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LS condition for filled Julia sets in $\mathbb{C}$

Frédéric PROTIN

Abstract In this article we derive an inequality of Lojasiewicz-Siciak type for certain sets arising in the context of the complex dynamics in dimension 1. More precisely, if we denote by $\text{dist}$ the euclidian distance in $\mathbb{C}$, we show that the Green function $G_K$ of the filled Julia set $K$ of a polynomial such that $\bar{K} \neq \emptyset$ satisfies the so-called LS condition $G_{A} \geq c \cdot \text{dist}(\cdot, K)^{c'}$ in a neighborhood of $K$, for some constants $c, c' > 0$. Relatively few examples of compact sets satisfying the LS condition are known. Our result highlights an interesting class of compact sets fulfilling this condition. For instance, this is the case for the filled Julia sets of quadratic polynomials of the form $z \mapsto z^2 + a$, provided that the parameter $a$ is hyperbolic, Siegel or parabolic. In particular, this latter case provides an example of a non semi-algebraic compact set in $\mathbb{C}$ which has a cusp and satisfies however the LS condition. In order to prove our main result, we define and study the set of obstruction points to the LS condition. We also prove, in dimension $n \geq 1$, that for a polynomially convex and $L$-regular compact set of non empty interior, these obstruction points are rare, in a sense which will be specified.

Keywords LS condition · Green function · pluricomplex Green function · complex dynamics · filled Julia set · potential theory

Mathematics Subject Classification (2000) 37F50 · 37F10 · 31C99 · 32U35

1 Introduction

We call pluricomplex Green function $G_A$ of a compact set $A \subset \mathbb{C}^n$, $n \geq 1$, the plurisubharmonic function defined as

$$G_A := \sup \{ v \in PSH(\mathbb{C}^n) : \| v \|_A_\leq 0, \ v(z) \leq \frac{1}{2} \log(1 + \| z \|^2) + O(1) \}.$$
where sup* denotes the upper semi-continuous regularization of the upper envelope, and \( PSH(C^n) \) denotes the set of plurisubharmonic functions in \( C^n \). The set \( A \) is called \( L\)-regular if \( G_A \) is continuous. In this case, the set \( \{G_A = 0\} \) is the polynomially convex envelope \( \hat{A} \) of \( A \). We also consider, for an open bounded set \( U \subset C^n \), the Green function of \( A \subset U \) relative to \( U \) defined by

\[
G_{A,U} := \sup^* \{ v \in PSH(U) : v \leq 1, v|_A \leq 0 \}.
\]

The reader should pay attention to the fact that in [9] the Green function of \( A \) relative to \( U \) is defined as \( G_{A,U} - 1 \).

Let \( U_a := \{G_A < a\} \) for \( a \in \mathbb{R}^+ \setminus \{0\} \). If \( A \) is not pluripolar and \( \hat{A} \subset U_a \), then a relation between \( G_A \) and \( G_{A,U_a} \) holding in \( U_a \) is given by Proposition 5.3.3 in [9]:

\[
G_A = G_{A,U_a}.
\]  

(1)

A compact \( A \subset C^n \) is said to satisfy the LS condition if there exists an open set \( U \) containing it and two constants \( c, c' > 0 \) such that its pluricomplex Green function \( G_A \) verifies the following regularity condition :

\[
\forall z \in U, G_A(z) \geq c \cdot \text{dist}(z, A)^{c'},
\]

where \( \text{dist} \) denotes the euclidean distance (see for instance [4] or [2]).

On a compact set \( A \subset C^n \) verifying the LS condition, as well as the HCP condition (i.e. the Hölderian continuity of \( G_A \), for example a semi-algebraic compact set), we have the rapid approximation property of continuous functions by polynomials. Relatively few examples of compacts satisfying the LS condition are known. Some examples are given in [13]. Let us also note that Pierzchala showed in [14] that a compact verifying the LS condition is polynomially convex. Białas and Kosek [3] construct such sets using holomorphic dynamics.

Along the same vein, we show that the so-called filled Julia sets in \( C \) satisfy the LS condition. More precisely, our main goal is to show the following result concerning the filled Julia set of a polynomial \( f : C \to C \), i.e. the set of points \( z \in C \) whose orbit \( (f^n(z))_n \) is bounded :

**Theorem.** The filled Julia set of a polynomial \( f : C \to C \) of degree \( \geq 2 \), if its interior is non empty, satisfies the LS condition.

Recall that a compact set in \( C \) is polynomially convex if and only if its complement is connected, so the filled Julia set of a polynomial is polynomially convex. The differentials operators \( \partial \) and \( \overline{\partial} \) will be understood in the sense of currents. Recall that a continuous function \( u \) from an open set of \( C^n \) into \( \mathbb{R} \) is pluriharmonic (harmonic if \( n = 1 \)) if and only if \( \partial \overline{\partial} u = 0 \) (see for example Theorem 2.28 in [12]).

In Section 2, we recall some definitions and elementary facts about holomorphic dynamics in one dimension, and we give a useful lemma concerning the regularity of filled Julia sets. More precisely, this lemma shows that the filled Julia set \( K \) of a polynomial of degree \( d \geq 2 \) with non-empty interior satisfies \( \overline{K} = K \). In Section 3, we define in \( C^n \), \( n \geq 1 \), the set of obstruction points to the LS condition, and
we prove that the complement of this set is big, in a sense which will be specified.
We also study explicitly the LS condition on several examples of compact sets in \( \mathbb{C} \). Section 4 is devoted to the proof of the main theorem previously stated.

2 Dynamics in \( \mathbb{C} \)

We start by recalling some definitions related to one-dimensional holomorphic dynamics. Let us consider a polynomial \( f : \mathbb{C} \to \mathbb{C} \) of degree \( d \geq 2 \).

We call \textit{Fatou set} of \( f \), denoted \( \mathcal{F} \), the largest open subset in which the family of iterations \( f^n \) is equicontinuous.
The \textit{Julia set} of \( f \), denoted \( \mathcal{J} \), is the complement of \( \mathcal{F} \) in \( \mathbb{C} \). Let us note for what follows that \( \mathcal{J} \) is not a polar set.

We call \textit{filled Julia set} of \( f \) the set \( \mathcal{K} \) of points \( z \in \mathbb{C} \) whose orbit \( (f^n(z))_n \) is bounded. Note that \( \mathcal{K} \) is compact, as \( \infty \) is a superattractive fixed point of \( f \), hence belonging to \( \mathcal{F} \). The complement of \( \mathcal{K} \) is the basin of attraction of infinity. We have \( \partial \mathcal{K} = \mathcal{J} \) and \( G_{\mathcal{K}} = G_{\mathcal{J}} \).

There are many situations where the set \( \mathcal{K} \) is of non-empty interior. Consider, for instance, the case where \( f(z) = z^2 + a \) with \( a \in \mathbb{C} \). By Sullivan’s classification theorem (see e.g. Theorem 2.1 in \cite{5} or Theorem 3.2 of \cite{11}), we can distinguish three cases where \( \mathcal{K} \neq \emptyset \). The first case is when \( a \) is chosen in the interior of the Mandelbrot set in such a way that \( f \) is \textit{hyperbolic} in the sense of \cite{5} p. 89. By Theorem 4.7 in \cite{11}, \( f \) is hyperbolic if and only if some iterate \( f^k \) of \( f \) has a fixed point \( z_0 \in \mathbb{C} \) for which \( |(f^k)'(z_0)| < 1 \). The second case is when \( a \) is chosen on the boundary of the Mandelbrot set such that some iterate \( f^k \) of \( f \) has a fixed point \( z_0 \in \mathbb{C} \) for which \( (f^k)'(z_0) \) is a root of the unity. By Theorem 6.5.10 of \cite{1} and Theorem 4.8 of \cite{11}, this corresponds to the \textit{parabolic} case in the Sullivan’s classification. By Theorem 4.8 of \cite{11}, the last case is when \( a \) is chosen on the boundary of the Mandelbrot set such that \( \mathcal{K} \) contains a \textit{Siegel disk} and all its preimages in the sense of Definition 7.1.1 of \cite{1}.

We construct the subharmonic function \( G : \mathbb{C} \to \mathbb{R}^+ \), limit in \( L^1_{\text{loc}} \) of the sequence \( (\log(1 + |f^n|)/d^n)_n \) (see \cite{8} for a general construction). It is known that \( G \) is continuous (and even H"olderian \cite{10}, see also Theorem 3.2 of \cite{5}), harmonic in \( \mathcal{F} \), that it verifies \( G(z) = 0 \) if and only if \( z \in \mathcal{K} \), and also that \( G(z) - \log |z| = O(1) \) at infinity. By uniqueness, \( G \) is therefore the pluricomplex Green function of \( \mathcal{K} \) (and of \( \mathcal{J} \)). It satisfies by construction the invariance property

\[ G \circ f = d \cdot G. \] (2)

The measure \( \frac{1}{2} \partial \partial G \) is a probability measure of support exactly \( \mathcal{J} \) (see e.g. \cite{7}). We will use the following preliminary lemma about filled Julia sets.

\textbf{Lemma 1} \textit{The filled Julia set} \( \mathcal{K} \) \textit{of a polynomial of degree} \( d \geq 2 \) \textit{with non-empty interior satisfies} \( \overline{\mathcal{K}} = \mathcal{K} \).

\textbf{Proof} Suppose, by contradiction, that there exists \( x \in \partial \mathcal{K} \) having a neighborhood \( U \) which does not intersect \( \mathcal{K} \). Then there exists \( n_0 \in \mathbb{N} \) such that \( \mathcal{K} \subset f^{n_0}(U) \) (see for example Theorem 4.2.5. of \cite{1}). But this contradicts the fact that \( f^{n_0}(U \cap \mathcal{K}) \subset \partial \mathcal{K} \). Thus every open subset of \( \mathbb{C} \) intersecting \( \mathcal{J} = \partial \mathcal{K} \) also intersects \( \mathcal{K} \). In other words, \( \overline{\mathcal{K}} = \mathcal{K} \). \( \square \)
3 Study of the obstruction to the LS condition

For \( n \geq 1 \), let

\[
O_c := \{ z \in \mathbb{C}^n : \text{dist}(z, A) < 1, G_A(z) < c \cdot \text{dist}(z, A)^{1/c} \}. \tag{3}
\]

Note that the sequence of open sets \( O_c \) is increasing with \( c \) for \( c < 1 \). The LS condition is satisfied by a compact non-pluripolar set \( A \subset \mathbb{C}^n \), L-regular and polynomially convex, if and only if the set

\[
I := \bigcap_{c>0} \overline{O_c} \subset \partial A \tag{4}
\]

is empty. We call \( I \) the set of obstruction points to the LS condition.

**Example 1** ([3]) If \( A \) is the union of two disks of radius 1, tangent to each other at the origin, then it does not satisfy the LS condition; the set of obstruction points to the LS condition is \( I = \{ 0 \} \neq \emptyset \).

**Example 2** The previous set \( A \) is mapped by the function \( g : z \rightarrow z^2 \) onto a filled cardioid \( C \), and we have \( g^{-1}(C) = A \). We deduce from Theorem 5.3.1 of [9] that the set of obstruction points to the LS condition for \( C \) is \( I = \{ 0 \} \neq \emptyset \).

**Example 3** (see also [4]). For \( \varepsilon \in ]0,1[ \) fixed, consider the sets \( L_\varepsilon := \{(1+i)t, t \in [-\varepsilon, \varepsilon]\} \), \( L'_\varepsilon := \{(1-i)t, t \in [-\varepsilon, \varepsilon]\} \), and \( X_\varepsilon := L_\varepsilon \cup L'_\varepsilon \subset B(0,2\varepsilon) \). We show that \( X_\varepsilon \) satisfies the LS condition, i.e. \( I = \emptyset \). Indeed, the function \( g : \mathbb{C} \rightarrow \mathbb{C} \) defined by \( g(z) = \frac{1}{2}z^2 \) maps \( X_\varepsilon \) onto \( [-\varepsilon^2, \varepsilon^2] \). On the other hand, \( g^{-1}([-\varepsilon^2, \varepsilon^2]) = X_\varepsilon \). Theorem 5.3.1 of [9] implies \( G_{X_\varepsilon} = G_{[-\varepsilon^2, \varepsilon^2]} \circ g \). Since the segment \( [-\varepsilon^2, \varepsilon^2] \) is convex, it satisfies the LS condition (see [4]). Moreover, it follows from Theorem 1 in [6] that \( \forall z \in \mathbb{C} \),

\[
\text{dist} \left( g(z), [-\varepsilon^2, \varepsilon^2] \right) \geq \frac{1}{4}|z| \text{dist} \left( z, X_\varepsilon \right) \geq \frac{1}{4} \text{dist} \left( z, X_\varepsilon \right)^2.
\]

We deduce that \( X_\varepsilon \) also satisfies the LS condition.

The following result provides more insight into the structure of the complement of \( O_c \). We prove it for any \( n \geq 1 \). Recall that, given an open set \( U \subset \mathbb{C}^n \), a set \( E \subset U \) is called pluripolar if for each \( a \in E \) there exist a neighborhood \( V \subset U \) of \( a \) and a pluriharmonic function \( v : V \rightarrow \mathbb{R} \cup \{-\infty\} \) such that \( E \cap V \subset \{ v = -\infty \} \).

**Proposition 1** Let \( A \subset \mathbb{C}^n \), \( n \geq 1 \), be a non-pluripolar, L-regular and polynomially convex compact set. Suppose that the pluricomplex Green function \( G_A \) is pluriharmonic outside of \( A \) (harmonic if \( n = 1 \)).

Then, there exists \( c_0 > 0 \) such that \( \forall c \in [0, c_0], \partial A \) is included in the boundary of the open set \( \{ z \in \mathbb{C}^n : G_A(z) > c \cdot \text{dist}(z, A)^{1/c} \} \).

**Proof** Let \( \mu \) denote the positive measure \( \frac{i}{\pi} \partial G_A \wedge \omega^{n-1} \) on \( \mathbb{C}^n \), where

\[
\omega := \frac{i}{2\pi} \partial \overline{\partial} \log(1 + |z|^2)
\]

is the Fubini-Study form. Note that the support of the measure \( \mu \) is exactly \( \partial A \). Indeed, \( \text{supp}(\mu) \subset \partial A \) since \( \frac{i}{\pi} \partial \overline{\partial} G_A = 0 \) in \( \mathbb{C}^n \setminus \partial A \) by hypothesis. On the other
hand, if there existed $x \in \partial A \setminus \text{supp}(\mu)$, then $G_A$ would be (pluri)harmonic in a neighborhood of $x$, hence null in this neighborhood, which can not happen because $A$ is polynomially convex.

Let us suppose by contradiction that $\forall c_0 > 0, \exists x \in [0, c_0], \exists r > 0, B(x, r) \cap \{z \in \mathbb{C}^n, G_A(z) > c \cdot \text{dist}(z, A)^{1/c}\} = \emptyset.$

Thus we can take $c' \in \left[0, \frac{1}{4n}\right], x' \in \partial A$, and $r' > 0$, such that

$$G_A(z) \leq c' \cdot \text{dist}(z, A)^{1/n}, \forall z \in B(x', r').$$

Denote $r_0 := \frac{c'}{2}$. Let us establish the following Chern-Levine-Nirenberg-type inequality : $\forall r < r_0, \forall x \in B(x', r_0) \cap \partial A,$

$$\mu(B(x, r)) \leq k \cdot r^{-2n} \sup_{B(x, 2r)} G_A \leq c' \cdot k \cdot (2r)^{-2n}, \quad (5)$$

for some constant $k > 0$ independent of $r, r_0, x$ and $c'$. Let indeed $\xi : \mathbb{C}^n \to \mathbb{R}^+$ be a positive test function $\equiv 1$ in $B(0, 1)$ and having its support in $B(0, 2)$. There exists a decreasing sequence $(G_n)_n$ of $C^\infty$ pluri-subharmonic functions converging towards $G_A$ (Theorem 2.9.2 in [9]). Theorem 3.4.3 in [9] and Stokes’ theorem imply that $\forall r < r_0, \forall x \in B(x', r_0) \cap \partial A,$

$$\mu(B(x, r)) \leq \int_{\mathbb{C}^n} \xi \left(\frac{z_1}{r}, ..., \frac{z_n}{r}\right) d\mu(z_1, ..., z_n) = \lim_{m \to +\infty} \int_{\mathbb{C}^n} \frac{G_m}{r^{2n}} (\partial \xi) \left(\frac{z_1}{r}, ..., \frac{z_n}{r}\right) \wedge \omega^{n-1}.$$ 

Then the monotone convergence theorem implies that $\forall r < r_0, \forall x \in B(x', r_0) \cap \partial A,$

$$\mu(B(x, r)) \leq \int_{\mathbb{C}^n} \frac{G_A}{r^{2n}} (\partial \xi) \left(\frac{z_1}{r}, ..., \frac{z_n}{r}\right) \wedge \omega^{n-1} \leq k \cdot r^{-2n} \sup_{B(x, 2r)} G_A,$$

where $k$ depends only on the sum of the supremum norms of the coefficients of the differential form $\partial \xi$. Therefore, (5) holds.

With the notation $\nu := \frac{\mu}{\mu(B(x', r_0))} 1_{B(x', r_0)}$, where $1_{B(x', r_0)}$ is the characteristic function of $B(x', r_0)$, the measure $\nu$ is a probability measure, and we can rewrite (5) : $\forall r > 0, \forall x \in B(x', r_0) \cap \partial A,$

$$\nu(B(x, r)) \leq \frac{c' \cdot k}{\mu(B(x', r_0))} \cdot (2r)^{-2n}.$$ 

Then, by Frostman Lemma (see for example Lemma 10.2.1 in [1]), the Hausdorff dimension of $\partial A \cap B(x_0, r_0)$ is strictly greater than $2n$ for our choice $c' < \frac{1}{17}$, which gives a contradiction. (Recall that Frostman Lemma ensures that, if $m$ is a probability measure on a metric space $E$ verifying $m(B(x, r)) < q \cdot r^n$ for all $x \in E, r > 0$, with fixed $q > 0, \alpha > 0$, then the Hausdorff dimension of $E$ is greater than $\alpha$).

We thus conclude that $\exists c_0 > 0, \forall c \in [0, c_0], \forall x \in \partial A, \forall r > 0$:

$$B(x, r) \cap \{z \in \mathbb{C}^n, G_A(z) > c \cdot \text{dist}(z, A)^{1/c}\} \neq \emptyset,$$

which proves the statement. \qed
4 Proof of the main theorem

We will need the following result of Poletsky (Corollary p. 170 in [15], see also [16], or Theorem 2.2.10 and Corollary 2.2.13 in [18]), generalized by Rosay ([17]). Let $U$ be a connected complex manifold of dimension $n \geq 1$. We denote by $\mathcal{H}_{a,U}$ the set of holomorphic functions $h : V_h \to U$ from a neighbourhood $V_h$ of $\overline{A} = \{|z| \leq 1\} \subset \mathbb{C}$ (possibly depending on $h$) into $U$ such that $h(0) = z$. We also denote by $PSH(U)$ the set of plurisubharmonic functions defined on $U$.

**Proposition 2** Let $u : U \to \mathbb{R}$ be an upper semi-continuous function. With the previous notations, the function defined by

$$\hat{u}(z) := \frac{1}{2\pi} \inf_{f \in \mathcal{H}_{a,U}} \int_0^{2\pi} u(f(e^{i\theta}))d\theta,$$

if it is not everywhere equal to $-\infty$, belongs to $PSH(U)$ and verifies $\hat{u} \leq u$. Moreover, this function $\hat{u}$ is maximal among all the functions in $PSH(U)$ verifying this inequality.

**Remark 1** We deduce from Proposition 2 the following property of antisubharmonic functions, i.e. functions with subharmonic opposite. Let $B := B(a,r) \subset \mathbb{C}$ be an open ball, $u : \overline{B} \to \mathbb{R}$ a continuous function, antisubharmonic in $B$. Then $\hat{u} : B \to \mathbb{R}$ is an harmonic function, with the same boundary values as $u$, in the sense that $\lim_{z \to z_0} \hat{u} = u(z_0)$ for $z_0 \in \partial B$.

Indeed, given a continuous function $g : \overline{B} \to \mathbb{R}$, denote by $\tilde{g} : B \to \mathbb{R}$ the solution of the Dirichlet problem in $B$ with boundary condition $\hat{u}|_{\partial B}$, that is to say, the unique continuous function defined on $\overline{B}$ which is harmonic in $B$ and equal to $g$ on $\partial B$. Then $v := \max(\hat{u}, \tilde{g})$ is a subharmonic function with the same values as $u$ on $\partial B$. Since $u$ is antisubharmonic, we have $\hat{u} \leq u$. Thus

$$\hat{u} \leq v \leq u.$$

Since $\hat{u}$ is maximal among the subharmonic functions which are $\leq u$ in $B$ and equal to $u$ on $\partial B$, we conclude that $\hat{u} = v$, and hence $\hat{u} = \tilde{g}$.

Thanks to Theorem 3.1.4 in [9], the conclusion is the same if $B$ is a ball in $\mathbb{C}^n$, when substituting the expression "harmonic function" by "maximal plurisubharmonic function", and the expression "antisubharmonic function" by "antiplurisubharmonic function".

Let $U \subset \mathbb{C}^n$, $n \geq 1$, be a bounded open set. Denote by $\lambda$ the normalized Lebesgue measure on the unit circle $\partial U \subset \mathbb{C}$. Denote also by $A_{\lambda,U}$ the set of measures of the form $h_{\lambda}, \lambda(\cdot) := \lambda(h^{-1}(\cdot))$, where $h : V_h \to U$ is an holomorphic function defined in a neighborhood $V_h$ (possibly depending on $h$) of the closed unit disk $\overline{U}$, such that $h(0) = z$. Note that the Dirac measure $\delta_z$ belongs to $A_{\lambda,U}$ (this corresponds to the case where the function $h$ is constant, equal to $z$). An immediate consequence of Proposition 2 is the following corollary, where $1_G$ denotes the characteristic function of $G \subset \mathbb{C}^n$:

**Corollary 1** Let $U \subset \mathbb{C}^n$ be a bounded open set, and $A \subset U$ a $L$-regular non-pluripolar compact set satisfying $\overline{A} = A$. Then

$$\frac{1}{2\pi} \inf_{f \in \mathcal{H}_{a,U}} \int_0^{2\pi} -1_A \circ f(e^{i\theta})d\theta = -\sup_{\mu \in A_{\lambda,U}} \mu(\hat{A}) = G_{A,U}(z) - 1.$$
Recall that we denote by $K$ the filled Julia set of a polynomial application $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$, and $\text{dist}(\cdot, \cdot)$ the euclidean distance on $\mathbb{C}^n$. Let us prove the main result stated in the introduction:

**Theorem 1** Let $K \subset \mathbb{C}$ be the filled Julia set of a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$, of non-empty interior. Then $K$ satisfies the LS condition.

**Proof** For $b \in \mathbb{R}^+ \setminus \{0\}$, denote $U_b := \{G_K < b\} \subset \mathbb{C}$. For $l \in \mathbb{R}^+ \setminus \{0\}$, denote also $K_l := \{z \in \mathbb{C} \mid \text{dist}(z, K) \leq l\}$. Then choose $a > 0$ such that $K_2 \subset f^{-1}(U_a)$. Note that $f^{-1}(U_a) = U_{\delta} \subset \subset U_a$ by (2). Denote by $\mathcal{C}_a$ the annulus $U_a \setminus f^{-1}(U_a)$. There exists $\delta \in (0, 1]$ such that

$$G_{K_2, U_a} \geq \delta G_{K, U_a} \text{ on } \mathcal{C}_a. \quad (6)$$

There exists also $c \in (0, 1]$ such that $c \cdot \text{dist}(\cdot, K) < 1$ on $U_a$. We change slightly (3) and define $O_c$ by

$$O_c := \{z \in \mathbb{C}^n : \text{dist}(z, K) < 1, G_A(z) < c \cdot \epsilon^2 \cdot \text{dist}(z, K)^{1/c}\},$$

that do not change the set $I$ defined in (4). Take $c \in \left[0, \frac{\delta}{2a}\right]$ sufficiently small to have $O_c \subset f^{-1}(U_a)$ and $(\frac{1}{\epsilon})^c < 2$, as well as

$$G_K \geq c \cdot \epsilon^2 \cdot \text{dist}(\cdot, K)^{\frac{1}{c}} \text{ on } \mathcal{C}_a. \quad (7)$$

We have $\forall \epsilon \in [0, 2], \forall y \in U_a$,

$$c \cdot \text{dist}(y, K)^{\frac{1}{c}} \geq \inf_{\mu_y \in \mathcal{A}_{y, U_a}} \int_{U_a} c \cdot \text{dist}(\cdot, K)^{\frac{1}{c}} d\mu_y$$

$$\geq \inf_{\mu_y \in \mathcal{A}_{y, U_a}} \int_{U_a \setminus K} c \cdot \text{dist}(\cdot, K)^{\frac{1}{c}} d\mu_y$$

$$\geq \left(\min_{U_a \setminus K} c \cdot \text{dist}(\cdot, K)^{\frac{1}{c}}\right) \inf_{\mu_y \in \mathcal{A}_{y, U_a}} \int_{U_a \setminus K} d\mu_y$$

$$= c \epsilon^2 \epsilon \cdot G_\mathcal{K}, U_a(y).$$

The first inequality comes from the fact that the Dirac measure $\delta_y$ belongs to $\mathcal{A}_{y, U_a}$. The last inequality comes from Lemma 1, whose application is allowed by Lemma 1, and from Corollary 4.5.9 in [9]. Then taking $\epsilon = (\frac{1}{\epsilon})^c < 2$, we obtain in $U_a$:

$$c \cdot \text{dist}(\cdot, K)^{\frac{1}{c}} \geq \frac{\epsilon}{c} G_\mathcal{K}, U_a. \quad (8)$$

**Now suppose, by contradiction, that** $O_c \neq \emptyset$ (see Equation (3) for definition). Recall that $c < \frac{\delta}{2a}$. We can then choose $x \in O_c \setminus \{G_K < \frac{\epsilon \epsilon^2}{a} \cdot \text{dist}(\cdot, K)^{\frac{1}{c}}\}$.

**Let us control the growth of the iterates of $f$**. Note that $z \in O_c$ implies a “slow growth” of $(f^n(z))_n$, in the sense that $\forall n \geq 1$ such that $f^n(z) \in \{\text{dist}(\cdot, K) < 1\} \setminus O_c$, we have

$$\frac{1}{d^n} c \cdot \epsilon^2 \cdot \text{dist}(f^n(z), K)^{\frac{1}{c}} \leq G_K(z) < c \cdot \epsilon^2 \cdot \text{dist}(z, K)^{\frac{1}{c}},$$
and hence

$$\dist(f^n(z), K) < d^{n^c} \dist(z, K).$$  \(9\)

Moreover, by a similar reasoning, Equation (7) implies that for all \(z \in O_c\) and \(n \geq 1\) such that \(f^n(z) \in U_a \setminus \{ \dist(\cdot, K) < 1 \}\), we have

$$\dist(f^n(z), K) < d^{n^c} \dist(z, K).$$  \(10\)

Finally, by (11) and (10), for all \(z \in O_c\) and \(n \geq 1\) such that \(f^n(z) \in U_a \setminus O_c\), we have

$$\dist(f^n(z), K) < (d^n)^c \dist(z, K).$$  \(11\)

**Conclusion.** Recall that we have choosed \(x \in O_c \setminus \{ G_K < \frac{2a^2}{\delta} \dist(z, K)^\frac{1}{2} \}\). Since \(U_a \setminus K = \bigcup_{i \geq 0} f^{-i}(C_a)\) by (2), there exists \(N > 0\) such that \(f^N(x) \in C_a\). Equations (11), (8), (6), (1), then (2), give

$$c \cdot \dist(x, K)^\frac{1}{2} \geq \frac{c}{dN} \dist(f^N(x), K)^\frac{1}{2} \geq \frac{1}{cdN} G_{K, U_a} \circ f^N(x) \geq \frac{\delta}{cdN} G_{K, U_a} \circ f^N(x) = \frac{\delta}{ca} G_K(x).$$

But this contradicts our assumption \(x \notin \{ G_K < \frac{2a^2}{\delta} \dist(z, K)^\frac{1}{2} \}\). We conclude that \(O_c = \emptyset\). In other words, \(K\) satisfies the LS condition. \(\Box\)

**Remark 2** We note that if \(f\) is assumed to be **hyperbolic**, that is to say if \(f\) do not have critical points in \(J\), there exist a constant \(b > 0\) and a neighborhood of \(K\) in which

$$\dist(f(\cdot), K) \geq b \cdot \dist(\cdot, K).$$  \(12\)

Indeed, it is sufficient to establish this inequality outside \(K\). Let then \(V\) be a neighborhood of \(K\) in which \(|f'| \geq a\) for some \(a > 0\), let \(z \in V \setminus K\), and \(z_0 \in J\) such that \(f(z_0) \in J\) achieves the distance \(\dist(f(z), J)\). Then Theorem 1 of [6] shows the existence of a constant \(k > 0\) (depending only on the degree of \(f\)) and of a point \(z_1 \in J = \partial K\), such that

$$\dist(f(z), K) = \dist(f(z), f(z_0)) \geq a \cdot k \cdot \dist(z, z_1) \geq a \cdot k \cdot \dist(z, K).$$

In the particular case where \(b \geq 1\) in (12), we obtain a simpler proof of Theorem 1, and a more quantitative estimation for \(c\) in Equation (3). Indeed, suppose \(O_c \neq \emptyset\) with \(O_c \subset \subset V\). We can choose \(x \in O_c\) such that \(f(x) \notin O_c\). Then, (11) together with (12) give

$$c > \frac{\log b}{\log d}.$$
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