Effect of initial stresses on the elastic waves in transversely isotropic thermoelastic materials

Lalawmpuia Tochhawng | Singh SS

Department of Mathematics and Computer Science, Mizoram University, Aizawl, India

Correspondence
Singh SS, Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram 796004, India.
Email: saratca32@yahoo.co.uk

Funding information
Science and Engineering Research Board, EMR/2017/001723

In the present article, we have studied the reflection/transmission of elastic waves in initially stressed transversely isotropic thermoelastic materials. Three quasi type coupled longitudinal (QL), transverse (QT), and thermal waves are found to propagate in initially stressed transversely isotropic thermoelastic materials. For incident QL and QT-waves at a plane interface, boundary conditions were implemented for obtaining the coefficients of reflection/transmission, the distribution of energy in the reflected and transmitted waves are also discussed. We have observed that the results vary with direction of incidence as well as the parameters due to elasticity, thermal, and initial stresses. Numerical computations have been performed and analyzed the impact of initial stresses on the results. It has been observed critical angles at $\theta_0 = 30^\circ$ and $58^\circ$ for the reflected and transmitted QL-waves for incident QT-wave.

KEYWORDS
energy ratios, initial stresses, longitudinal and transverse waves, reflection/transmission coefficients, thermoelastic materials

1 | INTRODUCTION

Thermoelasticity discusses heat conduction, strains, and thermal stresses in the materials with the inverse effect of temperature distribution. The study of thermoelastic materials has been implemented in many important fields such as seismology, soil dynamics, physical sciences, aeronautics, atomic smasher, and nuclear reactors. With the help of thermoelastic admittance matrix, the effect of the thermoelastic system to mechanical or thermal is presented. Lord and Shulman considered the relations between temperature and strain to generalize the theory of thermoelastic materials. The theory of thermoelastic materials was linearized by Green and Lindsay, they also proved the uniqueness for the theorem. Dhaliwal and Wang presented the theory of linear generalized dipolar thermoelastic materials having initial stress. The uniqueness theorem and reciprocity relation are proved for the generalized thermoelasticity under initial stresses. Researchers like Dhaliwal and Sherief, Chandrasekharai, and Hetnarski also contributed in the development of the theory of thermoelasticity.

Two quasiwaves, namely, longitudinal (QL) and transverse (QT) and a purely transverse wave can propagate in the transversely isotropic heat-conducting elastic material. McCarthy discussed the general properties of accelerated waves in generalized thermoelastic materials. Acceleration waves are isothermal or isentropic for the elastic materials with heat conduction. Frequency equation of surface waves in thermoelastic anisotropic materials is obtained. The velocity of propagation of the body waves in a thermally conducting transversely isotropic elastic solid are depicted numerically.
Singh obtained the propagation speeds for quasi-P and S waves in the generalized thermoelastic materials considering two relaxation times. He also generalized the theory of thermoelasticity for transversely isotropic heat conducting materials under initial stress to analyze the reflected waves. For isothermal and thermally insulated, the nature of Rayleigh wave has been investigated in a generalized thermoelastic half-space. In a void containing viscoelastic heat conducting material, Tomar et al. found four basic waves having different propagation speeds.

For identification of valuable materials buried inside the earth as well as for inspecting hydrocarbons, the technique of wave propagation has been employed. For incident P and SV-waves, Sinha and Sinha investigated the properties of reflection coefficients in thermoelastic solid half-space. The propagation of P-wave is affected due to presence of thermal coupling in thermoelastic materials with two relaxation times, but SV-wave remains unaffected. Tomar and Singh obtained the reflection/transmission coefficients for incident longitudinal wave and they also investigated for the incident transverse wave. At the plane interface of two different thermoelastic materials containing voids, Singh investigated the reflection/transmission phenomena of waves due to incident longitudinal and transverse waves. Some useful results of wave propagation are Achenbach, Sinha and Elsibai, Othman and Song, Kumar et al., Othman and Atwa, Sharma and Bhargava, Pal et al., Guo and Wei, Palacidi et al., Singh, Sahu et al., Chatterjee et al., Abbas et al., Zorammuana and Singh, Abdalla et al., Othman et al., Singh and Lianngenga, Barak and Kaliraman, Khurana and Tomar, Lianngenga and Singh, Kumar et al., Singh and Lalawmpuia, and Pal and Kanoria.

The fundamental equations for the present problem are inspected in Section 2. The formulation of the problem and wave propagation are explained in Section 3. The stress, heat, and displacements are continuous at the plane interface and this analysis has been performed in Section 4. The coefficients and energy distributions due to reflected and transmitted waves are discussed analytically in Sections 5 and 6, respectively. Some particular cases are obtained in Sections 7. Section 8 is concerned on numerical computation and the final section is the conclusion of the problem.

### 2 BASIC EQUATIONS

Following Wang et al., the constitutive relations for prestressed bodies with generalized thermoelasticity are

\[ \sigma_{ij} = c_{ijmn}e_{mn} + e_{jk}P_{ki} - \beta_{ij}T, \]  
\[ \rho \eta = \rho C_e T + \beta_{ij}e_{ij}, \]  
\[ q_i + \tau q_i = -a_i T - K_{ij}T_j - h_{ijk}e_{jk}, \]  
\[ e_{ij} = \frac{1}{2}(u_{ij} + u_{ji}), \quad (i,j,k,m,n = 1,2,3) \]

where \( \sigma_{ij}, q_i, \eta, P_{ki}, \) and \( e_{ij} \) are stress, thermal flux, entropy, prestress, and strain tensors, respectively, \( c_{ijmn} \) and \( \beta_{ij}, K_{ij}, a_i, h_{ijk} \) are elastic and thermal coefficients, respectively, the temperature is change from \( T_0 \) to \( T, u_i \) is the component of displacement vector of the material with density \( \rho \), thermal relaxation time \( \tau \), and specific heat \( C_e \).

For the generalized thermoelastic materials under initial stresses with body force \( F_i \) and internal heat source \( S \), the equations of motions are given as

\[ \rho \ddot{u}_i = \sigma_{ij} + \rho F_i, \]  
\[ \rho T_0 \dot{\eta} = -q_{li} + \rho S. \quad (i,j = 1,2,3). \]

Without body forces as well as heat sources and using Equations (1) to (3) in Equations (5) and (6), we get

\[ (d_{ijmn}e_{mn} - \beta_{ij}T)_{ij} = \rho \ddot{u}_i, \]

\[ T_0 \left( 1 + \tau \frac{d}{dt} \right) \left( \beta_{ij}\dot{e}_{ij} + \rho C_e \dot{T} \right) = (K_{ij}T_j + h_{ijk}e_{jk} + a_iT)_{ij}. \]
where $d_{ijmn} = c_{ijmn} + \delta_{jn} e_{mi}$ and $\delta_{jn}$ is the Kronecker's delta. Note that $a_1 = 0$ and $h_{ijk} = 0$ for the uniform temperature prestressed bodies.

3 | WAVE PROPAGATION

We consider Cartesian coordinates with $x$ and $y$-axes lying horizontally and $z$-axis as vertically. Two half-spaces $M : 0 \leq z < \infty$ and $M' : -\infty < z \leq 0$ of transversely isotropic thermoelastic medium under initial stresses are assumed to analyze wave propagation in $xz$-plane. The diagrammatic structure of the problem is shown in Figure 1.

For half-space $M$, equations of motions are\(^1\)

$$d_{11}u_{1,11} + (d_{13} + d_{44})u_{3,13} + d_{44}u_{1,33} - \beta_1 T_{1,1} = \rho \ddot{u}_1, \quad (9)$$

$$d_{44}u_{3,11} + (d_{13} + d_{44})u_{1,13} + d_{33}u_{3,33} - \beta_3 T_{3,3} = \rho \ddot{u}_3, \quad (10)$$

$$T_0 \left( 1 + \tau \frac{\partial}{\partial t} \right) (\beta_1 \ddot{u}_{1,1} + \beta_3 \ddot{u}_{3,3} + d \ddot{T}) = K_1 T_{1,11} + K_3 T_{3,33}, \quad (11)$$

where $u = (u_1, 0, u_3)$, $d_{11} = c_{11} + P_{11}$, $d_{13} = c_{13}$, $d_{33} = c_{33} + P_{33}$, $d_{44} = c_{44} + P_{11}$, $K_1 = K_{11}$, $K_3 = K_{33}$, $\beta_1 = \beta_{11} = (d_{11} + d_{12}) a_1 + d_{13} a_3$, $\beta_3 = \beta_{33} = 2d_{13} a_1 + d_{33} a_3$, $\tau = \rho C_v$, $a_1$ and $a_3$ are linear thermal expansion coefficients.

Similarly, for $M'$,

$$d_{11}' u_{1,11}' + (d_{13}' + d_{44}') u_{3,13}' + d_{44}' u_{1,33}' - \beta_1' T_{1,1}' = \rho' \ddot{u}_1', \quad (12)$$

$$d_{44}' u_{3,11}' + (d_{13}' + d_{44}') u_{1,13}' + d_{33}' u_{3,33}' - \beta_3' T_{3,3}' = \rho' \ddot{u}_3', \quad (13)$$

$$T_0' \left( 1 + \tau' \frac{\partial}{\partial t} \right) (\beta_1' \ddot{u}_{1,1}' + \beta_3' \ddot{u}_{3,3}' + d' \ddot{T}') = K_1' T_{1,11}' + K_3' T_{3,33}', \quad (14)$$

where $\beta_1' = \beta_{11}' = (d_{11}' + d_{12}') a_1' + d_{13}' a_3'$, $\beta_3' = \beta_{33}' = 2d_{13}' a_1' + d_{33}' a_3'$, $\alpha_1'$ and $\alpha_3'$ are due to thermal expansion, $u' = (u_1', 0, u_3')$, $d' = \rho' C_v'$, $d_{11}' = c_{11}' + P_{11}'$, $d_{13}' = c_{13}'$, $d_{33}' = c_{33}' + P_{33}'$, $d_{44}' = c_{44}' + P_{11}'$, $K_1' = K_{11}'$ and $K_3' = K_{33}'$.

For the incident, reflected, and transmitted waves, we have

$$\langle u_1^{(n)}, u_3^{(n)}, T^{(n)} \rangle = \langle A_n d_1^{(n)}, A_n d_3^{(n)}, I k_n F n A_n \rangle e^{ik_n [x \cos \theta_n + y \sin \theta_n - v_n t]}, n = 0, 1, 2, 3, 4, 5, 6 \quad (15)$$

where $A_n$ is the amplitude constant, $\langle d_1^{(n)}, 0, d_3^{(n)} \rangle$ and $\langle p_1^{(n)}, 0, p_3^{(n)} \rangle$ are unit displacement and propagation vectors, respectively, $k_n$ is wavenumber, $v_n$ is phase velocity.\(^1 \) Note that $n = 0$ represents incident a QL or a QT wave, $n = 1, n = 2,$ and

---

**Figure 1** Problem figure
\[ F^{(n)} = \begin{cases} (d_3 + d_{13} P_3^{(r)} d_3^{(r)} - \rho_3^2 d_3^{(r)} - \beta_3 F^{(r)}) + d_{13} k_0 P_1^{(0)} d_1^{(r)}, & n = 0, 1, 2, 3 \\ (d_3 + d_{13} P_3^{(r)} d_3^{(r)} - \rho_3^2 d_3^{(r)} - \beta_3 P_1^{(0)} d_1^{(r)}, & n = 4, 5, 6. \end{cases} \]

Using Snell's law, we can have\(^{22}\)

\[ \frac{k_0}{k_r} \sin \theta_0 = \sin \theta_r \text{ for } r = 1, 2, 3, 4, 5, 6. \] (16)

4 | BOUNDARY CONDITIONS

The stress tractions, heat flow, and displacement components are continuous at \( z = 0 \). We have

(i) Continuity of normal traction:

\[ \sum_{r=0}^{3} d_{33} u_{3,3}^{(r)} + d_{13} u_{1,1}^{(r)} - \beta_3 T^{(r)} = \sum_{r=0}^{6} d'_{33} u_{3,3}^{(r)} + d'_{13} u_{1,1}^{(r)} - \beta'_{3} T^{(r)}. \]

(ii) Continuity of shear traction:

\[ \sum_{n=0}^{3} d_{44}(u_{1,3}^{(r)} + u_{3,1}^{(r)}) = \sum_{n=0}^{6} d'_{44}(u_{1,3}^{(r)} + u_{3,1}^{(r)}). \]

(iii) Continuity of heat flow:

\[ \sum_{r=0}^{3} T^{(r)} = \sum_{n=4}^{3} T^{(r)} + \sum_{r=0}^{3} \frac{\partial T^{(r)}}{\partial z} = \sum_{n=4}^{6} \frac{\partial T^{(r)}}{\partial z}. \]

(iv) Continuity of displacement components:

\[ \sum_{r=0}^{3} u_{1}^{(r)} = \sum_{r=0}^{3} u_{1}^{(r)}, \sum_{r=0}^{3} u_{3}^{(r)} = \sum_{r=4}^{6} u_{3}^{(r)}. \]

These boundary conditions may be reduced to

\[ \sum_{r=0}^{3} [k_r (d_{33} P_3^{(r)} d_3^{(r)} - \beta_3 F^{(r)}) + d_{13} k_0 P_1^{(0)} d_1^{(r)}] A_r, = \]

\[ \sum_{r=0}^{6} [k_r (d_{33} P_3^{(r)} d_3^{(r)} - \beta_3 F^{(r)}) + d_{13} k_0 P_1^{(0)} d_1^{(r)}] A_r = 0, \] (17)

\[ \sum_{r=0}^{3} d_{44}(k_r P_3^{(r)} d_3^{(r)} + k_0 P_1^{(0)} d_4^{(r)}) A_n - \sum_{r=4}^{6} d'_{44}(k_r P_3^{(r)} d_3^{(r)} + k_0 P_1^{(0)} d_4^{(r)}) A_r = 0, \] (18)
\[
\sum_{r=0}^{3} k_r F^{(r)} A_r - \sum_{r=4}^{6} k_r F^{(r)} A_n = 0, \quad \sum_{r=0}^{3} k_r^2 P_3^{(r)} F^{(r)} A_r - \sum_{r=4}^{6} k_r^2 P_3^{(r)} F^{(r)} A_n = 0.
\] (19)

\[
\sum_{r=0}^{3} d_1^{(r)} A_r - \sum_{r=4}^{6} d_1^{(r)} A_n = 0, \quad \sum_{r=0}^{3} d_3^{(r)} A_r - \sum_{r=4}^{6} d_3^{(r)} A_n = 0.
\] (20)

Equations (17) to (20) will help to find the reflection and transmission coefficients of the reflected and transmitted waves.

5 | AMPLITUDE RATIO

The matrix representation of Equations (17) to (20) is given as

\[
AZ = B,
\] (21)

where \(A\) is a \(6 \times 6\) matrix, \(B\) and \(Z\) are \(6 \times 1\) matrices with the following elements

\[
a_{1r} = \begin{cases} 
k_r (d_{33} p_3^{(r)} d_3^{(r)} - \beta_3 F^{(r)}) + d_{13} k_0 p_1^{(0)} d_1^{(r)}, & r = 1, 2, 3 \\
- k_r (d_{33} p_3^{(r)} d_3^{(r)} - \beta_3 F^{(r)}) - d_{13} k_0 p_1^{(0)} d_1^{(r)}, & r = 4, 5, 6 \\
\end{cases}
\]

\[
a_{2r} = \begin{cases} 
d_{44} (k_r p_4^{(r)} d_1^{(r)} + k_0 p_1^{(0)} d_1^{(r)}), & r = 1, 2, 3 \\
-d_{44} (k_r p_4^{(r)} d_1^{(r)} + k_0 p_1^{(0)} d_1^{(r)}), & r = 4, 5, 6 \\
\end{cases}
\]

\[
a_{3r} = \begin{cases} 
k_r F^{(r)}, & r = 1, 2, 3 \\
- k_r F^{(r)}, & r = 4, 5, 6 \\
\end{cases}
\]

\[
a_{4r} = \begin{cases} 
d_1^{(r)}, & r = 1, 2, 3 \\
-d_1^{(r)}, & r = 4, 5, 6 \\
\end{cases}
\]

\[
a_{5r} = \begin{cases} 
d_1^{(r)}, & r = 1, 2, 3 \\
-d_3^{(r)}, & r = 4, 5, 6 \\
\end{cases}
\]

\[
a_{6r} = \begin{cases} 
k_r^2 F^{(r)}, & r = 1, 2, 3 \\
-k_r^2 F^{(r)}, & r = 4, 5, 6 \\
\end{cases}
\]

\[
b_1 = -k_0 (d_{33} p_3^{(0)} d_3^{(0)} + d_{13} p_1^{(0)} d_1^{(0)} - \beta_3 F^{(0)}),
\]

\[
b_2 = -d_{44} k_0 (p_3^{(0)} d_1^{(0)} + p_1^{(0)} d_1^{(0)}), \quad b_3 = -k_0 F^{(0)}, \quad b_4 = -d_1^{(0)},
\]

\[
b_5 = -d_3^{(0)}, \quad b_6 = -k_0^2 p_3^{(0)} F^{(0)}, \quad Z_r = \frac{A_r}{A_0}.
\]

Equation (21) is solved for \(Z_r\) due to incident QL and QT-waves.

6 | ENERGY RATIO

We have considered partition of energy at \(z = 0\) and the rate of transmission is given by Reference 24

\[
E^* = \langle \tau_{z3} \cdot u_3 \rangle + \langle \tau_{z1} \cdot u_1 \rangle.
\] (22)

Using Equation (22), the energy ratios waves are

\[
E_i = \frac{\eta_i}{\eta_0} Z_i^2, \quad (i = 1, 2, 3, 4, 5, 6)
\] (23)
where

\[ \eta_i = \begin{aligned}
&d_3^{(i)}(d_3^{(i)}k_i^d_3^{(i)} P_3^{(i)} + d_1 k_0 d_3^{(i)} P_1^{(0)} - \beta_3 k_i F^{(i)}) + \\
&d_4^{(i)}(k_i d_4 P_3^{(i)} + k_0 d_4 P_1^{(0)}), \quad i = 0, 1, 2, 3 \\
&d_3^{(i)}(d_3^{(i)}k_i^d_3^{(i)} P_3^{(i)} + d_1 k_0 d_3^{(i)} P_1^{(0)} - \beta_3' k_i F^{(i)}) + \\
&d_4^{(i)}(k_i d_4 P_3^{(i)} + k_0 d_4 P_1^{(0)}), \quad i = 4, 5, 6.
\end{aligned} \]

Note that \( E_r \) for \( r = 1, 2, 3 \) represent energy ratios of the reflected QL, QT, and T-mode waves, respectively, and \( r = 4, 5, 6 \) represent for the transmitted QL, QT, and T-mode waves, respectively.

7 PARTICULAR CASES

CASE I: If \( P_{11} = P_{33} = P'_{11} = P'_{33} = 0 \), then \( d_0 = c_0 \) and \( d_0' = c_0' \). Equations (21) and (23) have the following modified values

\[ a_{ij} = \begin{cases} 
  k_i (c_3^{(j)} P_3^{(i)} d_3^{(j)} - \beta_3 F^{(j)}) + c_1 k_0 P_1^{(0)} d_1^{(j)}, & j = 1, 2, 3 \\
  -k_i (c_3^{(j)} P_3^{(i)} d_3^{(j)} - \beta_3 F^{(j)}) - c_1 k_0 P_1^{(0)} d_1^{(j)}, & j = 4, 5, 6
\end{cases} \]

\[ a_{2j} = \begin{cases} 
  c_4 k_i P_3^{(i)} d_3^{(j)} + k_0 P_1^{(0)} d_3^{(j)}, & j = 1, 2, 3 \\
  -c_4 k_i P_3^{(i)} d_3^{(j)} + k_0 P_1^{(0)} d_3^{(j)}, & j = 4, 5, 6
\end{cases} \]

\[ b_1 = -k_0 (c_3^{(j)} P_3^{(i)} d_3^{(j)} + c_1 k_0 P_1^{(0)} d_1^{(j)} - \beta_3 F^{(j)}), \quad b_2 = -c_4 k_0 (d_3^{(j)} + P_1^{(0)} d_3^{(j)}). \]

\[ \eta_i = \begin{aligned}
&d_3^{(i)}(c_3^{(j)}k_i d_3^{(i)} P_3^{(i)} + c_1 k_0 d_3^{(i)} P_1^{(0)} - \beta_3 k_i F^{(i)}) + \\
&d_4^{(i)}(k_i d_4 P_3^{(i)} + k_0 d_4 P_1^{(0)}), \quad i = 0, 1, 2, 3 \\
&d_3^{(i)}(c_3^{(j)}k_i d_3^{(i)} P_3^{(i)} + c_1 k_0 d_3^{(i)} P_1^{(0)} - \beta_3' k_i F^{(i)}) + \\
&d_4^{(i)}(k_i d_4 P_3^{(i)} + k_0 d_4 P_1^{(0)}), \quad i = 4, 5, 6.
\end{aligned} \]

CASE II: If \( M' \) is stress free, then

\[ Z_1 = \frac{b_1 (a_{22} a_{63} - a_{23} a_{62}) - a_{12} (b_2 a_{63} - a_{23} b_5) + a_{13} (b_2 a_{62} - a_{23} b_5)}{a_{12} (a_{22} a_{63} - a_{23} a_{62}) - a_{12} (a_{21} a_{63} - a_{23} a_{61}) + a_{13} (a_{21} a_{62} - a_{23} a_{61})}, \]

\[ Z_2 = \frac{a_{11} (b_2 a_{63} - a_{23} b_5) - (b_2 a_{63} - a_{23} a_{62}) + a_{13} (a_{21} a_{62} - a_{23} a_{61})}{a_{11} (a_{22} a_{63} - a_{23} a_{62}) - a_{12} (a_{21} a_{63} - a_{23} a_{61}) + a_{13} (a_{21} a_{62} - a_{23} a_{61})}, \]

\[ Z_3 = \frac{a_{11} (a_{22} b_6 - a_{23} a_{62}) - a_{12} (a_{21} b_6 - a_{23} a_{61}) + a_{13} (a_{21} a_{62} - a_{23} a_{61})}{a_{11} (a_{22} a_{63} - a_{23} a_{62}) - a_{12} (a_{21} a_{63} - a_{23} a_{61}) + a_{13} (a_{21} a_{62} - a_{23} a_{61})}. \]

These ratios exactly match Singh.\(^{15}\)

The distribution of energy \( E_1, E_2, \) and \( E_3 \) of the reflected waves are given by Equation (23).

CASE III: If \( M' \) is stress free and \( P_{11} = P_{33} = 0 \). Equation (24) will be modified with the following changes

\[ a_{1j} = k_i(c_3^{(j)} P_3^{(i)} d_3^{(j)} - \beta_3 F^{(j)}) + c_1 k_0 P_1^{(0)} d_1^{(j)}, \]

\[ a_{2j} = c_4 k_i P_3^{(i)} d_3^{(j)} + k_0 P_1^{(0)} d_1^{(j)}, \quad (j = 1, 2, 3) \]

\[ b_1 = -k_0 (c_3^{(j)} P_3^{(i)} d_3^{(j)} + c_1 k_0 P_1^{(0)} d_1^{(j)} - \beta_3 F^{(j)}), \]

\[ b_2 = -c_4 k_0 (d_3^{(j)} + P_1^{(0)} d_3^{(j)}). \]

The results are exactly same as Sharma.\(^{13}\)

The energy ratios \( E_1, E_2, \) and \( E_3 \) are also given by Equation (23) with the modified value of

\[ \eta_i = d_3^{(i)}(c_3^{(j)}k_i d_3^{(i)} P_3^{(i)} + c_1 k_0 d_3^{(i)} P_1^{(0)} - \beta_3 k_i F^{(i)}) + c_4 d_3^{(i)}(k_i d_4 P_3^{(i)} + k_0 d_4 P_1^{(0)}), \quad (i = 0, 1, 2, 3). \]
8 | NUMERICAL RESULTS

For evaluating the coefficients and energy distributions due to reflected and transmitted waves for incident QL and QT waves, we have used the relevant parametric values given in Table 1 (see Reference 9).

The unit propagation and displacement vectors are (for incident quasilongitudinal wave)

\[(p_1^{(0)}, p_3^{(0)}) = (\sin \theta_0, 0, \cos \theta_0), \quad (d_1^{(0)}, d_3^{(0)}) = (\sin \theta_0, 0, \cos \theta_0),\]

(for incident quasitransverse wave)

\[(p_1^{(0)}, p_3^{(0)}) = (\sin \theta_0, 0, \cos \theta_0), \quad (d_1^{(0)}, d_3^{(0)}) = (\cos \theta_0, 0, -\sin \theta_0),\]

(for reflected waves)

\[(p_1^{(1)}, p_3^{(1)}) = (\sin \theta_1, 0, -\cos \theta_1), \quad (d_1^{(1)}, d_3^{(1)}) = (\sin \theta_1, 0, -\cos \theta_1),\]

\[(p_1^{(2)}, p_3^{(2)}) = (\sin \theta_2, 0, -\cos \theta_2), \quad (d_1^{(2)}, d_3^{(2)}) = (-\cos \theta_2, 0, -\sin \theta_2),\]

\[(p_1^{(3)}, p_3^{(3)}) = (\sin \theta_3, 0, -\cos \theta_3), \quad (d_1^{(3)}, d_3^{(3)}) = (\sin \theta_3, 0, -\cos \theta_3),\]

(for transmitted waves)

\[(p_1^{(4)}, p_3^{(4)}) = (\sin \theta_4, 0, \cos \theta_4), \quad (d_1^{(4)}, d_3^{(4)}) = (\sin \theta_4, 0, \cos \theta_4),\]

\[(p_1^{(5)}, p_3^{(5)}) = (\sin \theta_5, 0, \cos \theta_5), \quad (d_1^{(5)}, d_3^{(5)}) = (\cos \theta_5, 0, -\sin \theta_5),\]

\[(p_1^{(6)}, p_3^{(6)}) = (\sin \theta_6, 0, \cos \theta_6), \quad (d_1^{(6)}, d_3^{(6)}) = (\sin \theta_6, 0, \cos \theta_6).\]

Figures 2 to 5 are due to incident QL wave, while Figures 6 to 9 represent for the incident QT wave. It may be noted that \(P = P_{11} = P_{33}, P' = P'_{11} = P'_{33},\) and \(\omega = 5.\)

| Cobalt (M) | Value | Zinc (M') | Value | Units |
|------------|-------|-----------|-------|-------|
| \(\rho\)   | \(8.836 \times 10^3\) | \(\rho'\)  | \(7.14 \times 10^3\) | kgm\(^{-3}\) |
| \(c_{11}\) | \(3.071 \times 10^{11}\) | \(c'_{11}\) | \(1.628 \times 10^{11}\) | Nm\(^{-2}\) |
| \(c_{12}\) | \(1.650 \times 10^{11}\) | \(c'_{12}\) | \(0.362 \times 10^{11}\) | Nm\(^{-2}\) |
| \(c_{13}\) | \(1.027 \times 10^{11}\) | \(c'_{13}\) | \(0.508 \times 10^{11}\) | Nm\(^{-2}\) |
| \(c_{33}\) | \(3.581 \times 10^{11}\) | \(c'_{33}\) | \(0.627 \times 10^{11}\) | Nm\(^{-2}\) |
| \(c_{44}\) | \(0.755 \times 10^{11}\) | \(c'_{44}\) | \(0.385 \times 10^{11}\) | Nm\(^{-2}\) |
| \(\beta_1\) | \(7.04 \times 10^6\) | \(\beta'_{1}\) | \(5.75 \times 10^6\) | Nm\(^{-2}\)degree\(^{-1}\) |
| \(\beta_3\) | \(6.90 \times 10^6\) | \(\beta'_{3}\) | \(5.17 \times 10^6\) | Nm\(^{-2}\)degree\(^{-1}\) |
| \(C_e\)   | \(4.27 \times 10^7\) | \(C'_{e}\)  | \(3.9 \times 10^2\) | Jkg\(^{-1}\)degree\(^{-1}\) |
| \(K_1\)   | \(0.690 \times 10^2\) | \(K'_{1}\)  | \(1.24 \times 10^2\) | Wm\(^{-1}\)degree\(^{-1}\) |
| \(K_3\)   | \(0.690 \times 10^2\) | \(K'_{3}\)  | \(1.24 \times 10^2\) | Wm\(^{-1}\)degree\(^{-1}\) |
| \(T_0\)   | \(298\) | \(T'_{0}\) | \(296\) | K |
| \(\tau_0\) | \(0.05\) | \(\tau'_{0}\) | \(0.06\) | |
8.1 Incident QL-wave

Figure 2 explained the change in reflection coefficients with the change in $\theta_0$ at different values of $P$ and $P'$. We have observed that $|Z_1|$ in Figure 2A increases when $\theta_0$ is increased. In Figure 2B, all the curves corresponding to $|Z_2|$ increase initially and decrease when the value of $\theta_0$ get larger. Thereafter, Curve I, Curve II, and Curve III increase to the maximum values at $\theta_0 = 71^\circ$, $\theta_0 = 70^\circ$, and $\theta_0 = 68^\circ$, respectively, and then decrease again. All the curves in Figure 2C for the amplitude ratio $Z_3$ increase to the maximum values at $\theta_0 = 30^\circ$(Curve I), $\theta_0 = 28^\circ$(Curve II), and $\theta_0 = 26^\circ$(Curve III), which decrease with the increase of $\theta_0$. Note that the minimum and maximum effects of initial stresses on $|Z_1|$ are near grazing and normal incidence, respectively, while the minimum effect on $|Z_3|$ is near normal incidence.

The variation of the transmission coefficients are depicted in Figure 3. We have observed that $|Z_4|$ decrease with the increase of $\theta_0$, while $|Z_5|$ and $|Z_6|$ are similar to those of $|Z_2|$ and $|Z_3|$, respectively. All the curves in Figure 3B meet at a point $\theta_0 = 60^\circ$. Herein, also the effect of initial stresses on $|Z_4|$ is maximum when $\theta_0$ is close to normal angle of incidence.

The energy distribution on the reflected and transmitted waves are shown in Figures 4 and 5, respectively. In Figure 4A, $|E_1|$ increases when the value of $\theta_0$ is getting more. The effects of $P$ and $P'$ on $|E_1|$ are minimum and maximum at the grazing and normal angle of incidence, respectively. All the curves in Figure 4B for $|E_2|$ increase initially and meet at $\theta_0 = 52^\circ$, which then increase to the maximum values at $\theta_0 = 77^\circ$(Curve I), $\theta_0 = 78^\circ$(Curve II), and $\theta_0 = 79^\circ$(curve III).

After these points, all the curves decrease with the rise of $\theta_0$. The values of $|E_3|$ in Figure 4C increase to the maximum values for Curve I, Curve II, and Curve III are observed at $\theta_0 = 35^\circ$, $\theta_0 = 31^\circ$, and $\theta_0 = 28^\circ$, respectively, and all decrease with the higher value of $\theta_0$. We have observed that the minimum effect of $P$ and $P'$ on $|E_3|$ is near normal angle of incidence. In Figure 5, the value of $|E_4|$ falls when the value of $\theta_0$ is increased, while $|E_5|$ and $|E_6|$ show similar pattern with $|E_2|$ and $|E_3|$, respectively. The sum of the energy ratios is close to one.

8.2 Incident QT-wave

Figures 6 and 7 are corresponding to the coefficients for reflection and transmission, respectively. In Figure 6A, the values of $|Z_1|$ have parabolic paths in the regions, Curve I: $0^\circ \leq \theta_0 \leq 28^\circ$, Curve II: $0^\circ \leq \theta_0 \leq 24^\circ$, Curve III: $0^\circ \leq \theta_0 \leq 21^\circ$
**FIGURE 3**  Transmission coefficients with $\theta_0$

**FIGURE 4**  Energy distribution for reflected waves with $\theta_0$
**Figure 5** Energy distribution for transmitted waves with $\theta_0$

**Figure 6** Reflection coefficients with $\theta_0$
**Figure 7** Transmission coefficients with $\theta_0$

**Figure 8** Energy distribution for reflected waves with $\theta_0$. 
and then increase with the increase of \( \theta_0 \). In Figure 6B, \( |Z_2| \) starts decreasing to the minimum value, which then increase with the higher value of \( \theta_0 \). The values of \( |Z_3| \) in Figure 6C increase initially and decrease slightly, which increase and decrease again when the value of \( \theta_0 \) is getting larger. In Figure 7A, \( |Z_4| \) increases to the maximum value and then decreases with the rise in the value of \( \theta_0 \), while \( |Z_3| \) in Figure 7B decreases the value of \( \theta_0 \) is increased. It is observed that \( |Z_6| \) has similar pattern with \( |Z_3| \). In this case, we have observed critical angles \( \theta_0 = 30^\circ \) for \( |Z_1| \) and \( \theta_0 = 58^\circ \) for \( |Z_4| \). We have observed that the effects of \( P \) and \( P' \) on \( |Z_3| \) and \( |Z_6| \) are minimum near normal as well as grazing angle of incidence.

The variation of energy ratios corresponding to reflected and transmitted waves are depicted in Figures 8 and 9, respectively. In Figure 8A, all the curves show that \( |E_1| \) increases to some point and then drops to the minimum value, which then rises with the increase of \( \theta_0 \). The values of \( |E_2| \) in Figure 8B increase with the rise in the value of \( \theta_0 \). We notice that the effect of initial stresses is very small near the normal incidence. It is observed that \( |E_3| \) in Figure 8C increases up to certain value and then decreases with the increase of \( \theta_0 \). In Figure 9, we have observed that the variation of \( |E_4| \) and \( |E_6| \) have similar pattern with \( |Z_4| \) and \( |Z_6| \), respectively. The values of \( |E_5| \) in Figure 9B decrease initially and then increase up to certain value, which decreases again when the value of \( \theta_0 \) is increased. Herein, we have noticed critical angles, \( \theta_0 = 30^\circ \) for \( |E_1| \) and \( \theta_0 = 58^\circ \) for \( |E_4| \). The law of conservation of energy is also hold for this case. It is observed that the effect of initial stresses are very small near the grazing and normal incidence in most of the amplitude and energy ratios for incident QL and QT waves.

**9 | CONCLUSION**

For incident QL and QT-waves at the interface between two different half-spaces of initially stressed transversely isotropic thermoelastic materials, the reflected and transmitted waves are analyzed. The formula corresponding to the coefficient of reflection/transmission and energy ratios are obtained with the help of appropriate boundary conditions. These formulas are computed numerically for a particular model. We have the following concluding remarks:

(i) The reflection/transmission coefficients and the energy distributions are found to depend on angle of incidence, elastic, thermal and initial stress parameters.
(ii) The ratios $|Z_1|$ and $|E_1|$ increase, while $|Z_4|$ and $|E_4|$ decrease with the rise in the value of $\theta_0$ for the incident QL-wave. 

(iii) For incident QL-wave, the maximum and minimum effects of initial stresses on $|Z_1|$, $|Z_4|$, $|E_1|$, and $|E_4|$ are found near normal and grazing angle of incidence. 

(iv) The effect of initial stresses on $|Z_3|$, $|Z_6|$, $|E_3|$, and $|E_6|$ is minimum near normal angle of incidence for incident QL-wave and they have similar pattern. 

(v) The reflected and transmitted QL-waves have critical angles at 30° and 58°, respectively, for the incident QT wave. 

(vi) The addition of all the energy distributions for the incident QL and QT-waves is close to unity. 

(vii) We have recovered the results of Singh\textsuperscript{15} and Sharma\textsuperscript{13} in the special cases of the present problem. 

ACKNOWLEDGEMENT 

S. S. Singh acknowledges the Department of Science and Technology (SERB), New Delhi for their financial support through Grant No. EMR/2017/001723 to complete this work. The author (Lalawmpuia T) thanks to CSIR, New Delhi for providing junior research fellow (JRF). 

CONFLICT OF INTEREST 

The authors declare no conflict of interest. 

ORCID 

Singh SS @ https://orcid.org/0000-0003-4039-4019 

REFERENCES 

1. Biot M. Thermoelectricity and irreversible thermo-dynamics. *J Appl Phys*. 1956;27:249-253. 
2. Lord HW, Shulman Y. A generalized dynamical theory of thermoelasticity. *J Mech Phys Solids*. 1967;15:299-309. 
3. Green AE, Lindsay KA. Thermoelectricity. *J Elast*. 1972;2:1-7. 
4. Dhaliwal RS, Wang J. A generalized theory of thermoelectricity for prestressed bodies with microstructure. *Int J Solids Struct*. 1993;30:3467-3473. 
5. Wang J, Dhaliwal RS, Majumdar SR. Some theorems in the generalized theory of thermoelectricity for prestressed bodies. *Indian J Pure Appl Math*. 1997;28:267-276. 
6. Dhaliwal RS, Sherief HH. Generalized thermoelectricity for anisotropic media. *Q Appl Math*. 1980;33:1-8. 
7. Chandrasekharaihaiah DS. Thermoelectricity with second sound. *Appl Mech Rev*. 1986;39:355-376. 
8. Hetnarski RB. *Encyclopedia of Thermal Stresses*. Netherlands: Springer; 2014. 
9. Chadwick P, Seet LTC. Wave propagation in a transversely isotropic heat conducting elastic material. *Mathematica*. 1970;17:255-274. 
10. McCarthy MF. Wave propagation in generalized thermoelectricity. *Int J Eng Sci*. 1972;10:593-602. 
11. Chadwick P. Basic properties of plane harmonic waves in a prestressed heat-conducting elastic material. *J Ther Stresses*. 1979;2:193-214. 
12. Sharma JN, Sidhu RS. On the propagation of plane harmonic waves in anisotropic generalized thermoelectricity. *Int J Eng Sci*. 1986;24:1511-1516. 
13. Sharma JN. Reflection of thermoelectric waves from the stress-free insulated boundary of an anisotropic half-space. *Indian J Pure Appl Math*. 1988;19:294-304. 
14. Singh B. Wave propagation in an anisotropic generalized thermoelectric solid. *Indian J Pure Appl Math*. 2003;34:1479-1485. 
15. Singh B. Wave propagation in an initially stressed transversely isotropic thermoelectric solid half-space. *Appl Math Comput*. 2010;217:705-715. 
16. Sharma MD. Propagation and attenuation of Rayleigh waves in generalized thermoelectric media. *J Seismol*. 2013;18:61-79. 
17. Tomar SK, Bhagwan J, Steeb H. Time harmonic waves in a thermo-viscoelastic material with voids. *J Vib Control*. 2013;20(8):1119-1136. 
18. Sinha AN, Sinha SB. Reflection of thermoelastic waves at a solid half-space with thermal relaxation. *J Phys Earth*. 1974;22:237-244. 
19. Sinha SB, Elsibai KA. Reflection and refraction of thermoelastic waves at an interface of two semi-infinite media with two thermal relaxation times. *J Ther Stresses*. 1997;20:129-146. 
20. Tomar SK, Singh J. Transmission of longitudinal waves through a plane interface between two dissimilar porous elastic solid half-spaces. *Appl Math Comput*. 2005;169:671-688. 
21. Singh J, Tomar SK. Reflection and transmission of transverse waves at a plane interface between two different porous elastic solid half-spaces. *Appl Math Comput*. 2006;176:364-378. 
22. Singh SS. Reflection and transmission of couple longitudinal waves at a plane interface between two dissimilar half-spaces of thermo-elastic materials with voids. *Appl Math Comput*. 2011;218:3359-3371. 
23. Sharma JN. Transverse wave at a plane interface in thermo-elastic materials with voids. *Meccanica*. 2013;48:617-630. 
24. Achenbach JD. *Wave Propagation in Elastic Solids*. New York, NY: North-Holland Publishing Company; 1976. 
25. Sinha SB, Elsibai KA. Reflection of thermoelastic waves at a solid half-space with two relaxation times. *J Ther Stresses*. 1996;19(8):749-762.
26. Othman MIA, Song YQ. Reflection of plane waves from an elastic solid half-space under hydrostatic initial stress without energy dissipation. *Int J Solids Struct*. 2007;44(17):5651-5664.

27. Othman MIA, Song YQ. Reflection of plane waves from a thermo-micro-stretch elastic solid under the effect of rotation. *Can J Phys*. 2014;92(6):488-496.

28. Kumar R, Miglani A, Kumar S. Reflection and transmission of plane waves between two different fluid saturated porous half spaces. *Bull Pol Acad Sci Tech Sci*. 2011;59(2):227-234.

29. Othman MIA, Atwa SY. Thermoelastic plane waves for an elastic solid half-space under hydrostatic initial stress of type III. *Meccanica*. 2012;47(6):1337-1347.

30. Sharma K, Bhargava RR. Propagation of thermoelastic plane waves at an imperfect boundary of thermal conducting viscous liquid/generalized thermoelastic solid. *Afr Mat*. 2014;25:81-102.

31. Pal PC, Kumar S, Mandal D. Wave propagation in an inhomogeneous anisotropic generalized thermoelastic solid. *J Ther Stresses*. 2014;37:817-831.

32. Guo X, Wei P. Effects of initial stress on the reflection and transmission waves at the interface between two piezoelectric half spaces. *Int J Solids Struct*. 2014;51:3735-3751.

33. Placidi L, Rosi G, Giorgio I, Madeo A. Reflection and transmission of plane waves at surfaces carrying material properties and embedded in second-gradient materials. *Math Mech Solids*. 2014;19(5):555-578.

34. Singh SS. Transmission of elastic waves in anisotropic nematic elastomers. *ANZIAM J*. 2015;56:381-396.

35. Sahu SA, Paswan B, Chattopadhyay A. Reflection of plane waves through isotropic medium sandwiched between two highly anisotropic half-paces. *Waves Random Complex Media*. 2016;26(1):42-67.

36. Chatterjee M, Dhua S, Chattopadhyay A, Sahu SA. Reflection and refraction for three-dimensional plane waves at the interface between distinct anisotropic half-spaces under initial stresses. *Int J Geomech*. 2016;16(4):1-23.

37. Abbas IA, Abdalla AENN, Alzahrani FS, Spagnuolo M. Wave propagation in a generalized thermoelastic plate using eigenvalue approach. *J Ther Stresses*. 2016;39(11):1367-1377.

38. Zorarmuama C, Singh SS. Elastic waves in thermoelastic saturated porous medium. *Meccanica*. 2016;51:593-609.

39. Abdalla AENN, Alshaikh F, Del Vescovo D, Spagnuolo M. Plane waves and eigenfrequency study in a transversely isotropic magneto-thermoelastic medium under the effect of a constant angular velocity. *J Ther Stresses*. 2017;40(9):1079-1092.

40. Othman MIA, Abo-Dahab SM, Alsebaey ONS. Reflection of plane waves from a rotating magneto-thermoelastic medium with two-temperature and initial stress under three theories. *Mech Mech Eng*. 2017;21(2):217-232.

41. Singh SS, Lianngenga R. Reflection and transmission of wave propagation in micropolar thermoelastic materials with voids. *Appl Math Model*. 2017;49:487-497.

42. Barak MS, Kaliraman V. Reflection and transmission of elastic waves from an imperfect boundary between micropolar elastic solid half space and fluid saturated porous solid half space. *Mech Adv Mater Struct*. 2019;26(14):1226-1233.

43. Khurana A, Tomar SK. Waves at interface of dissimilar nonlocal micropolar elastic half-spaces. *Mech Adv Mater Struct*. 2019;26(10):825-833.

44. Lianngenga R, Singh SS. Effect of thermal and micro-inertia on the refraction of elastic waves in micropolar thermoelastic materials with voids. *Int J Comp Meth Eng Sci Mech*. 2018;19(4):240-252.

45. Kumar R, Kaushal P, Sharma R. Transversely isotropic magneto-visco thermoelastic medium with vacuum and without energy dissipation. *JSME*. 2018;10(2):416-434.

46. Singh SS, Lalawmpuia T. Stoneley and Rayleigh waves in thermoelastic materials with voids. *J Vib Control*. 2019;25(14):2053-2062.

47. Pal P, Kanoria M. Thermoelastic wave propagation in a transversely isotropic thick plate under Green–Naghdi theory due to gravitational field. *J Ther Stresses*. 2017;40(4):470-485.

---

**How to cite this article:** Tochhawng L, SS. Effect of initial stresses on the elastic waves in transversely isotropic thermoelastic materials. *Engineering Reports*. 2020;2:e12104. [https://doi.org/10.1002/eng2.12104](https://doi.org/10.1002/eng2.12104)