ON JACQUET-LANGLANDS ISOGENY OVER FUNCTION FIELDS

MIHRAN PAPIKIAN

Abstract. We propose a conjectural explicit isogeny from the Jacobians of hyperelliptic Drinfeld modular curves to the Jacobians of hyperelliptic modular curves of $D$-elliptic sheaves. The kernel of the isogeny is a subgroup of the cuspidal divisor group constructed by examining the canonical maps from the cuspidal divisor group into the component groups.

1. Introduction

Let $N$ be a square-free integer, divisible by an even number of primes. It is well-known that the new part of the modular Jacobian $J_0(N)$ is isogenous to the Jacobian of a Shimura curve; see [33]. The existence of this isogeny can be interpreted as a geometric incarnation of the global Jacquet-Langlands correspondence over $\mathbb{Q}$ between the cusp forms on $GL(2)$ and the multiplicative group of a quaternion algebra [24]. Jacquet-Langlands isogeny has important arithmetic applications, for example, to level lowering [35]. In this paper we are interested in the function field analogue of the Jacquet-Langlands isogeny.

Let $F_\mathbb{F}_q$ be the finite field with $q$ elements, and let $F = \mathbb{F}_q(T)$ be the field of rational functions on $\mathbb{F}_q[T]$. The set of places of $F$ will be denoted by $|F|$. Let $A := \mathbb{F}_q[T]$. This is the subring of $F$ consisting of functions which are regular away from the place generated by $1/T$ in $\mathbb{F}_q[T]$. The place generated by $1/T$ will be denoted by $\infty$ and called the place at infinity; it will play a role similar to the archimedean place for $\mathbb{Q}$. The places in $|F| - \infty$ are the finite places.

Let $v \in |F|$. We denote by $F_v$, $O_v$ and $\mathbb{F}_v$ the completion of $F$ at $v$, the ring of integers in $F_v$, and the residue field of $F_v$, respectively. We assume that the valuation $\text{ord}_v : F_v \to \mathbb{Z}$ is normalized by $\text{ord}_v(\pi_v) = 1$, where $\pi_v$ is a uniformizer of $O_v$. The degree of $v$ is $\text{deg}(v) = [\mathbb{F}_v : \mathbb{F}_q]$. Let $q_v := q^{\text{deg}(v)} = \#F_v$. If $v$ is a finite place, then with an abuse of notation we denote the prime ideal of $A$ corresponding to $v$ by the same letter.

Given a field $K$, we denote by $\bar{K}$ an algebraic closure of $K$.

Let $R \subset |F| - \infty$ be a nonempty finite set of places of even cardinality. Let $D$ be the quaternion algebra over $F$ ramified exactly at the places in $R$. Let $X^D_F$ be the modular curve of $D$-elliptic sheaves; see [22]. This curve is the function field analogue of a Shimura curve parametrizing abelian surfaces with multiplication by a maximal order in an indefinite division quaternion algebra over $\mathbb{Q}$. Denote
§

Drinfeld modular curve defined in

played by Drinfeld modular curves. With an abuse of notation, let a square-free ideal of $\mathbb{A}$ be the Drinfeld modular curve defined in [21]. Let $J_0(R)$ be the Jacobian of $X_0(R)_F$. The same strategy as over $\mathbb{Q}$ shows that $J^R$ is isogenous to the new part of $J_0(R)$ (see Theorem [7.1] and Remark [7.4]). The proof relies on Tate’s conjecture, so it provides no information about the isogenies $J^R \to J_0(R)^{\text{new}}$ beyond their existence. In this paper we carefully examine the simplest non-trivial case, namely $R = \{x, y\}$ with $\deg(x) = 1$ and $\deg(y) = 2$. (When $R = \{x, y\}$ and $\deg(x) = \deg(y) = 1$, both $X^R_F$ and $X_0(R)_F$ have genus 0.)

**Notation 1.1.** Unless indicated otherwise, throughout the paper $x$ and $y$ will be two fixed finite places of degree 1 and 2, respectively. When $R = \{x, y\}$, we write $X^{xy}_F$ for $X^R_F$, $J^{xy}$ for $J^R$, $X_0(xy)_F$ for $X_0(R)_F$, and $J_0(xy)$ for $J_0(R)$.

The genus of $X^{xy}_F$ is $q$, which is also the genus of $X_0(xy)_F$. Hence $J_0(xy)$ and $J^{xy}$ are $q$-dimensional Jacobian varieties, which are isogenous over $F$. We would like to construct an explicit isogeny $J_0(xy) \to J^{xy}$. A natural place to look for the kernel of an isogeny defined over $F$ is in the cuspidal divisor group $C$ of $J_0(xy)$. To see which subgroup of $C$ could be the kernel, one needs to compute, besides $C$ itself, the component groups of $J_0(xy)$ and $J^{xy}$, and the canonical specialization maps of $C$ into the component groups of $J_0(xy)$. These calculations constitute the bulk of the paper. Based on these calculations, in [7] we propose a conjectural explicit isogeny $J_0(xy) \to J^{xy}$, and prove that the conjecture is true for $q = 2$. We note that $X^{xy}_F$ is hyperelliptic, and in fact for odd $q$ these are the only $X^{xy}_F$ which are hyperelliptic [31]. The curve $X_0(xy)_F$ is also hyperelliptic, and for levels which decompose into a product of two prime factors these are the only hyperelliptic Drinfeld modular curves [36]. Hence this paper can also be considered as a study of hyperelliptic modular Jacobians over $F$ which interrelates [31] and [36].

The approach to explicating the Jacquet-Langlands isogeny through the study of component groups and cuspidal divisor groups was initiated in the classical context by Ogg. In [27], Ogg proposed in several cases conjectural explicit isogenies between the modular Jacobians and the Jacobians of Shimura curves (as far as I know, these conjectures are still mostly open, but see [14] and [29] for some advances).

We summarize the main results of the paper.

- The cuspidal divisor group $C \subset J_0(xy)(F)$ is isomorphic to

$$C \cong \mathbb{Z} / (q + 1)\mathbb{Z} \oplus \mathbb{Z} / (q^2 + 1)\mathbb{Z}.$$  

- The component groups of $J_0(xy)$ and $J^{xy}$ at $x$, $y$, and $\infty$ are listed in Table 1. ($J_0(xy)$ and $J^{xy}$ have good reduction away from $x$, $y$ and $\infty$, so the component groups are trivial away from these three places.)
If we denote the component group of \( J_0(xy) \) at \(*\) by \( \Phi_* \), and the canonical map \( \mathcal{C} \rightarrow \Phi_* \) by \( \phi_* \), then there are exact sequences
\[
0 \rightarrow \mathbb{Z}/(q + 1)\mathbb{Z} \rightarrow \mathcal{C} \xrightarrow{\phi_*} \Phi_x \rightarrow \mathbb{Z}/(q + 1)\mathbb{Z} \rightarrow 0,
\]
\[
0 \rightarrow \mathbb{Z}/(q^2 + 1)\mathbb{Z} \rightarrow \mathcal{C} \xrightarrow{\phi_*} \Phi_y \rightarrow \mathbb{Z}/(q^2 + 1)\mathbb{Z} \rightarrow 0,
\]
\[
\phi_\infty : \mathcal{C} \xrightarrow{\phi_*} \Phi_\infty \quad \text{if } q \text{ is even},
\]
\[
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{C} \xrightarrow{\phi_\infty} \Phi_\infty \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \quad \text{if } q \text{ is odd}.
\]

* The kernel \( \mathcal{C}_0 \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z} \) of \( \phi_y \) maps injectively into \( \Phi_x \) and \( \Phi_\infty \).

Conjecture 7.3 then states that there is an isogeny \( J_0(xy) \rightarrow J^{xy} \) whose kernel is \( \mathcal{C}_0 \). As an evidence for the conjecture, we prove that the quotient abelian variety \( J_0(xy)/\mathcal{C}_0 \) has component groups of the same order as \( J^{xy} \). This is a consequence of a general result (Theorem 1.3), which describes how the component groups of abelian varieties with toric reduction change under isogenies. Finally, we prove Conjecture 7.3 for \( q = 2 \) (Theorem 7.12); the proof relies on the fact that \( J_0(xy) \) in this case is isogenous to a product of two elliptic curves. Two other interesting consequences of our results are the following. First, we deduce the genus formula for \( X_0^2 \) proven in \([30]\) by a different argument (Corollary 6.3). Second, assuming \( q \) is even and Conjecture 7.3 is true, we are able to tell how the optimal elliptic curve with conductor \( xy/\infty \) changes in a given \( F \)-isogeny class when we change the modular parametrization from \( X_0(xy)_F \) to \( X_0^{xy} \) (Proposition 7.10).

2. Preliminaries

### 2.1. Drinfeld modular curves.

Let \( K \) be an \( A \)-field, i.e., \( K \) is a field equipped with a homomorphism \( \gamma : A \rightarrow K \). In particular, \( K \) contains \( \mathbb{F}_q \) as a subfield. The \( A \)-characteristic of \( K \) is the ideal \( \ker(\gamma) < A \). Let \( K\{\tau\} \) be the twisted polynomial ring with commutation rule \( \tau s = s^g \tau, \ s \in K \). A rank-2 Drinfeld \( A \)-module over \( K \) is a ring homomorphism \( \phi : A \rightarrow K\{\tau\}, \ a \mapsto \phi_a \) such that \( \deg_\tau \phi_a = -2\text{ord}_\Delta(a) \) and the constant term of \( \phi_a \) is \( \gamma(a) \). A homomorphism of two Drinfeld modules \( u : \phi \rightarrow \psi \) is \( u \in K\{\tau\} \) such that \( \phi_a u = u\psi_a \) for all \( a \in A \); \( u \) is an isomorphism if \( u \in K^\times \). Note that \( \phi \) is uniquely determined by the image of \( T \):
\[
\phi_T = \gamma(T) + g\tau + \Delta\tau^2,
\]
where \( g \in K \) and \( \Delta \in K^\times \). The \( j \)-invariant of \( \phi \) is \( j(\phi) = g^{q+1}/\Delta \). It is easy to check that if \( K \) is algebraically closed, then \( \phi \cong \psi \) if and only if \( j(\phi) = j(\psi) \).

Treating \( \tau \) as the automorphism of \( K \) given by \( k \mapsto k^q \), the field \( K \) acquires a new \( A \)-module structure via \( \phi \). Let \( a \triangleleft A \) be an ideal. Since \( A \) is a principal ideal domain, we can choose a generator \( a \in A \) of \( a \). The \( A \)-module \( \phi[a] = \ker \phi_a(K) \) does not depend on the choice of \( a \) and is called the \( a \)-torsion of \( \phi \). If \( a \) is coprime to the \( A \)-characteristic of \( K \), then \( \phi[a] \cong (A/a)^2 \). On the other hand, if \( p = \ker(\gamma) \neq 0 \), then \( \phi[p] \cong (A/p) \) or \( 0 \); when \( \phi[p] = 0 \), \( \phi \) is called supersingular.

**Lemma 2.1.** Up to isomorphism, there is a unique supersingular rank-2 Drinfeld \( A \)-module over \( \overline{\mathbb{F}}_q \): it is the Drinfeld module with \( j \)-invariant equal to 0. Up to isomorphism, there is a unique supersingular rank-2 Drinfeld \( A \)-module over \( \mathbb{F}_q^2 \), and its \( j \)-invariant is non-zero.

**Proof.** This follows from \([9\ (5.9)]\) since \( \deg(x) = 1 \) and \( \deg(y) = 2 \). \(\square\)
Let \( \text{End}(\phi) \) denote the centralizer of \( \phi(A) \) in \( \hat{K}\{\tau\} \), i.e., the ring of all homomorphisms \( \phi \to \phi \) over \( \hat{K} \). The automorphism group \( \text{Aut}(\phi) \) is the group of units \( \text{End}(\phi)^\times \).

**Lemma 2.2.** If \( j(\phi) \neq 0 \), then \( \text{Aut}(\phi) \cong \mathbb{F}_q^\times \). If \( j(\phi) = 0 \), then \( \text{Aut}(\phi) \cong \mathbb{F}_q^2 \). 

*Proof.* If \( u \in \hat{K}^\times \) commutes with \( \phi_T = \gamma(T) + g\tau + \Delta \tau^2 \), then \( u^q - 1 = 1 \) and \( u^{q-1} = 1 \) if \( g \neq 0 \). This implies that \( u \in \mathbb{F}_q^\times \) if \( j(\phi) \neq 0 \), and \( u \in \mathbb{F}_q^2 \) if \( j(\phi) = 0 \). On the other hand, we clearly have the inclusions \( \mathbb{F}_q^\times \subset \text{Aut}(\phi) \) and, if \( j(\phi) = 0 \), \( \mathbb{F}_q^2 \subset \text{Aut}(\phi) \). This finishes the proof. 

**Lemma 2.3.** Let \( p \nmid A \) be a prime ideal and \( \mathbb{F}_p := A/p \). Let \( \phi \) be a rank-2 Drinfeld \( A \)-module over \( \mathbb{F}_p \). Let \( n \nmid A \) be an ideal coprime to \( p \). Let \( C_n \) be an \( A \)-submodule of \( \phi[n] \) isomorphic to \( A/n \). Denote by \( \text{Aut}(\phi,C_n) \) the subgroup of automorphisms of \( \phi \) which map \( C_n \) to itself. Then \( \text{Aut}(\phi,C_n) \cong \mathbb{F}_q^\times \) or \( \mathbb{F}_q^2 \). The second case is possible only if \( j(\phi) = 0 \).

*Proof.* The action of \( \mathbb{F}_q^\times \) obviously stabilizes \( C_n \), hence, using Lemma 2.2, it is enough to show that if \( \text{Aut}(\phi,C_n) \neq \mathbb{F}_q^\times \), then \( \text{Aut}(\phi,C_n) \cong \mathbb{F}_q^2 \). Let \( u \in \text{Aut}(\phi,C_n) \) be an element which is not in \( \mathbb{F}_q \). Then \( \text{Aut}(\phi) = \mathbb{F}_q[u] \cong \mathbb{F}_q^2 \), where \( \mathbb{F}_q[u] \) is considered as a finite subring of \( \text{End}(\phi) \). It remains to show that \( \alpha + u\beta \) stabilizes \( C_n \) for any \( \alpha, \beta \in \mathbb{F}_q \) not both equal to zero. But this is obvious since \( \alpha \) and \( u\beta \) stabilize \( C_n \) and \( C_n \cong A/n \) is cyclic. 

One can generalize the notion of Drinfeld modules over an \( A \)-field to the notion of Drinfeld modules over an arbitrary \( A \)-scheme \( S \) [8]. The functor which associates to an \( A \)-scheme \( S \) the set of isomorphism classes of pairs \((\phi,C_n)\), where \( \phi \) is a Drinfeld \( A \)-module of rank 2 over \( S \) and \( C_n \cong A/n \) is an \( A \)-submodule of \( \phi[n] \), possesses a coarse moduli scheme \( Y_\ell(n) \) that is affine, flat and of finite type over \( A \) of pure relative dimension 1. There is a canonical compactification \( X_0(n) \) of \( Y_\ell(n) \) over \( \text{Spec}(A) \); see [8, §9] or [11]. The finitely many points \( X_0(n)(\hat{F}) - Y_\ell(n)(\hat{F}) \) are called the cusps of \( X_0(n) \).

Denote by \( \mathbb{C}_\infty \) the completion of an algebraic closure of \( F_\infty \). Let \( \Omega = \mathbb{C}_\infty - F_\infty \) be the Drinfeld upper half-plane; \( \Omega \) has a natural structure of a smooth connected rigid-analytic space over \( F_\infty \). Denote by \( \Gamma_0(n) \) the Hecke congruence subgroup of level \( n \): 

\[
\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A) \mid c \in n \right\}.
\]

The group \( \Gamma_0(n) \) naturally acts on \( \Omega \) via linear fractional transformations, and the action is discrete in the sense of [8, p. 582]. Hence we may construct the quotient \( \Gamma_0(n) \backslash \Omega \) as a 1-dimensional connected smooth analytic space over \( F_\infty \).

The following theorem can be deduced from the results in [8]:

**Theorem 2.4.** \( X_0(n) \) is a proper flat scheme of pure relative dimension 1 over \( \text{Spec}(A) \), which is smooth away from the support of \( n \). There is an isomorphism of rigid-analytic spaces \( \Gamma_0(n) \backslash \Omega \cong Y_0(n)_{F_\infty} \).

There is a genus formula for \( X_0(n) \), which depends on the prime decomposition of \( n \); see [16] Thm. 2.17. By this formula, the genera of \( X_0(x)_F \), \( X_0(y)_F \) and \( X_0(xy)_F \) are 0, 0 and \( q \), respectively.
2.2. Modular curves of $D$-elliptic sheaves. Let $D$ be a quaternion algebra over $F$. Let $R \subset |F|$ be the set of places which ramify in $D$, i.e., $D \otimes F_v$ is a division algebra for $v \in R$. It is known that $R$ is finite of even cardinality, and, up to isomorphism, this set uniquely determines $D$; see [42]. Assume $R \neq \emptyset$ and $\infty \notin R$. In particular, $D$ is a division algebra. Let $C := \mathbb{P}^1_{\mathbb{P}^1_v}$. Fix a locally free sheaf $D$ of $\mathcal{O}_C$-algebras with stalk at the generic point equal to $D$ and such that $D_v := D \otimes_{\mathcal{O}_C} \mathcal{O}_v$ is a maximal order in $D_v := D \otimes_F F_v$.

Let $S$ be an $\mathbb{F}_q$-scheme. Denote by $\text{Frob}_S$ its Frobenius endomorphism, which is the identity on the points and the $q$th power map on the functions. Denote by $C \times S$ the fibered product $C \times_{\text{Spec}(\mathbb{F}_q)} S$. Let $z : S \to C$ be a morphism of $\mathbb{F}_q$-schemes. A $D$-elliptic sheaf over $S$, with pole $\infty$ and zero $z$, is a sequence $E = (E_i,j_i,t_i)_{i \in \mathbb{Z}}$, where each $E_i$ is a locally free sheaf of $\mathcal{O}_{C \times S}$-modules of rank 4 equipped with a right action of $D$ compatible with the $\mathcal{O}_C$-action, and where

$$j_i : E_i \to E_{i+1},$$

$$t_i : E_i := (\text{Id}_C \times \text{Frob}_S)^* E_i \to E_{i+1}$$

are injective $\mathcal{O}_{C \times S}$-linear homomorphisms compatible with the $D$-action. The maps $j_i$ and $t_i$ are sheaf modifications at $\infty$ and $z$, respectively, which satisfy certain conditions, and it is assumed that for each closed point $w$ of $S$, the Euler-Poincaré characteristic $\chi(E_0|_{C \times w})$ is in the interval $[0,2]$; we refer to [26] §2 and [22] §1 for the precise definition. Moreover, to obtain moduli schemes with good properties at the closed points $w$ of $S$ such that $z(w) \in R$ one imposes an extra condition on $E$ to be “special” [22, p. 1305]. Note that, unlike the original definition in [26], $\infty$ is allowed to be in the image of $S$; here we refer to [11] §4.4 for the details. Denote by $\Ell^D(S)$ the set of isomorphism classes of $D$-elliptic sheaves over $S$. The following theorem can be deduced from some of the main results in [26] and [22].

**Theorem 2.5.** The functor $S \mapsto \Ell^D(S)$ has a coarse moduli scheme $X^R$, which is proper and flat of pure relative dimension 1 over $C$ and is smooth over $C - R - \infty$.

**Remark 2.6.** Theorems [23] and [25] imply that $J_0(R)$ and $J^R$ have good reduction at any place $v \in |F| - R - \infty$; cf. [2] Ch. 9.

### 3. Cuspidal divisor group

For a field $K$, we represent the elements of $\mathbb{P}^1(K)$ as column vectors $\begin{pmatrix} u \\ v \end{pmatrix}$ where $u, v \in K$ are not both zero and $\begin{pmatrix} u \\ v \end{pmatrix}$ is identified with $\begin{pmatrix} \alpha u \\ \alpha v \end{pmatrix}$ if $\alpha \in K^\times$. We assume that $\text{GL}_2(K)$ acts on $\mathbb{P}^1(K)$ on the left by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}.$$

Let $\mathfrak{n} \ll A$ be an ideal. The cusps of $X_0(\mathfrak{n})_F$ are in natural bijection with the orbits of $\Gamma_0(\mathfrak{n})$ acting from the left on $\mathbb{P}^1(F)$.

**Lemma 3.1.** If $\mathfrak{n}$ is square-free, then there are $2^s$ cusps on $X_0(\mathfrak{n})_F$, where $s$ is the number of prime divisors of $\mathfrak{n}$. All the cusps are $F$-rational.

**Proof.** See Proposition 3.3 and Corollary 3.4 in [11]. □
For every \( m|n \) with \( (m,n/m) = 1 \) there is an Atkin-Lehner involution \( W_m \) on \( X_0(n)_F \), cf. [36]. Its action is given by multiplication from the left with any matrix \( \begin{pmatrix} ma & b \\ n & m \end{pmatrix} \) whose determinant generates \( m \), and where \( a,b,m,n \in A \), \( (n) = n \), \( (m) = m \).

From now on assume \( n = xy \). Recall that we denote by \( x \) and \( y \) the prime ideals of \( A \) corresponding to the places \( x \) and \( y \), respectively. With an abuse of notation, we will denote by \( x \) also the monic irreducible polynomial in \( A \) generating the ideal \( x \), and similarly for \( y \). It should be clear from the context in which capacity \( x \) and \( y \) are being used. With this notation, \( X_0(xy)_F \) has 4 cusps, which can be represented by

\[
[\infty] := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [0] := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [x] := \begin{pmatrix} 1 \\ x \end{pmatrix}, \quad [y] := \begin{pmatrix} 1 \\ y \end{pmatrix},
\]

cf. [36] p. 333 and [15] p. 196.

There are 3 non-trivial Atkin-Lehner involutions \( W_x, W_y, W_{xy} \) which generate a group isomorphic to \((\mathbb{Z}/2\mathbb{Z})^2\): these involutions commute with each other and satisfy

\[
W_xW_y = W_{xy}, \quad W_x^2 = W_y^2 = W_{xy} = 1.
\]

By [36] Prop. 9, none of these involutions fixes a cusp. In fact, a simple direct calculation shows that

\[
W_{xy}([\infty]) = [0], \quad W_{xy}([x]) = [y];
\]
\[
W_x([\infty]) = [y], \quad W_x([0]) = [x];
\]
\[
W_y([\infty]) = [x], \quad W_y([0]) = [y].
\]

Let \( \Delta(z) \), \( z \in \Omega \), denote the Drinfeld discriminant function; see [11] or [15] for the definition. This is a holomorphic and nowhere vanishing function on \( \Omega \). In fact, \( \Delta(z) \) is a type-0 and weight-(\(q^2 - 1\)) cusp form for \( \text{GL}_2(A) \). Its order of vanishing at the cusps of \( X_0(n)_F \) can be calculated using [15]. When \( n = xy \), [15] (3.10) implies

\[
\text{ord}_{[\infty]}\Delta = 1, \quad \text{ord}_{[0]}\Delta = q_xq_y, \quad \text{ord}_{[x]}\Delta = q_y, \quad \text{ord}_{[y]}\Delta = q_x.
\]

The functions

\[
\Delta_x(z) := \Delta(xz), \quad \Delta_y(z) := \Delta(yz), \quad \Delta_{xy}(z) := \Delta(xyz)
\]

are type-0 and weight-(\(q^2 - 1\)) cusp forms for \( \Gamma_0(xy) \). Hence the fractions \( \Delta/\Delta_x \), \( \Delta/\Delta_y \), \( \Delta/\Delta_{xy} \) define rational functions on \( X_0(xy)_{\infty} \). We compute the divisors of these functions.

The matrix \( W_{xy} = \begin{pmatrix} 0 & 1 \\ xy & 0 \end{pmatrix} \) normalizes \( \Gamma_0(xy) \) and interchanges \( \Delta(z) \) and \( \Delta_{xy}(z) \). Thus by [36] and [38]

\[
\text{ord}_{[\infty]}\Delta_{xy} = q_xq_y, \quad \text{ord}_{[0]}\Delta_{xy} = 1, \quad \text{ord}_{[x]}\Delta_{xy} = q_x, \quad \text{ord}_{[y]}\Delta_{xy} = q_y.
\]

A similar argument involving the actions of \( W_x \) and \( W_y \) gives

\[
\text{ord}_{[\infty]}\Delta_x = q_x, \quad \text{ord}_{[0]}\Delta_x = q_y, \quad \text{ord}_{[x]}\Delta_x = q_xq_y, \quad \text{ord}_{[y]}\Delta_x = 1;
\]
\[
\text{ord}_{[\infty]}\Delta_y = q_y, \quad \text{ord}_{[0]}\Delta_y = q_x, \quad \text{ord}_{[x]}\Delta_y = 1, \quad \text{ord}_{[y]}\Delta_y = q_xq_y.
\]
From these calculations we obtain
\[
\text{div}(\Delta/\Delta_{xy}) = (1 - q_x q_y)[\infty] + (q_x q_y - 1)[0] + (q_x - q_y)[x] + (q_x q_y)[y] = (q^3 - 1)((0) - [\infty]) + (q^2 - q)([x] - [y]),
\]
and similarly,
\[
\text{div}(\Delta/\Delta_x) = (q - 1)([y] - [\infty]) + (q^3 - q^2)([0] - [x]),
\]
\[
\text{div}(\Delta/\Delta_y) = (q^2 - 1)([x] - [\infty]) + (q^3 - q)([0] - [y]).
\]

Next, by [15, p. 200], the largest positive integer \(k\) such that \(\Delta/\Delta_{xy}\) has a \(k\)th root in the field of modular functions for \(\Gamma_0(x)\) is \((q - 1)^2/(q - 1) = (q - 1)\). We can apply the same argument to \(\Delta/\Delta_x\) as a modular function for \(\Gamma_0(x)\) to deduce that \(\Delta/\Delta_x\) has \((q - 1)^2/(q - 1)\)th root. Similarly, \(\Delta/\Delta_y\) has \((q - 1)(q^2 - 1)/(q - 1)\)th root. Therefore, the following relations hold in \(\text{Pic}^0(X_0(xy))_F\):

\[
(q^2 + q + 1)((0) - [\infty]) + q([x] - [y]) = 0
\]
\[
([y] - [\infty]) + q^2([0] - [x]) = 0
\]
\[
([x] - [\infty]) + q([0] - [y]) = 0.
\]

There is one more relation between the cuspidal divisors which comes from the fact that \(X_0(xy)_F\) is hyperelliptic. By a theorem of Schweizer [36, Thm. 20], \(X_0(xy)_F\) is hyperelliptic, and \(W_{xy}\) is the hyperelliptic involution. Consider the degree-2 covering
\[
\pi : X_0(xy)_F \to X_0(xy)_F/W_{xy} \cong \mathbb{P}^1.
\]
Denote \(P := \pi([\infty]), Q := \pi([x]).\) Since \(W_{xy}([\infty]) \neq [x], P \neq Q\). There is a function \(f\) on \(\mathbb{P}^1_k\) with divisor \(P - Q\). Now
\[
\text{div}((\pi^*f)) = \pi^*(\text{div}(f)) = \pi^*(P - Q)
\]
\[
= ([\infty] + W_{xy}([\infty])) - ([x] + W_{xy}([x])) = [\infty] + [0] - [x] - [y].
\]
This gives the following relations in \(\text{Pic}^0(X_0(xy)_F)\)

\[
[\infty] + [0] - [x] - [y] = 0.
\]

Fixing \([\infty] \in X_0(xy)(F)\) as an \(F\)-rational point, we have the Abel-Jacobi map \(X_0(xy)_F \to J_0(xy)\) which sends a point \(P \in X_0(xy)_F\) to the linear equivalence class of the degree-0 divisor \(P - [\infty]\).

**Definition 3.2.** Let \(c_0, c_x, c_y \in J_0(xy)(F)\) be the classes of \([0] - [\infty], [x] - [\infty],\) and \([y] - [\infty],\) respectively. These give \(F\)-rational points on the Jacobian since the cusps are \(F\)-rational. The **cuspidal divisor group** is the subgroup \(\mathcal{C} \subset J_0(xy)\) generated by \(c_0, c_x,\) and \(c_y\).

From (3.4) and (3.4), we obtain the following relations:
\[
(q^2 + q + 1)c_0 + q c_x - q c_y = 0
\]
\[
q^2 c_0 - q^2 c_x + c_y = 0
\]
\[
q c_0 + c_x - q c_y = 0
\]
\[
c_0 - c_x - c_y = 0.
\]

**Lemma 3.3.** The **cuspidal divisor group** \(\mathcal{C}\) is generated by \(c_x\) and \(c_y\), which have orders dividing \(q + 1\) and \(q^2 + 1\), respectively.
Proof. Substituting $c_0 = c_x + c_y$ into the first three equations above, we see that $C$ is generated by $c_x$ and $c_y$ subject to relations:

$$(q + 1)c_x = 0$$
$$(q^2 + 1)c_y = 0.$$ 

The following simple lemma, which will be used later on, shows that the factors $(q^2 + 1)$ and $(q + 1)$ appearing in Lemma 3.3 are almost coprime.

**Lemma 3.4.** Let $n$ be a positive integer. Then

$$\gcd(n^2 + 1, n + 1) = \begin{cases} 
1, & \text{if } n \text{ is even;} \\
2, & \text{if } n \text{ is odd.}
\end{cases}$$

**Proof.** Let $d = \gcd(n^2 + 1, n + 1)$. Then $d$ divides $(n^2 + 1) - (n + 1) = n(n - 1)$. Since $n$ is coprime to $n + 1$, $d$ must divide $n - 1$, hence also must divide $(n + 1) - (n - 1) = 2$. For $n$ even, $d$ is obviously odd, so $d = 1$. For $n$ odd, $n + 1$ and $n^2 + 1$ are both even, so $d = 2$. □

### 4. Néron models and component groups

#### 4.1. Terminology and notation.

The notation in this section will be somewhat different from the rest of the paper. Let $R$ be a complete discrete valuation ring, with fraction field $K$ and algebraically closed residue field $k$.

Let $A_K$ be an abelian variety over $K$. Denote by $A$ its Néron model over $R$ and denote by $A_0$ the connected component of the identity of the special fiber $A_k$ of $A$. There is an exact sequence

$$0 \to A_0 \to A \to \Phi_A \to 0,$$

where $\Phi_A$ is a finite (abelian) group called the component group of $A_K$. We say that $A_K$ has semi-abelian reduction if $A_0$ is an extension of an abelian variety $A'_k$ by an affine algebraic torus $T_A$ over $k$ (cf. [2, p. 181]):

$$0 \to T_A \to A_0 \to A'_k \to 0.$$ 

We say that $A_K$ has toric reduction if $A_0 = T_A$. The character group

$$M_A := \text{Hom}(T_A, \mathbb{G}_m, k)$$

is a free abelian group contravariantly associated to $A$.

Let $X_K$ be a smooth, proper, geometrically connected curve over $K$. We say that $X_K$ is a semi-stable model of $X_K$ over $R$ if (cf. [2, p. 245]):

(i) $X$ is a proper flat $R$-scheme.

(ii) The generic fiber of $X$ is $X_K$.

(iii) The special fiber $X_k$ is reduced, connected, and has only ordinary double points as singularities.

We will denote the set of irreducible components of $X_k$ by $C(X)$ and the set of singular points of $X_k$ by $S(X)$. Let $G(X)$ be the dual graph of $X$: The set of vertices of $G(X)$ is the set $C(X)$, the set of edges is the set $S(X)$, the end points of an edge $x$ are the two components containing $x$. Locally at $x \in S(X)$ for the étale topology, $X$ is given by the equation $uv = \pi^{m(x)}$, where $\pi$ is a uniformizer of $R$. The integer $m(x) \geq 1$ is well-defined, and will be called the thickness of $x$. One
obtains from $G(X)$ a graph with length by assigning to each edge $x \in S(X)$ the length $m(x)$.

4.2. Raynaud’s theorem. Let $X_K$ be a curve over $K$ with semi-stable model $X$ over $R$. Let $J_K$ be the Jacobian of $X_K$, let $J$ be the Néron model of $J_K$ over $R$, and $\Phi := J_K/J'_K$. Let $\bar{X} \to X$ be the minimal resolution of $X$. Let $B(\bar{X})$ be the free abelian group generated by the elements of $C(\bar{X})$. Let $B^0(\bar{X})$ be the kernel of the homomorphism

$$B(\bar{X}) \to \mathbb{Z}, \quad \sum_{C_i \in C(\bar{X})} n_i C_i \mapsto \sum n_i.$$

The elements of $C(\bar{X})$ are Cartier divisors on $\bar{X}$, hence for any two of them, say $C$ and $C'$, we have an intersection number $(C \cdot C')$. The image of the homomorphism

$$\alpha : B(\bar{X}) \to B(\bar{X}), \quad C \mapsto \sum_{C_i \in C(\bar{X})} (C \cdot C') C'$$

lies in $B^0(\bar{X})$. A theorem of Raynaud [2, Thm. 9.6/1] says that $\Phi$ is canonically isomorphic to $B^0(\bar{X})/\alpha(B(\bar{X}))$.

The homomorphism $\phi : J_K(K) \to \Phi$ obtained from the composition

$$J_K(K) = J(R) \to J_k(k) \to \Phi$$

will be called the canonical specialization map. Let $D = \sum_Q n_Q Q$ be a degree-0 divisor on $X_K$ whose support is in the set of $K$-rational points. Let $P \in J_K(K)$ be the linear equivalence class of $D$. The image $\phi(P)$ can be explicitly described as follows. Since $X$ and $\bar{X}$ are proper, $X(K) = X(R) = \bar{X}(R)$. Since $\bar{X}$ is regular, each point $Q \in X(K)$ specializes to a unique element $c(Q)$ of $C(\bar{X})$. With this notation, $\phi(P)$ is the image of $\sum_Q n_Q c(Q) \in B^0(\bar{X})$ in $\Phi$.

We apply Raynaud’s theorem to compute $\Phi$ explicitly for a special type of $X$. Assume that $X_k$ consists of two components $Z$ and $Z'$ crossing transversally at $n \geq 2$ points $x_1, \ldots, x_n$. Denote $m_i := m(x_i)$. Let $r : \bar{X} \to X$ denote the resolution morphism; it is a composition of blow-ups at the singular points. It is well-known that $r^{-1}(x_i)$ is a chain of $m_i - 1$ projective lines. More precisely, the special fiber $\bar{X}_k$ consists of $Z$ and $Z'$ but now, instead of intersecting at $x_i$, these components are joined by a chain $E_{i1}, \ldots, E_{im-1}$ of projective lines, where $E_i$ intersect $E_{i+1}, E_1$ intersects $Z$ at $x_i$ and $E_{m-1}$ intersects $Z'$ at $x_i$. All the singularities are ordinary double points.

Assume $m_1 = m_n = m \geq 1$ and $m_2 = \cdots = m_{n-1} = 1$ if $n \geq 3$.

If $m = 1$, then $X = \bar{X}$, so $B^0(\bar{X})$ is freely generated by $z := Z - Z'$. In this case Raynaud’s theorem implies that $\Phi$ is isomorphic to $B^0(\bar{X})$ modulo the relation $nz = 0$.

If $m \geq 2$, let $E_1, \ldots, E_{m-1}$ be the chain of projective lines at $x_1$ and $G_1, \ldots, G_{m-1}$ be the chain of projective lines at $x_n$, with the convention that $Z$ in $\bar{X}_k$ intersects $E_1$ and $G_1$, cf. Figure 1. The elements $z := Z - Z', e_i := E_i - Z', g_i := G_i - Z'$, $1 \leq i \leq m - 1$ form a $Z$-basis of $B^0(\bar{X})$. By Raynaud’s theorem, $\Phi$ is isomorphic to $B^0(\bar{X})$ modulo the following relations:

- if $m = 2$, $-nz + e_1 + g_1 = 0$, $z - 2e_1 = 0$, $z - 2g_1 = 0$;
Figure 1. $\tilde{X}_k$ for $n = 5$ and $m = 4$

if $m = 3$,  
\[-nz + e_1 + g_1 = 0, \quad z - 2e_1 + e_2 = 0, \quad z - 2g_1 + g_2 = 0,\]
\[e_1 - 2e_2 = 0, \quad g_1 - 2g_2 = 0;\]

if $m \geq 4$  
\[-nz + e_1 + g_1 = 0, \quad z - 2e_1 + e_2 = 0, \quad z - 2g_1 + g_2 = 0, \]
\[e_i - 2e_{i+1} + e_{i+2} = 0, \quad g_i - 2g_{i+1} + g_{i+2} = 0, \quad 1 \leq i \leq m - 3,\]
\[e_{m-2} - 2e_{m-1} = 0, \quad g_{m-2} - 2g_{m-1} = 0.\]

**Theorem 4.1.** Denote the images of $z, e_i, g_i$ in $\Phi$ by the same letters, and let $\langle z \rangle$ be the cyclic subgroup generated by $z$ in $\Phi$. Then for any $n \geq 2$ and $m \geq 1$

(i) $\Phi \cong \mathbb{Z}/m(m(n-2)+2)\mathbb{Z}$.

(ii) If $m \geq 2$, then $\Phi$ is generated by $e_{m-1}$. Explicitly, for $1 \leq i \leq m - 1$,
\[e_i = (m - i)e_{m-1},\]
\[g_i = (i(nm + 1) - (2i - 1)m)e_{m-1},\]
\[z = me_{m-1}.\]

(iii) $\Phi/\langle z \rangle \cong \mathbb{Z}/m\mathbb{Z}$.

**Proof.** When $m = 1$ the claim is obvious, so assume $m \geq 2$. By [2] Prop. 9.6/10], $\Phi$ has order
\[\sum_{i=1}^{n} \prod_{j \neq i} m_j = m^2(n-2) + 2m.\]

From the relations
\[e_{m-2} - 2e_{m-1} = 0,\]
\[e_i - 2e_{i+1} + e_{i+2} = 0, \quad 1 \leq i \leq m - 3,\]
\[z - 2e_1 + e_2 = 0\]
it follows inductively that $e_i = (m - i)e_{m-1}$ for $1 \leq i \leq m - 1$, and $z = me_{m-1}$. Next, from the relations
\[-nz + e_1 + g_1 = 0 \quad \text{and} \quad z - 2g_1 + g_2 = 0\]
we get $g_1 = (nm - m + 1)e_{m-1}$ and $g_2 = (2nm - 3m + 2)e_{m-1}$. Finally, if $m \geq 4$, the relations $g_i - 2g_{i+1} + g_{i+2} = 0, 1 \leq i \leq m - 3$, show inductively that
\[g_i = (i(nm + 1) - (2i - 1)m)e_{m-1}, \quad 1 \leq i \leq m - 1.\]

This proves (i) and (ii), and (iii) is an immediate consequence of (ii). $\square$
Remark 4.2. Note that by the formula in Theorem 4.1
\[ g_{m-1} = (m^2(n-2) + 2m - (m(n-2) + 1))e_{m-1} = -(m(n-2) + 1)e_{m-1}. \]
It is easy to see that \( m(n-2) + 1 \) is coprime to the order \( m(m(n-2) + 2) \) of \( \Phi \). Hence \( g_{m-1} \) is also a generator. This is of course not surprising since the relations defining \( \Phi \) remain the same if we interchange \( e_i \)'s and \( g_i \)'s.

4.3. Grothendieck's theorem. Grothendieck gave another description of \( \Phi \) in [20]. This description will be useful for us when studying maps between the component groups induced by isogenies of abelian varieties.

Let \( A_K \) be an abelian variety over \( K \) with semi-abelian reduction. Denote by \( \hat{A}_K \) the dual abelian variety of \( A_K \). As discussed in [20], there is a non-degenerate pairing \( u_A : M_A \times M_A \to \mathbb{Z} \) (called monodromy pairing) having nice functorial properties, which induces an exact sequence
\[ 0 \to M_A \xrightarrow{u_A} \text{Hom}(M_A, \mathbb{Z}) \to \Phi_A \to 0. \]

Let \( H \subset A_K(K) \) be a finite subgroup of order coprime to the characteristic of \( k \). Since \( A(R) = A_K(K) \), \( H \) extends to a constant étale subgroup-scheme \( \mathcal{H} \) of \( A \). The restriction to the special fiber gives a natural injection \( \mathcal{H}_K \cong H \twoheadrightarrow A(k) \), cf. [2] Prop. 7.3/3]. Composing this injection with \( A_k \to \Phi_A \), we get the canonical homomorphism \( \phi : H \to \Phi_A \). Denote \( H_0 := \ker(\phi) \) and \( H_1 := \text{im}(\phi) \), so that there is a tautological exact sequence
\[ 0 \to H_0 \to H \xrightarrow{\phi} H_1 \to 0. \]
Let \( B_K \) be the abelian variety obtained as the quotient of \( A_K \) by \( H \). Let \( \varphi_K : A_K \to B_K \) denote the isogeny whose kernel is \( H \). By the Néron mapping property, \( \varphi_K \) extends to a morphism \( \varphi : A \to B \) of the Néron models. On the special fibers we get a homomorphism \( \varphi_k : A_k \to B_k \), which induces an isogeny \( \varphi^0_k : A^0_k \to B^0_k \) and a homomorphism \( \varphi^0_k : \Phi_A \to \Phi_B \). The isogeny \( \varphi^0_k \) restricts to an isogeny \( \varphi^*_k : T_A \to T_B \), which corresponds to an injective homomorphisms of character groups \( \varphi^* : M_B \to M_A \) with finite cokernel.

Theorem 4.3. Assume \( A_K \) has toric reduction. There is an exact sequence
\[ 0 \to H_1 \to \Phi_A \xrightarrow{\varphi^*} \Phi_B \to H_0 \to 0. \]

Proof. The kernel of \( \varphi_k \) is \( \mathcal{H}_k \cong H \). It is clear that \( \ker(\varphi_k) = H_1 \). Let \( \hat{\varphi}_K : \hat{B}_K \to \hat{A}_K \) be the isogeny dual to \( \varphi_K \). Using (4.1), one obtains a commutative diagram with exact rows (cf. [34] p. 89):
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M_A & \longrightarrow & \text{Hom}(M_A, \mathbb{Z}) & \longrightarrow & \Phi_A & \longrightarrow & 0 \\
& & \downarrow{\varphi^*} & & \downarrow{\text{Hom}(\varphi^*, \mathbb{Z})} & & \downarrow{\varphi^*} & \\
0 & \longrightarrow & M_B & \longrightarrow & \text{Hom}(M_B, \mathbb{Z}) & \longrightarrow & \Phi_B & \longrightarrow & 0.
\end{array}
\]
From this diagram we get the exact sequence
\[ 0 \to \ker(\varphi_k) \to M_B/\varphi^*(M_A) \to \text{Ext}^1_{\mathbb{Z}}(M_A/\varphi^*(M_B), \mathbb{Z}) \to \text{coker}(\varphi_k) \to 0. \]
Using the exact sequence \( 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \), it is easy to show that
\[ \text{Ext}^1_{\mathbb{Z}}(M_A/\varphi^*(M_B), \mathbb{Z}) \cong \text{Hom}(M_A/\varphi^*(M_B), \mathbb{Q}/\mathbb{Z}) =: (M_A/\varphi^*(M_B))^\vee, \]
so there is an exact sequence of abelian groups
\[ 0 \to \ker(\varphi_\Phi) \to M_B/\hat{\varphi}^*(M_A) \to (M_A/\varphi^*(M_B))^\vee \to \coker(\varphi_\Phi) \to 0. \]

Since we have not used the assumption that $A_K$ has toric reduction. Under this assumption, $B_K$ also has toric reduction, and $H_0$ is the kernel of $\varphi_1 : T_A \to T_B$. Hence $(M_A/\varphi^*(M_B))^\vee \cong H_0$. Next, [5 Thm. 8.6] implies that $M_B/\hat{\varphi}^*(M_A) \cong H_1$. Thus, we can rewrite (4.2) as
\[ 0 \to \ker(\varphi_\Phi) \to H_1 \to H_0 \to \coker(\varphi_\Phi) \to 0. \]

Since $\ker(\varphi_\Phi) = H_1$, this implies that $\coker(\varphi_\Phi) \cong H_0$. □

5. Component groups of $J_0(xy)$

5.1. Component groups at $x$ and $y$. We return to the notation in Section 3. As we mentioned in \[2.1\] $X_0(xy)$ is smooth over $A[1/xy]$.

**Proposition 5.1.**

(i) $X_0(xy)_{F_v}$ has a semi-stable model over $O_v$ such that $X_0(xy)_{F_v}$ consists of two irreducible components both isomorphic to $X_0(y)_{F_v} \cong \mathbb{P}^1_{F_v}$ intersecting transversally in $q + 1$ points. Two of these singular points have thickness $q + 1$, and the other $q - 1$ points have thickness 1.

(ii) $X_0(xy)_{F_v}$ has a semi-stable model over $O_v$ such that $X_0(xy)_{F_v}$ consists of two irreducible components both isomorphic to $X_0(x)_{F_v} \cong \mathbb{P}^1_{F_v}$ intersecting transversally in $q + 1$ points. All these singular points have thickness 1.

**Proof.** The fact that $X_0(xy)_{F}$ has a model over $O_x$ and $O_y$ with special fibers of the stated form follows from the same argument as in the case of $X_0(v)_{F}$ over $O_v$ ($v \in |F| - \infty$) discussed in [11, §5]. We only clarify why the number of singular points and their thickness are as stated.

(i) The special fiber $X_0(xy)_{\mathbb{F}_v}$ consists of two copies of $X_0(y)_{\mathbb{F}_v}$. The set of points $Y_0(y)_{\mathbb{F}_v}$ is in bijection with the isomorphism classes of pairs $(\phi, C_y)$, where $\phi$ is a rank-2 Drinfeld $A$-module over $\mathbb{F}_v$ and $C_y \cong A/y$ is a cyclic subgroup of $\phi$. The two copies of $X_0(y)_{\mathbb{F}_v}$ intersect exactly at the points corresponding to $(\phi, C_y)$ with $\phi$ supersingular; more precisely, $(\phi, C_y)$ on the first copy is identified with $(\phi^{(x)}, C_y^{(x)})$ on the second copy where $\phi^{(x)}$ is the image of $\phi$ under the Frobenius isogeny and $C_y^{(x)}$ is subgroup of $\phi^{(x)}$ which is the image of $C_y$, cf. [11].

Now, by Lemma 2.1, up to an isomorphism over $\mathbb{F}_v$, there is a unique supersingular Drinfeld module $\phi$ characteristic $x$ and $j(\phi) = 0$. It is easy to see that $\phi$ has $q_y + 1 = q^2 + 1$ cyclic subgroups isomorphic to $A/y$, so the set $S = \{(\phi, C_y) \mid C_y \subset \phi[y]\}$ has cardinality $q^2 + 1$. By Lemma 2.2, $\Aut(\phi) \cong \mathbb{F}_q^\times$. This group naturally acts $S$, and the orbits are in bijection with the singular points of $X_0(xy)_{\mathbb{F}_v}$. Since the genus of $X_0(xy)_{F_v}$ is $q$, the arithmetic genus of $X_0(xy)_{\mathbb{F}_v}$ is also $q$ due to the flatness of $X_0(xy) \to \Spec(A)$; see [21 Cor. III.9.10]. Using the fact that the genus of $X_0(y)_{F_v}$ is zero, a simple calculation shows that the number of singular points of $X_0(xy)_{\mathbb{F}_v}$ is $q + 1$, cf. [21 p. 298]. Next, by Lemma 2.3, the stabilizer in $\Aut(\phi)$ of $(\phi, C_y)$ is either $\mathbb{F}_q^\times$ or $\mathbb{F}_q^{\times 2}$. Let $s$ be the number of pairs $(\phi, C_y)$ with stabilizer $\mathbb{F}_q^\times$. Let $t$ be the number of orbits of pairs with stabilizers $\mathbb{F}_q^{\times 2}$; each such orbit consists of $\#(\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}) = q + 1$ pairs $(\phi, C_y)$. Hence we have
\[ (q + 1)t + s = q^2 + 1 \quad \text{and} \quad t + s = q + 1. \]
This implies that \( t = q - 1 \) and \( s = 2 \). Finally, as is explained in [11], the thickness of the singular point corresponding to an isomorphism class of \((\phi, C_y)\) is equal to \( \#(\text{Aut}(\phi, C_y) / \mathbb{F}_v^s) \).

(ii) Similar to the previous case, \( X_0(xy)_{\mathbb{F}_v} \) consists of two copies of \( X_0(x)_{\mathbb{F}_v} \cong \mathbb{P}^1_{\mathbb{F}_v^s} \). The two copies of \( X_0(x)_{\mathbb{F}_v} \) intersect exactly at the points corresponding to the isomorphism classes of pairs \((\phi, C_x)\) with \( \phi \) supersingular. Again by Lemma 2.3, up to an isomorphism over \( \mathbb{F}_v \), there is a unique supersingular \( \phi \) and \( j(\phi) \neq 0 \). Hence, by Lemma 2.3, \( \text{Aut}(\phi, C_x) \cong \mathbb{F}_v^s \) for any \( C_x \). There are \( q_x + 1 = q + 1 \) cyclic subgroups in \( \phi \) isomorphic to \( A/x \). The rest of the argument is the same as in the previous case. □

**Theorem 5.2.** Let \( \Phi_v \) denote the group of connected components of \( J_0(xy) \) at \( v \in |F| \). Let \( Z \) and \( Z' \) be the irreducible components in Proposition 5.1, with the convention that the reduction of \([\infty] \) lies on \( Z' \). Let \( z = Z - Z' \).

(i) \( \Phi_x \cong \mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z} \).

(ii) \( \Phi_y \cong \mathbb{Z}/(q + 1)\mathbb{Z} \).

(iii) Under the canonical specialization map \( \phi_x : C \to \Phi_x \) we have

\[
\phi_x(c_x) = 0 \quad \text{and} \quad \phi_x(c_y) = z.
\]

In particular, \( q^2 + 1 \) divides the order of \( c_y \).

(iv) Under the canonical specialization map \( \phi_y : C \to \Phi_y \) we have

\[
\phi_y(c_x) = z \quad \text{and} \quad \phi_y(c_y) = 0.
\]

In particular, \( q + 1 \) divides the order of \( c_x \).

**Proof.** (i) and (ii) follow from Theorem 4.1 and Proposition 5.1.

(iii) The cusps reduce to distinct points in the smooth locus of \( X_0(xy)_{\mathbb{F}_v} \), cf. [11]. Since by Theorem 4.1 we know that \( z \) has order \( q^2 + 1 \) in the component group \( \Phi_x \), it is enough to show that the reductions of \([y]\) and \([\infty]\) lie on distinct components \( Z \) and \( Z' \) in \( X_0(xy)_{\mathbb{F}_v} \), but the reductions of \([x]\) and \([\infty]\) lie on the same component.

The involution \( W_x \) interchanges the two components \( X_0(y)_{\mathbb{F}_v} \), cf. [11] (5.3)). Since \( W_x([\infty]) = [y] \), the reductions of \([\infty]\) and \([y]\) lie on distinct components. On the other hand, \( W_y \) acts on \( X_0(xy)_{\mathbb{F}_v} \) by acting on each component \( X_0(y)_{\mathbb{F}_v} \) separately, without interchanging them. Since \( W_y([\infty]) = [x] \), the reductions of \([\infty]\) and \([x]\) lie on the same component.

(iv) The argument is similar to (iii). Here \( W_y \) interchanges the two components \( X_0(x)_{\mathbb{F}_v} \) of \( X_0(xy)_{\mathbb{F}_v} \) and \( W_x \) maps the components to themselves. Hence \([\infty]\) and \([y]\) lie on one component and \([0]\) and \([x]\) on the other component. □

**Theorem 5.3.** The cuspidal divisor group

\[
\mathcal{C} \cong \mathbb{Z}/(q + 1)\mathbb{Z} \oplus \mathbb{Z}/(q^2 + 1)\mathbb{Z}
\]

is the direct sum of the cyclic subgroups generated by \( c_x \) and \( c_y \), which have orders \((q + 1)\) and \((q^2 + 1)\), respectively. (Note that \( \mathcal{C} \) is cyclic if \( q \) is even, but it is not cyclic if \( q \) is odd.)

**Proof.** By Lemma 3.3 and Theorem 5.2 \( \mathcal{C} \) is generated by \( c_x \) and \( c_y \), which have orders \((q + 1)\) and \((q^2 + 1)\), respectively. If the subgroup of \( \mathcal{C} \) generated by \( c_x \) non-trivially intersects with the subgroup generated by \( c_y \), then, by Lemma 3.3 \( q \) must be odd and \( \frac{q+1}{2}c_x = \frac{q^2+1}{2}c_y \). Applying \( \phi_y \) to both sides of this equality, we get \( \frac{q+1}{2}z = 0 \), which is a contradiction since \( z \) generates \( \Phi_y \cong \mathbb{Z}/(q + 1)\mathbb{Z} \). □
Remark 5.4. The divisor class $c_0$ has order $(q + 1)(q^2 + 1)$ (resp. $(q + 1)(q^2 + 1)/2$) if $q$ is even (resp. odd).

5.2. Component group at $\infty$. To obtain a model of $X_0(xy)_{F_\infty}$ over $O_\infty$, instead of relying on the moduli interpretation of $X_0(xy)$, one has to use the existence of analytic uniformization for this curve; see [28, §4.2]. As far as the structure of the special fiber $X_0(xy)_{\mathbb{F}_\infty}$ is concerned, it is more natural to compute the dual graph of $X_0(xy)_{\mathbb{F}_\infty}$ directly using the quotient $\Gamma_0(xy) \setminus \mathcal{T}$ of the Bruhat-Tits tree $\mathcal{T}$ of $\text{PGL}_2(F_\infty)$. For the definition of $\mathcal{T}$, and more generally for the basic theory of trees and groups acting on trees, we refer to [10].

The quotient graph $\Gamma_0(xy) \setminus \mathcal{T}$ was first computed by Gekeler [10, (5.2)]. For our purposes we will need to know the relative position of the cusps on $\Gamma_0(xy) \setminus \mathcal{T}$ and also the stabilizers of the edges. To obtain this more detailed information, and for the general sake of completeness, we recompute $\Gamma_0(xy) \setminus \mathcal{T}$ in this subsection using the method in [10].

Denote
\[
G_0 = \text{GL}_2(\mathbb{F}_q)
\]
and
\[
G_i = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \text{GL}_2(A) \mid \text{deg}(b) \leq i \right\}, \quad i \geq 1.
\]
As is explained in [10], $\Gamma_0(xy) \setminus \mathcal{T}$ can be constructed in “layers”, where the vertices of the $i$th layer (in [10] called type-$i$ vertices) are the orbits
\[
X_i := G_i \setminus \mathbb{P}^1(A/xy)
\]
and the edges connecting type-$i$ vertices to type-$(i + 1)$ vertices, called type-$i$ edges, are the orbits
\[
Y_i := (G_i \cap G_{i+1}) \setminus \mathbb{P}^1(A/xy).
\]
There are obvious maps $Y_i \to X_i$, $Y_i \to X_{i+1}$ and $X_i \to X_{i+1}$ which are used to define the adjacencies of vertices in $X_i$ and $X_{i+1}$; see [16, 1.7]. The graph $\Gamma_0(xy) \setminus \mathcal{T}$ is isomorphic to the graph with set of vertices $\bigsqcup_{i \geq 0} X_i$ and set of edges $\bigsqcup_{i \geq 0} Y_i$ with the adjacencies defined by these maps.

Note that $\mathbb{P}^1(A/xy) = \mathbb{P}^1(F_x) \times \mathbb{P}^1(F_y)$. We will represent the elements of $\mathbb{P}^1(A/xy)$ as couples $[P; Q]$ where $P \in \mathbb{P}^1(F_x)$ and $Q \in \mathbb{P}^1(F_y)$. With this notation, $G_i$ acts diagonally on $[P; Q]$ via its images in $\text{GL}_2(F_x)$ and $\text{GL}_2(F_y)$, respectively.

The group $G_0$ acting on $\mathbb{P}^1(A/xy)$ has 3 orbits, whose representatives are
\[
\left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right], \quad \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right], \quad \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} x \\ 1 \end{pmatrix} \right],
\]
where in the last element we write $x$ for the image in $F_y$ of the monic generator of $x$ under the canonical homomorphism $A \to A/y$. The orbit of $\left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$ has length $q + 1$, the orbit of $\left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ has length $q(q + 1)$, and the orbit of $\left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} x \\ 1 \end{pmatrix} \right]$ has length $q(q^2 - 1)$, cf. [10, Prop. 2.10]. Next, note that $G_0 \cap G_1$ is the subgroup $B$ of the upper-triangular matrices in $\text{GL}_2(F_q)$. The $G_0$-orbit of $\left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$ splits into two $B$-orbits with representatives:
\[
(5.1) \quad \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \text{ and } \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].
\]
The lengths of these $B$-orbits are 1 and $q$, respectively. The $G_0$-orbit of $\left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right]$ splits into three $B$-orbits with representatives:

\begin{equation}
\left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right], \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right], \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right].
\end{equation}

The lengths of these $B$-orbits are $q$, $q$, $q(q-1)$, respectively. Finally, the $G_0$-orbit of $\left[ \begin{array}{c} x \\ 0 \\ 1 \end{array} \right]$ splits into $(q+1)$ $B$-orbits each of length $q(q-1)$. The previous statements can be deduced from Proposition 2.11 in [16]. It turns out that the elements of $P^1(F_x) \times P^1(F_y)$ listed in (5.1) and (5.2) combined form a complete set of $G_1$-orbit representatives. For $i \geq 1$, the set of $G_i$-orbit representatives obviously contains a complete set of $G_{i+1}$-orbit representatives. A small calculation shows that

\begin{equation}
\left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right], \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right], \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right]
\end{equation}

is a complete set of $G_i$-orbit representatives for any $i \geq 2$. Moreover, the elements $\left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right]$ and $\left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]$ are in the same $G_2$-orbit. We recognize the elements in (5.3) as the cusps $[\infty], [0], [x], [y]$, respectively. Overall, the structure of $\Gamma_0(xyz) \setminus \mathcal{T}$ is described by the diagram in Figure 2. In the diagram the broken line $\cdots$ indicates that there are $(q-1)$ distinct edges joining the corresponding vertices, and an arrow $\rightarrow$ indicates an infinite half-line.

Now we compute the stabilizers of the edges. Let $e$ be an edge in $\Gamma_0(xyz) \setminus \mathcal{T}$ of type $i$. Let

\[ O(e) = (G_i \cap G_{i+1})[P; Q] \]

be its corresponding orbit in $(G_i \cap G_{i+1}) \setminus P^1(A/xyz)$. Then for a preimage $\tilde{e}$ of $e$ in $\mathcal{T}$ we have

\[ \#\text{Stab}_{\Gamma_0(xyz)}(\tilde{e}) = \#\text{Stab}_{G_i \cap G_{i+1}}([P; Q]) = \frac{\#(G_i \cap G_{i+1})}{\#O(e)}. \]

Using this observation, we conclude from our previous discussion that the edges connecting $\left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right]$ in $X_0$ to any vertex in $X_1$ have preimages whose stabilizers have order $#B/\text{gcd}(q-1) = q - 1$. The preimages of the edges connecting $\left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \in X_0$ to $\left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \in X_1$ and $\left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \in X_1$ have stabilizers of orders $q - 1$ and $(q-1)^2$, respectively. (Note that if a stabilizer has order $(q-1)$ then it is equal to the center $Z(\Gamma_0(xyz)) \cong F_q^2$ of $\Gamma_0(xyz)$, as the center is a subgroup of any stabilizer.) The valency of a vertex $v$ in a graph without loops is the number of distinct edges having $v$ as an endpoint. (A loop is an edge whose endpoints are the same.) Consider the vertex $v = \left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \in X_1$. Its valency is $(q+1)$. Let $\tilde{v}$ be a preimage of $v$ in $\mathcal{T}$. Since the valency of $\tilde{v}$ is also $q+1$, $\text{Stab}_{\Gamma_0(xyz)}(\tilde{v})$ acts trivially on all edges having $\tilde{v}$ as an endpoint. Hence the stabilizer of any such edge is equal to $\text{Stab}_{\Gamma_0(xyz)}(\tilde{v})$. We already determined that the stabilizer of a preimage of an edge connecting $v$ to a type-0 vertex is $F_q^3$. This implies that the stabilizer in $\Gamma_0(xyz)$ of a preimage of the edge connecting $v$ to $\left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] \in X_2$ is
also $\mathbb{F}_q^\times$. Finally, consider the vertex $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in X_1$. Its valency is 3. Let $S, S_1, S_2, S_3$ be the orders of stabilizers in $\Gamma_0(xy)$ of a preimage $\tilde{w}$ of $w$ in $T$, and the edges connecting $w$ to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in X_0$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in X_0$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in X_2$, respectively. From our discussion of the lengths of orbits of type-0 edges, we have $S_1 = (q-1)^2$ and $S_2 = (q-1)$. Obviously, $S_i$’s divide $S$. On the other hand, counting the lengths of orbits of $\text{Stab}_{\Gamma_0(xy)}(\tilde{w})$ acting on the set of (non-oriented) edges in $T$ having $\tilde{w}$ as an endpoint, we get

$$q + 1 = \frac{S}{S_1} + \frac{S}{S_2} + \frac{S}{S_3} = \frac{S}{(q-1)^2} + \frac{S}{(q-1)} + \frac{S}{S_3}.$$  

This implies $S = S_3 = (q-1)^2$. To summarize, in Figure 2 a wavy line $\sim$ indicates that a preimage of the corresponding edge in $T$ has a stabilizer in $\Gamma_0(xy)$ of order $(q-1)^2$. The edges connecting $\begin{pmatrix} 1 \\ 0 \\ x \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ to any other vertex have preimages in $T$ whose stabilizers in $\Gamma_0(xy)$ are isomorphic to $\mathbb{F}_q^\times$.

Now from [25] §4.2 one deduces the following. The quotient graph $\Gamma_0(xy) \setminus T$, without the infinite half-lines, is the dual graph of the special fiber of a semi-stable model of $X_0(xy)_{\mathbb{F}_q}$ over $\text{Spec}(\mathcal{O}_\infty)$. The special fiber $X_0(xy)_{\mathbb{F}_q}$ has 6 irreducible components $Z, Z', E, E', G, G'$, all isomorphic to $\mathbb{P}^1_{\mathbb{F}_q}$, such that $Z$ and $Z'$ intersect

---

**Figure 2.** $\Gamma_0(xy) \setminus T$
in $q - 1$ points, $E$ intersects $Z$ and $E'$, $E'$ intersects $Z'$ and $E$, $G$ intersects $Z$ and $G'$, $G'$ intersects $Z'$ and $G$. Moreover, all intersection points are ordinary double singularities. By [28] Prop. 4.3, the thickness of the singular point corresponding to an edge $e \in \Gamma_0(xy) \setminus T$ is

$$\#(\text{Stab}_{\Gamma_0(xy)}(\tilde{e})/\mathbb{F}_q^*)$$

hence all intersection points on $Z$ or $Z'$ have thickness 1, but the intersection points of $E$ and $E'$, and of $G$ and $G'$ have thickness $(q - 1)$, cf. Figure 3. From the structure of $\Gamma_0(xy) \setminus T$, one also concludes that the reductions of the cusps are smooth points in $X_0(xy)_{f_{\infty}}$. Moreover, $[\infty], [0], [x], [y]$ reduce to points on $E, E', G, G'$ respectively.

Blowing up $X_0(xy)_{f_{\infty}}$ at the intersection points of $E, E'$, and $G, G'$, $(q - 2)$-times each, we obtain the minimal regular model of $X_0(xy)_F$ over $\text{Spec}(\mathcal{O}_{\infty})$. This is a curve of the type discussed in [4,2] with $m = n = (q + 1)$, and we enumerate its irreducible components so that $E_1 = E, E_q = E', G_1 = G, G_q = G'$.

**Theorem 5.5.** Let $\phi_\infty : \mathcal{C} \to \Phi_\infty$ denote the canonical specialization map.

1. $\Phi_\infty \cong \mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z}$.
2. $\phi_\infty(c_x) = (q^2 + 1)e_q$ and $\phi_\infty(c_y) = -q(q + 1)e_q = (q^3 + 1)e_q$.
3. If $q$ is even, then $\phi_\infty : \mathcal{C} \to \Phi_\infty$ is an isomorphism.
4. If $q$ is odd, then there is an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{C} \xrightarrow{\phi_\infty} \Phi_\infty \to \mathbb{Z}/2\mathbb{Z} \to 0.$$ 

**Proof.** Part (i) is an immediate consequence of the preceding discussion and Theorem 4.1. We have determined the reductions of the cusps at $\infty$, so using Theorem 4.1 we get

$$\phi_\infty(c_x) = g_1 - e_1 = (q^2 + q + 1)e_q - qe_q = (q^2 + 1)e_q$$

and

$$\phi_\infty(c_y) = g_q - e_1 = -q^2e_q - qe_q = -q(q + 1)e_q,$$

which proves (ii). Since $\gcd(q^2 + 1, q(q + 1)) = 1$ (resp. 2) if $q$ is even (resp. odd), cf. Lemma 3.3 the subgroup of $\Phi_\infty$ generated by $\phi_\infty(c_x)$ and $\phi_\infty(c_y)$ is $\langle e_q \rangle$ (resp. $\langle 2e_q \rangle$) if $q$ is even (resp. odd). On the other hand, we know that $e_q$ generates $\Phi_\infty$. Therefore, if $q$ is even, then $\phi_\infty$ is surjective, and if $q$ is odd, then the cokernel of $\phi_\infty$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The claims (iii) and (iv) now follow from Theorem 4.8. □
Remark 5.6. We note that (iii) and a slightly weaker version of (iv) in Theorem 5.5 can be deduced from Theorem 5.3 and a result of Gekeler [14]. In fact, in [14], p. 366 it is proven that for an arbitrary \( n \) the kernel of the canonical homomorphism from the cuspidal divisor group of \( X_0(n) \) to \( \Phi_{\infty} \) is a quotient of \((\mathbb{Z}/(q - 1)\mathbb{Z})^{c-1}\), where \( c \) is the number of cusps of \( X_0(n) \). In our case, this result says that \( \ker(\phi_{\infty}) \) is a quotient of \((\mathbb{Z}/(q - 1)\mathbb{Z})^{3}\). Now suppose \( q \) is even. Then \( C \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z} \). Since for even \( q \), \( \gcd(q-1,(q^2+1)(q+1)) = 1 \), \( \phi_{\infty} \) must be injective. But by (i), \( \#\Phi_{\infty} = (q^2+1)(q+1) = \#C \), so \( \phi_{\infty} \) is also surjective. When \( q = 2 \), the fact that \( \#\Phi_{\infty} = 15 \) and \( \phi_{\infty} \) is an isomorphism is already contained in [14, (5.3.1)].

Now suppose \( q \) is odd. Then \( C \cong \mathbb{Z}/(q^2+1)\mathbb{Z} \). Since \( \ker(\phi_{\infty}) \) is cyclic but \( C \) is not, \( \ker(\phi_{\infty}) \) is not trivial, hence it is either \( \mathbb{Z}/2\mathbb{Z} \) or \((\mathbb{Z}/2\mathbb{Z})^2 \). (Theorem 5.5 implies that the second possibility does not occur.)

Notation 5.7. Let \( C_0 \) be the subgroup of \( C \) generated by \( c_y \).

Corollary 5.8. The cyclic group \( C_0 \) has order \( q^2 + 1 \). Under the canonical specializations \( C_0 \) maps injectively into \( \Phi_x \) and \( \Phi_{\infty} \), and \( C_0 \) is the kernel of \( \phi_y \).

Proof. The claims easily follow from Theorems 6.1, 6.3, and 6.6. \( \square \)

6. Component groups of \( J^{xy} \)

6.1. A class number formula. Let \( H \) be a quaternion algebra over \( F \). Let \( \mathcal{R} \subset |F| \) be the set of places where \( H \) ramifies. Assume \( \infty \in \mathcal{R} \). Denote \( \mathcal{R} = \mathcal{R} - \infty \). Note that \( \mathcal{R} \neq \emptyset \) since \( \#\mathcal{R} \) is even.

Let \( \Theta \) be a hereditary \( A \)-order in \( H \). Let \( I_1, \ldots, I_h \) be the isomorphism classes of left \( \Theta \)-ideals. It is known that \( h(\Theta) := h \), called the class number of \( \Theta \), is finite. For \( i = 1, \ldots, h \) we denote by \( \Theta_i \) the right order of the respective \( I_i \). (For the definitions see [42].) Denote

\[
M(\Theta) = \sum_{i=1}^{h} (\Theta_i^x : F_q^x)^{-1}.
\]

It is not hard to show that each \( \Theta_i^x \) is isomorphic to either \( F_q^x \) or \( F_{q^2}^x \); see [17], p. 383. Let \( U(\Theta) \) be the number of right orders \( \Theta_i \) such that \( \Theta_i^x \cong F_{q^2}^x \). In particular,

\[
h(\Theta) = M(\Theta) + U(\Theta) \left( 1 - \frac{1}{q+1} \right).
\]

Definition 6.1. For a subset \( S \) of \(|F|\), let

\[
\text{Odd}(S) = \begin{cases} 1, & \text{if all places in } S \text{ have odd degrees;} \\ 0, & \text{otherwise.} \end{cases}
\]

Let \( S \subset |F| - \infty \) be a finite (possibly empty) set of places such that \( \mathcal{R} \cap S = \emptyset \). Let \( n \triangleleft A \) be the square-free ideal whose support is \( S \). Let \( \Theta \) be an Eichler \( A \)-order of level \( n \). (When \( S = \emptyset \), \( \Theta \) is a maximal \( A \)-order in \( H \).) The formulae that follow are special cases of (1), (4) and (6) in [17]:

\[
M^S(H) := M(\Theta) = \frac{1}{q^2-1} \prod_{v \in \mathcal{R}} (q_v - 1) \prod_{w \in S} (q_w + 1),
\]
\[ U^S(H) := U(\Theta) = 2^{\# R + \# S - 1} \text{Odd}(R) \prod_{w \in S} (1 - \text{Odd}(w)). \]

Denote
\[ h^S(H) = M^S(H) + U^S(H) \frac{q}{q + 1}. \]

6.2. Component groups at \( x \) and \( y \). Let \( D \) and \( R \) be as in \[2.2\]. Recall that we assume \( \infty \notin R \). Fix a place \( w \in R \). Let \( D^w \) be the quaternion algebra over \( F \) which is ramified at \((R - w) \cup \infty \). Fix a maximal \( A \)-order \( \Theta \) in \( D^w \), and denote
\[
A^w = A[w^{-1}]; \\
\mathcal{O}^w = \mathcal{O} \otimes_A A^w; \\
\Gamma^w = \{ \gamma \in (\mathcal{O}^w)^\times \mid \text{ord}_w(\text{Nr}(\gamma)) \in 2\mathbb{Z} \};
\]
here \( w^{-1} \) denotes the inverse of a generator of the ideal in \( A \) corresponding to \( w \), and \( \text{Nr} \) denotes the reduced norm on \( D^w \).

By fixing an isomorphism \( D^w \otimes_F F_w \cong M_2(F_w) \), one can consider \( \Gamma^w \) as a subgroup of \( \text{GL}_2(F_w) \) whose image in \( \text{PGL}_2(F_w) \) is discrete and cocompact. Hence \( \Gamma^w \) acts on the Bruhat-Tits tree \( T^w \) of \( \text{PGL}_2(F_w) \). It is not hard to show that \( \Gamma^w \) acts without inversions, so the quotient graph \( \Gamma^w \setminus T^w \) is a finite graph without loops. We make \( \Gamma^w \setminus T^w \) into a graph with lengths by assigning to each edge \( e \) of \( \Gamma^w \setminus T^w \) the length \( \#(\text{Stab}_{\Gamma^w}(\tilde{e})/\mathbb{F}_q) \), where \( \tilde{e} \) is a preimage of \( e \) in \( T^w \). The graph with lengths \( \Gamma^w \setminus T^w \) does not depend on the choice of isomorphism \( D^w \otimes_F F_w \cong M_2(F_w) \), since such isomorphisms differ by conjugation.

As follows from the analogue of Cherednik-Drinfeld uniformization for \( X^R_{F_w} \), proven in this context by Hausberger [22], \( X^R_{F_w} \) is a twisted Mumford curve: Denote by \( \mathcal{O}^{(2)}_{w} \) the quadratic unramified extension of \( \mathcal{O}_w \) and denote by \( F^{(2)}_{w} \) the residue field of \( \mathcal{O}^{(2)}_{w} \). Then \( X^R_{F_w} \) has a semi-stable model \( X^R_{\mathcal{O}^{(2)}_{w}} \) over \( \mathcal{O}^{(2)}_{w} \) such that the irreducible components of \( X^R_{\mathcal{O}^{(2)}_{w}} \) are projective lines without self-intersections, and the dual graph \( G(X^R_{\mathcal{O}^{(2)}_{w}}) \), as a graph with lengths, is isomorphic to \( \Gamma^w \setminus T^w \).

On the other hand, as is done in [23] for the quaternion algebras over \( \mathbb{Q} \), the structure of \( \Gamma^w \setminus T^w \) can be related to the arithmetic to \( D^w \): The number of vertices of \( \Gamma^w \setminus T^w \) is \( 2h^0(D^w) \), the number of edges is \( h^0(D^w) \), each edge has length 1 or \( q + 1 \), and the number of edges of length \( q + 1 \) is \( U^w(D^w) \) (the notation here is as in \[6.1\]). Hence, using the formulae in \[6.1\] we get the following:

**Proposition 6.2.** \( X^R_{F} \) has a semi-stable model \( X^R_{\mathcal{O}^{(2)}_{w}} \) over \( \mathcal{O}^{(2)}_{w} \) such that \( X^R_{\mathcal{O}^{(2)}_{w}} \) is a union of projective lines without self-intersections. The number of vertices of the dual graph \( G(X^R_{\mathcal{O}^{(2)}_{w}}) \) is
\[
\frac{2}{q^2 - 1} \prod_{v \in R - w} (q_v - 1) + 2^{\# R - 1} \text{Odd}(R - w) \frac{q}{q + 1};
\]
the number of edges is
\[
\frac{(q_{w} + 1)}{q^2 - 1} \prod_{v \in R - w} (q_v - 1) + 2^{\# R - 1} \text{Odd}(R - w)(1 - \text{Odd}(w)) \frac{q}{q + 1}.
\]
The edges of \( G(\mathcal{X}_{\mathcal{O}_w}^{(2)}) \) have length 1 or \( q + 1 \). The number of edges of length \( q + 1 \) is
\[
2^{#R-1}\text{Odd}(R-w)(1-\text{Odd}(w)).
\]
This proposition has an interesting corollary:

**Corollary 6.3.** Let \( g(R) \) be the genus of \( \mathcal{X}^{(2)}_{\mathcal{O}_w} \). Then
\[
g(R) = 1 + \frac{1}{q^2 - 1} \prod_{v \in R} (q_v - 1) - \frac{q}{q + 1}2^{#R-1}\text{Odd}(R).
\]

**Proof.** Let \( h_1 \) be the dimension of the first simplicial homology group of \( G(\mathcal{X}_{\mathcal{O}_w}^{(2)}) \) with \( \mathbb{Q} \)-coefficients. Let \( V, E \) be the number of vertices and edges of this graph, respectively. By Euler’s formula, \( h_1 = E - V + 1 \). Proposition 6.2 gives formulae for \( V \) and \( E \) from which it is easy to see that \( h_1 \) is given by the above expression. Since the irreducible components of \( \mathcal{X}^{(2)}_{\mathcal{O}_w} \) are projective lines, it is not hard to show that \( h_1 \) is the arithmetic genus of \( \mathcal{X}^{(2)}_{\mathcal{O}_w} \); cf. [21, p. 298]. On the other hand, \( \mathcal{X}^{(2)}_{\mathcal{O}_w} \) is flat over \( \mathcal{O}_w^{(2)} \), so the genus \( g(R) \) of its generic fiber is equal to the arithmetic genus of the special fiber; see [21] p. 263). (Note that the special role of \( w \) in the formulae for \( V \) and \( E \) disappears in \( g(R) \), as expected. This formula for \( g(R) \) was obtained in [30] by a different argument.) \( \square \)

**Theorem 6.4.** Let \( \Phi_v' \) denote the group of connected components of \( J^{xy} \) at \( v \in |F| \).

(i) \( \Phi_x' \cong \mathbb{Z}/(q + 1)\mathbb{Z} \);
(ii) \( \Phi_y' \cong \mathbb{Z}/(q^2 + 1)(q + 1)\mathbb{Z} \).

**Proof.** In general, the information supplied by Proposition 6.2 is not sufficient for determining the graph \( G(\mathcal{X}_{\mathcal{O}_w}^{(2)}) \) uniquely. Nevertheless, in the case when \( R = \{x, y\} \) Proposition 6.2 does uniquely determine \( G(\mathcal{X}_{\mathcal{O}_w}^{(2)}): \) \( G(\mathcal{X}_{\mathcal{O}_w}^{xy}) \) is a graph without loops, which has 2 vertices, \( q + 1 \) edges, and all edges have length 1. Similarly, \( G(\mathcal{X}_{\mathcal{O}_w}^{xy}) \) is a graph without loops, which has 2 vertices, \( q + 1 \) edges, two of the edges have length \( q + 1 \) and all others have length 1. Hence, in both cases, the dual graph is the graph with two vertices and \( q + 1 \) edges connecting them, cf. Figure 4.

Now Theorem 4.1 can be used to conclude that the component groups are as stated. \( \square \)

6.3. **Component group at \( \infty \).** Here we again rely on the existence of analytic uniformization. Let \( \Lambda \) be a maximal \( A \)-order in \( D \). Let
\[
\Gamma^\infty := \Lambda^\times.
\]
Since \( D \) splits at \( \infty \), by fixing an isomorphism \( D \otimes F_\infty \cong M_2(F_\infty) \), we get an embedding \( \Gamma^\infty \hookrightarrow \text{GL}_2(F_\infty) \). The group \( \Gamma^\infty \) is a discrete, cocompact subgroup
of $\GL_2(F_\infty)$, well-defined up to conjugation. Let $T^\infty$ be the Bruhat-Tits tree of $\PGL_2(F_\infty)$. The group $T^\infty$ acts on $T^\infty$ without inversions, so the quotient $\Gamma^\infty \backslash T^\infty$ is a finite graph without loops which we make into a graph with lengths by assigning to each edge $e$ of $\Gamma^\infty \backslash T^\infty$ the length $\#(\text{Stab}_{\Gamma^\infty}(e)/\pi^2_q)\), where $\tilde{e}$ is a preimage of $e$ in $T^\infty$. By a theorem of Blum and Stuhler [1, Thm. 4.4.11],

$$(X^R_{\infty})^{\text{an}} \cong \Gamma^\infty \backslash \Omega.$$

From this one deduces that $X^R_F$ has a semi-stable model $X^R_{\infty}$ over $O_\infty$ such that the dual graph of $X^R_{\infty}$, as a graph with lengths, is isomorphic to $\Gamma^\infty \backslash T^\infty$, cf. [25]. The structure of $\Gamma^\infty \backslash T^\infty$ can be related to the arithmetic of $D$; see [32].

**Proposition 6.5.** $X^R_F$ has a semi-stable model $X^R_{\infty}$ over $O_\infty$ such that the special fiber $X^R_{\ell, \infty}$ is a union of projective lines without self-intersections. The number of vertices of the dual graph $G(X^R_{\infty})$ is

$$ \frac{2}{q-1}(q(R)-1) + \frac{q}{q-1}2^{#R-1}\text{Odd}(R);$$

the number of edges is

$$ \frac{q+1}{q-1}(q(R)-1) + \frac{q}{q-1}2^{#R-1}\text{Odd}(R).$$

All edges have length 1.

**Proof.** See Proposition 5.2 and Theorem 5.5 in [32].

**Theorem 6.6.** $\Phi^\prime_{\infty} \cong \mathbb{Z}/(q+1)\mathbb{Z}$.

**Proof.** Applying Proposition 6.5 in the case $R = \{x, y\}$, one easily concludes that $X^R_F$ has a semi-stable model over $O_\infty$ whose dual graph looks like Figure 4, it has 2 vertices, $q + 1$ edges, and all edges have length 1. The structure of $\Phi^\prime_{\infty}$ now follows from Theorem 6.1.

7. **Jacquet-Langlands isogeny**

Let $D$ and $R$ be as in [22]. Let $X := X^R_F$, $X' := X_0(R)_F$, $J := J^R$, $J' := J_0(R)$. Fix a separable closure $F^\sep$ of $F$ and let $G_F := \text{Gal}(F^\sep/F)$. Let $\ell$ be the characteristic of $F$ and fix a prime $\ell \neq p$. Denote by $V_\ell(J)$ the Tate vector space of $J$; this is a $Q_\ell$-vector space of dimension $2q(R)$ naturally equipped with a continuous action of $G_F$. Let $V_\ell(J)^*$ be the linear dual of $V_\ell(J)$.

**Theorem 7.1.** There is a surjective homomorphism $J' \to J$ defined over $F$.

**Proof.** Let $A = \prod_{v \in |F|} F_v$ denote the adele ring of $F$ and let $A^\infty = \prod_{v \in |F| - \infty} F_v$; so $A = A^\infty \times F_\infty$. Fix a uniformizer $\pi_\infty$ at $\infty$. Let $A(D^\times(F) \backslash D^\times(A)/\pi^2_\infty)$ be the space of $\mathbb{Q}_\ell$-valued locally constant functions on $D^\times(A)/\pi^2_\infty$ which are invariant under the action of $D^\times(F)$ on the left. This space is equipped with the right regular representation of $D^\times(A)/\pi^2_\infty$. Since $D$ is a division algebra, the coset space $D^\times(F) \backslash D^\times(A)/\pi^2_\infty$ is compact and decomposes as a sum of irreducible admissible representations $\Pi$ with finite multiplicities $m(\Pi) > 0$, cf. [26, §13]:

$$(7.1) \quad A_D := A(D^\times(F) \backslash D^\times(A)/\pi^2_\infty) = \bigoplus_{\Pi} m(\Pi) \cdot \Pi.$$

Moreover, as follows from the Jacquet-Langlands correspondence and the multiplicity-one theorem for automorphic cuspidal representations of $\GL_2(A)$, the multiplicities
Taking the following property: If \( \Pi \in \mathfrak{g} \) and \( \n\) is a Hecke character of \( \mathbb{A}_F^{\times} \) and \( \n\) is the reduced norm on \( \mathbb{D}^{\times} \), cf. [29, Lem. 14.8]. If \( \Pi \) is infinite dimensional, then \( \Pi \) is infinite dimensional for every \( v \notin R \).

Let \( \psi_v \) be a character of \( \mathbb{F}_v^{\times} \). Denote by \( \mathcal{S}_v \otimes \psi_v \), the unique irreducible quotient of the induced representation
\[
\text{Ind}_{B}^{\mathbb{G}_{L_2}}(\pi_1 \otimes \pi_2, \psi_v \otimes \pi_2),
\]
where \( B \) is the subgroup of upper-triangular matrices in \( \mathbb{G}_{L_2} \). The representation \( \mathcal{S}_v \otimes \psi_v \) is called the special representation of \( \mathbb{G}_{L_2}(\mathbb{F}_v) \) twisted by \( \psi_v \). If \( \psi_v = 1 \), then we simply write \( \mathcal{S}_v \).

For \( v \in R \), let \( \mathcal{D}_v \) be the maximal order in \( \mathbb{D}(\mathbb{F}_v) \). Let
\[
\mathcal{K} := \prod_{v \in \mathbb{P}} \mathcal{D}_v \times \prod_{v \in |F| - R - \infty} \text{GL}_2(\mathcal{O}_v) \subset \mathbb{D}(\mathbb{A}_F^{\times}).
\]
Taking the \( \mathcal{K} \)-invariants in Theorems 14.9 and 14.12 in [29], we get an isomorphism of \( \mathbb{G}_F \)-modules
\[
V_{\ell}(J)^{\ast} \otimes \mathbb{Q}_\ell, \mathbb{Q}_\ell) = \bigoplus_{\Pi \in \mathcal{A}_v} \mathcal{K} \otimes \sigma(\Pi),
\]
where \( \sigma(\Pi) \) is a 2-dimensional irreducible representation of \( \mathbb{G}_F \) over \( \mathbb{Q}_\ell \) with the following property: If \( (\Pi^{\mathcal{K}})^{\ast} \neq 0 \), then for all \( v \in |F| - R - \infty \), \( \sigma(\Pi) \) is unramified at \( v \) and there is an equality of \( L \)-functions
\[
L(s - 1/2 \Pi_v) = L(s, \sigma(\Pi)_v);
\]
where \( \sigma(\Pi)_v \) denotes the restriction of \( \sigma(\Pi) \) to a decomposition group at \( v \). This uniquely determines \( \sigma(\Pi) \) by the Chebotarev density theorem [39, Ch. I, pp. 8-11]. Next, we claim that the dimension of \( (\Pi^{\mathcal{K}})^{\ast} \) is at most one. Indeed, if \( v \in |F| - R - \infty \), then \( \Pi_v^{\mathfrak{g}_L(\mathcal{O}_v)} \) is at most one-dimensional by [39, Thm. 4.6.2]. On the other hand, note that \( \mathcal{D}_v^{\ast} \) is normal in \( \mathbb{D}^{\times}(\mathbb{F}_v) \) and \( \mathbb{D}^{\times}(\mathbb{F}_v) / \mathcal{D}_v^{\ast} \cong \mathbb{Z} \) for \( v \in R \). Hence \( \Pi_v^{\mathcal{D}_v^{\ast}} \neq 0 \) implies \( \Pi_v = \psi_v \otimes \n_{\mathbb{O}} \) for some unramified character of \( \mathbb{F}_v^{\times} \) (\( \psi_v \) is unramified because the reduced norm maps \( \mathcal{D}_v^{\ast} \) surjectively onto \( \mathcal{O}_v^{\times} \)).

Let \( \mathcal{I}_v \) be the Iwahori subgroup of \( \mathbb{G}_2(\mathcal{O}_v) \), i.e., the subgroup of matrices which maps to \( B(\mathbb{F}_v) \) under the reduction map \( \mathbb{G}_L(\mathcal{O}_v) \to \mathbb{G}(\mathbb{F}_v) \). Let
\[
\mathcal{I} = \prod_{v \in \mathbb{P}} \mathcal{I}_v \times \prod_{v \in |F| - R - \infty} \text{GL}_2(\mathcal{O}_v) \subset \mathcal{G}_2(\mathbb{A}_F^{\times}).
\]
Let \( \mathcal{A}_0 := \mathcal{A}_0(\mathbb{G}_L(F) \setminus \mathbb{G}_L(\mathbb{A})) \) be the space of \( \mathbb{Q}_\ell \)-valued cusp forms on \( \mathbb{G}_L(\mathbb{A}) \); see [17, §4] or [3, §3.3] for the definition. Taking the \( \mathcal{I} \)-invariants in Theorem 2 of [3], we get an isomorphism of \( \mathbb{G}_F \)-modules
\[
V_{\ell}(J)^{\ast} \otimes \mathbb{Q}_\ell, \mathbb{Q}_\ell) = \bigoplus_{\Pi \in \mathcal{A}_0} (\Pi^{\mathcal{I}})^{\ast} \otimes \rho(\Pi),
\]
where $\rho(\Pi)$ is 2-dimensional irreducible representation of $G_F$ over $\mathcal{Q}_\ell$ with the following property: If $(\Pi^\infty)^{\mathcal{I}} \neq 0$, then for all $v \in |F| - R - \infty$, $\rho(\Pi)$ is unramified at $v$ and

$$L(s - \frac{1}{2}, \Pi_v) = L(s, \rho(\Pi)_v).$$

In this case, $(\Pi^\infty)^{\mathcal{I}}$ is finite dimensional, but its dimension might be larger than one (due to the existence of old forms).

The global Jacquet-Langlands correspondence [24] Ch. III associates to each infinite dimensional automorphic representation $\Pi$ of $D^\times(\mathbb{A})$ a cuspidal representation $\Pi' = JL(\Pi)$ of $GL_2(\mathbb{A})$ with the following properties:

(1) if $v \notin R$ then $\Pi_v \cong \Pi'_v$;

(2) if $v \in R$ and $\Pi_v \cong \psi_v \circ \text{Nr}$ for a character $\psi$ of $F_v^\times$, then

$$\Pi'_v \cong \text{Sp}_v \otimes \psi_v.$$ 

As we observed above, for $\Pi \in \mathcal{A}_D$ such that $(\Pi^\infty)^{\mathcal{I}} \neq 0$, the characters $\psi_v$ at the places in $R$ are unramified. Thus, for $v \in R$, $\Pi'_v$ is a twist of $\text{Sp}_v$ by an unramified character. On the other hand, the representations of the form $\text{Sp}_v \otimes \psi_v$, with $\psi_v$ unramified, can be characterized by the property that they have a unique 1-dimensional $\mathcal{I}_v$-fixed subspace; see [4]. Hence if $(\Pi^\infty)^{\mathcal{I}} \neq 0$, then $((\Pi')^\infty)^{\mathcal{I}} \neq 0$.

Now using (7.2) and (7.3), one concludes that $V_l(J)$ is isomorphic with a quotient of $V_l(J')$ as a $G_F$-module. On the other hand, by a theorem of Zarhin (for $p > 2$) and Mori (for $p = 2$)

(7.4) \[ \text{Hom}_F(J', J) \otimes \mathbb{Q}_\ell \cong \text{Hom}_{G_F}(V_l(J'), V_l(J)). \]

Thus, there is a surjective homomorphism $J' \to J$ defined over $F$. \hfill \Box

Corollary 7.2. $J_0(xy)$ and $J^{xy}$ are isogenous over $F$.

Proof. Since $\dim(J^{xy}) = q = \dim(J_0(xy))$, the claim follows from Theorem 7.1. \hfill \Box

Conjecture 7.3. There exists an isogeny $J_0(xy) \to J^{xy}$ whose kernel is $C_0$.

As an initial evidence for the conjecture, note that $J_0(xy)/C_0$ has component groups at $x, y, \infty$ of the same order as those of $J^{xy}$. This follows from Theorem 4.3 Corollary 5.8 and Table 1 in the introduction. We will show below that Conjecture 7.3 is true for $q = 2$.

Remark 7.4. The statement of Theorem 7.1 can be refined. The abelian variety $J$ has toric reduction at every $v \in R$, so it is isogenous to an abelian subvariety of $J'$ having the same reduction property. The new subvariety of $J'$, $J^{\text{new}}$, defined as in the case of classical modular Jacobians (cf. [35], [13] p. 248)), is the abelian subvariety of $J'$ of maximal dimension having toric reduction at every $v \in R$. Hence $J$ is isogenous to a subvariety of $J^{\text{new}}$. By computing the dimension of $J^{\text{new}}$, one concludes that $J$ and $J^{\text{new}}$ are isogenous over $F$.

Remark 7.5. There is just one other case, besides the case which is the focus of this paper, when $J$ and $J'$ are actually isogenous. As one easily shows by comparing the genera of modular curves $X_R$ and $X_0(R)$, the genera of these curves are equal if and only if $R = \{x, y\}$ and $\{\deg(x), \deg(y)\} = \{1, 1\}, \{1, 2\}, \{2, 2\}$. Assume $\deg(x) = \deg(y) = 2$. Then the genus of both $X^{xy}$ and $X_0(xy)$ is $q^2$, but neither of these curves is hyperelliptic. The curve $X_0(xy)$ again has 4 cusps which can be
represented as in $\mathbb{E}$ Calculations similar to those we have carried out in earlier sections lead to the following result:

1. The cuspidal divisor group $C$ is generated by $c_0$ and $c_x$. The order of $c_0$ is $q^2 + 1$. The order of $c_x$ is divisible by $q^2 + 1$ and divides $q^3 - 1$. The order of $c_y$ is divisible by $q^2 + 1$ and divides $q^4 - 1$.
2. $\Phi_x \cong \Phi'_x \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z}$.
3. $\Phi_y \cong \Phi'_y \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z}$.
4. The canonical map $\phi_x : C \to \Phi_x$ is surjective, and $\phi_x(c_0) = z$, $\phi_x(c_x) = 0$, $\phi_x(c_y) = z$.
5. The canonical map $\phi_y : C \to \Phi_y$ is surjective, and $\phi_y(c_0) = z$, $\phi_y(c_x) = z$, $\phi_y(c_y) = 0$.

The fact that $X_0(xy)$ is not hyperelliptic complicates the calculation of $C$: just the relations between the cuspidal divisors arising from the Drinfeld discriminant function are not sufficient for pinning down the orders of $c_x$ and $c_y$, cf. $\mathbb{E}$ Next, the calculations required for determining $\Phi_\infty$, $\Phi'_\infty$, and $\phi_\infty$ appear to be much more complicated than those in $\mathbb{E}$ and $\mathbb{E}$ Nevertheless, based on the facts that we are able to prove, and in analogy with the case $\deg(x) = 1$, $\deg(y) = 2$, we make the following prediction: The cuspidal divisor group $C \cong (\mathbb{Z}/(q^2 + 1)\mathbb{Z})^2$ is the direct sum of the cyclic subgroups generated by $c_x$ and $c_y$ both of which have order $q^2 + 1$, and there is an isogeny $J_0(xy) \to J^{xy}$ whose kernel is $C$.

**Definition 7.6.** It is known that every elliptic curve $E$ over $F$ with conductor $n_E = n \cdot \infty$, $n < A$, and split multiplicative reduction at $\infty$ is isogenous to a subvariety of $J_0(n)$; see $\mathbb{L}$. This follows from $\mathbb{L}$, $\mathbb{L}$, and the fact that the representation $\rho_E : G_F \to \text{Aut}(V(E)^*)$ is automorphic (i.e., $\rho_E = \rho(\Pi)$ for some $\Pi \in \mathcal{A}_0$). The multiplicity-one theorem can be used to show that in the $F$-isogeny class of $E$ there exists a unique curve $E'$ which is isomorphic to a one-dimensional abelian subvariety of $J_0(n)$, thus maps “optimally” into $J_0(n)$. We call $E'$ the $J_0(n)$-optimal curve. Theorem $\mathbb{L}$ and Remark $\mathbb{L}$ imply that $E$ with square-free conductor $R \cdot \infty$ and split multiplicative reduction at $\infty$ is also isogenous to a subvariety of $J_R$. Moreover, in the $F$-isogeny class of $E$ there is a unique elliptic curve $E''$ which is isomorphic to a one-dimensional abelian subvariety of $J_R$. We call $E''$ the $J_R$-optimal curve.

**Notation 7.7.** Let $E$ be an elliptic curve over $F$ given by a Weierstrass equation

$$E : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$  

Let $E^{(p)}$ be the elliptic curve given by the equation

$$E^{(p)} : Y^2 + a_1^pXY + a_3^pY = X^3 + a_2^pX^2 + a_4^pX + a_6^p.$$  

There is a Frobenius morphism $\text{Frob}_p : E \to E^{(p)}$ which maps a point $(x_0, y_0)$ on $E$ to the point $(x_0^p, y_0^p)$ on $E^{(p)}$. It is clear that the $j$-invariants of these elliptic curves are related by the equation $j(E^{(p)}) = j(E)^p$. If $E$ has semi-stable reduction at $v \in |F|$, then $\Phi_{E,v} \cong \mathbb{Z}/n\mathbb{Z}$, where $\Phi_{E,v}$ denotes the component group of $E$ at $v$ and $n = -\text{ord}_v(j(E)) \geq 1$. In this case, $\Phi_{E^{(p)},v} \cong \mathbb{Z}/pn\mathbb{Z}$.

**Definition 7.8.** An elliptic curve $E$ over $F$ with $j$-invariant $j(E) \notin \mathbb{F}_q$ is said to be Frobenius minimal if it is not isomorphic to $\overline{E}(p)$ for some other elliptic curve $\overline{E}$ over $F$. It is easy to check that this is equivalent to $j(E) \notin F^p$, cf. $\mathbb{B}$.
Proof. (i) There is a method due to Gekeler and Reversat \cite[Cor. 3.19]{12} which can be used to determine $\#\Phi_{E,\infty}$ of the $J_0(n)$-optimal curve in a given isogeny class. This method is based on the study of the action of Hecke algebra on $H_1(\Gamma_0(n)\setminus \mathcal{T}, \mathbb{Z})$. For $\deg(n) = 3$ the Gekeler-Reversat method can be further refined \cite[Cor. 1.2]{38}. Applying this method for $n = xy$, one obtains $\#\Phi_{E,\infty} = 3$ (resp. $\#\Phi_{E,\infty} = 5$) for

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Equation & x & y & $\infty$ \\
\hline
$E_1$ & $Y^2 + TXY + Y = X^3 + T^3 + 1$ & 3 & 3 & 3 \\
$E'_1$ & $Y^2 + TXY + Y = X^3 + T^2(T^3 + 1)$ & 9 & 1 & 1 \\
$E''_1$ & $Y^2 + TXY + Y = X^3$ & 1 & 1 & 9 \\
\hline
\end{tabular}
\caption{Isogeny class I}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Equation & x & y & $\infty$ \\
\hline
$E_2$ & $Y^2 + TXY + Y = X^3 + X^2 + T$ & 5 & 1 & 5 \\
$E'_2$ & $Y^2 + TXY + Y = X^3 + X^2 + T^3 + T^2 + T$ & 1 & 5 & 1 \\
\hline
\end{tabular}
\caption{Isogeny class II}
\end{table}

For $q$ even, Schweizer has completely classified the elliptic curves over $F$ having conductor of degree 4 in terms of explicit Weierstrass equations; see \cite{37}. We are particularly interested in those curves which have conductor $xy\infty$ and split multiplicative reduction at $\infty$.

**Theorem 7.9.** Assume $q = 2^s$. Elliptic curves over $F$ with conductor $xy\infty$ exist only if there exists an $E_q$-automorphism of $F$ that transforms the conductor into $(T + 1)(T^2 + T + 1)\infty$. In particular, $s$ must be odd.

If $s$ is odd, then there exists two isogeny classes of elliptic curves over $F$ with conductor $(T + 1)(T^2 + T + 1)\infty$ and split multiplicative reduction at $\infty$. The Frobenius minimal curves in each isogeny class are listed in Tables 2 and 3. The last three columns in the tables give the orders of the component groups $\Phi_{E,v}$ of the corresponding curve $E$ at $v = x, y, \infty$.

Proof. Theorem 4.1 in \cite{37}. \hfill $\square$

Next, \cite[Prop. 3.5]{37} describes explicitly the isogenies between the curves in classes I and II: There is an isomorphism of étale group-schemes over $F$

$$E_1[3] \cong H_1 \oplus H_2,$$

where $H_1 \cong \mathbb{Z}/3\mathbb{Z}$ and $H_2 \cong \mu_3$. The subgroup-scheme $H_1$ is generated by $(T+1, 1)$ and $H_2$ is generated by $(T^2, sT^3 + s^2)$, where $s$ is a third root of unity. Then $E_1/H_1 \cong E'_1$ and $E_1/H_2 \cong E''_1$. (It is well-known that an elliptic curve over $F$ with conductor of degree 4 has rank 0, so in fact $E_1(F) = H_1 \cong \mathbb{Z}/3\mathbb{Z}$.) Similarly, the subgroup-scheme $H_3$ of $E_2$ generated by $(1, 1)$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$, $E_2/H_3 \cong E'_2$, and $E_2(F) = H_3 \cong \mathbb{Z}/5\mathbb{Z}$.

**Proposition 7.10.** Assume $q = 2^s$ and $s$ is odd.

(i) $E_1$ and $E_2$ are the $J_0(xy)$-optimal curves in the isogeny classes I and II.

(ii) $E'_2$ is the $J^{xy}$-optimal curve in the isogeny class II.

(iii) If Conjecture 7.3 is true, then $E_1$ is the $J^{xy}$-optimal curve in the isogeny class I.

Proof. (i) There is a method due to Gekeler and Reversat \cite[Cor. 3.19]{12} which can be used to determine $\#\Phi_{E,\infty}$ of the $J_0(n)$-optimal curve in a given isogeny class. This method is based on the study of the action of Hecke algebra on $H_1(\Gamma_0(n)\setminus \mathcal{T}, \mathbb{Z})$. For $\deg(n) = 3$ the Gekeler-Reversat method can be further refined \cite[Cor. 1.2]{38}. Applying this method for $n = xy$, one obtains $\#\Phi_{E,\infty} = 3$ (resp. $\#\Phi_{E,\infty} = 5$) for
the $J_0(xy)$-optimal elliptic curve $E$ in the isogeny class I (resp. II). Since there is a unique curve with this property in each isogeny class, we conclude that $E_1$ and $E_2$ are the $J_0(xy)$-optimal elliptic curves. (For $q = 2$, this is already contained in [12, Example 4.4].)

(ii) Assume $q$ is arbitrary. Let $E$ be an elliptic curve over $F$ which embeds into $J^{xy}$. Since $J^{xy}$ has split toric reduction at $\infty$, [29, Cor. 2.4] implies that the kernel of the natural homomorphism

$$
\Phi_{E, \infty} \to \Phi'_E \cong \mathbb{Z}/(q + 1)\mathbb{Z}
$$

is a subgroup of $\mathbb{Z}/(q - 1)\mathbb{Z}$. Hence $\#\Phi_{E, \infty}$ divides $(q^2 - 1)$. First, this implies that $\#\Phi_{E, \infty}$ is coprime to $p$, so $E$ must be Frobenius minimal in its isogeny class. Second, if $q = 2^s$ and $s$ is odd, then 5 does not divide $(q^2 - 1)$, so $E_2$ is not $J^{xy}$-optimal. This leaves $E'_2$ as the only possible $J^{xy}$-optimal curve in the isogeny class II.

(iii) Let $E$ be the $J^{xy}$-optimal curve in the isogeny class I. By the discussion in (ii), this curve is one of the curves in Table 2. Suppose there is an isogeny $\varphi : J_0(xy) \to J^{xy}$ whose kernel is $C_6$. Restricting $\varphi$ to $E_1 \to J_0(xy)$, we get an isogeny $\varphi' : E_1 \to E$ defined over $F$ whose kernel is a subgroup of $C_6 \cong \mathbb{Z}/(q^2 + 1)\mathbb{Z}$. Note that 3 does not divide $q^2 + 1$. On the other hand, any isogeny from $E_1$ to $E'_1$ or $E''_1$ must have kernel whose order is divisible by 3. This implies that $\varphi'$ has trivial kernel, so $E = E_1$.

Remark 7.11. In the notation of the proof of Proposition 7.10, consider the restriction of $\varphi$ to $E_2 \to J_0(xy)$. By part (ii) of the proposition, there results an isogeny $\varphi'' : E_2 \to E_2'$ whose kernel is a subgroup of $\mathbb{Z}/(q^2 + 1)\mathbb{Z}$. Since 5 divides $q^2 + 1$ when $s$ is odd, part (ii) of Proposition 7.10 is compatible with Conjecture 7.3.

Theorem 7.12. Conjecture 7.3 is true for $q = 2$.

Proof. Assume $q = 2$. By Proposition 7.10 $E_1$ and $E_2$ are the $J_0(xy)$-optimal curves. Since the genus of $X_0(xy)$ is 2, it is hyperelliptic (this is true for general $q$ by Schweizer’s theorem which we used in [29]. The genus being 2 also implies that a quotient of $X_0(xy)$ by an involution has genus 0 or 1. The Atkin-Lehner involutions form a subgroup in $\text{Aut}(X_0(xy))$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Since the hyperelliptic involution is unique, each $E_1$ and $E_2$ can be obtained as a quotient of $X_0(xy)$ under the action of an Atkin-Lehner involution. Thus, there are degree-2 morphisms $\pi_i : X_0(xy) \to E_i$, $i = 1, 2$. In fact, one obtains the closed immersions $\pi_i^* : E_i \to J_0(xy)$ from these morphisms by Picard functoriality. Let $\pi_i^* : J_0(xy) \to E_i$ be the dual morphism. It is easy to show that the composition $\pi_i^* \circ \pi_i^* : E_i \to E_i$ is the isogeny given by multiplication by $2 = \deg(\pi_i)$. This implies that $E_1$ and $E_2$ intersect in $J_0(xy)$ in their common subgroup-scheme of 2-division points $S := \pi_1^*(E_1)[2] = \pi_2^*(E_2)[2]$, so

$$
J_0(xy)(F) = H_1 \oplus H_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} = C.
$$

Let $\psi : J_0(xy) \to E_1 \times E_2$ be the isogeny with kernel $S$. Note that $S$ is characterized by the non-split exact sequence of group-schemes over $F$:

$$
0 \to \mu_2 \to S \to \mathbb{Z}/2\mathbb{Z} \to 0.
$$

By Proposition 7.10 $E'_2$ is the $J^{xy}$-optimal elliptic curve in the isogeny class II. Let $E$ be the $J^{xy}$-optimal elliptic curves in class I. From the proof of Proposition 7.10 we know that $E$ is Frobenius minimal, so it is one of the curves listed in Table
There are also Atkin-Lehner involutions acting on $X^{xy}$ and they form a subgroup in $\text{Aut}(X^{xy})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$; see [31]. Now exactly the same argument as above implies that $E$ and $E_2'$ intersect in $J^{xy}$ along their common subgroup-scheme of 2-division points $S' \cong S$. Let $\nu : J^{xy} \to E \times E_2'$ be the isogeny with kernel $S'$. Let $\hat{\nu} : E \times E_2' \to J^{xy}$ be the dual isogeny.

The following argument is motivated by [19]. Consider the composition

$$\phi : J_0(xy) \xrightarrow{\psi} E_1 \times E_2 \xrightarrow{\phi_1 \times \phi_2} E \times E_2' \xrightarrow{\hat{\nu}} J^{xy},$$

where $\phi_1$ is either the identity morphism or has kernel $H_1$, $H_2$, and $\phi_2$ has kernel $H_3$. Since $\phi_1 \times \phi_2$ has odd degree, this morphism maps the kernel of $\hat{\psi}$ to the kernel of $\hat{\nu}$. Indeed, both are the “diagonal” subgroups isomorphic to $S$ in the corresponding group-schemes $(E_1 \times E_2)[2]$ and $(E \times E_2')[2]$. More precisely, $\mathcal{H} := \ker(\hat{\psi})$ is uniquely characterized as the subgroup-scheme of $\mathcal{G} := (E_1 \times E_2)[2]$ having the following properties: $\mathcal{H}^0$ is the image of the diagonal morphism $\mu_2 \to \mu_2 \times \mu_2 = \mathcal{G}^0$ and the image of $\mathcal{H}$ in $\mathcal{G}^e$ under the natural morphism $\mathcal{G} \to \mathcal{G}^e$ is the image of the diagonal morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A similar description applies to $\ker(\hat{\nu}) \subset (E \times E_2')[2]$. Thus, there is an isogeny $\phi' : J_0(xy) \to J^{xy}$ such that $\phi = \phi'[2]$ and $\ker(\phi') \cong \ker(\phi_1 \times \phi_2)$. We conclude that $J^{xy}$ is isomorphic to the quotient of $J_0(xy)$ by one of the following subgroups

$$H_3, \quad H_1 \oplus H_3, \quad H_2 \oplus H_3.$$

Now note that $H_1$ and $H_3$ under the specialization map $\phi_\infty$ inject into $\Phi_\infty$, but $H_2$ maps to 0 (indeed, $H_2 \cong \mu_3$ has non-trivial action by $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ whereas $\Phi_\infty$ is constant). Hence Theorem [13] implies that the quotients of $J_0(xy)$ by the subgroups listed above have component groups at $\infty$ of orders 3, 1, 9, respectively. Since $\Phi_\infty \cong \mathbb{Z}/3\mathbb{Z}$, we see that $J^{xy}$ is the quotient of $J_0(xy)$ by $H_3$ which is $C_0$ in this case.

References

[1] A. Blum and U. Stuhler, Drinfeld modules and elliptic sheaves, in Vector bundles on curves - New directions (M. S. Narasimhan, ed.), Lect. Notes Math., vol. 1649, Springer, 1997, pp. 110–188.

[2] S. Bosch, W. Lütkebohmert, and M. Raynaud, Néron models, Springer-Verlag, 1990.

[3] D. Bump, Automorphic forms and representations, Cambridge University Press, 1998.

[4] W. Casselman, On some results of Atkin and Lehner, Math. Ann. 201 (1973), 301–314.

[5] B. Conrad and W. Stein, Component groups of purely toric quotients, Math. Res. Lett. 8 (2001), 745–766.

[6] P. Deligne, Les constantes des équations fonctionnelles des fonctions $L$, in Modular functions of one variable II (P. Deligne and W. Kuyk, eds.), Lect. Notes Math., vol. 349, Springer, 1973, pp. 501–508.

[7] M. Denert and J. Van Geel, The class number of hereditary orders in non-Eichler algebras over global function fields, Math. Ann. 282 (1988), 379–393.

[8] V. Drinfeld, Elliptic modules, Math. USSR Sbornik 23 (1974), 561–592.

[9] E.-U. Gekeler, Zur arithmetik von Drinfeld-moduln, Math. Ann. 262 (1983), 167–182.

[10] E.-U. Gekeler, Automorphe Formen über $\mathbb{F}_q(T)$ mit kleinem Führer, Abh. Math. Sem. Univ. Hamburg 55 (1985), 111–146.

[11] E.-U. Gekeler, Über Drinfeldsche Modulkurven vom Hecke-Typ, Compositio Math. 57 (1986), 219–236.

[12] E.-U. Gekeler, Analytic construction of Weil curves over function fields, J. Théor. Nombres Bordeaux 7 (1995), 27–49.
[13] E.-U. Gekeler, Jacquet-Langlands theory over $K$ and relations with elliptic curves, in Drinfeld modules, modular schemes and applications (E.-U. Gekeler, M. van der Put, M. Reversat, and J. Van Geel, eds.), World Scientific, 1997, pp. 224–257.

[14] E.-U. Gekeler, On the cuspidal divisor class group of a Drinfeld modular curve, Doc. Math. 2 (1997), 351–374.

[15] E.-U. Gekeler, On the Drinfeld discriminant function, Compositio Math. 106 (1997), 181–202.

[16] E.-U. Gekeler and U. Nonnengardt, Fundamental domains of some arithmetic groups over function fields, Internat. J. Math. 6 (1995), 689–708.

[17] E.-U. Gekeler and M. Reversat, Jacobians of Drinfeld modular curves, J. Reine Angew. Math. 476 (1996), 27–93.

[18] S. Gelbart, Automorphic forms on adele groups, Princeton University Press, 1975.

[19] J. González and V. Rotger, Equations of Shimura curves of genus two, Int. Math. Res. Not. 14 (2004), 661–674.

[20] A. Grothendieck, Modèles de Néron et monodromie, SGA 7, Exposé IX, Lect. Notes Math., vol. 288, Springer-Verlag, 1972, pp. 313–523.

[21] R. Hartshorne, Algebraic geometry, Springer-Verlag, 1977.

[22] T. Hausberger, Uniformisation des variétés de Laumon-Rapoport-Stuhler et conjecture de Drinfeld-Carayol, Ann. Inst. Fourier 55 (2005), 1285–1371.

[23] D. Helm, On maps between modular Jacobians and Jacobians of Shimura curves, Israel J. Math. 160 (2007), 61–117.

[24] H. Jacquet and R. Langlands, Automorphic forms on $GL(2)$, Lect. Notes Math., vol. 114, Springer-Verlag, 1970.

[25] A. Kurihara, On some examples of equations defining Shimura curves and the Mumford uniformization, J. Fac. Sci. Univ. Tokyo 25 (1979), 277–300.

[26] G. Laumon, M. Rapoport, and U. Stuhler, D-elliptic sheaves and the Langlands correspondence, Invent. Math. 113 (1993), 217–338.

[27] A. Ogg, Mauvaise réduction des courbes de Shimura, Sém. Théor. Nombres Paris (1983/1984), 199–217.

[28] M. Papikian, On component groups of Jacobians of Drinfeld modular curves, Ann. Inst. Fourier 54 (2004), 2163–2199.

[29] M. Papikian, Analogue of the degree conjecture over function fields, Tran. Amer. Math. Soc. 359 (2007), 3483–3503.

[30] M. Papikian, Genus formula for modular curves of D-elliptic sheaves, Arch. Math. 92 (2009), 237–250.

[31] M. Papikian, On hyperelliptic modular curves over function fields, Arch. Math. 92 (2009), 291–302.

[32] M. Papikian, Local diophantine properties of modular curves of D-elliptic sheaves, J. Reine Angew. Math., to appear.

[33] K. Ribet, Sur les variétés abéliennes à multiplications réelles, C. R. Acad. Sc. Paris 291 (1980), 121–123.

[34] K. Ribet, On the component groups and the Shimura subgroup of $J_0(N)$, Séminaire Théor. Nombres Bordeaux (1987/1988), Exposé 6.

[35] K. Ribet, On modular representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms, Invent. Math. 100 (1990), 431–476.

[36] A. Schweizer, Hyperelliptic Drinfeld modular curves, in Drinfeld modules, modular schemes and applications (E.-U. Gekeler, M. van der Put, M. Reversat, and J. Van Geel, eds.), World Scientific, 1997, pp. 330–343.

[37] A. Schweizer, On elliptic curves over function fields of characteristic two, J. Number Theory 87 (2001), 31–53.

[38] A. Schweizer, Strong Weil curves over $F_q(T)$ with small conductor, J. Number Theory 131 (2011), 285–299.

[39] J.-P. Serre, Abelian l-adic representations and elliptic curves, W.A. Benjamin, 1968.

[40] J.-P. Serre, Trees, Springer, 2003.

[41] M. van der Put and J. Top, Algebraic compactification and modular interpretation, in Drinfeld modules, modular schemes and applications (E.-U. Gekeler, M. van der Put, M. Reversat, and J. Van Geel, eds.), World Scientific, 1997, pp. 141–166.
[42] M.-F. Vignéras, *Arithmétique des algèbres de quaternions*, Lect. Notes Math., vol. 800, Springer-Verlag, 1980.

Department of Mathematics, Pennsylvania State University, University Park, PA 16802

E-mail address: papikian@math.psu.edu