Two theorems of Jhon Bell and Communication Complexity

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Abstract

John Bell taught us that quantum mechanics can not be reproduced by non-contextual and local Hidden variable theory. The impossibility of replacing quantum mechanics by non-contextual Hidden Variable Theory can be turned to an impossible coloring pseudo-telepathy game to be played by two distant players. The game can not be won without communication in the classical world. But if the players share entangled state (quantum correlation) the game can be won deterministically using no communication. This again shows that though quantum correlation can not be used for communication, two parties can not simulate quantum correlation without classical communication. The motivation of the article is to present the earlier works on Hidden Variable Theory and recently developed pseudo-telepathy problem in a simpler way, which may be helpful for the beginners in this area.

Introduction

Quantum mechanics is a mathematical theory to describe the physical world and at the same time it is a probabilistic theory. But this is not surprising. What is surprising, if we take Copenhagen interpretation for granted, is that this probability is not the probability of some dynamical variable having a particular value in some state, but that represents probability of finding a particular value if that dynamical variable is measured. This interpretation generated numerous debates and following
Einstein[1] many people used to think that in future quantum mechanics will be replaced by some more fundamental theory in which description of state will be complete in the sense that every dynamical variables will have well defined values and measurement merely reveals those pre-existing values.

If the question is put forward in this way; Can quantum mechanics be replaced by some complete theory in the above sense? At least the uncertainty principle and complementarity principle do not resolve this question in the negative. Uncertainty relation tells that if one prepares an ensemble corresponding to a quantum state, then all members of that ensemble can not have well defined values for the observables (like position and momentum) involved in the uncertainty relation but it is mute regarding the question whether individual member can have well defined values for observables [2]. Again complementarity principle prohibits joint measurement of certain observables (like position and momentum, polarization measurement in different directions) without addressing the question whether the system has well defined values of those observables.

For a long time, there was a general belief among quantum physicists that quantum mechanics can not be replaced by some complete theory, also popularly known as Hidden Variable Theory(HVT), due to Von Neumann’s impossibility proof (who imposed an unwarranted constraint on HVT). In sixties we got two theorems due to J.S.Bell and Kochen and Specker [3,4,5]. This two theorems showed that quantum mechanics can not be replaced by some classes of HVT, namely local and non-contextual HVT.

Here we shall present the impossibility of replacing quantum mechanics by some non-contextual Hidden Variable Theory by an example in four dimensional Hilbert space [6]. Using this example we shall construct a game to be played by two distant players who are deprived of having any type communication. The impossibility proof itself will provide the reason why in classical world the players can not win the game with certainty. But we shall see that they can win it by some protocol (not involving classical communication) if they share some quantum correlation beforehand[7]. This, in a fundamental way, proves the celebrated Bell’s theorem that no local Hidden Variable Theory can replace quantum mechanics[5].

Description of Hilbert Space Quantum Mechanics

1. Every system is associated with a Hilbert space.
2. States are associated with vectors in the Hilbert space.
3. Observables are associated with self adjoint operator acting on the corresponding Hilbert space.
Consider a self adjoint operator $A$ acting on the finite dimensional ($n$, say) Hilbert space $H$. Operator $A$ will have $n$ eigenvalues (different in general) $a_1, a_2, \ldots, a_n$ and let the corresponding eigenfunctions are $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$. Equivalently $A$ can be written as (also known as spectral representation)

$$A = \sum a_i |\psi_i\rangle \langle \psi_i|$$

where $|\psi_i\rangle \langle \psi_i|$ is a one dimensional projection operator. In general a projection operator $P$ (say) has two eigenvalues 1 or 0 as $P$ is idempotent ($P^2 = P$).

**Measurement**

1. Possible measurement result is one of the eigenvalue of the observable measured on the system.

2. If the initial state of the system is $|\psi\rangle$, and the observable that is measured is $A$, then the probability that the measurement result will be $a_k$ is given by

$$\text{Prob}_{|\psi\rangle}(a_k) = Tr[|\psi\rangle \langle \psi| |\psi_k\rangle \langle \psi_k|] = |\langle \psi | \psi_k \rangle|^2$$

As the eigenvalue does not appear in the probability expression this can be equivalently expressed in the following way; If measurement is performed in the basis $\{|\psi_i\rangle\}$, which means to decide which of the projector of the complete set $\{|\psi_i\rangle \langle \psi_i| : \sum |\psi_i\rangle \langle \psi_i| = I\}$ is true, then the probability that the value of the projector $|\psi_k\rangle \langle \psi_k|$ will be 1 (true) is given by the same quantity as

$$\text{Prob}_{|\psi\rangle}(|\psi_k\rangle \langle \psi_k| = 1) = |\langle \psi | \psi_k \rangle|^2$$

In general, if one performs a measurement to decide whether the projector $P$ is true (1) or false (0), then the probability that measurement result of $P$ will be true, is given by $Tr[|\psi\rangle \langle \psi| P] = \langle \psi | P | \psi \rangle$

3. After the measurement the state will collapse on the corresponding eigen-state i.e. if the result is $a_k$ (or equivalently $|\psi_k\rangle \langle \psi_k| = 1$) then final state will be $|\psi_k\rangle$. In general, if the measurement is to decide whether a projector $P$ is true or false, and if the result corresponds to the truth of $P$, then the final states will be $\frac{P|\psi\rangle}{|P|\psi\rangle}$

**Unitary dynamics**

The future development of a state is given by the unitary dynamics where the unitary operator $U$ ($UU^\dagger = U^\dagger U = I$) is determined by the Hamiltonian acting on
the system. The dynamical equation is given by the celebrated Schrödinger equation

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle$$

Unitary operator preserves the scalar product between any two vectors i.e. \( <\psi_1|\psi_2> = <\psi_1|UU^\dagger|\psi_2> \). If two vectors \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are orthogonal then \( U|\psi_1\rangle \) and \( U|\psi_2\rangle \) are also orthogonal.

**Is there underlying deterministic theory?**

If we prepare two systems in the same state \( |\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \) \((c_1, c_2 \neq 0)\) and on both the observable \( A \) is measured, the result, in general, will not be identical as probability for the results \( a_1 \) and \( a_2 \) are \( |\langle \psi|\psi_1\rangle|^2 \) \( (= |c_1|^2) \) and \( |\langle \psi|\psi_2\rangle|^2 \) \( (= |c_2|^2) \) respectively, both being non-zero. This indeterminism is fundamental, because according to quantum mechanics, the initial quantum states were truly identical. Having a underlying deterministic theory would imply that actually two states were not identical, they differ in some additional parameters (popularly known as hidden variables) which remain unknown to us. If description of state is to be completed, these parameter has to be taken into account and then the theory will behave in a deterministic way.

**Programme of Hidden Variable Theory**

There exists a complete theory where states assign values to all observables (equivalently yes/no to all projectors) and quantum state is some kind of statistical mixture of these finer states. Schematically, in quantum mechanics;

$$\text{Prob}_{|\psi\rangle}(P = 1) = \langle \psi|P|\psi\rangle \neq 0 \text{ or } 1$$

in general.

Now if the state \( |\psi\rangle \) is completed with the additional variable say \( \lambda \), then the completed state \( |\psi, \lambda\rangle \) (not a vector in the Hilbert space) will be able to assign values to all projectors when value of \( \lambda \) is specified. It is the incapacity of quantum theory that it treats two different state \( |\psi, \lambda_1\rangle \) and \( |\psi, \lambda_2\rangle \) \((\lambda_1 \neq \lambda_2)\) as same state. So we want a complete theory(HVT) where if the value of \( \lambda \) is known then probability of every proposition for that completed state becomes 0 or 1, and hence we can talk in terms of proposition being true or false (projector having value 0 or 1) instead of probability. Hence

$$\text{Prob}_{|\psi, \lambda\rangle}(P = 1) = V_{|\psi, \lambda\rangle}(P) = 0 \text{ or } 1$$
In this picture quantum state $|\psi\rangle$ is simply a statistical mixture of completed states $|\psi, \lambda\rangle$ and quantum probability arises due to ignorance on the value of $\lambda$. So if the distribution of $\lambda$ is given by $\rho(\lambda)$ with $\int \rho(\lambda)d\lambda = 1$, then the quantum probability will arise due to ignorance on $\lambda$, which can be expressed mathematically as

$$\langle\psi|P|\psi\rangle = \int \rho(\lambda)V_{|\psi, \lambda\rangle}(P)d\lambda$$

Is it possible to have a HVT even in principle?

Bell showed that if the dimension of the Hilbert space is two, then one can construct a HVT which reproduces QM so far as standard measurement represented by projector is concerned[3].

For two dimensional Hilbert space, there is a one to one association between states and points on the surface on a unit sphere. For any state $|\psi\rangle$, there is a unit vector $n$, such that the projector on $|\psi\rangle$ can be written as,

$$|\psi\rangle\langle\psi| = \frac{1}{2} [I + n.\sigma]$$

where $n$ is the unit vector, $\sigma$'s are $2 \times 2$ Pauli matrices and $\sigma.n = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$ and $I$ is an unit operator on 2 dimensional Hilbert space. Now consider a projector $P$ where

$$P = \frac{1}{2} [I + m.\sigma]$$

Then

$$Prob_{|\psi\rangle}(P = 1) = \frac{1}{4} Tr[(I + n.\sigma)(I + m.\sigma)] = \frac{1}{2} [1 + n.m]$$

which can take values from 0 to 1 depending on the scalar product of $m$ and $n$.

**Construction of HVT in two dimension**

Let us now consider completed (HVT) state $|\psi, \lambda\rangle$ or $|n, \lambda\rangle$, where the additional variable $\lambda$ varies from $-\frac{1}{2}$ to $\frac{1}{2}$, and distribution of $\lambda$ is uniform.

Now our completed state has to assign values 0 or 1 to all projectors. Let the state assign values to the projectors in the following way,

$$V_{|n, \lambda\rangle}[P] = \frac{1}{2} [1 + Sign(\lambda + \frac{1}{2} |n.m| Sign(n.m))]$$

One can easily check that it reproduces the quantum probability if one integrates
over $\lambda$ [8]. Let us now calculate quantum probability for the case for which $n.m = -ve$

$$Prob_{|\psi\rangle}(P = 1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho(\lambda)V_{|n,\lambda\rangle}(P)d\lambda$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} [1 + Sign(\lambda + \frac{1}{2} |n.m| Sign(n.m))]d\lambda$$

$$= \frac{1}{2} \left[ \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - 1) d\lambda + \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 + 1) d\lambda \right]$$

$$= \frac{1}{2} [1 - |m.n|]$$

$$= \frac{1}{2} [1 + (m.n)]$$

Similarly one can check it, for the case when $n.m = +ve$

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**Is HVT possible in higher dimension?**

After successful HVT model in two dimension, the natural question arises whether the construction is possible in higher dimension. In higher dimension there is further question of context which did not arise for two dimensional Hilbert space. For simplicity, consider a three dimensional Hilbert space where $\{|\psi_i\rangle\}_{i=1,2,3}$ is a orthogonal basis. Then HVT assigns 1(truth) or 0(false) value to all the projector with the following restriction:

1. $V(|\psi_i\rangle\langle\psi_i|) = 0 \text{ or } 1$

2. As the projectors are mutually exclusive, at a time one of them can be true(1) and the rest will be false(0) which means $\sum V(|\psi_i\rangle\langle\psi_i|) = 1$

Now consider another orthogonal basis $\{|\psi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$ where $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|\psi_2\rangle + |\psi_3\rangle)$ and $|\phi_3\rangle = \frac{1}{\sqrt{2}}(|\psi_2\rangle - |\psi_3\rangle)$. Obviously measurement in this basis is different from measurement in the basis considered earlier. But the projector $|\psi_i\rangle\langle\psi_i|$ is common. Does the HVT assign same value (0 or 1) to this common projector ignoring the different measurement context. If the answer is yes, then the HVT is called non-contextual(One should note that this question did not arise in two dimension as there can not be two different orthogonal basis having one vector in common).

Then it has been shown that this kind of Hidden Variable Theory namely non-contextual HVT is impossible if the dimension of the Hilbert space is more than or equal to three [3,4].

We shall present a proof of this impossibility for four dimensional Hilbert space[6].
We consider the following 18 (unnormalized) vectors appearing in 9 sets of orthogonal basis and assign value (0 or 1) to the projectors on the corresponding vectors under the restriction (a),(b) along with the non-contextual assumption. For short, we have used the symbol $V(|\psi\rangle)$ to mean the value of the projector $V(|\psi\rangle\langle\psi|)$

$$V(0001) + V(0010) + V(1100) + V(1 - 100) = 1$$

$$V(0001) + V(0100) + V(1010) + V(10 - 10) = 1$$

$$V(1 - 11 - 1) + V(1 - 1 - 11) + V(1100) + V(0011) = 1$$

$$V(1 - 11 - 1) + V(1111) + V(10 - 10) + V(010 - 1) = 1$$

$$V(0010) + V(0100) + V(1001) + V(100 - 1) = 1$$

$$V(1 - 1 - 11) + V(1111) + V(100 - 1) + V(01 - 10) = 1$$

$$V(11 - 11) + V(111 - 1) + V(1 - 100) + V(0011) = 1$$

$$V(11 - 11) + V(-1111) + V(1010) + V(010 - 1) = 1$$

$$V(111 - 1) + V(-1111) + V(1001) + V(01 - 10) = 1$$

Add these nine equations. The left hand side will be even as every vector has appeared twice, while the right hand side is obviously odd.

**Entangled state**

Let us now consider two systems A and B, both associated with d-dimensional Hilbert space and let $\{|i\rangle_{A(B)}\}_{i=1..d}$ is a orthogonal basis for A(B). Consider the state

$$|\phi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_B$$

One can see that this state $|\phi_{AB}\rangle$ can not be written in the form of product state $|\chi\rangle_A \otimes |\gamma\rangle_B$. 

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A state of joint system that can not be written as product of subsystem states, is called entangled state. Consider a Unitary $d \times d$ matrix $U$ acting on $d$-dimensional Hilbert space. Then $\{U|i\rangle_{A(B)}\}_{i=1,d}$ is a new basis. Again $U^*$ (the new operator formed by replacing each element by its complex conjugate) is also a unitary operator which also generate another orthogonal basis if acted on a orthogonal basis. Then the state $|\phi_{AB}\rangle$ has the following interesting property;

$$U_A \otimes U_B^* |\phi_{AB}\rangle = |\phi_{AB}\rangle$$

In case $U = U^*$, we get

$$|\phi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_B = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} U|i\rangle_A \otimes U|i\rangle_B$$

So $|\phi_{AB}\rangle$ can be written in terms of any orthogonal basis if it is related to $\{|i\rangle\}$ by a unitary operator $U$ such that $U = U^*$.

**Bi-partite pseudo-telepathy**

Consider the previous 9 sets of orthogonal basis in four dimension and denote them by $S^1, S^2, . . . S^9$, where $S^J$ contains the following four orthogonal vectors $|u_1\rangle^J, |u_2\rangle^J, |u_3\rangle^J, |u_4\rangle^J$ where, $|u_1\rangle^1 = |u_1\rangle^2 = (0001)$, $|u_2\rangle^1 = |u_1\rangle^5 = (0010)$, etc. Two players, say Alice and Bob, are far away and (1) Alice is given any one ($S^k$, say) of the nine sets randomly, and (2) Bob is given a vector ($|u_m\rangle^k$, say) randomly from the set given to Alice.

**Winning condition**

Alice has to assign value (0 or 1) to her four vectors and Bob also has to assign value to his single vector in such a way that

1. Exactly one of Alice's vector should receive the value 1.

2. Alice and Bob have to assign same value to their single common vector

with the condition that they will not be allowed to have any classical communication after the game starts. They will win the game if in every repetition of the game they win it with certainty.
Players in the classical world

Any classical deterministic winning strategy would mean precisely assigning values to all the 18 vectors in a non-contextual way which is already been forbidden by the Bell and Kochen-Specker theorem. No previously shared correlation can help them in this regard.

If one tries to satisfy all the equations by assigning non-contextual values to the maximum possible no. of vectors, then one would see that 17 vectors can be assigned non-contextual values and value assignment for one vector has to be contextual i.e. one vector out of 18 has to take value 1 when it occurs in one basis and 0 when it occurs in another basis.

Let us consider a contextual solution where the vector \((0001)\) takes value 1 when it occurs in \(S^1\) and 0 when in \(S^2\). Now one can have simplest strategy like this. Alice sends bit 0 to Bob when he is given the basis set \(S^1\) and bit 1 when it is not \(S^1\), so that they can win the game by assigning value to the vector \((0001)\) in a contextual way. So 1 cbit is more than sufficient to win the game in the classical world.

Quantum strategy

Let Alice and Bob share the following entangled state

\[
|\psi\rangle_{AB} = \frac{1}{2} \left( |u_1\rangle_A^1 |u_1\rangle_B^1 + |u_2\rangle_A^1 |u_2\rangle_B^1 + |u_3\rangle_A^1 |u_3\rangle_B^1 + |u_4\rangle_A^1 |u_4\rangle_B^1 \right)
\]

But all the 18 vectors involved here are real and hence all the other 8 orthogonal basis sets \(S^2, S^3, \ldots, S^9\) are related to this one \(\{|u_i\rangle\}^1\) by unitary operator whose matrix elements are real. So the state can be written in any of the basis keeping the form of the state intact (overall phase is omitted) i.e.

\[
|\psi\rangle_{AB} = \frac{1}{2} \sum_{i=1}^{4} (|u_i\rangle_A^k |u_i\rangle_B^k)
\]

for all \(k\).

**Alice**: She measures on her system in the basis of the set, she has been given. On whatever state she collapses due to measurement, she assigns value 1 to that vector and assigns 0 to the rest three.

**Bob**: He chooses any basis set containing the vector given to him and measure in that basis. If he collapses on the vector given to him, he assigns 1 and 0 otherwise.

With this strategy they can win the game deterministically.

Consider one example where Alice has been given the set \(S^m\) and Bob has been given the vector \(|u_p\rangle^m\). Now the shared state can be written in the basis \(S^m\) as

\[
|\psi\rangle_{AB} = \frac{1}{2} \sum_{i=1}^{4} (|u_i\rangle_A^m |u_i\rangle_B^m)
\]
Now according to the protocol described above, Alice will measure in the basis \((|u_1\rangle_m, |u_2\rangle_m, |u_3\rangle_m, |u_4\rangle_m)\). Let due to the measurement Alice’s system collapses on the vector \(|u_q\rangle_m\). Because of the correlation Bob’s system will also collapses on \(|u_q\rangle_m\). Then due to Bob’s measurement, the probability that his system will collapse on the vector \(|u_p\rangle_m\), given to him, is given by

\[
Prob_{Bob}(|u_p\rangle_m) = |\langle u_p | u_q \rangle|^2 = \delta_{pq}
\]

(Actually there is no temporal order between Alice’s and Bob’s measurement in the protocol. The result will remain same whoever performs the measurement earlier.) This clearly shows that Bob will assign value 1 to the vector given to him only when he collapses on that vector and it can happen if and only if Alice also collapses on that vector. So the protocol will work in every cases without error.

What is the teaching?

1. One should note that, though there is a successful protocol with entanglement, Bob can not know which set his vector belongs to (recall that every vector has appeared in two different basis sets). If he could know it that would imply signalling or real telepathy. Entanglement can offer advantage for only a class of problems of this type, solving which itself does not imply signalling (transfer of real information from one party to another instantaneously) or violation of causality.

2. The game can not be won deterministically in the classical world without classical communication. So it shows that classical correlation (shared randomness) can not generate the effect of quantum correlation if no classical communication is allowed. This is the essence of most celebrated Bell’s theorem, which says that no local Hidden Variable theory can reproduce all the correlations of quantum mechanics.

3. Communication complexity is an area that aims at quantifying the amount of communication necessary to solve distributed computational problem. Quantum communication complexity uses the power of entanglement to reduce the amount of communication that would be classically required. Here we discuss pseudo-telepathy problem to exhibit the potential power of entanglement to penetrate in the area of communication complexity of course under the condition of no violation of causality. This kind of example clearly indicates that for cleverly chosen distributed computational problem, entanglement can be
used to reduce the amount of bits of classical communication needed to solve it in the classical world.

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