Fuzzy strong $\phi$-b-normed linear space for fuzzy bounded linear operators

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Abstract

In this paper, concept of fuzzy continuous operator, fuzzy bounded linear operator are introduced in fuzzy strong $\phi$-b-normed linear spaces and their relations are studied. Idea of operator fuzzy norm is developed and completeness of BF(X,Y) is established.

Keywords: Fuzzy normed linear space, t-norm, fuzzy strong $\phi$-b-normed linear space, fuzzy bounded linear operator, operator fuzzy norm.

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1 Introduction

The problem of defining fuzzy norm on a linear space was first initiated by Katsaras [1] and afterwards Felbin [2]. Cheng and Mordeson [3], came up with their definition of fuzzy norm approaching from different perspective. Following the definition of fuzzy norm introduced by Cheng and Mordeson, Bag and Samanta [4] defined a fuzzy norm with a view to establish a complete decomposition of fuzzy norm into crisp norms. However, for doing so, they had to restrict the underlying t-norm in the triangle inequality of fuzzy norm to be the ‘min’. Next they attempted to deal with the problem by considering general t-norm and established many fundamental results of functional analysis in fuzzy settings [5–8].

Following the concept of Bag and Samanta type fuzzy normed linear space, several authors developed the concept of generalized fuzzy normed linear spaces viz. fuzzy cone normed linear space [9], G-fuzzy normed linear space [10], etc. In our earlier papers [11,12], concept of fuzzy strong $\phi$-b-normed linear space is introduced and developed many basic results in such spaces. In Bag and Samanta type fuzzy normed linear space, scalar multiplication is given by $N(cx, t) = N(x, \frac{t}{|c|})$. But in fuzzy strong $\phi$-b-normed linear space, scalar multiplication is given by $N(cx, t) = N(x, \frac{t}{\phi(|c|)})$ where $\phi$ is a real valued function satisfying some properties.

In this paper, we have been able to proceed further. The concept of fuzzy bounded linear operators, fuzzy continuous operators, operator norm for fuzzy bounded linear operators and spaces of fuzzy bounded linear operators (denoted by BF(X,Y)) are introduced in fuzzy strong $\phi$-b-normed linear spaces. Lastly completeness of BF(X,Y) is proved.

The organization of the article is in the following.

Section 2 contains some preliminary results. In Section 3, fuzzy bounded linear operators, fuzzy continuous operators are defined and some related results are studied. Section 4 consists of the study on operator fuzzy norm on fuzzy strong $\phi$-b-normed linear spaces.
2 Preliminaries

We provide some basic definitions and results which are used in this paper.

Definition 2.1. [4] A binary operation \( \ast : [0,1] \times [0,1] \rightarrow [0,1] \) is called a t-norm if it satisfies the following conditions:

(i) \( \ast \) is associative and commutative;
(ii) \( \alpha \ast 1 = \alpha \ \forall \alpha \in [0,1] \);
(iii) \( \alpha \ast \gamma \leq \beta \ast \delta \) whenever \( \alpha \leq \beta \) and \( \gamma \leq \delta \ \forall \alpha, \beta, \gamma, \delta \in [0,1] \).

If \( \ast \) is continuous then it is called continuous t-norm.

The following are examples of some t-norms.

(i) Standard intersection: \( \alpha \ast \beta = \min\{\alpha, \beta\} \).
(ii) Algebraic product: \( \alpha \ast \beta = \alpha \beta \).
(iii) Bounded difference: \( \alpha \ast \beta = \max\{0, \alpha + \beta - 1\} \).
(iv) Drastic intersection: \( \alpha \ast \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases} \).

Definition 2.2. [5] Let \( X \) be a linear space over a field \( \mathbb{F} \). A fuzzy subset \( N \) of \( X \times \mathbb{R} \) is called fuzzy norm on \( X \) if \( \forall x, y \in X \) the following conditions hold:

(NI) \( \forall t \in \mathbb{R} \) with \( t \leq 0 \), \( N(x, t) = 0 \);
(NII) \( \forall t \in \mathbb{R}, t > 0, N(x, t) = 1 \) \( \iff \) \( x = \theta \);
(NIII) \( \forall t \in \mathbb{R}, c \in \mathbb{R}, t > 0, N(cx, t) = N(x, \frac{t}{|c|}) \);
(NIV) \( \forall s, t \in \mathbb{R}, N(x + y, s + t) \geq N(x, s) \ast N(y, t) \);
(NV) \( N(x, \cdot) \) is a non-decreasing function of \( t \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

Then the pair \( (X, N) \) is called fuzzy normed linear space.

Remark 2.3. Bag and Samanta [4] assumed that,

(NVI) : \( N(x, t) > 0 \) \( \implies \) \( x = \theta \).

Definition 2.4. [4] Let \( (X, N) \) be a fuzzy normed linear space.

(i) A sequence \( \{x_n\} \) is said to be convergent if \( \exists x \in X \) such that \( \lim_{n \to \infty} N(x_n - x, t) = 1, \forall t > 0 \).
Then \( x \) is called the limit of the sequence \( \{x_n\} \) and denoted by \( \lim x_n \).
(ii) A sequence \( \{x_n\} \) in a fuzzy normed linear space \( (X, N) \) is said to be Cauchy if \( \lim_{n \to \infty} N(x_n + p - x_n, t) = 1, \forall t > 0 \) and \( p = 1, 2, \cdots \).
(iii) \( A \subseteq X \) is said to be a closed set if for any sequence \( \{x_n\} \) in \( A \) converging to \( x \in X \) implies \( x \in A \).
(iv) \( A \subseteq X \) is said to be the closure of \( A \), denoted by \( \bar{A} \) if for any \( x \in \bar{A} \), there is a sequence \( \{x_n\} \subseteq A \) such that \( \{x_n\} \) converges to \( x \).
(v) \( A \subseteq X \) is said to be compact if any sequence \( \{x_n\} \subseteq A \) has a subsequence converging to an element in \( A \).

Remark 2.5. For a t-norm, \( \ast : [0,1] \times [0,1] \rightarrow [0,1] \), Bag and Samanta [8] assumed

(T1) for \( \alpha > 0 \), \( \alpha \ast \alpha > 0 \).
Definition 2.6. Let \((X, N)\) be a fuzzy normed linear space and \(\alpha \in (0, 1)\).

(i) A sequence \(\{x_n\}\) in \(X\) is said to be \(\alpha\)-fuzzy convergent if \(\exists x \in X\) such that

\[
\lim_{n \to \infty} \bigcap \{ t > 0 : N(x_n - x, t) > 1 - \alpha \} = 0.
\]

If \(\{x_n\}\) is \(\alpha\)-fuzzy convergent for each \(\alpha \in (0, 1)\) then \(\{x_n\}\) is called level fuzzy \((l\text{-fuzzy})\) convergent.

(ii) A sequence \(\{x_n\}\) in \(X\) is said to be \(\alpha\)-fuzzy Cauchy if \(\exists x \in X\) such that

\[
\lim_{n,m \to \infty} \bigcap \{ t > 0 : N(x_n - x_m, t) > 1 - \alpha \} = 0.
\]

If \(\{x_n\}\) is \(\alpha\)-fuzzy Cauchy for each \(\alpha \in (0, 1)\) then \(\{x_n\}\) is called level fuzzy \((l\text{-fuzzy})\) Cauchy.

(iii) A subset \(F\) of \(X\) is said to be \(\alpha\)-fuzzy complete if every \(\alpha\)-fuzzy Cauchy sequence is \(\alpha\)-fuzzy convergent to some point in \(F\).

If \(F\) is \(\alpha\)-fuzzy complete for each \(\alpha \in (0, 1)\) then \(F\) is called level fuzzy \((l\text{-fuzzy})\) complete.

(iv) A subset \(F\) of \(X\) is said to be \(\alpha\)-fuzzy closed if for any sequence \(\{x_n\}\) in \(F\) is \(\alpha\)-fuzzy convergent to \(x \in X\) implies \(x \in F\).

If \(F\) is \(\alpha\)-fuzzy closed for each \(\alpha \in (0, 1)\) then \(F\) is called level fuzzy \((l\text{-fuzzy})\) closed.

(v) A set \(S\) is said to be \(\alpha\)-fuzzy closure of \(F \subset X\) if for each \(x \in S\), there exist a \(\alpha\)-fuzzy convergent sequence converging to \(x\).

It is denoted by \(l\text{-}F\).

(vi) A subset \(F\) of \(X\) is said to be \(\alpha\)-fuzzy compact if for any sequence \(\{x_n\}\) in \(F\), there exists a subsequence of \(\{x_n\}\) which is \(\alpha\)-fuzzy convergent to some point in \(F\).

If \(F\) is \(\alpha\)-fuzzy compact for each \(\alpha \in (0, 1)\) then \(F\) is called level fuzzy \((l\text{-fuzzy})\) compact.

(vii) A subset \(F\) of \(X\) is said to be \(\alpha\)-fuzzy bounded if there exists \(t(\alpha) > 0\) such that

\[
N(x, t) > 1 - \alpha, \forall x \in F.
\]

If \(F\) is \(\alpha\)-fuzzy bounded for each \(\alpha \in (0, 1)\) then \(F\) is called level fuzzy \((l\text{-fuzzy})\) bounded.

Remark 2.7. In [7], \(\bigcap \{ t > 0 : N_1(x, t) \geq \alpha \}, \alpha \in (0, 1)\) is denoted by \(d_\alpha\). By (NV1), it follows that for \(x \neq \theta\), \(d_\alpha > 0\) \(\forall \alpha \in (0, 1)\).

Definition 2.8. Let \(\phi\) be a function defined on \(\mathbb{R}\) to \(\mathbb{R}\) with the following properties

\((\phi1)\) \(\phi(-t) = \phi(t), \forall t \in \mathbb{R};\)

\((\phi2)\) \(\phi(1) = 1;\)

\((\phi3)\) \(\phi\) is strictly increasing and continuous on \((0, \infty);\)

\((\phi4)\) \(\lim_{\alpha \to 0} \phi(\alpha) = 0\) and \(\lim_{\alpha \to \infty} \phi(\alpha) = \infty.\)

The following are examples of such functions.

(i) \(\phi(\alpha) = |\alpha|, \forall \alpha \in \mathbb{R}\)

(ii) \(\phi(\alpha) = |\alpha|^p, \forall \alpha \in \mathbb{R}, p \in \mathbb{R}^+\)

(iii) \(\phi(\alpha) = \frac{2^{2n}}{|\alpha|^{2^n}}, \forall \alpha \in \mathbb{R}, n \in \mathbb{N}\)

Definition 2.9. Let \(X\) be a linear space over the field \(\mathbb{R}\) and \(K \geq 1\) be a given real number. A fuzzy subset \(N\) of \(X \times \mathbb{R}\) is called fuzzy strong \(\phi\)-b-norm on \(X\) if for all \(x, y \in X\) the following conditions hold:

(bN1) \(\forall t \in \mathbb{R}\) with \(t \leq 0\), \(N(x, t) = 0;\)

(bN2) \(\forall t \in \mathbb{R}, t > 0, \ N(x, t) = 1\) \(\text{iff} \ x = \theta;\)

(bN3) \(\forall t \in \mathbb{R}, t > 0, \ N(cx, t) = N(x, \frac{t}{\phi(c)})\) \(\text{if} \ \phi(c) \neq 0;\)
Lemma 2.10. \[12\] Let \( X, N, \phi, K, * \) be a fuzzy strong \( b \)-normed linear space. Then (2) = \( X, N, \phi, K, * \). Hence, (2) = \( X, N, \phi, K, * \). Then (2) = \( X, N, \phi, K, * \). Then (2) = \( X, N, \phi, K, * \).

Proposition 3.2. \[12\] Let \( N(x, t) \) be a non-decreasing function of \( t \) and \( \lim_{t \to \infty} N(x, t) = 1 \).

3 Fuzzy bounded linear operator

In this section definition of fuzzy bounded linear operator and fuzzy continuous operator in fuzzy \( b \)-normed linear space are given and their relation is studied.

Definition 3.1. Let \( T : (X, N_1, \phi, K, *_1) \to (Y, N_2, \phi, K, *_2) \) be a linear operator where \( X \) and \( Y \) are fuzzy strong \( b \)-normed linear spaces. \( T \) is said to be fuzzy bounded if for each \( \alpha \in (0, 1) \), \( \exists M_\alpha > 0 \) such that

\[
N_1(x, \frac{t}{M_\alpha}) \geq 1 - \alpha \implies N_2(Tx, Ks) \geq \alpha \quad \forall s > t \quad \forall t > 0.
\]

(1)

Proposition 3.2. Let \( T : X \to Y \) be a linear operator where \( X \) and \( Y \) are fuzzy strong \( b \)-normed linear spaces. If \( T \) is fuzzy bounded, then the relation (1) is equivalent to the relation

\[
\bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha \} \leq M_\alpha \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \} \quad \forall x \in X.
\]

(2)

Proof. First we show that (1) \( \implies \) (2).

Let \( r > M_\alpha \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \} \)

\[ \implies \frac{r}{M_\alpha} > \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \} \]

\[ \implies \exists \frac{r'}{M_\alpha} \text{ such that } \frac{r}{M_\alpha} > \frac{r'}{M_\alpha} > \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \} \]

\[ \implies \exists \frac{r'}{M_\alpha} \text{ such that } N_1(x, \frac{r'}{M_\alpha}) \geq 1 - \alpha \]

\[ \implies N_2(Tx, K\frac{r'}{M_\alpha}) \geq \alpha \quad \text{(by (1))} \]

\[ \implies \bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha \} \leq r \]

\[ \implies \bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha \} \leq M_\alpha \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \} \quad \forall x \in X. \]

So, (1) \( \implies \) (2).

Now we prove (2) \( \implies \) (1).

Assume that \( N_1(x, \frac{r}{M_\alpha}) \geq 1 - \alpha \). So,

\[ \bigwedge \{ r > 0 : N_1(x, r) \geq 1 - \alpha \} \leq \frac{t}{M_\alpha} \]

\[ \implies M_\alpha \bigwedge \{ r > 0 : N_1(x, r) \geq 1 - \alpha \} \leq t \]

\[ \implies \bigwedge \{ r > 0 : N_2(Tx, Kr) \geq \alpha \} \leq t \quad \text{(by (2))} \]

\[ \implies \text{for any } s > t, \quad \bigwedge \{ r > 0 : N_2(Tx, Kr) \geq \alpha \} < s \]

\[ \implies N_2(Tx, Ks) \geq \alpha. \]

Hence, (2) \( \implies \) (1).
**Remark 3.3.** We denote the collection of all linear operators defined from a fuzzy strong $\phi$-b-normed linear space $(X, N_1, \phi, K, *_1)$ to another $(Y, N_2, \phi, K, *_2)$ by $L(X, Y)$ and for fuzzy bounded linear operators we denote the collection by $BF(X, Y)$.

**Lemma 3.4.** Let $(X, N, \phi, K, *)$ be a fuzzy strong $\phi$-b-normed linear space and the underlying $t$-norm $*$ be continuous at $(1, 1)$. Then for each $\alpha \in (0, 1)$, $\exists \beta \geq \alpha$ such that

$$\bigwedge \{s + t > 0 : N(x + y, K(s + t)) \geq \alpha \} \leq \bigwedge \{s > 0 : N(x, s) \geq \beta \} + \bigwedge \{t > 0 : N(y, t) \geq \beta \}, \forall x, y \in X$$

**Proof.** Since $*$ is continuous at $(1, 1)$, for each $\alpha \in (0, 1)$, we can find $\beta \in (0, 1)$ such that $\beta \ast \beta \geq \alpha$. Again $\beta \geq \beta \ast \beta \geq \alpha \implies \beta \geq \alpha$. Now,

$$\bigwedge \{s > 0 : N(x, s) \geq \beta \} + \bigwedge \{t > 0 : N(y, t) \geq \beta \} = \bigwedge \{s + t > 0 : N(x, s) \geq \beta, N(y, t) \geq \beta \} \geq \bigwedge \{s + t > 0 : N(x, s) \ast N(y, t) \geq \beta \ast \beta \} \geq \bigwedge \{s + t > 0 : N(x + y, s + Kt) \geq \alpha \}$$

Hence $\forall x, y \in X$,

$$\bigwedge \{s + t > 0 : N(x + y, s + Kt) \geq \alpha \} \leq \bigwedge \{s > 0 : N(x, s) \geq \beta \} + \bigwedge \{t > 0 : N(y, t) \geq \beta \} \quad (3)$$

Now, $K \geq 1 \implies Ks \geq s \implies Ks + Kt \geq s + Kt \implies K(s + t) \geq s + Kt$. So,

$$\{s + t > 0 : N(x + y, s + Kt) \geq \alpha \} \subset \{s + t > 0 : N(x + y, K(s + t)) \geq \alpha \} \implies \bigwedge \{s + t > 0 : N(x + y, s + Kt) \geq \alpha \} \geq \bigwedge \{s + t > 0 : N(x + y, K(s + t)) \geq \alpha \}$$

From (3) we get,

$$\bigwedge \{s + t > 0 : N(x + y, K(s + t)) \geq \alpha \} \leq \bigwedge \{s > 0 : N(x, s) \geq \beta \} + \bigwedge \{t > 0 : N(y, t) \geq \beta \}, \forall x, y \in X \quad \Box$$

**Theorem 3.5.** $BF(X, Y)$ (set of all fuzzy bounded linear operators) is a subspace of $L(X, Y)$ (set of all linear operators) where $(X, N_1, \phi, K, *_1)$ and $(Y, N_2, \phi, K, *_2)$ are fuzzy strong $\phi$-b-normed linear spaces and $e_2$ is continuous at $(1, 1)$.

**Proof.** We take $T_1, T_2 \in BF(X, Y)$.

Now by Lemma 3.4, for non-zero scalars $k_1, k_2$, we have

$$\bigwedge \{s + t > 0 : N_2((k_1 T_1 + k_2 T_2)x, K(s + t)) \geq \alpha \} \leq \bigwedge \{s > 0 : N_2(k_1 T_1 x, s) \geq \beta \} + \bigwedge \{t > 0 : N_2(k_2 T_2 x, t) \geq \beta \}, \forall x \in X$$

where $\beta$ depends on $\alpha$ and $\beta \geq \alpha$.

Therefore,

$$\bigwedge \{s + t > 0 : N_2((k_1 T_1 + k_2 T_2)x, K(s + t)) \geq \alpha \} \leq \phi(k_1) \bigwedge \{s > 0 : N_2(T_1 x, s) \geq \beta \} + \phi(k_2) \bigwedge \{t > 0 : N_2(T_2 x, t) \geq \beta \}, \forall x \in X \quad (4)$$

Since $T_1, T_2$ are fuzzy bounded, $\exists M^1_{\beta(s)}$, $M^2_{\beta(s)} > 0$ such that

$$\bigwedge \{s > 0 : N_2(T_1 x, s) \geq \beta \} \leq M^1_{\beta(s)} \bigwedge \{s > 0 : N_1(x, s) \geq 1 - \beta \}, \forall x \in X$$

and

$$\bigwedge \{t > 0 : N_2(T_2 x, t) \geq \beta \} \leq M^2_{\beta(s)} \bigwedge \{t > 0 : N_1(x, t) \geq 1 - \beta \}, \forall x \in X.$$
Now from (4) we have,

$$\bigwedge\{s + t > 0 : N_2((k_1T_1 + k_2T_2)x, K(s + t)) \geq \alpha\} \leq \phi(k_1)M_1^{\beta(\alpha)} \bigwedge\{s > 0 : N_1(x, s) \geq 1 - \beta\} + \phi(k_2)M_2^{\beta(\alpha)} \bigwedge\{t > 0 : N_1(x, t) \geq 1 - \beta\}, \forall x \in X$$

Let $M_\alpha = \phi(k_1)M_1^{\beta(\alpha)} + \phi(k_2)M_2^{\beta(\alpha)}$. Then from above we get,

$$\bigwedge\{s + t > 0 : N_2((k_1T_1 + k_2T_2)x, K(s + t)) \geq \alpha\} \leq M_\alpha \bigwedge\{t > 0 : N_1(x, t) \geq 1 - \beta\}, \forall x \in X \quad (5)$$

Since $\beta \geq \alpha$, so $1 - \alpha \geq 1 - \beta$. Thus,

$$\{t > 0 : N_1(x, t) \geq 1 - \alpha\} \subset \{t > 0 : N_1(x, t) \geq 1 - \beta\} \Rightarrow \bigwedge\{t > 0 : N_1(x, t) \geq 1 - \alpha\} \geq \bigwedge\{t > 0 : N_1(x, t) \geq 1 - \beta\}$$

So from (5) we have,

$$\bigwedge\{s + t > 0 : N_2((k_1T_1 + k_2T_2)x, K(s + t)) \geq \alpha\} \leq M_\alpha \bigwedge\{t > 0 : N_1(x, t) \geq 1 - \alpha\}, \forall x \in X \Rightarrow k_1T_1 + k_2T_2 \in BF(X, Y)$$

Therefore $BF(X, Y)$ is a subspace of $L(X, Y)$.

**Definition 3.6.** An operator $T : (X, N_1, \phi, K, \ast_1) \to (Y, N_2, \phi, K, \ast_2)$ is said to be fuzzy continuous at $x \in X$ if for every sequence $\{x_n\}$ in $X$ with $x_n \to x$ implies $Tx_n \to Tx$. That is $\lim\limits_{n \to \infty} N(x_n - x, t) = 1, \forall t > 0 \Rightarrow \lim\limits_{n \to \infty} N(Tx_n - Tx, t) = 1, \forall t > 0$.

**Theorem 3.7.** Let $T : (X, N_1, \phi, K, \ast_1) \to (Y, N_2, \phi, K, \ast_2)$ be a linear operator where $X$ and $Y$ are fuzzy strong $\phi$-$b$-normed linear spaces. If $T$ is fuzzy continuous at a point $x_0 \in X$, then $T$ is fuzzy continuous everywhere in $X$.

**Proof.** Proof is straightforward (same as in [7]).

**Theorem 3.8.** Let $T : (X, N_1, \phi, K, \ast_1) \to (Y, N_2, \phi, K, \ast_2)$ be a linear operator where $X$ and $Y$ are fuzzy strong $\phi$-$b$-normed linear spaces. If $T$ is fuzzy bounded then it is fuzzy continuous but not conversely.

**Proof.** First we suppose that $T$ is fuzzy bounded. So for each $\alpha \in (0, 1), \exists M_\alpha$ such that

$$\bigwedge\{t > 0 : N_2(Tx, Kt) \geq \alpha\} \leq M_\alpha \bigwedge\{t > 0 : N_1(x, t) \geq 1 - \alpha\}, \forall x \in X$$

Let $\{x_n\}$ be a sequence in $X$ such that $x_n \to x$. Thus,

$$\lim\limits_{n \to \infty} N_1(x_n - x, t) = 1, \forall t > 0.$$ 

Let $\epsilon > 0$ be given. So for each $\alpha \in (0, 1), \exists$ a positive integer $N(\alpha, \epsilon)$ such that

$$N_1(x_n - x, \frac{\epsilon}{2KM_\alpha}) > 1 - \alpha, \forall n \geq N(\alpha, \epsilon)$$

$$\Rightarrow \bigwedge\{t > 0 : N_1(x_n - x, t) \geq 1 - \alpha\} \leq \frac{\epsilon}{2KM_\alpha}, \forall n \geq N(\alpha, \epsilon), \forall \alpha \in (0, 1)$$

$$\Rightarrow M_\alpha \bigwedge\{t > 0 : N_1(x_n - x, t) \geq 1 - \alpha\} \leq \frac{\epsilon}{2K} < \frac{\epsilon}{K}, \forall n \geq N(\alpha, \epsilon), \forall \alpha \in (0, 1)$$

$$\Rightarrow N_2(Tx_n - Tx, \epsilon) \geq \alpha, \forall n \geq N(\alpha, \epsilon), \forall \alpha \in (0, 1)$$

$$\Rightarrow \lim\limits_{n \to \infty} N_2(Tx_n - Tx, \epsilon) = 1$$

Since $\epsilon > 0$ is arbitrary, we have $\lim\limits_{n \to \infty} N_2(Tx_n - Tx, t) = 1, \forall t > 0$. Hence $T$ is fuzzy continuous on $X$.

Converse result may not be true which is justified by the following example.
Example 3.9. Let us consider a normed linear space \( (X, ||\cdot||) \). We define two functions as in the following:

\[
N_1(x, t) = \begin{cases} 
1 & \text{if } t > ||x|| \\
\frac{1}{2} & \text{if } 0 < t < ||x|| \\
0 & \text{if } t \leq 0
\end{cases}
\]

\[
N_2(x, t) = \begin{cases} 
1 & \text{if } t \geq ||x|| \\
\frac{||x||}{t} & \text{if } t > ||x|| \\
0 & \text{if } t < ||x||
\end{cases}
\]

Then \( N_1, N_2 \) are fuzzy norm on \( X \) where * is taken as ‘min’ (please see Observation 1.2 [13]). So \( N_1 \) and \( N_2 \) are strong fuzzy \( \phi \)-b-norm on \( X \) where * is ‘min’, \( \phi(\alpha) = |\alpha| \) \( \forall \alpha \in \mathbb{R} \) and \( K = 1 \).

Now define an linear operator \( T : (X, N_1) \to (X, N_2) \) by \( Tx = 2x \) \( \forall x \in X \). Then it can be shown that \( T \) is fuzzy continuous but not fuzzy bounded (please see Example 3.1 [7]).

**Theorem 3.10.** Let \( T : X \to Y \) be a linear operator where \( (X, N_1, \phi, K, *_1) \) and \( (Y, N_2, \phi, K, *_2) \) be two fuzzy strong \( \phi \)-b-normed linear spaces. If \( X \) is finite dimensional then \( T \) is fuzzy bounded.

**Proof.** Let \( \dim X = n \) and \( \{x_1, x_2, \ldots, x_n\} \) be a basis of \( X \).

Choose \( x(\neq \theta) \in X \). Then \( x = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \) for some suitable scalars \( \{\beta_1, \beta_2, \cdots, \beta_n\} \).

So,

\[
T x = T(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n) \\
= T(\beta_1 x_1) + T(\beta_2 x_2) + \cdots + T(\beta_n x_n)
\]

Let \( s = \sum_{i=1}^{n} |\beta_i| \). Clearly, \( s \neq 0 \).

Let us choose \( \alpha \in (0, 1) \) arbitrary. Now,

\[
\bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1), \frac{t}{n}) \geq \alpha \right\} + \bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_2 x_2), \frac{t}{K} + n) \geq \alpha \right\} + \cdots
\]

\[
\bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha \right\}
\]

\[
= \bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1), \frac{t}{n}) \geq \alpha \right\} + \bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_2 x_2), \frac{t}{K} + n) \geq \alpha \right\} + \cdots
\]

\[
\bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha \right\}
\]

\[
\geq \bigcap \left\{ \frac{t}{n} + \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1) + sT(\beta_2 x_2), \frac{t}{n} + n) \geq \alpha \right\}
\]

\[
+ \bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha \right\}
\]

... ...

\[
\geq \bigcap \left\{ \frac{t}{n} + \cdots + \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1) + \cdots + sT(\beta_n x_n), \frac{t}{n} + \cdots + \frac{t}{n}) \geq \alpha \right\}
\]

\[
= \bigcap \left\{ t > 0 : N_2(T(\beta_1 x_1 + \cdots + \beta_n x_n), \frac{t}{\phi(s)}) \geq \alpha \right\}
\]

Hence we obtain,

\[
\bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1), \frac{t}{n}) \geq \alpha \right\} + \cdots + \bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha \right\} \geq
\]

\[
\bigcap \left\{ t > 0 : N_2(T(\beta_1 x_1 + \cdots + \beta_n x_n), \frac{t}{\phi(s)}) \geq \alpha \right\}
\]

(6)

Let, \( N_\alpha = \max\{\bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1), \frac{t}{n}) \geq \alpha \right\}, \cdots, \bigcap \left\{ \frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha \right\} \} \).

Now from (6), we get

\[
nN_\alpha \geq \bigcap \left\{ t > 0 : N_2(T x, \frac{t}{\phi(s)}) \geq \alpha \right\}
\]

which implies

\[
\frac{nN_\alpha}{\phi(s)} \geq \bigcap \left\{ t > 0 : N_2(T x, t) \geq \alpha \right\}
\]

(7)
From Lemma 2.10, \( \exists c_\alpha > 0 \) such that
\[
\bigwedge \{ t > 0 : N_1(\beta_1 x_1 + \cdots + \beta_n x_n, Kt) \geq 1 - \alpha \} \geq \frac{c_\alpha}{\phi(s)} \tag{8}
\]

From 7 and 8, we have
\[
\frac{nN_\alpha}{c_\alpha} \bigwedge \{ t > 0 : N_1(x, Kt) \geq 1 - \alpha \} \geq \bigwedge \{ t > 0 : N_2(Tx, t) \geq \alpha * \cdots * \alpha \}
\]
that is
\[
\bigwedge \{ t > 0 : N_2(Tx, t) \geq \alpha * \cdots * \alpha \} \leq M_\alpha \bigwedge \{ t > 0 : N_1(x, Kt) \geq 1 - \alpha \} \quad \text{where } M_\alpha = \frac{nN_\alpha}{c_\alpha}
\]
that is
\[
K \bigwedge \{ \frac{t}{K} > 0 : N_2(Tx, K \cdot \frac{t}{K}) \geq \alpha * \cdots * \alpha \} \leq \frac{M_\alpha}{K} \bigwedge \{ Kt > 0 : N_1(x, Kt) \geq 1 - \alpha \}
\]
that is
\[
K \bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha * \cdots * \alpha \} \leq \frac{M_\alpha}{K} \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \}.
\]
Therefore finally we obtain,
\[
\bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha * \cdots * \alpha \} \leq M_\alpha \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \} \tag{9}
\]
where \( M_\alpha = \frac{M_\alpha}{K} \).
Since \( \alpha \geq \alpha * \cdots * \alpha \), so from 9 we have
\[
\bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha * \cdots * \alpha \} \leq M_\alpha \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha * \cdots * \alpha \} \quad \forall x \in X
\]
or
\[
\bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \beta(\alpha) \} \leq M_\alpha \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \beta(\alpha) \} \quad \forall x \in X.
\]
where \( \beta(\alpha) = \alpha * \cdots * \alpha \).
Since \( \alpha \in (0, 1) \) is arbitrary, thus \( T \) is fuzzy bounded.

\[
\square
\]

4 Operator fuzzy norm in \( BF(X, Y) \)

In this section, we define operator fuzzy norm in \( BF(X, Y) \) and finally study the completeness of \( BF(X, Y) \).

**Theorem 4.1.** Let \( (X, N_1, \phi, K, *)_1 \) and \( (Y, N_2, \phi, K, *)_2 \) be two fuzzy strong \( \phi \)-b-normed linear spaces and \( *_2 \) be lower semi-continuous. Let \( BF(X, Y) \) denotes the set of all fuzzy bounded linear operators defined from \( X \) to \( Y \). Then the mapping \( N : BF(X, Y) \times \mathbb{R} \rightarrow [0, 1] \) defined by
\[
N(T, s) = \begin{cases}
\bigvee \{ \alpha \in (0, 1) : \bigvee_{x \in X \setminus \{\emptyset\}} \bigwedge \{ \frac{t}{d_{1-\alpha}} > 0 : N_2(Tx, t) \geq \alpha \} \leq s \} & \text{for } (T, s) \neq (0, 0) \\
0 & \text{for } (T, s) = (0, 0)
\end{cases}
\]
is a fuzzy norm in \( BF(X, Y) \) with respect to underlying t-norm \( *_2 \).

**Proof.** First we show that for \( T \in BF(X, Y) \), \( \bigvee_{x \in X \setminus \{\emptyset\}} \bigwedge \{ \frac{t}{d_{1-\alpha}} > 0 : N_2(Tx, t) \geq \alpha \} \) exists for each \( \alpha \in (0, 1) \) and monotonically increasing with respect to \( \alpha \).
Since \( T \in BF(X, Y) \), for each \( \alpha \in (0, 1) \), \( \exists M_\alpha > 0 \) such that
\[
\bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha \} \leq M_\alpha \bigwedge \{ t > 0 : N_1(x, t) \geq 1 - \alpha \} \quad \forall x \in X
\]
\[
\implies \bigwedge \{ t > 0 : N_2(Tx, Kt) \geq \alpha \} \leq M_\alpha d_{1-\alpha} \quad \forall x \in X
\]
\[
\implies \bigwedge \{ \frac{t}{d_{1-\alpha}} > 0 : N_2(Tx, Kt) \geq \alpha \} \leq M_\alpha \quad \forall x(\neq \emptyset) \in X.
\]
Let \( Kt = r \). Then we get,
\[
\bigwedge \{ \frac{r}{Kd_{1-\alpha}} > 0 : N_2(Tx, r) \geq \alpha \} \leq M_\alpha \quad \forall x(\neq \emptyset) \in X
\]
\[
\implies \bigwedge \{ \frac{r}{d_{1-\alpha}} > 0 : N_2(Tx, r) \geq \alpha \} \leq KM_\alpha \quad \forall x(\neq \emptyset) \in X.
\]
Thus \( \bigvee_{x \in X \setminus \{\theta\}} \bigwedge \left\{ \frac{t}{d_{1-\alpha}} > 0 : N_2(Tx, t) \geq \alpha \right\} \) exists for each \( \alpha \in (0, 1) \).

Next part, that is monotonically increasing with respect to \( \alpha \) is same as the proof in Theorem 4.1 \[7\]. Now we verify that \( N \) satisfies (NI)-(NV).

(i) Proof of (NI) and (NII) are same as in Theorem 4.1 \[7\].

(ii) For any scaler \( \lambda > 0 \), we have

\[
N(\lambda T, s) = \bigvee \{ \alpha \in (0, 1) : \bigvee_{x \in X \setminus \{\theta\}} \bigwedge \left\{ \frac{t}{d_{1-\alpha}} > 0 : N_2((\lambda T)x, t, \frac{t}{\phi(\lambda)}) \geq \alpha \right\} \leq s \}
\]

\[
= \bigvee \{ \alpha \in (0, 1) : \bigvee_{x \in X \setminus \{\theta\}} \bigwedge \left\{ \frac{t}{d_{1-\alpha}} > 0 : N_2(Tx, t) \geq \alpha \right\} \leq s \}
\]

\[
= \bigvee \{ \alpha \in (0, 1) : \bigvee_{x \in X \setminus \{\theta\}} \bigwedge \left\{ \frac{t}{d_{1-\alpha}} > 0 : N_2(Tx, t) \geq \alpha \right\} \leq \frac{s}{\phi(\lambda)} \}
\]

\[
= N(T, \frac{s}{\phi(\lambda)})
\]

So, (NIII) holds.

(iii) We have to show that

\[
N(T_1 + T_2, s + Kt) \geq N(T_1, s) \ast_2 N(T_2, t) \quad \forall s, t \in \mathbb{R}.
\]

If possible suppose that the above relation does not hold. So \( \exists s_0, t_0 \in \mathbb{R} \) such that

\[
N(T_1 + T_2, s_0 + Kt_0) < N(T_1, s_0) \ast_2 N(T_2, t_0).
\]

Choose \( \alpha_0 \in (0, 1) \) such that

\[
N(T_1 + T_2, s_0 + Kt_0) < \alpha_0 < N(T_1, s_0) \ast_2 N(T_2, t_0). \quad (10)
\]

Since \( \ast_2 \) is lower semi-continuous, \( \exists \alpha_1, \alpha_2 \in (0, 1) \) where \( N(T_1, s_0) > \alpha_1 \) and \( N(T_2, t_0) > \alpha_2 \) such that \( \alpha_1 \ast_2 \alpha_2 > \alpha_0 \). Now,

\[
N(T_1, s_0) > \alpha_1 \implies \bigvee_{x \in X \setminus \{\theta\}} \bigwedge \left\{ \frac{s}{d_{1-\alpha_1}} > 0 : N_2(T_1x, s) \geq \alpha_1 \right\} \leq s_0
\]

\[
\implies \bigwedge \left\{ \frac{s}{d_{1-\alpha_1}} > 0 : N_2(T_1x, s) \geq \alpha_1 \right\} \leq s_0 \quad \forall x(\neq \theta) \in X.
\]

Similarly,

\[
K \bigwedge \left\{ \frac{t}{d_{1-\alpha_2}} > 0 : N_2(T_2x, t) \geq \alpha_2 \right\} \leq Kt_0 \quad \forall x(\neq \theta) \in X
\]

\[
\implies \bigwedge \left\{ \frac{Kt}{d_{1-\alpha_2}} > 0 : N_2(T_2x, \frac{Kt}{K}) \geq \alpha_2 \right\} \leq Kt_0 \quad \forall x(\neq \theta) \in X
\]

\[
\implies \bigwedge \left\{ \frac{t}{d_{1-\alpha_2}} > 0 : N_2(T_2x, \frac{t}{K}) \geq \alpha_2 \right\} \leq Kt_0 \quad \forall x(\neq \theta) \in X.
\]

Since \( \alpha_1 \geq \alpha_1 \ast_2 \alpha_2 > \alpha_0 \), so \( 1 - \alpha_0 > 1 - \alpha_1 \). Similarly, \( 1 - \alpha_0 > 1 - \alpha_2 \). Thus we get,

\[
\bigwedge \left\{ \frac{s + t}{d_{1-\alpha_0}} > 0 : N_2(T_1x, s) \geq \alpha_1, \quad N_2(T_2x, \frac{t}{K}) \geq \alpha_2 \right\} \leq s_0 + Kt_0 \quad \forall x(\neq \theta) \in X.
\]
Therefore,
\[
\bigwedge \left\{ \frac{s + t}{d_{1 - \alpha_0}} > 0 : N_2((T_1 + T_2)x, s + K \frac{t}{K}) \geq \alpha_1 \ast_2 \alpha_2 \right\} \leq s_0 + K t_0 \ \forall x(\neq \theta) \in X
\]
\[
\Rightarrow \bigwedge \left\{ \frac{s + t}{d_{1 - \alpha_0}} > 0 : N_2((T_1 + T_2)x, s + t) \geq \alpha_1 \ast_2 \alpha_2 \right\} \leq s_0 + K t_0 \ \forall x(\neq \theta) \in X
\]
\[
\Rightarrow \bigvee_{x \in X \setminus \{ \theta \}} \bigwedge \left\{ \frac{s + t}{d_{1 - \alpha_0}} > 0 : N_2((T_1 + T_2)x, s + t) \geq \alpha_0 \right\} \leq s_0 + K t_0
\]
\[
\Rightarrow N(T_1 + T_2, s_0 + K t_0) \geq \alpha_0.
\]
This contradicts the relation (10).

Hence (NIV): \( N(T_1 + T_2, s + K t) \geq N(T_1, s) \ast_2 N(T_2, t) \ \forall s, t \in \mathbb{R} \) holds.

(iv) The proof of the condition (NV) is same as the proof of the (NV) in Theorem 4.1 [7].

\[\square\]

**Proposition 4.2.** Let \((X, N, \phi, K, *)\) be a fuzzy strong \(\phi\)-b-normed linear spaces and \(*\) be lower semi-continuous. Then limit of every l-fuzzy convergent sequence in \(X\) is unique.

**Proof.** Let \(\beta \in (0, 1)\). By the lower semi-continuity of \(*\), \(\exists \alpha \in (0, 1)\) such that \((1 - \alpha) * (1 - \alpha) > (1 - \beta)\).

Let \(\{x_n\}\) be an l-fuzzy convergent sequence in \(X\) which converges to two different limits \(x\) and \(y\). So,

\[
\lim_{n \to \infty} \bigwedge \left\{ t > 0 : N(x_n - x, t) > 1 - \alpha \right\} = 0, \ \forall \alpha \in (0, 1)
\]

and

\[
\lim_{n \to \infty} \bigwedge \left\{ t > 0 : N(x_n - y, t) > 1 - \alpha \right\} = 0, \ \forall \alpha \in (0, 1).
\]

Then for \(\epsilon > 0\), \(\exists\) positive integers \(N_1(\alpha, \epsilon)\) and \(N_2(\alpha, \epsilon)\) such that

\[
\bigwedge \left\{ t > 0 : N(x_n - x, t) > 1 - \alpha \right\} < \frac{\epsilon}{2K}, \ \forall n \geq N_1(\alpha, \epsilon)
\]

and

\[
\bigwedge \left\{ t > 0 : N(x_n - y, t) > 1 - \alpha \right\} < \frac{\epsilon}{2}, \ \forall n \geq N_2(\alpha, \epsilon).
\]

Let \(N_0 = \max\{N_1, N_2\}\). Then,

\[
N(x_n - x, \frac{\epsilon}{2K}) > 1 - \alpha, \ \forall n \geq N_0(\alpha, \epsilon)
\]

and

\[
N(x_n - y, \frac{\epsilon}{2}) > 1 - \alpha, \ \forall n \geq N_0(\alpha, \epsilon).
\]

Now,

\[
N(x - y, \epsilon) = N(x_n - y - x_n, x_n + \frac{\epsilon}{2} + K \cdot \frac{\epsilon}{2K})
\]

\[
\geq N(x_n - y, \frac{\epsilon}{2}) \ast N(x_n - x, \frac{\epsilon}{2K})
\]

\[
>(1 - \alpha) \ast (1 - \alpha) > (1 - \beta).
\]

Since, \(\epsilon > 0\) and \(\beta \in (0, 1)\) are arbitrary, so it implies that

\[
N(x - y, t) > \alpha, \ \forall t > 0, \ \alpha \in (0, 1)
\]

\[
\Rightarrow N(x - y, t) = 1, \ \forall t > 0
\]

\[
\Rightarrow x - y = \theta
\]

\[
\Rightarrow x = y.
\]

This completes the proof. \[\square\]
Theorem 4.3. Let \((X, N_1, \phi, K, \ast_1)\) and \((Y, N_2, \phi, K, \ast_2)\) be two fuzzy strong \(\phi\)-\(b\)-normed linear spaces and \(\ast_2\) be lower semi-continuous. If \((Y, N_2, \phi, K, \ast_2)\) is \(l\)-fuzzy complete then \(BF(X, Y)\) is also \(l\)-fuzzy complete with respect to the underlying \(t\)-norm \(\ast_2\).

Proof. Let \(\{T_n\}\) be an \(l\)-fuzzy Cauchy sequence in \(BF(X, Y)\), then \(\{T_nx\}\) is an \(l\)-fuzzy Cauchy sequence in \(Y\) for each \(x \in X\) (Proof is similar to the proof of Theorem 5.1 \[7\]).

Since \(Y\) is \(l\)-fuzzy complete, so for each \(x \in X\), \(\exists y \in Y\) such that \(\lim_{n \to \infty} T_nx = y\).

Thus we can define a function \(T\) given by \(\lim_{n \to \infty} T_nx = Tx\).

So, \(\lim_{n \to \infty} \bigcap \{t > 0 : N_2(T_nx - Tx, t) > 1 - \alpha\} = 0 \ \forall x \in X, \ \forall \alpha \in (0, 1)\). (11)

First we show that \(T\) is linear.

Now (11) implies that, for \(\epsilon > 0\), \(\exists N_1(\alpha, \epsilon)\) such that

\[
\bigcap \{t > 0 : N_2(T_n x - Tx, t) > 1 - \alpha\} \leq \frac{\epsilon}{2} \ \forall x \in X, \forall n \geq N_1(\alpha, \epsilon)
\]

\[
\implies N_2(T_n x - Tx, \frac{\epsilon}{2}) \geq 1 - \alpha \ \forall x \in X, \forall n \geq N_1(\alpha, \epsilon).
\]

Similarly,

\[
N_2(T_n y - Ty, \frac{\epsilon}{2K}) > 1 - \alpha \ \forall y \in X, \forall n \geq N_2(\alpha, \epsilon).
\]

Thus,

\[
N_2(T_n(x + y) - (Tx + Ty), \frac{\epsilon}{2}) > (1 - \alpha) \ast_2 (1 - \alpha), \ \forall n \geq \max\{N_1, N_2\}.
\]

So we get,

\[
N_2(T_n(x + y) - (Tx + Ty), \frac{\epsilon}{2}) > (1 - \alpha) \ast_2 (1 - \alpha), \ \forall n \geq \max\{N_1, N_2\}.
\]

Let \(\beta \in (0, 1)\). Then \(\exists \alpha = \alpha(\beta) \in (0, 1)\) such that \((1 - \alpha) \ast_2 (1 - \alpha) > (1 - \beta)\).

Thus we have,

\[
N_2(T_n(x + y) - (Tx + Ty), \epsilon) > (1 - \beta) \ \forall n \geq \max\{N_1, N_2\}
\]

\[
\implies \bigcap \{t > 0 : N_2(T_n(x + y) - (Tx + Ty), t) > 1 - \beta\} \leq \epsilon \ \forall n \geq \max\{N_1, N_2\}
\]

\[
\implies \lim_{n \to \infty} \bigcap \{t > 0 : N_2(T_n(x + y) - (Tx + Ty), t) > 1 - \beta\} = 0.
\]

Since \(\beta \in (0, 1)\) is arbitrary, it follows that \(\{T_n(x + y)\}\) is \(l\)-fuzzy convergent and converges to \((Tx + Ty)\). So,

\[
\lim_{n \to \infty} T_n(x + y) = Tx + Ty\quad \text{that is} \quad T(x + y) = Tx + Ty.
\]

Again for any scalar \(\lambda\) we have,

\[
\lim_{n \to \infty} T_n(\lambda x) = T(\lambda x) \ \forall x \in X
\]

\[
\implies \lim_{n \to \infty} \{\lambda T_n x\} = T(\lambda x) \ \forall x \in X
\]

\[
\implies \lambda \lim_{n \to \infty} T_n x = T(\lambda x) \ \forall x \in X
\]

\[
\implies T(\lambda x) = \lambda (Tx) \quad \forall x \in X.
\]

Hence \(T\) is linear.

Now we show that \(T\) is fuzzy bounded.

Let \(\gamma \in (0, 1)\). By the lower semi-continuity of \(\ast_2\), \(\exists \alpha = \alpha(\gamma) \in (0, 1)\) such that \((1 - \alpha) \ast_2 (1 - \alpha) > \gamma)\).

Now \(\{T_n\}\) is an \(l\)-fuzzy Cauchy sequence in \(BF(X, Y)\), so

\[
\lim_{m,n \to \infty} \bigcap \{t > 0 : N_2(T_m - T_n, t) > 1 - \alpha\} = 0, \ \forall \alpha \in (0, 1).
\]
Thus for a given \( \epsilon > 0 \) and for \( \alpha \in (0, 1) \), there exists positive integer \( N(\alpha, \epsilon) \) such that

\[
\bigwedge \{ t > 0 : N(T_m - T_n, t) > 1 - \alpha \} < \frac{\epsilon}{2} \quad \forall m, n \geq N(\alpha, \epsilon)
\]

which implies

\[
\bigvee_{x \in X \setminus \{ \theta \}} \bigwedge \{ \frac{\epsilon}{d_0} > 0 : N_2(T_n x - T_m x, s) \geq 1 - \alpha \} \leq \frac{\epsilon}{2} \quad \forall m, n \geq N(\alpha, \epsilon).
\]  \hspace{1cm} (12)

From (12), for \( \alpha = (\alpha(\gamma)) \in (0, 1), \epsilon > 0, \exists N'(\alpha(\gamma), \epsilon) \in \mathbb{N} \) such that

\[
\bigwedge \{ \frac{\epsilon}{d_0} > 0 : N_2(T_n x - T_m x, s) \geq 1 - \alpha \} < \frac{\epsilon}{2} \quad \forall m, n \geq N'(\alpha(\gamma), \epsilon), \quad x(\neq \theta) \in X
\]

which implies

\[
N_2(T_n x - T_m x, \frac{d_0 \epsilon}{2}) \geq 1 - \alpha, \quad \forall m, n \geq N'(\alpha(\gamma), \epsilon, x), \quad x(\neq \theta) \in X.
\]  \hspace{1cm} (13)

From (11), it follows that, for \( \alpha = \alpha(\gamma) \in (0, 1), \epsilon > 0, \exists (\neq \theta) \in X, \exists N''(\alpha(\gamma), \epsilon, x) \in \mathbb{N} \) such that

\[
\bigwedge \{ t > 0 : N_2(T_m x - T x, t) > 1 - \alpha \} < \frac{d_0 \epsilon}{2K} \quad \forall m \geq N''(\alpha(\gamma), \epsilon, x).
\]  \hspace{1cm} (14)

Let \( N_0(\alpha(\gamma), \epsilon, x) = \max\{ N'(\alpha(\gamma), \epsilon, x), N''(\alpha(\gamma), \epsilon, x) \} \). Then from (13) and (14) we have,

\[
N_2(T_n x - T x, d_0 \epsilon) = N_2(T_n x - T_m x + T_m x - T x, \frac{d_0 \epsilon}{2} + K \cdot \frac{d_0 \epsilon}{2K})
\]

\[
\geq N_2(T_n x - T_m x, \frac{d_0 \epsilon}{2}) \ast_2 N_2(T_m x - T x, \frac{d_0 \epsilon}{2K})
\]

\[
\geq (1 - \alpha) \ast_2 (1 - \alpha) \geq \gamma \quad \forall n \geq N_0(\alpha(\gamma), \epsilon), \quad \forall x(\neq \theta) \in X.
\]

which implies

\[
\bigwedge \{ t > 0 : N_2(T_n x - T x, t) \geq \gamma \} \leq \epsilon d_0 \leq \epsilon d_1 - \gamma \quad \forall n \geq N_0(\alpha(\gamma), \epsilon).
\]  \hspace{1cm} (15)

Hence

\[
\bigwedge \{ t > 0 : N_2(T_n x - T x, K t) \geq \gamma \} \leq \frac{\epsilon}{K} \bigwedge \{ s > 0 : N_1(x, s) \geq 1 - \gamma \} \quad \forall n \geq N_0(\alpha(\gamma), \epsilon).
\]  \hspace{1cm} (16)

This shows that \( T_n - T \) is fuzzy bounded \( \forall n \geq N_0(\alpha(\gamma), \epsilon) \).

Since \( BF(X, Y) \) is a linear space, so \( T = (T - T_n) + T_n \in BF(X, Y) \). Thus \( T \) is fuzzy bounded.

Again from (15), we get

\[
\bigvee_{x \in X \setminus \{ \theta \}} \bigwedge \{ \frac{t}{d_1 - \gamma} > 0 : N_2(T_n x - T x, t) \geq \gamma \} \leq \epsilon \quad \forall m, n \geq N_0(\alpha(\gamma), \epsilon)
\]

\[
\Rightarrow N(T_n - T, \epsilon) \geq \gamma \quad \forall n \geq N_0(\alpha(\gamma), \epsilon)
\]

\[
\bigwedge \{ s > 0 : N(T_n - T, s) \geq \gamma \} \leq \epsilon \quad \forall n \geq N_0(\alpha(\gamma), \epsilon)
\]

Since \( \epsilon > 0 \) is arbitrary, so from above we have,

\[
\lim_{n \to \infty} \bigwedge \{ s > 0 : N(T_n - T, s) \geq \gamma \} = 0.
\]

Since \( \gamma \in (0, 1) \) is arbitrary, so it follows that

\[
\lim_{n \to \infty} \bigwedge \{ s > 0 : N(T_n - T, s) \geq \alpha \} = 0 \quad \forall \alpha \in (0, 1).
\]

So \( \{ T_n \} \) is l-fuzzy convergent and converges to \( T \in BF(X, Y) \). Thus \( BF(X, Y) \) is l-fuzzy complete.
Conclusion: In this paper, we are able to extend some concepts of fuzzy normed linear spaces to fuzzy strong $\phi$-b-normed linear spaces. Definitions of fuzzy bounded linear operator and fuzzy continuous operator are given and their relation is studied. Idea of operator norm in BF(X,Y) is introduced and completeness of BF(X,Y) is established.

There are huge scope for development in continuation with the results of this manuscript such as four fundamental theorems of functional analysis, uniform convex and strictly convex spaces in fuzzy setting.

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