Ribbons characterize magnetohydrodynamic magnetic fields better than lines: a lesson from dynamo theory

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ABSTRACT

Blackman and Brandenburg argued that magnetic helicity conservation in dynamo theory can in principle be captured by diagrams of mean field dynamos when the magnetic fields are represented by ribbons or tubes, but not by lines. Here, we present such a schematic ribbon diagram for the α² dynamo that tracks magnetic helicity and provides distinct scales of large-scale magnetic helicity, small-scale magnetic helicity, and kinetic helicity involved in the process. This also motivates our construction of a new ‘2.5 scale’ minimalist generalization of the helicity-evolving equations for the α² dynamo that separately allows for these three distinct length-scales while keeping only two dynamical equations. We solve these equations and, as in previous studies, find that the large-scale field first grows at a rate independent of the magnetic Reynolds number $R_M$ before quenching to an $R_M$-dependent regime. But we also show that the larger the ratio of the wavenumber where the small-scale current helicity resides to that of the forcing scale, the earlier the non-linear dynamo quenching occurs, and the weaker the large-scale field is at the turnoff from linear growth. The harmony between the theory and the schematic diagram exemplifies a general lesson that magnetic fields in magnetohydrodynamic are better visualized as two-dimensional ribbons (or pairs of lines) rather than single lines.

Key words: accretion, accretion discs – dynamo – MHD – stars: magnetic field – galaxies: magnetic fields.

1 INTRODUCTION

Dynamo theory has long been a topic of active research in astrophysical and geophysical magnetohydrodynamics (MHD), and describes how magnetic energy can be amplified and/or sustained in the presence of a turbulent diffusion that would otherwise rapidly dissipate the field (e.g. Glatzmaier 2002; Brandenburg & Subramanian 2005a; Shukurov 2005; Kulsrud & Zweibel 2008; Charbonneau 2013; Blackman 2014). In particular, large-scale dynamo (LSD) theory in astrophysics is aimed at understanding the in situ physics of magnetic field growth, saturation, and sustenance on time or spatial scales large compared to the turbulent scales of the host rotator. Galaxies, stars, planets, compact objects, and accretion engines (via their jets) often show direct or indirect evidence for large-scale magnetic fields. The extent to which magnetic fields are a fossil vestige of the formation of the object, or primarily generated via LSDs, is itself a question of active research, but observations of the Sun for example (e.g. Schrijver & Zwaan 2000; Wang & Sheeley 2003) prove that LSDs do operate in nature since a frozen-in field could not exhibit sign reversals. LSDs are also commonly seen in accretion disc or shearing box simulations (e.g. Brandenburg et al. 1995; Lesur & Ogilvie 2008; Davis, Stone & Pessah 2010; Gressel 2010; Guan & Gammie 2011; Simon, Hawley & Beckwith 2011; Sorathia et al. 2012; Ebrahimi & Bhattacharjee 2014; Suzuki & Inutsuka 2013), with observed large-scale field reversals occurring on time-scales of the order of 10 orbits.

The presence of MHD turbulence makes the study of LSDs highly non-linear, exacerbating the importance of numerical simulations. However, large-scale spatial symmetry and the slow evolution of large-scale fields compared to turbulent fluctuation time-scales motivate a mean field approach in which statistical, spatial, or temporal averages are taken, and the evolution of the mean field studied (Moffatt 1978; Parker 1979; Krause & Rädler 1980). For many decades, mean field dynamo theory was studied assuming a prescribed flow that was not affected by the growing magnetic field. This is a problem because the field does exert Lorentz forces back on the flow. Moreover, unless the back reaction is incorporated into the theory, it is impossible to make a dynamical prediction of LSD saturation. The 21st century has brought progress in this endeavour as the growth and saturation of LSDs seen in simple closed box simulations (Brandenburg 2001) are reasonably matched by newer mean field dynamo theories that follow the time-dependent dynamical evolution of magnetic helicity (e.g. Blackman & Field 2002, for reviews see Brandenburg & Subramanian 2005a; Blackman 2014). These newer 21st century mean field dynamos are discrete scale

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theories that capture the basic principles of the inverse transfer of the magnetic helicity first derived in the spectral model of Pouquet, Frisch & Leorat (1976), which revealed the fundamental importance of magnetic helicity evolution for large-scale field growth.

The simplest example of a mean field dynamo for which this saturation theory has been tested dynamically (Blackman & Field 2002; Field & Blackman 2002) and compared to simulations (Brandenburg 2001) is the $\alpha^2$ dynamo in a closed or periodic box. In this dynamo, the system is forced with kinetic helicity in a closed or periodic system at some wavenumber, say $k_1 = 5$ where the box wavenumber is $k_1 = 1$. The large-scale field growth at $k_1$ is captured by an equation which is coupled to the equation for small-scale magnetic helicity because the sum of small- and large-scale magnetic helicity is conserved up to resistive terms. The growth of large-scale helicity thus also implies the growth of small-scale helicity of the opposite sign. Because the growth driver turns out to be the difference between small-scale kinetic and small-scale current helicities (the latter related to the small-scale magnetic helicity), the growth of the small-scale magnetic helicity offsets the kinetic helicity driver, eventually halting the growth. While $\alpha^2$ dynamos are simple theoretical models for study, they are also important for the generation of large-scale fields in stars that lack strong differential rotation.

Blackman & Brandenburg (2003) argued that textbook diagrams of mean field dynamos do not properly capture the near conservation of magnetic helicity because they display a growth of large-scale magnetic helicity without any corresponding oppositely signed small-scale helicity to compensate. If instead the field is displayed as a ribbon or tube, any writhe of the large-scale field is naturally accompanied by a corresponding twisting of the small-scale field. On a related theme, Pfister & Gekelman (1991) showed that magnetic helicity conservation during a magnetic reconnection of linked loops can be captured when the field lines are represented as ribbons but not as lines. See also Bellan (2000) in this context.

In this paper, we revisit the $\alpha^2$ dynamo to show more specifically how to visually represent its large-scale and small-scale magnetic helicity growth. The resulting diagram has also stimulated us to introduce a minimalist generalization of the helicity-evolving $\alpha^2$ dynamo equations which allows for scale separation between the large-scale, forcing scale, and scale of the small-scale current helicity, whilst keeping only two dynamical equations. Normally, the latter two scales are taken to be the same, but there is evidence from simulations that they can be different (Park & Blackman 2012).

In Section 2, we discuss the different conceptual representations of magnetic helicity. In Section 3, we discuss the schematic diagram of the $\alpha^2$ dynamo. In Section 4, we derive the equations for the $\alpha^2$ dynamo based on previous work but with the new scale separation feature. We solve these equations in Section 5 and discuss their solutions and the correspondence with the schematic diagram. We conclude in Section 6.

2 MAGNETIC HELICITY AS A MEASURE OF LINKAGE, TWIST, OR WRITHE

Magnetic helicity is defined as the volume integral of the dot product of the vector potential $A$ and the magnetic field $B = \nabla \times A$, i.e. $\int A \cdot B \, dV$. This is a measure of magnetic linkage (e.g. Moffatt 1978; Berger & Field 1984). Consider two linked flux tubes (or ribbons whose flux is contained within an imaginary tube) of cross-sectional area vectors $dS_1$ and $dS_2$, respectively. Then $\Phi_1 = \int B_1 \cdot dS_1$ is the magnetic flux in tube 1, where $B_1$ is its magnetic field. Similarly, the flux of the second tube is $\Phi_2 = \int B_2 \cdot dS_2$, where $B_2$ is its magnetic field. If the fields in each tube are of constant magnitude and parallel to $dS_1$ and $dS_2$, respectively, we can write the magnetic helicity as the sum of contributions from each of the two flux tube volumes to obtain

$$\int A \cdot B \, dV = \iint A_1 \cdot B_1 \, dS_1 + \iint A_2 \cdot B_2 \, dS_2,$$

where we have factored the volume integrals into products of line and surface integrals, with the line integrals taken along the direction parallel to $B_1$ and $B_2$. Since the magnitudes of $B_1$ and $B_2$ are constant in the tubes, we can pull $B_1$ and $B_2$ out of each of the two line integrals on the right of equation (1) to write

$$\int A \cdot B \, dV = \int A_1 \, dl_1 \int B_1 \, dS_1 + \int A_2 \, dl_2 \int B_2 \, dS_2 = \Phi_1 \Phi_2 + \Phi_1 \Phi_2 = 2 \Phi_1 \Phi_2,$$

where we used Gauss’ theorem to replace $\int A_1 \cdot dl_1 = \Phi_1$, the magnetic flux of tube 2 that is linked through tube 1, and similarly $\int A_2 \cdot dl_2 = \Phi_2$. If the tubes are not linked, then the line integral would vanish and there would be no magnetic helicity.

Helicity can also be characterized as a measure of magnetic ‘twist’ and ‘writhe’. While a more extensive discussion of Figs 1 and 2 will come later, note that the schematic in the top left of Fig. 2 shows a single closed magnetic ribbon with zero net helicity. In evolving the loop from the top left to the top right, four right-handed writhe have been introduced. Each of these writhe can be perceived as a loop through which a rigid pole can be threaded along the direction of the electromotive force (EMF; as shown in the second panels of Figs 1 and 2) and around which the ribbons winds in the right-handed sense. However, no dissipation was involved in the evolution, and the overall ribbon remains closed. Under these conditions, magnetic helicity is conserved in MHD, so one left-handed twist along each of the writhe loops appears as a consequence of magnetic helicity conservation: one unit of writhe has the same amount of magnetic helicity as one full twist of the ribbon.

Having shown that linkage of flux is a measure of helicity, and argued that writhe and twist are equivalent, it remains to establish the correspondence between linkage and either writhe or twist. Seeing the equivalence between linkage and twist is wonderfully aided by the use of a strip of paper to represent a magnetic flux ribbon, along with scissors and tape (e.g. Bellan 2000; Blackman 2014). Give a long strip of paper a full 360° right-handed twist around its long axis (by twisting clockwise at the top with your right hand while holding the bottom with your left hand) and fasten the ends so that it is now a twisted closed loop. Now, consider that this ribbon could have been composed of two adjacent ribbons pressed together side by side. Separating these adjacent ribbons is achieved using scissors. Cut along the centre line of the strip all the way around. The result is two linked ribbons, each of 1/2 the width of the original and each with one right-handed twist. The amount of helicity of the system remains conserved but has been transformed into a different form: if the original unseparated ribbon had a magnetic flux $\Phi$, then each of the half ribbons has flux $\Phi/2$. From the above discussion of linkage, the linkage of these two new ribbons gives a magnetic helicity $2\Phi^2/4 = \Phi^2/2$. But the original uncut ribbon had a single right-handed twist with total flux $\Phi$. For the helicity of the initial twisted ribbon to equal that of the two linked twisted ribbons, any twisted ribbon must contribute a helicity equal to its flux squared. Helicity is thus conserved: $\Phi^2$ is the helicity associated with the initial twisted ribbon. This equals the sum of helicity from the linkage of the two half-thickness ribbons $\Phi^2/2$ and that from the
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Figure 1. Two-stage schematic for magnetic field structure in an $\alpha^2$ dynamo driven by negative kinetic helicity in the conventional 20th century approach with the magnetic field represented as lines. The first panel shows a large untwisted toroidal magnetic field ribbon. The second panel shows the action of kinetic helicity on the initial ribbon. The small-scale negative kinetic helicity produces each of the four small-scale poloidal loops. Each small loop incurs a writhe (or overlap) of positive (=right-handed) magnetic helicity. The direction of the mean EMF is shown in the second panel. It is parallel to the mean poloidal current and highlights that both top and bottom loops have right-handed writhe. In the third panel, the two intermediate-scale poloidal loops encircling the small-scale loops represent the resultant mean poloidal field averaged separately over each pair of loops. These intermediate-scale loops are linked to the initial large-scale toroidal loop. Since a single linked pair of ribbons has two units of magnetic helicity, we see that the two poloidal loops linking the toroidal loop have a total of four units of right-handed magnetic helicity. This helicity in the ‘large-scale field’ has come from zero initial magnetic helicity, and thus cannot be correct for MHD at large $R_M$ which conserves magnetic helicity. Therefore, this diagram does not account for the missing small-scale magnetic helicity of opposite sign (compare to Fig. 2).

right-handed twists in each of these two ribbons $2 \times (\Phi/2)^2 = \Phi^2/2$.

To summarize: one unit of twist helicity for a ribbon or tube of magnetic flux $\Phi$ is equal to one unit of writhe helicity for the same ribbon, and both are separately equal to 1/2 of the helicity resulting from the linkage of two untwisted flux tubes of flux $\Phi$. These three different, but equivalent ways of thinking about magnetic helicity are instructive for extracting the physical ideas in what follows.

3 DIAGRAMMATRIC DESCRIPTION OF AN $\alpha^2$ DYNAMO

Traditional 20th century textbook dynamos (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Ruzmaikin, Sokolov & Shukurov 1988) do not conserve magnetic helicity and this has rendered them unable to predict how dynamos saturate. Their deficiency in this regard is conspicuous even in their diagrammatic representations when the large-scale magnetic field is represented as a 1D line. The basic notion that ribbons better represent the magnetic field for LSDs has been emphasized before (Blackman & Brandenburg 2003), but here we incorporate this principle more specifically into a full description and schematic of the $\alpha^2$ dynamo. The absence of differential rotation makes the $\alpha^2$ dynamo even simpler than the common $\alpha - \Omega$ dynamo, but it is still very much sufficient to illustrate the essential principles.

Fig. 1 shows the amplification of large-scale magnetic fields by the $\alpha^2$ dynamo based on the traditional representation of the field as a 1D line. From a single initial toroidal loop, four small-scale helical eddies, each with typical velocity $\mathbf{v}$ and kinetic helicity $\mathbf{v} \cdot (\nabla \times \mathbf{v}) < 0$, create four small-scale poloidal loops with right-handed writhe. The result is a toroidal EMF $\mathbf{E}_g = (\mathbf{v} \cdot \mathbf{b})$ of the same sign inside all of the loops, regardless of whether they are above or below the initial toroidal loop. The vectors in the small-scale loops are shown explicitly just for two loops to avoid clutter, and the resultant EMF vector is shown. The result is two net large-scale poloidal magnetic field loops (shown in blue) in the third panel. But since the magnetic helicity is a measure of linkage, there is a problem: the first panel has only a single red
Figure 2. Correction of Fig. 1 to include magnetic helicity conservation. The top panel shows a large untwisted toroidal magnetic field ribbon. The bottom shows the action of kinetic helicity on the initial ribbon. The small-scale negative kinetic helicity produces each of the four small-scale poloidal loops. The triangular arrows indicate the magnetic field direction. Each small loop incurs a writhe (or overlap) of positive (=right-handed) magnetic helicity. Since magnetic helicity is conserved, each of these four loops also has a negative (=left-handed) twist along the field ribbon. The two intermediate-scale poloidal loops encircling the small-scale loops represent the resultant mean poloidal field averaged separately over each pair of loops. The intermediate-scale poloidal loops have accumulated two units of magnetic twist, one from each of the small loops that they encircle. These intermediate-scale poloidal loops are also linked to the initial large-scale toroidal loop. Since a single linked pair of ribbons has two units of magnetic helicity, we see that the two poloidal loops linking the toroidal loop have a total of four units of right-handed magnetic helicity in linkage which exactly balances the sum of 2+2 left-handed units of twists on these poloidal loops. In general, the small scale of the twists need not correspond to the same scale as the velocity driving the small-scale writhes, though the calculations of Section 4 assume such for simplicity. As in Fig. 1, the poles threading the loops in the second panel are parallel to the EMF and the mean poloidal current, and serve as a visual tool to clarify that both top and bottom loops have right-handed writhe. The three white lines in the bottom panel indicate the approximate relevant length-scales from smallest to largest $k^{-1}_2$ (ribbon width), $k^{-1}_1$ (inner loop size), and $k^{-1}_f$ (outer loop size), respectively, which appear in equations (21)–(23).

The solution to this conundrum is shown in Fig. 2. The analogous diagrams as in Fig. 1 are shown with the field represented by ribbons instead of lines. Conservation of helicity is now maintained: the second panel shows how the right-handed writhe of the small poloidal loops is compensated by the opposite (left-handed) twist.
helicity along the loops. The linkage that results in the third panel is in turn compensated by the exact opposite amount of small-scale twist helicity along the loops. Specifically, the intermediate-scale poloidal loops have acquired two units of magnetic twist, one from each of the small loops that they encircle. These intermediate-scale loops are also linked to the initial large-scale toroidal loop. Since a single linked pair of ribbons has two units of magnetic helicity, we see that the two poloidal loops linking the toroidal loop have a total of four units of right-handed linkage magnetic helicity which exactly balances the sum of 2+2 left-handed units of twist on these poloidal loops. The comparison of Figs 1 and 2 is also harmonious with the comparison of figs 8 and 9 of Pfister & Gekelman (1991) showing the non-conservation of magnetic helicity of reconnecting loops when they are treated as lines and the conservation when they are treated as ribbons.

Visualizing the field as a 2D ribbon rather than a 1D line also illustrates the conceptual key to understanding non-linear quenching of the $\alpha^2$ dynamo. In a system driven by kinetic helicity, the back reaction on the driving flow comes from the buildup of the small-scale twist. Field lines with small-scale twists become harder to bend because the small-scale current $j$ associated with the twist leads to a Lorentz force $j \times \overrightarrow{B}$, which pushes back against the small-scale driving flow. Only the helical part of the small-scale field enters the back reaction for flows which have at most weak anisotropy. This fact is consistent with the equations discussed in the following section, the original spectral approach of Bouquet et al. (1976), and numerical simulations (Brandenburg 2001; Park & Blackman 2012). As discussed further in Section 5, once the small-scale helicity builds up significantly, the $\alpha^2$ dynamo growth rate slows to the point where the near-exact conservation of magnetic helicity is violated and resistive terms become important. But, it is the helicity conserving phase which causes the back reaction that leads to the transition.

Much recent work on astrophysical dynamos has been focused on how to sustain the EMF such that the quenching from a buildup of small-scale helicity is avoided and the EMF is instead sustained by global or local helicity fluxes (the distinction is reviewed in Blackman 2014). In Fig. 2, global helicity fluxes would mean removing the constraint that the ribbon is closed and opening the boundaries over the region where the EMF is calculated to allow fluxes to remove the small-scale twists. Blackman & Brandenburg (2003) showed a schematic of how this might work in the context of coronal mass ejections of the Sun, with the flux ejection both being important for sustaining a fast dynamo and also providing energy for particle acceleration in the corona. Opening up the boundaries to allow the small-scale twist (as opposed to large-scale writhe) to preferentially leak away allows larger saturated field strengths and a longer regime of $\alpha\Omega$-independent growth for the $\alpha^2$ dynamo (Blackman 2003b). For the $\alpha = \Omega$ dynamo that includes magnetic helicity dynamics, the role of global or local helicity fluxes is particularly germane because the large-scale field can decay catastrophically in their absence in some models (e.g. Shukurov et al. 2006; Sur, Shukurov & Subramanian 2007) after an initial transient growth phase. See also Hubbard & Brandenburg (2012) for a discussion of possibly less catastrophic decay even without global fluxes and Ebrahimi & Bhattacharjee (2014) for direct numerical evidence of local fluxes sustaining an LSD in sheared system. The role of helicity fluxes as sustainers of the EMF as a matter of principle (e.g. Blackman & Field 2000) and specifically for sheared rotators (Vishniac & Cho 2001; Brandenburg & Subramanian 2005a; Shukurov et al. 2006; Vishniac 2009; Hubbard & Brandenburg 2011; Käpylä & Korpi 2011; Ebrahimi & Bhattacharjee 2014; Vishniac & Shapoval 2014) is a topic of active research. Note that the role of helicity fluxes in sustaining laboratory plasma dynamos in fusion devices has long been studied (e.g. Strauss 1985, 1986; Bhattacharjee & Hamerl 1986).

Fig. 2 also highlights that the scale of the twists along the ribbon can be different from the scale of the turbulent forcing, the latter of which may be better represented by the radius of the writhe loop. The reason for such scale separation in a real system would likely depend on the particular circumstance, such as the ratio of vertical to radial density gradient scales. However to give a sense of how such scale separation might arise even for an isotropic (but reflection asymmetric) system, recall that the diagram is intended to capture helical forcing with $(\nabla \cdot \overrightarrow{\omega}) < 0$, where $\overrightarrow{\omega} \equiv \nabla \times \overrightarrow{v}$ is the eddy vorticity. The velocity gradient scale and thus the scale of kinetic helicity would be that of the eddy itself. But depending on the vorticity profile within the eddy, magnetic flux near the eddy core can be dissipated via the large field gradients there and preferentially amplified by shear towards the eddy’s edge (Weiss 1966). Neighbouring eddies would have a similar effect and could overall drive the system into a kind of sponge with the magnetic field filling fraction potentially as small as the ratio of the average magnetic pressure to the average thermal pressure (Blackman 1996). Once a vortex has partly expelled magnetic flux into a thinner structure (i.e. ribbon), two scales in the magnetic structure now emerge as a result of the additional component of velocity parallel to the vorticity. First, the writh of the ribbon would be determined by the overall size of the eddy (the ‘hole’ where there is less magnetic flux). Secondly, this velocity component will ‘roll’ the ribbon as it writhes, twisting it with helicity sign opposite to the writh. Since the ribbon would be thin compared to the eddy scale, the resulting twist would be of a smaller scale than the overall eddy (forcing) and writh scales.

### 4 Equations of the $\alpha^2$ Dynamo

To make the connection between the diagram of the $\alpha^2$ dynamo of Fig. 2 and the dynamo equations, we follow standard derivations of the mean field equations for the $\alpha^2$ dynamo that incorporate magnetic helicity conservation (closely following Blackman & Subramanian 2013), but deviate where noted below, to allow a scale separation between the forcing scale and the small-scale current helicity buildup scale.

The electric field is

$$E = -\nabla \Phi - \frac{1}{c} \partial_t \overrightarrow{A}, \quad (3)$$

where $\Phi$ is the scalar potential. Taking an average (spatial, temporal, or ensemble) and denoting averaged values by the overbar, we have

$$\overline{E} = -\nabla \overline{\Phi} - \frac{1}{c} \partial_t \overline{\overrightarrow{A}}. \quad (4)$$

Subtracting equation (4) from equation (3) gives the equation for the fluctuating electric field

$$e = -\nabla \phi - \frac{1}{c} \partial_t \overrightarrow{a}, \quad (5)$$

where $\phi$ and $a$ are the fluctuating scalar and vector potentials. Using $\overline{\nabla} \cdot \overrightarrow{\omega} = \omega \cdot (\overline{\overrightarrow{A}} \times \overline{\overrightarrow{E}})$, where the latter two terms result from Maxwell’s equation $\partial_t \overrightarrow{B} = -c \nabla \times \overrightarrow{E}$, and the identity $\overline{\overrightarrow{A}} \cdot \overrightarrow{E} = \overline{\overrightarrow{E}} \cdot \overrightarrow{B} = \overline{\overrightarrow{B}} \cdot \nabla \cdot (\overline{\overrightarrow{A}} \times \overline{\overrightarrow{E}})$, we take the dot product of equation (4) with $\overrightarrow{B}$ and obtain

$$\partial_t (\overline{\overrightarrow{A}} \cdot \overrightarrow{B}) = -2c \overline{\overrightarrow{E}} \cdot \overrightarrow{B} - \nabla \cdot (c \overline{\overrightarrow{B}} + c \overline{\overrightarrow{E}} \times \overline{\overrightarrow{A}}). \quad (6)$$
Similarly, by dotting equation (5) with \( \mathbf{b} \), the evolution of the mean helicity density associated with fluctuating fields is
\[
\partial_t \mathbf{a} \cdot \mathbf{b} = -2\varepsilon \mathbf{e} \cdot \mathbf{b} - \nabla \cdot (\mathbf{c} \phi \mathbf{b} + \mathbf{c} \times \mathbf{a}).
\]
To eliminate the electric fields from equations (6) and (7), we use Ohm’s law with a resistive term to obtain
\[
\mathbf{E} = -\mathbf{V} \times \mathbf{B}/c + \eta \mathbf{J},
\]
where \( \mathbf{J} = \frac{\partial \mathbf{b}}{\partial t} \) is the current density and \( \eta \) is the resistivity. Taking the average gives
\[
\mathbf{E} = -\overline{\mathbf{E}}/c - \nabla \times \mathbf{B}/c + \eta \overline{\mathbf{J}},
\]
where \( \overline{\mathbf{E}} = \mathbf{v} \times \mathbf{B} \) is the turbulent EMF. Subtracting equation (9) from equation (8) gives
\[
\mathbf{e} = (\mathbf{E} - \mathbf{v} \times \mathbf{B} - \nabla \times \mathbf{E})/c + \eta \mathbf{f}.
\]
Plugging equation (9) into equation (6) and equation (10) into equation (7) and now globally averaging (indicated by brackets) and assuming divergence terms then vanish then gives
\[
\frac{1}{2} \partial_t (\mathbf{a} \cdot \mathbf{b}) = -\mathbf{b} \cdot \nabla \mathbf{E} - \nu_M (\overline{\mathbf{b} \cdot \nabla \times \mathbf{b}}),
\]
where \( \nu_M \equiv (\eta c^2/4\pi) \) and
\[
\frac{1}{2} \partial_t (\mathbf{A} \cdot \mathbf{B}) = \mathbf{b} \cdot \nabla \mathbf{E} - \nu_M (\overline{\mathbf{B} \cdot \nabla \times \mathbf{B}}).
\]
(Note that such globally averaging only eliminates divergence terms if the system goes to zero at infinity or otherwise is appropriately symmetric, such as a triply periodic simulation box. A shearing sheet’s shear velocity goes to infinity at \( \pm \infty \), so \( \alpha - \Omega \) dynamos in shearing sheets cannot be so simply treated.)

To obtain an expression for \( \mathbf{E} \), we use the ‘minimal tau’ closure for incompressible MHD (Blackman & Field 2002; Brandenburg & Subramanian 2005a) to replace triple correlations by a damping term on the grounds that \( \mathbf{E} \) should decay in the absence of \( \mathbf{B} \). For a system which can be reflection asymmetric but at most only weakly anisotropic, this gives
\[
\partial_t \mathbf{E} = (\mathbf{v} \cdot \mathbf{b}) + (\mathbf{v} \cdot \partial_t \mathbf{b}) = \frac{\alpha}{\tau} \mathbf{B} - \beta \nabla \times \mathbf{B} - \mathbf{E}/\tau,
\]
where \( \tau \) is a damping time and
\[
\alpha \equiv \frac{\tau}{3} \left( \frac{\mathbf{b} \cdot \nabla \times \mathbf{b}}{4\pi \rho} - (\mathbf{v} \cdot \nabla \mathbf{v}) \right) \quad \text{and} \quad \beta \equiv \frac{\tau}{3} (\mathbf{v}^2).
\]
The \( \alpha \) above is the \( \alpha \) effect that drives \( \alpha^2 \) and \( \alpha - \Omega \) dynamos, while the \( \beta \) above is the turbulent diffusivity.

The time evolution of \( \mathbf{E} \) can be retained as a separate equation, but simulations of magnetic field evolution in forced isotropic helical turbulence have shown both that the simulations are well matched when the left-hand side of equation (13) is ignored and that \( \tau \sim \frac{1}{\sqrt{\nu}} \), the eddy turnover time associated with the forcing scale (Field & Blackman 2002; Brandenburg & Subramanian 2005b). Rearranging equation (13) with this approximation then gives
\[
\mathbf{E} = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B}.
\]
Equations (6) and (7) then become
\[
\frac{1}{2} \partial_t (\mathbf{a} \cdot \mathbf{b}) = -\alpha (\mathbf{b}^2) + \beta (\mathbf{b} \cdot \nabla \times \mathbf{b}) - \nu_M (\overline{\mathbf{b} \cdot \nabla \times \mathbf{b}}),
\]
and
\[
\frac{1}{2} \partial_t (\mathbf{A} \cdot \mathbf{B}) = \alpha (\mathbf{b}^2) - \beta (\mathbf{B} \cdot \nabla \times \mathbf{B}) - \nu_M (\overline{\mathbf{B} \cdot \nabla \times \mathbf{B}}).
\]
The energy associated with the small-scale magnetic field does not enter \( \mathbf{E} \), so it does not enter equations (11) and (12). It enters as a higher order correction (Subramanian 2003) which we presently ignore.

However, upon plugging equation (14) into equations (11) and (12), the energy associated with the large-scale field \( \mathbf{B}^\prime \) does enter. Therefore, we need a separate equation for the energy associated with the energy of the mean field. To obtain this equation, we dot \( \phi \mathbf{B} = -c \mathbf{v} \times \mathbf{E} \) with \( \mathbf{B} \) and ignore the flux terms to obtain
\[
\frac{1}{2} \partial_t (\mathbf{b}^2) = -c (\mathbf{B} \cdot \nabla \times \mathbf{E}) = -c (\nabla \times \mathbf{v} \times \mathbf{B})
\]
\[
= (\mathbf{E} \cdot \nabla \times \mathbf{B}) - \nu_M (\overline{\nabla \times \mathbf{B}^2}) = \alpha (\mathbf{B} \cdot \nabla \times \mathbf{B})
\]
\[
- \beta (\overline{\nabla \times \mathbf{B}^2}) = \nu_M (\overline{\nabla \times \mathbf{B}^2}),
\]
where the latter two similarities follow from using equations (9) and (14), and \( \mathbf{v} = 0 \).

Equations (15)–(17) form a set of equations that can be solved in a two-scale model. To facilitate study of the essential implications of this coupled system for a closed or periodic system, we adopt a discrete scale approximation (Blackman & Field 2002; Brandenburg & Subramanian 2005a) and indicate large-scale mean magnetic quantities with subscript ‘1’, small-scale magnetic quantities with ‘2’, and the kinetic forcing scale with the subscript \( f \). We assume that the wavenumber \( k_1 \) associated with the spatial variation scale of large-scale quantities satisfies \( k_1 \ll k_2 \) and \( k_1 \ll k_1 \). In the standard two-scale model for the \( \alpha^2 \) dynamo, the kinetic forcing wavenumber \( k_1 = k_2 \). Here, we allow \( k_1 \) and \( k_2 \) to be distinct, a small generalization motivated by our schematic diagrams that provides some conceptual versatility.

Applying these scaling approximations to the closed or periodic system, we then use \( \mathbf{B} \cdot \nabla \mathbf{v} = k_2^2 (\mathbf{B} \cdot \mathbf{A}) \equiv k_2^2 H_1 \), where \( H_1 \) is the magnetic helicity of the large-scale magnetic field, \( \mathbf{B}^\prime \) \( = k_1 H_1 \), along with \( \mathbf{b} \cdot \nabla \mathbf{v} = k_3^2 (\mathbf{a} \cdot \mathbf{b}) = k_3^2 H_2 \), where \( H_2 \) is the magnetic helicity of the small-scale field, and we assume that \( H_1 > 0 \) for the cases discussed.

Then equations (15) and (16) become (Blackman & Subramanian 2013)
\[
\partial_t H_1 = \left( \frac{2\tau_1}{3} \right) \left( \frac{k_3^2 H_2}{4\pi \rho} - H_1 \right) B_1^2 - \frac{2\tau_1}{3} v_1^2 k_2^4 H_1 - 2\nu_M k_1^2 H_1,
\]
and
\[
\partial_t H_2 = \left( \frac{2\tau_1}{3} \right) \left( \frac{k_2^2 H_1}{4\pi \rho} - H_2 \right) B_1^2 + \frac{2\tau_1}{3} v_1^2 k_3^2 H_1 - 2\nu_M k_2^2 H_2,
\]
and
\[
\partial_t B_1^2 = \left( \frac{2\tau_1}{3} \right) \left( \frac{k_3^2 H_2}{4\pi \rho} - H_1 \right) k_2^2 H_1 - \frac{2\tau_1}{3} v_1^2 k_3^2 B_1^2 - 2\nu_M k_2^2 B_1^2,
\]
where \( H_1 \equiv (\mathbf{v} \cdot \nabla \mathbf{v}) \simeq |f_h| k_1 v_1^2 \) is the kinetic helicity density associated with fluctuating fields, which for dimensionless \( f_h = 1 \) would be maximally helical.

Note that since \( \tau \) is the damping time from the minimal \( \tau \) approximation (rather than an eddy turnover time), \( k_1 \) does not yet explicitly enter the equations until we specify its value. We now non-dimensionalize these equations by scaling lengths in units of \( k_1^{-1} \) and time in units of \( \tau \), where we assume \( \tau = (k_1^{-1})^{-1} \). This latter assumption is made on dimensional grounds, but has been numerically validated (Brandenburg & Subramanian 2005b).

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We define
\[ h_1 = \frac{k_i H_1}{4\pi \rho_v}, \quad h_2 = \frac{k_i H_2}{4\pi \rho_v}, \quad h_v = \frac{H_v}{k_i v_i}, \quad R_M = \frac{v_i}{v_m k_i} \]
and \( M_1 = \frac{B_1}{4\pi \rho_v}. \)

Equations (18)–(20) can then be respectively written as
\[
\partial_t h_1 = \frac{2}{3} \left( \frac{k_i^2}{k_f^2} h_2 - h_v \right) M_1 + \frac{2}{3} \left( \frac{k_i}{k_f} \right)^2 h_1 - \frac{2}{R_M} \left( \frac{k_i}{k_f} \right)^2 h_1, \tag{21}
\]
\[
\partial_t h_2 = \frac{2}{3} \left( \frac{k_i^2}{k_f^2} h_2 - h_v \right) M_1 + \frac{2}{3} \left( \frac{k_i}{k_f} \right)^2 h_1 - \frac{2}{R_M} \left( \frac{k_i}{k_f} \right)^2 h_2, \tag{22}
\]
and
\[
\partial_t M_1 = \frac{2}{3} \left( \frac{k_i^2}{k_f^2} h_2 - h_v \right) \left( \frac{k_i}{k_f} \right)^2 h_1 - \frac{2}{3} M_1 \left( \frac{k_i}{k_f} \right)^2 - \frac{2}{R_M} M_1 \left( \frac{k_i}{k_f} \right)^2. \tag{23}
\]

5 SOLUTIONS

Equation (23) can be ignored if the initial total magnetic energy is small and the forcing is sufficiently helical, for then only the helical large-scale magnetic energy grows significantly and the magnetic energy \( M_1 \) that appears in equations (21) and (22) is that of the helical field, and can be replaced by \( M_1 = \frac{h_1}{h_f} h_1 \). We employ this here and solve equations (21) and (22) for \( k_2 = 3 k_1 \) and \( R_M = 5000 \) for the cases with \( k_2 = k_1 \) and \( k_2 = 3 k_1 \), respectively. The results are shown in Fig. 3. The top row shows solutions for \( h_1 \) and \( h_2 \) at early and late times, respectively, when \( k_2 = k_1 \). The bottom row shows solutions for \( h_1 \) (thick blue line) and \( h_2 \) (thin purple lines) at early and late times, respectively, when \( k_2 / k_1 = 3 \). In comparing the two right-hand panels, we see that although \( h_1 \) saturates at the same value, \( h_2 \) saturates at a value reduced by a factor \( (k_2/k_1)^2 \) compared to the top. This can be seen analytically from setting the left-hand sides of equations (21) and (22) to zero for the steady-state solution, which shows that at saturation, \( h_1 = -(k_2/k_1)^2 h_2 \). Plugging this relation back in equation (21) for the steady state and dropping the \( R_M \) term because it is small, we then obtain \( h_2 = \frac{k_i}{k_f^2} (1 - \frac{k_i}{k_f}) = 20/k_f^2 \) and \( h_1 = \frac{k_i}{k_f^2} (1 - \frac{k_i}{k_f}) = 20 \).

We can estimate the transition from the kinematic growth regime to the \( R_M \)-dependent growth regime shown in the left-hand panels of Fig. 3 as follows. Before the \( R_M \) terms enter in equation (22), the magnetic helicity of the large and small scales must be essentially conserved, and so \( h_1 \approx -h_2 \) until the end of this regime. The deviation from approximate conservation of magnetic helicity becomes substantial when the sum of the terms independent of \( R_M \) depletes to such an extent that the \( R_M \) terms become important. For large \( R_M \), we can thus estimate the transition by just balancing the \( R_M \)-independent terms on the right-hand side setting \( h_1 = -h_2 \). The result gives \( h_1 \approx -h_2 \approx \frac{k_i}{k_f^2} (1 - k_i/k_f) = 20/k_f^2 \). This explains why the transition points for both \( h_1 \) and \( h_2 \) are reduced as a function of increasing \( k_2/k_1 \).

The ultimate steady state is sustained against resistive decay by the kinetic helicity forcing. If the forcing is removed at some point during the steady-state regime, some of the large-scale magnetic helicity would annihilate all of the small-scale magnetic helicity, leaving a residual magnetic helicity on the large scale which would then decay resistively. The complementary cases of (i) driven magnetic relaxation, where magnetic helicity (rather than kinetic helicity) is injected and relaxes to large scales, and (ii) large-scale helical field decay, which occurs only on resistive time-scales for sufficient large initial field strength are reviewed in Blackman (2014). See also Bhat, Blackman & Subramanian (2014) for simulations of case (ii).

The allowance for a scale distinction between \( k_2 \) and \( k_1 \) whilst still using just two dynamical equations is an intermediate approach between those which have restricted the solution of equations to strictly two scales and those which have added a third time evolution equation for magnetic helicity on the smallest scale in addition to that on the forcing scale (Blackman 2003a). In a sense this model can then be thought of as a 2.5 scale model. As discussed in Section 3, the diagrammatic representation of the \( \alpha^2 \) dynamo of Fig. 2 allows for this distinction as well as the tracking of magnetic helicity because the field is treated as a 2D ribbon. There is evidence from 3D numerical simulations that the small-scale current helicity in \( \alpha^2 \) dynamo simulations is not in fact peaked on the forcing scale but on smaller scales (Park & Blackman 2012). Despite the fact that the full spectrum of scales is present in simulations, the large-scale, the forcing scale, and the current helicity weighted wavenumber emerge as natural scales from which to build a discrete scale mean field theory. There is good opportunity for further numerical simulation studies of the \( \alpha^2 \) dynamo to assess the efficacy of equations (21)–(23) and also to dynamically predict the ratio \( k_2/k_1 \) for a given choice of \( k_i \).

A subtlety regarding the scale separation of \( k_2 \) and \( k_1 \) is that the time constant \( \tau \) entering equations (18) and (19) is just the damping constant appearing in the last term of equation (13), resulting from the ‘minimal tau’ closure. This contrasts the first-order smoothing approximation (FOSA; e.g. Moffatt 1978), where the corresponding \( \tau \) in the \( \alpha \) effect emerges only as a single approximation to the two correlation times in the separate time integrals \( \int f(t) \cdot \nabla \times \psi(t') dt' \) and \( \int f(t) \cdot b(t') dt' \) for the kinetic helicity and current helicity terms, respectively. In FOSA, even if \( f \) is not peaked on the forcing scale, \( \tau \) could be \( \tau \) scales inverse with \( k \), for example as \( k^{-2/3} \) in the Kolmogorov model. FOSA could also lead to a circumstance in which the \( \tau \)’s multiplying the two terms of the \( \alpha \) effect could be different. The present formalism of allowing for \( k_2 \# k_1 \) can also be alternatively interpreted to accommodate this possibility.

Finally, note that because Fig. 2 shows exact helicity conservation, it is most rigorously applicable in the time regime of solution to equations (21)–(23), before the \( R_M \) terms become significant (and thus the regime in the panels of Fig. 3 before the curves diverge). This is appropriate because it is the buildup of the small-scale helicity that in turn eventually saturates the dynamo, and it is because of this nearly helicity conserving phase that the subsequent \( R_M \)-dependent phase emerges.

6 CONCLUSIONS

We have shown how considering the magnetic field to be a 2D ribbon rather than 1D line leads to a schematic diagram that correctly captures the conservation of magnetic helicity for the \( \alpha^2 \) dynamo during its evolution to saturation unlike traditional dynamo diagrams that treat the magnetic field as a line.
Magnetic fields as ribbons not lines

Figure 3. Solutions to equations (21)–(23) for the fully helical forcing case \( h_1 = -1 \) for all times and initially \( h_1(0) = 0.001 \) and \( h_2(0) = 0 \) with zero non-helical magnetic energy. In this case, equation (23) can be ignored as it is redundant with equation (21). For all panels, \( k_f = 5 \) and \( R_M = 5000 \). The top row and the bottom row (c and d) differ in that \( k_2/k_f = 1 \) for the top row and \( k_2/k_f = 3 \) for the bottom row. The left-hand panel in each row is the early time solution of \( h_1 \) (thick line) and \(-h_2\) (thin line), where the growth is independent of \( R_M \). The right-hand panels show the late-time evolution where the magnitude of the small-scale magnetic helicity \( h_2 \) has grown enough to offset the kinetic helicity driving sufficiently so that the \( R_M \) terms become important. The asymptotic saturation value of \( h_1 \) is independent of \( k_2 \) but the transition value of \( h_1 \) when \( R_M \) becomes important is reduced by \((k_f/k_2)^2\) as is the saturation value of \( h_2 \). That \( k_1 \) and \( k_2 \) need not be equal is also captured in Fig. 2.

In addition, the diagram illustrates the need to allow for a distinction of three scales: (i) the helical forcing scale, (ii) the small-scale magnetic helicity, and (iii) the large-scale magnetic helicity. Towards this end, we introduced a simple generalization to the two-scale equations of the \( \alpha^2 \) dynamo to allow for the fact that the forcing scale and scale of current helicity buildup may not be equal without having to introduce a third dynamical equation. Solving these equations shows that when the scale of the current helicity buildup is much smaller than that of the forcing scale (though still well above the resistive scale), the quenching of the dynamo is exacerbated when compared to the case in which these latter two scales are equal. This provides a simpler framework to study the separation of these scales than has been considered before (Park & Blackman 2012).

In short, the near conservation of magnetic helicity in MHD dynamos for a large \( R_M \) closed system is well captured visually when magnetic fields are represented by ribbons but poorly captured when the fields are represented as lines. The associated diagram of the \( \alpha^2 \) dynamo also captures the potential richness of an additional scale separation which, as we have shown, affects predictions of dynamo saturation when incorporated into the theory. There is opportunity for future work to assess the efficacy of the 2.5 scale model of the \( \alpha^2 \) dynamo herein with simulations and to develop a theory to predict \( k_2/k_f \) for a given \( k_f \).

Because dynamos are so representative of how magnetic fields and flows interact in MHD, the lesson learned seemingly has very broad implications for high-\( R_M \) MHD beyond that of dynamo theory: magnetic fields are better visualized as 2D ribbons than 1D.
lines. Magnetic reconnection provides another example which bears this out (Pfister & Gekelman 1991).

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