Two-component Yang-Baxter maps associated to integrable quad equations, hypergeometric integrals, and the star-triangle relations

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Abstract

It is shown how Yang-Baxter maps may be constructed using solutions of the classical star-triangle relations associated to hypergeometric integrals and integrable quad equations. The classical star-triangle relation implies a classical vertex form of the Yang-Baxter equation which may be reinterpreted as a set-theoretical form of the Yang-Baxter equation for Yang-Baxter maps. The resulting Yang-Baxter maps have both two-component variables and two-component parameters and can be expressed as a QRT-like composition of separate maps for each component. Using explicit solutions of the classical star-triangle relations a new family of sixteen Yang-Baxter maps is derived.

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1 Introduction

The Yang-Baxter equation is a central equation for integrability of models of statistical mechanics where it can be used to solve a model through the use of the commuting transfer matrix method of Baxter [10]. There are known to be several different forms of such Yang-Baxter equations [11, 29, 47], which correspond to different types of integrable models of statistical mechanics. One of the forms that is central to the ideas of this paper is known as the star-triangle relation [9, 13], which implies integrability of an associated 2-dimensional lattice “spin” model of statistical mechanics. The latter star-triangle relation first appeared in statistical mechanics for the 2-dimensional Ising model [42], and several other important solutions have since been found [6, 14, 15, 25–27, 31, 34, 35, 56, 64] for different lattice models that generalise the Ising model.

There has recently been established new connections between the above star-triangle relations and integral formulas arising in the theory of hypergeometric functions. Examples of the latter include the well-known Euler beta integral and its various generalisations [28]. More specifically, it has been shown [13, 15, 37] that different univariate hypergeometric beta integrals can be written in one of the following two forms

\[ \int d\sigma_0 \overline{W}_{q-r}(\sigma_1, \sigma_0) W_{p-r}(\sigma_2, \sigma_0) \overline{W}_{p-q}(\sigma_3, \sigma_0) = W_{q-r}(\sigma_2, \sigma_3) \overline{W}_{p-r}(\sigma_1, \sigma_3) W_{p-q}(\sigma_2, \sigma_1), \quad (1) \]

\[ \int d\sigma_0 \overline{V}_{q-r}(\sigma_1, \sigma_0) V_{p-r}(\sigma_2, \sigma_0) \overline{W}_{p-q}(\sigma_3, \sigma_0) = V_{q-r}(\sigma_2, \sigma_3) \overline{V}_{p-r}(\sigma_1, \sigma_3) \overline{W}_{p-q}(\sigma_2, \sigma_1). \quad (2) \]

In these forms, the functions \( W_{p-q}(\sigma_1, \sigma_2), \overline{W}_{p-q}(\sigma_1, \sigma_2), V_{p-q}(\sigma_1, \sigma_2), \overline{V}_{p-q}(\sigma_1, \sigma_2) \) can be interpreted as Boltzmann weights for a lattice model of statistical mechanics with continuous valued spins \( \sigma_i \), and rapidity parameters \( p, q, r \), and the equations (1) and (2) are the star-triangle relations for these models. These equations provide some of the most general known forms of star-triangle relations known in the literature, and include important models such as the Ising and Chiral Potts models as special cases [15].

The hypergeometric integrals in star-triangle relation form (1) and (2) are also closely related to another type of integrable system. Namely, it has been shown that certain quasi-classical limits of (1) and (2) result in partial difference equations (quad equations) that satisfy integrability in the form of multi-dimensional consistency [18, 41]. For all of the integrable quad equations that appear in the Adler-Bobenko-Suris (ABS) classification [1, 3], the counterpart continuous star-triangle relations (1) and (2) have now been established [13, 15, 37]. Through this connection, it is natural to regard the star-triangle relations (1) and (2) as quantum versions of type-Q and type-II ABS equations respectively, where the latter are obtained as the equation of the saddle-point in a quasi-classical expansion of the star-triangle relations.

The purpose of this paper is to show how yet another class of integrable equations, known as Yang-Baxter maps (or set-theoretical solutions of the Yang-Baxter equation) [23, 24, 60], may be constructed from classical counterparts of the star-triangle relations (1) and (2). Using the classical star-triangle relations it will be shown how to systematically generate solutions (Yang-
Baxter maps) for the following two types of functional Yang-Baxter equations

\begin{align}
R_{jk}(\beta, \gamma) \circ R_{ik}(\alpha, \gamma) \circ R_{ij}(\alpha, \beta) & = R_{ij}(\alpha, \beta) \circ R_{ik}(\alpha, \gamma) \circ R_{jk}(\beta, \gamma), \\
U_{jk}(\beta, \gamma) \circ U_{ik}(\alpha, \gamma) \circ R_{ij}(\alpha, \beta) & = R_{ij}(\alpha, \beta) \circ U_{ik}(\alpha, \gamma) \circ U_{jk}(\beta, \gamma),
\end{align}

where in these equations the Yang-Baxter maps

\begin{align}
R_{ij}(\alpha, \beta) & : X \times X \to X \times X, \\
U_{ij}(\alpha, \beta) & : X \times X \to X \times X,
\end{align}

act non-trivially on the \(i\)-th and \(j\)-th components of the Cartesian product \(X \times X \times X\) of a set \(X\), and \(\alpha, \beta, \gamma\) are parameters. The Yang-Baxter equation (3) is the usual form for Yang-Baxter maps, while equation (4) is less common but has been studied before as case of the “entwining” Yang-Baxter equation [32, 40].

This paper will be concerned with the case of \(X = \CP^1 \times \CP^1\), as opposed to the more common \(X = \CP^1\), and for this reason the Yang-Baxter maps (5) which solve (3) and (4) will be referred to as two-component maps. The Yang-Baxter maps will also depend on two-component complex parameters \(\alpha, \beta \in \mathbb{C}^2\). It should be emphasised that the maps obtained here are not two-component extensions of some simpler one-component Yang-Baxter maps (as far as the author is aware), and the two-component cases are the simplest form of Yang-Baxter maps that will arise from the construction proposed in this paper.

In terms of two-component variables \(\xi_i = (y_i, z_i) \in \CP^1 \times \CP^1\), the Yang-Baxter maps \(R_{ij}, U_{ij} : (\xi_i, \xi_j) \mapsto (\xi'_i, \xi'_j)\) found in this paper can typically be written in the following form

\begin{align}
y'_i & = \Upsilon_1(\alpha, \beta)(z_j, y_j | y_j), \\
y'_j & = \Upsilon_2(\alpha, \beta)(z_i, y_i | y_i),
\end{align}

where each of the \(\Upsilon_1(\alpha, \beta)(x, y | z), \Upsilon_2(\alpha, \beta)(x, y | z), Z_1(\alpha, \beta)(x, y | z), Z_2(\alpha, \beta)(x, y | z)\), are rational bilinear functions of \(x\) and \(y\), but with non-polynomial dependence on \(z\) and the components of \(\alpha\) and \(\beta\). From the rational bilinearity of the maps of the form (6) it follows straightforwardly that they satisfy the so-called quadrirational property, such that the inverse maps for the first, second, or both components, are also rational bilinear.

It also follows from the form of the maps (6) that they can be naturally split up into the following sequence for the components

\begin{align}
(\xi_i, \xi_j) \mapsto (\xi'_i, \xi'_j) : ((y_i, z_i), (y_j, z_j)) \mapsto ((y'_i, z'_i), (y'_j, z'_j)).
\end{align}

This resembles the mapping sequence of the QRT maps [48, 49] which may be split up into so-called horizontal and vertical “switches”. For the Yang-Baxter maps, the analogue of the first switch would be \((y_i, y_j) \mapsto (y'_i, y'_j)\) and the analogue of the second switch would be \((z_i, z_j) \mapsto (z'_i, z'_j)\). However, besides this observation it is not known if there is a deeper connection to QRT maps, and unlike the QRT maps the switches for the Yang-Baxter maps are not involutions. Instead, due to a certain symmetry the Yang-Baxter maps \(R_{ij}(\alpha, \beta)\) (but not \(U_{ij}(\alpha, \beta)\)) will be found to satisfy the reversibility property for Yang-Baxter maps, such that \(R_{ij}(\alpha, \beta) \circ R_{ij}(\beta, \alpha) = \text{Id}\).

One of the main motivations of this paper came from a recent quantum group approach to Yang-Baxter maps [17, 58]. The work [17] focused on the case of the quantum group \(U_q(sl_2)\) to derive a single Yang-Baxter map for (3) associated to the star-triangle relation (2) for a
hypergeometric integral that is a hyperbolic counterpart of Barnes’s first lemma. Interestingly, the method of this paper provides a different Yang-Baxter map for the same integral, as a solution to (4) instead of (3). The reason for this is basically due to a different construction of the vertex Yang-Baxter equation associated to this hypergeometric integral. In this paper the starting point will be the classical form of star-triangle relation, and thus no quantum group structures will be considered, but the potential connections to quantum groups would be an interesting problem to consider in future.

There have also been different types of connections previously found between Yang-Baxter maps and the ABS quad equations, e.g. [4, 33, 39, 43–46]. The method of this paper provides a new connection that wasn’t considered previously, where the Yang-Baxter maps are essentially constructed through the use of the Yang-Baxter equations involving the Lagrangian/three-leg functions for ABS quad equations. This method also results in new Yang-Baxter maps. However, the connection between integrable quad equations and Yang-Baxter maps made in this paper is in some sense more complete than the previously found connections, because the method of this paper also covers the extended classification result of [1] which contains additional \( \varepsilon = 0, 1 \) deformations of the \( H \)-type quad equations from [3] that appear to not have been considered before in connection with Yang-Baxter maps. It is unknown if there is a relation between the Yang-Baxter maps of this paper and the previously found Yang-Baxter maps that are related to integrable quad equations, and it would be interesting to investigate potential connections in future.

The main results of this paper are the new method to construct Yang-Baxter maps from solutions of the classical star-triangle relations, and the use of this method to derive a new family of sixteen Yang-Baxter maps solving the functional Yang-Baxter equations (3) and (4).

The layout of this paper is as follows. In Section 2.1 classical star-triangle relations and its solutions will be introduced, which are based on previous results [13–15, 37] for the continuous spin solutions of the star-triangle relations and the quasi-classical limit. In Section 2.2, it will be shown how to use the solutions of the classical star-triangle relations to define the classical R-matrices for solutions of new classical vertex forms of the Yang-Baxter equation. In Section 3, it is shown how to convert the classical R-matrices into Yang-Baxter maps and how the functional Yang-Baxter equation is related to the classical Yang-Baxter equations from Section 2.2. Finally, using explicit solutions of the classical star-triangle relations given in Section 4.1, a new family of sixteen Yang-Baxter maps is given in Section 4.2, which are also listed individually in Appendix A.

2 Classical star-triangle relations and Yang-Baxter equations

In this section the classical forms of the star-triangle relations and Yang-Baxter equations will be introduced that will play key roles for this paper.

2.1 Classical star-triangle relations

The classical star-triangle relations may be written in terms of six complex-valued functions

\[
L_\alpha(x_i, x_j), \quad \hat{L}_\alpha(x_i, x_j), \quad \Lambda_\alpha(x_i, x_j), \quad \hat{\Lambda}_\alpha(x_i, x_j),
\]

that each depend on two complex variables \( x_i, x_j \) and a complex parameter \( \alpha \).
Defining the two functions

\[ A_1^{(ST R)}(x_a, x_b, x_c, x_d; u, v, w) = \Lambda_{v-w}(x_a, x_d) + \Lambda_{u-w}(x_b, x_d) + \hat{\Lambda}_{u-v}(x_d, x_c) - \Lambda_{v-w}(x_b, x_a) - \Lambda_{u-w}(x_a, x_c) - \hat{\Lambda}_{u-v}(x_a, x_d), \]

\[ A_2^{(ST R)}(x_a, x_b, x_c, x_d; u, v, w) = \Lambda_{v-w}(x_d, x_a) + \Lambda_{u-w}(x_d, x_b) + \hat{\Lambda}_{u-v}(x_c, x_d) - \Lambda_{v-w}(x_c, x_b) - \Lambda_{u-w}(x_c, x_a) - \hat{\Lambda}_{u-v}(x_b, x_a), \]

the classical star-triangle relations are given by the equations

\[ A_1^{(ST R)}(x_a, x_b, x_c, x_d; u, v, w) = 0, \]

\[ A_2^{(ST R)}(x_a, x_b, x_c, x_d; u, v, w) = 0, \]

where for (11) the variables \( x_a, x_b, x_c, x_d \) and parameters \( u, v, w \) are constrained to satisfy

\[ \frac{\partial}{\partial x} \left( A_1^{(ST R)}(x_a, x_b, x_c, x_d; u, v, w) \right)_{x=x_d} = 0, \]

and for (12) the variables \( x_a, x_b, x_c, x_d \) and parameters \( u, v, w \) are constrained to satisfy

\[ \frac{\partial}{\partial x} \left( A_2^{(ST R)}(x_a, x_b, x_c, x_d; u, v, w) \right)_{x=x_d} = 0. \]

The classical star-triangle relations have a useful graphical interpretation shown in Figure 1.

Figure 1: The classical star-triangle relations (11) (top) and (12) (bottom).

If the six functions in (8) satisfy the equations (11)–(14), they will be referred to as a solution of the classical star-triangle relations. The six functions need not be different. Generally, it is difficult to find non-trivial solutions to (11)–(14) that are valid for all complex
\(u, v, w, x_a, x_b, x_c, x_d \in \mathbb{C}\). The main reason is that the interesting solutions typically depend on the complex logarithm, and a solution that is valid in one region of the complex plane will not continue to hold across a branch cut.

To account for this it will be enough in this paper that the classical star-triangle relations (11) and (12) respectively hold up to terms of the form

\[
\sum_{I \in \{a,b,c,d\}} \left(2\pi i k_I x_I + \pi i \varepsilon_I x_I\right) + C(u,v,w),
\]

for some integers \(k_I \in \mathbb{Z}\), \(\varepsilon_I \in \{0,1\}\) \((I \in \{a,b,c,d\})\), and constant \(C(u,v,w)\) with respect to the variables \(x_I\). The \(k_I\) may vary as the \(u,v,w, x_I \) vary, while the \(\varepsilon_I\) are fixed for all \(u,v,w, x_I\).

In this case several interesting non-trivial solutions of the classical star-triangle relations can be found for all complex \(u,v,w, x_I\), away from branch cuts. For this paper the terms of the form (15) can mostly be ignored, since the equations of interest for the Yang-Baxter maps proposed in Section 3 will involve exponentials of the derivatives of expressions constructed from the classical star-triangle relations, to which the additional terms of the form (15) can contribute at most a sign.

It will be assumed that the following symmetries are satisfied

\[
\mathcal{L}_\alpha(x_i, x_j) = \mathcal{L}_\alpha(x_j, x_i) + C(\alpha), \quad \overline{\mathcal{L}}_\alpha(x_i, x_j) = \overline{\mathcal{L}}_\alpha(x_j, x_i) + C(\alpha),
\]

and similarly for \(\hat{\mathcal{L}}_\alpha\) and \(\overline{\hat{\mathcal{L}}}_\alpha\), where \(C(\alpha)\) is a constant with respect to \(x_i\) and \(x_j\). With (16) the ordering of variables for the functions \(\mathcal{L}_\alpha, \overline{\mathcal{L}}_\alpha, \hat{\mathcal{L}}_\alpha, \overline{\hat{\mathcal{L}}}_\alpha\) can be ignored in the classical star-triangle relations (11)–(14), since a change in ordering will only result in an additional term of the form (15). On the other hand, the functions \(\Lambda\) and \(\overline{\Lambda}\) are not assumed to satisfy symmetries of the form (16), and thus the ordering of variables for these functions must be kept as written in (11)–(14). This is the reason that these functions are associated to directed edges in Figure 1.

There is also another symmetry that is assumed to be satisfied

\[
\mathcal{L}_{-\alpha}(x_i, x_j) = -\mathcal{L}_\alpha(x_i, x_j) + 2\pi i (k_i x_i + k_j x_j) + C(\alpha),
\]

\[
\overline{\mathcal{L}}_{-\alpha}(x_i, x_j) = -\overline{\mathcal{L}}_\alpha(x_i, x_j) + 2\pi i (k_i x_i + k_j x_j) + C(\alpha),
\]

and similarly for \(\hat{\mathcal{L}}_{-\alpha}\) and \(\overline{\hat{\mathcal{L}}}_{-\alpha}\), where \(k_i, k_j\) are some integers and \(C(\alpha)\) is a constant with respect to \(x_i\) and \(x_j\). The symmetries (17) wont be needed for this section, but will be useful for establishing certain properties of the Yang-Baxter maps that will be derived in Section 3.

### 2.2 Classical R-matrices and classical Yang-Baxter equations

Let \(u, v, w\), denote the two-component parameters

\[
u = (u_1, u_2), \quad v = (v_1, v_2), \quad w = (w_1, w_2),
\]

and let \(x, x', \tilde{x}, \tilde{x}\), denote the triplets of variables

\[
x = (x_i, x_j, x_k), \quad x' = (x'_i, x'_j, x'_k), \quad \tilde{x} = (\tilde{x}_i, \tilde{x}_j, \tilde{x}_k), \quad \tilde{x} = (\hat{x}_i, \hat{x}_j, \hat{x}_k).
\]

The classical Yang-Baxter equations may be written in terms of the three complex-valued functions denoted as

\[
R_{uv}^{(1)}(x_i, x_j, x'_i, x'_j), \quad R_{uv}^{(2)}(x_i, x_j, x'_i, x'_j), \quad R_{uv}^{(3)}(x_i, x_j, x'_i, x'_j),
\]
where each of the functions depend on the complex variables and parameters indicated in the arguments and subscripts, respectively.

Defining the function

$$\mathcal{A}^{YBE}(x, x', \hat{x}, \bar{x}; u, v, w) = R_{uv}^{(1)}(x_i, x_j, \hat{x}_i, \hat{x}_j) + R_{uw}^{(2)}(\hat{x}_i, x_k, x'_j, \hat{x}_k) + R_{vw}^{(3)}(\hat{x}_j, \hat{x}_k, x'_l, x'_k)$$

\[ -R_{uv}^{(1)}(\hat{x}_i, \hat{x}_j, x'_l, x'_k) - R_{uw}^{(2)}(\hat{x}_i, \hat{x}_k, x'_l) - R_{vw}^{(3)}(x_j, x_k, \hat{x}_j, \hat{x}_k), \]

(21)

the classical Yang-Baxter equation (CYBE) is given by the expression

$$\mathcal{A}^{YBE}(x, x', \hat{x}, \bar{x}; u, v, w) = 0,$$  

(22)

where the variables $x, x'$, $\hat{x}$, $\bar{x}$ and parameters $u, v, w$ are constrained to satisfy the six equations

$$\frac{\partial}{\partial X} \mathcal{A}^{YBE}(x, x', \hat{x}, \bar{x}; u, v, w) = 0, \quad X \in \{\hat{x}_i, \hat{x}_j, \hat{x}_k, \hat{x}_l, \hat{x}_m, \hat{x}_n\}. \quad \text{(23)}$$

In analogy with solutions of the quantum Yang-Baxter equation, if the three functions in (20) satisfy the equations (22)–(23), they will be referred to as classical R-matrices. The three classical R-matrices (20) which satisfy (22)–(23) need not all be different. Similarly to the case for the classical star-triangle relations, it will be enough for our purposes that the CYBE is satisfied up to some term of the form

$$\sum_{J \in \{i, j, k\}} \left(2\pi i(k_j x_J + k'_j x'_J + \hat{k}_j \hat{x}_J + \hat{k}'_j \hat{x}_J) + \pi i(\varepsilon_j x_J + \varepsilon'_j x'_J + \hat{\varepsilon}_j \hat{x}_J + \hat{\varepsilon}'_j \hat{x}_J)\right) + C(u, v, w),$$

(24)

for some integers $k_j, k'_j, \hat{k}_j, \hat{k}'_j \in \mathbb{Z}, \varepsilon_j, \varepsilon'_j, \hat{\varepsilon}_j, \hat{\varepsilon}'_j \in \{0, 1\}$ $(J \in \{i, j, k\})$, and constant $C(u, v, w)$ with respect to the variables $x_J, x'_J, \hat{x}_J, \hat{x}_J$. The $k_j, k'_j, \hat{k}_j, \hat{k}'_j$ may vary as $x_J, x'_J, \hat{x}_J, \hat{x}_J$ and $u, v, w$ vary, while the $\varepsilon_j, \varepsilon'_j, \hat{\varepsilon}_j, \hat{\varepsilon}'_j$ are fixed for all $x_J, x'_J, \hat{x}_J, \hat{x}_J, u, v, w$.

In addition to the six equations (23), there are six other equations that are obtained by taking the derivative of the CYBE with respect to each of the six boundary variables. These six additional equations are

$$\frac{\partial}{\partial X} \mathcal{A}^{YBE}(x, x', \hat{x}, \bar{x}; u, v, w) = 0, \quad X \in \{x_i, x_j, x_k, x'_l, x'_j, x'_k\}. \quad \text{(25)}$$

Altogether the equations for the derivatives (23) and (25) can be regarded as a generalised system of discrete Laplace-type equations on a cuboctahedron, because the individual equations are a more general form of the so-called discrete Laplace-type equations associated to type-Q ABS equations [18, 19]. In this paper these equations will sometimes be referred to simply as the cuboctahedron equations. The cuboctahedron equations (23) and (25) underpin the method for constructing the Yang-Baxter maps that will be presented in Section 3.
2.2.1 Classical R-matrices from solutions of the classical star-triangle relations

In terms of (8), define the four functions

\[
R_{uv}(x_i, x_j, x'_i, x'_j) = \mathcal{L}_{u_1-v_1}(x'_i, x_j) + \mathcal{L}_{u_2-v_2}(x_i, x'_j) + \mathcal{L}_{u_1-v_2}(x_i, x_j) + \mathcal{L}_{u_2-v_1}(x'_i, x'_j). \quad (26)
\]

\[
\hat{R}_{uv}(x_i, x_j, x'_i, x'_j) = \hat{\mathcal{L}}_{u_1-v_1}(x'_i, x_j) + \hat{\mathcal{L}}_{u_2-v_2}(x_i, x'_j) + \hat{\mathcal{L}}_{u_1-v_2}(x_i, x_j) + \hat{\mathcal{L}}_{u_2-v_1}(x'_i, x'_j), \quad (27)
\]

\[
U_{uv}(x_i, x_j, x'_i, x'_j) = \Lambda_{u_1-v_1}(x'_i, x_j) + \Lambda_{u_2-v_2}(x_i, x'_j) + \Lambda_{u_1-v_2}(x_i, x_j) + \Lambda_{u_2-v_1}(x'_i, x'_j), \quad (28)
\]

\[
\hat{U}_{uv}(x_i, x_j, x'_i, x'_j) = \Lambda_{u_1-v_1}(x_j, x'_i) + \Lambda_{u_2-v_2}(x'_j, x_i) + \Lambda_{u_1-v_2}(x_j, x_i) + \Lambda_{u_2-v_1}(x'_j, x'_i). \quad (29)
\]

Following from the classical star-triangle relations pictured in Figure 1, these functions have a graphical representation as is shown in Figure 2 for (26) and (28).

![Graphical representation of classical R-matrices](image)

**Theorem 2.1.** If the six functions \( \mathcal{L}_\alpha, \hat{\mathcal{L}}_\alpha, \overline{\mathcal{L}}_\alpha, \Lambda_\alpha, \overline{\Lambda}_\alpha \), are a solution to the classical star-triangle relations (11)–(14) up to a term of the form (15), then the four combinations of classical R-matrices from (26)–(29) given by

\[
\begin{align*}
(1a) & \quad R^{(1)}_{uv} = R^{(2)}_{uv} = R^{(3)}_{uv} = R_{uv}, \\
(1b) & \quad \hat{R}^{(1)}_{uv} = \hat{R}^{(2)}_{uv} = \hat{R}^{(3)}_{uv} = \hat{R}_{uv}, \\
(2a) & \quad \hat{R}^{(1)}_{uv} = R_{uv}, \quad R^{(2)}_{uv} = R^{(3)}_{uv} = U_{uv}, \\
(2b) & \quad \hat{R}^{(1)}_{uv} = \hat{R}_{uv}, \quad \hat{R}^{(2)}_{uv} = \hat{R}^{(3)}_{uv} = \hat{U}_{uv},
\end{align*}
\]

are solutions to the CYBE (22)–(23) up to a term of the form (24).

For convenience, in the remainder of this paper the different combinations of classical R-matrices given in Theorem 2.1 will respectively be referred to as type-(1a), type-(1b), type-(2a), and type-(2b) solutions of the CYBE.

**Proof.** A straightforward way to prove Theorem 2.1 is through the use of graphical representations of the CYBE (22), where repeated applications of the classical star-triangle relations of
Figure 3 may be used to transform the left hand side of the CYBE into the right hand side, or vice versa. The graphical representation of the type-(2a) solution of the CYBE is shown in Figure 3, and a sequence of transformations to show that this CYBE holds is shown in Figure 4. The type-(1a), type-(1b), and type-(2b) cases follow from identical transformations by applying the appropriate forms of the classical star-triangle relations for each case.

![Diagram of CYBE solution](image)

**Figure 3:** Type-(2a) solution of the CYBE using the graphical representation of Figure 2. The diagram for the type-(2b) case would be the same but with the reverse orientation of each directed edge. For the type-(1a) and type-(1b) cases there would be no directed edges, and the classical R-matrices would all be of the form of either \((\ref{eq:15})\) or \((\ref{eq:16})\).

**Remark 2.2.** In Figure 4, each function of the type \(L_{u-v}\) or \(\overline{L}_{u-v}\) is transformed twice via the classical star-triangle relations. They first get transformed to \(\hat{L}_{u-v}\) or \(\hat{\overline{L}}_{u-v}\), and then back to \(L_{u-v}\) or \(\overline{L}_{u-v}\). This is the reason why the type-(2a) and type-(2b) solutions to the CYBE only involve one of the classical R-matrices \((\ref{eq:15})\) or \((\ref{eq:16})\), and not both of them.

**Remark 2.3.** The expression for the CYBE of Figure 3 is new, but the manipulations with the star-triangle relations of Figure 4 are well-known for quantum integrable systems and have previously been employed in various different contexts \([12, 16, 21, 22, 30, 36]\).

For particular solutions of the CYBE it will eventually be important to determine the values of \(\varepsilon_I, \varepsilon_J, \hat{\varepsilon}_I, \hat{\varepsilon}_J (J \in \{i, j, k\})\) appearing in the additional terms \((\ref{eq:24})\), as this will contribute an overall sign to the Yang-Baxter maps given in the next section. This may be done by simply keeping track of the values of the \(\varepsilon_I (I \in \{a, b, c, d\})\) appearing in \((\ref{eq:15})\) for the classical star-triangle relations, which are used in the deformations of the CYBE in Figure 4.

**Proposition 2.4.** For a given solution of the classical star-triangle relations \((\ref{eq:11})–(\ref{eq:14})\), let \(\varepsilon_I^{(1)} (I \in \{a, b, c, d\})\) be the additional terms from \((\ref{eq:15})\) needed to satisfy \((\ref{eq:11})\) and \((\ref{eq:13})\), and let \(\varepsilon_I^{(2)} (I \in \{a, b, c, d\})\) be the additional terms from \((\ref{eq:15})\) needed to satisfy \((\ref{eq:12})\) and \((\ref{eq:14})\). Then the values of \(\varepsilon_I, \varepsilon_J, \hat{\varepsilon}_I, \hat{\varepsilon}_J (J \in \{i, j, k\})\) for the additional terms in \((\ref{eq:24})\) needed to satisfy the corresponding CYBE from Theorem 2.1 are given by

\[
\varepsilon_i = \varepsilon_i' = \varepsilon_b^{(1)} + \varepsilon_c^{(2)} \pmod 2, \quad \varepsilon_j = \varepsilon_j' = \varepsilon_a^{(1)} + \varepsilon_c^{(2)} \pmod 2, \quad \varepsilon_k = \varepsilon_k' = \varepsilon_a^{(2)} + \varepsilon_b^{(2)} \pmod 2, \\
\hat{\varepsilon}_i = \hat{\varepsilon}_i = \varepsilon_a^{(1)} + \varepsilon_d^{(2)} \pmod 2, \quad \hat{\varepsilon}_j = \hat{\varepsilon}_j = \varepsilon_b^{(1)} + \varepsilon_d^{(2)} \pmod 2, \quad \hat{\varepsilon}_k = \hat{\varepsilon}_k = \varepsilon_c^{(1)} + \varepsilon_d^{(1)} \pmod 2.
\]

\[(\ref{eq:34})\]
Figure 4: Deformations of the CYBE of Figure 3, with the use classical star-triangle relation of Figure 1. The deformations for the type-(2b) solution of the CYBE are of the same form, but with the reverse orientation of each directed edge. For the type-(1a) and type-(1b) solution of the CYBE there would be no directed edges.

3 From classical R-matrices to Yang-Baxter maps

In this section it will be seen how to derive Yang-Baxter maps from the solutions of the classical Yang-Baxter equation (CYBE) of Theorem 2.1.

3.1 Two-component Yang-Baxter maps

From the four classical R-matrices (26)–(29), two different types of Yang-Baxter maps will be obtained, which will be denoted by

\[ R_{ij}(\alpha, \beta): X \times X \rightarrow X \times X, \quad U_{ij}(\alpha, \beta): X \times X \rightarrow X \times X, \]  

(35)

where the parameters are \( \alpha, \beta \in \mathbb{C}^2 \). For this paper, the set \( X \) is always taken as

\[ X = \mathbb{C}P^1 \times \mathbb{C}P^1, \]  

(36)

and thus it is appropriate to refer to (35) as two-component maps, as opposed to the more common expressions for Yang-Baxter maps where \( X = \mathbb{C}P^1 \).
The Yang-Baxter maps (35) will provide solutions to two different forms of the functional Yang-Baxter equation (FYBE) given by

\begin{align}
R_{jk}(\beta, \gamma) &\circ R_{ik}(\alpha, \gamma) \circ R_{ij}(\alpha, \beta) = R_{ij}(\alpha, \beta) \circ R_{ik}(\alpha, \gamma) \circ R_{jk}(\beta, \gamma), \
U_{jk}(\beta, \gamma) &\circ U_{ik}(\alpha, \gamma) \circ R_{ij}(\alpha, \beta) = R_{ij}(\alpha, \beta) \circ U_{ik}(\alpha, \gamma) \circ U_{jk}(\beta, \gamma).
\end{align}

(37) (38)

These equations respectively express the equality of two different maps \( X \times X \times X \to X \times X \times X \), where as usual, an individual map \( R_{ij} \) or \( U_{ij} \) only acts non-trivially on the \( i \)-th and \( j \)-th sets. The first expression for the FYBE (37) will be derived from the more symmetric type-(1a) and type-(1b) solutions of the CYBE given in Theorem 2.1, and the second expression for the FYBE (38) will be derived from the type-(2a) and type-(2b) solutions of the CYBE.

Defining the two-component variables

\[ \xi_i, \xi_j \in \mathbb{C}P^1 \times \mathbb{C}P^1, \]

(39)

the Yang-Baxter maps (35) may be visualised as having the variables and parameters on edges of a square as shown in Figure 5. Then a graphical representation of the FYBE (38) is pictured in Figure 6 (the FYBE (37) is simply obtained form this Figure by replacing the \( U_{ij} \) maps with \( R_{ij} \) maps). The left hand side of (38) maps \( (\xi_i, \xi_j, \xi_k) \mapsto (\xi_i^{(l)}, \xi_j^{(l)}, \xi_k^{(l)}) \), and the right hand side of (38) maps \( (\xi_i, \xi_j, \xi_k) \mapsto (\xi_i^{(r)}, \xi_j^{(r)}, \xi_k^{(r)}) \), and the FYBE (38) states that these two different sequences of maps should be equivalent, such that \( (\xi_i^{(l)}, \xi_j^{(l)}, \xi_k^{(l)}) = (\xi_i^{(r)}, \xi_j^{(r)}, \xi_k^{(r)}) \).

Figure 5: Graphical representation of Yang-Baxter maps (35), which involve two-component variables \( \xi_i, \xi_j \) and two-component parameters \( \alpha, \beta \) on edges. The single- and double-line arrows distinguish the Yang-Baxter maps \( R_{ij}(\alpha, \beta) \) and \( U_{ij}(\alpha, \beta) \), respectively. Parallel edges are assigned the same parameter.

### 3.2 Classical R-matrix to Yang-Baxter map

In this section the details will be given for constructing the Yang-Baxter maps for the FYBE’s (37) and (38) from the type-(2a) solution of the CYBE. The cases of other solutions of the CYBE follow from similar computations and so won’t be considered here separately.

Let \( \tau \) be a parameter that can take one of the two values 0 or 1. Recall that the classical R-matrices (26)–(29) satisfy the CYBE up to the additional terms of the form (24). For convenience it will be assumed here that for solutions of the CYBE,

\[ \varepsilon_i = \varepsilon'_i = \tilde{\varepsilon}_i = \tilde{\varepsilon}'_i = \varepsilon_j = \varepsilon'_j = \tilde{\varepsilon}_j = \tilde{\varepsilon}'_j = \tau, \]
\[ \varepsilon_k = \varepsilon'_k = \tilde{\varepsilon}_k = \tilde{\varepsilon}'_k = 0. \]

(40)
First, consider the classical R-matrix (26) and let the derivatives with respect to its four variables be denoted

\[
R_{uv,i}(x'_i, x_j | x_i) = \frac{\partial R_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x_i}, \quad R_{uv,i'}(x_j, x'_i | x'_j) = -\frac{\partial R_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x'_i},
\]
\[
R_{uv,j}(x'_i, x_i | x_j) = \frac{\partial R_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x_j}, \quad R_{uv,j'}(x_i, x'_j | x'_j) = -\frac{\partial R_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x'_j}.
\]

From the definition (26), each of the above derivatives are a sum of two terms that involve a derivative of \( L_{u-v}(x_i, x_j) \) and a derivative of \( \overline{L}_{u-v}(x_i, x_j) \).

Similarly, the respective derivatives of the classical R-matrix (28) will be denoted by

\[
U_{uv,i}(x'_i, x_j | x_i) = \nu \frac{\partial U_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x_i}, \quad U_{uv,i'}(x_j, x'_i | x'_j) = \nu \frac{\partial U_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x'_i},
\]
\[
U_{uv,j}(x'_i, x_i | x_j) = \nu \frac{\partial U_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x_j}, \quad U_{uv,j'}(x_i, x'_j | x'_j) = \nu \frac{\partial U_{uv}(x_i, x_j; x'_i, x'_j)}{\partial x'_j},
\]

where \( \nu \) and \( \overline{\nu} \) depend on the value of \( \overline{\nu} \) from (40) as

\[
\nu = (-1)^{\overline{\nu}}, \quad \overline{\nu} = (-1)^{\overline{\nu} + 1}.
\]

Next, a change of variables is usually necessary to put the equations (41) and (42) into a suitable form for the respective Yang-Baxter maps. For (41), this change of variables takes the following form

\[
y'_i = f(x'_i), \quad y'_j = f(x'_j), \quad y_i = f(x_i), \quad y_j = f(x_j), \quad \alpha_1 = h(u_1), \quad \alpha_2 = h(u_2), \quad \beta_1 = h(v_1), \quad \beta_2 = h(v_2).
\]

Here \( f(x) \) is chosen so that the exponentials of both \( R_{uv,j}(x'_i, x_i | x_j) \) and \( R_{uv,i}(x'_j, x_j | x_i) \) may be rewritten as a rational bilinear expression of \( y'_i \) and \( y'_j \), respectively. Except for the elliptic
case, it will turn out that the choices for \( f(x) \) also make \( R_{uv,i}(x'_i, x_i | x_j) \) and \( R_{uv,j}(x'_j, x_j | x_i) \) rational bilinear functions of \( y_i \) and \( y_j \), respectively. This is because the dependence of the expressions (41) on the first two arguments on the left hand sides are typically the same up to signs. The function \( h(x) \) is chosen so that the above rational bilinear expressions are algebraic in \( \alpha \) and \( \beta \).

Similarly to (44), the change of variables for (42) takes the form

\[
y'_i = f(x'_i), \quad y'_j = g(x'_j), \quad y_i = f(x_i), \quad y_j = g(x_j),
\]

\[
\alpha_1 = h(u_1), \quad \alpha_2 = h(u_2), \quad \beta_1 = h(v_1), \quad \beta_2 = h(v_2),
\]

where \( f(x) \) is the same function from (44), and \( g(x) \) is another function which is chosen so that the exponentials of \( U_{uv,i}(x'_j, x_j | x_i) \) and \( U_{uv,j}(x'_i, x_i | x_j) \) may be written as rational bilinear functions of \( y'_i \) and \( y'_j \), respectively. For the explicit cases considered in this paper (besides the elliptic case), there may be found a choice of \( f(x), g(x), h(x) \), which are compatible with the aforementioned requirements for transforming (44) and (45).

After the change of variables (44), the exponentials of \( R_{uv,i}, R_{uv,j}, R_{uv,i'}, R_{uv,j'} \) will be rational bilinear functions of two out of the four variables \( y_i, y_j, y'_i, y'_j \). Let the latter functions be respectively denoted by \( Y_{1,\alpha\beta}^{R}(y'_i, y'_j | y_i) \), \( Y_{1,\alpha\beta}^{R}(y'_i, y_i | y_j) \), \( Z_{1,\alpha\beta}^{R}(y_j, y'_j | y_i) \), \( Z_{2,\alpha\beta}^{R}(y_i, y'_i | y_j) \), where each function is rational bilinear in their first two arguments (the superscript \( R \) is used to indicate association with the classical R-matrix \( R_{uv}(x_i, x_j; x'_i, x'_j) \)). The expressions \( Y_{1,\alpha\beta}^{R}, Y_{2,\alpha\beta}^{R} \), \( Z_{1,\alpha\beta}^{R}, Z_{2,\alpha\beta}^{R} \) will in fact be the components of the desired Yang-Baxter map, and these are labelled by the new variables \( z_i, z_j, z'_i, z'_j \), as follows

\[
z_i = Y_{1,\alpha\beta}^{R}(y'_i, y'_j | y_i) = e^{R_{uv,i'}(x'_i, x_i | x_j)}, \quad z'_i = Z_{1,\alpha\beta}^{R}(y_j, y'_j | y'_i) = e^{R_{uv',j}(x'_i, x'_j | x_j)},
\]

\[
z_j = Y_{1,\alpha\beta}^{R}(y'_i, y_i | y_j) = e^{R_{uv,j}(x'_i, x_i | x_j)}, \quad z'_j = Z_{2,\alpha\beta}^{R}(y_i, y'_i | y'_j) = e^{R_{uv',j'}(x'_i, x'_j | x'_j)}. \tag{46}
\]

The exponentials will eliminate unwanted sums of logarithms. However, for some degenerate additive cases the derivatives of the classical R-matrix in (46) do not involve the complex logarithm, and for these cases no exponentials would be required.

The expressions (46) provide an implicit form of the desired Yang-Baxter map. To arrive at the final form, the two equations on the left hand side of (46) can be uniquely solved (due to rational bilinearity) for the variables \( y'_i \) and \( y'_j \). The resulting expressions for \( y'_i \) and \( y'_j \), will be written as

\[
y'_i = \Upsilon_{1,\alpha\beta}^{R}(z_j, y_i | y_j), \quad y'_j = \Upsilon_{2,\alpha\beta}^{R}(z_i, y_j | y_i),
\]

where \( \Upsilon_{1,\alpha\beta}^{R}(x, y | z) \) and \( \Upsilon_{2,\alpha\beta}^{R}(x, y | z) \) are both rational bilinear functions of \( x \) and \( y \).

Then the final expression for the Yang-Baxter map \( R_{ij}(\alpha, \beta) \): \((\xi_i, \xi_j) \mapsto (\xi'_i, \xi'_j)\) is given in terms of (46) and (47) as

\[
y'_i = \Upsilon_{1,\alpha\beta}^{R}(z_j, y_i | y_j), \quad z'_i = Z_{1,\alpha\beta}^{R}(y_j, y'_j | y'_i),
\]

\[
y'_j = \Upsilon_{2,\alpha\beta}^{R}(z_i, y_j | y_i), \quad z'_j = Z_{2,\alpha\beta}^{R}(y_i, y'_i | y'_j). \tag{48}
\]

The Yang-Baxter map (48), can be naturally split up into the following QRT-like sequence

\[
(\xi_i, \xi_j) \mapsto (\xi'_i, \xi'_j) : ((y_i, z_i), (y_j, z_j)) \mapsto ((y'_i, z_i), (y'_j, z_j)) \mapsto ((y'_i, z'_i), (y'_j, z'_j)), \tag{49}
\]
where the variables \( y_i, y_j \) are mapped before the variables \( z_i, z_j \). However, unlike QRT maps, the separate maps for the \( y_i, y_j \) and \( z_i, z_j \) do not define involutions.

The expression for the Yang-Baxter map \( U_{ij}(\alpha, \beta) : (\xi_i, \xi_j) \rightarrow (\xi'_i, \xi'_j) \) follows similarly. That is, after the change of variables (44) and (45) define the following functions (c.f. (46))

\[
\begin{align*}
z_i &= Y^{(U)}_{2, \alpha \beta}(y'_j, y_j | y_i) = e^{U_{uv, i}(x'_j, x_j | x_i)}, \\
z_j &= Y^{(U)}_{1, \alpha \beta}(y'_i, y_i | y_j) = e^{U_{uv, j}(x'_i, x_i | x_j)},
\end{align*}
\]

\[
\begin{align*}
z'_i &= Z^{(U)}_{1, \alpha \beta}(y_j, y'_j | y'_i) = e^{U_{uv, j}(x_j, x'_j | x'_i)}, \\
z'_j &= Z^{(U)}_{2, \alpha \beta}(y_i, y'_i | y'_j) = e^{U_{uv, i}(x_i, x'_i | x'_j)},
\end{align*}
\]

(50)

where \( Y^{(U)}_{1, \alpha \beta}(x, y | z), Y^{(U)}_{2, \alpha \beta}(x, y | z), Z^{(U)}_{1, \alpha \beta}(x, y | z), Z^{(U)}_{2, \alpha \beta}(x, y | z), \) are each rational bilinear functions of \( x \) and \( y \). Rewriting in terms of the variables \( y'_i, y'_j, z'_i, z'_j \), gives the final expression for the Yang-Baxter map \( U_{ij}(\alpha, \beta) : (\xi_i, \xi_j) \rightarrow (\xi'_i, \xi'_j) \), as

\[
\begin{align*}
y'_i &= Y^{(U)}_{1, \alpha \beta}(z_j, y_j | y_i), \\
y'_j &= Y^{(U)}_{2, \alpha \beta}(z_i, y_i | y_j),
\end{align*}
\]

\[
\begin{align*}
z'_i &= Z^{(U)}_{1, \alpha \beta}(y_j, y'_j | y'_i), \\
z'_j &= Z^{(U)}_{2, \alpha \beta}(y_i, y'_i | y'_j),
\end{align*}
\]

(51)

where \( Y^{(R)}_{1, \alpha \beta}(x, y | z) \) and \( Y^{(R)}_{2, \alpha \beta}(x, y | z) \) are appropriately defined rational bilinear functions of \( x \) and \( y \). This map may also be split up in into the sequence (49).

### 3.3 Properties of Yang-Baxter maps and FYBE

The following property is a straightforward consequence of the rational bilinear expressions for the Yang-Baxter maps (48) and (51).

**Proposition 3.1** (Quadrirationality). The maps (48) and (51) can be uniquely solved to define rational maps

\[
(\xi_i, \xi_j) \rightarrow (\xi'_i, \xi'_j) \rightarrow (\xi''_i, \xi''_j) \rightarrow (\xi'''_i, \xi'''_j)
\]

(52)

**Remark 3.2.** The first of (52) is the inverse map, and the remaining two maps are sometimes referred to as companion maps.

**Proposition 3.3** (Reversibility). If (17) is satisfied then the map (48) satisfies

\[
R_{ji}(\beta, \alpha) \circ R_{ij}(\alpha, \beta) = Id,
\]

(53)

where \( Id \) is the identity map.

Using the graphical representation of Figure 5, the reversibility property (53) is depicted in Figure 7.

**Proof.** In terms of variables, the left hand side of (53) defines a map

\[
R_{ji} \circ R_{ij} : (\xi_i, \xi_j) \rightarrow (\xi''_i, \xi''_j) \rightarrow (\xi'''_i, \xi'''_j).
\]

(54)

Thus (53) will hold if \( \xi'''_i = \xi_i \) and \( \xi'''_j = \xi_j \).
First it will be shown that the first components of $\xi''_i$ and $\xi_i$ are equal. By the definitions (46), (41), and (26), the first component $y''_i$ of the variable $\xi''_i$ is determined from a change of variables (44) of the equation

$$z'_j = \exp \left( \frac{\partial}{\partial x'_j} \left( \mathcal{L}_{v_2-u_2}(x'_j, x''_i) + \mathcal{L}_{v_2-u_1}(x'_j, x'_i) \right) \right).$$

(55)

On the other hand, by (46), (41), and (26), the first component $y_i$ of $\xi_i$ satisfies the equation (with the same change of variables (44))

$$z'_j = \exp \left( - \frac{\partial}{\partial x'_j} \left( \mathcal{L}_{u_2-v_3}(x_i, x'_j) + \mathcal{L}_{u_1-v_2}(x'_j, x'_i) \right) \right).$$

(56)

Equating (55) and (56) and using the relations (17) it follows that $y''_i = y_i$, i.e., the first components of $\xi''_i$ and $\xi_i$ are equal. Then by symmetry, the first components of $\xi''_j$ and $\xi_j$ will also be equal.

Then it remains to show that the second components of $\xi''_i$ and $\xi''_j$, are equal to the second components of $\xi_i$ and $\xi_j$, respectively. By the definition (46), (41), and (26), the second component $z''_i$ of the variable $\xi''_i$ is up to the change of variables (44) given by

$$z''_i = \exp \left( - \frac{\partial}{\partial x''_i} \left( \mathcal{L}_{v_2-u_1}(x''_i, x''_i) + \mathcal{L}_{v_2-u_2}(x'_j, x''_i) \right) \right).$$

(57)

On the other hand, by (46), (41), and (26), the second component $z_i$ of the variable $\xi_i$ is given by

$$z_i = \exp \left( \frac{\partial}{\partial x'_i} \left( \mathcal{L}_{u_1-v_2}(x_i, x'_i) + \mathcal{L}_{u_2-v_2}(x'_i, x'_i) \right) \right).$$

(58)

By the relations (17), and also using $x_j = x''_j$ (the first components of the variables $\xi_j$ and $\xi''_j$ are equal as was determined above), the equation (58) is found to be equivalent to the equation (57). This means that $z''_i = z_i$, i.e., the second components of $\xi''_i$ and $\xi_i$ are equal. By symmetry, the second components of $\xi''_j$ and $\xi_j$ will also be equal.

Remark 3.4. The functions $\Lambda_{u-v}(x_i, x_j)$ and $\overline{\Lambda}_{u-v}(x_i, x_j)$ do not satisfy analogues of the antisymmetry relations (17), and thus the reversibility property (53) should not be expected to hold for the Yang-Baxter map $U_{ij}(\alpha, \beta)$ in (51).
The relation between the CYBE (22)–(23) and the FYBE’s (37) and (38) is as follows.

**Theorem 3.5.** The maps (48) and (51) satisfy the FYBE’s (37) and (38) if and only if the classical R-matrices (26) and (28) are type-(1a) and type-(2a) solutions of the CYBE (22)–(23).

Theorem 3.5 is arguably the most important property of the maps (48) and (51) by which it is appropriate to call them Yang-Baxter maps.

**Proof.** Only the details for the case of the FYBE (38) will be considered (corresponding to a type-(2a) solution of the CYBE), since (37) follows from the case of (38) when all Yang-Baxter maps are of the same type (corresponding to a type-(1a) solution of the CYBE).

For the first part of the proof, it will be shown that the FYBE (38) satisfied with the maps (48) and (51) implies that the exponentials of the twelve cuboctahedron equations (23) and (25) are satisfied with the classical R-matrices (26) and (28) for a type-(2a) solution to the CYBE (22). This in turn implies that the CYBE holds up to a term of the form (24).

Consider the two mappings

\[ U_{jk}(\beta, \gamma) \circ U_{ik}(\alpha, \gamma) \circ R_{ij}(\alpha, \beta) : (\xi_i, \xi_j, \xi_k) \mapsto (\xi_i', \xi_j', \xi_k') \mapsto (\xi_i^{(l)}, \xi_j^{(l)}, \xi_k^{(l)}), \]  
\[ R_{ij}(\alpha, \beta) \circ U_{ik}(\alpha, \gamma) \circ U_{jk}(\beta, \gamma) : (\xi_i, \xi_j, \xi_k) \mapsto (\xi_i^*, \xi_j^*, \xi_k^*) \mapsto (\xi_i^{(r)}, \xi_j^{(r)}, \xi_k^{(r)}). \]  

Assume that the FYBE (38) is satisfied such that (59) and (60) are equivalent. For convenience, three new variables \( \xi_i', \xi_j', \xi_k' \) will be defined to be the final mapped variables as \( \xi_i' = \xi_i^{(l)} = \xi_i^{(r)} \), \( \xi_j' = \xi_j^{(r)} \), \( \xi_k' = \xi_k^{(l)} = \xi_k^{(r)} \).

The first components of the three initial variables \( \xi_i, \xi_j, \xi_k \) of the FYBE, may be identified with the variables \( x_i, x_j, x_k \) of the CYBE (22). On the left hand side of the FYBE these variables are mapped as \( (\xi_i, \xi_j, \xi_k) \mapsto (\xi_i', \xi_j', \xi_k') \), and on the right hand side as \( (\xi_i, \xi_j, \xi_k) \mapsto (\xi_i^*, \xi_j^*, \xi_k^*) \). The different mappings are determined through the respective expressions for \( z_i, z_j, z_k \) (the second components of \( \xi_i, \xi_j, \xi_k \)), which are defined differently on the left and right hand sides of the FYBE according to (46), (50), (44), and (45). Equating these different definitions of the second components \( z_i, z_j, z_k \), respectively gives the following three equations

\[ \exp\left( \frac{\partial}{\partial x_i} R_{uv}(x_i, x_j, x_k, x_i') \right) = \exp\left( \frac{\partial}{\partial x_i} U_{uv}(x_i, x_j, x_k, x_i') \right), \]  
\[ \exp\left( \frac{\partial}{\partial x_j} R_{uv}(x_i, x_j, x_k, x_j') \right) = \exp\left( \frac{\partial}{\partial x_j} U_{vw}(x_j, x_k, x_j, x_j') \right), \]  
\[ \exp\left( \frac{\partial}{\partial x_k} U_{uw}(x_i, x_k, x_j, x_k') \right) = \exp\left( \frac{\partial}{\partial x_k} U_{vw}(x_j, x_k, x_j, x_k') \right). \]  

Equations (61) may be seen to be equivalent (mod 2\( \pi \)) to three of the cuboctahedron equations given in (25) after identifying the following two sets of variables

\[ (x_i', x_j', x_k', x_i^*, x_j^*, x_k^*) = (\hat{x}_i, \hat{x}_j, \hat{x}_k, \bar{x}_i, \bar{x}_j, \bar{x}_k). \]  

The first components of \( \xi_i', \xi_j', \xi_k' \) may respectively be identified with the three variables \( x_i', x_j', x_k' \) of the CYBE. The second components of \( \xi_i', \xi_j', \xi_k' \) are defined through the equations
(46), (50), (44), and (45). Equating the different definitions of the second components of the variables \(\xi_i^{(l)}, \xi_j^{(l)}, \xi_k^{(l)}\) and \(\xi_i^{(r)}, \xi_j^{(r)}, \xi_k^{(r)}\), respectively, gives the following three equations

\[
\exp\left(\frac{\partial}{\partial x_i} U_{uw}(x_{i}^{\dagger}, x_{k}, x_{i}', x_{k}')\right) = \exp\left(\frac{\partial}{\partial x_i} R_{uv}(x_{i}^{\dagger}, x_{j}, x_{i}', x_{j}')\right),
\]

\[
\exp\left(\frac{\partial}{\partial x_j} U_{vw}(x_{j}^{\dagger}, x_{k}', x_{j}', x_{k}')\right) = \exp\left(\frac{\partial}{\partial x_j} R_{uv}(x_{i}^{\dagger}, x_{j}', x_{i}', x_{j}')\right),
\]

\[
\exp\left(\frac{\partial}{\partial x_k} U_{vw}(x_{j}^{\dagger}, x_{k}', x_{j}', x_{k}')\right) = \exp\left(\frac{\partial}{\partial x_k} U_{uw}(x_{i}, x_{k}', x_{i}', x_{k}')\right).
\]

These may be seen to be equivalent (mod 2\(\pi\)) to the remaining three cuboctahedron equations given in (25) after using the identification given in (62).

Next consider the composition of maps (59) for the left hand side of the FYBE (38). The map \(R_{ij}(\alpha, \beta) : (\xi_i, \xi_j) \rightarrow (\xi_i^{(l)}, \xi_j^{(l)})\) determines \(\xi_i^{(l)}\) and \(\xi_j^{(l)}\), which are subsequently mapped by \(U_{ik}(\alpha, \gamma)\) and \(U_{jk}(\beta, \gamma)\), respectively. Now from the definitions (46) and (50), the variables \(\xi_i^{(l)}\) and \(\xi_j^{(l)}\) that come from the map \(R_{ij}(\alpha, \beta)\) are defined differently from the corresponding variables that are subsequently mapped by \(U_{ik}(\alpha, \gamma)\) and \(U_{jk}(\beta, \gamma)\). However, the act of composition of the respective maps requires that the two different definitions are equated. Then equating the two different definitions of the second components of \(\xi_i^{(l)}\) and \(\xi_j^{(l)}\) respectively gives the following two equations

\[
\exp\left(-\frac{\partial}{\partial x_i} R_{uv}(x_{i}, x_{j}, x_{i}', x_{j}')\right) = \exp\left(\frac{\partial}{\partial x_i} U_{uw}(x_{i}^{\dagger}, x_{k}, x_{i}', x_{k}')\right),
\]

\[
\exp\left(-\frac{\partial}{\partial x_j} R_{uv}(x_{i}, x_{j}, x_{i}', x_{j}')\right) = \exp\left(\frac{\partial}{\partial x_j} U_{uw}(x_{i}^{\dagger}, x_{k}, x_{i}', x_{k}')\right).
\]

Similarly, the map \(U_{ik}(\alpha, \gamma) : (\xi_i^{(l)}, \xi_k^{(l)}) \rightarrow (\xi_i^{(l)}, \xi_k^{(l)})\) determines \(\xi_k^{(l)}\), which is subsequently mapped by \(U_{jk}(\beta, \gamma)\). By the definition of the second component of \(\xi_k^{(l)}\), the composition of the two maps requires that the following equation is satisfied

\[
\exp\left(-\frac{\partial}{\partial x_k} U_{uw}(x_{i}^{\dagger}, x_{k}, x_{i}', x_{k}')\right) = \exp\left(\frac{\partial}{\partial x_k} U_{vw}(x_{j}^{\dagger}, x_{k}, x_{j}', x_{k}')\right).
\]

The three equations (64)–(65) may be seen to be equivalent (mod 2\(\pi\)) to three of the cuboctahedron equations in (23) after using the identification (62).

In an analogous way, the composition of maps (60) for the right hand side of the FYBE (38) can be shown to imply the following three equations

\[
\exp\left(-\frac{\partial}{\partial x_i} U_{uw}(x_{i}, x_{k}', x_{i}', x_{k}')\right) = \exp\left(\frac{\partial}{\partial x_i} R_{uv}(x_{i}^{\dagger}, x_{j}', x_{i}', x_{j}')\right),
\]

\[
\exp\left(-\frac{\partial}{\partial x_j} U_{vw}(x_{j}, x_{k}', x_{j}', x_{k}')\right) = \exp\left(\frac{\partial}{\partial x_j} R_{uv}(x_{i}^{\dagger}, x_{j}', x_{i}', x_{j}')\right),
\]

\[
\exp\left(-\frac{\partial}{\partial x_k} U_{vw}(x_{j}, x_{k}', x_{j}', x_{k}')\right) = \exp\left(\frac{\partial}{\partial x_k} U_{uw}(x_{i}, x_{k}', x_{i}', x_{k}')\right).
\]

which are equivalent (mod 2\(\pi\)) to the remaining three cuboctahedron equations in (23). Then since the twelve cuboctahedron equations (23) and (25) are satisfied in the forms (61)–(66), it
follows that the classical R-matrices (26) and (28) are solutions to the CYBE (22) up to the additional terms of the form (24).

For the second part of the proof, it will be shown that the type-(2a) solution of the CYBE (22)–(23) implies that the FYBE (38) is satisfied with the maps (48) and (51).

As was shown above, the composition of maps (59) for the left hand side of the FYBE (38) implies that three of the cuboctahedron equations in (23) are satisfied in the form

\[
\begin{align*}
\exp \left( -\frac{\partial}{\partial x_i} R_{uv}(x_i, x_j, x_i^+, x_j^+) \right) &= \exp \left( \frac{\partial}{\partial x_i} U_{uw}(x_i^+, x_k, x_i^l, x_k^l) \right), \\
\exp \left( -\frac{\partial}{\partial x_j} R_{uv}(x_i, x_j, x_i^+, x_j^+) \right) &= \exp \left( \frac{\partial}{\partial x_j} U_{vw}(x_j^+, x_k, x_j^l, x_k^l) \right), \\
\exp \left( -\frac{\partial}{\partial x_k} U_{uw}(x_i^+, x_k, x_i^l, x_k^l) \right) &= \exp \left( \frac{\partial}{\partial x_k} U_{vw}(x_j^+, x_k, x_j^l, x_k^l) \right),
\end{align*}
\]

(67)

if the following variables are identified

\[
(x_i^+, x_j^+, x_k^+, x_i^l, x_j^l, x_k^l) = (\hat{x}_i, \hat{x}_j, \hat{x}_k, x_i', x_j', x_k').
\]

(68)

Note that this is true regardless of whether the FYBE (38) is satisfied or not, as it is simply a consequence of the definition and the action of composition of the maps. Assume that the classical R-matrices involved in these cuboctahedron equations are a solution to the CYBE (22)–(23). Then the remaining nine cuboctahedron equations in (23) and (25) that are implied by this CYBE may be used to show that the FYBE (38) is satisfied.

The first map \( U_{jk}(\beta, \gamma) : (\xi_j, \xi_k) \mapsto (\xi_j^*, \xi_k^*) \) in (60) for the RHS of the FYBE (38) determines \( \xi_j^* \) and \( \xi_k^* \). By the definition (50), the first components of these variables are respectively determined from two variables \( x_j^* \) and \( x_k^* \) which satisfy

\[
\begin{align*}
z_k &= \exp \left( \frac{\partial}{\partial x_k} U_{vw}(x_j, x_k, x_j^*, x_k^*) \right), \\
z_j &= \exp \left( \frac{\partial}{\partial x_j} U_{vw}(x_j, x_k, x_j^*, x_k^*) \right).
\end{align*}
\]

(69)

The \( z_k \) and \( z_j \) are respectively the second components of \( \xi_k \) and \( \xi_j \). Note that \( z_k \) is independent of \( x_k^* \) and that \( z_j \) is independent of \( x_j^* \). On the other hand, the definitions of \( z_k \) and \( z_j \) in (59) for the LHS of the FYBE (38) are different, and are given by

\[
\begin{align*}
z_k &= \exp \left( \frac{\partial}{\partial x_k} U_{uw}(x_i^+, x_k, x_i^l, x_k^l) \right), \\
z_j &= \exp \left( \frac{\partial}{\partial x_j} R_{uv}(x_i, x_j, x_i^+, x_j^+) \right).
\end{align*}
\]

(70)

Then after equating (69) and (70), and recalling (68), it is seen that \( x_j^* \) and \( x_k^* \) are respectively solutions (mod \( 2\pi i \)) to the third and second cuboctahedron equations in (25). Thus the variables \( \hat{x}_j \) and \( \hat{x}_k \) of the CYBE may be identified with the variables \( x_j^* \) and \( x_k^* \) in (60).

Next, the map \( U_{ik}(\alpha, \gamma) : (\xi_i, \xi_k) \mapsto (\xi_i^*, \xi_k^{(r)}) \) in (60) for the RHS of the FYBE (38) determines \( \xi_i^* \) and \( \xi_k^{(r)} \). By the definition (50), the first components of these variables are respectively determined from two variables \( x_i^* \) and \( x_k^{(r)} \) which satisfy

\[
\begin{align*}
z_k^* &= \exp \left( \frac{\partial}{\partial x_k^*} U_{uw}(x_i, x_k^*, x_i^*, x_k^{(r)}) \right), \\
z_i &= \exp \left( \frac{\partial}{\partial x_i} U_{uw}(x_i, x_k^*, x_i^*, x_k^{(r)}) \right).
\end{align*}
\]

(71)
The $z_k^*$ and $z_i$ are respectively the second components of $\xi_k^*$ and $\xi_i$. Note that $z_k^*$ is independent of $x_k^{(r)}$ and that $z_i$ is independent of $x_i^{(r)}$. On the other hand, the map $U_{jk}(\beta, \gamma)$ in (60) for the RHS of the FYBE (38), and the map $R_{ij}(\alpha, \beta)$ in (59) for the LHS of the FYBE (38), provide different expressions for $z_k^*$ and $z_i$ respectively, which are given by

$$z_k^* = \exp\left(-\frac{\partial}{\partial x_k^{*\gamma}} U_{vw}(x_j, x_k, x_j^{*\gamma}, x_k^{*\gamma})\right), \quad z_i = \exp\left(-\frac{\partial}{\partial x_i} R_{uv}(x_i, x_j, x_i^{(r)}, x_j^{(r)})\right).$$

(72)

Then after equating (71) and (72), and using $\tilde{x}_j = x_j^*$ and $\tilde{x}_k = x_k^*$ (as shown in the previous step), it is seen that $x_i^*$ and $x_k^{(r)}$ are respectively solutions (mod $2\pi i$) to the sixth cuboctahedron equation of (23) and the first cuboctahedron equation of (25) (recalling also (68)). Thus the variables $\tilde{x}_i$ and $x_i^{(l)} = x_k^*$ of the CYBE may be identified with the variables $x_i^*$ and $x_k^{(r)}$ in (60).

Finally, the map $R_{ij}(\alpha, \beta): (\xi_i^*, \xi_j^*) \mapsto (\xi_i^{(r)}, \xi_j^{(r)})$ in (60) for the RHS of the FYBE (38) determines $\xi_i^{(r)}$ and $\xi_j^{(r)}$. By the definition (46) and (50), the first components of these variables are respectively determined from two variables $x_i^{(r)}$ and $x_j^{(r)}$ which satisfy

$$z_j^* = \exp\left(-\frac{\partial}{\partial x_j^{*\gamma}} U_{vw}(x_i, x_j, x_j^{*\gamma}, x_k^{*\gamma})\right), \quad z_i^* = \exp\left(-\frac{\partial}{\partial x_i} U_{vw}(x_i, x_j, x_i^{(r)}, x_j^{(r)})\right).$$

(73)

The $z_j^*$ and $z_i^*$ are respectively the second components of $\xi_j^*$ and $\xi_i^*$. Note that $z_j^*$ is independent of $x_j^{(r)}$ and that $x_i^*$ is independent of $x_i^{(r)}$. On the other hand, the latter variables were first mapped by $U_{jk}(\beta, \gamma)$ and $U_{ik}(\alpha, \gamma)$, where they were defined through the different equations

$$z_j^* = \exp\left(-\frac{\partial}{\partial x_j^{*\gamma}} U_{vw}(x_j, x_k, x_j^{*\gamma}, x_k^{*\gamma})\right), \quad z_i^* = \exp\left(-\frac{\partial}{\partial x_i} U_{vw}(x_i, x_j, x_i^{*\gamma}, x_k^{*\gamma})\right).$$

(74)

Then after equating (73) and (74), and using $\tilde{x}_j = x_j^*$, $\tilde{x}_k = x_k^*$, $\tilde{x}_i = x_i^*$, and $x_i^{(l)} = x_k^{(r)}$ (as have been shown in the previous two steps), it is seen that $x_i^{(r)}$ and $x_j^{(r)}$ are respectively solutions (mod $2\pi i$) to the fifth and fourth cuboctahedron equations of (23). Thus the variables $x_i^{(l)} = x_i^{(r)}$, and $x_j^{(l)} = x_j^{(r)}$ of the CYBE may be identified with the variables $x_i^{(r)}$ and $x_j^{(r)}$ in (60).

Thus it has been shown that for the FYBE (38) with the maps (48) and (51), the first components of the variables $\xi_i^{(l)}, \xi_j^{(l)}, \xi_k^{(l)}$, are equal to the first components of the variables $\xi_i^{(r)}, \xi_j^{(r)}, \xi_k^{(r)}$.

The final step is to show that the second components of these variables are also equal. As was determined above, the first components of the variables involved on both sides of the FYBE (38) coincide (up to the change of variables (44) and (45)) with the variables of the CYBE (22) and nine of the cuboctahedron equations in (23) and (25). Using the definitions (46), (50), and (41), (42), the remaining three cuboctahedron equations in (25) are seen to be equivalent to the following three equations

$$z_i^{(l)} = z_i^{(r)} , \quad z_j^{(l)} = z_j^{(r)} , \quad z_k^{(l)} = z_k^{(r)} ,$$

(75)

where the latter variables may be identified with the second components of $\xi_i^{(l)}, \xi_i^{(r)}, \xi_j^{(l)}, \xi_j^{(r)}, \xi_k^{(l)}, \xi_k^{(r)}$, respectively.
Therefore $\xi_i^{(l)} = \xi_i^{(r)}$, $\xi_j^{(l)} = \xi_j^{(r)}$, $\xi_k^{(l)} = \xi_k^{(r)}$, and the maps (48) and (51) satisfy the FYBE (38).

There are some simple transformations of the Yang-Baxter maps (48) and (51) under which the FYBE’s (37) and (38) remain satisfied. First, the FYBE’s will still hold if the opposite signs are chosen for each of the expressions appearing on the right hand sides of (41) and (42). According to (46) and (50), this corresponds to setting $z_i \to (z_i)^{-1}$, $z_j \to (z_j)^{-1}$, $z'_i \to (z'_i)^{-1}$, $z'_j \to (z'_j)^{-1}$, in the expressions for the Yang-Baxter maps (48) and (51), respectively (or $z_i \to -z_i$, $z_j \to -z_j$, $z'_i \to -z'_i$, $z'_j \to -z'_j$, for degenerate additive cases). The computations for the proof of Theorem 3.5 are invariant under this change in sign.

The second transformation is less trivial. Let $\phi_{u,v}(x)$ be a function satisfying

$$\phi_{u,v}(x) + \phi_{-u,-v}(x) = 0. \quad (76)$$

Theorem 3.5 will still hold under the following gauge transformations for (41)

$$R_{uv,i}(x'_j, x_j \mid x_i) \to R_{uv,i}(x'_j, x_j \mid x_i) + \phi_{u_1,u_2}(x_i),$$
$$R_{uv,i'}(x_j, x'_j \mid x_i) \to R_{uv,i'}(x_j, x'_j \mid x_i) + \phi_{u_2,u_1}(x'_i),$$
$$R_{uv,j}(x'_i, x_i \mid x_j) \to R_{uv,j}(x'_i, x_i \mid x_j) + \phi_{v_1,v_2}(x_j),$$
$$R_{uv,j'}(x_i, x'_i \mid x'_j) \to R_{uv,j'}(x_i, x'_i \mid x'_j) + \phi_{v_2,v_1}(x'_j), \quad (77)$$

with identical transformations applied to $U_{uv,i}(x'_j, x_j \mid x_i)$, $U_{uv,i'}(x_j, x'_j \mid x'_i)$, $U_{uv,j}(x'_i, x_i \mid x_j)$, $U_{uv,j'}(x_i, x'_i \mid x'_j)$, respectively.

According to (46) and (50), this is equivalent to a change in the variables for the FYBE’s (37) and (38) of the form

$$z_i \to z_i \Phi_{\alpha_1,\alpha_2}(y_i), \quad z'_i \to z'_i \Phi_{\alpha_2,\alpha_1}(y'_i),$$
$$z_j \to z_j \Phi_{\beta_1,\beta_2}(y_j), \quad z'_j \to z'_j \Phi_{\beta_2,\beta_1}(y'_j), \quad (78)$$

where $\Phi_{\alpha,\beta}(x)$ satisfies

$$\Phi_{\alpha,\beta}(x) \Phi_{-\alpha,-\beta}(x) = 1. \quad (79)$$

Note that for the degenerate algebraic (additive) cases the changes would instead be of the form

$$z_i \to z_i + \phi_{\alpha_1,\alpha_2}(y_i), \quad z'_i \to z'_i + \phi_{\alpha_2,\alpha_1}(y'_i),$$
$$z_j \to z_j + \phi_{\beta_1,\beta_2}(y_j), \quad z'_j \to z'_j + \phi_{\beta_2,\beta_1}(y'_j). \quad (80)$$

### 4 New Yang-Baxter maps

Using different solutions of the classical star-triangle relations associated to hypergeometric integrals and discrete integrable quad equations [13,37], the construction described in the previous section will be used to derive a new family of two-component Yang-Baxter maps.

Throughout this section the ± symbol will be used to indicate a sum of two terms involving + and − respectively, e.g.,

$$f(x \pm y) = f(x + y) + f(x - y),$$
$$f(\pm x \pm y) = f(x + y) + f(x - y) + f(-x + y) + f(-x - y). \quad (81)$$

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4.1 Solutions to the classical star-triangle relations

The solutions to the classical star-triangle relations in the elliptic and hyperbolic cases will be given in terms of the dilogarithm function, defined for \( z \in \mathbb{C} - [1, \infty) \) by

\[
\text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt.
\] (82)

The solutions to the classical star-triangle relations in the rational and algebraic cases will be given in terms of a function denoted here as \( \gamma(z) \), which is defined in terms of the complex logarithm using the principal branch as

\[
\gamma(z) = iz\text{Log}(iz), \quad |\text{Arg}(iz)| < \pi.
\] (83)

For the elliptic case the Weierstrass functions will also be used, where \( \sigma(z) \) denotes the Weierstrass sigma function and \( \zeta(z) \) denotes the Weierstrass zeta function, which both depend on the elliptic invariants \( g_2, g_3 \) or associated half-periods \( \omega_1, \omega_2 \) [61]. These functions are related to each other and to the Weierstrass elliptic function \( \wp(z) \), by

\[
\frac{\partial}{\partial z} \text{Log}\sigma(z) = \zeta(z), \quad \frac{\partial}{\partial z} \zeta(z) = -\wp(z).
\] (84)

4.1.1 Elliptic case

The elliptic cases may be derived from quantum star-triangle relations that are equivalent to elliptic hypergeometric integrals [54, 55], including the master solution of the star-triangle relation [15] and its continuous/discrete spin generalisation [35, 38, 63]. In these cases the solutions to the classical star-triangle relations arise from the leading order quasi-classical asymptotics of the elliptic gamma function [51], and may be written in terms of

\[
\mathcal{L}_\alpha^{(1)}(x_i, x_j) = -\frac{\alpha((2x - \frac{x_i}{2})^2 + (2y - \frac{x_j}{2})^2)}{\pi\tau} - \phi(2i\alpha; 2\tau) + \phi(\pm x_i \pm x_j - \alpha; \tau),
\]

\[
\mathcal{T}_\alpha^{(1)}(x_i, x_j) = \mathcal{L}_{-\frac{x_i}{2}-\frac{x_j}{2}} - \mathcal{L}_\alpha^{(1)}(x_i, x_j),
\]

where \( \tau \) is a complex parameter satisfying \( \text{Im}(\tau) > 0 \), and

\[
\phi(z) = \frac{1}{2} \sum_{j=0}^{\infty} \left( \text{Li}_2(-e^{2iz}e^{i\pi(2j+1)}) - \text{Li}_2(-e^{-2iz}e^{i\pi(2j+1)}) \right).
\] (86)

**Proposition 4.1.** Let the functions in (8) be given in terms of (85) as

\[
\mathcal{L}_\alpha(x_i, x_j) = \hat{\mathcal{L}}_\alpha(x_i, x_j) = \Lambda_\alpha(x_i, x_j) = \mathcal{L}_\alpha^{(1)}(x_i, x_j),
\]

\[
\mathcal{T}_\alpha(x_i, x_j) = \hat{\mathcal{T}}_\alpha(x_i, x_j) = \Lambda_\alpha(x_i, x_j) = \mathcal{T}_\alpha^{(1)}(x_i, x_j).
\] (87)

Then (8) is a solution to the classical star-triangle relations (11)–(14) up to terms of the form (15).

**Proof.** The classical star-triangle relations (11) and (12) defined with the combination of functions given in (87) are equivalent, and thus only one of these relations needs to be shown. Let
\(A_1^{\text{STR}}(x_a, x_b, x_c, x_d; u, v, w)\) be the left hand side of the classical star-triangle relation (11) defined with (87). If the solutions to the exponential of the equation (13) also satisfy the following three equations

\[
\exp\left(\frac{i}{\partial x_I} A_1^{\text{STR}}(x_a, x_b, x_c, x_d; u, v, w)\right) = 1, \quad I \in \{a, b, c\},
\]

then the classical star-triangle relation (11) will be satisfied up to the terms of the form (15). The respective derivatives are explicitly given by

\[
\begin{align*}
\exp\left(\frac{i}{\partial x_d} A_1^{\text{STR}}(x_a, x_b, x_c, x_d; u, v, w)\right) &= \frac{\psi(x_d, x_a, \alpha_1)\psi(x_d, x_c, \alpha_3)}{\psi(x_d, x_b - \frac{\pi \tau}{2}, \alpha_1 + \alpha_3)}, \\
\exp\left(\frac{i}{\partial x_a} A_1^{\text{STR}}(x_a, x_b, x_c, x_d; u, v, w)\right) &= \frac{\psi(x_a, x_d, \alpha_1)\psi(x_a, x_b - \frac{\pi \tau}{2}, \alpha_3)}{\psi(x_b - \frac{\pi \tau}{2}, x_a, \alpha_1 + \alpha_3)}, \\
\exp\left(\frac{i}{\partial x_b} A_1^{\text{STR}}(x_a, x_b, x_c, x_d; u, v, w)\right) &= \frac{\psi(x_b - \frac{\pi \tau}{2}, x_c, \alpha_1)\psi(x_b - \frac{\pi \tau}{2}, x_a, \alpha_3)}{\psi(x_c, x_a, \alpha_1 + \alpha_3)}, \\
\exp\left(\frac{i}{\partial x_c} A_1^{\text{STR}}(x_a, x_b, x_c, x_d; u, v, w)\right) &= \frac{\psi(x_c, x_d, \alpha_3)\psi(x_c, x_b - \frac{\pi \tau}{2}, \alpha_1)}{\psi(x_c, x_a, \alpha_1 + \alpha_3)},
\end{align*}
\]

where \(\alpha_1 = v - w, \alpha_3 = u - v, \) and \(\psi(x_i, x_j, \alpha)\) is defined in terms of \(\sigma(z)\) by

\[
\psi(x_i, x_j, \alpha) = \frac{\sigma\left(\frac{2\omega_1(x_i + x_j + \alpha)}{\pi}\right)\sigma\left(\frac{2\omega_1(x_i - x_j + \alpha)}{\pi}\right)}{\sigma\left(\frac{2\omega_1(x_i + x_j - \alpha)}{\pi}\right)\sigma\left(\frac{2\omega_1(x_i - x_j - \alpha)}{\pi}\right)}.
\]

The Weierstrass invariants or half-periods involved in (88) and (89) are defined through the parameter \(\tau\) through the relation \(\tau = \frac{2\omega_1}{iw}\).

With (89), the exponential of (13) and the three equations (88) may be recognised as four equivalent versions of the three-leg forms for the elliptic discrete integrable quad equation known as \(Q4\) [2,3]. More specifically, under the following change of variables

\[
X_a = \wp\left(\frac{2\omega_1 x_a}{\pi}\right), \quad X_c = \wp\left(\frac{2\omega_1 x_c}{\pi}\right), \quad X_b = \wp\left(\frac{2\omega_1 x_b}{\pi} - \omega_2\right), \quad A_1 = \wp\left(\frac{2\omega_1 a_1}{\pi}\right), \quad A_3 = \wp\left(\frac{2\omega_1 a_3}{\pi}\right),
\]

the exponential of (13) and the three equations (88) are each equivalently written as the following multilinear polynomial equation

\[
\begin{align*}
\kappa_0 X_a X_b X_c X_d + \kappa_1 (X_a X_b X_c + X_a X_b X_d + X_a X_c X_d + X_b X_c X_d) + \kappa_2 (X_a X_c + X_b X_d) \\
+ \kappa_3 (X_a X_d + X_b X_c) + \kappa_4 (X_a X_b + X_c X_d) + \kappa_5 (X_a + X_b + X_c + X_d) + \kappa_6 = 0,
\end{align*}
\]

where the coefficients are given by

\[
\begin{align*}
\kappa_0 &= A_1' - A_3', \quad \kappa_1 = A_1 A_2' - A_3 A_1', \quad \kappa_2 = A_3^2 A_1' - A_2^2 A_3', \\
\kappa_3 &= \frac{A_1' A_2' (A_1' - A_3')}{2(A_3 - A_1)} + A_2^2 A_1' + \left(2A_1^2 - \frac{g_2}{4}\right) A_3', \\
\kappa_4 &= \frac{A_1' A_3' (A_1' - A_3')}{2(A_3 - A_1)} - A_3^2 A_1' - \left(2A_3^2 - \frac{g_2}{4}\right) A_1', \\
\kappa_5 &= \frac{g_3 \kappa_0}{2} - \frac{g_2 \kappa_1}{4}, \quad \kappa_6 = \frac{g_2^2 \kappa_0}{16} - g_3 \kappa_1,
\end{align*}
\]

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and $A'_1$ and $A'_3$ denote derivatives of the corresponding change of variables in (104) given by

$$A_1 = \varphi' \left( \frac{2\omega \alpha_1}{\omega} \right), \quad A_3 = \varphi' \left( \frac{2\omega \alpha_3}{\omega} \right). \quad (94)$$

This implies that the classical star-triangle relation (11) holds up to a term of the form (15) on solutions of the $Q4$ integrable quad equation in three-leg forms (88) and (13). Since there are no overall minus signs in (89), $\varepsilon_I = 0$ for each $I \in \{a, b, c, d\}$ in the additional terms (15).

**Remark 4.2.** The functions (85) were originally derived from the master solution of the star-triangle relation given by Bazhanov and Sergeev [15], where they used an integral of the logarithm of the Jacobi theta function instead of the infinite sum of dilogarithms in (86). Using modular properties of the Jacobi theta functions, they showed that the equations (13) and (88) can be written in terms of an alternate form of $Q4$ that uses Jacobi functions in place of the Weierstrass functions in (91).

### 4.1.2 Hyperbolic cases

The hyperbolic cases are derived from the quantum star-triangle relations that are equivalent to hyperbolic hypergeometric integrals. Examples includes the Faddeev-Volkov model [14] and various generalisations [27,56]. Solutions to the classical star-triangle relations in these cases arise from quasi-classical asymptotics of the hyperbolic gamma function (equivalently non-compact quantum dilogarithm), which may be written in terms of the following combinations of the dilogarithm (82)

$$L_\alpha^{(1)}(x_i, x_j) = \text{Li}_2(-e^{x_i \pm x_j \pm i\alpha}) - 2\text{Li}_2(-e^{i\alpha}) + x_i^2 + x_j^2 - \frac{\alpha^2}{2} + \frac{x^2}{6},$$

$$L_\alpha^{(2)}(x_i, x_j) = \text{Li}_2(-e^{\pm(x_i - x_j) + i\alpha}) - 2\text{Li}_2(-e^{i\alpha}) + \frac{(x_i - x_j)^2}{2},$$

$$L_\alpha^{(3)}(x_i, x_j) = \text{Li}_2(-e^{x_i \pm x_j \pm i\alpha}) + \frac{(x_i \pm i\alpha)^2 + x^2}{2},$$

$$L_\alpha^{(4)}(x_i, x_j) = \text{Li}_2(-e^{x_i \pm x_j \pm i\alpha}) + x_i^2 + x_j^2 - \frac{\alpha^2}{2} + i\alpha(x_i + x_j) + \frac{\pi^2}{6},$$

$$L_\alpha^{(5)}(x_i, x_j) = -x_i x_j,$$

**Proposition 4.3.** Let the functions in (8) be given in terms of (95) according to one of the five rows of Table 1. Then (8) is a solution to the classical star-triangle relations (11)–(14) up to terms of the form (15).

The proof is basically the same as for Proposition 4.1 and can be shown case-by-case. Following from the results of [13,37], for each of the five solutions of the classical star-triangle relations given in Table 1, from top to bottom row the equations (13) and (14) both correspond to the integrable ABS quad equations known as $Q3_{(\delta=1)}$, $Q3_{(\delta=0)}$, $H3_{(\delta=1;\varepsilon=1)}$, $H3_{(\delta=0;\varepsilon=1-\delta)}$, and $H3_{(\delta=0;\varepsilon=0)}$.

The solutions of the classical star-triangle relations listed in Table 1 are denoted as one of types I, II, or III. Type-I solutions are given in terms of up to two different functions which both satisfy (16) and (17) (thus (87) would also be a type-I solution of the CSTR). For type-II solutions the functions $\Lambda_\alpha$ and $\overline{\Lambda}_\alpha$ satisfy (16) but do not satisfy (17). For type-III solutions the functions $\Lambda_\alpha$ and $\overline{\Lambda}_\alpha$ do not satisfy (16) or (17). For type-I and type-II solutions the equations (11) and (12) are equivalent, while for type-III solutions they are different.
Let the functions in Table 2: Assignment of functions to (8) by the Barnes Lemmas \[ \text{hypergeometric integrals. Probably the most well-known examples of such integrals are given} \]

The rational cases are derived from quantum star-triangle relations that are equivalent to rational hypergeometric integrals. Probably the most well-known examples of such integrals are given by the Barnes Lemmas \[ \text{hyperbolic Askey-Wilson [52, 57] and Saalschütz integrals, Hyperbolic Barnes's first lemma [37], and Hyperbolic Barnes's 2F1 integral [37].} \]

### 4.1.3 Rational cases

The rational cases are derived from quantum star-triangle relations that are equivalent to rational hypergeometric integrals. Probably the most well-known examples of such integrals are given by the Barnes Lemmas \[ \text{hyperbolic Askey-Wilson [52, 57] and Saalschütz integrals, Hyperbolic Barnes's first lemma [37], and Hyperbolic Barnes's 2F1 integral [37].} \]

The functions appearing in Table 1 have been derived \[13, 37\] from the quasi-classical asymptotics of hyperbolic hypergeometric integrals, which may be interpreted as quantum counterparts of the classical star-triangle relations. Namely, from top to bottom row the functions arise from asymptotics of the hyperbolic beta function \[57\], hyperbolic Saalschütz integral \[59\], hyperbolic Askey-Wilson \[52, 57\] and Saalschütz integrals, Hyperbolic Barnes’s first lemma \[37\], and Hyperbolic Barnes’s 2F1 integral \[37\].

### Proposition 4.4

Let the functions in (8) be given in terms of (96) according to one of the four rows of Table 2. Then (8) is a solution to the classical star-triangle relations (11)–(14) up to terms of the form (15).
The functions appearing in Table 2 have been derived from the quasi-classical asymptotics of quantum star-triangle relations related to rational hypergeometric integrals \([13, 37]\). Namely, from top to bottom row the functions arise from asymptotics of the rational (Askey) beta function \([5]\), Barnes’s second lemma \([8]\), de Branges-Wilson integral \([20, 62]\) and Barnes’s second lemma, and Barnes’s first lemma \([7]\).

Also following from the results of \([13, 37]\), for each of the four solutions of the classical star-triangle relations given in Table 2, the equations (13) and (14) are equivalent up to a point transformation of variables to an integrable quad equation in the ABS list \([1, 3]\). Namely, from top to bottom row the equations (13) and (14) both correspond to \(Q^2, Q_{1(\delta=1)}, H^2_{2(\varepsilon=1)}\) and \(H^2_{2(\varepsilon=0)}\) \([13, 37]\).

4.1.4 Algebraic cases

The algebraic cases are derived from quantum star-triangle relations that are equivalent to classical hypergeometric integrals. Probably the most well-known examples of such integrals are the Euler beta function and the integral representation of Gauss’s hypergeometric function. Solutions of the classical star-triangle relations in these cases don’t arise from the asymptotics of a specific special function, and this makes defining the quasi-classical expansion less systematic than in the previous cases, as may be seen from the cases derived in \([37]\). The resulting functions that solve the classical star-triangle relations (11) and (12) may be written in terms of the following

\[
\mathcal{L}_\alpha^{(1)}(x_i, x_j) = \lambda(\pm (x_i - x_j) + i\alpha), \quad \mathcal{T}_\alpha^{(1)}(x_i, x_j) = \mathcal{L}_\alpha^{(1)}(x_i, x_j) + \lambda(2i\alpha),
\]

\[
\mathcal{L}_\alpha^{(2)}(x_i, x_j) = -2\alpha \log(x - y), \quad \mathcal{T}_\alpha^{(2)}(x_i, x_j) = \mathcal{L}_\alpha^{(2)}(x_i, x_j) + \lambda(2i\alpha),
\]

\[
\Lambda_\alpha^{(1)}(x_i, x_j) = (ix_j - \alpha) \log(x_i), \quad \Lambda_\alpha^{(2)}(x_i, x_j) = xizi + \alpha.
\]

Note that \(\mathcal{L}_\alpha^{(1)}\) and \(\mathcal{T}_\alpha^{(1)}\) in (97) are respectively equivalent to \(\mathcal{L}_\alpha^{(2)}\) and \(\mathcal{T}_\alpha^{(2)}\) in (96).

**Proposition 4.5.** Let the functions in (8) be given in terms of (97) according to one of the three rows of Table 3. Then (8) is a solution to the classical star-triangle relations (11)–(14) up to terms of the form (15).

| \(\mathcal{L}_\alpha(x_i, x_j)\) | \(\mathcal{T}_\alpha(x_i, x_j)\) | \(\hat{\mathcal{L}}_\alpha(x_i, x_j)\) | \(\hat{\mathcal{T}}_\alpha(x_i, x_j)\) | \(\Lambda_\alpha(x_i, x_j)\) | \(\hat{\Lambda}_\alpha(x_i, x_j)\) |
|------------------|------------------|------------------|------------------|------------------|------------------|
| I \(\mathcal{L}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{L}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{L}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(2)}(x_i, x_j)\) |
| II \(\mathcal{L}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{L}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{L}_\alpha^{(2)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(2)}(x_i, x_j)\) |
| III \(\mathcal{L}_\alpha^{(1)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(1)}(x_i, x_j)\) | \(\mathcal{L}_\alpha^{(1)}(x_i, x_j)\) | \(\mathcal{T}_\alpha^{(1)}(x_i, x_j)\) | \(\Lambda_\alpha^{(1)}(x_i, x_j)\) | \(\Lambda_\alpha^{(1)}(-x_i, -x_j)\) |

Table 3: Assignment of functions to (8) in terms of (97) for three different solutions of the classical star-triangle relations (11)–(14).

The functions appearing in Table 2 have been derived from the quasi-classical asymptotics of quantum star-triangle relations related to classical hypergeometric integrals \([13, 37]\). Namely, from top to bottom row the functions arise from asymptotics of a Selberg-type integral \([50, 53]\), a special case of Barnes’s \(2F_1\) formula\(^1\) \([7, 61]\), and Euler beta function.

\(^1\)This is only for the classical star-triangle relation (11), because the classical star-triangle relation (12) didn’t arise from the analysis of \([37]\) and its hypergeometric/quantum counterpart is presently unknown.
Also following from the results of [13, 37], for each of the three solutions of the classical star-triangle relations given in Table 3, the equations (13) and (14) are equivalent up to a point transformation of variables to an integrable quad equation in the ABS list [1,3]. Namely, from top to bottom row the equations (13) and (14) both correspond to $Q_1(\delta=0)$, $H_1(\varepsilon=1)$, and $H_1(\varepsilon=0)$ [13,37].

4.2 Yang-Baxter maps

In this section the method of Section 3.2 will be used to derive a new set of two-component Yang-Baxter maps associated to each of the solutions of the classical star-triangle relations given in Section 4.1. It turns out that (apart from the elliptic case) only type-II and type-III solutions of the classical star-triangle relations need to be considered, because the type-(1a) and type-(1b) solutions of the CYBE constructed from these cases, are equivalent to the same solutions of the CYBE constructed from type-I solutions of the classical star-triangle relations.

As was stated in the previous section, the different solutions of the classical star-triangle relations have been derived from the quasi-classical asymptotics of quantum star-triangle relations related to hypergeometric integrals and discrete integrable quad equations. The Yang-Baxter maps that are derived here will inherit these same connections because they are based on the solutions of the classical star-triangle relations, through the deformations used for the CYBE in Figure 4. The hypergeometric integral and ABS equation associated to the solutions of the FYBE’s (37) and (38) are listed in Tables 4 and 5 respectively.

| Hypergeometric       | ABS    | $R_{ij}(\alpha,\beta)$ |
|----------------------|--------|------------------------|
| Elliptic Beta        | $Q4$   | (108)                  |
| Hyperbolic Beta      | $Q3(\delta=1)$ | (116)             |
| Hyperbolic Saalschütz| $Q3(\delta=0)$ | (112)          |
| Rational Beta        | $Q2$   | (130)                  |
| Barnes’s second lemma| $Q1(\delta=1)$ | (126)          |
| Selberg Type         | $Q1(\delta=0)$ | (135)        |

Table 4: Hypergeometric integral and ABS equation associated to Yang-Baxter maps $R_{ij}(\alpha,\beta)$ for the FYBE (37).

The Yang-Baxter maps given in this section will accordingly be labelled with

$$\text{Quad equation} \quad \text{(Hypergeometric integral)} \quad \text{[Lattice model]} \quad (98)$$

according to the ABS quad equation that they correspond to and also the hypergeometric integral corresponding to the quantum star-triangle relation as was given in [37]. Furthermore, if the latter quantum star-triangle relation was previously investigated in connection with an integrable lattice model of statistical mechanics, this lattice model is also given. For type-III solutions of the classical star-triangle relations there will be multiple hypergeometric integrals and quad equations, since these cases also cover the Yang-Baxter maps associated to type-I solutions of the classical star-triangle relations (the Yang-Baxter maps listed in Table 4).
In the following, the notation \( \dot{x} \) where \( x \) is a variable or parameter, and \( \wp(x) \) is the Weierstrass elliptic function with elliptic invariants \( g_2, g_3 \) or associated half-periods \( \omega_1, \omega_2 \).

### 4.2.1 Elliptic case

In the following, the notation \( \dot{x} \) and \( \ddot{x} \) will be used to respectively denote

\[
\begin{align*}
\dot{x}^2 &= 4x^3 - g_2x - g_3, \\
\ddot{x} &= \frac{x^2}{4(x - \wp(\omega_2))} - x - \wp(\omega_2),
\end{align*}
\]

where \( x \) is a variable or parameter, and \( \wp(x) \) is the Weierstrass elliptic function with elliptic invariants \( g_2, g_3 \) or associated half-periods \( \omega_1, \omega_2 \).

#### 4.2.1.1 Q4 (Elliptic beta integral) [Bazhanov-Sergeev model]

The classical R-matrix for this case is given by (26), with the functions of (85). The expression for \( R_{ij}(x_j', x_j | x_i) \) in (41) is obtained as a derivative of the latter classical R-matrix with respect to \( x_i \), resulting in

\[
\begin{align*}
R_{ij}(x_j', x_j | x_i) &= -i \pi + \frac{16x_i(u_1 - u_2)\zeta(\omega_1)\omega_1}{\pi^2} + \frac{2i\omega_1(u_1 - u_2)(\pi - 4x_i)}{\pi \omega_2} \\
&\quad + \log \frac{\sigma(2\pi)(x_i - x_j' - u_2 + v_2)}{\sigma(2\pi)(x_i - x_j' + u_2 - v_2)} + \log \frac{\sigma(2\pi)(x_i - x_j + u_1 - v_2 + \omega_2)}{\sigma(2\pi)(x_i - x_j - u_1 + v_2 + \omega_2)} \\
&\quad + \log \frac{\sigma(2\pi)(x_i + x_j' - u_2 + v_2)}{\sigma(2\pi)(x_i + x_j' + u_2 - v_2)} + \log \frac{\sigma(2\pi)(x_i + x_j + u_1 - v_2 - \omega_2)}{\sigma(2\pi)(x_i + x_j - u_1 + v_2 - \omega_2)},
\end{align*}
\]

where \( \sigma(z) \) and \( \zeta(z) \) are the Weierstrass sigma and zeta functions respectively (see (84)). The other expressions in (41) can also be obtained from this case by comparing the respective dependencies on the variables. For example, \( R_{ij}(x_i', x_i | x_j) \) is equivalent to the right hand side of (100) with

\[
\begin{align*}
x_i &\leftrightarrow x_j, \quad x_j' \rightarrow x_i', \quad u_2 \rightarrow -v_1, \quad u_1 \rightarrow -v_2, \quad v_2 \rightarrow -u_1,
\end{align*}
\]

and

\[
R_{ij}(x_j', x_j | x_i') \text{ is obtained from (100) with}
\]

\[
\begin{align*}
x_i &\rightarrow x_i', \quad x_j \leftrightarrow x_j', \quad u_1 \leftrightarrow u_2, \quad v_2 \rightarrow v_1,
\end{align*}
\]
and $R_{uv,j'}(x_i, x'_j | x'_j)$ is obtained from (100) with
\[
x_i \leftrightarrow x'_j, \quad x_j \rightarrow x'_j, \quad u_1 \rightarrow -u_1, \quad u_2 \rightarrow -v_2, \quad v_2 \rightarrow -u_2.
\] (103)

The terms on the first line in (100) can be ignored, since by (101), (102), and (103), the first term will only contribute an overall sign to each Yang-Baxter map, and the second and third terms will only appear in the form of the gauge transformation given in (77).

The change of variables according to (44) is given by
\[
f(x) = \varphi(2\omega_1 x \pi^{-1}), \quad h(x) = \varphi(2\omega_1 x \pi^{-1}).
\] (104)

Then using the change of variables (104) for $R_{uv,i}(x'_j, x_j | x_i)$ in (100) gives an expression for $z_i$ in (46) as
\[
z_i = \left( \frac{\sigma(x_i + u_1)\sigma(x_i - u_2)}{\sigma(x_i - u_1)\sigma(x_i + u_2)} \right)^2 E(y_j, \beta_1, \beta_2, \alpha_1)^2 \frac{G(y_i, y'_j, \alpha_1, \beta_2)G(y_i, y'_j, \beta_1, \alpha_1)}{G(y_i, y'_j, \beta_2, \alpha_1)G(y_i, y'_j, \alpha_1, \beta_1)},
\] (105)

where
\[
A(\alpha, \beta) = \frac{(\hat{\alpha} + \hat{\beta})^2}{4(\alpha - \beta)^2} - \alpha - \beta, \quad B(x, \alpha, \beta) = \hat{\alpha} \beta + \hat{\beta} \alpha - (\hat{\alpha} + \hat{\beta})x,
\]
\[
E(x, \alpha, \beta, \gamma) = \frac{1}{E(x, \beta, \alpha, \gamma)} = \frac{(B(x, \alpha, \gamma) + (\alpha - \gamma)\hat{x})(B(x, \beta, \gamma) - (\beta - \gamma)\hat{x})}{(B(x, \alpha, \gamma) - (\alpha - \gamma)\hat{x})(B(x, \beta, \gamma) + (\beta - \gamma)\hat{x})},
\]
\[
F(x, \alpha, \beta) = 2\hat{x} \frac{B(A(\alpha, \beta), \alpha)}{\alpha - \beta} + 2g_3 + (A(\alpha, \beta) + x)(g_2 - 4xA(\alpha, \beta)),
\]
\[
G(x, y, \alpha, \beta) = F(x, \alpha, \beta) + 4y(A(\alpha, \beta) - x)^2.
\]

The expressions for $z_j$, $z'_j$, and $z''_j$ in (46) take a similar form (for $z'_j$ and $z''_j$ see (108)). Note that from (101), (102), and (103), the factor involving Weierstrass sigma functions in (105) is in the form of the gauge transformation of (78) and thus can effectively be set to 1. Also note that $z_i$ is a ratio of polynomials of degree one in $y'_j$, but degree two in $y_j$. Similarly the expressions for $z_j$, $z'_j$, and $z''_j$, from (46), will be ratios of polynomials of degree 1 and degree 2 in their first and second arguments respectively, instead of both degree 1. Such a quadratic dependence for these variables will only appear for this elliptic case. The consequence of this is that the resulting Yang-Baxter map for the elliptic case is birational rather than quadriational.

Finally, let $\Upsilon_1(z_j, y_j | y_j)$, $\Upsilon_2(z_j, y_j | y_j)$, be the following expressions
\[
\Upsilon_1 = \frac{z_i G(y_i, y_j, \beta_2, \alpha_1)F(y_j, \alpha_1, \beta_1)E(y_j, \beta_1, \beta_2, \alpha_1)G(y_j, y_i, \alpha_1, \beta_2) - G(y_j, y_i, \alpha_1, \beta_2)F(y_j, \beta_1, \alpha_1)}{4(y_j - A(\alpha_1, \beta_1))^2 G(y_j, y_i, \alpha_1, \alpha_2) - z_i G(y_j, y_i, \beta_2, \alpha_1)E(y_j, \beta_1, \beta_2, \alpha_1)G(y_j, y_i, \alpha_1, \beta_2)},
\]
\[
\Upsilon_2 = \frac{z_i G(y_i, y_j, \beta_2, \alpha_1)F(y_j, \alpha_2, \beta_2)E(y_j, \alpha_1, \alpha_2, \beta_2)G(y_j, y_i, \alpha_1, \beta_2) - G(y_j, y_i, \alpha_1, \beta_2)F(y_j, \alpha_2, \alpha_2)}{4(y_i - A(\alpha_2, \beta_2))^2 G(y_i, y_j, \alpha_1, \alpha_2) - z_i G(y_i, y_j, \beta_2, \alpha_1)E(y_i, \alpha_1, \alpha_2, \beta_2)},
\] (107)

Equations (107) are obtained by solving $z_j$ and $z_i$ in (105), for $y'_j$ and $y'_j$ respectively.

Then using the method of Section 3.2, the Yang-Baxter map $R_{ij}(\alpha, \beta)$ is given by
\[
y'_i = \Upsilon_1, \quad z'_i = E(\Upsilon_1, \alpha_2, \alpha_1, \beta_1)^2 \frac{G(\Upsilon_1, y_i, \alpha_1, \beta_1)G(\Upsilon_1, \Upsilon_2, \beta_1, \alpha_2)}{G(\Upsilon_1, y_j, \beta_1, \alpha_1)G(\Upsilon_1, \Upsilon_2, \alpha_2, \beta_1)},
\]
\[
y'_j = \Upsilon_2, \quad z'_j = E(\Upsilon_2, \beta_2, \beta_1, \alpha_2)^2 \frac{G(\Upsilon_2, y_j, \beta_2, \alpha_2)G(\Upsilon_1, \Upsilon_2, \beta_1, \alpha_2)}{G(\Upsilon_2, y_i, \beta_2, \alpha_2)G(\Upsilon_2, \Upsilon_1, \alpha_2, \beta_1)}.
\] (108)

According to Theorem 3.5, this is a solution of the FYBE (37).
4.2.2 Hyperbolic cases

In the following, for any variable or parameter \( x \), the notation \( \tau \) will be used to denote

\[
\tau = x + \sqrt{x^2 - 1}.
\]  

(109)

4.2.2.1 \( Q_{3(\delta)} \) and \( H_{3(\delta=1; \varepsilon=1)} \) (Hyperbolic Beta, Hyperbolic Askey-Wilson, and Hyperbolic Saalschütz integrals) [Faddeev-Volkov-type models]. This case results in two different solutions of the FYBE (38), constructed from the functions involved in the type-III solution of the classical star-triangle relations in Table 1, which also includes the functions involved in the two type-I solutions in Table 1. The first set of classical R-matrices for this case are given by (26) and (28), and the second set of classical R-matrices for this case are given by (27) and (29).

For the case of (26) and (28) the change of variables used in (44) and (45) is given by

\[
f(x) = e^x, \quad g(y) = \cosh(x), \quad h(x) = e^{ix}.
\]  

(110)

Let \( \Upsilon_1(z_j, y_i | y_j) \) and \( \Upsilon_2(z_i, y_j | y_i) \) (these are explicit forms of (47) derived from (26)), and \( \Upsilon_3(z_i, y_j | y_i) \) and \( \Upsilon_4(z_i, y_j | y_i) \) (derived from (28)) be defined by

\[
\Upsilon_1 = y_i \frac{\alpha_1(\alpha_1 y_i + \beta_2 y_j) + \beta_1 z_j(\alpha_1 y_j + \beta_2 y_i)}{\beta_1(\alpha_1 y_i + \beta_2 y_j) + \alpha_1 z_j(\alpha_1 y_j + \beta_2 y_i)}, \quad \Upsilon_3 = \frac{\beta_1 \beta_2 y_j(j - 1) + \alpha_1 y_i(z_j^2 - 1)}{\alpha_1 y_i y_j(j - 1) + \beta_2 (z_j - y_j^2)}
\]

(111)

\[
\Upsilon_2 = y_i \frac{\alpha_2(\alpha_1 y_i + \beta_2 y_j) + \beta_2 z_i(\alpha_1 y_j + \beta_2 y_i)}{\beta_2(\alpha_1 y_i + \beta_2 y_j) + \alpha_2 z_i(\alpha_1 y_j + \beta_2 y_i)}, \quad \Upsilon_4 = y_j z_i + y_i \frac{\alpha_2 + \alpha_1 z_i}{2\beta_2} + \beta_2 \frac{\alpha_1 + \alpha_2 z_i}{2\alpha_1 \alpha_2 y_i}.
\]

Then using the method of Section 3.2, the Yang-Baxter map \( R_{ij}(\alpha, \beta) \) derived from (26) is given by

\[
y_i' = \Upsilon_1, \quad z_i' = \frac{(\alpha_1 y_j - \beta_1 \Upsilon_1)(\alpha_2 \Upsilon_1 + \beta_1 \Upsilon_2)}{(\alpha_1 \Upsilon_1 - \beta_1 y_j)(\alpha_2 \Upsilon_2 + \beta_1 \Upsilon_1)}
\]

(112)

\[
y_j' = \Upsilon_2, \quad z_j' = \frac{(\alpha_2 y_i - \beta_2 \Upsilon_2)(\alpha_2 \Upsilon_2 + \beta_1 \Upsilon_1)}{(\alpha_2 \Upsilon_2 - \beta_2 y_i)(\alpha_2 \Upsilon_1 + \beta_1 \Upsilon_2)}
\]

and the Yang-Baxter map \( U_{ij}(\alpha, \beta) \) derived from (28) is given by

\[
y_i' = \Upsilon_3, \quad z_i' = \frac{-\alpha_1 \beta_1^2 + \alpha_2 \beta_2 y_3^2 + 2\alpha_2 \beta_1 \Upsilon_4 \Upsilon_3}{\alpha_2 \beta_1^2 + \alpha_2^2 \Upsilon_3^2 - 2\alpha_1 \beta_1 y_j y_3}
\]

(113)

\[
y_j' = \Upsilon_4, \quad z_j' = \frac{(\alpha_2 y_i - \beta_2 \Upsilon_4)(\beta_1 + \alpha_2 \Upsilon_3 \Upsilon_4)}{(\alpha_2 \Upsilon_3 + \beta_1 \Upsilon_4)(\alpha_2 y_j \Upsilon_4 - \beta_2)}
\]

According to Theorem 3.5, the Yang-Baxter map (112) is a solution of the FYBE (37), and the Yang-Baxter maps (112) and (113) are a solution of the FYBE (38). The CSTR associated to (112) is the second row of Table 1 for the hyperbolic Saalschütz integral and \( Q_{3(\delta=0)} \), and the CSTR associated to (113) is the third row of Table 1 for the hyperbolic Saalschütz and hyperbolic Askey-Wilson integrals and \( H_{3(\delta=1; \varepsilon=1)} \).

For the case of (27) and (29) the change of variables used in (44) and (45) is given by

\[
f(x) = \cosh(x), \quad g(x) = e^x, \quad h(x) = e^{ix}.
\]  

(114)
Let $\Upsilon_1(z_j, y_i | y_j)$ and $\Upsilon_2(z_i, y_j | y_i)$ (these are explicit forms of (47) derived from (27)), and $\Upsilon_3(z_j, y_i | y_j)$ and $\Upsilon_4(z_i, y_j | y_i)$ (derived from (29)) be defined by

$$
\Upsilon_1 = \frac{\Psi^2_j(1-z_j)(\alpha_1^2 + \frac{\beta_1^2 \beta_2^2}{\alpha_1^2}) + (\Psi^4_j - z_j)\beta_2^2 + (1-z_j\Psi^4_j)\beta_1^2 - 2\alpha_1\beta_2 y_j \Psi_j^2 (z_j - \Psi^4_j + (z_j\Psi^4_j - 1)\frac{\beta_1^2}{\alpha_1^2})}{2\alpha_1\beta_1 \Psi_j^2 (\beta_2^2 (\Psi^4_j - z_j) + \alpha_1^2(1-z_j\Psi^4_j) - 2\alpha_1\beta_2 y_j \Psi_j^2 (z_j - 1))},
$$

$$
\Upsilon_2 = \frac{\Psi^2_i(1-z_i)(\beta_1^2 + \frac{\alpha_1^2 \alpha_2^2}{\beta_1^2}) + (\Psi^4_i - z_i)\alpha_2^2 + (1-z_i\Psi^4_i)\alpha_1^2 - 2\alpha_1\beta_2 \Psi_i \Psi_j (z_i - \Psi^4_i + (z_i\Psi^4_i - 1)\frac{\alpha_2^2}{\beta_1^2})}{2\alpha_1\beta_2 \Psi_i (\beta_1^2 (\Psi^4_i - z_i) + \alpha_1^2(1-z_i\Psi^4_i) - 2\alpha_1\beta_2 \Psi_i \Psi_j (z_i - 1))},
$$

$$
\Upsilon_3 = y_i z_j + \frac{\beta_1 + \beta_2 z_j}{2\alpha_1 y_j} + \alpha_1 y_j \frac{\beta_2 + \beta_1 z_j}{2\beta_1 \beta_2}, \quad \Upsilon_4 = \frac{\beta_2 \beta_2 \Psi_i (z_i - 1) + \alpha_1 y_j (z_i \Psi^4_i - 1)}{\alpha_2 \alpha_1 \Psi_i (z_i - 1) + \beta_2 (z_i - \Psi^4_i)},
$$

(115)

Then using the method of Section 3.2, the Yang-Baxter map $R_{ij}(\alpha, \beta)$ derived from (27) is given by

$$
y'_i = \Upsilon_1, \quad z'_i = \frac{(\alpha_1^2 + \beta_1^2 \Upsilon_1 - 2\alpha_1\beta_1 \Upsilon_1 y_j)(\beta_1^2 + \alpha_2^2 \Upsilon_1^2 + 2\alpha_1^2 \Upsilon_1 \Upsilon_2)}{\beta_1^2 + \alpha_1^2 \Upsilon_1^2 - 2\alpha_1 \beta_1 \Upsilon_1 \Upsilon_2}(\alpha_1^2 \beta_1 \Upsilon_1 \Upsilon_2),
$$

$$
y'_j = \Upsilon_2, \quad z'_j = \frac{(\alpha_2^2 + \beta_2^2 \Upsilon_2 - 2\alpha_2\beta_2 \Upsilon_2 y_i)(\beta_2^2 + \alpha_1^2 \Upsilon_2^2 + 2\alpha_2^2 \Upsilon_2 \Upsilon_1)}{\beta_2^2 + \alpha_2^2 \Upsilon_2^2 - 2\alpha_2 \beta_2 \Upsilon_2 \Upsilon_1}(\beta_2^2 \beta_2 \Upsilon_2 \Upsilon_1),
$$

(116)

and the Yang-Baxter map $U_{ij}(\alpha, \beta)$ derived from (29) is given by

$$
y'_i = \Upsilon_3, \quad z'_i = \frac{(\alpha_1 y_j - \beta_1 \Upsilon_3)(\beta_1 + \alpha_2 \Upsilon_4 \Upsilon_3)}{(\alpha_2 \Upsilon_4 + \beta_1 \Upsilon_3)(\alpha_1 y_j \Upsilon_3 - \beta_1)},
$$

$$
y'_j = \Upsilon_4, \quad z'_j = -\frac{\beta_2 \beta_1^2 + \alpha_2^2 \Upsilon_4^2 + 2\alpha_1 \beta_1 \Upsilon_4 \Upsilon_3}{\beta_1 \beta_2^2 + \alpha_2^2 \Upsilon_4^2 - 2\alpha_2 \beta_2 y_i \Upsilon_4}.
$$

(117)

According to Theorem 3.5, the Yang-Baxter map (116) is a solution of the FYBE (37), and the Yang-Baxter maps (116) and (117) are a solution of the FYBE (38). The CSTR associated to (116) is the first row of Table 1 for the hyperbolic beta integral and $Q_{3 (\delta = 1)}$, and the CSTR associated to (117) is the third row of Table 1 for the hyperbolic Saalschütz and hyperbolic Askey-Wilson integrals and $H_{3 (\delta = 1; \epsilon = 1)}$.

4.2.2.2 $H_{3 (\delta = 0, 1; \epsilon = 1 - \delta)}$ (Hyperbolic Barnes’s first lemma). The classical R-matrices for this case are given by (26) and (28) with the functions of the first type-II solution of the classical star-triangle relation given in Table 1. The Yang-Baxter map for (26) was already derived in (112).

The classical R-matrix (28) for this case is given by

$$
U_{uv}(x_i, x_j; x'_i, x'_j) = i (\pi(x_i + x_j + x'_i + x'_j) + (x_i - x'_i)(u_1 - u_2) + (x_j - x'_j)(v_1 - v_2))
$$

$$
+ \text{Li}_2(-e^{x_i+x_j+i(u_1-u_2)}) - \text{Li}_2(e^{x_i+x'_j+i(u_2-v_2)})
$$

$$
+ \text{Li}_2(-e^{x'_i+x'_j+i(u_2-v_1)}) - \text{Li}_2(e^{x'_i+x_j+i(u_1-v_1)}) + r(u, v).
$$

(118)

where $r(u, v)$ is a constant with respect to $x_i, x_j, x'_i, x'_j$. For this case the change of variables used in (45) is given by

$$
f(x) = e^x, \quad g(x) = e^x, \quad h(x) = e^{ix}.
$$

(119)
Then using the method of Section 3.2, the Yang-Baxter map $U_{ij}(\alpha, \beta)$ derived from (28) is given by

$$
y_i' = y_i z_j + \frac{\beta_1 + \beta_2 z_j}{\alpha_1 y_j}, \quad z_i' = z_i + \frac{(\alpha_1 + \alpha_2 z_i)(\beta_1 + \beta_2 z_j)}{\alpha_1 \alpha_2 y_i y_j z_j},
$$

$$
y_j' = y_j z_i + \frac{\alpha_1 + \alpha_2 z_i}{\alpha_1 \alpha_2 y_i}, \quad z_j' = z_j + \frac{(\alpha_1 + \alpha_2 z_i)(\beta_1 + \beta_2 z_j)}{\alpha_1 \alpha_2 y_i y_j z_i}. \tag{120}
$$

According to Theorem 3.5, this is a solution of the FYBE (38) in combination with the Yang-Baxter map (112).

4.2.2.3 $H3(\delta=0; \varepsilon=0)$ (Hyperbolic Barnes’s $2F_1$ integral). The classical R-matrices for this case are given by (26) and (28) with the functions of the second type-II solution of the classical star-triangle relation given in Table 1. The Yang-Baxter map for (26) was already derived in (112).

The classical R-matrix (28) is given by

$$
U_{uv}(x_i, x_j; x_i', x_j') = (x_i - x_i')(x_j' - x_j). \tag{121}
$$

For this case the change of variables used in (45) is given by

$$
f(x) = e^x, \quad g(x) = e^x, \quad h(x) = e^{ix}. \tag{122}
$$

Then using the method of Section 3.2, the Yang-Baxter map $U_{ij}(\alpha, \beta)$ derived from (28) is given by

$$
y_i' = z_j y_i, \quad z_i' = z_i, \quad y_j' = z_i y_j, \quad z_j' = z_j. \tag{123}
$$

According to Theorem 3.5, this is a solution of the FYBE (38) in combination with the Yang-Baxter map (112).

4.2.3 Rational cases

4.2.3.1 $Q2, Q1(\delta=1),$ and $H2(\varepsilon=1)$ (Askey Rational Beta integral, de Branges-Wilson integral, and Barnes’s second lemma). This case results in two different solutions of the FYBE (38), constructed from the functions involved in the type-III solution of the classical star-triangle relations in Table 2, which also includes the functions involved in the two type-I solutions in Table 2. The first set of classical R-matrices for this case are given by (26) and (28), and the second set of classical R-matrices for this case are given by (27) and (29).

For the case of (26) and (28) the change of variables used in (44) and (45) is given by

$$
f(x) = x, \quad g(x) = x^2, \quad h(x) = ix. \tag{124}
$$

Let Υ₁(z_j, y_i | y_j) and Υ₂(z_i, y_j | y_i) (these are explicit forms of (47) derived from (26)), and
\[ Υ_3(y_i, y_j, z_j) \text{ and } Υ_4(y_i, y_j, z_i) \] (derived from (28)), be defined by

\[
Υ_1 = y_j + (α - β_1) \left( \frac{(z_j + 1)(y_i - y_j) + (1 - z_j)(α_1 - β_2)}{(1 - z_j)(y_i - y_j) + (z_j + 1)(α_1 - β_2)} \right),
\]
\[
Υ_2 = y_i + (α_2 - β_2) \left( \frac{(z_i + 1)(y_i - y_j) + (z_i - 1)(α_1 - β_2)}{(1 - z_i)(y_i - y_j) - (z_i + 1)(α_1 - β_2)} \right),
\]
\[
Υ_3 = (β_1 - α_1) + \left( \frac{(z_j - 1)y_j + (z_j + 1)(y_i + α_1 - β_2)\sqrt{y_j}}{(z_j - 1)(y_i + α_1 - β_2) + (z_j + 1)\sqrt{y_j}} \right),
\]
\[
Υ_4 = (β_2 - α_2) + \left( \frac{(z_i - 1)y_i + (z_i + 1)(y_j + α_1 - β_2)\sqrt{y_i}}{(z_i - 1)(y_j + α_1 - β_2) + (z_i + 1)\sqrt{y_i}} \right).
\]

Then using the method of Section 3.2, the Yang-Baxter map \( R_{ij}(α, β) \) derived from (26) is given by

\[
y_i' = Υ_1, \quad z_i' = \frac{(α - β_1 + (y_j - Υ_1))(α_2 - β_1 + (Υ_1 - Υ_2))}{(α_1 - β_1 - (y_j - Υ_1))(α_2 - β_1 - (Υ_1 - Υ_2))},
\]
\[
y_j' = Υ_2, \quad z_j' = \frac{(α_2 - β_2 + (y_i - Υ_2))(α_2 - β_1 - (Υ_1 - Υ_2))}{(α_2 - β_2 - (y_i - Υ_2))(α_2 - β_1 + (Υ_1 - Υ_2))},
\]

and the Yang-Baxter map \( U_{ij}(α, β) \) derived from (28) is given by

\[
y_i' = Υ_3, \quad z_i' = \frac{Υ_4 - (α_2 - β_1 + Υ_3)^2}{y_j - (α_1 - β_1 + Υ_3)^2},
\]
\[
y_j' = Υ_4, \quad z_j' = \frac{(α_2 - β_2 + y_i + Υ_4)(α_2 - β_1 + Υ_3 + Υ_4)}{(α_2 - β_2 + y_i + Υ_4)(α_2 - β_1 + Υ_3 - Υ_4)}.
\]

According to Theorem 3.5, the Yang-Baxter map (126) is a solution of the FYBE (37), and the Yang-Baxter maps (126) and (127) are a solution of the FYBE (38). The CSTR associated to (126) is the second row of Table 2 for Barnes’s second lemma and \( Q_{1(δ=1)} \), and the CSTR associated to (127) is the third row of Table 2 for the de Branges-Wilson integral and Barnes’s second lemma and \( H_2(ε=1) \).

For the case of (27) and (29) the change of variables used in (44) and (45) is given by

\[
f(x) = x^2, \quad g(x) = x, \quad h(x) = ix.
\]

Let \( Υ_1(y_i, y_j, z_j) \) and \( Υ_2(y_i, y_j, z_i) \) (these are explicit forms of (44) derived from (27)), and \( Υ_3(y_i, y_j, z_j) \) and \( Υ_4(y_i, y_j, z_i) \) (derived from (29)), be defined by

\[
Υ_1 = y_j + (α_1 - β_1)^2 \left( 1 - \frac{2\sqrt{y_j}((y_j - y_i + (α_1 - β_2)^2)(z_j + 1) + 2(α_1 - β_2)(z_j - 1)\sqrt{y_j})}{α_1 - β_1((y_j - y_i + (α_1 - β_2)^2)(z_j - 1) + 2(α_1 - β_2)(z_j + 1)\sqrt{y_j})} \right),
\]
\[
Υ_2 = y_i + (α_2 - β_2)^2 \left( 1 - \frac{2\sqrt{y_i}((y_i - y_j + (α_1 - β_2)^2)(z_i + 1) + 2(α_1 - β_2)(z_i - 1)\sqrt{y_i})}{α_2 - β_2((y_i - y_j + (α_1 - β_2)^2)(z_i - 1) + 2(α_1 - β_2)(z_i + 1)\sqrt{y_i})} \right),
\]
\[
Υ_3 = (α_1 - β_1 + y_j)^2 + (z_j(y_i - (α_1 - β_2 + y_j)^2),
\]
\[
Υ_4 = (β_2 - α_2) + (z_i - 1)y_i + (z_i + 1)(y_j + α_1 - β_2)\sqrt{y_i}.
\]
Then using the method of Section 3.2, the Yang-Baxter map $R_{ij}(\alpha, \beta)$ derived from (27) is given by

$$y_i' = Y_1, \quad z_i' = \frac{(y_j - (\alpha_1 - \beta_1 - \sqrt{Y_1})^2)(Y_2 - (\alpha_2 - \beta_1 + \sqrt{Y_1})^2)}{(y_j - (\alpha_1 - \beta_1 + \sqrt{Y_1})^2)(Y_2 - (\alpha_2 - \beta_1 - \sqrt{Y_1})^2)},$$

$$y_j' = Y_2, \quad z_j' = \frac{(y_i - (\alpha_2 - \beta_2 - \sqrt{Y_2})^2)(Y_1 - (\alpha_2 - \beta_1 + \sqrt{Y_2})^2)}{(y_i - (\alpha_2 - \beta_2 + \sqrt{Y_2})^2)(Y_1 - (\alpha_2 - \beta_1 - \sqrt{Y_2})^2)},$$

and the Yang-Baxter map $U_{ij}(\alpha, \beta)$ derived from (29) is given by

$$y_i' = Y_3, \quad z_i' = \frac{(\alpha_1 - \beta_1 + y_j - \sqrt{Y_3})(\alpha_2 - \beta_1 + Y_4 + \sqrt{Y_3})}{(\alpha_1 - \beta_1 + y_j + \sqrt{Y_3})(\alpha_2 - \beta_1 + Y_4 - \sqrt{Y_3})},$$

$$y_j' = Y_4, \quad z_j' = \frac{Y_3 - (\alpha_2 - \beta_1 + Y_4)^2}{y_i - (\alpha_2 - \beta_2 + Y_4)^2}.$$

According to Theorem 3.5, the Yang-Baxter map (130) is a solution of the FYBE (37), and the Yang-Baxter maps (130) and (131) are a solution of the FYBE (38). The CSTR associated to (130) is the first row of Table 2 for the Askey rational beta integral and $Q_2$, and the CSTR associated to (131) is the third row of Table 2 for the de Branges-Wilson integral and Barnes’s second lemma and $H^2_{(\varepsilon=1)}$.

4.2.3.2 $H^2_{(\varepsilon=0)}$ (Barnes’s first lemma). The classical R-matrices for this case are given by (26) and (28) with the functions of the type-II solution of the classical star-triangle relations given in Table 2. The Yang-Baxter map for (26) was already derived in (126).

The Yang-Baxter map $U_{ij}(\alpha, \beta)$ derived from (28) is given by

$$y_i' = -y_j - (\alpha_1 - \beta_1) + z_j(\alpha_1 - \beta_2 + y_i + y_j), \quad z_i' = z_j^{-1}(z_i + z_j - 1),$$

$$y_j' = -y_i - (\alpha_2 - \beta_2) + z_i(\alpha_1 - \beta_2 + y_i + y_j), \quad z_j' = z_i^{-1}(z_i + z_j - 1).$$

According to Theorem 3.5, the Yang-Baxter map (132) is a solution of the FYBE (38) in combination with the Yang-Baxter map (126).

It is interesting to note that this is the only case in this paper for which $\varepsilon = 1$ in (40), which is a consequence of the form of the solution to the CSTR that has been used.

4.2.4 Algebraic cases

4.2.4.1 $Q^1_{(\delta=0)}$ and $H^1_{(\varepsilon=1)}$ (Selberg-type integral and Barnes’s $2F_1$ integral). This case results in two different solutions of the FYBE (38), constructed from the functions featured in the type-III solution of the classical star-triangle relations in Table 3, which also contains the functions involved in the second type-I solution in Table 2 and the type-I solution in Table 3. The first set of classical R-matrices for this case are given by (26) and (28), and the second set of classical R-matrices for this case are given by (27) and (29). The Yang-Baxter map for (27) was already derived in (126).

For these cases the change of variables used in (44) and (45) is given by

$$f(x) = x, \quad g(x) = ix, \quad h(x) = 2x.$$
Let \( \Upsilon_1(z_j, y_i \mid y_j) \) and \( \Upsilon_2(z_i, y_j \mid y_i) \) (these are explicit forms of (47) derived from (26)) be defined by
\[
\Upsilon_1 = y_j + \frac{(y_i - y_j)(\alpha_1 - \beta_1)}{z_j(y_i - y_j) + (\alpha_1 - \beta_2)}, \quad \Upsilon_2 = y_i + \frac{(y_i - y_j)(\alpha_2 - \beta_2)}{z_i(y_i - y_j) - (\alpha_1 - \beta_2)}. \tag{134}
\]

Then using the method of Section 3.2, the Yang-Baxter map \( R_{ij}(\alpha, \beta) \) derived from (26) is given by
\[
y_i' = \Upsilon_1, \quad z_i' = \frac{\beta_1 - \alpha_2}{\Upsilon_1 - \Upsilon_2} \frac{\alpha_1 - \beta_1}{y_j - \Upsilon_1}, \quad y_j' = \Upsilon_2, \quad z_j' = \frac{\alpha_2 - \beta_1}{\Upsilon_1 - \Upsilon_2} \frac{\alpha_2 - \beta_2}{y_i - \Upsilon_2}. \tag{135}
\]

According to Theorem 3.5, this is a solution of the FYBE (37). The CSTR associated to (135) is the first row of Table 3 for a Selberg-type integral and \( Q_{1(\delta=0)} \).

The Yang-Baxter map \( U_{ij}(\alpha, \beta) \) derived from (28) is given by
\[
y_i' = -z_j y_i, \quad z_i' = -z_i z_j^{-1}, \quad y_j' = y_j + y_i z_i + \alpha_1 - \alpha_2, \quad z_j' = z_j. \tag{136}
\]

According to Theorem 3.5, the Yang-Baxter maps (135) and (136) are a solution of the FYBE (38). The CSTR associated to (136) is the second row of Table 3 for a special case of Barnes’s \( 2F_1 \) formula and \( H_1(\varepsilon=1) \).

The Yang-Baxter map \( U_{ij}(\alpha, \beta) \) derived from (29) is given by
\[
y_i' = y_i + y_j z_j + \beta_1 - \beta_2, \quad z_i' = z_i, \quad y_j' = -z_i y_j, \quad z_j' = -z_j z_i^{-1}. \tag{137}
\]

According to Theorem 3.5, the Yang-Baxter maps (126) and (137) are a solution of the FYBE (38). The CSTR associated to (137) is the second row of Table 3 for a special case of Barnes’s \( 2F_1 \) formula and \( H_1(\varepsilon=1) \).

4.2.4.2 \( H_1(\varepsilon=0) \) (Euler beta function). For this case the classical R-matrices are given by (26) and (28) with the functions of the type-II solution of the classical star-triangle relations given in Table 3. The Yang-Baxter map for (26) was already derived in (135).

Using the method of Section 3.2, the Yang-Baxter map \( U_{ij}(\alpha, \beta) \) derived from (28) is given by
\[
y_i' = y_i + z_j, \quad z_i' = z_i, \quad y_j' = y_j + z_i, \quad z_j' = z_j. \tag{138}
\]

According to Theorem 3.5, the Yang-Baxter maps (135) and (138) are a solution of the FYBE (38).

5 Conclusion

In this paper a new method to construct Yang-Baxter maps has been given, through the use of solutions of the classical star-triangle relations associated to the asymptotics of hypergeometric
integrals and integrable quad equations. In Section 2 it was shown how the classical star-triangle relations imply new vertex forms of the classical Yang-Baxter equations, and in Section 3 it was shown how the latter construction may be reinterpreted in terms of Yang-Baxter maps which satisfy the two types of functional Yang-Baxter equations given in (37) and (38). In Section 4, this method was utilised with explicit solutions of the classical star-triangle relations to derive a new family of sixteen Yang-Baxter maps. These are new Yang-Baxter maps which have both two-component variables and two-component parameters, and solve the two types of functional Yang-Baxter equations (37) and (38).

For this paper the solutions of the classical vertex form of the Yang-Baxter equations were based on the classical star-triangle relations, but in principle there could exist independent solutions. An example at the quantum level is how the 6-vertex model is independent of the star-triangle relations. It is worth investigating the analogue of such solutions at the classical level that would likely lead to different Yang-Baxter maps. The quadrirationality of the Yang-Baxter maps relies on determining a change of variables that transforms the derivatives of the classical R-matrices into rational bilinear expressions. This could be done for all cases given in Section 4 except for the elliptic case which was only rational linear, and thus defined a birational Yang-Baxter map. It is expected that a quadrirational expression should also exist, which might be obtained by using a slightly different change of variables or by using Jacobi functions. There is also potential for constructing more general multicomponent extensions of the Yang-Baxter maps given in this paper, by generalising the method to the case of classical Yang-Baxter equations associated to multivariate hypergeometric integral identities, such as those found in [16]. It would also be interesting to determine whether there is a quantum group interpretation of the Yang-Baxter maps of this paper, along the lines developed in [17,58], or the existence of any connections to the previously found Yang-Baxter maps associated to integrable quad equations, such as in [4,33,39,43–46].

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Appendix A  List of new Yang-Baxter maps

For convenience, a list of Yang-Baxter maps that were derived in Section 4.2 is given below.

Following the notation of Section 4, for any variable or parameter $x$, the notation $\dot{x}$ and $\ddot{x}$ are used to denote

$$\dot{x}^2 = 4x^3 - g_2x - g_3, \quad \ddot{x} = \frac{\dot{x}^2}{4(x - \wp(\omega_2))} - x - \wp(\omega_2),$$

(A.1)

where $\wp(x)$ is the Weierstrass elliptic function with elliptic invariants $g_2, g_3$, or associated half-periods $\omega_1, \omega_2$ [61].

The notation $\overline{x}$ is used to denote

$$\overline{x} = x + \sqrt{x^2 - 1}.$$  

(A.2)

Recall that Yang-Baxter maps from Section 4.2 are labelled by

$$\text{Quad equation (Hypergeometric integral) [Lattice model]}$$  

(A.3)
according to the ABS quad equation and hypergeometric integral identified with the star-triangle relations from [37]. If it has appeared before the associated lattice model is also given.

### A.1 Symmetric cases

The Yang-Baxter maps $R_{ij}(\alpha, \beta)$ given below are solutions to the FYBE (37).

#### A.1.1 $Q4$ (Elliptic beta integral) [Bazhanov-Sergeev model]

\[
A(\alpha, \beta) = \frac{(\hat{\alpha} + \hat{\beta})^2}{4(\alpha - \beta)^2} - \alpha - \beta, \quad B(x, \alpha, \beta) = \hat{\alpha} \beta + \hat{\beta} \alpha - (\hat{\alpha} + \hat{\beta})x, \\
E(x, \alpha, \beta, \gamma) = \frac{1}{E(x, \beta, \alpha, \gamma)} = \frac{(B(x, \alpha, \gamma) + (\alpha - \gamma)\hat{x})(B(x, \beta, \gamma) - (\beta - \gamma)\hat{x})}{(B(x, \alpha, \gamma) - (\alpha - \gamma)\hat{x})(B(x, \beta, \gamma) + (\beta - \gamma)\hat{x})},
\]

\[
F(x, \alpha, \beta) = 2\hat{x} \frac{B(A(\alpha, \beta), \alpha, \beta)}{\alpha - \beta} + 2g_3 + (A(\alpha, \beta) + x)(g_2 - 4xA(\alpha, \beta)), \\
G(x, y, \alpha, \beta) = F(x, \alpha, \beta) + 4y(A(\alpha, \beta) - x)^2.
\]

#### A.1.2 $Q3(\beta=1)$ (Hyperbolic beta integral) [Generalised Faddeev-Volkov model]

\[
\Upsilon_1 = \frac{z_jG(y_j, y_i, \beta_2, \alpha_1)F(y_j, \alpha_1, \beta_1, \beta_2, \alpha_1)^2 - G(y_j, y_i, \alpha_1, \beta_2)F(y_j, \beta_1, \alpha_1)}{4(y_j - A(\alpha_1, \beta_1))^2(G(y_j, y_i, \alpha_1, \beta_2) - z_jG(y_j, y_i, \beta_1, \alpha_2)E(y_j, \beta_1, \beta_2, \alpha_1)^2)},
\]

\[
\Upsilon_2 = \frac{z_jG(y_j, y_i, \beta_2, \alpha_1)F(y_j, \alpha_2, \beta_2, \alpha_2, \beta_2)^2 - G(y_j, y_i, \alpha_1, \beta_2, \alpha_2)F(y_j, \beta_2, \alpha_2)}{4(y_j - A(\alpha_2, \beta_2))^2(G(y_j, y_i, \alpha_1, \beta_2, \alpha_2) - z_jG(y_j, y_i, \beta_2, \alpha_2)E(y_j, \alpha_1, \alpha_2, \beta_2)^2)}.
\]

\[
y_i' = \Upsilon_1, \quad z_i' = \frac{G(\Upsilon_1, y_j, \alpha_1, \beta_1)G(\Upsilon_1, \bar{\Upsilon}_2, \beta_1, \alpha_2)}{G(\Upsilon_1, y_j, \beta_1, \alpha_1)G(\Upsilon_1, \bar{\Upsilon}_2, \alpha_2, \beta_1)},
\]

\[
y_j' = \Upsilon_2, \quad z_j' = \frac{G(\Upsilon_2, y_i, \alpha_2, \beta_2)G(\Upsilon_2, \bar{\Upsilon}_1, \alpha_2, \beta_1)}{G(\Upsilon_2, y_i, \beta_2, \alpha_2)G(\Upsilon_2, \bar{\Upsilon}_1, \alpha_1, \beta_2)}.
\]

### A.2 Non-symmetric cases

#### A.2.1 $Q4$ (Elliptic beta integral) [Bazhanov-Sergeev model]

\[
A(\alpha, \beta) = \frac{(\hat{\alpha} + \hat{\beta})^2}{4(\alpha - \beta)^2} - \alpha - \beta, \quad B(x, \alpha, \beta) = \hat{\alpha} \beta + \hat{\beta} \alpha - (\hat{\alpha} + \hat{\beta})x, \\
E(x, \alpha, \beta, \gamma) = \frac{1}{E(x, \beta, \alpha, \gamma)} = \frac{(B(x, \alpha, \gamma) + (\alpha - \gamma)\hat{x})(B(x, \beta, \gamma) - (\beta - \gamma)\hat{x})}{(B(x, \alpha, \gamma) - (\alpha - \gamma)\hat{x})(B(x, \beta, \gamma) + (\beta - \gamma)\hat{x})},
\]

\[
F(x, \alpha, \beta) = 2\hat{x} \frac{B(A(\alpha, \beta), \alpha, \beta)}{\alpha - \beta} + 2g_3 + (A(\alpha, \beta) + x)(g_2 - 4xA(\alpha, \beta)), \\
G(x, y, \alpha, \beta) = F(x, \alpha, \beta) + 4y(A(\alpha, \beta) - x)^2.
\]

#### A.2.2 $Q3(\beta=1)$ (Hyperbolic beta integral) [Generalised Faddeev-Volkov model]

\[
\Upsilon_1 = \frac{z_iG(y_i, z_i, \alpha_1, \beta_2, \alpha_1)F(y_i, \alpha_1, \beta_1, \beta_2, \alpha_1)^2 - G(y_i, z_i, \alpha_1, \beta_2)F(y_i, \beta_1, \alpha_1)}{4(y_i - A(\alpha_1, \beta_1))^2(G(y_i, z_i, \alpha_1, \beta_2) - z_iG(y_i, z_i, \beta_1, \alpha_2)E(y_i, \beta_1, \beta_2, \alpha_1)^2)},
\]

\[
\Upsilon_2 = \frac{z_iG(y_i, z_i, \beta_2, \alpha_1)F(y_i, \alpha_2, \beta_2, \alpha_2, \beta_2)^2 - G(y_i, z_i, \alpha_1, \beta_2, \alpha_2)F(y_i, \beta_2, \alpha_2)}{4(y_i - A(\alpha_2, \beta_2))^2(G(y_i, z_i, \alpha_1, \beta_2, \alpha_2) - z_iG(y_i, z_i, \beta_2, \alpha_2)E(y_i, \alpha_1, \alpha_2, \beta_2)^2)}.
\]

\[
y_i' = \Upsilon_1, \quad z_i' = \frac{G(\Upsilon_1, y_i, \alpha_1, \beta_1)G(\Upsilon_1, \bar{\Upsilon}_2, \beta_1, \alpha_2)}{G(\Upsilon_1, y_i, \beta_1, \alpha_1)G(\Upsilon_1, \bar{\Upsilon}_2, \alpha_2, \beta_1)},
\]

\[
y_j' = \Upsilon_2, \quad z_j' = \frac{G(\Upsilon_2, y_i, \alpha_2, \beta_2)G(\Upsilon_2, \bar{\Upsilon}_1, \alpha_2, \beta_1)}{G(\Upsilon_2, y_i, \beta_2, \alpha_2)G(\Upsilon_2, \bar{\Upsilon}_1, \alpha_1, \beta_2)}.
\]
A.1.3 \( Q_{3(\delta=0)} \) (Hyperbolic Saalschütz integral) [Faddeev-Volkov model]

\[
\begin{align*}
\Upsilon_1 &= y_j \frac{\alpha_1(\alpha_1 y_j + \beta_2 y_j) + \beta_1 z_j(\alpha_1 y_j + \beta_2 y_j)}{\beta_1(\alpha_1 y_j + \beta_2 y_j) + \alpha_1 z_j(\alpha_1 y_j + \beta_2 y_j)}, \\
\Upsilon_2 &= y_j \frac{\alpha_2(\alpha_1 y_j + \beta_2 y_j) + \beta_2 z_j(\alpha_1 y_j + \beta_2 y_j)}{\beta_2(\alpha_1 y_j + \beta_2 y_j) + \alpha_2 z_j(\alpha_1 y_j + \beta_2 y_j)}.
\end{align*}
\] (A.9)

\[
\begin{align*}
y_j' &= \Upsilon_1, \\
z_j' &= \frac{(\alpha_1 y_j - \beta_1 \Upsilon_1)(\alpha_2 \Upsilon_1 + \beta_1 \Upsilon_2)}{(\alpha_1 \Upsilon_1 - \beta_1 y_j)(\alpha_2 \Upsilon_2 + \beta_1 \Upsilon_1)}, \\
y_j' &= \Upsilon_2, \\
z_j' &= \frac{(\alpha_2 y_j - \beta_2 \Upsilon_2)(\alpha_2 \Upsilon_2 + \beta_1 \Upsilon_1)}{(\alpha_2 \Upsilon_2 - \beta_2 y_j)(\alpha_2 \Upsilon_1 + \beta_1 \Upsilon_2)}.
\end{align*}
\] (A.10)

A.1.4 \( Q_2 \) (Rational Beta integral)

\[
\begin{align*}
\Upsilon_1 &= y_j + (\alpha_1 - \beta_1)^2 \left(1 - \frac{2\sqrt{y_j}}{\alpha_1 - \beta_1} \frac{(y_j - y_i + (\alpha_1 - \beta_2)^2)^{z_j+1}}{(y_j - y_i + (\alpha_1 - \beta_2)^2)^{z_j+1}} + (\alpha_1 - \beta_2)(z_j - 1)\sqrt{y_j}\right), \\
\Upsilon_2 &= y_i + (\alpha_2 - \beta_2)^2 \left(1 - \frac{2\sqrt{y_i}}{\alpha_2 - \beta_2} \frac{(y_i - y_i + (\alpha_2 - \beta_2)^2)^{z_i+1}}{(y_i - y_i + (\alpha_2 - \beta_2)^2)^{z_i+1}} + (\alpha_1 - \beta_2)(z_i - 1)\sqrt{y_i}\right).
\end{align*}
\] (A.11)

\[
\begin{align*}
y_j' &= \Upsilon_1, \\
z_j' &= \frac{(y_j - (\alpha_1 - \beta_1 - \sqrt{\Upsilon_1})^2)(\Upsilon_2 - (\alpha_2 - \beta_1 + \sqrt{\Upsilon_1})^2)}{(y_j - (\alpha_1 - \beta_1 + \sqrt{\Upsilon_1})^2)(\Upsilon_2 - (\alpha_2 - \beta_1 - \sqrt{\Upsilon_1})^2)}, \\
y_j' &= \Upsilon_2, \\
z_j' &= \frac{(y_i - (\alpha_2 - \beta_2 - \sqrt{\Upsilon_2})^2)(\Upsilon_1 - (\alpha_2 - \beta_1 + \sqrt{\Upsilon_2})^2)}{(y_i - (\alpha_2 - \beta_2 + \sqrt{\Upsilon_2})^2)(\Upsilon_1 - (\alpha_2 - \beta_1 - \sqrt{\Upsilon_2})^2)}.
\end{align*}
\] (A.12)

A.1.5 \( Q_{1(\delta=1)} \) (Barnes’s second lemma)

\[
\begin{align*}
\Upsilon_1 &= y_j + (\alpha_1 - \beta_1)\frac{(z_j + 1)(y_i - y_j) + (1 - z_j)(\alpha_1 - \beta_2)}{(1 - z_j)(y_i - y_j) + (z_j + 1)(\alpha_1 - \beta_2)}, \\
\Upsilon_2 &= y_i + (\alpha_2 - \beta_2)\frac{(z_i + 1)(y_i - y_j) + (z_i - 1)(\alpha_1 - \beta_2)}{(1 - z_i)(y_i - y_j) - (z_i + 1)(\alpha_1 - \beta_2)}.
\end{align*}
\] (A.13)

\[
\begin{align*}
y_j' &= \Upsilon_1, \\
z_j' &= \frac{(\alpha_1 - \beta_1 + (y_j - \Upsilon_1))(\alpha_2 - \beta_1 - (\Upsilon_1 - \Upsilon_2))}{(\alpha_1 - \beta_1 - (y_j - \Upsilon_1))(\alpha_2 - \beta_1 - (\Upsilon_1 - \Upsilon_2))}, \\
y_j' &= \Upsilon_2, \\
z_j' &= \frac{(\alpha_2 - \beta_2 + (y_i - \Upsilon_2))(\alpha_2 - \beta_1 - (\Upsilon_1 - \Upsilon_2))}{(\alpha_2 - \beta_2 - (y_i - \Upsilon_2))(\alpha_2 - \beta_1 - (\Upsilon_1 - \Upsilon_2))}.
\end{align*}
\] (A.14)

A.1.6 \( Q_{1(\delta=0)} \) (Selberg-type integral) [Zamolodchikov fish-net model]

\[
\begin{align*}
\Upsilon_1 &= y_j + \frac{(y_i - y_j)(\alpha_1 - \beta_1)}{z_j(y_i - y_j) + (\alpha_1 - \beta_2)}, \\
\Upsilon_2 &= y_i + \frac{(y_i - y_j)(\alpha_2 - \beta_2)}{z_i(y_i - y_j) - (\alpha_1 - \beta_2)}.
\end{align*}
\] (A.15)

\[
\begin{align*}
y_j' &= \Upsilon_1, \\
z_j' &= \frac{\beta_1 - \alpha_2}{\Upsilon_1 - \Upsilon_2} - \frac{\alpha_1 - \beta_1}{y_j - \Upsilon_1}, \\
y_j' &= \Upsilon_2, \\
z_j' &= \frac{\alpha_2 - \beta_1}{\Upsilon_1 - \Upsilon_2} - \frac{\alpha_2 - \beta_2}{y_i - \Upsilon_2}.
\end{align*}
\] (A.16)
A.2 Asymmetric cases

The Yang-Baxter maps $U_{ij}(\alpha, \beta)$ given below are solutions to the FYBE (38), together with one of the above Yang-Baxter maps $R_{ij}(\alpha, \beta)$ for the symmetric cases.

A.2.1 $H_{3(\delta=1; \varepsilon=1)}$ (Hyperbolic Askey-Wilson integral and Saalschütz integrals) case 1

$$
\Upsilon_1 = \frac{\beta_1 \beta_2 y_j (z_j - 1) + \alpha_1 y_i (z_j y_j^2 - 1)}{\alpha_1 \alpha_1 y_i y_j (z_j - 1) + \beta_2 (z_j - y_j^2)}, \quad \Upsilon_2 = y_j z_i + y_i \frac{\alpha_2 + \alpha_1 z_i}{2 \beta_2} + \beta_2 \frac{\alpha_1 + \alpha_2 z_i}{2 \alpha_1 \alpha_2 y_i}, \quad (A.17)
$$

$$
y_i' = \Upsilon_1, \quad z_i' = -\frac{\alpha_1 \beta_1^2 + \alpha_2^2 \Upsilon_i^2 + 2 \alpha_2 \beta_1 \Upsilon_1 \Upsilon_2}{\alpha_i \beta_1^2 + \alpha_2^2 \Upsilon_i^2 - 2 \alpha_1 \alpha_2 y_j \Upsilon_i}, \quad (A.18)
$$

$R_{ij}(\alpha, \beta)$ is given by (A.10) ($Q_{3(\delta=0)}$ case).

A.2.2 $H_{3(\delta=1; \varepsilon=1)}$ (Hyperbolic Askey-Wilson integral and Saalschütz integrals) case 2

$$
\Upsilon_1 = y_i z_j + \frac{\beta_1 + \beta_2 z_j}{2 \alpha_1 y_j} + \alpha_1 y_j \frac{\beta_2 + \beta_1 z_j}{2 \beta_1 \beta_2}, \quad \Upsilon_2 = \frac{\beta_2 \beta_2 y_j (z_i - 1) + \alpha_1 y_j (z_i y_i^2 - 1)}{\alpha_2 \alpha_1 y_i y_j (z_i - 1) + \beta_2 (z_i - y_i^2)}. \quad (A.19)
$$

$$
y_i' = \Upsilon_1, \quad z_i' = \frac{(\alpha_1 y_j - \beta_1 \Upsilon_1)(\beta_1 + \alpha_2 \Upsilon_1 \Upsilon_2)}{(\alpha_2 \beta_2 + \beta_1 \Upsilon_1)(\alpha_1 y_j \Upsilon_1 - \beta_1)}, \quad (A.20)
$$

$R_{ij}(\alpha, \beta)$ is given by (A.8) ($Q_{3(\delta=1)}$ case).

A.2.3 $H_{3(\delta=0; 1; \varepsilon=1-\delta)}$ (Hyperbolic Barnes’s first lemma)

$$
\Upsilon_1 = y_i z_j + \frac{\beta_1 + \beta_2 z_j}{\alpha_1 y_j}, \quad \Upsilon_2 = y_j z_i + \frac{\alpha_1 + \alpha_2 z_i}{\alpha_1 \alpha_2 y_i}. \quad (A.21)
$$

$$
y_i' = \Upsilon_1, \quad z_i' = \frac{\Upsilon_1 \Upsilon_2 + \beta_2 \alpha_2^{-1}}{y_j \Upsilon_1 - \beta_1 \alpha_2^{-1}}, \quad (A.22)
$$

$R_{ij}(\alpha, \beta)$ is given by (A.10) ($Q_{3(\delta=0)}$ case).
A.2.4 $H_{3\left(\delta=0;\varepsilon=0\right)}$ (Hyperbolic Barnes’s $\, _2F_1$ integral)

\[ \Upsilon_1 = z_j y_i, \quad \Upsilon_2 = z_i y_j. \]  
(A.23)

\[
y'_i = \Upsilon_1, \quad z'_i = \Upsilon_2 y_j^{-1};
\]
\[
y'_j = \Upsilon_2, \quad z'_j = \Upsilon_1 y_i^{-1}. \]  
(A.24)

$R_{ij}(\alpha, \beta)$ is given by (A.10) ($Q_{3\left(\delta=0\right)}$ case).

A.2.5 $H_{2\left(\varepsilon=1\right)}$ (de Branges-Wilson integral and Barnes’s second lemma) case 1

\[ \Upsilon_1 = (\beta_2 - \alpha_2 + y_i)^2 + z_i(y_j - (\alpha_1 - \beta_2 + y_i)^2). \]  
(A.25)

\[
y'_i = \Upsilon_1, \quad z'_i = \frac{(\alpha_2 - \beta_2 + y_i - \sqrt{\Upsilon_2})(\alpha_2 - \beta_1 + \Upsilon_1 + \sqrt{\Upsilon_2})}{(\alpha_2 - \beta_2 + y_i + \sqrt{\Upsilon_2})(\alpha_2 - \beta_1 + \Upsilon_1 - \sqrt{\Upsilon_2})};
\]
\[
y'_j = \Upsilon_2, \quad z'_j = \frac{(\alpha_2 - \beta_2 + y_i - \sqrt{\Upsilon_1})(\alpha_2 - \beta_1 + \Upsilon_2 + \sqrt{\Upsilon_1})}{(\alpha_2 - \beta_2 + y_i + \sqrt{\Upsilon_1})(\alpha_2 - \beta_1 + \Upsilon_2 - \sqrt{\Upsilon_1})}. \]  
(A.26)

$R_{ij}(\alpha, \beta)$ is given by (A.14) ($Q_{1\left(\delta=1\right)}$ case).

A.2.6 $H_{2\left(\varepsilon=1\right)}$ (de Branges-Wilson integral and Barnes’s second lemma) case 2

\[ \Upsilon_1 = (\alpha_1 - \beta_1 + y_j)^2 + z_j(y_i - (\alpha_1 - \beta_2 + y_j)^2), \]
\[ \Upsilon_2 = (\beta_2 - \alpha_2) + \frac{(z_i - 1)y_i + (z_i + 1)(y_j + \alpha_1 - \beta_2)\sqrt{y_i}}{(z_i - 1)(y_j + \alpha_1 - \beta_2) + (z_i + 1)\sqrt{y_i}}. \]  
(A.27)

\[
y'_i = \Upsilon_1, \quad z'_i = \frac{(\alpha_1 - \beta_1 + y_j - \sqrt{\Upsilon_1})(\alpha_2 - \beta_1 + \Upsilon_2 + \sqrt{\Upsilon_1})}{(\alpha_1 - \beta_1 + y_j + \sqrt{\Upsilon_1})(\alpha_2 - \beta_1 + \Upsilon_2 - \sqrt{\Upsilon_1})};
\]
\[
y'_j = \Upsilon_2, \quad z'_j = \frac{\Upsilon_1 - (\alpha_2 - \beta_1 + \Upsilon_2)^2}{y_i - (\alpha_2 - \beta_2 + \Upsilon_2)^2}. \]  
(A.28)

$R_{ij}(\alpha, \beta)$ given by (A.12) ($Q_{2}$ case).

A.2.7 $H_{2\left(\varepsilon=0\right)}$ (Barnes’s first lemma)

\[ \Upsilon_1 = - y_j - (\alpha_1 - \beta_1) + z_j(\alpha_1 - \beta_2 + y_i + y_j), \]
\[ \Upsilon_2 = - y_i - (\alpha_2 - \beta_2) + z_i(\alpha_1 - \beta_2 + y_i + y_j). \]  
(A.29)

\[
y'_i = \Upsilon_1, \quad z'_i = \frac{\alpha_2 - \beta_1 + \Upsilon_1 + \Upsilon_2}{\alpha_1 - \beta_1 + y_j + \Upsilon_1};
\]
\[
y'_j = \Upsilon_2, \quad z'_j = \frac{\alpha_2 - \beta_1 + \Upsilon_1 + \Upsilon_2}{\alpha_2 - \beta_2 + y_i + \Upsilon_2}. \]  
(A.30)

$R_{ij}(\alpha, \beta)$ is given by (A.14) ($Q_{1\left(\delta=1\right)}$ case).
A.2.8  $H_{1(\varepsilon=1)}$ (Barnes’s $_2F_1$ integral) case 1

\[ \Upsilon_1 = -y_iz_j, \quad \Upsilon_2 = y_j + z_iy_i + \alpha_1 - \alpha_2. \]  
(A.31)

\[
\begin{align*}
    y'_i &= \Upsilon_1, \\
    z'_i &= (\Upsilon_2 - y_j - \alpha_1 + \alpha_2) \Upsilon_1^{-1}, \\
    y'_j &= \Upsilon_2, \\
    z'_j &= \Upsilon_1 y_i^{-1},
\end{align*}
\]  
(A.32)

$R_{ij}(\alpha, \beta)$ is given by (A.16) ($Q_{1(\delta=0)}$ case).

A.2.9  $H_{1(\varepsilon=1)}$ (Barnes’s $_2F_1$ integral) case 2

\[ \Upsilon_1 = y_i + z_jy_j + \beta_1 - \beta_2, \quad \Upsilon_2 = -y_jz_i. \]  
(A.33)

\[
\begin{align*}
    y'_i &= \Upsilon_1, \\
    z'_i &= -\Upsilon_2y_j^{-1}, \\
    y'_j &= \Upsilon_2, \\
    z'_j &= (\Upsilon_1 - y_i - \beta_1 + \beta_2) \Upsilon_2^{-1},
\end{align*}
\]  
(A.34)

$R_{ij}(\alpha, \beta)$ is given by (A.14) ($Q_{1(\delta=1)}$ case).

A.2.10  $H_{1(\varepsilon=0)}$ (Euler beta function)

\[ \Upsilon_1 = z_j + y_i, \quad \Upsilon_2 = z_i + y_j. \]  
(A.35)

\[
\begin{align*}
    y'_i &= \Upsilon_1, \\
    z'_i &= \Upsilon_2 - y_j, \\
    y'_j &= \Upsilon_2, \\
    z'_j &= \Upsilon_1 - y_i,
\end{align*}
\]  
(A.36)

$R_{ij}(\alpha, \beta)$ is given by (A.16) ($Q_{1(\delta=0)}$ case).

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