Low-Rank Covariance-Assisted Downlink Training and Channel Estimation for FDD Massive MIMO Systems

Jun Fang, Xingjian Li, Hongbin Li, Senior Member, IEEE, and Feifei Gao

Abstract—We consider the problem of downlink training and channel estimation in frequency division duplex (FDD) massive MIMO systems, where the base station (BS) equipped with a large number of antennas serves a number of single-antenna users simultaneously. To obtain the channel state information (CSI) at the BS in FDD systems, the downlink channel has to be estimated by users via downlink training and then fed back to the BS. For FDD large-scale MIMO systems, the overhead for downlink training and CSI uplink feedback could be prohibitively high, which presents a significant challenge. In this paper, we study the behavior of the minimum mean-squared error (MMSE) estimator when the channel covariance matrix has a low-rank or an approximate low-rank structure. Our theoretical analysis reveals that the amount of training overhead can be substantially reduced by exploiting the low-rank property of the channel covariance matrix. In particular, we show that the MMSE estimator is able to achieve exact channel recovery in the asymptotic low-noise regime, provided that the number of pilot symbols in time is no less than the rank of the channel covariance matrix. We also present an optimal pilot design for the single-user case, and an asymptotic optimal pilot design for the multi-user scenario. Lastly, we develop a simple model-based scheme to estimate the channel covariance matrix, based on which the MMSE estimator can be employed to estimate the channel. The proposed scheme does not need any additional training overhead. Simulation results are provided to verify our theoretical results and illustrate the effectiveness of the proposed estimated covariance-assisted MMSE estimator.

Index Terms—Massive MIMO systems, downlink training and channel estimation, channel covariance matrix, low rank structure, MMSE estimator.

I. INTRODUCTION

Massive multiple-input multiple-output (MIMO), also known as large-scale or very-large MIMO, is a promising technology to meet the ever growing demands for higher throughput and better quality-of-service of next-generation wireless communication systems [1], [2]. Massive MIMO systems are those that are equipped with a large number of antennas at the base station (BS) simultaneously serving a much smaller number of single-antenna users sharing the same time-frequency slot. By exploiting the asymptotic orthogonality among channel vectors associated with different users, massive MIMO systems can achieve almost perfect inter-user interference cancelation with a simple linear precoder and receive combiner [3], and thus have the potential to enhance the spectrum efficiency by orders of magnitude. In addition to higher throughput, massive MIMO systems can also improve the energy efficiency and enable the use of inexpensive, low-power components [4].

To reach the full potential of massive MIMO, accurate downlink channel state information (CSI) is required at the base station (BS) for precoding and other operations. Downlink channel estimation for massive MIMO systems has been extensively studied over the past few years. Most of existing studies, e.g. [1], [3], [5], [6] assume a time division duplex (TDD) mode in which channel reciprocity between opposite links (downlink and uplink) can be exploited to facilitate the acquisition of the downlink CSI at the BS. Nevertheless, it was pointed out that the reciprocity of the wireless channel may not hold exactly due to calibration errors in the downlink/uplink RF chains [7]. Also, it is noted that current wireless cellular systems are still primarily based on the frequency division duplex (FDD). To make the massive MIMO technique backward compatible with current systems, it is of great necessity to study downlink channel estimation for FDD massive MIMO systems.

For FDD systems, the reciprocity between downlink and uplink channels no longer holds. To obtain the channel state information at the transmitter (CSIT), the BS needs to transmit training signals to users, and each user, after acquiring the downlink CSI through the training phase, feeds back the CSI to the BS. The problem lies in that the required amount of overhead for downlink training grows linearly with the number of transmit antennas at the BS. This may not be an issue for conventional MIMO scenarios with only a small number of antennas. However, for massive MIMO systems where the number of transmit antennas at the BS is large, the overhead for the downlink training and uplink feedback could become prohibitively high. Therefore reducing the overhead for downlink training and uplink CSIT feedback has been a central issue in FDD massive MIMO systems. A multitude of efforts has been directed towards this goal over the past few years, e.g. [8]–[15]. Specifically, in [8]–[10], the sparsity of the channel on the virtual angular domain has been leveraged to formulate downlink channel estimation as a compressed
sensing problem, based on which the overhead for downlink training and uplink feedback can be substantially reduced. Recent experiments and studies (e.g. [5], [16]) show that for a typical cellular configuration with a tower-mounted BS, the angular spread of the incoming/outgoing rays at the BS is usually small, and as a result, the channel has a sparse or an approximate sparse representation on the virtual angular domain.

Besides compressed sensing-based techniques [8], [9], another line of research approaches the overhead reduction issue for FDD massive MIMO by implicitly or explicitly exploiting the low-rank structure of the channel covariance matrix, e.g. [11]–[15]. Low-rank channel covariance matrix also arises as a result of a small angular spread of the incoming/outgoing rays at the BS. Due to the narrow angular spread, different paths between the BS and the user are highly correlated, and consequently, the channel covariance matrix has a low-rank or an approximate low-rank structure with only a few dominant eigenvectors [5], [17]. In [18], it was shown that even for conventional MIMO scenarios, the dimension of the optimal pilot can be reduced if there are only a few dominant eigenvectors associated with the channel covariance matrix. Covariance-aided pilot design was also considered in [13], [14] for FDD massive MIMO systems, where open-loop and closed-loop training strategies were developed to reduce the overhead of the downlink training phase by exploiting the spatial correlation as well as the temporal correlation of the channel. In [11], [12], the dimensionality of the effective channels is reduced via a prebeamforming matrix that depends only on the channel second-order statistics (i.e. channel covariance matrix), based on which a joint spatial division and multiplexing (JSDM) scheme [11] and a beam channel covariance matrix), based on which a joint spatial division and multiplexing (JSDM) scheme [11]–[15]. Low-rank channel covariance matrix also arises as a result of a small angular spread of the incoming/outgoing rays at the BS. Due to the narrow angular spread, different paths between the BS and the user are highly correlated, and consequently, the channel covariance matrix has a low-rank or an approximate low-rank structure with only a few dominant eigenvectors [5], [17]. In [18], it was shown that even for conventional MIMO scenarios, the dimension of the optimal pilot can be reduced if there are only a few dominant eigenvectors associated with the channel covariance matrix. Covariance-aided pilot design was also considered in [13], [14] for FDD massive MIMO systems, where open-loop and closed-loop training strategies were developed to reduce the overhead of the downlink training phase by exploiting the spatial correlation as well as the temporal correlation of the channel. In [11], [12], the dimensionality of the effective channels is reduced via a prebeamforming matrix that depends only on the channel second-order statistics (i.e. channel covariance matrix), based on which a joint spatial division and multiplexing (JSDM) scheme [11] and a beam division multiple access scheme [12] were proposed to achieve significant savings in both the downlink training and the CSIT uplink feedback.

In this paper, we continue the direction of covariance-aided downlink training and channel estimation for FDD massive MIMO systems. Specifically, we study the asymptotic behavior of the minimum mean-squared error (MMSE) estimator when the channel covariance matrix has a low-rank structure. Our theoretical results reveal that with a low-rank channel covariance matrix, the MMSE estimator employing a random (not necessarily optimal) pilot can obtain a perfect channel recovery in the limit of vanishing noise, provided that the length of the pilot (i.e. the number of symbols in time) is no less than the rank of the covariance matrix. We also examine asymptotically optimal pilot design for the multi-user scenario. An overlayed training strategy similar to the JSDM scheme is proposed and shown to be asymptotically optimal in terms of estimation errors when users have mutually non-overlapping angles of arrival (AoAs). The optimal design suggests that the minimum MSE can be achieved as long as the length of pilot is no less than the rank of the channel covariance matrix. In addition, based on the one-ring model, we develop a simple model-based scheme to estimate the channel covariance matrix. The proposed scheme does not require any additional training overhead. Simulation results show that the proposed estimated covariance-assisted MMSE estimator achieves a substantial performance improvement over the compressed sensing-based methods.

The rest of this paper is organized as follows. In Section II we introduce the system model and basic assumptions. The asymptotic behavior of the MMSE estimator in the limit of vanishing noise is examined in Section III. An optimal pilot design for the single-user scenario and an asymptotic optimal pilot design for the multi-user scenario are studied in Sections IV and V respectively. In Section VI we develop a simple model-based scheme to estimate the channel covariance matrix, and construct a MMSE estimator to estimate the channel. Simulation results are provided in Section VII followed by concluding remarks in Section VIII.

II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider the problem of downlink training and channel estimation in a frequency division duplex (FDD) massive MIMO system, where the base station (BS) equipped with a large number of antennas serves a number of single-antenna users simultaneously. To simplify our problem, we consider the single-user scenario. The extension of our results to the multi-user scenario is straightforward, and the pilot design for the multi-user case will be discussed in Section V. We assume the channel vector $h \in \mathbb{C}^M$ is a flat Rayleigh fading channel under a narrowband assumption, where $M$ denotes the number of transmit antennas at the BS. The extension to the wideband frequency-selective channel is straightforward when an OFDM transmission scheme is adopted. The signal received by the user can be expressed as

$$ y_t = x_t^T h + w_t \quad \forall t = 1, \ldots, T $$

(1)

where $x_t \in \mathbb{C}^M$ is the transmitted pilot symbol vector at time $t$, and $w_t$ denotes the additive white Gaussian noise with zero mean and variance $\sigma^2$. Define $y \triangleq [y_1 \ y_2 \ \ldots \ y_T]^T$, $X \triangleq [x_1 \ x_2 \ \ldots \ x_T]^T$, and $w \triangleq [w_1 \ w_2 \ \ldots \ w_T]^T$. The data model (1) can be rewritten as

$$ y = Xh + w $$

(2)

In this paper, we consider the classical one-ring model that has been widely adopted (e.g. [5], [9], [11]) to characterize the massive MIMO channel, where the BS is assumed to be located in an elevated position with few scatterers around, and the propagation between the BS and the user is mainly characterized by rich local scatterers around the user (see Fig. 1).
Assuming the propagation consists of $P$ i.i.d. paths, we have
\[
\mathbf{h} = \frac{1}{\sqrt{P}} \sum_{p=1}^{P} \alpha_p \mathbf{a}(\theta_p)
\]  
where $\alpha_p \sim \mathcal{CN}(0, \sigma^2)$ denotes the fading coefficient associated with the $p$th path, and $\mathbf{a}(\theta_p)$ is the steering vector. For a uniform linear array, it is given as
\[
\mathbf{a}(\theta_p) \equiv [1, e^{-j(2\pi \chi) \cos(\theta_p)}, \ldots, e^{-j((M-1)(2\pi \chi) \cos(\theta_p))}]^T
\]
in which $\chi$ is the signal wavelength, $d$ denotes the distance between neighboring antenna elements, and $\theta_p \in [0, \pi]$ is the azimuth angle of arrival (AoA) of the $p$th path. In the onering mode, the user is surrounded by rich local scatterers with a radius $r$ that is relatively small compared to the distance between the BS and the user, $D$. Thus the angular spread at the BS, approximately given as $\delta = \arctan(r/D)$, is small.

To estimate the channel from the received signal $\mathbf{y}$ (c.f. (2)), it is usually required that the number of pilot symbols (in time), $T$, is no less than the number of transmitted antennas $M$, i.e. $T \geq M$. When $M$ is large, the overhead for downlink training and uplink channel state information (CSI) feedback becomes prohibitively high. Hopefully, due to the narrow angular spread at the BS, the steering vectors $\{\mathbf{a}(\theta_p)\}$ of these $P$ paths are highly correlated, and thus the channel covariance matrix $\mathbf{R} = E[\mathbf{h} \mathbf{h}^H]$ has an approximate low-rank structure. This low-rank structure can be utilized to reduce the overhead for downlink training for FDD systems, see, e.g. [11], [13], [14].

In this paper, we study the behavior of the minimum mean-squared error (MMSE) estimator when the channel covariance matrix has a low rank structure. We conduct a quantitative analysis to investigate how much training overhead reduction can be achieved by exploiting the low-rank structure of the channel covariance matrix. Assume $\mathbf{h}$ is zero-mean complex Gaussian with covariance matrix $\mathbf{R}$, the MMSE estimate of the channel $\mathbf{h}$ is given as
\[
\hat{\mathbf{h}} = \mathbf{R} X^H (X R X^H + \sigma^2 \mathbf{I})^{-1} \mathbf{y}
\]  
Note that the MMSE estimator, with the aid of the statistical information of the channel, does not require an invertible pilot matrix $X$ (i.e. $T \geq M$) for channel estimation. The mean-squared error (MSE) associated with the MMSE estimate is given by
\[
\text{MSE} = E \left[ \| \hat{\mathbf{h}} - \mathbf{h} \|^2 \right] = \text{tr} \left( \mathbf{R} - \mathbf{R} X^H (X R X^H + \sigma^2 \mathbf{I})^{-1} X \mathbf{R} \right)
\]  
III. ASYMPTOTIC BEHAVIOR OF THE MMSE

In this section, we first study the behavior of the MMSE estimator in the asymptotic low-noise regime, i.e. $\sigma^2 \rightarrow 0$. Our asymptotic analysis shows that a perfect channel recovery from a small number of symbols is possible when the channel covariance matrix has a low-rank structure. Our main results are summarized as follows.

**Theorem 1:** Consider the channel estimation problem described in (2), where $\mathbf{h} \sim \mathcal{N}(0, \mathbf{R})$ and the rank of the channel covariance matrix $\mathbf{R}$ is $r = \text{rank}(\mathbf{R})$. Define $\Phi \triangleq \mathbf{R}^H X \mathbf{R}^H$. Let $\Phi = \mathbf{V} \mathbf{\Sigma} \mathbf{V}^H$ denote the eigenvalue decomposition (EVD) of $\Phi$, where $\mathbf{V} \triangleq [v_1 \ldots v_M]$ is a unitary matrix consisting of eigenvectors of $\Phi$, and $\mathbf{\Sigma} = \text{diag}(\gamma_1, \ldots, \gamma_r, 0, \ldots, 0)$ is a diagonal matrix with $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_r > 0$. Suppose the pilot signal $X$ is randomly generated, and the number of symbols, $T$, is no less than $r$, i.e. $T \geq r$; then the MSE of the MMSE estimate of $\mathbf{h}$ is given by
\[
E \left[ \| \hat{\mathbf{h}} - \mathbf{h} \|^2 \right] = \sum_{i=1}^{r} (1 + \gamma_i/\sigma^2)^{-1} v_i^H R v_i
\]
and the MSE approaches zero in the limit of vanishing noise, that is,
\[
\lim_{\sigma^2 \rightarrow 0} E \left[ \| \hat{\mathbf{h}} - \mathbf{h} \|^2 \right] = 0
\]

**Proof:** Using the Woodbury identity, the MSE (6) can be rewritten as
\[
E \left[ \| \hat{\mathbf{h}} - \mathbf{h} \|^2 \right] = \text{tr} \left( \mathbf{R}^H (\mathbf{I} - \mathbf{R}^H X X^H (X R X^H + \sigma^2 \mathbf{I})^{-1} \mathbf{R}^H) \mathbf{R}^H \right)
\]
\[
= \text{tr} \left( \mathbf{R}^H (\mathbf{I} - \mathbf{R}^H (X R X^H + \sigma^2 \mathbf{I})^{-1} \mathbf{R}^H) \mathbf{R}^H \right)
\]
\[
= \text{tr} \left( \mathbf{R}^H (\mathbf{I} - \mathbf{R}^H \mathbf{R}^H \mathbf{R}^H) \mathbf{R}^H \right)
\]
\[
= \sum_{i=1}^{r} (1 + \gamma_i/\sigma^2)^{-1} v_i^H R v_i + \sum_{i=r+1}^{M} v_i^H R v_i
\]
We can see that the first term in (8) vanishes as $\sigma^2 \rightarrow 0$. We now examine under what conditions the second term in (8) reduces to zero. Let
\[
\mathbf{R} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^H
\]
de note the reduced EVD of $\mathbf{R}$, where $\mathbf{U} \in \mathbb{C}^{M \times r}$ and $\mathbf{\Lambda} \in \mathbb{C}^{r \times r}$. We can write
\[
\mathbf{R}^H X = \mathbf{U} \mathbf{\Lambda}^H \mathbf{U}^H X H = \mathbf{U} \mathbf{C}
\]
where $\mathbf{C} \triangleq \mathbf{\Lambda}^H \mathbf{U}^H X H \in \mathbb{C}^{r \times T}$. When $T \geq r$ and the pilot symbols of $X$ are randomly generated according to some distribution, the matrix $\mathbf{C}$ has a full row rank with probability one, i.e. $\text{rank}(\mathbf{C}) = r$. Thus we have
\[
\text{Range}(\mathbf{R}^H X) = \text{Range}(\Phi) = \text{Range}(\mathbf{U})
\]
where $\text{Range}(\mathbf{A})$ denotes the column space spanned by the column vectors of $\mathbf{A}$. From (11), we can immediately arrive at
\[
v_i^H R = v_i^H \mathbf{U} = v_i^H \Phi = 0 \quad \forall i = r+1, \ldots, M
\]
Hence the second term in (8) disappears provided that the length of the pilot in time is no less than the rank of the channel covariance matrix, i.e. $T \geq r$, and eventually we
reach the conclusion that the MSE of the MMSE estimate of $h$ approaches zero in the limit of vanishing noise, that is,

$$\lim_{\sigma^2 \to 0} E \left[ \| \hat{h} - h \|_2^2 \right] = 0$$

The proof is completed here.

**Discussions:** The significance of Theorem 1 lies in that, in the limit of vanishing noise, it establishes sufficient conditions for the MMSE estimator to achieve exact channel recovery from only a small number of pilot symbols. We note that another line of research [8], [9] for FDD downlink training and channel estimation exploits the sparsity of the channel on the virtual angular domain and formulates the channel estimation as a compressed sensing problem:

$$y = Xh + w = XA\hat{h} + w$$  \hspace{1cm} (13)

where $A$ is a basis for the virtual angular domain. For the uniform linear array case, the basis $A$ is a discrete Fourier transform (DFT) matrix. $\hat{h}$ is a sparse vector to be estimated.

This class of approaches are justified by compressed sensing theories, which assert that a sparse signal can be perfectly recovered from compressive measurements, provided that the measurement matrix satisfies a certain RIP condition [19]. Our recovered from compressive measurements, provided that the measurement matrix satisfies a certain RIP condition [19]. Our theorem here can be regarded as a counterpart result for the MMSE estimator, and provides a justification for using the MMSE estimator for channel estimation from a small number of pilot symbols.

It is also interesting to compare conditions required by the MMSE estimator and those by compressed sensing techniques to achieve perfect channel recovery. First recall the following lemma that characterizes the number of dimensions of a subspace spanned by a number of steering vectors with a bounded support of angles of arrival (AoAs):

**Lemma 1:** Define

$$\alpha(x) \triangleq [1, e^{-j\pi x}, \ldots, e^{-j\pi(M-1)x}]^T$$  \hspace{1cm} (14)

and $A \triangleq \text{span}\{\alpha(x), x \in [-1, 1]\}$. Given $b_1, b_2 \in [-1, 1]$ and $b_1 < b_2$, define $B \triangleq \text{span}\{\alpha(x), x \in [b_1, b_2]\}$, then

$$\dim(A) = M$$

$$\dim(B) \sim (b_2 - b_1)M/2 \text{ when } M \text{ grows large}$$  \hspace{1cm} (15)

**Proof:** See [5, Lemma 1].

Consider the one-ring model with the multipath angle of arrival $\theta$ distributed on a bounded support, i.e. $\theta \in [\theta_{\min}, \theta_{\max}]$. From Lemma 1 the rank of the channel covariance matrix $R$ is upper bounded by

$$\text{rank}(R) \leq \eta M \text{ as } M \to \infty$$  \hspace{1cm} (16)

where $\eta$ is defined as

$$\eta \triangleq \left| \cos(\theta_{\min}) - \cos(\theta_{\max}) \right|/\chi$$  \hspace{1cm} (17)

in which $\chi$ denotes the distance between neighboring antennas and $\chi$ is the signal wavelength. Another important property from Lemma 1 is that, when $M \to \infty$, the channel $h$ has a sparse representation on a virtual angular domain with $r = \text{rank}(R)$ nonzero coefficients.

With the above results, we are now ready to make a fair comparison between conditions required by the MMSE estimator and, respectively, by the compressed sensing methods for exact recovery of the channel. For the MMSE estimator, from Theorem 1 we know that as few as $T = r$ symbols are needed to perfectly recover the channel. On the other hand, for compressed sensing-based methods, it has been shown that the number of required measurements for exact recovery is of order $T = O(r \log(M/r))$ using polynomial-time optimization solvers or greedy algorithms [19]. If the computational complexity is not a concern, then at least $T = 2r$ measurements are required for exact recovery via the $\ell_0$-minimization. From the above discussion, we can see that the MMSE estimator requires fewer symbols than compressed sensing techniques for exact channel recovery. This result puts the covariance-aided methods into a favorable position for FDD downlink training and channel estimation.

**IV. OPTIMAL PILOT SEQUENCE DESIGN**

Our analysis in the previous section reveals that as few as $T = r$ symbols in time are required to guarantee perfect channel recovery in the asymptotic low-noise regime, i.e. $\sigma^2 \to 0$. Nevertheless, assuming a noiseless scenario is unrealistic in practical systems. Therefore it is meaningful to study the behavior of the MMSE estimator for a non-vanishing $\sigma^2$. For the case $\sigma^2 \neq 0$, we would like to examine whether a larger value of $T$ leads to a better estimation accuracy, or if $T = r$ is sufficient to attain a minimum MSE. To answer this question, we first need to impose a power constraint on the pilot signal, i.e. $\text{tr}(XX^H) \leq P$; otherwise a fair comparison between pilots of different lengths is impossible. Note that different pilots of the same length also result in different MSEs. Hence simply comparing the MSEs attained by two arbitrary pilots of different lengths does not provide any meaningful answers. To make sense, we have to compare the MSEs attained by optimally devised pilots for different values of $T$, and see if increasing $T$ will result in a lower MSE. This requires us to examine the following optimization problem

$$\min_X \text{MSE} = \text{tr} \left( R - RX^H(XX^H + \sigma^2 I)^{-1}XR \right)$$

s.t. $\text{tr}(XX^H) \leq P$  \hspace{1cm} (18)

The solution of the above optimization problem is summarized as follows.

**Theorem 2:** Let $R = U_0\Lambda_0U_0^H$ denote the EVD of $R$, where $\Lambda_0 = \text{diag}(\lambda_1, \ldots, \lambda_M)$ is a diagonal matrix with its diagonal entries arranged in a decreasing order and $U_0 \in \mathbb{C}^{M \times M}$ is a unitary matrix. The optimal solution to (18) is then given by

$$X = [\Delta \ 0]U_0^H$$  \hspace{1cm} (19)

where $\Delta = \text{diag}(\delta_1, \ldots, \delta_T)$ with $\delta_i$ given as

$$\delta_i = \begin{cases} \sqrt{\mu - \sigma^2 \lambda_i^{-1}} & \text{if } \mu \geq \sigma^2 \lambda_i^{-1} \text{ and } \lambda_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (20)

in which $\mu$ is determined by the constraint $\sum_{i=1}^T \delta_i^2 = P$.

1 Here $R = U_0\Lambda_0U_0^H$ is used to distinguish itself from the truncated EVD $R = U^HAU^H$. 
by setting the number of symbols equal to the rank of the channel covariance matrix. We have the following properties associated with all users, and an iterative algorithm was developed to solve the maximization problem. In this section, a different criterion is considered, where the objective is to minimize the sum of MSEs associated with all users, i.e.

\[
\min_X \sum_{k=1}^{K} \text{MSE}_k
\]

\[
= \sum_{k=1}^{K} \text{tr} \left( R_k - R_k X^H (X R_k X^H + \sigma^2 I)^{-1} X R_k \right)
\]

\[
\text{s.t.} \quad \text{tr}(X X^H) \leq P
\]

where \( R_k \) and \( \text{MSE}_k \) denote the channel covariance matrix and the MSE associated with the \( k \)th user, respectively. Also, to simplify the problem, we assume the noise variances across different users are identical, i.e. \( \sigma_1^2 = \ldots = \sigma_K^2 = \sigma^2 \). Finding an analytical solution to the above optimization is difficult. Nevertheless, we will show that an asymptotically optimal training sequence can be devised given that users have mutually non-overlapping angles of arrival (AoAs). Here the asymptotic optimality means that the solution approaches the optimal one as the number of antennas at the BS goes to infinity.

Before proceeding, we first introduce the following properties which were proved in [5, 21] and reveal the eigenstructure properties of the channel covariance matrices. Consider the channel \( h \) generated by the one ring model with a bounded support of angle of arrival \( \theta \in [\theta_{\text{min}}, \theta_{\text{max}}] \). Let \( R \) denote the channel covariance matrix. We have the following properties regarding the channel covariance matrices.

**Property 1** [5, Lemma 3]: In the asymptotic regime of large number of antennas, steering vectors \( a(\vartheta) \) with \( \vartheta \notin [\theta_{\text{min}}, \theta_{\text{max}}] \) fall in the null space of the covariance matrix \( R \), i.e.

\[
\text{null}(R) \supset \text{span}\{ a(\vartheta) / \sqrt{M}, \forall \vartheta \notin [\theta_{\text{min}}, \theta_{\text{max}}] \}, \text{as } M \to \infty
\]
Property 2 [21, Lemma 1]: For the uniform linear array (ULA) case, when $M \to \infty$, the eigenvector matrix of the channel covariance matrix $R$ can be well approximated by a unitary discrete Fourier transform (DFT) matrix.

Property 3: From the above two properties, we naturally arrive at the following property: The column vectors in the DFT matrix whose angular coordinates are located outside the support of angle of arrival form an orthonormal basis for the null space of $R$. Meanwhile, those column vectors in the DFT matrix whose angular coordinates lie within the support of angle of arrival form an orthonormal basis for $R$. More precisely, let

$$F = \frac{1}{\sqrt{M}} \begin{bmatrix} \alpha(\omega_1) & \alpha(\omega_2) & \ldots & \alpha(\omega_M) \end{bmatrix}$$

(30)

denote the DFT matrix, in which $\omega_m = -1 + 2(m-1)/M, \forall m$, and $\alpha(\omega_m)$ is defined in (14). Let $R = U \Lambda U^H$ denote the truncated eigenvalue decomposition, where $U \in \mathbb{C}^{M \times r}$, and $\Lambda \in \mathbb{C}^{r \times r}$. Then as $M \to \infty$, $U$ is composed of column vectors of $F$ whose angular coordinates $\{\omega_i\}$ lie within the support of AoA, i.e.

$$U = [\alpha(\omega_{i_1}) \ldots \alpha(\omega_{i_r})]$$

(31)

where $\omega_i \in [2d\cos(\theta_{\min})/\chi, 2d\cos(\theta_{\max})/\chi]$ for $i = i_1, \ldots, i_r$.

We now discuss how to devise an asymptotically optimal pilot sequence for (28). Let

$$R_k = U_k \Lambda_k U_k^H$$

(32)

denote the truncated eigenvalue decomposition, where $U_k \in \mathbb{C}^{M \times r_k}$ and $\Lambda_k \in \mathbb{C}^{r_k \times r_k}$. $r_k$ denotes the rank of $R_k$. For simplicity, we assume $r_k = r, \forall k$. Inspired by the above properties, we propose an overlayed pilot sequence that is a superposition of a set of pilot sequences $\{X_k\}$

$$X = \sum_{k=1}^{K} X_k$$

(33)

where $X_k$ denotes the pilot sequence optimally designed for user $k$, i.e. given a power constraint $\text{tr}(X_k X_k^H) = P_k^*$. $X_k$ is given by Theorem 2, i.e.

$$X_k = \Delta_k^H U_k^H$$

(34)

in which $\Delta_k$ is a diagonal matrix with its diagonal elements optimized according to a water-filling power allocation scheme as described in Theorem 2.

We now show that the asymptotically optimal solution to (28) has a form of (33). Note that any pilot sequence $X$ can be expressed in terms of the DFT matrix as follows

$$X = ZF^H$$

(35)

where $Z \in \mathbb{C}^{T \times M}$ is a matrix to be optimized. Recalling Properties 2 and 3, we have

$$XR_k = ZF^HR_k = (Z_k U_k^H + \bar{Z}_k \bar{U}_k^H) R_k = Z_k U_k^H R_k$$

(36)

where $(\bar{a})$ comes from the fact that we can partition the DFT matrix into two parts $F = [U_k \bar{U}_k]$, in which $U_k$ is an orthonormal basis of $R_k$ and $\bar{U}_k$ is an orthonormal basis for the null space of $R_k$. Accordingly, $Z$ can be partitioned into two parts: $Z = [Z_k \bar{Z}_k]$, where $Z_k \in \mathbb{C}^{T \times r_k}$ is a submatrix of $Z$ consisting of $r$ column vectors. Substituting (36) into the objective function (28), we have

$$\sum_{k=1}^{K} \text{tr} \left( R_k - R_k X_k^H (X R_k X_k^H + \sigma^2 I)^{-1} X R_k \right)$$

$$= \sum_{k=1}^{K} \text{tr} \left( R_k - R_k U_k Z_k^H (Z_k U_k^H R_k U_k Z_k^H + \sigma^2 I)^{-1} Z_k U_k^H R_k \right)$$

(37)

Since users have mutually non-overlapping AoAs, each matrix $Z_k$ is constructed by $r$ unique columns of $Z$ that are not shared by other matrices $Z_k, \forall k \neq k$. Therefore the optimization (28) can be decomposed into $K$ independent problems, with $Z_k$ optimized in each individual problem

$$\min_{Z_k} \text{tr}(R_k - R_k U_k Z_k^H (Z_k U_k^H R_k U_k Z_k^H + \sigma^2 I)^{-1} Z_k U_k^H R_k)$$

s.t. $\text{tr}(Z_k Z_k^H) = P_k^*$

(38)

where $P_k^*$ is the optimal power allocated to the $k$th user.

From Theorem 2, we know that setting $T = r$ is sufficient to achieve a minimum MSE and the optimal $Z_k$ is a diagonal matrix

$$Z_k^* = \Delta_k^H$$

(39)

with its diagonal elements determined according to a water-filling power allocation scheme (see Theorem 2) such that the constraint $\text{tr}(Z_k Z_k^H) = P_k^*$ is satisfied. For those columns of $Z$ that are not included in $\{Z_k\}_{k=1}^{K}$, since they make no difference to the objective function value, they should be set to zero in order to save the transmit power. Therefore the asymptotically optimal pilot signal $X$ can be written as

$$X = \sum_{k=1}^{K} Z_k^* U_k^H = \sum_{k=1}^{K} \Delta_k^H U_k^H$$

(40)

which is a superposition of a set of pilot sequences, with each pilot sequence optimally designed for each individual user.

Remark 1: The above overlayed pilot design has an intuitive explanation. Given that the AoAs of all users are distinct, from Property 1, we know that the channel of each user is asymptotically orthogonal to the channel covariance matrices associated with other users as $M \to \infty$, i.e. $h_k^H R_{k'} = 0, \forall k \neq k'$. As a result, we have $X_k^H h_k = 0, \forall k \neq k'$ for the pilot sequence $\{X_k\}$ devised in (34). Hence from the user’s perspective, only the optimal pilot signal will be received, while other non-optimal pilot signals are filtered when propagating through the channel.

Remark 2: The proposed overlayed downlink training scheme bears a resemblance to the joint spatial division and

\[ \text{Remark 2:} \]
multiplexing (JSDM) strategy [11], where a prebeamforming matrix is employed to reduce the dimension of the channel to be estimated. In particular, the prebeamforming matrix suggested by [11] is a concatenation of \( \{U_k\}_{k=1}^K \). Although both the proposed overlayed training scheme and the JSDM scheme use the eigenvectors of the channel covariance matrices for downlink training, the rationale behind these two schemes are different. The JSDM scheme is shown to be asymptotically optimal in terms of the achievable capacity, whereas the asymptotic optimality of the proposed overlayed training scheme is established from the channel estimation perspective. Finally, we remark that a coordination strategy can be used to make sure that users to be served in the same time-frequency slot are well separated in the AoA domain, similarly as discussed in [5], [11].

VI. Estimated Covariance-Assisted MMSE

The MMSE estimator assumes perfect knowledge of the downlink channel covariance matrix. This knowledge, however, is unavailable and needs to be estimated in practice. If the covariance matrix is estimated by the user, it needs to be fed back to the BS through some control channel, which involves a significant amount of overhead. One way to overcome this difficulty is to estimate the downlink channel covariance matrix from the uplink covariance matrix, e.g. [22], [23]. This approach, however, still requires a certain amount of specific uplink training. In this section, we develop a simple approach, however, still requires a certain amount of specific uplink training. In this section, we develop a simple scheme to estimate the channel covariance matrix based on the one ring model. A MMSE estimator is then constructed based on the estimated covariance matrix. Our simulation results indicate that the covariance estimation scheme is effective and can obtain notable improvement in estimation performance.

According to the one-ring model [3], the covariance matrix of \( h \) can be written as

\[
R_{mn} = \frac{\xi^2 P}{P} \sum_{i=1}^P E[\alpha(\theta_p)\alpha(\theta_p)^H] = \xi^2 E[\alpha(\theta)\alpha(\theta)^H]
\]  

(41)

To calculate \( E[\alpha(\theta)\alpha(\theta)^H] \), we need to know the distribution of \( \theta \). Here we assume \( \theta \) is uniformly distributed with mean angle \( \theta \) and angular spread \( \nu \). Thus the \((m,n)\)th entry of \( R \) can be expressed as

\[
R_{mn} = \frac{\xi^2}{2\nu} \int_{\theta-\nu}^{\theta+\nu} e^{-j2\pi \frac{(m-n)d}{\lambda} \cos(\theta)} d\theta
\]  

(42)

The above integration, however, is difficult to calculate. Noting that the angular spread \( \nu \) is usually small, we can use the Taylor expansion of \( \cos(\theta) \) to approximate the integral. We have

\[
\cos(\theta) \approx \cos(\hat{\theta}) - \sin(\hat{\theta})(\theta - \hat{\theta})
\]  

(43)

Substituting (43) into (42), we arrive at

\[
R_{mn} \approx \frac{\xi^2}{2\nu} e^{jA_{mn}} \sin(\hat{\theta}) \int_{\theta-\nu}^{\theta+\nu} e^{-j2\pi \frac{(m-n)d}{\lambda} \cos(\theta)} d\theta
\]

\[
= \xi^2 e^{jA_{mn}} \cos(\hat{\theta}) \sin(A_{mn} \sin(\hat{\theta}))
\]  

(44)

where \( A_{mn} \triangleq 2\pi(m-n)d/\lambda \), and \( \sin(x) \triangleq \sin(x)/x \) is the sinc function. Therefore, the covariance matrix \( R \) can be approximated as a parametric matrix with parameters \( \hat{\theta} \) and \( \nu \). Note that the parameter \( \xi^2 \) in (44) can be ignored since as a scaling factor, it is independent of the signal subspace of \( R \). Thus the channel covariance estimation problem is simplified to find the mean angle \( \hat{\theta} \) and the angular spread \( \nu \). There are several ways to estimate these two parameters. Here we introduce a compressed sensing-based method. Recalling that the channel with a narrow angular spread has an approximate sparse representation on the angular domain, i.e.

\[
h = Ah
\]  

(45)

where \( A \) is an \( M \times M \) unitary matrix determined by the array geometry at the base station. For the uniform linear array, \( A \) becomes the DFT matrix consisting of columns characterized by different angular coordinates. \( \hat{h} \) is an approximately sparse vector, of which the \( n \)th element is contributed by the paths around the \( n \)th angular coordinate. Due to the narrow angular spread, a majority of the channel energy is concentrated on a few consecutive angular coordinates. Hence the mean angle and angular spread can be coarsely estimated from the sparse signal \( \hat{h} \). More precisely, the angular coordinate which has the largest magnitude can be estimated as the mean angle, i.e.

\[
\hat{\theta} = \begin{cases} 
\arccos \left[ \frac{\xi}{\sqrt{M}} \left( \frac{\nu}{\xi^2} \right) \right] & \text{if } s \leq \frac{M}{e} + 1 \\
\arccos \left[ \frac{\xi}{\sqrt{M}} \left( \frac{s-1}{M} \right) \right] & \text{otherwise}
\end{cases}
\]  

(46)

where \( s \) is the index of the angular coordinate which has the largest magnitude, i.e. the \( s \)th element of \( \hat{h} \) has the largest magnitude. The angular spread can be estimated as a symmetric interval around the mean angle, say, \( [\hat{\theta} - \nu, \hat{\theta} + \nu] \), with a majority of the channel energy (say, 90%) included in this interval. Now it remains to estimate the sparse vector \( \hat{h} \). As indicated earlier in this paper, the estimation of \( \hat{h} \) can be formulated into a sparse signal recovery problem:

\[
y = Xh + w = XAh + w
\]  

(47)

and can be efficiently solved via greedy or convex optimization methods. After \( \hat{h} \) is recovered, the mean angle and the angular spread can be obtained by using the aforementioned procedure, and an estimate of the channel covariance matrix can be computed by substituting the estimated mean angle and angular spread into (44). Finally, a MMSE estimate of \( h \) can be obtained.

For clarity, we summarize our proposed estimated covariance-assisted MMSE scheme in Algorithm 1.

Remark 1: Although \( h \) can be directly estimated from (47) via compressed sensing techniques, the MMSE estimator with the help of the estimated channel covariance matrix can provide a better estimation accuracy, as demonstrated by our simulation results. Our proposed MMSE estimator can be employed either at the mobile station (i.e. user) or at the BS to estimate the channel. If the channel is estimated by the mobile station, the full CSI needs to be fed back to the BS, which causes a large amount of uplink overhead when \( M \) is large. An alternative approach is to let the mobile station simply feed back the received signal \( y \) to the BS, and let the BS form an estimate of the channel based on \( y \). This approach requires less uplink overhead since the dimension of \( y \) is usually smaller.
Algorithm 1: Estimated Covariance-Assisted MMSE (EC-MMSE)

Given the received signal $y \in \mathbb{C}^T$ and the pilot signal $X \in \mathbb{C}^{T \times M}$.

1. Recover $\tilde{h}$ from $y = XA\tilde{h} + w$ via compressed sensing techniques, where $A$ is a DFT matrix for the uniform linear array case.

2. Estimate the mean angle $\bar{\theta}$ and angular spread $\bar{\nu}$ based on $\tilde{h}$, then obtain an estimate of the channel covariance matrix, $\hat{R}$, via (44).

3. Construct a MMSE estimator $\hat{h} = \hat{R}X^H(X\hat{R}X^H + \sigma^2I)^{-1}y$ to estimate the channel $h$.

The channel covariance matrix is assumed perfectly known by the MMSE estimator. Fig. 2 depicts the normalized mean-squared errors (NMSEs) of the MMSE estimator vs. the reciprocal of the noise variance, where we consider both the optimal pilot sequence devised according to Theorem 2 and a random pilot sequence whose entries are i.i.d. normal random variables. Note that the random pilot sequence has to be multiplied by a scaling factor to satisfy a power constraint $\text{tr}(XX^H) \leq P$ that is also imposed on the optimal pilot. In Fig. 2(a), we randomly generate an exact low-rank channel covariance matrix $R$ whose rank is set equal to 15. While for Fig. 2(b), the channel covariance matrix is generated according to the one-ring model, where the AoAs are assumed to be uniformly distributed over an interval $[\theta - \nu, \theta + \nu]$, with the mean angle and the angular spread given respectively by $\theta = \pi/6$ and $\nu = \pi/10$, the total number of i.i.d. paths is set to $P = 100$, and $\sigma^2$ follows a complex Gaussian distribution with zero mean and variance $\xi^2 = 1$. A numerical average is utilized to compute (41) and obtain the channel covariance matrix for the one-ring model. Numerical results show that the covariance matrix has an approximate low-rank structure with about 12 dominant eigenvalues. To examine the impact of the number of pilot symbols on the estimation performance, we consider three different choices of $T$ in our simulations, namely, $T = 20 > \text{rank}(R)$, $T = \text{rank}(R)$, and $T = 10 < \text{rank}(R)$. From Fig. 2, we observe that when the number of symbols $T$ is no less than the rank of the channel covariance matrix, the NMSE of the MMSE estimator approaches zero in the limit of vanishing noise, i.e. $\sigma^2 \rightarrow 0$, whatever an optimal pilot sequence or a random pilot sequence is employed. On the other hand, when $T < \text{rank}(R)$, there exists an error floor for both the optimal and random pilots, that is, once the error floor is reached, a decrease in the noise power does not bring any additional estimation performance improvement. This result corroborates our theoretical analysis in Section III. Also, given a power constraint, the optimal pilot sequences for $T > \text{rank}(R)$ and $T = \text{rank}(R)$ are identical. Thus the NMSEs achieved by optimal pilot sequences remain unaltered for these two cases.

Next, we evaluate the performance of the EC-MMSE estimator proposed in Section VII. In our simulations, channels are randomly generated according to the one-ring model described above. Fig. 3(a) depicts the NMSEs of respective methods as a function of the signal-to-noise ratio (SNR), where we set $T = 20$ and the SNR is defined as $10 \log(\|Xh\|^2 / T \sigma^2)$. Results are averaged over 1000 independent runs, with the pilot sequence $X$ and the channel $h$ randomly generated for each run. In each run, the noise variance $\sigma^2$ is adjusted to meet a pre-specified SNR. A compressed sensing method and a MMSE estimator which has access to the true covariance matrix are also included for comparison. For the compressed sensing method, a fast iterative shrinkage-thresholding algorithm (FISTA) is employed to estimate the channel based on (47). The EC-MMSE estimator is built on the compressed sensing method: after the virtual channel $\tilde{h}$ is estimated via the FISTA, we

VII. SIMULATION RESULTS

We now carry out experiments to validate our theoretical results and to illustrate the performance of the estimated covariance-assisted MMSE estimator (referred to as EC-MMSE) proposed in Section VII. Throughout our simulations, unless otherwise explicitly specified, we assume a uniform linear array with $M = 64$ antennas, and the distance between neighboring antenna elements is set to a half of the wavelength of the signal.

We first examine the behavior of the MMSE estimator in the asymptotic low-noise regime when the channel covariance matrix has a low-rank or an approximate low-rank structure.
estimate the mean angle and the angular spread, then obtain an estimate of the channel covariance matrix, and finally construct the MMSE estimator. In our simulations, the angular spread is estimated as a symmetric interval around the estimated mean angle, with 95% of the channel energy concentrated on the interval. From Fig. 3(a), we see that our proposed scheme achieves a notably higher accuracy compared to the compressed sensing method. This result shows that the estimated covariance matrix, although imperfect, can still provide a substantial performance improvement. Fig. 3(b) plots the NMSEs of respective schemes vs. the number of symbols $T$, where we set SNR = 20dB. This result again demonstrates the advantage of the proposed EC-MMSE estimator over the compressed sensing method. To better illustrate the performance, we plot the histogram in Fig. 4 to show the distribution of the NMSE for the EC-MMSE and the compressed sensing method, respectively. From Fig. 4, we see that the proposed EC-MMSE estimator yields an accurate channel estimate (with an NMSE within the range $[0, 0.02]$) with a high probability, whereas the NMSEs associated with the compressed sensing method spread across the range $[0.04, 0.4]$ with a high probability.

Also, to examine the robustness of the proposed EC-MMSE estimator against the model mismatch, in our simulations, we assume that the angle of arrival follows a Gaussian distribution, with the mean and the standard deviation set to be $\bar{\theta} = \pi/6$ and $\sigma_{\theta} = \pi/30$, respectively. Note that in the proposed EC-MMSE estimator, a uniform AoA distribution is assumed to estimate the channel covariance matrix. In Fig. 5(a) and Fig. 5(b), we plot the NMSEs of respective schemes as a function of the SNR and the number of symbols, respectively, where we set $T = 20$ for Fig. 5(a) and SNR = 20dB for
Fig. 4. Histogram of the NMSE associated with the EC-MMSE estimator and the compressed sensing method

Fig. 5. Gaussian AoA: NMSEs of respective schemes vs. SNR and number of symbols $T$

Fig. 5(b). Results are averaged over 1000 independent runs, with the pilot sequence and the channel randomly generated for each run. In each run, the noise variance is adjusted to meet a pre-defined SNR. From Fig. 5 we see that the proposed EC-MMSE estimator achieves superior performance even the presumed AoA distribution is different from the true one. The reason, as already explained in the previous section, is that the eigenvectors of the channel covariance matrix are less dependent on the AoA distribution. Therefore our scheme which assumes a uniform AoA distribution can still reliably estimate the signal subspace spanned by dominant eigenvectors, and as a result, the EC-MMSE estimator still outperforms the compressed sensing method by a big margin.

Lastly, to more thoroughly evaluate the performance of the proposed EC-MMSE estimator, we examine its robustness against estimation errors of the mean angle and the angular spread. Since in the EC-MMSE scheme, the channel covariance matrix is obtained based on the estimated mean angle and angular spread, estimation errors of the mean angle and the angular spread will impair the estimation quality of the covariance matrix, which, in turn, affects the estimation accuracy of the EC-MMSE estimator. In Fig. 6(a), we plot the NMSE of the EC-MMSE estimator as the estimated mean angle deviates from the true one, where we set $T = 15$, $\sigma^2 = 0.1$, and the angular spread is assumed perfectly known. Results are averaged over $10^3$ independent runs, and for each run, the pilot sequence is randomly generated to meet a pre-specified power constraint, and the channel is randomly generated according to the one-ring model described in the second paragraph of this section. We see that the EC-MMSE estimator exhibits some robustness against the mean angle mismatch: the EC-MMSE estimator incurs mild performance
from the true one is small, say, $|\hat{\theta} - \bar{\theta}| < 3^\circ$. Nevertheless, a large deviation would result in a significant performance degradation. Fig. 6(b) depicts the behavior of the proposed EC-MMSE estimator when the estimated angular spread deviates from the true angular spread, where the mean angle is assumed perfectly estimated. From Fig. 6(b), it can be observed that the EC-MMSE estimator is robust to an overestimation of the angular spread, but is sensitive to the underestimation errors: it suffers from a substantial performance loss when the estimated angular spread is smaller than the true one. Hence it is safer to overestimate than to underestimate the angular spread.

**VIII. CONCLUSIONS**

We considered the problem of downlink training and channel estimation for FDD massive MIMO systems. Since the required amount of overhead for downlink training grows linearly with the number of transmit antennas at the BS, reducing the overhead for downlink training and uplink feedback has been a central issue in FDD massive MIMO systems. In this paper, we exploited the low-rank structure of the channel covariance matrix to reduce the overhead for downlink training. We studied the asymptotic behavior of the MMSE estimator when the channel covariance matrix has a low-rank structure. Our analysis shows that the MMSE estimator can achieve an exact channel recovery in the asymptotic low-noise regime, provided that the number of pilot symbols in time is no smaller than the rank of the channel covariance matrix. We also examined the optimal pilot sequence design for the single-user case, and an asymptotic optimal pilot sequence design for the multi-user scenario. We also develop a training-free scheme to estimate the channel covariance matrix. Simulation results show that a MMSE estimator based on the estimated covariance matrix achieves a substantial performance improvement as compared with the compressed sensing method, and is robust against the AoA distribution mismatch and the angular spread estimation error.

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