On chiral extrapolations of charmed meson masses
and coupled-channel reaction dynamics

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Abstract

We perform an analysis of QCD lattice data on charmed meson masses. The quark-mass dependence of the data set is used to gain information on the size of counter terms of the chiral Lagrangian formulated with open-charm states with $J^P = 0^-$ and $J^P = 1^-$ quantum numbers. Of particular interest are those counter terms that are active in the exotic flavour sextet channel. A chiral expansion scheme where physical masses enter the extrapolation formulae is developed and applied to the lattice data set. Good convergence properties are demonstrated and an accurate reproduction of the lattice data based on ensembles of PACS-CS, MILC, ETMC and HSC with pion and kaon masses smaller than 600 MeV is achieved. It is argued that a unique set of low-energy parameters is obtainable only if additional information from HSC on some scattering observables is included in our global fits. The elastic and inelastic s-wave $\pi D$ and $\eta D$ scattering as considered by HSC is reproduced faithfully. Based on such low-energy parameters we predict 15 phase shifts and inelasticities at physical quark masses but also for an additional HSC ensemble at smaller pion mass. In addition we find a clear signal for a member of the exotic flavour sextet states in the $\eta D$ channel, below the $\bar{K}D_s$ threshold. For the isospin violating strong decay width of the $D_{s0}^{*}(2317)$ we obtain the range (104 – 116) keV.
I. INTRODUCTION

Systems with one heavy and one light quark play a particularly important role in the spectroscopy of QCD [1-4]. Two distinct approximate symmetries characterize the spectrum of open-charm mesons. While in the limit of an infinitely heavy charm quark the heavy quark spin-symmetry arises, the opposite limit with vanishing masses for the up, down and strange quark mass leads to the flavour SU(3) chiral symmetry. The approximate chiral symmetry of the up, down and strange quarks guides the construction of effective field theory approaches based on the chiral Lagrangian. There are two complementary approaches feasible. Either one may construct an effective chiral Lagrangian formulated in terms of heavy-quark multiplet fields [1, 2, 5] or one may start with an effective chiral Lagrangian with fully relativistic fields, wherein the low-energy constants are correlated by constraints from the heavy-quark spin symmetry [6, 7]. The former approach may be more economic in applications where the coupled-channel unitarity constraint is implemented by means of partial summation techniques [8, 13].

A striking prediction of the leading order chiral interaction of the Goldstone bosons with the D mesons with either \( J^P = 0^- \) or \( J^P = 1^- \) is an attractive short-range force in the exotic flavour sextet channel [6, 8, 9]. The strength of this interaction is somewhat reduced as compared to a corresponding force in the conventional flavour triplet channel that can be successfully used to describe the lowest scalar and axial-vector states in the open-charm meson spectrum [6, 8, 9, 11, 16]. Whether the chiral force in the flavour sextet sector leads to the formation of exotic open-charm meson states is an open issue. The possible existence of such an exotic flavour sextet multiplet of states depends on the precise form of chiral correction terms [6, 9].

In this work we wish to study the size of such chiral counter terms. First rough studies [6, 9] suffer from limited empirical constraints. Additional information from first QCD lattice simulation on a set of s-wave scattering lengths was used in a series of later works [10-13]. Results are obtained that in part show unnaturally large counter terms and/or illustrate some residual dependence on how to set up the coupled-channel computation. Here we follow a different path and try to use the recent data set on the quark-mass dependence of the D meson ground-state masses [17, 23]. This dynamics is driven in part by the counter terms that also have significant impact on the open-charm coupled-channel systems as discussed
above. One may hope to obtain results that are less model dependent in this case.

However, it is well known that chiral perturbation theory formulated with three light
flavours does not always show a convincing convergence pattern [24–31]. How is this for the
case at hand? Only few studies are available in which this issue is addressed for open-charm
meson systems. In a recent work the authors presented a novel chiral extrapolation scheme
for the quark-mass dependence of the baryon octet and decuplet states that is formulated
in terms of physical masses [32–35]. It is the purpose of our study to adapt this scheme to
the open-charm sector of QCD and apply it to the available lattice data set. This requires
in particular to consider the D mesons with $J^P = 0^-$ and $J^P = 1^-$ quantum numbers on
equal footing. For a given set of low-energy constants each set of the four D meson masses
has to be determined numerically as a solution of a non-linear system.

The work is organized as follows. In section II the part of the chiral Lagrangian that
is relevant here is recalled. It follows a section where the one-loop contributions to the D
meson masses are derived in a finite box. We do not consider discretization effects in our
study. In section III and IV power counting in the presence of physical masses is discussed.
The application to available lattice data sets is presented in sections V and VI. Lattice data
taken on ensembles of PACS-CS, MILC, ETMC and HSC are considered. In section VII we
present our predictions for phase shifts and in-elasticities based on a parameter set obtained
form the considered lattice data. In addition the fate of possible exotic states but also the
isospin violating strong decay width of the $D_{s0}^*(2317)$ is discussed. With a summary and
outlook the paper is closed.
II. THE CHIRAL LAGRANGIAN WITH OPEN-CHARM MESON FIELDS

We recall the chiral Lagrangian formulated in the presence of two anti-triplets of \( D \) mesons with \( J^P = 0^- \) and \( J^P = 1^- \) quantum numbers \([1, 2]\). In the relativistic version the Lagrangian was developed in \([6, 8, 9]\). The kinetic terms read

\[
\mathcal{L}_{\text{kin}} = \left( \partial_\mu D \right) \left( \partial^\mu \bar{D} \right) - M^2 D \bar{D} - \left( \partial_\mu D^{\alpha} \right) \left( \partial^\nu \bar{D}_\nu \alpha \right) + \frac{1}{2} (M + \Delta)^2 D^{\alpha} \bar{D}_\alpha
\]

\[- f^2 \text{tr} \left\{ U_\mu U^{\mu} \right\} + \frac{1}{2} f^2 \text{tr} \left\{ \chi_+ \right\},
\]

(1)

where

\[
U_\mu = \frac{1}{2} e^{-i \frac{\Phi}{2}} \left( \partial_\mu e^{i \frac{\Phi}{2}} \right) e^{-i \frac{\Phi}{2}}, \quad \Gamma_\mu = \frac{1}{2} e^{-i \frac{\Phi}{2}} \partial_\mu e^{i \frac{\Phi}{2}} + \frac{1}{2} e^{i \frac{\Phi}{2}} \partial_\mu e^{-i \frac{\Phi}{2}},
\]

\[
\chi_\pm = \frac{1}{2} \left( e^{i \frac{\Phi}{2}} \chi_0 e^{i \frac{\Phi}{2}} \pm e^{-i \frac{\Phi}{2}} \chi_0 e^{-i \frac{\Phi}{2}} \right), \quad \chi_0 = 2 B_0 \text{diag}(m_u, m_d, m_s),
\]

\[
\partial_\mu \bar{D} = \partial_\mu \bar{D} + \Gamma_\mu \bar{D}, \quad \partial_\mu D = \partial_\mu D - D \Gamma_\mu.
\]

(2)

Following \([6]\) we represent the \( 1^- \) field in terms of an antisymmetric tensor field \( D_{\mu\nu} \). The covariant derivative \( \partial_\mu \) involves the chiral connection \( \Gamma_\mu \), the quark masses enter via the symmetry breaking fields \( \chi_\pm \) and the octet of the Goldstone boson fields is encoded into the \( 3 \times 3 \) matrix \( \Phi \). The parameter \( f \) is the chiral limit value of the pion-decay constant.

Finally, given our particular renormalization scheme, the parameters \( M \) and \( M + \Delta \) give the masses of the \( D \) and \( D^* \) mesons in that limit with \( m_u = m_d = m_s = 0 \).

We continue with first order interaction terms

\[
\mathcal{L}^{(1)} = 2 g_P \left\{ D_{\mu\nu} U^\mu \left( \partial^\nu \bar{D} \right) - \left( \partial^\nu D \right) U^\mu \bar{D}_\nu \right\}
\]

\[- \frac{i}{2} \tilde{g}_P \epsilon^{\mu\nu\alpha\beta} \left\{ D_{\mu\nu} U_\alpha \left( \partial^\beta \bar{D}_\beta \right) + \left( \partial^\beta D_\beta \right) U_\alpha \bar{D}_\mu \right\},
\]

(3)

which upon an expansion in powers of the Goldstone boson fields provide the 3-point coupling constants of the Goldstone bosons to the \( D \) mesons. While the decay of the charged \( D^* \)-mesons \([6]\) implies

\[
|g_P| = 0.57 \pm 0.07,
\]

(4)

the parameter \( \tilde{g}_P \) in \((3)\) can not be extracted from empirical data directly. The size of \( \tilde{g}_P \) can be estimated using the heavy-quark symmetry of QCD \([1, 2]\). At leading order one expects \( \tilde{g}_P = g_P \).
Second order terms of the chiral Lagrangian were first studied in [6, 9], where the focus was on counter terms relevant for s-wave scattering of Goldstone bosons with the $D$ mesons. A list of eight terms with dimension less parameters $c_i$ and $\tilde{c}_i$ was identified. This list was extended by further terms relevant for p-wave scattering in [36]. A complete collection of relevant terms is

$$\mathcal{L}^{(2)} = - (4 c_0 - 2 c_1) D \bar{D} \text{tr} \chi_+ - 2 c_1 D \chi_+ \bar{D}$$

$$+ 4 (2 c_2 + c_3) D \bar{D} \text{tr} \left( U_\mu U^{\mu \dagger} \right) - 4 c_3 D U_\mu U^{\mu \dagger} \bar{D}$$

$$+ \frac{1}{M^2} (4 c_4 + 2 c_5) (\partial_\mu D)(\partial_\nu \bar{D}) \text{tr} \left[ U^\mu, U^{\nu \dagger} \right] + \frac{1}{M^2} 2 c_5 (\partial_\mu D)[U^\mu, U^{\nu \dagger}]_+ (\partial_\nu \bar{D})$$

$$+ i c_6 \epsilon^{\mu \nu \rho \sigma} \left( D [U_\mu, U^{\nu \dagger}]_+ \bar{D} \rho \sigma - D \rho \sigma [U^{\mu \dagger}, U_\mu]_+ \bar{D} \right)$$

$$+ (2 \tilde{c}_0 - \tilde{c}_1) D^{\mu \nu} \bar{D}_{\mu \nu} \text{tr} \chi_+ + \tilde{c}_1 D^{\mu \nu} \chi_+ \bar{D}_{\mu \nu}$$

$$- (4 \tilde{c}_2 + 2 \tilde{c}_3) D^{\alpha \beta} \bar{D}_{\alpha \beta} \text{tr} \left( U_\mu U^{\mu \dagger} \right) + 2 \tilde{c}_3 D^{\alpha \beta} U_\mu U^{\mu \dagger} \bar{D}_{\alpha \beta}$$

$$- \frac{1}{(M + \Delta)^2} (2 \tilde{c}_4 + \tilde{c}_5) (\partial_\mu D^{\alpha \beta})(\partial_\nu \bar{D}_{\alpha \beta}) \text{tr} \left[ U^\mu, U^{\nu \dagger} \right] +$$

$$+ \frac{1}{(M + \Delta)^2} \tilde{c}_5 (\partial_\mu D^{\alpha \beta}) \left[ U^\mu, U^{\nu \dagger} \right]_+ (\partial_\nu \bar{D}_{\alpha \beta}) - 4 \tilde{c}_6 D^{\mu \alpha} [U_\mu, U^{\nu \dagger}]_+ \bar{D}_{\nu \alpha}, \quad (5)$$

where the parameter $M$ and $M + \Delta$ are the $D$ and $D^*$ meson masses as evaluated at $m_u = m_d = m_s = 0$. In the limit of a very large charm-quark mass it follows $M \to \infty$ but $\Delta \to 0$. All parameters $c_i$ and $\tilde{c}_i$ are expected to scale linearly in the parameter $M_0$. As illustrated in [6] it holds $\tilde{c}_i = c_i$ in the heavy-quark mass limit.

A first estimate of some parameters can be found in [6] based on large-$N_c$ arguments. Since at leading order in a $1/N_c$ expansion single-flavour trace interactions are dominant, the corresponding couplings should go to zero in the $N_c \to \infty$ limit, suggesting

$$c_0 \simeq \frac{c_1}{2}, \quad c_2 \simeq -\frac{c_3}{2}, \quad c_4 \simeq -\frac{c_5}{2}$$

$$\tilde{c}_0 \simeq \frac{\tilde{c}_1}{2}, \quad \tilde{c}_2 \simeq -\frac{\tilde{c}_3}{2}, \quad \tilde{c}_4 \simeq -\frac{\tilde{c}_5}{2}. \quad (6)$$

In the combined heavy-quark and large-$N_c$ limit we are left with 4 free parameter only, $c_1, c_3, c_5, c_6$. For two of them approximate ranges

$$c_1 \simeq 0.44 - 0.47, \quad c_3 + c_5 \simeq 1.0 - 1.4 \quad (7)$$

were obtained previously in [6]. While the parameter $c_1$ can be estimated from the $D$ meson masses, the parameter $c_3$ is constrained by the empirical $\pi D$ invariant mass spectrum [6, 9]. A complementary estimate was explored in [10], where the parameter $c_3 + c_5$ was adjusted.
to first QCD lattice computations for s-wave scattering lengths of the Goldstone bosons with the $D$ mesons. It is remarkable that their range for $c_3 + c_5 \simeq 1$ is quite consistent with the earlier estimates [6, 9] based on the empirical $\pi D$ invariant mass spectrum. The $c_3$ parameter is of crucial importance for the physics of two exotic sextets of $J^P = 0^+$ and $J^P = 1^+$ resonances. Such multiplets are predicted by the leading order chiral Lagrangian [1], which entails in particular the Tomozawa-Weinberg coupled-channel interactions of the Goldstone bosons with the $D$ mesons [8]. The latter predicts weak attraction in the flavour sextet channel. If used as the driving term in a coupled-channel unitarization exotic signals appear. A reliable estimate of the correction terms proportional to $c_3$ and $\tilde{c}_3$ is important in order to arrive at a detailed picture of this exotic sector of QCD [6, 9].

We close this section with a first construction of the symmetry breaking counter terms proportional to the product of two quark masses:

$$L^{(4)} = -d_1 D \chi_+^2 \bar{D} - d_2 D \chi_+ \bar{D} \text{tr} (\chi_+) - d_3 D \bar{D} \text{tr} (\chi_+^2) - d_4 D \bar{D} (\text{tr} \chi_+)^2$$

$$+ \frac{1}{2} \tilde{d}_1 D^{\mu \nu} \chi_+^2 \bar{D}_{\mu \nu} + \frac{1}{2} \tilde{d}_2 D^{\mu \nu} \chi_+ \bar{D}_{\mu \nu} \text{tr} (\chi_+) + \frac{1}{2} \tilde{d}_3 D^{\mu \nu} \bar{D}_{\mu \nu} \text{tr} (\chi_+^2)$$

$$+ \frac{1}{2} \tilde{d}_4 D^{\mu \nu} \bar{D}_{\mu \nu} (\text{tr} \chi_+)^2.$$  \(8\)

Such terms are relevant in the chiral extrapolation of the $D$ meson masses. For the pseudo-scalar mesons we provide the tree-level contributions to the polarization $\Pi_H^{(2)}$ and $\Pi_H^{(4-\chi)}$ of the $D$ and $D_s$ mesons. We use a convention with

$$M_{H|0-}^2 = M^2 + \Pi_H^{(2)} + \Pi_H^{(4-\chi)} + \cdots, \quad M_{H|1-}^2 = (M + \Delta)^2 + \Pi_H^{(2)} + \Pi_H^{(4-\chi)} + \cdots,$$

$$\Pi_H^{(2)} = 2 B_0 (4 c_0 - 2 c_1) (m_s + 2 m) + 4 B_0 c_1 m,$$

$$\Pi_D^{(4-\chi)} = 4 B_0^2 \left( d_1 + 2 d_2 + 2 d_3 + 4 d_4 \right) m^2 + 4 B_0^2 \left( d_3 + d_4 \right) \left( m_s^2 + 4 B_0^2 \left( d_2 + 4 d_4 \right) m m_s,$$

$$\Pi_{D_s}^{(2)} = 2 B_0 \left( 4 c_0 - 2 c_1 \right) (m_s + 2 m) + 4 B_0 c_1 m,$$

$$\Pi_{D_s}^{(4-\chi)} = 4 B_0^2 \left( 2 d_3 + 4 d_4 \right) m^2 + 4 B_0^2 \left( d_1 + 2 d_2 + d_3 + 4 d_4 \right) m_s^2 + 4 B_0^2 \left( 2 d_2 + 4 d_4 \right) m m_s,$$  \(9\)

where we consider the isospin limit with $m_w = m_d = m$. Analogous expressions hold for the vector mesons polarization $\Pi_{H|1-}^{(2)}$ and $\Pi_{H|1-}^{(4-\chi)}$ where the replacements $c_i \to \tilde{c}_i$ and $d_i \to \tilde{d}_i$ are to be applied to [9]. With $\tilde{c}_i = c_i$ and $\tilde{d}_i = d_i$ and $\Delta \to 0$ the heavy-quark spin symmetry is recovered exactly.

We need to mention a technical issue. The propagator $S^{\alpha \beta}_{\mu \nu}(p)$ of our $1^-$ fields involves four Lorentz indices, which are pairwise antisymmetric. Either interchanging $\alpha \leftrightarrow \beta$ or
\( \mu \leftrightarrow \nu \) generates a change in sign. A mass renormalization from a loop contribution arises from a particular projection \( \Pi(p^2) \) of the polarization tensor \( \Pi_{\alpha\beta}(p) \) with

\[
\Pi(p^2) = \frac{-1}{(d-1)p^2} \left( g_{\mu\alpha} p_{\nu} p_{\beta} - g_{\mu\beta} p_{\nu} p_{\alpha} - g_{\nu\alpha} p_{\mu} p_{\beta} + g_{\nu\beta} p_{\mu} p_{\alpha} \right) \Pi^{\mu\nu,\alpha\beta}(p),
\]

where \( d \) is the space-time dimension. This is the part which is used in (9) and will be used also in the following.
The chiral Lagrangian of section I is used to compute the D-meson masses at the one-loop level. In order to prepare for a comparison of QCD lattice data this computation is done in a finite box of volume $V$. A direct application of the relativistic chiral Lagrangian in the conventional $\overline{MS}$ scheme does lead to a plethora of power-counting violating contributions. There are various ways to arrive at results that are consistent with the expectations of power counting rules [32, 37–39].

We follow here the $\chi$-MS approach developed previously for the chiral dynamics of baryons [32, 40, 41], which is based on the Passarino-Veltman reduction scheme [42]. Recently this scheme was generalized for computations in a finite box [34]. This implies that all finite box effects are exclusively determined by the volume dependence of a set of universal scalar loop functions as discussed and presented in [34]. Our results will be expressed in terms of Clebsch coefficients $G_{QR}^{(H)}$, $G_{HQ}^{(S)}$, $G_{HQ}^{(V)}$ and a set of generic loop functions. While the index $H$ or $R$ runs either over the triplet of pseudo-scalar or vector $D$ mesons the index $Q$ runs over the octet of Goldstone bosons (see Tab. I and Tab. II). In our case there will be two tadpole integrals $\tilde{I}_Q^{(0)}$ and $\tilde{I}_Q^{(2)}$ from the Goldstone bosons and the scalar bubble-loop integral $\tilde{I}_{QR}$. In addition there may be tadpole contributions $\tilde{I}_R^{(n)}$ involving an intermediate $D$ meson. In order to render the power counting manifest it suffices to supplement the Passarino-Veltman reduction scheme by a minimal and universal subtraction scheme [32]:

- any tadpole integral involving a heavy particle is dropped
- the scalar bubble-loop integral requires a single subtraction.

The required loop functions have been used and detailed in a previous work [32, 34] for finite box computations. For the readers’ convenience we recall the loop functions in the infinite box limit [32, 34] with

$$\tilde{I}_Q^{(0)} = \tilde{I}_Q = \frac{m_Q^2}{\left(4\pi^2\right)^2} \log \left(\frac{m_Q^2}{\mu^2}\right), \quad \tilde{I}_Q^{(2)} = \frac{1}{4} m_Q^2 \tilde{I}_Q,$$

$$\tilde{I}_{QR} = \frac{1}{16 \pi^2} \left\{ \gamma_R^H - \left(\frac{1}{2} + \frac{m_Q^2 - M_R^2}{2 M_H^2}\right) \log \left(\frac{m_Q^2}{M_R^2}\right) \right.\right.$$  

$$+ \frac{p_{QR}}{M_H} \left( \log \left(1 - \frac{M_R^2 - 2 p_{QR} M_H}{m_Q^2 + M_R^2}\right) - \log \left(1 - \frac{M_R^2 + 2 p_{QR} M_H}{m_Q^2 + M_R^2}\right) \right) \right\}.$$
with \[ p_{QR}^2 = \frac{M_H^2}{4} - \frac{M_R^2 + m_Q^2}{2} + \frac{(M_R^2 - m_Q^2)^2}{4 M_H^2}, \] (11)

where we note that in the infinite volume limit the two tadpole integrals \( \bar{I}_Q^{(0)} \) and \( \bar{I}_Q^{(2)} \) turn dependent and can no longer be discriminated in that case. The finite volume corrections for \( \bar{I}_Q^{(0)}, \bar{I}_Q^{(2)} \) and \( \bar{I}_{QR} \) are detailed in [34].

We point at the presence of the additional subtraction term \( \gamma_R^H = \gamma^H_R(M, \Delta) \) with

\[ \gamma^H_R = - \lim_{m, m_s \to 0} \frac{M_R^2 - M_H^2}{M^2_R} \log \left| \frac{M_R^2 - M_H^2}{M^2_R} \right|, \] (12)

as suggested recently in [34] in the analogous case of a baryon self-energy computation. The subtraction term depends on the chiral limit values \( M \) and \( M + \Delta \) of the \( D \) and \( D^* \) meson masses only. It was not yet imposed in earlier computations [32–34]. As was discussed in [34] the request of such a term comes from a study of the chiral regime with

\[ m_Q \ll \Delta \quad \text{with} \quad Q \in \{\pi, K, \eta\}. \] (13)

Within a counting scheme with \( m_Q \sim \Delta \sim Q \) there is no need for any additional subtractions beyond the ones enforced by the \( \chi \)-MS approach. However, to arrive at consistent results for \( m_Q \ll \Delta \) this subtraction is instrumental. While for \( \Delta \sim m_Q \sim Q \) and \( \gamma_R^H = 0 \) the scalar bubble scales with \( \bar{I}_{QR} \sim Q \) as expected from dimensional analysis, in the chiral regime with \( m_Q \ll \Delta \) and \( m_Q \sim Q \) one would expect \( \bar{I}_{QR} \sim Q^2 \sim m_Q^2 \). This expectation turns true only, for \( \gamma_R^H \neq 0 \) as chosen in (12).

We are now well prepared to collect all contributions to the D-meson self energies at the one-loop level. Consider the bubble loop and tadpole contributions. The Passarino-Veltman

\[
\begin{align*}
G^{(D)}_{\pi D^*} &= 2 \sqrt{3} \tilde{g}_P \\
G^{(D)}_{\eta D^*} &= \frac{2}{\sqrt{3}} \tilde{g}_P \\
G^{(D)}_{K D^*} &= 2 \sqrt{2} \tilde{g}_P \\
G^{(D)}_{K^* D} &= 2 \sqrt{3} \tilde{g}_P \\
G^{(D)}_{\pi D} &= \frac{2}{\sqrt{3}} \tilde{g}_P \\
G^{(D)}_{\eta D} &= \frac{4}{\sqrt{3}} \tilde{g}_P \\
G^{(D)}_{K D} &= 2 \sqrt{2} \tilde{g}_P \\
G^{(D)}_{K^* D} &= 2 \sqrt{3} \tilde{g}_P \\
G^{(D)}_{K^* D^*} &= 4 \tilde{g}_P \\
G^{(D)}_{\pi^0 D} &= 2 \sqrt{3} \tilde{g}_P \\
G^{(D)}_{\eta^0 D} &= \frac{4}{\sqrt{3}} \tilde{g}_P \\
G^{(D)}_{K^0 D} &= 2 \sqrt{2} \tilde{g}_P \\
G^{(D)}_{K^{0*} D} &= \frac{4}{\sqrt{3}} \tilde{g}_P \\
G^{(D)}_{K^{0*} D^*} &= 4 \tilde{g}_P \\
G^{(D)}_{K^0 D^*} &= 2 \sqrt{2} \tilde{g}_P \\
G^{(D)}_{K^{0*} D^*} &= 4 \tilde{g}_P \\
G^{(D)}_{K^{0*} D^*} &= \frac{4}{\sqrt{3}} \tilde{g}_P \\
G^{(D)}_{K^{0*} D^*} &= 2 \sqrt{2} \tilde{g}_P \\
G^{(D)}_{K^{0*} D^*} &= 2 \sqrt{2} \tilde{g}_P \\
G^{(D)}_{K^{0*} D^*} &= 4 \tilde{g}_P \\
\end{align*}
\]

TABLE I. Coefficients \( G^{(H)}_{QR} \)
The expansion of the loop function as they are given confirms the leading chiral power as expected.

The results (14, 16) deserve a detailed discussion. First let us emphasize that a chiral reduction scheme in combination with the \( \chi \)-MS approach leads to the following expressions

\[
\Pi^{\text{bubble}}_{H \in [0^-]} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G^{(H)}_{QR}}{2f} \right)^2 \left\{ -\frac{1}{4} \left( M_H^2 - M_R^2 + m_Q^2 \right) \bar{I}_Q - M_H^2 p_{QR}^2 \bar{I}_Q \right\},
\]

\[
\Pi^{\text{tadpole}}_{H \in [0^-]} = \frac{1}{4f^2} \sum_{Q \in [8]} \left( G^{(\chi)}_{HQ} \bar{I}_Q - G^{(S)}_{HQ} m_Q^2 \bar{I}_Q - G^{(V)}_{HQ} M^2 \bar{I}^{(2)}_Q \right),
\]

\[
\Pi^{\text{bubble}}_{H \in [1^-]} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G^{(H)}_{QR}}{2f} \right)^2 \left\{ -\frac{1}{12} \left( M_H^2 - M_R^2 + m_Q^2 \right) \bar{I}_Q - \frac{1}{3} M_H^2 p_{QR}^2 \bar{I}_Q \right\}
\]

\[
+ \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G^{(H)}_{QR}}{2f} \right)^2 \left\{ \frac{M_H^2 + 2 M_R^2}{12 M_R^2} \bar{I}_Q^{(2)} - \frac{(M_H^2 + M_R^2)^2}{6 M_R^2} p_{QR}^2 \bar{I}_Q \right\}
\]

\[
- \left( \frac{M_H^2 - M_R^2}{24 M_H^2 M_R^2} \right) \left( \frac{M_H^2 + M_R^2}{M_H^2} \right)^2 \left( \frac{M_R^4 + 6 M_H^2 M_R^2 - 3 M_H^2}{24 M_H^2 M_R^2} \right) m_Q^2 \bar{I}_Q \right\},
\]

\[
\Pi^{\text{tadpole}}_{H \in [1^-]} = \frac{1}{4f^2} \sum_{Q \in [8]} \left( G^{(\chi)}_{HQ} \bar{I}_Q - G^{(S)}_{HQ} m_Q^2 \bar{I}_Q - G^{(V)}_{HQ} (M + \Delta)^2 \bar{I}^{(2)}_Q \right),
\]

where the loop functions are expressed in terms of physical meson masses. The sums in (14) extend over intermediate Goldstone bosons \( Q \) and pseudo-scalar or vector \( D \) mesons \( R \) with either \( R \in [0^-] \) or \( R \in [1^-] \). The Clebsch coefficients \( G^{(H)}_{QR} \) are specified in Tab. I. In the contributions from the tadpole diagrams the sums in (15) extend over the intermediate Goldstone bosons \( Q \). The coefficients \( G^{(\chi)}_{HQ}, G^{(S)}_{HQ}, G^{(V)}_{HQ} \) are listed in Tab. II.

The results (14, 16) deserve a detailed discussion. First let us emphasize that a chiral expansion of the loop function as they are given confirms the leading chiral power as expected from dimensional counting rules. All power-counting violating contributions are subtracted.
owing to the $\chi$-MS approach. Here we adopted the conventional counting rules

$$m_Q \sim Q \quad \text{and} \quad M_{1-} - M_{0-} \sim \Delta \sim Q, \quad (18)$$

which is expected to be effective for $\Delta \sim m_Q$. Our results \[14\,\,16\) are model dependent, as there are various subtraction schemes available to obtain loop expression that are compatible with dimensional counting rules. Most prominently there is the infrared regularization of Becher and Leutwyler \[38\] and the minimal subtraction scheme proposed by Gegelia and collaborators \[39\]. Following our previous work on the chiral extrapolation of the baryon masses we will attempt to extract a model independent part of such loop expressions. This goes in a few consecutive steps. The driving strategy behind this attempt is to keep the physical masses inside the loop function.

Consider first the terms that are proportional to the tadpole loop function $\bar{I}_Q$. There are two distinct classes of terms. The coefficient in front of any $\bar{I}_Q$ is either proportional to $m_Q^2$ or to $M^2_R - M^2_H$. The terms proportional to $m_Q^2 \bar{I}_Q$ or also to $\bar{I}^{(2)}_Q$ in \[14\,\,16\] have the same form as the corresponding structures in \[15\,\,17\] and therefore renormalize the low-energy parameters $c_n$ and $\tilde{c}_n$ with

$$c_r^2 = c_2 + \frac{1}{8} g_p^2, \quad c_r^4 = c_4, \quad \tilde{c}_r^4 = \tilde{c}_4 - \frac{1}{8} \bar{g}_p^2,$$

$$c_r^3 = c_3 - \frac{1}{4} g_p^2, \quad c^5_r = c_5, \quad \tilde{c}_r^5 = \tilde{c}_5 + \frac{1}{4} \bar{g}_p^2,$$

$$\tilde{c}_r^2 = \tilde{c}_2 + \frac{1}{12} \bar{g}_p^2 + \frac{1}{24} g_p^2,$$

$$\tilde{c}_r^3 = \tilde{c}_3 - \frac{1}{6} \bar{g}_p^2 - \frac{1}{12} g_p^2. \quad (19)$$

We conclude that the terms proportional to $m_Q^2 \bar{I}_Q$ or $\bar{I}^{(2)}_Q$ in \[14\,\,16\] may be dropped if we use the renormalized low-energy parameters $c^r_n$ and $\tilde{c}^r_n$ in the tadpole contributions $\bar{I}_Q$ and also in $\bar{I}^{(2)}$. Note, however, that by doing so some higher order terms proportional to

$$\left(1 - \frac{M_R^2}{M_H^2}\right)^n m_Q^2 \bar{I}_Q \rightarrow 0, \quad (20)$$

with $n \geq 1$ are neglected in $\Pi_{H \in [1-]}$. We argue that the latter terms would cause a renormalization scale dependence that can not be absorbed into the available counter terms at the considered accuracy level. In order to avoid a model dependence such terms should be dropped.

We are left with the terms proportional to $(M_R^2 - M_H^2) \bar{I}_Q$. If the charm meson masses are decomposed into their chiral moments the leading renormalization scale dependence of such
terms can be absorbed into the $Q^2$ counter terms $c_{0,1}$ and $	ilde{c}_{0,1}$. Similarly the components of order $Q^4$ can be matched with counter terms $d_n$ and $\tilde{d}_n$. Most troublesome, however, are the subleading contributions proportional to $m_Q^5 \log \mu$ in such a strict chiral expansion of the vector $D$ meson masses. There is no counter term available to remove such a scale dependence. In fact, only within a two-loop computation this issue is resolved in a conventional approach. Instead we keep the charm meson masses unexpanded in the terms $(M^2_R - M^2_H) \bar{I}_Q$ and follow the strategy proposed in [35]. For those terms we provide the following decomposition

$$
(M^2_R - M^2_H) \bar{I}_Q = \left( M^2_R - M^2_H \right) \frac{m^2_Q}{(4\pi)^2} \log \frac{m^2_Q}{M^2_R} + \frac{M^2_R - M^2_H}{M^2_R} m^2_Q \frac{\bar{I}_R}{\to 0},
$$

where the second term depending on the heavy-meson tadpole $\bar{I}_R$ can be systematically dropped without harming the chiral Ward identities. We end up with the following renormalized bubble-loop expressions

$$
\Pi^{\text{bubble}}_{H \in [0^-]} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G^{(H)}_{QR}}{2 f} \right)^2 \left\{ \alpha^H_{QR} - M^2_H p^2_{QR} \bar{I}_Q + \frac{1}{4} (M^2_R - M^2_H) \frac{m^2_Q}{(4\pi)^2} \log \frac{m^2_Q}{M^2_R} \right\}
$$

$$
\Pi^{\text{bubble}}_{H \in [1^-]} = \sum_{Q \in [8]} \sum_{R \in [0^-]} \left( \frac{G^{(H)}_{QR}}{2 f} \right)^2 \left\{ \frac{1}{3} \alpha^H_{QR} - \frac{1}{3} M^2_H p^2_{QR} \bar{I}_Q + \frac{1}{12} (M^2_R - M^2_H) \frac{m^2_Q}{(4\pi)^2} \log \frac{m^2_Q}{M^2_R} \right\}
$$

$$
+ \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G^{(H)}_{QR}}{2 f} \right)^2 \left\{ - \frac{(M^2_H + M^2_R)^2}{6 M^2_R} p^2_{QR} \bar{I}_Q + \frac{(M^2_H + M^2_R)^2}{24 M^2_R M^2_R} (M^2_R - M^2_H) \frac{m^2_Q}{(4\pi)^2} \log \frac{m^2_Q}{M^2_R} \right\},
$$

which will be the basis for our following studies. Note yet the additional subtraction terms $\alpha^H_{QR}$ in (22). Such terms were suggested in [35] for the analogous case of a baryon self-energy computation. In order to arrive at consistent results for $m_Q \ll \Delta$ the terms $\alpha^H_{QR}$ are instrumental:

$$
\alpha^H_{QR} = \frac{\alpha_1 \Delta^2}{32 \pi^2} \left\{ \left( M^2_H - M^2 \right) \left( \frac{\Delta \partial}{\partial M} - \frac{\Delta \partial}{\partial \Delta} - \frac{M + \Delta}{M} \right) \\
+ \left( M^2_R - (M + \Delta)^2 \right) \frac{M}{M + \Delta} \left( \frac{\Delta \partial}{\partial \Delta} + 1 \right) \right\} \gamma_1 + \frac{\Delta M m^2_Q}{16 \pi^2} \alpha_1 \gamma_2,
$$

13
\[
\alpha_{QR}^{H\in[1^{-}]} = \frac{\tilde{\alpha}_1 \Delta^2}{32\pi^2} \left\{ \left( M_H^2 - (M + \Delta)^2 \right) \frac{M}{M + \Delta} \left( \frac{\Delta \partial}{\partial \Delta} + 1 \right) \right.
\]
\[
+ \left. \left( M_R^2 - M^2 \right) \left( \frac{\Delta \partial}{\partial M} - \frac{\Delta \partial}{\partial \Delta} - \frac{M + \Delta}{M} \right) \right\} \tilde{\gamma}_1 + \frac{\Delta M m_Q^2}{16\pi^2} \tilde{\alpha}_1 \tilde{\gamma}_2, \quad (23)
\]

where the functions \( \alpha_i, \tilde{\alpha}_i \) and \( \gamma_i, \tilde{\gamma}_i \) depend on the ratio \( \Delta/M \) only. They are listed in Appendix A and Appendix B. While the rational functions \( \alpha_i \) and \( \tilde{\alpha}_i \) all approach the numerical value one in the limit \( \Delta \to 0 \), the functions \( \gamma_i \) and \( \tilde{\gamma}_i \) show a logarithmic divergence in that limit. We summarize the convenient implications of our subtraction scheme:

- the chiral limit values of the \( D \) meson masses are not renormalized
- the low-energy parameters \( c_{0,1} \) and \( \tilde{c}_{0,1} \) are not renormalized
- the wave-function factor of the \( D \) mesons are not renormalized in the chiral limit

We close this section with a brief discussion on the role of the renormalization scale \( \mu \). Given our scheme a scale dependence arises from the tadpole terms only. Such terms need to be considered in combination with the tree-level contribution \( \Pi_H^{(4-x)} \). This leads to the condition

\[
\mu^2 \frac{d}{d \mu^2} d_i = -\frac{1}{4} \left( \frac{\Gamma_d}{(4\pi f)^2} \right),
\]

\[
\Gamma_{d_1} = \frac{1}{6} \left( 4c_1 + 12c_3 + 3c_5 \right), \quad \Gamma_{d_2} = \frac{1}{9} \left( 44c_1 - 52c_3 - 13c_5 \right),
\]

\[
\Gamma_{d_3} = \frac{1}{18} \left( 240c_0 - 84c_1 + 240c_2 + 68c_3 + 60c_4 + 17c_5 \right),
\]

\[
\Gamma_{d_4} = \frac{1}{27} \left( 264c_0 - 132c_1 + 264c_2 + 140c_3 + 66c_4 + 35c_5 \right), \quad (24)
\]

where identical results hold for the \( \tilde{c}_i \) and \( \tilde{d}_i \) coupling constants. However, it is evident that scale invariant results follow with (24) only if the meson masses in the tadpole contributions are approximated by the leading order Gell-Mann Oakes Renner relations with \( m_{\pi}^2 = 2B_0 m \) and \( m_K^2 = B_0 (m + m_s) \) for instance. This is unfortunate since we wish to use physical masses inside all loop contributions. Recalling our previous work [35] there may be an efficient remedy of this issue. Indeed the counter term contributions can be rewritten in terms of physical masses such that scale invariance follows without insisting on the Gell-Mann Oakes Renner relations for the meson masses. Such a rewrite is most economically achieved in...
TABLE III. A rewrite of $\Pi_H^{(4-\chi)}$ in (9).

| $\Pi_H^{(4-\chi)}$ | $H = D$ | $H = D_s$ |
|---------------------|---------|---------|
| $m_\pi^4$           | $-9 d_2^c + 18 d_3^c$ | $-18 d_2^c + 18 d_3^c$ |
| $m_K^4$             | $-18 d_2^c + 24 d_3^c$ | $-12 d_2^c + 24 d_3^c$ |
| $m_\eta^4$          | $-5 d_2^c + 6 d_3^c$ | $-2 d_2^c + 6 d_3^c$ |
| $B_0 m m_\pi^2$     | $9 d_1^c$ | $18 d_1^c$ |
| $B_0 (m + m_s) m_K^2$ | $9 d_1^c$ | $6 d_1^c$ |
| $B_0 m m_\eta^2$    | $d_1^c$ | $2 d_1^c$ |
| $B_0 m_s m_\eta^2$  | $4 d_1^c$ | $0$ |
| $B_0^2 (2 m + m_s)^2$ | $4 d_4^c$ | $4 d_4^c$ |

terms of suitable linear combinations of the low-energy constants

\[ d_1^c = \frac{-1}{23} (26 d_1 + 9 d_2), \quad d_3^c = \frac{1}{345} \left( 43 d_1 + 60 d_2 + 69 d_3 \right), \]
\[ d_2^c = \frac{1}{276} \left( -132 d_1 + 18 d_2 \right), \quad d_4^c = \frac{1}{45} \left( -11 d_1 + 15 d_2 - 33 d_3 + 45 d_4 \right). \]  

With Tab. III our rewrite is specified in detail. We assure that replacing the meson masses in the table by their leading order expressions the original expressions as given in (9) are recovered identically. We note a particularity: at leading order the effects of $c_0$ in $G_H^{(\chi)}$ cannot be discriminated from $c_2$ in $G_H^{(S)}$. Scale invariance requires to consider the particular combinations $c_2 + c_0$ in $G_H^{(S)}$ and in turn use $c_0 = 0$ in $G_H^{(\chi)}$.  

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IV. SELF CONSISTENT SUMMATION APPROACH

The renormalized loop functions depend on the physical masses of the \( D \) mesons. In a conventional chiral expansion scheme the meson masses inside the loop would be expanded to a given order so that a self-consistency issue does not arise. This is fine as long as the expansion is rapidly converging. For a slowly converging system such a summation scheme is of advantage even though this may bring in some model dependence [32–35].

Let us be specific on how the summation scheme is set up in detail. There is a subtle point emphasized recently in [35] which needs some discussions. The coupling constant \( g_P \) was determined in [6] from the pion-decay width of the \( D^* \) meson using a tree-level decay amplitude. Alternatively the decay width can be extracted from the \( D^* \) meson propagator in the presence of the one-loop polarization \( \Pi^{\text{bubble}}_D \). The latter has imaginary contributions proportional to same coupling constant \( g_P^2 \) that reflect the considered decay process. In the absence of wave-function renormalization effects one would identify a Breit-Wigner width by

\[
M_{D^*} \Gamma_{D^* \to D \pi} = -\Im \Pi^{\text{bubble}}_D, \tag{26}
\]

where the loop function is evaluated at the \( D^* \) meson mass \( M_{D^*} \). Both determinations would provide identical results. However, in the presence of a wave-function renormalization effect from the loop function

\[
Z_H - 1 = \frac{\partial}{\partial M_{D^*}^2} \Pi_H, \tag{27}
\]

this would no longer be the case. Following [35] we will therefore use the following form of the Dyson equation

\[
M_H^2 - \Pi_H^{(0)} - \Pi_H^{(2)} - \Pi_H^{(4-\chi)} - \Pi_H^{\text{tadpole}} - \Pi_H^{\text{bubble}} / Z_H = 0, \tag{28}
\]

where we take \( \Pi_H^{(0)} = M^2 \) and \( \Pi_H^{(0)} = (M + \Delta)^2 \) for the pseudo-scalar and vector \( D \) mesons respectively. The second order terms \( \Pi_H^{(2)} \) are the tree-level contributions \([9]\) proportional to the quark masses as written in terms of the parameters \( c_0, c_1 \) and \( \tilde{c}_0, \tilde{c}_1 \). The fourth order terms \( \Pi_H^{(4-\chi)} \) are the tree-level contributions \([9]\) proportional to the product of two quark masses. Here the parameters \( d_i \) and \( \tilde{d}_i \) are probed. We recall that the wave-function renormalization \( Z_H \) has a quark-mass dependence which cannot be fully moved into the counter terms of the chiral Lagrangian.
We provide a first numerical estimate of the importance of the various terms in (28). We put \( \bar{\Pi}^{(4-\chi)}_{H} = \bar{\Pi}^{\text{tadpole}}_{H} = 0 \) since the associated counter terms are not known reliably. Insisting on the large-\( N_c \) relations (29)

\[
2 c_0 = c_1, \quad 2 \tilde{c}_0 = \tilde{c}_1 ,
\]

we adjust the four parameters \( c_1, \tilde{c}_1 \) and \( M, \Delta \) to the four isospin averaged pseudo-scalar and vector \( D \) meson masses [43]. The results of this procedure are collected in the second last column of Tab. IV. In the third column we show the size of the loop contribution \( \bar{\Pi}^{\text{bubble}}_{H} \) and the wave function renormalization factor \( Z_H \). From those numbers we conclude that the loop terms are as important as the contributions of the \( Q^2 \) counter terms (shown in the second column). Note also the significant size of the wave function factor for the strange \( D \) mesons. It is instructive to compare the values of the four parameters \( c_1, \tilde{c}_1 \) and \( M, \Delta \) with their corresponding values that follow in a scenario where all loop effects are neglected. Such values are shown in the last column of Tab. IV. A reasonable spread of the parameters as compared to the initial scenario is observed.

While with (28) we arrive at a renormalization scale invariant and self consistent approach for a chiral extrapolation of the \( D \) meson masses that considers all counter terms relevant at \( N^3\text{LO} \), there is an important issue remaining. Is it possible to decompose the renormalized loop function \( \bar{\Pi}^{\text{bubble}}_{H} \) into its chiral moments and therewith shed more light on the convergence properties of such a chiral expansion. It is known that a conventional chiral expansion has not too convincing convergence properties at physical values of the strange quark mass. Does a resumed scheme that is formulated in terms of physical meson masses

| \( H \) | \( \bar{\Pi}^{(2)}_{H}/(2 M_H) \) | \( \bar{\Pi}^{\text{bubble}}_{H}/(2 M_H) \) | \( Z_H \) | \( \text{with bubble} \) | \( \text{tree level} \) |
|---|---|---|---|---|---|
| \( D \) | 4.7 MeV | -50.2 MeV | 1.108 | \( M \) | 1907.4 MeV | 1862.7 MeV |
| \( D_s \) | 106.2 MeV | -65.5 MeV | 1.418 | \( \Delta \) | 191.7 MeV | 141.3 MeV |
| \( D^* \) | 5.0 MeV | -113.4 MeV | 1.163 | \( c_1 \) | 0.440 | 0.426 |
| \( D^*_s \) | 114.1 MeV | -166.1 MeV | 1.643 | \( \tilde{c}_1 \) | 0.508 | 0.469 |

TABLE IV. The loop functions [22] are evaluated with the coupling constants \( g_P = \tilde{g}_P \approx 0.57 \) and the physical isospin averaged meson masses. In addition the large-\( N_c \) relations [29] are assumed.
V. POWER-COUNTING DECOMPOSITION OF THE LOOP FUNCTION

At sufficiently small quark masses a linear dependence of the $D$ meson masses is expected as recalled in \(\text{(9)}\). The associated slope parameters $c_0, c_1$ and $\tilde{c}_0, \tilde{c}_1$ are scale independent. This is an effect of chiral order $Q^2$. With increasing quark masses additional terms in the chiral expansion turn relevant. While there is no controversy on how to count the $Q^4$ contributions $\tilde{\Pi}_H^{(4-\chi)}$ and $\tilde{\Pi}_H^{\text{tadpole}}$, it is less obvious how to further decompose the loop contribution $\tilde{\Pi}_H^{\text{bubble}}$ into its power counting moments. The loop functions depend on the physical masses $m_Q, M_H$ and $M_R$. In any power counting ansatz based on chiral dynamics we would assign

$$\frac{m_Q}{M_R} \sim Q \sim \frac{m_Q}{M_H},$$

for the ratios of the Goldstone boson masses over the D meson masses. The mass differences of either pseudo-scalar or vector mesons

$$\frac{M_H - M_R}{m_Q} \sim Q, \quad \frac{M_R - M_H}{M_H} \sim Q^2 \quad \text{for} \quad H \parallel R,$$

can also be counted without controversy. In \(\text{(31)}\) we use a notation $H \parallel R$ requesting $H, R \in [0^-]$ or $H, R \in [1^-]$. Less obvious is how to treat the mass differences of a pseudo-scalar and a vector $D$ meson.
TABLE V. The loop functions (22) are evaluated with the coupling constants $g_P = \tilde{g}_P \simeq 0.57$ and the physical isospin averaged meson masses. A decomposition according to (30, 31) and (32) is performed.

There are different schemes possible. Technically most straight forward is the extreme assumption

$$\frac{M_R - M_H}{m_Q} \sim Q, \quad \frac{M_R - M_H}{M_H} \sim Q^2$$

for $H \perp R$, (32)

which can be motivated in the limit of a large charm quark mass where $\Delta \to 0$ and therewith $\Delta \ll m_\pi$. In (32) we use a notation $H \perp R$ implying that either $H \in [0^-]$ and $R \in [1^-]$ or $H \in [1^-]$ and $R \in [0^-]$. While the counting ansatz (32) is expected to be faithful for $m_Q = m_K$ it is not so useful for $m_Q = m_\pi$. However, since the loop corrections are typically dominated by contributions involving the kaon and eta meson masses, such an assumption should have some qualitative merits nevertheless. The leading order terms are readily worked out with

$$\bar{\Pi}_{H \in [0^-]}^{\text{bubble}} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}^{(H)}}{2f} \right)^2 \left\{ \alpha_{QR}^H + X_{QR}^{(H)} \right. \right.$$

$$\left. + \frac{\gamma_R^H}{16 \pi^2} M_H^2 m_Q^2 \left[ 1 - \left( \frac{m_Q}{2M_H} - \frac{M_R - M_H}{m_Q} \right)^2 \right] \right\} + O(Q^6),$$

$$\bar{\Pi}_{H \in [1^-]}^{\text{bubble}} = \frac{2}{3} \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}^{(H)}}{2f} \right)^2 X_{QR}^{(H)} + \frac{1}{3} \sum_{Q \in [8]} \sum_{R \in [0^-]} \left( \frac{G_{QR}^{(H)}}{2f} \right)^2 \left\{ \alpha_{QR}^H + X_{QR}^{(H)} \right. \right.$$

$$\left. + \frac{\gamma_R^H}{16 \pi^2} M_H^2 m_Q^2 \left[ 1 - \left( \frac{m_Q}{2M_H} - \frac{M_R - M_H}{m_Q} \right)^2 \right] \right\} + O(Q^6),$$

$$X_{QR}^{(H)} = M_H^2 \frac{m_Q^2}{16 \pi^2} \left( \frac{m_Q^2}{M_H^2} + 2 \frac{M_R - M_H}{M_H} \right) - M_H^2 \frac{m_Q^2}{32 \pi^2} \left( \frac{m_Q^2}{M_H^2} - 3 \frac{M_R - M_H}{M_H} \right) \log \frac{m_Q^2}{M_R^2}.$$
FIG. 2. $D$ meson masses in the flavour limit as a function of the pion mass. The power-counting decomposed loop functions of (35, 36, 37) are used with the parameter set of Tab. IV.

\[ + M_H \frac{m_Q^3}{16 \pi^2} \left[ - \pi + \frac{3 \pi}{2} \left( \frac{m_Q}{2 M_H} - \frac{M_R - M_H}{m_Q} \right)^2 \right], \]  

(33)

accurate to order $Q^5$. The coefficients $\alpha_{QR}^H$ and $\gamma_{QR}^H$ were given already in (23) and (11).

In Tab. V we decompose the loop function into third, fourth and fifth order numerical values. The results are compared with the exact numbers already shown in Tab. IV. While we observe a qualitative reproduction of the full loop function, owing to contributions from intermediate pion states, there is no convergence observed as expected. By construction, the counting rule (32) fails in the chiral regime where all quark masses, in particular the strange quark mass approach zero. This is illustrated by Fig. 1 where we plot the loop function $\Pi_H$ in the flavour limit with $m_\pi = m_K = m_\eta$. Here the $D$ meson masses $M_D = M_{D_s}$ and $M_{D^*} = M_{D^*_s}$ are obtained as the solution of the set of Dyson equation (28) where the full loop expression (22) is assumed. The parameter set of Tab. IV is applied which is based on the scenario $\Pi_H^{(4-\chi)} = \Pi_H^{\text{tadpole}} = 0$. While for large pion masses the hierarchy of dashed and dotted lines systematically approach the solid line, this is not the case for pion masses smaller than $m_\pi \leq \Delta \sim 200$ MeV.

How to improve on the counting rule (32). Before presenting a universal approach we consider yet two further interim power counting scenarios. First we work out the extreme chiral region where all Goldstone boson masses are significantly smaller than $\Delta \sim 200$ MeV. In this case the counting rules

\[ \frac{m_Q}{\Delta} \sim Q, \quad \frac{\Delta}{M} \sim Q^0, \]  

(34)
are used. Since the extreme chiral region is not realized in nature, such an assumption is not expected to provide any significant results for quantities measurable in experimental laboratories.

Since at some stage lattice QCD simulations may be feasible at such low strange quark masses we provide the corresponding expressions for the loop function nevertheless. Here we decompose all meson masses into their chiral moments in application of a strict chiral expansion. At third order

$$\Pi^{\text{bubble}-3}_{H \in [0^-]} = 0, \quad \Pi^{\text{bubble}-3}_{H \in [1^-]} = -\frac{2}{3} \pi \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}}{8 \pi f} \right)^2 m_Q^3 \left( M + \Delta \right), \quad (35)$$

the vector $D$ mesons pick up a contribution only. At fourth order the expressions turn more complicated. We do not expand in powers of $\Delta/M$ because there are terms present proportional to $\log \Delta/M$, and also because we do not want to pollute the strict chiral expansion by a further scale assumption. The algebra required is somewhat involved and we organize it by a series of suitable dimensionless coefficients $\alpha_n, \gamma_n$ and $\tilde{\alpha}_n, \tilde{\gamma}_n$ that depend on the ratio $\Delta/M$ only. While the coefficients $\gamma_n, \tilde{\gamma}_n$ characterize the chiral expansion of the scalar bubble functions, the $\alpha_n, \tilde{\alpha}_n$ result from a chiral expansion of the coefficients in front of the scalar loop functions. Altogether we derive the compact expressions

$$\Pi^{\text{bubble}-4}_{H \in [0^-]} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}}{8 \pi f} \right)^2 \left\{ \gamma_d^{(1)} m_Q^2 \Pi_R^{(2)} + \gamma_d^{(2)} m_Q^2 \Pi_H^{(2)} + \tilde{\gamma}_d^{(3)} \Pi_R^{(2)} \Pi_R^{(2)} + \tilde{\gamma}_d^{(4)} \Pi_H^{(2)} \Pi_H^{(2)} \right. \right.$$

$$+ \tilde{\gamma}_d^{(5)} \Pi_R^{(2)} \Pi_H^{(2)} + \frac{M}{\Delta} m_Q^4 \left[ (\alpha_2 \gamma_2 - \alpha_1 \gamma_4) + (\alpha_2 \gamma_3 - \alpha_1 \gamma_5) \log \frac{m_Q^2}{(M + \Delta)^2} \right] \},$$

$$\Pi^{\text{bubble}-4}_{H \in [1^-]} = \sum_{Q \in [8]} \sum_{R \in [0^-]} \left( \frac{G_{QR}}{8 \pi f} \right)^2 \left\{ \tilde{\gamma}_d^{(1)} m_Q^2 \Pi_R^{(2)} + \tilde{\gamma}_d^{(2)} m_Q^2 \Pi_H^{(2)} + \tilde{\gamma}_d^{(3)} \Pi_R^{(2)} \Pi_R^{(2)} + \tilde{\gamma}_d^{(4)} \Pi_H^{(2)} \Pi_H^{(2)} \right. \right.$$

$$+ \tilde{\gamma}_d^{(5)} \Pi_R^{(2)} \Pi_H^{(2)} + \frac{M}{3 \Delta} m_Q^4 \left[ (\tilde{\alpha}_2 \tilde{\gamma}_2 - \tilde{\alpha}_1 \tilde{\gamma}_4) + (\tilde{\alpha}_2 \tilde{\gamma}_3 - \tilde{\alpha}_1 \tilde{\gamma}_5) \log \frac{m_Q^2}{M^2} \right] \},$$

$$+ \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}}{8 \pi f} \right)^2 \left\{ \frac{2}{3} m_Q^2 \left( m_Q^2 - \Pi_R^{(2)} + \Pi_H^{(2)} \right) \right. \right.$$

$$- \frac{1}{4} \left( 2 m_Q^2 + 3 \Pi_R^{(2)} - 3 \Pi_H^{(2)} \right) m_Q^2 \log \frac{m_Q^2}{(M + \Delta)^2} \right\}. \quad (36)$$

The dimension less coefficients $\gamma_d^{(n)}$ and $\tilde{\gamma}_d^{(k)}$ are expressed in terms of the basic coefficients $\alpha_n, \gamma_n$ and $\tilde{\alpha}_n, \tilde{\gamma}_n$ in Appendix A and B. Again they depend on the ratio $\Delta/M$ only. We
FIG. 3. $D$ meson masses in the flavour limit as a function of the pion mass. The power-counting decomposed loop functions of (39, 40, 57, 62) are used with the parameter set of Tab. IV. 

Note that the rational functions $\alpha_n$ and $\tilde{\alpha}_n$ approach one in the limit $\Delta/M \to 0$. In contrast the $\gamma_n$ and $\tilde{\gamma}_n$ have contributions proportional to $\log \Delta/M$ and do not approach one in the heavy-quark mass limit. All terms in (36) that are proportional to $\gamma_d(n)$ or $\tilde{\gamma}_d(n)$ can be viewed as a renormalization of the low-energy parameters $d_n$ and $\tilde{d}_n$. This is illustrated in Appendix A and B, where explicit expressions are provided. We note that the fifth order terms can also be readily constructed. For the vector $D$ mesons we derive

$$\Pi_{H \in [1]}^{\text{bubble}} = \sum_{Q \in [8]} \sum_{R \in [1]} \left( G_{QR}^{(H)} \right)^2 \frac{\pi m_Q}{8 \pi f} \left\{ 3m_Q^4 + m_Q^2 \left( 2 \Pi_H^{(2)} - 6 \Pi_R^{(2)} \right) \right\} + \cdots,$$

where the dots stand for additional terms extracted from (36) with the replacement $\Pi_H^{(2)} \to \Pi_H^{(3)}$. For the pseudo-scalar $D$ mesons the corresponding expression follow from (36) with the replacement $\Pi_H^{(2)} \to \Pi_H^{(3)}$ only.

We plot the loop function $\Pi_H$ in the flavour limit with $m_\pi = m_K = m_\eta$ and $M_D = M_{D_s} = M$ and $M_{D^*} = M_{D^*_s} = M + \Delta$. Here we use our first estimate for the low-energy parameters $c_{0,1}$ and $\tilde{c}_{0,1}$ as displayed in the next to last column of Tab. IV. From Fig. 2 we conclude that for pion masses smaller than $\Delta$ the successive orders (dashed, dotted and dash-dotted lines) approach the exact solid line convincingly. Unlike the consequences of the power-counting ansatz (32) as illustrated in the previous Fig. 1 this is clearly not the case for (34) in the large pion mass domain with $m_\pi > \Delta$. 

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Neither of the extreme counting assumptions (32) nor (34) generates an expansion scheme that converges for physical up, down and strange quark masses. A step forward may be provided by the following conventional ansatz

\[ \Delta \sim m_Q \sim Q, \quad \Delta_Q = \sqrt{\Delta^2 - m_Q^2} \sim Q, \quad \frac{\Delta}{M} \sim Q, \quad (38) \]

suggested originally by Banerjee and collaborators [44, 45] for the chiral expansion of baryon masses. Even though the authors demonstrated in a recent work [35] that such an expansion is not suitable to arrive at a meaningful expansion for the baryon octet and decuplet masses at physical values of the up, down and strange quark masses, it deserves a closer study whether it may prove significant for a chiral expansion of the D meson masses. The counting rules (38) lead to somewhat more complicated expressions. Again we derive the third, fourth and fifth order terms. We find

\[
\Pi^{\text{bubble-3}}_{H \in [0^-]} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}^{(H)}}{8\pi f} \right)^2 \left\{ - \Delta_Q^3 M \left( \log (\Delta + \Delta_Q) - \log (\Delta - \Delta_Q) \right) 
- \Delta M \left( \Delta_Q^2 - \frac{1}{2} m_Q^2 \right) \log \frac{m_Q^2}{4\Delta^2} - \frac{1}{2} \Delta M m_Q^2 \right\},
\]

\[
+ \sum_{Q \in [8]} \sum_{R \in [0^-]} \left( \frac{G_{QR}^{(H)}}{8\pi f} \right)^2 \frac{1}{3} \left\{ \Delta_Q^3 M \left( \log (\Delta - \Delta_Q) - \log (-\Delta + \Delta_Q) \right) 
+ \Delta M \left( \Delta_Q^2 - \frac{1}{2} m_Q^2 \right) \log \frac{m_Q^2}{4\Delta^2} + \frac{1}{2} \Delta M m_Q^2 \right\}, \quad (39)
\]

and

\[
\Pi^{\text{bubble-4}}_{H \in [0^-]} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}^{(H)}}{8\pi f} \right)^2 \left\{ \frac{1}{4} \left( -3 \Delta^2 + 4 m_Q^2 - 4 \Pi_H^{(2)} + 4 \Pi_R^{(2)} \right) m_Q^2
- \frac{3}{2} \Delta^2 \left( \Delta_Q^2 - \Pi_H^{(2)} + \Pi_R^{(2)} \right) - \frac{1}{2} m_Q^2 \right\} \log \frac{m_Q^2}{4\Delta^2}
- \frac{1}{4} \left( 2 m_Q^2 + 3 \Pi_H^{(2)} - 3 \Pi_R^{(2)} \right) m_Q^2 \log \frac{m_Q^2}{M^2} - \Delta_Q M \left( \log (\Delta + \Delta_Q) - \log (\Delta - \Delta_Q) \right) 
- \log \frac{\Delta}{M} \left( \frac{3 \Delta}{2 M} \left( \Delta_Q^2 - \Pi_H^{(2)} + \Pi_R^{(2)} \right) \right) \right\},
\]

\[
\Pi^{\text{bubble-4}}_{H \in [1^-]} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{G_{QR}^{(H)}}{8\pi f} \right)^2 \frac{1}{3} \left\{ - \pi \Delta m_Q^3 + \left( m_Q^2 - \Pi_H^{(2)} + \Pi_R^{(2)} \right) m_Q^2 \right\}.
\]
with $\Delta_Q$ of (38). Since the fifth order contributions are quite lengthy they are delegated to Appendix A and B. In Tab. VI we decompose the loop function into third, fourth and fifth order numerical values. The results are compared with the exact numbers already shown in Tab. IV. The conclusions of that table are unambiguous: the power counting ansatz (38) is not suitable for a chiral extrapolation of the D meson masses. We note that (38) neither reproduces the results of (32) nor those of (34). We further demonstrate our claim by a plot of the loop function $\Pi_H$ in the flavour limit with $m_\pi = m_K = m_\eta$ as was done in the previous figures [1] and [2]. Fig. 3 demonstrates that for $m_\pi > \Delta$ no quantitative reproduction of the solid line is obtained.

We finally present our counting ansatz that is expected to be applicable from small to medium size quark masses uniformly. It is an adaptation of the framework developed recently for the chiral extrapolation of the baryon octet and decuplet masses [35] and implements the driving idea to formulate the expansion coefficients in terms of physical masses. It is

| $H$  | $\Pi_H^{\text{bubble}}/(2 M_H)$ | $\Pi_H^{\text{bubble}-3}/(2 M_H)$ | $\Pi_H^{\text{bubble}-4}/(2 M_H)$ | $\Pi_H^{\text{bubble}-5}/(2 M_H)$ |
|------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $D$  | -50.2 MeV                       | -67.7 MeV                       | 15.0 MeV                        | -8.9 MeV                        |
| $D_s$| -65.6 MeV                       | -152.8 MeV                      | 27.8 MeV                        | 26.6 MeV                        |
| $D^*$| -113.4 MeV                      | -111.7 MeV                      | -57.1 MeV                       | 18.6 MeV                        |
| $D^*_s$| -166.1 MeV                      | -252.0 MeV                      | 84.3 MeV                        | -69.5 MeV                       |

TABLE VI. The loop functions (22, 40) are evaluated with the coupling constants $g_P = \tilde{g}_P \simeq 0.57$ and the physical isospin averaged meson masses. A decomposition according to (38) is performed.
supposed to interpolate the two extreme counting rules (32) and (34). The counting rules are

\[ \frac{M_R - M_H}{m_Q} \sim Q, \quad \frac{M_R - M_H}{M_H} \sim Q^2 \quad \text{for} \quad H \parallel R, \]
\[ \frac{M_R - M_H}{m_Q} \sim Q^0, \quad \frac{M_R - M_H \pm \Delta_H}{M_H} \sim Q^2 \quad \text{for} \quad H \perp R, \]

\[ \Delta_Q = \sqrt{(M_H - M_R)^2 - m_Q^2} \sim Q \quad \text{with} \quad \Delta_H = \Delta M_H \lim_{m_{u,d,s} \to 0} \frac{1}{M_H}, \quad (41) \]

where the sign ± is chosen such that the last ratio in (41) vanishes in the chiral limit. The implications of (41) are more difficult to work out. The counting rules (41) as they are necessarily imply

\[ Q \sim \frac{\Delta_H}{M_H} = \begin{cases} \frac{\Delta}{M} & \text{for} \quad H \in [0^-] \\ \frac{\Delta}{M+\Delta} & \text{for} \quad H \in [1^-] \end{cases}, \quad (42) \]

which is at odds with the assumption in (34). Therefore we supplement (41) by the request that the implications of (41) are recovered in the chiral regime. This requires a particular summation of terms proportional to \((\Delta/M)^n\) with \(n = 1, 2, 3, \ldots\).

There is yet another issue pointed out in [35]. The chiral expansion of the scalar bubble function is characterized by an alternating feature. We recall from [35] the following approximation hierarchy

\[ (4\pi)^2 I_{QR} = -\left\{1 - \frac{1}{8} x^2 - \frac{1}{128} x^4 - \frac{1}{1024} x^6 + O(x^8)\right\} \pi \sqrt{x^2} \]
\[ + \left\{1 - \frac{1}{12} x^2 - \frac{1}{120} x^4 - \frac{1}{840} x^6 + O(x^8)\right\} x^2 \]
\[ - \frac{1}{2} x^2 \log x^2, \quad (43) \]

where we denoted \(x = m_Q/M_H\) and \(M_R = M_H\). As was discussed in [35] the terms with even and odd powers in \(x\) have opposite signs always. This implies a systematic cancellation effect amongst terms proportional to \(x^n\) and \(x^{1+n}\), where the effect is most striking for \(n = 1\). Therefore it is useful to always group such terms together. Even though the need of such an reorganization is not very strong for the D meson systems under consideration we adapt this strategy in the following. Note that the convergence domain of (43) was proven to be limited by \(|x| < 2\) only, a surprisingly large convergence circle. Given this scheme accurate results can be obtained by a few leading order terms. We construct the third order contributions
from the one-loop diagrams.

\[
\Pi_{H^{\text{bubble-3}}} = \sum_{Q \in [8], R \in [1^+]} \left( \frac{G_{QR}(H)}{4\pi f} \right)^2 \frac{\alpha_1}{4} \left\{ \delta_7 \Delta M m_Q^2 + \delta_6 \Delta^2 M (M_R - M_H - \Delta_H) + M_H \left[ \Delta_Q^2 \left( \gamma_1 \Delta_H - \delta_1 (M_R - M_H) \right) - \frac{2 M + \Delta}{2 M} \left( (M_R - M_H) \left( \Delta_Q^2 - \frac{1}{2} m_Q^2 \right) \log \frac{m_Q^2}{M_R^2} \right) + \Delta_Q^3 \left[ \log (M_R - M_H + \Delta_Q) - \log (M_R - M_H - \Delta_Q) \right) \right] + \frac{m_Q^2}{\Delta_H} \left( - \delta_2 \Delta_Q^2 + \delta_3 m_Q^2 \log \frac{m_Q^2}{M_R^2} \right) \right\},
\]

\[
\Pi_{H^{\text{bubble-3}}} = \sum_{Q \in [8], R \in [1^+]} \left( \frac{G_{QR}(H)}{4\pi f} \right)^2 \frac{M_H}{6} \left\{ \frac{m_Q^2}{M_H} \left( 1 - \log \frac{m_Q}{M_R} \right) - \pi m_Q \right\} \left( m_Q^2 - (M_R - M_H)^2 \right) + \frac{M}{M + \Delta} \left[ \Delta_Q^2 \left( \gamma_1 \Delta_H - \delta_1 (M_R - M_H) \right) - \frac{M (2 M + \Delta)}{2 (M + \Delta)^2} \left( (M_H - M_R) \left( \Delta_Q^2 - \frac{1}{2} m_Q^2 \right) \log \frac{m_Q^2}{M_R^2} \right) + \Delta_Q^3 \left[ \log (M_R - M_H - \Delta_Q) - \log (M_R - M_H + \Delta_Q) \right) \right] + \frac{m_Q^2}{\Delta_H} \left( - \delta_2 \Delta_Q^2 + \delta_3 m_Q^2 \log \frac{m_Q^2}{M_R^2} \right) \right\}, \tag{44}
\]

with \( \Delta_Q \) and \( \Delta_H \) as introduced in (41). The dimension less coefficients \( \alpha_i, \gamma_i, \delta_i \) and \( \bar{\alpha}_i, \bar{\gamma}_i, \bar{\delta}_i \) depend on the ratio \( \Delta / M \) only. They are detailed in Appendix A and Appendix B. The contributions proportional to \( \alpha_i \delta_j \) and \( \bar{\alpha}_i \bar{\delta}_j \) in (44) are constructed to ensure that the terms proportional to \( (m_Q^2 / \Delta) \) and \( (m_Q^2 / \Delta) \log m_Q^2 \) are recovered exactly.

We advance to the fourth order terms. The following explicit expressions are obtained

\[
\Pi_{H^{\text{bubble-4}}} = \sum_{Q \in [8], R \in [1^+]} \left( \frac{G_{QR}(H)}{4\pi f} \right)^2 \left\{ - \frac{1}{4} \alpha_1 M \Delta^2 \delta_6 + \frac{M_H}{4} \left[ \alpha_1 \Delta^2 \frac{\partial \Delta \gamma_1}{\partial \Delta} + \Delta_Q^2 \beta_4 \right. \right.
\]

\[
- \frac{M_R - M_H}{\Delta} \left( \Delta_Q^2 \frac{m_Q^2}{2} \right) \log \frac{m_Q^2}{M_R^2}
\]

\[
+ \Delta_Q^3 \left[ \log (M_R - M_H + \Delta_Q) - \log (M_R - M_H - \Delta_Q) \right) \right] \right\}
\]
and the physical isospin averaged meson masses. A decomposition according to (41) is performed.

| $H$ | $\Pi_{H}^{\text{bubble}}/(2 M_H)$ | $\Pi_{H}^{\text{bubble}-3}/(2 M_H)$ | $\Pi_{H}^{\text{bubble}-4}/(2 M_H)$ | $\Pi_{H}^{\text{bubble}-5}/(2 M_H)$ |
|-----|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $D$ | -50.2 MeV                       | -48.5 MeV                       | -2.8 MeV                         | 1.1 MeV                         |
| $D_s$ | -65.6 MeV                       | -88.3 MeV                       | 20.1 MeV                         | 2.9 MeV                         |
| $D^*$ | -113.4 MeV                      | -99.5 MeV                       | -17.1 MeV                        | 3.1 MeV                         |
| $D_{s^*}$ | -166.1 MeV                     | -197.5 MeV                      | 26.3 MeV                         | 6.6 MeV                         |

TABLE VII. The loop functions \(22\) are evaluated with the coupling constants \(g_P = \tilde{g}_P \simeq 0.57\) and the physical isospin averaged meson masses. A decomposition according to \(41\) is performed. This leads to \(44\ 45\ 58\ 63\).

\[
\Pi_{H}^{\text{bubble}-4} = \sum_{Q \in \{8\}}^{1} \sum_{R \in \{1\}}^{\infty} \left( \frac{G_{QR}^{(H)}}{4\pi f} \right)^2 \frac{M_H}{3} \left\{ 1 + \frac{\log m_R^2}{2} \right\} (M_R - M_H - \Delta_H),
\]

\[
\sum_{Q \in \{8\}}^{1} \sum_{R \in \{0\}}^{\infty} \left( \frac{G_{QR}^{(H)}}{4\pi f} \right)^2 \left\{ \frac{1}{12} \alpha_1 M \Delta^2 \delta_0 + \frac{M_H}{12} \right\} (M_R - M_H - \Delta_H),
\]

with $\Delta_Q$ and $\Delta_H$ already introduced in \(44\).

In Tab. VII we decompose the loop function into third, fourth and fifth order numerical values. The results are compared with the exact numbers already shown in Tab. IV. The conclusions of that table are unambiguous: the power counting ansatz \(41\) is well justified for a chiral extrapolation of the D meson masses. We note that the fifth order contributions to the $D$ meson masses are on average about 3 MeV only. Our novel expansion scheme is characterized by a rapid convergence property. All $D$ meson masses are reproduced at
FIG. 4. $D$ meson masses in the flavour limit as a function of the pion mass. The power-counting decomposed loop functions of (44, 45, 58, 63) are used with the parameter set of Tab. IV.

the few MeV level. We further substantiate our claim by Fig. 4, which shows the loop function $\bar{\Pi}_H$ in the flavour limit with $m_\pi = m_K = m_\eta$. The figures are in correspondence to the previous figures 1, 2 and 3 and demonstrate that for any reasonable pion mass, say $0 \leq m_\pi < 600$ MeV, a quantitative reproduction of the solid line is obtained. We conclude that it is justified to identify the full loop expressions as the loop function to be used at chiral order $Q^4$ without any significant error from the incomplete 5th order terms.
VI. FIT TO QCD LATTICE DATA

In this section we will determine the low-energy constants $c_i$ and $d_i$ of the chiral Lagrangian from lattice QCD simulations of the $D$ meson masses. Open-charm mesons have been extensively studied on different QCD lattices [10, 18, 20–22, 46–54]. For a recent review we refer to [55]. There exists a significant data set for $D$-meson masses at various unphysical quark masses. We consider data sets where the pion and kaon masses are smaller than about 600 MeV only. Once we determined the LECs in our mass formula, the $D$-meson masses can be computed at any values for the up, down and strange quark masses, sufficiently small as to justify the application of the chiral extrapolation.

Though in principle such an analysis can be done at different chiral orders, we do so using the subtracted loop expressions (22) in (28) with the scalar loop functions as worked out previously for the finite box case in [34]. It is a matter of convenience to perform our fits using the full one-loop functions rather than any truncated form. Therewith the finite volume corrections specific to the various chiral moments, whose explicit derivation would require further tedious algebra, are not required. This strategy is justified since we have demonstrated with Tab. VII that the full loop function is reproduced quite accurately by its $N^3$LO approximation, with a residual uncertainty for the $D$ meson masses of about 3 MeV only. It is emphasized that such a point of view relies heavily on our reorganized chiral expansion approach, which is formulated in terms of physical meson masses.

While for instance in [18, 20] the extrapolation towards the physical point was the focus the purpose of our study is the extraction of the low-energy constants of the chiral Lagrangian. Therefore a different strategy is used in our work. We use the empirical $D$-meson masses as an additional constraint in our analysis. For a given pion and kaon mass we infer the quark masses from the one-loop mass formulae for the pseudo Goldstone bosons to be used in our expressions for the $D$ meson masses. Assuming that the lattice data can be properly moved to the physical charm quark mass the low-energy constants are obtained by a global fit to the QCD lattice data set. Altogether there are about 80 data points considered in our analysis.

A comprehensive published data set is from Mohler and Woloshyn [18, 53] based on the PACS-CS ensembles [17]. The Fermilab approach is employed in implementing the valence charm-quark [56, 57]. In this approach, heavy-quark mass dependent counter terms are
TABLE VIII. Meson masses and energy levels in units of the lattice spacing $a$ as taken from [18, 53] and [17]. Statistical errors are given only. The results are based on ensembles from PACS for which their estimate of the lattice spacing is $a = 0.0907(13)$ fm.

| $a m_\pi$ | $a m_K$ | $a E_D$ | $a E_{D_s}$ | $a E_{D^*}$ | $a E_{D^*_s}$ |
|-----------|---------|---------|-------------|-------------|-------------|
| $32^3 \times 64$ | 0.0717(32) | 0.2317(6) | 0.7765(12) | 0.8197(24) | 0.8447(27) | 0.8850(24) |
| $32^3 \times 64$ | 0.1359(140) | 0.27282(103) | 0.78798(82) | 0.83929(26) | 0.85776(122) | 0.90429(43) |
| $32^3 \times 64$ | 0.17671(129) | 0.26729(110) | - | 0.82848(40) | - | 0.89015(69) |
| $32^3 \times 64$ | 0.18903(79) | 0.29190(67) | 0.79580(61) | 0.84000(36) | 0.86327(99) | 0.90429(60) |

Recently the group of Marc Wagner analyzed a large set of ensembles from the European Twisted Mass Collaboration (ETMC) [20, 21]. Our analysis requires the $D$ meson masses evaluated at the physical charm quark mass. We are grateful to the authors of [20] for making available unpublished results, which allow us to independently extrapolate their lattice data.

![FIG. 5](image_url)  

FIG. 5. The interpolation of charmonium masses to determine $\mu_c$, at given $a = 0.0885$ fm. The ensemble is chosen with $a m_\pi = 0.1240$. The physical values of $a M_{\eta_c}$ and $a M_{J/\psi}$ are indicated by the dashed lines.
to the physical charm quark mass. For each ensemble, the four $D$-meson masses but also the $\eta_c$ and $J/\Psi$ masses are computed at two different values of the charm valence-quark mass $\mu_c$. As a consequence of the discretization procedure there are corresponding pairs of meson masses that turn degenerate in the continuum limit. We use the notation $(\pm, \mp)$ and $(\pm, \pm)$ from [20, 21]. In this work we focus on the $(\pm, \mp)$ states and use the masses of the partner

| $a$ [fm]  | $a m_\pi$       | $a m_K$          | $a \mu_c$ | $a M_{\eta_c}$ | $a M_{J/\Psi}$ |
|----------|-----------------|------------------|-----------|-----------------|-----------------|
| $48^3 \times 96$ | 0.0619 | 0.0703(4) | 0.1697(3) | 0.2230 | 1.0595(2) | 1.1006(3) |
|          |              |                  |           | 0.1919 | 0.9570(2) | 1.0003(4) |
| $48^3 \times 96$ | 0.0619 | 0.0806(3) | 0.1738(5) | 0.2227 | 1.0579(2) | 1.0989(4) |
|          |              |                  |           | 0.1727 | 0.8915(2) | 0.9364(5) |
| $48^3 \times 96$ | 0.0619 | 0.0975(3) | 0.1768(3) | 0.2230 | 1.0591(1) | 1.1002(3) |
|          |              |                  |           | 0.1727 | 0.8919(1) | 0.9370(3) |
| $32^3 \times 64$ | 0.0815 | 0.1074(5) | 0.2133(4) | 0.2230 | 1.3194(2) | 1.3835(4) |
|          |              |                  |           | 0.1727 | 1.1567(2) | 1.2233(4) |
| $32^3 \times 64$ | 0.0815 | 0.1549(2) | 0.2279(2) | 0.2230 | 1.3251(1) | 1.3903(2) |
|          |              |                  |           | 0.1727 | 1.1573(1) | 1.2253(2) |
| $24^3 \times 48$ | 0.0815 | 0.1935(4) | 0.2430(4) | 0.2230 | 1.3179(3) | 1.3837(4) |
|          |              |                  |           | 0.1727 | 1.1582(3) | 1.2273(4) |
| $32^3 \times 64$ | 0.0885 | 0.1240(4) | 0.2512(3) | 0.2772 | 1.3869(1) | 1.4649(3) |
|          |              |                  |           | 0.2270 | 1.2241(2) | 1.3042(4) |
| $32^3 \times 64$ | 0.0885 | 0.1412(3) | 0.2569(3) | 0.2768 | 1.3859(1) | 1.4636(3) |
|          |              |                  |           | 0.2389 | 1.2642(1) | 1.3430(3) |
| $24^3 \times 48$ | 0.0885 | 0.1440(6) | 0.2589(4) | 0.2768 | 1.3863(2) | 1.4645(4) |
|          |              |                  |           | 0.2389 | 1.2645(2) | 1.3442(5) |
| $24^3 \times 48$ | 0.0885 | 0.1988(3) | 0.2764(3) | 0.2929 | 1.4273(2) | 1.5069(4) |
|          |              |                  |           | 0.2299 | 1.2353(2) | 1.3172(5) |

**TABLE IX.** Meson masses in units of the lattice spacing $a$ based on the ensembles of the ETM collaboration. The values in the table are provided to us by the authors of [20]. Statistical errors are given only. The data correspond to three different $\beta_{QCD} = 1.90, 1.95, 2.10$ values for which in [58] an estimate of the lattice scale is provided with $a = 0.0934(37), 0.0820(37), 0.0644(26)$ fm respectively.
TABLE X. $D$ and $J/\psi$ meson masses in units of the lattice scale $a$. The charm-quark mass is determined to reproduce the physical $J/\psi$ mass. This leads to $a\mu_c = 0.2535, 0.1902$ and $0.1829$ for the three groups of ensembles. Statistical errors are given only.

| $a m_\pi$   | $a m_K$   | $a M_D$   | $a M_{D_s}$ | $a M_{D_s^*}$ | $a M_{D_s^{**}}$ | $a M_{J/\psi}$ |
|-------------|-----------|-----------|-------------|--------------|-----------------|----------------|
| 0.0703(4)   | 0.1697(3) | 0.5905(52)| 0.6236(56)  | 0.6466(86)   | 0.6770(28)      | 0.9715(20)     |
| 0.0806(3)   | 0.1738(5) | 0.5906(64)| 0.6234(57)  | 0.6506(26)   | 0.6763(11)      | 0.9697(21)     |
| 0.0975(3)   | 0.1768(3) | 0.5913(50)| 0.6229(57)  | 0.6486(28)   | 0.6764(15)      | 0.9703(21)     |
| 0.1074(5)   | 0.2133(4) | 0.7840(122)| 0.8159(147)| 0.8568(44)   | 0.8905(34)      | 1.2791(55)     |
| 0.1549(2)   | 0.2279(2) | 0.7895(128)| 0.8183(144)| 0.8678(47)   | 0.8950(39)      | 1.2828(55)     |
| 0.1935(4)   | 0.2430(4) | 0.7934(148)| 0.8175(151)| 0.8745(38)   | 0.8965(41)      | 1.2818(58)     |
| 0.1240(4)   | 0.2512(3) | 0.8514(181)| 0.8953(206)| 0.9356(28)   | 0.9806(45)      | 1.3890(75)     |
| 0.1412(3)   | 0.2569(3) | 0.8544(168)| 0.8972(208)| 0.9363(41)   | 0.9802(45)      | 1.3895(75)     |
| 0.1440(6)   | 0.2589(4) | 0.8552(159)| 0.8978(208)| 0.9403(23)   | 0.9844(45)      | 1.3906(77)     |
| 0.1988(3)   | 0.2764(3) | 0.8599(184)| 0.8950(219)| 0.9487(60)   | 0.9841(66)      | 1.3882(79)     |

states $(\pm, \pm)$ only as a rough estimate for the size of the discretization error. In the vicinity of the physical charm quark mass a linear behavior

$$a M_H = \alpha_H + \beta_H a \mu_c,$$  

is expected to hold for all hadron masses. Since the chosen charm quark masses are close to the physical one the ansatz (46) should be justified to sufficient accuracy. The parameters $\alpha_H$ and $\beta_H$ can be extracted from the data provided to us by Kalinowski and Wagner. In Tab. IX we show their results for the $\eta_c$ and $J/\psi$ masses together with their preferred lattice spacing values $a$. Corresponding results for the $D$ meson masses are listed at the end of Appendix B. It remains the task to determine the physical value for $\mu_c$. Since one would neither expect a significant dependence of the $\eta_c$ nor of the $J/\psi$ meson mass on the precise value of the up, down and strange quark masses, one may contemplate to use either of the two masses to obtain a good estimate for $\mu_c$. Both scenarios are scrutinized in the following based on the data of Kalinowski and Wagner. To fix the charm quark mass we always choose
TABLE XI. $D$ and $\eta_c$ meson masses in units of the lattice scale $a$. The charm-quark mass was determined to reproduce the physical $\eta_c$ meson mass. This leads to $a\mu_c = 0.2618, 0.1957$ and $0.1852$ for the three groups of ensembles. Statistical errors are given only.

| $a\,m_\pi$ | $a\,m_K$ | $a\,M_D$ | $a\,M_{D_s}$ | $a\,M_{D^*}$ | $a\,M_{\eta_c}$ |
|-------------|-----------|-----------|--------------|--------------|-----------------|
| 0.0703(4)   | 0.1697(3) | 0.5947(52)| 0.6279(56)   | 0.6506(86)   | 0.6809(28)      |
| 0.0806(3)   | 0.1738(5) | 0.5949(64)| 0.6277(57)   | 0.6546(26)   | 0.6803(11)      |
| 0.0975(3)   | 0.1768(3) | 0.5955(50)| 0.6271(57)   | 0.6526(28)   | 0.6804(15)      |
| 0.1074(5)   | 0.2133(4) | 0.7946(122)| 0.8263(147) | 0.8664(44)   | 0.9001(34)      | 1.2312(212)    |
| 0.1549(2)   | 0.2279(2) | 0.8004(128)| 0.8291(144) | 0.8777(47)   | 0.9049(39)      | 1.2342(217)    |
| 0.1935(4)   | 0.2430(4) | 0.8039(148)| 0.8278(151) | 0.8840(38)   | 0.9059(41)      | 1.2314(219)    |
| 0.1240(4)   | 0.2512(3) | 0.8677(181)| 0.9114(206) | 0.9506(28)   | 0.9953(45)      | 1.3370(296)    |
| 0.1412(3)   | 0.2569(3) | 0.8708(168)| 0.9132(208) | 0.9511(41)   | 0.9949(45)      | 1.3379(299)    |
| 0.1440(6)   | 0.2589(4) | 0.8714(159)| 0.9137(208) | 0.9545(24)   | 0.9990(45)      | 1.3382(302)    |
| 0.1988(3)   | 0.2764(3) | 0.8753(184)| 0.9102(219) | 0.9627(60)   | 0.9980(66)      | 1.3325(310)    |

The ensemble with the lightest up and down quark masses. In addition the lattice spacing $a$ as recalled in Tab. [IX] is assumed. A typical example for this procedure is shown in Fig. 5 where a sizable uncertainty for the extracted value of $\mu_c$ is observed.

How such an uncertainty propagates into the masses of the $D$ mesons is shown in Tab. [XI] and Tab. [X] which are based on the charm quark masses from the $\eta_c$ and the $J/\psi$ meson respectively. As expected this uncertainty in the charm quark mass is reduced for the ensembles that correspond to even smaller lattice spacings with $a = 0.0815$ fm and $a = 0.0619$ fm. This can be inferred by a comparison of Tab. [XI] and Tab. [X] While the center value of the masses in Tab. [XI] and Tab. [X] are derived from the $(\pm, \mp)$ states of Appendix B, the shown error bars entail an estimate for the total error including the statistical error and the uncertainty from the discretization procedure. We take half of the splittings of the two modes, $(\pm, \mp)$ and $(\pm, \pm)$, for the latter.

It is immediate from Tab. [XI] and Tab. [X] that the $D$ meson masses are quite sensitive to the precise charm quark mass used but also to the lattice scale $a$ assumed. We note that, for
TABLE XII. $D$ meson masses based on ensembles of MILC [63] as used by LHPC [60]. The results are recalled from [10] in units of the lattice spacing $a$. The lattice spacing is $a \simeq 0.12$ fm.

|          | $am_\pi$         | $am_K$         | $aM_D$         | $aM_{Ds}$        |
|----------|------------------|----------------|----------------|------------------|
| $20^3 \times 64$ | 0.1842(7)       | 0.3682(5)      | 1.2081(13)     | 1.2637(10)       |
| $20^3 \times 64$ | 0.2238(5)       | 0.3791(5)      | 1.2083(11)     | 1.2635(10)       |
| $20^3 \times 64$ | 0.3113(4)       | 0.4058(4)      | 1.2226(13)     | 1.2614(12)       |
| $20^3 \times 64$ | 0.3752(5)       | 0.4311(5)      | 1.2320(11)     | 1.2599(12)       |

instance, there exist two distinct values for the lattice spacing for the coarsest ensembles: The value $a = 0.0885(36)$ fm obtained from the pion decay constant [59] and $a = 0.0920(21)$ fm obtained from the nucleon mass [58]. We conclude that it may be of advantage to determine the lattice scale and the charm-quark mass from the $D$ meson masses directly. Such a procedure is expected to minimize the discretization errors for the $D$ meson masses. This is what we will do in the following. All information required for such a strategy is provided with Tab. XI and Tab. X, from which the parameters $\alpha_H$ and $\beta_H$ in (46) can be read off.

There are yet three further sources of QCD lattice data on the $D$ meson masses, which we will discuss briefly [10, 19, 22]. The two data sources [10, 19] are partial to the extent that not all four $D$ meson masses are provided. Only the pseudo-scalar masses are computed. The results of [10] rely on previous studies by the LHP collaboration [60], who use a mixed action framework with domain-wall valence quarks but staggered sea-quark ensembles generated by MILC [61–63]. For the charm quark they use a relativistic heavy-quark action motivated by the Fermilab approach [56, 57]. In Tab. XII we summarize the relevant masses that are considered in our study.

The results of the HPQCD Collaboration [19] are based on MILC ensembles together with a highly improved staggered valence quark (HISQ) action. The HISQ action has since been used very successfully in simulations involving the charm quark such as for charmonium and for $D$ and $D_s$ meson decay constants. In Tab. XIII we collect the relevant masses in units of the lattice spacing for the configurations on three coarse and two fine lattices.

Most recently the Hadron Spectrum Collaboration (HSC) computed the excited open-charm meson spectrum in a finite QCD box [22, 23]. Results for the for the D mesons masses
based on an ensemble with a pion mass of about 390 MeV are published in [23] and recalled in Tab. XIV. For an additional ensemble at smaller pion masses studies are on going [22].

We note that the charm-quark mass in [10], [19] and [23] was not adjusted to the \( D \) meson masses. While in [10] the spin average of the physical \( J/\Psi \) and \( \eta_c \) meson mass was used, in [19] the charm quark mass was tuned to the physical \( \eta_c \) mass. In both cases we cannot exclude uncertainties significant to our analysis. In order to minimize any bias from a possibly imprecise charm-quark mass determination we consider only mass differences from Tab. XII, Tab. XIII and Tab. XIV in our fits. In addition we fine tune the lattice scales. As we have seen in case of the ETMC results such a procedure reduces any possible bias significantly.

We introduce a universal parameter \( \Delta_c \) of the form

\[
a M_H \rightarrow a M_H + (1 + \epsilon_H) a \Delta_c, \quad \text{with} \quad \epsilon_H \simeq 0,
\]

which is supposed to fine tune the choice of the charm quark mass. In principle the values of \( \epsilon_H \) depends on the type of \( D \) meson considered but also the \( \beta_{QCD} \) value of the ensemble considered. The value \( (1 + \epsilon_H) a \Delta_c \) is to be added to \( a M_H \) as collected in Tabs. XI, XIV.

For the ETMC masses the magnitude of \( \epsilon_H \) can be extracted from Tab. X and Tab. XI, where we insist on the normalization condition that \( \epsilon_H = 0 \) for the \( D \) meson on the ensemble with the lightest pion mass. Then values for \( |\epsilon_H| \) of about 0.1 arise in some cases at most. Such an estimate is not available for the other collaborations. For these other cases we put \( \epsilon_H = 0 \), which would arise in the heavy-quark mass limit. We would argue that a precise determination of \( a \Delta_c \) and therewith the physical charm quark mass for a
given ensemble requires the quantitative control of the chiral extrapolation formulæ for the $D$ meson masses.

We do not implement discretization effects in our chiral extrapolation approach since this would introduce a significant number of further unknown parameters into the game. For each lattice group such effects have to be worked out in the context of our chiral extrapolation scheme. As a consequence a fully systematic error analysis is not possible yet in our present study. Here we follow the strategy suggested in [34, 35] where the statistical error given by the lattice groups is supplemented by a systematic error in mean quadrature. We perform fits at different ad-hoc values for the systematic error. Once this error is sufficiently large the $\chi^2$ per data point should be close to one. In our current studies we arrive at the estimate of 5-10 MeV. In anticipation of our analysis of the lattice data set we collect the result of four representative fits. Their characteristics and defining assumptions will be discussed in more detail in the next sections.

For a given ensemble the statistical errors in the lattice data are correlated. However, since the statistical error for any meson mass considered here is typically much smaller than our estimate for the systematic error such a correlation is of no relevance in our study. In contrast, the choice of the charm quark mass and the lattice scale setting, both of which we treat in detail, is a significant effect.

Our fit procedure goes as follows. For a given lattice ensemble we take the pion and kaon masses as given in lattice units and then determine from the one-loop expressions (28) in [35] the quark masses for that ensemble. They depend on the three particular linear combinations of the low-energy constants of Gasser and Leutwyler [67]. One combination can be fixed by the request that the $\eta$ meson mass is reproduced at physical quark masses. The other two are determined by our fit to lattice data. With those the quark mass ratio $m_s/m$ is determined. This is analogous to [35] where those low-energy constants are determined from

\begin{table}  
\centering  
\begin{tabular}{|c|cc|c|c|c|c|c|}
\hline  
\text{ensemble} & \text{a}_t m_\pi & \text{a}_t m_K & \text{a}_t M_D & \text{a}_t M_{D_s} & \text{a}_t M_{D^*} & \text{a}_t M_{D_s^*} \\
\hline  
$24^3 \times 128$ & 0.06906(13) & 0.09698(9) & 0.33265(7) & 0.34426(6) & 0.35415(17) & 0.36508(88) \\
$32^3 \times 256$ & 0.03928(18) & 0.08344(7) & - & - & - & - \\
\hline  
\end{tabular}  
\caption{D meson masses from HSC in units of the temporal lattice spacing \[23, 66\]. The lattice spacing is $3.5 a_t = 0.123(4)$ fm. It holds $a = a_s \simeq 3.5 a_t$.}  
\end{table}
TABLE XV. Results for Fit 1 - Fit 4. The low-energy constants $L_n$ are at the renormalization scale $\mu = 0.77$ GeV. The offset parameters $a \Delta_c$ is introduced in (47). We use $f = 92.4$ MeV throughout this work. A more detailed discussion of the four fit scenarios is given in Section VII and VIII.

|                  | Fit 1 | Fit 2 | Fit 3 | Fit 4 |
|------------------|-------|-------|-------|-------|
| $a_{\text{PACS-CS}}$ [fm] | 0.0934 | 0.0940 | 0.0935 | 0.0928 |
| $a \Delta_{c,\text{PACS-CS}}$ | 0.1067 | 0.1110 | 0.1119 | 0.1023 |
| $a_{\text{LHPC}}$ [fm] | 0.1291 | 0.1267 | 0.1291 | 0.1291 |
| $a \Delta_{c,\text{LHPC}}$ | 0.0359 | 0.0087 | 0.0443 | 0.0381 |
| $a_{\beta=6.76} \Delta_{c,\text{HPQCD}}$ [fm] | 0.1367 | 0.1359 | 0.1336 | 0.1367 |
| $a \Delta_{c,\text{HPQCD}}$ | 0.1500 | 0.1494 | 0.1184 | 0.1500 |
| $a_{\beta=7.09} \Delta_{c,\text{HPQCD}}$ | 0.0953 | 0.0991 | 0.0970 | 0.0992 |
| $a \Delta_{c,\text{HPQCD}}$ | 0.0936 | 0.1336 | 0.1049 | 0.1282 |
| $a_{\beta=1.90} \Delta_{c,\text{ETMC}}$ | 0.1018 | 0.0996 | 0.1025 | 0.1027 |
| $a \Delta_{c,\text{ETMC}}$ | 0.0983 | 0.0747 | 0.1041 | 0.1086 |
| $a_{\beta=1.95} \Delta_{c,\text{ETMC}}$ | 0.0934 | 0.0925 | 0.0928 | 0.0943 |
| $a \Delta_{c,\text{ETMC}}$ | 0.0908 | 0.0817 | 0.0817 | 0.1005 |
| $a_{\beta=2.10} \Delta_{c,\text{ETMC}}$ | 0.0695 | 0.0704 | 0.0695 | 0.0699 |
| $a \Delta_{c,\text{ETMC}}$ | 0.0629 | 0.0728 | 0.0608 | 0.0659 |
| $a_{\text{HSC}}$ [fm] | 0.1211 | 0.1243 | 0.1242 | 0.1242 |
| $a \Delta_{c,\text{HSC}}$ | 0.0050 | 0.0337 | 0.0328 | 0.0343 |

$10^3 (L_4 - 2L_6)$ | -0.1395 | -0.1112 | -0.1102 | -0.1575 |
$10^3 (L_5 - 2L_8)$ | 0.0406 | -0.0940 | -0.0235 | -0.0370 |
$10^3 (L_8 + 3L_7)$ | -0.5130 | -0.5127 | -0.4950 | -0.5207 |
$m_s/m$ | 26.547 | 26.187 | 26.596 | 26.600 |

A fit to the lattice data on baryon masses. In Tab. XV we show our results for four distinct fit scenarios, which are reasonably close to the results of [35]. The quark mass ratio $m_s/m$ as given in the last row of the table is compatible with the latest result of ETMC [59] with $m_s/m = 26.66(32)$. In Tab. XV also the lattice scale parameters $a$ together with the offset charm-quark mass parameters $\Delta_c$ are presented. All fits reproduce the D meson masses of all ensembles recalled in this work quite well. The table illustrates that the offset parameters
FIG. 6. D meson masses from Fit 1 compared to results based on lattice ensembles from PACS-CS and LHPC [10, 18, 53]. The yellow symbols present our predictions for the case where no lattice values are available yet.

are almost always non negligible. Our values for the lattice scale can be compared with the ones advocated by the various lattice groups as recalled in the tables of this section. Any deviation from such values may be viewed as a reflection of significant discretization effects. Those depend on the specifics of the scale setting. The aim of our work is to minimize such discretization effects in the open-charm meson sector of QCD. We find interesting that in particular our values for ETMC are amazingly close to those lattice scales obtained in our previous analysis of the baryon masses from the identical lattice ensembles [35].

The quality of the data description is illustrated at hand of Fit 1 for which we offer a comparison with the lattice data in Fig. 6. A more quantitative comparison with $\chi^2$ values will be provided in the next section. In all figures open symbols correspond to results from our chiral extrapolation approach. They lie always on top of the lattice points, which are shown with either green, blue or red filled symbols. In case that for a considered lattice ensemble there is no lattice result for the considered D meson mass available our theory prediction is presented with a yellow filled symbol.

In Fig. 6 we scrutinize the lattice results of [10, 18, 53] as recalled in Tab. VIII and Tab. XII. Note that the strange quark mass varies along the different pion masses of the figure. The D meson masses are shown in units of GeV, where the lattice scales for the two groups are taken from Tab. XV. In addition the effect of the fine tuned charm quark mass in terms
of the appropriate $\Delta_c$ values in Tab. XV is considered. From Fig. 6 we conclude that all masses from [10, 18, 53] are recovered well with an uncertainty of less than 10 MeV. The figures include predictions of 5 meson masses shown with yellow symbols for which there do not exist so far corresponding values from the lattice collaborations. Note that in some cases the lattice data point is fully covered with our chiral extrapolation symbol. This signals an almost perfect reproduction of the lattice point.

We continue with Fig. 7 where the predictions of ETMC are compared to our results. Here the meson masses are shown in lattice units. This permits an efficient presentation of the results at three distinct $\beta_{QCD}$ values. The data set of ETMC is of particular importance for the chiral extrapolation since it offers masses for the $J^P = 0^-$ and $J^P = 1^-$ states consistently. The figure illustrates that such data can be reproduced accurately for all $\beta_{QCD}$ values. Note that the effect of a fine tuned charm quark mass is considered again in terms
FIG. 8. D meson masses from Fit 1 in lattice units as compared to results from HPQCD and HSC [19, 23, 47]. The yellow symbols present our predictions for the case where no lattice values are available yet. Note that we show the HPQCD data in units of their spatial lattice spacing but the HSC data in units of 3.5 times their temporal lattice spacing.

It remains a discussion of Fig. 8 which combines results from HPQCD and HSC [19, 23, 47]. Again the meson masses are shown in lattice units with $\Delta_c$ from Tab. XV. The reproduction of the lattice data is again impressive. The reader is pointed to the fact that we predict 13 masses with yellow symbols for which there are not yet values available from the lattice groups. Of particular interest are the mass predictions for the second ensemble of HSC as recalled in Tab. XIV. For this ensemble the authors are informed that the HSC is currently computing various scattering observables. We will return to this issue below.

The section is closed with a brief discussion of the quark masses. Given the different
FIG. 9. The quark mass ratios $m_s/m$ are shown for the various lattice ensembles considered. Closed symbols show the values from the lattice collaborations, open symbols our results. Fit scenarios of Tab. XV their values can be computed for any lattice ensemble for which the pion and kaon mass are measured on a specified lattice volume, where again here we ignore discretization effects. Within a chiral Lagrangian approach only ratios of the quark masses can be determined. This is so since only products of $B_0 m$ or $B_0 m_s$ occur. In Fig. 9 such ratios are confronted with corresponding ratios from the various lattice groups. While our values are given by open symbols the lattice results by closed symbols. We follow here our convention that the open symbols are always on top of the closed symbols. An amazingly consistent pattern occurs. We note that the determination of the quark mass ratios depends on the action used, and may be quite involved due to non-trivial renormalization effects. Most straight forward are the results from HPQCD and ETMC [47, 59, 68] where it is stated that the quark-mass ratio remains unrenormalized. The PACS and LHPC col-
aborations made significant efforts to control their non-trivial renormalization effects in the quark masses [10, 17]. As shown in our figure all quark-mass ratios appear consistent with a universal set of chiral low-energy parameters as given in Tab. XV. All four fit scenarios lead to almost indistinguishable results for the quark masses. The small spread in the low-energy constants is not significant.
VII. LOW-ENERGY CONSTANTS FROM QCD LATTICE DATA

We report on our efforts to adjust the low-energy parameters to the D meson masses as evaluated by the various lattice groups. Our first observation is that the available data set is not able to determine a unique parameter set without additional constraints. Therefore it would be highly desirable to evaluate the D meson masses with $J^P = 0^-$ and $J^P = 1^-$ quantum numbers on further QCD lattice ensembles with unphysical pion and kaon masses.

Typically solutions can be found with similar quality in the lattice data reproduction but quite different values for the low-energy parameters. This problem is amplified by the unknown size of the underlying systematic error from discretization effects. Almost always the size of the statistical errors given by the lattice groups is negligible, and it is expected that the systematic error is dominating the total error budget. In turn it is unclear whether a parameter set with a better $\chi^2$ value is more realistic than a solution with a worse $\chi^2$. The D meson masses may be over fitted.

To actually perform the fits is a computational challenge. For any set of the low-energy parameters four coupled non-linear equations are to be solved on each lattice ensemble considered. We apply the evolutionary algorithm of GENEVA 1.9.0-GSI [69] with runs of a population size 4000 on 100 parallel CPU cores.

In Tab. XVI we collect four distinct fit scenarios which are constrained by additional input from first lattice results on some scattering observable. All four fit scenarios incorporate the s-wave scattering lengths of [10] into their $\chi^2$ functions. In addition Fit 2-4 are adjusted to the scattering phases shifts of [23]. In Fit 3 and Fit 4 the subleading counter terms (49) are activated. All parameter sets reproduce the D meson masses with a $\chi^2/N$ close to one given an estimate for the systematic error in the range 5-10 MeV. In all fit scenarios the four low-energy constants $c_{0,1}$ and $\tilde{c}_{0,1}$ are adjusted to recover the isospin averaged physical D meson masses with $J^P = 0^-$ and $J^P = 1^-$ quantum numbers from the PDG [43]. This implies that deviations from leading order large-$N_c$ or heavy-quark symmetry sum rules are considered for $c_{0,1}$ and $\tilde{c}_{0,1}$. In turn we must not impose the heavy quark-symmetry relations $d_n = \tilde{d}_n$ for all $n = 1, ..., 4$. Scale invariant expressions request $d_1^x \neq \tilde{d}_1^x$ and $d_3^x \neq \tilde{d}_3^x$ but permit the assumptions $d_2^x = \tilde{d}_2^x$ and $d_4^x = \tilde{d}_4^x$ (see (25)). All four fit scenarios are based on the latter. In addition we note that while Fit 1 and Fit 3 impose the leading order large-$N_c$
TABLE XVI. The low-energy constants from a fit to the pseudo-scalar and vector D meson masses based on QCD lattice ensembles of the PACS-CS, MILC, ETMC and HSC as described in the text. Each parameter set reproduces the isospin average of the empirical D meson masses from the PDG.

|        | Fit 1   | Fit 2   | Fit 3   | Fit 4   |
|--------|---------|---------|---------|---------|
| $M$ [GeV] | 1.8762  | 1.9382  | 1.9089  | 1.8846  |
| $\Delta$ [GeV] | 0.1873  | 0.1876  | 0.1834  | 0.1882  |
| $c_0$    | 0.2270  | 0.3457  | 0.2957  | 0.3002  |
| $\tilde{c}_0$ | 0.2089  | 0.3080  | 0.2737  | 0.2790  |
| $c_1$    | 0.6703  | 0.9076  | 0.8765  | 0.8880  |
| $\tilde{c}_1$ | 0.6406  | 0.9473  | 0.8420  | 0.8583  |
| $c_2^r = \tilde{c}_2^r$ | -0.5625 | -2.1893 | -1.6224 | -1.3046 |
| $c_3^r = \tilde{c}_3^r$ | 1.1250  | 4.4956  | 3.2448  | 2.9394  |
| $c_4^r = \tilde{c}_4^r$ | 0.3644  | 2.0012  | 1.2436  | 0.9122  |
| $c_5^r = \tilde{c}_5^r$ | -0.7287 | -4.1445 | -2.4873 | -2.1393 |
| $\tilde{d}_c^1$ [GeV$^{-2}$] | 1.8331  | 1.6937  | 1.6700  | 1.9425  |
| $\tilde{d}_c^2$ [GeV$^{-2}$] | 1.6356  | 1.6586  | 1.4701  | 1.7426  |
| $\tilde{d}_c^2 = \tilde{d}_c^2$ [GeV$^{-2}$] | 1.0111  | 0.9954  | 0.8684  | 1.0032  |
| $\tilde{d}_c^3$ [GeV$^{-2}$] | 0.1556  | 0.0679  | 0.1531  | 0.1109  |
| $\tilde{d}_c^3$ [GeV$^{-2}$] | 0.2571  | 0.1640  | 0.2597  | 0.2143  |
| $\tilde{d}_c^4 = \tilde{d}_c^4$ [GeV$^{-2}$] | 0.8072  | 1.6392  | 0.8607  | 1.1255  |

relations

\[
    c_2^r = -\frac{c_3^r}{2}, \quad c_4^r = -\frac{c_5^r}{2}, \quad \tilde{c}_2^r = -\frac{\tilde{c}_3^r}{2}, \quad \tilde{c}_4^r = -\frac{\tilde{c}_5^r}{2}, \quad (48)
\]

the remaining scenarios Fit 2 and Fit 4 keep those parameters unrelated.

The quality with which the four scenarios reproduce the D meson masses from the lattice ensembles is summarized in Tab. [XVII] From the fact that all chi-squared values are close to
TABLE XVII. The table shows the impact of an ad-hoc systematic error (that is added to the statistical error in mean quadrature) on the chisquare values of the various lattice data sets. The set of lattice data fitted is described in the text. The corresponding low-energy parameters of Fit 1-4 are given in Tab. XVI.

|                | Fit 1 | Fit 2 | Fit 3 | Fit 4 | systematic error |
|----------------|-------|-------|-------|-------|------------------|
| $\chi^2_{\text{PACS-CS}/N}$ | 0.5054 | 0.8721 | 0.5329 | 0.4824 | 10 MeV          |
|                | 1.6153 | 2.6456 | 1.9222 | 1.6726 | 5 MeV           |
| $\chi^2_{\text{LHPC}/N}$ | 0.0999 | 1.6006 | 0.3911 | 0.1574 | 10 MeV          |
|                | 0.3659 | 5.9049 | 1.4524 | 0.5851 | 5 MeV           |
| $\chi^2_{\text{HPQCD}/N}$ | 0.9430 | 0.9131 | 1.2962 | 1.0606 | 10 MeV          |
| $\beta \approx 6.76$ | 3.7132 | 3.5877 | 5.1052 | 4.1814 | 5 MeV           |
| $\chi^2_{\text{HPQCD}/N}$ | 0.2468 | 0.2688 | 0.3393 | 0.4172 | 10 MeV          |
| $\beta \approx 7.09$ | 0.9798 | 1.0662 | 1.3459 | 1.6495 | 5 MeV           |
| $\chi^2_{\text{ETMC}/N}$ | 0.4584 | 1.2096 | 0.9919 | 0.8367 | 10 MeV          |
| $\beta = 1.90$ | 1.1053 | 2.8710 | 2.5727 | 2.1517 | 5 MeV           |
| $\chi^2_{\text{ETMC}/N}$ | 0.6546 | 1.5087 | 1.0253 | 0.8279 | 10 MeV          |
| $\beta = 1.95$ | 1.6217 | 3.6038 | 2.5556 | 2.0590 | 5 MeV           |
| $\chi^2_{\text{ETMC}/N}$ | 0.1860 | 0.4915 | 0.4431 | 0.3572 | 10 MeV          |
| $\beta = 2.10$ | 0.4061 | 1.1424 | 0.9964 | 0.7943 | 5 MeV           |
| $\chi^2_{\text{HSC}/N}$ | 0.1425 | 0.1710 | 0.4735 | 0.2622 | 10 MeV          |
|                | 0.3757 | 0.5893 | 1.8550 | 0.9965 | 5 MeV           |

one for an ad-hoc systematic error in between 5 and 10 MeV we arrive at our estimate of an intrinsic systematic error of 5-10 MeV for the D meson masses. All low-energy parameters are in qualitative agreement with the first rough estimates in (7). On the other hand we find significant tension with the low-energy parameters as obtained in [11, 36, 70, 71]. The parameters of Fit 2 are reasonably close to the two sets claimed in [10] with the notable
exception of $c_1$ which differs by about a factor 2. Despite the considerable variations in the low-energy constants we deem all four parameter sets acceptable from the perspective of describing the D meson masses. We repeat that it is unclear whether Fit 1 should be trusted more, only because it would be compatible with a discretization error slightly smaller than the one for Fit 4. After all a 5 MeV systematic error would be an astonishingly small value.

We take up the additional constraints considered. In [10] a set of s-wave pion and kaon scattering lengths was computed on 4 different lattice ensembles as recalled in Tab. XII. Since only for the first two ensembles the kaon mass is smaller than our cutoff choice of 600 MeV, we include into our $\chi^2$ function only the scattering lengths from the first two ensembles of that table. The scattering lengths are computed in the infinite volume limit based on the parameter sets collected in Tab. XVI.

We apply the coupled-channel framework pioneered in [6, 8, 9] which is based on the flavour SU(3) chiral Lagrangian. It relies on the on-shell reduction scheme developed in [11, 72] which can be justified if the interaction is of short range nature or the long-range part is negligible [73, 74]. Fortunately this appears to be the case for the s-wave interactions of the Goldstone bosons off any of the D mesons. In these and the current works the coupled-channel interaction is approximated by tree-level expressions. Coupled-channel unitarity is implied by a particular summation scheme formulated in terms of scalar loop functions evaluated with physical meson masses and relativistic kinematics.

An alternative chain of works based on a somewhat different treatment of the coupled-channel effects are [10, 13, 36, 71]. We did a careful comparison of the three available sources for the flavour structure of the coupled-channel interaction [9, 11]. We find two discrepancies amongst the original work [9] and [10] where we do take into account the different phase conventions used in the two works for the isospin states. The two discrepancies are in the

| $\chi^2_{s\text{-wave scattering length}}/N$ | Fit 1 | Fit 2 | Fit 3 | Fit 4 |
|-------------------------------------------|-------|-------|-------|-------|
|                                           | 0.9184 | 1.3849 | 2.2596 | 2.0597 |

TABLE XVIII. Chisquare values from Fit 1-4 for the s-wave scattering length of [10]. The first two ensembles of Tab. XII with a kaon mass smaller than 600 MeV are considered in the chisquare function. The corresponding low-energy parameters of Fit 1-4 are given in Tab. XVI.
FIG. 10. S-wave scattering length from Fit 4 as compared to predictions from [10]. The blue (red) data points show the scattering lengths for the ensembles where the kaon mass is smaller (larger) than 600 MeV. The yellow points provide the physical value for the scattering lengths.

\((I, S) = (1/2, 0)\) sector. One is traced as a misprint, in \(C_{WT}\) of Tab. 2 of [9], in which the two entries 13 and 22 need to be interchanged (see [8]). The second one we attribute to a misprint in [10]. Unfortunately, we were not able to relate to the flavour coefficients shown in [11]. As compared to [9] and [10] there are more than 10 unresolved contradictions.

In Tab. XVIII we collect the \(\chi^2/N\) values that characterize how well we reproduce the s-wave scattering length of [10] in our four fit scenarios. Note that we use here our estimates for the lattice scales \(a_{\text{LHPC}}\) as shown in Tab. XV. The table is complemented by Fig. 10 where a direct comparison of our results with the lattice data is provided for Fit 4. In the figure the lattice data points, shown by filled symbols, are confronted with open symbols that represent our results. The error bars in the latter points reflect an estimate of the systematic uncertainty in our computation of the scattering lengths, where we should state that the \(\chi^2\) values in Tab. XVIII are computed always in terms of the center value of our prediction. Our systematic error estimate is implied by a variation of the matching scale \(\mu_M\) around its natural value [6, 8, 9]. The error bars are implied by \(\Delta\mu_M = \pm 100\) MeV with \(\mu_M \rightarrow \mu_M + \Delta\mu_M\). For a detailed discussion why \(\Delta\mu_M\) cannot be chosen much larger
without jeopardizing the approximate implementation of crossing symmetry we refer to the original works [41, 72]. It is important to recall that dialing the matching scale slightly off its natural value does not affect our self consistent determination of the D meson masses. The latter is a convenient tool to estimate the uncertainties of the unitarization process.

In the upper panels of Fig. 10 we show the channels that are dominated by a repulsive Tomozawa-Weinberg interaction term [8]. In terms of a flavour SU(3) multiplet classification they belong to a flavour 15plet, that can not be reached within the traditional quark-model picture. A minimal four quark state configuration is required. In contrast in the lower panels, channels are presented that belong to the exotic flavour sextet sector in which the leading Tomozawa-Weinberg interaction shows a weak attraction [8]. As pointed out in [3, 6, 8, 9] depending on the size of chiral correction terms exotic resonance states may be formed by the chiral dynamics. Final state interactions distort the driving leading order term and ultimately generate the more complicated quark mass dependence as seen in the figure. We discriminate results based on ensembles with a kaon mass larger or smaller than 600 MeV by distinct colored symbols. With red symbols we indicate that the kaon mass is larger than our cutoff value, and therefore chiral dynamics is not expected to be reliable. A fair reproduction of all relevant scattering lengths is seen in Fig. 10. Our predictions for the scattering lengths at the physical point are also included by the additional yellow filled points farthest to the left.

We would conclude that with the constraints set by scattering lengths of [10] we cannot rule out any of our four fit scenarios in Tab. XVI.
In this section we finally present an additional constraint on the low-energy parameters that provide a clear criterion which of the four fit scenarios is most reliable and should be used in applications. Recently HSC computed $\pi D$ phase shifts in both isospin channels. The results are based on the ensemble recalled in Tab. XIV. Given our four parameter sets we can compute those observable at the given unphysical pion and kaon masses. We do this for all four parameter sets.

It is necessary to explain how we compare with those lattice results. Ultimately one should compute the various discrete levels the collaboration computed and then apply the Lüscher method \cite{75, 76} to extract the coupled-channel scattering amplitudes. This requires an ansatz for the form of the reaction amplitudes. In the case of a single channel problem this can be analyzed in a model independent manner. In turn for $\pi D$ scattering in the $I = 3/2$ channel we can compare our results with the single energy phase shifts as taken from Fig. 20 of \cite{23} at different center-of-momentum energies $E = \sqrt{s} - m_\pi - M_D$. They are to be confronted with the four lines from our four fit scenarios. In the figure of Tab. XIX we see that the two red lines are significantly off the lattice data points, where with those Fit 1 and 2 are presented. This is the case even though in Fit 2 an attempt was made to reproduce the $\pi D$ phase shifts from \cite{23}. Note that in Fit 1 we ignored any of the latter. We assure that our conclusions are stable against a reasonable variation of the matching scale in this sector.

Based on this observation we made our ansatz for the scattering amplitudes more quantitative by the consideration of an additional set of low-energy constants relevant at chiral order three. Such terms were constructed in \cite{77, 78} to take the form

\begin{equation}
\mathcal{L}_3 = 4 g_1 D [\chi^-, \partial^\nu \bar{D} D] - 4 g_2 D ([U_\mu, [\partial^\nu, U^\mu]_-]_+ + [U_\mu, [\partial^\mu, U_\nu]_-]_-) \partial^\nu \bar{D} D
- 4 g_3 D [U_\mu, [\partial^\nu, U_\rho]_-]_- [\partial^\mu, [\partial^\nu, \partial^\rho]_+ D + h.c. .
\end{equation}

Our motivation to consider such terms is slightly distinct to the one followed in \cite{77, 78}. From the previous work \cite{9} we expect the light vector meson degrees of freedom to play a crucial role for the considered physics. Ultimately we would like to consider them as active degrees of freedom. This is beyond the scope of the current work. Here we consider the low-energy constants as a phenomenological tool to more accurately integrate out the light
vector meson degrees of freedom. In scenario Fit 3 and Fit 4 the contributions of the $g_n$ are worked into the coupled-channel interaction. Their values are displayed in Tab. XIX which consecutively lead to a significantly improved reproduction of the scattering phase shift.

We proceed by the coupled-channel $\pi D$ system with $I = 1/2$ for which its determination of the three phase shifts and in-elasticities is more involved. Some model dependence may enter the analysis. In [23] an estimate of the latter was accessed by allowing a quite large set of different forms of the ansatz for the coupled-channel amplitudes. That then leads to two error bands in their plotted phase shifts and in-elasticity parameters. The smaller one shows the statistical uncertainty, the larger one includes also the systematic error. In Fig. 9 and Fig. 10 of [23] it is shown in addition, on how many levels their results are based on in a given energy bin. Above the $\pi D$ and below the $KD_s$ threshold there are three clusters of levels. We take their center and translate those into single energy phase shifts and in-elasticities with error bars taken from the estimated uncertainties. In Fig. 11 those 'lattice data' points are shown and confronted with our results from the four fit scenarios. In addition a fourth lattice data point at energies above the $KD_s$ threshold is also included in the figure, but shown in red symbols. We do have some reservation towards those points, since the number of close-by energy levels is quite scarce. This is particularly troublesome since here it is a true three channel system that would need more rather than a fewer number of levels to unambiguously determine the scattering amplitude. In turn, the particular choice of ansatz is expected to play a much more significant role in the determination of the red lattice data points. We conclude that the error bars must be significantly underestimated for those points.
FIG. 11. Phase shifts with \((I, S) = (1/2, 0)\) from Fit 1-4 as compared to lattice data from [23].

While the solid lines are from Fit 2 and 4, the dashed lines with respect to Fit 1 and 3. The two red lines present the disfavoured scenarios from Fit 1 and 2. We apply the somewhat unusual convention of the lattice group where the phase shift at threshold is normalized to zero even in the presence of a bound state.

Fig. 11 confirms our conclusions from the previous Tab. XIX that only Fit 3 and Fit 4 may be expected to be faithful. The \(\pi D\) and \(\eta D\) phase shift points are highly discriminative amongst the 4 fit scenarios. Fit 3 and Fit 4 describe the lattice data in Fig. 11 significantly better than Fit 1 and Fit 2. Since Fit 4 is doing better in the D meson masses, but also in the s-wave scattering lengths one may identify Fit 4 to be the most promising candidate for making reliable predictions.

There is a further piece of information provided by HSC in the given ensemble. The mass \(M_B\) of a bound state just below the \(\pi D\) threshold is predicted. It is a member of the conventional flavour anti-triplet, which formation was predicted by chiral dynamics unambiguously [8, 9]. Within the given error it is not distinguishable from the \(\pi D\) threshold value. The following bound is derived from data published by HSC

\[
\epsilon_B = \frac{m_{\pi} + M_D}{M_B} - 1 < 0.001,
\]  

(50)
FIG. 12. Predictions for phase shifts from Fit 4 for the physical point but also for pion and kaon masses as shown in Tab. XIV.

at the one sigma level. We compute this value in the four fit scenarios with

\[ 10^3 \epsilon_B = \begin{cases} 
Fit 1 & 8.0 \\
Fit 2 & 5.4 \\
Fit 3 & 4.3 \\
Fit 4 & 5.7 
\end{cases} \]  

where we find discrepancies for the bound state mass of the order of our resolution of 5-10 MeV. As a consistency check we exploit the uncertainties in the unitarization process, by tuning the matching scale to meet the condition (50) for Fit 1 through Fit 4. This is achieved for instance with \( \Delta \mu_M \approx 69 \text{ MeV} \) and \( \Delta \mu_M \approx 86 \text{ MeV} \) in Fit 3 and Fit 4 respectively, where we emphasize that with \( \Delta \mu_M \) the determination of the D meson masses is not affected. Then we reconsider the phase shifts and in-elasticities and find that all together the impact of such a change of the matching scale is quite moderate. While now Fit 1 goes almost perfectly through the three blue lattice data points for the \( \pi D \) phase shift, the lines of Fit 3 and Fit 4 are slightly below those points. The crucial observation is that the significant disagreement with the single blue \( \eta D \) phase shift is persistent in the Fit 1 scenario and therefore Fit 4 must remain our favorite choice.

We wish to make one comment on Fit 1 since it is particularly interesting despite its deficiencies: a clear signal of a member of the exotic sextet state is visible in the \( \pi D \) phase
FIG. 13. Predictions for phase shifts from Fit 4 for the physical point but also for pion and kaon masses as shown in Tab. XIV.

shift. It shows a significant variation a little right from the last blue lattice point. We deem it unfortunate that exactly in this region there is not yet sufficient consolidated lattice points available which may rule out our first fit scenario unambiguously. Note furthermore that our Fit 1 scenario, which did not take any of the scattering observables from HSC into account,
is disfavoured mainly by one feature of the HSC results in the \((I, S) = (1/2, 0)\) sector. The single blue value for the \(\eta D\) phase shift. It would be interesting to make the ansatz used by HSC for the coupled-channel amplitude more flexible and allow for an exotic state coupling dominantly to the \(\eta D\) channel. One may speculate that this exercise could show that the claimed uncertainty for this lattice point is underestimated significantly. If this happens our Fit 1 scenario may come into the game again. This may be so even though HSC appears to reject our Fit 1 scenario based on their results in the \((I, S) = (3/2, 0)\) sector. Here the reader should be cautioned that we cannot fully rule out that the phenomenological treatment of the third order effects is fooling us. More detailed studies are required to substantiate our conclusions.

In the following we take our best fit scenario Fit 4 and provide a thorough documentation of its consequences. In Fig. 12 and Fig. 13 all phase shift and in-elasticity parameters are shown for all possible combinations of \((I, S)\). In Fig. 12 we present the channels in which no exotic signals are expected. Indeed, the evolution from the two HSC ensembles of Tab. XIV with unphysical quark masses to the physical point is smooth and unspectacular. While the solid black lines correspond to the physical point, the dashed and dotted lines to the two HSC cases, where the dashed lines are with respect to the upper ensemble of Tab. XIV. We refrain from including our estimate of the systematic uncertainty from a variation of the matching scale, because, first of all it is a small effect and second it obscures the clarity of the figures.

We advance to the exotic sectors with \((I, S) = (0, -1)\) and \((1, 1)\). With the upper two panels of Fig. 13 we demonstrate that here the evolution from the two HSC ensembles to the physical point is still smooth but quantitatively more significant, particularly in the two-channel system with \((I, S) = (1, 1)\). The corresponding amplitudes are characterized by strong cusp effects at threshold. The latter reflect some weak attraction present in those channels being members of the flavour sextet.

Most striking are our predictions for the quark-mass dependence of the \((I, S) = (1/2, 0)\) sector, which we present with the lower two panels of Fig. 13. The line conventions are identical to the ones used in the previous figures. The largest effect is seen in the \(\pi D\) phase shift. Going from the HSC ensembles to the physical point it even changes sign. Here we see a clear signal for a member of the exotic flavour sextet state. The \(\pi D\) phase shift passes through 90° in between the \(\eta D\) and \(\bar{K}D_s\) thresholds. We checked that the amplitudes
\[ \eta D \to \eta D \text{ but also } KD_s \to KD_s \] show a well defined resonance structure, with a width significantly smaller than the 300 – 400 MeV of the flavour anti-triplet partner at lower masses. We find this to be a spectacular confirmation of the leading order prediction of this state advocated since 15 years ago by one of the authors (see [3]). It is amusing to see that the clear signature of this state at the physical point may not be seen at the studied HSC ensemble with unphysically large pion masses. Most exciting is the most recent claim in [79] that this state can be seen in data from LHCb [80, 81].
IX. ISOSPIN VIOLATING DECAY OF $D_{s0}^{*}(2317)$ FROM QCD LATTICE DATA

A most striking prediction of chiral dynamics is the formation of the $D_{s0}^{*}(2317)$ as a coupled-channel hadronic molecule with significant components in the $\bar{K}D$ and $\eta D_s$ two-body states [8]. At leading order in a chiral expansion the coupled-channel interaction is predicted by the Tomozawa-Weinberg term that is parameterized only by the pion-decay or kaon-decay constants, $f_\pi$ or $f_K$, driven into their chiral flavour SU(3) limit with $f_{\pi,K} \rightarrow f$.

This term dominates the s-wave coupled-channel force of the Goldstone bosons with the pseudo-scalar and vector D mesons. The force is short ranged: it may be visualized in terms of a vector meson t-channel exchange process with properly adjusted coupling constants. In contrast to a widespread confusion in the field there are hadronic molecular states that are not driven by a long-range force as provided by an exchange process involving the pion. The challenge is to control and predict such short range forces.

The original work [8] was taken up by many authors [6, 9, 11, 14, 71, 79, 82, 83] who confirm this universal picture. The challenge is to make this approach more quantitative by controlling chiral correction terms. A first attempt was made in [6, 9] based on rough assumptions on the $\pi D$ invariant mass distributions. A more sophisticated approach was pursued in [10, 12] where first QCD lattice data on some s-wave scattering lengths were used. With the significantly improved and extended lattice data set the determination of the low-energy constants, as achieved in our work, is expected to be more controlled and reliable.

In this section we focus on a particular property of the $D_{s0}^{*}(2317)$, its isospin violating hadronic decay width. Since its mass is below the $KD$ threshold and it carries isospin zero it can decay into the $\pi D$ channel only via isospin violating processes. Estimates of that width within typical quark-model approaches predict such a width of less than 10 keV [84]. This is contrasted by estimates from chiral-coupled channel approaches. Here, already the leading order Tomozawa-Weinberg predicts a width of about 75 keV as demonstrated first in [6]. A corresponding computation with similar physics input but less stringent framework arrived at a similar value [82]. This is to be compared to the significantly larger values of about 140 keV in [6] and later with even an error estimate of $(133 \pm 22)$ keV [10]. The latter two works implemented chiral correction terms, where the more sophisticated approach [10] was based on additional constraints from some early lattice data.
The results of our study for the decay width is collected in Tab. XX for all four fit scenarios. They are based on the framework as detailed in [6]. Since the mass of the $D_{s0}^{*}(2317)$ was not tuned in any of our fits we again use the uncertainty in the unitarization and adjust the matching scale as to recover the precise mass of the $D_{s0}^{*}(2317)$. This is achieved with $50 \text{ MeV} < \Delta \mu_M < 100 \text{ MeV}$ in the four scenarios. Beside the low-energy constants determined in our work the computation of the width parameter depends crucially on the mixing angle $\epsilon$ of the $\pi_0 - \eta$ system. According to [67] it is determined by the quark masses as follows

$$\frac{\sin(2\epsilon)}{\cos(2\epsilon)} = \sqrt{3} \frac{m_d - m_u}{2m_s - m_u - m_d}. \quad (52)$$

While in [6] the value $\epsilon = 0.010(1)$ was taken from [67] an updated estimate $\epsilon = 0.0129(7)$ was used in [10]. Here we consider the impact of a recent and more precise lattice determination of the quark masses by ETMC [59]. This leads to a significantly lower estimate $\epsilon = 0.0122(18)$ which our faithful results in Tab. XX are based on.

Since we argued that the lattice data of HSC rule out Fit 1 and Fit 2, we estimate the isospin violating hadronic width of the $D_{s0}^{*}(2317)$ with $(104 - 116) \text{ keV}$, somewhat lower than the previous claimed value of $(133 \pm 22) \text{ keV}$ [10].

\[
\begin{array}{|c|c|c|c|c|}
\hline
\Gamma_{D_{s0}^{*}(2317) \rightarrow \pi_0 D_s} [\text{keV}] & \text{Fit 1} & \text{Fit 2} & \text{Fit 3} & \text{Fit 4} & \epsilon \\
\hline
61.1 & 54.1 & 88.6 & 80.1 & 0.0100 \\
74.6 & 68.4 & 115.8 & 104.4 & 0.0122 \\
\hline
\end{array}
\]

TABLE XX. Prediction for the isospin violating decay width of the $D_{s0}^{*}(2317)$ in the four fit scenarios of Tab. XIX.
X. SUMMARY AND CONCLUSIONS

We studied the chiral extrapolation of charmed meson masses based on the three-flavour chiral Lagrangian formulated with pseudo-scalar and vector charmed fields. Here the recent approach by the authors constructed for the chiral extrapolation of the baryon ground state masses was adapted to the charm sector successfully, where good convergence properties for the chiral extrapolation are observed. Within the framework the chiral expansion is formulated in terms of physical masses. While an attempt was made to remove all model dependence a residual scheme dependence cannot be ruled out at this stage. All D meson masses arise in a manifest scale invariant manner. The framework was applied to lattice data such that an almost unique set of low-energy constants was established. While we considered finite volume effects systematically, we did not implement discretization effects. In turn, a fully systematic error analysis was outside the realm of our present study.

The low-energy parameters were adjusted to QCD lattice data at $N^3$LO, where large-$N_c$ sum rules or relations that follow in the heavy charm-quark mass limit were used systematically. We considered lattice data based on ensembles of PACS-CS, MILC, ETMC and HSC with pion and kaon masses smaller than 600 MeV. Besides taking into account constraints from the D meson masses from the various lattice groups, we also considered first results on scattering observables in particular from HSC. Only with the latter, in particular their estimate of the $\eta D$ phase shift, we arrive at a rather well defined parameter set, in terms of which we make predictions. The data set on the D meson masses together with constraints from s-wave scattering lengths is not sufficient to nail down the set of low-energy constants.

We computed 15 phase shifts and in-elasticities at physical quark masses but also for an additional HSC ensemble. Such results can be scrutinized by lattice QCD with available computing resources and technology. In addition we predict the isospin violating strong decay width of the $D^{*}_{0}(2317)$ to be $(104 - 116)$ keV. Given our favorite set of low-energy parameters we find a clear signal for a member of the exotic flavour sextet states in the $\eta D$ channel, below the $\bar{K}D_s$ threshold.

To further substantiate the claimed chiral low-energy parameters it is necessary to take additional data on QCD lattices in particular at unphysical quark masses. Our predictions are relevant for the PANDA experiment at FAIR, where the width of the $D^{*}_{0}(2317)$ may be accessible by a scan experiment [85]. Also the invariant $\eta D$ mass distribution, in which we
expect a signal from an exotic flavour sextet state, may be accessed by the efficient detection of neutral particles with the available calorimeter.

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APPENDIX A

In this Appendix we collect all dimension less coefficients that are needed in the various power counting decompositions of the renormalized loop function \((14)\). Here we focus on the pseudo-scalar \(D\) mesons for which we find

\[
\alpha_1 = \frac{(2M + \Delta)^2}{4M^2}, \quad \alpha_2 = \frac{2M^2 + 2 \Delta M + \Delta^2}{2M^2}, \quad \alpha_3 = 1,
\]

\[
\gamma_1 = \frac{2M + \Delta}{M} \log \frac{\Delta (2M + \Delta)}{(M + \Delta)^2},
\]

\[
\gamma_2 = -\frac{2M^2 + 2 \Delta M + \Delta^2}{M (2M + \Delta)} \log \frac{\Delta (2M + \Delta)}{(M + \Delta)^2} - \frac{M}{2M + \Delta}, \quad \gamma_3 = \frac{M}{2M + \Delta},
\]

\[
\gamma_4 = -\frac{2M (M + \Delta)^2}{(2M + \Delta)^3} \log \frac{\Delta (2M + \Delta)}{(M + \Delta)^2} + \frac{M^3}{2(2M + \Delta)^3}, \quad \gamma_5 = \frac{M (M + \Delta)^2}{(2M + \Delta)^3},
\]

\[
\delta_1 = \gamma_1 - \frac{2M + \Delta}{M} \log \frac{2\Delta}{(M + \Delta)},
\]

\[
\delta_2 = \gamma_2 + \frac{2M^2 + 2 \Delta M + \Delta^2}{M (2M + \Delta)} \log \frac{2\Delta}{M} + \frac{2M + \Delta}{4M}
\]

\[
+ 2 (\delta_3 - \gamma_3) \log \frac{M + \Delta}{M},
\]

\[
\delta_3 = \gamma_3 - \frac{2M^2 + 2 \Delta M + \Delta^2}{2M (2M + \Delta)}, \quad \delta_5 = 0,
\]

\[
\delta_4 = \gamma_4 + \frac{2M (M + \Delta)^2}{(2M + \Delta)^3} \log \frac{2\Delta}{M}
\]

\[
- \frac{4M^2 + \Delta (4M + 5 \Delta)}{32M (2M + \Delta)} + 2 (\delta_5 - \gamma_5) \log \frac{M + \Delta}{M},
\]

\[
\delta_6 = \frac{2M + \Delta}{2M} \frac{\partial}{\partial \Delta} \frac{2M \Delta}{2M + \Delta} (\gamma_1 - \delta_1) + \delta_1, \quad \delta_7 = \gamma_2 + \frac{1}{2} (\gamma_1 - \delta_1) \frac{\Delta^2}{(2M + \Delta)^2},
\]

\[
\beta_1 = \Delta \frac{\partial}{\partial \Delta} \frac{\alpha_1}{2M} \frac{2M + \Delta}{2M},
\]

\[
\beta_2 = \Delta^2 \frac{\partial}{\partial \Delta} \frac{\alpha_1}{\Delta} \frac{\delta_2}{\Delta}, \quad \beta_3 = \Delta^2 \frac{\partial}{\partial \Delta} \frac{\alpha_1}{\Delta} \frac{\alpha_1 \delta_3}{\Delta},
\]

\[
\beta_4 = \Delta \gamma_1 \frac{\partial}{\partial \Delta} \frac{\alpha_1}{\Delta}, \quad \beta_5 = \Delta \frac{\partial}{\partial \Delta} \frac{\alpha_1}{\Delta} \frac{\delta_1}{\Delta},
\]

\[
\beta_6 = \frac{\Delta^2 \partial^2}{\partial \Delta \partial \Delta} \left( \frac{\alpha_1}{2M} \frac{2M + \Delta}{2M} \right), \quad \beta_7 = \Delta \frac{\Delta^2 \partial^2}{\partial \Delta \partial \Delta} \frac{\alpha_1 \delta_2}{\Delta},
\]

\[
\beta_8 = \Delta \frac{\Delta^2 \partial^2}{\partial \Delta \partial \Delta} \frac{\alpha_1 \delta_3}{\Delta}, \quad \beta_9 = \gamma_1 \frac{\Delta^2 \partial^2}{\partial \Delta \partial \Delta} \frac{\alpha_1}{\Delta},
\]

60
\[ \beta_{10} = \frac{\Delta^2 \partial^2}{\partial \Delta \partial \Delta} \alpha_1 \delta_1, \quad \beta_{11} = -\frac{1}{4} \alpha_1 \frac{M}{2M+\Delta} + \left( \alpha_1 - \alpha_2 \right) \frac{(2M+\Delta)M}{2\Delta^2}. \]  

(53)

While the \( \alpha_i \) characterize the chiral expansion of the coefficients in front of \( \bar{I}_{QR} \) and \( \bar{I}_Q \) in (16), the \( \gamma_i \) and \( \delta_i \) follow from a chiral expansion of \( \bar{I}_{QR} \) with \( M_H = M \) and \( M_R = M + \Delta \) and \( m_Q < \Delta \). The coefficients \( \beta_i \) are required in (44, 45, 58).

We turn to the chiral domain (34), in which the bubble-loop contributions to the D meson masses generate a renormalization of the low-energy parameters \( d_i \). Such terms are proportional to the product of two quark masses (9). We provide detailed results with

\[
\Pi^{(4-\chi)}_H \rightarrow \sum_{Q \in [8]} \sum_{R \in [1^{-}]} \left( \frac{G_{QR}^H}{8 \pi f} \right)^2 \left\{ \gamma_d^{(1)} m_Q^2 \Pi_R^{(2)} + \gamma_d^{(2)} m_Q^2 \Pi_H^{(2)} + \gamma_d^{(3)} \Pi_R^{(2)} \Pi_H^{(2)} \right. \\
+ \gamma_d^{(4)} \Pi_H^{(2)} \Pi_H^{(2)} + \gamma_d^{(5)} \Pi_R^{(2)} \Pi_H^{(2)} \bigg\},
\]

\[ d_i \rightarrow \frac{1}{4} g_P^2 \sum_{k=1}^5 \frac{\Gamma_d^{(k)}}{(4\pi f)^2} \gamma_d^{(k)}, \]

(54)

and

\[
\gamma_d^{(1)} = \frac{M}{2(M+\Delta)} \left[ \frac{\partial}{\partial \Delta} \left( \alpha_2 \Delta \gamma_1 - \alpha_1 \Delta \gamma_2 \right) - \Delta \gamma_1 \frac{\partial \alpha_2}{\partial \Delta} \right],
\]

\[
\gamma_d^{(2)} = \frac{\Delta}{2M} \left[ \frac{\partial}{\partial M} \left( \alpha_2 M \gamma_1 - \alpha_1 M \gamma_2 \right) - \frac{1}{M} \gamma_1 \frac{\partial}{\partial M} \left( \alpha_2 M^2 \right) \right] - \frac{M + \Delta}{M} \gamma_d^{(1)},
\]

\[
\gamma_d^{(3)} = -\frac{M}{4(M+\Delta)^2} \left( \frac{\partial \alpha_1 \Delta^2}{\partial \Delta} \right) \left( \frac{\partial \gamma_1 \Delta}{\partial \Delta} \right) - \frac{\alpha_1 \Delta^2 M}{4(M+\Delta)} \frac{\partial}{\partial \alpha_1} \left[ \frac{1}{2(M+\Delta)} \left( \frac{\partial \gamma_1 \Delta}{\partial \Delta} \right) \right],
\]

\[
\gamma_d^{(4)} = -\frac{1}{8M^2} \left( \frac{\partial}{\partial M} - \frac{\partial}{\partial \Delta} \right)^2 \left( \alpha_1 M \Delta^3 \gamma_1 \right) + \frac{\gamma_1 \Delta}{8M^3} \left( \frac{\partial}{\partial M} - \frac{\partial}{\partial \Delta} \right)^2 \left( \alpha_1 M^2 \Delta^2 \right)
+ \frac{1}{8M^3} \left( \frac{\partial}{\partial M} - \frac{\partial}{\partial \Delta} \right)^2 \left( \alpha_1 M \Delta^3 \gamma_1 \right) - \frac{\gamma_1 \Delta}{8M^4} \left( \frac{\partial}{\partial M} - \frac{\partial}{\partial \Delta} \right)^2 \left( \alpha_1 M^2 \Delta^2 \right),
\]

\[
\gamma_d^{(5)} = -\frac{\partial}{\partial M} \frac{1}{2M} \frac{1}{2(M+\Delta)} \frac{\partial}{\partial \Delta} \left( \alpha_1 M \Delta^3 \gamma_1 \right)
+ \frac{\gamma_1 \Delta}{2M^2} \frac{\partial}{\partial M} \frac{1}{2(M+\Delta)} \frac{\partial}{\partial \Delta} \left( \alpha_1 M^2 \Delta^2 \right) - 2 \frac{M + \Delta}{M} \gamma_d^{(3)},
\]

(55)

and
$$\Gamma_{d_1}^{(1)} = -\frac{32}{3} \bar{c}_1, \quad \Gamma_{d_2}^{(1)} = \frac{16}{9} (15 \bar{c}_0 - 2 \bar{c}_1), \quad \Gamma_{d_3}^{(1)} = 8 \bar{c}_1, \quad \Gamma_{d_4}^{(1)} = \frac{88}{9} (2 \bar{c}_0 - \bar{c}_1),$$
$$\Gamma_{d_1}^{(2)} = \frac{40}{3} c_1, \quad \Gamma_{d_2}^{(2)} = \frac{16}{9} (15 c_0 - 2 c_1), \quad \Gamma_{d_3}^{(2)} = 0,$$
$$\Gamma_{d_4}^{(2)} = \frac{88}{9} (2 c_0 - c_1), \quad \Gamma_{d_1}^{(3)} = -\frac{32}{3} c_1^2, \quad \Gamma_{d_2}^{(3)} = -\frac{64}{3} (2 \bar{c}_0 - \bar{c}_1), \quad \Gamma_{d_3}^{(3)} = 32 c_1^2,$$
$$\Gamma_{d_4}^{(3)} = \frac{64}{3} (16 c_0^2 - 10 \bar{c}_1 c_0 + c_1^2), \quad \Gamma_{d_1}^{(4)} = \frac{512}{3} (2 c_0 - c_1), \quad \Gamma_{d_2}^{(4)} = 0, \quad \Gamma_{d_3}^{(4)} = \frac{256}{3} (2 c_0 - c_1)^2,$$
$$\Gamma_{d_4}^{(4)} = -\frac{32}{3} c_1 \bar{c}_1, \quad \Gamma_{d_2}^{(5)} = -\frac{64}{3} (8 c_1 \bar{c}_0 - (c_0 + 2 c_1) \bar{c}_1), \quad \Gamma_{d_3}^{(5)} = 0,$$
$$\Gamma_{d_4}^{(5)} = \frac{32}{3} (2 c_0 - c_1) (16 \bar{c}_0 - 5 \bar{c}_1). \quad (56)$$

We turn to the conventional counting ansatz \([38]\), for which the third and fourth order contributions to the D meson polarization tensor are already given with \([39, 40]\). The fifth-order term is

$$\bar{\Pi}_{H\in[0^-]}^{\text{bubble} = 5} = \sum_{Q \in [8]} \sum_{R \in [1^-]} \left( \frac{C_{QR}^{(H)}}{8 \pi f} \right)^2 \left\{ \left[ -\frac{3}{4} (\Pi_H^{(3)} - \Pi_R^{(3)}) m_Q^2 \right. \right.$$
$$- \frac{\Delta}{2M} \left( m_Q^4 + 2m_Q^2 (2\Pi_H^{(2)} - \Pi_R^{(2)}) + (\Pi_H^{(2)} - \Pi_R^{(2)})^2 \right) \] \log \frac{m_Q^2}{M^2}$$
$$- \left( \Pi_H^{(3)} - \Pi_R^{(3)} \right) \frac{\Delta Q}{8M} \left( 14m_Q^4 - m_Q^2 (3\Delta^2 - 16\Pi_H^{(2)} + 36\Pi_R^{(2)}) \right)$$
$$- 10 (\Pi_H^{(2)} - \Pi_R^{(2)})^2 \right) \frac{\Delta Q^2}{M} \left( \log (\Delta - \Delta Q) \right)$$
$$- \log (\Delta - \Delta Q) \bigg[ -\frac{3}{2} \Delta M \left( \Pi_H^{(3)} - \Pi_R^{(3)} \right) + \frac{3}{4} \Delta Q^4$$
$$+ \frac{1}{2} \Delta Q^2 (3\Pi_R^{(2)} - 5\Pi_H^{(2)}) + \frac{3}{4} (\Pi_H^{(2)} - \Pi_R^{(2)})^2$$
$$+ \frac{3}{8} m_Q^2 \left( \Delta_Q^2 + 2 (\Pi_R^{(2)} - 3\Pi_H^{(2)}) + \frac{(\Pi_H^{(2)} - \Pi_R^{(2)})^2}{\Delta_Q^2} \right) \bigg]$$
$$+ \left[ \frac{3}{2} \Delta \left( \Pi_H^{(3)} - \Pi_R^{(3)} \right) - \frac{\Delta}{4M} \left( 3\Delta Q^4 - 2m_Q^2 - 2(\Delta^2 + m_Q^2) (\Pi_H^{(2)} + \Pi_R^{(2)}) \right) \right] \log \frac{m_Q^2}{4\Delta^2} \bigg\}.$$  \(57\)

It remains to specify the fifth order term with respect to the novel counting ansatz \([41]\). We
\[
\begin{align*}
\Pi_{H \in [0-]}^\text{bubble-5} &= \sum_{Q \in [8]} \sum_{R \in [1-]} \left( \frac{m_Q}{4\pi f} G_{QR}^{(H)} \right)^2 \left\{ -\frac{\alpha_1}{4} \frac{M^2}{2M + \Delta} \frac{\Delta^3}{4M^2} \log \frac{4\Delta^2}{(M + \Delta)^2} \\
&+ \frac{\alpha_1}{8} \frac{\Delta^2}{m_Q^2} (M_R - M_H - \Delta) \frac{M}{M + \Delta} \frac{\partial}{\partial \Delta} (\gamma_1 \Delta) \\
&+ \frac{M_H}{4} \left[ (\alpha_1 - \alpha_2) \left( \frac{2M + \Delta}{2M} \frac{m_Q^2}{\Delta^2_H} (M_R - M_H) \log \frac{m_Q^2}{M_R^2} - (\delta_1 - \gamma_1) \frac{\Delta^2}{\Delta_H} \\
&- \delta^1 \frac{\Delta^2}{\Delta^2_H} (M_R - M_H - \Delta) \right) - \frac{\beta_{11}}{M_R^2} \left( (M_R - M_H)^3 \log \frac{m_Q^2}{M_R^2} \right) \\
&+ \Delta^3_Q \left[ \log (M_R - M_H + \Delta_Q) - \log (M_R - M_H - \Delta_Q) \right] \right] \\
&+ \frac{m_Q^2 \Delta^2}{\Delta^2_H} \frac{\Delta^2}{\Delta^2_H} \left[ (\alpha_2 - \alpha_1) \left( \delta_2 + \delta_3 \log \frac{m_Q^2}{M_R^2} - \alpha_1 (\delta_4 + \delta_5 \log \frac{m_Q^2}{M_R^2}) \right) \right] \\
&+ \frac{M_H}{8} (M_R - M_H - \Delta) \left[ \beta_0 \frac{\Delta^2}{m_Q^2 \Delta_H} - \beta_{10} \frac{\Delta^2}{m_Q^2 \Delta^2_H} (M_R - M_B) \\
&- \frac{\beta_6}{m_Q^2 \Delta^2_H} (M_R - M_H) \left( \Delta^2_Q - \frac{m_Q^2}{2} \right) \log \frac{m_Q^2}{M_R^2} \\
&+ \Delta^3_Q \left[ \log (M_R - M_H + \Delta_Q) - \log (M_R - M_H - \Delta_Q) \right] \right] \\
&+ \left( - \beta_7 \frac{\Delta^2}{\Delta^2_H} + \beta_8 \frac{m_Q^2 \Delta^2}{\Delta^2_H \log \frac{m_Q^2}{M_R^2}} \right) \right\} . \quad (58)
\end{align*}
\]
APPENDIX B

In this Appendix we collect all dimension less coefficients that are needed in the various power counting decompositions of the renormalized loop function (16). Here we focus on the vector $D$ mesons for which we find

\[
\tilde{\alpha}_1 = \frac{(2M + \Delta)^2}{4M^2}, \quad \tilde{\alpha}_2 = \frac{2M^2 + 2\Delta M + \Delta^2}{2M^2}, \quad \tilde{\alpha}_3 = 1,
\]

\[
\tilde{\gamma}_1 = -\frac{M(2M + \Delta)}{(M + \Delta)^2} \log \frac{\Delta (2M + \Delta)}{M^2},
\]

\[
\tilde{\gamma}_2 = \frac{M}{2M + \Delta} + \frac{2M^2 + 2\Delta M + \Delta^2}{(2M + \Delta)(M + \Delta)^2} \log \frac{\Delta (2M + \Delta)}{M^2}, \quad \tilde{\gamma}_3 = -\frac{M}{2M + \Delta},
\]

\[
\tilde{\gamma}_4 = \frac{M(2M + \Delta)^2}{2(2M + \Delta)^3} + \frac{2M^3}{(2M + \Delta)^3} \log \frac{\Delta (2M + \Delta)}{M^2}, \quad \tilde{\gamma}_5 = -\frac{M^3}{(2M + \Delta)^3},
\]

\[
\tilde{\delta}_1 = \tilde{\gamma}_1 + \frac{M(2M + \Delta)}{(M + \Delta)^2} \log \frac{2\Delta}{M},
\]

\[
\tilde{\delta}_2 = \tilde{\gamma}_2 - \frac{M(2M^2 + 2\Delta M + \Delta^2)}{(2M + \Delta)(M + \Delta)^2} \log \frac{2\Delta}{M + \Delta} - \frac{M(2M + \Delta)}{4(M + \Delta)^2} - 2(\tilde{\delta}_3 - \tilde{\gamma}_3) \log \frac{M + \Delta}{M},
\]

\[
\tilde{\delta}_3 = \tilde{\gamma}_3 + \frac{M(2M^2 + 2\Delta M + \Delta^2)}{2(M + \Delta)^2(2M + \Delta)}, \quad \tilde{\delta}_5 = 0,
\]

\[
\tilde{\delta}_4 = \tilde{\gamma}_4 - \frac{2M^3}{(2M + \Delta)^3} \log \frac{M + \Delta}{M} + \frac{M(4M^2 + 4\Delta M + 5\Delta^2)}{32(M + \Delta)^2(2M + \Delta)} - 2(\tilde{\delta}_5 - \tilde{\gamma}_5) \log \frac{M + \Delta}{M},
\]

\[
\tilde{\delta}_6 = \frac{2M + \Delta}{2M} \frac{\partial}{\partial \Delta} \frac{2(M + \Delta)}{2M + \Delta} \Delta (\tilde{\gamma}_1 - \tilde{\delta}_1) + \tilde{\delta}_1, \quad \tilde{\delta}_7 = \tilde{\gamma}_2 + \frac{1}{2} \left( \frac{\tilde{\gamma}_1 - \tilde{\delta}_1}{(2M + \Delta)^2} \right)^2,
\]

\[
\tilde{\beta}_1 = \frac{M + \Delta}{M} \frac{\Delta \partial}{\partial \Delta} \frac{\partial}{\partial \Delta} \frac{\tilde{\alpha}_1}{2(M + \Delta)^3},
\]

\[
\tilde{\beta}_2 = \frac{\Delta^2}{\partial \Delta} \frac{\partial}{\partial \Delta} \frac{\tilde{\alpha}_1 \tilde{\delta}_2}{\Delta}, \quad \tilde{\beta}_3 = \frac{\Delta^2}{\partial \Delta} \frac{\partial}{\partial \Delta} \frac{\tilde{\alpha}_1 \tilde{\delta}_3}{\Delta},
\]

\[
\tilde{\beta}_4 = \frac{\Delta}{M} \frac{(M + \Delta)^2}{M} \frac{\tilde{\alpha}_1}{M + \Delta} \frac{\partial}{\partial \Delta} \frac{\partial}{\partial \Delta} \frac{M^2 \tilde{\alpha}_1}{(M + \Delta)^2}, \quad \tilde{\beta}_5 = \frac{M + \Delta}{M} \frac{\Delta \partial}{\partial \Delta} \frac{\tilde{\alpha}_1 \tilde{\delta}_1}{M + \Delta},
\]

\[
\tilde{\beta}_6 = D_{\Delta \Delta} \frac{2M + \Delta M^2}{2(M + \Delta)^3} \tilde{\alpha}_1, \quad \tilde{\beta}_7 = \frac{\Delta}{M + \Delta} D_{\Delta \Delta} \frac{M \tilde{\alpha}_1 \tilde{\delta}_2}{\Delta},
\]

\[
\tilde{\beta}_8 = \frac{\Delta}{M + \Delta} D_{\Delta \Delta} \frac{M \tilde{\alpha}_1 \tilde{\delta}_3}{\Delta}, \quad \tilde{\beta}_9 = \frac{\Delta}{M + \Delta} D_{\Delta \Delta} \frac{M^2}{(M + \Delta)^2} \tilde{\alpha}_1,
\]

\[
\tilde{\beta}_9 = \tilde{\gamma}_1 M + \frac{\Delta}{M} D_{\Delta \Delta} \frac{M^2}{(M + \Delta)^2} \tilde{\alpha}_1,
\]

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\[ \tilde{b}_{10} = D_{\Delta \Delta} \frac{M}{M + \Delta} \tilde{\alpha}_1 \delta_1, \quad \tilde{b}_{11} = -\frac{1}{4} \tilde{\alpha}_1 \frac{M}{2M + \Delta} + \left( \tilde{\alpha}_1 - \tilde{\alpha}_2 \right) \frac{(2M + \Delta)M}{2\Delta^2}, \]

with
\[
D_{\Delta \Delta} = \frac{(M + \Delta)^2}{M^2} \left( \frac{\Delta^2 \partial^2}{\partial \Delta \partial \Delta} + \frac{2\Delta}{M + \Delta} \frac{\Delta \partial}{\partial \Delta} \right). \tag{59}
\]

While the \( \tilde{\alpha}_i \) characterize the chiral expansion of the coefficients in front of \( \bar{I}_{QR} \) and \( \bar{I}_Q \) in \([16]\), the \( \tilde{\gamma}_i \) and \( \tilde{d}_i \) follow from a chiral expansion of \( \bar{I}_{QR} \) with \( M_H = M + \Delta \) and \( M_R = M \) and \( m_Q < \Delta \). The coefficients \( \tilde{b}_i \) are required in \([44, 45, 63]\).

We turn to the chiral domain \([34]\), in which the bubble-loop contributions to the \( D^* \) meson masses generate a renormalization of the low-energy parameters \( \tilde{d}_i \). Such terms are proportional to the product of two quark masses \([9]\). We provide detailed results with
\[
\Pi_H^{(d-\chi)} \rightarrow \sum_{Q \in [8]} \sum_{R \in [0-]} \left( \frac{G_H^{Q,R}}{8 \pi f} \right)^2 \left\{ \tilde{\gamma}_d^{(1)} m_Q^2 \Pi^{(2)}_R + \tilde{\gamma}_d^{(2)} m_Q^2 \Pi^{(2)}_H + \tilde{\gamma}_d^{(3)} \Pi^{(2)}_R \Pi^{(2)}_R \right. \\
+ \tilde{\gamma}_d^{(4)} \Pi^{(2)}_H \Pi^{(2)}_H + \tilde{\gamma}_d^{(5)} \Pi^{(2)}_R \Pi^{(2)}_H \left. \right\},
\]

\[
\tilde{d}_i \rightarrow \frac{1}{4} g^2_p \sum_{k=1}^5 \frac{\tilde{\Gamma}^{(k)}_{d_i}}{(4\pi f)^2} \tilde{\gamma}_d^{(k)} \tag{60}
\]

with
\[
\tilde{\gamma}_d^{(1)} = \frac{\Delta}{6M} \left[ \frac{\partial}{\partial M} \left( \tilde{\alpha}_2 M \tilde{\gamma}_1 - \tilde{\alpha}_1 M \tilde{\gamma}_2 \right) - \frac{1}{M} \tilde{\gamma}_1 \frac{\partial}{\partial M} \left( \tilde{\alpha}_2 M^2 \right) \right] - \frac{M + \Delta}{M} \tilde{\gamma}_d^{(2)},
\]
\[
\tilde{\gamma}_d^{(2)} = \frac{M}{6(M + \Delta)} \left[ \frac{\partial}{\partial \Delta} \left( \tilde{\alpha}_2 \Delta \tilde{\gamma}_1 - \tilde{\alpha}_1 \Delta \tilde{\gamma}_2 \right) - \Delta \tilde{\gamma}_1 \frac{\partial}{\partial \Delta} \tilde{\alpha}_2 \right],
\]
\[
\tilde{\gamma}_d^{(3)} = -\frac{1}{24M^2} \left( \frac{\partial}{\partial M} - \frac{\partial}{\partial \Delta} \right)^2 \left( \tilde{\alpha}_1 M \Delta^3 \tilde{\gamma}_1 \right) + \frac{\tilde{\gamma}_1 \Delta}{24M^3} \left( \frac{\partial}{\partial M} - \frac{\partial}{\partial \Delta} \right)^2 \left( \tilde{\alpha}_1 M^2 \Delta^2 \right) \\
+ \frac{1}{24M^3} \left( \frac{\partial}{\partial \Delta} - \frac{\partial}{\partial \Delta} \right) \left( \tilde{\alpha}_1 M \Delta^3 \tilde{\gamma}_1 \right) - \frac{\tilde{\gamma}_1 \Delta}{24M^2} \left( \frac{\partial}{\partial M} - \frac{\partial}{\partial \Delta} \right) \left( \tilde{\alpha}_1 M^2 \Delta^2 \right),
\]
\[
\tilde{\gamma}_d^{(4)} = -\frac{M}{12(M + \Delta)^2} \left( \frac{\partial}{\partial \Delta} \right) \left( \frac{\partial}{\partial \Delta} \right) \left( \frac{\partial}{\partial \Delta} \right) - \frac{\tilde{\alpha}_1 \Delta^2 M}{12(M + \Delta)} \frac{\partial}{\partial \Delta} \left[ -\frac{1}{2(M + \Delta)} \left( \frac{\partial}{\partial \Delta} \right) \right],
\]
\[
\tilde{\gamma}_d^{(5)} = -\frac{1}{6M} \frac{\partial}{\partial \Delta} \left( \frac{\partial}{\partial \Delta} \right) + \frac{1}{6M^2} \frac{\partial}{\partial \Delta} \left( \tilde{\alpha}_1 M \Delta^3 \tilde{\gamma}_1 \right) \\
+ \frac{\tilde{\gamma}_1 \Delta}{6M^2} \frac{\partial}{\partial \Delta} \left( \tilde{\alpha}_1 M^2 \Delta^2 \right) - 2 \frac{M + \Delta}{M} \tilde{\gamma}_d^{(4)}. \tag{61}
\]

where the other \( \Gamma^{(k)}_{d_i} \) with \( k = 1, 2, ..., 5 \) follow from the corresponding \( \Gamma^{(k)}_{d_i} \) in \([56]\) upon the interchange \( c_i \leftrightarrow \bar{c}_i \).
We turn to the conventional counting ansatz (38), for which the third and fourth order contributions to the $D^*$ meson polarization tensor are already given with (39, 40). The fifth-order term is

$$
\bar{\Pi}^{\text{bubble-5}}_{H\in[1^-]} = \sum_{Q\in[8]} \sum_{R\in[1^-]} \left( \frac{G^{(H)}_{QR}}{8\pi f} \right)^2 \frac{1}{12} \left\{ -8 \left( \Pi^{(3)}_H - \Pi^{(3)}_R \right) m_Q^2 \\
+ \frac{4\Delta}{M} m_Q^2 (2m_Q^2 - 3\Pi^{(2)}_R + 3\Pi^{(2)}_H) - 6 \left( \Pi^{(3)}_H - \Pi^{(3)}_R \right) m_Q^2 \log \frac{m_Q^2}{M^2} \\
+ \pi \frac{m_Q}{M} \left[ 3m_Q^4 + 2m_Q^2 (\Pi^{(2)}_H - 3\Pi^{(2)}_R) + 3 (\Pi^{(2)}_R - \Pi^{(2)}_H)^2 \right] \right\} \\
+ \sum_{Q\in[8]} \sum_{R\in[0^-]} \left( \frac{G^{(H)}_{QR}}{8\pi f} \right)^2 \frac{1}{3} \left\{ - \frac{3}{4} \left( \Pi^{(3)}_H - \Pi^{(3)}_R \right) m_Q^2 \\
\frac{\Delta}{2M} \left( m_Q^4 + 2m_Q^2 (2\Pi^{(2)}_H - \Pi^{(2)}_R) + (\Pi^{(2)}_H - \Pi^{(2)}_R)^2 \right) \right\} \log \frac{m_Q^2}{M^2} \\
\left( \Pi^{(3)}_H - \Pi^{(3)}_R \right) m_Q^2 + \frac{\Delta}{8M} \left[ - 6m_Q^4 + m_Q^2 (3\Delta^2 - 4\Pi^{(2)}_R + 24\Pi^{(2)}_H) \\
+ 10 (\Pi^{(2)}_H - \Pi^{(2)}_R)^2 \right] + \frac{\Delta Q}{M} \left( \log (-\Delta - \Delta Q) \\
- \log (-\Delta + \Delta Q) \right) \left[ \frac{3}{2} \Delta M \left( \Pi^{(3)}_H - \Pi^{(3)}_R \right) + \frac{3}{4} \Delta^2 Q \right] \\
+ \frac{1}{2} \Delta^2 Q \left( 3\Pi^{(2)}_R - 5\Pi^{(2)}_H \right) + \frac{3}{4} (\Pi^{(2)}_H - \Pi^{(2)}_R)^2 \\
+ \frac{3}{8} m_Q^2 \left( \Delta^2 + 2(\Pi^{(2)}_R - 3\Pi^{(2)}_H) + \frac{(\Pi^{(2)}_H - \Pi^{(2)}_R)^2}{\Delta^2} \right) \right] \\
+ \left[ \frac{3}{2} \Delta^2 \left( \Pi^{(3)}_H - \Pi^{(3)}_R \right) + \frac{\Delta}{4M} 3\Delta Q - 2m_Q^4 - 2(\Delta^2 + m_Q^2) (\Pi^{(2)}_H + \Pi^{(2)}_R) \\
+ (8\Delta^2 + \Pi^{(2)}_R - \Pi^{(2)}_H) (\Pi^{(2)}_R - \Pi^{(2)}_H) \right] \log \frac{m_Q^2}{4\Delta^2} \right\}. \\
(62)
$$

It remains to specify the fifth order term with respect to the novel counting ansatz (41). We find

$$
\bar{\Pi}^{\text{bubble-5}}_{H\in[1^-]} = \sum_{Q\in[8]} \sum_{R\in[1^-]} \left( \frac{m_{QR} G^{(H)}_{QR}}{4\pi f} \right)^2 \frac{1}{3} \left\{ \frac{3\pi}{16} \frac{m_Q^3}{M_H} - \frac{m_Q^4}{M^2} \left( \frac{1}{6} - \frac{1}{8} \log \frac{m_Q}{M_R} \right) \\
(M_R - M_H)^2 \left( \frac{\pi M_H}{4 m_Q} + 1 + \frac{3}{2} \log \frac{m_Q}{M_R} \right) \right\}
$$

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\[ + \sum_{Q \in \{8\}} \left( \frac{m_Q}{4\pi f^2} C_{Q_R}^{(H)} \right)^2 \left\{ \tilde{\alpha}_1 \frac{M^2}{12 (2M + \Delta)} \frac{\Delta^3}{4(M + \Delta)^2} \log \frac{4\Delta^2}{M^2} \right. \\
- \frac{\tilde{\alpha}_1}{24} \frac{\Delta^2}{m_Q^2} (M_R - M_H + \Delta_H) \frac{2M + \Delta}{M} \frac{\partial}{\partial \Delta} (\tilde{\gamma}_1 \Delta) \\
+ \frac{M_H}{12} \frac{M}{M + \Delta} \left[ - \left( \tilde{\alpha}_1 - \tilde{\alpha}_2 \right) \left( M (2M + \Delta) \frac{m_Q^2}{2(2M + \Delta)^2} \frac{\Delta^2}{\Delta_H^2} (M_H - M_R) \log \frac{m_Q^2}{M_R^2} \right) \right. \\
+ (\tilde{\delta}_1 - \tilde{\gamma}_1) \frac{\Delta^2}{\Delta_H^2} - \tilde{\delta}_1 \frac{\Delta^2}{\Delta_H^2} (M_R - M_H + \Delta_H) \left. \right] \\
+ \frac{\tilde{\beta}_1}{M_H^2} \left( (M_H - M_R)^3 \log \frac{m_Q^2}{M_R^2} \right) \\
+ \Delta_H^3 \left[ \log (M_R - M_H - \Delta_Q) - \log (M_R - M_H + \Delta_Q) \right) \\
+ \frac{m_Q^2 \Delta^2}{\Delta_H^3} \left( (\tilde{\alpha}_2 - \tilde{\alpha}_1) \left( \tilde{\delta}_2 + \tilde{\delta}_3 \log \frac{m_Q^2}{M_R^2} \right) - \tilde{\alpha}_1 \left( \tilde{\delta}_4 + \tilde{\delta}_5 \log \frac{m_Q^2}{M_R^2} \right) \right) \left] \right. \\
+ \frac{M_H}{24} (M_R - M_H + \Delta_H)^2 \left[ \tilde{\beta}_9 \frac{\Delta^2}{m_Q^2 \Delta_H^2} - \tilde{\beta}_10 \frac{\Delta^2}{m_Q^2 \Delta_H^2} (M_H - M_R) \right. \\
+ \frac{\tilde{\beta}_6}{m_Q^2 \Delta_H^2} \left( (M_H - M_R) \left( \Delta^2 - \frac{m_Q^2}{2} \right) \log \frac{m_Q^2}{M_R^2} \right) \\
+ \Delta_H^3 \left[ \log (M_R - M_H - \Delta_Q) - \log (M_R - M_H + \Delta_Q) \right) \\
+ \left( - \tilde{\beta}_7 \frac{\Delta^2}{\Delta_H^3} + \tilde{\beta}_8 \frac{\Delta^2}{\Delta_H^3} \log \frac{m_Q^2}{M_R^2} \right) \left] \right. \\
\right\} \],

with \( m_{QR}^2 = m_Q^2 - (M_R - M_H)^2 \). \hspace{1cm} (63)
| $a m_\pi$ | $a m_K$ | $a \mu_c$ | $a \text{discr.}$ | $a m_D$ | $a m_{D_s}$ | $a m_{D_s}$ | $a m_{D_s^*}$ |
|---------|---------|-----------|------------------|--------|-------------|-------------|-------------|
| 0.0703(4) | 0.1697(3) | 0.2230 ($\pm$, $\mp$) | 0.6655(12) | 0.6981(4) | 0.7161(18) | 0.7456(10) |
| 0.0703(4) | 0.1697(3) | 0.1919 ($\pm$, $\mp$) | 0.6072(11) | 0.6402(3) | 0.6621(18) | 0.6923(10) |
| 0.0806(3) | 0.1738(5) | 0.2227 ($\pm$, $\mp$) | 0.6661(19) | 0.6983(4) | 0.7209(26) | 0.7452(12) |
| 0.0975(3) | 0.1768(3) | 0.2230 ($\pm$, $\mp$) | 0.6666(16) | 0.6980(5) | 0.7183(23) | 0.7458(13) |
| 0.1074(5) | 0.2133(4) | 0.2230 ($\pm$, $\mp$) | 0.8473(10) | 0.8780(5) | 0.9140(31) | 0.9474(10) |
| 0.1549(2) | 0.2279(2) | 0.2230 ($\pm$, $\mp$) | 0.8543(5) | 0.8824(3) | 0.9268(11) | 0.9536(7) |
| 0.1935(4) | 0.2430(4) | 0.2230 ($\pm$, $\mp$) | 0.8559(8) | 0.8784(5) | 0.9309(18) | 0.9521(13) |

TABLE XXI. Masses for the $D$ mesons in units of the lattice scale $a$. The values in the table are provided to us by the authors of [20].
| $a m_{\pi}$ | $a m_K$ | $a \mu_c$ | discr. | $a m_D$ | $a m_{D_s}$ | $a m_{D^*}$ | $a m_{D_s^*}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1240(4) | 0.2512(3) | 0.2772 (±, ±) | 0.8979(9) | 0.9412(2) | 0.9782(16) | 1.0225(7) |
|       |       | 0.2270 (±, ±) | 0.7994(8) | 0.8441(2) | 0.8880(16) | 0.9338(7) |
|       |       | 0.2772 (±, ±) | 0.9154(14) | 0.9610(3) | 0.9759(15) | 1.0185(7) |
|       |       | 0.2270 (±, ±) | 0.8181(12) | 0.8655(3) | 0.8859(15) | 0.9289(8) |
| 0.1412(3) | 0.2569(3) | 0.2768 (±, ±) | 0.9002(10) | 0.9420(3) | 0.9776(20) | 1.0213(9) |
|       |       | 0.2389 (±, ±) | 0.8258(9) | 0.8692(3) | 0.9104(20) | 0.9545(9) |
|       |       | 0.2768 (±, ±) | 0.9162(13) | 0.9623(4) | 0.9743(19) | 1.0169(9) |
|       |       | 0.2389 (±, ±) | 0.8433(12) | 0.8904(4) | 0.9067(18) | 0.9501(9) |
| 0.1440(6) | 0.2589(4) | 0.2768 (±, ±) | 0.9006(8) | 0.9425(3) | 0.9801(23) | 1.0252(8) |
|       |       | 0.2389 (±, ±) | 0.8268(12) | 0.8697(2) | 0.9153(19) | 0.9589(8) |
|       |       | 0.2768 (±, ±) | 0.9160(13) | 0.9627(3) | 0.9813(16) | 1.0208(7) |
|       |       | 0.2389 (±, ±) | 0.8432(11) | 0.8911(3) | 0.9145(15) | 0.9544(7) |
| 0.1988(3) | 0.2764(3) | 0.2929 (±, ±) | 0.9327(8) | 0.9668(5) | 1.0148(17) | 1.0496(12) |
|       |       | 0.2299 (±, ±) | 0.8164(13) | 0.8520(4) | 0.9092(16) | 0.9449(11) |
|       |       | 0.2929 (±, ±) | 0.9500(12) | 0.9879(5) | 1.0098(44) | 1.0434(20) |
|       |       | 0.2299 (±, ±) | 0.8358(10) | 0.8746(5) | 0.9026(41) | 0.9381(18) |

TABLE XXII. Masses for the $D$ mesons in units of the lattice scale $a$. The values in the table are provided to us by the authors of [20].
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