The Chern-Gauss-Bonnet formula with nonnegative $Q$ curvature

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Abstract: Let $(\mathbb{R}^n, e^{x^2}|dx|^2)$ be a complete even dimensional manifold with nonnegative $Q$ curvature and $Q_g e^{nu} \in L^1(\mathbb{R}^n)$. Suppose $C_0 e^{-|x|^\beta} \leq Q_g(x) \leq C_0 |x|^{\gamma}$ at infinity for some $\gamma > 0$ and $\beta \in (0, 1)$, then

$$\int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy \leq c_n$$

and

$$\chi(\mathbb{R}^n) - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy = \lim_{r \to \infty} \frac{|\partial B_r(0)|_{g}^{\frac{n}{2}}}{n\omega_{n-1}|B_r(0)|_g},$$

where $c_n = 2^{n-2}(\frac{n-2}{2})!\pi^{n/2}$ and $\omega_{n-1} = |S^{n-1}|$. Especially, in four dimension, we give a simple proof without the upper bound about $Q_g$.

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1 Introduction

The Gauss-Bonnet formula plays an important role in differential geometry and is related to many geometric concepts. Its classical version can be expressed as follows: If $(M, g)$ is a compact without boundary, then

$$\int_M K_g dV_g = 2\pi \chi(M),$$

where $K_g$ is the Gauss curvature. For complete surfaces, Huber [13] and Finn [11] proved that if $K_g$ is absolutely integral, then

$$\int_M K_g dV_g \leq 2\pi \chi(M) \quad (1.1)$$

and

$$\chi(M) - \frac{1}{2\pi} \int_M K_g dV_g = \sum_{i=1}^{k} \nu_i, \quad (1.2)$$

where $\nu_i$ is the isoperimetric ratio at the $i$-th end of $(M, g)$, i.e.,

$$\nu_i = \lim_{r \to \infty} \frac{|\partial B_r|_g}{4\pi |B_r|_g}.$$

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where $B_r(0)$ contains a compact set $K_i$ and $M\setminus K_i$ is diffeomorphic to $\mathbb{R}^2\setminus B_1(0)$.

Subsequently, Chang, Qing and Yang [3, 4] successfully generalized (1.1) and (1.2) to four dimensions. They showed that if $(\mathbb{R}^4, e^{2u}|dx|^2)$ is complete with absolutely integrable $Q_g e^{4u}$, and supposed that $R_g$ is nonnegative at infinity, then
\[
\int_{\mathbb{R}^4} Q_g(x)e^{4u(x)}dx \leq 4\pi^2
\]  
and
\[
1 - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} Q_g(x)e^{4u(x)}dx = \lim_{r \to \infty} \frac{\|\partial B_r(0)\|^2_{Q_g}}{4(2\pi)^{n/2}|B_r(0)|_{g}}.
\]  
(1.4)

For $Q$ curvature, there are many papers concern it, we only list some famous references [1, 5, 6, 9].

Recently, Wang [21] has made significant progress on isoperimetric inequalities in higher dimensions. Using a Sobolev inequality with weights, he has shown that assuming $(\mathbb{R}^n, e^{2u}|dx|^2)$ is complete and normal, i.e.,
\[
u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y)e^{nu(y)}dy + C,
\]
where $c_n = 2^{n-2} (\frac{n-2}{2})! \pi^{n/2} = \frac{1}{2\omega_n} Q_g e^{nu}$ and $\omega_n = |\mathbb{S}^n|$. Moreover if
\[
\alpha_1 := \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy < c_n
\]
and
\[
\alpha_2 := \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy < +\infty,
\]
then there exist $C = C(n, \alpha_1, \alpha_2)$ such that for any bounded smooth domain $\Omega$ in $\mathbb{R}^n$,
\[
|\Omega|_g \leq C(n, \alpha_1, \alpha_2)|\partial \Omega|_{\gamma}^{\frac{n}{n-1}}.
\]

By studying the $n$-Laplace equation, a general Huber type theorem have been established by S. Ma and Qing [19].

In this paper, we consider the manifold is $(\mathbb{R}^n, g = e^{2u}|dx|^2)$ with nonnegative $Q$ curvature, $n = 2m$ and denote
\[
\alpha = \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g dV_g = \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy
\]  
(1.5)
and
\[
f(y) = Q_g(y)e^{nu(y)},
\]  
(1.6)
then we know
\[
(-\Delta)^m u = 2Q_g e^{nu} \quad \text{in} \quad \mathbb{R}^n.
\]  
(1.7)

Now we state our first theorem, the following Theorem 1.1 confirms their idea on the Page 526 of Chang, Qing and Yang [3].

**Theorem 1.1** Suppose $(\mathbb{R}^4, e^{2u}|dx|^2)$ is a complete and satisfies
\[
\begin{cases}
Q_g \geq 0 \text{ and } 0 < C_0 e^{-|x|^{\beta}} \leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } 0 < \beta < 1, \\
Q_g(x)e^{4u(x)} \in L^1(\mathbb{R}^n), \\
u(x) = o(|x|^2).
\end{cases}
\]
Then the metric is normal, i.e,
\[
u(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y)e^{4u(y)}dy + C.
\]
Remark 1.1 Combining the results of [3] and [21], we know the following conclusions are valid under the above assumption.

1. \[
\int_{\mathbb{R}^4} Q_g(y)e^{4u(y)}dy \leq 4\pi^2
\]

2. \[
1 - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} Q_g(y)e^{4u(y)}dy = \lim_{r \to \infty} \frac{\|\partial B_r(0)\|_g^4}{4(2\pi)^{4/3}\|B_r(0)\|_g^4}
\]

3. If \(\int_{\mathbb{R}^4} Q_g(y)e^{4u(y)}dy < 4\pi^2\), there exist \(C = C(\alpha)\) such that for any bounded smooth domain \(\Omega\) in \(\mathbb{R}^4\)

\[|\partial \Omega| \leq C(\alpha)|\partial \Omega|_g^4\]

From the viewpoint of PDE, the above theorem can be expressed as follows: Under some curvature decay assumption, the solution of the PDE (1.7) has some asymptotic behavior. In two dimension, Cheng and Lin [2] demonstrated that if \(u\) solves

\[-\Delta u = K(x)e^{2u(x)}\]

with \(K(x)e^{2u(x)}\) is absolutely integral and

\[e^{-|x|^\beta} \leq K(x) \leq |x|^\gamma\]

for some \(\beta \in (0,1)\) or

\[|x|^{-\gamma} \leq -K(x) \leq |x|^{-\gamma}\]

for \(|x| \gg 1\), then

\[
u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \left(\frac{|y|}{|x-y|}K(y)e^{2u(y)}\right)dy + C\quad (1.8)
\]

and

\[
u(x) = -\alpha \log |x| + o(\log |x|).\quad (1.9)
\]

In four dimension, if \(Q_g\) is a constant in (1.7), Lin [17] firstly gave a systematic approach to study the higher order critical exponent elliptic equation. In high dimension, the solutions are classified by Chang and Yang [7] and Wei and Xu [22] under \(u(x) = o(|x|^2)\) constraint. Afterwards, L. Martinazzi [18] clarified the details and gave many nice improvements. For more general \(Q\), you can refer [8, 12].

In this paper, our motivations are twofold. First, we hope to generalize the results of Chang, Qing and Yang [3] to high dimension without the scalar curvature assumption. Second, we want to give the almost sharp condition to characterize the normal metrics. So, what are the appropriate assumptions about \(Q_g\)? From many excellent papers mentioned above, we conclude the following assumptions:

1. \(e^{-|x|^\beta} \leq Q_g(x)\) for some \(\beta \in (0,1)\) if \(|x| \gg 1\). The necessity of this assumption can be seen in the following example, which was proposed by Chang, Qing and Yang [3]. Let \(u(x) = \log \frac{2}{1+|x|^2} + |x|^2\), you can check the completeness of the metric, then

\[
\frac{1}{c_n} \int_{\mathbb{R}^n} Q_g dV_g = \frac{1}{c_n} \int_{\mathbb{R}^n} (-\Delta)^m u = 2 > 1.
\]

In the above example \(Q_g = e^{-n|x|^3}Q_{g,n}\), which means that the decay of \(Q_g\) does not seem to be too fast. Moreover, in two dimension, Cheng and Lin [2] have pointed out that \(\beta < 1\) is necessary for (1.8) and (1.9). Moreover, you can find many evidences in this paper (especially in Lemma 3.3, Remark 4.1 and Remark 4.2).
(2) \( Q_{g}e^{nu} \in L^{1}(\mathbb{R}^{n}) \). This is a natural assumption for nonnegative \( Q_{g} \) curvature. In fact, this condition will imply that \( Q_{g} \) has some decay at infinity. Formally speaking,

\[
Q_{g}e^{nu} \sim \frac{Q_{g}}{|x|^{n}} \in L^{1}(\mathbb{R}^{n}).
\]

Now, we begin to state the second theorem.

**Theorem 1.2** Suppose \((\mathbb{R}^{n}, g = e^{2u}|dx|^2)\) is complete and \( u \) is a solution of

\[
(-\Delta)^{m}u(x) = 2Q_{g}(x)e^{nu(x)}
\]

in \( \mathbb{R}^{n} \) and satisfies

\[
\begin{cases}
C_{0}e^{-\beta|x|^2} \leq Q_{g}(x) \leq C_{0}|x|^\gamma \text{ for } |x| \gg 1, \text{ where } \gamma > 0 \text{ and } \beta \in (0, 1). \\
Q_{g} \geq 0 \text{ and } Q_{g}(x)e^{nu(x)} \in L^{1}(\mathbb{R}^{n})
\end{cases}
\]

Then,

(1) \( u \leq C \), which \( C \) may depend on \( u \).

(2) The metric is normal, i.e.,

\[
u(x) = \frac{1}{c_{n}} \int_{\mathbb{R}^{n}} \log \frac{|y|}{|x-y|}Q_{g}(y)e^{nu(y)}dy + C. \quad (1.10)
\]

and

\[
u(x) = -\alpha \log |x| + o(\log |x|) \quad (1.11)
\]

for \( |x| \gg 1 \).

(3)

\[
\int_{\mathbb{R}^{n}} Q_{g}(y)e^{nu(y)}dy \leq c_{n}. \quad (1.12)
\]

**Remark 1.2** The (1.10) and (1.11) are the high dimension versions with respect to (1.8) and (1.9). We also note that if \( R_{g} \) is nonnegative at infinity, then the inequality (1.12) has been proved by H. Fang [10].

We consider the following typical examples which satisfies all our assumptions. Let

\[
u_{\theta}(x) = \theta \log \frac{2}{1 + |x|^2} \quad \text{for } \theta \in (0, \frac{1}{2}],
\]

then \( g_{\theta} = e^{2nu}|dx|^2 \) is complete. Clearly,

\[
Q_{g_{\theta}} = \frac{1}{2} e^{-nu_{\theta}} (-\Delta) u_{\theta} = \theta Q_{g_{\theta^{n}}} \left( \frac{2}{1 + |x|^2} \right)^{n(1-\theta)}
\]

and

\[
\int_{\mathbb{R}^{n}} Q_{g_{\theta}}e^{nu_{\theta}} = \theta \omega_{n} Q_{g_{\theta^{n}}} \leq c_{n}.
\]

Following the strategy of [3], we first suppose that \( u \) is radial function and try to establish (1.13) and (1.14). Next, we reduce the general case into the radial case. Finally, we prove that:
Theorem 1.3 Let \((\mathbb{R}^n, e^{2u}|dx|^2)\) be a complete even dimensional manifold with nonnegative \(Q\) curvature and \(Q_\Delta e^{nu} \in L^1(\mathbb{R}^n)\). Suppose \(C_0 e^{-|x|^{\beta}} \leq Q_\Delta(x) \leq C_0 |x|^\gamma\) at infinity for some \(\gamma > 0\) and \(\beta \in (0, 1)\), then
\[
\int_{\mathbb{R}^n} Q_\Delta(y)e^{nu(y)}dy \leq c_n \tag{1.13}
\]
and
\[
1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_\Delta(y)e^{nu(y)}dy = \lim_{r \to \infty} \frac{|\partial B_r(0)|_{g}^{\frac{n}{n-1}}}{\omega_{n-1} |B_r(0)|_{g}}, \tag{1.14}
\]

Remark 1.3 With the same assumption as Theorem 1.3 and suppose \(\int_{\mathbb{R}^n} Q_\Delta(y)e^{nu(y)}dy < c_n\), the isoperimetric inequality also holds, i.e., there exists \(C(n, \alpha)\) such that for any bounded smooth domain \(\Omega\) in \(\mathbb{R}^n\)
\[
|\Omega| \leq C(n, \alpha)|\partial \Omega|^\frac{n}{n-1}.
\]

The organization of this paper is as follows. In Section 2, we establish some crucial estimates for the integral equation. In Section 3, we give a simple proof of normal metrics under almost sharp condition. In Section 4, we focus on how to prove high-dimensional normal metric. We obtain important upper bound estimates (The (1) in Theorem 1.2) and asymptotic formula (The (2) in Theorem 1.2). In Section 5, following the method of [3], we first establish the radial version of Theorem 1.3, then we reduce the general case into the radial case.

2 Preliminaries

2.1 Weak harnack inequality of singular integral

In this section, we consider the integral equation
\[
v_k(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\sigma}dy.
\]

Firstly, we introduce the conception of \(A_1\) weight. A nonnegative function \(w \in A_1\), for any ball \(B \subset \mathbb{R}^n\)
\[
\int_B w \leq C \inf_B w
\]
or equivalent to
\[
M(w)(x) = \sup_{r>0} \int_{B_r(x)} w \leq Cw(x) \quad \text{a.e. } x \in \mathbb{R}^n.
\]
For the basic knowledge of \(A_p\) weight, you can refer the Chapter 5 of [20].

Lemma 2.1 If \(0 < \sigma < n\), then \(\frac{|x|}{|y|^{\sigma}} \in A_1\) weight.

Proof. For any \(B_r(x)\), if \(r < \frac{|x|}{2}\), then for any \(y \in B_r(x)\) we have \(\frac{|x|}{2} < |y| < \frac{3|x|}{2}\). Thus,
\[
\int_{B_r(x)} \frac{1}{|y|^{\sigma}}dy \leq C \frac{1}{|x|^{\sigma}}.
\]
If \(r > \frac{|x|}{2}\), then
\[
\int_{B_r(x)} \frac{1}{|y|^{\sigma}}dy \leq \frac{1}{|B_r(x)|} \int_{B_r(0)} \frac{1}{|y|^{\sigma}}dy \leq \frac{C}{r^{\sigma}} \leq \frac{C}{|x|^{\sigma}}.
\]
Then we know maximal function
\[ M \left( \frac{1}{|y|^{\sigma}} \right) (x) = \sup_{r > 0} \int_{B_r(x)} \frac{1}{|y|^{\sigma}} dy \leq C \frac{1}{|x|^{\sigma}}. \]
i.e. \( \frac{1}{|x|^{\sigma}} \in A_1. \)

**Lemma 2.2** Suppose \( f \) is a nonnegative function and \( 0 < k \leq n - 2 \), then \( v_k(x) \in A_1 \) weight.

\[
 C \int_{B_r} v_k \leq \inf_{\partial B_r} v_k \leq \int_{\partial B_r} v_k. \tag{2.1}
\]

**Proof.** For any \( r > 0 \), we have
\[
 \int_{B_r(x)} v_k(y) dy = \int_{B_r(x)} \int_{\mathbb{R}^n} \frac{f(z)}{|y - z|^k} dz dy = \int_{B_r(x)} \frac{1}{|y - z|^k} dy \int_{\mathbb{R}^n} f(z) dz 
\leq C \int_{\mathbb{R}^n} \frac{f(z)}{|x - z|^k} dz = C v_k(x).
\]
The above inequality follows by Lemma 2.1,
\[
 \int_{B_r(x)} \frac{1}{|y - z|^k} dy = \int_{B_r(0)} \frac{1}{|y + x - z|^k} dy \leq \frac{C}{|x - z|^k}.
\]
Then, \( M(v_k)(x) \leq C v_k(x) \), i.e. \( v_k \in A_1 \). From easy calculation, we know
\[- \Delta v_k(x) = k(n + k - 2) \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{k+2}} dy > 0 \quad \text{for} \quad 0 < k < n - 2.
\]
And,
\[- \Delta v_k = (n - 2) \omega_{n-1} f \geq 0 \quad \text{for} \quad k = n - 2.
\]
Then (2.1) follows by maximum principle and the definition of \( A_1 \) weight.

2.2 estimate of singular integral

For even number \( n = 2m \), we consider the integral equation
\[
 v(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|x - y|}{|y|} f(y) dy \tag{2.2}
\]
where \( c_n = 2^{2m-2} (m-1)! \pi^m \) and \( f \in L^1(\mathbb{R}^n) \) is a nonnegative continuous function. Denote
\[
 \alpha = \frac{1}{c_n} \int_{\mathbb{R}^n} f(y) dy,
\]
obviously we have
\[
 (-\Delta)^m v(x) = -2f(x) \quad \text{in} \quad \mathbb{R}^n.
\]

**Lemma 2.3** Suppose \( v \) satisfies (2.2), then for \( |x| \gg 1 \)
\[
 v(x) \leq \alpha \log |x| + C,
\]
where \( C \) depends on \( \max_{B_1} f \).
Lemma 2.4 Suppose $v$ satisfies (2.2) and $f$ is a nonnegative continuous function, then for $|x| \gg 1$

$$v(x) \geq (\alpha - \epsilon) \log |x| - \frac{1}{cn} \int_{|y - x| < 1} \log \frac{1}{|x - y|} f(y)dy.$$ 

Proof. Similarly we decompose $\mathbb{R}^n$ into $\mathbb{R}^n = A_1 \cup A_2 \cup A_3 \cup A_4$, where

$$A_1 = \{ y ||y| < |x|/2 \}, \quad A_2 = \{ y ||y - x| < |x|/2 \},$$
$$A_3 = \{ y ||y| > |x|/2, |y - x| > |x|/2 \}.$$ 

Then

$$\int_{\mathbb{R}^n} \log \frac{|x-y|}{|y|} f(y)dy = \sum_{i=1}^{4} \int_{A_i} \log \frac{|x-y|}{|y|} f(y)dy.$$ 

For $A_1$, we suppose $2R_0 \ll |x|$ then

$$I \geq \int_{A_1} \log \frac{|x|}{|y|} f(y)dy$$

$$\geq \left( \int_{|y|<R_0} f(y)dy \right) \log |x| - (\log(2R_0) + C) \int_{|y|<R_0} f(y)dy. \quad (2.4)$$

In $A_2$, $|y| > R_0$ and $\frac{1}{2} |x| < |y| < \frac{3}{2} |x|$ then

$$II = \int_{A_2} \log |x-y| f(y)dy - \int_{A_2} \log |y| f(y)dy$$

$$\geq \int_{|y-x|<1} \log |x-y| f(y)dy - \left( \int_{A_2} f(y)dy \right) \log |x| - C.$$
\[
\geq \int_{|y-x|<1} \log |x-y| f(y)dy - \left( \int_{|y|>R_0} f(y)dy \right) \log |x| - C. \tag{2.5}
\]

For \(A_3\), then
\[
III = \int_{A_3} \log |x-y| f(y)dy - \int_{A_3} \log |y| f(y)dy \\
geq \left( \int_{A_3} f(y)dy \right) \log \left( \frac{|x|}{2} \right) - \left( \int_{A_3} f(y)dy \right) \log(2|x|) \\
geq - \left( \int_{|y|>R_0} f(y)dy \right) \log |x| - C. \tag{2.6}
\]

For \(A_4\), we know \(\frac{1}{4}|y| \leq |x-y| \leq 3|y|\), then
\[
IV \geq -C \int_{A_4} f(y)dy. \tag{2.7}
\]

Combating with (2.3), (2.4), (2.5), (2.6) and (2.7), we choose \(R_0\) sufficiently big, then we complete the proof.

**Corollary 2.1** If \(f \in L^1(\mathbb{R}^n)\), then for \(R_0 \gg 1\)
\[
\int_{\mathbb{R}^n \setminus B_{R_0}(0)} v^- \leq C.
\]

**Proof.** By Lemma 2.4, we know
\[
\int_{\mathbb{R}^n \setminus B_{R_0}(0)} v^- (x)dx \leq C \int_{\mathbb{R}^n \setminus B_{R_0}(0)} \int_{|y-x|<1} \log \frac{1}{|x-y|} f(y)dydx \\
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{|y-x|<1} \log \frac{1}{|x-y|} f(y)dydx \\
\leq C \int_{B_1(y)} \log \frac{1}{|x-y|} dx \int_{\mathbb{R}^n} f(y)dy \leq C. \tag{2.7}
\]

**Lemma 2.5** Suppose \(v\) is given by (2.2), then
\[
(-\Delta)^k v(x) = \frac{d_k}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{2k}} dy \quad \text{for} \quad k=1, \ldots, m-1,
\]
where \(d_{k+1} = 2k(n-2k-2)d_k\) and \(d_1 = -(n-2)\).

**Proof.** For \(h \ll 1\), we know
\[
\left| \frac{v(x+he_i) - v(x)}{h} \right| \leq C \int_{\mathbb{R}^n} \log \left( 1 + \frac{|x+he_i-y| - |x-y|}{|x-y|} \right) f(y)dy \\
\leq C \int_{\mathbb{R}^n} \frac{|x+he_i-y| - |x-y|}{|x-y|} f(y)dy \\
\leq C \int_{\mathbb{R}^n} \frac{|h+2(x_i-y_i)|}{|x+he_i-y|+|x-y|} f(y)dy \\
\leq C \int_{\mathbb{R}^n} \frac{1+2|x-y|}{|x-y|^2} f(y)dy < +\infty.
\]

So the Lebesgue dominated convergence theorem gives
\[
\frac{\partial v}{\partial x_i} = \lim_{h \to 0} \frac{v(x+he_i) - v_i(x)}{h} = \frac{1}{c_n} \int_{\mathbb{R}^n} \frac{x_i-y_i}{|x-y|^2} f(y)dy.
\]
Through a similar argument, we further prove
\[
\frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{1}{c_n} \int_{\mathbb{R}^n} \left[ \frac{\delta_{ij}}{|x-y|^2} - \frac{2(x_i - y_i)(x_j - y_j)}{|x-y|^4} \right] f dy.
\]
So, we arrive
\[
-\Delta v(x) = -\frac{(n-2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^2} dy.
\]
Since \((-\Delta)^k \log |x-y| = d_k |x-y|^{-2k}\), we obtain
\[
(-\Delta)^k v(x) = \frac{d_k}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{2k}} dy.
\]  
\[\square\]

In the following, we focus on estimate the term \(\int_{\partial B_1(x)} (-\Delta)^k v(y) d\sigma(y)\), which will work in Lemma 4.5.

**Lemma 2.6** For \(n > k > 2\) and \(x \in \mathbb{R}^n\), there holds
\[
\int_{\mathbb{R}^n \setminus B_4(x)} \frac{1}{|x-y|^{n-2} |y-z|^k} dy \leq C_1 \chi_{B_2}(z) + \frac{C_2 \chi_{B_2^c}(z)}{|x-z|^{k-2}}.
\]

**Proof.** For simplicity, we let \(I = \int_{\mathbb{R}^n \setminus B_4(x)} \frac{1}{|x-y|^{n-2} |y-z|^k} dy\). First, if \(|x - z| \leq 2\), then \(|y - z| > |x - y| - |x - z| \geq 2\). Thus, we know
\[
|x - y| \leq |x - z| + |y - z| \leq 2|y - z|
\]
and
\[
|y - z| \leq |x - y| + |x - z| \leq \frac{3}{2}|x - y|.
\]
So,
\[
I \leq C \int_{\mathbb{R}^n \setminus B_4(x)} \frac{1}{|x-y|^{n-2} |y-z|^k} \leq C_1.
\]

Second, if \(|x - z| \geq 2\), splitting \(\mathbb{R}^n \setminus B_4(x)\) into \(\mathbb{R}^n \setminus B_4(x) = A_1 \cup A_2 \cup A_3\), where
\[
A_1 = \{y \mid |x - y| \geq 4, |y - z| \leq |x - z|/2\},
\]
\[
A_2 = \{y \mid 4 \leq |x - y| \leq |x - z|/2\},
\]
\[
A_3 = \{y \mid |x - z|/2 < |x - y|, |x - z|/2 < |y - z|\}.
\]
And we note that \(A_1, A_2\) may be null set. There two cases will happen.

Case 1: \(|x - z| \geq 8\), then \(A_1, A_2 \neq \emptyset\). We have
\[
I = \int_{A_1} + \int_{A_2} + \int_{A_3} \frac{1}{|x-y|^{n-2} |y-z|^k} dy
\]
\[
:= I_1 + I_2 + I_3.
\]
In \(A_1\), we know
\[
|x - y| \leq |x - z| + |y - z| \leq \frac{3}{2}|x - z|
\]
and
\[
|x - z| \leq |x - y| + |y - z| \leq |x - y| + \frac{|x - z|}{2}.
\]
So,
\[ I_1 \leq \frac{C}{|x-z|^{n-2}} \int_{|y-z|<|x-z|/2} \frac{1}{|y-z|^k} \leq \frac{C}{|x-z|^{k-2}}. \]

Similarly, you can get
\[ I_2 \leq \frac{C}{|x-z|^{k-2}}. \]

In \( A_3 \),
\[ |x-y| \leq |x-z| + |y-z| \leq 3|y-z| \]
and
\[ |y-z| \leq |x-z| + |x-y| \leq 3|x-y|. \]

Thus, we obtain
\[ I_3 \leq C \frac{1}{|x-z|^{n-2}} \int_{|y-z|>|x-z|/2} \frac{1}{|y-z|^{n+k-2}} dy \leq \frac{C}{|x-z|^{k-2}}. \]

Case 2: \( 2 \leq |x-z| \leq 8 \). Then we get
\[
I \leq \int_{B_3} \frac{1}{|x-y|^{n-2}} \frac{1}{|y-z|^k} dy = \int_{B_1} + \int_{B_2} + \int_{B_3} \frac{1}{|x-y|^{n-2}} \frac{1}{|y-z|^k} dy,
\]
where
\[ B_1 = \{ y | |x-y| \leq 1 \} \]
\[ B_2 = \{ y | |y-z| \leq 1 \} \]
\[ B_3 = \{ y | |x-z| > 1, |y-z| > 1 \}. \]

Similarly, you can get
\[ I \leq C, \]
so we complete the proof.

**Lemma 2.7** For any \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \), then for any \( R > 0 \) we have
\[
\int_0^R \frac{1}{|\partial B_r|} \int_{B_r} u(x) dx dr = \frac{1}{(n-2)\omega_{n-1}} \int_{B_R} \left( \frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \right) u(x) dr.
\]

**Proof.** By element calculation, we know
\[
\int_0^R \frac{1}{|\partial B_r|} \int_{B_r} u(x) dx dr = \int_0^R \frac{1}{|\partial B_r|} \int_{\partial B_r} u(x) d\sigma dr
\]
\[
= \int_0^R \int_{\partial B_r} u(x) d\sigma ds \int_s^R \frac{1}{|\partial B_r|} dr
\]
\[
= \frac{1}{(n-2)\omega_{n-1}} \int_{B_R} \left( \frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \right) u(x) dx. \]

**Lemma 2.8** If \( v \) is defined by (2.2) and \( f := -\frac{1}{\Delta} (-\Delta)^m v \in L^1(\mathbb{R}^n) \), then there exist \( C \) which is independent of \( x \) such that
\[ 0 < \int_{\partial B_1(x)} (-\Delta)^i v(y) d\sigma(y) \leq C, \quad \text{for} \quad i = 1, \cdots, m-1, \]
where \( x \in \mathbb{R}^n \).
**Proof.** We argue it by induction. For $k = m - 1$, we know

$$- \int_{\partial B_k(x)} \frac{\partial}{\partial r} (-\Delta)^{m-1} v = \frac{1}{|\partial B_k(x)|} \int_{B_k(x)} (-\Delta)^m v,$$

Integral two sides from 0 to 4, then

$$RHS = (-\Delta)^{m-1} v(x) - \int_{\partial B_k(x)} (-\Delta)^{m-1} v = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{(-\Delta)^m v(y)}{|x-y|^{n-2}} dy - \int_{\partial B_k(x)} (-\Delta)^{m-1} v$$

and by Lemma 2.7 we arrive

$$LHS = \frac{1}{(n-2)\omega_{n-1}} \int_{B_k(x)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{4^{n-2}} \right) (-\Delta)^m v(y) dy.$$

So, we get

$$\int_{\partial B_k(x)} - (-\Delta)^{m-1} v = \frac{1}{(n-2)\omega_{n-1}} \left( \int_{\mathbb{R}^n \setminus B_k(x)} \frac{(-\Delta)^m v(y)}{|x-y|^{n-2}} dy + \frac{1}{4^{n-2}} \int_{B_k(x)} (-\Delta)^m v(y) dy \right),$$

we get $k = m - 1$. If we have know $k$ is right, we hope to prove $k - 1$ is also true where $m - 1 \geq k \geq 2$. Similarly, we know

$$\int_{\partial B_k(x)} - (-\Delta)^{k-1} v = \frac{1}{(n-2)\omega_{n-1}} \left( \int_{\mathbb{R}^n \setminus B_k(x)} \frac{(-\Delta)^k v(y)}{|x-y|^{n-2}} dy + \frac{1}{4^{n-2}} \int_{B_k(x)} (-\Delta)^k v(y) dy \right) \quad (2.8)$$

By Lemma 2.5 and Lemma 2.6, we conclude that

$$\int_{\mathbb{R}^n \setminus B_k(x)} \frac{(-\Delta)^k v(y)}{|x-y|^{n-2}} dy = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_k(x)} \frac{f(z)}{|x-y|^{n-2}|y-z|^{2k}} dy dz \leq C \int_{B_k(x)} f(z) dz + C \int_{\mathbb{R}^n \setminus B_k(z)} \frac{f(z)}{|x-z|^{n+2k-2}} dz \leq C. \quad (2.9)$$

For the second term, we know

$$- (-\Delta)^k v(y) = C \int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^{2k}} dy = C v_{2k}(y) \in A_1.$$

By Lemma 2.2, we obtain

$$\int_{B_k(x)} - (-\Delta)^k v = C \int_{B_k(x)} v_{2k} \leq \inf_{B_k(x)} v_{2k} = C \inf_{\partial B_k(x)} v_{2k} \leq C \int_{\partial B_k(x)} v_{2k} = C \int_{\partial B_k(x)} - (-\Delta)^k v \leq C, \quad (2.10)$$

where we use the induction for $k$. From (2.8) , (2.9) and (2.10), we complete the argument. □

Lastly, we list two Lemmas which are used to proof Lemma 4.5.
Lemma 2.9 (Hardy-Littlewood-Sobolev) If \( p, q \geq 1 \), \( 0 < \lambda < n \) and \( \frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2 \), we suppose \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^q(\mathbb{R}^n) \) then

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}
\]

Lemma 2.10 If \( \Omega \) is a bounded domain and \( h \) solves

\[
\begin{cases}
(-\Delta)^m h = 2f & \text{in } \Omega \\
h = (-\Delta) h = \cdots = (-\Delta)^{m-1} h = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( f \in L^1(\Omega) \). Then for any \( \delta \in (0, nc_n) \), there exist \( C_\delta \) such that

\[
\int_{\Omega} \exp \left( \frac{\delta \|h(x)\|}{\|f\|_{L^1(\Omega)}} \right) dx \leq C_\delta (\text{diam } \Omega)^n
\]

Proof. Let \( R = \text{diam } \Omega \) and \( \tilde{f} \) is a zero extension of \( f \) and we assume \( 0 \in \Omega \). Set

\[
w(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{2R}{|x-y|} |\tilde{f}(y)| dy,
\]

By the Lemma 2.5, we know

\[
(-\Delta)^k w(x) \geq 0 \quad \text{in } \Omega
\]

for \( k = 1, \ldots, m-1 \) and

\[
(-\Delta)^m w(x) \geq 2|f(x)| \quad \text{in } \Omega.
\]

By the maximum principle, we have \( |h(x)| \leq w(x) \). Applying the Jensen inequality and noticing that \( w(x) \leq 0 \) for \( |x| > 3R \), we obtain

\[
\int_{\Omega} \exp \left( \frac{\delta \|h(x)\|}{\|f\|_{L^1(\Omega)}} \right) dx \leq \int_{\Omega} \exp \left( \frac{\delta}{c_n} \int_{B_{3R}} \log \frac{2R}{|x-y|} \frac{|\tilde{f}(y)|}{\|f\|_{L^1(\Omega)}} dy \right) dx
\]

\[
\leq C \left( \frac{3R}{\|f\|_{L^1(\Omega)}} \right)^\delta \int_{B_{3R}} \frac{|\tilde{f}(y)|}{|x-y|^{\frac{n}{\lambda}}} dy dx
\]

\[
\leq CR^n.
\]

The last inequality follows by the Lemma 2.9.

3 A simple proof of normal metrics in four dimension

For \( (\mathbb{R}^4, g = e^{2u} |dx|^2) \), we consider the equation

\[
(-\Delta)^2 u(x) = 2Q_g(x)e^{4u(x)} \quad \text{in } \mathbb{R}^n.
\]

The main assumption:

\[
\begin{cases}
Q_g \geq 0 \text{ and } 0 < C_0 e^{-|x|^\beta} \leq Q_g(x) \text{ for } |x| \gg 1, \text{ where } 0 < \beta < 1, \\
Q_g(x)e^{nu(x)} \in L^1(\mathbb{R}^n), \\
u(x) = o(|x|^2).
\end{cases}
\]

Lemma 3.1 Let \( u \) is a solution of (3.1) and (3.2). Then we have

\[
-\Delta u > 0, \quad \text{for } x \in \mathbb{R}^n.
\]
Proof. Let \( w(x) = -\Delta u(x) \), first we claim that \( w(x) \geq 0 \). If there exist \( x_0 \in \mathbb{R}^n \) such that \( w(x_0) < 0 \), then we let

\[
\bar{f}(r) = \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} f(y) d\sigma(y).
\]

Hence, we know \( \bar{w}(0) = w(x_0) < 0 \) and

\[
-\Delta \bar{w} = \bar{w}, \quad -\Delta \bar{w} \geq 0.
\]

Due to \( (r^3 \bar{w}'(r))' \leq 0 \) and \( \bar{w}'(0) = 0 \), we arrive

\[
\bar{w}(r) \leq \bar{w}(0) < 0.
\]

Then \( -\frac{1}{r^3}(r^3 \bar{u}'(r))' = \bar{u}(r) \leq \bar{w}(0) \), we know

\[
\bar{u}'(r) \geq -\frac{\bar{w}(0)}{4} r.
\]

Thus, for \( r \geq r_1 \),

\[
\bar{u}(r) \geq c_1 r^2.
\]

Hence, by Jensen inequality, we obtain

\[
\int_{\partial B_r(x_0)} Q_g e^{nu} \geq C_0 e^{-r^3} \int_{\partial B_r(x_0)} e^{nu} \geq C_0 e^{-r^3} e^{\bar{w}(r)} \geq C_0 e^{nc_1 r^2 - r^3}. \tag{3.4}
\]

From (3.4) we know,

\[
\int_0^{+\infty} \int_{\partial B_r(x_0)} Q_g e^{nu} d\sigma dr \geq C \int_0^{+\infty} r^{n-1} e^{nc_1 r^2 - r^3} dr.
\]

This is a contradiction with \( Q_g(x) e^{nu(x)} \in L^1(\mathbb{R}^n) \). So, we know \( w(x) \geq 0 \), strong maximum principle implies \( w(x) > 0 \).

Lemma 3.2 There holds

\[-\Delta u(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^4} \frac{Q_g(y) e^{4u(y)}}{|x-y|^2} dy, \quad \text{for} \quad x \in \mathbb{R}.
\]

Proof. Let \( w = u + v \), we know \( \Delta^2 w = 0 \). Then for any \( x_0 \in \mathbb{R}^4 \)

\[
\Delta w(x_0) = \int_{B_r(x_0)} \Delta w = \frac{1}{|B_r(x_0)|} \int_{\partial B_r(x_0)} \frac{\partial w}{\partial v}.
\]

So, integrating two sides, we obtain

\[
\frac{r^2}{8} \Delta w(x_0) = \int_{\partial B_r(x_0)} w - w(x_0).
\]

By Jensen inequality and (3.5), we know

\[
\exp \left( \frac{r^2}{2} \Delta w(x_0) \right) = e^{-4w(x_0)} \exp \left( \int_{\partial B_r(x_0)} 4w \right)
\]
\[ \leq e^{-4w(x)} \int_{B_{r}(x)} e^{4w} \leq e^{-4w(x)} e^{\theta} C_{0} \int_{\partial B_{r}(x)} Q_{G} e^{4u} \]
\[ \leq C e^{-4w(x)} e^{4\alpha-3} e^{\theta} \int_{\partial B_{r}(x)} Q_{G} e^{4u}. \]

Then we get for \( R_{0} \gg 1, \)
\[ \int_{R_{0}}^{+\infty} r^{-4\alpha} \exp \left( \frac{r^{2}}{2} \Delta w(x) - r^{2} \right) dr < \infty. \quad (3.6) \]

Thus, using (3.6) we obtain \( \Delta w(x) \leq 0. \) So, the harmonic function theory implies that \( \Delta w(x) \equiv -C_{2} \leq 0 \) i.e,
\[ -\Delta u(x) = \frac{1}{2\pi^{2}} \int_{\mathbb{R}^{4}} \frac{Q_{g}(y)e^{4u(y)}}{|x-y|^{2}} dy + C_{2}. \]

where \( C_{2} \geq 0. \) If \( C_{2} > 0, \) then \( \Delta u(x) \leq -C_{2}. \) We consider the sphere average of \( u, \)
\[ \bar{u}(r) = \frac{1}{\partial B_{r}(0)} u. \]

We note that \( \frac{1}{r^{3}} (r^{3} \bar{u}'(r))' \leq -C_{2}, \) therefore \( \bar{u}(r) \leq u(0) - \frac{C_{2}r^{2}}{6}. \) It is a contradiction with \( u(x) = o(|x|^{2}). \quad \square \)

**Lemma 3.3** Suppose \( u \) satisfies (3.2) and (3.1), then
\[ u(x) = -\frac{1}{4\pi^{2}} \int_{\mathbb{R}^{4}} \log \frac{|x-y|}{|y|} Q_{g}(y)e^{4u(y)} dy + C. \]

**Proof.** Applying Lemma 2.4, Lemma 2.3 and Lemma 3.2, we know \( \Delta(u+v) = 0 \) and \( u+v = o(|x|^{2}). \) Then, we claim that: \( |u+v| = o(|x|^{2}). \)

Denote \( w = u+v, \) without loss of generality we assume \( w(0) = 0. \) Then for any \( \epsilon > 0, \) there exist \( R_{\epsilon} \) such that
\[ w(x) \leq \epsilon |x|^{2} \quad \text{for} \quad |x| > R_{\epsilon}. \]

For \( B_{R}(0) \) and \( R > 2R_{\epsilon}, \) using Harnack inequality we know
\[ \sup_{B_{R/2}(0)} (\epsilon R^{2} - w) \leq C(n) \inf_{B_{R/2}(0)} (\epsilon R^{2} - w) \leq C(n)\epsilon R^{2}, \]

then we obtain
\[ \sup_{B_{R/2}(0)} (-w) \leq C(n)\epsilon R^{2}. \]

Now we conclude that \( |w(x)| \leq C(n)\epsilon |x|^{2} \) for \( |x| > R_{\epsilon}. \) Standard elliptic estimate implies that for any fixed \( x_{0} \in \mathbb{R}^{n} \)
\[ |\nabla^{3}w(x)|_{(x_{0})} \leq \frac{C(n)}{R^{7}} \int_{B_{R}(x_{0})} |u| \leq \frac{C(n)\epsilon}{R} \to 0 \]
as \( R \to \infty, \) we get \( w(x) = p_{0} + p_{1}(x) + p_{2}(x), \) where \( p_{i}(x) \) is a homogeneous polynomial of degree \( i. \) In fact, \( p_{2}(x) \equiv 0. \) We suppose \( p_{2}(x) = |x|^{2}p_{2}(\theta), \) where \( \theta = \frac{x}{|x|} \in \mathbb{S}^{3}. \) If \( p_{2}(\theta_{0}) > 0 \) for some \( \theta_{0} \in \mathbb{S}^{3}, \) then
\[ C(n)\epsilon |x|^{2} \geq w(|x|\theta_{0}) \geq \frac{p_{2}(\theta_{0})}{2}|x|^{2} \quad \text{for} \quad |x| \gg 1. \]
This is impossible, so \( p_2 \leq 0 \). Similarly you can get \( p_2 \geq 0 \), then we know \( p_2 \equiv 0 \). Hence, 
\[
w = u(x) + v(x) = \sum_{i=1}^{4} a_i x_i + C \text{ and }
\]
\[
\frac{C_0}{|x|^{4\alpha}} e^{\sum_{i=1}^{4} a_i x_i + 4C - |x|^3} = C_0 e^{-|x|^3} e^{4u(x)} \leq Q_g(x)e^{4u(x)} \quad \text{for } |x| \gg 1.
\]

Since \( \int_{\mathbb{R}^4} Q_g(y)e^{4u(y)}dy < +\infty \), then \( a_i = 0 \) for \( i = 1, \cdots, 4 \), we complete the proof. \( \square \)

According to the Theorem 1.3 of Chang, Qing and Yang [3] and the result of Wang [21], we can get the following Theorem.

**Theorem 3.1** Suppose \( (\mathbb{R}^4, e^{2u}|dx|^2) \) is a complete and satisfies

\[
\begin{align*}
Q_g \geq 0 \quad &\text{and} \quad 0 < C_0 e^{-|x|^\beta} \leq Q_g(x) \quad \text{for } |x| \gg 1, \quad \text{where } 0 < \beta < 1. \\
Q_g(x)e^{4u(x)} \in L^1(\mathbb{R}^n), \\
u(x) = o(|x|^2).
\end{align*}
\]

Then
\[
\int_{\mathbb{R}^4} Q_g(y)e^{4u(y)}dy \leq 4\pi^2
\]

and
\[
1 - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} Q_g(y)e^{4u(y)}dy = \lim_{r \to \infty} \frac{||\partial B_r(0)||_g^2}{4(2\pi)^{1/3}|B_r(0)|_g}.
\]

Moreover, if \( \int_{\mathbb{R}^4} Q_g(y)e^{4u(y)}dy < 4\pi^2 \), there exist \( C = C(\alpha) \) such that for any bounded smooth domain \( \Omega \) in \( \mathbb{R}^4 \),
\[
|\Omega|_g \leq C(\alpha)|\partial \Omega|_g^{\frac{1}{2}}.
\]

## 4 normal metrics in even dimension

For \( (\mathbb{R}^n, g = e^{2u}|dx|^2) \), we consider the equation
\[
(-\Delta)^mu(x) = 2Q_g(x)e^{nu(x)} \quad \text{in } \mathbb{R}^n \quad (4.1)
\]

The main assumption:
\[
\begin{align*}
C_0 e^{-|x|^\beta} \leq Q_g(x) \leq C_0 |x|^\gamma &\quad \text{for } |x| \gg 1, \quad \text{where } \gamma > 0 \quad \text{and} \quad \beta \in (0, 1). \\
Q_g \geq 0 \quad \text{and} \quad Q_g(x)e^{nu(x)} \in L^1(\mathbb{R}^n)
\end{align*}
\]

In the following, inspired by L. Martinazzi [18] and Cheng, Lin [2], we prove the normal metric by three steps. Some similar ideas have appeared in [10, 14, 15].

**Step 1** \( u(x) = -v(x) + p(x) \) and \( p(x) \leq C \).

We begin with mean value Theorem and elliptic estimates for polyharmonic functions.

**Lemma 4.1** (Pizzetti formula, [18, Lemma 3]) Suppose \( \Delta^mh = 0 \) in \( B_{2R}(x_0) \), then
\[
\int_{B_{R}(x_0)} h(y)dy = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h(x_0),
\]

where \( c_i = \frac{n!}{n+2i(2i)!} \left( \frac{n-2}{2i} \right)^n \) for \( i \geq 1 \).
Proof. You can refer the Page of 310 of [18]. □

**Lemma 4.2** (L. Martinazzi, [18, Proposition 4]) If \( \Delta^m h = 0 \) in \( B_4(0) \), then for any \( \beta \in [0,1) \), \( p \geq 1 \) and \( k \geq 1 \), we have
\[
||h||_{W^{k,p}(B_1(0))} \leq C(k,p)||h||_{L^1(B_4(0))},
\]
\[
||h||_{C^{k,\beta}(B_1(0))} \leq C(k,p)||h||_{L^1(B_4(0))}.
\]

**Proof.** See the Page of 311 of [18]. □

**Lemma 4.3** If \( u \) solves (4.1) and satisfies (4.2), then
\[
u(x) + v(x) = p(x),
\] (4.3)

where \( p(x) \) is a polynomial of degree at most \( n - 2 \).

**Proof.** Let \( h = u + v \), for any fixed \( x_0 \in \mathbb{R}^n \) by the elliptic estimate 4.2 we have
\[
|\nabla^{n-1} h(x_0)| \leq \frac{C}{R^{n-1}} \int_{B_R(x_0)} |h|
= -\frac{C}{R^{n-1}} \int_{B_R(x_0)} h + \frac{2C}{R^{n-1}} \int_{B_R(x_0)} h^+
\]

Thanks to Lemma 4.1, we know
\[
\frac{C}{R^{n-1}} \int_{B_R(x_0)} h = O(R^{-1})
\]

Now we focus on the second part, applying (2.3) and Jensen inequality we obtain
\[
\frac{1}{R^{n-1}} \int_{B_R(x_0)} h^+ \leq \frac{1}{R^{n-1}} \int_{B_R(x_0)} u^+ + C \log \frac{R}{R^{n-1}}
\leq C \log \int_{B_R(x_0)} e^{nu^+} + C \log \frac{R}{R^{n-1}}.
\] (4.4)

Since
\[
\int_{B_R(x_0)} e^{nu^+} \leq \int_{B_R(x_0)} e^{nu} + 1
\leq C(R_0,x_0) + CR^n + C \int_{B_R \setminus B_{R_0}(x_0)} e^{|x|^3} e^{nu}
\leq C(R_0,x_0) + CR^n + Ce^{R^3}
\] (4.5)

for \( R \gg 1 \), from (4.4) and (4.5) we obtain
\[
\frac{1}{R^{n-1}} \int_{B_R(x_0)} h^+ \leq \frac{C}{R^{n-1}} + \frac{C \log R}{R^{n-1}} \to 0
\] (4.6)
as \( R \to \infty \). □

**Remark 4.1** If \( \beta = 1 \), then \( p(x) \) may be a linear function. At this time, you will estimate
\[
|\nabla h(x_0)| \leq C + \frac{C \log R}{R} \leq C.
\]

**Lemma 4.4** Let \( p(x) \) is given by (4.3) and \( (\mathbb{R}^n, e^{2n(|x|^2)}) \) is complete, then
\[
\sup_{x \in \mathbb{R}^n} p(x) < +\infty.
\]
Proof. If not, then we assume \( \sup_{\mathbb{R}^n} p(x) = +\infty \). Denote 
\[
p(x) = p_0 + |x|p_1(\theta) + \cdots + |x|^kp_k(\theta),
\]
where \( \theta = \frac{\nu}{|\nu|} \in S^{n-1} \) and \( k = \deg p \leq n - 2 \). We consider \( q_i := \max_{\theta \in S^{n-1}} p_i(\theta) \), firstly we know there exist \( 1 \leq k_0 \leq k \) such that \( q_{k_0} > 0 \). Otherwise \( q_i \leq 0 \) for \( i = 1, \ldots, k \) then \( p(x) \leq p_0 \). This is a contradiction with \( \sup_{\mathbb{R}^n} p(x) = +\infty \). Without loss of generality, we assume 
\[
q_{m_1}, \cdots, q_{m_{j_0}} > 0,
\]
where \( 1 \leq m_1 < m_2 < \cdots < m_{j_0} \leq k \). And let 
\[
V_{m_i} = \{ \theta \in S^{n-1} | p_{m_i}(\theta) > 0 \}
\]
for \( i = 1, \cdots, j_0 \). Moreover, for any \( \theta \in \bigcup_{1 \leq i \leq j_0} V_{m_i} \) there exist its maximum index \( m_{i_0} \), i.e, \( \theta \in V_{m_{i_0}} \) and \( \theta \not\in V_{m_i} \) for any \( i > i_0 \). Then we know there are two cases will happen:

1. There exists \( \theta_0 \in \bigcup_{1 \leq i \leq j_0} V_{m_i} \), such that 
\[
p_{m_{i_0}}(\theta_0) > 0 \quad \text{and} \quad p_l(\theta_0) = 0 \quad \text{for all} \quad l \geq m_{i_0} + 1. \tag{4.7}
\]
So, by (4.7) we obtain 
\[
\lim_{r \to \infty} \frac{p(r, \theta_0)}{r^{m_{i_0}}} = C(\theta_0) > 0.
\]
Thus, for any \( r > R_0 \) there exist \( x_r \) such that \( |x_r| = r, \frac{x_r}{|x_r|} = \theta_0 \in S^{n-1}, \) where \( \theta_0 \) is independent of \( r \). And 
\[
p(x) \geq c_0 r^{m_{i_0}} \quad \text{for} \quad x \in B_{1/r^{n-3}}(x_r),
\]
this follows by \( |\nabla p(x)| \leq C|x|^{k-1} \leq C|x|^{n-3} \). Then, thanks to Lemma 2.4, we know 
\[
u(x) = -v(x) + p(x) \geq c_0 r^{m_{i_0}} - \alpha \log r - C \geq \frac{c_0}{2} r^{m_{i_0}} \quad \text{for} \quad x \in B_{1/r^{n-3}}(x_r).
\]
But we get 
\[
\int_{\mathbb{R}^n \setminus B_{R_0}} Q_d e^{\nu u} \geq \int_{R_0}^{+\infty} \int_{B_r \cap B_{3-n}(x_r)} Q_d e^{\nu u} d\sigma dr \\
\geq C \int_{R_0}^{+\infty} e^{-\frac{c_0 r^{m_{i_0}} - \beta \alpha r^{n}(n-1)}{r^{n-3}}} = +\infty.
\]
The last equality follows by \( m_{i_0} \geq 1 \) and \( \beta \in (0, 1) \).

2. For any \( \theta \in \bigcup_{1 \leq i \leq j_0} V_{m_i} \), 
\[
p_{m_\theta}(\theta) > 0 \quad \text{there exist some} \quad l_\theta \geq m_{i_\theta} + 1, \quad p_{l_\theta}(\theta) < 0.
\]
In fact, for any \( l \geq m_{i_\theta} + 1 \) we also have \( p_l(\theta) \leq 0 \). By the definition of maximum index, we know \( p_{m_\theta}(\theta) \leq 0 \) for \( i > i_\theta \). Obviously, \( p_l(\theta) \leq 0 \) for \( l \neq m_{i_\theta + 1}, \cdots, m_{j_\theta} \). So, we obtain 
\[
\lim_{r \to \infty} \frac{p(r, \theta)}{r^{l_\theta}} = -C(\theta) < 0, \quad \text{for any} \quad \theta \in \bigcup_{1 \leq i \leq j_0} V_{m_i}.
\]
For any fixed $\theta$, we consider the length of curve near infinity, denote $$III = \int_{R_0 + 1}^{+\infty} e^{u(r, \theta)} dr,$$
where $R_0 \gg 1$. From Lemma 2.4, we know
$$III \leq C \int_{R_0 + 1}^{+\infty} \frac{e^{\rho(r, \theta)}}{r^{\alpha - \epsilon_0}} II(r) dr,$$
where
$$II(r) = \exp \left( \frac{1}{c_n} \int_{|r\theta - y| < 1} \log \frac{1}{|r\theta - y|} f(y) dy \right).$$
Since for any $r > R_0 + 1$, then $y \in \mathbb{R}^n \setminus B_{R_0}$. We rewrite $II(r)$ term as
$$II(r) = \exp \left( \int_{\mathbb{R}^n \setminus B_{R_0}} \sigma(R_0) \log \frac{1}{|r\theta - y|} f(y) dy \right),$$
where $\sigma(R_0) = \frac{1}{c_n} ||f||_{L^1(\mathbb{R}^n \setminus B_{R_0})} < \frac{1}{2}$. For $dv(y) = \frac{f(y)}{||f||_{L^1(\mathbb{R}^n \setminus B_{R_0})}} dy$, we apply the Jensen inequality to get
$$II(r) \leq \int_{\mathbb{R}^n \setminus B_{R_0}} \exp \left( \sigma(R_0) \log \frac{1}{|r\theta - y|} f(y) \right) \frac{f(y)}{||f||_{L^1(\mathbb{R}^n \setminus B_{R_0})}} dy.$$
Plug (4.9) into (4.8) and by the Fubini’s Theorem, we conclude
$$III \leq C \int_{\mathbb{R}^n \setminus B_{R_0}} (III_1(y) + III_2(y)) \frac{f(y)}{||f||_{L^1(\mathbb{R}^n \setminus B_{R_0})}} dy,$$
where
$$III_1(y) = \int_{I_y \cap (R_0 + 1, +\infty)} \frac{e^{\rho(r, \theta)}}{r^{(\alpha - \epsilon_0)} |r\theta - y|^{\sigma(R_0)}} dr,$$
$$III_2(y) = \int_{(R_0 + 1, +\infty) \setminus I_y} \frac{e^{\rho(r, \theta)}}{r^{(\alpha - \epsilon_0)}} dr$$
and
$$I_y = \{ r\theta | R_0 + 1 < r < +\infty \} \cap B_1(y).$$
For any fixed $y \in \mathbb{R}^n \setminus B_{R_0}$, denoted the center of $I_y$ by $y^*$, i.e., $y^* \in \{ r\theta | R_0 + 1 < r < +\infty \}$ and $(y^* - y) \cdot \theta = 0$. Note that for any fixed $y \in \mathbb{R}^n$, $|r\theta - y^*| \leq |r\theta - y|$ and $|I_y| \leq 2$, so we know
$$III_1 \leq \int_{|y^*| - 1, |y^*| + 1} \frac{1}{r^{(\alpha - \epsilon_0)} |r\theta - y^*|^{\sigma(R_0)}} dr \leq C(\theta).$$
(4.11)
Clearly,
$$III_2 = \int_{(R_0 + 1, +\infty) \setminus I_y} \frac{e^{\rho(r, \theta)}}{r^{(\alpha - \epsilon_0)}} dr \leq C(\theta),$$
(4.12)
then (4.10), (4.11) and (4.12) imply $III < +\infty$. This is a contradiction with the completeness of $(\mathbb{R}^n, e^{2u} |dx|^2)$. \qed
Remark 4.2 For $\beta < 1$ is essential in Case (1), since $p(x) = \sum_{i=1}^{n} a_i x_i + a_0$ may happen.

For any $k \in \mathbb{N}^+$, we consider the Green’s function of $(-\Delta)^k$ on $B_4(0)$. Let

$$\begin{cases} (-\Delta)^k G = \delta_0 & \text{in } B_4(0) \\ G = 0, -\Delta G = 0, \cdots, (-\Delta)^{k-1} G = 0 & \text{on } \partial B_4(0), \end{cases}$$

in fact $G$ is a radial function. For any $\Delta^k h(x) = 0$ in $B_8(0)$, then

$$h(0) = \int_{B_4} (-\Delta)^k G h = \int_{B_4} (-\Delta)^{k-1} G (-\Delta) h - \int_{\partial B_4} h \frac{\partial}{\partial r} (-\Delta)^{k-1} G$$

$$= - \int_{\partial B_4} h \frac{\partial}{\partial r} (-\Delta)^{k-1} G - \int_{B_4} (-\Delta) h \frac{\partial}{\partial r} (-\Delta)^{k-2} G = \cdots$$

$$= \sum_{i=0}^{k-1} \int_{\partial B_4} (-\Delta)^i h \left( -\frac{\partial}{\partial r} \right) (-\Delta)^{k-1-i} G$$

(4.13)

Then we claim that $\Delta^{k-1-i} G = c_i > 0$ on $\partial B_4(0)$ for $i = 0, 1, \cdots, k-1$. Let $h_i$ solves

$$\begin{cases} (-\Delta)^k h_i = 0 & \text{in } B_4(0) \\ h_i = 0, \cdots, (-\Delta)^{i-1} h_i = 1, \cdots, (-\Delta)^{k-1} h_i = 0 & \text{on } \partial B_4(0), \end{cases}$$

we know $h_i(x) > 0$ in $B_4(0)$, then

$$h_i(0) = \int_{\partial B_4} -\frac{\partial}{\partial r} (-\Delta)^{k-1-i} G = |\partial B_1| c_i > 0.$$

**Step 2** $v(x) = \alpha \log |x| + o(\log |x|)$ for $|x| \gg 1$.

Lemma 4.5 For any $\epsilon > 0$, there exist $R_{\epsilon} \gg 1$ if $|x| > R_{\epsilon}$,

$$\int_{B_4(x)} \frac{1}{|x-y|} Q_d(y) e^{n(y)} \leq \epsilon \log |x|$$

and

$$\alpha \log |x| + C \geq v(x) \geq (\alpha - \epsilon) \log |x| - C.$$  \hspace{1cm} (4.14)

**Proof.** For $|x| > R_0 \gg 1$, then we solve the PDE

$$\begin{cases} (-\Delta)^m l = 2f & \text{in } B_4(x) \\ l = (-\Delta) l = \cdots = (-\Delta)^{m-1} l = 0 & \text{on } \partial B_4(x) \end{cases}$$

By Lemma 2.10, we know there exist $q$ sufficiently big such that

$$\int_{B_4(x)} e^{q l(y)} dy \leq C$$  \hspace{1cm} (4.15)

Let $h(y) = -(v(y) + l(y))$, then is solves

$$\begin{cases} (-\Delta)^m h = 0 & \text{in } B_4(x) \\ h = -v, (-\Delta) h = \Delta v, \cdots, (-\Delta)^{m-1} h = - (-\Delta)^{m-1} v & \text{on } \partial B_4(x) \end{cases}$$
Then, let $G$ be the Green’s function of $(-\Delta)^{m-1}$ at $x$, i.e,

$$
\begin{align*}
(-\Delta)^{m-1}G &= \delta_x & \text{in } B_4(x) \\
G &= 0, -\Delta G = 0, \cdots, (-\Delta)^{m-2}G = 0 & \text{on } \partial B_4(x).
\end{align*}
$$

By (4.13) and Lemma 2.8 we know

$$
-\Delta h(x) = \sum_{i=0}^{m-2} \int_{\partial B_4(x)} (-\Delta)^{i+1} h \left( \frac{\partial}{\partial r} \right) (-\Delta)^{m-2-i} G = \sum_{i=1}^{m-1} c_{i-1} \int_{\partial B_4(x)} (-\Delta)^i h = \sum_{i=1}^{m-1} c_{i-1} \int_{\partial B_4(x)} (-\Delta)^i v \leq C
$$

For small $r_0$, $z \in B_{r_0}(x)$, let $G_z$ be the Green’s function at $z$, i.e,

$$
\begin{align*}
(-\Delta)^{m-1}G_z &= \delta_z & \text{in } B_4(x) \\
G_z &= 0, -\Delta G_z = 0, \cdots, (-\Delta)^{m-2}G_z = 0 & \text{on } \partial B_4(x).
\end{align*}
$$

Then we know

$$
0 < \frac{c_i}{2} < -\frac{\partial}{\partial \nu} (-\Delta)^{m-2-i} G_z < 2c_i & \text{on } \partial B_4(x)
$$

for $i = 0, 1, \cdots, m-2$. The Green’s function at $x$ can be obtained by translating the Green’s function at the zero point. Hence, we note that $r_0$ is independent of $x$! Similarly, you can get

$$
-\Delta h(z) \leq C & \text{in } B_{r_0}(x).
$$

Using the Corollary 2.1, we have

$$
\int_{B_4(x)} h^+ \leq \int_{B_4(x)} v^- \leq C
$$

Standard elliptic estimate implies that

$$
\sup_{x \in B_{\alpha/2}(x)} h \leq \int_{B_4(x)} h^+ + C \sup_{x \in B_{r_0}(x)} -\Delta h \leq C.
$$

By Lemma 4.4, we know

$$
u = p - v = p + h + l \leq C + l & \text{in } B_{\alpha/2}(x).
$$

Now, for any $q > s > 1$ and $\beta > 0$ we conclude

$$
\int_{B_{|z| - \beta}(x)} (Q_{\nu} e^{\alpha s})^\beta \leq C \int_{B_{|z| - \beta}(x)} Q_{\nu} e^{\beta s} \leq C|x|^\gamma \int_{B_{|z| - \beta}(x)} e^{\beta s} \leq C|x|^\gamma \left( \int_{B_{|z| - \beta}(x)} e^{\gamma s} \right)^{\frac{\beta}{\gamma}} \leq C|x|^\gamma \left( \int_{B_{|z| - \beta}(x)} e^{\gamma s} \right)^{\frac{\beta}{\gamma}}
$$

where we can choose $\beta = \frac{\gamma s}{n(q-s)}$. For (4.16) and $s = 2$, we obtain

$$
\int_{B_1(x)} \log \frac{1}{|x-y|} Q_{\nu}(y) e^{su(y)} dy \leq \left( \int_{B_{|z| - \beta}(x)} \left( \log \frac{1}{|x-y|} \right)^2 \right)^{\frac{1}{2}} \left( \int_{B_{|z| - \beta}(x)} (Q_{\nu} e^{su(y)})^2 \right)^{\frac{1}{2}}
$$
\[
+ \int_{B_1(x) \setminus B_{|x| - \alpha}(x)} \log \frac{1}{|x - y|} Q_g(y) e^{nu(y)} dy \\
\leq C|\theta|^{-\frac{\beta(\alpha - 1)}{2}} + \beta \log |x| \int_{B_1(x) \setminus B_{|x| - \alpha}(x)} Q_g(y) e^{nu(y)} dy \\
\leq \epsilon \log |x| + C
\]

(4.17)

where the last inequality we use the \( \int_{\mathbb{R}^n} Q_g e^{nu} < +\infty \) and \( |x| \gg 1 \). Combated with Lemma 2.3, Lemma 2.4 and (4.17), we get the estimate (4.14).

**Step 3** \( u(x) = -v(x) + C. \)

**Theorem 4.1** Suppose \((\mathbb{R}^n, e^{2u}|dx|^2)\) is complete and \( u \) is a solution of

\[
(-\Delta)^m u(x) = 2Q_g(x) e^{nu(x)} \quad \text{in} \quad \mathbb{R}^n.
\]

and satisfies

\[
\begin{cases}
C_0 e^{-|\theta|^\beta} \leq Q_g(x) \leq C_0 |x|^\gamma & \text{for } |x| \gg 1, \text{ where } \gamma > 0 \text{ and } \beta \in (0, 1). \\
Q_g \geq 0 \text{ and } Q_g(x) e^{nu(x)} \in L^1(\mathbb{R}^n)
\end{cases}
\]

(4.19)

Then,

1. \( u \leq C \), which \( C \) may depend on \( u \).
2. The metric is normal, i.e.,

\[
u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x - y|} Q_g(y) e^{nu(y)} dy + C.
\]

(4.20)

For any \( \epsilon > 0 \), there exist \( R_\epsilon \gg 1 \) such that

\[-(\alpha - \epsilon) \log |x| - C \leq u(x) \geq -\alpha \log |x| - C
\]

(4.21)

for \( |x| > R_\epsilon \).

3. \[
\int_{\mathbb{R}^n} Q_g(y) e^{nu(y)} dy \leq c_n.
\]

(4.22)

**Proof.** For (1), by (4.14) \( \lim_{|x| \to \infty} v(x) = +\infty \), combated with 4.4, we get

\[
u(x) = -v(x) + p(x) \leq C.
\]

For (2), since \( u(x) = -v(x) + p(x) \) and \( \deg p \) is even. Let \( p(x) = p_0 + |x| p_1(\theta) + \cdots + |x|^k p_k(\theta) \), where \( \theta = \frac{x}{|x|} \in S^{n-1} \) and \( k = \deg p \leq n - 2 \). Applying Lemma 4.4, we know

\[
sup_{x \in \mathbb{R}^n} p(x) < +\infty,
\]

then \( \sup_{\theta \in S^{n-1}} p_k(\theta) \leq 0 \). And there exist \( \theta_0 \in S^{n-1} \) such that \( p_k(\theta_0) < 0 \),

\[
\lim_{r \to \infty} \frac{p(r, \theta_0)}{r^k} = p_k(\theta_0) < 0.
\]

But this implies that

\[
\int_{R_0} e^{u(r, \theta_0)} dr \leq C \int_{R_0} \frac{e^{-c_0 r^k}}{r^k} dr < +\infty,
\]

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this is a contradiction with the completeness of \((\mathbb{R}^n, e^{2u}|dx|^2)\). The asymptotic formula follows by Lemma 2.3, Lemma 2.4 and Lemma 4.5.

For (3), if \(\alpha = \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy > 1\), then we know for \(|x| \gg 1\), we have
\[
e^{nu} \leq Ce^{-\nu} \leq Ce^{-n(\alpha-\epsilon)\log|x|}
\]
so we get \(\text{vol}_g(\mathbb{R}^n) = \int_{\mathbb{R}^n} e^{nu(x)}dx < +\infty\). This is a contradiction with the completeness of \((\mathbb{R}^n, e^{2u}|dx|^2)\).

**Lemma 4.6** Suppose the same condition as Theorem 4.1, then
\[
R_g(x) \geq -n \Delta u(x)e^{-2u(x)} > 0.
\]

**Proof.** By the conformal change of scalar curvature, we obtain
\[
R_g(x) = e^{-2u}(n-1)(-2\Delta u(x) - (n-2)|\nabla u(x)|^2)
= e^{-2u+2C}(n-1)(2\Delta v(x) - (n-2)|\nabla v(x)|^2).
\]

Clear,
\[
2\Delta v(x) - (n-2)|\nabla v(x)|^2
= \frac{2(n-2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^2} - \frac{(n-2)}{c_n^2} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x-y|^2} f(y) \right)^2.
\]

Notice that Theorem 4.1 (3) implies that \(\int_{\mathbb{R}^n} f(y)dy \leq c_n\), then
\[
\frac{(n-2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^2} - \frac{(n-2)}{c_n^2} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x-y|^2} f(y) \right)^2
\geq \frac{(n-2)}{c_n^2} \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^2} - \frac{(n-2)}{c_n^2} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x-y|^2} f(y) \right)^2
\geq 0.
\]

The final step follows by Cauchy-Schwarz inequality. From (2) in Theorem 4.1, we arrive
\[
-\Delta u = \frac{(n-2)}{c_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^2}.
\]

Combated with (4.23), (4.24), (4.25) and (4.26), we finishing the proof.

## 5 Chern-Gauss-Bonnet formula in even dimension

In this section, we prove the Chern-Gauss-Bonnet formula by two steps. Firstly, we assume that \(u(x) = u(|x|)\), since we have normal condition (4.20) and asymptotic behavior (4.21), we will take a very direct approach to prove it. Secondly, using Lemma 5.3, we can reduce the general case into radial case. In this section, we assume:

For \((\mathbb{R}^n, e^{2u}|dx|^2)\) is complete and satisfies
\[
\begin{cases}
C_0e^{-|x|^\beta} \leq Q_g(x) \leq C_0|x|^\gamma \text{ for } |x| \gg 1, \text{ where } \gamma > 0 \text{ and } \beta \in (0, 1).
Q_g \geq 0 \text{ and } Q_g(x)e^{nu(x)} \in L^1(\mathbb{R}^n)
\end{cases}
\]

Then we know the metric is normal, i.e,
\[
u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y)e^{nu(y)}dy + C.
\]
5.1 Radial symmetric case

Lemma 5.1 If \( u(x) = u(|x|) \), then

\[
\lim_{r \to \infty} ru'(r) = -\frac{1}{c_n} \int_{\mathbb{R}^n} f(y)dy.
\]

Proof. Since

\[
ru'(r) + \frac{1}{c_n} \int_{\mathbb{R}^n} f(y)dy = -\frac{1}{c_n} \int_{\mathbb{R}^n} \frac{y \cdot (x - y)}{|x - y|^2} f(y)dy,
\]

we only need to prove \( \int_{\mathbb{R}^n} \frac{y \cdot (x - y)}{|x - y|^2} f(y)dy \to 0 \) as \( |x| \to \infty \). By easy calculation, we know

\[
I := \int_{\mathbb{R}^n} \frac{y \cdot (x - y)}{|x - y|^2} f(y)dy = \int_{0}^{+\infty} \int_{\partial B_s(0)} \frac{y \cdot (x - y)}{|x - y|^2} f(y) d\sigma(y) dr
\]

\[
= \int_{0}^{+\infty} f(s) s \int_{\partial B_s(0)} \frac{y \cdot (x - y)}{|x - y|^2} d\sigma(y) ds
\]

\[
= \int_{0}^{+\infty} f(s) s \int_{\partial B_s(0)} \frac{\partial \log |x - y|}{\partial \nu} d\sigma(y) ds. \tag{5.3}
\]

We claim that for \( x \notin \partial B_s(0) \), we have

\[
\int_{\partial B_s(0)} \frac{\partial \log |x - y|}{\partial \nu} = \int_{B_s(0)} \Delta \log |x - y| dy = (n - 2) \int_{B_s(0)} \frac{1}{|x - y|^2} dy. \tag{5.4}
\]

First, if \( x \notin B_s(0) \) the above identity is trivial. Suppose \( x \in B_s(0) \), then for any \( s - |x| > \rho > 0 \), we obtain

\[
\int_{\partial B_{s}(0)} \frac{\partial \log |x - y|}{\partial \nu} - \int_{\partial B_{s}(x)} \frac{\partial \log |x - y|}{\partial \nu} = \int_{B_{s}(0) \setminus B_{s}(x)} \Delta \log |x - y| dy.
\]

Since

\[
\int_{\partial B_{s}(x)} \frac{\partial \log |x - y|}{\partial \nu} = \omega_{n-1} \rho^{n-2} \to 0
\]

and

\[
\int_{B_{s}(x)} \Delta \log |x - y| dy = (n - 2) \int_{B_{s}(x)} \frac{1}{|x - y|^2} dy = \omega_{n-1} \rho^{n-2} \to 0.
\]

Thus, combated (5.3) with (5.4), we know

\[
I = (n - 2) \int_{0}^{+\infty} f(s) s \int_{B_s(0)} \frac{1}{|x - y|^2} dy ds.
\]

For any \( \epsilon > 0 \), let \( |x| > \frac{1}{\epsilon} \) we get

\[
I \leq C \int_{0}^{+\epsilon |x|} + \int_{\epsilon |x|}^{+\infty} f(s) s \int_{B_s(0)} \frac{1}{|x - y|^2} dy ds
\]

\[
\leq \frac{C}{(1 - \epsilon)^2 |x|^2} \int_{0}^{+\epsilon |x|} f(s) s^{n+1} ds + C \int_{\epsilon |x|}^{+\infty} f(s) s \int_{B_s(0)} \frac{1}{|y|^2} dy dr
\]

\[
\leq C \epsilon^2 \int_{0}^{+\infty} f(s) s^{n-1} ds + C \int_{1/\epsilon}^{+\infty} f(s) s^{n-1} ds
\]

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\[ \leq C \varepsilon^2 \|f\|_{L^1(\mathbb{R}^n)} + C \int_{\mathbb{R}^n \setminus B_{1/\epsilon}(0)} f(y) dy, \]

where the second inequality follows by Rearrangement inequality in Chapter 3 of [16],

\[ \int_{B_r(0)} \frac{1}{|x-y|^2} dy \leq \int_{B_r(0)} \frac{1}{|y|^2} dy \leq Cs^{n-2}. \]

Let \( \epsilon \to 0 \), we know \( I \to 0 \).

**Lemma 5.2** If \( (\mathbb{R}^n, e^{2u}|dx|^2) \) is complete, \( u(x) = u(|x|) \) then

\[ 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_\delta(y)e^{nu(y)} dy = \lim_{r \to \infty} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n\omega_{n-1}} \left( \int_{B_r} e^{nu} \right) \right) \]

(5.5)

**Proof.** Applying (4.21) in Theorem 4.1, then for \( |x| \gg 1 \) we have

\[ -\alpha \log |x| - C \leq u(x) \leq -(\alpha - \epsilon) \log |x| + C. \]

Since \( \alpha \leq 1 \), then

\[ \int_{B_r} e^{nu} \to \infty \quad \text{and} \quad \frac{d}{dr} \int_{B_r} e^{nu} = \int_{\partial B_r} e^{nu} > 0. \]

(5.6)

Clearly,

\[ \frac{d}{dr} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n\omega_{n-1}} \right)^{\frac{1}{n-1}} = \frac{n\omega_{n-1}}{n-1} \frac{d}{dr} \int_{B_r} e^{nu} \]

\[ = \frac{n}{n-1} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n\omega_{n-1}} \right)^{\frac{1}{n-1}} \left[ (n-1) \int_{\partial B_r} \frac{d}{dr} e^{(n-1)u} + \frac{n-1}{r} \int_{\partial B_r} e^{(n-1)u} \right] \]

\[ = \int_{\partial B_r} e^{(n-1)u} \left( \frac{\int_{\partial B_r} \frac{d}{dr} e^{(n-1)u} + \frac{n-1}{r} \int_{\partial B_r} e^{(n-1)u}}{\int_{\partial B_r} e^{nu(r)}} \right) \]

\[ = \frac{e^{u(r)}\omega_{n-1}^{\frac{1}{n-1}}r}{\omega_{n-1}^{\frac{1}{n-1}}|\partial B_r|e^{nu(r)}} = 1 + ru'(r). \]

Now, from Lemma 5.1, we get

\[ \lim_{r \to \infty} \frac{\frac{d}{dr} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n\omega_{n-1}} \right)^{\frac{1}{n-1}}}{n\omega_{n-1} \frac{d}{dr} \int_{B_r} e^{nu}} = 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_\delta(y)e^{nu(y)} dy. \]

(5.7)

Combating (5.6), (5.7) and L'Hopital's rule imply that

\[ 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_\delta(y)e^{nu(y)} dy = \lim_{r \to \infty} \left( \frac{\int_{\partial B_r} e^{(n-1)u}}{n\omega_{n-1}} \right)^{\frac{1}{n-1}} \left( \int_{B_r} e^{nu} \right). \]

\[ \square \]
4.1

we get

Then

\[ \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy \leq c_n \]

and

\[ 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy = \lim_{r \to \infty} \frac{|\partial B_r(0)|_{\tilde{g}}}{nu_{n-1}|B_r(0)|_{\tilde{g}}}. \]

**Proof.** See the Theorem 4.1 and Lemma 5.2.

5.2 General case

For \( u \in C^\infty(\mathbb{R}^n) \), we define the spherical average

\[ \bar{u}(r) = \int_{\partial B_r(0)} u(y)d\sigma(y) \]

The following Lemma 5.3 is proved by Chang, Qing and Yang [3] in four dimension, but there is no difference for high dimension, so we omit the details.

**Lemma 5.3 (Chang Qing and Yang, [3, Lemma 3.2])** Suppose \( e^{2u}|dx|^2 \) is normal metric. Then for any \( k > 0 \),

\[ \frac{1}{|\partial B_r|} \int_{\partial B_r} e^{ku} = e^{k\bar{u}(r)}e^{o(1)}. \]

**Proof.** See the Page 523 of [3].

**Lemma 5.4** If \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) is complete and satisfies

\[ \begin{cases} C_0e^{-|x|^{\beta}} \leq Q_g(x) \leq C_0|x|^{\gamma} & \text{for } |x| \gg 1, \text{ where } \gamma > 0 \text{ and } \beta \in (0, 1), \\ Q_g \geq 0 \text{ and } Q_g(x)e^{nu(x)} \in L^1(\mathbb{R}^n) \end{cases} \] (5.8)

then \((\mathbb{R}^n, \tilde{g} = e^{u(r)}|dx|^2)\) also satisfies the above properties. Moreover \( \alpha_g = \alpha_{\tilde{g}} \), namely,

\[ \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(x)e^{nu(x)}dx = \frac{1}{c_n} \int_{\mathbb{R}^n} Q_{\tilde{g}}(x)e^{nu(x)}dx. \] (5.9)

**Proof.** Since \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) is complete, then for any \( \theta \in S^{n-1} \),

\[ \int_0^{+\infty} e^{u(r, \theta)} = +\infty. \]

And we know \( \int_{\partial B_r} e^{u}d\sigma(x) = \int_{\partial B_{r(0)}} e^{u(r, \theta)}d\sigma(\theta) \), by Fubini’s theorem, then we know

\[ +\infty = \int_{\partial B_{r(0)}} \int_0^{+\infty} e^{u(r, \theta)}drd\sigma(\theta) = \int_0^{+\infty} \int_{\partial B_{r(0)}} e^{u(x)}d\sigma(x)dr. \]

Combated with Lemma 5.3 we get \( \int_{\partial B_r} e^{u(x)}d\sigma(x) = e^{\bar{u}(r)}e^{o(1)}, \) then

\[ \int_0^{+\infty} e^{\bar{u}(r)}dr = +\infty. \] (5.10)
i.e, \((\mathbb{R}^n, \bar{g} = e^{n\bar{u}(r)}|dx|^2)\) is complete. Here it is apparent that

\[
2Q_g(r) = e^{-n\bar{u}(r)} (-\Delta)^m \bar{u} = e^{-n\bar{u}(r)} \int_{\partial B_r(0)} (-\Delta)^m u
\]

\[
= e^{-n\bar{u}(r)} \int_{\partial B_r(0)} 2Q_g e^{nu} \geq 0. \tag{5.11}
\]

By Lemma 4.6, we know

\[
\frac{C_1}{|x|^\alpha} \leq e^{nu(x)} \leq \frac{C_2}{|x|^{n(\alpha - \epsilon)}}.
\]

So,

\[
Q_g(r) \geq C r^{n(\alpha - \epsilon)} r^{-\alpha} e^{-r^\beta} \geq C e^{-r^{\beta'}} \quad \text{for} \quad r \gg 1 \quad \text{and} \quad 1 > \beta' > \beta. \tag{5.12}
\]

Similarly,

\[
Q_g(r) \leq C r^{n\alpha} r^{-(n\alpha - \epsilon)} r^\gamma = C r^{\gamma + n\epsilon} \quad \text{for} \quad r \gg 1. \tag{5.13}
\]

For the last property,

\[
\int_{\mathbb{R}^n} Q_g(x)e^{nu(x)}dx = \int_{\mathbb{R}^n} (-\Delta)^m \bar{u}(x)dx
\]

\[
= \int_0^{+\infty} \int_{\partial B_r} (-\Delta)^m \bar{u} d\sigma dr = \int_0^{+\infty} |\partial B_r| (-\Delta)^m \bar{u} dr
\]

\[
= \int_0^{+\infty} |\partial B_r| \int_{\partial B_r} (-\Delta)^m ud\sigma dr = \int_{\mathbb{R}^n} (-\Delta)^m u = \int_{\mathbb{R}^n} Q_g e^{nu} \tag{5.14}
\]

With (5.10), (5.11), (5.12) (5.13) and (5.14), we complete the argument. \(\square\)

**Remark 5.1** The \((\mathbb{R}^n, \bar{g})\) may have different \(\beta\) and \(\gamma\) with \((\mathbb{R}^n, g)\). We also note that the formula (5.9) is very important, it determines whether we can translate the general case into the radial symmetric case. Namely, the \(\alpha\) in Lemma 5.2 and the \(\alpha\) in Theorem 5.2 are the same.

**Theorem 5.2** Suppose \((\mathbb{R}^n, g = e^{2u}|dx|^2)\) is complete and satisfies

\[
\begin{cases}
C_0 e^{-|x|^\beta} \leq Q_g(x) \leq C_0 |x|^\gamma & \text{for} \ |x| \gg 1, \ \text{where} \ \gamma > 0 \ \text{and} \ \beta \in (0, 1), \\
Q_g \geq 0 \ \text{and} \ Q_g(x)e^{nu(x)} \in L^1(\mathbb{R}^n)
\end{cases}
\]

then

\[
1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy = \lim_{r \to \infty} \frac{|\partial B_r(0)|^{\frac{n-1}{n}}}{n\omega_{n-1}|\partial B_r(0)|}[\frac{\|\bar{u}\|_g}{n\omega_{n-1}|\partial B_r(0)|}]^{\frac{n-1}{n}}
\]

**Proof.** Similarly, by \(\alpha \leq 1\) we know

\[
\int_{B_r} e^{nu} \to \infty \quad \text{and} \quad \frac{d}{dr} \int_{B_r} e^{nu} = \int_{\partial B_r} e^{nu} > 0. \tag{5.15}
\]

Clearly,

\[
\frac{d}{dr} \left( \int_{B_r} e^{(n-1)\bar{u}(r)} \right)^{\frac{n}{n-1}} \frac{n}{n\omega_{n-1}} \frac{d}{dr} \int_{B_r} e^{nu}
\]

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\[ (\int_{\partial B_r} e^{(n-1)u(r)})^{\frac{1}{n-1}} \left( \frac{1}{\omega_{n-1}} \int_{\partial B_r} e^{nu} \right) \frac{\partial u(r)}{\partial r} (n-1) \bar{u}(r) + \frac{1}{r} \int_{\partial B_r} e^{(n-1)u(r)} = \frac{1}{\omega_{n-1}} \int_{\partial B_r} e^{nu} \left( \frac{\partial u(r)}{\partial r} (n-1) \bar{u}(r) + 1 \right) \bar{u}(r) \] \[ \] \[ (5.16) \]

By the Lemma 5.2, we obtain

\[ \lim_{r \to \infty} \left( \frac{1}{\omega_{n-1}} \int_{\partial B_r} e^{nu} \right)^{\frac{1}{n-1}} \left( \frac{\partial u(r)}{\partial r} (n-1) \bar{u}(r) + \frac{1}{r} \int_{\partial B_r} e^{(n-1)u(r)} \right) = 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy. \]

(5.17)

Lemma 5.3 implies that

\[ \lim_{r \to \infty} \int_{\partial B_r} e^{ku(r)} = 1 \quad \text{for} \quad k > 0. \]

(5.18)

Combined with (5.16), (5.17) and (5.18), we have

\[ \lim_{r \to \infty} \frac{d}{dr} \left( \int_{\partial B_r} e^{(n-1)u(r)} \right)^{\frac{1}{n-1}} \frac{1}{\omega_{n-1} \int_{\partial B_r} e^{nu}} \frac{\partial u(r)}{\partial r} = 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy. \]

Again (5.15) and L'Hopital's rule give that

\[ \lim_{r \to \infty} \left( \int_{\partial B_r} e^{(n-1)u(r)} \right)^{\frac{1}{n-1}} \frac{\frac{1}{\omega_{n-1}} \int_{\partial B_r} e^{nu}} = 1 - \frac{1}{c_n} \int_{\mathbb{R}^n} Q_g(y)e^{nu(y)}dy. \]

(5.19)

The final result follows by (5.17) and (5.19).

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