LIEB–THIRRING INEQUALITIES FOR COMPLEX FINITE GAP JACOBI MATRICES

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ABSTRACT. We establish Lieb–Thirring power bounds on discrete eigenvalues of Jacobi operators for Schatten class complex perturbations of periodic and more generally finite gap Jacobi matrices.

1. Introduction

In this paper we consider bounded non-selfadjoint Jacobi operators on $\ell^2(\mathbb{Z})$ represented by tridiagonal matrices

$$J = \begin{pmatrix}
\ddots & \ddots & \ddots & \\
 & a_0 & b_1 & c_1 \\
 & a_1 & b_2 & c_2 \\
 & & a_2 & b_3 & c_3 \\
 & & & \ddots & \ddots & \ddots
\end{pmatrix}$$

(1.1)

with bounded complex parameters $\{a_n, b_n, c_n\}_{n \in \mathbb{Z}}$. Our goal is to obtain Lieb–Thirring inequalities for complex perturbations of periodic and, more generally, almost periodic Jacobi operators with absolutely continuous finite gap spectrum. We are motivated by the recent progress on Lieb–Thirring inequalities for Jacobi matrices [1, 9, 10, 11, 13], and in particular by the finite gap results [14, 5, 6, 8, 4]. See also [12].

Before explaining our new results, let us briefly go through what is already known. The spectral theory for perturbations of the free Jacobi matrix, $J_0$, (i.e., the case of $a_n = c_n \equiv 1$ and $b_n \equiv 0$) is well-understood, see [18]. Let $\mathcal{E} = \sigma(J_0) = [-2, 2]$ and suppose $J$ is a selfadjoint Jacobi matrix (i.e., $a_n = c_n > 0$) such that $\delta J = J - J_0$ is a compact operator, that is, $J$ is a compact selfadjoint...
perturbation of \( J_0 \). Hundertmark and Simon [13] proved the following Lieb–Thirring inequalities,

\[
\sum_{\lambda \in \sigma_d(J)} \text{dist}(\lambda, E)^{p-\frac{1}{2}} \leq L_{p,E} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p, \quad p \geq 1, \tag{1.2}
\]

with some explicit constants \( L_{p,E} \) independent of \( J \). Here, \( \sigma_d(J) \) is the discrete spectrum of \( J \). It was also shown in [13] that the inequality is false for \( p < 1 \).

More recently, (1.2) was extended to selfadjoint perturbations of periodic and almost periodic Jacobi matrices with absolutely continuous finite gap spectrum [14, 5, 6, 4]. When \( E \) is a finite gap set (i.e., a finite union of disjoint, compact intervals), the role of \( J_0 \) as a natural background operator is taken over by the so-called isospectral torus, denoted \( T_E \). See, e.g., [2, 3, 19] for a deeper discussion of this object. For \( J' \in T_E \) and a compact selfadjoint perturbation \( J = J' + \delta J \), Frank and Simon [6] proved (1.2) for \( p = 1 \) while the case of \( p > 1 \) is established in [4]. The constant \( L_{p,E} \) is now independent of \( J \) and \( J' \) and only depends on \( p \) and the underlying set \( E \).

It is also worth mentioning a general result of Kato [15]. Specialized to the case of selfadjoint perturbations of Jacobi matrices from \( T_E \), it states that

\[
\sum_{\lambda \in \sigma_d(J)} \text{dist}(\lambda, E)^{p} \leq \| \delta J \|^p \leq \sum_{n=-\infty}^{\infty} (4|\delta a_n| + |\delta b_n|)^p, \quad p \geq 1, \tag{1.3}
\]

where \( \| \cdot \|_p \) denotes the Schatten norm. In contrast to the Lieb–Thirring bounds, the power on the eigenvalues in (1.3) is the same as on the perturbation. Kato’s inequality is optimal for perturbations with large sup norm. On the other hand, the Lieb–Thirring bound with \( p = 1 \) is optimal for perturbations with small sup norm (cf. [13]). A fact that seemingly went unnoticed is that one can combine (1.2) and (1.3) into one ultimate inequality which is optimal for both large and small perturbations (at least when \( p = 1 \)). This inequality takes the form

\[
\sum_{\lambda \in \sigma_d(J)} \text{dist}(\lambda, E)^{p-\frac{1}{2}} (1 + |\lambda|) \frac{1}{2} \leq C_{p,E} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p, \quad p \geq 1, \tag{1.4}
\]

where the constant \( C_{p,E} \) is independent of \( J \) and \( J' \).

In recent years, several results have also been established for non-selfadjoint perturbations of selfadjoint Jacobi matrices [1, 9, 10, 8, 11]. For compact non-selfadjoint perturbations \( J = J_0 + \delta J \) of the free Jacobi matrix \( J_0 \), a near generalization (with an extra \( \varepsilon \)) of the Lieb–Thirring bound (1.2) was obtained by Hansmann and Katriel [10] using the complex analytic approach developed in [1]. Their non-selfadjoint version of the Lieb–Thirring inequalities takes the
following form: For every $0 < \varepsilon < 1$,
\[
\sum_{z \in \sigma_d(J)} \frac{\text{dist}(z, [-2, 2])^{p+\varepsilon}}{|z^2 - 4|^\frac{1}{2}} \leq L_{p, \varepsilon} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \quad p \geq 1,
\] (1.5)
where the eigenvalues are repeated according to their algebraic multiplicity and the constant $L_{p, \varepsilon}$ is independent of $J$. Whether or not this inequality continues to hold for $\varepsilon = 0$ is an open problem.

For perturbations of Jacobi matrices from the isospectral tori associated with finite gap sets $E$, an eigenvalue power bound was obtained in [8]. Shortly thereafter, this was superseded by a generalization of Kato’s inequality due to Hansmann [11]. In the special case of a compact non-selfadjoint perturbation $J = J' + \delta J$ of $J' \in \mathcal{T}_E$, Hansmann’s inequality reads
\[
\sum_{z \in \sigma_d(J)} \text{dist}(z, E)^p \leq K_p \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \quad p > 1,
\] (1.6)
where the eigenvalues are repeated according to their algebraic multiplicity and $K_p$ is a universal constant that depends only on $p$.

The purpose of the present article is to obtain a near generalization of the Lieb–Thirring bound (1.4) for compact non-selfadjoint perturbations $J = J' + \delta J$ of $J' \in \mathcal{T}_E$, Hansmann’s inequality reads
\[
\sum_{z \in \sigma_d(J)} \text{dist}(z, E)^p \leq K_p \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \quad p \geq 1,
\] (1.7)
where the eigenvalues are repeated according to their algebraic multiplicity and the constant $L_{\varepsilon, p, E}$ is independent of $J'$ and $J$. We point out that (1.7) is new even for perturbations of the free Jacobi matrix $J_0$ since, unlike (1.5), it is nearly optimal not only for small but also for large perturbations. As with (1.5), it is an open problem whether or not (1.7) remains true for $\varepsilon = 0$.

2. **Schatten Norm Estimates**

In this section we establish the fundamental estimates that are needed to prove our main result, Theorem 3.3. Throughout, $S_p$ will denote the Schatten class and $\| \cdot \|_p$ the corresponding Schatten norm for $p \geq 1$. To clarify our application of complex interpolation, we occasionally use $\| \cdot \|_\infty$ to denote the operator norm.

**Theorem 2.1.** Suppose $J'$ is a selfadjoint Jacobi matrix and $D \geq 0$ is a diagonal matrix of Schatten class $S_p$ for some $p \geq 1$. Denote by $d\rho_n$ the
spectral measure of \((J', \delta_n)\), that is, the measure in the Herglotz representation of the \(n\)th diagonal entry of \((J' - z)^{-1}\),

\[
\langle \delta_n, (J' - z)^{-1}\delta_n \rangle = \int_{\sigma(J')} \frac{d\rho_n(t)}{t - z}, \quad z \in \mathbb{C} \setminus \sigma(J').
\] (2.1)

Then

\[
\|D^{1/2}(J' - z)^{-1}D^{1/2}\|_p^p \leq \frac{\sqrt{2} \|D\|_p^p}{\sup_{n \in \mathbb{Z}} \int_{\sigma(J')} |t - z|} \int_{\sigma(J')} \frac{d\rho_n(t)}{\text{dist}(z, \sigma(J'))^{p-1}}
\] (2.2)

for \(z \in \mathbb{C} \setminus \sigma(J')\).

**Proof.** We consider first the case \(p = 1\). Let \(\{P(t)\}_{t \in \mathbb{R}}\) denote the projection-valued spectral family of the selfadjoint operator \(J'\). Recall that for any measurable and bounded function \(f\) on \(\sigma(J')\), the measure in the functional calculus,

\[
f(J') = \int_{\sigma(J')} f(t) dP(t).
\] (2.3)

Taking \(f(t) = 1/(t - z)\) in (2.3), substituting into (2.1), and recalling that the measure in the Herglotz representation is unique yield

\[
\langle \delta_n, dP(t)\delta_n \rangle = d\rho_n(t).
\] (2.4)

Applying (2.3) to \(f(t) = 1/|t - z|\) and using (2.4) then imply

\[
\langle \delta_n, (J' - z)^{-1}\delta_n \rangle = \int_{\sigma(J')} \frac{d\rho_n(t)}{|t - z|}, \quad z \in \mathbb{C} \setminus \sigma(J').
\] (2.5)

We also note that if, in addition, the function \(f(t)\) in (2.3) is nonnegative, then \(f(J')\) is a bounded, selfadjoint, and nonnegative operator.

Fix \(z \in \mathbb{C} \setminus \sigma(J')\). In the following we assume without loss of generality that \(\text{Im}(z) \geq 0\). Define the nonnegative functions

\[
f(t) = \text{Im}\left(\frac{1}{t - z}\right) \quad \text{and} \quad f_{\pm}(t) = \pm \text{Re}\left(\frac{1}{t - z}\right) \chi_{(\text{Re}(z), \pm \infty)}(t),
\] (2.6)

and note that

\[
f_+(t) - f_-(t) + if(t) = \frac{1}{t - z},
\] (2.7)

\[
f_+(t) + f_-(t) + f(t) = |\text{Re}\left(\frac{1}{t - z}\right)| + |\text{Im}\left(\frac{1}{t - z}\right)| \leq \frac{\sqrt{2}}{|t - z|}.
\] (2.8)

Then we have \(f(J') \geq 0\), \(f_{\pm}(J') \geq 0\), and

\[
(J' - z)^{-1} = f_+(J') - f_-(J') + if(J'),
\] (2.9)

\[
f_+(J') + f_-(J') + f(J') \leq \sqrt{2} |J' - z|^{-1}.
\] (2.10)
Using (2.9), the triangle inequality, and the fact that for nonnegative operators the trace norm coincides with the trace, we obtain the estimate
\[
\|D^{1/2}(J' - z)^{-1} D^{1/2}\|_1 \\
\leq \|D^{1/2} f_+(J') D^{1/2}\|_1 + \|D^{1/2} f_-(J') D^{1/2}\|_1 + \|D^{1/2} f(J') D^{1/2}\|_1 \\
= \text{tr}(D^{1/2} f_+(J') D^{1/2}) + \text{tr}(D^{1/2} f_-(J') D^{1/2}) + \text{tr}(D^{1/2} f(J') D^{1/2}).
\] (2.11)

Let \(D_{n,n}\) denote the diagonal entries of \(D\). Since \(D\) is nonnegative and diagonal, we have
\[
\sum_{n \in \mathbb{Z}} D_{n,n} = \text{tr}(D) = \|D\|_1.
\] (2.12)

Hence, by linearity of the trace it follows from (2.11), (2.10), and (2.5) that
\[
\|D^{1/2}(J' - z)^{-1} D^{1/2}\|_1 \\
\leq \sqrt{2} \text{tr}(D^{1/2} |J' - z|^{-1} D^{1/2}) \\
= \sqrt{2} \sum_{n \in \mathbb{Z}} D_{n,n} \int_{\sigma(J')} \frac{d \rho_n(t)}{|t - z|} \\
\leq \sqrt{2} \|D\|_1 \sup_{n \in \mathbb{Z}} \int_{\sigma(J')} \frac{d \rho_n(t)}{|t - z|}.
\] (2.13)

This is exactly the case \(p = 1\) in (2.2).

To obtain (2.2) for \(p > 1\), we use complex interpolation. Define the map
\[
\zeta \mapsto T(\zeta) = D^{\zeta p/2} |J' - z|^{-1} D^{\zeta p/2}
\] (2.14)
from the strip \(0 \leq \text{Re}(\zeta) \leq 1\) into the space of bounded operators. Then for any \(u, v \in \ell^2(\mathbb{Z})\), the scalar function
\[
\zeta \mapsto \langle u, T(\zeta)v \rangle
\] (2.15)
is continuous on the strip \(0 \leq \text{Re}(\zeta) \leq 1\), analytic in its interior, and bounded. In addition, since \(\|D^{iy}\| \leq 1\) and
\[
D^{x+iy} = D^{iy} D^x = D^x D^{iy}
\] for all \(x \geq 0, y \in \mathbb{R}\), it follows that
\[
\|T(iy)\|_{\infty} \leq \| |J' - z|^{-1} \| \leq \frac{1}{\text{dist}(z, \sigma(J'))}, \quad y \in \mathbb{R},
\] (2.17)
and by [17, Theorem 2.7] and (2.13),
\[
\|T(1 + iy)\|_1 \leq \|D^{p/2} |J' - z|^{-1} D^{p/2}\|_1 \\
\leq \sqrt{2} \|D^{p}\|_1 \sup_{n \in \mathbb{Z}} \int_{\sigma(J')} \frac{d \rho_n(t)}{|t - z|}, \quad y \in \mathbb{R}.
\] (2.18)
Thus, by the complex interpolation theorem (see [17, Theorem 2.9], [7, Theorem III.13.1]), we have
\[
\|T(x)\|_{1/x} \leq \sup_{y \in \mathbb{R}} \|T(iy)\|_{\infty}^{1-x} \sup_{y \in \mathbb{R}} \|T(1+iy)\|_{1}, \quad 0 < x < 1. \tag{2.19}
\]

Taking \(x = 1/p\), raising both sides to the power \(p\), and noting that \(T(1/p) = D^{1/2}(J' - z)^{-1}D^{1/2}\) and \(\|D^{p}\|_1 = \|D\|^p\) finally yields (2.2).

In what follows, \(E \subset \mathbb{R}\) will denote a finite gap set, that is,
\[
E = \bigcup_{n=1}^{N} [\alpha_n, \beta_n], \quad \alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_N < \beta_N, \quad N \geq 1, \tag{2.20}
\]
and \(\partial E\) will be the set of endpoints of \(E\), that is,
\[
\partial E = \{\alpha_n, \beta_n\}_{n=1}^{N}. \tag{2.21}
\]

For a probability measure \(d\rho\) supported on \(E\), we define the associated \(m\)-function by
\[
m(z) = \int_{E} \frac{d\rho(t)}{t-z}, \quad z \in \mathbb{C} \setminus E. \tag{2.22}
\]

The measure \(d\rho\) is called reflectionless (on \(E\)) if
\[
\text{Re}[m(x + i0)] = 0 \quad \text{for a.e.} \ x \in E. \tag{2.23}
\]

Reflectionless measures appear prominently in spectral theory of finite and infinite gap Jacobi matrices (see, e.g., [2, 16, 18, 19]). In particular, the isospectral torus associated to \(E\) is the set of all Jacobi matrices \(J'\) that are reflectionless on \(E\) (i.e., the spectral measure of \((J', \delta_n)\) is reflectionless for every \(n \in \mathbb{Z}\)) and for which \(\sigma(J') = E\). It is well known (see for example [19]) that \(d\rho\) is a reflectionless probability measure on \(E\) if and only if \(m(z)\) is of the form
\[
m(z) = \frac{-1}{\sqrt{(z - \beta_N)(z - \alpha_1)}} \prod_{j=1}^{N-1} \frac{z - \gamma_j}{\sqrt{(z - \beta_j)(z - \alpha_{j+1})}}, \tag{2.24}
\]
for some \(\gamma_j \in [\beta_j, \alpha_{j+1}], \ j = 1, \ldots, N - 1\).

We now provide an estimate for the variant of the \(m\)-function for \(d\rho_n\) that appear in Theorem 2.1.

**Theorem 2.2.** Let \(E \subset \mathbb{R}\) be a finite gap set and suppose \(d\rho\) is a reflectionless probability measure on \(E\). Then for every \(p > 1\),
\[
\int_{E} \frac{d\rho(t)}{|t-z|^p} \leq \frac{K_{p,E}}{\text{dist}(z, E)^{p-1} \text{dist}(z, \partial E)^{\frac{1}{2}} (1 + |z|)^{\frac{1}{2}}}, \quad z \in \mathbb{C} \setminus E, \tag{2.25}
\]
where the constant $K_{p,E}$ is independent of $dp$. In addition, for every $\varepsilon > 0$,

$$\int_{E} \frac{dp(t)}{|t-z|} \leq \frac{K_{\varepsilon,E}}{\text{dist}(z,E)^{p} \text{dist}(z,\partial E)^{\frac{1}{2}}(1 + |z|)^{\frac{1}{2} - \varepsilon}}, \quad z \in \mathbb{C} \setminus E,$$

(2.26)

where the constant $K_{\varepsilon,E}$ is independent of $dp$.

**Proof.** Denote the bands of $E$ as in (2.20). Since $dp$ is reflectionless on a finite gap set, it follows from the Stieltjes inversion formula and (2.24) that $dp$ is absolutely continuous with density given by

$$w(t) = \frac{1}{\pi} \text{Im}[m(t + i0)] = \frac{1}{\pi} \prod_{j=1}^{N-1} \frac{|t - \gamma_j|}{\sqrt{|t - \beta_j| |t - \alpha_j|}},$$

(2.27)

for some $\gamma_j \in [\beta_j, \alpha_{j+1}]$, $j = 1, \ldots, N - 1$. Fix $1 \leq k \leq N$ and rearrange the terms in (2.27) as follows

$$w(t) = \frac{1}{\pi} \prod_{j=1}^{k-1} \frac{|t - \gamma_j|}{\sqrt{|t - \alpha_k| |t - \beta_k|}} \prod_{j=k+1}^{N} \frac{|t - \gamma_j|}{\sqrt{|t - \beta_j| |t - \alpha_j|}}.$$  

(2.28)

Since $\alpha_j < \beta_j \leq \gamma_j \leq \alpha_{j+1} < \beta_{j+1}$ for $j = 1, \ldots, N - 1$, the two products in (2.28) are at most 1 for every $t \in [\alpha_k, \beta_k]$ and thus

$$w(t) \leq \frac{1}{\pi} \frac{1}{\sqrt{|t - \alpha_k| |t - \beta_k|}}, \quad t \in [\alpha_k, \beta_k].$$

(2.29)

Applying this estimate for the individual bands of $E$ implies that

$$\int_{E} \frac{dp(t)}{|t-z|^{p}} \leq \frac{1}{\pi} \sum_{k=1}^{N} \int_{\alpha_k}^{\beta_k} \frac{1}{|t-z|^{p}} \frac{dt}{\sqrt{|t - \alpha_k| |t - \beta_k|}}, \quad z \in \mathbb{C} \setminus E.$$

(2.30)

By [10] Lemma 11, each integral in the sum can be estimated by

$$\int_{\alpha_k}^{\beta_k} \frac{1}{|t-z|^{p}} \frac{dt}{\sqrt{|t - \alpha_k| |t - \beta_k|}} \leq \frac{K_{p}}{\text{dist}(z,[\alpha_k, \beta_k])^{p-1} \sqrt{|z - \alpha_k| |z - \beta_k|}}.$$

(2.31)

Since the function $z \mapsto \text{dist}(z,\partial E)(1 + |z|)/|z - \alpha_k| |z - \beta_k|$ is continuous on $\mathbb{C} \setminus \{\alpha_k, \beta_k\}$ and bounded near $\alpha_k$, $\beta_k$, and $\infty$, it is bounded on $\mathbb{C} \setminus E$, and therefore

$$\int_{\alpha_k}^{\beta_k} \frac{1}{|t-z|^{p}} \frac{dt}{\sqrt{|t - \alpha_k| |t - \beta_k|}} \leq \frac{K_{p,E}}{\text{dist}(z,\partial E)^{p-1} \text{dist}(z,\partial E)^{\frac{1}{2}}(1 + |z|)^{\frac{1}{2}}}.$$  

(2.32)

Combining (2.32) with (2.30) yields (2.26).
In order to obtain (2.26), note that since \( E \) is a bounded set we have the trivial bound
\[
\frac{|t - z|}{1 + |z|} \leq \frac{|t| + |z|}{1 + |z|} \leq K_E, \quad t \in E, \quad z \in \mathbb{C} \setminus E.
\] (2.33)
This inequality yields the estimate
\[
\int_E \frac{d\rho(t)}{|t - z|} = \int_E \frac{|t - z|^\varepsilon d\rho(t)}{|t - z|^{1+\varepsilon}} \leq K_E^\varepsilon (1 + |z|)^\varepsilon \int_E \frac{d\rho(t)}{|t - z|^{1+\varepsilon}}, \quad z \in \mathbb{C} \setminus E
\] (2.34)
and (2.26) hence follows from (2.25). \( \square \)

3. Lieb–Thirring Bounds

We start this section by recalling some results on the distribution of zeros of analytic functions with restricted growth towards the boundary of the domain of analyticity. Let \( a_+ \) denote the maximum of \( a \) and 0. The following theorem for analytic functions on the unit disk is an alternative form of the extension [10, Theorem 4] of the earlier result [1, Theorem 0.2].

**Theorem 3.1.** Let \( S \subset \partial \mathbb{D} \) be a finite collection of points and suppose \( h(z) \) is an analytic function on \( \mathbb{D} \) such that \( |h(0)| = 1 \) and for some \( K, \alpha, \beta, \gamma \geq 0 \),
\[
\log |h(z)| \leq \frac{K|z|^\gamma}{(1 - |z|)^\alpha \text{dist}(z, S)^\beta}, \quad z \in \mathbb{D}.
\] (3.1)
Then for every \( \varepsilon > 0 \), there exists a constant \( C_{\alpha, \beta, \gamma, \varepsilon} \) independent of \( h(z) \) such that the zeros of \( h(z) \) satisfy
\[
\sum_{z \in \mathbb{D}, h(z) = 0} \frac{(1 - |z|)^{\alpha+1+\varepsilon} \text{dist}(z, S)^{(\beta-1+\varepsilon)+}}{|z|^{(\gamma-\varepsilon)+}} \leq C_{\alpha, \beta, \gamma, \varepsilon} K,
\] (3.2)
where each zero is repeated according to its multiplicity.

In [8], an analogous result on the distribution of zeros of analytic functions on \( \Omega = \overline{\mathbb{C}} \setminus E \) was obtained via a reduction to the unit disk case. For our purposes we will need the following extension of [8, Theorem 0.1] where an additional decay assumption at infinity is imposed in exchange for a stronger conclusion. The extension follows from the reduction to the unit disk case developed in [8] combined with the above version (Theorem 3.1) of the unit disk result. We omit the proof as it is a straightforward modification of the one presented in [8].
Theorem 3.2. Let \( E \subseteq \mathbb{R} \) be a finite gap set and suppose \( f(z) \) is an analytic function on \( \Omega = \mathbb{C} \setminus E \) such that \( |f(\infty)| = 1 \) and for some \( K, p, q, r \geq 0 \),

\[
\log |f(z)| \leq \frac{K}{\text{dist}(z, E)^p \text{dist}(z, \partial E)^q (1 + |z|)^r}, \quad z \in \Omega. \tag{3.3}
\]

Then for every \( \varepsilon > 0 \), there exists a constant \( C_{p,q,r,\varepsilon} \) independent of \( f(z) \) such that the zeros of \( f(z) \) satisfy

\[
\sum_{z \in \Omega, f(z) = 0} \text{dist}(z, E)^{p'} \text{dist}(z, \partial E)^{q'} (1 + |z|)^{r'} \leq C_{p,q,r,\varepsilon} K, \tag{3.4}
\]

where \( p' = p + 1 + \varepsilon, q' = \frac{1}{2}(p + 2q - 1 + \varepsilon)_{+} - p' \), \( r' = (p + q - r - \varepsilon)_{+} - p' - q' \), and each zero is repeated according to its multiplicity.

We are now ready to present our finite gap version of the Lieb–Thirring inequalities for non-selfadjoint perturbations of Jacobi matrices from the isospectral torus \( T_E \).

Theorem 3.3. Let \( E \subseteq \mathbb{R} \) be a finite gap set and suppose \( J, J' \) are two-sided Jacobi matrices such that \( J' \in T_E \) and \( J = J' + \delta J \) is a compact perturbation of \( J' \). Then for every \( p \geq 1 \) and any \( \varepsilon > 0 \),

\[
\sum_{z \in \sigma_d(J)} \frac{\text{dist}(z, E)^{p+\varepsilon} (1 + |z|)^{1/2}}{\text{dist}(z, \partial E)^{1/2}} \leq L_{\varepsilon, p, E} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \tag{3.5}
\]

where the eigenvalues are repeated according to their algebraic multiplicity and the constant \( L_{\varepsilon, p, E} \) is independent of \( J \) and \( J' \).

Proof. Suppose that \( \delta J \in \mathcal{S}_p \) for some \( p \geq 1 \) and define

\[
f(z) = \det_{[p]}(I + (J' - z)^{-1} \delta J), \tag{3.6}
\]

where \([p]\) is the smallest integer \( \geq p \). This regularized perturbation determinant is analytic on \( \Omega = \mathbb{C} \setminus E \). More importantly, the zeros of \( f \) coincide with the discrete eigenvalues of \( J \) and the multiplicity of the zeros matches the algebraic multiplicity of the corresponding eigenvalues (see \([10], [7, \text{Chapter IV. } \S 3])\).

Define \( D \geq 0 \) to be the diagonal matrix with the entries

\[
D_{n,n} = \max\{|\delta a_{n-1}|, |\delta a_n|, |\delta b_n|, |\delta c_{n-1}|, |\delta c_n|\}, \quad n \in \mathbb{Z}. \tag{3.7}
\]

A straightforward verification shows that \( \delta J = D^{1/2} B D^{1/2} \), where \( B \) is a bounded tridiagonal matrix whose entries lie in the unit disk. This in particular means that \( \|B\| \leq 3 \). Using the cyclicity property of the determinant, we see that

\[
f(z) = \det_{[p]}(I + D^{1/2}(J' - z)^{-1} D^{1/2} B) \tag{3.8}
\]
and hence by [17, Theorem 9.2(b)],
\[
\log |f(z)| \leq K_p \|D^{1/2}(J' - z)^{-1}D^{1/2}B\|^p_p \leq 3K_p \|D^{1/2}(J' - z)^{-1}D^{1/2}\|^p_p
\] (3.9)
for some constant \(K_p\). It now follows from Theorems 2.1 and 2.2 (with \(\varepsilon/2\) instead of \(\varepsilon\)) that
\[
\log |f(z)| \leq \frac{K_{\varepsilon,p,E} \|D\|^p_p}{\text{dist}(z, E)^{\varepsilon/2 - 1}\text{dist}(z, \partial E)^{1/2}(1 + |z|)^{1-\varepsilon}}, \quad z \in \Omega.
\] (3.10)
Applying Theorem 3.2 (with \(\varepsilon/2\) instead of \(\varepsilon\)) and noting that \((1 - 3\varepsilon)/2 \leq r'\) and \(D_{n,n}^p \leq |\delta a_{n-1}|^p + |\delta a_n|^p + |\delta b_n|^p + |\delta c_{n-1}|^p + |\delta c_n|^p\) then yield (3.5). □

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