Asymptotic behavior of Patil’s approximants in Hardy spaces: The real case

By Gomari Buanani.Naufal
Université Claude Bernard-Lyon Institut Girard desargues. 43, Bd du 11 novembre 1918, 69622 Villeurbanne, France.
gomari@desargues.univ-lyon1.fr

Abstract:

In this paper we consider a robust identification problem for a linear dynamical control system with limited-frequency intervals. In mathematical terms, this is the problem of recovering functions in Hardy spaces. Our purpose is to bound Patil’s approximants in the upper half plane case, out of a bounded real interval \( I \). To this end, we deal with residu techniques and give a class of functions to provide boundedness of these approximants on the complement of this interval.

Key Words: Hardy spaces, Asymptotics, Hilbert transform, Toeplitz operator, Wiener-Hopf operator.

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1 Introduction

Let $\mathbb{D}$ be the open unit disc and $T$ its boundary. $H^p(\mathbb{D})$ denotes the Hardy space of analytic functions on $\mathbb{D}$.

In 1972, Patil in [9] has given an algorithm to recapture a $H^p(\mathbb{D})$, $1 < p < \infty$ function $F$ from its values on $E$, a positive Lebesgue measure subset of $T$. Let $f$ be the boundary function of $F$ and $g$ its restriction on $E$.

In fact, using Toeplitz operator techniques he has found a sequence of functions $g_{\lambda}$ uniformly converging on compact subsets of $\mathbb{D}$ to $F$ and also strongly in $H^p(\mathbb{D})$.

The same approximants had been obtained thanks to a Carlman’s fruitful idea by constructing a “quenching” function, enabling us to eliminate in the Cauchy formula, integration over $T \setminus E$. In fact, we first construct an auxiliary function $\varphi \in H^\infty(\mathbb{D})$ satisfying:

1. $|\varphi(\xi)| = 1$ ae on $T \setminus E$.
2. $|\varphi(\xi)| > 1$ in $\mathbb{D}$.

To do so, we solve a suitable Dirichlet problem. If $u$ is the bounded harmonic solution in $\mathbb{D}$ such that:

$$u(x, y) = \begin{cases} 1 & \text{a.e on } E \\ 0 & \text{on } T \setminus E. \end{cases}$$

then we set $\varphi(z) = e^{u+i\nu}$ where $\nu$ is the harmonic function conjugate with $u$. Thus, we obtain the following formula:

$$f(z) = \lim_{m \to +\infty} \frac{1}{2\pi i} \int_E \left( \frac{\varphi(\xi)}{\varphi(z)} \right)^m \frac{f(\xi)}{\xi - z} d\xi$$

L.Baratchart, J.Leblond and J.R Partington have exhibited the same approximants applying optimization methods [1,2]. Their work was initiated by them and D.Alpay in the $H^2$ case, the parametre here is a Lagrange multiplier in [1,2].

Mukherjee in [11], has dealt with the upper half plane case by exhibiting the corresponding sequences $h_{\lambda}$ via the Cayley transform and $g_{\lambda}$ using Wiener-Hopf operator. He has shown that results obtained by Patil remain valid in this case. We wonder if in the case of the upper half plane we can hope better approximation of $f$ out of $I$ on the boundary, where $I$ is a real bounded interval.

An open question is the almost everywhere convergence of $g_{\lambda}$ to $f$ on $T \setminus I$. 

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Our aim is to show that in the upper half plane, under appropriate conditions, 
\( g_\lambda(x) = O(1) \) with respect to \( \lambda \) out of \( I \). We show also that even if \( g \) is the trace of a \( H^2 \) function on \( I \), the conjecture is false.

We divide mainly our work into four sections. The first one is the introduction. The second one deals with nontangential limits of \( g_\lambda \) and \( h_\lambda \) according to whether the interval is symmetrical around 0 or not. In the third part, we give asymptotic behavior of \( g_\lambda \) with respect to \( \lambda \) and exhibit a class of functions such that the trace of \( g_\lambda \) remains bounded on \( \mathbb{R} \setminus I \). Examples are given in the fourth section.

Notations:

1. \( \mathbb{C}_+ \) the open upper half plane.
2. \( \chi_I \) the characteristic function on \( I \).
3. p.v \( \int \) the principal value of the integral.
4. \( O(f) \) a function not exceeding \( f \) with respect to \( \lambda \) when \( \lambda \to +\infty \).

2 Boundary value functions

Let recall the two main theorems of recovering functions in the both cases \( \mathbb{D} \) and \( \mathbb{C}_+ \).

Theorem 1 ([9, thm 1])

Let \( E \subset \mathbb{T} \) with \( m(E) > 0 \). Suppose that \( g \) is the restriction of \( f \) to \( E \). For each \( \lambda > 0 \) define analytic functions \( h_\lambda, g_\lambda \) on \( \mathbb{D} \) by:

\[
g_\lambda(z) = \lambda h_\lambda(z) \frac{1}{2\pi i} \int_K \frac{h_\lambda(w)g(w)}{w - z} dw \quad z \in \mathbb{D},
\]

\[
h_\lambda(z) = \exp \left( -\frac{1}{4\pi} \ln(1 + \lambda) \int_K \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right) \quad z \in \mathbb{D}.
\]

Then as \( \lambda \to \infty \); \( g_\lambda \to f \) unifomly on compact subsets of \( \mathbb{D} \). Moreover for \( 1 < p < \infty \) we also have \( ||g_\lambda - f||_p \to 0 \) as \( \lambda \to \infty \).
**Theorem 2 ([11, thm 1])**

Let $I \subset \mathbb{R}$ with $m(I) > 0$. Suppose that $F \in H^2(\mathbb{C}_+)$ and $g$ is the restriction of $f$ to $I$. For each $\lambda > 0$ we define analytic functions on the upper half plane by:

$$h_\lambda(z) = \exp \left( -\frac{1}{2\pi i} \ln(1 + \lambda) \int_I \frac{1 + tz}{(t-z)(1+t^2)} dt \right) \Im(z) > 0,$$

$$g_\lambda(z) = \lambda h_\lambda(z) \int_I \frac{\overline{h}(t)g(t)}{t-z} dt \quad \Im(z) > 0.$$

then as $\lambda \to \infty$, $g_\lambda \to F$ uniformly on compact subsets of the upper half plane.

Moreover $\|g_\lambda - f\|_2 \to 0$ as $\lambda \to \infty$.

We treat two cases $I = ]-a, a[ \,$ where $a > 0$ and $I = ]a, b[ \,$ where $0 \leq a < b$.

### 2.1 Symmetrical case

Let $I = ]-a, a[ \,$ where $a > 0$, $g$ as in theorem 2.

**Proposition 1** If $z = x + iy \in \mathbb{C}_+$ then $h_\lambda(z) \to h_\lambda(x)$ as $y \to 0$. Where

$$h_\lambda(x) = (1 + \lambda \chi_I(x)) \frac{1}{2\pi i} e^{G_1(x)} \ln \left( \frac{a-x}{a+x} \right),$$

$G_1(x) = \frac{1}{2\pi} \ln(1 + \lambda) \ln \left( \frac{a-x}{a+x} \right)$.

**Remark** : One can find this proposition in [8]. We give a correct proof of this.

**Proof** :

We know that

$$h_\lambda(z) = \exp \left( -\frac{1}{2\pi i} \ln(1 + \lambda) \int_{-a}^{a} \frac{1 + tz}{(t-z)(1+t^2)} dt \right) \quad \text{for } \Im(z) > 0.$$

We wish $\lim_{y \to 0} h_\lambda(z)$. We have

$$\int_{-a}^{a} \frac{1 + tz}{(t-z)(1+t^2)} dt = \int_{-a}^{a} \frac{1}{t-z} dt - \frac{1}{2} \int_{-a}^{a} \frac{2t}{1+t^2} dt.$$

On one hand $\int_{-a}^{a} \frac{2t}{1+t^2} dt = 0$.

On the other hand,

$$\int_{I} \frac{1}{t-z} dt = \int_{I} \frac{t-x}{(t-x)^2 + y^2} dt + i \int_{I} \frac{y}{(t-x)^2 + y^2} dt.$$
It is easy to see that,
\[
\int_I \frac{t-x}{(t-x)^2+y^2} dt \to \ln \left| \frac{a-x}{a+x} \right| \quad \text{as } y \to 0,
\]
and
\[
\int_I \frac{y}{(t-x)^2+y^2} dt \to \chi_I(x) \pi \quad \text{as } y \to 0,
\]
so
\[
h_\lambda(z) \to \exp \left( -\frac{1}{2\pi i} \ln(1+\lambda) \left[ \ln \left| \frac{a-x}{a+x} \right| + i\chi_I(x) \pi \right] \right) \quad \text{as } y \to 0.
\]
Finally,
\[
h_\lambda(x) = (1 + \lambda \chi_I(x))^\frac{1}{2} e^{iG_1(x)} \times 1, \quad G_1(x) = \frac{1}{2\pi} \ln(1+\lambda) \ln \left| \frac{a-x}{a+x} \right|.
\]

**Proposition 2** \( g_\lambda(z) \to g_\lambda(x) \) a.e as \( y \to 0 \) and
\[
g_\lambda(x) = \frac{1}{2} \lambda (h_\lambda \overline{h_\lambda} g \chi_I)(x) + \frac{i}{2\pi} \lambda h_\lambda(x) \text{ p.v.} \int_I \frac{(\overline{h_\lambda g})(t)}{x-t} dt.
\]

**Proof:**
We have:
\[
g_\lambda(z) = \lambda h_\lambda(z) \frac{1}{2\pi i} \int_I \frac{(\overline{h_\lambda g})(t)}{t-z} dt \quad \Im(z) > 0.
\]
It is known that if \( f \) is in \( L^2(\mathbb{R}) \), we can construct an analytic function \( F \) on \( \mathbb{C}_+ \) by the integral formula:
\[
F(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} dt \quad \text{and} \quad F(z) \to f(x) + iHf(x) \quad \text{a.e as } y \to 0
\]
where \( H \) is the Hilbert transform.

One can verify that the result remains valid if \( f \) is a complex valued function.

Let \( f = \overline{h_\lambda g \chi_I} \), we see that:
\[
g_\lambda(z) = \lambda h_\lambda(z) \frac{1}{2\pi i} \int_I \frac{f(t)}{t-z} dt \quad \text{and as } y \to 0,
\]
\[
g_\lambda(z) \to \lambda h_\lambda(x) \frac{1}{2}(f + iHf)(x) \quad \text{a.e.}
\]
Indeed,
\[
g_\lambda(x) = \frac{1}{2} \lambda (h_\lambda \overline{h_\lambda g \chi_I})(x) + \frac{i}{2\pi} \lambda h_\lambda(x) \text{ p.v.} \int_I \frac{(\overline{h_\lambda g})(t)}{x-t} dt \quad \text{a.e.}
\]
Usefull formulas of $g_\lambda$ for almost all $x$ are:

$$g_\lambda(x) = \begin{cases} \frac{\lambda}{2(1+\lambda)} g(x) + \frac{i\lambda}{2\pi(1+\lambda)} e^{iG_1(x)} p.v \int_I \frac{e^{-iG_1(t)} g(t)}{x-t} dt & \text{for } x \in I \\ \frac{\lambda}{2\pi(1+\lambda)^2} e^{iG_1(x)} p.v \int_I \frac{e^{-iG_1(t)} g(t)}{x-t} dt & \text{for } x \notin I \end{cases}$$

Remark:

1. Note that we do not need to integrate in the upper half plane as this is the case in [7].
2. When $x \notin I$, observe that we can omit the p.v notation since in this case $p.v \int_I = \int_I$.

2.2 Nonsymmetrical case

Let $I = [a, b]$ where $0 \leq a < b$, $g$ as in theorem 2.

**Proposition 3**

$$h_\lambda(z) \to (1+\lambda \chi_I(x)) \frac{1}{2\pi} e^{iG_2(x)} as \ y \to 0, \text{where } G_2(x) = \frac{1}{2\pi} \ln(1+\lambda) \left[ -\frac{1}{2} \ln \frac{1+b^2}{1+a^2} + \ln \frac{b-x}{a-x} \right].$$

**Proof**: What is changing here is the value of the integral $\int_a^b \frac{2t}{1+t^2} dt$. Going back to the method of proposition 1, we have

$$\int_a^b \frac{2t}{1+t^2} dt = \ln \left( \frac{1+b^2}{1+a^2} \right)$$

and

$$h_\lambda(z) \to (1+\lambda \chi_I(x)) \frac{1}{2\pi} e^{iG_2(x)} as \ y \to 0,$$

Finally,

$$h_\lambda(x) = \begin{cases} (1+\lambda) \frac{1}{2\pi} e^{iG_2(x)} & \text{for } x \in I \\ e^{iG_2(x)} & \text{for } x \notin I \end{cases}$$

where

$$G_2(x) = \frac{1}{2\pi} \ln(1+\lambda) \left[ -\frac{1}{2} \ln \frac{1+b^2}{1+a^2} + \ln \frac{b-x}{a-x} \right].$$

For $g_\lambda$ we find the same expression, by substituting $G_1(x)$ in place of $G_2(x)$.  

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**Proposition 4** \( g_{\lambda}(z) \to g_{\lambda}(x) \text{ a.e as } y \to 0 \) and for almost all \( x \):

\[
g_{\lambda}(x) = \begin{cases} 
\frac{\lambda}{2(1+\lambda)} g(x) + \frac{i\lambda}{2\pi(1+\lambda)} e^{iG_2(x)} p.v \int_I \frac{e^{-iG_2(t)g(t)}}{x-t} dt & \text{for } x \in I \\
\frac{i\lambda}{2\pi(1+\lambda)} e^{iG_2(x)} \int_I \frac{e^{-iG_2(t)g(t)}}{x-t} dt & \text{for } x \notin I
\end{cases}
\]

3 Asymptotic behavior

Suppose \( I = ]-a, a[ \) and \( x \notin \{-a, a\} \).

In this section we wish to estimate \( g_{\lambda}(x) \) when \( x \in J = \mathbb{R} \setminus I \).

Let \( \varphi(u) = \frac{1 - e^{-u}}{1 + e^{-u}} \), a change of variable and \( \mathcal{H} \) be the following conditions :

\[
\mathcal{H} \left\{ \begin{array}{l}
go_{\lambda}(z) \text{ has no singularities in the strip } 0 \leq \Im(z) < \pi \text{ and does not accumulate on } \Im(z) = \pi \quad C_1. \\
\exists \delta \in ]0, 1[, \text{ such that } go_{\lambda}(z) = O(e^{\delta \Re(z)}) \\
as \Re(z) \to \pm \infty \forall z \in \Gamma_j, j \in \{1, 2, 3\} \quad C_2.
\end{array} \right.
\]

\[\Gamma = \bigcup_j \Gamma_j \cup [-R, R] \text{ is the following contour and } b \in ]\pi, \frac{3\pi}{2}]:\]

**Definition :**

\( \mathcal{M} \) is the class of functions verifying \( \mathcal{H} \).

**Theorem 3** Let \( I = ]-a, a[ \), \( g \in \mathcal{M} \) we have : \( g_{\lambda}(x) = O(1) \).
Proof: Let $g$ be in $\mathcal{M}$.

In order to simplify notations, suppose $g\varphi$ has no singularities on $\Im(z) = \pi$.

We have
\[ g_\lambda(x) = \frac{\lambda}{2\pi(1+\lambda)^2} e^{i G_1(x)} \int_I \frac{e^{-i G_1(t)} g(t)}{x-t} dt, \]
where
\[ G_1(x) = \frac{1}{2\pi} \ln(1+\lambda) \ln \left| \frac{a-x}{a+x} \right|. \]

Put $\xi = \frac{\ln(1+\lambda)}{2\pi}$, $u = -\ln \left| \frac{a-t}{a+t} \right|$ and $\alpha = \frac{a+x}{x-a}$, so,
\[ |g_\lambda(x)| = \frac{\lambda a}{\pi(1+\lambda)^{\frac{3}{2}} |a-x|} \left| \int_\mathbb{R} \frac{e^{i\xi u} e^u g\varphi(u)}{(e^u+1)(e^u+\alpha)} du \right|. \]

Note that:
\[ \frac{\lambda a}{\pi(1+\lambda)^{\frac{3}{2}} (a-x)} = O(e^{\xi \pi}). \quad (I) \]

Denote $k(u, \xi)$, the quotient $\frac{e^{i\xi u} e^u}{(e^u+1)(e^u+\alpha)}$.

We use the residu theorem to estimate $\left| \int_\mathbb{R} k(u, \xi) g\varphi(u) du \right|$.

Firstly, remark that singularities of the integral are of the form:
\[ i(\pi + 2k\pi) \text{ and } i(\pi + 2k\pi) + \ln(\alpha) \]
and
\[ \oint k(z, \xi) g\varphi(z) dz = (\text{Res}(kg, i\pi) + \text{Res}(kg, i\pi + \ln(\alpha))) 2\pi i. \]

Calculations of both residus show that:
\[ \text{Res}(kg, i\pi) = \frac{e^{-\xi \pi}}{\alpha - 1} g\varphi(i\pi) \text{ and } \text{Res}(kg, i\pi + \ln(\alpha)) = \frac{e^{-\xi \pi} e^{i\xi \ln(\alpha)}}{1 - \alpha} g\varphi(i\pi + \ln(\alpha)). \]

On the other hand:
\[ \oint k(z, \xi) g\varphi(z) dz = \int_{-R}^R k(u, \xi) g\varphi(u) du + \int_0^b k(R+iy, \xi) g\varphi(R+iy) dy \]
\[ + \int_{-R}^{-R} k(u+ib, \xi) g\varphi(u+ib) du + \int_b^0 k(-R+iy, \xi) g\varphi(-R+iy) dy. \]

When $R \to +\infty$ the first integral is the one we deal with. If we set $I_1$, $I_2$, $I_3$ and $I_4$ the four integrals, we have:
\[ I_1 = \frac{e^{-\xi \pi}}{\alpha - 1} g_0 \varphi(i\pi) + \frac{e^{-\xi \pi} e^{i\xi \ln(\alpha)}}{1 - \alpha} g_0 \varphi(i\pi + \ln(\alpha)) - I_2 - I_3 - I_4. \]

Put
\[ c_1(\alpha) = \frac{g_0 \varphi(i\pi)}{\alpha - 1} \quad \text{and} \quad c_2(\alpha, \xi) = \frac{e^{i\xi \ln(\alpha)}}{1 - \alpha} g_0 \varphi(i\pi + \ln(\alpha)), \]
so
\[ |I_1| \leq |c_1(\alpha) + c_2(\alpha, \xi)| e^{-\xi \pi} + |I_2| + |I_3| + |I_4|. \]

Noting \( c_2(\alpha, \xi) = c_2(\alpha) \), we get:
\[ |I_1| \leq (|c_1(\alpha)| + |c_2(\alpha)|) e^{-\xi \pi} + |I_2| + |I_3| + |I_4| \tag{II} \]

Remark: Note that we can choose \( x \) such that \( g_0 \varphi \) does not vanish in \( \ln(\alpha) + i\pi \), so \( c_2(\alpha, \xi) \neq 0 \).

Moreover, \( |I_2| = \int_0^b k(R + iy, \xi) g_0 \varphi(R + iy) dy \) and \( g \) satisfies \( C_1 \), then
\[ |g_0 \varphi(R + iy)| \leq M e^{\delta R} \] where \( M > 0 \). Thus,
\[ |I_2| \leq M e^{\delta R} \int_0^b |k(R + iy, \xi)| dy \]
\[ \leq M \frac{e^{\delta R}}{(e^R - 1)(e^R - \alpha)} \int_0^b e^{\xi y} dy \]
when \( R \to +\infty \) \( |I_2| \to 0 \).

In a similar fashion we show that when \( R \to +\infty \) \( |I_4| \to 0 \).

For \( I_3 \), we see that
\[ |I_3| \to \left| \int_{\mathbb{R}} k(u + ib, \xi) g_0 \varphi(u + ib) du \right| \text{ as } R \to +\infty. \]

Without loss of generality, let \( b = \frac{3\pi}{2} \) and if \( \tilde{I}_3 \) denotes \( \lim_{R \to +\infty} I_3 \), we get:
\[ |\tilde{I}_3| \leq e^{-3\pi \xi} \int_{\mathbb{R}} e^{\delta |t| e^t} dt \]
by the substitution \( t = \ln(r) \), we see that
\[ |\tilde{I}_3| \leq e^{-3\pi \xi} \int_0^{+\infty} e^{\ln(r^\delta)} \frac{r^{-\delta}}{|-ir + 1| - ir + \alpha} dr \]
\[ \leq e^{-3\pi \xi} \left[ \int_0^1 \frac{r^{-\delta}}{|-ir + 1| - ir + \alpha} dr + \int_1^{+\infty} \frac{r^\delta}{|-ir + 1| - ir + \alpha} dr \right] \]
\[ \leq e^{-3\pi \xi} \left[ \int_0^1 \frac{r^{-\delta}}{1 + r^2} dr + \int_1^{+\infty} \frac{r^\delta}{1 + r^2} dr \right]. \]

Finally,
\[ |\tilde{I}_3| = O(e^{-3\pi \xi}). \]

With (I) and (II) this ends the proof.

**Remark:**

1. Calculations show that the nonsymmetrical case give the same results.

2. To bound \( \tilde{I}_3 \), the most important is \( b \neq \pi \).

3. In the last proof, we supposed that \( g \circ \varphi \) has no singularities on \( \Im (z) = \pi \).

   If not, \( g \circ \varphi \) has for instance \( n \) ples, \( i\pi + \gamma_j \). Since we can use the residu theorem, we get:

   \[
   \operatorname{Res}(k \circ \varphi ; i\pi + \gamma_j) = \frac{e^{i\xi (i\pi + \gamma_j)} e^{i\pi + \gamma_j}}{(e^{i\pi + \gamma_j} + 1) (e^{i\pi + \gamma_j} + \alpha)} \lim_{u \to i\pi + \gamma_j} (u - i\pi - \gamma_j)^{m_j} g \circ \varphi (u) = e^{-\xi \pi} c_j (\alpha, \xi, \gamma_j).
   \]

   \( m_j \) is the ordre of the pôle \( i\pi + \gamma_j \). Finally:

   \[
   \oint_{\Gamma} k(z, \xi) g \circ \varphi (z) dz = 2i \pi e^{-\xi \pi} \sum_{j=1}^{n} c_j (\alpha, \xi, \gamma_j).
   \]

   Then we use the same methode to obtain the desired boundedness.

**Theorem 4** Suppose that \( g \) satisfies \( C_2 \) and \( g \circ \varphi \) is meromorphic in the strip \( \Omega = \{0 < \Im (z) < \pi \} \) whose poles are in a finite number in this open set.

Then \( g_\lambda (x) \to +\infty \) as \( \lambda \to +\infty \).

**Proof:** Employing the same methode as in the last theorem and without loss of generality, assume that \( g \circ \varphi (z) \) has two poles, other than those of \( k \), in \( \Omega \), say \( \beta_3 \) and \( \beta_4 \). We have,

\[
\oint_{\Gamma} k(z, \xi) g(z) dz = (\operatorname{Res}(kg, i\pi) + \operatorname{Res}(kg, i\pi + \ln (\alpha)) + \operatorname{Res}(kg, \beta_3) + \operatorname{Res}(kg, \beta_4)) 2\pi i
\]

but \( \operatorname{Res}(kg, i\pi) \) and \( \operatorname{Res}(kg, i\pi + \ln (\alpha)) \) are known. If \( j \in \{3, 4\} \) and \( m_j \) is the order of the pole \( \beta_j \), we see that:
\[
\begin{align*}
\text{Res}(kg, \beta_j) &= \frac{e^{i\xi \Re(\beta_j)}e^{-\xi \Im(\beta_j)}}{(1 + e^{-\beta_j})(1 + \alpha e^{-\beta_j})} \lim_{z \to \beta_j} g \varphi (z)(z - \beta_j)^{m_j}, \\
\text{Res}(kg, \beta_j) &= c_j(\beta_j, \xi)e^{-\xi \Im(\beta_j)}
\end{align*}
\]

where,
\[
c_j(\beta_j, \xi) = \frac{e^{i\xi \Re(\beta_j)}}{(1 + e^{-\beta_j})(1 + \alpha e^{-\beta_j})} \lim_{z \to \beta_j} g \varphi (z)(z - \beta_j)^{m_j}
\]

If \(|c_j(\beta_j, \xi)| = c_j(\beta_j)|
we see that \(|\text{Res}(kg, \beta_j)| = c_j(\beta_j)e^{-\xi \Im(\beta_j)}|.

On the other hand,
\[
\int_I k(z, \xi)g(z)dz = I_1 + I_2 + I_3 + I_4,
\]

therefore
\[
I_1 = c_3(\beta_3, \xi)e^{-\xi \Im(\beta_3)} + c_4(\beta_4, \xi)e^{-\xi \Im(\beta_4)} + [c_1(\alpha) + c_2(\alpha, \xi)]e^{-\xi \pi} - I_2 - I_3 - I_4.
\]

Put \(\tilde{I}_j = \lim_{R \to \infty} I_j. \) Since \(|\tilde{I}_2| = 0\) and \(|\tilde{I}_4| = 0\) we have,
\[
|\tilde{I}_1| \geq e^{-\xi \pi}|c_1(\alpha) + c_2(\alpha, \xi) + c_3(\beta_3, \xi)e^{-\xi(\Im(\beta_3) - \pi)} + c_4(\beta_4, \xi)e^{-\xi(\Im(\beta_4) - \pi)}| - |\tilde{I}_3|.
\]

From the last theorem, \(\exists M > 0\) such that \(-|\tilde{I}_3| \geq -e^{-\xi \pi}M\), then
\[
e^{\xi \pi}|\tilde{I}_1| \geq |c_1(\alpha) + c_2(\alpha, \xi) + c_3(\beta_3, \xi)e^{-\xi(\Im(\beta_3) - \pi)} + c_4(\beta_4, \xi)e^{-\xi(\Im(\beta_4) - \pi)}| - e^{(\pi - b)\xi}M.
\]

Note that \(e^{(\pi - b)\xi} \to 0\) as \(\xi \to +\infty. \) If for instance \(\Im(\beta_3) \geq \Im(\beta_4)\) we deduce :
\[
e^{\xi \pi}|\tilde{I}_1| \geq e^{(\pi - \Im(\beta_3))\xi}|(c_1(\alpha) + c_2(\alpha, \xi))e^{(-\pi + \Im(\beta_3))\xi} + c_3(\beta_3, \xi) + c_4(\beta_4, \xi)e^{(\Im(\beta_4) - \Im(\beta_3))\xi}|.
\]

Finally this last expression \(\to +\infty\) as \(\xi \to +\infty.
\]

Q.E.D.

**Corollary 1** If \(g \varphi (z)\) verifies \(C_2\) then \(C_1\) is a necessary and sufficient condition to obtain the boundedness of \(g_\lambda (x)\).

Since \(x\) is arbitrarily taken, it can lie in a neighborhood \(V\) of positive measure, where the bounds of \(I\) are not contained in \(V\). We get the following :

**Corollary 2** If \(g \varphi (z)\) verifies \(C_2\) and not \(C_1\) then \(g_\lambda (x) \to +\infty\) as \(\lambda \to +\infty \forall x \in V.\)

So we cannot hope the ae convergence of \(g_\lambda\) on \(\mathbb{T} \setminus I\).

Example 2 illustrates this.
4 Applications

Let $g(x) = g_1(x) + ig_2(x)$.

**Example 1**

\[
\begin{align*}
    g_1(x) &= 0 & \text{for } x \in ]-\infty, -a[ \\
    g_1(x) &= \sqrt{a^2 - x^2} & \text{for } x \in ]-a, +a[ \\
    g_1(x) &= 0 & \text{for } x \in ]a, +\infty[ \\
    g_2(x) &= -x - \sqrt{x^2 - a^2} & \text{for } x \in ]-\infty, -a[ \\
    g_2(x) &= -x & \text{for } x \in ]-a, +a[ \\
    g_2(x) &= -x + \sqrt{x^2 - a^2} & \text{for } x \in ]a, +\infty[ \\
\end{align*}
\]

In $]-a, +a[$, $g(x) = \sqrt{a^2 - x^2} - ix$.

To simplify calculations, assume $a = 1$. We assert that $g \in \mathcal{M}$. In fact

\[
g_1 \circ \varphi(t) = \sqrt{1 - \left(\frac{1 - e^{-t}}{1 + e^{-t}}\right)^2} = \frac{2e^{-t}}{1 + e^{-t}}
\]

and

\[
g_2 \circ \varphi(t) = \frac{e^{-t} - 1}{e^{-t} + 1}
\]

First, we are going to see that $g_1 \circ \varphi$ and $g_2 \circ \varphi$ satisfy $C_2$.

Observe that we have easily the equivalence:

$g \circ \varphi$ verifies $C_2$ if and only if $g_1 \circ \varphi$ and $g_2 \circ \varphi$ verify the same condition.

It is clear that boundedness from above on $\Gamma_1$ and on $\Gamma_2$ of $g_1$ and $g_2$ are sufficient to do the job.

**Case $g_1$ on $\Gamma_1$**:

We have to find $M$ and $\delta \in ]0, 1[$ such that:

\[
\left|\frac{2e^{-\left(\frac{-R+y_2}{2}\right)}}{1 + e^{R-y_2}}\right| < Me^{\delta R}
\]
where \( R \) is some constant \( > 0 \) and \( \forall y \in [0, b] \).

For this, one can for instance do

\[
\left| \frac{2e^{-\left(\frac{R+b+i}{2}\right)}y}{1+e^{R-yi}} \right| < \frac{2e^R}{1-e^R} < Me^{\delta R}.
\]

Case \( g_1 \) on \( \Gamma_2 \):

As before we must find \( M \) and \( \delta \in ]0,1[ \) such that :

\[
\left| \frac{2e^{-\left(\frac{ib+x}{2}\right)}}{1+e^{-x+ib}} \right| < Me^{\delta|x|} \quad \forall x \in \mathbb{R}.
\]

That is to say :

\[
\frac{2e^{-\frac{x}{2}}}{e^{-2x} + 2\cos(b)e^{-x} + 1} < e^{\delta|x|}.
\]

The first side, \( h(x) \), is a positive function \( C^\infty \) on \( \mathbb{R} \) whose denominator does not vanish. Furthermore, \( \lim_{x \to +\infty} h(x) = 0 \) and so does at \( -\infty \). Therefore there exists a maximum (function of \( b \)) such that \( h(x) < M(b) \), where \( \pi < b \leq \frac{3\pi}{2} \).

Case \( g_2 \) on \( \Gamma_1 \):

It is easy to see that we can have \( \delta \in ]0,1[ \) and \( M \) such that :

\[
\left| \frac{e^{-(-R+iy)} - 1}{e^{-(-R+iy) + 1}} \right| \leq Me^{-\delta R}.
\]

Case \( g_2 \) on \( \Gamma_2 \):

A simple triangular inequality application provides us the desired constants.

Note that the fact that \( g_0 \phi \) vérifie \( C_1 \) is an easy exercise.

Example 2

\[
g_1(x) = \frac{1}{1+x^2} \quad \forall x \in \mathbb{R}
\]

\[
g_2(x) = \frac{-x}{1+x^2} \quad \forall x \in \mathbb{R}
\]

After change of variable,

\[
g_0 \phi(t) = \frac{1}{2} \left( \frac{(1+e^{-t})^2}{1+e^{-2t}} - \frac{1-e^{-2t}}{1+e^{-2t}} \right)
\]
This example illustrates theorem 4. In fact, \( \text{Res}(kg, i\pi) = e^{-\xi \pi} c(\alpha) \), where \( c(\alpha) \) is a complex number.

We see that \( |g_\lambda(x)| \to +\infty \) as \( \lambda \to +\infty \).

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