PRODUCTS OF DIFFERENTIATION AND COMPOSITION OPERATORS FROM THE BLOCH SPACE AND WEIGHTED DIRICHLET SPACES TO MORREY TYPE SPACES

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Abstract. In this paper, we characterize the boundedness, compactness and essential norm of products of differentiation and composition operators from the Bloch space and weighted Dirichlet spaces to analytic Morrey type spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ denote the space of all functions analytic on $\mathbb{D}$. For $a \in \mathbb{D}$, $g(z,a) = \log \frac{1}{|\sigma_a(z)|}$ is Green’s function on $\mathbb{D}$, where $\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}$ is the Möbius transformation of $\mathbb{D}$. For a subarc $I \subseteq \partial\mathbb{D}$, let $S(I)$ be the Carleson box based on $I$ with

$$S(I) = \{ z \in \mathbb{D} : 1 - I \leq |z| < 1, \frac{z}{|z|} \in I \},$$

where $|I| = \frac{1}{2\pi} \int_I |d\xi|$ is the normalized length of the subarc $I$ of $\partial\mathbb{D}$. If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. Let $\mu$ be a nonnegative Borel measure on $\mathbb{D}$. We say that $\mu$ is a Carleson measure on $\mathbb{D}$ if

$$\|\mu\|^2 = \sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty.$$

Here and henceforth $\sup_{I \subseteq \partial\mathbb{D}}$ indicates the supremum taken over all subarcs $I$ of $\partial\mathbb{D}$.

As usual, $H^\infty$ is the set of bounded analytic functions in $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathcal{B}$, if (see [22])

$$\|f\|_\beta = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$
\(B\) is a Banach space under the norm \(\|f\|_B = |f(0)| + \|f\|_A\). The little Bloch space, denoted by \(B_0\), is the closed subspace of \(B\) consisting of functions \(f\) with 
\[ \lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0. \]
For \(0 < p < \infty\) and \(\alpha > -1\), the weighted Bergman space, denoted by \(A^p_\alpha\), is the set of all functions \(f \in H(\mathbb{D})\) satisfying 
\[ \|f\|_{A^p_\alpha} = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty, \]
where \(dA\) is the normalized Lebesgue area measure in \(\mathbb{D}\) such that \(A(\mathbb{D}) = 1\). The weighted Dirichlet space \(D^p_\alpha\) consists of those \(f \in H(\mathbb{D})\) such that \(f' \in A^p_\alpha\). Hence, for \(f \in D^p_\alpha\) we have 
\[ \|f\|_{D^p_\alpha} = \|f(0)\|^p + \|f'\|_{A^p_\alpha} < \infty. \]
It is well known that \(A^p_\alpha = D^p_{\alpha + p}\) (see, e.g., [22]).

For \(0 < p < \infty\), the Hardy space \(H^p\) consists of all \(f \in H(\mathbb{D})\) such that 
\[ \|f\|_{H^p} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty. \]

Let \(K : [0, \infty) \to [0, \infty)\) be a right-continuous and nondecreasing function. The analytic Morrey type space, denoted by \(H^2_K\), is the space of all analytic functions \(f \in H^2\) on \(\mathbb{D}\) such that 
\[ \|f\|_{H^2_K}^2 = \sup_{I \subseteq \partial \mathbb{D}} \frac{1}{K(|I|)} \int_I |f(\zeta) - f(I)|^2 \frac{|d\zeta|}{2\pi} < \infty, \]
where 
\[ f(I) = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}, I \subseteq \partial \mathbb{D}. \]
See [20] for more information of the Morrey type space \(H^2_K\). When \(K(t) = t\), it gives the \(BMOA\) space. It is well known that \(f \in BMOA\) if and only if 
\[ \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) < \infty. \]
When \(K(t) = t^\lambda (\lambda \in (0, 1))\), \(H^2_K\) is the Morrey space \(L^{2,\lambda}\), which was studied by Wu and Xie in [18].

Let \(S(\mathbb{D})\) denote the set of all analytic self-maps of \(\mathbb{D}\). Let \(\varphi \in S(\mathbb{D})\). Let \(Z\) denote the set of nonnegative integer. For \(f \in H(\mathbb{D})\), the composition operator \(C_\varphi\) on \(\mathbb{D}\) is defined by 
\[ C_\varphi(f) = f \circ \varphi. \]
The operator \(C_{\varphi,D^n}\) is defined by \(C_{\varphi,D^n}f = f^{(n)} \circ \varphi\), where \(n \in \mathbb{Z}\). If \(n = 0\), we get the composition operator \(C_\varphi\). If \(n = 1\), we get the operator \(C_{\varphi,D}\), which was studied in [4, 5, 6, 7, 8, 14, 15, 17, 24, 25]. In [13], Smith and Zhao characterized the boundedness and compactness of \(C_\varphi : B \to \mathcal{Q}_p\). In [19], Wulan characterized the boundedness and compactness of \(C_\varphi : B \to \mathcal{Q}_K\). In [9], Lindström etc. gave an asymptotic formula for the essential norm of the operator \(C_\varphi : B \to \mathcal{Q}_p\). In [12], Rättyä gave an asymptotic formula for
the essential norm of a composition operator $C_\varphi : \mathcal{D}_\alpha^p \to \mathcal{Q}_p$. Recall that the essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators $K$ mapping $X$ into $Y$, that is,

$$\|T\|_{e,X \to Y} = \inf\{\|T - K\|_{X \to Y} : K \text{ is compact}\},$$

where $X$ and $Y$ are Banach spaces, $\| \cdot \|_{X \to Y}$ is the operator norm.

In this paper, we study the boundedness, compactness and essential norm of products of differentiation and composition operators $C_\varphi D^n$ from the Bloch space and weighted Dirichlet spaces to analytic Morrey type spaces.

Throughout this paper we need some constraints on $K$. Let $\varphi_K$ be defined by

$$\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$  

By [3], we may suppose that $K$ is defined on $[0, 1]$ and extend its domain to $[0, \infty)$ by setting $K(t) = K(1)$ for $t > 1$, $K(t) \approx K(2t)$ and

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \quad \text{and} \quad \int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty.$$  

We shall also use the following standard notation: $f \lesssim g$ means that there is a constant $C$ independent of the relevant variables such that $f \leq Cg$, and $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$.

### 2. Characterization of the operator $C_\varphi D^n : \mathcal{B} \to H^2_K$

**Lemma 2.1 ([20]).** Let $K$ satisfy the conditions in (2). Then the following are equivalent:

(a) $f \in H^2_K$;

(b) $\sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_\mathbb{D} |f'(z)|^2 g(z, a) dA(z) < \infty$;

(c) $\sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_\mathbb{D} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) < \infty$;

(d) $\sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z) < \infty$.

**Remark 2.1.** By Lemma 2.1, for $f \in H^2_K$, we have

$$\|f\|^2_{H^2_K} \approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_\mathbb{D} |f'(z)|^2 g(z, a) dA(z)$$

$$\approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_\mathbb{D} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z)$$

$$\approx \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z).$$
Theorem 2.1. Let $f \in B$ for all $z \in D$.

Proof. From Theorem 5.15 in [22], we see that the dual space of $B_0$ is $A_1$, the Bergman space. Proposition 1.2 in [2] dictates that $f_j$ converges weakly to 0 in $B_0$ if and only if $\sup_j \|f_j\|_B < \infty$ and $f_j \to 0$ pointwise in $D$. Now consider the sequence $\{f_j\}$ as belonging to $B$. It is easy to see that weak convergence in $B$ is equivalent to weak convergence in $B_0$. In one direction, restrict an arbitrary functional on $B$ to a functional on $B_0$; in the other direction, use the Hahn-Banach theorem to extend an arbitrary functional on $B_0$ to a functional on $B$.

Lemma 2.2 ([22]). For every positive integer $n$, $f \in B$ if and only if $\sup_{z \in D}(1 - |z|^2)^n|f^{(n)}(z)| < \infty$. Moreover, the following asymptotic relationship holds

$$\|f\|_B \approx \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{z \in D}(1 - |z|^2)^n|f^{(n)}(z)|.$$ 

The following lemma is widely known, but we can not find a proof for it. Here we give a complete proof.

Lemma 2.3. A sequence $\{f_j\}$ in $B_0$ converges weakly to 0 in $B$ if and only if $\sup_j \|f_j\|_B < \infty$ and $f_j \to 0$ pointwise in $D$.

Proof. Assume that $C_\varphi D^n : B \to H^*_K$ is bounded, we have

$$\sup_{a \in D} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_D \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) < \infty;$$

$$\sup_{a \in D} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_D \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |\sigma_a(z)|^2) dA(z) < \infty;$$

$$\sup_{I \subseteq D} \frac{1}{K(|I|)} \int_{S(I)} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |z|^2) dA(z) < \infty.$$ 

Proof. (a) $\Rightarrow$ (b). Assume that $C_\varphi D^n : B \to H^*_K$ is bounded, we have

$$\|C_\varphi D^n f\|_{H^*_K} \leq \|C_\varphi D^n\|\|f\|_B$$ 

for all $f \in B$. By [23] we may choose two Bloch functions $f_1$ and $f_2$ satisfying

$$\frac{1}{(1 - |z|^2)^{n+1}} \approx |f_1^{(n+1)}(z)| + |f_2^{(n+1)}(z)|, \quad z \in D.$$ 

So that

$$\frac{|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{n+1}} \approx |(f_1^{(n)} \circ \varphi)'(z)| + |(f_2^{(n)} \circ \varphi)'(z)|.$$
By elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\), we get
\[
\frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} \lesssim 2|(f_1^{(n)} \circ \varphi)'(z)|^2 + 2|(f_2^{(n)} \circ \varphi)'(z)|^2,
\]
which implies that
\[
\sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_B \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z)
\]
\[
\lesssim \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_B \left(|(f_1^{(n)} \circ \varphi)'(z)|^2 + |(f_2^{(n)} \circ \varphi)'(z)|^2\right) g(z, a) dA(z)
\]
\[
\lesssim \|C_\varphi D^n\|^2(\|f_1\|^2 + \|f_2\|^2) < \infty,
\]
as desired.

(b)\Rightarrow(a). Let \(f \in B\). By the assumption, Remark 2.1 and Lemma 2.2, we have
\[
\|C_\varphi D^n f\|^2_{H^2_K} \simeq \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_B \frac{|(f^{(n)} \circ \varphi)'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z)
\]
\[
= \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_B \frac{|f^{(n+1)}(\varphi(z))^2(1 - |\varphi(z)|^2)^{2(n+1)}}{(1 - |\varphi(z)|^2)^{2(n+1)}} |\varphi'(z)|^2 g(z, a) dA(z)
\]
\[
\lesssim \|f\|^2_{B^2} \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_B \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z)
\]
\[
< \infty.
\]
Thus \(C_\varphi D^n : B \to H^2_K\) is bounded.

(a)\Rightarrow(d). Assume that \(C_\varphi D^n : B \to H^2_K\) is bounded. Fix an arc \(I \subset \partial D\) and consider the test function
\[
f_{m_2, \theta}(z) = \sum_{k=1}^{\infty} 2^k \left(\frac{2k + 2m_2 - 1}{2k + 2m_2 - n} e^{i\theta}\right)^2 z^{2k + 2m_2 - n},
\]
for \(m_2 \in \mathbb{N}\) such that \(2^{m_2} - n \geq 0\) and \(\theta \in [0, 2\pi)\). It is easy to check that \(\|f_{m_2, \theta}\|_B < \infty\). By Fubini’s theorem we have
\[
\int_0^{2\pi} \left\|C_\varphi D^n f_{m_2, \theta}\right\|^2_{H^2_K} \frac{d\theta}{2\pi} \lesssim \frac{1}{K(|I|)} \int_{S(I)} |\varphi'(z)|^2 (1 - |z|^2)^2 \left\{ \int_0^{2\pi} |f_{m_2, \theta}^{(n+1)}(\varphi(z))^2| d\theta \right\} dA(z)
\]
for all \(m_2 \in \mathbb{N}\). Parseval’s formula gives
\[
\int_0^{2\pi} |f_{m_2, \theta}^{(n+1)}(\varphi(z))^2| \frac{d\theta}{2\pi} = |\varphi(z)|^{2(2m_2 - n)} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} 2^{k(n+1)} e^{2k\theta} \varphi(z)^{2k - 1} \right|^2 \frac{d\theta}{2\pi}
\]
\[
= |\varphi(z)|^{2m_2 + 1 - 2n} \sum_{k=1}^{\infty} 2^{2k(n+1)} |\varphi(z)|^{2(k - 1)}.\]
By the formula (3.8) in [10], it is obvious that when \(|\varphi(z)| > \frac{1}{\sqrt{2}}\), we have
\[
\sum_{k=1}^{\infty} 2^{2k(n+1)} |\varphi(z)|^{2(2^{k} - 1)} \gtrsim \frac{1}{(1 - |\varphi(z)|^{2})^{2(n+1)}}.
\]
Hence we obtain
\[
(3) \quad \frac{1}{K(I)} \int_{S(I) \cap \{|\varphi(z)| > \frac{1}{\sqrt{2}}\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^{2})^{2(n+1)}}(1 - |z|^2)dA(z) < \infty
\]
for any \(I \subset \partial \mathbb{D}\). Since \(C_{\varphi}D^{n} : \mathbb{B} \rightarrow H_{K}^{2}\) is bounded, applying the operator \(C_{\varphi}D^{n}\) to \(z^{n+1}\), we obtain \(\varphi \in H_{K}^{2}\). Thus
\[
(4) \quad \frac{1}{K(I)} \int_{S(I) \cap \{|\varphi(z)| \leq \frac{1}{\sqrt{2}}\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^{2})^{2(n+1)}}(1 - |z|^2)dA(z) < \infty
\]
for any \(I \subset \partial \mathbb{D}\). Inequalities (3) and (4) show that
\[
\sup_{I \subset \partial \mathbb{D}} \frac{1}{K(I)} \int_{S(I) \cap \{|\varphi(z)| \leq \frac{1}{\sqrt{2}}\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^{2})^{2(n+1)}}(1 - |z|^2)dA(z) < \infty.
\]
The proof of (d)\(\Rightarrow\)(a) is similar to (b)\(\Rightarrow\)(a), (a)\(\Rightarrow\)(c) is similar to (a)\(\Rightarrow\)(b) and (c)\(\Rightarrow\)(a) is similar to (b)\(\Rightarrow\)(a). Hence the proof of these are omitted. The proof is completed. \(\Box\)

**Theorem 2.2.** Let \(\varphi\) be an analytic self-map of \(\mathbb{D}\), \(n \in \mathbb{Z}\) and \(K\) satisfy the conditions in (2). Suppose that \(C_{\varphi}D^{n} : \mathbb{B} \rightarrow H_{K}^{2}\) is bounded. Then
\[
\|C_{\varphi}D^{n}\|_{\mathbb{B} \rightarrow H_{K}^{2}} \approx \sqrt{A} \approx \sqrt{B} \approx \sqrt{U}.
\]
Here
\[
A = \limsup_{r \rightarrow 1} \sup_{a \in \mathbb{B}} K(1 - |a|^2) \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^{2})^{2(n+1)}}g(z,a)dA(z),
\]
\[
B = \limsup_{r \rightarrow 1} \sup_{a \in \mathbb{B}} K(1 - |a|^2) \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^{2})^{2(n+1)}}(1 - |\varphi(z)|^2)dA(z),
\]
\[
U = \limsup_{r \rightarrow 1} \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(I)} \int_{S(I) \cap \{|\varphi(z)| \geq r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^{2})^{2(n+1)}}(1 - |z|^2)dA(z).
\]

**Proof.** First we prove that
\[
\|C_{\varphi}D^{n}\|_{\mathbb{B} \rightarrow H_{K}^{2}} \gtrsim \sqrt{A}.
\]
Let \(\{\lambda_{m}\} \subset (1/2, 1)\) such that \(\lambda_{m} \rightarrow 1\) as \(m_{1} \rightarrow \infty\). Define
\[
\int_{m_{1}, m_{2}, \theta}(z) = z^{2m_{2}} \sum_{k=1}^{\infty} (2k^{2} + 2m_{2}) \cdot \cdots \cdot (2k^{2} + 2m_{2} - n) \left(\lambda_{m_{1}} e^{i\theta}\right)^{2k} z^{2k}
\]
\[
= \frac{1}{\lambda_{m_{1}}} \sum_{k=1}^{\infty} \frac{2k^{2}}{2k^{2} + 2m_{2} - 1} \cdot \cdots \cdot \frac{2k^{2} + 2m_{2} - n}{2k^{2} + 2m_{2} - n} \left(\lambda_{m_{1}} e^{i\theta}\right)^{2k} z^{2k}
\]
\[
= \frac{1}{\lambda_{m_{1}}} \sum_{k=1}^{\infty} \frac{2k^{2}}{2k^{2} + 2m_{2} - 1} \cdot \cdots \cdot \frac{2k^{2} + 2m_{2} - n}{2k^{2} + 2m_{2} - n} \left(\lambda_{m_{1}} e^{i\theta}\right)^{2k} z^{2k+2m_{2}}
\]
for $m_1, m_2 \in \mathbb{N}$ such that $2^{m_2} - n \geq 0$ and $\theta \in [0, 2\pi)$. Since
\[
0 \leq \lim_{k \to \infty} \left| \frac{2^k}{2^k + 2^{m_2}} \cdot \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (\lambda_{m_1} e^{i\theta})^{2^k} \right|
= \lim_{k \to \infty} \frac{2^k}{2^k + 2^{m_2}} \cdot \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (\lambda_{m_1})^{2^k}
\leq \lim_{k \to \infty} (\lambda_{m_1})^{2^k} = 0,
\]
the function $f_{m_1, m_2, \theta}$ belongs to $B_0$ by Theorem 1 of [21]. Moreover,
\[
\sup_{k \in \mathbb{N}} \left| \frac{2^k}{2^k + 2^{m_2}} \cdot \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (\lambda_{m_1} e^{i\theta})^{2^k} \right|
= \sup_{k \in \mathbb{N}} (\lambda_{m_1})^{2^k} \leq 1.
\]
The proof of Theorem 1 in [21] shows that there exists a positive constant $M$ such that $\|f_{m_1, m_2, \theta}\|_{\mathcal{B}} \leq M$ for all $m_1, m_2 \in \mathbb{N}$ such that $2^{m_2} - n \geq 0$ and $\theta \in [0, 2\pi)$. Define $g_{m_1, m_2, \theta} = f_{m_1, m_2, \theta}/M$. Then the sequence $\{g_{m_1, m_2, \theta}\}_{m_2=1}^{\infty}$ is contained in the closed unit ball of $B_0$. Moreover, $g_{m_1, m_2, \theta}$ tends to zero uniformly on compact subsets of $D$ for every $m_1$ and $\theta$ as $m_2 \to \infty$, and therefore $g_{m_1, m_2, \theta}$ tends to zero weakly as $m_2 \to \infty$ by Lemma 2.3. It follows that for any compact operator $T : B \to \mathcal{H}_K$,
\[
\|C_{\varphi} D^n - T\|_{B \to \mathcal{H}_K} \geq \limsup_{m_2 \to \infty} \sup_{m_1, \theta} \|C_{\varphi} D^n (g_{m_1, m_2, \theta})\|_{\mathcal{H}_K}
\geq \limsup_{m_2 \to \infty} \sup_{m_1, \theta} \|C_{\varphi} D^n (g_{m_1, m_2, \theta})\|_{\mathcal{H}_K} - \limsup_{m_2 \to \infty} \|T (g_{m_1, m_2, \theta})\|_{\mathcal{H}_K}
= \limsup_{m_2 \to \infty} \sup_{m_1, \theta} \|C_{\varphi} D^n (g_{m_1, m_2, \theta})\|_{\mathcal{H}_K}.
\]
Therefore, from the definition of the essential norm, we get
\[
\|C_{\varphi} D^n\|_{e, B \to \mathcal{H}_K}^2 = \frac{1}{T} \left| \|C_{\varphi} D^n - T\|_{B \to \mathcal{H}_K} \right|^2
\geq \frac{1}{M^2} \limsup_{m_1, \theta \to \infty} \sup_{a \in D} \left| 1 - |a|^2 \right| \int_D |f_{m_1, m_2, \theta} (\varphi(z))|^2 |\varphi'(z)|^2 g(z, a) dA(z).
\]
Given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that
\[
M^2 \|C_{\varphi} D^n\|_{e, B \to \mathcal{H}_K}^2 + \varepsilon \geq \frac{1}{(1 - |a|^2) K} \int_D |f_{m_1, m_2, \theta} (\varphi(z))|^2 |\varphi'(z)|^2 g(z, a) dA(z)
\]
for all $a$, $\theta$ and $m_1$ when $m_2 \geq N$. Let $a \in D$ be fixed. Integrating with respect to $\theta$, using Fubini’s theorem and Parseval’s formula, we obtain
\[
2\pi (M^2 \|C_{\varphi} D^n\|_{e, B \to \mathcal{H}_K}^2 + \varepsilon)
\geq \frac{1}{|1 - |a|^2|} \int_0^{2\pi} |f_{m_1, m_2, \theta} (\varphi(z))|^2 |\varphi'(z)|^2 g(z, a) dA(z)
\]
\[ \begin{align*}
&= \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{D} |\varphi(z)|^{2(2m+2-n)} \int_{0}^{2\pi} \left( \sum_{k=1}^{\infty} 2^{k(n+1)} \varepsilon^{2k+1} \right) |\lambda_{m_{1}} \varphi(z)|^{2^{k+1}-1} d\theta \\
&\times |\varphi'(z)|^{2} g(z,a) dA(z) \\
&= \frac{1 - |a|^2}{K(1 - |a|^2)} \\
&\times \int_{D} |\varphi(z)|^{2m+1-2n} \left( \sum_{k=1}^{\infty} 2^{k(n+1)} |\lambda_{m_{1}} \varphi(z)|^{2^{k+1}-1} \right) |\varphi'(z)|^{2} g(z,a) dA(z).
\end{align*} \]

By the formula (3.8) in [10], there exists a positive constant \( C \) such that

\[ \sum_{k=1}^{\infty} 2^{k(n+1)} |\lambda_{m_{1}} \varphi(z)|^{2^{k+1}-1} \geq \frac{C}{(1 - |\lambda_{m_{1}} \varphi(z)|^{2^{k+1}}} \]

for all \( z \in D \) with \( |\varphi(z)| > 1/2 \). Thus by Fatou’s Lemma, we get

\[ 2\pi (M^2 \| C_{\varphi} D^{n} \|_{e,B-H_{K}^{2}} + \varepsilon) \]

\[ \geq \liminf_{m_{1} \to \infty} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{D} |\varphi(z)|^{2m+1-2n} \left( \frac{|\varphi'(z)|^{2}}{(1 - |\lambda_{m_{1}} \varphi(z)|^{2^{k+1}}} g(z,a) dA(z) \right) \]

\[ \geq \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{D} |\varphi'(z)|^{2} g(z,a) dA(z) \]

\[ \geq \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{D} |\varphi(z)|^{2m+1} \left( \frac{|\varphi'(z)|^{2}}{(1 - |\lambda_{m_{1}} \varphi(z)|^{2^{k+1}}} g(z,a) dA(z) \right) \]

for all \( \varepsilon > 0 \). Therefore

\[ \| C_{\varphi} D^{n} \|_{e,B-H_{K}^{2}} \geq \sqrt{A} \]

A similar argument in the proof above shows that

\[ \| C_{\varphi} D^{n} \|_{e,B-H_{K}^{2}} \geq \sqrt{B} \quad \| C_{\varphi} D^{n} \|_{e,B-H_{K}^{2}} \geq \sqrt{C} \]

Next we prove that

\[ \| C_{\varphi} D^{n} \|_{e,B-H_{K}^{2}} \leq \sqrt{A} \]

For \( j \in \mathbb{N} \), define \( K_{j}(f) = K_{\psi_{j}}(f) \), where \( \psi_{j}(z) = \frac{j}{j+1} \), \( i.e., \ K_{j} f(z) = f(\frac{j}{j+1} z) \), \( z \in D \). Since the operator \( K_{j} \) is compact on \( B \) for all \( j \in \mathbb{N} \) (see [9]), and \( C_{\varphi} D^{n} : B \to H_{K}^{2} \) is bounded, it follows that

\[ \| C_{\varphi} D^{n} \|_{e,B-H_{K}^{2}} \leq \| C_{\varphi} D^{n} - C_{\varphi} D^{n} K_{j} \|_{B-H_{K}^{2}} \leq \| C_{\varphi} D^{n}(I - K_{j}) \|_{B-H_{K}^{2}} \]


\[
\begin{align*}
\approx & \sup_{\|f\|_{\|\cdot\| \leq 1}} \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int f - f \circ \psi_j)^{(n+1)}(\varphi(z)) |\varphi'(z)|^2 g(z,a) dA(z) \\
\leq & \sup_{\|f\|_{\|\cdot\| \leq 1}} \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) |\varphi'(z)|^2 g(z,a) dA(z) \\
& + \sup_{\|f\|_{\|\cdot\| \leq 1}} \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) |\varphi'(z)|^2 g(z,a) dA(z) \\
= & I_1 + I_2
\end{align*}
\]

for all \( r \in (0, 1) \) and \( j \in \mathbb{N} \), where \( \text{Id}(f) = f \) and

\[
I_1 = \sup_{\|f\|_{\|\cdot\| \leq 1}} \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) |\varphi'(z)|^2 g(z,a) dA(z)
\]

and

\[
I_2 = \sup_{\|f\|_{\|\cdot\| \leq 1}} \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) |\varphi'(z)|^2 g(z,a) dA(z).
\]

Since \( C_\varphi D^n : B \to H_K^2 \) is bounded, from the proof of Theorem 2.1 we see that \( \varphi \in H_K^2 \), and hence

\[
\tilde{K} = \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int |\varphi'(z)|^2 g(z,a) dA(z) < \infty.
\]

Since \( f - f \circ \psi_j \) and its derivative tend to zero uniformly in a compact subset of \( \mathbb{D} \) as \( j \to \infty \), it follows that

\[
I_1 \leq \tilde{K} \limsup_{j \to \infty} \sup_{\|f\|_{\|\cdot\| \leq 1}} \sup_{\|\varphi(\cdot)\| \leq r} \|f - f \circ \psi_j\|_{(n+1)} = 0.
\]

Now we estimate \( I_2 \). Since

\[
\|f - f \circ \psi_j\|_B \leq \|f\|_B + \|f \circ \psi_j\|_B \leq 2 \|f\|_B \leq 2,
\]

by Lemma 2.2 we get

\[
I_2 \leq \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z,a) dA(z).
\]

Consequently,

\[
\begin{align*}
\|C_\varphi D^n\|_{c,B \to H_K^2}^2 & \leq \limsup_{j \to \infty} \|C_\varphi D^n - C_\varphi D^n K_j\|_{B \to H_K^2}^2 \leq \limsup_{j \to \infty} I_1 + \limsup_{j \to \infty} I_2 \\
& \leq \sup_{a \in B} \frac{1 - |a|^2}{K(1 - |a|^2)} \int \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z,a) dA(z)
\end{align*}
\]

for all \( r \in (0, 1) \). Thus \( \|C_\varphi D^n\|_{c,B \to H_K^2}^2 \leq \sqrt{A} \). A similar argument shows that \( \|C_\varphi D^n\|_{c,B \to H_K^2}^2 \leq \sqrt{B} \).
Finally we prove that \(|C_\varphi D^n||_{C_2B \rightarrow H^2_K} \lesssim \sqrt{T}\). By Lemma 2.1 we have
\[
|C_\varphi D^n(Id - K_j)f||_{H^2_K}^2 \\
\approx \sup_{I \subset \partial \mathcal{B}} \frac{1}{K(|I|)} \int_{S(I)} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z)
\]
\[
\leq \sup_{I \subset \partial \mathcal{B}} \frac{1}{K(|I|)} \int_{S(I) \cap \{ |\varphi(z)| \leq r \}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z)
+ \sup_{I \subset \partial \mathcal{B}} \frac{1}{K(|I|)} \int_{S(I) \cap \{ |\varphi(z)| > r \}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z)
\]
for any \(r \in (0, 1)\). By Lemma 2.2 and (5) we obtain
\[
\sup_{I \subset \partial \mathcal{B}} \frac{1}{K(|I|)} \int_{S(I) \cap \{ |\varphi(z)| > r \}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z)
\]
\[
\lesssim \sup_{I \subset \partial \mathcal{B}} \frac{1}{K(|I|)} \int_{S(I) \cap \{ |\varphi(z)| \leq r \}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{(n+1)}(1 - |z|^2)} dA(z)
\]
for any \(r \in (0, 1)\) and \(f \in \mathcal{B}\) with \(\|f\|_{\mathcal{B}} \leq 1\). Now we only need to prove that
\[
\sup_{\|f\|_{\mathcal{B}} \leq 1} \frac{1}{K(|I|)} \int_{S(I) \cap \{ |\varphi(z)| \leq r \}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z) \rightarrow 0
\]
as \(j \rightarrow \infty\). We put \(v = \varphi(z)\) and denote the radial segment by \([\frac{1}{j+1}v, v]\). We obtain that
\[
\left| f^{(n+1)}(v) - f^{(n+1)}(\frac{j}{j+1}v) \right| \leq \frac{1}{j+1} |v||f^{(n+2)}(\xi(v))|
\]
for some \(\xi(v) \in [\frac{1}{j+1}v, v]\). An application of Cauchy’s estimate on the circle with center at \(\xi(v)\) and radius \(R \in (0, 1 - r)\) shows that
\[
|f^{(n+2)}(\xi(v))| \leq \frac{(n + 2)!}{R^{n+2}} \max_{|z| = R+r} |f(z)|.
\]
From the last two inequalities and the fact that \(\varphi \in H^2_K\), we get
\[
\sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{I \subset \partial \mathcal{B}} \frac{1}{K(|I|)} \int_{S(I) \cap \{ |\varphi(z)| \leq r \}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2
\]
\[
\times |\varphi'(z)|^2 (1 - |z|^2) dA(z)
\]
\[
\lesssim \frac{(n + 2)!r^2}{R^{n+2}(j+1)^2} \left( \log \frac{1}{1 - (R + r)} \right)^2 \|\varphi\|_{H^2_K} \rightarrow 0
\]
as \(j \rightarrow \infty\). Thus, we have
\[
\|C_\varphi D^n||_{C_2B \rightarrow H^2_K} \leq \lim \inf_{j \rightarrow \infty} \|C_\varphi D^n - C_\varphi D^n K_j\|_{B \rightarrow H^2_K}
\]
\[
= \lim \inf_{j \rightarrow \infty} \|C_\varphi D^n(Id - K_j)f||_{H^2_K}^2
\]
\[
\lesssim \sup_{I \subset \partial \mathcal{B}} \frac{1}{K(|I|)} \int_{S(I) \cap \{ |\varphi(z)| > r \}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{(n+1)}(1 - |z|^2)} dA(z)
\]
for any \( r \in (0, 1) \). Letting \( r \to 1 \), we obtain that \( \|C_\varphi D^n\|_{\mathcal{B} \to H^2_K} \lesssim \sqrt{U} \). The proof is completed. \( \square \)

From Theorem 2.2, we immediately get the following result.

**Theorem 2.3.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \), \( n \in \mathbb{Z} \) and \( K \) satisfy the conditions in (2). Suppose that \( C_\varphi D^n : \mathcal{B} \to H^2_K \) is bounded. Then the following statements are equivalent.

(a) \( C_\varphi D^n : \mathcal{B} \to H^2_K \) is compact;

(b) \[
\limsup_{r \to 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) = 0;
\]

(c) \[
\limsup_{r \to 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |\varphi_a(z)|^2) dA(z) = 0;
\]

(d) \[
\limsup_{r \to 1} \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |z|^2) dA(z) = 0.
\]

3. Characterization of the operator \( C_\varphi D^n : \mathcal{D}^p_\alpha \to H^2_K \)

In this section, we study the boundedness, compactness and the essential norm of the operator \( C_\varphi D^n : \mathcal{D}^p_\alpha \to H^2_K \). Hence, we first state some lemmas which will be used in the proofs of the main results in this section. The following result is Luecking’s characterization of Carleson measure in terms of functions in the Dirichlet type spaces (see [11]). In comparison with the original result, \( f^{(n)} \) has been replaced by \( f^{(n+1)} \) since this appears to be convenient for the purposes of the paper.

**Lemma 3.1** ([11]). Let \( \mu \) be a positive measure on \( \mathbb{D} \), \( 0 < p \leq 2 \) and \(-1 < \alpha < \infty \). Then \( \mu \) is a bounded \( \frac{2(2+\alpha)}{p} + 2n \)-Carleson measure if and only if there is a positive constant \( C \), depending only on \( \alpha, p \) and \( n \) such that

\[
\int_{\mathbb{D}} |f^{(n+1)}(z)|^2 d\mu(z) \leq C \|f\|_{\mathcal{D}^p_\alpha}^2
\]

for all \( f \in \mathcal{D}^p_\alpha \). Moreover, if \( \mu \) is a bounded \( \frac{2(2+\alpha)}{p} + 2n \)-Carleson measure, then \( C = C_1 C_2 \), where \( C_1 > 0 \) depends only on \( \alpha, p \) and \( n \) and

\[
C_2 = \sup_{I} \frac{\mu(S(I))}{|I|^{\frac{2(2+\alpha)}{p} + 2n}}.
\]
It is well-known that the bounded $t$-Carleson measure can be characterized by a global integral condition (see [1]), namely,

$$(6) \quad \sup_I \frac{\mu(S(I))}{|I|^t} = \sum_{k \in \mathbb{Z}} \left| a_k \right|^t d\mu(z), \quad 0 < t < \infty.$$ 

The following lemma is a partial boundary version of this result.

**Lemma 3.2** ([12]). Let $0 < r < 1, 1 \leq t < \infty$ and let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then

$$\sup_I \frac{\mu(S(I))}{|I|^t} \leq \sup_{|z| > r} \left| \int_{|z|}^{|z|} |\phi(z)|^t d\mu(z) \right|,$$

where $|\Delta(0, r)| = \{z : |z| < r\}$.

**Lemma 3.3** ([12]). Let $g$ and $u$ be positive measurable functions on $\mathbb{D}$, and let $\varphi \in S(\mathbb{D})$. Then

$$\int_{\mathbb{D}} (g \circ \varphi)(z)|\varphi'(z)|^2|u(z)|^2 dA(z) = \int_{\mathbb{D}} g(w)U(\varphi, w)dA(z),$$

where $U(\varphi, w) = \sum_{z \in \varphi^{-1}(w)} u(z)$ for $w \in \mathbb{D} \setminus \{\varphi(0)\}$.

For an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathbb{D}$, define

$$T_j f(z) = \sum_{k=0}^j a_k z^k, R_j f(z) = \sum_{k=j+1}^{\infty} a_k z^k.$$

**Lemma 3.4.** Let $1 < p < \infty, n \in \mathbb{Z}$ and $-1 < \alpha < \infty$. For each $w \in \mathbb{D}$, positive integer $j$ and $f \in D^p_\alpha$,

$$\left| (R_j f(w))^{(n+1)} \right| \leq \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \|f\|_{D^p_\alpha} \sum_{k=j}^{\infty} \frac{\Gamma(\alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} |w|^k,$$

where $\Gamma$ denotes the Gamma function.

**Proof.** Since $R_j f \in D^p_\alpha$, then $R_{j-1} f' \in A^p_\alpha$, we have

$$(R_j f)'(w) = \int_{\mathbb{D}} (R_j f)'(z)K_w(z)dA_\alpha(z),$$

where $K_w(z)$ is the Bergman Kernel function. Thus

$$(R_j f)^{(n+1)}(w) = \int_{\mathbb{D}} R_{j-1} f'(z) \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \frac{z^n}{(1 - \bar{w}z)^{\alpha + 2 + n}} dA_\alpha(z).$$

The orthogonality of monomials $z^\gamma$ with respect to $dA_\alpha$ shows

$$\int_{\mathbb{D}} R_{j-1} f'(z) \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \frac{z^n}{(1 - \bar{w}z)^{\alpha + 2 + n}} dA_\alpha(z)$$

$$= \int_{\mathbb{D}} f'(z) \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} R_{j-1} \left( \frac{z^n}{(1 - \bar{w}z)^{\alpha + 2 + n}} \right) dA_\alpha(z).$$
By Hölder inequality, we get
$$\left| (R_j f)^{(n+1)}(w) \right| \leq \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \int_D |f' (z)| dA_n(z) \approx \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \int_D |f' (z)| \sum_{k=0}^{\infty} \frac{\Gamma\left(k+n+\alpha+2\right)}{k!\Gamma(n+\alpha+2)} |\partial^{k+n} w| z^k dA_n(z)$$
$$\approx \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \int_D |f' (z)| \sum_{k=0}^{\infty} \frac{\Gamma\left(k+n+\alpha+2\right)}{k!\Gamma(n+\alpha+2)} |\partial^{k+n} w| z^k dA_n(z)$$
$$\leq \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \|f\|_{D^p} \int D \left( \sum_{k=0}^{\infty} \frac{\Gamma\left(k+n+\alpha+2\right)}{k!\Gamma(n+\alpha+2)} |\partial^{k+n} w| z^k \right)^{q} dA_n(z)$$
$$\leq \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \|f\|_{D^p} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+n+\alpha+2\right)}{k!\Gamma(n+\alpha+2)} |\partial^{k+n} w| z^k.$$  □

**Theorem 3.1.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$, $1 < p \leq 2, -1 < \alpha < \infty, n \in \mathbb{Z}$. Assume that $K$ satisfy the conditions in (2). Then the following statements are equivalent.

(a) $C_\varphi D^n : D^p_\alpha \to H^2_K$ is bounded,
(b) $$\sup_{a,b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_D |\sigma_b^\prime (\varphi (z))| \left( \frac{2(2+n)}{p} + 2n \right) |\varphi' (z)|^2 |\partial_a (z)| \partial_a (z) dA(z) < \infty;$$
(c) $$\sup_{a,b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_D |\sigma_b^\prime (\varphi (z))| \left( \frac{2(2+n)}{p} + 2n \right) |\varphi' (z)|^2 (1 - |\sigma_a (z)|^2) dA(z) < \infty.$$

**Proof.** The proof of (a)⇒(b) is similar to (a)⇒(b). Hence we only prove (a)⇒(b).

(b)⇒(a). Let $f \in D^p_\alpha$. By Lemma 3.1, (6) and Lemma 3.3 we have

$$\|C_\varphi D^n f\|_{H^2_K} \approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_D |f^{(n)}(z)\varphi(z)|^2 g(z,a) dA(z)$$
$$= \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_D |f^{(n+1)}(w)|^2 d\mu_a(w)$$
$$\leq \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \frac{\|f^{(n+1)}\|_{D^p_\alpha}^2}{|\sigma_b^\prime (w)| \left( \frac{2(2+n)}{p} + 2n \right) |\varphi' (z)|^2 |\partial_a (z)|}$$
$$\approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_{b \in \mathbb{D}} \int_D |\sigma_b^\prime (w)| \left( \frac{2(2+n)}{p} + 2n \right) |\varphi' (z)|^2 |\partial_a (z)| dA(z)\|f\|_{D^p_\alpha}^2$$
$$= \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_{b \in \mathbb{D}} \int_D |\sigma_b^\prime (\varphi (z))| \left( \frac{2(2+n)}{p} + 2n \right) |\varphi' (z)|^2 |\partial_a (z)| dA(z)\|f\|_{D^p_\alpha}^2$$
Let $Theorem 3.2.$

and $|b| for all $n \in \mathbb{Z}$. Assume that $C_\varphi D^n : \mathbb{D}^p \to H^2_K$ is bounded. Let $b \in \mathbb{D}$. Set 

$$f_b(z) = \int_0^z \left(1 - \frac{|b|^2}{1 - bw}\right)^{\frac{4+2\alpha}{p}} dw, \quad z \in \mathbb{D}.$$ 

Then $\|f_b\|_{\mathbb{D}^p} = 1$ for all $b \in \mathbb{D}$. Let $\zeta \in \partial \mathbb{D}$ be the center of arc $I \subset \partial \mathbb{D}$ and \( b = (1 - |f|)\zeta \in \mathbb{D}. \) Then 

$$f_b^{(n+1)}(z) = \Gamma\left(\frac{4+2\alpha}{p} + n\right) \frac{(1 - |b|^2)^{\frac{4+2\alpha}{p}} b^n}{(1 - bz)^{\frac{4+2\alpha}{p} + n}}$$ 

and $|f_b^{(n+1)}(z)| \geq \frac{1}{(1 - |b|)^{\frac{4+2\alpha}{p} + 2n}}$, $z \in S(I)$. Thus 

$$\infty > \|C_\varphi D^n\|^2 \|f_b\|_{\mathbb{D}^p}^2 \geq \|C_\varphi D^n f_b\|_{H^2_K}^2 \approx \sup_{a,b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(f_b^{(n)} \circ \varphi)(z)|^2 g(z,a) dA(z)$$ 

$$= \sup_{a,b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 g(z,a) dA(z)$$ 

$$\approx \sup_{a,b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(w)|^2 d\mu_a(w)$$ 

$$\leq \sup_{a,b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{S(I)} \frac{1}{(1 - |b|)^{\frac{4+2\alpha}{p} + 2n}} d\mu_a(w)$$ 

for all $b \in \mathbb{D}$. By (6) and Lemma 3.3 we have 

$$\infty > \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \frac{\int_{S(I)} d\mu_a(w)}{1 - |f|}$$ 

$$\approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(w)|^{\frac{4+2\alpha}{p}} dw d\mu_a(w)$$ 

$$= \sup_{a,b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{4+2\alpha}{p}} + 2n |\varphi'(z)|^2 g(z,a) dA(z).$$ 

This completes the proof of this theorem. \( \square \)

**Theorem 3.2.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$, $1 < p \leq 2, -1 < \alpha < \infty, n \in \mathbb{Z}$. Assume that $K$ satisfy the conditions in (2). Suppose that $C_\varphi D^n$ :
$D_\alpha^p \rightarrow H^2_{\mathbb{K}}$ is bounded. Then
\[ \|C_\varphi D^a\|_{c,D_\alpha^p \rightarrow H^2_{\mathbb{K}}} \approx \sqrt{P} \approx \sqrt{Q}. \]

Here
\[ P = \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\alpha'(\varphi(z))|^{\frac{2(\alpha + 1)}{p} + 2n} |\varphi(z)|^2 g(z, a) dA(z), \]
\[ Q = \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\alpha'(\varphi(z))|^{\frac{2(\alpha + 2n)}{p} + 2n} |\varphi(z)|^2 (1 - |\sigma(z)|^2) dA(z). \]

**Proof.** We only need to prove that $\|C_\varphi D^a\|_{c,D_\alpha^p \rightarrow H^2_{\mathbb{K}}} \approx \sqrt{P}$. Since the proof for $\|C_\varphi D^a\|_{c,D_\alpha^p \rightarrow H^2_{\mathbb{K}}} \approx \sqrt{Q}$ is similar.

First we prove that $\|C_\varphi D^a\|_{c,D_\alpha^p \rightarrow H^2_{\mathbb{K}}} \geq \sqrt{P}$. Let $b \in \mathbb{B}$. Set
\[ f_b(z) = \int_0^z \left( \frac{1 - |b|^2}{(1 - bw)^2} \right)^{\frac{\alpha + 2}{2}} dw, \quad z \in \mathbb{D}. \]

We have $\|f_b\|_{D_p^b} = 1$ and $f_b \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|b| \to 1$. Since $D_\alpha^p$ is reflexive, we see that $f_b \to 0$ weakly in $D_\alpha^p$ as $|b| \to 1$. Thus $\|J(f_b)\|_{H^2_{\mathbb{K}}} \to 0$ as $|b| \to 1$ for every compact operator $J : D_\alpha^p \to H^2_{\mathbb{K}}$. Hence
\[ \|C_\varphi D^a - J\|^2_{D_\alpha^p \rightarrow H^2_{\mathbb{K}}} \geq \limsup_{|b| \rightarrow 1} \|C_\varphi D^a(f_b) - J(f_b)\|^2_{H^2_{\mathbb{K}}} \]
\[ \geq \limsup_{|b| \rightarrow 1} \|C_\varphi D^a(f_b)\|^2_{H^2_{\mathbb{K}}} - \limsup_{|b| \rightarrow 1} \|J(f_b)\|^2_{H^2_{\mathbb{K}}} \]
\[ = \limsup_{|b| \rightarrow 1} \|C_\varphi D^a(f_b)\|^2_{H^2_{\mathbb{K}}} \]
for every compact operator $J : D_\alpha^p \to H^2_{\mathbb{K}}$. By (3) and Lemma 3.3 we have
\[ \limsup_{|b| \rightarrow 1} \|C_\varphi D^a(f_b)\|^2_{H^2_{\mathbb{K}}} \approx \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(f_b^{(n)} \circ \varphi)'(z)|^2 g(z, a) dA(z) \]
\[ = \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 g(z, a) dA(z) \]
\[ = \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(w)|^2 d\mu_a(w) \]
\[ = \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \left( \frac{\Gamma(\frac{4+2a}{p} + n) \left(1 - |b|^2\right)^{\frac{\alpha + 2}{2}} b^n}{\Gamma(\frac{4+2a}{p}) \left(1 - bw\right)^{\frac{2n + 2n}{p} + n}} \right)^2 d\mu_a(w) \]
\[ \approx \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{S(t)} \left(1 - |b|^2\right)^{\frac{2n + 2n}{p} + 2n} d\mu_a(w) \]
\[ \approx \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{B}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'(w)|^{\frac{2(\alpha + 2n)}{p} + 2n} d\mu_a(w) \]
Hence, from the definition of the essential norm, we obtain
\[ \| C_\varphi D^n \|_{e, D^k_\infty \rightarrow H^2_K} = \inf \| C_\varphi D^n - J \|_{D^k_\infty \rightarrow H^2_K}^2 \gtrsim P. \]

Next we prove that \( \| C_\varphi D^n \|_{e, D^k_\infty \rightarrow H^2_K} \lesssim \sqrt{P} \). For an analytic function \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) in \( \mathbb{D} \), since \( T_j \) is compact on \( D^k_\infty \), we have
\[
\| C_\varphi D^n \|_{e, D^k_\infty \rightarrow H^2_K} = \| C_\varphi D^n(T_j + R_j) \|_{e, D^k_\infty \rightarrow H^2_K} \\
\leq \| C_\varphi D^n T_j \|_{e, D^k_\infty \rightarrow H^2_K} + \| C_\varphi D^n R_j \|_{e, D^k_\infty \rightarrow H^2_K} \\
= \| C_\varphi D^n R_j \|_{e, D^k_\infty \rightarrow H^2_K} \\
\leq \| C_\varphi D^n R_j \|_{D^k_\infty \rightarrow H^2_K}.
\]

Hence
\[
\| C_\varphi D^n \|_{e, D^k_\infty \rightarrow H^2_K} \leq \lim_{j \to \infty} \| C_\varphi D^n R_j \|_{D^k_\infty \rightarrow H^2_K}.
\]

Therefore, by Lemma 3.3, we get
\[
\| C_\varphi D^n \|_{e, D^k_\infty \rightarrow H^2_K} \leq \lim_{j \to \infty} \inf \| C_\varphi D^n R_j \|_{D^k_\infty \rightarrow H^2_K}^{2} \\
\leq \lim_{j \to \infty} \inf \sup_{\| f \|_{D^k_\infty} \leq 1} \| C_\varphi D^n (R_j f) \|_{H^2_K}^{2} \\
\approx \lim_{j \to \infty} \inf \sup_{\| f \|_{D^k_\infty} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(R_j f)(^{(n)} \circ \varphi)'(z)|^2 g(z, a) dA(z) \\
(7) = \lim_{j \to \infty} \inf \sup_{\| f \|_{D^k_\infty} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(R_j f)(^{(n+1)})(w)|^2 d\mu_a(w).
\]

Let \( r \in (0, 1) \). For each \( a \in \mathbb{D} \) and \( f \in D^k_\infty \), by Lemma 3.4 we have
\[
\int_{|w| \leq r} |(R_j f)(^{(n+1)})(w)|^2 d\mu_a(w) \\
\lesssim \| f \|_{D^k_\infty}^2 \int_{|w| \leq r} \left( \sum_{k=j}^{\infty} \frac{\Gamma(k + \alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} |w|^k \right)^2 d\mu_a(w) \\
\leq \| f \|_{D^k_\infty}^2 \left( \sum_{k=j}^{\infty} \frac{\Gamma(k + \alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} \right)^2 \int_{|w| \leq r} d\mu_a(w).
\]

Since \( \varphi \in H^2_K \), by Lemma 3.3 we have
\[
\sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{|w| \leq r} d\mu_a(w) = \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{|w(z)| \leq r} |\varphi'(z)|^2 g(z, a) dA(z) \\
\approx \| C_\varphi D^n (z^{n+1}) \|_{H^2_K}^2 < \infty.
\]
It is well known that \( \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+2+n)}{\Gamma(\alpha+2+n)} r^k \approx \frac{1}{(1-r)^{\alpha+2+n}} \) for any \( r \in (0,1) \). Hence

\[
\liminf_{j \to \infty} \sup_{\|f\|_{D^n_p} \leq 1} e^{aD} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{|w| \leq r} \|R_j f\|^{(n+1)}(w) \, d\mu_a(w) = 0.
\]

We now estimate \( \int_{|w| > r} \|R_j f\|^{(n+1)}(w) \, d\mu_a(w) \). By Lemmas 3.1, 3.2 and 3.3 we obtain

\[
\int_{|w| > r} \|R_j f\|^{(n+1)}(w) \, d\mu_a(w) \\
\leq \|R_j f\|_{D^n_p} \sup_{|h| \geq r} \int_{D} |\sigma'_b(\varphi(z))|^2 g(z,a) \, dA(z).
\]

Using (7), (8) and (9), for any \( r \in (0,1) \) we get

\[
\|C_p D^n\| \leq \liminf_{j \to \infty} \sup_{\|f\|_{D^n_p} \leq 1} e^{aD} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{|w| > r} \|R_j f\|^{(n+1)}(w) \, d\mu_a(w) \\
\leq \sup_{\|f\|_{D^n_p} \leq 1} \|f\|_{D^n_p} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \sup_{|h| \geq r} \int_{D} |\sigma'_b(\varphi(z))|^2 g(z,a) \, dA(z).
\]

Taking the limit as \( r \to 1 \), we get the desired result. Thus we have

\[
\|C_p D^n\| \approx \sqrt{P}.
\]

The proof is complete. \( \square \)

From Theorem 3.2, we immediately get the following result.

**Theorem 3.3.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \), \( 1 < \rho \leq 2, -1 < \alpha < \infty, n \in \mathbb{Z} \). Assume that \( K \) satisfy the conditions in (2). Suppose that \( C_p D^n : D^n_p \to H^2_K \) is bounded. Then the following statements are equivalent.

(a) \( C_p D^n : D^n_p \to H^2_K \) is compact;

(b) \[
\limsup_{|b| \to 1} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{D} |\sigma'_b(\varphi(z))|^2 g(z,a) \, dA(z) = 0;
\]

(c) \( \|C_p D^n\| < \infty \).
\[ \limsup_{|a| \to 1} \sup_{a \in D} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_D |\sigma'_a(z)||\varphi'(z)|^2(1 - |\sigma_a(z)|^2) dA(z) = 0. \]

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