Introduction to quantum Fisher information

Dénes Petz
Alfréd Rényi Institute of Mathematics, H-1051 Budapest, Réaltanoda utca 13-15, Hungary

Catalin Ghinea
Department of Mathematics and its Applications Central European University, 1051 Budapest, Nádor utca 9, Hungary

Abstract

The subject of this paper is a mathematical transition from the Fisher information of classical statistics to the matrix formalism of quantum theory. If the monotonicity is the main requirement, then there are several quantum versions parametrized by a function. In physical applications the minimal is the most popular. There is a one-to-one correspondence between Fisher informations (called also monotone metrics) and abstract covariances. The skew information and the $\chi^2$-divergence are treated here as particular cases.

Keywords: Quantum state estimation, Fisher information, Cramér-Rao inequality, monotonicity, covariance, operator monotone function, skew information.

Introduction

Parameter estimation of probability distributions is one of the most basic tasks in information theory, and has been generalized to quantum regime [20, 22] since the description of quantum measurement is essentially probabilistic. First let us have a look at the classical Fisher information.

Let $(X, \mathcal{B}, \mu)$ be a probability space. If $\theta = (\theta^1, \ldots, \theta^n)$ is a parameter vector in a neighborhood of $\theta_0 \in \mathbb{R}^n$, then we should have a smooth family $\mu_\theta$ of probability measures with probability density $f_\theta$:

$$\mu_\theta(H) = \int_H f_\theta(x) \, d\mu(x) \quad (H \in \mathcal{B}).$$

---

1E-mail: petz@math.bme.hu.
2E-mail: ghinea_catalin@ceu-budapest.edu
The Fisher information matrix at \( \theta_0 \) is

\[
J(\mu_\theta; \theta_0)_{ij} := \int_X f_{\theta_0}(x) \frac{\partial}{\partial \theta^i} \log f_\theta(x) \bigg|_{\theta = \theta_0} \frac{\partial}{\partial \theta^j} \log f_\theta(x) \bigg|_{\theta = \theta_0} \, d\mu(x)
\]

(1)

Note that \( \log f_\theta(x) \) is usually called the log likelihood and its derivative is the score function.

The Fisher information matrix is positive semidefinite. For example, if the parameter \( \theta = (\theta_1, \theta_2) \) is two dimensional, then the Fisher information is a \( 2 \times 2 \) matrix. From the Schwarz inequality

\[
J(\mu_\theta; \theta_0)^2 \leq \int_X \left[ \frac{1}{\sqrt{f_{\theta_0}(x)}} \frac{\partial f_{\theta_0}(x)}{\partial \theta^1} \right]^2 d\mu(x) \int_X \left[ \frac{1}{\sqrt{f_{\theta_0}(x)}} \frac{\partial f_{\theta_0}(x)}{\partial \theta^2} \right]^2 d\mu(x)
\]

Therefore the matrix \( J(\mu_\theta; \theta_0) \) is positive semidefinite.

Assume for the sake of simplicity, that \( \theta \) is a single parameter. The random variable \( \hat{\theta} \) is an unbiased estimator for the parameter \( \theta \) if

\[
\mathbb{E}_\theta(\hat{\theta}) := \int \hat{\theta}(x) f_\theta(x) \, d\mu(x) = \theta
\]

for all \( \theta \). This means that the expectation value of the estimator is the parameter. The Cramér-Rao inequality

\[
\text{Var}(\hat{\theta}) := \mathbb{E}_\theta((\hat{\theta} - \theta)^2) \geq \frac{1}{J(\mu_\theta; \theta)}
\]

gives a lower bound for the variance of an unbiased estimator. (For more parameters we have an inequality between positive matrices.)

In the quantum formalism a probability measure is replaced by a positive matrix of trace 1. (Its eigenvalues form a probability measure, but to determine the so-called density matrix a basis of the eigenvectors is also deterministic.) If a parametrized family of density matrices \( D_\theta \) is given, then there is a possibility for the quantum Fisher information. This quantity is not unique, the possibilities are determined by linear mappings. The analysis of the linear mappings is the main issue of the paper. In physics \( \theta \in \mathbb{R} \) mostly, but if it is an \( n \)-tuple, then Riemannian geometries appear. A coarse-graining gives a monotonicity of the Fisher informations and this is the second main subject of the present overview.

Fisher information has a big literature both in the classical and in the quantum case. The reference of the papers is not at all complete here. The aim is to have an introduction.
1 A general quantum setting

The Cramér-Rao inequality belongs to the basics of estimation theory in mathematical statistics. Its quantum analog appeared in the 1970's, see the book [20] of Helstrom and the book [22] of Holevo. Although both the classical Cramér-Rao inequality and its quantum analog are mathematically as trivial as the Schwarz inequality, the subject takes a lot of attention because it is located on the boundary of statistics, information and quantum theory. As a starting point we give a very general form of the quantum Cramér-Rao inequality in the simple setting of finite dimensional quantum mechanics. The paper [43] is followed here.

For $\theta \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ a statistical operator $\rho(\theta)$ is given and the aim is to estimate the value of the parameter $\theta$ close to 0. Formally $\rho(\theta)$ is an $n \times n$ positive semidefinite matrix of trace 1 which describes a mixed state of a quantum mechanical system and we assume that $\rho(\theta)$ is smooth (in $\theta$). Assume that an estimation is performed by the measurement of a self-adjoint matrix $A$ playing the role of an observable. $A$ is called locally unbiased estimator if

$$\frac{\partial}{\partial \theta} \text{Tr} \rho(\theta) A \bigg|_{\theta=0} = 1.$$  \hfill (2)

This condition holds if $A$ is an unbiased estimator for $\theta$, that is

$$\text{Tr} \rho(\theta) A = \theta \quad (\theta \in (-\varepsilon, \varepsilon)).$$  \hfill (3)

To require this equality for all values of the parameter is a serious restriction on the observable $A$ and we prefer to use the weaker condition (2).

Let $[K, L]_\rho$ be an inner product (or quadratic cost function) on the linear space of self-adjoint matrices. This inner product depends on a density matrix and its meaning is not described now. When $\rho(\theta)$ is smooth in $\theta$, as already was assumed above, then

$$\frac{\partial}{\partial \theta} \text{Tr} \rho(\theta) B \bigg|_{\theta=0} = [B, L]_{\rho(0)}$$  \hfill (4)

with some $L = L^*$. From (2) and (4), we have $[A, L]_{\rho(0)} = 1$ and the Schwarz inequality yields

$$[A, A]_{\rho(0)} \geq \frac{1}{[L, L]_{\rho(0)}}.$$  \hfill (5)

This is the celebrated inequality of Cramér-Rao type for the locally unbiased estimator.

The right-hand-side of (5) is independent of the estimator and provides a lower bound for the quadratic cost. The denominator $[L, L]_{\rho(0)}$ appears to be in the role of Fisher information here. We call it quantum Fisher information with respect to the cost function $[\cdot, \cdot]_{\rho(0)}$. This quantity depends on the tangent of the curve $\rho(\theta)$. If the densities $\rho(\theta)$ and the estimator $A$ commute, then

$$L = \rho_0^{-1} \frac{d\rho(\theta)}{d\theta} \quad \text{and} \quad [L, L]_{\rho(0)} = \text{Tr} \rho_0^{-1} \left( \frac{d\rho(\theta)}{d\theta} \right)^2 = \text{Tr} \rho_0 \left( \rho_0^{-1} \frac{d\rho(\theta)}{d\theta} \right)^2.$$
Now we can see some similarity with (1).

The quantum Fisher information was defined as \([L, L]_{\rho(0)}\), where

\[
\frac{\partial}{\partial \theta} \rho(\theta) \bigg|_{\theta=0} = L.
\]

This \(L\) is unique, but the quantum Fisher information depends on the inner product \([\cdot, \cdot]_{\rho(0)}\). This is not unique, there are several possibilities to choose a reasonable inner product \([\cdot, \cdot]_{\rho(0)}\). Note that \([A, A]_{\rho(0)}\) should have the interpretation of “variance” (if \(\text{Tr} \rho_0 A = 0\)).

Another approach is due to Braunstein and Caves [4] in physics, but Nagaoka considered a similar approach [34].

1.1 From classical Fisher information via measurement

The observable \(A\) has a spectral decomposition

\[
A = \sum_{i=1}^{k} \lambda_i E_i.
\]

(Actually the property \(E_i^2 = E_i\) is not so important, only \(E_i \geq 0\) and \(\sum_i E_i = I\). Hence \(\{E_i\}\) can be a so-called POVM as well.) On the set \(X = \{1, 2, \ldots, k\}\) we have probability distributions

\[
\mu_\theta(\{i\}) = \text{Tr} \rho(\theta) E_i.
\]

Indeed,

\[
\sum_{i=1}^{k} \mu_\theta(\{i\}) = \text{Tr} \rho(\theta) \sum_{i=1}^{k} E_i = \text{Tr} \rho(\theta) = 1.
\]

Since

\[
\mu_\theta(\{i\}) = \frac{\text{Tr} \rho(\theta) E_i}{\text{Tr} DE_i} \text{Tr} DE_i
\]

we can take

\[
\mu(\{i\}) = \text{Tr} DE_i
\]

where \(D\) is a statistical operator. Then

\[
f_\theta(\{i\}) = \frac{\text{Tr} \rho(\theta) E_i}{\text{Tr} DE_i}
\]

and we have the classical Fisher information defined in (1):

\[
\sum_i \frac{\text{Tr} \rho(\theta) E_i}{\text{Tr} DE_i} \left[ \frac{\text{Tr} \rho(\theta) E_i}{\text{Tr} DE_i} : \frac{\text{Tr} \rho(\theta) E_i}{\text{Tr} DE_i} \right]^2 \text{Tr} DE_i = \sum_i \left[ \frac{\text{Tr} \rho(\theta) E_i}{\text{Tr} \rho(\theta)} \right]^2
\]
(This does not depend on $D$.) In the paper [4] the notation

$$F(\rho; \{E(\xi)\}) = \int \frac{[\Tr \rho(\theta) E(\xi)]^2}{\Tr \rho(\theta) E(\xi)} d\xi$$

is used, this is an integral form, and for Braunstein and Caves the quantum Fisher information is the supremum of these classical Fisher informations [4].

**Theorem 1.1** Assume that $D$ is a positive definite density matrix, $B = B^*$ and $\Tr B = 0$. If $\rho(\theta) = D + \theta B + o(\theta^2)$, then the supremum of

$$F(\rho; \{E_i\}) = \sum_i \frac{[\Tr B E_i]^2}{\Tr D E_i}$$

over the measurements $A = \sum_{i=1}^k \lambda_i E_i$ is

$$\Tr B J_D^{-1} (B), \quad \text{where} \quad \mathbb{J}_D C = (DC + CD)/2.$$  

**Proof:** The linear mapping $\mathbb{J}_D$ is invertible, so we can replace $B$ in (7) by $\mathbb{J}_D (C)$. We have to show

$$\sum_i \frac{[\Tr J_D(C) E_i]^2}{\Tr D E_i} = \frac{1}{4} \sum_i (\Tr CDE_i)^2 + (\Tr DCE_i)^2 + 2(\Tr CDE_i)(\Tr DCE_i) \leq \Tr DC^2.$$ 

This follows from

$$(\Tr CDE_i)^2 = \left( \Tr (E_i^{1/2} CD^{1/2})(D^{1/2} E_i^{1/2}) \right)^2 \leq \Tr E_iCDC \Tr D^{1/2} E_i D^{1/2} = \Tr E_iCDC \Tr D E_i.$$ 

and

$$(\Tr DCE_i)^2 = \left( \Tr (E_i^{1/2} D^{1/2})(D^{1/2} CE_i^{1/2}) \right)^2 \leq \Tr D^{1/2} E_i D^{1/2} \Tr D^{1/2} CE_i CD^{1/2} = \Tr E_iCDC \Tr D E_i.$$ 

So $F(\rho(0); \{E_i\}) \leq \Tr DC^2$ holds for any measurement $\{E_i\}$.

Next we want to analyze the condition for equality. Let $\mathbb{J}_D^{-1} B = C = \sum_k \lambda_k P_k$ be the spectral decomposition. In the Scwarz inequalities the condition of equality is

$$D^{1/2} E_i^{1/2} = c_i D^{1/2} CE_i^{1/2}$$

which is

$$E_i^{1/2} = c_i CE_i^{1/2}.$$
So $E_i^{1/2} \leq P_{j(i)}$ for a spectral projection $P_{j(i)}$. This implies that all projections $P_i$ are the sums of certain $E_i$’s. (The simplest measurement for equality corresponds to the observable $C$.) □

Note that $J_D^{-1}$ is in Example 1. It is an exercise to show that for

\[ D = \begin{bmatrix} r & 0 \\ 0 & 1-r \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \]

the optimal observable is

\[ C = \begin{bmatrix} a/r & 2b \\ 2b & -a/1-r \end{bmatrix}. \]

The quantum Fisher information is a particular case of the general approach of the previous session, $J_D$ is in Example 1 below, this is the minimal quantum Fisher information which is also called **SLD Fisher information**. The inequality between (7) and (8) is a particular case of the monotonicity, see [40, 42] and Theorem 1.2 below.

If $D = \text{Diag}(\lambda_1, \ldots, \lambda_n)$, then

\[ F_{\text{min}}(D; B) := \text{Tr} B J_D^{-1}(B) = \sum_{ij} \frac{2}{\lambda_i + \lambda_j} |B_{ij}|^2. \]

In particularly,

\[ F_{\text{min}}(D; i[D, X]) = \sum_{ij} \frac{2(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |X_{ij}|^2 \]

and for commuting $D$ and $B$ we have

\[ F_{\text{min}}(D; B) = \text{Tr} D^{-1}B^2. \]

The minimal quantum Fisher information corresponds to the inner product

\[ [A, B]_{\rho} = \frac{1}{2} \text{Tr} \rho(AB + BA) = \text{Tr} A J_{\rho}(B). \]

Assume now that $\theta = (\theta^1, \theta^2)$. The formula is still true. If

\[ \partial_i \rho(\theta) = B_i, \]

then the classical Fisher information matrix $F(\rho(0); \{E_k\})_{ij}$ has the entries

\[ F(\rho(0); \{E_k\})_{ij} = \sum_k \frac{\text{Tr} B_i E_k \text{Tr} B_j E_k}{\text{Tr} \rho(0) E_k} \quad (9) \]

and the quantum Fisher information matrix is

\[ \begin{bmatrix} \text{Tr} B_1 J_D^{-1}(B_1) & \text{Tr} B_1 J_D^{-1}(B_2) \\ \text{Tr} B_2 J_D^{-1}(B_1) & \text{Tr} B_2 J_D^{-1}(B_2) \end{bmatrix}. \quad (10) \]
Is there any inequality between the two matrices?

Let \( \beta(A) = \sum_k E_k A E_k \). This is a completely positive trace preserving mapping. In the terminology of Theorem 2.3 the matrix (10) is \( J_1 \) and

\[
J_2 = F(\rho(0); \{E_k\}).
\]

The theorem states the inequality \( J_2 \leq J_1 \).

### 1.2 The linear mapping \( J_D \)

Let \( D \in \mathbf{M}_n \) be a positive invertible matrix. The linear mapping \( J^f_D : \mathbf{M}_n \to \mathbf{M}_n \) is defined by the formula

\[
J^f_D = f(L_D R_D^{-1})R_D,
\]

where \( f : \mathbb{R}^+ \to \mathbb{R}^+ \),

\[
L_D(X) = DX \quad \text{and} \quad R_D(X) = XD.
\]

(The operator \( L_D R_D^{-1} \) appeared in the modular theory of von Neumann algebras.)

**Lemma 1.1** Assume that \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous and \( D = \text{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \).

Then

\[
(J^f_D B)_{ij} = \lambda_j f \left( \frac{\lambda_i}{\lambda_j} \right) B_{ij}
\]

Moreover, if \( f_1 \leq f_2 \), then \( 0 \leq J^{f_1}_D \leq J^{f_2}_D \).

**Proof:** Let \( f(x) = x^k \). Then

\[
J^f_D B = D^k B D^{1-k}
\]

and

\[
(J^f_D B)_{ij} = \lambda_i^k \lambda_j^{1-k} B_{ij} = \lambda_j f \left( \frac{\lambda_i}{\lambda_j} \right) B_{ij}.
\]

This is true for polynomials and for any continuous \( f \) by approximation. \( \square \)

It follows from the lemma that

\[
\langle A, J^f_D B \rangle = \langle B^*, J^f_D A^* \rangle
\]

if and only if

\[
\lambda_j f \left( \frac{\lambda_i}{\lambda_j} \right) = \lambda_i f \left( \frac{\lambda_j}{\lambda_i} \right),
\]

which means \( xf(x^{-1}) = f(x) \). Condition (11) is equivalent to the property that \( \langle X, J^f_D Y \rangle \in \mathbb{R} \) when \( X \) and \( Y \) are self-adjoint.

The functions \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) used here are the **standard operator monotone functions** defined as
(i) if for positive matrices $A \leq B$, then $f(A) \leq f(B)$,

(ii) $xf(x^{-1}) = f(x)$ and $f(1) = 1$.

These functions are between the arithmetic and harmonic means [27, 44]:

$$\frac{2x}{x + 1} \leq f(x) \leq \frac{1 + x}{2}.$$

Given $f$,

$$m_f(x, y) = yf\left(\frac{x}{y}\right)$$

is the corresponding mean and we have

$$(J_f^D B)_{ij} = m_f(\lambda_i, \lambda_j) B_{ij}. \quad (12)$$

Hence

$$J_f^D B = X \circ B$$

is a Hadamard product with $X_{ij} = m_f(\lambda_i, \lambda_j)$. Therefore the linear mapping $J_f^D$ is positivity preserving if and only if the above $X$ is positive.

The inverse of $J_f^D$ is the mapping

$$\frac{1}{f(L_D R_D^{-1})} R_D^{-1}$$

which acts as $B \mapsto Y \circ B$ with $Y_{ij} = 1/m_f(\lambda_i, \lambda_j)$. So $(J_f^D)^{-1}$ is positivity preserving if and only if $Y$ is positive.

A necessary condition for the positivity of $J_f^D$ is $f(x) \leq \sqrt{x}$, while the necessary condition for the positivity of $(J_f^D)^{-1}$ is $f(x) \geq \sqrt{x}$. So only $f(x) = \sqrt{x}$ is the function which can make both mappings positivity preserving.

**Example 1** If $f(x) = (x + 1)/2$ (arithmetic mean), then

$$J_f^D B = \frac{1}{2} (DB + BD) \quad \text{and} \quad J_f^{-1} B = \int_0^\infty \exp(-tD/2)B \exp(-tD/2) \, dt.$$ 

This is from the solution of the equation $DB + BD = 2B$. \[Q.E.D.\]

**Example 2** If $f(x) = 2x/(x + 1)$ (harmonic mean), then

$$J_f^D B = \int_0^\infty \exp(-tD^{-1}/2)B \exp(-tD^{-1}/2) \, dt$$

and

$$J_f^{-1} B = \frac{1}{2} (D^{-1}B + BD^{-1}).$$

This function $f$ is the minimal and it generates the maximal Fisher information which is also called right information matrix. \[Q.E.D.\]
Example 3  For the logarithmic mean

\[ f(x) = \frac{x - 1}{\log x} \]  

we have

\[ J_D(B) = \int_0^1 D^t B D^{1-t} \, dt \quad \text{and} \quad J_D^{-1}(B) = \int_0^\infty (D + t)^{-1} B (D + t)^{-1} \, dt \]

This function induces an important Fisher information. □

Example 4  For the geometric mean \( f(x) = \sqrt{x} \) and

\[ J_D(B) = D^{1/2} B D^{1/2} \quad \text{and} \quad J_D^{-1}(B) = D^{-1/2} B D^{-1/2}. \]

\[ J_D^{f} \] is the largest if \( f \) is the largest which is described in Example 1 and the smallest is in Example 2.

Theorem 1.2  Let \( \beta : M_n \to M_m \) be a completely positive trace preserving mapping and \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a matrix monotone function. Then

\[ \beta^*(J_{\beta(D)}^f)^{-1} \beta \leq (J_D^f)^{-1} \]  

and

\[ \beta J_D^f \beta^* \leq J_{\beta(D)}^f. \]  

Actually (14) and (15) are equivalent and they are equivalent to the matrix monotonicity of \( f \).

In the rest \( f \) is always assumed to be a standard matrix monotone function. Then \( \text{Tr} J_D B = \text{Tr} D B \).

Example 5  Here we want to study \( J_D^f \), when \( D \) can have 0 eigenvalues. Formula (12) makes sense. For example, if \( D = \text{Diag} (0, \lambda, \lambda, \mu) \) \((\lambda, \mu > 0, \lambda \neq \mu)\), then

\[
J_D^f B = \begin{bmatrix}
0 & m(0, \lambda)B_{12} & m(0, \mu)B_{13} & m(0, \mu)B_{14} \\
m(0, \lambda)B_{21} & \lambda B_{22} & m(\lambda, \mu)B_{23} & m(\lambda, \mu)B_{24} \\
m(0, \mu)B_{31} & m(\lambda, \mu)B_{32} & \mu B_{34} & \mu B_{34} \\
m(0, \mu)B_{41} & m(\lambda, \mu)B_{42} & \mu B_{43} & \mu B_{43}
\end{bmatrix}.
\]

If \( f(0) > 0 \), then this matrix has only one 0 entry. If \( f(0) = 0 \), then

\[
J_D^f B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \lambda B_{22} & m(\lambda, \mu)B_{23} & m(\lambda, \mu)B_{24} \\
0 & m(\lambda, \mu)B_{32} & \mu B_{34} & \mu B_{34} \\
0 & m(\lambda, \mu)B_{42} & \mu B_{43} & \mu B_{43}
\end{bmatrix}.
\]
and the kernel of $J_D$ is larger. We have
\[ \langle B, J^f_D B \rangle = \sum_{ij} m_f(\lambda_i, \lambda_j)|B_{ij}|^2 \]
and some terms can be 0 if $D$ is not invertible.

The inverse of $J^f_D$ exists in the generalized sense
\[ [(J^f_D)^{-1}B]_{ij} = \begin{cases} \frac{1}{m_f(\lambda_i, \lambda_j)}B_{ij} & \text{if } m_f(\lambda_i, \lambda_j) \neq 0, \\ 0 & \text{if } m_f(\lambda_i, \lambda_j) = 0. \end{cases} \]
(This is the Moore-Penrose generalized inverse.)

It would be interesting to compare the functions which non-zero at 0 with the others.

## 2 Fisher information and covariance

Assume that $f$ is a standard matrix monotone function. The operators $J^f_D$ are used to define Fisher information and the covariance. (The latter can be called also quadratic cost.) The operator $J^f_D$ depends on the function $f$, but $f$ will be not written sometimes.

Let $A = A^* \in M_n$ be observables and $D \in M_n$ be a density matrix. The covariance of $A$ and $B$ is
\[ \text{Cov}_D^f(A, B) := \langle A, J^f_D(B) \rangle - (\text{Tr} DA)(\text{Tr} DB). \] (16)

Since
\[ \text{Cov}_D^f(A, A) = \langle (A - I\text{Tr} DA), J^f_D(A - I\text{Tr} DA) \rangle \]
and $J_D \geq 0$, we have for the variance $\text{Var}_D^f(A) := \text{Cov}_D^f(A, A) \geq 0$.

The monotonicity (15) gives
\[ \text{Var}_D^f(\beta^* A) \leq \text{Var}_D^f(A). \]
for a completely positive trace preserving mapping $\beta$.

The usual **symmetrized covariance** corresponds to the function $f(t) = (t + 1)/2$:
\[ \text{Cov}_D(A, B) := \frac{1}{2} \text{Tr} \left( D(A^* B + B A^*) \right) - (\text{Tr} DA^*)(\text{Tr} DB). \]

Let $A_1, A_2, \ldots, A_k$ be self-adjoint matrices and let $D$ be a statistical operator. The covariance is a $k \times k$ matrix $C(D)$ defined as
\[ C(D)_{ij} = \text{Cov}_D^f(A_i, A_j). \] (17)
$C(D)$ is a positive semidefinite matrix and positive definite if the observables $A_1, A_2, \ldots, A_k$ are linearly independent. It should be remarked that this matrix is only a formal analogue of the classical covariance matrix and it is not related to a single quantum measurement [33].

The variance is defined by $J_D$ and the Fisher information is formulated by the inverse of this mapping:

$$
\gamma_D(A, B) = \text{Tr} A J_D^{-1}(B^*).
$$

(18)

Here $A$ and $B$ are self-adjoint. If $A$ and $B$ are considered as tangent vectors at the footpoint $D$, then $\text{Tr} A = \text{Tr} B = 0$. In this approach $\gamma_D(A, B)$ is a an inner product in a Riemannian geometry [22]. It seems that this approach is not popular in quantum theory. It happens also that the condition $\text{Tr} D = 1$ is neglected and only $D > 0$. Then formula (18) can be extended [26].

If $DA = AD$ for a self-adjoint matrix $A$, then

$$
\gamma_D(A, A) = \text{Tr} D^{-1} A^2
$$

does not depend on the function $f$. (The dependence is characteristic on the orthogonal complement, this will come later.)

**Theorem 2.1** Assume that $(A, B) \mapsto \gamma_D(A, B)$ is an inner product for $A, B \in M_n$, for positive definite density matrix $D \in M_n$ and for every $n$. Suppose the following properties:

(i) For commuting $D$ and $A = A^*$ we have $\gamma_D(A, A) = \text{Tr} D^{-1} A^2$.

(ii) If $\beta : M_n \to M_m$ is a completely positive trace preserving mapping, then

$$
\gamma_{\beta(D)}(\beta(A), \beta(A)) \leq \gamma_D(A, A).
$$

(19)

(iii) If $A = A^*$ and $B = B^*$, then $\gamma_D(A, B)$ is a real number.

(iv) $D \mapsto \gamma_D(A, B)$ is continuous.

Then

$$
\gamma_D(A, B) = \langle A, (J_D^f)^{-1} B \rangle
$$

(20)

for a standard matrix monotone function $f$.

**Example 6** In quantum statistical mechanics, perturbation of a density matrix appears. Suppose that $D = e^H$ and $A = A^*$ is the perturbation

$$
D_t = \frac{e^{H+tA}}{\text{Tr} e^{H+tA}} \quad (t \in \mathbb{R}).
$$
The quantum analog of formula (1) would be

$$-\text{Tr} D_0 \frac{\partial^2}{\partial t^2} \log D_t \bigg|_{t=0}.$$ 

A simple computation gives

$$\int_0^1 \text{Tr} e^{sH} A e^{(1-s)H} A ds - (\text{Tr} DA)^2$$

This is a kind of variance. □

Let \( \mathcal{M} := \{D_\theta : \theta \in G\} \) be a smooth \( m \)-dimensional manifold of \( n \times n \) density matrices. Formally \( G \subset \mathbb{R}^m \) is an open set including 0. If \( \theta \in G \), then \( \theta = (\theta_1, \theta_2, \ldots, \theta_m) \).

The Riemannian structure on \( \mathcal{M} \) is given by the inner product \([18]\) of the tangent vectors \( A \) and \( B \) at the foot point \( D \in \mathcal{M} \), where \( \mathbb{J}_D : \mathcal{M}_n \to \mathcal{M}_n \) is a positive mapping when \( \mathcal{M}_n \) is regarded as a Hilbert space with the Hilbert-Schmidt inner product. (This means \( \text{Tr} A \mathbb{J}_D(A)^* \geq 0 \).)

Assume that a collection \( A = (A_1, \ldots, A_m) \) of self-adjoint matrices is used to estimate the true value of \( \theta \). The expectation value of \( A_i \) with respect to the density matrix \( D \) is \( \text{Tr} DA_i \). \( A \) is an unbiased estimator if

$$\text{Tr} D_\theta A_i = \theta_i \quad (1 \leq i \leq n).$$

(21)

(In many cases unbiased estimator \( A = (A_1, \ldots, A_m) \) does not exist, therefore a weaker condition is more useful.)

The Fisher information matrix of the estimator \( A \) is a positive definite matrix

$$J(D)_{ij} = \text{Tr} L_i \mathbb{J}_D(L_j), \quad \text{where} \quad L_i = \mathbb{J}^{-1}_D(\partial_i D_\theta).$$

Both \( C(D) \) and \( J(D) \) depend on the actual state \( D \).

The next theorem is the the Cramér-Rao inequality for matrices. The point is that the right-hand-side does not depend on the estimators.

**Theorem 2.2** Let \( A = (A_1, \ldots, A_m) \) be an unbiased estimator of \( \theta \). Then for the above defined matrices the inequality

$$C(D_\theta) \geq J(D_\theta)^{-1}$$

holds.

**Proof:** In the proof the block-matrix method is used and we restrict ourselves for \( m = 2 \) for the sake of simplicity and assume that \( \theta = 0 \). Instead of \( D_0 \) we write \( D \).
The matrices $A_1, A_2, L_1, L_2$ are considered as vectors and from the inner product $(A, B) = \text{Tr} A \mathbb{J}_D(B)^*$ we have the positive matrix

$$X := \begin{bmatrix} \text{Tr} A_1 \mathbb{J}_D(A_1) & \text{Tr} A_1 \mathbb{J}_D(A_2) & \text{Tr} A_1 \mathbb{J}_D(L_1) & \text{Tr} A_1 \mathbb{J}_D(L_2) \\ \text{Tr} A_2 \mathbb{J}_D(A_1) & \text{Tr} A_2 \mathbb{J}_D(A_2) & \text{Tr} A_2 \mathbb{J}_D(L_1) & \text{Tr} A_2 \mathbb{J}_D(L_2) \\ \text{Tr} L_1 \mathbb{J}_D(A_1) & \text{Tr} L_1 \mathbb{J}_D(A_2) & \text{Tr} L_1 \mathbb{J}_D(L_1) & \text{Tr} L_1 \mathbb{J}_D(L_2) \\ \text{Tr} L_2 \mathbb{J}_D(A_1) & \text{Tr} L_2 \mathbb{J}_D(A_2) & \text{Tr} L_2 \mathbb{J}_D(L_1) & \text{Tr} L_2 \mathbb{J}_D(L_2) \end{bmatrix}.$$  

From the condition (21), we have

$$\text{Tr} A_i \mathbb{J}_D(L_i) = \frac{\partial}{\partial \theta_i} \text{Tr} D \theta A_i = 1$$

for $i = 1, 2$ and

$$\text{Tr} A_i \mathbb{J}_D(L_j) = \frac{\partial}{\partial \theta_j} \text{Tr} D \theta A_i = 0$$

if $i \neq j$. Hence the matrix $X$ has the form

$$\begin{bmatrix} C(D) & I_2 \\ I_2 & J(D) \end{bmatrix},$$

where

$$C(D) = \begin{bmatrix} \text{Tr} A_1 \mathbb{J}_D(A_1) & \text{Tr} A_1 \mathbb{J}_D(A_2) \\ \text{Tr} A_2 \mathbb{J}_D(A_1) & \text{Tr} A_2 \mathbb{J}_D(A_2) \end{bmatrix}$$

and

$$J(D) = \begin{bmatrix} \text{Tr} L_1 \mathbb{J}_D(L_1) & \text{Tr} L_1 \mathbb{J}_D(L_2) \\ \text{Tr} L_2 \mathbb{J}_D(L_1) & \text{Tr} L_2 \mathbb{J}_D(L_2) \end{bmatrix}.$$  

The positivity of (22) implies the statement of the theorem. □

We have have the orthogonal decomposition

$$\{ B = B^* : [D, B] = 0 \} \oplus \{ i[D, A] : A = A^* \}$$

(23)

of the self-adjoint matrices and we denote the two subspaces by $\mathcal{M}_D$ and $\mathcal{M}_D'$, respectively.

**Example 7** The Fisher information and the covariance are easily handled if $D$ is diagonal, $D = \text{Diag} (\lambda_1, \ldots, \lambda_n)$ or formulated by the matrix units $E(ij)$

$$D = \sum_i \lambda_i E(ii).$$

The general formulas in case of diagonal $D$ are

$$\gamma_D(A, A) = \sum_{ij} \frac{1}{\lambda_j f(\lambda_i/\lambda_j)} |A_{ij}|^2, \quad \text{Cov}_D(A, A) = \sum_{ij} \lambda_j f(\lambda_i/\lambda_j) |A_{ij}|^2.$$
Moreover,
\[
\gamma^f_D(i[D, X], i[D, X]) = \sum_{ij} \frac{(\lambda_i - \lambda_j)^2}{\lambda_j f(\lambda_i / \lambda_j)} |X_{ij}|^2.
\] (24)

Hence for diagonal \(D\) all Fisher informations have simple explicit formula.

The description of the commutators is more convenient if the eigenvalues are different. Let
\[
S_1(ij) := E(ij) + E(ji), \quad S_2(ij) := -iE(ij) + iE(ji)
\]
for \(i < j\). (They are the generalization of the Pauli matrices \(\sigma_1\) and \(\sigma_2\).) We have
\[
i[D, S_1(ij)] = (\lambda_i - \lambda_j)S_2(ij), \quad i[D, S_2(ij)] = (\lambda_j - \lambda_i)S_1(ij).
\]

In Example 1 we have \(f(x) = (1 + x)/2\). This gives the minimal Fisher information described in Theorem 1.1
\[
\gamma_D(A, B) = \int_0^\infty \text{Tr} \ A \exp(-tD/2)B \exp(-tD/2) \, dt.
\]
The corresponding covariance is the symmetrized \(\text{Cov}_D(A, B)\). This is maximal among the variances.

From Example 2 we have the maximal Fisher information
\[
\gamma_D(A, B) = \frac{1}{2} \text{Tr} \ D^{-1}(AB + BA)
\]
The corresponding covariance is a bit similar to the minimal Fisher information:
\[
\text{Cov}_D(A, B) = \int_0^\infty \text{Tr} \ A \exp(-tD^{-1}/2)B \exp(-tD^{-1}/2) \, dt - \text{Tr} \ DA \text{Tr} \ DB.
\]

Example 3 leads to the Boguliubov-Kubo-Mori inner product as Fisher information [41, 42]:
\[
\gamma_D(A, B) = \int_0^\infty \text{Tr} \ A(D + t)^{-1}B(D + t)^{-1} \, dt
\]
It is also called BKM Fisher information, the characterization is in the paper [14] and it is also proven that this gives a large deviation bound of consistent superefficient estimators [17].

Let \(\mathcal{M} := \{\rho(\theta) : \theta \in G\}\) be a smooth \(k\)-dimensional manifold of invertible density matrices. The quantum score operators (or logarithmic derivatives) are defined as
\[
L^f_i(\theta) := (\mathcal{J}_{\rho(\theta)})^{-1}(\partial_{\theta_i}\rho(\theta)) \quad (1 \leq i \leq m)
\] (25)
and
\[
J(\theta)_{ij} := \text{Tr} \ L^f_i(\theta)\mathcal{J}_{\rho(\theta)}(L_j(\theta)) = \text{Tr} (\mathcal{J}_{\rho(\theta)}^{-1}(\partial_{\theta_i}\rho(\theta)))(\partial_{\theta_j}\rho(\theta)) \quad (1 \leq i, j \leq k)
\] (26)
is the quantum Fisher information matrix (depending on the function $f$). The function $f(x) = (x + 1)/2$ yields the symmetric logarithmic derivative (SLD) Fisher information.

**Theorem 2.3** Let $\beta : M_n \to M_m$ be a completely positive trace preserving mapping and let $\mathcal{M} := \{\rho(\theta) \in M_n : \theta \in G\}$ be a smooth $k$-dimensional manifold of invertible density matrices. For the Fisher information matrix $J_1(\theta)$ of $\mathcal{M}$ and for Fisher information matrix $J_2(\theta)$ of $\beta(\mathcal{M}) := \{\beta(\rho(\theta)) : \theta \in G\}$ we have the monotonicity relation

$$J_2(\theta) \leq J_1(\theta).$$

**Proof:** We set $B_i(\theta) := \partial_\theta \rho(\theta)$. Then $\mathbb{J}^{-1}_{\beta(\rho(\theta))} \beta(B_i(\theta))$ is the score operator of $\beta(\mathcal{M})$ and we have

$$\sum_{ij} J_2(\theta)_{ij} a_i \overline{a}_j = \text{Tr} \mathbb{J}^{-1}_{\beta(\rho(\theta))} \beta \left( \sum_i a_i B_i(\theta) \right) \beta \left( \sum_j \overline{a}_j B_j(\theta) \right)$$

$$= \left\langle \sum_i a_i B_i, (\mathbb{J}^{-1}_{\beta(\rho(\theta))})^{\beta} \sum_j \overline{a}_j B_j(\theta) \right\rangle$$

$$\leq \left\langle \sum_i a_i B_i, \mathbb{J}^{-1}_{\rho(\theta)} \sum_j \overline{a}_j B_j(\theta) \right\rangle$$

$$= \text{Tr} \mathbb{J}^{-1}_{\rho(\theta)} \left( \sum_i a_i B_i(\theta) \right) \left( \sum_j \overline{a}_j B_j(\theta) \right)$$

$$= \sum_{ij} J_1(\theta)^{ij} a_i \overline{a}_j,$$

where (14) was used. □

The monotonicity of the Fisher information matrix in some particular cases appeared already in the literature: [38] treated the case of the Kubo-Mori inner product and [4] considered the symmetric logarithmic derivative and measurement in the role of coarse graining.

**Example 8** The function

$$f_{\beta}(t) = \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)}$$

is operator monotone if $0 < \beta < 2$. Formally $f(1)$ is not defined, but as a limit it is 1. The property $xf(x^{-1}) = f(x)$ also holds. Therefore this function determines a Fisher information [39]. If $\beta = 1/2$, then the variance has a simple formula:

$$\text{Var}_D A = \frac{1}{2} \text{Tr} D^{1/2}(D^{1/2} A + AD^{1/2}) A - (\text{Tr} DA)^2.$$

□

15
Example 9 The functions $x^{-\alpha}$ and $x^{\alpha-1}$ are matrix monotone decreasing and so is their sum. Therefore

$$f_\alpha(x) = \frac{2}{x^{-\alpha} + x^{\alpha-1}}$$

is a standard operator monotone function.

$$\gamma^f_\sigma(\rho, \rho) = 1 + \text{Tr} \left( \rho - \sigma \right) \sigma^{-\alpha}(\rho - \sigma)\sigma^{\alpha-1}$$

may remind us to the abstract Fisher information, however now $\rho$ and $\sigma$ are positive definite density matrices. In the paper [46]

$$\chi^2_\alpha(\rho, \sigma) = \text{Tr} \left( \rho - \sigma \right) \sigma^{-\alpha}(\rho - \sigma)\sigma^{\alpha-1}$$

is called quantum $\chi^2$-divergence. (If $\rho$ and $\sigma$ commute, then the formula is independent of $\alpha$. ) Up to the constant 1, this is an interesting and important particular case of the monotone metric. The general theory [19] implies the monotonicity of the $\chi^2$-divergence.

\[\Box\]

3 Extended monotone metrics

As an extension of the papers [5, 40] Kuamagai made the following generalization [26]. Now $H_n^+$ denotes the strictly positive matrices in $M_n$. Formally $K_\rho(A, B) \in \mathbb{C}$ is defined for all $\rho \in H_n^+$, $A, B \in M_n$ and $n \in \mathbb{N}$ and it is assumed that

(i) $(A, B) \mapsto K_\rho(A, B)$ is an inner product on $M_n$ for every $\rho \in H_n^+$ and $n \in \mathbb{N}$.

(ii) $\rho \mapsto K_\rho(A, B)$ is continuous.

(iii) For a trace-preserving completely positive mapping $\beta$

$$K_{\beta(\rho)}(\beta(A), \beta(A)) \leq K_\rho(A, A)$$

holds.

In the paper [26] such $K_\rho(A, B)$ is called extended monotone metric and the description is

$$K_\rho(A, B) = b(\text{Tr} \rho)\text{Tr} A^*\text{Tr} B + c(A, (J_\rho f)^{-1}(B)),$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is matrix monotone, $f(1) = 1$, $b : \mathbb{R}^+ \to \mathbb{R}^+$ and $c > 0$. Note that

$$(A, B) \mapsto b(\text{Tr} \rho)\text{Tr} A^*\text{Tr} B \quad \text{and} \quad (A, B) \mapsto c(A, (J_\rho f)^{-1}B)$$

satisfy conditions (ii) and (iii) with constant $c > 0$. The essential point is to check

$$b(\text{Tr} \rho)\text{Tr} A^* \text{Tr} A + c(A, (J_\rho f)^{-1}A) \geq 0.$$

16
In the case of $1 \times 1$ matrices this is

$$b(x)|z|^2 + \frac{c}{x}|z|^2 \geq 0$$

which gives the condition $xb(x) + c > 0$. If this is true, then

$$ \left( \sum \lambda_i \right) b \left( \sum \lambda_i \right) \left| \sum A_{ii} \right|^2 + c \left( \sum \lambda_i \right) \sum \frac{1}{m_f(\lambda_i, \lambda_j)} |A_{ij}|^2 \geq -c \left| \sum A_{ii} \right|^2 + c \left( \sum \lambda_i \right) \sum \frac{1}{\lambda_i} |A_{ii}|^2.$$ 

The positivity is the inequality

$$ \left( \sum \lambda_i \right) \sum \frac{1}{\lambda_i} |A_{ii}|^2 \geq \left| \sum A_{ii} \right|^2$$

which is a consequence of the Schwarz inequality.

### 4 Skew information

The Wigner-Yanase-Dyson skew information is the quantity

$$I_p(D, A) := -\frac{1}{2} \text{Tr} [D^p, A][D^{1-p}, A] \quad (0 < p < 1).$$

Actually, the case $p = 1/2$ is due to Wigner and Yanase [47] and the extension was proposed by Dyson. The convexity of $I_p(D, A)$ in $A$ is a famous result of Lieb [30].

It was observed in [39] that the Wigner-Yanase-Dyson skew information is connected to the Fisher information which corresponds to the function (28). For this function we have

$$\gamma_D(i[D, A], i[D, A]) = \frac{1}{2\beta(1-\beta)} \text{Tr} ([\rho^\beta, A][\rho^{1-\beta}, A]).$$

Apart from a constant factor this expression is the skew information proposed by Wigner and Yanase [47]. In the limiting cases $p \to 0$ or $1$ we have the function (13) corresponding to the Kubo-Mori-Bogoliubov case.

Let $f$ be a standard function and $A = A^* \in \mathbb{M}_n$. The quantity

$$I_f^A(A) := \frac{f(0)}{2} \gamma_f^A(i[D, A], i[D, A])$$
was called **skew information** in [16] in this general setting. So the skew information is nothing else but the Fisher information restricted to $\mathcal{M}_D$, but it is parametrized by the commutator. Skew information appeared twenty years before the concept of quantum Fisher information. Skew information appears in a rather big literature, for example, connection with uncertainty relations [3, 10, 9, 13, 25, 31, 32].

If $D = \text{Diag} (\lambda_1, \ldots, \lambda_n)$ is diagonal, then

$$
\gamma^f_D(i[D, A], i[D, A]) = \sum_{ij} \frac{(\lambda_i - \lambda_j)^2}{\lambda_j f'(\lambda_i/\lambda_j)} |A_{ij}|^2.
$$

This implies that the identity

$$
I^f_D(A) = \text{Cov}_D(A, A) - \text{Cov}^f_D(A, A)
$$

holds if $\text{Tr} DA = 0$ and

$$
\tilde{f}(x) := \frac{1}{2} \left( (x + 1) - (x - 1)^2 \frac{f(0)}{f'(x)} \right).
$$

It was proved in [8] that for a standard function $f : \mathbb{R}^+ \to \mathbb{R}$, $\tilde{f}$ is standard as well. Another proof is in [45] which contains the following theorem.

**Theorem 4.1** Assume that $X = X^* \in \mathcal{M}$ and $\text{Tr} DX = 0$. If $f$ is a standard function such that $f(0) \neq 0$, then

$$
\frac{\partial^2}{\partial t \partial s} S_F(D + ti[D, X], D + si[D, X])\bigg|_{t=s=0} = f(0) \gamma^f_D(i[D, X], i[D, X])
$$

for the standard function $F = \tilde{f}$.

All skew informations are obtained from an $f$-divergence (or quasi-entropy) by differentiation.

**Example 10** The function

$$
f(x) = \left( \frac{1 + \sqrt{x}}{2} \right)^2
$$

gives the Wigner-Yanase skew information

$$
I^{WY}(D, A) = I_{1/2}(D, A) = -\frac{1}{2} \text{Tr} [D^{1/2}, A]^2.
$$

The skew information coming from the minimal Fisher information and it is often denoted as $I^{SLD}(D, A)$. The simple mean inequalities

$$
\left( \frac{1 + \sqrt{x}}{2} \right)^2 \leq \frac{1 + x}{2} \leq 2 \left( \frac{1 + \sqrt{x}}{2} \right)^2
$$

imply

$$
I^{WY}(D, A) \leq I^{SLD}(D, A) \leq 2I^{WY}(D, A).
$$
Acknowledgement

This work is supported by the Hungarian Research Grant OTKA 68258.

References

[1] S. Amari, *Differential-geometrical methods in statistics*, Lecture Notes in Stat. **28** (Springer, Berlin, Heidelberg, New York, 1985)

[2] S. Amari and H. Nagaoka, *Methods of information geometry*, Transl. Math. Monographs **191**, AMS, 2000.

[3] A. Andai, Uncertainty principle with quantum Fisher information, J. Math. Phys. **49**(2008), 012106.

[4] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, Phys. Rev. Lett. **72**(1994), 3439–3443.

[5] L.L. Campbell, An extended Centcov characterization of the information metric, Proc. Amer. Math. Soc. **98** (1986), 135–141.

[6] J. Dittmann, On the Riemannian geometry of finite dimensional state space, Seminar Sophus Lie 3(1993), 73–87.

[7] E. Fick and G. Sauermann, *The quantum statistics of dynamic processes* (Springer, Berlin, Heidelberg) 1990.

[8] P. Gibilisco, D. Imparato and T. Isola, Uncertainty principle and quantum Fisher information II, J. Math. Phys. **48**(2007), 072109.

[9] P. Gibilisco, D. Imparato and T. Isola, A volume inequality for quantum Fisher information and the uncertainty principle, J. Statist. **130**(2007), 545–559.

[10] P. Gibilisco and T. Isola, Uncertainty principle and quantum Fisher information, Ann. Inst. Stat. Math, **59** (2007), 147–159.

[11] P. Gibilisco, F. Hiai and D. Petz, Quantum covariance, quantum Fisher information and the uncertainty principle, IEEE Trans. Inform. Theory **55**(2009), 439–443.

[12] P. Gibilisco, D. Imparato and T. Isola, Inequalities for quantum Fisher information, Proc. Amer. Math. Soc. **137**(2009), 317–327.

[13] P. Gibilisco, D. Imparato and T. Isola, A Robertson-type uncertainty principle and quantum Fisher information, Lin. Alg. Appl. **428**(2008), 1706–1724.

[14] M. Grasselli and R.F. Streater, Uniqueness of the Chentsov metric in quantum information theory, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **4** (2001), 173-182.

[15] F. Hansen, Characterizations of symmetric monotone metrics on the state space of quantum systems, Quantum Inf. Comput., **6**(2006), 597–605.
[16] F. Hansen, Metric adjusted skew information, Proc. Natl. Acad. Sci. USA. 105(2008), 9909–9916.

[17] M. Hayashi, Two quantum analogues of Fisher information from a large deviation viewpoint of quantum estimation, J. of Physics A: Mathematical and General, 35(2002), 7689–7727.

[18] M. Hayashi and K. Matsumoto, Asymptotic performance of optimal state estimation in quantum two level system, J. Math. Phys. 49(2008), 102101.

[19] M. Hayashi, Quantum information. An introduction, Springer-Verlag, Berlin, 2006.

[20] C. W. Helstrom, Quantum detection and estimation theory, Academic Press, New York, 1976.

[21] F. Hiai and D. Petz, Riemannian geometry on positive definite matrices related to means, Lin. Alg. Appl. 430(2009), 3105–3130.

[22] A. S. Holevo, Probabilistic and statistical aspects of quantum theory, North-Holland, Amsterdam, 1982.

[23] A. Jencová, Geodesic distances on density matrices, J. Math. Phys. 45 (2004), 1787–1794.

[24] O. Johnson, Information theory and the central limit theorem, Imperial College Press, 2004.

[25] H. Kosaki, Matrix trace inequality related to uncertainty principle, Internat. J. Math. 16(2005), 629–645.

[26] W. Kuamagai, A characterization of extended monotone metrics, to be published in Lin. Alg. Appl.

[27] F. Kubo and T. Ando, Means of positive linear operators, Math. Ann. 246(1980), 205–224.

[28] S. Kullback and R.A. Leibler, On information and sufficiency, Ann. Math. Statistics, 22(1951), 79–86.

[29] S. Kullback, Information theory and statistics, John Wiley and Sons, New York; Chapman and Hall, Ltd., London, 1959.

[30] E. H. Lieb, Convex trace functions and the Wigner-Yanase-Dyson conjecture, Advances in Math. 11(1973), 267–288.

[31] S. Luo and Z. Zhang, An informational characterization of Schrödinger’s uncertainty relations, J. Stat. Phys. 114(2004), 1557–1576.

[32] S. Luo and Q. Zhang, On skew information, IEEE Trans. Inform. Theory, 50(2004), 1778–1782.

[33] S. Luo, Covariance and quantum Fisher information, Theory Probab. Appl. 53(2009), 329–334.

[34] H. Nagaoka, On Fisher information on quantum statistical models, in Asymptotic Theory of Quantum Statistical Inference, 113–124, ed. M. Hayashi, World Scientific, 2005.
[35] M. Ohya and D. Petz, *Quantum entropy and its use*, Springer-Verlag, Heidelberg, 1993. Second edition 2004.

[36] D. Petz, Quasi-entropies for states of a von Neumann algebra, Publ. RIMS. Kyoto Univ. 21(1985), 781–800.

[37] D. Petz, Quasi-entropies for finite quantum systems, Rep. Math. Phys., 23(1986), 57-65.

[38] D. Petz, Geometry of canonical correlation on the state space of a quantum System, J. Math. Phys. 35(1994), 780–795.

[39] D. Petz and H. Hasegawa, On the Riemannian metric of $\alpha$-entropies of density matrices, Lett. Math. Phys. 38(1996), 221–225.

[40] D. Petz, Monotone metrics on matrix spaces, Linear Algebra Appl. 244(1996), 81–96.

[41] D. Petz and Cs. Sudár, Geometries of quantum states, J. Math. Phys. 37(1996), 2662–2673.

[42] D. Petz and Cs. Sudár, Extending the Fisher metric to density matrices, in Geometry of Present Days Science, eds. O.E. Barndorff-Nielsen and E.B. Vendel Jensen, 21–34, World Scientific, 1999.

[43] D. Petz, Covariance and Fisher information in quantum mechanics, J. Phys. A: Math. Gen. 35(2003), 79–91.

[44] D. Petz, *Quantum information theory and quantum statistics*, Springer, Berlin, Heidelberg, 2008.

[45] D. Petz and V.E.S. Szabó, From quasi-entropy to skew information, Int. J. Math. 20(2009), 1421–1430.

[46] K. Temme, M. J. Kastoryano, M. B. Ruskai, M. M. Wolf and F. Verstraete, The $\chi^2$-divergence and mixing times of quantum Markov processes, [arXiv:1005.2358](https://arxiv.org/abs/1005.2358).

[47] E.P. Wigner and M.M. Yanase, Information content of distributions, Proc. Nat. Acad. Sci. USA 49(1963), 910–918.