Wigner’s little group as a generator of gauge transformations

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Abstract

The role of Wigner’s little group, as an abelian gauge generator in different contexts, is studied.

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In his paper On unitary representations of the inhomogeneous Lorentz group, published in 1932 [1], Eugene Paul Wigner introduced the concept of little group and used it to classify the elementary particles on the basis of their helicity/spin quantum numbers. The topic of little group was a personal favorite of Prof. Wigner. During the last years of his life, he wrote a series of seven papers in collaboration with Y. S. Kim elaborating the geometrical meaning of the little group [2]. Meanwhile there were also studies regarding gauge generating aspects of the little group [3, 4]. Recently, we have found some more interesting and hitherto unknown facets of the little group in generating the gauge transformations in various abelian gauge theories, including topologically massive ones [5, 6, 7, 8, 9]. In this talk, I describe our work in this direction, highlighting the major results [2]. Further details can be found from the references given.

Wigner’s little group is defined as the subgroup of homogeneous Lorentz group that leaves the energy-momentum vector of a particle invariant: \( W_{\mu,\nu} k_{\mu} = k_{\mu} \). In 3+1 dimensions, the little group for a massive particle is the rotation group \( SO(3) \). On the other hand, for a massless particle, the little group is the Euclidean group \( E(2) \) which is a semi-direct product of \( SO(2) \) and \( T(2) \) - the group of translations in the 2-dimensional plane. As is well known, both the rotational groups \( SO(3) \) and \( SO(2) \) determine the classification of particles on the basis of

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2Notation: Greek alphabets \( \mu, \nu \) etc denote the space-time indices in 3+1 dimensions, letters \( a, b, c \) etc stands for 2+1 dimensions and those from the middle of the alphabet, i.e. \( i, j, k \) etc stands for 4+1 dimensions.
their spin quantum numbers. One can obtain the little group $E(2)$ as a particular limit of the rotation group $SO(3)$ by Inonu-Wigner group contraction. However, while the significance of rotational groups was evident, the role of translational group remained a mystery for a long time. Weinberg and Han et. al. noticed that the translational group acts as a gauge generator in Maxwell theory. Following Weinberg, one can find the explicit representation of Wigner’s little group which leaves invariant the 4-momentum $k^\mu = (\omega, 0, 0, \omega)^T$ of a photon of energy $\omega$ moving in the $z$-direction, to be $W_4(p, q; \phi) = W(p, q)R(\phi)$, where

$$W(p, q) = W_4(p, q; 0) = \begin{pmatrix} 1 + \frac{p^2 + q^2}{2} & p & q & -\frac{p^2 + q^2}{2} \\ p & 1 & 0 & -p \\ q & 0 & 1 & -q \\ \frac{p^2 + q^2}{2} & p & q & 1 - \frac{p^2 + q^2}{2} \end{pmatrix}$$

is a particular representation of the translational subgroup $T(2)$ of the little group and $R(\phi)$ represents a $SO(2)$ rotation about the $z$-axis. Note that the representations $W(p, q)$ and $R(\phi)$ of the translation and rotation groups satisfy the relations $W(p, q)W(\bar{p}, \bar{q}) = W(p + \bar{p}, q + \bar{q})$ and $R(\phi)R(\bar{\phi}) = R(\phi + \bar{\phi})$.

We begin the discussion by showing that (1) acts as a gauge generator for the Maxwell theory. The free Maxwell theory has the equation of motion $\partial_\mu F^{\mu\nu} = 0$ which follows from the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gauge field $A^\mu(x)$ for a single mode can be written as $A^\mu(x) = \epsilon^\mu(k)e^{ik\cdot x}$ suppressing the positive frequency part without any loss of generality. In terms of the polarization vector $\epsilon^\mu$, the gauge transformation $A^\mu(x) \rightarrow A^\mu_\prime = A^\mu + \partial_\mu f$ (where $f(x)$ is an arbitrary scalar function) is expressed as $\epsilon_{\mu}(k) \rightarrow \epsilon^\prime_\mu = \epsilon_\mu(k) + if(k)k^\mu$ where $f(x)$ has been written as $f(x) = f(k)e^{ik\cdot x}$. The equation of motion, in terms of the polarization vector, will now be given by $k^2 \epsilon^\mu - k^\mu k_{\nu} \epsilon^{\nu} = 0$. The massive excitation corresponding to $k^2 \neq 0$ leads to the solution $\epsilon^\mu \propto k^\mu$ which can be gauged away. For massless excitations ($k^2 = 0$), the Lorentz condition $k_{\mu} \epsilon^{\mu} = 0$ follows immediately from the momentum space equation of motion. Taking $k^\mu = (\omega, 0, 0, \omega)^T$, corresponding to a photon of energy $\omega$ propagating in the $z$ direction, and using the Lorentz condition, one can easily show that $\epsilon^\mu(k)$ is gauge equivalent to the maximally reduced form $\tilde{\epsilon}^\mu(k) = (0, \epsilon^1, \epsilon^2, 0)^T$ displaying the two transverse degrees of freedom corresponding to $\epsilon^1$ and $\epsilon^2$. Under the action of the translational group $T(2)$ in (1), this polarization vector transforms as follows: $\epsilon^\mu \rightarrow \epsilon^\prime^\mu = W^\mu_\nu(p, q)\epsilon^{\nu} = \epsilon^\mu + \left(\frac{p^1 + q^2}{\omega}\right)k^\mu$. It is now obvious that, this can be identified as a gauge transformation by choosing $f(k)$ suitably. This shows that the translational subgroup $T(2)$ of Wigner’s little group for massless particles acts as gauge generator in free Maxwell theory. Similar conclusions hold also for Kalb-Ramond theory in 3+1 dimensions.

Next, consider the $B \wedge F$ theory, which is a topologically massive gauge theory described by the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{m}{6} \epsilon^{\mu\nu\lambda\rho}H_{\mu\nu\lambda}A_\rho$ based on the vector field $A_\mu$, from which $F_{\mu\nu}$ is constructed and the antisymmetric tensor field $B_{\mu\nu}$, which defines $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$.

\(^3\)This method of obtaining the maximally reduced form of polarization vector/tensor of a theory is henceforth called the ‘plane wave method’.
It is invariant under the combined gauge transformations $A_{\mu}(x) \rightarrow A_{\mu} + \partial_{\mu} f$ and $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_{\mu} F_{\nu}(x) - \partial_{\nu} F_{\mu}(x)$ where $f(x)$ and $F_{\mu}(x)$ are arbitrary functions. Employing the plane wave method one can show that the massless excitations of $B \wedge F$ theory are gauge artifacts and the maximally reduced form of the polarization vector and tensor corresponding to the massive physical excitations are given by

$$\{\varepsilon^{\mu\nu}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c & -b \\ 0 & -c & 0 & a \\ 0 & b & -a & 0 \end{pmatrix}, \quad \varepsilon^\mu = -i \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} \quad (2)$$

with the duality relation $\varepsilon^{\mu\nu} \varepsilon_{\mu} = 0$ between them. Correspondingly, the momentum vector is $p^\mu = (m, 0, 0, 0)^T$. Now, unlike in the case of Maxwell and KR theories, here the representation $W(p, q) \quad (3)$ fails to be the generator of gauge transformations.

However one can arrive at the representation of the group that generates gauge transformations in $B \wedge F$ theory by considering the action of the matrix

$$D(p, q, r) = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3)$$

(where $p, q, r$ are real parameters) on the polarization vector and tensor in (2):

$$\varepsilon^\mu \rightarrow \varepsilon'^\mu = D^\mu_{\nu}(p, q, r)\varepsilon^\nu = \varepsilon^\mu - \frac{i}{m} (pa + qb + rc)p^\mu \quad (4)$$

$$\{\varepsilon_{\mu\nu}\} \rightarrow \{\varepsilon'_{\mu\nu}\} = D(p, q, r)\{\varepsilon_{\mu\nu}\} D^T(p, q, r) = \{\varepsilon_{\mu\nu}\} + (\Delta \varepsilon)_{\mu\nu} \quad (5)$$

where $(\Delta \varepsilon)_{\mu\nu} = k_{\mu} F_{\nu}(k) - k_{\nu} F_{\mu}(k)$ and $F_{\mu}(k)$ is an arbitrary function. It is obvious that these reproduce the gauge transformations in the $B \wedge F$ theory.

The group, of which $D(p, q, r)$ is a representation, can be found by noticing that $D(p, q, r) \times D(p', q', r') = D(p + p', q + q', r + r')$ which is the composition rule for the 3-dimensional translational group $T(3)$. Moreover, the generators $P_1 = \frac{\partial D(p, 0, 0)}{\partial p}$, $P_2 = \frac{\partial D(0, q, 0)}{\partial q}$ and $P_3 = \frac{\partial D(0, 0, r)}{\partial r}$ of $D(p, q, r)$ are the same as those of $T(3)$ as can be seen from their Lie algebra $[P_1, P_2] = [P_2, P_3] = [P_3, P_1] = 0$. Thus the gauge generator for $B \wedge F$ theory is the representation of the translational group $T(3)$. One may also notice that there are three different embeddings of $T(2)$ within $T(3)$ and they preserve the 4-momentum of massless particles moving in the three different spatial directions.

Now we describe a method by which one can derive the gauge generating representation of translational group for topologically massive theories from the corresponding representation for ordinary gauge theories living in one higher space-time dimensions. The starting point of this dimensional descent method is to note that one can interpret a massive particle in $d$-dimensions as a massless particle in $d + 1$-dimensions, with the mass being considered as the momentum.
component along the additional dimension $[10]$. In its content, dimensional descent is related to the Inonu-Wigner group contraction.

For example the momentum and polarization vectors of Maxwell theory in 5-dimensions are given by $p^i = (\omega, 0, 0, 0, \omega)^T$ and $\varepsilon^i = (0, a_1, a_2, a_3, 0)^T$. With the identification of $\omega$ with the mass $m$ and the deletion of the last columns (by the applying the projection operator $\mathcal{P} = \text{diagonal}(1, 1, 1, 1, 0)$), these vectors respectively becomes the rest frame momentum and polarization vectors of Proca model in 4-dimensions which is equivalent to $B \wedge F$ theory $[5, 7]$. The 5-dimensional analogue $W_5(p, q, r)$ of (1) generate gauge transformations in Maxwell theory in that space-time dimension $[7]$. That is, $\varepsilon'^i = W_5(p, q, r)^i_j \varepsilon^j = \varepsilon^i + \frac{pq_1 + qr_2 + rp_3}{\omega} p^i$. Using the projection operator $\mathcal{P}$ one can project out the extra 5th dimension: $\delta \tilde{\varepsilon}^\mu = \mathcal{P} \delta \varepsilon^i = \frac{pq_1 + qr_2 + rp_3}{\omega} \delta \varepsilon^\mu$. This is precisely the gauge transformation of the polarization vector in $B \wedge F$ theory in 4-dimensions. From the form of this transformation, one can readily read off the matrix representation of the group that generate the gauge transformation of the $B \wedge F$ theory in 4-dimensions to be (3). Dimensional descent from 4 to 3 dimensions yields analogous results for 2+1 dimensional Maxwell-Chern-Simons theory $[6, 7]$.

The above considerations are also valid for linearized gravity theories, both the usual as well as the topologically massive ones $[8]$ . The role of Wigner’s little group in generating the star gauge invariance in noncommutative gauge theories can be a possible extension of the present work. It is also of interest to construct and study the equivalent of the little group in Ads space where there is a novel gauge transformation connected to partially massless theories, as reported in $[11]$.

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