General Gauge Field Theory

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[Abstract] In this paper, we will construct a gauge field model, in which the masses of
gauge fields are non-zero and the local gauge symmetry is strictly preserved. A $SU(N)$
gauge field model is discussed in details in this paper. In the limit $\alpha^{-}\rightarrow 0$ or $\alpha^{-}\rightarrow \infty$, the
gauge field model discussed in this paper will return to Yang-Mills gauge field model. This
theory could be regarded as theoretical development of Yang-Mills gauge field theory.

1 Introduction

In 1954, Yang and Mills founded the non-Abel gauge field theory [1 ]. Since then,
the gauge field theory has developed greatly, and has been widely applied to elementary
particle theory. Now, we believe that, four kinds of fundamental interactions in nature are
all gauge interactions and can be described by gauge field theories. It is generally believed
that the principle of local gauge invariance should play a fundamental role in interaction
theory. In the sixties, the gauge field theory was applied to electro-weak interactions,
and the $SU(2)_L \times U(1)_Y$ unified electro-weak gauge theory was founded [2-4 ]. In the
seventies, gauge field theory was applied to strong interaction, and the $SU(3)_c$ quantum
chromodynamics was founded.

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But, we know that, if the local gauge symmetry is strictly preserved in Yang-Mills gauge field theory, the masses of gauge fields should be zero. However in the forties, physicists have realized that the intermediate bosons, which transmit weak interaction, should have very big masses [5]. A possible way to solve this problem is to introduce the concept of spontaneously symmetry breaking [6-8] and Higgs mechanism [9-13]. Higgs mechanism plays an important role in the standard model. The gauge bosons $W^\pm$ and $Z^0$, which are predicted by the standard model theoretically, are discovered by experiment in the eighties, but the Higgs particle, which is necessary for the standard model, is not found until now. Does Higgs particle exist in nature? If there were no Higgs particle, how could the intermediate bosons $W^\pm$ and $Z^0$ obtain masses? In the theoretical point of view, can we construct a gauge field model, in which we need no Higgs mechanism, but the masses of gauge bosons are non-zero.

In this paper, we will construct a gauge field model, which has strict local gauge symmetry and massive gauge fields. In order to do this, two sets of gauge fields are needed. In this paper, We will give the lagrangian of the model first, then prove the gauge symmetry of the model and deduce the conserved charges which correspond to the gauge symmetry. After that, we will construct the eigenvectors of mass matrix, and deduce the equations of motion of all fields. The case that matter fields are boson fields is also discussed in this paper. A more general gauge field model is given in chapter eight. After that, we discuss the Yang-Mills limit of the present theory. Finally, we present some discussions.

2 The Lagrangian of The Model

For the sake of generality, we let the gauge group be $SU(N)$ group, and for the sake of simplicity, we select fermion fields as matter fields. Suppose that $N$ fermion fields $\psi_l(x)$ ($l = 1, 2, \ldots, N$) form a multiplet of matter fields. let

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}$$

(2.1)

$\psi(x)$ is a N-component vector. All $\psi(x)$ form a space of fundamental representation of $SU(N)$ group. In this representative space, the representative matrices of the generators of $SU(N)$ group are denoted by $T_i$ ($i = 1, 2, \ldots, N^2 - 1$). They satisfy:

$$[T_i, T_j] = i f_{ijk} T_k$$

(2.2)

$$Tr(T_i T_j) = \delta_{ij} K.$$  

(2.3)

where $f_{ijk}$ are structure constants of $SU(N)$ group, $K$ is a constant which is independent of indices $i$ and $j$ but depends on the representation of the group. Generators $T_i$ are Hermit and traceless:

$$T_i^\dagger = T_i$$

(2.4)
The representative matrix of a general element of the $SU(N)$ group can be written as:

$$ U = e^{-i\alpha^i T_i} $$

with $\alpha^i$ the real group parameters. $U$ is a unitary $N \times N$ matrix

$$ U^\dagger U = 1 = UU^\dagger $$

In global gauge transformation, all $\alpha^i$ are independent of space-time coordinates, where in local gauge transformation, $\alpha^i$ are functions of space-time coordinates.

Two kinds of vector fields $A_\mu(x)$ and $B_\mu(x)$ will be introduced in this paper. $A_\mu(x)$ and $B_\mu(x)$ are vectors in the adjoint representation of $SU(N)$ group. They can be expressed as linear combinations of generators:

$$ A_\mu(x) = A^i_\mu(x) T_i $$
$$ B_\mu(x) = B^i_\mu(x) T_i. $$

$A^i_\mu(x)$ and $B^i_\mu(x)$ are component fields of gauge fields $A_\mu(x)$ and $B_\mu(x)$ respectively.

Corresponds to two kinds of gauge fields, there are two kinds of gauge covariant derivatives in the theory:

$$ D_\mu = \partial_\mu - igA_\mu $$
$$ D_{b\mu} = \partial_\mu + i\alpha gB_\mu. $$

The strengths of gauge fields $A_\mu(x)$ and $B_\mu(x)$ are defined as

$$ A_{\mu\nu} = \frac{1}{ig} [D_\mu , D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu , A_\nu] $$
$$ B_{\mu\nu} = \frac{1}{i\alpha g} [D_{b\mu} , D_{b\nu}] = \partial_\mu B_\nu - \partial_\nu B_\mu + i\alpha g[B_\mu , B_\nu]. $$

respectively. Similarly, $A_{\mu\nu}$ and $B_{\mu\nu}$ can also be expressed as linear combinations of generators:

$$ A_{\mu\nu} = A^i_{\mu\nu} T_i $$
$$ B_{\mu\nu} = B^i_{\mu\nu} T_i. $$

Using relations (2.2) and (2.10a,b), we could obtain

$$ A^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gf^{ijk} A^j_\mu A^k_\nu $$
$$ B^i_{\mu\nu} = \partial_\mu B^i_\nu - \partial_\nu B^i_\mu - \alpha gf^{ijk} B^j_\mu B^k_\nu. $$
The lagrangian density of the model is

\[ L = -\bar{\psi}(\gamma^\mu D_\mu + m)\psi - \frac{1}{4K} Tr(A^{\mu^\nu}A_{\mu^\nu}) - \frac{1}{4K} Tr(B^{\mu^\nu}B_{\mu^\nu}) - \frac{1}{16} K Tr[(A^\mu + \alpha B^\mu)(A_\mu + \alpha B_{\mu})] \]  

(2.13)

where \( \alpha \) is a constant. In this paper, the space-time metric is selected as \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \), \((\mu, \nu = 0, 1, 2, 3)\). According to relation (2.3), the above lagrangian density \( L \) can be rewritten as:

\[ L = -\bar{\psi}(\gamma^\mu (\partial_\mu - igA_i^\mu T_i) + m)\psi - \frac{1}{4} A^{i\mu\nu}A_{i\mu\nu} - \frac{1}{4} B^{i\mu\nu}B_{i\mu\nu} \\
- \frac{1}{2(1+\alpha^2)} (A^{\mu} + \alpha B^{\mu})(A_{\mu} + \alpha B_{\mu}) \]  

(2.14)

It is easy to see that, except for the mass term and the term concerned with gauge field \( B_\mu \), the above lagrangian density is the same as that of Yang-Mills theory.

### 3 Global Gauge Symmetry and Conserved Charges

Now we discuss the gauge symmetry of the lagrangian density \( L \). First, we discuss the global gauge symmetry and the corresponding conserved charges. In global gauge transformation, the matter field \( \psi \) transforms as:

\[ \psi \rightarrow \psi' = U\psi, \]  

(3.1)

where \( U \) is a \( N \times N \) transformation matrix which is defined by eq(2.6). \( U \) is independent of space-time coordinates. That is

\[ \partial_\mu U = 0. \]  

(3.2)

The corresponding global gauge transformations of gauge fields \( A_\mu \) and \( B_\mu \) are

\[ A_\mu \rightarrow U A_\mu U^\dagger \]  

(3.3a)

\[ B_\mu \rightarrow U B_\mu U^\dagger \]  

(3.3b)

respectively. It is easy to prove that

\[ D_\mu \rightarrow U D_\mu U^\dagger \]  

(3.4a)

\[ D_{b\mu} \rightarrow U D_{b\mu} U^\dagger \]  

(3.4b)

\[ A_{\mu^\nu} \rightarrow U A_{\mu^\nu} U^\dagger \]  

(3.5a)

\[ B_{\mu^\nu} \rightarrow U B_{\mu^\nu} U^\dagger \]  

(3.5b)

Using all the above transformation relations, we can prove that every term in eq(2.13) is gauge invariant. So, the whole lagrangian density has global gauge symmetry.
Let \( \alpha^i \) in eq(2.6) be the first order infinitesimal parameters, then the transformation matrix \( U \) can be rewritten as:

\[
U \approx 1 - i \alpha^i T^i. \tag{3.6}
\]

The first order infinitesimal changes of fields \( \psi, \overline{\psi}, A_\mu \) and \( B_\mu \) are

\[
\delta \psi = -i \alpha^i T^i \psi \tag{3.7a}
\]

\[
\overline{\delta \psi} = i \alpha^i \overline{\psi} T^i \tag{3.7b}
\]

\[
\delta A_\mu = \alpha^i f^{ijk} A^j_\nu T^k_\mu \tag{3.8a}
\]

\[
\delta B_\mu = \alpha^i f^{ijk} B^j_\nu T^k_\mu \tag{3.8b}
\]

respectively. From eqs(3.8a,b) and eqs(2.8a,b), we can obtain

\[
\delta A^k_\mu = \alpha^i f^{ijk} A^j_\mu \tag{3.9a}
\]

\[
\delta B^k_\mu = \alpha^i f^{ijk} B^j_\nu T^k_\mu \tag{3.9b}
\]

The first order variation of the lagrangian density is

\[
\delta \mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \overline{\psi}} \overline{\delta \psi} + \frac{\partial \mathcal{L}}{\partial \partial_\mu A^k_\nu} \delta A^k_\nu + \frac{\partial \mathcal{L}}{\partial \partial_\mu B^k_\nu} \delta B^k_\nu \right) \tag{3.10}
\]

where

\[
J^i_\mu = \overline{\psi} \gamma_\mu T^i \psi - f^{ijk} A^j_\mu A^k_\nu - f^{ijk} B^j_\mu B^k_\nu. \tag{3.11}
\]

The conserved current can also be written as

\[
J^i_\mu = \overline{\psi} \gamma_\mu T^i \psi + \overline{[A^\nu, A^k_\mu]} + \overline{[B^\nu, B^k_\mu]} \tag{3.12}
\]

Because the lagrangian density \( \mathcal{L} \) has global gauge symmetry, the currents \( J^i_\mu \) are conserved currents. They satisfy

\[
\partial_\mu J^i_\mu = 0. \tag{3.13}
\]

The corresponding conserved charges are

\[
Q^i = \int d^3 x J^i_0 \tag{3.14}
\]

After quantization, \( Q^i \) are the generators of gauge transformation. At the same time, we note that, no matter what is the value of parameter \( \alpha \), gauge fields \( A_\mu \) and \( B_\mu \) contribute the same terms to the conserved currents and conserved charges.
4 Local Gauge Symmetry

If $U$ in eq(3.1) depends on space-time coordinates, the transformation of eq(3.1) is a local gauge transformation. In this case,

$$\partial_{\mu} U \neq 0, \quad \partial_{\mu} \alpha^i \neq 0 \quad (4.1)$$

The corresponding transformations of gauge fields $A_{\mu}$ and $B_{\mu}$ are

$$A_{\mu} \rightarrow U A_{\mu} U^{\dagger} - \frac{1}{i g} U \partial_{\mu} U^{\dagger} \quad (4.2a)$$

$$B_{\mu} \rightarrow U B_{\mu} U^{\dagger} + \frac{1}{i g} U \partial_{\mu} U^{\dagger} \quad (4.2b)$$

respectively.

Using above relations, it is easy to prove that

$$D_{\mu} \rightarrow U D_{\mu} U^{\dagger} \quad (4.3a)$$

$$D_{\mu} \rightarrow U D_{\mu} U^{\dagger}. \quad (4.3b)$$

Therefore,

$$A_{\mu \nu} \rightarrow U A_{\mu \nu} U^{\dagger} \quad (4.4a)$$

$$B_{\mu \nu} \rightarrow U B_{\mu \nu} U^{\dagger} \quad (4.4b)$$

$$D_{\mu} \psi \rightarrow U D_{\mu} \psi \quad (4.5)$$

$$A_{\mu} + \alpha B_{\mu} \rightarrow U (A_{\mu} + \alpha B_{\mu}) U^{\dagger} \quad (4.6)$$

Using all these transformation relations, we could prove that the lagrangian density $L$ defined by eq(2.13) is invariant under local gauge transformations. Therefore the model has strict local gauge symmetry.

5 The Masses of Gauge Fields

If we select $A_{\mu}$ and $B_{\mu}$ as basis, the mass matrix on this basis is

$$M = \frac{1}{1 + \alpha^2} \begin{pmatrix} \mu^2 & \alpha \mu^2 \\ \alpha \mu^2 & \alpha^2 \mu^2 \end{pmatrix}, \quad (5.1)$$

and the mass term of gauge fields can be written as:

$$(A_\mu, B_\mu) M \begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix}. \quad (5.2)$$
The particles generated from gauge interaction should be eigenvectors of mass matrix and the corresponding masses of these particles should be eigenvalues of mass matrix. $M$ has two eigenvalues, they are

$$m_1^2 = \mu^2, \quad m_2^2 = 0.$$  \hspace{1cm} (5.3)

The corresponding eigenvectors are

$$\frac{1}{\sqrt{1 + \alpha^2}} \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \quad \frac{1}{\sqrt{1 + \alpha^2}} \begin{pmatrix} -\alpha \\ 1 \end{pmatrix},$$  \hspace{1cm} (5.4)

respectively. If we define

$$\cos \theta = \frac{1}{\sqrt{1 + \alpha^2}}, \quad \sin \theta = \frac{\alpha}{\sqrt{1 + \alpha^2}}.$$  \hspace{1cm} (5.5)

Then, eq(5.4) changes into

$$\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix},$$  \hspace{1cm} (5.6)

Define

$$C_\mu = \cos \theta A_\mu + \sin \theta B_\mu,$$  \hspace{1cm} (5.7a)

$$F_\mu = -\sin \theta A_\mu + \cos \theta B_\mu.$$  \hspace{1cm} (5.7b)

It is easy to know that $C_\mu$ and $F_\mu$ are eigenstates of mass matrix, they describe those particles generated from gauge interaction. The inverse transformations of (5.7a,b) are

$$A_\mu = \cos \theta C_\mu - \sin \theta F_\mu$$  \hspace{1cm} (5.8a)

$$B_\mu = \sin \theta C_\mu + \cos \theta F_\mu.$$  \hspace{1cm} (5.8b)

Then the lagrangian density $\mathcal{L}$ given by (2.14) changes into:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$  \hspace{1cm} (5.9)

where

$$\mathcal{L}_0 = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi - \frac{1}{4} C_0^{i\mu} C_0^{i\nu} - \frac{1}{4} F_0^{i\mu\nu} F_0^{i\nu} - \frac{\mu^2}{2} C^{i\mu} C^{i\nu}.  \hspace{1cm} (5.10a)$$

$$\mathcal{L}_I = ig\bar{\psi}\gamma^\mu (\cos \theta C_\mu - \sin \theta F_\mu)\psi$$

$$-\frac{\cos^2 \theta}{2} g f_{ijk} C_0^{i\mu} C_0^{j\nu} F_0^{k\rho} + \frac{\sin^2 \theta}{2} g f_{ijk} F_0^{i\mu\nu} F_0^{j\rho} F_0^{k\sigma} + g \sin \theta f_{ijk} C_0^{i\mu} C_0^{j\nu} C_0^{k\rho}$$

$$-\frac{1}{2} g f_{ijk} f^{ilm} C_0^{i\mu} C_0^{j\nu} C_0^{k\rho} C_0^{l\sigma}$$

$$-\frac{1}{2} g f_{ijk} f^{ilm} F_0^{i\mu} F_0^{j\nu} F_0^{k\rho} F_0^{l\sigma} + g^2 \sin \theta \cos \theta f_{ijk} f^{ilm} C_0^{i\mu} C_0^{j\nu} C_0^{k\rho} F_0^{l\sigma}$$

$$-\frac{1}{2} g f_{ijk} f^{ilm} (C_0^{i\mu} C_0^{j\nu} F_0^{k\rho} F_0^{l\sigma} + C_0^{i\mu} F_0^{j\nu} F_0^{k\rho} C_0^{l\sigma} + C_0^{i\mu} F_0^{j\nu} C_0^{k\rho} F_0^{l\sigma} + C_0^{i\mu} F_0^{j\nu} F_0^{k\rho} C_0^{l\sigma}).$$  \hspace{1cm} (5.10b)

In the above relations, we have used the following notations:

$$C_0^{i\mu\nu} = \partial_\mu C_0^{i\nu} - \partial_\nu C_0^{i\mu}.$$  \hspace{1cm} (5.11a)
From eq(5.10a), it is easy to see that the mass of gauge field $C_\mu$ is $\mu$ and the mass of gauge field $F_\mu$ is zero. That is

$$m_c = \mu, \quad m_F = 0.$$ (5.12)

Please note that, up to now, the gauge symmetry is strictly preserved. Therefore, without Higgs mechanism, gauge fields can have non-zero masses. Strict gauge symmetry does not mean that the gauge fields are all massless.

6 Equation of Motion

The Euler-Lagrange equation of motion for fermion field can be deduced from eq(5.9):

$$[\gamma^\mu(\partial_\mu - ig\cos\theta C_\mu + ig\sin\theta F_\mu) + m]\psi = 0.$$ (6.1)

If we deduce the Euler-Lagrange equations of motion for gauge fields from eq(5.9), we will obtain very complicated expressions. For the sake of simplicity, we deduce the equations of motion for gauge fields from eq(2.13). In this case, the equations of motion for gauge fields $A_\mu$ and $B_\mu$ are:

$$D^\mu A_{\mu\nu} - \frac{\mu^2}{1+\alpha^2}(A_{\nu} + \alpha B_{\nu}) = ig\bar{\psi}\gamma_\nu T^i \psi T^i$$ (6.2a)

$$D^\mu B_{\mu\nu} - \frac{\alpha\mu^2}{1+\alpha^2}(A_{\nu} + \alpha B_{\nu}) = 0$$ (6.2b)

respectively. In the above relations, we have used two simplified notations:

$$D^\mu A_{\mu\nu} = [D^\mu, A_{\mu\nu}]$$ (6.3a)

$$D^\mu B_{\mu\nu} = [D^\mu, B_{\mu\nu}]$$ (6.3b)

Eqs(6.2a,b) can be expressed in terms of component fields $A^i_\mu$ and $B^i_\mu$:

$$\partial^\mu A^i_{\mu\nu} - \frac{\mu^2}{1+\alpha^2}(A^i_{\nu} + \alpha B^i_{\nu}) = ig\bar{\psi}\gamma_\nu T^i \psi + gf^{ijk}A^j_{\mu\nu}A^{k\nu}$$ (6.4a)

$$\partial^\mu B^i_{\mu\nu} - \frac{\alpha\mu^2}{1+\alpha^2}(A^i_{\nu} + \alpha B^i_{\nu}) = -\alpha gf^{ijk}B^j_{\mu\nu}B^{k\nu}$$ (6.4b)

The equations of motion for gauge fields $C_\mu$ and $F_\mu$ can be easily obtained from eqs(6.4a,b). In other words, $\cos\theta \cdot (6.4a) - \sin\theta \cdot (6.4b)$ gives the equation of motion for massive vector field $C_\mu$, and $-\sin\theta \cdot (6.4a) + \cos\theta \cdot (6.4b)$ gives the equation of motion for massless vector field $F_\mu$. 
Please note that the above equations of motion are quite different from those of Yang-Mills gauge theory. But if $\alpha$ is small enough, two gauge theories will give similar results. Let
\[
\alpha \ll 1, \quad (6.5)
\]
then, in the leading term,
\[
\begin{align*}
\cos \theta & \approx 1, \quad \sin \theta \approx 0, \\
A_\mu & \approx C_\mu, \quad B_\mu \approx F_\mu
\end{align*} \quad (6.6)
\]
In this case, eqs(6.1) and (6.2a,b) change into
\[
\begin{align*}
\gamma^\mu (\partial_\mu - igC_\mu) + m)\psi &= 0 \quad (6.8) \\
D^\mu C^\mu = \mu^2 C_\nu = ig\bar{\psi}\gamma_\nu T^i\psi T^i \\
D^\mu F_\mu &= 0 \quad (6.9)
\end{align*}
\]
respectively. Except for a mass term in eq(6.9), eqs(6.8-9) are the same as those in Yang-Mills gauge theory.

From eq(6.2a) or (6.2b), we can obtain a supplementary condition. Using eq(6.1), we can prove that
\[
[D^\lambda, - ig\bar{\psi}\gamma_\lambda T^i \psi T^i] = 0. \quad (6.11)
\]
Let $D^\nu$ act on eq(6.2a) from the left, and let $D^\nu_b$ act on eq(6.2b) from the left, applying eq(5.5) and the following two identities:
\[
\begin{align*}
[D^\lambda, [D^\nu, A_{\nu\lambda}]] &= 0 \quad (6.12a) \\
[D^\lambda_b, [D^\nu_b, B_{\nu\lambda}]] &= 0, \quad (6.12b)
\end{align*}
\]
we could obtain the following two equations
\[
\begin{align*}
[D^\nu, A_\nu + \alpha B_\nu] &= 0 \quad (6.13a) \\
[D^\nu_b, A_\nu + \alpha B_\nu] &= 0 \quad (6.13b)
\end{align*}
\]
respectively. These two equations are essentially the same, they give a supplementary condition. If we expressed eqs(6.13a,b) in terms of component fields, these two equations will give the same expression:
\[
\partial^\nu (A^i_\nu + \alpha B^i_\nu) + \alpha g f^{ijk} A^j_\nu B^{k\nu} = 0. \quad (6.14)
\]
When $\nu = 0$, eqs(6.4a,b) don’t give dynamical equations of motion for gauge fields, because they contain no time derivative terms. They are just constrains. Originally, gauge fields $A^i_\mu$ and $B^i_\mu$ have $8(N^2 - 1)$ degrees of freedom, but they satisfy $2(N^2 - 1)$ constrains and have $(N^2 - 1)$ gauge degrees of freedom, therefore, gauge fields $A^i_\mu$ and $B^i_\mu$ have $5(N^2 - 1)$ independent dynamical degrees of freedom altogether. This result coincides with our experience: a massive vector field has 3 independent degrees of freedom and a massless vector field has 2 independent degrees of freedom.
7 The Case That Matter Fields Are Scalar Fields

In the above discussions, all matter fields are spinor fields. Now, we consider the case when matter fields are scalar fields. Suppose that there are $N$ scalar fields $\varphi_l(x)$ ($l = 1, 2, \cdots N$) which form a multiplet of matter fields:

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_N(x) \end{pmatrix}$$  \hspace{1cm} (7.1)

All $\varphi(x)$ form a representative space of $SU(N)$ group. In gauge transformation, $\varphi(x)$ transforms as:

$$\varphi(x) \rightarrow \varphi'(x) = U\varphi(x)$$  \hspace{1cm} (7.2)

The lagrangian density is

$$\mathcal{L} = -[(\partial_\mu - igA_\mu)\varphi]^+(\partial^\mu - igA^\mu)\varphi - V(\varphi) - \frac{1}{4}Tr(A^{\mu\nu}A_{\mu\nu}) - \frac{1}{4}Tr(B^{\mu\nu}B_{\mu\nu}) - \frac{2\mu^2}{2(1+\alpha^2)}Tr([A_\mu + \alpha B_\mu](A_\mu + \alpha B_\mu))$$  \hspace{1cm} (7.3)

The above lagrangian density can be expressed in terms of component fields:

$$\mathcal{L} = -[(\partial_\mu - igA_\mu T_i)\varphi]^+(\partial^\mu - igA^{\mu i}T_i)\varphi - V(\varphi) - \frac{1}{4}A^{\mu\nu}A_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{\mu^2}{2(1+\alpha^2)}(A^{\mu i} + \alpha B^{\mu i})(A^{i}_\mu + \alpha B^{i}_\mu)$$  \hspace{1cm} (7.4)

The general form for $V(\varphi)$ which is renormalizable and gauge invariant is

$$V(\varphi) = m^2 \varphi^+\varphi + \lambda(\varphi^+\varphi)^2.$$  \hspace{1cm} (7.5)

It is easy to prove that the lagrangian density $\mathcal{L}$ defined by eq(7.3) has local $SU(N)$ gauge symmetry. The Euler-Lagrange equation of motion for scalar field $\varphi$ is:

$$(\partial^\mu - igA^\mu)(\partial_\mu - igA_\mu)\varphi - m^2\varphi - 2\lambda\varphi(\varphi^+\varphi)^2 = 0$$  \hspace{1cm} (7.6)

If $N^2 - 1$ scalar fields $\varphi_l(x)$ ($l = 1, 2, \cdots N^2 - 1$) form a multiplet of matter fields

$$\varphi(x) = \varphi_l(x)T_l,$$  \hspace{1cm} (7.7)

then, the gauge transformation of $\varphi(x)$ should be

$$\varphi(x) \rightarrow \varphi'(x) = U\varphi(x)U^+.$$  \hspace{1cm} (7.8)

All $\varphi(x)$ form a space of adjoint representation of $SU(N)$ group. In this case, the gauge covariant derivative is

$$D_\mu\varphi = \partial_\mu\varphi - ig[A_\mu, \varphi],$$  \hspace{1cm} (7.9)

and the gauge invariant lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = -\frac{1}{4}Tr[(D^\mu\varphi)^+(D_\mu\varphi)] - V(\varphi) - \frac{1}{4}A^{\mu\nu}A_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{\mu^2}{2(1+\alpha^2)}(A^{\mu i} + \alpha B^{\mu i})(A^{i}_\mu + \alpha B^{i}_\mu).$$  \hspace{1cm} (7.10)
8 A More General Model

In the above discussions, we have constructed a gauge field model which has rigorous SU($N$) gauge symmetry and massive gauge bosons. In the above model, only gauge field $A_\mu$ interacts with matter fields $\psi$ or $\varphi$, gauge field $B_\mu$ doesn’t interact with matter fields. Now, we will construct a more general gauge field model, in which both gauge fields interact with matter fields. And in a proper limit, this model will return to the above model. As an example, we only discuss the case when matter fields are spinor fields. The case when matter fields are scalar fields can be discussed similarly.

In chapter 4, we have prove that, under local gauge transformations, $D_\mu$ and $D_{b\mu}$ transform covariantly. It is easy to prove that $\cos^2 \phi D_\mu + \sin^2 \phi D_{b\mu}$ is the most general gauge covariant derivative which transforms covariantly under local SU($N$) gauge transformations

$$\cos^2 \phi D_\mu + \sin^2 \phi D_{b\mu} \longrightarrow U(\cos^2 \phi D_\mu + \sin^2 \phi D_{b\mu})U^+, \quad (8.1)$$

where $\phi$ is constant. Then the following lagrangian has local SU($N$) gauge symmetry

$$\mathcal{L} = -\overline{\psi}[\gamma^\mu(\cos^2 \phi D_\mu + \sin^2 \phi D_{b\mu}) + m]\psi - \frac{1}{4K}Tr(A^{\mu\nu}A_{\mu\nu}) - \frac{1}{4K}Tr(B^{\mu\nu}B_{\mu\nu}) - \frac{\mu^2}{2K(1+\alpha^2)}Tr[(A^{\mu} + \alpha B^{\mu})(A_{\mu} + \alpha B_{\mu})] \quad (8.2)$$

Let $\mathcal{L}_\psi$ denote the part for fermions:

$$\mathcal{L}_\psi = -\overline{\psi}[\gamma^\mu(\cos^2 \phi D_\mu + \sin^2 \phi D_{b\mu}) + m]\psi. \quad (8.3)$$

Using eqs(2.9a,b), we can change $\mathcal{L}_\psi$ into

$$\mathcal{L}_\psi = -\overline{\psi}[\gamma^\mu(\partial_\mu - ig\cos^2 \phi A_\mu + i\alpha g\sin^2 \phi B_\mu) + m]\psi. \quad (8.4)$$

From the above lagrangian, we know that both gauge fields $A_\mu$ and $B_\mu$ couple with matter field $\psi$. Substitute eqs(5.8a,b) into eq(8.4), we get

$$\mathcal{L}_\psi = -\overline{\psi}[\gamma^\mu(\partial_\mu - ig\frac{\cos^2 \theta - \sin^2 \phi}{\cos \theta}C_\mu + i\alpha g\sin \theta F_\mu) + m]\psi. \quad (8.5)$$

The equation of motion for fermion field $\psi$ is

$$[\gamma^\mu(\partial_\mu - ig\frac{\cos^2 \theta - \sin^2 \phi}{\cos \theta}C_\mu + i\alpha g\sin \theta F_\mu) + m]\psi = 0. \quad (8.6)$$

The equations of motion for gauge fields $A_\mu$ and $B_\mu$ now change into:

$$D^\mu A_{\mu\nu} = -\frac{\mu^2}{1+\alpha^2}(A_{\nu} + \alpha B_{\nu}) = ig\cos^2 \phi \overline{\psi}\gamma_\nu T^i\psi T^i \quad (8.7a)$$

$$D^\mu B_{\mu\nu} = i\alpha \mu^2 \frac{1}{1+\alpha^2}(A_{\nu} + \alpha B_{\nu}) = -i\alpha g\sin^2 \phi \overline{\psi}\gamma_\nu T^i\psi T^i. \quad (8.7b)$$
If $\phi$ vanish, the lagrangian density (8.2) will become the original lagrangian density (2.13), the equations of motion (8.7a,b) will return to eqs(6.2a,b), and eq(8.6) will return to eq(6.1). So, the model discussed in the above chapters is just a special case of the model we discuss now.

9 U(1) Case

If the symmetry of the model is U(1) group, we will obtain a U(1) gauge field model. We also use $A_\mu$ and $B_\mu$ to denote gauge fields and $\psi$ to denote a multiplet of fermion fields. In U(1) case, the strengths of gauge fields are

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$ (9.1a, 9.1b)

Two gauge covariant derivatives are the same as (2.9a,b) but with different content. The lagrangian density of the model is:

$$L = -\overline{\psi} \left[ \gamma^\mu (\cos^2 \phi D_\mu + \sin^2 \phi D_{\mu \theta}) + m \right] \psi - \frac{1}{4} A_\mu A_\nu A^{\mu\nu} - \frac{1}{4} B_\mu B_\nu B^{\mu\nu} - \frac{\mu^2}{2(1+\alpha^2)} (A_\mu + \alpha B_\mu)(A_\mu + \alpha B_\mu)$$ (9.2)

The local U(1) gauge transformations are

$$\psi \rightarrow e^{-i\theta} \psi,$$ (9.3a)

$$A_\mu \rightarrow A_\mu - \frac{1}{g} \partial_\mu \theta,$$ (9.3b)

$$B_\mu \rightarrow B_\mu + \frac{1}{\alpha g} \partial_\mu \theta.$$ (9.3c)

Then, $A_{\mu\nu}$, $B_{\mu\nu}$ and $A_\mu + \alpha B_\mu$ are all U(1) gauge invariant. That is

$$A_{\mu\nu} \rightarrow A_{\mu\nu},$$ (9.4a)

$$B_{\mu\nu} \rightarrow B_{\mu\nu},$$ (9.4b)

$$A_\mu + \alpha B_\mu \rightarrow A_\mu + \alpha B_\mu.$$ (9.4c)

Using all these results, it is easy to prove that the lagrangian density given by eq(9.2) has local U(1) gauge symmetry.

Substitute eqs(5.8a,b) into eq(9.2), the lagrangian density $L$ changes into

$$L = -\overline{\psi} \left[ \gamma^\mu (\partial_\mu - ig \cos^2 \theta \sin^2 \phi C_\mu + ig \sin \theta F_\mu) + m \right] \psi - \frac{1}{4} C^{\mu\nu} C_{\mu\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\mu^2}{2} C^\mu C_\mu$$ (9.5)

Now, we see that there is a massive Abel gauge field as well as a massless Abel gauge field. They all have gauge interaction with matter field. In this case, U(1) gauge interaction is transmitted by two different kinds of gauge fields.
10 Yang-Mills Limit

In chapter six, we have discussed the Yang-Mills limit of equations of motion. Now, starting from the lagrangian density of the model, we will discuss the Yang-Mills limit of the model. There are two kinds of Yang-Mills limit of the present gauge field theory.

The first kind of Yang-Mills limit corresponds to very small parameter $\alpha$. Let

$$\alpha \to 0,$$  \hspace{1cm} (10.1)

then

$$\cos \theta = 1 \hspace{0.5cm}, \hspace{0.5cm} \sin \theta = 0.$$  \hspace{1cm} (10.2)

From eqs(5.7a,b), we know that the gauge field $A_\mu$ is just gauge field $C_\mu$ and the gauge field $B_\mu$ is just the gauge field $F_\mu$. That is

$$C_\mu = A_\mu \hspace{0.5cm}, \hspace{0.5cm} F_\mu = B_\mu.$$  \hspace{1cm} (10.3)

In this case, the lagrangian density (2.14) becomes

$$L = -\bar{\psi}[\gamma^\mu(\partial_\mu - igC_\mu^iT^i) + m]\psi$$

$$-\frac{1}{4}C^{i\mu\nu}C_{i\mu\nu} - \frac{1}{4}F^{i\mu\nu}F_{i\mu\nu} - \frac{\mu^2}{2}C^{i\mu\nu}C_{i\mu\nu}.$$  \hspace{1cm} (10.4)

Please note that massless gauge fields do not interact with matter fields. So, the $\alpha \to 0$ limit corresponds to the case that gauge interaction is mainly transmitted by massive gauge fields. In other words, the above lagrangian describe those gauge interactions of which the masses of intermediate bosons are non-zero. It is known that electroweak interactions belong to this category. Except for a mass term of gauge fields, the above lagrangian density is the same as that of the Yang-Mills theory. But if $\alpha$ strictly vanishes, the lagrangian does not have gauge symmetry and the theory is not renormalizable.

The second kind of Yang-Mills limit corresponds to very big parameter $\alpha$. Let

$$\alpha \to \infty,$$  \hspace{1cm} (10.5)

then

$$\cos \theta = 0 \hspace{0.5cm}, \hspace{0.5cm} \sin \theta = 1.$$  \hspace{1cm} (10.6)

From eqs(5.7a,b), we know that the gauge field $B_\mu$ is just the gauge field $C_\mu$ and the gauge field $A_\mu$ is just the gauge field $-F_\mu$. That is

$$C_\mu = B_\mu \hspace{0.5cm}, \hspace{0.5cm} F_\mu = -A_\mu.$$  \hspace{1cm} (10.7)

Then, the lagrangian density (2.14) becomes

$$L = -\bar{\psi}[\gamma^\mu(\partial_\mu + igF_\mu^iT^i) + m]\psi$$

$$-\frac{1}{4}F^{i\mu\nu}F_{i\mu\nu} - \frac{1}{4}C^{i\mu\nu}C_{i\mu\nu} - \frac{\mu^2}{2}C^{i\mu\nu}C_{i\mu\nu}.$$  \hspace{1cm} (10.8)
In this case, massive gauge fields do not interact with matter fields. So, this limit corresponds to the case when gauge interaction is mainly transmitted by massless gauge fields. Similarly, the lagrangian density (10.8) do not have strict gauge symmetry.

In the model of particles’ interaction which describes the gauge interaction of real world, the parameter $\alpha$ should be finite,

$$0 < \alpha < \infty.$$  \hspace{1cm} (10.9)

In this case, both massive gauge fields and massless gauge fields interact with matter fields, and gauge interaction is transmitted by both of them.

11 Discussion

In chapter six, we have said that when $\alpha \ll 1$, the equations of motion given by gauge fields model discussed in this paper are similar to those of Yang-Mills gauge theory except for a mass term. So, we could anticipate that these two gauge theories will give similar dynamical behaviors in describing particles’ interaction.

If we apply this model to strong interaction [14], we will obtain two sets of gluons: one set is massive and another set is massless. Because of color confinement, all colored gluons are confined. The important thing is that there may exist three sets of glueballs. If we apply this model to electroweak interactions, we will obtain two sets of intermediate gauge bosons: one set is massive which has already been found by experiment and another set is massless. [15]. In this new electroweak model, there is no Higgs particle. Because the parameter $\alpha$ is unknown, there exists no contradiction between the theory discussed in this paper and experiment.

The new theory predicts many new massless, electric neutral vector particles. In the high energy experiment, it is hard to distinguish between all these massless, electric neutral vector particles and $\gamma$ photon. Experimental physicists have found that $\gamma$ photon takes part in strong interaction and electroweak interactions [16]. This phenomenon means that there are some massless, electric neutral vector particles mixed in $\gamma$ photon.

Because the model has strict gauge symmetry, we can anticipate that this gauge field theory is renormalizable [17].

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