MEAN VALUE TYPE INEQUALITIES FOR QUASINEARLY
SUBHARMONIC FUNCTIONS

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ABSTRACT. The mean value inequality is characteristic for upper semicontinuous functions to be subharmonic. Quasinearly subharmonic functions generalize subharmonic functions. We find the necessary and sufficient conditions under which subsets of balls are big enough for the catheterization of nonnegative, quasinearly subharmonic functions by mean value inequalities. Similar result is obtained also for generalized mean value inequalities where, instead of balls, we consider arbitrary bounded sets which have nonvoid interiors and instead of the volume of ball some functions depending on the radius of this ball.

1. Subharmonic functions and generalizations. Some definitions and results

1.1. Notation. Our notation is rather standard, see e.g. [12][18][20] and the references therein. If \( E \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), then we write \( \delta E(x) := \inf \{ |x - y| : y \in E^c \} \), where \( E^c = \mathbb{R}^n \setminus E \). The Lebesgue measure in \( \mathbb{R}^n \) is denoted by \( m_n \). We write \( B^r(x, r) \) for the open ball in \( \mathbb{R}^n \) with center \( x \) and radius \( r \). Recall that \( m_n(B^r(x, r)) = v_n r^n \), where \( v_n := m_n(B^n(0, 1)) \). We denote by \( \text{Int} D, \text{cl} D \) and \( \partial D \) the interior, the closure and the boundary of a set \( D \subset \mathbb{R}^n \), i.e., \( \overline{B^r(x, r)} \) is the closed ball with center \( x \) and radius \( r \). Note also that our constants \( C \) and \( K \) are nonnegative, mostly \( \geq 1 \), and may vary from line to line.

1.2. Subharmonic functions and generalizations. Let \( \Omega \) be an open set in \( \mathbb{R}^n, n \geq 2 \). Let \( u : \Omega \to [-\infty, +\infty) \) be Lebesgue measurable. We adopt the following definitions:

(i) \( u \) is subharmonic if \( u \) is upper semicontinuous and if

\[
(1) \quad u(x) \leq \frac{1}{v_n r^n} \int_{B^r(x, r)} u(y) \, dm_n(y)
\]

for all balls \( B^r(x, r) \subset \Omega \). A subharmonic function may be \( \equiv -\infty \) on any component of \( \Omega \). See [13] p. 1], [? p. 18], [9 p. 9], and [1 p. 60].

(ii) \( u \) is nearly subharmonic if \( u^+ \in \mathcal{L}^{1}_{\text{loc}}(\Omega) \) and inequality (1) holds for all balls \( \overline{B^r(x, r)} \subset \Omega \). Observe that this definition is slightly more general than the usual one, compare [19] p. 51, with the standard definitions [13 p. 20], [? p. 30], and [9 p. 14].

(iii) Let \( K \geq 1 \). Then \( u \) is \( K \)-quasinearly subharmonic if \( u^+ \in \mathcal{L}^{1}_{\text{loc}}(\Omega) \) and inequality

\[
\inf_{M \geq 0} u^+_M(x) \leq \frac{K}{v_n r^n} \int_{B^r(x, r)} u^+_M(y) \, dm_n(y)
\]

holds for all \( M \geq 0 \) and for all balls \( \overline{B^r(x, r)} \subset \Omega \). Here \( u^+_M := \max \{ u, -M \} + M \).

The function \( u \) is quasinearly subharmonic if \( u \) is \( K \)-quasinearly subharmonic for some \( K \geq 1 \). For the definition and properties of quasinearly subharmonic functions, see e.g. [6][10][12][14][20], and the references therein. We write \( \mathcal{QNS}(\Omega) \) for the set of all nonnegative quasinearly subharmonic functions on the open set \( \Omega \subset \mathbb{R}^n \).

1.3. Proposition. (cf. [19], Proposition 2.1, pp. 54-55) The following statements hold:

(i) A subharmonic function is nearly subharmonic but not conversely.

(ii) A function is nearly subharmonic if and only if it is 1-quasinearly subharmonic.

(iii) A nearly subharmonic function is quasinearly subharmonic but not conversely.

1991 Mathematics Subject Classification. Primary 31B05, 31C05; Secondary 31C45.

Key words and phrases. Subharmonic, nearly subharmonic, quasinearly subharmonic, mean value.
(iv) If \( u : \Omega \to [0, +\infty) \) is Lebesgue measurable, then \( u \) is \( K \)-quasinearly subharmonic if and only if
\[
(2) \quad u(x) \leq \frac{K}{m_n(B^r(x,r))} \int_{B^r(x,r)} u(y) \, dm_n(y)
\]
for all closed balls \( \overline{B^r(x,r)} \subseteq \Omega \).

Note that if \( u \) is \( K \)-quasinearly subharmonic and nonnegative in \( \Omega \), then (2) holds also for every open ball \( B^r(x,r) \subseteq \Omega \).

Let \( A \) be a subset of the open half-line \( (0, \infty) \) such that 0 is a limit point of \( A \) and let \( u : \Omega \to [-\infty, +\infty) \) be an upper-semicontinuous function on an open set \( \Omega \subseteq \mathbb{R}^n \). The classical Blascke–Privalov theorem, see, for example, [3, Chapter II, §2] implies that \( u \) is subharmonic if inequality (1) holds whenever \( r \in A \) and \( B^r(x,r) \subseteq \Omega \). Moreover, the simple examples show that if nonnegative \( u \in L^1_{loc}(\Omega) \), then the fulfillment of (2) for all \( (x,r) \in \Omega \times A \) with \( B^r(x,r) \subseteq \Omega \) does not, generally, imply \( u \in QNS(\Omega) \). A legitimate question to raise in this point is in finding the sets \( A \subseteq (0, \infty) \) for which every nonnegative \( u \in L^1_{loc}(\Omega) \) is quasinearly subharmonic if (2) holds for \( (x,r) \in \Omega \times A \) whenever \( \overline{B^r(x,r)} \subseteq \Omega \).

1.4. Definition. Let \( \Omega \) be an open set in \( \mathbb{R}^n \). A set \( A \subseteq (0, \infty) \) is favorable for \( \Omega \) (favorable for the characterization of nonnegative, quasinearly subharmonic functions in \( \Omega \)) if for every nonnegative \( u \in L^1_{loc}(\Omega) \) the following conditions are equivalent:

(i) \( u \in QNS(\Omega) \);

(ii) There is \( K = K(u,A,\Omega) \geq 1 \) such that for all \( x \in \Omega \) the inequality
\[
(3) \quad u(x) \leq \frac{K}{V_{nr}^n} \int_{B^r(x,r)} u(y) \, dm_n(y)
\]
holds whenever \( r \in A \) and \( \overline{B^r(x,r)} \subseteq \Omega \).

We can characterize the favorable subsets of \( (0, \infty) \) by the following way.

1.5. Theorem. The following three statements are equivalent for every \( A \subseteq (0, \infty) \):

(i) \( A \) is favorable for all open sets \( \Omega \subseteq \mathbb{R}^n \);

(ii) The characteristic function
\[
\chi_{\Gamma}(x) = \begin{cases} 
1 & \text{if } x \in \Gamma \\
0 & \text{if } x \in \Omega \setminus \Gamma 
\end{cases}
\]
is quasinearly subharmonic for all open sets \( \Omega \subseteq \mathbb{R}^n \) and all Lebesgue measurable sets \( \Gamma \subseteq \Omega \) if and only if there is a constant \( K = K(\Gamma,\Omega,n) \) such that the inequality
\[
(4) \quad m_n(B^r(x,r)) \leq Km_n(\Gamma \cap B^r(x,r))
\]
holds for \( (x,r) \in \Omega \times A \) whenever \( \overline{B^r(x,r)} \subseteq \Omega \).

(iii) There exists \( C = C(A) > 1 \) such that
\[
\left[ \frac{x}{C} \right] \cap A \neq \emptyset
\]
for every \( x \in (0, \infty) \).

We shall prove the equivalence (i) \( \equiv \) (ii) in Theorem [2,5] below. Observe also that the implication (i) \( \Rightarrow \) (ii) is trivial and that (ii) \( \Rightarrow \) (iii) follows directly from the proof of Theorem [2,5]. The quasidisks give the important example of the sets \( \Gamma \) such that (4) holds in a bounded domain \( \Omega \subseteq \mathbb{R}^2 \) whenever \( B^r(x,r) \subseteq \Omega \). It is a particular case of the Gehring–Martio result which proves (4) for the so-called quasixtremal distance domains in \( \mathbb{R}^n \), \( n \geq 2 \). See [3] Lemma 2.13.

The following result closely connected to Theorem [1,5] follows from Theorem [2,14] formulated in the second section of the paper.
1.6. **Theorem.** Let $f$ be a positive function on $(0, \infty)$. The following three statements are equivalent.

(i) For all open sets $\Omega \subseteq \mathbb{R}^n$, Lebesgue measurable functions $u : \Omega \to [0, \infty)$ are quasinearly subharmonic if and only if there are constants $K = K(u, \Omega, n) \geq 1$ such that

$$u(x) \leq \frac{K}{(f(r))^n} \int_{B^n(x,r)} u(y) \, dm_n(y)$$

for all closed balls $B^n(x,r) \subset \Omega$.

(ii) For all open sets $\Omega \subseteq \mathbb{R}^n$ and all Lebesgue measurable sets $\Gamma \subseteq \Omega$ the characteristic functions $\chi_\Gamma$ are quasinearly subharmonic if and only if there are constants $K = K(\Gamma, \Omega, n)$ such that the inequality

$$(f(r))^n \leq Km_n(B^n(x,r) \cap \Gamma)$$

holds for all closed balls $B^n(x,r) \subset \Omega$ with $x \in \Gamma$.

(iii) There are a set $A \subseteq (0, \infty)$ and a constant $c > 1$ such that:

(iii) The inequality $\int_1^c r \, dm_n(B^n(x,r) \cap \Omega) \leq Km_n(B^n(x,r))$ holds for all $r \in (0, \infty)$;

(iii) $\ln A$ is an $\varepsilon$-net in $\mathbb{R}$ for some $\varepsilon > 0$;

(iii) The inequality

$$\frac{1}{c^2} \leq f(r)$$

holds for all $r \in A$.

Note that condition (iii) of Theorem 1.5 holds if and only if the set $\ln(A) := \{ \ln x : x \in A \}$ is an $\varepsilon$-net in $\mathbb{R}$ for some $\varepsilon > 0$. A characterization in terms of porosity for the sets $A$ which are favorable for bounded open sets $\Omega \subseteq \mathbb{R}^n$ is proved in Theorem 2.12 below.

2. **Generalized mean value inequalities**

Inequality (2), characteristic for quasinearly subharmonic functions, can be generalized by some distinct ways. Our first theorem characterizes nonnegative quasinearly subharmonic functions via mean values over some sets more general than just balls.

2.1. **Similarities of the Euclidean space.** Let $\Omega$ and $D$ be subsets of $\mathbb{R}^n$ with marked points $p_\Omega \in \Omega$ and $p_D \in D$. In what follows we always suppose that $\text{Int} D \neq \emptyset$ and $p_D \in \text{Int} D$. Denote by $\text{Sim}(p_D, p_\Omega)$ the set of all similarities $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(p_D) = p_\Omega$ and $h(D) \subseteq \Omega$. Recall that $h$ is a similarity if there is a positive number $k = k(h)$, the similarity constant of $h$, such that

$$|h(x) - h(y)| = k|x - y|$$

for all $x, y \in \mathbb{R}^n$. The group of all similarities of the Euclidean space $\mathbb{R}^n$ is sometimes denoted as $\text{SM}(\mathbb{R}^n)$, see e.g. [5, 2.1.14], and we also adopt this designation. Observe that each similarity $h \in \text{SM}(\mathbb{R}^n)$ can be written in the form

$$h(x) = k(h) T x + a, \quad x \in \mathbb{R}^n,$$

where $k(h) > 0$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear mapping and $a \in \mathbb{R}^n$.

2.2. **Theorem.** Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 2$, let $D$ be a bounded, Lebesgue measurable set with the marked point $p_D \in \text{Int} D$ and let $u : \Omega \to [0, \infty)$ be a function from $L^1_{\text{loc}}(\Omega)$. Then $u$ is quasinearly subharmonic if and only if there is $C \geq 1$ such that

$$u(x) \leq \frac{C}{m_n(h(D))} \int_{h(D)} u(y) \, dm_n(y)$$

for every point $x_\Omega$ and all $h \in \text{Sim}(p_D, x_\Omega)$. If $u$ is $K$-quasinearly subharmonic, then $C = C(D, p_D, K, n)$ and, conversely, if (5) holds, then $u$ is $K$-quasinearly subharmonic with $K = K(D, pp, C, n)$.

**Proof.** Write

$$R_D := \sup_{y \in D} |p_D - y|, \quad \text{and} \quad r_D := \delta_{\text{Int}(D)}(p_D).$$

Suppose that $u$ is quasinearly subharmonic, i.e., there is $K \geq 1$ such that (2) holds for all $B^n(x,r) \subseteq \Omega$. Let $x_\Omega$ be an arbitrary point of $\Omega$ and let $h \in \text{Sim}(p_D, x_\Omega)$. The last membership relation implies the inclusions

$B^n(x_\Omega, k(h)r_D) \subseteq \Omega$ and $h(D) \subseteq B^n(x_\Omega, k(h)R_D)$.
where $k(h)$ is the similarity constant of $h$. Consequently we obtain
\[
\frac{1}{m_n(h(D))} \int_{h(D)} u(y) \, dm_n(y) \geq \frac{1}{v_n(k(h)R_D)^n} \int_{h(D)} u(y) \, dm_n(y) \\
\quad \geq \left( \frac{r_D}{R_D} \right)^n \frac{1}{v_n(k(h)R_D)^n} B^p(x_\Omega, k(h)R_D) \int_{h(D)} u(y) \, dm_n(y) \geq \left( \frac{r_D}{R_D} \right)^n \frac{u(x_\Omega)}{K}.
\]
Thus if $f$ is $K$-quasinearly subharmonic, then (5) holds with
\[
C = \frac{K(R_D)^n}{(r_D)^n}.
\]
Conversely, suppose that (5) holds with some $C \geq 1$ for all $x_\Omega$ and all $h \in Sim(p_D, x_\Omega)$. Let $h$ be an arbitrary similarity with $k(h) = \frac{m_D}{R_D}$ and with $h(p_D) = x_\Omega$. Then we have $h \in Sim(p_D, x_\Omega)$ and $B^p(x_\Omega, k(h)R_D) \subseteq h(D) \subseteq B^p(x_\Omega, m_D)$. Consequently
\[
\frac{C}{m_n(B^p(x_\Omega, m_D))} \int_{B^p(x_\Omega, m_D)} u(y) \, dm_n(y) \geq \frac{Cm_n(h(D))}{m_n(B^p(x_\Omega, m_D))} \int_{h(D)} u(y) \, dm_n(y) \geq u(x_\Omega) \frac{m_n(h(D))}{m_n(B^p(x_\Omega, m_D))}.
\]
Since
\[
\frac{m_n(B^p(x_\Omega, m_D))}{m_n(h(D))} = \frac{v_n(m(D)^n)}{v_n(m(D)^n)} \frac{v_n(k(h))^n}{(k(D))^n m_n(D)} = \frac{v_nR_D^n}{m_n(m(D))},
\]
inequality (1) holds with
\[
K = C \frac{v_nR_D^n}{m_n(m(D))}.
\]
\[\Box\]

2.2.1. **Remark.** The standard notion of quasinearly subharmonicity is defined by the condition (5), where $D = B^p(0, 1)$, $p_D = 0$ and the considered similarities $h$ are of the form $h(x) = r_Dx + x_\Omega$. The point of Theorem 2.2 is that the definition and its consequences are, however, much more general: Instead of just $D = B^p(0, 1)$ and $p_D = 0$ one may consider arbitrary bounded sets $D$ with nonvoid interior $Int D$.

2.2.2. **Remark.** Inequality (5) remains valid for each nonnegative quasinearly subharmonic function if we use bi-Lipschitz mappings $h$ instead of similarities, but in this more general case the constant in (5) depends on the Lipschitz constant of $h$. See Lemma 2.1 in [7].

Inequality (5) remains also valid for unbounded sets $D$ if $m_n(D) < \infty$.

2.3. **Proposition.** Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 2$, $D$ a Lebesgue measurable set with $m_n(D) < \infty$, $p_D$ a point of $Int(D)$ and let $u : \Omega \to [0, \infty)$ be a $K$-quasinearly subharmonic function. Then there is a constant $C = C(D, p_D, K, n)$ such that (5) holds for all $x_\Omega \in \Omega$ and $h \in Sim(p_D, x_\Omega)$.

**Proof.** If $D$ is bounded, then this proposition follows from Theorem 2.2. Suppose $D$ is an unbounded. Let $t > 1$ be a constant. It is easy to show that there is a ball $B^p(p_D, r_t)$ with a sufficiently large radius $r_t$ such that
\[
tm_n(D \cap B^p(p_D, r_t)) \geq m_n(D).
\]
Write
\[
D_t := D \cap B^p(p_D, r_t) \quad \text{and} \quad p_{D_t} := p_D.
\]
Note that $D_t$ satisfies all conditions of Theorem 2.2 and that $p_{D_t} \in Int(D_t)$. Consequently there is $K \geq 1$ such that the inequality
\[
u(x_\Omega) \leq \frac{K}{m_n(h(D_t))} \int_{h(D_t)} u(y) \, dm_n(y)
\]
holds for all $x_\Omega$ and $h \in Sim(p_{D_t}, x_\Omega)$. If $h \in Sim(p_D, x_\Omega)$, then we have $h \in Sim(p_{D_t}, x_\Omega)$ and $h(D_t) \subseteq h(D)$. Since
\[
\frac{m_n(D_t)}{m_n(D)} = \frac{m_n(h(D_t))}{m_n(h(D))},
\]

for all \( h \in SM(\mathbb{R}^n) \), \( f \) implies the inequality
\[
\frac{1}{m_n(h(D))} \int_{h(D)} u(y) \, dm_n(y) \leq \frac{t}{m_n(D)} \int_{h(D)} u(y) \, dm_n(y).
\]
Thus (5) holds for all \( h \in Sim(p_D, x_\Omega) \) with \( C = tK \).

2.3.1. Remark. If \( Sim(p_D, x_\Omega) = \emptyset \) for all \( x_\Omega \), then Proposition 2.3 is vacuously true.

Let \( \varphi : SM(\mathbb{R}^n) \rightarrow (0, \infty) \) be a function such that the equality
\[
\varphi(h) = \varphi(is \circ h)
\]
holds for all \( h \in SM(\mathbb{R}^n) \) and for all isometries \( is : \mathbb{R}^n \rightarrow \mathbb{R}^n \). Then we have \( \varphi(h_1) = \varphi(h_2) \) whenever \( k(h_1) = k(h_2) \), that is there is a function \( f : (0, \infty) \rightarrow (0, \infty) \) such that the equality
\[
(8) \quad \varphi(h) = f(k(h))
\]
is fulfilled for all \( h \in SM(\mathbb{R}^n) \) with \( k(h) \) equals the similarity constant of \( h \). For instance, if \( D \) is a bounded nonvoid subset of \( \mathbb{R}^n \), we can put \( \varphi(h) = diam(h(D)) \). Other examples can be found in (2.15)–(2.18).

Let \( D \) be a measurable subset of \( \mathbb{R}^n \) with a marked point \( p_D \in Int D \). For every open set \( \Omega \subseteq \mathbb{R}^n \) define a subset \( Q(f, D, \Omega) \subseteq L^1_u(\Omega) \) by the rule:
\[
u \in Q(f, D, \Omega) \text{ if and only if } u \geq 0 \text{ and } u \in L^1_u(\Omega) \text{ and there is } K = K(u) \geq 1 \text{ such that the inequality}
\]
\[
(9) \quad u(x_\Omega) \leq \frac{K}{f(k(h)) \cdot m_n(D)} \int_{h(D)} u(y) \, dm_n(y)
\]
holds for every \( x_\Omega \in \Omega \) and all \( h \in Sim(p_D, x_\Omega) \).

It is clear that \( QNS(\Omega) = Q(f, D, \Omega) \) if \( D \) satisfies the conditions of Theorem 2.2 and \( f(k(h)) = k(h)(m_n(D))^{\frac{1}{n}} \).

2.4. Proposition. Let \( D \) be a bounded, Lebesgue measurable subset of \( \mathbb{R}^n \) with a marked point \( p_D \in Int D \) and let \( \varphi : SM(\mathbb{R}^n) \rightarrow (0, \infty), f : (0, \infty) \rightarrow (0, \infty) \) be functions such that (5) holds for all \( h \in SM(\mathbb{R}^n) \). Then the inclusion
\[
(10) \quad QNS(\Omega) \subseteq Q(f, D, \Omega)
\]
is valid for all open sets \( \Omega \subseteq \mathbb{R}^n \) if and only if there is \( c \geq 1 \) such that the inequality
\[
(11) \quad f(k) \leq ck
\]
holds for all \( k \in (0, \infty) \).

Proof. Suppose that inclusion (10) holds for all open sets \( \Omega \subseteq \mathbb{R}^n \). Let \( \Omega \) be an open half-space of \( \mathbb{R}^n \). Then for every \( k_0 \in (0, \infty) \) there is a similarity \( h_0 \) with the similarity constant \( k(h_0) = k_0 \) such that \( h_0(D) \subseteq \Omega \). The constant function \( u_1, u_1(x) \equiv 1 \) for \( x \in \Omega \), belongs to \( QNS(\Omega) \). Hence, by (10), \( u_1 \in Q(f, D, \Omega) \) and it follows from (9) that
\[
1 = u_1(h_0(p_D)) \leq \frac{K}{(f(k_0))^{\frac{1}{n}}} \int_{h_0(D)} u_1(x) \, dm_n(x) = \frac{Km_n(h_0(D))}{(f(k_0))^{\frac{1}{n}}} = \frac{K(k_0)^{n}m_n(D)}{(f(k_0))^{\frac{1}{n}}}.
\]
Consequently (10) implies (11) for all \( k \in (0, \infty) \) with
\[
c = (K(u_1)m_n(D))^{\frac{1}{n}} \lor 1.
\]

Conversely suppose that (11) holds for all \( k \in (0, \infty) \). Then using Theorem 2.2 we obtain the following inequalities for every open set \( \Omega \subseteq \mathbb{R}^n \), every \( u \in QNS(\Omega) \), every \( x_\Omega \in \Omega \) and every \( h \in Sim(p_D, x_\Omega) \):
\[
u(x_\Omega) \leq \frac{C(u)}{m_n(h(D))} \int_{h(D)} u(x) \, dm_n(x) = \frac{C(u)(f(k(h)))^{n}}{(k(h))^{n}m_n(D)(f(k(h)))^{n}} \int_{h(D)} u(x) \, dm_n(x)
\]
\[
\leq \frac{C(u)c^n}{m_n(D)(f(k(h)))^{n}} \int_{h(D)} u(x) \, dm_n(x).
\]
Hence (9) holds with
\[ K = \frac{C(u) c^n}{m_n(D)} \vee 1. \]
Thus (10) is valid for all open sets \( \Omega \subseteq \mathbb{R}^n \).

Before passing to the equality
\[ Q(f, D, \Omega) = QNS(\Omega) \]
we consider one relevant question.

### 2.5. Theorem

Let \( A \) be a subset of \((0, \infty)\). Then \( A \) is favorable for all open sets \( \Omega \subseteq \mathbb{R}^n \) if and only if the following statement holds.

(s) There exists \( C = C(A) > 1 \) such that
\[ \left[ \frac{x}{C}, x \right] \cap A \neq \emptyset \]
for every \( x \in (0, \infty) \).

The following lemma will be used in the proof of Theorem 2.5.

### 2.6. Lemma

Let \( A \subseteq (0, \infty) \). Statement (s) of Theorem 2.5 does not hold with this \( A \), if and only if there are disjoint open intervals \((a_m, b_m)\), \( a_m < b_m \), \( m = 1, 2, \ldots \), in \((0, \infty) \setminus A\) such that
\[ \lim_{m \to \infty} a_m = 0 \]
and either
\[ \lim_{m \to \infty} a_m = \lim_{m \to \infty} b_m = 0 \]
or
\[ \lim_{m \to \infty} a_m = \lim_{m \to \infty} b_m = \infty. \]

**Proof.** If statement (s) holds, then using (12) we obtain that
\[ \frac{a}{b} \geq \frac{1}{C(A)} \]
for every open interval \((a, b)\) in \((0, \infty) \setminus A\). This inequality contradicts (13).

Conversely, suppose that statement (s) of Theorem 2.5 does not hold and that 0 and \( \infty \) are limit points of \( A \). Then for every natural \( i \geq 2 \) there is \( x \in (0, \infty) \) such that
\[ \left( \frac{x}{i}, x \right) \cap A = \emptyset. \]
Let \( \bar{A} \) be the closure of \( A \) in \((0, \infty)\). Write \((a_i, b_i)\) for the connected component of \((0, \infty) \setminus \bar{A}\) which contains \((\frac{x}{i}, x)\). Since both 0 and \( \infty \) are the limit points of \( A \) we have
\[ 0 < a_i < b_i < \infty. \]
Passing to convergent, in \([0, \infty]\), subsequences \( \{a_{in}\}_{m \in \mathbb{N}} \) and \( \{b_{in}\}_{m \in \mathbb{N}} \) it is easy to see that limits \( \lim_{m \to \infty} a_{in} \) and \( \lim_{m \to \infty} b_{in} \) are 0 or \( \infty \) and that the equalities
\[ \lim_{m \to \infty} a_{in} = 0 \quad \text{and} \quad \lim_{m \to \infty} b_{in} = \infty, \]
cannot be true simultaneously. Renaming \( a_m := a_{in} \) and \( b_m := b_{in} \) we obtain the desirable sequence of intervals in \((0, \infty) \setminus A\).

If at least one of the points 0 and \( \infty \) is not a limit point of \( A \), then there is \( \varepsilon > 0 \) such that
\[ A \subset (0, \varepsilon] \quad \text{or} \quad A \subset [\varepsilon, \infty). \]
Each of these inclusions implies evidently the existence of desired intervals in \((0, \infty) \setminus \bar{A}\). \( \square \)
Proof of Theorem 2.5. We shall first prove that \( A \) is favorable for all open sets \( \Omega \subseteq \mathbb{R}^n \) if statement (s) holds.

Suppose that (s) is true. Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and let \( u \in L^1_{\text{loc}}(\Omega) \) be a nonnegative function which satisfies condition (ii) of Definition 1.4. It is enough to show that \( u \in \OmegaNS(\Omega) \). To prove this, consider an arbitrary \( B^n(x_\Omega, r_0) \subseteq \Omega \). By statement (s) there is \( r_1 \in A \) such that

\[
\frac{r_0}{C} \leq r_1 \leq r_0
\]

where the constant \( C = C(A) > 1 \). Using this double inequality and condition (ii) of Definition 1.4 we obtain

\[
u(x_\Omega) \leq \frac{K}{v_n(r_1)} \int_{B^n(x_\Omega, r_1)} u(y) \, dm_n(y) \leq \frac{KC^n}{v_n(r_0)} \int_{B^n(x_\Omega, r_0)} u(y) \, dm_n(y).
\]

Statement (iv) of Proposition 1.3 implies that

\[
u_{\Omega}(u) = \int_{\Omega} u(z) \, dz
\]

and write

\[
u_{\Omega}(C) = \int_{\Omega} C \, dz
\]

Moreover, passing, if necessary, to a subsequence we may assume that

\[
\sum_{m=1}^{\infty} b_m < \infty.
\]

For the sake of simplicity, we shall describe our constructions only on the plane but in such a way that a generalization to the dimensions \( n \geq 3 \) is a trivial matter.

Define the points \( z_m \in \mathbb{C}, \ m = 1, 2, \ldots \), as

\[
z_m := \begin{cases} 
0 & \text{if } m = 1 \\
2 \sum_{i=1}^{m-1} b_i & \text{if } m \geq 2
\end{cases}
\]

and write

\[
R_1 := \{ z \in \mathbb{C} : 0 < \Re(z) < 2b_1, |\Im(z)| < a_2 \}
\]

and

\[
R_m := \{ z \in \mathbb{C} : 2 \sum_{i=1}^{m-1} b_i < \Re(z) < 2 \sum_{i=1}^{m} b_i, |\Im(z)| < a_{m+1} \}
\]

for \( m \geq 2 \). Using (16) we see that \( B^2(z_m, b_m) \) are open, pairwise disjoint balls and that \( R_m \) are open, pairwise disjoint rectangles. The desired domain \( \Omega \) is, by definition, the union

\[
\bigcup_{m=1}^{\infty} \left( B^2(z_m, b_m) \cup R_m \right),
\]

see Fig. 1. Let us define now a function \( u \) as the characteristic function of the set

\[
X := \bigcup_{m=1}^{\infty} B^2(z_m, b_m),
\]

i.e.,

\[
u(z) := \begin{cases} 
1 & \text{if } z \in X \\
0 & \text{if } z \in \Omega \setminus X
\end{cases}
\]

(21)
where the infimum is taken over the set of all balls $B^2(z_m, \frac{b_m}{N_0})$.

Using the last inequality, (23) and (16) we obtain

\[ \int_{B^2(z_m, \frac{b_m}{N_0})} u(z) \, dm_2(z) = \frac{4N_0^2(a_m)^2}{(b_m)^2}, \]

statement (iv) of Proposition 1.3 and limit relation (18) imply $u \notin QNS(\Omega)$. It remains to show that there is $K$ such that (3) holds whenever $r \in A$ and $B^2(x, r) \subseteq \Omega$. If $x \in \Omega \setminus X$, then (3) is trivial and we must consider only $x \in X$. The last membership relation implies that there exists $m = m_x$ such that

\[ x \in B^2(z_m, a_m). \]

Let us consider all $r \in A$ such that

\[ B^2(x, r) \subseteq \Omega. \]

From (17) follows that either $r \geq b_m$, or $r \leq a_m$. If $r \geq b_m$, then we have

\[ B^2(x, r) \supseteq B^2(z_m, \frac{b_m}{N_0}). \]

Indeed, the triangle inequality and (16) imply

\[ |y - x| \leq |x - z_m| + |z_m - y| \leq a_m + \frac{b_m}{N_0} < \frac{3}{4} b_m < r \]

for all $y \in B^2(z_m, \frac{b_m}{N_0})$. Inclusion (25) and definition of $\Omega$ show that $B^2(x, r) \not\subseteq \Omega$ if $r \geq b_m$. Consequently if (24) holds, then

\[ r \leq a_m. \]

Using the last inequality, (23) and (16) we obtain

\[ B^2(x, r) \subseteq B^2\left(z_m, \frac{b_m}{N_0}\right), \quad m = m_x, \]

for these $x$ and $r$. Hence the equality

\[ \frac{1}{m_2(B^2(x,r))} \int_{B^2(x,r)} u(y) \, dm_2(y) = \frac{m_2(B^2(z_m, a_m) \cap B^2(x,r))}{m_2(B^2(x,r))} \]

holds for such $x$ and $r$. Write

\[ C = \inf_{m \in A} \frac{m_2(B^2(z_m, a_m) \cap B^2(x,r))}{m_2(B^2(x,r))} \]

where the infimum is taken over the set of all balls $B^2(x, r)$ with $x \in B^2(z_m, a_m)$ and with $r \leq a_m$. If $r$ is fixed and $x_1, x_2 \in B^2(z_m, a_m)$, then the inequality $|x_1 - z_m| \geq |x_2 - z_m|$ implies

\[ m_2(B^2(z_m, a_m) \cap B^2(x_1, r)) \leq m_2(B^2(z_m, a_m) \cap B^2(x_2, r)). \]

Thus we have

\[ C = \inf_{r \leq a_m} \frac{m_2(B^2(z_m, a_m) \cap B^2(z_m + a_m, r))}{m_2(B^2(z_m + a_m, r))}. \]
The right-hand side of the last formula is invariant under the similarities. Consequently using the similarity
\[ C \ni z \mapsto - \frac{1}{r} (z - z_m) \in C \]
we see that
\[ (29) \quad C = \inf_{r \leq a_m} \frac{m_2(B^2(0, \frac{r}{\sqrt{r}}) \cap B^2(\frac{r}{\sqrt{r}}, 1))}{m_2(B^2(\frac{r}{\sqrt{r}}, 1))} = \inf_{r \geq 1} \frac{m_2(B^2(0, r) \cap B^2(r, 1))}{m_2(B^2(r, 1))} = \frac{1}{\pi} \inf_{r \geq 1} m_2(B^2(0, r) \cap B^2(r, 1)) = \frac{1}{\pi} m_2(B^2(-1, 1) \cap B^2(0, 1)) = \frac{2}{3} \cdot \frac{\sqrt{3}}{2\pi}. \]

The last equality, (26) and (28) imply that

\[ \int_{B^2(x, r)} u_3(y) dm_2(y) \geq u_3(x) \]

whenever \( r \in A \) and \( B^2(x, r) \subseteq \Omega \).

Thus the theorem is proved in the case where limit relations (13) and (14) hold. Similar constructions can be realized if (13) and (15) hold and we omit them here. \(\Box\)

Statement (s) of Theorem 2.5 has a useful reformulation. For \( A \subseteq (0, \infty) \) define
\[ \ln(A) := \{ \ln x : x \in A \} \]
with \( \ln(\emptyset) := \emptyset \). Then \( \ln(A) \) is a subset of \( \mathbb{R} \). Recall that a set \( X \subseteq \mathbb{R} \) is an \( \varepsilon \)-net in \( \mathbb{R} \) if \( \varepsilon > 0 \), if
\[ \mathbb{R} = \bigcup_{x \in X} B^1(x, \varepsilon). \]

2.7. Proposition. Let \( A \) be a subset of \( (0, \infty) \). Then statement (s) in Theorem 2.5 is valid with this \( A \) if and only if there is \( \varepsilon > 0 \) such that \( \ln(A) \) is an \( \varepsilon \)-net in \( \mathbb{R} \).

Proof. If (s) holds, then \( \ln(A) \) is an \( \varepsilon \)-net with \( \varepsilon = \ln C \) where \( C \) is the constant in (12). If (s) does not hold, then Lemma 2.6 implies that \( \ln(A) \) is not an \( \varepsilon \)-net for any \( \varepsilon > 0 \). \(\Box\)

Using this proposition and analysing the first part of the proof of Theorem 2.5 we obtain the following

\[ \frac{1}{\sqrt{2\pi}} \int_{B^2(x, r)} u_3(y) dm_2(y) \geq u_3(x) \]

\[ \text{Figure 2. The center of the fixed small ball } B \text{ lies on the boundary spheres of the large balls } B_n. \]
2.8. **Proposition.** Let $A$ be a subset of $(0, \infty)$. The following three statements are equivalent.

(i) $A$ is favorable for all domains of $\mathbb{R}^n$.

(ii) $A$ is favorable for all open sets of $\mathbb{R}^n$.

(iii) There is $\epsilon > 0$ such that $\ln(A)$ is an $\epsilon$-net in $\mathbb{R}$.

The condition for the set $A \subseteq (0, \infty)$, to be favorable for all *bounded* domains $\Omega$ can be presented in terms of porosity of $A$, so recall a definition.

2.9. **Definition.** Let $A \subseteq (0, \infty)$. The right hand porosity of $A$ at zero is the quantity

$$ p_0(A) := \limsup_{h \to 0^+} \frac{l(h, A)}{h} $$

where $l(h, A)$ is the length of the longest interval in $[0, h] \setminus A$, $h > 0$.

2.9.1. **Remark.** It is easy to see that $0 \leq p_0(A) \leq 1$ for each $A \subseteq \mathbb{R}$. A variety of computations directly related to the notion of porosity can be found in [21, p. 183–212].

2.10. **Definition.** Let $A \subseteq (0, \infty)$. The right porosity index of $A$ at 0, $i_0(A)$, is defined to be the supremum of all real numbers $r$ for which there is a sequence of open intervals $(a_n, b_n)$ such that $\lim_{n \to \infty} a_n = \text{diam}(A)$ and $(a_n, b_n) \subseteq (0, \infty) \setminus A$ and

$$ r < \frac{b_n - a_n}{a_n} $$

for each $n \in \mathbb{N}$.

If no such numbers $r$ exist, then following the usual conversion we define $i_0(A) := 0$.

The following lemma is a particular case of Lemma A2.13 from [21] p. 185.

2.11. **Lemma.** The equality

$$ i_0(A) = \frac{p_0(A)}{1 - p_0(A)} $$

holds for each $A \subseteq (0, \infty)$.

2.12. **Theorem.** Let $A$ be a subset of $(0, \infty)$. Then $A$ is favorable for all bounded domains $\Omega \subseteq \mathbb{R}^n$ if and only if $p_0(A) < 1$.

**Proof.** It follows from Lemma 2.11 that $p_0(A) = 1$ if and only if $i_0(A) = 1$. Using the definition of porosity index $i_0(A)$ we can prove that the equality $i_0(A) = \infty$ implies the existence of disjoint intervals $(a_n, b_n) \subseteq (0, \infty) \setminus A$, $n = 1, 2, \ldots, $ such that equations (13) and (14) hold. It was shown in the proof of Theorem 2.5 that if (13) and (14) hold then there are a domain $\Omega \subseteq \mathbb{R}^n$ and a nonnegative $u \in L^1(\Omega) \setminus QNS(\Omega)$ such that (4) holds whenever $r \in A$ and $\overline{B}(x, r) \subseteq \Omega$. It remains to observe that inequality (19) implies $\text{diam}(\Omega) < \infty$. Thus if $A$ is favorable for all bounded domains $\Omega \subseteq \mathbb{R}^n$, then $p_0(A) < 1$.

Now note that if $p_0(A) < 1$, then the set $(-\infty, R) \cap \ln(A)$ is an $\epsilon$-net, $\epsilon = \epsilon(R)$, in $(-\infty, R)$ for each $R \in \mathbb{R}$. Hence, reasoning as in the first part of the proof of Theorem 2.5 we can prove the implication

$$(p_0(A) < 1) \Rightarrow (A \text{ is favorable for every bounded domain } \Omega \subseteq \mathbb{R}^n).$$

2.12.1. **Remark.** As in Proposition 2.8 it is easy to prove that $A$ is favorable for all bounded domains of $\mathbb{R}^n$ if and only if $A$ is favorable for all bounded open subsets of $\mathbb{R}^n$. Theorem 2.12 remains valid even for unbounded domains and open sets $\Omega \subseteq \mathbb{R}^n$ if

$$ \sup_{x \in \Omega} (\delta_\Omega(x)) < \infty. $$
2.12.2. **Remark.** In complete analogy with Definition 2.10 we may define the quantity \( i_0(A) \), the left porosity index of \( A \) at \(+\infty\), after which Proposition 2.7 can be reformulated as:

Let \( A \subset (0, \infty) \). The following statements are equivalent:

(i) \( A \) is favorable for all domains \( \Omega \subset \mathbb{R}^n \);

(ii) The indexes \( i_0(A) \) and \( i_\infty(A) \) are less than infinity,

\[
i_0(A) \vee i_\infty(A) < \infty;
\]

(iii) There is \( \varepsilon > 0 \) such that \( \ln(A) \) is an \( \varepsilon \)-net in \( \mathbb{R}^1 \).

Theorem 2.5, Proposition 2.7 and Theorem 2.12 imply the following.

2.13. **Corollary.** Let \( A \) be a subset of \((0, \infty)\) and let \( \alpha, \beta \) be positive constants. Then the set \( A \) is favorable for all domains \( \Omega \subset \mathbb{R}^n \) (for all bounded domains \( \Omega \subset \mathbb{R}^n \)) if and only if the set

\[
\alpha A^\beta := \{ \alpha \beta^k : x \in A \}
\]

has the same property.

**Proof.** One just directly observe that if condition (s) holds for the set \( A \) with a constant \( C \), then condition (s) holds for the set \( \alpha A^\beta \) with the constant \( C^\beta \). \( \square \)

Now we are ready to characterize the function \( f : (0, \infty) \to (0, \infty) \) for which the equality

\[
Q(f, D, \Omega) = QNS(\Omega)
\]

is fulfilled for all open sets \( \Omega \subset \mathbb{R}^n \).

2.14. **Theorem.** Let \( D \) be a bounded, Lebesgue measurable subset of \( \mathbb{R}^n \) with a marked point \( p_D \in \text{Int} D \) and let \( \varphi : SM(\mathbb{R}^n) \to (0, \infty) \), \( f : (0, \infty) \to (0, \infty) \) be functions such that (3) holds for all \( h \in SM(\mathbb{R}^n) \). Then equality (30) holds for all open sets \( \Omega \subset \mathbb{R}^n \) if and only if there are \( A \subseteq (0, \infty) \) and \( c > 1 \) such that:

(i) the inequality \( f(k) \leq ck \) holds for all \( k \in (0, \infty) \),

(ii) \( \ln(A) \) is an \( \varepsilon \)-net in \( \mathbb{R} \) for some \( \varepsilon > 0 \),

(iii) the inequality

\[
\frac{1}{c} k \leq f(k)
\]

holds for all \( k \in A \).

**Proof.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( A \subseteq (0, \infty) \) and \( c > 1 \). Assume that \( \ln(A) \) is an \( \varepsilon \)-net in \( \mathbb{R} \) for some \( \varepsilon > 0 \) and that (31) holds with this \( c \) for all \( k \in A \). Then using (29) and (31) we obtain

\[
u(h,x) \leq \frac{c^n K(u)}{(k(h))^m} \int_{h(D)} u(y) dm_n(y)
\]

for every \( u \in Q(f, D, \Omega) \) and every \( h(x) \) whenever \( h \in \text{Sim}(p_D, x_0) \) and \( k(h) \in A \). As in the proof of Theorem 2.2 write

\[
R_D = \sup_{y \in D} |p_D - y|.
\]

Let \( B^0(x_0, r_0) \) be a ball such that \( r_0 = R_D k_0 \), with \( k_0 \in A \) and \( B^0(x_0, r_0) \subset \Omega \). Then each similarity \( h \) such that \( k(h) = k_0 \) and \( h(p_D) = x_0 \) belongs to \( \text{Sim}(p_D, x_0) \) and satisfies \( h(D) \subseteq B^0(x_0, r_0) \). Consequently (32) implies

\[
u(h,x_0) \leq \frac{c^n K(u)(R_D)^n}{(k_0 R_D)^m} \int_{B^0(x_0, r_0)} u(y) dm_n(y) = \frac{c^n K(u) v_n(R_D)^n}{m_n(B^0(x_0, r_0))} \int_{B^0(x_0, r_0)} u(y) dm_n(y)
\]

for every \( u \in Q(f, D, \Omega) \) and every \( B^0(x_0, r_0) \subset \Omega \) whenever \( r_0 \in R_D A \). Corollary 2.13 implies that the set \( R_D A \) is favorable for \( \Omega \). Hence \( Q(f, D, \Omega) \subseteq QNS(\Omega) \). Taking into account Proposition 2.4 we see that conditions (i)–(iii) of the present theorem imply equality (30) for all open sets \( \Omega \subset \mathbb{R}^n \).

Conversely, suppose that (30) holds for all open sets \( \Omega \subset \mathbb{R}^n \) but for every \( r > 0 \) the set \( \ln(A) \), where

\[
A_r := \{ k \in (0, \infty) : f(k) \geq rk \},
\]
is not an $\varepsilon$-net for any $\varepsilon > 0$. It is clear that $A_1 \subseteq A_2$ if $t_1 \geq t_2$. Applying Proposition 2.7 and Lemma 2.6 to the sets $A_2, A_3, \ldots$ we obtain a sequence $\{a_m\}_{m=2}^{\infty}$ of positive numbers $a_m$ such that $A_m \cap (a_m, ma_m) = \emptyset$ for each $m \geq 2$, i.e.,

(34)  
$$f(k) < \frac{1}{m}$$

if $a_m < k < ma_m$, and that

(35)  
$$(a_{m_1}, m_1a_{m_1}) \cap (a_{m_2}, m_2a_{m_2}) = \emptyset$$

whenever $m_1 \neq m_2$. Passing, if necessary, to a subsequence we may assume that $\{a_m\}_{m=2}^{\infty}$ and $\{ma_m\}_{m=2}^{\infty}$ are monotone and convergent in $[0, \infty]$ sequences. This assumption and (35) imply either the equalities

(36)  
$$\lim_{m \to \infty} a_m = \lim_{m \to \infty} ma_m = 0$$
or the equalities

(37)  
$$\lim_{m \to \infty} a_m = \lim_{m \to \infty} ma_m = \infty.$$

As in the proof of Theorem 2.5 we consider only the case when (36) holds and the dimension $n = 2$. We shall construct a domain $\Omega \subseteq \mathbb{R}^2$ and a nonnegative $u \in L^1_{\text{loc}}(\Omega)$ such that

$$u \in Q(f, D, \Omega) \setminus \text{QNS}(\Omega).$$

To this end, note that (30) implies (10), so using Proposition 2.4 we can find $c \geq 1$ such that

(38)  
$$f(k) \leq ck$$

for every $k \in (0, \infty)$. Let us define a function $f_1 : (0, \infty) \to (0, \infty)$ by the rule

(39)  
$$f_1(k) := \begin{cases} \frac{c}{m} & \text{if } a_m < k < ma_m, m = 2, 3, \ldots \\ \frac{c}{k} & \text{if } k \in (0, \infty) \setminus \bigcup_{m=2}^{\infty}(a_m, ma_m) \end{cases}$$

where $c \geq 1$ is the constant from inequality (37). Inequalities (34) and (37) imply $f(k) \leq f_1(k)$ for all $k \in (0, \infty)$. Hence from the definition of the set $Q(f, D, \Omega)$ follows the inclusion

$$Q(f, D, \Omega) \supseteq Q(f_1, D, \Omega).$$

Thus it is sufficient to find a domain $\Omega \subseteq \mathbb{R}^2$ and a nonnegative $u \in L^1_{\text{loc}}(\Omega)$ such that

$$u \in Q(f_1, D, \Omega) \setminus \text{QNS}(\Omega).$$

Let us define

$$\Omega := \bigcup_{m=N_0 + 1}^{\infty} \left( B^2\left(z_m, \frac{ma_m}{N_0}\right) \cup R_m \right), \quad u(x) := \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in \Omega \setminus X \end{cases}$$

where

$$X := \bigcup_{m=N_0 + 1}^{\infty} B^2(z_m, a_m), \quad z_m := 2 \sum_{i=1}^{m-1} a_i,$$

$$R_m := \{z \in \mathbb{C} : 2 \sum_{n=1}^{m-1} na_n < \text{Re}(z) < 2 \sum_{n=1}^{m} na_n, \ |\text{Im}(z)| < a_{m+1}\}.$$

The parameter $N_0$ is free here and we will specify this parameter later. It is relevant to remark that the domain $\Omega$ is obtained from the domain depicted on Fig. 1, by deleting of the balls $B^2(z_1, b_1)$, $B^2(z_2, b_2)$, $B^2(z_3, b_3)$, $B^2(z_{N_0}, b_{N_0})$ and the rectangles $R_1, \ldots, R_{N_0}$ and putting $b_m := ma_m$ in the rest of balls and rectangles. As in the proof of Theorem 2.5 we have $u \notin \text{QNS}(\Omega)$. It still remains to prove that $u \in Q(f_1, D, \Omega)$. The last relation holds if and only if there exists $K(u) \geq 1$ such that

(40)  
$$(f_1(k(h)))^2 \leq K(u) \int_{h(D)} u(y) dm_2(y)$$

for all $h \in \text{Sim}(p_0, x_\Omega)$ with $x_\Omega \in X$.

Let $x_\Omega \in X$. It follows from the definitions of $\Omega$ and $X$ that there is $m \geq N_0 + 1$ for which

$$x_\Omega \in B^2(z_m, a_m).$$
We claim that the inequality
\begin{equation}
(40) \quad k(h)r_D \leq \frac{2ma_m}{N_0}.
\end{equation}
holds for every \( h \in \text{Sim}(p_D,x\Omega) \) with \( r_D = \delta_{\text{int}(D)}(p_D) \).

Let us prove it. Since \( h \in \text{Sim}(p_D,x\Omega) \) we have
\[ h(B^2(p_D,r_D)) \subseteq \Omega. \]
The last inclusion implies
\[ \partial \Omega \cap h(B^2(p_D,r_D)) = \emptyset \]
because \( \Omega \cap \partial \Omega = \emptyset \) for the open sets. The intersection
\[ \partial B^2\left(z_m, \frac{ma_m}{N_0}\right) \cap \partial \Omega = \left\{ z \in \mathbb{C} : |z - z_m| = \frac{ma_m}{N_0}\right\} \cap \partial \Omega \]
is not empty, see Fig. 1. Consequently there is \( \xi \in \partial B^2(z_m, \frac{ma_m}{N_0}) \setminus h(B^2(p_D,r_D)) \). Hence
\[ |x_\Omega - \xi| \geq k(h)r_D. \]
Using the triangle inequality we obtain
\[ |x_\Omega - \xi| \leq |x_\Omega - z_m| + |z_m - \xi| = |x_\Omega - z_m| + \frac{ma_m}{N_0}. \]
Consequently
\[ k(h)r_D \leq |x_\Omega - z_m| + \frac{ma_m}{N_0}. \]
Since \( x_\Omega \in B^2(z_m,a_m) \) we have \( |x_\Omega - z_m| \leq a_m \). It follows directly from the definition of \( \Omega \) that \( m \geq N_0 \). Hence
\[ k(h)r_D \leq a_m + \frac{ma_m}{N_0} \leq \frac{2ma_m}{N_0}. \]
Inequality \( (40) \) follows.

Since \( h(D) \supseteq h(B^2(p_D,r_D)) \), the inequality
\begin{equation}
(41) \quad (f_1(k(h)))^2 \leq K(u) \int_{B^2(x\Omega,k(h)r_D)} u(y) \, dm_2(y)
\end{equation}
implies \( (39) \), so it is sufficient to prove \( (41) \). The following two cases are possible: \( k(h) \in (0,a_m] \) and \( k(h) \in (a_m,\infty) \). Before analyzing these cases note that \( f_1(k) \leq ck \) for every \( k \in (0,\infty) \) because \( \frac{1}{m} \leq \frac{1}{2} \) and \( c \geq 1 \) in definition \( (38) \). Hence in the first case we can replace \( (41) \) by
\begin{equation}
(42) \quad c^2 \leq \frac{K(u)}{(k(h))^2} \int_{B^2(x\Omega,k(h)r_D)} u(y) \, dm_2(y).
\end{equation}
It is clear that
\begin{equation}
(43) \quad \frac{K(u)}{(k(h))^2} \int_{B^2(x\Omega,k(h)r_D)} u(y) \, dm_2(y) \geq \frac{K(u)\pi (r_D \wedge 1)^2}{\pi k(h)(r_D \wedge 1)^2} \int_{B^2(x\Omega,k(h)(r_D \wedge 1))} u(y) \, dm_2(y).
\end{equation}
Since \( k(h) \in (0,a_m] \), we see that
\[ k(h)(r_D \wedge 1) \leq a_m. \]
Hence, as it was shown in the proof of Theorem \( 2.3 \) in the case under consideration we have
\[ \frac{1}{\pi k(h)(r_D \wedge 1)^2} \int_{B^2(x\Omega,k(h)(r_D \wedge 1))} u(y) \, dm_2(y) \geq \frac{2}{3} - \frac{\sqrt{3}}{2\pi}. \]
The last estimation and \( (43) \) show that \( (42) \) holds if
\[ c^2 = K(u)\pi (r_D \wedge 1)^2 \left( \frac{2}{3} - \frac{\sqrt{3}}{2\pi} \right). \]
Consider now the case \( k(h) \in (a_m,\infty) \). Inequality \( (40) \) shows that
\[ k(h) \leq \frac{2ma_m}{N_0r_D}. \]
Let us specify $N_0$ as the smallest positive integer $N$ satisfying the inequality $\frac{2}{Nd} > 1$. Then we obtain the double inequality

$$a_m < k(h) < ma_m.$$  

This inequality and (38) show that

$$f_1(k(h)) = \frac{k}{m} \leq a_m.$$  

Consequently we can prove (41) as in the case $k(h) \in (0, a_m]$. □

Let us consider now some examples of functions $\varphi$ and $f$ for which equality (8) holds.

2.15. Example. Let $\psi$ be a positive bounded periodic function on $\mathbb{R}$. Write

$$\mu(x) := \frac{1}{2} \left( x + \frac{1}{x} \right)$$  

for $x > 0$ and define

(44)  

$$f(k) := k\psi(\mu(k)).$$  

Using some routine estimations we see that conditions (i)--(iii) from Theorem 2.14 are satisfied by the function $f$ if we take

$$A = \mu^{-1}\{ x \in (0, \infty) : \psi(x) \geq \frac{1}{2}M \}, \quad c = M \sqrt{\frac{2}{M}}$$  

where

$$M = \sup_{y \in \mathbb{R}} \psi(y).$$

An important special case of the preceding example is the constant function $\psi$. Then $f$ is linear on $(0, \infty)$ and conditions (i)--(iii) from Theorem 2.14 are evidently hold. In this simplest case the function $\varphi : Sm(\mathbb{R}^n) \to (0, \infty)$ connected with $f$ can be obtained in distinct ways depending on the geometrical properties of the set $D$.

In all following examples $D$ is a bounded Lebesgue measurable subset of $\mathbb{R}^n$ with $Int D \neq \emptyset$ and $h \in Sm(\mathbb{R}^n)$.

2.16. Example. Let $d$-dimensional Hausdorff measure $\mathcal{H}^d$, $n - 1 \leq d \leq n$, of the boundary $\partial D$ be finite and nonzero, $0 < \mathcal{H}^d(\partial D) < \infty$. Write

$$\varphi(h) = (\mathcal{H}^d(\partial(h(D))))^{\frac{1}{d}}.$$  

2.17. Example. Let $D$ be a set with the finite Caccioppoli–de Gorgi perimeter $P$, see, for instance, [2, Chapter 3, §3]. Write

$$\varphi(h) = (P(h(D)))^{\frac{1}{n-1}}.$$  

2.18. Example. Let $D \subseteq \mathbb{R}^2$ be a simply connected domain with the rectifiable boundary $\partial D$, $0 < \mathcal{H}^1(\partial D) < \infty$. Suppose that the domain $D$ is not a disk. Write

$$\varphi(h) = ((\mathcal{H}^1(h(\partial D)))^2 - 4\pi m_2(h(D)))^{\frac{1}{2}}.$$  

In this case the inequality $\varphi(h) > 0$ follows from the Classical Isoperemetric Inequality, see, for instance, [2 Chapter 1, §1].

This list of examples can be simply extended by involving the analytic capacity, the transfinite diameter, the Menger curvature etc. for the definition of the function $\varphi$. The homogeneity under dilatations $x \mapsto \alpha x$, $x \in \mathbb{R}^n$, $\alpha > 0$, the invariance under isometries, finiteness and positiveness are sufficient for this purpose.

Acknowledgment. The first author was partially supported by the Academy of Finland.
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