Capacity of a Class of State-Dependent Relay Channels

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Abstract

The class of orthogonal relay channels, in which the orthogonal channels connecting the source terminal to the relay and the destination depend on a state sequence, is considered. It is assumed that the state sequence is fully known at the destination while it is not known at the source or the relay. The capacity of this class of relay channels is characterized, and shown to be achieved by the partial decode-compress-and-forward (pDCF) scheme. Then the capacity of certain binary and Gaussian state-dependent orthogonal relay channels are studied in detail, and it is shown that the compress-and-forward (CF) and partial-decode-and-forward (pDF) schemes are suboptimal in general. To the best of our knowledge, this is the first single relay channel model for which the capacity is achieved by pDCF, while pDF and CF schemes are both suboptimal. Furthermore, it is shown that the capacity of the considered orthogonal state-dependent relay channels is in general below the cut-set bound. The conditions under which pDF or CF suffices to meet the cut-set bound, and hence, achieve the capacity, are also derived.

Index Terms

Capacity, channels with state, relay channel, decode-and-forward, compress-and-forward, partial decode-compress-and forward.

I. INTRODUCTION

We consider a state-dependent orthogonal relay channel, in which the channels connecting the source to the relay and the destination are orthogonal, and are governed by a state sequence, which is assumed to be known only at the destination. We call this model the state-dependent orthogonal relay channel with state information available at the destination, and refer to it as the ORC-D model. See Figure 1 for an illustration of the ORC-D channel model.

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Many practical communication scenarios can be modelled by the ORC-D model. For example, consider a cognitive network with a relay, in which the transmit signal of the secondary user interferes simultaneously with the received primary user signals at both the relay and the destination. After decoding the secondary user message, the destination obtains information about the interference affecting the source-relay channel, which can be exploited to decode the primary transmitter’s message, which may not be decoded at the relay. Similarly, consider a mobile network with a relay (e.g., a femtostation), in which the base station (BS) operates in the full-duplex mode, and transmits on the downlink channel to a user, in parallel to the uplink transmission of a femtocell user, causing interference for the first user’s transmission at the femtostation. While the relay has no prior information about this interfering signal, the BS already knows it (if decoding of the secondary user’s message is successful), which can be used to decode the primary user’s message.

The best known transmission strategies for the three terminal relay channel are the decode-and-forward (DF), compress-and-forward (CF) and partial decode-compress-and-forward (pDCF) schemes, which were all introduced by Cover and El Gamal in [2]. In DF the relay decodes the source message and forwards it to the destination together with the source terminal. DF is generalized by the partial decode-and-forward (pDF) scheme in which the relay decodes and forwards only a part of the message. In the ORC-D model, pDF would be optimal when the channel state information is not available at the destination [3]; however, when the state information is known at the destination, fully decoding and re-encoding the message transmitted on the source-relay link renders the channel state information at the destination useless. Hence, we expect that pDF is suboptimal for ORC-D in general.

In CF, the relay does not decode any part of the message, and simply compresses the received signal and forwards the compressed bits to the destination using Wyner-Ziv coding followed by separate channel coding. Using CF in the ORC-D model allows the destination exploit its knowledge of the state sequence; and hence, it can decode messages that may not be decodable by the relay. However, CF also forwards some noise to the destination, and therefore, may be suboptimal in certain scenarios. For example, as the dependence of the source-relay channel on the state sequence weakens, i.e., when the state information becomes less informative, CF performance is expected to degrade.

pDCF combines both schemes: part of the source message is decoded by the relay, and forwarded, while the remaining signal is compressed and forwarded to the destination. Hence, pDCF can optimally adapt its transmission to the dependence of the orthogonal channels on the state sequence. Indeed, we show that pDFC achieves the capacity in the ORC-D channel model, while pure DF and CF are in general suboptimal. The main results of the paper are summarized as follows:
• We derive an upper bound on the capacity of the ORC-D model, and show that it is achievable by the pDCF scheme. This characterizes the capacity of this class of relay channels.

• Focusing on the multi-hop binary and Gaussian models, we show that applying either only the CF or only the DF scheme is in general suboptimal.

• We show that the capacity of the ORC-D model is in general below the cut-set bound. We identify the conditions under which pure DF or pure CF meets the cut-set bound. Under these conditions the cut-set bounds is tight, and either DF or CF scheme is sufficient to achieve the capacity.

While the capacity of the general relay channel is still an open problem, there have been significant achievements within the last decade in understanding the capabilities of various transmission schemes, and the capacity of some classes of relay channels has been characterized. For example, DF is shown to be optimal for physically degraded relay channels and inversely degraded relay channels in [2]. In [3], the capacity of the orthogonal relay channel is characterized, and shown to be achieved by the pDF scheme. It is shown in [4] that pDF achieves the capacity of semi-deterministic relay channels as well. CF is shown to achieve the capacity in deterministic primitive relay channels in [5]. While all of these capacity results are obtained by using the cut-set bound for the converse proof [6], the capacity of a class of modulo-sum relay channels is characterized in [7], and it is shown that the capacity, achievable by the CF scheme, can be below the cut-set bound. The pDCF scheme is shown to achieve the capacity of a class of diamond relay channels in [8].

The state-dependent relay channel has recently attracted considerable attention in the literature. Key to the investigation of the state-dependent relay channel model is whether the state sequence controlling the channel is known at the nodes of the network, the source, relay or the destination in a causal or non-causal manner. The relay channel in which the state information is non-causally available only at the source is considered in [9], [10], and both causally and non-causally available state information is considered in [11]. The model in which the state is non-causally known only at the relay is studied in [12] while causal and non-causal knowledge is considered in [13]. Similarly, the relay channel with state causally known at source and relay is considered in [14] and state non-causally known at source, relay and destination in [15]. The compound relay channel with informed relay and destination are discussed in [16] and [17]. The state-dependent relay channel with structured state has been considered in [18] and [19]. To the best of our knowledge, this is the first work that focuses on the state-dependent relay channel in which the state information is available only at the destination.

The rest of the paper is organized as follows. In Section II we provide the system model and our main result. Section III is devoted to the proof of the achievability and converse of the main result. In section IV, we provide
two examples showing the suboptimality of pDF and CF schemes, while in Section V we show that the capacity is in general below the cut-set bound, and we provide conditions under which pure DF and CF schemes meet the cut-set bound. Finally, Section VII concludes the paper.

We use the following notation in the rest of the paper: $X^j_i \triangleq (X_i, X_{i+1}, \ldots, X_j)$ for $i < j$, $X^n \triangleq (X_1, \ldots, X_n)$ for the complete sequence, $X_{n+1}^n \triangleq \emptyset$, and $Z^{n\setminus i} \triangleq (Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n)$.

II. System Model and Main Result

We consider the class of orthogonal relay channels depicted in Figure 1. The source and the relay are connected through a memoryless channel characterized by $p(y_R|x_1, z)$, while the source and the destination are connected through an orthogonal memoryless channel characterized by $p(y_2|x_2, z)$. Both memoryless channels depend on an independent and identically distributed (i.i.d.) state sequence $\{Z\}_{i=1}^n$, which is available at the destination. The relay and the destination are connected by a memoryless channel $p(y_1|x_R)$, which is independent of the state sequence $z^n$. The input and output alphabets are denoted by $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_R$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}_R$, and the state alphabet is denoted by $Z$.

Let $W$ be the message to be transmitted to the destination with the assistance of the relay. The message $W$ is assumed to be uniformly distributed over the set $\mathcal{W} = \{1, \ldots, N\}$. An $(M, n, \nu_n)$ code for this channel consists of an encoding function at the source:

$$f : \{1, \ldots, M\} \to \mathcal{X}_1^n \times \mathcal{X}_2^n,$$  

(1)

a set of encoding functions $\{f_{r,i}\}_{i=1}^n$ at the relay, whose output at time $i$ depends on the symbols it has received.
up to time \(i-1\):

\[
X_{Ri} = f_{r,i}(Y_{R1}, ..., Y_{R(i-1)}), \quad i = 1, ..., n,
\]

(2)

and a decoding function at the destination

\[
g : \mathcal{Y}_1^n \times \mathcal{Y}_2^n \times \mathcal{Z}^n \rightarrow \{1, ..., M\}.
\]

(3)

The probability of error, \(\nu_n\), is defined as

\[
\nu_n \triangleq \frac{1}{M} \sum_{w=1}^{M} \Pr\{g(Y_1^n, Y_2^n, Z^n) \neq w | W = w\}.
\]

(4)

The joint probability mass function (pmf) of the involved random variables over the set \(\mathcal{W} \times \mathcal{Z}^n \times \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{X}_R^n \times \mathcal{Y}_R^n \times \mathcal{Y}_1^n \times \mathcal{Y}_2^n\) is given by

\[
p(w, z^n, x_1^n, x_2^n, x_R^n, y_1^n, y_2^n) = p(w) \prod_{i=1}^{n} p(z_i)p(x_{1i}, x_{2i}|w). \\
p(y_{Ri}|z_i, x_{1i})p(x_{Ri}|y_{Ri}^{i-1})p(y_{1i}|x_{Ri})p(y_{2i}|x_{2i}, z_i).
\]

A rate \(R\) is said to be achievable if there exists a sequence of \((2^{nR}, n, \nu_n)\) codes such that \(\lim_{n \to \infty} \nu_n = 0\).

The capacity \(C\) of this class of state-dependent orthogonal relay channels, denoted as ORC-D, is defined as the supremum of the set of achievable rates.

We define \(R_0\) as the capacity of the link connecting the relay to the destination, and \(R_1\) as the capacity of the direct link connecting the source to the destination when the channel state sequence is available at the destination:

\[
R_0 \triangleq \max_{p(x_R)} I(X_R; Y_1), \quad R_1 \triangleq \max_{p(x_2)} I(X_2; Y_2|Z),
\]

(5)

and let \(p^*(x_R)\) and \(p^*(x_2)\) be the channel input distributions achieving \(R_0\) and \(R_1\), respectively.

Let us define \(\mathcal{P}\) as the set of all joint pmfs given by

\[
\mathcal{P} \triangleq \{p(u, x_1, z, y_R, \hat{y}_R) : p(u, x_1, z, y_R, \hat{y}_R) = p(u, x_1)p(z)p(y_R|x_1, z)p(\hat{y}_R|y_R, u)\},
\]

(6)

where \(U\) and \(\hat{Y}_R\) are auxiliary random variables defines over the alphabets \(\mathcal{U}\) and \(\hat{Y}_R\), respectively.

The main result of this work, provided in the next theorem, is the capacity of the class of relay channels described above.
Theorem 1. The capacity of the ORC-D relay channel is given by

$$C = \sup_P R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R | U Z),$$

subject to

$$R_0 \geq I(U; Y_R) + I(Y_R; \hat{Y}_R | U Z),$$

(7)

where $|\mathcal{U}| \leq |\mathcal{X}_1| + 3$ and $|\hat{Y}_R| \leq |\mathcal{U}| |Y_R| + 1$.

Proof: The achievability part of the theorem is proven in Section III-A, while the converse proof is given in Section III-B.

In the next section, we show that the capacity of this class of state-dependent relay channels is achieved by the pDCF scheme. To the best of our knowledge, this is the first single relay channel model for which the capacity is achieved by pDCF, while the partial decode-and-forward (pDF) and compress-and-forward (CF) schemes are both suboptimal in general. In addition, the capacity of this relay channel is in general below the cut-set bound [6]. These issues are discussed in more detail in Sections IV and V.

III. PROOF OF THEOREM 1

We first show in Section III-A that the capacity region in Theorem 1 is achievable by pDCF. Then, we derive the converse for Theorem 1 in Section III-B.

A. Achievability

We derive the rate achievable by pDCF scheme for ORC-D using the achievable rate expression for the pDCF scheme proposed in [2] for the general relay channel. The discrete memoryless relay channel consists of four finite sets $\mathcal{X}$, $\mathcal{X}_R$, $\mathcal{Y}$ and $\mathcal{Y}_R$ and a set of probability distribution $p(y, y_R | x, x_R)$. In this setup, $X$ corresponds to the source input to the channel, $Y$ to the channel output available at the destination, while $Y_R$ is the channel output available at the relay, and $X_R$ is the channel input symbol chosen by the relay. We note that the three terminal relay channel in [2] reduces to the ORC-D-channel by setting $X^n = (X_1^n, X_2^n)$ and $Y^n = (Y_1^n, Y_2^n, Z^n)$, and

$$p(y, r | x_1, x_R) = p(y_1, y_2, y_R, z | x_1, x_2, x_R) = p(z) p(y_R | x_1, z) p(y_1 | x_R) p(y_2 | x_2).$$

In pDCF for the general relay channel, the source applies message splitting, and the relay decodes only a part of the message. The part to be decoded by the relay is transmitted through the auxiliary random variable $U^n$, while the rest of the message is superposed onto this through channel input $X^n$. Block Markov encoding is used for transmission. The relay receives $Y_R^n$ and decodes only the part of the message that is conveyed by $U^n$. The
remaining signal $Y^n_R$ is compressed into $\hat{Y}_R^n$. The decoded message is forwarded through $V^n$, which is correlated with $U^n$, and the compressed signal is superposed onto $V^n$ through the relay channel input $X^n_R$. At the destination the received signal $Y^n$ is used to recover the message. See [2] for details. The achievable rate of the pDCF scheme is given below.

**Theorem 2.** (Theorem 7, [2]) The capacity of a relay channel $p(y, y_R|x, x_R)$ is lower bounded by the following rate:

$$R_{pDCF} = \sup \min \{ I(X; Y, Y_R|X_R, U) + I(U; Y_R|X_R, V), I(X, X_R; Y) - I(\hat{Y}_R; Y_R|X, X_R, U, Y) \},$$

s.t. $I(\hat{Y}_R; Y|X, X_R, U) \leq I(X_R; Y|V)$, (8)

where the supremum is taken over all joint pmf’s of the form

$$p(v)p(u|v)p(x|u)p(x_1|v)p(y, y_R|x, x_R)p(\hat{y}_R|x_R, y_R, u).$$

Since ORC-D is a special case of the general relay channel model, the rate $R_{pDCF}$ is achievable in an ORC-D as well. The capacity achieving pDCF scheme for the state-dependent channel from (8) is obtained by setting $V = \emptyset$ and generating $X^n_R$ and $X^n_1$ independent of the rest of variables with distribution $p^*(x_R)$ and $p^*(x_1)$, respectively, as given in the next lemma.

**Lemma 1.** For the class of relay channels characterized by the ORC-D model, the capacity expression $C$ defined in [2] is achievable by the pDCF scheme.

**Proof:** See Appendix I. ■

The optimal pDCF scheme for ORC-D applies independent coding over the source-destination and the source-relay-destination branches. The source applies message splitting. Part of the message is transmitted over the source-destination branch and decoded at the destination using $Y^n_2$ and $Z^n$. In the relay branch, the part of the message to be decoded at the relay is transmitted through $U^n$, while the rest of the message is superposed onto this through the channel input $X^n_R$. At the relay the part conveyed by $U^n$ is decoded from $Y^n_R$, and the remaining signal $Y^n_R$ is compressed into $\hat{Y}_R^n$ using binning and assuming that $Z^n$ is available at the decoder. Both $U^n$ and the bin index corresponding to $\hat{Y}_R^n$ are transmitted over the relay-destination channel using $X^n_R$. At the destination, $X^n_R$ is decoded from $Y^n_1$, and $U^n$ and the bin index are recovered. Then, the decoder looks for the part of message transmitted over the relay branch jointly typical with $\hat{Y}_R^n$ within the corresponding bin and $Z^n$. 


B. Converse

The proof of the converse consists of two parts. First we derive a single-letter upper bound on the capacity, and then, using the single-letter expression of the upper bound we provide an alternative expression for this bound, which coincides with the rate achievable by pDCF.

Lemma 2. The capacity of the class of relay channels characterized by the ORC-D model is upper bounded by

\[
R_{up} = \sup_{P} \min \{ R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R|UZ), R_1 + R_0 - I(\hat{Y}_R; Y_R|X_1UZ) \}. \tag{9}
\]

Proof: See Appendix II.

As stated in the next lemma, the upper bound \( R_{up} \) given in Lemma 2 is equivalent to the capacity expression \( C \) given in Theorem 1. Since the achievable rate meets the upper bound, this concludes the proof of Theorem 1.

Lemma 3. The upper bound on the achievable rate \( R_{up} \) given in Lemma 2 is equivalent to the capacity expression \( C \) in Theorem 1.

Proof: See Appendix III.

IV. THE MULTIHOP RELAY CHANNEL WITH STATE: SUBOPTIMALITY OF PURE PDF AND CF SCHEMES

We have seen in Section III that the pDCF scheme is capacity-achieving for the class of relay channels characterized by the ORC-D model. In order to prove the suboptimality of the pure DF and CF schemes for this class of relay channels, we consider a simplified system model, called the multihop relay channel with state information available at the destination (MRC-D), which is obtained by simply removing the direct channel from the source to the destination, i.e., \( R_1 = 0 \).

The capacity of this multihop relay channel model and the optimality of pDCF follows directly from Theorem 1. However, the single-letter capacity expression depends on the joint pmf of \( X_1, Y_R, X_R \) and \( Y_1 \) together with the auxiliary random variables \( U \) and \( \hat{Y}_R \). Unfortunately, the numerical characterization of the optimal joint pmf of these random variables is very complicated for most channels. A simple and computable upper bound on the capacity can be obtained from the cut-set bound \[20\]. For MRC-D, the cut-set bound is given by

\[
R_{CS} = \min \{ R_0, \max_{p(x_1)} I(X_1; Y_R|Z) \}. \tag{10}
\]

Next, we characterize the rates achievable by the DF and CF schemes for MRC-D. Since they are special cases
Fig. 2. The parallel binary symmetric MRC-D with parallel source-relay links. The destination has side information about only one of the source-relay links.

of the pDCF scheme, their achievable rates can be obtained by particularizing the achievable rate of pDCF for this setup.

1) DF Scheme: If we consider a pDCF scheme that does not perform any compression at the relay, i.e., $\hat{Y}_R = \emptyset$, we obtain the rate achievable by the pDF scheme. Note that the optimal distributions of $X_R$ is given by $p^*(x_r)$. Then, we have

$$R_{pDF} = \min \{R_0, \sup_{p(x_1,u)} I(U;Y_R)\}. \quad (11)$$

From the Markov chain $U - X_1 - Y_R$, we have that $I(U;Y_R) \leq I(X_1;Y_R)$, where the equality is achieved by $U = X_1$. That is, the performance of pDF is maximized by letting the relay decode the whole message. Therefore, the maximum rate achievable by pDF and DF for MRC-D coincide, and is given by

$$R_{DF} = R_{pDF} = \min \{R_0, \max_{p(x_1)} I(X_1;Y_R)\}. \quad (12)$$

2) CF Scheme: If the pDCF scheme does not perform any decoding at the relay, i.e., $U = \emptyset$, pDCF reduces to CF. Then, the achievable rate for the CF scheme in MRC-D is easily seen to be given by

$$R_{CF} = \sup_{p(x_1)} I(X_1;\hat{Y}_R|Z)$$

s.t. $R_0 \geq I(\hat{Y}_R;Y_R|Z)$,

over $p(x_1)p(z)p(y_R|x_1,z)p(\hat{y}_R|y_R)$. \quad (13)

A. Multihop Parallel Binary Symmetric Channel

In this section we consider a special MRC-D as shown in Figure 2, which we call the parallel binary symmetric MRC-D. For this setup, we characterize the optimal performance of the DF and CF schemes, and show that in
general pDCF outperforms both, and that in some cases the cut-set bound is tight and coincides with the channel capacity. This example proves the suboptimality of both DF and CF on their own for the ORC-D.

In this scenario, the source-relay channel consists of two parallel binary symmetric channels. We have $X_1 = (X_1^1, X_1^2)$, $Y_R = (Y_R^1, Y_R^2)$ and $p(y_R|x_R,z) = p(y_R^1|x_R^1,z)p(y_R^2|x_R^2)$ characterized by

$$Y_R^1 = X_1^1 \oplus N_1 \oplus Z, \quad \text{and} \quad Y_R^2 = X_1^2 \oplus N_2,$$

where $N_1$ and $N_2$ are i.i.d. Bernoulli random variables with $\Pr\{N_1 = 1\} = \Pr\{N_2 = 1\} = \delta$, i.e., $N_1 \sim \text{Ber}(\delta)$ and $N_2 \sim \text{Ber}(\delta)$. We consider a Bernoulli distributed state $Z$, $Z \sim \text{Ber}(p_z)$, which affects one of the two parallel channels, and is available at the destination. We have $X_1^1 = Y_1^1 = Y_R^1 = N_1 = N_2 = Z = \{0, 1\}$.

From (10), the cut-set bound is given by

$$R_{CS} = \min \{ R_0, \max_{p(x_1^1 x_1^2)} I(X_1^1 X_1^2; Y_R^1 Y_R^2 | Z) \}$$

$$= \min \{ R_0, 2(1 - h_2(\delta)) \}, \quad (14)$$

where $h_2(\cdot)$ is the binary entropy function defined as $h_2(p) \triangleq -p \log p - (1-p) \log(1-p)$.

The maximum DF rate is achieved by $X_1^1 \sim \text{Ber}(1/2)$ and $X_1^2 \sim \text{Ber}(1/2)$, and is found to be

$$R_{DF} = \min \{ R_0, \max_{p(x_1^1 x_1^2)} I(X_1^1 X_1^2; Y_R^1 Y_R^2) \}$$

$$= \min \{ R_0, 2 - h_2(\delta \ast p_z) - h_2(\delta) \}, \quad (15)$$

where $\alpha \ast \beta \triangleq \alpha(1 - \beta) + (1 - \alpha)\beta$.

Following (13), the rate achievable by the CF scheme in the parallel binary symmetric MRC-D is given by

$$R_{CF} = \max I(X_1^1 X_1^2, \hat{Y}_R | Z),$$

s.t. $R_0 \geq I(Y_R^1 Y_R^2; \hat{Y}_R | Z)$

$$\text{over } p(z)p(x_1^1 x_1^2)p(y_R^1 | z, x_1^1)p(y_R^2 | x_2)p(\hat{y}_R | y_R^1 y_R^2). \quad (16)$$

Let us define $h_2^{-1}(q)$ as the inverse of the entropy function $h_2(p)$ for $q \geq 0$. For $q < 0$, we define $h_2^{-1}(q) = 0$.

As we show in the next lemma, the achievable CF rate in (16) is maximized by transmitting independent channel inputs over the two parallel links to the relay by setting $X_1^1 \sim \text{Ber}(1/2)$, $X_1^2 \sim \text{Ber}(1/2)$, and by independently compressing each of the channel outputs $Y_R^1$ and $Y_R^2$ as $\hat{Y}_R^1 = Y_R^1 \oplus Q_1$ and $\hat{Y}_R^2 = Y_R^2 \oplus Q_2$, respectively, where
Q₁ \sim \text{Ber}(h^{-1}_2(1 - R_0/2)) and Q₂ \sim \text{Ber}(h^{-1}_2(1 - R_0/2)). Note that for \( R_0 \geq 2 \), the channel outputs can be compressed errorlessly. The maximum achievable CF rate is given in the following lemma.

**Lemma 4.** The maximum rate achievable by CF in the parallel binary symmetric MRC-D is given by

\[
R_{CF} = 2 \left( 1 - h_2 \left( \delta \star h_2^{-1} \left( 1 - \frac{R_0}{2} \right) \right) \right).
\]  

**(Proof):** See Appendix [IV].

Now, we consider the pDCF scheme for the parallel binary symmetric MRC-D. Although we have not been able to characterize the optimal choice of \((U, \hat{Y}_R, X_1^1, X_2^1)\) in general, we provide an achievable scheme that outperforms both DF and CF schemes and meets the cut-set bound in some regimes. Let \( X_1^1 \sim \text{Ber}(1/2) \) and \( X_2^1 \sim \text{Ber}(1/2) \) and \( U = X_2^1 \), i.e., the relay decodes the channel input \( X_2^1 \), while \( Y_1^R \) is compressed using \( \hat{Y}_R = Y_1^R + Q \), where \( Q \sim \text{Ber}(h^{-1}_2(2 - h_2(\delta) - R_0)) \). The rate achievable by this scheme is given in the following lemma.

**Lemma 5.** A lower bound on the achievable pDCF rate in the parallel binary symmetric MRC-D is given by

\[
R_{pDCF} \geq \min \{R_0, 2 - h_2(\delta) - h_2 \left( \delta \star h_2^{-1} (2 - h_2(\delta) - R_0) \right) \}.
\]

**(Proof):** See Appendix [V].

We notice that for \( p_z \leq h^{-1}_2 (2 - h_2(\delta) - R_0) \), or equivalently, \( \delta \leq h^{-1}_2 (2 - h_2(p_z) - R_0) \), the proposed pDCF is outperformed by DF. In this regime, pDCF can achieve the same performance by decoding both channel inputs, reducing to DF.

Comparing the cut-set bound expression in (14) with \( R_{DF} \) in (15) and \( R_{CF} \) in (17), we observe that DF achieves the cut-set bound if \( R_0 \leq 2 - h(\delta \star p_z) - h(\delta) \) while \( R_{CF} \) coincides with the cut-set bound if \( R_0 \geq 2 \). On the other hand, the proposed suboptimal pDCF scheme achieves the cut-set bound if \( R_0 \geq 2 - h_2(\delta) \), i.e., for \( \delta \geq h^{-1}_2(2 - R_0) \).

Hence, the capacity of the parallel binary symmetric MRC-D in this regime is achieved by pDCF, while both DF and CF are suboptimal, as stated in the next lemma.

**Lemma 6.** If \( R_0 < 2 \) and \( \delta \geq h^{-1}_2 (2 - R_0) \), pDCF achieves the capacity of the parallel binary symmetric MRC-D, while pure CF and DF are both suboptimal under these constraints. For \( R_0 \geq 2 \), both CF and pDCF achieve the capacity.

The achievable rates of DF, CF and pDCF, together with the cut-set bound are shown in Figure [3] with respect to \( \delta \) for \( R_0 = 1.2 \) and \( p_z = 0.15 \). We observe that in this setup, DF outperforms CF in general, while for
Fig. 3. Achievable rates and the cut-set upper bound for the parallel binary symmetric MRC-D with respect to the binary noise parameter \( \delta \), for \( R_0 = 1.2 \) and \( p_z = 0.15 \).

\[ \delta \leq h_2^{-1}(2 - R_0 - h_2(p_z)) = 0.0463, \]
DF outperforms pDCF as well. We also observe that pDCF meets the cut-set bound for \( \delta \geq h_2^{-1}(2 - R_0) = 0.2430 \), characterizing the capacity in this regime, and proving the suboptimality of both the DF and CF schemes when they are used on their own.

**B. Multihop Binary Symmetric Channel**

In order to gain further insights into the proposed pDCF scheme, we look into the binary symmetric MRC-D, in which, there is only a single channel connecting the source to the relay, given by

\[ Y_R = X_1 \oplus N \oplus Z, \quad (18) \]

where \( N \sim \text{Ber}(\delta) \) and \( Z \sim \text{Ber}(p_z) \).

Similarly to Section IV-A, the cut-set bound and the maximum achievable rates for DF and CF are found as

\[ R_{CS} = \min\{R_0, 1 - h_2(\delta)\}, \quad (19) \]

\[ R_{DF} = \min\{R_0, 1 - h_2(\delta \ast p_z)\}, \quad (20) \]

\[ R_{CF} = 1 - h_2(\delta \ast h_2^{-1}(1 - R_0)), \quad (21) \]
where $R_{DF}$ is achieved by $X_1 \sim \text{Ber}(1/2)$, and $R_{CF}$ can be shown to be maximized by $X_1 \sim \text{Ber}(1/2)$ and $\hat{Y}_R = Y_R \oplus Q$, where $Q \sim \text{Ber}(h^{-1}_2(1-R_0))$ similarly to Lemma 4. Note that, for $Y_R$ independent of $Z$, i.e., $p_z = 0$, DF achieves the cut-set bound while CF is suboptimal. However, CF outperforms DF whenever $p_z \geq h^{-1}_2(1-R_0)$.

For the pDCF scheme, we consider binary $(U, X_1, \hat{Y}_R)$, with $U \sim \text{Ber}(p)$, a superposition codebook $X_1 = U \oplus W$, where $W \sim \text{Ber}(q)$, and $\hat{Y}_R = Y_R \oplus Q$ with $Q \sim \text{Ber}(\alpha)$. As stated in the next lemma, the maximum achievable rate of this pDCF scheme is obtained by reducing it to either DF or CF, depending on the values of $p_z$ and $R_0$.

**Lemma 7.** For the binary symmetric MRC-D, pDCF with binary $(U, X_1, \hat{Y}_R)$ achieves the following rate.

$$R_{DCF} = \max\{R_{DF}, R_{CF}\} = \begin{cases} 
\min\{R_0, 1 - h_2(\delta \ast p_z)\} & \text{if } p_z < h^{-1}_2(1-R_0), \\
1 - h_2(\delta \ast h^{-1}_2(1-R_0)) & \text{if } p_z \geq h^{-1}_2(1-R_0).
\end{cases}$$

This result justifies the pDCF scheme proposed in Section IV-A for the parallel binary symmetric MRC-D. Since the channel $p(y_2^2|x_2)$ is independent of the channel state $Z$, the largest rate is achieved if the relay decodes $X_1^2$ from $Y_R^2$. However, for channel $p(y_1^1|x_1, z)$, which depends on $Z$, the relay either decodes $X_1^1$, or compress $Y_R^1$, depending on $p_z$.

**C. Multihop Gaussian Channel with State**

Next, we consider an AWGN multihop channel, which we denote by Gaussian MRC-D, in which the source-relay link is characterized by $Y_R = X_1 + V$, while the destination has access to correlated state information $Z$. We assume that $V$ and $Z$ are zero mean jointly Gaussian random variables with a covariance matrix

$$C_{ZV} = \begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix}.$$  

(22)

The channel input at the source has to satisfy the power constraint $E[|X_1^n|^2] \leq nP$. Finally, the relay and the destination are connected by a noiseless link of rate $R_0$ (see Figure 4 for the channel model).
In this case, the cut-set bound is given by

\[ R_{CS} = \min \left\{ R_0, \frac{1}{2} \log \left( 1 + \frac{P}{1 - \rho^2} \right) \right\}. \] (23)

It easy to characterize the optimal DF rate, achieved by a Gaussian input, as follows:

\[ R_{DF} = \min \left\{ R_0, \frac{1}{2} \log(1 + P) \right\}. \] (24)

For CF and pDCF, we consider the achievable rate when the random variables \((X_1, U, \hat{Y}_R)\) are constrained to be jointly Gaussian, which is a common assumption in evaluating achievable rates, yet potentially suboptimal. For CF, we generate the compression codebook using \(\hat{Y}_R = Y_R + Q\), where \(Q \sim N(0, \sigma_q^2)\). Optimizing over \(\sigma_q^2\), the maximum achievable rate is given by

\[ R_{CF} = R_0 - \frac{1}{2} \log \left( \frac{P + 2^{2R_0}(1 - \rho^2)}{P + 1 - \rho^2} \right). \] (25)

For pDCF, we let \(U \sim N(0, \alpha P_1)\), and \(X_1 = U + T\) to be a superposition codebook where \(T\) is independent of \(U\) and distributed as \(T \sim N(0, \bar{\alpha} P_1)\), where \(\bar{\alpha} \triangleq 1 - \alpha\). We generate a quantization codebook using the test channel \(\hat{Y}_R = Y_R + Q\) as in CF. Next lemma shows that with this choice of random variables, pDCF reduces either to pure DF or pure CF, similarly to the multihop binary model in Section IV-B.

**Lemma 8.** The optimal achievable rate for pDCF with jointly Gaussian \((X_1, U, \hat{Y}_R)\) is given by

\[ R_{pDCF} = \max\{R_{DF}, R_{CF}\} = \begin{cases} \min \{ R_0, 1/2 \log(1 + P) \} & \text{if } \rho^2 \leq 2^{-2R_0}(1 + P), \\ R_0 - 1/2 \log \left( \frac{P + 2^{2R_0}(1 - \rho^2)}{P + 1 - \rho^2} \right) & \text{if } \rho^2 > 2^{-2R_0}(1 + P). \end{cases} \]

**Proof:** See Appendix VI.

In Figure 5 the achievable rates are compared with the cut-set bound. It is shown that DF achieves the best rate when the correlation coefficient \(\rho\) is low, i.e., when the destination has low quality channel state information, while CF achieves higher rates for higher values of \(\rho\). It is seen that pDCF achieves the best of the two transmission schemes. Note also that for \(\rho = 0\) DF achieves the cut-set bound, while for \(\rho = 1\) CF achieves the cut-set bound.

Although this example proves the suboptimality of the DF scheme for the channel model under consideration, it does not necessarily lead to the suboptimality of the CF scheme as we have constrained the auxiliary random variables to Gaussian.
V. COMPARISON WITH THE CUT-SET BOUND

In the examples considered in Section IV, we have seen that for certain conditions, the choice of certain random variables allows us to show that the cut-set bound and the capacity coincide. For example, we have seen that for the parallel binary symmetric MRC-D the proposed pDCF scheme achieves the cut-set bound for $\delta \geq h_2^{-1}(2 - R_0)$, or Gaussian random variables meet the cut-set bound for $\rho = 0$ or $\rho = 1$ in the Gaussian MRC-D. An interesting question is whether the capacity expression in Theorem I always coincides with the cut-set bound or not; that is, whether the cut-set bound is tight for the relay channel model under consideration.

To address this question, we consider the multihop binary channel in (18) for $Z \sim \text{Ber}(1/2)$. The capacity $C$ of this channel is given in the following lemma.

Lemma 9. The capacity of the binary symmetric MRC-D with $Y_R = X_1 \oplus N \oplus Z$, where $N \sim \text{Ber}(\delta)$ and $Z \sim \text{Ber}(1/2)$, is achieved by CF and pDCF, and is given by

$$C = 1 - h_2(\delta \ast h_2^{-1}(1 - R_0)).$$

(26)

Proof: See Appendix VII

From (19), the cut-set bound is given by $R_{CS} = 1 - h_2(\delta)$. It then follows that in general the capacity is below
the cut-set bound. Note that for this setup, $R_{DF} = 0$ and pDCF reduces to CF, i.e., $R_{pDCF} = R_{CF}$. See Figure 6 for comparison of the capacity with the cut-set bound for varying $\delta$ values.

CF suffices to achieve the capacity of the binary symmetric MRC-D for $Z \sim \text{Ber}(1/2)$. While in general pDCF outperforms DF and CF, in certain cases these two schemes are sufficient to achieve the cut-set bound, and hence, the capacity. For the ORC-D model introduced in Section [II], the cut-set bound is given by

$$R_{CS} = R_1 + \min\{R_0, \max_{p(x_1)} I(X_1; Y_R|Z)\}. \quad (27)$$

Next, we present four cases for which the cut-set bound is achievable, and hence, is the capacity:

1) If $I(Z; Y_R) = 0$, the setup reduces to the class of orthogonal relay channels studied in [21], for which the capacity is known to be achieved by pDF.

2) If $H(Y_R|X_1 Z) = 0$, i.e., $Y_R$ is a deterministic function of $X_1$ and $Z$, the capacity, given by

$$R_1 + \min\{R_0, \max_{p(x_1)} I(X_1; Y_R|Z)\},$$

is achievable by CF.

3) If $\max_{p(x_1)} I(X_1; Y_R) \geq R_0$, the capacity, given by $C = R_1 + R_0$, is achievable by pDF.

4) Let $\arg \max_{p(x_1)} I(X_1; Y_R|Z) = \bar{p}(x_1)$. If $R_0 > H(\bar{Y}_R|Z)$ for $\bar{Y}_R$ induced by $\bar{p}(x_1)$, the capacity, given by

$$R_1 + I(\bar{X}_1; \bar{Y}_R|Z),$$

is achievable by CF.

**Proof:** See Appendix [VIII].

These cases can be observed in the examples from Section [IV]. For example, in the Gaussian MRC-D, with $\rho = 0$, $Y_R$ is independent of $Z$, and thus, DF meets the cut-set bound as stated in case 1. Similarly, for $\rho = 1$ CF meets the cut-set bound since $Y_R$ is a deterministic function of $X_R$ and $Z$, which corresponds to case 2.

For the parallel binary symmetric MRC-D in Section [IV-A], pDCF achieves the cut-set bound if $\delta \geq h_2^{-1}(2 - R_0)$ due to the following reasoning. Since $Y_R^1$ is independent of $X_1^1$, from case 1, DF should achieve the cut-set bound. Once $X_1^1$ is decoded, the available rate to compress $Y_2$ is given by $R_0 - I(X_1^1; Y_1^1) = R_0 - 1 + h_2(\delta)$, and the entropy of $Y_2$, given the channel state at the destination, is given by $H(Y_2|Z) = 1 - h_2(\delta)$. For $\delta \geq h_2^{-1}(2 - R_0)$ we have $R_0 - I(X_1^1; Y_1^1) \geq H(Y_2|Z)$. Therefore the relay can compress $Y_2$ losslessly, and transmit to the destination. This corresponds to case 4. Thus, the capacity characterization in the parallel binary symmetric MRC-D is due to a combination of cases 1 and case 4.
VI. Conclusion

We have considered a class of orthogonal relay channels, in which the source and the relay are connected with a channel that depends on a state sequence, known at the destination. We have characterized the capacity of this class of relay channels, and shown that it is achieved by the partial decode-compress-and-forward (pDCF) scheme. This is the first three-terminal relay channel model for which the pDCF is shown to be capacity achieving while partial decode-and-forward (pDF) and compress-and-forward (CF) schemes are both suboptimal in general. We have also shown that, in general, the capacity of this channel is below the cut-set bound.

APPENDIX I

Proof of Lemma 1

In the rate expression and joint pmf in Theorem 2 we set $X^n = (X_1^n, X_2^n)$, $Y^n = (Y_1^n, Y_2^n, Z^n)$, $V = \emptyset$, and generate $X^n_R$ and $X^n_1$ independent of the rest of the random variables with distributions $p^*(x_R)$ and $p^*(x_1)$, which
maximize the mutual information terms in (5), respectively. Under this set of distributions we have

\[
I(X; Y \hat{Y}_R | X_R U) = I(X_1 X_2; Y_1 Y_2 \hat{Y}_R Z | X_R, U)
\]

\[
\overset{(a)}{=} I(X_1 X_2; Y_2 \hat{Y}_R | X_R U Z)
\]

\[
\overset{(b)}{=} I(X_2; Y_2 | Z) + I(X_1; \hat{Y}_R | U Z)
\]

\[
= R_1 + I(X_1; \hat{Y}_R | U Z),
\]

\[
I(U; Y_R | X_R V) = I(U; Y_R | X_R) \overset{(c)}{=} I(U; Y_R),
\]

\[
I(X X_R; Y) = I(X_1 X_2 X_R; Y_1 Y_2 Z)
\]

\[
\overset{(d)}{=} I(X_2 X_R; Y_1 Y_2 | Z)
\]

\[
\overset{(e)}{=} I(X_R; Y_1) + I(X_2; Y_2 | Z) = R_0 + R_1,
\]

\[
I(\hat{Y}_R; Y_R | X_X U Y) = I(\hat{Y}_R; Y_R | X_R X_2 U Y_1 Y_2 Z)
\]

\[
\overset{(f)}{=} I(\hat{Y}_R; Y_R | X_R X_2 U Y_2 Z)
\]

\[
\overset{(g)}{=} I(\hat{Y}_R; Y_R | X_1 U Z),
\]

\[
I(\hat{Y}_R; Y_R | Y X_R U) = I(\hat{Y}_R; Y_R | Y_1 Y_2 Z X_R U)
\]

\[
\overset{(h)}{=} I(\hat{Y}_R; Y_R | U Z),
\]

\[
I(X_R; Y | V) = I(X_R; Y_1 Y_2 Z) = I(X_R; Y_1) = R_0,
\]

where (a) is due to the Markov chain \((X_1 X_2) - X_R - Y_1\); (b), (c), (e), (f), (g), (h) are due to the independence of \((U, X_1)\) and \(X_R\), and (d) is due to the Markov chain \((Y_1 Y_2) - (X_2 X_R Z) - X_1\).

Then, (5) reduces to the following rate

\[
R = \sup_{\mathcal{P}} \min \{ I(U; Y_R) + R_1 + I(X_1; \hat{Y}_R | U Z), R_1 + R_0 - I(\hat{Y}_R; Y_R | X_1 U Z) \},
\]

\[
\text{s.t. } R_0 \geq I(\hat{Y}_R; Y_R | U Z).
\]  

(28)

By denoting by the joint distributions in \(\mathcal{P}\) such the the minimum in \(R\) is achieved for the first argument, i.e.,

\[
R_0 - I(\hat{Y}_R; Y_R | X_1 U Z) \geq I(U; Y_R) + I(X_1; \hat{Y}_R | U Z),
\]

(29)

and arranging using the chain rule for the mutual information, we have that the rate achievable by pDCF is lower
bounded by

\[
R \geq \sup_{P} R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R | UZ)
\]

s.t. \( R_0 \geq I(U; Y_R) + I(X_1 Y_R; \hat{Y}_R | UZ), \) \hspace{1cm} (30)

\[
R_0 \geq I(\hat{Y}_R; Y_R | UZ).
\]

From (30), we have

\[
R_0 \geq I(U; Y_R) + I(X_1 Y_R; \hat{Y}_R | UZ)
\]

\[
\overset{(a)}{=} I(U; Y_R) + I(\hat{Y}_R; Y_R | UZ)
\]

\[
\geq I(\hat{Y}_R; Y_R | UZ), \hspace{1cm} (32)
\]

where (a) is due to the Markov chain \( \hat{Y}_R - (U Y_R) - (X_1 Z) \). Hence, (30) implies (31), i.e., the latter condition is redundant, and \( R \geq C \). Therefore the capacity expression \( C \) in (7) is achievable by pDCF. This concludes the proof.

APPENDIX II

PROOF OF LEMMA 2

Consider any sequence of \( (2^n R, n, \nu_n) \) codes such that \( \lim_{n \to \infty} \nu_n = 0 \). We need to show that \( R \leq R_{up} \).

Let us define \( U_i \triangleq (Y_{R_{i+1}}^{n_{i+1}} X_{1_{i+1}}^{n_{i+1}} Z_{1_{i+1}}^{n_{i+1}}) \) and \( \hat{Y}_{R_{i+1}} \triangleq (Y_{n_{i+1}}^{n_{i+1}}) \). For such \( \hat{Y}_{R_{i+1}} \) and \( U_i \), the following Markov chain holds

\[
\hat{Y}_{R_{i+1}} - (U_i, Y_{R_{i+1}}) - (X_{1_{i+1}}, X_{2_{i+1}}, Z_{1_{i+1}}, Y_{1_{i+1}}, Y_{2_{i+1}}, X_{R_{i+1}}).
\]

From Fano’s inequality, we have

\[
H(W|Y_1^n Y_2^n Z^n) \leq n \epsilon_n,
\]

such that \( \epsilon_n \to 0 \) as \( n \to \infty \).

First, we derive the following set of inequalities related to the capacity of the source-destination channel.

\[
n R = H(W)
\]

\[
\overset{(a)}{=} I(W; Y_1^n Y_2^n | Z^n) + H(W|Y_1^n Y_2^n Z^n)
\]

\[
\overset{(b)}{\leq} I(X_1^n X_2^n; Y_1^n Y_2^n | Z^n) + n \epsilon_n,
\]
where (a) follows from the independence of \( Z^n \) and \( W \) and (b) follows from Fano’s inequality in (34).

We also have the following inequalities:

\[
I(X_2^n; Y_2^n | Z^n) = \sum_{i=1}^{n} H(Y_{2i}|Z^n) - H(Y_{2i}|X_{2i})
\]

\[
\leq \sum_{i=1}^{n} H(Y_{2i}|Z_i) - H(Y_{2i}|X_{2i}) = \sum_{i=1}^{n} I(X_{2i}; Y_{2i}|Z_i)
\]

\[
\leq n I(X_{2Q}; Y_{2Q'}|Q')
\]

\[
\leq n I(X_{2Q}; Y_{2Q'}) \leq nI
\]

\[
\leq nR_1,
\]

(36)

where (a) follows since conditioning reduces entropy, (b) follows by defining \( Q' \) as a uniformly distributed random variable over \( \{1, \ldots, n\} \) and \( (X_{2Q}, Y_{2Q'}) \) as a pair of random variables satisfying \( \Pr\{X_{2i} = x_2, Y_{2i} = y_2\} = \Pr\{X_{2Q} = x_2, Y_{2Q'} = y_2|Q = i\} \) for \( i = 1, \ldots, n \), (c) follows from the Markov chain relation \( Q' - X_{2Q'} - Y_{2Q'} \), and (d) follows from the definition of \( R_1 \) in (5).

Then, we can bound the achievable rate as,

\[
nR = I(W; Y_1^n Y_2^n Z^n) + H(W|Y_1^n Y_2^n Z^n)
\]

\[
\leq I(W; Y_1^n Y_2^n Z^n) + n\epsilon_n
\]

\[
\leq I(W; Y_2^n | Z^n) + I(W; Y_1^n | Y_2^n Z^n) + n\epsilon_n
\]

\[
\leq I(X_2^n; Y_2^n | Z^n) + I(W; Y_1^n | Y_2^n Z^n) + n\epsilon_n
\]

\[
\leq nR_1 + H(Y_1^n | Y_2^n Z^n) - H(Y_1^n | W Z^n) + n\epsilon_n
\]

\[
\leq nR_1 + H(Y_1^n | Z^n) - H(Y_1^n | W X_1^n Z^n) + n\epsilon_n
\]

\[
= nR_1 + I(X_1^n; Y_1^n | Z^n) + n\epsilon_n
\]

\[
\leq nR_1 + H(X_1^n) - H(X_1^n | Y_1^n Z^n) + n\epsilon_n
\]

\[
= nR_1 + \sum_{i=1}^{n} H(X_{1i}|X_{1i+1}^n) - H(X_{1i}|Y_1^n Z^n) + n\epsilon_n
\]

\[
\leq nR_1 + \sum_{i=1}^{n} \left[ I(Y_{R1}^{i-1} Z^n; Y_{Ri}) + H(X_{1i}|X_{1i+1}^n) - H(X_{1i}|Y_1^n Z^n) + n\epsilon_n
\]
\[ nR_1 + \sum_{i=1}^{n} \left[ I(Y_R^{i-1} Z^n | X_{i+1}^n, Y_R) - I(X_{i+1}^n; Y_R | Y_R^{i-1} Z^n) + H(X_{i+1} | Y_R^{i-1} Z^n) X_{i+1}^n \right] \\
+ I(X_{i+1}; Y_R^{i-1} Z^n | X_{i+1}^n) - H(X_{i+1}^n | Y_R^n Z^n) + n\epsilon_n \\
\leq nR_1 + \sum_{i=1}^{n} \left[ I(Y_R^{i-1} Z^n | X_{i+1}^n; Y_R) + H(X_{i+1} | Y_R^{i-1} Z^n) X_{i+1}^n \right] - H(X_{i+1}^n | Y_R^n Z^n) + n\epsilon_n \\
= nR_1 + \sum_{i=1}^{n} \left[ I(U_i; Y_R) + H(X_{i+1} | U_i) - H(X_{i+1} | U_i, Z_i Y_R) \right] + n\epsilon_n \\
\leq nR_1 + \sum_{i=1}^{n} \left[ I(U_i; Y_R) + I(X_{i+1}; Y_R) \right] + n\epsilon_n, \\
\tag{37}
\]

where (a) is due to Fano’s inequality; (b) is due to the chain rule and the independence of \( Z^n \) from \( W \); (c) is due to the data processing inequality, (d) is due to the Markov chain relation \( Y_R^n - W, Z^n - Y_R^n \) and (36), (e) is due to the fact that conditioning reduces entropy, and that \( X_{i+1}^n \) is a deterministic function of \( W \); (f) is due to the Markov chain relation \( Y_R^n - X_{i+1}^n - W \); (g) is due to the independence of \( Z^n \) and \( X_{i+1}^n \); (i) follows because

\[ \sum_{i=1}^{n} I(X_{i+1}; Y_R^{i-1} Z^n | X_{i+1}^n) \overset{(l)}{=} \sum_{i=1}^{n} I(X_{i+1}; Y_R^{i-1} | X_{i+1}^n Z^n) \overset{(m)}{=} \sum_{i=1}^{n} I(X_{i+1}^n; Y_R | Y_R^{i-1} Z^n), \tag{37} \]

where (l) is due to the independence of \( Z^n \) and \( X_{i+1}^n \); and (m) is the conditional version of Csiszár’s equality \(20\). Inequality (j) is due to the following bound,

\[
H(X^n_1 | Y^n_1 Z^n) = \sum_{i=1}^{n} H(X_{i+1} | X_{i+1}^n Z^n Y_1^n) \\
\geq \sum_{i=1}^{n} H(X_{i+1} | Y_R^{i-1} X_{i+1}^n Z^n Y_1^n) \\
\overset{(n)}{=} \sum_{i=1}^{n} H(X_{i+1} | Y_R^{i-1} X_{i+1}^n Z^n Y_{i+1}^n) \\
= \sum_{i=1}^{n} H(X_{i+1}; U_i Z_i Y_R), 
\tag{38}
\]

where (n) is follows from the Markov chain relation \( X_{i+1} - (Y_R^{i-1} X_{i+1}^n Z^n Y_{i+1}^n) - Y_{i+1} \), and noticing that \( X_{Ri} = f_{r,i}(Y_R^{i-1}) \). Finally, (k) is due to the fact that \( Z_i \) independent of \((X_{i+1}, U_i)\).
We can also obtain the following sequence of inequalities

\[ nR_0 + nR_1 \geq I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2 | Z^n) \]

\[ \geq H(Y^n_2 | Z^n) - H(Y^n_2 | X^n_2 Z^n) + H(Y^n_1 | Y^n_2 Z^n) - H(Y^n_1 | X^n_1) \]

\[ = H(Y^n_1 Y^n_2 | Z^n) - H(Y^n_2 | X^n_2 Z^n) - H(Y^n_1 | X^n_1) \]

\[ = H(Y^n_1 Y^n_2 | Z^n) - H(Y^n_2 | X^n_2 Y^n_1 R| Z^n) - H(Y^n_1 | X^n_1 Y^n_1 X^n_2 Y^n_2 R X^n_1; Z^n) \]

\[ = H(Y^n_1 Y^n_2 | Z^n) - H(Y^n_2 | X^n_2 Y^n_1 R X^n_1; Z^n) + H(Y^n_1 | X^n_1 Y^n_1 X^n_2 Y^n_2 R X^n_1; Z^n) \]

where (a) follows from the definitions of \( R_0 \) and \( R_1 \) in (5); (b) is due to the fact that conditioning reduces entropy; (c) is due to the Markov chains \( Y^n_1 - (X^n_2 Z^n) - (X^n_1 Y^n_1 R) \) and \( Y^n_2 - X^n_2 Y^n_2 R Y^n_2 Z^n \); (d) is follows since \( X^n_1 \) is a deterministic function of \( Y^n_2 \); (e) is due to the expression in (5); (f) is due to the Markov chain \( (Y^n_2 Y_1^n) - (X^n_2 Y^n_2 Z^n) - (X^n_2 Y^n_2) \) and; (g) is due to the Markov chain \( (X^n_1 Y_1^n) - (X^n_2 Y^n_2) - (X^n_2 Y^n_2) \).

A single letter expression can be obtained by using the usual time-sharing random variable arguments. Let \( Q \) be a time sharing random variable uniformly distributed over \( \{1, \ldots, n\} \), independent of all the other random variables. Also, define a set of random variables \((X^n_1, Y^n_2, U^n_1, \hat{Y}^n_2, Z^n_Q)\) satisfying

\[ \Pr\{X^n_1 = x_1, Y^n_2 = y_R, U^n_1 = u, \hat{Y}^n_2 = \hat{y}_R, Z^n_Q = z | Q = i\} \]

\[ = \Pr\{X^n_1 = x_1, Y^n_2 = y_R, U^n_1 = u, \hat{Y}^n_2 = \hat{y}_D, Z^n_1 = z\} \quad \text{for } i = 1, \ldots, n. \]

Define \( U = (U^n_1, Q) \), \( \hat{Y}^n_R = \hat{Y}^n_1, X^n_1 = X^n_1, Y^n_2 = Y^n_2 \) and \( Z = Z^n_Q \). We note that the pmf of the tuple
\( (X_1, Y_R, U, \hat{Y}_R, Z) \) belongs to \( \mathcal{P} \) in (6) as follows:

\[
p(u, x_1, y_R, z, \hat{y}_R) = p(q, u_Q, x_{1Q}, y_{RQ}, z_Q, \hat{y}_{RQ}) \\
= p(q, u_Q, x_{1Q}) p(z_q y_{RQ} \hat{y}_{RQ}|q, u_Q x_{1Q}) \\
= p(q, u_Q, x_{1Q}) p(z_q|q, u_Q, x_{1Q}) p(y_{RQ}|q, u_Q, x_{1Q}, z_Q) \\
= p(q, u_Q, x_{1Q}) p(z) p(y_{RQ}|q, u_Q, x_{1Q}, z_Q) p(\hat{y}_{RQ}|q, u_Q, x_{1Q}, z_Q, y_{RQ}) \\
= p(q, u_Q, x_{1Q}) p(z) p(y_R|x_1, z) p(\hat{y}_{RQ}|q, u_Q, y_{RQ}) \\
= p(u, x_1) p(z) p(y_R|x_1, z) p(\hat{y}_R|u, y_R),
\]

where (a) follows since the channel state \( Z^n \) is i.i.d. and thus \( p(z_q|q, u_Q, x_{1Q}) = p(z_q|q) = p(z) \), (b) follows since \( p(y_{RQ}|q, u_Q, x_{1Q}, z_Q) = p(y_{RQ}|q, x_{1Q}, z_Q) = p(y_R|x_1, z) \), (c) follows from the Markov chain in (33).

Then, we get the single letter expression,

\[
R \leq R_1 + \frac{1}{n} \sum_{i=1}^{n} [I(U_i; Y_{R_i}) + I(X_{1i}; \hat{Y}_{R_i}|U_i, Z_i)] + \epsilon_n \\
= R_1 + I(U_Q; Y_{RQ}|Q) + I(X_{1Q}; \hat{Y}_{RQ}|U_Q Z_Q Q) + \epsilon_n \\
\leq R_1 + I(U_Q; Y_{RQ}) + I(X_{1Q}; \hat{Y}_{RQ} Q|U_Q Z_Q) + \epsilon_n \\
= R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R|U Z) + \epsilon_n,
\]

and

\[
R_0 + R_1 \geq R + \frac{1}{n} \sum_{i=1}^{n} I(\hat{Y}_{R_i}; Y_{R_i}|X_{1i} U_i Z_i) - n\epsilon_n \\
= R + I(\hat{Y}_{RQ}; Y_{RQ}|X_{1Q} U_Q Z_Q Q) - n\epsilon_n \\
= R + I(\hat{Y}_R; Y_R|X_1 U Z) - n\epsilon_n.
\]

The cardinality of the bounds on the alphabets of \( U \) and \( \hat{Y}_R \) can be found using the usual techniques [20]. This completes the proof.
APPENDIX III

PROOF OF LEMMA \[\text{3}\]

Now, we will show that the expression of $R_{up}$ in (9) is equivalent to the expression $C$ in (7). First we will show that $C \leq R_{up}$ as follows. Consider the subset of pmfs in $P$ such that

$$R_0 + R_1 - I(\hat{Y}_R; Y_R | X_1 U Z) \geq R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R | U Z)$$

(39)

holds. Then, similarly to (32) in Appendix II this condition is necessitates

$$R_0 \geq I(U; Y_R) + I(Y_R; \hat{Y}_R | U Z).$$

(40)

Hence, we have $C \leq R_{up}$.

Then, it remains to show that $C \geq R_{up}$. As $R_1$ can be extracted from the supremum, it is enough to show that, for each $(X_1, U, Z, Y_R, \hat{Y}_R)$ tuple with a joint pmf $p_e \in P$ satisfying

$$R(p_e) \leq I(U; Y_R) + I(X_1; \hat{Y}_R | U Z),$$

$$R(p_e) \triangleq R_0 - I(\hat{Y}_R; Y_R | X_1 U Z),$$

(41)

there exist random variables $(X_1^*, U^*, Z, Y_R^*, \hat{Y}_R^*)$ with joint pmf $p_e^* \in P$ that satisfy

$$R(p_e) = I(U^*; Y_R) + I(X_1^*; \hat{Y}_R^* | U^* Z)$$

and

$$R(p_e) \leq R_0 - I(\hat{Y}_R^*; Y_R | X_1^* U^* Z).$$

(42)

This argument is proven next.

Let $B$ denote a Bernoulli random variable with parameter $\lambda \in [0, 1]$, i.e., $B = 1$ with probability $\lambda$, and $B = 0$ with probability $1 - \lambda$. We define the triplets of random variables:

$$(U', X'_1, \hat{Y}'_R) = \begin{cases} (U, X_1, \hat{Y}_R) & \text{if } B = 1, \\ (X_1, X_1, \emptyset) & \text{if } B = 0, \end{cases}$$

(43)

and

$$(U'', X''_1, \hat{Y}''_R) = \begin{cases} (X_1, X_1, \emptyset) & \text{if } B = 1, \\ (\emptyset, \emptyset, \emptyset) & \text{if } B = 0. \end{cases}$$

(44)
We first consider the case $R(p_e) > I(X_1; Y_R)$. Let $U^* = (U', B)$, $X^*_1 = X'_1$, $\hat{Y}_R^* = (\hat{Y}'_R, B)$. For $\lambda = 1$,

$$I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z)$$

$$= I(U; Y_R) + I(X_1; \hat{Y}_R|UZ) > R(p_e),$$

and for $\lambda = 0$,

$$I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z) = I(X_1; Y_R) < R(p_e).$$

As $I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z)$ is a continuous function of $\lambda$, by the intermediate value theorem, there exists a $\lambda \in [0, 1]$ such that $I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z) = R(p_e)$. We denote the corresponding joint distribution by $p^*_e$.

We have

$$I(\hat{Y}_R^*; Y_R|X^*_1U^*Z) = I(\hat{Y}'_R; Y_R|X'_1U'ZB)$$

$$= \lambda I(\hat{Y}_R; Y_R|X_1UZ)$$

$$\leq I(\hat{Y}_R; Y_R|X_1UZ),$$

which implies that $p^*_e$ satisfies $(42)$ since

$$R(p_e) = R_0 - I(\hat{Y}_R; Y_R|X_1UZ)$$

$$\leq R_0 - I(\hat{Y}'_R; Y_R|X'_1U'Z).$$

(46)

Next we consider the case $R(p_e) \leq I(X_1; Y_1)$. We define $U^* = (U'', B)$, $X^*_1 = X''_1$ and $\hat{Y}_R^* = (\hat{Y}'_R, B)$. Then, for $\lambda = 1$,

$$I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z) = I(X_1; Y_R) \geq R(p_e),$$

and for $\lambda = 0$,

$$I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z) = 0 < R(p_e).$$

(47)

Once again, as $I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z)$ is a continuous function of $\lambda$, by the intermediate value theorem, there exists a $\lambda \in [0, 1]$ such that $I(U^*; Y_R) + I(X^*_1; \hat{Y}_R^*|U^*Z) = R(p_e)$. Again, we denote this joint distribution
by $p_e^*$. On the other hand, we have $I(\hat{Y}_R^*; Y_R|X_1^*U^*Z) = 0$, which implies that

$$R(p_e) = R_0 - I(\hat{Y}_R; Y_R|X_1UZ) \leq R_0 = R_0 - I(\hat{Y}_R^*; Y_R|X_1^*U^*Z).$$

That is, $p_e^*$ also satisfies (42).

We have shown that for any joint pmf $p_e \in \mathcal{P}$ satisfying (41), there exist another joint pmf, $p_e^*$, that satisfies (42). For a distribution satisfying (42) we can write

$$R_0 > I(U^*; Y_R) + I(X_1^*; \hat{Y}_R^*|U^*Z) + I(\hat{Y}_R^*; Y_R|X_1^*U^*Z)$$

$$= I(U^*; Y_R) + I(Y_RX_1^*; \hat{Y}_R^*|U^*Z)$$

$$\overset{(a)}{=} I(U^*; Y_R) + I(\hat{Y}_R^*; Y_R|U^*Z)$$

where $(a)$ is due to Markov chain $X_1^* - (Y_RZU^*) - \hat{Y}_R^*$. This concludes the proof.

**APPENDIX IV**

**Proof of Lemma 4**

Before deriving the maximum achievable rate by CF in Lemma 4 we provide some definitions that will be used in the proof.

Let $X$ and $Y$ be a pair of discrete random variables, where $\mathcal{X} = \{1, 2, ..., n\}$ and $\mathcal{Y} = \{1, 2, ..., m\}$, for $n, m < \infty$. Let $p_Y \in \Delta_m$ denote the distribution of $Y$, where $\Delta_k$ denotes the $(k-1)$-dimensional simplex of probability $k$-vectors. We define $T_{XY}$ as the $n \times m$ stochastic matrix with entries $T_{XY}(j, i) = \Pr\{X = j | Y = i\}$. Note that the joint distribution $p(x, y)$ is characterized by $T_{XY}$ and $p_Y$.

Next, we state the conditional entropy bound from [22], which lower bounds the conditional entropy between two variables. Note the relabeling of the variables to fit our model.

**Definition 1** (Conditional Entropy Bound). Let $p_Y \in \Delta_m$ be the distribution of $Y$ and $T_{XY}$ denote the channel matrix relating $X$ and $Y$. Then, for $q \in \Delta_m$ and $0 \leq s \leq H(Y)$, define the function

$$F_{T_{XY}}(q, s) \triangleq \inf_{p(w|y); X \rightarrow Y \rightarrow W, \text{ s.t. } H(Y|W)=s, \text{ } p_Y=q} H(X|W).$$
That is, $F_{T_{XY}}(q, s)$ is the infimum of $H(X|W)$ given a specified distribution $q$ and the value of $H(Y|W)$. Many properties of $F_{T_{XY}}(q, s)$ are derived in [22], such as its convexity on $(q, s)$ [22, Theorem 2.3] and its non-decreasing monotonicity in $s$ [22, Theorem 2.5].

Consider a sequence of $N$ random variables $Y = (Y_1, ..., Y_N)$ and denote by $q_i$ the distribution of $Y_i$, for $i = 1, ..., N$, by $q^{(N)}$ the joint distribution of $Y$ and by $q = \frac{1}{N} \sum_{i=1}^{N} q_i$ the average distribution. Note that $Y_1, ..., Y_N$ can have arbitrary correlation. Define the sequence $X = (X_1, ..., X_N)$, in which $X_i$, $i = 1, ..., N$, is jointly distributed with each $Y_i$ through the stochastic matrix $T_{XY}$ and denote by $T^{(N)}_{XY}$ the Kronecker product of $N$ copies of the stochastic matrix $T_{XY}$.

Then, the theorem given in [22, Theorem 2.4] can be straightforwardly generalized to non i.i.d. sequences as given in the following lemma.

**Lemma 10.** For $N = 1, 2, ...,$ and $0 \leq Ns \leq H(Y)$, we have

$$F_{T^{(N)}_{XY}}(q^{(N)}, Ns) \geq NF_{T_{XY}}(q, s),$$

where equality holds for i.i.d. $Y_i$ components following $q$.

**Proof:**

Let $W, X, Y$ be a Markov chain, such that $H(Y|W) = Ns$. Then, using the standard identity we have

$$H(Y|W) = \sum_{k=1}^{N} H(Y_k|Y_1^{k-1}, W),$$

(51)

$$H(X|W) = \sum_{k=1}^{N} H(X_k|X_1^{k-1}, W).$$

(52)

Letting $s_k = H(Y_k|Y_1^{k-1}, W)$, we have

$$\frac{1}{N} \sum_{k=1}^{N} s_k = s.$$  (53)

Also, from the Markov chain $X_k - (Y_1^{k-1}, W) - X_1^{k-1}$, we have

$$H(X_k|X_1^{k-1}, W) \geq H(X_k|Y_1^{k-1}, X_1^{k-1}, W)$$

(54)

$$= H(X_k|Y_1^{k-1}, W).$$

(55)
Applying the conditional entropy bound in (49) we have
\[ H(X_k|Y_1^{k-1}, W) \geq F_{T_{XY}}(q_k, s_k). \]  
(56)

Combining (52), (54) and (56) we have
\[ H(X|W) \geq \sum_{k=1}^{N} F_{T_{XY}}(q_k, s_k) \geq NF_{T_{XY}}(q, s), \]
where the last inequality follows from the convexity of \( F_T(q, s) \) in \( q \) and \( s \) and (53).

If we let \( W, Y, X \), be \( N \) independent copies of the random variables \( W, X, Y \), that achieve \( F_{T_{XY}}(q, s) \), we have
\[ H(Y|W) = Ns \text{ and } H(X|W) = F_{T_{XY}^{(N)}}(q^N) = NF_{T_{XY}}(q, s). \]
Therefore, \( F_{T_{XY}^{(N)}}(q^N) \leq NF_{T_{XY}}(q, s) \) and the equality holds for i.i.d. components of \( Y \).

Now, we look into the binary symmetric channel \( Y = X \oplus N \) where \( N \sim \text{Ber}(\delta) \). Due to the binary modulo-sum operation, we have \( X = Y \oplus N \), and we can characterize the channel \( T_{XY} \) of this model as
\[ T_{XY} = \begin{bmatrix} 1 - \delta & \delta \\ \delta & 1 - \delta \end{bmatrix}. \]  
(57)

When \( Y \) and \( X \) are related through channel \( T_{XY} \) in (57), \( F_{T_{XY}}(q, s) \) is characterized as follows [22].

**Lemma 11.** Let \( Y \sim \text{Ber}(q) \), i.e., \( q = [q, 1-q] \), and \( T_{XY} \) be given as in (57). Then the conditional entropy bound is
\[ F_{T_{XY}}(q, s) = h_2(\delta * h_2^{-1}(s)), \text{ for } 0 \leq s \leq h_2(q). \]

In the following, we use the properties of \( F_{T_{XY}}(q, s) \) to derive the maximum rate achievable by CF in the parallel binary symmetric MRC-D. From (16), we have
\[ I(Y_1^1; \hat{Y}_R|Z) = I(X_1^1 \oplus N_1 \oplus Z, X_1^2 \oplus N_2; \hat{Y}_R|Z) \]
\[ = I(X_1^1 \oplus N_1, X_1^2 \oplus N_2; \hat{Y}_R|Z). \]

Let us define \( \hat{Y}_R \triangleq X_1^1 \oplus N_1 \) and \( \tilde{Y}_R \triangleq (Y_1^1, Y_1^2) \), and the channel input \( X \triangleq (X_1^1, X_1^2) \). Note that the distribution of \( \hat{Y}_R \), given by \( q^{(2)} \), determines the distribution of \( X \) via \( T_{XY}^{(2)} \), the Kronecker product of \( T_{XY} \) in (57). Then, we
can rewrite the achievable rate for CF in (16) as follows

$$R_{CF} = \max_{p(x)p(z)p(y_R|x)p(\hat{y}_R|y_R,z)} I(X, \hat{Y}_R|Z)$$

s.t. $R_0 \geq I(\hat{Y}_R; \hat{Y}_R|Z)$ \hspace{1cm} (58)

Next, we derive a closed form expression for $R_{CF}$. First, we note that if $R_0 \geq 2$, we have $H(\hat{Y}_R) \leq R_0$ and $R_{CF} = 2(1 - h(\delta))$, i.e., CF meets the cut-set bound.

For fixed $q^{(2)}$, if $H(\hat{Y}_R) \leq R_0 \leq 2$, the constraint in (58) is satisfied by any $\hat{Y}_R$, and can be ignored. Then, due to the Markov chain $X - \hat{Y}_R - \hat{Y}_R Z$, and the data processing inequality, the achievable rate is upper bounded by

$$R_{CF} \leq I(X, \hat{Y}_R) = H(\hat{Y}_R) - 2h(\delta) \leq R_0 - 2h(\delta).$$ \hspace{1cm} (59)

For $R_0 \leq H(\hat{Y}_R) \leq 2$, the achievable rate by CF is upper bounded as follows.

$$R_{CF} \leq \max_{p(x)p(z)p(y_R|x)p(\hat{y}_R|y_R,z)} H(X) - H(X|Z\hat{Y}_R)$$

s.t. $H(Y_R|Z\hat{Y}_R) \geq H(\hat{Y}_R) - R_0$

$$\leq \max_{p(x)p(y_R|x)p(\hat{y}_R|y_R)} H(X) - H(X|W)$$

s.t. $H(\hat{Y}_R|W) \geq H(\hat{Y}_R) - R_0$

$$= \max_{p(x)p(y_R|x)} \left[ H(X) - \min_{p(w|\hat{Y}_R)} H(X|W) \right]$$

s.t. $H(\hat{Y}_R|W) \geq H(\hat{Y}_R) - R_0$

$$\leq \max_{p(x)p(y_R|x), 0 \leq s \leq H(Y_R)} \left[ H(X) - F_{\hat{X}\hat{Y}}^{(2)}(q^{(2)}, s) \right]$$

s.t. $s \geq H(\hat{Y}_R) - R_0$

$$\leq \max_{p(x)p(y_R|x)} \left[ H(X) - F_{\hat{X}\hat{Y}}^{(2)}(q^{(2)}, H(\hat{Y}_R) - R_0) \right]$$

$$\leq \max_{p(x)p(y_R|x)} \left[ H(X) - 2F_{\hat{X}\hat{Y}}(q, (H(\hat{Y}_R) - R_0)/2) \right]$$

$$\leq \max_{p(x)p(y_R|x)} \left[ H(X) - 2h_2(\delta \ast h_2^{-1}(H(\hat{Y}_R) - R_0)/2)) \right]$$

s.t. $0 \leq (H(\hat{Y}_R) - R_0)/2 \leq h_2(q)$

$$\leq \max_{p(x)p(y_R|x)} \left[ H(X) - 2h_2(\delta \ast h_2^{-1}(H(\hat{Y}_R) - R_0)/2)) \right]$$

s.t. $R_0 \leq H(\hat{Y}_R) \leq 2 + R_0$
where \((a)\) follows from the independence of \(Z\) from \(X\) and \(\bar{Y}_R\), \((b)\) follows since optimizing over \(W\) can only increase the value compared to optimizing over \((Z, \bar{Y}_R)\), \((c)\) follows from the definition of the conditional entropy bound in \((49)\), \((d)\) follows from the nondecreasing monotonicity of \(F_{T_{XY}}(q^2, s)\) in \(s\), and \((e)\) follows from Lemma 10 and \(q \equiv [q, 1 - q] = \frac{1}{2}(q_1 + q_2)\) is the average distribution of \(Y\). Equality \((f)\) follows from the definition of \(F_{T_{XY}}(q, s)\) for the binary symmetric channel, and \((g)\) follows since we are increasing the optimization domain since \(h_2(q) \leq 1\).

Now, we lower bound \(H(\bar{Y}_R)\). Since conditioning reduces entropy, we have \(H(\bar{Y}_R) \geq H(\bar{Y}_R|N_1N_2) = H(X)\), and then we can lower bound \(H(\bar{Y}_R)\) as follows:

\[
\max\{H(X), R_0\} \leq H(\bar{Y}_R) \leq 2.
\]  

Then, we have

\[
R_{CF} \overset{(a)}{=} \max_{p(x)} \left[H(X) - 2h_2(\delta \ast h_2^{-1}(H(\bar{Y}_R) - R_0)/2))\right] \\
\text{s.t. } \max\{H(X), R_0\} \leq H(\bar{Y}_R) \leq 2
\]

\[
\overset{(b)}{\leq} \max_{p(x)} \left[H(X) - 2h_2(\delta \ast h_2^{-1}((\max\{H(X), R_0\} - R_0)/2))\right] \\
\text{s.t. } \max\{H(X), R_0\} \leq H(\bar{Y}_R) \leq 2
\]

\[
\overset{(c)}{=} \max_{0 \leq \alpha \leq 1} \left[2\alpha - 2h_2(\delta \ast h_2^{-1}((\max\{2\alpha, R_0\} - R_0)/2))\right] \\
\text{s.t. } \max\{R_0, 2\alpha\} \leq 2,
\]

where \((a)\) follows since there is no loss in generality by introducing \((60)\) since it is satisfied by any \((X, \bar{Y}_R)\) following \(p(x, \bar{Y}_R)\), \((b)\) follows from \((60)\) and \(F_{T_{XY}}(q, s)\) being non-decreasing in \(s\), and \((c)\) follows from defining \(H(X) \overset{\Delta}{=} 2\alpha\), for \(0 \leq \alpha \leq 1\).

Then, for \(2\alpha \leq R_0\), we have

\[
R_{CF} \leq \max_{0 \leq \alpha \leq R_0/2} \left[2\alpha - 2h_2(\delta)\right] = R_0 - 2h_2(\delta),
\]

and for \(2\alpha > R_0\), we have

\[
R_{CF} \leq \max_{R_0/2 < \alpha \leq 1} \left[2\alpha - 2h_2(\delta \ast h_2^{-1}(\alpha - R_0/2))\right].
\]

Now, we solve \((62)\). Let us define \(f(u) \overset{\Delta}{=} h_2(\delta \ast h_2^{-1}(u))\) for \(0 \leq u \leq 1\). Then, we have the following lemma
Lemma 12 ([23]). Function $f(u)$ is convex for $0 \leq u \leq 1$.

Then, we define $g(\alpha) \triangleq \alpha - h_2(\delta \ast h_2^{-1}(\alpha - R_0/2))$, such that $R_{CF} \leq \max_{R_0/2 < \alpha \leq 1} 2g(\alpha)$. We have that $g(\alpha)$ is concave in $\alpha$, since it is a shifted version by $\alpha$, which is linear with the composition of the concave function $-f(u)$ and the affine function $\alpha - R_0/2$.

Proposition 1. $g(\alpha)$ is monotonically increasing for $R_0/2 \leq \alpha \leq 1 + R_0/2$.

Proof: Using the chain rule for composite functions, we have

$$
\frac{d^2 g(\alpha)}{d\alpha^2} = -f''(\alpha - R_0/2),
$$

(63)

where $f''(u) \triangleq \frac{d^2 f}{du^2}(u)$.

Since $g(\alpha)$ is convex and is defined over a convex region, it follows that its unique maximum is achieved either for $f''(\alpha - R_0/2) = 0$, or at the boundaries of the region. It is shown in [23, Lemma 2] that $f''(u) > 0$ for $0 < u < 1$. That means that the maximum is achieved either at $u = 0$ or at $u = 1$, or equivalently, for $\alpha = R_0/2$ or $\alpha = 1 + R_0/2$. Since $g(R_0/2) = R_0/2 - h_2(\delta)$ and $g(1 + R_0/2) = R_0/2$, i.e., $g(R_0/2) < g(1 + R_0/2)$, it follows that $g(\alpha)$ is monotonically increasing in $\alpha$ for $R_0/2 \leq \alpha \leq 1 + R_0/2$.

From Proposition 1 it follows that for $R_0/2 \leq \alpha \leq 1$, $g(\alpha)$ achieves its maximum at $\alpha = 1$. Then, for $2\alpha > R_0$, we have

$$
R_{CF} \leq 2(1 - h_2(\delta \ast h_2^{-1}(1 - R_0/2))).
$$

(64)

Thus, from (61) and (64), we have that for $R_0 \leq H(\bar{Y}_R)$

$$
R_{CF} \leq 2 \max\{R_0/2 - h_2(\delta), 1 - h_2(\delta \ast h_2^{-1}(1 - R_0/2))\}
$$

$$
= 2(1 - h_2(\delta \ast h_2^{-1}(1 - R_0/2))),
$$

(65)

where the equality follows from Proposition 1 by noting that the first element in the maximum coincides with $g(R_0/2) = R_0/2 - h_2(\delta)$, and the second one coincides $g(1)$.

Finally, $R_{CF}$ is upper bounded by the maximum over the joint distributions satisfying $H(\bar{Y}_R) \leq R_0$ given in (59) and the upper bound for the joint distributions satisfying $R_0 \leq H(\bar{Y}_R)$ given in (65). Since (59) coincides
with $g(R_0/2)$, $R_{CF}$ is upper bounded when $R_0 \leq H(\hat{Y}_R)$ as in (65).

Next, we show that the upper bound on the rate in (65) is achievable by considering the following variables

$$X_1 \sim \text{Ber}(1/2), \quad X_2 \sim \text{Ber}(1/2), \quad \hat{Y}_R = (\hat{Y}_R^1, \hat{Y}_R^2)$$

$$\hat{Y}_R^1 = Y_1^1 \oplus Q_1, \quad Q_1 \sim \text{Ber}(h_2^{-1}(1 - R_0/2)).$$

$$\hat{Y}_R^2 = Y_2^2 \oplus Q_2, \quad Q_2 \sim \text{Ber}(h_2^{-1}(1 - R_0/2)).$$

Let $Q_i \sim \text{Ber}(\nu)$ for $i = 1, 2$. Then from the constraint in (16) we have

$$I(Y_R^1, Y_R^2; \hat{Y}_R|Z) = H(\hat{Y}_R|Z) - H(\hat{Y}_R|Y_R^1Y_R^2Z)$$

$$= H(X_1^1 \oplus N_1 \oplus Q_1, X_2^2 \oplus N_2 \oplus Q_2) - H(Q_1, Q_2)$$

$$= 2 - 2h_2(\nu),$$

where $a$ follows since $X_i^1 \sim \text{Ber}(1/2)$, $i = 1, 2$ and from the independence of $Q_1$ and $Q_2$. We have $2h_2(\nu) \geq 2 - R_0$, and thus, $\nu \geq h_2^{-1}(1 - R_0/2)$.

Then, the achievable rate in (16) is given by

$$I(X; \hat{Y}_R|Z) = H(\hat{Y}_R|Z) - H(\hat{Y}_R|XZ)$$

$$= H(X_1^1 \oplus N_1 \oplus Q_1, X_2^2 \oplus N_2 \oplus Q_2) - H(N_1 \oplus Q_1, N_2 \oplus Q_2)$$

$$= 2 - 2h(\delta \star \nu)$$

$$\leq 2 - 2h_2(\delta \star h_2^{-1}(1 - R_0/2)),$$

where the last inequality follows from the bound on $\nu$. This completes the proof.

**APPENDIX V**

**PROOF OF LEMMA 5**

From (7), the achievable rate for the proposed pDCF scheme is given by

$$R_{pDCF} = I(X_1^1; Y_R^1) + I(X_2^2; \hat{Y}_R|Z)$$

s.t. $R_0 \geq I(X_1^1; Y_R^1) + I(Y_R^2; \hat{Y}_R|Z)$. 

First, we note that the constraint is always satisfied for the choice of variables:

\[
I(X_1^1; Y_1^1) + I(Y_1^2; \hat{Y}_1^1 | Z) = H(Y_1^1) - H(N_1) + H(X_1^2 \oplus N_2 \oplus Q) - H(Q)
\]

\[
= 1 - h_2(\delta) + 1 - h_2(h_2^{-1}(2 - h(\delta) - R_0))
\]

\[
= R_0,
\]

(66)

where \(H(Y_1^1) = 1\) since \(X_1^1 \sim \text{Ber}(1/2)\) and \(H(X_1^2 \oplus N_2 \oplus Q) = 1\) since \(X_1^2 \sim \text{Ber}(1/2)\). Then, similarly the achievable rate is given by

\[
R_{pDCF} = I(X_1^1; Y_1^1) + I(X_1^2; \hat{Y}_1^1 | Z)
\]

\[
= H(Y_1^1) - H(N_1) + H(X_1^2 \oplus N_2 \oplus Q) - H(V \oplus Q)
\]

\[
= 1 - h_2(\delta) + 1 - h_2(\delta \cdot h_2^{-1}(2 - h(\delta) - R_0)),
\]

which completes the proof.

**APPENDIX VI**

**PROOF OF LEMMA 8**

By evaluating (7) with the considered Gaussian random variables, we get

\[
R = \frac{1}{2} \log \left( 1 + \frac{\alpha P}{\bar{\alpha} P + 1} \right) \left( 1 + \frac{\bar{\alpha} P}{(1 - \rho^2) + \sigma_q^2} \right)
\]

s.t. \(R_0 \geq \frac{1}{2} \log \left( 1 + \frac{\alpha P}{\bar{\alpha} P + 1} \right) \left( 1 + \frac{\bar{\alpha} P + (1 - \rho^2)}{\sigma_q^2} \right).

We can rewrite the constraint on \(R_0\) as,

\[
\sigma_q^2 \geq f(\alpha) \triangleq \frac{(P + 1)\bar{\alpha} P + 1 - \rho^2}{2^{2R_0} \bar{\alpha} P + 1 - (P + 1)}.
\]

(67)

Since \(R\) is increasing in \(\sigma_q^2\), it is clear that the optimal \(\sigma_q^2\) is obtained by \(\sigma_q^2 = f(\alpha)\), where \(\alpha\) is chosen such that \(f(\alpha) \geq 0\). It is easy to check that \(f(\alpha) \geq 0\) for

\[
\alpha \in \left[ 0, \min \left\{ (1 - 2^{-2R_0}) \left( 1 + \frac{1}{P} \right), 1 \right\} \right].
\]

(68)
Now, we substitute $\sigma_n^2 = f(\alpha)$ in (67), and write the achievable rate as a function of $\alpha$ as

$$R(\alpha) = \frac{1}{2} \log G(\alpha),$$

where

$$G(\alpha) \triangleq \left(1 + \frac{\alpha P}{\bar{\alpha}P + 1}\right) \left(1 + \frac{\bar{\alpha}P}{(1 - \rho^2) + f(\alpha)}\right) = \frac{2^{2R_0}(1 + P)(1 - \rho^2 + \bar{\alpha}P)}{(1 - \rho^2)^2} \frac{\bar{\alpha}P(1 + \bar{\alpha}P)}{2^{2R_0}(1 + \bar{\alpha}P) + \bar{\alpha}P(1 + P)}.$$  (70)

We take the derivative of $G(\alpha)$ with respect to $\alpha$:

$$G'(\alpha) \triangleq \frac{2^{2R_0} P(1 + P)(1 - \rho^2)(P + 1 - 2^{2R_0} \rho^2)}{[P(1 + P)\bar{\alpha} + 2^{2R_0}(1 + \bar{\alpha}P)(1 - \rho^2)]^2}.$$

We note that if $\rho^2 \geq 2^{-R_0}(P + 1)$, then $G'(\alpha) < 0$, and hence, $G(\alpha)$ is monotonically decreasing. Achievable rate $R$ is maximized by setting $\alpha^* = 0$. When $\rho^2 < 2^{-R_0}(P + 1)$, we have $G'(\alpha) > 0$, and hence $\alpha^* = \min \left\{ (1 - 2^{-R_0}) \left(1 + \frac{1}{P}\right), 1 \right\} = 1$, since we have $(1 - 2^{-R_0}) \left(1 + \frac{1}{P}\right) \geq (1 + \frac{1 - \rho^2}{P}) > 1$.

**APPENDIX VII**

**PROOF OF LEMMA 9**

In order to characterize the capacity of the binary symmetric MRC-D, we find the optimal distribution of $(U, X_1, \hat{Y}_R)$ in Theorem 1 for $Z \sim \text{Ber}(1/2)$. First, we note that $U$ is independent of $Y_R$ since

$$I(U; Y_R) \leq I(X_1; Y_R) = 0,$$

where the inequality follows from the Markov chain $U - X_1 - Y_R$, and the equality follows since for $Z \sim \text{Ber}(1/2)$ the channel output of the binary channel $Y_R = X_1 \oplus N \oplus Z$ is independent of the channel input $X_1$. Then, the capacity region in (7) is given by

$$C = \sup \left\{ I(X_1; \hat{Y}_R | U Z) : R_0 \geq I(Y_R; \hat{Y}_R | U Z) \right\},$$

over $p(u, x_1)p(z)p(y_R | x_1, z)p(\hat{y}_R | y_R, u).$  (72)
Let us define $\bar{Y} \triangleq X_1 \oplus N$. The capacity is equivalent to
\[
C = \sup_{p(u, x_1)p(z)p(\hat{y}|x_1)p(\hat{y}_R|\bar{y}, u, z)} \{ I(X_1; \bar{Y}_R|UZ) : H(\bar{Y}|U) \geq H(\bar{Y}|U) - R_0 \},
\]
where we have used the fact that $\bar{Y}$ is independent from $Z$.

For any joint distribution for which $0 \leq H(\bar{Y}|U) \leq R_0$, the constraint in (73) is also satisfied. In that case, we can find the following upper bound on the capacity. It follows from the Markov chain $X_1 - \bar{Y} - \hat{Y}_R$ given $U, Z$, and the data processing inequality, that
\[
C \leq \max_{p(u, x_1)} \{ I(X_1; \bar{Y}|ZU) : H(\bar{Y}|U) \leq R_0 \}
\]
(74)
\[
\leq R_0 - h_2(\delta).
\]

We next consider the joint distributions for which $R_0 \leq H(\bar{Y}|U)$. Let $p(u) = \Pr[U = u]$ for $u = 1, ..., |U|$, and we can write
\[
I(X_1; \hat{Y}_R|UZ) = H(X_1|U) - \sum_u p(u)H(\bar{Y}_R|ZU),
\]
(75)
and
\[
I(\bar{Y}_R; \hat{Y}_R|UZ) \overset{(a)}{=} I(\bar{Y}; \hat{Y}_R|UZ)
\]
\[
\overset{(b)}{=} H(\bar{Y}|U) - \sum_u p(u)H(\bar{Y}_R|ZU),
\]
(76)
where $(a)$ follows from the definition of $\bar{Y}$, and $(b)$ follows from the independence of $Z$ from $\bar{Y}$ and $U$.

For each $u$, the channel input $X_1$ corresponds to a binary random variable $X_u \sim \text{Ber}(\nu_u)$, where $\nu_u \triangleq \Pr[X_1 = 1|U = u] = p(1|u)$ for $u = 1, ..., |U|$. The channel output for each $X_u$ is given by $\bar{Y}_u = X_u \oplus N$. We denote by $q_u \triangleq \Pr[Y_u = 1] = \Pr[Y_R = 1|U = u]$. Similarly, we define $\hat{Y}_u$ as $\hat{Y}_R$ for each $u$ value. Note that for each $u$, $X_u - \bar{Y}_u - \hat{Y}_u$ form a Markov chain.

Then, we have $H(X_1|u) = h_2(\nu_u)$ and $H(\bar{Y}|u) = h_2(\delta \ast \nu_u)$. We define $s_u \triangleq H(\bar{Y}_R|ZU)$, such that $0 \leq s_u \leq
\( H(\bar{Y}_u) \). Substituting (75) and (76) in (73) we have

\[
\mathcal{C} = \max_{p(u, x_1)p(\bar{Y}_R|y_R, u)} \left[ H(X_1|U) - \sum_u p(u)H(X_1|\bar{Y}_RZu) \right]
\]

s.t. \( R_0 \geq H(\bar{Y}|U) - \sum_u p(u)H(\bar{Y}|\bar{Y}_RZu) \)

\[= \max_{p(u, x_1)} \left[ H(X_1|U) - \sum_u p(u)F_{TXY}(q_u, s_u) \right]
\]

s.t. \( R_0 \geq H(\bar{Y}|U) - \sum_u p(u)s_u, \ 0 \leq s_u \leq H(\bar{Y}_u) \)

\[\leq \max_{p(u, x_1)} \left[ H(X_1|U) - h_2\left( \delta \ast h_2^{-1}\left( \sum_u p(u)s_u \right) \right) \right]
\]

s.t. \( \sum_u p(u)s_u \geq H(\bar{Y}|U) - R_0 \),

where (a) follows from the definition of \( F_{TXY}(q, s) \) for channel \( \bar{Y}_u = X_u \oplus N \), which for each \( u \) has a matrix \( T_{XY} \) as in (57), (b) follows from the expression of \( F_{TXY}(q, s) \) for the binary channel \( T_{XY} \) in Lemma 11, (c) follows from noting that \( -h_2(\delta \ast h_2^{-1}(s_u)) \) is concave on \( s_u \) from Lemma 12 and applying Jensen’s inequality. We also drop the conditions on \( s_u \), which can only increase \( \mathcal{C} \).

Then, similarly to the proof of Lemma 4 we have \( H(\bar{Y}|U) \geq H(\bar{Y}|UV) = H(X_1|U) \), and we can upper bound the capacity as follows

\[
\mathcal{C} \leq \max_{p(x_1, s)} \left[ H(X_1|U) - h_2\left( \delta \ast h_2^{-1}\left( \sum_u p(u)s_u \right) \right) \right]
\]

s.t. \( \sum_u p(u)s_u \geq \max\{H(X_1|U), R_0\} - R_0 \)

\[\leq \max_{0 \leq \alpha \leq 1} \alpha - h_2(\delta \ast h_2^{-1}(\max\{\alpha, R_0\} - R_0)) \tag{77}
\]

where we have defined \( \alpha \equiv H(X_1|U) \).

The optimization problem can be solved similarly to the proof in Appendix IV as follows. If \( 0 \leq \alpha \leq R_0 \), we have \( \bar{s} \geq 0 \) and

\[
\mathcal{C} \leq \max_{0 \leq \alpha \leq R_0} \alpha - h_2(\delta) = R_0 - h_2(\delta). \tag{78}
\]
For $R_0 \leq \alpha \leq 1$, we have

$$C \leq \max_{R_0 \leq \alpha \leq 1} \alpha - h_2(\delta \ast h_2^{-1}(\alpha - R_0)).$$

(79)

Then, it follows from a scaled version of Proposition 1 that the upper bound is maximized for $\alpha = 1$. Then, by noticing that (78) corresponds to the value of the bound in (79) for $\alpha = R_0$, it follows that

$$C \leq 1 - h_2(\delta \ast h_2^{-1}(1 - R_0)).$$

(80)

This bound is achievable by CF. This completes the proof.

APPENDIX VIII

PROOF OF THE CUT-SET BOUND OPTIMALITY CONDITIONS

Cases 1 and 2 are straightforward since under these assumptions, the ORC-D studied here becomes a particular case of the channel models in [21] and [5], respectively.

To prove case 3 we use the following arguments. For any channel input distribution to the ORC-D, we have

$$I(X_1; Y_R|Z) = H(X_1|Z) - H(X_1|Y_R, Z) \geq H(X_1) - H(X_1|Y_R)$$

$$= I(X_1; Y_R),$$

(81)

where we have used the independence of $X_1$ and $Z$ and the fact that conditioning reduces entropy. Then, the condition $\max_{p(x_1)} I(X_1; Y_R) \geq R_0$, implies $\max_{p(x_1)} I(X_1; Y_R|Z) \geq R_0$; and hence, the cut-set bound is given by $R_{CS} = R_1 + R_0$, which is achievable by DF scheme.

In case 4, the cut-set bound is given by $R_1 + \min\{R_0, I(\bar{X}_1; \bar{Y}_R|Z)\} = R_1 + I(\bar{X}_1; \bar{Y}_R|Z) \geq R_0 \geq H(\bar{Y}_R|Z)$. CF achieves the capacity by letting $X_1$ be distributed with $\bar{p}(x_1)$, and choosing $\hat{Y}_R = \bar{Y}_R$. This choice is always possible as the CF constraint

$$R_0 \geq I(\bar{Y}_R; \bar{Y}_R|Z) = H(\bar{Y}_R|Z) - H(\bar{Y}_R|Z, \bar{Y}_R) = H(\bar{Y}_R|Z),$$

always holds. Then, the achievable rate for CF is $R_{CF} = R_1 + I(\bar{X}_1; \bar{Y}_R|Z) = R_1 + I(\bar{X}_1; \bar{Y}_R|Z)$, which is the capacity.
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