Nonlinear effects for Bose Einstein condensates in optical lattices

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We present our experimental investigations on the subject of dynamical nonlinearity-induced instabilities and of nonlinear Landau-Zener tunneling between two energy bands in a Rubidium Bose-Einstein condensate in an accelerated periodic potential. These two effects may be considered two different regimes (for small and large acceleration) of the same physical system and studied with the same experimental protocol. Nonlinearity introduces an asymmetry in Landau-Zener tunneling; as a result, tunneling from the ground state to the excited state is enhanced whereas in the opposite direction it is suppressed. When the acceleration is lowered, the condensate exhibits an unstable behaviour due to nonlinearity. We also carried out a full numerical simulation of both regimes integrating the full Gross-Pitaevskii equation; for the Landau-Zener effect we also used a simple two-level model. In both cases we found good agreement with the experimental results.

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I. INTRODUCTION

Cold atoms and, more recently, Bose-Einstein condensates (BECs) in optical lattices have attracted increasing interest since their first realization [1]. In particular, the formal similarity between the wavefunction of a BEC inside the periodic potential of an optical lattice and electrons in a crystal lattice have triggered theoretical and experimental efforts alike. Many phenomena from condensed matter physics, such as Bloch oscillations and Landau-Zener tunneling have since been shown to be observable also in optical lattices [2-4]. In a recent experiment, a BEC in an optical lattice even made possible the observation of a quantum phase transition that had, up to then, only been theoretically predicted for condensed matter systems [5]. However, an important difference between electrons in a crystal lattice and a BEC inside the periodic potential of an optical lattice is the strength of the self interaction and hence the magnitude of the nonlinearity of the system. Electrons in a metal are almost noninteracting whereas atoms inside a BEC interact strongly. A perturbation approach is appropriate in the former case while in the latter the full nonlinearity must be taken into account. From this feature new physics is expected. Most experiments to date have been carried out in the regime of shallow lattice depth, for which the system is well described by the mean field Gross-Pitaevskii equation with a periodic potential. Moreover, the nonlinearity induced by the mean-field of the condensate has been shown, both theoretically and experimentally, to give rise to instabilities [6-13], in certain regions of the Brillouin zone. These instabilities are not present in the corresponding linear system, i.e. the electron system.

In this paper we review and summarize our experimental and theoretical results on the subject of nonlinear Landau-Zener tunneling and nonlinearity-induced instabilities in a Bose-Einstein condensate interacting with an external periodic potential. These two phenomena represent the most dramatic manifestations of nonlinearity in two different regimes of the system. In order to study these phenomena we have used a single experimental procedure. The underlying idea is to linearly scan the Brillouin zone, by applying a constant acceleration to the periodic potential, and cross the band edge. Then the condensate is released and an absorption picture of the condensate is taken after a time of flight, reflecting the momentum distribution at the time of release. By varying the nonlinearity of the system with a fixed (large) acceleration, we can study the nonlinear contributions to the Landau-Zener effect. We shall denote this regime as the “Landau-Zener” regime. On the contrary, by varying the acceleration from very small to intermediate values with a fixed nonlinearity, we can study the stability of the condensate and the effects of nonlinearity on the dynamics. We will refer to this regime as the “instability” regime.

This paper is organized as follows. After describing our theoretical approach in section II, we explain our experimental techniques in section III. Section IV presents a discussion of our results on the Landau-Zener tunneling, and the experimental and conceptual difficulties encountered in obtaining them. For the interpretation of the nonlinear Landau-Zener tunneling we re-examine and critically compare the effective potential concept, introduced into previous investigations, with our present results. Furthermore we interpret the asymmetry in the nonlinear Landau-Zener effect on the basis of different
chemical potentials calculated for the ground band and for the first excited band. Section IV discusses our experimental and theoretical results on the condensate instabilities. Finally our conclusions and perspectives for future developments and improvements are given in section VI.

II. THEORY

The motion of a Bose-Einstein condensate in an accelerated 1D optical lattice is described by the Gross-Pitaevskii equation

\[ i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2M} \left( -i\hbar \frac{\partial}{\partial x} - M a_L \right)^2 \psi + \frac{V_0}{2} \cos(2k_L x) \psi + \frac{4\pi \hbar^2 a_s}{M} |\psi|^2 \psi \]  

(1)

where \( M \) is the atomic mass, \( k_L = \pi/d \) is the optical lattice wavenumber with optical lattice step \( d \). \( V_0 \) is the periodic potential depth, \( E_R = \hbar^2 k_L^2 / 2M \) is the recoil energy. We introduce the dimensionless parameter \( s \)

\[ V_0 = s E_R \]  

(2)

denoting the lattice depth in units of the recoil energy. The \( s \)-wave scattering length \( a_s \) determines the nonlinearity of the system, with the two-body coupling constant given by \( 4\pi \hbar^2 a_s / M \). Equation (1) is written in the co-moving frame of the lattice, so the inertial force \( M a_L \) appears as a momentum modification. The wavefunction \( \psi \) is normalized to the total number of atoms in the condensate and we define \( n_0 \) as the average uniform atomic density. By defining the dimensionless quantities \( \bar{x} = 2k_L x, \bar{t} = 8E_R t / \hbar, \bar{\psi} = \psi / \sqrt{n_0}, v = s / 16, \alpha = M a_L / 16 E_R k_L, q_B = 1 / 2 \). The nonlinearity is characterized through the parameter \( C \)

\[ C = \frac{\pi n_0 a_s}{k_L^2}. \]  

(3)

Therefore eq. (1) is cast in the following form \[ \ref{eq:1}, \ref{eq:2}, \ref{eq:3} \] :

\[ i \frac{\partial \psi}{\partial \bar{t}} = \frac{1}{2} \left( -i \frac{\partial}{\partial \bar{x}} + \alpha \bar{t} \right)^2 \psi + v \cos(\bar{x}) \psi + C |\psi|^2 \psi \]  

(4)

where we have replaced \( \bar{x} \) with \( x \), etc. In the neighborhood of the Brillouin zone edge, at quasimomentum \( q = q_B \), we approximate the wave function by a superposition of two plane waves with complex coefficients (the two level model illustrated in \[ \ref{fig:1} \] assuming that only the ground state and the first excited state are populated \[ \ref{fig:1} \]. We then substitute in eq. (4)

\[ \psi(x, t) = a(t)e^{iqx} + b(t)e^{i(q-1)x}, \]  

(5)

with \( |a(t)|^2 + |b(t)|^2 = 1 \). Comparing the coefficients of \( e^{iqx} \) and \( e^{i(q-1)x} \), linearizing the kinetic terms and dropping the irrelevant constant energy \( 1/8 + C[1 + (|a|^2 + |b|^2)/2] \), eq. (4) assumes the form

\[ i \frac{\partial}{\partial \bar{t}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha \sigma_3 + v \sigma_1 \\ -C \left( \frac{|a|^2 - |b|^2}{2} \right) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \]  

(6)
where $\sigma_i (i = 1, 2, 3)$ are the Pauli matrices. Each solution of eq. (6) has an associated conserved energy $\epsilon[\psi]$

$$\epsilon[\psi] = \alpha t <\psi|\frac{\sigma_3}{2}|\psi> + v <\psi|\frac{\sigma_1}{2}|\psi> +$$

$$- C \left( <\psi|\frac{\sigma_3}{2}|\psi> \right)^2 \text{ with } |\psi> \equiv \left( \begin{array}{c} a \\ b \end{array} \right) \quad (7)$$

It must be stressed that the energy and the hamiltonian eigenvalue do not coincide, since we are dealing with a nonlinear system. The energy is the quantity that is conserved along the system trajectories while the hamiltonian eigenvalue is not. In nonlinear systems the hamiltonian eigenvalue is usually called the chemical potential (indicated with $\mu$). The connection between energy $\epsilon$ and chemical potential $\mu$ for a condensate within an optical lattice was discussed in \cite{21, 22, 23}. Those quantities are equivalent in the case of negligible atomic interactions, i.e. $C \sim 0$. The Bloch bands may describe either the energy or the chemical potential as a quasimomentum function. In the following we choose to consider the Bloch bands as the curves representing the chemical potential $\mu$ as a function of the quasimomentum (fig. 1).

The adiabatic Bloch bands of eq. (6) have a swallow tail structure (i.e. they develop a loop and become multi-valued) at the edge of the Brillouin zone for $C \geq v$ \cite{18, 21, 22, 23}, a regime not explored in our experiments. In fig. 2(a) we plot the band gap at the BZ edge between the first excited and ground band, both consistent with the total density of the condensate in each of them; moreover we define $\Delta\mu$ as the chemical potential difference between the first excited and ground band, both calculated self-consistently assuming the total density of the condensate in each of them; in the adiabatic case $\Delta\mu$ is obtained in the following way [18, 21, 22, 23]. A regime not explored in our experiments.

In fig. 2(a) we plot the band gap at the BZ edge between the lower band and the first excited band in the nonlinear case, calculated in regimes that are relevant for the Landau-Zener experiments. We defined the gap in the non linear case as the chemical potential difference between the two bands at the edge of the BZ. However a definition of this kind is manifold: we define $\Delta\mu$ as the difference in chemical potential between the first excited and ground band, both calculated self-consistently assuming the total density of the condensate in each of them; moreover we define $\Delta\mu_{10}$ as the chemical potential difference between the first excited and ground band, both consistent with the total density of the condensate in the mean field term (i.e. using the ground state wavefunction in the mean field term) and $\Delta\mu_{01}$ as the chemical potential difference between the first excited and ground band, both consistent with the total density in the first excited band (i.e. using the excited state wavefunction in the mean field term). In order to describe a transition from the ground band to the upper one, we believe that the gap $\Delta\mu_{10}$ is better suited, and conversely the gap $\Delta\mu_{01}$ should better describe the tunneling in the opposite direction.

III. EXPERIMENTAL SETUP

Our experimental apparatus for creating BECs of $^{87}$Rb atoms was described in \cite{24}. The main feature of our apparatus relevant for the present work is the triaxial time-averaged orbiting potential (TOP) trap with trapping frequencies $\nu_x : \nu_y : \nu_z$ in the ratio $2 : 1 : \sqrt{2}$. Our trap is, therefore, almost isotropic. The optical lattice is created by two laser beams with parallel linear polarizations and wavelength $\lambda$, as described in \cite{25}. The two beams are derived from the first diffraction orders of two acousto-optic modulators that are phase-locked but with independent frequencies, allowing us to introduce a frequency difference $\Delta\nu$ between them. The resulting periodic potential has a variable lattice constant $d$ depending on the intersecting angle between the two laser beams. The smallest lattice constant, $0.39 \mu m$, is obtained in the counterpropagating configuration and can be increased up to $1.2 \mu m$ when the two lasers intersect at about 38 degrees. The depth $V_0$ of the periodic potential (depending on the laser intensity and detuning from the atomic resonance of the rubidium atoms) can be varied from $0 E_R$ up to approximatively $3 E_R$. It must be noted that $E_R$ is the true recoil energy and depends on the angle at which the two laser beams intersect. The beams are detuned to the red side of the rubidium atomic resonance by 30 GHz. In this way, a periodic potential with lattice recoil energy $E_R/h = 455$ Hz is created. In addition, by linearly chirping the frequency difference $\Delta\nu$, the lattice is accelerated with $a_L = d \frac{d\Delta\nu}{dt}$. In our experiments,
we used accelerations ranging from \( a_L = 0.3 \text{ m/s}^2 \) to \( a_L = 5 \text{ m/s}^2 \).

The experimental protocol for ‘moving’ the condensate across the Brillouin zone is as follows. After creating BECs with roughly \( 10^4 \) atoms, we adiabatically relax the magnetic trap frequency to \( \nu_z = 42 \text{ Hz} \). Thereafter, the intensity of the lattice beams is ramped up from 0 \( E_R \) to a value corresponding to a lattice depth of approximately \( 2 E_R \). Once the final lattice depth is reached, the lattice is accelerated for a time \( t \). Finally, both the magnetic trap and the optical lattice are switched off, and the condensate is observed by absorption imaging after a time-of-flight of 21 ms.

IV. LANDAU-ZENER TUNNELING

A. Modelling

Evaluating the transition probability in the adiabatic approximation for the transition from an initial state to a final one separated by an energy gap, we find the linear LZ formula for the tunneling probability \( r \)

\[
 r = e^{-\frac{\alpha^2}{2v}} 
\]  
expressing the state population changes in terms of the rate \( \alpha \) at which the diagonal terms of the linear Hamiltonian change their value, and of the off-diagonal interaction strength \( v \) \cite{24}. It must be noted that since we are considering two states only, \( v \) represents the energy gap, too. The transition probability is symmetric in the linear case, i.e. the tunneling rate from the lower level to the upper one is the same as in the opposite direction.

In the nonlinear regime, as the nonlinearity parameter \( C \) grows, the lower to upper tunneling probability grows as well until an adiabaticity breakdown occurs at \( C = v \) where the swallow tail structure appears \cite{15}. The upper to lower tunneling probability, on the other hand, decreases with increasing nonlinearity \cite{25}. We derive the tunneling rate from the numerical integration of eq. (6).

In fig. 3 we plot the lower to upper tunneling rate (initial \( \alpha_t \sigma = (1, 0) \) in eq. (5)) and the upper to lower tunneling rate (initial \( \alpha_t \sigma = (0, 1) \) in eq. (5)) of the Bose-Einstein condensate as a function of the nonlinear parameter \( C \). We see that for \( C = 0 \) the rate is the same for both tunneling directions whereas for \( C \neq 0 \) the two rates are different. We confirm the presence of tunneling asymmetry by integrating eq. (6) directly (taking into account the full experimental protocol), finding qualitative agreement with the prediction of the two-state model.

From an analytical point of view the nonlinear regime is interpreted straightforwardly by writing eq. (6) as

\[
 i \frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix} = \left[ \alpha t \sigma_3 \frac{\sigma_3}{2} + v \frac{\sigma_1}{2} \right] \begin{pmatrix} a \\ b \end{pmatrix} + \frac{C^2}{2} \begin{pmatrix} 0 & 2a^*b \\ 2ab^* & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} 
\]  
for a transition from the lower state to the upper state. For tunneling in the opposite direction, \( a^*b \) simply changes sign. The explicit expression for \( v_{eff} \) is

\[
v_{eff} = v \sqrt{1 \pm \frac{2C}{\sqrt{\alpha^2 t^2 + v^2}} \cos \left( \frac{\alpha}{\alpha^2 t^2 + v^2} \right) + \frac{C^2}{\alpha^2 t^2 + v^2}} 
\]  
Within the spirit of the adiabatic approximation we put
\( \alpha = 0 \) obtaining

\[
\nu_{\text{eff}} = \nu \sqrt{1 + \frac{2C}{v} + \frac{C^2}{v^2}} = \nu \left( 1 + \frac{C}{v} \right)
\]  

(12)

where the upper and lower signs correspond to initial conditions of excited/ground states. For the ground energy band the following effective potential was introduced by Choi and Niu \[15\] and experimentally tested by Morsch et al. \[16\]:

\[
\nu_{\text{eff}} = \frac{\nu}{1 + 4C}
\]  

(13)

For small \( C \) values we modify the LZ formula of eq. (8) to include nonlinear corrections, replacing the potential \( v \) by the effective potential \( \nu_{\text{eff}} \) \[20\].

Equations (12) and (13) define a different role of the nonlinearity. Equation (13) states that the ratio \( \nu_{\text{eff}}/v \) does not depend on the potential magnitude \( v \), but only on the magnitude of the nonlinear parameter \( C \). Whereas eq. (12) predicts a deviation from the linear case according to the ratio between the magnitude of the nonlinearity and the bare potential strength. In the present description, the energy scale determined by the bare potential strength \( v \) appears to be irrelevant at the center of the band; the only relevant energy scale is fixed by the mean field interaction strength \( C \). On the contrary, the energy scale determined by \( v \) acquires more and more significance toward the band edge. At the band edge the two energy scales have the same relevance and \( \nu_{\text{eff}}/v \) only depends on the ratio of them. The swallow tail threshold \( C/v = 1 \) represents a critical value for the system since the effective potential (12) vanishes for a transition from the lower band to the upper band while it doubles for a transition in the opposite direction. Therefore when \( C/v > 1 \), it is no longer possible to interpret the nonlinearity through an effective potential.

The difference in the tunneling rates in the two directions may be also derived from the difference between the

[FIG. 4: Ratio between the effective potential \( \nu_{\text{eff}} \) and the bare potential \( v \) as a function of the nonlinear parameter \( C \). The dotted lines are eq. (12) for both tunneling directions. The shaded regions are presented in the text. The squared marks are the experimental values calculated using the LZ tunneling rate of eq. (8) and data from \[16\] for a transition from the lower band to the upper band (open marks) and in the opposite direction (dotted marks). The circular marks are data from \[3\] with optical lattice step \( d = 1.18 \mu m \) (filled marks) and \( d = 0.39 \mu m \) (open marks), as explained in section III. All calculations and experimental data are for \( v = 0.1375 \) corresponding to \( s = 2.2 \) and \( \alpha = 0.03636 \) corresponding to \( a_L = 2.925 \text{ms}^{-2} \).]
energy gaps $\Delta_{\mu_{10}}$ and $\Delta_{\mu_{01}}$ (fig. 2b) using eq. 3 to link the transition rates to the energy gaps (fig. 2b). As a consequence, the tunneling rate is enhanced in one case and it is suppressed in the other one. These tunneling rates are in very good agreement with those predicted by the effective potential of eq. 12.

B. Experimental observations

Landau-Zener tunneling between the two lowest energy bands of a condensate inside an optical lattice was investigated in the following way (see fig. 1). Initially, the condensate was loaded adiabatically into one of the two bands, either in the ground band or into the excited band. Subsequently, the lattice was accelerated in such a way that, at the BZ edge, a finite probability for tunneling into the other band resulted. After the tunneling event, the two bands had populations reflecting the Landau-Zener tunneling rate. In order to experimentally determine the number of atoms in the two bands, we then increased the lattice depth and decreased the acceleration. In this way, successive crossings of the band edge resulted in a much reduced Landau-Zener tunneling probability between the ground band and the first excited band (of order a few percent), as illustrated in fig. 1. The fraction of the condensate that populated the ground band after the first tunneling event, therefore, remained in that band, whereas the population of the first excited band underwent tunneling to the second excited band with a large probability (around 90 percent) as the gap between these two bands is smaller than the gap between the two lowest bands. Once the atoms underwent tunneling into the second excited band, they essentially behaved as free particles. For both tunneling directions, the tunneling rate $r$ was derived from the ratio between the number of atoms experiencing the tunneling process and the total number of atoms measured from the absorption picture.

In order to verify the experimental procedure for measuring the tunneling rate in both directions and also to verify that in the linear regime the Landau-Zener tunneling was symmetric, we measured the two tunneling rates as a function of the lattice depth for a condensate in a weak magnetic trap and hence a small value of the interaction parameter $C$. In this case, both tunneling rates were essentially the same and agreed well with the linear Landau-Zener prediction, as reported in 10. By contrast, when $C$ was increased, the two tunneling rates began to differ. For instance for the parameter $C = 0.095(9)$, the measured tunneling rate from the ground to excited state was $r = 0.72 \pm 0.10$, whereas we measured $r = 0.37 \pm 0.05$ in the opposite direction, proving the tunneling asymmetry.

The effective potential description allows us to present within a unified picture the nonlinear Landau-Zener tunneling rates measured in refs. 3, 10, as plotted in fig. 3 together with the theoretical predictions. We derive a qualitative agreement with the theoretical predictions of the non-linear Landau-Zener model, whereas quantitatively there are significant deviations. We believe these to be partly due to experimental imperfections. In particular, the sloshing (dipolar oscillations) of the condensate inside the magnetic trap can lead to the condensate not being prepared purely in one band due to non-adiabatic mixing of the bands if the initial quasimomentum is too close to a band-gap. Furthermore the amplitude of the tunneling rates measured in refs. [3, 16], as plotted in fig. 4, points out a strong dependence of the effective potential on the quasimomentum. If the condensate trapped in the periodic potential has a finite extension in momentum space, the overall effect of the effective potential on the tunneling probability is not restricted to a single quasimomentum but represents a mean effect over all the quasimomenta of the condensate. These effects could be responsible for the experimental points not falling exactly on the lines corresponding to the effective potential evaluated at the band edge. In any case the experimental data must fall within the shaded areas corresponding to the band in which the condensate was loaded, as in fig. 4.

Because of the elastic force of the magnetic harmonic trap, it is important in the experiment not to drag the condensate too far from the rest position. If the condensate is dragged too far, the dragged part starts to feel the restoring force due to the harmonic potential and hence does not feel a constant force anymore. For this reason it is not possible to study large $C$ values by varying only the harmonic trap frequency. For our experimental parameters, $C \approx 0.11$ was found to be the largest acceptable value, corresponding to a harmonic trap frequency of about 50 Hz. Furthermore, a numerical simulation of the experiment showed that for large values of $C$, for which the magnetic trap frequency was large, the measured tunneling rates were significantly modified by the presence of the trap. However, we verified in the simulation that when $C$ was varied without varying the trap frequency, the asymmetric tunneling effect persisted. In future experiments, one might study large $C$ values by increasing the atomic density in the condensate by using an additional optical trap, in order to increase the radial trapping frequency or, alternatively, by using Feshbach resonances to vary the atomic scattering length $a_s$. The results of a numerical simulation using the latter method are reported in fig. 4 (b).

V. INSTABILITIES

A. Experimental observations

In the previous experiment we investigated the dependence of the tunneling probability as a function of the nonlinearity magnitude with fixed acceleration. We have also investigated the stability of the condensate as a function of the acceleration with (roughly) fixed nonlinearity 12. When the condensate acquired a quasimomen-
tum close to the band edge, the unstable solutions of eq. (1) grow exponentially in time, leading to a loss of phase coherence of the condensate along the direction of the optical lattice. In our experiment, the time the condensate spent in the ‘critical region’ where unstable solutions existed, was varied through the lattice acceleration. When the acceleration was small the condensate moved across the Brillouin zone more slowly and hence the growth of the unstable modes became more important. Figs. 5c and 5d show typical integrated profiles of the interference pattern obtained for a lattice acceleration $a_{L} = 0.3 \text{m s}^{-2}$. Here, the condensate reached the same point close to the Brillouin zone edge as in Figs. 5a and 5b, but because of the longer time it spent in the unstable region, the interference pattern was almost completely washed out. It is also evident that the radial expansion of the condensate was considerably enhanced when the Brillouin zone was scanned with a small acceleration.

In order to characterize more quantitatively our experimental findings on the instability, we defined two observables for the time-of-flight interference pattern. By integrating the profile in a direction perpendicular to the optical lattice direction, we obtained a two-peaked curve (see fig. 5(a)) for which we defined a visibility $V$ reflecting the phase coherence of the condensate. $V$ is close to 1 for perfect coherence, whereas $V \to 0$ for an incoherent condensate. The second observable we defined, was the width $w_{R}$ of a Gaussian fit to the interference pattern integrated along the lattice direction over the extent of one of the peaks (see fig. 5(b) and (d)).

In [12] we measured $V$ and $w_{R}$ as a function of the final quasimomentum value reached for different values of the acceleration. Figure 6 shows clearly that for large accelerations, both $V$ and $w_{R}$ remained reasonably stable when the edge of the Brillouin zone is crossed. In contrast, for $a_{L} = 0.3 \text{m s}^{-2}$ one sees a drastic change in both quantities as the quasimomentum approaches the value $q_{B}$. For those accelerations, the condensate spent a sufficiently long time in the BZ unstable region and hence lost its phase coherence, resulting in a sharp drop of the visibility. At the same time, the radial width of the interference pattern increased. This increase is evidence for an instability in the transverse directions. For $a_{L} = 0.3 \text{m s}^{-2}$, the interference patterns for quasimomenta larger than unity were so diffuse that it was not possible to measure either the visibility or the radial width in a meaningful way.

In [12] we estimated an instability growth rate of $10^{3} \text{s}^{-1}$ from the time spent by the condensate in the unstable region of the Brillouin zone. This experimental value agrees with the theoretical prediction of $2000 \text{s}^{-1}$ estimated by Fallani et al. [13].
mentum) between their experimental results and their theory is good. Their experiment was subsequently simulated by Modugno et al. [11] who predicted growth rates of the unstable modes in the range 2000 – 3000 s\(^{-1}\). In that work the growth rate is expressed in units of the transverse magnetic trap frequency. This choice could be interpreted as meaning that the instability depends on the radial modes. It has, however, been clarified by one of the authors of that paper [30] that this is not the case and that the choice of the transverse frequency as the frequency unit was made purely for reasons of convenience. In fact, the dependence of the growth rate on the transverse frequency needs to be studied in more depth. We should point out that all the theoretical analyses predict the occurrence of the instabilities at the microscopic level only, while the experimental observations reflect the macroscopic changes of the condensate features, for instance the visibility in our case or the atom loss rate in the case of [13].

![Graph](image-url)

FIG. 7: Results of a one-dimensional numerical simulation for the visibility \( V \) versus the condensate final quasimomentum \( q \) in the conditions of the experiment for acceleration \( a = 0.3 \text{ m s}^{-2} \) and for different values of the nonlinear parameter \( C \). The open squares, circles and triangles correspond to \( C = 0.008 \) (the value for our experiment), \( C = 0.004 \) and \( C = 0 \), respectively. The dashed lines connect the theoretical points to guide the eye. The closed symbols are the experimental values of the visibility as reported in fig.[30] for \( a = 0.3 \text{ m s}^{-2} \).

B. Modelling

We compared our experimental results to a simple 1-D numerical simulation. Figure 6 shows the results of a numerical integration of the one-dimensional Gross-Pitaevskii equation with the parameters of our experiment. The visibility was calculated in the same way as was done for the experimental interference patterns. It is clear from this simulation that it is, indeed, the nonlinearity that is responsible for the instability at the edge of the Brillouin zone. When \( C \) was set to 0 in the numerical simulation, the visibility remains unaltered when the BZ edge is crossed, whereas for finite values of \( C \) the visibility decreases as the quasimomentum \( q_B \) was approached. Furthermore, the larger the value of \( C \), the more pronounced was the decrease in visibility near the band edge. For \( C = 0.008 \), corresponding to the value realized in our experiment, the onset of the instability was located just below a quasimomentum of 0.8\( q_B \). Experimentally, we found that the visibility started decreasing consistently beyond a quasimomentum of \( 0.8 q_B \), agreeing reasonably well with the results of the simulation. The presence of experimental points with visibility less than unity in the quasimomentum region below 0.6\( q_B \) can be explained by considering the initial sloshing of the condensate inside the harmonic trap. This oscillation introduced a sensitive error in the quasimomentum determination which was estimated to be of the order of 0.3\( q_B \).

The determination of the quasimomentum corresponding to the onset of instability is a subject that has been examined in the literature, and in a recent experimental investigation the quasimomentum scanning across the Brillouin zone was stopped at different values in order to verify the instability growth [13]. Our measured values for the instability onset are in agreement with those measured in that reference.

VI. CONCLUSIONS

We have numerically simulated and experimentally studied the dynamics of a Bose Einstein condensate inside a periodic potential in two different regimes. In the Landau-Zener regime, we investigated the tunneling between two energy bands in a periodic potential and found that, in the presence of a nonlinear interaction term, an asymmetry in the tunneling rates arises. Experimentally, we measured these tunneling rates for different values of the interaction parameter and found qualitative agreement with the simulations. In the instability regime, we studied the stability of a BEC in the vicinity of the band edge, finding good agreement between experimental results and the theoretical expectation of unstable behavior. These observations confirmed that Bose-Einstein condensates may be used to simulate a variety of nonlinear physics configurations. Future experiments could probe the complicated and time-dependent tunneling behaviour due to the changing tunneling rate for multiple crossings of the zone edge.

To conclude, we note that the phenomenon of asymmetric tunneling should be a rather general feature of quantum systems exhibiting a nonlinearity. For instance, calculating the energy shift due to a nonlinearity for two adjacent levels of a harmonic oscillator, one finds that both levels are shifted upwards in energy, the shift being proportional to the population of the respective level. The energy difference between the levels, therefore, decreases if only the lower state is populated and increases if all the population is in the upper level. Furthermore the asymmetric Landau-Zener tunneling rate was applied to
interpret the photoassociation of a Bose-Einstein condensate with the surprising result that at small crossing rates the no-transition probability is directly proportional to the rate at which the resonance is crossed.

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