Combinatorics of the Modular Group II
The Kontsevich integrals

C. Itzykson and J.-B. Zuber

Abstract We study algebraic aspects of Kontsevich integrals as generating functions for intersection theory over moduli space and review the derivation of Virasoro and KdV constraints.

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* Laboratoire de la Direction des Sciences de la Matière du Commissariat à l’Energie Atomique
0. Introduction

The study of two-dimensional gravity has uncovered a rich mathematical structure including Virasoro constraints, KdV flows, $N = 2$ twisted supersymmetry, etc. The remarkable contributions of Witten [1] and Kontsevich [2] to its topological interpretation have reduced an intersection problem on the moduli space of curves to the computation of a matrix integral over $N \times N$ hermitian matrices

$$Z(\Lambda) = \frac{\int dM \exp \left(-\text{tr} \left( \frac{\Lambda M^2}{2} - i \frac{M^3}{6} \right) \right)}{\int dM \exp \left(-\text{tr} \frac{\Lambda M^2}{2} \right)}$$

While many aspects of this connection have been already discussed by these authors [3][4], our endeavour has been to study this integral in a purely algebraic context, combining reviews of former work and new results. We apologize to the expert reader who may skip sec. 1 where we sketch Kontsevich’s construction and parts of sec. 3 which present a short account of Sato’s work on $\tau$–functions. In sec. 2 we study the integral (0.1) as an expansion in powers of the traces tr$\Lambda^{-r}$. After taking a suitable large $N$ limit, we prove the crucial property that the asymptotic expansion of this integral does not depend on the even traces tr$\Lambda^{-2r}$. This result which followed in Kontsevich’s work from topological considerations is derived here in a purely combinatorial fashion. The arguments although straightforward are unfortunately rather intricate since they cannot directly apply to the finite $N$ integral, for which only $N$ of the traces are algebraically independent. The integral is thus subject to several constraints:

(i) The equations of the KdV hierarchy pertaining to a differential operator of second order follow from Sato’s work and this independence with respect to the even traces. These provide evidence of the equivalence of Kontsevich’s model with the one–matrix model considered by Witten in [1].

(ii) The Virasoro highest weight constraints follow from the matrix Airy’s equation satisfied by the numerator of (0.1) (sec. 4) [3].

As for the standard matrix models, a systematic genus expansion is possible, the leading term of which had been obtained by Witten [1] and from another point of view by Makeenko and Semenoff [5] relying on earlier work by Kazakov and Kostov [6] (sec. 5). An analysis of the resulting expressions in a certain singular limit (sec. 6) provides effective ways to resum families of intersection numbers and derive explicit formulae. This step introduces a Painlevé equation and its perturbations.
The integral (0.1) admits a generalization in which the cubic potential is replaced by a suitable higher degree polynomial \[ p(x) \]. The corresponding topological interpretation presented in a recent paper \[ \text{[3]} \] involves an intersection theory on a finite covering of moduli space. On the other hand, it is most likely equivalent to the multimatrix integrals. Our previous discussion extends to these cases without any difficulty of principle (sec. 7) although the calculations soon become very cumbersome.

1. Intersection numbers.

Witten conjectured in \[ \text{[1]} \] that the logarithm of the partition function of the general one-matrix model \[ \text{[9][10]} \], expressed in terms of suitable deformation parameters \( t_i \), could be expanded as

\[
\ln Z = \sum_{k_0, \ldots, k_i} \langle \tau_{k_0} \ldots \tau_{k_i} \rangle \frac{t_{k_0}^{k_0}}{k_0!} \cdots \frac{t_{k_i}^{k_i}}{k_i!} \ldots
\]

where the bracketed rational coefficients admitted the following interpretation as intersection numbers. Let \( \mathcal{M}_{g,n} \) be the moduli space of complex dimension \( 3g - 3 + n \geq 0 \) of algebraic curves of genus \( g \) with \( n \) marked points \( x_1, \ldots, x_n \) and \( \overline{\mathcal{M}}_{g,n} \) a suitable compactification. The cotangent spaces at \( x_i \) define line bundles \( \mathcal{L}_i \) with first Chern class \( c_1(\mathcal{L}_i) \) interpreted as 2–forms over \( \mathcal{M}_{g,n} \). For integral non-negative \( df \)'s such that \( 3g - 3 + n = \sum_f df \) the integral

\[
\int_{\mathcal{M}_{g,n}} c_1(\mathcal{L}_1)^{d_1} \cdots c_1(\mathcal{L}_n)^{d_n}
\]

(independent of the ordering since 2–forms commute and powers are exterior powers) is a rational positive number when one considers \( \mathcal{M}_{g,n} \) as an orbifold, \( i.e. \) the quotient of a contractible ball (Teichmüller space) by the mapping class group. If \( k_i = \# \{ df = i, \ i \geq 0 \} \) (so that \( \sum_{i \geq 0} i k_i = 3g - 3 + n \)) then Witten’s conjecture was that \( \langle \tau_{k_0}^{k_0} \ldots \tau_{k_i}^{k_i} \ldots \rangle \) defined by (1.1) is given by the integral (1.2). (We have skipped a number of essential technicalities which make the above definitions sensible). An intuitive picture of the line bundles \( \mathcal{L}_i \) over \( \mathcal{M}_{g,n} \) is not straightforward but at least one can see that \( \langle \tau_0^3 \rangle = 1 \), since \( \mathcal{M}_{0,3} \) is a point! Even to find that \( \langle \tau_1 \rangle = \frac{1}{24} \) from the definition is non trivial.

These intersection numbers being topological invariants, Kontsevich has been able to reduce them to more manageable expressions using a cell decomposition of \( \mathcal{M}_{g,n} \) inherited from the physicists’ “fat–graph” expansion of Hermitian matrix integrals. One considers connected fat (\( i.e. \) double-line) graphs with vertices of valency three or more, genus \( g \) and
n faces (dual to the n punctures). One assigns to each (double) edge \( e \) a positive length \( \ell_e \) and to each face \( f \) a perimeter \( p_f = \sum_{e \subset f} \ell_e \), where \( e \subset f \) denotes the incidence relations. The set of such fat graphs with assigned \( \ell \)'s is a decorated combinatorial model for \( \mathcal{M}_{g,n} \).

Cells have “dimensions” over the reals obtained by counting the number of independent lengths for fixed values of the perimeters, namely \( E - n \) where \( E \) is the number of edges. If \( V_p \) denotes the number of \( p \)-valent vertices, one has

\[
E - n - \sum_{p \geq 3} V_p = 2g - 2
\]

\[
2E = \sum_{p \geq 3} pV_p \geq 3 \sum V_p.
\]

(1.3)

Thus

\[
E - n \leq 2(3g - 3 + n)
\]

(1.4)

with equality (top dimension) if and only if all vertices are trivalent.

Let \( k \) be the number of edges bordering a face \( f \) and \( \ell_1, \ell_2, \ldots, \ell_k \) be their successive lengths as we circle around the boundary counterclockwise (the faces come with a positive orientation), up to cyclic permutation. One introduces the 2–form

\[
\omega = \sum_{1 \leq a < b \leq k-1} d\left( \frac{\ell_a}{p_f} \right) \wedge d\left( \frac{\ell_b}{p_f} \right)
\]

(1.5)

invariant under a rescaling and a cyclic permutation of the \( \ell \)'s, which is the first Chern class of a universal \( S_1 \) bundle with \( k \)-gons as fibers over a combinatorial model for \( \mathcal{M}_{g,n} \).

For fixed values of the perimeters \( p_f \) the formula obtained for the intersection numbers is then

\[
\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int \prod_f \omega_f^{d_f}
\]

(1.6)

the integral being on top-dimensional cells (we omit a discussion of the compatibility of orientations with the complex structure of \( \mathcal{M}_{g,n} \)). A generating function is obtained by computing

\[
\int_0^\infty \prod_f \left( dp_f e^{-\lambda_f p_f} \right) \int \frac{(\sum p_f^2 \omega_f)^{3g-3+n}}{(3g-3+n)!}
\]

(1.7)
where the integral sign stands both for the integral over a cell and a sum over cells of dimension \(3g - 3 + n\) weighted by the inverse of the order of their automorphism group (orbifold integration). On the one hand this is

\[
\sum_{d_1 + \ldots + d_n = 3g - 3 + n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \int \prod_{f=1}^{n} dp_f \left( e^{-\lambda_f p_f} \right) \frac{p_{2d_f}}{d_f!} = 2^{3g - 3 + n} \sum_{d_1 + \ldots + d_n = 3g - 3 + n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \prod_{f=1}^{n} (2d_f - 1)! \lambda_f^{(2d_f + 1)}.
\]

On the other hand one can proceed to a direct evaluation using

\[
\frac{1}{(3g - 3 + n)!} \prod_{f} dp_f \wedge \left( \sum_{f} \sum_{1 \leq a < b \leq k_f - 1} \sum_{e_a, e_b \subset f} d\ell_a \wedge d\ell_b \right)^{3g - 3 + n} = 2^{5g - 5 + 2n} \prod_{e} (2d_f - 1)!! \frac{\lambda_f}{\lambda_f + \lambda_f'}
\]

up to an overall orientation which is henceforth ignored. The computation of the above Jacobian (which depends on the structure of the graph only through \(g\) and \(n\)) is a delicate matter for which we refer to [2]. Inserting this expression into the integral one notes that each edge of length \(\ell\) is shared by two faces \(f, f' \supset e\), the corresponding integral contributing a factor \(\frac{2}{\lambda_f + \lambda_f'}\), while the ratio of powers of 2 reads \(2^{5g - 5 + 2n} / 2^{3g - 3 + n} = 2^{2g - 2 + n} = 2^{E - V}\). By comparison one obtains Kontsevich’s main identity with \(\Gamma_{g,n}\) the set of all face-labelled trivalent connected fat–graphs \(\gamma\) of genus \(g\) and \(n\) faces

\[
\sum_{\gamma \in \Gamma_{g,n}} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \prod_{f=1}^{n} \frac{(2d_f - 1)!}{\lambda_f^{2d_f + 1}} = \sum_{\gamma \in \Gamma_{g,n}} \frac{2^{-V}}{|\text{Aut}_\gamma|} \prod_{e \in \gamma} \frac{2}{\lambda_f + \lambda_f'}
\]

where as above \(V \equiv V_{\gamma}\), \(e\) denote the edges of the graph and \(2/(\lambda_f + \lambda_f')\) is the propagator attached to the edge \(e\) bordering \(f\) and \(f'\).

The right hand side of this expression is suggestive of the Feynman expansion of the logarithm of the matrix integral \((0.1)\) over \(N \times N\) Hermitian matrices \((N \to \infty)\). Let \(\Lambda\) stand for a diagonal matrix \(\text{diag}(\lambda_0, \ldots, \lambda_{N-1})\) and introduce the infinite set \(t_0, t_1, \ldots\) defined as

\[
t_i(\Lambda) = -(2i - 1)!! \text{tr} \Lambda^{-2i-1}.
\]

1 By convention \((-1)!! = 1\).
By summing the above expression over $g$ and $n$, one has
\[
F(t, \Lambda) = \sum_{n \geq 1} \frac{1}{n!} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle t_{d_1}(\Lambda) \ldots t_{d_n}(\Lambda)
\]
where the last equality is in terms of the Feynman graph expansion of the integral (1.1) and $\sum_{\Gamma_N}$ refers to the summation over fat graphs with $N$ faces and all possible distinct assignments of variables $\lambda$ to their faces. We have noted that $(-1)^n = i^V$ since the relation $2E = 3V$ for trivalent graphs implies that the number of vertices $V$ is even $= 2p$ and from $V - E + n = 2p - 3p + n \equiv 0 \mod 2$, it follows that $n$ is of the parity of $p$. Formula (1.1) leads to a propagator $2/(\lambda_f + \lambda_{f'})$, while the coupling at each vertex is $i^2$. Thus the above reads (after letting $N \to \infty$)
\[
\ln Z(\Lambda) = \left\langle e^{\sum_{d=1}^{\infty} \tau_{d}t_{d}} \right\rangle
\]
in agreement with (1.1). In the next section, we discuss the precise mechanism of the $N \to \infty$ limit by which from finite $\Lambda$ and finitely many independent $\text{tr}\Lambda^{-2i+1}$ infinitely many independent variables $t$ are generated.

The first non-trivial graphs in the Feynman expansion yield after some rearrangement
\[
F = \ln Z = \frac{t_0^3}{3!} + \frac{t_1^3}{24} + \frac{i_0^2 t_1}{3!} + \frac{1}{24} \left( t_0 t_2 + \frac{t_1^2}{2} \right) + \ldots
\]

2. The Kontsevich integral.

2.1. The main theorem

With the normalized measure
\[
d\mu_\Lambda(M) = \frac{dM \exp -\frac{1}{2} \text{tr}\Lambda M^2}{\int dM \exp -\frac{1}{2} \text{tr}\Lambda M^2}, \quad dM \equiv \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\text{Re}M_{ij} d\text{Im}M_{ij}
\]
over $N \times N$ hermitian matrices $M$, $\Lambda$ standing at first for a positive definite such matrix, we wish to study the matrix Airy function
\[
Z^{(N)} = \int d\mu_\Lambda(M) \exp \frac{i}{6} \text{tr}M^3 = \sum_{k \geq 0} Z_k^{(N)}(\Lambda)
\]
\[
Z_k^{(N)}(\Lambda) = \left( \frac{(-1)^k}{(2k)!} \right) \int d\mu_\Lambda(M) \left( \frac{\text{tr}M^3}{6} \right)^{2k}
\]
or rather its asymptotic expansion in traces of powers of $\Lambda^{-1}$ for large $N$. A generalization is presented in sec. 7.

More precisely the terms $Z_k^{(N)}(\Lambda)$ can be expressed as polynomials with rational coefficients in the variables

$$\theta_r = \frac{1}{r} \text{tr} \Lambda^{-r}.\tag{2.3}$$

Then $Z_k^{(N)}$ is homogeneous of degree $3k$, if we set

$$\deg \theta_r = r.\tag{2.4}$$

Only the first $N$ of the $\theta_r$ are algebraically independent for an $N \times N$ matrix $\Lambda$. However

**Lemma 1** Considered as a function of $\theta_r \equiv \{\theta_r\}$, $Z_k^{(N)}(\Lambda)$ is independent of $N$ for $3k \leq N$ and depends only on $\theta_r$, $1 \leq r \leq 3k$.

This allows one to define unambiguously the series

$$Z(\theta_r) = \sum_{k \geq 0} Z_k(\theta_r)\tag{2.5}$$

where

$$Z_k(\theta_r) = Z_k^{(N)}(\theta_r), \quad N \geq 3k\tag{2.6}$$

without any further reference to $N$. The set of variables $\theta_r$ is denumerable but each $Z_k$ ($Z_0 = 1$) depends on finitely many of them. This almost evident lemma follows from eq. (2.43) below.

**Theorem 1** (Kontsevich)

(i) \[ \frac{\partial Z}{\partial \theta_{2r}}(\theta_r) = 0 \quad r \geq 1 \tag{2.7} \]

(ii) $Z(\theta_r)$ is a $\tau$-function for the Korteweg-de Vries equation. Namely if

$$t_r = -(2r + 1)!! \quad \theta_{2r+1} = -(2r - 1)!! \text{tr} \Lambda^{-2r-1}\tag{2.8}$$

$$u = \frac{\partial^2}{\partial t_0^2} \ln Z$$

then \[ \frac{\partial u}{\partial t_1} = \frac{\partial}{\partial t_0} \left( \frac{1}{12} \frac{\partial^2 u}{\partial t_0^2} + \frac{1}{2} u^2 \right) \tag{2.9a} \]

and more generally \[ \frac{\partial}{\partial t_n} u = \frac{\partial}{\partial t_0} R_{n+1}. \tag{2.9b} \]
In (2.9), the $R_n$ denote the Gelfand-Dikii differential polynomials (derivatives are taken with respect to $t_0$)

\[ R_2 = \frac{u^2}{2} + \frac{u''}{12} \]
\[ R_3 = \frac{u^3}{6} + \frac{uu''}{12} + \frac{u'^2}{24} + \frac{u^{(4)}}{240} \]
\[ R_4 = \frac{u^4}{24} + \frac{uu'^2}{24} + \frac{u'^u''}{24} + \frac{uu^{(4)} + u'u'''}{240} + \frac{(u'')^2}{160} + \frac{u^{(6)}}{6720} \]
\[ \cdots \]
\[ R_n = \frac{u^n}{n!} + \cdots \]  

(2.10)

computed from

\[ (2n + 1)R'_{n+1} = \frac{1}{4}R''_{n} + 2uR'_{n} + u'R_{n}. \]  

(2.11)

Proof of the first part of theorem 1 follows in Kontsevich’s approach from the identities (1.10)–(1.13), relying on topological considerations, namely on the introduction of the combinatorial model for $\mathcal{M}_{g,n}$. The rest of this section is devoted to a purely algebraic proof of this result.

2.2. Expansion of $Z$ on characters and Schur functions

The integral (2.1) in the case $N = 1$, denoted for short $z(\lambda)$,

\[ z(\lambda) = \left\{ \int_{-\infty}^{\infty} dm \, e^{-\frac{1}{8}\lambda m^2 + \frac{i}{8}m^3} \right\}^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} dm \, e^{-\frac{1}{2}\lambda m^2} \right\}^{\frac{1}{2}} \]

(2.12)

admits the asymptotic expansion

\[ z(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^{-3k} \]  

(2.13a)

\[ c_k = \left( -\frac{2}{9} \right)^k \frac{\Gamma(3k + \frac{1}{2})}{(2k)! \sqrt{\pi}} \]  

(2.13b)

and satisfies the differential equation equivalent to Airy’s equation

\[ (D^2 - \lambda^2)z(\lambda) = 0 \]  

(2.14)

\[ D = -e^{\frac{1}{2} \lambda^3} \lambda^\frac{1}{2} \left( \frac{1}{\lambda} \frac{\partial}{\partial \lambda} \right) e^{-\frac{1}{2} \lambda^3} \lambda^{-\frac{1}{2}} \]
\[ = \lambda + \frac{1}{2\lambda^2} - \frac{1}{\lambda} \frac{\partial}{\partial \lambda}. \]  

(2.15)
For $N$ arbitrary, we can make a shift of variable $M \to M - i\Lambda$ in the numerator of
the integral (2.2.) This gives

$$Z^{(N)}(\Lambda) = \frac{1}{2^{N(N-1)}} \Pi_r \frac{3}{2} \Pi_{r<s} (\lambda_r + \lambda_s) \exp \frac{\Lambda^3}{3} \int dM \exp i \text{tr} \left( \frac{M^3}{6} + \frac{MA^2}{2} \right),$$

(2.16)

with $\lambda_i$ the eigenvalues of $\Lambda$. We then integrate over "angles", i.e. over the unitary group
with the action $M \to U M U^{-1}$. The result [11] (which we now realize had been established
long before by Harish-Chandra [12]) yields

\[
\int dM \exp i \text{tr} \left( \frac{M^3}{6} + \frac{MA^2}{2} \right) =
\]

\[
= (2\pi)^{N^2} \int \prod_{0 \leq r \leq N-1} \frac{dm_r}{(2\pi)^{\frac{3}{2}}} e^{i \left( \frac{1}{2} m_r^3 + \frac{1}{2} m_r \lambda_r^2 \right)} \prod_{0 \leq r < s \leq N-1} \left( \frac{m_s - m_r}{i \frac{\lambda_r^2}{2} - i \frac{\lambda_s^2}{2}} \right) \text{ } \tag{2.17}
\]

with $m_0, m_1, \ldots, m_{N-1}$ the eigenvalues of the matrix $M$. We denote the Vandermonde
determinant

\[
\left| \begin{array}{cccc}
x_0^0 & x_0^1 & \cdots & x_0^{N-1} \\
x_1^0 & x_1^1 & \cdots & x_1^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
x_{N-1}^0 & x_{N-1}^1 & \cdots & x_{N-1}^{N-1}
\end{array} \right| = \prod_{0 \leq r < s \leq N-1} (x_s - x_r) \tag{2.18}
\]

by $|x^0, x^1, \ldots, x^{N-1}|$, with the understanding that in each row one substitutes successively
$x_0, x_1, \ldots, x_{N-1}$ for the variable $x$. Inserting (2.17) into (2.16), we find

\[
Z^{(N)}(\Lambda) = \frac{|D^0 z, D^1 z, \ldots, D^{N-1} z|}{|\lambda^0, \lambda^1, \ldots, \lambda^{N-1}|}, \tag{2.19}
\]

with $D$ defined as in (2.15). The asymptotic expansion involves only inverse powers of $\Lambda$.
We set

\[
x_r = \lambda_r^{-1} \quad \theta_k = \frac{1}{k} \sum_{0 \leq r \leq N-1} x_r^k, \tag{2.20}
\]

and consider henceforth the function $z$ as given by the formal series

\[
z(x) = \sum_{0}^{\infty} c_k x^{3k}. \tag{2.21}
\]
We have $D^2 z = \lambda^2 z$, and one readily sees that

\[
D^{2k} z = \lambda^{2k} z \mod (D^{2k-1} z, \ldots, D^0 z) \\
D^{2k+1} z = \lambda^{2k+1} \overline{z} \mod (D^{2k} z, \ldots, D^0 z),
\]

with

\[
\overline{z} = \frac{1}{\lambda} D z \\
= \left[1 + x^3 \left(\frac{1}{2} + x \frac{d}{dx}\right)\right] z = \sum_0^\infty d_k x^{3k} \\
d_k = \frac{1 + 6k}{1 - 6k} c_k.
\]

As a consequence, we rewrite (2.19) as

\[
Z^{(N)}(\Lambda) = \left| \lambda^0 z, \lambda^1 \overline{z}, \lambda^2 z, \ldots \right| \\
\left| \lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^{N-1} \right|,
\]

where the last term in any row of the upper determinant is $\lambda^{N-1} z$ if $N$ is odd, and $\lambda^{N-1} \overline{z}$ if $N$ is even. Finally we cancel from numerator and denominator the product $(\lambda_0 \cdots \lambda_{N-1})^{N-1}$, and express $Z^{(N)}$ in terms of the variables $x_r = \lambda_r^{-1}$ as

\[
Z^{(N)} = \frac{x^{N-1} z, x^{N-2} \overline{z}, \ldots}{|x^{N-1}, x^{N-2}, \ldots|}.
\]

Then we substitute the expansions of $z$ and $\overline{z}$ to obtain the series

\[
Z^{(N)} = \sum_{0 \leq n_0, n_1, \ldots, n_{N-1}} c_n^{(0)} c_n^{(1)} \cdots c_n^{(N-1)} \left| x^{3n_0 + N-1}, x^{3n_1 + N-2}, \ldots, x^{3n_{N-1}} \right| \\
\left| x^{N-1}, x^{N-2}, \ldots, x^0 \right|,
\]

with the convention that

\[
c_n^{(2p)} = c_n \quad c_n^{(2p+1)} = d_n.
\]

When the indices $f_0 = 3n_0$, $f_1 = 3n_1$, $\cdots$, $f_{N-1} = 3n_{N-1}$ form a non-increasing sequence, we recognize in the above ratio of determinants the symmetric function in $x_0, \cdots, x_{N-1}$ which corresponds to the polynomial character of the general linear group $GL(N)$ specified by the Young tableau with rows of length $f_0$, $f_1$, $\cdots$, $f_{N-1}$.
The latter admits a natural extension outside the standard Weyl chamber, as an antisymmetric function of the unordered exponents

\[ l_{N-1} = f_0 + N - 1, \quad \cdots \quad l_0 = f_{N-1} \ . \]

Henceforth we refer to this extension when we write this character as

\[ \text{ch}_{l_{N-1}, \ldots, l_0} = \frac{|x^{l_{N-1}}, \ldots, x^0|}{|x^{N-1}, \ldots, x^0|} \ . \]

We wish to express this quantity in terms of the traces \( \theta_k = \sum_{i=0}^{N-1} x_i^k \). This follows from the standard identities for characters which we now recall [13]. Thinking of \( X \) as the diagonal matrix \( X \equiv \Lambda^{-1} = \text{diag}(x_0, x_1, \cdots, x_{N-1}) \) we set \( s_0 = p_0 = 1 \) and for \( k \geq 1 \)

\[ s_k(X) = \text{tr} \wedge^k X \quad \text{det}(1 + uX) = \sum_{0}^{N} u^k s_k(X) \quad (2.30a) \]

\[ p_k(X) = \text{tr} \otimes_{\text{sym}}^k X \quad \frac{1}{\text{det}(1 - uX)} = \sum_{0}^{\infty} u^k p_k(X) \quad (2.30b) \]

\[ \theta_k = \frac{1}{k} \text{tr} X^k. \quad (2.30c) \]

The quantities \( s_k, \ 1 \leq k \leq N, \) are the elementary symmetric functions corresponding to the vertical Young tableaux up to \( N \) lines, whereas the \( p_k \)’s are the traces of symmetric tensor products, i.e. they correspond to Young tableaux with only one row.

We have from the definition

\[ \exp \sum_{1}^{\infty} u^n \theta_n(X) = \sum_{0}^{\infty} u^k p_k(X) , \quad (2.31) \]
which expresses the $p_k$’s as homogeneous polynomials of degree $k$ of the $\theta_n$’s ($\deg \theta_n = n$), ignoring the relations among traces. This justifies the definition of (elementary) Schur functions $p_n$ via

$$\exp \sum_{1}^{\infty} u^n \theta_n = \sum_{0}^{\infty} u^k p_k(\theta) ,$$  \hfill (2.32)

without reference to any $N \times N$ matrix, and where now the $p$’s are functions of the $\theta$’s. When both $p$ and $\theta$ refer to the same matrix $X$ we recover the previous definitions (2.30) and (2.31). Explicitly we write

$$p_r(\theta) = \sum_{\nu_1 + 2\nu_2 + \ldots = r} \theta_{1}^{\nu_1} \theta_{2}^{\nu_2} \ldots .$$  \hfill (2.33)

Eq. (2.32) entails

$$\frac{\partial p_r(\theta)}{\partial \theta_k} = \frac{\partial^k p_r(\theta)}{\partial \theta^k_1} = p_{r-k}(\theta) ,$$  \hfill (2.34)

where $p_n$ vanishes for $n < 0$. When expanding the matrix elements along successive columns, Cauchy’s determinental formula

$$\det \left| \frac{1}{1-x_t y_s} \right|_{0 \leq r, s \leq N-1} = \frac{|x^{N-1}, \ldots, x^0| \cdot |y^{N-1}, \ldots, y^0|}{\prod_{r, s}(1-x_t y_s)}$$  \hfill (2.35)

yields

$$\sum_{l_0, \ldots, l_{N-1}} y_0^{l_{N-1}} \cdots y_0^{l_0} \frac{|x^{l_{N-1}}, \ldots, x^{l_0}|}{|x^{N-1}, \ldots, x^0|} = \frac{|y^{N-1}, \ldots, y^0|}{\prod_{r, s}(1-x_t y_s)} .$$  \hfill (2.36)

Therefore if $X \equiv \text{diag}(\{x_r\})$, then

$$\text{ch}_{l_{N-1}, \ldots, l_0}(X) = \text{coeff. of } y_0^{l_{N-1}} \cdots y_0^{l_0} \text{ in }$$

$$\begin{vmatrix}
  y_0^{N-1} & \cdots & y_0^0 \\
  \det(1-y_0 X) & \cdots & \det(1-y_0 X) \\
  \vdots & \ddots & \vdots \\
  y_{N-1}^{N-1} & \cdots & y_{N-1}^0 \\
  \det(1-y_{N-1} X) & \cdots & \det(1-y_{N-1} X) 
\end{vmatrix} .$$  \hfill (2.37)

Expanding $[\det(1-y X)]^{-1}$ according to (2.30), we obtain the classical formula (Jacobi-Schur)

$$\text{ch}_{l_{N-1}, \ldots, l_0}(X) = \frac{|x^{l_{N-1}}, \ldots, x^{l_0}|}{|x^{N-1}, \ldots, x^0|} = \begin{vmatrix}
  p_{l_{N-1}-(N-1)}(X) & \cdots & p_{l_{N-1}}(X) \\
  \vdots & \ddots & \vdots \\
  p_{l_0-(N-1)}(X) & \cdots & p_{l_0}(X) \\
\end{vmatrix} ,$$  \hfill (2.38)
valid for any ordered or unordered sequence \( l_{N-1}, \cdots, l_0 \). Terms along the diagonal read \( p_{f_0}, p_{f_1}, \cdots, p_{f_{N-1}} \), and indices increase (decrease) by successive units as one moves from a diagonal term to the right (left). We abbreviate this expression as

\[
\operatorname{ch}_{N-1+f_0, \cdots, f_{N-1}}(\theta) = \left| \begin{array}{cccc}
p_{f_0} & * & \cdots & * \\
* & p_{f_1} & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & p_{f_{N-1}} \\
\end{array} \right| ,
\]  

(2.39)

substituting for the elementary Schur polynomials their expressions in terms of the variables \( \theta \). We conclude that

\[
Z^{(N)} = \sum_{n_0, \cdots, n_{N-1}} c_{n_0}^{(0)} c_{n_1}^{(1)} \cdots c_{n_{N-1}}^{(N-1)} \left| \begin{array}{cccc}
p_{3n_0} & & & \\
& \ddots & & \\
& & \ddots & \\
p_{3n_{N-1}} & & & 1 \\
\end{array} \right|
\]

(2.40)

yields an expression of \( Z^{(N)} \) in terms of the infinitely many variables \( \theta \). (which can henceforth be treated as independent). It follows from eq. (2.40) that

\[
Z_k^{(N)} = \sum_{n_0 + \cdots + n_{N-1} = k} c_{n_0}^{(0)} c_{n_1}^{(1)} \cdots c_{n_{N-1}}^{(N-1)} \left| \begin{array}{cccc}
p_{3n_0} & & & \\
& \ddots & & \\
& & \ddots & \\
p_{3n_{N-1}} & & & 1 \\
\end{array} \right|
\]

(2.41)

where each character is of degree 3k. This is obvious for the diagonal term and, as one readily ascertains, holds also for non-diagonal terms.

We are now in position to prove the lemma, which is trivially true for \( Z_0^{(N)} = 1 \). Suppose \( 0 < 3k \leq N \) and for a given term in (2.41) let \( \delta \) be its “depth”, i.e. the smallest integer \( \leq N \) such that \( r \geq \delta \Rightarrow n_r = 0 \). From (2.39) it follows that the corresponding term reads

\[
\left| \begin{array}{cccc}
p_{3n_0} & & & \\
& \ddots & & \\
& & \ddots & \\
p_{3n_{\delta-1}} & & & 1 \\
\end{array} \right| ; \quad n_0 + \cdots + n_{\delta-1} = k.
\]

The last column of the \( \delta \times \delta \) determinant reads \( (p_{3n_0+\delta-1}, \cdots, p_{3n_{\delta-1}})^T \) where the subscripts are \( \delta \) positive integers with a sum equal to \( 3k + \sum_{0}^{\delta-1} r \). If \( 3k < \delta \), this is smaller than the sum of the first \( \delta \) positive integers. From Dirichlet’s box principle, two of the subscripts among \( 3n_0 + \delta - 1, 3n_1 + \delta - 2, \cdots, 3n_{\delta-1} \) have to be equal, which results in two identical lines in the determinant which therefore vanishes. Hence \( \delta \) has to be smaller than or equal to \( 3k \), showing that

\[
N \geq 3k \quad \Rightarrow \quad Z_k^{(N)} = Z_k^{(3k)} \equiv Z_k
\]

(2.42)

which concludes the proof and allows a definition of the formal series \( Z \), without reference to \( N \).
2.3. Proof of the first part of the Theorem.

(i) The first part of the theorem will be proved for each $Z^3_k$ which we take equal to $Z^3_k$. Differentiating each line successively in the determinental characters using the crucial formula (2.34) we find

\[
\begin{aligned}
2r > 3k & \quad \frac{\partial Z_k}{\partial \theta_{2r}} = 0 \\
2r \leq 3k & \quad \begin{cases}
\frac{\partial Z_k}{\partial \theta_{2r}} = \sum_{s=0}^{3k-1} Z_{k,(s)} \\
Z_{k,(s)} = \sum_{n_0+\ldots+n_{3k-1}=k} c_n^{(0)} c_n^{(1)} \cdots c_n^{(3k-1)}
\end{cases}
\end{aligned}
\]  

(2.43)

where subscripts in the $s$-th row of each determinant have been decreased by $2r$ units. For $0 \leq s \leq 3k - 1 - 2r$ the subscripts in line $s$ and $s + 2r$ only differ by the interchange of $n_s$ and $n_{s+2r}$. The determinental character is therefore antisymmetric in the interchange of indices $n_s$ and $n_{s+2r}$ whereas in the product of $c$'s, due to (2.27) $\ldots c_n^{(s)} \ldots c_n^{(s+2r)} \ldots \equiv \ldots c_n^{(s)} \ldots c_n^{(s)} \ldots$ is symmetric in these indices. As a consequence, $Z_{k,(s)}$ vanishes for $s \leq 3k - 1 - 2r$ and we need only consider terms with $s > 3k - 1 - 2r$, i.e. when the derivative acts on one of the last $2r$ lines and we cannot use the above argument relying on the periodicity $c_n^{(r)} = c_n^{(r+2)}$.

(ii) Therefore fix $s$ such that $3k - 2r \leq s \leq 3k - 1$. The only possibly non-vanishing terms in $Z_{k,(s)}$ are those whose depth $\delta$ defined as above to be the smallest integer such that $r \geq \delta \Rightarrow n_r = 0$, satisfies the inequality $3k - 2r \leq s \leq \delta - 1 \leq 3k - 1$. In this case they read

\[
\begin{vmatrix}
c_n^{(0)} c_n^{(1)} \cdots c_n^{(\delta-1)} \\
p_{3n_0} \\
\vdots \\
p_{3n_s-2r} \\
p_{3n_{\delta-1}} \\
p_{3n_{\delta-1}}
\end{vmatrix}, n_{\delta-1} > 0, n_0 + \ldots + n_{\delta-1} = k. \quad (2.44)
\]

The $\delta$ indices of the $p$'s in the last column of the determinant before subtracting $2r$ from the indices of the $s$-th row are all positive integers and have a sum equal to $3k - \delta + \sum_1^\delta t$, where $0 \leq 3k - \delta \leq 2r - 1$. We now make use of the following
Lemma 2.

If from a set of $\delta$ positive integers, with sum exceeding the one of the first $\delta$ positive integers by an amount $\Delta \geq 0$, one decreases one by $\Delta' > \Delta$, then in the new sequence two terms coincide or one is a non positive integer.

Think of the original set as occupied integral levels. Let $r_0 + 1$ be the first unoccupied one ($r_0 \geq 0$) and $r_1$ the greatest occupied one. It follows from the hypothesis that $r_1 - r_0 \leq \Delta + 1$. If one decreases one element of the set by an amount $\Delta' \geq \Delta + 1$, it therefore becomes less than or equal to $r_0$, which proves the lemma.

Applying this result to the above circumstance ($\Delta = 2r - 1, \Delta' = 2r$), we deduce that the only possibly non-vanishing terms in $Z_{k,(s)}$, $3k - 2r \leq s \leq 3k - 1$ occur when $3n_s = 2r - (\delta - 1 - s)$ with $0 \leq \delta - 1 - s \leq 2r - 1$. Thus $2r - (\delta - 1 - s)$ takes the possible values $1, \ldots, 2r$. If $r = 1$ this is never a multiple of 3. Thus we have already obtained

$$\frac{\partial Z}{\partial \theta_2} = 0.$$  \hfill (2.45)

We can even say more. Let $a$ be the integral part of $(\delta - 1)/3$ and consider in the last column starting from the bottom the $a + 1$ positive subscripts

$$3n_{\delta - 1}, 3(n_{\delta - 1} - 3), \ldots, 3(n_{\delta - 1} - 3a + a).$$

For the corresponding character to be non-zero, these have to be all distinct. Hence their sum is larger than or equal to $3 \sum_{\alpha=0}^{a}(\alpha + 1)$. The inequality

$$3 \sum_{\alpha=0}^{a} (n_{\delta - 1 - 3\alpha} + \alpha) \geq 3 \sum_{\alpha=0}^{a+1} \alpha$$  \hfill (2.46)

implies that $\sum_{\alpha=0}^{a} n_{\delta - 1 - 3\alpha} \geq a + 1$. Should a non-vanishing term arise in $Z_{k,(s)}$, there would exist an index $n_s$ such that from the preceding observation

$$3n_s + \delta - 1 - s = 2r$$  \hfill (2.47)

with

$$3k - 2r \leq s \leq \delta - 1.$$  \hfill (2.48)

Thus $3n_s + \delta - 1 = 2r + s \geq 3k$, or equivalently $n_s + a \geq k$. If $2r$ is not a multiple of 3, i.e.

$$r \not\equiv 0 \mod 3,$$  \hfill (2.49)
it follows that
\[ \delta - 1 - s \neq 0 \mod 3. \quad (2.50) \]

This means that the subscript \( 3n_s + \delta - 1 - s \) does not belong to the sequence \( 3n_{\delta - 1 - 3\alpha} + 3\alpha \). Since the sum of all \( n \)'s is \( k \) we should have
\[ k \geq \sum_{0}^{a} n_{\delta - 1 - 3\alpha} + n_s \geq a + 1 + n_s \quad (2.51) \]

whereas from the above \( a + 1 + n_s \geq k + 1 \), a contradiction. Thus in general
\[ r \neq 0 \mod 3 \quad \frac{\partial Z}{\partial \theta_2 r} = 0. \quad (2.52) \]

(iii) The remaining cases are those for which we take a derivative with respect to \( \theta_6 r \). We shall need a relation between the coefficients \( c_n, d_n \) which did not play any specific role until now. The series \( z(x) \) of (2.21) is a solution of the following differential equation in the variable \( x \)
\[ x^4 z'' + 2(2x^3 + 1)z' + \frac{5}{4}x^2 z = 0, \quad (2.53) \]

while \( \overline{z}(x) \) of (2.23) satisfies
\[ \overline{z}(x) = \sum_{0}^{\infty} d_n x^{3n} = \left(1 + \frac{x^3}{2}\right) z + x^4 z'. \quad (2.54) \]

**Lemma 3**
\[ z(x)\overline{z}(-x) + z(-x)\overline{z}(x) = 2 \quad (2.55) \]

To see this, substitute \( \overline{z} \) in terms of \( z \) and compute the derivative with respect to \( x \), making use of the differential equation for both \( z(x) \) and \( z(-x) \), to find that it vanishes.

In terms of the series expansions this reads
\[ \sum_{s=0}^{2n} (-1)^s c_{2n-s} d_s = 0 \quad n > 0. \quad (2.56) \]

It was convenient to use generalized characters until now, but we can also recast the expansion of \( Z \) in terms of standard characters indexed by Young tableaux \( Y \) \( (f_0 \geq f_1 \geq \ldots \geq f_{\delta - 1} > 0) \) at the price of having more complicated coefficients. From eqs. (2.38) and (2.40) this reads
\[ Z = \sum_{Y, \ |Y| = 0, \mod 3} \xi_Y \text{ch}_Y(\theta). \quad (2.57) \]
with

\[ \text{ch}_Y(\theta_r) = \det \{ p_{f_i+j-i}(\theta_r) \} \quad 0 \leq i, j \leq \delta - 1 \]  \tag{2.58} 

and

\[ \xi_Y = \det \xi_{i,j} \]  \tag{2.59} 

\[ \xi_{i,j} = \begin{cases} 
0 & \text{if } f_i + j - i \neq 0 \text{ mod } 3 \\
\epsilon^{(j)}_{i}(f_i+j-i) & \text{if } f_i + j - i = 0 \text{ mod } 3
\end{cases} \] .

In the above, \( i \) (\( j \)) is the row (column) index. The diagonal subscripts in \( \text{ch}_Y \) are no longer multiples of 3, instead running down the diagonal they form a non-decreasing sequence, the last one being positive (and \( \delta \) being the number of rows of the corresponding Young tableau). The quantity \( Z_k \) is obtained by restricting the sum to \( |Y| = 3k \).

When taking a derivative with respect to \( \theta_{6r} \), the first contributing term is \( Z_{2r} \). (Recall that terms in \( Z_k \) are homogeneous of degree \( 3k \)). Let us therefore investigate first \( \frac{\partial Z_{2r}}{\partial \theta_{6r}} \). The only Young tableaux such that \( |Y| = 6r \) for which \( \partial \text{ch}_Y / \partial \theta_{6r} \neq 0 \) are of the “Fermi-Bose” type \((t + 1)(1)^s \), \( s + t + 1 = 6r \)

\[
\begin{array}{c}
| & | & | & | & | & | \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& s & & & & & \\
| & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& t + 1 & | & & & & \\
\end{array}
\]  \tag{2.60}

in which case \( \partial \text{ch}_Y / \partial \theta_{6r} = (-1)^s \). Applying formula \((2.57)\) we get

\[
\begin{align*}
\frac{\partial Z_{2r}}{\partial \theta_{6r}} &= 2 \sum_{s=0}^{2r-1} (-1)^s \begin{vmatrix}
1 & d_1 & \cdots & c_{s}^{(s)} \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & c_1^{(s)} \\
\end{vmatrix} \\
&\quad + \sum_{s=0}^{2r-1} (-1)^s \begin{vmatrix}
1 & c_{2r-s} & \cdots & c_{2r-s+1}^{(s+1)} \\
0 & c_1 & \cdots & c_{s}^{(s+1)} \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & c_1^{(s+1)} \\
\end{vmatrix} .
\end{align*}
\]  \tag{2.61}
Expanding each of these determinants along the first column (where \(c\) and \(d\) with negative index are set equal to zero), we find

\[
\frac{\partial Z_{2r}}{\partial \theta_{6r}} = \sum_{j=0}^{r} \left[ (1 - 3j)c_{2j-1} \Delta_{2r-(2j-1)} + 3jc_{2j} \Delta_{2r-2j} \right] \\
+ \sum_{j=1}^{r} \left[ 3jd_{2j} \Delta_{2r-2j} - (3j - 2)d_{2j+1} \Delta_{2r-2j-1} \right],
\]

(2.62)

where \(\Delta_0 = \Delta_0 = 1\) and

\[
\Delta_s = \begin{vmatrix} d_1 & c_2 & \ldots & c_s^{(s)} \\ 1 & c_1 & \ldots & c_{s-1}^{(s)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & c_1^{(s)} \end{vmatrix} \quad \Delta_s = \begin{vmatrix} c_1 & d_2 & \ldots & c_s^{(s+1)} \\ 1 & d_1 & \ldots & c_{s-1}^{(s+1)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & c_1^{(s+1)} \end{vmatrix}.
\]

(2.63)

Taking into account the identity (2.56) satisfied by the coefficients \(c\) and \(d\) we readily see by a recursive argument that \(\Delta\) and \(\Delta_s\) reduce to

\[
\Delta_s = d_s \quad \Delta_s = c_s
\]

(2.64)

Thus

\[
\frac{\partial Z_{2r}}{\partial \theta_{6r}} = (3r - 1) \sum_{j=0}^{2r} (-1)^j c_j d_{2r-j} = 0.
\]

(2.65)

(iv) It remains finally to examine

\[
\frac{\partial Z_{2r+k}}{\partial \theta_{6r}} \quad k > 0
\]

Let us look at the expression (2.43) with the required modification \(k \to k+2r, 2r \to 6r\). As before when taking derivatives of the row of index \(s\) we need only take into account those terms for which \(s \geq 3k\), the others vanishing due to the antisymmetry of the characters. This being assumed we consider for fixed \(s\) a specific term in the sum (2.43) with character of depth \(\delta > 3k\) so that we can omit the rows and columns of label larger than \(\delta - 1\) in the computation of the corresponding determinental character. The labels in the last column before derivation are \(3n_0 + \delta - 1, \ldots, 3n_{\delta-1}\), no pair of them equal. According to a previous analysis using lemma 2, to get a non-trivially vanishing derivative, the quantity \(3n_s+\delta−1−s (s \geq 3k)\) has to equal \(6r\). This means that \(\delta−1−s = 3\sigma, \text{ and } n_{\delta-1-3\sigma} = 2r−\sigma\).
Since by definition $n_{\delta - 1} > 0$, from the preceding reasoning we must have $\sum_{\alpha=0}^{\sigma-1} n_{\delta - 1 - 3\alpha} \geq \sigma$, if this sum is non-empty (i.e. $\sigma > 0$). Then

$$\sum_{\alpha=0}^{\sigma} n_{\delta - 1 - 3\alpha} \geq 2r , \quad (2.66)$$

an inequality which remains obviously true when $\sigma = 0$, in which case $n_{\delta - 1} = 2r$. A fortiori

$$\sum_{\rho=3k}^{\delta-1} n_{\rho} \geq 2r , \quad (2.67)$$

since the latter sum includes the previous one ($\delta - 1 - 3\sigma \geq 3k$). Since $\sum_{0}^{\delta-1} n_{\rho} = 2r + k$ from the homogeneity property of $Z_{2r+k}$, we have the complementary inequality

$$\sum_{0}^{3k-1} n_{\rho} \leq k . \quad (2.68)$$

Both inequalities must in fact be equalities. Indeed among the integers of the form $\delta - 1 - 3\alpha$ ($\alpha \geq 0$) such that $\delta - 1 - 3\alpha \geq k$, consider the largest one, say $\beta$, for which $n_{\delta - 1 - 3\alpha} > 0$. We have $\beta \geq \sigma$ and again appealing to a previous reasoning

$$\sum_{0 \leq \rho < \delta - 1 - 3\beta} n_{\rho} \geq k , \quad (2.69)$$

since there are at least $k$ integers equal to $\delta - 1 \mod 3$ between 0 and $3k - 1 \leq \delta - 1 - 3\beta$. Hence

(i) $\sum_{\alpha=0}^{\sigma} n_{\delta - 1 - 3\alpha}$ has to equal $2r$ otherwise we would violate homogeneity;

(ii) $\beta$ has to be equal to $\sigma$ for the same reason (no other $n_{\delta - 1 - 3\alpha}$ except those entering the previous sum are $> 0$);

(iii) and finally should $n_{\rho_0}, \rho_0 \geq 3k, \rho_0 \neq \delta - 1 \mod 3$, be positive, then again $\sum_{\rho=\rho_0 \mod 3}^{\rho_0} n_{\rho}$ would be larger than $k$ (since the sum includes $\rho_0 \geq 3k$), a contradiction. We conclude that we can replace (2.68) and (2.69) by equalities. This means that we can write

$$\frac{\partial Z_{2r+k}}{\partial \theta_{6r}} = \sum_{s \geq 3k} \sum_{n_0+\cdots+n_{3k-1}=k} c_{n_0}^{(0)} \cdots c_{n_{6r+3k-1}}^{(6r+3k-1)} \left| \begin{array}{c} p_{3n_0} \\ \vdots \\ p_{3n_3k+6r-1} \end{array} \right|$$

(2.70)
Let us group together all contributions corresponding to a fixed choice of indices \( n_0, \ldots, n_{3k-1} \) with sum equal to \( k \). Taking into account the antisymmetry in the last \( 6r \) rows we see that the analysis reduces to our previous computation of \( \partial Z_{2r}/\partial \theta_{6r} \). To be precise, the column labelled \( 3k \) will correspond to coefficients labelled \( c \) or \( d \) according to \( k \) even or odd so that if \( k \) is even

\[
\frac{\partial Z_{2r+k}}{\partial \theta_{6r}} = \sum_{n_0 + \cdots + n_{3k-1} = k} c^{(0)}_{n_0} \cdots c^{(3k-1)}_{n_{3k-1}} \left| \begin{array}{ccc} p_{3n_0} & \cdots & \frac{\partial Z_{2r}}{\partial \theta_{6r}} \\ \vdots \end{array} \right| \right|_{p_{3n_{3k-1}}}.
\]  

(2.71)

If \( k \) is odd, we get the same formula with \( \partial Z_{2r}/\partial \theta_{6r} \) replaced by the same expression with \( c \)'s and \( d \)'s interchanged. In both cases the expression vanishes since \( c \)'s and \( d \)'s play a symmetric role in the vanishing of \( \partial Z_{2r}/\partial \theta_{6r} \) as it relies on the identity (2.55) invariant in the interchange of \( z \) and \( \overline{z} \).

We have at last fully proved the first part of Kontsevich’s theorem from a purely algebraic standpoint

\[
\frac{\partial Z_{k}}{\partial \theta_{2r}} = 0
\]  

(2.72)

Even though the proof appears a little long, the steps are completely elementary relying on the second order differential equation satisfied by \( z \) and Weyl antisymmetry of characters. Without repeating in detail each step, it will go through in the generalized case considered in sec. 7.

It may also be worth remarking that by retracing the above discussion, this property is very likely to imply the (Airy-like) differential equation. We now turn to the second part of the theorem.

3. From Grassmannians to KdV

Expert readers will have recognized the connection between the expansion (2.57) of \( Z \) and Sato’s approach to soliton equations and \( \tau \)- functions. The latter relies on a clever reinterpretation of the familiar Plücker relations of projective geometry in terms of properties of associated (pseudo-)differential operators [14]. In more physical terms, this involves the characterization of those submanifolds that correspond to pure Slater determinants (as opposed to their linear combinations) in a many body fermionic space.

Let us begin with a short review of the subject following Sato. Let \( V \) be a vector space of dimension \( N \) equipped with a basis \( e_0, \ldots, e_{N-1} \). The field of constants is arbitrary but
one may think of $\mathbb{R}$ or $\mathbb{C}$. A linear subspace generated by $m$ vectors $\xi^{(0)}, \ldots, \xi^{(m-1)}$ is intrinsically described by the antisymmetric multivector

$$\xi^{(0)} \wedge \cdots \wedge \xi^{(m-1)} = \sum_{0 \leq l_0 \leq \cdots \leq l_{m-1} \leq N-1} \xi_{l_0,\ldots,l_{m-1}} e_{l_0} \wedge \cdots \wedge e_{l_{m-1}}$$

with antisymmetric components

$$\xi_{l_0,\ldots,l_{m-1}} = \det \xi_{i,j}, \quad 0 \leq i, j \leq m - 1, \quad \xi^{(k)} = \xi_{i,k} e_i$$

All relations to be written being homogeneous we may consider the above quantities as being homogeneous components of the corresponding $m - 1$ dimensional linear subspace in the projective space $PV(N - 1)$ of dimension $N - 1$. A familiar case is the description of lines ($m = 2$) in projective three space ($N = 4$). An $(m - 1)$-dimensional subspace in $PV(N - 1)$ depends on $m(N - m)$ parameters\footnote{This is $m \times N$, the number of components of the vectors $\xi^{(k)}$, minus $m^2$, the dimension of the linear group $GL(m)$ acting linearly on this vector without modifying the subspace.} (four in the above example, for instance the intersections of the line with two planes) while the number of coordinates in (3.1) (taking into account homogeneity) is $\binom{N}{m} - 1$ (i.e. 5 in the example). They must therefore satisfy some (non-linear but homogeneous) relations. These are the Plücker relations. In the aforementioned example this is the classical quadratic relation expressing that the geometry of lines in the projective 3–dimensional space is equivalent to the geometry of points on a quadric in 5–dimensional projective space.

To derive typical Plücker relations, we demand that a linear combination

$$\eta = \sum x_k \xi^{(k)}$$

lies in the subspace generated by the $\xi$'s, i.e. that

$$0 = \eta \wedge \xi^{(0)} \wedge \cdots \wedge \xi^{(m-1)} = \sum_{0 \leq l_0 \leq \cdots \leq l_{m-1} \leq N-1} \tilde{\xi}_{l_0,\ldots,l_{m-1}} e_{l_0} \wedge \cdots \wedge e_{l_{m-1}}$$

$$\tilde{\xi}_{l_0,\ldots,l_{m-1}} = \sum_{i=0}^{m} (-1)^i \sum_{r=0}^{m-1} x_r \xi^{(r)}_{i_0,\ldots,\hat{i},\ldots,i_{m}}.$$  

Choose the coefficients $x_r$ as the minors in the last line of

$$\xi_{k_0,\ldots,k_{m-2},l} \equiv \left| \begin{array}{ccc} \xi^{(0)}_{k_0} & \cdots & \xi^{(m-1)}_{k_0} \\
\vdots & \ddots & \vdots \\
\xi^{(0)}_{k_{m-2}} & \cdots & \xi^{(m-1)}_{k_{m-2}} \\
\xi^{(0)}_{l} & \cdots & \xi^{(m-1)}_{l} \end{array} \right| = \sum_{r=0}^{m-1} x_r \xi^{(r)}_l$$  

(3.5)
which only depend on the choice of \( k_0, \cdots, k_{m-2} \) and not on the index \( l \). Hence we get the Plücker relations in the form

\[
\sum_{i=0}^{m} \xi_{k_0, \cdots, k_{m-2}, l, i} \xi_{l, i_{\bar{r}}, \cdots, l_m} = 0. \tag{3.6}
\]

The reader might have fun to find the relations among these relations and so on. In any case by turning the argument around these relations do characterize \( m \)-dimensional vector subspaces in \( V \) or the \((m-1)\)-dimensional ones in \( PV \) which form the \((N, m)\) Grassmannian (obviously not a vector space but rather an intersection of quadrics).

For our purposes we will need a generalization of (3.3) which follows from the observation that we could equally well replace \( \eta \) by some combination

\[
\eta' = \sum_{k<k'} x_{k, k'} \xi^{(k)} \wedge \xi^{(k')}, \tag{3.7}
\]

(or for that matter by any higher superposition of exterior products). Leaving the general case aside, we also have

\[
\sum_{i<j} (-1)^{i+j-1} \xi_{k_0, \cdots, k_{m-3}, l, i, l_j} \xi_{l_0, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, l_{m+1}} \tag{3.8}
\]

and so on.

The relevance of this discussion to the present problem arises from the determinental expressions for the function \( Z \) as expressed in eqs. (2.26) and (2.57). Indeed it was Sato’s idea to associate to points of the Grassmannian a \( \tau \)-function obtained by replacing exterior products of basis vectors by the corresponding antisymmetric generalized Schur functions. In our case one can view vector subspaces as those generated by the functions \( z, Dz, D^2z, \cdots \) so that Kontsevich’s integral appears as a realization of Sato’s idea. The task is now to translate equivalents of the Plücker relations in terms of \( Z \).

It will be easier to state this in the finite \( N \) case we started from, since by letting \( N \) become arbitrarily large we will recover the required results term by term in the asymptotic series. Thus we return to formula (2.40) understanding by \( p_n(\theta) \) the unconstrained Schur functions which we recast in the following form

\[
Z^{(N)} = \begin{vmatrix}
\frac{\partial^{N-1} f_{N-1}}{\partial \theta_1^{N-1}} & \cdots & \frac{\partial f_{N-1}}{\partial \theta_1} & f_{N-1} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial^{N-1} f_0}{\partial \theta_1^{N-1}} & \cdots & \frac{\partial f_0}{\partial \theta_1} & f_0
\end{vmatrix}, \tag{3.9}
\]
where

\begin{align*}
  f_{N-1}(\theta) &= \sum c_n^{(0)} p_{3n+N-1}(\theta) \\
  f_1(\theta) &= \sum c_n^{(1)} p_{3n+N-2}(\theta) \\
  &\vdots \\
  f_0(\theta) &= \sum c_n^{(N-1)} p_{3n}(\theta).
\end{align*}

In the above we have used eq. (2.34) which is meaningful for Schur functions with independent \( \theta \) arguments. The precise meaning of (3.10) for unconstrained \( \theta \)'s is therefore that it extracts from the complete formula (2.57) only those terms corresponding to Young tableaux which have at most \( N \) rows. By letting \( N \to \infty \) we reach any desired term.

The expression (3.9) singles out the variable \( \theta_1 \), the others playing the role of parameters. For the time being we simplify the notations by referring to \( Z^{(N)} \) as \( Z \) until at the end we restore the correct subscript. Looking at (3.9) we see that it takes the form of a Wronskian of the components \( f_0, \ldots, f_{N-1} \) of a vector denoted \( f \). It is therefore natural to attach it to an \( N \)-th order differential operator \( \Delta_N \) such that for an arbitrary function \( F \) of \( \theta_1 \) (\( d \equiv \frac{d}{d\theta_1} \))

\[
\Delta_N F = \sum_{r=0}^{N} w_r(\theta_1) d^{N-r} F = Z^{-1} \begin{vmatrix}
  \frac{\partial^N F}{\partial \theta_1^N} & \cdots & F \\
  \frac{\partial^N f_{N-1}}{\partial \theta_1^N} & \cdots & f_{N-1} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial^N f_0}{\partial \theta_1^N} & \cdots & f_0
\end{vmatrix}
\]  

(3.11)

Strictly speaking the coefficients \( w \) should also carry the subscript \( N \). In order to obtain a smooth transcription as \( N \to \infty \), Sato uses rather than the differential operator \( \Delta_N \) an equivalent pseudodifferential operator \( W \) defined as

\[
\Delta_N = W_N d^N, \\
W_N = \sum_{r=0}^{N} w_r d^{-r}
\]

(3.12)

(3.13)

with \( w_0 = 1 \) and

\[
w_1 = Z^{-1} \left( -\frac{\partial}{\partial \theta_1} \right) Z.
\]

(3.14)

Expanded in power series in the \( \theta \)'s, \( w_1 \) will have terms of fixed degree independent of \( N \) for \( N \) large enough and the remark applies to the successive coefficients \( w_p \) (infinitely
many as \(N \to \infty\) justifying that we drop eventually all reference to \(N\). Equation (3.14) admits a rather neat generalization as

\[
w_r = \frac{1}{Z} p_r \left(-\frac{\partial}{\partial \theta}\right) Z, \tag{3.15}
\]

where

\[
\frac{\partial}{\partial \theta_r} \equiv \frac{\partial}{\partial \theta_1}, \frac{1}{2} \frac{\partial}{\partial \theta_2}, \ldots, \frac{1}{k} \frac{\partial}{\partial \theta_k}, \ldots \tag{3.16}
\]

(which is meaningful since we consider the \(\theta\)'s as independent variables). Indeed let us form the generating function

\[
Z^{-1} \left( \sum_{r \geq 0} y^r p_r \left(-\frac{\partial}{\partial \theta_r}\right) \right) Z = Z^{-1} \left( \exp - \sum_{r=1}^{\infty} \frac{y^r}{r} \frac{\partial}{\partial \theta_r} \right) Z. \tag{3.17}
\]

Acting on any column vector of \(Z\), the operator \(\frac{\partial}{\partial \theta_r}\) is equivalent to \(\frac{\partial^r}{\partial \theta_1^r}\). This gives

\[
Z^{-1} \left( \sum_{r \geq 0} y^r p_r \left(-\frac{\partial}{\partial \theta_r}\right) \right) Z = Z^{-1} \det \left| (1-yd)^{N-1} f, \ldots, (1-yd)^{N} f \right|. \tag{3.18}
\]

Let us compare this with the quantity

\[
\sum_{r=0}^{N} w_r y^r = Z^{-1} \det \left| \begin{array}{ccc} y^0 & \cdots & y^N \end{array} \right|. \tag{3.19}
\]

The \((N+1) \times (N+1)\) determinant on the right hand side can be computed by subtracting the first column multiplied by \(y\) from the second, the second column multiplied by \(y\) from the third and so on, with the cofactor of the only non-vanishing element in the first row equal to the previous expression proving eq. (3.13).

One is familiar with the commutation relations involving the operators \(d^{-r}\)

\[
d^{-r} a = \sum_{k \geq 0} (-1)^k \binom{r+k-1}{k} a^{(k)} d^{-r-k} a^{(k)}
\]

\[
ad^{-r} = \sum_{r \geq 0} \binom{r+k-1}{k} d^{-r-k} a^{(k)} \tag{3.20}
\]

where \(a^{(k)} \equiv (d^k a)\). These formulae enable one to give a meaning, again dropping the index \(N\), to the “dual” pseudo–differential operator

\[
W^* = \sum d^{-r} w_r^* = \sum \left( -\frac{\partial}{\partial \theta_r} \right) Z \tag{3.21}
\]
satisfying
\[ W^* = W^{-1} \quad (3.22) \]

We relegate the (cumbersome) proof of this latter fact which relies on Plücker formulae to an appendix of this section.

The crux of the matter is the basic equation
\[ \frac{\partial W}{\partial \theta_n} = Q_n W - W d^n \quad (3.23) \]

with \( Q_n \) a normalized differential operator of order \( n \), \( Q_n = d^n + \ldots \), given by
\[ Q_n = \left( W d^n W^{-1} \right)_+ \quad (3.24) \]

the subscript \( + \) refers to the differential part of \( W d^n W^{-1} \), as follows from the fact that \( \frac{\partial W}{\partial \theta_n} \) is of order \( d^{-1} \).

Multiplying it on the right by \( d^N \), eq. (3.23) is equivalent to
\[ \frac{\partial \Delta_N}{\partial \theta_n} = Q_n \Delta_N - \Delta_N d^n \quad Q_n = \left( \Delta_N d^n \Delta_N^{-1} \right)_+ \quad (3.25) \]

where \( \Delta_N^{-1} = d^{-N} W^{-1} \). Again \( N \) is assumed much larger than \( n \) fixed, in fact as large as we want. Among these equations one is trivially verified namely the one for \( n = 1 \) where \( Q_1 = d \). To prove (3.25) choose \( Q_n \) as indicated so that the combination \( Q_n \Delta_N - \Delta_N d^n \) is a differential operator of order \( N - 1 \) as it should be if the formula is to make sense. It will then hold if both sides agree when operating on \( N \) linearly independent functions which (see eq. (3.11)) we naturally choose as being \( f_0, \ldots, f_{N-1} \) or equivalently on any linear combination which we denote by \( f \). Thus \( \Delta_N f = 0 \) and we want to prove that
\[ \left( \frac{\partial \Delta_N}{\partial \theta_n} \right) f + \Delta_N \frac{d^n f}{d\theta_1^n} = 0. \quad (3.26) \]

But \( \frac{d^n f}{d\theta_1^n} \) through the definition (3.10) is equal to \( \frac{\partial f}{\partial \theta_n} \) reducing the above expression to
\[ \frac{\partial}{\partial \theta_n} (\Delta_N f) = 0. \]

Thus eqs. (3.23) and (3.24) hold true and yield in general the so-called KP hierarchy of compatible integrable systems \( \frac{\partial^2 W}{\partial \theta_{n1} \partial \theta_{n2}} = \frac{\partial^2 W}{\partial \theta_{n2} \partial \theta_{n1}} \) for the coefficients of the pseudo-differential operator \( W \). An equivalent form is as follows. Set
\[ L = W d W^{-1} = d + O(d^{-1}) \]
\[ L^n = W d^n W^{-1} \quad Q_n = \left( L^n \right)_+ . \quad (3.27) \]
Then from (3.23) we have
\[
\frac{\partial W^{-1}}{\partial \theta_n} = - W^{-1} Q_n + d^n W^{-1}
\]
\[
\frac{\partial L}{\partial \theta_n} = [Q_n, L],
\]  
(3.28)
where in writing these formulae we have implicitly taken the $N \to \infty$ limit. As a consequence of (3.28) we have the zero curvature conditions
\[
\frac{\partial Q_m}{\partial \theta_n} - \frac{\partial Q_n}{\partial \theta_m} + [Q_m, Q_n] = 0.
\]
(3.29)
According to equations (3.15) and (3.21) the coefficients in $W$ and $W^{-1}$ do not depend on $\theta_{2r}$. Hence the differential operators $Q_{2r} \equiv (L^{2r})_+$ commute with $L$ as well as any of its powers
\[
[Q_{2r}, Q_{2r'}] = 0.
\]
(3.30)
Using the notations of theorem 1, $t_r = -(2r + 1)!! \theta_{2r+1}$, $u = \frac{\partial^2}{\partial \theta_1^2} \ln Z = \frac{\partial^2}{\partial t_0^2} \ln Z$ and (3.15), (3.21) and (3.24)
\[
Q_2 = d^2 + 2u,
Q_3 = d^3 + 3ud + \frac{3}{2} \frac{\partial u}{\partial \theta_1} \equiv (Q_2^\frac{3}{2})_+
\]
(3.31)
Setting $m = 2$ and $n = 3$ in (3.29) we conclude that the first non trivial equation in the hierarchy reads
\[
\frac{\partial Q_2}{\partial \theta_3} = [Q_3, Q_2]
\]
(3.32)
i.e.
\[
\frac{\partial u}{\partial t_1} = \frac{\partial}{\partial t_0} \left( \frac{1}{12} \frac{\partial^2 u}{\partial t_0^2} + \frac{1}{2} u^2 \right)
\]
(3.33)
as claimed in the second part of Theorem 1. Higher equations involve $\frac{\partial}{\partial t_2}, \cdots$ and are of the form (2.94), (2.11).

In fact, the commutation of $L$ and $Q_2$ implies that $L^2 = Q_2$, i.e. that $L^2$ is a differential operator. To prove this, one may appeal to a lemma [13] that asserts that the space of operators that commute with $Q_2$ is spanned by the powers of the pseudodifferential operator square root of $Q_2$, with constant coefficients. Thus
\[
L = Q_2^\frac{3}{2} + \sum_{l=0}^{\infty} \alpha_l \left( Q_2^\frac{3}{2} \right)^{-l}.
\]
(3.34)
Both $L$ and $Q^1_2$, however, are functionals of $Z$ with the limit $d$ as $Z \to 1$. It follows that all the constants $\alpha$ vanish and

$$L = Q^1_2.$$  \hfill (3.35)

This implies that the KdV flows (3.28) are generated by the

$$Q_{2r+1} = (L^{2r+1})_+ = (Q^r_2 + \frac{1}{2})_+.$$  

Notice that the above identity (3.35) means that $L$, which a priori depends on all the derivatives of $Z$, is actually a functional of the sole $u = \partial^2 \ln Z / \partial \theta_1^2$.

**Appendix**

It would seem that Plücker relations have not entered directly the discussion. One place where they play a hidden role is in the computation of the inverse $W^{-1} = W^*$. Of course if we need only the first few terms as in (3.31) one can obtain them by a direct calculation.

For completeness we give a recursive proof of eq. (3.22). The integer $N$ being fixed we start with the expressions (3.9)–(3.11) and consider $f$ in (3.10) as a column vector function of independent $\theta$’s with $d \equiv \frac{\partial}{\partial \theta_1}$ and $f^{(r)} \equiv \frac{\partial f}{\partial \theta_1^r}$. Dropping the index $N$

$$\Delta = W d^N,$$

$$W = \sum_{0}^{N} w_r d^{-r}$$  \hfill (3.36)

$$w_r = Z^{-1} p_r \left(- \frac{\partial}{\partial \theta_1}\right) Z = Z^{-1} (-1)^r \begin{vmatrix} f^{(N)} \cdots \hat{f}^{(N-r)} \cdots f \end{vmatrix}$$

using a shorthand notation for determinants. The kernel of $\Delta$ is the finite dimensional vector space generated by the components of $f$. We distinguish a flag $(f_0), (f_0, f_1), (f_0, f_1, f_2), \cdots$ and associate to it a sequence of determinants

$$Z^{(1)} = f_0 \quad Z^{(2)} = \begin{vmatrix} f_1' & f_1 \\ f_0' & f_0 \end{vmatrix}, \cdots, Z^{(N)} \equiv Z$$  \hfill (3.37)

in terms of which one can write a factorized form (the Miura transformation)

$$\Delta = \left( \frac{Z^{(N)}}{Z^{(N-1)}} d \frac{Z^{(N-1)}}{Z^{(N)}} \right) \cdots \left( \frac{Z^{(2)}}{Z^{(1)}} d \frac{Z^{(1)}}{Z^{(2)}} \right) \left( Z^{(1)} d \frac{1}{Z^{(1)}} \right).$$  \hfill (3.38)

It is clear that applied to $Z^{(1)} = f_0$, $\Delta$ gives 0 while if $\Delta_k$ is the product of the first $k$ factors starting from the right and if we assume $\Delta_k f_0 = \Delta_k f_1 = \cdots = \Delta_k f_{k-1} = 0$, then $\Delta_k f_k = \frac{Z^{(k+1)}}{Z^{(k)}}$, hence

$$\Delta_{k+1} f_k = \left( \frac{Z^{(k+1)}}{Z^{(k)}} d \frac{Z^{(k)}}{Z^{(k+1)}} \right) f_k = 0$$
proving the above factorization. The identity to be established is therefore

\[ W^{-1} = d^N \Delta^{-1} = d^N \left( Z^{(1)} d^{-1} \frac{1}{Z^{(1)}} \right) \left( \frac{Z^{(2)}}{Z^{(1)}} d^{-1} \frac{Z^{(1)}}{Z^{(2)}} \right) \cdots \left( \frac{Z^{(N)}}{Z^{(N-1)}} d^{-1} \frac{Z^{(N-1)}}{Z^{(N)}} \right) \]

\[ = \sum_{r \geq 0} d^{-r} w_r^* \]  

(3.39)

with coefficients \( w_r^* \) given by

\[ w_r^* = Z^{-1} p_r \left( \frac{\partial}{\partial \theta} \right) Z. \]  

(3.40)

Upon taking a generating function

\[ \sum_{r \geq 0} y^r w_r^* = Z^{-1} \exp \left( \sum_{n=1}^{\infty} y^n \frac{\partial}{\partial \theta} \right) Z = Z^{-1} \left| \frac{1}{1 - y \frac{\partial}{\partial \theta}} f^{(N-1)} \cdots \frac{1}{1 - y \frac{\partial}{\partial \theta}} f \right| \]  

(3.41)

where upper indices on \( f \) label derivatives. We have recognized that acting on each column of \( Z \) the shift operator is equivalent to

\[ \exp \sum_{n=1}^{\infty} y^n \left( \frac{\partial}{\partial \theta} \right)^n = \frac{1}{1 - y \frac{\partial}{\partial \theta}}. \]  

If in the above determinant we subtract from the last column the preceding one multiplied by \( y \) and so on, we get

\[ w_r^* = Z^{-1} | f^{(N-1+r)} , f^{(N-2)} \cdots f \|. \]  

(3.42)

Comparing (3.36) and (3.39), we see that the determinental numerators have a natural pictorial description in terms of Young tableaux. The first determinant is a vertical Young tableau, the second a horizontal one. This parallels the correspondence between \( p_r(-\theta) = (-1)^r s_r(\theta) \) (recall (2.30)) and \( p_r(\theta) \), and the formula to be established is similar to the identity \( \det(1 - X) \det(1 - X)^{-1} = 1 \).

In any case with this expression for \( w_r^* \) we return to (3.39) and note that (3.42) says that \( w_0^* = 1 \) in agreement with (3.39) for every \( N \) whereas, should \( N = 1, W^{-1} \) reduces to

\[ d(f_0 d^{-1} f_0^{-1}) = \sum_{r \geq 0} d^{-r} \frac{f_0^{(r)}}{f_0} \]  

(3.43)

again in agreement with (3.42). We therefore assume that (3.42) holds for any \( r \) if \( N' < N \) and for \( r' \leq r \) when the size of determinants is \( N \), and establish it for \( w_{N,r+1}^* \) reinstating
the index $N$. We have

$$W_{N}^{-1} = dW_{N}^{-1} \frac{Z^{(N)}}{Z^{(N-1)}} d^{-1} \frac{Z^{(N-1)}}{Z^{(N)}}$$

$$= \sum_{k \geq 0} d^{1-k} w_{N-1,k}^{*} \frac{Z^{(N)}}{Z^{(N-1)}} d^{-1} \frac{Z^{(N-1)}}{Z^{(N)}}$$

$$= \sum_{r \geq 0} d^{-r} \sum_{k+l=r} \left( w_{N-1,k}^{*} \frac{Z^{(N)}}{Z^{(N-1)}} \right)^{(l)} \frac{Z^{(N-1)}}{Z^{(N)}}. \quad (3.44)$$

Write

$$w_{N,r}^{*} = \frac{v_{N,r}}{Z^{(N)}}. \quad (3.45)$$

We have

$$v_{N,r} = Z^{(N-1)} \sum_{k+l=r} \left( v_{N-1,k} \frac{Z^{(N)}}{Z^{(N-1)} Z^{(N-1)}} \right)^{(l)} \cdot \quad (3.46)$$

We want to show that

$$v_{N,\rho} = |f^{(N-1+\rho)}, f^{(N-2)} \ldots f| \quad (3.47)$$

assuming it to be true for $N' < N$ (where we only keep the components $f_0, \ldots, f_{N'-1}$) and also for $\rho \leq r$ to prove that it holds for $r + 1$. Take a derivative of the above identity

$$v'_{N,r} = \frac{Z^{(N-1)}}{Z^{(N-1)} v_{N,r}} + Z^{(N-1)} \sum_{k+l=r} \left( v_{N-1,k} \frac{Z^{(N)}}{Z^{(N-1)} Z^{(N-1)}} \right)^{(l+1)} \cdot \quad (3.48)$$

The last sum differs from $v_{N,r+1}$ by the missing term $v_{N-1,r+1} \frac{Z^{(N)}}{Z^{(N-1)}}$. Hence

$$v_{N,r+1} = v'_{N,r} + v_{N-1,r+1} \frac{Z^{(N)}}{Z^{(N-1)}} \frac{Z^{(N-1)}}{Z^{(N-1)} v_{N,r}}. \quad (3.49)$$

Denote the column vector $(f_0, \ldots, f_{N-2})^T$ by $\varphi$ (of dimension $N - 1$). According to the recursive hypothesis this reads

$$v_{N,r+1} = |f^{(N+r)}, f^{(N-2)}, \ldots, f| + \frac{\gamma}{Z^{(N-1)}} \quad (3.50)$$

with

$$\gamma = |\varphi^{(N-1+r)}, \varphi^{(N-1)}, \varphi^{(N-2)} \ldots, \varphi| f^{(N-1+r)}, f^{(N-1)}, f^{(N-2)} \ldots, f$$

$$- |\varphi^{(N-1+r)}, \varphi^{(N-1)}, \varphi^{(N-2)} \ldots, \varphi| f^{(N-1+r)}, f^{(N-1)}, f^{(N-2)} \ldots, f$$

$$+ |\varphi^{(N-1+r)}, \varphi^{(N-1)}, \varphi^{(N-2)} \ldots, \varphi| f^{(N-1+r)}, f^{(N-1)}, f^{(N-2)} \ldots, f| \quad (3.51)$$
where in each term the first determinant is \((N - 1) \times (N - 1)\) dimensional, the second \(N \times N\). We have to show that the combination \(\gamma\) vanishes since we wish to prove that \(v_{N,r+1}\) is the first term of the r.h.s. of (3.50). One easily checks that \(\gamma = 0\) if \(N = 2\), so we henceforth assume \(N > 2\). To reduce the vanishing of \(\gamma\) to one of Plücker’s identities, expand the \(N \times N\) determinants involving \(f\) and its derivatives according to its first line. For each term of the form \(f_{N-1}^{(k)}\) the coefficient is a combination of \(\varphi\)-determinants which vanishes by virtue of the Plücker relations (3.8), completing the proof of formulas (3.21) and (3.22).

4. Matrix Airy equation and Virasoro highest weight conditions.

The differential equation (2.14) generalizes to the \(N\)-dimensional case as follows. Call \(Y(\Lambda)\) the integral appearing in (2.16) for finite \(N\)

\[
Y(\Lambda) = \int dM \exp \left( \frac{M^3}{6} + \frac{M\Lambda^2}{2} \right). \tag{4.1}
\]

The function \(Y\) satisfies for each index \(k\)

\[
0 = \int dM \frac{d}{dM_{kk}} \exp \left( \frac{i}{6} (3\text{tr}M\Lambda^2 + M^3) \right) \tag{4.2}
\]

i.e.

\[
0 = \left\langle \sum_{l, l \neq k} M_{kl}M_{lk} + M_{kk}^2 + \lambda_k^2 \right\rangle \tag{4.3}
\]

where \(\langle \cdot \rangle\) denotes an integral taken with respect to the weight \(dM \exp i\text{tr} \left( \frac{M^3}{6} + \frac{M\Lambda^2}{2} \right)\). The insertion of a diagonal factor \(M_{kk}\) can be achieved by acting with the derivative operator \(-i\frac{1}{\lambda_k} \frac{\partial}{\partial \lambda_k}\) on \(Y\). To deal with non-diagonal insertions we express the invariance of the integral \(Y\) under an infinitesimal change of variable of the form

\[
M \rightarrow M + i\epsilon [X, M], \quad \text{with} \quad X_{ab} = \delta_{ak}\delta_{bl}M_{kl}. \tag{4.4}
\]

The Jacobian is \(1 + i\epsilon (M_{ll} - M_{kk})\), while the term \(\text{tr}M^3\) is invariant. Thus

\[
0 = \left\langle M_{ll} - M_{kk} + \frac{i}{2} (\lambda_k^2 - \lambda_l^2) M_{kl}M_{lk} \right\rangle \tag{4.5}
\]
with no summation implied. Inserting this into (4.3) leads to

\[ 0 = \lambda_k^2 + \langle M_{kk}^2 \rangle - 2i \sum_{l,l \neq k} \frac{\langle M_{kl} - M_{lk} \rangle}{\lambda_k^2 - \lambda_l^2}. \]  

(4.6)

This yields the matrix Airy equations

\[
\begin{bmatrix}
\lambda_k^2 - \left( \frac{1}{\lambda_k} \frac{\partial}{\partial \lambda_k} \right)^2 - 2 \sum_{l,l \neq k} \frac{1}{\lambda_k^2 - \lambda_l^2} \left( \frac{1}{\lambda_k} \frac{\partial}{\partial \lambda_k} - \frac{1}{\lambda_l} \frac{\partial}{\partial \lambda_l} \right) \\
\end{bmatrix} Y = 0
\]

(4.7)

which can be turned into equivalent equations for \( Z \) itself

\[
\begin{bmatrix}
\frac{1}{\lambda_k^2} \left( \sum_{l} \frac{1}{\lambda_l} \right)^2 + \frac{1}{4 \lambda_k^4} + 2 \sum_{l,l \neq k} \frac{1}{\lambda_k^2 - \lambda_l^2} \left( \frac{1}{\lambda_k} \frac{\partial}{\partial \lambda_k} - \frac{1}{\lambda_l} \frac{\partial}{\partial \lambda_l} \right) \\
-2 \left( 1 + \frac{1}{\lambda_k^2} \sum_{l} \frac{1}{\lambda_k + \lambda_l} \right) \frac{\partial}{\partial \lambda_k} + \left( \frac{1}{\lambda_k} \frac{\partial}{\partial \lambda_k} \right)^2 \\
\end{bmatrix} Z = 0.
\]

(4.8)

In the limit \( N \to \infty \) we know from sec. 3 that \( Z \) admits an expansion in terms of odd traces

\[ t_n = -(2n-1)!! \sum_{l} \frac{1}{\lambda_l^{2n+1}}. \]

(4.9)

The differential equations (4.7) can be expanded in inverse powers of \( \lambda_k \) in the form

\[ 2 \sum_{m \geq -1} \frac{1}{(\lambda_k^2)^{m+2} L_m Z = 0}. \]

(4.10)

Explicitly

\[ L_{-1} = \frac{1}{2} t_0^2 + \sum_{k \geq 0} t_{k+1} \frac{\partial}{\partial t_k} - \frac{\partial}{\partial t_0} \]

For an alternative derivation one can transform the “equation of motion” (4.3) into a matrix differential equation, assuming at first the argument \( T \equiv \Lambda^2 \) to be an arbitrary (i.e. not necessarily diagonal) Hermitian matrix. Recognizing that the integral is invariant under conjugation of \( T \), hence only a function of its eigenvalues \( \{ t_a \} \) one then uses

\[ \frac{\partial}{\partial T_{kl}} = \sum_a \frac{\partial t_a}{\partial T_{kl}} \frac{\partial}{\partial t_a} = \sum_a \frac{\text{Min}_{kl}(t_a - T) \frac{\partial}{P'(t_a)}}{\partial t_a} \]

where \( P(x) = \det(x-T) \) and \( \text{Min}_{kl} \) denotes the \((k,l)\) minor in the corresponding matrix.
\[ L_0 = \frac{1}{8} + \sum_{k \geq 0} (2k + 1) t_k \frac{\partial}{\partial t_k} - 3 \frac{\partial}{\partial t_1} \] (4.11)

\[ L_1 = \sum_{k \geq 1} (2k + 1)(2k - 1) t_{k-1} \frac{\partial}{\partial t_k} + \frac{1}{2} \frac{\partial^2}{\partial t_0^2} - 15 \frac{\partial}{\partial t_2} \]

\[ L_2 = \sum_{k \geq 2} (2k + 1)(2k - 1)(2k - 3) t_{k-2} \frac{\partial}{\partial t_k} + 3 \frac{\partial^2}{\partial t_0 \partial t_1} - 105 \frac{\partial}{\partial t_3} \]

\[ \ldots \ldots \]

\[ L_m = \sum_{k \geq m} \frac{(2k + 1)!!}{(2(k - m) - 1)!!} t_{k-m} \frac{\partial}{\partial t_k} + \frac{1}{2} \sum_{k+l=m-1} (2k + 1)!!(2l + 1)!! \frac{\partial^2}{\partial t_k \partial t_l} \\
- (2m + 3)!! \frac{\partial}{\partial t_{m+1}} + \frac{t_0^2}{2} \delta_{m+1,0} + \frac{1}{8} \delta_{m,0} \]

In eq. (4.10), each coefficient has to vanish so that

**Theorem 2** (Kontsevich)

\[ Z \text{ satisfies and is determined by the highest weight conditions} \]

\[ L_m Z = 0 \quad m \geq -1. \] (4.12)

The operators \( L_m \) obey (part of) the Virasoro (or rather the Witt) algebra, namely

\[ [L_m, L_n] = (m - n) L_{m+n} \quad m, n \geq -1 \] (4.13)

generated by \( L_{-1}, L_0, L_1, L_2 \). Note that only the first two involve first order derivatives.

**5. Genus expansion.**

As in standard matrix models there exists a genus expansion for \( F \)

\[ \ln Z = F = \sum_{g \geq 0} F_g \] (5.1)

One way to obtain it is from the Airy system. One inserts appropriate extra factors of \( N \) and studies the large \( N \) limit, paying attention to corrections, according to a method applied to leading order by Kazakov and Kostov [3] and revived by Makeenko and Semenoff [5]. In the spirit of our paper we follow a slightly different approach based on the KdV
equations and Virasoro constraints. In genus $g$, $F_g$ collects in the expansion (2.1) all terms such that

$$F_g(t, \tau) = \sum_{k_i} \left( \prod_{i \geq 0} \left( \frac{(\tau_i t_i)^k_i}{k_i!} \right) \right)$$

We set

$$u_g(t) = \frac{\partial^2 F_g}{\partial t^2}.$$  \hspace{1cm} (5.3)

In the KdV hierarchy (eq. (2.9)) the leading term in the semi-classical or genus expansion corresponding to the term $u_0$ is obtained from power counting by ignoring in the differential polynomial $R_n$ all terms involving derivatives. It then reduces to

$$n \geq 0 \quad \frac{\partial u_0}{\partial t_n} = \frac{\partial}{\partial t_0} \frac{u_0^{n+1}}{(n+1)!}.$$  \hspace{1cm} (5.4)

For $n = 0$ this is vacuous and has to be supplemented by the first Virasoro condition (4.10) (for $m = -1$) which amounts to

$$\frac{\partial u_0}{\partial t_0} = 1 + \sum_{k \geq 0} t_{k+1} \frac{\partial u_0}{\partial t_k}.$$  \hspace{1cm} (5.5)

Inserting (5.4) into (5.3) gives

$$\frac{\partial}{\partial t_0} \left\{ u_0 - \sum_{n \geq 0} t_n \frac{u_0^n}{n!} \right\} = 0.$$  \hspace{1cm} (5.6)

Define

$$I_k(u_0, t, \tau) = \sum_{p \geq 0} t_{k+p} \frac{u_0^p}{p!}.$$  \hspace{1cm} (5.7)

Equation (5.6) suggests

**Lemma 4**

$u_0(t, \tau)$ satisfies the implicit equation

$$u_0 - I_0(u_0, t, \tau) = 0.$$  \hspace{1cm} (5.8)

To check this, multiply (5.3) by $\frac{u_0^n}{n!}$ and use (5.4) to obtain

$$\frac{\partial u_0}{\partial t_n} = \frac{u_0^n}{n!} + \sum_{k \geq 0} t_{k+1} \frac{\partial}{\partial t_k} \frac{u_0^{n+1}}{(n+1)!}.$$  \hspace{1cm} (5.9)
while
\[
\frac{\partial I_0(u_0, t_0)}{\partial t_n} = \frac{u_0^n}{n!} + \sum_{k \geq 0} t_{k+1} \frac{\partial}{\partial t_n} \frac{u_0^{k+1}}{(k+1)!} .
\] (5.10)

Subtracting and using (5.4) again, we find
\[
\frac{\partial}{\partial t_n} [u_0 - I_0(u_0, t_0)] = \sum_{k \geq 0} t_{k+1} \left( \frac{u_0^n}{n!} \frac{\partial}{\partial t_k} u_0 - \frac{u_0^k}{k!} \frac{\partial}{\partial t_n} u_0 \right)
\] (5.11)
\[
= \sum_{k \geq 0} t_{k+1} \left( \frac{u_0^n}{n!} \frac{\partial}{\partial t_0} \frac{u_0^{k+1}}{(k+1)!} - \frac{u_0^k}{k!} \frac{\partial}{\partial t_0} \frac{u_0^{n+1}}{(n+1)!} \right) = 0 .
\]

The difference \(u_0 - I_0(u_0, t_0)\) is thus a constant. Since it vanishes at \(t_0 = 0\) it is identically zero, completing the proof. To find \(F_0\) we have to integrate twice eq. (5.8) with respect to \(t_0\). From (5.2) the boundary conditions are given by the vanishing of \(F_0\) and \(\frac{\partial}{\partial t_0} F_0\) when \(t_0 = 0\). This yields with \(u_0(t_0)\) implicitly given by (5.8)
\[
F_0 = \frac{u_0^3}{6} - \sum_{k \geq 0} \frac{u_0^{k+2}}{k + 2} \frac{t_k}{k!} + \frac{1}{2} \sum_{k \geq 0} \frac{u_0^{k+1}}{k + 1} \sum_{a + b = k} \frac{t_a t_b}{a! b!} .
\] (5.12)

**Remarks**

(i) As is generally the case if we extend \(F_0\) by considering \(u_0\) (originally equal to \(\frac{\partial^2 F_0}{\partial t_0^2}\)) as an auxiliary independent parameter, we find that the stationarity condition
\[
\frac{\partial F_0}{\partial u_0} = \frac{1}{2} (u_0 - I_0(u_0, t_0))^2 = 0
\] (5.13)
yields equation (5.8).

(ii) The expression for \(F_0\) is equivalent to the one given by Makeenko and Semenoff [5] using (infinitely many) eigenvalues \(\lambda_k\)
\[
F_0 = \frac{1}{3} \sum_k \lambda_k^2 - \frac{1}{3} \sum_k (\lambda_k - 2s)^{\frac{3}{2}} - s \sum_k (\lambda_k^2 - 2s)^{\frac{3}{2}}
+ \frac{s^3}{6} - \frac{1}{2} \sum_{k, l} \ln \frac{\sqrt{\lambda_k^2 - 2s} + \sqrt{\lambda_l^2 - 2s}}{\lambda_k + \lambda_l}
\] (5.14)
with the condition
\[
\frac{\partial F_0}{\partial s} = \frac{1}{2} \left( s + \sum_k \frac{1}{\sqrt{\lambda_k^2 - 2s}} \right) = 0
\] (5.15)
only upon identification of \(s\) with \(u_0\), of \(I_p\) with \(-(2p - 1)!! \sum_k \frac{1}{(\lambda_k^2 - 2s)^{p+\frac{1}{2}}} \) and \(t_n\) as in (4.9).
(iii) From (5.12) or (5.8) we can readily find the first few terms in the expansion of \( F_0 \) which up to a factor \( t_0^3 \) only involves the combinations \( t_0^{k-1} t_k \).

\[
F_0 = \frac{t_0^3}{3!} + t_1 \frac{t_0^3}{3!} + \left( t_2 \frac{t_0^4}{4!} + 2 \frac{t_1^2 t_0^3}{2! 3!} \right) + \left( t_3 \frac{t_0^5}{5!} + 3 t_1 t_2 \frac{t_0^4}{4!} + 6 \frac{t_0^3 t_1^2}{3! 3!} \right) \\
+ \left[ t_4 \frac{t_0^6}{6!} + \left( \frac{6 t_2^2}{2!} + 4 t_1 t_3 \right) \frac{t_0^5}{5!} + 24 \frac{t_0^3 t_1^4}{3! 4!} + 12 t_2 \frac{t_1^2 t_0^4}{2! 4!} \right] \\
+ \left[ t_5 \frac{t_0^7}{7!} + (5 t_1 t_4 + 10 t_2 t_3) \frac{t_0^6}{6!} + 120 \frac{t_0^3 t_1^5}{3! 5!} + \left( 30 t_1 \frac{t_2^2}{2!} + 20 t_3 \frac{t_1^2}{2!} \right) \frac{t_0^5}{5!} + 60 t_2 \frac{t_1^3 t_0^4}{3! 4!} \right] \\
+ \left[ t_6 \frac{t_0^8}{8!} + \left( 20 \frac{t_2^3}{2!} + 6 t_1 t_5 + 15 t_2 t_4 \right) \frac{t_0^7}{7!} + 720 \frac{t_0^3 t_1^4}{3! 6!} + \left( 90 \frac{t_2^3}{3!} + 30 t_4 \frac{t_2^2}{2!} + 60 t_1 t_2 t_3 \right) \frac{t_0^6}{6!} \right. \\
\left. + \left( 120 t_3 \frac{t_1^3}{3!} + 180 \frac{t_2^3}{2! 2!} \right) \frac{t_0^5}{5!} + 360 t_2 \frac{t_1^4 t_0^4}{3! 4!} \right] + \ldots \tag{5.16}
\]

In genus zero all coefficients are positive integers (as opposed to fractional) due to the smoother structure of \( \mathcal{M}_{0,n} \), \( n \geq 3 \). Indeed we have the obvious

**Lemma 5**

The class of formal power series in \( t_0, t_1, \ldots \) which vanish at \( t_0 = 0 \) with non-negative integral derivatives at the origin is stable under

(i) addition

(ii) product

(iii) composition

To apply this to \( u_0 \) (and hence to \( F_0 \)) we remark that the sequence

\[
f_0 = t_0, \quad f_n = \sum_{k=0}^{\infty} f_{n-k} \frac{t_k}{k!} \tag{5.17}
\]

has each of its derivatives at the origin which stabilizes to the corresponding one of \( u_0 \) after finitely many steps.

To obtain the next terms we split the Virasoro constraints expressed on \( \ln Z \) as follows

\[
m = -1
\]

\[
\frac{t_0^2}{2} \delta_{g,0} + \sum_{k \geq 0} t_{k+1} \frac{\partial F_g}{\partial t_k} - \frac{\partial F_g}{\partial t_0} = 0
\]

\[
m = 0
\]

\[
\frac{1}{8} \delta_{g,1} + \sum_{k \geq 0} (2k + 1) t_k \frac{\partial F_g}{\partial t_k} - 3 \frac{\partial F_g}{\partial t_1} = 0 \tag{5.18}
\]

\[
m = 1
\]
\[
\sum_{k \geq 1} (2k + 1)(2k - 1)t_k \frac{\partial F_g}{\partial t_k} - 15 \frac{\partial F_g}{\partial t_2} + \frac{1}{2} \frac{\partial^2 F_{g-1}}{\partial t_0^2} + \frac{1}{2} \sum_{g_1 + g_2 = g} \frac{\partial F_{g_1}}{\partial t_0} \frac{\partial F_{g_2}}{\partial t_0} = 0
\]

while a similar splitting of the KdV equation (2.9) yields

\[
\frac{\partial u_g}{\partial t_1} = \frac{\partial}{\partial t_0} \left( \frac{1}{12} \frac{\partial^2 u_{g-1}}{\partial t_0^2} + \frac{1}{2} \sum_{g_1 + g_2 = g} u_{g_1} u_{g_2} \right). \tag{5.19}
\]

For genus one this equation reads

\[
\frac{\partial}{\partial t_0} \left\{ \left( \frac{\partial}{\partial t_1} - u_0 \frac{\partial}{\partial t_0} \right) F_1 - \frac{1}{12} \frac{\partial^2 u_0}{\partial t_0^2} \right\} = 0. \tag{5.20}
\]

We have

\[
\frac{\partial u_0}{\partial t_0} = \frac{1}{1 - I_1}, \quad \frac{\partial u_0}{\partial t_1} = \frac{u_0}{1 - I_1},
\]

\[
p \geq 1 \quad \frac{\partial I_p}{\partial t_0} = \frac{I_{p+1}}{1 - I_1}, \quad \left( \frac{\partial}{\partial t_1} - u_0 \frac{\partial}{\partial t_0} \right) I_p = \delta_{p,1}
\]

hence we can rewrite

\[
\frac{1}{12} \frac{\partial^2 u_0}{\partial t_0^2} = \frac{1}{12} \frac{I_2}{(1 - I_1)^3} = \frac{1}{24} \frac{\partial}{\partial t_1} \frac{I_2}{(1 - I_1)^2} = \frac{\partial}{\partial I_1} \frac{\partial}{\partial t_0} \left( \frac{1}{24} \ln \frac{1}{1 - I_1} \right). \tag{5.22}
\]

If \( F_1 \) is a function of \( t \) only through \( I_1, I_2, \ldots \), we can rewrite

\[
\left( \frac{\partial}{\partial t_1} - u_0 \frac{\partial}{\partial t_0} \right) \frac{\partial}{\partial t_0} F_1 = \frac{\partial}{\partial I_1} \frac{\partial}{\partial t_0} F_1 \tag{5.23}
\]

so that equation (5.20) becomes

\[
\frac{\partial}{\partial t_0} \frac{\partial}{\partial I_1} \frac{\partial}{\partial t_0} \left\{ F_1 - \frac{1}{24} \ln \frac{1}{1 - I_1} \right\} = 0. \tag{5.24}
\]

This suggests that

\[
F_1 = \frac{1}{24} \ln \frac{1}{1 - I_1}, \tag{5.25}
\]
in agreement with the above hypothesis so that eq. (5.19) is satisfied. A straightforward computation shows that the Virasoro conditions are satisfied, proving (5.25).

**Remark**

It is not unexpected that the genus one (or “one-loop”) result involves as usual a logarithm. Expanding $F_1$

$$24F_1 = t_1 + \left( \frac{t_1^2}{2!} + t_0 t_2 \right) + \left( \frac{2t_1^3}{3!} + t_2 t_1^2 + 2t_0 t_1 t_2 \right)$$

$$+ \left( \frac{6t_1^4}{4!} + t_2 t_1^3 + 4t_0^2 t_2^2 + 6t_0 t_2 t_1^2 + 3t_1 t_3 t_0 \right)$$

$$+ \left( \frac{24t_1^5}{5!} + t_2 t_1^4 + 24t_0 t_2 t_1^3 + (4t_1 t_4 + 7t_2 t_3) t_0^3 \right) + 16t_1 t_2^2 t_0 + 12t_3 t_1^2 t_0 \right)$$

$$+ \left[ 120t_0^6 + t_6 t_0^2 t_2 + 120t_0 t_2^4 \right] + \left( \frac{14t_1^2}{2!} + 5t_1 t_5 + 11t_2 t_4 \right) t_0^4 + 48t_0^3 t_1^3 + 60t_3 t_1^2 t_0$$

$$+ \left( 20t_4 t_1^2 + 35t_1 t_2 t_0 \right)$$

$$+ \left[ 720t_0^7 + t_7 t_0^6 + 720t_0 t_2 t_1^5 + (6t_1 t_6 + 16t_2 t_5 + 25t_3 t_4) \right] t_0^5 + 360t_3 t_1^4 t_0$$

$$+ \left( 84t_1 t_2^3 + 118t_3 t_2^2 + 30t_5 t_1^2 + 66t_1 t_2 t_4 \right) t_0^4 + 4!$$

$$+ 288t_1 t_0^3 t_2^3 + \left[ 120t_4 t_3^3 + 480t_0^2 t_2^2 \right] t_1^3 + 210t_2 t_3 t_0^3] + \ldots$$

(5.26)

All intersection numbers are of the form $\frac{1}{24} \times$ (a positive integer) since $I_1$ and $-\ln(1 - I_1)$ belong to the class of functions referred to in Lemma 5.

For higher genus the Ansatz

$$F_g = \sum_{2 \leq k \leq 3g-2} \langle \tau_2^{l_2} \tau_3^{l_3} \ldots \tau_{3g-2}^{l_{3g-2}} \rangle \frac{1}{(1 - I_1)^{2(g-1)}} \sum_{l_3} \frac{I_2^{l_2} I_3^{l_3} \ldots I_{3g-2}}{l_2! l_3! \ldots l_{3g-2}!},$$

(5.27)

which is a finite sum of monomials in $I_k/(1 - I_1)^{2k+1}$, the number of which is $p(3g - 3)$ (with $p(n)$ the number of partitions of $n$), is consistent with the KdV equation (5.13). Inserted into the latter, it allows one to compute the coefficients with the result

$$F_2 = \frac{1}{5760} \left[ 5 \frac{I_4}{(1 - I_1)^3} + 29 \frac{I_3 I_2}{(1 - I_1)^4} + 28 \frac{I_2^3}{(1 - I_1)^5} \right].$$

(5.28)

Hence

$$\langle \tau_2 \rangle = \frac{1}{1152}, \quad \langle \tau_2 \tau_3 \rangle = \frac{29}{5760}, \quad \langle \tau_3 \rangle = \frac{7}{240}$$

(5.29)
in agreement with Witten [1]. The other intersection numbers can be derived by expanding (5.28) in \( t \).

For genus 3 we find

\[
F_3 = \frac{1}{2903040} \left[ 35 \frac{I_7}{(1 - I_1)^5} + 539 \frac{I_6 I_2}{(1 - I_1)^6} + 1006 \frac{I_5 I_3}{(1 - I_1)^6} + 4284 \frac{I_5 I_2^2}{(1 - I_1)^7} \\
+ 607 \frac{I_4^2}{(1 - I_1)^6} + 13452 \frac{I_4 I_3 I_2}{(1 - I_1)^7} + 22260 \frac{I_4 I_2^3}{(1 - I_1)^8} + 2915 \frac{I_3^2}{(1 - I_1)^7} \\
+ 43050 \frac{I_3 I_2^2}{(1 - I_1)^8} + 81060 \frac{I_3 I_2}{(1 - I_1)^9} + 34300 \frac{I_2^4}{(1 - I_1)^10} \right],
\]

which yields the table

\[
\begin{align*}
\langle \tau_7 \rangle &= \frac{1}{82944} \\
\langle \tau_6 \tau_2 \rangle &= \frac{77}{414720} \\
\langle \tau_5 \tau_3 \rangle &= \frac{503}{1451520} \\
\langle \tau_5 \tau_2^2 \rangle &= \frac{17}{5760} \\
\langle \tau_4^2 \rangle &= \frac{607}{1451520} \\
\langle \tau_6 \rangle &= \frac{1225}{144} \\
\langle 4 \tau_3 \rangle &= \frac{111920}{144} \\
\langle 3 \rangle &= \frac{53}{1152} \\
\langle 2 \rangle &= \frac{583}{96768} \\
\langle \tau_3^2 \rangle &= \frac{205}{3456} \\
\langle \tau_2^4 \rangle &= \frac{193}{288} \\
\langle \tau_2^6 \rangle &= \frac{1}{144} \\
\langle \tau_3^2 \rangle &= \frac{1}{144}
\end{align*}
\]

Table I

There is no difficulty to pursue these computations as far as one wishes.

**Remarks**

(i) It follows from Lemma 5 that all series coefficients in (5.27) reexpressed in terms of \( \prod l_k! / l_k! \) are non-negative integers up to the finitely many prefactors. All intersection numbers of a fixed genus when written as irreducible fractions have therefore a lowest common multiplier (l.c.m.) \( D_g \): \( D_0 = 1 \), \( D_1 = 24 \), and for \( g > 1 \), \( D_g \) is the lowest common denominator of the finitely many intersection numbers appearing in (5.27), when written as irreducible fractions. We conjecture that for \( 1 < g' \leq g \), the order of any automorphism group of an algebraic curve of genus \( g' \) (bounded by \( 84(g-1) \)) divides \( D_g \). Thus \( D_3 = 2903040 \) is divisible by 168 (the order of the largest automorphism group of a genus 3 curve) and by 48 (the same for genus 2).

(ii) The term in \( F_g \), \( g > 1 \) which has the highest power of \( (1 - I_1) \) in the denominator has the form

\[
\frac{\langle \tau_2^{3g-3} \rangle}{(3g-3)! (1 - I_1)^{5g-5}} \]

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In the next section we develop a formalism to resum these terms as well as subleading ones akin to the “double scaling limit” of standard matrix models [10] (one recognizes the same string exponents and the same ingredients). At the other extreme the term with the lowest power of $1 - I_1$ in the denominator is

$$
\langle \tau_{3g-2} \rangle \frac{I_{3g-2}}{(1 - I_1)^{2g-1}} \langle \tau_{3g-2} \rangle = \frac{1}{(24)^g g!}.
$$

(5.31)

The last equality (also valid for $g = 1$) follows from (5.19) by keeping terms with the lowest power of $(1 - I_1)^{-1}$. It implies that $(24)^g g!$ divides $D_g$.

6. Singular behaviour and Painlevé equation.

The expression (2.18) exhibits a singular behaviour as $I_1 \to 0$. In a first step, we can keep in the KdV equation the dominant terms by considering that

$$
I_2 \approx \text{constant}, \quad I_k \approx 0 \text{ for } k \geq 3. \quad (6.1)
$$

This is consistent with the derivatives of the $I$’s: $\partial I_k/\partial t_0 = I_{k+1}/(1 - I_1)$ and $\partial I_k/\partial t_1 = u_0 I_{k+1}/(1 - I_1)$ for $k \geq 2$. In this approximation, the genus $g$ contribution to the specific heat $u_g = \partial^2 F_g/\partial t^2_0$ is of the form

$$
u_g = \partial^2 F_g/\partial t^2_0 = \alpha_g \frac{I_{3(g-1)+2}^2}{(1 - I_1)^{5(g-1)+4}}.
$$

(6.2)

We introduce the scaling variable

$$
Z = \frac{(1 - I_1)}{I_2^{3/5}}
$$

(6.3)

and separate the genus zero contribution by setting $u = u_0 + \tilde{u}$, $\tilde{u} = \sum_{g \geq 1} u_g$. The KdV equation (2.94) then reads

$$
\frac{\partial \tilde{u}}{\partial t_1} = \tilde{u} \left( \frac{1}{1 - I_1} + \frac{\partial \tilde{u}}{\partial t_0} \right) + \frac{1}{4} \frac{I_2^2}{(1 - I_1)^5} + \frac{1}{12} \frac{\partial^3 \tilde{u}}{\partial t_0^3}
$$

$$
= \tilde{u} \left( \frac{1}{1 - I_1} + \frac{\partial \tilde{u}}{\partial I_1} \right) + \frac{1}{4} \frac{I_2^2}{(1 - I_1)^5} + \frac{1}{12} \left( \frac{I_2}{1 - I_1} \frac{\partial \tilde{u}}{\partial I_1} \right)^3. \quad (6.4)
$$

A rescaling

$$
\tilde{u} = I_2^{-2/5} \psi(z)
$$

(6.5)

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leads to the equation
\[
\frac{\partial \psi}{\partial z} + \frac{\psi}{z} \left[1 - \frac{\partial \psi}{\partial z}\right] + \frac{1}{4z^5} - \frac{1}{12} \left(\frac{1}{z} \frac{\partial}{\partial z}\right)^3 \psi = 0.
\] (6.6)

This equation will be generalized below. In this particular case of the behaviour (6.1), one can transform it into the Painlevé equation: we set \( t = 2^{-2/5}z^2, \psi(z) = z + 2^{1/5} \phi(t) \) and find
\[
\frac{1}{3} \phi'' + \phi^2 - t = 0.
\] (6.7)

which has the asymptotic expansion
\[
\phi = \sum \frac{\phi_g}{t^{3(g-1)+2}}, \quad \phi_0 = -1,
\] (6.8)

where the successive terms satisfy
\[
\phi_{g+1} = \frac{25g^2 - 1}{24} \phi_g + \frac{1}{2} \sum_{m=1}^{g} \phi_{g+1-m} \phi_m.
\] (6.9)

Hence in this regime
\[
\sum_{g \geq 2} F_{g}^{\text{sing}} = \sum_{g \geq 2} \frac{\langle \tau_2^{3g-3} \rangle}{(3g-3)! (1-I_1)^{5(g-1)}} I_2^{3(g-1)}
\]
\[
\langle \tau_2^{3g-3} \rangle = \frac{2^g(3g-3)!}{(5g-5)(5g-3)} \phi_g
\] (6.10)

\[
g = 2 \quad \langle \tau_2^3 \rangle = \frac{7}{240}, \quad g = 3 \quad \langle \tau_2^6 \rangle = \frac{1225}{144}, \quad g = 4 \quad \langle \tau_2^9 \rangle = \frac{1816871}{48}, \ldots
\] (6.11)

This discussion may be extended to the regime in which all (or a finite number of) the \( I \)'s are retained and tend to zero according to the following scaling law
\[
z = \frac{(1 - I_1)}{I_2^{3/5}}
\]
\[
v_q = \frac{I_q(1 - I_1)^{q-2}}{I_2^{q-1}} \quad q \geq 3
\] (6.12)

\[
F_{g}^{\text{sing}} = z^{-5(g-1)} \sum_{\Sigma_{2 \leq k \leq 3g-2} (k-1)l_k = 3g-3} \langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \rangle \prod_{q=3}^{3g-2} \frac{l_q!}{v_q}
\]
The KdV equation is then rephrased as

\[
\left[ z \frac{\partial}{\partial z} + \sum_{q \geq 3} (q - 2)v_q \frac{\partial}{\partial v_q} + 1 \right] \psi =
\]

\[
= \frac{1}{z} \psi \left( \Delta + \frac{2}{5} v_3 \right) \psi - \frac{3 + v_3}{12z^4} + \frac{1}{12z^5} \left[ \Delta - 4 - \frac{8}{5} v_3 \right] \left[ \Delta - 2 - \frac{3}{5} v_3 \right] \left[ \Delta + \frac{2}{5} v_3 \right] \psi
\]

where the same change of function as in (6.5) has been carried out, and \( \Delta \) denotes the differential operator

\[
\Delta = (1 + \frac{3}{5} v_3) z \frac{\partial}{\partial z} + \sum_{q \geq 3} \left( - v_{q+1} + (q - 2)v_q + (q - 1)v_q v_3 \right) \frac{\partial}{\partial v_q}.
\]

The contribution to a given genus \( g \) involves only a finite number of terms in the sum: \( q \leq 3g - 2 \). Moreover the differential operators in (6.13) respect the grading in powers of \( z \),

\[
\psi_g(z, v, \cdot) = z^{-5(g-1)-4} \psi_g(v, \cdot),
\]

thus determining the polynomials \( \psi_g(v, \cdot) \) recursively. For instance, if only \( z \) and \( v \equiv v_3 \) are retained, we have for \( g > 1 \) (and \( \partial_v \equiv \partial/\partial v \))

\[
[5(g - 1) + 3 - v \partial_v] \psi_g(v) = \sum_{g' = 1}^{g-1} \psi_{g-g'}(v) \left[ 4 + 2v + (g' - 1)(5 + 3v) - (1 + 2v)v \partial_v \right] \psi_{g'}(v)
\]

\[
+ \frac{1}{12} \left[ 8 + 4v + (g - 2)(5 + 3v) - (1 + 2v)v \partial_v \right] \times
\]

\[
[6 + 3v + (g - 2)(5 + 3v) - (1 + 2v)v \partial_v] \left[ 4 + 2v + (g - 2)(5 + 3v) - (1 + 2v)v \partial_v \right] \psi_{g-1}(v),
\]

which yields

\[
24 \psi_1(v) = 2 + v
\]

\[
1152 \psi_2(v) = 196 + 352v + 109v^2
\]

\[
82944 \psi_3(v) = 117600 + 362564v + 324660v^2 + 84699v^3 + 3043v^4
\]

\[
3981312 \psi_4(v) = 1906157232 + 7865959024v + 11212604992v^2 + 6581090736v^3
\]

\[
+ 1465796801v^4 + 83580341v^5.
\]

40
From this one extracts the first intersection numbers of the form \( \langle \tau_2^l \tau_3^l \rangle \). One recovers for \( g \leq 3 \) results obtained in (5.29) or in Table I, and in genus \( g = 4 \) the results

\[
\begin{align*}
\langle \tau_2^9 \rangle &= \frac{1816871}{48} \\
\langle \tau_2^7 \tau_3 \rangle &= \frac{3326267}{1728} \\
\langle \tau_2^5 \tau_3^2 \rangle &= \frac{728465}{6912} \\
\langle \tau_2^3 \tau_3^3 \rangle &= \frac{43201}{6912} \\
\langle \tau_2^7 \tau_4 \rangle &= \frac{134233}{331776} .
\end{align*}
\]

Table II

We hope we have amply demonstrated the practical use of these expansions.

7. Generalization to higher degree potentials

The cubic potential in the integral (5.1) may be generalized to a potential of degree \( p + 1 \), as noticed by several authors [7], [2], [3]. The case \( p = 2 \) is the one discussed previously. Let us consider first the one-variable integral analogous to (2.12), also denoted \( z(\lambda) \). The normalizations are adjusted in such a way as to make the quadratic term positive definite, in order to have a well-defined asymptotic expansion

\[
z(\lambda) = \frac{\int \frac{e^{p^2 + 1}}{X_{(p+1)}} \left[ (m+(-i)^{p+1})^{p+1} \right]_{\text{lin}} \, dm}{\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4}m^2 \lambda^{p+1}}} \, dm} ,
\]

where the subscript "\( \text{lin} \)" denotes the sum of terms of degree \( \geq 2 \) in the polynomial. By considerations similar to those of sect. 2.1, it is easy to see that \( z(\lambda) \) admits an asymptotic expansion in inverse powers of \( \lambda^{p+1} \)

\[
z(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^{-(p+1)k} \]

with \( c_0 = 1 \). It satisfies a differential equation of order \( p + 1 \)

\[
(D^p - \lambda^p)z(\lambda) = 0
\]
\[ D = \lambda^{p-1} e^{-\frac{p}{2} \frac{p}{(p+1)} (-\lambda)^{p+1}} \left( 2(-1)^{p+1} \frac{\partial}{\partial \lambda^p} \right) \lambda^{p-1} e^{\frac{p}{2} \frac{p}{(p+1)} (-\lambda)^{p+1}} \]

\[ = \lambda + \frac{(-1)^p}{p} \left( \frac{p-1}{\lambda^p} - \frac{2}{\lambda^{p-1}} \frac{\partial}{\partial \lambda} \right) . \quad (7.4) \]

The corresponding matrix integral

\[ Z^{(N)}(\Lambda) = \int dMe^{\frac{p^2}{2} \frac{p}{(p+1)}} \text{tr} \left[ (M + (-i)^{p+1} \Lambda)^{p+1} \right]_{\text{lin}} \]

\[ \int dMe^{-\frac{4}{2} \text{tr} \sum_{k=0}^{p-1} \Lambda^k M \Lambda^{p-1-k}} \] \quad (7.5)

may then be handled as in equations (2.17)–(2.24). One considers the set of functions \( z^{(j)} \) defined by

\[ z^{(j)}(\lambda) = \lambda^{-j} D^j z(\lambda), \quad j \geq 0 . \quad (7.6) \]

From eq. (7.3), it follows that

\[ D^{r(p-1)+j} z = \lambda^{r(p-1)+j} z^{(j)} \mod (D^0 z, \ldots, D^{r(p-1)+j-1} z) . \]

Thus the analogue of (2.24) reads

\[ Z^{(N)}(\Lambda) = \left| \begin{array}{c} \lambda^0 z^{(0)}, \ldots, \lambda^{p-1} z^{(p-1)}, \lambda^p z^{(0)}, \ldots \end{array} \right| \frac{\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^{N-1}}{\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^{N-1}} . \quad (7.7) \]

The \( p \) functions \( z = z^{(0)}, \ldots, z^{(p-1)} \) have asymptotic expansions

\[ z^{(j)}(\lambda) = \sum_{k=0}^{\infty} \epsilon_k^{(j)} \lambda^{-k(p+1)} \], \quad (7.8)

with coefficients \( \epsilon_k^{(j)} \); in the sequel the latter are regarded as periodic in \( j \) of period \( p \: \epsilon_k^{(j+p)} = \epsilon_k^{(j)} \). One then proceeds as in sec. 2.2, introducing Schur functions, with the result (analogous to (2.41)) that

\[ Z_k^{(N)}(\Lambda) = \sum_{n_0 + \cdots + n_{N-1} = k} c_{n_0}^{(0)} c_{n_1}^{(1)} \cdots c_{n_{N-1}}^{(N-1)} \left| \begin{array}{c} p(p+1)n_0 \cdots \cdots \cdots \left. p(p+1)n_{N-1} \end{array} \right| \] \quad (7.9)

is independent of \( N \) for \( N \geq (p+1)k \).

The steps of sec. 2.2 may then be followed sequentially to prove that \( Z_k \), computed now for \( N = (p+1)k \), is independent of the \( \theta_{rp} = \text{tr} \Lambda^{-rp} \). When differentiating \( Z_k \) with respect to \( \theta_{rp} \), the only non-trivially non-vanishing terms are those for which the derivative
acts on one of the last $rp$ lines, where $r$ is a multiple of $p + 1$. The discussion of such a case then appeals to the following identity generalizing (2.56)

$$\det [\omega^{ij} z^{(j)}(\omega^i)\lambda]_{0 \leq i,j \leq p-1} = \text{constant}$$

where $\omega$ is a $p$-th root of unity. This is proved by differentiating the determinant, using the relations (7.6) between the functions $z^{(j)}$. This implies a family of identities on the coefficients

$$C_{kp} \equiv \sum_{n_i=kp}^{n_i+1} \epsilon_{\pi} c_{n_0}^{(0)} \ldots c_{n_{p-1}}^{(p-1)} = 0$$

where the summation runs over the configurations of indices $n_i$ that may be written as indicated, with $\pi$ a permutation of the $p$ integers $0, \ldots, p - 1$.

On the other hand, as before, the only contributions to $\frac{\partial Z}{\partial \theta_{rp(p+1)}}$ come from Young tableaux with a square-rule shape (as in eq. (2.60)), and one finds that

$$\frac{\partial Z}{\partial \theta_{rp(p+1)}} = \sum_{l=0}^{rp-1} (-1)^l \sum_{i=0}^{rp-1-l} \left( 2c_{rp-l}^{(i)} \Delta_{i}^{(i+j+1)} + \sum_{j=1}^{p-1} c_{rp-j}^{(i+j)} \Delta_{i}^{(i+j+1)} \right),$$

where the $\Delta$'s are determinants generalizing those of (2.63):

$$\Delta_{i}^{(j)} = \begin{vmatrix} c_{1}^{(j)} & c_{2}^{(j+1)} & \ldots & c_{s}^{(j+s-1)} \\ 1 & c_{1}^{(j+1)} & \ldots & c_{s}^{(j+s-1)} \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 1 & c_{1}^{(j+s-1)} \end{vmatrix}.$$  

The expression (7.12) may be recast in the form

$$\frac{\partial Z}{\partial \theta_{rp(p+1)}} = \sum_{s=1}^{r} \sum_{i=0}^{p-1} (-1)^{(s-1)p+i} \left[ \sum_{j=0}^{p-1-i} \left( (r-s+1)(p+1) - i \right) + \sum_{j=p-i}^{p-1} c_{(r-s+1)p-i}^{(j)} \Delta_{i}^{(j+1)} \right].$$

(7.14)

It appears that the combination of $c$ and $\Delta$ in the summand in (7.14), namely $c_{sp-i}^{(j)} \Delta_{i}^{(j+1)}$, is, up to a sign, the coefficient of $c_{sp-i}^{(j)}$ in the constraint $C_{sp}$ of eq. (7.11)

$$c_{sp-i}^{(j)} \Delta_{i}^{(j+1)} = (-1)^{(p-1)(p-2)-i} \sum_{j=0}^{p-1} \epsilon_{\pi} c_{n_0}^{(0)} \ldots c_{n_{p-1}}^{(p-1)}$$

(7.15)

with the sum $\sum'$ subject to the same constraints as in (7.11) and to $n_j = sp - i$. Using this fact and after some reshuffling, one finds that $\frac{\partial Z}{\partial \theta_{rp(p+1)}}$ is proportional to the constraint $C_{rp}$ and thus vanishes,

$$\frac{\partial Z}{\partial \theta_{rp(p+1)}} = (-1)^{(p-1)(p-2)} r(p+1) C_{rp} = 0.$$  

(7.16)
The last part of the argument is carried out as in the end of sec. 2.2, thus completing the proof of the independence of $Z$ with respect to the $\theta_{rp}$.

One then proceeds as in sec. 3, deriving the higher KdV hierarchies associated with a differential operator $Q_p$ of order $p$ depending on $p - 1$ functions, and as in sec. 4, writing the generalized Airy equation satisfied by $Z$. For example, in the case $p = 3$, we have

$$\left\{ -\frac{1}{8}t_j + D^3_j + \sum_{k,k \neq j} \left[ \frac{1}{t_j - t_k} (D_j - D_k) + \frac{1}{t_j - t_k} (D_j^2 - D_k^2) \right] \
+ \sum_{k,l \neq j \neq k} \left( \frac{1}{(t_j - t_k)(t_j - t_l)} D_j + \text{circ.} \right) \right\} Z = 0 \quad (7.17)$$

with the following notations

$$t_j = \lambda_j^3$$
$$\partial_j = \frac{\partial}{\partial t_j}$$
$$D_j = e^{-\frac{2}{9} \sum_k \lambda_k^4} \prod_{k,l} (\lambda_k^2 + \lambda_k \lambda_l + \lambda_l^2)^{\frac{1}{2}} \partial_j e^{\frac{2}{9} \sum_k \lambda_k^4} \prod_{k,l} (\lambda_k^2 + \lambda_k \lambda_l + \lambda_l^2)^{-\frac{1}{2}}$$
$$= \partial_j + a_j \quad (7.18)$$
$$a_j = \frac{1}{2} \lambda_j - \frac{1}{3 \lambda_j^2} \sum_k \frac{2 \lambda_j + \lambda_k}{\lambda_j^2 + \lambda_k \lambda_j + \lambda_k^2}$$

From this system of equations, the strong reader will be able to extract the expression of the generators of the $W_3$ algebra, in a way similar to (4.10), and to calculate the analogues of the genus expansion and of the singular behavior discussed in sec. 5 and 6 ...

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