Short Survey of Darboux Transformations

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Abstract
A selective chronological survey of Darboux transformations as related to supersymmetric quantum mechanics, intertwining operators and inverse scattering techniques is presented. Short comments are appended to each quotation and basic concepts are explained in order to provide a useful primer.

Contents
Chronological set 1: DTs (as covariance of Sturm-Liouville problems)
Chronological set 2: Intertwining (transformation) operators (the more general operator language)
Chronological set 3: Related inverse quantum scattering approaches

Some abbreviations:
1. Darboux transformation(s) - DT(s) 2. Factorization method - FM
3. Inverse Quantum Scattering - IQS 4. One-dimensional - 1D
5. Quantum mechanics - QM 6. Schroedinger equation - SE
7. Supersymmetry, supersymmetric - SUSY

Possible combinations are allowed.
CHRONOLOGICAL SET 1
Darboux Transformations
(covariance of Sturm-Liouville problems)

1882: *Sur une proposition relative aux équations linéaires*

This proposition, which history proved to be a notable theorem, is provided by G. Darboux in Compt. Rend. Acad. Sci. 94, 1456 [physics/9908003], and is also to be found in his course “Théorie des Surfaces”, II, p. 210 (Gauthier-Villars, 1889). Darboux’s result was not used and recognized as important for a long time, and in fact has been only mentioned as an exercise in 1926 by Ince.

1926: *Ordinary Differential Equations* (Dover)

In his famous Dover book, E.L. Ince publishes Darboux’s theorem in a section of “Miscellaneous Examples” at page 132, together with other two particular examples of the theorem related to the free particle and Pöschl-Teller potential (the latter belonging to Ince), Exercises 5, 6 and 7, respectively. The exercises in Ince follow closely the formulation of Darboux (see §1).

§1: Exercise 5 at page 132 of Ince’s textbook
Prove that if the general solution $u = u(x)$ of the equation

$$\frac{d^2 u}{dx^2} = (\phi(x) + h)u$$

is known for all values of $h$, and that any particular solution for the particular value $h = h_1$ is $u = f(x)$, then the general solution of the equation

$$\frac{d^2 y}{dx^2} = \left( f(x) \frac{d^2}{dx^2} (f^{-1}(x)) + h - h_1 \right) y$$

for $h \neq h_1$ is

$$y = u'(x) - u(x) \frac{f'(x)}{f(x)}$$

1940-1941: *The factorization of the hypergeometric equation*

In Proc. Roy. Irish Acad. 47 A, 53 (1941) [physics/9910003], E. Schrödinger factorizes the hypergeometric equation, finding that there are several ways of factorizing it. This was a byproduct of his FM (published during 1940-1941 in the same review) originating “from a, virtually, well-known treatment of the oscillator”, i.e., an approach that can be traced back to Dirac’s creation and
annihilation operators for the harmonic oscillator and to older factorization ideas in the mathematical literature.

**1951: The FM**

L. Infeld and T.E. Hull present the classification of their factorizations of linear second order differential equations in Rev. Mod. Phys. **23**, 21.

**1955: Associated Sturm-Liouville systems**

M.M. Crum publishes an important iterative generalization of Darboux’s result in Quart. J. Math. **6**, 121 (physics/9908019), without any mention of Darboux (see §2).

**§2: Crum’s iteration**

Let $\psi_1, \psi_2, ... \psi_N$ be solutions of a given Schrödinger equation $-D^2 \psi_i + u \psi_i = \lambda \psi_i$, for fixed, arbitrary constants $\lambda = \lambda_1, \lambda_2, ... \lambda_N$, respectively. Define the Wronskian determinant $W$ of $k$ functions $f_1, f_2, ..., f_k$ by

$$W(f_1, ..., f_2) = \text{det} A, \quad A_{ij} = \frac{d^{i-1} f_j}{dx^{i-1}}, \quad i, j = 1, 2, ..., k.$$ 

Then, the transformations (Crum’s formulas)

$$\psi[N] = W(\psi_1, ..., \psi_N, \psi)/W(\psi_1, ..., \psi_N)$$

$$u[N] = u - 2D^2 \ln W(\psi_1, ..., \psi_N)$$

are covariant ones, i.e., one can write the following SL equation:

$$-D^2 \psi[N] + u[N] \psi[N] = \lambda \psi[N].$$

Darboux’s result of 1882 may be seen as the case $N = 1$. In other words, Crum presented the successive (iterative) DTs in compact formulas.

**1957: Krein’s approach to generalized DT**

M.G. Krein publishes extensions of Crum’s iteration in D.A.N. SSSR **113**, 970, using a different perspective and with more elaborations. In particular he provides an important theorem on the sign of the Wronskian.

**1979: Concept of DT**
V.B. Matveev uses for the first time the expression DT (directly in the title) in his series of papers in Lett. Math. Phys. 3, 217, 425, in which he extended the range of applicability of the concept to differential-difference and difference-difference evolution equations.

1981: Dynamical breaking of SUSY

In section 6, “Some models”, of his renowned paper Nucl. Phys. B 185, 513, E. Witten introduces SUSY QM as a toy model for supersymmetry breaking in quantum field theories. As a matter of fact, the whole SUSY breaking paradigm is nothing but a classification of vacuum states based on the SUSY QM toy model. The SUSY breaking is considered as a sort of “phase transition” with the order parameter being the Witten index, defined as the grading operator $\tau = (-1)^{N_f}$, where $N_f$ is the fermion number operator. For the 1D SUSY QM Witten’s index is the third Pauli matrix $\sigma_3$, which is +1 for the bosonic sector and -1 for the fermionic sector of the 1D quantum problem at hand. It is also quite common to call a particular Riccati solution as a (Witten) “superpotential”. Papers that now are standard references are published during 1982-1984. It is known that H. Nicolai was the first to write the simple SUSY QM matrix algebra in 1976, whereas a relativistic graded algebra has been first written by Yu.A. Gol’fand and E.P. Likhtman in JETP Lett. 13, 323 (1971) and since then supersymmetric gauge theories have been main stream research.

1981: DTs and nonlinear evolution equations

V.B. Matveev and M. Salle apply for the first time DT to nonlinear equations in D.A.N. SSSR 261, 533. At present, there are hundreds of papers in the area.

§3: DTs and Bäcklund Transformations

A well-known transformation in nonlinear physics is the Bäcklund transformation (BT) [A.V. Bäcklund, Math. Ann. 9 (1876), 297; 19 (1882), 387], that has been first applied to sine-Gordon equation. When BT operators are applied to N soliton solutions one gets N+1 soliton solutions. In the Korteweg-de Vries case, DTs may be seen as BTs for instantaneous solitons, since instantaneous KdV sech$^2$ solitons may be considered at the same time as Schrödinger potentials of a well-known SUSY QM problem.

On the other hand, in the Lax representation of KdV equation, the first Lax operator is of Schroedinger type, a fact showing the importance of Darboux covariance in nonlinear physics.

1983: Derivation of exact spectra of the SE by means of SUSY
In JETP Lett. 38, 356, L.E. Gendenshtein introduces the important concept of shape invariance (SI) in SUSY QM.

§4: The SI property

SI is a property of some classes of potentials with respect to their parameter(s), say $a$, and reads

$$V_{n+1}(x, a_n) = V_n(x, a_{n+1}) + R(a_n)$$

where $R$ is a remainder. This property assures a fully algebraic scheme for the spectrum and wavefunctions. Fixing $E_0 = 0$, the excited spectrum is given by

$$E_n = \sum_{k=2}^{n+1} R(a_k)$$

and the wavefunctions are obtained from

$$\psi_n(x, a_1) = \prod_{k=1}^{n} A^+(x, a_k) \psi_0(x, a_{n+1})$$

1984: *FM and new potentials with the oscillator spectrum*

In J. Math. Phys. 25, 3387, B. Mielnik provides the first application of the general Riccati solution to the harmonic oscillator, noticing also the similarity to the Abraham-Moses class of isospectral potentials in the area of inverse scattering. In the same year, D. Fernández gives a second application to the atomic hydrogen spectrum in Lett. Math. Phys. 8, 337, whereas M.M. Nieto clarifies further the inverse scattering aspect of Mielnik’s construction in Phys. Lett. B 145, 208. The procedure may be seen as a double Darboux transformation in which the general Riccati (superpotential) solution is involved (see §7). Thereby it will be denoted by the acronym DDGR (double Darboux general Riccati).

1984-1985: *FM and DTs for multidimensional Hamiltonians*

In papers published in Teor. Mat. Fiz. 61 and subsequent Phys. Lett. A versions, A.A. Andrianov, N.V. Borisov and M.V. Ioffe discover the relation between SUSY QM and DTs while playing with matrix Hamiltonians in SUSY QM.

1985: *Exactness of semiclassical bound state energies for SUSY QM*
In Phys. Lett. B 150, 159, A. Comtet, A. Bandrauk and D.K. Campbell were the first to study the WKB features of SUSY QM introducing a SUSY QM WKB formula \( \int_{a}^{b} |E - W^2(y)|^{1/2} dy = n \pi \hbar \), where \( W \) is Witten’s superpotential and \( a \) and \( b \) are turning points. This research line has a rich publishing output.

1988: PARASUSY QM

V.A. Rubakov and V.P. Spiridonov introduce PARASUSY QM in Mod. Phys. Lett. A 3, 1337, involving supercharges of order-three nilpotency. Later, J. Beckers and N. Debergh found a different algebra of the same type [Nucl. Phys. B 340, 767 (1990)]. There are many generalizations, making this ten-year topic quite active and interesting from the application point of view.

1991: DTs and Solitons (Springer)

V.B. Matveev and M. Salle publish the first (excellent) book (112 pp, 197 refs.) focusing on DTs and their relation with soliton (mathematical) physics. The DTs are defined as covariant properties of the SE. Many types of DTs (not mentioned here) are presented in this book in a concise manner.

§5: Darboux Covariance

The statement of the Darboux theorem can be interpreted as the *Darboux covariance* of a Sturm-Liouville equation

\[-\psi_{xx} + u \psi = \lambda \psi\]

by which one should understand that the following DT

\[\psi \rightarrow \psi[1] = (D - \sigma_1) \psi = \psi_x - \sigma_1 \psi = \]

\[\frac{W(\psi_1, \psi)}{\psi_1}\]

\[u \rightarrow u[1] = u - 2\sigma_1 x = u - 2D^2 \ln \psi_1\]

where \( \sigma_1 = \psi_1 x \psi^{-1} \), i.e. it is the logarithmic derivative, and \( W \) is the Wronskian determinant passes the SL equation to the (Darboux isospectral) form

\[-\psi_{xx}[1] + u[1] \psi[1] = \lambda \psi[1]\]

When DTs are applied iteratively one gets Crum’s result. One can also say that the two SL equations are related by a DT.
1994: *A modification of Crum’s method*

In Theor. Math. Phys. 101, 1381, V.E. Adler discusses a double Darboux transform for consecutive eigenfunctions $\psi_k$ and $\psi_{k+1}$ as transformation functions, producing a *nonsingular* potential missing the two levels $E = k$ and $E = k + 1$ in its discrete spectrum (thus anharmonic). According to Samsonov this result is implicit in Krein’s theorem on the sign of the Wronskian. Very recently, D.J. Fernández, V. Hussin and B. Mielnik [Phys. Lett. A 244, 309, (1998)] gave an interesting combination of Adler’s method and DDGR.

1994: *Coherent states for isospectral oscillator Hamiltonians*

In J. Phys. A 27, 3547, D.J. Fernández, V. Hussin and L.M. Nieto provide the first discussion of coherent states for Darboux transformed systems. In this important topic there are recent significant results due to Bagrov and Samsonov [J. Phys. A 29, 1011 (1996)] and Samsonov [J. Math. Phys. 39, 967 (1998)]. In the latter paper, Samsonov showed that in some cases the distortion of the phase space due to DT is calculable.

1995: *SUSY and QM*

In Phys. Rep. 251, 267, F. Cooper, A. Khare and U. Sukhatme publish the latest (at this time) review on SUSY QM. Even though this is a comprehensive work with 265 references, they cite 16 omitted topics and are forced to accept the following: “So much work has been done in the area of SUSY QM in the last 12 years that it is almost impossible to cover all the topics in such a review”. For comparison, what may be considered (in a limited sense) as the first SUSY QM review paper written by L.E. Gendenshtein and I.V. Krive in 1985 has 65 references in various areas.

1996: *SUSY Methods in Quantum and Statistical Physics* (Springer)

G. Junker publishes his Habilitation at Erlangen as a book (124 pp) with more than 300 references.

1996: *DTs for time-dependent SE (TDSE)*

In Phys. Lett. A 210, 60, V.G. Bagrov and B.F. Samsonov provide an important extension of DTs to nonstationary SEs by means of intertwining. A direct (less general) factorization approach for the time-dependent Pauli equation has been given by V.A. Kostelecký, V.I. Man’ko, M.M. Nieto, and D.
Truax, in Phys. Rev. A 48, 951 (1993); see also, V.M. Tkachuk, J. Phys. A 31, 1859 (1998).

1997: DT for Dirac eqs with (1+1) potentials

A.V. Yurov studies DT for Dirac equations in Phys. Lett. A 225, 51.

1997: DT of the SE

In Phys. Part. Nucl. 28, 374 (1997) [Fiz. Elem. Chastits At. Yadra 28, 951 (1997)] V.G. Bagrov and B.F. Samsonov publish a first review of the results in QM obtained by means of DT. There are 83 references.

§6: DTs and SUSY QM (Matveev and Salle)

Proposition: Witten’s SUSY QM is equivalent to a single DT.

Proof: Consider two Schrödinger equations

\[-D^2 \psi + u \psi = \lambda \psi\]
\[-D^2 \phi + v \phi = \lambda \phi\]

related by DT, i.e., \( v = u[1] \) and \( \phi = \psi[1] \).

Notice now that the function \( \phi_1 = \psi_1^{-1} \) satisfies the Darboux-transformed equation for \( \lambda = \lambda_1 \).

If now one uses the second (transformed) equation as initial one and perform the DT with the generating function \( \phi_1 \), one just go back to the initial \( u \) equation. Thus, one can define an inverse DT, that follows in a clear way from the direct one:

\[ u = v - 2D^2 \ln \phi_1 = v[-1] = v - 2D^2 \ln \psi_1^{-1} \]

\[ \psi = \left( \phi_x - \frac{\phi_1 x}{\phi_1} \right) (\lambda_1 - \lambda) = \left( \phi_x + \frac{\psi_1 x}{\psi_1} \phi_1 \right) (\lambda_1 - \lambda) \]

If the \( \sigma \) notation for the logarithmic derivative is introduced, i.e.,

\[ \sigma = \frac{\psi_1 x}{\psi_1} = -\frac{\phi_1 x}{\phi_1} \]

the Riccati (SUSY QM) representation of the Darboux pair of Schrödinger potentials is obtained

\[ u = v[-1] = \sigma_x + \sigma^2 + \lambda_1 \]
It is now easy to enter the SUSY QM concept of supercharge operators. For that, one employs the factorization operators

\[ B^+ = -D + \sigma, \quad B^- = D + \sigma \]

They effect the wavefunction part of the direct and inverse DT, respectively. Moreover

\[ B^+ B^- = -D^2 + v - \lambda_1 \]
\[ B^- B^+ = -D^2 + u - \lambda_1 \]

Thus, the commutator \([B^+, B^-] = v - u = -2D^2 \ln \psi_1\) gives the Darboux difference in the shape of the Darboux-related potentials. Introducing the Hamiltonian operators

\[ H^+ = B^- B^+ + \lambda_1 \]
\[ H^- = B^+ B^- + \lambda_1 \]

one can also interpret the \(B\) operators as factorization ones and write the famous matrix representation of SUSY QM, as well as the simplest possible superalgebra.

The factorizing operators in matrix representation are called *supercharges* in SUSY QM, and are nilpotent operators

\[ Q^- = A_- \sigma_+ = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} \]

and

\[ Q^+ = A_+ \sigma_- = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \]

\( \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( \sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) are Pauli matrices. In this realization, the matrix form of the Hamiltonian operator reads

\[ \mathcal{H} = \begin{pmatrix} A^+ A^- & 0 \\ 0 & A^- A^+ \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \]

defining the partner Hamiltonians as diagonal elements of the matrix one. They are partners in the sense that they are isospectral, apart from the ground state \(\phi_{gr.-}\) of \(H_-\), which is not included in the spectrum of \(H_+\).

**Remark:** Denoting \(\Lambda_1(x) = -2D^2 \ln \psi_1(x) + \lambda_1\), one can see that for a Gaussian of width \(\sigma^2\) one gets \(\Lambda_1 = 2\sigma^{-2} + \lambda_1\), and in the case of Gaussians of constant width (harmonic ground states) \(\Lambda_1\) is just a constant shift of the factorization constant.
DDGR offers an interesting possibility to construct families of potentials strictly isospectral with respect to the initial (bosonic) one, if one asks for the most general superpotential (i.e., the general Riccati solution) such that \( V_+ (x) = w_+^2 + \frac{dw_+}{dx} \), where \( V_+ \) is the fermionic partner potential. It is easy to see that one particular solution to this equation is \( w_+ = w(x) \), where \( w(x) \) is the common Witten superpotential. One is led to consider the following Riccati equation \( w_+^2 + \frac{dw_+}{dx} = w_+^2 + \frac{d w_0}{dx} \), whose general solution can be written down as \( w_g(x) = w_0(x) + \frac{1}{v(x)} \), where \( v(x) \) is an unknown function. Using this ansatz, one obtains for the function \( v(x) \) the following Bernoulli equation
\[
\frac{dv(x)}{dx} - 2v(x)w_p(x) = 1,
\]
that has the solution
\[
v(x) = \frac{\mathcal{I}_0(x) + \lambda}{u_0^2(x)},
\]
where \( \mathcal{I}_0(x) = \int_c^x u_0^2(y) dy \), \( (c = -\infty \) for full line problems and \( c = 0 \) for half line problems, respectively), and \( \lambda \) is an integration constant thereby considered as a free DDGR parameter. Thus, \( w_g(x) \) can be written as follows
\[
w_g(x; \lambda) = w_p(x) + \frac{d}{dx} \left[ \ln(\mathcal{I}_0(x) + \lambda) \right] = w_p(x) + \sigma_0(\lambda) = -\frac{d}{dx} \left[ \ln \left( \frac{u_0(x)}{\mathcal{I}_0(x) + \lambda} \right) \right].
\]
Finally, one easily gets the \( V_-(x; \lambda) \) family of potentials
\[
V_-(x; \lambda) = w_g^2(x; \lambda) - \frac{dw_g(x; \lambda)}{dx} = V_-(x) - 2\sigma_0(x) = V_-(x) - 2\sigma_0, x \lambda = V_-(x) - 4u_0(x)u_0'(x) \frac{1}{\mathcal{I}_0(x) + \lambda} + \frac{2u_0^4(x)}{(\mathcal{I}_0(x) + \lambda)^2}.
\]
All \( V_-(x; \lambda) \) have the same supersymmetric partner potential \( V_+(x) \) obtained by deleting the ground state. They are asymmetric double-well potentials that may be considered as a sort of intermediates between the bosonic potential \( V_-(x) \) and the fermionic partner \( V_+(x) = V_-(x) - 2\sigma_0, x \lambda \). From Eq. (3c)
one can infer the ground state wave functions for the potentials $V_-(x; \lambda)$ as follows

$$u_0(x; \lambda) = f(\lambda) \frac{u_0(x)}{I_0(x) + \lambda},$$

where $f(\lambda)$ is a normalization factor that can be shown to be of the form $f(\lambda) = \sqrt{\lambda(\lambda + 1)}$. One can now understand the double Darboux feature of the DDGR by writing the parametric family in terms of their unique “fermionic” partner

$$V_-(x; \lambda) = V_+(x) - 2 \frac{d^2}{dx^2} \ln \left( \frac{1}{u_0(x; \lambda)} \right),$$

which shows that the DDGR transformation is of the inverse Darboux type, allowing at the same time a two-step (double Darboux) interpretation, namely, in the first step one goes to the fermionic system and in the second step one returns to a deformed bosonic system.

**CHRONOLOGICAL SET 2**

**Intertwining (Transformation) Operators**

*(the more general operator language)*

**1938:** Intertwining discovered

J. Delsarte introduces the notion of transformation (transmutation) operator in *Sur certaines transformationes fonctionelles relative aux équations lineaires aux dérivées partielles du 2nd ordre*, Comp. Rend. Acad. Sci. (Paris) 206, 1780 [physics/9909061], [see also, *Sur une extension de la formule de Taylor*, J. Math. Pures et Appl. 17, 213 (1938)], but it is only 18 years later that he with J.L. Lions expound more on the method. J. Delsarte, Colloq. Int. Nancy, p. 29 (1956); J.L. Lions, ibidem, p. 125 (1956); J. Delsarte, J.L. Lions, Comment. Math. Helv. 32, 113 (1956).

§8 What is intertwining?

Two operators $L_0$ and $L_1$ are said to be intertwined by an operator $T$ if

$$L_1T = TL_0.$$

(1)

If the eigenfunctions $\varphi_0$ of $L_0$ are known, then from the intertwining relation one can show that the (unnormalized) eigenfunctions of $L_1$ are given by $\varphi_1 = T\varphi_0$. The main problem in the intertwining transformations is to construct the transformation operator $T$. 1D QM is one of the simplest examples of intertwining relations since Witten’s transformation operator $T_{qm} = T_1$ is
just a first spatial derivative plus a differentiable coordinate function (the superpotential) that should be a logarithmic derivative of the true bosonic zero mode (if it exists), but of course higher-order transformation operators can be constructed without much difficulty.

Thus, within the realm of 1D QM, writing $T_1 = D - \frac{u'}{u}$, where $u$ is a true bosonic zero mode, one can infer that the adjoint operator $T_1^\dagger = -D - \frac{u'}{u}$ intertwines in the opposite direction, taking solutions of $L_1$ to those of $L_0$

$$\varphi_0 = T_1^\dagger \varphi_1.$$  \hspace{1cm} (2)

In particular, for standard 1D QM, $L_0 = H_-$ and $L_1 = H_+$ and although the true zero mode of $H_-$ is annihilated by $T_1$, the corresponding (unnormalized) eigenfunction of $H_+$ can nevertheless be obtained by applying $T_1$ to the other independent zero energy solution of $H_-$. 

1973: *Theory of generalized shift operators* (Nauka)

This is one of the remarkable books of B.M. Levitan edited by the publishing house Nauka.

1978: *Applications of a commutation formula*

P.A. Deift presents applications of the so-called “commutation formula" (that can be found, e.g., in the book of S. Sakai, *$C^*$-Algebras and $W^*$-Algebras*, Springer, 1971). In the paper of Deift in Duke Math. J. 45, 267 (1978), section 4 contains the application of the commutation formula to ordinary differential operators. Many of the results in that section have been reproduced later in SUSY QM style, e.g. by Sukumar.

1986-1987: *Isometric operators, isospectral Hamiltonians, and SUSY QM*

In Phys. Rev. D 33, 2267, D 36, 1103, D.L. Pursey uses intertwining in his combined procedures of generating families of strictly isospectral Hamiltonians, starting from the Marchenko inverse scattering equation.

1991: *Intertwining of exactly solvable Dirac eqs. with 1D potentials*

In Phys. Rev. A 43, 4602, A. Anderson applies matrix intertwining relations to the Dirac equation showing that their structure is described by an N=4 superalgebra.

1995-1998: *Intertwining widely used*
Intertwining is already well known to many active authors in SUSY QM, who are playing with higher-order generalizations. But, as always, the most important (at least for standard quantum mechanics) are the simplest cases, namely the Darboux first-order intertwining operators.

§9: Using intertwining: DTs for TDSE (Bagrov and Samsonov)

The DT for the TDSE are based on the intertwining relation

\[ T \left( i \frac{\partial}{\partial t} - H_0 \right) = \left( i \frac{\partial}{\partial t} - H_1 \right) T, \quad (1) \]

where

\[ H_i = - \frac{\partial^2}{\partial x^2} + V_i(x,t), \quad i = 0, 1, \quad (2) \]

and \( T \) is a first-order diff. transformation operator of the form

\[ T = L_1(x,t) \partial_x + L_0(x,t). \quad (3) \]

It follows immediately from the intertwining relation that if \( \psi_0 \) solves the TDSE with Hamiltonian \( H_0 \), then \( \psi_1 = T \psi_0 \) will solve the TDSE with Hamiltonian \( H_1 \). It is also easily verified that the intertwining relation will be satisfied if and only if

\[ T = L_1(\partial_x + \chi_x), \quad V_1 = V_0 + 2\chi_{xx} + i(\log L_1)_t, \quad (4) \]

where \( e^{-\chi} \) is a solution of the TDSE with potential \( V_0 \), and \( L_1 = L_1(t) \) is an arbitrary function. The transformed potential \( V_1(x,t) \) is a real-valued function if and only if

\[ \text{Im} \chi_{xxx} = 0 \]

and

\[ |L_1| = \exp \left[ -2 \int_{t_0}^{t} \text{Im} \chi_{xx}(x, s) \, ds \right]. \]

Without loss of generality, one can assume that \( L_1 \) is real and positive, and is therefore given by the right-hand side of the above equation.

Just as in the time-independent case, the DT for the TDSE can be inverted. Thus, if \( \psi_1 \) is a solution of the TDSE with potential \( V_1 \) given by Eq.(4), the function

\[ \psi_0(x,t) = \frac{e^{-\chi(x,t)}}{L_1(t)} \left[ \int_{x_0}^{x} e^{\chi(y,t)} \psi_1(y,t) \, dy + c_0(t) \right] \quad (5) \]

with \( c_0(t) \) given by

\[ c_0(t) = iL_1(t) \int_{t_0}^{t} \frac{e^{\chi(x_0,s)}}{L_1(s)} \left( \psi_{1,x}(x_0,s) - \chi_x(x_0,s) \psi_1(x_0,s) \right) \, ds \]
is a solution of the TDSE with potential $V_0$. As remarked by F. Finkel et al. in math-ph/9809013 if the factor $L_1$ is taken as unity, the mapping $\psi_1 \mapsto \psi_0$ given by Eq.(5) reduces to the non-local transformation considered by Bluman and Shtelen in J. Phys. A 29, 4473 (1996).

**CHRONOLOGICAL SET 3**

**Related Inverse Quantum Scattering (IQS) Approaches**

In the case of classical dynamics, the inverse problem just means to determine the force acting on a macroscopic body from the features of the trajectory. In this sense, one of the oldest inverse problem has been Newton’s problem of determining the force on the planets from the Kepler properties (laws) of their movement.

In the realm of differential operators, an inverse spectral method generally means to determine the (linear) operator from some given spectral data. Roughly speaking, in the case of a Sturm-Liouville operator, this would reduce to getting the potential. Thus, there is a certain parallel to the classical case. As well known, in the IQS approach one uses integral transformation operators. The key point for the construction of the integral operators are the famous Gel’fand-Levitan equation and Marchenko equation. The IQS research started in the second half of the 1940s and the 1950s have been a real boom.

**1954-1956: Krein’s approach to IQS**

M.G. Krein reports a new IQS approach in three D.A.N. notes, DAN SSSR 97, 21 (1954); 105, 433 (1955); 111, 1167 (1956).

§10: Krein’s IQS

As explained to us by Chadan and Sabatier in their book (1977, 1989), Krein’s approach is the following way to solve the $l = 0$ inverse scattering problem. The SE

$$-D^2 y + V(r)y = k^2 y ,$$

where $D = \frac{d}{dr}$, is substituted by the equivalent system

$$Dy + A(r)y = kz$$

$$-Dy + A(r)z = ky .$$

The two functions $V$ and $A$ are connected by the Riccati equation

$$V = -DA + A^2 .$$
From the analytical properties of the Schrödinger regular solution $\phi(k, r)$ and the Wiener-Paley theorem, one can obtain the following representation

$$\phi(k, r) = k^{-1} \text{Im} \left[ e^{ikr} \left( 1 + \int_0^{2r} \Gamma_{2r}(t)e^{-ikt} \, dt \right) \right],$$

where $A(r) = 2\Gamma_{2r}(2r)$. Moreover, for any fixed value of $r$, the function $\Gamma_{2r}(t)$ is a solution of the Fredholm integral equation

$$\Gamma_{2r}(t) + H(t) + \int_0^{2r} \Gamma_{2r}(s)H(s - t) \, ds = 0$$

where, $H(t)$ is related to the Jost function $F(k)$ in the following way

$$H(t) = \frac{1}{\pi} \int_0^\infty \left[ |F(k)|^{-2} - 1 \right] \cos kt \, dk$$

The scheme to solve the inverse scattering problem according to Krein is first to build the function $H(t)$, then to get $\Gamma_{2r}(t)$, leading immediately to $A(r)$, and finally obtaining the potential $V(r)$ from the Riccati equation.

1980: Changes in potentials due to changes in the point spectrum: Anharmonic oscillators with exact solutions

Using the Gel’fand-Levitan equation, P.B. Abraham and H.E. Moses introduce a procedure of deleting and adding bound states, which is “almost” equivalent to DDGR [Phys. Rev. A 22, 1333].

§11: Comparison between strictly isospectral techniques within IQS

According to A. Khare and U. Sukhatme [Phys. Rev. A 40, 6185 (1989)], the difference between DDGR, Abraham-Moses, and Pursey schemes when used in the IQS context to generate one-parameter strictly isospectral potentials on the base of an initial zero mode $\psi_0$ lies merely in the employed function used for the (inverse) steps of readding the zero-mode as follows ($I = \int_x^{-\infty} \psi_0^2(y) \, dy$):

For DDGR, the function is $\phi = \psi_0^{-1}$, being a node-free solution of the first step transformed equation.

For Pursey’s method, it is $v = \psi_0/I$, such that $v$ is a solution of the first step zero-energy SE that vanishes at $+\infty$.

For the Moses-Abraham procedure, one should use $u = \psi_0/(1 - I)$, where $u$ is a solution of the first step zero-energy SE vanishing at $-\infty$.

One can obtain five independent families of strictly isospectral potentials when combining the three procedures. However, only the DDGR method leads to
reflection and transmission amplitudes identical to those of the original potential, showing the complete degeneracy produced by such a construction.

1985: First issue of Inverse Problems

Birth year of the IOP review Inverse Problems.

There are many good books in the IQS field, e.g., those of K. Chadan and P.C. Sabatier (1977, 1989), B.M. Levitan (1984), V.A. Marchenko (1986), B.N. Zakhariev and A.A. Suzko (1985). According to Levitan, the generalized use of intertwining operators in IQS is due to the works of V.A. Marchenko.

1985: SUSY QM and the Inverse Scattering Method

In J. Phys. A 18, 2937, C.V. Sukumar provides an important contribution to the understanding of the connections between the two approaches.

1993: Bound states in the continuum (BSICs) from SUSY QM

In Phys. Rev. A 48, 3525, J. Pappademos, U. Sukhatme, A. Pagnamenta are the first to apply DDGR to get BSICs. They also study two-parameter families of SUSY BSIC potentials.

1995: SUSY transformations of real potentials on the line

In J. Phys. A 28, 5079, J.-M. Sparenberg and D. Baye present an exhaustive work on SUSY iterations showing the power of the method in generating isospectral potentials.

1994: Exactly solvable models for the SE from generalized DTs

In J. Phys. A 27, 2605, W.A. Schnizer and H. Leeb show in the IQS context that integral transformations with a degenerate kernel are equivalent to differential ones.

1995: On the equivalence of the integral and the differential exact solution generation methods for the 1D SE

In J. Phys. A 28, 6989, B. Samsonov shows the same equivalence in a very transparent way, giving an integral form of Crum’s iteration.
Conclusion

A scientific review covering 116 years, even when the chronological style is used, cannot have any claim of completeness and people reading it may see a lot of missing material in general due to the natural biases acting on the author(s). In my case, a strong bias has been a dead line forcing me to leave out many active authors and their results in a number of important topics not quoted above, such as the connections between partially algebraic quantum problems (quasi-exactly solvable ones) and SUSY QM, or intricate connections going directly to the core of more sophisticated theories. Nevertheless, even from this brief chronological collection one can get a partial grasp of what has been done and a feeling of what could be done in the future.

Acknowledgment

This work has been partially supported by CONACyT (Mexico) Project No. 458100-5-25844E. The author wishes to thank L.J. Boya and B. Mielnik for remarks.