Intermittency and Bose-Einstein correlations

I.V. Andreev$^1$, M. Biyajima$^2$, I.M. Dremin$^1$, N. Suzuki$^3$

$^1$Lebedev Physical Institute, Moscow, Russia
$^2$Department of Physics, Faculty of Liberal Arts, Shinshu University, Matsumoto 390, Japan
$^3$Matsusho Gakuen Junior College, Matsumoto 390-12, Japan

Abstract

The role of Bose-Einstein correlations in a widely discussed intermittency phenomenon is reviewed. In particular, it is shown that particle correlations of different origin are better displayed when analysed as functions of appropriately chosen variables. Correspondingly, if the shape of Bose-Einstein contribution is chosen to be Gaussian in 3-momentum transferred, it provides the power-like law in 4-momentum squared and is smeared out in (pseudo)rapidity.

1 Introductory survey and definitions.

The study of fluctuations and correlations in hadron production at high energies has found considerable interest in recent years. The $q$-particle inclusive densities $\rho_q(p_1, \ldots, p_q)$ or rather the factorial moments $\langle n(n-1)\ldots(n-q+1) \rangle$ estimated in different phase space regions $\delta$ were studied in a variety of reactions ranging from $e^+e^-$ to nucleus-nucleus scattering.

The concept of intermittency has been introduced in order to describe enhanced fluctuations observed for individual events in the density distributions of hadrons (for a recent review see [1]). Originally the definition of intermittency was the strict power laws of the normalized factorial moments taken as a function of bin size [2] at small bin sizes (self-similarity of the
moments). Later on it has become customary by calling intermittency any increase of the factorial moments with decreasing phase space intervals without regard to scaling behaviour. The extended version indicates just positive correlations in momentum space which increase at smaller bins. Such an increase of factorial moments demonstrates that the multiplicity distributions become wider in small phase bins i.e. fluctuations are stronger.

Being important by itself, it shadows a crucial feature of the original proposal, namely, the search for self-similarity in the processes which could be related [3] to the fractal structure of the pattern of particle locations in the three-dimensional phase space. Therefore using the term ”intermittency” in a wide sense here, we still keep in mind that the scaling regime is of primary interest. It gave name to the whole effect when was first proposed in study of turbulence [4].

We shall discuss some possible sources of correlations. Our main concern is to show that they contribute differently when the various projections of multiparticle phase space are considered. The proper choice of the variable can emphasize the particular correlation and, vice versa, conceal the contribution of others. Therefore, by choosing different variables we study the correlations of different origin weighted with different weights.

Various explanations with different physical origin of correlations have been proposed. We shall discuss them in more detail in Section 3, but here let us just mention some of them. The analogy to turbulence has led in the original proposal to the term ”intermittency” and to the phenomenological multiplicative cascade model of the phenomenon. On more strict grounds, it could be related to the parton shower in quantum chromodynamics (QCD) as described in Section 3.5. The observed phenomenon of hadron jets by itself indicates strong positive correlations. Thus, the scaling regime could appear also at the stage of transition from partons (quarks, gluons) to the observable particles (mostly hadrons). Therefore the ideas of phase transition have been elaborated too. Even a more trivial dynamical reason could be connected with abundant production of resonances which, surely, imply the correlation of relative energies of final particles. Some unknown sources of completely new dynamics have been looked for (e.g. stochastic dynamics, instabilities etc.).

Beside those ”dynamical” effects there exists the well-known symmetry property of the interaction which necessarily contributes to the enhancement of correlations in small phase-space volumes. We mean Bose-Einstein corre-
lations due to the symmetrization of the wave functions of identical particles.

In this review, we shall deal mostly with the problem of what and how the Bose-Einstein effect contributes to the observed increase of moments of multiplicity distributions in ever smaller phase space regions when various variables are chosen to tag the phase space bin. Other topics, mentioned above, will be considered to the extent they are necessary for clarification of some issues related to our main purpose.

Bose-Einstein (BE) statistics leads to specific positive correlations of identical bosons (in multiple production processes those are mainly like-charged pions). BE correlations are stronger for particles having smaller difference of three-dimensional momenta. So BE correlations lead to intermittency effect (in the broad sense of the word). Indeed it was first considered in [5], and was asserted a few years ago [6, 7, 8] that BE correlations may be responsible for the intermittency effect. However ineffective methods of the data analysis did not give a possibility to reach convincing conclusions that time.

Early investigation of the intermittency was concentrated on the study of one-dimensional rapidity dependence of the traditional factorial moments in decreasing rapidity intervals. Then it was realized that intermittency effect in multiparticle production, if any, would naturally occur also in two and three dimensions when two or all three components of momenta of outgoing particles are registered.

Furthermore high-order inclusive density $\rho_q(p_1, ..., p_q)$ contains contributions from lower order correlations. Being interested in genuine multiparticle correlations one is led to investigate (connected) correlation functions $C_q(p_1, ..., p_q)$ with lower order correlations subtracted (the correlation function $C_q$ vanishes when a subgroup of $m < q$ particles is statistically independent from the other $q-m$ particles).

The normalized factorial moment estimated in the phase space region $\delta$ is defined as

$$F_q(\delta) = \frac{F_q^{(u)}(\delta)}{\langle n \rangle^q} = \frac{1}{\langle n \rangle^q} \int_\delta dp_1 \ldots \int_\delta dp_q \rho_q(p_1, \ldots, p_q) =$$

$$= \frac{\langle n(n-1)\ldots(n-q+1) \rangle}{\langle n \rangle^q},$$

where $F_q^{(u)}$ are called unnormalized moments. The normalization is chosen so that at integer ranks $q$ one gets $F_q \equiv 1$ for Poisson distribution. In Eq. (1)
$dp_j$ is the differential momentum space volume

$$dp_j = d^3p_j/2E_j(2\pi)^3$$

or any other differential interval of interest. Here $\delta$ will be used as a general notation for a selected phase space volume which differs for different selection procedures. In one-dimensional analysis this is usually the interval of rapidity $\delta y$ or pseudorapidity $\delta \eta$ where rapidity is defined as $y = \frac{1}{2} \ln \frac{E+p_l}{E-p_l}$ and pseudorapidity as $\eta = \frac{1}{2} \ln \frac{p^+}{p^-} = \ln \tan \frac{\theta}{2}$ with $E, p, p_l, \theta$ being the energy, the momentum, the longitudinal momentum and the angle of particle emission.

In practice, an averaging over cells of the original phase space is performed, so that traditional normalized factorial moments in $d$ dimensions are determined as

$$F_q(\delta) = \frac{1}{M_d} \sum_{m=1}^{M_d} F_{q,m}(\delta), \quad d = 1, 2, 3,$$

where the sum runs over $M^d$ $dq$-dimensional boxes having the same size. Self-similarity of the moments $F_q(\delta)$ taken as a function of momentum cell size suggests a power law behaviour

$$F_q(\delta) = \left(\frac{\Delta}{\delta}\right)^{\phi_q} F_q(\Delta),$$

where $\phi_q > 0$ are known as intermittency indices.

General interrelation of the inclusive densities $\rho_q$ and the correlation functions $C_q$ is provided through the inclusive generating functional

$$G[Z] = \sum_{q=0}^{\infty} \frac{1}{q!} \prod_{j=1}^{q} \int dp_j \rho_q(p_1, \ldots, p_q) Z(p_1) \ldots Z(p_q),$$

where $Z(p_j)$ are auxiliary functions. The inclusive densities are then given by differentiation

$$\rho_q(p_1, \ldots, p_q) = \frac{\delta^q G}{\delta Z(p_1) \ldots \delta Z(p_q)}|_{Z=0}.$$

The correlation functions $C_q$ are defined through the generating functional $G$ in the following way (cluster expansion):

$$G[Z] = \exp(\int dp_1 \rho_1(p) Z(p) + \sum_{q=2}^{\infty} \frac{1}{q!} \prod_{j=1}^{q} \int dp_j C_q(p_1, \ldots, p_q) Z(p_1) \ldots Z(p_q))$$

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being "the exponents" of the inclusive densities,
\[ C_q(p_1, \ldots, p_q) = \left. \frac{\delta^q \ln G}{\delta Z(p_1) \ldots \delta Z(p_q)} \right|_{z=0}. \quad (8) \]

A comparison of (5) and (7) leads to relationships
\[ \rho_2(p_1, p_2) = \rho_1(p_1)\rho_1(p_2) + C_2(p_1, p_2), \quad (9) \]
\[ \rho_3(p_1, p_2, p_3) = \rho_1(p_1)\rho_1(p_2)\rho_1(p_3) + \sum_{i \neq j \neq k; i=1}^{3} \rho_1(p_i)C_2(p_j, p_k) + C_3(p_1, p_2, p_3) \quad (10) \]

etc, or the inverse ones:
\[ C_2(p_1, p_2) = \rho_2(p_1, p_2) - \rho_1(p_1)\rho_1(p_2), \quad (11) \]
\[ C_3(p_1, p_2, p_3) = \rho_3(p_1, p_2, p_3) - \sum_{i \neq j \neq k; i=1}^{3} \rho_1(p_i)\rho_2(p_j, p_k) \]
\[ + 2\rho_1(p_1)\rho_1(p_2)\rho_1(p_3) \quad (12) \]

etc. Let us note that the correlation functions have the above mentioned physical interpretation if the number of produced particles exceeds (not clear a priori to what extent) the order of the correlation function.

Integrated (and normalized) correlation functions are known as cumulant moments
\[ K_q(\delta) = \frac{\left. K^{(u)}_q(\delta) \right|_{(n)^q}}{(n)^q} = \frac{1}{(n)^q} \int dp_1 \ldots \int dp_q C_q(p_1, \ldots, p_q). \quad (13) \]
Cumulants of Poisson distribution are identically equal to zero.

The generating function of the moments arises if one puts the functions \( Z(p_j) \) in generating functional (5) to be a constant \( z \),
\[ g(z) = \sum_{q=0}^{\infty} \frac{1}{q!} F_q^{(u)} z^q = \exp(\langle n \rangle z + \sum_{q=2}^{\infty} \frac{1}{q!} K_q^{(u)} z^q) = \sum_{n=0}^{\infty} P_n (1 + z)^n. \quad (14) \]
The moments and particle number distribution \( P_n \) can be found by differentiation of the generating function:
\[ \frac{d^q g(z)}{dz^q} \big|_{z=0} = F_q^{(u)} = \langle n \rangle^q F_q = \langle n(n-1) \ldots (n-q+1) \rangle, \quad (15) \]
\begin{align}
\frac{d^n \ln g(z)}{dz^n} \bigg|_{z=0} &= K_q^{(u)} = \langle n \rangle^q K_q, \\
\frac{d^n g(z)}{dz^n} \bigg|_{z=-1} &= P_n n!.
\end{align}

Relationships between factorial moments and cumulant moments follow from (14) or from (10) and (12):

\begin{align}
F_2 &= 1 + K_2, \\
F_3 &= 1 + 3K_2 + K_3, \\
F_4 &= 1 + 6K_2 + 3K_2^2 + 4K_3 + K_4, \ldots
\end{align}

for a fixed cell in momentum space. An averaging over cells, such as that in (3),

\[ K_q = \frac{1}{M^d} \sum_{m=1}^{M^d} K_{q,m}(\delta) \]

requires the corresponding averaging of Eqs. (20).

Let us return to the intermittency study. The cumulant moments (21), representing genuine multiparticle correlations, suffer from arbitrary binning and low statistics. The accuracy in the measurement of the factorial moments is also unsatisfactory for small bin sizes. That has led to investigation of more general density and correlation integrals which give the possibility to use the available statistics in an optimal way and to perform the correlation analysis in any convenient variable.

The general density integral is defined as

\[ F_{q}^{\Omega}(\delta) = \frac{1}{N_q} \prod_{j=1}^{q} \int dp_j \rho_j(p_1, \ldots, p_q) \Omega_q(\delta; p_1, \ldots, p_q), \]

and the correlation integral as

\[ K_{q}^{\Omega} = \frac{1}{N_q} \prod_{j=1}^{q} \int dp_j C_q(p_1, \ldots, p_q) \Omega_q(\delta; p_1, \ldots, p_q), \]

where the “window function” \( \Omega_q \) is determined to be nonzero in some prescribed interval of the \( q \)-particle phase space. The normalization to uncorrelated background is suggested in (22), (23):

\[ N_q = \prod_{j=1}^{q} \int dp_j \rho_j(p_j) \Omega_q(\delta; p_1, \ldots, p_j). \]
The general definitions (22), (23) give a possibility not to split the $q$-particle phase space in an artificial way (as it was the case in Eqs. (3), (21)) ensuring better statistics. This in turn gives a possibility to measure $F_q^\Omega$ and $K_q^\Omega$ for like-charged and unlike-charged particles separately providing direct evidence for the BE correlations.

2 Experimental survey.

The extensive review of experimental data is given in [1]. Here, we mention some typical findings which help us exemplify our treatment, especially, those which reveal BE-contributions.

2.1 One-dimensional moments (rapidity variable).

Consider first the one-dimensional analysis in rapidity variable. Factorial moments (3) for like-charged particles were measured in $\pi^+p$ and $K^+p$ interactions at 250 GeV/c by NA22 Collaboration [9]. They are characterized by intermittency indices $\phi_q$ according to the fit

$$F_q(\delta y) = a_q(\delta y)^{-\phi_q}. \quad (25)$$

Though the bin size dependence turns out to be far from power-like in the whole interval measured, but it could be fitted by power laws separately at large ($\delta > 1$) and small ($\delta < 1$) bins. We remind that the latter region was suspected for such a law in original proposal (see Table 1 and Fig. 1).

The values of the moments are smaller but the increase with decreasing bin sizes is stronger when a like-charge sample is used instead of all charged sample. To what extent the BE correlations (presumably dominating the correlation of like-sign particles) are responsible for the full intermittency

|                | $\phi_2$       | $\phi_3$       | $\phi_4$       | $\phi_5$       |
|----------------|----------------|----------------|----------------|----------------|
| all charged    | $0.008 \pm 0.002$ | $0.043 \pm 0.006$ | $0.16 \pm 0.02$ | $0.39 \pm 0.06$ |
| negatives only | $0.007 \pm 0.003$ | $0.06 \pm 0.02$  | $0.29 \pm 0.06$ |                |
is not clear from the data in the rapidity variable. Anyhow they are not dominating.

Analogous analysis of the factorial moments was performed in $p\bar{p}$ collisions at $s^{1/2} = 630$ GeV by UA1 Collaboration [10], see Fig. 2. The authors conclude that the BE effect is weak in this variable. (Their early data of intermittency were analysed by means of the negative binomial distribution and the pure birth stochastic theory in Ref.[11].)

### 2.2 Higher-dimensional analysis of the factorial moments.

The intermittency effect is more pronounced in two and especially in three dimensions. The impressive results on the second factorial moment were reported in $\mu N$ interactions at 280 GeV/c by NA9 Collaboration [12]. The moments $F_2$ for unlike-charged and for negative particles were measured in one and three dimensions. The number of boxes is $M$ and $M^3$ correspondingly. The results are given in Fig. 3. The authors claim that the strong intermittency signal observed in $F_2^{--}$ in three dimensions has to be attributed exclusively to BE correlations since such a signal is not present in $F_2^{+-}$. This conclusion is supported by the fact that the Lund model (not containing BE correlations) is in rough agreement with the data for $F_2^{+-}$ but in complete disagreement for $F_2^{--}$. In one (rapidity) dimension the effects are much weaker. Nevertheless a considerable difference between data and the Lund prediction is seen for $F_2^{--}$ in one dimension too whereas for $F_2^{+-}$ the difference is small. This also supports the noticeable role of BE correlations.

Similar results for $F_2(\delta)$ in three dimensions were also presented in $\pi^+p$ and $K^+p$ interactions by NA22 Collaboration [9]. The box volume dependence was fitted with the form

$$K_2(\delta) = F_2(\delta) - 1 = c + a\delta^{-b}, \quad \delta = 1/M^3. \quad (26)$$

A striking difference for unlike- and like-charged pairs was found. While the $(+ -)$ pairs are dominated by long-range correlations (large $c$), these are smaller or absent in the case $(- -)$ and $(+ +)$. Correspondingly, the parameter $a$ is compatible with zero for $(+ -)$, but relatively large for $(- -)$ and $(+ +)$. This again supports BE interpretation of the intermittency in three dimensions.
2.3 Density integrals. Three dimensional analysis.

For higher orders \((q \geq 3)\) the traditional normalized factorial moments have large statistical errors when evaluated for small bin sizes. So nowadays the general correlation analysis is usually performed (see Eqs. (22)-(24)) permitting not to split the phase space interval under consideration. Using the density and correlation integral method, one must introduce the distance \(\delta_{ij}\) for each pair of particles and confine the permissible \(q\)-particle phase space. Usually the ”distance” is defined as four-momentum difference squared, \(\delta_{ij} = Q_{ij}^2 = (p_i - p_j)^2\). The permissible phase space is most commonly confined either by GHP condition [13], \(\delta_{ij} \leq \delta\) for all \(i, j \leq q\) or by so called star topology, \(\delta_{ij} \leq \delta\) for \(j = 2, \ldots, q\), or by snake topology, \(\delta_{j,j+1} < \delta\) for \(j = 1, \ldots, q - 1\).

A comparison of GHP-integral with the conventional normalized factorial moments was performed in \(\pi^+p\) and \(K^+p\) interactions at 250 GeV/c by NA22 Collaboration. The three-dimensional ”distance” between two particles \(i\) and \(j\) was defined using \(y, \varphi\) and \(p_t\) variables in a rather specific way as

\[
d_{ij} = \max(|y_i - y_j|, |\varphi_i - \varphi_j|, |p_{t,i} - p_{t,j}|)^3. \tag{27}
\]

The size \(\delta\) of \(q\)-tuple was defined by the smallest box volume that encloses the \(q\)-tuple. The determination of the density integrals can now be compared with moving around a box in the full phase space under consideration and counting the number of the \(q\)-tuples fitted into the box.

In Fig. 4 conventional moments \(F_2\) and \(F_3\) are compared to the density integral version \(F_{2GHP}^q\) and \(F_{3GHP}^q\). As anticipated, in Fig. 4 one indeed observes that statistical errors in the \(F_{qGHP}^q\) are strongly reduced. This, in principle, allows the analysis to be carried down in much smaller box volumes. It, furthermore, allows a comparison of different charge combinations.

For the second-order integral \(F_{2GHP}^2(\delta)\) the fit (26) was used as it was the case for conventional factorial moments considered in the previous section. The results are shown in Table 2 (we omitted the constant \(b\) when the constant \(a\) was found to be compatible with zero) being qualitatively the same as for ordinary three-dimensional factorial moments.

The above results again support the conclusion that the intermittency in three dimensions is strongly enhanced due to BE correlations.

For the higher order GHP density integrals the modified power law as-
Table 2: Results of fits (26) to the data on $F_{2}^{GHP}$

|         | unlike charged | negatives only | positives only |
|---------|----------------|----------------|----------------|
| $a$     | 0.0006 ± 0.0009 | 0.0115 ± 0.0003 | 0.04 ± 0.02   |
| $b$     | 0               | 0.469 ± 0.004   | 0.37 ± 0.06   |
| $c$     | 0.380 ± 0.006   | 0              | 0.08 ± 0.03   |

Table 3: The slopes $\alpha_{q}/\alpha_{2}$ obtained by fitting (28) to the data

|         | all charged | negatives only | positives only |
|---------|-------------|----------------|----------------|
| $\alpha_{3}/\alpha_{2}$ | 3.81 ± 0.09 | 4.3 ± 0.2 | 5.3 ± 0.2 |
| $\alpha_{4}/\alpha_{2}$ | 8.0 ± 0.03 | 9.4 ± 0.5 | 15.0 ± 0.5 |
| $\alpha_{5}/\alpha_{2}$ | 11.8 ± 0.3 | 13 ± 0.1 | 24 ± 1 |

Assumption

$$\ln F_{q}^{GHP}(\delta) = a_{q} + \frac{\alpha_{q}}{\alpha_{2}} F_{2}(\delta)$$

(28)

can be fitted to the data though (28) only holds approximately. The results are presented in Table 3.

As can be seen from the Table, the growth in decreasing three-dimensional phase-space volumes is faster for higher order density integrals and for like-charged particles.

Let us mention here the analysis [14] done by the same NA22 Collaboration using the opening angle $\theta_{ij}$ between two particles

$$\theta_{ij} = \arccos(\mathbf{p}_{i}\mathbf{p}_{j}/||\mathbf{p}_{i}||||\mathbf{p}_{j}||)$$

(29)

with $\mathbf{p}_{i}$ and $\mathbf{p}_{j}$ being the three-momenta of particles $i$ and $j$. An angular distance measure for more than two particles is defined as the maximal relative angle between all the pairs chosen. Therefore, the numerator of the factorial moment of rank $q$ is determined by counting, for each event, the number of $q$-tuples that have a pairwise angular opening smaller than a given value and then averaging over all events.
The fitted values of intermittency indices appeared to be very low (sometimes negative) and strongly dependent on the production angle of the particles. We argue in Section 3.5 that such an analysis reminds of two-dimensional analysis in relative pseudorapidity and azimuthal angle but with the bin size which depends on the production angle. Therefore it is not very useful for comparison with analytical calculations for parton jets. Nevertheless, it is possible to compare the experimental findings with the FRITIOF Monte Carlo model. It shows that BE correlations should be incorporated into the model to get the agreement.

2.4 \( Q^2 \)-analysis of the density integrals.

NA22 Collaboration has also presented the data on \( Q^2 \)-dependent GHP density integrals, that is the integration over particle densities \( \rho_q \) was confined by the factor

\[
\Omega^GHP_q(Q^2) = \prod_{i<j} \Theta(Q^2 - Q^2_{ij}).
\]  

(30)

The data are shown in Fig. 5 and Table 4 where the parameters of the power law fit

\[
F^GHP_q(Q^2) = a_q(Q^2)^{-\phi_q}
\]  

(31)

are given. One can see that evaluation of the GHP integral for different charge combinations yields effective intermittency indices a factor 1.2 (\( \phi_5 \)) to 1.6 (\( \phi_2 \)) larger for the negatives-only sample than for all-charged sample. This indicates the important role of BE correlations as displayed on \( Q^2 \)-scale though more definitive conclusions can be hardly reached from the data.

Let us note in this connection that \( p_t \)-dependence of the second and third order GHP density integrals taken as a function of \( Q^2 \) appears to be opposite for all-charged and negative-only particles in this experiment, see Table 5.

According to the Table, the intermittency effect is weaker at low \( p_t \) than at high \( p_t \) for negatives. On the contrary, intermittency is stronger when small \( p_t \) particles are selected from all charged. This hampers easy interpretation of the all-charged data only, without proper analysis of the different charge combinations.

The related results on density integrals \( F_2(Q^2) \) and \( F_3(Q^2) \) were presented in \( \mu N \) interaction at 490 GeV/c, using data from the E665 experiment at the
Table 4: Results of fits to the data presented in Fig.5 according to Eq.(31).

|        | all charged | negatives only | positives only |
|--------|-------------|----------------|----------------|
| $a_2$  | 1.219 ± 0.003 | 1.131 ± 0.002 | 1.026 ± 0.002 |
| $\phi_2$ | 0.051 ± 0.001 | 0.081 ± 0.001 | 0.067 ± 0.001 |
| $a_3$  | 1.751 ± 0.007 | 1.38 ± 0.01  | 1.15 ± 0.05   |
| $\phi_3$ | 0.177 ± 0.002 | 0.253 ± 0.004 | 0.227 ± 0.003 |
| $a_4$  | 2.90 ± 0.02  | 1.88 ± 0.02  | 1.41 ± 0.01   |
| $\phi_4$ | 0.358 ± 0.006 | 0.47 ± 0.01  | 0.45 ± 0.01   |
| $a_5$  | 5.45 ± 0.08  | 2.78 ± 0.07  | 1.89 ± 0.04   |
| $\phi_5$ | 0.56 ± 0.01  | 0.66 ± 0.02  | 0.66 ± 0.02   |

Table 5: Intermittency indices for different $p_t$ regions

|                | $p_t < 0.15$ GeV/c | $p_t > 0.15$ GeV/c |
|----------------|--------------------|--------------------|
| all-charged    | $\phi_2$           | 0.046 ± 0.002      | 0.032 ± 0.001      |
|                | $\phi_3$           | 0.136 ± 0.004      | 0.107 ± 0.003      |
| negatives-only | $\phi_2$           | 0.053 ± 0.002      | 0.081 ± 0.001      |
|                | $\phi_3$           | 0.17 ± 0.01        | 0.269 ± 0.006      |
Tevatron of Fermilab [15]. This time the star topology was used to confine the \( q \)-particle phase space, that is the confining factor was taken in the form

\[
\Omega_q(Q^2) = \prod_{j=2}^{q} \Theta(Q^2 - Q_{1j}^2).
\]

(32)

The Fig. 6 shows a log-log plot of the density integrals for different charge combinations. \( F_2(Q^2) \) rises more steeply with \( 1/Q^2 \) for (– – ) than for (+ – ) pairs. The same is true for \( F_3(Q^2) \) for (– – – ) triplets as compared to (ccc) triplets. This different behaviour indicates important role of BE correlations between like-sign particles.

No significant energy \( W \)-dependence of the \( F_2 \) slope was observed for \( Q^2 \geq 0.01 \) GeV\(^2\); for smaller \( Q^2 \) the \( F_2 \) slope of (– – ) pairs seems to be somewhat larger for the high-\( W \) sample. In fact, the slopes \( d \ln F_2/d \ln(1/Q^2) \) of the \( F_2 \) integrals show close agreement for NA9 and E665 experiments [12, 15] in spite of the somewhat different \( \langle W \rangle \) values and the experimental differences of the two experiments.

\( Q^2 \)-analysis of the density integrals was also performed in \( p\bar{p} \) reaction at \( s^{1/2} = 630 \) GeV by UA1 Collaboration [10]. The authors however used quite another definition to confine the permissible \( q \)-particle phase space,

\[
\sum_{j_1 < j_2} q_{j_1,j_2}^2 < Q^2
\]

(33)

proposed in Ref. [16], which is much stronger than Eqs.(30),(32). The results are given in Fig. 7 for all-charged and like-sign particles up to order \( q=5 \). The \( Q^2 \)-dependence of the density integrals is close to linear one in a log-log plot. The corresponding intermittency indices are listed in the Table 6.

The comparison in Fig. 7 and Table 6 shows once again that the slope parameters \( \phi_q \) are bigger for like-sign particles than for all charged particles and the condition

\[
\phi_q(\text{like – sign}) \approx 2\phi_q(\text{all})
\]

(34)

is fulfilled approximately in the \( Q^2 \)-representation with the confinement condition (33). Let us note, that the relationship (34) was suggested in Ref. [7] as an indication on BE origin of the intermittency effect (though it was advocated for rapidity variable in [7]).

At the same time only small differences between different charge combinations are visible in the pseudorapidity analysis of the same UA1 data,
Table 6: The results of fitting of normalized density integrals as functions of \( Q^2 \) to a power law in \( p\bar{p} \) interactions at \( s^{1/2} = 630 \) GeV

| slope parameters     | \( \phi_2 \)         | \( \phi_3 \)         | \( \phi_4 \)         | \( \phi_5 \)         |
|----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| all charged particles| 0.0348 ± 0.0006       | 0.078 ± 0.001        | 0.213 ± 0.004        | 0.338 ± 0.019        |
| like-sign particles  | 0.0522 ± 0.0009       | 0.147 ± 0.001        | 0.443 ± 0.01         | 0.855 ± 0.051        |

see Fig. 2. The Figures 2 and 7 demonstrate that the manifestation of BE correlations is strongly dependent on the variable used.

### 2.5 Second order correlation function.

A detailed study of the second order normalized correlation functions \( C_2(Q^2) \) (sometimes known as differential correlation integral) was undertaken in \( p\bar{p} \) interaction by UA1 Collaboration [10]. Fig. 8b shows a comparison of the samples of the like-sign pairs with unlike-sign pairs and all charged particles. In \( Q^2\)-variable one observes a dominance of unlike-sign pair correlation for \( 0.03 \text{ GeV}^2 \leq Q^2 \leq 1 \text{ GeV}^2 \) which is at least partly due to resonances and particle decay (i.e. there is a broad peak at \( Q^2 \approx 0.5 \text{ GeV}^2 \) which is due to \( \rho \)-meson decays and a peak at \( Q^2 \approx 0.17 \text{ GeV}^2 \) which is due to \( K_S^0 \) decays). However at small \( Q^2 \leq 0.03 \text{ GeV}^2 \) this function is nearly constant. Contrarily, the like-sign particle correlation function is rising very rapidly at very small \( Q^2 \) up to \( Q^2 = 0.001 \text{ GeV}^2 \) exhibiting approximately power-law behaviour.

A comparison with the same analysis in pseudorapidity variable \( \delta \eta \) (Fig. 8a) shows once more the significance of the choice of the proper variable in correlation analysis: the two body correlation function of all charged particles is dominated by unlike-sign particle correlations when analysed in \( \delta \eta \) but dominated by the like-sign correlation function when analysed at very small \( Q^2 \). Similar results on \( C_2(Q^2) \) were reported in \( \pi^+p \) and \( K^+p \) interactions by NA22 Collaboration.

The correlation function \( C_2(Q^2) \) was also investigated in \( e^+e^- \)-annihilation
at 91 GeV ($Z^0$-boson) by DELPHI Collaboration (see [17]). In the range $Q^2 \leq$ 0.03 GeV$^2$ the function $C_2(Q^2)$ of DELPHI ($e^+e^-$) and that of UA1 ($p\bar{p}$) show quite a similar shape, see Fig. 9. Since there is a rise for smaller $Q^2$ only for the same charged pairs, BE correlation is evidently responsible for this behaviour.

However, the $C_2$ of DELPHI is also rising for smaller $Q^2$ in the interval $Q^2 \geq$ 0.03 GeV$^2$ both for same and opposite pairs, where the UA1 data show a comparatively small rise. Therefore in $e^+e^-$ annihilation some other mechanism must be responsible for this power law behaviour which is manifest even in oppositely charged pairs. Jet evolution or hadronization may play the role.

2.6 Correlation integrals.

A further insight into the problem of correlations would be provided by investigation of cumulant moments $K_q(\delta)$ describing true multiparticle correlations of the order $q$ with lower order contributions subtracted (cumulants vanish whenever one of the particles involved is statistically independent of the others). The cumulants relevant to intermittency study are integrals over (connected) correlation functions taken in decreasing phase space volumes. An experimental investigation of the correlation integrals is difficult because they suffer from low statistics. It was performed recently using star topology confinement in correlation integrals and $q_{ij}^2$ as a ”distance” between particles, that is inserting the factor (32) into the correlation integrals.

Results on correlation integral $K_3(Q^2)$ for all charged particles and negatives only in $\mu N$ interactions [15] are shown in Fig. 10a. The behaviour of $K_3(Q^2)$ is approximately power-like in $Q^2$ variable. In order to find the origin of the three-particle correlations in Fig. 10a, $\mu N$ Monte Carlo (MC) events were generated according to the Lund model without BE correlations. The MC predictions for $K_3(Q^2)$ are shown in Fig. 10b. For $(- - -)$ triplets, $K_3(Q^2)$ of the MC events is rather independent of $1/Q^2$ in contrast to the data. This shows that the rise of $K_3(Q^2)$ in the data is very likely due to three-particle BE correlations which were not incorporated into the Lund MC used. For $(ccc)$-triplets the situation is more complicated since in the data both BE correlations (weaker than in $(- - -)$) and resonance decays contribute. In the MC (without BE but with resonance decays) $K_3(Q^2)$ is smaller than in the data but rises due to resonance decays.
Irreducible higher-order correlations were also established up to fifth order in multiparticle production in $\pi^+ p$ and $K^+ p$ collisions at 250 GeV/c by EHS/NA22 Collaboration [18]. The star integral method has provided a clear improvement over the earlier analysis based on the same data. The charge dependence of the correlation was studied in a comparison of like-charged (Fig. 11a) and unlike-charged (Fig. 11b) particle combinations. Both charge combinations show non-zero genuine higher order correlations and an increase of the correlation functions with decreasing interval $Q^2$.

The correlations among unlike-charged combinations (i.e. combinations to which resonances contribute) are relatively strong near $Q^2 \sim 1$ GeV$^2$ but the increase for smaller $Q^2$ is relatively slow. Correlations among like-charged particles are small at $Q^2 \sim 1$ GeV$^2$ but increase rapidly to reach, or even cross, those of the unlike-charged combinations at lower $Q^2$. This difference diminishes with increasing order $q$. The effective slopes of the power-like scaling law for the various charge combinations fitted in the range $0 \leq \ln(1\text{GeV}^2/Q^2) \leq 5$ are given in the Table 7.

One can see that like-charged particles show faster growth of the correlation integrals with decreasing $Q^2$-intervals thus indicating the important role of irreducible higher order BE correlations.

### 2.7 Summary of experiment.

Correlations between different charge combinations in multiple production processes were recently measured in a variety of reactions ($p\bar{p}, \mu N, e^+ e^-$,
\( \pi^+p, K^+p \). The data on two- and many-particle correlations taken as a function of permissible phase space volume were presented (intermittency study). This new development became possible due to investigation of the density and correlation integrals which give a possibility to use the available statistics in an optimal way and introduce any convenient variable.

From this analysis it becomes clear that the increase of correlations with decreasing 3-dimensional phase space volume is essentially due to correlations between like-charged particles. An evident candidate for these like-sign correlations is the BE statistics.

The intermittency effect (in its wide sense) depends strongly on variable chosen. It is poorly seen in rapidity-analysis and much more pronounced in three-dimensional- and \( Q^2 \)-analyses. The last variable is one of the most popular nowadays but it brings its own problems as will be discussed below. As we show, the effect in higher dimensions becomes stronger just due to the trivial decrease of the available phase space (smaller denominator in normalized factorial moments) while the steep rise at smaller \( Q^2 \) is determined by the peaked contribution of BE correlations as exposed in that variable.

3 Correlation studies in different variables.

3.1 Variables and windows.

For a proper interpretation of the intensity and correlation integrals \( F^\Omega_q \) and \( K^\Omega_q \) one has to look more carefully on their structure. In particular it is important to realize what kind of windows they provide.

In general, the density and correlation functions \( \rho_q(p_1, \ldots, p_q) \) and \( C_q(p_1, \ldots, p_q) \) depend on \( 3q \) independent variables, say, \( 3(q-1) \) independent momentum differences \( q_{ij} = p_i - p_j \) and three components of their average momentum \( p = \frac{1}{q} \sum_{j=1}^{q} p_j \). The \( 3(q-1) \) differences must be confined in some or other way to provide an elementary cell in the phase space (the window) and the remaining \( 3 \) variables show the position of the window. The dependence on the cell size is the object of the intermittency study. The dependence on the cell position is a subject of an averaging. The last is necessary to get an acceptable statistics. This averaging is explicit in traditional (vertical) factorial moments (3) and implicit in the density and correlation integrals (22), (23). Let us note that the cell size must be independent of the
order of the moments; otherwise they will not be the (averaged) moments of any distribution.

The choice of appropriate variables is of great importance in correlation study. As it could be seen from the experimental survey the intermittency effect is poorly seen in rapidity variable and it is clearly seen in 3 dimensions and in $Q^2$-variable. In BE correlations one is led to consider the differences of three-momenta of the particles $q_{ij} = p_i - p_j$ as input variables because the strength of BE effect is determined just by these differences. Considering other variables one has to translate the BE correlation from $q_{ij}$ to these variables to compare the effect with the data.

3.2 $Q^2$-confinement.

Let us consider in more details the $Q^2$-variable which is intensively used in the experimental study of the intermittency. The ”distance” between two particles is defined now as

$$q_{ij}^2 = (p_i - p_j)^2 - (E_i - E_j)^2 = q^2 - q_0^2,$$

(35)
and the factor confining permissible phase-space volume contains the corresponding (step-) $\Theta$-function; $\Theta(Q^2 - q_{ij}^2)$. It is not difficult to see that the above $\Theta$-function confines an ellipsoid

$$q_t^2 + \frac{M^2}{E^2} q_l^2 = Q^2,$$

(36)
where $M$ is the invariant mass, $M^2 = Q^2 + 4m^2$, and $E$ is the total energy of the pair. So the confined volume in momentum space is

$$V = \frac{4\pi}{3} Q^3 \frac{E}{M}.$$  

(37)

The longitudinal phase space (along the direction of the total momentum $p_i + p_j$) is rising here with $E$ and $Q$,

$$L_l = \frac{Q E}{M}.$$  

(38)

On the one hand the $E$-dependence of $V$ means that the $Q^2$-confined cells (bins) have different sizes in momentum space being much larger for fast
particles. So with $Q^2$ fixed one already has an averaging over momentum cell sizes. If the correlation functions have the momentum difference as a characteristic scale (as it is the case for BE correlations) then the above feature is rather unpleasant because the intermittency study suggests an investigation of correlations taken as a function of the cell size.

On the other hand the particles in increased momentum intervals may well become uncorrelated and the integral over their correlation function is already saturated at some fixed momentum scale whereas the cell size $L_l$ is still rising with $Q$ due to large $E/M$ factor. In this case the normalized correlation integral will decrease with increasing $Q^2$ (increase with decreasing $Q^2$) approximately according to the power law in some $Q^2$ interval. Just this mechanism was found responsible [19] for the steep rise of the two-particle correlation function of the like-charged particles with decreasing $Q^2$. As a conclusion, $Q^2$ variable is not an appropriate variable for the physical interpretation of the intermittency effect if the correlation between particles has a characteristic scale in momentum difference $q_{ij}^2$, as it is the case for BE correlations (at the same time $Q^2$ being related to invariant mass squared, $Q^2 = M^2 - 4m^2$, is an appropriate variable for correlations arising due to resonance decay). The steep and approximately power law rise of the normalized correlation integrals with decreasing $Q^2$ does not necessarily reflect the corresponding behaviour of the original correlation function $C_q(p_1, \ldots, p_q)$, being a kinematical effect inherent in $Q^2$ variable.

### 3.3 Three-momentum difference confinement.

In general, different mechanisms responsible for particle correlation bring their own natural variables (the variable is ”natural” if the correlation function $C_q$ has a scale parameter related to this variable). If one considers BE correlations then the natural variable is three-momentum difference $q_{ij} = p_i - p_j$ (the energy difference is also involved for nonstationary particle sources and a modification of the variables is necessary for expanding sources). The correlation function $C_q(p_1, \ldots, p_q)$ depends also on total momentum $P = \sum_{j=1}^{q} p_j$ (or on average momentum $p = P/q$).

The essential point is that these two dependences have different momentum scales: the characteristic momentum difference scale $p_d \sim 1/R$ ($R$ is an effective source size for BE correlations) is noticeably smaller than the characteristic transverse and longitudinal momentum scales $p_t$ and $p_l$. This
means that the correlation function is nonzero in a "tube" in $3q$-dimensional space with its axis directed along the line $p_1 = \ldots = p_q$ and with cross-section of the tube of the order $p_d^{3(q-1)}$. The length of the tube is of the order $p_l$ in longitudinal (along the beam direction) and $p_t$ in transverse direction ($p_l \gg p_t$ in experiment).

Let us now confine the 3-momentum differences imposing the condition $|q_{ij}| \leq \delta$ (the window). The result for the normalized correlation integral $K_q$ (see (23)) depends strongly on the relationship of $\delta$ and the scale parameters $p_d \ll p_t \ll p_l$.

a) The region of very small $\delta < p_d$, where the correlation function is maximal, is not still accessible in experiments.

b) In the region $p_d < \delta < p_t$ the integration over momentum differences in $K_q$ is saturated with the scale $p_d$. All corresponding integrations are cut by $\delta$ in the normalization factor $N_q$. As a result, we are left with a dependence of the form

$$K_q = \frac{K_q^{(w)}}{N_q} \sim \left(\frac{p_d^3}{\delta^3}\right)^{q-1}.$$  \hspace{1cm} (39)

c) In the region $p_t < \delta < p_l$ only integrations over longitudinal momenta are cut off and we have a dependence

$$K_q \sim \left(\frac{p_t^3}{p_l^2 \delta}\right)^{q-1}.$$  \hspace{1cm} (40)

d) In the full phase-space ($\delta \sim p_l$) the normalized correlation integral takes its minimal value

$$K_q \sim \left(\frac{p_l^3}{p_t^2 p_l}\right)^{q-1}.$$  \hspace{1cm} (41)

The above rough estimation shows that the normalized correlation integral obeys approximately power-law $\delta$-dependence in some subintervals of the window width $\delta$. The dependence is steeper for small windows (the effective intermittency index in the region b) is three times larger than in the region c)) and for higher order of the moments.

We conclude that the intermittency effect (in the broad sense of the word) is strong and ensures approximately power-like behaviour in three-dimensional analysis in momentum difference variables. No specific physical phenomena were involved to get the above power-law behaviour. It is just
connected with the trivial dependence of moments on the available phase space and does not involve any dynamical "anomalous" dimension as it is the case for QCD jets discussed in the next section. The transition from "$q^3$-regime" in (39) to "$q$-regime" in (40) is also due to the transverse momentum limitation which is an inherent property of multiparticle production. Actually, the dependence on $\delta$ is mostly determined by the "kinematical" normalization factor in the denominator which shows how large is the average multiplicity within the phase space window provided by the confinement condition.

3.4 An illustrative example (BE correlations).

To get somewhat more detailed picture of the intermittency effect induced by BE correlations let us consider the cumulant moments for BE correlations in a simple environment. We suggest that the particles are created by some random gaussian currents [20]. ( A generating functional for BE correlations with chaotic and coherent components is formulated in Ref.[21].) Then the particle densities $\rho_q(p_1, \ldots, p_q)$ and the correlation functions $C_q(p_1, \ldots, p_q)$ can be expressed through a single quantity $F(p_i, p_j)$ which is an averaged current correlator. In particular

$$\rho_1(p) = F(p, p), \quad (42)$$

and the (unnormalized) correlation integral (23) is

$$K_q^{(u)}(\delta) = (q - 1)! \prod_{j=1}^{q} \int dp_j F(p_1, p_2)F(p_2, p_3) \ldots F(p_q, p_1)\Omega_q(\delta; p_1, \ldots, p_q). \quad (43)$$

We represent $F$-function in the form

$$F(p_1, p_2) = [\rho_1(p_1)\rho_1(p_2)]^{1/2}d_{12}(p_1 - p_2), \quad d(0) = 1 \quad (44)$$

suggesting that the function $d_{ij}$ depends only on the momentum difference. Remembering that the correlation scale $p_d$ inherent in $d_{ij}$ is much smaller than characteristic particle momenta $p_t, p_l$, we take the densities $\rho_1(p_j)$ in Eq. (43) in coinciding points to get

$$K_q^{(u)}(\delta) = (q - 1)! \int dp d_{12}^{q-1}(p_{\rho_1}(p)) \prod_{j=1}^{q-1} d(p_j - p_{j+1})d_{12} \ldots d_{q_1}\Omega_q(\delta; p_1, \ldots, p_q) \quad (45)$$
with $q - 1$-fold integration over differences $p_j - p_{j+1}$.

Being interested in qualitative results we may cancel one of $d$-functions in Eq. (45) taking $d_{q1} = d_{q1}(0) = 1$. This leads to inessential numerical misrepresentation of the integral retaining its qualitative behaviour because one of $q$ $d$-functions in (45) does not serve as a direct cut factor. On an equal footing we may neglect a variation of one-particle density $\rho_1(p)$ in the region where the density is substantial, this region being confined by $p_t$ and $p_l$ as above. The momentum difference window may be defined by a confinement of successive momentum differences,

$$\Omega_q(\delta) = \Theta(\delta - q_{12})\Theta(\delta - q_{23})\ldots\Theta(\delta - q_{q-1,q})$$ (46)

(the snake topology). As a result, the correlation integral (45) takes a simple form

$$K_q^{(u)}(\delta) \approx (q - 1)! \int^{(p_t,p_l)} dp_1^q(\int_{\delta} dq d\rho(q))^{q-1}.$$ (47)

The normalization factor $N_q$ in Eq. (23) taken in the same approximation is equivalent to $\langle n \rangle^q$,

$$N_q \approx \int^{(p_t,p_l)} dp_1^q(\int_{\delta} dq) q^{-1} \approx \langle n \rangle^q,$$ (48)

and the correlation integral (47) can be written in the form

$$K_q^{(u)}(\delta) \approx (q - 1)! \langle n \rangle^q k^{1-q}$$ (49)

with

$$k = \frac{\int_{\delta}^{(p_t,p_l)} dp}{\int_{\delta} dp d\rho(p)} \approx \frac{\int_{\delta}^{(p_t,p_l)} dp d^2 p_t}{\int_{\delta}^{(p_t,p_l)} dp d_3 p_l} \geq 1.$$ (50)

The cumulant moments (49) correspond to generating function (14) of the form

$$\ln g(z) = -k \ln(1 - \frac{\langle n \rangle z}{k}).$$ (51)

This is the generating function of the negative binomial distribution (NBD)

$$P_n = \frac{\Gamma(n + k) \Gamma(k)}{\Gamma(n + 1)} \left(1 + \frac{\langle n \rangle}{k}\right)^{-k} \left(1 - \frac{\langle n \rangle}{\langle n \rangle}\right)^{-n}.$$ (52)
(let us note that the linked pair approximation also leads to NBD [22]). The parameter "k" of NBD is given by Eq. (50). It decreases with decreasing of permissible interval δ. In a rough step function approximation it reduces to

\[ k = 1 \text{ for } \delta < p_d, \quad (53) \]
\[ k \approx \delta^3/p_d^3 \text{ for } p_d < \delta < p_t, \quad (54) \]
\[ k \approx p_t^2 \delta/p_d^3 \text{ for } p_t < \delta < p_l, \quad (55) \]
\[ k \approx p_l^2 p_t / p_d^3 \text{ for } \delta \sim p_l \quad (56) \]

in an accordance with estimations (39)-(41) of the normalized correlation integral.

From the above example one can clearly see the role of BE-correlations in the intermittency effect. For very narrow windows (δ → 0, k → 1) we have a geometrical distribution (GD) characteristic for thermal ("totally chaotic") excitation of the particle source. This distribution, having normalized cumulant moments \( K_q = (q - 1)! \) and normalized factorial moments \( F_q = q! \), is the widest (having maximal moments) distribution possible in the present scheme, where BE correlations were estimated for a single fixed scale particle source. To get larger values of moments (for experimental evidence see Fig. 3, Ref. [12]) one has to introduce additional fluctuations or to revise the particle source form.

For wide enough windows (δ → p_l) one has \( k \gg 1 \) leading to the relatively narrow Poisson distribution \( PD = \lim_{k \to \infty} NBD \). This means that BE correlations which we consider here are not effective in large phase space volume. The change of the window width from zero to infinity interpolates between GD and PD and this is a general feature of the present scheme. NBD is one, especially simple, kind of the interpolation and it is not surprising that NBD appears as a possible approximation. More accurate estimations of the correlation integral (43) lead to similar conclusions. Additional coefficients \( \gamma_q \) appear in the cumulant moments (49) and a single parameter \( k \) varies slightly for different orders \( q \), but the qualitative behaviour (49), (50) remains unchanged.

3.5 The (pseudo) rapidity confinement.

Let us turn now to the rapidity analysis which initiated the whole story of intermittency in particle physics. We show that pseudorapidity (which coin-
cides with rapidity for relativistic particles) is the most suitable variable to study correlations providing jet-like structure of high energy events. Moreover, such events give rise to (quasi)intermittent (in strict sense!) behaviour of correlations in this variable.

Really, we will speak about the one-dimensional angular confinement when the bin of a definite size $\delta \theta$ in a polar angle is chosen. The pseudorapidity size $\delta \eta$ is proportional to $\delta \theta$

$$\delta \equiv \delta \eta \approx \delta \theta / \theta$$

with a weight given by a location $\theta$ of the center of the bin that is trivially accounted when averaging over all locations is done. The dimension of the analyzed bin enters the results in a very simple (even trivial) manner as it is shown below.

Let us consider in QCD the correlation within the gluon jet emitted by a quark produced in $e^+e^-$-annihilation [23, 24, 25, 26]. We are interested in correlations among the partons (mostly gluons) created during the evolution of the initial gluon and fitting the pseudorapidity bin $\delta \eta$. They belong to some subjet inside the primary jet which is separated from others due to the angular ordering in QCD. Surely, the prehistory of a jet as a whole is important for the subjet under consideration as is shown in Fig. 12.

Here

1. the primary quark emits the hard gluon with energy $E$ in the direction of the angular interval $\delta$, but not necessarily hitting the window,

2. the emitted gluon produces the jet of partons with parton splitting angles larger than the window size,

3. among those partons there exists such a parton (subjet) with energy $k$ which hits the window,

4. all decay products of that parton subjet cover exactly the bin $\delta$.

This picture dictates the rules of calculation of the $q$-th correlator of the whole jet. One should average the $q$-th correlator of the subjet $F_q^{(u)}$ over all possible ways of its production i.e. convolute it with the inclusive spectra of
such partons $D^{(\delta)}$ in the whole jet and with the probability of creation of the jet $\alpha_s K^G_{\delta}$. Analytically, it is represented by

$$F_q^{(u)}(E_0\delta) \sim \int^{E_0} \frac{dE}{E} \frac{\alpha_s}{2\pi} K^G_{\delta}(E/E_0) \int^E \frac{dk}{k} D^{(\delta)} F_{q}^{(u)}(k\delta),$$  \hspace{1cm} (58)

where $E_0$ is the primary energy, $E$ is the jet energy, and $k$ is the energy of the subjet hitting the window. Since the unnormalized moments increase with energy while the parton spectrum decreases, the product $D^{(\delta)}(k) F_q^{(u)}(k\delta)$ has a maximum at some energy $k_{\text{max}}$, and the integral over momenta may be calculated by the steepest descent method. Leaving aside the details of calculations (see [25]), we describe the general structure of the correlator for the fixed coupling constant $\gamma_0 = (6\alpha_s/\pi)^{1/2} = \text{const}$

$$F_q^{(u)}(\delta) \sim \Delta \Omega(\delta)^{-\gamma_0/q}(\delta)^{q\gamma_0},$$  \hspace{1cm} (59)

where the three factors represent the phase space volume, the energy spectrum factor and the $q$-th power of the average multiplicity. To get the normalized factorial moment $F_q(\delta)$ one should divide (59) by the $q$-th power of that part of the mean multiplicity of the whole jet which appears inside the window $\delta$ i.e. by the share of the total average multiplicity corresponding to the phase space volume $\Delta \Omega$:

$$\Delta n(\delta) \sim \Delta \Omega \langle n \rangle.$$  \hspace{1cm} (60)

If the analysis has been done in the $d$-dimensional space, the phase space volume is proportional to

$$\Delta \Omega \sim \delta^d,$$  \hspace{1cm} (61)

where $\delta$ corresponds to the minimal linear size of the $d$-dimensional window. The last statement stems from the singular behaviour of parton propagators in quantum chromodynamics (see [25]). That is why the factorial moments may be represented as products of the purely kinematical factor depending on the dimension of the analysed space and of the dynamical factor which is not related to the dimension and defined by the coupling constant

$$F_q \sim (\delta)^{-d(q-1)}(\delta)^{(q^2-1)\gamma_0/q}.$$  \hspace{1cm} (62)

At small angular windows $\delta$ the intermittency indices are given by

$$\phi_q = d(q - 1) - \frac{q^2 - 1}{q} \gamma_0.$$  \hspace{1cm} (63)
This formula is only valid for moderately small bins when the condition
\( \alpha_S \ln(\Delta/\delta) < 1 \) is fulfilled. For extremely small windows, one should take
into account that the QCD coupling constant is running. Then the constant
\( \gamma_0 \) should be replaced by the effective value \( \langle \gamma \rangle \) which depends logarithmi-
cally on the width of the window \( \delta \). As a result (see [25]), numerical values of
the intermittency indices for very small bins become noticeably smaller than
in the fixed coupling regime, especially for the low-rank moments. Moreover,
the simple power-law behaviour is modified by the logarithmic correction
terms and the intermittency indices depend on the value of the interval chosen.
The resulting curve of \( \ln F_q(\delta) \) as a function of \( -\ln \delta \) has two branches
and qualitatively reminds those shown in Fig. 1. The rather steep linear in-
crease at the moderately small bins with the slope (63) is replaced at smaller
windows by much slower quasi-linear increase. It is easy to calculate the
location of the transition point to another regime and to show that at higher
values of \( q \) it shifts to smaller bin sizes in accordance with trends in Fig. 1.
We have described the results of the double logarithmic approximation of
QCD. Higher-order terms have been treated in [25]. They do not spoil the
general conclusions providing the corrections of the order of 10 per cents.

It is interesting to note that higher dimension analysis just adds an in-
teger number to the trivial part of the intermittency index and does not
change its non-trivial "anomalous" dimension. Thus the increase of inter-
mittency indices in higher dimension is trivial and has nothing to do with jet
dynamics but is a consequence of phase space factors (mostly in the normal-
ization denominator). The important difference from BE-effect is that the
numerator of the moments provides the non-trivial "anomalous" part of the
intermittency index which is absent in BE-treatment. It is common in all
dimensions.

The above results may be restated in terms of fractals. The power-like
behaviour of factorial moments points out to fractal properties of particles
(partons) distributions in the phase space. According to the general theory of
fractals (see [1] and references therein), the intermittency indices are related
to fractal (Renyi) dimensions \( D(q) \) by the formula
\[
\phi_q = (q - 1)(d - D(q)),
\]
wherefrom one gets (see (63)):
\[
D(q) = \frac{q + 1}{q} \gamma_0 = \gamma_0 + \frac{\gamma_0}{q}.
\]

26
The first term corresponds to monofractal behaviour and is due to the average multiplicity increase. The second one provides multifractal properties and is related to the descent of the energy spectrum as discussed above. It is clearly seen that the fractality in quantum chromodynamics has a purely dynamical origin \( D(q) \sim \gamma_0 \) related to the cascade nature of the process while the kinematical factor in (64) has an integer dimension. The attempts to relate the fractal properties in the momentum space to the fractal structure of colliding objects in the ordinary space were tried also ([27]). The fractality in momentum space can be also formulated as the fractal nature of the subsequent available phase space at each branching of the gluon jet ([28, 29]).

Coming back to our problem we would like to stress that the angular variable is the most convenient one to analyse correlations originated by the jet-like structure. At first sight the opening angle of the jet seems even a more convenient (and "natural") variable. However, it is more difficult to incorporate this angle into above theoretical study than just the usual polar angle \( \theta \). It is easy to show that the relative angle of two partons \( \theta_{12} \) is connected with their polar emission angles \( \theta_1, \theta_2 \), and with their relative distances in pseudorapidity \( \eta_{12} \) and azimuthal angle \( \varphi_{12} \) by the formula

\[
\theta_{12}^2 \approx \theta_1 \theta_2 (\eta_{12}^2 + \varphi_{12}^2)
\]

at \( \theta_1 \ll 1, \theta_2 \ll 1 \). Therefore the analysis in the relative angle variable corresponds to two-dimensional analysis in \( \eta_{12} \) and \( \varphi_{12} \) when the size of the two-dimensional interval depends on polar angles of emitted partons and should be larger at small polar angles. It would produce higher average multiplicities in the denominator of factorial moments and suppress their values at small relative angles what is observed in experiment [14].

It exemplifies our statement about "natural" variables for each mechanism responsible for correlations in multiparticle production.

4 Discussion and conclusions.

The very first intermittency studies were aimed at a scaling law in behaviour of factorial moments with hope to find out new collective effects in high energy interactions. Later it was recognized that it is just a part of the day-to-day work on correlation properties of multiparticle production that does
not diminish the importance of the above problem but helps also to disen-
tangle contributions of different known mechanisms to particle correlations.
According to our present-day theoretical prejudices we can name at least four
of them. At the initial stage the quark-gluon jets appear. If described by
the perturbative QCD they should give rise to the (quasi) intermittent be-
haviour of factorial moments as functions of the (pseudo)rapidity bin size. At
a simplified level of the hypothesis on local parton-hadron duality it should
be valid for final hadrons as well. However, the transition from partons to
hadrons could be not as simple as that, and it is sometimes considered as a
phase transition imposing its own critical indices. In addition to it, the final
stage interactions giving rise to known resonant states, surely, play impor-
tant role. The final stage of creating hadrons asks for symmetry properties
such as Bose-Einstein symmetrization to be respected. If one is interested in
looking for new non-traditional sources of correlations (stochasticity, insta-
bilities etc.) one should, first, to show that they contribute to correlations
differently compared to considered "traditional" sources.

As we mentioned above, each of them should be described in its own
natural variable connected to its characteristic scale. We tried to show
that resonances bring with them the mass scale of squared 4-momenta, BE-
symmetrization is better revealed in 3-dimensional momentum analysis, while
the jet-like structure asks for angular (or (pseudo)rapidity) variable with a
scale determined by the corresponding "length" of the shower related to the
(running) coupling constant.

When analysed in "unnatural" variables, these mechanisms can produce
the dependences which are not typical for them and mask (or imitate) some
other effects due to the determinant of the transformation. It happens,
for example, with BE-contribution when looked in $Q^2$-variable. It becomes
strongly peaked at small $Q^2$ and imitates power-like law. One should not
attempt to fit it by "traditional" Gaussian dependence when looked in that
variable as well as one should not claim that it produces intermittency in a
strict sense.

Unfortunately, that example demonstrates also, that the contribution of
some mechanism in its "unnatural" variable is not necessarily smeared out
but, on the contrary, can produce rather steep dependence provoking mis-
leading conclusions.

The choice of (pseudo)rapidity as a "natural" variable in the original
paper [2] was just related to traditions in theoretical approach and to exper-
imental facilities. It appeared to be natural for jets but not for resonances and BE-correlations which seem to be smeared out in that variable. Separate study of the quantitative contributions of different mechanisms plotted as functions of the same variable asks for Monte-Carlo calculations. However, main qualitative ingredients are seen from above analytical approaches.

In our opinion, the strong “kinematical” phase space dependence provided by the denominator of the normalized moments spoils the analysis introducing strongly increasing (at small $\delta$) factor depending on the dimension of the analysed bins. To unify the intermittency indices it looks appropriate to leave just $\langle n \rangle$ in the denominator of moments that cancels all kinematical factors at the expense of introducing energy dependence.

To conclude, we have shown that different effects are better displayed if their ”natural” variables are chosen. Some proposals are discussed above, but that study is just at the very initial stage and we call for further elaborated criteria beside those considered in the present paper. What concerns the title of the paper, we can say that Bose-Einstein correlations do contribute to intermittency in a wide sense. However, intermittency in its initial meaning exists even for unlike-charged particles and should be ascribed (probably) to jet-like parton cascades but not to Bose-Einstein correlations.

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Figure captions

Fig. 1. Factorial moments of order $q = 2, 3, 4$ for the all charged sample, and the restriction to the positive-only and negative-only samples in $\pi^+p$ and $K^+p$ interactions [9].

Fig. 2. The rise of the factorial moments and density integrals with decreasing $\delta \eta$ in $p\bar{p}$ collisions [10].

Fig. 3. Log-log plot of the second factorial moment $F_2$ in one dimension $(a, b)$ and three dimensions $(c, d)$ for unlike $(a, c)$ and negative $(b, d)$ charges. The full dots show the data, the open circles are Lund predictions [12].

Fig. 4. Comparison of a), b) conventional factorial moments $F_2$ and $F_3$ in three dimensions to c), d) $F_2^{GHP}$ and $F_3^{GHP}$ obtained from density integral method in GHP-topology. Solid lines in a) and c) correspond to fits according to (25) [9].

Fig. 5. The $\ln F_q^{GHP}$ as function of $-\ln(Q^2/1\text{GeV}^2)$. Note that the abscissa value 0.65 corresponds to the peak of $\rho$-meson and 1.77 is the value corresponding to the $K^0_S$ mass [9].

Fig. 6. Log-log plot of a) $F_2(Q^2)$ for $(-\cdots)$ and $(+-\cdots)$ pairs and $F_3(Q^2)$ for $(-\cdots\cdots)$ and $(ccc)$ triplets vs $1/Q^2$ [15].

Fig. 7. The rise of the density integrals with decreasing $Q^2$ in $p\bar{p}$ collisions [10].

Fig. 8. The normalized two-body correlation function for different charge combinations a) as a function of $\delta \eta$, b) as a function of $Q^2$ [10].

Fig. 9. A comparison of $p\bar{p}$ collider data (UA1) with $e^+e^-$ at the $Z^0$ pole (DELPHI) for the second order correlation function [17].

Fig. 10. Log-log plot of $K_3(Q^2)$ for $(--\cdots)$ and $(ccc)$ triplets a) from data and b) from the Lund Monte Carlo program including resonances but without BE correlations [15].

Fig. 11. $\ln K_q(Q^2)$ as a function of $-\ln Q^2$ a) for like-charged and b) for unlike-charged particle combinations, compared to the expectations from FRITIOF model [18].

Fig. 12. The subjet hitting the window $\delta \equiv \theta$ originates from a parton which appeared in the evolution of the primary gluon emitted by a parent quark.
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