Bifurcations of a predator–prey system with cooperative hunting and Holling III functional response

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Abstract In this paper, we consider the dynamics of a predator–prey system of Gause type with cooperative hunting among predators and Holling III functional response. The known work numerically shows that the system exhibits saddle-node and Hopf bifurcations except homoclinic bifurcation for some special parameter values. Our results show that there are a weak focus of multiplicity three and a degenerate equilibrium with double zero eigenvalues (i.e., a cusp of codimension two) for general parameter conditions and the system can exhibit various bifurcations as perturbing the bifurcation parameters appropriately, such as the transcritical and the pitchfork bifurcations at the degenerate boundary equilibrium, the saddle-node and the Bogdanov–Takens bifurcations at the degenerate positive equilibrium and the Hopf bifurcation around the weak focus. The comparative study demonstrates that the dynamics are far richer and more complex than that of the system without cooperative hunting among predators. The analysis results reveal that appropriate intensity of cooperative hunting among predators is beneficial for the persistence of predators and the diversity of ecosystem.

Keywords Predator–prey system · Cooperative hunting · Holling III functional response · Weak focus of multiplicity three · Bifurcation

1 Introduction

Mathematical models have played a central role in population biology. To understand the interactions among populations, many mathematical modeling efforts have been made through investigating the food chain dynamics. One of the most classical mathematical models is the Gause-type predator–prey system taking the following form [23],

\[
\begin{align*}
\frac{dx}{dt} &= rx(1 - \frac{x}{K}) - y\Phi(x, y), \\
\frac{dy}{dt} &= cy\Phi(x, y) - dy,
\end{align*}
\]

(1.1)

where \(x\) and \(y\) are the prey density and predator density respectively, \(r\) represents the per capita intrinsic growth rate of prey, \(K\) stands for the carrying capacity of prey, \(c\) is the conversion rate, \(d\) represents the mortality rate of predators and \(\Phi(x, y)\) is the functional response referring to the consumption rate of a predator to prey. The functional response usually depends on many factors, such as the population density, the encounter rate and...
the handling time. The following functional responses have been extensively used in modeling population dynamics.

(i) Lotka–Volterra functional response

\[ \Phi(x, y) = ex, \]

where \( e \) is the encounter rate, which is an unbounded and linear function \([26,32]\).

(ii) Holling II functional response

\[ \Phi(x, y) = \frac{ex}{1 + ehx}, \]

where \( e \) is the encounter rate and \( h \) is the handling time, which is a bounded and monotonically increasing nonlinear function \([17]\).

(iii) Holling III functional response

\[ \Phi(x, y) = \frac{ex^2}{1 + ehx}, \]

where \( e \) and \( h \) have the same biological meanings, which also is a bounded and monotonically increasing nonlinear function like Holling II functional response. Although both Holling II and III functional responses are approaching an asymptote, the former is decelerating and the latter is sigmoid \([17]\).

(iv) Holling IV functional response

\[ \Phi(x, y) = \frac{ex}{1 + ehx + eh_0x^2}, \]

where \( e \) and \( h \) still have the same biological meanings and \( h_0 \) describes how handling time increases with prey density due to group defense, which is a bounded and nonmonotonic function \([13]\).

System (1.1) with various functional responses has attracted a lot of attention from mathematicians and theoretical biologists and displays interesting dynamics despite their complete dependence on prey density. System (1.1) with Lotka–Volterra functional response has relatively simple dynamics except the rise of the unique positive equilibrium from a transcritical bifurcation since the linear functional response is sufficiently simple. System (1.1) with nonlinear functional responses has relatively complicated dynamical properties. For example, the system with Holling II functional response called Rosenzweig–MacArthur model \([29]\) can exhibit a unique positive equilibrium and a unique stable limit cycle around the unstable positive equilibrium, which are induced by the transcritical bifurcation and the Hopf bifurcation respectively. Therefore, both of predators and prey coexist at either the coexistence equilibrium or the limit cycle. The appearance of the stable oscillation demonstrates exactly the paradox of enrichment \([10,18,20]\). Kazarnovskii and van den Driessche \([22]\) considered the system with Holling III functional response and used the Hopf theory to determine the bifurcation of the small amplitude periodic solution, the bifurcation diagram and the corresponding phase portraits of which were further exhibited in the book of Bazykin \([4]\). Furthermore, Chen and Zhang \([8]\) also studied the global dynamics of the system with Holling III functional response and obtained parameter conditions for the global stability of the unique positive equilibrium and the uniqueness of the limit cycle around the positive equilibrium. Some researchers investigated the system with Holling IV functional response (the parameters chosen such that the denominator of the functional response does not vanish for non-negative prey density) and obtained that the system can display complicated bifurcation phenomena, such as the degenerate Hopf bifurcation and the Bogdanov–Takens bifurcation of codimensions two and three \([19,34,39]\).

The independence of functional response of predator density is not always true in the ecosystem since the independence means that one predator affects the growth rate of its prey independently of its conspecifics. The two inverse ecological phenomena, interference and facilitation (or cooperative hunting) among predators, therefore need the predator-dependent functional response to reflect. The predator interference phenomenon first received the attention of scholars and was modeled by the ratio-dependent \([1,3]\) and the Beddington–DeAngelis \([5,14,25]\) functional responses, etc. Similarly, the cooperative hunting among predators also is a common phenomenon in nature \([15]\), which even is an effective mechanism to promote the evolution and diversity of species \([27]\). A lot of recent studies paid attention to the cooperative hunting among predators \([2,6,11,12,21,28,30,35]\). Alves and Hilker \([2]\) considered the following four types of predator-dependent functional response to model the cooperative hunting among predators based on the above-mentioned four types of functional response.

(a) Lotka–Volterra functional response with cooperative hunting

\[ \Phi(x, y) = (e + ay)x. \]
(b) Holling II functional response with cooperative hunting
\[ \Phi(x, y) = \frac{(e+ay)x}{1+h(e+ay)x}. \]
(c) Holling III functional response with cooperative hunting
\[ \Phi(x, y) = \frac{(e+ay)x^2}{1+h(e+ay)x^2}. \]
(d) Holling IV functional response with cooperative hunting
\[ \Phi(x, y) = \frac{(e+ay)x}{1+h(e+ay)x}. \]

The parameter \( a \) is the intensity of cooperative hunting among predators. The modeling motivation is that the encounter rate of predators with their prey is no longer a constant due to the cooperative hunting among predators, but rather an increasing function of predator density, which is analogous to the encounter-driven functional response put forward by Berec [6]. Alves and Hilker [2] investigated the two-parameter bifurcation of system (1.1) with the four types of functional response (a)–(d) respectively by numerical simulations and came to the conclusion that cooperative hunting among predators is a mechanism to produce the Allee effect in predators. Their numerical results showed that system (1.1) with Lotka–Volterra, Holling II and Holling IV types functional response (i.e., (a), (b) and (d)) can display Bogdanov–Takens bifurcation of codimension two for some special parameters, but the system with Holling III type functional response (i.e., (c)) cannot exhibit Bogdanov–Takens bifurcation of codimension two, exactly, there is neither a Bogdanov–Takens bifurcation point nor a homoclinic bifurcation curve in the two-parameter bifurcation diagram. Whereafter, Zhang and Zhang [37] further considered system (1.1) with Lotka–Volterra type functional response, and discussed the location of equilibria qualitatively and gave analytical conditions for not only the Bogdanov–Takens bifurcation of codimension two at positive equilibrium but also the transcritical and pitchfork bifurcations at boundary equilibrium. For system (1.1) with Holling II type functional response, which actually is the special case of Berec’s system [6]. Yao [36] further gave the parameter conditions for the qualitative properties of equilibria and analysed the bifurcations at nonhyperbolic equilibria including saddle-node, transcritical, pitchfork and Hopf bifurcations and Bogdanov–Takens bifurcation of codimension two by applying the pseudo-division reduction and complete discrimination system of parametric polynomial. The comparative studies revealed that system (1.1) with cooperative hunting has even richer and more complicated dynamics than the system without cooperative hunting for both Lotka–Volterra type and Holling II type functional responses.

In this paper, we consider system (1.1) with Holling III type functional response and analyse whether the system can also exhibit complicated dynamics. The system takes the following form
\[
\begin{align*}
\frac{dx}{dt} &= r x \left(1 - \frac{x}{\tilde{K}}\right) - \frac{(e+ay)x^2y}{1+h(e+ay)x^2}, \\
\frac{dy}{dt} &= \frac{c(e+ay)x^2y}{1+h(e+ay)x^2} - dy.
\end{align*}
\] (1.2)

By the scaling \( \tilde{x} := \sqrt{\frac{c}{d}} x, \tilde{y} := \sqrt{\frac{c}{d}} y, \sigma := \frac{a}{v} \sqrt{\frac{dc}{e}}, \) \( \kappa := \sqrt{\frac{ce}{d}} K, \tilde{h} := \frac{hd}{e} \) and \( \tau := df, \) system (1.2) can be simplified as
\[
\begin{align*}
\frac{dx}{dt} &= \sigma x \left(1 - \frac{x}{\tilde{K}}\right) - \frac{(1+\alpha)x^2y}{1+h(1+\alpha)x^2}, \\
\frac{dy}{dt} &= \frac{(1+\alpha)x^2y}{1+h(1+\alpha)x^2} - y,
\end{align*}
\] (1.3)

where we still denote \( \tilde{x}, \tilde{y}, \tilde{h} \) and \( \tau \) as \( x, y, h \) and \( t \) respectively, and \( \alpha \) represents the intensity of cooperative hunting among predators. Alves and Hilker [2] displayed the two-parameter bifurcation diagram when varying the cooperative hunting \( \alpha \) and intrinsic growth rate of prey \( \sigma \). By choosing \( h = 0.9 \) and \( \kappa = 0.8 \), in which the Hopf bifurcation curve does not touch the saddle-node bifurcation curve and the homoclinic bifurcation curve does not appear, which means that system (1.3) does not undergo a Bogdanov–Takens bifurcation for the special parameter values. Nevertheless, we can prove that the system can exhibit not only a Bogdanov–Takens bifurcation of codimension two at the degenerate positive equilibrium but also a degenerate Hopf bifurcation at the nonhyperbolic positive equilibrium. The time-rescaling transformation shows that system (1.3) and the following quartic polynomial system are orbitally equivalent
\[
\begin{align*}
\frac{dx}{dt} &= x(\sigma (x - \alpha) + h(1+\alpha)x^2) - k(1+\alpha)xy), \\
\frac{dy}{dt} &= kxy((1-h) - h(1+\alpha)x^2 - 1).
\end{align*}
\] (1.4)

The paper is organized as follows. In Sect. 2, we prove qualitatively that system (1.4) has at most four equilibria and give the parameter conditions for the existence and their qualitative properties. In Sect. 3, we distinguish the topological types of the nonhyperbolic equilibria and analyse various possible bifurcations around
them, such as a transcritical bifurcation and a pitchfork bifurcation at the degenerate boundary equilibrium, a saddle-node bifurcation and a Bogdanov–Takens bifurcation of codimension two at the degenerate positive equilibrium and a Hopf bifurcation at the weak focus of multiplicity at most three. Finally, we make numerical simulations to demonstrate our theoretical results and end the paper with a brief discussion in Sect. 4.

2 Equilibria and their properties

In this section we discuss the existence of equilibria of system (1.4) and their qualitative properties. We first give the following partition of parameters \((\alpha, \kappa, \sigma)\) as

\[
\mathbb{R}^3_+ := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3 : \alpha > 0, \kappa > 0, \sigma > 0\}
\]

where

\[
\mathcal{P}_1 := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ : \alpha < \alpha_2, \kappa < \kappa_1\},
\]

\[
\mathcal{P}_2 := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ : \alpha = \alpha_1, \kappa \leq \kappa_1\},
\]

\[
\mathcal{P}_3 := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ : \alpha = \alpha_2, \kappa < \kappa_1\},
\]

\[
\mathcal{P}_4 := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ : \alpha < \alpha_1, \kappa \leq \kappa_1\},
\]

\[
\mathcal{P}_5 := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ : \kappa > \kappa_1\},
\]

\[
\mathcal{P}_6 := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ : \alpha > \frac{2}{\sqrt[3]{\kappa}}, \kappa = \kappa_1\},
\]

\[
\mathcal{P}_7 := \{(\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ : \alpha > \alpha_2, \kappa < \kappa_1\}
\]

with \(\kappa_1 := \frac{1}{\sqrt{1-\kappa}}\) and

\[
\alpha_1 := -\frac{9\kappa^2(\kappa - 1) + 16}{81\kappa^3(h - 1)} - \frac{1 + \sqrt{3}i}{2} \sqrt{-\frac{A}{2} + \sqrt{\Delta}}
\]

\[
+ \frac{1 - \sqrt{3}i}{2} \sqrt{\frac{A}{2} - \sqrt{\Delta}}.
\]

\[
\alpha_2 := \frac{-9\kappa^2(\kappa - 1) + 16}{81\kappa^3(h - 1)} + \frac{1 - \sqrt{3}i}{2} \sqrt{-\frac{A}{2} + \sqrt{\Delta}}
\]

\[
+ \frac{1 + \sqrt{3}i}{2} \sqrt{\frac{A}{2} - \sqrt{\Delta}},
\]

\[
A := \frac{1}{531441\kappa^6(h - 1)^4} + 776760(h - 1)^3\kappa^6 + 1492992(\kappa - 1)^2\kappa^4
\]

\[
+(3538944h - 3538944\kappa^2 + 2097152),
\]

\[
\Delta := \frac{64}{14348907} (h\kappa^2 - \kappa^2 + 1)(3h^2 - 3\kappa^2 + 128)^3
\]

\[
\kappa^6(h - 1)^4 - \kappa^6(h - 1)^4. \tag{2.1}
\]

The following theorem shows the various parametric conditions for the existence of equilibria of system (1.4) and their corresponding qualitative properties.

\[\text{Theorem 1} \text{ System (1.4) has at most four equilibria as follows. (i) System (1.4) always has two boundary equilibria } E_0(0, 0) \text{ and } E_\kappa(\kappa, 0). \text{ Moreover, } E_0(0, 0) \text{ is a saddle and } E_\kappa(\kappa, 0) \text{ is a stable node (or a saddle or degenerate) if } \kappa > 0 \text{ and } h \geq 1 \text{ or } 0 < \kappa < \kappa_1 \text{ and } 0 \leq h < 1 \text{ (or } \kappa > \kappa_1 \text{ and } 0 < h < 1 \text{ or } \kappa = \kappa_1 \text{ and } 0 \leq h < 1). \text{ (ii)} \text{ System (1.4) has at most two positive equilibria. More concretely, (ii.a) system (1.4) has no positive equilibrium if } h \geq 1 \text{ and } (\alpha, \kappa, \sigma) \in \mathbb{R}^3_+ \text{ or } 0 < h < 1 \text{ and } (\alpha, \kappa, \sigma) \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_4; \text{ (ii.b) system (1.4) has a unique positive equilibrium } E_1(x_1, y_1) \text{ if } 0 < h < 1 \text{ and } (\alpha, \kappa, \sigma) \in \mathcal{P}_3 \cup \mathcal{P}_5 \text{ and system (1.4) has a unique positive equilibrium } E_\kappa(x_\kappa, y_\kappa) \text{ if } 0 < h < 1 \text{ and } (\alpha, \kappa, \sigma) \in \mathcal{P}_2 \cup \mathcal{P}_6 \text{ and system (1.4) has a unique positive equilibrium } E_\kappa(x_\kappa, y_\kappa). \text{ Further-}
\]

\[
E_{\kappa}(x_\kappa, y_\kappa) \text{ is a stable node or focus (or an unstable node or focus or a center or weak focus) if } T(x_1) < 0 \text{ (or } T(x_1) = 0, \text{ or } T(x_1) > 0) \text{ then } E(x_\kappa, y_\kappa) \text{ is a saddle and } E(x_\kappa, y_\kappa) \text{ is degenerate, where}
\]

\[
y_\kappa = \sigma x_1(1 - \frac{1}{\kappa}) \quad (i = 1, 2, \ast)
\]

\[
T(x_1) := -\kappa(h - 1)^2x_1^2 - 2h\sigma x_1 + \kappa(\sigma(2h - 1) - h + 1).
\]

\[\text{Proof} \text{ The following algebraic equations determine all the equilibria of system (1.4)}
\]

\[
\begin{align*}
\{ x(\sigma(\kappa - x)(1 + h(1 + \alpha y)x^2) - k(1 + \alpha y)x) &= 0, \\
\kappa y((1 - h)(1 + \alpha y)x^2 - 1) &= 0.
\end{align*}
\]

\[\tag{2.2}
\]

Note that system (1.4) always has two boundary equilibria \(E_0(0, 0)\) and \(E_\kappa(\kappa, 0)\) for all permissible parameters. The positive equilibria of system (1.4) lie on the curve

\[
y = \sigma x(1 - \frac{x}{\kappa}), \quad 0 < x < \kappa. \tag{2.3}
\]

We obtain the following function by substituting curve (2.3) into the second equation in (2.2)

\[
F(x) := -\alpha(\sigma(2h - 1) - h + 1)x^4 + 2 \alpha \sigma h(1 + \alpha y)x^3 + \kappa(2h - 1)x^2 + \kappa, \tag{2.4}
\]

whose zeros in the interval \((0, \kappa)\) determine the abscissas of positive equilibria of system (1.4). It is obvious that \(F(x)\) has zeros only if \(h \neq 1\). Clearly, the derivative of \(F(x)\) is

\[
F'(x) = (h - 1)x(-4\alpha \sigma x^2 + 3\kappa \alpha \sigma x + 2\kappa). \tag{2.5}
\]
The unique positive zero of \( F'(x) = 0 \) is
\[
x_* := \frac{3\kappa \alpha \sigma + \sqrt{\kappa \alpha \sigma (9\kappa \alpha \sigma + 32)}}{8\alpha \sigma}. \tag{2.6}
\]
If \( h > 1 \), then \( F(x) \) increases first and then decreases for \( x \in (0, +\infty) \) and \( F(\kappa) = \kappa [\kappa^2 (h - 1) + 1] > 0 \), which implies that \( F(x) \) has no zero for \( x \in (0, \kappa) \). If \( 0 < h < 1 \), then \( F(x) \) decreases first and then increases for \( x \in (0, +\infty) \). In order to determine the number of zeros of \( F(x) \) in the interval \((0, \kappa)\) for this case, we need consider the signs of \( F(x_*) \) and \( F(\kappa) \) and \( \kappa \) for \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathbb{R}_+^3\). The discussion is divided into the following three subcases. (I) If \( F(x_*) > 0 \), i.e., \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_1 \), then \( F(x) \) has no positive zero. (II) If \( F(x_*) = 0 \) and \( x_* \geq \kappa \), i.e., \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_2 \), then \( F(x) \) has no positive zero in the interval \((0, \kappa)\); If \( F(x_*) = 0 \) and \( x_* < \kappa \), i.e., \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_3 \), then \( F(x) \) has a unique positive zero \( x_0 \) in the interval \((0, \kappa)\). (III) If \( F(x_*) < 0 \), \( F(\kappa) \geq 0 \) and \( x_* > \kappa \), i.e., \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_4 \), then \( F(x) \) has a unique positive zero \( x_1 \) in the interval \((0, \kappa)\); If \( F(x_*) < 0 \), \( F(\kappa) < 0 \), \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_5 \), then \( F(x) \) has a unique positive zero \( x_1 \) in the interval \((0, \kappa)\); If \( F(x_*) < 0 \), \( F(\kappa) > 0 \) and \( x_* < \kappa \), i.e., \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_6 \), then \( F(x) \) has a unique positive zero \( x_2 \) in the interval \((0, \kappa)\). Furthermore, \( \alpha_1 \) and \( \alpha_2 \) \((\alpha_1 < \alpha_2) \) given in (2.1) are two larger positive zeros of
\[
f(\alpha) := -27\kappa^3 \sigma^3 (h - 1) \alpha^3 - 16\sigma^2 (9\kappa \kappa^2 - 9\kappa^2 + 16) \alpha^2 - 4\sigma \kappa (h - 1) (\kappa \kappa^2 - \kappa^2 + 32) \alpha - 16 \kappa^2 (h - 1)^2,
\]
which determines the sign of \( F(x_*) \). Therefore, system (1.4) has no positive equilibrium if \( h \geq 1 \) and \((\alpha, \kappa, \sigma) \in \mathbb{R}_+^3 \), or \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_4 \); System (1.4) has either a unique positive equilibrium \( E_1(x_1, y_1) \) if \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_3 \cup \mathcal{P}_6 \) or a unique positive equilibrium \( E_*(x_*, y_*) \) if \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_3 \); System (1.4) has two positive equilibria \( E_1(x_1, y_1) \) and \( E_2(x_2, y_2) \) if \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_7 \), where \( y_i = \sigma x_i (1 - \frac{1}{\kappa}) \) \((i = 1, 2, *) \).

In the following, we study the qualitative properties of the equilibria. By a direct computation, the Jacobian matrix of system (1.4) at any equilibrium has the form
\[
J := \begin{pmatrix}
J_{11} & x^2 (ah \sigma_1 x - ah \sigma_2 x^2 - 2ah y - \kappa) \\
2x(1 - h) (xy + 1) y & \kappa \alpha (h - 1)^2 y
\end{pmatrix}
\]
with
\[
J_{11} := \sigma \{1 + h(\alpha + 1) x^2\} (\kappa - 2x)
+ 2x (h \kappa \sigma x - h \sigma x^2 - \kappa y) (\alpha y + 1).
\]
We use the symbols \( T \) and \( D \) to denote the trace and determinant of Jacobian matrix \( J \) respectively. It is obvious that \( E_0 \) is a saddle since \( D\big|_{E_0} = -\sigma \kappa^2 < 0 \). The computation yields that
\[
D\big|_{E_0} = \kappa \sigma^2 (h^2 + 1) (\kappa^2 (h - 1) + 1),
\]
\[
T\big|_{E_0} = -\kappa \sigma (h^2 + 1) + \kappa^2 (h - 1) + 1,
\]
\[
T\big|_{E_0} - 4D\big|_{E_0} = \kappa \sigma (h^2 + 1) + \kappa^2 (1 - h) - 1^2,
\]
we therefore have \( D\big|_{E_0} < 0 \) if \( 0 < h < 1 \) and \( \kappa > \kappa_1 \) implying that \( E_1 \) is a saddle, \( D\big|_{E_0} = 0 \) if \( 0 < h < 1 \) and \( \kappa_1 < \kappa < \kappa_1 \) implying that \( E_1 \) is degenerate, \( D\big|_{E_0} > 0 \), \( T\big|_{E_0} < 0 \) and \( T\big|_{E_0} - 4D\big|_{E_0} \geq 0 \) if \( h \geq 1 \) or \( 0 < h < 1 \) and \( 0 < \kappa < \kappa_1 \) implying that \( E_1 \) is a stable node. At the positive equilibria \( E_i, i = 1, 2, * \), we use the branch \( 1 + h(\alpha + 1)x^2 = (1 + \alpha y)x^2 \) in \( J_{11} \) and the pseudo-division reduction to obtain determinant \( D\big|_{E_i} \) and trace \( T\big|_{E_i} \) of the Jacobian matrix \( J \) as follows
\[
D\big|_{E_i} = \frac{\sigma (-\alpha \sigma_1 x^2 + \alpha \kappa \sigma_1 x(y_1 - \kappa)) x^3}{\kappa_2 \alpha \sigma^2 (1 - h)} F'(x_1),
\]
\[
T\big|_{E_i} = \frac{\alpha \sigma^2 \kappa \kappa (1 - h)}{\kappa_2 \alpha \sigma^2 (1 - h)} F'(x_1) \tag{2.7}
\]
where \( F'(x_1) \) is the derivative of \( F(x) \) at \( x_1 \) given by (2.5). It is obvious that the sign of \( D\big|_{E_i} \) is opposite to that of \( F'(x_1) \) and the sign of \( T\big|_{E_i} \) is same as that of \( T(x_1) \), which is the factor of numerator of \( T\big|_{E_i} \) given in Theorem 1. Because \( F'(x_1) < 0 \), \( F'(x_2) > 0 \) and \( F'(x_*) = 0 \), we obtain \( D\big|_{E_1} > 0 \), \( D\big|_{E_2} < 0 \) and \( D\big|_{E_*} = 0 \), which imply that \( E_2 \) is a saddle, \( E_* \) is degenerate and \( E_1 \) is a stable node or focus (or an unstable node or focus or a center or weak focus) if \( T(x_1) < 0 \) (or \( T > 0 \)).

**Remark 1** System (1.4) has more equilibria than the system without cooperative hunting among predators (i.e., \( \alpha = 0 \) in system (1.4)). More concretely, the two systems have the same boundary equilibrium \( E_0 \) and \( E_\kappa \) and they have the same qualitative properties as shown in Theorem 1. Nevertheless, the two systems have different positive equilibrium. System (1.4) has at most
two positive equilibria as in Theorem 1, but the system without cooperative hunting has a unique positive equilibrium \((\kappa_1, \sigma \kappa_1 (1 - \frac{\kappa_1}{\kappa}))\) for \(\kappa > \kappa_1\) and \(0 < h < 1\), which is a stable node or focus (or an unstable node or focus or a center or weak focus) if \(\kappa > \kappa_1\) and \(0 < h \leq \frac{1}{2}\) or \(\kappa_1 < \kappa < \frac{2h_0}{2h-1}\) and \(\frac{1}{2} < h < 1\) (or \(\kappa > \frac{2h_0}{2h-1}\) and \(\frac{1}{2} < h < 1\) or \(\kappa = \frac{2h_0}{2h-1}\) and \(\frac{1}{2} < h < 1\)), and has no positive equilibrium if \(\kappa > 0\) and \(h \geq 1\) or \(0 < \kappa \leq \kappa_1\) and \(0 < h < 1\). Therefore, the main effect of the cooperative hunting among predators is on the coexistence of species.

3 Bifurcations

From Theorem 1, we know that system (1.4) has two degenerate equilibria \(E_k\) and \(E_s\) and a center or weak focus \(E_1\). In this section, we further identify the topological types of these nonhyperbolic equilibria and display all bifurcations at them. Concretely, system (1.4) may exhibit transcritical and pitchfork bifurcations around equilibrium \(E_k\), saddle-node and Bogdanov–Takens bifurcations around equilibrium \(E_s\) and Hopf bifurcation around equilibrium \(E_1\).

3.1 Transcritical and pitchfork bifurcations

In this subsection, we identify the topological type of the degenerate equilibrium \(E_k\) and show that both transcritical and pitchfork bifurcations may occur at \(E_k\). In fact, Theorem 1 shows that \(E_k\) is degenerate if \(\kappa = \kappa_1\) and \(0 < h < 1\) with \(D|_{E_k} = 0\) and \(T|_{E_k} < 0\). The following theorem indicates that \(E_k\) is a saddle-node and system (1.4) may undergo transcritical and pitchfork bifurcations at \(E_k\).

**Theorem 2** For \(\kappa = \kappa_1\) and \(0 < h < 1\), \(E_k\) is a saddle-node of system (1.4). Moreover, (i) when \(\alpha \neq \frac{2}{\sigma \kappa_1}\), a transcritical bifurcation happens at \(E_k\) if \(\kappa\) varies from \(\kappa > \kappa_1\) to \(\kappa < \kappa_1\); (ii) when \(\alpha = \frac{2}{\sigma \kappa_1}\), a pitchfork bifurcation happens at \(E_k\) if \(\kappa\) varies from \(\kappa > \kappa_1\) to \(\kappa < \kappa_1\).

**Proof** Let \(\epsilon = \kappa - \kappa_1\) and consider \(|\epsilon|\) sufficiently small. Suspending system (1.4) with the variable \(\epsilon\) and using the linear transformation \(x = u + v + \kappa_1 + \epsilon\) and \(y = -\sigma v\) together with the time-rescaling \(\tau = \frac{\epsilon t}{\kappa_1}\) to translate \(E_k\) to \((0, 0)\) and diagonalize the linear part of the suspended system, we obtain

\[
\begin{align*}
\frac{du}{d\tau} &= -\frac{(\alpha \kappa \sigma -2(h-1))u^2}{\kappa_1} + \frac{2(h-1)}{\kappa_1}uv + \frac{2(h-1)}{\kappa_1}ue - \frac{2(\sigma + h_0 - \kappa_1)(h-1)}{\kappa_1}u^3 - \frac{2(\alpha + h_0 - \kappa_1)(h-1)}{\kappa_1}u^2v \\
&\quad - \frac{3(\alpha + h_0 - \kappa_1)(h-1)}{\kappa_1}u^2\epsilon - \frac{(h-1)^2}{\kappa_1}uv^2 - \frac{4(h-1)^2}{\kappa_1}u^2v + \frac{3(h-1)^2}{\kappa_1}u^2\epsilon + O\left(\| (u, v, \epsilon) \|^4\right), \\
\frac{dv}{d\tau} &= \epsilon - \frac{2(h+1)^2 - 3(\sigma-1)(h-1)^2u^2 + 2(\sigma-1)(h-1)u}{(h-1)(3\sigma - \kappa_1 + \alpha \sigma + 3\alpha(\sigma - 1))}\frac{u}{\kappa_1}v \\
&\quad - \frac{4(\alpha + h_0 - \kappa_1)(h-1)}{\kappa_1}u^2v - \frac{\alpha \kappa_1(\sigma - 1)(h-1) + 2\alpha(\alpha + h_0 - \kappa_1)(h-1)}{\kappa_1}u^2\epsilon \\
&\quad - \frac{4h_0(\alpha + h_0 - \kappa_1)(h-1)}{\kappa_1}u^2v - \frac{3h(h-1)u^2 - 6h(h-1)u^2\epsilon + O\left(\| (u, v, \epsilon) \|^4\right)}{\kappa_1}
\end{align*}
\]

where \(\tau\) is still denoted as \(t\). Theorem 1 of [7] shows that system (3.1) has a \(C^\infty\) center manifold \(W^c : v = h_1(u, \epsilon)\) near \((u, v, \epsilon) = (0, 0, 0)\), which is tangent to the plane \(v = 0\) at \((u, v, \epsilon) = (0, 0, 0)\). Let

\[
v = h_1(u, \epsilon) = a_1 u^2 + b_1 \epsilon^2 + c_1 u \epsilon + O\left(\| (u, \epsilon) \|^3\right).
\]

By the invariant property of center manifold (3.2) to the solutions of (3.1), we differentiate both sides of (3.2) and obtain \(\dot{\nu} = h_{1u}u + h_{1\epsilon} \epsilon\). Substituting (3.1) in the equality and comparing the coefficients of \(u^2, \epsilon^2\) and \(u \epsilon\), we get \(b_1 = 0\) and

\[
\begin{align*}
a_1 &= \frac{\alpha \sigma (h-1) \kappa_1^2 + \alpha \sigma (h_0 + h - 1) \kappa_1 - 2 \alpha h_0 + 2h + \sigma - 2}{\kappa_1 \sigma}, \\
c_1 &= \frac{2(\sigma-1)(h-1)}{\kappa_1 \sigma}.
\end{align*}
\]

Thus, restricted to the center manifold (3.2), system (3.1) can be written as
\[
\frac{du}{dt} = \frac{2(h - 1)}{\kappa_1 \sigma} u \epsilon - \frac{(\alpha \kappa_1 \sigma - 2)(h - 1)}{\sigma \kappa_1} u^2 \\
- \frac{(h - 1)(-4(h - 1)(\sigma h - h - 3\sigma + 1)\kappa_1 + 3\sigma \alpha^2)}{\sigma^2 \kappa_1} u^2 \epsilon \\
- \frac{3(h - 1)^2}{\sigma} u^2 \\
+ \frac{(h - 1)((h - 1)(4h \sigma - 4h - 3\sigma + 4)\kappa_1 + 2\alpha \sigma(\sigma h - h - 2\sigma + 1))}{\sigma^2 \kappa_1} u^3 + O(\|u\| \|\epsilon\|^4). \tag{3.3}
\]

If \(\alpha \neq \frac{2}{\sigma \kappa_1}\), then the coefficient \(\frac{(\alpha k_1 \sigma - 2)(h - 1)}{\sigma \kappa_1} \neq 0\) in system (3.3), the origin is the unique equilibrium of system (3.3) as \(\epsilon = 0\) and the other equilibrium arises from the origin as \(\epsilon \neq 0\). Therefore, the boundary equilibrium \(E_0\) is a saddle-node as \(\kappa = \kappa_1\) and system (1.4) undergoes a transcritical bifurcation [16] at \(E_0\).

Furthermore, if \(\alpha < \frac{2}{\sigma \kappa_1}\) and \(\kappa\) varies from \(\kappa < \kappa_1\) to \(\kappa > \kappa_1\), then a stable node \(E_0\) changes into a saddle \(E_1\) and a stable node \(E_2\); if \(\alpha > \frac{2}{\sigma \kappa_1}\) and \(\kappa\) varies from \(\kappa > \kappa_1\) to \(\kappa < \kappa_1\), then a saddle \(E_0\) changes into a stable node \(E_1\) and a saddle \(E_2\). If \(\alpha = \frac{2}{\sigma \kappa_1}\), then \(\frac{5\sigma(1-h)}{\sigma^2 \kappa_1} = 0\) and the coefficient of \(u^3\) is \(\frac{5\sigma(1-h)}{\sigma^2 \kappa_1} \neq 0\) in (3.3) and the origin is the unique equilibrium as \(\epsilon = 0\) and the other two equilibria arise from the origin as \(\epsilon > 0\). Therefore, \(E_0\) is a saddle-node as \(\kappa = \kappa_1\) and system (1.4) undergoes a pitchfork bifurcation [16] at \(E_0\) as \(\kappa\) varies from \(\kappa < \kappa_1\) to \(\kappa > \kappa_1\) such that a stable node \(E_0\) changes into a saddle \(E_1\) and a saddle node \(E_2\). This completes the proof of the theorem. \(\square\)

### 3.2 Saddle-node bifurcation

In this subsection, we prove the topological type of the degenerate equilibrium \(E_0\) and show that the saddle-node bifurcation may occur at \(E_0\) if \(D|\theta = 0\) and \(T|\theta \neq 0\). From \(F(x_\ast) = F'(x_\ast) = 0\), we can express \(\alpha\) and \(\kappa\) by \(x_\ast\), \(\sigma\) and \(h\) as follows

\[
\alpha = \alpha_3 := \frac{2((h - 1)x_\ast^2 + 2)}{(h - 1)x_\ast^2 + 3}, \tag{3.4}
\]

\[
\kappa = \kappa_2 := \frac{2x_\ast((h - 1)x_\ast^2 + 2)}{(h - 1)x_\ast^2 + 3},
\]

with \(0 < x_\ast \leq 1\) and \(0 < h < 1\) or \(x_\ast > 1\) and \(\frac{x_\ast^2 - 1}{x_\ast^2} < h < 1\). Substituting (3.4) into \(T|\theta\), we obtain that \(T|\theta < 0\) if \(\sigma > \sigma_1\), \(T|\theta > 0\) if \(\sigma < \sigma_1\) and \(T|\theta = 0\) if \(\sigma = \sigma_1\) with

\[
\sigma_1 := \frac{(h - 1)((h - 1)x_\ast^2 + 2)((h - 1)x_\ast^2 + 1)}{(h - 1)^2x_\ast^2 + h - 2}.
\]

**Theorem 3** For \(\alpha = \alpha_3\), \(\kappa = \kappa_2\) and \(\sigma \neq \sigma_1\) with \(0 < x_\ast \leq 1\) and \(0 < h < 1\) or \(x_\ast > 1\) and \(\frac{x_\ast^2 - 1}{x_\ast^2} < h < 1\), \(E_0\) is a saddle-node of system (1.4) and a saddle-node bifurcation occurs at \(E_0\) if \(\alpha\) varies from \(\alpha < \alpha_3\) to \(\alpha > \alpha_3\).

**Proof** Let \(\epsilon = \alpha - \alpha_3\) and consider \(|\epsilon|\) sufficiently small. Translating \(E_0\) to the origin \((0, 0)\) and suspending system (1.4) with the variable \(\epsilon\) we obtain the following system

\[
\begin{align*}
\frac{dx}{d\tau} &= a_{100} x + a_{010} y + a_{001} \epsilon + a_{200} x^2 \\
&+ a_{110} x y + a_{201} x \epsilon + a_{011} \epsilon y + a_{011} \epsilon \epsilon y \\
\frac{dy}{d\tau} &= b_{100} x + b_{010} y + b_{001} \epsilon + b_{200} x^2 \\
&+ b_{110} x y + b_{101} x \epsilon + b_{011} \epsilon y + b_{011} \epsilon \epsilon y + b_{011} \epsilon \epsilon \epsilon y \\
\frac{d\epsilon}{d\tau} &= 0,
\end{align*}
\]

where the coefficients \(a_{ijk}\) and \(b_{ijk}\) are listed as follows.

\[
\begin{align*}
a_{100} &:= -\frac{2(a_\ast \sigma (h - 1)x_\ast^2 + h - 2)(h - 1)}{(h - 1)^2x_\ast^2 + 3}, \\
a_{010} &:= -\frac{4(a_\ast (h - 1)x_\ast^2 - 2[(h - 1)^2x_\ast^2 + h - 2])}{(h - 1)^2x_\ast^2 + 3}, \\
a_{001} &:= -\frac{4(h - 1)x_\ast^2 + (h - 1)^2x_\ast^2 + 1}{2(h - 1)^2x_\ast^2 + 3}, \\
a_{200} &:= -\frac{4(h - 1)^2x_\ast^2 - 3h - 4}{(h - 1)^2x_\ast^2 + 3}, \\
a_{110} &:= -\frac{4(h - 1)^2x_\ast^2 + 2}{(h - 1)^2x_\ast^2 + 3}, \\
a_{011} &:= -\frac{4(h - 1)^2x_\ast^2 - 2}{(h - 1)^2x_\ast^2 + 3}(h - 1)^2x_\ast^2 + 3).
\end{align*}
\]
where \( p_{ijk} \) and \( q_{ijk} \) are listed in the “Appendix”. System (3.5) similarly has a two-dimensional center manifold \( W^c : v = h_2(u, \epsilon) \) near \((u, v, \epsilon) = (0, 0, 0)\) as follows

\[ v = h_2(u, \epsilon) = a_2u^2 + b_2\epsilon^2 + c_2ue + O(||(u, \epsilon)||^3) \] 

with coefficients \( a_2, b_2 \) and \( c_2 \) given in the “Appendix”. Restricted to the center manifold (3.6), system (3.5) can be written as

\[ \frac{dw}{dt} = d_0(\epsilon) + d_1(\epsilon)u + d_2(\epsilon)u^2 + O(||u||^3), \]

where

\[
\begin{align*}
\frac{d\epsilon}{dt} &= \alpha(\epsilon) + u^2 + O(||u||^3), \\
\frac{d\xi}{dt} &= \frac{4d_0(\epsilon)(d_1(\epsilon) - d_2(\epsilon))}{4d_2(\epsilon)} + \frac{4d_0(\epsilon)}{d_2(\epsilon)}
\end{align*}
\]

We can check that \( d_0(0) = d_1(0) = 0 \) and \( d_2(0) \neq 0 \) for \( \sigma \neq \sigma_1 \) as well as \( 0 < x_s \leq 1 \) and \( 0 < h < 1 \) or \( x_s > 1 \) and \( \frac{x_s^2 - 1}{x_s^2} < h < 1 \). Concretely, we have \( d_2(0) > 0 \) for \( \sigma > \sigma_1 \) and \( d_2(0) < 0 \) for \( \sigma < \sigma_1 \). Therefore, the origin is a saddle-node and the time-rescaling \( \tau := d_2(\epsilon)t \) to system (3.7), we obtain

\[ \frac{dw}{d\tau} = \zeta(\epsilon) + u^2 + O(||u||^3), \]

where \( \zeta(\epsilon) := \frac{4d_0(\epsilon)(d_1(\epsilon) - d_2(\epsilon))}{4d_2(\epsilon)} + \frac{4d_0(\epsilon)}{d_2(\epsilon)} \). The computation shows \( \zeta(0) = 0 \) and \( \zeta'(0) = \frac{d_0(0)}{d_2(0)} = \frac{(h-1)(x_s^2 - x_s^2 + 1)\alpha x_s^2}{2(x_s^2 - x_s^2 + 6)(x_s^2 - x_s^2 + 2)} < 0 \) for \( \sigma \neq \sigma_1 \) as well as \( 0 < x_s \leq 1 \) and \( 0 < h < 1 \) or \( x_s > 1 \) and \( \frac{x_s^2 - 1}{x_s^2} < h < 1 \). Hence, the origin is the equilibrium of (3.8) for \( \epsilon = 0 \) and two nonzero equilibria arise as \( \epsilon > 0 \). Therefore, for \( \alpha = \alpha_3, \kappa = \kappa_2 \) and \( \sigma \neq \sigma_1 \) with \( 0 < x_s \leq 1 \) and \( 0 < h < 1 \) or \( x_s > 1 \) and \( \frac{x_s^2 - 1}{x_s^2} < h < 1 \), equilibrium \( E_s \) is a saddle-node and the node \( E_1 \) and the saddle \( E_2 \) arise from the saddle-node bifurcation [16] at \( E_s \) as \( \alpha \) changes from \( \alpha < \alpha_3 \) to \( \alpha > \alpha_3 \). The proof is completed. \( \square \)
3.3 Bogdanov–Takens bifurcation

As shown in Sect. 3.2, the equilibrium \( E_\alpha \) is degenerate with \( D|_{E_\alpha} = 0 \) and \( T|_{E_\alpha} = 0 \) if \( \alpha = \alpha_* := \alpha_3, \sigma = \sigma_* := \sigma_1 \) and \( \kappa = \kappa_2 \) with \( 0 < x_* \leq 1 \) and \( 0 < h < 1 \) or \( x_* > 1 \) and \( x_*^2 \frac{1}{x_*^2} < h < 1 \). The following theorem shows that the codimension of the degenerate equilibrium \( E_\alpha \) is two (i.e., a cusp of codimension two) and system (1.4) may exhibit the Bogdanov–Takens bifurcation of codimension two at \( E_\alpha \) under a small parameter perturbation around \( (\alpha_*, \sigma_*) \).

**Theorem 4** If \( \alpha = \alpha_*, \sigma = \sigma_*, \) and \( \kappa = \kappa_2 \) with \( 0 < x_* \leq 1 \) and \( 0 < h < 1 \) or \( x_* > 1 \) and \( \frac{x_*^2}{x_*^2} < h < 1 \), then the equilibrium \( E_\alpha \) is a degenerate equilibrium of codimension two (i.e., a cusp of codimension two) and system (1.4) undergoes a Bogdanov–Takens bifurcation of codimension two around \( E_\alpha \) as the pair of parameters \( (\alpha, \sigma) \) varies near \( (\alpha_*, \sigma_*) \). Concretely, there is a small neighborhood \( U \) of \( (\alpha_*, \sigma_*) \) in the \( (\alpha, \sigma) \)-parameter space and four curves

\[
S^{N+}(\alpha, \sigma) := \{(\alpha, \sigma) \in U : \alpha = \alpha_*, \sigma = \sigma_* \} + O(|\sigma - \sigma_*|^3), \sigma < \sigma_* \}
\]

\[
S^{N-}(\alpha, \sigma) := \{(\alpha, \sigma) \in U : \alpha = \alpha_*, \sigma = \sigma_* \} + O(|\sigma - \sigma_*|^3), \sigma > \sigma_* \}
\]

\[
\mathcal{H}_{\alpha} := \{(\alpha, \sigma) \in U : \alpha = \alpha_*, \sigma = \sigma_* \} + O(|\sigma - \sigma_*|^3), \sigma > \sigma_* \}
\]

\[
\mathcal{H}_{\sigma} := \{(\alpha, \sigma) \in U : \alpha = \alpha_*, \sigma = \sigma_* \} + O(|\sigma - \sigma_*|^3), \sigma > \sigma_* \}
\]

**Proof** Let \( (\epsilon_1, \epsilon_2) := (\alpha - \alpha_*, \sigma - \sigma_*) \) be a parameter vector near \( (0, 0) \). Translating \( E_\alpha \) of system (1.4) to the origin, using the linear transformations

\[
x = \frac{-h^2x_*^2(2h^2/3 - 1) - x_2^2 - 1}{h(3h^2 - x_2^2 + 1)}u + v, \quad y = u
\]

and time-rescaling

\[
t = \frac{(-h^2x_*^2(2h^2/3 - 1) - x_2^2 - 1)(h^2 - x_2^2 + 1)^2}{2x_*(-h^2x_*^2 + 1)(h^2 - x_2^2 + 1)}
\]

and expanding the system at origin, we obtain

\[
\begin{aligned}
\frac{du}{dt} &= c_{00}(\epsilon_1, \epsilon_2)u + c_{01}(\epsilon_1, \epsilon_2)v + c_{11}(\epsilon_1, \epsilon_2)uv + O(\|u, v\|^3), \\
\frac{dv}{dt} &= d_{00}(\epsilon_1, \epsilon_2)u + d_{01}(\epsilon_1, \epsilon_2)v + d_{11}(\epsilon_1, \epsilon_2)uv + O(\|u, v\|^3), \\
\end{aligned}
\]

where \( c_{ij} \) and \( d_{ij} \) are given in the “Appendix”. Additionally, the following near-identity transformation

\[
u_1 \equiv c_{00}(\epsilon_1, \epsilon_2)u + c_{01}(\epsilon_1, \epsilon_2)v + c_{11}(\epsilon_1, \epsilon_2)uv + O(\|u, v\|^3)
\]

brings system (3.9) to the Kukles from

\[
\begin{aligned}
\frac{du_1}{dt} &= v_1, \\
\frac{dv_1}{dt} &= e_{00}(\epsilon_1, \epsilon_2)u_1 + e_{01}(\epsilon_1, \epsilon_2)v_1 + e_{10}(\epsilon_1, \epsilon_2)u_1 v_1 + e_{11}(\epsilon_1, \epsilon_2)u_1^2 v_1 + O(\|u_1, v_1\|^3),
\end{aligned}
\]

where these coefficients \( e_{ij} \) are given in the “Appendix”. Since the calculation yields

\[e_{11}(0, 0) = \frac{3h^2x_*^2 - 2x_*^2(4x_*^2 - 3)h^2 + (x_*^2 - 1)(x_*^2 - 3)h^2 - 2x_*^2(2x_*^2 - 2)[(-h^2x_*^2 + (2h^2/3 - 1)h - x_2^2 + 2)]}{x_*(-h^2x_*^2 + 1)(h^2 - x_2^2 + 1)^2} > 0\]

under the conditions of Theorem 4, we use the transformation

\[
u_2 \equiv u_1 + \frac{e_{01}(\epsilon_1, \epsilon_2)}{e_{11}(\epsilon_1, \epsilon_2)}v_1, \quad v_2 \equiv v_1
\]

to eliminate the term \( e_{01}(\epsilon_1, \epsilon_2)v_1 \) in system (3.10) and obtain

\[
\begin{aligned}
\frac{du_2}{dt} &= v_2, \\
\frac{dv_2}{dt} &= f_{00}(\epsilon_1, \epsilon_2)u_2 + f_{01}(\epsilon_1, \epsilon_2)v_2 + f_{10}(\epsilon_1, \epsilon_2)u_2 v_2 + f_{11}(\epsilon_1, \epsilon_2)u_2^2 v_2 + O(\|u_2, v_2\|^3),
\end{aligned}
\]

where

\[f_{ij} = \frac{c_{ij}}{e_{11}(\epsilon_1, \epsilon_2)} \quad \text{and} \quad f_{i0} = \frac{d_{i0}}{e_{11}(\epsilon_1, \epsilon_2)} \quad \text{and} \quad f_{0j} = \frac{d_{0j}}{e_{11}(\epsilon_1, \epsilon_2)}
\]

and

\[f_{11} = \frac{c_{11}}{e_{11}(\epsilon_1, \epsilon_2)} \quad \text{and} \quad f_{01} = \frac{d_{01}}{e_{11}(\epsilon_1, \epsilon_2)} \quad \text{and} \quad f_{10} = \frac{d_{10}}{e_{11}(\epsilon_1, \epsilon_2)}
\]

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where
\[
\begin{align*}
f_{00}(e_1, e_2) & := e_{00}(e_1, e_2) - \frac{e_{10}(e_1, e_2)e_{01}(e_1, e_2)}{e_{11}(e_1, e_2)} + \frac{e_{20}(e_1, e_2)e_{01}^2(e_1, e_2)}{e_{11}^2(e_1, e_2)}, \\
f_{10}(e_1, e_2) & := e_{10}(e_1, e_2) - \frac{2e_{20}(e_1, e_2)e_{01}(e_1, e_2)}{e_{11}(e_1, e_2)}.
\end{align*}
\]

Under the near-identity and time-rescaling transformations
\[
u_3 := u_2, \quad v_3 := v_2 - e_{02}(e_1, e_2)u_2v_2,
\]
\[
J_0 := \frac{-x^2_1(h^2x^2_2-x^2_3+6)|h^2x^2_2+(-2x^2_3+1)h+x^2_2-2|^2}{4(h-1)^2(h^2x^2_2-x^2_3+1)(h^2x^2_2-x^2_3+2)(3h^2x^2_2-2x^2_3(4x^2_3-3)h^2+(x^2_2-1)(7x^2_3-5)h-2x^2_2)}
\]

Thus, system (3.11) becomes
\[
\begin{align*}
\frac{d\nu_3}{d\tau} &= v_3, \\
\frac{d\nu_3}{d\tau} &= \mu_1(e_1, e_2) + \mu_2(e_1, e_2)u_3 + A(e_1, e_2)u^2_3 + B(e_1, e_2)u_3v_3,
\end{align*}
\]

where
\[
\begin{align*}
\mu_1(e_1, e_2) & := f_{00}(e_1, e_2), \\
\mu_2(e_1, e_2) & := f_{10}(e_1, e_2) - 2e_{02}(e_1, e_2)f_{00}(e_1, e_2), \\
A(e_1, e_2) & := e_{20}(e_1, e_2) + 2e_{02}(e_1, e_2)e_{02}(e_1, e_2)f_{00}(e_1, e_2) - f_{10}(e_1, e_2), \\
B(e_1, e_2) & := e_{11}(e_1, e_2),
\end{align*}
\]

Since
\[
A(0, 0) = \frac{(h^2x^2_2+(-2x^2_3+1)h+x^2_2-2)^2(h^2x^2_2-x^2_3+6)}{2x_1(h-1)^3(h^2x^2_2-x^2_3+1)^3} < 0
\]

and \(B(0, 0) = e_{11}(0, 0) > 0\) under the conditions of Theorem 4, with the rescaling
\[
u_4 := \frac{B(e_1, e_2)}{A(e_1, e_2)}u_3, \quad v_4 := \frac{B(e_1, e_2)}{A(e_1, e_2)}v_3 \quad \text{and} \quad \tau := \frac{B(e_1, e_2)}{A(e_1, e_2)}\tau,
\]

system (3.12) can be reduced to the following system
\[
\begin{align*}
\frac{d\nu_4}{d\tau} &= v_4, \\
\frac{d\nu_4}{d\tau} &= \beta_1(e_1, e_2) + \beta_2(e_1, e_2)u_4 + u^2_4 - u_4v_4,
\end{align*}
\]

where
\[
\beta_1(e_1, e_2) := \frac{B^4(e_1, e_2)}{A^3(e_1, e_2)}A(e_1, e_2),
\]

Under the conditions of Theorem 4, the parameter transformation (3.14) is a homeomorphism in a small neighborhood of the origin and parameters \(\beta_1(e_1, e_2)\) and \(\beta_2(e_1, e_2)\) are independent. The results in section 8.4 of [24] indicate that system (3.13) undergoes the Bogdanov–Takens bifurcation of codimension two. More specifically, system (3.13) undergoes a saddle-node bifurcation as \((\beta_1(e_1, e_2), \beta_2(e_1, e_2))\) crossing \(\mathcal{SN}^+ \cup \mathcal{SN}^-\), where
\[
\mathcal{SN}^+ = \{ (\beta_1(e_1, e_2), \beta_2(e_1, e_2)) \in U : \beta_1(e_1, e_2) > 0 \},
\]

\[
\mathcal{SN}^- = \{ (\beta_1(e_1, e_2), \beta_2(e_1, e_2)) \in U : \beta_1(e_1, e_2) < 0 \},
\]

a Hopf bifurcation as \((\beta_1(e_1, e_2), \beta_2(e_1, e_2))\) crossing \(\mathcal{H}\), where
\[
\mathcal{H} = \{ (\beta_1(e_1, e_2), \beta_2(e_1, e_2)) \in U : \beta_1(e_1, e_2) = 0, \beta_2(e_1, e_2) < 0 \},
\]

and a homoclinic bifurcation as \((\beta_1(e_1, e_2), \beta_2(e_1, e_2))\) crossing \(\mathcal{H}\), where
\[
\mathcal{H}\mathcal{L} = \{ (\beta_1(e_1, e_2), \beta_2(e_1, e_2)) \in U : \beta_1(e_1, e_2) = -\frac{6}{25}\beta_2^2(e_1, e_2) + O(\beta_2^2(e_1, e_2)^3), \beta_2(e_1, e_2) < 0 \}.
\]

In the following, we express the four bifurcation curves \(\mathcal{SN}, \mathcal{H}, \mathcal{H}\mathcal{L}\) in terms of \(e_1\) and \(e_2\). From (3.14), we solve \(e_1\) and \(e_2\) as follows
\[
e_1 = g_{10}\beta_1 + g_{01}\beta_2 + O(\| (\beta_1, \beta_2) \|^2),
\]

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\[
\epsilon_2 = h_0 \beta_1 + h_{01} \beta_2 + O(\| (\beta_1, \beta_2) \|^2),
\]
where the coefficients \( g_{ij} \) and \( h_{ij} \) are given in the “Appendix”. For the saddle-node bifurcation curves \( SN \), we consider \( \Gamma := \beta_1(\epsilon_1, \epsilon_2) - \frac{\partial \beta_1}{\partial \epsilon_1}(\epsilon_1, \epsilon_2) = 0 \).

Since
\[
\frac{\partial \epsilon_1}{\partial \epsilon_2} = \frac{2\epsilon_1(1-\epsilon_1^2) - 2\epsilon_2(1-\epsilon_2^2) - \epsilon_1(1-\epsilon_1^2) + \epsilon_2(1-\epsilon_2^2)}{(1-\epsilon_1^2)(1-\epsilon_2^2)} \neq 0
\]
at the origin under the conditions of Theorem 4, the implicit function theorem indicates that there exists a unique function \( \epsilon_1(\epsilon_2) \) such that \( \epsilon_1(0) = 0 \) and \( \Gamma(\epsilon_1(\epsilon_2), \epsilon_2) = 0 \), which can be obtained as an expansion
\[
\epsilon_1(\epsilon_2) = f_{11} \epsilon_2 + f_{12} \epsilon_2^2 + O(|\epsilon_2|^3),
\]
where the coefficients \( f_{11} \) and \( f_{12} \) are given in the “Appendix”. Furthermore, we get \( \epsilon_2 = h_0 \beta_2 + O(|\beta_2|^2) \) restricted on the curve \( \Gamma = 0 \) and the coefficient of \( \beta_2 \) is negative, which implies that \( \epsilon_2 > 0 \) if \( \beta_2 < 0 \). Therefore, we obtain the bifurcation curve \( \mathcal{H}_c \) as
\[
\mathcal{H}_c = \{ (\epsilon_1, \epsilon_2) \in U : \epsilon_1 = f_{31} \epsilon_2 + f_{32} \epsilon_2^2 + O(|\epsilon_2|^3), \epsilon_2 > 0 \}.
\]

With the linear transformation \( \epsilon_1 = \alpha - \alpha_1 \) and \( \epsilon_2 = \sigma - \sigma_1 \), we can rewrite the above four bifurcation curves as in Theorem 4. The proof is completed. \( \square \)

### 3.4 Hopf bifurcation

Theorem 1 shows that \( E_1 \) is a center or weak focus as \( T(x_1) = 0 \) under the conditions \( 0 < h < 1 \) and \((\alpha, \kappa, \sigma) \in \mathcal{P}_7\). In this subsection, we are devoted to the center-focus determination and obtain that the multiplicity of weak focus \( E_1 \) is at most three and the Hopf bifurcation occurs.

In order to have a Hopf bifurcation at \( E_1 \), we look for some parameter conditions such that \( E_1 \) is a nonhyperbolic equilibrium satisfying \( F(x_1) = 0, T(x_1) = 0 \) and \( D|_{E_1} > 0 \), where
\[
D|_{E_1} = \frac{(h-1)\kappa^2 + 3\kappa - 2\kappa_1((h-1)x_1^2 + 2)}{(k-x_1)(1-kx_1^2)}.
\]

Thus, we can express \( \alpha \) and \( \sigma \) by \( \kappa, h \) and \( x_1 \) as follows
\[
\alpha = \frac{(2h-1)k-2hx_1}{x_1((h-1)^2x_1^2-k)}, \quad \sigma = \frac{k(h-1)((h-1)x_1^2 + 1)}{(2h-1)(k-2hx_1)} \tag{3.15}
\]
and \( \kappa, h \) and \( x_1 \) satisfy \((\kappa, h, x_1) \in \mathcal{P}\) with
\[
\mathcal{P} := \{ (\kappa, h, x_1) \in \mathbb{R}_+^3 : \kappa > \kappa_2, h \leq \frac{1}{2},
1 - \frac{1}{x_1} < 2 \sqrt{2} \text{ or } \kappa > h > \frac{2hx_1}{2h-1} \}< \frac{2hx_1}{2h-1},
\]
\[
\frac{1}{2} < h < 1, x_1 \leq \sqrt{2} \text{ or } \kappa < \frac{2hx_1}{2h-1} < h < 1, x_1 > \sqrt{2}
\tag{3.16}
\]
where \( \kappa_2 \) is given in Sect. 3.2. Then under the critical conditions (3.15) and (3.16), the linearized system of (1.4) at equilibrium \( E_1 \) has a pair of purely imaginary eigenvalues \( \pm i\omega \) with \( \omega := \sqrt{D|_{E_1}} \). Hence we consider the Hopf bifurcation at \( E_1 \) under conditions (3.15) and (3.16) and obtain the following result.
Theorem 5 Under conditions (3.15) and \((\kappa, h, x_1) \in \mathcal{P}\), equilibrium \(E_1\) is a weak focus of multiplicity at most three. More concretely, \(E_1\) is a weak focus of multiplicity one, two and three as \((\kappa, h, x_1) \in \mathcal{P}_1, (\kappa, h, x_1) \in \mathcal{P}_2\) and \((\kappa, h, x_1) \in \mathcal{P}_3\) respectively, where \( \mathcal{P}_1 := \mathcal{P} \backslash (\mathcal{P}_2 \cup \mathcal{P}_3), \mathcal{P}_2 := \{(\kappa, h, x_1) \in \mathcal{P} : f_1 = 0, f_2 \neq 0\}, \mathcal{P}_3 := \{(\kappa, h, x_1) \in \mathcal{P} : f_1 = f_2 = 0\}\) with \(f_1\) and \(f_2\) given in the “Appendix”. Thus, at most three limit cycles arise from the Hopf bifurcation at equilibrium \(E_1\).

Proof Translating \(E_1\) to \((0, 0)\) and using the linear transformation
\[
x = \frac{(2h-1)\kappa - 2hx_1}{2\kappa(x-x_1)((h-1)x_1^2+1)(h-1)} u - \frac{(2h-1)\kappa - 2hx_1}{2\kappa(x-x_1)(h-1)} v\quad \text{and} \quad y = \frac{1}{\kappa} v
\]
and the time-rescaling \(\tau := \omega t\) with \(\omega\) given above to normalize the linear part of system (1.4), we can change system (1.4) into the following system
\[
\begin{align*}
\frac{du}{dt} &= -v + a_{20}u^2 + a_{02}v^2 + a_{11}uv + a_{30}u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + a_{40}u^4 + a_{31}u^3v + a_{22}u^2v^2 + a_{13}uv^3 + a_{04}v^4 + a_{41}u^4v + a_{32}u^3v^2 + a_{23}u^2v^3 + a_{14}uv^4 + a_{05}v^5, \\
\frac{dv}{dt} &= u + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + b_{22}u^2v^2 + b_{13}uv^3 + b_{04}v^4,
\end{align*}
\]
where these coefficients are given in the “Appendix”. In the following, we are devoted to identifying a weak focus from a center and determining the multiplicity of the weak focus by using the successive function method [38]. We obtain the first three focal values \(l_i(i = 1, 2, 3)\), where
\[
\begin{align*}
l_1 &:= \frac{\kappa f_1}{16(h-1)^2(\kappa-x_1)^2\omega^2((hx_1^2-x_1^3+3)\kappa-2x_1(hx_1^2-x_1^3+2))x_1}, \\
l_2 &:= \frac{\kappa^2 f_2}{3072(h-1)^4(\kappa-x_1)^4\omega^6((hx_1^2-x_1^3+3)\kappa-2x_1(hx_1^2-x_1^3+2))(2h\kappa-2hx_1-\kappa)^2x_1^2}, \\
l_3 &:= \frac{\kappa^3 f_3}{7077880(h-1)^6(\kappa-x_1)^6\omega^{12}((hx_1^2-x_1^3+3)\kappa-2x_1(hx_1^2-x_1^3+2))(2h\kappa-2hx_1-\kappa)^4x_1^3},
\end{align*}
\]
where \(f_1\) and \(f_2\) are given in the “Appendix” and the polynomial \(f_3\) of 1755 terms is too long to display. Because the denominators of the three focal values are positive, the real zeros and signs of \(l_i(i = 1, 2, 3)\) are determined by the factors \(f_i(i = 1, 2, 3)\) respectively. To determine the multiplicity of weak focus \(E_1\), we need to compute the two algebraic varieties \(V(f_1, f_2)\) and \(V(f_1, f_2, f_3)\) in \(\mathcal{P}\), which are the sets of common zeros of two polynomials \(f_i(i = 1, 2)\) and three polynomials \(f_i(i = 1, 2, 3)\) respectively [9, 33]. The tedious symbolic computation of \(V(f_1, f_2)\) and \(V(f_1, f_2, f_3)\) is displayed in the “Appendix”. It is shown that \(V(f_1, f_2, f_3) = \emptyset\) but \(V(f_1, f_2) \neq \emptyset\) in \(\mathcal{P}\), which implies that the multiplicity of weak focus \(E_1\) is at most three and at most three limit cycles arise from the Hopf bifurcation at \(E_1\). The proof is completed. \(\square\)

Theorem 5 shows that the degenerate Hopf bifurcation at the weak focus of multiplicity three produces at most three hyperbolic limit cycles. Conversely, these limit cycles may coalesce into a multiple limit cycle through the multiple limit cycle bifurcation, or destroy through either the Hopf bifurcation or the homoclinic bifurcation or even the multiple limit cycle bifurcation as perturbing the parameters.

Remark 2 System (1.4) has more complicated bifurcation phenomena than the system without cooperative hunting among predators (i.e., \(\alpha = 0\) in system (1.4), because the system without cooperative hunting only undergoes the transcritical bifurcation at the saddle-node \(E_k\) as \(\kappa\) varies across the bifurcation value \(\kappa = \kappa_1\) and the Hopf bifurcation around the weak focus of multiplicity one \((\kappa_1, \kappa_1(1 - \kappa_1))\) if \(\kappa = \frac{2h\kappa_1}{2\kappa_1 - 1}\) and \(\frac{1}{2} < h < 1\) and at most one limit cycle bifurcates from the positive equilibrium [4, 8].

4 Simulations and discussions

The cooperative hunting among predators is a ubiquitous behavior in ecological system and plays an important role in determining the dynamics of predator–prey system. In this paper, we consider the dynamics of the predator–prey system with cooperative hunting among predators and Holling III functional response. We first give the parametric conditions for the existence and the qualitative properties of the equilibria (Theorem 1), and then discuss the various bifurcations of the system around the nonhyperbolic equilibria such as the transcritical and the pitchfork bifurcations at the degenerate boundary equilibrium (Theorem 2), the saddle-node
and the Bogdanov–Takens bifurcations at the degenerate positive equilibrium (Theorems 3 and 4) and the Hopf bifurcation around the weak focus (Theorem 5). Remarks 1 and 2 indicate that system (1.4) has richer and more complicated dynamics than the system without cooperative hunting, because the later at most has three equilibria and only undergoes the transcritical and the Bogdanov–Takens bifurcations at the degenerate equilibrium (cusp) $E_*$ when the intensity of cooperative hunting among predators is so weak that it cannot sustain the survival of predators, which thereby causes the survival of prey and the extinction of predators. Theorem 4 indicates that the neighborhood $U$ of point $(\sigma_*, \alpha_*)$ can be divided into four regions, i.e., $U = SN^+ \cup SN^- \cup H \cup H^L \cup I_1 \cup I_2 \cup I_3 \cup I_4$ with

\[
\begin{align*}
I_1 &:= \{(\sigma, \alpha) \in U : \alpha < \alpha_* + f_{11}(\sigma - \sigma_*) + f_{12}(\sigma - \sigma_*^2) + O(|\sigma - \sigma_*|^3)\}, \\
I_2 &:= \{(\sigma, \alpha) \in U : \alpha_* + f_{11}(\sigma - \sigma_*) + f_{12}(\sigma - \sigma_*^2) + O(|\sigma - \sigma_*|^3) < \alpha < \alpha_* + f_{21}(\sigma - \sigma_*) + f_{22}(\sigma - \sigma_*^2) + O(|\sigma - \sigma_*|^3), \sigma > \sigma_*\}, \\
I_3 &:= \{(\sigma, \alpha) \in U : \alpha_* + f_{21}(\sigma - \sigma_*) + f_{22}(\sigma - \sigma_*^2) + O(|\sigma - \sigma_*|^3), \sigma < \sigma_*\}, \\
I_4 &:= \{(\sigma, \alpha) \in U : \alpha > \alpha_* + f_{31}(\sigma - \sigma_*) + f_{32}(\sigma - \sigma_*^2) + O(|\sigma - \sigma_*|^3), \sigma > \sigma_*\}.
\end{align*}
\]

The corresponding dynamical properties of system (1.4) near $E_*$ for the parameters $\sigma$ and $\alpha$ in the neighborhood $U$ of point $(\sigma_*, \alpha_*)$ are displayed in Table 1.

In the section, we provide some numerical simulations to demonstrate our theoretical results and give some biological interpretations for the case $h < 1$. If taking $h = 0.5$ and $\kappa = 1.2$, then $E_*$ is a saddle-node when $\alpha = 6$ and $\sigma = 1$ (see Fig. 1a), $E_*$ is a degenerate equilibrium of codimension two (i.e., a cusp) when $\sigma = \sigma_* = 0.3$ and $\alpha = \alpha_* = 20$ (see Fig. 1b). Moreover, the predator extinction equilibrium $E_\kappa$ is a stable node in both cases. Thus, the numerical simulations indicate that if both the environmental capacity of prey and the intensity of cooperative hunting among predators are relatively small (i.e., $\kappa < \kappa_1$ and $\alpha \leq \alpha_2$), then the intensity of cooperative hunting is so weak that it cannot sustain the survival of predators, which thereby causes the survival of prey and the extinction of predators.
ing phase portraits are shown in Fig. 2 for \( h = 0.5 \) and \( \kappa = 1.2 \). The bifurcation diagram is displayed in Fig. 2a. The following explains the various phase portraits of Bogdanov–Takens bifurcation. If \( (\sigma, \alpha) = (0.35, 17.1) \in \mathcal{I}_1 \), then system (1.4) has no any positive equilibrium (see Fig. 2b), which also shows that the prey can survive but the predators go extinct because the intensity of cooperative hunting is too weak to sustain the survival of predators. If \( (\sigma, \alpha) = (0.35, 17.5) \in \mathcal{I}_2 \), then system (1.4) has two positive equilibria arising from the saddle-node bifurcation, and one is the stable focus \( E_1 \) and the other one is the saddle \( E_2 \) (see Fig. 2c), which means that the predators and their prey can coexist at stable coexistence equilibrium or predator extinction equilibrium for different initial values. If \( (\sigma, \alpha) = (0.35, 18) \in \mathcal{I}_3 \), then system (1.4) has a stable limit cycle around the unstable focus \( E_1 \) induced by the Hopf bifurcation and the saddle \( E_2 \) (see Fig. 2d). If \( (\sigma, \alpha) = (0.305, 19.6855) \in \mathcal{H} \mathcal{L} \), then system (1.4) has the saddle \( E_2 \) and a homoclinic orbit around the unstable focus \( E_1 \) (see Fig. 2e). The disappearance of the limit cycle and the appearance of the homoclinic orbit are due to the homoclinic bifurcation. If \( (\sigma, \alpha) = (0.35, 18.6) \in \mathcal{I}_4 \), then system (1.4) still has two positive equilibria, and more specifically one is the unstable focus \( E_1 \) and the other one is the saddle \( E_2 \) (see Fig. 2f). Figure 2d, f show that if the intensity of cooperative hunting is too large, then the predators also go extinct, because the system undergoes Hopf bifurcation leading that the stable coexistence equilibrium loses stability and the limit cycle emerges, or the system undergoes homoclinic bifurcation leading to the disappearance of the limit cycle. The cooperative hunting therefore can turn detrimental to predators if the prey density is relatively small and the intensity of cooperative hunting is too large because of the increased predation pressure of predators. Because the Hopf bifurcation at the weak focus \( E_1 \) of multiplicity three depends on more bifurcation parameters and even is very sensitive to the perturbation of parameters, we display one limit cycle and two limit cycles around \( E_1 \) arising from the Hopf bifurcation through numerical simulations in Fig. 3.

(i) If \( h = 0.5, \kappa = 0.8, \alpha = 54.902 \) and \( \sigma = 0.68 \), then system (1.4) has four equilibria, i.e., the saddles \( E_0 \) and \( E_2 \), the stable node \( E_\kappa \) and the unstable focus \( E_1 \) with the trace \( T|_{E_1} = 0.02 \) and the first order Lyapunov quantity \( l_1 = -21.2827 \), which indicates that system (1.4) has a stable limit cycle around the unstable focus \( E_1 \) (see Fig. 3a);

(ii) If \( h = 0.45, \kappa = 133.7629, \alpha = 0.3555 \) and \( \sigma = 2.319 \), then system (1.4) has two limit cycles around the unstable focus \( E_1(1, 2.3016) \) with the trace \( T|_{E_1} = 5.1927 \times 10^{-25} \) and the first two Lyapunov quantities \( l_1 = -2.6823 \times 10^{-19} \) and \( l_2 = 1.2546 \times 10^{-15} \). Two orbits spiraling inward and outward respectively and the unstable focus \( E_1 \) form two annular regions. From Poincaré–Bendixson Theorem [38], there are two closed orbits in the two annular regions. We plot the two orbits starting from \( P_1(1, 2) \) and \( P_2(1, 2.2) \) respectively in Fig. 3b. However, it is very blurry to distinguish whether the two orbits starting from \( P_1 \) and \( P_2 \) spiral outward or inward as \( t \to +\infty \) from Fig. 3b. Therefore, by zooming in the orbits near \( P_1 \) and \( P_2 \) in Fig. 3c and d respectively, we obtain that the orbit from \( P_1 \) spirals outward while the orbit from \( P_2 \) spirals inward as \( t \to +\infty \). There-

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**Table 1** Dynamical behaviors near \( E_* \)

| \((\sigma, \alpha)\) ∈ | Positive equilibria and properties | Closed or homoclinic orbits |
|------------------------|---------------------------------|-----------------------------|
| \( \mathcal{I}_1 \)    | No equilibria                   | No                          |
| \( \mathcal{S}N^- \)   | \( E_* \) (saddle node)        | No                          |
| \( \mathcal{I}_2 \)    | \( E_1 \) (stable focus or node) \( E_2 \) (saddle) | No                          |
| \( \mathcal{H} \)      | \( E_1 \) (stable weak focus) \( E_2 \) (saddle) | No                          |
| \( \mathcal{I}_3 \)    | \( E_1 \) (unstable focus) \( E_2 \) (saddle) | A stable limit cycle        |
| \( \mathcal{H} \mathcal{L} \) | \( E_1 \) (unstable focus) \( E_2 \) (saddle) | A homoclinic orbit          |
| \( \mathcal{I}_4 \)    | \( E_1 \) (unstable focus or node) \( E_2 \) (saddle) | No                          |
| \( \mathcal{S}N^+ \)   | \( E_* \) (saddle node)        | No                          |
| \( (\sigma_*, \alpha_*) \) | \( E_* \) (degenerate equilibrium, i.e., cusp) | No                          |
Fig. 2 The bifurcation diagram and corresponding phase portraits of system (1.4) when $h = 0.5$ and $\kappa = 1.2$.  

(a) Bifurcation diagram with $(\sigma_*, \alpha_*) = (0.3, 20)$.  

(b) No positive equilibrium when $(\sigma, \alpha) = (0.35, 17.1) \in \mathcal{I}_1$.  

(c) Stable focus $E_1$ and saddle $E_2$ when $(\sigma, \alpha) = (0.35, 17.5) \in \mathcal{I}_2$.  

(d) A stable limit cycle around unstable focus $E_1$ and saddle $E_2$ when $(\sigma, \alpha) = (0.35, 18) \in \mathcal{I}_3$.  

(e) A homoclinic orbit around unstable focus $E_1$ and saddle $E_2$ when $(\sigma, \alpha) = (0.305, 19.6855) \in \mathcal{HL}$.  

(f) Unstable focus $E_1$ and saddle $E_2$ when $(\sigma, \alpha) = (0.35, 18.6) \in \mathcal{I}_4$.  


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Bifurcations of a predator–prey system

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\[ SN^+ \]

\[ SN^- \]

\[ \mathcal{I}_1 \]

\[ \mathcal{I}_2 \]

\[ \mathcal{I}_3 \]

\[ \mathcal{HL} \]

\[ (\sigma_*, \alpha_*) \]

\[ (\sigma, \alpha) \]

\[ (\sigma, \alpha) = (0.3, 20) \]

\[ (\sigma, \alpha) = (0.35, 17.1) \in \mathcal{I}_1 \]

\[ (\sigma, \alpha) = (0.35, 17.5) \in \mathcal{I}_2 \]

\[ (\sigma, \alpha) = (0.35, 18) \in \mathcal{I}_3 \]

\[ (\sigma, \alpha) = (0.305, 19.6855) \in \mathcal{HL} \]

\[ (\sigma, \alpha) = (0.35, 18.6) \in \mathcal{I}_4 \]
Therefore, there is an unstable limit cycle lying in the annular region bounded by the two orbits starting from $P_1$ and $P_2$, and a stable limit cycle lying in the annular region bounded by the orbit starting from $P_2$ and the unstable focus $E_1$. These numerical simulations indicate that if only the environmental capacity of prey is relatively small but the intensity of cooperative hunting is relatively large (i.e., $\kappa < \kappa_1$ and $\alpha > \alpha_2$), then the cooperative hunting can promote the probability of survival of predators, which implies that both of predators and prey can coexist at the positive equilibrium or limit cycles for different initial values. Therefore, the cooperative hunting can be beneficial to the predators and the system exhibits two types of bistability phenomenon, i.e., either a stable coexistence equilibrium together with a stable predator extinction equilibrium or a stable limit cycle together with a stable predator extinction equilibrium. This phenomenon actually is the Allee effect in the predators [31]. Our theoretical results indicate that the cooperative hunting among predators can lead to the changes of existence and stabilities for invariant sets, such as the equilibria, the limit cycles and the homoclinic orbit, which are induced by various bifurcations. The cooperative hunting among predators therefore also has a destabilizing effect on the dynamics in the system with Holling III functional response just as in the systems with Lotka–Volterra or Holling II or Holling IV functional response.
responses respectively [2,36,37]. By the bifurcation analysis, we provide some thresholds to control the dynamical behaviours of the predator–prey system, which are the critical values to promote the coexistence of predators and their prey. Hence, the analysis results reveal that appropriate intensity of cooperative hunting among predators is beneficial for the persistence of predators and the diversity of ecosystem.

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**Data availability** All data generated or analysed during this study are included in this published article.

**Declarations**

**Conflict of interest** The authors declare that they have no competing interests.

**Ethics approval and consent to participate** The research in this paper does not involve any ethical research.

**Consent for publication** All authors agree to publish this paper.

**Appendix**

This section can be found at Appendix of preprint for the paper on arXiv.org, which is available at arXiv:2111.13632.

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