Approximate Explicit Stationary Solutions to a Vlasov Equation for Planetary Rings

Armando Majorana
Department of Mathematics and Computer Science
University of Catania, Viale A. Doria 6, 95125 Catania, Italy

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Abstract. In this paper we consider a Vlasov or collisionless Boltzmann equation describing the dynamics of planetary rings. We propose a simple physical model, where the particles of the rings move under the gravitational Newtonian potential of two primary bodies. We neglect the gravitational forces between the particles. We use a perturbative technique, which allows to find explicit solutions at the first order and approximate solutions at the second order, by solving a set of two linear ordinary differential equations.

1. Introduction. The gravitational N-body problem is one of the oldest problems in physics. The N bodies interact classically through Newton's Law of Universal Gravitation. Then the equations of motion are [1]

\[ \ddot{r}_i = -G \sum_{j=1, j \neq i}^{N} \frac{m_j}{|r_i - r_j|^3} r_i - r_j, \quad (i = 1, 2, ..., N) \]  

where \( m_i \) is the mass of the body \( P_i \), \( r_i \) is its position vector relative to some inertial frame, and \( G \) is the universal constant of gravitation. These equations provide a reasonable and well-accepted mathematical model with numerous applications in astrophysics, including the motion of planets, asteroids, comets and other bodies in the Solar System. The number \( N \) of the bodies can be very large; for instance, the planetary rings are composed of a large number of small bodies with sizes from specks of dust to small moons.

The system (1) is nonlinear and strongly coupled. There are two basic factors which lead to difficulty in integrating the set of ordinary differential equations (1). First, close encounters lead to instabilities. Secondly, since the force on each particle depends on the position of all other particles the time needed in calculating the force increases as the square of the number of particles being integrated. Computing time, therefore, increases at least as \( N^2 \) and this is the basic limitation of studying system of large \( N \). Moreover, very accurate numerical methods (see, for instance, Ref. [1]) are needed for long-time integrations. Relevant examples suggest the use of time periods greater than a few million years to describe important features of the Solar System.

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In general, for huge \( N \), an alternative approach might be a statistical description through a transport (kinetic) equation, where the unknown is a function depending on time, space coordinates and velocity. Since, now, the deterministic description of the ensemble of particles is replaced by a statistical description, then some particular aspects of the dynamics are lost. For instance, close encounters, the possibility to few particles to escape from the main cluster, and other phenomena cannot recovered from the solutions of a kinetic equation. This is not an disadvantage because, we are only loosing information on a very small part of particles employed in the simulation.

The most famous kinetic equation is the Boltmann equation [5], which describes the evolution of a distribution of monatomic particles. Unfortunately, from a computational point of view, Boltzmann equation is a non-linear integro-differential equation and solving it in six-dimensional phase space requires an extremely large memory and computational time. So, alternative kinetic equations [5], as collisionless Boltzmann equation, Bhatnagar-Gross-Krook (BGK) models, where a relaxation term replaces the collision integral, and other models are been considered.

The problem to write a kinetic equation “equivalent” to Eq. (1) is not trivial, because we must take into account the following aspects.

\( a_1 \) Systems of particles interacting through long-range forces behave very differently from those in which particles interact through short-range forces.

\( a_2 \) Particles interacting via gravitational forces exhibit some unusual behavior, like non-ergodicity and negative specific heat.

\( a_3 \) Equilibrium distribution functions for a system of particles subject only to gravitational forces cannot be described using the usual equilibrium statistical mechanics, and new methods must be developed.

The classical kinetic theory was well developed for a gas with particles interacting via short-range forces, where the assumption of binary collisions is reasonable and well accepted. If this hypothesis is not valid, than (see, for instance Ref. [5]) the BBGKY hierarchy of equations attributed to Bogoliubov, Born, Green, Kirkwood, and Yvon, can be considered, but the high complexity of the model strongly limits the applications. For example, in [6] the authors consider the equilibrium statistical mechanics of systems with long-range interactions, by means of a static BBGKY-like hierarchy. In another recent paper [4], the BBGKY hierarchy was used to develop an equilibrium kinetic equation for a weakly non ideal inhomogeneous gravitational system.

Concerning the point \( a_2 \), often these systems are characterized by a negative specific heat [6, 14, 19] and a broken ergodicity.

The last is the most important point, since a kinetic equation must agree with the statistical mechanics in equilibrium regime. This is also the most serious difficulty in writing a collisional kinetic equation. This is clear, for instance in interesting recent paper [18], where the case of an unbound two-dimensional self-gravitating system is investigated. The authors show that for a finite number \( N \) of particles, relaxation to equilibrium proceeds in two steps. First, the system relaxes to a quasi-stationary state, in which it stays for a time proportional \( N \), after which it crosses over to the normal thermodynamic equilibrium with the Maxwell-Boltzmann velocity distribution. As a simple consequence, for \( N \rightarrow +\infty \) the thermodynamic equilibrium is never reached and the system becomes trapped in a non-ergodic stationary state. Moreover, stationary states does not have a Maxwell-Boltzmann velocity distribution and explicitly depend on the initial conditions.
Nevertheless, in the past, also for gravitational systems, the Boltzmann equation [3], alternative kinetic equations, as collisionless Boltzmann equation [15], Bhatnagar-Gross-Krook (BGK) models, where a relaxation term replaces the collision integral, and other models [16] have been considered.

In my opinion, the most reasonable kinetic model is given by Vlasov or collisionless Boltzmann equation. The particle-particle interaction can be taken into account, in the framework of a mean field theory, assuming that the most important contribution to the interactions of a generic particle with its neighboring particles is determined by the mean field due to the neighboring particles. In this case we must add Poisson’s equation for the gravitational potential; so, we obtain the well-known Vlasov-Poisson system.

In the last decade many papers have shown both interesting analytical results and refined numerical schemes. For instance, recently, accurate numerical solutions of the Vlasov-Poisson model for self-gravitating systems are proposed in [7] and [20].

In this paper we will restrict our attention to a Vlasov or collisionless Boltzmann equation describing the dynamics of planetary rings. Saturn's rings are the largest and best studied. The literature is very rich in papers devoted to analytical, numerical and computational studies on this topic. In particular, the stability and the structure of Saturn's rings was studied by Griv et al. [9]-[11] using both collisionless and BGK models.

The aim of this paper is to show a large class of explicit analytical solutions. It is possible that some of these are physically meaningful solutions, but in any case these can be useful to check the accuracy of new numerical schemes.

2. Basic equations. A simple mathematical model, describing the dynamics of a planetary rings, is given by the following $N$ circular restricted three body problems.

A large set of small bodies, are subject to the attraction of the Sun and a planet. The primary bodies move in a plane in circular orbits about their center of mass. The total mass of the small bodies is negligible compared to the primary body masses. Then the presence of the small bodies does not disturb the circular motion of the two large bodies.

We denote by $r_S$ and $m_S$ the position and the mass of the Sun; $r_P$ and $m_P$, are the position a the mass of the planet. Hence, the equations of motion are

$$m_i \ddot{r_i} = -G \left[ \frac{m_i m_S}{|r_i - r_S|^3} (r_i - r_S) + \frac{m_i m_P}{|r_i - r_P|^3} (r_i - r_P) \right] \quad (i = 1, 2, ..., N) . \quad (2)$$

Since we are considering a large number of small bodies moving under the gravitational potential of the primary bodies, at any time $t$ a full description of the state of this system can be given by specifying the number of small bodies $f(t, r, \xi) dr d\xi$, having positions in the small volume $dr$ centered on $r$ in the small velocity range $d\xi$ centered on $\xi$. The function $f(t, r, \xi)$ is called the distribution function of the system. Obviously, we require that $f \geq 0$ almost everywhere, since we do not allow particles with negative mass.

We assume that the following Vlasov equation describes the evolution of the distribution function $f(t, r, \xi)$

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_r f - \nabla_r \Phi \cdot \nabla_\xi f = 0 , \quad (3)$$

where

$$\Phi(t, r) = -G \left[ \frac{m_S}{|r - r_S|} + \frac{m_P}{|r - r_P|} \right] \quad (4)$$
is gravitational potential of the primary bodies. Eq. (3), without the potential due to the presence of the Sun but coupled with Poisson’s equation, was considered by many authors (see, for instance, the review paper [15]), but, to the best of our knowledge, Eq. (3) has never been studied.

It is useful to introduce a suitable reference frame, where the primary bodies are at rest. We follow the same changes of spatial coordinates as made in the classical restricted three body problem [17], but in our case we must perform related transformations for the velocities.

We denote by \((X,Y,Z)\) the component of the vector \(r\). Now, we introduce a uniformly rotating coordinate system with origin at the mass center of the primary bodies, so that the Sun and the planet are located on the \(x\) axis with coordinates \((x_S,0,0)\) with \(x_S > 0\), and \((x_P,0,0)\), respectively. This implies the following transformation of variables

\[
\begin{align*}
X &= x \cos(\omega t) - y \sin(\omega t) \\
Y &= x \sin(\omega t) + y \cos(\omega t) \\
Z &= z \\
\xi_1 &= c_1 \cos(\omega t) - c_2 \sin(\omega t) - \omega x \sin(\omega t) - \omega y \cos(\omega t) \\
\xi_2 &= c_1 \sin(\omega t) + c_2 \cos(\omega t) + \omega x \cos(\omega t) - \omega y \sin(\omega t) \\
\xi_3 &= c_3
\end{align*}
\]

where \((x,y,z)\) are the new spatial coordinates, \((c_1,c_2,c_3)\) the component of the particle’s velocity and \(\omega\) is the constant angular velocity of the primary bodies.

It is useful to introduce cylindrical coordinates (centered in \(P\)) given by

\[
\begin{align*}
x &= x_P + r \cos \theta \\
y &= r \sin \theta \\
z &= z \\
c_1 &= u_r \cos \theta - u_\theta \sin \theta \\
c_2 &= u_r \sin \theta + u_\theta \cos \theta \\
c_3 &= u_z
\end{align*}
\]

In terms of the new variables, the distribution function \(f(t,r,\xi)\) is replaced by the new unknown \(F(t,r,\theta,z,u_r,u_\theta,u_z)\), and Eq.(3) writes

\[
\frac{\partial F}{\partial t} + u_r \frac{\partial F}{\partial r} + u_\theta \frac{\partial F}{\partial \theta} + u_z \frac{\partial F}{\partial z} + \left[ \omega^2 (x_P \cos \theta + r) + 2 \omega u_\theta + \frac{u_\theta^2}{r} - \frac{\partial V}{\partial r} \right] \frac{\partial F}{\partial u_r} - \left[ \omega^2 x_P \sin \theta + 2 \omega u_r + \frac{u_r u_\theta}{r} + \frac{1}{r} \frac{\partial V}{\partial u_\theta} \right] \frac{\partial F}{\partial u_\theta} - \frac{\partial F}{\partial u_z} = 0, \tag{5}
\]

where, now, the gravitational potential is

\[
V(r,\theta,z) = -G \left[ \frac{m_S}{\sqrt{(x_P - x_S)^2 + r^2 - 2 |x_P - x_S| r \cos \theta + z^2}} + \frac{m_P}{\sqrt{r^2 + z^2}} \right]. \tag{6}
\]

The aim of this paper is the study of the kinetic model given by Eqs. (5)-(6).

3. **A 2D approximate model.** Assuming that all the particles move on the plane of the primary bodies, Eqs. (5)-(6) reduce to

\[
\frac{\partial F}{\partial t} + u_r \frac{\partial F}{\partial r} + u_\theta \frac{\partial F}{\partial \theta} + \left[ \omega^2 (x_P \cos \theta + r) + 2 \omega u_\theta + \frac{u_\theta^2}{r} - \frac{\partial V}{\partial r} \right] \frac{\partial F}{\partial u_r} - \left[ \omega^2 x_P \sin \theta + 2 \omega u_r + \frac{u_r u_\theta}{r} + \frac{1}{r} \frac{\partial V}{\partial u_\theta} \right] \frac{\partial F}{\partial u_\theta} = 0, \tag{7}
\]

where the gravitational potential is

\[
V(r,\theta) = -G \left[ \frac{m_S}{\sqrt{(x_P - x_S)^2 + r^2 - 2 |x_P - x_S| r \cos \theta + z^2}} + \frac{m_P}{\sqrt{r^2 + z^2}} \right].
\]
with
\[
V(r, \theta, z) = -G \left[ \frac{m_S}{\sqrt{(x_P - x_S)^2 + r^2 - 2|x_P - x_S| r \cos \theta}} + \frac{m_P}{r} \right].
\] (8)

Moreover, if we assume that \( r \ll |x_P - x_S| \), we can simplify the Sun gravitational potential. This assumption is often very reasonable; for instance, if we consider the Saturn’s rings, the mean distance of rings from the center of the planet is small with respect to the distance Sun-Saturn.

By using a MacLaurin expansion, we have
\[
V(r, \theta, z) \approx -G m_S \left[ \frac{1}{|x_P - x_S|} + \cos \theta \frac{3 \cos^2 \theta - 1}{|x_P - x_S|^3 r^2} - \frac{G m_P}{r} \right].
\] (9)

Since
\[
\omega^2 x_P = -G m_S \frac{|x_P - x_S|^3}{|x_P - x_S|^2},
\]
we have
\[
\omega^2 x_P \cos \theta - \frac{\partial V}{\partial r} = G m_S \frac{3 \cos^2 \theta - 1}{|x_P - x_S|^3} \frac{r}{r^2} - \frac{G m_P}{r^2}
\]
\[
\omega^2 x_P \sin \theta + \frac{1}{r} \frac{\partial V}{\partial \theta} = G m_S \frac{3 \sin \theta \cos \theta}{|x_P - x_S|^3} r.
\]

Therefore, from Vlasov equation (7) we obtain the approximate kinetic equation
\[
\frac{\partial F}{\partial t} + u_r \frac{\partial F}{\partial r} + \frac{u_\theta}{r} \frac{\partial F}{\partial \theta} + \left[ \omega^2 r + 2 \omega u_\theta + \frac{u_\theta^2}{r} + \varepsilon (3 \cos^2 \theta - 1) r - \frac{\mu}{r^2} \right] \frac{1}{r^2} \frac{\partial F}{\partial u_r} = 0,
\] (10)

where, we have defined
\[
\varepsilon = \frac{G m_S}{|x_P - x_S|^3} \quad \text{and} \quad \mu = G m_P.
\]

Often the parameter \( \varepsilon \) is small. For instance, in the case of Saturn’s rings, using the following units of measure
\[
5 \times 10^5 \text{ km (length)}, 5.68319 \times 10^{26} \text{ kg (Saturn’s mass)} \quad \text{and} \quad 3600 \text{ s (time)},
\]
then the maximum distance of the rings from the center of Saturn is approximately 0.96, and we have
\[
\frac{\mu}{r} \geq 3.93 \times 10^{-3} \quad (\text{for } r < 1) \quad \text{and} \quad \varepsilon \approx 5.92 \times 10^{-10}.
\]

In this case, if we must solve Eq. (10) numerically, then, for large time integration, it is required a very accurate scheme, which takes into account the effects of the small term \( \varepsilon \) in Eq. (10).

### 3.1. A perturbative technique.

In order to overcome this difficulty, we suggest a simple perturbative expansion, assuming that
\[
F(t, r, \theta, u_r, u_\theta) \approx F_0(t, r, \theta, u_r, u_\theta) + \varepsilon F_1(t, r, \theta, u_r, u_\theta),
\] (11)

where \( F_0 \) and \( F_1 \) are two new completely independent unknowns.
Now, if we define the linear differential operator

\[
L(K) = \frac{\partial K}{\partial t} + u_r \frac{\partial K}{\partial r} + \frac{u_\theta}{r} \frac{\partial K}{\partial \theta} + \left( \omega^2 r + 2 u_\theta \omega + \frac{u_\theta^2}{r^2} - \frac{\mu}{r^2} \right) \frac{\partial K}{\partial u_r} - \left( 2 u_r \omega + \frac{u_r u_\theta}{r} \right) \frac{\partial K}{\partial u_\theta} \tag{12}
\]

then, Eq. (10) gives the following set of partial differential equations

\[
L(F_0) = 0, \tag{13}
\]

\[
L(F_1) = \left( 1 - 3 \cos^2 \theta \right) r \frac{\partial F_0}{\partial u_r} + \left( 3 \cos \theta \sin \theta \right) r \frac{\partial F_0}{\partial u_\theta}. \tag{14}
\]

We note the splitting of the equations, and, of course, before we solve Eq. (13), analytically or numerically, and then Eq. (14). It is evident that Eq. (13) is the exact Vlasov equation for an ensemble of particles moving in the gravitational field of a central mass.

4. Analytical solutions of Eq. (13). Here, we use the method of the characteristic curves for solving the first partial differential equation (13). This method allows to find analytical or numerical solutions of linear partial differential equations for fixed initial condition, and sometimes families of explicit solutions. In our case, we get a set of ordinary differential equations, which correspond to Eq. (13).

\[
\begin{align*}
\frac{dr}{dt} &= u_r \\
\frac{d\theta}{dt} &= \frac{u_\theta}{r} \\
\frac{du_r}{dt} &= \omega^2 r + 2 u_\theta \omega + \frac{u_\theta^2}{r^2} - \frac{\mu}{r^2} \\
\frac{du_\theta}{dt} &= -2 u_r \omega - \frac{u_r u_\theta}{r}
\end{align*} \tag{15}
\]

In the appendix A, we give mathematical details of the analysis of the system (15). Here, we show a family of exact solutions of Eq. (13). To this aim, we define

\[
\varphi_1(r, u_\theta) = r u_\theta + \omega r^2, \quad \varphi_2(r, u_r, u_\theta) = \frac{1}{2} (u_r)^2 + \frac{1}{2} (u_\theta + \omega r)^2 - \frac{\mu}{r}. \tag{16}
\]

The two functions \( \varphi_1 \) and \( \varphi_2 \) correspond to the conservation of momentum and energy in the two body problem. This is a consequence of the fact that Eq. (13) includes only the interactions central body - particles.

It is very easy to verify the following result.

**Proposition 1.** If \( \Psi \) is differentiable function, then

\[
F_0(r, u_r, u_\theta) = \Psi(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)) \tag{17}
\]

satisfies Eq. (13).

This result recalls Jean’s Theorem and the existence of stationary spherically symmetric solutions of a Vlasov equations for stellar dynamics [2], [13].
The stationary solutions (17) do not depend on \( \theta \). Since, in Eq. (13), the gravitational potential depends only on the distance \( r \) from the origin, spherically symmetric solutions have a clear physical meaning. These explicit solutions can be useful to check the accuracy of numerical schemes for solving Eq. (13).

In the following, it is useful to consider functions, which depend on the variables \( u_r \) and \( u_\theta \) only through the functions \( \varphi_1 \) and \( \varphi_2 \). We point out that the transformation
\[
\begin{align*}
\alpha &= \varphi_1(r, u_\theta) \\
\beta &= \varphi_2(r, u_r, u_\theta)
\end{align*}
\]
is invertible, for \( u_r \geq 0 \) or \( u_r \leq 0 \), and
\[
\begin{align*}
(r, \alpha, \beta) &= \left( r, \frac{\alpha}{r} - \omega r, \frac{\beta + 2}{r} \right) \\
(u_\theta, u_r) &= \left( \frac{\alpha}{r}, \frac{\beta}{r} - 2 \right)
\end{align*}
\]
So, this transformation can be made ‘without loss of generality’, considering two problems: one for \( u_r \geq 0 \) and the other for \( u_r < 0 \). We note that, if
\[
K(t, r, \theta, u_r, u_\theta) = K(t, r, \varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)),
\]
then
\[
L(K) = \frac{\partial K}{\partial t} + u_r \frac{\partial K}{\partial r} + \frac{u_\theta}{r} \frac{\partial K}{\partial \theta}.
\] (18)

5. The equation for \( F_1 \). Using two elementary trigonometric formulas, Eq. (14) becomes
\[
L(F_1) = -\frac{1}{2} (1 + 3 \cos 2\theta) r \frac{\partial F_0}{\partial u_r} + \frac{3}{2} (\sin 2\theta) r \frac{\partial F_0}{\partial u_\theta}.
\] (19)

If the solution \( F_0 \) does not depend on \( \theta \), then we can look for solutions of this kind
\[
F_1(t, r, \theta, u_r, u_\theta) = \mathcal{A}(t, r, u_r, u_\theta) + \mathcal{B}(t, r, u_r, u_\theta) \cos 2\theta + \mathcal{C}(t, r, u_r, u_\theta) \sin 2\theta.
\]
This yields the set of equations
\[
L(\mathcal{A}) = -\frac{1}{2} r \frac{\partial F_0}{\partial u_r}(t, r, u_r, u_\theta),
\] (20)
\[
L(\mathcal{B}) = -\frac{u_\theta}{r} \mathcal{C} - \frac{3}{2} r \frac{\partial F_0}{\partial u_r}(t, r, u_r, u_\theta),
\] (21)
\[
L(\mathcal{C}) = \frac{u_\theta}{r} \mathcal{B} + \frac{3}{2} r \frac{\partial F_0}{\partial u_r}(t, r, u_r, u_\theta),
\] (22)
where \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) are the new unknowns, that do not contain the variable \( \theta \); this is the main advantage to consider the system of partial differential equations given by Eqs. (20)-(22), instead of the full Eq. (19).

5.1. Analytical solutions of Eq. (20). We first consider only this equation, because \( \mathcal{B} \) and \( \mathcal{C} \) do not appear in Eq. (20), and we look for exact analytical solutions of this equation.

Proposition 2. If \( \Psi(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)) \) is a solution of Eq. (13), then
\[
\mathcal{A}(t, r, u_r, u_\theta) = \Psi_4(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)) - \frac{1}{4} r^2 \frac{\partial \Psi}{\partial \varphi_2}(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)),
\] (23)
satisfies Eq. (20), for every differentiable function $\Psi_A$.

**Proof.** Since the solution $\Psi$ of Eq. (13) depends on the variables $u_r$ and $u_\theta$ only through the functions $\varphi_1$ and $\varphi_2$, then Eq. (20) writes

$$L(\mathcal{A}) = -\frac{1}{2} r u_r \frac{\partial \Psi}{\partial \varphi_2}(\varphi_1, \varphi_2).$$

Now, we assume that $\mathcal{A}$ depends on the variables $u_r$ and $u_\theta$ only through the functions $\varphi_1$ and $\varphi_2$, and if we look for stationary solutions, then we must solve the equation

$$u_r \frac{\partial \mathcal{A}}{\partial r} = -\frac{1}{2} r u_r \frac{\partial \Psi}{\partial \varphi_2}(\varphi_1, \varphi_2). \tag{24}$$

Since every solutions of Eq. (24) must be even function with respect to the variable $u_r$, we can use the new variables $\alpha = \varphi_1(r, u_\theta)$ and $\beta = \varphi_2(r, u_r, u_\theta)$ instead of $u_r$ and $u_\theta$ without any difficulty. Hence, Eq. (24) is a simple ordinary differential equation and the general solution is given by (23).

5.2. **The system** (21)-(22). We assume that

$$F_0(t, r, u_r, u_\theta) = \Psi(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)),$$

and we look for stationary solutions of this kind

$$B(t, r, u_r, u_\theta) = u_r \mathcal{B}(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)), \quad C(t, r, u_r, u_\theta) = u_r \mathcal{C}(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)).$$

Therefore, Eqs. (21)-(22) write

$$u_r \frac{\partial \mathcal{B}}{\partial r} = -2 \frac{u_\theta u_r}{r} \mathcal{C} - \frac{3}{2} r u_r \frac{\partial \Psi}{\partial \varphi_2}(\varphi_1, \varphi_2), \tag{25}$$

$$(u_r)^2 \frac{\partial \mathcal{C}}{\partial r} + \left(\omega^2 r + 2 u_\theta \omega + \frac{u_\theta^2}{r} - \frac{\mu}{r^2}\right) \mathcal{C} = 2 \frac{u_\theta}{r} \mathcal{B}$$

$$+ \frac{3}{2} r \left[ r \frac{\partial \Psi}{\partial \varphi_1}(\varphi_1, \varphi_2) + (u_\theta + \omega r) \frac{\partial \Psi}{\partial \varphi_2}(\varphi_1, \varphi_2) \right]. \tag{26}$$

Again, since the functions $\mathcal{B}$ and $\mathcal{C}$ are even with respect to the variable $u_r$, then we can use $\alpha$ and $\beta$ instead of $u_r$ and $u_\theta$. Taking into account that

$$\omega^2 r + 2 u_\theta \omega + \frac{u_\theta^2}{r} - \frac{\mu}{r^2} = \frac{1}{r^3} (\varphi_1)^2 - \frac{\mu}{r^2} \quad \text{and} \quad u_\theta = \frac{\alpha}{r} - \omega r,$$

we obtain the equations

$$\frac{\partial \mathcal{B}}{\partial r} + 2 \left(\frac{\alpha}{r^2} - \omega\right) \mathcal{C} = -r q, \tag{27}$$

$$(2 \beta + 2 \frac{\mu}{r} - \frac{\alpha^2}{r^2}) \frac{\partial \mathcal{C}}{\partial r} + \left(\frac{\alpha^2}{r^3} - \frac{\mu}{r^2}\right) \mathcal{C} - 2 \left(\frac{\alpha}{r^2} - \omega\right) \mathcal{B} = r^2 p + \alpha q, \tag{28}$$

where

$$p = \frac{3}{2} \frac{\partial \Psi}{\partial \alpha}(\alpha, \beta) \quad \text{and} \quad q = \frac{3}{2} \frac{\partial \Psi}{\partial \beta}(\alpha, \beta).$$

Eqs. (27)-(28) are linear ordinary differential equations, where $r$ is the only variable, because $\alpha$ and $\beta$ play only the role of parameters. In general, these equations can be solved numerically with suitable initial conditions, using standard routines.
Series solutions can be achieved easily. Let be

\[ B(r, \alpha, \beta) = \sum_{k=2}^{+\infty} b_k(\alpha, \beta) r^k \quad \text{and} \quad C(r, \alpha, \beta) = \sum_{k=3}^{+\infty} c_k(\alpha, \beta) r^k, \]

we obtain the two equations

\[
\sum_{k=2}^{+\infty} k b_k r^{k-1} + \sum_{k=3}^{+\infty} 2 (\alpha r^{k-2} - \omega r^k) c_k = -q r, \quad (29)
\]

\[
\sum_{k=3}^{+\infty} \left[ 2 \beta k r^{k-1} + \mu (2k - 1) r^{k-2} + \alpha^2 (1 - k) r^{k-3} \right] c_k + \sum_{k=2}^{+\infty} 2 (\omega r^k - \alpha r^{k-2}) b_k = pr^2 + \alpha q. \quad (30)
\]

It is a simple matter to prove that Eqs. (29)-(30) imply

\[
\begin{align*}
2 b_2 + 2 \alpha c_3 &= -q \quad (n = 1) \\
3 b_3 + 2 \alpha c_4 &= 0 \quad (n = 2) \\
(n + 1) b_{n+1} + 2 \alpha c_{n+2} - 2 \omega c_n &= 0 \quad (n \geq 3) \\
-2 \alpha^2 c_3 - 2 \alpha b_2 &= \alpha q \quad (n = 0). \quad (n = 0) \\
5 \mu c_3 - 3 \alpha^2 c_4 - 2 \alpha b_3 &= 0 \quad (n = 1) \\
6 \beta c_3 + 7 \mu c_4 - 4 \alpha^2 c_5 + 2 \omega b_2 - 2 \alpha b_4 &= p \quad (n = 2) \\
2 \beta (n + 1) c_{n+1} + \mu (2n + 3) c_{n+2} - \alpha^2 (n + 2) c_{n+3} + 2 \omega b_n - 2 \alpha b_{n+2} &= 0 \quad (n \geq 3)
\end{align*}
\]

If \( \alpha \neq 0 \), then we have

\[
\begin{align*}
b_2 &= b_* (\text{arbitrary}), \quad b_3 = \mu \frac{q + 2 b_*}{\alpha^2}, \\
b_4 &= \frac{q + 2 b_*}{\alpha^4} \left( \frac{1}{2} \alpha^2 \beta + \frac{7}{4} \mu \omega^2 - \frac{1}{3} \omega \alpha^3 \right) + \frac{p - 2 \omega b_*}{6 \alpha}, \\
c_3 &= -\frac{q + 2 b_*}{2 \alpha}, \quad c_4 = -3 \mu \frac{q + 2 b_*}{2 \alpha^3}, \\
c_5 &= \frac{q + 2 b_*}{\alpha^5} \left( \frac{1}{6} \omega \alpha^3 - \alpha^2 \beta - \frac{7}{2} \mu^2 \right) + \frac{2 \omega b_* - p}{3 \alpha^2}, \\
\end{align*}
\]

and, for \( n \geq 3 \),

\[
\begin{align*}
c_{n+3} &= \frac{\mu (n + 2)(2n + 3)}{n(n + 4) \alpha^2} c_{n+2} + \frac{2 \beta (n + 1)(n + 2) - 4 \alpha \omega}{n(n + 4) \alpha^2} c_{n+1} + \frac{2 (n + 2) \omega}{n(n + 4) \alpha^2} b_n, \\
b_{n+2} &= -\frac{2 \mu (2n + 3)}{n(n + 4) \alpha} c_{n+2} + \frac{2 (n + 2) \omega \alpha - 4 \beta (n + 1)}{n(n + 4) \alpha} c_{n+1} - \frac{4 \omega}{n(n + 4) \alpha} b_n.
\end{align*}
\]
If $\alpha = 0$, then we obtain

$$b_2 = -\frac{q}{2}, \quad b_3 = 0, \quad c_3 = 0, \quad c_4 = \frac{p + \omega q}{7 \mu}, \quad c_5 = -\frac{8 \beta (p + \omega q)}{63 \mu^2},$$

$$c_{n+2} = \frac{-1}{(2n + 3) \mu} \left[ 2 \beta (n + 1) c_{n+1} + \frac{4 \omega^2}{n} c_{n-1} \right] \quad (n \geq 4),$$

$$b_{n+1} = \frac{2 \omega}{n + 1} c_n \quad (n \geq 3).$$

6. **Examples.** Here, we show simple explicit examples of solutions $F_0(r, u_r, u_\theta)$ of Eq. (13), and we plot the density of mass $\rho(r)$, which, in terms of the variables $u_r$ and $u_\theta$, writes

$$\int_{\mathbb{R}} \int_{\mathbb{R}} F_0(r, u_r, u_\theta) \, du_r \, du_\theta.$$ 

If $F_0(r, u_r, u_\theta) = \Psi(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta))$, then we have

$$\rho(r) = \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi \left( r (u_\theta + \omega r) + \frac{1}{2} (u_r)^2 + \frac{1}{2} (u_\theta + \omega r)^2 - \frac{\mu}{r} \right) \, du_r \, du_\theta$$

$$= 2 \int_0^{+\infty} \int_{\mathbb{R}} \Psi \left( s, \frac{1}{2} (u_r)^2 + \frac{1}{2} s^2 - \frac{\mu}{r} \right) \, du_r \, ds.$$

We choose the following two solutions of Eq. (13)

$$\Psi_1(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)) = \exp \left[ -\frac{1}{2} (\varphi_1(r, u_\theta) - 5)^2 - 2 (\varphi_2(r, u_r, u_\theta) - 5)^2 \right],$$

$$\Psi_2(\varphi_1(r, u_\theta), \varphi_2(r, u_r, u_\theta)) = \exp \left[ -\frac{1}{2} (\varphi_1(r, u_\theta) - 10)^2 - 2 (\varphi_2(r, u_r, u_\theta) - 5)^2 \right].$$

**Figure 1.** The density of mass of $\Psi_1$. 
The parameter $\mu$ is set equal to 0.01. Here, we show the density of mass of these solutions. Of course, the densities depend only on the radius $r$, but to make clear the figures, we choose to use a two-dimensional space centered on the planet. The shape of the density recalls a planetary ring. A linear combination of solutions of this type gives a set of rings.

7. Conclusions and final remarks. In this paper we have considered a kinetic model to describe the distribution of small bodies in planetary rings. A suitable comoving reference frame was introduced, where the energy potential did not depend on time. This allows to find approximate, but explicit, stationary solutions of the kinetic equation. To this scope, we have used a simple perturbative technique and, at the lowest order, we have found a Vlasov equation, where it appeared only the main part of the energy potential. The other equation contained only perturbative potentials.

This perturbative scheme suggests the possibility to include other small terms in the Vlasov equation (3) as the gravitational potential due to the presence of other bodies (planets, moons, ..). This is necessary for finding the effects of the resonances, which play a fundamental role in the distribution of the matter in the Saturn’s ring [8, 12].

If we include the gravitational particle-particle interactions, i.e. the self-gravitating potential, then we must add Poisson’s equation to the model. In this case it is reasonable that the integral, with respect to the velocity, of $F_0$ gives the density; so, the equation for $F_0$ and Poisson’s equation will not contain the other unknown $F_1$. Of course we must add a constrain on $F_1$; in fact its integral, with respect to the velocity, is always null. Moreover, if, for instance as in the case of Saturn’s rings, the self-gravitating potential is a perturbative term, then, first we solve the equation for $F_0$, then we use the solution for solving Poisson’s equation and at the end we consider last equation. In this paper the self-gravitating potential was neglected, since it does not allow us to finding explicit solutions for $F_1$.

Another aspect to be considered is the stability of the first order solutions. This study might be useful to select physically meaningful solutions.
Appendix A. We consider the set of ordinary differential equations

\[
\begin{aligned}
\frac{dr}{dt} &= u_r \\
\frac{d\theta}{dt} &= \frac{u_\theta}{r} \\
\frac{du_r}{dt} &= \omega^2 r + 2 u_\theta \omega + \frac{u_\theta^2}{r} - \frac{\mu}{r^2} \\
\frac{du_\theta}{dt} &= -2 u_r - \frac{u_r u_\theta}{r}
\end{aligned}
\]  

\tag{31}

From the last equation of Eq. (31), taking into account the first equation, we derive

\[
\frac{du_\theta}{dt} + 2 \omega \frac{dr}{dt} + \frac{u_\theta}{r} \frac{dr}{dt} = 0 \quad \Leftrightarrow \quad \frac{d(r u_\theta)}{dt} + \omega \frac{d r^2}{dt} = 0.
\]

Hence, we have

\[r u_\theta + \omega r^2 = k_1, \tag{32}\]

where \(k_1\) is a constant. Now, we consider Eq. (31)_3. Using Eq. (32), we have

\[
\frac{du_r}{dt} = \omega^2 r + 2 \left( \frac{k_1}{r} - \omega r \right) \omega + \frac{1}{r} \left( \frac{k_1}{r} - \omega r \right)^2 - \frac{\mu}{r^2} \quad \Leftrightarrow \quad \frac{du_r}{dt} = \frac{k_1^2}{r^3} - \frac{\mu}{r^2}.
\]

Taking into account Eq. (31)_1, we obtain

\[
\frac{d^2 r}{dt^2} = \frac{k_1^2}{r^3} - \frac{\mu}{r^2} \quad \Leftrightarrow \quad \frac{d}{dt} \left[ \frac{1}{2} \left( \frac{dr}{dt} \right)^2 \right] = \frac{d}{dt} \left( -\frac{1}{2} \frac{k_1^2}{r^2} + \frac{\mu}{r} \right),
\]

that is

\[
\frac{1}{2} \left( \frac{dr}{dt} \right)^2 + \frac{1}{2} \frac{k_1^2}{r^2} - \frac{\mu}{r} = k_2,
\]

where \(k_2\) is another constant. It is useful to eliminate the constant \(k_1\) by means of Eq. (32) and to use Eq. (31)_1. We obtain

\[
\frac{1}{2} (u_r)^2 + \frac{1}{2} (u_\theta + \omega r)^2 - \frac{\mu}{r} = k_2. \tag{33}
\]

The physical meaning of Eqs. (32)-(33) is clear, because the equations of the characteristic curves describe the motion of a particle under the influence of the gravitational force of a central body in a comoving frame. To solve the full system (31), we can use this simple flowchart.

1. We solve the equation \(\frac{1}{2} \left( \frac{dr}{dt} \right)^2 + \frac{1}{2} \frac{k_1^2}{r^2} - \frac{\mu}{r} = k_2\) and we find \(r(t)\).
2. Since \(u_r = \frac{dr}{dt}\), we obtain \(u_r(t)\) by differentiating.
3. The equation \(u_\theta = \frac{k_1}{r} - \omega r\) gives \(u_\theta(t)\), immediately.
4. We solve the differential equation \(\frac{d\theta}{dt} = \frac{u_\theta}{r}\) to have \(\theta(t)\).
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E-mail address: majorana@dmi.unict.it