Renewal equations for option pricing

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Abstract

In this paper we will develop an original approach, based in the use of renewal equations, for obtaining pricing expressions for financial instruments whose underlying asset can be solely described through a simple continuous-time random walk (CTRW). This enhances the potential use of CTRW techniques in finance. We solve these equations for different contract specifications in a particular but exemplifying case. We recover the celebrated results for the Wiener process under certain limits.

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The continuous-time random walk (CTRW) formalism, introduced in the physics literature by Montroll and Weiss [1], is a way to generalize ordinary random walks by letting the steps and the time elapsed between them be random magnitudes. In this sense CTRW is related to several other well-known extensions of random walks in continuous time, like semi-Markov processes or Markov renewal processes [2], the oldest of which is perhaps the pure birth Poisson process [3, 4].

In the most common version of the CTRW, any realization of the process \( X(t) \) consists of a series of step functions: it changes at random times \( t_0, t_1, t_2, \cdots \) while it remains fixed in place between successive steps. The interval between these successive steps is the random variable \( \Delta t_n = t_n - t_{n-1} \) called sojourn or waiting time. At the conclusion of the \( n \)th sojourn \( X(t) \) experiences a random change, or jump, given by

\[
\Delta X_n = X(t_n) - X(t_{n-1}).
\]

Both waiting times \( \Delta t_n \) and random jumps \( \Delta X_n \) are assumed to be (mutually) independent and identically distributed random variables described by their probability density functions (PDF’s) which we denote by \( \psi(t) \) and \( h(x) \) respectively.

The key objective within the CTRW framework is to obtain the so-called propagator, the transition PDF of \( X(t) \), defined by

\[
p(x - x_0, t - t_0)dx = \Pr\{x < X(t) \leq x + dx | X(t_0) = x_0\}.
\]

As is well known, the propagator obeys the following renewal equation [5, 6]:

\[
p(x, t) = \delta(x) \int_t^\infty dt' \psi(t') + \int_0^t dt' \psi(t') \int_{-\infty}^{\infty} h(y)p(x - y, t - t')dy.
\]

This integral equation can be easily solved in the Fourier-Laplace space:

\[
\hat{\hat{p}}(\omega, s) = \frac{1}{s} \frac{1 - \hat{\psi}(s)}{1 - \hat{h}(\omega)\hat{\psi}(s)},
\]

where \( \hat{\hat{p}}(\omega, s) \) is the joint Fourier-Laplace transform of function \( p(x, t) \), \( \hat{h}(\omega) \) is the Fourier transform of \( h(x) \), and \( \hat{\psi}(s) \) is the Laplace transform of \( \psi(t) \).

In fact, two distinctive features of CTRW’s are, on the one hand, that most of the statistical properties of the process can be expressed in the form of renewal equations and,
on the other hand, that it is convenient for solving them to consider integral transforms. The reader can find in Ref. [7] a classical introduction to renewal theory.

In this paper we will develop an original approach based in the use of renewal equations for obtaining pricing expressions for financial instruments whose underlying can be solely described through a CTRW. This enhances the potential use of CTRW techniques in finance.

Specifically, we will assume that $X(t) = \ln S(t)$, where $S(t)$ is the stock price at $t$, and concentrate our attention in the study of derivatives [8] whose value depends on the present state of the market, $x_0 = \ln S_0$. We allow the price to be a function of the sample path to some extend as well. A typical example are European options, since for them the price $C(x_0, t_0)$ must obey the renewal equation

$$C(x, t) = C(x, T) e^{-(rT - t) \int_0^\infty dt' \psi(t' - t)} + \int_t^T dt' \psi(t' - t) e^{-r(t' - t) \int_{-\infty}^{+\infty} h(y - x) C(y, t') dy},$$

where $r$ is the risk-free interest rate, and $C(x, T)$ is the payoff function, the value of the option at expiration $T$, which is known. Note that, depending on the contract specifications, the option price will be constrained by additional conditions.

Eq. (2) will describe a valid option price from the financial point of view only if one can define an equivalent risk-neutral market measure for which

$$\mathbb{E}[S(t) e^{-r(t-t_0)} | \mathcal{F}(t_0)] = S_0$$

holds, where with $\mathcal{F}(t)$ we denote all the available information up to time $t$. If we translate this constraint into Eq. (1) we will find that waiting times must be exponentially distributed, $\psi(t) = \lambda e^{-\lambda t}$, with the following risk-neutral intensity $\lambda$:

$$\lambda = \frac{r}{h(\omega = -i) - 1}.$$

It is very convenient to work with the backward version of Eq. (2) by introducing $C(x, \bar{t}) = C(x, T - \bar{t})$ and $\bar{t} = T - t$:

$$C(x, \bar{t}) = C(x, 0) e^{-(\lambda + r) \bar{t}} + \lambda \int_0^{\bar{t}} d\bar{t}' e^{-(\lambda + r)(ar{t} - \bar{t}')} \int_{-\infty}^{+\infty} h(y - x) C(y, \bar{t}') dy.$$

$C(x, \bar{t})$ fulfills the classical Merton’s equation [9] for jump-diffusive market models once one removes the contribution of the Wiener process to the asset evolution. In fact there is an
extensive literature—see [10] for a review—on the issue of financial processes with random jumps, but the usual approach identifies such changes with abnormal market behavior, e.g. a crash, whereas diffusion determines the normal evolution. Within our framework we obtain option prices for the pure CTRW, with no additional stochastic process needed. In fact, as we will show below, we can recover the celebrated results for the Wiener process under certain limits.

Let us move Eq. (3) into the Laplace domain,

\[ \hat{C}(x, s) = \frac{1}{\lambda + r + s} \left\{ C(x, 0) + \lambda \int_{-\infty}^{+\infty} dy h(y-x) \hat{C}(y, s) \right\}, \]

but not into the Fourier domain because \( \hat{\tilde{C}}(\omega, s) \) does not necessarily exist. In any case, one must specify \( h(x) \) before explicit results can be obtained. We have chosen the following asymmetric exponential,

\[ h(x) = \frac{\gamma\rho}{\gamma + \rho} \left[ e^{-\rho x} 1_{x \geq 0} + e^{\gamma x} 1_{x < 0} \right], \]

because it is a tractable case with interesting properties. After this choice the risk-free value of \( \lambda \) reads

\[ \lambda = r \frac{(\rho - 1)(\gamma + 1)}{\gamma - \rho + 1}, \]

and therefore \( \rho \geq 1, \) and \( \gamma \geq \rho - 1. \) Eq. (5) makes the transformation of integral Eq. (4) into

\[ \frac{1}{\lambda + r + s} \left\{ \partial_{xx}^2 C(x, 0) + (\gamma - \rho) \partial_x C(x, 0) - \gamma \rho C(x, 0) \right\}, \]

possible. The general solution of this equation is

\[ \hat{C}(x, s) = A_+(s)e^{\beta_+ x} + A_-(s)e^{\beta_- x} + \frac{1}{\lambda + r + s} C(x, 0) \]

\[ - \frac{\lambda \gamma \rho}{(\lambda + r + s)^2} \int_0^x dy C(y, 0) \left[ \frac{e^{\beta_+(x-y)} - e^{\beta_-(x-y)}}{\beta_+ - \beta_-} \right], \]

\[ \beta_{\pm} = -\frac{\gamma - \rho}{2} \pm \frac{1}{2} \sqrt{(\gamma + \rho)^2 - \frac{4\lambda \gamma \rho}{\lambda + r + s}} \geq 0, \]

provided that \( C(x, 0) \) is a smooth-enough function. In practice, payoff functions show at least one point where the second derivative is not well defined, \( x^* = \ln K. \) Then we have different
solutions for the different regions. Consider for instance call options, where \( C(x, 0) = 0 \) for \( x < x^* \). Since the option price must fulfill in this case that \( \lim_{x \to -\infty} C(x, \tilde{t}) = 0 \), we will have, when \( x < x^* \),

\[
\hat{C}(x, s) = A_+(s)e^{\beta_+ x}.
\]

The value of the payoff function for \( x \geq x^* \) is different for different option flavors. Binary calls are ascertained by \( C(x, 0) = 1 \) and the following boundary condition: \( \lim_{x \to +\infty} C(x, \tilde{t}) = e^{-r \tilde{t}} \). Then

\[
\hat{C}(x, s) = A_-(s)e^{\beta_- x} + \frac{1}{r + s}.
\]

Note that, like the process itself, option prices are discontinuous in general, and therefore we must use Eq. (4) in order to determine functions \( A_\pm(s) \):

\[
A_\pm(s) = -\frac{\beta_x}{\beta_+ - \beta_- (\lambda + r + s)(r + s)}e^{\beta_\pm x^*}.
\]

Now we can perform the Laplace inversion to get:

\[
C(x, \tilde{t}) = \left[1_{x \geq x^*} + 2 \int_0^\infty du I_1(2u) \exp \left(-\frac{u^2}{\lambda \tilde{t}}\right) \right. \\
\times \mathcal{N} \left( \frac{x - x^*}{2u} \sqrt{\frac{\gamma - \rho}{\lambda \tilde{t}}} + \frac{\gamma - \rho + \sigma^2}{\sqrt{\gamma \rho \lambda \tilde{t}}} u \right) e^{-(\lambda + r)\tilde{t}}
\]

where \( \mathcal{N}(\cdot) \) is the cumulative distribution function of a zero-mean unit-variance Gaussian PDF, and \( I_n(\cdot) \) is a \( n \)-order modified Bessel function of the first kind.

Pay attention to the \( 1_{x \geq x^*} \) term. It counts for the finite possibility that the system keeps in place until expiration. In Fig. 1 we can see the how the relative contribution of this term diminishes for larger values of \( \lambda \). Indeed, the discontinuity disappears when considering continuous trading, \( \lambda \to \infty \). We can approach to this limit by letting \( \rho \to \infty \) and \( \gamma \to \infty \), but in such a way that the difference remains finite \( \infty > \gamma - \rho + 1 = \varepsilon > 0 \). In fact we can identify \( \sigma = \sqrt{2r/\varepsilon} \) as the volatility of the market, since

\[
m_1(t - t_0) \equiv \mathbb{E}[X(t) - x_0|\mathcal{F}(t_0)] \\
= \frac{\gamma - \rho}{\gamma \rho} \lambda(t - t_0) \to \left(r - \frac{1}{2} \sigma^2\right)(t - t_0),
\]

\[
m_2(t - t_0) \equiv \mathbb{E}[(X(t) - x_0)^2|\mathcal{F}(t_0)] - [m_1(t - t_0)]^2 \\
= \frac{\gamma^2 - \gamma \rho + \rho^2}{\gamma \rho} \lambda t \to \sigma^2 t, \text{ and}
\]

\[
C(x, \tilde{t}) \to e^{-r \tilde{t}} \mathcal{N} \left[ \frac{x - x^* + (r - \sigma^2/2)\tilde{t}}{\sigma \sqrt{\tilde{t}}} \right],
\]

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FIG. 1: (Color online) Price for an European binary call option. We have set \( \gamma = \rho - 1 + 2r/\sigma^2 \), and we have chosen typical market values for \( r = 4\% \), \( \sigma = 10\% \), and \( T - t_0 = 0.25 \) years.

the well-known results for a Wiener process \[8\].

Consider now the case of a vanilla call for which \( C(x, 0) = e^x - K \) when \( x \geq x^* \), and

\[
\lim_{x \to +\infty} \frac{C(x, t)}{e^x - Ke^{-rt}} = 1.
\]

the solution of the differential equation for \( x \geq x^* \) is:

\[
\hat{C}(x, s) = A_-(s)e^{\beta-\lambda x} + \frac{e^x}{s} - \frac{K}{r+s},
\]

and functions \( A_{\pm}(s) \) are in this case:

\[
A_{\pm}(s) = \frac{1}{\lambda + r + s} \left[ \frac{\lambda}{r + s} \beta_+ + \frac{\lambda + r}{s} (1-\beta_+) \right] \frac{e^{(1-\beta_{\pm})x^*}}{\beta_+ - \beta_-}. 
\]
The Laplace inversion of $\hat{C}(x, s)$ is cumbersome

$$C(x, \bar{t}) = \left\{ 2 \int_0^\infty du I_1(2u) \left[ e^x \mathcal{M}_{\rho-1}^{\gamma+1} \left( x - x^*; \sqrt{\frac{2u^2}{\gamma \rho \lambda \bar{t}}} \right) - K \mathcal{M}_\rho(x - x^*; \sqrt{\frac{2u^2}{\gamma \rho \lambda \bar{t}}}) \right] + \left[ e^x - K \right] \mathbf{1}_{x \geq x^*} \right\} e^{-(\lambda + r)\bar{t}},$$

$$\mathcal{M}_\rho^c(c; \xi) = e^{-ab\xi^2 / 2} N \left( \frac{a - b}{2} \frac{\xi}{\sqrt{\xi}} + c \right),$$

but still readable. In particular one can foresee how the classical Black-Scholes solution [11] appears in the continuous-trade limit, see Fig 2. Also in this figure we observe that in this case the no-trade limit, which corresponds to $\rho \to 1$, leads to a non-trivial result:

$$C(x, \bar{t}) = e^x \left( 1 - e^{-r\bar{t}} \right) + \left[ e^x - K \right] e^{-r\bar{t}} \mathbf{1}_{x \geq x^*}.$$  

The previous results for European calls can be used in order to obtain put prices, $P(x, \bar{t}) =$
\[ P(x, T - t), \] because the so-called put-call parity stands also in our case:

\[ P(x, t) + C(x, t) = e^{-rt} \quad \text{(binary)}, \]

\[ P(x, t) - C(x, t) = Ke^{-rt} - e^x \quad \text{(vanilla)}. \]

This statement can be easily proven by using Eq. (6) for \( \hat{F}(x, s) = \hat{P}(x, s) \pm \hat{C}(x, s) \) because \( F(x, 0) \) is regular.

Let us present next the renewal formulas for American options. When dealing with American derivatives it is more convenient that we focus our attention on puts rather than calls because when the stock pays no dividends American calls will be never exercised before expiration and therefore become European options. The (backward) renewal equation that follow American put options is

\[
P(x, t) = \lambda \int_0^t d\bar{t} e^{-\lambda(\bar{t} - t)} \left[ \int_{-\infty}^{z(\bar{t})} h(y - x) P(y, 0) dy + P(x, 0) e^{-\lambda(\bar{t})} \mathbf{1}_{x \leq z_0} \right],
\]

where \( z(\bar{t}) \) fulfills \( P(z(\bar{t}), \bar{t}) = P(z(\bar{t}), 0) \), and obviously \( z_0 \equiv \lim_{\bar{t} \to 0} z(\bar{t}), z_0 \leq x^* \). The core of the problem lies in the fact that in general one must find \( P(x, t) \) and \( z(\bar{t}) \) simultaneously [12]. However there are exceptions to this rule, as in the case of binary puts, because for them \( z(\bar{t}) = x^* \). As as result, the Laplace transform of \( P(x, t) \) can be computed

\[
\hat{P}(x, s) = \frac{\lambda}{\lambda + r + s} \left\{ \frac{1}{s} + \int_{-\infty}^\infty dy h(y - x) \left[ \hat{P}(x, s) - \frac{1}{s} \right] \right\},
\]

as well as the equivalent differential equation when \( h(x) \) is described by Eq. (5):

\[
\partial_{xx}^2 \hat{P}(x, s) + (\gamma - \rho) \partial_x \hat{P}(x, s) - \frac{r + s}{\lambda + r + s} \gamma \rho \hat{P}(x, s) = 0.
\]

Here we must solely investigate the solution for \( x > x^* \), since \( P(x, t) = 1 \) for \( x \leq x^* \). The upper boundary condition \( \lim_{x \to +\infty} P(x, t) = 0 \) leads to:

\[
\hat{P}(x, s) = A(s) e^{\beta_+ x},
\]

and we must use again the integral equation to get \( A(s) \), since the price is discontinuous, i.e. \( \hat{P}(x^*, s) \neq 1/s \),

\[
A(s) = \frac{\gamma + \beta_+ e^{-\beta_+ x^*}}{\gamma} \frac{s}{s}.
\]
FIG. 3: (Color online) Price for an American binary put option. We have kept the same parameters as in Fig. 1. The inset shows a different decay behavior for every level of market activity.

From this expression we can directly obtain the value of perpetual American puts because:

\[ P(x, \bar{t} \to \infty) = \lim_{s \to 0} s \hat{P}(x, s) = \frac{\rho - 1}{\gamma} e^{-(\gamma - \rho + 1)(x - x^*)}, \]

a result to be published elsewhere [13]. In spite of the apparent simplicity of \( A(s) \) the expression of \( P(x, \bar{t}) \) is very intricate:

\[
P(x, \bar{t}) = \sqrt{\frac{2\rho \lambda \bar{t}}{\gamma}} \int_0^\infty d\xi I_1 \left( \sqrt{2\gamma \rho \lambda \bar{t}} \xi \right) \times \left\{ \mathcal{L}_{\rho - 1}^{\gamma + 1}(\xi) + \mathcal{L}_{\gamma + 1}^{\rho - 1}(\xi) - \mathcal{L}_{\gamma + \rho}^{0}(\xi) \right\} e^{-(\lambda + r)\bar{t}},
\]

\[
\mathcal{L}_{\rho}^{a}(\xi) = b \exp \left[ (a - \rho)(x^* - x) \right] \mathcal{M}_b^a(x^* - x; \xi).
\]

In the continuous-trade limit we can simplify the previous expression and recover once again the Wiener result [14], but in the rest of the cases the wisest procedure is perhaps to invert numerically the Laplace solution, as we have done in the confection of Fig. 3. In fact, we have followed this approach in all the previous figures of the paper.
We will finish the article with some discussion concerning the problem of American vanilla put options. It is notorious that no closed expression is known in this case, even when the evolution of the return is driven by a Wiener process. Therefore we will present partial results only. For instance, we can compute the value of $z_0$. When $h(x)$ follows Eq. (5) we will have

$$Z_0 = e^{z_0} = K \left[ \frac{\gamma + \rho + 1}{\gamma + 1} \right]^{\frac{1}{\rho}} \leq K.$$  

In particular if $\rho = 1$ we have $Z_0 = K$. And we recover the same result for the continuous-trade limit $\lambda \to \infty$.

We may also compute perpetual put prices, because for them we will have a renewal equation for $x > z^*$:

$$P_{\infty}(x) = \lambda \int_0^t d\tilde{t} e^{-(\lambda+\rho)(t-\tilde{t})} \left[ \int_{z^*}^{\infty} dy h(y-x) P_{\infty}(y) + \int_{-\infty}^{z^*} dy h(y-x) (K - e^y) \right],$$

where $z^* \equiv \lim_{\tilde{t} \to \infty} z(\tilde{t})$, and $P_{\infty}(x) \equiv \lim_{\tilde{t} \to \infty} P(x, \tilde{t})$. The ordinary differential equation for $P_{\infty}(x)$ is

$$P''_{\infty}(x) + (\gamma - \rho)P'_{\infty}(x) + (\gamma - \rho + 1)P_{\infty}(x) = 0,$$

and the solution is [13]:

$$P_{\infty}(x) = \frac{\rho - 1}{\gamma} \left[ K - \frac{\gamma}{\gamma + 1} e^{z^*} \right] e^{(\gamma-\rho+1)(z^*-x)},$$

where we have used the integral equation as well. The value of $z^*$ is obtained by demanding $P_{\infty}(z^*) = K - e^{z^*}$:

$$Z^* = e^{z^*} = K \frac{(\gamma + 1)(\gamma - \rho + 1)}{\gamma(\gamma - \rho + 2)}.$$

In the $\rho \to \infty$ limit we will obtain once again the Wiener results [15],

$$Z^* = \frac{2r}{2r + \sigma^2} K, \quad P_{\infty}(x) = \frac{\sigma^2}{2r + \sigma^2} K e^{2r(z^*-x)/\sigma^2}.$$
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