DECOMPOSITION OF THE ADJOINT
REPRESENTATION OF THE SMALL QUANTUM $sl_2$.

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1. Introduction.

1.1. Given a finite type root datum and a primitive root of unity $q = \sqrt[4]{1}$, G. Lusztig has defined in [Lu] a remarkable finite dimensional Hopf algebra $u$ over the cyclotomic field $\mathbb{Q}(\sqrt[4]{1})$. It is called a restricted quantum universal enveloping algebra, or a small quantum group.

Recall that for a Hopf algebra $A$ with a coproduct $\Delta$, antipode $S$, and counit $\varepsilon$ the adjoint representation is defined in the following way. $A$ is an $A$-bimodule with respect to left and right multiplication. Using the antipode we can consider $A$ as $A \otimes A$-module. Combining this with the coproduct we get a new structure of $A$-module on $A$. It is called the adjoint representation and denoted by $\text{ad}$.

1.2. In this note we study the adjoint representation of $u$ in the simplest case of the root datum $sl_2$.

The semisimple part of this representation is of big importance in the study of local systems of conformal blocks in WZW model for $\hat{sl}_2$ at level $l - 2$ in arbitrary genus. The problem of distinguishing the semisimple part is closely related to the problem of integral representation of conformal blocks (see [BFS]).

We find all the indecomposable direct summands of $\text{ad}$ with multiplicities. To formulate the answer let us recall a few notations from the representation theory of $u$. The representation $\text{ad}$ is naturally $\mathbb{Z}$-graded, so we consider the category $\mathcal{C}$ of $\mathbb{Z}$-graded $u$-modules. The simple modules in this category are parametrized by their highest weights which can assume arbitrary integer values. The simple module with a highest weight $\lambda \in \mathbb{Z}$ is denoted by $L(\lambda)$, and its indecomposable projective cover is denoted by $P(\lambda)$.

1.3. Main theorem. Let $l \geq 3$ be an odd integer. The adjoint representation $\text{ad}$ is isomorphic to the direct sum of the modules $P(0), P(2), \ldots, P(l - 3), L(-l + 1), L(-l + 3), \ldots, L(2l - 4), L(2l - 2)$ with the following multiplicities:

Date: December 1995.
(i) the multiplicity of \( P(l - 1) \simeq L(l - 1) \) is \( l \);
Let \( i \in [0, \ldots, \frac{l-3}{2}] \).

(ii) the multiplicity of \( P(2i) \) is \( \frac{i+1}{2} + i \);
(iii) the multiplicity of \( L(2i) \) is \( l - 1 - 2i \);
(iii) the multiplicities of \( \hat{L}(2l - 2 - 2i) \) and \( \hat{L}(-2 - 2i) \) are \( \frac{l-1}{2} - i \).

1.4. G. Lusztig has defined (see e.g. [Lu]) a quantum enveloping algebra with divided powers \( U \) containing \( u \) as a Hopf subalgebra. Its finite dimensional irreducible representations are parametrized by their highest weights which can assume arbitrary nonnegative integer values. The simple module with a highest weight \( \lambda \in \mathbb{N} \) is denoted by \( \hat{L}(\lambda) \), and its indecomposable projective cover is denoted by \( \hat{P}(\lambda) \). There is a natural restriction functor from the category of finite dimensional \( U \)-modules to \( \mathcal{C} \). It appears that \( \text{ad} \) lies in its essential image.

Corollary. The structure of \( \mathbb{Z} \)-graded \( u \)-module on \( \text{ad} \) can be lifted to the structure of \( U \)-module isomorphic to the direct sum of the modules \( \hat{P}(0), \hat{P}(2), \ldots, \hat{P}(l-3); \hat{L}(0), \hat{L}(2), \ldots, \hat{L}(2l-4), \hat{L}(2l-2) \) with the following multiplicities:

(i) the multiplicity of \( \hat{P}(l - 1) \simeq \hat{L}(l - 1) \) is \( l \);
Let \( i \in [0, \ldots, \frac{l-3}{2}] \).

(ii) the multiplicity of \( \hat{P}(2i) \) is \( \frac{i+1}{2} + i \);
(iii) the multiplicity of \( \hat{L}(2i) \) is \( l - 1 - 2i \);
(iii) the multiplicity of \( \hat{L}(2l - 2 - 2i) \) is \( \frac{l-1}{2} - i \).

1.5. The author is very grateful to Michael Finkelberg; it is hard to overestimate his help with writing down the present work. Also I am happy to express my sincere gratitude to Thomas Kerler who taught me everything I know about \( \text{ad} \).

2. Notations

2.1. In this section we recall the necessary facts about \( u \) and its representations, following mainly [Lu].

Let \( l \) be an odd integer, \( l > 1 \). Let \( q \in \mathbb{C} \) be a primitive \( l \)-th root of unity. Let \( (i)_q = \frac{q^i - q^{-i}}{q - q^{-1}} \).

Let \( u \) be an associative algebra over \( \mathbb{C} \) with generators \( E, F, K, K^{-1} \) and relations:

\[
KK^{-1} = K^{-1}K = 1;
\]

\[
KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F;
\]
\[ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}; \]
\[ E^l = F^l = 0, K^l = 1. \]

The algebra \( u \) is finite-dimensional and \( \dim u = l^3 \).

Let \( \omega \) be an automorphism of the algebra \( u \) given on generators by the formulas:
\[ \omega(E) = F, \quad \omega(F) = E, \quad \omega(K) = K^{-1}. \]

The algebra \( u \) is Hopf algebra with respect to coproduct \( \Delta \), antipode \( S \) and counit \( \varepsilon \) given by the formulas:
\[ \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K; \]
\[ S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}; \]
\[ \varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = 1. \]

2.2. Let \( C \) be a category of finite-dimensional \( \mathbb{Z} \)-graded \( u \)-modules \( V = \bigoplus_{i \in \mathbb{Z}} V^i \) such that the following conditions hold:
(a) \( E \) is operator of degree 2, i.e. \( E \) acts from \( V^i \) to \( V^{i+2} \);
(b) \( F \) is operator of degree \(-2\);
(c) \( K \) acts on \( V^i \) by multiplication by \( q^i \).

The morphisms in category \( C \) are morphisms of \( u \)-modules compatible with \( \mathbb{Z} \)-grading.

2.3. We introduce the duality \( D \) on category \( C \). If \( V \in C \), then \( D(V) \) is \( V^* \) as a vector space. The action of \( x \in u \) on \( D(V) \) is given by the formula \((xf)(v) = f(\omega S(x)v)\), where \( f \in D(V), v \in V, S \) is the antipode.

2.4. Let us define the adjoint representation \( \text{ad} \in C \) (see e.g. [LM]). Let \( x \) be an element of \( u \). The adjoint action of generators is given by the following formulas:
\[ \text{ad}(E)x = Ex - KxK^{-1}E = K[K^{-1}E, x], \]
\[ \text{ad}(F)x = FxK - xFK = [F, x]K, \]
\[ \text{ad}(K)x = KxK^{-1}. \]

2.5. Now we introduce \( \mathbb{Z} \)-grading on adjoint representation. We put \( \deg(E) = 2, \deg(F) = -2, \deg(K) = 0 \) and \( \deg(ab) = \deg(a) + \deg(b) \) for any \( a, b \in u \) such that \( \deg(a), \deg(b) \) are defined. Note that all the weights of \( \text{ad} \) are even integers in the interval \([2 - 2l, \ldots, 2l - 2]\).

2.6. It is known (see e.g. [LM] or [BFS]) that \( D(\text{ad}) \simeq \text{ad} \).
3. u-modules.

3.1. It is easy to check that an element

\[ X = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} \]  

lies in the center of algebra \( u \) (see e.g. [Ke]). The element \( X \) is called Casimir element. It satisfies the following equation of degree \( l \) (see loc.cit.):

\[ P(X) := \prod_{j \in \mathbb{Z}/l\mathbb{Z}} (X - b_j) = 0, \]

where \( b_j = \frac{q^{j+1} + q^{-j-1}}{(q - q^{-1})^2} \) (\( q' = 1 \), so \( q^j \) is well defined). In particular \( b_j = b_j' \) if \( j + j' = l - 2 \). The root \( b_{-1} = \frac{2}{(q - q^{-1})^2} \) of \( P \) has multiplicity 1, and the rest roots \( b_j \) have multiplicity 2.

3.2. Let \( j \in \mathbb{Z}/l\mathbb{Z} \). Let \( C_j \) be a full subcategory of \( C \) such that \( (X - b_j) \) acts nilpotently on objects of \( C_j \). In what follows we will identify \( C_j \) and \( C_{j'} \) if \( j + j' = l - 2 \).

3.3. Let us fix \( H' \)— a maximal subset of \( \mathbb{Z}/l\mathbb{Z} \) with the following property: if \( \{j, j'\} \) is two-element subset of \( H' \) then \( j + j' \neq l - 2 \). We have \( H' = \{-1\} \cup H \), where \( H = H' - \{-1\} \).

3.4. For any \( j \in H \) we define the integers \( J, J' \) by the following properties:

1. \( 0 \leq J < J' < l \);
2. \( J + J' = l - 2 \);
3. \( (J - j)(J' - j) \equiv 0 \) (mod \( l \)).

3.5. The category \( C \) is a direct sum of subcategories \( C_j \) where \( j \) runs through \( H' \).

3.6. We denote by \( u^\pm \subset u \) the subalgebra generated by \( K, E \) (resp. \( K, F \)). For \( \lambda \in \mathbb{Z} \) denote by \( C^\pm_\lambda \) the one-dimensional \( \mathbb{Z} \)-graded \( u^\pm \)-module of weight \( \lambda \) such that \( K \) acts as \( q^\lambda \) and \( E \) (resp. \( F \)) acts as zero on it. We denote by \( M^\pm(\lambda) \) the \( \mathbb{Z} \)-graded \( u \)-module \( u \otimes_{u^\pm} C^\pm_\lambda \). The modules \( M^\pm(\lambda) \) are called Verma modules. Let \( M^\pm(\lambda) \ni v^\pm(\lambda) := 1 \otimes 1 \). Let \( V \in C, v \in V^\lambda \) and \( E \cdot v = 0 \) (resp. \( Fv = 0 \)). Then \( v \) is called an upper singular (resp. a lower singular) vector in \( V \), and there exists a unique morphism \( \phi : M^\pm(\lambda) \to V \) such that \( \phi(v^\pm(\lambda)) = v \).
3.7. For each \( \lambda \in \mathbb{Z} \) there is a unique up to isomorphism simple module \( L(\lambda) \in \mathcal{C} \) with highest weight \( \lambda \). The modules \( L(\lambda_1) \) and \( L(\lambda_2) \) are isomorphic iff \( \lambda_1 \equiv \lambda_2 (\text{mod} \ l) \). Indecomposable projective cover of \( L(\lambda) \) will be denoted by \( P(\lambda) \). We have \( D(L(\lambda)) \cong L(\lambda) \) and \( D(P(\lambda)) \cong P(\lambda) \). In particular \( P(\lambda) \) is injective; \( \dim \text{Hom}(L(\lambda), P(\lambda)) = 1 \).

3.8. The set of isomorphism classes of simple objects in category \( \mathcal{C}_{-1} \) is \( \{L(\lambda), \lambda \equiv -1 (\text{mod} \ l)\} \). As \( \mathfrak{u} \)-modules without grading all the \( L(\lambda) \) are isomorphic to one and the same \( \mathfrak{u} \)-module \( \text{St} \) (Steinberg module). It has dimension \( l \). The category \( \mathcal{C}_{-1} \) is semisimple. In particular \( P(\lambda) = L(\lambda) \).

3.9. Let \( j \in H \). The set of isomorphism classes of simple modules in \( \mathcal{C}_j \) is \( \{L(\lambda), \lambda \equiv J(\text{mod} \ l) \text{ or } \lambda \equiv J'(\text{mod} \ l)\} \). The modules \( L(\lambda), \lambda \equiv J(\text{mod} \ l) \) (resp. \( \lambda \equiv J'(\text{mod} \ l) \)) are isomorphic as \( \mathfrak{u} \)-modules. Their dimension is \( J + 1 \) (resp. \( J' + 1 \)).

The projective module \( P(\lambda) \) admits a filtration \( P(\lambda) \supset W(\lambda) \supset L(\lambda) \supset 0 \) such that \( P(\lambda)/W(\lambda) \cong L(\lambda), W(\lambda)/L(\lambda) \cong L(\lambda') \oplus L(\lambda'') \) where \( \lambda' \neq \lambda'', \lambda' \equiv \lambda'' \equiv -2 - \lambda \) \((\text{mod} \ l)\), \( |\lambda - \lambda'| < 2l > |\lambda - \lambda''| \). In particular \( \dim P(\lambda) = 2l \).

3.10. It is easy to see from 3.3, 3.8 and 3.9 that all the simple subquotients of adjoint representation have the type \( L(\lambda) \) where \( \lambda \) is an even integer from the interval \([1 - l, \ldots, 2l - 2]\). Hence a projective module \( P(\lambda) \) can be a subquotient of \( \text{ad} \) only if \( \lambda \in 2\mathbb{Z} \cap [0, \ldots, l - 1] \). In particular each subcategory \( \mathcal{C}_j \) contains only one isomorphism class of such projectives.

3.11. **Lemma.** Suppose \( V \in \mathcal{C} \) is indecomposable and the action of Casimir element \( X \) on \( V \) is not semisimple. Then there exists \( \lambda \neq -1 (\text{mod} \ l) \) such that \( V \cong P(\lambda) \).

**Proof.** Casimir acts nonsemisimply on regular representation (see (3)). It follows that action on projective modules \( P(\lambda), \lambda \neq -1 (\text{mod} \ l) \) is not semisimple. It is easy to see that the space of eigenvectors of Casimir in \( P(\lambda) \) is \( W(\lambda) \).

Let \( W \subset V \) be a maximal submodule of \( V \) such that \( X \) acts on \( W \) semisimply. Choose \( 0 \neq \varphi \in \text{Hom}(V, L) \) where \( L = L(\lambda) \) for some \( \lambda \in \mathbb{Z} \) such that \( \text{Ker} \varphi \) contains \( W \). We have a morphism \( \psi \in \text{Hom}(P, V) \) where \( P = P(\lambda) \) such that the diagram is commutative:

\[
\begin{array}{ccc}
P & \rightarrow & V \\
\downarrow \psi & & \downarrow \varphi \\
0 & \rightarrow & L
\end{array}
\]
If Ker $\psi \neq 0$ then $X$ acts on Im $\psi$ semisimply. Therefore $W$ is not maximal. We have a contradiction. If Ker $\psi = 0$ then we have injection $P \hookrightarrow V$. From 3.7 follows that $P$ is direct summand of $V$. The proof is complete. $\square$

4. THE BLOCKS OF ADJOINT REPRESENTATION

In what follows we always will identify $S_j$ and $S_j'$ where $j + j' = l - 2$ and $S$ is an object, map, etc. Also we will identify $S_J$ and $S_j$ if $J \in \mathbb{Z}, J \equiv j \pmod{l}$.

4.1. The regular action of Casimir $X$ (by multiplication) is an endomorphism of adjoint representation. This gives a decomposition of adjoint representation into blocks $\text{ad} = \bigoplus_{j \in H'} \text{ad}_j$ where $(X - b_j)$ acts nilpotently on $\text{ad}_j$. Let $pr_j$ denote a projection onto $\text{ad}_j$.

4.2. Let $j \in H$. Let $M_j = \text{Ker}(X - b_j), N_j = \text{ad}_j \cap \text{Im}(X - b_j)$. Each $\text{ad}_j$ admits a filtration $\text{ad}_j \supset M_j \supset N_j \supset 0$. The rest of this section is a computation of associated graded of this filtration. Evidently $\text{ad}_j / M_j \simeq N_j$. It remains to compute $N_j, M_j / N_j$. It is convenient to put $N_{-1} = \text{ad}_{-1}$.

4.3. Recall (see 2.2) that $\text{ad}^0 \subset \text{ad}$ denotes the zero weight space. Let $\text{ad}^0_j = \text{ad}_j \cap \text{ad}^0$ (for all $j \in H'$), $N_j^0 = N_j \cap \text{ad}^0, M_j^0 = M_j \cap \text{ad}^0$ (for $j \in H$) and $N_{-1}^0 = \text{ad}_{-1}^0$. We will compute the action of $\text{ad}(X)$ on $\text{ad}^0_j$. We start with a computation of action of $\text{ad}(X)$ on $N_j^0$.

4.4. Lemma. We have

(a) $\dim \text{ad}_j = 2l^2$ if $j \in H$ and $\dim \text{ad}_{-1} = l^2$;
(b) if $j \in H$ then $\dim N_j = \dim \text{ad}_j / M_j = (J + 1)^2 + (J' + 1)^2$ and $\dim M_j / N_j = 4(J + 1)(J' + 1)$.
(c) $\dim \text{ad}^{2m}_j = l(l - |m|)$ for all $m \in \mathbb{Z}$ such that $|m| < l$;
(d) $\dim \text{ad}^0_j = 2l$ for $j \in H$ and $\dim N_j^0 = l$ for all $j \in H'$.
(e) $\dim \text{ad}^{2m}_j \geq 2(l - |m|)$ if $j \in H$ and $|m| \geq l - 1 - J$.

Proof. (a), (b), (c), (d) are trivial. Let us prove (e). Suppose $m > 0$. It is easy to see from consideration of $u$-action on Verma modules $M^+(J)$ and $M^+(J')$ that $E^m$ acts nontrivially at least on $2(l - m)$ weights. By standard arguments with Vandermond determinant we obtain that the desired dimension is at least $2(l - m)$. The proof for $m < 0$ is similar. $\square$
4.5. The subspace $\mathbf{ad}^0 \subset \mathfrak{u}$ is a subalgebra of $\mathfrak{u}$. It is generated as algebra by $K$ and $X$ (see [Ke]). Moreover $\mathbf{ad}^0$ is a free module over a subalgebra generated by $K$ (see loc. cit.). In particular we have $M_j^0 = N_j^0$.

We have

$$ad(X)K^i = \frac{q^{2i-1} + q^{1-2i}}{(q - q^{-1})^2} K^i - (q^i - q^{-i})^2 X K^{i+1} + (i)q(i+1)qK^{i+2}$$

(3)

4.6. For $j \in H$ the elements $(X - b_j)pr_jK^i, i = 1, \ldots, l$ (resp. $pr_{-1}K^i, i = 1, \ldots, l$ for $j = -1$) form a basis of $N_j^0$. In this basis $\text{ad}(X)$ acts as a lower-triangular matrix:

$$A(j) = \begin{pmatrix}
    b_0 & 0 & 0 & \ldots & 0 \\
    (q - q^{-1})b_j & b_2 & 0 & \ldots & 0 \\
    (1)q(2)_q & (q^2 - q^{-2})b_j & b_4 & \ldots & 0 \\
    0 & (2)q(3)_q & (q^3 - q^{-3})b_j & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & b_0
\end{pmatrix}$$

(4)

4.6.1. Remark. The vectors $pr_jK^i, i = 1, \ldots, l; (X - b_j)pr_jK^i, i = l + 1, \ldots, 2l$ form a basis of $\mathbf{ad}_j^0 (j \in H)$.

4.7. Let $k \in 2\mathbb{Z} \cap [0, \ldots, l - 1]$. The eigenvalues of this matrix are $b_k$ (with multiplicity 2 if $k \neq l - 1$ and multiplicity 1 if $k = l - 1$). Let $k \neq l - 1$. It is obvious that there exists 1 or 2 eigenvectors of $A(j)$ corresponding to eigenvalue $b_k$. We have 2 eigenvectors iff the determinant $d(j, k)$ of matrix (see Lemma 3.1)

$$D(j, k) = \begin{pmatrix}
    (q^{k/2+1} - q^{-k/2-1})b_j & b_{k+2} - b_k & 0 & \ldots \\
    (k/2 + 1)q(k/2 + 2)_q & (q^{k/2+2} - q^{-k/2-2})b_j & b_{k+4} - b_k & \ldots \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & \ldots & \ldots
\end{pmatrix}$$

(5)

is equal to zero. It is easy to see that this determinant is a polynomial in $b_j^2$ of degree $\frac{l-1-k}{2}$. Since $b_j^2 = b_j^2 \Rightarrow b_j = b_{j'}$, the polynomial $d(j, k)$ vanishes for at most $\frac{l-1-k}{2}$ values of $j \in H'$.

4.8. In this section we prove the following proposition:

**Proposition.** For any $j \in H$ we have the following decomposition:

$$N_j \cong \bigoplus_{i=0}^{j} (L(2i) \oplus L(2i)) \oplus \bigoplus_{i=j+1} P(2i) \oplus L(l - 1).$$
Proof. The proof proceeds by induction: we start from \( j = \frac{l-3}{2} \), then proceed to \( j = \frac{l-5}{2} \) etc.

It follows from Lemma 4.3 that for any \( j \in H' \) the module \( N_j \) contains as subquotients \( L(0), L(2), \ldots, L(l - 3) \) with multiplicities 2 and \( L(l - 1) \) with multiplicity 1 (since only these modules have nontrivial zero weight space).

Lemma. Let \( j + 1 = \frac{l-1}{2} \). Then \( N_j \simeq L(0) \oplus L(0) \oplus \ldots \oplus L(l - 3) \oplus L(l - 1) \).

Proof. Let us compute the dimensions. We have \( \dim N_j = (\frac{l-1}{2})^2 + (\frac{l-1}{2})^2 = \frac{l^2}{2} \) (see Lemma 4.4(b)). On the other hand by the above we have \( \dim N_j \geq 2 \dim L(0) + 2 \dim L(2) + \ldots + 2 \dim L(l - 3) + \dim L(l - 1) = 2 \cdot 1 + 2 \cdot 3 + \ldots + 2 \cdot (l - 2) + l = \frac{l^2 + l}{2} \). It follows that in this case \( N_j \) is a direct sum of \( L(0), L(2), \ldots, L(l - 3) \) with multiplicities 2 and \( L(l - 1) \) with multiplicity 1 (since \( \text{Ext}^1(L(\lambda), L(\mu)) = 0 \) \( \forall \lambda, \mu \in \{0, 2, \ldots, l - 1\} \)).

The Lemma implies that all eigenvalues of \( A(\frac{l-3}{2}) \) are semisimple. It follows from Lemma 4.3 that eigenvalue \( b_{l-3} \) is not semisimple for all the rest \( j \). Therefore for \( j \neq \frac{l-3}{2} \) the corresponding \( N_j \) contains projective submodule \( P(l - 3) \) (see Lemma 3.11).

Now let \( j + 1 = \frac{l-3}{2} \). Then \( \dim N_j = \frac{l^2 + l}{2} \). On the other hand \( \dim N_j \geq 2 \cdot 1 + \ldots + 2 \cdot (l - 4) + 2l + l = \frac{l^2 + l}{2} \). So in this case \( N_j \) is a direct sum of \( L(0), L(2), \ldots, L(l - 5) \) with multiplicities 2, \( L(l - 1) \) with multiplicity 1, and \( P(l - 3) \). As above it follows that all the rest \( N_j \) contains projective submodules \( P(l - 5) \) and \( P(l - 3) \) etc. The Proposition is proved. □

4.8.1. Corollary. We have:

\[
\text{ad}_{-1} = N_{-1} = \bigoplus_{i=0}^{l+1} P(2i).
\]

Proof. It follows from the proof of the Proposition 4.8 that all the eigenvalues of the matrix \( A(-1) \), except for \( b_{l-1} \), are not semisimple. Hence the result follows from the Lemma 3.11 and computation of dimensions. □

4.9. The Corollary 4.8.1 gives a decomposition of \( \text{ad}_{-1} \). So in what follows we will assume that \( j \neq -1 \) i.e. \( j \in H \).

Lemma. The module \( M_j/N_j \) is a direct sum of modules \( L(\lambda) \) with multiplicity 2 where \( \lambda \) is even and satisfies one of the following conditions:

(i) \( \lambda \) lies in the interval \( [2(l - J + 1), \ldots, 2l - 2] \);
(ii) \( \lambda \) lies in the interval \([-2J - 2, \ldots, -2]\).

**Proof.** Recall that \( M_j^0 = N_j^0 \). Hence \( M_j/N_j \) contains only subquotients \( L(\lambda) \) where either \( \lambda > l \) or \( \lambda < 0 \). By the Lemma 4.4(e), Proposition 4.8 and Corollary 4.8.1, we have \( \dim \text{ad}_j^{2m} \geq 2(l - |m|) \) if \( j \in H \) and \( \dim \text{ad}_j^{2m} \geq l - |m| \) for any \( m \geq \frac{l+1}{2} \). It follows from the Lemma 4.4(c) that \( \dim \text{ad}_j^{2m} = 2(l - |m|) \) for any \( j \in H \) and \( m \geq \frac{l+1}{2} \).

Hence for any \( m \geq \frac{l+1}{2} \) we have exactly two upper (resp. lower) singular vectors of weight \( 2m \) (resp. \(-2m\)). It follows that \( \text{ad}_j \) has two simple subquotients with highest weight \( 2m \) and two simple subquotients with lowest weight \(-2m\) for any \( m \geq \frac{l+1}{2} \). Thus \( \text{ad}_j \) has the following subquotients: \( L(2i) \) where \( i \in \{0, \ldots, \frac{l-3}{2}\} \) with multiplicities 4; \( L(l - 1) \) with multiplicity 2; \( L(-2 - 2i) \) and \( L(2l - 2 - 2i) \), where \( i \in \{0, \ldots, \frac{l-3}{2}\} \), with multiplicities 2. The computation of dimensions shows that these modules are all the subquotients of \( \text{ad}_j \). It follows from Proposition 4.8 that \( [N_j : L(\lambda)] = 1 \) if \( \lambda = -2 - 2i \) and \( \lambda = 2l - 2 - 2i \), where \( i \in \{J + 1, \ldots, \frac{l-3}{2}\} \). Since \( \text{ad}_j/M_j \simeq N_j \), any simple subquotient of \( M_j/N_j \) is of the type \( L(\lambda) \) where \( \lambda \in [2(l - J + 1), \ldots, 2l - 2] \cup [-2J - 2, \ldots, -2] \) is even. But for any \( \lambda, \mu \) satisfying such conditions we have \( \text{Ext}^1(L(\lambda), L(\mu)) = 0 \). The result follows. \( \square \)

4.9.1. **Remark.** It follows from the proof of the Lemma 4.3 that \( \dim \text{ad}_j^{2m} = 2(l - |m|) \) for any \( j \in H \) and \( m \in \mathbb{Z}, |m| < l \).

4.10. Let \( k \) be an even integer and \( k \in [0, l - 1] \). Let \( \text{ad}_j(k) \) be a summand corresponding to the subcategory \( \mathcal{C}_k \) in \( \text{ad}_j \). Let \( M_j(k) = M_j \cap \text{ad}_j(k) \) and \( N_j(k) = N_j \cap \text{ad}_j(k) \). Let us summarize the results of the present section.

4.10.1. \( \text{ad}_{-1}(k) = P(k) \).

4.10.2. If \( j \in H \) then
(a) \( \text{ad}_j(l - 1) = L(l - 1) \oplus L(l - 1) \);
(b) if \( k \geq 2J + 2 \) then \( \text{ad}_j(k) = P(k) \oplus P(k) \);
(c) if \( k \leq 2J \) then \( \text{ad}_j(k) \) admits a filtration \( \text{ad}_j(k) \supset M_j(k) \supset N_j(k) \supset 0 \) with the following associated graded factors:

\[
N_j(k) \simeq L(k) \oplus L(k);
\]

\[
M_j(k)/N_j(k) \simeq L(2l - 2 - k) \oplus L(2l - 2 - k) \oplus L(-2 - k) \oplus L(-2 - k);
\]

\[
\text{ad}_j(k)/M_j(k) \simeq L(k) \oplus L(k).
\]
5. The proof of the main theorem

5.1. Let us find the multiplicities of projective submodules in \( \mathbf{ad}_j \).
Recall (see Remark 4.6.1) that the vectors \( pr_j K^i, i = 1, \ldots, l; (X - b_j)pr_j K^i, i = l + 1, \ldots, 2l \) form a basis of \( \mathbf{ad}^0_j \). In this basis \( \mathbf{ad}(X) \) acts as a block matrix

\[
A'(j) = \begin{pmatrix}
A(j) & 0 \\
B & A(j)
\end{pmatrix}
\]

where \( A(j) \) is a matrix (4) and \( B \) is a matrix (see (3))

\[
\begin{pmatrix}
0 & 0 & 0 & \ldots \\
-(q - q^{-1})^2 & 0 & 0 & \ldots \\
0 & -(q^2 - q^{-2})^2 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

5.2. Let \( b_k \) be a nonsemisimple eigenvalue of \( A(j) \). Then a summand corresponding to the subcategory \( C_k \) in \( \mathbf{ad}_j \) is a sum of 2 copies of projective \( P(k) \).

5.3. Let \( b_k \ (k \neq -1) \) be a semisimple eigenvalue of \( A(j) \), i.e. \( k \leq 2J \).

5.3.1. Lemma. In this case \( \mathbf{ad}_j \) contains projective module from category \( C_k \).

Proof. It is enough to prove that the matrix \( A'(j) \) has exactly 3 eigenvectors with eigenvalue \( b_k \) or, equivalently, that the matrix \( A'(j) - b_k \) has corank 3. Let us denote by \( A'(j, k) \) the matrix \( A'(j, k) - b_k \) with the \( i \)-th and \( (i + l) \)-th columns divided by positive numbers \( -(q^i - q^{-i})^2 \) for any \( i \in [1, \ldots, l-1] \). In order to apply the Lemma 5.2 let us put \( A' = A'(j, k) \). Then corresponding matrix \( D \) in notations of the Lemma 5.2 is the matrix \( D(j, k) \) with columns divided by some positive numbers. In order to check the semisimplicity of the matrix \( D \) put \( q = \exp(\pi i \frac{l+1}{l}) \). Then conditions of Lemma 5.3 hold. Indeed, off-diagonal entries of \( D(j, k) \) are either \( (t)q(t + 1)q \) or \( b_{k+2t} - b_k = (t)q(t + k + 1)q \). In both cases this entry is \( (t_1)q(t_2)q \) where \( t_1, t_2 \in [1, \ldots, l - 1] \) and one of \( t_1, t_2 \) is even and another is odd. But if \( q = \exp(\pi i \frac{l+1}{l}) \) and \( t \in [1, \ldots, l - 1] \) then \( (t)q > 0 \Leftrightarrow t \) is odd. Finally note that the entries of \( D \) are the entries of \( D(j, k) \) divided by some positive numbers. The Lemma is proved.

5.3.2. The above Lemma implies that \( \mathbf{ad}_j(k) \) is a sum of a projective module \( P(k) \) and some module \( Y(k, j) \). It follows from 4.10 that \( Y(k, j) \) admits a filtration of length 3 with the following associated graded factors: \( L(k); L(-2 - k) \oplus L(2l - 2 - k); L(k) \).
5.3.3. For any $0 \leq s < l$ an element $pr_j(K^{-1}E)^s$ is an upper singular vector in $\text{ad}_j$. Let us prove that if $s \leq \frac{l-1}{2}$ then $(X - b_j) pr_j(K^{-1}E)^s \neq 0$. Consider the action of this element on $P(J')$. Then $(X - b_j)$ is a surjection onto $L(J') \subset P(j')$ and the desired result follows from the fact that $\dim L(J') \geq \frac{l+1}{2}$. Similarly the vector $pr_j F^s$ is a lower singular vector in $\text{ad}_j$, and $pr_j F^s \not\in M_j$ if $s \leq \frac{l-1}{2}$. For $s = \frac{k}{2}$ we obtain that $\text{ad}_j$ contains a submodule $L(k)$ which does not lie in $M_j$. Indeed, the submodules generated by $pr_j(K^{-1}E)^s$ and $pr_j F^s$ coincide since $\text{ad}_j(K) / M_j(k) \cong L(k) \oplus L(k) \oplus L(-2 - k)$.

5.3.4. It follows that $Y(k, j) = Z(k, j) \oplus L(k)$ where $Z(k, j)$ contains a submodule $L(k)$. Hence $\text{ad}(k) = \bigoplus_{j \in H'} \text{ad}_j(k)$ is a direct sum of a few copies of $P(k)$ and $Y(k) = \bigoplus_{j \in H'} Y(k, j)$ where all the subquotients $L(k)$ of $Y(k)$ are the submodules of $Y(k)$. Now from the autoduality (see 2.6) we see that all the subquotients $L(k)$ are direct summands of $Y(k)$. Thus $Y(k)$ is a direct sum of its simple subquotients. Hence in the case 1.10.2 (c) (i.e. if $k \leq 2J$) we have that

$$\text{ad}_j(k) = P(k) \oplus L(k) \oplus L(k) \oplus L(2l - 2 - k) \oplus L(-2 - k)$$

This completes the proof of the Main Theorem. □

6. Three matrix lemmas

The results of this section were used in the previous sections.

6.1. Let $A$ be a $r \times r$ lower-triangular matrix:

$$A = \begin{pmatrix}
\alpha_1 & 0 & 0 & \ldots & 0 \\
\beta_1 & \alpha_2 & 0 & \ldots & 0 \\
\gamma_1 & \beta_2 & \alpha_3 & \ldots & 0 \\
0 & \gamma_2 & \beta_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \beta_{r-1} & \alpha_r
\end{pmatrix} \quad (6)$$

Lemma. Let $\alpha_i = \alpha_j = \alpha$ for some $i < j$ and $\alpha_k \neq \alpha$ for $k \neq i, j$. The matrix $A$ has 2 different eigenvectors with eigenvalue $\alpha$ iff the determinant of $(j - i) \times (j - i)$ matrix
\[ D = \begin{pmatrix} \beta_i & \alpha_{i+1} - \alpha & 0 & \ldots & 0 \\ \gamma_i & \beta_{i+1} & \alpha_{i+2} - \alpha & \ldots & 0 \\ 0 & \gamma_{i+1} & \beta_{i+2} & \ldots & 0 \\ 0 & 0 & \gamma_{i+2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \gamma_{j-2} & \beta_{j-1} \end{pmatrix} \] (7)

vanishes.

**Proof.** Clear. \( \blacksquare \)

6.2. Suppose \( \alpha_i = \alpha_j \) iff \( i + j = r + 1 \). Let \( A' \) be the following matrix

\[ A' = \begin{pmatrix} A & 0 \\ B' & A \end{pmatrix} \]

where

\[ B' = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

**Lemma.** Suppose the matrix \( A \) above has 2 eigenvectors with eigenvalue \( \alpha_i \). Suppose the matrix \( D \) is semisimple of corank 1. Then the matrix \( A' - \alpha_i \) has corank 3.

**Proof.** It suffices to consider the case \( i = 1, j = r, \alpha_1 = \alpha_r = 0 \). Deleting two rows and columns consisting of zeros we obtain a matrix

\[ D' = \begin{pmatrix} D & 0 \\ E & D \end{pmatrix} \]

where \( E \) is a unit matrix. We have to prove that corank of \( D' \) is equal to 1. Let \( \text{Im}(D) \) (resp. \( \text{Im}(D') \)) denote the linear space generated by columns of \( D \) (resp. \( D' \)). Let \( pr : \text{Im}(D') \rightarrow \mathbb{C}^{r-1} \) be a map forgetting the last \( r - 1 \) coordinates. Then \( pr(\text{Im}(D')) = \text{Im}(D) \) has dimension \( r - 2 \). Let us prove that \( \text{Ker}(pr) \) has dimension \( r - 1 \). Indeed, \( \text{Ker}(pr) \supset \text{Im}(D) \) and \( \text{Ker}(pr) \) contains the kernel of operator \( D \). Since \( D \) is semisimple \( \text{Im}(D) \not\supset \text{Ker}(D) \). The result follows. \( \Box \)

6.3. **Lemma.** Suppose \( D \) is a real matrix, and all the off-diagonal entries are negative. Then \( D \) is semisimple.

**Proof.** It is easy to see that conjugating matrix \( D \) by some diagonal matrix we can obtain a symmetric matrix. \( \Box \)
References

[BFS] R. Bezrukavnikov, M. Finkelberg, V. Schechtman, Localization of u-modules. V. The modular structure on the category $\mathcal{FS}$, Preprint, to appear.

[Ke] T. Kerler, Mapping class group actions on quantum doubles, Comm. Math. Phys. 168 (1995), 353-388.

[Lu] G. Lusztig, Finite-dimensional Hopf algebras arising from quantized universal enveloping algebras, J. of AMS 3 (1990), 257-296.

[LM] V. Lyubashenko, S. Majid, Braided groups and quantum Fourier transform, Preprint DAMTP/91-26.

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