Spherical structures on torus knots and links *

Alexander Kolpakov, Alexander Mednykh

Abstract

The present paper considers two infinite families of cone-manifolds endowed with spherical metric. The singular strata is either the torus knot $t(2n + 1, 2)$ or the torus link $t(2n, 2)$. Domains of existence for a spherical metric are found in terms of cone angles and volume formulae are presented.

Key words: Spherical geometry, cone-manifold, knot, link.

1 Introduction

A three-dimensional cone-manifold is a metric space obtained from a collection of disjoint simplices in the space of constant sectional curvature $k$ by isometric identification of their faces in such a combinatorial fashion that the resulting topological space is a manifold (also called the underlying space for a given cone-manifold).

Such the metric space inherits the metric of sectional curvature $k$ on the union of its 2- and 3-dimensional cells. In case $k = +1$ the corresponding cone-manifold is called spherical (or admits a spherical structure). By analogy, one defines euclidean ($k = 0$) and hyperbolic ($k = −1$) cone-manifolds.

The metric structure around each 1-cell is determined by a cone angle that is the sum of dihedral angles of corresponding simplices sharing the 1-cell under identification. The singular locus of a cone-manifold is the closure of all its 1-cells with cone angle different from $2\pi$. For the further account we suppose that every component of the singular locus is an embedded circle with constant cone angle along it.

A particular case of cone-manifold is an orbifold with cone angles $2\pi/m$, where $m$ is an integer (cf. [1]).

The present paper considers two infinite families of cone-manifolds with underlying space the three-dimensional sphere $\mathbb{S}^3$. The first family consists of cone-manifolds with singular locus the torus knot $t(2n + 1, 2)$ with $n \geq 1$. In the rational census [2] these knots are denoted by $(2n + 1)/1$. The second family of cone-manifolds consists of those with singular locus a two-component torus link $t(2n, 2)$ with $n \geq 2$. These links are two-bridge and correspond to the links

*The work is performed under auspices of the Swiss National Science Foundation no. 200020-113199/1, “Scientific Schools”-5682.2008.1 and RFBR no. 06-01-00153.
2n/1 in the rational census. The simplest examples of such the knots and links are the trefoil knot 3/1 and the link 4/1. In the Rolfsen table [2] one finds them as the knot 3_1 and the link 4_1.

By the Theorem of W. Thurston [3], the manifold $S^3\setminus S^1$ does not admit a hyperbolic structure. However, it admits two other geometric structures [4]: $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\text{PSL}}(2, \mathbb{R})$. It follows from the paper [5] that the spherical dodecahedron space (i.e. Poincaré homology sphere) is a cyclic 5-fold covering of $S^3$ branched over $3_1$. Thus, the orbifold $3_1(\frac{2\pi}{5})$ with singular locus the trefoil knot and cone angle $\frac{2\pi}{5}$ is spherical. Due to the Dunbar’s census [6], orbifold $3_1(\frac{2\pi}{n})$ is spherical if $n \leq 5$, Nil-orbifold if $n = 6$ and $\widetilde{\text{PSL}}(2, \mathbb{R})$-orbifold if $n \geq 7$. Spherical structure on the cone-manifold $3_1(\alpha)$ with underlying space the three-dimensional sphere $S_3$ is studied in [7].

The consideration of two-bridge torus links is carried out starting with the simplest one possessing non-abelian fundamental group, namely $4_1$.

The previous investigation on spherical structures for cone-manifolds is carried out mainly in the papers [8, 9, 10]. The present paper develops a method to analyse existence of a spherical metric for two-bridge torus knot and link cone-manifolds. Also, the lengths of singular geodesics are calculated and the volume formulae are obtained (cf. Theorem 1 and Theorem 2).

2 Projective model $S^3_\lambda$

The purpose of the present section is to construct the projective model $S^3_\lambda$ that one can use to study geometric properties of two-bridge torus knots and links and to build up holonomy representation for the corresponding cone-manifolds. Other projective models for homogeneous geometries are described in [11].

Consider the set $\mathbb{C}^2 = \{ (z_1, z_2) : z_1, z_2 \in \mathbb{C} \}$ as a four-dimensional vector space over $\mathbb{R}$. We denote it by $\mathbb{C}^2_\mathbb{R}$ and equip with Hermitian product

$$\langle (z_1, z_2), (w_1, w_2) \rangle_H = (z_1, z_2)^T H (w_1, w_2),$$

where

$$H = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

is a symmetric matrix with $-1 < \lambda < +1$.

The natural inner product is associated to the Hermitian form above:

$$\langle (z_1, z_2), (w_1, w_2) \rangle = \text{Re} (\langle (z_1, z_2), (w_1, w_2) \rangle_H)$$

and the respective norm is

$$\| (z_1, z_2) \| = |z_1|^2 + |z_2|^2 + \lambda (\overline{z_1}z_2 + z_1\overline{z_2}).$$

Call two elements $(z_1, z_2)$ and $(w_1, w_2)$ in $\mathbb{C}^2_\mathbb{R} = \mathbb{C}^2 \setminus (0, 0)$ equivalent if there is $\mu > 0$ such that $(z_1, z_2) = (\mu w_1, \mu w_2)$. We denote this equivalence relation as $(z_1, z_2) \sim (w_1, w_2)$. 

2
Identify the factor-space $\mathbb{C}^2_\mathbb{R} / \sim$ with the three-dimensional sphere

$$S^3_\lambda = \{(z_1, z_2) \in \mathbb{C}^2_\mathbb{R} : \|(z_1, z_2)\| = 1 \},$$

endowed with the Riemannian metric

$$ds^2 = |dz_1|^2 + |dz_2|^2 + \lambda(dz_1 d\bar{z}_2 + d\bar{z}_1 dz_2).$$

By means of equality

$$ds^2 = \frac{1+\lambda}{2} |dz_1 + dz_2|^2 + \frac{1-\lambda}{2} |dz_1 - dz_2|^2,$$

the linear transformation

$$\xi_1 = \sqrt{\frac{1+\lambda}{2}} (z_1 + z_2), \quad \xi_2 = \sqrt{\frac{1-\lambda}{2}} (z_1 - z_2)$$

provides an isometry between $(S^3_\lambda, ds^2)$ and $(S^3, ds^2)$, where $ds^2 = |d\xi_1|^2 + |d\xi_2|^2$ is the standard metric of sectional curvature $+1$ on the unit sphere $S^3 = \{(\xi_1, \xi_2) \in \mathbb{C}^2 : |\xi_1|^2 + |\xi_2|^2 = 1 \}$.

Let $P, Q$ be two points in $S^3_\lambda$. The spherical distance between $P$ and $Q$ is a real number $d_\lambda(P, Q)$ that is uniquely determined by the conditions $0 \leq d_\lambda(P, Q) \leq \pi$ and $\cos d_\lambda(P, Q) = \langle P, Q \rangle$.

3 Torus knots $T_n$

Let $T_n, n \geq 1$ be the torus knot $t(2n+1, 2)$ embedded in $S^3$. The knot $T_n$ is the two-bridge knot $(2n + 1)/1$ in the rational census (Fig. 1). Let $T_n(\alpha)$ denote a cone-manifold with singular locus $T_n$ and the cone angle $\alpha$ along it.

![Figure 1: Knot (2n+1)/1](image)

The aim of the present section is to investigate cone-manifolds $T_n(\alpha), n \geq 1$ to find out the domain of sphericity in terms of the cone angle and to derive the volume formulae.

Two lemmas precede the further exposition:
Lemma 1 For every $0 < \alpha < 2\pi$ and $-1 < \lambda < +1$ the linear transformations
\[
A = \left( \begin{array}{cc} 1 & 0 \\ -2 i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} & e^{i\alpha} \end{array} \right)
\]
and
\[
B = \left( \begin{array}{cc} e^{i\alpha} & -2 i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} \\ 0 & 1 \end{array} \right)
\]
are isometries of $S^3_\lambda$.

Proof. For the further account let us assume that the multiplication of vectors by matrices is to the right. A linear transformation $L$ of the space $\mathbb{C}_R^2$ preserves the corresponding Hermitian form if and only if for every pair of vectors $P, Q \in \mathbb{C}_R^2$ it holds that
\[
\langle P, Q \rangle_H = PHLQ^T = PLHL^TQ^T = \langle PL, QL \rangle_H.
\]
The condition above is equivalent to
\[
H = LHL^T.
\]
In particular,
\[
\cos d_\lambda(P, Q) = \langle P, Q \rangle = \langle PL, QL \rangle = \cos d_\lambda(PL, QL),
\]
that means $L$ preserves the spherical distance between $P$ and $Q$.
Let $L = A$ and $L = B$ in series, one verifies that $A$ and $B$ preserve the Hermitian norm on $\mathbb{C}_R^2$ and, consequently, the spherical distance on $S^3_\lambda$. □

Lemma 2 Let $A$ and $B$ be the same matrices as in the affirmation of Lemma 1. Then for all integer $n \geq 1$ one has
\[
(AB)^nA - B(AB)^n = 2 U_{2n}(\Lambda) e^{i(2n+1)\pi/2} \sin \frac{\alpha}{2} M,
\]
where $M$ is a non-zero $2 \times 2$-matrix and $U_{2n}(\Lambda)$ is the second kind Chebyshev polynomial of power $2n$ in variable $\Lambda = \lambda \sin \frac{\alpha}{2}$.

Proof. As far as $-1 < \lambda < +1$, one obtains
\[
-1 < \Lambda = \lambda \sin \frac{\alpha}{2} < +1.
\]
Substitute
\[
\Lambda = \cos \theta,
\]
with the unique $0 < \theta < \pi$.
Then matrices $A$ and $B$ are rewritten in the form
\[
A = \left( \begin{array}{cc} 1 & 0 \\ -2 i e^{i\frac{\alpha}{2}} \cos \theta & e^{i\alpha} \end{array} \right),
\]
and
\[ B = \left( \begin{array}{cc} e^{i\alpha} & -2 ie^{\frac{i\pi}{2} \cos \theta} \\ 0 & 1 \end{array} \right). \]

On purpose to diagonalize the matrix \( AB \), use
\[ V = \left( \begin{array}{cc} ie^{-i\frac{\pi}{2}} e^{-i\theta} & i e^{-i\frac{\pi}{2}} e^{i\theta} \\ 1 & 1 \end{array} \right), \]

and obtain
\[ D = V^{-1}(AB)V = \left( \begin{array}{cc} -e^{i\alpha} e^{2i\theta} & 0 \\ 0 & -e^{i\alpha} e^{-2i\theta} \end{array} \right). \]

Note, that \( V \) might be not an isometry, but it is utile for computation. Thus
\[ (AB)^n A - B(AB)^n = (V D^n V^{-1})A - B(V D^n V^{-1}) = \]
\[ = 2 \frac{\sin(2n + 1) \theta}{\sin \theta} e^{i \frac{(2n + 1) \pi}{2 \alpha}} \sin \frac{\alpha}{2} \left( \begin{array}{cc} -1 & \lambda \\ -\lambda & 1 \end{array} \right) = \]
\[ = 2 U_{2n}(\cos \theta) e^{i \frac{(2n + 1) \pi}{2 \alpha}} \sin \frac{\alpha}{2} M = 2 U_{2n}(\Lambda) e^{i \frac{(2n + 1) \pi}{2 \alpha}} \sin \frac{\alpha}{2} M, \]

with the matrix
\[ M = \left( \begin{array}{cc} -1 & \lambda \\ -\lambda & 1 \end{array} \right) \]
as the present Lemma claims. \( \square \)

The main theorem of the section follows:

**Theorem 1** The cone-manifold \( T_n(\alpha) \), \( n \geq 1 \) is spherical if
\[ \frac{2n - 1}{2n + 1} \pi < \alpha < 2\pi - \frac{2n - 1}{2n + 1} \pi. \]

The length of its singular geodesic (i.e. the length of the knot \( T_n \)) equals
\[ l_\alpha = (2n + 1) \alpha - (2n - 1) \pi. \]

The volume of \( T_n(\alpha) \) is
\[ \text{Vol} T_n(\alpha) = \frac{1}{2n + 1} \left( \frac{2n + 1}{2} \alpha - \frac{n - 1}{2} \pi \right)^2. \]

**Proof.** The fundamental group of the knot \( T_n \) is presented as
\[ \pi_1(S^3 \setminus T_n) = \langle a, b | (ab)^n a = b(ab)^n \rangle, \]
with generators \( a \) and \( b \) as at Fig. 1.

Since the cone-manifold \( T_n(\alpha) \) admits a spherical structure, then there exists a holonomy mapping \([1]\), that is a homomorphism
\[ h : \pi_1(S^3 \setminus T_n) \longrightarrow \text{Isom} S^3, \]

5
We will choose $h$ in respect with geometric construction of the cone-manifold. All the further computations to find the length of the knot $T_n$ and the volume of the cone-manifold $T_n(\alpha)$ are performed making use of the corresponding fundamental polyhedron $\mathcal{P}_n$ (Fig. 2). The construction algorithm for the polyhedron is given in [12].

The combinatorial polyhedron $\mathcal{P}_n$ has vertices $P_i, i \in \{1, \ldots, 4n + 2\}$ and edges $P_i P_{i+1}, i \in \{1, \ldots, 4n + 2\}$, with $P_{4n+3} = P_1$, also $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$. Let $N, S$ denote the middle points (the North and the South poles of $\mathcal{P}_n$) on the edges $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$, respectively. Then, consider also edges $NP_i, SP_i, i \in \{1, \ldots, 4n + 2\}$.

Without loss in generality, choose the holonomy representation such that

$$h(a) = A, \ h(b) = B,$$

where $A$ and $B$ are matrices from Lemma 1.

The generators of the fundamental group for $T_n$ under the holonomy mapping $h$ correspond to isometries acting on $\mathcal{P}_n$. These isometries identify its faces by means of rotation about the edge $P_1 P_{2n+2}$ for the top “cupola” of $\mathcal{P}_n$ and rotation about $P_2 P_{2n+3}$ for the bottom one (see, Fig. 2). Then the edges $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$ knot itself to produce $T_n$ (cf. [12, 13]).

In order to construct the polyhedron $\mathcal{P}_n$ assume that its edge $P_1 P_2$ is given by

$$P_1 = (1, 0), \ P_2 = (0, 1).$$

Then one has

$$\cos d_A(P_1, P_2) = \langle P_1, P_2 \rangle = \lambda,$$

i.e. the spherical distance between the points $P_1$ and $P_2$ can vary from 0 to $\pi$. Thus, prescribing certain coordinates to the end-points of the edge $P_1 P_2$ we do not loss in generality of the consideration.

Note, that the axis of the isometry $A$ from Lemma 1 contains $P_1$ and the axis of $B$ contains $P_2$. The aim of the construction for the polyhedron $\mathcal{P}_n$ is to bring its edges $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$ to be axes of the respective isometries $A$ and $B$.

The other vertices $P_i$ has to be images of $P_1$ and $P_2$ under action of $A$ and $B$. The polyhedron $\mathcal{P}_n$ is said to be proper if

(a) inner dihedral angles along $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$ are equal to $\alpha$;

(b) the following curvilinear faces are identified by $A$ and $B$:

$$A : NP_1 P_2 \ldots P_{2n+2} \rightarrow NP_1 P_{4n+2} \ldots P_{2n+3} P_{2n+2},$$

$$B : SP_2 P_1 P_{4n+2} \ldots P_{2n+3} \rightarrow SP_2 P_3 \ldots P_{2n+3};$$

(c) sum of the inner dihedral angles $\psi_i$ along $P_i P_{i+1}, i \in \{1, \ldots, 4n + 1\}$ equals $2\pi$;

(d) sum of the dihedral angles $\phi_i$ for corresponding tetrahedra $N S P_i P_{i+1}, i \in \{1, \ldots, 4n + 1\}$ at their common edge $N S$ is $2\pi$.
(e) all the tetrahedra $NSP_i P_{i+1}$ with $i \in \{1, \ldots, 4n + 2\}$, $P_{4n+3} = P_1$ are non-degenerated and coherently oriented.

By the orientation of a tetrahedron $NSP_i P_{i+1}$ one means the sign of the Gram determinant $\det(S, N, P_i, P_{i+1})$ for corresponding quadruple $S, N, P_i, P_{i+1} \in C^2_R$, where $i \in \{1, \ldots, 4n + 2\}$, $P_{4n+3} = P_1$. A tetrahedron is non-degenerated if $\det(S, N, P_i, P_{i+1}) \neq 0$. Thus, claim (e) is satisfied if all the Gram determinants are non-zero and of the same sign.

If $\alpha = \frac{2\pi}{m}$, $m \in \mathbb{N}$, then due to the Poincaré Theorem [14, Theorem 13.5.3] claims (a) – (e) imply that the group generated by the isometries $A$ and $B$ is discreet and its presentation is

$$\Gamma = \langle A, B | (AB)^n A = B(AB)^n, A^m = B^m = \text{id} \rangle.$$

The metric space $S^3 / \Gamma \cong \mathbb{T}_n \left( \frac{2\pi}{m} \right)$ is a spherical orbifold, and $\mathcal{P}_n$ is its fundamental polyhedron. If $m \not\in \mathbb{N}$ then the group generated by $A$ and $B$ might be non-discreet. However, the identification for the faces of $\mathcal{P}_n$ is of the same fashion as if it were $m \in \mathbb{N}$ and as the result one obtains the cone-manifold $\mathbb{T}_n (\alpha)$. By means ofLemma 1 and construction of $\mathcal{P}_n$ claims (a) and (b) are satisfied. For the holonomy mapping $h$ to exist the following relation should be satisfied:

$$h((ab)^n a) - h(b(ab)^n) = (AB)^n A - B(AB)^n = 0.$$

By Lemma 2, the condition above is satisfied if and only if

$$U_{2n}(\Lambda) = 0,$$
where $\Lambda = \lambda \sin \frac{\alpha}{2}$.

Thus, the parameter $\lambda$ of the metric $d_{s^2}$ is determined completely by a root of the polynomial $U_{2n}(\Lambda)$. From the above formula, $\lambda$ is related to the cone angle $\alpha$ by means of the equality

$$\lambda = \frac{\Lambda}{\sin \frac{\alpha}{2}}.$$

The roots of $U_{2n}(\Lambda)$ are given by the following formula:

$$\Lambda_k = \cos \frac{k\pi}{2n+1},$$

with $k \in \{1, \ldots, 2n\}$.

The parameter $\lambda$ for the metric $d_{s^2}$ has to be chosen in order the polyhedron $P_n$ be proper and the metric itself be spherical.

Note, that the edges $P_iP_{i+1}$, $i \in \{1, \ldots, 4n+2\}$, $P_{4n+3} = P_1$ are equivalent under action of the group $\Gamma = \langle A, B \rangle$. Thus, the relation $(AB)^nA = B(AB)^n$ implies the equality

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi,$$

where $k$ is an integer.

Show that one can choose $\lambda$ for the equality $k = 1$ to hold for all $\alpha$ in the affirmation of the Theorem. Due to the paper [15], every two-bridge knot cone-manifold with cone angle $\pi$ is a spherical orbifold. In this case all the vertices $P_i$ of the fundamental polyhedron belong to the same circle and all the dihedral angles $\psi_i$ and $\phi_i$ are equal to each other [12]:

$$\phi_i = \psi_i = \frac{\pi}{2n+1}.$$

As far as $\cos d_{s^2}(N, S) = \cos d_{s^2}(P_t, P_{t+1}) = \lambda$, then in case $\alpha = \pi$ one obtains

$$\lambda = \frac{\Lambda_k}{\sin \frac{\alpha}{2}} = \cos \theta$$

for certain $k \in \{1, \ldots, 2n\}$ and then

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2(2n+1)\theta.$$

Using the formula for the roots of $U_{2n}(\Lambda)$ obtain that

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi$$

if $\alpha = \pi$. Thus, claim (c) for the polyhedron $P_n$ with $\alpha = \pi$ is satisfied if $k = 1$. As far as the parameter $\alpha$ varies continuously and sum of the angles $\psi_i$
represents a multiple of $2\pi$, one has that

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2\pi$$

for all $\alpha$.

By analogy, show that with

$$\lambda = \frac{A_1}{\sin \frac{\alpha}{2}}$$

the equality

$$\sum_{i=1}^{2(2n+1)} \phi_i = 2\pi$$

holds, that means claim (d) is also satisfied.

Verify that under conditions of the Theorem the metric $ds^2$ is spherical. This claim is equivalent to the inequality

$$-1 < \lambda < +1.$$ 

Note, that for

$$\frac{2n-1}{2n+1} \pi < \alpha < 2\pi - \frac{2n-1}{2n+1} \pi$$

it follows

$$\sin \frac{\alpha}{2} > \sin \frac{(2n-1)\pi}{2(2n+1)}.$$ 

As far as $\sin \frac{\alpha}{2} > 0$ and $A_1 = \sin \frac{(2n-1)\pi}{2(2n+1)} > 0$, one has

$$0 < \lambda < 1.$$ 

By analogy with Lemma 1 verify that

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an isometry of $ds^2$.

Fixed point sets of $A$ and $B$ in $S_\lambda^3$ are circles

$$\text{Fix } A = \{ (z_1, 0) : z_1 \in \mathbb{C}, |z_1| = 1 \}$$

and

$$\text{Fix } B = \{ (0, z_2) : z_2 \in \mathbb{C}, |z_2| = 1 \},$$

correspondingly. The geometric meaning of $C$ is that it maps the first fixed circle to the other. Thus, the relation $B = CAC^{-1}$ holds.

The following equalities

$$P_{2k+1} = P_{1}(AB)^k, \quad k \in \{0, \ldots, n\},$$
\[ P_{2k} = P_2(AB)^{k-1}, \quad k \in \{1, \ldots, n+1\} \]

and
\[
\begin{align*}
P_{2k+1} &= P_1(BA)^{2n-k+1}, \quad k \in \{n+1, \ldots, 2n\}, \\
P_{2k} &= P_2(BA)^{2n-k+2}, \quad k \in \{n+2, \ldots, 2n+1\},
\end{align*}
\]

follow from the identification scheme of the edges of \( \mathcal{P}_n \).

Define the auxiliary function
\[ \varepsilon(m) = \frac{m}{2} \alpha - \frac{4n-m}{2} \pi. \]

By analogy with the proof of Lemma 2 it follows that
\[
(AB)^k = C(BA)^k C^{-1} =
\begin{pmatrix}
\sin(2k-1)\theta & e^{i\varepsilon(2k)} \\
\sin(2k)\theta & e^{i\varepsilon(2k-1)}
\end{pmatrix},
\]

where \( \theta = \frac{\pi}{2n+1} \).

Suppose \( N \) and \( S \) to be middle-points of the edges \( P_1P_{2n+2} \) and \( P_2P_{2n+3} \), respectively. Then
\[
N = (e^{i\varepsilon(2n+1)/2}, 0), \quad S = (0, e^{i\varepsilon(2n+1)/2}).
\]

For the lengths \( l_\alpha \) of the singular geodesic one has
\[
\cos \frac{l_\alpha}{4} = \langle P_1, N \rangle = \langle P_1C, NC \rangle = \langle P_2, S \rangle.
\]

Thus
\[
\cos \frac{l_\alpha}{4} = \cos \left( \frac{(2n+1)\alpha - (2n-1)\pi}{4} \right).
\]

By construction of the polyhedron \( \mathcal{P}_n \), the inequality \( 0 < l_\alpha < 4\pi \) holds. Then it follows
\[
l_\alpha = (2n+1)\alpha - (2n-1)\pi.
\]

Given the coordinates of the vertices \( P_i \) and the poles \( N \) and \( S \) of the polyhedron \( \mathcal{P}_n \), verify claim (e).

For every four points \( A, B, C, D \in \mathbb{C}_R^2 \), where
\[
A = (A_1, A_2), \quad B = (B_1, B_2), \quad C = (C_1, C_2), \quad D = (D_1, D_2),
\]

their Gram determinant is
\[
\det(A, B, C, D) := \det \begin{pmatrix}
\Re A_1 & \Im A_1 & \Re A_2 & \Im A_2 \\
\Re B_1 & \Im B_1 & \Re B_2 & \Im B_2 \\
\Re C_1 & \Im C_1 & \Re C_2 & \Im C_2 \\
\Re D_1 & \Im D_1 & \Re D_2 & \Im D_2
\end{pmatrix}.
\]
Each tetrahedron $\text{NSP}_i P_{i+1}$ with $i \in \{1, \ldots, 2n+1\}$ is isometric to $\text{NSP}_{2n+i+1} P_{2n+i+2}$, $i \in \{1, \ldots, 2n+1\}$, $P_{4n+3} = P_1$ by means of the isometry $C$ defined above. Thus, we consider only the tetrahedra $\text{NSP}_i P_{i+1}$ with $i \in \{1, \ldots, 2n+1\}$. Split them into two groups: the tetrahedra $\text{NSP}_{2k+1} P_{2k+2}$ with $k \in \{0, \ldots, n\}$ and the tetrahedra $\text{NSP}_{2k} P_{2k+1}$ with $k \in \{1, \ldots, n\}$.

Substitute $\alpha = \beta + \pi$ and proceed with straightforward calculations:

\[
\Delta_k^{(1)}(\beta) = \det(S, N, P_{2k+1}, P_{2k+2}) = \cos^2 \frac{L_1 \beta}{4} - U^2_{2k-1}(\cos \theta) \sin^2 \frac{\beta}{2} =
\]

\[
= T^2_{L_1}(\cos \frac{\beta}{4}) - U^2_{2k-1}(\cos \theta) \sin^2 \frac{\beta}{2},
\]

where $k \in \{0, \ldots, n\}$, $L_1 = |2n - 4k + 1|$, $\theta = \frac{\pi}{2n+1}$, $\beta \in [-2\theta, 2\theta]$:

\[
\Delta_k^{(2)}(\beta) = \det(S, N, P_{2k}, P_{2k+1}) = \cos^2 \frac{L_2 \beta}{4} - U^2_{2k-2}(\cos \theta) \sin^2 \frac{\beta}{2} =
\]

\[
= T^2_{L_2}(\cos \frac{\beta}{4}) - U^2_{2k-1}(\cos \theta) \sin^2 \frac{\beta}{2},
\]

where $k \in \{1, \ldots, n\}$, $L_2 = |2n - 4k + 3|$, $\theta$ and $\beta$ the same as above. The first kind Chebyshev polynomial of degree $k \geq 0$ is denoted by $T_k$. Assume that

\[
U_{-1}(\cos \theta) = 0, \quad U_0(\cos \theta) = 1
\]

for the sake of brevity.

All the functions $\Delta_k^{(j)}(\beta), j \in \{1, 2\}$ are even on the interval $[-2\theta, 2\theta]$. Then one considers them only on the interval $[0, 2\theta]$. Note, that the polynomial $T^2_{L_j}(\cos \beta)$ monotonously decreases and the function $\sin^2 \frac{\beta}{2}$ monotonously increases with $\beta \in [0, 2\theta]$. Moreover, $T^2_{L_j}(\cos 0) = T^2_{L_j}(1) = 1$. Then it follows that $\Delta_k^{(j)}(\beta) > 0$ with $\beta \in (-2\theta, 2\theta)$. Also, one has $\Delta_k^{(j)}(\pm 2\theta) = 0$.

Then for all $\beta \in (-2\theta, 2\theta)$ (i.e. for all $\alpha$ in the affirmation of the Theorem)

\[
\det(S, N, P_i, P_{i+1}) > 0
\]

where $i \in \{1, \ldots, 4n+2\}$, $P_{4n+3} = P_1$. Thus, claim (e) for the polyhedron $\mathcal{T}_n$ is satisfied.

Use the Schl"afli formula [16] to obtain the volume formula for $\mathcal{T}_n(\alpha)$. One has

\[
d\text{Vol} \mathcal{T}_n(\alpha) = \frac{L_\alpha}{2} d\alpha = \frac{(2n + 1)\alpha - (2n - 1)\pi}{2} d\alpha.
\]

Note, that $\text{Vol} \mathcal{T}_n(\alpha) \to 0$ with $\alpha \to \frac{2n - 1}{2n + 1} \pi$. In this case $d\lambda(P_i, P_{i+1}) \to 0$, where $i \in \{1, \ldots, 4n+2\}$, $P_{4n+3} = P_1$ and the fundamental polyhedron collapses to a point. Thus

\[
\text{Vol} \mathcal{T}_n(\alpha) = \frac{1}{2n+1} \left( \frac{2n + 1}{2} \alpha - \frac{2n - 1}{2} \pi \right)^2.
\]

\[
\square
\]

**Remark 1.** The domain of the spherical metric existence in Theorem 1 was indicated before in [10, Proposition 2.1].
4 Torus links $\mathbb{L}_n$

Let $\mathbb{L}_n, n \geq 2$ be a torus link $t(2n, 2)$ with two components. The corresponding link in the rational census is $2n/1$ (Fig. 3). The fundamental group of $\mathbb{L}_n$ is presented as

$$\pi_1(S^3 \setminus \mathbb{L}_n) = \langle a, b \mid (ab)^n = (ba)^n \rangle.$$ 

Let $\mathbb{L}_n(\alpha, \beta)$ denote a cone-manifold with singular locus the link $\mathbb{L}_n$ and the cone angles $\alpha, \beta$ along its components.

For every $\alpha, \beta \in (0, 2\pi)$ and $\lambda \in (-1, +1)$, we denote

$$A = \begin{pmatrix} 1 & 0 \\ -2i e^{i/2} \lambda \sin \frac{\alpha}{2} & e^{i\alpha} \end{pmatrix}$$

and

$$B = \begin{pmatrix} e^{i\beta} & -2i e^{i/2} \lambda \sin \frac{\beta}{2} \\ 0 & 1 \end{pmatrix}.$$ 

By Lemma 1, linear transformations $A$ and $B$ are isometries of $S^3_{\lambda}$.

**Lemma 3** For every integer $n \geq 2$ the following equality holds

$$(AB)^n - (BA)^n = 4 U_{n-1}(\Lambda) \lambda e^{i(n+\beta-\beta)/2} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} M,$$

where $M$ is a non-zero $2 \times 2$ matrix and $U_{n-1}(\Lambda)$ is the second kind Chebyshev polynomial of degree $n - 1$ in variable

$$\Lambda = (1 - \lambda^2) \cos \frac{\alpha - \beta}{2} + \lambda^2 \cos \frac{\alpha + \beta}{2}.$$ 

**Proof.** By analogy with Lemma 2. □

With Lemma 3 the main theorem of the section follows:
Theorem 2 The cone-manifold $L_n(\alpha, \beta), n \geq 2$ is spherical if

\[-2\pi \left(1 - \frac{1}{n}\right) < \alpha - \beta < 2\pi \left(1 - \frac{1}{n}\right),\]

\[2\pi \left(1 - \frac{1}{n}\right) < \alpha + \beta < 2\pi \left(1 + \frac{1}{n}\right).\]

The lengths $l_\alpha, l_\beta$ of its singular geodesics (i.e. lengths of the components for $L_n$) are equal to each other and

\[l_\alpha = l_\beta = \frac{\alpha + \beta}{2} n - \pi (n - 1).\]

The volume of $L_n(\alpha, \beta)$ is

\[\text{Vol} L_n(\alpha, \beta) = \frac{1}{2n} \left(\frac{\alpha + \beta}{2} n - (n - 1)\pi\right)^2.\]

Proof. One continues the proof by analogy with Theorem 1. Suppose that $L_n(\alpha, \beta)$ is spherical. Then there exists a holonomy mapping [1]:

\[h : \pi_1(S^3 \setminus L_n) \mapsto \text{Isom } S^3,\]

\[h(a) = A, h(b) = B.\]

Also,

\[h((ab)^n) - h((ba)^n) = (AB)^n - (BA)^n = 0.\]

By means of Lemma 3 the equality above holds either if $\lambda = 0$, or if

\[\Lambda = (1 - \lambda^2) \cos \frac{\alpha - \beta}{2} + \lambda^2 \cos \frac{\alpha + \beta}{2}\]

is a root of the equation $U_{n-1}(\Lambda) = 0$.

In case $\lambda = 0$ the image of $h$ is abelian, because of the additional relation $AB = BA$. With $n \geq 2$ this leads to a degenerate geometric structure. Thus, one has to choose the parameter $\lambda$ for the metric $ds^2$ using roots of the Chebyshev polynomial $U_{n-1}(\Lambda)$.

The fundamental polyhedron $F_n$ for the cone-manifold $L_n(\alpha, \beta)$ is depicted at Fig. 4. Suppose its vertices $P_1$ and $P_2$ to be

\[P_1 = (1, 0), P_2 = (0, 1).\]

The axes of isometries $A$ and $B$ correspond to the edges $P_1P_{2n+1}$ and $P_2P_{2n+2}$. Points $N$ and $S$ are respective middles of the edges $P_1P_{2n+1}$ and $P_2P_{2n+2}$. Those are called North and South poles of the polyhedron.

The polyhedron $F_n$ is said to be proper if

(a) respective inner dihedral angles along the edges $P_1P_{2n+1}$ and $P_2P_{2n+2}$ are equal to $\alpha$ and $\beta$;
Figure 4: The fundamental polyhedron $F_n$ for $L_n(\alpha, \beta)$

(b) curvilinear faces of the polyhedron are identified by $A$ and $B$:
\[ A: NP_1P_2\ldots P_{2n+1} \rightarrow NP_1P_{4n}\ldots P_{2n+2}P_{2n+1}, \]
\[ B: SP_2P_1P_{4n}\ldots P_{2n+2} \rightarrow SP_2P_3\ldots P_{2n+2}; \]

(c) sum of the inner dihedral angles $\psi_i$ along the edges $P_iP_{i+1}$, $i \in \{1,\ldots,4n-1\}$ equals $2\pi$;

(d) sum of the dihedral angles $\phi_i$ for tetrahedra $NSP_iP_{i+1}$, $i \in \{1,\ldots,4n-1\}$ at their common edge $NS$ equals $2\pi$;

(e) all the tetrahedra $NSP_iP_{i+1}$ with $i \in \{1,\ldots,4n\}, P_{4n+1} = P_1$ are non-degenerated and coherently oriented.

In order to choose the parameter $\lambda$ for the corresponding metric consider the fundamental polyhedron $F_n$ with $\alpha = \beta = \pi$. Then all its vertices belong to the same circle and all the dihedral angles $\psi_i$ of the tetrahedra $NSP_iP_{i+1}$ along the edges $P_iP_{i+1}$ are equal to $\psi = \frac{\pi}{2n}$ [12]. Also the dihedral angles $\phi_i$ of the tetrahedra $NSP_iP_{i+1}$ along their common edge $NS$ are equal to each other:
\[ \phi_i = \phi = \frac{\pi}{2n}. \]

In this case $\lambda = \langle P_1, P_2 \rangle = \cos \phi$ and
\[ \Lambda = -\cos 2\phi = \cos \frac{(n-1)\pi}{n}. \]
All the roots of $U_{n-1}(\Lambda)$ are given by the formula

$$\Lambda_k = \cos \frac{k\pi}{n}, \quad k \in \{1, \ldots, n-1\},$$

so one choose the root $\Lambda_k$ with $k = n - 1$. Then, by analogy with Theorem 1, equalities

$$\sum_{i=1}^{4n} \psi_i = 2\pi$$

and

$$\sum_{i=1}^{4n} \phi_i = 2\pi$$

are satisfied at the point $\alpha = \beta = \pi$ of the domain

$$\mathcal{D} = \left\{ (\alpha, \beta) : |\alpha - \beta| < 2\pi \left( 1 - \frac{1}{n} \right), |\alpha + \beta - 2\pi| < \frac{2\pi}{n} \right\},$$

depicted at Fig. 5.

In terms of the parameter $\lambda$, that defines the metric $d\sigma_\lambda^2$, one has

$$\lambda^2 = \frac{\cos \frac{\alpha-\beta}{2} + \cos \frac{\pi}{n}}{\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2}}.$$ \hspace{1cm} \text{(15)}$$

As for all $(\alpha, \beta) \in \mathcal{D}$ the inequality $0 < \lambda^2 < 1$ is satisfied, the metric $d\sigma_\lambda^2$ is spherical regarding the corresponding domain. By analogy with Theorem 1 one can show that claims (a) – (d) for the polyhedron $\mathcal{F}_n$ are satisfied in the interior of $\mathcal{D}$.

The lengths $l_\alpha$ and $l_\beta$ of singular geodesics for the cone-manifold $\mathbb{L}_n(\alpha, \beta)$ meet the relations

$$\cos \frac{l_\alpha}{2} = \langle P_1, N \rangle,$$

$$\cos \frac{l_\beta}{2} = \langle P_2, S \rangle.$$

By analogy with the proof of Theorem 1 one obtains

$$l_\alpha = l_\beta = \frac{\alpha + \beta}{2} n - \pi(n-1).$$

Given the coordinates of the vertices for the fundamental polyhedron verify claim (e) for all $(\alpha, \beta)$ in the domain $\mathcal{D}$.

Make use of the Schläfli formula [16] to obtain the volume of $\mathbb{L}_n(\alpha, \beta)$:

$$d\text{Vol} \mathbb{L}_n(\alpha, \beta) = \frac{l_\alpha}{2} d\alpha + \frac{l_\beta}{2} d\beta = \left( \frac{\alpha + \beta}{2} n - \pi(n-1) \right) d\left( \frac{\alpha + \beta}{2} \right).$$

Note, that with

$$\alpha = \beta \rightarrow \pi \frac{n-1}{n}$$

the fundamental polyhedron $\mathcal{F}_n$ collapses to a point (i.e. the volume tends to 0). The last affirmation of the Theorem follows. □
Figure 5: The domain $\mathcal{D}$ of sphericity for $L_n(\alpha, \beta)$

**Remark 2** Under condition $\alpha = \beta$ the inequality from the affirmation of Theorem 2 coincides with the inequality from [10, Proposition 2.2].

**Remark 3** Note, that the lengths of the singular geodesics for $L_n(\alpha, \beta)$ are equal even if $\alpha \neq \beta$.

**References**

[1] *Thurston W. P.* The geometry and topology of 3–manifolds. Princeton: Lecture Notes, 1977–78. Available on-line.

[2] *Rolfsen D.* Knots and links. Berkeley: Publish or Perish Inc., 1976. Google Books.

[3] *Thurston W. P.* Hyperbolic geometry and 3–manifolds. Cambridge: Cambridge Univ. Press, 1982. (London Math. Soc. Lect. Note Ser.; 48, 9–25).

[4] *Neumann W. P.* Notes on geometry and 3-manifolds, with appendix by Paul Norbury, in: Low Dimensional Topology, Böröczky, Neumann, Stipsicz, Eds. // Bolyai Society Mathematical Studies. 1999. V. 8. P. 191–267. Available on-line.

[5] *Seifert H., Weber C.* Die beiden Dodecaederräme // Math. Z. 1933. V. 37. P. 237–253. Available on-line.

[6] *Dunbar W. D.* Geometric orbifolds // Rev. Mat. Univ. Complut. Madrid. 1988. V. 1. P. 67–99. Available on-line.
[7] Derevnin D., Mednykh A., Mulazzani M. Geometry of trefoil cone–manifold // Preprint. 2007.

[8] Hilden H. M., Lozano M. T., Montesinos-Amilibia J. M. On a remarkable polyhedron geometrizing the figure eight cone manifolds // J. Math. Sci. Univ. Tokyo. 1995. V. 2. P. 501–561. Available on-line.

[9] Mednykh A., Rasskazov A. Volumes and degeneration of cone-structures on the figure–eight knot // Tokyo J. of Math. 2006. V. 29. N. 2. P. 445–464.

[10] Porti J. Spherical cone structures on 2–bridge knots and links // Kobe J. of Math. 2004. V. 21. N. 1. P. 61–70. Available on-line.

[11] Molnár E. The projective interpretation of the eight 3–dimensional homogeneous geometries // Beiträge zur Algebra und Geometrie. 1997. V. 38. N. 2. P. 261–288. Available on-line.

[12] Mednykh A., Rasskazov A. On the structure of the canonical fundamental set for the 2-bridge link orbifolds // Preprint. 1996. Available on-line.

[13] Minkus J. The branched cyclic coverings of 2-bridge knots and links // Mem. Amer. Math. Soc. 1982. V. 35. N. 255. Google Books.

[14] Ratcliffe J. Foundations of hyperbolic manifolds. New York: Springer-Verlag, 1994. (Graduate Texts in Math.; 149). Google Books.

[15] Hodgson C., Rubinstein J. H. Involutions and isotopies of lens spaces, Knot theory and manifolds (Vancouver, B.C., 1983). Berlin: Springer-Verlag, 1985. (Lecture Notes in Math.; 1144, 60–96).

[16] Hodgson C. Degeneration and regeneration of hyperbolic structures on three–manifolds // Princeton: Thesis, 1986.

Alexander Kolpakov
Novosibirsk State University
630090, Pirogova str., bld. 2
Novosibirsk, Russia
kolpakov.alexander@gmail.com

Alexander Mednykh
Sobolev Institute of Mathematics, SB RAS
630090, Koptyug avenue, bld. 4,
Novosibirsk, Russia
mednykh@math.nsc.ru