Some new periodic Golay pairs

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Abstract

Periodic Golay pairs are a generalization of ordinary Golay pairs. They can be used to construct Hadamard matrices. A positive integer \( v \) is a (periodic) Golay number if there exists a (periodic) Golay pair of length \( v \). Taking into the account the results obtained in this note and yet unpublished new result \cite{9}, there are only seven known periodic Golay numbers which are definitely not Golay numbers, namely 34, 50, 58, 68, 72, 74, 82. We construct here periodic Golay pairs of lengths 74, 122, 164, 202, 226. It is apparently unknown whether 122, 164, 202, 226 are Golay numbers. The smallest length for which the existence of periodic Golay pairs is undecided is now 90.

1 Preliminaries

Let \( v \) be a positive integer and \( \mathbb{Z}_v = \{0, 1, \ldots, v - 1\} \) the ring of integers modulo \( v \). Let \( A = a_0, a_1, \ldots, a_{v-1} \) be a binary, i.e., \( \{±1\}\)-sequence of length \( v \). The periodic autocorrelation function \( \text{PAF}_A : \mathbb{Z}_v \to \mathbb{Z} \) and the nonperiodic autocorrelation function \( \text{NAF}_A : \mathbb{Z} \to \mathbb{Z} \) of \( A \) are defined by

\[
\text{PAF}_A(s) = \sum_{i=0}^{v-1} a_ia_{i+s} \pmod{v},
\]

\[
\text{NAF}_A(s) = \sum_{i \in \mathbb{Z}} a_ia_{i+s}.
\]

(In the nonperiodic case we set \( a_i = 0 \) when \( i < 0 \) or \( i \geq v \).) Note that

\[
\text{PAF}_A(s) = \text{NAF}_A(s) + \text{NAF}_A(v - s), \quad s \in \mathbb{Z}_v.
\]

A Golay pair is an ordered pair \((A, B)\) of binary sequences of length \( v \) such that \( \text{NAF}_A(s) + \text{NAF}_B(s) = 0 \) for \( s \neq 0 \). Similarly, a periodic Golay pair is an ordered pair \((A, B)\) of binary sequences of length \( v \) such that \( \text{PAF}_A(s) + \text{PAF}_B(s) = 0 \) for \( s \neq 0 \). It follows from (1) that any Golay pair is also a periodic Golay pair. Periodic Golay pairs are also known as complementary binary sequences. They can be used to construct Hadamard matrices of order \( 2v \) (see subsection 1.1 below). Thus, if \( v > 1 \) is a periodic Golay number then \( v \) must be even.

Periodic Golay pairs can be viewed as a particular case of supplementary difference sets (SDS). Let us recall some facts which will be used in our constructions, see e.g. \cite{6} Proposition 1.

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First we have to recall the definition of SDS. Let $k_1, \ldots, k_t$ be positive integers and $\lambda$ an integer such that $\lambda(v-1) = \sum_{i=1}^{t} k_i(k_i - 1)$, and let $X_i$ be a subset of $\mathbb{Z}_v$ with cardinality $|X_i| = k_i$, $i \in \{1, 2, \ldots, t\}$.

**Definition 1** We say that $(X_1, \ldots, X_t)$ are supplementary difference sets with parameters $(v; k_1, \ldots, k_t; \lambda)$, if for every nonzero element $c \in \mathbb{Z}_v$ there are exactly $\lambda$ ordered pairs $(a, b)$ such that $a - b = c \pmod{v}$ and $\{a, b\} \subseteq X_i$ for some $i \in \{1, 2, \ldots, t\}$.

For convenience, we shall refer to the sets $X_1, \ldots, X_t$ as the base blocks of this SDS. To any SDS with parameters $(v; k_1, \ldots, k_t; \lambda)$ we attach an additional parameter $n$ defined by $n = k_1 + \cdots + k_t - \lambda$. (SDSs with $t = 1$ are known as cyclic difference sets.)

There is a bijection from the set of all binary sequences of length $v$ to the set of all subsets of $\mathbb{Z}_v$ which assigns to the sequence $A$ the subset $\{i \in \mathbb{Z}_v : a_i = -1\}$. If $(A, B)$ is a periodic Golay pair, then the corresponding pair of subsets $(X, Y)$ of $\mathbb{Z}_v$ is an SDS whose parameters $(v; r, s; \lambda)$ satisfy the equation $v = 2n$. (Recall that $n = r + s - \lambda$ in this case.) The converse is also true. Moreover, if $a = v - 2r$ and $b = v - 2s$ then $a^2 + b^2 = 2v$. In particular, $v$ must be a sum of two squares.

### 1.1 Periodic Golay pairs and Hadamard matrices

One of the reasons that periodic Golay pairs are useful is that they can be used to construct Hadamard matrices. Let us detail the relevant construction. Suppose that two binary sequences $A, B$ of length $v$ that form a periodic Golay pair are given. Then one can construct the associated $v \times v$ circulant matrices $C_A, C_B$ whose first rows are the sequences $A, B$ respectively. These two circulant matrices satisfy the matrix equation

$$C_A C_A^t + C_B C_B^t = (2v)I_v$$

where $t$ denotes transposition and $I_v$ denotes the $v \times v$ unit matrix. Using these two circulant matrices, a Hadamard matrix of order $2v$ can be constructed as

$$H_{2v} = \begin{bmatrix} C_A & C_B \\ -C_B^t & C_A^t \end{bmatrix}.$$  (2)

We refer the reader to [14] for more details and additional Hadamard matrix constructions.

### 2 Known Golay and periodic Golay numbers

If $\alpha, \beta, \gamma$ are nonnegative integers, then it is well known that $2^\alpha 10^\beta 26^\gamma$ is a Golay number. No additional Golay numbers have been found so far. Moreover, there are no other Golay numbers in the range $1, 2, \ldots, 100$ (see [2]).

We have already mentioned in Section 1 that the periodic Golay number $v > 1$ must be even and a sum of two squares. The following two important necessary conditions for Golay and periodic Golay numbers have been proved more than twenty years ago.
Theorem 1 (Eliahou-Kervaire-Saffari [12]) A Golay number is not divisible by any prime \( p \equiv 3 \pmod{4} \).

Theorem 2 (Arasu-Xiang [1, Corollary 3.6]) If \( v = p^t u > 1 \) is a periodic Golay number, \( p \equiv 3 \pmod{4} \) is a prime number, and \( (p, u) = 1 \) then \( u \geq 2p^{t/2} \).

(Since \( v \) is a sum of two squares, the exponent \( t \) is an even integer.)

By applying this theorem to the integers in the range 1, 2, ..., 500, we deduce that the numbers 18, 36, 98, 162, 242, 324, 392, 484, 490 are not periodic Golay although each of them is even and a sum of two squares.

It has been shown very recently [10, Theorem 1] that the product of a Golay number and a periodic Golay number is again a periodic Golay number. More precisely, the authors of that paper show how to “multiply” an ordinary Golay pair of length \( g \) and a periodic Golay pair of length \( d \) to obtain a periodic Golay pair of length \( gd \). Consequently, there are infinitely many periodic Golay numbers which are not of the form \( 2^a 10^b 26^c \). We point out that this “multiplication theorem” is an easy consequence of [11, Theorems 13 and 14].

As noted in the abstract, there are currently only seven periodic Golay numbers for which we know that they are not Golay numbers. More specifically we have that:

- The first periodic Golay pair whose length, 34, is not a Golay number has been found in 1998 by Đoković [3]. (In fact two non-equivalent such pairs were found.)
- Periodic Golay pairs of length 50 have been found by Đoković [4] and Kotsireas and Koukouvinos [13].
- Periodic Golay pairs of length 58 have been found by Đoković and Kotsireas [6].
- Several periodic Golay pairs of length 68 have been constructed recently in [8]. Such pairs can be constructed also by using the method described in [10].
- Periodic Golay pairs of length 72 have been constructed recently [9].
- For examples of periodic Golay pairs of length 74 see the next section.
- Periodic Golay pair of length 82 has been found in 2008 by Vollrath [15] (see also the next section).

3 New periodic Golay pairs

We have constructed several new periodic Golay pairs, and we deduce that 74, 122, 164, 202, 226 are periodic Golay numbers. As 74 < 100 we know that 74 is not a Golay number. It is apparently unknown whether 122, 164, 202, 226 are Golay numbers. We also construct periodic Golay pairs of length 82, not equivalent to the known one [15]. In all cases below, the non-equivalence is established by the method described in [4].
The SDSs \((X, Y)\) listed below are given by using the following compact notation. The parameter set is \((v; r, s; \lambda)\) with \(n = r + s - \lambda\) and \(v = 2n\). In each case, we make use of a nontrivial subgroup \(H\) of the group of units \(\mathbb{Z}_v^*\) of the ring \(\mathbb{Z}_v\). This subgroup acts on \(\mathbb{Z}_v\) by multiplication modulo \(v\). The orbit of \(H\) containing \(j \in \mathbb{Z}_v\) is given by \(H \cdot j = \{hj \pmod{v} : h \in H\}\). The base blocks \(X\) and \(Y\) are composed of orbits of \(H\). Thus we have

\[X = \bigcup_{j \in J} H \cdot j, \quad Y = \bigcup_{k \in K} H \cdot k,\]

where \(J, K \subseteq \mathbb{Z}_v\). Instead of listing the elements of \(X\) and \(Y\) we shall list only their index sets \(J\) and \(K\), respectively. We note that, as a representative \(j\) of an orbit \(H \cdot j\) we always choose the smallest integer in that orbit.

A short description of the method we used to construct the required SDSs and some computational details are given in section 4.

### 3.1 Periodic Golay pairs of length 74

The parameter set is \((74; 36, 31; 30)\), with \(n = 37\). Using the subgroup \(H = \{1, 47, 63\}\) of \(\mathbb{Z}_{74}^*\), we give two non-equivalent solutions:

\[
J = \{1, 4, 6, 7, 9, 12, 22, 23, 28, 29, 34, 42\} \\
K = \{1, 2, 4, 6, 9, 12, 17, 21, 22, 37, 55\}
\]

\[
J = \{1, 2, 3, 6, 7, 12, 22, 23, 28, 29, 34, 55\} \\
K = \{2, 4, 5, 7, 9, 10, 17, 21, 34, 37, 42\}
\]

### 3.2 Periodic Golay pairs of length 82

The parameter set is \((82; 45, 36; 40)\), with \(n = 41\). Using the subgroup \(H = \{1, 37, 51, 57, 59\}\) of \(\mathbb{Z}_{82}^*\), we give two non-equivalent solutions:

\[
J = \{1, 2, 11, 12, 15, 17, 22, 23, 30\} \\
K = \{1, 4, 10, 12, 17, 22, 23, 41\}
\]

\[
J = \{1, 2, 3, 6, 8, 12, 17, 23, 30\} \\
K = \{3, 5, 6, 12, 17, 22, 30, 41\}
\]
3.3 Periodic Golay pairs of length 122

The parameter set is \((122; 56, 55; 50)\), with \(n = 61\). Using the subgroup \(H = \{1, 9, 81, 95, 119\}\) of \(Z_{122}^*\), we give one solution:

\[
J = \{1, 3, 6, 8, 10, 13, 16, 21, 23, 25, 52, 61\} \\
K = \{3, 4, 6, 7, 13, 19, 24, 25, 46, 51, 52\}
\]

3.4 Periodic Golay pairs of length 164

The parameter set is \((164; 81, 73; 72)\), with \(n = 82\). Using the subgroup \(H = \{1, 37, 57, 133, 141\}\) of \(Z_{164}^*\), we give three non-equivalent solutions:

\[
J = \{4, 5, 6, 10, 11, 12, 16, 20, 23, 25, 30, 33, 34, 46, 51, 60, 65, 123\} \\
K = \{3, 4, 11, 13, 16, 19, 20, 23, 30, 33, 41, 44, 46, 53, 66, 82, 123\}
\]

\[
J = \{1, 3, 4, 6, 10, 12, 13, 16, 22, 23, 25, 33, 34, 39, 44, 46, 123\} \\
K = \{2, 4, 5, 8, 10, 11, 12, 16, 17, 33, 34, 39, 41, 51, 65, 82, 123\}
\]

\[
J = \{4, 5, 6, 8, 11, 13, 20, 22, 24, 30, 33, 34, 39, 43, 44, 65, 123\} \\
K = \{2, 8, 12, 13, 20, 23, 25, 30, 39, 41, 43, 46, 51, 60, 65, 82, 123\}
\]

Additional periodic Golay pairs of length 164 can be constructed by “multiplying” Golay pairs of length 2 with the known periodic Golay pairs of length 82.

3.5 Periodic Golay pair of length 202

The parameter set is \((202; 100, 91; 90)\), with \(n = 101\). Using the subgroup \(H = \{1, 87, 95, 137, 185\}\) of \(Z_{202}^*\), we give one solution:

\[
J = \{2, 4, 9, 11, 12, 13, 18, 20, 22, 24, 25, 26, 38, 41, 50, 51, 53, 55, 67, 76\} \\
K = \{1, 3, 4, 6, 8, 9, 11, 12, 16, 17, 20, 25, 39, 41, 48, 52, 67, 76, 101\}
\]

3.6 Periodic Golay pairs of length 226

The parameter set is \((226; 106, 105; 98)\), with \(n = 113\). Using the subgroup \(H = \{1, 49, 109, 129, 141, 143, 219\}\) of \(Z_{226}^*\), we give two non-equivalent solutions:

\[
J = \{1, 3, 4, 5, 6, 9, 10, 15, 16, 36, 40, 41, 43, 78, 99, 113\} \\
K = \{5, 8, 12, 13, 15, 21, 22, 24, 26, 33, 34, 40, 43, 78, 99\}
\]
\[ J = \{3, 7, 12, 13, 16, 18, 20, 21, 22, 40, 41, 43, 55, 78, 99, 113\} \]
\[ K = \{1, 2, 3, 5, 10, 13, 24, 26, 36, 39, 40, 41, 43, 78, 99\} \]

4 Algorithm description

The algorithm that we used to find the new periodic Golay pairs is a straightforward adaptation of the algorithm we used in [5] to construct D-optimal matrices. First we select a subgroup \( H \) of \( \mathbb{Z}_v^\star \) and enumerate the orbits of its action on \( \mathbb{Z}_v \). To construct an SDS with parameters \((v; r, s; \lambda)\) we first generate two files of subsets \( X \) and \( Y \) of size \( r \) and \( s \), respectively, of \( \mathbb{Z}_v \) such that the corresponding binary sequences \( A \) and \( B \) pass the PSD test, i.e. satisfy the inequalities

\[
\text{PSD}_A(i) \leq 2v, \quad \text{PSD}_B(i) \leq 2v, \quad i = 1, \ldots, v/2.
\]

See [5, 6] for the precise definition of the PSD function. The subsets \( X \) and \( Y \) are constructed as suitable unions of the orbits of the action described above. Subsequently we look for a match in the two files, i.e., for two subsets \( X \) and \( Y \) of size \( r \) and \( s \) such that the the pair \((X, Y)\) satisfies the condition stated in Definition 1.

Here are some more specific computational details pertaining to the solution for \( v = 202 \) shown in 3.5. First we ran a program for 7 days to generate a list of about 30 million subsets \( X \) of \( \mathbb{Z}_{202} \) of size 100. All of these sets were made up of 20 \( H \)-orbits each of size 5. Another 7-day run of the same program generated about 26 million subsets \( Y \) of size 91. Each of these subsets was the union of 18 orbits of size 5 and the singleton orbit \( \{101\} \). In both runs we collected only the subsets for which the corresponding binary sequences pass the PSD test, see [5, 6]. For each of the sets, say \( X \), we recorded in a separate file the multiplicities of the nonzero differences \( a - b \) (mod 202) with \( a, b \in X \). Since these searches were not exhaustive, we applied the transformations \( X \to h \cdot X \), \( h \in H \), to the output of the first run and for each of the resulting sets we recorded the difference multiplicities. This resulted in a much bigger file containing about 300 million cases. In one of the files we replaced the multiplicities \( m \) with \( \lambda - m = 90 - m \), and then ran a program to find matching lines in the two multiplicity files. The search produced only two matches but they gave equivalent SDSs. Thus we obtained only one solution.

5 Closing comments

As ordinary Golay pairs are also periodic Golay pairs, the Golay numbers are also periodic Golay numbers. The known Golay numbers are exactly the integers \( v \) admitting the factorization \( v = 2^\alpha 10^\beta 26^\gamma \), where the exponents \( \alpha, \beta, \gamma \) are nonnegative integers. However, as mentioned earlier, there exist infinitely many periodic Golay numbers which do not admit such factorization. Since all Golay numbers in the range 1, 2, \ldots, 100 are known [2], we deduce that 34, 50, 58, 68, 72, 74, 82 are the only periodic Golay numbers for which we are presently sure that they are not Golay numbers.
By using the construction \([2]\) and the “multiplication theorem” mentioned in section \([2]\) we deduce the existence of Hadamard matrices of order \(2gv\), where \(g\) is a Golay number and \(v\) a periodic Golay number.

If \(v > 1\) is a periodic Golay number then \(v\) is even, it is a sum of two squares and satisfies the Arasu-Xiang condition of Theorem \([2]\). We list all numbers in the range 1, 2, \ldots, 300 which satisfy these three necessary conditions and for which the question whether they are periodic Golay numbers remains open:

\[
90, 106, 130, 146, 170, 178, 180, 194, 212, 218, 234, 250, 274, 290, 292, 298.
\]

This list may be useful to readers interested in constructing new periodic Golay pairs or finding new periodic Golay numbers.

The results of the preprint of this note \([7]\) (posted on the arXiv), have been already used in \([10]\) for the construction of orthogonal and nearly orthogonal designs for computer experiments.

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