Research Article

Existence and Stability of Square-Mean S-Asymptotically Periodic Solutions to a Fractional Stochastic Diffusion Equation with Fractional Brownian Motion

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In this paper, a generalized Gronwall inequality is demonstrated, playing an important role in the study of fractional differential equations. In addition, with the fixed-point theorem and the properties of Mittag–Leffler functions, some results of the existence as well as asymptotic stability of square-mean S-asymptotically periodic solutions to a fractional stochastic diffusion equation with fractional Brownian motion are obtained. In the end, an example of numerical simulation is given to illustrate the effectiveness of our theory results.

1. Introduction

Originated in 1695, fractional calculus has been widely applied in physics, chemistry, economics, biology, and other fields. Recent decades have witnessed the rapid development of fractional calculus, with the emergence of many related researches [1–9]. The dynamic behavior of some complex processes in reality can be explained by fractional differential equations. For example, anomalous diffusion phenomena can be described with fractional diffusion equations. Compared with the traditional diffusion equations (first order), with fractional diffusion equations, subdiffusion or supdiffusion phenomenon can be described when its order is between 0 and 1 or between 1 and 2, respectively.

In addition, the diffusion phenomenon in real life is often affected by random factors, promoting the generation of fractional stochastic diffusion equations. However, this research has not aroused much concern until recent years. In [10], a class of nonautonomous fractional stochastic reaction–diffusion equations was studied, obtaining the regularity of random attractors. The Galerkin method was applied by Wang [11] to investigate the existence of tempered pullback random attractors for nonautonomous fractional reaction–diffusion equations with multiplicative noise. Chen [12] studied the stochastic time-fractional diffusion equations with multiplicative white noise, obtaining the Hölder continuity of the solution. Peng and Huang [13] established the existence of mild solutions for a nonlocal backward problem for fractional stochastic diffusion equations.

We also notice that some researches focus on the stability of solutions to fractional stochastic differential equations of order $\alpha \in (0, 1)$. Li and Wang [14] studied the existence and asymptotic behavior of solutions to fractional stochastic delay evolution equations with integral term and Wiener process by using fractional resolvent operator theory and the Schauder fixed-point theorem. Mathiyalagan and Balachandran [15] studied the finite-time stochastic stability of fractional-order singular systems with time delay and white noise utilizing the Gronwall approach and stochastic analysis technique. In [16], applying the Laplace transform

In summary, the study of fractional stochastic diffusion equations has become a significant and active field in recent years. The focus of this paper is to establish the existence and asymptotic stability of square-mean S-asymptotically periodic solutions to a fractional stochastic diffusion equation with fractional Brownian motion. The research results will provide a theoretical basis for the further study of fractional stochastic systems.

In this paper, we prove some important theorems and obtain the existence and asymptotic stability of square-mean S-asymptotically periodic solutions. The main results are achieved by using the fixed-point theorem and the properties of Mittag–Leffler functions.
method, the authors obtained the existence, uniqueness, and Hyers–Ulam stability of solutions to a class of linear fractional differential equations involving Mittag–Leffler kernel. The resolvent operator technique and contraction mapping principle were used in [17] to study the existence and uniqueness of mild solution to fractional neutral stochastic integrodifferential equations involving impulses driven by fractional Brownian motion (FBM), and a new impulsive-integral inequality was used to obtain the exponential stability for these equations. Moreover, the existence and asymptotic stability in the p-th moment of mild solutions to a class of fractional stochastic partial differential equations with Wiener process was investigated by Zhang et al. [18]. Because the form of the equations in this paper is different from those in the above studies, the methods to prove the stability in these studies cannot be directly applied to this paper.

Inspired by the above researches, in order to study the stability and periodicity of anomalous diffusion phenomena affected by random factors, we consider the fractional stochastic diffusion equation involving Dirichlet boundary conditions:

\[
\begin{align*}
\partial_t^\alpha u(x, t) + u(x, t) &= f(t, u(x, t)) + D(t, u(x, t)) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial B^H_{\omega}(x, t)}{\partial t} &= u(x, t) \quad \text{on } \partial \Omega \times (0, \infty), \\
\end{align*}
\]

FBM. It is worth mentioning that the standard Brownian motion, without long memory, cannot represent all types of noise. A good long-term memory noise could be described by FBM of Hurst parameter \( H \in ((1/2), 1) \) [20]. For instance, the continuous disturbance and long-term dependence in the financial market model can be considered as a kind of FBM [21], with the impact of nuclear waste on the environment being seen as FBM in ecological models. Other research studies on FBM can be referred to [22–30]. (2) Stability of the S-asymptotically periodic solutions is studied by means of a new generalized Gronwall inequality.

The paper is organized as follows: the readers are allowed to review Section 2 for the necessary basic knowledge, followed by some results of the existence and uniqueness of S-asymptotically periodic solutions in Section 3. Subsequently, the asymptotic stability of S-asymptotically periodic solutions is studied in Section 4, with a numerical simulation example in Section 5.

2. Preliminaries

For the sake of convenience in writing, throughout this paper, by \( \infty \), we mean \(+\infty\). \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) denotes a complete filtered probability space, and \( U \) and \( H \) are two separable Hilbert spaces. The space of bounded linear operators from \( U \) into \( H \) is written as \( L(U, H) \). For convenience, the same notation \( \| \cdot \| \) is applied to denote norms in \( U, H \) and \( L(U, H); \), \( (\cdot, \cdot) \) is applied to denote the inner product of \( U \) and \( H \). Moreover, \( L^2(\Omega; H) \) is the space of all strongly measurable and square-integrable \( H \)-valued random variables under the Banach norm \( (E\| \cdot \|^2)^{1/2} \).

A stochastic process \( u: [0, \infty) \rightarrow L^2(\Omega; H) \) is called stochastically bounded if \( \sup_{t \geq 0} E\| u(t) \|^2 < +\infty \), called stochastically continuous if \( \lim_{t \rightarrow s} E\| u(t) - u(s) \|^2 = 0 \) for all \( t, s \geq 0 \) and called square-mean S-asymptotically \( \omega \)-periodic if

\[
\lim_{t \rightarrow -\infty} E\| u(t + \omega) - u(t) \|^2 = 0,
\]

where \( \omega > 0 \) is a constant.
Denote by $\text{SBC}([0, \infty); L^2(\Omega; H))$ the space of all stochastically bounded and continuous processes from $[0, \infty)$ into $L^2(\Omega; H)$ and its norm

$$\|\mu\|_{\infty} = \left(\sup_{t \geq 0} E\|\mu(t)\|^2\right)^{1/2},$$

where $E\|\mu(t)\|^2 = \int_{\Omega} \|\mu(t)\|^2 dP$. Then, $\text{SBC}([0, \infty); L^2(\Omega; H))$ is a Banach space.

We use $\text{SAP}_{n}([0, \infty); L^2(\Omega; H))$ to denote the space of square-mean $\alpha$-asymptotically $\omega$-periodic stochastic process from $[0, \infty)$ into $L^2(\Omega; H)$. Then, $\text{SAP}_{n}([0, \infty); L^2(\Omega; H))$ is a Banach space with the sup norm $\| \cdot \|_{\infty}$ and is a linear closed subspace of $\text{SBC}([0, \infty); L^2(\Omega; H))$.

In the following, we introduce the definition and properties of FBM. We denote by $\{\beta^H(t)\}_{t \in \mathbb{R}} (H \in (0, 1))$ a two-sided one-dimensional FBM [23]. Then, $\beta^H$ is a continuous-centered Gaussian process, whose variance function is

$$R_H(t, s) = E[\beta^H(t)\beta^H(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \in \mathbb{R}. \tag{6}$$

In addition, if $W$ is a Wiener process, then

$$\beta^H(t) = \int_0^t K_H(t, s) dW(s), \quad K_H(t, s) = C_Hs^{(1/2)-H} \int_s^t (t-s)^{H-(3/2)}s^{H-(1/2)} ds, \tag{7}$$

for $t > s$, where $C_H = \sqrt{H(2H-1)/B(2-2H, H-(1/2))}$, with $B$ denoting the beta function.

Let $Q \in L(H, H)$ be an operator with $T_e(Q) = \sum_{n=1}^{\infty} \lambda_n \xi_n \otimes \xi_n$, and $Q_{k,l} = \lambda_l \xi_l$. For constants $\lambda_l \geq 0 (n = 1, 2, \ldots)$ and a complete orthonormal basis $\{\xi_n\}_{n=1}^{\infty}$ in $H$. The infinite dimensional FBM on $H$ can be expressed by

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_{Qn}^H(t) Q^{1/2} \xi_n = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_{Qn}^H(t) e_n, \quad t \geq 0, \tag{8}$$

where $Q$ is the covariance operator and $\{\beta_{Qn}^H(t)\}_{n=1}^{\infty}$ are two-sided one-dimensional FBMs, which are mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$.

Let $L^2_{\xi}(U, H)$ be the collection of all $Q$-Hilbert–Schmidt operators $\xi: U \rightarrow H$, where $\xi Q^{1/2}$ is a Hilbert–Schmidt operator, and the norm is

$$\|\xi\|_{L^2_{\xi}} = \text{tr}(\xi Q^* \xi) = \sum_{n=1}^{\infty} |\langle \xi \eta_n, \xi e_n \rangle|^2 < \infty. \tag{9}$$

For convenience, set $L^2_{\xi}(H) = L^2_{\xi}(H, H, \|\cdot\|)$. The space $L^2_{\xi}(U, H)$ is a separable Hilbert space whose inner product $\langle \phi, \psi \rangle_{L^2_{\xi}} = \sum_{n=1}^{\infty} \langle \phi e_n, \psi e_n \rangle$. Then, we define the stochastic integral of $\phi$ with respect to $B_{Qn}^H(t)$ by

$$\int_0^t \phi(s) dB_Q^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n dB_{Qn}^H(s) = \sum_{n=1}^{\infty} \int_0^t \phi(s) Q^{1/2} e_n dB_{Qn}^H(s), \tag{10}$$

where $\{\phi(t)\}_{t \in [0, T]}$ is the deterministic function with values in $L^2_{\xi}(U, H)$.

Now, we recall the Mittag–Leﬄer function and the probability density functions which play important roles in fractional differential equations [31].

**Lemma 1.** The Mittag–Leﬄer functions

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha(k+1))}, \quad t \in \mathbb{R}, \tag{11}$$

where $\Gamma$ is the Gamma function, have the following properties:

1. $E_\alpha(t), E_{\alpha,\alpha}(t) > 0$ [32, 33].
2. $(E_\alpha(t))' = (1/\alpha)E_{\alpha,\alpha}(t) [34].$
3. $\lim_{t \rightarrow \infty} E_\alpha(t) = \lim_{t \rightarrow -\infty} E_{\alpha,\alpha}(t) = 0 [32, 35].$

We notice that the Mittag–Leﬄer function $E_\alpha(t)$ is a generalization of exponential function $e^t$, which is $E_1(t)$. Set $u(t)(x) = u(x, t)$, $f(t, u(t))(x) = f(t, x, u(x, t))$, and $(B_Q^H u)(t)(x) = B_Q^H u(x, t)$. Let $\mathbb{H} = L^2(\Omega)$, and we define $A: D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ by

$$D(A) = H^2(\Omega) \cap H^1_0(\Omega),$$

$$(Au)(t)x = \mathcal{A}u(x, t). \tag{12}$$

We suppose that $-A$ generates an exponentially stable $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ satisfying

$$\|T(t)\| \leq M e^{-\delta t}, \quad \forall t \in [0, \infty), \tag{13}$$

where $M > 0$ and $\delta > 0$ are constants.

Thus, (1) can be transformed into

$$D_Q^H u(t) + Au(t) = f(t, u(t)) + D(t, u(t)) \frac{d_B^H(t)}{dt}, \quad t \in (0, \infty),$$

$u(0) = u_0, \tag{14}$$

where $D_Q^H$ is the Caputo-fractional derivative. Later, in the paper, $u_0 \in L^2(\Omega; H)$. 

**Remark 1.** We see that (13) is much easy to be verified. For example, let
\[\Omega = [0, \pi],\]
\[\mathbb{H} = L^2[0, \pi],\]
\[A = -\frac{\partial^2}{\partial x^2},\]
\[D(A) = \{u \in \mathbb{H} | u, u' \text{ are absolutely continuous, } u'' \in \mathbb{H}, u(0) = u(\pi) = 0\}.\]

Then, \(A\) has eigenvalues \(n^2 (n \in \mathbb{N})\), whose normalized eigenvectors \(w_n(t) = \sqrt{2/\pi} \sin(nt) (n \in \mathbb{N})\), \(-A\) generates an analytic, compact, and exponentially stable semigroup \(\{T(t)\}_{t \geq 0}\), and
\[
T(t)u = \sum_{n=1}^{\infty} e^{-nt} (u, w_n) w_n, \quad \|T(t)\| \leq e^{-t}, \quad t \in [0, \infty).
\]

**Definition 1.** A stochastic continuous process \(u: [0, \infty) \rightarrow L^2(\Omega; \mathbb{H})\) is said to be a mild solution of equation (14) if
\[
u(t) = U(t)u_0 + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, u(s)) ds + \int_0^t (t-s)^{\alpha-1} V(t-s) D(s, u(s)) dB^H(s),
\]
where
\[
U(t) = \int_0^\infty \zeta_\alpha(\theta) T(t^\theta) d\theta, \quad V(t) = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) T(t^\theta) d\theta,
\]
and the probability density function \([36, 37]\)
\[
\zeta_\alpha(\theta) = \frac{1}{\alpha} \theta^{1-\alpha/2} \rho_\alpha \left( \theta^{(1/\alpha)} \right),
\]
\[
\rho_\alpha(\theta) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \theta^{-\alpha n - 1} \Gamma(\alpha n + 1) \sin(\alpha n \theta), \quad \theta \in (0, \infty).
\]

Later, in this paper, we need the following results.

**Remark 2**

1. \(\zeta_\alpha(\theta) \geq 0\), for \(\theta \in (0, \infty)\).
2. \(\int_0^\infty \theta^\gamma \zeta_\alpha(\theta) d\theta = \Gamma(1 + \gamma) / \Gamma(1 + \alpha \gamma)\), for \(\gamma \in (-1, +\infty)\) \([36, 38]\).
3. \(\int_0^\infty e^{-t\theta} \zeta_\alpha(\theta) d\theta = (\alpha/\theta) E_{\alpha,\alpha}(-\theta)\), for \(t \in \mathbb{R}\) \([37]\).
4. \(\int_0^\infty e^{-t\theta} \theta \zeta_\alpha(\theta) d\theta = (\alpha/\theta^2) E_{\alpha,\alpha}(-\theta),\) for \(t \in \mathbb{R}\) \([36]\).

**Lemma 2**

1. \(U(t)\) and \(V(t)\) are strongly continuous for \(t \geq 0\) \([38]\).
2. If \(\{T(t)\}_{t \geq 0}\) satisfy (14), then \(\|U(t)\| \leq ME_{\alpha,-\delta t^\alpha}\) and \(\|V(t)\| \leq ME_{\alpha,\alpha,-\delta t^\alpha}\) for \(t \geq 0\) \([2]\).
3. \(\|V(t)\| \leq (M/\delta t^\alpha)\) for \(t > 0\).

**Proof.** The proof of (3) is as follows. In fact, for \(t > 0\), in view of \(e^{-\nu} < (1/\nu)\) with \(\nu > 0\), we have
\[
\|V(t)\| \leq M \int_0^\infty \theta \zeta_\alpha(\theta) T(t^\theta) d\theta
\]
\[
\leq M \int_0^\infty \theta \zeta_\alpha(\theta) e^{-\delta t^\theta} d\theta
\]
\[
\leq M \int_0^\infty \zeta_\alpha(\theta) e^{-\delta t^\theta} d\theta
\]
\[
\leq \frac{M}{\delta t^\alpha}.
\]

In order to get the stability of the square-mean S-asymptotically \(\omega\)-periodic solution, we need the following generalized Gronwall inequality for fractional differential equations.

**Lemma 3.** Let \(u_0, \lambda_1, \lambda_2 \in \mathbb{R}\) be two constants. If a continuous function \(u: [0, +\infty) \rightarrow \mathbb{R}\) satisfies
\[
u(t) \leq E_{\alpha}(\lambda_1 t^\alpha) u_0 + \lambda_2 \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda_1 (t-s)^\alpha) u(s) ds,
\]
then
\[
u(t) \leq E_{\alpha}((\lambda_1 + \lambda_2 t^\alpha) u_0).
\]

**Proof.** We find that the solution of the equation \([31]\)
\[
D^\alpha u(t) = \lambda_1 u(t) + \lambda_2 u(t), \quad t \in (0, +\infty),
\]
\[
u(0) = u_0,
\]
is given by
\[
u(t) = E_{\alpha}(\lambda_1 t^\alpha) u_0 + \lambda_2 \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda_1 (t-s)^\alpha) u(s) ds,
\]
or
\[
u(t) = E_{\alpha}((\lambda_1 + \lambda_2 t^\alpha) u_0).
\]

In view of the uniqueness of solution to (23), we get (22).
Remark 3. Compared with the generalized Gronwall inequality in [19], $E_n(\lambda_1^a)u_0$ does not have to be a nondecreasing function, and $\lambda_2$ is not necessarily nonnegative. This is very important to prove the stability of the solution.

Next, we give a result which is very useful for the estimations of fractional stochastic integral with FBM.

Lemma 4 (see [39]). Let $H \in (1/2, 1)$ and $h: [0, T] \rightarrow L^H_0(U, V)$ satisfy
\[ \int_a^b \|\phi(s)\|_{L^2_2}^2 ds < \infty, \quad \text{for } \forall a, b \in [0, T] \text{ with } b > a, \] (25)
then the corresponding sum given in (10) is well defined and we obtain
\[ E\left\|\int_a^b h(s)dB^H_Q(s)\right\|^2 \leq 2H(b-a)^{2H-1} \int_a^b h(s)^2_2 ds. \] (26)

3. Existence Uniqueness of Square-Mean S-Asymptotically \(a\)-Periodic Solutions

Lemma 5. Let $u \in SAP_\omega([0, \infty), L^2(\Omega; \mathbb{H}))$,
\[
(\Gamma_1u)(t + \omega) - (\Gamma_1u)(t) = (U(t + \omega) - U(t))u_0 + \int_0^t (t-s)^{\alpha-1}V(t-s)f(s, u(s))ds,
\] (27)
and if $f: [0, \infty) \times L^2(\Omega; \mathbb{H}) \rightarrow L^2(\Omega; \mathbb{H})$ satisfies.

(H1) For $x$ in every bounded subset $K$ of $L^2(\Omega; \mathbb{H})$, $f(t, x)$ is $S$-asymptotically $a$-periodic in $t$. Moreover, there exists a positive constant $L$ such that
\[ E\left\|f(t, y) - f(t, z)\right\|^2 \leq LE\|y - z\|^2, \] (28)
for $t \in [0, \infty)$ and $y, z \in L^2(\Omega; \mathbb{H})$.

Then, $(\Gamma_1u) \in SAP_\omega([0, \infty), L^2(\Omega; \mathbb{H}))$.

Proof
Step 1: for $\forall u(t) \in SAP_\omega([0, \infty), L^2(\Omega; \mathbb{H}))$, we prove that
\[
\lim_{t \rightarrow \infty} E\|(\Gamma_1u)(t + \omega) - (\Gamma_1u)(t)\|^2 = 0. \] (29)

Firstly, we have
\[
(\Gamma_1u)(t + \omega) - (\Gamma_1u)(t) = (U(t + \omega) - U(t))u_0 + \int_0^t (t-s)^{\alpha-1}V(t-s)f(s, u(s))ds \]
(30)

Then,
\[
E\|(\Gamma_1u)(t + \omega) - (\Gamma_1u)(t)\|^2 \leq 4E\left\|U(t + \omega) - U(t)\right\|u_0\|^2
+ 4E\left\|\int_0^t (t-s)^{\alpha-1}V(t-s)f(s, u(s))ds\right\|^2
+ 4E\left\|\int_0^t (t-s)^{\alpha-1}V(t-s)f(s, u(s))ds\right\|^2
+ 4E\left\|\int_0^t (t-s)^{\alpha-1}V(t-s)f(s, u(s))ds\right\|^2
=: I_1(t) + I_2(t) + I_3(t) + I_4(t).
\] (31)
In view of Lemma 2, it is obvious that
\[
\lim_{t \to \infty} I_1(t) = 0. \tag{32}
\]
By combining Hölder inequality with \((H_1)\) and using Lemma 2, we have
\[
I_2(t) \leq 4 \int_{-\infty}^{0} (t - s)^{\alpha - 1} \|V(t - s)\|ds \int_{-\infty}^{0} (t - s)^{\alpha - 1} \|V(t - s)\|E \|f(s + \omega, u(s + \omega))\|^2 ds
\]
\[
\leq 8 \left( L \|u\|^2_{\infty} + \sup_{t \geq 0} E \|f(t, 0)\|^2 \right) \left( \int_{-\infty}^{0} (t - s)^{\alpha - 1} \|V(t - s)\| ds \right)^2
\]
\[
\leq 8M^2 \left( L \|u\|^2_{\infty} + \sup_{t \geq 0} E \|f(t, 0)\|^2 \right) \left( \int_{-\infty}^{0} (t - s)^{\alpha - 1} E_{\alpha,a}(-\delta(t - s))^a ds \right)^2
\]
\[
= \frac{8M^2}{\delta^2} \left( L \|u\|^2_{\infty} + \sup_{t \geq 0} E \|f(t, 0)\|^2 \right) \left( E_{\alpha}(-\delta^a) - E_{\alpha}(-\delta(t + \omega)^a) \right)^2.
\]

The last formula and Lemma 1 yield that
\[
\lim_{t \to \infty} I_2(t) = 0. \tag{34}
\]
Since \((H_1)\) implies \( \lim_{t \to \infty} E \|f(t + \omega, u(t + \omega)) - f(t, u(t + \omega))\|^2 = 0 \), then for \( \varepsilon > 0 \), there exists \( T_\varepsilon > 0 \) such that
\[ E \|f(t + \omega, u(t + \omega)) - f(t, u(t + \omega))\|^2 \leq \varepsilon \] whenever \( t > T_\varepsilon \). Then, we have
\[
I_3(4) \leq 4 \int_{0}^{t} (t - s)^{\alpha - 1} \|V(t - s)\|ds \int_{0}^{t} (t - s)^{\alpha - 1} \|V(t - s)\|E \|f(s + \omega, u(s + \omega)) - f(s, u(s + \omega))\|^2 ds
\]
\[
\leq 4M^2 \int_{0}^{t} (t - s)^{\alpha - 1} E_{\alpha,a}(-\delta(t - s)^a) ds \int_{0}^{t} (t - s)^{\alpha - 1} E_{\alpha,a}(-\delta(t - s)^a) \cdot E \|f(s + \omega, u(s + \omega)) - f(s, u(s + \omega))\|^2 ds
\]
\[
\leq \frac{4M^2}{\delta^2} \left( 1 - E_{\alpha}(-\delta^a) \right) \left( \int_{0}^{T_\varepsilon} (t - s)^{\alpha - 1} E_{\alpha,a}(-\delta(t - s)^a) \cdot E \|f(s + \omega, u(s + \omega)) - f(s, u(s + \omega))\|^2 ds
\]
\[
+ \int_{T_\varepsilon}^{t} (t - s)^{\alpha - 1} E_{\alpha,a}(-\delta(t - s)^a) E \|f(s + \omega, u(s + \omega)) - f(s, u(s + \omega))\|^2 ds \right)
\]
\[
\leq \frac{4M^2}{\delta^2} \left( 1 - E_{\alpha}(-\delta^a) \right) \left( 4 \left( L \|u\|^2_{\infty} + \sup_{t \geq 0} E \|f(t, 0)\|^2 \right) \left( E_{\alpha}(-\delta(t - T_\varepsilon)^a) - E_{\alpha}(-\delta^a) \right) + \varepsilon \left( 1 - E_{\alpha}(-\delta(t + T_\varepsilon)^a) \right) \right).
\]
(35)

Due to Lemma 1, it is obvious that
\[
\lim_{t \to \infty} I_3(t) = 0. \tag{36}
\]

\[
I_4(4) \leq 4L \int_{0}^{t} (t - s)^{\alpha - 1} \|V(t - s)\|ds \int_{0}^{t} (t - s)^{\alpha - 1} \|V(t - s)\|E \|u(s + \omega) - u(s)\|^2 ds.
\]
(37)
Using a strategy similar to the one in the proof of (36), we get

$$\lim_{t \to \infty} I_4(t) = 0. \quad (38)$$

Then, \( \lim_{t \to \infty} E\| (\Gamma_1 u) (t + \omega) - (\Gamma_1 u) (t) \|^2 = 0 \).

Step 2: For \( \forall u(t) \in SAP_\omega([0, \infty), L^2(\Omega; \mathbb{H})) \), we prove that \( (\Gamma_1 u)(t) \) is stochastically bounded and continuous. On the one hand, for a given \( t_0 \geq 0 \), we have

\[
E\| (\Gamma_1 u) (t) - (\Gamma_1 u) (t_0) \|^2 \leq 3E\| U(t) - U(t_0) \| u_0^2 + 3E \int_{t_0}^{t} (t_0 - s)^{a-1} V(t_0 - s) f(s + t - t_0, u(s + t - t_0)) ds \leq 3E \int_{0}^{t_0} (t_0 - s)^{a-1} V(t_0 - s) f(s + t - t_0, u(s + t - t_0)) ds \leq 3E \int_{0}^{t_0} (t_0 - s)^{a-1} V(t_0 - s) f(s + t - t_0, u(s + t - t_0)) ds + 3E \int_{0}^{t_0} (t_0 - s)^{a-1} V(t_0 - s) f(s + t - t_0, u(s + t - t_0)) ds.
\]

which means that \( \lim_{t \to t_0} K_2(t) = 0 \).

Lemma 2 (1) implies that \( \lim_{t \to t_0} K_1(t) = 0 \). Arguing similarly as in (33), we see that

\[
K_2(t) \leq \frac{6M^2}{\delta^2} \left( LE\|a\|_\infty + \sup_{t \geq 0} E\| f(t, 0) \|^2 \right) \cdot (E_a(\delta t_n^a) - E_a(\delta t^a))^2,
\]

\[
K_3(t) \leq 6M^2 \int_{0}^{t_0} (t_0 - s)^{a-1} E_{a,a}(\delta(t_0 - s)^a) ds \int_{0}^{t_0} (t_0 - s)^{a-1} E_{a,a}(\delta(t_0 - s)^a) \left( E\| f(s + t - t_0, u(s + t - t_0)) - f(s, u(s + t - t_0)) \|^2 + E\| f(s, u(s + t - t_0)) - f(s, u(s)) \|^2 \right) ds \leq \frac{6M^2}{\delta^2} \left( 1 - E_a(\delta t_n^a) \right) \int_{0}^{t_0} (t_0 - s)^{a-1} E_{a,a}(\delta(t_0 - s)^a) \left( E\| f(s + t - t_0, u(s + t - t_0)) - f(s, u(s + t - t_0)) \|^2 + LE\| u(s + t - t_0) - u(s) \|^2 \right) ds.
\]

For an arbitrary sequence of real numbers \( \{t_n\} \) with \( t_n \to t_0 \) as \( n \to \infty \), for \( \forall u(t) \in SAP_\omega([0, \infty), L^2(\Omega; \mathbb{H})) \), we have

\[
E\| f(s + t_n - t_0, u(s + t_n - t_0)) - f(s, u(s + t_n - t_0)) \|^2 + LE\| u(s + t_n - t_0) - u(s) \|^2 \to 0, \quad \text{as } n \to \infty,
\]

which is due to \((H_1)\). Hence,

\[
E\| f(s + t_n - t_0, u(s + t_n - t_0)) - f(s, u(s + t_n - t_0)) \|^2 + LE\| u(s + t_n - t_0) - u(s) \|^2 < 1,
\]

for every \( n \) sufficiently large. In view of

\[
\int_{0}^{t_0} (t_0 - s)^{a-1} E_{a,a}(\delta(t_0 - s)^a) ds = \frac{1}{\delta} \left( 1 - E_a(\delta t_n^a) \right) < \frac{1}{\delta} < \infty,
\]

(41)

(42)
then it follows from Lebesgue’s dominated convergence theorem that

\[
\lim_{n \to \infty} \int_0^{t_0} (t_0 - s)^{a-1} E_{t_0, t_0} (-\delta(t_0 - s)^a) \left( E\|f(s + t_n - t_0, u(s + t_n - t_0)) - f(s, u(s + t_n - t_0))\|^2 + LE\|u(s + t_n - t_0) - u(s)\|^2 \right) ds = 0.
\]

(45)

Additionally, according to the arbitrariness of \(t_n\), we have that

\[
\lim_{t \to t_0} \int_0^{t_0} (t_0 - s)^{a-1} E_{t_0, t_0} (-\delta(t_0 - s)^a) \left( E\|f(s + t - t_0, u(s + t - t_0)) - f(s, u(s + t - t_0))\|^2 + LE\|u(s + t - t_0) - u(s)\|^2 \right) ds = 0,
\]

(46)

which gives \(\lim_{t \to t_0^+} K_\delta(t) = 0\). Then, we know that \((\Gamma_1 u)(t)\) is stochastically continuous.

On the other hand,

\[
E\|\Gamma_1 u\| \leq 2E\|U(t)u\|^2 + 2E\int_0^t (t - s)^{a-1} V(t - s)f(s, u(s)) ds
\]

\[
\leq 2M^2 E\|u\|^2 + 2M^2 \int_0^t (t - s)^{a-1} E_{s, s} (-\delta(t - s)^a) ds \cdot \int_0^t (t - s)^{a-1} E_{s, s} (-\delta(t - s)^a) E\|f(s, u(s))\|^2 ds
\]

\[
\leq 2M^2 E\|u\|^2 + \frac{4M^2}{\delta^2} (1 - E_{s, s} (-\delta^a)) \left( LE\|u\|^2 + \sup_{t > 0} E\|f(t, 0)\|^2 \right)
\]

\[
\leq 2M^2 E\|u\|^2 + \frac{4M^2}{\delta^2} \left( LE\|u\|^2 + \sup_{t > 0} E\|f(t, 0)\|^2 \right) < \infty,
\]

which implies that \((\Gamma_1 u)(t)\) is stochastically bounded.

By Steps 1-2, we obtain \(\Gamma_1 u \in SAP(\Omega; \mathbb{H})\).

Lemma 6. Let \(u \in SAP(\Omega; \mathbb{H})\) and

\[
(\Gamma_1 u)(t) = \int_0^t (t - s)^{a-1} V(t - s)D(s, u(s)) dB^H_t (s).
\]

(48)

If \(D: [0, \infty) \times L^2(\Omega; \mathbb{H}) \to L^0^2(\mathbb{H})\) satisfies

\((H_4)\) For \(x\) in every bounded subset \(K\) of \(L^2(\Omega; \mathbb{H})\), \(D(t, x)\) is stochastically bounded and continuous in \(t\). Furthermore, for \(\forall \varepsilon > 0\) and \(K\), there exists \(T > 0\) such that \(t^{2(a+H-1)/2} \|D(t + \omega, y) - D(t, y)\|_2^2 < \varepsilon\) for \(t > T\) and \(y \in K\).

\((H_5)\) There exists a positive constant \(L_1\) such that

\[
\lim_{t \to \infty} E\|(\Gamma_1 u)(t + \omega) - (\Gamma_1 u)(t)\|^2 = 0.
\]

(50)

Proof

Step 1: for \(\forall u(t) \in SAP(\Omega; \mathbb{H})\), we prove that

\[
\lim_{t \to \infty} E\|(\Gamma_1 u)(t + \omega) - (\Gamma_1 u)(t)\|^2 = 0.
\]

(50)

Let \(B^H(\tau) = B^H(\tau + \omega) - B^H(\tau)\) for each \(\tau \in \mathbb{R}\). Then, it is easy to find that \(B^H(\tau)\) is identically distributed like \(B^H(\tau)\). Next, for \(u \in SAP(\Omega; \mathbb{H})\), we see that
\[
(\Gamma_2 u) (t + \omega) - (\Gamma_2 u) (t) = \int_0^{t+\omega} (t + \omega - s)^{a-1} V (t + \omega - s) D(s, u(s)) d\beta^H_Q(s)
- \int_0^{t} (t - s)^{a-1} V (t - s) D(s, u(s)) d\beta^H_Q(s)
= \int_0^{t} (t - \tau)^{a-1} V (t - \tau) D(\tau + \omega, u(\tau + \omega)) d\beta^H_Q(\tau + \omega)
- \int_0^{t} (t - s)^{a-1} V (t - s) D(s, u(s)) d\beta^H_Q(s)
= \int_{-\omega}^{t} (t - \tau)^{a-1} V (t - \tau) D(\tau + \omega, u(\tau + \omega)) d\beta^H_Q(\tau)
- \int_0^{t} (t - s)^{a-1} V (t - s) D(s, u(s)) d\beta^H_Q(s). \tag{51}
\]

Then,
\[
E\left\| (\Gamma_2 u) (t + \omega) - (\Gamma_2 u) (t) \right\|^2 \leq 2E\left\| \int_{-\omega}^{0} (t - s)^{a-1} V (t - s) D(s + \omega, u(s + \omega)) d\beta^H_Q(s) \right\|^2
+ 2E\left\| \int_{0}^{t} (t - s)^{a-1} V (t - s) (D(s + \omega, u(s + \omega)) - D(s, u(s))) d\beta^H_Q(s) \right\|^2
=: 2I(t) + 2J(t). \tag{52}
\]

Moreover, \((H_3) - (H_4)\), Lemmas 2 and 4 yield that
\[
I(t) \leq 2Hw^{2H-1} \int_{-\omega}^{0} (t - s)^{2a-2} \|V(t - s)\|^2 \|D(s + \omega, u(s + \omega))\|^2_{L_2} ds
\leq \frac{2Hw^{2H-1}M^2\alpha^2}{\delta^2} \int_{-\omega}^{0} (t - s)^{-2} \|D(s + \omega, u(s + \omega))\|^2_{L_2} ds
= \frac{2Hw^{2H-1}M^2\alpha^2}{\delta^2 t^2} \int_{0}^{\omega} \|D(s, u(s))\|^2_{L_2} ds \tag{53}
\leq \frac{4Hw^{2H-1}M^2\alpha^2}{\delta^2 t^2} \left( \int_{0}^{\omega} \left( \|D(s, 0)\|^2_{L_2} + L_1 s^{2(1-H-a)} E\|u(s)\|^2 \right) ds \right)
\leq \frac{4Hw^{2H-1}M^2\alpha^2}{\delta^2 t^2} \left( \omega \sup_{t \geq 0} \|D(t, 0)\|^2_{L_2} + \frac{L_1 \omega^{3-2H-2a}}{3 - 2H - 2a} \|u\|^2_{L_2} \right).
\]

Then,
\[
\lim_{t \to \infty} I(t) = 0, \tag{54}
\]

\[
J(t) \leq 4HT^{2H-1}_\epsilon \int_{0}^{T_\epsilon} (t - s)^{2a-2} \|V(t - s)\|^2 \|D(s + \omega, u(s + \omega)) - D(s, u(s))\|^2_{L_2} ds
+ 4H (T_\epsilon - T_\epsilon^{2H-1}) \int_{T_\epsilon}^{T_\epsilon} (t - s)^{2a-2} \|V(t - s)\|^2 \|D(s + \omega, u(s + \omega)) - D(s, u(s))\|^2_{L_2} ds
= 4HT^{2H-1}_\epsilon J_1(t) + 4HJ_2(t), \tag{55}
\]
where \( \lim_{t \to -\infty} J_1(t) = 0 \) can be gotten similarly to (53). Moreover, without loss of generality, we may suppose for \( \forall \varepsilon > 0, \ E\|u(t + \omega) - u(t)\|^2 < \varepsilon \), for \( t > T_\varepsilon \), owing to \( u \in \text{SAP}_\omega([0, \infty), L^2(\Omega; \mathbb{H})) \):

\[
J_2(t) \leq 2(t - T_\varepsilon)^{2H-1} \int_{T_\varepsilon}^{t} (t - s)^{2\alpha - 2} \|V(t - s)\|^2 \left( \|D(s + \omega, u(s + \omega)) - D(s, u(s))\|^2 \right) ds
\]

\[
\leq 2M^2(t - T_\varepsilon)^{2H-1} \int_{T_\varepsilon}^{t} (t - s)^{2\alpha - 2} \left( E_{\alpha,a}(-\delta(t - s)^a) \right)^2 \left( \|D(s + \omega, u(s + \omega)) - D(s, u(s))\|^2 \right) ds
\]

\[
+ \|D(s, u(s + \omega)) - D(s, u(s))\|^2 |_{\mathbb{H}}
\]

\[
= \frac{2M^2 \epsilon(1 + L_1)}{(\Gamma(\alpha))^2} (t - T_\varepsilon)^{2H-1} \int_{0}^{t - T_\varepsilon} \mu^{2\alpha - 2}(1 - \mu)^{2(1 - a - H)} d\mu
\]

\[
= \frac{2M^2 \epsilon(1 + L_1)B(2\alpha - 1, 3 - 2(\alpha + H))}{(\Gamma(\alpha))^2} \varepsilon,
\]

where \( B \) is the beta function. Thus, \( \lim_{t \to -\infty} J_2(t) = 0 \). Therefore, \( \lim_{t \to -\infty} E\|\Gamma_2(u)(t + \omega) - (\Gamma_2(u)(t))\|^2 = 0. \)

Step 2: For \( \forall u \in \text{SAP}_\omega([0, \infty), L^2(\Omega; \mathbb{H})) \), we prove that \((\Gamma_2 u)(t)\) is stochastically bounded and continuous.

For a given number \( t_0 \geq 0 \) and \( t > t_0 \), we get

\[
E\|\Gamma_2(u)(t) - (\Gamma_2(u)(t_0))\|^2 \leq 2E\int_{t_0}^{t} \|V(t - s)^{2\alpha - 1} D(s + t - t_0, u(s + t - t_0)) d\mathcal{B}_Q^H(s)\|^2
\]

\[
+ 2E\int_{t_0}^{t} (t - s)^{2\alpha - 1} D(s + t - t_0, u(s + t - t_0)) \|D(s, u(s))\| d\mathcal{B}_Q^H(s)
\]

\[
:= 2N_1(t) + 2N_2(t).
\]

It follows from \((H_3)\) and Lemmas 2 and 4 that

\[
N_1(t) \leq \frac{2HM^2}{(\Gamma(\alpha))^2} (t - t_0)^{2H - 1} \int_{t_0}^{t} (t - s)^{2\alpha - 2} \|D(s + t - t_0, u(s + t - t_0))\|^2 ds
\]

\[
\leq \frac{2HM^2}{(\Gamma(\alpha))^2} (t - t_0)^{2H - 1} \int_{0}^{t - t_0} (t - s)^{2\alpha - 2} \|D(s, u(s))\|^2 ds
\]

\[
\leq \frac{4HM^2}{(\Gamma(\alpha))^2} (t - t_0)^{2H - 1} \int_{0}^{t - t_0} (t - s)^{2\alpha - 2} ds \sup_{t \geq 0} \|D(t, 0)\|^2 |_{\mathbb{H}}
\]

\[
+ \frac{4HM^2 L_1}{(\Gamma(\alpha))^2} (t - t_0)^{2H - 1} \int_{0}^{t - t_0} (t - s)^{2\alpha - 2} s^{2(1 - H - a)} ds \|u\|^2 |_{\mathbb{H}}
\]

\[
\leq \frac{4HM^2 \sup_{t \geq 0} \|D(t, 0)\|^2 |_{\mathbb{H}}^2}{(\Gamma(\alpha))^2} (t - t_0)^{2H - 1} \left( (t - t_0)^{2\alpha - 1} - t_0^{2\alpha - 1} \right)
\]

\[
+ \frac{4HM^2 L_1 \|u\|^2 |_{\mathbb{H}}^2}{(\Gamma(\alpha))^2 (2\alpha - 1)} (t - t_0)2\alpha - 2a.
\]
This means \( \lim_{t \to t_0} N_1(t) = 0 \). Next, we find that

\[
N_2(t) \leq 2HM^{2H-1} \int_0^t (t-s)^{2\alpha-2} \left( E_{a,a} (-\delta(t_0-s)^a) \right)^2 \| D(s+t-t_0, u(s+t-t_0)) - D(s,u(s)) \|^2_{L^2} ds
\]

\[
\leq 4HM^{2H-1} \int_0^t (t-s)^{2\alpha-2} \left( E_{a,a} (-\delta(t_0-s)^a) \right)^2 \left( \| D(s+t-t_0, u(s+t-t_0)) - D(s,u(s+t-t_0)) \|^2_{L^2} + L_1 s^{2(1-H-a)} \| u(s+t-t_0) - u(s) \|^2 \right) ds.
\]

Note that \( N_2(t) = 0 \) for \( t_0 = 0 \), and for \( t_0 > 0 \), we have

\[
\int_0^t (t-s)^{2\alpha-2} \left( E_{a,a} (-\delta(t_0-s)^a) \right)^2 ds = \int_0^{t/2} r^{2\alpha-2} \left( E_{a,a} (-\delta r^a) \right)^2 dr + \int_{t/2}^t r^{2\alpha-2} \left( E_{a,a} (-\delta r^a) \right)^2 dr
\]

\[
\leq \frac{1}{(\Gamma(\alpha))^2} \int_0^{t/2} s^{2\alpha-2} ds + \left( \frac{t_0}{2} \right)^{2\alpha-2} \int_{t/2}^t r^{2\alpha-2} \left( \frac{1}{\delta^2} \right)^2 dr
\]

\[
\leq \frac{1}{(\Gamma(\alpha))^2} \frac{1}{2\alpha-1} \left( \frac{t_0}{2} \right)^{2\alpha-1} \frac{1}{\delta^2} \left( 2\alpha-1 \right) \left( \frac{t_0}{2} \right)^{2\alpha-2} + \frac{2 - 2^{2\alpha-2}}{\delta^2 (2\alpha-1) t_0} < \infty,
\]

\[
t_0^{2H-1} \int_0^t (t-s)^{2\alpha-2} s^{2(1-a-H)} ds = B(2\alpha-1, 3-2\alpha-2H).
\]

Then, by a similar argument to that used in (46), we deduce \( \lim_{t \to t_0} N_2(t) = 0 \). Moreover, we apply \((H_1) - (H_4)\) and Lemma 2 and use Lemma 4 to conclude that

\[
E \| (\Gamma_2 u)(t) \| \leq 2HM^{2H-1} \int_0^t (t-s)^{2\alpha-2} \| V(t-s) \|^2 \| D(s,u(s)) \|^2_{L^2} ds
\]

\[
\leq \frac{4M^2 H^{2H-1}}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} \left( \| D(s,0) \|^2_{L^2} + L_1 s^{2(1-H-a)} E \| u(s) \|^2 \right) ds
\]

\[
\leq \frac{4L_1 M^2 H \| u \|^2_{L^2}}{\Gamma(\alpha)^2} \int_0^t (1-\mu)^{2\alpha-2} \mu^{2(1-H-a)} ds
\]

\[
= \frac{4L_1 M^2 H \| u \|^2_{L^2} B(2\alpha-1, 3-2H-2\alpha)}{\Gamma(\alpha)^2}.
\]

Applying this and the above arguments, we conclude that \((\Gamma_2 u)(t)\) is stochastically bounded and continuous. By combining Steps 1 and 2, we obtain \( \Gamma_2 u \in \text{SAP}_\omega ([0,\infty), L^2(\Omega; H)). \)
Now we state our main results.

**Theorem 1.** If \((H_1)-(H_4)\) are satisfied, then equation (14) has a unique mild solution \(u \in \text{SAP}_\omega ([0, \infty), L^2 (\Omega; \mathbb{H}))\) provided with

\[
2M \left( \frac{2L_1 HB (2 \alpha - 1, 3 - 2 (\alpha + H))}{(\Gamma (\alpha))^2} + \frac{L}{\delta^2} \right) < 1. \tag{62}
\]

**Proof.** We define \(\Gamma\) by

\[
\Gamma [u] (t) = U (t) u_0 + \int_0^t (t-s)^{\alpha-1} V (t-s) f (s, u (s)) ds + \int_0^t (t-s)^{\alpha-1} V (t-s) D (s, u (s)) ds B^H_Q (s).
\]

It follows from Lemmas 5 and 6 that \(\Gamma u \in \text{SAP}_\omega ([0, \infty), L^2 (\Omega; \mathbb{H}))\). For all \(t \geq 0, u, v \in \text{SAP}_\omega ([0, \infty), L^2 (\Omega; \mathbb{H}))\), we get

\[
E[\| \Gamma u(t) - (\Gamma v) (t) \|^2] \leq 2E \left\| \int_0^t (t-s)^{\alpha-1} V (t-s) (f (s, u(s)) - f (s, v(s))) ds + \int_0^t (t-s)^{\alpha-1} V (t-s) (D(s, u(s)) - D(s, v(s))) ds B^H_Q (s) \right\|^2
\]

\[
\leq 2LM^2 \sup_{t \geq 0} \| u(t) - v(t) \|^2 \left( \int_0^t (t-s)^{\alpha-1} E_{\alpha,a} (-\delta (t-s)^a) ds \right)^2
\]

\[
= \frac{2M^2 L}{\delta^a} \| u - v \|^2_{\infty} \left( \int_0^t (t-s)^{\alpha-1} E_{\alpha,a} (-\delta (t-s)^a) ds \right)^2
\]

\[
\leq \frac{2M^2 L}{\delta^a} \| u - v \|^2_{\infty}.
\]

On the other hand, we get

\[
J (t) \leq 4M^2 H t^{2H-1} \int_0^t (t-s)^{2a-2} \left( E_{\alpha,a} (-\delta (t-s)^a) \right)^2 \cdot \| D(s, u(s)) - D(s, v(s)) \|^2_{L^2} ds
\]

\[
\leq \frac{4M^2 L_1 H t^{2H-1}}{(\Gamma (\alpha))^2} \int_0^t (t-s)^{2a-2} \frac{1}{s^{(1-a-H)}} ds \| u - v \|^2_{\infty}
\]

\[
= \frac{4M^2 L_1 H}{(\Gamma (\alpha))^2} \int_0^1 (1-r)^{2a-2} r^{(1-a-H)} dr \| u - v \|^2_{\infty}
\]

\[
= \frac{4M^2 L_1 HB (2 \alpha - 1, 3 - 2 (\alpha + H))}{(\Gamma (\alpha))^2} \| u - v \|^2_{\infty}.
\]
Thus,

$$
\| (\Gamma u) - (\Gamma v) \|_\infty^2 \leq 2M^2 \left( \frac{2L_1HB(2\alpha - 1, 3 - 2(\alpha + H))}{(\alpha)} + \frac{L}{\delta^2} \right) \| u - v \|_\infty^2. 
$$

(67)

If (62) holds, \( \Gamma \) is a contraction mapping. Using the Banach fixed-point theorem, we have that \( \Gamma \) has a unique fixed point in SAP\( \omega \). This completes the proof.

4. Asymptotic Stability of Square-Mean S-Asymptotically \( \omega \)-Periodic Solutions

Theorem 2. Assume that \((H_1)\) and \((H_2')\)

\[ D: [0, \infty) \rightarrow L^2(\Omega; \mathbb{H}) \]

is stochastically and continuous. Furthermore, for \( \forall \epsilon > 0 \), there exists \( T_* > 0 \) such that

\[ t^{2(\alpha+H-1)} \| D(t + \omega) - D(t) \|_{L^2}^2 < \epsilon \]

and

\[ \| D(t) \|_{L^2}^2 \leq t^{2(1-\alpha-H)} \]

for \( t > T \) hold. If \( \delta^2 > 2M^2L \), there exists a unique asymptotically stable S-asymptotically \( \omega \)-periodic solution \( u^\ast \) in square-mean sense to equation (14) with \( D(t, \cdot) = D(t) \).

Proof. From the proof process of Theorem 1, we get the existence and uniqueness of the S-asymptotically \( \omega \)-periodic solution \( u^\ast (t) \) similarly. In addition, for \( \forall u_1 \in L^2(\Omega; \mathbb{H}) \), equation (14) has a unique mild solution \( u(t) \) with the new initial value \( u(0) = u_1 \). And then from (17) and Lemmas 2 and 4, we have

\[
E \| u(t) - u^\ast (t) \|^2 \leq 2E \| U(t)u_1 - U(t)u_0 \|^2 
\]

\[ + 2E \int_0^t (t - s)^{\alpha-1} V(t - s)(f(s, u(s)) - f(s, u^\ast (s))) ds \]

\[ \leq 2M^2E \| u_1 - u_0 \|^2 + 2 \int_0^t (t - s)^{\alpha-1} \| V(t - s) \| ds \]

\[ \cdot \int_0^t (t - s)^{\alpha-1} \| V(t - s) \| \| f(s, u(s)) - f(s, u^\ast (s)) \|^2 ds \]

\[ \leq 2M^2E \| u_1 - u_0 \|^2 + 2M^2L \int_0^t (t - s)^{\alpha-1} E \| u(s) - u^\ast (s) \|^2 ds \]

\[ - E \| u_1 - u_0 \|^2 + \frac{2M^2L}{\delta} \int_0^t (t - s)^{\alpha-1} E \| u(s) - u^\ast (s) \|^2 ds \]

\[ = E \left( \frac{2M^2L}{\delta} - \delta \right) t^\alpha \frac{2M^2E \| u_1 - u_0 \|^2}{\delta^2} \cdot t^{2(\alpha+H-1)} \| D(t) \|_{L^2}^2 \leq t^{2(1-\alpha-H)} \]

(68)

If \( \delta^2 > 2M^2L \), by Lemma 1, we obtain that the square-mean S-asymptotically \( \omega \)-periodic solution \( u^\ast \) to equation (14) with \( D(t, \cdot) = D(t) \) is asymptotically stable in square-mean sense.

Now the proof could be finished.

5. Numerical Simulation

Example 1. We consider the following fractional stochastic equation with FBM:

\[ D_0^{0.85} u(t) + 4u(t) = \sin u(t) + \sin \sqrt{t} + \sin \frac{1}{t} Q(t) dt, \quad t \in (0, +\infty), \]

\[ u(0) = 0.22, \]

(69)
where $D_0^{0.85}$ is the Caputo fractional derivative and $B_{Q}^{0.85}(t)$ is one-dimensional FBM with $H = 0.6$ (Figure 1).

We notice that $-4$ generates an exponentially stable semigroup $\{e^{-4t}\}_{t \geq 0}$, $\delta = 4$, and $M = 1$. Set $f(t, u(t)) = \sin u(t) + \sin \sqrt{t}$ and $D(t) = \sin (1/t)$. Then, by the Lagrange differential mean value theorems, we obtain that $(H_1)$ and $(H_2)$ are satisfied with $L = 1$. It is easy to see that $\delta^2 > 2M^2L$. From Theorem 2, we conclude that equation (69) has a unique square-mean $S$-asymptotically $\omega$-periodic solution (Figure 2), which is asymptotically stable in square-mean sense.

From the above example, we find that although there is no periodic solution for the fractional-order differential equation with finite lower limit [2], the $S$-asymptotically periodic solution can be found and is stable even for fractional stochastic differential equation.

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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