The dimensional dependence of naked singularity formation in spherical gravitational collapse

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Abstract
The complete spectrum of the endstates—naked singularities or black holes—of gravitational collapse is analyzed for a wide class of \(N\)-dimensional spacetimes in spherical symmetry, which includes and generalizes the dust solutions and the case of vanishing radial stresses. The final fate of the collapse is shown to be fully determined by the local behavior of a single scalar function and by the dimension \(N\) of the spacetime. In particular, the ‘critical’ behavior of the \(N = 4\) spacetimes, where a sort of phase transition from the black hole to the naked singularity can occur, is still present if \(N = 5\) but does not occur if \(N > 5\), independently of the initial data of the collapse. Physically, the results turn out to be related to the kinematical properties of the considered solutions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Understanding singularities has always been one of the most intriguing issues in general relativity since its beginning. The mathematical prediction that gravitational collapse may lead to singularity formation hugely increased the attention over the study of the last stage of a heavy stars’ life. Problems related to strong density regions of spacetimes are in need of an ultimate answer already and, over all, a satisfactory formulation of Penrose’s cosmic censorship conjecture [25]: the causal character, and the endstate, of singularities arising from a dynamical process such as an indefinite collapse is still one of the favorite test beds for relativity. A great amount of work in this direction has been done in the case of spherically symmetric four-dimensional spacetimes: a number of collapsing models have been analytically studied where, under suitable assumptions, the arising singularity is not completely hidden behind a horizon, also when the latter forms. For instance, the pioneering
work of Christodoulou [2] showed that it suffices to remove the homogeneity assumption from the paradigm of gravitational collapse leading to a black hole—i.e. the Oppenheimer–Snyder solution. These cases of naked singularities have been intensively explored, in particular Tolman–Bondi–Lemaître dust clouds (see [14] and references therein), and vanishing radial stress models [13, 19].

Recently, a class of new solutions has been found [6], including the above as particular cases, where naked singularities generically appear as an outcome of collapse. Physically, they describe the gravitational collapse of a class of anisotropic elastic materials, and are characterized by a particular choice of the equation of state that, in a certain coordinate system, allows us to reduce Einstein field equations to a quadrature. In this paper, we find a natural extension of this class of solutions to the case of general N-dimensional gravitational theory. The importance of higher dimensional models goes up e.g. to Kaluza–Klein theories, superstring theory and braneworld models—see in particular [12, 26], where a description of the world with more than four non-compact dimensions is proposed.

In this perspective, the present study is motivated by a number of earlier and more recent works on spherically symmetric higher dimensional spacetimes: [1, 16] extend earlier well-known results and properties of the four-dimensional scalar field collapse; the Vaidya–AdS four dimensional solution is generalized in [18, 22] to higher dimensions adding extra gravity terms to the action functional. Far from being exhaustive, more references on the subject of higher dimensional collapse are [3, 8, 9, 11, 17]. In particular, the class of solutions that we find again extends vanishing radial stress models as dust [4, 24]. We will find the complete spectrum of endstates, analyzing if and how it is modified by the dimension of the spacetime N. In particular, naked singularities will be proved to survive in any larger dimension, despite earlier results contained in [10, 20]—see the discussion at the end.

It is worth noting that some criticism arose of singularities occurring in astrophysical sources modeled with continuous media in the past, due to the fact that one can construct situations in which Newtonian systems made out of continua develop singularities. As a consequence, singularities in these models cannot be considered an exclusive product of general relativity. It is difficult, however, to assess to what extent this phenomenon denies validity to continuous models, although a simple remark once made by Seifert [27] may be of help: on taking this point of view, one could discard the big bang of the standard model as being an artifact of Newtonian gravity, since the Friedmann equation holds—formally unchanged—also for the Newtonian cosmological models.

This paper is organized as follows: section 2 is devoted to derivation and the class of exact solutions, and to briefly illustrating some particular cases. Physical reasonability conditions will also be imposed on the solution, together with conditions that will ensure formation of singularities, whose endstate will be analyzed in section 3. In section 4 we will show how to complete the model, matching the solution with a suitable exterior spacetime. In the final section we discuss the results found, relating them to kinematical properties of the spacetime.

2. The solution in area–radius coordinates

The general spherically symmetric line element in comoving coordinates \((t, r, \theta^i), i = 1, \ldots, N - 2\), is given by

\[
d s^2 = -e^{2\nu(t,r)} \, dt^2 + \eta(t,r)^{-1} \, dr^2 + R(t,r)^2 \, \Omega_{N-2}^2, \tag{1}
\]

where \(\Omega_{N-2}^2 \equiv \sum_{i=1}^{N-2} \left( \prod_{j=1}^{i-1} \sin^2 \theta^j \right) (d\theta^i)^2\). The source of the gravitational field will be given by an elastic material under isothermal conditions. Generalizing the \(N = 4\) case, the property of the source is encoded in a state function depending on the space–space part of
the metric, that is—using spherical symmetry assumption—\( w = w(r, R, \eta) \)\cite{15, 19}. The stress–energy tensor is given by

\[
T = -\epsilon \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + p_r \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + p_t \left( d\theta^i \otimes \frac{\partial}{\partial \theta^i} \right),
\]

(2)

where introducing the matter density

\[
\rho = (N-2)(8\pi E)^{-1}\sqrt{\eta}R^{-2}
\]

\((E\) is an arbitrary function of \(r\)), the internal energy \(\epsilon\) and the stresses \(p_r\) and \(p_t\) are given in terms of the state function by

\[
\epsilon = \rho w, \quad p_r = 2\rho \eta \frac{\partial w}{\partial \eta}, \quad p_t = -\frac{1}{N-2} \rho R \frac{\partial w}{\partial R}.
\]

(3)

Although the comoving coordinates usually yield the natural system to describe the physical evolution of the collapse, for our purposes, however, it will be convenient to introduce the area–radius coordinate system \((r, R, \theta^i)\), first introduced by Ori\cite{23} in the study of four-dimensional charged dust, in such a way that (1) becomes

\[
ds^2 = -A dr^2 - 2B dR dr - u^{-2} dR^2 + R^2 d\Omega_{N-2}^2,
\]

(4)

with \(A, B, u\) being unknown functions of \((r, R)\). In this way the internal energy will depend only on one field variable, \(\eta\), and on the two coordinates \(r, R\). We introduce the function

\[
\Delta = B^2 - Au^{-2} = \eta^{-1}u^{-2},
\]

so that the Einstein field equations \(G^r_r = 8\pi T^r_r, G^R_R = 8\pi T^R_R\) and \(G^r_R = 8\pi T^r_R\) can be expressed in terms of \(A, \Delta\) and \(u\) as follows:

\[
\left(1 - \frac{N}{2}\right) [(N-3)(1 - A/\Delta) - R(A/\Delta)_r] = 8\pi R^2 p_r,
\]

(5)

\[
\left(1 - \frac{N}{2}\right) R^{-1}(A/\Delta)_r = -8\pi \sqrt{\Delta + Au^{-2}u^{-2}(\epsilon + p_r)},
\]

(6)

\[
\sqrt{u^2 + A/\Delta(\sqrt{\Delta})_r} - (u^{-1})_r = 0.
\]

(7)

Equation (5) can be integrated to give \(A/\Delta\) in terms of \(p_r(r, R, \eta)\). Therefore, if one removes dependence on the comoving field variables, assuming that \(p_r\) in (3) satisfies

\[
\frac{\partial p_r(r, R, \eta)}{\partial \eta} = 0,
\]

or equivalently

\[
w = h(r, R) + \ell(r, R)\eta^{-1/2}
\]

(8)

with \(h, \ell\) arbitrary, then one obtains

\[
\Delta = \Delta(1 - 2\Psi R^{3-N})
\]

(9)

where, in view of (3) and (8), \(\Psi\) is the function

\[
\Psi(r, R) = F(r) + \frac{1}{E(r)} \int_{R_0(r)}^R \ell(r, \sigma) d\sigma,
\]

(10)

with \(F(r)\) an arbitrary function of \(r\), and \(R_0(r)\) describing \(r\) at initial (comoving) time that will be chosen equal to \(r\) hereafter. The function \(\Psi\) (10) is the Misner–Sharp mass of the system, defined by the relation \(1 - 2\Psi R^{3-N} = g(\nabla R, \nabla R)\). Now, introducing

\[
Y(r, R) = E(r)\Psi(r, R)h(r, R)^{-1},
\]

(11)
the field equations (6) and (7), in view of (3), (8) and (10), simply become respectively

$$u^2 = 2\Psi R^{3-N} - 1 + Y^2$$  \hspace{1cm} (12)

and $$(\sqrt{\Delta})_R + Y^{-1}(u^{-1})_r = 0$$ that can be integrated, using the initial condition, to give

$$\sqrt{\Delta}(r, R) = \int_r^R \frac{u_r(r, \sigma)}{Y(r, \sigma)u(r, \sigma)^2} \, d\sigma + \frac{1}{Y(r, r)u(r, r)}.$$  \hspace{1cm} (13)

Then, we conclude that the class of exact solutions found expresses all the metric unknown functions in (4) in terms of two arbitrary functions $(\Psi, Y)$ of $r$ and $R$.

We stress the fact that the constitutive function $w(r, R, \eta)$ as an equation of state, introduced at the beginning of this section, uniquely and completely carries on the physical properties of the matter, regardless of possible anisotropies. Isotropy of the matter is characterized when $w$ can be written as a function of the matter density $\rho$ only. In this case, $p_r = p_t$ and both can be seen as a function of the energy density $\epsilon$ only, as one can easily calculate from (3). When $w$ fails to be a function of $\rho$ only, anisotropy comes into play, but any other relation is not needed to close the system, because of equations (3). Another way to see this is to observe that equations (3) identically imply the conservation law arising from one of the Bianchi identities written in comoving coordinates, and again this is of course an outcome of having assumed that the source is an elastic continuum under isothermal conditions. Of course, the requirement given by (8) is exactly the state function characterizing the class of functions considered, and the fact that the arbitrary functions can be viewed in terms of the kinematical properties of the continuum is a very well-known consequence of the structure of the field equations within the assumed symmetries and holds for all the models of this kind.

2.1. Examples

The components of the stress–energy tensor are generically nonzero, as readily calculated from (3)–(10), and are given by

$$\epsilon = \frac{N - 2}{8\pi R^{N-2}} \left[ \sqrt{\eta} \Psi, r \right] + \Psi, R,$$  \hspace{1cm} (14)

$$p_r = -\frac{(N - 2) \Psi, R}{8\pi R^{N-2}},$$  \hspace{1cm} (15)

$$p_t = -\frac{\sqrt{\eta}}{8\pi R^{N-3}} \left( \Psi, R \right) \left[ \frac{Y, r}{Y} - \frac{Y, r \partial Y}{Y^2 \partial R} + \frac{\Psi, R}{\sqrt{\eta}} \right].$$  \hspace{1cm} (16)

From these expressions we can recognize some particular cases:

(i) dust spacetimes [10], occurring when both $\Psi$ and $Y$ are functions of $r$ only;
(ii) vanishing radial stress solutions [20] that occur when $\Psi$ is a function of $r$ only, but $Y$ may also depend on $R$;
(iii) acceleration-free solutions, when $Y$ is a function of $r$ only but $\Psi$ may also depend on $R$ (note that the norm of the acceleration is simply given by $Y, R$).

2.2. Energy condition and shell focusing singularity occurrence

On the above class of solutions, some conditions will be imposed, as requirements on $\Psi$ and $Y$. Since we would like to obtain global gravitational collapsing models, we will consider the interior metric (4) as defined on a right neighborhood $[0, r_b]$ of $r = 0$, for some $r_b > 0$, and match the above solutions at $r = r_b$ with some exterior spacetime to be defined later (see
section 4). For this reason, in the following we will consider \( \Psi \) and \( Y \) as defined on the set \( \{(r, R) : r \in [0, r_b], R \in [0, r]\} \).

As a physical reasonability condition, WEC on the metric (4) will be required, but in view of (14)–(16), it suffices that

\[
\Psi_{,r} \geq 0, \quad \Psi_{,R} \geq 0
\]

\[
(N - 2)\Psi_{,r} Y^{-1} \geq R(\Psi_{,r} Y^{-1}), R, \quad (N - 2)\Psi_{,R} \geq R\Psi_{RR}.
\]

Moreover, we impose the condition of decreasing initial energy, i.e. \( \epsilon_0(r) := \epsilon(r, r) \) must be a decreasing function of \( r \):

\[
\Psi_{,r}(r, r) + \Psi_{,RR}(r, r) + 2 \Psi_{,rR}(r, r) \leq \frac{(N - 2)}{r}[\Psi_{,r}(r, r) + \Psi_{,R}(r, r)].
\]

The functions \( \Psi \) and \( Y \) must be chosen in such a way that the spacetime is regular at initial (comoving) time, and a (shell focusing) singularity forms, for each shell \( r \in [0, r_b] \), in a finite amount of time. Therefore, the first shell crossing singularity formation must be avoided, and to this end it must be required that \( \sqrt{\Delta} > 0 \) when \( R \geq 0 \). By inspection of (13), sufficient conditions for this to happen are given by

\[
\sqrt{\Delta}(r, 0) > 0, \quad u_{,r}(r, R) > 0, \quad \forall r \in [0, r_b], R \in [0, r].
\]

Moreover, it must be observed that the use of the \((r, R, \theta^i)\) coordinate system has the obvious advantage of parametrizing the singularity with the straight line \( R = 0 \), but the drawback is that both the regular and the singular center are mapped into the point \( r = R = 0 \), and then it does not make a distinction between them, unless one does not consider the inverse function \( t = t(r, R) \). The function \( \dot{R} \), the derivative of \( R \) w.r.t comoving time, satisfies the identity \( u = -\dot{R} e^{-\nu} \), that can be formally integrated to give \( t(r, R) = \int_{\sigma}^{\sigma} e^{-\nu} u(r, \sigma) d\sigma \).

Although the integrand yet contains an unknown function in the comoving coordinates, a key remark at this stage is to observe that \( e^{-\nu} \) is bounded in a neighborhood of the center, which allows us to express the above conditions in terms of \( \Psi \) and \( Y \): it suffices that the function \( R^{N-3}u^2 \) is Taylor-expandable at the center \( (r = R = 0) \), with the expression given by

\[
R^{N-3}u^2 = \sum_{i+j=N-1} h_{ij}r^i R^j + \sum_{i+j=N-1+p} h_{ij}r^i R^j + \cdots. \tag{17}
\]

In particular, for the center to become singular in a finite time, it must be required that \( (h_{N-1,0}, h_{N-2,1}, \ldots, h_{1,N-2}) \neq 0 \). Hereafter, we will suppose, as already done in [6],

\[
\alpha := h_{N-1,0} \neq 0.
\]

Although this is a generic assumption, the results we are going to state can also be extended to the degenerate case \( \alpha = 0 \), as done in [29] for \( N = 4 \).

3. Naked singularity versus black hole formation

The endstate of the singularity for these models will be studied. First, let us observe that the central singularity is the only one that can be naked. Indeed, under the above assumptions, the apparent horizon \( R_h(r) \) is such that \( R_h(r) = \alpha \frac{1}{\sqrt{\nu}} r^{\frac{N-3}{2}} + o(r^{\frac{N-3}{2}}) \), and moreover, if \( t_h(r) \) and \( t_s(r) \) are the comoving times when the shell labeled \( r \) becomes trapped and singular, respectively, then \( \lim_{r \to 0} t_h(r) - t_s(r) = 0 \).

To analyze the endstate of the central singularity we will study the existence of null radial geodesics \( R_{g}(r) \) emanating from the (singular) center, such that \( R_{g}(r) > R_h(r) \) in a right
neighborhood of \( r = 0 \). To do this, we will use a remarkable property of \( R_b(r) \), to be a supersolution of the null radial geodesic equation

\[
\frac{dR}{dr} = u \sqrt{\Delta(Y - u)}. \tag{18}
\]

Therefore, to have the existence of such a \( R_y(r) \) as above, we will actually look for supersolutions of (18) of the form \( R_r(r) = x r^{\frac{N-2}{N-1}} \), with \( x > \alpha \frac{\sqrt{3}}{N-1} \), that therefore emanate from the singular center—so that \( R_y \) also will. Incidentally, this also explains why it suffices to look for radial curves: indeed, the projection of a nonradial geodesic on the plane would be a supersolution of (18), so if the singularity is nonradially naked, it is also radially naked.

As it happens for the \( N = 4 \) case, the endstate of the singularity is related to the Taylor expansion of the function

\[
\sqrt{\Delta}(r, 0) = \xi r^{n-1} + o(r^{n-1}), \tag{19}
\]

but also the dimension \( N \) of the spacetime will now play a crucial role. Indeed, the condition for the existence of \( R_y \) as above is equivalent to the existence of \( x > \alpha \frac{\sqrt{3}}{N-1} \) satisfying

\[
\frac{N-1}{N-3} x r^{\frac{N-2}{N-1}} < \left( 1 - \sqrt{\frac{\alpha}{x^{N-3}}} \right) \left( \frac{\alpha}{x^{N-3}} x r^{n-1} + x r^{\frac{N-2}{N-1}} \right). \tag{20}
\]

The above inequality gives the complete spectrum of the endstates since it provides a necessary and sufficient condition for the singularity to be naked. Indeed, if \( N = 4 \) one recovers the well-known results of [6] that the inequality holds—and hence the singularity is naked—if \( n = 1, 2 \), and if \( n = 3 \) a critical case happens when the endstate is related to the value of \( \xi \) in (19), since it must be \( 2 \xi > (26 + 5\sqrt{3}) \alpha \) for the singularity to be naked. In larger dimensions, the singularity is naked if \( n = 1, \forall N \), and if \( n = 2, N = 5 \), provided \( 2 \xi > 27\sqrt{3} \alpha \). In all other cases a black hole forms. Then we observe that the critical behavior, when a phase transition from a black hole to a naked singularity occurs, depending on the value of \( \xi \) is a feature of dimensions \( N = 4 \) and \( N = 5 \), and is forbidden at larger dimensions. As one can see, the contribution of the dimension \( N \), when it is larger than 4, basically enters in the behavior of the apparent horizon, which behaves like \( r^{1/2(N-3)} \), which is no longer an integer power of \( r \) as \( N \geq 6 \), and it always leads upon the ‘kinematical’ contribution of \( N \)—i.e. the last term in (20). Since the contribution of \( \sqrt{\Delta}(r, 0) \) is always an integer power of \( r \)—see below—this fact results in the lack of critical case when \( N \geq 6 \).

4. Exterior spacetime and matching conditions

In this section we will see how to complete the model, matching the interior solution studied so far with an exterior spacetime, and requiring that Israel–Darmois junction conditions hold along the matching hypersurface \( \Sigma = \{ r = r_b \} \). From (15) we observe that radial pressure \( p_r \) does not vanish in general along \( \Sigma \), so we cannot expect to match the solution with a Schwarzschild exterior. In this case a natural choice for the exterior metric can be given by generalized Vaidya solutions [11, 28], which for generic \( N \) read

\[
d^2 s^2 = -\left( 1 - \frac{2 M(V, R)}{R^{N-3}} \right) dV^2 - 2 dR dV + R^2 d\Omega_{N-2}^2,
\]

and Israel–Darmois junction conditions simply become requirements on the mass function \( M(V, R) \) on the junction hypersurface. To find the conditions, it is convenient to work with the general interior metric written in comoving coordinates (1). Parametrizing \( \Sigma \) with coordinates \( (\tau, \theta') \leftarrow (\tau, r_b, \theta') \), the first and second fundamental forms of \( \Sigma \) w.r.t. this metric read

\[
\|s\|^2_{\Sigma} = -e^{2\nu} d\tau^2 + R^2 d\Omega_{N-2}^2. \tag{21}
\]
Asymptotic behavior of the apparent horizon with respect to the dimension near the center. In the $(r, R)$ plane, the shaded region represents the evolution of the solution. With fixed value for $\alpha$, the higher the $N$, the bigger the trapped region lying between the $r$-axes and the horizon.

\[ \Pi^\Sigma_{\text{int}} = -\frac{1}{2} \left( e^{2\nu'} \left( \frac{2}{r^2} \frac{d}{\Omega^2_{N-2}} \right) - R \frac{d}{\Omega^2_{N-2}} \right), \]  

(22)

where a dash and a dot denote derivatives w.r.t. $r$ and $t$ respectively, and all functions are intended evaluated in $(r, r_b)$. Injection of $\Sigma$ into the exterior spacetime reads in coordinates as $(V(\tau), Y(\tau), \theta^i)$, where $V(\tau), Y(\tau)$ must be determined. The first fundamental form of $\Sigma$ takes the form

\[ \Pi^\Sigma_{\text{ext}} = -\left( 1 - \frac{2M(V(\tau), Y(\tau))}{Y(\tau)^{N-3}} \right) \left( V(\tau)^2 + 2V(\tau)Y(\tau) \right) d\tau^2 + Y(\tau)^2 d\Omega^2_{N-2}, \]  

(23)

Comparing (21) with (23) gives

\[ Y(\tau) = R(\tau, r_b), \]  

(24)

\[ \left( 1 - \frac{2M}{Y^{N-3}} \right) V^2 + 2VV = e^{2\nu}, \]  

(25)

Comparing angular terms in (22) and (26) and using (25) gives

\[ \chi = R^2 \eta - (\dot{R} e^{-\nu})^2, \]  

(27)
which is the continuity of the Misner–Sharp mass across $\Sigma$. Therefore we find the differential equation for $V(\tau)$:

$$\dot{V}(\tau) = \frac{e^\nu}{R' \eta + R e^{-\nu}}. \quad (28)$$

At this stage, it remains to compare $d\tau^2$ terms in the second fundamental forms. But, with some algebra, the above relations together with the field equation $\dot{R} = \lambda R' + \nu R$ simply reduce the condition to

$$M, V(\tau), Y(\tau) = 0. \quad (29)$$

Therefore, we conclude that the generalized Vaidya solutions can always be matched with a spherically symmetric interior metric (1) along $\Sigma$, provided that conditions (24) and (28) hold, and the mass function $M(V, Y)$ satisfies (27) and (29) on $\Sigma$.

The above fact can easily be translated in an area–radius formalism: it suffices to parametrize $\Sigma$ with coordinates $(\sigma, \theta_i) \mapsto (r_b, \sigma, \theta_i)$. In this case the injection of $\Sigma$ in the exterior spacetime reads $(V(\sigma), \sigma, \theta_i)$, where $V(\sigma)$, in view of (24)–(29), becomes

$$\frac{dV}{d\sigma} = \frac{1}{u(r_b, \sigma)}(u(r_b, \sigma) - Y(r_b, \sigma)), \quad (30)$$

and the mass function satisfies

$$M(V(\sigma), \sigma) = \Psi_1(r_b, \sigma), \quad M, Y (V(\sigma), \sigma) = \Psi_1, R(r_b, \sigma). \quad (31)$$

It can be observed that an interesting subclass of the above exterior metric is given by the anisotropic generalizations of de Sitter spacetime [5], which is obtained by taking $M = M(Y)$. Obviously, in this case condition (29) is trivially satisfied, and (31) simply reduces to requiring continuity of the mass across the junction hypersurface (see also [7]).

5. Discussion and conclusions

There have been previous works trying to explain the endstate in terms of the kinematical properties of the spacetime, in particular the shear at initial time [10, 20]. In the following we are going to address this point, relating the indices $n, \xi$ coming from (19) to all kinematical properties (see also [4, 21]). The function $\sqrt{\Delta}(r, 0)$ can be split into the sum of $I_1(r) + I_2(r)$, where

$$I_1(r) := \frac{1}{Y(r, r)} \frac{1}{Y(r, r)} \int_0^r \frac{1}{u(r, \sigma)} d\sigma,$$

$$I_2(r) := \int_0^r \left( \frac{1}{Y(r, \sigma)} - \frac{1}{Y(r, r)} \right) \left( \frac{1}{u(r, \sigma)} \right) d\sigma.$$

The behavior of these quantities near the center can be studied to find that $I_1(r) = par^{p-1} + o(r^{p-1})$, where $p$ is given in (17), and $a \in \mathbb{R}$ depends on the coefficients $h_{ij}$ of orders $N - 1$ and $N - 1 + p$.

Introducing the polynomials $P_k(\tau) = \sum_{j=0}^k h_{k-j,j} \tau^j$, the value of $a$ is given by

$$a = - \int_0^1 \frac{P_{N-1+p}(\tau) \tau^{(N-3)/2}}{2P_{N-1}(\tau)^{3/2}} d\tau.$$

On the other hand, $I_2(r) = b r^a + o(r^q)$, where $b \in \mathbb{R}$ and $q$ is the order of the first nonvanishing term of $Y, R (R, R)$ expansion at the center. Then, $n$ in (19) is given by the smallest one between $p$ and $q + 1$. Now, the shear of the solution can be controlled by the scalar

$$\sigma = \frac{1}{2} \sigma^{\mu\nu} \sigma_{\mu\nu} = - u \sqrt{(N/2 - 1)/(N - 1)} (\log(R u \sqrt{\Delta}^{-1})), R.$$
and on the initial slice $R = r$ behaves like $p\sigma_0 r^p + o(r^p)$, where $\sigma_0 = \sqrt{(N/2 - 1)/(N - 1)}(2 + 1)$, and it rules the quantity $I_2(r)$, but again knowledge of the initial acceleration could not be enough to establish the value of $q$. We can conclude that the evolutions of both acceleration and shear influence the endstate of the gravitational collapse, but none of them can be considered a stand-alone responsible, as the function $\sqrt{\Delta(r, 0)}$ is, together with the dimension $N$.

We observe that, if $N \geq 6$, the singularity is naked only when $n = 1$. This is not in contrast to [10, 20], where the dust and vanishing radial stress solutions are shown to produce a black hole when $N \geq 6$. Indeed, the special cases considered in those papers are acceleration free, or more generally such that $\sqrt{\Delta}(r, 0)$ behaves like $I_1(r)$ anyway, and so the endstate is related to the first nonvanishing power of $r^{N-3}u^2$, after the $(N - 1)$th order. In the cases produced in [10, 20] the expansion for both $\Psi(r)/r^{N-1}$ and $Y^2$ is assumed to contain only even-order terms, which excludes the possibility $n = 1$. Instead, in the case studied in the present paper a more general situation is considered, when $r^{N-3}u^2$ may contain both odd- and even-order terms, but only terms of order $N - 1 + 2k$, $k \in \mathbb{N}$, when restricted to the initial slice $R = r$. In other words, solutions may be produced, when $\Psi/r^{N-1}$ and $Y^2$ are even at initial time, but later they evolve to allow also for odd-order terms. All in all, the conclusion stated for $N = 4$ in [6] confirmed at higher dimensions is that the formation of naked singularities or black holes weakly depends on the initial data, but is essentially a local phenomenon, depending on the Taylor expansion of a kinematical invariant near the center. The contribution of the dimension is basically related to the behavior of the apparent horizon, which forbids occurrence of critical cases when $N \geq 6$, and restricts, but still allows for naked singularity formation at any dimension.

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