CONDITIONS FOR RATIONAL WEAK MIXING

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Abstract. We exhibit rationally ergodic, weakly mixing measure preserving transformations which are not subsequence rationally weakly mixing and give a condition for smoothness of renewal sequences.

Rational weak mixing of a measure preserving transformation was defined and considered in [A1] to which we refer for definitions.

We show by example in §1 that weak rational ergodicity and weak mixing does not imply subsequence rational weak mixing. The main examples are dyadic towers with super-growth sequences (defined below). We also give zero type examples. See [DGPS] for related examples.

A Markov shift is conservative, ergodic iff the associated stochastic matrix is irreducible and recurrent, and in this case is rationally ergodic. It is weakly mixing iff its associated renewal sequences are aperiodic, and (subsequence) rationally weakly mixing iff the associated renewal sequences are (subsequence) smooth (see [A1]).

It is not known whether every aperiodic, recurrent renewal sequence is subsequence smooth, or whether smoothness implies Orey’s strong ratio limit property. See §8 in [A1].

Smoothness of the renewal sequence $u$ would follow e.g. from the property $\sum_{n \geq 1} |u_n - u_{n+1}| < \infty$. This is known for positively recurrent, aperiodic, renewal sequences and is conjectured in for all aperiodic renewal sequences (see [K] §1.6(iv)).

In §2, we give a sufficient condition establishing smoothness e.g. when $\sum_{n \geq 1} u_n$ is 1-regularly varying. This condition entails the property $\sum_{n \geq 1} (u_n - u_{n+1})^2 < \infty$.

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1.1 Dyadic towers over the dyadic adding machine.

Let $\Omega := \{0, 1\}^N$, and let $P \in \mathcal{P}(\Omega)$ be symmetric product measure: $P = \prod \left( \frac{1}{2}, \frac{1}{2} \right)$, and let $\tau : \Omega \to \Omega$ be the dyadic odometer defined by
\[ \tau(1, \ldots, 1, 0, \omega_{\ell+1}, \ldots) = (0, \ldots, 0, 1, \omega_{\ell+1}, \ldots) \]
where $\ell = \ell(\omega) := \min\{n \geq 1 : \omega_n = 0\}$. An increasing sequence $q \in \mathbb{N}^N(\uparrow)$ is a growth sequence as in [A2] if $q_n > \sum_{1 \leq k < n} q_k$.

The dyadic cocycle $\varphi : \Omega \to \mathbb{N}$ associated to the growth sequence $q \in \mathbb{N}^N(\uparrow)$ is defined by
\[
(\varphi) \quad \varphi(\omega) := q_\ell(\omega) - \sum_{k=1}^{\ell(\omega)-1} q_k = \sum_{n \geq 1} q_n (\tau(\omega)_k - \omega_k).
\]

The dyadic tower with the growth sequence $q$ is the tower over $(\Omega, \mathcal{B}(\Omega), P, \tau)$ with height function $\varphi$, namely $(X, \mathcal{B}(X), m, T)$ with
\[
X := \{(x, n) \in \Omega \times \mathbb{N} : 1 \leq \varphi(x)\}, \quad m(A \times \{n\}) := P(A \cap [\varphi \geq n]),
\]
\[
T(x, n) = \begin{cases} (x, n+1) & \varphi(x) \geq n+1; \\ (\tau x, 1) & \varphi(x) = n. \end{cases}
\]

Rational ergodicity of $T$. Recall from [A2] that $(X, \mathcal{B}, m, T)$ is (boundedly) rationally ergodic with return sequence $a_n(T) \asymp c(n)$ where $c(n) = \min\{k \geq 1 : q_k \geq n\}$.

Weak mixing of $T$.

By [AN], $T$ is weakly mixing iff $G_2(q) = \{0\}$ where
\[
G_2(q) := \{t \in T : \sum_{n \geq 1} \|q_n t\|^2 < \infty\}
\]
where $\|x\| := \min_{n \in \mathbb{Z}} |x - n|$.

By lemma 3 and theorem 2 in [P] (see also [AHL]), if $q_{n+1} = a_n q_n + 1$ where $a_n \in \mathbb{N}$, $\sum_n \frac{1}{a_n} = \infty$, then $G_2(q) = \{0\}$ and $T$ is weakly mixing. See also [AFS].

Negation of subsequence rational weak mixing.

Let $q \in \mathbb{N}^N(\uparrow)$ be a growth sequence. For $\epsilon \in \mathcal{E} := \{\eta \in \{-1, 0, 1\}^N : \eta_n \to 0\}$, let
\[
N_\epsilon := \sum_{k \geq 1} \epsilon_k q_k.
\]
It is easy to see that $\epsilon \mapsto N_\epsilon$ ($\mathcal{E} \to \mathbb{Z}$) is injective if $q$ is a super growth sequence in the sense that $q_n > 2 \sum_{1 \leq k < n} q_k$.  

§1 Examples

For $\epsilon \in \mathcal{E}$, we have that $N_\epsilon > 0$ iff $\epsilon_{\kappa_{\text{max}}} = 1$ where $\kappa_{\text{max}}(\epsilon) := \max \{k \geq 1 : \epsilon_k \neq 0\}$.

Write $\mathcal{E}^+ := \{\epsilon \in \mathcal{E} : \epsilon_{\kappa_{\text{max}}} = 1\}$ and $\|\epsilon\| := \sum_{n \geq 1} |\epsilon_n|$.

We claim that

A dyadic tower with a super growth sequence cannot be subsequence, rationally weakly mixing.

Using the above, it is easy to construct super growth sequences $q \in \mathbb{N}^\downarrow$ with $G_2(q) = \{0\}$ and hence with weakly mixing dyadic towers.

The claim (憲) will follow from

$$(\star) \quad m(\Omega \cap T^{-n}\Omega) = \begin{cases} \frac{1}{2^n} & n = N_\epsilon, \ \epsilon \in \mathcal{E}; \\ 0 & \text{else.} \end{cases}$$

**Proof of (憲) (see [HK])**

Let $N \geq 1$ and $x \in \Omega$, then $T^N x \in \Omega$ iff $\exists \ N \geq 1$ so that $\varphi_N(x) = N$.

By (憲),

$$N = \varphi_N(x) = \sum_{k \geq 1} q_k (T^N(\omega)_k - \omega_k) = \sum_{k \geq 1} \epsilon_k q_k =: N_\epsilon$$

for some $\epsilon \in \mathcal{E}^+$.

We now show that $m(\Omega \cap T^{-N_\epsilon}\Omega) = \frac{1}{2^{|\epsilon|}}$.

If $n_1 < n_2 < \cdots < n_k$, $m_1 < m_2 < \cdots m_{\ell}$ and

$$\epsilon_{n_i} = 1, \ \epsilon_{m_j} = -1 \ \& \ \epsilon_n = 0 \ \text{else},$$

then

$$\Omega \cap T^{-N_\epsilon}\Omega = \{\omega \in \Omega : \omega_{n_i} = 1 \ \forall \ i \ \& \ \epsilon_{m_j} = 0 \ \forall \ j\}$$

and $T^{N_\epsilon}\omega = G\omega$ where $G : \Omega \to \Omega$ is defined by

$$G(x)_k = \begin{cases} 1 - x_k & k \in \{n_i\}_i \cup \{m_j\}_j; \\ x_k & \text{else.} \end{cases}$$

Thus

$$m(\Omega \cap T^{-N_\epsilon}\Omega) = m(\{\omega \in \Omega : \omega_{n_i} = 1 \ \forall \ i \ \& \ \epsilon_{m_j} = 0 \ \forall \ j\}) = \frac{1}{2^{|\epsilon|}}. \ \checkmark(\star)$$

**Proof of (憲)**

It suffices to show that

$$\lim_{n \to \infty} \frac{1}{2^{c(n)}} \sum_{k=1}^n |u_k - u_{k+q_1}| > 0$$

where $u_n = u_n(\Omega) := m(\Omega \cap T^{-n}\Omega)$.

To see this, we restrict summation to $k = N_\epsilon$ where $\epsilon \in \mathcal{E}^+$ & $\epsilon_1 = 0$; noting $N_\epsilon \leq n$ iff $\kappa_{\text{max}} \leq c(n)$.
Here
\[ u_{N_\epsilon} = \frac{1}{2|\epsilon|} \]
\[ u_{N_\epsilon + q_1} = \frac{1}{2|\epsilon| + 1} \]
and
\[ u_{N_\epsilon} - u_{N_\epsilon + q_1} = \frac{1}{2} \cdot u_{N_\epsilon}. \]

Thus
\[ \sum_{k=1}^{n} |u_k - u_{k+q_1}| \geq \sum_{\epsilon \in E^*, \epsilon_1 = 0, \kappa_{\text{max}} \leq c(n)} |u_{N_\epsilon} - u_{N_\epsilon + q_1}| \]
\[ = \frac{1}{2} \sum_{\epsilon \in E^*, \epsilon_1 = 0, \kappa_{\text{max}} \leq c(n)} u_{N_\epsilon} \]
\[ = \frac{1}{2} \sum_{\epsilon \in E^*, \epsilon_1 = 0, \kappa_{\text{max}} \leq c(n)} \frac{1}{2|\epsilon|}. \]

Now
\[ \sum_{\epsilon \in E^*, \epsilon_1 = 0, \kappa_{\text{max}} \leq c(n)} \frac{1}{2|\epsilon|} = \sum_{\emptyset \neq F \subset N \cap [2, c(n)]} \sum_{\epsilon \in E^*, \text{supp} \epsilon = F} \frac{1}{2|F|}. \]

For fixed $F$,
\[ \# \{ \epsilon \in \mathcal{E}, \text{supp} \epsilon = F \} = 2^{|F|}. \]

Thus
\[ \sum_{k=1}^{n} |u_k - u_{k+q_1}| \geq \frac{1}{2} \sum_{\epsilon \in E^*, \epsilon_1 = 0, \kappa_{\text{max}} \leq c(n)} \frac{1}{2|\epsilon|} \]
\[ = \frac{1}{2} \# \{ \emptyset \neq F \subset N \cap [2, c(n)] \} \]
\[ \geq \frac{1}{8} \cdot 2^{c(n)}. \]  

### 1.2 A zero type example.

Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing, dyadic tower with a super-growth sequence (as above) and let $(Y, \mathcal{C}, \nu, S)$ be a conservative, aperiodic ergodic Markov shift with $A \in M(S), \; u(A)$ Kaluza and with $S \times T$ conservative.

The transformation $(Y \times X, \mathcal{C} \otimes \mathcal{B}, \nu \times \mu, S \times T)$ is zero-type, rationally ergodic and weakly mixing, but not subsequence, rationally weakly mixing.

**Proof**

Evidently $S \times T$ is zero type.

To see that $S \times T$ is weakly mixing, let $R$ be an ergodic, probability preserving transformation. It follows (from the weak mixing of $T$) that
$R \times T$ is conservative, ergodic. Since $S$ is the natural extension of a mildly mixing transformation, we have that
\[
(S \times T) \times R = S \times (T \times R)
\]
is conservative, ergodic. This shows that $S \times T$ is weakly mixing.

Next, we claim that $S \times T$ is rationally ergodic with $A \times \Omega \in R(S \times T)$.

**Proof** It suffices to consider the one sided Markov shift $(Y, C, \nu, \sigma)$ and show that $\sigma \times T$ is rationally ergodic with $A \times \Omega \in R(\sigma \times T)$.

Write $u_k := \nu(A \cap \sigma^{-n}A)$, then $v_k := u_k - u_{k+1} > 0$ and on $A \times \Omega$, the transfer operator is given by
\[
\overline{\sigma \times T^n}(1_A \times \Omega)(y, \omega) = \overline{\sigma^n}(1_A)(y)1_\Omega(T^{-n}\omega) = u_n 1_\Omega(T^{-n}\omega).
\]

Thus, for $(y, \omega) \in A \times \Omega$,
\[
\sum_{1 \leq k \leq n} \overline{\sigma \times T^k}(1_A \times \Omega)(y, \omega) = \sum_{1 \leq k \leq n} u_k 1_\Omega(T^{-k}\omega) \\
= \sum_{1 \leq k \leq n} u_k(S_k^{(T^{-1})}(1_\Omega)(\omega) - S_{k-1}^{(T^{-1})}(1_\Omega)(\omega)) \\
= \sum_{1 \leq k \leq n} v_k S_k^{(T^{-1})}(1_\Omega)(\omega) + u_n S_n^{(T^{-1})}(1_\Omega)(\omega) \\
\leq M \sum_{1 \leq k \leq n} v_k a_k^{(T)}(\Omega) + Mu_n a_n^{(T)}(\Omega) \\
= M \sum_{1 \leq k \leq n} u_k \mu(\Omega \cap T^{-k}\Omega) + 2Mu_n a_n^{(T)}(\Omega) \\
\leq 3Ma_n^{(S \times T)}(A \times \Omega).
\]

Rational ergodicity of $S \times T$ and $A \times \Omega \in R(S \times T)$ follow from this.

To see that $S \times T$ is not rationally weakly mixing, consider $E, F \in R(S \times T)$ given by
\[
E := A \times \Omega, \quad F := A \times T^{-q_1}\Omega,
\]
then for $n = N_\epsilon$ with $\epsilon_1 = 0$, we have
\[
\nu \times \mu(E \cap (S \times T)^{-n}E) - \nu \times \mu(F \cap (S \times T)^{-n}F) \\
= u_n(\mu(\Omega \cap T^{-n}\Omega) - \mu(\Omega \cap T^{-(n+q_1)}\Omega)) \\
= \frac{1}{2}u_n \mu(\Omega \cap T^{-n}\Omega).
\]
Noting that
\[ a_n(S \times T) \sim \sum_{\varepsilon \in \mathcal{E}^+, N_\varepsilon \leq n} u_{N_\varepsilon} \frac{1}{2|\varepsilon|}, \]
we see that
\[ \sum_{1 \leq k \leq n} |\nu \times \mu(E \cap (S \times T)^{-k}E) - \nu \times \mu(F \cap (S \times T)^{-k}F)| \]
\[ = \sum_{1 \leq k \leq n} u_n|\mu(\Omega \cap T^{-n}\Omega) - \mu(\Omega \cap T^{-(n+q_1)}\Omega)| \]
\[ \geq \sum_{\varepsilon \in \mathcal{E}^+, \varepsilon_1 = 0, N_\varepsilon \leq n} u_{N_\varepsilon} \frac{1}{2|\varepsilon|} \]
\[ \sim a_n(S \times T). \]

§2 Smoothness of renewal sequences

Suppose that \( u = u(f) = (u_0, u_1, \ldots) \) is an aperiodic, recurrent, renewal sequence with lifetime distribution \( f \in \mathcal{P}(\mathbb{N}) \).

The renewal sequence \( u \) is called smooth if
\[ \sum_{k=1}^n |u_k - u_{k+1}| = o(a_n) \text{ as } n \to \infty \text{ where } a_n = a_n(u) := \sum_{k=1}^n u_k. \]

It follows from [GL] that if \( u = (u_0, u_1, \ldots) \) is an aperiodic, recurrent, renewal sequence and \( a(u) \) is \( \alpha \)-regularly varying with \( \alpha \in (0, 1) \), then \( u \) is smooth (see [A1]). The case \( \alpha = 1 \) follows from the next proposition (which is related to proposition 8.3 in [A1]).

For \( u = u(f) \), let \( c_N := f([N, \infty)) \) and \( L(N) := \sum_{k=1}^N c_k \).

**Proposition** If \( u = (u_0, u_1, \ldots) \) is an aperiodic, recurrent, renewal sequence with
\[ (\star) \quad \lim_{N \to \infty} \frac{Nc_N}{L(N)} < \frac{1}{\sqrt{5} + 1}, \]
then \( u \) is smooth.

**Remark** If \( a(u) \) is \( t \)-regularly varying for some \( t > \frac{1}{\sqrt{5} + 1} \), then by the renewal equation and Karamata theory, \( L(N) \propto \frac{N}{a(u)(N)} \) is \( (1-t) \)-regularly varying, \( Nc_N \sim (1-t)L(N) \) and (\( \star \)) holds.

**Proof** We show first that
\[ (\diamondsuit) \quad \sum_{k=1}^\infty (u_k - u_{k+1})^2 < \infty. \]
Let $u = u(f)$ and let $c_N := f([N, \infty))$, $M(N) := \sum_{1 \leq n \leq N} n f_n$ and $V(N) := \sum_{1 \leq n \leq N} n^2 f_n$.

By (⋆), $\exists R < \frac{1}{\sqrt{5}+1}$ so that $NC_N \leq RL(N)$ for large $N$. It follows that

$$NC_N \leq RL(N) = R(M(N) + NC_N)$$

whence

(⊗§)

$$NC_N \leq \frac{R}{1 - R} \cdot M(N)$$

for large $N$ where $\frac{R}{1 - R} < \frac{1}{\sqrt{5}}$.

In particular

$$M(N) \asymp L(N) \quad \text{as } N \rightarrow \infty;$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2 M(n)^2} < \infty.$$  

We’ll use these to prove (⊗).

By Parseval’s formula, and the renewal equation,

$$\int_{-\pi}^{\pi} \frac{|\theta|^2 d\theta}{|1 - f(\theta)|^2} < \infty \iff \sum_{n=1}^{\infty} (u_n - u_{n+1})^2 < \infty$$

where $f(\theta) := \sum_{n=1}^{\infty} f_n e^{i n \theta}$. By aperiodicity, $\sup_{\epsilon \leq |\theta| \leq \pi} |f(\theta)| < 1 \ \forall \ \epsilon > 0$ whence (using symmetry)

$$\int_{-\pi}^{\pi} \frac{|\theta|^2 d\theta}{|1 - f(\theta)|^2} < \infty \iff \int_{0}^{\epsilon} \frac{\theta^2 d\theta}{|1 - f(\theta)|^2} < \infty \text{ for some } \epsilon > 0.$$  

For $|\theta| < \frac{\pi}{2}$,

$$1 - f(\theta) = \sum_{n=1}^{\infty} f_n (2 \sin^2 \left( \frac{n\theta}{2} \right) - i \sin(n\theta))$$

$$= \left( \sum_{1 \leq n \leq \frac{\pi}{2\pi}} + \sum_{n > \frac{\pi}{2\pi}} \right) f_n (2 \sin^2 \left( \frac{n\theta}{2} \right) - i \sin(n\theta))$$

and

$$|1 - f(\theta)| \geq \left| \sum_{1 \leq n \leq \frac{\pi}{2\pi}} - \sum_{n > \frac{\pi}{2\pi}} \right|.$$
Now

\[
| \sum_{1 \leq n \leq \frac{2\pi}{\theta}} | = \sqrt{\left( \sum_{1 \leq n \leq \frac{2\pi}{\theta}} f_n \sin^2 \left( \frac{n\theta}{2} \right) \right)^2 + \left( \sum_{1 \leq n \leq \frac{2\pi}{\theta}} f_n \sin(n\theta) \right)^2 }
\]

\[
\geq \left| \sum_{1 \leq n < \frac{2\pi}{\theta}} f_n \sin(n\theta) \right|
\]

\[
\geq \frac{2}{\pi} \cdot |\theta| \sum_{1 \leq n < \frac{2\pi}{\theta}} n f_n
\]

\[
= \frac{2}{\pi} \cdot |\theta| M \left( \frac{2}{\pi |\theta|} \right);
\]

and

\[
| \sum_{n > \frac{2\pi}{\theta}} | = | \sum_{n > \frac{2\pi}{\theta}} f_n \left( 2\sin^2 \left( \frac{n\theta}{2} \right) - i \sin(n\theta) \right) |
\]

\[
\leq \sqrt{\left( \sum_{n > \frac{2\pi}{\theta}} f_n 2\sin^2 \left( \frac{n\theta}{2} \right) \right)^2 + \left( \sum_{n > \frac{2\pi}{\theta}} f_n \sin(n\theta) \right)^2 }
\]

\[
\leq \sqrt{5} c \frac{2}{\pi |\theta|}.
\]

By assumption \( \exists \Delta > 0 \) so that for \( |\theta| < \Delta \),

\[
c \frac{2}{\pi |\theta|} \leq R' \frac{2|\theta|}{\pi} M \left( \frac{2}{\pi |\theta|} \right)
\]

where \( R' = \frac{R}{1-R} = \frac{1}{\sqrt{5}} (1 - \eta) \) for some \( \eta > 0 \); whence

\[
|1 - f(\theta)| \geq \frac{2}{\pi} \cdot |\theta| M \left( \frac{2}{\pi |\theta|} \right) - \sqrt{5} c \frac{2}{\pi |\theta|}
\]

\[
\geq \frac{2}{\pi |\theta|} M \left( \frac{2}{\pi |\theta|} \right) (1 - \sqrt{5} R')
\]

\[
= \frac{2\eta}{\pi |\theta|} M \left( \frac{2}{\pi |\theta|} \right).
\]
Let $N \geq 1$, $\Delta > \frac{\pi}{N}$, then
\[
\int_0^{\frac{\pi}{N}} \frac{\theta^2 d\theta}{|1 - f(\theta)|^2} \ll \int_0^{\frac{\pi}{N}} \frac{\theta^2 d\theta}{(\theta M(\frac{2}{\pi\theta}))^2}
\]
\[
= \int_0^{\frac{\pi}{N}} \frac{d\theta}{M(\frac{2}{\pi\theta})^2}
\]
\[
= \sum_{n=1}^{\infty} \int_{\frac{n}{\pi}}^{\frac{n+1}{\pi}} \frac{d\theta}{M(\frac{2}{\pi\theta})^2}
\]
\[
\approx \sum_{n=1}^{\infty} \frac{1}{n^2 M(n)^2} < \infty. \quad \forall \theta(\bullet)
\]

To see smoothness, by assumption
\[
\log L(N) \sim \sum_{k=1}^{N} \frac{c_k}{L(k)} \leq \sum_{k=1}^{N} \frac{1}{\sqrt{3k}} + O(1) = \frac{1}{\sqrt{3}} \log N + O(1)
\]
whence
\[
L(N) = O(N^{1/\sqrt{3}}).
\]
Using the renewal equation, $a_u(n) \asymp \frac{n}{L(n)}$, whence $a_u(n) \gg N^{1-\frac{1}{\sqrt{3}}}$ and
\[
\frac{\sqrt{n}}{a_u} \ll \frac{1}{N^{1/2} \sqrt{\frac{1}{3}}} \quad n \to \infty
\]
By CSI
\[
\frac{1}{a_u(n)} \sum_{k=1}^{n} |u_k - u_{k+1}| \leq \frac{\sqrt{n}}{a_u(n)} \cdot \sqrt{\sum_{k=1}^{\infty} (u_k - u_{k+1})^2} \quad n \to \infty
\]

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