Finite covers of random 3-manifolds

Abstract A 3-manifold is Haken if it contains a topologically essential surface. The Virtual Haken Conjecture posits that every irreducible 3-manifold with infinite fundamental group has a finite cover which is Haken. In this paper, we study random 3-manifolds and their finite covers in an attempt to shed light on this difficult question. In particular, we consider random Heegaard splittings by gluing two handlebodies by the result of a random walk in the mapping class group of a surface. For this model of random 3-manifold, we are able to compute the probabilities that the resulting manifolds have finite covers of particular kinds. Our results contrast with the analogous probabilities for groups coming from random balanced presentations, giving quantitative theorems to the effect that 3-manifold groups have many more finite quotients than random groups. The next natural question is whether these covers have positive betti number. For abelian covers of a fixed type over 3-manifolds of Heegaard genus 2, we show that the probability of positive betti number is 0.

In fact, many of these questions boil down to questions about the mapping class group. We are led to consider the action of the mapping class group of a surface $\Sigma$ on the set of quotients $\pi_1(\Sigma) \to Q$. If $Q$ is a simple group, we show that if the genus of $\Sigma$ is large, then this action is very mixing. In particular, the action factors through the alternating group of each orbit. This is analogous to Goldman’s theorem that the action of the mapping class group on the $SU(2)$ character variety is ergodic.

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1 Introduction

Here, we study various notions of a random 3-manifold, and try to understand the distribution of topological and group-theoretic properties for such manifolds. Our primary motivation is to try to determine the underlying issues behind the Virtual Haken Conjecture and related problems about properties of finite covers of 3-manifolds. While any hyperbolic 3-manifold has many finite covers—its fundamental group is residually finite—we do not know what most of the covering groups are, much less the properties that we may reasonably expect these covers to have. The reason that the fundamental group of a hyperbolic 3-manifold $M$ is residually finite is that it is a finitely generated group of matrices, and this gives many quotients of $\pi_1(M)$ of the form $\text{PSL}_2 F$, where $F$ is a finite field. Lubotzky has shown [Lub2] that such quotients have measure zero among all quotients of $\pi_1(M)$, but his proof provides little insight into what those other quotients might be. For instance, is it reasonable to expect $\pi_1(M)$ to have a quotient which is an alternating group $A_n$? Should there be many such quotients? Also, for a particular kind of finite quotient, how likely is it that the associated cover be Haken or have positive betti number? Although there have been many partial results on the Virtual Haken Conjecture, these questions have been hard to address in general by direct deductive reasoning.

Another way of thinking about these questions is from a probabilistic point of view. Since the set of homeomorphism types of compact 3-manifolds is countably infinite, there is no uniform, countably-additive, probability measure on this set. Thus the first issue is to define a plausible context in which we can discuss probability. The density of 3-manifolds with any given property will depend on the order in which we consider them, unless the property is either true or false for all but a finite set of 3-manifolds. It seems best to us to analyze orders of enumeration that are plausible and tractable, while acknowledging that there may be other equally plausible and tractable orders that give different answers. In Section 2, we outline several reasonable models for a random 3-manifold, and then in most of the rest of this paper concentrate on a model of random 3-manifolds which comes from looking at Heegaard splittings generated by random walks in mapping class groups.

1.1 Random Heegaard splittings

Every closed orientable 3-manifold has a Heegaard splitting, that is, it can be constructed by gluing two handlebodies of genus $g$ together using a homeomorphism between their boundaries. We will look at 3-manifolds of a fixed Heegaard genus $g$, and consider gluings obtained by a random word in a finite set of generators for the mapping class group of the surface of genus $g$. In principle, the density of a particular property of the manifolds obtained in this way could depend on the choice of generators for the mapping class group. Indeed, this is plausible since in a non-amenable group such as this one, the correlation between random words of large length in different sets of generators usually tends to 0. However, the properties we analyze have limiting densities independent of this choice of generators. In particular, for a finite group $Q$ the probability that the manifold obtained from
a random genus $g$ Heegaard splitting has a cover with covering group $Q$ is well-defined (Prop.
6.1), and we denote this probability as $p(Q, g)$. When $Q$ is a simple group, we are able to
compute the limit of these probabilities as the genus $g$ goes to infinity:

**Theorem 7.1** Let $Q$ be a non-abelian finite simple group. Then as the genus $g$
goes to infinity, the probability of a $Q$-cover converges:

$$p(Q, g) \to 1 - e^{-\mu} \quad \text{where} \quad \mu = \frac{|H_2(Q; \mathbb{Z})|}{|\text{Out}(Q)|}.$$  

Moreover, the limiting distribution of the number of $Q$-covers converges to the Poisson distribution with mean $\mu$.

For example, if $Q = \text{PSL}_2 \mathbb{F}_p$ where $p$ is an odd prime then $\mu = 1$. Thus for
large genus the probability of a PSL$_2 \mathbb{F}_p$ cover is about $1 - e^{-1} \approx 0.6321$. Hyperbolic
3-manifolds must have infinitely many covers of this form, namely the congruence quotients. However, at least naively, one expects far fewer congruence quotients than given by Theorem 7.1. Another interesting example is the case where $Q = A_n$ is an alternating group; here again $\mu = 1$ and the probability of an $A_n$ cover is greater than 63%. Moreover, we show that covers with different groups $Q_i$ do not correlate with one another, at least for large genus. As a consequence, we can prove results such as the following, which is a special case of

**Theorem 7.7** Let $\varepsilon > 0$. For all sufficiently large $g$, the probability that the 3-manifold obtained from a random genus-$g$ Heegaard splitting has an $A_n$-cover with $n \geq 5$ is at least $1 - \varepsilon$. Moreover, the same is true if we require some fixed number $k$ of such covers.

These results should be contrasted with the analogous results for finitely presented groups with an equal number of generators and relators, where the relators are chosen at random. There, the probability of a $A_n$-quotient goes to zero like $1/n!$ as $n$ goes to infinity, rather than remaining constant (Theorem 3.10). Thus one way of interpreting our results is that they affirm the belief that 3-manifolds have many finite quotients. In Section 5, we give some heuristic ways to understand why this should be true, working from a more naive point of view. These stem from the fact that the group relators given by a Heegaard splitting come from disjoint embedded curves on a surface. In particular, we highlight special features of attaching the last two 2-handles in forming a 3-manifold that suggest many extra finite quotients.

### 1.2 Mapping class group

The proof of Theorem 7.1 boils down to understanding the action of the mapping class group of a surface on a certain finite set. Let $\Sigma_g$ be a closed surface of genus $g$, and let $\mathcal{M}_g$ be its mapping class group. Consider the set $\mathcal{A}_g$ of epimorphisms of $\pi_1(\Sigma_g)$ onto our fixed simple group $Q$, modulo automorphisms of $Q$. Then $\mathcal{M}_g$ acts on $\mathcal{A}_g$ via the induced automorphisms of $\pi_1(\Sigma_g)$, and we show:
1.3 Theorem Let $Q$ be a non-abelian finite simple group. Then for all sufficiently large $g$, the orbits of $\mathcal{A}_g$ under the action of $\mathcal{M}_g$ correspond bijectively to $H_2(Q;\mathbb{Z})/\text{Out}(Q)$. Moreover, the action of $\mathcal{M}_g$ on each orbit is by the full alternating group of that orbit.

For a finite group $Q$ which is not necessarily simple the orbits can be classified in the same way (Theorem 6.20); that the action on an orbit is by the full alternating group is special to simple groups $Q$ (Theorem 7.4). You can view Theorem 1.3 as saying that when the genus is large the action of $\mathcal{M}_g$ on $\mathcal{A}_g$ is nearly as mixing as possible. As such, it is directly analogous to Goldman’s theorem that the action of $\mathcal{M}_g$ on the $\text{SU}(2)$-character variety is ergodic for any genus $\geq 2$ [Gol]. Perhaps surprisingly, the proof of Theorem 1.3 uses the Classification of Simple Groups even for concrete cases such as $Q = A_n$. However, Theorem 7.1 which is a corollary of Theorem 1.3 also follows from a weaker version which does not use the Classification.

What about other types of finite groups? For abelian groups, we give a complete picture of the distribution of $H_1(M)$ for a 3-manifold coming from a random Heegaard splitting (Section 8). For a general finite group $Q$, we do not know how to show the existence of a limiting distribution as the genus goes to infinity, but we can show that the expected number of $Q$-quotients does converge (Theorem 6.21).

1.4 Virtual positive betti number

These results give a good picture about the number of different types of covers in many cases, so we now turn to the main question at hand:

1.5 Virtual Haken Conjecture Let $M$ be a closed irreducible 3-manifold with $\pi_1(M)$ infinite. Then $M$ has a finite cover which is Haken, i.e. contains an incompressible surface.

This conjecture was first proposed by Waldhausen in the 1960s [Wal]. It is often motivated as a way of reducing questions to the case of Haken manifolds, where one has the most topological tools available. However, we prefer to view it as an intrinsic question about the topology $M$ itself: does $M$ contain an immersed incompressible surface? If so, can we lift it to an embedded surface in some finite cover? From a more algebraic point of view, one of the fundamental tasks of 3-manifold topology is to understand the special properties of their fundamental groups, as compared to finitely presented groups in general; thus it is natural to ask: does $\pi_1(M)$ always contain the fundamental group of a closed surface? (Having $\pi_1(M)$ contain a surface group is equivalent to $M$ having an immersed incompressible surface, but is a priori weaker than having a finite cover which is Haken. The difference is another subtle and interesting question about $\pi_1(M)$, namely subgroup separability.)

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1 Not in published version: Jordan Ellenberg points out that Conway and Parker studied the analogous question for the braid groups, which are mapping class groups of punctured discs. In particular, they proved a version of our Theorem 6.20 in that context. The theorem of Conway and Parker was written down and improved by Fried and Volklein [FV], who used it to study the inverse Galois problem.
Perelman has announced a proof of Thurston’s Geometrization Conjecture using Hamilton’s Ricci flow \cite{Per1,Per2}; this should reduce the Virtual Haken Conjecture to the (generic) case when $M$ is hyperbolic. For hyperbolic $M$, the Virtual Haken Conjecture fits nicely into a more general question of Gromov: must a 1-ended word-hyperbolic group contain a surface group? In any event, it seems to us that it would be very hard to prove the Virtual Haken Conjecture without first establishing Geometrization; for instance, the only known way to show that an “easy to understand” atoroidal Haken manifold $M$ has a non-trivial finite cover is to hyperbolize it to see that $\pi_1(M)$ is in fact a group of matrices, hence residually finite!

Deciding whether a 3-manifold is Haken is difficult, so in the rest of this paper we focus on the stronger version of the conjecture which asks that the finite cover has positive first betti number. (In the case of arithmetic 3-manifolds, this is also the version that relates directly to the theory of automorphic forms.) In our prior work \cite{DT2}, we found that this conjecture holds for all $\approx 11,000$ of the small volume hyperbolic 3-manifolds in the Hodgson-Weeks census. One of our goals here is to determine whether the patterns we observed there are in some sense generic, or are a consequence of special properties of that sample. For simple quotients, the results above give even larger probabilities for such covers than those we observed in \cite{DT2} (see Section 6.5 for a quantitative comparison).

However, for the crucial question of whether simple covers have positive betti number, a different picture emerges for our random manifolds here than we saw in \cite{DT2}. In \cite{DT2 §5}, we found that covers with a particular fixed finite simple group $\mathcal{Q}$ had positive betti number with probabilities between 52–98% depending on $\mathcal{Q}$. However, for our Heegaard splitting notion of random our experimental evidence strongly suggests that these probabilities are 0. Moreover, we can show

**9.1 Theorem** Let $\mathcal{Q}$ be a finite abelian group. The probability that the 3-manifold obtained from a random Heegaard splitting of genus 2 has a $\mathcal{Q}$-cover with $\beta_1 > 0$ is 0.

1.6 Potential uses of random 3-manifolds

In combinatorics, studying random objects is done not just for the intrinsic interest and beauty of the subject but also for the applications. For instance, constructing explicit infinite families of expander graphs is quite difficult; the first such construction was based on the congruence quotients of $\text{PSL}_2\mathbb{Z}$ and uses Selberg’s $3/16$ Theorem (see e.g. \cite{Lub1}). On the other hand, proofs of existence and practical construction can be done by looking at certain classes of random graphs and showing that the desired property occurs with non-zero probability. Closer to the study of 3-manifolds, Gromov initiated the study of groups coming from certain types of random group presentations. These have been used to produce many examples of word-hyperbolic groups with additional properties, such as having Property T \cite{Zak} \cite{Gro2}. Very recently, Belolipetsky and Lubotzky have used random techniques to show that given $n$ and a finite group $G$ there exists a hyperbolic $n$-manifold whose full isometry group is exactly $G$ \cite{BL}.

Perhaps similar techniques could be applied to questions about 3-manifolds. In particular, to construct 3-manifolds with a certain list of properties, one could try
to show that these properties occur with positive probability for a suitable model of random 3-manifold. For such applications, the fact that a random 3-manifold is an ill-defined concept becomes a strength rather than a weakness, since by varying the model one can change the characteristics of the resulting manifolds.

Finally, another point of view on random 3-manifolds is that they provide a quantitative context in which to understand one of the central questions in 3-dimensional topology: how do 3-manifold groups differ from finite presented groups in general? As we mentioned, our results show that from the point of view of random Heegaard splittings, 3-manifold groups have many more finite quotients than finitely presented groups in general. Recent work of the first author and Dylan Thurston shows a similar sharp divergence behavior with respect to fibering over the circle, where here a group “fibers” if is an algebraic mapping torus \([DT1]\). Surprisingly, the 3-manifolds studied there were much less likely to fiber than similar finitely presented groups.

1.7 Outline of contents

In Section 2 we discuss several different models of random 3-manifolds and some of their basic properties. In Section 3 we discuss groups coming from random balanced presentations, both as a warm up for the 3-manifold case and to provide a point of comparison. We compute the probabilities that such random groups have a particular abelian or simple quotient. Our results about random balanced presentation fit most naturally into the context of profinite groups as we discuss in Section 4. Also in Section 4 we define a profinite generalization of random Heegaard splittings. In Section 5 we discuss some reasons why 3-manifolds should have many finite quotients, working from a more naive heuristic point of view than in later sections.

The remainder of the paper, Sections 6–9, focuses on the specific model of random Heegaard splittings and on the finite covers of the corresponding 3-manifolds. Section 6 contains as much of the picture as we could develop for an arbitrary finite covering group \(Q\). In Section 7 we give much more detailed results in the case when \(Q\) is simple. Similarly, Section 8 is devoted to the case when \(Q\) is abelian. Finally, Section 9 discusses the homology of a cover of a random 3-manifold.

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2 Models of random 3-manifolds

In this section we give several different models of random 3-manifolds, and outline some elementary properties about them. In each case, the idea is to filter 3-manifolds in such a way so that number of 3-manifolds with bounded complexity is finite.
2.1 Random triangulations

Arguably the most natural notion of a random 3-manifold comes from filtering by the number of tetrahedra in a minimal triangulation. To sidestep the difficult problem of determining minimal triangulations, we can make the triangulations themselves the basic objects. Let \( T_3(n) \) be the set of oriented triangulations of closed 3-manifolds with \( n \) tetrahedra. Here, a triangulation is just an assemblage of 3-simplices with their faces glued in pairs, and need not be a simplicial complex in the classical sense. In the probabilistic setting, we are interested in the properties of the manifolds in \( T_3(n) \) as \( n \) tends to infinity. For instance, does the probability that \( M \in T_3(n) \) is hyperbolic go to 1 as \( n \to \infty \)? Unfortunately, it seems very difficult to prove anything about the manifolds in \( T_3(n) \), or even generate random elements of \( T_3(n) \) for large \( n \). As we will explain (Proposition 2.8), the problem is that if we start with \( n \) tetrahedra and glue their faces in pairs we almost never get a 3-manifold.

2.2 Random surfaces

We Will start with the 2-dimensional case, since one gets a good picture there and it helps explain the problem in dimension 3. Let \( T_2(n) \) be the set of oriented triangulations of (not necessarily connected) surfaces; as with studying random graphs, it is convenient to make these labeled triangulations where each triangle is assigned a number in \( 1, 2, \ldots, n \) and also has an identification with the standard 2-simplex. Since you can not build a surface from an odd number of triangles, we will always assume that \( n \) is even. Another point of view on a triangulated surface is to look at the dual 1-skeleton. This is a trivalent graph with labeled vertices where the incoming edges at each vertex are labeled by \( 1, 2, 3 \) according to the label of the edge that they are dual to. Conversely, such a labeled trivalent graph gives a triangulation in \( T_2(n) \). This triangulation is unique since there is only one way to glue a pair of sides on two oriented triangles compatible with the orientations.

We can generate elements of \( T_2(n) \) with the uniform distribution easily, indeed in time linear in \( n \); simply start with \( n \) triangles and pick pairs of sides at random and glue. Note that the dual 1-skeleton of a random element of \( T_2(n) \) is a uniformly distributed random trivalent graph with \( n \) vertices. Thus we can directly apply results about random regular graphs to study properties of \( T_2(n) \) (see [Wor] for a survey of regular random graphs). For instance, it follows that the probability that \( \Sigma \in T_2(n) \) is connected goes to 1 as \( n \to \infty \).

2.3 Euler characteristic

We will discuss further consequences of the structure of the dual graph later, but first we will explain why the expected genus of a random surface is close to the maximum possible. With a slightly different model, that of gluing sides of an \( n \)-gon, the genus distribution is needed to compute the Euler characteristic of the moduli space of Riemann surfaces of a fixed genus. For this reason, it was studied in detail by Harer, Zagier, and Penner [HZ] [Pen] [Zag].
To compute the Euler characteristic of $\Sigma \in \mathcal{T}_2(n)$, we just need to know the number of vertices $v$ as $\chi(\Sigma) = -n/2 + v$. So what is the expected number of vertices? Take the point of view of randomly gluing triangles together, and think of how the links of the final vertices are built up by the gluing process. We start with $3n$ link segments in the corners of the $n$ triangles. These link segments have an orientation induced from the orientation on the triangles. At each stage, we have some number of arcs and circles built up out of these segments. At each gluing of triangles, two pairs of endpoints of arcs are glued together, respecting the orientations.

If we were not gluing the link arcs two at a time, we would have exactly the same situation as counting the number of cycles of a random permutation (see e.g. [Fel, §X.6(b)]). We will first describe what would happen with this simplification and discuss the full picture below. With this simplification, at the $k$th arc gluing we have $(3n - k + 1)$ choices of where to glue the positive end of the given link arc, and exactly one of these choices creates a closed link circle. Thus we expect $1/(3n - k + 1)$ final vertices to be created per gluing, and the expected total number of vertices is $\sum_{k=1}^{3n} 1/k = \log(3n) \approx \log(n)$. Note that this says that the expected genus is about $n/4 - \log(n)$ where the maximum genus possible with $n$ triangles is $\lfloor n/4 \rfloor$. Thus the expected genus is quite close to the maximal one. In particular, the probability that $\Sigma \in \mathcal{T}_2(n)$ has any fixed genus $g$ goes to zero as $n \to \infty$.

Unfortunately, we do not know how to prove that the expected number of vertices is $\log(n)$; as this is somewhat tangential, we content ourselves with the following upper bound, which gives many of the same qualitative results:

**2.4 Theorem** The expected number of vertices for a random surface $\Sigma \in \mathcal{T}_2(n)$ is at most $(3/2) \log(n) + 6$.

**Proof** First, we will explain where the problem is with the argument we gave above. Call an unglued edge of a triangle bad if the two link arcs which intersect it are actually the same arc. Gluing a bad edge creates a link circle if and only if it is glued to another bad edge. Thus, the expected number of circles created by such a gluing depends on the number of preexisting bad edges, which could conceivably be large.

We can deal with this problem as follows. At each stage, we always pick a good edge as the first edge in the pair to glue, if there is one. Because we allow the first chosen edge to be glued to any edge, good or bad, every $\Sigma$ is generated by this process and we have not changed the distribution on $\mathcal{T}_2(n)$. However, we have made counting easier. Let $G_k$ be the random variable which is the number of link circles created at the $k$th stage by a gluing which contains at least one good edge. Let $B_k$ be the number of bad edges created at the $k$th gluing. (Both of these variables are set to 0 once we have exhausted the good edges.) The number of vertices in the final surface is equal to $\sum G_k$ plus half the number of bad edges left at the end of the good gluings. The number of bad edges at the end is at most the number created during the whole process. Thus as expectations always add, the expected number of vertices is bounded by

$$\sum_{k=1}^{3n/2} E(G_k) + \frac{1}{2} \sum_{k=1}^{3n/2} E(B_k).$$
We claim that if there is a good edge left, then $E(G_k) = 2/(3n - 2k + 1)$, where the denominator is the number of choices for an edge to glue to. There are two cases to consider, depending on whether the link arcs of our chosen good edge have other endpoints on the same edge, but the probability is the same in both cases. Similarly, you can see $E(B_k) \leq 2/(3n - 2k + 1)$. Combining, we get that the expected number of vertices is less than $3 \sum_{k=1}^{3n/2} 1/(2k - 1) \leq 3/2 \log(n) + 6$. 

We conclude with an outline of how to turn the problem of precisely computing the expected Euler characteristic into a problem about the character theory of the symmetric group. Suppose we are to create a surface from $n$ triangles. Label the oriented sides of the triangles by $1, 2, \ldots, 3n$. A pairing of the sides can be thought of as a fixed-point free involution $\pi$ in the symmetric group $S_{3n}$. Label the oriented vertex links of the triangles by saying that such a vertex link has the same number as the edge which contains its positive endpoint. We label each triangle so that the 3-cycle $(3k - 2, 3k - 1, 3k)$ rotates the edges of the $k$th triangle by $1/3$ of a turn in the direction of the vertex links. Let $\sigma = (123)(456)(789)\ldots \in S_{3n}$ be the element which rotates the sides of each triangle in this way. Then the vertex link $k$ is glued to the vertex link $(\sigma \pi)(k)$. Thus the number of vertices of the surface corresponding to the gluing permutation $\pi$ is just the number of cycles of $\sigma \pi$. Hence if $C$ is the conjugacy class of fixed-point free involutions in $S_{3n}$, then the average number of vertices is:

$$\frac{1}{|C|} \sum_{\pi \in C} \text{num cycles in } \sigma \pi.$$

There are different ways of attacking problems of this kind, and an elementary approach is to use the character theory of the symmetric group, see [Jac] [Zag]. For other approaches, based on random matrices, see [HZ] [Pen] [IZ].

2.5 Local structure

An interesting property of random regular graphs is that most vertices have neighborhoods which are embedded trees. More precisely, fix a radius $r$ and let the number of vertices $n$ get large. Then with probability approaching 1, the proportion of vertices which have neighborhoods which are embedded trees of radius $r$ is very near 1. The distribution of short cycles in a random regular graph is also understood, and as the following theorem shows, the distribution is essentially independent of the size of the graphs [Wor, Thm 2.5] :

**2.6 Theorem** (Bollobás) Consider regular graphs where the vertices have valence $d$. Let $X_{i,n}$ be the random variable which is the number of cycles of length $i$ in a random such graph with $n$ vertices. Then for $i$ less that some fixed $k$, the $X_{i,n}$ limit as $n \to \infty$ to independent Poisson variables with means $\lambda_i = \frac{(d-1)^i}{i!}$.

One consequence is that if we fix $r$ and pick $\Sigma \in T_2(n)$ with $n$ large, there is a non-zero (albeit small) probability that the shortest cycle in the dual 1-skeleton has length $\geq r$. That is, the “combinatorial injectivity radius” of a random triangulated surface is large a positive proportion of the time.
2.7 Triangulations of 3-manifolds

Now we return to trying to understand random triangulations in $T_3(n)$. Unlike the surface case, we can not study this question by studying random gluings of tetrahedra:

**2.8 Proposition** Let $X$ be the cell complex resulting from gluing pairs of faces of $n$ tetrahedra at random. Then the probability that $X$ is a 3-manifold goes to 0 as $n \to \infty$.

*Proof* The link of a vertex in $X$ is always a surface. Moreover, $X$ is a 3-manifold if and only if the combinatorial link of every vertex is a sphere (the “only if” direction follows for Euler characteristic reasons). Intuitively, since we are gluing the tetrahedra at random, a link surface should be a random surface in the above sense. If this were the case, then the probability that the link is a sphere goes to 0 as $n \to \infty$, and so $X$ would almost never be a manifold. However, every time we glue a pair of tetrahedra, we are gluing 3 pairs of link surface pieces at once in a correlated way.

We will finesse this issue by using the fact that if $X$ is a manifold then the average valence of an edge is uniformly bounded, and contrast this with the fact that the dual 1-skeleton of $X$ is a random 4-valent graph. For the first point, Euler’s formula implies that the average valence of a vertex in a triangulation of $S^2$ is less than 6. The average valence of an edge in $X$ is equal to the average valence of a vertex in the vertex links; thus when $X$ is a manifold the average edge valence is less than 6. In particular, since every edge has positive valence, this implies that at least 1/6 of the edges have valence $\leq 6$. Let $\Gamma$ be the dual 1-skeleton of $X$. An edge of valence $k$ in $X$ gives a cycle in $X$ of length $k$. Thus if $X$ is a 3-manifold, the number of distinct cycles in $\Gamma$ of length $\leq 6$ is a definite multiple of the number of vertices. But $\Gamma$ is a random 4-valent graph, and by Theorem 2.6, the distribution of the number of cycles of length $\leq 6$ is essentially independent of $n$. Thus as $n \to \infty$, the probability that $X$ is a manifold goes to 0.

**2.9 Remark** The proof just given also shows that the probability that a 4-valent graph is the 1-skeleton of some triangulation of a 3-manifold goes to 0 as the number of vertices goes to infinity.

All the properties of random surfaces that we described were consequences of the fact that the uniform distribution on $T_2(n)$ was generated by randomly gluing triangles. In the 3-manifold case, we are deprived of this tool, and it seems difficult to say anything at all about a random element of $T_3(n)$. For instance, we do not even know the expected number of vertices for an $M \in T_3(n)$, much less whether we should expect $M$ to be irreducible or hyperbolic.

Even if one could not say much theoretically, it would be very useful to be able to generate elements of, say, $T_3(100)$ with the uniform distribution, even approximately or heuristically. It would also be interesting to understand the complexity of uniformly generating elements of $T_3(n)$; perhaps there is simply no polynomial-time algorithm to do so. It is interesting to note that while spheres make up a vanishingly small proportion of $T_2(n)$ as $n \to \infty$, it is actually possible to generate triangulations of $S^2$ in time linear in the number of triangles [PS].
Every closed orientable 3-manifold has a Heegaard splitting, that is, can be obtained by gluing together the boundaries of two handlebodies. Considering such descriptions gives us another notion of a random 3-manifold, and this is the one we focus on in this paper. The version that we will use here takes the following point of view, based on the mapping class group. Fix a genus- \( g \) handlebody \( H_g \), and denote \( \partial H_g \) by \( \Sigma \). Let \( \mathcal{M}_g \) be the mapping class group of \( \Sigma \). Given \( \phi \in \mathcal{M}_g \), let \( N_\phi \) be the closed 3-manifold obtained by gluing together two copies of \( H_g \) via \( \phi \). Fix generators \( T \) for \( \mathcal{M}_g \). A random element \( \phi \) of \( \mathcal{M}_g \) of complexity \( L \) is defined to be the result of a random walk in the generators \( T \) of length \( L \). Then we define the manifold of a random Heegaard splitting of genus \( g \) and complexity \( L \) to be \( N_\phi \), where \( \phi \) is a random element of \( \mathcal{M}_g \) of complexity \( L \). We are then interested in the properties of such random \( N_\phi \) as \( L \to \infty \). A priori, this might depend on the choice of generators for \( \mathcal{M}_g \). We will show, however, that certain properties do have well-defined limits independent of this choice (Sections 6–9).

Random Heegaard splittings are much more tractable than random triangulations, in part because every random walk in \( \mathcal{M}_g \) actually gives a 3-manifold. Also, for many problems we can reduce the question to a 2-dimensional one, that is, a question about the mapping class group. The disadvantage is that the need to fix the Heegaard genus feels artificial from some points of view. For instance, it means that the injectivity radius of a hyperbolic structure on \( N_\phi \) is uniformly bounded above [Whi].

A very natural question is how often does the same 3-manifold appear as we increase \( L \)? For instance, there are arbitrarily long walks \( \phi \) in \( \mathcal{M}_g \) for which \( N_\phi = S^3 \). Thus you might worry that some small number of manifolds dominate the distribution, and so our notion of random is not very meaningful. However, we will show later that if \( \mathcal{F} \) is any finite set of 3-manifolds, then the probability that \( N_\phi \in \mathcal{F} \) goes to 0 as \( L \to \infty \). This follows from Corollary 8.5, which shows that \( H_1( N_\phi, \mathbb{Z}) \) is almost always finite, but the expected size grows with \( L \).

The next obvious question is: what is the probability that \( N_\phi \) is hyperbolic? We believe

**2.11 Conjecture** As \( L \to \infty \) the probability that \( N_\phi \) is hyperbolic goes to 1. Moreover, the expected volume of \( N_\phi \) grows linearly in \( L \).

One expects that the hyperbolic geometry of \( N_\phi \) away from the cores of the handlebodies should be close to that of a “model manifold” of the type used in the proof of the Ending Lamination Conjecture. Despite this heuristic picture, a proof of Conjecture 2.11 is likely to be quite difficult. One approach would be to try to show that the expected distance of the Heegaard splitting defining \( N_\phi \) (in the sense of Hempel [Hem2]) is greater than \( 2^{\frac{L}{2}} \). Namazi’s results connecting Heegaard splittings to hyperbolic geometry are also relevant here [Nam].

Finally, there are other notions of random Heegaard splittings you could consider. For instance you could think of specifying a random Heegaard diagram of

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2 Joseph Maher has recently announced [Mah] a proof that the probability of a Heegaard splitting having distance less than a fixed \( C \) goes to 0 as \( L \to \infty \); this would establish the first part of Conjecture 2.11 assuming geometrization.
complexity $L$ by choosing one uniformly from among the finite number of such where the number of intersections of pairs of defining curves is $\leq L$.

2.12 Universal link related notions

There are links $L$ in $S^3$ such that every closed orientable 3-manifold is a cover of $S^3$ branched over $L$. One such link is the figure-8 knot \[HLM\]. Let $K$ be this knot and $M$ be its exterior. There are only finitely many branched covers of $(S^3, K)$ of degree $\leq L$, since such a cover corresponds to a finite-index subgroup of $\pi_1(M)$. Thus another notion of random 3-manifold is to choose uniformly among all conjugacy classes of subgroups of $\pi_1(M)$ of index $\leq L$ and build the corresponding manifold. As with random triangulations, it is unclear if there are even efficient ways to generate such covers experimentally. While enumerating all subgroups of index $\leq L$ is certainly algorithmic \[Sim\], the number of such subgroups grows super-exponentially in our case. This is because $\pi_1(M)$ virtually surjects onto a free group on two-generators. So if we wanted to sample branched covers of index $\leq L$ for large $L$, we would need some way of picking out the subgroups without enumerating all of them. With current technology, it would be difficult to enumerate all subgroups of $\pi_1(M)$ for indices beyond the low 20s. For more about 3-manifolds from the point of view of branched covers of the figure-8 knot, see \[Hem1\].

2.13 Random knots based notions

Another notion of random 3-manifold would be to take a Dehn surgery point of view. That is, one could take some notion of a random knot or link in $S^3$ and do Dehn surgery on it, where the Dehn surgery parameters are confined to some finite range at each stage. For this, one would need a good notion of a random knot or link. One could use models based on choosing a random braid and either taking the closed braid or making a bridge diagram. Or you could look at all planar diagrams with a fixed number of crossings. These can be efficiently generated \[PS\]. Another reasonable notion is to build a knot out of a uniformly distributed collection of unit length sticks stuck end to end (see, e.g. \[DPS\]). These can be generated efficiently, but have the disadvantage that they are typically satellite knots \[Jun\].

3 Random balanced presentations

For a finite presentation of a group, the deficiency is the difference $g - r$ between the number of generators and the number of relators. In the case of a closed 3-manifold group, the natural presentations coming from cell divisions or Heegaard splitting have deficiency 0. Deficiency 0 presentations are also called balanced. If a group has a presentation with positive deficiency, then it already has positive first betti number, so deficiency 0 is the borderline case for the virtual positive betti number property of a finitely presented group. In this section, we study groups defined by random presentations of deficiency 0, and otherwise ignore the constraints coming from the topology of 3-manifolds. In particular, we compute the
probabilities that they admit epimorphisms to certain finite groups. In later sections, we will contrast these results with those specific to 3-manifold groups.

First let us choose a suitable meaning for a “random presentation” by giving a definition of a random relator. Consider the free group \(F_g\) on \(g\) generators \(a_1, \ldots, a_g\). Given an integer \(n > 0\), consider all unreduced words of length \(n\) where each letter is either a generator \(a_i^{\pm 1}\) or the identity; there are \((2g + 1)^n\) such words. A random relator of length \(n\) is such a word selected at random, with each word equally likely. If we fix a number \(g\) of generators and number \(r\) of relations, a random presentation of complexity \(n\) is the group \(G = \langle F_g | R_1, \ldots, R_r \rangle\), where each \(R_i\) is a random relator of complexity \(n\). Such random presentations have been studied extensively by Gromov and others. In particular, Gromov showed that the probability that \(G\) is word-hyperbolic goes to 1 as \(n \to \infty\) \cite{Gro1,Ol}.

In the rest of this section, we consider the probabilities that such random groups have different kinds of finite quotients, focusing on the case of deficiency 0. What we do here fits well into the context of profinite groups, as we describe later in Section 4, and that point of view provides additional motivation for this section.

### 3.1 Quotients of a fixed type

Let \(Q\) be a finite group. We want to consider the probability that a random \(g\)-generator \(r\)-relator group \(G\) has an epimorphism onto \(Q\). We begin by showing that this probability makes sense, and, in later subsections, calculate it for certain classes of \(Q\). First, consider a fixed epimorphism \(f: F_g \to Q\); what is the probability that \(f\) extends to \(G\), equivalently that \(f(R_i) = 1\) for all \(i\)? One way to think of \(R_i\) is as the result of a random walk of length \(n\) in the Cayley graph of \(F_g\). In this random walk, each edge is equally likely as the next step, and there is a \(1/(2g + 1)\) probability of not moving at each stage. The image \(f(R_i)\) is thus the result of the analogous random walk in the Cayley graph of \(Q\) with respect to the generators \(\{f(a_i)\}\). The next lemma says that as \(n \to \infty\), the result of such a random walk on a finite graph is nearly uniformly distributed; thus the probability that \(f(R_i) = 1\) converges to \(1/|Q|\) as \(n \to \infty\).

**3.2 Lemma** Let \(\Gamma\) be a connected finite graph. Consider random walks on \(\Gamma\) with fixed transition probabilities. Suppose that at each vertex the probability of taking any given edge is positive, as is the probability that the walk stays at the vertex. Then the distribution of the position of the walk after \(n\) steps converges to the uniform distribution as \(n \to \infty\).

The reason for requiring a positive probability for pausing at each stage is to avoid parity issues, as happens when \(\Gamma\) is a cycle of even length; therefore, we will always include the identity in the set of generators when constructing a Cayley graph. The lemma is completely standard, but as we use it repeatedly we include a proof.

**Proof** Consider the vector space \(F\) of functions from the vertices of \(\Gamma\) to \(\mathbb{C}\). Let \(L\) be the linear transformation of \(F\) which averages a function over the radius one
neighborhood of a vertex, weighted according to the transition probabilities. That
is, if \( f : \Gamma \to C \) then
\[
L(f)(v) = \sum_{w \in B_1(v)} \text{(Probability of transition from } v \text{ to } w) f(w).
\]

Let \( \delta \in F \) be the characteristic function for the initial point of the walk. Then the
probability distribution for the position of the walk after \( n \) steps is \( L^n \delta \). Note that
\( L \) is a non-negative linear matrix, and that if \( n \) is greater than the diameter of \( \Gamma \) then \( L^n \) has strictly positive entries. Constant functions are eigenvectors of \( L \), and
by the Perron-Frobenius theorem, the successive images by \( L \) of any non-negative
and non-zero function converge to this eigenspace. In particular, \( L^n \delta \) converges to
the uniform measure.

Now let us use the same idea to show that the probability that a random \( G \) has
a \( Q \)-quotient is well-defined. More precisely, let \( p(Q, g, r, n) \) be the probability that
the group of a \( g \)-generator, \( r \)-relator presentation of complexity \( n \) has a \( Q \)-quotient
(note there are only finitely many such presentations). Then

**3.3 Proposition** The probabilities \( p(Q, g, r, n) \) converge as the complexity \( n \) of the
presentation goes to infinity. Moreover, the distribution of the number of quotients
also converges.

In discussing the distribution of the quotients, it is natural to consider two epimorphisms to \( Q \) as the same if they differ by an automorphism of \( Q \), and so we
will adopt this convention in our counts. This is equivalent to counting normal
subgroups with quotient \( Q \).

**Proof** Let \( \mathcal{E} \) be the set of epimorphisms of the free group \( F_g \) onto \( Q \), modulo
automorphisms of \( Q \). We will fix one representative in each equivalence class in
\( \mathcal{E} \), and so regard the elements as actual epimorphisms from \( F_g \) to \( Q \). Consider
the group \( Q^\mathcal{E} \), and let \( P : F_g \to Q^\mathcal{E} \) be the induced homomorphism where the \( f \)-coordinate
of \( P(w) \) is \( f(w) \). Let \( S \leq Q^\mathcal{E} \) be the image of \( F_g \) under \( P \), and let \( \Gamma' \) be
the Cayley graph of \( S \) with respect to the generators \( \{ P(a_i) \} \). Suppose \( R \) is a word
in \( F_g \). Then the \( f \in \mathcal{E} \) which kill \( R \) are exactly those where the \( f \)-coordinate of
\( P(R) \) is 1. If \( R \) is a random relator with high complexity, then by Lemma 3.2 the
element \( P(R) \) in \( S \) is nearly uniformly distributed in \( S \). Thus the probability that
some \( f \in \mathcal{E} \) kills \( R \) is approximately the ratio
\[
\alpha = \frac{| \{ s \in S \mid s_f = 1 \text{ for some } f \} |}{|S|}.
\]

As the relators are chosen independently, the probabilities \( p(Q, g, r, n) \) converge\(^3\)
to \( \alpha' \) as \( n \to \infty \). Similarly, the probability that we have a fixed number \( k \) of \( Q \)-quotients converges for each \( k \).

\(^3\) Not in published version: Michael Bush kindly points out that the limiting probability
\( p(Q, g, r, \infty) \) is not usually \( \alpha' \) as we incorrectly claim here. The problem is that for there to be
a homomorphism of the resulting group, the \( r \) relators must all map under \( P \) to elements which
are the identity at the same component \( f \). However, the \( p(Q, g, r, n) \) still have a limit, and the
correct expression for \( p(Q, g, r, \infty) \) can be computed by counting the corresponding subset of \( S' \).
The lemma itself is correct, and the erroneous formula for \( p(Q, q, r, \infty) \) is not used elsewhere in
this paper.
We will call the limiting probability \( p(Q, g, r) \). As we saw, it only depends on the finite sets \( \mathcal{E} \) and \( S \), so we turn now to understanding them. First, homomorphisms from \( F_g \) to \( Q \) which are not necessarily onto are parameterized by \( Q^g \). The set \( \mathcal{E} \) is the quotient of the subset of \( Q^g \) consisting of \( g \)-tuples which generate \( Q \), under the diagonal action of \( \text{Aut}(Q) \). As \( \text{Aut}(Q) \) acts freely on this proper subset, we get that \( |\mathcal{E}| < |Q|^g/|\text{Aut}(Q)| \). This over-estimate will actually be close to \( |\mathcal{E}| \) if \( g \) is large; as \( g \to \infty \) the proportion of \( g \)-tuples in \( Q^g \) which do not generate goes to 0. Understanding \( S \) in general is complicated as it is typically not all of \( Q^g \). However, it is easy to compute the expected (average) number of \( Q \)-quotients of such random \( G \). Note that for fixed \( f \in \mathcal{E} \) the probability that the \( i \)-th relator is in the kernel of \( f \) is \( 1/|Q| \). As the relators are chosen independently, the probability that \( f \) extends to our random group with \( r \) relators is \( 1/|Q|^r \). Thus the expected number of such quotients coming from \( f \) is \( 1/|Q|^r \); as expectations add, the expected number of \( Q \)-quotients for \( G \) is \( |\mathcal{E}|/|Q|^r \). For any non-negative integer-valued random variable, the chance it is positive is less than or equal to its expectation. Thus \( p(Q, g, r) \leq |\mathcal{E}|/|Q|^r < |Q|^{g-r}/|\text{Aut}(Q)| \). Now, we are most interested in balanced presentations, and this gives:

**3.4 Theorem** Let \( Q \) be a finite group. The probability \( p(Q, g, g) \) that a random \( g \)-generator balanced group has a epimorphism to \( Q \) is \( < 1/|\text{Aut}(Q)| \).

Now the number of finite groups with \( |\text{Aut}(Q)| \) bounded is finite [LN], and so the theorem implies that \( p(Q, g, g) \to 0 \) as \( |Q| \to \infty \). Thus the larger \( Q \) is, the less likely a random balanced \( G \) is to have \( Q \) as a quotient. In the rest of this section we refine our picture for the classes of abelian and simple groups.

### 3.5 Non-abelian quotients

We start with the case of a non-abelian simple group \( Q \), where we develop a complete picture. As in Section 3.1 consider the set \( \mathcal{E} \) of epimorphisms from \( F_g \) to \( Q \), modulo \( \text{Aut}(Q) \). Most collections of \( g > 1 \) elements of a finite simple group \( Q \) generate it, especially if \( g > 2 \) or if \( Q \) is not too small. To get a rough idea of the probability that a random collection of \( g \) elements of \( Q \) generates \( Q \), consider the contrary hypothesis. If the elements fail to generate, then there is some maximal subgroup \( H \) of \( Q \) that contains them all. For a particular \( H \), the chance that \( g \) elements lie in \( H \) is \( 1/|Q| : H|^g \). The sum over all maximal subgroups gives an upper bound for the proportion that do not generate, substantially less than 1 in all but a few small cases. These upper bounds with \( g = 2 \) in a few of the small cases are \( (A_5, .53), (A_6, .57), (A_7, .35), (A_8, .34), (A_9, .18), (\text{PSL}_2\mathbb{F}_7, .41), (\text{PSL}_2\mathbb{F}_8, .17), (\text{PSL}_2\mathbb{F}_9, .57), (\text{PSL}_2\mathbb{F}_{11}, .28) \) and \( (\text{PSL}_2\mathbb{F}_{13}, .11) \). As the size of the simple group gets larger, the probability of 2 elements generating goes to 1; see the references in [Pak §1.1].

The automorphism group of a non-abelian finite simple group contains \( Q \) itself as the group of inner automorphisms; the quotient group is the outer automorphism group, which is generally rather small. The upper bound we gave earlier is thus \( |\mathcal{E}| \leq |Q|^g/|\text{Aut}(Q)| = |Q^{g-1}|/|\text{Out}(Q)| \). The preceding paragraph indicates that this bound is actually quite accurate except for small \( Q \) and \( g \). As in Section 3.1 we now have that the expected number of \( Q \) quotients of a \( g \)-generator
balanced group is \(|\mathfrak{g}|/Q^g|\) and that this is a bound on the probability \(p(Q,g,g)\) for having a \(Q\)-quotient. Thus we have

\[
p(Q,g,g) \leq \frac{|\mathfrak{g}|}{|Q|^g} \leq \frac{1}{|Q||\text{Out}(Q)|} \leq \frac{1}{|Q|}. \tag{3.6}
\]

In order to compute \(p(Q,g,g)\) exactly, we need to understand the image \(S\) of the induced product map \(F_g \to Q^g\) used in the proof of Proposition 3.3. In this case \(S\) is actually all of \(Q^g\):

3.7 Lemma (Hal) Consider epimorphisms \(f_i : F_g \to Q_i\), where each \(Q_i\) is a non-abelian finite simple group. Suppose no pair \((f_i, f_j)\) are equivalent under an isomorphism of \(Q_i\) to \(Q_j\). Then the product map \(F_g \to \prod Q_i\) is surjective.

It is important in this lemma that \(Q_i\) be non-abelian. For instance, if we take \(Q = \mathbb{Z}/2\), then \(|\mathfrak{g}| = 2^g - 1\) and so \(Q^g\) has \(2^{2^g-1}\) elements. In contrast, the image of \(F_g \to (\mathbb{Z}/2)^g\) is generated by \(g\) elements and thus has size at most \(2^g\).

Proof: This lemma was first proved by P. Hall [Hal]. As it is crucial for us, and unfamiliar to most topologists, we include a proof.

We begin with case \(n = 2\). Suppose \(F_g \to Q_1 \times Q_2\) is not surjective. Let \(S\) be the image; we will show that \(S\) is the graph of an isomorphism between \(Q_1\) and \(Q_2\) compatible with the \(f_i\). Consider the projection \(\pi : S \to Q_1\), and let \(\overline{Q}_2\) denote the subgroup \(\{1\} \times Q_2\) in \(Q_1 \times Q_2\). Let \(K\) be the kernel of \(\pi\), that is \(K = S \cap \overline{Q}_2\). Note that the conjugation action of a \(g \in \overline{Q}_2\) on \(\overline{Q}_2\) can also be induced by conjugating by some \(s \in S\), as the projection \(S \to Q_2\) is onto; therefore, as \(K\) is normal in \(S\) it must be normal in \(\overline{Q}_2\). As \(Q_2\) is simple, \(K\) must either be \(1\) or \(\overline{Q}_2\). In the latter case, \(S\) contains \(\overline{Q}_2\) which implies \(S = Q_1 \times Q_2\) as \(\pi\) is onto. Thus \(K = 1\) and \(\pi\) is an isomorphism. Similarly, the projection \(\pi' : S' \to Q_2'\) is an isomorphism. Thus \(f_1\) and \(f_2\) are equivalent under the isomorphism \(\pi' \circ \pi^{-1}\).

The \(n = 2\) case did not use that the \(Q_i\) are non-abelian. That hypothesis is used in the form of

3.8 Claim Let \(N \leq Q_1 \times \cdots \times Q_k\) be a normal subgroup, where the \(Q_i\) are non-abelian simple groups. Then \(N\) is a direct product of a subset of the factors.

As before, let \(\overline{Q}_i\) denote the copy of \(Q_i\) in the product. To see the claim, first observe that \(N \cap \overline{Q}_i\) is either \(1\) or all of \(\overline{Q}_i\). If the latter case, mod out by \(\overline{Q}_i\) to get a case with smaller \(k\). So we can assume \(N \cap \overline{Q}_i = 1\) for each \(i\). But then \([N, \overline{Q}_i]\) \leq N \cap \overline{Q}_i\), as both subgroups are normal, and so \([N, \overline{Q}_i] = 1\). But then \(N\) is central, and thus trivial, proving the claim.

To conclude the proof of the lemma, choose the smallest \(n\) such that \(F_g \to Q_1 \times \cdots \times Q_n\) is not surjective, and let \(S\) be the image. Then as in the \(n = 2\) case, the projection \(\pi : S \to Q_1 \times \cdots \times Q_{n-1}\) is an isomorphism. Let \(\alpha : Q_1 \times \cdots \times Q_{n-1} \to Q_n\) be the composition of \(\pi^{-1}\) with projection onto \(Q_n\). By the claim, the kernel \(N\) of \(\alpha\) is a direct product of some of the factors. After reordering, we can assume \(N = 1 \times Q_2 \times \cdots \times Q_{n-1}\). But then the map \(F_g \to Q_1 \times Q_n\) is not surjective, and we are back in the \(n = 2\) case. \(\square\)
We saw above that the limiting probability of getting exactly $k$ quotients with group $Q$ is simply the density of $s \in S$ with exactly $k$ of the coordinates equal to 1. Thus as $S = Q^E$ the limiting distribution is the binomial distribution:

$$\{ |Q\text{-quotients}| = k \} = \binom{n}{k} p^k (1 - p)^{n-k}$$

(3.9)

where $p = 1/|Q|^g$ and $n = |E|$. This binomial distribution is well-approximated by the Poisson distribution. Recall that the Poisson distribution with mean $\mu > 0$ is a probability distribution on $\mathbb{Z}_{\geq 0}$ where $k$ has probability $\mu^k/k! e^{-\mu}$. Roughly, the Poisson distribution describes the number $k$ of occurrences of a preferred outcome in a large ensemble of events where, individually, the outcome is rare and independent, but in aggregate the expected number of occurrences is $\mu > 0$. For instance, it is the limit of the binomial distribution we have here, if $\mu = |E|/|Q|^g$ is kept constant and $n = |E| \to \infty$. The difference between (3.9) and the Poisson distribution is usually negligible even in small cases. For instance, $p(A_5, 2, 2)$ is 0.0052646... whereas the Poisson approximation is $1 - e^{-\mu} \approx 0.0052638$. Summarizing, we have:

**3.10 Theorem** Let $Q$ be a non-abelian finite simple group. Let $n$ be the number of epimorphisms from the free group $F_g$ to $Q$ (modulo $\text{Aut}(Q)$), and let $\mu = n/|Q|^g$. The probability that a random $g$-generator balanced group has a $Q$-quotient is

$$p(Q, g, g) = 1 - (1 - |Q|^{-g}) |Q|^g \mu \approx 1 - e^{-\mu}.$$ 

and the distribution of the number of quotients is nearly Poisson with mean $\mu$.

Moreover, as $g$ goes to infinity $\mu \to 1/|\text{Aut}(Q)|$, and the distributions limit to the Poisson distribution with mean $1/|\text{Aut}(Q)|$.

We end this subsection with Table 3.11 which summarizes the situation for the first few finite simple groups. As you can see, all the probabilities are very low; we will see that this is not the case for 3-manifold groups.

**3.12 Probability of some simple quotient**

Now, let us consider the more global question: What are the chances that a finitely presented group of deficiency 0 admits an epimorphism to some non-abelian finite simple group? First consider a finite collection $\mathcal{C}$ of simple groups. Let $G$ be a random $g$-generator balanced group with complexity $n$. For a fixed $Q \in \mathcal{C}$, Theorem 3.4 implies that the probability of $G$ having a $Q$-quotient is $< 1/|\text{Aut}(Q)| \leq 1/|Q|$, as long as $n$ is large enough. As there are finitely many $Q$, we get that for large $n$ the probability that $G$ has a $Q$-quotient for some $Q \in \mathcal{C}$ is less than

$$\sum_{Q \in \mathcal{C}} 1/|\text{Aut}(Q)| \leq \sum_{Q \in \mathcal{C}} 1/|Q|.$$ 

(3.13)

In fact, quotients for different $Q$ are independent (this follows from Lemma 3.7 just as in the proof that $S = Q^E$ in the context of Theorem 3.10). Therefore, we could replace (3.13) by $1 - \prod (1 - 1/|\text{Aut}(Q)|)$, but the former will do for us here.
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Table 3.11 This table gives values and bounds for the expected number of epimorphisms from a random deficiency 0 group to a finite simple group Q. The simple group Q is listed in the first column. The second column is the order of Q, and the third column Gen pairs is the number of pairs of elements that generate Q, up to automorphisms of Q. Column Out gives the order of the outer automorphism group of Q. Column Exp 2-gen is the expected number of epimorphisms to Q among groups with 2 generators and 2 relators. Column Exp n-gen is 1/|Aut(Q)|, which is an upper bound for the expectation independent of the number of generators, the limit of the expectation as the number n of generators goes to infinity, and a good approximation to the expectation when n > 2.

| Quotient | Order | Gen pairs | Out | Exp 2-gen | Exp n-gen |
|----------|-------|-----------|-----|-----------|-----------|
| A5       | 60    | 19        | 2   | .005278   | .008333   |
| PSL₂(𝔽₇)| 168   | 57        | 2   | .002020   | .002976   |
| A6       | 360   | 53        | 4   | .00409    | .00694    |
| PSL₂(𝔽₈)| 504   | 142       | 3   | .00559    | .00661    |
| PSL₂(𝔽₁₁)| 660  | 254       | 2   | .00583    | .00758    |
| PSL₂(𝔽₁₃)| 1092| 495       | 2   | .00415    | .00458    |
| PSL₂(𝔽₁₇)| 2448| 1132      | 2   | .00189    | .00204    |
| A₇       | 2520  | 916       | 2   | .00144    | .00198    |
| PSL₂(𝔽₁₉)| 3420| 1570      | 2   | .00134    | .00146    |
| PSL₂(𝔽₂₆)| 4080| 939       | 4   | .00056    | .00061    |
| PSL₂(𝔽₈)| 5616  | 2424     | 2   | .00077    | .00089    |
| PSL₂(𝔽₂₃)| 6072| 2881      | 2   | .00078    | .00083    |
| PSL₂(𝔽₂₅)| 7800| 1822      | 4   | .00030    | .00032    |
| M₁₁      | 7920  | 6478     | 1   | .00103    | .00126    |
| PSL₂(𝔽₂₇)| 9828| 1572      | 6   | .00016    | .00017    |
| PSL₂(𝔽₂₉)| 12180| 5825     | 2   | .00039    | .00041    |
| PSL₂(𝔽₃₁)| 14880| 7135     | 2   | .00032    | .00034    |
| A₈       | 20160| 7448     | 2   | .00018    | .00024    |
| PSL₃(𝔽₄)| 20160| 1452     | 12  | .00004    | .00004    |
| PSL₂(𝔽₇)| 25308| 12291    | 2   | .00019    | .00020    |
| U₄(𝔽₂)  | 25820| 11505    | 2   | .00017    | .00019    |
| Sz(𝔽₈)  | 29120| 9534     | 3   | .00011    | .00011    |
| PSL₂(𝔽₃₂)| 32736| 6330    | 5   | .00006    | .00006    |

If we were to formally carry out this calculation for the collection of all finite simple groups, we would get that the probability of having a non-abelian simple quotient is less than \(\sum 1/|\text{Aut}(Q)| \leq \sum 1/|Q|\), where the sum is over all such groups. By the Classification of Finite Simple Groups, eventually nearly all non-abelian simple groups up to a given size are of the form PSL₂(𝔽₉). As PSL₂(𝔽₉) has size \((q-1)q(q+1)/2\) for odd q and twice that for even q, the sum of 1/|Aut(Q)| over all non-abelian finite simple groups Q is finite. It is not even very large: approximately 0.015. However, this does not give a proof that many random balanced groups have no non-abelian simple quotients; for a fixed group G, the relators certainly do not map to uniformly distributed random elements of Q as \(|Q| \to \infty\). For one thing, a relator of length R is confined to the ball of radius R in the Cayley graph of Q, and this ball has fewer than \((2g)^R-1)/(2g-1)\) elements. If the relators were uniformly distributed in these balls, then the probability of an epimorphism to Q would be bounded below (although very small), so one would expect there to eventually be an epimorphism of G to some phenomenally large simple group Q. But this argument is also invalid, since among all finite simple groups with a choice of a sequence of g generators, there are only finitely many...
isomorphism classes of balls of radius $R$, so we have only finitely many chances to find an epimorphism.

The real question, which appears to be a difficult issue, is how many different isomorphism classes of balls of radius $R$ exist among all non-abelian finite simple groups. It seems reasonable to us that most of these balls fall into patterns with relatively few new variations when $\log [Q]$ is large compared to $R$. A good estimate of this sort could imply that most groups of deficiency 0 have only finitely many epimorphisms to finite simple groups, often no such homomorphisms. Since a random group in our sense is word hyperbolic, this would imply that there are word hyperbolic groups that are not residually finite.

In any case, for a typical deficiency 0 group that has no quotients among the first few non-abelian simple groups, it is clear that if it has any such quotient, the index must be so astronomically large as to be far beyond brute force computation. From the calculation above, a random balanced presentation with 3 or more generators has about 1.5% probability to admit an epimorphism to a non-abelian simple group of manageable size, and a 2-generator group has about 1.3% probability. To test our thinking, we made 1000 random 2-generator presentations with statistics similar to the census manifolds used in our paper [DT2], and computed all epimorphisms to the first few non-abelian simple groups. Only 15 of these groups had any such quotients, and only 4 had more than one such quotient. This fits reasonably well with the estimate of 1.3% above.

3.14 Abelianization

Let us begin by looking at the abelianization of a random balanced group from a different, more global, perspective. The abelianization is the quotient of $\mathbb{Z}^g$ by the subgroup generated by the abelianization of each relator. In other words, we can make a matrix $M_R$ whose columns correspond to the relators so that the $(i,j)$ entry is the exponent sum of the occurrences of generator $g_i$ in relator $R_j$. If the abelianization is infinite, the determinant of $M_R$ is 0, otherwise the determinant of $M_R$ is the order of the abelianization. For a random relator $R_j$ of length $n$, the corresponding column is just the result of a suitable random walk in the integer lattice $\mathbb{Z}^g$. Individual entries can also be thought of as generated by random walks, and like all 1-dimensional walks their absolute value is proportional to $\sqrt{n}$. Thus the typical determinant grows large as $r$ grows large—indeed, it grows as $n^{\delta/2}$ (to see this rigorously, note that the distribution of $(1/\sqrt{n})M_R$ converges to that of matrices with independent Gaussian entries).

However, for our purposes it is more important to determine the probability that a random presentation admits a finite-sheeted covering of a given type, as we did in the previous subsection. Any finite abelian group is the product of its $p$-Sylow subgroups, so we focus in on just one prime. We Will think about this from the point of view of the $p$-adic integers $\mathbb{Z}_p$. As rational integers which are coprime to $p$ have inverses in $\mathbb{Z}_p$, the $p$-Sylow subgroup of the cokernel of our matrix $M_R$ is the same as the quotient of $\mathbb{Z}_p^g$ by the $\mathbb{Z}_p$-submodule generated by the columns of $M_R$. We are interested in asymptotics as the complexity $n$ of our presentation goes to infinity, and so we want to understand the limiting distribution of the $M_R$. More precisely, let $m_n$ be the probability measure on elements of $\mathbb{F}_g$.
coming from random walks of length $n$. This gives us a measure on the set of balanced presentations with $g$-generators, with finite support. Then one has

**3.15 Lemma** The push-forward of the measure $m_n$ to the space of $p$-adic $g \times g$ matrices converges weakly to the uniform distribution, i.e. in the limit the entries are elements of $\mathbb{Z}_p$ chosen uniformly and independently with respect to Haar measure.

**Proof** The Haar measure on $\mathbb{Z}_p$ can be understood by thinking of $\mathbb{Z}_p$ as the inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$. Showing that the weak limit is Haar measure is tantamount to checking that, for each $k$, the distribution of the entries modulo $p^k$ converges to the uniform distribution as $n \to \infty$. The mod $p^k$ abelianization of a random relator is the same as going for a random walk in the Cayley graph of $(\mathbb{Z}/p^k\mathbb{Z})^g$. As always, that distribution becomes the uniform one as $n \to \infty$, proving the lemma. \hfill \square

This $p$-adic point of view is equivalent to considering $(\mathbb{Z}/p^k\mathbb{Z})^g$ modulo the subgroup generated by a random sequence of $g$ elements, and looking at the limiting distribution of quotient groups as $k$ goes to infinity; however, it has the advantage of giving us a concrete limiting object to calculate with.

First, let us compute the distribution for the orders of the $p$-Sylow subgroups. This is just the largest power of $p$ which divides $\det(M_R)$; if $|\cdot|_p$ denotes the $p$-adic norm, this is the same as $1/\det(M_R)|_p$. Thus, we need to understand the distribution of $\det(M_R)$, where $M_R$ is a $g \times g$ matrix with entries in $\mathbb{Z}_p$, chosen uniformly. The easy case is when $g = 1$, for then $\det(M_R)$ is just uniformly distributed. As an element in $\mathbb{Z}/p^k\mathbb{Z}$ has a $p^{-k}$ chance of being 0, an element in $\mathbb{Z}_p$ has a $p^{-k}$ chance of being in $p^k\mathbb{Z}_p$. Thus the chance that $z \in \mathbb{Z}_p$ has $|z|_p = p^{-k}$ is $c_k = p^{-k}(1 - 1/p)$. A useful way to encode the sequence $\{c_k\}$ is to use a generating function:

$$
\sum_{k=0}^{\infty} c_k t^k = \frac{p - 1}{p - t}.
$$

In the general case we will show:

**3.16 Proposition** Let $d_k$ be the asymptotic probability that the order of the $p$-Sylow subgroup of the abelian group defined by a random $g$-generator balanced presentation is $p^k$. The generating function for the sequence $\{d_k\}$ is

$$
\frac{(p - 1)(p^2 - 1)\cdots(p^g - 1)}{(p - t)(p^2 - t)\cdots(p^g - t)}.
$$

Except for small primes $p$, this is close to the distribution for the case $g = 1$. Thus, except for small $p$, the probability that the $p$-Sylow subgroup is non-trivial is close to $1/p$. This is the same as asking that the group surject onto $\mathbb{Z}/p$, and by Theorem 3.3, we already knew this probability was $< 1/|\text{Aut } Q| = 1/(p - 1)$. Thus in this case the general estimate is close to correct. We now prove the proposition.

**Proof** A vector in $\mathbb{Z}_p^g$ has probability $1/p^{kg}$ to be in $p^k\mathbb{Z}_p^g$. This tells us the distribution of the maximal $p$-adic norms of an element in the first column of $M_R$: the probability that the maximal $p$-adic norm equals $1/p^k$ is a geometric progression, with generating function $(p^g - 1)/(p^g - t)$. If the first column equals $p^kW$ where $k$ is as
large as possible, then $\mathbb{Z}_p^g / \langle W \rangle$ is isomorphic to $\mathbb{Z}_p^{g-1}$. Moreover, the remaining columns map to independent random elements of this module. Thus $|\det(M_R)|_p$ is the product of $p^{-k}$ with $|\det(N)|_p$ where $N$ is a random $(g-1) \times (g-1)$ matrix. Therefore we can get the generating function for $|\det(M_R)|_p$ by multiplying the generating functions for these two things. Inducting on $g$ completes the proof. □

3.17 Remark It is worth noting that the proof shows that the Sylow subgroups for distinct primes $p$ and $q$ are independent, essentially because the quotient maps from $\mathbb{Z}$ to $\mathbb{Z}/p$ and $\mathbb{Z}/q$ induces a surjection $\mathbb{Z} \to \mathbb{Z}/p \times \mathbb{Z}/q$; thus a random walk in $\mathbb{Z}$ pushes forward to the (nearly) uniform distribution on $\mathbb{Z}/p \times \mathbb{Z}/q$.

Now, we will delve further and determine the typical isomorphism type for the $p$-part of the homology. One way to describe the isomorphism class of an abelian $p$-group $A$ is to specify the sequence of ranks of $\rho_i(A)$ of $p^iA/(p^{i+1}A)$. For example, the group $(\mathbb{Z}/p)^2 \oplus \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^5$ corresponds to $4, 2, 1, 1, 0$, with all subsequent terms also 0. Introducing a variable $t_k$ to denote an instance of $(\mathbb{Z}/p)^k$, then an isomorphism class corresponds to a monomial in the $t_k$; the example corresponds to $t_4t_2t_1^3$. With this notation, there is a fairly nice and straightforward computation for the power series in $t_1, \ldots, t_6$ whose coefficients give the asymptotic probability that a $g$-generator, $g$-relator group has the particular isomorphism type of $p$-Sylow subgroup of its abelianization; this series is a rational function $\text{AFP}_g$. This is a bit of a digression for studying 3-manifolds, so we will content ourselves with stating the formulae for 1, 2, and 3-generator groups:

\[
\text{AFP}_1 = \frac{p-1}{p-t_1}
\]

\[
\text{AFP}_2 = \frac{(-1 + p)^2(1 + p)(p^2 + t_1)}{(p-t_1)(p^4-t_2)}
\]

\[
\text{AFP}_3 = \frac{(-1 + p)^3(1 + 2p + 2p^2 + p^3)(p^6 + p^4t_1 + p^6t_1 + p^2t_2 + p^3t_2 + t_1t_2)}{(p-t_1)(p^4-t_2)(p^9-t_3)}
\]

For instance, the 2-Sylow subgroup of the abelianization of a random 2-generator 2-relator group has probability 3/8 to be trivial, 9/32 to be $\mathbb{Z}/2$, 9/64 to be $\mathbb{Z}/4$, 3/128 to be $(\mathbb{Z}/2)^2$, etc. Independently, the 3-Sylow subgroup has probability 16/27 to be trivial, 64/243 to be $\mathbb{Z}/3$, 64/729 to be $\mathbb{Z}/9$, 16/2187 to be $(\mathbb{Z}/3)^2$, and so on. To see how to compute $\text{AFP}_g$, note that in our case, where $A$ is the quotient of $\mathbb{Z}_p^g$ by the subgroup generated by a random sequence of $g$ elements, the probability that $\rho_0(A) = k$ is the probability that $g$ elements of $(\mathbb{Z}/p)^g$ generate a subgroup of rank $g-k$. Similarly, when $\rho_k(A) = h$, the conditional probability that $\rho_{k+1} = k$ is the probability that a random sequence of $h$ elements of $(\mathbb{Z}/p)^h$ generate a subgroup of rank $h-k$.

4 The profinite point of view

In the last section, when we studied the finite quotients of a “typical” balanced group, we worked with asymptotic probabilities $p(Q,g,g)$, which were limits of finite probabilities as the size of the presentation increases. In the case of abelian
groups, we saw that these probabilities could be thought of as probabilities on a certain $p$-adic object, where the notion of probability came from the natural Haar measure (Section 3.14). In this section, we explain how this picture holds true in general by considering random quotients of profinite free-groups; this helps clarify why we got well-defined probabilities such as $p(Q,g,g)$. At the end, we discuss a natural analog of a Heegaard splitting in the profinite context.

4.1 Profinite completions

We begin with a brief sketch of the theory of profinite groups and completions (for more, see e.g. [Wil], [RZ], and [DdSMS]). Let $G$ be a finitely generated group. The *profinite completion* $\hat{G}$ of $G$ is a compact topological group defined as the inverse limit of the system of all finite quotients of $G$. (Note that whenever $Q_1$ and $Q_2$ are any two finite quotients, both quotients factor through the image of $G$ in the product map to $Q_1 \times Q_2$, so the set of finite quotients does form an inverse system.) If $G$ has only finitely many finite quotients, then $\hat{G}$ is a finite group (possibly trivial). Otherwise, $\hat{G}$ has the topology of a Cantor set, whose stages of refinement give particular finite quotients. The natural map $G \to \hat{G}$ is injective if and only if $G$ is residually finite. To reconstruct the finite quotients of $G$, take small open and closed neighborhoods $V$ of the identity in $G$, form the subgroup $W$ generated by $V$, and then pass to the intersection of the finitely many conjugates of $W$ to obtain an open and closed neighborhood $X$ which is a normal subgroup. The quotient $G/X$ is a finite group, and the finite quotients obtained in this way from any neighborhood basis of 1 are cofinal among all finite quotients of $G$. In general, a *profinite group* is any compact topological group that has a neighborhood basis of the identity consisting of open and closed subgroups. Equivalently, a profinite group is a group that is the inverse limit of finite groups. Since a profinite group $\hat{G}$ is a compact topological group, it has a unique bi-invariant probability measure, its Haar measure. This measure is the inverse limit of the counting measures on its finite quotients. Thus, any property of elements of $\hat{G}$ has a well-defined probability (provided the set of such elements is measurable).

4.2 Profinite presentations

In the profinite context, a finitely presented group is the following. Consider the profinite completion $\hat{F}_g$ of the free group on $g$ generators. Given a finite set of elements $\{R_1, \ldots, R_r\}$ of $\hat{F}_g$, let $K$ be the topological closure of the normal subgroup they generate. The quotient topological group $\hat{G} = \hat{F}_g/K$ is the group of the *profinite presentation* with $g$ generators and relations $\{R_i\}$. Now focus on the set $\mathcal{B}_g$ of all $g$-generator balanced profinite presentations, which is just the product of $g$ copies of $\hat{F}_g$, one for each relator. As such, it has a natural probability measure coming from the product of Haar measures on each factor; equivalently, we are thinking of each relator as being chosen independently at random. Thus we can talk about the probability that $\hat{G} \in \mathcal{B}_g$ has some particular property. In the case of the property of having a epimorphism to a finite group $Q$, this is really the same question we encountered before:
4.3 Theorem Let $Q$ be a finite group. Let $\mathcal{G}$ be the group defined by a randomly chosen $g$-generator balanced profinite presentation. Then the probability that $\mathcal{G}$ has a epimorphism to $Q$ is $p(Q, g, g)$.

The quickest way to see this would be to repeat the proof of Proposition 3.3 in this context, and see that one gets the same answer. We will phrase it a little differently to make clear why we get the same answer—after all, the set of regular (non-profinite) presentations has measure 0 in $\mathcal{B}_g$, and so it is hardly given that asymptotic probabilities of regular presentations are the same as the corresponding probabilities for profinite presentations.

Consider random walks on $F_g$ of length $n$, and let $m_n$ be the probability measure on $F_g$ given by the endpoints of such walks. We can also think of $m_n$ as a measure on $F_g$. Then we have:

4.4 Lemma The measures $m_n$ converge weakly to Haar measure on $F_g$.

Proof On a totally disconnected space such as $F_g$, locally constant functions are uniformly dense among continuous functions. So it suffices to check that for a locally constant function $f: F_g \to \mathbb{R}$, the integrals of $f$ with respect to $m_n$ converge to the integral of $f$ with respect to Haar measure. Since every locally constant function on $F_g$ is the pullback from a function on some finite quotient, this lemma follows from Lemma 3.2.

If $S \subset \mathcal{B}_g$ is both open and closed, then its characteristic function is continuous. Hence, if we look at regular (non-profinite) presentations Lemma 4.4 implies

$$\lim_{n \to \infty} P\{ G \in S \mid G \text{ a random balanced group of complexity } n \} = \mu(S),$$

where $\mu$ is the natural measure on $\mathcal{B}_g$. For instance, the property of having an epimorphism to a fixed finite group $Q$ is both open and closed; thus Theorem 4.3 follows from Lemma 4.4.

Passing to random profinite presentations makes it possible to estimate the probability that a group has no non-abelian simple quotients at all. For regular presentations, we weren’t able to show this probability was positive, but the formal calculation in Section 3.12 actually applies in the profinite context. In particular, the subset of $\mathcal{B}_g$ consisting of groups which surject onto $Q$ has measure less than $1/|\text{Aut } Q|$. As $\sum 1/|\text{Aut } Q|$ is finite and indeed about 0.015 we have:

4.5 Theorem Let $\mathcal{G}$ be the group defined by a random $g$-generator profinite balanced presentation. Then with probability 1, the group $\mathcal{G}$ has only finitely many non-abelian finite simple quotients. If $g \geq 3$, the probability that $\mathcal{G}$ has no such quotients is about 98.5%; if $g = 2$, about 98.7%.

The abelian quotients of a random balanced $\mathcal{G}$ can be understood directly from Section 3.14. Usually, but not almost always, the abelianization $A$ of a random balanced profinitely presented group is the inverse limit of cyclic groups. This is equivalent to the condition that there is no prime $p$ such that $A$ admits a continuous epimorphism to $\mathbb{Z}/p \times \mathbb{Z}/p$. Most of the exceptions are for $p = 2$, with most of the remaining exceptions for $p = 3$; the probability for the existence of such a homomorphism is only about $1/p^4$ for larger $p$. Among balanced profinitely presented groups with 1 through 5 generators, the probabilities that all finite abelian quotients are cyclic are about $1.0, 0.924, 0.885, 0.865, 0.856$. The limiting value for a large number of generators is about 0.847.
4.6 Profinite generalizations of 3-manifold groups

In this subsection, we define a class of profinite groups that includes the profinite completions of all 3-manifold groups; this class comes with a natural probability measure. While we will not make direct reference to these ideas elsewhere in this paper, they provide a natural context for the results of Sections 6-8, just as groups with balanced profinite presentations do for the results of Section 3.

Consider a Heegaard diagram of a 3-manifold, and let $S_g$ be the fundamental group of the Heegaard surface. Looking at the fundamental groups of the two handlebodies gives us a diagram of groups

$$F_g \leftarrow S_g \rightarrow F_g.$$  

There is a corresponding diagram of profinite completions:

$$\widehat{F}_g \leftarrow \widehat{S}_g \rightarrow \widehat{F}_g.$$  

The profinite completion of the fundamental group of the 3-manifold is the quotient of $\widehat{S}_g$ by the topological closure $K$ of the normal closure of the kernel of the two homomorphisms.

Since $S_g$ is finitely generated and residually finite, there is a neighborhood basis for 1 in $\widehat{S}_g$ that consists of invariant subgroups of finite index, that is, subgroups invariant under all automorphisms. For any invariant subgroup, the mapping class group $M_g$ acts as an automorphism of the quotient group. Therefore $M_g$ is also residually finite, and furthermore, the action of $M_g$ on $\widehat{S}_g$ extends to a continuous action of $\overline{M}_g$.

4.7 Remark  These actions are not necessarily faithful, in particular it is not for $g = 1$. The torus case is the first case of the congruence subgroup problem: does every finite index subgroup of $SL_k\mathbb{Z}$ contain a principal congruence subgroup? (A principal congruence subgroup is the kernel of a reduction mod $n$ to $SL_k\mathbb{Z}/n\mathbb{Z}$.) For $SL_2\mathbb{Z}$ the answer is no, basically since $SL_2\mathbb{Z}$ is virtually a free group and thus it is easy to find quotients which are simple groups not isomorphic to $PSL_2\mathbb{F}_p$. (Tangentially, the answer to the congruence subgroup problem is yes for $k \geq 3$.) It is unknown if the action of $\overline{M}_g$ is faithful in genus greater than 1.

This picture gives us some justification in considering profinite Heegaard diagrams $\overline{F}_g \leftarrow \overline{S}_g \rightarrow \overline{F}_g$ which are limits of diagrams of actual 3-manifolds; in other words, they are obtained by gluing two copies of the standard map $\overline{S}_g \rightarrow \overline{F}_g$ by an element of $\overline{M}_g$. Associated with such a diagram is a locally compact, totally disconnected topological group, which we will refer to as a profinitefold group: the quotient of $\overline{S}_g$ by the smallest normal, closed subgroup $K$ containing the kernels of the two homomorphisms to $\overline{F}_g$. (This construction is special to dimension 3, so we will not bother with a dimension indicator such as “3-profinitefold group”.)

Let $T_g$ be the subgroup of $\overline{M}_g$ consisting of homeomorphisms of the surface that extend to homeomorphisms of the handlebody. Two elements $f_1, f_2 \in \overline{M}_g$ define equivalent Heegaard diagrams if $\overline{F}_g \backslash f_1 / T_g = \overline{F}_g \backslash f_2 / T_g$. Similarly, it makes sense to define two Heegaard profinite diagrams to be equivalent if the gluing automorphisms are in the same double coset in $\overline{T}_g \overline{M}_g / \overline{T}_g$. Haar measure on $\overline{M}_g$ pushes forward to a measure on this double coset space. This gives a probability
measure on the set of profinitefold groups, which we will use to make sense of statements about random profinitefold groups.

The first homology of any finite sheeted cover of any irreducible 3-manifold \( M \) can be reconstructed from \( \pi_1(M) \): if \( \Gamma \) is the fundamental group of this finite sheeted cover, the profinite completion of the abelianization of \( G \) is the same as the abelianization of \( \Gamma \), which is the corresponding subgroup of finite index in \( \pi_1(M) \). In particular, the abelianization of \( \Gamma \) is infinite if and only if the abelianization of \( \Gamma \) admits a continuous epimorphism to \( \mathbb{Z} \).

In the profinite context, first consider a group \( \overline{G} \) defined by a random balanced profinite presentation. We claim that with probability 1, \( \overline{G} \) has no continuous homomorphism to \( \mathbb{Z} \). In this context, Proposition 3.16 says that the probability that \( \overline{G} \) has a continuous homomorphism to \( \mathbb{Z}/p \) is about \( 1/p \). As these probabilities are independent as we vary \( p \) (see Remark 3.17), the probability that we have one to all \( \mathbb{Z}/p \) is zero.

Turning now to the case of profinitefold groups, the natural analog of the virtually positive betti conjecture is

**4.8 Question** Does the profinitefold group \( \overline{G} \) defined by almost every profinite Heegaard diagram have a subgroup of finite index with a continuous epimorphism to \( \mathbb{Z} \)?

It is too weak a condition merely to require that \( \overline{G} \) have an infinite abelianization. In fact, the abelianization of almost every profinitefold group is indeed infinite, because the first homology group of a 3-manifold is typically a finite group that is large if the manifold is complicated.

There are uncountably many isomorphism classes of Heegaard profinite diagrams up to isomorphism, so the countable set coming from profinite completions of actual Heegaard diagrams forms a set of measure 0. Thus, Question 4.8 and the question of whether all 3-manifolds with infinite fundamental group have virtually positive betti number does not appear to have any easy logical implication one way or the other — the divergence between them involves different orders of taking limits. Nevertheless, they are intuitively and heuristically connected, and so it would be quite interesting to settle Question 4.8.

### 5 Quotients of 3-manifold groups

Group presentations coming from Heegaard splittings of 3-manifolds differ substantially from random deficiency-0 presentations because the relators, rather than being generic elements in the free group, are given by a \( g \)-tuple of disjoint simple closed curves on a genus \( g \) handlebody. Indeed, 3-manifold presentations are a vanishingly small proportion of all deficiency-0 presentations since the number of simple closed curves with word length \( R \) grows polynomially in \( R \) rather than exponentially. Geometrically, the curves’ embeddedness forces the words to be far from independent, and typically there are many repeating syllables at varying scales (for a graphical illustration of this, see [DT1, Fig. 1.5]).

In this section, we try to explain why these geometric properties force there to be more finite quotients than for a general deficiency-0 group. Later we will examine this question from the point of view of random Heegaard splittings (Sections 6–9), but in this section we take a more naive heuristic point of view. In
particular, we try to explain why a given quotient \( f: F_g \to Q \) is much more likely to extend over the last 2 relators.

5.1 Last relator

One reason to expect 3-manifold groups to have more finite quotients than random deficiency-0 presentations has to do with the last relator. To describe this topologically, if we attach 2-handles to the handlebody along \( g-1 \) of the curves, we obtain a 3-manifold \( M \) whose boundary is a torus; the remaining curve is a simple closed curve on the torus. Thus, if an epimorphism \( F_g \to Q \) satisfies the first \( g-1 \) relators, the remaining relator is restricted to an abelian subgroup \( A \) of \( S \) that can be generated by at most 2 elements. Assuming that the distribution in \( A \) is nearly uniform, this suggests that there is approximately a \( 1/|A| \) chance that the last relator is satisfied, as compared to a \( 1/|Q| \) chance for a general relator. Actually, the situation is more complicated because the last relator is a simple closed curve on the torus \( \partial M \).

Consider a torus \( T \) and a finite quotient \( f: \pi_1(T) \to A \). Simple closed curves on \( T \) correspond to primitive elements of \( \pi_1(T) = \mathbb{Z}^2 \), and so we are interested in the probability that a primitive element lies in the kernel of \( f \). If \( A \) is cyclic of order \( a \), then one can change basis so that \( f \) is the factor-preserving map \( \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/a \oplus \mathbb{Z}/ab \); thus every element of the kernel is divisible by \( a \), and so there are no primitive elements in the kernel.

On the other hand, if \( A \) is non-cyclic then you can change basis for \( \pi_1(T) = \mathbb{Z}^2 \) so that \( f \) is the factor-preserving map \( \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/a \oplus \mathbb{Z}/ab \); thus every element of the kernel is divisible by \( a \), and so there are no primitive elements in the kernel.

Returning to our original situation, suppose we are attaching the last of \( g \) relators and want to know if a given epimorphism \( F_g \to Q \) extends over this final handle. This leads us to ask: what is the distribution of possible subgroups \( A \) of \( Q \) which are the image of the fundamental group of the remaining torus \( T \)? Not all 2-generator abelian subgroups can occur. For instance, the image \( H_2(A) \to H_2(Q) \) must be trivial, since the torus \( T \) is the boundary of a 3-manifold and \( H_2(T) \to H_2(A) \) is surjective. This condition reduces the number of non-cyclic \( A \) we need to consider (though it need not eliminate them completely), which is good since those never extend to the resulting manifold.

For example, in \( A_5 \) the subgroups isomorphic to \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) are eliminated by this criterion, and so the relevant abelian subgroups are just the cyclic subgroups, which have orders 1, 2, 3 or 5. If each type of cyclic subgroup occurs equally often, this would lead to the guess that the last relator would be satisfied 35% of the time. But even if the cyclic group of order 5 occurs much more frequently than the others, this would still give an estimate that the last relator would be satisfied 16.7% of the time, far more than the 1.7% predicted for a random relator. If one looks at a random map from \( \mathbb{Z}^2 \to A_5 \), then the cyclic group of order 5 is indeed the most common image, occurring about half the time.
5.2 Genus 2

There is also a special argument that sometimes applies to the next-to-last relator, which works in particular for epimorphisms to PSL$_2\mathbb{F}_q$. The surface $\Sigma$ of genus 2 has a special homeomorphism of order 2, the hyperelliptic symmetry $\tau$, that is centralized by the entire mapping class group of $\Sigma$. The quotient $\Sigma/\tau$ orbifold is a sphere with 6 elliptic points of order 2. Any simple closed curve on $\Sigma$ can be isotoped to be set-wise invariant under $\tau$. If the curve is non-separating, then it is mapped to itself with reversed orientation. If we fix a hyperbolic metric on $\Sigma$ which is invariant under $\tau$, then the geodesic representative of a non-separating curve passes through exactly 2 of the 6 fixed points of $\tau$.

The consequences of this are easiest to describe for the boundary of a genus 2 handlebody $H$: Any non-separating simple closed curve on $\partial H$ describes a circular word in the free group $F_2 = \pi_1(H)$ that is the same read backward or forward. This is because the hyperelliptic symmetry $\tau$ of $\partial H$ extends over $H$; the induced action $\tau_* : \pi_1(H) \to \pi_1(H)$ sends standard generators $\{a,b\}$ of $\pi_1(H)$ to their inverses $\{a^{-1},b^{-1}\}$. As mentioned, $\tau$ sends a non-separating curve on $\partial H$ to itself with reversed orientation. Thus if $w$ is a word in $\pi_1(H)$ which is the image of a non-separating curve, we have that $\tau_*(w)$ is conjugate to $w^{-1}$; if $w$ is regarded as a circular word this is the same as saying that it is the same read backward or forward. It may or may not be possible to conjugate the linear word $w$ to read the same backward and forward. This depends on which pair of the 6 fixed points of $\tau$ the curve passes through, as we now explain. Pick dual discs for our chosen basis of $\pi_1(H)$ which are invariant under $\tau$; these contain 4 of the fixed points of $\tau$. Running around the geodesic representing $w$ looking at intersections with the discs reads off the word $w$. If the geodesic goes through one of the middle 2 fixed points of $\tau$ which are not near the dual disks, then reading off starting at one of those points results in a linearly palindromic $w$. If instead we start reading from a fixed point of $\tau$ on one of the discs, then we get a $w$ so that $\tau^2(w) = sw^{-1}s^{-1}$ where $s$ is one of the generators. Thus, it is always possible to conjugate $w$ so that $\tau_*(w) = sw^{-1}s^{-1}$ where $s \in \{1,a^{\pm 1},b^{\pm 1}\}$. In this case, we will say that $w$ is in standard form.

In the case of an epimorphism $\pi_1(H) \to \text{PSL}_2\mathbb{F}_q$, we will show that the involution $\tau_*$ on $\pi_1(H)$ pushes forward to one on the image group. We will use this fact to greatly restrict the possibilities for the image of a non-separating curve in PSL$_2\mathbb{F}_q$. In particular, we will show:

5.3 Theorem Let $H$ be a handlebody of genus 2, and $\rho : \pi_1(H) \to \text{PSL}_2\mathbb{F}_q$ be an epimorphism where $q$ is an odd prime power. If $w$ is a word in standard form coming from a non-separating embedded curve in $\partial H$, then the image of $w$ under $\rho$ lies in a subset of PSL$_2\mathbb{F}_q$ of size at most $(1/2)(q^2 + q + 2)$.

For comparison, the order of PSL$_2\mathbb{F}_q$ is $(1/2)q(q + 1)(q - 1)$. Note also there is always a non-separating curve on $\Sigma$ in the kernel of $\rho$, as there is one in the kernel of $\rho : \pi_1(\Sigma) \to \pi_1(H)$. Thus from this theorem one would naively expect that a non-separating curve is about $q$ times more likely to be in the kernel of $\rho$ than a random word in $\pi_1(H)$. In the theorem, the subset of PSL$_2\mathbb{F}_q$ mentioned depends on the standard form of $w$, more precisely on the $s$ such that $\tau_*(w) = sw^{-1}s^{-1}$. If you prefer a statement which is independent of $s$, just multiply the size of the subset by $5$ (the number of possibilities for $s$).
Before proving Theorem 5.3 let us further contrast the picture it gives with that of random words in \( \pi_1(H) \). Consider 3-manifolds \( M \) obtained by attaching a single 2-handle to \( H \) along a non-separating curve (these are examples of tunnel-number one 3-manifolds). For comparison, look at two-generator, one-relator groups where the generator is chosen at random. For such a random group, we can work out the probability of a \( Q = \text{PSL}_2\mathbb{F}_q \) quotient just as we did before; there are \( \approx |Q|/|\text{Out } Q| \) epimorphisms from \( F_2 \) onto \( Q \), and each factors over the relator with probability \( 1/|Q| \). Thus the number of \( Q \)-quotients should be roughly Poisson distributed with mean \( 1/|\text{Out } Q| \). If we specialize to the case that \( q \) is prime, then \( |\text{Out } Q| = 2 \) and so the probability of a \( Q \)-quotient for the random group is \( \approx 1 - e^{-1/2} \approx 39\% \). In particular, this probability is essentially independent of \( Q \). In contrast, Theorem 5.3 suggests that the number of quotients of a 3-manifold group should be Poisson distributed with mean \( \approx q/|\text{Out } Q| \). In the case where \( q \) is prime, this leads to the probability of a \( Q \) cover being \( 1 - e^{-q/2} \), which goes to 1 as \( \text{Out } q \to \infty \). The last column of Table 6.3 gives some data on this, using random curves coming from our notion of a random Heegaard splitting. It suggests that, at least qualitatively, this last prediction of Theorem 5.3 really does hold.

To prove Theorem 5.3 we first show that \( \tau \), pushes forward to the image group \( \text{PSL}_2\mathbb{F}_q \).

5.4 Lemma Let \( A \) and \( B \) be elements of \( \text{PSL}_2\mathbb{F}_q \) where \( q \) is an odd prime power. Suppose that \( A \) and \( B \) do not have a common fixed point when acting on \( \pi_1(\mathbb{F}_q) \). Then there exists an element \( T \) in \( \text{PGL}_2\mathbb{F}_q \) of order 2 such that

\[
TAT^{-1} = A^{-1} \quad \text{and} \quad TBT^{-1} = B^{-1}.
\]

Proof In general, the trace of an element in \( \text{PGL}_2\mathbb{F}_q \) depends on the lift to \( \text{GL}_2\mathbb{F}_q \). However, the elements \( T \) in \( \text{PGL}_2\mathbb{F}_q \) which have order 2 are exactly those where the trace of any lift is 0. Now lift \( A \) and \( B \) to elements of \( \text{SL}_2\mathbb{F}_q \), and consider the equations

\[
\text{tr}(T) = 0, \text{tr}(TA) = 0, \quad \text{and} \quad \text{tr}(TB) = 0
\]

where \( T \) is a \( 2 \times 2 \) matrix over \( \mathbb{F}_q \) (possibly singular). As these are homogeneous linear equations, there is a non-zero solution, call it \( T \). We claim that \( T \) must be non-singular. If not, change to a basis where the first vector spans the kernel of \( T \), and so

\[
T = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}
\]

where \( t \neq 0 \).

But then \( \text{tr}(TA) = 0 \) forces \( A \) to be upper-triangular. As the same is true for \( B \), we have a contradiction in that \( A \) and \( B \) have a common fixed point in \( \pi_1(\mathbb{F}_q) \). In \( \text{PGL}_2\mathbb{F}_q \), the elements \( T, TA, \) and \( TB \) have order 2, which proves the lemma.

Now consider an epimorphism \( \rho : \pi_1(H) \to \text{PSL}_2\mathbb{F}_q \). By the lemma, there exists a \( T \in \text{PGL}_2\mathbb{F}_q \) so that the involution of \( \text{PSL}_2\mathbb{F}_q \) induced by \( T \) is the push-forward of the involution \( \tau \) on \( \pi_1(H) \). So if \( w \) is a word in standard form coming from a non-separating curve, then there is a \( U \) of order 2 in \( \text{PGL}_2\mathbb{F}_q \) such that \( U\rho(w)U^{-1} = \rho(w)^{-1} \), where \( U \) is one of \( T, A \pm T, \) or \( B \pm T \). Thus just knowing that \( w \) comes from a non-separating curve implies that \( \rho(w) \) is sent to its inverse by
an involution of $\text{PSL}_2\mathbb{F}_q$ which is completely determined by $\rho$ and the symmetry points of $w$. The next lemma shows that this restricts $\rho(w)$ to a proper subset of $\text{PSL}_2\mathbb{F}_q$, and completes the proof of Theorem 5.3.

5.5 Lemma Let $q$ be an odd prime power, and $U$ an element of $\text{PGL}_2\mathbb{F}_q$ of order 2. Then the number of $W \in \text{PSL}_2\mathbb{F}_q$ such that $UW^{-1} = W^{-1}$ is

$$\frac{1}{2} \left( q^2 + (2\varepsilon_- - 1)q + 2\varepsilon_+ \right),$$

where $\varepsilon_\pm$ is 1 if $\pm \det U$ is a square in $\mathbb{F}_q$ and 0 otherwise.

Proof Requiring that $UW^{-1} = W^{-1}$ is the same as saying that $UW$ has order 1 or 2. In the former case, $W = U$ and this contributes to our count only when $U \in \text{PSL}_2\mathbb{F}_q$, that is, when $\varepsilon_+ = 1$. The latter case is the same as counting solutions to the equations,

$$\text{tr}(UW) = 0 \quad \text{and} \quad \det(W) = 1$$

where here we have lifted everything to $\text{GL}_2\mathbb{F}_q$. One can write out these equations explicitly (one is linear and the other quadratic), and it is not hard to see that the number of solutions is equal to $q^2 + (2\varepsilon_- - 1)q$. Passing to $\text{PSL}_2$ from $\text{SL}_2$ reduces the number of such $W$ by half. Combining, we get the count claimed for $W$. \qed

5.6 Second to last relator

Having worked out the simpler handlebody case, we return to our original question about attaching the second to last relator. Again, let $\Sigma$ be a surface of genus 2, and set

$$G = \pi_1(\Sigma) = \langle a, b, c, d \mid [a, b] = [c, d] \rangle,$$

where our convention is that $[a, b] = aba^{-1}b^{-1}$. If we choose the base point for $\pi_1(\Sigma)$ to be the second leftmost fixed point of $\tau$, then $\tau$ acts on $G$ in the following way

$$\tau(a) = a^{-1}, \tau(b) = b^{-1}, \tau(c) = xc^{-1}x^{-1}, \quad \text{and} \quad \tau(d) = xd^{-1}x^{-1}, \quad (5.7)$$

where $x = a^{-1}b^{-1}dc$. By taking arcs from the base point to the other fixed points of $\tau$, we see that every non-separating simple closed curve on $\Sigma$ can be represented by an element $w \in G$ such that $\tau(w) = sws^{-1}$ where $s$ is in $S = \{1, a, b, x, xd\}$. Such a $w$ is said to be in standard form.

Now consider a homomorphism $G \to \text{PSL}_2\mathbb{F}_q$. The next theorem gives a criterion for when $\tau$ pushes forward to an automorphism of $\text{PSL}_2\mathbb{F}_q$.

5.8 Theorem Let $G$ be the fundamental group of a surface of genus 2. Let $\tau$ be the automorphism of $G$ coming from the hyperelliptic involution. Consider an irreducible homomorphism $f : G \to \text{PSL}_2\mathbb{F}_q$ where $q$ is odd. If $f$ lifts to a homomorphism into $\text{SL}_2\mathbb{F}_q$, then there is an element $T \in \text{PGL}_2\mathbb{F}_q$ such that $TfT^{-1} = f \circ \tau$. 
Now let $M$ be a 3-manifold with boundary $\Sigma$, and suppose $f$ is the restriction of a homomorphism $\pi_1(M) \to \text{PSL}_2\mathbb{F}_q$. The obstruction to lifting $f$ to $\text{SL}_2\mathbb{F}_q$ is an element of $H^2(G, \mathbb{Z}/2\mathbb{Z})$, which vanishes as $f$ extends over $M$. So if $f$ is irreducible, then it has the above symmetry, and this restricts the image under $f$ of a non-separating simple closed curve similar to as before. If $w \in \tilde{G}$ is a standard form representative for such a curve, then $f(w)$ must satisfy $Tf(w)T^{-1} = f(s)f(w)^{-1}f(s)^{-1}$, where $s$ is one of the five elements of $S$ and $T \in \text{PGL}_2\mathbb{F}_q$ is the element inducing the symmetry. If instead $f$ is reducible, its image lies in a proper subgroup of $\text{PSL}_2\mathbb{F}_q$.

Thus, in either case, the image of $f(w)$ is restricted. So as long as there is some non-separating simple closed curve in the kernel of $f$, one would expect that the probability that $f$ extends over the second to last relator should be much higher than for a random word in a free group. However, as in the torus case, there could be situations where there are no such curves in $\ker f$. Unlike the torus case, we do not know any examples where this actually occurs. Now let us prove the theorem.

**Proof** Before beginning the proof itself, let us rephrase the question in order to make the algebra that follows seem more natural. Let $X$ be the hyperbolic orbifold $\Sigma/\tau$. We have

$$1 \to G \to \pi_1(X) \to \mathbb{Z}/2 \to 1,$$

where $\pi_1(X)$ can be obtained by adding an element $t$ to $G$ subject to the requirement that conjugating by $t$ induces the action of $\tau$ given in (5.7). Thus the problem at hand is given $f: G \to \text{PSL}_2\mathbb{F}_q$, does $f$ extend to a homomorphism $\pi_1(X) \to \text{PGL}_2\mathbb{F}_q$? The separating curve in $\Sigma$ representing $[a, b]$ maps down to a curve $\gamma$ which separates $X$ into two regions containing 3 cone points each; in $\pi_1(X)$ we have $\gamma^2 = [a, b]$. Roughly, Lemma 5.4 says that we can extend $f$ over each half of $X$, and the issue is whether these agree along $\gamma$. It turns out that $f$ lifting to $\text{SL}_2\mathbb{F}_q$ guarantees this.

Let $\tilde{f}$ denote the lift of $f$ to $\text{SL}_2\mathbb{F}_q$, and set $A = \tilde{f}(a), B = \tilde{f}(b)$, etc. To prove the theorem, it suffices to find a $T \in \text{GL}_2\mathbb{F}_q$ such that

$$\text{tr}(T) = \text{tr}(TA) = \text{tr}(TB) = \text{tr}(X^{-1}T) = \text{tr}(X^{-1}TC) = \text{tr}(X^{-1}TD) = 0 \quad (5.9)$$

as then all of the above elements have order 2 in $\text{PGL}_2\mathbb{F}_q$ and so $T f T^{-1} = f \circ \tau$. The main case is when $f$ is irreducible when restricted to both of the subgroups $\langle a, b \rangle$ and $\langle c, d \rangle$, and we begin there. Think of (5.9) as homogeneous linear equations in the entries of a $2 \times 2$ matrix $T$. By the proof of Lemma 5.4 any non-zero $T$ which satisfies the first 3 trace conditions is necessarily non-singular. Thus we just need to prove that the dimension of the solution space of (5.9) is positive dimensional. To check this, we are free to enlarge our base field from $\mathbb{F}_q$ to its algebraic closure $k$. Now over $k$, the equations (5.9) have a non-zero solution if and only if there is one with $\det(T) = 1$, since any non-zero determinant has a square root in $k$. So henceforth, we try to solve equations (5.9) for $T \in \text{SL}_2 k$.

By Lemma 5.4, we can choose $T, U \in \text{SL}_2 k$ so that

$$\text{tr}(T) = \text{tr}(TA) = \text{tr}(TB) = 0 \quad \text{and} \quad \text{tr}(U) = \text{tr}(UC) = \text{tr}(UD) = 0.$$

Our goal is show $U = X^{-1}T$. As we are working in $\text{SL}_2$ rather than $\text{PSL}_2$ we have $T^2 = -1$ not 1, and the same for $TA, U$, etc. Thus we have

$$(BAT)^2 = -[B, A] = -[D, C] = (DCU)^2.$$
So $BAT$ and $DCU$ have the same square, which is not $-1$ because if $A$ and $B$ commuted we would be in the reducible case. Any $S \neq -1$ in $SL_2k$ has at most 2 square roots. Thus, after replacing $U$ with $-U$ if necessary, we have $BAT = DCU$ which implies $U = X^{-1}T$, as required. This completes the proof of the theorem in the case when $f$ is irreducible when restricted to both $\langle a,b \rangle$ and $\langle c,d \rangle$.

Now suppose instead $f$ is reducible when restricted to $\langle a,b \rangle$. This is equivalent to $\text{tr}(ABA^{-1}B^{-1}) = 2$, and the same reasoning implies that $f$ must be reducible when restricted to $\langle c,d \rangle$ as well. Now $ABA^{-1}B^{-1}$ can not be parabolic, as then $f$ itself would be reducible. Therefore, we are down to the case where $ABA^{-1}B^{-1} = 1$, i.e. both $\langle A,B \rangle$ and $\langle C,D \rangle$ are abelian. Now by changing generators of $G$ we can assume that $\langle A,C \rangle$ do not have a common fixed point in $\mathbb{P}^1$. By Lemma 5.4 there is a $T \in GL_2(\mathbb{F}_q)$ such that

$$
\text{tr}(T) = \text{tr}(TA) = \text{tr}(TCBA) = 0.
$$

Now as $B$ commutes with $A$, the above also forces $\text{tr}(TB) = \text{tr}(TBA) = \text{tr}(BAT) = 0$. Rewriting $\text{tr}(TCBA) = 0$ as $\text{tr}(BAT \cdot C) = 0$, the commuting of $C$ and $D$ gives $\text{tr}(BAT \cdot Y) = 0$ for all $Y \in \langle C,D \rangle$. Expanding $X$ in (5.2), it follows that all those equations hold, completing the proof. $\square$

### 6 Covers of random Heegaard splittings

In this section, we address the following question: Fix a finite group $Q$ and genus $g$. Consider a manifold $M$ obtained from a random genus-$g$ Heegaard splitting. What is the probability that $M$ has a cover with covering group $Q$? What is the distribution of the number of covers? In this section, we will begin looking at these by showing that the answers to both these question are well-defined (Prop. 6.1). We will then give three examples which illustrate some of the key issues in computing these probabilities. In later sections, we will compute these probabilities exactly for abelian groups (Section 8), and we will give a complete characterization of these probabilities for non-abelian simple groups in the limit when $g$ is large (Section 7).

Fix a genus-$g$ handlebody $H_g$, and denote $\partial H_g$ by $\Sigma$. Let $\mathcal{M}_g$ be the mapping class group of $\Sigma$. Given $\phi \in \mathcal{M}_g$, let $N_\phi$ be the closed 3-manifold obtained by gluing together two copies of $H_g$ via $\phi$. Our notion of a random Heegaard splitting of genus $g$ is as follows. Fix generators $T$ for $\mathcal{M}_g$. A random element $\phi$ of $\mathcal{M}_g$ of complexity $L$ is defined to be the result of a random walk in the generators $T$ of length $L$. Then we define the manifold of a random Heegaard splitting of genus $g$ and complexity $L$ to be $N_\phi$, where $\phi$ is a random element of $\mathcal{M}_g$ of complexity $L$.

We begin with the following proposition which shows that the questions we are interested in make sense in the context of random Heegaard splittings:

#### 6.1 Proposition

Fix a Heegaard genus $g$ and a finite group $Q$. Let $N$ be the manifold of a random Heegaard splitting of complexity $L$, and let $p(L)$ be the probability that $\pi_1(N)$ has an epimorphism onto $Q$. Then $p(L)$ converges to a limit $p(Q,g)$ as $L$ goes to infinity. Moreover, $p(Q,g)$ is independent of the choice of generators for $\mathcal{M}_g$, and the probability distribution of the number of epimorphisms also converges.
Proof Consider the collection $\mathcal{A}$ of all epimorphisms from $\pi_1(\Sigma)$ to $Q$, up to automorphisms of $Q$. Let $\phi$ be in $\mathcal{M}_g$, and consider the associated 3-manifold $N_\phi$. Identify $\Sigma$ with the boundary of the first copy of $H_\phi$ in $N_\phi$, so that a homomorphism $f$ in $\mathcal{A}$ factors through to one of $\pi_1(N_\phi)$ if and only if $f$ and $f \circ \phi^{-1}$ both extend over $H_\phi$.

Let $\mathcal{E} \subset \mathcal{A}$ consist of those homomorphisms which do extend over $H_\phi$. Before proving the full proposition, let us consider the simpler question: Given an $f$ in $\mathcal{E}$, what is the probability that it extends over the copy of $H_\phi$ in $N_\phi$? That is, how often does $f \circ \phi^{-1}$ also lie in $\mathcal{E}$? Consider the (left) action of $\mathcal{M}_g$ on $\mathcal{A}$ by precomposition: $\phi \cdot f = f \circ \phi^{-1}$. Thus we are interested in the probability that $\phi \cdot f$ lies in $\mathcal{E}$. Now look at the Cayley graph of this action of $\mathcal{M}_g$ on $\mathcal{A}$. That is, consider the graph whose vertices are $\mathcal{A}$ and whose edges correspond to the action of the chosen generators $T$ of $\mathcal{M}_g$. If we do a random walk in $\mathcal{M}_g$, the corresponding sequence of $\phi \cdot f$ moves around this graph according to the labels on the edges. As in Lemma 3.2, the distribution of the $\phi \cdot f$ converges to the uniform measure on the orbit of $f$ in $\mathcal{A}$. Let $C$ be the orbit of $f$ in $\mathcal{A}$. Since the $\phi \cdot f$ are nearly uniformly distributed in $C$ for large $i$, we see that the probability that $f$ extends to $N_\phi$ converges to $|C \cap \mathcal{E}|/|C|$. Note that this limit depends only on the orbit $C$ and not on the choice of generators for $\mathcal{M}_g$.

Returning to the question of the probability of $N_\phi$ having an epimorphism to $Q$, consider the action of $\mathcal{M}_g$ on subsets of $\mathcal{A}$. Again, if we look at the images of $\mathcal{E}$ under $\phi$, these are nearly uniformly distributed in the orbit of $\mathcal{E}$ in the power set of $\mathcal{A}$; thus $p(L)$ converges to the proportion of subsets in the orbit of $\mathcal{E}$ which intersect $\mathcal{E}$. Finally, the distribution of the number of epimorphisms also converges, to the corresponding finite averages over the orbit of $\mathcal{E}$ under $\mathcal{M}_g$.

Next, we will give three detailed examples in genus 2 where we calculate these probabilities exactly. These illustrate the some of the main issues and techniques that arise later in Sections 8.4, 7.

6.2 Example: $\mathbb{Z}/2$

For our first genus 2 example, let us begin with $Q = \mathbb{Z}/2$. In this case, $\mathcal{A}$ is isomorphic to the non-zero elements of $H^1(\Sigma; \mathbb{Z}/2)$. Thus $|\mathcal{A}| = 15$, and similarly $|\mathcal{E}| = 3$. In order to compute the probabilities, we need to understand the image of $\mathcal{M}_2$ in the symmetric group of $\mathcal{A}$. While the action is transitive, its image is much smaller than all of $\text{Sym}(\mathcal{A})$: it is the 4-dimensional symplectic group over $\mathbb{F}_2$, which we will call $G$. As we do a random walk in $\mathcal{M}_2$, the images under $\mathcal{M}_2 \to \text{Sym}(\mathcal{A})$ converge to the uniform distribution on $G$. Thus $p(\mathbb{Z}/2, 2)$ is the same as the probability that $(g \cdot \mathcal{E}) \cap \mathcal{E} \neq \emptyset$ for a random $g \in G$.

First, let us compute the expected number of $\mathbb{Z}/2$-quotients. To begin, focus on whether a fixed $f \in \mathcal{E}$ extends to $\pi_1(N_\phi)$. For $g \in G$, we have $f \in g \cdot \mathcal{E}$ if and only if $g^{-1} \cdot f \in \mathcal{E}$. Since the action of $G$ is transitive, $\{g^{-1} \cdot f\}_{g \in G}$ is uniformly distributed in $\mathcal{A}$. Therefore, the probability that $f \in g \cdot \mathcal{E}$ is $|\mathcal{E}|/|\mathcal{A}| = 1/5$. Thus the expected number of $\mathbb{Z}/2$ quotients from $f$ is $1/5$. As expectations add, the overall expected number of quotients is $|\mathcal{E}|^2/|\mathcal{A}| = 3/5$. Computing $p(\mathbb{Z}/2, 2)$ is more complicated because there may be correlations between different $f$ in $\mathcal{E}$.
6.3 Example: $A_5$

Next, let us consider the smallest non-abelian simple group $Q = A_5$. In this case, $|\mathcal{A}| = 2016$ and $|\mathcal{E}| = 19$. The action of $\mathcal{M}_2$ on $\mathcal{A}$ has two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$ of size 1440 and 576 respectively, and $\mathcal{E}$ is completely contained in $\mathcal{A}_1$. In order to compute $p(A_5, 2)$, we only need to understand the action on $\mathcal{A}_1$. It turns out that this is the full alternating group $\text{Alt}(\mathcal{A}_1)$. As before, we get that $p(A_5, 2)$ is the probability that $(g \cdot \mathcal{E}) \cap \mathcal{E} \neq \emptyset$ for a random $g \in \text{Alt}(\mathcal{A}_1)$. Now $\text{Alt}(\mathcal{A}_1)$ acts transitively on subsets of $\mathcal{A}_1$ of size 19, as $19 \leq |\mathcal{A}_1| - 2$. Thus $g \cdot \mathcal{E}$ is uniformly distributed over all subsets of $\mathcal{A}_1$ of size 19. Hence the probability that $(g \cdot \mathcal{E}) \cap \mathcal{E} \neq \emptyset$ is just the probability that a randomly chosen subset of 19 elements of $\mathcal{A}_1$ intersects $\mathcal{E}$. Thus

$$p(A_5, 2) = 1 - \frac{1421}{19} \approx 22.43\%.$$ 

As in the previous example, the expected number of $A_5$ quotients is $|\mathcal{E}|^2 / |\mathcal{A}_1| \approx 0.2507$.

The case where $\mathcal{M}_g$ acts as the alternating group of an orbit is one that we will find in general for non-abelian simple groups, so it is worth discussing here the connection to the Poisson distribution. Recall that the Poisson distribution with mean $\mu > 0$ is a probability distribution on $\mathbb{Z}^+$ where $k$ has probability $\frac{\mu^k e^{-\mu}}{k!}$. Roughly, the Poisson distribution describes the number of occurrences of a preferred outcome in a large ensemble of events where, individually, the outcome is rare and independent, but in aggregate the expected number of occurrences is $\mu > 0$. In our context, we have a set $\mathcal{A}$ of size $n$ which contains a marked subset $\mathcal{E}$ of size $a$; we then pick another subset $\mathcal{E}'$ of size $a$ and want to know the size of $\mathcal{E} \cap \mathcal{E}'$. If $n$ is large, the distribution of $|\mathcal{E} \cap \mathcal{E}'|$ is essentially Poisson with mean $\mu = a^2 / n$. In particular, the probability of at least one intersection is $1 - e^{-\mu}$. In the case of $A_5$, this approximation gives the probability of an $A_5$ cover at 22.17%.

It is worth mentioning that the alternating group action here makes it very easy to compute $p(Q, g)$ from just the sizes of the orbits of $\mathcal{A}$ and $\mathcal{E}$, and that this is not true in general. For instance, returning to the previous example, $p(\mathbb{Z}/2, 2) = 7/15 \approx 0.4667$ but the probability that $|\mathcal{E}| = 3$ items chosen from $|\mathcal{A}| = 15$ things intersects a fixed set of size 3 is larger: $47/91 \approx 0.5165$. The reason for the difference is that the action of $\mathcal{M}_2$ is not 3-transitive, and there are positive correlations between different elements of $\mathcal{E}$ factoring through to $\pi_1(\mathcal{N}_g)$; in particular, because the number of $\mathbb{Z}/2$-quotients is $|H^1(\mathcal{N}_g, \mathbb{Z}/2)| = 1 = 2^n - 1$, if we have more that one $\mathbb{Z}/2$-quotient then we have 3 of them.
6.4 Example: \( \text{PSL}_2 \mathbb{F}_{13} \)

Let us look at a more complicated example. Consider \( Q = \text{PSL}_2 \mathbb{F}_{13} \), a group of order 1092. In this case, \( |A| = 623520 \) and \( |E| = 495 \). In this case there are four orbits of sizes 235680, 94080, 278400, and 15360. Only the first two orbits intersect \( E \) in subsets of size 307 and 188 respectively; set \( E_i = E \cap A_i \). The action on the first two orbits \( A_1 \) and \( A_2 \) are again by the full alternating groups \( \text{Alt}(A_1) \). Since the two alternating groups have different orders, by Lemma 3.7 the map \( M_2 \to \text{Alt}(A_1) \times \text{Alt}(A_2) \) is surjective. Therefore \( g \cdot E_1 \) and \( g \cdot E_2 \) are independent of each other, and

\[
p(\text{PSL}_2 \mathbb{F}_{13}, 2) = 1 - \prod_{i=1}^{2} \left( \frac{|A_i| - |E_i|}{|E_i|} \right) / \left( \frac{|A_i|}{|E_i|} \right) \approx 54.02\%.
\]

Further, the expected number of quotients is \( \sum_{i=1}^{2} \frac{|E_i|^2}{|A_i|} \approx 0.7756 \). Although \( E \) is contained in two orbits, the overall distribution of quotients is still nearly Poisson since the sum of two independent Poisson variables is also Poisson.

6.5 Example: Small simple groups

As we will see in Lemma 6.10, for simple groups the size of \( A_g \) grows like \( |Q|^{2g-2} \). Thus even for genus 2, it is difficult to compute the action of \( M_g \) on \( A_g \). However, the proof of Proposition 6.1 suggests a way to approximate \( p(Q, g) \) by looking only at \( |Q|^{g-1} \) epimorphisms. Namely, we first compute \( E_g \) and consider \( \phi \in M_g \). The time average of \( (\phi \cdot E_g) \cap E_g \) will then converge to \( p(Q, g) \). For genus 2, we did this for the simple groups of order less than 7000. The results are shown in Table 6.6 and we compare them to the results of our earlier experiment [DT2], as well as the limit of \( p(Q, g) \) as \( g \to \infty \).

6.7 The general picture

The rest of this section is devoted to what we can say in general about \( p(Q, g) \) for an arbitrary group, with particular emphasis on the limiting picture as the genus \( g \) goes to infinity. We would like to say that the probability distributions on the number of covers for each \( g \) converge to a limiting probability distribution as \( g \to \infty \); in particular, this would suggest some robustness in our notion of random Heegaard splitting and that, perhaps, we should expect to find this same limiting distribution with other notions of random 3-manifold. While we will build up quite a bit of information about \( A_g, \mathcal{E}_g \) and \( A_g / M_g \) for large \( g \), we do not know how to prove the existence of a limiting distribution in general. Instead, we will show here that the expected number of covers does have a limit as \( g \to \infty \) (Theorem 6.21). In the special cases of abelian and simple groups, we are able to obtain a complete asymptotic picture (see Sections 8 and 7). The case of abelian groups is essentially independent of the rest of this section, so if that is your primary interest you can skip ahead to Section 8.
### Table 6.6

This table gives the percentage of genus 2 manifolds which have particular finite simple quotients, and compares this data to other samples. The first 4 columns list the quotient group \( Q \) and some of its basic properties. The number \( p(Q, 2) \) is the probability that a manifold coming from a random Heegaard splitting of genus 2 will have a cover with covering group \( Q \); as discussed in Section 6.5, these numbers are actually an approximation coming from looking at a random walk in \( \mathcal{M}_2 \) of length \( 10^6 \). The Census column is the corresponding probability for those genus 2 manifolds in the Hodgson-Weeks census that we studied in \[DT2, \S 5\]. The number \( p(Q, \infty) \) is the limit of \( p(Q, g) \) as \( g \to \infty \); by Theorem 7.1 we have

\[
p(Q, \infty) = 1 - e^{-|H_2|/|\text{Out}|}.
\]

Finally, we looked at attaching a single 2-handle to a genus 2 handlebody along a non-separating curve to give a tunnel number one manifold with torus boundary; as before, the attaching curve is chosen by a random walk in \( \mathcal{M}_2 \). The last column records the probability that the resulting manifold has a \( Q \) cover. This data is interesting to compare with the discussion in Section 5.2.

Examples 6.2–6.4 illustrate the key issues that we encounter here. As usual, let \( H_g \) be a handlebody of genus \( g \), \( \Sigma_g = \partial H_g \), and \( \mathcal{M}_g \) be the mapping class group of \( \Sigma_g \). We want to compute the probability \( p(Q, g) \) that a 3-manifold associated to a random genus-\( g \) Heegaard splitting has a cover with group \( Q \). As we saw in the examples, what we need to know is how many epimorphisms \( \pi_1(\Sigma_g) \to Q \) there are, and how \( \mathcal{M}_g \) acts on them. As we will see, the answer to the first question is easy (Section 6.8), and it is the second question that requires more work. For the latter question, for a general \( Q \) we will only be able to classify the orbit set \( \mathcal{A}_g / \mathcal{M}_g \) (Sections 6.12–6.19). The basic idea is this: given a homomorphism \( f: \pi_1(\Sigma_g) \to Q \) in \( \mathcal{A}_g \) we get a map from \( \Sigma_g \) to the classifying space \( BQ \), and thus an element of \( H_2(Q, \mathbb{Z}) \). This homology class is invariant under the action of \( \mathcal{M}_g \) on \( \mathcal{A}_g \). The key is to show that for large \( g \) this class is the only invariant of the action of \( \mathcal{M}_g \) on \( \mathcal{A}_g \) (Theorem 6.20). Finally, this section concludes with a more group-theoretic point of view on some aspects of this section as they relate to \( \mathcal{E}_g \) (Section 6.24).

#### 6.8 Counting \( \mathcal{A} \) and \( \mathcal{E} \)

In this subsection, we determine the number of elements of \( \mathcal{A} \) and \( \mathcal{E} \) for a fixed group \( Q \) as the genus gets large. For results where the genus is fixed but the (typically simple) group \( Q \) gets large see \[LS1\] and \[LS2\]. If \( Q \) is a group, \( \bar{Q} \) will
denote the commutator subgroup. The following lemma computes for us the probability that a random homomorphism $F_{2g} \to Q$ factors through $\pi_1(\Sigma_g)$.

6.9 Lemma Let $Q$ be a finite group. If $(a_1, b_1, \ldots, a_g, b_g) \in Q^{2g}$ is chosen uniformly at random, the probability that $\prod_{i=1}^{g} [a_i, b_i] = 1$ converges to $1/|Q|$ as $g \to \infty$.

Proof The set $T = \{ [a, b] \}_{a, b \in Q}$ generates $Q$. Thus choosing $(a_i, b_i) \in Q^{2g}$ uniformly at random is the same as choosing a string of $g$ elements of $T$. That is, $\prod_{i=1}^{g} [a_i, b_i]$ is the result of a random walk in $Q$ with respect to $T$. As such, it converges to the uniform distribution on $Q$. Therefore, the probability of it being 1 converges to $1/|Q|$ as $g \to \infty$.

For non-abelian simple groups, the set $T$ above is very large, conjecturally all of $Q = Q'$. In this case, the random walk is on a (weighted) complete graph with vertices $Q$. Thus it will converge to the uniform distribution very quickly.

Let $\mathcal{A}_g$ be the set of all epimorphisms of $\pi_1(\Sigma_g)$ to $Q$, modulo automorphisms of $Q$.

6.10 Lemma Let $Q$ be a finite group. Then $|\mathcal{A}_g| \sim |Q|^{2g}/(|Q'| |\text{Aut}(Q)|)$ as $g \to \infty$.

Proof Consider $\pi_1(\Sigma_g)$ in its standard presentation with $2g$ generators. A $2g$-tuple $(a_i, b_i) \in Q^{2g}$ gives a possible homomorphism $\pi_1(\Sigma_g) \to Q$. If the $2g$-tuple is randomly chosen, the probability that the entries generate $Q$ goes to 1 as $g \to \infty$. By the preceding lemma, the probability that it induces a homomorphism of $\pi_1(\Sigma_g)$ is $1/|Q'|$. Combining the above, this says that the number of elements of $Q^{2g}$ which give epimorphisms is asymptotic to $|Q|^{2g}/|Q'|$. Since the action of $\text{Aut}(Q)$ on epimorphisms is free, we get the claimed formula for $|\mathcal{A}_g|$.

Let $\mathcal{B}_g$ be the set of all epimorphisms of $\pi_1(\Sigma_g)$ to $Q$, modulo automorphisms of $Q$.

6.11 Lemma Let $Q$ be a finite group. Then $|\mathcal{B}_g| \sim |Q|^{3g}/|\text{Aut}(Q)|$.

6.12 Associated homology classes

For us, an important invariant of an epimorphism $f: \pi(\Sigma_g) \to Q$ is the associated homology class $c_f$ in $H_2(Q, \mathbb{Z})$ coming from the induced map on classifying spaces $\Sigma_g \to BQ$. In the context of finite groups, $H_2(Q, \mathbb{Z})$ is usually called the Schur multiplier, and it is an important invariant there, especially in the study of simple groups (see e.g. [Wie]). In this section all homology groups will have $\mathbb{Z}$ coefficients, so we will stop including this in the notation. It is not quite true that this homology class is well-defined for elements of $\mathcal{A}_g$, but we can associate an element of $H_2(Q)/\text{Out}(Q)$. The issue here is that $\text{Aut}(Q)$ may act non-trivially on $H_2(Q)$; after all, elements in $\mathcal{A}_g$ are equivalence classes of epimorphisms modulo the action of $\text{Aut}(Q)$. Now inner automorphisms of $Q$ induce self-maps of $BQ$ which are homotopic to the identity, and so such automorphisms act trivially on $H_2(Q)$. Thus associated to an $[f] \in \mathcal{A}_g$ we get a well-defined homology class $c_f$ in $H_2(Q)/\text{Out}(Q)$. For simple groups, it is often the case that $H_2(Q)$ is trivial or $\mathbb{Z}/2$; the action of $\text{Out}(Q)$ must be trivial in this case. The first simple group where the
action is non-trivial is $A_6$, where $H_2(Q) = \mathbb{Z}/6$ and $\text{Out}(Q) \to \text{Aut}(H_2(Q)) = \mathbb{Z}/2$ is surjective. In general, the map $\text{Out}(Q) \to \text{Aut}(H_2(Q))$ need not be surjective, as the example of $\text{PSL}_2\mathbb{F}_4$ shows.

Now the map $\mathscr{A}_g \to H_2(Q)/\text{Out}(Q)$ is invariant under the action of $\mathcal{M}_g$, since acting by a mapping class just re-launches the surface and so does not change the image of the map $\Sigma_g \to BQ$. Thus we get a well-defined map $\mathscr{A}_g/\mathcal{M}_g \to H_2(Q)/\text{Out}(Q)$. Later, we will show that this map is a bijection for large $g$. For this reason, we are interested in the number of epimorphisms in each homology class:

**6.13 Lemma** Let $Q$ be a finite group. Fix $[c] \in H_2(Q,\mathbb{Z})/\text{Out}(Q)$. Let $k$ be the number of elements of $H_2(Q,\mathbb{Z})$ which are in the equivalence class $[c]$. Then the ratio

$$\frac{|\{f \in \mathscr{A}_g \mid f(c) = [c]\}|}{|\mathscr{A}_g|}$$

converges to $k/|H_2(Q,\mathbb{Z})|$ as $g \to \infty$.

**Proof** First, we explain how to compute $c_f$ directly using Hopf’s description of $H_2$ of a group (see e.g. [Bro, §I.5]). Let $G$ be a finitely generated group, and express it as a quotient of a free group $G = F/R$. Then $H_2(G)$ can be naturally identified with $(F' \cap R)/[F,R]$. From this point of view, $H_2(G)$ is a subgroup of leftmost term of the exact sequence

$$1 \to R/[F,R] \to F/[F,R] \to G \to 1;$$

(6.14)

note that the leftmost term is central in the middle term. When $G$ is finite, $R/[F,R]$ is the direct sum of $(F' \cap R)/[F,R]$ and a free abelian group $A$ (see e.g. [Wie]). Taking the quotient by $A$ we get a exact sequence of finite groups

$$0 \to H_2(G) \to S \to G \to 1,$$

where $S$ is called a Schur cover of $G$. Here, $H_2(G)$ is central in $S$ and contained in $S'$. The Schur cover need not be unique unless $G$ is perfect; in that case $S \cong F'/[F,R]$ and $S$ is the universal central extension of $G$.

Now, let $f : \pi_1(\Sigma_g) \to Q$ be a homomorphism. Consider standard generators $a_i, b_i$ of $\pi_1(\Sigma_g)$; in the Hopf picture, $H_2(\Sigma)$ is generated by the standard relator $[a_i, b_i]$. Fix a Schur cover $S$ of $Q$. Consider an induced map of the sequences (6.14) for $\pi_1(\Sigma_g)$ and $Q$. Thus $c_f$ is given by the following procedure: Pick lifts $s_i, t_i \in S$ of the images $f(a_i), f(b_i)$, and then $c_f = \prod[s_i, t_i]$.

Now we determine the proportion of elements of $\mathscr{A}_g$ which have a fixed homology class. It is easier here to work directly with epimorphisms $f : \pi_1(\Sigma_g) \to Q$ before quotienting out by $\text{Aut}(Q)$. As in Lemma[6.10] consider a random $2g$-tuple $(s_i, t_i)$ of elements of $S$ and let $c = \prod[s_i, t_i]$. The images of $(s_i, t_i)$ in $Q$ are also uniformly distributed, and therefore the image of $c$ in $Q$ is nearly uniformly distributed in $Q'$ if $g$ is large. Recall that $H_2(Q)$ lies in $S'$. As we know $c$ is nearly uniformly distributed in $S'$, if we restrict to those $(s_i, t_i)$ which induce a homomorphism $\pi_1(\Sigma_g) \to Q$, the probability that $c$ is some particular element of $H_2(Q)$ is essentially $1/|H_2(Q)|$. Modding out by $\text{Aut} Q$ gives the probability claimed in the original statement.
6.15 Stabilization

Now, we want to discuss our main topological tool for understanding $\mathcal{A}_g$ when $g$ is large. Let $f: \pi_1(\Sigma_g) \to Q$ be in $\mathcal{A}_g$. A stabilization of $f$ is an element $f'$ of $\mathcal{A}_{g+h}$ obtained by viewing $\Sigma_{g+h}$ as $\Sigma_g \# \Sigma_h$ and setting $f'$ to be $f$ on $\Sigma_g$ and the trivial homomorphism on $\Sigma_h$, that is, $f'$ is the composition of $f$ with the map on $\pi_1$ induced by the quotient map $\Sigma_g \# \Sigma_h \to \Sigma_g$. Note here we are not fixing a particular identification of $\Sigma_{g+h}$ with $\Sigma_g \# \Sigma_h$, so each $f \in \mathcal{A}_g$ usually gives rise to many elements in $\mathcal{A}_{g+h}$. So that stabilization respects the associated classes in $H_2(Q)/\text{Out}(Q)$, we do require the identification of $\Sigma_{g+h}$ with $\Sigma_g \# \Sigma_h$ to be orientation preserving. Looking at it another way, an $f'$ in $\mathcal{A}_{g+h}$ is the result of an $h$-fold stabilization if there is an essential subsurface $S$ of $\Sigma_{g+h}$ which is a once-punctured $\Sigma_h$, and where $f'$ is the trivial homomorphism on $\pi_1(S)$.

First, we will show that for large enough genus every element of $\mathcal{A}_g$ is a stabilization:

**6.16 Proposition** Let $Q$ be a finite group. If $g > |Q|$ then every $f: \pi_1(\Sigma_g) \to Q$ is a stabilization.

Before giving the proof, let us point out an important consequence. If $f_1, f_2 \in \mathcal{A}_g$ are in the same orbit under $\mathcal{M}_g$, then so are their stabilizations in $\mathcal{A}_{g+1}$ under the action of $\mathcal{M}_{g+1}$. Thus we get a map of orbit sets $\mathcal{A}_g/\mathcal{M}_g \to \mathcal{A}_{g+1}/\mathcal{M}_{g+1}$. The proposition says that for $g > |Q|$ these maps are surjective; thus once some threshold is crossed no new orbits appear, though existing orbits can merge. But since there are only finitely many orbits, this merging process must eventually stop. Thus we have

**6.17 Corollary** Let $Q$ be a finite group. Then for large enough $g$, the number of $\mathcal{M}_g$ orbits in $\mathcal{A}_g$ is constant.

It is worth noting that this argument gives no indication of when $|\mathcal{A}_g/\mathcal{M}_g|$ stabilizes, even though Proposition 6.16 gives an explicit bound. Later, we will show the stable number of orbits is just $|H_2(Q,\mathbb{Z})/\text{Out}(Q)|$. We now prove the proposition.

**Proof** View $\Sigma_g$ as $g$ punctured-torus spokes attached to a central hub which is a $g$-times punctured sphere. Consider standard $2g$ generators $\{a_i, b_i\}$ of $\pi_1(\Sigma_g)$, where each $(a_i, b_i)$ pair are generators of the $i^{th}$ spoke. If we orient everything symmetrically, any element $w_i = a_1 \cdot a_2 \cdots a_i$ of $\pi_1(\Sigma_g)$, with $1 \leq i \leq g$, can be represented by an embedded non-separating simple closed curve in $\Sigma_g$.

Now fix $f \in \mathcal{A}_g$. We need to find an essential punctured torus on which $f$ is the trivial homomorphism. By the pigeon hole principle, there exists $i < j$ such that $f(w_i) = f(w_j)$. Therefore, $f(a_{i+1} \cdots a_j) = 1$, and so we can find one non-separating simple closed curve in the kernel of $f$. To turn this into an entire handle where $f$ is trivial, consider a maximal disjoint collection $c_1, \ldots, c_k$ of such non-separating curves in the kernel of $f$. By changing the basis of $\pi_1(\Sigma)$, we can make these $c_i$ be the curves $b_1, \ldots, b_k$ in our preferred basis. Then as before, there exists some $w = a_{i+1} \cdots a_j$ in the kernel of $f$. By maximality, the curve $w$ must intersect one of our $c_i$, say $c_1$. As $w$ and $c_1$ intersect in a single point, they have a regular neighborhood which is a punctured torus whose fundamental group is mapped trivially under $f$. Thus $f$ is a stabilization.
To complete our understanding of stabilization, we characterize the stable equivalence classes of epimorphisms. Two epimorphisms $\alpha$: $\pi_1(\Sigma) \to Q$ and $\beta$: $\pi_1(\Sigma_b) \to Q$ are called stably equivalent if they have a common stabilization. In particular, $c_\alpha = c_\beta$ in $H_2(Q)$. The following theorem of Livingston [Liv] shows that this homological condition is sufficient as well as necessary.

**6.18 Theorem** ([Liv]) Let $Q$ be a finite group, and consider two epimorphisms $\alpha$: $\pi_1(\Sigma) \to Q$ and $\beta$: $\pi_1(\Sigma_b) \to Q$. Then $\alpha$ and $\beta$ are stably equivalent if and only if $c_\alpha = c_\beta$ in $H_2(Q, \mathbb{Z})$.

Since this result is crucial to our characterization of $\mathcal{A}_g/\mathcal{M}_g$ and is quick to prove, we include a complete proof, following [Liv]. Unfortunately, the proof gives no control over the amount of stabilization required. One example where stabilization is needed is $\text{PSL}_2\mathbb{F}_{13}$ where there are multiple orbits of $\mathcal{A}_2$ which correspond to 0 in $H_2$ (see Example 6.4). For some meta-cyclic groups, Edmonds showed that no stabilization is required [Edm]. Zimmermann [Zim] gave a quite different, purely algebraic, proof of this theorem which might be useful for an analysis of the degree of stabilization required for particular $Q$. We now give the proof of the theorem.

**Proof** Suppose $\alpha$ and $\beta$ are as above, with $c_\alpha = c_\beta$ in $H_2(Q, \mathbb{Z})$. Now bordism is the same as homology in dimension 2, so in particular, the two induced maps $\Sigma_g \to BQ$ and $\Sigma_b \to BQ$ can be extended over some cobordism between $\Sigma_g$ and $\Sigma_b$ [Con] [CF]. Thus there exists a 3-manifold $M$ with two boundary components $\partial_1 M = \Sigma_g$ and $\partial_2 M = \Sigma_b$, and an epimorphism $f$: $\pi_1(M) \to Q$ which restricts to $\pi_1(\partial_1 M)$ as $\alpha$ and $\beta$ respectively. Take a relative handle decomposition of $M$ with all the 2-handles added after the 1-handles; that is $M$ is $(\partial_1 M) \times I$, plus some 1-handles, plus some 2-handles, plus $(\partial_2 M) \times I$. Let $M_1$ be $(\partial_1 M) \times I$ and the 1-handles, and let $M_2$ be the 2-handles and $(\partial_2 M) \times I$. Thus $M$ is the union of $M_1$ and $M_2$ along a surface $\Sigma$. To conclude, we will show that the restriction of $f$ to $\pi_1(\Sigma)$ is a stabilization of both $\alpha$ and $\beta$.

First consider $\alpha$ and $M_1$. Because $\alpha$ is surjective, we can slide the attaching maps of the 1-handles of $M_1$ around so that all their cores map trivially under $f$. This shows $(\Sigma, f)$ is a stabilization of $\alpha$. If we flip the handles over, the same reasoning applies to $M_2$ and $\beta$. Thus $\alpha$ and $\beta$ have a common stabilization. 

**6.19 Characterization of $\mathcal{A}_g/\mathcal{M}_g$ for large $g$**

Recall from Section 6.12 that we have a natural map $\mathcal{A}_g/\mathcal{M}_g \to H_2(Q)/\text{Out}(Q)$. We will now show:

**6.20 Theorem** Let $Q$ be a finite group. For all large $g$, the map

$$\mathcal{A}_g/\mathcal{M}_g \to H_2(Q, \mathbb{Z})/\text{Out}(Q)$$

is a bijection.

**Proof** First, by Corollary 6.17, the size of $\mathcal{A}_g/\mathcal{M}_g$ is constant for large $g$, and the stabilization maps $\mathcal{A}_g/\mathcal{M}_g \to \mathcal{A}_{g+1}/\mathcal{M}_{g+1}$ are bijections. Moreover, these stabilization maps are compatible with the maps to $H_2(Q)$. Theorem 6.18 implies...
that any two elements of \( \mathcal{A}_g \) which represent the same class in \( H_2(Q)/\text{Out}(Q) \) become the same in some \( \mathcal{A}_{g+h}/\mathcal{M}_{g+h} \); thus \( \mathcal{A}_g/\mathcal{M}_g \to H_2(Q)/\text{Out}(Q) \) is injective for all large \( g \). Lemma 6.13 shows that every class is realized for large enough \( g \), so the map \( \mathcal{A}_g/\mathcal{M}_g \to H_2(Q)/\text{Out}(Q) \) is surjective as well (alternatively, this follows because bordism is the same as homology in dimension 2).

Thus \( \mathcal{A}_g/\mathcal{M}_g \to H_2(Q)/\text{Out}(Q) \) is a bijection for all large \( g \), as claimed. \( \square \)

Unfortunately, this theorem and its constituent parts do not seem to be enough to show the existence of a limiting distribution of the number of quotients as the genus \( g \) goes to infinity. However, it is easy to show that the number of expected quotients does converge:

6.21 Theorem  Let \( Q \) be a finite group. Let \( E(Q,g) \) be the expected number of covers with covering group \( Q \) of a 3-manifold associated to a random Heegaard splitting of genus \( g \). Then as the genus \( g \) goes to infinity

\[
E(Q,g) \to \frac{|Q'||H_2(Q,\mathbb{Z})|}{|\text{Aut}(Q)|}.
\]

Proof  Let \( g \) be large enough so that \( \mathcal{A}_g/\mathcal{M}_g \to H_2(Q)/\text{Out}(Q) \) is a bijection. Let \( A^0_g \) be those elements \( f \) of \( \mathcal{A}_g \) which have \( c_f = 0 \) in \( H_2(Q)/\text{Out}(Q) \); by choice of \( g \), \( A^0_g \) is a single orbit of \( \mathcal{M}_g \). By Lemmas 6.10 and 6.13 we have

\[
|A^0_g| \sim \frac{|Q|^2 g}{|Q'||\text{Aut}(Q)||H_2(Q,\mathbb{Z})|}.
\]

As usual, let \( \mathcal{E}_g \) be those epimorphisms which extend over the handlebody. By Lemma 6.11 we have \( |\mathcal{E}_g| \sim |Q|^g/\text{Aut}(Q) \). Since \( \mathcal{E}_g \subset A^0_g \) and the action of \( \mathcal{M}_g \) on \( A^0_g \) is transitive, as in Example 6.2 we have that

\[
E(Q,g) = \frac{|\mathcal{E}_g|^2}{|\mathcal{A}_g|} \sim \frac{|Q'||H_2(Q)|}{|\text{Aut}(Q)|},
\]

as desired. \( \square \)

6.22 Characterization of \( \mathcal{A}'_g/\mathcal{M}_g \) for large \( g \)

In Section 7 we will need a slight variant of Theorem 6.20 and its precursors in Section 6.15. For a finite group \( Q \) and genus \( g \), define \( \mathcal{A}'_g \) to be the set of epimorphisms \( \pi_1(\Sigma) \) onto \( Q \); unlike \( \mathcal{A}_g \), we do not mod out by the action of \( \text{Aut}(Q) \). As with \( \mathcal{A}_g \), elements in \( \mathcal{A}'_g \) have associated homology classes, but now these are well-defined in \( H_2(Q,\mathbb{Z}) \). In this context, we have the following analog of Theorem 6.20:

6.23 Theorem  Let \( Q \) be a finite group. For all large \( g \), the map \( \mathcal{A}'_g/\mathcal{M}_g \to H_2(Q,\mathbb{Z}) \) is a bijection.
The proof of this is the same as for Theorem 6.20 after modifying Corollary 6.17 and Lemma 6.13 by replacing \( A_g \) by \( A'_g \) and \( H_2(Q)/\text{Out}(Q) \) by \( H_2(Q) \). In case you are wondering why we do not work with \( A' \) throughout, the reason is that the action of \( M_g \) on \( A'_g \) can not be highly transitive in general, e.g., the full alternating group of each component; this is because \( |E'_g \cap (\phi \cdot E'_g)| \) must always be an integer multiple of \( \text{Aut}(Q) \).

**6.24 Orbit of \( E_g \) under \( M_g \)**

In the case of \( \text{PSL}_2\mathbb{F}_{13} \) (Example 6.4) we saw that the action of \( M_2 \) on \( E_2 \) was not transitive, and there were two distinct orbits. However, one consequence of Theorem 6.20 is that for any \( Q \) the action of \( M_g \) on \( E_g \) is always transitive when \( g \) is large enough. In fact, something more is true: the action of the stabilizer of \( E_g \) is transitive for large \( g \). Moreover, when \( Q \) is a non-abelian simple group, the image of the stabilizer of \( E_g \) in \( \text{Sym}(E_g) \) contains \( \text{Alt}(E_g) \). Both of these facts follow immediately from the well-developed theory of the action of \( \text{Out}(F_g) \) on epimorphisms \( F_g \to Q \). We outline the needed results below, following the survey \[Pak\], to which we refer the reader for further results along these lines. The material in this subsection is not strictly necessary for the rest of this paper, but has two merits. The first is that it gives explicit bounds, in terms of \( Q \), of when these phenomena occur. The second is that the strong result in the case of simple groups allows us to avoid the use of the Classification of Simple Groups for some of the results in Section 7.

Before continuing, recall that \( \text{Out}(F_g) \) is generated by three kinds of elements: replacing a generator by its product with another generator (on either the left or the right), permuting two generators, and inverting a generator. If we think of elements of \( E_g \) as \( g \)-tuples of elements of \( Q \) (up to \( \text{Aut}(Q) \)), then these generators carry over to natural operations on such tuples. Now let us state the main fact of this subsection:

**6.25 Proposition** Let \( Q \) be a finite group. Then for all large \( g \), the action of \( \text{Out}(F_g) \) on \( E_g \) is transitive. Hence the action of \( M_g \) on \( E_g \subset A_g \) is also transitive for large \( g \).

How big does \( g \) need to be for the conclusion to hold? For simple groups, a conjecture of Wiegold posits that \( g \geq 3 \) suffices, and this is known for some classes of such groups \[Pak\] Thm. 2.5.6. In general, the bound in the proof we will give is proportional to \( \log |Q| \). Now, let us prove the proposition, following Prop. 2.2.2 of \[Pak\], which in turn follows \[DSC\].
Proof. Let \( d \) be the minimal number of generators of \( Q \). Let \( \bar{d} \) be the maximal size of a minimal generating set for \( Q \); that is, a generating set for which no proper subset generates. We will show that the action of \( \text{Out}(F_g) \) is transitive provided that \( g \geq d + \bar{d} \), which we now assume. Let \( q_1, \ldots, q_d \) be a minimal generating set for \( Q \). Consider \( f_0 = (q_1, \ldots, q_d, 1, \ldots, 1) \) in \( \mathcal{E}_g \). Let \( (r_1, \ldots, r_g) \) be any element in \( \mathcal{E}_g \), which we will move to \( f_0 \) by elements in \( \text{Out}(F_g) \). Since there is some subset of the \( r_i \) of size \( \bar{d} \) which generate \( Q \), permute the \( r_i \) so that \( r_{d+1}, \ldots, r_g \) generates \( Q \). Now by using elements of \( \text{Out}(F_g) \) which multiply \( (r_1, \ldots, r_d) \) by elements of \( (r_{d+1}, \ldots, r_g) \), we can turn \( (r_1, \ldots, r_g) \) into \( (q_1, \ldots, q_d, r_{d+1}, \ldots, r_g) \). Repeating this using the first \( d \) places instead lets us get to \( f_0 \) by further elements of \( \text{Out}(Q) \).

Thus the action of \( \text{Out}(F_g) \) on \( \mathcal{E}_g \) is transitive.

Before moving on, let us note that \( \bar{d} \) is no more than the length of a properly increasing sequence of subgroups
\[
\{1\} = Q_0 < Q_1 < \cdots < Q_k = Q.
\]
Since the index of one \( Q_i \) in the next is at least 2, we get that \( \bar{d} \leq \log_2 |Q| \). Thus in Proposition 6.25 we can take \( g \geq 2 \log_2 |Q| \) as \( d + \bar{d} \leq 2 \bar{d} \). Again, see [Pak] for sharper results about particular groups.

Since we are particularly interested in non-abelian simple groups, the following theorem is of interest.

**6.26 Theorem.** Let \( Q \) be a non-abelian finite simple group and \( g \geq 3 \). Then there is some orbit \( \mathcal{E}' \) of \( \mathcal{E}_g \) where \( \text{Out}(F_g) \) acts on \( \mathcal{E}' \) as either the full symmetric or alternating group.

In particular, when \( \text{Out}(F_g) \) acts transitively, the image of the homomorphism \( \text{Out}(F_g) \to \text{Sym}(\mathcal{E}_g) \) is either the whole thing or has index two. Theorem 6.26 was proved by Gilman [Gil] for \( g \geq 4 \) and improved to \( g = 3 \) by Evans [Eva]. The statements they give seem to put restrictions on the simple group, but their conditions actually hold for all finite simple groups; see the discussion in [Pak, Thm. 2.4.3]. The proof of Theorem 6.26 is fairly short and elementary but we will not reproduce it here; see [Gil] for more. We should mention, though, that as we stated it the result is more complex because it uses that every finite simple group is generated by two elements, one of which has order two; as such it uses part of the Classification of Finite Simple Groups. However, for the way we will use Theorem 6.26 we only need that the conclusion holds for all large \( g \), and this is exactly what Gilman’s elementary argument shows.

### 7 Covers where the quotient is simple

This section is devoted to the proof of the following theorem, which gives a complete asymptotic picture in the case of a non-abelian simple group

**7.1 Theorem.** Let \( Q \) be a finite non-abelian simple group. Then as the genus \( g \) goes to infinity,
\[
p(Q, g) \to 1 - e^{-\mu} \quad \text{where} \quad \mu = |H_2(Q, \mathbb{Z})|/|\text{Out}(Q)|.
\]
Moreover, the limiting distribution on the number of \( Q \)-covers converges to the Poisson distribution with mean \( \mu \).
As an example, if \( Q = \text{PSL}_2\mathbb{F}_p \), where \( p \) is an odd prime then \( \mu = 1 \). Thus for large genus the probability of a \( \text{PSL}_2\mathbb{F}_p \) cover is about \( 1 - e^{-1} \approx 0.6321 \). Unfortunately, the proof does not give information about the rate of convergence, and we will discuss this issue in detail later.

It is interesting to compare this result with the corresponding result for a general random group coming from a balanced presentation. Theorem 5.10 shows that as the number of generators goes to infinity, the limiting distribution is also Poisson. However, the mean is much smaller, namely \( 1/|\text{Aut}(Q)| \leq 1/|Q| \). In particular, the mean goes to 0 as the size of the groups increases. In contrast, we just saw that for 3-manifold groups the limiting mean of the number of \( \text{PSL}_2\mathbb{F}_p \) covers is independent of the size of the group.

To prove the theorem, the thing that we need beyond Section 6 is a better understanding of the action of the mapping class group \( \mathcal{M}_g \) on the set of epimorphisms \( \mathcal{A}_g \) (notion is as in Section 6). In particular, we will show that for a fixed non-abelian simple group \( Q \) the action is eventually by the full alternating group \( \text{Alt}(Q) \). Theorem 7.4 shows that as the number of generators goes to infinity, the limiting distribution is also Poisson. However, the mean is much smaller, namely \( 1/|\text{Alt}(Q)| \leq 1/|Q| \). In particular, the mean goes to 0 as the size of the groups increases. In contrast, we just saw that for 3-manifold groups the limiting mean of the number of \( \text{PSL}_2\mathbb{F}_p \) covers is independent of the size of the group.

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**7.2 Lemma** Let \( Q \) be a non-abelian finite simple group, with \( c \in H_2(Q) / \text{Out}(Q) \). Let \( k \) be a positive integer. Then for all large \( g \), the action of \( \mathcal{M}_g \) on \( \mathcal{A}_g^c \) is \( k \)-transitive.

Proof Consider the direct product \( Q^k \) of \( k \) copies of \( Q \). We will use \( \mathcal{A}_g(Q) \) and \( \mathcal{A}_g(Q^k) \) to denote epimorphisms to \( Q \) and \( Q^k \) respectively. As in Section 6.19 we will also consider the set \( \mathcal{A}_g'(Q^k) \) of epimorphisms to \( Q^k \) where we do not mod out by \( \text{Out}(Q^k) \). By Theorem 6.23 for all large \( g \), the map \( \mathcal{A}_g'(Q^k) / \mathcal{M}_g \to H_2(Q^k) \) is a bijection; henceforth we will assume this holds.

Now consider \( k \)-tuples \( ([f_1], \ldots, [f_k]) \) and \( ([h_1], \ldots, [h_k]) \) of distinct elements of \( \mathcal{A}_g'(Q) \). Let \( \tilde{c} \) be a lift of \( c \) to \( H_2(Q, \mathbb{Z}) \), and choose our representatives

\[
f_i, h_i : \pi_1(\Sigma_g) \to Q
\]

so that all the \( c_{f_i} \) and \( c_{h_i} \) are equal to \( \tilde{c} \) in \( H_2(Q) \). Now consider the induced product maps \( f, h : \pi_1(\Sigma_g) \to Q^k \). Since \( Q \) is simple and the \( f_i \) represent distinct classes in \( \mathcal{A}_g' \), the homomorphism \( f \) is also surjective (Lemma 3.7). Similarly \( h \) is also an epimorphism, and so both \( f \) and \( h \) are elements of \( \mathcal{A}_g'(Q) \).

As \( Q \) is non-abelian, \( H_1(Q, \mathbb{Z}) = 0 \), and so \( H_2(Q^k) \) is the direct sum \( (H_2(Q))^k \). Thus \( c_f = c_h = \tilde{c}^k \) in \( H_2(Q^k) \). By our choice of genus, there must be a \( \phi \in \mathcal{M}_g \) such that \( h = \phi \cdot f \). Then \( \phi \) carries \( ([f_1], \ldots, [f_k]) \) to \( ([h_1], \ldots, [h_k]) \) and so the action of \( \mathcal{M}_g \) is \( k \)-transitive. \( \square \)

We will also note the following variant of the preceding lemma, whose proof is essentially identical.
7.3 Lemma Consider a finite set $Q_i$ of non-abelian finite simple groups, and fix $c_i \in H_2(Q_i)/\Out(Q_i)$. Suppose there are no isomorphisms from $Q_i$ to $Q_j$ which take $c_i$ to $c_j$. Then the action of $\mathcal{M}_g$ on $\mathfrak{A}_g^{c_1}(Q_1) \times \cdots \times \mathfrak{A}_g^{c_n}(Q_n)$ is transitive.

Now we use Lemma 7.2 to prove:

7.4 Theorem Let $Q$ be a non-abelian finite simple group, with a fixed class $c$ in $H_2(Q)/\Out(Q)$. Then for all large $g$, the action of $\mathcal{M}_g$ on $\mathfrak{A}_g^c$ is by the full alternating group $\Alt(\mathfrak{A}_g^c)$. Moreover, the actions on different $\mathfrak{A}_g^c$ are independent, in the sense that the map $\mathcal{M}_g \to \prod \Alt(\mathfrak{A}_g^c)$ is surjective.

You can view this theorem as saying that the action of $\mathcal{M}_g$ on $\mathfrak{A}_g$ is nearly as mixing as possible, at least when the genus is large. As such, it is analogous to Goldman’s theorem that the action of $\mathcal{M}_g$ on the SU(2)-character variety is ergodic for any genus $\geq 2$ [Col]. While the proof here does not give an explicit bound on when this mixing behavior occurs, we suspect that it typically occurs as soon as $g \geq 3$.

Proof One consequence of the Classification of Simple Groups is that a group which acts 6-transitively on a finite set $\Omega$ must contain $\Alt(\Omega)$ (see e.g. [DM] Thm. 7.3A). By Theorem 7.2, the action of $\mathcal{M}_g$ on $\mathfrak{A}_g^c$ is 6-transitive for all large $g$; hence the action contains $\Alt(\mathfrak{A}_g^c)$. Since $\mathcal{M}_g$ is a perfect group for genus at least 2, the image of $\mathcal{M}_g \to \Sym(\mathfrak{A}_g^c)$ must be $\Alt(\mathfrak{A}_g^c)$ and not $\Sym(\mathfrak{A}_g^c)$, proving the first part of the theorem.

For the independence, if $\mathcal{M}_g \to \prod \Alt(\mathfrak{A}_g^c)$ is not surjective then there are distinct $c$ and $d$ such that $\mathcal{M}_g \to \Alt(\mathfrak{A}_g^c) \times \Alt(\mathfrak{A}_g^d)$ is not surjective. Then there is a bijection $\alpha : \mathfrak{A}_g^c \to \mathfrak{A}_g^d$ which is compatible with the $\mathcal{M}_g$ action on both of these sets; in particular, the action of $\mathcal{M}_g$ on pairs $(f, h) \in \mathfrak{A}_g^c \times \mathfrak{A}_g^d$ is not transitive. But that contradicts Lemma 7.3, finishing the proof. □

You might wonder if we really need the Classification of Simple Groups to prove this theorem. In the case of the orbit $\mathfrak{A}_g^0$, which is what we need for Theorem 7.1 we can replace the Classification by some much less difficult results. First choose $g$ large enough so that the action on $\mathfrak{A}_g^0$ is 2-transitive. Let $G$ be the image of $\mathcal{M}_g \to \Sym(\mathfrak{A}_g^0)$. Gilman showed (see Theorem 6.26) that the action of the stabilizer of $\mathfrak{e}_g$ in $G$ is all of $\Alt(\mathfrak{e}_g)$ for large $g$. Thus $G$ has subgroups $K \lhd G \leq G$ so that $H/K$ is isomorphic to $\Alt(\mathfrak{e}_g)$; in this context, $G$ is said to have a section which is $\Alt(\mathfrak{e}_g)$. Then Theorem 5.5B of [DM] says that since $G$ is 2-transitive, either $G$ contains $\Alt(\mathfrak{A}_g^0)$ or $|\mathfrak{e}_g| < 6 \log |\mathfrak{A}_g^0|$. The latter cannot happen for large $g$ by the results of Section 6.8 and so $M_g$ acts on $\mathfrak{A}_g^0$ by the full alternating group. Both Gilman’s result and the theorem about permutation groups are fairly elementary, and, while certainly not trivial, they are orders of magnitude easier than the Classification.

The main result of this section now follows easily:

Proof of Theorem 7.1 We need to show that the distribution of the number of $Q$-covers converges to the Poisson distribution with mean $|H_2(Q)|/|\Out(Q)|$. By the
proceeding theorem, choose \( g \) large enough so that the action of \( \mathcal{M}_g \) on \( \mathcal{A}_g^0 \) is by \( \text{Alt}(\mathcal{A}_g^0) \). Moreover, by Lemmas 6.10 and 6.11 we have that
\[
|\mathcal{A}_g^0| \sim \frac{|Q|^{2g-2}}{|\text{Out}(Q)||H_2(Q)|} \quad \text{and} \quad |\mathcal{E}_g| \sim \frac{|Q|^{g-1}}{|\text{Out}(Q)|}.
\]
Since the action of \( \text{Alt}(\mathcal{A}_g^0) \) on \( \mathcal{A}_g^0 \) is \((|\mathcal{A}_g^0| - 2)\)-transitive, the action is in particular \( |\mathcal{E}_g| \)-transitive. Thus the number of \( Q \)-covers is distributed exactly as the number of intersections of \( \mathcal{E}_g \) with a randomly chosen subset of \( \mathcal{A}_g^0 \) of size \( |\mathcal{E}_g| \).

By the above, the expected number of covers is \(|\mathcal{E}_g|^2/|\mathcal{A}_g^0| \sim |H_2(Q)|/|\text{Out}(Q)|\). Thus the distribution of \( Q \)-covers converges to a Poisson distribution with mean \(|H_2(Q)|/|\text{Out}(Q)|\), as desired.

There is another proof of Theorem 7.1 that is worth mentioning, which bypasses Theorem 7.4 and instead relies directly on Lemma 7.2 and some elementary probability theory. Let \( X_g \) denote the random variable whose distribution is the number of \( Q \)-covers for genus \( g \). For comparison purposes, set \( Y_g \) to be the random variable which is the number of intersections of a randomly chosen subset of \( \mathcal{A}_g^0 \) of size \( |\mathcal{E}_g| \) with \( \mathcal{E}_g \). Let \( \mu = |H_2(Q)|/|\text{Out}(Q)| \). We know that the \( Y_g \) limit to the Poisson distribution with mean \( \mu \); we want to show the same for \( X_g \).

Fix a positive integer \( k \). Lemma 7.2 says that the action of \( \mathcal{M}_g \) on \( \mathcal{A}_g^0 \) is \( k \)-transitive for all large \( g \). Think of \( X_g \) as a sum of variables \( I_f \) for \( f \in \mathcal{E}_g \), which are indicator functions for whether a particular \( f \) extends. The \( k \)-transitivity of the action implies that if we look at \( k \) variables \( I_{f_1}, \ldots, I_{f_k} \), then these are distributed exactly as if we picked a \( \phi \cdot \mathcal{E}_g \) as a random subset of \( \mathcal{A}_g^0 \) of size \( |E_g| \). In other words, \( I_{f_1}, \ldots, I_{f_k} \) have the same distribution if we decomposed \( Y_g \) instead of \( X_g \).

Thus the \( k^{th} \)-moment of \( X_g \) is equal to that of \( Y_g \). Therefore the \( k^{th} \)-moments of the \( X_g \) converge to the \( k^{th} \)-moments of the Poisson distribution with mean \( \mu \). This is enough to show that the \( X_g \) converge to that Poisson distribution (see e.g. [Boi] Thm. 19); see also [AS] Ch. 8 for related statements.

### 7.5 Sequences of simple covers

In trying to understand the set of all finite covers of a 3-manifold \( M \), a natural question is whether \( \pi_1(M) \) has a quotient which is in some particular class of groups, such as alternating groups, or groups of the form \( \text{PSL}_2\mathbb{F}_p \). If \( M \) is hyperbolic, then the congruence quotients give an infinite number of \( \text{PSL}_2\mathbb{F}_p \)-covers. Moreover, Lubotzky has shown that congruence subgroups have density zero among all finite-index subgroups of \( \pi_1(M) \) [Lub2]. However, even the answer to the following question is unknown:

**7.6 Question** Does every hyperbolic 3-manifold have a cover with group \( A_n \) for some \( n \geq 5 \)?

While this question for hyperbolic surfaces is trivially yes (since those groups surject onto a free group of rank 2), the 2-dimensional case becomes interesting if we generalize to orbifolds. In this case, Everitt [Eve] has shown that the fundamental group of any hyperbolic 2-orbifold surjects onto \( A_n \) for all large \( n \); the hard
cases here are the $S^2(p,q,r)$ orbifolds, and there Everitt manages to build explicit covers which realize these groups. Recently, Liebeck and Shalev have given another proof which studies the question from a very different probabilistic point of view [LS1].

Theorem 7.1 suggests that we might expect the answer to Question 7.6 to be yes because $|H_2(A_n)|/|\text{Out}(A_n)| = 1$ for all large $n$. Thus if having covers with different groups are independent random events, we naively expect a random 3-manifold to have at least one, indeed infinitely many, $A_n$ covers. In this direction, we can show:

7.7 Theorem Let $\varepsilon > 0$. Then for all sufficiently large $g$, the probability that the 3-manifold obtained from a random genus-$g$ Heegaard splitting has an $A_n$-cover for some $n \geq 5$ is at least $1 - \varepsilon$. Moreover, the same is true if we require that there be at least some fixed number $k$ of such covers.

What about similar questions where we consider a collection $\{Q_i\}_{i=0}^\infty$ of non-abelian simple groups? For this we will show:

7.8 Theorem Let $\{Q_i\}_{i=0}^\infty$ be a sequence of distinct non-abelian finite simple groups; set $\mu_i = |H_2(Q_i)|/|\text{Out}(Q_i)|$. Suppose further that $\sum \mu_i = \infty$. Fix $\varepsilon > 0$. Then for all sufficiently large $g$, the probability that the 3-manifold obtained from a random genus-$g$ Heegaard splitting has an $Q_i$-cover for some $i$ is at least $1 - \varepsilon$. Moreover, the same is true if we require that there be at least some fixed number $k$ of such covers.

Since $|H_2(A_n)|/|\text{Out}(A_n)| = 1$ for all large $n$, this theorem immediately implies Theorem 7.7. Another class where it applies is $Q_i = \text{PSL}_2^\mathbb{F}_{p_i}$ where $p_i$ is prime; an example of where it does not is $Q_i = \text{PSL}_2^\mathbb{F}_{p,2}$ where $p$ is a fixed prime. In fact, it is not hard to work out exactly which sequences $\{Q_i\}$ satisfy the hypothesis from the Classification of Finite Simple Groups (see [CCNPW] §3). In particular, any sequence $\{Q_i\}$ must either contain an infinite number of $A_n$ or an infinite number of Chevalley groups. In the former case, Theorem 7.8 applies. For the Chevalley case, these are groups of Lie type, and let $q_i = p_i^{f_i}$ be the field of definition of $Q_i$. Then $\mu_i = c_i/f_i$ where $c_i$ is a non-zero rational number which is universally bounded above and below (see [CCNPW] §3, especially Table 5). Thus Theorem 7.8 applies in this case if and only if $\sum 1/f_i$ diverges. Thus Theorem 7.8 applies to $\text{PSL}_2^\mathbb{F}_{p,2}$ but not $\text{PSL}_2^\mathbb{F}_{p,2}$. Now we will give the proof of the theorem.

Proof of Theorem 7.8 The key lemma is the following, which says that covers with different $Q_i$ are essentially independent.

7.9 Lemma Let $Q_1, \ldots, Q_n$ be non-abelian finite simple groups. Then for all large $g$, the map $\mathcal{M}_g \to \prod \text{Alt}(\alpha_{g,1}^Q(Q_i))$ is surjective.

First, let us see why this lemma implies the theorem. For a fixed genus $g$, let $X_i$ denote the random variable corresponding to the number of $Q_i$ covers. Fix a positive integer $n$. Choose $g$ large enough so that the lemma holds for $X_1, \ldots, X_n$, and so that the expectations $E(X_i)$ are very nearly $\mu_i$ for $1 \leq i \leq n$. From the lemma, the $X_i$ are independent random variables which are nearly Poisson with mean $\mu_i$. Thus their sum $X_1 + \cdots + X_n$ is nearly Poisson with mean $M_n = \mu_1 + \cdots + \mu_n$. 

Proof of Lemma 7.9 Let $n \geq 1$ be a prime power, $G$ a finite simple group, and $n \in G$. Then $G$ is almost simple. 

Proof: Let $K$ be a Sylow $n$-subgroup of $G$, and let $H$ be a subgroup of $G$ containing $K$. Since $K$ maps onto $G/K$, there is a homomorphism $\chi: G \to \text{Aut}(K)$ such that $H = \ker(\chi)$. Let $L = \text{Im}(\chi)$. Then $L$ is a subgroup of $\text{Aut}(K)$, and since $n$ is a prime power, $L$ is a $n$-group. If $L$ is not a $n$-group, then $L$ contains a $n$-group $M$ such that $|M| = |L|$ and $M < L$. Then $\chi(M) < L$, so that $\chi$ is not surjective, a contradiction. Thus $L$ is a $n$-group. Since $L$ is a normal subgroup of $\text{Aut}(K)$, $L$ is a $n$-group. Thus $G$ is almost simple.
\[ \mu_n. \text{In particular, the probability of having a cover with group one of } Q_1, \ldots, Q_n \text{ is about } 1 - e^{-\mu_n}. \text{ Since } \sum \mu_i \text{ diverges, by increasing } n \text{ we can make this probability as close to 1 as we like (of course, } g \text{ may well have to increase as we change } n). \text{ Similarly, we can require some fixed number } k \text{ of } Q_i \text{ covers. This proves the theorem modulo the lemma.}

As for the lemma, by Theorem 7.4 gives a } g \text{ so that } \mathcal{M}_g \to \text{Alt}(\omega_g^0(Q_i)) \text{ is surjective for } i \leq n. \text{ If the product } \mathcal{M}_g \to \prod \text{Alt}(\omega_g^0(Q_i)) \text{ is not surjective, then because the factors are simple, there must be a pair } i, j \text{ such that } |\omega_g^0(Q_i)| = |\omega_g^0(Q_j)| \text{ and a bijection } \omega_g^0(Q_i) \to \omega_g^0(Q_j) \text{ which is compatible with the } \mathcal{M}_g \text{ action on both of these sets. In particular, the action of } \mathcal{M}_g \text{ on pairs } (f, h) \in \omega_g^0(Q_i) \times \omega_g^0(Q_j) \text{ is not transitive. But that would contradict Lemma 7.3 thus proving Lemma 7.9 and hence the theorem.}

8 Homology of random Heegaard splittings

In this section, we work out the distribution of the homology of a 3-manifold coming from a random Heegaard splitting of genus } g. \text{ First, let us set up the point of view we will take in this section. As always, let } H_g \text{ be a genus-} g \text{ handlebody, } \Sigma_g \text{ its boundary, and } \mathcal{M}_g \text{ the mapping class group of } \Sigma_g. \text{ Given } \phi \in \mathcal{M}_g \text{ let } M_\phi \text{ be the resulting 3-manifold. Let } J \text{ be the kernel of the map } H_1(\Sigma_g; \mathbb{Z}) \to H_1(H_g; \mathbb{Z}). \text{ Then } H_1(M_\phi; \mathbb{Z}) \text{ is the quotient of } H_1(\Sigma_g; \mathbb{Z}) \text{ by the subgroup } \langle J, \phi^{-1}(J) \rangle. \text{ Note that counting algebraic intersections between two cycles gives } H_1(\partial H_g; \mathbb{Z}) \text{ a natural symplectic form. With respect to this symplectic structure } J \text{ is a Lagrangian subspace. The natural map } \mathcal{M}_g \to \text{Sp}_{2g} \mathbb{Z} \text{ coming from the action of mapping classes on } H_1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g} \text{ is surjective. Thus understanding the distribution of the homology of } M_\phi \text{ boils down to considering the distribution of Lagrangians under the action of } \text{Sp}_{2g} \mathbb{Z}.

8.1 Symplectic groups over } \mathbb{F}_p

As with the case of random finitely presented groups, we will work one prime at time. We will work in the following framework, where } p \text{ is a fixed prime (or prime power). Let } J \text{ be a vector space over } \mathbb{F}_p, \text{ of dimension } g, \text{ and let } K \text{ be its dual. A nice way of thinking about the symplectic vector space over } \mathbb{F}_p, \text{ of dimension } 2g \text{ is to consider } V = J \oplus K \text{ with symplectic form given by } \langle (j_1, k_1), (j_2, k_2) \rangle = k_1(j_2) - k_2(j_1). \text{ Notice that both } J \text{ and } K \text{ are Lagrangian subspaces of } V. \text{ Now set } G = \text{Sp}V = \text{Sp}_{2g} \mathbb{F}_p. \text{ Identify } V \text{ with } H_1(\Sigma_g; \mathbb{F}_p) \text{ so that the kernel of the map to } H_1(H_g; \mathbb{F}_p) \text{ is the Lagrangian } J. \text{ The action of } \mathcal{M}_g \text{ on homology gives a surjection } \mathcal{M}_g \to G. \text{ If } \phi \in \mathcal{M}_g \text{ is the result of a long random walk then its image in } G \text{ is nearly uniformly distributed. Now by the analog of the Gram-Schmidt process for a symplectic vector space, it is easy to show that the action of } G \text{ on the set of Lagrangians is transitive. Thus } \phi^{-1}_* (J) \text{ is nearly uniformly distributed among all Lagrangians. Hence the asymptotic probability that } \dim(H_1(M_\phi; \mathbb{F}_p)) = k \text{ is simply the ratio of the number of Lagrangians } L \text{ in } V \text{ such that } \dim(J \cap L) = k \text{ to the total number of Lagrangians.}
8.2 Counting Lagrangians

First, let us count the total number of Lagrangians in \( V \); we will do this by computing the size of the stabilizer \( S \leq G \) of the fixed Lagrangian \( J \). Notice we have a natural homomorphism \( S \to \text{GL}(J) \). This map is surjective, for if \( A \in \text{GL}(J) \) then recalling that \( K = J^* \), we have that the map \( A \oplus (A^*)^{-1} \) of \( V \) respects the symplectic form and hence is in \( S \). An element \( s \) of the kernel of \( S \to \text{GL}(J) \) is determined by its restriction to \( K \). Write the restriction as \( B_1 \oplus B_2 \) where \( B_1 \) maps \( K \) into \( J \) and \( B_2 \) maps \( K \) into \( K \). For each \( j \in J \) and \( k \in K \) we have \( \langle k, j \rangle = \langle s(k), s(j) \rangle = \langle B_1(k) + B_2(k), j \rangle = \langle B_2(k), j \rangle \) and hence \( B_2 = \text{Id} \) on \( K \). Moreover, the requirement that \( s \) preserve the symplectic form is then equivalent to \( \langle B_1(k_1), k_2 \rangle = \langle B_1(k_2), k_1 \rangle \) for each \( k_1, k_2 \in K \). Such a \( B_1 \) is called symmetric. If we write \( B_1 \) with respect to a pair of dual basis for \( J \) and \( K \) then this is equivalent to the resulting matrix being symmetric. Thus

\[
|S| = |\{ g \times g \text{ symmetric matrices } \}| \cdot |\text{GL}(J)| = p^{g(g+1)/2} \prod_{k=1}^{g} (p^g - p^{k-1})
\]

\[
= p^{g^2} \prod_{k=1}^{g} (p^k - 1).
\]

To complete the count of Lagrangians, we also need to know the order of \( G \) itself. Fix a basis for \( V \) consisting of a basis \( \{ e_i \} \) for \( J \) and the dual basis \( \{ e^i \} \) for \( K \). To build an element of \( G \), there are \( p^{2g} - 1 \) choices for the image \( v \) of \( e_1 \), and as the image \( w \) of \( e^1 \) must satisfy \( \langle v, w \rangle = 1 \), there are \( p^{2g-1} \) choices for \( w \). The other basis vectors must be sent into \( \text{span}(v, w) \). Thus inductively one has

\[
|G| = \prod_{k=1}^{g} (p^{2k} - 1) p^{2k-1} = p^{g^2} \prod_{k=1}^{g} (p^{2k} - 1).
\]

Thus, the number of Lagrangians in \( V \) is

\[
\frac{|G|}{|S|} = \prod_{k=1}^{g} \frac{p^{2k} - 1}{p^g - 1} = \prod_{k=1}^{g} \left( p^k + 1 \right).
\]

8.3 Counting transverse Lagrangians

Now we will calculate the probability that \( H_1(M_g; \mathbb{F}_p) = 0 \). To do this, we need to count the number of Lagrangians \( L \) which are transverse to our base Lagrangian \( J \). Given such an \( L \), projection of \( L \) onto \( K \) is surjective; thus we can view it as the graph of a map \( B: K \to J \). The requirement that the graph \( L \) is Lagrangian is equivalent to

\[
\langle k_1 + B(k_1), k_2 + B(k_2) \rangle = 0 \iff \langle B(k_1), k_2 \rangle = \langle B(k_2), k_1 \rangle \quad \text{for all } k_1, k_2 \in K.
\]

That is, \( B \) is symmetric in the same sense as before. Thus the number of Lagrangians transverse to \( J \) is equal to the number of symmetric \( g \times g \) matrices over \( \mathbb{F}_p \), that is \( p^{g(g+1)/2} \). Combining this with our count of Lagrangians gives the following:
8.4 Theorem  Fix a Heegaard genus $g$. Then the asymptotic probability that

$$H_1(M_\phi;\mathbb{F}_p) = 0$$

is

$$\prod_{k=1}^{g} \frac{1}{1 + p^{-k}}$$

as we let the length of the random walk generating $\phi \in \mathcal{M}_g$ go to infinity.

Notice that for a fixed genus $g$ the larger $p$ is the closer this probability is to 1. Thus the asymptotic probability that $H_1(M_\phi;\mathbb{Q})$ vanishes is 1, as you would expect. On the other hand, if we consider a pair of primes $p$ and $q$ then Lemma 3.7 implies that the induced map

$$\mathcal{M}_g \to \text{Sp}_{2g}(\mathbb{F}_p) \times \text{Sp}_{2g}(\mathbb{F}_q)$$

is surjective. Thus homology with $\mathbb{F}_p$ versus $\mathbb{F}_q$ coefficients are asymptotically independent variables. Moreover, the sum of the probabilities that $H_1(M_\phi;\mathbb{F}_p) \neq 0$ diverges (albeit slowly, like the harmonic series). Thus the probability that some $H_1(M_\phi;\mathbb{F}_p) \neq 0$, where $p$ is larger than some fixed $C$, goes to 1 as the complexity of $\phi$ goes to infinity. Hence the expected size of $H_1(M_\phi;\mathbb{Z})$ goes to infinity as the complexity of $\phi$ increases. Summarizing:

8.5 Corollary  Fix a Heegaard genus $g$. Then with asymptotic probability 1 the rational homology $H_1(M_\phi;\mathbb{Q})$ vanishes. However, for a fixed $C$ the probability that $|H_1(M_\phi;\mathbb{Z})| > C$ goes to 1 as the complexity of $\phi \to \infty$.

Notice that the second conclusion implies that while a fixed manifold $N$ occurs with many different $\phi$, as the complexity of $\phi$ goes to infinity the probability that $N = M_\phi$ goes to zero. Thus asymptotic probabilities are, in particular, not highly disguised statements about some finite set of manifolds.

It is also interesting to compare Theorem 8.4 to the corresponding results for the group $\Gamma$ of random $g$-generator balanced presentations. In that case, by Proposition 3.16 the probability that $H_1(\Gamma;\mathbb{F}_p) = 0$ is

$$\prod_{k=1}^{g} \left(1 - p^{-k}\right).$$

Comparing term by term shows that this is less than the same probability for $\mathcal{M}_\phi$; thus the homology of the random group is more likely to be non-zero than for the random 3-manifold. Note also that in both cases the probabilities have the same limit (namely 0) as $p$ goes to infinity. Both of these facts are the reverse of what we saw for non-abelian simple groups.

8.6 General distribution

To complete the picture, we need to find the probability that $\dim(H_1(M_\phi;\mathbb{F}_p)) = d$ for general $d$. To do this, we start by parameterizing those Lagrangians $L$ where $\dim(L \cap J) = d$. Let us fix $A = L \cap J$. Any Lagrangian $L$ intersecting $J$ in $A$ is contained in $A^\perp = J \oplus \text{Ann}(A)$, where $\text{Ann}(A)$ is the subset of $K = J^\perp$ which annihilates $A$. Note that $A^\perp/A = (J/A) \oplus \text{Ann}(A)$ inherits a natural symplectic form, and $L$ projects to a Lagrangian in $A^\perp/A$ which is transverse to $J/A$. Indeed,
Lagrangians in \( A^\perp \) which are transverse to \( J/A \) exactly parameterize Lagrangians in \( V \) which intersect \( J \) in \( A \). Thus the latter are parameterized by \( (g-d) \times (g-d) \) symmetric matrices. Hence the number of Lagrangians intersecting \( J \) in a \( d \) dimensional subspace is:

\[
\frac{|\{(g-d) \times (g-d) \text{ symmetric matrices}\}| \cdot |\{\text{dim } d \text{ subspaces of } J\}|}{p^{(g-d+1)(g-d)/2} \prod_{k=1}^{d} \frac{p^{g-k+1} - 1}{p^k - 1}}.
\]

If we set \( c_d \) to be the probability that \( H_1(M_\phi; \mathbb{F}_p) \) has dimension \( d \), then a concise way of summarizing this distribution is

\[
c_0 = \prod_{k=1}^{g} \left(1 + p^{-k}\right)^{-1} \quad \text{and} \quad c_d = \prod_{k=1}^{d} \frac{1 - p^{-g+k-1}}{p^k - 1} \quad \text{for } 1 \leq d \leq g.
\]

8.7 Large genus limit

As with the case of simple groups, the distributions of \( \dim H_1(M_\phi; \mathbb{F}_p) \) have a well-defined limit distribution as the genus \( g \) goes to infinity. Namely, it is the probability distribution given by

\[
\tilde{c}_0 = \prod_{k=1}^{\infty} \left(1 + p^{-k}\right)^{-1} \quad \text{and} \quad \frac{\tilde{c}_d}{\tilde{c}_0} = \prod_{k=1}^{d} \left(p^k - 1\right)^{-1} \quad \text{for } 1 \leq d.
\]

The reader can check that this is really a probability distribution, i.e. that it has unit mass. One way to do this is to use that the finite approximates have unit mass, and then observe that \( c_d/c_0 \) is an increasing function of \( g \).

8.8 \( p \)-adic point of view

It is possible to work out the \( p \)-adic distribution of homology, analogous to Section 3.14. However, we will not go into this here. The key technical tool needed is the ability to count non-singular \( g \times g \) matrices over \( \mathbb{F}_p \), which is done in [Car] and [Mac]; see also [Sta, §4].

9 Homology of finite-sheeted covers

In this section, we try to determine the probability that a cover has \( \beta_1 > 0 \), where the covering group \( Q \) is fixed. As before we work in the context of random Heegaard splittings. Our original goal was to find groups where this probability is
positive, and so demonstrate that the Virtual Haken Conjecture is true in many cases, perhaps with probability 1. Unfortunately, our results are in the other direction; in particular the main result in this section is:

**9.1 Theorem** Let \( Q \) be a finite abelian group. The probability that the 3-manifold obtained from a random Heegaard splitting of genus 2 has a \( Q \)-cover with \( \beta_1 > 0 \) is 0.

The restriction to genus 2 is almost certainly artificial and simply due to a technical difficulty in the general case that we were unable to overcome.

What about when \( Q \) is non-abelian? We did some computer experiments, and could not find any groups \( Q \) for which \( Q \)-covers seemed to have \( \beta_1 > 0 \) with positive probability. However, these experiments are not completely convincing, even for the groups examined; as you will see in the proof of Theorem 9.1 one needs to consider certain subgroups of the mapping class group of truly staggering index, even for fairly small \( Q \). Thus one can not rule out experimentally some very small probability that a \( Q \)-cover has \( \beta_1 > 0 \). We still suspect that the probability of \( \beta_1 > 0 \) is 0 for all \( Q \), and would be surprised if there was a \( Q \) for which the probability of \( \beta_1 > 0 \) was greater than, say, \( 10^{-4} \).

Let us now contrast these results with our earlier experimental results in [DT2]. There, we examined about 11,000 small volume hyperbolic 3-manifolds. Of these, more than 8,000 had 2-generator fundamental groups, and therefore almost certainly have Heegaard genus 2. About 1.7% of these genus 2 manifolds had a \( \mathbb{Z}/2 \) cover with \( \beta_1 > 0 \), and at least 1.7% had a \( \mathbb{Z}/3 \) cover with \( \beta_1 > 0 \). Overall, 7.3% of the genus 2 manifolds have an abelian cover with \( \beta_1 > 0 \) (of the full sample of 11,000 manifolds, this rises to 9.6%). In comparing this with Theorem 9.1 however, it is important to keep in mind that the covering group \( Q \) is fixed in the theorem. In particular, we do not know the answer to the following question, even experimentally:

**9.2 Question** Let \( M \) be obtained from a random Heegaard splitting of genus \( g \). What is the probability that the maximal abelian cover of \( M \) has \( \beta_1 > 0 \)?

Compared to the experiments in §5 of [DT2] which dealt with non-abelian simple covers, the contrast becomes much more marked. In [DT2] §5, we found for a fixed such group \( Q \) that a very large proportion of the covers (\( > 50\% \)) had \( \beta_1 > 0 \); indeed, the expectation for \( \beta_1 \) grew linearly with \( |Q| \). However, in our experiments for random genus 2 Heegaard splittings we found that the probability seems to be 0 for the first few group (\( A_5 \), \( \text{PSL}_2 \mathbb{F}_7 \), \( A_6 \), and \( \text{PSL}_2 \mathbb{F}_8 \)), and we expect that this pattern continues for all simple groups.

**9.3 Homology and subgroups of the mapping class group**

The goal of this section is to prove Theorem 9.1. While Theorem 9.1 restricts to genus 2 and abelian covering groups, with the exception of Lemma 9.6, this section will be done without these restrictions. So from now on, we fix a Heegaard genus \( g \geq 2 \) and an arbitrary finite group \( Q \). Let \( H \) be a handlebody of that genus, and \( \Sigma = \partial H \). Let \( \mathcal{M} \) denote the mapping class group of \( \Sigma \), with a fixed generating set \( T \). We focus on a single epimorphism \( f: \pi_1(\Sigma) \to Q \) which extends over \( H \). Proposition 6.1 shows that for \( \phi \) the result of a random walk in \( \mathcal{M} \), the probability
that \( f \) extends to the 3-manifold \( N_\phi \) converges to a positive number as the length of the walk goes to infinity. We want to show that the probability that the corresponding cover of \( N_\phi \) has \( \beta_1 > 0 \) is 0; as there are only finitely many choices for \( f \) this will suffice to prove Theorem 9.4.

Recall from the proof of Proposition 6.1 that \( f \) extends over \( N_\phi \) if and only if \( \phi \cdot f = f \circ \phi^{-1} \) extends over \( H \). We will consider each of the possibilities for \( \phi \cdot f \) individually. We start with the case that \( \phi \cdot f = f \) in the set \( \mathcal{A} \) of all such epimorphisms, up to \( \text{Aut}(Q) \). This case is a little simpler, and it forms the core for the argument in general. Let \( K \) be the kernel of \( f \): \( \pi_1(\Sigma) \to Q \). We are focusing on the subgroup

\[
\mathcal{M}_f = \{ \phi \in \mathcal{M} \mid \phi \cdot f = f \text{ in } \mathcal{A}, \text{ or equivalently } \phi_* (K) = K \},
\]

which has finite index in \( \mathcal{M} \). The mapping classes in \( \mathcal{M}_f \) are exactly those which lift to the cover \( \tilde{\Sigma} \to \Sigma \) corresponding to \( K \), and let \( \tilde{\mathcal{M}}_f \) be the group of all such lifts, modulo isotopy within this restricted class of homeomorphisms of \( \tilde{\Sigma} \). Regarding \( Q \) as the group of covering translations of \( \tilde{\Sigma} \), we have the exact sequence

\[
1 \to Q \to \tilde{\mathcal{M}}_f \to \mathcal{M}_f \to 1.
\]

Here, we are using that \( g \geq 2 \), as in the torus case \( Q \) need not inject into \( \tilde{\mathcal{M}}_f \). Note that \( Q \) is not central unless \( \mathcal{M}_f \) acts trivially on \( \pi_1(\Sigma)/K \).

If \( \phi \in \mathcal{M}_f \) then the \( Q \)-cover corresponding to \( f \) will be denoted \( \tilde{N}_\phi \). If \( \tilde{\phi} \in \tilde{\mathcal{M}}_f \) is a lift of \( \phi \), then \( \tilde{N}_\phi = \tilde{N}_{\tilde{\phi}} \). The covering group \( Q \) acts on the homology \( H_1(\tilde{N}_{\phi}; \mathbb{Z}) \), and we can consider the decomposition of this action into irreducible (over \( \mathbb{Q} \)) representations. This is the same decomposition as considering \( L = H_1(\tilde{N}_{\phi}; \mathbb{Z})/(\text{torsion}) \), which can be regarded as a lattice in \( H_1(\tilde{N}_{\phi}; \mathbb{Q}) \), and looking at the sublattices on which \( Q \) acts irreducibly. (The direct sum of these sublattices has finite index in \( L \), but is not necessarily all of \( L \).) We will examine \( L \) by looking at the submodules individually. Equivalently, for each irreducible representation \( \rho: Q \to \text{GL}(V_i) \) where \( V_i \) is a lattice, we consider the homology \( H_1(\tilde{N}_{\phi}, V_i) \) with coefficients twisted by \( \rho \).

To understand the homology of \( \tilde{N}_{\phi} \), we first consider the action of the covering group \( Q \) on \( V = H_1(\tilde{\Sigma}; \mathbb{Z}) \). Again, the action decomposes into a sum of (rationally) irreducible representations. We group these representations by isomorphism type, and so express a finite index sublattice of \( V \) as \( V_0 \oplus V_1 \oplus \cdots \oplus V_n \), where the action on \( V_i \) is a direct sum of copies of a single representation, which differs for distinct indices \( i \). The action of \( \tilde{\mathcal{M}}_f \) on \( V \) preserves the decomposition into \( V_i \) because \( Q \) is a normal subgroup of \( \tilde{\mathcal{M}}_f \).

Now consider the cover \( \tilde{H} \) of the handlebody \( H \) corresponding to \( f \), and set \( W \) to be the kernel of \( V \to H_1(\tilde{H}; \mathbb{Z}) \). Then we have

\[
H_1(\tilde{N}_{\phi}) = V / \left\langle W, \phi_*^{-1}(W) \right\rangle.
\]

As the action of \( Q \) on \( \tilde{\Sigma} \) extends over \( \tilde{H} \), the kernel \( W \) is a \( Q \)-invariant subset of \( V \). Hence it also decomposes into \( W_0 \oplus W_1 \oplus \cdots \oplus W_n \) where \( W_i = W \cap V_i \).

**9.4 Lemma** Each \( W_i \) is half-dimensional in \( V_i \).
Proof Basically, we can compute both dimensions explicitly using the Euler characteristic. Consider an irreducible rational $Q$-module $T \cong \mathbb{Q}^d$ corresponding to some $V_i$. Suppose $X$ is a finite CW complex with an epimorphism $\pi_1(X) \to Q$. Then the Euler characteristic of the twisted homology $H_*(X, T)$ is just $\dim(T)\chi(X)$, as is clear from the chain complex used to compute the former. Moreover, $H^0(X, T)$ and $H_0(X, T)$ both vanish because $T$ is irreducible; in general, $H^0(X, T)$ is the submodule of $T$ consisting of invariant vectors, and $H_0(X, T)$ is the module of co-invariants ($= V/\langle qv - v \mid q \in Q, v \in T \rangle$) [Bro, §III.1].

Consider the homology $H_*(X, T)$ of the handlebody $H$. As $H$ is homotopy equivalent to a bouquet of $g$ circles, we have that the only non-zero homology is in dimension 1. Thus

$$\dim H_1(H, T) = -\chi(H_*(H, T)) = (g - 1) \dim T$$

and so $H_1(\tilde{H}, \mathbb{Q})$ contains exactly $(g - 1)$ copies of $T$.

Next, consider $H_*(\Sigma, T)$. Poincaré duality implies $H_2(\Sigma, T) \cong H^0(\Sigma, T) = 0$ [Bro, VIII.10]. Thus again we have

$$\dim H_1(\Sigma, T) = -\chi(H_*(\Sigma, T)) = (2g - 2) \dim T.$$ 

and so $V_i$ consists of $(2g - 2)$ copies of $T$. Counting dimensions now shows that $W_i$ must consist of $(g - 1)$ copies of $T$, proving the lemma.

We now focus attention on one summand $V_i$, looking at the homology piece by piece. As we only care about rational homology, set

$$U_i = \left( V_i/\left< W_i, \bar{\phi}_*^{-1}(W_i) \right> \right) \otimes \mathbb{Q}.$$ 

Consider the homomorphism $\rho: \mathcal{M} \to \text{GL}(V_i)$ induced by the projection $V \to V_i$. We want to show that $U_i$ almost surely vanishes, or equivalently that the image of $\rho$ almost surely takes $W_i$ to a complementary subspace. More precisely we need:

9.5 Claim Given $\varepsilon > 0$ there exists a $C_0$ such that the following holds. If $\phi$ is the result of a random walk in $\mathcal{M}$ of length $C \geq C_0$ then the probability

$$P\left\{ \phi \in \mathcal{M} \text{ and } U_i \neq 0 \right\} < \varepsilon.$$ 

We will deduce the above claim with the help of the next lemma, but first some notation. Let $P$ be the orbit of $V_i$ in the half-dimensional Grassmannian $\text{Gr}(W_i)$. Let $B$ be the $X \in P$ such that $X$ is not transverse to $V_i$. We will want to work mod a fixed large prime $p$. We denote reduction mod $p$ by a bar, e.g. $\overline{V}_i = V_i/pV_i$. We are interested in $\overline{\mathcal{M}}: \overline{\mathcal{M}} \to \text{GL}(\overline{V}_i)$, whose image we call $\overline{G}$. We want to understand the action of $\overline{G}$ on the half-dimensional Grassmannian $\text{Gr}(\overline{W}_i)$. Let $\overline{P} \subset \text{Gr}(\overline{V}_i)$ be the orbit of $\overline{W}_i$ under $\overline{G}$; equivalently, $\overline{P}$ is the image of $P$ under $V_i \to \overline{V}_i$. We set $\overline{\mathcal{B}}$ to be the reduction of $B$ mod $p$. Thus for $\overline{\phi} \in \overline{\mathcal{M}}$ we have $U_i \neq 0$ implies that $\overline{\mathcal{B}}(\overline{\phi})(\overline{V}_i) \in \overline{\mathcal{B}}$. Now we can state:

9.6 Lemma Assume that $Q$ is cyclic and $g = 2$. Then the ratio $|\overline{\mathcal{B}}|/|\overline{P}|$ goes to 0 as $p \to \infty$. 

For a finite abelian group, any irreducible representation over \( \mathbb{C} \) is 1-dimensional and the image consists of roots of unity; hence it factors through some cyclic quotient. It easily follows that all \( \mathbb{Q} \)-irreducible representations also factor through cyclic quotients. Thus Lemma 9.6 is sufficiently general for its role in proving Theorem 9.1. Before proving the lemma, let us deduce Claim 9.5 from it. Let \( \Gamma \) be the stabilizer of \( W_i \) in \( \mathcal{P} \), which is a subgroup of \( \mathcal{M} \) of index \( |\mathcal{P}| \). Let \( \Gamma \) be the image of \( \Gamma \) in \( \mathcal{M} \). As \( Q \) fixes \( W_i \) it is contained in \( \Gamma \) and so \( |\mathcal{M} : \Gamma| = [\mathcal{M} : \Gamma] \).

Now, if \( \phi \in \mathcal{M} \) then \( U \neq 0 \) implies that \( \phi \) is in one of \( |\mathcal{B}| \) cosets of \( \Gamma \) in \( \mathcal{M} \). As always, if \( \phi \) is the result of a long random walk in \( \mathcal{M} \) then the location of \( \phi \) in the finite coset space \( \mathcal{M} / \Gamma \) is nearly uniform. Thus for long walks

\[
P\{ \phi \in \mathcal{M} \text{ and } U_i \neq 0 \} \leq \frac{|\mathcal{B}|}{|\mathcal{M} : \Gamma|} = \frac{1}{|\mathcal{M} : \mathcal{M} / \Gamma|} \frac{|\mathcal{B}|}{|\mathcal{M} : \Gamma|} = \frac{1}{|\mathcal{M} : \mathcal{M} / \Gamma|} \frac{|\mathcal{B}|}{|\mathcal{P}|}
\]

By the lemma, we can make the rightmost term as small as we want, which proves Claim 9.5. We now turn to the proof of the lemma.

**Proof of Lemma 9.6.** We think of the handlebody \( H \) as the union of two solid tori \( H_1 \) and \( H_2 \) glued along a disc. Then \( \pi_1(H) \) is freely generated by \( \{x_1, x_2\} \) where each \( x_i \) generates \( \pi_1(H_i) \). Given our \( f: \pi_1(\Sigma) \to Q \) which extends over \( H \), we can choose this decomposition so that the induced map \( f: H \to Q \) is such that \( f(x_1) \) generates \( Q \) and \( x_2 \) generates the kernel of \( f \). The corresponding \( Q \)-cover of \( H \) is easy to describe; it consists of the \( |Q| \)-fold cyclic cover of \( H_1 \) with \( |Q| \) copies of \( H_2 \) glued on like spokes around a central hub.

Now we construct an element \( \phi \in \mathcal{M} \) which allows us to estimate the size of \( \mathcal{P} \) from below. Let \( T_j \) be the torus with one hole that is \( \Sigma \cap H_j \). Choose simple closed curves \( \alpha_j \) and \( \beta_j \) in \( T_j \) which meet in one point, and where \( \alpha_j \) bound a disc in \( H_j \). Consider the element \( \phi \in \mathcal{M} \) which is a Dehn twist in \( \beta_j \) followed by \( |Q| \) Dehn twists in \( \beta_1 \). A lift \( \tilde{\phi} \) of \( \phi \) is easy to describe: the preimage of \( \beta_j \) consists of \( |Q| \) curves, one on each \( T_j \) spoke, while the preimage of \( \beta_1 \) is a single curve running once round the \( \tilde{T}_1 \) hub; the lift \( \tilde{\phi} \) is simply the Dehn twist along this disjoint collection of curves. The important thing for us is that \( \tilde{\phi} \) takes the Lagrangian kernel of

\[
H_1(\Sigma; \mathbb{Z}) \to H_1(\tilde{H}; \mathbb{Z})
\]

to something transverse with itself; moreover, the images of this Lagrangian under proper powers \( \tilde{\phi}^n \) are all distinct, mutually transverse subspaces.

Now consider the particular summand \( V_i \) of \( H_1(\Sigma; \mathbb{Z}) \) in the case at hand. For \( \phi \), we know that the orbit of the Lagrangian \( W_i \) inside \( V_i \) is infinite. Therefore, by choosing \( p \) large we can make \( \mathcal{P} \) as large as we want. Turning now to understanding \( \mathcal{P} \), by the Euler characteristic calculation of Lemma 9.4, \( W_i \) consists of a single irreducible \( Q \)-module. Therefore, for \( \psi \in \mathcal{M} \) the intersection \( \psi(W_i) \cap W_i \) is either \( \{0\} \) or all of \( W_i \) since this intersection is \( Q \)-invariant. Thus \( B \) consists solely of \( W_i \) and \( |\mathcal{B}| = 1 \). Since we can make \( |\mathcal{P}| \) as large as we want, the Lemma follows.
This completes the proof of Theorem 9.1 in the case that \( \phi \cdot f = f \). We now turn to the general case, where \( \phi \cdot f = g \) for an arbitrary \( g : \pi_1(\Sigma) \to \mathbb{Q} \) extending over \( H \). Consider

\[
\mathcal{M}_{f,g} = \{ \phi \in \mathcal{M} \mid \phi \cdot f = g \mbox{ in } \mathcal{A}, \mbox{ or equivalently } \phi_*(K) = \ker g \},
\]

which is no longer a subgroup but is a coset of \( \mathcal{M}_f \). Fix \( \phi_0 \) such that \( \phi_0 \cdot g = f \).

Given \( \phi \in \mathcal{M}_{f,g} \), we have \( \phi_0 \circ \phi \in \mathcal{M}_f \). Schematically, we have:

\[
\begin{array}{ccc}
\tilde{\Sigma}_f & \xrightarrow{\phi_0 \circ \phi} & \tilde{\Sigma}_f \\
\downarrow & & \downarrow \\
\Sigma & \xrightarrow{\phi_0 \circ \phi} & \Sigma
\end{array}
\]

where we have distinguished the \( \mathbb{Q} \)-covers of \( \Sigma \) corresponding to \( f \) and \( g \) by subscripts. Let \( V_f = H_1(\tilde{\Sigma}_f; \mathbb{Z}) \) and let \( W_f \) be the kernel of \( V_f \to H_1(\tilde{H}_f; \mathbb{Z}) \); similarly, let \( V_g \) and \( W_g \) be the corresponding lattices for \( \tilde{\Sigma}_f \). Then moving from the right hand column of the diagram to the middle we get

\[
H_1(\tilde{N}_f) = V_g / \langle W_g, \tilde{\phi}_*(W_f) \rangle \cong V_f / \langle \phi_0*(W_g), (\tilde{\phi}_0 \circ \tilde{\phi})*_{\phi}(W_f) \rangle.
\]

Now the subspace \( \phi_0*(W_g) \) is \( \mathbb{Q} \)-invariant, so as before we can break this question into separate questions for each summand \( V_{f,i} \) of \( H_1(\tilde{\Sigma}_f; \mathbb{Z}) \) corresponding to an irreducible representation. Again, we are interested in the orbit \( P \) of the Lagrangian \( W_{f,i} \subset V_{f,i} \) under the elements of \( \mathcal{M}_f \). The only difference is that the set \( B \) should now be taken to be those \( X \in P \) which are not transverse to \( \phi_0*(W_{g,i}) \) (rather than to \( W_{f,i} \)). If we now assume that the genus is 2, then, as in the proof of Lemma 9.6 the \( \mathbb{Q} \) actions on \( \phi_0*(W_{g,i}) \) and elements of \( P \) are irreducible. Thus \( B \) has either 0 or 1 element depending on whether \( \phi_0*(W_{g,i}) \in P \). The general case can now be completed in exactly the same way as before. This finishes the proof of Theorem 9.1.

### 9.7 Possible generalizations

It would be nice to remove the restrictions that \( Q \) is abelian and that the genus is 2 from the hypotheses of Theorem 9.1. These two conditions are actually quite separate from the point of view of the proof, so we discuss them each in turn.

First, the fact that \( Q \) is abelian (or really, cyclic) ensured that the cover \( \tilde{\Sigma} \to \Sigma \) was concrete enough to exhibit an element \( \phi \in \mathcal{M}_f \) showing that each \( P \) is infinite. (Note that the \( \phi \) given in Lemma 9.6 easily generalizes to any genus.) Such \( \phi \) could probably also be found for other groups, especially small cases like \( S_3 \). More ambitiously, if one wanted to do a whole class of simple groups, e.g. alternating groups, one would be badly hampered if one could not remove the genus restriction — after all, Section 7 only applies in that case.
The genus 2 hypothesis is used at only one point — to show \(|\mathcal{B}| = 1\) and thus allowing us to make \(|\mathcal{B}| / |\mathcal{P}|\) small simply by knowing that \(P\) is infinite. For higher genus, we would need a more detailed picture of the image

\[ \mathcal{M}_f \to \text{Sp}(V) \]

where \(V = H_1(\tilde{\Sigma}; \mathbb{Z})\) in order to compare the relative sizes of \(\mathcal{B}\) and \(\mathcal{P}\).

However, a more abstract point of view might also work to circumvent this issue. Consider a finitely generated subgroup \(\Gamma\) of \(\text{Sp}_{2n}(\mathbb{Z})\), which we think of as sitting inside \(G = \text{Sp}_{2n}(\mathbb{R})\). Fix a standard integral Lagrangian \(W\) in \(\mathbb{R}^{2n}\) and set

\[ D = \{ g \in G \mid g(W) \cap W \neq \{0\} \} \]

which is a proper real-algebraic subvariety (but not subgroup) of \(G\). Now fix generators of \(\Gamma\), and consider the probability that a random walk of length \(N\) in \(\Gamma\) lies in \(D\). It seems very reasonable that, as long as \(\Gamma\) does not have a finite-index subgroup which is contained in \(D\), then this probability goes to 0 as \(N \to \infty\).

Indeed one can prove this with some additional hypothesis on the Zariski closure \(\overline{\Gamma}\) of \(\Gamma\). What one needs is the following. Consider the mod \(p\) reduction \(\Gamma(\mathbb{F}_p)\) of \(\overline{\Gamma}\), which has a natural structure as an algebraic variety over \(\mathbb{F}_p\). By hypothesis, \(D(\mathbb{F}_p)\) is a proper subvariety. It follows that

\[ \frac{|D(\mathbb{F}_p)|}{|\overline{\Gamma}(\mathbb{F}_p)|} \to 0 \quad \text{as} \quad p \to \infty \]

as each has roughly as many points as the projective space over \(\mathbb{F}_p\) of the appropriate dimension (this is a weak form of the Weil Conjecture). Thus if one has that the mod \(p\) reduction map

\[ \Gamma \to \overline{\Gamma}(\mathbb{F}_p) \quad (9.8) \]

is surjective for arbitrarily large \(p\), this would prove the desired claim.

It is known that (9.8) holds when \(\Gamma\) is a lattice in \(\overline{\Gamma}(\mathbb{F}_p)\), or if \(\overline{\Gamma}(\mathbb{F}_p)\) is simple and simply connected. (The latter is the celebrated Nori-Weisfeiler Strong Approximation Theorem [Wei], see also [LS, §Windows] for a discussion.) It is unclear if either of these hypotheses hold in our setting.

References

AS. N. Alon and J. H. Spencer. \textit{The probabilistic method}. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, second edition, 2000.

BL. M. Belolipetsky and A. Lubotzky. Finite Groups and Hyperbolic Manifolds. \textit{Invent. Math.} \textbf{162} (2005), 459–472. [arXiv:math.GR/0406607]

Bol. B. Bollobás. \textit{Random graphs}. Academic Press, London, 1985.

Bro. K. S. Brown. \textit{Cohomology of groups}, volume 87 of \textit{Graduate Texts in Mathematics}. Springer-Verlag, New York, 1982.

Car. L. Carlitz. Representations by quadratic forms in a finite field. \textit{Duke Math. J.} \textbf{21} (1954), 123–137.

CF. P. E. Conner and E. E. Floyd. \textit{Differentiable periodic maps}. Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Band 33. Springer, Berlin, 1964.
Con. P. E. Conner. Differentiable periodic maps, volume 738 of Lecture Notes in Mathematics. Springer, Berlin, second edition, 1979.

CCNPW. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. Atlas of finite groups. Oxford University Press, Eynsham, 1985.

DSC. P. Diaconis and L. Saloff-Coste. Walks on generating sets of groups. Invent. Math. 134 (1998), 251–299.

DPS. Y. Diao, N. Pippenger, and D. W. Sumners. On random knots. In Random knotting and linking (Vancouver, BC, 1993), volume 7 of Ser. Knots Everything, pages 187–197. World Sci. Publishing, River Edge, NJ, 1994.

DdSMS. J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. Analytic pro-p groups, volume 61 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, second edition, 1999.

DM. J. D. Dixon and B. Mortimer. Permutation groups, volume 163 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1996.

DT1. N. M. Dunfield and D. P. Thurston. A random tunnel number one 3-manifold does not fiber over the circle. Preprint, 2005. [arXiv:math.GT/0510129]

DT2. N. M. Dunfield and W. P. Thurston. The Virtual Haken Conjecture: experiments and examples. Geom. Topol. 7 (2003), 399–441. [arXiv:math.GT/0209214]

Edm. A. L. Edmonds. Surface symmetry. II. Michigan Math. J. 30 (1983), 143–154.

Eva. M. J. Evans. T-systems of certain finite simple groups. Math. Proc. Cambridge Philos. Soc. 113 (1993), 9–22.

Eve. B. Everitt. Alternating quotients of Fuchsian groups. J. Algebra 223 (2000), 457–476.

Fel. W. Feller. An introduction to probability theory and its applications. Vol. I. Third edition. John Wiley & Sons Inc., New York, 1968.

FV. M. D. Fried and H. Völklein. The inverse Galois problem and rational points on moduli spaces. Math. Ann. 290 (1991), 771–800.

Gil. R. Gilman. Finite quotients of the automorphism group of a free group. Canad. J. Math. 29 (1977), 541–551.

Gol. W. M. Goldman. Ergodic theory on moduli spaces. Ann. of Math. (2) 146 (1997), 475–507.

Gro1. M. Gromov. Asymptotic invariants of infinite groups. In Geometric group theory, Vol. 2 (Sussex, 1991), volume 182 of London Math. Soc. Lecture Note Ser., pages 1–295. Cambridge Univ. Press, 1993.

Gro2. M. Gromov. Random walk in random groups. Geom. Funct. Anal. 13 (2003), 73–146.

Hal. P. Hall. The Eulerian functions of a group. Quart. J. Math. 7 (1936), 134–151.

HZ. J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. Invent. Math. 85 (1986), 457–485.

Hem1. J. Hempel. The lattice of branched covers over the figure-eight knot. Topology Appl. 34 (1990), 183–201.

Hem2. J. Hempel. 3-manifolds as viewed from the curve complex. Topology 40 (2001), 631–657.

HLM. H. M. Hilden, M. T. Lozano, and J. M. Montesinos. On knots that are universal. Topology 24 (1985), 499–504.

IZ. C. Itzykson and J.-B. Zuber. Matrix integration and combinatorics of modular groups. Comm. Math. Phys. 134 (1990), 197–207.

Jac. D. M. Jackson. Counting cycles in permutations by group characters, with an application to a topological problem. Trans. Amer. Math. Soc. 299 (1987), 785–801.

Jun. D. Jungreis. Gaussian random polygons are globally knotted. J. Knot Theory Ramifications 3 (1994), 455–464.

LN. W. Ledermann and B. H. Neumann. On the order of the automorphism group of a finite group. I. Proc. Roy. Soc. London. Ser. A. 233 (1956), 494–506.

LS1. M. W. Liebeck and A. Shalev. Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks. J. Algebra 276 (2004), 552–601.

LS2. M. W. Liebeck and A. Shalev. Fuchsian groups, finite simple groups and representation varieties. Invent. Math. 159 (2005), 317–367.

Liv. C. Livingston. Stabilizing surface symmetries. Michigan Math. J. 32 (1985), 249–255.
Finite covers of random 3-manifolds

Lub1. A. Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994.

Lub2. A. Lubotzky. *Subgroup growth and congruence subgroups*. *Invent. Math.* 119 (1995), 267–295.

LS. A. Lubotzky and D. Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.

Mac. J. MacWilliams. *Orthogonal matrices over finite fields*. *Amer. Math. Monthly* 76 (1969), 152–164.

Mah. J. Maher. Random walks on the mapping class group. Preprint 2006. arXiv:math.GT/0604433

Nam. H. Namazi. *Heegaard splittings and hyperbolic geometry*. PhD thesis, Yale, 2005.

Ol. A. Y. Ol’shanskii. Almost every group is hyperbolic. *Internat. J. Algebra Comput.* 2 (1992), 1–17.

Pak. I. Pak. What do we know about the product replacement algorithm? In *Groups and computation, III* (Columbus, OH, 1999), volume 8 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 301–347. de Gruyter, Berlin, 2001.

Pen. R. C. Penner. Perturbative series and the moduli space of Riemann surfaces. *J. Differential Geom.* 27 (1988), 35–53.

Per1. G. Perelman. The entropy formula for the Ricci flow and its geometric applications, Preprint 2002. arXiv:math.DG/0211159

Per2. G. Perelman. Ricci flow with surgery on three-manifolds, Preprint 2003. arXiv:math.DG/0303109

PS. D. Poulalhon and G. Schaeffer. Optimal Coding and Sampling of Triangulations. Preprint, 2003.

RZ. L. Ribes and P. Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag*, Berlin, 2000.

Sim. C. C. Sims. *Computation with finitely presented groups*. Cambridge University Press, 1994.

Sta. R. P. Stanley. Spanning trees and a conjecture of Kontsevich. *Ann. Comb.* 2 (1998), 351–363. arXiv:math.CO/9806055

Wal. F. Waldhausen. The word problem in fundamental groups of sufficiently large irreducible 3-manifolds. *Ann. of Math.* (2) 88 (1968), 272–280.

Wei. B. Weisfeiler. Strong approximation for Zariski-dense subgroups of semisimple algebraic groups. *Ann. of Math.* (2) 120 (1984), 271–315.

Whi. M. E. White. Injectivity radius and fundamental groups of hyperbolic 3-manifolds. *Comm. Anal. Geom.* 10 (2002), 377–395.

Wie. J. Wiegold. The Schur multiplier: an elementary approach. In *Groups—St. Andrews 1981 (St. Andrews, 1981)*, volume 71 of *London Math. Soc. Lecture Note Ser.*, pages 137–154. Cambridge Univ. Press, 1982.

Wil. J. S. Wilson. *Profinite groups*, volume 19 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1998.

Wor. N. C. Wormald. Models of random regular graphs. In *Surveys in combinatorics, 1999 (Canterbury)*, volume 267 of *London Math. Soc. Lecture Note Ser.*, pages 239–298. Cambridge Univ. Press, Cambridge, 1999.

Zag. D. Zagier. On the distribution of the number of cycles of elements in symmetric groups. *Nieuw Arch. Wisk.* (4) 13 (1995), 489–495.

Zim. B. Zimmermann. Surfaces and the second homology of a group. *Monatsh. Math.* 104 (1987), 247–253.

Zuk. A. Żuk. Property (T) and Kazhdan constants for discrete groups. *Geom. Funct. Anal.* 13 (2003), 643–670.