Nonlinear Amplitude Maxwell-Dirac Equations. Optical Leptons

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Abstract

We apply the method of slowly-varying amplitudes of the electrical and magnet fields to integro-differential system of nonlinear Maxwell equations. The equations are reduced to system of differential Nonlinear Maxwell amplitude Equations (NME). Different orders of dispersion of the linear and nonlinear susceptibility can be estimated. This method allow us to investigate also the optical pulses with time duration equal or shorter to the relaxation time of the media. The electric and magnetic fields are presented as sums of circular and linear components. Thus, NME is written as a set of Nonlinear Dirac Equations (NDE). Exact solutions of NDE with classical orbital momentum $\ell = 1$ and opposite directions of the spin (opposite charge) $j = \pm 1/2$ are obtained. The possible generalization of NME to higher number of optical components and higher number of $\ell$ and $j$ is discussed. Two kind of Kerr type media: with and without linear dispersion of the electric and the magnet susceptibility are consider. The vortex solutions in case of media with dispersion admit finite energy while the solutions in case of a media without dispersion admit infinite energy.

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1 Introduction

In the last four decades there has been considerable interest in the nonlinear generalizations of the quantum field equations [1 2 3] and
in the possibility of obtaining exact stationary solitary solutions of the field equations [4,5]. As a rule, the nonlinearity has been introduced in an *ad-hoc* fashion in the Klein-Gordon equation and also for all four spinor components of the Dirac equations. For the usual case of a cubic nonlinearity, exact localized solutions are not found. Our present work, reported in this paper, shows that the optical analogy of nonlinear Dirac equations leads to a nonlinear part only the first coupled equation of Nonlinear Dirac Equations. This result allows to solve the NDE by separation of variables and to obtain solutions representing optical vortices with classical momenta one and spin one-half.

The initial investigation of optical vortices began with a scalar theory, based on the well-known 2D+1 paraxial Nonlinear Schroedinger equation (NSE) [6,7]. The existence of optical vortices was predicted in the self-focusing regime, but as it was shown in many papers, the solutions obtained are modulationally unstable. In spite of this, various interactions (attraction, repulsion, fusion) have been observed. The scalar paraxial approximation is valid for slowly varying amplitudes of the electrical field in weakly nonlinear media. As it was pointed out in Refs.[8,9] this theory is not valid for a very intense narrow pulses.

The first generalization of the scalar paraxial theory of optical vortices is based on investigation of so called spatio-temporal scalar evolution equations [10,11,12]. For all cited theories no exact solutions have been found, but numerical and energy momentum techniques are used. The existence of exact stable vortex solutions of these types of nonlinear equations was finally discovered with the vector generalization of the 3D+1 Nonlinear Schrodinger equation [13]. It also have been shown numerically that these vortices are stable at distances comparable to those where localized solutions of the one component scalar equation self-focusing rapidly. All of these exact vortices are a combination of linearly-polarized components and have spin $\ell = 1$. To extend the theory to vortices with spin $j = 1/2$ we return to an analogy between the Maxwell and Dirac equations. As it is shown in Ref.[14] this analogy is possible only if the electrical and magnet components are represented as a sum of linear and circularly polarized components.
2 Maxwell’s equations with non-stationary linear and nonlinear polarization

Consider the Maxwell’s equations in the next two cases:

1. A source-free medium with non-stationary linear and nonlinear electric polarization and non-stationary magnetic polarization (the case with dispersion).

2. A source-free medium with stationary linear and nonlinear electric polarization and stationary magnetic polarization (the case without dispersion).

For these cases, the Maxwell’s equations can be written:

\[ \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \]  

(1)

\[ \nabla \times \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t}, \]  

(2)

\[ \nabla \cdot \vec{D} = 0, \]  

(3)

\[ \nabla \cdot \vec{B} = \nabla \cdot \vec{H} = 0, \]  

(4)

\[ \vec{D} = \vec{P}_{\text{lin}} + 4\pi \vec{P}_{\text{n\ell}}. \]  

(5)

The linear magnetic polarization corresponds is [18,19]:

\[ \vec{B} = \vec{H} + 4\pi \vec{M}_{\text{lin}}, \]  

(6)

where \( \vec{E} \) and \( \vec{H} \) are the electric and magnetic intensity fields, \( \vec{D} \) and \( \vec{B} \) are the electric and magnetic induction fields, \( \vec{P}_{\text{lin}}, \vec{P}_{\text{n\ell}} \) are the linear and nonlinear polarization of the medium respectively and \( \vec{M}_{\text{lin}} \) is the linear magnetic polarization. The magnetic polarization (magnetization) \( \vec{M}_{\text{lin}} \) is written as the product of the linear magnetic susceptibility \( \eta^{(1)} \) and the magnetic field \( \vec{H} \). The nonstationary linear electric polarization can be written as:
\[ \vec{P}_{lin} = \int_{-\infty}^{t} \left( \delta(t - \tau) + 4\pi \chi^{(1)}(t - \tau) \right) \vec{E}(\tau, x, y, z) d\tau \]

\[ = \int_{-\infty}^{t} \varepsilon_0(t - \tau) \vec{E}(\tau, x, y, z) d\tau, \quad (7) \]

where \( \chi^{(1)} \) and \( \varepsilon_0 \) are the linear electric susceptibility and the dielectric constant respectively. Similar expression describes the dependence of \( \vec{B} \) on \( \vec{H} \) in the case of nonstationary linear magnetic polarization [15]:

\[ \vec{B} = \int_{-\infty}^{t} \left( \delta(t - \tau) + 4\pi \eta^{(1)}(t - \tau) \right) \vec{H}(\tau, x, y, z) d\tau \]

\[ = \int_{-\infty}^{t} \mu_0(t - \tau) \vec{H}(\tau, x, y, z) d\tau, \quad (8) \]

where \( \eta^{(1)} \) and \( \mu_0 \) are the linear magnetic susceptibility and magnetic permeability respectively. The magnetic susceptibility of the main part of the dielectrics ranges from \( 10^{-6} - 10^{-4} \) and usually decreases with the increasing of the frequency. In the following, we will study such media with nonstationary cubic nonlinear polarization, where the nonlinear polarization in the case of one carrying frequency can be expressed as:

\[ \vec{P}_{nl}^{(3)} = \frac{3}{4} \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{t} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) \times \vec{E}(\tau_1, x, y, z) \vec{E}^*(\tau_2, x, y, z) \vec{E}(\tau_3, x, y, z) d\tau_1 d\tau_2 d\tau_3. \quad (9) \]

The causality request the next conditions on the response functions:

\[ \varepsilon(t - \tau) = 0; \quad \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) = 0, \]

\[ t - \tau < 0; \quad t - \tau_i < 0; \quad i = 1, 2, 3. \quad (10) \]

That why we can prolonged the upper integral boundary to infinity and to use standard Fourier presentation [18]:

\[ 4 \]
\[
\int_{-\infty}^{t} \varepsilon_0(\tau - t) \exp(i\omega\tau)d\tau = \int_{-\infty}^{+\infty} \varepsilon_0(\tau - t) \exp(i\omega\tau)d\tau \tag{11}
\]

\[
\int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{t} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) d\tau_1 d\tau_2 d\tau_3 = \int_{-\infty}^{+\infty} \int_{+\infty}^{+\infty} \int_{+\infty}^{+\infty} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) d\tau_1 d\tau_2 d\tau_3. \tag{12}
\]

The spectral presentation of linear optical susceptibility \(\hat{\varepsilon}_0(\omega)\) is connected to the nonstationary optical response function by the next Fourier transform:

\[
\hat{\varepsilon}_0(\omega) \exp(i\omega t) = \int_{-\infty}^{+\infty} \varepsilon_0(t - \tau) \exp(i\omega\tau)d\tau, \tag{13}
\]

Similar expression for the spectral presentation of the non-stationary magnetic response \(\hat{\mu}_0(\omega)\):

\[
\hat{\mu}_0(\omega) \exp(-i\omega t) = \int_{-\infty}^{+\infty} \mu_0(t - \tau) \exp(-i\omega\tau)d\tau. \tag{14}
\]

and nonlinear optical susceptibility \(\chi^{(3)}\)

\[
\hat{\chi}^{(3)} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) \exp(i(\omega(\tau_1 + \tau_2 + \tau_3)))d\tau_1 d\tau_2 d\tau_3 \tag{15}
\]
can be written. It is important to point here the remark of Akhmanov at all in [16], that such nonstationary representation applied to slowly varying amplitudes of electrical and magnet fields is valid as well when the optical pulse duration of the pulses \(t_0\) is greater than the characteristic response time of the media \(\tau_0\) (\(t_0 \gg \tau_0\)), as when the time duration of the pulses are equal or less than the time response of the media (\(t_0 \leq \tau_0\)). We will discuss this possibility in the process of deriving of the amplitude equations.
3 Deriving of the amplitude equations

In this section we derive the slowly varying amplitude approximation in the standard way, as it was done in [17], [18]. For the case of Maxwell equations with linear and nonlinear dispersion (1) - (6) we define the electric and magnetic field amplitudes with the relations:

\[ \vec{E}(x, y, z, t) = \vec{A}(x, y, z, t) \exp(i(\omega_0 t - g(x, y, z))), \]  
\[ \vec{H}(x, y, z, t) = \vec{C}(x, y, z, t) \exp(-i(\omega_0 t - q(x, y, z))), \]

where \( \vec{A}, \vec{C}, \omega_0, g \) and \( q \) are the amplitudes of the electric and magnetic fields, the optical frequency and the real spatial phase functions respectively. In case of monochromatic and quasi-monochromatic fields the Stokes parameters can be constructed from transverse components of the wave field [19, 20]. This leads to two component vector fields in a plane, transverse to the direction of propagation. For electromagnetic fields with spectral bandwidth (our case) the two dimensional coherency tensor cannot be used and the Stokes parameters cannot be found directly. As it was shown by T.Carozzi and all in [20], using high order of symmetry (SU(3)), in this case six independent Stokes parameters can be found. This corresponds to three component vector field. Here is investigated this case. The increasing of the spectral bandwidth of the vector wave, increases also the depolarization term (component, normal to the standard Stokes coherent polarization plane).

We use the Fourier representation of the response functions (13)-(15) and of the amplitude functions \( \vec{A} \) (and \( \vec{C} \)) to obtain next expressions for the first derivatives in time of the linear polarization, nonlinear polarization and magnetic induction fields:

\[ \frac{1}{c} \frac{\partial \vec{P}_{lin}(x, y, z, t)}{\partial t} = i \exp(i(\omega_0 t - g(x, y, z))) \]
\[ \times \int_{-\infty}^{+\infty} \frac{\omega \hat{\varepsilon}_0(\omega)}{c} \vec{A}(x, y, z, \omega - \omega_0) \exp(i(\omega - \omega_0)t) \, d\omega, \]  
\[ \frac{4\pi}{c} \frac{\partial \vec{P}_{nlin}(x, y, z, t)}{\partial t} = i \exp(i(\omega_0 t - g(x, y, z))) \]
\[ \times \int_{-\infty}^{+\infty} \frac{3\pi\omega^{(3)}(\omega)}{c} |\vec{A}(x, y, z, \omega - \omega_0)|^2 \vec{A}(x, y, z, \omega - \omega_0) \]
\[ \times \exp \left( i(\omega - \omega_0)t \right) d\omega. \quad (19) \]

\[ - \frac{1}{c} \frac{\partial \vec{B}(x, y, z, t)}{\partial t} = i \exp \left( i(\omega_0 t - q(x, y, z)) \right) \]
\[ \times \int_{-\infty}^{+\infty} \frac{\omega^{\mu_0}(\omega)}{c} \vec{C}(x, y, z, \omega - \omega_0) \exp \left( -i(\omega - \omega_0)t \right) d\omega. \quad (20) \]

At this point we restrict the spectrum of the amplitude of electrical and magnet fields by writing the wave vectors \( k_{1,nl} \) in a Taylor series:

\[ k_1(\omega) = \frac{\omega \hat{e}_0(\omega)}{c} = k_1^0(\omega_0) + \]
\[ + \frac{\partial (k_1(\omega_0))}{\partial \omega} (\omega - \omega_0) + \frac{1}{2} \frac{\partial^2 (k_1(\omega_0))}{\partial \omega^2} (\omega - \omega_0)^2 + .... = \]
\[ = k_1^0(\omega_0) + \frac{1}{v_1}(\omega - \omega_0) + \frac{1}{2} k_1^0(\omega - \omega_0)^2 + ...., \quad (21) \]

\[ k_{nl}(\omega) = \frac{3\pi\omega^{(3)}(\omega)}{c} = \]
\[ = k_{nl}^0(\omega_0) + \frac{\partial (k_{nl}(\omega_0))}{\partial \omega} (\omega - \omega_0) + .... = \]
\[ = k_{nl}^0(\omega_0) + \frac{1}{v_{nl}}(\omega - \omega_0) + ...., \quad (22) \]

\[ k_2(\omega) = \frac{\omega \mu_0(\omega)}{c} = k_2^0(\omega_0) \]
\[ \frac{\partial (k_2(\omega_0))}{\partial \omega} (\omega - \omega_0) + \frac{1}{2} \frac{\partial^2 (k_2(\omega_0))}{\partial \omega^2} (\omega - \omega_0)^2 + \ldots = \]

\[ k_2^0(\omega_0) + \frac{1}{v_2}(\omega - \omega_0) + \frac{1}{2} k''_2(\omega - \omega_0)^2 + \ldots, \quad (23) \]

where \( v_i \) and \( k''_i; i = 1, nl, 2 \) have dimensions of group velocity, and nonlinear addition to the group velocity and dispersion respectively. The nonlinear wave vector is expressed:

\[ k_{nl}^0 = 3\pi \omega_0 \hat{\chi}^{(3)}(\omega_0)/c = \frac{\omega_0 \hat{\epsilon}_0}{c} \frac{3\pi \hat{\chi}^{(3)}(\omega_0)}{\hat{\epsilon}_0} = k_1 n_2, \quad (24) \]

where

\[ n_2(\omega_0) = \frac{3\pi \hat{\chi}^{(3)}(\omega_0)}{\hat{\epsilon}_0}, \quad (25) \]

is the nonlinear refractive index. It is important to note here that we do not use any approximation of the response function. There is only one requirement of the spectral presentations (15), (13) and (14) of the response functions: to admit first and second order derivatives in respect to frequency (to be smooth functions). The restriction is only in respect of the relation between the main frequency \( \omega_0 \) and time duration of the envelope functions \( t_0 \) determinate from the relations (21), (22) and (23) (conditions for slowly varying amplitudes). Putting eqn. (24) in (18), (23) in (20) and (22) in (19), and keeping in mind the expressions for time derivatives of the spectral presentation of the amplitude functions, the first derivatives (18), (19) and (20) are presented in form:

\[ \frac{1}{c} \frac{\partial \bar{P}^{lin}(x, y, z, t)}{\partial t} = \left( ik_1^0 \bar{A} + \frac{1}{v_1} \frac{\partial \bar{A}}{\partial t} - \frac{1}{2} \frac{\partial^2 \bar{A}}{\partial t^2} \right) \times \exp (i(\omega_0 t - g(x, y, z))), \quad (26) \]

\[ \frac{4\pi}{c} \frac{\partial \bar{P}^{nl}(x, y, z, t)}{\partial t} = \left( ik_1^0 n_2 |\bar{A}|^2 \bar{A} + \frac{1}{v_{nl}} \frac{\partial |\bar{A}|^2 \bar{A}}{\partial t} \right) \times \exp (i(\omega_0 t - g(x, y, z))), \quad (27) \]
Finally, from the Maxwell equations \((1)-(6)\), using eq. \((26)\), eq. \((27)\) and eq. \((28)\), and using in fact that \(\nabla \cdot \vec{D} \approx \nabla \cdot \vec{E} \approx 0\) \([13]\), we obtain the next system of Nonlinear Maxwell vector amplitude Equations (NME):

\[
\nabla \times \vec{A} = ik_2^0 \vec{C} - \frac{1}{v_2} \frac{\partial \vec{C}}{\partial t} - \frac{ik_2^0}{2} \frac{\partial^2 \vec{C}}{\partial t^2},
\]

\[\text{(29)}\]

\[
\nabla \times \vec{C} = ik_1^0 \vec{A} + \frac{1}{v_1} \frac{\partial \vec{A}}{\partial t} - \frac{ik_1^0}{2} \frac{\partial^2 \vec{A}}{\partial t^2} + ik_1^0 n_2 \left( \vec{A} \cdot \vec{A}^* \right) \vec{A} + \left( \frac{n_2}{v_1} + k_1 \frac{\partial n_2}{\partial \omega} \right) \frac{\partial \left( \vec{A} \cdot \vec{A}^* \right)}{\partial t},
\]

\[\text{(30)}\]

\[
\nabla \cdot \vec{A} = 0,
\]

\[\text{(31)}\]

\[
\nabla \cdot \vec{C} = 0,
\]

\[\text{(32)}\]

if the gradient of the spatial phase functions \(g\) and \(q\) satisfied the relations:

\[
\nabla g \times \vec{A} = 0,
\]

\[\text{(33)}\]

and

\[
\nabla q \times \vec{C} = 0.
\]

\[\text{(34)}\]

The phase functions whichever satisfied \((33)-(34)\) are determinate in Section \((10)\).
We investigate the case, when our vector fields are presented as a sum of circular and linear polarizing components:

\[ \vec{A} = \vec{A}_{\text{lin}} + \vec{A}_{\text{cir}}, \quad (35) \]
\[ \vec{C} = \vec{C}_{\text{lin}} + \vec{C}_{\text{cir}}. \quad (36) \]

The nonlinear polarization admit different nonlinear refractive indexes in the case of linear and circular polarization \([21]\) (\(n_{\text{lin}}^2 \neq n_{\text{cir}}^2\)). We will include this difference in the our rescaled equations, defining the next rescaled dependant variables:

\[ \vec{A} = A_{0\text{lin}} \vec{A}'_{\text{lin}} + A_{0\text{cir}} \vec{A}'_{\text{cir}}, \quad (37) \]
\[ \vec{C} = C_0(\vec{C}'_{\text{lin}} + \vec{C}'_{\text{cir}}), \quad (38) \]
\[ (A_{0\text{lin}})^2 = \frac{n_{\text{cir}}^2}{n_{\text{lin}}^2}(A_{0\text{cir}})^2, \quad (39) \]

and independent variables;

\[ x = r_0 x'; y = r_0 y'; z = r_0 z', t = t_0 t'. \quad (40) \]

In additions the next constants are determinate:

\[ \alpha_i = k_i^0 r_0; \quad \beta_i = k_i^0 r_0 / 2t_0^2; \quad \gamma_1 = r_0 k_i n_{\text{cir}}^2 |A_0|^2; \]
\[ \gamma_2 = n_{\text{cir}}^2 |A_0|^2; \quad \gamma_3 = v_1 n_{\text{cir}}^2 |A_0|^2 / c; \]
\[ \delta = v_1 / v_2 \leq 1; \quad v_1 \approx r_0 / t_0; \quad i = 1, 2. \]

The NME (29)-(32) in rescaled variables are transformed into the following (the primes have been omitted for clarity):

\[ \nabla \times \vec{A} = i\alpha_2 \vec{C} - \delta \frac{\partial \vec{C}}{\partial t} - i\beta_2 \frac{\partial^2 \vec{C}}{\partial t^2}, \quad (41) \]

\[ \nabla \times \vec{C} = i\alpha_1 \vec{A} + \frac{\partial \vec{A}}{\partial t} - i\beta_1 \frac{\partial^2 \vec{A}}{\partial t^2} + i\gamma_1 (\vec{A} \cdot \vec{A}^*) \vec{A} \]
\[ + (\gamma_2 + \gamma_3) \frac{\partial (\vec{A} \cdot \vec{A}^*)}{\partial t}, \quad (42) \]
4 \text{ Dirac representation of NME}

To solve the NME (45)-(48), we apply the method of separation of variables. The slowly varying amplitude vector of the electric field \( \vec{A} \) and the magnetic field \( \vec{C} \) are represented as:

\[
\vec{A}(x, y, z, t) = \vec{F}(x, y, z) \exp(i\Delta \alpha t),
\] (49)
\[ \vec{C} (x, y, z, t) = \vec{G} (x, y, z) \exp (i\Delta \alpha t). \]  

Substituting these forms into the NME (45)-(48) we obtain:

\[ \nabla \times \vec{F} = -i\nu_2 \vec{G}, \]  

\[ \nabla \times \vec{G} = i\nu_1 \vec{F} + i\gamma_1 (\vec{F} \cdot \vec{F}^*) \vec{F}, \]

\[ \nabla \cdot \vec{F} = 0, \]  

\[ \nabla \cdot \vec{G} = 0, \]

where \( \nu_1 = \alpha_1 + \Delta \alpha; \nu_2 = \delta \Delta \alpha - \alpha_2 > 0. \) In a Cartesian coordinate system, the vector equations (51)-(54) are reduced to a scalar system of eight nonlinear wave equations. When the electric and magnetic fields are represented as a sum of a linear polarization component and a circular polarized one it is possible to reduce eqns. (51) - (54) to a system of four nonlinear equations. Substituting:

\[ \psi_1 = iF_l = iF_z, \]

\[ \psi_2 = F_c = iF_x - F_y, \]

\[ \psi_3 = G_l = -G_z, \]

\[ \psi_4 = G_c = -G_x - iG_y, \]

into the nonlinear system (51)-(54), we obtain a stationary nonlinear Dirac system of equations (NDE):
\[
\left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \Psi_4 + \frac{\partial}{\partial z} \Psi_3 = -i \left( \nu_1 + \gamma_1 \sum_{i=1}^{2} |\Psi_i|^2 \right) \Psi_1, \tag{59}
\]

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \Psi_3 - \frac{\partial}{\partial z} \Psi_4 = -i \left( \nu_1 + \gamma_1 \sum_{i=1}^{2} |\Psi_i|^2 \right) \Psi_2, \tag{60}
\]

\[
\left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \Psi_2 + \frac{\partial}{\partial z} \Psi_1 = -i \nu_2 \Psi_3, \tag{61}
\]

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \Psi_1 - \frac{\partial}{\partial z} \Psi_2 = -i \nu_2 \Psi_4. \tag{62}
\]

This substitution allows to reduce the system of eight equations (51)-(54) to a system of four scalar complex equations (59)-(62). The system (59)-(62) is the optical analog of the nonlinear Dirac equations (NDE). Note that the optical NDE are significantly different from the NDE in the field theory. The nonlinear part appears only in the first two coupled equations of the system.

5 Vortex solutions with orbital momentum l=1 and spin j=1/2

The symmetries of the NDE (59)-(62) are used to obtain exact vortex solutions. The NDE (59)-(62) have both spherical and spinor symmetry only in the case where the nonlinear part does not manifest the angular dependence on the radial variable \(\sum_{i=1}^{2} |\Psi_i(r, \theta, \varphi)|^2 = F(r)\). This type of solution can be found using with the following technique. Using Pauli matrices, we write the NDE system (59)-(62) as:

\[
(\vec{\sigma} \cdot \vec{P}) \phi = -i \left( \nu_1 + \gamma_1 \sum_{i=1}^{2} |\eta_i|^2 \right) \eta, \tag{63}
\]

\[
(\vec{\sigma} \cdot \vec{P}) \eta = -i \nu_2 \phi, \tag{64}
\]
where

\[ \vec{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \],

are the Pauli matrices, \( \vec{P} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \) is the differential operator and

\[ \eta = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}; \phi = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}, \]

are the corresponding spinors. After substituting eqn.(64) into eqn.(63) we obtain:

\[ \left( \vec{\sigma} \cdot \vec{P} \right) \left( \vec{\sigma} \cdot \vec{P} \right) \eta = -\nu_2 \left( \nu_1 + \gamma_1 \sum_{i=1}^{2} |\eta_i|^2 \right) \eta. \quad (65) \]

When there is no external electric or magnetic field, the operator on the left-hand side of eqn. (65) is the Laplacian operator \( \Delta \):

\[ P^2 = \left( \vec{\sigma} \cdot \vec{P} \right) \left( \vec{\sigma} \cdot \vec{P} \right) = \Delta. \quad (66) \]

From (65) and (66) we obtain:

\[ \nu_2 \nu_1 \eta + \nu_2 \gamma_1 \sum_{i=1}^{2} |\eta_i|^2 \eta + \Delta \eta = 0. \quad (67) \]

In case of spherical representation of the spinor equations (65), there are two possibilities, \( l = 0 \) and \( l = 1 \), that will permit only a radial dependence of the nonlinear part:

\[ \sum_{i=1}^{2} |\eta_i (r, \theta, \phi)|^2 = F(r). \quad (68) \]

For the \( l = 0 \) case, the equations (67) are transformed to equations with radial parts:

\[ \nu_2 \nu_1 \eta + \nu_2 \gamma_1 \sum_{i=1}^{2} |\eta_i|^2 \eta + \frac{\partial^2 \eta}{\partial r^2} + \frac{2 \partial \eta}{r \partial r} = 0. \quad (69) \]

The scalar variant of these equations has been investigated in many papers but exact localized solutions have not been found. In the case \( l = 1 \) we look for spinors in the next two forms:

\[ \eta = \begin{pmatrix} \tilde{\eta} (r) \cos (\theta) \\ \tilde{\eta} (r) \sin (\theta) \exp (i\varphi) \end{pmatrix}. \quad (70) \]
and

\[ \eta = \begin{pmatrix} \tilde{\eta}(r) \sin(\theta) \exp(-i\varphi) \\ -\tilde{\eta}(r) \cos(\theta) \end{pmatrix}. \]  

(71)

As it will be seen later this corresponds to two opposite directions of the own orbital momentum \( j = \pm 1/2 \) (opposite charge). After substituting solutions (70)–(71) into equations (67) the following equation describing the radial dependence is obtained:

\[ \nu_2 \nu_1 \tilde{\eta} + \nu_2 \gamma_1 |\tilde{\eta}|^2 \tilde{\eta} + \frac{\partial^2 \tilde{\eta}}{\partial r^2} + \frac{2}{r} \frac{\partial \tilde{\eta}}{\partial r} - \frac{2}{r^2} \tilde{\eta} = 0. \]  

(72)

The angular parts are the standard spherical harmonics with \( l = 1 \). This system has exact vortex de Broglie soliton solutions \([4]\) in the form:

\[ \tilde{\eta}(r) = \frac{\sqrt{2}}{i} \exp \left( \frac{i\sqrt{\nu_1 \nu_2} r}{r} \right), \]  

(73)

if \( \nu_2 \gamma_1 = 1 \). The complete solutions for these two cases are written:

\[ \eta = \begin{pmatrix} \sqrt{2} \exp(i\sqrt{\nu_1 \nu_2} r) \cos(\theta) \\ \sqrt{2} \exp(i\sqrt{\nu_1 \nu_2} r) \sin(\theta) \exp(i\varphi) \end{pmatrix}, \]  

(74)

\[ \phi = \begin{pmatrix} \sqrt{2} \left( \frac{i\sqrt{\nu_1 \nu_2} \exp(i\sqrt{\nu_1 \nu_2} r)}{r} + \exp i\left(\sqrt{\nu_1 \nu_2} r\right) \right) \\ 0 \end{pmatrix}, \]  

(75)

and

\[ \eta = \begin{pmatrix} \sqrt{2} \exp(i\sqrt{\nu_1 \nu_2} r) \sin(\theta) \exp(i\varphi) \\ -\sqrt{2} \exp(i\sqrt{\nu_1 \nu_2} r) \cos(\theta) \end{pmatrix}, \]  

(76)

\[ \phi = \begin{pmatrix} 0 \\ -\sqrt{2} \left( \frac{i\sqrt{\nu_1 \nu_2} \exp(i\sqrt{\nu_1 \nu_2} r)}{r} + \exp i\left(\sqrt{\nu_1 \nu_2} r\right) \right) \end{pmatrix}. \]  

(77)
There is another, more direct way for separating the variables of the spinor equations (63)-(64). To illustrate this, we represent the NDE (59)-(62) in spherical variables:

\[
\exp (-i\phi) \left( \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} - \frac{i}{r \sin \theta \frac{\partial}{\partial \varphi}} \right) \Psi_4
+ \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \Psi_3 = -i \left( \nu_1 + \gamma_1 \sum_{i=1}^{2} |\Psi_i|^2 \right) \Psi_1, \tag{78}
\]

\[
\exp (i\phi) \left( \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} + \frac{i}{r \sin \theta \frac{\partial}{\partial \varphi}} \right) \Psi_3
- \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \Psi_4 = -i \left( \nu_1 + \gamma_1 \sum_{i=1}^{2} |\Psi_i|^2 \right) \Psi_2, \tag{79}
\]

\[
\exp (-i\phi) \left( \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} - \frac{i}{r \sin \theta \frac{\partial}{\partial \varphi}} \right) \Psi_2
+ \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \Psi_1 = -i\nu_2 \Psi_3, \tag{80}
\]

\[
\exp (i\phi) \left( \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} + \frac{i}{r \sin \theta \frac{\partial}{\partial \varphi}} \right) \Psi_1
- \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \Psi_2 = -i\nu_2 \Psi_4. \tag{81}
\]

We make the following two ansatzes for solutions to the system of nonlinear equations (78)-(81):

\[
\Psi_1 = a(r) \cos (\theta)
\Psi_2 = a(r) \sin (\theta) e^{i\varphi}
\Psi_3 = -ib(r)
\Psi_4 = 0.
\tag{82}
\]

and

\[
\Psi_1 = a(r) \sin (\theta) e^{i\varphi}
\Psi_2 = -a(r) \cos (\theta)
\Psi_3 = 0
\Psi_4 = -ib(r).
\tag{83}
\]
Put (82) or (83) in equations (78)-(81) we separate the variables. The following system of equations describing the radial dependence of the amplitudes are obtained:

\[
\frac{\partial a (r)}{\partial r} + \frac{2}{r} a (r) = -\nu_2 b (r), \quad (84)
\]

\[
\frac{\partial b (r)}{\partial r} = \nu_1 a (r) + \gamma_1 |a (r)|^2 a (r). \quad (85)
\]

In the system of equations (84)-(85) the nonlinearity appears only in the radial part, while for the angular part we have standard spherical spinors with spin \( j = \pm 1 \). Solving eqns. (84) and (85) we straightforwardly show that when \( \nu_2 \gamma = 1 \) these equations admit the localized de Broglie solitons of eqns. (74)-(75).

6 Hamiltonian representation of the NDE. First integrals for vortex solutions with spin \( j = \pm 1/2 \)

It is not difficult to show that for the NDE system of eqns. (59)-(62) a Hamiltonian of the form:

\[
H = (\vec{\sigma} \cdot \vec{P}) + \sum_{i=1}^{2} |\Psi_i|^2, \quad (86)
\]

can be written. Using this, eqns. (59)-(62) can be rewritten in the form:

\[
H \Psi = \varepsilon \Psi, \quad (87)
\]

where \( \varepsilon = (-i\nu_1, -i\nu_1, -i\nu_2, -i\nu_2) \) is the energy operator. Here we investigate the case where the nonlinear part of the equation is represented as a number of spinors with a scalar sum that depends only on the radial component.

\[
\sum_{i=1}^{2} |\Psi_i (r, \theta, \phi)|^2 = F (r). \quad (88)
\]

We also introduce here the well known orbital momentum operator \( \vec{L} \), own orbital (spin) momentum \( \vec{S} \), and the full momentum \( \vec{J} \), as well as:
\[ \vec{L} = \vec{r} \times \vec{P} \]  
\( (89) \)

\[ \frac{1}{2}\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \]  
\( (90) \)

\[ \vec{J} = \vec{L} + \frac{1}{2}\vec{S} \]  
\( (91) \)

It straightforward to show that the Hamiltonian (86) of eqn. (87) commutes with the operators \( \vec{J}^2 \) and \( J_z \) (the z- projections must be x or y). Using these symmetries and the condition that the nonlinearity is of Kerr type, we can solve the NDE equations (87) by a separation of variables technique. We look for solutions in the form:

\[ \Psi_1 = a(r) \Omega_{jLM} \]  
\( (92) \)

\[ \Psi_2 = a(r) \Omega_{jLM} \]  
\( (93) \)

\[ \Psi_3 = ib(r) \Omega_{j'M} \]  
\( (94) \)

\[ \Psi_4 = ib(r) \Omega_{j'M}, \]  
\( (95) \)

where \( \Omega_{jlm} \) is the spherical spinor, \( l + l' = 1 \), and \( a(r) \) and \( b(r) \) are arbitrary radial functions. Using the symmetries of (87) and the fact, that the nonlinear parts depend on \( r \), we separate variables and obtain the following system of equations for the radial part:

\[ \frac{\partial a(r)}{\partial r} + \frac{1 + \chi}{r} a(r) = -\nu_2 b(r) \]  
\( (96) \)

\[ \frac{\partial b(r)}{\partial r} + \frac{1 - \chi}{r} b(r) = \nu_1 a(r) + \gamma |a(r)|^2 a(r), \]  
\( (97) \)

where

\[ \chi = l(l + 1) - j(j + 1) - 1/4. \]  
\( (98) \)

Exclude \( b(r) \) from the system (96)-(97), we obtained the next equation for \( a(r) \):
\[ \nu_1 \nu_2 a(r) + \frac{\partial^2 a}{\partial r^2} + \frac{2}{r} \frac{\partial a}{\partial r} - \frac{(1 + \chi) \chi}{r^2} a + \nu_2 \gamma |a|^2 a = 0. \]  

(99)

Formally this equation admit exact "de Broglie" soliton solutions for arbitrary number of \( \chi \). But as we remember our solutions are limited by the conditions (88) the nonlinear part to depends only on the radial components. The condition (88) for a number \( \chi \geq 1 \) can be fulfilled also for a higher number of field on different frequencies. This case including also the parametric processes. The case of one carrying frequencies correspond to localized solutions with \( \chi = 1 \) and angular components \( l = 1 \) and \( j = \pm 1/2 \). In this case the system (96)-(97) becomes:

\[ \frac{\partial a(r)}{\partial r} + \frac{2}{r} a(r) = -\nu_2 b(r) \]  

(100)

\[ \frac{\partial b(r)}{\partial r} = \nu_1 a(r) + \gamma |a(r)|^2 a(r). \]  

(101)

As was shown above, this system has exact radial solutions of the form (73)-(75):

7 Experimental conditions

There are some differences between the nonlinear conditions for localized solutions of the Vector Nonlinear Schrodinger Equation (VNLS) and localized conditions for the Nonlinear Maxwell Equations (NME). The nonlinear parameter for the VNLS is written:

\[ \gamma_{\text{vnls}} = k^2_0 r^2_0 n_2 |A_0|^2 = 1. \]  

(102)

For localized solutions in optical region the constant \( \alpha^2 \) ranges from:

\[ \alpha^2 = k^2_0 r^2_0 \approx 10^4 - 10^6, \]  

(103)

which corresponds to a required nonlinear refractive index change of order of:

\[ n_2 |A_0|^2 \approx 10^{-4} - 10^{-6}. \]  

(104)
On the other hand, the nonlinear condition for localized solutions of the normalized NDE (45)-(48) is:

\[ \gamma_{NME} = \nu_2 \gamma_1 \approx (\delta \Delta \alpha - \alpha_2) k_1 r_0 n_2 |A_0|^2 = 1. \]  

(105)

For the typical value of the constant \( \nu_2 \approx 1 \) the next required nonlinear refractive index change appears:

\[ n_2 |A_0|^2 \approx 10^{-2} - 10^{-3}. \]  

(106)

Another difference between these two cases is that the solutions of the VNLS are comprised of linearly polarized components and that the dispersion of the nonlinear medium plays an important role. This leads to the fact that vortex solutions of the VNLS may be observed only in special dispersion regions of nonlinear media. The vortices of the NDE are a combination of linear and circular polarization components, and do not have this marked dependence on the dispersion and may be observed in the transparency region of a nonlinear media.

8 Vortex solutions in a nonlinear Kerr type media without dispersion

The case without linear dispersion of the electrical and magnetic susceptibility will correspond to \( \chi^{(1)} = \text{const} \) and \( \eta^{(1)} = \text{const} \). We suppose now that the amplitude functions do not depend from time and will look for 3D+1 monochromatic electric and magnet field of kind:

\[ \vec{E}(x, y, z, t) = \vec{M}(x, y, z) \exp(i \omega_0 t), \]  

(107)

\[ \vec{H}(x, y, z, t) = \vec{N}(x, y, z) \exp(i \omega_0 t), \]  

(108)

where \( \vec{M}, \vec{N} \) and \( \omega_0 \) are the amplitudes of the electric, magnetic fields and the optical frequency respectively.

Substituting the relations (107) and (108) one obtain the next amplitude equations in the case of dispersionless media:

\[ \nabla \times \vec{M} = i\alpha_2 \vec{N}, \]  

(109)
\[ \nabla \times \vec{N} = i\alpha_1 \vec{M} + \gamma (\vec{M} \cdot \vec{M}^*) \vec{M}, \]  
(110)\[
\nabla \cdot \vec{M} = 0 \]  
(111)\[
\nabla \cdot \vec{N} = 0 \]  
(112)\[
\]
Again the electric and magnetic fields are represented as a sum of a linear polarization component and a circular polarized one. Substituting:

\[ \psi_1 = iM_l = iM_z \]  
(113)\[
\psi_2 = M_c = iM_x - M_y \]  
(114)\[
\psi_3 = N_l = -iN_z \]  
(115)\[
\psi_4 = N_c = -N_x - iN_y, \]  
(116)\[
\]
into the nonlinear system (109)-(112), we obtain the same kind of stationary nonlinear Dirac system of equations (NDE) as (59)-(62):

\[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \Psi_4 + \frac{\partial}{\partial z} \Psi_3 = -i \left( \kappa_1 + \zeta \sum_{i=1}^{2} |\Psi_i|^2 \right) \Psi_1 \]  
(117)\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \Psi_3 - \frac{\partial}{\partial z} \Psi_4 = -i \left( \kappa_1 + \zeta \sum_{i=1}^{2} |\Psi_i|^2 \right) \Psi_2 \]  
(118)\[
\left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \Psi_2 + \frac{\partial}{\partial z} \Psi_1 = -i\kappa_2 \Psi_3 \]  
(119)\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \Psi_1 - \frac{\partial}{\partial z} \Psi_2 = -i\kappa_2 \Psi_4, \]  
(120)\[
\]
but with different coefficients: \(\kappa_1 = \frac{\omega_0 \varepsilon_0}{c} ; \kappa_2 = \frac{\omega_0 \mu_0}{c} ; \zeta = \frac{4\pi\varepsilon_0 \chi(3)}{c}\). Naturally, the NDE (117)-(120) admit the same kind of solutions (74)-(75).


9 Conditions for finiteness of the energy of the vortex solutions

In this section we will investigate the localized energy in the both cases; with and without of linear dispersion. This corresponds to two kind of optical vortices; with and without spectral bandwidth. In a dielectric media without dispersion the expression for the linear part of energy density is:

\[ W_{\text{lin}} = \frac{1}{8\pi} \left( \varepsilon |\vec{E}|^2 + \mu |\vec{H}|^2 \right), \quad (121) \]

where \( \varepsilon \) and \( \mu \) are constants. Substituting the vortex solutions (74)-(75) in (121) and integrating over 3D space one obtain that quasi-monochromatic vortices in a dielectric media without dispersion admit infinite energy. Now we come to the case of slowly varying amplitude approximation and to calculation of energy of the vortex solutions of a media with linear electric and magnet dispersion. To prove the finiteness of energy of the vortex solutions (74)-(75) we start with the equations for averaged in time balance of energy density of electrical and magnet field [15]:

\[ \langle \frac{\partial W}{\partial t} \rangle = \frac{1}{16\pi} \left( \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial t} + \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{B}^* \cdot \frac{\partial \vec{B}}{\partial t} \right), \quad (122) \]

where \( \vec{D} = \vec{D}_{\text{lin}} + 4\pi \vec{P}_{\text{nlin}} \) is a sum of the linear induction and the nonlinear induction of the electrical field. The calculations of the averaged energy of the optical waves in a dispersive media are worked out considering the first order of slowly varying amplitude approximation of electrical induction (the same as in the NDE). The result comes to the old result of Brillouin 1921 for energy density of electrical field:

\[ \langle W_{\text{lin}} \rangle = \frac{1}{8\pi} \left( \frac{\partial (\omega \varepsilon_0)}{\partial \omega} |\vec{A}|^2 + \frac{\partial (\omega \mu_0)}{\partial \omega} |\vec{C}|^2 \right). \quad (123) \]

The conditions of electric constant:

\[ \varepsilon_0 > 0; \quad \frac{\partial (\omega \varepsilon_0)}{\partial \omega} > 0 \quad (124) \]
is fulfilled in most of the dielectrics. The conditions of the magnetic constant can be present in two cases. In the first one:

\[ \hat{\mu}_0 > 0; \frac{\partial (\omega \hat{\mu}_0)}{\partial \omega} > 0. \]

(125)

This condition corresponds to the case, when there are not included any magnet relaxation processes. To include these processes is possible only for ultrashort optical pulses with time duration of dimension of local structure of one paramagnetic media. The real part of magnet susceptibility in this case is symmetric function in respect to the paramagnetic resonance and the derivative in respect to the frequency is always negative.

\[ \hat{\mu}_0 > 0; \frac{\partial \hat{\mu}_0(\omega)}{\partial \omega} < 0. \]

(126)

Under appropriate conditions this leads to:

\[ \hat{\mu}_0 > 0; \frac{\partial (\omega \hat{\mu}_0(\omega))}{\partial \omega} < 0. \]

(127)

We can find spectral region and dispersion parameters when the next condition will be satisfied:

\[ \frac{\partial (\omega \hat{\mu}_0)}{\partial \omega} = -\frac{\partial (\omega \hat{\varepsilon}_0)}{\partial \omega}. \]

(128)

Thus, the linear (infinity) part of energy density is zero. The nonlinear part of averaged energy density is expressed in \[22\] and for the vortex solutions \[(74)-(75)\] becomes:

\[ \langle W_{\text{lin}} \rangle = n_2 |\Psi|^2 \Re(\Psi^2) + \omega_0 \frac{\partial n_2}{\partial \omega_0} |\Psi|^2 \Re(\Psi^2). \]

(129)

These results give the conditions for the finiteness of energy of the vortices. Integrating \(W_{\text{lin}}\) in the 3D space we obtain a finite value proportional to the main frequency \(\omega_0\).
10 Spatial phase functions, Poynting vector and flow of energy

The kind of the phase functions which satisfied (33)-(34) is determinate for vortex solutions (82) with spin $j = 1/2$. Using again the relations between the spinors of NDE and the amplitude functions (55)-(58) we have:

\[
F_x = \frac{\psi_2 - \psi_2^*}{2i}; \quad F_y = \frac{\psi_2 + \psi_2^*}{2}; \quad F_z = \frac{\psi_1 - \psi_1^*}{2i},
\]

\[
G_x = -\frac{\psi_4 + \psi_4^*}{2}; \quad G_y = \frac{\psi_4 - \psi_4^*}{2i}; \quad G_z = -\frac{\psi_3 - \psi_3^*}{2i}.
\]

Substituting the solutions (82) with spin $j = 1/2$ in (130)-(131) for arbitrary real $a(r)$ and $b(r)$ we obtain:

\[
F_x = -a(r)\frac{x}{r}; \quad F_y = -a(r)\frac{y}{r}; \quad F_z = -a(r)\frac{z}{r},
\]

\[
G_x = 0; \quad G_y = 0; \quad G_z = b(r).
\]

We rewrite again the conditions for the spatial phase functions:

\[
\nabla g \times \vec{F} = 0; \quad \nabla q \times \vec{G} = 0.
\]

These relations for solutions of kind (132)-(133) are satisfied only when:

\[
g = k_0 r \text{ or } g = k_0 f(r),
\]

and

\[
q = k_0 z \text{ or } q = k_0 f(z),
\]

where $k_0$ is the carrying wave number. The spatial phase functions of kind $g = k_0 r$ and $q = k_0 z$ correspond to spectral limited pulses which satisfied additional relations:
The spatial phase functions of kind \( q = k_0 f(r) \) and \( q = k_0 f(z) \) correspond to phase modulated pulses and for them the relations (137) is not satisfied. The Poynting vector can be expressed by the amplitude functions of the electrical and magnet field:

\[
\vec{S} = \vec{E}(x,y,z,t) \times \vec{H}(x,y,z,t) = \\
\exp\left(i \left( W(t) - K(r) \right) \right) \vec{F}(x,y,z,t) \times \vec{G}(x,y,z,t)
\]

where \( W \) and \( K \) are scalar phase functions. Substituting the solutions with spin \( j = 1/2 \) in above expression we find that:

\[
\vec{S} = \exp\left(i \left( W(t) - K(r) \right) \right) \left(-a(r)b(r)\frac{y}{r}; a(r)b(r)\frac{x}{r}; 0 \right).
\]

We see that the Poynting vector \( \vec{S} \) is one circulation vector for solutions with spin \( j = 1/2 \) and its divergency is zero:

\[
\nabla \cdot \vec{S} = 0.
\]

The relation (140) determine that the energy flow through arbitrary closed surface around our vortex solutions with spin \( j = 1/2 \) is zero. The relation (139) show that flow of energy of our solutions circulate in \( x,y \) plane. Now we can generalize the above results for solutions with spin \( j = 1/2 \): The vortex solutions with spin \( j = 1/2 \) without external fields are immovable and electromagnetic energy oscillated in \( x,y \) plane. The electrical field oscillating spherically, in \( 'r' \) direction, while the magnet field oscillating in \( z \) direction. In the same way was calculated the Poynting vector for solutions with spin \( j = -1/2 \). For them we obtain that \( \nabla \cdot \vec{S} \neq 0 \) and we expect that they are not stable. One exact investigation of stability request investigation also the perturbation of the Poynting theorem and will discussed in one next paper.

\section{Conclusion}

In this paper we derive a set of Nonlinear Maxwell amplitude Equations (NME) for nonlinear optical media with and without dispersion.
of the electric and magnetic susceptibility. We have shown that in cases of linear and circularly polarized components of the electric and magnetic fields, the NME reduces to the Nonlinear Dirac system of equations (NDE). The equations are represented in a spinor form. Using the method of separation of variables, exact vortex solutions for both cases have been obtained. The optical vortex solutions admit classical orbital momentum \( l = 1 \) and classical own momentum \( j = \pm 1/2 \). Here I would like to explain more about the differences between solutions, obtained by separation of variables of the usual linear Dirac equation with potential depending only on 'r' (for example hydrogen atom) and solutions of the NDE. In case of linear Dirac equations with potential to higher order spherical spinors correspond higher order of radial spherical Bessel functions. On the other hand for NDE to higher order of spherical spinors \((\ell = 1, 2, ..)\) correspond higher number of field components and higher value of localized energy. For all radial solutions of NDE the zero spherical radial Bessel function \( \sin \alpha r \) are valid. The optical vortices in a media without dispersion admit infinite energy integral. The energy integral of the vortex solutions is finite only in some special cases of paramagnetic media with suitable conditions on linear electric and magnet dispersion. Using the Poynting vector for solutions with spin \( j = 1/2 \) we find that the energy flow through arbitrary closed surface around our vortex solutions is zero and the localized energy of our solutions circulate in \( x, y \) plane. Other important result is, that the vortex solutions with spin \( j = 1/2 \) without external fields are immovable. The electrical field in the vortices oscillating spherically, while the magnet field oscillating in z direction. The initial investigations on stability of these solutions show the following: While the vortices with spin \( j = 1/2 \) are stable, the vortices with opposite spin (charge) \( j = -1/2 \) are not.

All of the above results will be discussed later in relation with nonlinear field theory. However one direct approach to the plasma physics can be found. We consider the old idea presented by Gaponov and Miller in [23]: Confinement of charged particles in potentials of high frequency electromagnetic fields. In our article are presented solutions inspired by this idea. The optical vortices are of such type of high frequency fields with zero intensity in the centrum. In a cold plasma it is possible to obtain trapping of ions in the field of vortices. Thus, conditions for ruled nuclear fusion can be reached.
References

[1] Nonlinear Quantum Field Theory, edited by D. Ivanenko, Inostrannia Literatura, 1959.
[2] V. G. Makhankov. Phys.Rep. 35C.1(1971).
[3] R. Rajaraman, Solitons and Instantons, North-Holland Publishing Co. Amsterdam, The Netherlands.1982.
[4] A. O. Barut, Geom. and Algebr. Aspects of Nonlinear Field Theory, 37 (1989).
[5] W.I.Fushchych, W.M.Shtelen, N.I. Serov, Symmetry Analysis and Exact Solution of Nonlinear Equations of Mathematical Physics, Kyiv, Naukova Dumka, 1989.
[6] G. A. Swartzlander, Jr., C. T. Law, Phys. Rev. Lett. 69, 2503 (1992)
[7] N. N. Rozanov, V. A. Smirnov, and N. V. Vyssotina, Chaos Solitons Fractals 4, 1767 (1994).
[8] M. D. Feit and J. A. Fleck, Jr., JOSA B, 5, 633-640 (1988).
[9] J. A. Powell, J. V. Moloney, A. C. Newell, JOSA B, 10, 1230-1241 (1993).
[10] Y. Silberberg, Opt. Lett., 15, 1282 (1990).
[11] A. B. Blagoeva, S. G. Dinev, A. A. Dreisehnh, and A. Naidenov, IEEE J. Quantum. Electron.QE-27, 2060 (1991).
[12] D. E. Edmundson and R. H. Enns, Opt.Lett. 18,1609 (1993).
[13] D. R. Andersen and L. M. Kovachev, JOSA B, 19, 376-384(2002).
[14] Quantum Mechanic, A. Dacev.( Nauka Izkustvo, 1973).
[15] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, (Nayka, Moskow, 1978).
[16] C. A. Akhmanov, V. A. Vysloukh, A. C. Chirkin, Optics of femtosecond optical pulses(Nayka, Moskow, 1988).
[17] V. I. Karpman, Nonlinear Waves in Dispersive Media, (Nauka, Moskow, 1973).
[18] J. V. Moloney and A. C. Newell, Nonlinear Optics, (Addison-Wesley Publ. Comp., 1991).
[19] M. Born and E. Wolf, *Principes of Optics*, 6th (corr) ed., (Cambridge University Press, Cambridge, England, 1998).

[20] T. Carozzi, R. Karlsson, and J. Bergman, Phys. Rev. E, 61, 2024-2028 (2000).

[21] R. W. Boyd, *Nonlinear optics*, (Acad. Press Inc., 1992).

[22] Y. R. Shen *The Principles of Nonlinear Optics*, (John Wiley, Sons, Inc., 1984).

[23] A. V. Gaponov and M. A. Miller, JETP, 34, 242-243(1958).