Two diverse models of embedding class one

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Abstract
Embedding theorems have continued to be a topic of interest in the general theory of relativity since these help connect the classical theory to higher-dimensional manifolds. This paper deals with spacetimes of embedding class one, i.e., spacetimes that can be embedded in a five-dimensional flat spacetime. These ideas are applied to two diverse models, a complete solution for a charged wormhole admitting a one-parameter group of conformal motions and a new model to explain the flat rotation curves in spiral galaxies without the need for dark matter.

Keywords: Wormholes, Dark matter, Conformal symmetry, Embedding class one

Highlights:
• The embedding of curved spacetime in higher-dimensional flat spacetime is discussed.
• The focus is on two diverse models of embedding class one.
• The first model is a complete wormhole solution including the junction conditions.
• The second is a new model to explain the flat galactic rotation curves without the need for dark matter.

1 Introduction
Embedding theorems have a long history in the general theory of relativity, as exemplified by the induced-matter theory discussed in Ref. [1]. Of particular interest to us are spacetimes of embedding class one. Here we recall that an \( n \)-dimensional Riemannian space is said to be of class \( m \) if \( n + m \) is the lowest dimension of the flat space in which the given space can be embedded. It is well known that the exterior Schwarzschild solution is a Riemannian space of class two. Following Refs. [2], we assume a spherically symmetric metric of class two that will be reduced to class one by a suitable transformation discussed in Sec. [2] Other useful references are [3, 4, 5].

These ideas will be applied to two very different models, a complete solution for a charged wormhole admitting a one-parameter group of conformal motions and a new model to explain the flat rotation curves in spiral galaxies without the need for dark matter.

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2 The embedding

The discussion in Ref. [2] begins with the static and spherically symmetric line element (in units in which \( G = c = 1 \))

\[
 ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{1}
\]

where \( \lambda \) and \( \nu \) are functions of the radial coordinate \( r \). It is shown that this metric of class two can be reduced to class one and thereby embedded in the five-dimensional flat spacetime

\[
 ds^2 = -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2 + (dz^5)^2. \tag{2}
\]

This reduction is accomplished by the following transformation:

\[
 z^1 = r \sin \theta \cos \phi, \quad z^2 = r \sin \theta \sin \phi, \quad z^3 = r \cos \theta, \quad z^4 = \sqrt{K} e^{\frac{\nu}{2}} \coth \frac{t}{\sqrt{K}}, \quad \text{and} \quad z^5 = \sqrt{K} e^{\frac{\nu}{2}} \sinh \frac{t}{\sqrt{K}}. \tag{3}
\]

The differentials of these components are

\[
 dz^1 = dr \sin \theta \cos \phi + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi, \tag{3}
\]

\[
 dz^2 = dr \sin \theta \sin \phi + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi, \tag{4}
\]

\[
 dz^3 = dr \cos \theta - r \sin \theta d\theta, \tag{5}
\]

\[
 dz^4 = \sqrt{K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \coth \frac{t}{\sqrt{K}} dr + e^{\frac{\nu}{2}} \sinh \frac{t}{\sqrt{K}} dt, \tag{6}
\]

and

\[
 dz^5 = \sqrt{K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \sinh \frac{t}{\sqrt{K}} dr + e^{\frac{\nu}{2}} \cosh \frac{t}{\sqrt{K}} dt, \tag{7}
\]

where the prime denotes differentiation with respect to the radial coordinate \( r \). To facilitate the substitution into Eq. (2), we first obtain the expressions for \(-(dz^1)^2-(dz^2)^2-(dz^3)^2\) and \(-(dz^4)^2+(dz^5)^2\):

\[
 -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 = -dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{8}
\]

\[
 -(dz^4)^2 + (dz^5)^2 = e^\nu dt^2 - \frac{K e^\nu}{4} \nu'^2 dr^2. \tag{9}
\]

Substituting Eqs. (8) and (9) in Eq. (2), we obtain the metric

\[
 ds^2 = e^\nu dt^2 - \left( 1 + \frac{K e^\nu}{4} \nu'^2 \right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{10}
\]

So metric (10) is equivalent to metric (11) if

\[
 e^\lambda = 1 + \frac{K e^\nu}{4} \nu'^2, \quad K > 0. \tag{11}
\]

This condition is equivalent to the condition derived by Karmarkar [6] in terms of the Riemann curvature tensor components:

\[
 R_{1414} = \frac{R_{1212} R_{3434} + R_{1224} R_{1334}}{R_{2323}}, \quad R_{2323} \neq 0.
\]

(See Ref. [7] for details.)
3 Conformal Killing vectors

As noted above, we assume a static spherically symmetric spacetime admitting a one-parameter group of conformal motions, which are motions along which the metric tensor remains invariant up to a scale factor. Equivalently, there exist conformal Killing vectors such that

\[ \mathcal{L}_\xi g_{\mu\nu} = g_{\eta\nu} \xi^\eta \partial_\mu \psi + g_{\mu\eta} \xi^\eta \partial_\nu \psi = \psi(r) g_{\mu\nu}, \]  
(12)

where the left-hand side is the Lie derivative of the metric tensor and \( \psi(r) \) is the conformal factor \([8, 9]\). The metric tensor \( g_{\mu\nu} \) is conformally mapped into itself along the vector \( \xi \), which generates the conformal symmetry. This type of symmetry has been used to describe relativistic stellar-type objects, as discussed in Refs. \([10, 11]\). Additional new geometric and kinematical insights are described in Refs. \([12, 13, 14, 15, 16]\). Two earlier studies assumed non-static conformal symmetry \([9, 16]\). Another significant observation is that the Kerr black hole is conformally symmetric \([17]\).

To study the effect of conformal symmetry, it is convenient to use line element (1) with the opposite signature \([18, 19]\):

\[ ds^2 = -e^{-\lambda}(r) dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  
(13)

Using this form, the Einstein field equations become

\[ e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi \rho, \]  
(14)

\[ e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = 8\pi p_r, \]  
(15)

and

\[ \frac{1}{2} e^{-\lambda} \left[ \frac{1}{2}(\nu')^2 + \nu'' - \frac{1}{2} \lambda' \nu' + \frac{1}{r} (\nu' - \lambda') \right] = 8\pi p_t. \]  
(16)

Following Herrera and Ponce de León \([10]\), we can simplify the analysis by requiring that \( \xi^\alpha U_\alpha = 0 \), where \( U_\alpha \) is the four-velocity of the perfect fluid distribution, so that fluid flow lines are mapped conformally onto fluid flow lines. From the assumption of spherical symmetry, it now follows that \( \xi^0 = \xi^2 = \xi^3 = 0 \) \([10]\). Eq. (12) then yields the following results:

\[ \xi^1 \nu' = \psi, \]  
(17)

\[ \xi^1 = \frac{\psi r}{2}, \]  
(18)

and

\[ \xi^1 \lambda' + 2 \xi^1 = \psi. \]  
(19)

From Eqs. (17) and (18), we then obtain \( \nu' = 2/r \) and thus

\[ e^{\nu} = Cr^2, \]  
(20)
where $C$ is an integration constant. Now from Eq. (18) we get

$$\xi_{1,1} = \frac{1}{2}(\psi' r + \psi).$$

Substituting in Eq. (19) and using $\nu' = 2/r$, simplification yields

$$\lambda' = -2\frac{\psi'}{\psi}.$$

Finally, solving for $\lambda$, we have

$$e^\lambda = \left(\frac{B}{\psi}\right)^2,$$

(21)

where $B$ is another integration constant. When substituting into Eqs. (14)-(16), it becomes apparent that $B$ is merely a scale factor, so that we may assume that $B = 1$. We then get

$$e^{-\lambda} = \psi^2$$

(22)

and the Einstein field equations can be rewritten as follows:

$$\frac{1}{r^2} (1 - \psi^2) - \frac{(\psi^2)'}{r} = 8\pi \rho,$$

(23)

$$\frac{1}{r^2} (3\psi^2 - 1) = 8\pi p_r,$$

(24)

and

$$\frac{\psi^2}{r^2} + \frac{(\psi^2)'}{r} = 8\pi p_t.$$  

(25)

4  Wormhole structure

Wormholes are handles or tunnels in spacetime connecting widely separated regions of our Universe or entirely different universes. While there were a number of forerunners, actual physical structures suitable for interstellar travel was first proposed by Morris and Thorne [20]. Such wormholes can be described by the static and spherically symmetric line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(26)

using units in which $c = G = 1$. Here $\Phi = \Phi(r)$ is called the redshift function, which must be everywhere finite to avoid an event horizon. The function $b = b(r)$ is called the shape function since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram [20]. The spherical surface $r = r_0$ is the throat of the wormhole. The shape function must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$ and $b'(r_0) \leq 1$, called the flare-out condition. For a Morris-Thorne wormhole, this condition can only be satisfied by violating the null energy condition, thereby becoming the primary condition for the existence of a traversable wormhole.
The discussion in Ref. [20] was based on the following strategy: specify the geometric conditions required for a traversable wormhole and then either manufacture or search the Universe for matter or fields that will produce the corresponding energy-momentum tensor. One of our goals in this paper is to reverse this strategy by showing that the conditions described are sufficient for producing a complete solution, i.e., for obtaining both $\Phi = \Phi(r)$ and $b = b(r)$, as well as the necessary junction conditions.

5 Charged wormholes

The motivation for a wormhole with a constant charge $Q$, first proposed by Kim and Lee [21], was provided by the Reissner-Nordström black hole

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

suggesting that

$$e^{\lambda(r)} = \left(1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}\right)^{-1}.$$ \hfill (27)

Charged wormholes are also discussed in Refs. [22, 23].

Since we are assuming conformal symmetry, we have $e^\nu = C r^2$ from Eq. (20). Moreover, from Eq. (22),

$$\psi^2 = 1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}. \hfill (28)$$

We also assume that $b = b(r)$ satisfies the usual conditions for a shape function: letting $r = r_1$ be the radius of the throat, we require that $b(r_1) = r_1$ and $b(r_1) < 1$, while $b(r) < r$ for $r > r_1$.

6 Wormholes from a metric of embedding class one

Returning to Sec. 2, we know from Eq. (11) that in view of Eq. (27)

$$e^{\lambda(r)} = \frac{1}{1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}} = 1 + \frac{1}{4}Ke^\nu(r)[\nu'(r)]^2. \hfill (29)$$

Moreover, from Eq. (20), $e^\nu = C r^2$, we have

$$1 = \left(1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}\right)\left(1 + \frac{1}{4}K(C r^2)\frac{4}{r^2}\right)$$

since $(\nu')^2 = 4/r^2$. Hence

$$\frac{1}{1 + KC} = 1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}.$$ (Observe that since $b(r_1) = r_1$, $Q^2$ cannot be zero.) Solving for $b(r)$, we get

$$b(r) = r \left(1 + \frac{Q^2}{r^2} - \frac{1}{1 + KC}\right). \hfill (30)$$
The condition \( b(r_1) = r_1 \) now leads to \( 1 + KC = r_1^2/Q^2 \) and

\[
C = \frac{1}{K} \left( \frac{r_1^2}{Q^2} - 1 \right).
\] (31)

The result is

\[
b(r) = r \left( 1 + \frac{Q^2}{r^2} - \frac{Q^2}{r_1^2} \right).
\] (32)

This result, in turn, leads to

\[
b'(r_1) = 1 - \frac{2Q^2}{r_1^2} < 1,
\] (33)

provided that \( r_1^2 > 2Q^2 \).

The flare-out condition is thereby satisfied, but unlike a Morris-Thorne wormhole, satisfying this condition does not automatically result in a violation of the null energy condition (NEC). To see why, recall that the NEC states that for the energy-momentum tensor \( T_{\alpha\beta} \), \( T_{\alpha\beta}k^\alpha k^\beta \geq 0 \) for all null vectors \( k^\alpha \). Consider the radial outgoing null vector \((1, 1, 0, 0)\). Then if \( b = b(r) \) were the shape function of a regular Morris-Thorne wormhole, Eq. (26), we would have \( 8\pi\rho + 8\pi p_r(r_1) = b'(r_1) - b(r)/r + 2\Phi' \left( 1 - \frac{b(r)}{r} \right) \).

So at \( r = r_1 \), \( 8\pi\rho(r_1) + 8\pi p_r(r_1) < 0 \) since \( b'(r_1) < 1 \). The problem is that our metric has been altered due to the embedding, Eq. (11), i.e., \( e^\lambda(r) = 1 + \frac{1}{4}Ke^{\nu(r)}(\nu')^2 \). As a result, we are no longer dealing with the same null vector: since \( b(r) \) has changed, so has \( 8\pi\rho + 8\pi p_r(r_1) \). On the other hand, we also made use of Eq. (27), \( e^\lambda(r) = (1 - b(r)/r + Q^2/r^2)^{-1} \), which gives us the effective shape function used in Ref. [23]:

\[
b_{\text{eff}}(r) = b(r) - \frac{Q^2}{r}.
\] (34)

This form suggests that Eq. (30) be modified as follows:

\[
b(r) = r \left( 1 + \frac{Q^2}{r^2} - \frac{1}{KC} \right) + \frac{Q^2}{r}.
\] (35)

This modification needs to be justified by showing that \( b(r) \) in Eq. (35) satisfies all the required conditions. To that end, let us denote the throat by \( r = r_0 \), so that \( b(r_0) = r_0 \). This condition yields \( 1 + KC = r_0^2/2Q^2 \) and

\[
C = \frac{1}{K} \left( \frac{r_0^2}{2Q^2} - 1 \right).
\] (36)

Substituting in Eq. (35), we get

\[
b(r) = r \left( 1 + \frac{Q^2}{r^2} - \frac{2Q^2}{r_0^2} \right) + \frac{Q^2}{r}.
\] (37)
Finally, the flare-out condition is also met:

\[ b'(r_0) = 1 - \frac{4Q^2}{r_0^2} < 1. \]

(Since we want \( b'(r_0) \) to be positive, we also require that \( r_0^2 > 4Q^2 \).)

As noted earlier, we still need to check the violation of the null energy condition: From Eqs. (11), (22), and (27),

\[ e^{-\lambda(r)} = 1 - \frac{b(r)}{r} + \frac{Q^2}{r_0^2} \psi^2(r). \]

Substituting \( b(r) \), we get

\[ \psi^2(r) = 1 - \left( 1 + \frac{Q^2}{r^2} - \frac{2Q^2}{r_0^2} \right) - \frac{Q^2}{r^2} r_0^2 = -\frac{2Q^2}{r^2} + \frac{3Q^2}{r_0^2}. \]  (38)

Thus

\[ \psi^2(r_0) = \frac{Q^2}{r_0^2}. \]  (39)

Also,

\[ (\psi^2(r))' = \frac{4Q^2}{r^3}. \]  (40)

Returning now to the Einstein field equations (23) and (24),

\[ 8\pi (\rho + p_r)|_{r=r_0} = \frac{1}{r^2} \left[ 2\psi^2(r) - \frac{\psi^2(r)'}{r} \right]_{r=r_0} = \frac{1}{r_0^2} \left( \frac{2Q^2}{r_0^2} \right) - \frac{4Q^2}{r_0^4} = -\frac{2Q^2}{r_0^4} < 0. \]  (41)

So the null energy condition is indeed violated.

Returning now to Eq. (36), it is clear that the free parameter \( K \) can be used to determine the constant \( C \) and hence \( e^\nu \), which, in turn yields the redshift function.

To complete the solution, we still need to consider the following: we can see from Eq. (20), \( e^\nu = Cr^2 \), that our wormhole spacetime is not asymptotically flat. So the wormhole material must be cut off at some \( r = a \) and joined to an exterior Schwarzschild spacetime,

\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \]  (42)

Thus

\[ e^{\nu(a)} = Ca^2 = 1 - \frac{2M}{a}, \]  (43)

where \( M \) is the mass of the wormhole as seen by a distant observer and \( C \) is obtained from Eq. (36). It follows that the cut-off \( r = a \) is implicitly determined by the equation

\[ \frac{1}{K} \left( \frac{r_0^2}{2Q^2} - 1 \right) a^2 = 1 - \frac{2M}{a}. \]  (44)

So given \( M \), \( Q^2 \), and \( r_0 \), the free parameter \( K \) determines the radius of the junction surface. Eq. (44) will have a real solution if \( K \) is sufficiently large. (Plausible values might be \( 10^9 \text{ m}^2 \) and \( 10^{17} \text{ m}^2 \) arising in the discussion of compact stellar objects in Ref. [5].)
7 Flat galactic rotation curves

One goal in many modified gravitational theories is to explain the peculiar behavior of galactic rotation curves without postulating the existence of dark matter, whether this be noncommutative geometry [24, 25] or $f(R)$ modified gravity [26]. The basic problem is that test particles move with constant tangential velocity $v^\phi$ in a circular path sufficiently far from the galactic core. Taking the observed rotation curves as input, it is well known that

$$e^\nu = B_0 r^l,$$  \hspace{1cm} (45)

where $l = 2v^{2\phi}$ and $B_0$ ia an integration constant [27]. In addition, it is shown in Ref. [28] that in the presumed dark-matter dominated region, $v^\phi \approx 300 \text{ km/s} = 10^{-3}$ for a typical galaxy. So $l = 0.000001$ [29].

The existence of flat rotation curves indicates that the matter in the galaxy increases linearly in the outward radial direction. To recall the reason for this, suppose $m_1$ is the mass of a star, $v^\phi$ the constant tangential velocity, and $m_2$ the mass of everything else. Multiplying $m_1$ by the centripetal acceleration yields

$$m_1 \frac{v^{2\phi}}{r} = m_1 m_2 \frac{G}{r^2},$$  \hspace{1cm} (46)

where $G$ is Newton’s gravitational constant. The result is (since $G = 1$)

$$m_2 = rv^{2\phi}.$$  \hspace{1cm} (47)

Eq. (47) essentially characterizes the dark-matter hypothesis, but, as noted above, other explanations are possible. As a starting point, suppose we consider [30]

$$e^{\lambda(r)} = \frac{1}{1 - 2m(r)/r}.$$  \hspace{1cm} (48)

Then $m(r) = \frac{1}{2v}(1 - e^{-\lambda})$. Since $e^{-\lambda} \to 1$ as $r \to \infty$, we can assume that $m(r)$ is approximately constant over a large range of $r$. In other words, $m(r) = Cr$ for some constant $C$. Unfortunately, by Eq. (47), $C$ has to be approximately equal to $v^{2\phi}$. This obstacle can be overcome by the embedding theory in Sec. [2] Using Eq. (11),

$$m(r) = \frac{1}{2} r \left( 1 - \frac{1}{1 + \frac{1}{4} K e^\nu(\nu')^2} \right),$$  \hspace{1cm} (48)

the free parameter $K$ gives us the extra degree of freedom to produce the correct values, provided, of course, that $e^\nu$ and $(\nu')^2$ are indeed approximately constant. To that end, let us return to Refs. [27]-[29], which deal with typical galaxies, including our own. Suppose we assume for now that $B_0 = 1$ in Eq. (45). So with our own galaxy in mind, let us consider the range from 8 kps to 50 kps, associated with flat galactic rotation curves. Then $r^l$ ranges from

$$(26000 \times 9.46 \times 10^{15})^{0.000001} \approx 1.000047$$
to
\[(6.25 \times 26\,000 \times 9.46 \times 10^{15})^{0.000001} \approx 1.000049.\]

These calculations show that the value of \(B_0\) has little effect, so that \(e''\) does remain approximately constant.

The values for \((\nu')^2\) are much less robust. However, of great help in this situation is that \(B_0\) drops out entirely: from \(e'' = B_0 r^4\), we have
\[
(\nu')^2 = \frac{l^2}{r^2},
\]

So we can obtain an adequate approximation for the above range, while in the range 16 kps to 30 kps, the resulting values for \((\nu')^2\) are essentially fixed: \(4.1 \times 10^{-54}\) m\(^{-2}\) and \(1.2 \times 10^{-54}\) m\(^{-2}\), respectively. For these values, \(K \approx 10^{48}\) m\(^2\).

With this choice of \(K\), Eq. (48) reduces to \(m(r) = \nu^{2\nu} r\), thereby producing another alternative to the dark-matter hypothesis. This outcome may be viewed as the analogue of the induced-matter theory in Ref. [1], i.e., one could maintain that the five-dimensional flat spacetime impinges on our Universe to produce the effect that we normally interpret as dark matter.

Remark: The existence of two models from the same embedding theory invites the following speculation: according to Brownstein and Moffet [31], a significant amount of dark matter is missing in the Bullet Cluster 1E0657-558. This is also the cluster that has supposedly shown that dark matter actually exists. The main argument in Ref. [31] is that this phenomenon can be explained by means of a modified gravitational theory, to which we could add the present embedding theory. On the other hand, if the dark-matter hypothesis is to be retained and if some of the dark matter is indeed missing, then the existence of a conformally symmetric charged wormhole may be the preferred explanation: the Bullet Cluster consists of two colliding galaxies moving at very high velocities, so that the dark matter could be literally driven into the wormhole.

8 Conclusion

It is well known that a curved spacetime can be embedded in a higher-dimensional flat spacetime. A spacetime is said to be of class \(m\) if \(n + m\) is the lowest dimension of the flat space in which the given space can be embedded. Following Ref. [2], we assume a spherically symmetric metric of class two that can be reduced to class one by a suitable transformation.

These ideas have been applied to two completely different models, a new solution for a charged wormhole admitting a one-parameter group of conformal motions and a new model to explain the flat rotation curves in spiral galaxies without the need for dark matter. The existence of the latter model can be attributed to the free parameter \(K\) in the embedding theory. In the former case, the free parameter \(K\) plays an equally critical role in obtaining a complete wormhole solution: \(K\) helps determine the redshift and shape functions, as well as the radius of the junction interface that joins the interior solution to an exterior Schwarzschild spacetime.
References

[1] P.S. Wesson, J. Ponce de León, J. Math. Phys. 33 (1992) 3883.
[2] S.K. Maurya, M. Govender, Eur. Phys. J. C 77 (2017) 347.
[3] S.K. Maurya, Y.K. Gupta, S. Ray, D. Deb, Eur. Phys. J. C 77 (2017) 45.
[4] S.K. Maurya, Y.K. Gupta, S. Ray, D. Deb, Eur. Phys. J. C 76 (2016) 693.
[5] S.K. Maurya, D. Deb, S. Ray, P.K.F. Kuhfittig, arXiv: 1703.08436.
[6] K.R. Karmarkar, Proc. Ind. Acad. Sci. 27 (1948) 56.
[7] P. Bhar, S.K. Maurya, Y.K. Gupta, T. Manna, Eur. Phys. J. A 52 (2016) 312.
[8] R. Maartens, C.M. Mellin, Class. Quantum Grav. 13 (1996) 1571.
[9] C.G. Böhmer, T. Harko, F.S.N. Lobo, Phys. Rev. D 76 (2007) 084014.
[10] L. Herrera, J. Ponce de León, J. Math. Phys. 26 (1985) 778.
[11] L. Herrera, J. Ponce de León, J. Math. Phys. 26 (1985) 2018.
[12] M. Mars, J.M.M. Senovilla, Class. Quantum Grav. 10 (1993) 1633.
[13] S. Ray, A.A. Usmani, M. Kalam, K. Chakraborty, Indian J. Phys. 82 (2008) 1191.
[14] F. Rahaman, M. Jamil, M. Kalam, K. Chakraborty, A. Ghosh, Astrophys. Space Sci. 325 (2010) 137.
[15] F. Rahaman, S. Ray, I. Karar, H.I. Fatima, S. Bhowmick, G.K. Ghosh, arxiv: 1211.1228 [gr-qc].
[16] C.G. Böhmer, T. Harko, F.S.N. Lobo, Class. Quantum Grav. 25 (2008) 075016.
[17] A. Castro, A. Maloney, A. Strominger, Phys. Rev. D 82 (2010) 024008.
[18] F. Rahaman, S. Ray, G.S. Khadekar, P.K.F. Kuhfittig, I. Karar, Int. J. Theor. Phys. 54 (2015) 699.
[19] P.K.F. Kuhfittig, Indian J. Phys. 90 (2010) 877.
[20] M.S. Morris, K.S. Thorne, Amer. J. Phys. 56 (1988) 395.
[21] S.-W. Kim, H. Lee, Phys. Rev. D 63 (2001) 064014.
[22] P.K.F. Kuhfittig, Central Eur. J. Phys. 9 (2011) 144.
[23] P.K.F. Kuhfittig, J. Appl. Math. Phys. (JAMP) 4 (2016) 2117.
[24] R. Rahaman, P.K.F. Kuhfittig, K. Chakraborty, A.A. Uamani, S. Ray, Gen. Rel. Gravit. 44 (2012) 905.
[25] P.K.F. Kuhfittig, V.D. Gladney, J. Mod. Phys. 5 (2014) 1931.

[26] C.G. Böhmer, T. Harko, F.S.N. Lobo, Astropart. Phys. 29 (2008) 386.

[27] K.K. Nandi, I. Valitov, N.G. Migranov, Phys. Rev. D 62 (2009) 047301.

[28] T. Matos, F.S. Guzman, D. Nunez, Phys. Rev. D 62 (2008) 061301R.

[29] K.K. Nandi, A.I. Filippov, F. Rahaman, S. Ray, A.A. Usmani, M. Kalam, A. DeBenedictis, Mon. Not. R. Astron. Soc. 399 (2009) 2079.

[30] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, W. Freeman and Company, New York, 1973, page 608.

[31] J.R. Brownstein, J.W. Moffet, Mon. Not. Roy. Astron. Soc. 382 (2007) 29.