2D black holes and effective actions

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Abstract

We compare the canonical quantization and the effective action method to derive expectation values of the stress energy tensor for scalar fields conformally coupled to a 2D Schwarzschild black hole spacetime. Particular attention is devoted to the thermal equilibrium Hartle-Hawking state where the striking disagreement of the results may be reconduced to the incomplete knowledge of the effective action. We show how to reconcile the two procedures and find physically meaningful analytical approximate expressions for the stress tensor in the Hartle-Hawking state.

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1 Introduction

The study of quantum fields propagating in lower-dimensional spacetime is usually regarded as an amusing playground which allows many interesting features of ordinary 4D physics to be inferred by simple technical tools. Within this perspective many efforts have been devoted to the study of conformally coupled scalar fields propagating in a 2D Schwarzschild spacetime

\[ ds^2 = -(1 - 2M/r) dt^2 + (1 - 2M/r)^{-1} dr^2. \] (1.1)

These kind of studies are supposed to give some hints on the quantum properties of the real 4D black hole at least in the “s-wave sector”. The basis objects of investigation are the renormalized expectation values \( \langle T_{\mu \nu} \rangle \) of the stress tensor operator for these quantum fields. These quantities can be calculated in the canonical quantization scheme by mode sums and regularization or, in a more elegant fashion, by functional variation of an effective action. The latter method, requiring the knowledge of the effective action for an arbitrary background metric, is far more reaching for applications (see for example backreaction calculations). It is obvious that the two procedures cannot lead to unequal results.

We first review the well known Polyakov model [1], which describes massless and minimally coupled 2d scalar fields. Its purpose is mainly pedagogical, because in this simple context it is easy to show the agreement of the two procedures, canonical quantization vs. effective action. Particular attention however is needed to show how thermal state expectation values can be retrieved by use of effective action as emphasized by the authors of Ref. [2]. We then turn our attention to a more sophisticated theory, the so called dimensional reduction model, which has received much interest in the literature [3] (for a recent review see [4]). Here things become trickier since the results coming from canonical quantization [5] do not seem to match those coming from the effective action. It is widely believed that such problems arise because of our incomplete knowledge of the exact effective action (see for instance the first of Refs. [3] and also [6], [7]). What is known is the so called anomaly induced effective action \( S_{AI} \), obtained by integrating the conformal anomaly. This procedure allows the effective action to be known up to a Weyl invariant functional. Based on this observation we propose two ways to construct analytical expressions for the stress tensor in agreement with the results inferred from canonical quantization. The first is based on the anomalous transformation law of the effective action under conformal transformations and is constructed using an ansatz à la Brown-Ottewill [8]. The second method requires a minimal addition to \( S_{AI} \) in the form of a nonlocal Weyl invariant term. Finally, we state our conclusions.

2 The “Polyakov” model

Let us start our discussion by considering a conformal invariant minimally coupled scalar field \( \hat{f} \) whose action is
leading to the field equation

\[ \Box \hat{f} = 0. \]  

In the 2D Schwarzschild spacetime of eq. (1.1) one can expand the field operator \( \hat{f} \) in terms of Eddington-Finkelstein modes \( \{ e^{-i\omega u}, e^{-i\omega v} \} \) where

\[ u = t - r^*, \quad v = t + r^* \]

\[ r^* = r + 2M \ln |r/2M - 1|. \]

This is the so called “Boulware vacuum” construction and leads after renormalization to

\[ \langle B|T^t_t|B \rangle = \frac{1}{12\pi f} \left[ \frac{2M}{r^3} - \frac{7M^2}{2r^4} \right], \]

\[ \langle B|T^r_r|B \rangle = -\frac{1}{12\pi f} \left[ \frac{M^2}{2r^4} \right], \]

where \( f = 1 - 2M/r \). The \( \langle B|T^a_a|B \rangle \) component vanishes identically. From the above equations one recovers the usual trace anomaly

\[ \langle B|T^a_a|B \rangle = \frac{R}{24\pi} = \frac{M}{6\pi r^3}. \]

The Boulware vacuum describes vacuum polarization outside a static spherically symmetric body. Asymptotically this state coincides with the usual Minkowski vacuum of flat space-time quantum field theory. However it has a rather pathological behaviour at the event horizon. Regularity of the stress tensor there in a regular frame requires \( (T^t_t - T^r_r)/f \) to be finite [9]. This condition is not fulfilled by \( \langle B|T^a_a|B \rangle \).

A quantum state yielding a regular \( T^a_a \) on the horizon can be obtained by using a different set of modes as basis for the field operator. Choosing the Kruskal modes \( (e^{-i\omega U}, e^{-i\omega V}) \) where

\[ U = -4Me^{-u/AM}, \quad V = 4Me^{v/AM} \]

one performs the so called “Hartle-Hawking vacuum” construction leading to

\[ \langle H|T^t_t|H \rangle = \frac{1}{12\pi f} \left[ \frac{2M}{r^3} - \frac{7M^2}{2r^4} \right] - \frac{1}{384\pi M^2 f}, \]

\[ \langle H|T^r_r|H \rangle = -\frac{1}{12\pi f} \left[ \frac{M^2}{2r^4} \right] + \frac{1}{384\pi M^2 f}, \]

while as before

\[ \langle H|T^a_a|H \rangle = \frac{M}{6\pi r^3}. \]
which stresses the state independence of the trace anomaly. One can check that $\langle H | T^a_b | H \rangle$ does indeed satisfy the regularity condition on the horizon. Asymptotically $\langle H | T^a_b | H \rangle$ is not vanishing, approaching the form

$$\frac{\pi T_H^2}{6} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(2.12)

describing thermal radiation at the Hawking temperature $T_H = 1/(8\pi M) = 1/2\pi \beta_H$. $|H\rangle$ is indeed a thermal state (the corresponding propagator being periodic in imaginary time with period $2\pi \beta_H$) which describes a black hole in thermal equilibrium with its radiation.

A better insight into these results can be obtained by using the effective action formalism. For the above theory the calculation of the effective action is usually made by integrating the trace anomaly. The result is the well known Polyakov action [1]

$$S_P = -\frac{1}{96\pi} \int d^2 x \sqrt{-g} R \frac{1}{\Box} R.$$  

(2.13)

However, as emphasized in Ref. [2], writing the effective action in the form (2.13) one looses the information about the state of the quantum field and in particular it is not clear how (2.13) can reproduce thermal radiation. The integration of the conformal anomaly does not give the absolute value of the effective action, but rather the difference between the effective actions for two conformally related ($g_{ab} = e^{2\sigma} \hat{g}_{ab}$) manifolds

$$\Gamma(g) = \Gamma(\hat{g}) - \frac{1}{24\pi} \int d^2 x \sqrt{-\hat{g}} \left[ (\hat{\nabla} \sigma)^2 + \hat{R} \sigma \right].$$

(2.14)

The corresponding relation for the stress energy tensor is

$$T_{ab}(g) = T_{ab}(\hat{g}) - \frac{1}{48\pi} \left( -4 \hat{\nabla}_\mu \hat{\nabla}_\nu \sigma + 4 \hat{\nabla}_\mu \sigma \hat{\nabla}_\nu \sigma + \hat{g}_{\mu\nu} (4 \Box \sigma - 2(\hat{\nabla} \sigma)^2) \right).$$

(2.15)

By writing the 2D Schwarzschild metric as

$$ds^2 = -(1 - 2M/r)(dt^2 - dr^2) = -e^{2\sigma} ds^2$$

(2.16)

one can take $\hat{g}$ as flat Minkowski space and setting $T_{ab}(\hat{g}) = 0$ one gets from eq. (2.15)

$$T^t_t = \frac{1}{12\pi f} \left[ \frac{2M}{r^3} - \frac{7M^2}{2r^4} \right]$$

(2.17)

which coincides with eq. (2.5) and similarly for the $T^r_r$ component. So using as reference the usual Minkowski vacuum one obtains as conformal image the Boulware state. To obtain a thermal state at temperature $T$ ($= 1/2\pi \beta$) one writes the (Euclidean) Schwarzschild metric as [2]

$$ds^2 = e^{2\sigma} ds^2$$

(2.18)
where now
\[ e^{2\sigma} = \frac{\beta_H^2 f}{z^2} \]  
(2.19)

and
\[ ds^2 = dz^2 + (\alpha z^2) d\tilde{\tau}^2 , \]  
(2.20)

\[ z = \beta_H e^{\frac{1}{\beta_H}} \int \frac{dv}{f} , \]  
(2.21)

where \( \alpha = \beta/\beta_H \), \( \tau = \beta \tilde{\tau} \) is the euclidean time \((\tau = it)\) \(0 \leq \tilde{\tau} \leq 2\pi\).

For \( \beta = \beta_H \), \( ds^2 \) is the metric of the flat disk \( D \) of radius \( z_0 \), whereas for \( \beta \neq \beta_H \) \( ds^2 \) is the metric of a flat cone \( C_\alpha \) with deficit angle \( \delta = 2\pi(1 - \alpha) \). Let us first consider the regular instanton \((\beta = \beta_H)\). From eq. (2.15) we now obtain

\[ T_{tt} = T_{tt}(D) + \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + \frac{1}{24\pi \beta_H^2} . \]  
(2.22)

Assuming the quantum field on the disk to be in the state for which \( T^{a}_{b}(D) = 0 \) we see that eq. (2.22) coincides with \( \langle H|T_t^t|H \rangle \) of eq. (2.9). Similarly for the \( T^r_r \) component. So the conformal image of the vacuum on the disk is the Hartle-Hawking state.

For the cone \((\beta \neq \beta_H)\) one has

\[ T_{tt}(C) = \frac{(1 - \alpha^2)}{24\pi \beta^2} \]  
(2.23)

and by eq. (2.22) one has

\[ T_{tt} = \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + \frac{1}{24\pi \beta^2} \]  
(2.24)

which asymptotically describes thermal radiation at the temperature \( T = 1/2\pi \beta \). Note that only for \( \beta = \beta_H \) we have regularity of the stress tensor at the horizon.

One can calculate \( T_{\mu\nu} \) directly in terms of the black hole metric by using a local expression of \( \Gamma(g) \) in terms of quantities defined only with respect to the black hole metric \( g_{\mu\nu} \).

Introducing an auxiliary field \( \psi \) by

\[ \Box \psi = R \]  
(2.25)

one can write the effective action \( \Gamma[g] \) as

\[ \Gamma(g) = -\frac{1}{48\pi} \int d^2x \sqrt{-g} \left( \frac{\nabla^2 \psi}{2} + \psi R \right) + \Gamma_0 \]  
(2.26)

where \( \Gamma_0 \) is a conformally invariant functional. Neglecting \( \Gamma_0 \) (we will come back to this point), the stress tensor is expressed in terms of \( \psi \) as

\[ T_{\mu\nu} = \frac{1}{48\pi} \left( 2\nabla_\mu \nabla_\nu \psi - \nabla_\mu \psi \nabla_\nu \psi + g_{\mu\nu}(-2R + \frac{1}{2}(\nabla \psi)^2) \right) . \]  
(2.27)

For the Schwarzschild metric eq. (2.25) can be solved by

\[ \psi = -\ln(f) + br^* . \]  
(2.28)
Note that the second term on the r.h.s. of eq. (2.28) is a homogeneous solution of the auxiliary field equation (2.25). $b$ is an arbitrary (for the moment) constant. Inserting in eq. (2.27) one obtains

$$T_{tt} = \frac{1}{12\pi} \left[ -\frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + \frac{b^2}{96\pi}. \quad (2.29)$$

The correct thermal behaviour at infinity is obtained by setting $b = 2/\beta$ reproducing all previous results. Note that near the horizon ($r \rightarrow 2M$)

$$\psi \rightarrow \psi_c = -2(1 - \frac{\beta_H}{\beta}) \ln z \quad (2.30)$$

which is regular for $\beta = \beta_H$, whereas for $\beta \neq \beta_H$ $\psi_c$ coincides with the solution of the cone equation

$$\Box_c \psi_c = R_c, \quad (2.31)$$

where

$$R_c = 2 \frac{(1 - \alpha)}{\alpha} \delta(z). \quad (2.32)$$

So far we have neglected any contribution of $\Gamma_0$ to the stress tensor. But being $\Gamma_0$ conformally invariant it contributes to the stress tensor just with a conserved traceless tensor. These conditions plus time independence are so stringent that the resulting tensor must be proportional to

$$\frac{1}{f} \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \quad (2.33)$$

for a static black hole spacetime.

Consider now what has been the effect of adding the homogeneous solution to the eq. (2.25). By adding a homogeneous solution $\psi_0$ ($\Box \psi_0 = 0$), $T_{\mu\nu}(\psi)$ transforms as follows

$$T_{\mu\nu}(\psi) \rightarrow T_{\mu\nu}(\psi + \psi_0) = T_{\mu\nu}(\psi) + [T_{\mu\nu}(\psi_0) - \frac{Rg_{\mu\nu}}{24\pi}] + \frac{1}{48\pi} \left[ -\nabla_{(\mu} \psi \nabla_{\nu)} \psi_0 + g_{\mu\nu}(\nabla \psi)(\nabla \psi_0) \right]. \quad (2.34)$$

Note that $T^a_0(\psi + \psi_0) = T^a_0(\psi)$. The two additional tensors are conserved and traceless giving

$$T^a_b = \frac{b^2}{96\pi f} \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right). \quad (2.35)$$

Comparing this with eq. (2.33), one realizes that one can simply neglect $\Gamma_0$ in eq. (2.26) and add a homogeneous solution to the auxiliary field eq. (as we did) obtaining the general expression for the stress tensor. The overall coefficient is fixed requiring the thermal radiation behaviour at infinity (i.e. $b = 2/\beta$). So for the Polyakov theory one obtains the correct thermal contribution just by adding the homogeneous solution to the auxiliary field eq. (2.25).
3 The dimensional reduction model: canonical quantization

We come now to the second, more interesting but intriguing example. Let us consider the following scalar dilaton action

\[ S_{m}^{(2)} = -\frac{1}{4\pi} \int d^{2}x \sqrt{-g} e^{-2\phi} (\nabla \hat{f})^{2}. \]  

(3.1)

Unlike the previous case ( eq. (2.1) ) the scalar field \( \hat{f} \) is now coupled not only to 2D gravity but also to a dilaton field. The action eq. (3.1) is however still conformal invariant. This action can be obtained by dimensional reduction of the following 4D action

\[ S_{m}^{(4)} = -\frac{1}{(4\pi)^{2}} \int d^{4}x \sqrt{-g^{(4)}} (\nabla \hat{f})^{2} \]  

(3.2)

describing a minimally coupled 4D scalar field. The reduction yielding eq. (3.1) is performed by assuming the 4D metric to be spherically symmetric

\[ ds_{(4)}^{2} = g_{ab}(x^{a}) dx^{a} dx^{b} + e^{-2\phi(x^{a})} d\Omega^{2}, \]  

(3.3)

where \( a, b = 1, 2, \) \( d\Omega^{2} \) is the line element of the transverse unit sphere, \( e^{-\phi} \) is the radius and \( \hat{f} = \hat{f}(x^{a}) \). Because of this feature, the model of eq. (3.1) has acquired considerable interest [3]-[7]. It is supposed to give a better hint on the physics of a real 4D black hole when compared to the model based on \( S_{m}^{(2)} \) (2.1).

The presence of the dilaton in the action (3.1) makes the field equation for \( \hat{f} \) more complicated. Instead of the simple D’Alembert equation (2.2) seen before, we have now

\[ \nabla^{a}(e^{-2\phi} \nabla_{a} \hat{f}) = 0. \]  

(3.4)

For the 2D Schwarzschild spacetime this eq. becomes (writing \( \hat{f} = e^{-i\omega t}R(r)/r \), where \( r = e^{-\phi} \))

\[ \frac{d^{2}R}{dr^{2}} + \frac{2M}{r^{3}} (1 - 2M/r)R - w^{2}R = 0. \]  

(3.5)

The canonical quantization procedure starts by finding a complete set of solutions to the above equation of motion. Plane waves are no longer solutions as the effective potential acts as a reflecting barrier. Normal modes of eq. (3.5) are not known analytically in explicit form. However from their asymptotic behaviours near the horizon

\[ \hat{R} \sim e^{i\omega r} + \hat{A}(w) e^{-i\omega r}, \]

\[ \hat{R} \sim \hat{B}(w) e^{-i\omega r}, \]  

(3.6)

and at infinity,

\[ \hat{R} \sim \hat{B}(w) e^{i\omega r}, \]

\[ \hat{R} \sim e^{-i\omega r} + \hat{A}(w) e^{i\omega r}, \]  

(3.7)
where \(A\) and \(B\) are the reflection and transmission coefficients (see Ref. [10]), the following expressions for \(\langle T_{ab}\rangle\) can be derived without recursion to any regularization procedure [5]

\[
\langle B|T^0_1|B\rangle_{r\to 2M} \sim \frac{1}{384\pi M^2 f} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\langle H|T^0_1|H\rangle_{r\to \infty} \sim \frac{1}{384\pi M^2 f} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

where \(a, b = t, r\) and \(f = 1 - 2M/r\).

Furthermore using a WKB approximation for the modes, performing a large \(w\) expansion and regularizing the stress tensor by point splitting and then performing renormalization one can obtain approximate analytic expressions for \(\langle T_{ab}\rangle\) in the two states [5]. For the Boulware (zero temperature) state we have

\[
\langle B|T_{tt}|B\rangle_{WKB} = -\frac{1}{2\pi} \left( -\frac{fM}{6r^3} + \frac{M^2}{12r^4} - \frac{f^2}{4r^2} \ln \left( \frac{m^2 f}{4\lambda^2} \right) \right),
\]

\[
\langle B|T_{rr}|B\rangle_{WKB} = \frac{1}{2\pi f^2} \left( -\frac{fM}{2r^3} - \frac{M^2}{12r^4} + \frac{f^2}{4r^2} \ln \left( \frac{m^2 f}{4\lambda^2} \right) \right),
\]

where \(m^2\) is a renormalization scale and \(\lambda\) an infrared cutoff. These are fixed by requiring that for \(M = 0\) one recovers the Minkowski results, namely \(\langle T_{ab}\rangle = 0\). This yields \(m^2 = 4\lambda^2\). For the thermal equilibrium state we have

\[
\langle T_{tt}\rangle_{WKB} = \frac{\pi T^2}{6} - \frac{1}{2\pi} \left( -\frac{fM}{6r^3} + \frac{M^2}{12r^4} - \frac{f^2}{4r^2} (2\gamma + \ln \left[ \frac{m^2 f}{4\lambda^2} \right]) \right),
\]

\[
\langle T_{rr}\rangle_{WKB} = \frac{\pi T^2}{6f^2} + \frac{1}{2\pi f^2} \left( -\frac{fM}{2r^3} - \frac{M^2}{12r^4} + \frac{f^2}{4r^2} (2\gamma + \ln \left[ \frac{m^2 f}{4\lambda^2} \right]) \right),
\]

where \(\gamma\) is Euler number. In both states the trace

\[
\langle T^a_a\rangle = -\frac{M}{3\pi r^3}
\]

is the same. This is not an approximation but an exact result, being just a realization of the 2D conformal anomaly which gives the stress tensor a state independent anomalous trace proportional to the \(a_1\) Seeley-De Witt coefficient. For the theory described by the action (3.1) it is

\[
\langle T^a_a\rangle = (24\pi)^{-1}[R - 6(\nabla \phi)^2 + 6\Box \phi]
\]

from which eq. (3.14) follows.

The crucial problem with eqs. (3.12) and (3.13) is the logarithmic term, which would imply the nonregularity on the horizon of the Hartle-Hawking stress tensor \((T = T_H)\) in a free falling frame. Logarithmic divergences of this kind are unfortunately a constant feature in approximations schemes based on the WKB expansion in the case where the Ricci tensor is nonvanishing (see the analogous problem in 4D in Ref. [11]). This is an
artifact of the WKB approximation which breaks down near \( r = 2M \). As shown in Ref. [5] \( \langle H | T_{\mu\nu} | H \rangle \) is indeed regular on the horizon and no \( f^2 \ln f \) terms are present. Finally, on the horizon we have

\[
\langle H | T_t^t | H \rangle_{r=2M} = \langle H | T_r^r | H \rangle_{r=2M} = -\frac{1}{48\pi M^2}.
\]

Unfortunately the calculations of the modes are rather involved and one has not been able to find till now an analytic expression which smoothly matches the regular behavior on the horizon with the \( r >> 2M \) behavior of eqs. (3.12) and (3.13).

4 The dimensional reduction model: effective action formalism

The crucial question of this paper is how one can reproduce the results of section 3 inferred by the canonical quantization procedure using the effective action formalism and in particular whether one can find an (at least approximate) analytical expression for the \( T_{\mu\nu} \) in the Hartle-Hawking state which has the required behaviour at infinity and at the horizon.

By integrating the trace anomaly eq. (3.15) one can write the effective action \( \Gamma(g) \) for the model in the following form

\[
\Gamma(g) = -\frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ \frac{1}{48} R \frac{1}{\Box} R - \frac{1}{4} (\nabla \phi)^2 \frac{1}{\Box} R + \frac{1}{4} \phi R \right] + \Gamma_0 = S_{AI}(g) + \Gamma_0(g) \quad (4.1)
\]

where we have indicated with \( S_{AI} \) the anomaly induced contribution and \( \Gamma_0 \) as usual is an arbitrary conformal invariant (not necessarily local) functional. The difference of \( S_{AI} \) with respect to the Polyakov action are the two terms (one nonlocal and the other local) containing the dilaton.

An interesting attempt to find a workable expression of \( \Gamma(g) \) including finite temperature corrections has been made in Ref. [12]. Using the high frequency approximation, a new approximation scheme for quantum fields in (static) curved space, one has obtained the following form of \( \Gamma(g) \)

\[
\Gamma_{HFA}(g) = \int d^2x \sqrt{-g} \left[ -\frac{1}{24\pi \chi^2 \beta^2} + \frac{(\nabla \eta)^2}{6\pi} - \frac{1}{8\pi} \ln \left( \frac{m^2 \beta^2 \chi^2}{4e^{-2\gamma}} \right) \left( \frac{R}{6} - (\nabla \phi)^2 + \Box \phi \right) \right],
\]

where \( \eta = -\frac{1}{4} \ln \chi^2 \) and \( \chi^2 \) is the norm of the Killing vector.

For zero temperature \( \Gamma_{HFA} \) is just \( S_{AI} \) specialised to static 2D spacetimes.

Coming to the finite temperature case, it is interesting to note that neglecting the dilaton terms in \( \Gamma_{HFA}(g) \), one has an interesting expression for the finite temperature Polyakov effective action, starting from which the results of section 2 can be derived in an alternative and elegant way.

Coming back to the actual problem, variation of \( \Gamma_{HFA} \) with respect to the metric gives

\[
\langle T_{\mu\nu} \rangle_{HFA} = \frac{1}{24\pi \beta^2 \chi^2} \left[ g_{\mu\nu} - \frac{2 \chi_{\mu} \chi_{\nu}}{\chi^2} \right] - \frac{1}{6\pi} g_{\mu\nu} (\nabla \eta)^2 + \frac{1}{3\pi} \nabla_{\mu} \eta \nabla_{\nu} \eta - \frac{1}{6\pi} \frac{\chi_{\mu} \chi_{\nu}}{\chi^2} \Box \eta +
\]
\[ \frac{1}{4\pi} \chi_\mu \chi_\nu \left[ \frac{R}{6} - (\nabla \phi)^2 + \Box \phi \right] - \frac{1}{6\pi} [\nabla_\mu \nabla_\nu \eta - g_{\mu\nu} \Box \eta] - \frac{1}{2\pi} [\nabla_\mu \phi \nabla_\nu \eta + \nabla_\nu \phi \nabla_\mu \eta - g_{\mu\nu} \nabla \phi \nabla \eta] + \frac{1}{8\pi} \ln \left( \frac{m^2 \beta^2 \chi^2}{4e^{-2\gamma}} \right) [g_{\mu\nu} (\nabla \phi)^2 + 2 \nabla \phi \nabla \phi] . \] (4.3)

For the Schwarzschild spacetime \( \chi^2 = (1 - 2M/r) \) and as shown in Ref. [12] these expressions coincide exactly with eqs. (3.12) and (3.13) obtained by the canonical formalism under the large \( w \) WKB approximation. So the resulting \( T_{\mu\nu} \) has the correct asymptotic behaviour, but presents for \( \beta = \beta_H \) the unphysical log divergence on the horizon. Therefore \( \Gamma_{HFA} (g) \) does not represent a valuable solution to our problem.

Let us come back to the expression (4.1). For conformally related metrics \( g_{ab} = e^{2\sigma} \hat{g}_{ab} \) one has the following relation which, we stress, is an exact identity

\[ \Gamma(g) = \Gamma(\hat{g}) - \frac{1}{24\pi} \int d^2 x \sqrt{-\hat{g}} \left[ (\hat{\nabla} \sigma)^2 + \hat{R} \sigma \right] - \frac{1}{4\pi} \int d^2 x \sqrt{-\hat{g}} \left[ \sigma (\hat{\Box} \phi - (\hat{\nabla} \phi)^2) \right] . \] (4.4)

The corresponding relation for the stress tensor is

\[ T_{\mu\nu}(g) = T_{\mu\nu}(\hat{g}) - \frac{1}{48\pi} \left[ -4 \hat{\nabla}_\mu \hat{\nabla}_\nu \sigma + 4 \hat{\nabla}_\mu \sigma \hat{\nabla}_\nu \sigma + \hat{g}_{\mu\nu} (4\Box \sigma - 2(\hat{\nabla} \sigma)^2) \right] + \frac{1}{2\pi} \left[ \frac{\hat{\nabla}_\sigma (\hat{\nabla} \phi)}{2} - \frac{1}{2} \hat{g}_{\mu\nu} (\hat{\nabla} \phi) (\hat{\nabla} \sigma) + \sigma \hat{\partial}_\mu \phi \hat{\partial}_\nu \phi - \frac{1}{2} \hat{g}_{\mu\nu} \sigma (\hat{\nabla} \phi)^2 \right] . \] (4.5)

The idea is now to proceed as in the Polyakov model (see section 2). Writing the Schwarzschild metric as

\[ ds^2 = -(1 - 2M/r)(dt^2 - dr^2) \] (4.6)

one gets

\[ T_{tt}(g) = T_{tt}(\hat{g}) + \frac{1}{12\pi} \left[ - \frac{2M}{r^3} + \frac{7M^2}{2r^4} \right] + \frac{1}{2\pi} \left[ - \frac{Mf}{2r^3} + f^2 \ln f \right] \] (4.7)

and similarly

\[ T_{rr}(g) = T_{rr}(\hat{g}) - \frac{1}{12\pi f^2} \left[ \frac{M^2}{2r^4} - \frac{1}{2\pi f} \left[ \frac{M}{2r^3} - f \ln f \right] \right] \] (4.8)

where \( \hat{g} \) is Minkowski spacetime.

The striking difference with the previous case is that here one is not allowed to set \( T_{ab}(\hat{g}) = 0 \) since it is true that \( \hat{g} \) is flat space, but our matter field \( f \) is now coupled not only to the metric but to the dilaton as well (\( \phi = - \ln(r) \)) whose expression in terms of \( r^* \) is rather complicated. In particular, from the anomaly one has

\[ T(\hat{g}) = T_{a}^{\phi}(\hat{g}) = \frac{1}{24\pi} (6 \hat{\nabla} \phi - 6(\hat{\nabla} \phi)^2) = - \frac{Mf}{2\pi r^3} \] (4.9)

which is indeed not vanishing. Writing

\[ T_{ab}(g) = T_{ab}(\hat{g}) - \frac{1}{2} \hat{g}_{ab} T(\hat{g}) + \frac{1}{2} \hat{g}_{ab} T(\hat{g}) \] (4.10)
and (à la Brown-Ottewill [8]) neglecting the traceless contribution, we can write an approximate analytic expression for the zero temperature stress tensor for the dimensional reduction theory as

\[
\langle B|T_{tt}|B \rangle_{BO} = -\frac{1}{2\pi} \left( \frac{M}{3r^3} - \frac{7M^2}{12r^4} \right) + \frac{f^2 \ln(f)}{8\pi r^2},
\]

\[\text{(4.11)}\]

\[
\langle B|T_{rr}|B \rangle_{BO} = \frac{1}{2\pi f^2} \left( -\frac{fM}{r^3} - \frac{M^2}{12r^4} \right) + \frac{\ln(f)}{8\pi r^2}.
\]

\[\text{(4.12)}\]

As in eqs. (3.10), (3.11) we see that asymptotically \(\langle T_{\mu\nu} \rangle \to 0\). Moreover, also the expression (3.8) is correctly reproduced. Despite these good properties, we see that the polynomial parts in the above eqs. are quite different from those we have obtained using the WKB approximation scheme. Performing the same calculation for the thermal state (using eqs. (2.18)-(2.21) ) we find

\[
T_{tt}(g) = T_{tt}(\hat{g}) + \frac{f^2}{384 \pi M^2} \left[ 1 + \frac{4M}{r} + \frac{60M^2}{r^2} - 48\frac{M^2}{r^2} \ln \frac{r}{2M} \right] - \frac{Mf}{12\pi r^3},
\]

\[\text{(4.13)}\]

\[
T_{rr}(g) = T_{rr}(\hat{g}) + \frac{1}{384 \pi^3} \left[ 1 + \frac{4M}{r} + \frac{60M^2}{r^2} - 48\frac{M^2}{r^2} \ln \frac{r}{2M} \right] + \frac{M}{12\pi r^3 f},
\]

\[\text{(4.14)}\]

where now \(\hat{g}\) is the flat disc for \(\beta = \beta_H\) of the flat cone for \(\beta \neq \beta_H\). Again \(T_{ab}(\hat{g}) \neq 0\). Neglecting as before the traceless part of \(T_{ab}(\hat{g})\) we can approximate the Hartle-Hawking stress tensor, à la Brown-Ottewill, as following

\[
\langle H|T_{tt}|H \rangle_{BO} = \frac{f^2}{384 \pi M^2} \left[ 1 + \frac{4M}{r} + \frac{60M^2}{r^2} - 48\frac{M^2}{r^2} \ln \frac{r}{2M} \right] + \frac{fM}{6\pi r^3},
\]

\[\text{(4.15)}\]

\[
\langle H|T_{rr}|H \rangle_{BO} = \frac{1}{384 \pi M^2} \left[ 1 + \frac{4M}{r} + \frac{60M^2}{r^2} - 48\frac{M^2}{r^2} \ln \frac{r}{2M} \right] - \frac{M}{6\pi r^3 f}.
\]

\[\text{(4.16)}\]

This stress tensor has extremely nice features. Comparison with \(\langle H|T_{\mu\nu}|H \rangle_{WKB}\) of eqs. (3.12)-(3.13) (with \(T = T_H\)) shows the correct asymptotic behavior not only in the leading \(T_H^2/6\pi\) term but also in the \(1/r\) term which is quite remarkable. Furthermore this \(T_{\nu}^\mu\) has the property of being regular on the horizon where the limiting value of eq. (3.16) are reached. This is our first proposal for \(T_{\mu\nu}\) in the Hartle-Hawking state. How good this approximation is cannot be said because of our ignorance of the traceless part of \(T_{ab}(\hat{g})\). Numerical computation or a better analytical approximation of \(T_{\mu\nu}(\hat{g})\) near the horizon are required.

## 5 The dimensional reduction model: local fields formulation

As for the Polyakov case one can find a local expression for the effective action depending on the background metric \(g_{\mu\nu}\) and (now) two auxiliary fields \(\psi\) and \(\chi\) [13]

\[
\Gamma(g) = -\frac{1}{96\pi} \int d^2x \sqrt{-g} \left[ 2R(\psi - 6\chi) + (\nabla\psi)^2 - 12\nabla\psi \nabla\chi - 12\psi(\nabla\phi)^2 + 12R\phi \right] + \Gamma_0
\]

11
\begin{align}
&= S_{AI}^{loc} + \Gamma_0 \tag{5.1}
\end{align}

where \( \psi \) and \( \chi \) satisfy
\begin{align}
\Box \psi &= R, \quad \Box \chi = (\nabla \phi)^2, \tag{5.2}
\end{align}

\( S_{AI}^{loc} \) is the local form of \( S_{AI} \) and \( \Gamma_0 \) is an arbitrary conformal invariant functional. If we neglect \( \Gamma_0 \) the stress tensor corresponding to \( S_{AI}^{loc} \) is
\begin{align}
\langle T_{\mu\nu} \rangle &= -\frac{1}{48\pi} \left[ 2\nabla_\mu \nabla_\nu \psi - \nabla_\mu \psi \nabla_\nu \chi - g_{\mu\nu}(2R - \frac{1}{2}(\nabla \psi)^2) \right] - \frac{1}{4\pi} \left[ \frac{g_{\mu\nu}}{2}((\nabla \phi)^2 \psi + \nabla \psi \nabla \chi - 2(\nabla \phi)^2) + \frac{1}{2} \nabla(\phi \nabla_\nu \psi - \nabla_\mu \nabla \phi) \right] + \frac{1}{4\pi} (g_{\mu\nu} \Box \phi - \nabla_\mu \nabla_\nu \phi). \tag{5.3}
\end{align}

The solution for the auxiliary field \( \psi \) is (see eqs. (2.25) and (2.28)) is
\begin{align}
\psi &= -\ln f + br^* \tag{5.4}
\end{align}

where we set as before \( b = 2/\beta \). The other eq. (5.2) is solved by
\begin{align}
\chi &= -\frac{1}{2} \ln(r/2M - 1) - \frac{1}{2} \ln r/l. \tag{5.5}
\end{align}

The resulting stress tensor reads
\begin{align}
T_{tt} &= -\frac{1}{24\pi} \left[ \frac{2M}{r^3} - \frac{3M^2}{r^4} - \frac{1}{\beta^2} \right] - \frac{1}{2\pi} \left[ \frac{f^2}{4r^2} \frac{2r}{\beta} + \frac{4M}{\beta} \ln(r - 2M)/l - \ln f \right] \\
&+ \frac{1}{4M\beta r^2} (-4Mr + 4M^2) + \frac{fM}{6\pi r^3}], \tag{5.6}
\end{align}

whereas
\begin{align}
T_{rr} &= -\frac{1}{24\pi f^2} \left[ \frac{2M}{r^3} - \frac{3M^2}{r^4} - \frac{1}{\beta^2} \right] - \frac{1}{2\pi} \left[ \frac{1}{4r^2} \left( \frac{2r}{\beta} + \frac{4M}{\beta} \ln(r - 2M)/l - \ln f \right) \\
&+ \frac{1}{4M\beta r^2 f^2} (-4Mr + 4M^2) - \frac{M}{6\pi r^3 f}. \tag{5.7}
\end{align}

It is rather disappointing to see that in the regular instanton case \( \beta = \beta_H \), which would correspond to the Hartle-Hawking case, the stress tensor
\begin{align}
T_{tt}|_{\beta = \beta_H} &= -\frac{1}{24\pi} \left[ \frac{2M}{r^3} - \frac{3M^2}{16M^2} \right] - \frac{1}{2\pi f^2} \left[ \frac{f}{2M} + \ln r/l \right] \\
&+ \frac{1}{32M^2 r^2 f^2} (-4Mr + 4M^2) + \frac{Mf}{6\pi r^3 f}], \tag{5.8}
\end{align}

\begin{align}
T_{rr}|_{\beta = \beta_H} &= -\frac{1}{24\pi f^2} \left[ \frac{2M}{r^3} - \frac{3M^2}{16M^2} \right] - \frac{1}{2\pi f^2} \left[ \frac{r}{2M} + \ln r/l \right] \\
&+ \frac{1}{32M^2 r^2 f^2} (-4Mr + 4M^2) - \frac{M}{6\pi r^3 f}], \tag{5.9}
\end{align}

\[ \text{Page 12} \]
although having the correct asymptotic behaviour is not regular on the horizon. One can circumvent this negative result by including a homogeneous solution to the second of eqs. (5.2). This, as shown in the second of Refs. [13], allows to construct a $T_{\mu\nu}|_{\beta=\beta_H}$ which is regular on the horizon, unfortunately the homogeneous solution modifies also the asymptotic behaviour giving an unphysical negative Hawking radiation. The failure of the local fields formalism to give meaningful results has to be found in the completely arbitrary assumption we made to neglect $\Gamma_0$ in (5.1). Unlike the previous case (Polyakov), now the inclusion of homogeneous solutions to the auxiliary field equations does not reproduce completely the contribution coming from the unknown part of the effective action. The reason is that now the corresponding $T_{\mu\nu}$ is not conserved (see eq. (1.1) in Appendix A) and the staticity and traceless conditions are not sufficient to fix this contribution unambiguously. Unfortunately for the moment no closed workable expression for $\Gamma_0$ is known (see however the interesting work of Ref. [7]). In the following we will show how a simple form of $\Gamma_0$ is sufficient to lead to a regular $T_{\mu\nu}|_{\beta_H}$.

6 A “phenomenological approach”

Let us start by rewriting $S_{AI}^{loc}$ in a more symmetric form

$$S_{AI}^{loc} = S_1(\tilde{\psi}) + S_2(\tilde{\chi}) + S_3$$

where

$$S_1(\tilde{\psi}) = \int d^2x \sqrt{-g} \left[ \frac{1}{2} \tilde{\psi} \Box \tilde{\psi} + \frac{1}{\sqrt{48\pi}} \left( R - 6(\nabla\phi)^2 \right) \tilde{\psi} \right],$$

$$S_2(\tilde{\chi}) = \int d^2x \sqrt{-g} \left[ -\frac{1}{2} \tilde{\chi} \Box \tilde{\chi} + \sqrt{\frac{3}{4\pi}}(\nabla\phi)^2 \tilde{\chi} \right]$$

and finally

$$S_3 = \int d^2x \sqrt{-g} \left[ -\frac{1}{8\pi} \phi R \right].$$

The auxiliary fields $\tilde{\psi}$ and $\tilde{\chi}$ satisfy the equations of motion

$$\Box \tilde{\psi} = -\frac{1}{\sqrt{48\pi}} \left( R - 6(\nabla\phi)^2 \right),$$

$$\Box \tilde{\chi} = \sqrt{\frac{3}{4\pi}}(\nabla\phi)^2.$$  

$\tilde{\psi}$ and $\tilde{\chi}$ are related to the fields $\psi$ and $\chi$ of the previous section by

$$\tilde{\psi} = -\frac{1}{\sqrt{48\pi}}(\psi - 6\chi), \quad \tilde{\chi} = \sqrt{\frac{3}{4\pi}}\chi.$$  

The energy momentum tensor can be similarly splitted in three terms

$$\langle T_{\mu\nu} \rangle = T^1_{\mu\nu}(\tilde{\psi}) + T^2_{\mu\nu}(\tilde{\chi}) + T^3_{ab}$$
where
\[ T^1_{\mu\nu}(\tilde{\psi}) = \partial_\mu \tilde{\psi} \partial_\nu \tilde{\psi} + \frac{2}{\sqrt{48\pi}} \nabla_\mu \nabla_\nu \tilde{\psi} + \frac{12}{\sqrt{48\pi}} \partial_\mu \phi \partial_\nu \phi \tilde{\psi} 
+ g_{\mu\nu} \left( \left( \nabla \tilde{\psi} \right)^2 - \frac{2}{\sqrt{48\pi}} \Box \tilde{\psi} - \frac{6}{\sqrt{48\pi}} \left( \nabla \phi \right)^2 \tilde{\psi} \right), \] (6.9)

\[ T^2_{\mu\nu}(\tilde{\chi}) = -\partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - 2\sqrt{\frac{3}{4\pi}} \partial_\mu \phi \partial_\nu \phi \tilde{\chi} 
+ g_{\mu\nu} \left( \frac{1}{2} \left( \nabla \tilde{\chi} \right)^2 + \sqrt{\frac{3}{4\pi}} \left( \nabla \phi \right)^2 \tilde{\chi} \right), \] (6.10)

and
\[ T^3_{ab} = -\frac{1}{4\pi} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi \right). \] (6.11)

One should note that \( T^{(2)}_{\mu\nu}(\chi) \) is traceless, being \( S_2 \) conformally invariant. On the other hand we have that the trace \( \langle T \rangle \equiv \langle T^\mu_\mu \rangle \) is given by
\[ T \equiv \langle T^\mu_\mu \rangle = T^1_\mu(\tilde{\psi}) + T^3_\mu = -\frac{1}{\sqrt{12\pi}} \Box \tilde{\psi} + \frac{1}{4\pi} \Box \phi = \frac{1}{24\pi} (R - 6(\nabla \phi)^2 + 6\Box \phi). \] (6.12)

Let us now give a closer look at the local form of \( S^{loc}_{AI} \) eq. (6.1). We already remarked that the second nonlocal term is conformally invariant. \( \Gamma(g) \) is obtained by functional integration of the trace anomaly. This procedure determines \( \Gamma(g) \) up to conformal invariant terms. Therefore terms like \( S_2 \), whose nonlocal expression is
\[ \frac{3}{8\pi} \int d^2x \sqrt{-g} \left( -\frac{1}{2} \Box \chi + l_1 \left( \nabla \phi \right)^2 \chi \right), \] (6.13)

are completely arbitrary, in particular its overall coefficient \( \frac{3}{8\pi} \). Given our present ignorance we shall try to mimic the effect of this unknown part by adding to \( S^{loc}_{AI} \) an additional nonlocal term proportional to (6.13) which being Weyl invariant does not alter the trace anomaly. As a consequence the nonlocal effective action we shall consider is the following
\[ S_{AI}^{local} + \int d^2x \sqrt{-g} \left( \frac{l_1^2}{2} - \frac{3}{8\pi} \right) \left( \nabla \phi \right)^2 \frac{1}{\Box} \left( \nabla \phi \right)^2, \] (6.14)

where \( l_1 \) is an arbitrary parameter (see a similar procedure in 4D in [14]). This should mimic the state dependence of the effective action, the parameter \( l_1 \) taking different values according to the state of the quantum field. In the local formulation \( S_2 \) is modified to
\[ S_2(\chi) = \int d^2x \sqrt{-g} \left[ -\frac{1}{2} \Box \tilde{\chi} \tilde{\chi} + l_1 \left( \nabla \phi \right)^2 \tilde{\chi} \right] \] (6.15)

and accordingly
\[ T^2_{\mu\nu}(\chi) = -\partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - 2l_1 \partial_\mu \phi \partial_\nu \phi \tilde{\chi} + g_{\mu\nu} \left( \frac{1}{2} \left( \nabla \tilde{\chi} \right)^2 + l_1 \left( \nabla \phi \right)^2 \tilde{\chi} \right). \] (6.16)
In the Schwarzschild spacetime the general sol. to the auxiliary field eqs. reads
\[ \tilde{\psi} = -\frac{1}{\sqrt{48\pi}} \left[ Ar + (2MA + 2) \ln\left(\frac{r}{2M} - 1\right) + 4 \ln \frac{r}{l} \right] \] (6.17)

and
\[ \tilde{\chi} = \sqrt{\frac{3}{4\pi}} \left[ Br + (2MB - \frac{1}{2}) \ln\left(\frac{r}{2M} - 1\right) - \frac{1}{2} \ln \frac{r}{l} \right], \] (6.18)

where \( A \) and \( B \) are integration constants corresponding to the homogeneous solution and \( l \) is an arbitrary scale. The corresponding expression for \( \langle T_{\mu\nu} \rangle \) becomes (we shall work in \( u, v \) Eddington-Finkelstein coordinates)
\[ \langle T_{uu} \rangle = \langle T_{vv} \rangle = \frac{1}{192\pi} \left( fA + \frac{2AM + 2}{r} + \frac{4f}{r} \right)^2 + \frac{1}{96\pi} \left( 2AM + 2 + \frac{4f^2}{r^2} \right), \] (6.19)
\[ \langle T_{uv} \rangle = \frac{1}{12\pi} \left( 1 - \frac{2M}{r} \right) \frac{M}{r^3}. \] (6.20)

We now proceed to determine these constants.

The Boulware vacuum is required to coincide with the Minkowski vacuum when \( M = 0 \). Vanishing of the logarithmic part
\[ \lim_{M \to 0} -\frac{1}{16\pi} \left[ 2 \ln\left(\frac{r}{2M} - 1\right) + 4 \ln \frac{r}{l} \right] + \frac{l_1^2}{4} \left[ \ln\left(\frac{r}{2M} - 1\right) + \ln \frac{r}{l} \right] = 0 \] (6.21)

requires \( l = 2M \) and \( l_1^2 = 3/4\pi \) (i.e. the extra term in (6.14) vanishes). The resulting \( \langle B|T_{ab}|B \rangle \) is
\[ \langle B|T_{uu}|B \rangle = \langle B|T_{vv}|B \rangle = \frac{1}{24\pi} \left[ -\frac{M}{r^3} + \frac{3M^2}{2r^4} \right] + \frac{1}{16\pi} \left( 1 - \frac{2M}{r} \right)^2 \frac{1}{r^2} \ln\left( 1 - \frac{2M}{r} \right), \] (6.22)
\[ \langle B|T_{uv}|B \rangle = \frac{1}{12\pi} \left( 1 - \frac{2M}{r} \right) \frac{M}{r^3}. \] (6.23)

In \((t, r)\) coords. this corresponds exactly to \( \langle B|T_{\mu\nu}|B \rangle_{WK} \) of eqs. (3.10), (3.11) once we fix \( m^2 = 4\lambda^2 \).

We come now to the thermal Hartle-Hawking case. Regularity on the horizon \(^3\) of the log and polynomial parts require
\[ 2AM + 2 = -8\pi l_1^2 (2MB - \frac{1}{2}), \]
\[ (2AM + 2)(AM + 2) = 24\pi l_1^2 (2MB - \frac{1}{2})^2. \] (6.24)

\(^3\)Working in \( u, v \) coords. we mention that regularity in a free falling frame in both the future and past horizon is achieved by requiring \( \langle T_{uu} \rangle / f^2 < \infty, \langle T_{vv} \rangle / f^2 < \infty \) and \( \langle T_{uv} \rangle / f < \infty \).
The asymptotic condition of a thermal bath at temperature $T_H = (8\pi M)^{-1}$ yields

\[ \frac{A^2}{192\pi} - \frac{l_1^2 B^2}{4} = \left(768\pi M^2\right)^{-1} = T_H^2 / 12\pi. \]  

(6.25)

These equations admit, as general solution, a unique value for $l_1$ given by

\[ l_1 = \frac{1}{2\sqrt{\pi}} \]  

(6.26)

and two possibilities for the constants $A, B$, that is $A = -\frac{1}{4M}$, $B = \frac{1}{4M}$ (i.e. $\tilde{\psi}$ and $\tilde{\chi}$ regular at $r = 2M$) and $A = -\frac{1}{2M}$, $B = 0$. Both cases give the same Hartle-Hawking stress tensor

\[ \langle H | T_{uu} | H \rangle = \langle H | T_{vv} | H \rangle = \frac{f^2}{768\pi M^2} \left[ 1 + \frac{4M}{r} + \frac{36M^2}{r^2}(1 - 4\ln \frac{r}{l}) \right], \]  

(6.27)

\[ \langle H | T_{uv} | H \rangle = \frac{1}{12\pi} \left(1 - \frac{2M}{r}\right) \frac{M}{r^3}. \]  

(6.28)

$l$ remains as an arbitrary scale. In $(t, r)$ coordinates these expressions become (where we fix $l = 2M$)

\[ \langle H | T_{tt} | H \rangle = \frac{f^2}{384\pi M^2} \left[ 1 + \frac{4M}{r} + \frac{36M^2}{r^2}(1 - 4\ln \frac{r}{2M}) \right] + \frac{fM}{6\pi r^3}, \]  

(6.29)

\[ \langle H | T_{rr} | H \rangle = \frac{1}{384\pi M^2} \left[ 1 + \frac{4M}{r} + \frac{36M^2}{r^2}(1 - 4\ln \frac{r}{2M}) \right] - \frac{M}{6\pi r^3}. \]  

(6.30)

Let us compare eqs. (6.29), (6.30) with the analytical expression for $\langle H | T_{ab} | H \rangle_{WKB}$ obtained by canonical quantization eqs. (3.12), (3.13) (with $T = T_H$). Being by construction both expressions regular on the horizon we have the correct limiting value of $\langle H | T_{tt} | H \rangle$ and $\langle H | T_{rr} | H \rangle$ at $r = 2M$ given by (3.16). Asymptotically the correct limiting behaviour has been imposed by our “phenomenological” construction. However one can see that also the $1/r$ term has the correct coefficient as provided by the WKB approximation (see eqs. (3.12), (3.13)). This is definitely a nontrivial outcome of the model. Finally, comparing with $\langle H | T_{\mu\nu} | H \rangle_{BO}$ of section 4 we see that the only difference are in the numerical coeffs. of the terms $\frac{1}{r^2}$ and $\frac{1}{r^2} \ln \frac{r}{2M}$ inside the square parenthesis.

### 7 Conclusions

The theoretical relevance of the model described by the action (3.1) lies in its intimate connection to real 4D physics. It is therefore crucial that the predictions made by the analysis of this model be trustworthy. In this spirit any proposed effective action $\Gamma(q)$ which for a Schwarzschild black hole does not nicely reproduce at least the asymptotic behaviour described by eq. (3.9) and is regular on the horizon should be regarded with suspicion. Using the conformal transformation law of the effective action we were able to
find, à la Brown-Ottewill [8], an approximation for the analytical expression of the $\langle T_{\mu\nu}\rangle$ in Boulware and Hartle-Hawking states. Another expression can be obtained by adding to $S_{AI}$ a conformal invariant functional (see eq. (6.14). We mention that a term of this type arises in the perturbative expansion of the effective action in powers of $P = (\nabla\phi)^2 - \Box\phi$ proposed in [7]. In particular we have shown that for the Boulware vacuum this extra term vanishes and $S_{AI}$ reproduces exactly the WKB form of $\langle B|T_{ab}|B \rangle$. For the Hartle-Hawking state the extra term is crucial to have the correct asymptotic behaviour and regularity on the horizon at the same time. The expressions we have proposed give the correct value of $\langle H|T_t^t|H \rangle$ and $\langle H|T_r^r|H \rangle$ on the horizon and agree with the WKB form of $\langle H|T_{\mu\nu}|H \rangle$ even to the $O(1/r)$ term for $r \to \infty$. Finally, for the quantum states considered one can construct the 2D analogue of the 4D pressure term $\langle P \rangle \equiv \langle T_{\theta\theta} \rangle$ using the nonconservation equations for the 2D stress tensor $\langle T_{ab} \rangle$. The results are given in Appendix A.

A Calculation of the pressure terms

For the theory of eq. (3.1) the stress tensor is not conserved ([15], [4]). Indeed $\langle T_{\mu\nu} \rangle$ satisfies the following nonconservation eqs.

$$\nabla_\mu \langle T^\mu_\nu \rangle = -\frac{1}{\sqrt{-g}} \left\langle \frac{\delta S}{\delta \phi} \nabla_\nu \phi \right\rangle.$$  \hspace{1cm} (1.1)

This is nothing else but the 4D conservation eqs. $\nabla_\mu \langle T^\mu_\nu \rangle = 0$, where the 4D tangential pressure in the local field formulation of section 6 is represented by

$$\langle P \rangle = \frac{1}{8\pi r^2} \frac{\delta S}{\delta \phi} = \frac{1}{4\pi r^2} \left[ l_1 (\Box \phi \tilde{\chi} + f \partial_r \phi \partial_r \tilde{\chi}) - \frac{6}{\sqrt{48\pi}} (\Box \phi \tilde{\psi} + f \partial_r \phi \partial_r \tilde{\psi}) + \frac{R}{16\pi} \right].$$  \hspace{1cm} (1.2)

Its general expression in the Scharschchild spacetime is

$$\langle P \rangle = \frac{1}{4\pi r^2} \left[ l_1^2 \left( 1 - \frac{4M}{r} \right) \left( B(r + 2M \ln(\frac{r}{2M} - 1)) - \frac{1}{2} \ln(\frac{r}{2M} - 1) - \frac{1}{2} \ln \frac{r}{l} \right) \right.
\quad - \left. l_1^2 \frac{f}{r} \left( B + \frac{2MB - 1/2}{r - 2M} - \frac{1}{2r} \right) \right.
\quad + \left. \frac{1}{8\pi r^2} \left( A(r + 2M \ln(\frac{r}{2M} - 1)) + 2 \ln(\frac{2M}{2M} - 1) + 4 \ln \frac{r}{l} \right) \right.
\quad - \left. \frac{f}{8\pi r} \left( A + \frac{2AM + 2}{r - 2M} + \frac{4}{r} \right) + \frac{M}{4\pi r^3} \right].$$  \hspace{1cm} (1.3)

Substituting the values found for $A, B, l_1$ in the states considered we find $\langle P \rangle$ in Boulware and Hartle-Hawking states

$$\langle B|P|B \rangle = \frac{1}{32\pi^2} \left[ \frac{4M}{r^5} - \frac{(1 - 4M/r)}{r^4} \ln(1 - \frac{2M}{r}) \right],$$  \hspace{1cm} (1.4)
\[ \langle H|P|H \rangle = \frac{1}{32\pi^2} \left[ \frac{8M}{r^5} - \frac{2}{r^4} + 3\left(1 - \frac{4M}{r^4}\right) \ln \frac{r}{2M} \right]. \]  \hfill (1.5)

In particular, we find that \( < B|P|B > \) exhibits a logarithmic divergence (\( \sim \ln(1 - 2M/r) \)) at the horizon, whereas \( < H|P|H > \) is regular there. On the other hand substituting the \( \langle T_{\mu\nu} \rangle_{BO} \) approximations of section 4 one obtains

\[ \langle B|P|B \rangle_{BO} = \frac{1}{8\pi r} \left[ -(1 - \frac{4M}{r}) \frac{\ln(f)}{4\pi r^3} + \frac{M}{4\pi r^4} + \frac{1}{2\pi f} \left( \frac{3M}{r^4} - \frac{7M^2}{r^5} \right) \right], \]  \hfill (1.6)

\[ \langle H|P|H \rangle_{BO} = \frac{1}{768\pi^2 Mr^3} \left[ 1 + \frac{4M}{r} + \frac{108M^2}{r^2} - \frac{48M^2}{r^2} \ln \frac{r}{2M} \
- f(1 + \frac{30M}{r} - \frac{24M}{r} \ln \frac{r}{2M} + \frac{12M}{r}) \right]. \]  \hfill (1.7)

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