1 Introduction

QCD at non-zero temperature and density is a rich subject discussed in many talks at this meeting: from the introductory lesson describing the theoretical discovery of the quark gluon plasma to the subtle interrelations of gauge fields dynamics and charmonium suppressions, from the detailed analysis of the exotic phases at large baryon density to the fascinating aspects of the QCD phase diagram with many flavor, from the time-honored theoretical subjects of the interplay between topology and confinement to the challenging aspects of the interrelations of chiral and axial symmetries. These notes mostly aimed at the non-specialist might hopefully provide, besides some introductory material, a path among general idea as well as among the many new results presented in the lively topical session on the QCD phases.

The material is organised as follows: Section 2 reviews the imaginary time formulation of field theory thermodynamics, and gives the functional integral representation of the partition function $Z$. The representation of $Z$ naturally leads to a theoretical suggestion: the concept of universality which is singled out in a small Section on its own, because it is so important. Section 4 discusses the QCD chiral transition, Section 5 the QCD deconfinement transition, and Section 6 how to put together the two, eventually describing the complex phenomenology of the real world high temperature QCD phase transition. The equation of state is briefly mentioned here. Section 6 introduces the method used to obtain all of the results of the previous Sections: the lattice regularization. We shall see that introducing temperature is straightforward while a non–zero density poses specific problems. We will only mention the main proposals to circumvent such problems, referring to the most recent reviews for details and recent results.
2 Formulation

Here we will concern ourselves with the path integral representation of the partition function. The basic property of equilibrium field theory is that one single function $Z$ (the grand canonical partition function):

$$Z = Z(V, T, \mu)$$ (1)

determines completely the thermodynamic state of a system according to:

$$P = T \frac{\partial \ln Z}{\partial V}$$ (2)

$$N = T \frac{\partial \ln Z}{\partial \mu}$$ (3)

$$S = \frac{\partial T \ln Z}{\partial T}$$ (4)

$$E = -PV + TS + \mu N$$ (5)

while physical observables $< O >$ can be computed as

$$< O > = Tr\hat{O}\hat{\rho}/Z$$ (6)

Any of the excellent books on statistical field theory and thermodynamics can provide a more detailed discussion of these points. I would like to underscore, very shortly, that the problem is to learn how to represent $Z$ at non-zero temperature and baryon density, and how to design a calculational scheme.

2.1 $Z$ at non-zero temperature and chemical potential

$Z$ is the trace of the density matrix of the system $\hat{\rho}$

$$Z = Tr\hat{\rho}$$ (7)

$$\hat{\rho} = e^{(-H-\mu \hat{N})/T}$$ (8)

$H$ is the Hamiltonian, $T$ is the temperature and $\hat{N}$ is any conserved number operator.

2.2 Chemical potential, relativistic and non-relativistic

It is worth stressing the main differences between the non relativistic and relativistic meaning of a chemical potential $\mu$, eqs (5) and (8).

In a non relativistic setting, the chemical potential tells us the energy ‘cost’ of adding an extra particle to the system: we thus have a different
chemical potential for each type of particle. The term which is added to the Hamiltonian is \( \mu_k N_k \), \( N_k \) being indeed the number of particles belonging to the \( k \)-th species.

In a relativistic setting, particles can be created and destroyed, thus losing their individuality, so to speak: it only makes sense to have a chemical potential coupled to the 0-th component of a conserved current.

In practice, we will mostly concern ourselves with the current \( J_\mu = \bar{\psi} \gamma_\mu \psi \), i.e. \( J_0 = \bar{\psi} \gamma_0 \psi = \bar{\psi} i \psi \) will be the density of fermion (baryon) number, or, more precisely, the difference between the number of fermions and antifermions: \( \int J_0 = N - \tilde{N} \). The chemical potential will then appear in the Lagrangian as a term \( \mu J_0 \) and we recognize immediately that a change of sign for the chemical potential corresponds to an exchange particles–antiparticles.

A second difference between non-relativistic and relativistic approach concerns the magnitude itself of \( \mu \), which in a relativistic theory contains the particle rest mass.

From a technical point of view, a finite density is easily handled in a non–relativistic setting. From a relativistic field theoretic perspectives there are instead important technical problems. We will discuss these problems later on, and we just anticipate here that the explicit breaking of the particle–antiparticle symmetry induced by a chemical potential induces the loss of positivity of the effective gluonic Action–i.e. the breakdown of the condition which makes lattice calculations possible.

2.3 Temperature

Consider the transition amplitude for returning to the original state \( \phi_a \) after a time \( t \):

\[
< \phi_a | e^{-iHt} | \phi_a > = \int d\pi \int_{\phi(x,0)=\phi_a(x)}^{\phi(x,t)=\phi_a(x)} d\phi e^{i \int_0^t dt \int d^3x (\pi(x,t) \frac{\partial \phi(x,t)}{\partial t} - H(\pi,\phi))}
\]  

(9)

Compare now the above with expression (2) for \( Z \), and make the trace explicit:

\[
Z = Tr e^{-\beta(H - \mu \tilde{N})} = \int d\phi_a < \phi_a | e^{-\beta(H - \mu N^a)} | \phi_a >
\]  

(10)

We are naturally lead to the identification

\[
\beta \equiv \frac{1}{T} \rightarrow it
\]  

(11)

We note - anticipating the discussions of Section below – that studying nonzero temperature on a lattice is straightforward: one just takes ad-
vantage of the finite temporal extent of the lattice, while keeping the space
directions much larger than any physical scale in the system.

By introducing the integral \( S(\phi, \psi) \) of the Lagrangian density (from now
on we will always use \( 1/T \) as the upper extreme for the time integration)
\[
S(\phi, \psi) = \int_0^{1/T} dt \int d^3 x \mathcal{L}(\phi, \psi)
\]  
(12)
\( Z \) is written as
\[
Z = \int d\phi d\psi e^{-S(\phi, \psi)}
\]  
(13)
The only missing ingredient are the boundary conditions for the fields: ba-
sically, they follow from the (anti)commuting properties of the (fermi)bose
fields which imply
\[
\hat{\phi}(\vec{x}, 0) = \hat{\phi}(\vec{x}, \beta)
\]  
(14)
for the bosons and
\[
\hat{\psi}(\vec{x}, 0) = -\hat{\psi}(\vec{x}, \beta)
\]  
(15)
for the fermions.

Fermions and bosons obey antiperiodic and periodic boundary conditions,
respectively, in the time direction.

The expression above, together with the boundary conditions just intro-
duced, is the key to field theory thermodynamics.

3 Universality

It is intuitive that when the smallest significant length scale of the system \( l \gg 1/T \) the system becomes effectively d–dimensional. Moreover, the description
of the system can be effectively ‘coarse grained’, ignoring anything which
happens on a scale smaller than \( l \).

This can become true when the system is approaching a continuous transi-
tion: the correlation length of the system \( \xi \) is diverging. In such situation
all the physics is dominated by long wavelength modes. Not only the system
gets effectively reduced, but the coarse graining procedure become doable. As
an effect of this procedure, systems which are very different one from another
might well be described by the same model, provided that the long range
physics is regulated by the same global symmetries: this is the idea of uni-
versality which provides the theoretical framework for the study of the QCD
transition in two interesting (albeit non physical) cases which we review in
the next two sections.
4 QCD chiral transition \((m_q = 0)\)

Let us recall the symmetries of the QCD action with \(N_f\) flavors of massless quarks, coupled to a \(SU(N_c)\) color group:

\[
SU(N_c) \times SU(N_f) \times SU(N_f) \times Z_{A}(N_f)
\]  

\((16)\)

\(SU(N_c)\) is the gauge color symmetry. \(SU(N_f) \times SU(N_f) \times Z_{A}(N_f)\) is the flavor chiral symmetry, after the breaking of the classical \(U_{A}(1)\) symmetry to the discrete \(Z_{A}(N_f)\).

We want to study the realization and pattern(s) of breaking of the chiral symmetries and we would like to know the interrelation of the above with the possibility of quark liberation predicted at high temperature and density.

4.1 Ordinary conditions: zero temperature and density

In normal conditions (zero temperature and density) the \(SU(N_f)_L \times SU(N_f)_R\) chiral symmetry is spontaneously broken to the diagonal \(SU(N_f)_{L+R}\). Let us note the isomorphism:

\[
SU(2) \times SU(2) \equiv O(4)
\]  

\((17)\)

which shows that the symmetry is the same as the one of an O(4) ferromagnet. The relevant degrees of freedom are the three pions, and the sigma particle, and the effective potential is a function of \(\sigma^2 + |\pi|^2\) in the chiral space. Once a direction in the chiral sphere is selected (say in the \(\sigma\) direction) chiral symmetry is spontaneously broken in that direction, according to the equivalent patterns:

\[
SU(2)_R \times SU(2)_L \rightarrow SU(2)_{L+R}
\]  

\((18)\)

\[
O(4) \rightarrow O(3)
\]  

\((19)\)

Massless Goldstone particles (in this case, the three pions) appear in the direction orthogonal to the one selected by the spontaneous breaking.

4.2 Increasing \(T\)

Disorder increases with temperature. Then, one picture of the high \(T\) QCD transition can be drawn by using a ferromagnetic analogy of the chiral transition: \(\bar{\psi}\psi\) can be thought of as a spin field taking values in real space, but whose orientation is in the chiral sphere. Chiral symmetry breaking occurs when \(\langle \bar{\psi}\psi \rangle \neq 0\), i.e. it corresponds to the ordered phase. By increasing \(T\), \(\langle \bar{\psi}\psi \rangle \rightarrow 0\), and the O(4) symmetry should be restored.
Combining this symmetry analysis with the general idea of dimensional reduction, Pisarski and Wilczek proposed that the high temperature transition in two flavor QCD should be in the universality class of the O(4) sigma model in three dimensions. At high temperature when symmetry is restored there will be just one global minimum for zero value of the fields, and pion and sigma become eventually degenerate.

We have however to keep in mind possible sources of violation of this appealing scenario and, all in all, one has to resort to numerical simulations to measure the critical exponents, and verify or disprove the O(4) universality. In turn, this gives information on the issues raised for instance in as well as on the possible restoration of the axial anomaly, see the discussion in Meggiolaro’s review.

In practice, one measures the chiral condensate as a function of the coupling parameter $\beta$, which in turns determines the temperature of the system. This gives the exponent $\beta_{mag}$ according to

$$< \bar{\psi} \psi > = B(\beta - \beta_c)^{\beta_{mag}}$$  \hspace{1cm} (20)

The exponent $\delta$ is extracted from the response at criticality:

$$< \bar{\psi} \psi > = A m^{1/\delta}; \beta = \beta_c$$  \hspace{1cm} (21)

The results for the critical exponents compare favorably with the O(4) results $\beta_{mag} = .38(1), \delta = 4.8(2)$, and definitively rule out mean fields exponents (which would have characterized a weak first order transition). However, the results can still be compatible with O(2) exponents, which would signal the persistence of some lattice artifact, and of course it is still possible that the final answer do not fit any of the above predictions, for instance if we were just observing some crossover phenomenon.

In conclusion the symmetry analysis of the (two flavor) QCD transition gives a definite predictions for the value of the critical exponents, which is possibly (slightly) violated by the numerical results. The numerical relevance of this violation (i.e. possible systematic errors) as well as the its physical implications are an interesting open problem.

The contribution by E. Meggiolaro covers in more detail these points.

4.3 Increasing density

Till two or three years ago, we thought that asymptotic freedom was the main physical agent behind the pattern of chiral symmetries at high density. Now it has been recognized that, at least at zero temperature, the main mechanisms are instability at the Fermi surface leading to color superconductivity. The
review by Nardulli and Sannino cover these points which, unfortunately, are not yet amenable to a lattice study. At non–zero density the lattice approach (see contribution by M. D’Elia for developments presented at this meeting) is still limited to rather high temperature, where color superconductivity is most likely lost anyway.

5 QCD deconfinement transition \((m_q = \infty)\)

When \(m_q = \infty\) quarks are static and do not contribute to the dynamics: hence, the dynamic of the system is driven by gluons alone, i.e. we are dealing with a purely Yang-Mills model:

\[
S = F_{\mu\nu} F^{\mu\nu} \tag{22}
\]

In addition to the local gauge symmetry, the action enjoys the global symmetry associated with the center of the group, \(Z(N_c)\). The order parameter is the Polyakov loop \(P\)

\[
P = e^{i \int_0^{1/T} A_0 dt} \tag{23}
\]

In practice, \(P\) is the cost of a static source violating the \(Z(N_c)\) global symmetry.

The interquark potential \(V(R,T)\) (\(R\) is the distance, \(T\) is the temperature) is

\[
e^{-V(R,T)/T} \propto \langle \bar{P}(\vec{0}) P(\vec{R}) \rangle \tag{24}
\]

Confinement can then be read off the behavior of the interquark potential at large distance. When \(V(\vec{R}) \propto \sigma \vec{R}\) it would cost an infinite amount of energy to pull two quarks infinitely apart. Above a certain critical temperature \(V(\vec{R})\) becomes constant at large distance: i.e. the string tension is zero, confinement is lost. The implication of this is that \(|P|^2 = V(\infty, T)\) is zero in the confining phase, different from zero otherwise. \(P\) plays then a double role, being the order parameter of the center symmetry, and an indicator of confinement. We learn that in Yang Mills models there is a natural connection between confinement and realization of the \(Z(N_c)\) symmetry. Hence, the confinement / deconfinement transition in Yang Mills systems is amenable to a symmetry description. By applying now the same dimensional reduction argument as above, we conclude that the Universality class expected of the three color model is the same as the one of a three dimensional model with \(Z(3)\) global symmetry: Indeed, work by the Columbia, Tsukuba and APE group in the mid 80’s – see again Nicola Cabibbo’s talk for comments on this – found that
The transition turns out to be ‘almost’ second order, i.e. very weakly first order, like the 3d three state Potts model.

The same reasoning tells us that the two color model is in the universality class of the three dimensional $Z(2)$ (Ising) model. This prediction has been checked with a remarkable precision by Engels and collaborators, and it is a spectacular confirmation of the general idea of universality and dimensional reduction.

6 Summary and Open Questions for the QCD High $T$ Transition

What do we know about the real world: two (nearly) massless quark $m_q << \Lambda_{QCD}$, and one more heavy?

We can approach then the 'real' world from two sides, either decreasing the mass from infinity, or increasing the quark mass from zero.

6.1 Approaching the physical point from infinite mass

Remember that in the infinite mass limit QCD reduces to the pure gauge (Yang Mills) model. Yang Mills systems have a deconfining transition associated with the realization of the global $Z(N_c)$ symmetry. This places the system in the Ising 3d universality class for two colors, and makes the transition weakly first order (near second, in fact) for three colors. General universality arguments are perfectly fulfilled by the deconfining transition.

The $Z(N_c)$ symmetry is broken by the kinetic term of the action when the quarks are dynamic ($m_q < \infty$) : this particular symmetry description of deconfinement only holds for infinite quark mass. When light quarks enter the game, the global $Z(N_c)$ symmetry observed at infinite mass is lost, and the simple description of confinement in terms of such symmetry is not possible any more. It should however remain true that color forces at large distance should decrease with temperature: the main mechanism, already at work at $T = 0$, is the recombination of an (heavy) quark and antiquark with pairs generated by the vacuum: $\bar{Q}Q \rightarrow \bar{q}Q + q\bar{Q}$. At high temperature it becomes easier to produce light $\bar{q}q$ pairs from the vacuum, hence it is easier to 'break' the color string between an (heavy) quark and antiquark $Q\bar{Q}$. In other words, we expect enhanced screening of the color forces, which should be sharp at a phase transition (or rapid crossover). It is however worth mentioning that, even if the string 'breaks' bound states might well survive giving rise to a complicated, non-perturbative dynamics above the critical temperature. The physical scale of these phenomena is the larger physical scale in the system,
i.e. the pion radius.

6.2 Approaching the physical point from zero mass

For zero bare mass the phase transition is chiral. For three colors, two flavors is second order with $T_c \approx 170$ MeV. The prediction from dimensional reduction + universality –$O(4)$ exponents– is compatible with the data, but the agreement is not perfect.

If the agreement were confirmed, that would be an argument in favor of the non-restoration of the $U_A(1)$ symmetry at the transition, which is also suggested by the behavior of the masses spectrum. Remember in fact that the chiral partner of the pion is the $f_0$, which is in turn degenerate with the scalar $a_0$ with $U_A(1)$ is realized. All in all, $U_A(1)$ non–restoration across the chiral transition corresponds to $m_\pi \simeq m_{f_0} \neq m_{a_0}$ which is the pattern observed in lattice calculations (once more I refer to meggiolaro’s talk).

The transition with three (massless) flavor turns out to be first order. The question is than as to whether the strange quark should be considered ‘light’ or ‘heavy’. In general, the real world will be somewhere in between two and three light flavor, and to really investigate the nature of the physical phase transition in QCD one should work as close as possible to the realistic value of the quark masses, which is a very demanding numerical task.

6.3 What do we know on the real QCD phase diagram

Among the most prominent open questions, there is of course the behavior of ‘real’ QCD, with two light flavor, and a third one of the order of $\Lambda_{QCD}$, so how and when exactly the $N_f = 2$ scenario morfs with the $N_f = 3$? Also, why is $T_\chi$ much smaller that the pure gauge deconfining transition?

At a theoretical level the question is if it is possible to give an unified description of the two transitions, chiral and deconfining. This question is currently under active investigation: recent work suggests that a symmetry analysis of the deconfining transition can be extended also to theories with dynamical fermions. The physical argument is rooted in a duality transformation which allows the identification of magnetic monopoles as agent of deconfinement. The order parameter for deconfinement would that be the monopole condensate $\Phi$. An alternative approach uses percolation as the common agent driving chiral and confining transitions $\Psi$.

One unifying description and perhaps the most dramatic evidence of a phase transition away from the ‘simple’ limits $m_q = 0$ and $m_q = \infty$ comes from the equation of state: for $T \simeq 180$ Mev we observe, from lattice calculations, a sharp increase of the internal energy: the behavior of the internal energy is
a direct probe of the number of degrees of freedom, and indicates quark and
gluon liberation \[10\].

Finally, the work \[13\] arrives at an interesting picture of the phase diagram
of QCD by combining symmetry analysis and phenomenological considera-
tion. Particularly interesting is the prediction of an endpoint of a first order
line stemming from zero temperature chiral transition at finite density, which
should be experimentally observable.

7 Methods for QCD at finite temperature and density

Here we will concern ourselves with computational schemes for QCD. The
methods described here are those used to obtain the results reviewed above
on the chiral transition, deconfinement and equation of state in QCD. We
will give some details on these methods, we will explain why they are not
immediately applicable at finite density and we will close up with a brief
assessment of the current situation for finite density QCD.

The question is how to estimate the physical observables \[<O> = \text{Tr}O\hat{\rho}/Z\] starting from the representation of the partition function

\[
Z(\mu, T) = \int_0^{1/T} dt \int e^{-S_G + i(\partial + m + \mu \gamma_0)\psi} d\bar{\psi} d\psi dU
\]  

(25)

The need for two integrations (over bosons and over fermions) lends itself
naturally to two different paths: either integrate gluons first, or fermions.

In the first case (if gluons are integrated out first) we end up with a purely
effective fermionic model:

\[
Z(T, \mu, \bar{\psi}, \psi, U) \approx Z(T, \mu, \bar{\psi}, \psi)
\]  

(26)

As the integration over gluons cannot be done exactly, such effective models
are often built with the help of a symmetries’ analysis for QCD. The gauge
fields enters the the game under the guise of coefficients for such models.
Very important examples of this approach include instanton models, chiral
perturbation theory, four fermion models such as those discussed by Nardulli
and Sannino at this meeting.

The other way takes advantage of the bilinear nature of \(S\) in the fermionic
fields, yielding the exact expression:

\[
Z(T, \mu, U) = \int dU e^{-\left(S_g - \log(\det M)\right)}
\]  

(27)

\[a\] There indeed a a systematic approach to QCD based on the strong coupling expansion
which makes this integrations easy, but this is not in the scope of this introduction.
The above integral needs being regulated, and scheme for calculating it has to be devised. Both tasks are accomplished within the lattice approach.

7.1 Lattice field theory at $T, \mu \neq 0$

Temperature comes for free on a lattice: the lattice has a finite extent $N_t a$, hence temperature is given by $T = 1/N_t a$. The discretization can be carried out in complete analogy to $T = 0$, and most of the techniques developed there (see e.g. the recent review for an introduction to lattice field theory at zero temperature) apply at finite temperature as well.

A finite density of baryons $\mu B_J$ poses instead specific problems.

Recall first the 'natural' discretization of the matter fields $\phi(x)$ and their derivatives $\partial_\mu \phi(x)$ on a regular lattice with spacing $a$:

$$\phi_{\text{LATT}}(n_1, n_2, n_3, n_4) = \phi(n_1 a, n_2 a, n_3 a, n_4 a)$$

$$\Delta_\mu \phi_{\text{LATT}}(n_1, n_2, n_3, n_4) = (\phi(n_1 a, (n_\mu + 1)a, n_3 a, n_4 a) - \phi(n_1 a, (n_\mu a, n_3 a, n_4 a))/2a$$

This, for instance, is the correct prescription for the chiral condensate: $m\bar{\psi}\psi \rightarrow m\bar{\psi}_{\text{LATT}}\psi_{\text{LATT}}$ but not, as we will see in a moment, for the baryon density! $\mu\bar{\psi}\gamma_0\psi \rightarrow \mu\bar{\psi}_{\text{LATT}}\gamma_0\psi_{\text{LATT}}$ is not the correct lattice form, i.e. The naive discretization is not adequate for baryon density.

Let us then consider free fermions in the continuum

$$S = \int^\beta_0 \bar{\psi}_m u \partial_\mu \psi + m\bar{\psi}\psi + \mu \bar{\psi}\gamma_0\psi$$

The internal energy $\epsilon = \frac{1}{V} \frac{\partial}{\partial \beta} \ln Z = \frac{4}{(2\pi)^4} \int d^4p \frac{(p_0 + i\mu)^2 + p^2 + m^2}{\epsilon^2}$, after subtracting the vacuum energy, is finite at $T = 0$: $\lim_{T \to 0} \epsilon = \frac{\mu^4}{4\pi^2}$ and gives the expect result.

By use of the naive discretization, $\epsilon$ would instead diverge in the continuum ($a \to 0$): $L = \bar{\psi}_x \gamma_\mu \psi_{x + \mu a} + m\bar{\psi}_x \psi_x + \mu \bar{\psi}_x \gamma_0 \psi_x \epsilon \propto \frac{\mu^2}{a^2} \to a \to 0 \propto \infty$

The solution\[^b\]: $\mu$ is an external field

Note the analogy: $\bar{\psi}_m A_\mu \psi \leftrightarrow i\epsilon \bar{\psi}_0 \gamma_0 \psi$. It shows us that $\mu$ looks like an external field in the time direction. But we know how to 'put' external field

\[^b\] Remember again that $L(\mu) = L_0 + \mu J_0$, $J_0 = \bar{\psi}\gamma_0\psi$, i.e. $N - \bar{N} = \int J_0$
on a lattice: they live on the lattice links: for instance, in electrodynamics $A \to \theta = e^{iA}$ (and, for the gauge fields, this implements in a natural and elegant way gauge invariance).

These considerations suggest how to put finite density on a lattice:

$$L(\mu) = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_x + \bar{\psi}_x \gamma_0 e^{-\mu a} \psi_x$$  \hspace{1cm} (28)

Indeed, this turns out to be the correct prescription: unphysical divergences disappear and the continuum limit is reproduced.

There is also a very expressive physical interpretation: as we can see, forward propagation is encouraged and backward propagation is suppressed: we are indeed inducing an asymmetry particles-antiparticles.

The 0-th component of the current $J_0$ counts indeed the differences between backward and forward propagation:

$$J_0 = -\partial_\mu L = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_x + \bar{\psi}_x \gamma_0 e^{-\mu a} \psi_x$$  \hspace{1cm} (29)

and reproduces the correct continuum limit.

Chemical potential and boundary conditions

By use of an unitary transformation of the fields it is possible to re-express $L$ as

$$L(\mu) = L(0)$$  \hspace{1cm} (30)

with boundary conditions

$$\phi(x + N_T) = e^{\mu N_T} \phi(x)$$  \hspace{1cm} (31)
$$\psi(x + N_T) = -e^{\mu N_T} \psi(x)$$  \hspace{1cm} (32)

It is of some interest to consider the effect of the chemical potential on the baryonic propagators in the two cases – when the chemical potential is included into the Action, and when instead affects only the boundary conditions. In the first case, (at zero temperature, and ignoring feedbacks) a term $\exp(-\mu c)$ multiplies the baryonic propagator: this produces an apparent decrease of the baryon mass

$$m_B = m_B - 3\mu$$  \hspace{1cm} (33)

i.e. the baryon becomes massless at $\mu = \mu_c = m_B/3$. In the same situation, the chiral condensate remains constant till $\mu_c$, and then suddenly drops to zero. The behavior of the chiral condensate would be suggestive of a strong first order transition, while the behavior of the baryon mass would suggest a second order transition with $\nu = 1!$
In the other formulation (when the chemical potential only affects the boundary conditions), at zero temperature the baryon mass is constant till $\mu_c$: this is consistent with the behavior of the chiral condensate, and with the physical intuition that nothing should happen till the Fermi level is reached. Of course, nothing is wrong and the apparent contradiction is resolved by noticing that the apparent decrease of the baryon mass below $\mu_c$ merely reflects the change of the reference energy. The relationships above provide a link between the two pictures.

**QCD at nonzero $T$ and $\mu$ at a glance**

The continuum formulation:

$$L = L_{YM} + \bar{\psi}(i\gamma_\mu D_\mu + m)\psi + \mu\bar{\psi}\gamma_0\psi$$

(34)

$\mu$ is explicitly included via the coupling to $\mu J_0$, and the temperature is the reciprocal of the imaginary time.

On a lattice:

$$L = L_{YM} + \sum_{i=1}^{3} \bar{\psi}_x U_{\gamma_i} \psi_{x+i} - \bar{\psi}_{x+i} U^\dagger_{\gamma_i} \psi_x$$

$$+ m\bar{\psi}\psi + \bar{\psi}_x \gamma_0 e^{\mu a} U \psi_{x+0} - \bar{\psi}_{x+0} U^\dagger \gamma_0 e^{-\mu a} \psi_x$$

$\mu$ appears as a link term, and the temperature is again the reciprocal of the imaginary time. The two formulations coincide in the limit $a \to 0$.

7.2 From the formulation to the results

Let us write again:

$$S = S_{YM}(U) + \bar{\psi}M(U)\psi$$

(35)

By taking advantage of $S$ bilinearity in the fields $\psi, \bar{\psi}$ we can write

$$Z = \int dU e^{-S_{YM}(U)} detM(U) dU$$

(36)

It is convenient to define an ‘effective’ Action

$$S_{eff}(U) = S_{YM}(U) - ln(detM(U))$$

(37)

Averages of purely gluonic observables can be expressed as

$$< f(U) > = Z^{-1} \int dU e^{-S_{eff}(U)} f(U)$$

(38)
while fermion bilinears can be evaluated with the help of Grassman algebra

\[ <\bar{q}q> = Z^{-1} \int dU e^{-S_{\text{eff}}(U)} \text{Det}(U)^{-1} \]

(39)

In conclusion, if we know how to treat \( \int e^{-S_{\text{eff}}(U)} dU \) we have access to all of the gluonic observables, chiral condensate, meson and baryons propagators (masses, decay constant, etc.) etc. much in the same way as at \( T = 0 \).

In practical numerical works lattice discretization is combined with importance sampling: a configuration of gauge fields \([U]\) is a 'point' in a multidimensional integration space. A Markov chain of points is then created according to the prescription: \( P([U]) \propto e^{-S_{\text{eff}}([U])} \) Expectation values are then given by simple averages: \( <O> = \lim_{N \to \infty} 1/N \sum_{i=1}^{N} O(U) \).

The prescription above relies on importance sampling: \( S_{\text{eff}}([U]) \) must be positive \[ \] In QCD (eq. above) \( M^\dagger(\mu) = -M(-\mu) \) : importance sampling, hence MonteCarlo evaluation of physical observables, works for purely imaginary chemical potential, including, of course, \( \mu = 0 \).

For QCD with two color, however, as well as for several fermionic models, the positivity condition is met by the Action at nonzero \( \mu \). All in all, we have a rather elaborate pattern of warnings and possibilities.

7.3 Results and possibilities for QCD at \( \mu \neq 0 \)

I would classify the many questions one can ask along two main lines:

Firstly, what can be done in order to learn about the general behavior at nonzero baryon density?

Secondly, what can be done in order to learn about real QCD at nonzero baryon density?

I would say that to learn about the effects of a finite density of baryons, we can study two color QCD: here, not only can study fermionic observables, but it is also possible to observe the effects of a density of baryons on the gauge fields. This is indeed an active field of research, and I refer to the proceedings of the Lattice meetings for details.

How about real QCD? there is a growing consensus that at least the high temperature, small density region is accessible: by increasing temperature it is easier to fluctuate light baryons, hence to explore a region with a moderate baryon density. At least three different, complementary approaches stem from this consideration:direct evaluation of derivatives at \( \mu = 0 \), reweighting,

\[ c\] If this is not the case, one might think of using \( S^\dagger \) instead: this has to be used with care because it introduces extra degrees of freedom which might well be dangerous.
analytic continuation from purely imaginary chemical potential. This last approach has been discussed by M. D’Elia at this meeting.

I hope to have managed at least to describe some of the aspects of a rapidly evolving field, and to convey a feeling of its many reasons of interest. A discussion of the recent results is not in the scope of this cursory introduction; for this, and for a complete set of references, I refer to the latest reviews (not yet available at the time of the Frascati meeting).

References

1. N. Cabibbo, introductory remarks, this volume.
2. H. Satz, ibid.
3. G. Nardulli, ibid.; M. Mannarelli, ibid.
4. F. Sannino, ibid.; W. Schaefer, ibid.
5. L. Del Debbio, ibid.; A. Drago, ibid.; D. Cosmai, ibid. ; A. Papa, ibid.
6. E. Meggiolaro, ibid.
7. M. d’Elia, ibid.
8. Introductory material on the lattice approach to field theory can be found in J. B. Kogut, Rev.Mod.Phys.51 (1979) 659; M. Creutz, Quarks, Gluons and Lattice, Cambridge; C. Itzykson-Douffre, Statistical Field Theory, Cambridge, Chapter 7.
9. M. Alford, hep-ph/0102047; K. Rajagopal and F. Wilczek, in ‘At the Frontier of Particle Physics/Handbook of QCD’, M. Shifman, ed., (World Scientific); R. Rapp, T. Schafer, E.V. Shuryak, M. Velkovsky, Annals Phys.280 (2000)35.
10. For an introduction to numerical simulations of lattice QCD thermodynamics see e.g. C. DeTar, in Hwa, R.C. (ed.): Quark-gluon plasma, vol.2, pag. 1 or F. Karsch, hep-lat/0206034.
11. R. D. Pisarski and F. Wilczek, Phys.Rev. D29 (1984)338.
12. J. Engels, S. Mashkevich, T. Scheideler, G. Zinovev, Phys.Lett.B365 (1996) 219.
13. M. Stephanov, K. Rajagopal, E. Shuryak, Phys. Rev. Lett Phys.Rev.Lett.81 (1998) 4816; J. Berges and K. Rajagopal, Nucl.Phys. B538 (1999) 215; M.A. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov and J.J.M. Verbaarschot, Phys.Rev.D58 (1998) 096007; M. Stephanov, K. Rajagopal, E. Shuryak, Phys.Rev.D60 (1999) 114028.
14. Recent reviews on QCD in extreme conditions include: J. B. Kogut, Lattice2002; Z. Fodor, QM2002; K. Kanaya, QM2002; M. Alford, ICHEP2002.