WIENER-HOPF OPERATORS ON SPACES OF FUNCTIONS ON $\mathbb{R}^+$ WITH VALUES IN A HILBERT SPACE

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Abstract. A Wiener-Hopf operator on a Banach space of functions on $\mathbb{R}^+$ is a bounded operator $T$ such that $P^+S_aTS_a = T$, $a \geq 0$, where $S_a$ is the operator of translation by $a$. We obtain a representation theorem for the Wiener-Hopf operators on a large class of functions on $\mathbb{R}^+$ with values in a separable Hilbert space.

Key words: Wiener-Hopf operators, symbol, Fourier transformation, spectrum of translation operators

1. Introduction

This paper deals with Wiener-Hopf operators on Banach spaces of functions on $\mathbb{R}^+$ with values in a separable Hilbert space $H$. Let $E$ be a Banach space of functions on $\mathbb{R}^+$ such that $E \subset L^1_{loc}(\mathbb{R}^+)$. For $a \geq 0$, define the operator

$$S_a : E \rightarrow L^1_{loc}(\mathbb{R}^+),$$

by the formula $(S_a f)(x) = f(x - a)$, for almost every $x \in [a, +\infty[$ and $(S_a f)(x) = 0$, for $x \in [0, a[$. For $a \geq 0$, introduce

$$S_{-a} : E \rightarrow L^1_{loc}(\mathbb{R}^+),$$

defined by the formula $(S_{-a} f)(x) = f(x + a)$, for almost every $x \in \mathbb{R}^+$. Notice that $S_{-a}S_a = I$ but $S_aS_{-a} \neq I$. From now, we suppose that $S_aE \subset E$ and $S_{-a}E \subset E$, $\forall a \in \mathbb{R}^+$. The Wiener-Hopf operators on $E$ are the bounded operators

$$T : E \rightarrow E$$

satisfying

$$S_{-a}TS_a = T, \forall a \in \mathbb{R}^+.$$  

Denote by $P^+$ the operator

$$P^+ : L^1_{loc}(\mathbb{R}) \rightarrow L^1_{loc}(\mathbb{R}^+)$$

defined by

$$(P^+ f)(x) = f(x), a.e. on \mathbb{R}^+.$$  

The Wiener-Hopf operators which appear in theory of the signal and in control theory have been studied in a lot of papers. The problem we deal here is the existence of a symbol for
operators of this type. It is well-known that if $T$ is a Wiener-Hopf operator on $L^2(\mathbb{R}^+)$ there exists $h \in L^\infty(\mathbb{R})$ such that

$$Tf = P^+ F^{-1}(h \hat{f}), \forall f \in L^2(\mathbb{R}^+).$$

Here $F$ denotes the usual Fourier transformation from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. The function $h$ is called the symbol of $T$. Despite of the extensive literature related to Wiener-Hopf operators, there are not analogous representation theorem for Wiener-Hopf operators on general Banach spaces of functions even if the functions are with values in $\mathbb{C}$. Here we develop a theory of the existence of a $L^\infty$ symbol for every Wiener-Hopf operator in a very large class of spaces of functions on $\mathbb{R}^+$ with values in a separable Hilbert space. Moreover, we obtain a caracterisation of $\text{spec}(S_1) \cap (\text{spec}(S_{-1}))^{-1}$. The determination of the spectrum of a translation operator is an open question in general spaces of functions on $\mathbb{R}^+$ and it plays an important role in the scattering theory. We are motivated by the results of [5] proving the existence of a symbol for every Wiener-Hopf operator on a weighted space $L^2_\omega(\mathbb{R}^+)$ (see Example 1 for the definition). On the other hand, the methods exposed in [6] and [2] show that the existence of the symbol of a multiplier (a bounded operator commuting with the translations) on spaces of scalar functions on $\mathbb{R}$ implies an analogous result for the multipliers on a space of functions on $\mathbb{R}$ with values in an Hilbert space. The arguments in [6] and [2] have been based on the link between the scalar and the vector-valued cases. However the results concerning the symbol of a multiplier do not imply analogous results about Wiener-Hopf operator in the general case. It is well-known that for every Wiener-Hopf operator $T$ on $L^2(\mathbb{R}^+)$, there exists a multiplier $M$ on $L^2(\mathbb{R})$ such that $P^+ M = T$. Unfortunaly, a such result is not known even for Wiener-Hopf operators on a weighed space $L^2_\omega(\mathbb{R}^+)$. Despite some progress (see [2], [6]) in the study of the symbol of a multiplier on a space of functions on $\mathbb{R}$ with values in a Hilbert space, the analogous problem for Wiener-Hopf operators has been very few considered. Moreover, even in the case of the weighted spaces of functions on $\mathbb{R}^+$ with values in a Hilbert space the existence of the symbol of a Wiener-Hopf operator was an open problem still now. First in Section 2, we improve the results of [5] concerning the existence of the symbol of a Wiener-Hopf operator on $L^2_\omega(\mathbb{R}^+)$ replacing $L^2_\omega(\mathbb{R}^+)$ by a general Banach space of functions on $\mathbb{R}^+$ satisfying only three natural hypothesis given below. Next following the methods of [6] and [2] and using the results of Section 2, we obtain the existence of the symbol of a Wiener-Hopf operator on a very large class of spaces of functions on $\mathbb{R}^+$ with values in a separable Hilbert space. In Section 4 we explain how the setup considered here can by extended in several directions.

Let $E$ be a Banach space of functions on $\mathbb{R}^+$ with values in $\mathbb{C}$ satisfying the following three hypothesis.

(H1) We have $C^\infty_c(\mathbb{R}^+) \subset E \subset L^1_{\text{loc}}(\mathbb{R}^+)$, the inclusions are continuous and $C^\infty_c(\mathbb{R}^+)$ is dense in $E$.

(H2) For every $x \in \mathbb{R}$, $S_x E = E$ and $\sup_{x \in K} \|S_x\| < +\infty$, for every compact $K$ of $\mathbb{R}$. 


(H3) For all $a \in \mathbb{R}$, the operator $\Gamma_a$ defined by

$$(\Gamma_a f)(x) = e^{iax} f(x), \ a.e., \forall f \in E$$

is bounded on $E$ and

$$\sup_{a \in \mathbb{R}} \|\Gamma_a\| < +\infty.$$  

Notice that (H3) is trivial, if we have $\|f\| = \||f||$ in $E$. Let $C^\infty_K(\mathbb{R}^+)$ be the space of $C^\infty$ functions with a compact support included in $K$. For simplicity, we will write $S$ instead of $S_1$. Since the norm of $f$ given by $\sup_{a \in \mathbb{R}} \|\Gamma_a f\|$ is equivalent to the norm of $E$, we will assume from now that $\Gamma_a$ is an isometry for every $a \in \mathbb{R}$. Denote by $\rho(A)$ the spectral radius of a bounded operator $A$. Set

$$I_E = [-\ln \rho(S^{-1}), \ln \rho(S)]$$
and

$$U_E = \{z \in \mathbb{C}, \Im z \in I_E\}.$$ 

For $f \in E$, denote by $(f)_a$ the function defined by $(f)_a(x) = e^{ax} f(x)$, a.e. on $\mathbb{R}^+$. In Section 2 we obtain the following result which generalizes Theorem 1 in [5].

**Theorem 1.** Let $T \in W(E)$.
1) For every $a \in I_E$ we have $(Tf)_a \in L^2(\mathbb{R}^+)$, for $f \in C^\infty_c(\mathbb{R}^+)$.  
2) For every $a \in I_E$ there exists a function $\nu_a \in L^\infty(\mathbb{R})$ such that

$$(Tf)_a = P^+ \mathcal{F}^{-1}(\nu_a(\hat{f}))_a, \text{ for } f \in C^\infty_c(\mathbb{R}^+)$$

and we have $\|\nu_a\|_\infty \leq C\|T\|$, where $C$ is a constant dependent only on $E$.  
3) Moreover, if $\hat{I}_E \neq \emptyset$ (i.e. $\frac{1}{\rho(S^{-1})} < \rho(S)$), there exists a function $\nu \in \mathcal{H}^\infty(U_E)$ such that for every $a \in \hat{I}_E$ we have

$$\nu(x + ia) = \nu_a(x), \text{ almost everywhere on } \mathbb{R}.$$  

**Definition 1.** If $\hat{I}_E \neq \emptyset$, $\nu$ is called the symbol of $T$, and if $I_E = \{a\}$, then $\nu_a$ is the symbol of $T$.

Using Theorem 1, we also obtain the following spectral result.

**Theorem 2.** We have

$$\text{spec}(S) \cap \left(\text{spec}(S^{-1})\right)^{-1} = \left\{z \in \mathbb{C}, \frac{1}{\rho(S^{-1})} \leq |z| \leq \rho(S)\right\}.$$ 

This result is new even in the case of the spaces $L^2_\omega(\mathbb{R}^+)$. In particular, we conclude that if $\rho(S) > \frac{1}{\rho(S^{-1})}$ the spectrum of $S$ contains a disk. The proof of Theorem 2 is based on the existence of a symbol for every Wiener-Hopf operator and the construction of suitable cut-off function $f \in C^\infty_c(\mathbb{R}^+)$. This application was one of the motivations to search a symbol of a
Wiener-Hopf operator. Moreover, we extend below the same result for operators with values in a Hilbert space (see Theorem 4).

The main result of this paper is an analogous result for Wiener-Hopf operators on spaces of functions on $\mathbb{R}^+$ with values in a separable Hilbert space. Denote by $\langle u, v \rangle$ the scalar product of $u, v \in H$ and let $\|u\|_H$ be the norm of $u \in H$. Denote by $L^1_{\text{loc}}(\mathbb{R}^+, H)$ the space of functions $F : \mathbb{R}^+ \to H$ such that $(\mathbb{R}^+ \ni x \mapsto \|F(x)\|_H) \in L^1_{\text{loc}}(\mathbb{R}^+)$. Let $L(H)$ be the space of bounded operators on $H$. Introduce the vector space $C_c^\infty(\mathbb{R}^+) \otimes H$ generated by $fu$ for $f \in C_c^\infty(\mathbb{R}^+)$ and $u \in H$. Denote by $C_0(\mathbb{R}^+, H)$ the Banach space of all norm continuous functions $\Phi : \mathbb{R}^+ \to H$ such that for every $\epsilon > 0$, there exists a compact set $K_\epsilon$ such that $\|\Phi(x)\|_H = 0$, $\forall x \in \mathbb{R}^+ \setminus K_\epsilon$.

Let $E$ be a Banach space of functions on $\mathbb{R}^+$ with values in $\mathbb{C}$ satisfying (H1), (H2) and (H3). Denote by $\overline{E}$ the Banach space of functions $F : \mathbb{R}^+ \to H$ such that $(\mathbb{R}^+ \ni x \mapsto \|F(x)\|_H) \in E$. We will see in Section 3 that $C_c^\infty(\mathbb{R}^+) \otimes H$ is dense in $\overline{E}$. For illustration, we give below some examples.

**Example 1.** Let $E = L^p_\omega(\mathbb{R}^+)$, where $\omega$ is a weight on $\mathbb{R}^+$ and $p \in [1, +\infty]$. We recall that $\omega$ is a weight on $\mathbb{R}^+$ if $\omega$ is a non-negative measurable function on $\mathbb{R}^+$ such that for all $y \in \mathbb{R}^+$,

$$0 < \sup_{x \in \mathbb{R}^+} \frac{\omega(x + y)}{\omega(x)} < +\infty$$

and

$$0 < \sup_{x \in \mathbb{R}^+} \frac{\omega(x)}{\omega(x + y)} < +\infty.$$
equipped with the norm
\[ \|f\|_{\omega,p} = \left( \int_{\mathbb{R}^+} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}}. \]

It is easy to see that \( L^p_\omega(\mathbb{R}^+) \) satisfies the hypothesis (H1), (H2) and (H3). For the study of the Wiener-Hopf operators on \( L^2_\omega(\mathbb{R}^+) \) the reader may consult [5]. The space \( \mathcal{E} \) associated to \( L^p_\omega(\mathbb{R}^+) \) is the space usually denoted by \( L^p_\omega(\mathbb{R}^+, H) \) of functions
\[ F : \mathbb{R}^+ \rightarrow H \]
such that
\[ \int_{\mathbb{R}^+} \|F(x)\|^p_H \omega(x)^p dx < +\infty. \]

**Example 2.** Let \( A \) be a real-valued continuous function on \([0, +\infty[\), such that \( A(0) = 0 \) and let \( \frac{A(y)}{y} \) be non-decreasing for \( y > 0 \). Let \( L_A(\mathbb{R}^+) \) be the set of all complex-valued, measurable functions on \( \mathbb{R}^+ \) such that
\[ \int_{\mathbb{R}^+} A\left(\frac{|f(x)|}{t}\right) dx < +\infty, \]
for some positive number \( t \) and let
\[ \|f\|_A = \inf \left\{ t > 0 \mid \int_{\mathbb{R}^+} A\left(\frac{|f(x)|}{t}\right) dx \leq 1 \right\}, \]
for \( f \in L_A(\mathbb{R}^+) \). Then \( L_A(\mathbb{R}^+) \) is a Banach space called a Birnbaum-Orlicz space (see [1]). It is easy to check that \( L_A(\mathbb{R}^+) \) satisfies (H1), (H2) and (H3). If \( E = L_A(\mathbb{R}^+) \), the associated space \( \mathcal{E} \) is the set \( L_A(\mathbb{R}^+, H) \) of measurable functions
\[ F : \mathbb{R}^+ \rightarrow H \]
such that for some \( t > 0 \), we have
\[ \int_{\mathbb{R}^+} A\left(\frac{\|F(x)\|_H}{t}\right) dx < +\infty. \]

**Example 3.** Let \( A \) be a function satisfying the properties described in Example 2. Let \( \omega \) be a weight on \( \mathbb{R}^+ \). Define \( L_{A,\omega}(\mathbb{R}^+) \) as the space of measurable functions on \( \mathbb{R}^+ \) such that
\[ \int_{\mathbb{R}^+} A\left(\frac{|f(x)|}{t}\right) \omega(x) dx < +\infty, \]
for some positive number \( t \) and let
\[ \|f\|_{A,\omega} = \inf \left\{ t > 0 \mid \int_{\mathbb{R}^+} A\left(\frac{|f(x)|}{t}\right) \omega(x) dx \leq 1 \right\}. \]
for $f \in L_{A, \omega}(\mathbb{R}^+)$. Then $L_{A, \omega}(\mathbb{R}^+)$ is a Banach space called a weighted Orlicz space. It is easy to check that $L_{A, \omega}(\mathbb{R}^+)$ satisfies (H1), (H2) and (H3). If $E = L_{A, \omega}(\mathbb{R}^+)$, the associated space $\overline{E}$ is the set $L_{A, \omega}(\mathbb{R}^+, H)$ of measurable functions

$$F : \mathbb{R}^+ \rightarrow H$$

such that for some $t > 0$,

$$\int_{\mathbb{R}^+} A\left(\frac{\|F(x)\|_H}{t}\right) \omega(x) dx < +\infty.$$

For $a > 0$, we define the operators

$$S_a : E \rightarrow \overline{E}$$

and

$$S_{-a} : E \rightarrow \overline{E}$$

by

$$(S_a F)(x) = F(x - a), \text{ a.e. on } [a, +\infty[,$$

$$(S_a F)(x) = 0, \forall x \in [0, a[,$$

$$(S_{-a} F)(x) = F(x + a), \text{ a.e. on } \mathbb{R}^+.$$

For simplicity, we will write $S$ instead of $S_1$. For $F \in \overline{E}$, we denote by $\|F\|_H$ the function

$$\|F\|_H : \mathbb{R}^+ \ni x \rightarrow \|F(x)\|_H \in \mathbb{C}.$$

For fixed $a \in \mathbb{R}$, we see that for $F \in \overline{E}$, $F \neq 0$, we have

$$\frac{\|S_a F\|}{\|F\|} = \frac{\|S_a(\|F\|_H)\|}{\|\|F\|_H\|} \leq \|S_a\|.$$

We conclude that $S_a$ is bounded and $\|S_a\| \leq \|S_a\|$. If $\|f\| = \|\|f\|\|$, for every $f \in E$, obviously we get $\|S_a\| = \|S_a\|$. Introduce the operator

$$P^+ : L^1_{loc}(\mathbb{R}, H) \rightarrow L^1_{loc}(\mathbb{R}^+, H)$$

defined by the formula

$$(P^+ F)(x) = F(x), \text{ a.e. on } \mathbb{R}^+.$$

**Definition 2.** We call a Wiener-Hopf operator on $\overline{E}$ every bounded operator $T$ on $\overline{E}$ such that

$$T \Phi = S_{-a} TS_a \Phi, \forall a > 0, \forall \Phi \in \overline{E}.$$}

Denote by $W(\overline{E})$ the set of the Wiener-Hopf operators on $\overline{E}$.

The main result of this paper is the following.
Theorem 3. Let $E$ be a Banach space satisfying (H1), (H2) and (H3). Let $T \in W(E)$.
1) We have $(T\Phi)_a \in L^2(\mathbb{R}^+, H), \forall \Phi \in C_c^\infty(\mathbb{R}^+) \otimes H, \forall a \in I_E.$
2) There exists $V_a \in L^\infty(\mathbb{R}, L(H))$ such that
   $$(T\Phi)_a = P^+\mathcal{F}^{-1}(\mathcal{F}(\Phi)_a(\cdot)), \forall a \in I_E, \forall \Phi \in C_c^\infty(\mathbb{R}^+) \otimes H.$$ Moreover, $\text{ess sup}_{x \in \mathbb{R}} \|V_a(x)\| \leq C\|T\|$, where $C$ is a constant dependent only on $E$.
3) If $\hat{U}_E \neq \emptyset$, set
   $$\mathcal{V}(x + ia) = V_a(x), \forall a \in \hat{U}_E,$$ for almost every $x \in \mathbb{R}.$
   Then for $u, v \in H$, the function
   $$z \mapsto <u, \mathcal{V}(z) [v]>$$
is in $\mathcal{H}^\infty(\hat{U}_E)$ and $\sup_{z \in \hat{U}_E} \|\mathcal{V}(z)\| \leq C\|T\|.$

Remark 1. We will see later that $\rho(S) = \rho(S)$, $\rho(S_{-1}) = \rho(S_{-1})$ and
   $$I_E = [-\ln \rho(S_{-1}), \ln \rho(S)] = [-\ln \rho(S_{-1}), \ln \rho(S)].$$

We also obtain the following.

Theorem 4. We have
   $$\text{spec}(S) \cap \left(\text{spec}(S_{-1})\right)^{-1} = \left\{z \in \mathbb{C}, \frac{1}{\rho(S_{-1})} \leq |z| \leq \rho(S)\right\}.$$

The spectral characterization in Theorem 4 has not been known until now even in particular cases when $E$ is a weighted $L^p$ space with a simple weight.

2. Wiener-Hopf Operators on Banach Spaces of Scalar Functions on $\mathbb{R}^+$

In this section, we prove Theorem 1. We follow the arguments of [5] in our more general case. For the reader convenience we give the details of the steps which need some modifications. First, we show that every Wiener-Hopf operator is associated to a distribution. Denote by $C^\infty_0(\mathbb{R}^+)$ the space of functions of $C^\infty(\mathbb{R})$ with support in $]0, +\infty[.$ Set
   $$H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\},$$
the derivative of $f \in L^2(\mathbb{R})$ being computed in the sense of distributions.

Lemma 1. If $T \in W(E)$ and $f \in C^\infty_K(\mathbb{R}^+)$, then $(Tf)' = T(f').$
Consequently, Proposition 1.

If $E$ is a Wiener-Hopf operator, then there exists a distribution $\mu_T$ of order 1 such that

$$Tf = P^+(\mu_T * f),$$

for $f \in C_c^\infty(\mathbb{R}^+)$. 

The proof of Proposition 1 follows the arguments of that of Theorem 2 in [5] and we omit it. We just give the definition of $\mu_T$. We have

$$< \mu_T, f > = \lim_{x \to +\infty} (TS_x \tilde{f})(x),$$

for $f \in C_c^\infty(\mathbb{R})$, where $\tilde{f}$ is the function defined by $\tilde{f}(x) = f(-x)$, for $f \in C_c^\infty(\mathbb{R})$, $x \in \mathbb{R}$.

**Definition 3.** If $\phi \in C_c^\infty(\mathbb{R})$, we denote by $T_\phi$ the Wiener-Hopf operator such that

$$T_\phi f = P^+(\phi * f), \forall f \in C_c^\infty(\mathbb{R}^+).$$

**Proposition 2.** If $T \in W(E)$, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^+)$ such that

$$\lim_{n \to +\infty} \|T_{\phi_n} f - T f\| = 0, \forall f \in E$$

and

$$\|T_{\phi_n}\| \leq C\|T\|, \forall n \in \mathbb{N},$$

where $C$ is a constant depending only on $E$.

**Proof.** The proof follows the idea of the proof of Theorem 3 in [5], but here we must work with Bochner integrals and this leads to some difficulties. For the convivance of the reader we give the details. Let $T \in W(E)$ and set $\mathcal{T}(t) = \Gamma_t \circ T \circ \Gamma_{-t}$, $\forall t \in \mathbb{R}$. For $a > 0$, and $f \in E$ we have

$$\mathcal{T}(t)S_a f(x) = (T(t)S_a f)(x + a) = e^{it(x+a)} \left( T(f(s-a)e^{-its}) \right)(x + a) = e^{itx} \left( S_a T \left( f(s-a)e^{-its-a} \right) \right)(x) = e^{itx} (S_a T_{\Gamma_t} f)(x) \in (\mathcal{T}(t)f)(x), \text{ a.e.}$$
This shows that $T(t) \in W(E)$. Moreover, we have $\|T(t)\| = \|T\|$, for $t \in \mathbb{R}$ and $T(0) = T$.

The application $T$ is continuous from $\mathbb{R}$ into $W(E)$. For $n \in \mathbb{N}$, $\eta \in \mathbb{R}$, $x \in \mathbb{R}$, set

$$g_n(\eta) := \left(1 - \left|\frac{\eta}{n}\right|\right)\chi_{[-n,n]}(\eta)$$

and

$$\gamma_n(x) = \frac{1 - \cos(nx)}{\pi x^2 n}.$$ 

We have $\hat{\gamma}_n(\eta) = g_n(\eta)$, $\forall \eta \in \mathbb{R}$, $\forall n \in \mathbb{N}$. Clearly, $\|\gamma_n\|_{L^1} = 1$ for all $n$ and

$$\lim_{n \to +\infty} \int_{|x| \geq a} \gamma_n(x) dx = 0, \forall a > 0.$$ 

Set $Y_n := (T \ast \gamma_n)(0)$. Then for $f \in E$ we obtain

$$\lim_{n \to +\infty} \|Y_n f - T f\| = 0.$$ 

We claim that for $f \in C_c^\infty(\mathbb{R}^+)$, we have

$$(Y_n f)(y) = \int_\mathbb{R} (T(x) f)(y) \gamma_n(-x) dx, \forall y \in \mathbb{R}^+$$

From Lemma 1, we know that for fixed $x \in \mathbb{R}$ the function

$$\mathbb{R}^+ \ni y \longrightarrow (T(x) f)(y)$$

is $C^\infty$. Let $K_0$ be a compact subset of $\mathbb{R}^+$ and let $\psi \in C_c^\infty(\mathbb{R}^+)$. We see that

$$|\psi(y)(T(x) f)(y)| = |\psi(y)(\mu_T \ast \Gamma_{-x}(f))(y)| = |\psi(y) \mu_T \ast \Gamma_{-x}(f)(z-y)e^{-ix(z-y)}| \leq C(\mu)\|\psi\|_{L^\infty} (\|S_y \Gamma_{-x} f\|_{L^\infty} + \|S_y \Gamma_{-x} f\|'_{L^\infty})$$

$$\leq C(\mu)\|\psi\|_{L^\infty} (\|f\|_{L^\infty} + \|f\|'_{L^\infty}), \forall y \in K_0.$$

Consequently,

$$\int_\mathbb{R} \|\psi T(x) f\|_{L^\infty} \gamma_n(-x) dx < +\infty$$

and hence the integral

$$\int_\mathbb{R} \psi(T(x) f) \gamma_n(-x) dx$$

is a well-defined Bochner integral with values in $C_c^\infty(\mathbb{R}^+)$. The map

$$C_c^\infty(\mathbb{R}^+) \ni g \longrightarrow g(x) \in \mathbb{C},$$

is a continuous linear form for every $x \in \mathbb{R}^+$. Since Bochner integrals commute with continuous linear forms (see [3]) we have

$$\psi(y) (Y_n f)(y) = \psi(y) \int_\mathbb{R} (T(x) f)(y) \gamma_n(-x) dx, \forall y \in \mathbb{R}^+$$
and the claim \([2,1]\) is proved.

It is clear that

\[
\|Y_n f\| \leq \int_{\mathbb{R}} \|T(x)f\|\gamma_n(-x)dx \leq \|T\|\|f\|, \forall f \in E.
\]

Since \(\|T\| = \|T\|\), we get \(\|Y_n\| \leq \|T\|\), \(\forall n \in \mathbb{N}\).

Now consider the distribution associated to \(Y_n\). Let \(K\) be a compact subset of \(\mathbb{R}\) and let \(z_K \geq 1\) be such that \(K \subset [-\infty, z_K]\). Choose \(g \in C_c^\infty(\mathbb{R}^+)\) such that \(g\) is positive, \(\text{supp}\, g \subset [z_K - 1, z_K + 1]\) and \(g(z_K) = 1\). For \(f \in C_c^\infty(\mathbb{R})\), we have \(gT(S_{z_K}(\tilde{f}g_n)) \in H^1(\mathbb{R})\) and it follows from Sobolev’s lemma (see \([7]\)) that

\[
\|(TS_{z_K}(\tilde{f}g_n))(z_K)\| = |g(z_K)(TS_{z_K}(\tilde{f}g_n))(z_K)|
\]

\[
\leq C\left(\int_{|y-z_K|\leq 1} g(y)^2|(TS_{z_K}(\tilde{f}g_n)(y)|^2dy\right)^{\frac{1}{2}} + \left(\int_{|y-z_K|\leq 1} |(g(TS_{z_K}(\tilde{f}g_n))'(y)|^2dy\right)^{\frac{1}{2}},
\]

where \(C > 0\) is a constant. Taking into account \((H1)\), \(T\) may be considered as a bounded operator from \(C_c^\infty(\mathbb{R}^+)\) into \(L^1_{loc}(\mathbb{R}^+)\) and we have

\[
\|(TS_{z_K}(\tilde{f}g_n))(z_K)\|
\]

\[
\leq C(K)\|T\|\|\tilde{f}g_n\|_{\infty} + \|(\tilde{f}g_n)'\|_{\infty}
\]

\[
\leq \tilde{C}(K)(\|f\|_{\infty} + \|f'\|_{\infty}),
\]

where \(C(K)\) and \(\tilde{C}(K)\) are constants depending only on \(K\). Therefore

\[
|(TS_z(\tilde{f}g_n))(z)| \leq \tilde{C}(K)(\|f\|_{\infty} + \|f'\|_{\infty}), \forall z \geq z_K, \forall f \in C_c^\infty(\mathbb{R})
\]

and we conclude that \(\mu_{Tg_n}\) defined by

\[
< \mu_{Tg_n}, f > := \lim_{z \to +\infty} (TS_z(\tilde{f}g_n))(z)
\]

is a distribution of order 1. On the other hand, we have

\[
(Y_n f)(y) = \int_{\mathbb{R}} (T(-s)f)(y)\gamma_n(s)ds = \int_{\mathbb{R}} e^{-isy}(T(\Gamma_s f))(y)\gamma_n(s)ds
\]

\[
= \int_{\mathbb{R}} < \mu_{T,x}, f(y-x) e^{-isx} > \gamma_n(s)ds
\]

\[
= \int_{\mathbb{R}} < \mu_{T,x}, f(y-x)g_n(x) > = < \mu_{Tg_n, f}(y), \forall y \geq 0, \forall f \in C_c^\infty(\mathbb{R}^+)\).
\]

Finally, we obtain

\[
Y_n f = P^+(\mu_{Tg_n} * f), \forall f \in C_c^\infty(\mathbb{R}^+), \forall n \in \mathbb{N}.
\]

Since \(\text{supp}\, \mu_{Tg_n} \subset [-n, n]\), it is sufficient to obtain the Proposition 2 for \(T \in W(E)\) such that \(\mu_T\) is a distribution with compact support. Without lost of generality we assume that \(\mu_T\)
is with compact support. Let \((\theta_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^+)\) be a sequence such that \(\text{supp}\theta_n \subset [0, \frac{1}{n}]\), \(\theta_n \geq 0\),

\[
\lim_{n \to +\infty} \int_{x \geq a} \theta_n(x)dx = 0, \forall a > 0
\]

and \(\|\theta_n\|_{L^1} = 1\), for \(n \in \mathbb{N}\). For \(f \in E\) we have

\[
\lim_{n \to +\infty} \|\theta_n * f - f\| = 0.
\]

Set

\[
T_n f = T(\theta_n * f), \forall f \in E.
\]

We conclude that \((T_n)_{n \in \mathbb{N}}\) converges to \(T\) with respect to the strong operator topology and \(T_n = T\phi_n\), where \(\phi_n = \mu T * \theta_n \in C_c^\infty(\mathbb{R}^+)\). For \(f \in E\), we have

\[
\|T_n f\| = \left\|P^+ \left( \int_0^{\frac{1}{n}} \theta_n(y)S_y(\mu T * f)dy \right) \right\|
\]

\[
\leq \left\| \int_0^{\frac{1}{n}} \theta_n(y)P^+ (\mu T * S_y f)dy \right\|
\]

\[
\leq \int_0^{\frac{1}{n}} \theta_n(y)\|T\|\|S_y\|\|f\|dy, \forall f \in C_c^\infty(\mathbb{R}^+).
\]

Then we obtain

\[
\|T_n\| \leq \left( \int_0^{\frac{1}{n}} \theta_n(y)\|S_y\|dy \right)\|T\|, \forall n \in \mathbb{N}
\]

and this completes the proof of the proposition. \(\square\)

We need also the following lemma.

**Lemma 2.** For every \(\phi \in C_c^\infty(\mathbb{R}^+)\), we have

\[
|\hat{\phi}(\alpha)| \leq \|\phi\|, \forall \alpha \in U_E.
\]

**(Proof.** We use the fact that for a bounded operator \(A\) on \(E\), there exists a sequence \((f_n)_{n \in \mathbb{N}} \subset E\) such that:

\[
\lim_{n \to +\infty} \|Af_n - \rho(A)f_n\| = 0 \text{ and } \|f_n\| = 1, \forall n \in \mathbb{N}.
\]

Fix \(\lambda = \rho(S)\). Let \((f_{n,1})_{n \in \mathbb{N}}\) be a sequence of \(E\) such that

\[
\lim_{n \to +\infty} \|Sf_{n,1} - \rho(S)f_{n,1}\| = 0
\]

and

\[
\|f_{n,1}\| = 1, \forall n \in \mathbb{N}.
\]

For \(p \in \mathbb{N}^*\), observe that

\[
\lambda^\frac{1}{p} = \rho(\int S_{\frac{1}{p}}).
\]
Let \((f_{n,p})_{n \in \mathbb{N}} \subset E\) be a sequence such that
\[
\lim_{n \to +\infty} \left\| S_{\frac{1}{p}} f_{n,p} - \rho(S_{\frac{1}{p}} f_{n,p}) \right\| = 0
\]
and
\[
\left\| f_{n,p} \right\| = 1, \ \forall n \in \mathbb{N}.
\]
Notice that for all \(q \in \mathbb{N}^*, \) such that \(q \leq p\) we have:
\[
\left\| S_{\frac{1}{q}} f_{n,p} - \lambda_{\frac{1}{q}} f_{n,p} \right\| = \left\| (S_{\frac{1}{q}})^{\frac{1}{q}} f_{n,p} - (\lambda^{\frac{1}{q}})^{\frac{1}{q}} f_{n,p} \right\|
\leq \left( \prod_{u \in C, \ u \neq 1} \left\| S_{\frac{1}{q}} - u\lambda^{\frac{1}{q}} \right\| \right) \left\| S_{\frac{1}{q}} f_{n,p} - \lambda^{\frac{1}{q}} f_{n,p} \right\|.
\]
We have
\[
\prod_{u \in C, \ u \neq 1, \ u \neq 1} \left\| S_{\frac{1}{q}} - u\lambda^{\frac{1}{q}} \right\| \leq C,
\]
where \(C\) is a constant independent of \(n\) and hence we have
\[
\lim_{n \to +\infty} \left\| S_{\frac{1}{q}} f_{n,p} - \lambda_{\frac{1}{q}} f_{n,p} \right\| = 0.
\]
Consequently, by a diagonal extraction, we can construct \((f_n)_{n \in \mathbb{N}}\) such that:
\[
\lim_{n \to +\infty} \left\| S_{\frac{1}{p}} f_n - \lambda^{\frac{1}{p}} f_n \right\| = 0, \ \forall p \in \mathbb{N}^*
\]
and
\[
\left\| f_n \right\| = 1, \ \forall n \in \mathbb{N}.
\]
For all \(p \in \mathbb{N}^*\) and for all \(q \in \mathbb{N},\) we have
\[
S_{\frac{1}{p}} f_n - \lambda^{\frac{1}{p}} f_n = C_{q,p} \left( S_{\frac{1}{p}} - \lambda^{\frac{1}{p}} I \right) f_n,
\]
where \(C_{q,p}\) is a linear combination of translations. Then
\[
\left\| S_{\frac{1}{p}} f_n - \lambda^{\frac{1}{p}} f_n \right\| \leq \left\| C_{q,p} \right\| \left\| S_{\frac{1}{p}} f_n - \lambda^{\frac{1}{p}} f_n \right\|, \ \forall n \in \mathbb{N}
\]
and
\[
\lim_{n \to +\infty} \left\| S_{\frac{1}{p}} f_n - \lambda^{\frac{1}{p}} f_n \right\| = 0.
\]
On the other hand,
\[
\left\| S_{-\frac{1}{p}} f_n - \lambda^{-\frac{1}{p}} f_n \right\| \leq \left| \lambda^{-\frac{1}{p}} \right| \left\| S_{-\frac{1}{p}} \right\| \left\| \lambda^{\frac{1}{p}} f_n - S_{\frac{1}{p}} f_n \right\|, \ \forall n \in \mathbb{N}
\]
and
\[
\lim_{n \to +\infty} \left\| S_{-\frac{1}{p}} f_n - \lambda^{-\frac{1}{p}} f_n \right\| = 0, \ \forall p \in \mathbb{N}^*, \ \forall q \in \mathbb{N}.
\]
Since \( Q \) is dense in \( \mathbb{R} \), we deduce that

\[
\lim_{n \to +\infty} \| S_t f_n - \lambda^t f_n \| = 0, \quad \forall t \in \mathbb{R}.
\]

Now, fix \( \phi \in C_c(\mathbb{R}^+) \). Notice that

\[
\int_{\mathbb{R}} \phi(x) S_x f_n \, dx
\]

is a well-defined Bochner interval on \( E \) and

\[
T_\phi f_n = \int_{\mathbb{R}} \phi(x) (S_x f_n) \, dx.
\]  \hfill (2.4)

Indeed, let \( K \) be a compact subset of \( \mathbb{R}^+ \). We have \( T_\phi(C_c^\infty(\mathbb{R}^+)) \subset C_c^\infty(K + \text{supp}(\phi)(\mathbb{R}^+)) \) and the restriction of \( \int_{\mathbb{R}} \phi(x) S_x dx \) to \( C_c^\infty(K^+) \) can be considered as a Bochner integral on \( C_c^\infty(K^+) \) with values in \( C_c^\infty(K^+ + \text{supp}(\phi)(\mathbb{R}^+)) \). It is clear that for \( x \in \mathbb{R}^+ \), the map

\[
f \mapsto f(x)
\]

is a continuous linear form on \( C_c^\infty(K^+) \), for every compact \( K_0 \). Since Bochner integrals commute with continuous linear forms, we obtain, for \( g \in C_c(\mathbb{R}^+) \),

\[
(T_\phi g)(x) = (\phi \ast g)(x) = \int_{\mathbb{R}} \phi(y)g(x - y) \, dy = \int_{\text{supp}(\phi)} \phi(y)(S_y g)(x) \, dy
\]

\[
= \left( \int_{\text{supp}(\phi)} \phi(y)S_y g \right)(x), \quad \forall x \in \mathbb{R}^+
\]

and the formula (2.4) follows from the density of \( C_c(\mathbb{R}^+) \) in \( E \). Then, for all \( n \in \mathbb{N} \), we get

\[
\left| \int_{\mathbb{R}} \phi(x) \lambda^x \, dx \right| = \left\| \left( \int_{\mathbb{R}} \phi(x) \lambda^x \, dx \right) f_n \right\|
\]

\[
\leq \left\| \int_{\mathbb{R}} \phi(x) \lambda^x f_n \, dx - \int_{\mathbb{R}} \phi(x) S_x f_n \, dx \right\| + \left\| \int_{\mathbb{R}} \phi(x) S_x f_n \, dx \right\|
\]

\[
\leq \int_{\mathbb{R}} |\phi(x)||\lambda^x f_n - S_x f_n| \, dx + \| T_\phi \|.
\]

Taking into account the properties of \( (f_n)_{n \in \mathbb{N}} \) and the dominated convergence theorem, it follows that

\[
\lim_{n \to +\infty} \int_{\mathbb{R}} |\phi(x)||\lambda^x f_n - S_x f_n| \, dx = 0.
\]

Denote by \( C_r \) the circle of radius \( r \) and denote by \( D_r \) the line

\[
D_r = \{ z \in \mathbb{C} \mid \text{Im } z = r \}.
\]

We will write \( e^{iax} \) for the function

\[
x \mapsto e^{iax}.
\]
Since \( \|T_\phi\| = \|T_{\phi \alpha}\| \), for all \( \alpha \in \mathbb{R} \), we obtain
\[
\left| \int_{\mathbb{R}} \phi(x)x^\lambda dx \right| \leq \|T_\phi\|, \forall \lambda \in \mathcal{C}_p(S).
\]
We conclude that
\[
|\hat{\phi}(\alpha)| \leq \|T_\phi\|, \forall \phi \in \mathcal{C}_\infty^c(\mathbb{R}), \forall \alpha \in \mathcal{D}_{in,p}(S).
\]
Denote by \( E^* \) the dual space of \( E \) and denote by \( \|\cdot\|_* \) the norm of \( E^* \). For \( \lambda \in \mathcal{C}_{\mathbb{R}^{-1}} \), applying the same methods in \( E^* \), we obtain that there exists a sequence \((g_n)_{n \in \mathbb{N}} \subset E^*\) such that
\[
\lim_{n \to +\infty} \|(S_x)^* g_n - \lambda^2 g_n\|_* = 0
\]
and
\[
\|g_n\|_* = 1, \forall n \in \mathbb{N}.
\]
We notice that we have
\[
T^*_\phi = \left( \int_{\mathbb{R}^+} \phi(x)S_x dx \right)^* = \int_{\mathbb{R}^+} \phi(x)(S_x)^* dx, \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}), \quad (2.5)
\]
see [3]. Then we obtain as above that
\[
|\hat{\phi}(\alpha)| \leq \|T^*_\phi\| = \|T_\phi\|, \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}), \forall \alpha \in \mathcal{D}_{-in,p(S)}.
\]
From the Phragmen-Lindelöf theorem, it follows that
\[
|\hat{\phi}(\alpha)| \leq \|T_\phi\|, \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}), \forall \alpha \in U_E.
\]
\[\square\]

The proof of Theorem 1 follows from Proposition 2 and Lemma 2 exactly in the same way as in the proof of Theorem 1 in [4] and we omit the details.

In the proof of Theorem 2, we need the following technical lemma.

**Lemma 3.** Let \( \epsilon > 0, \eta_0 > 0 \) and \( V = \{\xi \in \mathbb{R}^+ : |\eta_0 - \xi| \leq \delta \} \subset \mathbb{R}^+ \) be fixed. Let \( C_0 > 0 \) be a fixed constant. For \( t_0 > 0 \) sufficiently large there exists a function \( f \in \mathcal{C}_c^\infty(\mathbb{R}^+) \) with the properties:
\[
\int_{\mathbb{R} \setminus V} |\hat{f}(\xi)| d\xi \leq \epsilon / C_0. \quad (2.6)
\]
\[
\int_{\mathbb{R}} |\hat{f}(\xi)| d\xi \leq 2\sqrt{2\pi}. \quad (2.7)
\]
\[
|f(t_0)| = 1. \quad (2.8)
\]
Proof. Introduce the function \( g \) with Fourier transform
\[
\hat{g}(\xi) = \frac{1}{a} e^{-\frac{(\xi - \eta_0)^2}{2a^2}} e^{-it_0 \xi},
\]
where \( a > 0 \) will be taken small enough below. We have
\[
\int_{\mathbb{R} \setminus V} |\hat{g}(\xi)| d\xi = \frac{1}{a} \int_{|\xi - \eta_0| \geq \delta} e^{-\frac{(\xi - \eta_0)^2}{2a^2}} d\xi
\leq e^{-\frac{\delta^2}{4a^2}} \frac{1}{a} \int_{\mathbb{R}} e^{-\frac{(\xi - \eta_0)^2}{4a^2}} d\xi \leq \frac{e}{2C_0}
\]
for \( a > 0 \) small enough. We fix \( a > 0 \) with this property. Obviously,
\[
\int_{\mathbb{R}} |\hat{g}(\xi)| d\xi = \int_{\mathbb{R}} e^{-\mu^2/2} d\mu = \sqrt{2\pi}.
\]
On the other hand,
\[
g(t) = \frac{1}{2\pi a} \int_{\mathbb{R}} e^{-\frac{(\xi - \eta_0)^2}{2a^2}} e^{i(t-t_0) \xi} d\xi
= \frac{1}{2\pi} e^{i(t-t_0)\eta_0} \int_{\mathbb{R}} e^{-\mu^2/2} e^{i(t-t_0)\mu} d\mu = e^{-\frac{a^2(t-t_0)^2}{2}} e^{i(t-t_0)\eta_0}
\]
and \( |g(t_0)| = 1 \).

Now we will take \( t_0 > 2 \) sufficiently large. Let \( \varphi \in C_c^\infty(\mathbb{R}) \) be a fixed function such that \( \varphi(t) = 0 \) for \( t \leq 1/2 \) and for \( t \geq 2t_0 - 1/2 \) and let \( \varphi(t) = 1 \) for \( 1 \leq t \leq 2t_0 - 1, \ 0 \leq \varphi \leq 1 \). Introduce the function \( f = \varphi g \in C_c^\infty(\mathbb{R}^+) \). The property (2.8) is trivial. We will show that (2.6) is satisfied for \( t_0 > 0 \) large enough depending on the choice of \( a > 0 \). The proof of (2.7) is similar and easier.

The function \( F = (\varphi - 1)g \) has a small Fourier transform. Moreover, given \( \epsilon > 0 \) we can take \( t_0 > 0 \) large enough in order to have
\[
|(1 + \xi^2) \hat{F}(\xi)| \leq \frac{\epsilon}{2\pi C_0} \tag{2.9}
\]
Indeed, for \( \xi^2 \hat{F}(\xi) \) we use an integration by parts with respect to \( t \) using the fact that
\[
\xi^2 e^{-it\xi} = -\partial_t^2 \left(e^{-it\xi}\right).
\]
On the support of \( (\varphi - 1) \) we have \( |t - t_0| > t_0 - 1 \). Thus after the integration by parts in the integral \( \int_{\mathbb{R}} e^{-it\xi}(1 + \xi^2) F(t) dt \) we are going to estimate an integral
\[
\int_{|t-t_0| \geq t_0 - 1} e^{-\frac{a^2(t-t_0)^2}{2}} |P(t)| dt
\]
with \( P \) a polynomial of degree not greater than 2.
To get (2.9), remark that this integral is bounded by
\[ C\left[\int_{-\infty}^{1-t_0} y^2 e^{-a^2 y^2/2}dy + \int_{t_0-1}^{\infty} y^2 e^{-a^2 y^2/2}dy\right] \]
and taking \( t_0 > 0 \) sufficiently large we arrange (2.9). Next we obtain
\[ \int_{R\backslash V} |\hat{f}(\xi)|d\xi \leq \int_{R\backslash V_t} |\hat{g}(\xi)|d\xi + \int_{R\backslash V_c} |\hat{F}(\xi)|d\xi \]
\[ \leq \frac{\epsilon}{2C_0} + \frac{\epsilon}{2\pi C_0} \int_{R} (1 + \xi^2)^{-1}d\xi \leq \frac{\epsilon}{C_0}. \]
The proof of the lemma is complete. \( \square \)

**Proof of Theorem 2.** First, we show that
\[ \left\{ z \in \mathbb{C}, \frac{1}{\rho(S-z)} \leq |z| \leq \rho(S) \right\} \subset spec(S). \tag{2.10} \]
Fix \( \lambda \notin \text{spec}(S) \). Then the operator \((S - \lambda I)^{-1}\) is a Wiener-Hopf operator and following 2) of Theorem 1, we get
\[ (S - \lambda I)^{-1}(f)_a = P^+ \mathcal{F}^{-1}(\nu_a(f)_a), \forall a \in I_E, \forall f \in C_c^\infty(\mathbb{R}^+), \]
where \( \nu_a \in L^\infty(\mathbb{R}) \). Replacing, \( f \) by \((S - \lambda I)g\), we obtain
\[ (g)_a = P^+ \mathcal{F}^{-1}\left(\nu_a \mathcal{F}\left((S - \lambda I)g\right)\right), \forall g \in C_c^\infty(\mathbb{R}^+). \]
Denote by \( e^{a+ix} \) the function \( x \rightarrow e^{a+ix} \).

It is easy to see that
\[ \mathcal{F}((Sg)_a)(t) = e^{-it} \mathcal{F}((g)_a)(t), \forall a \in I_E, \forall t \in \mathbb{R}, \forall g \in C_c^\infty(\mathbb{R}^+). \]
Consequently,
\[ (g)_a(t) = \mathcal{F}^{-1}[(e^{a-i} - \lambda)\nu_a(\widehat{g})_a](t), \forall a \in I_E, \forall t \in \mathbb{R}^+, \forall g \in C_c^\infty(\mathbb{R}^+). \tag{2.11} \]
We have
\[ \| \mathcal{F}^{-1}[(e^{a-i} - \lambda)\nu_a(\widehat{g})_a]\|_\infty \leq \| (e^{a-i} - \lambda)\nu_a(\widehat{g})_a\|_{L^1(\mathbb{R})}, \forall a \in I_E, \forall g \in C_c^\infty(\mathbb{R}^+). \tag{2.12} \]
Now, suppose that \( |\lambda| = e^b \), for some \( b \in I_E \). Choose a small \( \epsilon \in ]0,1[ \). It is easy to find an interval \( V_\epsilon \subset \mathbb{R}^+ \) such that
\[ |e^{b-it} - \lambda| \leq \frac{\epsilon}{2\|\nu_b\|_\infty}, \forall t \in V_\epsilon. \]
Taking into account Lemma 3, we can choose \( g \in C_c^\infty(\mathbb{R}^+) \) satisfying the following three conditions:
\[ 1) \int_{R\backslash V_\epsilon} |(g)_b(t)|dt \leq \frac{\epsilon}{4\|\nu_b\|_\infty} \]
2) $\int_{V_e} |(g)_{b}(t)| dt \leq 1$
3) There exists $t_0 \in \mathbb{R}^+$, such that $|(g)_{b}(t_0)| \geq \epsilon$.

Taking into account that (2.11) and (2.12) hold for $g \in C_c^\infty(\mathbb{R}^+)$, we get

$$|(g)_{b}(t_0)| \leq \int_{V_e} |e^{b-it} - \lambda\|\nu_b\|_\infty|(g)_{b}(t)| dt + \int_{\mathbb{R}\setminus V_e} |e^{b-it} - \lambda\|\nu_b\|_\infty|(g)_{b}(t)| dt \leq \epsilon.$$ 

Hence we obtain a contradiction so $|\lambda| \neq e^a$, $\forall a \in I_E$ and (2.10) follows.

We will prove now that

$$\left\{ z \in \mathbb{C}, \frac{1}{\rho(S)} \leq |z| \leq \rho(S-1) \right\} \subset \text{spec}(S-1). \quad (2.13)$$

Let $\lambda \notin \text{spec}(S-1)$. Then $(S-1 - \lambda I)^{-1} \in W(E)$. Indeed, for all $x \in \mathbb{R}^+$, we observe that

$$S_{-x}(S-1 - \lambda I)^{-1}S_x = (S-1 - \lambda I)^{-1}(S-1 - \lambda I)S_{-x}(S-1 - \lambda I)^{-1}S_x = (S-1 - \lambda I)^{-1}. $$

Hence, for all $g \in C_c^\infty(\mathbb{R}^+)$ and for each $a \in I_E$, we have

$$((S-1 - \lambda I)^{-1}g)_a = P^+ F^{-1}(h_a\hat{g}_a),$$

for some $h_a \in L^\infty(\mathbb{R})$ and

$$(f)_a = P^+ F^{-1}\left(h_a F(((S-1 - \lambda I)f)_a)\right), \forall f \in C_c^\infty(\mathbb{R}^+).$$

Then

$$F\left(((S-1 - \lambda I)f)_a\right)(t) = (e^{it-a} - \lambda)(f)_a(t), \text{ a.e. on } \mathbb{R}^+,$$

if we suppose that $\text{supp}(f) \subset [1, \infty[$. Repeating the argument of the proof of (2.10), we get a contradiction if $|\lambda| = e^{-a}$, for some $a \in I_E$. We conclude that

$$\left\{ z \in \mathbb{C}, \frac{1}{\rho(S)} \leq |z| \leq \rho(S-1) \right\} \subset \text{spec}(S-1).$$

It follows that, if $z \in \mathbb{C}$ is such that $\frac{1}{\rho(S-1)} \leq |z| \leq \rho(S)$ then $\frac{1}{z} \in \text{spec}(S-1)$ and we deduce that

$$\left\{ z \in \mathbb{C}, \frac{1}{\rho(S-1)} \leq |z| \leq \rho(S) \right\} \subset \text{spec}(S) \cap \left(\text{spec}(S-1)\right)^{-1}.$$ 

From the definition of the spectral radius we get immediately that

$$\text{spec}(S) \cap \left(\text{spec}(S-1)\right)^{-1} \subset \left\{ z \in \mathbb{C}, \frac{1}{\rho(S-1)} \leq |z| \leq \rho(S) \right\}$$

and the proof of Theorem 2 is complete. \(\square\)
Proposition 3. If $\phi \in C_c^\infty(\mathbb{R})$, then

$$\widehat{\phi}(U_E) \subset \text{spec}(T_\phi).$$

**Proof.** Fix $\lambda \notin \text{spec}(T_\phi)$. Then $(T_\phi - \lambda I)^{-1}$ is a Wiener-Hopf operator and we obtain as above

$$(g)_a = P^+ F^{-1}(\nu_a[\widehat{(\phi)} - \lambda \widehat{(g)}]_a), \forall g \in C_c^\infty(\mathbb{R}^+), \forall a \in I_E,$$

where $\nu_a \in L^\infty(\mathbb{R})$. Choosing a suitable $g \in C_c^\infty(\mathbb{R}^+)$, we obtain in the same way as in the proof of (2.10) a contradiction if

$$\widehat{(\phi)}_a(t) = \lambda,$$

for some $a \in I_E$ and some $t \in \mathbb{R}$ and the proposition follows immediately. □

3. Wiener-Hopf operators on Banach spaces of functions on $\mathbb{R}^+$ with values in a Hilbert space $H$

Now let $H$ be a separable Hilbert space. Denote by $< u, v >$ the scalar product of $u, v \in H$. Let $\|u\|_H$ be the norm of $u \in H$. In this section we prove Theorem 3. Let $E$ be the Banach space of functions from $\mathbb{R}^+$ into $H$ satisfying (H1), (H2) and (H3). Let $\overline{E}$ be a Banach space of functions $F : \mathbb{R}^+ \rightarrow H$

such that

$$\left( \mathbb{R}^+ \ni x \rightarrow \|F(x)\|_H \right) \in E.$$

We have the following two lemmas.

**Lemma 4.** The space $C_c^\infty(\mathbb{R}^+) \otimes H$ is dense in $\overline{E}$.

**Proof.** Let $\Phi \in \overline{E}$. Then there exists a positive sequence $(\phi_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^+)$ such that

$$\lim_{n \rightarrow +\infty} \|\phi_n - \|\Phi(.)\|_H\|_E = 0.$$

For almost every $x \in \mathbb{R}^+$, set

$$\Phi_n(x) = \phi_n(x) \frac{\Phi(x)}{\|\Phi(x)\|_H}, \text{ if } \Phi(x) \neq 0,$$

$$\Phi_n(x) = 0, \text{ if } \Phi(x) = 0.$$

We have

$$\|\Phi_n - \Phi\|_{\overline{E}} = \|\Phi_n(.) - \Phi(.)\|_H \|_E = \|\phi_n - \|\Phi(.)\|_H\|_E$$

and it is clear that

$$\lim_{n \rightarrow +\infty} \|\Phi_n - \Phi\|_{\overline{E}} = 0.$$

Since $C_c^\infty(\mathbb{R}^+) \otimes H$ is dense in $C_0(\mathbb{R}^+, H)$, the space $C_c^\infty(\mathbb{R}^+) \otimes H$ is dense in $\overline{E}$. □
Lemma 5. If $\Phi \in \overline{E}$ and $u \in H$, then the function defined by
$$\mathbb{R}^+ \ni x \mapsto \langle \Phi(x), u \rangle \in \mathbb{C}$$
is a element of $E$.

Proof. Let $\left( \sum_{n=1}^{N} \phi_n u_n \right)_{N \geq 0}$ be a sequence in $C^\infty_c(\mathbb{R}^+) \otimes H$ such that $\phi_n \in C^\infty_c(\mathbb{R}^+)$, $\forall n \in \mathbb{N}$ and
$$\lim_{N \to +\infty} \left\| \sum_{n=1}^{N} \phi_n u_n - \Phi \right\|_E = 0.$$Let $u \in H$. Then we have
$$\lim_{N \to +\infty} \left\| \sum_{n=1}^{N} \phi_n(.) u_n, u > - \langle \Phi(.), u \rangle \right\|_E$$
$$\leq \lim_{N \to +\infty} \left\| \sum_{n=1}^{N} \phi_n(.) u_n - \Phi(.) \right\|_H \left\| u \right\| = 0.$$Now, it is clear that
$$x \mapsto \langle \Phi(x), u \rangle \in \mathbb{C}$$is a element of $E$. $\square$

In the proof of Theorem 3 we will also use the following lemma.

Lemma 6. Let $G \in L^2(\mathbb{R}, H)$ and $v \in H$. Then we have
$$\mathcal{F}(\langle G(.), v \rangle)(x) = \langle \mathcal{F}(G)(x), v \rangle,$$for almost every $x \in \mathbb{R}$.

The reader may find the proof of Lemma 6 in [6]. Next we pass to the proof of our main result.

Proof of Theorem 3. Let $T \in W(\overline{E})$. Fix $u, v \in H$. Define $T_{u,v}$ on $E$ by the formula
$$(T_{u,v}f)(x) = \langle T(fu)(x), v \rangle, \forall f \in E, a.e.$$From Lemma 5, it follows that $T_{u,v}$ is an operator from $E$ into $E$. It is clear that
$$S_{-x} \langle S_x fu, v \rangle = \langle S_{-x}T(S_x fu), v \rangle = \langle T(fu), v \rangle, \forall x \in \mathbb{R}^+.$$Then we see that $T_{u,v} \in W(E)$. Following Theorem 1, for $a \in I_E$ there exists a function $\nu_{a,u,v} \in L^\infty(\mathbb{R})$ such that
$$(T_{u,v}f)_a = P^+ \mathcal{F}^{-1}(\nu_{a,u,v}(f)_a), \forall f \in C^\infty_c(\mathbb{R}^+)$$.
Let $B$ be an orthonormal basis of $H$ and let $O$ be the set of finite linear combinations of elements of $B$. We have
\[ |\nu_{a,u,v}(x)| \leq C \|T_{u,v}\|, \quad \forall x \in \mathbb{R} \setminus N_{u,v}, \]
where $N_{u,v}$ is a set of measure zero. Without loss of generality, we can modify $\nu_{a,u,v}$ on $N = \bigcup_{(u,v) \in O \times O} N_{u,v}$ in order to obtain
\[ |\nu_{a,u,v}(x)| \leq C \|M_{u,v}\| \leq C \|T\| \|u\| \|v\|, \quad \forall u, v \in O, \text{a.e.} \]
For fixed $x \in \mathbb{R} \setminus N$ we observe that $O \times O \ni (u,v) \rightarrow \nu_{a,u,v}(x) \in C$ is a sesquilinear and continuous form on $O \times O$ and since $O$ is dense in $H$, we conclude that there exists an unique map
\[ (H \times H \ni (u,v) \rightarrow \tilde{\nu}_{a,u,v}(x) \in C \]
such that
\[ \tilde{\nu}_{a,u,v}(x) = \nu_{a,u,v}(x), \quad \forall u, v \in O. \]
Consequently, there exists an unique map
\[ \mathcal{V}_a : \mathbb{R} \rightarrow \mathcal{L}(H) \]
such that
\[ \langle \mathcal{V}_a(x)[u], v \rangle = \tilde{\nu}_{a,u,v}(x), \forall u, v \in H, \text{a.e.} \]
It is clear that
\[ \| \mathcal{V}_a(x) \| = \sup_{\|u\|=1,\|v\|=1} | \langle V_a(x)[u], v \rangle | \leq C \|T\|, \text{a.e.} \]
Fix $a \in I_E$ and $f \in C^\infty_c(\mathbb{R}^+)$. It is obvious that we have $(\hat{f})_a(x)u \in H, \forall x \in \mathbb{R}$. Next for almost every $x \in \mathbb{R}^+$, we obtain
\[ \mathcal{F}^{-1}(\langle \mathcal{V}_a(.)[(\hat{f})_a(.)]u, v \rangle)(x) = \mathcal{F}^{-1}(\langle V_a(.)[u], v \rangle = (f)_a(.)u)(x) = \mathcal{F}^{-1}(\tilde{\nu}_{a,u,v}(.)(\hat{f})_a(.)u)(x) = (T_{u,v}f)_a(x). \]
Consequently,
\[ \mathcal{F}^{-1}(\langle \mathcal{V}_a(.)[(\hat{f})_a(.)]u, v \rangle)(x) = (T[fu](.), v >)_a(x), \quad (3.1) \]
for almost every $x \in \mathbb{R}^+$. Now, consider the function $\Psi_a$ on $\mathbb{R}^+$ defined for almost every $x \in \mathbb{R}^+$ by the formula
\[ \Psi_a(x) = \mathcal{V}_a(x)[(\hat{f})_a(x)u] \]
and observe that $\Psi_a \in L^2(\mathbb{R}^+, H)$. Indeed, we have
\[ \int_{\mathbb{R}^+} \| \mathcal{V}_a(x)[(\hat{f})_a(x)u] \|^2 dx \]
\[ \leq \int_{\mathbb{R}^+} \|V_a(x)\| \|\hat{(f)}_a(x)u\|^2 dx \]
\[ \leq C^2 \|T\|^2 \int_{\mathbb{R}^+} |\hat{(f)}_a(x)|^2 \|u\|^2 dx < +\infty. \]

This makes possible to apply Lemma 6, and we get
\[ \mathcal{F}^{-1}(\langle V_a(.)\hat{(f)}_a(.)u, v >)(x) = \mathcal{F}^{-1}(V_a(.)\hat{(f)}_a(.)u)(x), \]
for almost every \( x \in \mathbb{R}^+ \). It follows from (3.1) that we have
\[ (T[fu])_a(x) = \mathcal{F}^{-1}(V_a(.)\hat{(f)}_a(.)u)(x), \]
for almost every \( x \in \mathbb{R}^+ \) and this yields
\[ (T[fu])_a \in L^2(\mathbb{R}^+, H). \]

This completes the proof of 1) and 2). The proof of 3) uses the same argument as the proof of the assertion 3) of Theorem 1. \( \Box \)

**Proof of Theorem 4.** Fix \( \alpha \in \mathbb{C} \) and suppose that \( \alpha \not\in \text{spec}(S) \). Then we have \((S - \alpha I)^{-1} \in W(E)\) and from Theorem 3, we get
\[ ((S - \alpha I)^{-1}F)_a = \mathcal{F}^{-1}(V_a(.)\hat{(F)}_a(.)], \forall a \in I_E, \forall F \in C_c^\infty(\mathbb{R}^+) \otimes H. \]
Replacing \( F \) by \((S - \alpha I)G\), we get
\[ (G)_a(x) = \mathcal{F}^{-1}(V_a(.)\mathcal{F}[(S - \alpha I)G(.)](x) = \mathcal{F}^{-1}\left(V_a(.)[(e^{\alpha - i} - \alpha)(\hat{G})_a(.)]\right)(x). \]

We have
\[ \|G\|_{1, \infty} \leq \|V_a(.)[(e^{\alpha - i} - \alpha)(\hat{G})_a(.)]\|_{L^1(\mathbb{R})}, \forall a \in I_E. \]
Then if \( |\alpha| = e^a \), for some \( a \in I_E \) choosing a suitable \( G \in C_c^\infty(\mathbb{R}^+) \otimes H \) in the same way as in the proof of Theorem 2, we obtain a contradiction. Hence,
\[ \left\{ z \in \mathbb{C}, \frac{1}{\rho(S-1)} \leq |z| \leq \rho(S) \right\} \subset \text{spec}(S). \]

In the same way, we obtain
\[ \left\{ z \in \mathbb{C}, \frac{1}{\rho(S-1)} \leq |z| \leq \rho(S) \right\} \subset (\text{spec}(S-1))^{-1}. \]

It follows that
\[ \left\{ z \in \mathbb{C}, \frac{1}{\rho(S-1)} \leq |z| \leq \rho(S) \right\} \subset \text{spec}(S) \cap \left(\text{spec}(S-1)\right)^{-1}. \]

Taking into account that
\[ \text{spec}(S) \cap \left(\text{spec}(S-1)\right)^{-1} \subset \left\{ z \in \mathbb{C}, \frac{1}{\rho(S-1)} \leq |z| \leq \rho(S) \right\} \]
and
\[ \|S\| \leq \|S\|, \quad \|S_{-1}\| \leq \|S_{-1}\|, \]
(see Section 1), we observe that
\[ \rho(S) = \rho(S), \quad \rho(S_{-1}) = \rho(S_{-1}). \]
We deduce that
\[ \text{spec}(S) \cap \left( \text{spec}(S_{-1}) \right)^{-1} = \left\{ z \in \mathbb{C}, \frac{1}{\rho(S_{-1})} \leq |z| \leq \rho(S) \right\} \]
and the proof of Theorem 4 is complete. □

4. Generalizations

In this section we first deal with the Wiener-Hopf operators in a larger class of Banach spaces of functions on \( \mathbb{R}^+ \) with values in a separable Hilbert space. Let \( W \) be an operator-valued weight on \( \mathbb{R}^+ \). It means that \( W \) is an operator-valued weight on \( \mathbb{R}^+ \), and it satisfies the property
\[ 0 < \sup_{x \in \mathbb{R}^+} \frac{\|W(x + y)\|}{\|W(x)\|} < +\infty, \quad \forall y \in \mathbb{R}^+. \]  
(4.1)
This implies (see [4], [5]) that for every compact \( K \) of \( \mathbb{R}^+ \), we have
\[ \sup_{x \in K} \|W(x)\| < +\infty. \]
Notice that if \( H \) has a finite dimension, \( W \) is given by a matrix. We denote by \( L^p_W(\mathbb{R}^+, H) \) the space of measurable functions \( F \) on \( \mathbb{R}^+ \) with values in \( H \) such that
\[ \int_{\mathbb{R}^+} \|W(x)[F(x)]\|_H^p dx < +\infty. \]
For illustration we give a simple example.

**Example.** If \( H \) is the space \( \mathbb{R}^5 \), the operator-valued weight \( W \) defined for \( x \) by the matrix
\[
\begin{pmatrix}
  1 & e^x & e^{3x} & 1 & 1 \\
  1 + x & x & e^x & 1 & e^{3x} \\
  e^x & 1 & 1 & x & x + 1 \\
  1 & 1 & e^x & e^{2x} & 1 \\
  x & x & 1 + x & e^x & \frac{x^2}{2}
\end{pmatrix}
\]
is such that the condition (4.1) trivially holds.
The space $L^p_W(\mathbb{R}^+, H)$ is equipped with the norm
\[
\left( \int_{\mathbb{R}^+} \|W(x)[F(x)]\|_H^p \, dx \right)^{\frac{1}{p}}.
\]
Let $T$ be a Wiener-Hopf operator on $L^p_W(\mathbb{R}^+, H)$. We fix $u, v \in H$. Notice that for $u \in H$ and $f \in L^p_W(\mathbb{R}^+)$, we have $fu \in L^p_W(\mathbb{R}^+, H)$. Indeed,
\[
\int_{\mathbb{R}^+} \|W(x)[f(x)u]\|_H^p \, dx \leq \int_{\mathbb{R}^+} \|W(x)\|_H^p |f(x)|^p \|u\|_H^p \, dx < +\infty.
\]
Introduce the operator $T_{u,v}$ defined on $L^p_W(\mathbb{R}^+)$ by the formula
\[
(T_{u,v}f)(x) = \langle T(fu)(x), v \rangle, \text{ a.e., } \forall f \in L^p_W(\mathbb{R}^+).
\]
It is easy to see that
\[
\int_{\mathbb{R}^+} \|W(x)[f(x)u]\|_H^p \, dx \leq \int_{\mathbb{R}^+} \|W(x)\|_H^p |f(x)|^p \|u\|_H^p \, dx < +\infty.
\]
Consequently, $T_{u,v}$ is a Wiener-Hopf operator on $L^p_W(\mathbb{R}^+)$. Therefore $T_{u,v}$ has a symbol following Theorem 1. Applying the methods exposed in Section 3, we obtain that Theorem 3 holds also if we replace $E$ by $L^p_W(\mathbb{R}^+)$, for $1 \leq p < \infty$. Denote by $I_E$ (resp. $U_E$) the set $I_E$ (resp. $U_E$) for $E = L^p_W(\mathbb{R}^+)$. We recall that $I_E$ and $U_E$ are defined in the Introduction.

We have the following.

**Theorem 5.** Let $T$ be a Wiener-Hopf operator on $L^p_W(\mathbb{R}^+, H)$, for $1 \leq p < \infty$.
1) We have $(T\Phi)_a \in L^2(\mathbb{R}^+, H)$, $\forall \Phi \in C^\infty_c(\mathbb{R}^+) \otimes H$, $\forall a \in I_W$.
2) There exists $\mathcal{V}_a \in L^\infty(\mathbb{R}, \mathcal{L}(H))$ such that
\[
(T\Phi)_a = \mathcal{P}^+ F^{-1}(\mathcal{V}_a([\hat{\Phi}]_a)(\cdot)), \forall a \in I_W, \forall \Phi \in C^\infty_c(\mathbb{R}^+) \otimes H.
\]
Moreover, $\esssup_{x \in \mathbb{R}} \|\mathcal{V}_a(x)\| \leq C\|T\|$. 
3) If $U_W \neq \emptyset$, set
\[
\mathcal{V}(x+ia) = \mathcal{V}_a(x), \forall a \in \overset{\circ}{I}_W, \text{ for almost every } x \in \mathbb{R}
\]
We have $\sup_{z \in \overset{\circ}{U}_W} \|\mathcal{V}(z)\| \leq C\|T\|$ and for $u, v \in H$, the function
\[
z \rightarrow \mathcal{V}(z)u, v
\]
is analytic on $\overset{\circ}{U}_W$. 

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The results of Section 3 and Section 4 hold if we replace $H$ by a separable Banach space $B$ satisfying the following conditions:
1) $B$ has a countable basis.
2) The dual space of $B$ denoted by $B^*$ has a countable basis.
For example these conditions are satisfied if $B = l^p_\omega(Z)$, where $\omega$ is a weight on $Z$ and $1 \leq p < +\infty$. We recall that $\omega$ is a weight on $Z$, if $\omega$ is a positive sequence on $Z$ satisfying
$$0 < \sup_{k \in Z} \frac{\omega(k + n)}{\omega(k)} < +\infty, \forall n \in \mathbb{Z}.$$ 

It is easy to see that $B^* = l^{q*}_\omega(Z)$, where $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. The weight $\omega^*$ is given by the formula
$$\omega^*(n) = \frac{1}{\omega(-n)}, \forall n \in \mathbb{Z}.$$ 

Denote by $e_n$ the sequence defined by $e_n(k) = 0$ if $n \neq k$ and $e_n(n) = 1$. Considering the family $\{e_n\}_{n \in \mathbb{Z}}$ included in $l^p_\omega(Z)$ and in $l^{q*}_\omega(Z)$, it is trivial to see that les conditions 1) and 2) are satisfied.

Let $B$ be a Banach space satisfying 1) and 2). Let $E$ be a Banach space of functions on $\mathbb{R}^+$ satisfying (H1)-(H3). Denote by $<,>_B$ the duality between $B$ and $B^*$. Let $E$ be the space of functions $F: \mathbb{R}^+ \rightarrow B$ such that $\|F(.)(x)\|_B \in E$. Let $T$ be a Wiener-Hopf operator on $E$. Then using the operators $T_{u,v}$ defined by
$$(T_{u,v}f)(x) = < T(fu)(x), v>_B, \forall u \in B, \forall v \in B^*, \text{ a.e.}$$
and the arguments of the proof of Theorem 3, we obtain an extended version of Theorem 3 in the case of spaces of functions on $\mathbb{R}^+$ with values in $B$. For example Theorem 3 holds for the Wiener-Hopf operators on spaces of the form $L^p_{\omega_1}(\mathbb{R}^+, l^q_{\omega_2}(Z))$, for $1 \leq p < \infty$, $1 \leq q < \infty$, where $\omega_1$ (resp. $\omega_2$) is a weight on $\mathbb{R}^+$ (resp. $Z$). The arguments developed in this paper do not hold if we replace $l^q_{\omega_2}(Z)$ by $L^q_{\omega_2}(\mathbb{R})$, for $q \neq 2$. The existence of the symbol of a Wiener-Hopf operator on the family of spaces $L^p_{\omega_1}(\mathbb{R}^+, L^q_{\omega_2}(\mathbb{R}))$ for $q \neq 2$ is an interesting direction of investigation.

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