Lorentz Transformation in Flat 5\(D\)
Complex-Hyperbolic Space

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Abstract

The Lorentz transformation is derived in 5D flat pseudo-complex affine space or \(TT\) Space. The \(TT\) space or pseudo-Complex space accommodates one uncompactified time-like extra dimension. The effects of one extra time-like dimension are shown to affect the structure of the Lorentz transformation and the maximum allowable speed. The maximum allowable speed for particles living in \(TT\) space is shown to exceed the speed of light, \(c\), where \(c\) is the absolute speed in Minkowski space.

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1 Introduction

Non-Euclidean geometries are geometries with the exclusion of Euclid’s fifth postulate, the parallel postulate\(^1\). Minkowski and Einstein saw the potential application of one of these non-Euclidean geometries, the hyperbolic geometry. Subsequently, Einstein formulated his general theory of relativity on the hyperbolic space. The hyperbolic space with the flat metric \(g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) = \eta_{\mu\nu}\), is the well known Minkowski space. The standard approach in dealing with higher dimensional theories is to consider almost exclusively space-like extra dimensions. Large extra-dimensions have been used to address the hierarchy problem, whereas Higgs mass is proven to be finite\(^2\). The effects of the extra space-like dimensions have also been examined in the context of 4D superspace formalism\(^3\). However, there is no priory reason why extra time-like dimensions cannot exist. Time-like extra dimensions have been ignored due to serious conflicts with causality and unitarity\(^4\)[\(^5\)[\(^6\)]. However, these technicalities could be avoided if calculations are performed in 5D flat pseudo-complex affine space with metric function \(g_{AB} = [TT]_{AB} = \text{diag}(-1, -1, 1, 1, 1)\) or in short, \(TT\) space. The acronym, \(TT\), is associated with the two minuses in the metric function.

The letter is divided as follows. In section 2, the specific forms of elements, \(p^A \in M \subset TT\) space and \(\dot{p}_A \in W \subset T_{\dot{p}}(M)\), where \(M\) and \(W\) are some open subsets of \(TT\) space and its associated tangent space \(T_{P_A}(M)\), are derived via \(TT\) metric function. In section 3, the \(TT\) Lorentz transformation is obtained. In section 4, as an application in \(TT\) space, using the \(TT\) Lorentz transformation, a boost of \(v\) along the spatial direction between the two frames is perform. The maximum allowable speed for particles living in \(TT\) space is shown to be \(v_{\text{max}} < \sqrt{2}c\), which exceeds the absolute speed, \(c\), the speed of light in Minkowski space.
2 TT Space and The Associated Tangent Space

In this section, elements of 5D flat pseudo-complex affine space or TT space with metric $g_{AB} = [TT]_{AB} = \text{diag}(-1, -1, 1, 1, 1)$ and Lorentzian signature $(r - s) = n - 2s = 1$, are derived. Consider an element in our TT space, $p^A \in M \subset TT$, where $M$ is an open set contained in TT space and $p^A$ is a mapping

\[ p^A(\psi) : \mathbb{R} \to TT \subset \mathbb{C}^5 \]

\[ p^A(\psi) = p_0 + \begin{pmatrix} x^1(\psi) \\ x^2(\psi) \\ x^3(\psi) \\ x^4(\psi) \\ x^5(\psi) \end{pmatrix} \]

\[ p^A(\psi) = p_0 + x^A(\psi), \quad (1) \]

where $\mathbb{R}$ is the real, $\mathbb{C}^5$ is 5D complex space, $x^A(\psi)$ are coordinates of the point $p^A(\psi)$, and $p_0$ is defined to be the origin by the initial condition, accompanied by a parametrization parameter $\psi$. The range of the mapping function $p^A(\psi)$ can have non-real values because of the complex nature of the Lorentz transformations associated with the TT space. The 5-vector can
be computed by taking the total differential of $p^A(\psi)$

$$dp^A(\psi) = \sum_{B=1}^{5} \frac{\partial p^A}{\partial x^B} dx^B$$

$$dp^A(\psi) = \widehat{x}_1 dx^1(\psi) + \widehat{x}_2 dx^2(\psi) + \widehat{x}_3 dx^3(\psi) + \widehat{x}_4 dx^4(\psi) + \widehat{x}_5 dx^5(\psi)$$

$$dp^A(\psi) = \begin{pmatrix}
  dx^1(\psi) \\
  dx^2(\psi) \\
  dx^3(\psi) \\
  dx^4(\psi) \\
  dx^5(\psi)
\end{pmatrix}$$

(2)

Without loss of generality, the spatial component of $p^A(\psi)$ can be suppressed, i.e.

$$p^A(\psi) = p_0 + \begin{pmatrix}
  x^1(\psi) \\
  x^2(\psi) \\
  r^i(\psi)
\end{pmatrix},$$

(3)

and

$$r^i(\psi) = \begin{pmatrix}
  x^3(\psi) \\
  x^4(\psi) \\
  x^5(\psi)
\end{pmatrix},$$

(4)

where $i = 3, 4, 5$. The $TT$ space is a subset of $\mathbb{C}^5$, and is defined as

$$TT = \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 : -x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2 \}$$

(5)

$$TT = \{ (x_1, x_2, r_i) \in \mathbb{C}^5 : -x_1^2 - x_2^2 + r_i^2 = R^2 \},$$

(6)

where $R$ is the positive constant curvature of $TT$ affine space. The mapping

$$p^A(\psi) = p_0 + \begin{pmatrix}
  x^1(\psi) \\
  x^2(\psi) \\
  r^i(\psi)
\end{pmatrix},$$

(7)
with the initial condition,

\[ p^A(0) = p_0 + (x^1(0), x^2(0), r^i(0)) = (0, 0, R), \]  

where \( r^i(0) = R \), would yield \( p_0 = (0, 0, 0) \).

An element of tangent space to \( TT \) at \( p^A(\psi) \) can be obtained by taking the derivative of \( p^A(\psi) \), w.r.t \( \psi \),

\[ \dot{p}_A(\psi) = \frac{dp_A}{d\psi} = (\dot{x}_1, \dot{x}_2, \dot{r}_i), \]  

where \( \dot{r}_i = \frac{dr_i}{d\psi} \) and \( \dot{x}_j = \frac{dx_j}{d\psi} \). Taking the \( TT \) inner product is defined as \((\otimes)\), and the operation satisfies the \( TT \) metric function

\[ p^2 = p^A \otimes p_A = g_{AB} p^A p^B \]  

\[ p^2 = -x_1^2 - x_2^2 + r_i^2 = R^2. \]  

Isometries of a hyperboloid model associated with ordinary hyperbolic space\([7]\),

\[ L = \{(x_1, \ldots, x_n, x_{n+1}) : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 \text{ and } x_{n+1} > 0\}, \]  

where \( x_{n+1} \) is the time-like dimension. Transformations or linear maps of elements from the hyperboloid model which preserve the associated metric or the hyperbolic inner product

\[ dx_L^2 = dx_1^2 + \cdots + dx_n^2 + dx_{n+1}^2, \]  

would induce linear, Riemannian and topological isometries on that space. Analogously, Lorentzian transformations on elements \( p^A \in M \subset TT \) space, which preserve the \( TT \) inner product \((10)\), induces a pseudo-Riemannian isometry \([L]\) on \( TT \). The pseudo-Riemannian isometry \([L]^A_B\) is a diffeomorphism of \( TT \) that satisfies

\[ [L]_*(ds^2)(u, v) = ds^2[D[L](u), D[L](v)], \]  

5
where \([L]_\ast\) is the pullback, \(u\) and \(v\) are elements from the tangent space \(T_{p^A}(M)\) at point \(p^A\) and \(D[L]\) is the derivative maps on tangent vectors at \(p^A\) to tangent vectors at \([L](p^A)\).

Differentiating equation (10) w.r.t. parametrization parameter \(\psi\), we have

\[- x_1 \dot{x}_1 - x_2 \dot{x}_2 + r_i \dot{r}_i = 0.\]  

(15)

Equation (15) is equivalent to

\[p^A \ast \dot{p}_A = (x^1, x^2, r^i) \ast \left(\dot{x}_1, \dot{x}_2, \dot{r}_i\right)\]  

(16)

\[p^A \ast \dot{\dot{p}}_A = -x_1 \dot{x}_1 - x_2 \dot{x}_2 + r_i \dot{r}_i = 0.\]  

(17)

The vectors \(p^A\) and \(\dot{p}_A\) are said to be TT perpendiculars w.r.t. each other. By inspection, the specific forms of \(p^A\) and \(\dot{p}_A\) can be obtained from the following differential equations

\[\dot{r}_i = x_1 + x_2,\]  

(18)

\[\dot{x}_1 = r_i,\]  

(19)

\[\dot{x}_2 = r_i.\]  

(20)

Taking the derivative of equation (18) yields

\[\ddot{r}_i = \dot{x}_1 + \dot{x}_2 = 2r_i\]  

\[\ddot{r}_i = 2r_i.\]  

(21)

The solution to equation (21) takes a general form of

\[r_i(\psi) = Ae^{\sqrt{2}\psi} + Be^{-\sqrt{2}\psi},\]  

(22)

where \(A\) and \(B\) are arbitrary constants and to fixed by the initial condition (8). To determine the constants, take the derivative of equation (22),
substituting in for equation (18)

\[ \dot{r}_i(\psi) = \sqrt{2}Ae^{\sqrt{2}\psi} - \sqrt{2}Be^{-\sqrt{2}\psi} = x_1(\psi) + x_2(\psi). \quad (23) \]

Imposing the temporal initial condition (8) to equation (23), yields

\[ \dot{r}_i(0) = \sqrt{2}Ae^{\sqrt{2}0} - \sqrt{2}Be^{-\sqrt{2}0} = x_1(0) + x_2(0) = 0 \]
\[ \Rightarrow A - B = 0 \Rightarrow A = B. \quad (24) \]

The spatial solution (22) becomes

\[ r_i(\psi) = A(e^{\sqrt{2}\psi} + e^{-\sqrt{2}\psi}). \quad (25) \]

Imposing the spatial initial condition (8) on equation (25)

\[ r_i(0) = A(e^{\sqrt{2}0} + e^{-\sqrt{2}0}) = R \]
\[ \Rightarrow 2A = R \Rightarrow A = \frac{R}{2}. \quad (26) \]

The particular spatial solution is

\[ r_i(\psi) = R \left( \frac{e^{\sqrt{2}\psi} + e^{-\sqrt{2}\psi}}{2} \right) \]
\[ r_i(\psi) = R \cosh \left( \sqrt{2}\psi \right). \quad (27) \]

Using the spatial solution (27), the two temporal solutions for equations (19) and (20) can now be obtained

\[ \frac{dx_1(\psi)}{d\psi} = R \cosh \left( \sqrt{2}\psi \right) \Rightarrow x_1(\psi) = \frac{R}{\sqrt{2}} \sinh \left( \sqrt{2}\psi \right) + C_1, \quad (28) \]
\[ \frac{dx_2(\psi)}{d\psi} = R \cosh \left( \sqrt{2}\psi \right) \Rightarrow x_2(\psi) = \frac{R}{\sqrt{2}} \sinh \left( \sqrt{2}\psi \right) + C_2. \quad (29) \]
Imposing the temporal initial condition (8) on solutions (28) and (29), yields vanishing constants $C_1$ and $C_2$,

\[
x_1(\psi) = \frac{R}{\sqrt{2}} \sinh \left( \sqrt{2} \psi \right), \quad (30)
\]
\[
x_2(\psi) = \frac{R}{\sqrt{2}} \sinh \left( \sqrt{2} \psi \right). \quad (31)
\]

Parameterized by $\psi$, the temporal and spatial components of $p^A \in M \subset TT$ can be specified as

\[
p^\hat{A}(\psi) = \frac{R}{\sqrt{2}} \left( \sinh \left( \sqrt{2} \psi \right), \sinh \left( \sqrt{2} \psi \right), \sqrt{2} \cosh \left( \sqrt{2} \psi \right) \right), \quad (32)
\]

where $\hat{A} = 1, 2, 3$. The $TT$ perpendicular to $p^\hat{A}$ is

\[
p_A^\hat{A}(\psi) = R \left( \cosh \left( \sqrt{2} \psi \right), \cosh \left( \sqrt{2} \psi \right), \sqrt{2} \sinh \left( \sqrt{2} \psi \right) \right). \quad (33)
\]

The following are additional relations between $TT$ space and its associated tangent space $T_{p^\hat{A}}(M)$,

\[
p^2 = R^2 \quad (34)
\]
\[
p^2 = -2R^2 = -p \odot \ddot{p}, \quad (35)
\]

where

\[
\ddot{p}_A(\psi) = R\sqrt{2} \left( \sinh \left( \sqrt{2} \psi \right), \sinh \left( \sqrt{2} \psi \right), \sqrt{2} \cosh \left( \sqrt{2} \psi \right) \right) \quad (36)
\]

the second derivative of $p^\hat{A}(\psi)$.
3 Lorentz Transformation in $TT$ Space

In this section, the Lorentz transformation in $TT$ space is derived. Recall from section 2, that an element of $TT$ space is parametrized by a single parameter $\psi$

$$p^\hat{A}(\psi) = \frac{R}{\sqrt{2}} \left( \sinh(\sqrt{2}\psi), \sinh(\sqrt{2}\psi), \sqrt{2} \cosh(\sqrt{2}\psi) \right),$$

$$p^A(\psi) = (ct_1(\psi), ct_2(\psi), r_i(\psi)),$$  \hspace{1cm} (37) \hspace{1cm} (38)

where $t_1$, $t_2$, $r_i$ are the two time-like and three space-like dimensions and $\psi \in \mathbb{R}$. The $p^\hat{A}(\psi)$ and $p^A(\psi)$ are 3-D and 5-D vectors, respectively. The total differential of $p^A$ is

$$dp^A(\psi) = (cdt_1(\psi), cdt_2(\psi), dr_i(\psi)).$$

The Lorentz transformation in $TT$ space can be written in matrix equation as

$$dp'^A = [L]_B^A dp^B,$$  \hspace{1cm} (39)

where $dp^B$ is a 5-vector and $[L]_B^A$ is the $5 \times 5$ Lorentz transformation matrix. Elements of $[L]_B^A$ can be fixed by length-invariance condition

$$[L] [TT] [L]^T = [TT],$$  \hspace{1cm} (40)

where

$$[TT] = \begin{pmatrix}
    -1 & 0 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},$$
and $[L]^T$ is the complex conjugate-transposed matrix. To obtain the rotation matrix on the $r_4 - r_5$ plane, we rotate around the $r_3$ by an angle $\theta$. The rotation transformation

$$\begin{pmatrix} dx'_4 \\ dx'_5 \end{pmatrix} = R(\theta)_{r_4-r_5} \begin{pmatrix} dx_4 \\ dx_5 \end{pmatrix},$$

(41)

with

$$R(\theta)_{r_4-r_5} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

(42)

preserves the length invariance of the 5-vector $dp^A$. Likewise, rotation transformation on the $ct_1 - ct_2$ plane which preserves length invariance of the 5-vector $dp^A$ can be performed by

$$\begin{pmatrix} cdt'_1 \\ cdt'_2 \end{pmatrix} = R(\phi)_{ct_1-ct_2} \begin{pmatrix} cdt_1 \\ cdt_2 \end{pmatrix},$$

(43)

where

$$R(\phi)_{ct_1-ct_2} = \begin{pmatrix} -\cos \phi & +\sin \phi \\ -\sin \phi & -\cos \phi \end{pmatrix}.$$  

(44)

The boost transformation between two inertial frames by $\mathbf{\hat{v}} = v\mathbf{\hat{r}}_3$, where $\mathbf{\hat{r}}_3$ is a unit vector pointing in the $r_3$-direction. With this boost, $(r_4, r_5)$ coordinates are not affected, hence we could rewrite the 5-vector $dp^A$ as 3-vector $dp^{\hat{A}}$

$$dp^{\hat{A}} = (cdt_1 (\psi), cdt_2 (\psi), dr_3 (\psi)),$$

(45)

where $\hat{A} = 1, 2, 3$. The transformed 3-vector can be obtained by

$$dp^{\hat{A}} = [L]_{\hat{B}}^{\hat{A}} dp^{\hat{B}}$$

(46)

$$\begin{pmatrix} ct'_1 \\ ct'_2 \\ r'_3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} ct_1 \\ ct_2 \\ r_3 \end{pmatrix},$$
where $\hat{B} = 1, 2, 3$. The length invariance condition requires

\[
\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{pmatrix}
\begin{pmatrix}
  -1 & 0 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
  a & d & g \\
  b & e & h \\
  c & f & i \\
\end{pmatrix}
= 
\begin{pmatrix}
  -1 & 0 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 1 \\
\end{pmatrix}
\]

The length invariance condition yields the following equations

\[
- (a^2 + b^2) + c^2 = -1, \quad (48) \\
-(d^2 + e^2) + f^2 = -1, \quad (49) \\
-(g^2 + h^2) + i^2 = 1, \quad (50)
\]

and

\[
- ad - be + cf = -ag - bh + ci = -dg - eh + fi = 0, \quad (51)
\]

which can then be used to fix the boost transformation matrix $[L]_{\hat{B}}$. Using equations (48) to (50), the matrix becomes

\[
[L]_{\hat{B}} = \begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{\sqrt{2}} \cosh (\sqrt{2}\psi_1) & \frac{1}{\sqrt{2}} \cosh (\sqrt{2}\psi_1) & \sinh (\sqrt{2}\psi_1) \\
  \frac{1}{\sqrt{2}} \cosh (\sqrt{2}\psi_2) & \frac{1}{\sqrt{2}} \cosh (\sqrt{2}\psi_2) & \sinh (\sqrt{2}\psi_2) \\
  \frac{1}{\sqrt{2}} \sinh (\sqrt{2}\psi_3) & \frac{1}{\sqrt{2}} \sinh (\sqrt{2}\psi_3) & \cosh (\sqrt{2}\psi_3) \\
\end{pmatrix}, \quad (52)
\]
where $\psi_j \in \mathbb{R}$ are parametrization parameters, $j = 1, 2, 3$. The relationships among the parametrization parameters can be obtained through (51),

$$-ad - be + cf = 0$$

$$- \cosh(\sqrt{2}\psi_1) \cosh(\sqrt{2}\psi_2) + \sinh(\sqrt{2}\psi_1) \cosh(\sqrt{2}\psi_2) = 0$$

$$\cosh[\sqrt{2}(\psi_1 - \psi_2)] = 0. \quad (53)$$

Equation (53) can be satisfied if $\sqrt{2}(\psi_1 - \psi_2) = i\pi \left(n + \frac{1}{2}\right)$, thus we must have

$$\psi_2 = \psi_1 - i\frac{\pi}{\sqrt{2}} \left(n + \frac{1}{2}\right). \quad (54)$$

The relation between $\psi_1$ and $\psi_3$ is obtained by (51)

$$-ag - bh + ci = 0$$

$$- \cosh(\sqrt{2}\psi_1) \sinh(\sqrt{2}\psi_3) + \sinh(\sqrt{2}\psi_1) \cosh(\sqrt{2}\psi_3) = 0$$

$$\sinh[\sqrt{2}(\psi_1 - \psi_3)] = 0, \quad (55)$$

implies $\psi_3 = \psi_1$. The boost transformation matrix becomes

$$[L]_{\hat{A}B} = \begin{pmatrix}
\frac{1}{\sqrt{2}} \cosh(\sqrt{2}\psi_1) & \frac{1}{\sqrt{2}} \cosh(\sqrt{2}\psi_1) & \sinh(\sqrt{2}\psi_1) \\
\frac{1}{\sqrt{2}} \sinh(\sqrt{2}\psi_1) & \frac{1}{\sqrt{2}} \sinh(\sqrt{2}\psi_1) & \cosh(\sqrt{2}\psi_1) \\
\frac{1}{\sqrt{2}} \cosh(\sqrt{2}\psi_2) & \frac{1}{\sqrt{2}} \cosh(\sqrt{2}\psi_2) & \sinh(\sqrt{2}\psi_2)
\end{pmatrix} \quad (56)$$

where $z = \left(\psi_1 - i\frac{\pi}{\sqrt{2}} \left(n + \frac{1}{2}\right)\right)$. After little algebra, the Lorentz transformation matrix $[L]_{\hat{A}B}$ simplifies to

$$[L]_{\hat{A}B} = \begin{pmatrix}
\frac{1}{\sqrt{2}}C_1 & \frac{1}{\sqrt{2}}C_1 & S_1 \\
\frac{i(-1)^{n+1}}{\sqrt{2}}S_1 & \frac{i(-1)^{n+1}}{\sqrt{2}}S_1 & i(-1)^{n+1}C_1 \\
\frac{1}{\sqrt{2}}S_1 & \frac{1}{\sqrt{2}}S_1 & C_1
\end{pmatrix} \quad (57)$$
where
\[ C_1 = \cosh \left( \sqrt{2}\psi_1 \right), \]  
(58)
and
\[ S_1 = \sinh \left( \sqrt{2}\psi_1 \right). \]  
(59)

The Lorentz transformation equation for a 3-vector in \( TT \) space becomes

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} C_1 \\
\frac{1}{\sqrt{2}} S_1 \\
r_3
\end{pmatrix}
\begin{pmatrix}
\frac{1}{i} (\psi_1)^{n+1} + S_1 \\
S_1 \\
C_1
\end{pmatrix}
\begin{pmatrix}
ct \\
r_3
\end{pmatrix} =
\begin{pmatrix}
ct_1 \\
r_3
\end{pmatrix}, \quad (60)
\]

where \( i = \sqrt{-1} \) is an imaginary number. Let \( ct' = c(t_1' + t_2') \) and \( ct = c(t_1 + t_2) \), equation (60) reduces to

\[
\begin{pmatrix}
ct' \\
r_3'
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\sqrt{2}} C_1 + i (\psi_1)^{n+1} S_1 \\
S_1 + i (\psi_1)^{n+1} C_1
\end{pmatrix}
\begin{pmatrix}
ct \\
r_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
ct' \\
r_3'
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\sqrt{2}} z_1 \\
\frac{1}{\sqrt{2}} S_1 C_1
\end{pmatrix}
\begin{pmatrix}
ct \\
r_3
\end{pmatrix}, \quad (61)
\]

where
\[ z_1 = C_1 + i (\psi_1)^{n+1} S_1, \quad (62)\]

and
\[ z_2 = S_1 + i (\psi_1)^{n+1} C_1. \]
Recall equation (61),

\[
\begin{pmatrix}
  ct' \\
r'_3
\end{pmatrix}
= \begin{pmatrix}
  \frac{1}{\sqrt{2}} [ C_1 + i (-1)^{n+1} S_1 ] & S_1 + i (-1)^{n+1} C_1 \\
  \frac{1}{\sqrt{2}} S_1 & C_1
\end{pmatrix}
\begin{pmatrix}
  ct \\
r_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
  ct' \\
r'_3
\end{pmatrix}
= \begin{pmatrix}
  \frac{1}{\sqrt{2}} [ 1 + i (-1)^{n+1} T_1 ] & T_1 + i (-1)^{n+1} \\
  \frac{1}{\sqrt{2}} T_1 & 1
\end{pmatrix}
\begin{pmatrix}
  ct \\
r_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
  ct' \\
r'_3
\end{pmatrix}
= \gamma_5 \begin{pmatrix}
  \frac{1}{\sqrt{2}} \left[ 1 - i \frac{(-1)^{n+1}}{\sqrt{2}} \beta \right] & -\frac{1}{\sqrt{2}} \beta + i \left( 1 - (-1)^{n+1} \right) \\
  -\frac{1}{\sqrt{2}} \beta & 1
\end{pmatrix}
\begin{pmatrix}
  ct \\
r_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
  ct' \\
r'_3
\end{pmatrix}
= \gamma_5 \begin{pmatrix}
  \frac{1}{\sqrt{2}} \left[ 1 - i \frac{(-1)^{n+1}}{\sqrt{2}} \beta_5 \right] & -\beta_5 + i \left( 1 - (-1)^{n+1} \right) \\
  -\frac{1}{\sqrt{2}} \beta_5 & 1
\end{pmatrix}
\begin{pmatrix}
  ct \\
r_3
\end{pmatrix},
\]

where \( T_1 = \tanh (\sqrt{2} \psi_1) = -\frac{1}{\sqrt{2}} \beta = -\beta_5 \) and \( C_1 = \cosh (\sqrt{2} \psi_1) = \gamma_5 \). To simplify the Lorentz transformation matrix, we make the following substitutions

\[
z_{1,n} = \frac{\gamma_5}{\sqrt{2}} \left[ 1 - i (-1)^{n+1} \beta_5 \right], \quad (63)
\]

\[
z_{2,m} = \gamma_5 \left[ 1 + i (-1)^{-(m+1)} \beta_5 \right]. \quad (64)
\]

The operators \( z_{1,n} \) and \( z_{2,m} \) can be decomposed into even and odd parts

\[
z_{1,n} = z_{1,n=2k} + z_{1,n=2k+1} \quad (65)
\]

\[
z_{1,n} = z_{1,\text{even}} + z_{1,\text{odd}},
\]

and

\[
z_{2,m} = z_{2,m=2l} + z_{2,m=2l+1} \quad (66)
\]

\[
z_{2,m} = z_{2,\text{even}} + z_{2,\text{odd}},
\]
where \( k, l \in \mathbb{R} \). By inspection, we have the followings

\[
\begin{align*}
    z_{1,n=2k} &= \frac{\gamma_5}{\sqrt{2}} [1 + i\beta_5], \\
    z_{1,n=2k+1} &= \frac{\gamma_5}{\sqrt{2}} [1 - i\beta_5], \\
    z_{2,m=2l} &= \gamma_5 [1 - i\beta_5], \\
    z_{2,m=2l+1} &= \gamma_5 [1 + i\beta_5],
\end{align*}
\]

(67) \quad (68) \quad (69) \quad (70)

thus we have

\[
\begin{align*}
    z_{2,m=2l} &= \sqrt{2} z_{1,n=2k+1}, \\
    z_{2,m=2l+1} &= \sqrt{2} z_{1,n=2k}.
\end{align*}
\]

(71) \quad (72)

Equations (65) and (66) become

\[
\begin{align*}
    z_{1,n} &= z_{1,\text{even}} + z_{1,\text{odd}} = z_{1,\text{even}} + \frac{1}{\sqrt{2}} z_{2,\text{even}}, \\
    z_{2,m} &= z_{2,\text{even}} + z_{2,\text{odd}} = z_{2,\text{even}} + \sqrt{2} z_{1,\text{even}}.
\end{align*}
\]

(73) \quad (74)

Multiplying equation (73) by \( \sqrt{2} \) and then divide \( \sqrt{2} \) (73) by equation (74), yields the following relation

\[
    z_{2,m} = \sqrt{2} z_{1,n}.
\]

(75)

The Lorentz transformation matrix then simplifies to

\[
[L]_B^A = \begin{pmatrix}
    z_{1,n} & \sqrt{2} z_{1,n} \\
    -\frac{\gamma_5 \beta_5}{\sqrt{2}} & \gamma_5
\end{pmatrix}.
\]
The coordinate transformation matrix equation for a 3-vector $dp^A$ becomes

$$
(dp')^A = [L]^A_B dp^B
$$

$$
egin{pmatrix}
  cdt' \\
  dr'_3
\end{pmatrix} =
\begin{pmatrix}
  z_{1,n} & \sqrt{2}z_{1,n} \\
  -\frac{\gamma_5 \beta_5}{\sqrt{2}} & \gamma_5
\end{pmatrix}
\begin{pmatrix}
  cdt \\
  dr_3
\end{pmatrix}
$$

$$
egin{pmatrix}
  cdt' \\
  dr'_3
\end{pmatrix} =
\begin{pmatrix}
  \frac{\gamma_5}{\sqrt{2}} [1 - i (-1)^{n+1} \beta_5] & \gamma_5 [1 - i (-1)^{n+1} \beta_5] \\
  -\frac{\gamma_5 \beta_5}{\sqrt{2}} & \gamma_5
\end{pmatrix}
\begin{pmatrix}
  ct \\
  r_3
\end{pmatrix}
$$

with the aid of equations (63) and (75). The inverse $TT$ transformation to $[L]^A_B$ is obtained via the length invariant condition (40) and the following substitution

$$
z_{1,n} = \frac{\gamma_5}{\sqrt{2}} [1 - i (-1)^{n+1} \beta_5] = \frac{\gamma_5}{\sqrt{2}} \xi_n(\beta_5).
$$

(77)

With equation (77), the $TT$ transformation matrix (76) reduces to

$$
[L]^A_B = z_{1,n}
\begin{pmatrix}
  1 & \sqrt{2} \\
  -\beta_5 \xi_n(\beta_5) & \beta_5 \xi_n(\beta_5)
\end{pmatrix},
$$

where $\xi_n(\beta_5) = 1 - i (-1)^{n+1} \beta_5$ is a complex-valued function. Without loss of generality, the indices of the $TT$ transformation matrix are suppressed for clarity. Recall equation (40), we have

$$
[L] I_L [T]^T = I_L,
$$

(78)

where the $TT$ metric matrix

$$
I_L = \begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix},
$$

(79)
and \([L]^T\) is the complex conjugate-transposed of \([L]\). Multiplying both sides of equation (78) by the \([TT]\) metric or its 2D analogue \(I_L\), we then have

\[
[L] I_L [\overline{L}]^T I_L = I_L I_L = I,
\]

(80)

where

\[
I_L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

and

\[
I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let's define \([L] I_L = \hat{L}\), equation (80) yields

\[
\hat{L} \hat{L}^{-1} = I.
\]

(81)

The inverse \(\hat{L}^{-1}\) of \(\hat{L}\) can be obtained by standard methods and its closed analytical form yields

\[
\hat{L}^{-1} = \frac{1}{z_{1,n} (1 + \beta_5)} \begin{pmatrix} -1 & \xi_n(\beta_5) \\ \frac{1}{\xi_{n(\beta_5)}\beta_5} & \frac{1}{\sqrt{2\xi_n(\beta_5)}} \end{pmatrix},
\]

where \(\beta_5 = \frac{1}{\sqrt{2}}\). Using equations (80) and (81), the inverse matrix \(\hat{B} [L]^{-1}\), of the \(TT\) transformation matrix \([L]\hat{A}\), can finally be obtained

\[
\hat{L} \hat{L}^{-1} = I
\]

(82)

\[
[L] I_L \hat{L}^{-1} = I
\]

\[
[L] I_L \hat{L}^{-1} I_L = II_L = I_L.
\]

(83)
Therefore, the inverse $TT$ Lorentz transformation matrix yields

$$
\hat{A}^{-1} = I, \hat{B}^{-1} = \frac{1}{z_{1,n}(1 + \beta_5)} \left( -\frac{1}{\sqrt{2}} \beta_5, \frac{1}{\sqrt{2}} \xi_n(\beta_5) \right). \tag{84}
$$

The pseudo-Riemannian isometry properties on $TT$ space is evident when we perform the Lorentz transformations on the infinitesimal length squared

$$
dp \otimes dp \xrightarrow{LT} dp' \otimes dp' = dp^C \left( \hat{A}^{-1} \right) \cdot \left( \hat{B}^{-1} \right) dp_B
$$

which yield an invariant quantity under coordinate transformations.

As an application in $TT$ space, the maximum allowable speed for particles living in $TT$ space is derived. Recall the boost transformation of parameter $v$, along the $r_3$ direction

$$
r_3 \xrightarrow{v} r'_3 = 0 = \frac{1}{\sqrt{2}} \sinh \left( \sqrt{2} \psi_1 \right) ct + \cosh \left( \sqrt{2} \psi_1 \right) r_3. \tag{85}
$$

Solving equation (85) for the velocity $v_3 = \frac{r_3}{t_i}$

$$
\frac{1}{c} t = -\frac{1}{\sqrt{2}} \tanh \left( \sqrt{2} \psi_1 \right), \quad \frac{1}{c} 2t_1 = -\frac{1}{\sqrt{2}} \tanh \left( \sqrt{2} \psi_1 \right), \quad v_3 = -\sqrt{2} c \tanh \left( \sqrt{2} \psi_1 \right). \tag{86}
$$

In term of $\beta$,we have

$$
tanh \left( \sqrt{2} \psi_1 \right) = -\frac{1}{\sqrt{2}} v_3 = -\frac{1}{\sqrt{2}} \beta. \tag{87}
$$
Solving the hyperbolic identity for \( \cosh(\sqrt{2}\psi_1) \), we have

\[
\cosh^2(\sqrt{2}\psi_1) - \sinh^2(\sqrt{2}\psi_1) = 1 \\
1 - \tanh^2(\sqrt{2}\psi_1) = \cosh^{-2}(\sqrt{2}\psi_1), \\
\cosh(\sqrt{2}\psi_1) = \frac{1}{\sqrt{1 - \tanh^2(\sqrt{2}\psi_1)}} \\
\cosh(\sqrt{2}\psi_1) = \frac{1}{\sqrt{1 - \frac{1}{2} \left( \frac{v_3}{c} \right)^2}} = \gamma_5, \quad (88)
\]

where \( \gamma_5 \) is the 5D Lorentz factor in TT space. To find the maximum allowable speed in TT space, we examine the Lorentz factor, \( \gamma_5 \), and extract the following constrained inequality

\[
0 < 1 - \frac{1}{2} \left( \frac{v_3}{c} \right)^2. \quad (89)
\]

Solving the inequality (89) for \( v_{\text{max}} \), we have

\[
v_{\text{max}} < \pm \sqrt{2}c \approx \pm 1.4c, \quad (90)
\]

which exceeds the absolute speed \( c \) of Minkowski space. The maximum allowable speed can also be obtained from equation (45)

\[
dp^\hat{A} = (cdt_1(\psi), cdt_2(\psi), dr_3(\psi)), \\
dp^\hat{A} = (cdt_1, cdt_1, dr_3), \quad (91)
\]

since the two time-like dimensions are parametrized by the same hyperbolic function as derived in section 2. The invariant infinitesimal length squared
\( ds^2 \) can be computed by using the \( TT \) inner product on the 3-vector

\[
\begin{align*}
ds^2 &= [TT]_{\hat{A}\hat{B}} dp^\hat{A} dp^\hat{B} = dp^\hat{A} \otimes dp^\hat{A} \\
&= -c^2 dt_1^2 - c^2 dt_1^2 + dr_3^2 \\
ds^2 &= -2c^2 dt_1^2 + dr_3^2.
\end{align*}
\] (92) (93)

To obtain \( v_{\text{max}} \), the world-line of the particle must be on the light-cone, i.e. \( ds^2 = 0 \), hence

\[
\left( \frac{dr_3}{dt_1} \right)^2 = 2c^2
\]

\[
\frac{dr_3}{dt_1} = \pm \sqrt{2}c
\]

\[
v_{\text{max}} = v_3 = \pm \sqrt{2}c \approx \pm 1.4c,
\] (94)

where \( v_3 = \frac{dr_3}{dt_1} \) and \( v_{\text{max}} = v_3 < \sqrt{2}c \).

4 Summary

The specific forms of the elements, \( p^A \in M \subset TT \) space and \( \dot{p}_A \in W \subset T_T(M) \), where \( M \) and \( W \) are some open subsets of \( TT \) space and its associated tangent space \( T_P(M) \), were derived via \( TT \) metric function. The \( TT \) Lorentz transformation matrix was fixed or obtained by using the length invariant condition and the \( TT \) metric function. It was shown that the \( TT \) Lorentz transformation matrix transformed a 5D vector \( p^A \in R^5 \) into \( C^5 \). As an application in \( TT \) space, aid with the \( TT \) Lorentz transformation matrix, a boost of \( v \) along the spatial direction between the two inertial frames is performed. The resulting boost transformation yielded a maximum allowable speed, \( v_{\text{max}} \), for particles living in the \( TT \) space. The maximum speed is shown to be, \( v_{\text{max}} < \sqrt{2}c \), which exceeds the absolute speed, \( c \), where \( c \) is the speed of light in Minkowski space.
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