THE GEOMETRY OF THE PERMUTOHEDRAL VARIETY AS A HESSENBERG VARIETY

JAN-LI LIN

ABSTRACT. We construct a concrete isomorphism from the permutohedral variety to the regular semisimple Hessenberg variety associated to the Hessenberg function $h_+(i) = i + 1$, $1 \leq i \leq n - 1$. As applications, we first identify torus orbit closures of permutohedral variety as components of disconnected Hessenberg varieties. Using the isomorphisms we constructed, we are also able to describe the geometric structure of regular semisimple Hessenberg varieties associated to the Hessenberg function represented by $h_k = (2, 3, \cdots, k + 1, n, \cdots, n)$ for some $k \leq n - 3$.

1. Introduction

There are several different ways to describe the permutohedral variety $X$ of dimension $n - 1$, for instances:

(1) It is the toric variety associated to the normal fan of the permutohedron.
(2) It is the graph of the Cremona involution

$$J : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$$

$$[x_0 : \cdots : x_{n-1}] \mapsto [x_0^{-1} : \cdots : x_{n-1}^{-1}]$$

(3) It is an iterated blowup of $\mathbb{P}^{n-1}$ at all (strict transforms of) coordinate linear subspaces in a certain order. (see Section 2 for detail.)
(4) It is the regular semisimple Hessenberg variety $Hess(S, h_+)$ associated to the Hessenberg function $h_+$ defined by $h_+(i) = i + 1$ for $1 \leq i \leq n - 1$ and $h_+(n) = n$.

The isomorphisms among the first three descriptions are well-known. On the other hand, to the author’s knowledge, the isomorphism between the permutohedral variety and the Hessenberg variety is proved by general theory of toric varieties [DMPS92, Lemma 10 and Theorem 11]. The first goal of this article is to construct a concrete isomorphism from the iterated blowups on $\mathbb{P}^{n-1}$ to the Hessenberg variety. In the process, we further obtain isomorphisms from each step of the iterated blowups in (3) to a “Hessenberg type” subvariety of a partial flag variety, which we denote $Hess^{(k)}(S, h_+)$ (for details, see Section 3). Then, we apply the theory of toric varieties to explore certain geometric structure of the Hessenberg variety.

Since the relation between the cohomology of the base manifold and the blowup manifold is well known, the cohomology of $Hess^{(k)}(S, h_+)$ can be explicitly described. Moreover, there is a $\mathfrak{S}_n$-action on the cohomology of $Hess^{(k)}(S, h_+)$ for each $k = 1, \cdots, n - 1$ which is compatible with the dot action on the Hessenberg variety $Hess(S, h_+)$ defined by Tymoczko [Tym08]. In fact, the $\mathfrak{S}_n$-actions on cohomologies are induced from an $\mathfrak{S}_n$-action on the variety $Hess^{(k)}(S, h_+)$ itself, and we can see the action concretely by viewing $Hess^{(k)}(S, h_+)$ as an iterated blowup of $\mathbb{P}^{n-1}$. As a consequence, we can construct a permutation basis for the dot action on the cohomology groups.
recursively. The question of finding a permutation basis for the dot action on the cohomology of Hessenberg varieties has drawn people’s attention recently [Ailm19, CHL, Cho, HPT] because of its relation with the Stanley-Stembridge conjecture. The relation was observed by Shareshian and Wachs [SW16]. Thus also announced an important conjecture that was later proved by Brosnan and Chow [BC18], and independently by Guay-Paquet [GP]. In the case of the permutohedral variety, the Stanley-Stembridge conjecture is known to be true, and a permutation basis is also known. The known basis conjectured by Chow [Cho] and proved by Cho, Hong, and Lee [CHL] is based on the GKM theory on equivariant cohomology, then pass it on to the usual cohomology. The basis constructed in this paper is based on the geometry of $H^\text{ess}(k)(S, h_+)$ and is for the usual cohomology. It might be interesting to compare these two kinds of basis.

Finally, we use the intermediate isomorphisms at each step of the blowups to investigate semisimple Hessenberg varieties $H^\text{ess}(S, h_k)$ associated to the Hessenberg function $h_k = (2, 3, \cdots , k + 1, n, \cdots , n)$, $1 \leq k \leq n - 3$. We observe that $H^\text{ess}(S, h_k)$ has a fiber bundle structure over $H^\text{ess}(k)(S, h_+)$ with fibers isomorphic to the flag variety $\text{Flag}(\mathbb{C}^{n-k})$. We use this structure to obtain a description of the cohomologies of $H^\text{ess}(S, h_k)$, and finally we post a question about the dot action the cohomologies.

2. THE PERMUTOHEDRAL VARIETY AS ITERATED BLOWUPS OF $\mathbb{P}^{n-1}$

One can obtain the permutohedral variety by performing a sequence of blowups on $\mathbb{P}^{n-1}$, as follows.

1. First, we blowup the $n$ points $Z_1 = [1 : 0 : \cdots : 0], Z_2 = [0 : 1 : 0 \cdots : 0], \cdots , Z_n = [0 : \cdots : 0 : n]$. We denote the resulting variety and the projection map by $\pi_1 : X_1 \rightarrow \mathbb{P}^{n-1} := X_0$. We also have the exceptional divisors $E_i = \pi_1^{-1}(Z_i) \subset X_1$.

2. Next, we blowup the strict transforms (in $X_1$) of all the lines $Z_{i,j} \subset \mathbb{P}^{n-1}$ connecting $Z_i$ and $Z_j$ for $1 \leq i < j \leq n$. This gives us the second level space $\pi_2 : X_2 \rightarrow X_1$ and the exceptional divisors $E_{i,j} \subset X_2$. Notice that, although the lines $Z_{i,j}$ and $Z_{i,k}$ intersect at $Z_i$ in $\mathbb{P}^{n-1}$, the blowups in step 1 would separate their strict transforms. Therefore, the resulting space $X_2$ is independent of the order of blowups.

3. We repeat the above process until we reach codimension 2. More precisely, in the $k$-th step ($1 \leq k \leq n - 2$) we do the following. For $\alpha \subset [n]$ a subset of $k$ elements, let $Z_\alpha$ denote the linear subvariety of $\mathbb{P}^{n-1}$ generated by $\{Z_i | i \in \alpha\}$, and $\overline{Z}_\alpha$ the strict transform of $Z_\alpha$ in $X_{k-1}$. The blowups in the previous steps have the effect of blowing up all coordinate linear subspaces on $Z_\alpha$, thus $\overline{Z}_\alpha$ is a permutohedral variety of dimension $k - 1$. We blow up all the $\overline{Z}_\alpha$ in this step. The $\overline{Z}_\alpha$’s intersect with each other along coordinate subspaces of lower dimensions, hence are separated by previous blowups. Thus, the order to perform blowups in this step does not matter. This produces the space $X_k$, the map $\pi_k : X_k \rightarrow X_{k-1}$, as well as the exceptional divisors $E_\alpha \subset X_k$. The end result is a sequence of spaces and projection maps:

$$X_{n-2} \xrightarrow{\pi_{n-2}} X_{n-3} \xrightarrow{\pi_{n-3}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^{n-1}$$

The variety $X = X_{n-2}$ is the permutohedral variety. We call the intermediate variety $X_k$ ($1 \leq k \leq n - 3$) the pre-permutohedral variety of order $k$. It will be identified with certain Hessenberg style variety in a partial flag variety.
3. Permutohedral varieties as Hessenberg varieties

We consider regular semisimple Hessenberg varieties of type A. To set the notations, let $S$ be an $n \times n$ complex diagonal matrix with different diagonal entries $s_1, \ldots, s_n$ and $h : [n] \to [n]$ be a Hessenberg function, i.e., $h(i) \geq h(j)$ for all $n \geq i > j \geq 1$ ($h$ is non-decreasing) and $h(i) \geq i$ for $i = 1, \ldots, n$.

\[ \text{Hess}(S, h) = \{ \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid S(V_i) \subset V_{h(i)} \text{ for } i = 1, \ldots, n \} . \]

We also use the list notation $h = (h(1), \ldots, h(n))$ to denote a Hessenberg function. In this section, we focus on the specific Hessenberg function $h_+ = (2, 3, \ldots, n-1, n, n)$, i.e. $h_+(i) = \min(i+1, n)$ for $1 \leq i \leq n$. We start with an observation in linear algebra.

**Lemma 3.1.** let $S$ be an $n \times n$ complex diagonal matrix with different diagonal entries $s_1, \ldots, s_n$ and $v = (x_1, \ldots, x_n) \in \mathbb{C}^n$ be a (column) vector. If at least $k$ of the $x_i$’s are nonzero (i.e. there are no more than $n-k$ of the $x_i = 0$), then the vectors

\[ v, Sv, \ldots, S^{k-1}v \]

are linearly independent. If exactly $k$ of the $x_i$’s are nonzero, then the vectors

\[ v, Sv, \ldots, S^k v \]

are linearly dependent.

**Proof.** Assuming $x_i \neq 0$, then a linear relation

\[ a_0v + a_1Sv + \cdots + a_{k-1}S^{k-1}v = 0 \]

implies the relation (on the first coordinate)

\[ a_0 + a_1 s_i + \cdots + a_{k-1} s_i^{k-1} = 0. \]

That is, $s_i$ is a root of the polynomial $\sum_{i=0}^{k-1} a_i x^i$. The first part of the lemma is then a consequence of the fact that a polynomial equation of degree $k-1$ cannot have $k$ or more distinct roots.

For the second part of the lemma, assume that $x_1, \ldots, x_k$ are the non-zero $x_i$’s. Then the coefficients of the polynomial $(x-s_1) \cdots (x-s_k)$ give a non-trivial linear relation among $v, Sv, \ldots, S^k v$.

In particular, if $0 = (0, \ldots, 0)$ denotes the origin, and $v \in \mathbb{C}^n \setminus \{0\}$, then $v$ gives rise to a point $[v] \in \mathbb{P}^{n-1}$. We further denote

\[ \text{Ind} := \bigcup_{1 \leq i < j \leq n} (x_i = x_j = 0). \]

(The set $\text{Ind}$ is exactly the indeterminate set of the Cremona involution $J$ defined in the introduction.) If $v \in \mathbb{P}^{n-1} \setminus \text{Ind}$, then at least $n-1$ of the $x_i$ are nonzero. Thus, by the lemma, the vectors

\[ v, Sv, \ldots, S^{n-1}v \]

are linearly independent, and the following is a well-defined flag (i.e. the dimension of the vector spaces are correct).

\[ V_* = (\langle 0 \rangle \subset \langle v \rangle \subset \langle v, Sv \rangle \subset \cdots \subset \langle v, Sv, \ldots S^{n-2}v \rangle \subset \mathbb{C}^n). \]
Moreover, it is obvious that $V_\bullet \in \mathcal{Hess}(S, h_+)$. Conversely, suppose that $V_\bullet = (\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n)$ is a flag in $\mathcal{Hess}(S, h_+)$, and the one-dimensional space $V_1$, as an element of $\mathbb{P}^{n-1}$, satisfies $V_1 \not\in \text{Ind}$, then $V_\bullet$ must be in the form
\begin{equation*}
V_\bullet = (\{0\} \subset \langle v \rangle \subset \langle v, Sv \rangle \subset \cdots \subset \langle v, Sv, \cdots S^{n-2}v \rangle \subset \mathbb{C}^n)
\end{equation*}
for any nonzero $v \in V_1$. This defines an isomorphism
\begin{equation*}
\mathbb{P}^{n-1} \setminus \text{Ind} \longrightarrow \mathcal{U} \subset \mathcal{Hess}(S, h_+)
\end{equation*}
\begin{equation*}
v \longmapsto (\{0\} \subset \langle v \rangle \subset \langle v, Sv \rangle \subset \cdots \subset \langle v, Sv, \cdots S^{n-2}v \rangle \subset \mathbb{C}^n),
\end{equation*}
where $\mathcal{U}$ is the open subset of $\mathcal{Hess}(S, h_+)$ consists of all flags $V_\bullet = (\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n)$ such that $V_1 \not\in \text{Ind}$. We will extend this isomorphism to isomorphisms between blowups of $\mathbb{P}^{n-1}$ and Hessenberg type varieties defined in the next paragraph.

In order to do so, we introduce a type of partial flag variety, which we call the $k$-th order partial flag variety, for $1 \leq k \leq n - 1$,
\begin{equation*}
\mathcal{Flag}^{(k)}(\mathbb{C}^n) := \{V_\bullet = (V_0 := \{0\} \subset V_1 \subset \cdots \subset V_k) \mid \dim_{\mathbb{C}}(V_i) = i \text{ for } i = 1, \cdots, k\}.
\end{equation*}
We also define the corresponding $k$-th order Hessenberg type variety
\begin{equation*}
\mathcal{Hess}^{(k)}(S, h_+) := \left\{V_\bullet \in \mathcal{Flag}^{(k)}(\mathbb{C}^n) \mid SV_i \subset V_{i+1} \text{ for } i = 0, \cdots, k-1\right\}.
\end{equation*}
Notice that $\mathcal{Flag}^{(n-1)}(\mathbb{C}^n) = \mathcal{Flag}(\mathbb{C}^n)$ and $\mathcal{Hess}^{(n-1)}(S, h_+) = \mathcal{Hess}(S, h_+)$. The main goal of this section is to show the following.

**Proposition 3.2.** There is a natural isomorphism
\begin{equation*}
\mathcal{X}_{k-1} \overset{\cong}{\longrightarrow} \mathcal{Hess}^{(k)}(S, h_+)
\end{equation*}
for $k = 1, \cdots, n - 1$, where $\mathcal{X}_{k-1}$ ($2 \leq k \leq n - 2$) is the pre-permutohedral variety of order $k-1$.

**Proof.** It is clear that $\mathcal{X}_0 := \mathbb{P}^{n-1} \cong \mathcal{Hess}^{(1)}(S, h_+)$. For $V_\bullet = (V_0 := \{0\} \subset V_1 \subset V_2) \in \mathcal{Hess}^{(2)}(S, h_+)$, suppose that $0 \neq v = (x_1, \cdots, x_n) \in V_1$. By lemma 3.3 if at least two of the $x_i$’s are non-zero, then $v$ and $Sv$ are linearly independent, and $V_2 = \langle v, Sv \rangle$ is determined. If only one of the $x_i \neq 0$, then $V_1 = Z_i$ (see Section 2 for the definition of $Z_i$) for some $i = 1, \cdots, n$ and thus $SV_1 = V_1$. To determine $V_2$, we need an extra piece of information, which is the direction that give us the second dimension of $V_2$. This can be specified as a point in $\mathbb{P}(\mathbb{C}^n/V_1)$, which is canonically isomorphic to the exceptional divisor $E_i$ when we blowup $\mathbb{P}^{n-1}$ at $Z_i$. Therefore, after we blowup all the $Z_i$’s, then for those $V_1$ such that $SV_1 = V_1$, we also know what $V_2$ is. This gives the isomorphism $\mathcal{X}_1 \to \mathcal{Hess}^{(2)}(S, h_+)$.

We continue the blowup process inductively as follows. First, we introduce the “forgetful” morphism
\begin{equation*}
f^{(k)} : \mathcal{Hess}^{(k)}(S, h_+) \to \mathcal{Hess}^{(k-1)}(S, h_+) \text{ for } k = 2, \cdots, n - 2,
\end{equation*}
which sends a flag of $k$ vector spaces to the first $k - 1$ of them, and “forgets” the last vector space.

**Claim.** The morphism $f^{(k)}$ is a birational map.

**Proof.** Given $V_\bullet = (V_0 := \{0\} \subset V_1 \subset \cdots \subset V_{k-1}) \in \mathcal{Hess}^{(k-1)}(S, h_+)$, since $SV_{k-2} \subset V_{k-1}$, we know from linear algebra that $\dim_{\mathbb{C}}(\langle V_{k-1} \cup SV_{k-1} \rangle) = k$ or $k-1$. Moreover, $\dim_{\mathbb{C}}(\langle V_{k-1} \cup SV_{k-1} \rangle) = \cdots}$
where $v$ corresponds to the flag $s$ of the torus orbit closure. The standard reference for this representative, say $v$, is the generic situation in $kH^{(k-1)}(S, h_+)$, and a reflection of Lemma 3.1 tells us that, via the isomorphism $X_{k-2} \cong H^{(k-1)}(S, h_+)$, $V_v$ lies in the strict transform of $Z_\alpha$ for some $\alpha \in [n]$ with $k - 1$ elements.

If we blowup $X_{k-2}$ along the strict transform of $Z_\alpha$, then the exceptional divisor $E_\alpha$ is canonically identified with the projective normal bundle $\mathbb{P}(NZ_\alpha \subset X_{k-2})$. A point on $E_\alpha$ carries the information of $V_v$, together with a (projective) normal direction of $Z_\alpha$ in $\mathbb{C}^n$, i.e. an element in $\mathbb{P}(\mathbb{C}^n/Z_\alpha)$. This assigns a unique flag in $H^{(k)}(S, h_+)$.

If $\dim C(V_{k-1} \cup SV_{k-1}) = k - 1$, then $SV_{k-1} = V_{k-1}$, and $h_+$, and a reflection of Lemma 3.1 tells us that, via the isomorphism $X_{k-2} \cong H^{(k-1)}(S, h_+)$, $V_v$ lies in the strict transform of $Z_\alpha$ for some $\alpha \in [n]$ with $k - 1$ elements.

Suppose we have $v_1 = (1, 1, 0, 0, 0) \in \mathbb{C}^5$, then $V_1 = \langle v_1 \rangle \subset V_2 = \langle v_1, Sv_1 \rangle$ form the first two spaces in the flag but $SV_2 = V_2$. Thus we need to blowup $Z_{\{1,2\}}$. Notice that $V_2 = Z_{\{1,2\}} \subset \mathbb{C}^5$ and elements in $E_{\{1,2\}}$ over $v_1$ are in one-to-one correspondence to projectivized normal vectors for the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{C}^5$ at $v_1$, i.e., $\mathbb{P}(\mathbb{C}^5/\mathbb{C}^2)$. Suppose we pick $v_2 = (1, 1, 1, 0, 0)$ as a representative for an element in $\mathbb{P}(\mathbb{C}^5/\mathbb{C}^2)$, then that gives us $V_3 = \langle v_2, v_2 \rangle$ (one can check that $V_3$ is independent of the choice of $v_2$), but then we would have $SV_3 = V_3$ again. This means that the element represented by $v_2$ in $E_{\{1,2\}}$ is in the strict transform of the set $Z_{\{1,2,3\}}$. When we blowup $Z_{\{1,2,3\}}$, elements of $E_{\{1,2,3\}}$ over $v_2$ will then correspond to $\mathbb{P}(\mathbb{C}^5/\mathbb{C}^3)$. Picking a representative, say $v_3 = (1, 1, 1, 1)$, we will have $V_4 = \langle v_3, v_3 \rangle$ and $V_5 = \mathbb{C}^5$. This gives a flag $V_v = \langle 0 \rangle \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 \subset H^{(k)}(S, h_+)$. 

4. PERMUTOHEDRAL VARIETIES AS TORIC VARIETIES

From the isomorphism constructed in the previous section, we can see the toric structure of the Hessenberg variety. In particular, the torus $(\mathbb{C}^*)^{n-1}$ sits inside $H^{(k)}(S, h_+)$ as follows. The point $z = (z_1, \cdots, z_{n-1}) \in (\mathbb{C}^*)^{n-1}$ corresponds to the flag

$$V_v := \langle \langle 0 \rangle \subset \langle v \rangle \subset \langle v, Sv \rangle \subset \cdots \subset \langle v, Sv, \cdots S^{n-2}v \rangle \subset \mathbb{C}^n \rangle$$

where $v = (1, z_1, \cdots, z_{n-1})$. We denote this correspondence as an injective map $\phi : (\mathbb{C}^*)^{n-1} \hookrightarrow H^{(k)}(S, h_+)$ by $\phi(z) = V_v$.

One can also observe the algebraic group structure of $(\mathbb{C}^*)^{n-1} \subset H^{(k)}(S, h_+)$ as follows. Given $(z_1', \cdots, z_{n-1}') \in (\mathbb{C}^*)^{n-1}$ and $v' = (1, z_1', \cdots, z_{n-1}')$, the product of $(z_1, \cdots, z_{n-1})$ and $(z_1', \cdots, z_{n-1}')$ corresponds to the flag

$$\langle \langle 0 \rangle \subset \langle vv' \rangle \subset \langle vv', Sv(vv') \rangle \subset \cdots \subset \langle vv', Sv(vv'), \cdots S^{n-2}(vv') \rangle \subset \mathbb{C}^n \rangle,$$

where $vv'$ is the coordinate-wise product of $v$ and $v'$. That is, $\phi(z)\phi(z') := \phi(zz')$. Moreover, one can discover the action of $(\mathbb{C}^*)^{n-1}$ on $H^{(k)}(S, h_+)$ similarly.

Given that $H^{(k)}(S, h_+)$ is a toric variety, we would like to know the structure of the corresponding fan, and the geometry of each of the torus orbit closure. The standard reference for this part is [PL93]. First, recall the structure of the fan $\Delta_{p^{n-1}}$ corresponding to $\mathbb{P}^{n-1}$. Let $e_1, \cdots, e_{n-1}$
be the standard basis of $\mathbb{Z}^{n-1}$ and $e_n = -(e_1 + \cdots + e_n)$. The cones in $\Delta_{\mathbb{P}^{n-1}}$ are generated by proper subsets of $\{e_1, \ldots, e_n\}$. More precisely, for any proper subset $\alpha \subset [n]$, we denote $\sigma_\alpha$ to be the cone generated by $\{e_i | i \in \alpha\}$, i.e., $\sigma_\alpha = \{\sum_{i \in \alpha} a_i e_i | a_i \geq 0\}$. Then

$$\Delta_{\mathbb{P}^{n-1}} = \{\sigma_\alpha | \alpha \subset [n]\},$$

with the convention that $\sigma_\emptyset = \{0\}$.

As we showed earlier, the permutohedron variety $X$ can be obtained from $\mathbb{P}^{n-1}$ by blowing up all the torus invariant subvarieties of codimensions $\geq 2$. For the fan structure, this means we perform the barycentric subdivisions on all the top dimensional cones, i.e., all those $\sigma_\alpha$ with $|\alpha| = n-1$. We will describe the fan $\Delta_X$ corresponds to the permutohedral variety $X$ after we set up some notations.

For a non-empty proper subset $\alpha \subset [n]$, we define the vector $e_\alpha = \sum_{i \in \alpha} e_i$. The rays (i.e. one-dimensional cones) of $\Delta_X$ are generated by $e_\alpha$ for non-empty proper subset $\alpha \subset [n]$. A chain of proper subsets of $[n]$ is a sequence of strict inclusion

$$\mathcal{C} = (\alpha_0 := \emptyset \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_d)$$

of proper subsets of $[n]$. We call the number $d$ the length of $\mathcal{C}$, and denote $\ell(\mathcal{C}) = d$. For a chain $\mathcal{C}$ of proper subsets of $[n]$ of length $d$, we define $\sigma_\mathcal{C}$ to be the cone generated by $e_{\alpha_1}, \ldots, e_{\alpha_d}$. Thus the length of a chain $\mathcal{C}$ is equal to the dimension of the cone $\sigma_\mathcal{C}$. For $d = 1, \ldots, n-1$, the set $\Delta_X(d)$ of $d$-dimensional cones in $\Delta_X$ constitutes all the cones $\sigma_\mathcal{C}$ where $\mathcal{C}$ is a chain of proper subsets of $[n]$ of length $d$. With the convention $\Delta_X(0) = \{\{0\}\}$, we have a description of the fan

$$\Delta_X = \bigcup_{d=0}^{n-1} \Delta_X(d).$$

For each cone $\sigma \in \Delta_X$, there is an associated torus-invariant subvariety $V(\sigma) \subset X$, called the orbit closure associate to $\sigma$ (cf. [Ful93, p.51]). Two kinds of orbit closures are of particular interest:

- For $\ell(\mathcal{C}) = n-1$, chains of maximal length correspond to permutations of the set $[n]$. The orbit closure $V(\mathcal{C})$ is a fixed point under the torus action.
- For $\ell(\mathcal{C}) = 1$, chains of subsets of length one are of the form $\mathcal{C} = (\emptyset \subset \alpha_1)$. Assigning $\mathcal{C} \mapsto \alpha_1$ gives a one-to-one correspondence between chains of length one and nontrivial cones of $\Delta_{\mathbb{P}^{n-1}}$. There are two cases:
  1. For $|\alpha_1| \geq 2$, the orbit closure $V(\sigma_\mathcal{C}) \subset X$ is then the exceptional divisor when we blow up the (strict transform of the) orbit closure $V(\sigma_{\alpha_1})$ of $\mathbb{P}^{n-1}$.
  2. For $|\alpha_1| = 1$, $V(\sigma_\mathcal{C})$ is the strict transform of the divisor $V(\sigma_{\alpha_1}) \subset \mathbb{P}^{n-1}$.

4.1. Torus invariant subvarieties. For each cone $\sigma \in \Delta_X$, the associated orbit closure $V(\sigma)$ is a connected component of a disconnected Hessenberg variety. Conversely, for a Hessenberg function $h \leq h_+$ (i.e. $h(j) \leq h_+(j)$ for $j = 1, \ldots, n$), the corresponding Hessenberg variety $\mathcal{Hess}(S, h)$ is a union of orbit closures of $X \cong \mathcal{Hess}(S, h_+)$. We classify these subvarieties in this subsection.

**Proposition 4.1.** Let $\mathcal{C} = (\alpha_0 := \emptyset \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_d)$ be a chain of proper subsets of $[n]$, then $V(\sigma_\mathcal{C})$ is a connected component of the Hessenberg variety $\mathcal{Hess}(S, h)$ for the Hessenberg function

$$h(j) = \begin{cases} j & \text{if } j = n - |\alpha_i| \text{ for } i = 0, \cdots, d, \\
 j + 1 & \text{otherwise.} \end{cases}$$

\(^1\)Strictly speaking, it is not a variety in this case, but we will not bother give it another name.
Theorem 5.1. [Voi07, Theorem 7.31] Let $h = c_1(\mathcal{O}_E(1)) \in H^2(E; \mathbb{Z})$, we have isomorphism

$$H^p(X; \mathbb{Z}) \oplus \left( \bigoplus_{i=0}^{r-2} H^{p-2i-2}(Z; \mathbb{Z}) \right) \xrightarrow{\tau^* + \sum_j j_* (\cup h^i) \circ \tau^*_Z} H^p(\tilde{X}_Z; \mathbb{Z}).$$

The map on the second component is decomposed as follows.

$$H^{p-2i-2}(Z; \mathbb{Z}) \xrightarrow{\tau^*_Z} H^{p-2i-2}(E; \mathbb{Z}) \xrightarrow{\cup h^i} H^{p-2}(E; \mathbb{Z}) \xrightarrow{j_*} H^p(\tilde{X}_Z; \mathbb{Z}).$$

Here $j_*$ is the Gysin morphism which is defined as the Poincare dual of the map

$$j_* : H_{2n-p}(E; \mathbb{Z}) \to H_{2n-p}(\tilde{X}_Z; \mathbb{Z}).$$

Some remarks:

- The theorem is also valid for cohomology groups with $\mathbb{C}$ coefficients. In this paper, we always use complex coefficients, thus we will write $H^p(X)$ for $H^p(X; \mathbb{C})$ from now on.
• We denote \( H^*(X) = \oplus_p H^p(X) \) as a graded complex vector space. Moreover, define \( H^*_0(X) := H^{p+q}(X) \), i.e. the degree \( p \) part of \( H^*_0(X) \) is equal to the degree \( p+q \) part of \( H^*(X) \). The theorem can be written in term of the graded \( \mathbb{C} \) vector spaces as

\[
H^*(\tilde{X}_Z) \cong H^*(X) \oplus \left( \bigoplus_{i=1}^{r-1} H^*_{2i}(Z) \right).
\]

In the process of obtaining \( \tilde{X}_k \), we blowup strict transforms of coordinate linear subvarieties \( Z_\alpha \). The strict transforms \( \tilde{Z}_\alpha \) are themselves permutohedral varieties of lower dimension. We will use \( \tilde{X}^d \) to denote the permutohedral variety of dimension \( d \).

**Theorem 5.2.** The cohomology, as a graded complex vector space, of the pre-permutohedral variety \( \tilde{X}^{n-1}_k, 1 \leq k \leq n-2 \), is given by

\[
H^*(\tilde{X}^{n-1}_k) \cong H^*(\mathbb{P}^{n-1}) \oplus \left( \bigoplus_{j=1}^{k} \bigoplus_{\alpha \subset [n], |\alpha| = j} H^*_{2\ell}(\tilde{X}^{j-1}_\alpha) \right).
\]

**Proof.** We apply Theorem 5.1 with our setting:

- \( \tilde{X}^{n-1}_0 = \mathbb{P}^{n-1} \).
- For \( 1 \leq k \leq n-2 \), \( \tilde{X}^{n-1}_k \) is the blowup of \( \tilde{X}^{n-1}_{k-1} \) along all \( \tilde{Z}_\alpha \) with \( \alpha \subset [n] \) and \( |\alpha| = k \).
- \( \tilde{Z}_\alpha \) is isomorphic to \( \tilde{X}^{k-1} \), thus the codimension of \( \tilde{Z}_\alpha \) is \( n-k \).

This gives us

\[
H^*(\tilde{X}^{n-1}_k) \cong H^*(\tilde{X}^{n-1}_{k-1}) \oplus \left( \bigoplus_{\alpha \subset [n], |\alpha| = k} H^*_{2\ell}(\tilde{Z}_\alpha) \bigoplus \bigoplus_{|\alpha| = k} H^*_{2\ell}(\tilde{X}^{k-1}_\alpha) \right).
\]

The isomorphism in the theorem can be obtained by applying the above isomorphism inductively on \( k \).

In the following, we compute some examples in low dimensions.

**Example 5.1.** For the base case, \( H^*(\tilde{X}^{n-1}_0) \cong H^*(\mathbb{P}^{n-1}) \cong \mathbb{C}[\xi]/(\xi^n) \) where \( \xi \in H^2(\mathbb{P}^{n-1}) \) is the first Chern class of the hyperplane bundle. Next, let us consider \( \tilde{X}^{n-1}_1 \). This is the space obtained in the first step of the blowups. Here, we blowup the points \( Z_1, \ldots, Z_n \). The cohomology groups become

\[
\begin{align*}
H^0(\tilde{X}^{n-1}_1) &\cong H^0(\mathbb{P}^{n-1}) \cong \mathbb{C} \\
H^{2j}(\tilde{X}^{n-1}_1) &\cong H^0(\mathbb{P}^{n-1}) \oplus \bigoplus_{i=1}^{n} H^0(Z_i) \cong \mathbb{C} \oplus \mathbb{C}^n, \ 1 \leq j \leq n-2 \\
H^{2n-2}(\tilde{X}^{n-1}_1) &\cong H^{2n-2}(\mathbb{P}^{n-1}) \cong \mathbb{C}.
\end{align*}
\]

In particular, if \( n = 3 \), then \( \tilde{X}^2_1 \) is the blowup of \( \mathbb{P}^2 \) at the three points \([1 : 0 : 0], [0 : 1 : 0] \) and \([0 : 0 : 1] \). \( H^0(\tilde{X}^2_1) \cong \mathbb{C}; H^1(\tilde{X}^2_1) \cong \mathbb{C} \oplus \mathbb{C}^2 \) and \( H^2(\tilde{X}^2_1) \cong \mathbb{C} \).
Example 5.2. We investigate the cohomology of $\mathcal{X}_2^{n-1}$ next. In order to obtain the space, we blow up all the lines $Z_{\{i,j\}}$ where \{i, j\} $\subset$ [n] and $i \neq j$. Notice $Z_{\{i,j\}}$ is isomorphic to its strict transform $\overline{Z}_{\{i,j\}}$ since its dimension is one. We have $H^0(Z_{\{i,j\}})$ $\cong$ $\mathbb{C}$ and $H^2(Z_{\{i,j\}})$ $\cong$ $\mathbb{C}$. Therefore, by Theorem 5.2, we have

\[ H^0(\mathcal{X}_2^{n-1}) \cong H^0(\mathbb{P}^{n-1}) \cong \mathbb{C} \]
\[ H^2(\mathcal{X}_2^{n-1}) \cong H^0(\mathbb{P}^{n-1}) \bigoplus \bigoplus_{i\neq j, i,j\in[n]} H^0(Z_{\{i,j\}}) \cong \mathbb{C} \oplus \mathbb{C}^{(n \choose 2)} \]
\[ H^{2i}(\mathcal{X}_2^{n-1}) \cong H^0(\mathbb{P}^{n-1}) \bigoplus \bigoplus_{i=1}^{n} H^0(Z_i) \bigoplus \bigoplus_{i,j\in[n], \ i\neq j} (H^0(Z_{\{i,j\}})) \cong \mathbb{C} \oplus \mathbb{C}^{(n \choose 2)} \]
\[ H^{2n-4}(\mathcal{X}_2^{n-1}) \cong H^0(\mathbb{P}^{n-1}) \bigoplus \bigoplus_{i,j\in[n], \ i\neq j} H^2(Z_{\{i,j\}}) \cong \mathbb{C} \oplus \mathbb{C}^{(n \choose 2)} \]
\[ H^{2n-2}(\mathcal{X}_2^{n-1}) \cong H^{2n-2}(\mathbb{P}^{n-1}) \cong \mathbb{C}. \]

For $n = 4$, we can read the cohomologies of $\mathcal{X}_k^3$, $k = 0, 1, 2$, from the following table:

| $\mathcal{X}_0^3$ | $\mathcal{X}_1^3$ | $\mathcal{X}_2^3$ |
|-------------------|-------------------|-------------------|
| $H^0(\mathcal{X}_0^3)$ $\cong$ $H^0(\mathbb{P}^3)$ |                      |                      |
| $H^2(\mathcal{X}_0^3)$ $\cong$ $H^2(\mathbb{P}^3)$ $\oplus_{j=1}^3 H^0(Z_j)$ | $\oplus_{i\neq j, i,j\in[n]} H^0(Z_{\{i,j\}})$ |                      |
| $H^4(\mathcal{X}_0^3)$ $\cong$ $H^4(\mathbb{P}^3)$ $\oplus_{j=1}^3 H^0(Z_j)$ | $\oplus_{i\neq j, i,j\in[n]} H^2(Z_{\{i,j\}})$ |                      |
| $H^6(\mathcal{X}_0^3)$ $\cong$ $H^6(\mathbb{P}^3)$ |                      |                      |

Entries in the table record the direct summands needed to obtain the cohomology of the corresponding spaces. For example, one can read from the table that $H^2(\mathcal{X}_3^3) \cong H^2(\mathbb{P}^3) \oplus_{j=1}^3 H^0(Z_j)$ and $H^4(\mathcal{X}_2^3) \cong H^4(\mathbb{P}^3) \oplus_{j=1}^3 H^0(Z_j) \oplus_{i\neq j, i,j\in[n]} H^2(Z_{\{i,j\}})$. Notice that $Z_j$ here are points and $Z_{\{i,j\}}$ are lines.
Example 5.3. For $n = 5$, the table for cohomologies is in the following.

| $\mathcal{X}_n^1$ | $\mathcal{X}_n^2$ | $\mathcal{X}_n^3$ | $\mathcal{X}_n^4$ |
|-------------------|-------------------|-------------------|-------------------|
| $H^0(\mathcal{X}_n^1) \cong H^0(\mathbb{P}^4)$ | $H^0(\mathcal{X}_n^2) \cong H^0(\mathbb{P}^4)$ | $H^0(\mathcal{X}_n^3) \cong H^0(\mathbb{P}^4)$ | $H^0(\mathcal{X}_n^4) \cong H^0(\mathbb{P}^4)$ |
| $H^2(\mathcal{X}_n^1) \cong H^2(\mathbb{P}^4)$ | $\bigoplus_{j=1}^5 H^0(Z_j)$ | $\bigoplus_{i,j \in [n], \ i \neq j} H^0(Z_{i,j})$ | $\bigoplus_{\alpha \subset [n], |\alpha| = 3} H^0(\mathbb{Z}_\alpha)$ |
| $H^4(\mathcal{X}_n^1) \cong H^4(\mathbb{P}^4)$ | $\bigoplus_{j=1}^5 H^0(Z_j)$ | $\bigoplus_{i,j \in [n], \ i \neq j} (H^0(Z_{i,j}) \oplus H^2(Z_{i,j}))$ | $\bigoplus_{\alpha \subset [n], |\alpha| = 3} H^2(\mathbb{Z}_\alpha)$ |
| $H^6(\mathcal{X}_n^1) \cong H^6(\mathbb{P}^4)$ | $\bigoplus_{j=1}^5 H^0(Z_j)$ | $\bigoplus_{i,j \in [n], \ i \neq j} H^2(Z_{i,j})$ | $\bigoplus_{\alpha \subset [n], |\alpha| = 3} H^4(\mathbb{Z}_\alpha)$ |
| $H^8(\mathcal{X}_n^1) \cong H^8(\mathbb{P}^4)$ | | | |

Here, for $\alpha \subset [n]$ and $|\alpha| = 3$, $\mathbb{Z}_\alpha$ is isomorphic to $\mathbb{X}_2^3$ and the cohomologies are recorded in the previous example.

5.2. The dot action on pre-permutohedral varieties. The permutation group $\mathfrak{S}_n$ acts on pre-permutohedral varieties $\mathcal{X}_n^{k-1}$, $1 \leq k \leq n-2$, in the following way. First, for $v \in \mathbb{P}^{n-1} \setminus \mathrm{Ind}$, we have the corresponding flag

$$V_\bullet = \left( \langle 0 \rangle \subset \langle v \rangle \subset \langle v, S_v \rangle \subset \cdots \subset \langle v, S_v, \cdots S^{k-1}v \rangle \right) \in \mathcal{X}_n^{n-1}.$$ 

Given $w \in \mathfrak{S}_n$, the action of $w$ on $V_\bullet$ is induced by the action of $w$ on $v$. That is,

$$wV_\bullet = \left( \langle 0 \rangle \subset \langle vw \rangle \subset \langle vw, S(vw) \rangle \subset \cdots \subset \langle vw, S(vw), \cdots S^{k-1}(vw) \rangle \right) \in \mathcal{X}_n^{n-1}.$$ 

This defines a birational map on $\mathcal{X}_k^{n-1}$ which can be extended to an automorphism. This defines the action of $\mathfrak{S}_n$ on $\mathcal{X}_k^{n-1}$. The pullback on cohomology groups of this action is the dot action (or the dot prepresentation).

Theorem 5.3. There is a permutation basis for the dot action on the cohomology of the pre-permutohedral varieties $\mathcal{X}_k^{n-1}$, $1 \leq k \leq n-2$.

Proof. We will construct a permutation basis inductively on $n$ and $k$. This means we want to construct a set of basis such that $\mathfrak{S}_n$ acts on the set by permutation, and the stabilizer of each vector in the basis is a Young subgroup of $\mathfrak{S}_n$. First, $\mathcal{X}_0^{n-1} = \mathbb{P}^{n-1}$ and $\mathfrak{S}_n$ acts on its cohomology groups trivially.

Suppose we already constructed a permutation basis for cohomologies of the pre-permutohedral varieties of dimensions at most $n-2$, and $H^*(\mathcal{X}_j^{n-1})$ for $1 \leq j \leq k-1$. In particular, we have a permutation basis for $H^*(\mathcal{X}_j^{k-1})$. In order to extend this basis to a basis for $H^*(\mathcal{X}_k^{n-1})$, we need to
find a permutation basis for
\[
\bigoplus_{\alpha \subset [n], |\alpha| = k} ^{-n-k-1} \bigoplus_{i=1}^{n-k-1} H^*_{-2i}(Z_\alpha) \cong \bigoplus_{\alpha \subset [n], |\alpha| = k} ^{-n-k-1} \bigoplus_{i=1}^{n-k-1} H^*_{2i}(Z_\alpha).
\]

For a given \( i \), an element \( w \in \mathfrak{S}_n \) acts on \( \bigoplus_{\alpha \subset [n], |\alpha| = k} H^*_{-2i}(Z_\alpha) \), thus we can construct permutation basis of \( \bigoplus_{\alpha \subset [n], |\alpha| = k} H^*_{-2i}(Z_\alpha) \) for each \( i \) independently. We first fix an \( \alpha \), then fix a permutation basis for \( H^*_{-2i}(Z_\alpha) \), which exists by induction hypothesis. We claim that the orbit of this basis under the action of \( \mathfrak{S}_n \) will be the basis we want.

For an element of the chosen basis \( \xi \in H^*_{-2i}(Z_\alpha) \), we would like to show that the stabilizer of \( \xi \in \mathfrak{S}_n \) is a Young subgroup. For \( w \in \mathfrak{S}_n \), since \( w : H^*_{-2i}(Z_\alpha) \to H^*_{-2i}(Z_{w^{-1} \alpha}) \), in order for \( w \) to stabilize \( \xi \), we must have \( w^{-1} \alpha = \alpha \), i.e. \( w \in \mathfrak{S}_n \times \mathfrak{S}_n \). The stabilizer of \( \xi \) in \( \mathfrak{S}_n \) is then the groups \( G \times \mathfrak{S}_n \) where \( G \) is the stabilizer of \( \xi \in H^*_{-2i}(Z_\alpha) \) acted by \( \mathfrak{S}_n \). By induction hypothesis, \( G \) is a Young subgroup of \( \mathfrak{S}_n \). Therefore, the stabilizer of \( \xi \) in \( \mathfrak{S}_n \) is also a Young subgroup of \( \mathfrak{S}_n \).

\[\Box\]

6. The geometry of certain Hessenberg varieties

In this section, we consider Hessenberg varieties associated with Hessenberg functions of the type
\( h_k = (2, 3, \cdots, k + 1, n, \cdots, n) \), for some \( k \leq n - 3 \). That is,
\[
h_k(j) = \begin{cases} j + 1, & j = 1, \cdots, k \\ n, & j = k + 1, \cdots, n. \end{cases}
\]

From now on, we fix \( h \) to be such a Hessenberg function. We denote the corresponding Hessenberg variety \( \mathcal{Y} = \mathcal{Hess}(S, h_k) \). The is a morphism from \( \mathcal{Y} \) to the Hessenberg type variety \( X_{k-1} = \mathcal{Hess}^{(k)}(S, h_+) \) which remembers only the flags up to dimension \( k \). More precisely, we define
\[
f : \mathcal{Y} \to X_{k-1}
\]
as follows:
\[
(V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots \subset V_n) \mapsto (V_0 \subset V_1 \subset \cdots \subset V_k).
\]
The fiber of \( f \) over a partial flag \((V_0 \subset \cdots \subset V_k)\) can be identified with the flag variety \( \mathcal{F}lag(\mathbb{C}^n/V_k) \cong \mathcal{F}lag(\mathbb{C}^{n-k}) \). Hence \( \mathcal{Y} \) has a fiber bundle structure over \( X_{k-1} \).

Over \( \mathcal{Y} \), there is a tautological filtration
\[
\mathcal{V}_k \subset \mathcal{V}_{k+1} \subset \cdots \subset \mathcal{V}_n \cong \mathcal{Y} \times \mathbb{C}^n
\]
of vector subbundles of the trivial bundle over \( \mathcal{Y} \): the fiber of \( \mathcal{V}_j \) over a flag \((V_0 \subset \cdots \subset V_n)\) is the vector space \( V_j \). We then have the line bundles \( \mathcal{L}_j := \mathcal{V}_j/\mathcal{V}_{j-1} \), \( k + 1 \leq j \leq n \). Set \( x_j = -c_1(\mathcal{L}_j) \) to be the negative of the first Chern class of the line bundle \( \mathcal{L}_j \), then we have the following description of the cohomology ring \( H^*(\mathcal{Y}) \).

**Proposition 6.1.** Let \( R := H^*(X_{k-1}) \) denote the cohomology ring of \( X_{k-1} \), then \( H^*(\mathcal{Y}) = H^*(\mathcal{Y}; \mathbb{C}) \) is generated over \( R \) by the classes \( x_{k+1}, \cdots, x_n \).
Proof. The proof mimics the proof for the cohomology of the flag variety \cite[Proposition 10.2.3]{Ful93} and is based on basic facts about projective bundles. For more details, see \cite[Appendix B.4]{GH78} or \cite[Appendix B.4]{Ful97}. For a vector bundle $V$ over a variety $X$, let $\rho : P(V) \to X$ denote the corresponding projective bundle. There is a tautological bundle $L \subset \rho^*(V)$. Set $\xi = -c_1(L)$, then
\begin{equation}
H^*(P(V)) \cong H^*(X)/(\xi^n + a_1\xi^{n-1} + \cdots + a_r),
\end{equation}
where $a_i = c_i(V) \in H^{2i}(X)$.

In this proof we suppress all the notions of pullbacks of bundles. One can construct $Y$ from $X_{k-1}$ as a sequence of projective bundles. First, over $X_{k-1}$ there is a bundle $\mathcal{U} \to X_{k-1}$ of rank $n-k$ whose fiber over the flag $(V_0 \subset \cdots \subset V_k)$ is the vector space $\mathbb{C}^n/V_k$. The projective bundle $P(\mathcal{U})$ is the first bundle in the sequence. It gives the direction of the extra dimension of $V_{k+1}$ over the flag $(V_0 \subset \cdots \subset V_k)$. The tautological bundle $\mathcal{U}_1$ of $P(\mathcal{U})$ pulls back to the line bundle $L_{k+1}$ on $Y$.

Next, over $P(\mathcal{U})$, we have the bundle $\mathcal{U}/\mathcal{U}_1$ of rank $n-k-1$, and we construct the second projective bundle $P(\mathcal{U}/\mathcal{U}_1) \to P(\mathcal{U})$. The tautological bundle of $P(\mathcal{U}/\mathcal{U}_1)$ is of the form $\mathcal{U}_2/\mathcal{U}_1$ for some vector bundle $\mathcal{U}_2$ of rank 2 and $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}$ as bundles over $P(\mathcal{U}/\mathcal{U}_1)$. Moreover, the tautological bundle of $P(\mathcal{U}/\mathcal{U}_1)$ pulls back to $L_{k+2}$. One can continue this process and construct $P(\mathcal{U}/\mathcal{U}_j)$ as a projective bundle over $P(\mathcal{U}/\mathcal{U}_j)$, with tautological bundle of the form $\mathcal{U}_j/\mathcal{U}_{j-1}$, and so on. At the end, one arrives at the space $P(\mathcal{U}/\mathcal{U}_{n-k-1})$ which is isomorphic to the Hessenberg variety $Y$.

Finally, by the formula (6.1) and the fact that the tautological bundle of $P(\mathcal{U}/\mathcal{U}_j)$ pulls back to the line bundle $L_{k+j+1}$ on $Y$, we obtain the conclusion that $H^*(Y)$ is generated by $x_{k+1}, \cdots , x_n$ over $R$.

The dot action on $R$ was described in Section 5.2. Motivated by the fact that the dot action acts trivially on the flag variety, it is natural to make the following

**Conjecture 6.1.** The dot action acts trivially on the classes $x_j$, $k+1 \leq j \leq n$.

The conjecture might be evident to the experts, but the author is not able to prove it.

### References

[AHM19] Hiraku Abe, Tatsuya Horiguchi, and Mikiya Masuda, *The cohomology rings of regular semisimple Hessenberg varieties for $h = (h_1, n, \ldots , n)$*, J. Comb. **10** (2019), no. 1, 27–59.

[BC18] Patrick Brosnan and Timothy Y. Chow, *Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties*, Adv. Math. **329** (2018), 955–1001.

[CHL] Soojin Cho, Jaehyun Hong, and Eunjeong Lee, *Bases of the equivariant cohomologies of regular semisimple Hessenberg varieties*, available at [https://arxiv.org/abs/2008.12500](https://arxiv.org/abs/2008.12500).

[Cho] Timothy Chow, *The Erasing Marks Conjecture*, available at [http://timothychow.net/erasing.pdf](http://timothychow.net/erasing.pdf).

[DMPS92] F. De Mari, C. Procesi, and M. A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 529–534.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993.

[Ful97] , *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978.

[GP] Mathieu Guay-Paquet, *A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra*, available at [https://arxiv.org/abs/1601.05498](https://arxiv.org/abs/1601.05498).

[HPT] Megumi Harada, Martha Precup, and Julianna Tymoczko, *Toward permutation bases in the equivariant cohomology rings of regular semisimple Hessenberg varieties*, available at [https://arxiv.org/abs/2101.03191](https://arxiv.org/abs/2101.03191).
[SW16] John Shareshian and Michelle L. Wachs, *Chromatic quasisymmetric functions*, Adv. Math. 295 (2016), 497–551.

[Tef11] Nicholas Teff, *Representations on Hessenberg varieties and Young’s rule*, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 903–914 (English, with English and French summaries).

[Tym08] Julianna S. Tymoczko, *Permutation actions on equivariant cohomology of flag varieties*, Toric topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 365–384.

[Voi07] Claire Voisin, *Hodge theory and complex algebraic geometry. I*, Reprint of the 2002 English edition, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2007. Translated from the French by Leila Schneps.

Email address: jan-li@wustl.edu