Nonassociative black holes in R-flux deformed phase spaces and relativistic models of Perelman thermodynamics

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ABSTRACT: This paper explores new classes of black hole (BH) solutions in nonassociative and noncommutative gravity, focusing on features that generalize to higher dimensions. The theories we study are modelled on (co) tangent Lorentz bundles with a star product structure determined by R-flux deformations in string theory. For the nonassociative vacuum Einstein equations we consider both real and complex effective sources. In order to analyze the nonassociative vacuum Einstein equations we develop the anholonomic frame and connection deformation methods, which allows one to decoupled and solve these equations. The metric coefficients can depend on both space-time coordinates and energy-momentum. By imposing conditions on the integration functions and effective sources we find physically important, exact solutions: (1) 6-d Tangherlini BHs, which are star product and R-flux distorted to 8-d black ellipsoids (BEs) and BHs; (2) nonassociative space-time and co-fiber space double BH and/or BE configurations generalizing Schwarzschild-de Sitter metrics. We also investigate the concept of Bekenstein-Hawking entropy and find it applicable only for very special classes of nonassociative BHs with conventional horizons and (anti) de Sitter configurations. Finally, we show how analogs of the relativistic Perelman W-entropy and related geometric thermodynamic variables can be defined and computed for general classes of off-diagonal solutions with nonassociative R-flux deformations.

KEYWORDS: Black Holes in String Theory, Non-Commutative Geometry

ArXiv ePrint: 2207.05157

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1 Introduction

Nonassociative and noncommutative geometric and physical models have been subjects of great interest for almost 90 years, since the first works of nonassociative quantum mechanics [1, 2] and further developments in mathematical particle physics [3–9]. Much is known about nonassociative gravity, nonassociative gauge theories, nonassociative membrane theory and double field theory, from string theory [10–15]. Nonassociative structures also arise in the world volume of a D-brane (for open strings) and for flux compactification (for closed strings). Reviews of various aspects of nonassociative physics and gravity and a comprehensive list of references can be found in [16–21].

Self-consistent theories of nonassociative vacuum gravity were formulated for \( \star \)-product (i.e. star-product) deformations determined by R-flux backgrounds in string gravity [17, 18]. Such nonassociative and noncommutative geometric and gravity theories are modelled on a conventional phase space \( M = T^* V \), which for theories including general relativity (GR) is a cotangent bundle, \( T^* V \), on a spacetime Lorentz manifold, \( V \). The nonassociative, vacuum, gravitational equations, \( \text{Ric}^\star [\nabla^\star] = 0 \), were postulated as phase space \( \star \)-deformations of the standard Ricci tensor \( \text{Ric} \) in GR. The nonassociative tensor \( \text{Ric}^\star \) was constructed for a unique nonassociative Levi-Civita (LC) connection, \( \nabla^\star \), which is torsionless and compatible with the respective \( \star \)-deformed symmetric, \( g^\star \), and nonsymmetric, \( q^\star \), metric structures.\(^1\)

Nonassociative and noncommutative modified gravity theories (MGTs) have attracted attention as a type of bimetric gravity theory [22, 23] (see [24, 25] for reviews), with the second metric structure being nonsymmetric [26–29]. Earlier works on the development of nonassociative gravity are given in [19, 30]. There are also geometric models on a phase space \( M^\star \) enabled with \( \star \)-product structure which involves nonassociative generalizations of relativistic and supersymmetric/(non) commutative Finsler-Lagrange-Hamilton spaces [31–33]. In this paper, we do not use the explicit form of Finsler-like geometric objects and variables, but re-define the formulas for nonlinear quadratic elements in a form so that nonassociative phase space BH solutions can be generated as real configurations coming from star R-flux deformations. We also compute nontrivial phase space components of the nonsymmetric parts of the metric which can occur in nonassociative gravity.

\(^1\)Our conventions can be found in [19–21] and are explained in the next section and appendix A.1. In this work, the phase space dimension is \( \dim M = 8 \), with local coordinates labeled like \( ^\star u^{\mu^\star} = (x^{\alpha}, \; ^\star p_\alpha) \). Here “\( \star \)” on the left indicates a phase space with spacetime coordinates, \( x^{\alpha} = (x^i, t) \), and complex momentum coordinates, \( ^\star p_\alpha = ip_\alpha = (ip, iE) \), with \( i^2 = -1 \).
The works [17–20] raised two important questions in nonassociative gravity: (1) How to formulate and analyze fundamental properties of nonassociative modified Einstein equations? (2) How to construct in explicit form classes of exact, physically important solutions in nonassociative gravity and determine the physical meaning of such solutions? In regard to question (2) nonassociative versions of gravitational and matter field equations cannot be solved by standard methods. It is difficult to construct and study the physical properties of nonassociative BH and cosmological solutions (see [20, 21]).

The main hypothesis in our works on nonassociative and noncommutative gravity is that such viable physical theories can be formulated as nonholonomic real and/or (almost) complex deformations of GR to geometric models with extra-dimensional coordinates and generalized (almost) complex/ symplectic structures. We carry out constructions with a generalized metric, frame, and (non) linear connection structures on nonholonomic manifolds and/or (co) tangent bundles. The works [31–33] further laid out various extensions/generalizations involving supersymmetry, noncommutativity, and nonassociativity.

This is the fourth paper in a series of papers [19–21] devoted to constructing exact, physically important solutions in nonassociative geometry and (modified) gravity models determined by \( \ast \)-products induced by R-flux deformations in string theory. In [20], we developed, in nonassociative geometric form, the anholonomic frame and connection deformation method, AFCDM. Using geometric and analytic methods, we can decouple and solve in a general form, vacuum and non-vacuum gravitational and matter field equations and geometric flow evolution equations. For such solutions, the coefficients of the symmetric and nonsymmetric generic off-diagonal metrics and (non) linear connections depend, in general, on all phase space coordinates.

The AFCDM is different from the usual approach of constructing exact solutions which uses a diagonal ansatz for the metric, and has metric components that only depend on spacetime coordinates. In the usual approach the systems of nonlinear PDEs are transformed into systems of ODEs (ordinary differential equations), which can be integrated in some exact, parametric forms depending on integration constants. The physical meaning of these constants is defined by symmetry and asymptotic conditions (for the well known, exact solutions in GR, details can be found in [34–37]). Applying the AFCDM, we can integrate directly, in exact, parametric form, physically important systems of nonlinear PDEs. In [21], we constructed exact, parametric solutions, in \( \kappa \) and \( \hbar \), describing 4-d \( \ast \) R-flux distortions of Schwarzschild BHs, ellipsoid configurations defining black ellipsoids (BEs) and BHs with ellipsoidal thin accretion disks. We also used 4-d nonholonomic constraints of the quadratic elements for 8-d quasi-stationary nonassociative vacuum solutions found in [20].

In this paper we further develop our program of constructing BH and BE solutions in nonassociative gravity [20, 21]. In addition to previous solutions [21], we construct two new classes of BH configurations in 8-d nonassociative phase spaces. For this purpose, we use modified quasi-stationary integrals for metrics and (non) linear connections depend, in general, on all phase space coordinates.

\footnote{Briefly, a nonholonomic manifold is enabled with a non-integrable distribution, with a prescribed system of anholonomic frames, or certain Whitney/ direct sums defining some (non) linear connection structure, see references [19, 20] for details.}
an energy Killing symmetry. The main goal of this article is to generalize and apply the AFCDM in such a form which allows one to generate, in nonassociative form, classes of 8-d phase space BH solutions. We find that for our generic off-diagonal solutions it is only possible, in very special cases, to define the concept of a generalized Bekenstein-Hawking entropy. However, we show that one can define the concept of an entropy for these solutions using Perelman’s geometric flow thermodynamic variables [48]. Previous work on applying Perelman’s thermodynamics in relativistic theories can be found in [33].

These constructions of entropy for relativistic and noncommutative geometric flow models using Perelman thermodynamics were given in detail in the works [38–40]. In this paper, we consider Perelman-type geometric, entropic and thermodynamic values determined by real R-flux effective sources as in [18, 20, 21].

The paper is organized as follows: in section 2, we review the necessary geometric concepts, methods, and formulas of [19–21]. The nonassociative vacuum gravitational equations are provided in nonholonomic, dyadic variables with a shell by shell parametric decompositions of the 8-d phase space. We also present the quadratic elements for quasi-stationary phase spaces, in terms of gravitational polarizations and generating functions, which will be used for constructing BH solutions in later sections.

Section 3 is devoted to nonassociative $\star$-product R-flux deformations of Schwarzschild-de Sitter metrics with the metric coefficients having additional dependencies on energy and momentum. We provide explicit formulas for $\star$-$\kappa$-parametric deformations of Tangherlini’s BHs [41] into quasi-stationary configurations on nonassociative phase spaces. We construct examples of double BHs, BEs, configurations both on 4-d phase spacetime and in 4-d (co) fiber space with explicit energy dependence and related nonassociative and nonlinear symmetries. We then give two classes of exact solutions describing quasi-stationary BHs and BEs with an energy-like Killing symmetry. The first solution describes nonassociative R-flux distortions of 6-d Tangherlini like BHs to 8-d phase spaces. The second solution is for nonassociative, nonholonomic deformations of BHs into BEs with a fixed energy parameter.

Section 4 focuses on the entropy and geometric thermodynamics of BHs and BEs as quasi-stationary Ricci solitons encoding nonassociative R-flux data. We speculate on the phase space generalizations of the Bekenstein-Hawking entropy for nonassociative BH and BE solutions with hypersurface horizons. For more general classes of off-diagonal solutions, with nonassociative deformations of relativistic and phase space Ricci solitons, we conclude that the Bekenstein-Hawking approach is not applicable. However we show how relativistic generalizations of Grigori Perelman’s entropy and statistical/ geometric thermodynamics can be applied in those cases where the Bekenstein-Hawking approach to entropy does not exist. We give two examples of how to compute the generalized Perelman thermodynamic variables for quasi-stationary phase space R-flux deformed Tangherlini BHs and double 4-d BHs.

A summary and discussion of results, with conclusions, are given in section 5. The appendix contains a brief overview of coordinate and index conventions and useful formulas for nonassociative AFCDM and generating functions for various solutions.
2 Nonassociative phase space geometry of quasi-stationary star-deformed Einstein spaces

We are primarily interested in constructing BH solutions in nonassociative vacuum gravity defined on phase spaces. Such metrics may depend both on spacetime and co-fiber coordinates and encode ⋆-product and R-flux contributions. In this section we review a nonassociative geometric formalism [17, 18] reformulated in nonholonomic dyadic variables [19], which will allow us to construct exact and parametric solutions in the following sections. We follow the conventions and definitions on (non) associative phase space geometry and the AFCDM given in [20]. We consider extensions of generic off-diagonal ansatz which allow us to generalize quasi-stationary solutions for 4-d spacetimes [21] and generate new classes of vacuum 8-d nonassociative phase spaces.

2.1 Nonholonomic dyadic shell coordinates, s-frames, and star products

In this work, nonassociative geometric and physical theories are constructed on nonholonomic phase space modeled as a cotangent Lorentz bundle \( M = T^* s q V \) on a pseudo-Euclidean manifold of signature \((+++−)\). Such a phase space is enabled with a quasi-Hopf structure [4, 18] adapted to a nonholonomic dyadic decomposition into four oriented shells \( s = 1, 2, 3, 4 \) with conventional \((2+2)+(2+2)\) splitting of dimensions - briefly, an \( s \)-decomposition. Boldface and not boldface symbols carry the same meaning as in [19, 20] and footnote 4 from [21], and are discussed more in the following paragraph. Further necessary conventions and coordinate formulas are given in appendix A.1.

The \((2+2)+(2+2)\) nonholonomic shell splitting of a phase spaces can be defined by a nonlinear connection (i.e. N-connection) structure. The \( s \)-splitting is given as:

\[
\begin{align*}
\mathbb{s}N : s T^* V &= \mathbb{1} h T^* V \oplus \mathbb{2} v T^* V \oplus \mathbb{3} c T^* V \oplus \mathbb{4} c T^* V, \quad \text{and} \\
\mathbb{s}N : s T^* q V &= \mathbb{1} h T^* q V \oplus \mathbb{2} v T^* q V \oplus \mathbb{3} c T^* q V \oplus \mathbb{4} c T^* q V, \quad \text{for } s = 1, 2, 3, 4.
\end{align*}
\]

The double dash on the left again means phase space coordinates with complex momentum, while the single dash means phase space coordinates with real momentum. More details can be found in appendix A.1. The nonlinear s-connection (2.1) is characterized by a set of coefficients \( \mathbb{s}N = \{ \mathbb{s}N_{i,a}(u) \} \) which can be used for constructing N-elongated bases (N-/ \( s \)-adapted bases),

\[
\mathbb{s}e_{\alpha s} = \left( \mathbb{s}e_a = \frac{\partial}{\partial x^a}, \; \mathbb{s}e_b = \frac{\partial}{\partial p_b} \right) \quad \text{on } s T^* q V, \tag{2.2}
\]

and, dual \( s \)-adapted bases (also called \( s \)-cobases)

\[
\mathbb{s}e^{\alpha s} = \left( \mathbb{s}e^i = dx^i, \; \mathbb{s}e_a = dp_a + \mathbb{s}N_{i,a} dx^i \right) \quad \text{on } s T^* V. \tag{2.3}
\]

In similar form, we can construct \( s \)-bases \( e_{\alpha s} [\mathbb{s}N_{i,a}] \) and \( s \)-cobases \( e^{\alpha s} [\mathbb{s}N_{i,a}] \) as linear N-operators. Such bases are not integrable, i.e. nonholonomic (equivalently, anholonomic) [19, 20].
We now consider the action of $\mathbf{e}_s$ (here the single dash on the left means an Nelongated base like (2.2) depending on real momenta) on some functions $f(x,p)$ and $b(x,p)$, we can define the nonholonomic s-adapted star product $\star_s$:

\[
\begin{align*}
\mathcal{F}_s^{-1}(f,b) & = \left[ \exp \left( -\frac{1}{2} i\hbar (\mathbf{e}_s \otimes \mathbf{e}_s^s - \mathbf{e}_s^s \otimes \mathbf{e}_s) + \frac{i\ell^4}{12\hbar} R^{a_s j_a s}_s(p_{a_s} \mathbf{e}_{i_s} \otimes \mathbf{e}_{j_s} - \mathbf{e}_{j_s} \otimes p_{a_s} \mathbf{e}_{i_s}) \right) \right] f \otimes b \\
& = f \cdot b - \frac{i}{2\hbar} [\langle \mathbf{e}_{i_s}, f \rangle (\mathbf{e}_s^s b) - (\mathbf{e}_s^s f)(\mathbf{e}_{i_s} b)] + \frac{i\ell^4}{6\hbar} R^{a_s j_a s}_s p_{a_s} (\mathbf{e}_{i_s}, f)(\mathbf{e}_{j_s} b) + \ldots.
\end{align*}
\]

This nonassociative operator involves a constant $\ell$ characterizing the R-flux contributions determined by an antisymmetric $R^{a_s j_a s}_s$ background in string theory. The tensor product $\otimes$ can also be written in a s-adapted form $\otimes_s$. For a decomposition in parameters $\hbar$ and $\kappa = \ell^4/6\hbar$, the tensor products turn into usual multiplications as in the third line of (2.4).

To investigate associative geometric flows, commutative geometric flows, and physical models on phase space we work with (pseudo) Riemannian symmetric metrics on cotangent Lorentz bundle $T^*V$. Such a metric field is a tensor \( g = \{ g_{a_\beta} \} \in TT^*V \otimes TT^*V \) of local signature (+, +, +, −; +, +, +, −). We can write \( g \) in equivalent form as an s-metric \( \mathbf{g} = \{ \mathbf{g}_{a_\beta} \} \) with coefficients computed with respect to a nonholonomic s-base and using cofiber coordinates multiplied to complex identity $i$ if such an s-metric is on a phase space $\mathcal{M} = T^n_s V$. Considering symmetric tensor products of \( \mathbf{e} \mathbf{e}^s \in T^n T^*_s V \) (2.3), we can obtain an s-decomposition of the form

\[
\mathbf{g} = \mathbf{g}_s = (h_1 \mathbf{g}_1, v_2 \mathbf{g}_2, \ldots, v_3 \mathbf{g}_4) \in TT^*_n V \otimes_s TT^*_n V
\]

where \( \mathbf{g}_s = \{ h_1 \mathbf{g}_{1 \beta}, v_2 \mathbf{g}_{2 \beta}, \ldots, v_3 \mathbf{g}_{3 \beta}, v_4 \mathbf{g}_{4 \beta} \} \).

A star product and R-flux deformation in nonassociative geometry transforms a symmetric metric \( \mathbf{g}_s \) into a general nonsymmetric one \( \mathbf{g}_s \mathbf{q} \) with symmetric and nonsymmetric components, \( \mathbf{g}_s \mathbf{q} \). Nonholonomic s-adapted constructions \[19, 20\] can be used to split \( \mathbf{g}_s \mathbf{q} \) and \( \mathbf{q} \), as

\[
\mathbf{g}_s \mathbf{q} = (h_1 \mathbf{g}_1, v_2 \mathbf{g}_2, \ldots, v_3 \mathbf{g}_4) \mathbf{q} \]

\[
\mathbf{q} = (h_1 \mathbf{q}, v_2 \mathbf{q}, \ldots, v_3 \mathbf{q}, v_4 \mathbf{q})
\]

and nonsymmetric: \( \mathbf{q} = (h_1 \mathbf{q}, v_2 \mathbf{q}, \ldots, v_3 \mathbf{q}, v_4 \mathbf{q}) \)

and nonsymmetric: \( \mathbf{q} = (h_1 \mathbf{q}, v_2 \mathbf{q}, \ldots, v_3 \mathbf{q}, v_4 \mathbf{q}) \)

For any associative and commutative phase space s-metrics, \( \mathbf{g}_s \mathbf{q} \), we can construct two linear connection structures — the Levi-Civita (LC) connection and the canonical
We use hat labels for canonical geometric objects determined by \( ^{\hat{\mathcal{D}}}_{s} \) acting on tangent spaces of phase space, i.e. on \( T^T\) on \( \mathcal{V} \). Fundamental geometric objects like torsions, \( ^{\hat{\nabla}} T = 0 \) and \( ^{\hat{\nabla}} \hat{T} \neq 0 \), and curvature, \( ^{\hat{\nabla}} R = \{ ^{\hat{\nabla}} R^{\alpha}_{\beta \gamma \delta} \} \) and \( ^{\hat{\mathcal{R}}} = \{ ^{\hat{\mathcal{R}}}^{\alpha}_{\beta \gamma \delta} \} \), and other s-tensor objects can be defined and computed (see details in [19, 20]).

In [20], we studied in detail the nonassociative (non) symmetric and generalized connection structures on S-d phase spaces. A nonassociative symmetric metric s-tensor, as in (2.6), on a phase space with R-flux induced terms can be represented as

\[
^{\hat{\mathcal{A}}} q_{\alpha \beta} = ^{\hat{\mathcal{A}}} g_{\alpha \beta} = ^{\hat{\mathcal{A}}} g_{\alpha \beta} \ast \left( ^{\hat{\mathcal{A}}} e^{\alpha} \otimes ^{\hat{\mathcal{A}}} e^{\beta} \right), \text{ where } ^{\hat{\mathcal{A}}} g_{\alpha \beta}, \ ^{\hat{\mathcal{A}}} g_{\beta \alpha} \in \mathcal{A}^{s}.
\]

Such a nonassociative s-metric is compatible with star R-flux deformations of an s-connection \( ^{\hat{\mathcal{D}}} \to ^{\hat{\mathcal{D}}} ^{\hat{\ast}} \) if the condition \( ^{\hat{\mathcal{D}}} ^{\hat{\ast}} g = 0 \) is satisfied. Here, we note that with respect to s-adapted bases, \( ^{\hat{\mathcal{A}}} e_{\xi} \), and tensor products of their dual s-bases, a nonsymmetric s-metric structure (2.7) can be parameterized in a \([ (2 \times 2) + (2 \times 2) ] + [(2 \times 2) + (2 \times 2)] \) block form,

\[
^{\hat{\mathcal{A}}} q_{\alpha \beta} = ^{\hat{\mathcal{A}}} g_{\alpha \beta} = i k^{\hat{\mathcal{R}}}^{\alpha}_{\xi \alpha} e_{\xi} \ast ^{\hat{\mathcal{A}}} g_{\beta \gamma}.
\]

Formulas (A.2), (A.3) and (A.4) from appendix A.1 show how to split \( ^{\hat{\mathcal{A}}} q_{\alpha \beta} \) into symmetric and anti-symmetric parts.

Here we use the convention [19, 20] that on phase spaces, the star product (2.4) can be defined using nonholonomic dyadic decompositions with \( ^{\hat{\mathcal{A}}} e_{\alpha} \) and R-flux terms. This can be used to compute star deformations of canonical, s-adapted, geometric objects into nonassociative metric of the form:

\[
\begin{align*}
( ^{\hat{\mathcal{N}}} \ast \mathcal{N}, \ ^{\hat{\mathcal{A}}} \mathcal{A}, \ ^{\hat{\mathcal{D}}} \ast \mathcal{D} ) & \iff ( ^{\hat{\mathcal{N}}} \ast \mathcal{N}, \ ^{\hat{\mathcal{A}}} \mathcal{A}, \ ^{\hat{\mathcal{D}}} \ast \mathcal{D} ) \\
( ^{\hat{\mathcal{A}}} \ast \mathcal{A}, \ ^{\hat{\mathcal{D}}} \ast \mathcal{D} ) & \iff ( ^{\hat{\mathcal{A}}} \ast \mathcal{A}, \ ^{\hat{\mathcal{D}}} \ast \mathcal{D} )
\end{align*}
\]

where \( ^{\hat{\mathcal{D}}} \ast = ^{\hat{\mathcal{D}}} \ast + ^{\hat{\mathcal{Z}}} \ast \), for a nonholonomic splitting \( 4+4 \), and \( ^{\hat{\mathcal{D}}} \ast = ^{\hat{\mathcal{D}}} \ast + ^{\hat{\mathcal{Z}}} \ast \), for an s-splitting.

Following the convention of (2.10), we can compute nonassociative star deformations of LC- and canonical s-connections from (2.8) as follows:

\[
\begin{align*}
( ^{\hat{\mathcal{N}}} \ast \mathcal{N} ) & \rightarrow \begin{cases} ^{\hat{\mathcal{N}}} \nabla : ^{\hat{\mathcal{N}}} \nabla ^{\hat{\mathcal{N}}} g = 0; ^{\hat{\mathcal{N}}} \nabla ^{\hat{\ast}} T = 0, \text{ star LC-connection; } \\
^{\hat{\mathcal{D}}} \ast : ^{\hat{\mathcal{D}}} \ast ^{\hat{\mathcal{N}}} g = 0; h_1 ^{\hat{\nabla}} ^{\hat{\ast}} T = 0, v_2 ^{\hat{\nabla}} ^{\hat{\ast}} T = 0, c_3 ^{\hat{\nabla}} ^{\hat{\ast}} T = 0, c_4 ^{\hat{\nabla}} ^{\hat{\ast}} T = 0, \text{ canonical } \\
h_1 v_2 ^{\hat{\nabla}} ^{\hat{\ast}} T \neq 0, h_1 c_3 ^{\hat{\nabla}} ^{\hat{\ast}} T \neq 0, v_2 c_4 ^{\hat{\nabla}} ^{\hat{\ast}} T \neq 0, c_3 c_4 ^{\hat{\nabla}} ^{\hat{\ast}} T \neq 0, \text{ s-connection. }
\end{cases}
\end{align*}
\]
In these formulas, we can consider nonholonomic, dyadic horizontal and (co) vertical decompositions of the form \( {^s\tilde{D}^*} = (h_1 {^s\tilde{D}^*}, v_2 {^s\tilde{D}^*}, c_3 {^s\tilde{D}^*}, c_4 {^s\tilde{D}^*}) \), and adapted to nonlinear \( s \)-connection structures \( {^s\mathcal{N}} \).

There is a canonical distortion relation between two linear connections in (2.11), given by

\[
{^s\tilde{D}^*} = {^s\nabla} + {^s\tilde{Z}}. \tag{2.12}
\]

Without the star labels (2.12) takes the associative and commutative form in (2.8). In order to apply the AFCDM to construct physically interesting solutions it is more convenient to work with (non) associative nonholonomic dyadic canonical geometric quantities \((^s q, {^s\tilde{D}^*})\).

After constructing classes of nonholonomic solutions in explicit form, we can redefine the constructions in terms of star deformed LC-configurations using (2.12). Nonassociative LC-configurations can be extracted imposing zero s-torsion conditions,

\[
{^s\tilde{Z}} = 0, \quad \text{which is equivalent to} \quad {^s\tilde{D}^*} = {^s\nabla} \quad \text{for} \quad {^s\tilde{T}} = 0. \tag{2.13}
\]

In general, all type of solutions subjected, or not, to some conditions of type (2.13) contain certain nonzero anholonomy coefficients of frame structures.

### 2.2 Nonassociative vacuum Einstein equations and dyadic \( \kappa \)-parametric splitting

The nonassociative Riemannian, \( {^s\tilde{R}^*} = \{ {^s\tilde{R}^{*\alpha}_\beta_{\gamma\delta}} \} \), and Ricci, \( {^s\tilde{R}i^*} = \{ {^s\tilde{R}^*_{\alpha\beta}} \} \), s-tensors for the canonical s-connection \( {^s\tilde{D}^*} = \{ {^s\tilde{T}^*_{\alpha\beta\gamma\delta}} \} \) (2.12) are defined respectively by formulas (A.6) and (A.8) in appendix A.1. Such fundamental geometric objects can be decomposed in parametric form (A.7) with \([01,10,11] : = [h, \kappa] \) components containing nonassociative and noncommutative contributions from star product deformations which can be real or complex ones.

We can consider nonholonomic distributions on phase space when \( [00] {^s\tilde{R}i^*_{\alpha\beta}} = {^s\tilde{R}^*_{\alpha\beta}} \) are determined by associative and commutative s-adapted canonical s-connections [33]. As a result, the star s-deformed Ricci s-tensor can be expressed in parametric form,

\[
{^s\tilde{R}i^*} = \{ {^s\tilde{R}^*_{\beta\gamma\delta}} \} = q {^s\tilde{R}i^*} + \{ {^s\tilde{K}^{\gamma\delta}_{\alpha\beta}} \} = \{ {^s\tilde{R}^{\gamma\delta}_{\alpha\beta}} \} + \{ {^s\tilde{K}^*_{\alpha\beta}} \}, \tag{2.14}
\]

where \( {^s\tilde{K}i^*} = \{ {^s\tilde{K}^{\gamma\delta}_{\alpha\beta}} \} \) is determined by nonholonomic coefficients of frame structures.

For the canonical s-connection \( {^s\tilde{D}^*} \), the nonassociative phase space vacuum Einstein equations for (2.14), with a nonzero cosmological constant on at least one shell (\( q \lambda \neq 0 \) for some \( s_i \)), can be written in the form

\[
{^s\tilde{R}i^*_{\alpha\beta}} - \frac{1}{2} {^s\tilde{g}^{\alpha\beta}} {^s\tilde{R}^*} = s \lambda {^s\tilde{g}^{\alpha\beta}}, \tag{2.15}
\]

where the nonassociative scalar curvature \( {^s\tilde{R}^*} \) is defined by formulas (A.9). We can consider nonzero values of \( {^s\tilde{R}^*} = \Lambda(u^1) + 2 \Lambda(u^2) + 3 \Lambda(u^3) + 4 \Lambda(u^4) \), given
as a sum of effective cosmological constants on different shells $^s\Lambda(\frac{\mu}{a^4})$. It is possible to choose effective $^s\Lambda(\frac{\mu}{a^4})$ and $^s\lambda = ^s\lambda'$ in a form that $^s\lambda + \frac{1}{2}^s q_{\alpha\beta,s}w^sR_{\alpha\beta}c^s = 0$, when $^s q_{\alpha\beta,s} = "^s q_{\alpha\beta,s}^0 + "^s a_{\alpha\beta,s}$, The nonassociative symmetric and nonsymmetric components of $^s q_{\alpha\beta,s}$ and $^s a_{\alpha\beta,s}$ are computed following formulas (A.2), (A.3) and (A.4) being determined by $^s q_{\alpha\beta,s}^0 = "^s g_{\alpha\beta,s}$ and $^s a_{\alpha\beta,s}$ in (2.9).

Using (2.14) for the canonical Ricci s-tensor, we express the nonassociative vacuum gravitational field equations (2.15) in the form

$$\hat{\mathbf{R}}_{\beta\gamma,k} = "^s \mathbf{K}_{\beta\gamma,k}, \text{ for effective nonassociative sources} \quad \text{(2.16)}$$

$$^s \mathbf{Y}_{\beta\gamma,k} = [0] \mathbf{Y}_{\beta\gamma,k} + [1] \mathbf{K}_{\beta\gamma,k}[h,\kappa], \quad \text{where} \quad [0] \mathbf{Y}_{\beta\gamma,k} = "^s \Lambda(\frac{\mu}{a^4})"^s g_{\beta\gamma,k} \text{ and}$$

$$[1] \mathbf{K}_{\beta\gamma,k}[h,\kappa] = "^s \Lambda(\frac{\mu}{a^4})"^s q_{\beta\gamma,k}(\kappa) - \hat{\mathbf{K}}_{\beta\gamma,k}[h,\kappa]. \quad \text{(2.17)}$$

$[1] \mathbf{K}_{\beta\gamma,k}[h,\kappa]$ are effective parametric sources with coefficients proportional to $h, \kappa$ and $h\kappa$.

The effective sources in (2.16) can be parameterized for nontrivial real quasi-stationary 8-d configurations using coordinates $(x^k, "p_8)$, for $"p_8 = E$, with $"^s g_{\beta\gamma,k}|_{h=0} = "^s \mathbf{g}_{\beta\gamma,k}$, in such forms:

$${^s \mathbf{K}}_{\alpha\beta,k} = \left\{ ^s \mathbf{K}_{\alpha\beta,k}^{ij}(\kappa, x^k) = \left[ \frac{1}{2} \mathbf{Y}(x^k, x^3) + \frac{1}{4} \mathbf{K}(\kappa, x^k, x^3) \right] \delta_{i,j}^{k,l} \right\}, \quad \text{where}$$

$${^s \mathbf{K}}_{\alpha\beta,k}^{ij}(\kappa, x^k) = \left[ \frac{1}{2} \mathbf{Y}(x^k, x^3) + \frac{1}{4} \mathbf{K}(\kappa, x^k, x^3) \right] \delta_{i,j}^{k,l}, \quad \text{(2.19)}$$

$${^s \mathbf{K}}_{\alpha\beta,k}^{ij}(\kappa, x^k, "p_8) = \left[ \frac{1}{2} \mathbf{Y}(x^k, x^3) + \frac{1}{4} \mathbf{K}(\kappa, x^k, x^3) \right] \delta_{i,j}^{k,l}, \quad \text{where}$$

$${^s \mathbf{Y}}_{\alpha\beta,k} = - [11] \mathbf{R}_{\alpha\beta,k}, \quad \mathbf{R}_{\alpha\beta,k}^{ij}(\kappa, x^k) \text{ as in (A.8), for}$$

$${g}_{\alpha\beta,k} = \{ g_1(x^k), g_2(x^k), g_3(x^k, x^3), g_4(x^k, x^3), g_5(x^k, "p_8), g^6(x^k, "p_8), g^7(x^k, "p_8), g^8(x^k, "p_8) \}.$$
Prescribing four effective sources \( ^{\prime} \mathcal{K} (2.20) \) as generating sources, we constrain the nonholonomic gravitational dynamics. Such generating sources are related to conventional cosmological constants via nonlinear symmetries, when the nonsymmetric parts of the s-metrics and canonical Ricci s-tensors can be computed as R-flux deformations of some off-diagonal symmetric metric configurations.

2.3 Nonassociative parametric vacuum quasi-stationary s-metrics

In nonassociative nonholonomic phase space gravity, the system in (2.16) with a generating source (2.20) can be decoupled and solved in exact forms on 8-d quasi-stationary phase spaces. This can be performed if the AFCDM is applied \([20]\). In this subsection, we modify the constructions and formulas to be able to generate quasi-stationary and BH solutions with nonassociative star R-flux configurations.

2.3.1 Nonholonomic deformations encoding star R-flux sources

We consider a commutative prime s-metric \( ^{\prime} \mathbf{g} \) and s-connection structures of type (2.5) and (2.3)

\[
^{\prime} \mathbf{g} = \mathbf{g} = \tilde{g}_{\alpha_\nu, \beta_\sigma} (x^{\lambda}, p_\alpha) d^{\prime} x^{\alpha} \otimes d^{\prime} x^{\beta} = \tilde{e}^{\alpha} \otimes \tilde{e}^{\beta}, \quad (2.21)
\]

By prime we mean the metric from which we begin when we make various deformations. These prime metrics may be some well known solution, such as Schwarzschild. Here, we shall consider prime metrics which are exact solutions in 8-d nonholonomic phase space gravity \([33]\) subjected to nonassociative star R-flux extensions into another classes of solutions determined, respectively, by symmetric and nonsymmetric metrics,

\[
^{\prime} \mathbf{q}_{\mu, \nu} = \left( ^{\prime} \mathbf{q}_{\mu_1, \nu_1}, ^{\prime} \mathbf{q}_{\mu_2, \nu_2}, ^{\prime} \mathbf{q}_{\mu_3, \nu_3}, ^{\prime} \mathbf{q}_{\mu_4, \nu_4} \right) \text{ of type (A.3), and } ^{\prime} \mathbf{a}_{\mu, \nu} = (0, 0, ^{\prime} \mathbf{a}_{\mu_1, \nu_1}, ^{\prime} \mathbf{a}_{\mu_2, \nu_2}) \text{ of type (A.4).}^{4}
\]

The main goal of this article is to study nonassociative, nonholonomic deformations

\[
^{\prime} \mathbf{g} \rightarrow \mathbf{g} = \left[ \eta_{\alpha_\nu}, ^{\prime} \tilde{g}_{\alpha_\nu}, ^{\prime} \tilde{N}^a_{\alpha_\nu} = \left( ^\eta \eta_{\alpha_\nu} \right) \tilde{N}^a_{\alpha_\nu} \right] \quad (2.22)
\]

to target quasi-stationary s-metrics of type \( ^{\prime} \mathbf{g} \) (2.6). We shall use a “hat” on an s-metric, \( ^{\prime} \mathbf{g} \), if it defines an exact/ parametric solution of nonassociative vacuum gravitational equations (2.16). Such nonholonomic star s-deformations can be described in terms of so-called gravitational polarization (\( \eta \)-polarization) functions, when the target s-metrics are parameterized as

\[
^{\prime} \mathbf{g} = \left( ^{\prime} g_{\alpha_\nu, \beta_\sigma} (h, \kappa, x^{\lambda}, p_\alpha) d^{\prime} x^{\alpha} \otimes d^{\prime} x^{\beta} + ^{\prime} g_{\alpha_\nu, \beta_\sigma} (h, \kappa, x^{\lambda}, p_\alpha) \right) ^{\prime} \mathbf{e}^{\alpha_\nu} \otimes ^{\prime} \mathbf{e}^{\beta_\sigma} \quad (2.23)
\]

\[
= \left( ^{\prime} \eta_{\alpha_\nu, \beta_\sigma} (h, \kappa, x^{\lambda}, y^{\mu}, p_\alpha) + ^{\prime} \eta_{\alpha_\nu, \beta_\sigma} (h, \kappa, x^{\lambda}, y^{\mu}, p_\alpha) \right) ^{\prime} g_{\alpha_\nu, \beta_\sigma} (h, \kappa, x^{\lambda}, p_\alpha) d^{\prime} x^{\alpha} \otimes d^{\prime} x^{\beta} + ^{\prime} \eta_{\alpha_\nu, \beta_\sigma} (h, \kappa, x^{\lambda}, y^{\mu}, p_\alpha) \right) ^{\prime} \mathbf{e}^{\alpha_\nu} \otimes ^{\prime} \mathbf{e}^{\beta_\sigma} \quad (2.23)
\]

\[
^{\prime} \mathbf{e}^{\alpha_\nu} = \left( d^{\prime} x^{\alpha}, ^{\prime} \mathbf{e}^{\alpha_\nu} = d^{\prime} y^{\alpha} + ^{\prime} \eta_{\alpha_\nu} (h, \kappa, x^{\lambda}, y^{\mu}, p_\alpha) ^{\prime} \mathbf{e}^{\alpha_\nu} = \left( ^\eta \eta_{\alpha_\nu} \right) ^{\prime} \mathbf{e}^{\alpha_\nu} \right) \quad (2.23)
\]

\[
^{\prime} \tilde{N}^a_{\alpha_\nu} (h, \kappa, x^{\lambda}, y^{\mu}, p_\alpha) = \left( ^\eta \eta_{\alpha_\nu} \right) ^{\prime} \tilde{N}^a_{\alpha_\nu} \quad (2.23)
\]

\[\text{4We label prime s-metrics and related geometric s-objects with a small circle on the left/right/up of corresponding symbols.}\]
The AFCDM can be applied for generating quasi-stationary s-metric ansatz (2.23) with a small parameter (string constant) $\kappa$ with $0 \leq \kappa < 1$. Also we can use decompositions on a second small parameter, $h$, in order to distinguish between possible noncommutative and nonassociative effects. For simplicity, we only give formulas for linear approximations on $\kappa$ encoding $h$ into generation functions and sources and respective integration functions. Parametric $\kappa$-decompositions of the $\eta$-polarization functions (2.22) result in $s$-adapted coefficients of target metrics of the form:

$$
^n g_{i1}(\kappa, x^{k1}) = \eta_i \tilde{g}_{i1} = \zeta_{i1}(1 + \kappa \chi_{i1}) \tilde{g}_{i1},
$$

(2.24)

Explicit formulas for parametric generating functions defining quasi-stationary solutions (2.23) result in $s$-adapted coefficients of target metrics and related N-connection structure on $N_1$:

$$
^n g_{a2}(\kappa, x^{i1}, y^{j2}) = \eta_{a2} \tilde{g}_{a2} = \zeta_{a2}(1 + \kappa \chi_{a2}) \tilde{g}_{a2},
$$

$$
^n N_{i1}^{a2}(\kappa, x^{k1}, y^{l2}) = \eta_{i1} \tilde{N}_{i1}^{a2} = \zeta_{i1}^{a2}(1 + \kappa \chi_{i1}^{a2}) \tilde{N}_{i1}^{a2},
$$

$$
^n N_{i2a3}(\kappa, x^{k1}, y^{l2}, p^{m2}) = \eta_{i2a3} \tilde{N}_{i2a3} = \zeta_{i2a3}(1 + \kappa \chi_{i2a3}) \tilde{N}_{i2a3},
$$

$$
^n N_{i3a4}(\kappa, x^{k1}, y^{l2}, p^{m2}, p^{n2}) = \eta_{i3a4} \tilde{N}_{i3a4} = \zeta_{i3a4}(1 + \kappa \chi_{i3a4}) \tilde{N}_{i3a4},
$$

(2.25)

Briefly, we can write formulas for the coefficients of target s-metrics and related N-connection structure on $s_T$ in symbolic form:

$$
^n g_{a2} = \zeta_{a2}(1 + \kappa \chi_{a2}) \tilde{g}_{a2},
$$

$$
^n N_{i1}^{a2} = \zeta_{i1}^{a2}(1 + \kappa \chi_{i1}^{a2}) \tilde{N}_{i1}^{a2},
$$

$$
^n N_{i2a3} = \zeta_{i2a3}(1 + \kappa \chi_{i2a3}) \tilde{N}_{i2a3},
$$

$$
^n N_{i3a4} = \zeta_{i3a4}(1 + \kappa \chi_{i3a4}) \tilde{N}_{i3a4},
$$

Explicit formulas for parametric generating functions defining quasi-stationary solutions of $E$-dependent configurations, were given in appendix B.3 of [20]. Here we consider a different class of configurations for the case $E_0 = const$ on the shell $s = 4$, with explicit dependence on $p_7$ of generating functions, $\zeta^8$, $\chi^8$; generating source and cosmological
constant, \( ^{\mu}K \), \( ^{\mu}A_0 \); integration functions, \( ^{\mu}n_{k_4}, ^{\mu}n_{k_4} \); prescribed data for a prime solution, \( ^{\mu}g_7, ^{\mu}g_8 \); \( ^{\mu}N_{k_4} \), \( ^{\mu}N_{k_3} \):

\[
^{\mu}\zeta^7 = -\frac{4}{^{\mu}g_7} \left[ ^{\mu}\partial^2 (^{\mu}\zeta^8 \cdot ^{\mu}g_8^{1/2}) \right] \quad \text{and}
\]

\[
^{\mu}\chi^7 = -\frac{^{\mu}\partial^2 (^{\mu}\chi^8 \cdot ^{\mu}g_8^{1/2})}{4 \cdot ^{\mu}\partial^2 (^{\mu}\zeta^8 \cdot ^{\mu}g_8)} \quad \text{and}
\]

\[
^{\mu}\zeta_{i_3}^7 = \left( ^{\mu}N_{i_3} \right) (^{\mu}K)^{\mu} \partial^{\mu} (^{\mu}\zeta^8) \quad \text{and}
\]

\[
^{\mu}\chi_{i_3}^7 = \left( ^{\mu}N_{i_3} \right) (^{\mu}K)^{\mu} \partial^{\mu} (^{\mu}\zeta^8) \quad \text{and}
\]

\[
^{\mu}\chi_{i_3}^8 = \left( ^{\mu}N_{i_3} \right)^{-1} \left[ \frac{16}{^{\mu}n_{i_3}} \right] + 16 \left( ^{\mu}n_{i_3} \right) \quad \text{and}
\]

\[
\frac{16}{^{\mu}n_{i_3}} \quad \text{and}
\]

\[
^{\mu}\chi_{i_3}^8 = \frac{16}{^{\mu}n_{i_3}} \quad \text{and}
\]

These formulas transform into corresponding ones with \( E \)-dependencies if we change the indices \( 7 \leftrightarrow 8 \), \( ^{\mu}p_7 \rightarrow ^{\mu}p_8 = ^{\mu}E \), \( ^{\mu}\partial^7(\ldots) \rightarrow ^{\mu}\partial^8(\ldots) = (\ldots)^* \), see details in section 5 of [20] on how to derive analogous formulas for shells \( s = 1, 2, 3 \).

The \( \zeta \)- and \( \chi \)-coefficients for deformations (2.25) are of the form

\[
^{\mu}\eta_2 (\kappa, x^{k_1}) = ^{\mu}\zeta_2 (1 + ^{\mu}\chi_2 (x^{k_3}));
\]

\[
^{\mu}\eta_4 (\kappa, x^{k_1}, y^{k_3}) = ^{\mu}\zeta_4 (x^{k_1}, y^{k_3}) (1 + ^{\mu}\chi_4 (x^{k_4}, y^{k_3}));
\]

\[
^{\mu}\eta_5 (\kappa, x^{k_1}, y^{a_2}, \ p_6) = ^{\mu}\zeta_5 (x^{k_1}, y^{a_2}, \ p_6) (1 + ^{\mu}\chi_5 (x^{k_1}, y^{a_2}, \ p_6)); \quad (2.26)
\]

\[
^{\mu}\eta_8 (\kappa, x^{k_1}, y^{a_2}, \ p_{a_3}, ^{\mu}p_7) = ^{\mu}\zeta_8 (x^{k_1}, y^{a_2}, \ p_{a_3}, ^{\mu}p_7) (1 + ^{\mu}\chi_8 (x^{k_1}, y^{a_2}, \ p_{a_3}, ^{\mu}p_7)),
\]

for fixed \( ^{\mu}E_0 \), or

\[
^{\mu}\eta_7 (\kappa, x^{k_1}, y^{a_2}, \ p_{a_3}, ^{\mu}E) = ^{\mu}\zeta_7 (x^{k_1}, y^{a_2}, \ p_{a_3}, ^{\mu}E) (1 + ^{\mu}\chi_7 (x^{k_1}, y^{a_2}, \ p_{a_3}, ^{\mu}E)).
\]

We can generate similar solutions when

\[
^{\mu}\eta_6 (\kappa, x^{i_2}, ^{\mu}p_5) = ^{\mu}\zeta_6 (\kappa, x^{i_2}, ^{\mu}p_5) (1 + ^{\mu}\chi_6 (\kappa, x^{i_2}, ^{\mu}p_5)) \quad \text{instead of}
\]

\[
^{\mu}\eta_5 (\kappa, x^{i_2}, ^{\mu}p_6) = ^{\mu}\zeta_5 (\kappa, x^{i_2}, ^{\mu}p_6) (1 + ^{\mu}\chi_5 (\kappa, x^{i_2}, ^{\mu}p_6)).
\]

The resulting new classes of parametric solutions posses a Killing symmetry on \( p_6 \) which is different from off-diagonal configurations (2.26) with Killing symmetry on \( p_5 \).
2.3.2 Off-diagonal ansatz in radial coordinates with fixed/or running energy dependence

Introducing $\kappa$-coefficients (2.25) instead of $\eta$-coefficients in (2.23), we generate nonlinear quadratic elements for quasi-stationary s-metrics,

$$d\ s^2 = \tilde{g}_{\alpha\beta}(x^k, y^3, \quad p_{\alpha 3}, \quad p_{\alpha 4}; \ g_4, \quad g^{a_3}, \quad g^{a_4}, \quad \mathcal{K}, \quad \Lambda_{\alpha 0}, \kappa) d\ u^\alpha d\ u^\beta,$$

$$d\ s^2 = d\ 1 s^2(x^k, \ 1\mathcal{K}, \ 1\Lambda_{\alpha 0}, \kappa) + d\ 2 s^2(x^k, \ y^3, \ 2\mathcal{K}, \ 2\Lambda_{\alpha 0}, \kappa) + d\ 3 s^2(x^k, \ y^3, \ 3\mathcal{K}, \ 3\Lambda_{\alpha 0}, \kappa) + d\ 4 s^2(x^k, \ y^3, \ 4\mathcal{K}, \ 4\Lambda_{\alpha 0}, \kappa).$$

(2.27)

Here we study the total phase space quasi-stationary s-metric structures (2.27) describing extensions of the BH stationary solutions in nonassociative vacuum gravity with Killing symmetry on $g^4 = t$ on the shell $s = 2$.\(^5\)

The shell components of the quadratic nonlinear elements (2.27) define solutions of (2.16) if the coefficients of the s-metric are defined and parameterized as:

On shell $s = 1$,

$$d\ s^2 = g_1(x^{k1})[(dx^{k1})^2] = e^{\psi_0(x^{k1})(1 + \kappa \psi^{\mathcal{K}})}[(dx^{k1})^2 + (dx^2)^2],$$

where $\psi(x^{k1}) = \psi(0)(x^{k1}) + \kappa \psi^{\mathcal{K}}(x^{k1})$ is a solution of $\psi^{**} + \psi'' = 2 \mathcal{K}$. (2.28)

On shell $s = 2$,

$$d\ s^2 = g_{a2}(x^{k1}, y^3)(e^{a_2})^2$$

$$= \left\{ \begin{array}{l}
\frac{4}{3} \int \frac{dy^3}{(\mathcal{K})(\mathcal{C}_4)(\mathcal{C}_4)^{\circ}} \left( \begin{array}{c}
\mathcal{X}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Y}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Z}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{W}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\end{array} \right) \right\}^{\circ} \mathcal{G}_3
+ \kappa \left( \begin{array}{c}
\mathcal{X}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Y}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Z}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{W}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\end{array} \right) \right\}^{\circ} \mathcal{N}_4^{\circ}\mathcal{X}_1\mathcal{X}_1
+ \kappa \left( \begin{array}{c}
\mathcal{X}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Y}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Z}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{W}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\end{array} \right) \right\}^{\circ} \mathcal{N}_4^{\circ}\mathcal{X}_1\mathcal{X}_1
$$

$$- \kappa \left( \begin{array}{c}
\mathcal{X}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Y}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Z}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{W}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\end{array} \right) \right\}^{\circ} \mathcal{N}_4^{\circ}\mathcal{X}_1\mathcal{X}_1
$$

$$+ \kappa \left( \begin{array}{c}
\mathcal{X}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Y}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Z}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{W}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\end{array} \right) \right\}^{\circ} \mathcal{N}_4^{\circ}\mathcal{X}_1\mathcal{X}_1
$$

$$16 \mathcal{N}_4 \int \frac{dy^3}{(\mathcal{K})(\mathcal{C}_4)(\mathcal{C}_4)^{1/2} \mathcal{C}_4)} \left( \begin{array}{c}
\mathcal{X}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Y}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Z}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{W}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\end{array} \right) \right\}^{\circ} \mathcal{N}_4^{\circ}\mathcal{X}_1\mathcal{X}_1
$$

$$\times \left( \begin{array}{c}
\mathcal{X}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Y}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{Z}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\mathcal{W}_4 \mathcal{C}_4 \mathcal{C}_4 (\mathcal{C}_4)^{1/2} \mathcal{C}_4 \\
\end{array} \right) \right\}^{\circ} \mathcal{N}_4^{\circ}\mathcal{X}_1\mathcal{X}_1
$$

\(^5\)Such 8-d s-metrics have a Killing symmetry on $p_5$, or on $p_6$, on $s = 3$.\(^5\)
is a solution of the system:

\[
(3g)^\circ g_4^\circ = 2g_3 g_4 \frac{\partial}{\partial x^i} K(x^i, g^j) ,
\]

\[
2\beta w_j - \alpha_j = 0, \quad n_{k_1}^o, \quad 2\gamma n_{k_1}^c = 0 ,
\]

where

\[
\begin{aligned}
\alpha & = g_i^2 \partial_i (2\varpi), \quad 2\beta = g^2 (2\varpi)^\circ, \quad 2\gamma = (\ln \frac{|g_4^{3/2}|}{|g_3|})^o \\
\end{aligned}
\]

for generating function \( \frac{\partial}{\partial x^i} \Psi = \exp(2\varpi) \), \( 2\varpi = \ln |g_4^{3/2}|/|g_3 g_4| \).

On co-fiber space, for shell \( s = 3 \),

\[
d g^0 s^2 = "g^o(x^i, g^a, "p_6)("e_{i3})^2
\]

\[
= "\zeta^5 (1 + \kappa "\tilde{\chi}^5)^\circ \tilde{g}_5 \left\{ d "p_6
\right. \\
+ \left[ ("\tilde{N}_{i3})^{-1} \right]^{\frac{n_{i3}}{2}} + 16 \frac{n_{i3}}{2} \left[ \int d "p_6 \left( \frac{"\partial \tilde{g}_5 ["\zeta^5 \tilde{g}_5 (1/2)]}{\tilde{g}_5 \tilde{g}_5 ["\zeta^5 \tilde{g}_5 (1/2)]} \right) \right]
\]

\[
- \left\{ \frac{4}{"g_6} \int d "p_6 \left[ \frac{"\tilde{g}_6 ["\zeta^5 \tilde{g}_5 (1/2)]}{\tilde{g}_5 \tilde{g}_5 ["\zeta^5 \tilde{g}_5 (1/2)]} \right] \right\}
\]

\[
\left( \frac{"\partial \tilde{g}_6 }{\tilde{g}_6 \tilde{g}_6 ["\zeta^5 \tilde{g}_5 (1/2)]} \right)
\]

\[
\times ("\tilde{N}_{i3}) d x^3\]

is a solution of this system:

\[
"\partial \tilde{g}_6 (\frac{3}{3} \varpi \varpi) "\partial \tilde{g}_6 "g_5 = 2 "g_5 "g_6 \frac{\partial}{\partial x^i} K(x^i, "p_6),
\]

\[
"\partial \tilde{g}_6 ("n_{k_2}) + "\gamma "\partial \tilde{g}_6 ("n_{k_2}) = 0 , \quad 3\beta "w_j - "\alpha_j = 0 ,
\]

where

\[
\begin{aligned}
\\
(\frac{\partial}{\partial x^i}) \Psi = \exp(\frac{3}{3} \varpi), \quad \frac{3}{3} \varpi = \ln |"g_4^{3/2}|/|g_3 g_4| ,
\end{aligned}
\]

\((i_2, j_2, k_2 = 1, 2, 3, 4)\)
The solutions for shell $s = 4$ can either have a fixed energy-like coordinate or be dependent of the energy-like coordinate. We have

$$d_4 s^2 [\,^" E\,] = \,^" g^a_4 (x^a, y^a_2, \,^" p_{a3}, \,^" E)(\,^" e_{a_4})^2$$

$$(2.29)$$

$$= \,^" \zeta^7 (1 + \kappa \,^" \chi^7) \,^" \hat{g}^7 \left\{ d^n p_7$$

$$+ \left( \,^" \hat{N}_{i37} \right)^7 \right\}^{n_{i3} + 16 \,^" n_{i3}} \int d^n E \left[ \left( \left( \,^" \zeta^7 \,^" \hat{g}^7 \right)^{-1/4} \right)^2 \right]$$

$$- \kappa \int d^n E \left[ \left( \left( \,^" \zeta^7 \,^" \hat{g}^7 \right)^{-1/4} \right)^2 \right]$$

$$\times \left( \,^" \hat{N}_{i38} \right)^7 \left\{ d^n E \left[ \left( \,^" \hat{K} \right)^7 \right] \right\}.$$
allowing for the generation of exact solutions of nonlinear systems of partial differential
generalizations. This approach was carried out in N-adapted and s-adapted forms \[19, 20\]

vibrated by string and M-theory and various nonassociative and noncommutative geometric

\[\Psi(h, \kappa; x^{k_1}), \Psi(h, \kappa; x^{k_2})\]

also many of these simple solutions have purely

the Kerr-Newmann family of stationary vacuum metrics. There are also generalizations of

Constructing BH configurations is important in theories of gravity. BH solutions are exact

solutionsofvacuumgravitationalequations, whichhaveahorizonthatactsasaboundaryof

3 Nonassociative star R-flux distortions of 8-d and (4+4)-d BHs

We note that any generic, off-diagonal, target solution \(\tilde{g} = \tilde{g}_{\alpha_0, \beta_0}(x, p) \tilde{e}^\alpha \otimes \tilde{e}^\beta\) of type (2.27) determined by the above quasi-stationary conditions depends on \(h, \kappa\) for any R-flux nonassociative data from \(\tilde{K}\).

A nonassociative quasi-stationary phase space solution (2.27) is characterized by generating sources involving \(\eta\)-polarization functions (2.23) and \(\kappa\)-coefficients (2.25) with re-defined parametric generating functions (2.26). Using such nonlinear transforms, we can re-define the generating functions and generating sources into equivalent data with effective cosmological constants, when some s-metric coefficients or polarization functions can be used as generating functions,

\[
\left( s\Psi, \tilde{g}\tilde{K} \right) \leftrightarrow \left( s\Psi, \tilde{g}\tilde{K} \right) \leftrightarrow \left( s\Psi, \tilde{g}\tilde{K} \right) \leftrightarrow \left( s\Psi, \tilde{g}\tilde{K} \right) \leftrightarrow (2.32)
\]

We provide respective formulas and examples in appendix A.2.

3 Nonassociative star R-flux distortions of 8-d and (4+4)-d BHs

Constructing BH configurations is important in theories of gravity. BH solutions are exact solutions of vacuum gravitational equations, which have a horizon that acts as a boundary of a spatial region whose causal future is trapped. In GR, general BH solutions are determined by three parameters: charge, mass, and angular momentum. Such solutions are known as the Kerr-Newmann family of stationary vacuum metrics. There are also generalizations of BH solutions with non-trivial cosmological constants and/or non-trivial electro-magnetic/ gauge fields, \[34–37\]. Many of these stationary, vacuum solutions are characterized by a curvature singularity inside the horizon. Also many of these simple solutions have purely diagonal metrics with spherical/ cylindrical symmetry. In this way the vacuum Einstein equations are transformed into a system of nonlinear ordinary differential equations, ODEs.

Here we investigate nonassociative phase space generalizations of BH solutions encoding \(\star\)-products and R-flux contributions. The corresponding geometric constructions is carried out in 8-d \(\star\)-product deformed (co) tangent Lorentz bundles. This can be motivated by string and M-theory and various nonassociative and noncommutative geometric generalizations. This approach was carried out in N-adapted and s-adapted forms \[19, 20\] allowing for the generation of exact solutions of nonlinear systems of partial differential
equations, PDEs. For nonassociative higher dimension models, there are several substantial differences involving certain physical features which are known for nonassociative, nonholonomic 4-d configurations [21]. There are also similar features for 8-d, 10-d, 11-d theories of associative (super) string gravity, supergravity and (super) geometric flows [33, 43]:

- In the class of nonassociative vacuum gravity theories of type [17–21], we have non-trivial, real, effective sources \(q_{\tilde{K}}\) which via nonlinear symmetries (A.10) are related to respective shell cosmological constants \(s\Lambda_0\).

- Nonassociative s-metrics \(\gamma_{\alpha\beta}\) are characterized by symmetric \(\gamma_{\alpha\beta}(A.3)\) and nonsymmetric components \(\gamma_{\alpha\beta} (A.4)\), which are generic off-diagonal with respect to coordinate frames.

- Adapting the geometric constructions to nonholonomic distributions, we can define an infinite number of (non) linear connections which, in general, are not metric compatible and with nonzero torsion. For certain subclasses of such metric compatible connections, we can consider physically important invariant conditions and generalized conservation laws, elaborate on nonholonomic Clifford structures with nonassociative modified Einstein-Dirac-Yang-Mills-Higgs systems.

- The nonassociative gravitational field equations (2.16) consist a nonlinear system of PDEs for the coefficients of nontrivial N-connection structure and corresponding off-diagonal metrics depending (in general) on all spacetime and phase space coordinates. Such PDEs can not be transformed in general form into ODEs using certain "simplified" diagonal ansatz for metrics with high symmetry and dependence, for instance, on a radial/ cylindric coordinates.

- The geometric and physical properties of certain classes of nonsymmetric nonholonomic solutions is investigated, We begin with some prime metrics for BH spacetimes and phase space commutative configurations and elaborate on some small parametric deformations to nonassociative models. By prime metric we mean a well known solution to ordinary GR.

Here, we focus our research on quasi-stationary s-metrics with star R-flux deformations of certain BH metrics embedded into \(M_4 \times M_4\); with distortions of 6-d Tangherlini solutions to 8-d, or to double 4-d BH metrics, when the target s-metrics define solutions of nonassociative vacuum gravitational equations (2.16).

3.1 Prime metrics for higher dimensional and phase space BHs

Here we analyze two classes of prime, phase space BH metrics taken as diagonal associative solutions of nonassociative vacuum gravitational equations. The first class of solutions we consider is of the Tangherlini type [41, 42] but having a different signature and with phase space dimensions. The second class of solutions is for configurations with double BHs — with the base being spacetime coordinates and with the cofiber having momentum coordinates. Such prime metrics were used for generating associative and commutative
stationary BH solutions in relativistic Finsler-Lagrange-Hamilton gravity \[33\]. In the next subsections, we shall nonholonomically deform both classes of BH solutions to off-diagonal configurations generating exact and parametric solutions of nonassociative vacuum gravitational equations. Via effective sources, such symmetric and nonsymmetric metrics encode R-flux contributions and spacetime — cofiber correlations.

### 3.1.1 Tangerlini 6-d BHs embedded and distorted in curved 8-d phase space

For a 5-d phase space with local coordinates \((x^i, p_5, p_6)\) [when \(i = 1, 2, 3; x^4 = t\)] and metrics of signature \((+ \ldots +)\), we introduce a corresponding radial coordinate \('r',\)

\[
'r = \sqrt{(x^1)^2 + (x^2)^2 + (y^3)^2 + (p_5)^2 + (p_6)^2}
\]

or we can introduce a different radial coordinate with complex momentum variables

\[
''r = \sqrt{(x^1)^2 + (x^2)^2 + (y^3)^2 - \hbar^2[('p_5)^2 + ('p_6)^2]}.
\]

In the above we are working in units where the dimensions of position and momentum are equivalent — see also conventions on coordinates and indices \((A.1)\). The 6-d spherical coordinates are parameterized as \(x^1 = 'r, x^2 = \theta, y^3 = \varphi, x^4 = t; p_5, p_6\) and the shell \(s = 4\) is parameterized by momentum coordinates \(p_r, p_8 = E\). For the cofiber space we use a trivial embedding into a 8-d phase space. For the Tangerlini solution we find it necessary to define the radial, metric function

\[
h('r') = 1 - \frac{'\mu}{('r)^3} - \frac{'\kappa_6^2 \Lambda}{10} ('r)^2. \tag{3.1}
\]

Here \('\kappa_6\) is a constant determined by the effective gravitational constant in 6-d and the constant \('\mu = ('\kappa_6^2 \Gamma[5/2]/4\pi^{5/2}) 'M\) is for an effective mass \('M\) and with \(\Gamma[5/2]\), being the gamma function — more details can be found in \([20, 41, 42]\). Such an \(h('r')\) is used for constructing 6-d Schwarzschild-de Sitter solutions with an effective cosmological constant \('\Lambda\).

Static 6-d phase space s-metrics of type \(^s\hat{g} (2.21)\) can be generated by quadratic linear elements

\[
d{^s}s^2 = \hat{g}_{\alpha\beta}(x^i, p_\alpha)d \ 'u^\alpha d \ 'u^\beta
\]

\[
= h^{-1}( 'r)d( 'r)^2 - h( 'r)dt^2 + ( 'r)^2d\Omega^2_4 + (dp_r)^2 - dE^2 \tag{3.2}
\]

\[
= \hat{g}_{\alpha\beta}(x^i, p_\alpha)d \ ''u^\alpha d \ ''u^\beta
\]

\[
= h^{-1}( ''r)d( ''r)^2 - h( ''r)dt^2 + ( ''r)^2d\Omega^2_4 - h^2[(dp_r)^2 - dE^2],
\]

where the area of the 5-dimensional unit sphere is parameterized by coordinates on shells \(s = 1, 2, 3,\)

\[
d\Omega^2_4 = d\theta^2_3 + \sin^2 \theta_3[ d\theta^2_2 + \sin^2 \theta_2(d\theta^2_1 + \sin^2 \theta_1 d\varphi^2)],
\]

for \(\theta_1 = \theta\) is the standard polar angle. Such a diagonal metric is a trivial extension on coordinates \(u^7\) and \(u^8\). Equation (3.2) defines an exact vacuum solution of nonassociative
gravitational equations (2.16), with $\Lambda_0 = 2\Lambda_0 = 3\Lambda_0 = 1\Lambda$ and $4\Lambda_0 = 0$ on shells $s = 1, 2, 3$. In standard form, such a solution is associative and commutative. However, on phase space the solution may involve star R-flux information if it is constructed as a diagonal limit of a subclass of solutions determined by generating functions related to effective sources and shell cosmological constants via nonlinear symmetries (2.32). In general, for 8-d extensions and nontrivial 4-shell sources, such phase space modifications of the 6-d Tangherlini metrics [41] for higher dimension generalization of the Schwarzschild BH, are not exact solutions of the nonassociative vacuum equations. Nevertheless, it is possible to construct off-diagonal deformations (2.22) of any prime metric (3.2) to certain target quasi-stationary, nonassociative, vacuum phase spaces as we shall show in next subsections.

### 3.1.2 Spacetime and co-fiber phase space double BH configurations

Turning now to phase space BHs, we show that it is possible to construct double 4-d BH configurations on shells $s = 1, 2$ on the base spacetime and shells $s = 3, 4$ co-fiber spaces. We will use the spherical coordinate parametrization (3.2) with local coordinates $x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t$ for the spacetime coordinates and $p_5 = p_r, p_6 = p_\theta, p_7 = p_\varphi, p_8 = E$ for the momentum-energy phase space coordinates. More details about the conventions on coordinates and indices can be found in appendix (A.1). There are two conventional radial coordinates: one in coordinate space and one in momentum space define as

$$
\begin{align*}
  r &= \sqrt{(x^1)^2 + (x^2)^2 + (y^3)^2} \\
  p_r &= \sqrt{(p_5)^2 + (p_6)^2 + (p_7)^2},
\end{align*}
$$

(3.3)

In these formulas, prime indices indicate Cartesian coordinates e.g. $x^1 = r \sin \theta \cos \varphi$ etc. The label “p” on the left side indicates spherical type coordinates on co-fibers. The prime phase space metric is taken as in section 2.4.1 of [33],

$$
\begin{align*}
  ds^2 &= g_{\alpha\beta}(r, p_r)d^\alpha w^\alpha d^\beta w^\beta = g_{\alpha\beta}(r, p_r)d^\alpha u^\alpha d^\beta u^\beta \\
  &= f^{-1}(r)(d^\alpha d^\beta)^2 - f(r)dt^2 + p_\ell f^{-1}(p_r)(d^\alpha p_r)^2 - p_r^2 d\Omega^2 - p_\ell f(r)dt^2
\end{align*}
$$

(3.4)

with radial functions

$$
\begin{align*}
  f(r) &= 1 - \frac{\mu}{r} - \frac{\kappa_4^2 r^2}{3} \\
  p_\ell f(p_r) &= 1 - \frac{p_\mu}{p_r} - \frac{(p_\kappa)(p_\Lambda)(p_r)^2}{3},
\end{align*}
$$

(3.5)

and the 2-d solid angle given by

$$
\begin{align*}
  d\Omega^2 &= d\theta^2 + \sin^2 \theta d\varphi^2 \\
  dp_\ell \Omega^2 &= d\theta^2 + \sin^2(\theta) d(\varphi)^2, \text{ or } dp_\ell \Omega^2 = d\theta^2 + \sin^2(\theta) d(\varphi)^2.
\end{align*}
$$

In the above formulas $8\pi G_4 = \kappa_4^2 \Lambda$, where $G_4$ is the Newtonian gravitational constant, $\mu = \kappa_4^2 M \Gamma(3/2)/2\pi^{3/2}$ with $M$ being the BH mass and $\Gamma(3/2)$, is the Gamma function.
Further details can be found in [42]. In the co-fiber phase space one can write down similar relationship among constants e.g. $8\pi p G = p^2 k^2 p \Lambda$. The superscript $p$ indicates these are relationships in the co-fiber space. We will chose nonholonomic distributions to be of the form so that formulas labeled by "\(” and “\”” are equivalent.

A prime s-metric may define an exact vacuum solution of nonassociative gravitational equations (2.16), if the s-adapted nonholonomic distributions are subjected to nonlinear symmetries (2.32) when $1\Lambda_0 = 2\Lambda_0 = \Lambda$ and $3\Lambda_0 = 4\Lambda_0 = p \Lambda$. In this work we omit explicit computations of s-adapted components of nonsymmetric metrics and complex valued Ricci tensors for complex phase space double BH solutions. Such phase space and quasi-stationary solutions have different nonholonomic structures compared to similar solutions in Finsler-Lagrange-Hamilton gravity [33] where the effective sources/cosmological constants are not related to R-flux contributions. On the base Lorentz manifold, the radial function $f(r)$ from (3.4) defines a Schwarzschild - de Sitter configuration as in GR. A similar configuration is found for the co-fiber space, but is determined by $p f( p r)$.

### 3.2 Nonassociative quasi-stationary deformations of BH solutions with fixed energy parameter

Here we will construct explicit off-diagonal solutions of the type (2.27) for a fixed $E_0$ describing quasi stationary nonholonomic deformations of prime BH metrics (3.2) or (3.4). For simplicity, we consider only linear $\kappa$-dependent nonassociative contributions with the other generating functions and generating sources subject to nonlinear symmetries as in appendix A.2, see formulas (A.10). Such solutions have nonholonomic induced torsion but we can always extract LC-configurations. Prescribing generating sources $\gamma K$ (2.20) we impose certain nonholonomic constraints on distributions of nonassociative sources which allow us to find generic off-diagonal solutions in explicit form.

#### 3.2.1 Off-diagonal parametric R-flux deformations of Tangerlini BHs

In this subsection we consider the prime diagonal metric (3.2) but under a non-singular coordinate transformation $(r, \theta, \varphi, \theta_2, \theta_3, p_\gamma, E) \to (u^\beta)$, $\tilde{g}_{\alpha s}(u^\beta s) = \tilde{h}_s g_{\alpha s}[r, \theta, \varphi, \theta_2, \theta_3, p_\gamma]$, and with non-zero N-coefficients $\tilde{N}^{a s}_{r s - 1}(u^\beta s)$. In this way the non-zero s-adapted coefficients of metric (3.2) become

\[
\begin{align*}
\tilde{h}_s g_1 &= h^{-1}(r), \\
\tilde{h}_s g_2 &= (r)^2 \sin^2 \theta_2 \cdot \sin^2 \theta_3, \\
\tilde{h}_s g_3 &= (r)^2 \sin^2 \theta \cdot \sin^2 \theta_2 \cdot \sin^2 \theta_3, \\
\tilde{h}_s g_4 &= - h(r), \\
\tilde{h}_s g_5 &= (r)^2 \sin^2 \theta_3, \\
\tilde{h}_s g_6 &= (r)^2, \\
\tilde{h}_s g_7 &= 1, \\
\tilde{h}_s g_8 &= -1; \\
\tilde{h}_s \tilde{N}^{a s}_{r s - 1} &\neq 0, \text{ but } \tilde{h}_s \tilde{N}^{a s}_{r s - 1}(u^\beta s) \to \tilde{N}^{a s}_{r s - 1}(r, \theta, \varphi, \theta_2, \theta_3, p_\gamma, E) = 0.
\end{align*}
\]  

In these formulas, we use a left label $h$ in order to emphasize that such diagonalizable s-metrics are determined by a radial function $h(r)$ (3.1). We have used similar labels for the N-connection coefficients $\tilde{h}_s \tilde{N}^{a s}_{r s - 1}$. For this class of prime metrics, denoted with over-circles, the form of the metric components and N-coefficients are given by coordinate transforms and encode physical parameters of the prime BHs.
We now apply the parametric deformations of type (2.27), with a fixed energy parameter \(E_0 = \text{const.}\) to the metric and N-coefficients of (3.6). The result is:

\[
d^{\Delta} \tilde{s}^2 = d^{\Delta} \tilde{s}^2(\, r, \theta, \varphi, \theta_2, \theta_3, p_r; E_0; \ h(\, r), \ \kappa, \ \Lambda_0) = d^{\Delta} \tilde{s}^2 = (r, \theta, \varphi, \theta_2, \theta_3, p_r; E_0; \ h(\, r), \ \kappa, \ \Lambda_0) = d \tilde{s}^2(\, r, \varphi; \ h(\, r), \ \kappa, \ \Lambda_0) + d \Delta \tilde{s}^2(\, r, \varphi, \theta; \ h(\, r), \ \kappa, \ \Lambda_0) + d \Delta \tilde{s}^2(\, r, \theta, \varphi, \theta_2, \theta_3; \ h(\, r), \ \kappa, \ \Lambda_0)
\]

On shell 1 the components are:

\[
d \tilde{s}^2 = \tilde{g}_{i1}(x^i)[(dx^{i1})^2] = e^{\psi_0}(x^i)\left[1 + \kappa \psi \eta \chi(x^i)\right]\left[(dx^{i1})^2 + (dx^{i2})^2\right]
\]

On shell 2, the s-metric (3.7) is determined by

\[
d \Delta \tilde{s}^2 = g_{a_2}(\, r, \theta, \varphi|a^2(\, r, \theta, \varphi)|)^2
\]

with generating functions \(\kappa \psi \eta \chi\) and \(\kappa \psi \eta \chi\); R-flux effective source \(\kappa \theta\); integration functions \(n_{k_1}(\, r, \theta, \varphi)\) and \(n_{k_2}(\, r, \theta, \varphi)\) and components of the prime metric \(\tilde{g}_{31}(\, r, \theta)\) and \(\tilde{g}_{41}(\, r, \theta)\) from (3.6). The non-zero prime N-connection coefficients, \(\Delta \tilde{N}_{a_2}^{k_1}\), are introduced via coordinate transforms \(\Delta \tilde{N}_{k_1}^{a_2}(\, r, \theta, \varphi)\) and, in general, give rise to generic off-diagonal metrics.
For shell $s = 3$ on the co-fiber space, the nonlinear quadratic component of (3.7), with Killing symmetry on $\theta_2$ and ($(p_5, p_6) \rightarrow (\theta_2, \theta_3)$), is

$$d\, \hat{s}^2 = \hat{g}^{a3}(\ 'r, \theta, \varphi, \phi) [\ 'e_{a3}(\ 'r, \theta, \varphi, \phi)]^2$$

$$= \xi^5(1 + \kappa \chi^5) \hat{h}^{35} \left\{ d\theta_2 + \left[ (\ '\bar{\bar{N}}_{i2})^{-1} \left[ \hat{\eta}^{n_{i2}} \right. + 16 \hat{\eta}^{n_{i2}} \right] \int d\theta_3 \left( \hat{\partial}_{\partial\theta_3} \left[ \left( \xi^{53} \hat{h}^{35} \right)^{-1/4} \right] \right)^2 \right. + \left. 16 \hat{\eta}^{n_{i2}} \int d\theta_3 \left( \hat{\partial}_{\partial\theta_3} \left[ \left( \xi^{53} \hat{h}^{35} \right)^{-1/4} \right] \right) \right] \right\}$$

$$+ \kappa \hat{\eta}^{n_{i2}} \int d\theta_3 \left( \hat{\partial}_{\partial\theta_3} \left[ \hat{\partial}_{\partial\theta_3} \left[ \left( \xi^{53} \hat{h}^{35} \right)^{-1/4} \right] \right) \right)^2 + \int d\theta_3 \left( \hat{\partial}_{\partial\theta_3} \left[ \hat{\partial}_{\partial\theta_3} \left( \hat{\partial}_{\partial\theta_3} \left[ \xi^{53} \hat{h}^{35} \right] \right) \right] \right) \right\}$$

$$(\ '\bar{\bar{N}}_{i2})d\, x^{i2} = \left\{ - \frac{4}{\hat{h}^{35}} \int d\theta_3 \left[ \hat{\partial}_{\partial\theta_3} \left[ \hat{\partial}_{\partial\theta_3} \left[ \xi^{53} \hat{h}^{35} \right] \right] \right] \right\}$$

with generating functions $\xi^5(\ 'r, \theta, \varphi, \phi)$ and $\chi^5(\ 'r, \theta, \varphi, \phi)$; R-flux effective source $\xi^5(\ 'r, \theta, \varphi, \phi)$; integration functions $\hat{\eta}^{n_{i2}}(\ 'r, \theta, \varphi, \phi)$ and $\hat{\eta}^{n_{i2}}(\ 'r, \theta, \varphi, \phi)$; and prime metric components $\hat{h}^{35}(\ 'r, \theta, \varphi)$ and $\hat{h}^{35}(\ 'r, \theta, \varphi)$ from (3.6). The non-zero, prime N-connection coefficients, $\bar{\bar{N}}^{n_{i2}}_{k3}$, arise via coordinate transformations $\bar{\bar{N}}^{n_{i2}}_{k3}(\ 'r, \theta, \phi) \rightarrow \bar{\bar{N}}^{n_{i2}}_{k3}(\ 'r, \theta, \varphi, \phi)$ and lead to generic off-diagonal metrics.

Finally for shell 4, we consider solutions (3.7) with $E = E_0 = const$, i.e. with Killing symmetry on $p^8$, and variable $p_7$, on the co-fiber space. This yields the metric

$$d\, \hat{s}^2 = \hat{g}^{a3}(\ 'r, \theta, \varphi, \phi, \theta_3, 'p_7, 'E_0)[\ 'e_{a3}(\ 'r, \theta, \varphi, \phi, \theta_3, 'p_7, 'E_0)]^2$$

$$= - \frac{4}{\hat{h}^{35}} \int d\theta_3 \left[ \hat{\partial}_{\partial\theta_3} \left[ \hat{\partial}_{\partial\theta_3} \left[ \xi^{53} \hat{h}^{35} \right] \right) \right] + \kappa \left[ \hat{\partial}_{\partial\theta_3} \left[ \hat{\partial}_{\partial\theta_3} \left[ \xi^{53} \hat{h}^{35} \right] \right) \right]$$

$$(\ '\bar{\bar{N}}_{i3})d\, x^{i3} = \left\{ \hat{\partial}_{\partial\theta_3} \left[ \hat{\partial}_{\partial\theta_3} \left[ \xi^{53} \hat{h}^{35} \right] \right) \right] \right\}$$

$$(\ '\bar{\bar{N}}_{i3})d\, x^{i3} = \left\{ \hat{\partial}_{\partial\theta_3} \left[ \hat{\partial}_{\partial\theta_3} \left[ \xi^{53} \hat{h}^{35} \right] \right) \right] \right\}$$
\[ + \zeta^8 (1 + \kappa \chi^8) \hat{g}^8 \left\{ d^\nu \mathcal{E} + \left( \hat{N}_{i3} \right)^{-1} \left[ d^{i3} \right] \right\} \\
+ \frac{16}{2} n_{i3} \left[ \int d^4 p_r \left\{ \left( \hat{N}^8 \hat{g}^8 \right)^{-1/4} \right\} \right] \\
- \frac{16}{2} n_{i3} \left[ \int d^4 p_r \left( \hat{N}^8 \hat{g}^8 \right)^{-1/4} \right] \left[ \int d^4 p_r \left( \hat{N}^8 \hat{g}^8 \right)^{-1/4} \right] \\
+ \frac{16}{2} n_{i3} \left[ \int d^4 p_r \left( \hat{N}^8 \hat{g}^8 \right)^{-1/4} \right] \left[ \int d^4 p_r \left( \hat{N}^8 \hat{g}^8 \right)^{-1/4} \right] \\
\times \left( \hat{N}_{i3} \right) \right] \\
\]

with generating functions \( \zeta^8 (r, \theta, \varphi, \theta_2, \theta_3) \) and \( \chi^8 (r, \theta, \varphi, \theta_2, \theta_3) \); R-flux effective source \( \hat{N}^8 \hat{g}^8 \); integration functions \( n_{i3} \) and \( \hat{N}_{i3} \); and components of the prime metric \( \hat{N}_i^a \) and \( \hat{N}_i^a \) from (3.6). The non-zero, prime N-connection coefficients \( \hat{N}_i^a \) arise via the coordinate transformations \( \hat{N}_i^a \) and \( \hat{N}_i^a \) into (3.6), and result in generic off-diagonal metrics.

We can deform the solutions of (3.7) into ellipsoidal form by choosing the generating functions in (3.9) as:

\[ \chi_4 = \chi_4 (r, \theta, \varphi) = 2 \chi (r, \theta) \sin(\omega_0 \varphi + \varphi_0), \]

(3.12)

The function \( \chi (r, \theta) \) can be a smooth function, or a constant, and \( \omega_0 \) and \( \varphi_0 \) are some constants. On the spacetime manifold, this s-metric has an ellipsoidal horizon with eccentricity \( \kappa \) given by

\[ (1 + \kappa \chi_4) \hat{g}^8 = 1 - \frac{i \mu}{(r)^3} - \frac{\kappa^2}{10} (r)^2 + \kappa \chi_4 = 0 \]

Considering configurations with \( \frac{\kappa^2}{10} (r)^2 \approx 0 \), we can approximate \( r \approx i \mu^{1/3} / (1 - \kappa^2 \chi_4) \), which defines an ellipsoidal horizon.

Next the Tangerini BH metric in (3.7) can be deformed into a family of nonassociative, quasi-stationary BHs, with a fixed energy parameter. The deformed metric splits in a symmetric, \( \hat{q}_{\alpha \beta} \), and antisymmetric, \( \hat{a}_{\alpha \beta} \), s-metrics. The s-adapted coefficients of \( \hat{q}_{\alpha \beta} \), and \( \hat{a}_{\alpha \beta} \), can be computed in explicit form for any prescribed R-flux, \( \hat{R}^{\alpha \beta} \), by introducing the s-metric in (3.3) and the s-connection in (3.4) yielding

\[ \hat{q}_{\alpha \beta} = \frac{i \kappa}{2} \left( \hat{R}^{\alpha \beta} \right) \hat{e}_{\alpha} \hat{e}_{\beta} \]

and

\[ \hat{a}_{\alpha \beta} = \frac{i \kappa}{2} \left( \hat{R}^{\alpha \beta} \right) \hat{e}_{\alpha} \hat{e}_{\beta} \]

We omit the technical details for computing \( \hat{q}_{\alpha \beta} = \hat{q}_{\alpha \beta} \) and \( \hat{a}_{\alpha \beta} = \hat{a}_{\alpha \beta} \), given in (3.3). For quasi-stationary R-flux deformations considered in this section, such nonassociative s-metrics are induced and completely determined by \( \left( \hat{g}_{\alpha \beta}, \hat{N}_{\alpha \beta} \right) \), with coefficients computed above or in other forms with different generating functions. Ellipsoidal base spacetime configurations (3.12) are modelled for functions \( \chi_4 = \chi_4 (r, \theta, \varphi) \).
### 3.2.2 Spacetime and co-fiber phase space double BHs and BEs with fixed energy parameter

Starting with the prime metric (3.4) we can generate nonassociative configurations with metric components of the form \[^\prime g_{\alpha}\] \(( \omega^{\beta} ) = [^\prime g_{\alpha}, [^\prime u^{\beta}(r, \theta, \varphi; p_{r}, p_{\theta}, p_{\varphi})] \) and nontrivial \(N\)-coefficients \[^{\prime \prime}N_{i_{s-1}}\] \(( \omega^{\beta} )\) determined by some non-singular coordinate transforms \((r, \theta, \varphi, t, p_{r}, p_{\theta}, p_{\varphi}, E) \rightarrow [^\prime \omega^{\beta}, \) or \((r, \theta, \varphi, t, p_{r}, p_{\theta}, p_{\varphi}, E) \rightarrow [^\prime \omega^{\beta}].\) The metric components and non-zero \(N\)-coefficients are

\[
\begin{align*}
[^\prime \hat{g}_1] &= f^{-1}(r), \quad [^\prime \hat{g}_2] = r^2, \quad [^\prime \hat{g}_3] = r^2 \sin^2 \theta, \quad [^\prime \hat{g}_4] = -f(r), \\
[^\prime \hat{g}_5] &= p_n f^{-1}(p_r), \quad [^\prime \hat{g}_6] = (p_r)^2, \quad [^\prime \hat{g}_7] = (p_r)^2 \sin^2 \theta, \quad [^\prime \hat{g}_8] = -p_n f^{-1}(p_r);
\end{align*}
\]

\(^{\prime \prime}N_{i_{s-1}}\) \(\neq 0\), but \(^{\prime \prime}N_{i_{s-1}}\) \(\rightarrow [^\prime \hat{g}_{\alpha}, \) \(\rightarrow [^\prime \hat{g}_{\alpha}, \) \((r, \theta, \varphi, t, p_{r}, p_{\theta}, p_{\varphi}, E) = 0.\)

In these formulas, a left label \(f\) emphasizes that such diagonalizable \(s\)-metrics are determined respectively by two radial functions \(f^{-1}(r)\) and \(p_n f^{-1}(p_r)\). The nontrivial \(N\)-connection coefficients \(^{\prime \prime}N_{i_{s-1}}\) \(\neq 0\) can be generated by coordinate transformations and encode physical parameters of the prime BH metric. Substituting, shell by shell, the prime metric components, \(^{\prime \prime}g_{\alpha}\), from (3.13) into formulas (3.8), (3.9), (3.10), and (3.11), we can construct a nonlinear quadratic element of type (3.7). These parametric, generic off-diagonal solutions describe nonassociative, R-flux deformations of double BH configurations, where the first prime metric is on the base spacetime manifold and the second prime metric in the co-fiber subspace. In a more compact form, this class of solutions can be written in terms of \(^{\prime \prime}g_{\alpha}\) and \(^{\prime \prime}N_{i_{s-1}}\) as

\[
^{\prime \prime}g_{\alpha} = \text{diag}[g_1 = \eta_1(r) f^{-1}(r), g_2 = \eta_2(r, \theta, \varphi) r^2],
\]

\[
g_3 = -\left\{\frac{4}{r^2 \sin^2 \theta} \int d\varphi \left[\frac{2K(r, \theta, \varphi) \partial_{\varphi} \left[ \zeta_4(r, \theta, \varphi) \right]}{f(r)^{1/2}} \right] \right\} r^2 \sin^2 \theta,
\]

\[
g_4 = -\zeta_4(r, \theta, \varphi)[1 + \kappa \chi_4(r, \theta, \varphi)] f(r),
\]

\[
^{\prime \prime}g_5 = \zeta^5(p_r, p_{\theta}, p_{\varphi})[1 + \kappa \chi^5(p_r, p_{\theta}, p_{\varphi})] p_n f^{-1}(p_r),
\]

\[
^{\prime \prime}g_6 = -\left\{\frac{4}{r^2 \sin^2 \theta} \int d\varphi \left[\frac{\zeta^5(p_r, p_{\theta}, p_{\varphi})}{f(r)^{1/2}} \right] \right\} (p_r)^2,
\]
and for the N-connection

\[ \mathcal{N}_{i1}^3 = \left[ (\mathcal{N}_{+1}^3)^{-1} \right]_{(r, \theta, \phi)} \left[ \int d\phi \left( \frac{\partial_{\phi}[\mathcal{N}_4(r, \theta, \phi)]}{\int d\phi \partial_{\phi}[\mathcal{N}_4(r, \theta, \phi)]} \right)^2 \right]_{(r, \theta, \phi)} \]

(3.15)

\[ \mathcal{N}_{i25} = \left[ (\mathcal{N}_{+25})^{-1} \right]_{(r, \theta, \phi)} \left[ \int d\phi d\theta \left( \frac{\partial_{\phi}[\mathcal{N}_5(r, \theta, \phi)]}{\int d\phi d\theta \partial_{\phi}[\mathcal{N}_5(r, \theta, \phi)]} \right)^2 \right]_{(r, \theta, \phi)} \]

(3.15)
Here \( \chi \) is a smooth function (or constant), and the pairs \( (\omega, \varphi_0) \) and \( (p_0, \varphi_0) \) are constants. Both on the spacetime manifold, and on the typical fiber space, such an s-metric possesses two distinct ellipsoidal horizons with eccentricities, \( \kappa_4 \) and \( p_\kappa \), given by

\[
(1 + \kappa \chi_4) \, g_4 = 1 - \frac{\mu}{r^3} - \frac{\kappa_4^2 \Lambda}{6} r^2 + \kappa \chi_4 = 0 \quad \text{and} \quad (1 + p_\kappa \chi^8) \, g^8 = 1 - \frac{p_\mu}{p_r} - \frac{(p_\kappa \Lambda)(p_r)^2}{2} + p_\kappa \chi^8.
\]
Here $\zeta_4 \neq 0$ and $\zeta^8 \neq 0$. The physical meaning of these constants is explained just after formulas (3.5). For small parametric deformations with $\kappa^2_4 A r^2 / 6 \approx 0$ and $(p\kappa)^2(\rho\Lambda)(p\rho)^2 / 6 \approx 0$, we can make the approximations

$$r \approx \mu^{1/3} / \left(1 - \frac{K}{3} \chi_4\right) \quad \text{and} \quad p r \approx (p\mu)^{1/3} / \left(1 - \frac{p_K}{3} \chi^8\right).$$

These are parametric formulas for ellipsoidal horizons are defined by small gravitational R-flux polarizations. In the limit when the eccentricities go to zero, these double BE configurations transform into standard prime double BH metric given in (3.13).

We conclude this section by noting the following two points:

- We could also used AFCDM to construct such parametric, off-diagonal solutions describing nonassociative R-flux deformations of Tangherlini or double BH and BE metrics but with a Killing symmetry on $\rho^7$ and with explicit dependence on $p_8 = E$. Such quasi-stationary solutions were found in [33] in MGTs with MDRs (modified dispersion relations and generalized Finsler-Lagrange-Hamilton gravity).

- The nonlinear quadratic elements, with off-diagonal metric components given by (3.14), (3.15)) are characterized by nontrivial, nonholonomically induced s-torsion structures (2.11). Nevertheless, we can always extract zero torsion, LC-configurations for more special classes of generating functions and effective sources.

4 Generalized Bekenstein-Hawking and Perelman thermodynamics of nonassociative BHs

In this section we investigate the thermodynamics for the quasi-stationary, parametric solutions outlined in the previous section. We find results similar to those in section 5 of [33]. Here the effective sources and generating functions are chosen so as to encode nonassociative, R-flux deformations. We find that the entropy is completely determined by the 8-d phase space quantities $(\bar{g}_{\alpha\beta}, \bar{\kappa}_{\mu\nu})$. Also only for very special classes of small parametric deformations of the original, prime metrics is it possible to formulate thermodynamics using Bekenstein-Hawking entropy [44–47]. In general, nonassociative, R-flux deformations of metrics will result in nonholonomic, off-diagonal configurations without conventional horizons. It is this lack of a horizon that makes it impossible to define a Bekenstein-Hawking entropy for these spacetimes. In contrast we find that for all of these quasi-stationary solutions one can define an entropy by using the generalized Perelman entropy functionals for nonholonomic Ricci solitons [33, 39, 40, 43, 48]. These nonholonomic, Ricci solitons are defined as self-similar geometric flow configurations for a fixed flow parameter.

4.1 Phase space generalizations of the Bekenstein-Hawking entropy for prime and nonassociative target s-metrics

For parametric solutions with conventional spherical/ ellipsoidal horizons in (non) associative/ commutative phase space gravity, we can apply standard Bekenstein-Hawking BH thermodynamics. In this subsection, we study two such examples:
4.1.1 The entropy and temperature of Tangerlini BHs and BEs

The prime Tangerlini-like s-metric \( \tilde{g} \) given in (3.2) is a trivial embedding of a 6-d BH into a 8-d phase space. For this spacetime, we can define the Hawking temperature as that of a Schwarzschild BH with dimension \( n' = 4 + m' = 6 \) - see [33, 49]. This Hawking temperature is similar to that of a Schwarzschild BH in \( n' \geq 4 \) spacetime dimensions,

\[
T = \frac{m' + 1}{4\pi} \left( \frac{\Omega_{m'+2}}{4} \right)^{1/(m'+2)} S_0^{-1/(m'+2)}.
\]

Here the Bekenstein-Hawking entropy is given by

\[
S_0 = \frac{\Omega_{m'+2} \times (\hat{\tau}_0)^{m'+2}}{4}.
\]

The results in (4.1) and (4.2) differ from the standard temperature and entropy, by the number of dimensions and by the dependence of \( \hat{\tau} \) on both coordinate and momentum variables.

We now deform the metric in (3.7) by the function \( \chi_4 = 2\chi \sin(\omega_0\varphi + \varphi_0) \), with \( \chi = \text{const} \) and \( \hat{\tau}_0 = \left( \frac{2\chi \hat{M}}{4\pi} \right)^{1/3} = '\mu^{1/3} \) as in (3.12). This deformed metric has an ellipsoidal horizon with eccentricity \( \varepsilon = 2\chi\kappa/3 \), and is described by the parametric formula \( 'r \approx 'r_0 \sin(\omega_0\varphi + \varphi_0) \). For small \( \varepsilon \) and \( \hat{\tau}_0 \rightarrow \hat{\tau}_0(\varphi) \), we obtain the following anisotropic modifications of the Hawking temperature and Bekenstein-Hawking entropy,

\[
T(\varphi) = \frac{3}{4\pi} \left( \frac{\Omega_4}{4} \right)^{1/4} S_0^{-1/4}(\varphi) \quad \text{with} \quad S_0(\varphi) = \frac{\Omega_4 \times ('\mu)^{4/3}}{4}(1 + 4\chi\kappa/3 \sin(\omega_0\varphi + \varphi_0)).
\]

This example is a very special class of R-flux deformations described by parametric solutions with ellipsoidal horizons and integration constant \( \chi \). The results in (4.3) become the temperature and entropy in (4.1) and (4.2) if the nonassociative deformations are taken to zero, \( \chi\kappa \rightarrow 0 \).

4.1.2 Phase space thermodynamics for double 4-d Schwarzschild BHs and BEs

In this subsection we show that one can extend the concept of the Bekenstein-Hawking entropy to the double 4-d BH and BE configurations of (3.14) and (3.15).

For the two BEs from (4.3) the Bekenstein-Hawking temperature and entropy are

\[
T(\varphi) = \frac{1}{4\pi} \left( \frac{\Omega_2}{4} \right)^{1/2} \left[ S_0^{-1/2}(\varphi) + S_0^{-1/2}(p,\varphi) \right], \quad \text{with}
\]

\[
S_0(\varphi) = \frac{\Omega_2 \times ('\mu)^{4/3}}{4}(1 + 4\chi\kappa/3 \sin(\omega_0\varphi + \varphi_0)) \quad \text{and}
\]

\[
S_0(p,\varphi) = \frac{\Omega_2 \times ('\mu)^{4/3}}{4}(1 + 4\chi\kappa/3(p,\omega_0,\varphi + p,\varphi_0)),
\]

\[\text{JHEP05(2023)057}\]
where $\Omega_2 = \frac{\pi^2}{12}$, is the volume of the unit-sphere in 4-d. The horizon in the 4-d spacetime and the horizon in the 4-d co-fiber are computed using conventional mass parameters using (3.3) and (3.5). If $\chi \kappa \to 0$ the BEs transform into BH configurations, with the standard temperatures.

For the spacetimes considered in this subsection and the previous subsection, it was possible to define Bekenstein-Hawking thermodynamic variables. In the next section we examine generalizes spacetimes, with both regular coordinates and momentum coordinates, for which it is not to define Bekenstein-Hawking thermodynamics variables, but for which it is possible to define thermodynamic variables using Perelman’s entropy functional [48].

4.2 Perelman’s thermodynamic variables for nonassociative quasi-stationary vacuum configurations

A Hawking temperature and a Bekenstein-Hawking entropy can often not be defined for solutions of modified GR which do not possess horizons. In reference [33, 39, 40, 43] we studied statistical and geometric thermodynamics models which were derived from Perelman’s W-entropy functional [48] for relativistic flows of phase spaces. Perelman’s W-entropy can be applied to general spacetimes from various gravity theories [19, 20] for which Bekenstein-Hawking thermodynamics do not apply. Despite the difficulty of formulating a theory of nonassociative geometric flows we can apply Perelman’s thermodynamic variables to parametric, quasi-stationary solutions in nonassociative MGTs. For certain classes of generating functions and sources, which encode R-flux deformations, we can formulate to parametric, quasi-stationary solutions in nonassociative MGTs. For certain classes of generating functions and sources, which encode R-flux deformations, we can formulate statistical thermodynamic models which are determined by $(g_{\alpha\beta}, N_{a_{i-1}}'')$.

4.2.1 A statistical analogy for nonholonomic phase space Ricci flows

Section 5 of [48] proposes a geometric, thermodynamic model for geometric flows of Riemannian metrics on a closed manifold $V', \dim V' = n'$. These geometric flows describe the evolution of a metric $g_{\alpha'\beta'}(\tau) \simeq g_{\alpha'\beta'}(\tau, u')$, as a function of the positive definite, temperature-like parameter $\tau$. A similar statistical analogy can be formulated on a phase space endowed with $\tau$-parametric flows of the quantities $(g_{\alpha\beta}(\tau), N_{a_{i-1}}'(\tau))$, and for the flows of the canonical $s$-connection $\hat{\nabla}_{\tau}$, Ricci $s$-tensor, $\hat{\nabla}_{\tau}$, and Ricci $s$-scalar, $\hat{\nabla}_{\tau}$, see formulas (2.14).

The Perelman F-functional and W-functional can be defined for the canonical quantities $(g_{\alpha}(\tau), \hat{\nabla}_{\tau}(\tau))$ as [33, 39, 40, 43]:

$$\hat{\nabla}_{\tau}(\tau) = \int_{\hat{\Xi}} e^{-\frac{i}{2} \hat{\nabla}_{\tau}} \sqrt{\hat{\nabla}_{\tau} \gamma_{\alpha\beta}} \gamma_{\delta} \gamma_{\delta} u (\hat{\nabla}_{\tau} \hat{\nabla}_{\tau} + |\hat{\nabla}_{\tau} \hat{\nabla}_{\tau}|^2),$$

$$\hat{\nabla}_{\tau}(\tau) = \int_{\hat{\Xi}} (4\pi \tau)^{-\frac{1}{2}} e^{-\frac{i}{2} \hat{\nabla}_{\tau}} \sqrt{\hat{\nabla}_{\tau} \gamma_{\alpha\beta}} \gamma_{\delta} \gamma_{\delta} u [\tau(\hat{\nabla}_{\tau} \hat{\nabla}_{\tau} + \sum_{a} |\hat{\nabla}_{\tau} \hat{\nabla}_{\tau}|^2 + \hat{\nabla}_{\tau} - 8].$$

The W-functional from (4.4) can be treated as a “minus entropy” for projections on space-like hypersurfaces.

In above formulas, the integrals and normalizing functions $\hat{\nabla}_{\tau}(\tau, u)$ satisfy the condition

$$\int_{\hat{\Xi}} |\hat{\nabla}_{\tau} \hat{\nabla}_{\tau} \gamma_{\alpha\beta}| \gamma_{\delta} \gamma_{\delta} u := \int_{\hat{\Xi}} \int_{\hat{\Xi}} \int_{\hat{\Xi}} \int_{\hat{\Xi}} \int_{\hat{\Xi}} \int_{\hat{\Xi}} \int_{\hat{\Xi}} \int_{\hat{\Xi}} |\hat{\nabla}_{\tau} \hat{\nabla}_{\tau} \gamma_{\alpha\beta}| \gamma_{\delta} \gamma_{\delta} u = 1,$$
for integration measures \( \delta \tilde{\nu} = (4\pi\tau)^{-d} e^{-\delta s} \). The integration measures are chosen as in [48] but for \( 8 - d \) with shell splitting and adapting normalizing functions in order to simplify formulas for \( s \)-adapted coefficients. Defining a class of normalizing shell functions, we define relativistic flow models of geometric objects. The W-functional (4.4) can be treated as a “minus entropy” only for projections on space like hypersurfaces. We can formulate relativistic thermodynamics models for a solution of (2.16) if \( \delta g \simeq \{ \delta g_{\alpha\beta} \} \) (2.5) has a causal \((3+1)+(3+1)\) splitting. We write

\[
\delta g(\tau) = \delta g_{\alpha\beta}(\tau, \delta u) d^d e^\alpha d^d e^\beta \\
= q_1(\tau, x^k) dx^i \otimes dx^j + q_3(\tau, x^k, y^3) e^3 \otimes e^3 - \tilde{N}^2(\tau, x^k, y^3) e^4 \otimes e^4 \\
+ \delta q^3(\tau, x^k, y^3, p_{b_2}) \delta e_{a_2} \otimes \delta e_{a_2} + \delta q^7(\tau, x^k, y^3, p_{b_2}, p_{b_3}) \delta e_7 \otimes \delta e_7 \\
- \delta \tilde{N}^2(\tau, x^k, y^3, p_{b_2}, p_{b_3}) \delta e_8 \otimes \delta e_8,
\]

where \( \delta e^\alpha \) are \( N \)-adapted bases in total phase space. Such an ansatz for \( 8 \)-d phase space \( s \)-metrics are written as an extension of a couple of 3-d metrics, \( q_{ij} = \text{diag}(q_i) = (q_1, q_3) \) on a hypersurface \( \Sigma_t \) and \( \delta q^{ab} = \text{diag}(\delta q^3, \delta q^7) \) on a hypersurface \( \Sigma_E \), if

\[
q_1 = g_1, q_2 = g_2, q_3 = g_3, \tilde{N}^2 = -g_4 \quad \text{and} \quad \delta q^3 = \delta g^3, \delta q^6 = \delta g^6, \delta q^7 = \delta g^7, \delta \tilde{N}^2 = -\delta g^8.
\]

In these formulas, \( \tilde{N} \) is the lapse function on the base and \( \delta \tilde{N}^2 \) is the lapse function in the co-fiber.

Next we introduce a general statistical partition function

\[
\delta \tilde{Z}(\tau) = \exp\left[ \Omega \delta \tilde{F} + 8 \right] (4\pi\tau)^{-d} e^{-\delta \tilde{F}} \delta \text{W}(\tau),
\]

with the volume element

\[
\delta \text{W}(\tau) = \sqrt{|q_1(\tau)q_2(\tau)q_3(\tau)\tilde{N}^2(\tau)|} (\delta q^3(\tau) \delta q^6(\tau) \delta q^7(\tau) \delta \tilde{N}^2(\tau))
\]

\[
dx^1 dx^2 \delta y^3 \delta y^4 \delta u^5(\tau) \delta u^6(\tau) \delta u^7(\tau) \delta u^8(\tau).
\]

Using the quantities \( \delta \tilde{Z} \) from (4.6) and \( \delta \tilde{W} \) from (4.4), we can define the Perelman thermodynamic values of energy, entropy and fluctuations:

\[
\delta \tilde{E} = -2\int_{\Sigma_t} \delta \tilde{R} s c + \delta \tilde{D} \delta \tilde{s} f^2 - \frac{4}{\tau} (4\pi\tau)^{-d} e^{-\delta \tilde{F}} \delta \text{W}(\tau),
\]

\[
\delta \tilde{S} = -\int_{\Sigma_t} \tau (\delta \tilde{R} s c + \delta \tilde{D} \delta \tilde{s} f^2 + \delta \tilde{F} - 8) (4\pi\tau)^{-d} e^{-\delta \tilde{F}} \delta \text{W}(\tau),
\]

\[
\delta \tilde{\sigma} = 2 \tau \int_{\Sigma_t} \delta \tilde{R}_{\alpha\beta} c + \delta \tilde{D}_{\alpha} \delta \tilde{D}_{\beta} \delta \tilde{s} f - \frac{1}{2\tau} q_{ij} \left( \delta \tilde{F} - 8 \right)^2 (4\pi\tau)^{-d} e^{-\delta \tilde{F}} \delta \text{W}(\tau)
\]

In the next sections, we show how Perelman’s W-entropy (4.4) and related thermodynamic variables (4.8) can be computed for quasi-stationary, phase space, deformed BE and BH solutions.
4.2.2 Phase space integration functions, measures and effective cosmological constants

To formulate geometric thermodynamic models, which give quasi-stationary solutions of the nonholonomic Einstein phase space equations \( \dot{\mathbf{R}}_{\beta \gamma \alpha} (\tau_0) = 4 \mathbf{K}_{\beta \gamma \alpha} [\tau_0, h, \kappa] \) (2.16), we will chose constant values for the normalizing functions, \( \dot{\mathbf{f}} = \frac{\dot{\mathbf{f}}_0}{\dot{\mathbf{f}}_0} = \text{const} = 0 \). These quasi-stationary solutions define a subclass of noncommutative Ricci solitons, which are self-similar configurations under nonholonomic Ricci flows, when the effective temperature is fixed as \( \tau = \tau_0 \). We will study examples where the constants for integration functions will substantially simplify the computation of Perelman’s thermodynamic variables. Then, using general frame/coordinate transformation to an arbitrary system of reference, the solutions can be given in general covariant form.

We now compute \( \sqrt{\text{g}((\mathbf{\gamma}) \tau, \tau_0) / s} \text{\hat{E}} \), (4.6) and \( \sqrt{\text{g}((\mathbf{\gamma}) \tau, \tau_0) / s} \text{\hat{E}} \) (4.8). An s-metric \( \text{g}(\tau), \) a \( \tau \)-dependent solution of (2.16), is generated by shift and laps s-coefficients, with integration functions fixed to constant values:

\[
q_1(\tau, x^k) = q_2(\tau, x^k) = e^{\psi(\tau)}, q_3(\tau, x^k) = \frac{4 \{ \partial_{\bar{\beta}}[2 \Phi(\tau)] \}^2}{\int dy^k 2K(\tau) (\partial_{\bar{\beta}}[2 \Phi(\tau)]^2)^2}, \\
[\dot{N}(\tau)]^2 = -g_4(\tau) = -g_4(\tau, x^k, y^3) = -\frac{[2 \Phi(\tau)]^2}{2 \Lambda_0(\tau)}; \\
\dot{q}_5(\tau) = \dot{q}_5(\tau, x^k, y^3) = \frac{[3 \Phi(\tau)]^2}{3 \Lambda_0(\tau)}, \\
\dot{q}_6(\tau) = \dot{q}_6(\tau, x^k, y^3) = \frac{[4 \{ \partial_{\bar{\beta}}[\frac{1}{2} \Phi(\tau)] \}^2}{\int dy^k 4K(\tau) (\partial_{\bar{\beta}}[\frac{1}{2} \Phi(\tau)]^2)^2}}, \\
\dot{q}_7(\tau) = \dot{q}_7(\tau, x^k, y^3) = -\dot{q}_8(\tau, x^k, y^3) = \frac{[4 \Phi(\tau)]^2}{4 \Lambda_0(\tau)}.
\]

Inserting these values into (4.4) we obtain

\[
\mathcal{W}(\tau_0) = \int_0^{\tau_0} \int_{\sqrt{\text{g}}(\tau)} \frac{d\tau}{4} \left[ \sqrt{\frac{\text{g}[\Phi(\tau)]}{\Lambda(\tau)}} \left( \frac{\Lambda(\tau)}{\Lambda(\tau)} \right)^2 - 8 \right],
\]

where integration over the temperature parameter, \( \tau' < \tau_0 \), is defined so as to avoid singularity conditions and adapted to the spacetime shell configurations so as to give well-defined thermodynamic variables. In (4.9),

\[
\sqrt{\text{g}[\Phi(\tau)]} = e^{\psi(\tau)} \left[ 2 \Phi(\tau) \right] \frac{\partial_{\bar{\beta}}[2 \Phi(\tau)]^2}{\int dy^k 2K(\tau) (\partial_{\bar{\beta}}[2 \Phi(\tau)]^2)^2} \frac{1}{1/2} \times \frac{[\Phi(\tau)]^2}{3 \Lambda_0(\tau) \int dy^k 4K(\tau) (\partial_{\bar{\beta}}[\frac{1}{2} \Phi(\tau)]^2)^2} \frac{1}{1/2} \times \frac{[\Phi(\tau)]^2}{4 \Lambda_0(\tau) \int dy^k 4K(\tau) (\partial_{\bar{\beta}}[\frac{1}{4} \Phi(\tau)]^2)^2} \frac{1}{1/2}.
\]

\( \text{Here for simplicity we set } g_4^{[0]} = 0, \dot{g}_5^{[0]} = 0, \dot{g}_8^{[0]} = 0; n_{k1} = 0, n_{k2} = 0; \dot{n}_{k1} = 0, \dot{n}_{k2} = 0; \dot{n}_{k3} = 0, \dot{\dot{n}}_{k3} = 0. \)
Using the assumptions as in footnote 6, we find that the \( s \delta^8 \overset{\cdot}{u}(\tau) \) from (4.9) is
\[
\delta^8 \overset{\cdot}{u}(\tau) = dx^1 dx^2 dy^3 (\tau) \delta u^4(\tau) \overset{\cdot}{\delta} u_5(\tau) \overset{\cdot}{\delta} u_6(\tau) \overset{\cdot}{\delta} u_7(\tau) \overset{\cdot}{\delta} u_8(\tau) = dx^1 dx^2 [dy^3 + w_i(\tau) dx^i] dt dp_5 [dp_6 + w_{i_2}(\tau) dx^i_2] [dp_7 + w_{i_3}(\tau) d x^{i_3}] dE.
\]
In these formulas, the shell N-connection coefficients are split into \( w_{i_1,2,3} \) and \( n_{i_1,2,3} \) as:
\[
N_{i_1}^{a_2}(\tau) \rightarrow \left[ w_{i_1}(\tau) = \frac{\partial_1 \left( \int dy^3 2 K(\tau) \partial_3 \left[ 2 \Phi(\tau) \right]^2 \right)}{2 K(\tau) \partial_3 \left[ 2 \Phi(\tau) \right]^2}, \quad n_{i_1}(\tau) = 0 \right],
\]
\[
\overset{\cdot}{N}_{i_2 a_3}(\tau) \rightarrow \left[ \overset{\cdot}{n}_{i_2}(\tau) = 0, \quad \overset{\cdot}{w}_{i_2}(\tau) = \frac{\overset{\cdot}{\partial}_2 \left( \int dp_6 3 K(\tau) \partial_6 \left[ \frac{4}{3} \Phi(\tau) \right]^2 \right)}{3 K(\tau) \partial_6 \left[ \frac{4}{3} \Phi(\tau) \right]^2} \right],
\]
\[
\overset{\cdot}{N}_{i_3 a_5}(\tau) \rightarrow \left[ \overset{\cdot}{w}_{i_3}(\tau) = \overset{\cdot}{\partial}_3 \left( \int dp_7 3 K(\tau) \partial_7 \left[ \frac{4}{3} \Phi(\tau) \right]^2 \right), \quad \overset{\cdot}{n}_{i_4}(\tau) = 0 \right].
\]
For the above s-connection and N-connection coefficients and N-elongated differentials, the effective volume integration functional is
\[
\overset{\cdot}{\delta} V(\tau) = e^{\psi(\tau)} \left| 2 \Phi(\tau) \partial_3 \left[ 2 \Phi(\tau) \right]^2 \right| \times \left[ dy^3 + \frac{\partial_1 \left( \int dy^3 2 K(\tau) \partial_3 \left[ 2 \Phi(\tau) \right]^2 \right) dx^i_1}{2 K(\tau) \partial_3 \left[ 2 \Phi(\tau) \right]^2} \right] dx^1 dx^2 dt
\]
\[
\left| \frac{4}{3} \Phi(\tau) \partial_6 \left[ \frac{4}{3} \Phi(\tau) \right]^2 \right| \times \left[ dp_6 + \frac{\overset{\cdot}{\partial}_2 \left( \int dp_6 3 K(\tau) \partial_6 \left[ \frac{4}{3} \Phi(\tau) \right]^2 \right) dx^i_2}{3 K(\tau) \partial_6 \left[ \frac{4}{3} \Phi(\tau) \right]^2} \right]
\]
\[
\left| \frac{4}{3} \Phi(\tau) \partial_7 \left[ \frac{4}{3} \Phi(\tau) \right]^2 \right| \times \left[ dp_7 + \overset{\cdot}{\partial}_3 \left( \int dp_7 3 K(\tau) \partial_7 \left[ \frac{4}{3} \Phi(\tau) \right]^2 \right) \right] dp_5 dE.
\]
This volume integration functional allows us to compute all statistical and geometric variables by integrating over the \( \tau \) parameter and phase space coordinates. It can be restricted to LC-configurations by imposing additional nonholonomic constraints.

### 4.2.3 Perelman thermodynamic variables for quasi-stationary configurations and effective cosmological constants

The thermodynamic variables for a fixed temperature parameter \( \tau_0 \) and with \( \overset{\cdot}{\delta} \overset{\cdot}{V}(\tau) \) computed for quasi-stationary solutions are expressed in the form
\[
\overset{\cdot}{\mathcal{W}} = \int_{\tau'}^{\tau_0} \frac{dt}{(4\pi)^2} \int_{\Xi} \left( \tau \left[ \sum_s \overset{\cdot}{s} \Lambda(\tau) \right]^2 - 8 \right) \overset{\cdot}{\delta} V(\tau),
\]
\[
\overset{\cdot}{Z} = \exp \left[ \int_{\tau'}^{\tau_0} \frac{dt}{(2\pi)^2} \int_{\Xi} \overset{\cdot}{\delta} V(\tau) \right],
\]
\[
\overset{\cdot}{E} = \int_{\tau'}^{\tau_0} \frac{dt}{(4\pi)^2} \int_{\Xi} \left( \sum_s \overset{\cdot}{s} \Lambda(\tau) \right) - \frac{4}{\tau}, \quad \overset{\cdot}{\delta} V(\tau), \quad \overset{\cdot}{S} = \int_{\tau'}^{\tau_0} \frac{dt}{(4\pi)^2} \int_{\Xi} \left( \tau \left[ \sum_s \overset{\cdot}{s} \Lambda(\tau) \right] - 8 \right) \overset{\cdot}{\delta} V(\tau).
\]
(4.11)
These integrals can be computed for quasi-stationary solutions of the generalized vacuum equations, with the s-metrics being $\tau$-dependent. We consider only well-defined thermodynamic variables on certain classes of spacetimes. To study possible physical applications, we set $\frac{\partial}{\partial s}\Lambda(\tau) = \frac{\partial}{\partial s}\Lambda_0$.

### 4.2.4 Effective volume elements for quasi-stationary generating functions & sources

The effective volume element (4.10) is a parametric functional $\delta \mathcal{V}[\tau, \frac{\partial}{\partial s}\Lambda(\tau), \frac{\partial}{\partial s}\mathcal{K}(\tau); \psi(\tau), \frac{\partial}{\partial s}\Phi(\tau)]$ which allows us to compute thermodynamic variables (4.11) for a corresponding class of generating functions $\frac{\partial}{\partial s}\Phi(\tau)$. Using the nonlinear symmetries of (2.32) and the transforms of (A.10), we can redefine the generating functions to give deformations of the prime s-metrics into target quasi-stationary metric, $\frac{\partial}{\partial s}\hat{g}(\tau) \rightarrow \frac{\partial}{\partial s}g(\tau)$ (2.22). For the $\eta$-polarization functions and with decompositions on the small parameter $\kappa$, we find

\[
\frac{\partial}{\partial s}\Phi(\tau) = 2\sqrt{|2\Lambda(\tau)\eta_4(\tau)\hat{g}_4(\tau)|} \left[ 1 - \frac{\kappa}{2} \chi_4(\tau) \right],
\]

\[
\frac{\partial}{\partial s}\mathcal{V}(\tau) = 2\sqrt{|2\Lambda(\tau)\eta^6(\tau)\hat{g}^6(\tau)|} \left[ 1 - \frac{\kappa}{2} \chi^6(\tau) \right],
\]

\[
\frac{\partial}{\partial s}\Phi(\tau) = 2\sqrt{|\frac{1}{3}\Lambda(\tau)\eta^6(\tau)\hat{g}^6(\tau)|} \left[ 1 - \frac{\kappa}{2} \chi^6(\tau) \right].
\]

Inserting the generating functions (4.12) into (4.10), we find representations of the effective volume functionals

\[
\delta \mathcal{V} = \delta \mathcal{V}[\tau, \frac{\partial}{\partial s}\Lambda(\tau), \frac{\partial}{\partial s}\mathcal{K}(\tau); \psi(\tau), g_4(\tau), \frac{\partial}{\partial s}g_4(\tau), \frac{\partial}{\partial s}g^6(\tau), \frac{\partial}{\partial s}g^7(\tau)]
\]

\[= \delta \mathcal{V}[\tau, \frac{\partial}{\partial s}\Lambda(\tau), \frac{\partial}{\partial s}\mathcal{K}(\tau); \psi(\tau), \hat{g}_4(\tau), \frac{\partial}{\partial s}\hat{g}_4(\tau), \frac{\partial}{\partial s}\hat{g}^6(\tau), \frac{\partial}{\partial s}\hat{g}^7(\tau); \eta_4(\tau), \eta^6(\tau), \eta^6(\tau), \eta^6(\tau), \eta^6(\tau), \eta^6(\tau), \eta^6(\tau), \eta^6(\tau)] +
\]

\[\kappa \delta \mathcal{V}_1[\tau, \frac{\partial}{\partial s}\Lambda(\tau), \frac{\partial}{\partial s}\mathcal{K}(\tau); \psi(\tau), \hat{g}_4(\tau), \frac{\partial}{\partial s}\hat{g}_4(\tau), \frac{\partial}{\partial s}\hat{g}^6(\tau), \frac{\partial}{\partial s}\hat{g}^7(\tau); \chi_4(\tau), \chi^6(\tau), \chi^6(\tau), \chi^6(\tau), \chi^6(\tau), \chi^6(\tau), \chi^6(\tau), \chi^6(\tau), \chi^6(\tau), \chi^6(\tau)] +
\]

Under the above linear $\kappa$-decomposition, with $\delta \mathcal{V} = \delta \mathcal{V}_0 + \kappa \delta \mathcal{V}_1$, the first term describes a re-definition of the prime thermodynamic vacuum via $\zeta$-terms and the second term describes contributions of small $\chi$-terms. This linear $\kappa$-decomposition of the effective volume functionals also gives a linear $\kappa$-decomposition of the thermodynamic variables from (4.11),

\[
\frac{\partial}{\partial s}\tilde{W} = \frac{\partial}{\partial s}\tilde{W}_0 + \kappa \frac{\partial}{\partial s}\tilde{W}_1, \quad \frac{\partial}{\partial s}\tilde{Z} = \frac{\partial}{\partial s}\tilde{Z}_0 + \kappa \frac{\partial}{\partial s}\tilde{Z}_1, \quad \frac{\partial}{\partial s}\tilde{E} = \frac{\partial}{\partial s}\tilde{E}_0 + \kappa \frac{\partial}{\partial s}\tilde{E}_1, \quad \frac{\partial}{\partial s}\tilde{S} = \frac{\partial}{\partial s}\tilde{S}_0 + \kappa \frac{\partial}{\partial s}\tilde{S}_1.
\]

These values may be not well-defined in the framework of a relativistic thermodynamic model for some solutions. We consider such solutions as unphysical.
4.3 Examples of nonassociative BH deformations and Perelman’s thermodynamics

In this section, the generating and integration functions and effective sources are given in a form so that the statistical geometric thermodynamic variables are determined by the corresponding effective volume functionals and fixed effective cosmological constants. We show how these values are computed in parametric form for nonassociative R-flux, phase space Tangerlini BHs and double 4-d Schwarzschild BHs, AdS BHs and dS BHs.

4.3.1 The thermodynamic variables for quasi-stationary R-flux deformed Tangerlini BHs

Expressing the effective volume (4.13) in terms of \( \eta \)-polarizations and prime metrics \( ^\prime \hat{g}_{\alpha s}(r, \theta, \varphi, \theta_2, \theta_3, p_7) \) (3.6) for generating functions (4.12), we get the volume functional

\[
^\prime \delta \hat{V} = \delta \hat{V}[\tau, \ ^\prime \Lambda(\tau), \ ^\prime \bar{K}(\tau, \ ^\prime \hat{u}^{\beta s}(r, \theta, \varphi, \theta_2, \theta_3, p_7)); \psi(\tau, \ ^\prime r, \theta), \ ^\prime \hat{g}_4(\tau, \ ^\prime r, \theta), \ ^\prime \eta^6(\tau, \ ^\prime r, \theta, \varphi, \theta_2), \ ^\prime \eta^8(\tau, \ ^\prime r, \theta, \varphi, \theta_2), \ ^\prime \eta^4(\tau, \ ^\prime r, \varphi, \theta_2)]
\]

We can chose solutions with \( ^\prime \Lambda(\tau) = ^\prime \Lambda_0 = \Lambda_0 \) and write the formulas for thermodynamic variables (4.11) in the form

\[
^\prime \mathcal{W}_{[hn]} = \int_{^{\prime}r^0}^{\tau_0} \frac{d\tau}{32(2\pi)^2} \int_{^{\prime}S^{\prime}} \left( \Lambda_0^2 - 1 \right) ^\prime \delta \hat{V}(\tau),
\]

\[
^\prime \mathcal{Z}_{[hn]} = \exp \left[ \int_{^{\prime}r^0}^{\tau_0} \frac{d\tau}{64(2\pi)^2} \int_{^{\prime}S^{\prime}} ^\prime \delta \hat{V}(\tau) \right]
\]

\[
^\prime \mathcal{E}_{[hn]} = - \int_{^{\prime}r^0}^{\tau_0} \frac{d\tau}{32(2\pi)^2} \int_{^{\prime}S^{\prime}} \left( \Lambda_0 - \frac{1}{\tau} \right) ^\prime \delta \hat{V}(\tau),
\]

\[
^\prime \mathcal{S}_{[hn]} = - \int_{^{\prime}r^0}^{\tau_0} \frac{d\tau}{32(2\pi)^2} \int_{^{\prime}S^{\prime}} (\Lambda_0 - 1) ^\prime \delta \hat{V}(\tau).
\]

The above quantities encode a nonassociative R-flux into \( ^\prime \delta \hat{V} \). Prescribing generating sources \( ^\prime \bar{K} \), we construct statistical thermodynamic models for nonholonomic deformations of phase space Tangerlini BHs. The thermodynamic variables computed for \( ^\prime \delta \hat{V} \) are functionals of generating functions \( \eta_1, \eta^6 \) and \( \eta^8 \), which also define respective families of parametric solutions with s-metrics (3.7). The \( \eta \)-generating functions also determine symmetric, \( ^\prime \hat{g}_{\alpha \beta s} \), and antisymmetric, \( ^\prime \hat{a}_{\alpha \beta s} \), s-metrics (see formulas (A.3) and (A.4)) characterizing the nonassociative properties of phase space MGTs.

4.3.2 Phase space thermodynamics for quasi-stationary R-flux deformed double 4-d BHs

Here the prime s-metrics are taken in the form \( ^\prime \hat{g}_{\alpha s}(\tau, \ ^\prime \hat{u}^{\beta s}) \) (3.13) instead of \( ^\prime \hat{g}_{\alpha s}(\tau, \ ^\prime \hat{u}^{\beta s}) \) (3.6). We can compute the Perelman thermodynamic variables if the nonassociative R-flux \( \kappa \)-deformations result in target s-metrics and N-connection coefficients \( ^\prime \hat{g}_{\tau, \alpha s} \).
given by (3.14) and (3.15). For \( \chi \)-polarization functions we have,

\[
2 \Phi(\tau) = 2 \sqrt{2 \Lambda(\tau) g_4(\tau)} = 2 \sqrt{\kappa \Lambda_0} \eta_4(\tau) \dot{g}_4 \simeq 2 \sqrt{\kappa \Lambda_0} \zeta_4(\tau) \dot{g}_4 \left[ 1 - \frac{\kappa}{2} \chi_4(\tau, r, \theta, \varphi) \right],
\]

\[
3 \Phi(\tau) = 2 \sqrt{3 \Lambda(\tau) g^6(\tau)} = 2 \sqrt{\kappa \Lambda_0} \eta^6(\tau) \dot{g}^6
\]

\[
\simeq 2 \sqrt{\kappa \Lambda_0} \zeta^6(\tau) \dot{g}^6 \left[ 1 - \frac{\kappa}{2} \chi^6(\tau, r, \theta, \varphi, \rho, r, \rho, \varphi) \right],
\]

\[
4 \Phi(\tau) = 2 \sqrt{4 \Lambda(\tau) g^8(\tau)} = 2 \sqrt{\kappa \Lambda_0} \eta^8(\tau) \dot{g}^8
\]

\[
\simeq 2 \sqrt{\kappa \Lambda_0} \zeta^8(\tau) \dot{g}^8 \left[ 1 - \frac{\kappa}{2} \chi^8(\tau, r, \theta, \varphi, \rho, r, \rho, \varphi) \right],
\]

(4.16)

where

\[
1 \Lambda(\tau) = 2 \Lambda(\tau) \simeq \kappa \Lambda_0 = \text{const}, \quad 3 \Lambda(\tau) = 4 \Lambda(\tau) \simeq \kappa \Lambda_0 = \text{const}
\]

\[
\dot{g}_{a_4}(\tau, \dot{u}^{b_4}) \simeq \dot{g}_{a_4}(\dot{u}^{b_4}(r, \theta, \varphi; \rho, r, \rho, \varphi)).
\]

The effective volume functional generated by (4.16) is of the form

\[
\delta \dot{\mathcal{V}} = \delta \dot{\mathcal{V}}_0 + \kappa \delta \dot{\mathcal{V}}_1 \quad \text{(4.14)},
\]

with

\[
\delta \dot{\mathcal{V}}_1 = \delta \dot{\mathcal{V}}_1[\tau, \dot{u}^{b_4}(r, \theta, \varphi; \rho, r, \rho, \varphi); \chi_4(\tau, r, \theta, \varphi, \rho, r, \rho, \varphi)],
\]

(4.17)

Inserting the effective volume elements (4.17) into (4.11), we compute the linear \( \kappa \)-decomposition of the thermodynamic variables (4.14) as,

\[
\delta \dot{\mathcal{W}}^{(db)} = \delta \dot{\mathcal{W}}^{(db)}_0 + \kappa \delta \dot{\mathcal{W}}^{(db)}_1[\chi_4, \chi^6, \chi^8]
\]

\[
= \int_{\tau}^{\tau_0} \left[ \tau \left[ \kappa \Lambda_0 + c \Lambda_0 \right] \right] \frac{d\tau}{16(T^2)^4} \int_{\varphi \in \mathbb{R}} \left[ \tau \left[ \kappa \Lambda_0 + c \Lambda_0 \right] \right] \delta \dot{\mathcal{V}}_0(\tau)
\]

\[
+ \kappa \int_{\tau}^{\tau_0} \left[ \tau \left[ \kappa \Lambda_0 + c \Lambda_0 \right] \right] \frac{d\tau}{16(T^2)^4} \int_{\varphi \in \mathbb{R}} \left[ \tau \left[ \kappa \Lambda_0 + c \Lambda_0 \right] \right] \delta \dot{\mathcal{V}}_1(\tau, \chi_4, \chi^6, \chi^8),
\]

(4.18)

and

\[
s^{(db)} = s^{(db)}_0 \times s^{(db)}_1
\]

\[
= \exp \left[ \int_{\tau}^{\tau_0} \frac{d\tau}{(2\pi\tau^2)^4} \int_{\varphi \in \mathbb{R}} \delta \dot{\mathcal{V}}_0(\tau) \right] \times \exp \kappa \int_{\tau}^{\tau_0} \frac{d\tau}{(2\pi\tau^2)^4} \int_{\varphi \in \mathbb{R}} \delta \dot{\mathcal{V}}_1(\tau, \chi_4, \chi^6, \chi^8),
\]

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The thermodynamic variables in (4.15) and (4.18) can be computed in explicit form by integrating over the respective volume functionals $\delta_{h} V$ and $\delta_{c} V$. We do not provide such cumbersome formulas here. The geometric and thermodynamic values/variables provided above show a very different dependence of the nonholonomic deformations of the phase space Tangherlini and double BHs with respect to the temperature $T$, the effective cosmological constants, and on the possible terms coming from the generating functions $\eta_{4}$, $\eta^{6}$, $\eta^{8}$ or linear $\kappa$-deformations by $\chi_{4}$, $\chi^{6}$, $\chi^{8}$. Using a nonholonomic, relativistic generalization of Perelman’s thermodynamic approach to geometric flows and Ricci solitons, we can distinguish possible thermodynamic effects of various type modifications of phase space higher dimension BHs or double BHs solutions.

We now discuss some physical implications of (4.18) for nonassociative star product and R-flux quasi-stationary deformations of double BH configurations in phase spaces. For very special classes of generating functions with ellipsoidal symmetry of $(\chi_{4}, \chi^{6}, \chi^{8})$, the primary double BH metrics transform into generic off-diagonal ones for double BEs as explained in section 4.1.2. In such cases, we can define certain polarized and $\varphi$-anisotropic versions of the Bekenstein-Hawking temperature and entropy. Nevertheless, more general classes of nonassociative deformations involving momentum type variables and generic off-diagonal metrics result in double quasi-stationary s-metrics. This leads to a change of the Bekenstein-Hawking thermodynamic paradigm into a more general Perelman type even for small $\kappa$-deformations. The nonlinear symmetries of such solutions, given in formulas (A.10), allow one to introduce effective cosmological constants $h\Lambda_{0}$ and $c\Lambda_{0}$. Nevertheless, we can not consider solutions like double Schwarzschild-(A)dS configurations if general classes of effective sources are used. So, nonassociative star product and R-flux deformations result in substantial modifications of the base spacetime Lorentz structure if the solutions of nonassociative vacuum gravitational equations contain dependencies on momentum coordinates. We can prescribe the nonholonomic phase space structure when the entropy of $s \hat{S}^{(dbh)}$ and $\kappa s \hat{S}^{(dbh)}$ can be related to $S_{0}(\varphi)$ and $S_{0}(\varphi)$ from section 4.1.2. However, this is true only for very special diagonalizable phase space metrics. Even in such cases, there will be a difference between the geometric flow temperature $T$ and the double BE temperature $T(\varphi)$. Nonassociative deformations of double BH configurations are also characterized by geometric flow thermodynamic energies $s \hat{E}^{(dbh)}$ and $\kappa s \hat{E}^{(dbh)}$ which do not have analogs in the framework of the Bekenstein-Hawking thermodynamics.
5 Discussion, concluding remarks and open questions

We begin with a summary and discussion of the main results of this article. In this work we followed the approach to nonassociative gravity with $\ast$-product and R-flux deformations in string theory [17, 18]. This was generalized in nonholonomic form [19, 20] with the aim of constructing physically important, exact solutions. The possibility to decouple and integrate the nonassociative vacuum gravitational equations for quasi-stationary solutions, using the anholonomic frame and connection deformation method (AFCDM) was given in [20]. In another work [21], we showed how to construct parametric solutions on a base spacetime manifold defining BEs and encoding nonassociative R-flux effects. In this article, in addition to gaining a more complete understanding of physical effects of nonassociative R-flux deformations, we also studied certain new classes of 8-d parametric quasi-stationary solutions. This included the following new and original results:

1. In section 2 and the appendix, we outlined the necessary formulas for the AFCDM which allowed us to construct exact 8-d solutions, with a fixed energy parameter, in the conventional co-fiber space of the phase space. This co-fiber space was modeled as a cotangent Lorentz bundle nonholonomically deformed by a nonassociative star product determined by a R-flux nongeometric structure. Such nonholonomic geometric constructions are nonassociative versions of geometrical and physical models from [33]. The methods and solutions of the present work are very different from [33] which focused on generalized “rainbow” configurations with a dependence on a variable energy parameter $E$ and on a temperature parameter $\tau$. Here, we consider fixed values $E_0$ and $\tau_0$ and study possible relativistic momentum effects which via nonlinear symmetries (2.32) and also appendix A.2.

2. In section 3.1.1 we extended the Tangherlini type 6-d BH solutions [41, 42] to 8-d phase space. These new solutions are higher dimensional BHs with a conventional horizon and radius in phase space and which, via nonlinear symmetries and effective cosmological constants (see formulas (2.32) and appendix A.2), encode nonassociative data.

3. Another class of solutions for the nonassociative, vacuum gravitational equations is given in section 3.1.2. These describe double BHs — a 4-d BH on the base spacetime and another 4-d BH in a co-fiber (momentum) space. These two 4-d BHs are not independent because of the nonlinear symmetries of the generating functions and the effective sources.

4. In subsection 3.2 off-diagonal R-flux deformations of the prime 8-d metrics were given which resulted in quasi-stationary solutions with a fixed energy parameter. In general, it is not clear what physical properties these solutions, which encoded nonassociative R-flux data, may have. It is possible to prove certain stability conditions for such BE solutions.

5. In section 4.2 we generalized the Bekenstein-Hawking entropy approach to phase space BH and BE configurations and speculated how the corresponding tempera-
ture and associated thermodynamic values might encode nonassociative data. We concluded that this approach was limited solutions with conventional horizons. For more general configurations in nonassociative gravity we need a new type of geometric thermodynamic models, which was based on the concept of Perelman thermodynamic variables [48], as well as generalizations of Perelman’s thermodynamic variables [33, 39, 40, 43].

6. Also in section 4.2, the concept of W-entropy was generalized to 8-d phase spaces. We showed how a statistical analogy for nonholonomic phase space Ricci flows can be formulated in order to compute thermodynamic variables for quasi-stationary solutions of nonassociative Ricci solitons with fixed temperature parameter $\tau_0$. It was proved that Perelman-like thermodynamic variables for $\kappa$-dependent solutions are determined by certain temperature and hypersurface integrals with effective volume elements.

7. Section 4.3 gave two explicit examples of how the Perelman thermodynamic variables can be computed for general quasi-stationary R-flux deformations of phase space Tangherlini BHs and deformed double 4-d BHs. These two configurations are distinguished thermodynamically by different dependencies on the effective cosmological constants and temperature. This new geometric thermodynamic physics cannot be studied using standard Bekenstein-Hawking thermodynamics.

All prime and target phase space solutions constructed in this work are characterized additionally by nonassociative metrics $q_{\alpha\beta} = \tilde{q}_{\alpha\beta} + a_{\alpha\beta}$ (A.3) with symmetric and nonsymmetric components. For quasi-stationary solutions, such generic off-diagonal metrics can be computed as induced ones, determined by corresponding symmetric s-metrics. This was proven using the AFCDM in [20, 21] — see the respective discussions on “nonassociative hair” BEs and BHs at the end of subsections in 3.2. In general, such target phase space solutions have nontrivial, nonholonomically induced torsion but we can always extract zero torsion, LC-solutions.

But what about the legacy of using Perelman’s thermodynamics for Ricci flows to prove the Poincarè-Thorston conjecture for geometric flows of Riemannian metrics [48]? There are difficulties to proving certain nonassociative/ noncommutative analogs of the Poincarè conjecture in general forms — see discussions in [33, 39, 40, 43]. One could formulate a number of such topological and geometric models which depend on the type of nonassociative/noncommutative solutions studied, as in [1–6, 10, 11, 38]. Various nonassociative theories are different from those with $\star$-products and R-fluxes considered in this work. So, it is not clear how to formulate a general mathematical framework involving fundamental topology and geometric analysis, for all models of nonassociative and noncommutative spaces. Nevertheless, self-consistent generalizations of the statistic and geometric thermodynamics using Perelman’s F- and W-functionals are possible if one considers nonholonomic deformations with functionals of type (4.4). They encode nonassociative quantities in $\kappa$-dependent form and result in a Ricci soliton, i.e. modified vacuum Einstein equations,
of type (2.16). Applying the AFCDM, this system of nonlinear PDEs can be decoupled and integrated in very general form (as we proved in section 2.3, and with more details in [20]). We can associate and compute for such generic off-diagonal solutions respective Perelman-like thermodynamic variables (4.8). Thus, such modified gravity and modified thermodynamic theories can be formulated in a self-consistent, relativistic and physical form, even if there is not a rigorous mathematical version of the nonassociative Poincarè hypothesis. It should be noted also that if we can attribute physical significance to generalized classes of nonassociative solutions, then the concept of Bekenstein-Hawking entropy is not applicable. However, the W-entropy and related statistical Perelman thermodynamics can be applied for very general modified theories and their solutions.

The results of this paper support the main hypothesis from the Introduction in two senses: (1) we can construct physically interesting nonassociative, nonholonomic phase space BE and BH solutions; and (2) such configurations are characterized by corresponding statistical and geometric thermodynamic variables in a Bekenstein-Hawking approach or, in more general cases, in a modified Perelman theory for nonholonomic Ricci solitons. Nevertheless, there are number of important open questions that should be investigated in the future which are laid out in [19, 20]. Here we outline four of most important open questions:

- **Q1:** a full investigation of nonassociative gravity theories should involve models with nontrivial sources and other types of nonassociative/ noncommutative structures which are not necessarily determined by R-flux deformations but based on other types of star product or algebraic structures. One of the next steps is to study models with star R-flux nonholonomic deformations of Einstein-Yang-Mills-Higgs systems resulting in nonassociative gravity and matter field theories. Such nonassociative models should generalize the Einstein-Eisenhart-Moffat theories [26–28, 28, 30].

- **Q2:** nonassociative theories determined by a star product of type (2.4) on tensor products of cotangent bundles, are a class of nonassociative generalizations of the Finsler-Lagrange-Hamilton geometry. In (non) commutative/ supersymmetric variants, such models were given in [24, 31–33]. The importance of Finsler-like variables is that they can be reformulated as some equivalent almost Kaehler/ complex variables. An even deeper question is to formulate and study models of quantum gravity using Lagrange-Hamilton geometric methods encoding nonassociative and noncommutative structures.

- **Q3:** nonassociative gravity with star and R-flux structures was given originally in [17, 18]. This was enough to develop a new direction of the theory of nonassociative/ noncommutative Ricci flows. However, to study possible physical implications we need to work with nonholonomic dyadic shell structures as stated in [19] and section 4. In a formal nonassociative geometric form, we can generalize the Perelman F- and W-functionals (4.4), $\tilde{\mathcal{F}}(\tau) \rightarrow \hat{\mathcal{F}}(\tau)$ and $\hat{\mathcal{W}}(\tau) \rightarrow \hat{\mathcal{W}}(\tau)$, in terms of the nonassociative phase space canonical Ricci s-tensor, $\hat{\mathfrak{R}}_{\alpha\beta}^s$ and Ricci canonical
s-scalar, $\tilde{\mathcal{R}}^{sc*}$. In this way, we can derive a $\ast$-deformed Hamilton nonassociative geometric flow equation and define nonassociative versions of the Perelman thermodynamic generating function (4.6) and variables (4.8) i.e., $\tilde{s}\mathcal{E}(\tau) \rightarrow \tilde{s}\mathcal{E}^*(\tau)$ and $\tilde{s}\mathcal{S}(\tau) \rightarrow \tilde{s}\mathcal{S}^*(\tau)$, $\tilde{s}\mathcal{S}(\tau) \rightarrow \tilde{s}\mathcal{S}^*(\tau)$, $\tilde{s}\mathcal{S}(\tau) \rightarrow \tilde{s}\mathcal{S}^*(\tau)$, $\tilde{s}\mathcal{S}(\tau) \rightarrow \tilde{s}\mathcal{S}^*(\tau)$. These constructions can be motivated by finding exact/parametric solutions which for $\kappa$-dependent decompositions will result in geometric thermodynamic models that are similar to those considered in the previous section. Although it may be difficult to attribute a physical significance to nonassociative geometric thermodynamics we have shown that it is always possible to formulate well-defined commutative kinetic, diffusion and thermodynamic models encoding almost symplectic structures determined by R-flux deformations.

• Q4: finally we point to a promising avenue to extend the geometric flow information theory [39, 40] to some versions with nonassociative qubits and entanglement and conditional entropies. These new directions in modern quantum field theory/gravity / computers have deep roots in nonassociative quantum mechanics [1–3, 6] and motivations from string and M-theory [7–15, 31].

We plan to report on progress to answers for questions Q1-Q4 in future works.

Acknowledgments

This work is elaborated in the framework of a research program supported to a Fulbright senior fellowship of SV and hosted by DS at physics department at California State University at Fresno, USA. It develops for nonassociative geometry and gravity some former research projects on geometry and physics supported by fellowships and grants at the Perimeter Institute and Fields Institute (Ontario, Canada), CERN (Geneva, Switzerland) and Max Planck Institut für Physik / Werner Heisenberg Institut, München (Germany) and DAAD. The sections 3 and 4 are related also to research programs proposed for visiting at CASLMU München, and PNRR-Initiative 8 of the Romanian Ministry of RID. SV is also grateful to professors D. Lüst, N. Mavromatos, J. Moffat, Yu. A. Seti, P. Stavrinos, M.V. Tkach, and E. V. Veliev for respective hosting visits and collaborations.

A Nonholonomic deformations and nonassociative parametric solutions

A.1 Conventions on (non) associative coordinates, and s-adapted geometric objects

We follow the conventions from [19–21] on local spacetime and phase space coordinates and respective labels with boldface and non-boldface symbols:

$$\text{on } V \text{ and } sV: \quad x = \{x^i\} = x^s = \{x^{i_s}\} = (x^i, x^{a_1} \rightarrow y^{a_2}) = (x^{i_2}), \quad \text{with } x^4 = t, \quad (A.1)$$

where $i, j, \ldots = 1, 2, 3, 4$; shells: $s = 1$, when $i_1, j_1, \ldots = 1, 2$; $s = 2, a_1, b_2, \ldots = 3, 4$;
on $TV$ and $T_sV$: 
\[ u = (x, y) = \{ u^\alpha = (u^k = x^k, \ u^a = y^a) \} = \]
\[ s_u = (x, y) = \{ u^\alpha = (u^k = x^k, \ u^a = y^a) \} = (x^i, x^j, x^3 \to y^i, x^d \to y^j), \] shells $s = 1, 2, 3, 4$, where $a, b, \ldots = 5, 6, 7, 8$; $a_3, b_3, \ldots = (5, 6); a_4, b_4, \ldots = (7, 8);

on $T^*V$ and $T_s^*V$: 
\[ u = (x, p) = \{ u^\alpha = (u^k = x^k, \ \ p_a = p_0) \} \]
\[ = (x, p) = \{ u^\alpha = (u^k = x^k, \ \ p_a = p_0) \} = \]
\[ = (x, p) = \{ u^\alpha = (u^k = x^k, \ \ p_a = p_0) \} = (x^i, x^j, \ \ x^3 \to p_3, \ \ p_4), \] where \( x^3 = (x^i, x^j, \ p_3, \ p_4) = (x^i, x^j, \ p_3 = p_0, \ \ p_4 = p_0) \),

where \( x^3 = (x^i, x^j, \ p_3 = p_0, \ \ p_4 = p_0) \).

In the above formulas, the coordinate $x^i = y^i = t$ is time-like and $p_8 = E$ is energy-like. Boldface indices are used for spaces and geometric objects enabled with $N_-$/$s$-connection structure $\nabla_N$. An upper or lower left label "\( u \)" is used to distinguish coordinates with "complexified momenta" of real phase coordinates \( (u^\alpha, p_a) \) on $T^*V$. The formalism of $N$- and $s$-adapted labels and abstract index/symbol or frame coefficient notations is given in such a way that allows one to formulate a unified "symbolic" nonholonomic geometric calculus.

We can consider also $\kappa$-decompositions of a general nonsymmetric metric (2.7),
\[ \bar{q}_{\alpha_3 \beta_3} = \bar{q}_{\alpha_3 \beta_3}^{[0]} + \bar{q}_{\alpha_3 \beta_3}^{[1]}(\kappa) = \bar{q}_{\alpha_3 \beta_3} + q_{\alpha_3 \beta_3}, \] (A.2)
where \( \bar{q}_{\alpha_3 \beta_3} \) is the symmetric part and \( q_{\alpha_3 \beta_3} \) is the anti-symmetric part of star and R-flux deformations. In these formulas,
\[ \bar{q}_{\alpha_3 \beta_3} := \frac{1}{2} (\bar{g}_{\alpha_3 \beta_3} + \bar{g}_{\beta_3 \alpha_3}) = \bar{g}_{\alpha_3 \beta_3} - \frac{i\kappa}{2} (\bar{\gamma}^{\alpha_3 intersect} e_{\alpha_3} \bar{g}_{\gamma_{\alpha_3}} + \bar{\gamma}^{\beta_3 intersect} e_{\beta_3} \bar{g}_{\gamma_{\beta_3}}), \]
(A.3)
\[ \bar{q}_{\alpha_3 \beta_3}^{[0]} = \bar{q}_{\alpha_3 \beta_3}^{[0]}(\kappa), \]
\[ \bar{q}_{\alpha_3 \beta_3}^{[1]}(\kappa) = \frac{i\kappa}{2} (\bar{\gamma}^{\alpha_3 intersect} e_{\alpha_3} \bar{g}_{\gamma_{\alpha_3}} + \bar{\gamma}^{\beta_3 intersect} e_{\beta_3} \bar{g}_{\gamma_{\beta_3}}); \]
\[ q_{\alpha_3 \beta_3} := \frac{1}{2} (\bar{q}_{\alpha_3 \beta_3} - \bar{q}_{\beta_3 \alpha_3}) = \frac{i\kappa}{2} (\bar{\gamma}^{\alpha_3 intersect} e_{\alpha_3} \bar{g}_{\gamma_{\alpha_3}} - \bar{\gamma}^{\beta_3 intersect} e_{\beta_3} \bar{g}_{\gamma_{\beta_3}}), \]
(A.4)
\[ q_{\alpha_3 \beta_3}^{[1]}(\kappa) = \frac{i\kappa}{2} (\bar{q}_{\alpha_3 \beta_3}^{[1]}(\kappa) - \bar{q}_{\beta_3 \alpha_3}^{[1]}(\kappa)). \]
The nonholonomic distributions can be prescribed in the form \( q_{\alpha_3 \beta_3}^{[0]} = 0 \) for nonassociative star deformations of commutative theories with symmetric metrics. To compute inverse metrics and s-metrics for such nonassociative geometric models, we have to apply a more sophisticated procedure — see details in [18] and [20]. Respective nonsymmetric inverse s-metrics can be parameterized in the form
\[ q_{\alpha_3 \beta_3} = q_{\alpha_3 \beta_3}^{[0]} + q_{\alpha_3 \beta_3}^{[1]}(\kappa), \] (A.5)
when \( q^{\alpha \beta} \) is not the inverse to \( q_{\alpha \beta} \), and \( a^{\alpha \beta} \) is not inverse to \( a_{\alpha \beta} \). After certain classes of nonholonomic solutions have been constructed in explicit form, we can redefine the constructions in terms of star deformed LC-configurations using canonical s-distortions (2.12) and imposing additional nonholonomic constraints (2.13).

The nonassociative canonical Riemann s-tensor \( \hat{s}R^* \) can be defined and computed for the data \( \mathbf{g}^* = \{ \hat{s}q_{\alpha \beta} \}, \hat{s}D^* = \{ \hat{s}\Gamma_{*\alpha \beta \gamma}^\tau \} \) [see formulas (2.9), (A.2) and (2.12); and details in [20]], when

\[
\hat{s}R^* \hat{s}q_{\alpha \beta \gamma} = 1 \hat{s}R^* \hat{s}q_{\alpha \beta \gamma} + 2 \hat{s}R^* \hat{s}q_{\alpha \beta \gamma}, \quad \text{where}
\]

\[
\hat{s}R^* \hat{s}q_{\alpha \beta \gamma} = \hat{e}_{\gamma} \hat{s}\Gamma_{*\alpha \beta \gamma} + \hat{e}_{\beta} \hat{s}\Gamma_{*\alpha \gamma \beta} + \hat{s}\Gamma_{*\gamma \alpha \beta \gamma} (\delta_{\tau \gamma}^{\alpha} \hat{s}\Gamma_{*\beta \alpha \tau \gamma} - \delta_{\tau \beta}^{\alpha} \hat{s}\Gamma_{*\alpha \beta \tau \gamma})
\]

\[
+ \hat{s}\Gamma_{*\alpha \beta \gamma} \hat{s}\Gamma_{*\alpha \beta \gamma} \hat{s}\Gamma_{*\alpha \beta \gamma}.
\]

Using parametric decompositions of the star canonical s-connection in (A.6),

\[
\hat{s}\Gamma_{*\alpha \beta \gamma} = \hat{s}\Gamma_{*\alpha \beta \gamma} + \hat{s}\Gamma_{*\alpha \beta \gamma} + \hat{s}\Gamma_{*\alpha \beta \gamma} (h) + \hat{s}\Gamma_{*\alpha \beta \gamma} (k) + \hat{s}\Gamma_{*\alpha \beta \gamma} (hk) + O(h^2, k^2, \ldots),
\]

we can compute such parametric decompositions of the nonassociative canonical curvature tensor,

\[
\hat{s}R^* \hat{s}q_{\alpha \beta \gamma} = \hat{s}R^* \hat{s}q_{\alpha \beta \gamma} + \hat{s}R^* \hat{s}q_{\alpha \beta \gamma} (h) + \hat{s}R^* \hat{s}q_{\alpha \beta \gamma} (k) + \hat{s}R^* \hat{s}q_{\alpha \beta \gamma} (hk) + O(h^2, k^2, \ldots).
\]

Further h1-v2-c3-c4 decompositions are also possible for such formulas:

The nonassociative canonical Ricci s-tensor,

\[
\hat{s}Ric^* = \hat{s}Ric^* \hat{s}q_{\alpha \beta \gamma} \hat{s}e_{\alpha} \hat{s}e_{\beta} \hat{s}e_{\gamma}, \quad \hat{s}Ric_{*\alpha \beta \gamma} := \hat{s}Ric^* \hat{s}q_{\alpha \beta \gamma} \hat{s}e_{\alpha} \hat{s}e_{\beta} \hat{s}e_{\gamma}.
\]

is defined by contracting the first and fourth indices of the nonassociative canonical s-tensor,

\[
\hat{s}Ric_{*\alpha \beta \gamma} := \hat{s}Ric^* \hat{s}q_{\alpha \beta \gamma} \hat{s}e_{\alpha} \hat{s}e_{\beta} \hat{s}e_{\gamma}, \quad \hat{s}Ric_{*\alpha \beta \gamma} = \hat{s}Ric_{*\alpha \beta \gamma} (h) + \hat{s}Ric_{*\alpha \beta \gamma} (k).
\]

Because of nonholonomic structures, the canonical Ricci s-tensors are not symmetric for general (non) commutative and nonassociative cases. Using contractions with the inverse nonassociative and nonsymmetric s-metric \( \mathbf{g}^{\mu \nu \lambda} \) (A.5), we can compute the nonassociative nonholonomic canonical Ricci scalar curvature:

\[
\hat{s}Ric_{*\mu \nu \lambda} := \hat{s}q^{\mu \nu \lambda} \hat{s}Ric_{*\mu \nu \lambda} = \hat{s}q^{\mu \nu \lambda} \hat{s}a^{\mu \nu \lambda} (\hat{s}Ric_{*\mu \nu \lambda} + \hat{s}Ric_{*\mu \nu \lambda} (h) + \hat{s}Ric_{*\mu \nu \lambda} (k)).
\]
In these formulas, respective symmetric ($\ldots$) and anti-symmetric $[\ldots]$ operators are defined using the multiple $1/2$, when, for instance, $\tilde{\mathbf{R}}ic^{\mu_1\nu_1}_s = \tilde{\mathbf{R}}ic^{\mu_1\nu_1}_{\mu_2\nu_2} + \tilde{\mathbf{R}}ic^{\mu_2\nu_2}_{\mu_1\nu_1}$.

A.2 Nonassociative nonlinear symmetries of generating functions and effective sources

By straightforward computations (see details in section 5.4 of [20]), we can check that any s-metric define the same class of exact/parametric quasi-stationary solutions of nonassociative vacuum gravitational equations if the transforms (2.32) are subjected to such conditions:

\[
s = 2: \quad \frac{[(2\Psi)^2]}{\frac{\partial}{\partial K}} = \frac{[(2\Phi)^2]}{2\Lambda_0},
\]

i.e. $(2\Psi)^2 = (\frac{\partial}{\partial K})^{-1}\int dx^3(\frac{\partial}{\partial K})[(2\Psi)^2]$ and/or $(2\Phi)^2 = 2\Lambda_0 \int dx^3(\frac{\partial}{\partial K})^{-1}[(2\Psi)^2]$,

\[
[(2\Psi)^2] = -\int d^3g \frac{(\frac{\partial}{\partial K})g_1}{2\Lambda_0} \quad \text{and/or} \quad (2\Phi)^2 = -4\frac{\Lambda_0 g_4}{2},
\]

\[
[(2\Psi)^2] = -\int d^3g \frac{(\frac{\partial}{\partial K})g_1 \eta_4}{4\Lambda_0} \quad \text{and/or} \quad (2\Phi)^2 = -4\frac{\Lambda_0 \eta_4}{2} \hat{g}_4,
\]

\[
[(2\Psi)^2] = -\int d^3g \frac{(\frac{\partial}{\partial K})g_1 \chi_4}{4\Lambda_0} \quad \text{and/or} \quad (2\Phi)^2 = -4\frac{\Lambda_0 \chi_4}{2} \hat{g}_4;
\]

\[
s = 3: \quad \frac{\tilde{\mathbf{R}}ic^{\mu_1\nu_1}_s}{\frac{\partial}{\partial K}} = \frac{\tilde{\mathbf{R}}ic^{\mu_1\nu_1}_s}{3\Lambda_0},
\]

i.e. $(\frac{\partial}{\partial K}) = (\frac{\partial}{\partial K})^{-1}\int d^3p \frac{(\frac{\partial}{\partial K})[(\frac{\partial}{\partial K})] }{3\Lambda_0} \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = 3\Lambda_0 \int d^3p \frac{(\frac{\partial}{\partial K})^{-1}[(\frac{\partial}{\partial K})]}{3\Lambda_0}$

\[
\tilde{\mathbf{R}}ic^{\mu_1\nu_1}_s = -\int d^3p \frac{(\frac{\partial}{\partial K})g_6}{4\Lambda_0} \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = -4\frac{\Lambda_0 g_6}{4},
\]

\[
\tilde{\mathbf{R}}ic^{\mu_1\nu_1}_s = -\int d^3p \frac{(\frac{\partial}{\partial K})g_6 \eta_6}{4\Lambda_0} \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = -4\frac{\Lambda_0 \eta_6}{4} \hat{g}_6,
\]

\[
\tilde{\mathbf{R}}ic^{\mu_1\nu_1}_s = -\int d^3p \frac{(\frac{\partial}{\partial K})g_6 \chi_6}{4\Lambda_0} \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = -4\frac{\Lambda_0 \chi_6}{4} \hat{g}_6;
\]

\[
s = 4: \quad \frac{[(\frac{\partial}{\partial K})]}{4\Lambda} = \frac{[(\frac{\partial}{\partial K})]}{4\Lambda},
\]

i.e. $(\frac{\partial}{\partial K}) = (\frac{\partial}{\partial K})^{-1}\int d^3p E(\frac{\partial}{\partial K})[(\frac{\partial}{\partial K})] \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = 4\Lambda \int d^3p E(\frac{\partial}{\partial K})^{-1}[(\frac{\partial}{\partial K})]$,

\[
[(\frac{\partial}{\partial K})] = -\int d^3p E(\frac{\partial}{\partial K})g_7 \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = -4\frac{\Lambda_0 g_7}{4},
\]

\[
[(\frac{\partial}{\partial K})] = -\int d^3p E(\frac{\partial}{\partial K}) \eta_7 \hat{g}_7 \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = -4\frac{\Lambda_0 \eta_7}{4} \hat{g}_7,
\]

\[
[(\frac{\partial}{\partial K})] = -\int d^3p E(\frac{\partial}{\partial K}) \chi_7 \hat{g}_7 \quad \text{and/or} \quad (\frac{\partial}{\partial K}) = -4\frac{\Lambda_0 \chi_7}{4} \hat{g}_7;
\]

or, for $\frac{\partial}{\partial E} = \frac{\partial}{\partial E} = \text{const}$,
\[ s = 4 : \quad \frac{\partial^7 [ ( \frac{\eta}{4} \Psi )^2 ]}{4 \Lambda} = \frac{\partial^7 [ ( \frac{\chi}{4} \Phi )^2 ]}{4 \Lambda_0}, \]

i.e. \(( \frac{\eta}{4} \Psi )^2 = ( \frac{\chi}{4} \Lambda_0 )^{-1} \int d^4 \eta ( \frac{\chi}{4} \Lambda_0 )^{-1} [ ( \frac{\eta}{4} \Psi )^2 ]\) and/or \(( \frac{\chi}{4} \Phi )^2 = \frac{\chi}{4} \int d^4 \eta ( \frac{\chi}{4} \Lambda_0 )^{-1} [ ( \frac{\chi}{4} \Psi )^2 ]\)

\[ \frac{\partial^7 [ ( \frac{\eta}{4} \Psi )^2 ]}{4 \Lambda} = -\int d^4 \eta ( \frac{\chi}{4} \Lambda_0 )^{-1} \frac{\partial^7 [ ( \frac{\eta}{4} \Psi )^2 ]}{4 \Lambda_0} \] and/or \(( \frac{\chi}{4} \Phi )^2 = \frac{\chi}{4} \int d^4 \eta ( \frac{\chi}{4} \Lambda_0 )^{-1} [ ( \frac{\eta}{4} \Psi )^2 ]\)

\[ \frac{\partial^7 [ ( \frac{\eta}{4} \Psi )^2 ]}{4 \Lambda} = -\int d^4 \eta ( \frac{\chi}{4} \Lambda_0 )^{-1} \frac{\partial^7 [ ( \frac{\chi}{4} \Psi )^2 ]}{4 \Lambda_0} \] and/or \(( \frac{\chi}{4} \Phi )^2 = \frac{\chi}{4} \int d^4 \eta ( \frac{\chi}{4} \Lambda_0 )^{-1} [ ( \frac{\chi}{4} \Psi )^2 ]\)

Nonlinear symmetries given by the above formulas allow us to construct different classes of exact/parametric solutions or to express a known solution in other forms. Working with \(( \frac{\eta}{4} \Psi, \frac{\chi}{4} \Lambda_0 )\), we can generate nonassociative vacuum s-metrics encoding star R-flux contributions via \( \frac{\chi}{4} \Lambda_0 \) when \( \frac{\eta}{4} \Psi \) are prescribed in order to generate target quasi-stationary s-metrics. For \(( \frac{\eta}{4} \Phi, \frac{\chi}{4} \Lambda_0 )\), the sources are re-encoded in \( \frac{\chi}{4} \Phi \) but with some approximations to effective cosmological constants \( \frac{\chi}{4} \Lambda_0 \). Usually, the procedure for constructing solutions generated by \( \frac{\chi}{4} \Phi \) is simpler but there are additional considerations in order to choose generating and integration functions describing physically important target configurations. Choosing some coefficients of a s-metric as generating functions and working with \(( \frac{\eta}{4} \tilde{g}, \frac{\chi}{4} \Lambda_0 )\), we can construct new classes of solutions with certain prescribed properties and re-defined in some closed to “real physical” configurations. Finally, we note that for \(( \frac{\eta}{4} \tilde{g}_s, \frac{\chi}{4} (1 + \kappa \frac{\chi}{4} \Lambda_0 ) \tilde{g}_s, \frac{\chi}{4} \Lambda_0 )\) it is possible to embed self-consistently certain background configurations \( \frac{\chi}{4} \tilde{g}_s \) into some classes of more general vacuum solutions. For small \( \kappa \)-dependent deformations, we can determine physical properties and compute nonassociative classical and quantum physical effects.

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