A Comment on Emergent Gravity

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Abstract

This paper is a set of notes that we wrote concerning the first version of Emergent Gravity [gr-qc/0602022]. It is our version of an exercise that we proposed to some of our students. The idea was to find mathematical errors and inconsistencies on some recent articles published in scientific journals and in the arXiv, and we did.
1 Introduction

This paper is a set of notes that we wrote as a guide to a query that was proposed to some of our students: find mathematical errors and inconsistencies on some recent Physics papers published in scientific journals and/or posted in the arXiv using jargon of higher Mathematics. The paper here analyzed is the first version of Emergent Gravity [15]. Other papers will be analyzed elsewhere (see, e.g., [13], where we criticise [1]).

The reader is here informed that we sent the notes to the author of [15], which used them to prepare new versions of his paper. Originally it was not our intention to post the notes but we changed our mind due to the following reasons:

(i) our believe that notes may be eventually useful to many students and researchers;

(ii) because in the sixth ‘improved version’ it is written in the comments of the article: ‘6th draft due to math corrections by Prof Waldyr Rodrigues Jr UNICAMP and new empirical information from UCLA Dark Matter 2006 Conference.

Well, unfortunately despite the fact that in the ‘improved versions’ some of the wrong mathematical statements of the first version have been deleted, there are in our opinion new ones which need to be corrected. We have no responsibility for any one of the versions of that paper, we have not endorsed the paper for the arXiv.

2 Some Necessary Preliminaries

2.1 Tetrads and Cotetrads

1. The Collins Dictionary of Mathematics [2] defines the word tetrad as a set or sequence of four elements. And indeed, the prexif tetra comes from the Greek word for the number four. The meaning of tetrads (and cotetrads) in differential geometry will be explained next [12] and that meaning must be kept in mind, specially in the discussion in Section 7.

2. Let $M$ be a 4-dimensional manifold equipped with a Lorentzian metric tensor field $g \in \text{sec} \, T_0^0 M$. In the differential geometry of $M$ the word tetrad is used to denominate a set of four orthonormal vector fields $\{e_a\}$ defined in $U \subset M$. We code this information writing $e_a \in \text{sec} \, TU \subset \text{sec} \, TM$, $a = 0, 1, 2, 3$ and $g(e_a, e_b) = \eta_{ab} := \text{diag}(1, -1, -1, -1) \quad (1)$

3. Another way to code the above information is by writing that the set $\{e_a\} \in \text{sec} \, P_{SO_{1,3}} U \subset \text{sec} \, P_{SO_{1,3}} M$, i.e., set $\{e_a\}$ is a section of the orthonormal frame bundle, which is a principal bundle with structural group $\text{SO}_{1,3}$. For details, please consult, e.g., [11]. Then the set $\{e_a\}$ is also called an orthonormal

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1 On February 27, 2006 we already have six versions.

2 This will be discussed elsewhere.
(moving) frame. It is eventually important to recall a classical result (see, e.g., [3] that for a 4-dimensional Lorentzian manifold to admit spinor fields $\mathbf{P}_{SO_{1,3}} M$ must be trivial, i.e., must have global sections.

4. Sometimes it is useful to consider a set of vector fields $\{e^a\} \subset \text{sec} \mathbf{P}_{SO_{1,3}} U \subset \text{sec} \mathbf{P}_{SO_{1,3}} M$ such that

$$g(e^a, e_b) = \delta^a_b,$$ \hspace{1cm} (2)

which is called the reciprocal frame of the frame $\{e_a\}$.

5. We define the dual frame of the frame $\{e_a\}$ as the set of four covector fields (also called 1-form fields) $\{\varepsilon^a\}$, $\varepsilon^a \in \sec T^* U \subset \sec T^* M$, $a = 0, 1, 2, 3$. We also write that $\{\varepsilon^a\} \in \sec \mathcal{P}_{SO_{1,3}} U \subset \sec \mathcal{P}_{SO_{1,3}} M$, i.e., it is a section of the orthonormal coframe bundle. The set $\{\varepsilon^a\} \subset \sec \mathbf{P}_{SO_{1,3}} M$ is called an orthonormal coframe.

6. Recall that 1-forms are mappings $\sec TM \to \mathbb{R}$ and by definition the 1-forms $\varepsilon^a$, $a = 0, 1, 2, 3$ satisfy

$$\varepsilon^a(e_b) = \delta^a_b$$ \hspace{1cm} (3)

7. If $g \in \sec T^0_2 M$ is the metric of the cotangent bundle we have

$$g(\varepsilon^a, \varepsilon^b) = \eta^{ab} := \text{diag}(1, -1, -1, -1).$$ \hspace{1cm} (4)

8. The coframe $\{\varepsilon_a\} \subset \text{sec} \mathbf{P}_{SO_{1,3}} U \subset \text{sec} \mathbf{P}_{SO_{1,3}} M$ such that

$$g(\varepsilon_a, \varepsilon^b) = \delta^b_a$$ \hspace{1cm} (5)

is called the reciprocal coframe of the coframe $\{\varepsilon^a\}$.

9. Next introduce a coordinate chart\(^3\) $(\chi, U)$ of the maximal atlas of $M$ $(U \subset M)$ with coordinate functions $\{x^\mu\}$, $\mu = 0, 1, 2, 3$. Then, we have the set of coordinate vector fields\(^4\) $\{\partial_\mu\}$, where each one of the $\partial_\mu \in \sec TU \subset \sec TM$ is a basis for $TU$. We also write $\{\partial_\mu\} \in \sec \mathbf{F} U \subset \sec \mathbf{F} M$ and read that $\{\partial_\mu\}$ is a section of the frame bundle $\mathbf{F} M$.

10. The set of coordinate covector fields $\{dx^\mu\}$, where each one of the $dx^\mu \in \sec T^* U \subset \sec T^* M$ is a basis for $T^* U$. We also write $\{dx^\mu\} \in \sec FU \subset \sec FM$ and read that $\{dx^\mu\}$ is a section of the coframe bundle $\mathbf{F} M$.

11. Since $\{\partial_\mu\}$ is a basis for $TU$ we can expand any one of the vector fields $e_a$, $a = 0, 1, 2, 3$ as

$$e_a = e^\mu_a \partial_\mu$$ \hspace{1cm} (6)

where for each fixed $a$ the set $\{e^\mu_a\}$ are the components of the vector field $e_a$ in the basis $\{\partial_\mu\}$ and where for each fixed $a$ and fixed $\mu$, $e^\mu_a : \mathbb{R}^4 \supset \chi(U) \to \mathbb{R}$, i.e., is a real function. Of course, we need a set of 16 real functions to represent the tetrad $\{e_a\}$.

12. If we denote by $\{\partial^\mu\} \subset \sec \mathbf{F} U \subset \sec \mathbf{F} M$ the reciprocal frame of the frame $\{\partial_\mu\}$ we can write

$$e^a = e^a_\mu \partial^\mu,$$ \hspace{1cm} (7)

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\(^3\)Recall that $\chi : M \supset U \to \mathbb{R}^4$, $U \ni x \mapsto \chi(x) := (x^0(x), x^1(x), x^2(x), x^3(x)) \in \mathbb{R}^4$.

\(^4\)Recall that $\partial_\mu := \frac{\partial}{\partial x^\mu}$.
where for each fixed $\mathbf{a}$ the set \{\(e^a_\mu\)} are the components of the vector field $\mathbf{e}^a$ in the basis \(\{\partial^\mu\}\) and where for each fixed $\mathbf{a}$ and fixed $\mu$, $e^a_\mu : \mathbb{R}^4 \ni \chi(U) \to \mathbb{R}$, i.e., is a real function. Of course, we need a set of 16 real functions to represent the tetrad \(\{e^a\}\).

13. Note that we write as usual
\[ g(\partial_\mu, \partial_\nu) = g_{\mu\nu} = g(\partial_\nu, \partial_\mu), \]
where for each fixed $\mu, \nu$, $g_{\mu\nu} : U \to \mathbb{R}$, i.e., is a real function. Of course there are at most 10 independent $g_{\mu\nu}$ functions.

14. We have immediately that
\[ e^a_\mu e^b_\nu = \delta^a_b, \quad \varepsilon^a_\mu e^b_\nu = \delta^\mu_\nu, \]
and
\[ g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu. \]

15. Before proceeding it is worth to emphasize that for each $x \in U \subset M$, any one of the vectors $\partial_\mu|_x$, $\mathbf{e}^a|_x \in T_x U$, i.e., are elements of the same space, i.e., the tangent space $T_x U$, which is the fiber over $x$ of the tangent bundle $TM$. It is nonsense as written more than one time in [15] to say that: "the set \(\{\partial_\mu|_x\}\) is a basis of vectors in the base curved spacetime and that the set \(\{\mathbf{e}^a|_x\}\) belongs to the tangent fiber at the same local scattering coincidence $x \in U \subset M$.

16. If we denote by \(\{dx_\mu\}\) \(\in \sec FU \subset \sec FM\) the reciprocal coframe of the coframe \(\{dx^\mu\}\) we can write
\[ \varepsilon^a = e^a_\mu dx^\mu, \quad \varepsilon^a = e^a_\mu dx^\mu \]
where for each fixed $\mathbf{a}$ the set \(\{e^a_\mu\}\) (respectively \(\{e^a_\nu\}\)) contains the components of the covector field $\mathbf{\varepsilon}^a$ (respectively $\varepsilon^a_\mu$) in the basis \(\{dx^\mu\}\) (respectively \(\{dx_\nu\}\)) and where for each fixed $\mathbf{a}$ and fixed $\mu$ (respectively $\nu$), $e^a_\mu : \mathbb{R}^4 \ni \chi(U) \to \mathbb{R}$, $e^a_\nu : \mathbb{R}^4 \ni \chi(U) \to \mathbb{R}$, i.e., they are real functions. Of course, we need a set of 16 real functions $e^a_\mu$ to represent the cotetrad \(\{\varepsilon^a\}\). We also have
\[ g(dx_\mu, dx_\nu) = g_{\mu\nu} = g^\nu_\mu = g(dx_\nu, dx^\mu) \]
\[ g(\varepsilon^a_\mu, \varepsilon^b_\nu) = \eta^{ab} := \text{diag}(1, -1, -1, -1), \]
\[ g(dx^\mu, dx_\nu) = \delta^\nu_\mu, \]
\[ g_{\mu\nu} = \varepsilon^a_\mu \eta^{ab} \varepsilon^b_\nu, \quad g^\mu_\nu = \delta^\mu_\nu, \]
\[ g = g_{\mu\nu} dx^\mu \otimes dx_\nu = \eta_{ab} \varepsilon^a \otimes \varepsilon^b, \]
\[ g = g_{\mu\nu} \partial_\mu \otimes \partial_\nu = \eta^{ab} \mathbf{e}_{\mathbf{a}} \otimes \mathbf{e}_{\mathbf{b}}. \]

17. An observation similar to the one in 15 holds, e.g., for anyone of the 1-forms $dx^\mu|_x$ or $\varepsilon^a_\mu|_x$ which are elements of the same space, i.e., the cotangent space $T^*_x U$, which is the fiber over $x$ of the cotangent bundle $T^* M$. 

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18. Now, once the set of 32 real functions \(e_a^\mu\), \(e_b^\nu\) is known we can construct the following tensor field
\[
\varepsilon = e_a^\mu e_b^\nu \otimes dx^\mu = e_a^\mu \partial_\mu \otimes e_n \in \sec T^1_1 U \subset \sec T^1_1 M. \tag{13}
\]

Of course now, e.g., the set \(\{e_a^\mu\}\) with \(a = 0, 1, 2, 3\) and \(\mu = 0, 1, 2, 3\) contains the components of tensor field of type \((1, 1)\) in the hybrid basis \(\{e_a \otimes dx^\mu\}\) of \(T^1_1 U\).

Now, by definition the sections of \(\sec T^1_1 U\) are mappings \(\sec T^1 U \rightarrow \sec T^2 U\), i.e., we have that \(\varepsilon : \sec TU \rightarrow \sec TU\). Now, take an arbitrary vector field \(v \in \sec TU\). Write
\[
v = v^a \partial_a. \tag{14}
\]

Then, using the definition of \(\varepsilon\) we have
\[
\varepsilon(v) = e_a^\mu e_b^\nu \otimes dx^\mu (v^a \partial_\alpha) \\
= e_a^\mu v^a e_b^\nu \otimes dx^\mu (\partial_\alpha) \\
= e_a^\mu v^a e_b^\nu \partial_\alpha \\
= e_a^\mu v^a e_n = v^a e_n = v. \tag{15}
\]

Eq. (15) shows that \(\varepsilon\) is nothing more than the identity tensor in \(TU\). To call \(\varepsilon\) the Einstein-Cartan tetrad 1-form field seems to me a nonsense since \(\varepsilon\) is a vector valued 1-form field, and it is related to the pullback of soldering form of the theory of the linear connections\(^5\). For details, please consult \[5\] (one of the best books I ever read on differential geometry) or \[11\] which has a more soft mathematical presentation.

### 2.1.1 A Single Identity Operator Mislead as a ‘Tetrad’

19. Note that we can write from Eq. (15) that
\[
\varepsilon = e_a^a e_n \otimes e_b^b \tag{16}
\]

Note that there exists a chart of the maximal atlas of \(M\) with coordinate functions conveniently denoted by \(\{\xi^a\}\) such that at a given \(x \in U \subset M\) we can take
\[
d\xi^a|_x = \varepsilon^a|_x \tag{17}
\]

The coordinate functions \(\xi^a\), \(a = 0, 1, 2, 3\) are called local Lorentz coordinates in Physics textbooks.\(^6\)

20. Of course, if we write\(^7\)
\[
\varepsilon = e_a^\mu e_n \otimes dx^\mu = I + \ell \tag{18}
\]
\[
= I^a_\mu e_a \otimes dx^\mu + f^a_\mu e_n \otimes dx^\mu \tag{19}
\]
\[
= \tilde{I}^a_\mu e_a \otimes d\xi^\mu + \tilde{f}^a_\mu e_n \otimes d\xi^\mu, \tag{20}
\]

\(^5\)Some presentations on these issues, like the one by Rovelli [14] are very bad and adds confusion on the subject. We discuss the approach involving soldering forms in the Appendix.

\(^6\)In Mathematics text books they are called Riemann normal coordinates for \(U\), based on \(x \in U\).

\(^7\)Note that these are equations (1.1) and (1.2) in [19].
then at \( x \in U \) we have that
\[
I_\mu^a \big|_x = \text{diag}(1,1,1,1), \quad \ell_\mu^a \big|_x = 0.
\] (21)

21. Keep in mind that \( \ell \in \text{sec} T^1 M \), i.e., it is a vector valued 1-form field.

22. In [15] it is stated that when the ”intrinsically curved piece \( \ell_\mu^a \) of \( e \)” is null on a region \( U \subset M \) then \( U \) is flat. Since curvature refers to a well-defined property of a given connection defined on \( M \), the above statement has meaning only if the manifold \( M \) is equipped with a given connection, which is the Levi-Civita connection\(^8\) \( D \) of \( g \). We can introduce other general connections in \( M \) such that the statement is not true.

Moreover, the converse of the statement is not true. As example, imagine that \( (M,g,D) \) is Minkowski spacetime\(^9\). If we introduce, e.g., spherical coordinates in \( U \subset M \) then in that coordinates \( \ell_\mu^a \neq 0 \) on \( U \). However \( U \) is always flat, this last statement meaning that the Riemann curvature tensor of \( D \) is null in all points of \( M \).

23. A choice of a section \( \{e_a\} \) of \( P_{SO^*_3} M \) will be called a choice of gauge. If \( \{e'_a\} \in \text{sec} P_{SO^*_3} M \) is another choice of gauge we have
\[
e'_a = e_b L^b_a,
\] (22)
where the matrix \( L = (L^a_b) : M \supset U \to SO^*_3 \).

3 Connection 1-forms

24. Perhaps the most pedestrian way for introducing the connection 1-form fields \( \omega^a_b \ (a,b = 0,1,2,3) \) associated with an arbitrary metric compatible connection \( \nabla \) on the manifold \( M \) is the following. Introduce on \( U \subset M \) co-ordinate functions \( \{x^\mu\} \) and the following bases for \( TU \) and respective dual bases for \( T^*U \): \( \{\partial_\mu\}, \{\partial^\mu\} \in \text{sec} FM \), \( \{e_a\}, \{e^a\} \in \text{sec} P_{SO^*_3} M \subset \text{sec} FM \), \( \{\theta^\mu = dx^\mu\}, \{\theta_\mu = g_{\mu\nu}dx^\nu\} \in \text{sec} FM \), \( \{e^a\}, \{e_a\} \in \text{sec} P_{SO^*_3} M \subset \text{sec} FM \).

Now define the coefficients of the connection in the various bases introduced above by:

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\(^8\)A covariant derivative is a connection acting on some vector bundle associated with a given principal bundle where a connection field is defined. Details can be found, e.g., in [5,11].

\(^9\)More precisely, \( (M,g,D) \) is part of the structure \( (M \simeq \mathbb{R}^4, g, D, \tau^g, \uparrow) \) defining Minkowski spacetime. For details, see e.g., [11].
\[ \nabla_{\partial \mu} \partial \nu = \Gamma_{\mu \nu}^{\alpha} \partial \alpha, \quad \nabla_{\partial \sigma} \partial \mu = -\Gamma_{\sigma \mu}^{\alpha} \partial \alpha, \]
\[ \nabla_{e_a} e_b = \omega_{ab}^c e_c, \quad \nabla_{e_a} e^b = -\omega_{ac}^b e_c, \quad \nabla_{e_a} \varepsilon^b = -\omega_{ac}^b \varepsilon_c. \]

Recall that for the Levi-Civita connection of \( \mathbf{g} \) that we denote by \( D \) the connection coefficients \( \Gamma_{\mu \nu}^{\alpha} \) are symmetric but the connection coefficients \( \omega_{bc}^a \) of the same connection (in another basis) are antisymmetric, i.e., \( \omega_{bc}^a = -\omega_{cb}^a \).

25. The covariant differential \( \nabla \) of a vector field \( v \in \text{sec} T M \) is the mapping:
\[ \nabla : \text{sec} T M \rightarrow \text{sec} T M \otimes \text{sec} T^* M, \quad v \rightarrow \nabla v, \quad (24) \]
such that for any vector field \( X \in \text{sec} T M \) we have
\[ \nabla v(X) = \nabla_X v. \quad (25) \]

We now can easily verify that the covariant differential of a basis vector field \( e_a \) is given by
\[ \nabla e_b = e_a \otimes \omega_{b}^a, \quad (26) \]
where the \( \omega_{b}^a \) are the so called (gauge dependent) connection 1-form fields,
\[ \omega_{b}^a = \omega_{cb}^e \varepsilon^c. \quad (27) \]

We can immediately verify that
\[ \omega_{ab} := \eta_{ac} \omega_{b}^c = -\omega_{ba}, \quad (28) \]
a relation which is important for what follows.

3.1 Change of Gauge

26. Consider two frames \( \{e_a\}, \{e'_a\} \in \text{sec} P_{\text{SO}_{1,3}} U \subset \text{sec} P_{\text{SO}_{1,3}} M \subset \text{sec} F M \) related as in Eq. 22 and the respective dual coframes \( \{\varepsilon^a\}, \{\varepsilon'^a\} \in \text{sec} P_{\text{SO}_{1,3}} U \subset \text{sec} P_{\text{SO}_{1,3}} M \). It is useful to introduce the following matrix notation,
\[ e = (e_0, e_1, e_2, e_3), \quad e' = (e'_0, e'_1, e'_2, e'_3), \]
\[ \varepsilon^t = (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3)^t, \quad \varepsilon'^t = (\varepsilon'^0, \varepsilon'^1, \varepsilon'^2, \varepsilon'^3)^t \quad (29) \]

Under these conditions we can write Eq. 22 as
\[ e' = eL \quad (30) \]
Obviously we have also
\[ \varepsilon' = L^{-1} \varepsilon \]  

27. Interpret \( \mathbb{R}^4 \) as a vector space over the field of real numbers with the canonical basis

\[ E^0 = (1, 0, 0, 0), \quad E^1 = (0, 1, 0, 0), \quad E^2 = (0, 0, 1, 0), \quad E^3 = (0, 0, 0, 1). \]  

Consider another copy of (the vector space) \( \mathbb{R}^4 \), denoted here by \( ^*\mathbb{R}^4 \) with canonical basis

\[ ^*E_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad ^*E_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad ^*E_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad ^*E_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]  

Then, we can write

\[ e = e^a \otimes E^a, \quad \varepsilon = \varepsilon^b \otimes^* E^b \]  

and say that \( e \) is a \( \mathbb{R}^4 \)-valued vector field and \( \varepsilon \) is a \( \mathbb{R}^4 \)-valued 1-form field.

28. Recall that the set of \( 4 \times 4 \) real matrices \( \mathbb{R}(4) \) is given by the tensor product \( \mathbb{R}^4 \otimes^* \mathbb{R}^4 \), i.e., \( \mathbb{R}(4) = \mathbb{R}^4 \otimes^* \mathbb{R}^4 \). Next we define the tensor product \( e \otimes \varepsilon \) by

\[ e \otimes \varepsilon = (e^a \otimes E^a) \otimes (\varepsilon^b \otimes^* E^b) := \varepsilon^b (e^a) E^a \otimes^* E^b = \delta^b_a E^a \otimes^* E^b \]  

This is usually simplified by writing the ‘product’ of the matrices \( e \) and \( \varepsilon \) as meaning:

\[ e \otimes \varepsilon = (e_0, e_1, e_2, e_3) \begin{pmatrix} \varepsilon^0 \\ \varepsilon^1 \\ \varepsilon^2 \\ \varepsilon^3 \end{pmatrix} = \begin{pmatrix} \varepsilon^0 (e_0) & 0 & 0 & 0 \\ 0 & \varepsilon^1 (e_1) & 0 & 0 \\ 0 & 0 & \varepsilon^2 (e_2) & 0 \\ 0 & 0 & 0 & \varepsilon^3 (e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I. \]  

29. Let \( \{ E_{ab} \} \) be a set of matrices which is a basis of \( \mathbb{R}(4) \). Each matrix \( E_{ab} \) has a 1 in line \( a \), column \( b \) and zero in their other entries.

Put \( E^b_a = \eta^{bc} E_{ac} \) and define the matrix \( \omega \) of 1-form fields by

\[ \omega = \omega^a_b \otimes E^b_a = \omega^{ab} \otimes E_{ab} \]
Taking into account that $\omega^{ab} = -\omega^{ba}$ we can write
\[
\omega = \frac{1}{2} \omega^{ab} \otimes E_{ab} + \frac{1}{2} \omega^{ba} \otimes E_{ba} \\
= \frac{1}{2} \omega^{ab} \otimes (E_{ab} - E_{ba}) \\
= \frac{1}{2} \omega^{ab} \otimes G_{ab},
\]
(38)

where $G_{ab} := E_{ab} - E_{ba}$ are a set of antisymmetric matrices in $\mathbb{R}(4)$, and as it is well known forms a basis for a representation of the Lie algebra of $SO_{1,3}$ in $\mathbb{R}^4$.

We then can say that $\omega$ is a 1-form with values in the Lie algebra of $SO_{1,3}$.

30. With $e$ and $\omega$ defined as above we can write from Eq. (22),
\[
\nabla e = e \otimes \omega,
\]
(39)

which we write in simplified form as
\[
\nabla e = e \omega
\]
(40)

Since the covariant differential $\nabla$ must be well defined, i.e., independent of the basis used, we must have
\[
\nabla(e') = e' \omega' = \nabla(eL) = (\nabla e)L + e dL,
\]
(41)

or,
\[
eL \omega' = e \omega L + e dL,
\]
(42)

from where we immediately get
\[
\omega' = L^{-1} \omega L + L^{-1} dL.
\]
(43)

31. In [15] the connection 1-form fields are denoted by $\Sigma^{ab}$ instead of $\omega^{ab}$. Sarfatti calls the $\Sigma^{ab}$ Sarfatti the “spin connection 1-form”. This wording is misleading because it did not leave clear that $\{\Sigma^{ab}\}$ refers to a set of six different 1-form fields. Having introduced the set $\{\Sigma^{ab}\}$ Sarfatti presents in his Eq. (1.9) an object called $\Sigma$, written as:
\[
\Sigma = \Sigma^{ab}_\mu dx^\mu \partial_a \partial_b (1.9)
\]
and call it the “local invariant spin-connection 1-form in curved spacetime”. This denomination containing the word invariant is misleading, since $\Sigma$ is not a tensor field. There is a different $\Sigma$ for each choice of gauge as it is clear from Eq. (43) above. If the $\partial_a$ is interpreted as meaning the vector fields$^{10}$ $e_a$ as introduced above and if we introduce the Clifford bundle $\mathcal{C}(M, g)$ of multivector fields we can write as in $[10] [11]$
\[
\Sigma = \Sigma^{ab}_\mu dx^\mu \otimes e_a \otimes e_b \\
= \frac{1}{2} \Sigma^{ab}_\mu dx^\mu \otimes e_a \wedge e_b = \frac{1}{2} \Sigma^{ab}_\mu dx^\mu \otimes e_a e_b,
\]
(44)

$^{10}$Please, do not call the $\partial_a$ coforms...
where \(e_a e_b = g(e_a, e_b) + e_b \wedge e_a\) is the Clifford product of the vector fields \(e_a\) and \(e_b\) interpreted as sections of \(\mathcal{Cl}(M, g)\). As it is well known (see, e.g., [11]) the bivector fields \(e_{ab} = e_a e_b\) generate the Lie algebra of \(SO_{1,3} \simeq \text{SL}(2, \mathbb{C}) \simeq \text{Spin}_{1,3}\) and so we arrive again at another (equivalent) description of the connection 1-forms fields.

### 4 Vector-valued \(r\)-forms, Torsion and Curvature

32. In what follows we denote by \(\tau M = \sum_{r,s=0}^{\infty} T^r_s M\) the tensor bundle of \(M\) and by \(\bigwedge T^* M = \sum_{r=0}^{4} \bigwedge T^r M\) the exterior bundle of \(M\). Note that \(\bigwedge T^* M\) is the bundle of \(r\)-forms and we have the identification \(\bigwedge^1 T^* M = T^* M\) and \(\bigwedge^0 T^* M = \mathcal{F}(M)\) the set of differentiable functions on \(M\). Also, by \(\bigwedge TM = \sum_{r=0}^{4} \bigwedge T^r M\) we denote the exterior bundle of multivectors. We have also the identifications \(\bigwedge^1 TM = TM\) and \(\bigwedge^0 TM = \mathcal{F}(M)\).

A vector valued \(r\)-form \(\alpha\) is a section of the bundle \(TM \otimes \bigwedge T^r M\) which we write using the basis \(\{e_a\}\) of \(TM\) as

\[
\alpha = e_a \otimes \alpha^a, \tag{45}
\]

where \(\alpha^a \in \sec \bigwedge T^r M\).

33. We introduce next the exterior covariant differential of a vector valued \(r\)-form \(\alpha\) as the \((r+1)\)-form \(\nabla \alpha\) such that

\[
\nabla \alpha = \nabla(e_a \otimes \alpha^a) := (\nabla e_a) \otimes \wedge \alpha^a + e_a \otimes d\alpha^a. \tag{46}
\]

The product \(\otimes \wedge\) is defined by

\[
(\nabla e_b) \otimes \wedge \alpha^b = (e_a \otimes \omega^a_b) \otimes \wedge \alpha^b := e_a \otimes (\omega^a_b \wedge \alpha^b). \tag{47}
\]

Then, from Eq. (46) we get

\[
\nabla \alpha = e_a \otimes (d\alpha^a + \omega^a_b \wedge \alpha^b) \tag{48}
\]

34. Before continuing we observe that we denoted the exterior covariant differential by the same symbol as the covariant differential because it is in a sense an extension of this later object which was has been introduced above as a mapping (satisfying certain properties) sending \(\sec TM \to \sec TM \otimes \sec \bigwedge^1 T^* M\).

35. Observe that \(\varepsilon = e_a \otimes \varepsilon^a\) is a vector valued 1-form. Then, its exterior covariant differential is:

\[
\nabla \varepsilon = e_a \otimes (d\varepsilon^a + \omega^a_b \wedge \varepsilon^b) \tag{49}
\]
Now, recalling that
\[
d\varepsilon^a(e_b, e_c) = e_b(\varepsilon^a(e_c)) - e_c(\varepsilon^a(e_b)) - \varepsilon^a([e_b, e_c])
\]
\[
= -\varepsilon^a([e_b, e_c]) = -\varepsilon^a(\nabla_{e_b} e_c - \nabla_{e_c} e_b - \tau(e_b, e_c))
\]
\[
= -\varepsilon^a (\omega^d_{bc} e_d - \omega^d_{cb} e_d - \tau(e_b, e_c))
\]
\[
= -(\omega^a_{bc} - \omega^a_{cb}) + T^a_{bc}.
\]

(50)

(51)

(52)

(where we used that \(\tau \in \text{sec}T M \otimes \text{sec} \bigwedge^2 T^*M\) the vector valued torsion form is given by
\[
\tau = e_a \otimes \varepsilon^a = \frac{1}{2} e_a \otimes \omega^a_{bc} \wedge \varepsilon^c,
\]

(53)

where the \(\tau^a = \frac{1}{2} T^a_{bc} \varepsilon^b \wedge \varepsilon^c \in \text{sec} \bigwedge^2 T^*M\) are called the torsion 2-forms.

For eventual future reference we also recall that
\[
[e_b, e_c] = d^d_{bc} e_d,
\]

(54)

and that
\[
d\varepsilon^a := -\frac{1}{2} \omega^a_{bc} \wedge \varepsilon^c,
\]

(55)

from where we get
\[
T^a_{bc} = \omega^a_{bc} - \omega^a_{cb} - \varepsilon^a_{bc}.
\]

(56)

36. We then have,
\[
\tau^a = d\varepsilon^a + \omega^a_c \wedge \varepsilon^b,
\]

(57)

known as Cartan’s first structure equation.

37. Defining
\[
\tau^t = (\tau^0, \tau^1, \tau^2, \tau^3)
\]

(58)

we may write (in obvious notation)
\[
\tau = d\varepsilon + \omega \wedge \varepsilon.
\]

(59)

38. Observe that since \(\nabla e_b \in \text{sec}T M \otimes \bigwedge^1 T^*M\) then \(\nabla(\nabla e_b) \in \text{sec}T M \otimes \bigwedge^2 T^*M\) we have
\[
\nabla(\nabla e_b) = \nabla(e_a \otimes \omega^a_b) = (\nabla e_a) \otimes \omega^a_b + e_a \otimes d\omega^a_b
\]
\[
= (e_c \otimes \omega^c_a) \otimes \omega^a_b + e_a \otimes d\omega^a_b
\]
\[
= e_a \otimes (\omega^a_c \wedge \omega^c_b) + e_a \otimes d\omega^a_b
\]
\[
= e_a \otimes (d\omega^a_b + \omega^a_c \wedge \omega^c_b)
\]
\[
= e_a \otimes R^a_{cb},
\]

(60)

where the \(R^a_{cb} \in \text{sec} \bigwedge^2 T^*M\) are called curvature 2-forms. The equations which define \(R^a_{cb} (a, b = 0, 1, 2, 3)\), i.e.,
\[
R^a_{cb} = d\omega^a_b + \omega^a_c \wedge \omega^c_b,
\]

(61)
is known as Cartan’s second structure equation.

39. With the above ‘technology’ it is now an easy task to show that

\[ R^a_{\,bc} = \frac{1}{2} R^a_{\,bcd} \varepsilon^c \wedge \varepsilon^d, \]  \tag{62} \]

where \( R^a_{\,bcd} \) are the components of the Riemann curvature tensor in the "orthonormal frame" \( \{ e_a \otimes \varepsilon^b \otimes \varepsilon^c \otimes \varepsilon^d \} \) of \( T^2_1 U \subset T^3_1 M \).

40. Let us calculate \( \nabla \nabla \varepsilon = \nabla(\nabla(e_b \otimes \varepsilon^b)) \). We have:

\[
\nabla(\nabla(e_b \otimes \varepsilon^b)) = \nabla[(\nabla e_b) \otimes (\varepsilon^b + e_b \otimes \varepsilon^b)]
\]

\[
= \nabla[e_a \otimes (d\varepsilon^a + \omega^a_b \wedge \varepsilon^b)]
\]

\[
= (\nabla e_a) \otimes (d\varepsilon^a + \omega^a_b \wedge \varepsilon^b) + e_a \otimes [d\omega^a_b \wedge \varepsilon^b - \omega^a_b \wedge d\varepsilon^b]
\]

\[
= e_a \otimes (\omega^a_c \wedge \omega^b_c + d\omega^a_b) \wedge \varepsilon^b.
\]  \tag{63} \]

41. Also, a calculation of \( \nabla \nabla e \) gives (in obvious notation):

\[ \nabla \nabla e = e(d\omega + \omega \wedge \omega) \]  \tag{64} \]

5 \((p + q)\)-indexed \(r\)-forms

42. We already meet some indexed \(r\)-forms, namely the torsion 2-forms \( \tau^a \) and the curvature 2-forms \( R^a_{\,bcd} \). A general \((p + q)\)-indexed \(r\)-form is an object defined as follows. Suppose that \( X \in \sec T^{r+p}_q M \) and let

\[ X^{\mu_1, \ldots, \mu_p}_{\nu_1, \ldots, \nu_q}(e_1, \ldots, e_r) \in \sec \bigwedge^r T^* M, \]  \tag{65} \]

such that

\[ X^{\mu_1, \ldots, \mu_p}_{\nu_1, \ldots, \nu_q}(e_1, \ldots, e_r) = X(e_1, \ldots, e_r, e_{\nu_1}, \ldots, e_{\nu_q}, \varepsilon^{\mu_1}, \ldots, \varepsilon^{\mu_p}). \]  \tag{66} \]

43. The exterior covariant derivative (differential) \( D \) of a \((p + q)\)-indexed \(r\)-form \( X^{\mu_1, \ldots, \mu_p}_{\nu_1, \ldots, \nu_q} \) on a manifold with a general connection \( \nabla \) is the mapping:

\[ D : \sec \bigwedge^r T^* M \to \sec \bigwedge^{r+1} T^* M, \ 0 \leq r \leq 4, \]  \tag{67} \]

such that\(^{11}\)

\[ (r + 1)D X^{\mu_1, \ldots, \mu_p}_{\nu_1, \ldots, \nu_q}(e_0, e_1, \ldots, e_r) \]

\[ = \sum_{\nu = 0}^r (-1)^\nu \nabla_{e_\nu} X(e_0, e_1, \ldots, e_{\nu}, \ldots, e_r, e_{\nu_1}, \ldots, e_{\nu_q}, \varepsilon^{\mu_1}, \ldots, \varepsilon^{\mu_p}) \]

\[ - \sum_{0 \leq \nu, \mu \leq r} (-1)^{\nu + \mu} X(T(e_\nu, e_\mu), e_0, e_1, \ldots, e_{\nu}, \ldots, e_r, e_{\nu_1}, \ldots, e_{\nu_q}, \varepsilon^{\mu_1}, \ldots, \varepsilon^{\mu_p}). \]  \tag{68} \]

\(^{11}\)As usual the inverted hat over a symbol (in Eq. \(65\)) means that the corresponding symbol is missing in the expression.
Then, we may verify that
\[ D_{\mu \nu} = d_{\mu \nu} + \omega_{\mu} \wedge \nu_{\nu} + \omega_{\nu} \wedge \mu_{\mu} \]
(69)

44. Note that if Eq. (69) is applied on the connection 1-forms \( \omega^a_m \) we would get
\[ D\omega^a_m = d\omega^a_m + \omega^a_c \wedge \omega^c_m - \omega^c_m \wedge \omega^a_c \]
(70)

which appears in many textbooks, and in particular as Eq. (1.11) in [15] with the substitutions \( \omega^a_m \mapsto \Sigma^a_m \) and \( R^a_m \mapsto R^a_m \) is meaningless. The reason for many authors to write an equation like Eq. (70) is the wish to have an equation similar to one that appears in the fiber bundle theory formulation of the theory of connections. We are not going to recall that theory here. An interested reader may study, e.g., [5, 11].

6 The Einstein-Hilbert Lagrangian Density

45. We recall that the Hodge star operator is the mapping
\[ * : \sec \bigwedge^p T^* M \to \sec \bigwedge^{n-p} T^* M, \]
(71)
such that for any \( A, B \in \sec \bigwedge^p T^* M \)
\[ A \wedge * B = (A \cdot B)\tau_g, \]
(72)

where \( A \cdot B \) is the scalar product of \( p \)-forms\(^{12} \) and \( \tau_g \in \sec \bigwedge^4 T^* M \) is the volume 4-form, which can be written with the previously introduced notations as
\[ \tau_g = \sqrt{-\det g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \varepsilon^0 \wedge \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3. \]
(73)

It is useful to recall that
\[ * (\varepsilon^{i_1} \wedge ... \varepsilon^{i_p}) = \frac{1}{(4-p)!} \eta^{j_1...j_p} \epsilon_{j_1...j_4} \varepsilon^{j_{p+1}} \wedge ... \wedge \varepsilon^{j_4}, \]
(74)

where \( \varepsilon_{0123} = 1 \) and \( \epsilon_{j_1...j_4} = 1 \) if \( j_1...j_4 \) is an even permutation of \( (0123) \), \( \epsilon_{j_1...j_4} = -1 \) if \( j_1...j_4 \) is an odd permutation of \( (0123) \) and \( \epsilon_{j_1...j_4} = 0 \) if there are two equal digits in the string \( j_1...j_4 \).

46. The Einstein-Hilbert action (in geometrical units) with cosmological constant is
\[ S_{\text{gravity}} = \frac{1}{2} \int (R + \Lambda) \tau_g = \int L_{\text{gravity}} \]
(75)

\(^{12}\)If \( A = u_1 \wedge ... \wedge u_p \) and \( B = v_1 \wedge ... \wedge v_p \), \( u_i, v_i \in \sec \bigwedge^1 T^* M \), then \( A \cdot B = \det g(u_i, v_j) \).

See details in e.g., [11].
where \( R \in \text{sec} \bigwedge^0 T^* M \) is the scalar curvature and \( \Lambda \) is a constant called the cosmological constant and \( L_{\text{gravity}} \in \text{sec} \bigwedge^4 T^* M \) is the Lagrangian density also called density of action.

Note that we can write \( L_{\text{gravity}} \) in very different but equivalent forms, one of them very convenient for the application of the variational formalism. It is:

\[
L_{\text{gravity}} = \frac{1}{2} R_{cd} \wedge (\varepsilon^c \wedge \varepsilon^d) + \frac{1}{2} \Lambda \tau_g = \frac{1}{2} \star R_{cd} \wedge (\varepsilon^c \wedge \varepsilon^d) + \frac{1}{2} \Lambda \tau_g \tag{76}
\]

Note also that we can write correctly

\[
L_{\text{gravity}} = \frac{1}{4} \epsilon_{abcd} \left( R_{ab} \wedge \varepsilon^c \wedge \varepsilon^d + \frac{1}{12} \Lambda \varepsilon^a \wedge \varepsilon^b \wedge \varepsilon^c \wedge \varepsilon^d \right) \tag{77}
\]

47. Now, we may comment that Eq.(1.13) of [15] has no mathematical meaning at all, since besides representing \( L_{\text{gravity}} \) by \( \frac{\delta S_{\text{gravity}}}{\delta x^4} \), a nonsequitur, it is written that

\[
\frac{\delta S_{\text{gravity}}}{\delta x^4} = \star ( R_{ab} \wedge \varepsilon^c \wedge \varepsilon^d + \Lambda \varepsilon^a \wedge \varepsilon^b \wedge \varepsilon^c \wedge \varepsilon^d), \tag{1.13S}
\]

where the symbol \( \star \) is defined in a misleading and incorrect way.

We introduce the complete Lagrangian density as

\[
\mathcal{L} = L_{\text{gravity}} + L_{\text{matter}}. \tag{78}
\]

Now, we have

\[
\delta L_{\text{matter}} := \delta \varepsilon^a \wedge \frac{\partial L_{\text{matter}}}{\partial \varepsilon^a} = \frac{1}{2} \delta \varepsilon^a \wedge T_a
\]

where \( T_a = T_a^b \varepsilon_b \in \text{sec} \bigwedge^1 T^* M \), \( a, b = 0, 1, 2, 3 \) are the energy-momentum 1-form fields, such that \( T = T_a^b \varepsilon^a \otimes \varepsilon_b \in \text{sec} T^1_1 M \) is the energy-momentum tensor of matter.

Variation of the total action gives Einstein equations (see details, e.g., in [11]), which here we write as

\[
R_a - \frac{1}{2} \varepsilon^a (R + \Lambda) = T_a \tag{79}
\]

where

\[
R_a = R_a^b \varepsilon_b \in \text{sec} \bigwedge^1 T^* M, a, b = 0, 1, 2, 3 \tag{80}
\]

are the Ricci 1-forms, the \( R_a^b \) being the components of the Ricci tensor in the \( \{ \varepsilon^a \otimes \varepsilon_b \} \) of \( T^1_1 M \).
7  "Energy-Momentum Conservation" and Λ

48. An equation equivalent to Eq. (79) is the following one\(^{13}\):

\[-d \star S^a = \star T^a + \star t^a + \Lambda \star \varepsilon^a,\]  

(81)

where \(S^a \in \text{sec} \bigwedge^2 T^* M\) are the superpotentials and the \(t^a \in \text{sec} \bigwedge^1 T^* M\) are the 1-forms whose components are the energy momentum pseudo-tensor of the gravitational field in a given gauge. If you are interested in the explicit forms of these objects, please consult \[11\]. Here, the importance of Eq. (81) is that applying the differential operator \(d\) to both sides of Eq. (81) and moreover, if we suppose that \(\Lambda \in \text{sec} \bigwedge^0 T^* M\) is a scalar function instead of a constant, we get

\[d(\star T^a + \star t^a + \Lambda \star \varepsilon^a) = -d\Lambda \wedge \star \varepsilon^a\]  

(82)

Eq. (82) shows the existence of an “energy-momentum conservation law”\(^{14}\) only if \(d\Lambda = 0\). So, our conclusion is in contradiction with the statement in \[15\] which follows his Eq.(1.16).

Of course, in order to get an “energy-momentum conservation law” it is necessary to make \(\Lambda\) a dynamic field and write a complete Lagrangian for it. This will be not discussed here.

49. Introduction of a connection with torsion will not imply in the automatic validity of Einstein field equations, and so the discourse based on Eqs.(1.17) of \[15\] is ad hoc.

8  What is \(\ell\)

50. Sarfatti \[15\] originally defined the \(\ell\) field by his Eqs.(1.1) and (1.2) (see Eqs. (18), (19) and (20) above). As observed in 21 Eqs.(1.1) and (1.2) imply that \(\ell \in \text{sec} T^1_1 M\), i.e., it is a vector valued form field. However, later Sarfatti wrote: “Define the 1-form invariant curved spacetime tetrad field as

\[\ell = \sqrt{\frac{8G}{c^3}}((d\theta)\phi - \theta d\phi) = \ell_\mu dx^\mu.\]  

(1.28)

51. Since \(\theta, \phi \in \text{sec} \bigwedge^0 T^* M\) are functions according to Eq.(1.23) of \[15\] we have that the object defined by Eq.(1.28) must be a 1-form field, i.e., \(\ell \in \text{sec} \bigwedge^1 T^* M\). So, the object defined by Eq.(1.28) cannot be the same \(\ell\) as the object defined in Eq.(1.1) in \[15\], which as we already said is a section of to \(T^1_1 M\).

52. Besides this last observation it is necessary now to recall 1 and to emphasize here that the \(\ell\) in Eq.(1.28) is only one 1-form field. So, it cannot represent a tetrad which is a set of four 1-form fields.

\(^{13}\)Details in how to obtain this equation may be found, e.g., in \[11\].

\(^{14}\)For the reason of the “”, please consult \[11\].
53. The conceptual error just mentioned, shows clearly that Sarfatti did not grasped well the true mathematical meaning of the objects he uses. Another example of our unfortunately not very polite statement is the last formula in Eq.(1.31) of [15], namely

\[
\ell^a_\mu = \ell^a_\mu \frac{P^a L^\mu}{i\hbar},
\]

which is a completely nonsequitur one, once our author declared that “the \{P^a/i\hbar\} is the Lie algebra of spacetime translation group”. This is so because according to his Eq.(1.2) for fixed \(a\) and fixed \(\mu\) each one of the \(\ell^a_\mu\) are real valued mappings, i.e., \(\ell^a_\mu : \mathbb{R}^4 \supset \chi(U) \to \mathbb{R}\).

54. Besides that, also take notice that from Eq.(1.28) in [15] it follows that

\[
d\ell = -2\sqrt{\frac{\hbar G}{c^3}}d\theta \wedge d\phi
\]

instead of his Eq.(1.29), where it is missing the \(\) signal.

9 Quantization of Area

55. Of course, given observations 49-54 the claim of an original contribution in [15] declaring to have deduced gravity as an emergent phenomenon cannot be taken seriously. Our statement will be reinforced after the mathematical analysis of the topological part of [15].

56. The theory of topological defects in ordered media is a well developed subject (see. e.g., [6, 7, 8]) and to expose our criticisms to some topological considerations in [15] we need to recall some of its rudiments.

We suppose that an ordered medium can be regards as a region \(U\) of the spacetime manifold \(M\) described by a function

\[
\Psi : U \to O,
\]

called order parameter. In Eq. (84) \(O\) is called the parameter space or manifold of internal states. The specification of \(O\) depends on the particular theory of the field \(\Psi\).

57. The mapping \(\Psi\) is supposed to vary continuously through \(U\), except at some isolated worldlines and some appropriate hypersurfaces, where it is singular. These regions of lower dimensionality will be called defects in spacetime. We suppose that \(U\) can be foliated as \(U = \mathbb{R} \times X\) where \(X\) is 3-dimensional manifold (a spacelike hypersurface). In this case, in condensed matter physics, the order parameter is a mapping \(\Psi|_X : X \to O\) and the defects in the ‘space \(X\’\) are points, lines, surfaces where \(\Psi|_X\) is singular.

58. In the ‘model’ imagined in [15] the parameter space \(O\) is identified with the unit radius sphere \(S^2\). We will not discuss if this hypothesis is reasonable or not, let us accept it here.

59. For the theory to work, we need to cut out from the manifold \(U \subset M\) the points where \(\Psi\) is singular. If we admit, e.g., a single point defect, as it is the case imagined in [15] then we must take \(U = \mathbb{R} \times X\) with \(X = \mathbb{R}^3 - \{0\} \simeq \mathbb{R} \times S^2\),
where the $S'^2$ in the previous formula is a space isomorphic, but distinct from
the parameter space $S^2$ and where $\{0\}$ stands for the location of the point defect
in 3-dimensional space. Under these conditions the effective manifold modelling
spacetime where the theory rolls is:

$$U = \mathbb{R}^2 \times S'^2$$  \hspace{1cm} (85)

60. Before continuing we must comment that from the Physical point of
view to suppose that the condensate responsible for the existence of gravitation
has only a point defect seems to us an ad hoc assumption. Indeed author of [15]
did not present a single argument for it.

61. To continue we write

$$\Psi : U \to S^2,$$
$$X \ni x \mapsto y = \Psi(x) \in S^2,$$  \hspace{1cm} (86)

and introduce the usual spherical coordinate functions$^{15}$ $\theta, \varphi$ on the sphere $S^2$
with the usual domain, say$^{16}$ $V_1 \subset S^2$. We have then the following coordinate
representation of $\Psi$ in $V_1$

$$\Psi(x) = (\theta(y), \varphi(y))$$  \hspace{1cm} (87)

62. Now, the ‘area element’ of the parameter space$^{17}$ $S^2$, which mathematicians call the volume element of $S^2$ is given once we use the coordinate functions
$\theta, \varphi$ covering $V_1 \subset S^2$ by the 2-form $v \in \text{sec} \bigwedge T^* S^2$

$$v = \sin \theta d\theta \wedge d\varphi$$  \hspace{1cm} (88)

The area of the parameter space $S^2$ is then calculated as$^{18}$

$$A_{S^2} = \int_{S^2} v = \int_{S^2} \sin \theta d\theta \wedge d\varphi = 4\pi$$  \hspace{1cm} (89)

63. It is then automatically quantized (joke)!

64. The ‘area element’ 2-form $v$ is, of course, closed because all 3-forms in
$S^2$ are null.

65. Now, the pullback of $v$ under the mapping $\Psi$ is the 2-form $\nu \in \text{sec} \bigwedge^2 T^* U \subset \text{sec} \bigwedge^2 T^* M$ such that

$$\nu = \sin(\Psi^* \theta) d(\Psi^* \theta) \wedge d(\Psi^* \varphi),$$  \hspace{1cm} (90)

$^{15}$Please do not confound these variables with the ones used by Sarfatti, which are defined in his Eq.(1.23).

$^{16}$Of course to cover all $S^2$ it is necessary to introduce complementary spherical coordinate functions $\theta', \varphi'$ covering $V_2 \subset S^2$ and such that $V_1 \cap V_2 \neq \emptyset$. Of course $S^2 \subset V_1 \cup V_2$.

$^{17}$Please, notice that this space $S^2$ has nothing to do with any surface in the spacetime manifold $M$.

$^{18}$This is so because $S^2 - V_1$ is a set of zero measure.
with

\[(\Psi^*\theta, \Psi^*\varphi) = (\theta \circ \Psi, \varphi \circ \Psi)\]  \hspace{1cm} (91)

Now, let us restrict our considerations to \(\Psi\), which is the restriction of the mapping \(\Psi|_X\) to \(S^2\), i.e., the mapping

\[\Psi = \Psi|_X : X \supset S^2 \to O\]

The integral \(\int_{S^2} \upsilon\) such that

\[\Psi(S^2) = S^2\]  \hspace{1cm} (92)

is given by (see, e.g., [3, 4])

\[\int_{S^2} \upsilon = \deg(\Psi) \int_{S^2} \upsilon = 4\pi \deg(\Psi)\]  \hspace{1cm} (93)

where \(\deg(\Psi)\) denotes the Brouwer degree of mapping \(\Psi\), which we recall is the restriction of the mapping \(\Psi|_X : X \to S^2\) to \(S^2\). The Brouwer degree is an integer that is roughly speaking the number of times that each point \(y \in S^2\) is covered by the image of \(S^2\) under \(\Psi\), each covering counted positively or negatively depending on the orientation of \(\Psi\) in an open set of the point \(x = \Psi^{-1}(y)\). This is one of the doors from where homotopy theory makes its entrance in Physics.

66. The 2-form \(\upsilon\) is closed, but since it is defined in \(U = \mathbb{R}^2 \times S^2\) (and thus not diffeomorphic to \(\mathbb{R}^4\)) it is not exact. Then it must have period integrals according to de Rham theorem (see, e.g., [3, 6]). Looking at Eq. (93) we see that this is indeed necessary, for otherwise, if we could write globally \(\upsilon = dA\), \(A \in \sec 1 \wedge T^*U \subset \sec 1 \wedge T^*M\) then Stokes theorem would give

\[\int_{S^2} \upsilon = \int_{S^2} dA = \int_{\partial S^2} A = \int_{\emptyset} A = 0,\]  \hspace{1cm} (94)

contradicting Eq. (93).

67. We end our observations by remarking that in [15] the 2-form written

\[A = \sqrt{\frac{\hbar G}{c^3}} dl = 2 \frac{\hbar G}{c^3} d\theta \wedge d\varphi\]  \hspace{1cm} (1.29)

is supposed to be an area ‘flux density’. Of course, if \((\theta, \varphi)\) are interpreted as spherical coordinate functions for \(S^2\) this is not true, because, in order to be the 2-form ‘area element’ of a sphere a factor \(\sin \theta\) is missing. However, introduce coordinate functions\(^{19}\) \((a, b)\) on \(S^2\) covering \(V_1 \subset S^2\) and such that

\[a = -\cos \theta, \hspace{0.5cm} b = \varphi\]  \hspace{1cm} (95)

\(^{19}\)We leave aside the dimensional factor \(\sqrt{\frac{\hbar G}{c^3}}\) in what follows.
Then, we have immediately from Eq. (88) that
\[ \nu = -2 da \wedge db \quad (96) \]
which in \( V_1 \) can be written as the differential of the 1-form field \( a_{V_1} \in \text{sec} \bigwedge^1 T^*V_1 \subset \text{sec} \bigwedge^1 T^*S^2 \) such that
\[ a_{V_1} = b da - adb \quad (97) \]

68. Now, consider again \( \nu \in \text{sec} \bigwedge^2 T^*U \subset \text{sec} \bigwedge^2 T^*M \) which is the pull-back under the mapping \( \Psi \) of \( \nu \) in \( S^2 \).

69. Introduce Cartesian and spherical coordinate functions in \( X = \mathbb{R}^3 - \{0\} = \mathbb{R} \times S^2 \) with center in \( \{0\} \), the defect localization point. The spherical coordinates \( (r, \theta, \varphi) \) on \( S^2 \) are given by the restrictions of the functions \( (\theta, \varphi) \) on \( S^2 \). We now specify the restriction of the coordinate representation of the mapping \( \Psi \) as functions of the spherical coordinates \( (\theta, \varphi) \), i.e., we write:
\[
\Psi(\theta, \varphi) = \begin{pmatrix}
\Psi_1(\theta, \varphi) \\
\Psi_2(\theta, \varphi) \\
\Psi_3(\theta, \varphi)
\end{pmatrix} = \begin{pmatrix}
\sin \theta \cos n\varphi \\
\sin \theta \sin n\varphi \\
\cos \theta
\end{pmatrix}, \quad (98)
\]
with \( n \in \mathbb{Z} \). This means that \( (\theta, \varphi) = (\theta, n\varphi) = (\Psi^*\theta, \Psi^*\varphi) \).

We remark that as defined \( \Psi \) is a smooth mapping outside the poles. Moreover, \( \Psi \) maps \( S^2 \) \( n \) times around \( S^2 \). This is easily seen if we observe from Eq. (88) that \( \Psi \) maps any circle \( \theta = \theta_0 \) \( n \) times on the corresponding circle in \( S^2 \).

70. Write moreover as usual the following relations between the Cartesian and spherical coordinate functions on \( X = \mathbb{R}^3 - \{0\} \),
\[
x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \cos \varphi, \quad z = r \cos \theta,
\]
\[
r = \sqrt{x^2 + y^2 + z^2}. \quad (100)
\]

Then the non exact 2-form \( \nu \in \text{sec} \bigwedge^2 T^*U \subset \text{sec} \bigwedge^2 T^*M \) can be described as the differential of the following two 1-form fields on the regions \( U_1 \) and \( U_2 \),
\[
\begin{align*}
\mathcal{A}_1 &= -n(\cos \theta - 1)d\varphi \quad \text{on} \quad U_1 = \mathbb{R} \times (\mathbb{R}^3 - \{r = 0 \text{ or } z < 0\}), \\
\mathcal{A}_2 &= -n(\cos \theta + 1)d\varphi \quad \text{on} \quad U_2 = \mathbb{R} \times (\mathbb{R}^3 - \{r = 0 \text{ or } z > 0\}),
\end{align*} \quad (101)
\]
\[
U_1 \cap U_2 = S^1. \quad (102)
\]

If we write the representatives of the 2-form \( \nu \) on the same regions \( U_1 \) and \( U_2 \) as \( \nu_1 \) and \( \nu_2 \) we can write
\[
\int_{S^2} \nu = \int_{U_1} \nu_1 + \int_{U_2} \nu_2 = \int_{S^1} (\mathcal{A}_1 - \mathcal{A}_2) = n \int_{S^1} d\varphi = 4\pi n \quad (103)
\]
71. Finally we can write on $U_1 \cup U_2 - \{\text{worldline of the defect}\}$

$$\nu = n \sin \theta d\theta \wedge d\varphi$$

$$= \frac{n}{r} (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy), \quad (104)$$

72. Readers that know the $U(1)$ principal fiber bundle formulation of the magnetic monopole will recognize that apart for the correct physical units $\nu$ describes the field of a magnetic monopole. This is not a coincidence, of course, since the formulation of both problems (the point defect and the monopole one) have many common ingredients.

10 Conclusion

Sarfatti’s paper\textsuperscript{21} \cite{15}, we regret to say, is unfortunately a potpourri of nonsense Mathematics\textsuperscript{22}. The fact that he found endorsers which permitted him to put his article in the arXiv is a preoccupying fact. Indeed, the incident shows that endorsers did not pay attention to what they read, or worse, that there are a lot of people with almost null mathematical knowledge publishing Physics\textsuperscript{23} papers replete of nonsense Mathematics. We recall here that among others, author of \cite{15} confounded a single 1-form field (the one given by Eq.(1.28)) with (non trivial part) a tetrad, which is a set of four distinct 1-form fields, wrote in a wrong and misleading way the Einstein-Hilbert Lagrangian density, misleads the real nature of the connection 1-forms, wrote a misleading ‘conservation’ equation to deduce that the cosmological constant need not be a constant in General Relativity, supposed in an ad hoc way that Einstein’s equations also holds in a spacetime with torsion, and finally, used in a misleading way topological arguments. Also that author did not leave it clear what are the hypotheses he used. A careful reading of \cite{15} shows that his hypotheses are completely ad hoc assumptions, since in our view no arguments from Physics or Mathematics are given for them. Summing up, we must say that Sarfatti’s claim to have deduced Einstein’s equations as an emergent phenomena is an statement that cannot be taken seriously.

\textsuperscript{20}See, e.g., \cite{5}.

\textsuperscript{21}More specifically (as said in the abstract) the first version posted at the arXiv.

\textsuperscript{22}This paper has been written as a consequence of an exercise that we proposed to some of our students. Eventually, Sarfatti will not like it, and will probably say that we are very pedantic, but eventually (we hope) he will use it to write a better version of his paper. In any case, we would like that he be aware that in writing it we found also inspiration in Aristotle [who in his Nicomachean Ethics, book 1, Chapter 6 said in a similar situation where he could not agree with the presentations of some of his friends on a given subject that: ‘...piety requires us to honor truth above our friends’], and also in our (late) friend Pertti Lounesto that enlightened us for many years with his posters on errors and counterexamples to ‘theorems’ found in the literature on Clifford algebras.

\textsuperscript{23}And also mathematical papers, as e.g., \cite{1}. See our analysis of that bad paper published in Nonlinear Analysis in \cite{13}.
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