Finding Supercharacter Theories on Character Tables

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Abstract. We describe an easy way how to find supercharacter theories for a finite group, if its character table is known. Namely, we show how an arbitrary partition of the conjugacy classes or of the irreducible characters can be refined to the coarsest partition that belongs to a supercharacter theory. Our constructions emphasize the duality between superclasses and supercharacters. An algorithm is presented to find all supercharacter theories on a given character table. The algorithm is used to compute the number of supercharacter theories for some nonabelian simple groups with up to 26 conjugacy classes.

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1. Introduction

Supercharacter theories of finite groups were introduced by Diaconis and Isaacs [8] as approximations of a group’s ordinary character theory. Let $G$ be a finite group and write $\text{Irr} \, G$ to denote the set of irreducible complex characters of $G$. For any subset $X \subseteq \text{Irr} \, G$, let $\sigma_X$ be the character

$$\sigma_X = \sum_{\chi \in X} \chi(1) \chi.$$ 

Suppose that $\mathcal{X}$ is a partition of $\text{Irr} \, G$ and $\mathcal{K}$ is a partition of $G$. The pair $(\mathcal{X}, \mathcal{K})$ is called a supercharacter theory of $G$, if the following conditions hold:

(a) $|\mathcal{X}| = |\mathcal{K}|,$
(b) the characters $\sigma_X$ ($X \in \mathcal{X}$) are constant on the members of $\mathcal{K}$.

The members of $\mathcal{K}$ are called the superclasses of the theory $(\mathcal{X}, \mathcal{K})$, and the characters $\sigma_X$ (or certain rational multiples) the supercharacters.

We say that $\mathcal{K}$ is compatible with $\mathcal{X}$, if $\sigma_X$ is constant on the members of $\mathcal{K}$ for every $X \in \mathcal{X}$. Clearly, for every collection $\mathcal{X}$ of subsets of $\text{Irr} \, G$, there is a unique coarsest partition $\mathcal{K}$ of $G$ that is compatible with $\mathcal{X}$: Namely, $\mathcal{K}$ is the partition of $G$ whose members are the equivalence classes under the relation on $G$ defined by $g \sim h$ if and only if $\sigma_X(g) = \sigma_X(h)$ for all $X \in \mathcal{X}$. We write

$$\mathcal{K} = \text{ClPt}(\mathcal{X})$$

for this partition (as in class partition).
We always have $|\mathcal{X}| \leq |\text{ClPt}(\mathcal{X})|$, and the partition $\mathcal{X}$ belongs to a supercharacter theory if and only if equality holds. Thus in a supercharacter theory, $\mathcal{X}$ determines $\mathcal{K} = \text{ClPt}(\mathcal{X})$ as the coarsest partition of $G$ that is compatible with $\mathcal{X}$. This description appears in the paper of Diaconis and Isaacs [8, Theorem 2.2 (c)].

In this note (in Definition 3.4 below), we show how each collection $\mathcal{K}$ of normal subsets of $G$ determines a partition $\mathcal{X}$ of $\text{Irr} G$, thereby defining a map

$$\mathcal{K} \mapsto \text{IrPt}(\mathcal{K}) = \mathcal{X}$$

which is in some sense dual to the map $\text{ClPt}$ described above. With both these maps in hand, we can easily characterize partitions belonging to supercharacter theories, and we also get an easy way to refine an arbitrary partition of $\text{Irr} G$ or a $G$-invariant partition of $G$ to a partition belonging to a supercharacter theory. (A partition $\mathcal{K}$ of $G$ is called $G$-invariant if all its members are normal subsets of $G$.)

**Theorem A.** Let $\mathcal{K}$ a $G$-invariant partition of $G$, and $\mathcal{X}$ a partition of $\text{Irr} G$.

(a) $|\mathcal{K}| \leq |\text{IrPt}(\mathcal{K})|$, and equality holds if and only if $\left( \text{IrPt}(\mathcal{K}), \mathcal{K} \right)$ is a supercharacter theory.

(b) $|\mathcal{X}| \leq |\text{ClPt}(\mathcal{X})|$, and equality holds if and only if $\left( \text{ClPt}(\mathcal{X}), \mathcal{X} \right)$ is a supercharacter theory.

(c) The partition $\text{ClPt} \left( \text{IrPt}(\mathcal{K}) \right)$ refines $\mathcal{K}$, and these partitions are equal if and only if $\left( \text{IrPt}(\mathcal{K}), \mathcal{K} \right)$ is a supercharacter theory.

(d) The partition $\text{IrPt} \left( \text{ClPt}(\mathcal{X}) \right)$ refines $\mathcal{X}$, and these partitions are equal if and only if $\left( \mathcal{X}, \text{ClPt}(\mathcal{X}) \right)$ is a supercharacter theory.

Part (b) is just the result of Diaconis and Isaacs mentioned before, and is included just to emphasize duality with Part (a). While it is also known that $\mathcal{K}$ determines $\mathcal{X}$ when $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory, our description of $\mathcal{X}$ as $\text{IrPt}(\mathcal{K})$ is different from the one contained in the literature. These results are proved in Section 3.

The maps $\text{ClPt}$ and $\text{IrPt}$ yield an efficient way to refine a given partition of $G$ or of $\text{Irr} G$ to a partition belonging to a supercharacter theory: For example, Suppose we are given a $G$-invariant partition $\mathcal{K}$ of $G$. We apply the maps $\text{IrPt}$ and $\text{ClPt}$ repeatedly in turns. By Parts (a) and (b) of Theorem A, we have

$$|\mathcal{K}| \leq |\text{IrPt}(\mathcal{K})| \leq |\text{ClPt}(\text{IrPt}(\mathcal{K}))| \leq |\text{IrPt}(\text{ClPt}(\text{IrPt}(\mathcal{K})))| \leq \cdots .$$

When two consecutive partitions contain the same number of sets, then these two partitions form a supercharacter theory. Moreover, by Part (c), the corresponding superclass partition is the coarsest partition that refines $\mathcal{K}$ and belongs to a supercharacter theory. Computational evidence suggests that most often, a small number of steps suffices to reach a supercharacter theory. Also, both maps $\text{ClPt}$ and $\text{IrPt}$ can be computed easily and effectively from the character table.

Similarly, we can start with a partition $\mathcal{X}$ of $\text{Irr} G$ and refine it to a partition of $\text{Irr} G$ that belongs to a supercharacter theory.
So when we know the character table of a group $G$, we can easily decide whether a given nonempty, normal subset $S$ of $G$ is a superclass in some supercharacter theory on $G$: We apply IrPt and ClPt to the partition $\{S, G \setminus S\}$ of $G$, until either $S$ is not a member of the resulting partition of $G$, or we have found a supercharacter theory $(X, K)$ with $S \in K$. In the first case, $S$ can not be a superclass. In the second case, we have also found the coarsest supercharacter theory in which $S$ is a superclass.

Although the original motivation for developing supercharacter theories were cases where the full character table is difficult to compute, there has recently been some interest in understanding all possible supercharacter theories on given groups and character tables. Only for a few families of finite groups are all possible supercharacter theories known: Leung and Man [14, 15] classified supercharacter theories of finite cyclic groups in the language of Schur rings (cf. Hendrickson’s paper [10]). Wynn [19] and Lamar [13] both classified supercharacter theories of dihedral groups, and Lewis and Wynn [16, 19] considered supercharacter theories of Camina pairs, and in particular classified them for Frobenius groups of order $pq$. Recently, Burkett and Lewis [7] began a classification of supercharacter theories of $C_p \times C_p$.

Our results suggest an algorithm for determining all supercharacter theories on a given character table (Algorithm 4.3). In a first step, this algorithm runs through half of the nonempty, normal subsets $S$ of $G \setminus \{1\}$. As described above, we can at the same time decide whether such an $S$ can be a superclass, and compute the coarsest supercharacter theory $(X, K)$ with $S \in K$, if there is such a theory at all. In a second step, the algorithm forms meets of the supercharacter theories found in the first step, and adds the trivial supercharacter theory with class partition $K = \{\{1\}, G \setminus \{1\}\}$.

The algorithm runs through $2^{k(G)} - 2$ subsets of $G$, and thus the algorithm is applicable only for groups with few conjugacy classes. But our algorithm is more efficient than the one suggested by Hendrickson [11], or the modified version by Burkett, Lamar, Lewis and Wynn [6, 13]. (See the remarks at the end of Section 4 below.) A further, small improvement is possible by using automorphisms of the character table, which we discuss briefly in Section 5.

Using the character tables from the character table library of GAP [9], we have computed the number of supercharacter theories for some groups with up to 26 conjugacy classes. The results are summarized at in Section 6. For a group with about 24 conjugacy classes, the algorithm needs a few minutes on a standard desktop computer.

For example, consider the second Janko $J_2$. This group has 21 conjugacy classes. It is unfeasible (with current technology) to run through all the 51 724 158 235 372 $G$-invariant partitions of $G$ containing $\{1\}$ as a block, and check for each partition whether it belongs to a supercharacter theory. But our algorithm has only to run through $2^{19} - 1 = 524287$ $G$-invariant subsets of $G \setminus \{1\}$. (The second step of the algorithm is trivial for $J_2$ since only two supercharacter theories are found in the first step.) In about 100 seconds, the algorithm finds that the second Janko group $J_2$ has exactly three supercharacter theories. This confirms a conjecture by A. R. Ashrafi and F. Koorepazan-Moftakhar [4, Conjecture 2.8], According to the published version of this paper [5, remarks before Lemma 3.9], this fact has also been established by N. Thiem and J. P. Lamar in unpublished work, but their
algorithm needs about 3 hours.

2. Partitions and algebras of maps

We begin with a very general, but well known and elementary result. Let \( S \) be a finite set and let \( \mathcal{K} \) and \( \mathcal{L} \) be partitions of \( S \). Recall that \( \mathcal{K} \) is said to be finer than \( \mathcal{L} \), written \( \mathcal{K} \preceq \mathcal{L} \), when every block of \( \mathcal{K} \) (that is, every set \( K \in \mathcal{K} \)) is contained in a block of \( \mathcal{L} \). One also says that \( \mathcal{L} \) is coarser than \( \mathcal{K} \) in this case. The set of all partitions of \( S \) forms a partial ordered set under the relation of refinement, and in fact a lattice.

Now let \( F \) be a field and \( S \) a finite set. The set \( FS = Maps(S,F) \) of all functions \( f: S \to F \) forms a commutative \( F \)-algebra with 1 with respect to pointwise addition and multiplication. Let \( \mathcal{K} \) be a partition of \( S \). The set \( Maps_K(S,F) \) of all functions which are constant on the blocks of \( \mathcal{K} \) is a unital subalgebra of \( Maps(S,F) \). Conversely, every subalgebra of \( Maps(S,F) \) containing the all-1-function has this form. This is the content of the following (well-known) lemma:

2.1. Lemma. The map \( \mathcal{K} \mapsto Maps_K(S,F) \) defines a bijection between partitions \( \mathcal{K} \) of \( S \) and unital \( F \)-subalgebras of \( Maps(S,F) \). The inverse sends a subalgebra \( A \) to the partition corresponding to the equivalence relation on \( S \) defined by \( s \sim t \) if and only if \( f(s) = f(t) \) for all \( f \in A \).

The bijections are order reversing with respect to refinement and inclusion, that is, \( \mathcal{K} \preceq \mathcal{L} \) if and only if \( Maps_K(S,F) \supseteq Maps_L(S,F) \).

Proof. It is clear that \( \mathcal{K} \mapsto Maps_K(S,F) \mapsto \mathcal{K} \), because for \( K \in \mathcal{K} \), the characteristic function \( \delta_K \) with \( \delta_K(s) = 1 \) if \( s \in K \) and \( \delta_K(s) = 0 \) else is contained in \( Maps_K(S,F) \).

For the sake of completeness, we also prove the converse, although all this is well known. Let \( A \subseteq Maps(S,F) \) be a \( F \)-subalgebra containing 1 (the all-1-function), and let \( \mathcal{K} \) be the partition whose members are equivalence classes under the relation on \( S \) defined by \( s \sim t \) if and only if \( f(s) = f(t) \) for all \( f \in A \). Obviously, \( A \subseteq Maps_K(S,F) \).

Let \( K \in \mathcal{K} \) and fix \( s \in K \). For every \( t \in S \setminus K \), there is a function \( f \in A \) such that \( f(s) \neq f(t) \). Then the function \( g_t = \left( f(s) - f(t) \right)^{-1} \left( f(t) \cdot 1 \right) \) is also in \( A \) and we have \( g_t(s) = 1 \) and \( g_t(t) = 0 \). Since \( g_t \) is constant on \( K \), we have \( g_t(x) = 1 \) for all \( x \in K \). Multiplying all \( g_t \) for \( t \in S \setminus K \), we get the characteristic function \( \delta_K \) of \( K \). It follows that \( A \) contains the characteristic functions \( \delta_K \) for all \( K \in \mathcal{K} \), and so \( A = Maps_K(S,F) \).

The last statement of the lemma is easy to verify.

2.2. Corollary. Let \( B \subseteq Maps(S,F) \) be a set of maps. The unital subalgebra of \( Maps(S,F) \) generated by \( B \) is \( Maps_K(S,F) \), where \( K \) is the partition corresponding to the equivalence relation on \( S \) defined by \( s \sim t \) if and only if \( b(s) = b(t) \) for all \( b \in B \).

3. Class and character partitions

First we apply the results of the last section to the algebra of maps \( Maps(G, \mathbb{C}) \) from a finite group \( G \) into the field of complex numbers \( \mathbb{C} \). Actually everything takes place in
the subalgebra of class functions \( \text{cf}(G) \). These are the functions \( G \to \mathbb{C} \) that are constant on conjugacy classes.

For \( X \subseteq \text{Irr } G \), set

\[
\sigma_X = \sum_{\chi \in X} \chi(1)\chi
\]

as in the introduction.

**3.1. Definition.** For a set \( \mathcal{X} \) consisting of subsets of \( \text{Irr } G \), let \( \text{ClPt}(\mathcal{X}) \) be the partition of \( G \) whose members are the equivalence classes under the relation \( g \sim_X h \) if and only if \( \sigma_X(g) = \sigma_X(h) \) for all \( X \in \mathcal{X} \).

Recall from the introduction that we call a partition \( K \) of \( G \) compatible with \( \mathcal{X} \), if \( \sigma_X \) is constant on the members of \( K \) for every \( X \in \mathcal{X} \).

**3.2. Lemma.** Let \( \mathcal{X} \) be a collection of subsets of \( \text{Irr } G \).

(a) \( \text{ClPt}(\mathcal{X}) \) is the unique coarsest partition of \( G \) that is compatible with \( \mathcal{X} \).

(b) \( \text{Maps}_{\text{ClPt}(\mathcal{X})}(G, \mathbb{C}) \) is the unital subalgebra of \( \text{cf}(G) \) generated by the characters \( \sigma_X \) for \( X \in \mathcal{X} \).

If \( \mathcal{X} \) is a partition of \( \text{Irr } G \), then the following hold:

(c) \( |\mathcal{X}| \leq |\text{ClPt}(\mathcal{X})| \).

(d) \( \{1\} \in \text{ClPt}(\mathcal{X}) \).

(e) When \( \mathcal{Y} \) is another partition of \( \text{Irr } G \) with \( \mathcal{X} \preceq \mathcal{Y} \), then \( \text{ClPt}(\mathcal{X}) \preceq \text{ClPt}(\mathcal{Y}) \).

As explained in the introduction, supercharacter theories are characterized by equality in (c). The results of Lemma 3.2 are well known and recorded here for convenient reference, but also for motivation of the dual results to follow.

**Proof of Lemma 3.2.** Part (a) just rephrases the definition. Part (b) is immediate from Corollary 2.2. If \( \mathcal{X} \) is a partition, then the characters \( \sigma_X \) are linearly independent. Thus \( |\mathcal{X}| \leq \dim \text{Maps}_{\text{ClPt}(\mathcal{X})}(G, \mathbb{C}) = |\text{ClPt}(\mathcal{X})| \), which is (c). Part (d) follows since the regular character \( \rho_G = \sum_{X \in \mathcal{X}} \sigma_X \) is in the span of the characters \( \sigma_X \)’s. Finally, when \( \mathcal{X} \preceq \mathcal{Y} \), then every \( Y \in \mathcal{Y} \) is a union of blocks of \( \mathcal{X} \), and \( \sigma_Y \) is the sum of the corresponding \( \sigma_X \)’s. Then it is clear from the definition that \( g \sim_X h \) implies \( g \sim_Y h \), as claimed.

Next we want to describe how a \( G \)-invariant partition of \( G \) determines a partition of \( \text{Irr } G \). First, we need to introduce some notation. Recall that any \( \chi \in \text{Irr } G \) defines a central primitive idempotent \( e_\chi \) of the group algebra \( \mathbb{C}G \), namely

\[
e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g)g^{-1}.
\]

These idempotents yield a decomposition of the center of the group algebra: Namely, we have

\[
\mathbb{Z}(\mathbb{C}G) = \bigoplus_{\chi \in \text{Irr } G} \mathbb{Z}(\mathbb{C}G)e_\chi = \bigoplus_{\chi \in \text{Irr } G} \mathbb{C}e_\chi.
\]
Moreover, recall that
\[ \omega_\chi : \mathbb{Z}(CG) \to \mathbb{C}, \quad \omega_\chi(z) = \frac{\chi(z)}{\chi(1)} \]
is the central character associated to \( \chi \in \text{Irr} \, G \). It is defined by the property \( R(z) = \omega_\chi(z)I \) for \( z \in \mathbb{Z}(CG) \), where \( R \) is a representation affording \( \chi \). The central character \( \omega_\chi \) also describes the projection to the component \( \mathbb{C}e_\chi \) of \( \mathbb{Z}(CG) \). In other words, we have
\[ z = \sum_{\chi \in \text{Irr} \, G} \omega_\chi(z)e_\chi \quad \text{for all } z \in \mathbb{Z}(CG). \]

This yields the following lemma:

3.3. Lemma. For \( z \in \mathbb{Z}(CG) \), define
\[ \alpha_z : \text{Irr} \, G \to \mathbb{C}, \quad \alpha_z(\chi) = \frac{\chi(z)}{\chi(1)} \]
The map \( z \mapsto \alpha_z \) defines an algebra isomorphism
\[ \mathbb{Z}(CG) \cong \text{Maps}(\text{Irr} \, G, \mathbb{C}). \]
The inverse sends a map \( f : \text{Irr} \, G \to \mathbb{C} \) to the element \( \sum_\chi f(\chi)e_\chi \).

By Lemma 2.1, there is a natural correspondence between the unital subalgebras of \( \mathbb{Z}(CG) \cong \text{Maps}(\text{Irr} \, G, \mathbb{C}) \) and partitions of \( \text{Irr} \, G \). For \( X \subseteq \text{Irr} \, G \), we write
\[ e_X = \sum_{\chi \in X} e_\chi = \frac{1}{|G|} \sum_{g \in G} \sigma_X(g)g^{-1}. \]
Then the subalgebra \( A_X \) corresponding to a partition \( X \) of \( \text{Irr} \, G \) is given by \( A_X = \bigoplus_{X \in \mathcal{X}} \mathbb{C}e_X \) of \( \mathbb{Z}(CG) \).

We also use the following notation: For any subset \( K \) of \( G \), we write
\[ \hat{K} := \sum_{g \in K} g \in \mathbb{C}G. \]

3.4. Definition. Let \( \mathcal{K} \) be a collection of subsets of \( G \), such that each member of \( \mathcal{K} \) is a union of conjugacy classes of \( G \). Then define \( \text{IrPt}(\mathcal{K}) \) to be the partition of \( \text{Irr} \, G \) whose members are the equivalence classes of the relation on \( \text{Irr} \, G \) defined by \( \chi \sim_K \psi \) if and only if \( \chi(\hat{K})/\chi(1) = \psi(\hat{K})/\psi(1) \) for all \( K \in \mathcal{K} \).

We have the following result, which is completely dual to Lemma 3.2.

3.5. Lemma. Let \( \mathcal{K} \) be a collection of \( G \)-invariant subsets of \( G \).
(a) \( \text{IrPt}(\mathcal{K}) \) is the unique coarsest partition of \( \text{Irr} \, G \) such that for every \( K \in \mathcal{K} \), the map \( \alpha_{\hat{K}} \) is constant on the members of \( \text{IrPt}(\mathcal{K}) \).
(b) \( \sum_{X \in \text{IrPt}(\mathcal{K})} \mathbb{C}e_X \) is the subalgebra of \( \mathbb{Z}(CG) \) generated by the block sums \( \hat{K} \) for \( K \in \mathcal{K} \).
If $\mathcal{K}$ is a $G$-invariant partition of $G$, then also:

(c) $|\mathcal{K}| \leq |\operatorname{IrrPt}(\mathcal{K})|$.

(d) $\{1_G\} \in \operatorname{IrrPt}(\mathcal{K})$.

(e) When $\mathcal{L}$ is another $G$-invariant partition of $G$ with $\mathcal{K} \preceq \mathcal{L}$, then $\operatorname{IrrPt}(\mathcal{K}) \preceq \operatorname{IrrPt}(\mathcal{L})$.

Proof. Let $K \in \mathcal{K}$. Since $K$ is a union of conjugacy classes of $G$, we have $\hat{K} \in \mathbb{Z}(CG)$. Thus the class sums $\hat{K}$ for $K \in \mathcal{K}$ generate a subalgebra of $\mathbb{Z}(CG) \cong \mathbb{C}^{\operatorname{Irr}G}$, and this subalgebra determines a partition $\mathcal{X}$ of $\operatorname{Irr}G$. By Lemma 2.1 and Lemma 3.3, the members of $\mathcal{X}$ are the equivalence classes of the relation on $\operatorname{Irr}G$ defined by $\chi \sim_{\mathcal{K}} \psi$ if and only if $\alpha_K(\chi) = \alpha_K(\psi)$ for all $K \in \mathcal{K}$. By the definition of $\alpha$ in Lemma 3.3, we have $\chi \sim_{\mathcal{K}} \psi$ if and only if $\chi(\hat{K})/\chi(1) = \psi(\hat{K})/\psi(1)$ for all $K \in \mathcal{K}$. Thus $\mathcal{X} = \operatorname{IrrPt}(\mathcal{K})$, and (a) and (b) follow.

When $\mathcal{K}$ is a partition of $G$, then the sums $\hat{K}$ are linearly independent. Thus (c) follows from (b). To see (d), we use that for $z = \sum_{g \in G} g$, we have $\omega(\mathcal{K}) \neq 0$ if and only if $\chi = 1_G$. Part (e) is easy.

Notice that Part (b) yields another characterization of the partition $\operatorname{IrrPt}(\mathcal{K})$. This is the characterization given by Diaconis and Isaacs [8] in the case where $\mathcal{K}$ belongs to a supercharacter theory.

Partitions belonging to a supercharacter theory are characterized by equality in Part (c) of Lemma 3.2 or Lemma 3.5, respectively.

3.6. Lemma.

(a) Let $\mathcal{K}$ be a collection of $G$-invariant subsets of $G$. Then $\operatorname{ClPt}(\operatorname{IrrPt}(\mathcal{K}))$ is the coarsest partition $\mathcal{L}$ such that the linear span of the sums $\hat{L}$, where $L \in \mathcal{L}$, contains the unital subalgebra of $\mathbb{Z}(CG)$ generated by the sums $\hat{K}$, where $K \in \mathcal{K}$.

(b) Let $\mathcal{X}$ be a collection of subsets of $\operatorname{Irr}G$. Then $\operatorname{IrrPt}(\operatorname{ClPt}(\mathcal{X}))$ is the coarsest partition $\mathcal{Y}$ such that the linear span of the characters $\sigma_Y$, where $Y \in \mathcal{Y}$, contains the unital subalgebra of $\operatorname{cf}(G)$ generated by the characters $\sigma_X$, where $X \in \mathcal{X}$.

Proof. We begin with (a). Let $\mathcal{X} := \operatorname{IrrPt}(\mathcal{K})$. By Lemma 3.5 (b), the idempotents $e_X$, where $X \in \mathcal{X}$, form a basis of the subalgebra generated by the sums $\hat{K}$, $K \in \mathcal{K}$. But we have

$$e_X = \sum_{\chi \in \mathcal{X}} e_X = \frac{1}{|G|} \sum_{g \in G} \sigma_X(g)g.$$

By definition, $\operatorname{ClPt}(\mathcal{X}) = \operatorname{ClPt}(\operatorname{IrrPt}(\mathcal{K}))$ is the coarsest partition $\mathcal{L}$ of $G$ such that $\sum_{L \in \mathcal{L}} C\hat{L}$ contains the idempotents $e_X$, as claimed.

The proof of (b) is similar. First, let $K \subseteq G$ be a union of conjugacy classes, and let $\delta_K : G \to \mathbb{C}$ be its characteristic function. By the orthogonality relations, we have

$$\delta_K = \frac{1}{|G|} \sum_{\chi \in \operatorname{Irr}G} \overline{\chi(\hat{K})} \chi = \frac{1}{|G|} \sum_{\chi \in \operatorname{Irr}G} \omega(\hat{K}) \chi(1) \chi.$$

Now let $\mathcal{K} := \operatorname{ClPt}(\mathcal{X})$. By Lemma 3.2 (b), the functions $\delta_K$ where $K \in \mathcal{K}$, form a basis of the subalgebra generated by the characters $\sigma_X$, where $X \in \mathcal{X}$. By the above formula
for $\delta_K$, the linear span of the characters $\sigma_Y$, where $Y$ runs through a partition $\mathcal{Y}$ of $\text{Irr} G$, contains all $\delta_K$ for $K \in \mathcal{K}$, if and only if $\mathcal{Y} \subseteq \text{IrPt}(\mathcal{K}) = \text{IrPt}(\text{ClPt}(\mathcal{X}))$. This is the claim.  

3.7. Corollary. 
(a) Let $\mathcal{K}$ be a $G$-invariant partition of $G$. Then $\text{ClPt}(\text{IrPt}(\mathcal{K})) \subseteq \mathcal{K}$. 
(b) Let $\mathcal{X}$ be a partition of $\text{Irr} G$. Then $\text{IrPt}(\text{ClPt}(\mathcal{X})) \subseteq \mathcal{X}$.

3.8. Example. Let $\mathcal{N}$ be a collection of normal subgroups of the finite group $G$. Then we can apply Lemma 3.6 (a) to $\mathcal{N}$. We want to describe $L := \text{ClPt}(\text{IrPt}(\mathcal{N}))$.

Let $A$ be the unital subalgebra of $\mathbb{Z}(\mathbb{C}G)$ generated by the sums $\hat{N}$ for $N \in \mathcal{N}$. For $N, M \in \mathcal{N}$, we have $\hat{N}\hat{M} = |N \cap M|\hat{N}\hat{M} \in A$. Let $\mathcal{S} = \mathcal{S}(\mathcal{N})$ be the set of normal subgroups of the form $N_1 \cdots N_r$ with $N_i \in \mathcal{N}$, including the trivial subgroup as the empty product. Then for each $S \in \mathcal{S}$ there is some positive integer $n_S$ such that $n_S\hat{S} \in A$. On the other hand, the sums $\hat{S}$ with $S \in \mathcal{S}$ obviously span a subalgebra of $\mathbb{Z}(\mathbb{C}G)$.

By Lemma 3.6 (a), $L$ is the coarsest partition of $G$ such that every $S \in \mathcal{S}$ is a union of elements of $L$. Describing $L$ is now a matter of elementary set theory: Let $\mathcal{N}_1$ be the set of intersections of finitely many elements of $\mathcal{S}$, including $G$. This is then a set of normal subgroups of $G$ which is closed under taking intersections. (In general, $\mathcal{N}_1$ may not be closed under taking products.) The partition $L$ is the coarsest partition of $G$ such that every $N \in \mathcal{N}_1$ is a union of elements of $L$. Since $\mathcal{N}_1$ is closed under intersections, every $g \in G$ is contained in a unique minimal $N \in \mathcal{N}_1$. Let us write $g^{\mathcal{N}_1}$ for this normal subgroup. The (nonempty) fibers of the map $G \ni g \mapsto g^{\mathcal{N}_1} \in \mathcal{N}_1$ are then the blocks of the partition $L$. That is, every block $L \in L$ has the form

$$L = L_N = \{ g \in G \mid g^{\mathcal{N}_1} = N \} = N \setminus \bigcup_{M \in \mathcal{N}_1 \setminus N} M \quad \text{for some } N \in \mathcal{N}_1.$$ 

It follows that the sums $\hat{N}$ with $N \in \mathcal{N}_1$ and the sums $\hat{L}$ with $L \in L$ span the same $\mathbb{Z}$-submodule of $\mathbb{Z}(\mathbb{C}G)$.

When $\mathcal{N}_1$ is closed under products, then the linear span of the sums $\hat{N}$ with $N \in \mathcal{N}_1$ is actually a subalgebra. In general, this is not the case, but we can repeat the process until we have found a collection of normal subgroups that is closed under intersections and products. The corresponding partition of $G$ then belongs to a supercharacter theory. These supercharacter theories were described by F. Aliniaeifard [1].

3.9. Theorem. Let $\mathcal{X}$ be a partition of $\text{Irr} G$. Then the following are equivalent:
(a) $|\mathcal{X}| = |\text{ClPt}(\mathcal{X})|$. 
(b) The linear span in the space of class functions of the characters $\sigma_X$ for $X \in \mathcal{X}$ is a unital subalgebra of the algebra of all class functions. 
(c) $\mathcal{X} = \text{IrPt}(\text{ClPt}(\mathcal{X}))$.

Proof. The equivalence of (a) and (b) follows from (b) and (c) in Lemma 3.2, and Lemma 3.6 (b) implies that (b) and (c) are equivalent.  

\[ \square \]
When the equivalent conditions in Theorem 3.9 hold, then the pair \((\mathcal{X}, \text{ClPt}(\mathcal{X}))\) forms a supercharacter theory. Indeed, Condition (a) is actually equivalent to the definition given in the introduction.

We leave the following dual as an exercise for the reader:

3.10. **Theorem.** Let \(\mathcal{K}\) be a \(G\)-invariant partition of \(G\). Then the following are equivalent:
   - (a) \(|\mathcal{K}| = |\text{IrPt}(\mathcal{K})|\).
   - (b) The linear span in \(\mathbb{C}G\) of the sums \(\hat{K}\) for \(K \in \mathcal{K}\) is a unital subalgebra of \(\mathcal{Z}(\mathbb{C}G)\).
   - (c) \(\mathcal{K} = \text{ClPt}(\text{IrPt}(\mathcal{K}))\).

Condition (b) means that the superclass sums \(\hat{K}\) span a so-called Schur ring over \(G\). (The connection between Schur rings and supercharacter theories is explained with more details by Hendrickson [10].)

Notice that when \(\mathcal{X} = \text{IrPt}(\text{ClPt}(\mathcal{X}))\), then \(\text{ClPt}(\mathcal{X}) = \text{ClPt}(\text{IrPt}(\text{ClPt}(\mathcal{X})))\). Thus the conditions in Theorem 3.9 imply that the conditions of Theorem 3.10 hold for \(\mathcal{K} := \text{ClPt}(\mathcal{X})\), and conversely, the conditions in Theorem 3.10 imply the conditions in Theorem 3.9 for \(\mathcal{X} := \text{IrPt}(\mathcal{K})\).

We have by now proved all the claims in Theorem A from the introduction. We have also done all the work for the following characterizations of supercharacter theories:

3.11. **Theorem.** Let \(\mathcal{X}\) be a partition of \(\text{Irr}\ G\) and \(\mathcal{K}\) a \(G\)-invariant partition of \(G\). Then the following are equivalent:
   - (a) \(|\mathcal{K}| = |\mathcal{X}|\) and \(\mathcal{K} = \text{ClPt}(\mathcal{X})\).
   - (b) \(|\mathcal{K}| = |\mathcal{X}|\) and \(\mathcal{X} = \text{IrPt}(\mathcal{K})\).
   - (c) \(\mathcal{K} = \text{ClPt}(\mathcal{X})\) and \(\mathcal{X} = \text{IrPt}(\mathcal{K})\).
   - (d) The linear span in \(\mathbb{C}G\) of the elements \(\hat{K}\) for \(K \in \mathcal{K}\) is a subalgebra of \(\mathcal{Z}(\mathbb{C}G)\), and \(\mathcal{X} = \text{IrPt}(\mathcal{K})\).
   - (e) The linear span in the space of class functions of the characters \(\sigma_X\) for \(X \in \mathcal{X}\) is a subalgebra of the ring of all class functions and \(\mathcal{K} = \text{ClPt}(\mathcal{X})\).

**Proof.** (a), (b), (c) and (e) are equivalent by Theorem 3.9, and Theorem 3.10 yields the equivalence with (d). \(\square\)

3.12. **Remark.** The conditions of Theorem 3.11 are also equivalent to the following:
   - (a') \(|\mathcal{K}| = |\mathcal{X}|\) and \(\mathcal{K} \preceq \text{ClPt}(\mathcal{X})\).
   - (b') \(|\mathcal{K}| = |\mathcal{X}|\) and \(\mathcal{X} \preceq \text{IrPt}(\mathcal{K})\).
   - (c') \(\mathcal{K} \preceq \text{ClPt}(\mathcal{X})\) and \(\mathcal{X} \preceq \text{IrPt}(\mathcal{K})\).

**Proof.** Since we always have \(|\mathcal{X}| \leq |\text{ClPt}(\mathcal{X})|\), condition (a) is in fact equivalent to the weaker (a'). Similarly, (b) and (b') are equivalent. The equivalence of (c') and (c) follows from Corollary 3.7 (or otherwise). \(\square\)

The last theorem emphasizes the duality between superclasses and supercharacters. The duality between classes and characters is maybe obscured by the following difference: On the one side, we simply consider class sums \(\hat{K} = \sum_{g \in \mathcal{K}} g\), while on the other side, we do not add simply the irreducible characters in a subset \(X\), but the multiples \(\chi(1)\chi\).
When \((\mathcal{X}, K)\) is a supercharacter theory, and if \(K\) and \(L \subseteq K\) are superclasses, then \(\hat{K}\hat{L}\) is a nonnegative integer linear combination of superclass sums, as is not difficult to see [8, Corollary 2.3]. But the product of two supercharacters \(\sigma_X\) and \(\sigma_Y\) is in general only a rational combination of such supercharacters. However, there is an easy remedy for this problem.

3.13. Remark. Let \((\mathcal{X}, K)\) be a supercharacter theory of \(G\). For each \(X \in \mathcal{X}\), define \(d_X = \gcd\{\chi(1) | \chi \in X\}\) and \(\tau_X = (1/d_X)\sigma_X\). Then every character that is constant on members of \(K\) is a nonnegative integer linear combination of the \(\tau_X\). In particular, this holds true for a product \(\tau_X\tau_Y\).

\[\gamma = \sum_{X \in \mathcal{X}} a_X \tau_X.\]

One the other hand, for \(\chi \in X\) we have
\[(\gamma, \chi) = a_X (\tau_X, \chi) = a_X \frac{\chi(1)}{d_X} \in \mathbb{N},\]

since \(\gamma\) is a character. Choose \(k_\chi \in \mathbb{Z}\) such that \(d_X = \sum_{\chi \in X} k_\chi \chi(1)\). It follows
\[a_X = \frac{a_X}{d_X} \sum_{\chi \in X} k_\chi \chi(1) = \sum_{\chi \in X} k_\chi a_X \frac{\chi(1)}{d_X} \in \mathbb{Z}.\]

Thus \(a_X \in \mathbb{N}\), as claimed. \(\square\)

3.14. Question. How are the supercharacters \(\chi_\lambda\) of algebra groups as defined by Diaconis and Isaacs [8] related to the \(\tau_X\)? Is \(\tau_X = \chi_\lambda\) (where \(X\) is the set of constituents of \(\chi_\lambda\))?}

4. Computing all supercharacter theories of a finite group

Let \((\mathcal{X}, K)\) and \((\mathcal{Y}, L)\) be supercharacter theories of a group \(G\). We say that \((\mathcal{X}, K)\) is finer than \((\mathcal{Y}, L)\) or that \((\mathcal{Y}, L)\) is coarser than \((\mathcal{X}, K)\), when \(\mathcal{X}\) is finer than \(\mathcal{Y}\). It is clear from Theorem 3.11 that \(\mathcal{X} \preceq \mathcal{Y}\) if and only if \(K \preceq L\) (for supercharacter theories). (This has also been proved by Hendrickson [10].)

It follows that the partially ordered set of supercharacter theories embeds naturally into the partially ordered set of partitions of \(\text{Irr}\, G\), or of \(G\)-invariant partitions of \(G\).

The partitions of a given set form a lattice. Let \((\mathcal{X}, K)\) and \((\mathcal{Y}, L)\) be two supercharacter theories. Then \((\mathcal{X}, K) \vee (\mathcal{Y}, L) := (\mathcal{X} \vee \mathcal{Y}, K \vee L)\), is also a supercharacter theory [10, Prop. 3.3]. It follows that the supercharacter theories of a group form also a lattice. Thus for every partition \(\mathcal{X}\) of \(\text{Irr}\, G\), there is a unique coarsest partition \(\mathcal{Y}\) refining \(\mathcal{X}\), and such that \(\mathcal{Y}\) belongs to a supercharacter theory, and similarly for \(G\)-invariant partitions of \(G\). The results of the last section provide a convenient way to compute this supercharacter theory.
For example, start with an arbitrary $G$-invariant partition $K$ of $G$. Apply in turns the maps $\text{IrPt}$ and $\text{ClPt}$ to form partitions of $\text{Irr} \ G$ and of $G$. Thus we get two chains of partitions as follows (here and in the following, we abbreviate $I := \text{IrPt}$ and $C := \text{ClPt}$):

$$
\begin{array}{c}
K \geq CI(K) \geq \cdots \geq (CI)^n(K) \geq \cdots \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
I(K) \geq ICI(K) \geq \cdots \geq I(CI)^n(K) \geq \cdots
\end{array}
$$

The number of blocks in a partition increases along the arrows. The first two partitions that have the same numbers of blocks, form a supercharacter theory.

Of course, we can as well start with a partition of $\text{Irr} \ G$.

4.1. Corollary.
(a) Let $K$ be a $G$-invariant partition of $G$. Choose $n$ such that $(CI)^{n+1}(K) = (CI)^n(K)$. Then $(I(CI)^n(K), (CI)^n(K))$ is the coarsest supercharacter theory whose class partition refines $K$.
(b) Let $X$ be a partition of $\text{Irr} \ G$. Choose $n$ such that $(IC)^{n+1}(X) = (IC)^n(X)$. Then $((IC)^n(X), (IC)^n(X))$ is the coarsest supercharacter theory whose character partition refines $X$.

Proof. This follows from the theorems in the last section.

The number of steps to reach a supercharacter theory is in theory bounded by $k(G) - |K| = |\text{Irr} \ G| - |K|$. Usually, one needs much less steps since the number of sets in the involved partitions grows much faster than by one in a step. For example, consider the alternating group $G = A_7$ on 7 elements, a group with 9 conjugacy classes. There are 4140 partitions $K$ of $G$ into $G$-invariant sets and such that $\{1\}$ is a block of the partition. Of these, 3 are supercharacter theories, and another 3807 partitions are such that $\text{IrPt}(K)$ belongs to a supercharacter theory. There are 292 partitions where we need 2 steps to reach a supercharacter theory, 31 partitions where we need 3 steps, and only 7 partitions where we need 4 steps. This is the worst case. (Of course, we always need to compute one step more to see that we have actually reached a supercharacter theory.)

As another example, let $G$ be an elementary abelian group of order 16. Consider only partitions $K$ of the form $G = \{1\} \cup S \cup T$. Up to automorphisms of the group, there are only 22 such partitions, and of these, 5 are already supercharacter theories. For all these partitions $K$, already $CI(K)$ defines a supercharacter theory.

4.2. Corollary. Let $K \subseteq G$ be a normal subset. Then there is a supercharacter theory $(X, K)$ with $K \in K$ if and only if $K \in (CI)^n(\{K\})$ for all $n > 0$. If this is the case, then $(CI)^n(\{K\})$ for $n$ large belongs to the coarsest supercharacter theory with $K$ as superclass. (An analogous statement holds for subsets $X \subseteq \text{Irr} \ G$.)

Proof. This follows from Corollary 4.1 and Lemma 3.6.

In the following, we use the notations $G^\# := G \setminus \{1\}$ and $\text{Irr}^\# G := \text{Irr} \ G \setminus \{1\}$. 

4.3. Algorithm. To compute the set $S$ of all supercharacter theories of a given group $G$ with known character table, do the following steps:

1. For every nonempty, $G$-invariant subset $S \subset G^\#$, such that $S$ contains at most half of the nontrivial conjugacy classes of $G$, do the following: compute the partitions $I(\{S\})$, $CI(\{S\})$, $\ldots$, until either $S \notin (CI)^n(\{S\})$ for some $n$, or the number of blocks in these partitions stabilizes. In the latter case, we have found the coarsest supercharacter theory, $L(S)$, which has $S$ as one of its superclasses. Add each such $L(S)$ to the set $S$.

2. Form all possible meets of two or more members of $S$ (in the lattice of supercharacter theories) and add them to $S$.

3. Add the coarse supercharacter theory $M(G) := \left( \{1\}, \text{Irr}^\# G \right), \left( \{1\}, G^\# \right)$ to $S$.

To compute the meet of some supercharacter theories $(\mathcal{X}_i, \mathcal{K}_i)$, we first compute the meet $\mathcal{K} := \bigwedge_i \mathcal{K}_i$ in the partition lattice, and then use Corollary 4.1 to refine $\mathcal{K}$ to a supercharacter theory.

Also in practice, one works only on the character table, and represents $G$-invariant subsets $S \subset G$ by a list of class positions.

Proof of correctness of Algorithm 4.3. Let $\mathcal{K} = \{S_1, \ldots, S_r\}$ be a partition of the set $G$ into disjoint, nonempty sets, and assume that $r \geq 2$. Then, in the partition lattice, $\mathcal{K}$ equals the meet

$$\mathcal{K} = \{S_1, G \setminus S_1\} \wedge \cdots \wedge \{S_{r-1}, G \setminus S_{r-1}\}.$$ 

Notice that we have omitted exactly one set $S_r$ here.

Now let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory, and write $\mathcal{K} = \{S_1, \ldots, S_r\}$. Then $\{1\} \in \mathcal{K}$, say $S_1 = \{1\}$. Assume that $\mathcal{K} \neq M(G)$, that is, $r = |\mathcal{K}| \geq 3$. Clearly, we have $\mathcal{K} \geq L(S_1) \geq \{S_i, G \setminus S_i\}$ for each $S_i \in \mathcal{K}$. It follows that

$$\mathcal{K} = L(S_1) \wedge \cdots \wedge L(S_{r-1}).$$

(This is true both in the partition lattice and in the lattice of supercharacter theories.) There is at most one $S_i \in \mathcal{K}$ that contains more than half of the nontrivial conjugacy classes of $G$, and we may assume that this is $S_r$. Since every supercharacter theory refines $L(\{1\}) = M(G)$, we can omit $L(\{1\})$ from the above meet. It follows that $\mathcal{K}$ is a meet of partitions $L(S_i)$ that were found in the first step. Thus the algorithm works.

Let us compare this algorithm to the one proposed by Hendrickson [11, A.9]. Hendrickson first determines certain good subsets of $G$, by running through all nonempty, $G$-invariant subsets of $G$. In a second step, Hendrickson determines all partitions into good subsets belonging to supercharacter theories.

A dual version of Hendrickson’s algorithm was proposed by Lamar [13], building on ideas used by Burkett, Lamar, Lewis and Wynn [6] to show that $G = \text{Sp}(6, 2)$ has only
two supercharacter theories. Here one uses *good* subsets of $\text{Irr } G$ and *admissible* partial partitions of $\text{Irr } G$.

It follows from Lemma 3.6 that a subset $K$ of $G$ is good in Hendrickson’s sense if and only if $K \in CI(\{K\})$, and that a subset $X \subseteq \text{Irr } G$ is good in the sense of Burkett et al. [6] if and only if $X \in IC(\{X\})$. The original definitions were different, and the resulting algorithms to decide whether some subset is good are less effective. For example, to check whether a subset $X \subseteq \text{Irr } G$ is good, Burkett et al. check whether the characters $\chi(1)$ and $\psi(1)$ occur with the same coefficients in the powers $\sigma_X^k$ for all $\chi, \psi \in X$.

If $K \subseteq G$ is a superclass in some supercharacter theory, then $K$ is necessarily good, but not conversely. The first step of Algorithm 4.3 decides directly if a certain normal subset $S \subseteq G$ can be a superclass, and if so, finds the coarsest supercharacter theory of $G$ that contains $S$ as a superclass. In particular, the first step of our algorithm finds all possible superclasses of supercharacter theories, except the trivial superclass $G^\#$ and possibly some large superclasses containing more than $(k(G) - 1)/2$ conjugacy classes. (In principle, it is even possible that such a superclass is not contained in any supercharacter theory determined in the first step, but then it has to be contained in an even larger superclass.)

Of course, for some sets $S$, we will perhaps need more steps until we reach a supercharacter theory or see that $S$ is not a superclass. As mentioned before, usually very few steps suffice, but I do not understand how to predict this.

A variant of the second step in Algorithm 4.3 would be to try all possible refinements of the superclass partitions found in the first step, using only possible superclasses. However, it is often the case that the second step yields no or only few new supercharacter theories.

Of course, Algorithm 4.3 is still rather naive. For example, many sets $S$ will lead to the same supercharacter theory $L(S)$. One possible speedup is to use automorphisms of the character table (see below).

5. Table automorphisms

A *table automorphism* of the character table of some finite group $G$ is a pair $(\sigma, \tau)$, where $\sigma$ is a permutation of $\text{Irr } G$, and $\tau$ a permutation of $\text{Cl}(G)$, the conjugacy classes of $G$, and such that $\chi^\sigma(g^\tau) = \chi(g)$ for all $\chi \in \text{Irr } G$ and $g \in G$ (where $g^\tau$ denotes an element in $(gG)^\tau$). Since the character table as a matrix is invertible, it follows that $\sigma$ and $\tau$ determine each other uniquely, when $(\sigma, \tau)$ is a table automorphism. The set of all table automorphisms of a given character table forms a group, which acts on $\text{Irr } G$ and on $\text{Cl}(G)$. (To prevent possible confusion, we remark here that in the computer algebra system GAP [9], the attribute `AutomorphismsOfTable` gives only those column permutations of a character table that preserve power maps, in particular element orders.)

By Brauer’s permutation lemma [12, Theorem 6.32], the number of irreducible characters fixed by $\sigma$ equals the number of classes fixed by $\tau$ when $(\sigma, \tau)$ is a table automorphism.

The next lemma is the common generalization of two constructions of supercharacter theories described in the introduction of the Diaconis-Isaacs paper [8]. We record it here
5.1. Lemma. If $A$ is a subgroup of table automorphisms, then the partitions of $\text{Irr} \, G$ and $\text{Cl}(G)$ into orbits under $A$ form a supercharacter theory.

Proof. Namely, every table automorphism fixes the class of the identity of $G$ (and the trivial character), and thus every $\chi$ in an orbit $X$ has the same degree. It is easy to see that the sum of the characters in an orbit $X$ is constant along orbits of $A$ on $\text{Cl}(G)$. By Brauer’s permutation lemma, there are equal number of $A$-orbits on $\text{Irr} \, G$ and on $\text{Cl}(G)$.

5.2. Lemma. Let $(\sigma, \tau)$ be a table automorphism of the character table of $G$.

(a) $\text{ClPt}(X^\sigma) = \text{ClPt}(X)^\tau$ for any collection $X$ of subsets of $\text{Irr} \, G$.
(b) $\text{IrPt}(K^\tau) = \text{IrPt}(K)^\sigma$ for any collection $K$ of $G$-invariant subsets of $G$.
(c) If $(X, K)$ is a supercharacter theory of $G$, then so is $(X^\sigma, K^\tau)$.

Proof. The first and second item follows easily from the definitions. The last one is then clear in view of Theorem 3.11.

A special class of table automorphisms is induced by Galois automorphisms. Indeed, if $\sigma$ is a field automorphism of $\mathbb{C}$, then there is some integer $r$ coprime to $|G|$ such that $(\sigma, \tau)$ is a table automorphism, where $\tau$ is induced by the permutation $g \mapsto g^r$.

5.3. Lemma. Let $(\sigma, \tau)$ be a table automorphism of the character table of $G$, where $\sigma$ is induced by some field automorphism.

(a) $\text{ClPt}(X^\sigma) = \text{ClPt}(X)^\tau$ for any collection $X$ of subsets of $\text{Irr} \, G$.
(b) $\text{IrPt}(K^\tau) = \text{IrPt}(K)^\sigma$ for any collection $K$ of $G$-invariant subsets of $G$.
(c) [8, Theorem 2.2(e),(f)] If $(X, K)$ is a supercharacter theory of $G$, then $(X^\sigma, K^\tau) = (X, K)$.

Proof. Let $X \in X$. We have $\sigma_X^\sigma = \sigma \circ \sigma_X$ as function $G \to \mathbb{C}$. This shows (a).

Similarly, the partition $\text{IrPt}(K^\tau)$ is defined by maps $\alpha_{K^\tau} : \text{Irr} \, G \to \mathbb{C}$. We have

$$\alpha_{K^\tau}(\chi) = \frac{\chi(K^\tau)}{\chi(1)} = \frac{1}{\chi(1)} \sum_{g \in K} \chi(g^r) = \frac{1}{\chi(1)} \sum_{g \in K} \chi(g)^\sigma = \alpha_K(\chi)^\sigma,$$

and so $\alpha_{K^\tau} = \sigma \circ \alpha_K$. This yields (b), and (c) follows from (a) and (b).

The last two results can be used to speed up Algorithm 4.3.

6. Discussion

By using an implementation of Algorithm 4.3 into the computer algebra system GAP, all supercharacter theories for some nonabelian simple groups were computed. There is always the coarsest supercharacter theory $\mathcal{M}(G)$, and there are the supercharacter theories that can be constructed from a subgroup $A$ of the table automorphism group, as described for the sake of completeness and reference below.
Table 1: Number of supercharacter theories for some projective special linear groups $G = \text{PSL}(n,q)$

| $n$ | $q$ | $k(G)$ | $\text{Aut}(\text{CT}(G))$ | Supercharacter theories |
|---|---|---|---|---|
| 2  | 7  | 6   | $C_2$         | 4          | 2        |
| 8  | 9  | 7   | $C_3 \times C_3$ | 7          | 4        |
| 9  | 7  | 8   | $C_2 \times C_2$ | 7          | 4        |
| 11 | 8  | 9   | $C_2 \times C_2$ | 13         | 4        |
| 13 | 9  | 8   | $C_6$         | 13         | 4        |
| 16 | 17 | 7   | $C_8 \times C_4$ | 33         | 12       |
| 17 | 11 | 9   | $C_6 \times C_2$ | 25         | 8        |
| 19 | 12 | 9   | $C_6 \times C_2$ | 34         | 8        |
| 23 | 14 | 10  | $C_{10} \times C_2$ | 41         | 8        |
| 25 | 15 | 7   | $C_6 \times C_2 \times C_2$ | 81         | 16       |
| 27 | 16 | 9   | $C_6 \times C_6$ | 45         | 16       |
| 29 | 17 | 7   | $C_{12} \times C_2$ | 89         | 12       |
| 31 | 18 | 9   | $C_4 \times C_4 \times C_2$ | 161        | 18       |
| 37 | 21 | 18  | $C_{18} \times C_3$ | 76         | 12       |
| 41 | 23 | 12  | $C_{12} \times C_2 \times C_2$ | 307        | 24       |
| 43 | 24 | 12  | $C_{30} \times C_2$ | 100        | 16       |

| $n$ | $q$ | $k(G)$ | $\text{Aut}(\text{CT}(G))$ | Supercharacter theories |
|---|---|---|---|---|
| 3  | 3  | 12  | $C_4 \times C_2$ | 7          | 6        |
| 4  | 10 | 12  | $S_3 \times C_2 \times C_2$ | 23         | 20       |
| 7  | 22 | 12  | $S_3 \times C_{12} \times C_2$ | 121        | 60       |

Table 2: Number of supercharacter theories for some alternating groups

| $G$ | $k(G)$ | $\text{Aut}(\text{CT}(G))$ | Supercharacter theories |
|---|---|---|---|
| $A_4$ | 4 | $C_2$ | 3 | 2 |
| $A_5$ | 5 | $C_2$ | 3 | 2 |
| $A_6$ | 7 | $C_2 \times C_2$ | 7 | 4 |
| $A_7$ | 9 | $C_2$ | 3 | 2 |
| $A_8$ | 14 | $C_2 \times C_2$ | 5 | 4 |
| $A_9$ | 18 | $C_2 \times C_2$ | 5 | 4 |
| $A_{10}$ | 24 | $C_2 \times C_2$ | 5 | 4 |
Table 3: Number of supercharacter theories for some sporadic simple groups

| G     | k(G) | Aut(CT(G))       | Supercharacter theories |
|-------|------|------------------|------------------------|
| M_{11}| 10   | C_2 \times C_2  | 5                      |
| M_{12}| 15   | C_2 \times C_2  | 5                      |
| M_{22}| 12   | C_2 \times C_2  | 5                      |
| M_{23}| 17   | (C_2)^4         | 17                     |
| M_{24}| 26   | (C_2)^3         | 9                      |
| J_1   | 15   | C_6             | 5                      |
| J_2   | 21   | C_2             | 3                      |
| J_3   | 21   | C_6 \times C_2 \times C_2 | 17           |
| HS    | 24   | (C_2)^3        | 9                      |
| McL   | 24   | (C_2)^4        | 17                     |

\(a\) M_k: Mathieu groups, J_k: Janko groups, HS: Higman-Sims group, McL: McLaughlin group.

In Lemma 5.1. (For \(A = \{1\}\), this includes the other trivial supercharacter theory.) For a number of simple groups, these are actually all supercharacter theories. This includes all the sporadic simple groups with 26 or fewer conjugacy classes (the five Mathieu groups, the Janko groups \(J_1, J_2, J_3\), the Higman-Sims group \(HS\) and the McLaughlin group \(McL\)), the alternating groups \(A_n\) with \(n \leq 10\) and \(n \neq 6\), the Tits group \(T\) and the exceptional group \(G_2(3)\), and the unitary groups \(U_n(q)\) with \((n, q) = (3, 3), (3, 4), (3, 5), (4, 2), (4, 3)\). (In particular, \(J_2\) and \(U_4(2)\) have exactly 3 supercharacter theories.) On the other hand, for the projective special linear groups, there are usually more supercharacter theories. Table 1 contains the number of supercharacter theories of \(\text{PSL}(n, q)\) for some values of \(q\) and \(n = 2, 3\) (second last column). The last column indicates how many supercharacter theories can be obtained from a subgroup \(A\) of the table automorphism group \(\text{Aut}(CT(G))\), as in Lemma 5.1. Another simple group with nontrivial supercharacter theories not coming from table automorphisms is the Suzuki group \(Sz(8)\) which has 11 different supercharacter theories, of which only 8 come from table automorphisms.

These tables suggest a number of problems and conjectures:

**6.1. Conjecture.** For all \(n \geq 7\), every supercharacter theory \(\neq M(G)\) of the alternating group \(A_n\) can be constructed from a group of table automorphisms.

In view of the specifics of the representation theory of \(A_n\) and \(S_n\), this is equivalent to a conjecture by J. Lamar [13, Conjecture 3.35].

Closely related is the following conjecture:

**6.2. Conjecture.** For every \(n \geq 7\), the symmetric group \(S_n\) has exactly 4 supercharacter
(The normal subgroup $A_n$ of $S_n$ yields 2 more supercharacter theories besides the trivial ones via known constructions [10].)

**6.3. Problem.** Give a generic construction of some supercharacter theory of $\text{PSL}(n, q)$ (or $\text{Sz}(q)$) other than supercharacter theories from orbits of table automorphisms and the trivial supercharacter theory.

The last problem can of course be asked for any other class of groups. In fact, the original motivation of the theory was to construct a supercharacter theory for the unitriangular groups $\text{UT}(n, q)$, where the full character table is not available, and Diaconis and Isaacs constructed more generally a supercharacter theory of algebra groups [8]. Since then, similar constructions have been given for many other unipotent groups [2, 3, 17, 18].

When $G$ is not simple, then there are several constructions of supercharacter theories using normal subgroups [1, 10], and table automorphisms yield also supercharacter theories. Moreover, one may form meets and joins of these supercharacter theories. So in general, it may be difficult to decide whether some proposed construction of supercharacter theories yields something that can not be obtained from the other available methods.

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