INTEGRABLE TRILINEAR PDE’s

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Abstract
In a recent publication we proposed an extension of Hirota’s bilinear formalism to arbitrary multilinearities. The trilinear (and higher) operators were constructed from the requirement of gauge invariance for the nonlinear equation. Here we concentrate on the trilinear case, and use singularity analysis in order to single out equations that are likely to be integrable. New PDE’s are thus obtained, along with others already well-known for their integrability and for which we obtain here the trilinear expression.

1. Introduction

What is astonishing with integrable PDE’s is that there are so many of them. Infi-
nite hierarchies of equations are known to date and the domain is still expanding. Moreover, these equations are often associated to physical models [1]. Then, using integrability, one can construct particular solutions and compute the pertinent physical quantities for these models.

One of the essential tools in the study of integrable PDE’s, over the past 20 years, has been the bilinear formalism of Hirota [2]. The main advantage of this approach lies in the fact that it allows one to obtain multisoliton solutions in a straightforward way [3]. Still, the proof of integrability, based on the existence of an arbitrary number of solitons, may present considerable difficulties. It is often easier to test the integrability of a given equation using singularity analysis (Painlevé method)[4], especially since this criterion (absence of singular solutions that exhibit branching) can be implemented algorithmically.

In a recent work [5] we have presented an extension of Hirota’s bilinear formalism to higher multilinearities. This was motivated by the existence of integrable equations that cannot be bilinearized, like the Satsuma determinantal trilinear PDE’s [6]. (It must be made clear at this point that the nonexistence of a bilinear form is
not a rigorous statement; it simply means that no bilinear form has been obtained to date). Our method is based on the assumption of gauge invariance. In the first exploratory study [5] we studied only the leading part of one-dimensional trilinear equations with one dependent variable and found those that satisfy the Painlevé criterion. The aim of the present paper is to complete this work, and obtain the integrable higher dimensional generalizations with the non-leading parts. As we will show in the following sections some of the equations obtained are new, while others are well-known for their integrability without necessarily possessing a simple bilinear form.

2. From bilinear to trilinear operators.

A prerequisite to the application of the Hirota bilinear formalism is a dependent variable transformation that converts the nonlinear equation into a quadratic ‘pre-potential’ form. In order to make things more clear let us present the classical example of the KdV equation. Starting from

\[ u_{xxx} + 6u u_x + u_t = 0, \]  

(2.1)

we introduce the transformation \( u = 2 \partial_x^2 \log F \) and obtain (after one integration):

\[ F_{xxxx} F - 4F_{xxx} F_x + 3F_{xx}^2 + F_{xt} F - F_x F_t = 0. \]  

(2.2)

This last equation can be written in a particularly condensed form using the Hirota \( D \)-operator:

\[ (D_x^4 + D_x D_t) F \cdot F = 0, \]  

(2.3)

where the \( D \)-operator is defined by its antisymmetric derivative action on a pair of functions (the ‘dot product’):

\[ D_x^n f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1) g(x_2) \bigg|_{x_2 = x_1}. \]  

(2.4)

The crucial observation here is the relation of the ‘physical’ variable \( u = 2 \partial_x^2 \log F \) to the Hirota’s function \( F \): the gauge transformation \( F \rightarrow e^{px + \omega t} F \) leaves \( u \) invariant. It turns out that this is a general property of bilinear equations. In fact, one can define Hirota’s bilinear equations through the requirement of gauge invariance. This statement was proven in [5].

Having obtained the Hirota bilinear operators on the sole requirement of gauge invariance we investigated in [5] the possible extension of this formalism and the introduction of multilinear operators. This turned out to be possible. Our first step was the extension of the bilinear operators to the trilinear case. We found that one convenient basis for the representation of the gauge invariant \( N \)th order derivative operators is given by \( (\partial_1 - \partial_2)^n (\partial_1 - \partial_3)^{N-n} \) for \( n = 0, \ldots, N \), where the subscripts tell on which of the three functions each derivative acts. Thus the basic building blocks for the trilinear operators are again the Hirota bilinear \( D \)'s:
we must just specify the indices in this case. We thus have \( D_{12} \equiv \partial_{x_1} - \partial_{x_2} \), \( D_{23} \equiv \partial_{x_2} - \partial_{x_3} \), \( D_{31} \equiv \partial_{x_3} - \partial_{x_1} \), but, of course the three are not linearly independent: \( D_{12} + D_{23} + D_{31} = 0 \). Their action on a ‘triple dot product’ is analogous to the bilinear case:

\[
D_{ij} f \cdot g \cdot h = (\partial_{x_i} - \partial_{x_j})f(x_1)g(x_2)h(x_3)\big|_{x_1=x_2=x_3=x} = (f'g - fg')h.
\]

The choice of a particular pair of \( D \)'s as the basic trilinear operators breaks the symmetry between the three coordinates \( x_i \)'s. It is possible to restore this symmetry by introducing a different basis for the trilinear operators, \( T \) and \( T^* \):

\[
T = \partial_1 + j \partial_2 + j^2 \partial_3, \quad T^* = \partial_1 + j^2 \partial_2 + j \partial_3,
\]

where \( j \) is the cubic root of unity, \( j = e^{2i\pi/3} \). (Note that the star in \( T^* \) indicates complex conjugation for the coefficients in \( T \) but not for the independent variables). Note that \( T^n T^m F \cdot F \cdot F = 0 \), unless \( n - m \equiv 0 \) (mod 3), which is the equivalent to the bilinear property \( D^n F \cdot F = 0 \), unless \( n \equiv 0 \) (mod 2). Moreover, a reality condition is satisfied: \( T^n T^m F \cdot F = T^m T^n F \cdot F \).

The generalization to higher multilinear equations is straightforward. One can introduce the set of \( n(n-1)/2 \) operators \( D_{ij} \) acting on \( n \)-tuple dot-products \( D_{ij} f_1 \cdot f_2 \cdot \ldots \cdot f_n \). Of course only \( n-1 \) of the \( D_{ij} \)'s are independent, a convenient basis being the \( D_{1j}, j = 2, \ldots n \). As in the trilinear case, one can also construct ‘symmetric’ operators:

\[
M_n^m = \sum_{k=0}^{n-1} e^{2\pi ikm/n} \partial_{k+1}
\]

for \( 0 < m < n \). We have, for example, \( D = M_2^1, T = M_3^1, T^* = M_3^2 \) and so on.

3. General comments on multilinearization

Before dealing with specific cases let us present here some general considerations. Let us start with a nonlinear (in \( u \)) equation, of order \( k + 1 \) having the form \( u_t + \partial_x P(u, u_x, \ldots, u_{kx}) = 0 \). Several well known integrable equations belong to this class. Let us assume that the leading part of \( P \) is weight-homogeneous in \( u \) and \( \partial_x \), with \( u \) having the same weight as \( \partial_x^2 \). Then we can transform the equation into a multilinear expression through the transformation \( u = \alpha \partial_x^2 (\log F) \) and obtain, after one \( x \)-integration, generically an \((k+2)\)-multilinear equation. The scaling factor \( \alpha \) can then be chosen (perhaps in several ways) so as to make an \( F^2 \) term factor out: the resulting multilinear equation is at most \( k \)-linear. For example at order five \( (k = 4) \) we have three integrable equations, and we should expect, in principle, these equations to have quadrilinear forms. Some unexpected cancellations, however, do occur. Thus the Sawada-Kotera equation [7] has a bilinear expression, the Lax-5 equation [8] a trilinear form, but the Kaup-Kupershmidt equation [9] is quadrilinear.
In the process of multilinearization of a given nonlinear equation the following points should be noted:

1) Above we discussed only \( u = \alpha \partial^2 \log (F) \) substitutions. If \( u \) always appears in the equation with at least one derivative then we can get a gauge invariant form also with \( u = \alpha \partial \log (F) \), and if we always have at least two derivatives, \( u = \alpha \log (F) \) is sufficient.

2) Any bilinear expression multiplied by \( F \) is cubic and gauge invariant and therefore has trilinear form, for example

\[
F(D_x D_y) F \cdot F = \frac{2}{3} T_x T_y^* F \cdot F, \\
F(D_x^3 D_y) F \cdot F = \frac{2}{3} T_x^2 T_y^* F \cdot F, \\
F(D_x^5 D_y) F \cdot F = \frac{2}{27} (10 T_x^3 T_y^* T_y^* - T_x^5 T_y^*) F \cdot F, \\
F(D_x^7 D_y) F \cdot F = \frac{2}{81} (35 T_x^4 T_y^* T_y^* - T_x^7 T_y^* - 7 T_x^5 T_y^* T_y^*) F \cdot F. 
\]

Therefore known one-component bilinear equations reappear in this trilinear study.

3) Taking a derivative of an equation having an \( n \)-linear form yields an equation that is \( n + 1 \) multilinear, for example

\[
\partial_x \left( F^{-2} D_x D_y F \cdot F \right) = \frac{2}{3} F^{-3} T_x^2 T_y F \cdot F, \\
\partial_x \left( F^{-2} D_x D_y F \cdot F \right) = \frac{2}{9} F^{-3} (T_x^4 T_y^* + 2 T_x^3 T_y T_y^*) F \cdot F, \\
\partial_x \left( F^{-2} D_x^6 F \cdot F \right) = \frac{2}{3} F^{-3} T_x^5 T_y^* F \cdot F. 
\]

The integrability of the derivative form is another matter: if for example the \( x \)-derivative form is also integrable then the original equation would be integrable with inhomogeneous \( x \)-independent term. Thus we will later find the \( x \)-derivative of Ito equation to be integrable, but not its \( t \)-derivative, therefore Ito’s equation is integrable even with a inhomogeneous \( g(t) \)-term.

Since equations having a bilinear form will reappear in the trilinear analysis as mentioned before, let us collect here the integrable equations that can be bilinearized to the form \( P(D) F \cdot F = 0 \) [10,11]:

Kadomtsev-Petviashvili:

\[
(D_x^4 - 4 D_x D_t + 3 D_y^2) F \cdot F = 0, 
\]

Ito:

\[
(D_t D_x^3 + a D_x^2 + D_t D_y) F \cdot F = 0, 
\]

Hietarinta:

\[
(D_x (D_x^3 - D_t^3) + a D_x^2 + b D_t D_x + c D_t^2) F \cdot F = 0, 
\]

Sawada-Kotera-Ramani:

\[
(D_x^6 + 5 D_x^3 D_y - 5 D_y^2 + D_x D_t) F \cdot F = 0, 
\]
4. Singularity analysis of trilinear equations

One motivation behind the multilinear approach is to find new integrable equations. Eventually we would like to get something similar to the systematic classification of bilinear equations presented in [10,11]. In the following paragraphs we will limit ourselves to the singularity analysis of trilinear equations involving only one dependent variable, i.e. unicomponent equations $P(T_x, T_x^*, T_y, T_y^*, \ldots)F \cdot F \cdot F = 0$. (Let us recall here that in the bilinear case a complete classification of these simplest equations [10,11] was possible).

Since the dependent function $u$ in a nonlinear equation is related to the multilinear $F$ through $u = \alpha \partial^2 \log F$ it is clear that a zero in $F$ induces a pole-like behavior in $u$. Let us here consider a concrete example: $T^{3k}F \cdot F \cdot F = 0$. Putting, for the dominant part, $F \sim \phi^n$, where $\phi$ is the singular manifold we find that the only possible leading behaviors have $n = 0, 1, \ldots, k - 1$. The first corresponds to a nonsingular Taylor-like expansion, which is always possible. The second behavior $F \sim \phi$ corresponds to a simple zero of $F$ that would give a (double) pole in $u$.

Next the resonances $r$ can be obtained if we substitute $F = \phi^n(1 + \omega \phi^r)$ and collect terms linear in $\omega$. For the equation to pass the Painlevé test the resonances must be integers and no incompatibilities must arise at any resonance. As an example let us take $k = 2$, i.e., the operator $T^6$. The leading behavior $F \sim \phi$ leads to the resonances $r = -1, 0, 1, 2$ and the roots of $r^2 - 13r + 60 = 0$ which are complex. Thus $T^6F \cdot F \cdot F = 0$ does not possess the Painlevé property.

4.1 Leading behaviors

The Painlevé analysis for the leading part of the equations was performed in [5], with just one independent variable. This is sufficient for the computation of dominant singularities and resonances. Note that for a given $N = n + m$ there may exist several pairs $(n, m)$ such that $T^nT^mF \cdot F \cdot F$ is not identically zero, namely those for which $n \equiv m \mod 3$. The leading part of the general equation at order $N$ is then given by a linear combination of all the non-vanishing $T^nT^mF \cdot F \cdot F$’s. In each case we give below the precise combinations that lead to equations with the Painlevé property. The nonlinear forms of the leading parts of the equations are obtained by the standard substitution $F = e^g$ followed by $u = 2g''$. The results of [5] are summarized as follows:

- **N = 2**: operator : $TT^*$ leading part : $u$.

  In this case we can write the result also in bilinear form

  - **N = 3**: operator : $T^3$ leading part : $u'$.
  - **N = 4**: operator : $T^2T^s2$ leading part : $u'' + 3u^2$.

  This, of course, is just the leading part of the KdV equation in potential form, and can also be written in bilinear form, see (3.2).

  - **N = 5**: operator : $T^4T^*$ leading part : $u'' + 6u'u$.

  This is the derivative of the expression obtained at $N = 4$.

  - **N = 6**: operator : $\lambda T^6 + \mu T^3T^s3$. This is the first case where we have two possible $(n, m)$ pairs. The $\lambda, \mu$ combinations that pass the Painlevé test are the following
a) \((\lambda = 7, \mu = 20)\) leading part: \(u^{'''} + 10u''u + 5u'^2 + 10u^3\).

This is the leading part of the once integrated 5th order equation in the Lax hierarchy of KdV.

b) \((\lambda = -2, \mu = 20)\) leading part: \(u^{'''} + 15u''u + 15u^3\).

This is the leading part of the once integrated Sawada-Kotera-Ramani equation (3.11), and can also be written in bilinear form, see (3.3).

c) \((\lambda = -\mu)\) leading part: \(uu'' - u'^2 + u^3\).

This expression can be cast in determinantal equation

\[
\begin{vmatrix}
F^{'''} & F''' & F'' \\
F''' & F'' & F' \\
F'' & F' & F
\end{vmatrix} = 0,
\]

whose integrability was already noticed by Chazy [12].

\(N = 7\): operator: \(T^5T^* + \mu T^4T^*\) leading part: \(u^{(5)} + 15u^{''''}u + 15u''u' + 45u'u^2\).

This is the Sawada-Kotera equation, i.e., the derivative of the expression obtained in \(N = 6\), c.f. (3.7).

\(N = 8\): operator: \(\lambda T^7T^* + \mu T^6T^*\) and nonlinearization with \(u = 6g''\) instead of \(u = 2g''\) used before.

a) \((\lambda = 4, \mu = 5)\) leading part: \(u^{(6)} + 6u^{'''}u + 10u''u' + 5u'^2 + 10u''u^2 + 10u'^2u + \frac{5}{3}u^4\)

This would correspond to a 7th order equation which, we believe, leads to a new integrable case.

b) \((\lambda = 4, \mu = 14)\) leading part: \(u^{(6)} + 7u^{'''}u + 7u''u' + 7u'^2 + 14u''u^2 + 7u'^2u + \frac{7}{3}u^4\)

This is the leading part of a once integrated higher Sawada-Kotera equation.

c) \((\lambda = -\mu)\) leading part: \(u^{''''}u - 3u''u' + 2u'^2 + 4u''u^2 - 3u'^2u + \frac{3}{2}u^4\).

This new equation looks like an extension of Satsuma’s equation given at \(N=6\) above, but it cannot be written as a single determinant.

\(N = 9\): No cases passing the Painlevé test exist at this order.

\(N = 10\): operator: \(5T^8T^* + 4T^5T^*\) (and we take \(u = 30g''\)).

leading part: \(u^{(8)} + 2u^{(6)}u + 4u^{(5)}u' + 6u^{''''}u'' + 5u'^2 + \frac{6}{5}u^{''''}u'^2 + 4u''u'u + 2u'^2u + 2u''u'^2 + \frac{4}{15}u''u^3 + \frac{2}{5}u''u^2 + \frac{2}{375}u^5\)

This is also a new equation.

Furthermore, no integrable candidates were found at orders \(N = 11, 12\). The singularity analysis at these higher orders becomes progressively more difficult. (Already at \(N = 12\) there exist three non-vanishing \(n, m\) combinations). It is, thus, not possible to extend our investigation to very high orders (as was done in our study of bilinear equations [11]), but we do believe that no further integrable candidates exist at higher orders.

### 4.2 Resonance conditions

In order to investigate the resonance conditions we must specify precisely the PDE we are working with, i.e., also the nonleading parts. Let us state from the outset that we are not going to consider equations of the form \(N = 6c\) or \(N = 8c\). These
equations have a form that makes the expected leading singularity to vanish identically, which leads to certain complications in the singularity analysis: they will be discussed elsewhere. Also in every case examined, subleading parts of opposite parity (= odd vs. even number of $T$’s) to that of the leading were found to violate the Painlevé property, but we did not make an exhaustive study of this. The technical details of the resonance conditions study are not particularly interesting, the analysis was performed on a computer using the REDUCE language [13] for symbolic manipulation. Although in some cases the resonances are particularly high (in the case of $N = 10$ we have a resonance at $r = 30$ for the leading behavior $F \sim \phi^{14}$) we were able in all cases to check the resonance conditions.

For low $N$ the complete results are simple: for $N = 3$ one obtains only linear equations, and for $N = 4$ all the integrable equations could also be written in bilinear form.

For equations with $N \geq 5$ an important question was how to extend the leading-order trilinear operator to two dimensions and still keep the calculations manageable. Our choice has been to consider only ‘monomial’ leading parts, that is monomial in derivatives and not necessarily in $T$’s. (From our experience on the bilinear case we do not expect nonmonomial leading parts to play a role for $N \geq 5$).

$N = 5$:

The leading 1-dimensional operator is $T_x T_x^*$ and three monomial generalizations to 1+1 dimensions are possible: i) $T_x T_x^*$, ii) $T_x T_x^* + \lambda T_x^3 T_y T_x^*$ and iii) $T_x^3 T_y T_y + \lambda T_x^2 T_y T_x^*$.

The case i) supplemented by subleading terms leads only to the $x$-derivative of the KP equation (3.5-8):

\[(T_x T_x^* - 4T_x^2 T_t + 3T_x T_y^2)F \cdot F \cdot F = 0.\]

In the case ii) (discarding the $\lambda = -1$ case that does not have a standard leading singularity) there exist the three Painlevé candidates $\lambda = -4$, 2 and 8. The first case does not satisfy the resonance condition. The case $\lambda = 2$ yields just the $x$-derivative of the Ito equation (3.9):

\[(T_x^4 T_x^* + 2T_x^3 T_x^* T_t + 3T_x^2 T_y^2 + 3T_x^2 T_y)F \cdot F \cdot F = 0.\]

The final case $\lambda = 8$ is more interesting. The most general result passing the Painlevé test reads:

\[(T_g(T_x^4 + 8T_x^3 T_x^* + 9T_y^2) + 9T_x^2 T_t)F \cdot F \cdot F = 0. \tag{4.1}\]

Putting $F = e^{g/2}$ we obtain:

\[g_{xxyy} + 4g_{xxy}g_{xx} + 2g_{xy}g_{xxx} + g_{yyyy} + g_{xxt} = 0. \tag{4.2}\]

This equation is a generalization (the $g_{yyyy}$ term is new) of an equation obtained by Schiff [14] from a reduction of the self-dual Yang-Mills equations. Incidentally, for
this equation we have checked that a nontrivial three soliton solution does exist and
this is a further indication of its integrability. It is interesting to observe that this
equation reduces to the derivative KP of Case i) if the first factor $T_y$ is replaced by
$T_x$ in (4.1).

Finally the case iii) did not lead to any equations with the Painlevé property.

A general study of monomial leading parts was performed for all $N$’s higher than
5. To make a long story short, only a monomial leading part of the form $T_x^{N-1}T_y$
has acceptable resonances for $N = 6, 7$. However, even in these cases, the resonance
conditions are not satisfied. Thus, beyond $N = 5$ the leading part is not only
monomial but also 1-dimensional.

$N = 6$:
The leading operator is $\lambda T_x^6 + 20T_x^3T_y^*$ with $\lambda = 7$ or $-2$. For $\lambda = 7$ the equation
obtained is the fifth-order equation in the Lax hierarchy [8] plus the KdV:

$$ (20T_x^3T_y^3 + 7T_x^6 + \alpha T_x^2T_y^2 + 27T_x^*T_y)F \cdot F \cdot F = 0 \quad (4.3) $$

or in nonlinear form after one integration ($u = 2\partial_x^2 \log F$):

$$ u_{5x} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x + \alpha(u_{xxx} + 6uu_x) + u_y = 0 \quad (4.4) $$

Thus the Lax-5 equation does not possess a simple bilinear form like KdV but a
trilinear one. For $\lambda = -2$ the resulting equation is the 3-dimensional Sawada-
Kotera-Ramani equation,

$$ (10T_x^3T_y^3 - T_x^6 + 45\alpha(T_x^2T_y^*T_y - \alpha T_yT_y^*) + 9T_x^*T_y)F \cdot F \cdot F = 0, \quad (4.5) $$

which also has a bilinear form (3.11).

In the case $N = 7$ the only equation satisfying the Painlevé requirement is the
$x$-derivative of the Sawada-Kotera-Ramani equation,

$$ (3T_x^5T_y^2 + 5\alpha(T_x^4T_y^* - T_x^3T_y) - 15\alpha^2T_x^2T_y - 3T_x^2T_y)F \cdot F \cdot F = 0, \quad (4.6) $$

which means that the original equation is probably integrable even with an inho-
mogeneous $k(y, t)$ term.

For $N = 8$ only the case 8a admits additive terms. Moreover it turns out that
the first subleading term must also be 1-dimensional and it coincides with 6a. The
full equation in this case reads:

$$ \left( 4T_x^7T_y^* + 5T_x^4T_y^4 + \alpha(20T_x^3T_y^3 + 7T_x^6) + 9\beta T_x^2T_y^*T_y + \frac{9\alpha\beta}{2}T_xT_y^* \right) F \cdot F \cdot F = 0, \quad (4.7) $$

and its nonlinear form with $u = 6\partial_x \log F$ is:

$$ u_{7x} + 6u_{5x}u_x + 10u_{4x}u_{xx} + 5u_{xxx}^2 + 10u_{xxxx}u_x^2 + \frac{5}{3}u_x^4 $$

$$ + \alpha(3u_{5x} + 10u_{xxxx}u_x + 5u_{xxx}^2 + \frac{10}{3}u_x^3) $$

$$ + \beta(u_{xyy} + u_xu_y) + \frac{\alpha\beta}{2}u_y = 0 \quad (4.8) $$
To our knowledge this equation is a new integrable PDE.

Case $N = 10$: Again one equation satisfies the integrability criterion. It reads:

$$
(5T_x^8 T_x^2 + 4T_x^5 T_x^5 + \alpha(4T_x^7 T_x^7 + 5T_x^4 T_x^4) + \beta(20T_x^3 T_x^2 T_y^2 + 7T_x^5 T_y^2) + 6\alpha\beta T_x^2 T_x^* T_y^* + \frac{3\beta^2}{2} T_y T_y^*) F \cdot F \cdot F = 0 \quad (4.9)
$$

This equation, too, has not been encountered before as an integrable equation.

5. Conclusion

In the preceding paragraphs we have presented an extension of Hirota’s bilinear formalism that can encompass any degree of multilinearity. The main guide in our investigation has been the requirement that the equations be gauge-invariant. Since our objective is the study of integrability we have also presented a classification of one-component trilinear equations that pass the Painlevé test.

An interesting difference between the bilinear and the tri- (and multi-)linear cases is that now free parameters enter already at the leading part. In practice this means that the Painlevé analysis of the higher order unicomponent equations becomes increasingly difficult. Once the leading parts of these equations are fixed, we have studied the lower-order terms that can be added without destroying the Painlevé property. As a result new integrable equations were discovered. The fact that they have eluded discovery till now is understandable since these equations are of high order and there is no hope to discover them by chance: a solid guide and a systematic approach are needed.

Starting from a complete classification of unicomponent equations one can build up multicomponent ones following the approach we presented in [10,11] for the bilinear case. Further extensions can be presented and we can, of course, construct also higher multilinear equations (quadri-, penta-, etc.) equations.

Extension of our formalism to the discrete case is also possible. In this case the integrability requirement is just the property of singularity confinement [15]. Some preliminary results exist already in this direction.

Thus the multilinear extension to Hirota bilinear approach has a wide range of applicability. There are still many open problems, e.g., the computation of the multisoliton solutions of the trilinear equations. Hopefully also the $\tau$-function formalism of the Kyoto school [16] can be extended in this direction.

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