Feshbach resonance described by the boson fermion coupling

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(Dated: March 22, 2022)

We consider a possibility to describe the Feshbach resonance in terms of the Boson Fermion (BF) model. Using such model we show that after a gradual disentangling of the boson from fermion subsystem the resonant type scattering between fermions is indeed generated. We decouple the subsystems via: (a) the single step, and (b) the continuous canonical transformation. With the second one we investigate the feedback effects effectively leading to the finite amplitude of the scattering strength. We study them in detail in: the normal $T > T_c$ and superconducting $T ≤ T_c$ states.

I. INTRODUCTION

Superconductivity is a quantum state which appears at sufficiently low temperatures $T ≤ T_c$ in various systems like e.g. metals (Hg, Pb), alloys (Nb3Sn), copper oxides (La-Sr-Cu-O, Y-Ba-Cu-O), exotic compounds (MgB2, UGe2, ZrZn2), the liquid 3He, etc. Depending on a material, $T_c$ can range from $10^{-3}$ to more than hundred K and various underlying mechanisms could be responsible for superconductivity (like the phonon or magnetically mediated attraction between electrons, the BE condensation of tightly bound electron pairs, etc.). One should add to this list a recently obtained superfluidity in the binary mixtures of the trapped alkali atoms such as: rubidium [1], potassium [2] and lithium [3]. Transition temperatures $T_c ≈ 100$ nK became there experimentally accessible due to the “resonance superfluidity”. Atoms of different hyperfine configuration are scattered from each other with a strength depending on the external magnetic field $B$ in a following way $a = a_0 [1 − ΔB/(B − B_{res})]$. Near the resonance $B ≈ B_{res}$ atoms experience a considerably amplified attractive interaction which gives rise to $T_c$’s comparable with the Fermi temperature $T_F$.

From the theoretical point of view such controlled way of adjusting the effective interactions between atoms is very appealing. It opens new possibilities to explore for instance such fundamental problems like a crossover from the weak coupling BCS superconductivity to the BE superfluidity of tightly bound atom pairs. On a microscopic level, the resonant type interactions are however rather delicate to treat (for example near the resonance the usual perturbation theory can not be applied). So far, the most reasonable way of describing such interactions was proposed by Timmermans [4] in terms of the boson fermion (BF) model. This idea was recently intensively investigated by the JILA group [5] and independently by Griffin and Ohashi [6].

Alkali atoms of different hyperfine configurations are represented within the BF model via fermion fields which are characterized by a two labeled index, e.g. $σ = ↑, ↓$. These fermions are assumed to interact with a boson field $c$ which can be thought of as some bound molecules made of two atoms. According to the theory [4] the resonant type interaction arises when a total energy of two colliding fermions matches the energy of boson state, the so called Feshbach resonance [7].

Using the BF model in the above mentioned context it was shown that transition temperature $T_c$ can become extremely high, of the order 0.2 $T_F$ for a uniform gas and about 0.5 $T_F$ in an isotropic harmonic trap [8].

II. THE MODEL

For simplicity we consider here a model of free fermions coupled to the molecular boson field as described by the
following Hamiltonian \( H^{BF} \)
\[
H^{BF} = \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} + \sum_{q} (E_q - 2\mu) b_q^\dagger b_q
+ v \sum_{k, q} (b_q^\dagger c_{-k+q} c_{k+q} + \text{h.c.}) .
\] (1)

neglecting the fermion fermion interactions which eventually would be responsible for the background scattering \( \Delta Q \). We use the second quantization operators \( c_{k\sigma}^\dagger, c_{k\sigma} \) for the fermion state of energy \( \varepsilon_k \) which can exist in two possible hyperfine configurations symbolically denoted by \( \sigma = \uparrow, \downarrow \) and \( b_q^\dagger, b_q \) for the molecular boson state of energy \( E_q \). Fermion and boson subsystems are coupled through the isotropic interaction \( v \) (in the context of HTSC, it should be anisotropic \( \Delta Q \)). Since a pair of fermions can “dissociate” into the boson state it implies that bosons are carrying the double fermions’ charge. In order to satisfy the charge conservation we introduce the common chemical potential \( \mu \) and work in the grand canonical ensemble.

It is worthwhile to comment that within the BF model the mechanism of superconductivity is unconventional \( 8, 10, 14, 15 \). Due to interaction \( v \) bosons acquire a well established effective mass \( m_B = m_B^{-1} \neq 0 \), even if at the outset they are immobile \( (E_q = \Delta_B) \). At a critical temperature \( T_{BE} \) bosons undergo the BE condensation and simultaneously this triggers a superconducting ordering of the fermion subsystem \( T_{SC} = T_{BE} = T_c \) \( 11, 12 \).

Let us point out some of the unusual properties predicted theoretically on a basis of this BF scenario which are relevant for the HTSC materials but could possibly manifest somehow also in the mixtures of the trapped alkali atoms. As far as superconducting phase is concerned: (i) critical temperatures \( T_c \) is known to become extremely high because the effective pairing potential \( [\text{a level of the mean field theory estimated to be } |V(T)| = \frac{\Delta^2}{2\Delta T^2} \text{tanh} \left( \frac{\Delta^2}{2\Delta T^2} \right) \text{[11]} \] is very large in the, so-called, mixed regime of coexisting fermion and boson particles \( 11, 10 \) which occurs when \( \mu \rightarrow \Delta_B/2 \); (ii) in the mixed regime, high values of \( T_c \) are accompanied by the non-BCS ratio \( \Delta_{sc}(0)/k_B T_c \geq 4 \text{[11, 17]} \); (iii) the upper critical field \( H_{c2}(T) \) has a characteristic upward curvature \( d^2 H_{c2}(T_c)/dT^2 > 0 \text{[12]} \). In the normal phase above \( T_c \) it was shown that: (i) \( d\sigma \) resistivity is linear with respect to \( T \) up to very high temperatures \( 20 \), (ii) depending on a doping level and on temperature there is a change of sign of the Hall constant \( 21 \), (iii) in a temperature regime \( T^* > T > T_c \) the pseudogap (partial depletion of the fermion density of states) builds up near the Fermi energy \( 11, 22, 23 \), (iv) the single particle spectrum reveals a clear particle-hole asymmetry \( 12, 21, 21 \).

### III. THE SINGLE STEP TRANSFORMATION

For studying effective physics of the BF model it is convenient first to apply the standard canonical trans-
formation. We treat the boson fermion interaction as a perturbation \( H_{pert} = v \sum_{k, q} (b_q^\dagger c_{-k+q} + \text{h.c.}) \) and try to eliminate it from \( \Delta \) via the unitary transformation \( \epsilon^{\lambda} \). Choosing
\[
S = \sum_{k, q} \left( \frac{v b_q^\dagger c_{-k+q} + \text{h.c.}}{E_q - \varepsilon_k - \varepsilon_{-k+q} - \text{h.c.}} \right),
\] (2)

such that \( H_{pert} + [S, H_0] = 0 \), we obtain the transformed Hamiltonian \( \tilde{H} \equiv \epsilon^{\lambda} H \epsilon^{-\lambda} \) as
\[
\tilde{H} = \sum_{k, \sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} + \sum_{q} \left( \tilde{E}_q - 2\mu \right) b_q^\dagger b_q
+ \sum_{k, p, q} U_{k, p, q} c_{k\uparrow}^\dagger c_{p\downarrow}^\dagger c_{q\downarrow} c_{k+q-p\uparrow} + o(v^3) ,
\] (3)

where \( o(v^3) \) stands for other terms of the order \( v^3 \). The renormalized quantities \( \tilde{\varepsilon}_k, \tilde{E}_k \) and \( U_{k, p, q} \) present in \( \Delta \) are given by
\[
\tilde{\varepsilon}_k = \varepsilon_k + v^2 \sum_{q} \frac{f_{BE}(E_q)}{\varepsilon_k + \varepsilon_{-k+q} - E_q}
\] (4)
\[
\tilde{E}_q = E_q - v^2 \sum_{k} \frac{1 - I_{FD}(\varepsilon_k) - I_{FD}(\varepsilon_{-k+q})}{\varepsilon_k + \varepsilon_{-k+q} - E_q}
\] (5)
\[
U_{k, p, q} = \left[ \frac{v^2/2}{\varepsilon_k + \varepsilon_p - E_{k+p}} + \frac{v^2/2}{\varepsilon_q + \varepsilon_{k+p-q} - E_{k+p}} \right]
\] (6)

with \( f_{BE}, I_{FD} \) denoting the Bose-Einstein and Fermi-Dirac distributions respectively. In the \( q = p \) channel thus induced interaction between fermions of different \( \sigma \) simplifies to
\[
\frac{v^2}{\varepsilon_k + \varepsilon_p - E_{k+p}} \sigma_{k\uparrow}^\dagger \sigma_{k\uparrow}^\dagger \sigma_{p\downarrow}^\dagger \sigma_{p\downarrow}
\] (7)

which explicitly exhibits a divergence. We notice that a scattering potential has a resonant type character when total energy of two colliding fermions is equal to the energy of boson field (Feshbach resonance \( 3 \)).

One may argue that validity of canonical transformation is limited only to such states which are far from the resonance because otherwise the operator \( \Delta \) is ill defined. Our conclusion about the resonant scattering strength \( \Delta \) may then seem questionable. However, in the next section we shall prove that such resonance is not an artifact, it indeed exists although somewhat smeared.

#### IV. CONTINUOUS CANONICAL TRANSFORMATION

Instead of a single step transformation we will now disentangle the fermion from boson subsystems applying a sequence of infinitesimal transformations. The main idea behind is to proceed with a continuous canonical transformation \( S(l) \) (where \( l \) denotes some formal flow parameter which can vary between 0 and any other value) until
a given structure of the Hamiltonian \( H(l) = e^{S(l)}He^{-S(l)} \) is obtained. A virtue of this method is that one can freely manipulate with \( S(l) \) in order to get a constrained structure of \( H(l) \).

In this work we follow the algorithm outlined by Wegner 22 for constructing the generating operator \( \eta(l) \equiv dS(l)/dl \) of the continuous transformation. An arbitrary Hamiltonian \( H(l) = H_0(l) + H_{\text{pert}}(l) \) can be reduced to a semidiagonal structure by choosing \( \eta(l) = [H_0(l), H_{\text{pert}}(l)] \) which assures that \( \lim_{l \to \infty} H_{\text{pert}}(l) = 0 \). During the transformation \( H_0(l) \) part does evolve too, its \( l \)-dependent parameters have to be deduced from the general flow equation \( dH(l)/dl = [\eta(l), H(l)] \) 22. In practice, this flow equation can never be identically satisfied because the higher and higher order interactions (not present in \( H_0 \)) are generated from the commutator. Flow equation is often approximated by truncating the higher order interactions in a spirit of the perturbation theory, however the nonperturbative methods are in principle possible too 22.

In the previous work 14 we have already formulated the continuous canonical transformation for the BF model. The corresponding flow equations (16-21) of Ref. 14 were derived within a perturbational estimation up to order \( v^3 \). In this short report we want to study in some more detail the resonant-type interactions induced between fermions in the effective Hamiltonian \( \hat{H} = H(l \to \infty) \).

In a course of transformation the interactions between fermions evolve from zero, at \( l = 0 \), to some effective value \( \hat{U}_{k,p,q} \) at \( l = \infty \) when boson and fermion subsystems are finally decoupled from each other. Flow of the potential is governed by the equation 14

\[
\frac{dU_{k,p,q}(l)}{dl} = [\alpha_{k,p}(l) + \alpha_{q,k+p-q}(l)]v_{k,p}(l)v_{q,k+p-q}(l)
\]

(8)

where \( \alpha_{k,p}(l) = \varepsilon_k(l) + \varepsilon_p(l) - E_{k+p}(l) \) and the introduced momentum dependence of the boson-fermion coupling \( v_{k,p}(l)v_{q,k+p,q}^* \) arises from \( dH(l)/dl \). Equation 8 is convoluted with the following ones 14

\[
\frac{dv_{k,p}(l)}{dl} = -\alpha_{k,p}(l)v_{k,p}(l), \quad (9)
\]

\[
\frac{d\varepsilon_{k}(l)}{dl} = 2\sum_p \alpha_{k,p}(l)v_{k,p}^2(l)FBE(E_{k+p}(l)), \quad (10)
\]

\[
\frac{dE_q(l)}{dl} = -2\sum_k \alpha_{q-k,k}(l)v_{k-k,k}(l) \times [1 - 2f_{FD}(\varepsilon_{k-q}(l))]. \quad (11)
\]

These four \( l \)-dependent quantities \( v_{k,p}(l) \), \( \varepsilon_{k}(l) \), \( E_q(l) \), \( U_{k,p,q}(l) \) should be determined simultaneously. Mathematically it is a tremendous task, nevertheless we shall estimate \( U \) either in an approximate way or numerically and, in one special case, exactly.

A. Exact solution at \( T < T_c \)

Some exact statements can be done for the superconducting/superfluid phase of the BF model. For temperatures \( T \) smaller than a critical \( T_c \) there exists a finite fraction of condensed bosons \( n_B^0 = \langle b_{k}b^*_{k}\rangle_q = 0 \) and chemical potential is then located at the lowest boson energy \( \mu(T) = E_0/2 \). The flow equation (10)

\[
\frac{d\varepsilon_{k}(l)}{dl} \equiv 4\varepsilon_{k}(l)n_B^0v^2_{k,k}(l) \quad (12)
\]

due to \( d\varepsilon_{k}(l)/dl = -8\varepsilon_{k}(l)v^2_{k,k}(l) \). After integration by parts we get from (12)

\[
\xi_{k}(l)|_{0}^{\infty} - \int_{0}^{\infty} \frac{d\xi_{k}(l)}{dl}U_{k,-k}(l) = v^2 / 2 \quad (13)
\]

because \( \varepsilon_{k,-k}(\infty) = 0 \). Since according to (10) we have \( d\xi_{k}/dl \propto v^2 \) and from our previous estimation 14 also \( U_{k,p,q} \propto v^2 \) we finally conclude

\[
U_{k,-k}(\infty) = \frac{v^2 + o(v^4)}{2\text{sign} (\xi_{k}) \sqrt{\xi_{k}^2 + n_B^0 v^2}}. \quad (14)
\]

This equation shows that two colliding electrons with total energy \( \varepsilon_k + \varepsilon_{-k} = E_{k=0} \) have a resonant-like scattering strength (remember that in this case \( E_0 = 2\mu \)) and due to symmetry \( \varepsilon_{-k} = \varepsilon_k \). Amplitude of the resonance is now finite and is controlled by the superconducting gap \( \Delta_{sc}(T) = v\sqrt{n_B^0}(T) \).

B. Approximate solution for \( T \geq T_c \)

Effective interaction between fermions can be calculated approximately, for instance iteratively. In the first step we can neglect \( l \)-dependence of \( \alpha_{k,p}(l) \) in the flow equations 8 and 9 because fermion \( \varepsilon_{k}(l) \) is \( \varepsilon_k \) and boson energies \( E_q(l) \propto E_q \) are rather weakly renormalized during the transformation 14.

With an approximation \( \alpha_{k,p}(l) \approx \alpha_{k,p}(l = 0) \) we can easily obtain \( v_{k,p}(l) = v \exp [- (\varepsilon_k + \varepsilon_p - E_{k+p})^2 l] \).

When further substituted to equation 8 we get the following asymptotic value \( U_{k,p,q}(l \to \infty) \)

\[
\frac{v^2 (\varepsilon_k + \varepsilon_p + \varepsilon_q + \varepsilon_{k+p-q} - 2E_{k+p})}{(\varepsilon_k + \varepsilon_p - E_{k+p})^2 + (\varepsilon_k + \varepsilon_{k+p-q} - E_{k+p})^2} \quad (15)
\]
We obtained the effective potential \( U_{k,p,q} \) for several temperatures \( T \) using a fixed total charge concentration \( \sum_{\kappa} \langle c_{\kappa}^\dagger c_{\kappa}\rangle + 2 \sum_{q} \langle b_q^\dagger b_q\rangle = \text{const} \). Temperature had rather a negligible influence on a magnitude of the fermion fermion interaction for \( T > T_c \). Using a given \( \Delta_B \) value we adjusted the total charge concentration so, that number of bosons \( N_B = \sum_{q} f_{BE}(E_q) \) and fermions \( N_F = \sum_{k} f_{FD}(\varepsilon_k) \) were comparable. Chemical potential was located very close to the bottom of boson states \( \mu(T) \approx \Delta_B / 2 \). Figure 1 illustrates the behavior of \( U_{k,p,q}(T) \). Again, we notice an appearance of the resonant type interaction for \( \varepsilon_k + \varepsilon_p = \Delta_B \) but, in distinction from (7), amplitude of the resonance is finite.

**V. CONCLUSIONS**

In summary, we analyzed the resonant type interactions between fermions (experimentally achievable for example in the binary mixtures of \(^{85}\text{Rb}\) \(^{40}\text{K}\) or \(^{6}\text{Li}\) atoms \(^{3}\) in a presence of the magnetic field) using the microscopic boson fermion (BF) model. Within the BF scenario, fermions are coupled to the boson field (the long lived molecules composed of two atoms). This coupling is responsible for the many body effects and, in particular at \( T_c \) it drives a system to the superconducting/superfluid ordering (8). When approaching the critical temperature from above, several precursor features can be observed which show that the pairing and the long range pair coherence may occur in this model at different temperatures \( T^* \) and \( T_c \leq T^* \), respectively.

We determined the effective fermion fermion interaction induced by elimination of the boson fermion coupling via the canonical transformation. Boson and fermion subsystems were disentangled (i) by the single step transformation, and (ii) through a continuous sequence of transformations taking account of the feedback effects (15). In both methods we obtained the effective resonant type interactions between fermions. At temperatures \( T > T_c \) the resonance appeared to be more sharp (although of finite amplitude), while for \( T < T_c \) its amplitude reduced to a magnitude of the superconducting gap.

The resonant scattering seems to be a robust feature of the mixed boson-fermion system. It has an impact on several physical properties, for instance, it is responsible for the particle-hole asymmetric spectrum (21). Other aspect which is important in a context of the soft matter physics is that the microscopic BF model can serve as a convenient tool for description of the resonant type interactions \( a = a_0[1 - \Delta B / (B - B_{rev})] \) (1, 13). Many body effects can there be studied in a secure way (though still far from trivial) without a necessity to deal with any divergences.

Author kindly acknowledges valuable discussions with Profs J. Ranninger, F. Wegner and K.I. Wysokiński.

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