Deformed Minkowski spaces: classification and properties

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Abstract

Using general but simple covariance arguments, we classify the ‘quantum’ Minkowski spaces for dimensionless deformation parameters. This requires a previous analysis of the associated Lorentz groups, which reproduces a previous classification by Woronowicz and Zakrzewski. As a consequence of the unified analysis presented, we give the commutation properties, the deformed (and central) length element and the metric tensor for the different spacetime algebras.

1 Introduction

Following the approach of [1], we present here a classification of the possible deformed Minkowski spaces (algebras). Our analysis, which provides a common framework for the properties of the various Minkowski spacetimes, requires the consideration of the two ($SL_q(2)$ and $SL_h(2)$) deformations of $SL(2, \mathbb{C})$ and provides a characterization of the appropriate $R$-matrices defining the deformed Lorentz groups given in [2] (see also [3]).

It is well known that $GL(2, \mathbb{C})$ admits only two different deformations having a central determinant: one is the standard $q$-deformation [4, 5] and the other is the non-standard or ‘Jordanian’ $h$-deformation [6, 7, 8]. Both $GL_q(2)$ and $GL_h(2)$ have associated ‘quantum spaces’ in the sense of [9]. These deformations (which may be shown to be related by contraction [10]) are defined as the associative algebras generated by the entries $a, b, c, d$ of a matrix $M$, the commutation properties of which may be expressed by an ‘FRT’ [11] equation

$$R_{12}M_1M_2 = M_2M_1R_{12}$$

for a suitable $R$-matrix. Let us summarize their properties.
a) For $GL_q(2)$ the $R$-matrix in (3) is $(\lambda \equiv q - q^{-1})$

$$R_q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad \hat{R}_q \equiv \mathcal{P} R_q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad \mathcal{P} R_q \mathcal{P} = R_q^t,$$ (2)

where $\mathcal{P}$ is the permutation operator ($\mathcal{P} = \mathcal{P}^t$, $P_{ijkl} = \delta_{ij}\delta_{jk}$), and the commutation relations defining the quantum group algebra are

\begin{align*}
abla = qba, &\quad \lambda = qca, &\quad ad - da = \lambda bc, \\
bc = cb, &\quad bd = qdb, &\quad cd = qdc.
\end{align*} (3)

$Fun(GL_q(2))$ has a quadratic central element,

$$det_q M := ad - qbc;$$

$det_q M = 1$ defines $SL_q(2)$. The matrix $\hat{R}_q \equiv \mathcal{P} R_q$ satisfies Hecke’s condition

$$\hat{R}_q^2 - \lambda \hat{R}_q - I = 0, \quad (\hat{R}_q - qI)(\hat{R}_q + q^{-1}I) = 0,$$ (5)

and (we shall assume $q^2 \not= -1$ throughout) it has a spectral decomposition in terms of a rank three projector $P_{q+}$ and a rank one projector $P_{q-}$.

$$\hat{R}_q = q P_{q+} - q^{-1} P_{q-}, \quad \hat{R}_q^{-1} = q^{-1} P_{q+} - q P_{q-}, \quad [\hat{R}_q, P_{q\pm}] = 0, \quad P_{q\pm} \hat{R}_q P_{q\mp} = 0,$$ (6)

$$P_{q+} = \frac{I + q \hat{R}_q}{1 + q^2}, \quad P_{q-} = \frac{I - q^{-1} \hat{R}_q}{1 + q^{-2}} = \frac{1}{1 + q^{-2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-2} & -q^{-1} & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ (7)

The following relations have an obvious equivalent in the undeformed case:

$$\epsilon_q M_q \epsilon_q^{-1} = M^{-1}, \quad \epsilon_q = \begin{bmatrix} 0 & q^{-1/2} & 0 \\ -q^{1/2} & 0 & 0 \end{bmatrix} = -\epsilon_q^{-1}, \quad P_{q-ij,kl} = \frac{1}{[2]_q} \epsilon_q \epsilon_q \epsilon_q^{-1}.$$ (8)

The determinant of an ordinary $2 \times 2$ matrix may be defined as the proportionality coefficient in $(det M) P_- := P_- M_1 M_2$ where $P_-$ is given by (4) for $q=1$. In the $q \not= 1$ case the $q$-determinant (4) may be expressed as

$$(det_q M) P_{q-} := P_{q-M_1 M_2}, \quad (det_q M^{-1}) P_{q-} = M_{q^2}^{-1} M_1^{-1} P_{q-}, \quad (det_q M^{-1}) = (det_q M)^{-1}, \quad (det_q M) P_{q-} = M_{q^2}^{-1} M_1^{-1} P_{q-}.$$ (9)

b) For $GL_h(2)$ the $R$-matrix in (4) is the solution of the Yang-Baxter equation given by

$$R_h = \begin{bmatrix} 1 & -h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{R}_h \equiv \mathcal{P} R_h = \begin{bmatrix} 1 & -h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{P} R_h \mathcal{P} = R_h^{-1},$$ (10)
(or $R_{h}12R_{h}21 = I$, triangularity condition) for which (1) gives

$$
\begin{align*}
[a, b] &= h(\xi - a^2), & [a, c] &= hc^2, & [a, d] &= hc(d - a), \\
[b, c] &= h(ac + cd), & [b, d] &= h(d^2 - \xi), & [c, d] &= -hc^2
\end{align*}
$$

(so that $[a - d, c] = 0$ follows), where $\xi$ is the quadratic central element

$$
\xi \equiv \det hM = ad - cb - hcd;
$$

setting $\xi = 1$ reduces $GL_{h}(2)$ to $SL_{h}(2)$. The matrix $\hat{R}_{h}$ satisfies

$$
\hat{R}_{h}^2 = I, \quad (I - \hat{R}_{h})(I + \hat{R}_{h}) = 0.
$$

It has two eigenvalues ($1$ and $-1$) and a spectral decomposition in terms of a rank three projector $P_{h+}$ and a rank one projector $P_{h-}$

$$
\hat{R}_{h} = P_{h+} - P_{h-}, \quad P_{h\pm} \hat{R}_{h} = \pm P_{h\pm},
$$

$$
P_{h+} = \frac{1}{2}(I + \hat{R}_{h}), \quad P_{h-} = \frac{1}{2}(I - \hat{R}_{h}) = \frac{1}{2}
\begin{bmatrix}
0 & h & -h & -h^2 \\
0 & 1 & -1 & -h \\
0 & -1 & 1 & h \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

For $SL_{h}(2)$, the formulae equivalent to those in (8) are

$$
\epsilon_{h}M'\epsilon_{h}^{-1} = M^{-1}, \quad \epsilon_{h} = \begin{pmatrix} h & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{h}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix}, \quad P_{h-}^{-1} = \frac{-1}{2}\epsilon_{h}ij\epsilon_{h}^{-1}.
$$

Using $P_{h-}$, the deformed determinant and its inverse, $det_{h}M$ and $det_{h}M^{-1}$, (12) are also given by eqs. (13).

The quantum planes [9] associated with $SL_{q}(2)$ and $SL_{h}(2)$ are the associative algebras generated by two elements $(x, y) \equiv X$, the commutation properties of which (explicitly and in $R$-matrix form) are

a) for $SL_{q}(2)$ [4]

$$
xy = qyx \quad \longleftrightarrow \quad R_{q}X_{1}X_{2} = qX_{2}X_{1},
$$

b) for $SL_{h}(2)$ [4, 8]

$$
xy = yx + hy^2 \quad \longleftrightarrow \quad R_{h}X_{1}X_{2} = X_{2}X_{1}.
$$

These commutation relations are preserved under transformations by the corresponding quantum groups matrices $M$, $X' = MX$. This invariance statement, suitably extended to apply to the case of deformed Minkowski spaces, provides the essential ingredient for their classification.

From now on we shall often write $R_{Q}$, $P_{Q}$ ($Q = q, h$) to treat both deformations simultaneously. For instance, (17) and (18) may be jointly written as $R_{Q}X_{1}X_{2} = \rho X_{2}X_{1}$, where $\rho = (q, 1)$ is the appropriate eigenvalue of $R_{Q}$.

---

1 The $GL_{q}(2)$ and $GL_{h}(2)$ matrices also preserve, respectively, the ‘$q$-symplectic’ and ‘$h$-symplectic’ metrics $\epsilon_{q}$ (or $\epsilon_{q}^{-1}$) and $\epsilon_{h}^{-1}$.
2 Deformed Lorentz groups and associated Minkowski algebras

As is well known, the vector representation \( D^{\frac{1}{2}} = D^{\frac{1}{2} I} \otimes D^{0,\frac{1}{2}} \) of the restricted Lorentz group may be given by the transformation \( K' = AKA' \), \( A \in SL(2, C) \). The spacetime coordinates are contained in \( K = K^\dagger = \sigma^\mu x_\mu \), where \( \sigma^0 = I \) and \( \sigma^i \) are the Pauli matrices; the time coordinate may be identified as \( x^0 = \frac{1}{2} tr(K) \). Since \( detK = (x_0)^2 - x^i x_i = detK' \), the correspondence \( \pm A \to \Lambda \in SO(1, 3) \), where \( x'^\mu = \Lambda^\mu_\nu x^\nu \), realizes the covering homomorphism \( SL(2, C)/Z_2 = SO(1, 3) \). A first step to obtain a deformation of the Lorentz group is to replace the \( SL(2, C) \) group by the generator matrix \( M \) of \( SL_q(2) \).

In general, the full determination of a deformed Lorentz group requires the characterization of all possible commutation relations among the generators \((a, b, c, d)\) of \( M \) and \((a^*, b^*, c^*, d^*)\) of \( M^\dagger \), \( M \) being a deformation of \( SL(2, C) \). The \( R \)-matrix form of these may be expressed in full generality by

\[
\begin{align*}
R^{(1)} M_1 M_2 &= M_2 M_1 R^{(1)} , \\
M_1 R^{(3)} M_1 &= M_1 R^{(3)} M_1^\dagger , \\
M_2^\dagger R^{(2)} M_2 &= M_2 R^{(2)} M_1^\dagger , \\
R^{(4)} M_2^\dagger M_2 &= M_2^\dagger M_2 R^{(4)} ,
\end{align*}
\]

where \( R^{(3)} = R^{(2)} = \mathcal{P} R^{(3)} \mathcal{P} \) (or ‘reality’ condition\(^2\) for \( R^{(3)} \)) and \( R^{(4)} = R^{(1)} \) or \( R^{(4)} = (\mathcal{P} R^{(1)} \mathcal{P})^{\dagger} \) since the first eq. in \([19]\) is invariant under the exchange \( R^{(1)} \leftrightarrow \mathcal{P} R^{(1)} \mathcal{P} \).

Eqs. \([19]\), which also follow (see e.g. \([15]\)) from the bi-spinor (dotted and undotted) description of ‘quantum’ spacetime in terms of a deformed \( K \), will be taken as the starting point for the classification of the deformed Lorentz groups. In it, the matrix \( R^{(1)} \) characterizes the appropriate deformation of the \( SL(2, C) \) group \( (R^{(1)} = R_q) \), \( R^{(2)} \) (or \( R^{(3)} \)) defines how the elements of \( M \) and \( M^\dagger \) commute and it is not \( a \) priori fixed (but it must satisfy consistency relations with \( R^{(1)} \), see eq. \([20]\) below) and \( R^{(4)} \) gives the commutation relations for the complex conjugated generators contained in \( M^\dagger \). The specification of the deformed Lorentz group will be completed by the commutation properties of the generators with their complex conjugated ones \( i.e. \), by the determination of \( R^{(2)} = R^{(3)} \).

The commutation relations of the deformed Lorentz group algebra generators (entries of \( M \) and \( M^\dagger \)) are given by eqs. \([19]\). The consistency of these relations is assured if \( R^{(1)} \) (and \( R^{(4)} \)) obey the Yang-Baxter equation (YBE) and \( R^{(3)} \) and \( R^{(4)} \) satisfy the mixed consistency equations \([17, 18]\)

\[
R^{(1)}_{12} R^{(3)}_{13} R^{(3)}_{23} = R^{(3)}_{23} R^{(3)}_{13} R^{(1)}_{12} , \\
R^{(4)}_{12} R^{(2)}_{13} R^{(2)}_{23} = R^{(2)}_{23} R^{(2)}_{13} R^{(4)}_{12} ,
\]

(20)

\( (\)these two equations are actually the same since either \( R^{(4)} = R^{(1)} \) or \( R^{(4)} = (\mathcal{P} R^{(1)} \mathcal{P})^{\dagger} \) and \( R^{(2)} = R^{(3)} \)). It will be convenient to notice that the first

\(^2\)This reality condition can be given in a more general form \( R^{(3)} = \tau \mathcal{P} R^{(3)} \mathcal{P} \) for \(|\tau| = 1 \), however this phase factor can be eliminated by the redefinition \( R^{(3)} \to \tau^{1/2} R^{(3)} \) (cf. \([2]\)).
equation, considered as an ‘RTT’ equation, indicates that \( R^{(3)} \) is a representation of the deformed \( GL(2,C) \) group, i.e., the matrix \( R^{(3)} \) provides a \( 2 \times 2 \) representation of the entries \( M_{ij} \) of the generator matrix \( M \): 
\[
(M_{ij})_{\alpha\beta} = R^{(3)}_{\alpha,i} R^{(3)}_{\beta,j}.
\]
Thus, \( R^{(3)} \) may be seen as a matrix in which the \( 2 \times 2 \) blocks satisfy among themselves the same commutation relations that the entries of \( M \),

\[
R^{(3)} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

and the problem of finding all possible Lorentz deformations is equivalent to finding all possible \( R^{(3)} \) matrices with \( 2 \times 2 \) block entries satisfying (3) or (11) such that \( PR^{(3)}P = R^{(3)\dagger} \) (\( R^{(3)} = \hat{R}^{(3)} \)).

To introduce the deformed Minkowski algebra \( \mathcal{M}^{(j)} \) associated with a deformed Lorentz group \( L^{(j)} \) (where the index \( j \) refers to the different cases) it is natural to extend \( \mathcal{K}' = AKA^\dagger \) above to the deformed case by stating that in it the corresponding \( K \) generates a comodule algebra for the coaction \( \phi \) defined by

\[
\phi : K \mapsto K' = MKM^\dagger, \quad K'_{is} = M_{ij}M^\dagger_{js}K_{jl} \quad K = K^\dagger, \quad \Lambda = M \otimes M^*,
\]

where it is assumed that the matrix elements of \( K \), which now do not commute among themselves, commute with those of \( M \) and \( M^\dagger \). As in (17), (18) for \( q \)-two-vectors (rather, two-spinors) we now demand that the commuting properties of the entries of \( K \) are preserved by (21). The use of covariance arguments to characterize the algebra generated by the entries of \( K \) has been extensively used, and the resulting equations are associated with the name of reflection equations [17, 18] or, in a more general setting, braided algebras [19, 20] of which the former constitute the ‘algebraic sector’ (for an introduction to braided geometry see [21]); similar equations were also early introduced in [16]. Let us now extend the arguments given in [1] to classify the deformed Lorentz groups and their associated Minkowski algebras in an unified way.

This is achieved by describing the commutation properties of the entries of the hermitian matrix \( K \) generating a possible Minkowski algebra \( \mathcal{M} \) by means of a general reflection equation of the form

\[
R^{(1)} K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R^{(4)},
\]

where the \( R^{(i)} \) matrices \( (i = 1, ..., 4) \) are those introduced in (19). Indeed, writing equation (22) for \( K' = M K M^\dagger \), it follows that the invariance of the commutation properties of \( K \) under the associated deformed Lorentz transformation (22) is achieved if relations (19) are satisfied.

The deformed Minkowski length and metric, invariant under a Lorentz transformation (21) of \( L^{(j)} \), is defined through the quantum determinant of \( K \). Since the two matrices \( \hat{R}^{(1)} = P R_Q \) have spectral decompositions (8), (14) with a rank three projector \( P_{Q+} \) and a rank one projector \( P_{Q-} \), and
the determinants of $M, M^\dagger$ are central (eqs. (3), (12)), the $Q$-deformed and invariant (under (21)) determinant of the 2×2 matrix $K$ may now be given by
\[
(det_Q K) P_Q^- P_Q^\dagger = -\rho P_Q^- K_1 \tilde{R}^{(3)} K_1 P_Q^- .
\] (23)

It is easy to check that $(P_Q^- P_Q^\dagger)^2 = \left(\frac{\omega_\rho}{\sqrt{|q|}}\right)^2 P_Q^- P_Q^\dagger$, where $\omega_q = |q| + |q^{-1}|$, $\omega_h = 2 + h^2$ and $[2]_1 = 2$. In eq. (23), the subindex $Q$ in $det_Q K$ indicates that it depends on $q$ or $h$ (or on other parameters on which $R^{(3)}$ may depend) and $\rho (= (q, 1)$ as before) has been added by convenience. Since $\tilde{R}^{(3)}$ and $K$ are hermitian, $det_Q K$ is real (if $\rho$ is not real it may be factored out). We stress that the above formula provides a general expression for a central (see below) quadratic element which constitutes the deformed Minkowski length for all deformed spacetimes $\mathcal{M}^{(q)}$.

Similarly, it is possible to write in general the invariant scalar product of contravariant (transforming as the matrix $K$, eq. (21)) and covariant (transforming by $Y \mapsto Y' = (M^\dagger)^{-1} Y M^\dagger$) matrices (four-vectors) as the quantum trace of a matrix product $\tilde{R}^{(3)}$ (cf. 4). In the present general case, the deformed trace of a matrix $B$ is defined by
\[
tr_Q(B) := tr(D_Q B) \quad , \quad D_Q = \rho^2 tr_{(2)}(P((R_Q)^{-1})^t) ,
\] (24)

where $tr_{(2)}$ means trace in the second space. This deformed trace is invariant under the quantum group coaction $B \mapsto MBM^{-1}$ since the expression of $D_Q$ above guarantees that $D_Q = M^t D_Q(M^{-1})^t$ is fulfilled. In particular, the $D_Q$ matrices for $R_q$ and $R_h$ are found to be
\[
D_q = \left( \begin{array}{cc} q^{-1} & 0 \\ 0 & q \end{array} \right) \quad , \quad D_h = \left( \begin{array}{cc} 1 & -2h \\ 0 & 1 \end{array} \right) .
\] (25)

Let us now find the expression of the metric tensor. Using $\epsilon_Q$ (cf. (3), (4)) $(P_Q^-)_{ij,kl} = -\frac{1}{|q|} \epsilon_Q^{ij} \epsilon_Q^{kl}$ and $D_Q = -\epsilon_Q^{(\epsilon_Q^{-1})^t} (D_Q^t = M^t D_Q(M^{-1})^t$ now follows from $\epsilon_Q M^t \epsilon_Q^{-1} = M^{-1}$, eqs. (3) and (4)). The covariant $K^\epsilon_{ij}$ vector is
\[
K^\epsilon_{ij} = \tilde{R}^\epsilon_{Q,i,j,k} K_{kl} \quad , \quad \tilde{R}^\epsilon_Q = (1 \otimes (\epsilon_Q^{-1})^t) \tilde{R}^{(3)} (1 \otimes (\epsilon_Q^{-1})^t) ,
\] (26)

from which follows that the general Minkowski length and metric is given by
\[
l_Q \equiv det_Q K = \frac{\rho}{\omega_Q} tr_Q K K^\epsilon \equiv \rho^2 g_{Q,i,j,k} K_{ij} K_{kl} \quad , \quad g_{Q,i,j,k} = \frac{\rho^{-1}}{\omega_Q} D_Q^{si} \tilde{R}^\epsilon_{Q,j,s,k} .
\] (27)

This concludes the unified description of all cases. Let us now look at their classification and specific properties.
3 Characterization of the Lorentz deformations

First we use the reality condition $R^{(3)\dagger} = \mathcal{P} R^{(3)} \mathcal{P}$ to reduce the number of independent parameters in $R^{(3)}$. It implies

$$
R^{(3)} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{21}^* & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ a_{12}^* & c_{12} & a_{22}^* & c_{22} \\ b_{12}^* & c_{22} & b_{22}^* & d_{22} \end{bmatrix},
$$

(28)

where $a_{11}$, $d_{22}$, $b_{21}$, $c_{12}$ are real numbers and the rest are complex.

a) Deformed Lorentz groups associated with $SL_q(2)$

Let now $M \in SL_q(2)$ and $R^{(1)} = R_q$, eq. [2]. The problem of finding the $q$-Lorentz groups associated with the standard deformation is now reduced to obtaining all matrices $R^{(3)}$ satisfying (29). This means that the $2 \times 2$ matrices $A, B, C, D$ in (28) must satisfy the commutation relations in (3). This implies that (see [3]) $B^2 = C^2 = 0$, $AD \sim I_2$ and that either $B$ or $C$ are zero. Now

a1) $B = 0$ gives $R^{(3)} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & c_{12} & a_{22}^* & 0 \\ 0 & 0 & 0 & d_{22} \end{bmatrix}$ with $a_{11}, d_{22}, c_{12} \in R$.

From $AD \sim I_2$ it is easy to see (fixing first $a_{11} = 1$) that $d_{22} = a_{22}^*/a_{22}$; its reality then implies $d_{22} = \pm 1$, $d_{22} = 1$ when $a_{22} \in R$ and $d_{22} = -1$ for $a_{22} \in iR$. The relation $AC = q CA$ forces $a_{22} = q^{-1}$ or $c_{12} = 0$.

a2) $C = 0$ gives $R^{(3)} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & b_{21} & 0 \\ 0 & 0 & a_{22}^* & 0 \\ 0 & 0 & 0 & d_{22} \end{bmatrix}$ with $a_{11}, d_{22}, b_{21} \in R$, $a_{11} = 1$

as in the previous case, $d_{22} = \pm 1$ and $a_{22} \in R$ for $d_{22} = 1$ and $a_{22} \in iR$ for $d_{22} = -1$. Analogously, from $AB = q BA$ one obtains that $b_{21} = 0$ or $a_{22} = q$.

Thus, the solutions for $R^{(3)}$ are the following

$$
R^{(3)} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & r & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad q \in R, r \in R,
$$

(29)

$$
R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & \pm t & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix}, \quad + \text{ for } t \in R, \quad - \text{ for } t \in iR,
$$

(30)
b) Deformed Lorentz groups associated with

\[ R^{(3)} = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix}, \quad q \in R, \quad r \in R, \]  

\[ R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & r & -q^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad q \in iR, \quad r \in R, \]  

\[ R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q & r & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad q \in iR, \quad r \in R, \]  

Remarks:
- Notice that, as anticipated, the $Q$-'determinant' of all these $R^{(3)}$ matrices, computed as $\text{det}_QM$, is a scalar -hence commuting- $2 \times 2$ matrix.
- $R_q^1 = PR_qP$ iff $q \in R$. Hence, $R_q^{(4)} = R_{q^{12}}$ or $R_q^{-1}$. Thus $\tilde{M} \equiv (M^{-1})^\dagger$ provides a second copy of $SL_q(2)$, since then $R_q \tilde{M}_1 \tilde{M}_2 = \tilde{M}_2 \tilde{M}_1 R_q$.
- The case (29) for $r = q - q^{-1} = \lambda (R^{(3)} = R_q)$ is the quantum Lorentz group of $SU_q(2)$ $(L_q^{(1)}$ in the notation of [4]). If $r \neq \lambda$ we obtain a ‘gauged’ version of it: $R^{(3)} = e^{i \alpha \sigma_3} R_q e^{-i \alpha \sigma_3} (r = \lambda e^{2\alpha})$, where the subindex in $\sigma_3^2$ refers to the second space.
- The matrix (30) for $t = 1$ and $q \in R$ corresponds to $L_q^{(2)}$ in [4].
- The calculations leading to (30)-(33) require assuming $q^2 \neq 1$. However, the solutions for $q \in R$ are also valid in the limit $q = 1$ (see [4]); in this limit ($R^{(1)} = R^{(4)} = I_d$), the case (30) gives the deformed Lorentz group (twisted) of $SU_q(2)$. For $q = -1$, additional solutions appear and, although we shall not discuss these particular cases (see [4]), the associated Minkowski algebras may be obtained as in the general $q$ case.
- These results coincide with the classification in [4]: the solutions (30) correspond to eqs. (13) and (14) in [4]; similarly, (29), (31), (32) and (33) correspond to (74) ($q$ real), (15), (74) ($q$ imaginary) and (16) in that reference.

b) Deformed Lorentz groups associated with $SL_h(2)$

Let now $R^{(1)} = R_h$, eq. (10). For $h$ imaginary, $h \in iR$, the matrix $R_h$ satisfies the reality condition $R_h^* = R_h^{-1} (= PR_hP)$; this means that $\tilde{M} \equiv M^*$ defines a second copy of $SL_h(2)$ since $R_h \tilde{M}_1 \tilde{M}_2^* = M_2^* M_1^* R_h$. The value of $h \in C \setminus \{0\}$, however, is not important. Indeed, quantum groups related with two different values of $h \in C$ are equivalent and their $R$ matrices are related by a similarity transformation, thus, we can take $h \in R$ or even $h = 1$.

Since the entries of $M$ satisfy (11), the $2 \times 2$ blocks in $R^{(3)}$ (eq. (28)) will satisfy now these commutation relations. This leads to (see [4]) $C = 0$ so that,
taking the $h$-‘determinant’ of $R^{(3)}$ equal $I_2$, the set of commutation relations reduces to

$$AD = I_2, \quad [A, B] = h(I_2 - A^2) \quad . \quad (34)$$

Using them in (28) the following solutions for $R^{(3)}$ are found ($h \in R$)

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad r \in R \quad , \quad (35)$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & -h & 0 \\ -h & 1 & r & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h & 1 \end{bmatrix}, \quad h \in R, \quad r \in R \quad . \quad (36)$$

Remarks:
- In (36), for $r = h^2$ we have $R^{(3)} = (PR_hP)^4$. However, the parameter $r$ can be removed with an appropriate change of basis provided $h \neq 0$. For $h = 0$, this is not possible and constitutes a different case, eq. (35). This case is another example where the non-commutativity is solely due to $R^{(3)} \neq I_4$.
- The cases (35), (36) correspond to (20) and (21) [cf. (78)] in [2].

4 Minkowski algebras: classification and properties

We now present here, in explicit form, the commutation relations for the generators of the deformed Minkowski spacetimes; they follow easily from (22) using the previous $R^{(3)}$ matrices. We saw in (19) that $R^{(3)} = R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ and $R^{(4)} = R^{(1)} \dagger$ or $R^{(4)} = (\mathcal{P}R^{(1)}\mathcal{P})\dagger$ (these two possibilities are the same for $Q = h$). Clearly, eq. (22) allows for a factor in one side without impairing its invariance properties. This factor may be selected with the (natural) condition that the resulting Minkowski algebra does not contain generators $\alpha, \beta, \ldots$, with the Grassmann-like property $\alpha^2 = \beta^2 = \ldots = 0$. In terms of $P_{Q+}$, this tantamount to requiring that $P_{Q+}K_1\tilde{R}^{(3)}K_1P_{Q+}^\dagger$ must be non-zero. This leads to (cf. (22)) the equations

$$R_{Q}K_1R_{Q}^{(2)}K_2 = \pm K_2R^{(3)}K_1R_{Q}^{\dagger} \quad (+ \text{for } q, h \in R, - \text{for } q \in iR) \quad . \quad (37)$$

In the $q$-case we might also consider $R^{(4)} = (\mathcal{P}R^{(1)}\mathcal{P})\dagger$. However using Hecke’s condition for $R^{(1)}$ it is seen that this leads to the same algebra as (17) with the restriction $det_q K = 0$, so that this case may be considered as included in the previous one.

An important ingredient is the centrality of the $Q$-determinant (23), $(det_q K)K = K(det_q K)$, since it will correspond to the Minkowski length. Using twice (17) we find the following commutation property for three $K$ matrices

$$R_{Q_{13}}R_{Q_{23}}K_1R_{Q_{12}}^{(2)}K_2R_{Q_{13}}^{(2)}R_{Q_{23}}^{(2)}K_3 = K_3R_{Q_{13}}^{(3)}R_{Q_{23}}^{(3)}K_1R_{Q_{12}}^{(2)}R_{Q_{13}}^{\dagger}K_2R_{Q_{23}}^{\dagger} \quad . \quad (38)$$
Multiplying from the right by $P_{12} P^\dagger_{Q-12}$ and by $P_{Q-12}$ from the left and using that $R_Q$ and $R^{(3)}$ represent $GL_Q(2)$ and hence have a central $Q$-‘determinant’ represented by a scalar $2\times 2$ matrix we get

$$(\text{det}_Q R_Q)(\text{det}_Q R^{(3)})^\dagger(\text{det}_Q K)K = (\text{det}_Q R_Q)^\dagger(\text{det}_Q R^{(3)})K(\text{det}_Q K) \quad ,$$

(39)

The scalar $\text{det}_Q R^{(i)}$ matrices always cancel out in the cases below ($\text{det}_q R_q = qI_2$ and $\text{det}_h R_h = I_2$) assuring the centrality of $\text{det}_Q K$ (as it may be checked by direct computation).

### a) $q$-Minkowski spaces associated with $SL_q(2)$

1) Let us consider the case (29) for $r = \lambda$ (i.e., $R^{(3)} = R_q$, $q$ real). The commutation relations for the entries of $K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ are

$$\alpha \beta = q^{-2} \beta \alpha \quad , \quad [\delta, \beta] = q^{-1} \lambda \alpha \beta \quad , \quad \alpha \gamma = q^2 \gamma \alpha \quad , \quad [\gamma, \delta] = q^{-1} \lambda \gamma \alpha \quad ;$$

(40)

they characterize the algebra $\mathcal{M}_q^{(1)}$ ([12]-[14]; see also [21, 23, 24, 1]). The Minkowski length is given by (23),

$$\text{det}_q K = \alpha \delta - q^2 \gamma \beta \quad .$$

(41)

If $r \neq \lambda$, the commutation relations are slightly different; this, however, corresponds only to an appropriate election of the basis (‘gauged’ version of this Minkowski space).

2) Let $R^{(3)}$ be given by eq. (30). The centrality of the $q$-determinant implies that $q$ and $t$ are both real or both imaginary. The commutation relation for the entries of $K$ and the $q$-Minkowski length (eq. (23)) are (the sign $+$ is for $q, t \in \mathbb{R}$ and the $-$ for $q, t \in i\mathbb{R}$)

$$q \alpha \beta = \pm t \beta \alpha \quad , \quad t \alpha \gamma = \pm q \gamma \alpha \quad , \quad \alpha \delta = \delta \alpha \quad , \quad [\beta, \gamma] = \pm t \lambda \alpha \delta \quad , \quad \beta \delta = \pm q t \delta \beta \quad , \quad \delta \gamma = \pm q t \gamma \delta \quad ;$$

(42)

$$\text{det}_{q,t} K = \frac{q + q^{-1}}{q \pm q^{-1}}(-q \gamma \beta \pm t \alpha \delta) \quad .$$

(43)

Remarks:

- For $t = 1$, these commutation relations correspond to the Minkowski algebra $\mathcal{M}_q^{(2)}$ ([12, 23, 1]) which is isomorphic to the quantum algebra $GL_q(2)$.
- For $q = 1$ and $t$ real, we get the Minkowski space obtained in [22] (denoted $\mathcal{M}^{(3)}$ in [1]). This algebra and the corresponding deformed Poincaré algebra have been shown to be [27] a simple transformation (twisting) of the classical one. As a result, it is possible to remove the non-commuting character of the entries of $K$ [28].

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4The Minkowski space of [26] is also a $GL_q(2)$-like space, but different from the above.
3) Let us take \( R^{(3)} \) as in eq. (31) for \( r=-\lambda \) \( (R^{(3)} = \mathcal{P}R^{-1}_q\mathcal{P}) \). Then,

\[
\begin{align*}
[\alpha, \beta] &= q\lambda \beta \delta , & [\alpha, \gamma] &= -q\lambda \delta \gamma , & [\alpha, \delta] &= 0 , \\
[\beta, \gamma] &= q\lambda (\alpha - \delta)\delta , & \beta \delta &= q^2 \delta \beta , & \gamma \delta &= q^{-2} \delta \gamma , \\
\det_q K &= q^2 \alpha \delta - \beta \gamma .
\end{align*}
\]  

(44)

This algebra may also be identified with the algebra of spacetime derivatives in [14] (see also [23]).

4) Let \( R^{(3)} \) be now given by (32). The Minkowski algebra and the central length are given by

\[
\begin{align*}
\alpha \beta &= -q^{-2} \beta \alpha , & \beta \delta &= -r \beta \delta , & \alpha \gamma &= -q^2 \gamma \alpha , \quad \alpha \gamma + \gamma \alpha &= -r \delta \gamma , & [\alpha, \delta] &= 0 , \\
[\beta, \gamma] &= -q^{-1} \lambda \alpha \delta + r \alpha^2 , & [\beta, \delta] &= q \delta^2 , & \gamma \delta &= -q^{-2} \delta \gamma ,
\end{align*}
\]  

(46)

\[
\det_q K = \frac{-q[2]}{\lambda}(q^{-2} \alpha \delta + \gamma \beta) .
\]  

(47)

5) Finally, let \( R^{(3)} \) be as in eq. (33). Then,

\[
\begin{align*}
\alpha \beta + \beta \alpha &= -r \beta \delta , & \alpha \gamma + \gamma \alpha &= -r \delta \gamma , & [\alpha, \delta] &= 0 , \\
[\beta, \gamma] &= -q \lambda \alpha \delta + r \delta^2 , & \beta \delta &= -q^2 \delta \beta , & \gamma \delta &= -q^{-2} \delta \gamma ,
\end{align*}
\]  

(48)

\[
\det_q K = \frac{-q[2]}{\lambda}(q^2 \alpha \delta + \beta \gamma) .
\]  

(49)

b) Deformed Minkowski spaces associated with \( SL_h(2) \)

1) Let \( R^{(3)} \) be given first by eq. (35) and let \( R^{(1)} = R_h \), eq. (10). Using (17) with the plus sign and (23) we find (\( h \) real)

\[
\begin{align*}
[\alpha, \beta] &= -h \beta^2 - r \beta \delta + h \delta \alpha - h \beta \gamma + h^2 \delta \gamma , & [\alpha, \delta] &= h(\delta \gamma - \beta \delta) , \\
[\alpha, \gamma] &= h \gamma^2 + r \delta \gamma - h \alpha \delta + h \beta \gamma - h^2 \beta \delta , & [\beta, \delta] &= h \delta^2 , \\
[\beta, \gamma] &= h \delta(\gamma + \beta) + r \delta^2 , & [\gamma, \delta] &= -h \delta^2 ;
\end{align*}
\]  

(50)

\[
\det_h K = \frac{2}{h^2 + 2}(\alpha \delta - \beta \gamma + h \beta \delta) .
\]  

(51)

2) Let \( R^{(3)} \) be given now by eq. (36) with \( r = 0 \). In this case,

\[
\begin{align*}
[\alpha, \beta] &= 2h \alpha \delta + h^2 \beta \delta , & [\alpha, \delta] &= 2h(\delta \gamma - \beta \delta) , \\
[\alpha, \gamma] &= -h^2 \delta \gamma - 2h \delta \alpha , & [\beta, \delta] &= 2h \delta^2 , \\
[\beta, \gamma] &= 3h^2 \delta^2 , & [\gamma, \delta] &= -2h \delta^2 ;
\end{align*}
\]  

(52)

\[
\det_h K = \frac{2}{h^2 + 2}(\alpha \delta - \beta \gamma + 2h \beta \delta) .
\]  

(53)
c) Final remarks

For all the $Q$-spacetime algebras, time may be defined as proportional to $tr_Q K (= 2x^0$ in the undeformed case). The time generator obtained in this way is central only for $\mathcal{M}_q^{(1)}$ [12]–[14] and for the Minkowski algebra [14] (in fact, they are isomorphic: the entries of the covariant vector $K'$ for $\mathcal{M}_q^{(1)}$ satisfy the commutation relations [14] [1]).

The differential calculus on all the above Minkowski spaces may be easily discussed now along the lines of [1, 23]; one could also investigate the rôle played in it by the contraction relating [10] the $q$- and $h$-deformations. To conclude, let us mention that the additive braided group structure [19]–[21] of all these algebras, may be easily found. It suffices to impose that eq. (37) is also satisfied by the sum $K' + K$ of two copies $K$ and $K'$. Using Hecke’s condition ($R_{Q12} = R_{Q21}^{-1} + (\rho - \rho^{-1})P$) this gives

$$R_Q K_1' R^{(2)} K_2 = \pm K_2 R^{(1)} K_1' (PR_Q^P)^{-1}$$

(54)

which is clearly preserved by (21); for $\mathcal{M}_q^{(1)}$, it reproduces the result of [24].

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