Natural vibrations of clamped curved nano-beams and nano-arches

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ABSTRACT. Numerical solution of the clamped-clamped (C-C) nanoarches based on the non-local theory of elasticity is analysed. The present study is helpful to understand the mechanical behaviour of the nanostructures. The nanoarch under consideration has constant thickness and dimension of the cross-section and is weakened by crack-like defects. It is assumed that the crack is stationary and the mechanical behaviour of the nanoarch can be modelled by the Eringen’s non-local theory of elasticity. The influence of cracks on the natural vibration is prescribed with the aid of additional local compliance at the weakened cross-section. An algorithm to determine the eigen frequencies of nanoarches is developed. Present study reveals that there are significant effects of crack length, non-local parameters and radius of the arch on eigen frequency of the nanoarches. The final results are obtained numerically with the help of computer and presented graphically. The results are also compared with those presented by other researchers in the form of a table.

1. Introduction

Vibration is an integral part of most of the mechanical structures. Therefore, many researchers are currently investigating the frequency of different nanomechanical structures to tackle the effects of resonance by taking into account the non-local effect. Materials change their physical and thermal properties when their dimension goes up to the nano level. The classical theory of elasticity is unable to describe such changes in material properties. This is because, during the development of the classical theory of elasticity, the speculation of molecular objects was avoided. Therefore, the non-local theory of elasticity is applied to study the vibration of nanostructures and it has been accepted by many researchers. In the non-local theory of elasticity, it is assumed that the stress state of the body at a given point depends on the stress state of each point of the structure. However, within the classical theory of elasticity, the stress state of the body depends only on the given point. The non-local theory of elasticity was developed by Eringen and
his co-workers to model the surface waves and the screw dislocation. In the present study, the vibration of nanoarches with a defect is analyzed by formulating the problem implying a non-local effect which has been taken into account by many researchers. Paddieson \[3\] applied the simplified version of Eringen’s non-local theory of elasticity Eringen1\[1\] to study the behavior of nanobeams as an actuator in the minor structures. Reddy \[9\] has extended several beam theories when the constitutive relations are defined under the umbrella of the non-local theory of elasticity. The non-local theory is used by Lu et al. \[5\] and Li et al. \[4\] and Thai \[6\] to obtain the solution to the problems of the vibration behavior of nanobeams. Ganapathi et al. \[7\] study the vibration analysis of curved nanobeams based on the non-local Wang\[8\] theory of elasticity and inspect the effects of different structural theories on the vibration of nanobeams. The aforementioned literature illustrates the importance of the non-local theory of elasticity to analyze the dynamic behavior of the nanoscale structures.

Arches have great importance in the field of engineering because they support the load from above. In practice, it has been seen that crack like defects can generate local flexibility that ultimately cast an effect on the dynamic behavior of the whole system. Therefore, the study of nanoarches with modeled defects has great importance for the engineers to build nanorobots, and microcomputers. The results obtained from the research work will also be very useful for the engineers and the designers in the field of micro electromechanical system (MEMS) and nano electromechanical system (NEMS). Chiang \[10\] has previously has formulated the model to study rotary inertia due to the torsional vibration and effect of shear deformation of arches and derives the natural frequency of arches with different boundary conditions and central angles. The present work is the extension of the work done by Lellep \[11\] where they analysed the behavior of the natural frequency of arches with a crack-like defect of different angles and with crack. Whereas, in this research work, a simply supported nanoarch of constant thickness bearing a crack is to be analyzed.

2. Formulation of the problem

The constitutive equation defined by Hook’s law containing stress components $\sigma_{ij}$ and non-local stress components $\sigma_{ij}^n$ is defined as

$$\sigma_{ij}^n - \eta \nabla^2 \sigma_{ij}^n = \sigma_{ij}^c$$  \hspace{1cm} (1)

where $\nabla$ is the Laplacian operator and $\eta$ denotes material constant defined as

$$\eta = e_0 a$$  \hspace{1cm} (2)

Where $a$ is an internal characteristic length (e.g. C-C bond length, granular distance, lattice parameter) and $e_0$ is a calibration constant appropriate to
each material. The relationship between bending moment can be written by using (1) as

\[ M - \eta \frac{\partial M^2}{\partial x^2} = EI\kappa \]  

In (3), \( M, E, \) and \( I \) represent bending moment, Young’s modulus, and moment of inertia of the cross section respectively, and

\[ \kappa = -\frac{1}{R^2}(w + w'') \]  

In (4), \( R \) and \( w \) denote radius and transverse deflection, respectively.

Equilibrium equations for a beam element can be written by denoting shear force with \( Q \) and bending moment with \( M \)

\[ M'' + R^2(p - \mu \ddot{w}) = 0 \]  

Where \( Q = M' \), \( p \) stands for external load (which is considered zero in the case) and is the mass per unit length of the element. Whereas, dot and prime denote derivative with respect to time \( t \) and axial displacement \( x \). By solving Eqs. (3) and (5)

\[ \frac{-EI}{R^2} (w'''' + 2w'' + w) + \mu R^2(\eta w'' - \ddot{w}) = 0 \]  

Figure 1. Clamped-clamped nanoarch with defects.
For separating the spatial and temporal variables, the following shape of modal displacement function is assumed.

\[ w(x, t) = W(x) \sin \omega t \quad (7) \]

Where \( \omega \) is the natural frequency. By replacing (7) into (6) the following equation is obtained.

\[ W^{IV} + (2 + K \eta)W'' + (1 - K)W = 0 \quad (8) \]

Where

\[ K = \frac{\mu R^4 \omega^2}{EI} \]

Eq. (8) is a fourth order linear partial differential equation and it is solved by using separation of variable method. The solution of the above equation is written as follows.

\[ W = A_0 \cosh \mu x + B_0 \sinh \mu x + C_0 \cos \nu x + D_0 \sin \nu x; \quad (0, \alpha) \]
\[ W = A_1 \cosh \mu x + B_1 \sinh \mu x + C_1 \cos \nu x + D_1 \sin \nu x; \quad (0, \beta) \quad (9) \]

Where \( A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1 \) are constants and

\[ \mu = \pm \sqrt{\frac{(2 + K \eta) + A}{2}} \]
\[ \nu = \pm \sqrt{\frac{(2 + K \eta) + A}{2}} \]

\[ A = \sqrt{(2 + K \eta)^2 - 4(1 - K)} \]

The equation of the motion (8) is integrated with following boundary conditions when the nanoarch is simply supported.

\[ W(0) = W''(0) = 0 \]
\[ W(\beta) = w''(\beta) = 0 \quad (10) \]

By implying boundary conditions in (9), the solution of the equation is reduced in the form of

\[ W = B_0 \sinh \mu x + D_0 \sin \nu x; \quad (0, \alpha) \]
\[ W = A \sinh \mu (x - \beta) + C \sin \nu (x - \beta); \quad (0, \beta) \quad (11) \]

Where \( A = \frac{-A_1}{\sinh \nu \beta} \) and \( C = \frac{-C_1}{\sin \nu \beta} \)

Details will be written with the help of supervisor.

\[ [W'] = p_- (w''(\alpha) + W(\alpha)) \]
\[ [W] = 0 \]
\[ [M] = 0 \]
\[ [Q] = 0 \]

where

\[ p_- = 6 \pi h/(1 - \nu^2) f(s) \]
\[ f(s) = 1.86s^2 - 3.95s^3 + 16.37s^4 - 34.23s^5 + 76.81s^6 - 126.93s^7 + 172s^8 - 143.97s^9 + 66.56s^{10} \quad (12) \]
By using intermediate boundary conditions in (11), the following four equations are obtained.

\[ B_0 \sinh \mu \alpha + D_0 \sin \nu \alpha - A \sinh (\alpha - \beta) + C \sin (\alpha - \beta) = 0 \] (13)

\[-B_0 \mu \cosh \mu \alpha + p(1 + \mu^2) \sinh \mu \alpha - D_0 \nu \cos \nu \alpha + p(1 - \nu^2) \sin \nu \alpha + C \sin (\alpha - \beta) + A \sinh (\alpha - \beta) = 0 \] (14)

\[ B_0 \mu^2 \sinh \mu \alpha - D_0 \nu^2 \sin \nu \alpha - A \mu^2 \sinh (\alpha - \beta) + C \nu^2 \sin (\alpha - \beta) = 0 \] (15)

\[ B_0 \mu^3 \sinh \mu \alpha - D_0 \nu^3 \sin \nu \alpha - A \mu^3 \sinh (\alpha - \beta) + C \nu^3 \sin (\alpha - \beta) = 0 \] (16)

Eqs. (11), (12), (13), (14) are the system of linear equations whose solution exists if the determinant of the coefficients of the constants is equal to zero. This system is solved by using computer software MATLAB. As \( E' = E \) for the plane stress state and \( E' = \frac{E}{1-\nu^2} \) for plane strain state. One can use the function \( F(s) \) in the form

\[ F(s) = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.8s^4 \] (17)

Another shape function often used by researchers is

\[ F(s) = \sqrt{\tan \psi \frac{\psi}{\cos \psi}} [0.923 + 0.199(1 - \sin \psi)^4] \] (18)

where \( \psi = \frac{\pi s}{2} \).

3. Numerical results

Numerical results are presented for nano-arches simply supported at both ends in fig 2-6. Here the nano-arches with a single step whereas \( R = 30 \text{nm} \), \( h_0 = 10 \text{nm} \), \( h_1 = 20 \text{nm} \), the material constant are \( E = 7 \times 10^{11} \text{Pa} \), \( \nu = 0.3 \)

![Figure 2. Natural frequency Vs central angle of the nano-arch](image-url)
Figure 3. Natural frequency Vs central angle of the nano-arch

Figure 4. Natural frequency Vs material constant

In figure (2) the natural frequency of the nano-arch versus central angle is drawn by taking into account the material constant. It is clear from the figure (2) that natural frequency increases as the central angle $\beta$ is increased. However, the natural frequency decreases with the increase of material constant $\eta$.

In figures (3) again natural frequency against the central angle $\beta$ is depicted by taking into account the depth of the crack in the nanoarch. It can be seen from the figure (3) that the depth of the crack has a huge effect on the natural frequency because the frequency keeps increasing as the depth is increased. Nonetheless, the effect of $\beta$ on natural frequency is same as in figure (2). In figure (3) the nano-arch with the central angle $\beta = 1 \text{rad}$ is
treated. Different curves in figure (3) are correspond to the crack extensions $s = 0$, $s = 0.1$, $s = 0.2$, $s = 0.3$, $s = 0.4$, $s = 0.5$, $s = 0.6$, $s = 0.7$ respectively. It can be seen from the figure (3) that the lowest values of the natural frequency correspond to the arch without any cracks. In figure (4) the natural frequency is depicted against the material constant $\eta$. It is revealed from the figure that natural frequency decreases as we increase the material constant $\eta$. However, the natural frequency decreases with the decreasing value of radius $R$.

The relationship between the natural frequency and the radius $R$ of the nanoarch is shown in figure (5) by taking into account the crack location $\alpha$. The natural frequency decreases as we increase the radius of the nanoarch.
It reveals from the figure (5) that the smaller $\alpha$ the larger the natural frequency of the nano-arch.

In figure (6) natural frequency is drawn against the radius of the nanoarch but two different functions are compared.

### 4. Concluding remarks

A method of vibration analysis of nanoarches with simply supported end condition is developed in the frameworks of Eringen’s nonlocal theory of elasticity. Here, the nanoarch we have under consideration has constant thickness and is weakened by stable cracks or crack-like defects. The additional compliance produced by the defect is calculated according to the method of Dimarogonas. The calculations carried out revealed that defects affect the Eigen frequencies of nanoarches. It was shown that the maximal values of Eigen frequencies have nanoarches with the defects. The matter that cracks reduce natural frequencies of beams is recognized at the macro-level, as well. Moreover, the effects of central angle, radius, and material constant on the natural frequencies of the nanoarches with defects are also
analysed and plotted in different figures. Therefore, an algorithm to determine the Eigen frequencies of nanoarches is developed which will be useful to examine the mechanical behaviour of the nanostructures.

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