \( W \in \text{Ind}(\mathbb{W}^{\psi_{N M}}(\pi))^0 \) and \( W^\wedge \in \text{Ind}(\mathbb{W}^{\psi_{N M}}(\pi)) \), the left-hand side of the Main Identity (MI) for \( \bar{W} = A^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, \Phi^\vee, \cdot, -\frac{1}{2}) \in \mathbb{W}^{\psi_{\bar{N}}}_{\bar{N}}(\bar{\pi}) \) is equal to

\[
B(\Phi \star \frac{1}{2} W, M(\frac{1}{2})(\Phi^\vee \star \frac{1}{2} W^\wedge), \frac{1}{2}) = B(\Phi \star \frac{1}{2} W, \Phi^\vee \star -\frac{1}{2} M(\frac{1}{2})W^\wedge, \frac{1}{2}).
\]

**Proof.** The left-hand side of (MI) is

\[
\int_{N'} (\int_{N' \setminus G'} A^\psi(W, \Phi, \bar{g}, \frac{1}{2})\bar{W}(\bar{g}\bar{u}) \, dg) \psi_{\bar{N}}(\bar{u}) \, du.
\]

Using the Bruhat decomposition we can write the above as

\[
\int_{N'} (\int_{T'} \int_{N'} A^\psi(W, \Phi, \bar{w}_0'\bar{t}\bar{n}, \frac{1}{2})\bar{W}(\bar{w}_0'\bar{t}\bar{n}\bar{u})\delta_{B'}(t)\psi_{\bar{N}}(\bar{u}) \, dn dt) \, du.
\]

By Lemma 5.6 the integrand is compactly supported in \( t, n \). By the definition of the stable integral, the above is

\[
\int_{\Omega} (\int_{T'} \int_{N'} A^\psi(W, \Phi, \bar{w}_0'\bar{t}\bar{n}, \frac{1}{2})\bar{W}(\bar{w}_0'\bar{t}\bar{n}\bar{u})\delta_{B'}(t)\psi_{\bar{N}}(\bar{u}) \, dn dt) \, du
\]

provided that \( \Omega \in \mathcal{CSGR}(N') \) is sufficiently large. On the other hand by Lemma 5.1 this is equal to

\[
\int_{T'} \int_{N'} \delta_{B'}(t)A^\psi(W, \Phi, \bar{w}_0'\bar{t}\bar{n}, \frac{1}{2})(\int_{N'} \bar{W}(\bar{w}_0'\bar{t}\bar{n}\bar{u})\psi_{\bar{N}}(\bar{u}) \, du) \, dn dt
\]

provided, again, that \( \Omega \) is sufficiently large (independently of \( t \) and \( n \), since they are compactly supported by the assumption on \( W \)). Making a change of variable \( u \mapsto n^{-1}u \) we get

\[
\int_{T'} \int_{N'} \delta_{B'}(t)A^\psi(W, \Phi, \bar{w}_0'\bar{t}\bar{n}, \frac{1}{2})\psi_{\bar{N}}(n^{-1})(\int_{N'} \bar{W}(\bar{w}_0'\bar{t}\bar{u})\psi_{\bar{N}}(\bar{u}) \, du) \, dn dt.
\]
When $\tilde{W} = A^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, \Phi^\vee, \cdot, -\frac{1}{2})$, this is
\[
\int_{T^*} \left( \int_{N'} A^{\psi}(W, \Phi, \hat{w}_{U}, \hat{w}_{0}^{M^*}t_{n_1}', \frac{1}{2})\psi_{\tilde{N}}(n_1')^{-1} \, dn_1' \right) \\
\left( \int_{N'} A^{\psi^{-1}}(M(\frac{1}{2})W^\wedge, \Phi^\vee, \hat{w}_{U}, \hat{w}_{0}^{M^*}t_{n_2}'{2}, -\frac{1}{2})\delta_{B'}(t)\psi_{\tilde{N}}(n_2') \, dn_2' \right) \, dt.
\]
From (5.3), (4.1) and the fact that $\beta_{\psi}(g)\beta_{\psi^{-1}}(g) = 1$, we get the required identity. \qed

5.5. By Lemma 4.1, Proposition 5.8 and Lemma 5.7 we conclude

**Corollary 5.9.** Suppose that $\pi \in \text{Irr}_{\text{meta, temp}} M$ is good and $\tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi)$ is tempered. Then
\[
B(W, M(\frac{1}{2})W^\wedge, \frac{1}{2}) = c_{\pi} A_{\psi}(M^*W)A_{\psi^{-1}}(M^*W^\wedge)
\]
for all $W \in \text{Ind}(\mathbb{W}^{N}_{\Psi}(\pi))_2^\circ$ and $W^\wedge \in \text{Ind}(\mathbb{W}^{N}_{\Psi}(\pi))_0^\circ$. Moreover, the linear form $A_{\psi}(M^*W)$ does not vanish identically on $\text{Ind}(\mathbb{W}^{N}_{\Psi}(\pi))_2^\circ$.

In other words, taking into account the reduction step of §3.5 we have reduced Theorem 3.2 to the following statement.

**Proposition 5.10.** Assume $\pi \in \text{Irr}_{\text{meta, temp}} M$ is good and $\tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi)$ is tempered. Then for any $W \in \text{Ind}(\mathbb{W}^{N}_{\Psi}(\pi))_2^\circ$ and $W^\wedge \in \text{Ind}(\mathbb{W}^{N}_{\Psi}(\pi))_0^\circ$ we have
\[
B(W, M(\frac{1}{2})W^\wedge, \frac{1}{2}) = \epsilon_{\pi} A_{\psi}(M^*W)A_{\psi^{-1}}(M^*W^\wedge).
\]

The proposition will eventually be proved in §7 after further reductions, using the results of [LM14b] and [LM14c].

6. APPLICATIONS OF FUNCTIONAL EQUATIONS

In this section we will use the functional equations established in [LM14c] to give a different expression for $B(W, M(\frac{1}{2})W^\wedge, \frac{1}{2})$.

6.1. In order to state the main result we need some more notation. Let
\[
\epsilon_1 = \tilde{t}_M(\epsilon_{n,1}) \left( w_{M^*}^{n} \right) \in \mathbb{M}; \quad \epsilon_2 = \ell_M(\text{diag}(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \ldots)w_{M^*}^{n}) \in \mathbb{M}_n.
\]
As in [LM14b, §7.4], define
\[
\Delta(t) := |t_1|^{-n} \delta_{B'}^2(\varrho(t)), \quad t = \text{diag}(t_1, \ldots, t_{2n}) \in T_M.
\]
In particular when $t = \eta^{\vee}_{M}(t')$ with $t' \in T_{M^1}$, we have $\Delta(t) = \delta_{B'}(\varrho(t'))^{\frac{1}{2}}$.

Let $V_{R} = \{(t_n, n') : t_n \in \mathbb{R}, n \in \mathbb{N}_n \}$ and $\psi_{V_{R}}(t_n, n') = \psi_{N_{M^1}}(n)$. Let $\pi \in \text{Irr}_{\text{gen}} \mathbb{M}_n$.

For $W \in \text{Ind}(\mathbb{W}^{N}_{\Psi}(\pi))_2^\circ$, $t \in T_M$ and $s \in \mathbb{C}$ let
\[
E^{\psi}(W, s; t) := \Delta(t)^{-1} \int_{V_{R}} \int_{U} W_s(\varrho(t_{2}r_{1}) \varrho_{U}) \hat{\psi}(\varrho) \psi_{V_{R}}(r) \, d\varrho \, dr
\]
provided that the integral converges. Similarly define \( E^{\psi^{-1}}(W^\wedge, s; t) \) for \( W^\wedge \in \text{Ind}(\mathbb{W}^{\psi N_M^{-1}}(\pi)) \).

It is clear from the definition that (when defined)

\[
(6.2) \quad E^\psi(W, s; tz) = E^\psi(W, s; t) |\det z|^s \frac{1}{\Gamma(t)} , \quad z \in \mathbb{Z}_M.
\]

Let \( \mathbb{W}^{N_M^{-1}}(\pi)_2 \) be the linear subspace of \( \mathbb{W}^{N_M^{-1}}(\pi) \) consisting of \( W \) such that

\[
W(\cdot, \epsilon_1)|_{\mathcal{P}^*} \in C^\infty(N_M \setminus \mathcal{P}^*, \psi_{N_M}^{-1}) \quad \text{and} \quad W(\cdot, \epsilon_1)|_{\eta_M(T_M') \times V_R} \in C^\infty(\eta_M(T_M') \times V_R).
\]

We note that even if we assume that \( \pi \) is supercuspidal, \( \mathbb{W}^{N_M^{-1}}(\pi)_2 \) is a proper subspace of \( \mathbb{W}^{N_M^{-1}}(\pi) \) since the set \( N_M \cdot (\eta_M(T_M') \times V_R) \) is not closed.

Let \( \text{Ind}(\mathbb{W}^{N_M^{-1}}(\pi))_2^\circ \) be the linear subspace of \( \text{Ind}(\mathbb{W}^{N_M^{-1}}(\pi))_2 \) spanned by the functions which vanish outside \( P_{W, N} \) on the big cell are given by

\[
W(u' m u W u) = \delta_{\pi}(m) W^M(m) \phi(u), \quad m \in M, \; u, u' \in U
\]

with \( \phi \in C^\infty(U) \) and \( W^M \circ \varphi \in \mathbb{W}^{N_M^{-1}}(\pi)_2 \).

Let \( T'' = \eta_M(T_M') \times Z_M \). In this section we prove

**Proposition 6.1.** Let \( \pi \in \text{Irr}_{\text{temp}} M \). For \( -\Re s \gg 1 \) and any \( W \in \text{Ind}(\mathbb{W}^{N_M^{-1}}(\pi))_2 \), \( W^\wedge \in \text{Ind}(\mathbb{W}^{N_M^{-1}}(\pi))_2^\circ \), the integrals defining \( E^\psi(M(s)W, -s; t) \) and \( E^{\psi^{-1}}(W^\wedge, s; t) \) converge. Moreover, we have

\[
(6.3) \quad B(W, M(s)W^\wedge, s) = \int_{\eta_M(T_M')} E^\psi(M(s)W, -s; t) E^{\psi^{-1}}(W^\wedge, s; t) \frac{dt}{|\det t|}
\]

where the integrand is continuous and compactly supported.

**6.2.** We first show the convergence, continuity and compact support assertions of Proposition 6.1.

**Lemma 6.2.** Let \( \pi \in \text{Irr}_{\text{gen}} M \). For \( \Re s \gg 1 \) the integral (6.1) defining \( E^\psi(W, s; t) \) converges for any \( W \in \text{Ind}(\mathbb{W}^{N_M^{-1}}(\pi)) \) and \( t \in T_M \) uniformly for \( (s, t) \) in a compact set. Hence, \( E^\psi(W, s; t) \) is holomorphic for \( \Re s \gg 1 \) and continuous in \( t \). Moreover, if \( W^\wedge \in \text{Ind}(\mathbb{W}^{N_M^{-1}}(\pi))_2^\circ \) and \( t \in \eta_M(T_M') \) then \( E^{\psi^{-1}}(W^\wedge, s; t) \) is entire in \( s \) and compactly supported and locally constant in \( t \), uniformly in \( s \).

**Proof.** By the equivariance of \( W_s \) we have

\[
|W_s(\varphi(te_2)g)| = |W_s(\varphi(t)g)|.
\]

Moreover \( T_M \) normalizes the group \( V_R \). As \( U \) is normalized by \( M \), we are left to check (upon changing \( W \)) the convergence of

\[
\int_{V_R} \int_{\mathbb{C}} |W_s(\varphi(r)\tilde{u}(t))| \, d\tilde{u} \, dr.
\]
This expression is clearly locally constant in $t$, so we can ignore $t$. As $\phi(V_R) \subset w_0^M N_M^{\gen}(w_0^M)^{-1}$, the integral in $r$ and $v$ is a ‘partial integration’ of the double integral (5.4a) in Lemma 5.3 for $W'(g) = W(gw_0^M)$, thus converges by Remark 2.2.

When $W^\wedge \in \text{Ind}(\mathbb{W}^{\psi_{\gen}^{-1}}_{\gen}(\pi))$, the integrand in (6.1) (with $W^\wedge$ in place of $W$) is compactly supported in $t$ (when we restrict the domain to $t \in \eta_{\gen}(T_M^\wedge)$), $v$ and $r$, uniformly in $s$. The second statement follows. □

6.3. We apply a functional equation proved in [LM14c, Appendix B]. For $s \in \mathbb{C}$, $W \in \text{Ind}(\mathbb{W}^{\psi_{\gen}}_{\gen}(\pi))$, $W^\vee \in \text{Ind}(\mathbb{W}^{\psi_{\gen}^{-1}}_{\gen}(\pi))$ we define

$$
B(W, W^\vee, s) = \int_{N'_M} \int_{U} \int_{U} \left( \int_{N'_{M'}\backslash M'} W_s(\eta_M(gn)\bar{v}_1w_U) W_{-s}^\vee(\eta_M(g)\bar{v}_2w_U) \right)
\delta_P(\eta_M(g))^{-1} |\det g|^{1-n} \hat{\psi}_U(\bar{v}_1)\hat{\psi}_U(\bar{v}_2)^{-1} \psi_{N'_M}(n) dg \, d\bar{v}_1 \, d\bar{v}_2 \, dn
$$

whenever the iterated integral is absolutely convergent. (The second expression is obtained from the first one by a change of variables $g \mapsto gn^{-1}$ in the inner integral and $n \mapsto n^{-1}$ in the outer integral.)

Proposition 6.3. ([LM14c, Appendix B]) Let $\pi \in \text{Irr}_{\temp} M$. Then

1. For $\Re s \gg 1$, $B(W, W^\vee, s)$ is well defined for any $W \in \text{Ind}(\mathbb{W}^{\psi}_{\gen}(\pi))$, $W^\vee \in \text{Ind}(\mathbb{W}^{\psi_{\gen}^{-1}}_{\gen}(\pi))$.  

2. For $-\Re s \gg 1$, $B(W, W^\vee, s)$ is well defined for any $W \in \text{Ind}(\mathbb{W}^{\psi}_{\gen}(\pi))^\circ$, $W^\vee \in \text{Ind}(\mathbb{W}^{\psi_{\gen}^{-1}}_{\gen}(\pi))^\circ$.

3. For $-\Re s \gg 1$ we have

$$
B(W, M(s)W^\wedge, s) = B(M(s)W, W^\wedge, -s)
$$

for any $W \in \text{Ind}(\mathbb{W}^{\psi}_{\gen}(\pi))^\circ$, $W^\wedge \in \text{Ind}(\mathbb{W}^{\psi_{\gen}^{-1}}_{\gen}(\pi))^\circ$.

On the other hand, we have:

Lemma 6.4. Let $\pi \in \text{Irr}_{\gen} M$. Then for $-\Re s \gg 1$ we have

$$
B(W, W^\vee, s) = \int_{N'_M} \int_{U} \int_{U} \int_{N'_{M'}\backslash M'} W_s(\eta_M(gn)\bar{v}_1w_U) W_{-s}^\vee(\eta_M(g)\bar{v}_2w_U) \delta_P(\eta_M(g))^{-1} |\det g|^{1-n} \hat{\psi}_U(\bar{v}_1)\hat{\psi}_U(\bar{v}_2)^{-1} \psi_{N'_M}(n) dg \, d\bar{v}_1 \, d\bar{v}_2 \, dn
$$

for any $W \in \text{Ind}(\mathbb{W}^{\psi}_{\gen}(\pi))^\circ$, $W^\vee \in \text{Ind}(\mathbb{W}^{\psi_{\gen}^{-1}}_{\gen}(\pi))$, with the integral being absolutely convergent. Thus for $-\Re s \gg 1$ and any $W \in \text{Ind}(\mathbb{W}^{\psi}_{\gen}(\pi))^\circ$, $W^\vee \in \text{Ind}(\mathbb{W}^{\psi_{\gen}^{-1}}_{\gen}(\pi))$, we have $B(W, W^\vee, s) = B(W, W^\vee, s)$.
Theorem 6.6. \( \psi \) GL_6.4. An identity of Whittaker functions on \( A \) (6.6) where for any \( \eta \leq W \):

\[
\nu'(t)^3 \delta_{B_{M'}}(t) \int \psi^*_U(t) \bar{\psi}_U(t)^{-1} \psi_{N_{M'}}(n_2n_1^{-1}) dt \]

where the integral is absolutely convergent.

Making a change of variables \( n_1 \to n_2n_1 \) and noting that for \( m \in \mathbb{M}' \), \( \eta(g'(m))^\star = \eta_M(m^\star) \) we get that \( B(W, W^\vee, s) \) is equal to

\[
\int T_{M'} \int_{N_{M'}} \int U \int_{N_{M'}} \int_U W_s \left( \eta(w_0^{M'}t_{n_2}) \star \bar{v}_1 w_U \right) W_{-s}^\vee \left( \eta(w_0^{M'}t_{n_2}) \star \bar{v}_2 w_U \right)
\]

\[
\nu'(t)^3 \delta_{B_{M'}}(t) \int \psi^*_U(t) \bar{\psi}_U(t)^{-1} \psi_{N_{M'}}(n_1) dt
\]

It remains to use the Bruhat decomposition for \( \mathbb{M}' \) and to note that \( \delta_{P}(\eta_M(g)) = |det g|^{2n+1} \).

From Lemma 6.4 and Proposition 6.3 we immediately get:

Corollary 6.5. For \(-\Re s \gg 1 \) we have

\[
B(W, M(s)W^\vee, s) = B(M(s)W, W^\vee, -s)
\]

for any \( W \in \text{Ind}(\mathbb{W}^1_{N,M}(\pi))^\circ \), \( W^\vee \in \text{Ind}(\mathbb{W}^1_{N,M}(\pi))^\circ \).

6.4. An identity of Whittaker functions on \( \text{GL}_{2n} \). A key fact in the formal argument for the case \( n = 1 \) in [LM13b, §7] was that for any unitarizable \( \pi \in \text{Irr}_{\text{gen}} \text{GL}_2 \), the expression

\[
\int_F W_1((t_1)) W_2((t_1)) dt
\]

defines \( \text{GL}_2 \)-invariant bilinear form on \( \mathbb{W}^1_{N,M}(\pi) \times \mathbb{W}^1_{N,M}(\pi)^\vee \).

In the general case we encountered (in the definition of \( B \)) a similar integral

\[
A_n(W, W^\vee) = \int_{N_{M'} \setminus \mathbb{M}'} W(\eta_M(g)) W^\vee(\eta_M(g)) |det g|^{1-n} dg
\]

where \( W \in \mathbb{W}^1_{N,M}(\pi) \) and \( W^\vee \in \mathbb{W}^1_{N,M}(\pi)^\vee \). While this integral does not define an \( \mathbb{N} \)-invariant form, we have the following relations. Let \( w_{2n,n} := (I_n L_n) \).

Theorem 6.6. ([LM14c, Theorem 1.3]) Let \( \pi \in \text{Irr}_{\text{temp}} \mathbb{M} \). Then for any \( W \in \mathbb{W}^1_{N,M}(\pi) \), \( W^\vee \in \mathbb{W}^1_{N,M}(\pi)^\vee \) we have

\[
A_n(W, W^\vee) = \int_{R_n} \int_{R_n} \int_{N_{M'} \setminus \mathbb{M}'} W(\eta_M^\vee(\bar{\eta}_M(X)w_{2n,n})) W^\vee(\eta_M^\vee(\bar{\eta}_M(Y)w_{2n,n})) |det g|^{n-1} dg dX dY
\]
The integrals are absolutely convergent.

We remark that in the above Theorem we may replace \( w_{2n,n} \) by \( xw_{2n,n} \) where \( x \) is any element in \( \eta_M'(M') \) with \( |\det x| = 1 \). (In the sequel we will apply the theorem with \( w_{2n,n} := \begin{pmatrix} \iota_n' \iota_n \end{pmatrix} \) instead of \( w_{2n,n} \).)

We can slightly rephrase Theorem 6.6 as follows. Observe the following equivariance property of \( A_n \):

\[
A_n(\pi(n)W, \pi^\vee(n)W^\vee) = A_n(W, W^\vee)
\]

for all \( n \in N_M \). (This follows easily from [LM14c, (20)].) Let \( \epsilon_1 \) be an arbitrary element of the form \( w_{2n,n}' \epsilon_1' \) where \( \epsilon_1' \in N_M \) (e.g., \( \epsilon_1 = \epsilon_1 \) defined in §6.1). From the above equation and Theorem 6.6 we get that \( A_n(W, W^\vee) \) equals

\[
\int_{R_n} \int_{R_n} \int_{N_M'} \int_{T_M'} W \left( \eta_M'(g) \mathbf{\tau}_M(X) w_{2n,n}' \epsilon_1' \right) W^\vee \left( \eta_M'(g) \mathbf{\tau}_M(Y) w_{2n,n}' \epsilon_1' \right) |\det g|^{n-1} \, dg \, dX \, dY.
\]

Using Bruhat decomposition for \( g \), the integral can be rewritten as

\[
\int_{R_n} \int_{R_n} \int_{N_M'} \int_{T_M'} W \left( \eta_M'(tn^t) \mathbf{\tau}_M(X) \epsilon_1 \right) W^\vee \left( \eta_M'(tn^t) \mathbf{\tau}_M(Y) \epsilon_1 \right) |\det t|^{n-1} \delta_{B_M'}(t)^{-1} \, dt \, dn \, dX \, dY.
\]

Finally, by a change of variables in \( X \) and \( Y \) and using the equivariance of the Whittaker functions we get

\[
(6.7) \quad A_n(W, W^\vee) = \int_{R_n} \int_{R_n} \int_{N_M'} \int_{T_M'} W \left( \eta_M(t) \epsilon_2 \left( \begin{pmatrix} I_n & 0 \\ 0 & n^t \end{pmatrix} \right) \epsilon_1 \right) W^\vee \left( \eta_M(t) \epsilon_2 \left( \begin{pmatrix} I_n & 0 \\ 0 & n^t \end{pmatrix} \right) \epsilon_1 \right) |\det t|^{n-1} \delta_{B_M'}(t)^{-1} \, dt \, dn \, dX \, dY
\]

for an arbitrary \( \epsilon_2 \in N_M \).

6.5. We will apply the above functional equation in the following setting. Let \( \pi \in \operatorname{Irr}_{\text{temp}} M \). Define the bilinear form \( D(W, W^\vee) \) on \( \mathbb{W}^{\psi_{NM}}(\pi) \times \mathbb{W}^{\psi_{NM}^{-1}}(\pi^\vee) \) by

\[
(6.8) \quad D(W, W^\vee) := \int_{N_M'} A_n(\pi(\eta_M(n))W, W^\vee) \psi_{N_M'}(n) \, dn.
\]

It is shown in [LM14c, Appendix A] that the integral is absolutely convergent.

**Proposition 6.7.** Let \( \pi \in \operatorname{Irr}_{\text{temp}} M \) and \( \epsilon_2 \in N_M \). Assume that \( W^\vee \in \mathbb{W}^{\psi_{NM}^{-1}}(\pi^\vee) \). Then for any \( W \in \mathbb{W}^{\psi_{NM}}(\pi) \), \( D(W, W^\vee) \) is equal to the absolutely convergent integral

\[
\int_{V_R} \int_{V_R} \int_{T_M'} W \left( \eta_M(t) \epsilon_2 r_1 \epsilon_1 \right) W^\vee \left( \eta_M(t) \epsilon_2 r_2 \epsilon_1 \right) |\det t|^{n-1} \delta_{B_M'}(t)^{-1} \psi_{R}(r_1 r_2^{-1}) \, dt \, dr_2 \, dr_1
\]

where the integrand is compactly supported in all variables.
Proof. From (6.7) and (6.8), we get that $D(W, W^\vee)$ is equal to
\[
\int_{N_{M'}^*} \left( \int_{R_n} \int_{R_n} \int_{T_{M'}} W \left( \eta_{M}(t) \epsilon_2 \left( \frac{I_n}{X \ n_1^2} \right) \epsilon_1 \eta_{M}(n) \right) \right) W^\vee \left( \eta_{M}(t) \epsilon_2 \left( \frac{I_n}{Y \ n_2^2} \right) \epsilon_1 \right) |\det t|^{n-1} \delta_{B_{M'}^*}(t)^{-1} dt dn_2 dX dY \psi_{N_{M'}^*}(n) \ dn.
\]
Now, since $\mathfrak{t}_M(\epsilon_n, 1)$ commutes with $\eta_{M}^\vee((N_{M'}^*)^t)$, we have $\epsilon_1 \eta_{M}(n) = \eta_{M}^\vee(n_1^t) \epsilon_1$ with $n_1 \in N_{M'}^*$ and $\psi_{N_{M'}^*}(n) = \psi_{N_{M'}^*}(n_1)$. Thus, the above is
\[
\int_{N_{M'}^*} \left( \int_{R_n} \int_{R_n} \int_{T_{M'}} W \left( \eta_{M}(t) \epsilon_2 \left( \frac{I_n}{X \ n_1^2} \right) \epsilon_1 \right) \right) W^\vee \left( \eta_{M}(t) \epsilon_2 \left( \frac{I_n}{Y \ n_2^2} \right) \epsilon_1 \right) |\det t|^{n-1} \delta_{B_{M'}^*}(t)^{-1} dt dn_2 dX dY \psi_{N_{M'}^*}(n_1) \ dn_1.
\]
By the condition on $W^\vee$, the integrand is compactly supported in $t$, $Y$ and $n_2$. We now show that the integrand is also compactly supported in $X$, $n_1$. Note that $\eta_{M}^\vee(t) \epsilon_2 \eta_{M}(t)^{-1} \in N_M$ and therefore, by the equivariance of $W$,
\[
|W \left( \eta_{M}^\vee(t) \epsilon_2 g \right)| = |W \left( \eta_{M}^\vee(t)g \right)|
\]
for all $g$. As $\eta_{M}^\vee(t)$ normalizes $V_R$, we are left to show that the restriction of any Whittaker function $W$ to $V_R$ is compactly supported.

For $r \in V_R$, let $r^* = uak$ be the Iwasawa decomposition of $r^*$ with $a = \text{diag}(a_1, \ldots, a_{2n})$. Since the last row of $r^*$ is $(0, \ldots, 0, 1)$ we have $a_{2n} = 1$ and thus, over the support of $W$, $|a_i|$ is bounded above (since $W(g^*)$ is a Whittaker function). On the other hand, since $r^*$ is lower unitriangular, the entries of $r^*$ are bounded in norm by products of the $|a_i|$s, and therefore are bounded. Hence, $W$ is compactly supported when restricted to $V_R$. Therefore the integrand in the Proposition (as well as in (6.9)) is compactly supported. A change of variable $n_1 \mapsto n_1 n_2^{-1}$ gives the identity in the Proposition. \hfill \square

6.6. Recall $B(W, W^\vee, s)$ given by (6.4).

Proposition 6.8. Let $\pi \in \text{Irr}_{\text{temp}} M$. Then for $\Re s \gg 1$ we have
\[
B(W, W^\vee, s) = \int_{\eta_{M}(T_{M'})} E^\psi(W, s; t) E^{\psi^{-1}}(W^\vee, -s; t) \frac{dt}{|\det t|}
\]
for any $W \in \text{Ind}(\mathbb{W}^\psi_M(\pi))$ and $W^\vee \in \text{Ind}(\mathbb{W}^{\psi^{-1}}_M(\pi^\vee))^\circ$. Recall that (by Lemma 6.2) the integrand on the right-hand side is continuous and compactly supported.

Proof. For $(W, W^\vee) \in \text{Ind}(\mathbb{W}^\psi_M(\pi)) \times \text{Ind}(\mathbb{W}^{\psi^{-1}}_M(\pi^\vee))$, define
\[
D_G(W, W^\vee) := D(\delta_p^{-\frac{1}{2}} W \circ \varphi, \delta_p^{-\frac{1}{2}} W^\vee \circ \varphi).
\]
Clearly for any $s \in \mathbb{C}$,
\[
D_G(W, W^\vee) = D(\delta_p^{-\frac{1}{2}} W_s \circ \varphi, \delta_p^{-\frac{1}{2}} W_s^\vee \circ \varphi).
\]
By Proposition 6.3 $\mathcal{B}(W, W^\vee, s)$ is well-defined for $\Re s \gg 1$. We have
\[
\mathcal{B}(W, W^\vee, s) = \int_{\tilde{U}} \int_{\tilde{U}} D_G(I(s, \tilde{v}_1 w_U)W, I(-s, \tilde{v}_2 w_U)W^\vee)\hat{\psi}_U(\tilde{v}_1)\hat{\psi}_U^{-1}(\tilde{v}_2) d\tilde{v}_1 d\tilde{v}_2.
\]
From Proposition 6.7 and the fact that for $t \in T_{M'}$
\[
d_{B_s'}(t)^{-1}d_P(\eta_M^\vee(t))^{-1} |\det t|^{n-1} = d_{B_t'}(\rho'(t))^{-1} |\det t|^{-1},
\]
the above is
\[
\int_{\tilde{U}} \int_{\tilde{U}} (\int_{T_{M'}} \int_{V_R} \int_{V_R} d_{B_t'}(\rho'(t))^{-1} W_s(\rho(\eta_M^\vee(t)\epsilon_2 r_1 \epsilon_1)\tilde{v}_1 w_U) \hat{\psi}_U(\tilde{v}_1)\psi_{V_R}(r_1)) \nu_{V_s'}(\rho(\eta_M^\vee(t)\epsilon_2 r_1 \epsilon_1)\tilde{v}_2 w_U) \hat{\psi}_U^{-1}(\tilde{v}_2)\psi_{V_R}(r_2) dr_1 dr_2 \frac{dt}{|\det t|} d\tilde{v}_1 d\tilde{v}_2.
\]
To finish the proof of Proposition 6.8 it suffices to show the convergence of
\[
\int_{T_{M'}} d_{B_t'}(\rho'(t))^{-1} \int_{V_R} \int_{\tilde{U}} |W_s(\rho(\eta_M^\vee(t)\epsilon_1)\tilde{v}_1 w_U)| d\tilde{v}_1 dr_1 \int_{V_R} \int_{\tilde{U}} |W_{s'}(\rho(\eta_M^\vee(t)\epsilon_2 r_1 \epsilon_1)\tilde{v}_2 w_U)| d\tilde{v}_2 dr_2 \frac{dt}{|\det t|}.
\]
It follows from Lemma 6.2 that the four integrations over $V_R$ and $\tilde{U}$ converge for a fixed $t$. By the same Lemma, the resulting function in $t$ is of compact support. The argument in Lemma 6.2 also shows that the integrand is smooth in $t$, thus the integral converges. □

Combining Proposition 6.8 with Corollary 6.5 and (6.2) we get Proposition 6.1.

7. PROOF OF PROPOSITION 5.10

We are now in a position to use the results of [LM14b, §11] (with the parameter $a = -\frac{1}{2}$) to conclude the proof of Proposition 5.10 from Proposition 6.1.

7.1. First observe that for $r \in V_R$ we have
\[
\hat{\psi}_U(\rho(\epsilon_2 r_1)\tilde{v}) = \hat{\psi}_U(\tilde{v}), \quad \tilde{v} \in \tilde{U}
\]
where $\psi_U$ is the character on $U$ given by $\psi_U(\tilde{v}) := \psi(\tilde{v}_{2n+1,1})$ (independently of $r$). Making a change of variable
\[
\tilde{v} \mapsto \rho(n\epsilon_2 r_1)^{-1} \tilde{v} \rho(n\epsilon_2 r_1)
\]
in (6.1), we get when $\Re s \gg 1$:
\[
E^\psi(W, s; t) := \Delta(t)^{-1} \int_{V_R} \int_{\tilde{U}} W_s(\rho(\tilde{v}) \rho(\epsilon_2 r_1) w_U) \psi_U(\tilde{v}) \psi_{V_R}(r) d\tilde{v} dr.
\]
Proposition 7.1. Suppose that \( \pi \in \text{Irr}_{\text{meta, temp}} \mathbb{M} \). Then for any \( W \in \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi))_\mathbb{Z}^\omega \), \( W^\wedge \in \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi))_\mathbb{Z}^\omega \) we have

\[
B(W, M(\frac{1}{2})W^\wedge, \frac{1}{2}) = \epsilon_\pi^n A^\psi_{\epsilon}(M^*W) \int_{Z_{\mathbb{M}} \setminus T''} \psi^{-1}(W^\wedge, \frac{1}{2}; t) dt. 
\]

Proof. Note that Proposition 6.1 is not yet applicable at \( s = \frac{1}{2} \). We need some more analysis. For \( i = 1, \ldots, n-1 \) let \( T_i \) be the one-dimensional torus

\[ T_i := \{ \text{diag} (z^{-1}, \ldots, z^{-1}, z, \ldots, z) : z \in F^* \}. \]

Let \( S \) be the torus \( \prod_{i=1}^{n-1} T_i \) so that as algebraic groups \( Z_\mathbb{M} S = T'' \). By Lemma 6.2, for any \( W^\wedge \in \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi))_\mathbb{Z}^\omega \) there exists \( K_0 \in \mathcal{CSR}(T'') \) such that \( \psi^{-1}(W^\wedge, s; \cdot) \in C(T'')^{K_0} \) for all \( s \). Suppose that \( f \in C_c^{\infty}(S) \) is supported in \( S \cap K_0 \). Let \( f'(t) := f(t^{-1}) \) and \( f * g(\cdot) = \int_S f(t) g(t) dt \) for \( g \in C(T'') \). Let \( W \in \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi))_\mathbb{Z}^\omega \). By Proposition 6.1, for \( -\Re s \gg 1 \)

\[
B(W, M(s)W^\wedge, s) \int_S f(t) \Delta(t) dt = \int_{Z_{\mathbb{M}} \setminus T''} \psi(M(s)W, -s; t) f' * \psi^{-1}(W^\wedge, s; t) dt. 
\]

(Of course, \( \Delta(t) = 1 \) on the support of \( f \).) By [LM14b, Corollary 11.9], for any \( W \in \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi))_\mathbb{Z}^\omega \) and \( f \in C_c^{\infty}(S) \), \( f * \psi(W, s; t) \) extends to an entire function in \( s \) and is locally constant in \( t \).

From Lemma 6.2 we infer that both sides of (7.4) are meromorphic functions in \( s \) and the identity holds whenever \( M(s) \) is holomorphic. Specializing to \( s = \frac{1}{2} \) we get

\[
B(W, M(\frac{1}{2})W^\wedge, \frac{1}{2}) \int_S \Delta(t) f(t) dt = \int_{Z_{\mathbb{M}} \setminus T''} f * \psi(M(\frac{1}{2})W, -\frac{1}{2}; t) \psi^{-1}(W^\wedge, \frac{1}{2}; t) dt. 
\]

Let

\[
\hat{\omega} := \text{diag} (I_{n}, (\omega I_n), I_n) = \left( \begin{array}{cc} I_n & \omega \end{array} \right). 
\]

Note that \( \hat{\omega} \gamma_3 = \hat{\omega}(\epsilon_1) w_U \). Using the expression (7.2) for \( \psi(W, s; t) \) and the definition (4.2) of \( A^\psi_{\epsilon} \) we infer from [LM14b, Corollary 11.10] that

\[
f * \psi(M(\frac{1}{2})W, -\frac{1}{2}; t) = \epsilon_\pi^n A^\psi_{\epsilon}(M^*W) \int_S f(t') \Delta(t') dt'
\]

for any \( W \in \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi))_\mathbb{Z}^\omega \), \( t \in T'' \) and \( f \in C_c^{\infty}(S) \). The Proposition follows. \( \Box \)
Next we analyze the right-hand side of (7.3).

**Proposition 7.2.** Let $\pi \in \text{Irr}_{\text{meta, temp}} \mathbb{M}$. Then for any $W \in \text{Ind}(\mathbb{W}^{\text{NM}}_{\pi}(\pi))_q$ we have

$$\int_{Z_M \setminus T''} E_\psi(W, \frac{1}{2}; t) \frac{dt}{|\det t|} = \epsilon_\pi^{n+1} A_\pi(M^* W).$$

**Proof.** Let us explain how the proposition follows from the results of [LM14b]. Consider the integrand is compactly supported. We can assume without loss of generality that $W \subset M$ consisting of root subgroups $Z \subset M$. Let $V$ be the direct product of the commuting one-parameter unipotent group (see [LM14b, §6]) and $\psi(\phi) \in C^\infty_c(U)$. Thus,

$$I = I' \int_U \phi(v) \hat{\psi}_{V}^{-1}(v) \, dv$$

where

$$I' = \int_{\eta_{M}(T'_{M'})} \int_{V_R} \Delta(t)^{-1} |\det t|^{n} W^{\text{M}}(t e_2 r e_1) \psi_{V_R}(r) \, dr \, dt.$$ 

The integrand in $I'$ is compactly supported because $W^M \in \mathbb{W}^{\text{NM}}_{\pi}(\pi)_q$. Observe for $t = \text{diag}(I_n, t_1, \ldots, t_n) \in \eta_{M}(T'_{M'})$,

$$\psi_{N_{M}}(e_2 t^{-1}) = \psi((-1)^{n} \frac{1}{2} t^{-1}) = \Xi(t).$$

By the equivariance of $W^M$,

$$I' = \int_{\eta_{M}(T'_{M'})} \int_{V_R} \Delta(t)^{-1} |\det t|^{n} \Xi(t) W^{\text{M}}(t e_1) \psi_{V_R}(r) \, dr \, dt.$$ 

We will apply [LM14b, Lemma 11.6]. First recall some groups defined in [LM14b]. Let $Z \subset \mathbb{M}$ consisting of $m$ such that for all $1 \leq i, j \leq 2n$, $m_{i,j} = 1$, $m_{i,j} = 0$ when $j > i$ and $i + j \geq 2n$ or when $i > j$ and $i + j \leq 2n + 1$. Then $Z$ is a unipotent group (see [LM14b, §6]) and

$$\psi_Z(m) = \psi(1 + m_{1,2} + \ldots + m_{n-1,n} - m_{n+2,n+1} - \ldots - m_{2n,2n-1}), \quad m \in Z$$

is a character on it. Let $N_{\mathbb{M},\Delta} \subset \mathbb{M}$ be the direct product of the commuting one-parameter root subgroups $(N_{\mathbb{M},\Delta}^{m})^i : m \not\in (\mathbb{M},\Delta)^i$. Let $V_\Delta = Z \cap (N_{\mathbb{M},\Delta})^i$. Then $V_R = V_\Delta \times N_{\mathbb{M},\Delta}$, and $e_1$ commutes with elements in $N_{\mathbb{M},\Delta}$. We have

$$I' = \int_{N_{\mathbb{M},\Delta}} \int_{V_\Delta} \int_{\eta_{M}(T'_{M'})} \Delta(t)^{-1} |\det t|^{n} \Xi(t) W^{\text{M}}(t e_1) \psi_{V_R}(r) \, dr \, dr \, du.$$
Applying [LM14b, Lemma 11.6] to the integration over \( \eta_{\mathcal{M}}(T_{\mathcal{M}}) \times V_\Delta \). We get that as functions on \( N_{\mathcal{M}, \Delta} \):

\[
\int_{V_\Delta} \int_{\eta_{\mathcal{M}}(T_{\mathcal{M}})} \Delta(t)^{-1} |\det t|^n \Xi(t) W^M(tr \cdot \epsilon_1) \psi_{V_\Delta}(r) \, dt \, dr
\]

\[
= \int_{Z \cap H_\Delta \setminus Z} \Phi^M(\pi(n\epsilon_2 \cdot \epsilon_1)W^M) \psi_Z(n)^{-1} \, dn,
\]

where

\[
\Phi^M(W^M) := \int_{N_{\mathcal{M}} \cap H_\Delta \setminus P \cap H_\Delta} W^M(p) \, dp.
\]

Thus,

\[
I = \int_{N_{\mathcal{M}, \Delta}} \left( \int_{Z \cap H_\Delta \setminus Z} \int_U \Phi^M(\pi(n\epsilon_2 u\epsilon_1)W^M) \psi_Z(n)^{-1} \phi(v) \Phi^M(v) \, dv \, dn \right) \, du.
\]

From (7.5), \( (\delta_P^{-\frac{1}{2}} I(\frac{1}{2}, \varrho(m)(-v)w_U)W) \circ \varrho = \varphi(\ell(v)) \delta_P^{-\frac{1}{2}} (\varrho(m)) |\det m|^{-\frac{1}{2}} \pi(m)W^M \) for any \( v \in \mathfrak{g}_{2n}, m \in M \). Thus, \( I \) equals

\[
\int_{N_{\mathcal{M}, \Delta}} \left( \int_{Z \cap H_\Delta \setminus Z} \int_U \Phi^M(\delta_P^{-\frac{1}{2}} I(\frac{1}{2}, \varrho(\epsilon_2 \epsilon_1) \bar{u}w_U)W) \circ \varrho \psi_Z(n)^{-1} \Phi^M(\bar{u}) \, d\bar{u} \, dn \right) \, du.
\]

Making a change of variable

\[
\bar{u} \mapsto \varrho(\epsilon_2 \epsilon_1)^{-1} \bar{u} \varrho(\epsilon_2 \epsilon_1)
\]

on \( \bar{U} \) and using (7.1) we obtain

\[
I = \int_{N_{\mathcal{M}, \Delta}} \left( \int_{\varrho(Z \cap H_\Delta \setminus Z) \in \mathfrak{E}} \Phi^M(\delta_P^{-\frac{1}{2}} I(\frac{1}{2}, \varrho(\epsilon_2 \epsilon_1) w_U)W) \circ \varrho \psi^{-1}_\mathfrak{E}(v) \, dv \right) \, du
\]

where \( \mathfrak{E} = \varrho(Z) \times \bar{U} \) with

\[
\psi_{\mathfrak{E}}(\varrho(m)\bar{u}) = \psi_Z(m) \psi^{-1}_\mathfrak{E}(\bar{u}), \quad m \in Z, \bar{u} \in \bar{U}.
\]

Define

\[
L_W(g) := \int_{P \cap H \setminus H} \int_{N_{\mathcal{M}} \cap H \setminus P \cap H_\Delta} W(\varrho(p)hg) |\det p|^{-(n+1)} \, dp \, dh
\]

\[
= \int_{H \cap \bar{U}} \Phi^M(\delta_P^{-\frac{1}{2}} I(\frac{1}{2}, \bar{u}g)W) \circ \varrho \, d\bar{u}
\]

where \( H \) is the centralizer of \( \varrho(E) \) in \( G \). By [LM14b, Lemma 4.3] \( L_W \) is well defined for any \( W \in \operatorname{Ind}(\mathcal{W}^\omega_{\text{hom}}(\pi), \frac{1}{2}) \) and \( L_W \) is left \( H \)-invariant.

Integrating over \( H \cap \bar{U} \) first, we get

\[
I = \int_{N_{\mathcal{M}, \Delta}} \left( \int_{H \cap \mathfrak{E} \setminus \mathfrak{E}} L_W(\varrho(\epsilon_2 \epsilon_1) w_U) \psi_{\mathfrak{E}}^{-1}(v) \, dv \right) \, du.
\]
On the other hand by (4.2) and [LM14b, Corollary 11.10], we get

\[ A^\psi_e(M^*W) = \epsilon_\pi^{n+1} \int_{N_\Delta} \left( \int_{H \cap e \not\in e} L_W(v \varrho(\epsilon_2u) \hat{w}^\epsilon \epsilon_3) \psi_e^{-1}(v) \, dv \right) \, du. \]

Since \( \hat{w}^\epsilon \epsilon_3 = \varrho(\epsilon_1)w_U \), we get \( I = \epsilon_\pi^{n+1} A^\psi_e(M^*W) \). The proposition follows.

\[ \square \]

7.3. Central character of the descent. For the record we mention the following consequence of the analysis above.

**Proposition 7.3.** Let \( \pi \in \text{Irr}_{\text{gen}, \text{meta}} \mathfrak{M} \), and \( \tilde{\pi} = D_\psi(\pi) \). Then \( \tilde{\pi}(-I_{2n}) = \gamma_\psi((-1)^n)\epsilon_\pi \).

We note that we do not assume \( \tilde{\pi} \) is irreducible. When \( \tilde{\pi} \) is irreducible, \( \tilde{\pi}(-I_{2n})/\gamma_\psi((-1)^n) \) is the central sign of \( \tilde{\pi} \) introduced by Gan–Savin [GS12].

**Proof.** As in Lemma 4.1, for \( W \in \text{Ind}(\mathbb{W}^\psi \mathcal{M}(\pi)) \) and \( \Phi \in C^\infty_c(F^n) \) we have

\[ A^\psi(W, \Phi, -I_{2n}, s) = \gamma_\psi((-1)^n)A^\psi_e(\Phi \ast (W_s)(\cdot \text{diag}(I_n, -I_{2n}, I_n))). \]

As Whittaker functions in \( \tilde{\pi} \) has the form \( A^\psi(M(\frac{1}{2})W, \Phi, \cdot, -\frac{1}{2}) \), we are left to show

\[ A^\psi_e(M^*W) = \epsilon_\pi A^\psi_e(M^*W(\cdot \text{diag}(I_n, -I_{2n}, I_n))). \]

For the moment assume \( \pi \) is good and tempered. From (7.7) and the fact that \( \hat{w}^\epsilon \epsilon_3 = \varrho(\epsilon_1)w_U \), the right hand side is:

\[ \epsilon_\pi^n \int_{N_\Delta} \left( \int_{H \cap e \not\in e} L_W(v \varrho(\epsilon_2u\epsilon_1)w_U \text{diag}(I_n, -I_{2n}, I_n)) \psi_e^{-1}(v) \, dv \right) \, du. \]

Recall \( \epsilon_1 = \bar{\ell}_M(\epsilon_{1n}) \left( \begin{smallmatrix} w_{0}^M & I_n \\ 0 & w_{0}^M \end{smallmatrix} \right) \). By a change of variable in \( s \), we can replace \( \epsilon_1 \) above by the element \( \epsilon_1' \left( \begin{smallmatrix} w_{0}^M & I_n \\ 0 & w_{0}^M \end{smallmatrix} \right) \) where \( \epsilon_1' = \bar{\ell}_M(w_0^M \text{diag}(1, -1, \ldots, (-1)^{n-1})) \). As \( \epsilon_1' \) commutes with elements in \( N_\Delta \), we can rewrite (after a change of variable in \( s \)) the right hand side of (7.8) as:

\[ \epsilon_\pi^n \int_{N_\Delta} \left( \int_{H \cap e \not\in e} L_W(v \varrho(\epsilon_2\epsilon_1' \text{diag}(I_n, -I_n)) \varrho(u \left( \begin{smallmatrix} w_{0}^M \\ I_n \end{smallmatrix} \right))w_U) \psi_e^{-1}(v) \, dv \right) \, du. \]

By a similar reasoning, the left-hand side of (7.8) is

\[ \epsilon_\pi^{n+1} \int_{N_\Delta} \left( \int_{H \cap e \not\in e} L_W(v \varrho(\epsilon_2\epsilon_1') \varrho(u \left( \begin{smallmatrix} w_{0}^M \\ I_n \end{smallmatrix} \right))w_U) \psi_e^{-1}(v) \, dv \right) \, du. \]

Now observe that \( \epsilon_2\epsilon_1' \text{diag}(I_n, -I_n) = a\epsilon_2\epsilon_1' \) where

\[ a = \text{diag}(1, -1, \ldots, (-1)^{n-1} \frac{1}{2}, (-1)^{n-1} \frac{1}{2}, \ldots, -2, 2)w_0^M. \]

Since \( \varrho(a) \) stabilizes \( (\mathfrak{e}, \psi_e) \) and normalizes \( H \) with \( L_W(\varrho(a) \cdot) = \epsilon_\pi L_W(\cdot) \), we get (7.8). Finally, the same argument as in §3.5 using the classification result Theorem 3.6 gives the proposition for all \( \pi \in \text{Irr}_{\text{gen}, \text{meta}} \mathfrak{M} \).
7.4.

Proof of Proposition 5.10. From (7.3) and Proposition 7.2 we get that (5.11) holds for $(W, W^\wedge) \in \text{Ind}(\mathbb{W}^\psi_N^M(\pi))^2 \times \text{Ind}(\mathbb{W}^\psi_N^M(\pi))^2$, namely

$$B(W, M(\frac{1}{2})W^\wedge, \frac{1}{2}) = \epsilon_e A_e(\mathbb{M}^*W)A_e^{-1}(\mathbb{M}^*W^\wedge).$$

On the other hand, as in the proof of Lemma 5.7, it follows from the definition (6.1) and the fact that the image of the restriction map $\mathbb{W}^\psi_{N^M}(\pi) \to C(N_M\setminus G^\psi, \psi_{N_M}^1)$ contains $C^\infty_c(N_M\setminus G^\psi, \psi_{N_M}^1)$ that the linear map $\text{Ind}(\mathbb{W}^\psi_{N^M}(\pi))^2 \to C^\infty_c(T_{M'}^\psi)$ given by $W^\wedge \mapsto E(\psi^{-1}(W^\wedge, \frac{1}{2}: \cdot))$ is onto. Therefore, by Proposition 7.2 the linear form $A_e^\psi(\mathbb{M}^*W^\wedge)$ is nonvanishing on $\text{Ind}(\mathbb{W}^\psi_{N^M}(\pi))^2$. By Corollary 5.9 we conclude that (5.11) holds for all $W^\wedge$ (not necessarily in $\text{Ind}(\mathbb{W}^\psi_{N^M}(\pi))^2$). Proposition 5.10 follows.

By the discussion before Proposition 5.10, this concludes the proof of Theorem 3.2 and thus the proof of Theorem 1.3.

APPENDIX A. NON-VANISHING OF BESSEL FUNCTIONS

Let $G$ be a split group over a $p$-adic field. Let $B = A \times N$ be a Borel subgroup of $G$. Let $G^\circ = Bw_0B$ be the open Bruhat cell where $w_0$ is the longest element of the Weyl group. Fix a non-degenerate continuous character $\psi_N$ of $N$. For any $\pi \in \text{Irr}_{\text{gen}, \psi_N}(G)$, the Bessel function $\mathbb{B}_\pi = \mathbb{B}_{\pi}^\psi_N$ of $\pi$ with respect to $\psi_N$ is the locally constant function on $G^\circ$ given by the relation

$$\mathbb{B}_\pi = \mathbb{B}_{\pi}^\psi_N = \int_N^\text{st} W(gn)\psi_N(n)^{-1} \, dn = \mathbb{B}_{\pi}(g)W(e)$$

for any $W \in \mathbb{W}^\psi_N(\pi)$ (see [LM13a]). In this section we prove the following result.

Theorem A.1. For any tempered $\pi \in \text{Irr}_{\text{gen}, \psi_N}(G)$ the function $\mathbb{B}_\pi$ is not identically zero on $G^\circ$.

The argument is similar to the one in [IZ].

Fix a tempered $\pi \in \text{Irr}_{\text{gen}, \psi_N}(G)$ and realize it on its Whittaker model $\mathbb{W}^\psi_N(\pi)$. Similarly, realize $\pi^\vee$ on $\mathbb{W}^\psi_N(\pi^\vee)$. Thus, we get a pairing $(\cdot, \cdot)$ on $\mathbb{W}^\psi_N(\pi) \times \mathbb{W}^\psi_N(\pi^\vee)$.

Fix $W^\vee_0 \in \mathbb{W}^\psi_N(\pi^\vee)$ such that

$$\int_N^\text{st} (\pi(n)W, W^\vee_0)\psi_N(n)^{-1} \, dn = W(e)$$

for all $W \in \mathbb{W}^\psi_N(\pi)$. This is possible by [LM14a, Propositions 2.3, 2.10] Then for any $g \in G$ we have

$$W(g) = \int_N^\text{st} (\pi(ng)W, W^\vee_0)\psi_N(n)^{-1} \, dn.$$

---

5 The notation in appendix is different from the body of the paper.
Similarly, fix $W_0 \in \mathbb{W}_N^\psi(\pi)$ such that

\begin{equation}
(A.3) \quad \int_N^{st} (W_0, \pi^\vee(n)W^\vee)\psi_N(n) \, dn = W^\vee(e)
\end{equation}

for all $W^\vee \in \mathbb{W}_N^{-1}(\pi^\vee)$.

Set $\Phi(g) = (\pi(g)W_0, W_0^\vee)$. Let $N^{der}$ be the derived group of $N$. Also let $\Xi$ be the Harish-Chandra function on $G$ (see e.g. [Wal03]).

**Lemma A.2.** The function

\[ g \mapsto \int_{N^{der}} \int_{N^{der}} \Phi(n_1gn_2) \, dn_1 \, dn_2 \]

on $G^o$ in locally $L^1$ on $G$. Moreover,

\[ g \mapsto \int_{N^{der}} \int_{N^{der}} \Xi(n_1gn_2) \, dn_1 \, dn_2 \]

is locally $L^1$ on $G$.

**Proof.** The argument is exactly as in [IZ, Lemma A.4] using the convergence of $\int_{N^{der}} \Xi(n) \, dn$ [SV12, Lemma 6.3.1].

**Remark A.3.** Note that $g \mapsto \int_{N^{der}} \int_{N^{der}} \Xi(n_1gn_2) \, dn_1 \, dn_2$ is locally constant on $G^o$. Thus, its local integrability on $G$ implies its convergence for any $g \in G^o$.

For any $f \in C^\infty_c(G)$ let $L_f(W) = \int_G f(g)W(g) \, dg$, $W \in \mathbb{W}_N^\psi(\pi)$. Then $L_f \in \pi^\vee$. Let $L_f^*$ be the corresponding element in $\mathbb{W}_N^{-1}(\pi^\vee)$ and set $B_\pi(f) = L_f^*(e)$. The distribution $f \mapsto B_\pi(f)$ is called the **Bessel distribution**. It is non-zero: we can choose $f \in C^\infty_c(G)$ such that $L_f$ is non-trivial, and then, by translating $f$ if necessary we can arrange that $L_f^*(e) \neq 0$.

Note that by (A.3),

\[ B_\pi(f) = \int_N^{st} (W_0, \pi^\vee(n)L_f^\vee)\psi_N(n) \, dn = \int_N^{st} (\int_G f(g)W_0(gn^{-1}) \, dg)\psi_N(n) \, dn \]

and therefore by (A.2) we have

\begin{equation}
(A.4) \quad B_\pi(f) = \int_N^{st} (\int_G f(g) (\int_N^{st} \Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1} \, dn_1) \, dg) \psi_N(n_2) \, dn_2.
\end{equation}

Let $A^1$ be the maximal compact subgroup of $A$. Fix $\Omega_0 \in \mathcal{CSGR}(N)$ which is invariant under conjugation by $A^1$ (e.g., take $\Omega_0 = N \cap K$). Fix an element $a \in A$ such that $|\alpha(a)| > 1$ for all $\alpha \in \Delta_0$. Consider the sequence $\Omega_n = a^n\Omega_0a^{-n} \in \mathcal{CSGR}(N)$, $n = 1, 2, \ldots$. Any $\Omega_n$ is invariant under conjugation by $A^1$, $\Omega_1 \subset \Omega_2 \subset \ldots$ and $\cup \Omega_n = N$.

Let $A^d = A \cap G^{der}$. Consider the family

\[ A_n = \{ t \in A^d : |\alpha(t) - 1| \leq q^{-n} \text{ for all } \alpha \in \Delta_0 \} \in \mathcal{CSGR}(A^d) \]
which forms a basis of neighborhoods of 1 for \( A^d \). We only consider \( n \) sufficiently large so that the image of \( A_n \) under the homomorphism \( t \in A \mapsto (\alpha(t))_{\alpha \in \Delta_0} \in (F^*)^{\Delta_0} \) is \( \prod_{\alpha \in \Delta_0} (1 + \varpi^n \mathcal{O}) \) where \( \varpi \) is a uniformizer of \( \mathcal{O} \).

For any \( \alpha \in \Delta_0 \) choose a parameterization \( x_\alpha : F \to N_\alpha \) such that \( \psi_N \circ x_\alpha \) is trivial on \( \mathcal{O} \) but not on \( \varpi^{-1} \mathcal{O} \). Let \( N_\alpha \) be the group generated by \( \langle x_\alpha(\varpi^{-n} \mathcal{O}), \alpha \in \Delta_0 \rangle \) and the derived group \( N^{\text{der}} \) of \( N \). The following lemma is clear:

**Lemma A.4.** For any \( u \in N \) we have

\[
(\text{vol } A_n)^{-1} \int_{A_n} \psi_N(tut^{-1}) \, dt = \begin{cases} 
\psi_N(u) & u \in N_n, \\
0 & \text{otherwise.} 
\end{cases}
\]

Next we prove:

**Lemma A.5.** Suppose that \( f \) and \( \Phi \) are bi-invariant under \( A_n \). Then we have

\[
B_{\pi}(f) = \int_G f(g)\alpha_n(g) \, dg
\]

where

\[
\alpha_n(g) = \int_{N_n} \int_{N_n} \Phi(n_1 gn_2^{-1})\psi_N(n_1)^{-1}\psi_N(n_2) \, dn_2 \, dn_1.
\]

**Remark A.6.** Of course we cannot conclude from the lemma by itself that \( B_{\pi} \) is given by a locally \( L^1 \) function. (This is conjectured to be the case.)

**Proof.** We start with (A.4). Let \( m \) be such that

\[
B_{\pi}(f) = \int_{\Omega_m} \int_G \left( \int_{N_n} f(g)\Phi(n_1 gn_2^{-1})\psi_N(n_1)^{-1} \, dn_1 \right) \, dg \, \psi_N(n_2) \, dn_2.
\]

Then for any \( t \in A_n \) we have

\[
B_{\pi}(f) = \int_{\Omega_m} \int_G \left( \int_{N_n} f(gt)\Phi(n_1 gn_2^{-1}t)\psi_N(n_1)^{-1} \, dn_1 \right) \, dg \, \psi_N(n_2) \, dn_2
\]

\[
= \int_{\Omega_m} \int_G \left( \int_{N_n} f(gt)\Phi(n_1 gtn_2^{-1})\psi_N(n_1)^{-1} \, dn_1 \right) \, dg \, \psi_N(tn_2 t^{-1}) \, dn_2
\]

\[
= \int_{\Omega_m} \int_G \left( \int_{N_n} f(g)\Phi(n_1 gn_2^{-1})\psi_N(n_1)^{-1} \, dn_1 \right) \, dg \, \psi_N(tn_2 t^{-1}) \, dn_2.
\]

Averaging over \( t \in A_n \) we get

\[
B_{\pi}(f) = \int_{N_m \cap \Omega_m} \int_G \left( \int_{N_n} f(g)\Phi(n_1 gn_2^{-1})\psi_N(n_1)^{-1} \, dn_1 \right) \, dg \, \psi_N(n_2) \, dn_2
\]

by Lemma A.4. Thus, for \( m' \) sufficiently large (depending on \( f \) and \( m \)) we have

\[
B_{\pi}(f) = \int_{N_m \cap \Omega_m} \int_{G \cap \Omega_{m'}} f(g)\Phi(n_1 gn_2^{-1})\psi_N(n_1)^{-1} \, dn_1 \, dg \, \psi_N(n_2) \, dn_2.
\]
Once again, for any $t \in A_n$

$$B_{\pi}(f) = \int_{N_n \cap \Omega_m} \int_G \int_{\Omega_{m'}} f(tg) \Phi(tn_1 gn_2^{-1}) \psi_N(n_1)^{-1} \, dn_1 \, dg \, \psi_N(n_2) \, dn_2$$

$$= \int_{N_n \cap \Omega_m} \int_G \int_{\Omega_{m'}} f(tg) \Phi(n_1 tgn_2^{-1}) \psi_N(t^{-1}n_1 t)^{-1} \, dn_1 \, dg \, \psi_N(n_2) \, dn_2$$

$$= \int_{N_n \cap \Omega_m} \int_G \int_{\Omega_{m'}} f(g) \Phi(n_1 gn_2^{-1}) \psi_N(t^{-1}n_1 t)^{-1} \, dn_1 \, dg \, \psi_N(n_2) \, dn_2.$$  

As before, averaging over $A_n$ we get

$$B_{\pi}(f) = \int_{N_n \cap \Omega_m} \int_G \int_{N_n \cap \Omega_{m'}} f(g) \Phi(n_1 gn_2^{-1}) \psi_N(n_1)^{-1} \, dn_1 \, dg \, \psi_N(n_2) \, dn_2.$$  

Since $m$ and $m'$ can be chosen arbitrarily large, the lemma follows from the convergence of

$$\int_{N_n} \int_G \int_{N_n} |f(g)\Phi(n_1 gn_2^{-1})| \, dn_1 \, dg \, dn_2,$$

i.e., Lemma A.2. \qed

Analogously, we prove:

**Lemma A.7.** For $g \in G^o$, let $\mathbb{E}_{\pi}^A_{\g}(g) := \text{vol}(A_n)^{-2} \int_{A_n} \int_{A_n} \mathbb{P}_{\pi}(t_1 gt_2) \, dt_1 \, dt_2$. Suppose that $\Phi$ is bi-invariant under $A_n$. Then

$$\mathbb{E}_{\pi}^A_{\g}(g) W_0(e) = \alpha_n(g)$$

for all $g \in G^o$.

**Proof.** First note that for any compact subset $C$ of $G^o$ we have

$$\int_{\Omega_m} \left( \int_C \Phi(n_1 gn_2^{-1}) \psi_N(n_1)^{-1} \, dn_1 \right) \psi_N(n_2) \, dn_2 = \mathbb{P}_{\pi}(g) W_0(e)$$

for all $g \in C$ and $m$ sufficiently large. Indeed, by (A.2) the inner stable integral is $W_0(gn_2^{-1})$. Thus, the relation above follows from [LM13a] and (A.1).

Now, for any $t \in A_n$ we have

$$\mathbb{P}_{\pi}(gt) W_0(e) = \int_{\Omega_m} \left( \int_{N} \Phi(n_1 gtn_2^{-1}) \psi_N(n_1)^{-1} \, dn_1 \right) \psi_N(n_2) \, dn_2$$

$$= \int_{\Omega_m} \left( \int_{N} \Phi(n_1 gn_2^{-1}t) \psi_N(n_1)^{-1} \, dn_1 \right) \psi_N(t^{-1}n_2 t) \, dn_2$$

$$= \int_{\Omega_m} \left( \int_{N} \Phi(n_1 gn_2^{-1}) \psi_N(n_1)^{-1} \, dn_1 \right) \psi_N(t^{-1}n_2 t) \, dn_2.$$  

Thus, by Lemma A.4, we have

$$\text{vol}(A_n)^{-1} \int_{A_n} \mathbb{P}_{\pi}(gt) \, dt \ W_0(e) = \int_{N_n \cap \Omega_m} \left( \int_{N} \Phi(n_1 gn_2^{-1}) \psi_N(n_1)^{-1} \, dn_1 \right) \psi_N(n_2) \, dn_2.$$
For $g$ in a compact set $C$ of $G^0$ and for $m'$ sufficiently large (depending on $C$ and $m$) we can write this as

$$\int_{N_n \cap \Omega_m} \int_{\Omega_{m'}} \Phi(n_1 g n_2^{-1}) \psi_N(n_1)^{-1} \ dn_1 \ \psi_N(n_2) \ dn_2.$$  

Now, for any $t_1 \in A_n$ we have

$$\text{vol}(A_n)^{-1} \int_{A_n} \mathbb{B}_\pi(t_1 g t_2) \ dt_2 \ W_0(e)$$

$$= \int_{N_n \cap \Omega_m} \int_{\Omega_{m'}} \Phi(n_1 t_1 g n_2^{-1}) \psi_N(n_1)^{-1} \ dn_1 \ \psi_N(n_2) \ dn_2$$

$$= \int_{N_n \cap \Omega_m} \int_{\Omega_{m'}} \Phi(t_1 n_1 g n_2^{-1}) \psi_N(t_1 n_1 t_1)^{-1} \ dn_1 \ \psi_N(n_2) \ dn_2$$

$$= \int_{N_n \cap \Omega_m} \int_{\Omega_{m'}} \Phi(n_1 g n_2^{-1}) \psi_N(t_1 n_1 t_1)^{-1} \ dn_1 \ \psi_N(n_2) \ dn_2.$$  

Averaging over $t_1 \in A_n$ we get

$$\mathbb{B}_{\pi}^{A_n}(g) W_0(e) = \int_{N_n \cap \Omega_m} \int_{N_n \cap \Omega_{m'}} \Phi(n_1 g n_2^{-1}) \psi_N(n_1)^{-1} \psi_N(n_2) \ dn_1 \ dn_2.$$  

The lemma follows from the convergence of

$$\int_{N_n} \int_{N_n} |\Phi(n_1 g n_2^{-1})| \ dn_1 \ dn_2,$$

i.e., Remark A.3. 

Proof of Theorem A.1. Since $B_\pi(f) \neq 0$ we have $\alpha_n|_{G^0} \neq 0$ for $n \gg 1$ by Lemma A.5. Hence, by Lemma A.7, $\mathbb{B}_\pi^{A_n}|_{G^0} \neq 0$ and therefore $\mathbb{B}_\pi|_{G^0} \neq 0$. 

Remark A.8. Theorem A.1 and its proof are also valid for $\tilde{\text{Sp}}_n$. 

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