COMPOSITION FACTORS FOR THE SPRINGER RESOLUTION

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Abstract. Let \( \pi: \tilde{N} \to N \) be the Springer resolution of the nilpotent cone for \( \mathfrak{sl}_n(\mathbb{C}) \) and \( k \) be a field. What happens to \( \pi_* k_{\tilde{N}}^{[\dim \tilde{N}]} \) if the decomposition theorem fails for it? We show that in this case, some additional (with respect to the case \( \text{char } k = 0 \)) composition factors of this direct image in the (abelian) category of perverse sheaves may emerge. Their multiplicities are given by the coranks of the intersection forms associated to the inclusion of the Springer fibers in the corresponding resolutions of Slodowy slices.

1. Introduction

The recent Hodge theoretic proof of the decomposition theorem by de Cataldo and Migliorini [3], [4] relies on the non-degeneracy of some intersection forms. An extension of this argument [9, Theorem 3.7] allowed the authors to prove that over a field of characteristic \( p > 0 \), the decomposition theorem for semi-small morphisms holds if and only if the \( p \)-modular reductions of these forms are non-degenerate.

The aim of this paper is to study the Springer resolution \( \pi: \tilde{N} \to N \) as an example of such a semi-small morphism in the case when the decomposition theorem fails for \( \pi_* k_{\tilde{N}}^{[\dim \tilde{N}]} \). This direct image is a perverse sheaf regardless of the characteristic of \( k \). So we can study its composition factors in the the abelian category of perverse sheaves and compare them to the zero characteristic case.

The paper is organized as follows. In Section 2, we recall the main constructions connected with the Springer resolutions and the Slodowy slices. Our aim here is to construct a neighbourhood basis \( \{W_x(t)\}_{t \in (0, +\infty)} \) for a point \( x \) in the nilpotent cone satisfying certain properties (Lemma 2). The main result of this paper is Theorem 1, which is proved in Section 3. Our calculations here rely on the cohomologies of the complements to the Springer fibers in the resolutions of the corresponding Slodowy slices. Note that the cohomologies of similar spaces, which may depend substantially on the characteristic of the field of coefficients, were already considered in [10], where they were used in Deligne’s formula for the intermediate extension. Although we do not use this formula, we encounter similar cohomologies, when we calculate the composition of the form \( i^* f_* \) (see formula (11)). As a result, we get some additional composition factors of the direct image, which correspond to constant local systems as is shown in Section 4. These factors result exactly from the degeneracy of the intersection forms described in Section 2.5. On the other hand, if all these forms are non-degenerate, then the decomposition theorem holds (this also follows from Theorem 1) and no new composition factors appear.

Finally, note that for \( \mathfrak{sl}_2(\mathbb{C}) \) the inequalities of Theorem 1 turn to equalities, as the intermediate extension functor is not applied in this case. Moreover, the exact values of \( a_\lambda \) (see (2) for definition) for \( \mathfrak{sl}_2(\mathbb{C}) \) can be found in [10, Section 2.4]. However, we can not prove equalities in general, as the intermediate extension functor is not exact. It would be interesting to see any concrete examples of composition factors not described by Theorem 1.
2. Notation and setup

2.1. Partitions and dominance order. Throughout the paper, we fix an integer \( n > 1 \).
A partition \( \lambda \) of \( n \) is a sequence \( (\lambda_1, \ldots, \lambda_k) \) on integers such that \( \lambda_1 \geq \cdots \geq \lambda_k \) and \( \lambda_1 + \cdots + \lambda_k = n \). For convenience, we assume \( \lambda_i = 0 \) for \( i > k \). We denote by \( \Lambda \) the set of all partitions of \( n \). It possesses the following partial order called the dominance order:

\[
\lambda \succeq \mu \iff \lambda_1 + \cdots + \lambda_m \leq \mu_1 + \cdots + \mu_m \quad \text{for any} \; m.
\]

This order is connected with closures of orbits (Section 2.2), which prompts the following terminology: a subset \( A \subset \Lambda \) is closed if \( \mu \leq \lambda \) and \( \lambda \in A \) imply \( \mu \in A \) and is called open if its compliment \( \tilde{A} = \Lambda \setminus A \) is closed. Equivalently, a subset \( A \subset \Lambda \) is open if and only if \( \mu \geq \lambda \) and \( \lambda \in A \) imply \( \mu \in A \).

2.2. Springer resolution of nilpotent cone. We consider the special Lie algebra \( \mathfrak{g} := \mathfrak{sl}_n(\mathbb{C}) \) and the special linear group \( G := \text{SL}_n(\mathbb{C}) \), which acts on \( \mathfrak{g} \) by conjugation \( g \cdot x = gxg^{-1} \). The symmetric group \( S_n \) on \( n \) letters is the Weyl group of \( G \). We shall consider \( \mathfrak{g} \) in the metric topology.

The set \( \mathcal{N} \) of all nilpotent matrices \( x \in \mathfrak{g} \) is called the nilpotent cone of \( \mathfrak{g} \). This set is stable under the above action of \( G \) and hence can be decomposed into the union of orbits \( \mathcal{N} = \bigsqcup_{\lambda \in \Lambda} \mathcal{O}_\lambda \), where \( \mathcal{O}_\lambda \) is the set of all nilpotent matrices of Jordan type \( \lambda \). This stratification \( X = \{ \mathcal{O}_\lambda \}_{\lambda \in \Lambda} \) is a Whitney stratification (see [2]).

A well-known result originally due to M. Gerstenhaber [6] states that \( \mathcal{O}_\mu \subset \overline{\mathcal{O}_\lambda} \) if and only if \( \mu \leq \lambda \). Thus the set \( \mathcal{N}_A := \bigsqcup_{\lambda \in \Lambda} \mathcal{O}_\lambda \) is closed (resp. open) if \( A \) is closed (resp. open).

We briefly describe the Springer resolution for \( \mathcal{N} \). Let \( \mathcal{B} \) denote the set of all Borel subalgebras of \( \mathfrak{g} \). Let

\[
\tilde{\mathcal{N}} := \{(x, b) \in \mathcal{N} \times \mathcal{B} \mid x \in b\}.
\]

The projection \( \pi : \tilde{\mathcal{N}} \to \mathcal{N} \) to the first component is called the Springer resolution. On the other hand, the projection \( \tilde{\mathcal{N}} \to \mathcal{B} \) to the second component identifies \( \tilde{\mathcal{N}} \) with the cotangent bundle \( T^*\mathcal{B} \). It is well known that \( \pi \) is semi-small and proper.

For any subset \( A \subset \Lambda \), we set \( \tilde{\mathcal{N}}_A := \pi^{-1}(\mathcal{N}_A) \) and \( \pi_A := \pi|_{\tilde{\mathcal{N}}_A} : \tilde{\mathcal{N}}_A \to \mathcal{N}_A \). Similarly to \( \mathcal{N}_A \), the subset \( \tilde{\mathcal{N}}_A \) is closed (resp. open) if \( A \) closed (resp. open). Clearly, \( \pi_A \) is proper as \( \tilde{\mathcal{N}}_A \) is the full preimage. We denote by \( i_A : \mathcal{N}_A \to \mathcal{N} \) and \( \tilde{i}_A : \tilde{\mathcal{N}}_A \to \tilde{\mathcal{N}} \) the natural inclusions. The preimage \( \mathcal{B}_x := \pi^{-1}(x) \) of an element \( x \in \mathcal{N} \) is called the Springer fibre. For different points \( x \) in the same orbit, Springer fibres \( \mathcal{B}_x \) are isomorphic, as it follows from \( \mathfrak{g}\mathcal{B}_x = \mathcal{B}_{gx} \).

We shall write \( \dim X \) for the complex dimension of \( X \) (vector space, variety) and \( \dim_k V \) for the dimension of a vector space over \( k \). We set

\[
d_N := \dim \mathcal{N} = \dim \tilde{\mathcal{N}}; \quad b_\lambda := \dim \mathcal{B}_x \quad \text{and} \quad d_\lambda := \dim H^{2b_\lambda}(\mathcal{B}_x, \mathbb{C}) \quad \text{for} \; x \in \mathcal{O}_\lambda.
\]

The following precise descriptions (see [7]) are well known\(^1\):

\[
d_N = n(n-1), \quad \dim \mathcal{O}_\lambda = n^2 - \sum_i (\lambda'_i)^2, \quad b_\lambda = (d_N - \dim \mathcal{O}_\lambda)/2,
\]

\[
d_\lambda = \text{the number of standard } \lambda\text{-tableaux}.
\]

The universal coefficient theorem and the main result of [5] show that \( \dim_k H^{2b_\lambda}(\mathcal{B}_x, k) = d_\lambda \) and \( \dim_k H^m(\mathcal{B}_x, k) = 0 \) for odd \( m \). We shall use these facts throughout the paper without further mention.

\(^1\)where \( \lambda' \) is the partition conjugate to \( \lambda \).
2.3. Slodowy slice. We describe the main constructions of [11] that we need here. Take any \( x \in \mathcal{O}_\lambda \). By the Jacobson-Morozov lemma, there exists a Lie algebra homomorphism \( \omega : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g} \) mapping the matrix \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) to \( x \). Let \( y \) be the image of \( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( h \) be that of \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). These elements satisfy the relations \( [x, y] = h \), \( [h, y] = -2y \), \( [h, x] = 2x \).

The homomorphism \( \omega \) defines the representation \( \tilde{\omega} : \text{SL}_2(\mathbb{C}) \to \text{Aut}_\mathbb{C}(\mathfrak{g}) \) by exponentiating. The image of \( \tilde{\omega} \) consists of inner automorphisms of \( \mathfrak{g} \). Take an arbitrary \( t \in \mathbb{C}^* \) and consider the following Lie algebra automorphisms:

\[
\rho(t) := \tilde{\omega} \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right), \quad \sigma(t)(z) = tz.
\]

They clearly preserve the Jordan types of elements of \( \mathfrak{g} \).

The **Slodowy slice** is the following affine subspace of \( \mathfrak{g} \) (see [11, §2.4]):

\[
S_x := x + \ker \text{ad} \, y.
\]

As explained in [11, §2.5], the automorphism \( \mathfrak{j}(t) := \sigma(t^2)\rho(t^{-1}) \) stabilizes \( S_x \) as well as each \( S_x \cap \mathcal{O}_\mu \). We shall use the following notation \( \mathfrak{j}(t)(z) = t * z \) for brevity. Clearly, \( \lim_{t \to 0} t * z = x \) for any \( z \in S_x \). This property and the decomposition of the tangent space \( T_x(\mathfrak{g}) = T_x(\mathcal{O}_\lambda) \oplus T_x(S_x) \) show that \( S_x \cap \mathcal{O}_\lambda = \{ x \} \) and \( S_x \cap \mathcal{O}_\mu = \emptyset \) if \( \mu \neq \lambda \).

Consider the Cartan subalgebra \( \mathfrak{h} := \{ (a_1, \ldots, a_n) \in \mathbb{C}^n \mid a_1 + \cdots + a_n = 0 \} \) of \( \mathfrak{g} \) and its quotient \( \mathfrak{h}/S_n = \mathbb{C}^{n-1} \). The natural projection \( \phi : \mathfrak{h} \to \mathfrak{h}/S_n \) takes an \( n \)-tuple to the sequence of elementary symmetric polynomials \( (\sigma_2(a_1, \ldots, a_n), \sigma_3(a_1, \ldots, a_n), \ldots, \sigma_n(a_1, \ldots, a_n)) \).

In [11, 3.4], the following simultaneous resolution was considered:

\[
\begin{array}{ccc}
\tilde{S}_x & \xrightarrow{\psi_x} & S_x \\
\downarrow \theta_x & & \downarrow \chi_x \\
\mathfrak{h} & \xrightarrow{\phi} & \mathfrak{h}/S_n
\end{array}
\]

Here \( \tilde{S}_x = \{ (z, b) \in S_x \times \mathcal{B} \mid z \in \mathfrak{b} \} \), \( \psi_x \) erases the second component, \( \theta_x((z, b)) \) is the sequence of eigenvalues of \( z \) in the order defined by \( \mathfrak{b} \) and \( \chi_x(z) \) is the sequence of coefficients of the characteristic polynomial of \( z \) without the (zero) trace. We shall use the following notations for preimages: \( \tilde{S}_{x, h} = \theta_x^{-1}(h) \) and \( \tilde{S}_{x, \tilde{h}} = \chi_x^{-1}(\tilde{h}) \) for \( h \in \mathfrak{h} \) and \( \tilde{h} \in \mathfrak{h}/S_n \). Thus \( S_{x,0} = S_x \cap \mathcal{N} \), \( \tilde{S}_{x,0} = \pi^{-1}(S_x \cap \mathcal{N}) \). In the special case \( x = 0 \), we have \( S_x = \mathfrak{g} \) and \( \psi_x \) restricts to the Springer resolution \( \pi \).

One can easily extend the \(*)\)-action of \( \mathbb{C}^* \) from \( S_x \) to \( \tilde{S}_x \) by \( t * (z, b) = (t * z, \mathfrak{j}(t)(b)) \). Thus the map \( \psi_x : \tilde{S}_x \to S_x \) becomes \(*\)-equivariant. In what follows, we shall not use the action of the whole group \( \mathbb{C}^* \) but only of its subgroup \((0, +\infty)\).

2.4. Neighbourhood bases. We describe here a special neighbourhood basis of \( x \) in \( S_x \cap \mathcal{N} \). First, recall the following formalism from [11, §2.5]:

\[
t * \left( x + \sum_{i=1}^{r'} c_i e_i \right) = x + \sum_{i=1}^{r'} t^{n_i+1} c_i e_i,
\]

where \( e_1, \ldots, e_{r'} \) is some basis of \( \ker \text{ad} \, y \) defined only by \( x \) and \( y \), \( n_i \) are nonnegative integers, \( c_i \in \mathbb{C} \) and \( t \in (0, +\infty) \). Consider the following open neighbourhood of \( x \) in \( S_x \):

\[
U := \left\{ x + \sum_{i=1}^{r'} c_i e_i \mid |c_i| < 1 \right\}.
\]
Clearly, $\overline{t \ast U} \subset U$ for $t < 1$. Moreover, the closure $\overline{U}$ is compact and $\{t \ast U \mid t \in (0, +\infty)\}$ is a neighbourhood basis for $x$ in $S_x$. The intersection $V := U \cap N$ satisfies the following properties:

1. The closure $\overline{V}$ is compact.
2. $\{t \ast V \mid t \in (0, +\infty)\}$ is a neighbourhood basis for $x$ in $S_x \cap N$.
3. $\overline{t_0 \ast V} \subset t_1 \ast V$ for $t_0 < t_1$.

Lemma 1. For $t_0 < t_1$, the natural inclusion and the $*$-action by $t_1/t_0$ are homotopic maps from $\pi^{-1}(t_0 \ast V \setminus \{x\})$ to $\pi^{-1}(t_1 \ast V \setminus \{x\})$. Each such space is homeomorphic to $\tilde{S}_{x,0} \setminus \mathcal{B}_x$.

Proof. The required homotopy $F : \pi^{-1}(t_0 \ast V \setminus \{x\}) \times [t_0, t_1] \to \pi^{-1}(t_1 \ast V \setminus \{x\})$ is given by $F(q, s) := (s/t_0) \ast q$. To prove the second statement, we define the radius of a point $z \in S_x \cap N$ by

$$r(z) := \inf \{t \mid t \in (0, +\infty), z \in t \ast V\}.$$ 

From this definition, it is clear that $r(t \ast z) = r(z)$ for any $t \in (0, +\infty)$. Property (3) above shows that $r$ is continuous and that $r(z) < 1$ implies $z \in V$. By property (2), we get that $r(z) = 0$ if and only if $z = x$.

Now let $g : (0, 1) \to (0, +\infty)$ be the function defined by $g(a) := a/(1 - a)$. We define the homeomorphism $\omega : \pi^{-1}(V \setminus \{x\}) \to \tilde{S}_x \setminus \mathcal{B}_x$ by $u(q) = (g \circ r \circ \pi(q)) \ast q$. To calculate its inverse map, consider the function $l : (0, +\infty) \to (0, 1)$ given by $l(a) := (-a + \sqrt{a^2 + 4a})/2$. Then the inverse map is given by $u^{-1}(p) = (g \circ l \circ r \circ \pi(p))^{-1} \ast p \quad \Box$

Now we construct a neighbourhood basis for $x$ in $N$. By [2, Lemma 3.2.20], any set $(t \ast U) \cap N = t \ast V$ is a transverse slice (see [2, Definition 3.2.19]) to $\mathcal{O}_\lambda$ in $N$ for small enough $t$. There exists an open in $N$ neighbourhood $W_x(t)$ of $x$ and a homeomorphism $(\mathcal{O}_\lambda \cap W_x(t)) \times (t \ast V) \tilde{\rightarrow} W_x(t)$ that restricts to projections $p_2 : \{x\} \times (t \ast V) \tilde{\rightarrow} (t \ast V)$ and $p_1 : (\mathcal{O}_\lambda \cap W_x(t)) \times \{x\} \tilde{\rightarrow} \mathcal{O}_\lambda \cap W_x(t)$. We can clearly suppose that $W_x(t_0) \subset W_x(t_1)$ for $t_0 < t_1$ and that the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{O}_\lambda \cap W_x(t_1)) \times (t_1 \ast V) & \tilde{\rightarrow} & W_x(t_1) \\
\uparrow & & \uparrow \\
(\mathcal{O}_\lambda \cap W_x(t_0)) \times (t_0 \ast V) & \tilde{\rightarrow} & W_x(t_0)
\end{array}$$

Using this description and the fact that $\mathcal{O}_\lambda$ is smooth, we can choose $W_x(t)$ so that each subspace $\mathcal{O}_\lambda \cap W_x(t)$ is homeomorphic to the open ball $B_t$ with center 0 and radius $t$ in $\mathbb{C}^{\dim \mathcal{O}_\lambda}$ in a way respecting inclusion: for $t_0 < t_1$ the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{O}_\lambda \cap W_x(t_1) & \tilde{\rightarrow} & B_{t_1} \\
\uparrow & & \uparrow \\
\mathcal{O}_\lambda \cap W_x(t_0) & \tilde{\rightarrow} & B_{t_0}
\end{array}$$

Moreover, we can suppose that $\mathcal{O}_\lambda \cap W_x(t)$ form a neighbourhood basis for $x$ in $\mathcal{O}_\lambda$. Finally, applying [2, Corollary 3.2.21], we get $\pi^{-1}(W_x(t)) \cong (\mathcal{O}_\lambda \cap W_x(t)) \times \pi^{-1}(t \ast V)$. Subtracting $\pi^{-1}(\mathcal{O}_\lambda)$, we get the following result.

Lemma 2. For small enough $t$, there exists a homeomorphism

$$\pi^{-1}(W_x(t) \setminus \mathcal{O}_\lambda) \cong (\mathcal{O}_\lambda \cap W_x(t)) \times \pi^{-1}(t \ast V \setminus \{x\}) \cong \mathbb{C}^{\dim \mathcal{O}_\lambda} \times (\tilde{S}_{x,0} \setminus \mathcal{B}_x).$$
For $t_0 < t_1$, the natural inclusion $\pi^{-1}(W_x(t_0) \setminus O_\lambda) \subset \pi^{-1}(W_x(t_1) \setminus O_\lambda)$ is homotopic to a homeomorphism. The sets $W_x(t)$ form a neighbourhood basis for $x$ in $\mathcal{N}$.

\textbf{Proof.} The result follows from our choice of $W_x(t)$ and Lemma 1. \hfill \qed

If $x \in O_\lambda$ and we need to know $\mathcal{B}_x$, $S_x$, $\tilde{S}_{x,0}$, etc. only up to a homeomorphism, then we simply write $\mathcal{B}_x$, $S_x$, $\tilde{S}_{\lambda,0}$, etc. respectively.

\section*{2.5. Intersection form and cohomology of difference.} We shall use the following interpretation of the intersection form for Springer fibers. Recall from [11, 4.3] that $\mathcal{B}_\lambda$ is a deformation retract of $\tilde{S}_{\lambda,0}$. Therefore the map $H^{2b_\lambda}(\tilde{S}_{\lambda,0}, k) \to H^{2b_\lambda}(\mathcal{B}_\lambda, k)$ induced by the natural inclusion $i: \mathcal{B}_\lambda \to \tilde{S}_{\lambda,0}$ is an isomorphism. Hence we get the following sequence of $k$-linear maps:

\[ H^{2b_\lambda}(\mathcal{B}_\lambda, k)^\vee \xrightarrow{\sim} H^{2b_\lambda}(\tilde{S}_{\lambda,0}, k)^\vee \xrightarrow{\sim} H^{2b_\lambda}(\tilde{S}_{\lambda,0}, k) \xrightarrow{\text{Poincaré duality}} H^{2b_\lambda}(\mathcal{B}_\lambda, k). \tag{1} \]

The composition defines a bilinear product $(\cdot, \cdot)_\lambda : H^{2b_\lambda}(\mathcal{B}_\lambda, k)^\vee \times H^{2b_\lambda}(\mathcal{B}_\lambda, k)^\vee \to k$. Let $a_\lambda := d_\lambda - \text{rank}(\cdot, \cdot)_\lambda$ be the corank of this form.

The triviality of the odd degree cohomologies of Springer fibres allows us to reformulate $a_\lambda$ as the dimension of $\text{Hom}^{2b_\lambda}_c(\tilde{S}_{\lambda,0} \setminus \mathcal{B}_\lambda, k)$. Indeed, we get from this triviality and [8, III.7.6] the following exact sequence:

\[ 0 \to H^{2b_\lambda}_c(\tilde{S}_{\lambda,0} \setminus \mathcal{B}_\lambda, k) \to H^{2b_\lambda}_c(\tilde{S}_{\lambda,0}, k) \to \text{H}^{2b_\lambda}(\mathcal{B}_\lambda, k) \to H^{2b_\lambda+1}_c(\tilde{S}_{\lambda,0} \setminus \mathcal{B}_\lambda, k) \to 0, \]

whence

\[ a_\lambda = \dim_k \text{Hom}^{2b_\lambda}_c(\tilde{S}_{\lambda,0} \setminus \mathcal{B}_\lambda, k) = \dim_k \text{Hom}^{2b_\lambda+1}_c(\tilde{S}_{\lambda,0} \setminus \mathcal{B}_\lambda, k) \tag{2} \]

Finally, note that the product $(\cdot, \cdot)_\lambda$ is $S_n$-invariant. To explain it, we can define the action of $S_n$ on all spaces of (1). The most suitable construction for our case is given in [11, Foth Lecture]. We briefly recall this construction.

Let $\mathfrak{h}'$ be the subset of $\mathfrak{h}$ consisting of pairwise distinct tuples. We set $\mathfrak{h}'/S_n := \phi(\mathfrak{h}')$. So $\mathfrak{h}'/S_n$ is an open subvariety of $\mathfrak{h}'$ with compliment denoted by $Q$ (discriminant). Take any $x \in O_\lambda$. To simplify notation, we set

\[ E := S_x \setminus \chi_\tau^{-1}(Q), \quad X := \mathfrak{h}'/S_n, \quad F := \tilde{S}_{\lambda,0}, \quad \tau := \chi_\tau |_{S_x \setminus \chi_\tau^{-1}(Q)}. \]

Then we have the fibre bundle $\tau : E \to X$ with fiber $F$. A remarkable property of this bundle is that its pull-back by $\phi|_{\mathfrak{h}'} : \mathfrak{h}' \to X$ is trivial. The sheaves $R^i\tau^*k_E$ and $R^i\tau_*(k_E)$ are locally constant and we can define the monodromy actions of $\Pi_1(X)$ on $H^i_c(F, k)$ and $H^i(F, k)$ respectively. As $\Pi_1(\mathfrak{h}')$ acts trivially in view of the triviality of the pull-back, we get an action of $S_n \cong \Pi_1(X)/\Pi_1(\mathfrak{h}')$ on each cohomology space.

There are two ways to relate these actions. First, consider the natural morphism of functors $\alpha : \tau_! \to \tau_*$. Hence we obtain the morphism of local systems $(R^i\alpha)(k_E) : R^i\tau^*k_E \to R^i\tau_*k_E$. As the monodromy action is functorial, the resulting morphism of stalks $H^i_c(F, k) \to H^i(F, k)$ is a morphism of $\Pi_1(X)$-modules. This morphism is actually equal to $(R^i\beta)(k)$, where $\beta : \Gamma_c(F, \cdot) \to \Gamma_c(F, \cdot)$ is the natural morphism.

We already noticed above that the natural inclusion $i : \mathcal{B}_\lambda \to F$ induces an isomorphism $H^i(F, k) \cong H^i(\mathcal{B}_\lambda, k)$. We shall use it to define the action of $\Pi_1(X)$ on $H^i(\mathcal{B}_\lambda, k)$. We
get the following commutative diagram:

\[ \begin{array}{ccc}
H^i(F, k) & \xrightarrow{\text{induced by } \iota} & H^i_c(\mathcal{B}_\lambda, k) \\
(R^i\beta)(k) \downarrow & & \downarrow \\
H^i(F, k) & \xrightarrow{\sim} & H^i(\mathcal{B}_\lambda, k)
\end{array} \]

The bottom morphism is \( \Pi_1(X) \)-equivariant by definition and \( (R^i\beta)(k) \) is \( \Pi_1(X) \)-equivariant by our arguments above. Therefore the upper morphism \( H^i_c(F, k) \to H^i_c(\mathcal{B}_\lambda, k) = H^i(\mathcal{B}_\lambda, k) \) is also \( \Pi_1(X) \)-equivariant.

On the other hand, it is known that the pairing \( (R^i\tau_{kE})_x \times (R^{4b-i}\tau_{kE})_x \to k \) coming from the Poincaré duality for the smooth manifold \( F \) is locally constant. Hence it is constant along each curve \( \gamma : [0, 1] \to X \). Therefore, the Poincaré duality pairing \( H^i_c(F, k) \times H^{4b-i}(F, k) \to k \) is \( \Pi_1(X) \)-equivariant. This means that \( H^i_c(F, k) \cong H^{4b-i}(F, k)^\vee \cong H^{4b-i}(\mathcal{B}_\lambda, k)^\vee \) as \( \Pi_1(X) \)-modules. It remains to factor all these actions over \( \Pi_1(h') \).

2.6. Category of perverse sheaves. For a topological space \( X \), we denote by \( D^b(X, k) \) the bounded derived category of \( k \)-sheaves. For an object \( C \) of \( D^b(X, k) \), we denote by \( H^m(C) \) its \( m \)th cohomology sheaf. For a stratification \( S \) of \( X \) with locally closed equidimensional strata, we denote by \( D^b_S(X, k) \) the full subcategory of \( D^b(X, k) \) whose objects \( C \) are such that each cohomology sheaf \( H^m(C) \) is \( S \)-constructible (that is, the restriction \( \iota_S^*H^m(C) \) to every stratum \( S \in \mathcal{S} \) is locally constant).

We consider only the middle perversity \( p : S \to \mathbb{Z} \), which is defined by \( p(S) = -\dim S \), and denote by \( p_{T \geq 0} \) and \( p_{T < 0} \) the truncation functors defined by the \( t \)-structure relative to \( p \). We denote by \( p^!H^0 : D^b_S(X, k) \to D^b(X, k) \) the cohomological functor \( p_{T \geq 0}^*p_{T < 0} \). In the case of trivial stratification \( S = \{X\} \), we have the following relation:

\[ p^!H^0C = (H^{-\dim X}C)[\dim X]. \]

Considering the shifted functor \( p^!H^nC := p^!H^n(C[a]) \), we get in this case

\[ p^!H^aC = (H^{-\dim X}C[a])[\dim X] = (H^{-\dim X+a}C)[\dim X]. \] (3)

The core of the \( t \)-structure for \( D^b_S(X, k) \) relative to perversity \( p \) is the category of perverse sheaves \( P_S(X, k) \). Following [1], we set \( p^!T = p^!H \circ T \circ \varepsilon' \) for any exact functor between triangulated categories \( T : D^b_S(X', k) \to D^b_S(X, k) \), where \( \varepsilon' : P_S(X', k) \to D^b_S(X', k) \) is the natural inclusion. We get in this way, an additive functor from \( P_S(X', k) \) to \( P_S(X, k) \).

In what follows, we consider only the stratification \( X = \{\mathcal{O}_\lambda\}_{\lambda \in \Lambda} \) of \( \mathcal{N} \) and stratifications of unions of strata \( Z = \bigcup_{\lambda \in \Lambda'} \mathcal{O}_\lambda \) with stratifications \( X|_Z = \{\mathcal{O}_\lambda\}_{\lambda \in \Lambda'} \) induced by \( X \). In this case, use the notation \( P_X(Z, k) := P_{X|Z}(Z, k) \) for brevity.

The simple objects of the category \( P_Z(X, k) \) are the intersection cohomology complexes
\[ IC_Z(\mathcal{O}_\lambda \cap Z, \mathcal{L}) := i_*\mathcal{L}[^{\dim \mathcal{O}_\lambda}] \]
where \( \mathcal{L} \) is an irreducible local system on \( \mathcal{O}_\lambda \subset Z \) and \( i : \mathcal{O}_\lambda \to Z \) is the natural inclusion. In the next section, we are going to study the composition multiplicities of \( IC_Z(\mathcal{O}_\lambda \cap Z, \mathcal{L}) \) in objects \( C \) of \( P_X(Z, k) \), which we denote by \( [C : IC_Z(\mathcal{O}_\lambda \cap Z, \mathcal{L})] \).

In the proof of Theorem 1, we shall also need the dualizing functor \( \mathbb{D} := R\text{Hom}(\_ \_ \mathcal{D}) \), where \( \mathcal{D} \) is dualizing complex for the corresponding space.

3. Composition multiplicities and semisimplicity

Recall the coranks \( a_\lambda \) defined in Section 2.5. We actually need them defined in form (2). The main result of this paper is as follows.
Theorem 1. For any \( \lambda \in \Lambda \), we have
\[
\left[ \pi_* \kappa_{\widetilde{N}}[d_N] : \text{IC}_{\Lambda} (\overline{O}_{\lambda}, k) \right] \geq d_{\lambda} + a_{\lambda}.
\]
Moreover, \( \pi_* \kappa_{\widetilde{N}}[d_N] \) is semisimple in \( P_{X}(N, k) \) if and only if \( a_{\lambda} = 0 \) for any \( \lambda \in \Lambda \).

Proof. We denote for brevity \( C := \pi_* \kappa_{\widetilde{N}}[d_N] \). Take any nonempty open subset \( A \subset \Lambda \). As we have the cartesian diagram
\[
\begin{array}{ccc}
\widetilde{N}_{A} & \xrightarrow{i_A} & \widetilde{N} \\
\pi_{A} \downarrow & & \downarrow \pi \\
N_{A} & \xrightarrow{i_A} & N
\end{array}
\]
and \( \pi \) and \( \pi_A \) are proper, the base change gives
\[
i_A^* C = i_A^* \pi_* \kappa_{\widetilde{N}}[d_N] = \pi_A^* i_A^* \kappa_{\widetilde{N}}[d_N] = \pi_A^* \kappa_{\widetilde{N}_A}[d_N].
\]
(4)

We denote this complex by \( C_A \) and prove inductively on \( A \) with respect to the inclusion relation the following statement:

Inductive claim. For any \( \lambda \in A \), we have
\[
\left[ C_A : \text{IC}_{\Lambda}(\overline{O}_{\lambda} \cap N_A, k) \right] \geq d_{\lambda} + a_{\lambda}.
\]
(5)

Moreover, \( C_A \) is semisimple in \( P_{X}(N_A, k) \) if and only if \( a_{\lambda} = 0 \) for any \( \lambda \in A \).

The case \( A = \{ (n) \} \) is obvious, as \( \pi_{(n)} \) is a homeomorphism. Hence \( C_A = \kappa_{\overline{O}(n)}[d_N] = \text{IC}_{\overline{O}(n)}(O(n), k) \) is a simple module. It remains to notice that \( d_{\lambda} = 1 \) and \( a(n) = 0 \) as a cohomology of the empty space.

Now suppose that \( A \neq \{ (n) \} \). Choose a minimal partition \( \lambda \in A \) and consider the open set \( B := A \setminus \{ \lambda \} \). Let \( i : O_{\lambda} \to N_A \) and \( j : N_B \to N_A \) be the natural closed and open embeddings, respectively.

Step 1. We have the following exact sequence (see [1, Lemme 1.4.19]):
\[
0 \to p_{i_*} p_{H^{-1} i^*} C_A \to p_{j_*} p_{j^*} C_A \to C_A \to p_{i_*} p_{i^*} C_A \to 0
\]
(6)

Consider the following cartesian diagram:
\[
\begin{array}{ccc}
\widetilde{N}_{(\lambda)} & \xrightarrow{i_{(\lambda)}} & \widetilde{N} \\
\pi_{(\lambda)} \downarrow & & \downarrow \pi \\
O_{\lambda} & \xrightarrow{i_{(\lambda)}} & N
\end{array}
\]
Now again, the base change gives
\[
i^* C_A = (i_{A!})^* C = i_{(\lambda)^*} \pi_* \kappa_{\widetilde{N}}[d_N] = \pi_{(\lambda)^*} i_{(\lambda)^*} \kappa_{\widetilde{N}}[d_N] = \pi_{(\lambda)^*} \kappa_{\widetilde{N}_{(\lambda)}}[d_N].
\]
This formula and (3) imply
\[
p_{i^*} C_A = p_{H^0(\pi_{(\lambda)^*} \kappa_{\widetilde{N}_{(\lambda)}}[d_N])} \left( H^{d_{\dim \overline{O}_{\lambda}} - \dim \overline{O}_{\lambda}} \kappa_{\widetilde{N}_{(\lambda)}}[d_N] \right) \left( R^{2\dim \overline{O}_{\lambda}} \kappa_{\widetilde{N}_{(\lambda)}}[d_N] \right) \left[ \dim \overline{O}_{\lambda} \right] = \left( R^{2\dim \overline{O}_{\lambda}} \kappa_{\widetilde{N}_{(\lambda)}}[d_N] \right) \left[ \dim \overline{O}_{\lambda} \right].
\]

We denote \( L := R^{2\dim \overline{O}_{\lambda}} \kappa_{\widetilde{N}_{(\lambda)}}[d_N] \). This is a local system on \( O_{\lambda} \). As \( \pi \) is proper, we get
\[
L_c = H^{d_{\lambda}}(\mathcal{R}_c, k) = k^{d_{\lambda}}.
\]
(7)
A well-known argument shows that $L$ is a constant sheaf. We are going to give a brief outline of this proof. Let $\delta : [0, 1] \to G$ be a curve. We choose an arbitrary element $\xi \in \mathcal{O}_\lambda$ and define the map $\beta : G \to \mathcal{O}_\lambda$ by $\beta(g) = g\xi g^{-1}$. The composition $\gamma := \beta\delta$ is a curve in $\mathcal{O}_\lambda$. Let $x := \gamma(0)$ and $y := \gamma(1)$ be its initial and end points respectively.

We set $w := \delta(1)\delta(0)^{-1}$ and denote by $m_w : \tilde{\mathcal{N}} \to \mathcal{N}$ the action by $w$. As $\pi$ is $G$-equivariant, $m_w$ takes homeomorphically $\mathcal{B}_x$ to $\mathcal{B}_y$. A routine check involving the fact that homotopic maps induce the same maps on cohomology spaces\(^2\) yields the commutativity of the following diagram:

\[
\begin{array}{ccc}
H^{2b_h}(\mathcal{B}_x, k) & \xleftarrow{m_w^*} & H^{2b_h}(\mathcal{B}_y, k) \\
\downarrow \sim & & \downarrow \sim \\
(\gamma^*\mathcal{L})_0 & \longrightarrow & \Gamma([0, 1], \gamma^*\mathcal{L}) & \longrightarrow & (\gamma^*\mathcal{L})_1 & \sim & \longrightarrow & \mathcal{L}_y
\end{array}
\]

In the particular case $\delta(0) = \delta(1) = 1_G$, the curve $\gamma$ is a loop based at $\xi$, the bottom line of the above diagram is the monodromy along $\gamma$ and the upper line is identical. Hence $\gamma$ acts identically on $\mathcal{L}_\xi$.

As we have the exact sequence

\[
\Pi_1(G) \xrightarrow{\beta_0} \Pi_1(\mathcal{O}_\lambda) \longrightarrow C_G(\xi)/C_G^0(\xi)
\]

and the rightmost group is trivial in the case under consideration $G = \text{SL}_n(\mathbb{C})$, any loop $\gamma$ in $\mathcal{O}_\lambda$ based at $\xi$ can be lifted to a loop $\delta$ in $G$ based at $1_G$. Therefore $\Pi_1(\mathcal{O}_\lambda)$ acts trivially on $\mathcal{L}_\xi$ and $\mathcal{L}$ is a constant sheaf. This fact and (7) imply

\[
\mathcal{L} = H^{2b_h}(\mathcal{B}_x, k)_{\mathcal{O}_\lambda} = (k_{\mathcal{O}_\lambda})^{\otimes d_\lambda}.
\]

Now let us look at the left-hand side of (6). Similarly to the above calculations for $p^j i^* C_A$, we get

\[
p^{H^{-1}} i^* C_A = p^{H^{-1}} \pi_{(\lambda)^1} x_{\mathcal{N}_{(\lambda)}} [d_{\mathcal{N}}] = \left( R^{2d_\lambda-1} \pi_{(\lambda)} x_{\mathcal{N}_{(\lambda)}} \right) [\dim \mathcal{O}_\lambda].
\]

As $\pi_{(\lambda)}$ is proper, we have

\[
\left( R^{2d_\lambda-1} \pi_{(\lambda)} x_{\mathcal{N}_{(\lambda)}} \right) = H^{2d_\lambda-1}(\mathcal{B}_x, k) = 0
\]

for any $x \in \mathcal{O}_\lambda$. Thus we have proved that $p^{H^{-1}} i^* C_A = 0$.

Note here that $j^* C_A = p^j i^* C_A$, as $j^*$ is t-exact, and $p^j i^* C_A = (i_{A_j})^* C = i_B^* C = C_B$. So we can rewrite (6) as the following exact sequence:

\[
0 \longrightarrow p^j i_B C_B \longrightarrow C_A \longrightarrow \text{IC}_{\mathcal{N}_A}(\mathcal{O}_\lambda, k)^{\otimes d_\lambda} \longrightarrow 0. \tag{9}
\]

**Step 2.** We have the following exact sequences:

\[
\begin{array}{c}
0 \longrightarrow i^*_x p^{H^{-1}} i^*_s j_s C_B \longrightarrow p^j i_C C_B \longrightarrow j_s C_B \longrightarrow i^*_x p^i s^*_j C_B \longrightarrow 0 \\
0 \longrightarrow j^* C_B \longrightarrow 0
\end{array}
\tag{10}
\]

Our next aim is to calculate $p^{H^{-\varepsilon}} i^* j_s C_B$ for $\varepsilon \in \{0, 1\}$. Let $f$ denote the composition

\[
\tilde{\mathcal{N}}_B \xrightarrow{\pi_B} \mathcal{N}_B \xrightarrow{j} \mathcal{N}_A
\]

\(^2\)See Section 4 for a similar argument.
We have $C_B = \pi_{B*} k_{N_B} [d_N]$ as defined by (4), where $A$ is replaced by $B$. Thus applying (3), we get

$$\mathcal{H}^{\vee}_\ast i_{\ast} C_B = \mathcal{H}^{\vee}_\ast (i_{\ast} (j_{\ast} B_{\ast} k_{N_B} [d_N])) = \left( H^{\vee}_{-\dim \Omega} (i_{\ast} f_{\ast} k_{N_B} [d_N]) \right) [\dim \Omega]$$

$$= \left( i_{\ast} H^{d_{N',-\dim \Omega,\ast} f_{\ast} k_{N_B}} \right) [\dim \Omega] = \left( i_{\ast} R^{2\lambda_{\ast}} f_{\ast} k_{N_B} \right) [\dim \Omega].$$

By proposition [8, Proposition II.5.11], the sheaf $\mathcal{M} := R^{2\lambda_{\ast}} f_{\ast} k_{N_B}$ in the right-hand side is the sheaf associated to the presheaf $\mathcal{M}_{pre}$ defined by

$$U \mapsto H^{2\lambda_{\ast}} (f^{-1}(U), k)$$

for open $U \subset N_A$.

Take a point $x \in \Omega$ and recall the neighbourhood basis $\{W_x(t)\}_{t \in (0, +\infty)}$ defined in Section 2.3. As $N_A$ is open, we have $W_x(t) \subset N_A$ for small enough $t$. We clearly have $f^{-1}(W_x(t)) = \pi^{-1}(W_x(t) \setminus \Omega)$. The restriction of the presheaf $\mathcal{M}_{pre}$ has the form

$$i_{\ast} : H^{2\lambda_{\ast}} (\pi^{-1}(W_x(t_1) \setminus \Omega), k) \rightarrow H^{2\lambda_{\ast}} (\pi^{-1}(W_x(t_0) \setminus \Omega), k),$$

where $t_0 < t_1$ and $i : W_x(t_0) \rightarrow W_x(t_1)$ is the natural inclusion. The second statement of Lemma 2 shows that $i_{\ast}$ is an isomorphism. Hence by the first statement of Lemma 2, Poincaré duality and the Künneth formula, we get

$$\mathcal{M}_x = \mathcal{M}_{pre} = H^{2\lambda_{\ast}} (\pi^{-1}(W_x(t) \setminus \Omega), k) = H^{2\lambda_{\ast}} (\mathbb{C}^{dim \Omega} \times (\tilde{\mathcal{S}}_{\lambda, 0} \setminus \mathcal{B}_\lambda), k)$$

$$= H^{2\lambda_{\ast}} (\mathbb{C}^{dim \Omega} \times (\tilde{\mathcal{S}}_{\lambda, 0} \setminus \mathcal{B}_\lambda), k).$$

In Section 4, we shall prove that $i_{\ast} \mathcal{M}$ is a constant sheaf. From this fact and (12), it follows that $i_{\ast} \mathcal{M} = (k_{\Omega})^{\boxtimes \lambda}$. Substituting this to (11), we get $p^H^{\vee} i_{\ast} C_B = (k_{\Omega})^{\boxtimes \lambda}$. Finally, applying $i_{\ast}$, we obtain

$$i_{\ast} p^H^{\vee} i_{\ast} C_B = \text{IC}_{N_A} (\Omega, k)^{\boxtimes \lambda}. \quad (13)$$

Coming back to diagram (10), we get

$$p^H_{j_{\ast}} C_B : \text{IC}_{N_A} (\Omega, k) \geq a_{\lambda}. \quad (14)$$

On the other hand, we already know by induction that $[C_B : \text{IC}_{N_B} (\overline{\Omega} \cap \mathcal{N}_B, k)] \geq d_{\nu} + a_{\nu}$ for any $\nu \in B$. Applying the functor $j_{\ast}$, which is known to preserve monomorphisms and epimorphisms, we get $[j_{\ast} C_B : \text{IC}_{N_A} (\overline{\Omega} \cap N_A, k)] \geq d_{\nu} + a_{\nu}$. Hence and from (10), we obtain that

$$p^H_{j_{\ast}} C_B : \text{IC}_{N_A} (\overline{\Omega} \cap N_A, k) \geq d_{\nu} + a_{\nu}$$

for any $\nu \in B$. Plugging these inequalities and (14) into sequence (9), we get the desired inequality (5) for any $\lambda \in A$.

Step 3. Suppose that $C$ is semisimple. Then $C_B$ is also semisimple. We know that $p^H_{j_{\ast}} C_B$ does not have quotient objects in $i_{\ast} P_{\ast} (\Omega, k)$. By semisimplicity $p^H_{j_{\ast}} C_B$ also does not have subobjects in $i_{\ast} P_{\ast} (\Omega, k)$. Thus $p^H_{j_{\ast}} C_B = j_{\ast} C_B$ and by diagram (10), we get $i_{\ast} p^H^{\vee} i_{\ast} C_B = 0$. Hence $a_{\lambda} = 0$ by (13). By induction, we have $a_{\nu} = 0$ for any $\nu \in B$.

Step 4. First, note that from (4) it follows that $\mathfrak{D} C_A = C_A$.

Now suppose that $a_{\nu} = 0$ for any $\nu \in A$. By and (13) and (10), we have $p^H_{j_{\ast}} C_B = j_{\ast} C_B$. So sequence (9) can be rewritten as

$$0 \rightarrow j_{\ast} C_B \rightarrow C_A \rightarrow \text{IC}_{N_A} (\Omega, k)^{\boxtimes d_{\lambda}} \rightarrow 0. \quad (15)$$
By induction, \( C_B \) is semisimple. Hence so is \( j_\ast C_B \). It remains to show that the above sequence splits. We have the dual morphism \( \mathbb{D} \beta : \mathbf{IC}_{\mathcal{N}_A}(\mathcal{O}_\lambda, k)^{\text{sh}} \to C_A \). As \( \beta \) is an epimorphism, \( \mathbb{D} \beta \) is a monomorphism. The kernel of the composition \( \beta \circ \mathbb{D} \beta \) is zero as otherwise \( \ker \beta \) would have a subobject isomorphism to \( \mathbf{IC}_{\mathcal{N}_A}(\mathcal{O}_\lambda, k) \). This is however impossible as \( \ker \beta = \text{im } \alpha \cong j_\ast C_B \) and the latter complex cannot have such subobjects by definition of \( j_\ast \). By duality, the cokernel of \( \beta \circ \mathbb{D} \beta \) is also zero, whence \( \beta \circ \mathbb{D} \beta \) is an isomorphism. \( \square \)

4. Constancy of \( i^\ast \mathcal{M} \)

We choose an arbitrary element \( \xi \in \mathcal{O}_\lambda \) and define the map \( \beta : G \to \mathcal{O}_\lambda \) by \( \beta(g) = g\xi g^{-1} \). Consider a curve \( \delta : [0, 1] \to G \). We set \( u := \delta(0) \), \( v := \delta(1) \) and \( w := vu^{-1} \). The composition \( \gamma := \beta \delta \) is a curve in \( \mathcal{O}_\lambda \), whence also in \( \mathcal{N}_A \). Consider the points \( x := \gamma(0) \), \( y := \gamma(1) \) and choose some fixed Slodowy slice \( S_x \) for \( x \) (this means that we choose a fixed \( y \) in the Jacobson-Morozov lemma). Further, choose a neighbourhood basis \( \{W_x(t)\}_{t \in (0, +\infty)} \) for \( x \) as in Section 2.4.

We can translate \( S_x \) and \( W_x(t) \) along \( \gamma \) by

\[
S_{\gamma(a)} = \delta(a)u^{-1}S_xu\delta(a)^{-1} \quad \text{and} \quad W_{\gamma(a)}(t) = \delta(a)u^{-1}W_x(t)u\delta(a)^{-1}.
\]

(16)

Clearly, \( W_{\gamma(a)}(t) \) thus defined satisfy Lemma 2, where \( x \) is replaced by \( \gamma(a) \).

We can consider two pull-back functors \( \gamma^* \) and \( \gamma_*^{\text{pre}} \) in the categories of sheaves and presheaves respectively. Consider the following morphisms of sheafification (see [12]):

\[
\text{sh} : \mathcal{M}^{\text{pre}} \to \mathcal{M} \quad \text{sh} : \gamma_*^{\text{pre}} \mathcal{M} \to \gamma^* \mathcal{M}.
\]

Then the composition \( \text{sh} \circ \gamma_*^{\text{pre}} \text{sh} : \gamma_*^{\text{pre}} \mathcal{M} \to \gamma^* \mathcal{M} \) is also a morphism of sheafification.

Take a section \( s \in \gamma^* \mathcal{M}([0, 1]) \). There exists an open covering \( [0, 1] = \bigcup_{i \in J} \mathcal{V}_i \) such that \( s|_{\mathcal{V}_i} = \text{sh}(\mathcal{V}_i)(s_i) \) for some \( s_i \in \gamma_*^{\text{pre}} \mathcal{M}(\mathcal{V}_i) \). Now recall that

\[
\gamma_*^{\text{pre}} \mathcal{M}(\mathcal{V}_i) = \lim_{i \to \gamma(\mathcal{V}_i)} \mathcal{M}(\mathcal{U}_i).
\]

So we can assume that each \( s_i \) is represented by a section \( r_i \in \mathcal{M}(\mathcal{U}_i) \) for some open \( \mathcal{U}_i \subset \mathcal{N}_A \) containing \( \gamma(\mathcal{V}_i) \). Shrinking \( \mathcal{U}_i \) and refining the covering \( [0, 1] = \bigcup_{i \in J} \mathcal{V}_i \) if necessary, we can assume that \( r_i = \text{sh}(\mathcal{U}_i)(p_i) \) for some \( p_i \in \mathcal{M}^{\text{pre}}(\mathcal{U}_i) \) such that \( a_0 = 0 \in \mathcal{V}_1 \), \( a_{m+1} = 1 \in \mathcal{V}_m \), \( a_i \in \mathcal{V}_i \cap \mathcal{V}_{i-1} \) for \( i = 1, \ldots, m-1 \).

As \( \mathcal{N}_A \) is an open subspace of the locally compact space \( \mathcal{N} \), the closure \( \overline{W_x(t_0)} \) is compact and contained in \( \mathcal{N}_A \) for small enough \( t_0 \), which we fix. Hence the product \( \overline{W_x(t_0)} \times [a_i, a_{i+1}] \) is also compact. Consider the map \( \nu_i : \overline{W_x(t_0)} \times [a_i, a_{i+1}] \to \mathcal{N}_A \) defined by \( (z, a) \mapsto \delta(a)u^{-1}zu\delta(a)^{-1} \). It is obviously continuous.

We clearly have

\[
\bigcap_{q>1/t_0} \overline{W_x(1/q)} \times [a_i, a_{i+1}] = \{x\} \times [a_i, a_{i+1}],
\]

whence

\[
U_i \supset \gamma([a_i, a_{i+1}]) = \nu_i \left( \bigcap_{q>1/t_0} \overline{W_x(1/q)} \times [a_i, a_{i+1}] \right) = \bigcap_{q>1/t_0} \nu_i \left( \overline{W_x(1/q)} \times [a_i, a_{i+1}] \right).
\]

To see why the last equality holds, we use the following topological observation: if \( X \) is a countably compact topological space, \( X \supset Z_1 \supset Z_2 \supset \cdots \) is an infinite sequence of its closed subspace and \( f : X \to Y \) is a continuous map, then \( f(\bigcap_{q=1}^{+\infty} Z_q) = \bigcap_{q=1}^{+\infty} f(Z_q) \). The above inclusion and the compactness of each \( \overline{W_x(1/q)} \times [a_i, a_{i+1}] \) prove that \( \nu_i \left( \overline{W_x(1/q)} \times [a_i, a_{i+1}] \right) \subset
$U_i$ for some $q > 1/t_0$. Restricting this inclusion to $W_x(1/q) \times \{a\}$ and applying (16), we get $W_{\gamma(a)}(1/q) \subset U_i$ for any $a \in [a_i, a_{i+1}]$. Clearly, this $q$ can be chosen the same for all $i = 0, \ldots, m$.

For any $a \in [a_i, a_{i+1}]$, the composition of canonical morphisms

$$
\gamma^* \mathcal{M}([0, 1]) \to (\gamma^* \mathcal{M})_a \to \mathcal{M}_{\gamma(a)} \to \mathcal{M}_{\gamma(a)}^\pre \cong H^{2b_\lambda-\varepsilon}(\pi^{-1}(W_{\gamma(a)}(1/q) \setminus \mathcal{O}_\lambda), k),
$$

maps $s$ to the image of $p_i$ under the morphism

$$
iota_* : H^{2b_\lambda-\varepsilon}(f^{-1}(U_i), k) \to H^{2b_\lambda-\varepsilon}(\pi^{-1}(W_{\gamma(a)}(1/q) \setminus \mathcal{O}_\lambda), k),
$$

where $\iota : \pi^{-1}(W_{\gamma(a)}(1/q) \setminus \mathcal{O}_\lambda) \to f^{-1}(U_i)$ is the natural inclusion.

We claim that the following diagram is commutative

$$
\begin{array}{ccc}
H^{2b_\lambda-\varepsilon}(f^{-1}(U_i), k) & \xrightarrow{\iota_*} & H^{2b_\lambda-\varepsilon}(\pi^{-1}(W_{\gamma(a)}(1/q) \setminus \mathcal{O}_\lambda), k) \\
\downarrow m_{(a_i(a_{i+1}))^{-1}} & \cong & \downarrow m_{\gamma(a_{i+1})a_i} \\
\mathcal{M}_x & \cong & \Gamma([0, 1], \gamma^* \mathcal{M})_1 \\
\end{array}
$$

Together with exact sequence (8) it proves that $i^* \mathcal{M}$ is constant.

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