Standard and geometric approaches to quantum Liouville theory on the pseudosphere

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Abstract

We compare the standard and geometric approaches to quantum Liouville theory on the pseudosphere by performing perturbative calculations of the one and two point functions up to the third order in the coupling constant. The choice of the Hadamard regularization within the geometric approach leads to a discrepancy with the standard approach. On the other hand, we find complete agreement between the results of the standard approach and the bootstrap conjectures for the one point function and the auxiliary two point function.

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Introduction

Classical Liouville theory is well understood even if it can be solved explicitly only in special cases. Quantum Liouville theory was first developed in the Hamiltonian framework \cite{1,2}. Later, functional techniques were applied to the Euclidean version of the theory.

Within the functional formulation there exist the so-called standard approach \cite{3,4,5,6,7,8} and the geometric approach \cite{9}. In the first case one introduces the vertex function by adding to the Liouville action external currents as is usually done in quantum field theory. Within the geometric approach one starts from the regularized classical action in presence of boundary terms that represent the sources and then considers the fluctuations of the field around the classical background.

In this paper we shall compare the standard and the geometric formulations of quantum Liouville theory on the sphere and the pseudosphere with particular attention to the pseudosphere \cite{3}, where the perturbative results are directly compared. Both formulations need to be regularized and such a regularization process is crucial, since the results depend non trivially on the adopted regularization procedure.

In the geometric approach the action is defined through a limit process and such a structure is not so easy to use in explicit calculations. However it is possible, by introducing a background field and a source field, to rewrite the action in a form such that no limit procedure appears. This structure is not as elegant as the original one, but from it one can read directly the transformation properties of the off-shell action and consequently also of the correlation functions. This form can be used as a starting point for perturbative calculations of the correlation functions of vertex operators on the pseudosphere.

As first shown in \cite{9}, the appealing properties of the geometric action is to transform under conformal transformations as the vertex correlation functions of the quantum theory, generating quantum conformal dimensions $= (Q)$, which are those found in the Hamiltonian approach, provided the central charges are properly identified.

On the sphere, due to the bounds imposed by the Picard inequalities, the situation is more complex and in this paper we shall perform perturbative computations only on the pseudosphere.

With regard to the one-point function the main outcome will be the following: the results of the two formulations agree up to order $b^4$ included for the values of the first cumulant $G_1$, provided one properly identifies the coupling constant $b_g$ of the geometric approach as a function of the coupling constant $b$ of the standard approach and the same for the cosmological constants. In order to match the value of the second cumulant $G_2$,
one has to introduce by hand a coupling constant dependence in the source subtraction term of the geometric action. Such a \( b_3 \) dependent subtraction influences only the second cumulant and has no effects on the other cumulants \( (G_1 \text{ and } G_n \text{ with } n > 3) \). A significant test of the equivalence of the two perturbative expansions can be achieved by computing the third cumulant \( G_3 \). This is easily done to first order, yet some improvement in the computational technique is needed both in the standard and geometric approach, in order to get the third order coefficient. The explicit computation of the third cumulant to order \( b^3 \) within the standard approach has been given in [10] and it disagrees with the result of the geometric approach computed in Section 2 of the present paper.

From the general field theoretical point of view we find that within the geometric approach the asymptotic value of the vacuum expectation value of the Liouville field reproduces the classical background value. This does not happen within the standard approach, where the two asymptotic behaviors agree only qualitatively. On the other hand, within the standard approach the field \( e^{2b} \), which appears in the cosmological term, transforms like a \((1;1)\) primary field, while within the geometric approach the analogous field \( e^{2b^g} \) has not such a transformation property because its quantum conformal dimensions are \((1 \ b_2^g; 1 \ b_2^g) \). The different characters of the operators appearing in the cosmological terms of the two approaches has already noticed by Takhtajan [11]. Here we find a difference respectively at the second and third order in the perturbative expansion of the second and third cumulant and these differences cannot be matched consistently by a redefinition of the coupling constants.

The obtained results can be compared to the perturbative expansion of the formula conjectured by Zamolodchikov and Zamolodchikov (ZZ) for the one point function on the pseudosphere [3]. Complete agreement has been found with the perturbative computation within the standard approach up to the third order [10]. To gain deeper insight, we shall move further through the perturbative computation of the two point function on the pseudosphere. This allows not only a better comparison between the two approaches but also to compare the results with another conjecture [3, 4, 7], i.e. the two point function when one vertex is the degenerate field \( e^{b^g} \). The interest of this computation is that, by taking a proper ratio of two and one point functions, we can extract quantities not depending on the possible ambiguities in the subtraction terms of the geometric approach. Once more we find that the two point functions are different within the two approaches and that the perturbative results of the standard approach agree with the perturbative expansion of the formula conjectured through the bootstrap method.

A deeper inspection shows that the origin of the differences between the results obtained within the two approaches lies in the different regulators employed and not in the
way used to introduce the sources. Indeed, it can be shown that adopting the ZZ regulator both the approaches produce the same results, identifying exactly the couplings and the cosmological constants. The discrepancy in the structure of the unperturbed dimensions is matched by the different origin of the zero order in the second cumulant of the one point function. For all the other orders there is a one to one correspondence between the contributions.

1 Geometric action on the pseudosphere

The pseudosphere can be represented on the upper half plane or on the unit disk. We shall use mostly the disk representation.
At the quantum level the geometry is encoded by the boundary condition at 1, i.e. on the unit circle.
Within the geometric approach, we assume that the Liouville field behaves like
\[
\frac{Q}{2} \log (1 - zz)^2 + O(1) \quad \forall j \quad 1
\]
\[
\tau_n \log z_j \quad z_1^j + O(1) \quad z \quad z_n
\]
The N point vertex functions are defined as follows
\[
\langle z_1 \cdots z_N \rangle = e^{2 \left( \sum_{j=1}^N (z_j) + \sum_{n=1}^N (z^n) \right)} = \frac{D[\cdot] e^{S_{iN}[\cdot]}}{D[\cdot] e^{S_{i0}[\cdot]}} \quad \forall z
\]
where
\[
S_{iN}[\cdot] = \lim_{r \to 1} \left( \frac{1}{r} \sum_{j=1}^N \frac{\partial z_j}{\partial z} \log z_j \right)
\]
\[
\frac{Q}{2} \sum_{j=1}^N \frac{z_j}{1 - z_j^2} \frac{dz}{dz} + \frac{1}{2} \sum_{n=1}^N \frac{z_n}{1 - z_n^2} \frac{dz}{dz} + f(r^2 b_3)
\]
\[
\frac{1}{2} \sum_{n=1}^N \frac{z_n}{1 - z_n^2} \frac{dz}{dz} + \frac{X}{2} \sum_{n=1}^N \frac{z_n}{1 - z_n^2} \frac{dz}{dz} + f(r^2 b_3)
\]
The points \( z_n \), for \( n = 1 \cdots N \), are the positions of the sources; the domains of integration are \( r = \frac{1}{2} j \), and \( n = \frac{1}{2} j \), and the \( n \sum \) term, while \( f(r^2 b_3) \)
is a numerical subtraction term. \( S_{\phi} \) is the action in absence of sources, which is formally equal to the action of the standard approach.

The equation of motion is the Liouville equation in presence of sources

\[ \Theta_z \Theta_z = b^2 g e^{2b_0} \sum_{n=1}^{N} n^2 (z, z_n) : \]  \hspace{1cm} (5)

We decompose as the sum of two classical fields (\( B \) and \( 0 \)) and a quantum field \( M \)

\[ = M + 0 + B \]  \hspace{1cm} (6)

with the background field \( B \) having the asymptotics \( g_0 \), ie.

\[ B' = \frac{Q}{\bar{z}} \log (1 - \bar{z}z)^2 + c_B; + o(1) \]  \hspace{1cm} (7)

where \( c_B \) is a constant.

One could choose for the source field \( 0 \) a solution of the equation

\[ \Theta_z \Theta_z \phi = \sum_{n=1}^{N} n^2 (z, z_n) : \]  \hspace{1cm} (8)

similarly to what has been done on the sphere \( \text{(see [12] and Appendix)} \).

Requiring that \( 0 \) vanishes on the unit circle, one gets

\[ 0 = \sum_{n=1}^{N} n \log \left( \frac{z - z_n}{1 - \bar{z}z_n} \right)^2 : \]  \hspace{1cm} (9)

However, such \( 0 \) vanishes too slowly for \( \bar{z}z \leq 1 \), giving rise to ill-defined integrals in the perturbative calculation.

Thus, instead of \( 0 \) given by (9), we shall choose \( g_0 \) satisfying

\[ \Theta_z \Theta_z \phi = \sum_{n=1}^{N} n^2 (z, z_n) : \]  \hspace{1cm} (10)

The solution of this equation is

\[ g_0 (z; z_1, \ldots, z_N) = \sum_{n=1}^{N} g(z; z_n) \]  \hspace{1cm} (11)

being \( g(z; z') \) the propagator \( [2,3] \)

\[ g(z; z') = \frac{1}{2} \left( \frac{1 + \log}{1} + 2 \right) \]  \hspace{1cm} (12)
and \((z; z^0)\) the SU(1;1) invariant
\[
(z; z^0) = \frac{z}{1 - zz^0} z^0 \quad (13)
\]
which is related to the geodesic distance between \(z\) and \(z^0\).

The source field \(g_0\) converges to zero like \(O((1 - zz)^2)\) when \(|z| < 1\), which makes the perturbative integrals convergent at infinity.

A procedure similar to the one employed for the sphere (see [12] and Appendix) gives the geometric action on the pseudosphere with a generic background field \(B\) that satisfies the boundary conditions \(7\). The result is
\[
S_{N} = S_{\theta} [B] + S_{\omega \mu} [M; B] + \sum_{n,m} \frac{\omega^2}{2} \log (1 - zz) 2 \quad (14)
\]
where \(n_m = (z_n; z_m)\) and \(S_{\theta} [B]\) is the background action
\[
S_{\theta} [B] = \lim_{r \to 1} \frac{1}{2} \Theta_{z} \quad \Theta_{z} B + g_0 e^{2b_0} \quad g^2 z
\]
\[
\frac{Q}{2} \frac{z}{1 - zz} dz \quad \frac{z}{1 - zz} dz + f (z; b_0) \quad (15)
\]
while \(S_{\omega \mu} [M; B]\) is the action for the quantum field
\[
S_{\omega \mu} [M; B] = \frac{1}{2} \Theta_{z} M \quad \Theta_{z} M + g_0 e^{2b_0} \quad g e^{2b_0} (M + g_0) l
\]
\[
\frac{2}{M + g_0} \Theta_{z} \Theta_{z} B + g e^{2b_0} \cdot 2b_0 g_0 (g_0 + 2 M) \quad d^2 z : \quad (16)
\]
The classical background field \(\omega^2\) is given by
\[
\omega^2 (z) = \frac{1}{2b_0} \log \quad b_0 \quad g (1 - zz)^2 \quad (17)
\]
and it solves the Liouville equation
\[
\Theta_{z} \Theta_{z} = b_0 g e^{2b_0} \quad (18)
\]
with boundary conditions \(7\) if \(Q = 1 = b_0\). The field \(\omega^2\) enters into \(S_{\omega \mu} [M; B]\) because of the introduction of the singular field \(g_0 (z; z_1; \cdots; z_0)\) through the equation \(10\). Under SU(1;1) transformations
\[
z \quad \mapsto \quad w = \frac{az + b}{bz + a} \quad a^2 j \quad b^2 j = 1 \quad (19)
\]
the background field transforms as follows

\[ b_{0}(z) = b(z) \quad \frac{Q}{2} \log \frac{dw}{dz}^{2} \]  \hspace{1cm} (20) \]

while \( b_{0} \) and \( M \) are scalars. As a result, the background action [15] with \( Q = 1 = b_{0} \) is SU(1;1) invariant. The action \( S_{M,\mu}^{N,\nu} \left[ M_{\mu,\nu}^{(1)} b_{0} \right] \) is invariant as well and the terms that transform in [14] are

\[ \chi_{n}^{N} \left[ n \right] = 2 \chi_{n}^{N} \left[ n \right] \log (1 - z_{n}z_{n})^{2} \]  \hspace{1cm} (21) \]

Therefore the transformation law for the geometric action under SU(1;1) is

\[ S_{0,N}^{N} = S_{0,N}^{N} + \chi_{n}^{N} \left[ n \right] \log \frac{dw}{dz}^{2} \]  \hspace{1cm} (22) \]

where \( Q = 1 = b_{0} \).

It is in portant to observe that such a transform ation property for the action holds off shell, i.e. also when \( b_{0} \) is not a solution of the equation of motion.

With respect to the integration measure, we could choose the one induced by the distance

\[ \int \frac{1}{z_{n} - z_{n}} d^{2}z \]  \hspace{1cm} (23) \]

or by the distance

\[ \int \frac{1}{z_{n} - z_{n}} d^{2}z \]  \hspace{1cm} (24) \]

Both are invariant under the group SU(1;1), where \( b_{0} \) transforms like [20] with \( Q = 1 = b_{0} \), but the measure induced by [23] is not invariant under translation of the Liouville field, while the other one is.

The measure induced by [23] differs from the one induced by [24] by ultralocal terms. Such a difference should not be relevant in perturbative calculation [13]. We will work with [24], which gives rise to an integration measure that is invariant under translations in the field.

Thus, we have that

\[ \int D \left[ e^{S_{M,\mu}^{N,\nu} \left[ M_{\mu,\nu}^{(1)} b_{0} \right]} \right] = \int D \left[ e^{S_{M,\mu}^{N,\nu} \left[ M_{\mu,\nu}^{(1)} b_{0} \right]} \right] \]  \hspace{1cm} (25) \]

and it is invariant under SU(1;1), for every \( N \).

From the transformation law [22], one derives the quantum conformal dimensions of the Liouville vertex operators \( e^{2} \)

\[ \left( Q = 1 \right) = \frac{1}{b_{0}} \]  \hspace{1cm} (26) \]
On the classical background \( B = \frac{c_1}{B} \) given in \cite{17}, \( S_{\mu^\mu}^{\mu}_{\nu^\nu} \) can be written as follows

\[
S_{\mu^\mu}^{\mu} = Z \left( 1 - \theta_z \right) \theta_z + \frac{2 \frac{z}{m}}{(1 - zz)^2} d^2 z + \frac{Z}{k!} \frac{2b_y (M + g_0)^k}{b_y^2 (1 - zz)^2} d^2 z
\]

\[
= S_{\rho\rho} + \mathcal{S}_{\mu^\mu}^{\mu} \quad (27)
\]

where

\[
S_{\rho\rho}^{\mu} = \frac{1}{\theta_z \theta_z} + \frac{e^{2b_y}}{b_y^2 (1 - zz)^2} d^2 z
\]  

\[
= \frac{1}{\theta_z \theta_z} + \frac{2b_y (M + g_0)}{b_y^2 (1 - zz)^2} d^2 z
\]  

(28)

is formally identical to the action for the quantum field of the standard approach. The source field \( g_0(z; x_1, \ldots, x_N) \) is contained only in \( \mathcal{S}_{\mu^\mu}^{\mu} \).

The propagator \( g(z; z') = \frac{1}{M} \frac{1}{M} (z') \) in the geometric approach is the same as in the standard approach \cite{3}.

In order to compare the geometric approach with the standard one, the third form of (27) turns out to be the most useful.

2 One point function

In this section we shall provide the perturbative expansion of the one point function on the pseudosphere in the geometric approach and we shall compare it with the results of the standard approach \cite{3,14}.
We recall that, within the geometric approach, the expectation value of the Liouville vertex operator $V_i(z_i) = e^{2 \phi_i(z_i)}$ is given by

$$Z \begin{vmatrix} D \end{vmatrix} e^{\mathcal{S}_{i, \phi_i(z_i)}} = \frac{Z^C}{Z^C}$$

being $\mathcal{S}_{i, \phi_i(z_i)}$ the geometric action \[14\] with one source of charge $1$.

To perform a perturbative calculation, we choose $B = \frac{c_1}{B}$ given in \[14\], as done in \[3\]. Then the geometric action \[14\] with one source simplifies to

$$\mathcal{S}_{i, \phi_i(z_i)} = \mathcal{S}_{M; \phi_i(z_i)} + \mathcal{S}_{\phi_i(z_i); \phi_i(z_i)}$$

where $\mathcal{S}_{M; \phi_i(z_i)}$ is the action \[14\] with $N = 1$ and the source field is given by $g_0(z; z_i) = 2 \log(z; z_i)$.

The one point function \[30\] in the geometric approach can be written as

$$hV_i(z_i) = \frac{U_g}{(1-z_i)z_i}$$

where $Q = 1 = b_1$ and

$$Z \begin{vmatrix} D \end{vmatrix} e^{\mathcal{S}_{M; \phi_i(z_i)}} = \frac{Z^C}{Z^C}$$

where $U_g$ is taken as a function of charge $1$, has the correct normalization $U_g(1 = 0) = 1$ \[3\], being $\mathcal{S}_{\phi_i(z_i); \phi_i(z_i)} = 0$ when $1 = 0$. Moreover, the mean value in the second line of \[33\] does not depend on $g$.

As noticed in \[11\], one obtains the same central charge and the same quantum conformal dimensions of the standard approach by relating the coupling constant of the geometric approach $b_1$ to the coupling constant of the standard approach $b$ as follows

$$\frac{1}{b_1} = \frac{1}{b} + b$$

Thus, the comparison between the one point functions of the two theories reduces to the comparison between $U_g(1; b_1; g)$ in the geometric approach and $U(1; b; g)$ in the
standard approach [3].

Following [3], it is useful to consider, instead of $e^{2 \cdot (z_1)}$, the cumulant expansion

$$\log e^{2 \cdot (z_1)} = \sum_{n=1}^{\infty} \frac{(2 \cdot 1\!^n)}{n!} G_n g$$

$$= 2 \cdot \frac{1}{2b_g} \log (1 - z_1 z_2)^2 + \frac{(2 \cdot 1\!^2)}{2} \log (1 - z_1 z_1) + \log U_g :$$

(35)

Using the expression of $U_g$ given in [33], we get

$$G_1^g = h (z_1) i = \frac{1}{2b_g} \log (1 - z_1 z_1)^2 - \frac{1}{2b_g} \log \left[ \frac{b_g^2}{g} \right] - \frac{1}{2} D \left( \frac{1}{2} \right) E$$

(36)

$$G_2^g = h^2 (z_1) i h (z_1) i^2$$

$$= \log (1 - z_1 z_1) \cdot 1 + \frac{1}{2} \left( D \left( \frac{1}{2} \right) E \right. + D \left( \frac{1}{2} \right) i^2 \left. \right) - \frac{1}{2} D \left( \frac{1}{2} \right) E$$

(37)

$$G_3^g = h^3 (z_1) i - 3 h^2 (z_1) ih (z_1) i + 2 h (z_1) i^3$$

$$= \frac{1}{2} \left( \begin{array}{llll}
D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2 & D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2 \\
D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2 & D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2 \\
D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2 & D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2 \\
D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2 & D \left( \frac{1}{2} \right) E & D \left( \frac{1}{2} \right) i^2
\end{array} \right)$$

(38)

where

$$S_{\mu, \lambda}^{(k)} = \frac{\Theta}{\Theta_k} S_{\mu, \lambda}^{(k)}$$

and the mean values are taken with respect to the action [28].

Using [33] and considering the $O(\frac{1}{1!})$ contribution to [28], one obtains

$$G_1^g = \frac{c_l}{b_g} (z_1) + \frac{1}{b_g} \frac{\partial g_0^{(1)} (z; z_1)}{\partial (1 - z z_1)^2} \log (1 - z_1 z_1)^2 + \frac{1}{2b_g} \log (1 - z_1 z_1) + \log U_g :$$

(40)

where $g_0^{(1)} (z; z_1) = 2 g(z; z_1)$ is the derivative with respect to $z_1$ of the classical source $g_0 (z; z_1)$.

The graphs contributing to $G_1^g$ up to $O(b_g^3)$ included are shown below, where the dashed lines represent $g_0^{(1)} (z; z_1)$.
In the perturbative expansion divergent graphs appear due to the occurrence of $g(z;z)$; therefore the theory has to be regulated.

The proposal of [5] is to regulate such a propagator at coincident points with the Hadamard procedure (see also [14, 15, 16]), which amounts to set

$$ g(z;z) \quad \lim_{z \to z} g(z;z') + \frac{1}{2} \log + \log 2 : $$

This limit gives $g(z;z) = 1 + \log 2$; but for sake of generality we shall consider $g(z;z) = C$.

We notice that within the geometric approach $g(z;z)$ has to be a constant in $z$, which implies $U_g$ constant in $z_1$, otherwise the relation between the central charge

$$ c_g = 1 + 6 \cdot \frac{1}{b_g} \cdot \frac{2}{2} $$

and the quantum conformal dimensions of $\xi^{(z_1)}$, which are

$$ \xi = 1 \cdot \frac{1}{b_g} \cdot \frac{1}{1} $$

would be violated.

As already noticed in [11], expression (44) provides the cosmological term with quantum conformal dimensions $(1 \ b_g^2; 1 \ b_g^2)$. 
The $O(\ell_h)$ contribution to $G_1^q$ is given by $b_2 C$. Concerning the $O(\ell_h^3)$ contribution, we find that the three graphs contained in the second line of (41) sum up to zero and the same happens for the two graphs of the third line. The whole $O(\ell_h^3)$ contribution comes from the two graphs of the fourth line: their values are $(-15)=18$ and $2=3$ respectively. Thus, the expansion of $G_1^q$ in the coupling constant $b_2$ within the geometric approach, up to $O(\ell_h^3)$ included, is

$$G_1^q = \frac{1}{2b_2} \log b_2^2 g (1 - z_1 z_1)^2 b_2 C + \frac{b_2^3}{18} \frac{1}{6} + O(\ell_h^3) : (45)$$

Notice that, because of the cancellation between graphs explained just above, the regulator $C$ appears only in the contribution $O(\ell_h)$. The contribution of the term in the square brackets of (40) is given by $h_M(z_1)$. Indeed, varying $h_M$ in (48), we obtain for the expectation value of the equation of motion

$$\frac{1}{\theta_2 \theta_4} h_M i + g b_2 e^{2b_2} e^{i} h e^{2b_2} i = 0$$

which can be rewritten, using the equation of the propagator and the expression (47), as follows

$$h_M(z_1) = \frac{2}{b_2} \frac{Z}{(1 - z z)^2} he^{2b_2} i = 1 + 2b_2 h_M(z) i : (47)$$

Being $q_0(z) = 2 g(z z)$, we recognize in the term in square brackets of (40) the rhs. of the previous $W$ and identity.

Equation (47) has the same form of the $W$ and identity obtained by $ZZ$ within the standard approach [3] for the quantum field and verified up to $O(\ell_h^3)$ with their regulator.

Direct computation of $h_M(z_1) i$ with the $C$ (Hadamard) regulator up to $b_2^3$ included agrees with the rhs. of (45). On the other hand, we have checked the validity of the equation of motion (46) in the $C$ regularized theory up to $b_2^3$ included. Indeed, (47) follows directly from the independence of $h_M(z)$ on $z$ and from $he^{2b_2} i = 1$. The perturbative expansion (45) gives the constancy of $h_M(z)$ up to $b_2^3$ included, while we have checked that, with the $C$ regulator, $he^{2b_2} i = 1$ is valid up to $b_2^3$ included. We expect such an identity to be valid to all orders in the coupling constant $b_2$.

Therefore we obtain up to $b_2^3$ included

$$G_1^q = h_M(z_1) = \frac{c_1}{b_2} (z_1) + h_M(z_1) i : (48)$$

Notice that, due to (47), within the geometric approach $h_M(z_1)$ cannot depend on the position $z_1$ if we want to keep for $Q$ the value $1=b_2$. As seen above this requirement is
satisfied by using the SU (1;1) invariant regulator C [3].

The structure [48] was obtained also in the standard approach by [3], with the difference that \( h_{\mu}(z_1) \) is not \( z_1 \) independent because of the different choice of the regulator. In the standard approach that choice of regulator is necessary in order to provide the correct quantum conformal dimensions of the hamiltonian treatment [11].

\[
1 = \frac{1}{b} + b_1 \tag{49}
\]

Dependence on \( z_1 \) of \( G_2^g \) given in [48] compared to that of the standard approach [3] imposes the relation [39] between \( b_g \) and \( b \), while the agreement between the constant terms can be obtained by choosing \( g \) as a proper function of \( b \) and \( C \).

Concerning \( G_2^g \), from [37] we get

\[
G_2^g = \log(1 - z_1 z_1) - \frac{Z}{(1 + z z)^2} \he^{2b_g \times (z)} \ii \dd z \\
+ \frac{1}{b_g^2} \frac{Z}{(1 + z z)^2} \frac{g_0^{(1)}(z; z_1)}{g_0^{(1)}(z; z_1)} \he^{2b_g \times (z)} - \frac{1}{b_g^2} \frac{Z}{(1 + z z)^2} \he^{2b_g \times (z)} \ii \dd z \\
+ \frac{1}{b_g^2} \frac{Z}{(1 + z z)^2} \he^{2b_g \times (z)} \ii \dd z \\
+ O(b_g^2) \tag{50}
\]

Through the Ward identity [47], the last term reduces to \( h_{\mu}(z_1) i^2 \), which has already been computed. The graphs that contribute to \( G_2^g \) are given below.

\[
G_2^g = \log(1 - z_1 z_1) - \frac{Z}{(1 + z z)^2} \he^{2b_g \times (z)} \ii \dd z \\
+ \frac{1}{b_g^2} \frac{Z}{(1 + z z)^2} \frac{g_0^{(1)}(z; z_1)}{g_0^{(1)}(z; z_1)} \he^{2b_g \times (z)} - \frac{1}{b_g^2} \frac{Z}{(1 + z z)^2} \he^{2b_g \times (z)} \ii \dd z \\
+ \frac{1}{b_g^2} \frac{Z}{(1 + z z)^2} \he^{2b_g \times (z)} \ii \dd z \\
+ O(b_g^2) \tag{51}
\]
The first integral in (50) gives the two graphs in the first line of the \( O(\ell^2) \) contribution and they cancel out. Indeed, it can be shown that this integral vanishes at least up to the order \( \ell^5 \) included, in agreement with the already discussed Ward identity (47).

The second integral in (50) provides a nonvanishing \( O(\ell^2) \) contribution through the first two graphs in the second line, but the second of these graphs simplifies with the third one, which is the order \( O(\ell^2) \) of the last integral in (50), i.e. \( \eta \; \eta \; (z_1) i^2 \).

Thus

\[
G^g_2 = \log(1 + z_1 z_1) + \frac{5}{6} \frac{2}{18} + O(\ell^2) \quad (52)
\]

and it does not depend on \( C \).

Since \( h_2 \) is fixed by (34), \( G^g_2 \) disagrees with the result of the standard approach [3]:

\[
G_2 = \log(1 + z_1 z_1) + \frac{3}{2} \frac{2}{6} + O(\ell^2) \quad (53)
\]

Notice that the \( O(\ell^2) \) contribution to \( G^g_2 \) comes from the regularization of the geometric action, while in the standard approach the \( O(\ell^2) \) contribution to \( G_2 \) is the result of the ZZ regularization of the one-loop graph that contributes to this order.

One could modify the geometric action (4) by replacing the subtraction term \( \sum_n \frac{2}{n} \log(2 \; \eta^2) \) with a properly chosen \( h_2 \) dependent on \( n \), i.e. \( \eta \; \eta \; (z_1) i^2 \). This change would remove the discrepancy between \( G^g_2 \) and \( G_2 \); but still differences remain both in the comparison between \( G^g_3 \) and \( G_3 \), as we shall show in what follows, and in the comparison of the two-point functions, as we shall discuss in the next section.
The graphs contributing to $G^g_3$ in the geometric approach are shown below

$$G^g_3 = b_4 + b^3_3 + \ldots + O(b^5_3)$$

On the other hand, in the standard approach one gets

$$G^g_3 = b + b^3 + \ldots + O(b^5)$$
which is in contrast with \(55\), taking into the account the relation \(34\) between \( b \) and \( b_2 \). As a further check, we have considered the first perturbative order of the fourth cumulant. Within the standard approach the \( O ( b^7) \) contribution to \( G_4 \) is given by

\[
G_4 = b^2 \begin{array}{c}
\text{2} \\
\text{3}
\end{array} + 7 \begin{array}{c}
\text{4}
\end{array} + O ( \phi^4) = 2 b^2 + O ( \phi^4)
\tag{57}
\]

and it agrees with the perturbative expansion of \( G_4 \) up to this order.

Instead of regulating the theory by giving a finite value to \( g (z; z) \), one can proceed through the usual Pauli-Villars regulator technique, i.e. by introducing a regulator \( \phi^4 \) and replacing with \( + \) in the interaction lagrangian. We show here that the \( C \) and \( ZZ \) regulators arise from two Pauli-Villars regulators whose lagrangians possess different transformation properties.

Let us consider the following quadratic lagrangian

\[
\frac{1}{\Theta_z z} \frac{1}{\Theta_z z} \frac{1}{m (m - 1)} \left( \frac{1}{zz} \right)^2
\tag{58}
\]

and the equation for the Green function that follows from it

\[
\frac{2}{\Theta_z z} g (z; z) = \frac{2}{m (m - 1)} \frac{1}{zz} g (z; z) = \frac{1}{z (z - z^0)}
\tag{59}
\]

Such an equation is invariant under \( SU (1; 1) \). The explicit form of the Green function \( g (z; z^0) \) is

\[
g (z; z^0) = \frac{1}{2} \left( \frac{m}{2m} \right)^2 (1 - \frac{\gamma}{2} F_1 (m; m; 2m; 1))
\tag{60}
\]

where \( (z; z^0) \) is the invariant \(13\) and \( F_1 \) is the hypergeometric function.

Note that, for fixed \( z \neq 0 \), \( g (z; z^0) \neq 0 \) when \( m \neq 1 \). Because of this property, all the diagrams containing \( g (z; z^0) \) with \( z \neq z^0 \) vanish when \( m \neq 1 \).

The divergence of \( g (z; z^0) \) at coincident points \( z = z^0 \), i.e. \( 0 \), cancels out the one of \( g (z; z^0) \) as follows

\[
\lim_{z \to z^0} g (z; z^0) + g (z; z^0) = 1 + + (m)
\tag{61}
\]

where \( (x) = 0 (x) = (x) \) is the diagramma function.

Thus the Pauli-Villars regulator \(64\) generates a \( C \) regulator, where the constant \( C \)
diverges as \( \log(m) \) when \( m^! + 1 \).

A different Pauli-Villars regulating field can be introduced through the lagrangian

\[
\frac{1}{\theta_2} \frac{\theta_2}{\theta_2} - \frac{M^2}{4} \quad (62)
\]

that gives the following equation for the Green function

\[
- \frac{2}{\theta_2} \theta_2 g(z; z^0) \quad \frac{M^2}{2} g(z; z^0) = \quad (z; z^0) \quad (63)
\]

If we set

\[
g(z; z^0) = K_0 \left( 2M \, jz \quad z^0 j \right) + R(z; z^0) \quad (64)
\]

being \( K_0 \) the modiﬁed Bessel function of the second kind, then \( R(z; z^0) \) satisﬁes the homogeneous elliptic equation

\[
4 \theta_2 \theta_2 R(z; z^0) \quad M^2 R(z; z^0) = 0 \quad (65)
\]

with boundary condition

\[
R(z; z^0) = K_0 \left( 2M \, jz \quad z^0 j \right) \quad jz^! 1 \quad (66)
\]

that gives \( g(z; z^0) \) ! 0 when \( j z^! 1 \).

Using the maximum principle for this kind of equations \( \text{[17]} \), we obtain

\[
\max_{jw} K_0 \left( 2M \, jw \quad z^0 j \right) \quad (67)
\]

for every \( z \) and \( z^0 \) inside the disk.

Thus, for \( z \) and \( z^0 \) inside the unit disk, we have that \( R(z; z^0) \) ! 0 when \( M^! + 1 \), being \( K_0 \left( M \, jw \quad z^0 j \right) \) ! 0 for \( M^! + 1 \).

The field provides the regulating ﬁeld and, in this case, we nd

\[
\lim_{z^!! z} g(z; z^0) + g(z; z^0) = \log(1 \quad zz) \quad 1^+ + \log M + R(z; z) \quad (68)
\]

This reproduces the ZZ regularization \( \text{[3]} \) by a proper subtraction of the divergent term since, for every \( z \) inside the disk, \( R(z; z) = 0 \ (e^M) \).

In closing this section, we notice that, using \( \text{(14)} \), the decom position of \( S_{\mu, \mu} \left( M; b_1 b_2 \right) \) given in \( \text{(27)} \) and the expression for the one point function \( \text{(32)} \), one can write the N point vertex functions \( \text{[3]} \) as follows

\[
hV(z_1) \cdots V(z_n) i = \sum_{n=1}^{N} \quad \frac{e^{2 \theta_n}}{e^{2 \theta_n}} \quad g^{(n)} \quad (69)
\]
where the mean values in the ratio are taken with respect to $Z$. 
If we compute formally the limit of the previous expression e.g. for $j \ll j' \ll 1$, we can verify the cluster decay at large distance 
\[
\lim_{j \ll j' \ll 1} hV_1(z_1) \cdots V_N(z_N) i = hV_1(z_1) i hV_2(z_2) \cdots V_N(z_N) i
\]
that is the boundary condition used by ZZ \cite{3} to get the one point function through the bootstrap method.

3 Two point function

To gain further insight into the relation between the two approaches, we provide the perturbative computation of the two point function.
First we compute the complete one loop order and some results to two loop within the standard approach for generic $1$ and $2$. Then, we compare these results with the ones obtained within the geometric approach for generic $1$ and $2$.
A perturbative check of the exact formula conjectured for the auxiliary two point function $V(z_1) V_{b=2}(z) \cite{3,4,7}$ will be given at the end of this section.
As pointed out in \cite{3}, it is more efficient to compute the ratio
\[
g_{1;2}(\ ) = \frac{hV_1(z_1) V_2(z_2) i}{hV_1(z_1) i hV_2(z_2) i}
\] (71)
where $hV_1(z_1) V_2(z_2) i$ represents the full correlator and not only the connected component.
Taking into account once more the cumulant expansion
\[
\log g_{1;2}(\ ) = \sum_{k_1,k_2=1}^\infty \frac{(2_{1})^{k_1}}{k_1!} \frac{(2_{2})^{k_2}}{k_2!} M_{k_1,k_2}(\ )
\] (72)
we compute perturbatively $M_{k_1,k_2}(\ )$.
From (71), it can be easily seen that the background field does not contribute to $g_{1;2}(\ )$. 
The graphs contributing to $M_{1;1}$ up to one loop are

\[
M_{1;1} = \begin{array}{c}
\text{2} \\
+ b^2 \frac{6}{4} \\
+ O(b^3)
\end{array}
\]

As requested by the standard approach \[3\], we have computed them by using the ZZ regulator, obtaining

\[
M_{1;1} = h(z_1)(z_2)i + h(z_1)ih(z_2)i + !
= g(\ ) + b^2 \left( \frac{3}{2} + \frac{2 \log^2}{2(\alpha - \frac{1}{\gamma})} \right) \left( 1 + \log \left( \frac{1}{\gamma} \right) \right) + O(b^3) \tag{73}
\]

As expected, the limit of the $O(b^3)$ contribution to $M_{1;1}$ for $\gamma \to 0$ gives exactly the order $O(b^3)$ of $G_2$.

Concerning $M_{2;1}$, its first order expansion is given by the following graph

\[
M_{2;1} = b^2 \frac{6}{4} + \frac{7}{5} + O(b^3) \tag{75}
\]

and it gives \[3\]

\[
M_{2;1} = h^2(z_1)(z_2)i + h^2(z_1)ih(z_2)i + 2h(z_1)(z_2)i + 2h(z_1)ih(z_2)i + !
= b \log \left( \frac{\alpha}{\gamma} \right) \frac{1}{\gamma} + O(b^3) \tag{76}
\]

We have considered also the two loop graphs that give the first order of $M_{3;1}$

\[
M_{3;1} = b^2 \frac{6}{4} + \frac{7}{5} + O(b^3) \tag{77}
\]
The result is

\[ M_{3\lambda} = b^2 \left( \frac{(1 + ) \log^2}{(1)^3} \right) \left( 2 + O(\psi^3) \right) : \quad (78) \]

To compute the graphs contributing to \( O(\psi^2) \) in \( M_{1\lambda} \) and to \( O(\psi^2) \) in \( M_{3\lambda} \), we have employed the same technique developed in \[10\] rst giving (in analogy to the Gegenbauer method; see e.g. \[18\]) a harmonic expansion of the Green function \[12\] and then reducing the graphs, through angular integrations, to radial integrals.

Within the geometric approach, by using \((69)\) with \( N = 2 \), we introduce \( g^{1; 2}(\lambda) \) as we have done in \((71)\) for the standard approach, obtaining

\[ g^{1; 2}(\lambda) = \frac{\hbar V_1(z_1) V_2(z_2) i}{\hbar V_1(z_1) i V_2(z_2) i} = \psi^{1; 2} g(z_1, z_2) \psi \quad (79) \]

Notice that in this ratio the possible ambiguities in the subtraction terms of the geometric action, mentioned in Section 2, cancel out.

Again, we consider the cumulant expansion of \( g^{1; 2}(\lambda) \), which defines the functions \( M_{1\lambda}^{g_{k_1k_2}}(\lambda) \), as done in \((72)\) for the standard approach.

From \((79)\), we can get the functions \( M_{1\lambda}^{g_{k_1k_2}}(\lambda) \) within the geometric approach

\[ M_{1\lambda}^{g_{k_1k_2}} = g(\lambda) + \frac{1}{2^2} \left( \frac{D}{\psi^{1/2}} \right) E \left( \frac{D}{\psi^{1/2}} \right)^2 E \left( \frac{D}{\psi^{1/2}} \right)^3 E \left( \frac{D}{\psi^{1/2}} \right)^4 E \right) \]

\[ M_{2\lambda}^{g_{k_1k_2}} = \frac{1}{2^2} \left( \frac{D}{\psi^{1/2}} \right)^2 E \left( \frac{D}{\psi^{1/2}} \right)^2 E \left( \frac{D}{\psi^{1/2}} \right)^3 E \left( \frac{D}{\psi^{1/2}} \right)^4 E \right) \]

\[ M_{3\lambda}^{g_{k_1k_2}} = \cdots \]

where

\[ s_{k_1k_2}^{1/2} = \frac{\psi^{k_1} \psi^{k_2}}{\psi^{1/2} \psi^{1/2}} \quad (82) \]
The graphs contributing to $M_{g_{k_1,k_2}}$ up to $h^2_g$ included are shown below.

\[ M_{g_{1,1}} = \hspace{1cm} \]

\[ + h^2_g \frac{6}{4} \hspace{1cm} + \hspace{1cm} \]

\[ + O \left( \delta^3_g \right) : \]

\[ (83) \]

\[ M_{g_{2,1}} = h_g \frac{6}{4} \hspace{1cm} \]

\[ \frac{7}{5} + O \left( \delta^3_g \right) \]

\[ (84) \]

\[ M_{g_{3,1}} = h_g \frac{6}{4} \hspace{1cm} \]

\[ \frac{7}{5} + O \left( \delta^3_g \right) : \]

\[ (85) \]

Comparing $M_{g_{k_1,k_2}}$ with the functions $M_{k_1,k_2}$ obtained within the standard approach, we have found no differences for $M_{g_{2,1}}$ and $M_{g_{3,1}}$ up to $O \left( \delta^3 \right)$ included, while a discrepancy with the standard approach arises comparing $M_{g_{1,1}}$ with $M_{1,1}$. This difference is due to the fact that, using the Hadamard regularization, the second and the third graph of the $O \left( \delta^3_g \right)$ contribution to $M_{g_{1,1}}$ given in (83) sum up to zero, while, on the other hand, within the standard approach with $ZZ$ regulator, the second and the third graph in the $O \left( \delta^3 \right)$ contribution to $M_{1,1}$ shown in (83) provide a non-trivial function.

We have, up to order $h^2$:

\[ M_{g_{1,1}} = M_{1,1} + h^2 \frac{2}{3} 1 + \log(1) + \frac{\log(1)}{1} + \frac{(1 + \log(1)) \log(1)}{2(1 - 1)} \]

\[ + \frac{1 + \log(1)}{1} \log(1) + O \left( \delta^3 \right) : \]

\[ (86) \]

It is interesting to compare such results with the exact formula conjectured in [2,4,7] for $G_{g_{1,1}} (z^0,z) = V (z^0) V_{b=2} (z)$, by setting $1 = , 2 = b=2, z_1 = z^0$ and $z_2 = z$. 

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The conjectured formula can be written equivalently in the upper half plane as

$$G_{\text{b=2}}(0;\cdot) = \frac{0 + 2^0}{0 + 4^0} U(\cdot) U(\text{b=2}) (1 + 3b^2)^2 g_{\text{b=2}}(\cdot) \quad (87)$$

and in the unit disk representation as

$$G_{\text{b=2}}(z^0;z) = \frac{(1 + z^0)^2}{(1 + z^0)^4} U(\cdot) U(\text{b=2}) (1 + 3b^2)^2 g_{\text{b=2}}(\cdot) \quad (88)$$

where $U(\cdot)$ is the structure constant of the one point function conjectured in [3]. Notice that, to get (87) or equivalently (88), the ZZ boundary conditions (i.e. the cluster decay at large distance $B(\cdot)(\cdot) = U(\cdot)U(\text{b=2})$) have been used [3].

From

$$G_{\text{b=2}}(z^0;z) = V(z^0) V_{\text{b=2}}(z) g_{\text{b=2}}(\cdot) \quad (89)$$

where

$$g_{\text{b=2}}(\cdot) = b^2 F_1(1 + b^2; 2b; 2b + 2b^2; 1) \quad (90)$$

the cumulant expansion of $g_{\text{b=2}}(\cdot)$ in up to $O(b^3)$ is

$$\log g_{\text{b=2}}(\cdot) = b \log g + \frac{b^3}{2} \log^2 2 + 4 \log^2 b + \frac{1}{2} \log 2 \log^2 b + 2 \frac{1}{1} \log 2 \log^2 b + O(b^5)$$

$$+ 2 b^2 \left( \frac{2 \log^2}{(1)} + O(b^3) \right)$$

$$+ 3 b^3 \left( \frac{8}{3} + \frac{4}{3} \log^3 b + O(b^5) \right) + O(b^6) + O(b^7)$$

It agrees perfectly with our perturbative results obtained within the standard approach and, as discussed above, it disagrees with the ones found within the geometric approach.

4 Invariance under background transformations

In order to improve our understanding of the two approaches, we shall examine in this section the dependence of the results on the choice of the background. We shall find that both the approaches are background independent.

The perturbative quantization of the lagrangian [14] with the boundary conditions of the pseudosphere has been carried through starting from the classical background solution $\gamma^1_b$
and perturbatively expanding around it. This has been done both within the standard approach [10] and within the geometric approach.

In this section, we shall examine the independence of the results on the choice of the background. We shall provide this calculation both at the formal level, by using the equation of motion for \( M \), and at the perturbative level, i.e. expanding the results perturbatively in the background field around \( c_l^B \) and verifying that these results are independent on such variations. This happens both within the standard approach and within the geometric approach, independently on the choice of the regulators. We have obtained the same results by employing the related Pauli-Villars regulator fields, described in Section 2.

First we give a formal proof of the background invariance within the standard approach. Using the background field method, i.e. decomposing the Liouville field as the sum of a background field and a quantum correction, the action on the pseudosphere with a generic background becomes [10]

\[
S[\ ] = S_B[\ ] + S[\ ; B]\tag{92}
\]

where

\[
S_B[\ ] = \frac{1}{Z} \theta_z B \theta_z + e^{2b_B} d^2z\tag{93}
\]

and

\[
S[\ ; B] = \frac{1}{Z} \theta_z \theta_z + e^{2b_B} e^{2b} 1 - \frac{1}{2} \theta_z \theta_z B d^2z\quad\tag{94}
\]

Within the standard approach the correlation functions of the vertex operators are given by

\[
h V_{i}(z_1) : \cdots : V_{i}(z_N) \left| i \right. = \frac{Z}{D[\ ] e^{S[\ ; B]} c(\ )} = \frac{Z[J]}{Z[0]}\tag{95}\]

because the background contribution simplifies.

Under a variation \( B \) with compact support of the generic background field, we have

\[
h V_{i}(z_1) : \cdots : V_{i}(z_N) \left| i \right. = \frac{Z[J]}{Z[0]} \frac{Z[0]}{Z[0]} \frac{Z[J]}{Z[0]} :\tag{96}
\]
From (95), we get

\[ Z [J] = D [M] \frac{2}{\theta_2 \theta_2} \frac{2b}{e^{2b}} e^{2b} 1 \]

\[ + 2 \sum_{n=1}^{\infty} (z_n) \frac{2}{e^{2b}} e^{2b} 1 \]

\[ D [M] e^{S [M]} + \sum_{n=1}^{\infty} 2 (z_n) \]

\[ = Z \left( \frac{2}{e^{2b}} e^{2b} 1 \right) + \sum_{n=1}^{\infty} (z_n) \]

\[ = Z [J] \]

(97)

that, due to the equation of motion in presence of sources for the field, becomes

\[ Z [J] = Z [J] \frac{2}{\theta_2 \theta_2} \frac{2b}{e^{2b}} e^{2b} 1 \]

\[ + 2 \sum_{n=1}^{\infty} (z_n) \frac{2}{e^{2b}} e^{2b} 1 \]

\[ = Z [J] \]

(98)

i.e. the variation of \( Z [J] \) is given by a numerical factor that multiplies \( Z [J] \). The same structure and the same numerical factor appear for the variation of \( Z [0] \).

Thus, the first variation (98) of the correlation functions (95) under changes of the background vanishes. Since such a variation is null starting from a generic background field, we have the independence of the correlation functions from \( B \).

Obviously, the above general reasoning is only formal since the theory contains divergencies.

On the other hand, \( Z \) (9) proved the validity of the equations of motion on the background \( c_i \) up to order \( b^2 \) using their particular regulator and suggested their validity to all orders in the perturbation theory. Moreover starting from \( c_i \), one can explicitly verify that \( G_1 \) and \( G_2 \) do not change under a variation \( B \) with compact support, to the first order in \( B \). We did it up to \( O (b^2) \) included for \( G_1 \) and up to \( O (b^3) \) included for \( G_2 \).

A completely similar reasoning works for the geometric approach.

The correlation functions of the Liouville vertex operators are given by

\[ Z \left( \frac{2}{e^{2b}} e^{2b} 1 \right) + \sum_{n=1}^{\infty} (z_n) \]

\[ = Z \left( \frac{2}{e^{2b}} e^{2b} 1 \right) + \sum_{n=1}^{\infty} (z_n) \]

\[ = Z [J] \]

(99)

where \( S \left( [M] \right) \) is the geometric action (14) with a generic background satisfying the boundary conditions (17). Varying this action under a variation \( B \) of compact support of the background field, we get the variation of \( Z [J] \)

\[ Z_g [J] = D [M] \frac{2}{\theta_2 \theta_2} \frac{2b}{e^{2b}} e^{2b} 1 \]

\[ + Z [J] \frac{2}{e^{2b}} e^{2b} 1 \]

\[ = Z [J] \frac{2}{e^{2b}} e^{2b} 1 \]

(100)

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that, through the equation of motion for $M$, becomes

$$Z_g[J] = Z_g[J] - 2 \varphi_z \varphi_z - 2 \beta_3 \varphi e^{2\phi_3} - \delta B^2 z :$$  \hspace{1cm} (101)

Again, taking into account the variation of $Z_g[0]$, we have to the first order in $B$

$$hV_i(z_1) \cdots V_i(z_N) = \frac{Z_g[J]}{Z_g[0]} - \frac{Z_g[0]}{Z_g[0]} = 0$$  \hspace{1cm} (102)

also in the geometric approach.

Within the regulated theory with the invariant regulator $C$ in presence of the classical background $c^1_B$, we have veri$\mathcal{f}$d the validity of the equations of motion up to $O(\phi_3^4)$ included. Moreover, one can verify that $G_1^a$ and $G_2^a$ do not vary respectively up to $O(\phi_3^4)$ and $O(\phi_3^3)$ included.

5 Geometric approach with ZZ regulator

From the results obtained in the previous sections, one could get the impression that the origin of the differences between the standard and geometric approach consists in the way used to introduce the sources.

An important point, already noticed in [11] and remarked in Section 2, is that, the Hadamard regularization within the geometric approach provides for the quantum conformal dimensions of the cosmological term $(1; 1; 1; 1)$. The only way to find quantum dimensions $(1; 1)$ for the cosmological term is to adopt also within the geometric approach the ZZ regulator [3]. With such a regulator, one obtains the same results of the standard approach to all orders. Indeed, there is a one to one correspondence between the graphs of the two approaches, except for the $^2$ contribution to the quantum conformal dimensions, which is provided by a counterterm of the action within the geometric approach and by the one loop graph regularized through the ZZ regulator within the standard approach. Adopting the ZZ regulator within both approaches, the two treatments are identified for $b = b_3$ and $= q$, and consequently

$$h^{2 \phi_3^a} \delta e^{2 \phi_3^a} = \frac{D}{E} \frac{U(z_1)}{(1 - z_1 z_1)^{2b - 1}} = b^2 \frac{e^{2 \phi_3^a}}{(1 - z_1 z_1)^{2b - 1}}$$  \hspace{1cm} (103)

where the mean value $h:\cdots:i_{z_1}$ is taken with respect to [28]. The one point structure constant $U(z_1)$ is given by the exact formula conjectured in [3] through the application of the bootstrap method.
Another difference between the two approaches is the asymptotic behaviour of \( h (z_1)i \), defined as
\[
h (z_1)i = \frac{1}{2} \frac{\partial}{\partial z} \frac{e^{2 : (z_1)} :}{1 - 0}
\]
(104)
Within the geometric approach with the \( C \) regulator one obtains
\[
h (z_1)i' = \frac{1}{2b_9} \log (1 - z_1z_1)^2 + \text{const}
\]
(105)
that is exactly the boundary condition \([1]\) and \([7]\) with \( Q = 1 = b_9 \), imposed respectively on the Liouville field and on the background field \( b \).
This does not happen within the standard approach with the ZZ regulator \([3]\), where
\[
h (z_1)i' = \frac{1}{2} \frac{1}{b} + \frac{1}{b}\log (1 - z_1z_1)^2 + \text{const}:
\]
(106)
The asymptotics \(106\) reproduces the boundary behaviour of the background field
\[
cl (z) = \frac{1}{2b} \log b^2 (1 - zz)^2
\]
only qualitatively, i.e., with a different constant in front of the logarithm.
Thus, the main difference between the standard and the geometric approach is not so much in the way one uses to introduce the sources, but in the regulator chosen to make the divergent graphs finite, which reflects in itself the quantum nature of the cosmological term, deeply related to the boundary behaviour of \( h (z_1)i \).

Conclusions

In this paper, we have compared the standard and the geometric approaches to quantum Liouville theory on the sphere and on the pseudosphere. Detailed perturbative calculations up to three loops have been performed on the pseudosphere in both approaches.

The geometric approach, even if invariant under SU \((1;1)\) group like the one of \([2]\), gives different results with respect to the standard approach \([3]\). A feature of the geometric approach, already noticed in \([11]\), is that it provides the cosmological term with quantum conformal dimensions \((1 - b_9^2; 1 - b_9^2)\), while the standard approach gives \((1;1)\) for them. Moreover, while the perturbative calculations in the standard approach agree with the conformal bootstrap formulas of \([3,4,7]\), this does not happen for the results obtained within the geometric approach. Both theories exhibit background invariance, yet they are definitely different.
A deeper analysis shows that the real difference between the two approaches does not concern the general setting, i.e. geometric action vs. standard sources approach, but it lies in the process of regularization. Indeed, as remarked in Section 5, adopting the ZZ regulator in the perturbative expansion of the geometric action, we have obtained the same results of the standard approach, even if the \( b^2 \) contributions to the quantum conformal dimensions have different form at origins within the two approaches.

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Appendix

In this appendix, for sake of completeness, we shall give the transformation of the geometric action for the sphere \([9,11]\) into such a form that no limit process appears \([12]\) and therefore suitable to perform perturbative computations. This form can be compared with the one derived for the pseudosphere in Section II.

Again we write the Liouville field as a sum of a quantum field \( \phi \), a background field \( \phi^B \) and a source field \( \phi^0 \). Here we shall leave \( \phi \) generic, except for the boundary conditions, and \( \phi^0 \) shall be chosen as a solution of a linear equation with point sources.

The geometric action in presence of sources at the points \( z_1, \ldots, z_N \) can be written, using the notation of \([5]\), as follows

\[
S_{P_{1,N}}[\ ] = \lim_{\epsilon \to 0} \int_{R}^{z} \frac{1}{z_z} dz_z + g e^{2\theta_z} d^2 z + \frac{Q}{2} \sum_{i=1}^{n} \left( \frac{d z}{z_z} \frac{d z}{z_z} + 2 \log R^2 \right) + \frac{1}{2} \sum_{i=1}^{n} \left( \frac{d z}{z_z} \frac{d z}{z_z} + 2 \log R^2 \right) \tag{108}
\]

where \( d^2 z = (dz \wedge dz)^2 = 2 \) and the domains of integration are \( R = \int_{z}^{z} R_g, \ n = \int_{z}^{z} g \) and \( R^{\prime} = \int_{z}^{z} n \).

The field is assumed regular at 1 and to transform as \([5]\)

\[
(z; z) \rightarrow (w; w) = (z; z) \frac{Q}{2} \log \frac{dw}{dz} \tag{109}
\]
under holomorphic coordinate transformations \( z ! w(z) \), which implies

\[
(z;z)' = Q \log(zz) + O(1) \quad \text{for} \quad |z| \ll 1 : \quad (110)
\]

The asymptotic behaviour of near the sources is

\[
(z;z)' = n \log |z| z_n f + O(1) \quad \text{for} \quad z \neq z_n \quad (111)
\]
as for the pseudosphere.

The geometric action in (111) is obtained for \( Q = 1 = b_q \), but, to perform a comparison with the standard approach [5], we shall keep \( Q \) generic for a while.

From (108), one can extract the dependence of the correlation functions on the sphere by a proper constant shift of the Liouville field, as done in [5, 6, 7]. If we define the correlation functions on the sphere not dividing by \( Z_g(0) \), as suggested in [5], we obtain for such a dependence the following factor

\[
\left. \frac{(Q^p)}{g} \right|_{n}^{-b_y} : \quad (112)
\]

The equation of motion derived from the geometric action (108) is the Liouville equation in presence of sources

\[
\Theta_z \Theta_z = b_g e^{2b} \sum_{n=1}^{\infty} X_n^2 (z - z_n) : \quad (113)
\]

We write [12]

\[
\Theta_z \Theta_z = M + 0 + B \quad \text{(114)}
\]

where \( B \) is the background field, regular on the whole plane

\[
B = Q \log(zz) + b_B + O \left( \frac{1}{|z|} \right) \quad \text{for} \quad |z| \ll 1 \quad (115)
\]

and the source field \( 0 \) is

\[
0 = \sum_{n=1}^{\infty} X_n^2 (z - z_n) \quad (116)
\]

i.e. the solution of

\[
\Theta_z \Theta_z 0 = \sum_{n=1}^{\infty} X_n^2 (z - z_n) \quad \Theta_z \Theta_z B \quad (117)
\]

with \( ^\wedge \) given by

\[
^\wedge = \frac{1}{Q} \sum_{n=1}^{\infty} X_n^2 \quad (118)
\]
to have $\phi_0$ going to a constant at 1. As a consequence, also $\phi_M$ goes to a constant at 1. The equation of motion for $\phi_M$ is

$$\theta_{z} \phi_{z M} = b \phi e^{2b_0} + (\wedge 1) \theta_{z} \phi_B$$  \hfill (119) Perform ing a number of integrations by part, the geometric action (108) can be rewritten as

$$S_{P;N} = S_{P;B} + S_{P;M} \frac{2}{\pi} \int_0^{\infty} \theta_{z} \phi_B \, d^2z$$

$$+ \frac{\x}{n} \log jz_n \phi_n \phi_B + \frac{\x}{n} \phi_0 \phi_B (z_n) 2 \frac{\x}{n} \phi_B (z_n)$$  \hfill (120)

where $S_{P;B}$ is the background action [5]

$$S_{P;B} = \lim_{R \to 1} - \frac{1}{2} \theta_{z} \phi_B + g e^{2b_0} \, d^2z$$

$$+ \frac{Q}{2} \int_{\theta_B} dz + \frac{Q^2 \log R}{z} (121)$$

and $S_{P;M} \frac{2}{\pi} \phi_B (z)$ is the action for the quantum field $\phi_M$

$$S_{P;M} \frac{2}{\pi} \phi_B (z) =$$

$$\frac{Z}{\pi} \theta_{z} \phi_M \phi_M + g e^{2b_0} \, e^{2b_0} (\wedge 1) \frac{2 \phi (1)}{\pi} \theta_{z} \phi_B \, d^2z$$  \hfill (122)

Notice that, for $Q = 1 = b_0$, $e^{2b_0}$ is a $(1;1)$ density at the classical level, which can be achieved by assigning to $e^{2b_0}$ a classical $(1;1)$ nature and treating $\phi_0$ and $\phi_M$ as scalars. The background action $S_{P;B}$, the action $S_{P;M}$ for the quantum field $\phi_M$ and the term

$$\frac{2}{\pi} \phi_B (z)$$  \hfill (123)

are invariant under a $SL(2;C)$ transformation

$$z \rightarrow w = \frac{az + b}{cz + d} \quad a \ b \ c \ d \ 2 SL(2;C)$$  \hfill (124)

when $Q = 1 = b_0$. Thus, the transformation properties of $S_{P;N}$ can be read from the second line of (120). Taking into account that, being $\phi_0$ a scalar field, $\phi_0$ transforms like

$$\phi_0 = \phi_0 \phi_n \log jcz_n + d^2$$  \hfill (125)
one obtains
\[ S^{0}_{P^{1};N} \bigl[ 0 \bigr] = S^{0}_{P^{1};N} \bigl[ 1 \bigr] + \sum_{n=1}^{N} \log \frac{dw}{dz} n (Q n) \log \frac{2}{z - z_{n}} : (126) \]

where \( \log j dw = dz \sum_{n=1}^{N} 2 \log j cz_{n} = \frac{dz}{j^2} \frac{j}{z} + x \frac{dz}{j^2} \).

We remark that the geometric action \((120)\) has the transformation property \((126)\) under SL \((2; C)\) only for \(Q = 1 = b_{q}\).

The crucial property of the geometric action is to provide, through \((126)\) and the SL \((2; C)\) invariant measure, the quantum conformal dimensions
\[ = (Q ) = \frac{1}{b_{q}} \quad (127) \]

for the Liouville vertex operators \(e^{2 \lambda} \).

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