On The Heterotic Dipole At Strong Coupling

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We analyse the dipole solution of heterotic string theory in four dimensions. It has the structure of monopole and anti-monopole connected by flux line (string). Due to growing coupling near the poles, the length of the string diverges. However, exploiting the self-duality of heterotic string theory in four dimension, we argue that this string is correctly described in terms of dual variables.
1. Introduction:

There are various solitonic objects in superstring theory. Among those, most useful are the ones that are BPS saturated. BPS saturated solitons preserve certain fraction of supersymmetry. As a result, because of non-renormalisation theorems, certain quantum numbers associated to these objects do not receive corrections as we increase the string coupling. It is because of this key property, BPS saturated solutions in string theory play an important role in testing various duality conjectures in string theory [1]. However, in order to shed more light on dualities, one has to come out of these BPS objects. One kind of such non-BPS states have been in focus recently [2]. They are stable (even though non-renormalisation theorems are not applicable for these states) because they are the lowest states of certain quantum numbers. There is nothing to which they can decay.

On the other hand, in various works, the dualities of string theory have been exploited in order to understand instabilities associated with the non-BPS states. In [3] for example, the focus was on a D6 and anti-D6 brane pair of IIA string theory in ten dimension. This is an unstable object in IIA theory. When the separation between them is of the order of string length, there is tachyonic instability. This tachyonic instability, however, was argued to be absent in the strongly coupled version of IIA theory-the M theory.

In this note, we will be interested in certain non-BPS solitons in four dimensional heterotic string theory. They are dipole like solutions somewhat similar in nature to that of Gross-Perry dipole solutions [4] reduced to four dimension [5,6]. The heterotic dipole that we will be analysing has a string like structure (may be interpreted as a flux line) joining the two poles. However, measured in the string-frame, the length of this string diverges. Also we notice that, at the same time, the string coupling blows up near the poles as well. In section 3, we then analyse the solution from the dual frame. Heterotic string theory, at the classical level, is conjectured to be self-dual in four dimension under $SL(2, R)$ transformation. Thus the $SL(2, R)$ transformed dipole will continue to be a solution the heterotic string in four dimension. Exploiting this property, we analyse the nature of the dual-dipole. We find, among other things, that the flux line connecting the poles is finite in length.

2. Heterotic Dipole:

In this section we will discuss the dipole solution in heterotic string theory. The heterotic string in four dimension is described by the action

$$S = \int d^4 x \sqrt{-G} e^{-\Phi} [R_G + G^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} - F_{\mu \nu} F^{\mu \nu}]. \quad (1)$$
Here \( G_{\mu\nu} \) is the metric in the string frame. \( \Phi \) is the dilaton, \( H_{\mu\nu\rho} \) is the anti-symmetric tensor and \( F_{\mu\nu} \) is the electromagnetic field strength.

This action has a dipole solution as found in [5]. The field configurations for the dipole is given by:

\[
dS^2 = \left( \Delta + a^2 \sin^2 \theta \right)\left[ -dt^2 + \frac{\Delta \sin^2 \theta (r^2 - a^2 \cos^2 \theta)^2}{(\Delta + a^2 \sin^2 \theta)^2} d\phi^2 \right.
\]
\[
+ \frac{(r^2 - a^2 \cos^2 \theta)^2}{\Delta + (M^2 + a^2) \sin^2 \theta} \left( \frac{dr^2}{\Delta} + d\theta^2 \right),
\]
\[
\Phi = \log \frac{r^2 - a^2 \cos^2 \theta}{\Delta + a^2 \sin^2 \theta},
\]
\[
A_\phi = \frac{4aM \sin^2 \theta}{\Delta + a^2 \sin^2 \theta},
\]
\[
\Delta = r^2 - 2Mr - a^2.
\]

Here the solution is written in the Einstein-frame. The Einstein-frame metric \( (g_{\mu\nu}) \) is related to the string-frame metric as \( g_{\mu\nu} = e^{-\Phi} G_{\mu\nu} \).

First of all, we notice that the metric is asymptotically flat. The dilaton \( \Phi \) also goes to a constant as \( r \) becomes large. Furthermore, the asymptotic behaviour of the gauge field reveals the dipole nature of the solutions. The dipole moment can be read off and it is \( 4aM \) for the above field configuration.

To have further understanding of the metric, we would now look at the region \( \Delta = 0, \theta = 0 \) and \( \Delta = 0, \theta = \pi \) as they are the possible locations of singularities. This can be done by blowing up the coordinates near the corresponding points. For example, near \( \Delta = 0, \theta = 0 \), we define coordinates [3]

\[
(r_0 - 2M) \sin^2 \theta = \tilde{\rho}(1 - \cos \tilde{\theta}), \quad 2(r - r_0) = \tilde{\rho}(1 + \cos \tilde{\theta}).
\]

Here, \( r_0 = M + \sqrt{M^2 + a^2} \) is the root of \( \Delta = 0 \). Near the region \( \theta = 0 \) (keeping \( a \sin^2 \theta \) and \( r - r_0 \) finite), the field configurations reduce to:

\[
dS^2 = -(1 + \frac{M}{\tilde{\rho}})^{-1} dt^2 + (1 + \frac{M}{\tilde{\rho}})(d\tilde{\rho}^2 + \tilde{\rho}^2 d\tilde{\theta}^2 + \tilde{\rho}^2 \sin^2 \tilde{\theta} d\phi^2),
\]
\[
\Phi = \log(1 + \frac{M}{\tilde{\rho}}), \quad A_\phi = M(1 - \cos \tilde{\theta}).
\]

The Ricci scalar can easily be calculated for the metric and is given by

\[
R = \frac{M^2}{2\tilde{\rho}(\tilde{\rho} + M)^3}.
\]
Thus, near $\theta = 0$, the metric behaves as anti-monopole in heterotic string theory. On the other hand, we can blow up the coordinates near $r = r_0, \theta = \pi$. A similar computation as before reveals the structure of a monopole near the region $r = r_0, \theta = \pi$.

We will now analyse the heterotic dipole in string frame. Field configurations in string frame can be read off from (2) after making the necessary metric scaling.

$$dS^2 = -dt^2 + \frac{\Delta \sin^2 \theta (r^2 - a^2 \cos^2 \theta)^2}{(\Delta + a^2 \sin^2 \theta)^2} d\phi^2$$
$$+ \frac{(r^2 - a^2 \cos^2 \theta)^2}{\Delta + (M^2 + a^2) \sin^2 \theta} \left( \frac{dr^2}{\Delta} + d\theta^2 \right),$$

$$e^\Phi = \frac{r^2 - a^2 \cos^2 \theta}{\Delta + a^2 \sin^2 \theta},$$
$$A_\phi = \frac{4aMr \sin^2 \theta}{\Delta + a^2 \sin^2 \theta}.$$

From above we see that the non-zero electro-magnetic field strength components are

$$F_{r\phi} = -\frac{4aM(r^2 + a^2 \cos^2 \theta) \sin^2 \theta}{(\Delta + a^2 \sin^2 \theta)^2}, \quad F_{\theta\phi} = \frac{4aMr \Delta \sin 2\theta}{(\Delta + a^2 \sin^2 \theta)^2}.$$

The metric can again be analysed near $\theta = 0, \pi$ exactly as before in terms of coordinates $(t, \tilde{\rho}, \tilde{\theta}, \phi)$. Near $\theta = 0$ is takes the form:

$$dS^2 = -dt^2 + (1 + \frac{M}{\tilde{\rho}})^2 (d\tilde{\rho}^2 + \tilde{\rho}^2 d\tilde{\theta}^2 + \tilde{\rho}^2 \sin^2 \tilde{\theta} d\phi^2),$$

$$\Phi = \log(1 + \frac{M}{\tilde{\rho}}), \quad A_\phi = M(1 - \cos \tilde{\theta}).$$

The curvature scalars related to the metric near $\theta = 0$, for example, are

$$R = \frac{2M^2}{(\tilde{\rho} + M)^4}, \quad R_{\mu\nu}R^{\mu\nu} = \frac{2M^2(3\tilde{\rho}^2 + 2\tilde{\rho}M + M^2)}{(\tilde{\rho} + M)^8}.$$

Thus we notice that near $\tilde{\rho} = 0$ the curvature components are finite. The heterotic string coupling $g$ in our case is given by $e^\Phi$. From (8) we see that for large $r$, the coupling $g$ goes to constant which has been normalised to 1 for our solution. However, as we see from (8), when $\Delta$ vanishes, $g$ diverges at $\theta = 0, \pi$. Thus the solutions given above do not make sense. This is because, one expects, for large $g$, the action (1) itself would receive non-negligible corrections. We will discuss the consequences later. We also notice that $\int_0^{2\pi} \sqrt{G_{\phi\phi}} d\phi$ goes
to zero as \( r \to r_0 \). This means that at \( r = r_0 \), there is actually a string joining \( \theta = 0 \) and \( \theta = \pi \). The metric of the string that follows from (7) is

\[
ds^2 = -dt^2 + \frac{(r_0^2 - a^2 \cos^2 \theta)^2}{(M^2 + a^2) \sin^2 \theta} \, d\theta^2.
\]

One can thus calculate the length of the string as

\[
l = \int_0^\pi \sqrt{G_{\theta\theta}} \, d\theta = \frac{1}{\sqrt{M^2 + a^2}} \int_0^\pi \frac{r_0^2 - a^2 \cos^2 \theta}{\sin \theta} \, d\theta.
\]

This is clearly infinity. However, we have noticed earlier that the string coupling is divergent at both the ends of the string.

One can calculate the flux associated to the string by integrating \( A_\phi \) around the coordinate \( \phi \) which is

\[
\kappa = \int_0^{2\pi} A_\phi \, d\phi = \frac{8\pi M r_0}{a}.
\]

### 3. The \( SL(2, R) \) Transformed Solutions:

In the last section, we saw that the effective string connecting the monopole, anti-monopole system diverged as a consequence of string coupling becoming strong near the poles. What we can do however is to take the above configuration and slowly decrease the string coupling \( g \) by \( SL(2, R) \) transformation. Since, heterotic string theory is self-dual in four-dimension, \( SL(2, R) \) transformed string-frame metric and other fields would continue to be solution of the same theory. Furthermore, since a particular choice of \( SL(2, R) \) metric inverts the string coupling, the new coupling would be finite and small every where. We thus first discuss a special class of \( SL(2, R) \) transformed solution labeled by a parameter \( 0 \leq \delta \leq \infty \) (see [7] for the constructional detail ). For \( \delta = \infty \), we get the configuration where new string coupling is the inverse of the earlier. In what follows, we first construct the \( SL(2, R) \) transformed solutions and then discuss their behaviour in the light of section 2.

To do this, we have to remember that the heterotic string has an anti-symmetric three rank tensor field \( H_{\mu\nu\sigma} \) as shown explicitly in (1). For our earlier discussion, we set that \( H_{\mu\nu\sigma} \) to zero and hence it does not appear explicitly in the solutions (2)-(3). However, a generic \( SL(2, R) \) transformation will indeed excite this field.
Given the solutions in (7)-(8), we can easily generate a subclass of $SL(2, R)$ dual solutions. In the string frame, various fields of the new solutions are given by:

$$dS^2 = \frac{(r^2 - a^2\cos^2\theta)^2 + \delta^2(\Delta + a^2\sin^2\theta)^2}{(1 + \delta^2)(\Delta + a^2\sin^2\theta)(r^2 - a^2\cos^2\theta)}[-dt^2 + \frac{\Delta\sin^2\theta(r^2 - a^2\cos^2\theta)^2}{(\Delta + a^2\sin^2\theta)^2}d\phi^2$$

$$+ \frac{(r^2 - a^2\cos^2\theta)^2}{\Delta + (M^2 + a^2)\sin^2\theta}(\frac{dr^2}{\Delta} + d\theta^2)],$$

The dilaton and axion fields are respectively

$$e^\Phi' = \frac{(r^2 - a^2\cos^2\theta)^2 + \delta^2(\Delta + a^2\sin^2\theta)^2}{(1 + \delta^2)(\Delta + a^2\sin^2\theta)(r^2 - a^2\cos^2\theta)},$$

$$\Psi' = \frac{\delta[(r^2 - a^2\cos^2\theta)^2 - (\Delta + a^2\sin^2\theta)^2]}{(r^2 - a^2\cos^2\theta)^2 + \delta^2(\Delta + a^2\sin^2\theta)^2}.$$  

Furthermore, the electro-magnetic field components can be calculated from the formula

$$F'_{\mu\nu} = \frac{1}{\sqrt{1 + \delta^2}}(F_{\mu\nu} + \delta \tilde{F}_{\mu\nu}).$$

Here $\tilde{F}_{\mu\nu}$ is defined by

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \sqrt{-g} g_{\mu\rho} g_{\nu\sigma} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.$$  

As it is straightforward to compute various components of $F'_{\mu\nu}$ using the above formula, we do not display them here. However, we notice that for generic values of the $SL(2, R)$ parameter $\delta$, $F'_{\mu\nu}$ will have electric as well as magnetic components. In (16), we have defined $\Psi$ through

$$\sqrt{-g}e^{2\Phi} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \Psi = -g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} H_{\alpha\beta\gamma}.  \tag{19}$$

In terms of the coordinates $(t, \tilde{\rho}, \tilde{\theta}, \phi)$, (15) takes the form

$$dS^2 = -\frac{(\tilde{\rho} + M)^2 + \delta^2\tilde{\rho}^2}{(1 + \delta^2)(\tilde{\rho} + M)^2}dt^2 + \frac{(\tilde{\rho} + M)^2 + \delta^2\tilde{\rho}^2}{(1 + \delta^2)\tilde{\rho}^2}(d\tilde{\rho}^2 + \tilde{\rho}^2d\tilde{\theta}^2 + \tilde{\rho}^2\sin^2\tilde{\theta}d\phi^2).  \tag{20}$$

It is easy to check that all the curvature components associated to this metric are finite. We also notice that the new string coupling $g' = e^{\Phi'}$ has many nice features. Firstly, for any $\delta$, $g$ goes to 1 for large $r$. Secondly, for finite $\delta$, at $\theta = 0$ and $\pi$, $g'$ is finite. Thirdly, for $\delta = \infty$, $g' = g^{-1}$. Hence, the new coupling becomes very weak at $\theta = 0, \pi$.

At $r = r_0$, as before, the proper length along $\phi$ direction goes to zero. Thus there is still a string like structure ending at $\theta = 0$ and $\pi$. This string is now described by the metric

$$ds^2 = \frac{(r_0^2 - a^2\cos^2\theta)^2 + \delta^2a^4\sin^4\theta}{(1 + \delta^2)(r_0^2 - a^2\cos^2\theta)^2}[-dt^2 + \frac{(r_0^2 - a^2\cos^2\theta)^2}{(M^2 + a^2)\sin^2\theta}d\theta^2].  \tag{21}$$
One can as before calculate the length of the string
\[ l|_{\delta \to \infty} = \frac{2a^2}{\sqrt{M^2 + a^2}}. \tag{22} \]

In the limit \( \frac{M}{a} \to 0 \), we have from (22) \( l = 2a \).

To conclude, we summarize what we have done in this note. In section 2, we have analysed the heterotic dipole in four dimensions as it would be probed by a fundamental string. We noticed that there was a string (flux line) connecting the two poles of the soliton. However, due to the string coupling becoming strong near the poles, the length of the string turns out to be infinite. In section 3, we then analysed the dual version of the solution. This is constructed by exploiting the \( SL(2, R) \) self-duality property of the heterotic string in four dimensions. In the dual description, the string coupling is well-behaved all along the string joining the monopole, anti-monopole pair. As a consequence, we found the length of this string to be finite.

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References

[1] See for example J. Schwarz, *Lectures on Superstring and M-theory dualities*, hep-th/9607201; M. Duff, R. Khuri and J. Lu, Phys. Rep. 259 (1995) 213.
[2] A. Sen, JHEP 9809 (1998) 023.
[3] A. Sen, JHEP 9710 (1997) 002.
[4] D. Gross and M. Perry, Nucl. Phys. B226 (1983) 29.
[5] A. Davidson and E. Gedalin, Phys. Lett. B339 (1994) 304.
[6] A. Macias and T. Matos, Class. Quant. Grav. 13 (1996) 345.
[7] A. Sen, Nucl. Phys. B440 (1995) 421.