Systematic Improvement of Parton Showers with Effective Theory

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Abstract

We carry out a systematic classification and computation of next-to-leading order kinematic power corrections to the fully differential cross section in the parton shower. To do this we devise a map between ingredients in a parton shower and operators in a traditional effective field theory framework using a chain of soft-collinear effective theories. Our approach overcomes several difficulties including avoiding double counting and distinguishing approximations that are coordinate choices from true power corrections. Branching corrections can be classified as hard-scattering, that occur near the top of the shower, and jet-structure, that can occur at any point inside it. Hard-scattering corrections include matrix elements with additional hard partons, as well as power suppressed contributions to the branching for the leading jet. Jet-structure corrections require simultaneous consideration of potential $1 \rightarrow 2$ and $1 \rightarrow 3$ branchings. The interference structure induced by collinear terms with subleading powers remains localized in the shower.
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I. INTRODUCTION

For scattering problems involving strongly-interacting particles, we are often interested in final states with large multiplicities, sometimes including thousands of hadrons. To get to this level, we cannot rely solely on full fixed-order calculations. Tree-level event generators [1–4] only go up to 8–10 external particles as Monte Carlo for higher multiplicity phase space is increasingly intractable. At one-loop, the frontier is $2 \rightarrow 4$ processes, which have been done at the level of differential cross sections for $W + 3$ jets [5, 6] and $t\bar{t}bb$ [7]. At two-loops, there are $2 \rightarrow 1$ exclusive calculations for weak boson production by hadrons followed by decay ($W$ and $Z$ [8] and $W$ to leptons, and $H$ decaying to photons [10, 11]). Additionally, $e^+e^- \rightarrow 3$ jets to NNLO is known [12–15]. In any case, a strict fixed order counting is not suitable for exclusive observables with large multiplicities, nor for many inclusive observables where certain regions of phase space receive kinematic enhancement by large logarithms. If $Q$ is a hard scale in the process, then a subset of the amplitude gets enhanced so that its coefficient is $(\alpha_s \ln^2(Q/p))^m$, where $p \ll Q$ refers to a small scale that is induced by the choice of observable or cuts. Since we can resum these large logs by systematically treating real radiation, we can give a leading log (LL) description of these observables without performing multiloop computations. The soft and collinear limits that yield these large logs also allow us to simplify the amplitude. Therefore, capturing the dominant contributions to these observables and simulating processes with a large number of particles becomes feasible. This is a main goal of parton Shower Monte Carlo (SMC).

A final state SMC is based on the “strongly-ordered limit,” which describes the leading log contribution (accounting for soft emission by angular ordering or other approximations). In this kinematic configuration, each radiated particle comes off much more collinear to its parent than the previous one, a situation that can be formulated in terms of perpendicular momenta or virtualities, i.e.

$$q_0 \perp \gg q_1 \perp \gg q_2 \perp \gg \ldots,$$

or

$$q_0^2 \gg q_1^2 \gg q_2^2 \gg \ldots.$$  \hspace{1cm} (1)

Furthermore, and important for practical computation, in this limit each collinear emission is independent of the previous one. Thus, if we have calculated the differential cross section for $i$-parton emission, $d\sigma_i$, then we can obtain the $(i + 1)$-parton case as

$$d\sigma_{i+1} \propto \frac{P_{j \rightarrow kl}^{(0)}}{q_i^2} d\sigma_i,$$ \hspace{1cm} (2)

where $P_{j \rightarrow kl}^{(0)}$ is the leading order (LO) “splitting function” that captures the probability for the $i^{th}$ emitted parton, of type $j$, to split into two others, $kl$, and $q_i^2$ is its virtuality. We can therefore formulate the process in terms of a probabilistic Markov chain of $i \rightarrow 1$ particle splittings. The probabilities are determined by the functions $P_{j \rightarrow kl}^{(0)}$, which are the LO Altarelli-Parisi kernels. As an example, for $q \rightarrow qg$, after averaging and summing over spins,

$$P_{q \rightarrow qg}^{(0)} = \frac{\alpha_s}{2\pi} C_F \frac{1 + z^2}{1 - z},$$ \hspace{1cm} (3)

where $z$ is the longitudinal momentum fraction of the daughter with respect to the parent. This classical, probabilistic process gives rise to the SMC algorithms used by event generators.
such as Pythia \cite{16, 17} and Herwig \cite{18, 19} to model radiation. For a virtuality-ordered shower, such as the original version of Pythia, given some initial off-shellness, \( q^2_0 \), and an initial momentum fraction, \( x_0 \), SMCs generate the virtuality and the momentum fraction of the daughter particle after the splitting. The former is determined by a Sudakov factor, \( \Delta(q^2, q^2_0) \), which gives the probability of a parton to evolve from \( q^2_0 \) to \( q^2 \) without branching,

\[
\Delta(q^2, q^2_0) = \exp \left[ -\int_{q^2_0}^{q^2} \frac{dq'^2}{q'^2} \int dz \frac{\alpha_s}{2\pi} P_{jk}^{(0)}(z) \right]. \tag{4}
\]

The traditional LL parton shower makes the multiplicity problem tractable, but it has shortcomings related to the leading log approximation. Even though Eq. (2) is only correct in the collinear limit, the shower is used everywhere in order to generate events that cover the full phase space. In addition, since each collinear emission is independent from the previous one in the shower, the LL approximation does not include their spin or color correlations, nor any of their interference. The situation is different for soft gluons where the inclusion of color effects allows one to work in the simplifying limit of angular ordering.

The hierarchy of scales in the parton shower makes it amenable to an effective field theory treatment. Since the shower regime occurs for particles in the soft and collinear regions, we can describe it with Soft-Collinear Effective Theory (SCET) \cite{20–23}. Like any EFT, SCET comes with an expansion that allows, in principle, for systematic improvement. The first work on parton showers using SCET came in \cite{24, 25}, which we review in Sec. II B, where the authors showed how the splitting functions and corresponding Sudakov factors, along with the factorization of emissions emerge naturally. Furthermore, they could include virtual corrections by matching to QCD at higher order in \( \alpha_s \). Unfortunately, in reproducing the LL shower in SCET, they introduced many conventions whose extension to higher orders in the kinematic expansion is unclear. We therefore develop a modified approach to alleviate these difficulties.

Before discussing our setup, we give an overview of advances in the parton shower literature beyond the basic LL picture. The structure of these advances depends on what aspect of the shower one aims to improve. Possible motivations include accuracy at higher orders in \( \alpha_s \), higher order in logs, and higher order in powers of the kinematic expansions. We first introduce some terminology for higher order log resummation. If the resummation of large logs, \( L \), is at the cross section level

\[
d\sigma \sim \left[ \sum_k (\alpha_sL^2)^k \right]_{LL} + \left[ \sum_k \alpha_s L (\alpha_sL^2)^k \right]_{NLL} + \ldots \tag{5}
\]

then we will refer to it as LL, NLL, etc, as indicated. If the cross section transformed to an appropriate set of variables has a resummation of logs in the exponent

\[
\ln d\sigma \sim L \left[ \sum_k (\alpha_sL)^k \right]_{LL_{exp}} + \left[ \sum_k (\alpha_sL)^k \right]_{NLL_{exp}} + \ldots \tag{6}
\]

then we will attach a subscript “exp” to the orders to indicate this.

A major concern with parton showers is how one handles the merging with matrix element (ME) calculations that describe the initial underlying hard process. One can consider a
simple setup where one declares that a scale, $\mu_0$, divides collinear from hard radiation. Here, emissions above $\mu_0$ are described through tree-level ME calculations, and those beneath by running SMC. Each regime would get a reasonable treatment, but naively interfacing the two leaves leading-log sensitivity to $\mu_0$. This is because the LO (in $\alpha_s$) result contains no Sudakov log resummation. Methods for carrying out matrix element and parton shower merging including this information have been considered in Refs. [26–28] and are referred to as CKKW-L and MLM. In CKKW-L, one distributes the particles in an event according to the probabilities given by the exact tree-level matrix element, with $\mu_0^2$ as a lower cutoff related to the perpendicular momentum between any two particles. One then clusters the event using the $k_T$ algorithm [29] to determine the splitting scales, $q_{iT}^2$. With these in hand, one reweights the event by multiplication by appropriate Sudakov factors, as well as factors of $\alpha_s(q_{iT})/\alpha_s(Q)$, where $Q$ is some hard scale. We can then run a parton shower algorithm on these squared amplitudes, vetoing any splitting $q_{iT}$ harder than $\mu_0^2$ to avoid double counting. It was demonstrated that the $n$-jet rate depends on $\mu_0$ only beyond NLL order, with the first missing term being $\alpha_s^2 \ln^2(Q/\mu_0)$. CKKW-L has been built into Sherpa [30].

Another important effect concerns soft gluons, which are also kinematically enhanced. Collinear emissions reinforce the picture of partonic radiation as an isolated jet since they get distributed within some narrow cone about the original hard parton. A priori, soft gluons have no preferred direction and can communicate between elements of the shower. Fortunately, wide-angle radiation only observes the net color charge contained in the cone of emission. Therefore, the pattern of soft radiation far from the collinear jet is not sensitive to splittings that have taken place within it. This coherent branching and angular ordering can be accommodated by methods such as evolving the shower by decreasing angle monotonically, as is done in Herwig [31], or by enforcing it with a veto in a virtuality-ordered shower (the rightmost expression in Eq. (1)), which is an option in Pythia [16]. Accounting for coherence properties leads to LL resummation for the soft emissions [32–36]. Additional considerations treated in shower programs include putting $\alpha_s$ at the $k_T$ scale of each splitting, and encoding momentum conservation at each vertex, which give the parton shower information beyond an analytic LO/LL calculation. These along with the overall choice in evolution variable (mass, $k_\perp$, angle, etc.) are treated in different fashions by different SMC codes.

There are of course further corrections to include to go to NLO in $\alpha_s$, denoted NLO($\alpha_s$), NLL in kinematic logs, and/or NLO in power corrections to the strong ordering, denoted NLO($\lambda$). The most effort to date has gone to working out the NLO($\alpha_s$)/LL contribution to incorporate one-loop corrected amplitudes at the top of the shower. Adding $\alpha_s$ corrections involves the numerical challenge of combining real and virtual results which separately have IR divergences. The basic resolution is to extract the pole-portion of the real emission of $i$-partons and include it along with the virtual contributions to the $i - 1$ case. Unfortunately, this does not sum leading logs. One cannot blindly extend the CKKW procedure to NLO($\alpha_s$)/LL, as it leads to double-counting problems; the Sudakov factors in the reweighting contain a portion of the one-loop contributions. Separately adding the full one-loop result would clearly overcount.

There are two main solutions to the NLO($\alpha_s$)/LL merging problem in the context of standard $1 \to 2$ splittings. MC@NLO [37] works by means of subtraction, finding the places
where the Sudakovs will contribute at NLO($\alpha_s$), and removing the splitting function contribution. This approach is conceptually clear. Since the full amplitude and splitting function portions are calculated separately before subtraction, the latter for each SMC program, this is time-consuming. Furthermore, since the subtractions occur for the amplitude squared, one cannot guarantee positivity of the result and must deal with negatively weighted events. To avoid the computational difficulties of process-by-process subtraction and negative weights, an alternative is the POWHEG algorithm [38]. It keeps the IR-safe NLO($\alpha_s$) cross section manifest, and defines a Sudakov factor based on a modified splitting function to handle LL$_{\text{exp}}$ and a subset of NLL$_{\text{exp}}$ resummation for the hardest emission. In this way, it makes use of quantities already obtained in the fixed order NLO($\alpha_s$) calculation, requiring fewer additional steps for its implementation for each known process. The conservation of probability obeyed by the splittings and related Sudakov factors avoid double countings and give back $\sigma_{\text{NLO}}$ upon integration.

A separate set of approaches goes beyond the $1 \rightarrow 2$ formalism to consider the radiation’s effects on one or more “spectators.” The consideration of an additional parton in the pre-emission configuration has led to work known as dipole subtraction and dipole antennas. The former was initially developed in [39, 40]. It explicitly subtracts the IR divergence from real emission via a simplified “dipole” term. Refs. [41, 42] have proposed algorithms based on these techniques. There has also been development on the theoretical side of subtractions by Nagy and Soper [43-46], with the aim of including spin and color effects, while improving the efficiency of implementation [47]. The original use of antennas came in the ARIADNE program, which treats the $2 \rightarrow 3$ splitting as its basic unit [48-51] and allows for exact momentum conservation. There have since been more systematic attempts to extract the $2 \rightarrow 3$ “antenna” functions from QCD and implement them in a shower, e.g. VINCIA [52]. Ref. [53] even derives spin-dependent antenna functions, though its SMC implementation is yet to appear.

A different approach is the GenEvA framework [54, 55] which allows the issues of phase space double counting and combining matrix elements and log resummation to be treated independently. This is done using effective theory ideas for how to separate scales. In this setup, one manifestly avoids negative weights and double counting by using multiplicative merging. For example, GenEvA yields a calculation that is equivalent to POWHEG for the NLO($\alpha_s$)/LL matching and at the same time a CKKW-L type LO($\alpha_s$)/LL matching for additional emissions. In a similar fashion, the power suppressed matrix element computations and subleading no-branching probabilities derived here could be implemented in GenEvA, and work in this direction is commencing.

Another approach to go beyond LL is to incorporate the contribution of the $O(\alpha_s^2)$ corrections to the Altarelli-Parisi splitting kernels, $P_{qq}^{(1)}$. This was done to resum soft logs to NLL for semi-inclusive variables in DIS and Drell-Yan [56]. In order to conserve probability, these corrections must be correctly accounted for in both the probability for real emission in Eq. (3), as well as no-branching branching probabilities. This is related to why POWHEG only implements them for the hardest splitting, where they have information from the full fixed-order computation. The KRKMC group incorporates the subleading real emission contributions into fully exclusive partonic configurations in SMC [57-59]. Some of the subleading contributions take the form of $1 \rightarrow 3$ splittings, requiring a modification of the usual
$1 \rightarrow 2$ algorithm. Similar to CKKW, the KRKMC groups corrections take the form of a multiplicative reweighting. For a particular configuration of partons in phase space, they reweight by a factor that includes the insertion of $1 \rightarrow 3$ “defects” and loop-corrected $1 \rightarrow 2$ splittings that account for the effects of $P^{(1)}_{q \bar{q}}$. If $\rho$ is the fully differential cross section, they define a corrected weight for $n$ partons, $w_n$ as:

$$w_n = \frac{\rho_{\text{LO}}(k_1, \ldots, k_n) + \sum_{r=1}^{n/2} \rho_{\text{NLO}}(k_1, \ldots, k_n)}{\rho_{\text{LO}}(k_1, \ldots, k_n)},$$

(7)

where $r$ determines the number of defect insertions in any configuration. Since this reweighting involves splitting probabilities and not subleading no-branching probabilities, it does not clearly improve the level of log resummation.

In this work we set up an EFT framework to classify and study perturbative $\alpha_s$ corrections, higher order log resummation and/or kinematic power corrections to parton showers. While the ultimate goal is to facilitate the implementation of a NLL/NLO($\alpha_s$) parton shower algorithm accounting for the leading deviations from strong ordering, our task here is much more modest. We focus primarily on kinematic power corrections in the fully differential cross section for an arbitrary number of final state emissions. That is, our main goal is to compute

$$\frac{d\sigma^\text{LO}}{d\vec{p}_1^3 \cdots d\vec{p}_n^3} + \frac{d\sigma^\text{NLO(\lambda)}}{d\vec{p}_1^3 \cdots d\vec{p}_n^3}.$$

(8)

Here NLO(\lambda) is the next-to-leading order power correction in the cross section, which involves terms that are NLO(\lambda) and NNLO(\lambda) in the amplitude. Similarly to [24, 25], we use an operator approach based on SCET. A main issue to resolve is taking into account different possibilities for the kinematic configurations of subsequent emissions, to go beyond the strong ordering described in Eq (1). The hierarchy between regions is expressed by the power counting parameter $\lambda \ll 1$. We overcome this issue by setting up a tower of related soft collinear effective theories, called SCET$_i$, which also helps us deal with several technical obstacles. We formulate the shower description as a standard matching procedure between operators in different SCET$_i$. Power corrections are encoded by performing matching computations at subleading order in the kinematic expansion. These corrections modify the processes that initiate the shower, modify certain early branching probabilities, and open up the $1 \rightarrow 3$ splitting channel. Virtual perturbative $\alpha_s$ corrections are included by performing matching calculations beyond tree level between SCET$_i$ theories. Finally, corrections to the Sudakov no-branching probabilities are encoded through anomalous dimensions of leading and subleading operators at the appropriate order within different SCET$_i$’s. When we refer to a parton shower in the context of our calculations, we mean an explicit amplitude formula that would agree numerically with a corresponding shower algorithm. We will carry out the necessary computations for the power corrected matching equations, and a subset of the required calculations for anomalous dimensions occurring for operators beyond the LL shower. This analysis includes the leading corrections to the shower from interference.

\[1\] In particular we note that soft NLL resummation may only be feasible at leading orders in $1/N_c$. [60, 61].
and from spin correlations. As much as possible, we attempt to give pointers for additional computations that are needed in places where our analysis is incomplete. For example, to simplify things we have not treated color correlations since doing so increases the basis of operators and the number of computations, but does not change the conceptual setup.

The outline of our paper is as follows. We present a brief overview of SCET in Section II A. We review the Bauer-Schwartz SCET shower method in Section II B and discuss the technical obstructions to extending it to include power corrections. In Section II C we present our SCET framework to resolve these issues. In Section III we analyze the LL shower in the SCET framework, and show that the transition between SCETs, SCET \( \rightarrow \) SCET, can be encoded by operator replacement rules on single parton collinear fields. Soft emissions in SCET are discussed, and we summarize the correspondence between SCET objects and LL shower ingredients. In Section IV we use the SCET formulation to classify and compute various corrections to the shower to \( \mathcal{O}(\lambda^2) \) in the cross section. Two main categories of branching corrections emerge, which we refer to as “hard-scattering” and “jet-structure.” We also discuss ingredients needed for renormalization group evolution corresponding to no-branching probabilities, derive all the LL anomalous dimensions for our subleading operators. Additionally, we mention the issues involved in obtaining NLL resummation from our results. A summary of corrections in the SCET framework is presented as a table in section IV E including the type of corresponding ingredients needed in a subleading shower. We present in Eqs. (104)-(106) a parton shower reweighting factor that should allow one to implement our corrections. We also discuss the correspondence of these corrections with those currently included in other Monte Carlos. Conclusions are given in Section V. At the present time, we do not have an algorithmic implementation of our power suppressed shower results, but work in this direction is in progress.

Many details are relegated to the Appendices. Further details about SCET can be found in Appendix A. We describe finite reparametrization transformations in Appendix B which is an important symmetry that we use in our matching computations to disentangle kinematic coordinate conventions from kinematic power corrections. Details on the matching of QCD \( \rightarrow \) SCET, SCET \( \rightarrow \) SCET, and SCET \( \rightarrow \) SCET can be found in Appendices C D and E, respectively. A complete list of the operators needed to compute Eq. (8) in SCET is given in App. E. Appendix F contains a cross-check on our results, where we integrate a subset of our power suppressed terms to rederive the abelian terms in \( P_{q \rightarrow qg}^{(1)} \), namely the \( \mathcal{O}(\alpha_s) \) correction to the \( q \rightarrow qg \) splitting function [62].

Those readers looking to find a quick summary of our results should look in Secs. III C and IV E.

II. OBTAINING THE PARTON SHOWER WITH SCET

A. SCET Basics

Soft-Collinear Effective Theory is an effective field theory of QCD that describes the interactions of collinear and soft particles [20–23]. We present here the basic ideas needed for our analysis of the parton shower, including how collinear sectors are organized into equivalence classes by the power counting parameters. Further SCET concepts are reviewed
The momentum, $p$, of any particle can be decomposed along two light-cone vectors, $n$ and $\bar{n}$, with $n^2 = 0$, $\bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$, as

$$p^\mu = \bar{p}^\mu \frac{n^\mu}{2} + p_\perp^\mu + n \cdot p \bar{n}^\mu \frac{\bar{n}^\mu}{2},$$

where $\bar{p} = \bar{n} \cdot p$ and the particle’s invariant mass is $p^2 = n \cdot p \bar{p} + p_\perp^2$. We use a Minkowskian notation for $p_\perp^2 = -\bar{p}_\perp^2$, where $\bar{p}_\perp$ is Euclidean. SCET’s degrees of freedom include $n_i$-collinear fields for a set of distinct directions $\{n_i\}$, and soft fields. A particle is collinear to a direction $n$ if its momentum scales as:

$$(n \cdot p, \bar{p}, p_\perp) \sim (\lambda^2, 1, \lambda) \bar{p},$$

where $\bar{p} \sim Q$ is some hard scale in the process, and $\lambda \ll 1$ is the SCET power counting parameter. A particle is soft if

$$(n \cdot p, \bar{p}, p_\perp) \sim (\lambda^2, \lambda^2, \lambda^2) Q.$$  

(11)

Collinear and soft fields have virtuality $\sim Q^2 \lambda^2$ and $Q^2 \lambda^4$, respectively. We obtain SCET from QCD by expanding in powers of $\lambda$, integrating out hard modes, and dividing the remaining ones into collinear and soft fields. Our collinear and soft degrees of freedom also contain all the IR regions that can be obtained by a rescaling of $\lambda \to \lambda^i$, for $i > 1$. The leading order SCET Lagrangian is

$$L_{\text{SCET}}^{(0)} = L_s^{(0)} + \sum_{n \in \{n_i\}} L_n^{(0)},$$

(12)

where $L_n^{(0)}$ is defined in Eq. (A9) and has only interactions among particles collinear to the same $n$. $L_s^{(0)}$ is the Lagrangian for soft interactions discussed further in App. A. Particles collinear to different directions can interact either by the exchange of soft modes, or from their coupling to other sectors in external operators. Two collinear sectors in SCET, $n_1$ and $n_2$, are distinct if

$$n_1 \cdot n_2 \gg \lambda^2,$$

(13)

so any particle is collinear to at most one direction within a given SCET. The collinear sectors $\{n_i\}$ in SCET are really sets of equivalence classes of null vectors, $\{[n_i]\}$, where the equivalence class is $[n_j] = \{ n \in [n_j] | n \cdot n_j \leq \lambda^2 \}$. A class $[n_j]$ consists of all light-like vectors connected to $n_j^\mu$ by a type-I reparametrization invariance (RPI) transformation, $n_j^\mu \to n_j^\mu + \Delta^\mu_{n_j\perp}$, where the scaling of the transformation parameter is $\Delta_{n_j\perp} \sim \lambda$ (see App. B for a detailed discussion of RPI). Physically, the class $[n_j]$ corresponds to light-like vectors for particles whose momenta is in a cone centered on $\vec{n}_j$ with an opening angle $\sim \lambda$ (cf. Fig. 15).

\[\text{Our primary interest here is the perturbative structure of jets, so we use SCET}_1 \text{ theories with collinear and ultrasoft modes. For simplicity we will always use the phrase soft in place of ultrasoft.}\]
Thus, the defining concepts of a SCET-theory are its hard-scale $Q$, its collinear sectors $\{n_i\}$, and its power counting parameter $\lambda$ which governs the importance of operators and the size of the collinear sectors in phase space.

Most of our discussion will involve interactions with collinear fields, and we use the notation $\chi_n$ for quarks and $B_{n\perp}^\mu$ for gluons (definitions of these fields can be found in Eq. (A13), and they incorporate collinear Wilson lines built out of $\bar{n} \cdot A_n$ fields). We can match QCD onto a series of SCET operators organized by powers of $\lambda$. The key building blocks are: $\chi_n$, $B_{n\perp}^\mu$, and $P_{n\perp}^\mu$ (a type of derivative operator that yields the perpendicular momentum of an $n$-collinear field), each of which scale as $\lambda$ in the kinematic power counting. A general notation for the $i$-parton operators we will consider is:

$$O^{(j,k,\ell)}(n_1^{[\ell_1]}, \ldots, n_j^{[\ell_1+k]}) = \left[ \prod_{a=1}^{j/2} (P_{n_a\perp})^{\ell_a} \Gamma_n \right] \left[ \prod_{b=j/2+1}^j (P_{n_b\perp})^{\ell_b} \bar{\chi}_n \right] \left[ \prod_{c=1}^k (P_{n_c\perp})^{\ell_c} g B_{n_c\perp} \right],$$

where the number of partons is the sum of quarks and gluons, $j + k = i$, and the total number of $\perp$ derivatives is $\ell = \sum_{m=1}^{j+k} \ell_m$. In the operator argument, we list the index labels, $n_d$, of the parton fields on the RHS. The superscripts in the argument on the LHS denote the number of derivatives acting on the field with the corresponding direction. There may be a degeneracy among the index labels, $n_d$, and so the operator has at most $i$ distinct collinear directions. The scaling of these operators is $O^{(j,k,\ell)} \sim \lambda^{j+k+\ell}$. They are tensors in the space of spinors and Lorentz vectors, and the indices get contracted with structures contained in the Wilson coefficient $C$ for the operator. If $C O$ is a Lorentz scalar, then $j$ is even. Since the collinear fields carry a label referring to a specific light-cone vector, these operators describe particles in a specific region of phase space. SCET therefore distinguishes situations with the same particle content, but different kinematics, in a straightforward way.

For example, one can take an amplitude for three external particles: a quark, gluon, and antiquark. We can consider two different configurations, $|q_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle$ and $|q_{n_1} g_{n_1'} \bar{q}_{\bar{n}}\rangle$. In the first, shown in Fig. (I), the quark and the gluon are $n_0$-collinear, and the antiquark is collinear to a different direction, $\bar{n}$. Here the amplitude is described by operators with two distinct directions, say

$$O^{(2,0,0)}(n_0, \bar{n}) = \bar{\chi}_{n_0} \Gamma_{\chi_n} \sim \lambda^2,$$

$$O^{(2,1,0)}(n_0, n_0, \bar{n}) = \bar{\chi}_{n_0} g B_{n_0\perp}^\mu \Gamma'_{\chi_n} \sim \lambda^3,$$

where the form of the Dirac structures $\Gamma$ and $\Gamma'$ are not central to our discussion here. $O^{(2,0,0)}$ can emit $\bar{n} \cdot A_{n_0}$ gluons from the Wilson line in $\chi_{n_0}$, but requires a Lagrangian insertion to emit an $A_{n_0\perp}^\perp$ gluon. Schematically, the amplitude for a transverse gluon has contributions:

$$A^I = \int dx \langle 0| T \{ L_{n_0}^{(0)}(x) \bar{\chi}_{n_0} \Gamma_{\chi_n}(0) \} |q_{n_0} g_{n_0} \bar{q}_{\bar{n}} \rangle + \langle 0| \bar{\chi}_{n_0} g B_{n_0\perp}^\mu \Gamma'_{\chi_n}(0) |q_{n_0} g_{n_0} \bar{q}_{\bar{n}} \rangle.$$ (16)

In Fig. (II), each of the particles is collinear to a distinct direction, so no cone of size $\sim \lambda$ fits two of the momenta. In this case, the amplitude can only come from an operator with three distinct labels, such as $\bar{\chi}_{n_1} B_{n_1\perp}^\mu \Gamma''_{\chi_n}$:

$$A^{II} = \langle 0| \bar{\chi}_{n_1} g B_{n_1\perp}^\mu \Gamma''_{\chi_n} |q_{n_1} g_{n_1'} \bar{q}_{\bar{n}} \rangle.$$ (17)
FIG. 1: Different kinematic configurations of a final state with a quark, antiquark, and gluon are described by different SCET operators. In (I), the quark and the gluon are collinear to the direction $n_0$, represented by their sharing a common cone. In (II), the vectors $q'_1$ and $k'_1$ are too far apart to be collinear. The Feynman diagrams show that collinear particles can come from Lagrangian insertions, whereas non-collinear ones arise exclusively from higher-multiplicity operators. The Feynman diagram in (I) only depicts the first term on the RHS of Eq. (16).

The ability of SCET to cleanly separate contributions such as those in Eqs. (16) and (17) will be useful for formulating a complete set of power suppressed corrections to the parton shower.

B. Bauer-Schwartz Method

The original application of SCET to study and improve the parton shower was carried out in [24, 25] by Bauer & Schwartz. The main reasons why SCET is useful for this are:

- The SCET fields, soft and collinear quarks and gluons, have support in the infrared exactly where the parton shower amplitudes have their dominant contributions in phase space.

- Since SCET is improvable order-by-order in the kinematic expansion parameter, $\lambda$, one has the potential to systematically correct the shower.

We will give a short overview of the Bauer-Schwartz approach, and then discuss the complications that arise when trying to extend the analysis to NLO in the $\lambda$ expansion, namely NLO($\lambda$). In this section we will use notation that is not found elsewhere in the paper to retain consistency with [24, 25].
The procedure of [24, 25] starts by constructing $i$-parton operators, $O_i$, through matching SCET to QCD at a hard scale. For example, their $O_2$ will equal $O^{(2,0,0)}(n_1, n_2)$ in the notation of Eq. (14), and $O_3$ will be $O^{(2,1,0)}(n_1, n_2, n_3)$. As we run $O_i(\mu)$ down, the leading log renormalization group evolution (LL RGE) does not mix operators and the exponential evolution kernel encodes the no-branching probability. The evolution continues until another parton becomes apparent at a scale $\mu = p_T$.

If we have an $i$-parton operator, $O_i = O^{(j,i-j,0)}(n_1, \ldots, n_i)$ with all $n$’s distinct, then it has the RG solution $O_i(\mu) = U^{(j,i-j,0)}(Q, \mu)O_i(Q)$ with

$$U^{(j,i-j,0)}(Q, \mu) = \exp \left[ -\int_\mu^Q \frac{d\mu'}{\mu'} \gamma^{(j,i-j,0)}(\mu') \right],$$

(18)

where $\gamma^{(j,i-j,0)}$ is the operator’s anomalous dimension. The leading-log resummation effects of the Sudakov factor in the PS enter through one-loop operator running in SCET, as dictated by the cusp anomalous dimension. The one-loop cusp portion is especially easy to calculate in SCET as it depends solely on the number of collinear fields, even though the calculations have loops involving soft ones as well [24, 25],

$$\gamma_{LL, n_i}^{(n_i, n_0, 0)}(\mu) = -\frac{\alpha_s}{\pi} \left[ \frac{n_i}{2} C_F + \frac{n_0}{2} C_A \right] \log \frac{\mu^2}{Q^2}. $$

(19)

This form of the kernel gives a product of Sudakov factors which are the no-branching probabilities for each parton in the operator:

$$U_{LL, j-i,0}^{(j,i-j,0)}(Q, \mu) = \Delta_{\gamma}(Q, \mu) \Delta_{g^2}(Q, \mu).$$

(20)

Here, as in [27], one accounts for leading-log effects for any particle multiplicity by simply multiplying matrix elements by appropriate Sudakov factors.

As we run $O_i(\mu)$ down, another parton becomes apparent at a scale $\mu = p_T$. To account for this, Bauer-Schwartz devised a “threshold matching” of $O_i$ to a new, higher multiplicity operator, $O_{i+1}^{(i)}$, where the subscript still denotes the number of partons in the operator and the superscript tracks the parent operator. The general threshold matching equation is

$$[C_n^{(j)}(O_n^{(j)})]_{\mu=p_T+\epsilon} = [C_{n+1}^{(j)}(O_{n+1}^{(j)})]_{\mu=p_T-\epsilon}.$$  

(21)

After further running and threshold matching, we eventually have $O_n^{(i)}$ for various $n > i$. The $n - i$ particles emitted at increasingly lower scales by this process correspond to the parton showering of the original fields created at the hard scale by $O_i$. Additionally, they also showed that an appropriate list of SCET operators ($O_i$’s and $O_{i+1}^{(i)}$’s) can interpolate between fixed-order QCD and parton shower (PS) calculations of IR-safe observables. Furthermore, they derived the $O(\alpha_s)$ effects from matching QCD to SCET at one-loop for $O_3$.

That subsequent emissions reproduce the usual parton shower splitting function emerges easily from SCET. Consider an operator $O_i = \bar{\chi}_{n_0}\Omega$, where $\Omega$ is arbitrary and we have made explicit a single collinear quark field, $\bar{\chi}_{n_0}$. If we emit a collinear gluon from this quark, $q(q_0^\mu) \rightarrow q(q_1^\mu)g(k_1^\mu)$, the amplitude for the process is

$$A_{LO}^{X+qg} = \bar{u}_{n_0}(q_1)\rho \frac{\bar{q}_0}{q_0} \Omega,$$

(22)
where \( u_{n_0} \) is the collinear quark spinor, and \( \rho^\alpha \) is the combination of the SCET single gluon emission Feynman rule plus the \( \bar{\chi}_{n_0} \) Wilson line emission (the quark \( \mathcal{L}_{n}^{(0)} \) can be found in Eq. (A9)),

\[
\rho^\alpha = n_0^\alpha + \frac{(\bar{q}_1)_{n_0\perp} \gamma_{n_0\perp}^\alpha}{\bar{q}_1} + \frac{\gamma_{n_0\perp}^\alpha (\bar{q}_0)_{n_0\perp}}{\bar{q}_1} - \bar{n}^\alpha \left[ \frac{q_0^2}{k_1} \right. + \frac{(\bar{q}_1)_{n_0\perp} (\bar{q}_0)_{n_0\perp}}{\bar{q}_1} \left. \right].
\]

Note that \( \rho^\alpha \) in SCET comes entirely from \( \bar{\chi}_{n_0} \) without reference to anything residing in \( \Omega \). The subscript \((n_0\perp)\) refers to components perpendicular to \( n_0^\mu \) and \( \bar{n}^\mu \), which we denote by \( \perp \) for the remainder of this computation. The amplitude in Eq. (22) is gauge invariant and \( k_1^\alpha \rho_\alpha = 0 \). Squaring \( A_{LO}^{X+gq} \) and summing over spins we have \( \sum_{\text{spin}} \bar{u}_{n_0}(q_1) u_{n_0}(q_1) = \bar{q}_1 \gamma_0 / 2 \), and the gluon polarization sum denoted \( \sum_{\text{spin}} \epsilon_\alpha \epsilon_\beta^* = d_{\alpha\beta} \). Since \( \rho^\alpha \) commutes with \( \gamma_0 \), we get an answer proportional to \( \rho^\alpha \rho^\beta d_{\alpha\beta} \), where without loss of generality we can use a light-cone gauge, \( d_{\alpha\beta} = -g_{\alpha\beta} + (\bar{n}_\alpha k_1 \gamma_0 + k_1 \gamma_\alpha) / k_1 \). Crucially, this is a Dirac scalar:

\[
\rho^\alpha \rho^\beta d_{\alpha\beta} \equiv |\rho|^2 = 2 \left( \frac{2q_0^2}{k_1 \bar{q}_0} - \frac{q_1^2}{\bar{q}_1} + \frac{2q_0 \cdot q_1}{\bar{q}_0 \bar{q}_1} - \frac{q_0^2}{\bar{q}_0^2} \right) \times \mathbb{I}_4,
\]

where we have used the on-shell conditions \( q_0^2 = 0 \) and \( k_1^2 = 0 \).

In a frame where \( q_0^2 = 0 \) we have \( q_1 \perp = -k_1 \perp \) and \( \bar{q}_0 / q_0^2 = 1 / (n_0 \cdot q_0) \). Here \( n_0 \cdot q_0 = n_0 \cdot k_1 + n_0 \cdot q_1 = -k_1 \perp / (\bar{q}_0 z (1 - z)) \), where \( z \equiv \bar{q}_1 / q_0 \). Thus we have the simpler expression

\[
\rho^\alpha = n_0^\alpha + \frac{(\bar{q}_1)_{n_0\perp} \gamma_{n_0\perp}^\alpha}{\bar{q}_1}.
\]

which we have written in light-cone gauge without the Wilson line contribution \((\propto \bar{n}^\alpha q_0^2)\), and

\[
\rho^\alpha \rho^\beta d_{\alpha\beta} = 2 \left( \frac{2 n \cdot k_1}{k_1} - \frac{q_1^2}{\bar{q}_1} - \frac{q_0^2}{\bar{q}_0^2} \right) \times \mathbb{I}_4 = -2 \frac{k_1^2}{q_0^2} z^2 (1 + z)^2.
\]

Putting these properties together in the full amplitude squared we get

\[
|A_{LO}^{X+gq}|^2 = \frac{g^2 C_F}{(n_0 \cdot q_0)^2} \frac{\bar{q}_1}{2} \mathbb{I} [\bar{q}_0 \rho^\alpha \Omega \rho^\beta d_{\alpha\beta}] = \frac{g^2 C_F}{(n_0 \cdot q_0)^2} |\rho|^2 \mathbb{I} \left[\frac{\bar{q}_0}{2} \Omega \Omega^\dagger\right]
\]

\[
= g^2 C_F 2z \frac{(1 + z)^2}{k_1 \perp^2} \mathbb{I} \left[\frac{\bar{q}_0}{2} \Omega \Omega^\dagger\right].
\]

Thus, all information about the emission factors out to the front and is independent of the rest of the process encoded by \( \Omega \). Since the power expansion is built into SCET, there was no need to expand terms in the amplitude to obtain this result (unlike the analogous computation in full QCD). In order to recover Eq. (3), we still need to include the \( z \)-dependence from phase space, since \( F_{jk}^{(0)}(z) \) operates at the level of the cross section. Using \( d^3k/(2E_k) = d\bar{q} d^2q_{1\perp} / (2k) \), for \( q_1 \) and \( k_1 \) we have

\[
\frac{d\bar{q}_1 d^2q_{1\perp} d\bar{k}_1 d^2k_{1\perp}}{2\bar{q}_1} \rightarrow \frac{d\bar{q}_0 d^2q_{0\perp} d\bar{k}_0 d^2k_{1\perp}}{2\bar{q}_0} \frac{dz d^2k_{1\perp}}{2z(1 - z)},
\]

\[13\]
where the arrow means that we insert \(d^4q_0 \delta^{(4)}(q_0 - q_1 - k_1)\) and integrate \(d^3q_1\) along with \(d(n_0 \cdot q_0)\). Thus, we recover the expected \(1/(1-z)\) dependence from the measure. Combining pieces and performing the trivial azimuthal integral \(d\phi_{k_1}\), we get the expected expression:

\[
d\sigma_{X+qg} = dz \frac{d^2k_1}{k_1} P^{(0)}_{q\rightarrow qg}(z) d\sigma_{X+q},
\]

(29)

where \(P^{(0)}_{q\rightarrow qg}(z)\) is the quark splitting function in Eq. (3). Here \(d\sigma_{X+q}\) is the cross section for the rest of the process with emission of a momentum \(q_0\) quark, and the corresponding amplitude squared is \(\text{Tr} \left[ \bar{q}_0 \gamma_\mu q_0 \right]\). Whether \(\Omega\) represents a simple hard current or an entire chain of collinear splittings, we see that the \(q \rightarrow qg\) emission factors out with the expected soft-collinear double pole, as in Eq. (2).

In order to obtain their results, Bauer-Schwartz introduced choices and approximations at several points which obscure the path toward systematically computing NLO(\(\lambda\)) corrections. Indeed, they concluded that obtaining these corrections may be prohibitively difficult [25]. Some of the issues one encounters trying to work at higher orders are:

1. At NLO(\(\lambda\)), it becomes crucial to distinguish which simplifications correspond to approximations with power corrections, and which involve a choice of coordinates where a symmetry makes the final answer coordinate independent. For example, a collinear state typically has nonzero momentum components perpendicular to the index \(n\) of the field that annihilates it. Refs. [24, 25], however, dictated that collinear SCET fields in their operators only create particles whose momenta perfectly align with their index direction, \(n\):

\[
\chi_n|q\rangle = \delta_{n,nq}, \quad \text{where } n^\mu = q^\mu/E_q,
\]

(30)

leaving it ambiguous what amount of symmetry protects this choice. Eq. (30) enforces certain kinematical restrictions on final state particles, and requires that fermion fields be rotated to an appropriate \(n^\mu\) via \(\xi_n \rightarrow (\xi_n \cdot /4)\xi_n\).

2. At LO, it was possible to avoid a potential double counting between collinear and soft fields by dropping soft emission and Wilson line emission, and taking only collinear emissions with transverse polarization. The threshold matching procedure is designed to avoid double counting of collinear operators, such as a Lagrangian emission from \(\mathcal{O}_2\) and direct emission from \(\mathcal{O}_3^{(2)}\), since only one of these is allowed to operate at a time. However, the threshold matching in Eq. (21) makes the technical procedure for incorporating power corrections unclear.

3. Threshold matching contains another impediment to systematic improvement. Through this procedure, the initial operator \(\mathcal{O}_2\) has nonzero projection onto Fock states of any multiplicity, but the number of particles created by an operator is a scale-dependent question. The matching scales are determined by the strong ordering kinematics, \(p_1 \gg \ldots \gg p_m\). At the scale of an emission, say \(p_1\), one threshold matches to the operator \(\mathcal{O}_3^{(2)}\), which only adds one parton at a time. However, going to higher orders in the shower necessitates more general configurations.
In carrying out their method, Bauer-Schwartz carefully enumerated the above approximations. They affect the ability to include corrections in $\lambda$, but do not impact the terms necessary for a LL shower.

Building on the work of Refs. [24, 25], the main goal of the framework we develop in the next section is to overcome this list of issues so that we can determine power corrections to the shower using SCET.

C. Using SCET

The main feature of the parton shower is the ability to capture the dominant physics of particles emitted in kinematically hierarchical regions of phase space. Our goal is to formulate the SCET interface with the shower using a standard sequence of matching and running steps in different versions of SCET,

$$QCD \rightarrow SCET_1 \rightarrow SCET_2 \rightarrow \cdots \rightarrow SCET_N.$$  \hfill (31)

We refer to this as the SCET$_i$ procedure. The key distinction between a SCET at one stage and the next is the definition of the corresponding resolution parameters $1 \gg \lambda_1 \gg \lambda_2 \gg \cdots \gg \lambda_N$, where $\lambda_i$ sets the power counting for SCET$_i$. As we move down the chain, the corresponding SCET resolves smaller $\sim (Q\lambda_j)^2$ invariant masses and relative squared perpendicular momenta, and has a different meaning for its collinear sectors $\{[n_i]\}_{SCET_j}$. To keep track of this, we will attach a subscript to the operators to denote the SCET$_i$ in which its fields live,

$$O^{(j,k,\ell)}_i(n_1,\ldots,n_j+k).$$  \hfill (32)

Effectively with Eq. (31), we partition the momenta of partons in the shower history into classes,

$$\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_N,$$  \hfill (33)

where $\Omega_j$ contains the momenta of all propagators having $p^2 \sim (Q\lambda_j)^2$ or smaller, or an equivalent condition on relative perpendicular momenta. The allowed momenta in $\Omega_i$ correspond to the collinear modes of SCET$_i$. The sequence of SCET$_i$’s is truncated when we resolve a scale of order the parton shower cutoff, $Q\lambda^N_T = p_{cut}^T \simeq 1$ GeV, that is in SCET$_N$.

Note that we do not associate a large hierarchy to the hard scales $\bar{p}_i$ between SCET$_j$ and SCET$_{j+1}$. That is to say we do not associate the energy loss due to splitting with a power of $\lambda_j$. Instead if $Q$ is the scale of the primary hard interaction then we consider $\bar{p}_i \sim \eta^j Q$ in SCET$_j$, where $\eta \gg \lambda_j$ and for numerical estimates we can take $\eta \sim \frac{1}{2}$. (For each branching the geometric mean of the two daughters’ $\bar{p}$ fraction averages to 0.4 which is roughly one half.) Parametrically, the decrease in the parton energy is not as rapid as that for the perpendicular momenta encoded in the power counting parameter $\lambda_j$. In principle, we can account for $\eta$ as a separate factor. In practice, we will be most interested in tracking powers of $\lambda_j$ and will only include $\eta$ factors in places where the corresponding powers of two have a numerical impact on the implementation, or if we wish to disentangle the changes in
offshellness due to strong-ordering effects and those coming from the more modest decrease in $\bar{p}_i$.

The strongly ordered configuration of partons in Eq. (11) corresponds to removing a single $q_j^2$ in $\Omega_j$ as we pass from $\Omega_j \rightarrow \Omega_{j+1}$. However, with Eq. (33), nothing stops us from having multiple emissions at a single scale. If two mother particles, with $q_j^2$ and $q_{j+1}^2$, are associated to the same $\Omega_k$, then when we integrate out that scale in SCET_{k+1} this configuration just contributes to an operator with a different parton multiplicity from the strongly ordered one. Thus, with Eq. (33) there is no obstacle to considering corrections from an arbitrary assignment of $q_j^2$’s to $\Omega_k$’s. This resolves issue 3. of Sec. II B since we can treat emissions where the shower tree has momenta with the same parametric scaling in $\lambda$.

To carry out calculations in the SCET\textsubscript{i} framework, it is convenient and sufficient to take a specific definition of the power counting parameters, $\lambda_i = (\lambda)^i$. We want the hierarchy between neighboring splittings to stay the same throughout the shower so as not to privilege any portion of it. We will see in Sec. [IV D] that this democratic setup allows us to interpret part of our $O(\lambda)$ corrections to i-parton amplitudes as universal corrections to the splitting probability, given at LO by Eq. (3). As we go to lower scales, our definition of collinearity also changes, and by analogy to Eq. (10), fields collinear to $n$ within $\Omega_i$ have:

$$(n \cdot q_i, \bar{q}_i, q_{i\perp}) \sim (\lambda^{2i}, 1, \lambda^i) \bar{q}_i,$$

and virtuality $\sim (\bar{q}_i)^2 \lambda^{2i}$. In SCET\textsubscript{i}, $\mathcal{L}_n^{(0)}$ again only couples collinear fields in the same direction $n$. Since different SCET\textsubscript{i}'s have different definitions of collinearity, our description of identical physical processes changes when we switch to a theory with a lower scale. For convenience, we will use the same auxiliary vector $\bar{n}^\mu$ for any $n_j$-collinear field in any SCET\textsubscript{i}. If $\bar{n}$ is a valid auxiliary vector for $n$-collinear fields in SCET\textsubscript{1}, then it is readily apparent that it will be a valid choice for all subsequent collinear fields in SCET\textsubscript{i}'s that descend from an $n$-collinear mother in SCET\textsubscript{1}. Our default choice is stronger: given a set of light-like

vectors in $\{n_j\}$ in SCET\textsubscript{1} we take a light-like $\bar{n}$ that is parametrically close or aligned with the antiquark direction. We then adjust the magnitude of $n_0^2$ and of $\bar{n}_j$ so that $n_j^2 = 0$ and $\bar{n} \cdot n_j = 2$ (for a related discussion based on RPI see Appendix C).

We depict the different descriptions of the same physical configuration in Fig. 2, where the left panel is in SCET\textsubscript{i} and the right panel is in SCET\textsubscript{i+1}. In SCET\textsubscript{i}, the quark ($\bar{q}_i$) and gluon ($\vec{k}_1$) are $n_0$-collinear. This means that at LO they are emitted from a $qqg$ vertex in the LO SCET\textsubscript{i} Lagrangian (or a Wilson line interaction). Schematically, the amplitude for a $\perp$-polarized gluon looks like

$$A^{q\bar{q}g} = C^{(2,0,0)} \int dx \langle 0 | T \{ \mathcal{L}_{\text{SCET}_1}(x) O^{(2,0,0)} \} | q\bar{q}g \rangle,$$

namely like the first term in Eq. (16). The right-hand panel of Fig. 2 denotes the same configuration as seen by SCET_{i+1}. The scale of this theory is lower and the definition of collinearity stricter, so the quark and gluon are not collinear here. Therefore, the amplitude now comes from a three-parton operator,

$$A^{q\bar{q}g} = C^{(2,1,0)} \langle 0 | O^{(2,1,0)} | q\bar{q}g \rangle,$$

\footnote{From here on, we will drop the superscript (0) and the subscript $n$ from the collinear Lagrangian.}
FIG. 2: The same three-parton process as seen in two different SCETs, SCET\(_{i}\) and SCET\(_{i+1}\). Above: Kinematic configuration of the quarks and gluon. The solid cones represent the regions considered collinear to the vectors drawn. Below: Feynman diagrams for the corresponding amplitude. Note that in SCET\(_{i+1}\) we have removed a degree of freedom that propagates in SCET\(_{i}\). The amplitude thus comes from a higher dimension operator \(O^{(1)}_{i+1}\), rather than from a time-ordered product of \(L_{\text{SCET}}\) with \(O^{(0)}_{i}\), as it did in SCET\(_{i}\).

as in Eq. (17). We match SCET\(_{i}\) → SCET\(_{i+1}\) to calculate \(C^{(2,1,0)}\).

Given the above conventions and with the notation in Fig. 2 at hand, it is worth stating some simple kinematic relations that we will use later on. Take an \(n_0\)-collinear mother particle of momentum \(q^\mu_0 = q^\mu_1(\bar{n}^\mu/2) + \bar{q}_0(n_0^\mu/2)\). Let \(q_0\) decay to two onshell massless daughters, \(k_1\) and \(q_1\), with momentum fractions \(x\) and \((1-x)\), back-to-back \(\perp\)-momenta \(\vec{k}_\perp\), and light-like directions \(n_1\) and \(n'_1\), then

\[
k_1^\mu = k_1^\perp \bar{n}_1^\mu + \bar{k}_1 \frac{n_0^\mu}{2} + k_1^\perp = \bar{k}_1 \frac{n_1^\mu}{2}, \quad q_1^\mu = q_1^\perp \bar{n}_1^\mu + \bar{q}_1 \frac{n_0^\mu}{2} - k_1^\perp = \bar{q}_1 \frac{n_1^\mu}{2}. \tag{37}
\]

Note that our convention of using the same \(\bar{n}_1^\mu\) auxiliary vector ensures that in these decompositions the momentum multiplying \(n_0^\mu\) is the same as the momentum multiplying \(n_1^{(\perp)\mu}\). The collinearity of \(k_1\) and \(q_1\) can be determined by the size of \(k_\perp^2\), \(q_0^2\), or \(n_1 \cdot n'_1\), and the relation between these three choices is

\[
n_1 \cdot n'_1 = \frac{2 \vec{k}_\perp^2}{(\bar{q}_0)^2 x^2(1-x)^2} = \frac{2 q_0^2}{(\bar{q}_0)^2 x(1-x)}. \tag{38}
\]

Since we take \(\vec{k}_\perp^2/(\bar{q}_0)^2 \sim \lambda_i^2\) in SCET\(_{i}\), we have \(q_0^2/(\bar{q}_0)^2 \sim \lambda_i^2/\eta^2\) and \(n_1 \cdot n'_1 \sim \lambda_i^2/\eta^4\). Thus, all three choices are equivalent for counting powers of \(\lambda_i\), but differ with respect to how powers of the energy loss parameter \(\eta \sim 1/2\) appears.

After this introduction to SCET\(_{i}\), we now list some technical advantages of this framework for our analysis:
1. Collinear fields in SCET with different \( n \)-labels, as well as soft fields, do not overlap in Hilbert space. This allows us to separate an \( i \)-jet process with \( i \) distinguished partons, from an \( (i-1) \)-jet process with \( i \) partons, where two are collinear and unresolved. Lower-scale SCET\(_i\)’s distinguish configurations more finely based on their stricter definition of collinearity. This resolves issue 2, avoiding the double-counting of similar configurations, from Sec. II B. This SCET property also illuminates simplified structures in the power corrections, such as the form of the amplitude interference (cf. section IV D).

2. Soft modes communicate between collinear sectors and threaten the factorization of different jets. Fortunately, SCET constrains the interactions they have with collinear fields. In fact, one can decouple them using soft Wilson lines in the LO SCET Lagrangian. At LO, using the SCET\(_i\) soft Wilson lines, we maintain factorization, obtain angular ordering, and rederive the coherent branching of soft emissions (cf. section II B). Soft interactions which are power suppressed can also be systematically studied in SCET with Lagrangians available in the literature [64–66], which we give in Eq. (108).

3. In SCET\(_i\), we have a symmetry group RPI\(_i\) which corresponds to coordinate choices. In SCET\(_{i+1}\), only a subset of this, RPI\(_{i+1}\) \( \subset \) RPI\(_i\), remains a symmetry of the new theory. The kinematics in the coset portion RPI\(_i\)/RPI\(_{i+1}\) within SCET\(_i\) give a set of higher-dimension operators in SCET\(_{i+1}\), and describe configurations which would not otherwise be contained in the SCET\(_{i+1}\) Lagrangian (cf. section II B and Appendix B). This resolves issue 1 from Sec. II B making the difference clear between approximations and conventions chosen for simplicity.

4. In matching between SCET\(_i\) and SCET\(_{i+1}\), suppressed operators in the lower-scale theory are needed to reproduce the physics of the higher one. It can be proven that all higher order purely collinear operators can be built from quark fields \( (\chi_n) \), perpendicular gluon fields \( (B_{\perp n}) \), and the perpendicular momentum operators \( (P_{\perp n}) \) [67]. Thus the symmetries and equations of motion of SCET greatly simplify the operator basis one needs to consider at each order in \( \lambda \) (cf. section IV and Appendices C, D, and E).

The final SCET\(_N\) corresponds to the scale where the shower stops, i.e. where \( Q \eta^N \lambda^N \sim p_T^{\text{cut}} \). In SCET\(_N\), we only need the coefficients of the operators where all collinear partons have distinct \( n \)-labels, and which have no \( P_{n\perp} \)’s: \( C_N^{(j,k,0)} C_N^{(j,k,0)} \). Once we reach the physical resolution scale, it is only meaningful to have one collinear parton in each distinguished block of phase space. Using RPI\(_N\), we can set \( n_j^{\mu} = p_j^{\mu}/p_0^{\mu} \). This is as in Eq. (30), but we only do this when we run up against the physical limit that requires just one parton per equivalence class. At intermediate stages, we allow different fields to share \( n \)-labels, which also results in operators containing \( P_{n\perp} \). The coefficients \( C_N^{(j,k,0)} \) encode the history of the shower. They can be written entirely in terms of: dot products \( n_i \cdot n_j \), equivalent to products of final parton momenta, which carry the scaling in \( \lambda \); hard momenta \( \vec{p}_i \), the renormalization scale \( \mu \), and collinear cutoff parameters encoded in \( \Theta \)-functions.

As far as the shower is concerned, \( \lambda \) is merely a bookkeeping device which determines what pieces are needed beyond LO. One could try defining \( \lambda_1 = k_{1\perp}/Q \), \( \lambda_2 = k_{2\perp}/Q \), etc., but this
FIG. 3: Operators that reproduce strongly-ordered gluons are constructed through a series of matching computations with emissions in different SCET\textsubscript{j}. The horizontal dashed arrows refer to the radiation of a gluon from a time-ordered product of the SCET\textsubscript{j} Lagrangian with the operator creating fields at the point marked by ⊗. The diagonal solid arrows denote the matching onto a higher multiplicity operator in SCET\textsubscript{j+1}.

| $p^2 \sim Q^2$ | QCD          |
|---------------|--------------|
| $p^2 \sim Q^2 \lambda^2$ | SCET\textsubscript{1} |
| $p^2 \sim Q^2 \lambda^4$ | SCET\textsubscript{2} |
| $p^2 \sim Q^2 \lambda^6$ | SCET\textsubscript{3} |
| ... | ... |
| $p^2 \sim Q^2 \lambda^{2i}$ | SCET\textsubscript{i} |

is not ideal since there is a chance for events where $k_{1\perp} \sim Q$ or $k_{1\perp} \sim p_T^{\text{cut}}$. The organization in Eq. (33) instead exploits the fact that on average showers are strongly-ordered. Our expansion in $\lambda$ will then on average give a description of the most likely deviations from strong-ordering. Our goal in using the SCET\textsubscript{i} framework is to extract an amplitude suitable for reweighing the parton shower to this level of accuracy.\footnote{As a well-defined EFT, one certainly could also do standard factorized cross section computations in any SCET\textsubscript{i} if one wanted.} From the SCET side, we pass to the shower weights built from SCET\textsubscript{N} squared amplitudes (cf. Eqs. 104-107). They contain the information needed to describe a strongly-ordered shower and its leading kinematic corrections.

Before proceeding to our computations, it is worth commenting explicitly on which shower ingredients we do not compute. We only treat the case of a showering quark $q \rightarrow qg$ and in general take the abelian limit of QCD ($C_A = 0$). We have left out gluon splittings, $g \rightarrow q\bar{q}$ and $g \rightarrow gg$, from this analysis, though we expect that the extension to these cases should be straightforward. We have also not determined the effect of NLO($\lambda$) power corrections from subleading soft interactions, although we briefly examine the factorized structure of LO softs in section (III B). These items are all left to future investigations.
FIG. 4: The opening angle of the light grey (blue) cone is $\sim \lambda^{2i}$, and the opening angle of the dark grey (red) one is $\sim \lambda^{2(i+1)}$. The particle with momentum $p$ is collinear to both $n$ and $n'$ in SCET$_i$, but only to $n'$ in SCET$_{i+1}$. RPI$_i$ allows us to move the field label, $n$, to any location inside the appropriate cone for SCET$_i$ while keeping the theory invariant.

III. PARTON SHOWER IN SCET VIA OPERATOR REPLACEMENT

In the previous section, we presented our approach of using a series of EFTs, the SCET$_i$, to handle processes with a hierarchy of many scales. We will now use this technique to calculate the leading contribution to a series of collinear emissions, as occurs in the parton shower. Our ultimate goal is to incorporate corrections, but as a starting point we want to easily reproduce the strongly-ordered configuration of Eq. (1). We can do this if we declare that in a shower, the $i$th particle decomposes as:

$$(n \cdot q_i, \bar{q}_i, q_i) \sim (\lambda^{2i}, 1, \lambda^i) \bar{q}_i,$$  \hspace{1cm} (39)$$

and therefore has virtuality $q_i^2 \sim (\bar{q}_i)^2 \lambda^{2i}$ (cf. Fig. 8). This is exactly the same condition as Eq. (34), which we used to define the EFT, SCET$_i$.

To calculate the operators that describe $i$ emissions in the strongly-ordered limit, we will perform a series of matchings SCET$_i \to$ SCET$_{i+1}$. We will find that the most efficient way to describe the process at LO in $\lambda$ is to be in SCET$_{i+1}$ for $i$-parton radiation. Thus, we emit and match $i$-times in series, as shown by Fig. 3. At LO, we will show that one can implement this using an operator replacement rule. In the case of $q \to qg$ emission, it takes the form:

$$\chi_{n_1} \to c g B^\alpha_{n_2} \chi_{n_2},$$  \hspace{1cm} (40)$$

where $\chi_n$ and $B^\alpha_{n\perp}$ are the SCET fields associated with collinear quarks and gluons, respectively, and $c$ is the Wilson coefficient whose spin and color indices are suppressed. Though we do not compute them, there are similar $B^\alpha_{n_{1\perp}} \to c' \bar{\chi}_{n_2} \chi_{n_3} + c'' B^\beta_{n_{2\perp}} B^\gamma_{n_{3\perp}}$ rules as well. In SCET, each collinear field carries the label $n$, which gives its direction of collinearity. Note that the quark field on the LHS of (40) has a different one from those on the RHS. This relates to the stricter definition of collinearity in SCET$_{i+1}$ shown in Fig. 8. In order to perform the matching, we will make use of the reparametrization invariance (RPI) discussed in point 3. of Sec. II C to change fields’ $n$-labels.
A. Leading Shower Revisited

We first want to reproduce the strongly-ordered contribution to $i$-gluon radiation from the quark in an initial $\gamma^* \rightarrow q\bar{q}$ pair production. Our iterative matching procedure for multiple EFTs takes a particularly simple form at LO in $\lambda$. For our standard example, we take the process $e^+e^- \rightarrow \text{jets}$. Starting in QCD, we couple the quarks to another sector via the operator, $J_{\text{QCD}}^{\mu} = \bar{q} \Gamma^\mu q$. This allows us to avoid complications that come from the initial state such as backward evolution. In SCET$_1$ (which is equivalent to the usual SCET), matching to QCD at tree-level converts the quark coupling to the following operator at LO: $\bar{\chi}_{n_0} \Gamma^\mu \chi_{\bar{n}}$, which produces $q$ and $\bar{q}$ in different collinear directions. Details on the matching of QCD to SCET$_1$ are given in App. C. Using the notation in Eq. (14), we write the SCET$_1$ operator in the following way:

$$\bar{\chi}_{n_0} \Gamma^\mu \chi_{\bar{n}} = \left( C^{(2,0,0)}_{1,\text{LO}} \right)_{ij} \left( O^{(2,0,0)}_{1}(n_0, \bar{n}) \right)_{ij},$$

where

$$\left( O^{(2,0,0)}_{1}(n_0, \bar{n}) \right)_{ij} = (\bar{\chi}_{n_0})_i (\chi_{\bar{n}})_j,$$

$$\left( C^{(2,0,0)}_{1,\text{LO}} \right)_{ij} = (\Gamma^\mu)_{ij},$$

and $i$ and $j$ are spinor indices. The subscripts 1 in Eq. (42) indicate that the fields are defined in SCET$_1$. Our focus is on gluon emissions from the quark, and we always take the antiquark in the same direction, $\bar{n}$, therefore we drop it from the list of $n$-labels. Also, we will use the following shorthand notation for the most common operator,

$$O^{(2,k,0)}(n_1, n'_1, \ldots, n'_{k}, \bar{n}) \equiv O^{(k)}(n_1, n'_1, \ldots, n'_{k}),$$

where the subscript marks these as being in SCET$_i$. In the rest of the paper, we will often drop the spinor indices. Using the above convention, we write the operator in Eq. (41) as:

$$\bar{\chi}_{n_0} \Gamma^\mu \chi_{\bar{n}} = C^{(0)}_{1,\text{LO}} O^{(0)}_{1}(n_0).$$

The LO derivations are independent of the exact structure of $\Gamma^\mu$. In fact, even the antiquark is a spectator, and we could just as easily use $O^{(q)} = \bar{\chi}_{n_0} \Omega$, where $\Omega$ is arbitrary. However, as we will discuss in Sec. IV, matching QCD to SCET$_1$ at higher orders requires us to specify $\Omega$.

To calculate operators in SCET$_2$, we start with single gluon radiation. In this case, shown in Fig. 5, the emission amplitude is:

$$A_{\text{LO}}^{gg} = C^{(0)}_{1,\text{LO}} \langle 0 | \int dx T \{ L^{\text{SCET}1}(x) | O^{(0)}_{1}(n_0) \} | q_{n_0} g_{n_0} \bar{q}_{\bar{n}} \rangle$$

$$= g \bar{u}_{n_0}(q_1) \left( n_0^\alpha + (\hat{q}_1)_{n_0\perp} \gamma_\perp q_{\bar{n}} q_{\bar{n}} \right) \frac{\bar{q}_0}{q_0^\alpha} \Gamma^\mu v_{\bar{n}}(p_q),$$

All the amplitudes we write in this work refer only to the hadronic part of $e^+e^- \rightarrow \text{jets}$, thus $A_{\text{LO}}^{gg}$ is the amplitude of $\gamma^* \rightarrow q\bar{q}g$. 

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where we have labeled the collinear directions of the particles in the state $|q_{n_0}g_{\bar{n}_0}\bar{q}_{\bar{n}_0}\rangle$ for later convenience. The SCET$_1$ Lagrangian is given in Eq. (A9). Here we study the process in the center of mass frame with $p_\gamma = (Q, 0, 0, 0)$ and the quark ($q_0$) and antiquark ($p_{\bar{q}}$) along the directions $n_0 = (1, 0, 0, 1)$ and $\bar{n} = (1, 0, 0, -1)$, respectively:

$$p_\gamma^\mu = \frac{Q}{2} n_0^\mu + \frac{Q}{2} \bar{n}^\mu,$$

$$p_{\bar{q}}^\mu = \frac{n_0 \cdot p_{\bar{q}}}{2} \bar{n}^\mu,$$

$$q_0^\mu = \frac{\bar{q}_0}{2} n_0^\mu + \frac{n_0 \cdot q_0}{2} \bar{n}^\mu. \tag{47}$$

We decompose the emitted quark ($q_1$) and gluon ($k_1$) along the directions ($n_0, \bar{n}$),

$$q_1^\mu = \frac{\bar{q}_1}{2} n_0^\mu + (q_1)^\mu_{n_0 \perp} + \frac{n_0 \cdot q_1}{2} \bar{n}^\mu, \tag{48}$$

$$k_1^\mu = \frac{\bar{k}_1}{2} n_0^\mu + (k_1)^\mu_{n_0 \perp} + \frac{n_0 \cdot k_1}{2} \bar{n}^\mu.$$

The variables are illustrated in Fig. 5. By momentum conservation we have $(k_1)_{n_0 \perp} = -(q_1)_{n_0 \perp}$, $Q = \bar{q}_0 = \bar{k}_1 + \bar{q}_1$ and $n_0 \cdot p_{\bar{q}} = Q - n_0 \cdot q_1 - n_0 \cdot k_1$. We take all the external particles on-shell, thus $n_0 \cdot q_1 = -(q_1)_{n_0 \perp}/\bar{q}_1$ and similarly for $n_0 \cdot k_1$. As we discussed in Section II B, [24, 25] showed that single gluon emission in SCET reproduces the splitting function, Eq. (3), and factorization behavior, Eq. (27), of the standard parton shower. This simple behavior for a single radiation will reproduce the shower for an arbitrary number of gluons.

We now want to match the single emission to SCET$_2$ (cf. Fig. 6). There is a slight technical complication due to the different definitions of collinearity in the two theories, as illustrated by Fig. 4. In SCET$_i$, a collinear field with label $n$ can annihilate a state containing a particle whose momentum vector lies anywhere in a cone with angle $\sim \lambda_i$ about $n$. When we change to a lower-scale theory in a matching equation, we have to take care that the operators’ $n$-labels are appropriate for the desired amplitude. Using the terminology of Fig. 4, while any label vector in the light grey (blue) cone is sufficient for a particle with momentum $p$ in SCET$_i$, for SCET$_{i+1}$ we need one in the dark grey (red) cone. This is where RPI$_i$ comes in, as mentioned in Sec. II C. We use it in SCET$_i$ to transform all quantities in the amplitude (spinors and vectors) that depend on the label vectors, such that the label after rotation lies within a collinear cone with angle $\sim \lambda_i+1$ about the particle momentum.
FIG. 6: (Left panel) Single gluon emission in SCET\textsubscript{1} comes from the time-ordered product of the Lagrangian with a quark-creating operator, $A = \langle 0|T\{\mathcal{L}_{\text{SCET}\textsubscript{1}}^{(0)}\}|qgX\rangle$. (Right panel) For parent quarks with virtuality $\gg Q^2\lambda^4$, the gluon comes from the central vertex in SCET\textsubscript{2} via a higher-dimensional operator, $A = \langle 0|\mathcal{O}^{(1)}_2|qgX\rangle$.

The simplest convention is to choose the $n$-label to align perfectly with the particle. If desired, we could make any choice consistent with RPI\textsubscript{i+1} transformations. For the process under consideration, we define labels, $n_1, n'_1$ such that,

$$q_1 = \bar{q}_1 \frac{n_1}{2},$$

$$k_1 = \bar{k}_1 \frac{n'_1}{2}.$$  \hspace{1cm} (49)

In SCET\textsubscript{1}, we are free to use $n_0$ or $n_1$ to describe the $q_1$ quark and $k_1$ gluon because of the RPI\textsubscript{1} symmetry. Since $n_1$ is a valid index for the quark field in SCET\textsubscript{2}, we do the matching computation using the same spinor, $u_{n_1}(q_1)$, in both theories. In App. \textbf{B}, we derive the RPI transformations we use here and other rotation formulas. For now, we quote the results we need:

$$u_{n_0} = \frac{\tilde{q}_0 \tilde{n}}{4} u_{n_1},$$

$$n'_1 = \alpha_0 + \frac{2(q_1)_{n_0\perp}}{q_1^2} - \frac{(q_1)_{n_0\perp}^2}{q_1^2} \bar{n} \alpha,$$

$$n''_1 = \alpha_0 + \frac{2(k_1)_{n_0\perp}}{k_1^2} - \frac{(k_1)_{n_0\perp}^2}{k_1^2} \bar{n} \alpha.$$  \hspace{1cm} (50)

As required, the two different $n_i$-vectors’ directions lie within cones of size $\lambda$ about $n_0$. It is simple to check that in the new basis, $(q_1)_{n_1\perp} = q_1 - (n_1 \cdot p)\bar{n}/2 - \bar{q}_1 n_1/2 = 0$ and similarly for $(k_1)_{n'_1\perp}$. Acting on Eq. (46), we get:

$$A_{\text{LO}}^{g g g} = g \left[ q_0 \gamma^\mu u_{n_1} \right]^{\alpha_0} \left[ q_0 + \frac{(q_1)_{n_0\perp} \gamma^\alpha_{n'_1\perp}}{q_1} \right] \frac{\tilde{q}_0 \bar{n} \gamma^\mu v_n}{4},$$

where $q_0 = q_1 + k_1$. Having changed bases, we can easily write the SCET\textsubscript{2} operator that
reproduces Eq. (51), \( C^{(1)}_{2,\text{LO}} \mathcal{O}_2^{(1)}(n_1, n'_1) \), where:

\[
\mathcal{O}_2^{(1)}(n_1, n'_1) = (\chi_{n_1}) \, g \mathcal{G}_{n'_1}^{\alpha} (\chi_{n_1})_k ; \\
C^{(1)}_{2,\text{LO}}(n_1, n'_1) = U_{\text{LL}}^{(2,0,0)}(n_0; Q, \mu_1) \left[ \frac{\bar{q}_0}{q_0} \left( n_0^\alpha + \frac{\bar{g}_1(n_0)\gamma_{n'_1}^\alpha}{q_1} \right) \frac{\bar{p} \cdot \bar{p}}{4} \Gamma^\mu \right] \, \Theta_{\delta k}[n_1 \cdot n'_1].
\]

We note that we have also given the Wilson coefficient the \( n \)-labels of the operator it multiplies. In cases where it is clear, we will only explicitly label one of \( C \) or \( \mathcal{O} \). In addition to the expected tree-level amplitude term in brackets, we also give the RG kernel, \( U_{\text{LL}}^{(2,0,0)} \), and an angular phase-space cutoff, \( \Theta_{\delta k} \). We discuss each of them in turn.

The former comes from running the SCET\(_1\) operator \( \mathcal{O}_1^{(0)} \) from \( Q \) to the scale \( \mu_1 \sim \lambda Q \). When \( U_{\text{LL}} \) refers to an operator where all collinear directions are distinct, we will drop \( n \)'s from the notation. From Eq. (20), we have

\[
U_{\text{LL}}^{(2,0,0)}(n_0; Q, \mu_1) = \Delta_q(Q, \mu_1),
\]

where LL refers to the fact that we take the one-loop cusp anomalous dimension, which resums the leading logs of this running. As mentioned in Sec. II B [24, 25] showed this resummation to be equivalent to that of no-branching Sudakov factors of CKKW-L. We discuss the running of our operators in more detail in Sec. IV C.

The phase-space cutoff \( \Theta_{\delta k}[n_1 \cdot n'_1] \) encodes that \( n_1 \cdot n'_1 \lesssim \lambda^2/\eta^4 \) (the power of \( \eta^{-4} \) was discussed in Sec. II C). The SCET\(_2\) operator, \( \mathcal{O}_2^{(1)}(n_1, n'_1) \), can only distinguish that the quark and gluon are not collinear in SCET\(_2\), but does not know that they were collinear in SCET\(_1\). Thus, we put a cutoff on how far apart they are using \( n_1 \cdot n'_1 \) to ensure that this SCET\(_2\) operator cannot create them in a region of phase-space where they would have been non-collinear, even in SCET\(_1\). As an example, we could choose \( \Theta \) to be the usual step-function

\[
\Theta_{\delta k}[n_1 \cdot n'_1] = \begin{cases} 
1 & n_1 \cdot n'_1 \leq \delta_k \\
0 & n_1 \cdot n'_1 > \delta_k
\end{cases},
\]

\[
\hat{\Theta}_{\delta k} = 1 - \Theta_{\delta k}.
\]

In practice we will use a smooth version of the above step. For later convenience, we defined the complement, \( \hat{\Theta}_{\delta k} \). In working with SCET\(_1\) operators, we relate \( \delta_k \) to \( \lambda \). In general, the Wilson coefficient in SCET\(_1\) has to encode whether \( n_1 \cdot n_j \lesssim \lambda^{2(i-1)}/\eta^4 \) or \( n_1 \cdot n_j > \lambda^{2i-3}/\eta^4 \), in order to do it we will set \( \delta_i = \lambda^{2i-3}/\eta^4 \). This satisfies the necessary criteria since \( \lambda^{2i-2} \ll \lambda^{2i-3} \ll 1 \) (and recall that \( \eta \) is the parameter that accounts for the decrease in \( \bar{p} \) of a daughter relative to its mother). For \( C^{(1)}_{2,\text{LO}} \) above, this means \( \delta_2 = \lambda/\eta^4 \). At the end of Sec. II C, we discussed how \( \lambda \) gives us a way to parametrize strong-ordering and deviations from it. To this end, we did not need to assign it a numerical value beyond \( \lambda \ll 1 \). Here for the implementation, we do have to make an explicit choice as to where

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6 See Appendix D for more detail on this matching. Though we have written Eq. (52) to look as much like the SCET\(_1\) amplitude as possible, we can rewrite it purely in terms of external momenta, as in Eq. (D13).
FIG. 7: Plot of the smoothed Θ-function, \( \Theta(x)_{\Lambda,a} \), defined in Eq. (D21), taking \( \Lambda = 0.1 \) and \( a = 0.016 \). The parameter \( \Lambda \) determines the value of \( x \) where the function switches from 0 to 1, and \( 2a \) is the range in \( x \) over which the transition is made. Comparing this smoothed Θ-function to \( \Theta_\delta \) in Eq. (54), we have parametrically \( \Lambda \simeq \delta \) and \( a \ll \delta \). This plot is for the case \( \delta_3 = 0.1 \).

As our Θ functions turn over, and for this purpose we will use fixed values such as \( \lambda = 0.1 \) and \( \eta = 1/2 \). This means \( \delta_2 = 1.6 \) and since the \( \eta^{-4} \) is a common overall factor that all \( \delta_{k \geq 3} \leq 0.16 \). The smoothness of both Θ and our physical processes gives us great leeway in the choice for \( \lambda \), and we expect that any \( \lambda \simeq 0.1 \) will suffice (cf. Fig. 10).

Once we square and integrate our operators, we have certain practical considerations to take into account. For example, it is better to use a smoothed step. We give an example of such a function in Eq. (D21), and plot it in Fig. 7 where we choose an appropriate numerical value for \( \delta \). If one only wishes to recover the LL shower, then one should use \( \Theta = 1 \), as the errors induced by this do not affect the leading resummation. Furthermore, taking \( \Theta = 1 \) ensures that the LL shower can cover all of phase space. Once we include corrections, though, then it is important to keep different types of collinearity distinct and include non-trivial Θ’s. In the presence of corrections, there will always be amplitudes with a Θ and others with a \( \tilde{\Theta} \), which together cover all of phase space (see also Fig. 10).

Unlike standard SCET, where all the coefficients are of order \( \lambda^0 \), \( C_{2,\text{LO}}^{(1)} \) has an overall weight of \( \lambda^{-1} \). We get \( \lambda^{-2} \) from the SCET propagator, \( 1/q_0^2 \). The numerator is proportional to \( \lambda \) and comes from the vertex: \( \left( n_0^\alpha + (q_1)_0 n_{0\perp} n_{1\perp}^\alpha / q_1 \right) \). The second term is straightforwardly \( O(\lambda) \) from \( (q_1)_0 n_{0\perp} \). Since \( n_0^\alpha \) gets contracted with \( B_{n_{1\perp}}^\alpha \), it only contributes its perpendicular component in the \( n_{1\perp} \) frame. From Eq. (50), we see that \( (n_0)_n_{1\perp} \sim n_0 - n_{1\perp} \sim (k_1)_0 / k_1 \sim \lambda \).

\( C_{2,\text{LO}}^{(1)} \) is gauge invariant despite the presence of the \( \Theta_\delta \) function. This follows from writing Eq. (52) only in terms of scalar products of \( n \) vectors, (cf. Eq. (D13)), since collinear directions are invariant under collinear gauge transformations [23].

We note that we can obtain \( C_{2,\text{LO}}^{(1)} O_{2,\text{LO}}^{(1)} (n_1, n_{1\perp}) \) from the original two-parton operator, \( C_{1,\text{LO}}^{(0)} O_{1,\text{LO}}^{(0)} (n_0) \), in two steps: first we multiply it by the running factor

\[
U_{\text{LL}}^{(2,0)} (n_0; Q, \mu_1) = \Delta_q(Q, \mu_1),
\]

(55)
where the formulas for $U_{LL}^{(0)}$ are given in Eqs. (18,19). Secondly, we apply the replacement rule
\[ (\bar{c}_n)_i \to (c_{\text{LO}}^\alpha(n_0))_{ji} (\bar{c}_{n_1})_j g\bar{B}_{n'_1}^\alpha, \]
where $c_{\text{LO}}^\alpha$ is:
\[ c_{\text{LO}}^\alpha(n_0) = \frac{q_0}{q_0^2} (n_0^\alpha + \frac{(q_1)^{n_0} \gamma_{n'_1}^\alpha}{q_1}) \cdot \frac{\Theta}{4} \delta_2 [n_1 \cdot n'_1]. \]

The relation [56] is the operator statement of splitting in the parton shower. The scale $\mu_1$ defines the endpoint of running in the UV theory. As we evolve down, more partons become apparent. We can see this here by the presence of two fields where there had been one. It makes the basic aspects of the shower manifest. The replacement rule affects the quark alone, and so we see that the amplitude for splitting factorizes off from the rest of the process. The RG kernel reflects the no-branching probability. Lastly, we can interpret the vertex portion of $c_{\text{LO}}^\alpha$ as the “square root” of the splitting function. The spinor projector $(\phi_0^2/4)$ in Eq. (57) rotates the spin-sum from $\phi_1$ to $\phi_0$ in accordance with Eq. (2). The remaining part of $c_{\text{LO}}^\alpha$ after stripping off the $\Theta_\delta$ is:
\[ P_\alpha \equiv \frac{q_0}{q_0^2} \left( n_0^\alpha + \frac{(q_1)^{n_0} \gamma_{n'_1}^\alpha}{q_1} \right), \]

which squares to a trivial Dirac structure. Furthermore, even though $\rho_\alpha(q/q_0^2) \neq P_\alpha$ because of the RPI rotations we performed (where $\rho$ is defined in Eq. [25]), we have $|\rho|^2 (\overline{q_0}/q_0^2)^2 = |P|^2$ with respect to the gauge polarization sum, $d_{\alpha\beta}$, so
\[ |P|^2 = \frac{1 + z^2}{k_{1\perp}^2}. \]

Just as before, including the $z$-dependence from the measure and spin-sum, we recover the standard splitting function $\propto (1 + z^2)/(1 - z)$. Thus, $c_{\text{LO}}^\alpha$ weights the probability assigned to the expectation value of $C_{2,LO}^{(1)}O_2^{(1)}(n_1, n'_1)$ appropriately.

Having computed the LO result for a single gluon, it is straightforward to proceed to an arbitrary number of emissions. In SCET$_2$, we know that a two-gluon process comes from the $T$-product of the Lagrangian with $C_{2,LO}^{(1)}O_2^{(1)}(n_1, n'_1)$. Similarly to before, the amplitude has the contribution,
\[ A_{\text{LO}}^{ggg} = C_{2,LO}^{(1)}(n_1, n'_1)|dx T\{L_{\text{SCET}_2}(x)O_2^{(1)}(n_1, n'_1)|q_0, q_0', q_0'\}. \]

The vertex for gluon emission in the SCET$_2$ Lagrangian is identical to that in SCET$_1$. Thus, integrating out the parent of the Lagrangian-emitted gluon, we obtain a two-gluon SCET$_3$ operator, $C_{3,LO}^{(2)}O_3^{(2)}(n_2, n'_1, n'_2)$, similarly to before. Also like in the matching SCET$_1 \rightarrow$ SCET$_2$, we can obtain $C_{3,LO}^{(2)}O_3^{(2)}(n_2, n'_1, n'_2)$ from the SCET$_2$, $C_{2,LO}^{(1)}O_2^{(1)}(n_1, n'_1)$, by multiplying it by the running factor for $O_2^{(1)}$:
\[ U_{LL}^{(1)} = \Delta_\delta(\mu_0, \mu_2) \Delta_\delta(\mu_1, \mu_2)^{1/2}, \]
FIG. 8: (Left panel) Our kinematic convention for a strongly ordered process. Quark momenta are denoted by $q_i$ and gluon momenta by $k_i$. (Right panel) Power counting of the LO coefficient in SCET$_N$. The powers of $\lambda$ with negative exponents refer to the propagator contribution to the amplitude. Those with positive exponents refer to the perpendicular momentum of the gluon with respect to its parent, which appears in the SCET vertex Feynman rule.

with $\mu_0 \sim Q$ and applying the replacement rule:

$$\langle \bar{\chi}_{n_2} \rangle_i \rightarrow (c^\alpha_{LO}(n_1))_{ji} \langle \bar{\chi}_{n_3} \rangle_j g B^\alpha_{n_2}$$

$$c^\alpha_{LO}(n_1) = \frac{\bar{q}_1}{q_1^2 n_1^\alpha + \frac{(\bar{q}_2)_n (\gamma_{n_2}^\lambda)}{\bar{q}_2^2}} \frac{\bar{q}_1^2}{4} \Theta_3 \left[ n_2 \cdot n_2' \right],$$

where $n_2$ and $n_2'$ are directional proportional to the quark and second gluon momenta, defined in Eq. (42), and $\delta_3 = \lambda^3/\eta^4$. One can iterate this procedure to obtain the LO result for $(N - 1)$-gluon emission. If we use the replacement rule $N - 1$ times we go down to the SCET$_N$ operator $C^{(N-1)}_{N,LO} O^{(N-1)}_{N}(n_{N-1}, n'_1, \ldots, n'_{N-1})$, after which Lagrangian emissions are no longer distinguished as separate particles. We have:

$$O^{(N-1)}_{N}(n_{N-1}, n'_1, \ldots, n'_{N-1}) = \bar{\chi}_{n_N} \left( \prod_{k=1}^{N-1} g B^\alpha_{n_k} \right) \chi_{\bar{n}},$$

$$C^{(N-1)}_{N,LO}(n_{N-1}, n'_1, \ldots, n'_{N-1}) = \left( \prod_{k=1}^{N-1} U_{LL}^{(k-1)}(\mu_{k-1}, \mu_k) c^\alpha_{LO}(n_{k-1}) \right) \Gamma^\mu,$n

$$c^\alpha_{LO}(n_{k-1}) = \frac{\bar{q}_{k-1}}{q_{k-1} n_{k-1}^\alpha + \frac{(\bar{q}_k)_n (\gamma_{n_k}^\lambda)}{\bar{q}_k^2}} \frac{\bar{q}_{k-1}^2}{4} \Theta_3 \left[ n_k \cdot n_k' \right],$$

$$U^{(k-1)}_{LL}(\mu_{k-1}, \mu_k) = \Delta_q(\mu_{k-1}, \mu_k) \Delta_g(\mu_{k-1}, \mu_k)^{(k-1)/2}.$$
This accounts for soft interference effects and coherent branching, see Ref. [69] for a review. To investigate this scale choice in the SCET framework requires an examination of the logs in the one-loop matching computation for \( c_{1,\text{LO}} \), and consideration of soft gluons in SCET and SCET\(_{i+1} \). Having not carried out this computation ourselves, we rely on the previous literature. For our variables using Eq. (38) this implies

\[
\mu_k^2 = \left( \frac{q_k \bar{k}_k}{q_{k-1}} \right)^2 \frac{|n_k \cdot n_k'|}{2} \tag{64}
\]

(In contrast, the choice of invariant mass \( q_{k-1}^2 \) would have yielded \( \mu_k^2 = \left( \bar{q}_k \bar{k}_k \right) |n_k \cdot n_k'| / 2 \), but this leads to incomplete cancellations of soft divergences, and therefore problems with the resummation of soft logs [68].) The directions \( n_k \) and \( n_k' \) are aligned with the external quark, \( q_k \), and the gluon momenta, \( k_k \). They are related to \( n_{k-1} \) through an RPI transformation. We can extend the argument to calculate the scaling of \( C(1)_{2,\text{LO}} \) to the SCET\(_N \) coefficient in Eq. (63). Counting the contributions from the tree-level terms, \( c_{1,\text{LO}}^k \), \( C_{N,\text{LO}}^{(N-1)} \sim \prod_{i=1}^{N-1} 1/\lambda_i = \lambda^{-N(N-1)/2} \), cf. Fig. 8.

Similarly to the discussion above Eq. (59), we can extract the vertex part of \( c_{1,\text{LO}}^k \) to define \( P_k^\alpha \). We get that:

\[
|P_k^\alpha|^2 = \frac{1 + z_k^2}{(q_k^2)_{n_{k-1}^\perp}} \tag{65}
\]

where \( z_k \equiv \bar{q}_k / \bar{q}_{k-1} \). Thus, the amplitude squared goes like the factorized product of the appropriate \( 1 \to 2 \) splitting functions. Since \( \mathcal{O}_N^{(N-1)}(n_{N-1}, n'_1, \ldots, n'_{N-1}) \) is just built up from the repeated use of Eq. (56), we see that it requires no added information after we compute the first \( q \to qg \) splitting. Thus, what we need to pass to a shower algorithm comes just from single real and single virtual gluon computations, as we list below in Sec. IIIC in Table I. The collinear splitting needed for a LL shower is entirely handled by the replacement rule in Eq. (56).

Lastly, we note that at higher orders in SCET\(_N \), we will only ever need to compute the Wilson coefficient, \( C_{N}^{(N-1)} \), of \( \mathcal{O}_N^{(N-1)} \). Since each field in this theory has its own direction by the physical resolution constraint, we can use RPI\(_N \) to make all operators with \( P_n^\perp \) equal to zero.

\textsuperscript{7} It is straightforward to see that we do not have additional contributions at LO in \( \lambda \). Firstly, consider the possibility of operators that do not take the form of a single-field replacement rule. These would depend on the details of the hard process that produced the quark in the first place and could threaten the factorization of the shower. In fact, we will get such terms when we match QCD \( \to \) SCET\(_1 \), but they are always suppressed, as we discuss in Sec. IV. Returning to single-field replacement, let us consider matching SCET\(_1 \) \( \to \) SCET\(_2 \), as results in this case will generalize to all SCET\(_i \). Rule (56) sends \( \chi_n \to CB_{n_{3,\perp}}^\mu \chi_n \). At LO, we cannot get such a replacement involving multiple gluon fields, \( B_{n_{3,\perp}} \), as this implies that we have integrated out multiple, hard (\( \sim Q\lambda^2 \)) propagators. Such a contribution would not be strongly ordered, and is suppressed. In Sec. IV we will also see that we do have such contributions at higher orders.

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B. Soft Emissions

SCET describes soft degrees of freedom using soft quark and gluon fields: $q_s(x)$ and $A_s(x)$. In this work, we focus on fully differential cross sections where we can always distinguish collinear and soft modes. In an integrated cross section in SCET, we have to implement soft emissions with some form of zero-bin subtractions \cite{72} to avoid double counting between soft and collinear radiation. (In the shower literature a proper treatment of softs is also often implemented by subtraction methods \cite{39, 40, 43–46, 73}.) The collinear sector and the soft sector couple through the covariant derivative,

$$iD^\mu_s = i\partial^\mu + gA^\mu_s,$$  \hspace{1cm} (66)

acting on the collinear fields. At LO in $\lambda$, the collinear particles only couple to the $n \cdot A_s$ component of the soft gluons and the soft-collinear factorization guarantees that we can absorb this interaction into a Wilson line, $Y_n(x)$, along the direction of the collinear particle,

$$Y_n(x) = P \exp \left[ ig \int_{-\infty}^{0} ds \, n \cdot A_s(x+sn) \right].$$  \hspace{1cm} (67)

In SCET, this is accomplished by making field redefinitions \cite{23}, so that the new collinear fields no longer couple to soft gluons through their kinetic term, as we review in App. A.

The outcome for the composite fields considered here is that

$$\chi_n \to Y_n \chi_n, \quad B^\mu_n \to Y_n B^\mu Y_n^\dagger.$$

(68)

Note that here we consider nonabelian soft interactions, which is why the soft Wilson lines do not cancel for the $B^\mu_n$ field.

In matching SCET$_i$ to SCET$_{i+1}$, we will only consider external soft modes in SCET$_{i+1}$ with momenta $k \sim Q\lambda^{2(i+1)}$. These are contained as a subset of the softs in SCET$_i$. We do not consider particles with soft momenta $k \sim Q\lambda^{2i}$ that could not be encoded by onshell modes in SCET$_{i+1}$. Such modes are forced to have larger momenta than the soft fields in SCET$_{i+1}$, and they are not responsible for IR divergences. Any contributions from momenta of this type can be encoded in the Wilson coefficients of our SCET$_{i+1}$ operators.

In a given SCET$_i$, after making the field redefinition, the effect of soft gluons is encoded by Wilson lines $Y_n$ in the operators, with the form

$$\bar{\chi}^{(0)}_{nN} Y^\dagger_{nN} \prod_{k=1}^N Y^\dagger_{n'_k} B^{\alpha_k}_{n'_k} Y_{n_k} Y_{n_k} \Gamma_\mu Y_n \chi_n.$$

(69)

The angular ordering property and the coherent parton branching formalism for soft emissions with multiple hard partons emerge naturally from such operators in SCET$_{i+1}$. If we take the Fourier transform of $Y_n(x)$ we get

$$Y = 1 + \sum_{m=1}^{\infty} \sum_{\text{perms.}} \frac{(-g)^m}{m!} n \cdot A_1^{a_1} \cdots n \cdot A_m^{a_m} \frac{n \cdot (k_1 n \cdot (k_1 + k_2) \cdots n \cdot (\sum_{i=1}^{m} k_i )) T^{a_m} \cdots T^{a_1}}{T^{a_1}}$$

(70)
where $k_1$, $k_2$, ... $k_n$ are the momenta of the gluon fields. The eikonal structure of (70) leads to angular ordering. If a collinear particle with momentum $q_i$ in the $n_i$ direction emits a soft gluon of momentum $k_s$, the amplitude acquires a term proportional to

$$F_{\text{soft}} = \frac{n_i \cdot \varepsilon_s}{n_i \cdot k_s} = \frac{q_i \cdot \varepsilon_s}{q_i \cdot k_s} + O(\lambda), \quad (71)$$

where $\varepsilon_s$ is the polarization vector of the soft radiation and $q_i^\mu = \bar{q} n_i^\mu / 2$ up to power corrections. If $A_n(q_1, q_2, \cdots, q_n)$ is the amplitude to emit $n$ collinear particles with momenta $q_1, q_2, \cdots, q_n$ and $A_{n+1}$ the amplitude with one more emission, $k_s$, in the soft region, we get $A_{n+1}(q_1, q_2, \cdots, q_n, k_s) \sim A_n(q_1, q_2, \cdots, q_n) \sum_{i=1}^{n} C_i q_i \cdot \varepsilon_s / q_i \cdot k_s$, where $C_i$ is a color factor. For the cross section this implies

$$d\sigma_{n+1} = d\sigma_n \frac{dE_s}{E_s} \frac{d\Omega_s}{2\pi} \frac{\alpha_s}{2\pi} \sum_{i,j} C_{i,j} W_{i,j}, \quad (72)$$

where $d\Omega_s$ and $E_s$ are the element of solid angle and the energy of the emitted soft gluon, and $C_{i,j}$ is a color factor. Here

$$W_{i,j} = \frac{E_s^2 \bar{q}_i \cdot q_j}{q_i \cdot k_s \bar{q}_j \cdot k_s} \quad (73)$$

is known as the radiation function. Without color weights, the integration of $W_{i,j}$ over azimuthal angular variables would imply that soft gluons only contribute when the gluon is confined to the cones centered in the directions of particles $i$ and $j$, and are hence angular ordered.

To see how coherent branching emerges, we consider effects encoded by operators with exactly the same collinear field content in SCET$_i$ and SCET$_{i+1}$. Graphs involving soft gluons will agree, and there is no contribution to the matching. If we consider instead the collinear calculations that lead to the LO replacement rule $\bar{\chi}_{n_0} \to c_{\text{LO}} \bar{\chi}_{n_1} B_{n_1}^\perp$, then the soft gluons are encoded by

$$\text{SCET}_i \, : \, \bar{\chi}_{n_0} Y_{n_0}^\dagger, \quad \text{SCET}_{i+1} \, : \, c_{\text{LO}} \bar{\chi}_{n_1} Y_{n_1}^\dagger Y_{n_1} B_{n_1}^\perp Y_{n_1}^\dagger. \quad (74)$$

For soft gluons at wide angles relative to $n_0$, $n_1$, and $n_1'$, the effect of attachments to $Y_{n_1}^\dagger Y_{n_1}$ are power suppressed because soft emission from these two lines cancels up to terms that are power suppressed by $n_1 \cdot n_1' \sim \lambda^2 / \eta^4$. The remaining attachment to $Y_{n_1'}^\dagger$ looks the same as those to $Y_{n_0}^\dagger$ at leading power, since $n_0 \cdot n_1' \sim \lambda^2 / \eta^4$. Thus, wide angle soft gluons do not resolve the substructure revealed by matching to SCET$_{i+1}$ and effectively only couple to the overall color charge of the parent quark $\bar{\chi}_{n_0}$. Soft radiation that is close in angle to $n_1$ and $n_1'$ resolves the split into quark $\bar{\chi}_{n_1}$ and gluon $B_{n_1}^\perp$, compensating for the $n_1 \cdot n_1'$ suppression by additional collinear singularities in its propagator factors. Thus, the coherent branching formalism for soft gluons emerges naturally for amplitudes in our SCET$_i$ picture.

From the SCET point of view, it would be natural to distinguish soft and collinear radiation in the shower and treat them independently, being careful not to double count. For simplicity, all available shower codes treat them in a simultaneous fashion. Accounting for soft coherent branching in the shower typically leads to modifications of the Sudakov probability factors (see for example Ref. [74]), and affects the choice of evolution variable or adds additional vetoes. In the context of SCET, the implications of this were discussed recently in [75].
C. Summary for LO Parton Shower

In Table I, we summarize results for the mapping between the LL parton shower and our SCET\(_i\) picture at LO in \(\lambda\). In the first column, we put the elements needed for showering, and in the central column the translation to elements in the SCET\(_i\) setup. The usual splitting function is related to our replacement rule \(\bar{\chi}_{n_0} \rightarrow c_{\text{LO}} \bar{\chi}_{n_1} \mathcal{B}_{n_1}^\perp\), that in turn is related to the SCET\(_2\) coefficient of the operator \(\mathcal{O}_2^{(1)}\). The LL Sudakov comes from LL running factors related to the one-loop cusp anomalous dimension as in [24, 25]. At leading order, soft emission in SCET\(_i\) is taken into account by adding soft Wilson lines \(Y_n\) into our operators. This leads to angular ordering and coherent branching, which must be accounted for with modifications to the shower to account for the soft singular regions. Finally, showers are constructed with different choices of evolution variables and the choice effects the structure of power corrections. In SCET\(_i\), we have seen that we can write all coefficients in terms of the large momenta (\(\bar{q}\)) and dot product of \(n\) vectors (\(n_i \cdot n_j\)), which are natural variables in the SCET\(_i\) picture. One can convert these variables to \(k_T^2\), virtuality, or angles as desired. At LL this translation is straightforward.

| Shower Concepts          | Quantity in SCET\(_i\)       | Found In: |
|--------------------------|-------------------------------|-----------|
| Splitting function       | Replacement rule              | Eq.(56)   |
| LL Sudakov factor        | One-loop cusp anomalous       | Eq. (20)  |
|                          | dimension                     |           |
| Soft emission            | Soft amplitude                | Eq. (69)  |

TABLE I: Mapping between parton shower and SCET\(_i\) at LO/LL.

IV. SCET POWER CORRECTIONS TO THE SHOWER

As we have seen in the previous section, we reproduce the usual parton shower by matching collinear gluon emissions to increasingly lower-scale EFTs, the SCET\(_i\). Our goal is to catalog the leading power corrections (in \(\lambda\)) to the differential cross section for the emission of an arbitrary number of collinear gluons to a quark. By this we mean all amplitude terms to LO(\(\lambda\)) and NLO(\(\lambda\)), as well as those at NNLO(\(\lambda\)) that can interfere with LO(\(\lambda\)). As we will argue in Sec. [IVD] in most cases of interest, there is no LO(\(\lambda\))/NLO(\(\lambda\)) interference, and so we focus on the most important power suppressed terms which are NLO(\(\lambda\))×NLO(\(\lambda\)) and LO(\(\lambda\))×NNLO(\(\lambda\)). Just as in the strongly-ordered case, it is convenient to integrate down to SCET\(_{i+1}\) when describing the emission of \(i\)-gluons. We obtain these corrections by doing our matching computations at higher order. We will show that there are two distinct types of subleading matching, and they have a different physical interpretation:

- One type originates in matching QCD → SCET\(_1\) at higher orders. This generates a set of subleading terms that remain suppressed as we move down to lower-scale SCET\(_i\)’s.
We call them \textit{hard-scattering} power corrections as they involve the details of the hard-scale process that created our original partons. Also, they are most important for partons radiated closest to the hard vertex.

- The other type comes from the subleading matching \( \text{SCET}_i \to \text{SCET}_{i+1} \). They involve processes described by the \( \text{SCET}_i \) Lagrangian, but ones that get integrated out into higher dimension operators at lower scales. These corrections are ubiquitous. They do not depend on the hard-scattering details, and we can determine them for arbitrary \( \text{SCET}_i \to \text{SCET}_{i+1} \) once we have found them in \( \text{SCET}_1 \to \text{SCET}_2 \). Furthermore, they relate to known \( O(\alpha_s) \) corrections to the \( q \to qg \) splitting function, which exponentiate to sum part of NLL. For this reason, we call them \textit{jet-structure} corrections.

Determining the above to NLO\((\lambda)\) in the cross section will only involve single and double gluon emission. Thus, we will never need to compute in a lower-scale theory than \( \text{SCET}_3 \). We perform all the necessary QCD→\( \text{SCET}_1 \to \text{SCET}_2 \to \text{SCET}_3 \) matchings for these amplitudes in Appendices \( \text{C-E} \). Below, we discuss the final results for the corrections, with Sec. IV A focusing on hard-scattering and Sec. IV B on jet-structure. For these portions of the paper, the matching is only done at tree level, though formulas in the Appendices include one-loop RG kernels. We give the effects of LL running on correction terms in Sec. IV C along with a discussion of how to include NLL resummation for the LO \((\text{in } \lambda)\) Wilson coefficients. In Sec. IV D, we will study the amplitude squared and will see there is a great simplification of the interference structure in \( \text{SCET}_N \), and hence for NLO\((\lambda)\) power corrections in a shower. Lastly, we give in Sec. IV E the NLO counterpart to our LO table in Sec. III C. We describe how our corrections from subleading operators relate to improvement of the parton shower with higher order resummation of logs, corrections at higher order in \( \alpha_s \), as well as corrections to spin correlations and interference. These effects are summarized in a shower reweighting formula, Eq. (104).

A. Hard-Scattering Corrections

Just as in Sec. III A, we begin by examining the matching QCD→\( \text{SCET}_1 \) for single gluon emission collinear to the quark. For this case, all corrections are of the hard-scattering type. Beyond LO, we can have dependence on the process that creates the \( \bar{q}q \) pair. For concreteness, we will consider the coupling of QCD quarks to the vector current, \( J_{\text{QCD}}^\mu = \bar{q} \gamma^\mu q \). The matching is performed in the center of mass frame with the initial virtual photon having momentum, \( p_\gamma = (Q, 0, 0, 0) \). The full details of this matching calculation for QCD to \( \text{SCET}_1 \) are in Appendix \( \text{C} \). To reproduce the full QCD current, \( J_{\text{QCD}}^\mu \), we need an infinite tower of \( \text{SCET}_1 \) operators increasingly higher order in \( \lambda \). However, to get the required amplitude to NNLO\((\lambda)\), we only need four:

\[
A_{\text{to NNLO}}^{\bar{q}qg} = C_{1, \text{LO}}^{(0)}(n_0) \int dx \langle 0 | T(L_{\text{SCET}_1}(x) O_1^{(0)}) | q_0 g_0 \bar{q}_n \rangle \\
+ C_{1, \text{NLO}}^{(1)}(n_0, n_0) \langle 0 | O_1^{(1)} \langle q_0 g_0 \bar{q}_n \rangle + C_{1, T}^{(1)}(n_0, n_0) \langle 0 | T_1^{(1)} | q_0 g_0 \bar{q}_n \rangle \\
+ C_{1}^{(1)}(n_1, n'_1) \langle 0 | O_1^{(1)} | q_{n_1} g_{n'_1} \bar{q}_n \rangle,
\] (75)
FIG. 9: Matching QCD to SCET\textsubscript{1} to SCET\textsubscript{2} for one gluon emission which is either collinear to the quark or is in its own direction (SCET graphs for emission collinear to the antiquark are not shown). The figures represent operator structures that describe this process in each of the three theories. The QCD contribution is standard. In SCET\textsubscript{1}, we either emit a collinear gluon through the time-ordered product of the Lagrangian with a two-parton operator, or from three-parton operators. In SCET\textsubscript{2}, the emission relevant for us only arises from higher-dimension three-parton operators. In SCET\textsubscript{2}, the emission relevant for us only arises from higher-dimension three-parton operators.

where

\begin{align}
  \mathcal{O}^{(0)}_1(n_0) &= \bar{x}_{n_0} \chi_{\bar{n}}, \\
  \mathcal{O}^{(1)}_1(n_0, n_0) &= \bar{x}_{n_0} g B_{n_0 n_0}^{\alpha} \chi_{\bar{n}}, \\
  \mathcal{T}^{(1)}_1(n_0, n_0) &= \bar{x}_{n_0} \left[ \mathcal{P}^{\alpha} g B_{n_0 n_0}^{\alpha} \right] \chi_{\bar{n}}, \\
  \mathcal{O}^{(1)}_1(n_1, n'_1) &= \bar{x}_{n_1} g B_{n_1 n_1}^{\alpha} \chi_{\bar{n}}. 
\end{align}

Here we introduced a short-hand for the notation established in Eq. (14), \( \mathcal{T}^{(1)}_1(n_0, n_0) = \mathcal{O}^{(2,1,1)}_1(n_0, n_0^{[1]}) \). We give the expression for \( C^{(0)}_1(n_0) \) in Eq. (42). The amplitude from the operator \( \mathcal{O}^{(0)}_1(n_0) \) is shown in the first diagram in the SCET\textsubscript{1} column of Fig. 9 those from \( \mathcal{O}^{(1)}_1(n_0, n_0) \) and \( \mathcal{T}^{(1)}_1(n_0, n_0) \) in the second, and that for \( \mathcal{O}^{(1)}_1(n_1, n'_1) \) in the third.

We call \( \mathcal{O}^{(1)}_1(n_0, n_0) \) and \( \mathcal{T}^{(1)}_1(n_0, n_0) \) “two-jet” operators as they are labeled with two distinct collinear directions \((n_0, \bar{n})\) (we do not denote the antiquark direction explicitly, following the convention in Eq. 43). They describe a gluon collinear to the quark. We obtain the coefficients \( C^{(1)}_1(n_0, n_0) \) and \( C^{(1)}_1(n_0, n_0) \) by expanding the QCD amplitude in the limit of small gluon momentum transverse to the quark’s direction with the usual SCET proportionality: \((n_0 \cdot k_1, \bar{k}_1, k_{1 n_0} \perp) \sim (\lambda^2, 1, \lambda) Q\). \( C^{(1)}_1(n_0, n_0) \) and \( C^{(1)}_1(n_1, n'_1) \), are derived
above Eq. (C13) and given here:

\[ C_{1,\text{NLO}}^{(1)}(n_0, n_0) = \frac{1}{Q} (n_0^\mu - \bar{n}_0^\mu) \gamma_{n_0\perp}^\alpha, \]
\[ C_{1,\tau}^{(1)}(n_0, n_0) = \frac{1}{q_1 k_1} \gamma_{n_0\perp}^\mu \gamma_{n_0\perp}^\beta \gamma_{n_0\perp}^\alpha - \frac{2}{q_1 Q} g_1^{\beta\mu} \gamma_{n_0\perp}^\alpha. \]  

(77)

We use the same kinematic variables as in Fig. 5. For \( C_{1,\text{NLO}}^{(1)}(n_0, n_0) \) and \( C_{1,\tau}^{(1)}(n_0, n_0) \), the initial current is not a spectator, so neither term is simply proportional to the \( \gamma^\mu \) with which we started. This dependence on the details of the rest of the process is a characteristic feature of hard-scattering corrections. There are an additional set of two-jet configurations corresponding to the gluon collinear to the antiquark. These are trivial to obtain by charge conjugation.

The operator \( O_{1}^{(1)}(n_1, n'_1) \) is a three-jet configuration, as it describes three distinct directions. Whenever we have an operator where each field has its own index label, we can choose the \( n_i \) such that they are exactly aligned with the external particle momenta. We give the coefficient \( C_{1}^{(1)}(n_1, n'_1) \) in Eq. (C16).

Going to SCET\(_2\) for single gluon emission is straightforward. The basis of operators needed to reproduce the amplitude (75) is equal to (76), but with SCET\(_2\) fields: \( O_{2}^{(0)}(n_0) \), \( O_{2}^{(1)}(n_0, n_0) \), \( T_{2}^{(1)}(n_0, n_0) \), and \( O_{2}^{(1)}(n_1, n'_1) \). As the computations get more complicated with subsequent emissions, we wish to minimize our effort by only including those terms necessary to give the corrections to a shower Monte Carlo. This means we are only interested in the following:

1. We will need to keep those NNLO(\( \lambda \)) contributions that can interfere with LO(\( \lambda \)). These give terms at the same order as an NLO(\( \lambda \)) operator squared. We do not compute NNLO(\( \lambda \)) amplitude terms which have zero interference with the LO(\( \lambda \)) amplitude. A list of the necessary computations is found in App. D.

2. Our ultimate goal is not a complete SCET\(_i\) theory from which one can do computations, but an improved shower algorithm. In Table I we give a list of those ingredients needed to construct a map between SCET\(_i\) and a LL parton shower. We will augment the map with items needed for corrections (Eq. 104, Table II), but will not calculate contributions which only contain redundant information for the shower amplitude.

The latter point has important implications for the sorts of operator structures we need to consider. If we wanted to do computations in SCET\(_2\), then we would need all operators and Wilson coefficients to the order we are working. However, single gluon contributions in SCET\(_2\) where the gluon and the quark are collinear (inside a cone of angle \( \sim \lambda^2 \), i.e. \( O_2^{(1)}(n_0, n_0) \) or \( T_2^{(1)}(n_1, n_1) \)) correspond to a quark which does not split until after the scale of matching SCET\(_1\) \( \rightarrow \) SCET\(_2\). The corresponding no-branching probability, however, is already determined in SCET\(_1\) from the one-loop RG kernel. Thus, the coefficients of these operators in SCET\(_2\) are not required. We only need to calculate those single gluon contributions where each field has its own index label in SCET\(_2\), which means \( C_2^{(1)}(n_1, n'_1) \) for \( O_2^{(1)}(n_1, n'_1) \).
The matching equation for \( C_2^{(1)}(n_1, n'_1) \) in SCET\(_2\) is:

\[
C_2^{(1)}(n_1, n'_1)\langle 0| \mathcal{O}_2^{(1)} | q_{n_1}, g_{n'_1}, \bar{q}_n \rangle = C_2^{(0)}(n_0) \int d^4x \langle 0| \mathcal{L}_{\text{SCET}_1}(x) \mathcal{O}_1^{(0)} | q_{n_1}, g_{n'_1}, \bar{q}_n \rangle \\
+ C_1^{(1)}(n_0, n_0)\langle 0| \mathcal{O}_1^{(1)} | q_{n_1}, g_{n'_1}, \bar{q}_n \rangle + C_1^{(1)}(n_1, n'_1)\langle 0| \mathcal{O}_1^{(1)} | q_{n_1}, g_{n'_1}, \bar{q}_n \rangle \\
+ C_1^{(1)}(n_0, n_0)\langle 0| T_{(1)}^{(1)} | q_{n_1}, g_{n'_1}, \bar{q}_n \rangle.
\]

It is convenient to decompose \( C_2^{(1)}(n_1, n'_1) \) as

\[
C_2^{(1)}(n_1, n'_1) = C_2^{(1)\text{LO}}(n_1, n'_1) + C_2^{(1)\text{NLO}}(n_1, n'_1) + C_2^{(1)\text{NNLO}}(n_1, n'_1),
\]

where the four terms on the RHS of Eq. (79) correspond to each of the contributions on the RHS of Eq. (78). We calculated in \( C_2^{(1)\text{LO}} \) in Eq. (52) using RPI\(_1\) to rotate objects in the SCET\(_1\) amplitude such that they can come from SCET\(_2\) operators that annihilate the given external state. The second through fourth terms can be calculated in a similar manner. Their values are derived in Eqs. (D17)-(D19):

\[
C_2^{(1)H_a}(n_1, n'_1) = \frac{1}{Q} \left( \frac{k_1 n_{1}^{\mu} + \bar{q}_1 n_{1}'^{\mu}}{q_0} - \left( 1 + \frac{\bar{q}_1 k_1}{2 q_0^2} (n_1 \cdot n'_1) \right) n_{1}'^{\mu} \right) \Theta_{\delta_2}[n_1 \cdot n'_1],
\]

\[
C_2^{(1)H_b}(n_1, n'_1) = -\frac{2}{(n_1 \cdot n'_1) q_1 k_1} \gamma_\alpha \psi_\alpha \gamma_\mu \\
+ \left[ \frac{1}{(n \cdot p_q) q_1} \left( \gamma_\mu \gamma_\nu - \bar{q}_1 n_{11}^{\mu} \right) + \frac{2(n \cdot p_q)}{(n_1 \cdot n'_1) q_1 k_1} n_{11}^{\mu} \right] \Theta_{\tilde{\delta}_2}[n_1 \cdot n'_1],
\]

\[
C_2^{(1)H}(n_1, n'_1) = \left( \frac{1}{2Q} \left( \gamma_\mu n_{11}^{\mu} \sqrt{n_1 \cdot n'_1} \psi_1 + \bar{q}_1 \frac{1}{Q} (n_1 \cdot n'_1) \right) \Theta_{\delta_2}[n_1 \cdot n'_1] \right) \\
+ \frac{k_1}{Q^2} \left( \sqrt{n_1 \cdot n'_1} \gamma_\mu - n_{11}^{\mu} (n_1 \cdot n'_1) \frac{k_1^2 - \bar{q}_1^2}{2 Q^2} \right) \Theta_{\tilde{\delta}_2}[n_1 \cdot n'_1].
\]

Here \( n_1 \) and \( n'_1 \) are aligned with the direction of the quark and the gluon, and \( v_1 \) is defined in Eq. (B12). In Eq. (50), we have left off the running factors from evolution of the SCET\(_1\) operators. The terms in Eqs. (76) run differently. In particular, the two-jet and three-jet operators have different LL evolution. Therefore, it is important to decompose \( C_2^{(1)} \) as in Eq. (79), so that we can keep track of which SCET\(_1\) evolution factor to include for each. The running of these operators is discussed further in Section [V.C].

We also note the different \( \Theta \) dependence of the terms, where \( \Theta \) and \( \tilde{\Theta} \) we introduced in Eq. (54) and the surrounding discussion. We can read off from \( C_2^{(1)H_b} \) its origin as a three-jet term in SCET\(_1\), while the others come from two-jet operators. The \( \Theta \) functions are necessary because without them SCET\(_2\) operators, (e.g. \( \mathcal{O}_2^{(2)}(n_1, n'_1) \)) can only tell that the quark and gluon are not collinear according to the SCET\(_2\) definition. By including these phase space cutoffs, we can keep the distinct origins of different contributions manifest. By adopting a smoothed step function, as suggested in Sec. [II.A] and given in Eq. (D21), the amplitude squared for \( C_2^{(1)} \mathcal{O}_2^{(1)} \) will be continuous despite having different supports in different parts of phase space. An example of this is shown in Fig. 10. The full expression for the plot
FIG. 10: Merging of the two-jet and and three-jet squared amplitudes using a smooth theta function for the $\gamma^* \rightarrow q\bar{q}g$ process. Plots of the amplitude squared components from $C_2^{(1)} \mathcal{O}_2^{(1)}$: $|A_{q\bar{q}g}|^2_{\text{LO}} + |A_{q\bar{q}g}|^2_{\text{NLO,2-jet}}$ (short dashed green), $|A_{q\bar{q}g}|^2_{\text{NLO,3-jet}}$ (long dashed blue), and sum (solid red) versus $|k_1|_{\perp 0}$. The amplitudes are evaluated without running coefficients, and taking $\tilde{k}_1/\tilde{q}_0 = 0.4$. The $\delta_2$ parameter in the $\Theta$-function is 1.2, which for the above $\bar{p}$ fraction corresponds to $\eta = 0.5$, and $\lambda = 0.08$.

is given in Eqs. (D24) and (D25). To illustrate the effects of including hard-scattering corrections, in Fig. 11 we plot the ratios $R_{\text{LO}} = |A_{q\bar{q}g}|^2_{\text{LO}}/|A_{q\bar{q}g}|^2_{\text{QCD}}$ and $R_{\text{NLO}} = (|A_{q\bar{q}g}|^2_{\text{LO}} + |A_{q\bar{q}g}|^2_{\text{NLO,2-jet}})/|A_{q\bar{q}g}|^2_{\text{QCD}}$ versus the gluon perp momentum. Here, $|A_{q\bar{q}g}|^2_{\text{QCD}}$ is the QCD amplitude squared for one-gluon emission, $|A_{q\bar{q}g}|^2_{\text{LO}}$ is the SCET$_2$ amplitude squared for one-gluon emission from the LO coefficient $C_2^{(1)} \mathcal{O}_2^{(1)}$ (from Eq. 52), and $|A_{q\bar{q}g}|^2_{\text{NLO,2-jet}}$ is the NLO($\lambda$) amplitude squared for one-gluon emission in the two-jet region that comes from the coefficients $C_{2,\text{NLO}}^{(1)H,a}$ and $C_{2,\text{NNLO}}^{(1)H}$ (given in Eq. 80). As we expect, including corrections up to NNLO($\lambda$) in the amplitudes squared extends the region where tree-level SCET$_2$ and QCD agree. The advantage of using the one-gluon SCET$_2$ amplitude over QCD comes from factorization properties that affect interference as well as renormalization group evolution. For example the one-loop running in SCET$_2$ performs the LL Sudakov resummation.

With two-gluon emission, the SCET$_1$ graphs will include jet-structure corrections in addition to hard-scattering ones. It is straightforward to distinguish the types as the former result from taking time-ordered products of the SCET$_1$ Lagrangian with operators generated by the LO replacement rule, Eq. (56), while the latter will come only from terms involving a power suppressed SCET$_1$ operator. To fully identify the subleading contributions to two-gluon emission, we must match down to SCET$_2$ where the LO contribution is first uniquely identified. We already know that it comes from two applications of Eq. (56).

In Fig. 12, we show the contributions to two-gluon emission in QCD, SCET$_1$, SCET$_2$, and
FIG. 11: Plot of the ratios of the amplitudes squared for $\gamma^* \to q\bar{q}g$, namely $R_{LO} = |A_{q\bar{q}g}|_LO^2/|A_{q\bar{q}g}|_{QCD}^2$ (blue short dashed) and $R_{NLO} = (|A_{q\bar{q}g}|_LO^2 + |A_{q\bar{q}g}|_{NLO,2\text{-jet}}^2)/|A_{q\bar{q}g}|_{QCD}^2$ (red long dashed) versus $|k_1|_{n_0\perp}$, for $k_1/\bar{q}_0 = 0.4$. The amplitudes are evaluated without running factors.

FIG. 12: Matching QCD to SCET\textsubscript{1} to SCET\textsubscript{2} to SCET\textsubscript{3} for two gluons emitted collinear to the quark direction (SCET graphs for other gluon kinematic configurations not shown). Once again, we depict the operator structures that lead to this process in each of the theories. Gluons drawn away from the central vertex are emitted by the leading order Lagrangian in that theory, while those coming from the vertex are due to higher dimension operators.

SCET\textsubscript{3}. The first column in the SCET\textsubscript{1} category corresponds to the jet-structure corrections to be considered in the next section. In the second column we have a set of hard-scattering corrections from taking the $T$-product of the SCET\textsubscript{1} Lagrangian with the suppressed single
gluon operators we calculated above in Eqs. (77), $C_1^{(1)} \mathcal{O}_1^{(1)}$ and $C_1^{(1)} \mathcal{O}_1^{(1)}$.

In considering the basis of operators in SCET$_2$ we do not need operators such as $\mathcal{T}_2^{(1)}(n_1, n_1')$, since $\mathcal{P}_{n_1' \bot} \mathcal{B}_{n_2' \bot} = 0$, with $n_1'$ lying along the gluon momentum. We can use RPI$_2$ in SCET$_2$ to make a coordinate choice where they are not necessary. As mentioned above in the single gluon matching section, our interest is only in calculating those terms needed to improve a shower algorithms, which precludes us from considering operators such as $\mathcal{T}_2^{(1)}(n_0, n_0)$ or $\mathcal{O}_2^{(1)}(n_0, n_0)$, corresponding to an unbranched quark passing from SCET$_1$ into SCET$_2$. Therefore, for double gluon emission we only need to calculate the coefficients of the following operators:

$$\mathcal{O}_2^{(1)}(n_1, n_1') = \bar{\chi}_{n_1} g \mathcal{B}_{n_1' \bot}^3 \chi_{n_1},$$

$$\mathcal{O}_2^{(2)}(n_2, n_2, n_1') = \bar{\chi}_{n_2} g \mathcal{B}_{n_2' \bot}^3 g \mathcal{B}_{n_2' \bot}^3 \chi_{n_1},$$

$$\mathcal{O}_2^{(2)}(n_2, n_1', n_1') = \bar{\chi}_{n_2} g \mathcal{B}_{n_1' \bot}^3 g \mathcal{B}_{n_1' \bot}^3 \chi_{n_1},$$

$$\mathcal{O}_2^{(2)}(n_2, n_1', n_2') = \bar{\chi}_{n_2} g \mathcal{B}_{n_1' \bot}^3 g \mathcal{B}_{n_2' \bot}^3 \chi_{n_1}.$$  

Thus in SCET$_2$, we are interested in two-gluon operators where two fields can have the same label. When we pass to SCET$_3$, we can restrict our interest to only $\mathcal{O}_3^{(2)}(n_2, n_1', n_2')$.

We already gave the coefficients of $\mathcal{O}_2^{(0)}(n_1, n_1')$ needed to compute the leading power corrections in Eqs. (80) and (D17)-(D19). We get an NLO($\lambda$) contribution to the two gluon amplitude by computing the matrix element, $C_{2, NLO}^{(1)}(0) |T\{ \mathcal{L}_{SCET, 2} \mathcal{O}_2^{(1)} \}|qgggg$ (first SCET$_2$ column in Fig. 12). The contribution receives no further suppression as the gluon from $\mathcal{L}_{SCET}$ gives a tree-level vertex-propagator factor of $\lambda^{-2}$, just as with LO. There are also coefficients we need from two-gluon matching calculations for the operator $\mathcal{O}_2^{(2)}$ (second SCET$_2$ column in Fig. 12). Putting in the index structures, these include $C_{2, NLO}^{(2)}(n_2, n_1', n_2')$ for $\mathcal{O}_2^{(2)}(n_2, n_1', n_2')$, $C_{2, NLO}^{(2)}(n_2, n_1', n_1')$ for $\mathcal{O}_2^{(2)}(n_2, n_1', n_1')$ and $C_{2, NLO}^{(2)}(n_2, n_2, n_2') + C_{2, NNLO}^{(2, N)}(n_2, n_2, n_2')$ for $\mathcal{O}_2^{(2)}(n_2, n_2, n_2')$. We include NNLO($\lambda$) for the last one as only it interferes with LO($\lambda$). In the next subsection, we give the jet-structure corrections. All hard-scattering contributions to these structures just listed are beyond the order we need except for $C_{2, NNLO}^{(2)}(n_2, n_2, n_2')$, given by the matching equation:

$$C_{1, NLO}^{(1)}(n_0, n_0) \int dx(0)|T\{ \mathcal{L}_{SCET, 1}(x) \mathcal{O}_1^{(1)}(n_0, n_0)\}|q_{n_2} g_{n_2} g_{n_1'} \bar{q}_{n_1}$$

$$- C_{2, NLO}^{(2)}(n_2, n_1') \int dx(0)|T\{ \mathcal{L}_{SCET, 2}(x) \mathcal{O}_2^{(1)}(n_2, n_1')\}|q_{n_2} g_{n_2} g_{n_1'} \bar{q}_{n_1}$$

$$= C_{2, NNLO}^{(2)}(n_2, n_2, n_1')\langle 0|\mathcal{O}_2^{(2)}(n_2, n_2, n_1')|q_{n_2} g_{n_2} g_{n_1'} \bar{q}_{n_1}\rangle,$$

where we subtract the Lagrangian emission graph in SCET$_2$ from that in SCET$_1$ ($C_{1, NLO}^{(1)}$ is given in Eq. 77 and $C_{2, NLO}^{(2)}$ in Eq. 80). The result for $C_{2, NNLO}^{(2)}(n_2, n_2, n_1')$ is given in Eq. (D61). It is straightforward to see why $\mathcal{O}_2^{(2)}$ only gets hard-scattering at NNLO($\lambda$) and higher. By definition, hard-scattering has to involve a suppressed operator from the QCD $\rightarrow$ SCET$_1$ matching, and so we begin at NLO($\lambda$) at the lowest order. Including a second gluon, but demanding that we cannot write it as coming from a SCET$_2$ Lagrangian emission takes us to one order higher, namely NNLO($\lambda$).
All the contributions we have discussed so far have come from the hard-scattering, single-gluon, suppressed operators in SCET$_1$. There are also those with two gluons. That is to say a process where neither gluon comes from the SCET$_1$ Lagrangian, represented by the diagram in the third SCET$_1$ column in Fig. 12. One example is double $\perp$-gluon emission from the antiquark, as shown in the third QCD graph of Fig. 12. We know from applying Eq. (56) twice, that LO for this process is at $O(\lambda^{-1})$, counting only the tree-level vertex×propagator factors, as these are all we need to compare different $q\bar{q}gg$ processes. We readily see that double antiquark emission is $\sim \lambda^0$ as there are no small virtualities or emission angles for this term. Thus, they are N$^3$LO, and beyond this analysis. Besides antiquark vertices, we also have subleading emissions from the quark in QCD that arise from the suppressed SCET-spinor portion of the QCD quark propagator (cf. Appendix A). If both emissions come from the suppressed propagator, once again, this is $\sim \lambda^0$ at lowest order, and so we can neglect it. Mixed antiquark/suppressed spinor contributions are also N$^3$LO.

Thus, we do not need corrections to double emission collinear to the quark if they do not involve at least one SCET$_1$ Lagrangian insertion. We can extend this argument further. If there are no SCET$_1$ Lagrangian insertions, then the contribution goes like $O(\lambda^0)$, while LO goes like $O(\lambda^{-\frac{d+1}{2}})$. Thus, to the order we are working, we only need the single gluon hard-scattering corrections given by Eq. (80), plus Lagrangian insertions.

B. Jet-Structure Corrections

The jet-structure corrections only involve contributions from the SCET$_1$ Lagrangian. These arise from the graphs in the first SCET$_1$ column in Fig. 12. We specifically designed our leading order replacement rule in Eq. (56), so when used twice it only contains that part of double emission corresponding to the leading strongly-ordered limit. This occurs for the gluons having collinearities $\sim \lambda$, $\lambda^2$, respectively. However, SCET$_1$ describes other kinematic situations and in this section we compute the corrections from them.

The prescription for obtaining two-gluon jet-structure corrections is to compute the double gluon emission amplitude in SCET$_1$ coming from two Lagrangian insertions and take different limits on the relative collinearities of $n_2$, $n_2'$, and $n_1'$, where these labels refer to the null vectors exactly proportional the corresponding particle momenta. We can define:

$$A_{\text{NLO}}^{ggg} = C_{1,\text{LO}}^{(0)}(n_0) \int dx_1 dx_2 \langle 0|T\{\mathcal{L}_{\text{SCET}_1}(x_1)\mathcal{L}_{\text{SCET}_1}(x_2)\mathcal{O}_{1}^{(0)}\}|q_{n_2}g_{n_1'}g_{n_2'}\bar{q}_n\rangle, \quad (83)$$

and then calculate,

$$\lim_{n_2 \cdot n_2' \sim \lambda^2} A_{\text{NLO}}^{ggg} = C_{2,\text{NLO}}^{(2)}(n_2, n_1', n_2') \langle 0|\mathcal{O}_2^{(2)}|q_{n_2}g_{n_1'}g_{n_2'}\bar{q}_n\rangle, \quad (84)$$

$$\lim_{n_1' \cdot n_2' \sim \lambda^4} A_{\text{NLO}}^{ggg} = C_{2,\text{NLO}}^{(2)}(n_2, n_1', n_2') \langle 0|\mathcal{O}_2^{(2)}|q_{n_2}g_{n_1'}g_{n_2'}\bar{q}_n\rangle, \quad (85)$$

$$\lim_{n_2 \cdot n_2' \sim \lambda^4} A_{\text{NLO}}^{ggg} = C_{2,\text{LO}}^{(1)}(n_2, n_1') \langle 0|T\{\mathcal{L}_{\text{SCET}_2}\mathcal{O}_2^{(1)}\}|q_{n_2}g_{n_1'}g_{n_2'}\bar{q}_n\rangle$$

$$+ C_{2,\text{NLO}}^{(2)}(n_2, n_1', n_2) \langle 0|\mathcal{O}_2^{(2)}|q_{n_2}g_{n_1'}g_{n_2'}\bar{q}_n\rangle. \quad (86)$$

We note a few things about the above equations. Firstly, there is a correction to the LO Wilson coefficient obtained from the replacement rule (Eq. 56). We cannot get it purely
as a limit of $A_{\text{NLO}}^{3g}$, so we need to subtract off the LO contribution. Secondly, the limit in Eq. (84) does not lead to an expansion of any part of $A_{\text{NLO}}^{3g}$, as the scaling of the $n$-indices’ dot products is exactly that from SCET$_1$. Even though it just gives back the same expression as the SCET$_1$ amplitude, $A_{\text{NLO}}^{3g}$, the SCET$_2$ result for $C_{2,\text{NLO}}^{(2)J}(n_2, n_1, n_2')O_{J,\text{NLO}}^{(2)}$ tells us something more. This Wilson coefficient is proportional to $\tilde{\Theta}_{\delta_2}[n'_2 \cdot n_2']\tilde{\Theta}_{\delta_2}[n_2 \cdot n_2']$, where the $\tilde{\Theta}$’s only have support outside the phase space region of Eq. (85), as well as the strongly-ordered limit, Eq. (86), (see Eqs. (54) and (D21) for the definition of $\Theta$, $\tilde{\Theta}$). The full results for the Wilson coefficients shown in Eq. (86) can be found in Eqs. (D37), (D52), and (D59). At the amplitude level, given a particular phase space configuration for an external state, we will only ever need one of these terms for double gluon emission in SCET$_2$. Squaring the result is straightforward as there will be no interference between them.

We will now examine how to improve the matching of SCET$_i$ to SCET$_{i+1}$, and show that the jet-structure corrections computed here generalize to that case. We first notice that the first two operators above do not interfere with the one giving LO, as they have different index structures. The subleading term in Eq. (84) does inhabit the strongly-ordered region of phase space, but as we will argue in Sec. V.D LO($\lambda$)/NLO($\lambda$) interference cancels out of most observables of interest. Before proceeding, we note that our description of corrections to two-gluon emission we have found in the coefficient, $C_{3}^{(2)}$, for the operator $O_{3}^{(2)}(n_2, n_1', n_2') = \bar{x}_{n_2}gB_{n_1'}^\alpha gB_{n_2'}^\beta \chi_{\perp}$, as we do in Eqs. (E2). The same will hold for $i$-gluon emission in SCET$_{i+1}$. Our NLO($\lambda$) jet-structure operators therefore have the following form:

$$C_{3,\text{NLO}}^{(2)J I}(n_2, n_1, n_1')O_{J}^{(2)} = h_I^{\alpha\beta} \bar{x}_{n_2}gB_{n_1'}^\alpha gB_{n_2'}^\beta \Gamma^\mu \chi_{\perp}, \quad (87)$$

where $h_I^{\alpha\beta}$ is given by Eq. (E11). Here $I = \{1, 2, 3\}$, and we distinguish the coefficients $C_{3,\text{NLO}}^{(2)J I}$ depending on which SCET$_2$ operators they come from in order to properly account for their RG evolution in SCET$_2$.

When doing the LO matching for SCET$_i$ to SCET$_{i+1}$, we found that the replacement rule to go from SCET$_1$ to SCET$_2$ generalized to the case of $i$-gluon strongly-ordered emission. Similarly, we can take the above operator, Eq. (87), and recast it as a replacement rule for our original current insertion, $C_{1,\text{LO}}^{(0)}O_{1}^{(0)}$. It takes the form of a $1 \rightarrow 3$ replacement rule:

$$\bar{x}_{n_0} \rightarrow h_I^{\alpha\beta} \bar{x}_{n_2}gB_{n_1'}^\alpha gB_{n_2'}^\beta, \quad (88)$$

with contributions from $I = 1, 2, 3$.

If we want to consider the NLO($\lambda$) radiation of $i+1$ gluons, we can perform a very similar matching between SCET$_1$ and SCET$_{i+2}$ to the one above for SCET$_1 \rightarrow$ SCET$_3$ to obtain an operator $C_{i+2,\text{NLO}}^{(i+1)J}O_{i+2}^{(i+1)}$. Since the first $(i - 1)$ emissions are strongly ordered, they completely factor out. Thus, the amplitude for the emission of the final two gluons will be identical to that for simple two-gluon emission. We can therefore take the $(i - 1)$ gluon LO operator, $C_{i,\text{LO}}^{(i-1)J}O_{i}^{(i-1)}$, and use the replacement rule in Eq. (88), to obtain $C_{i+2,\text{NLO}}^{(i+1)J}O_{i+2}^{(i+1)}$. Our NLO($\lambda$) replacement rule corresponds to violating strong ordering at any location in the shower, either by taking the $j^{th}$ and $(j+1)^{th}$ gluons to have the same parametric collinearity
with respect to their parents, \( k_{j+1\perp} \sim k_{j\perp} \) (Eqs. 84 and 85); or by including the region of phase space where the propagator between them is hard even in SCET\(_1\), and so we get no collinear divergence as the quark and second gluon become collinear (86).

It is not difficult to see that this gives an NLO(\(\lambda\)) contribution for any \(j\). If we have \(i\)-gluon strongly-ordered emission, the tree-level factors, \( c_{\text{LO}}(n_{k-1}) \) (cf. Eq. 63) will go as \( \lambda^{-i(i+1)/2} \), where the \(j\)th gluon contributes \( \lambda^{-j} \). If we violate strong ordering as we mention above for any two gluons, the product of their vertices times propagators goes like \( \lambda^{-2j} \) instead of \( \lambda^{-(2j+1)} \). Thus, we can insert \( \bar{\chi}_{n_0} \rightarrow h_I \bar{\chi}_{n_2} g\mathcal{B}_{n'_1\perp} g\mathcal{B}_{n'_2\perp} \) instead of two successive \( \bar{\chi}_{n_0} \rightarrow c_{\text{LO}} \bar{\chi}_{n_2} \mathcal{B}_{n'_1\perp} \)'s in operator matching as a “defect” in strong ordering at any stage and obtain an NLO(\(\lambda\)) jet-structure correction. The \(\Theta\)-functions contained in the Wilson coefficients, \( C_{(i+1)J}^{(i)\text{LO}} \), allow us to read off at which step in the shower we violated strong-ordering.

In App. F we show that an integrated version of \( h_I^{q,g} \) is related to the splitting function at NLO in \(\alpha_s\), which serves as a cross-check on our computations.

C. Operator Running

Up until now, our discussion of matching has taken place mostly at tree-level. Connecting to the no-branching probabilities and log resummation in the parton shower however, requires that we include the anomalous dimensions needed for running. For this reason, our final expressions for Wilson coefficients in Apps. C-E include the necessary notation for evolution kernels. Identifying the power suppressed amplitudes as corresponding to perturbative corrections to more inclusive observables, it is natural to take only LL\(_{\text{exp}}\) evolution for power suppressed or \(\alpha_s\) suppressed corrections, and include NLL\(_{\text{exp}}\) evolution only for the leading shower terms. For the former, we assume (without carrying out the proof in SCET\(_i\)) that we must make the \(k_T^2\) choice for the scales \(\mu_k^2\) as in Eq. (64), and that this accounts for the difference between LL and LL\(_{\text{exp}}\). NLL\(_{\text{exp}}\) would require full one-loop, two-loop cusp, and NLL \(\alpha_s\) running, plus any modifications to the evolution induced by subleading soft effects. If subleading soft effects are neglected then in the terminology of [60, 61], this gives the full collinear NLL\(_{\text{exp}}\) resummation. The subleading logarithms coming from pure soft effects involve the exponentiation of nonabelian matrices. As mentioned earlier, we do not compute the effects of subleading soft SCET\(_i\) operators here. (In fact, for more than three hard, colored particles, the problem is quite non-trivial [60].)

In this section, we determine the LL\(_{\text{exp}}\) running for our subleading operators and discuss what is missing in our setup for a NLL\(_{\text{exp}}\) evolution kernel for emission anywhere in the shower. To set the stage, we consider SCET\(_1\) matched to QCD at the scale \(Q\) for the first
order power corrections. We then run down to $\mu$ in preparation for matching to SCET$_2$. The zero and single gluon operators in SCET$_1$ acquire the following running factors, $U$, (cf. the tree-level version in Eq. (77)):

$$C_0^{(0)}(n_0) = U^{(2,0,0)}(n_0; Q, \mu) \gamma_{n_0,\perp}^\mu,$$

$$C_{1,\text{NLO}}^{(1)}(n_0, n_0) = U^{(2,1,0)}(n_0, n_0; Q, \mu) \otimes \frac{n_0^\mu - \bar{n}_0^\mu}{Q} \gamma_{n_0,\perp}^\alpha,$$

$$C_{1,T}^{(1)}(n_0, n_0) = U_T^{(2,1,1)}(n_0, n_0; Q, \mu) \otimes \frac{1}{\bar{q}_1 k_1} \left( \gamma_{n_0,\perp}^\mu \gamma_{n_0,\perp}^\beta \gamma_{n_0,\perp}^\alpha - \frac{2}{\bar{q}_1 Q} g^{\mu\beta} \gamma_{n_0,\perp}^\alpha \right),$$

$$C_{1}^{(1)}(n_1, n'_1) = -U^{(2,1,0)}(n_1, n'_1; Q, \mu) \left( \frac{2}{n_1 \cdot n'_1 \bar{q}_1 k_1} \gamma_{n_0,\perp}^\alpha \gamma_{n_0,\perp}^\mu \right) + \left[ \frac{1}{(n \cdot p_q) k_1} \left( \gamma_T^{\mu} \bar{p}_T - \bar{q}_1 n'_1 T \right) + \frac{2(n \cdot p_q)}{(n_1 \cdot n'_1 \bar{q}_1 k_1)} \bar{n}_T^\mu \right] \gamma_{n_0,\perp}^\alpha,$$  \hspace{1cm} (89)

where the superscripts follow the convention in Eq. (11). We inserted the symbol $\otimes$ in the second and third line of Eq. (89) since an operator with multiple fields sharing the same collinear direction can convolve the momentum fraction of $\bar{p}$ between the corresponding RG kernel $U$ and momenta in the tree-level coefficient. This is because collinear fields that are in the same direction in SCET can exchange momentum while running down from $Q$ to $\mu$. The anomalous dimension of an operator is independent of which SCET$_i$ it is defined, but does depend on the field content and in particular how many different collinear directions are in the operator. Thus, the RG-kernel for $\bar{\chi}_{n_0} g B_{n_0,\perp}^{\alpha} \chi_{n_0}$ is different from that of $\bar{\chi}_{n_0} g B_{n_0,\perp}^{\alpha} \chi_{n_0}$. In Ref. [24, 25], the LL part of $U^{(j,i-j,0)}(Q, \mu)$ was related to the Sudakov form factor, Eq. (20) (up to accounting for the soft effects of angular ordering [73]). The cusp term in the anomalous dimension resums the LL, and comes from soft and collinear one-loop diagrams. The result from the soft diagrams is constrained by that of the collinear diagrams in order to cancel out infrared sensitivity that cannot be absorbed in local counterterms at the hard scale. Here we will use this same argument, but in reverse, in order to determine the LL$_{\text{exp}}$ anomalous dimension of various subleading operators.

Due to the soft-collinear factorization, the soft structure only depends on the number of collinear directions. After making the field redefinition, operators like $\bar{\chi}_{n_0} \chi_{n_0}$ and $\bar{\chi}_{n_0} g B_{n_0,\perp}^{\alpha} \chi_{n_0}$ both have $Y^+_{n_0} Y_{n_0}$, and so both have the same soft divergences. Hence they have the same one-loop cusp term and the same LL anomalous dimension from the sum of collinear and soft loops. Thus, the leading-log resummation only depends on the number of collinear index directions in the operator, and not on the number of active partons. (At leading power these concepts are the same, but it is not so for the power corrections.) We therefore have

$$U_{LL}^{(2,0,0)}(n_0) = U_{LL}^{(2,1,1)}(n_0, n_0) = U_{LL,T}^{(2,1,1)}(n_0, n_0),$$  \hspace{1cm} (90)

where we give $U_{LL}^{(2,0,0)}$ in Eqs. (18) and (19). Thus, at LL order we have the full set of evolution kernels for subleading collinear operators, and we account for these factors in the appendices. Since this is a LL effect, we expect soft radiation and angular ordering to be incorporated in a manner identical to the evolution factor in the LL shower.

An important consequence of this result for the LL evolution is that it justifies treating our hard-scattering corrections as improvements to the fixed-order, matrix-element calculation...
that goes into a shower algorithm. Correcting the two-jet amplitude with either $C_{1,NLO}^{(1)}$ or $C_{1,T}^{(1)}$, we see that the LL resummation is the same as that in the standard shower except that there is an extra parton already inside the leading jet. We thus get a shower correction just by using a matrix element improved by including our hard-scattering terms. This is unlike simply running a LL shower on higher order matrix elements, as different anomalous dimensions control different operators’ evolution. Some, like those just mentioned with only $n_0$ and $\bar{n}$ collinear directions, run like two-jet configurations, that is with a quark-antiquark Sudakov. Others, (e.g. $C_{1}^{(1)} O_{1}^{(1)}(n_{1}', n_{1})$) have three-parton running since they have three distinct collinear directions. This latter set corresponds to the usual implementation of fixed order corrections in parton showers, but the former is a novel type of shower improvement.

On the other hand, the effect of jet-structure corrections is not to modify the initial scattering process, but to go hand in hand with the NLL change to the leading operators’ running. Similarly to Eq. (63), we might anticipate the following Wilson coefficient for $O_{N}^{(N-1)}$ with evolution:

$$C_{N,NLL,1}^{(N-1)}(m) = \left[ \prod_{k=1, k \neq m}^{N-1} U_{LL}^{(k-1)}(\mu_{k-1}, \mu_{k}) c_{LO}^{\alpha_{k}}(n_{k-1}) \right] \Gamma_{\mu},$$

with a sum over all locations where the NLL evolution can be inserted:

$$C_{N,NLL,1}^{(N-1)} = \sum_{m=1}^{N-1} C_{N,NLL,1}^{(N-1)}(m).$$

One would expect to use $C_{N,NLL,1}^{(N-1)} O_{N}^{(N-1)}$ along with our real emission corrections (Eq. E13) to correct a shower to resum at NLL the ratios of all emission scales (cf. Eq. 104). The complication we face for the calculation of $U_{NLL}^{(m-1)}$ is that this correction to the evolution kernel must, in principle, be carried out in the same scheme used to distinguish the phase space regions for the jet-structure corrections, and hence can depend on the choice for the $\Theta$ functions. In particular, we could have non-trivial operator mixing on the edge where the cutoff makes a smooth transition between operators with different numbers of jets, and we have not yet performed the analysis that would determine whether this affects the resummation at NLL_{exp} order. Furthermore, it is possible that power suppressed soft effects will also have implications for the subleading evolution kernel, and may make the nonabelian generalization of Eq. (92) tricky. Our lack of an appropriate NLL_{exp} evolution factor for the shower is due to these two issues.

To setup the distinction between kinematic regions, we used Wilsonian type $\Theta$ functions, but from the point of view of evolution $\overline{MS}$ would be simpler. Although it is only indirectly relevant to our setup, it is nevertheless still interesting to consider how the NLL evolution kernel would arise in $\overline{MS}$. As we discuss below in App. [4] when integrated over phase space in dimensional regularization the jet-structure corrections give the real emission portion of $P_{qq}^{(1)}$, which is the $O(\alpha_s)$ correction to the Altarelli-Parisi splitting kernel. Combined with known SCET results for single-emission at one-loop, we can recover all of the abelian portion of $P_{qq}^{(1)}$. Obtaining this expression is important conceptually. It validates our formal
expansion in $\lambda$, showing that corrections to $\mathcal{O}(\lambda^2)$, along with a set of known one-loop diagrams, capture contributions needed for collinear NLL$_{\text{exp}}$ resummation. On the practical side, it provides a cross check on our computations.

With $P_{qq}^{(1)}$ in hand, we can extend the argument of [24, 25] that the Sudakov factor gives the LL part of the the RG kernel $U^{(2,0,0)}(Q, \mu)$ (Eq. 20) to the NLL level, looking at $U^{(2,0,0)}(Q, \mu)$ for running of the operator $\mathcal{O}_{1,\text{LO}}^{(0)}$. Using the Sudakov factor of [27] for quarks, we have:

$$\Delta_q(Q, \mu) = \exp \left\{ -\frac{C_F}{2\pi} \int_{\mu}^{Q} \frac{d\mu'}{\mu'} \alpha_s(\mu') \int_{1 - \frac{\sqrt{Q}}{\mu'}}^{1} \frac{dz}{1 - z} \left( 1 + z^2 \right) \right\}, \quad (93)$$

where we recognize $P_{qq}^{(0)}$, Eq. (3). Performing the $z$ integral and expanding in the limit of large $Q$ gives:

$$\Delta_q(Q, \mu) \approx \exp \left\{ \frac{C_F}{\pi} \int_{\mu}^{Q} \frac{d\mu'}{\mu'} \alpha_s(\mu') \left[ \log \left( \frac{\mu'^2}{Q^2} \right) + \frac{3}{2} \right] \right\}, \quad (94)$$

which is identical to $U^{(2,0,0)}(Q, \mu)$ at one-loop. The term in the exponent proportional to $\log(\mu'^2/Q^2)$ sums the leading logs in the parton shower. We also see that upon $\mu'$ integration, we get the double logarithm characteristic of the soft-collinear divergence of collinear splitting. Interpreting Eq. (94) as an RG kernel, this log piece is coming from the one-loop cusp anomalous dimension, $C_F$. The factor of $3/2$ is the remaining part of the one-loop anomalous dimension, and it sums part of the collinear NLL. In order to get the full collinear NLL$_{\text{exp}}$ summation, one also needs corrections corresponding to the two-loop cusp anomalous dimension. This is a known result in SCET for the operator $\bar{\chi}_n \chi_\bar{n}$, which we can relate to $P_{qq}^{(1)}$, by adding the subleading splitting function to the exponent of $\Delta_q(Q, \mu)$. We wish to stress, however, that the ultimate goal of improving parton showers through resummation is to include all next-to-leading-logs. In this paper, as mentioned previously we have not considered the effects of soft NLL, nor those related to the two-loop running of $\alpha_s$, which will affect collinear NLL. Our formulas in Apps. C-E include LL running for all subleading operators. In App. F we discuss the relation of our $1 \to 3$ splitting amplitude with $P_{qq}^{(1)}$ in \textit{MS}. The collinear-NLL-improved Sudakov corresponding to this is

$$\Delta_q^{\text{NLL}}(Q, \mu) = \exp \left\{ -\int_{\mu}^{Q} \frac{d\mu'}{\mu'} \int_{1 - \frac{\sqrt{Q}}{\mu'}}^{1} dz \left[ P_{qq}^{(0)}(z, \alpha_s(\mu')) + P_{qq}^{(1)}(z, \alpha_s(\mu')) \right] \right\}, \quad (95)$$

where $P_{qq}^{(0)}$ given in Eq. (3) and $P_{qq}^{(1)}$ in [62]. Once again, we integrate in $z$, expanding in

---

9 Since Eq. (94) resums the NLL contributions expanded in the cross section (cf. Eq. 5), Ref. [27] calls it the NLL Sudakov factor.

10 At a practical level, while we see full collinear NLL$_{\text{exp}}$ as coming from a straightforward extension of this work, pure soft NLL may only be possible at the leading orders in $1/N_c$ [60, 61].
large $Q$ to get:

$$\Delta_{q}^\text{NLL}(Q, \mu) = \exp \left\{ \int_{\mu}^{Q} \frac{d\mu'}{\mu'} \left[ \frac{\alpha_s(\mu')}{\pi} C_F \left( \log \left( \frac{\mu'^2}{Q^2} \right) + \frac{3}{2} \right) \right. \right.$$  

$$\left. + \frac{\alpha_s^2(\mu')}{4\pi^2} C_F \left( C_g \left( \frac{67}{9} - \frac{\pi^2}{3} \right) - \frac{20}{9} C_F T_F n_F \right) \log \left( \frac{\mu'^2}{Q^2} \right) \right] \right\}, \quad (96)$$

where the term $\propto \alpha_s^2$ reproduces the known result for the two-loop cusp anomalous dimension. While including this $\overline{\text{MS}}$ NLL effect for “no-branching” was already possible, our result in Eq. (E13) allows one to modify the differential cross section for real emission to include the effects of $P_{qq}^{(1)}$, as well. Without including both, one does not have a systematic improvement beyond LL. In [56], the authors were able to get NLL exp soft resummation by treating the subleading real and virtual effects in semi-inclusive observables for DIS and Drell-Yan. A full implementation in our framework with more exclusive observables must wait for computations that address the missing NLL ingredients mentioned above.

D. Squared Amplitudes and Interference Structures

As discussed previously, our series of matchings terminates with SCET$_N$, where each field has its own index direction. Further Lagrangian emission from these operators is physically meaningless, as the resolution scale is set $\sim O(\text{GeV})$, below which we stop computing in perturbation theory and pass to a hadronization routine. Thus, we match everything to the single operator $O_N^{(N-1)}(n_N, n'_1, \ldots, n'_{N-1})$ and all the information about the shower at LO and NLO is encoded in the Wilson coefficients. In this SCET$_N$, we square amplitudes and compute corrections to observables, as we detail in Sec. [IV.E]. As we saw in Secs. [IV.A] and [IV.B] for arbitrary $N$, we only needed one and two-gluon computations to obtain leading corrections in $\lambda$ to the differential cross section. Using the LO replacement rule (Eq. 56) will account for the rest of the multiplicity. Since the strongly-ordered emissions it describes have trivial interference, we should expect that squaring our results retains the simple picture we have for corrections at the amplitude level.

1. Interference for LO$^2$ and for Jet-Structure Corrections

It is a general statement about SCET fields with different $n$ index labels that they have no overlap in Hilbert space. As an example, we can take two different operators, $O_{n_1}$ and $O_{n_2}$ where all the fields in $O_{n_1}$ and $O_{n_2}$ are identical, except those labeled by $n_1$ and $n_2$ (e.g. $\bar{\chi}_{n_1}$ versus $\bar{\chi}_{n_2}$). For generality the field labeled by $n_2$ may or may not be in the same equivalence class as $n_1$. We thus have:[11]

$$\langle q_1, q_2, \ldots, q_m | O_{n_1}^{\dagger} | 0 \rangle \langle 0 | O_{n_2} | q_1, q_2, \ldots, q_m \rangle = \delta_{[n_1], [n_2]} \langle q_1, q_2, \ldots, q_m | O_{n_1}^{\dagger} | 0 \rangle \langle 0 | O_{n_2} | q_1, q_2, \ldots, q_m \rangle. \quad (97)$$

[11] By RPI, $n_1$ and $n_2$ do not have to be exactly equal, but must concur up to an angle of $O(\lambda^i)$ in SCET$_i$. 

45
FIG. 13: Amplitude squared for the LO SCET shower operator $C_{N, \text{LO}}^{(N-1)} \mathcal{O}_N^{(N-1)}$. Rather than drawing the less intuitive squared amplitude in SCET$_N$, we illustrate the process here with a cut SCET$_1$ Feynman diagram in order to emphasize the simple ladder structure.

This relation between $n_1$ and $n_2$ is simple when the difference is encoded in the collinear fields in operators. However, as discussed in Sec. [IV.A] we also have to deal with situations where this information ended up in Wilson coefficients when matching SCET$_i$ to SCET$_{i+1}$. It is to guarantee a relation like Eq. (97) that our Wilson coefficients contain $\Theta$-functions (cf. Eqs. 54 and D21), which will cutoff the overlap regions in phase space once we begin integrating. The amplitude squared is particularly simple in SCET$_N$, where we have only the operator $\mathcal{O}_N^{(N-1)}(n_N, n'_1, \ldots, n'_{N-1})$, and where each particle is defined in a different collinear direction.

SCET$_N$ (or SCET$_i$, in general) easily distinguishes which configurations are strongly-ordered by the structure of their Wilson coefficients. This means that we have no interference between $C_{N, \text{LO}}^{(N-1)} \mathcal{O}_N^{(N-1)}$ and $C_{N, \text{NLO}}^{(N-1)} \mathcal{O}_N^{(N-1)}$ where $C_{N, \text{LO}}^{(N-1)}$ is the LO SCET$_N$ coefficient given in Eq. (63), and $C_{N, \text{NLO}}^{(N-1)}$ is in Eq. (E12). Even though the $\mathcal{O}$’s are the same, the $\Theta$-functions in the $C$’s enforce different conditions, where the former is strongly ordered, while the latter is not. Thus, in the analog of Eq. (97), the Kronecker delta will give zero.

We get a further simplification when we square the NLO($\lambda$) contributions. Looking at $C_{N, \text{NLO}}^{(N-1)J}$ in detail, we have:

$$C_{N, \text{NLO}}^{(N-1)J} = \sum_{l=1}^{N-2} C_{N, \text{NLO}}^{(N-1)J} (l) \, ,$$

(98)
where
\[
C_{N,NLO}^{(N-1)J}(l) = \sum_{l=1}^{3} \left[ \left( \prod_{k=1}^{l-1} U_{LL}^{(k-1)}(\mu_{k-1}, \mu_k) c_{LO}^{\alpha_k}(n_{k-1}) \right) U_{LL}^{(l-1)}(\mu_{k-1}, \mu_k) \otimes h_{\lambda k}^{\alpha_k}(n_{l+1}, n'_l, n''_{l+1}) \right] \times \left( \prod_{k=l+1}^{N-1} U_{LL}^{(k-1)}(\mu_{k-1}, \mu_k) c_{LO}^{\alpha_k}(n_{k-1}) \right) \right] \Gamma^\mu.
\] (99)

In \( C_{N,NLO}^{(N-1)J}(l) \), we have made explicit that the \( l, (l+1) \)th gluons violate strong-ordering and come with the factor \( h_{\lambda k}^{\alpha_k} \) of the subleading splitting rule, Eq. (88). The sum in the last term over \( I \) counts the different types of NLO jet-structure terms given in Eq. (87). The \( c_{LO}^{\alpha_k} \) are defined in Eqs. (63) and (E7), and the \( U \)’s are running factors given in Eqs. (18)-(20). The complete explanation of the symbols in Eq. (99) can be found in the discussion around Eq. (E13). The convolution factor is explained below Eq. (89). Since different \( l \) correspond to a violation of strong-ordering at different points in the shower, each of the \( C_{i,NLO}^{(i+1)J}(l, l+1) \) encodes a different \( \Theta \) structure. Therefore, there is no interference for different values of \( l \), and we have that the amplitude squared to NLO(\( \lambda \)) for jet-structure corrections (we call corrections of \( \mathcal{O}(\lambda^2) \) at the amplitude squared level NLO(\( \lambda \))) is just the sum of squares of the individual operators:
\[
|A^{(N-1)gqJ}_{LO,NLO}|^2 = |A^{(N-1)gq}_{LO}|^2 + |A^{(N-1)gqJ}_{NLO}|^2,
\] (100)

where
\[
|A^{(N-1)gq}_{LO}|^2 = |C_{N,LO}^{(N-1)}|^2 |\langle 0|\mathcal{O}_N^{(N-1)}|q(N-1)gq\rangle|^2,
\] (101)
\[
|A^{(N-1)gqJ}_{NLO}|^2 = \sum_{l=1}^{N-2} |C_{N,NLO}^{(N-1)J}(l)|^2 |\langle 0|\mathcal{O}_N^{(N-1)}|q(N-1)gq\rangle|^2,
\]

and \( |q(N-1)gq\rangle \) indicates the state with \( N-1 \) gluon emission. The simplification even extends inside each of the terms, since the \( j \)th gluon only gets contracted with itself. Diagrammatically, this means there are zero nearest-neighbor crossings in the \( |LO|^2 \) diagram, as we see in Fig. 13 and a maximum of one in the \( |NLO|^2 \) diagram, Fig. 13. We thus only slightly modify the factorized emission formula, Eq. (2). Even for an arbitrary number of gluon emissions, we at most have to take into account a single defect that involves a full two-particle phase space.

We can see why terms that have non-trivial interference with more than two gluons are suppressed by looking at the propagators in the amplitude. The amplitude for \( i+1 \) emissions has a factor \( 1/q_i^2 \times 1/q_{i+1}^2 \times \cdots \times 1/q_{i+1}^2 \). The LO term comes form the strong-ordered region where \( q_i^2 \gg q_{i+1}^2 \gg \cdots \gg q_{i+n}^2 \), Eq. (1). The jet-structure NLO(\( \lambda \)) is given when \( q_j^2 \sim q_{j+1}^2 \), which allows the two gluons \( k_{j+1} \) and \( k_{j+2} \) to share the same region of the phase space and therefore interfere. To have an overlap of three or more gluons, we would need \( q_j^2 \sim q_{j+1}^2 \sim \cdots \sim q_{j+k}^2 \), which is clearly suppressed beyond NLO(\( \lambda \)).

2. Interference for Hard-Scattering Corrections

The corrections to the differential cross section to \( \mathcal{O}(\lambda^2) \) involve squaring the subleading hard-scattering amplitudes as well. Unlike the jet-structure case, these involve amplitude
FIG. 14: Contribution to the amplitude squared of the jet-structure piece at NLO. We show a cut SCET\(_1\) Feynman diagram to emphasize that the square of the SCET\(_N\) operator, \(C_{N,L\text{O}}^{(N-1)}O_N^{(N-1)}\), contains only a single deviation from the simple ladder structure appearing at LO in Fig. 13.

terms up to NNLO(\(\lambda\)). As we argued above, they only modify the gluons closest to the hard interaction. Thus, we will not need to sum over many terms as we do in Eq. (101). In fact, for hard-scattering corrections, we only need to worry about interfering SCET\(_N\) operators that arise from acting with the LO replacement rule Eq. (56) on either \(C_{1,L\text{O}}^{(0)}\), \(C_{1,N\text{LO}}^{(1)}\), and \(C_{1,T}^{(1)}\), given in Eqs. (12) and (17), or \(C_{2,N\text{NNLO}}^{(2)}\) given in Eq. (D65). Since the 3\(^{rd}\) through \(i^{th}\) gluons arise from the LO rule for all three coefficients, they proceed as in the \(|\text{LO}|^2\) case. The interference to look at in detail is that of the first two gluons. In SCET\(_N\), we have:

\[
|A_{qg\bar{q}}^{(N-1)g\bar{q}}|_{\text{NLO}}^2 = \left(|C_{N,L\text{O}}^{(N-1)}C_{N,N\text{LO}}^{(N-1)H} + C_{N,N\text{LO}}^{(N-1)H}C_{N,L\text{O}}^{(N-1)}| + |C_{N,N\text{LO}}^{(N-1)H}|^2 + |C_{N,L\text{O}}^{(N-1)}C_{N,N\text{NNLO}}^{(N-1)H} + C_{N,N\text{NNLO}}^{(N-1)H}C_{N,L\text{O}}^{(N-1)}||0|O_{N}^{(N-1)}|q(N-1)g\bar{q})|^2 \right)
\]

The Wilson coefficients are found in Eqs. (E6), (E8), and (E9), respectively. Nontrivial interference in Eq. (102) occurs between the first two-gluon emissions. The interference between LO and NLO(\(\lambda\)) simplifies in many cases of interest. For example for one-gluon emission,

\[
|A_{qg\bar{q}}^{qg\bar{q}}|_{\text{LO/NLO}}^2 = \frac{4q_{1\perp }\vec{p}_g}{q_0^2}n_\mu - \vec{n}_\mu
\]

If we can cleanly separate the initial and final states (e.g. \(e^+e^- \rightarrow \text{jets}\)), then by a classic proof involving the Ward identity (reproduced, for example, in [76]), once we have integrated over final state vector quantities (we can keep scalars such as \(z_i\) unintegrated), the resulting
differential observable depends on \( g^{\mu \nu} A_{LO/NLO}^{\mu \nu} \), which for Eq. (103) is zero. This is quite straightforward for leptonic initial states, and one may be able to extend it to certain hadronic ones as well.

One can account for these corrections by modifying the hard-scale matrix element and then running a parton shower modified to include the different no-branching probabilities for different phase space configurations of the same particle content. In the next section we discuss using a reweighting to implement these corrections.

E. Correction Summary at Subleading Order

In general, our corrections avoid double counting issues, because all contributions, whether LO, hard-scattering, or jet-structure corrections are kept separately with distinct \( \Theta \) structures. Given the SCET\(_N\) amplitude for \( N + 1 \) final state particles with corrections implemented both for the branching and for the no-branching, one can consider reweighting a LL shower in order to implement our results. For correcting the abelian emissions off a single quark line, this weight factor would take the following form:

\[
w = \frac{[J(N - 1, 0) + H(N - 1, 0)]}{A(N - 1, 0)}, \tag{104}\]

where \( A(N - 1, 0) \) is the LL amplitude squared for \( N - 1 \) emissions from the quark line, \( J(N - 1, 0) \) includes the LL result along with power corrections and subleading resummation associated with jet-structure corrections, and \( H \) contains hard scattering corrections. With our LO(\( \lambda \)) result,

\[
A(N - 1, 0) = |C_{N,LO}^{(N-1)}|^2 |\langle 0 | O_N^{(N-1)} \langle q(N - 1) g\bar{q} | 0 \rangle|^2, \tag{105}\]

but in general \( A \) could be whatever amplitude squared a particular shower algorithm has for a given configuration. Eq. (104) then reweights that particular shower to our NLO(\( \lambda \)) corrected result. An example of shower Monte Carlo with an analytic expression for \( A(N - 1, 0) \) is GenEvA [54]. For a leading log shower without an explicit formula for \( A(N - 1, 0) \) one can use Eq. (105), with the understanding that it is likely a good approximation to the shower output. For the terms in the numerator of Eq. (104) we have:

\[
J(N - 1, 0) = |C_{N,NLL,1}^{(N-1)} f_j |^2 \langle 0 | O_N^{(N-1)} \langle q(N - 1) g\bar{q} | 0 \rangle|^2 + |C_{N,NLO}^{(N-1)} f_j |^2 \langle 0 | O_N^{(N-1)} \langle q(N - 1) g\bar{q} | 0 \rangle|^2, \]

\[
H(N - 1, 0) = |C_{N,NLO}^{(N-1)} H |^2 \langle 0 | O_N^{(N-1)} \langle q(N - 1) g\bar{q} | 0 \rangle|^2 + \left(C_{N,NLO}^{(N-1)} H C_{N,NNLO}^{(N-1)} f_j |^2 \langle 0 | O_N^{(N-1)} \langle q(N - 1) g\bar{q} | 0 \rangle|^2 \right) + h.c., \tag{106}\]

where we give formulas for \( C_{N,LO}^{(N-1)} \) and \( O_N^{(N-1)} \) in Eq. (63), \( C_{N,NLL,1}^{(N-1)} f_j \) is discussed near Eqs. (91) and (92), \( C_{N,NLO}^{(N-1)} H \) is given in Eqs. (98) and (99), \( C_{N,NLO}^{(N-1)} H \) is given in Eq. (ES), and \( C_{N,NNLO}^{(N-1)} H \) is given in Eq. (E9). Our operators, \( O_N^{(N-1)} \), describe a process with \( N - 1 \) emissions off the quark line. The \( A(N - 1, 0) \) amplitude squared in Eq. (105) is contained within the first term in \( J(N - 1, 0) \). As discussed in Sec. IV C while we have worked out
the real emission terms \((C^{(N-1)J}_{\text{N,NLO}}})\) completely, we have yet to determine the subleading RG kernels needed for \((C^{(N-1)J}_{\text{N,NLL}}})\).

We introduce the \(A, J, H\) notation to describe more general abelian processes. \(A(j, k)\) gives the amplitude squared necessary for the LL shower of \(j\) gluons collinear to the quark, and \(k\) collinear to the antiquark. The correction, \(J(j, k)\), contains the virtual and real corrections necessary for NLL\(_\text{exp}\) resummation of collinear logs. Since it contains an implicit sum over insertions of a single defect, which can occur anywhere in the shower, it depends on the total number of collinear emissions. Including the hard-scattering contributions to NLO(\(\lambda\)) only requires modification of the first two emissions, after which one simply uses the LO replacement rule, Eq. (56). In the general case we denote it by \(H'\), which differs from the above by including corrections to antiquark emissions as well. These are easily obtained by charge conjugation. Thus, an abelian two-jet process with \(j+k\) gluons gets the following reweighting factor:

\[
w(j, k) = [J(j, k) + H'(j, k)] / A(j, k).
\]

(107)

These weight factors are positive definite. All contributing terms are squares of amplitudes, except for \(\text{LO}(\lambda) \times \text{NNLO}(\lambda)\) in \(H(N - 1, 0)\). This contributes in the same region of phase space as the \(\text{LO}(\lambda)\) amplitude squared, and the sum of these terms is positive. In the full nonabelian case, with the presence of gluon splittings, one must sum over possible shower histories in writing down the analog of Eq. (107). Algorithms for handling this complication can be found in \[1, 54\].

In Table II we list concepts that are addressed by our shower framework at subleading order, and associate these concepts with corresponding calculations in SCET\(_i\). This table provides a summary of our results which appear in the weights given in Eq. (107), as well as pointers for future calculations. Since it is easier, in the table we use the language of SCET\(_1\) and SCET\(_2\) to discuss the corrections, rather than referring to terms in the final SCET\(_N\). In SCET\(_N\), the features of the SCET\(_1\) operators that avoid double counting and allow the various contributions to be distinguished are encoded by \(\Theta\) functions in the Wilson coefficients, and the operator language makes the discussion easier. For the total differential cross section, we found at NLO(\(\lambda\)) two kinds of power corrections. This includes a set of matrix-element corrections called hard-scattering corrections (Sec. IV A), and a set of contributions that improve double real emissions that we called jet-structure corrections (Sec. IV B).

In the the hard-scattering category, we have overall three different kinds of corrections. The first is due the the SCET\(_1\) operator \(\bar{\chi}_n \eta_l \chi_{n'} \eta_{l'}\) that gives the SCET\(_2\) coefficient \(C^{(1)H,b}_{2,\text{NLO}}\) in Eq. (80). This is an improvement of the hard matrix element that takes into account the emission of an extra parton at the hard scale. The second is due to the SCET\(_1\) operators \(\bar{\chi}_n \eta_l \chi_{n'} \eta_{l'}\) and \(\bar{\chi}_n \eta_l [\mathcal{P}_l \eta_l] \chi_{n'} \eta_{l'}\) that give the SCET\(_2\) coefficients \(C^{(1)H,a}_{2,\text{NLO}}\) and \(C^{(1)H}_{2,\text{NNLO}}\) in Eq. (80). This correction also accounts for more partons, but it describes a situation where they are initially emitted close to the collinear quark. Therefore, they are corrections which improve the description of the first branching within the leading jet. It is important to note that because these two types of hard corrections occur in different regions of phase space they have different renormalization group evolution, and thus different Sudakov no-branching factors. The required LL Sudakov factors were determined in our analysis. For a full
| Category               | Shower Concepts                                      | Quantity in SCET$_i$                                                                 | Found In: |
|------------------------|------------------------------------------------------|-------------------------------------------------------------------------------------|-----------|
| Hard Scattering        | Hard matrix elements with more partons              | Wilson coeff. of $\bar{\chi}_{n_1}B_{n_1}^q\chi_{\bar{n}}$ in SCET$_i$            | Eq. (80) |
|                        | Power correction to initial branching within the leading jet | Wilson coeff. of $\bar{\chi}_{n_0}B_{n_0}^q\chi_{\bar{n}}$ and $\bar{\chi}_{n_1}[P_{\perp B_{n_0}}]\chi_{\bar{n}}$ | Eq. (80) |
|                        | $\mathcal{O}(\alpha_s)$ hard virtual correction    | One-loop matching for $\bar{\chi}_{n_1}\chi_{\bar{n}}$                            | Eq. (80) |
| Jet Structure          | $1 \to 3$ Splitting functions                      | Double gluon real emission in SCET$_i$                                            | Eq. (87) |
|                        | Combining $1 \to 2$ splittings with the various $1 \to 3$ splittings | Compute weights from SCET squared amplitudes                                      | Eq. (107) |
|                        | $\mathcal{O}(\alpha_s)$ virtual correction for LO $1 \to 2$ splitting | One-loop correction to $1 \to 2$ replacement rule                                | Left for future work |
| No Branching Probabilities | NLL Sudakov factor for leading branching             | NLL anomalous dimension for leading operators                                     | See Sec. [IV.C] |
|                        | LL Sudakovs for subleading branching                 | LL anomalous dimensions for subleading operators                                  | Eqs. (20, 90) |
| Soft Emission          | Subleading corrections from soft gluons             | Include effects of soft emission from subleading SCET soft Lagrangians            | Left for future work |

TABLE II: Mapping between concepts in an NLO parton shower algorithm and computations in SCET$_i$. For exclusive cross sections these ingredients would together yield results accurate to NLO in the power expansion ($\lambda$), and with corresponding NLL resummation.

NLO($\alpha_s$) treatment we also need a third type of hard scattering correction, the one-loop virtual corrections to the leading shower operator. For the required operator, $\bar{\chi}_{n_0}\chi_{\bar{n}}$ these types of corrections were discussed in Refs. [24, 25].

For the jet-structure corrections, there are several ingredients to consider. We derived a replacement rule for two emissions $1 \to 3$, Eq. (88), that involved three different types of terms. This correction takes into account emissions in a region of the phase space that is not strongly-ordered and automatically avoids double counting from multiple $1 \to 2$ emissions.\[^{12}\] In addition at NLO($\alpha_s$)/NLL we require the $\mathcal{O}(\alpha_s)$ virtual correction to the LO splitting rule. This would be derived from a one loop matching computation that should

\[^{12}\] The method by which we avoid double counting for two gluon emission should be obvious, coming directly from our implementation of the $\Theta$ functions. Since $\Theta + \tilde{\Theta} = 1$, the double $1 \to 2$ and $1 \to 3$ together cover all of phase space without double counting. For three emissions we have either i) three $1 \to 2$ emissions, ii) a $1 \to 3$ followed by a $1 \to 2$ emission, or iii) a $1 \to 2$ followed by a $1 \to 3$ emission. Here there is an apparent combinatoric issue, as ii) and iii) both provide corrections for the middle gluon in i). However they do so in nonoverlapping regions of phase space. The same is true for more than three emissions. We thank J. Thaler for asking this question.
be straightforward, but was not considered here.

We also discussed how no-branching Sudakov factors are associated with the operator RG kernels, and by extension their anomalous dimensions (Section IV.C. To NLL, we need the NLL Sudakov factor for leading branching and the LL Sudakovs for subleading branching. These are associated to the full one-loop and two-loop cusp anomalous dimensions for the leading operators, and one-loop cusp for the subleading ones. At LL, we have determined all the Sudakovs for subleading branching (Eqs. 20 and 90). We have not yet calculated the NLL Sudakov for leading branching in the scheme with \( \Theta \)-functions that is needed for our setup, as described in Sec. IV.C.

The last item in the table is the treatment of soft radiation at NLO. This can be achieved by considering time-ordered products for the matching of QCD to SCET\(_i\) and SCET\(_i\) to SCET\(_{i+1}\) that involve soft gluons and subleading soft Lagrangians that are known in SCET \([64,66]\) up to \( \mathcal{O}(\lambda^2) \). For the terms involving collinear quarks they read

\[
\mathcal{L}^{(1)}_{\xi \xi} = (\xi_n \cdot W) \frac{1}{2} \left(W^\dagger i \partial_n^\perp \frac{\beta}{2} \xi_n\right) + (\xi \cdot i \partial_n^\perp W) \frac{1}{2} i \partial_n^\perp \left(W^\dagger i \partial_n^\perp \frac{\beta}{2} \xi_n\right),
\]

\[
\mathcal{L}^{(2)}_{\xi \xi} = (\xi_n \cdot W) \frac{1}{2} i \partial_n^\perp \left(W^\dagger \xi_n\right) + (\xi \cdot i \partial_n^\perp W) \frac{1}{2} i \partial_n^\perp \left(W^\dagger \xi_n\right),
\]

\[
\mathcal{L}^{(1)}_{\xi q} = \xi_n \frac{1}{2} \frac{\beta}{2} \frac{1}{\gamma_n} \frac{1}{\gamma_n} \left[i \gamma_n \cdot D_n, i \gamma_n \cdot D_n + g n \cdot A_{n s}\right] W q_n + h.c., \quad \mathcal{L}^{(2b)}_{\xi q} = \xi_n \frac{1}{2} i \partial_n^\perp W q_n + h.c.,
\]

\[
\mathcal{L}^{(2a)}_{\xi q} = \xi_n \frac{1}{2} \frac{\beta}{2} \frac{1}{\gamma_n} \frac{1}{\gamma_n} \left[i \gamma_n \cdot D_n, i \gamma_n \cdot D_n + g n \cdot A_{n s}\right] W q_n + h.c.,
\]

while the analogous pure glue Lagrangians can be found in Ref. \([64]\). Here the expressions are prior to the soft field redefinition, and \( ig \mathcal{B}_n = [i \gamma_n \cdot D_n, i \gamma_n \cdot W q_n] \). One must then work out the effect that these NLO soft amplitudes have on interference. The associated soft calculations and investigations have also been left for future work.

We also briefly comment on how the corrections in Table I relate to those already implemented in parton shower codes in the literature. In most cases, the goal of these codes differed from the power suppressed corrections considered here. This makes a strict association impossible, but there is still a general correspondence that can be made. CKKW \([27]\) is a LO(\( \alpha_s \))/LL procedure whose goal is to merge matrix elements involving multiple partons with a parton shower in a manner that avoids double counting. In our language, this corresponds to the real emission hard-scattering corrections in the first row of Table I. The \( \bar{\chi}_{n_0} \chi_n \) and \( \bar{\chi}_{n_0} B_{n_0}^\perp \chi_n \) operators describe processes with different numbers of initial well-separated jets. In CKKW, a parameter \( y_{\text{cut}} \) is used to separate the extra emission in the matrix element from emissions in the shower. In our analysis, the contributions from showering \( \bar{\chi}_{n_0} \chi_n \) does not interfere with the direct contribution from \( \bar{\chi}_{n_0} B_{n_0}^\perp \chi_n \), and this is encoded by \( \Theta \) functions in the Wilson coefficient of SCET\(_X\). CKKW carries out this procedure for several matrix element emissions, while we have only considered one.

In MC@NLO \([37]\) and POWHEG \([38]\), virtual and real matrix element corrections at NLO(\( \alpha_s \)) are incorporated into the shower, with the goal of ensuring that it reproduces an associated cross section completely at NLO(\( \alpha_s \)). The implementation includes careful handling of the cancellation of real and virtual IR divergences. Our goal was to implement corrections at NLO(\( \lambda \)) and we discussed NLL, but for all emissions from the shower rather
than just the first jet needed for the NLO($\alpha_s$) cross section. At NLL, we would have only terms up to $\mathcal{O}(\alpha_s \log)$ in the total cross section, and hence this does not encode the entire NLO($\alpha_s$) result. In our language, the corrections that contribute to the NLO($\alpha_s$) cross section correspond to the hard scattering corrections in the first through third rows of Table II. In order to compute the NLO($\alpha_s$) cross section it is not necessary to distinguish between the terms in the first and second rows of the table, and these terms are indeed considered simultaneously in MC@NLO and POWHEG. The full NLO($\alpha_s$) virtual result are obtained in our language by including the items mentioned in the 3rd and 9th rows of Table II.

The work of KRKMC [57–59], on the other side, aims to improve the shower algorithm taking into account an exclusive version of the Altarelli-Parisi splitting function at NLO($\alpha_s$), $P_{qq}^{(1)}$. In our language, this corresponds to jet-structure corrections and we show in Appendix F how our replacement rule in Eq. (88) is also related to $P_{qq}^{(1)}$. Hence our $1 \to 3$ emission corresponds to an exclusive version of $P_{qq}^{(1)}$, though in a different scheme. Part of the corrections in $P_{qq}^{(1)}$ involve order $\alpha_s$ corrections to the $1 \to 2$ splitting function, which are taken into account by $\mathcal{O}(\alpha_s)$ virtual $1 \to 2$ matching corrections in our framework (6th row of Table II). In fact, in Sec. IV D, we saw that SCET$_i$ also leads one to view corrections to the shower as a “defect” insertion just as KRKMC. In addition to these splitting corrections, in our framework the amplitude also involves no-branching probabilities given by evolution kernels that appear in the weight factors, which do not appear in the KRKMC weights. Keeping track of the evolution also determines the appropriate scale for evaluating $\alpha_s$.

V. CONCLUSION

In this paper we developed a framework based on a tower of independent but related EFTs, the SCET$_i$, to study corrections to the parton shower. The work of [24, 25] showed how to formulate the LL parton shower in terms of SCET, and how virtual corrections are straightforward to incorporate by one-loop matching. Our SCET$_i$ framework extends these ideas in a manner that makes it easy to deal with: double counting, the issue of disentangling coordinate choices from kinematic power corrections, and the construction of a complete set of operators for corrections at a desired order. The interference structures, and hence the leading corrections that give spin and color correlations, also appear in a straightforward manner in the SCET$_i$ setup.

The SCET$_i$ are iteratively used to integrate out the characteristic scale, $Q\lambda^i$ for increasing $i$. This approach allows us to perform a systematic expansion which can correct both the hard-scale process that produces partons to setup initial conditions for the shower algorithm and the iterative shower itself. We described the parton shower through a set of operators $\mathcal{O}_i^{(j)}$ in SCET$_i$, and used standard matching procedures to make the transition from SCET$_i$ to SCET$_{i+1}$, where more partons become apparent. Performing the matching relied crucially on the RPI symmetry of SCET, and we extended the usual infinitesimal version to carry out the finite rotations that we needed. At LO, a simple operator replacement rule generates the LL shower, $\chi_{n_0} \to c_{LO}^{n_0} \chi_{n_0} g B_{n_0+1}^{n_0}$, where $c_{LO}$ is related to the standard LO splitting function. Also, angular ordering and coherent branching for LO soft emissions emerge naturally in the SCET$_i$ framework. A summary of ingredients required for the shower with power corrections

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at NLO($\lambda$) are given in Table II, including both calculations carried out here, as well as those left for future work. The main results of our paper are:

1. At NLO($\lambda$) we found two kinds of branching corrections: hard-scattering and jet-structure. The hard-scattering corrections depend on the hard process and appear near the top of the shower tree. They came from matching QCD to SCET$_1$ at higher order. Since they only occur at the top of the shower, one can treat these as a modified form of matrix-element corrections. A subset of these corrections correspond to the usual implementation of fixed-order matrix elements, while the remaining ones give power corrections to the initial branching in the LL shower. These two types require different Sudakov factors. This effect is apparent for the kinematic power corrections, but is beyond NLO($\alpha_s$) for the fixed order counting.

2. The jet-structure corrections are independent from what happens at the hard scale, hence they are universal for any process we want to study. They come from matching SCET$_i$ to SCET$_{i+1}$ at higher order for any $i$. They can appear anywhere in the shower tree and they take into account emissions in regions of the phase space that are not strongly-ordered. For these corrections we found that the NLO($\lambda$) operators are related to the LO operator via a replacement rule for two emissions: $\bar{\chi}_{n_0} \rightarrow h_{n_1}^{\alpha \beta} \chi_{n_2} g_{\nu_{n_1 \perp}} B_{\nu_{n_2 \perp}}^{\alpha}$. This NLO($\lambda$) rule automatically avoids double counting with the iteration of two LO operator replacements.

3. The SCET$_i$ picture allowed us to easily take into account interference for the NLO($\lambda$) power corrections. Once we reach the final SCET$_N$ theory, all the fields are labeled in a different collinear directions. Because in SCET we can only contract collinear fields that share the same collinear direction, in SCET$_N$ calculating the amplitude squared becomes very easy. Kinematic information that is encoded by the shower history from passing through earlier SCET$_i$’s is encoded by $\Theta$ functions in the final SCET$_N$ Wilson coefficients. We demonstrated that when emitting an arbitrary number of partons, the non-trivial part of the amplitude squared involves at most four fields.

A comparison of how these SCET$_i$ results relate to earlier parton shower literature that goes beyond LL is given in Sec. [IV.E]

The framework developed here allows for systematic improvement to arbitrary orders in the kinematic expansion. There are still several important steps to take, though, before this picture can lead to a practical implementation, including additional computations that we outlined in Sec. [IV.E]. We list here three topics which are natural next steps, and which we believe should be straightforward to approach:

1. This work has only considered $q \rightarrow qg$ splittings and an abelian theory. One should include the full nonabelian results and compute the coefficients required for gluon splitting as well. This is required to properly treat color correlation corrections in a manner determined by the NLO($\lambda$) interference pattern. For collinear particles we expect that one can include the dominant part of these effects by considering nearest-neighbor interference since this arises from the kinematic expansion, and thus leaves the rest of the shower as before.
2. Only a subset of the terms required for a full NLL$_{\text{exp}}$ resummation were considered here. We determined the LL$_{\text{exp}}$ evolution for subleading operators, but did not carry out the computation of the NLL$_{\text{exp}}$ evolution of the leading operator in a scheme that is consistent with our power corrections (we only considered it in MS). In order for a consistent treatment as a probabilistic process, the real emission probabilities and Sudakov no-branching corrections must go hand in hand. Furthermore, once these evolution factors are determined, the reweighting discussed in Sec. IV E must be tested in an actual shower Monte Carlo.

3. Since soft modes in SCET can communicate between different collinear jets, they carry the ability to spoil their factorization. Fortunately, this does not happen for their LO interactions, which yield angular ordering and coherent branching of soft gluons in SCET. It is open question as to what extent NLO soft couplings can be factorized in the shower tree and the necessary SCET computations were discussed but not carried out here. The treatment of soft NLO interactions in SCET in other contexts has always led to factorized structures, so we remain optimistic that such effects will be tractable for the shower.

Future investigation of these items is well warranted.

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Appendix A: More SCET basics

Soft-Collinear Effective Theory describes the interactions of collinear and soft quarks and gluons [20–23]. As we mentioned in Sec. IV A, to define the collinearity of a particle, the momentum is decomposed along two light-cone vectors, $n$ and $\bar{n}$, with $n^2 = 0$, $\bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$

$$p^\mu = n \cdot \bar{p} \frac{n^\mu}{2} + \bar{p} \frac{\bar{n}^\mu}{2} + p_\perp^\mu,$$  \hspace{1cm} (A1)

where $\bar{p} = n \cdot p$. A particle is collinear to the direction $n$ if its momentum scales as:

$$(n \cdot p, \bar{p}, p_\perp) \sim (\lambda^2, 1, \lambda) Q,$$  \hspace{1cm} (A2)

where $Q$ is the hard scale of the process, and $\lambda \ll 1$. A particle is soft if:

$$(n \cdot p, \bar{p}, p_\perp) \sim (\lambda^2, \lambda^2, \lambda^2) Q.$$  \hspace{1cm} (A3)
We obtain SCET from QCD by expanding in powers of $\lambda$ and integrating out modes harder than $\sim Q^2/\lambda$. Both Eqs. (A2) and (A3) imply that $p^2 = \bar{p} \cdot (n \cdot p) + p_\perp^2 \lesssim Q^2/\lambda^2$.

In addition to the expansion, we also want to divide the quark and gluon fields into separate soft and collinear modes. For the collinear case, the fields are indexed by $n$, and two collinear sectors are distinct if $n_i \cdot n_j \gg \lambda^2$. In addition, we introduce a momentum-space lattice for the $O(\lambda^6)$ and $O(\lambda)$ momenta in order to facilitate carrying out the multipole expansion with respect to the $O(\lambda^2)$ momenta. To divide the QCD fields in this way, we split the momentum of a collinear particle into a "large" part $\bar{p}$ and a residual one $k^\mu \sim \lambda^2$

$$p^\mu = \bar{p}^\mu + k^\mu, \quad \text{where} \quad \bar{p}^\mu \equiv n \cdot p - \frac{p_\perp}{2} + p_\perp.$$

We can pull out the large momenta $\bar{p}$ from the fermion field by the phase redefinition

$$\psi(x) = \sum_{\bar{p}, n} e^{-i\bar{p} \cdot x} \psi_{n, \bar{p}}.$$  

For a collinear particle along $n$, $\partial^\mu \psi_{n, \bar{p}}(x) \sim \lambda^2$. The four component field, $\psi_{n, \bar{p}}$, has two large components, $\xi_{n, \bar{p}}$, and two small components $\xi_{\bar{n}, \bar{p}}$, that can be separated using the following projectors:

$$\psi_{n, \bar{p}} = \frac{\not{\bar{p}} \not{n}}{4} \psi_{n, \bar{p}} + \frac{\not{\bar{p}} \not{\bar{n}}}{4} \psi_{n, \bar{p}} \equiv \xi_{n, \bar{p}} + \xi_{\bar{n}, \bar{p}}.$$  

These satisfy the relations,

$$\frac{\not{\bar{p}} \not{\bar{p}}}{4} \xi_{n, \bar{p}} = \xi_{n, \bar{p}}, \quad \not{\bar{p}} \xi_{n, \bar{p}} = 0,$$

$$\frac{\not{\bar{p}} \not{\bar{p}}}{4} \xi_{\bar{n}, \bar{p}} = \xi_{\bar{n}, \bar{p}}, \quad \not{\bar{p}} \xi_{\bar{n}, \bar{p}} = 0.$$  

Similarly, we can define a collinear gluon field, $A_{n, \bar{q}}^\mu(x)$. Pictorially, we can think of $\xi_{n, \bar{p}}(x)$ and $A_{n, \bar{q}}^\mu(x)$ as fields that create a particle whose three-momentum lies inside a cone with opening angle $\sim \lambda$ about the three-direction $\vec{n}$. $P_n^\mu$ is the momentum operator that picks up the large components of the momentum, $P_n^\mu \xi_{n, \bar{p}}(x) = \bar{p}^\mu \xi_{n, \bar{p}}(x)$. Collinear fields always appear with a sum over $\bar{p}$, and both label and residual momenta are separately conserved. Therefore it is often useful to abbreviate the notation as

$$\xi_n = \sum_{\bar{p}} \xi_{n, \bar{p}}, \quad A_n = \sum_{\bar{q}} A_{n, \bar{q}}.$$  

The SCET collinear Lagrangian, $\mathcal{L}_n$, describes the interaction between the collinear fields $\xi_n$ and $A_n^\mu(x)$. It is derived from the QCD Lagrangian by integrating out the field, $\xi_{\bar{n}}$. At LO, for the kinetic and purely collinear interaction terms we have [21, 22]:

$$\mathcal{L}_n^{(0)} = \bar{\xi} \left( i n \cdot \partial + g n \cdot A_n + i d_{n, \perp} W_n \frac{1}{P_n} W_n^\dagger i \bar{P}_n n_\perp \right) \frac{\not{\bar{p}}}{2} \xi_n,$$
where we intrinsically sum over the large, label momenta, $\tilde{p}$. The $in \cdot \partial$ derivative picks out the $O(\lambda^2)$ momenta. The collinear derivative, $D_n^\mu$, and collinear Wilson line, $W_n$, are defined as:

$$iD_n^\mu = \mathcal{P}_n^\mu + gA_n^\mu,$$

$$W_n(x) = \left[ \sum_{\text{perms.}} \exp \left( -\frac{g}{\mathcal{P}_n} \tilde{n} \cdot A_n(x) \right) \right]. \quad (A10)$$

The leading order coupling of collinear quarks to soft gluons is eikonal,

$$\mathcal{L}_{sn}^{(0)} = \xi_n g n \cdot A_s \frac{\tilde{\xi}_n}{2}, \quad (A11)$$

while the Lagrangian for purely soft quarks and gluons has the same form as full QCD. The LO collinear Lagrangian for gluons has similar properties and is given in Ref. [23]. The interactions between soft and collinear particles, such as the one in Eq. (A11), can be removed from the Lagrangian by the field redefinitions [23]:

$$\xi_n \rightarrow Y_n \xi_n, \quad A_n^\mu \rightarrow Y_n A_n^\mu Y_n^\dagger, \quad (A12)$$

where the soft Wilson line $Y_n$ is defined in Eq. (67). This causes soft interactions to be represented by Wilson lines in operators, as in Eq. (69).

Now that we have split up gluons according to a momentum-space lattice, the gauge structure of the theory has become more complex and involves global, collinear, and soft gauge transformations. Fortunately, with the collinear Wilson line, it is possible to construct fermion and gluon fields that are manifestly invariant under collinear gauge transformations. The definitions are:

$$\chi_n(x) = W_n^\dagger(x)\xi_n(x), \quad B_n^\mu(x) = \frac{1}{g} [W_n^\dagger(x)iD_n^\mu(x)W_n(x)], \quad (A13)$$

where the derivative in $B_n^\mu$ does not act outside of the brackets in its definition, and we always have $\tilde{n} \cdot B_n = 0$. In the $\tilde{n} \cdot A_n = 0$ light-cone gauge, $W_n = 1$ and $B_n^\mu = A_n^\mu$. One can construct collinear operators out of just three objects: the fermion field, $\chi_n$, the perpendicular gluon field, $B_n^\mu$, and the perpendicular momentum operator, $\mathcal{P}_{n\perp}^\mu$. All the other operators, like $n \cdot B_n$, or $n \cdot \partial$ can be written in terms of these three using the equation of motions [67].

**Appendix B: Finite RPI**

Even though SCET explicitly breaks Lorentz invariance, the symmetry returns at each order in $\lambda$ by reparametrization invariance (RPI). RPI is the version appropriate for SCET. As usual, we define $p$ as collinear to the direction $n$ in SCET if its components scale as $(n \cdot p, \tilde{p}, p_\perp) \sim (\lambda^{2i}, 1, \lambda^i)Q$, where $Q$ is the hard scale and $\lambda \ll 1$ (cf. Eq. (A1)). The vector $n$ has physical meaning as its 3-vector subset, $\vec{n}$, is the direction where most of the momentum is allocated. The direction $\vec{p}$ is therefore inside a cone of opening angle $\lambda^i$ around $\vec{n}$, (cf. Fig 15). By contrast, $\tilde{n}$ is an auxiliary vector only needed to decompose the momentum. The parameter $\lambda$ gives the amount of collinearity to $n$. The decomposition is
FIG. 15: In SCET, a particle is collinear to the direction $n$ if it is inside a cone centered in $\vec{n}$ and of opening angle $\lambda$.

not unique since we can shift $n$ by an amount $\lambda$ and the particle will still be collinear to it. This means that if we move $n$ inside the cone in Fig. [15], $p$ is still collinear to it. This is called a reparametrization invariance (RPI) transformation of type-I. Thus, if a particle is collinear to $n$, it is also collinear to any direction $n'$ related by a type-I transformation. To be more formal, we can divide the space of light-cone vectors, \{n_i\}, into equivalence classes, \{[n_i]\}, where $[n_j] = \{n \in [n_j] | n \cdot n_j \lesssim \lambda n_i^2\}$. The meaningful objects in SCET are the $[n_j]$.

By extension from SCET [63], two collinear sectors in SCET, $n_1$ and $n_2$, are distinct if

$$n_1 \cdot n_2 \gg \lambda^2,$$

(B1)

Just as in regular SCET, we can write the external state with the $n$-label to which each particle is collinear. For working in SCET, we give a subscript to indicate the appropriate definition of collinearity. For example, $|q_{n_1}\rangle_i$ is a state with one quark, collinear to $n_1$ that can be annihilated by any $\chi_n$ such that $n \cdot n_1 \ll \lambda^2$, or $n \in [n_1]$.

For each $\{n, \bar{n}\}$, the type-I RPI infinitesimal transformations are\textsuperscript{13}

\begin{equation}
(I) \begin{cases} 
n^\mu \rightarrow n^\mu + \Delta^\mu_{n\perp} \\
\bar{n}^\mu \rightarrow \bar{n}^\mu
\end{cases},
\end{equation}

(B2)

where $\Delta^\mu_{n\perp} \sim \lambda$ and $n \cdot \Delta_{n\perp} = \bar{n} \cdot \Delta_{n\perp} = 0$. These transformations preserve the relations $n^2 = 0$, $\bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$.\textsuperscript{14}

The general problem of matching SCET$_i$ → SCET$_{i+1}$ is our need to rotate the direction $n$ of objects in the amplitude (such as spinors and vectors) to $n'$ that is close enough to the particle momentum such that $p$ is collinear to $n'$ in SCET$_{i+1}$. Thus, RPI$_i$ is crucial for matching as it determines how formerly identical SCET$_i$ configurations wind up in different SCET$_{i+1}$ terms. Any transformation in RPI$_i$/RPI$_{i+1}$ is therefore of consequence. By contrast, the choice within SCET$_{i+1}$ is purely a convention we may use to our convenience (cf. Fig. [4]). For example, we can pick $n'$ as that direction $n_p$ such that $p$ has zero perpendicular momentum in the $n_p - \bar{n}$ frame:

$$p = \frac{\bar{n}^\mu p^\mu}{2} + n_p \cdot \frac{\bar{n}^\mu}{2}.$$

(B3)

\textsuperscript{13}Infinitesimal does not refer to the expansion in $\lambda$.

\textsuperscript{14}It is also possible to rotate $\bar{n} \rightarrow \bar{n} + \varepsilon_{\perp}$ where $\varepsilon_{\perp} \sim \lambda^0$, which is a type-II RPI transformation. Finally, a type-III transformation takes $n \rightarrow e^{\alpha}n$ and $\bar{n} \rightarrow e^{-\alpha}\bar{n}$.
This is satisfied for:

\[ n_\mu^p = n^\mu + 2 \frac{p_\perp^\mu}{\tilde{p}^2} - \tilde{n}^\mu \left( \frac{p_\perp}{\tilde{p}^2} \right)^2, \]  

(B4)

with \( p_\perp \) defined in the \( n \)-frame. Unlike Eq. (B2), this RPI transformation is finite. It is easy to check that \( n_\mu^p = 0 \), \( n_p \cdot \tilde{n} = 2 \) and that \( p_\perp^\mu = p^\mu - n_p \cdot p \tilde{n}^\mu/2 - \tilde{p} n_p^\mu/2 = 0 \).

We can derive similar relations for other quantities. To see how the quark field transforms, we use the RPI invariant fermion field [67]:

\[ \psi_n = \left( 1 + \frac{\not{D} \cdot \not{\bar{p}}}{\bar{n} \cdot D_n} \right) \xi_n. \]  

(B5)

Since (B5) is invariant under RPI, \( \psi_n = \psi_{n^p} \) and we can write,

\[ \left( 1 + \frac{\not{D} \cdot \not{\bar{p}}}{\bar{n} \cdot D_{n^p}} \right) \xi_n = \left( 1 + \frac{\not{D} \cdot \not{\bar{p}}}{\bar{n} \cdot D_{n^p}} \right) \xi_{n^p}. \]  

(B6)

Multiplying (B6) by the projector \( \not{n} \) we get the finite RPI relation

\[ \xi_n = \frac{\not{n} \not{p}}{4} \xi_{n^p}. \]  

(B7)

The relation (B7) is in agreement with the spinor equation (A7) in [24] upon setting \( \bar{n}_1 = \bar{n}_2 \). Objects with a full Lorentz index, like \( p^\mu \) or \( \gamma^\mu \), are RPI invariant as there is no reference to the light-cone vectors \( n \) and \( \bar{n} \). Those in the perpendicular direction though, such as \( p_\perp^\mu \) or \( \gamma_\perp^\mu \), are not, as \( \perp \) is defined with respect to \( n \) and \( \bar{n} \). Using the relation \( \gamma_\perp^\mu = \gamma^\mu - \bar{n}^\mu \not{p}/2 - n^\mu \not{\bar{p}}/2 \), we derive the expression

\[ \gamma_\perp^\mu = \gamma^\mu - \bar{n}^\mu \frac{\not{p}}{\tilde{p}^2} - n^\mu \frac{\not{\bar{p}}}{\tilde{p}^2}. \]  

(B8)

We now focus on those transformations needed for one-gluon emission. As in Sec. III A, we consider the case of a virtual quark with momentum \( q_0 \) emitting an external gluon and quark with momentum \( k_1 \) and \( q_1 \), respectively. In Fig (16)A, we portray this kinematics for one-gluon emission where the initial quark \( q_0 \) comes from a QCD current \( \bar{q} \gamma^\mu q \). We call \( n_0 \), \( n_1' \) and \( n_1 \) the directions where \( q_0 \), \( k_1 \) and \( q_1 \) zero have perpendicular component, that is:

\[ q_0 = \bar{q}_0 \frac{n_0^\mu}{2} + n_0 \cdot q_0 \frac{\tilde{n}^\mu}{2}, \]
\[ k_1 = \bar{k}_1 \frac{n_1^\mu}{2}, \]
\[ q_1 = \bar{q}_1 \frac{n_1^\mu}{2}. \]  

(B9)

Using Eq. (B4), we can relate \( n_1' \) and \( n_1 \) to \( n_0 \),

\[ n_1'^\mu = n_0^\mu - 2 \frac{(q_1)^\mu}{k_1} - \bar{n}^\mu \left( \frac{q_1}{k_1} \right)^2, \]
\[ n_1'^\mu = n_0^\mu + 2 \frac{(q_1)^\mu}{\bar{q}_1} - \bar{n}^\mu \left( \frac{q_1}{\bar{q}_1} \right)^2, \]  

(B10)
FIG. 16: Kinematic variables for one-gluon emission (A), two-gluon emission (B) and \(i\)-gluon emission (C). The \(n\)'s are defined such that the corresponding particle’s momentum has no perpendicular component along the directions \(n - \bar{n}\). Note that \(q_1\) is a different vector for single and double emissions.

where we have used the equality \((k_1)_{n_0,\perp} = -(q_1)_{n_0,\perp}\). Some useful relations are:

\[
\begin{align*}
n_1 \cdot n'_1 &= n_0 \cdot n'_1 \frac{q_0^2}{k_1^2} = n_0 \cdot n_1 \frac{\bar{q}_0^2}{\bar{q}_1^2} = -2 \frac{(q_1)^2}{q_0^2} \frac{\bar{q}_0^2}{k_1^2}, \\
n_0^\mu &= \frac{\bar{k}_1 n_1^\mu + q_1 n_1^\mu}{\bar{q}_0} - \bar{n}^\mu (n_1 \cdot n'_1) \frac{q_1 \bar{k}_1}{2 \bar{q}_0^2}, \\
(q_1)^\mu_{n_0,\perp} &= \frac{\bar{q}_1 \bar{k}_1}{\bar{q}_0} \sqrt{n_1 \cdot n'_1} \frac{v_1^\mu}{2} - \bar{n}^\mu (n_1 \cdot n'_1) \frac{q_1 \bar{k}_1 (\bar{k}_1^2 - q_1^2)}{4 \bar{q}_0^2}, \\
\gamma_{n_0,\perp} &= \gamma_{n'_1,\perp} - \bar{n}^\mu (q_1)^{n_0,\perp} \frac{\bar{q}_1 \bar{k}_1}{k_1} + \bar{n}^\mu (n_1 \cdot n'_1) \frac{\bar{q}_1^2}{2 \bar{q}_0^2} \bar{n}, \\
\end{align*}
\]  
\(\text{(B11)}\)

where

\[
v_1^\mu = \frac{n_1^\mu - n'_1^\mu}{\sqrt{n_1 \cdot n'_1}}, \quad (B12)
\]

and \(|v_1^2| = 2\). Another useful relation is

\[
q_0^2 = (q_1 + k_1)^2 = n'_1 \cdot n_1 \frac{\bar{q}_1 \bar{k}_1}{2}. \quad (B13)
\]

We can express all quantities of interest in terms of the vectors \(n'_1, n_1\) and the momenta \(\bar{q}_1\) and \(\bar{k}_1\).

In two-gluon emissions, the kinematic variables are assigned in Fig. 16(B). We define \(n_0,\)
\( n_1', n_2' \) and \( n_1 \) as follows (note that \( q_1 \) and \( n_1 \) are different from above):

\[
q_0 = \frac{\bar{q}_0 n_0^\mu}{2} + n_0 \cdot q_0 \frac{\bar{n}^\mu}{2},
\]

\[
k_1 = \frac{\bar{k}_1 n_1^\mu}{2},
\]

\[
q_1 = \frac{\bar{q}_1 n_1^\mu}{2} + n_1 \cdot q_1 \frac{\bar{n}^\mu}{2},
\]

\[
k_2 = \frac{\bar{k}_2 n_2^\mu}{2},
\]

\[
q_2 = \frac{\bar{q}_2 n_2^\mu}{2},
\]

(Eq. (B14))

Eq. (B10) is still valid, and we can similarly define \( n_2' \) and \( n_2 \) as:

\[
n_2'^\mu = n_1'^\mu - 2 \frac{(q_2)_{n_1\perp}^\mu}{k_2} - \bar{n}^\mu \frac{(q_2)_2^2}{\bar{q}_2^2},
\]

\[
n_2'' = n_1'' + 2 \frac{(q_2)_{n_1\perp}^\mu}{\bar{q}_2} - \bar{n}^\mu \frac{(q_2)_2^2}{\bar{q}_2^2},
\]

(Eq. (B15))

where \((k_2)_{n_1\perp} = -(q_2)_{n_1\perp}\). Also, Eq. (B11) is still valid, and we get a new set by sending \(0 \rightarrow 1\) and \(1 \rightarrow 2\):

\[
n_2 \cdot n_2' = n_1 \cdot n_2' \frac{q_1^2}{k_2} = n_1 \cdot n_2' \frac{q_1^2}{k_2} = -2 \frac{(q_2)_2^2 q_1^2}{\bar{q}_2^2 k_2^2},
\]

\[
n_1'' = \frac{\bar{k}_2 n_2'' + \bar{q}_2 n_2''}{\bar{q}_1} - \bar{n}^\mu (n_2 \cdot n_2') \frac{q_2 k_2}{2 \bar{q}_1},
\]

\[
(q_2)_{n_1\perp} = \frac{\bar{q}_2 k_2}{\bar{q}_1} \sqrt{n_2 \cdot n_2'} \frac{v_2''}{2} - \bar{n}^\mu (n_2 \cdot n_2') \frac{q_2 k_2}{2 \bar{q}_1},
\]

\[
\gamma_{n_1\perp} = \gamma_{n_2''\perp} - \bar{n}^\mu \frac{(q_2)_{n_1\perp}}{k_2} - (q_2)_{n_1\perp} \frac{\bar{q}_2 k_2}{k_2^2} + \bar{n}^\mu (n_2 \cdot n_2') \frac{q_2^2}{2 \bar{q}_1},
\]

where

\[
v_2'' = \frac{n_2'' - n_2'}{\sqrt{n_2 \cdot n_2'}},
\]

(Eq. (B17))

and \(|v_2''| = 2\). We can write \(n_1 \cdot n_1'\) and \(v_1\) in terms of \(n_2, n_1'\) and \(n_2'\) so that once again we only need to work with external quantities:

\[
n_1 \cdot n_1' = \frac{\bar{k}_2 (n_2' \cdot n_1') + \bar{q}_2 (n_2' \cdot n_1')}{\bar{q}_1} - (n_2 \cdot n_2') \frac{\bar{q}_2 k_1}{\bar{q}_0},
\]

(Eq. (B18))

\[
v_1'' = \frac{2 \bar{k}_2 \bar{q}_1 n_1'' + 2 \bar{q}_1 \bar{q}_2 n_1'' - \bar{k}_2 \bar{q}_2 (n_2 \cdot n_2') \bar{n}'' - 2 (\bar{q}_1)^2 n_1''}{2(\bar{q}_1)^2 \sqrt{n_1 \cdot n_1'}}.
\]

Eq. (B13) for two emissions is modified to:

\[
q_0^2 = (q_2 + k_1 + k_2)^2 = n_1' \cdot n_2 \frac{\bar{q}_2 k_1}{2} + n_2' \cdot n_2 \frac{\bar{q}_2 k_2}{2} + n_1' \cdot n_2' \frac{\bar{k}_1 k_2}{2}.
\]

(Eq. (B19))
Other useful relations are

\[
\gamma^\mu_{n_0\perp} = \gamma^\mu_{n_2\perp} - \bar{n}^\mu \left( \frac{(q_1)_{n_0\perp}}{k_1} + \frac{(q_2)_{n_1\perp}}{k_2} \right) - \frac{(q_1)_{n_0\perp}}{k_1} - \frac{(q_2)_{n_1\perp}}{k_2} \right) \bar{\psi}
\]

\[
q_1^2 = (k_2 + q_2)^2 = n_2^\prime \cdot n_2 \frac{k_2 q_2}{2},
\]

\[
(k_1 + q_2)^2 = n_1^\prime \cdot n_2 \frac{k_1 q_2}{2}. \quad (B20)
\]

For \( i \) gluon emissions, Fig. 16(C), \( n_1^\prime \) is parallel to the \( k \)-gluon, \( n_i \) parallel to the external quark, and \( n_k \) is the light cone vector such that the \( k \)th virtual quark has zero perpendicular momentum with respect to \( (n_k, \bar{n}) \). To calculate \( n_i, n_i^\prime \) we can iterate the formulas above up to \( i \) emissions. That is we can calculate \( n_i, n_i^\prime \) from \( n_{i-1} \) using Eq. (B10) with \( 0 \to (i - 1), \) \( 1 \to i \).

**Appendix C: Matching QCD to SCET**

To study the process of \( q \to qg \) emission, we match the QCD current,

\[
J_{\text{QCD}}^\mu = \bar{q} \gamma^\mu q, \quad (C1)
\]

to SCET operators for a final state with a quark, antiquark, and gluon. The particle momenta are \( q_1 \) for the quark, \( p_\bar{q} \) for the antiquark, and \( k_1 \) for the gluon, (cf. Fig. 17). We do the matching in the center of mass frame with

\[
p_\gamma = q_1 + p_\bar{q} + k_1 = (Q, 0, 0, 0). \quad (C2)
\]

SCET, being equivalent to the usual SCET, is formulated as an expansion in the parameter \( \lambda \). The current in Eq. (C1) matches onto an infinite series of SCET operators. We will perform the matching up to NNLO(\( \lambda \)) for one gluon emission, and focus only on the cases when the gluon is either collinear to the quark or has its own direction. Obtaining the result for gluon-antiquark collinearity from our work is a simple exercise in charge conjugation.

We can construct the SCET operators out of a few building blocks: the quark field \( \chi_n \), the gluon field \( B_{n\perp}^\alpha \) and the perpendicular momentum operator \( P_{n\perp}^\alpha \), plus Dirac structures. \( \chi_n, B_{n\perp}^\alpha \) and \( P_{n\perp}^\alpha \) all scale \( \sim \lambda \). The basis of SCET operators for one emission up to NNLO(\( \lambda \))
is \[67\]:

\[
\mathcal{O}_1^{(0)}(n_0) = \tilde{\chi}_{n_0} \chi_{\bar{n}} , \\
\mathcal{O}_1^{(1)}(n_0, n_0) = \tilde{\chi}_{n_0} g B_{n_0 \perp} \chi_{\bar{n}} , \\
\mathcal{T}_1^{(1)}(n_0, n_0) = \tilde{\chi}_{n_0} \left[ \mathcal{P}_{n_0 \perp} g B_{n_0 \perp} \right] \chi_{\bar{n}} , \\
\mathcal{O}_1^{(1)}(n_1, n_1') = \tilde{\chi}_{n_1} g B_{n_1' \perp} \chi_{\bar{n}} ,
\]

(C3)

Following the convention of Eq. \[14\], we do not write the antiquark direction as it is always \(\bar{n}\). \(\mathcal{O}_1^{(0)}\) is the LO operator and scales as \(\lambda^2\); \(\mathcal{O}_1^{(1)}(n_0, n_0)\) and \(\mathcal{O}_1^{(1)}(n_1, n_1')\) are NLO(\(\lambda\)) operators, scaling like \(\lambda^3\), and \(\mathcal{T}_1^{(1)} \sim \lambda^4\).

In SCET\(_1\), two particles are collinear if they are inside a cone with opening angle \(\sim \lambda\), equivalently \(p_1 \cdot p_2 \lesssim (Q \lambda)^2/\eta^4\). The factor \(\eta \sim 1/2\) represents the average \(\bar{p}\) fraction taken by the daughter from the mother as discussed in Sec. II C. Usually, we formulate this condition with dimensionless quantities, \(n_{p_1} \cdot n_{p_2} \lesssim \lambda^2/\eta^4\), where \(n_{p_i}\) is exactly proportional to the particle momentum. To distinguish a “two-jet” from a “three-jet” state, we label the external states with the direction to which the particles are collinear. A state \(|q_{n_0}\rangle_1\) indicates a state where a quark with momentum \(q_1\) is collinear to the direction \(n_0\), that is \((\bar{q}_1, n_0 \cdot q_1, (q_1)_{n_0 \perp}) \sim (1, \lambda^2, \lambda)Q\), and the subscript, 1, tells us the state can be annihilated by any operator, \(\chi_n\), where \(n\) and \(n_0\) are in the same SCET\(_1\) equivalence class, \(\{[n]\}\). As we will see, when we match to lower-scale SCET\(_i\), we will change this number appropriately. A two-jet state with a collinear quark and gluon, and an antiquark is given by \(|q_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle_1\). The fact that the quark and gluon share an index label implies that \(q_1 \cdot k_1 \lesssim (Q \lambda)^2/\eta^4\). A three-jet state is indicated by \(|q_{n_1} g_{n_1} \bar{q}_{\bar{n}}\rangle_1\), where each particle is collinear to a different direction. The operators \(\mathcal{O}_1^{(0)}(n_0)\), \(\mathcal{O}_1^{(1)}(n_0, n_0)\) and \(\mathcal{T}_1^{(1)}(n_0, n_0)\) can only create a two-jet state, whereas \(\mathcal{O}_1^{(1)}(n_1, n_1')\) is for three-jets. Multiplying the terms in (C3) by the Wilson coefficients, we have:

\[
J_{\text{QCD}}^\mu = C_{1,\text{LO}}^{(0)}(n_0)\mathcal{O}_1^{(0)} + C_{1,\text{NLO}}^{(1)}(n_0, n_0)\mathcal{O}_1^{(1)} + C_{1,\mathcal{T}}^{(1)}(n_1, n_1')\mathcal{T}_1^{(1)} \\
+ C^{(1)}(n_1, n_1')\mathcal{O}_1^{(1)} + \ldots ,
\]

(C4)

where the ellipses indicate higher order terms in \(\lambda\). When it is unambiguous, we will only write the \(n\)-labels in the Wilson coefficients, as above. We begin by looking at two-jet operators in detail. Here, because we are in the center of mass frame, the two jets are back to back. We define the kinematics as follows, the antiquark is exactly parallel to \(\bar{n} = (1, 0, 0, -1)\), while the quark and the gluon are collinear to \(n_0 = (1, 0, 0, 1)\), such that

\[\mathcal{T}_1^{(1)}(n_1, n_1')\] encodes redundant information that can be obtained with RPI. For example we can choose the directions \(n_1\) and \(n_1'\) to align perfectly with particle momenta such that \(\text{e.g.} \, \mathcal{P}_{n_1 \perp} \mathcal{B}_{n_1 \perp} = 0\). This is not possible for \(\mathcal{T}_1^{(1)}(n_0, n_0)\).
FIG. 17: Matching QCD to SCET$_1$ for the two-jet configuration: In the first column there are the two Feynman graphs for one-gluon emission in QCD, labeled by the 4-momenta. In the second column there are the two Feynman graphs in SCET$_1$ that reproduce the same amplitude in the case the quark and gluon are collinear along the direction $n_0$. The first graph comes from the operator $O^{(0)}_1(n_0)$ with the insertion of the SCET$_1$ Lagrangian, the second graph comes from the operators $O^{(1)}_1(n_0, n_0)$ and $T^{(1)}_1(n_0, n_0)$.

$q_0 = q_1 + k_1$ has no component perpendicular to $n_0$ and $\bar{n}$, and:

\[
\begin{align*}
    p_\bar{q}^\mu &= n_0 \cdot p_\bar{q} \frac{\bar{n}^\mu}{2}, \\
    q_1^\mu &= \bar{q}_1 \frac{n_0^\mu}{2} + n_0 \cdot q_1 \frac{\bar{n}^\mu}{2} + (q_1)^{\mu}_{n_0 \perp}, \\
    k_1^\mu &= \bar{q}_1 \frac{n_0^\mu}{2} + n_0 \cdot k_1 \frac{\bar{n}^\mu}{2} + (k_1)^{\mu}_{n_0 \perp},
\end{align*}
\]

where $(n_0 \cdot q_1, \bar{q}_1, q_1_{\perp})$ and $(n_0 \cdot k_1, \bar{k}_1, k_1_{\perp})$ scale as $(\lambda^2, 1, \lambda)$, and $(q_1)^{\mu}_{n_0 \perp} = -(k_1)^{\mu}_{n_0 \perp}$ by momentum conservation. The Wilson coefficients are defined through the equation

\[
\langle 0 | J_{\text{QCD}}^\mu | q_{n_0} g_{n_0} \bar{q}_{\bar{n}} \rangle_1 = C^{(0)}_{1, \text{LO}}(n_0, n_0) \int d^4x \langle 0 | T \{ L_{\text{SCET}_1}(x) O^{(0)}_1 \} | q_{n_0} g_{n_0} \bar{q}_{\bar{n}} \rangle_1 + C^{(1)}_1(n_0, n_0) \langle 0 | O^{(1)}_1 | q_{n_0} g_{n_0} \bar{q}_{\bar{n}} \rangle_1 + C^{(1)}_{1, T}(n_0, n_0) \langle 0 | T^{(1)}_1 | q_{n_0} g_{n_0} \bar{q}_{\bar{n}} \rangle_1. \tag{C6}
\]

Calculating the $C$’s for this two-jet process goes as follows. We decompose the QCD amplitude along $n_0$ and $\bar{n}$, using Eq. (C5), and we write the QCD spinor in terms of the SCET$_1$ spinor, Eq. (C8). Expanding in $\lambda$ up to NNLO, on the RHS we compute the amplitudes for the three SCET$_1$ terms. The coefficient $C^{(0)}_{1, \text{LO}}$ was already determined from matching QCD to SCET$_1$ for zero gluon emission, it is:

\[
C^{(0)}_{1, \text{LO}} = \gamma^\mu. \tag{C7}
\]

The coefficients $C^{(1)}_1$ and $C^{(1)}_{1, T}$ come from solving Eq. (C6) at NLO($\lambda$) and NNLO($\lambda$), respectively. Since $O^{(1)}_1$ and $T^{(1)}_1$ are at different orders in $\lambda$, there are no ambiguities.
In order to do the matching, we need the relation between the QCD and SCET spinors. Using Eq. (B5), we can write:

\[ u(p) = \left(1 + \frac{\bar{p} \cdot \hat{q}}{2 \bar{p}}\right) u_n(p), \quad (C8) \]

where \( u(p) \) is the QCD spinor and \( u_n(p) \) is the SCET one. It easy to see that the SCET spinor satisfies:

\[ \sum_s \bar{u}_n n_s u_s = 0, \quad n \]

\[ \bar{u}_n = u_n, \quad (C9) \]

The QCD amplitude for \( \gamma^* \rightarrow q\bar{q}g \) (shown in Fig. 17) is:

\[ A_{\text{QCD}}^{q\bar{q}g} = \bar{u}(q_1) i g \gamma^\alpha \frac{\hat{q}_0}{Q_0} \gamma^\mu v(p_\bar{q}) - \bar{u}(q_1) i g \gamma^\mu \gamma^\alpha \frac{\hat{p}_0}{P_0} v(p_\bar{q}). \quad (C10) \]

Using Eqs. (C5) & (C8) in (C10) and expanding to NNLO in \( \lambda \) we get:

\[ A_{\text{QCD}}^{q\bar{q}g} = A_{\text{LO}}^{q\bar{q}g} + A_{\text{NLO}}^{q\bar{q}g} + A_{\text{NNLO}}^{q\bar{q}g}, \quad (C11) \]

where,

\[ A_{\text{LO}}^{q\bar{q}g} = -g \bar{u}_{n_0} \left[ \left( n_0^\alpha + \frac{\hat{q}_1}{q_1} \bar{n}_0^\alpha \right) \frac{\hat{q}_0}{Q_0} + \frac{\bar{n}_0^\alpha}{k_1} \right] \gamma^\mu \gamma^\alpha v_n, \]

\[ A_{\text{NLO}}^{q\bar{q}g} = g \frac{n_0^\mu - \bar{n}_0^\mu}{Q_0} \bar{u}_{n_0} \left( \gamma^\alpha n_0^\perp - \frac{\bar{k}_1}{k_1} \bar{n}_0^\alpha \right) v_n, \]

\[ A_{\text{NNLO}}^{q\bar{q}g} = g \left( \frac{1}{\bar{q}_1} + \frac{1}{k_1} \right) \frac{1}{Q} \bar{u}_{n_0} \gamma^\mu (\bar{k}_1)_{n_0^\perp} \left( \gamma^\alpha_{n_0^\perp} - \frac{(\bar{k}_1)_{n_0^\perp}}{k_1} \bar{n}_0^\alpha \right) v_n - g \frac{2}{\bar{q}_1 Q} \bar{u}_{n_0} (k_1)^{\mu^\perp} \left( \gamma^\alpha_{n_0^\perp} - \frac{(\bar{k}_1)_{n_0^\perp}}{k_1} \bar{n}_0^\alpha \right) v_n. \quad (C12) \]

We already know \( C_{1,\text{LO}}^{(0)} \), and it is easy to determine the other two Wilson coefficients to reproduce \( A_{\text{QCD}}^{q\bar{q}g} \), they are:

\[ C_{1,\text{NLO}}^{(1)}(n_0, n_0) = \frac{1}{Q} \left( n_0^\mu - \bar{n}_0^\mu \right) \gamma^\alpha_{n_0^\perp}, \]

\[ C_{1,\text{T}}^{(1)}(n_0, n_0) = \frac{1}{\bar{q}_1 k_1} \gamma^\mu_{n_0^\perp} \gamma^\beta_{n_0^\perp} \gamma^\alpha_{n_0^\perp} - \frac{2}{\bar{q}_1 Q} g^\beta \gamma^\alpha_{n_0^\perp}, \quad (C13) \]

where we have used the relation \( \bar{q}_1 + \bar{k}_1 = Q \).

For the three-jet operator \( O_{1}^{(1)}(n_1, n'_1) \), the matching was already done in [67], but we will translate it to the notation used here. In this case, we need three distinct directions in
to decompose the particles. In the center of mass frame, $q$ that we gain by keeping track of them as separate contributions.

The Wilson coefficient will be $O$ \( \gamma \). The field content in SCET \( 2 \) will resum the large collinear logarithms of those configurations (cf. Sec. IV C). It is for this reason that we gain by keeping track of them as separate contributions.

Even though they are all in independent directions, we need only four independent vectors to decompose the particles. In the center of mass frame, $q_0 = q_1 + k_1$ is back to back with the antiquark, $p_q \propto \bar{n}$. We decompose $q_0$ along $(n_0, \bar{n})$ such that it has no component perpendicular to them: $q_0 = \bar{n} \cdot q_0 n_0/2 + n_0 \cdot q_0 \bar{n}/2$. Using Eq. (B10) we can define $n_1$ and $n_1'$ such that they are parallel to $q_1$ and $k_1$, respectively and such that the quark is decomposed along $(n_1, \bar{n})$ and the gluon along $(n_1', \bar{n})$, both without $\perp$ components. Unlike the two-jet case, where $(q_1)_{n_0 \perp} \lessapprox \lambda$, since the quark was collinear to $n_0$, here $(q_1)_{n_0 \perp} > \lambda$ in Eq. (B10). We have:

\[
\begin{align*}
q_1^\mu &= \bar{n} \cdot q_1 \frac{n_1^\mu}{2}, \\
p_q^\mu &= n_0 \cdot p_q \frac{n_1^\mu}{2}, \\
k_1^\mu &= \bar{n} \cdot k_1 \frac{n_1'^\mu}{2},
\end{align*}
\]

where $\bar{n} \cdot q_1$, $n_0 \cdot p_q$ and $\bar{n} \cdot k_1$ are $O(\lambda)$, and $n_1 \cdot n_1' > \lambda^2 / \eta^4$. With this setup $T^{(1)}_{\gamma}(n_1, n_1') = \bar{\chi}_{n_1} P_{n_1}^\perp B_{n_1}^\perp \chi_{\bar{n}} = 0$.

The matching is therefore given by:

\[
\langle 0 | J_\gamma^{(1)}(q_1, g_{n_1} q_{n_1})_1 = C^{(1)}_{1}(n_1, n_1') \langle 0 | O^{(1)}_1(q_1, g_{n_1'} q_{n_1'})_1, \tag{C15}
\]

and the Wilson coefficient is:

\[
C^{(1)}_{1}(n_1, n_1') = -\frac{2}{(n_1 \cdot n_1') q_1 k_1} \gamma^\alpha \gamma^\mu_{\gamma} + \left[ \frac{1}{(n_1 \cdot q_1) k_1} \left( \gamma^\mu_{\gamma} \phi_{\gamma} - \bar{q}_1 n_{1T} \right) + \frac{2(n_1 \cdot p_q) k_1}{(n_1 \cdot n_1') q_1 k_1} \bar{\gamma}^\alpha_{1 \perp} \right], \tag{C16}
\]

where the subscript $T$ applied to a generic four vector $f^\mu$ means: $f_T^\mu \equiv f^\mu - p_\gamma^\mu (f \cdot p_\gamma)/p_\gamma^2$, and $p_\gamma$ is defined in Eq. (C2).

Before moving on to lower scale SCET, we note that all the Wilson coefficients in SCET are of order $\lambda^0$. This will change with SCET \( 2 \) as these factors will determine the relative importance of different contributions. As we discussed at the very end of Sec. IV A we do not need to compute any suppressed two-gluon operators in SCET \( 1 \) to the order at which we are working. Their Wilson coefficient will be $O(\lambda^0)$. Matching this contribution to a two-gluon SCET \( 2 \) operator will leave this factor unchanged as there are no further emissions from it. The field content in SCET \( 2 \) will scale $\sim \lambda^5$. As shown in Eq. D7 though, LO in SCET \( 2 \) is at $\lambda^5$. 

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Lastly, we described the effects of adding running effects in Sec. [IVC]. In the next Appendix we will match SCET\(_1\) to SCET\(_2\). Before doing it we have to run the SCET\(_1\) operators from \(Q\) down to \(\mu_1\), where we have the first emission:

\[
C_0^{(0)}(n_0) = U^{(2,0,0)}(n_0; Q, \mu) \, \gamma^{\mu}_{\bar{n} n_0 T} \\
C_1^{(1)}(n_0, n_0) = U^{(2,1,0)}(n_0, n_0; Q, \mu) \otimes \frac{n_0^\mu - \bar{n}_\mu}{Q} \gamma^\alpha_{\bar{n} n_0 T} \\
C_1^{(1)}(n_0, n_0) = U^{(2,1,1)}_T(n_0, n_0; Q, \mu) \otimes \frac{1}{q_1 k_1} \left( \gamma^\mu_{\bar{n} n_0 T} \gamma^\beta_{\bar{n} n_0 T} \gamma^\alpha_{\bar{n} n_0 T} - 2 \frac{2}{q_1 Q} g^{\mu \beta} \gamma^\alpha_{\bar{n} n_0 T} \right) \\
C_1^{(1)}(n_1, n'_1) = U^{(2,1,0)}(n_1, n'_1; Q, \mu) \left( - \frac{2 (n_1 \cdot n'_1 \bar{q}_1 k_1) \gamma^\alpha_{\bar{n} n_0 T} \gamma^\mu_T}{(n_1 \cdot n'_1 \bar{q}_1 k_1) \bar{n}_T} \right). 
\]

For the definition of the running factors \(U^{(i,j,k)}(Q, \mu)\) see Eqs. (18)-(20), and (90). The convolution symbol, \(\otimes\), is only relevant beyond LL; that is beyond the level required here.

**Appendix D: Matching SCET\(_1\) to SCET\(_2\)**

1. **One-Gluon Emission**

We now match SCET\(_1\) to SCET\(_2\) for one and two-gluon emissions, starting with the former. The basis of SCET\(_2\) operators necessary for the matching up to NNLO(\(\lambda\)) is equal to Eq. (C3), but defined in SCET\(_2\): \(\mathcal{O}^{(0)}_2(n_0), \mathcal{O}^{(1)}_2(n_0, n_0), \mathcal{\bar{T}}^{(1)}_2(n_0, n_0), \mathcal{O}^{(1)}_2(n_1, n'_1)\). In the previous section, we matched QCD to SCET\(_1\) for one emission and found either a two-jet configuration (\(|q_{n_0} g_{n_0} \bar{q}_{n_1}\rangle\)) or three-jet configuration (\(|q_{n_1} g_{n'_1} \bar{q}_{n_2}\rangle\)), depending on the collinearity of the external particles. When we go to SCET\(_2\), our definition of collinearity becomes stricter. Particles with momenta \(p_1\) and \(p_2\) are collinear only if \(p_1 \cdot p_2 \lesssim Q^2 \lambda^4 / \eta^4\), where \(\eta \sim 1/2\) is the average energy loss factor between mother and daughters discussed in Sec. [IVC]. As a result of this change, a two-jet configuration in SCET\(_1\) can be matched both onto \(|q_{n_0} g_{n_0} \bar{q}_{n_2}\rangle\) and \(|q_{n_1} g_{n'_1} \bar{q}_{n_2}\rangle\) in SCET\(_2\). The three-jet configuration in SCET\(_1\) can, of course, only go to the three-jet state \(|q_{n_1} g_{n'_1} \bar{q}_{n_2}\rangle\) in SCET\(_2\). The matching is given by

\[
J^\mu_{\text{QCD}} = C^{(0)}_{1,\text{LO}}(n_0) \mathcal{O}^{(0)}_1 + C^{(1)}_{1,\text{NLO}}(n_0, n_0) \mathcal{O}^{(1)}_1 + C^{(1)}_2(n_1, n'_1) \mathcal{O}^{(1)}_1 + C^{(1)}_{1,\bar{T}}(n_0, n_0) \mathcal{\bar{T}}^{(1)}_1 + \cdots \tag{D1}
\]

\[
= C^{(0)}_2(n_0) \mathcal{O}^{(0)}_2 + C^{(1)}_2(n_0, n_0) \mathcal{O}^{(1)}_2 + C^{(1)}_2(n_1, n'_1) \mathcal{O}^{(1)}_2 + C^{(1)}_{2,\bar{T}}(n_0, n_0) \mathcal{\bar{T}}^{(1)}_2 + \cdots, \tag{D2}
\]

\(\text{16}\) As before, we do not consider operators like \(\mathcal{O}^{(1)}_2(n_0, \bar{n})\) that describe a gluon collinear to the antiquark.
where we give the decomposition into both SCET$_1$ and SCET$_2$ operators. The ellipses indicate higher order terms. If we close Eq. (D2) with the state $|q_{n_0} g_{n_0} \bar{q}_{n_0}\rangle_2$, we get

\begin{equation}
C^{(0)}_{1,\text{LO}}(n_0) \int dx^4 \langle 0| T\{ \mathcal{L}_{\text{SCET}_1}(x) \mathcal{O}^{(0)}_{1}\} |q_{n_0} g_{n_0} \bar{q}_{n_0}\rangle_2 \\
+ C^{(1)}_{1,\text{rNLO}}(n_0, n_0) \langle 0| \mathcal{O}^{(1)}_{1} |q_{n_0} g_{n_0} \bar{q}_{n_0}\rangle_2 + C^{(1)}_{1,T}(n_0, n_0) \langle 0| T^{(1)}_{1} |q_{n_0} g_{n_0} \bar{q}_{n_0}\rangle_2 \\
= C^{(0)}_{2}(n_0) \int dx^4 \langle 0| T\{ \mathcal{L}_{\text{SCET}_2}(x) \mathcal{O}^{(0)}_{2}\} |q_{n_0} g_{n_0} \bar{q}_{n_0}\rangle_2 \\
+ C^{(1)}_{2}(n_0, n_0) \langle 0| \mathcal{O}^{(1)}_{2} |q_{n_0} g_{n_0} \bar{q}_{n_0}\rangle_2 + C^{(1)}_{2,T}(n_0, n_0) \langle 0| T^{(1)}_{2} |q_{n_0} g_{n_0} \bar{q}_{n_0}\rangle_2 .
\end{equation}

(D3)

Since the structure of the operators in Eq. (D3) is the same on the LHS and RHS, we simply get:

\begin{equation}
C^{(0)}_{2}(n_0) = C^{(0)}_{1,\text{LO}}(n_0), \\
C^{(1)}_{2}(n_0, n_0) = C^{(1)}_{1,\text{NLO}}(n_0, n_0), \\
C^{(1)}_{2,T}(n_0, n_0) = C^{(1)}_{1,T}(n_0, n_0).
\end{equation}

(D4)

Acting on Eq. (D2) with the state $|q_{n_1} g_{n_1} \bar{q}_{n_1}\rangle_2$, we have:

\begin{equation}
C^{(0)}_{1,\text{LO}}(n_0) \int dx^4 \langle 0| T\{ \mathcal{L}_{\text{SCET}_1}(x) \mathcal{O}^{(0)}_{1}\} |q_{n_1} g_{n_1} \bar{q}_{n_1}\rangle_2 \\
+ C^{(1)}_{1}(n_1, n_1') \langle 0| \mathcal{O}^{(1)}_{1} |q_{n_1} g_{n_1} \bar{q}_{n_1}\rangle_2 + C^{(1)}_{1,T}(n_0, n_0) \langle 0| T^{(1)}_{1} |q_{n_1} g_{n_1} \bar{q}_{n_1}\rangle_2 \\
= C^{(1)}_{2}(n_1, n_1') \langle 0| \mathcal{O}^{(1)}_{2} |q_{n_1} g_{n_1} \bar{q}_{n_1}\rangle_2 .
\end{equation}

(D5)

We decompose $C^{(1)}_{2}(n_1, n_1')$ as

\begin{equation}
C^{(1)}_{2}(n_1, n_1') = C^{(1)}_{2,\text{LO}}(n_1, n_1') + C^{(1)H,a}_{2,\text{NLO}}(n_1, n_1') + C^{(1)H,b}_{2,\text{NLO}}(n_1, n_1') + C^{(1)H}_{2,\text{NNLO}}(n_1, n_1'),
\end{equation}

where $C^{(1)}_{2,\text{LO}}$ is the coefficient that reproduces the first term on the LHS of Eq. (D3), etc. All the SCET$_2$ coefficients in Eq. (D4) scale as $\lambda^0$, like in SCET$_1$, but we will see that those in Eq. (D6) scale differently, giving the hierarchy indicated in the subscript. We will show that:

\begin{equation}
\begin{aligned}
C^{(0)}_{2}(n_0)\mathcal{O}^{(0)}_{2} & \sim \lambda^4, & C^{(0)}_{2,\text{LO}}(n_1, n_1')\mathcal{O}^{(1)}_{2} & \sim \lambda^5, \\
C^{(1)}_{2}(n_0, n_0)\mathcal{O}^{(1)}_{2} & \sim \lambda^6, & C^{(1)H,a}_{2,\text{NLO}}(n_1, n_1')\mathcal{O}^{(1)}_{2} & \sim \lambda^6, \\
C^{(1)}_{2,T}(n_0, n_0)\mathcal{T}^{(1)}_{2} & \sim \lambda^8, & C^{(1)H,b}_{2,\text{NLO}}(n_1, n_1')\mathcal{O}^{(1)}_{2} & \sim \lambda^6, \\
& & C^{(1)H}_{2,\text{NNLO}}(n_1, n_1')\mathcal{O}^{(1)}_{2} & \sim \lambda^7.
\end{aligned}
\end{equation}

(D7)

In the second column we have only one operator $\mathcal{O}^{(1)}_{2}(n_1, n_1')$ and we have decomposed its coefficient according to Eq. (D6). The matching does not conserve the power counting, as collinear SCET$_1$ fields scale as $\lambda$, but in SCET$_2$ they go as $\lambda^2$. For example, we have that the LO operator in SCET$_1$ is $C^{(0)}_{1}\mathcal{O}^{(0)}_{1} \sim \lambda^2$, but for the LO operator in SCET$_2$ we have $C^{(0)}_{2}\mathcal{O}^{(0)}_{2} \sim \lambda^4$.  


If we want to calculate a cross section for a fixed number of external particles, then we need all the SCET$_2$ operators in Eq. (D2). Our interest, though, is in improving shower Monte Carlo, and so we only calculate operators needed for that (cf. discussion in Sec. III A). To reproduce the LL emission of two gluons, the only higher dimension operator we need is $C_{2, \text{LO}}^{(1)}(n_1, n'_1) \mathcal{O}_2^{(1)}$. The operators $\mathcal{O}_2^{(0)}(n_0)$ and $\mathcal{O}_2^{(1)}(n_0, n_0)$ only tell us about the no-branching probabilities already determined by the one-loop cusp anomalous dimension. For example, $\mathcal{O}_2^{(0)}(n_0)$ describes a quark which does not emit until after the scale of matching $k_{1\perp}$.

For this reason, we call $C_{2, \text{LO}}^{(1)}(n_1, n'_1) \mathcal{O}_2^{(1)}$ our LO operator. Naively, two-gluon contributions from $\mathcal{O}_2^{(0)}(n_0)$ and $\mathcal{O}_2^{(1)}(n_0, n_0)$ are lower order at tree-level, but this does not take into account the exponential suppression from running. The dominant contribution to showers comes from strong-ordering, not “every emission as collinear as possible.” Thus, we build our shower around $C_{2, \text{LO}}^{(1)}(n_1, n'_1) \mathcal{O}_2^{(1)}$. The coefficients $C_{2, \text{NLO}}^{(1), H}(n_1, n'_1)$ and $C_{2, \text{NNLO}}^{(1), H}(n_1, n'_1)$ give corrections for one emission. We therefore obtain a correction if we run a LL shower based on a matrix element computed with one of these suppressed terms.

We now turn to calculate the terms in Eq. (D6), in three steps: first we calculate the amplitudes in SCET$_1$ on the LHS of (D5); second we rotate it using the finite RPI$_1$ transformations defined in App. B, so that the necessary operators overlap with SCET$_2$ states; and third we calculate the Wilson coefficients necessary to match the two sides. We do it order by order and we start by calculating the coefficient $C_{2, \text{LO}}^{(1)}$. The first term of the LHS of (D5) is

$$A_{\text{LO}}^{qgq} = U^{(2,0,0)}(n_0; Q, \mu_1) g \tilde{r}_n \left( n_0^\alpha + \frac{(q_1^\alpha)_{n_0\perp}}{q_1} \right) \gamma_{n_0\perp} \xi_{\bar{n}} ,$$

(D8)

where $U^{(2,0,0)}(n_0)$ is the running factor (cf. Eqs. [18, 19] and [90], and $\mu_1 \sim \lambda Q$ is at the scale of the emission. In (D8), we have omitted the terms proportional to $\bar{n}^\alpha$ as they are unnecessary for matching. Gauge invariance constrains all appearances of $\bar{n} \cdot A_n$ to come from the Wilson lines in $\chi$ and $B$. The amplitude is written in terms of objects projected in the $n_0$ and $\bar{n}$ directions. As discussed in Appendix B, these directions are not suitable for a SCET$_2$ states, but we can use the formulas (B7) and write (D8) in terms of the directions $n_1$ and $n'_1$ where the quark and gluon have zero perpendicular component, this gives

$$A_{\text{LO}}^{qgq} = U^{(2,0,0)}(n_0; Q, \mu_1) g \tilde{r}_n \left( n_1^\alpha + 2 \frac{(q_1^\alpha)_{n_0\perp}}{k_1} + \frac{(\bar{q}_1^\alpha)_{n_0\perp}}{q_1} \right) \frac{q_0}{q_0} \gamma_{n_0\perp} \xi_{\bar{n}} .$$

(D9)

In (D9) we have rotated the spinor in the $n_1$ direction, $\gamma_{n_0\perp}$ in the $n'_1$ direction and we have written $n_0$ in terms of $n_1$, $n'_1$ and $(q_1)_{n_0\perp}$. We have dropped all the terms proportional to $\bar{n}^\alpha$ and we made use of relations $\not\! n \not\! n = 0$ and $\xi_{\not\! n}, \not\! n_1 = 0$. Since the gluon momentum is parallel to $n'_1\mu$, only the polarizations in the perpendicular direction with respect to $n'_1\mu$ are physical, thus we can neglect the term proportional to $n'_1\mu$ in Eq. (D9). The SCET$_2$ amplitude $\langle 0 | \bar{\chi}_{n_1} g B^{\alpha}_{n'_1\perp} | q_{n_1} g_{n'_1} \bar{q}_n \rangle_2$ is

$$\langle 0 | \bar{\chi}_{n_1} g B^{\alpha}_{n'_1\perp} | q_{n_1} g_{n'_1} \bar{q}_n \rangle_2 = g \bar{u}_{n_1} \epsilon_{n'_1\perp}^\alpha v_{\bar{n}} ,$$

(D10)

where in Eq. (D10) we have explicitly written the polarization vector for the gluon. From
Eq. (D9) and Eq. (D10), we can see that the LO Wilson coefficient is

$$C_{2, \text{LO}}^{(1)} = U^{(2,0,0)}(n_0; Q, \mu_1)c_{\text{LO}}^\alpha(n_0)\frac{\bar{q}_0}{q_0^\alpha}\gamma_{n_0\perp}, \quad (D11)$$

where

$$c_{\text{LO}}^\alpha(n_0) = \left(2\frac{(q_1)_{n_0\perp}^\alpha}{k_1} + \frac{(\bar{q}_1)_{n_0\perp}\gamma_{n_1'\perp}^\alpha}{\bar{q}_1} \right)\frac{\bar{q}_0}{4}\Theta_{s}[n_1 \cdot n_1']. \quad (D12)$$

The difference with Eq. (57) is that we replaced $n_0$ in terms of external vectors. $\Theta_{s}[n_1 \cdot n_1']$ is the phase space cutoff that guarantees $(n_1 \cdot n_1') \lesssim \lambda^2/\eta^4$. We will say more about it below. Since this comes from matching to a SCET$_1$ operator, $(q_1)_{n_0\perp} \sim \lambda$ and $\bar{q}_0^2 \sim \lambda^2$, thus $C_{2, \text{LO}}^{(1)}$ scales as $\lambda^{-1}$. Using formulas (D11), we can write (D11) only in terms of $n_1$ and $n_1'$, this gives

$$C_{2, \text{LO}}^{(1)} = U^{(2,0,0)}(n_0; Q, \mu_1) \left(\frac{\bar{q}_1}{Q}\sqrt{n_1 \cdot n_1'}v_1^\alpha + \frac{\bar{k}_1}{2Q}\sqrt{n_1 \cdot n_1'}\gamma_{n_1'\perp}^\alpha\right) \frac{2\bar{q}_0}{(n_1 \cdot n_1')\bar{q}_1k_1}$$

$$\times \left[\gamma_{n_1'\perp}^\mu - \bar{n}_1^\mu \frac{1}{2\bar{q}_0^\alpha}\sqrt{n_1 \cdot n_1'}\gamma_{n_1'\perp}^\alpha\right], \quad (D13)$$

where $v_1^\alpha$ is defined in Eq. (B12), $\bar{q}_1 + \bar{k}_1 = Q$. For $\mu_1$, as explained in Sec. IIIA, we take it at the scale of $(k_1)_{n_0\perp}$ as in Eq. (64) for $k = 1$. Since $|v_1^2| = 2$, the power counting of (D13) is given by the scalar product $n_1 \cdot n_1'$, that is $\mathcal{O}(\lambda^2)$. In a similar way, we can calculate $C_{2, \text{NLO}}^{(1)H,a}$ and $C_{2, \text{NNLO}}^{(1)H}$.

We have done the matching starting from the vector current $J_{q\text{CD}}^\mu = \bar{q}\slashed{\gamma}\mu q$. If we had started from a general structure, $\bar{q}\Gamma^\mu q$, the results (D11) for $C_{2, \text{LO}}^{(1)}$ would have been the same upon the substitution

$$\gamma_{n_0\perp}^\mu \rightarrow \Gamma^\mu. \quad (D14)$$

We can obtain $C_{2, \text{LO}}^{(1)}(n_1, n_1')C_2^{(2)}$ from the SCET$_1$ operator $\bar{\chi}_{n_0}\Gamma^\mu\chi_{n_1}$ by running down from $Q$ to $\mu_1$, multiplying by the factor $U^{(2,0,0)}(n_0; Q, \mu_1)$, and subsequently using the replacement rule

$$(\bar{\chi}_{n_0})_i \rightarrow (c_{\text{LO}}^\alpha(n_0))_{ji}(\bar{\chi}_{n_1})_j gB_{\alpha}^{n_1'\perp}. \quad (D15)$$

The coefficients $C_{2, \text{NLO}}^{(1)H,a}$, $C_{2, \text{NLO}}^{(1)H,b}$, $C_{2, \text{NNLO}}^{(1)H}$, however, are sensitive to the particular QCD current. This is why we refer to them as hard-scattering corrections, denoted by the superscript, $H$.

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17 The factor of $\eta \simeq \frac{1}{2}$ tracks the average energy loss between mother and daughter. In choosing appropriate values for the parameters $\delta_k$ in the numerical implementation of $\Theta$ it is important to track these $\eta$ factors in the scaling of $n_1 \cdot n_1'$.  

70
For the NLO(\(\lambda\)) and NNLO(\(\lambda\)) amplitudes in the second and third line of the LHS of Eq. (D5) we have

\[
A_{\text{NLO}}^{q\bar{q}} = U^{(2,1,0)}(n_0, n_0; Q, \mu_1) \frac{n_0^\mu - \bar{n}_0^\mu}{Q} g\tilde{u}_{n_0} \gamma_\alpha^\alpha v_{n_1} = \frac{n_0^\mu - \bar{n}_0^\mu}{Q} g\tilde{u}_{n_1} \gamma_\alpha v_{n_1},
\]

\[
A_{\text{NNLO}}^{q\bar{q}} = U^{(2,1,1)}(n_0, n_0; Q, \mu_1)
\times \left( \frac{1}{q_1} + \frac{1}{k_1} \right) \frac{1}{Q} g\tilde{u}_{n_0} \gamma_\alpha^\alpha(k_1)_{n_0}^{\gamma_\alpha} v_{n_1} - \frac{2}{q_1} g\tilde{u}_{n_1} (k_1)_{n_0}^{\gamma_\alpha} v_{n_1}
= U^{(2,1,1)}(n_0, n_0; Q, \mu_1)
\times \left( \frac{1}{q_1} + \frac{1}{k_1} \right) \frac{1}{Q} g\tilde{u}_{n_1} \gamma_\alpha^\alpha(k_1)_{n_0}^{\gamma_\alpha} v_{n_1} - \frac{2}{q_1} g\tilde{u}_{n_1} (k_1)_{n_0}^{\gamma_\alpha} v_{n_1}.\tag{D16}
\]

The SCET\(_2\) coefficients needed to reproduce the amplitudes in Eq. (D16) are:

\[
C_{2,\text{NLO}}^{(1)H,a} = U^{(2,1,0)}(n_0, n_0; Q, \mu_1) \otimes c_{2,\text{NLO}}^{H,a}(n_0, n_0),
\]

\[
C_{2,\text{NNLO}}^{(1)H} = U_T^{(2,1,1)}(n_0, n_0; Q, \mu_1) \otimes c_{2,\text{NNLO}}^{H}(n_0, n_0),
\]

where

\[
c_{2,\text{NLO}}^{H,a}(n_0, n_0) = U^{(2,1,0)}(n_0, n_0; Q, \mu_1) \frac{n_0^\mu - \bar{n}_0^\mu}{Q} \gamma_\alpha^\alpha \Theta_\delta_2[n_1 \cdot n_1']\tag{D18}
= U^{(2,1,0)}(n_0, n_0; Q, \mu_1) \frac{1}{Q} \left[ \frac{k_1 n_1^\mu + \bar{q}_1 n_1'^\mu}{q_0} - \left( 1 + \frac{\bar{q}_1 k_1}{2 q_0^2 (n_1 \cdot n_1')^\mu} \right) \bar{n}_0^\mu \right] \times \gamma_\alpha^\alpha \Theta_\delta_2[n_1 \cdot n_1'],
\]

\[
c_{2,\text{NNLO}}^{H}(n_0, n_0) = U_T^{(2,1,1)}(n_0, n_0; Q, \mu_1)[(\frac{1}{q_1} + \frac{1}{k_1}) \frac{1}{Q} \gamma_\alpha^\alpha(k_1)_{n_0}^{\gamma_\alpha} - \frac{2}{q_1} (k_1)_{n_0}^{\gamma_\alpha}] \times \Theta_\delta_2[n_1 \cdot n_1']
= -U_T^{(2,1,1)}(n_0, n_0; Q, \mu_1) \left( \frac{1}{2 Q^2} \left( \gamma_\alpha n_1^\mu \sqrt{n_1 \cdot n_1'} + \bar{n}_0^\mu \frac{\bar{q}_1}{Q} (n_1 \cdot n_1') \right) \gamma_\alpha^\alpha n_1'^\mu \right.
+ \frac{k_1}{Q^2} \left( \sqrt{n_1 \cdot n_1'} n_1^\mu \bar{n}_0^\mu (n_1 \cdot n_1') \left( \frac{k_1^2 - \bar{q}_1^2}{2 Q^2} \right) \gamma_\alpha^\alpha \right) \Theta_\delta_2[n_1 \cdot n_1'].
\]

The coefficients scale \(C_{2,\text{NLO}}^{(1)H,a} \sim \lambda^0\) and \(C_{2,\text{NNLO}}^{(1)H} \sim \lambda\). As discussed below Eq. (89), we have a convolution because SCET fields collinear to the same direction can exchange longitudinal momentum during the running. However, this convolution is only needed beyond the LL level considered here.

For the coefficient \(C_{2,\text{NLO}}^{(1)H,b}\), the matching comes from the SCET\(_1\) three-jet operator where \(n_1 \cdot n_1' \sim \lambda^0/\eta^4\). Since the \(n\)-labels in \(C_{1}^{(1)}(n_1, n_1')\) are already parallel to the external particles, we can simply write:

\[
C_{2,\text{NLO}}^{(1)H,b}(n_1, n_1') = C_{1}^{(1)}(n_1, n_1') \Theta_\delta_2[n_1 \cdot n_1'],\tag{D19}
\]

\[\text{18 With our conventions where } n_1 \cdot \bar{n} = 2, \text{ two well separated directions } n_1 \text{ and } n_2 \text{ really do give } n_1 \cdot n_2 \sim 16.\]
where $\hat{\Theta}_\delta[n_1 \cdot n'_1]$ only has support for $(n_1 \cdot n'_1) > \lambda/\eta^4$, where it is equal to 1. Knowing that for this term, $n_1 \cdot n'_1 \sim \lambda^0/\eta^4$, $C_{2,\text{NLO}}^{(1)H,b}(n_1, n'_1)$ scales $\sim \lambda^6$, and

$$C_{2,\text{NLO}}^{(1)H,b}(n_1, n'_1)O_2^{(1)} \sim \lambda^6. \tag{D20}$$

In keeping with our conventions, we keep track of dependence on $\eta \sim \frac{1}{2}$ for our $\Theta$ functions and their dot product arguments, where the various $2^n$ factors affect where the step function turns over. We do not include them in the power counting for operators, where $\lambda$ parametrizes strong-ordering and the deviations from it. Accounting for $\eta$ here is certainly possible, but in the end we always will compare amplitudes with the same number of external particles, so $\eta$ factors from operators will not play any role.

The operator $O_2^{(1)}(n_1, n'_1)$ only knows that $n_1 \cdot n'_1 > \lambda^4/\eta^4$, it is not able to distinguish its two-jet contributions, Eq. (D13) and Eq. (D17), from its three-jet one, Eq. (D19). This information must then be in the Wilson coefficients, and we have put it in the functions $\Theta$ and $\tilde{\Theta}$, first described in Sec. III.A. We can think of $\Theta_\delta[x]$ as usual theta function: $\Theta_\delta[x] = \theta[\delta_2 - x]$ and $\tilde{\Theta}_\delta[x] = 1 - \Theta_\delta[x]$, but for integrating phase space, this can lead to numerical problems. Instead, we can use a smoother theta function, such as the following, plotted in Fig. 7

$$\Theta_{\Lambda,a}(x) = \begin{cases} 0 & \text{if } x < \Lambda - a \\ -\frac{\text{Sign}(x-\Lambda)}{2} e^{2a\text{Sign}(x-\Lambda)} + \frac{\text{Sign}(x-\Lambda)+1}{2} & \text{if } \Lambda - a < x < \Lambda + a \\ 1 & \text{if } x > \Lambda + a \end{cases} \tag{D21}$$

The parameter $\Lambda$ determines where the function switches from 0 to 1, and $a$ governs how fast it does it. For the SCET$_2$ coefficients, we have $\Lambda \simeq \delta_2$. In order to have $n_i \cdot n'_1 \lesssim \lambda^2/\eta^4$, we need $\lambda^2/\eta^4 \ll \delta_2 < 1/\eta^4$, so we choose $\delta_2 = \lambda/\eta^4$. When we go down to lower SCET$_i$, in general the Wilson coefficient has to encode the that either $n_i \cdot n_j \leq \lambda^{2(i-1)}/\eta^4$ or $n_i \cdot n_j > \lambda^{2(i-1)}/\eta^4$, in order to do so, we will use $\Theta_{\delta_i}$ where $\delta_i = \lambda^{2i-3}/\eta^4$. To see how this $\Theta$ works, we look at the amplitude squared up to NLO($\lambda$). The LO amplitude squared is

$$|A^{q\bar{q}\gamma}|^2_{\text{LO}} = |C_{2,\text{LO}}^{(1)}(n_1, n'_1)|^2 G(q_1, k_1, k_2, p_q), \tag{D22}$$

where

$$G(q_1, k_1, k_2, p_q) = 2\langle q_{n_1} g_{n'_1} \bar{q}_n | O_1^{(1)} | n_1, n'_1 \rangle |0\rangle \langle 0| O_1^{(1)} | n_1, n'_1 \rangle q_{n_1} g_{n'_1} \bar{q}_n \rangle_2. \tag{D23}$$

The NLO($\lambda$) amplitude squared is

$$|A^{q\bar{q}\gamma}|^2_{\text{NLO}} = |A^{q\bar{q}\gamma}|^2_{\text{NLO, 2-jet}} + |A^{q\bar{q}\gamma}|^2_{\text{NLO, 3-jet}}, \tag{D24}$$

We perform some trivial azimuthal integrals in order to eliminate some terms that will drop out of typical observables. Also, by NLO($\lambda$) corrections for amplitudes squared, we mean suppressed by two powers of $\lambda$. Since there are no odd powers of $\lambda$ in the expansion, this means NLO($\lambda$) for the cross section.
where

\[
|A^{q\bar{q}g}|^2_{\text{NLO, 2-jet}} = \left(C^{(1)}_{2,\text{LO}}(n_1, n'_1)C^{(1)}_{2,\text{NLO}}(n_1, n'_1)
+ C^{(1)}_{2,\text{NNLO}}(n_1, n'_1)C^{(1)}_{2,\text{LO}}(n_1, n'_1)
+ |C^{(1)}_{2,\text{NLO}}(n_1, n'_1)|^2 \right) G(q_1, k_1, k_2, p_{\bar{q}}),
\]

\[
|A^{q\bar{q}g}|^2_{\text{NLO, 3-jet}} = \left|C^{(1)}_{2,\text{NLO}}(n_1, n'_1)\right|^2 \left|G(q_1, k_1, k_2, p_{\bar{q}})\right|^2.
\]

(D25)

In Fig. 11 we plot the ratios \( |A^{q\bar{q}g}|^2_{\text{LO}}/|A^{q\bar{q}g}|^2_{\text{QCD}} \) and \((|A^{q\bar{q}g}|^2_{\text{LO}} + |A^{q\bar{q}g}|^2_{\text{NLO, 2-jet}})/|A^{q\bar{q}g}|^2_{\text{QCD}} \) versus \((k_1)_{n_0\perp}\). We note that including NLO(\(\lambda\)) corrections extends the region where tree-level SCET\(_2\) and QCD agree. In Fig. 10, we plot the merging of the two-jet and three-jet amplitude squared using the theta function. Although we have not undertaken any systematic study of how our phase space cutoff enters observables, we take Fig. 10 as visual evidence of minimal sensitivity. Lastly, in Fig. 18 we plot \( |A^{q\bar{q}g}|^2_{\text{LO}} + |A^{q\bar{q}g}|^2_{\text{NLO}} \) with

![Amplitude Squared](image)

FIG. 18: Plot of the SCET\(_2\) amplitude squared up to NLO, \( |A^{q\bar{q}g}|^2_{\text{LO}} + |A^{q\bar{q}g}|^2_{\text{NLO}} \), with (green) and without (red) running factors versus \((k_1)_{n_0\perp}\) for \(\bar{k}_1/\bar{q}_0 = 0.4\).

and without running factors. As expected, the latter is suppressed relative to the former.

2. Two-Gluon Emission

We show the Feynman diagrams corresponding to the operators needed for matching two-gluon emission in Fig. 19. As discussed at the very end of Sec. IV A and in App. C, we do not need the two-gluon, SCET\(_1\) operator, \(\mathcal{O}_1^{(2)}(n_0, n_0, n_0)\) at this order. Thus, the ones in (C3) are sufficient.
The SCET\(_2\) basis has the following two gluon operators:

\[
\begin{align*}
O_2^{(2)}(n_2, n_1', n_2) &= \bar{x}_{n_2} g B^\alpha_{n_2 \perp} g B^{\beta}_{n_2 \perp} \chi_n, \\
O_2^{(2)}(n_2, n_1', n_1') &= \bar{x}_{n_2} g B^\alpha_{n_2 \perp} g B^{\beta}_{n_1' \perp} \chi_{n_1} \\
O_2^{(2)}(n_2, n_1', n_2') &= \bar{x}_{n_2} g B^\alpha_{n_1' \perp} g B^{\beta}_{n_2' \perp} \chi_{n_1} \\
O_2^{(2)}(n_0, n_0, n_0) &= \bar{x}_{n_0} g B^\alpha_{n_0 \perp} g B^{\beta}_{n_0 \perp} \chi_{n_0}.
\end{align*}
\]

The last operator in (D26) is not necessary for the matching at NNLO(\(\lambda\)). It can only be closed with states \(|q_{n_0} q_{n_0} q_{n_0} q_{n_0}\rangle\) having both gluons collinear in SCET\(_2\). Its coefficient can only come from the SCET\(_1\) operator \(O_1^{(2)}(n_0, n_0, n_0)\). Any contribution involving SCET\(_1\) Lagrangian emission that matches to a higher-dimension operator in SCET\(_2\) will necessarily have some partons in different SCET\(_2\) directions, e.g. \((n_0, n_0, n_1')\). Since \(C_1^{(2)}(n_0, n_0, n_0) \sim \lambda^0\), and the matching does not change this, \(O_2^{(2)}(n_0, n_0, n_0)\) contributes at \(N^3\text{LO}\). The Wilson coefficients of the operators (D26) are defined such that

\[
J^\mu_{\text{QCD}} = C_{1,\text{LO}}^{(0)}(n_0)O_1^{(0)} + C_{1}^{(1)}(n_0, n_0)O_1^{(1)} + C_{1,\tau}^{(1)}T_1^{(1)}
\]

\[
= C_{2}^{(0)}(n_0)O_2^{(0)} + C_{1}^{(1)}(n_0, n_0)O_2^{(2)} + C_{2,\tau}^{(2)}(n_0, n_0)T_2^{(2)}
\]

\[
+ C_{2}^{(1)}(n_1, n_1')O_2^{(1)} + C_{2}^{(2)}(n_2, n_1', n_2)O_2^{(2)}
\]

\[
+ C_{2}^{(2)}(n_2, n_1', n_1')O_2^{(2)} + C_{2}^{(2)}(n_2, n_1', n_2')O_2^{(2)} + \cdots
\]

FIG. 19: Matching SCET\(_1\) to SCET\(_2\) to SCET\(_3\) for two emissions to the two-jet configuration in SCET\(_1\). We organize here by column number: (1) QCD Feynman diagrams; (2) SCET\(_1\) diagrams from the operator \(O_1^{(0)}(n_0)\); (3) SCET\(_1\) diagrams from the operators \(O_1^{(1)}(n_0, n_0)\) and \(T_1^{(1)}(n_0, n_0)\); (4) SCET\(_1\) diagram from the operator \(O_1^{(2)}(n_0, n_0, n_0)\), this operator contributes only at \(N^3\text{LO}\) to the SCET\(_2\) matching; (5) SCET\(_2\) diagram from the operator \(O_1^{(1)}(n_2, n_1');\) (6) SCET\(_2\) diagrams from the operators \(O_2^{(2)}(n_2, n_1', n_2), O_2^{(2)}(n_2, n_1', n_2)\) and \(O_2^{(2)}(n_2, n_1', n_1')\); (7) SCET\(_3\) diagram from the operator \(O_2^{(2)}(n_1, n_1', n_1')\).
where we have written the QCD current in terms of SCET$_1$ and SCET$_2$ operators. The ellipses indicate higher order terms.

We divide the subleading Wilson coefficients in two categories: jet-structure and hard-scattering, labeling their contributions with the superscripts $J$ and $H$. As mentioned previously, the latter come from suppressed operators in the QCD $\to$ SCET$_1$ matching and depend on the details of the hard partons’ creation. The former are subleading terms from the SCET$_1$ Lagrangian that correct Eq. (56) as we match to lower-scale theories. They are completely independent of the initial hard process.

We have seen in the previous section that the LO single gluon coefficient $\times$ operator is $C^{(1)}_2(n_1, n'_1) O^{(1)}_2 \sim \lambda^5$, Eq. (D7). We are interested in calculating the amplitude squared to NLO($\lambda$). We therefore only need to calculate those NNLO($\lambda$) contributions that can interfere with the LO amplitude. These operators are of the form $O^{(2)}_2(n_2, n'_2, n'_1)$, as the others in Eq. (D26) are not strongly-ordered.$^{20}$

We now calculate the coefficients in (D26), starting with $C^{(2)}_2(n_2, n'_1, n'_2)$, which we decompose as:

\[
C^{(2)}_2(n_2, n'_1, n'_2) = C^{(2)J}_{2, \text{NLO}}(n_2, n'_1, n'_2) + C^{(2)H}_{2, \text{NLO}}(n_2, n'_1, n'_2), \tag{D28}
\]

where

\[
C^{(2)J}_{2, \text{NLO}}(n_2, n'_1, n'_2) = C^{(2)J}_{2, \text{NLO}}(n_2, n'_1, n'_2) \langle 0 \mid O^{(2)}_2 \mid q_n g_n' g_n' \bar{q}_n \rangle_2 = \tag{D29}
\]

\[
C^{(0)}_{1, \text{LO}}(n_0) \int dx_1 dx_2 dx_0 \{ \mathcal{L}_{\text{SCET}_1}(x_1) \mathcal{L}_{\text{SCET}_1}(x_2) O^{(1)}_1(n_0, n_0) \} \langle q_n g_n' g_n' \bar{q}_n \rangle_2
\]

and

\[
C^{(2)H}_{2, \text{NLO}}(n_2, n'_1, n'_2) = C^{(2)H}_{2, \text{NLO}}(n_2, n'_1, n'_2) \langle 0 \mid O^{(2)}_2 \mid q_n g_n' g_n' \bar{q}_n \rangle_2 = \tag{D30}
\]

\[
+ C^{(1)}_1(n_0, n_0) \int dx_0 \{ \mathcal{L}_{\text{SCET}_1}(x) O^{(1)}_1 \} \langle q_n g_n' g_n' \bar{q}_n \rangle_2
\]

\[
+ C^{(1)}_{1, T}(n_0, n_0) \int dx_0 \{ \mathcal{L}_{\text{SCET}_1}(x) T^{(1)}_1 \} \langle q_n g_n' g_n' \bar{q}_n \rangle_2
\]

\[
+ C^{(1)}_1(n_1, n'_1) \int dx_0 \{ \mathcal{L}_{\text{SCET}_1}(x) O^{(1)}_1 \} \langle q_n g_n' g_n' \bar{q}_n \rangle_2.
\]

We decompose $C^{(2)H}_{2, \text{NLO}}(n_2, n'_1, n'_2)$ as

\[
C^{(2)H}_{2, \text{NLO}}(n_2, n'_1, n'_2) = C^{(2)H, a}_{2, \text{NNLO}}(n_2, n'_1, n'_2) + C^{(2)H, b}_{2, \text{NNLO}}(n_2, n'_1, n'_2) + C^{(2)H, b}_{2, \text{NNLO}}(n_2, n'_1, n'_2), \tag{D31}
\]

where $C^{(2)H, a}_{2, \text{NNLO}}(n_2, n'_1, n'_2)$ is the coefficient that reproduces the the second line in Eq. (D30), etc.

Since $O^{(2)}_2(n_2, n'_1, n'_2)$ does not interfere with the LO operator, we only need the coefficient, $C^{(2)J}_{2, \text{NLO}}(n_2, n'_1, n'_2)$. We also calculate $C^{(2)H, a}_{2, \text{NNLO}}(n_2, n'_1, n'_2)$ though, because it will be useful.

---

$^{20}$ In principle, we also have $T^{(2)}_2(n_2, n_2, n'_1)$. However, the field content alone makes this $\lambda^{10}$, but all our correction operators have Wilson coefficients at $O(\lambda^{-2})$, so its contribution is beyond NNLO($\lambda$).
later. We prove below that these coefficients and their corresponding operators are of order \( \lambda^6 \) and \( \lambda^7 \), respectively (in Eq. D7, we show that LO is at \( \lambda^5 \)). \( \mathcal{C}_{2,\text{NLO}}^{(2)}(n_2, n_1', n_2')\) and \( \mathcal{C}_{2,\text{NNLO}}^{(2)}(n_2, n_1', n_2')\) come from two-jet operators in SCET$_1$. Thus, they both contain factors of \( \Theta_{\delta_2}[n_2' n_1'] \Theta_{\delta_2}[n_2' n_1'] \Theta_{\delta_2}[n_2' n_1'] \). We first described these phase space cutoffs in Sec. III A and made use of them in previous section on single-gluon matching. The subscript, \( \delta_2 \), constrains the argument to be \( \lesssim \lambda/\eta^4 \).

To calculate the coefficients, we proceed as with one-gluon emission: on the LHS of Eqs. D29 and D30 we calculate the SCET$_1$ amplitude and rotate it along the directions \( n_2, n_1', n_2' \) where the quark and the two gluons are aligned using the finite RPI$_1$ described in App. B; on the RHS we write the SCET$_2$ amplitude and calculate the Wilson coefficient necessary for the matching. We decompose the SCET$_1$ amplitude:

\[
A^{qgq}_{\text{NLO}} = C_{1}^{(0)}(n_0) \int dx_1 dx_2 \langle 0| T\{ \mathcal{L}_{\text{SCET}_1}(x_1) \mathcal{L}_{\text{SCET}_1}(x_2) \mathcal{O}_1^{(0)} \} | q_{n_2} g_{n_1'} g_{n_2'} q_{n_0} \rangle_2, \tag{D32}
\]

in

\[
A^{qgq}_{\text{NLO}} = A^{qgq}_{\text{NLO}, A} + A^{qgq}_{\text{NLO}, B} + A^{qgq}_{\text{NLO}, C}, \tag{D33}
\]

where \( A, B, C \) correspond to the three graphs in Fig. 20. Using the SCET$_1$ Feynman rules, we have:

\[
A^{qgq}_{\text{NLO}, A} = U^{(2,0,0)}(n_0; Q, \mu_1) g^2 \bar{u}_{n_0} \left[ n_0^\beta + \gamma_{n_0}^\alpha \left( \frac{\gamma_1}{q_1} \frac{\gamma_0}{q_2} \gamma_{n_0} \right) \right] \bar{q}_1 \bar{q}_2 \gamma_{n_0} v_n, \tag{D34}
\]

\[
A^{qgq}_{\text{NLO}, B} = U^{(2,0,0)}(n_0; Q, \mu_1) g^2 \bar{u}_{n_0} \left[ n_0^\beta + \gamma_{n_0}^\alpha \left( \frac{\gamma_2 + \gamma_1}{q_2 + k_1} \frac{\gamma_0}{q_2} \gamma_{n_0} \right) \right] \bar{q}_1 \bar{q}_2 \gamma_{n_0} v_n, \tag{D35}
\]

\[
A^{qgq}_{\text{NLO}, C} = U^{(2,0,0)}(n_0; Q, \mu_1) g^2 \bar{u}_{n_0} \left[ \frac{1}{q_2 + k_1} \gamma_{n_0}^\alpha \gamma_{n_0}^\beta + \frac{1}{q_2 + k_2} \gamma_{n_0}^\alpha \gamma_{n_0}^\beta \right] \bar{q}_0 \gamma_{n_0} v_n, \tag{D36}
\]

where \( q_1 = q_2 + k_2 \) and \( q_0 = q_2 + k_1 + k_2 \). As before, we do not write terms with \( \bar{n}^\alpha \) and \( \bar{n}^\beta \), as they are not necessary for the matching because the operator \( \bar{n} \cdot A_n \) is constrained by gauge
invariance to be only in Wilson lines. Now we rotate the amplitude (D34) to the directions
$n_2$ and $n'_2$ parallel to the quark and the two gluons, as described in Eq. (B14)

$$A^{qg}_{NLO, A} = U^{(2,0,0)}(n_0; Q, \mu_1) g^2 \bar{u}_{n_2} \left[ \frac{\bar{q}_2}{\bar{q}_1} \sqrt{n_2 \cdot n'_2} v_2^\beta + \frac{\bar{k}_2}{\bar{q}_1} \sqrt{n_2 \cdot n'_2} \frac{\gamma_2^\beta}{2} \gamma_{n'_2}^\alpha \right]$$ (D35)

$$A^{qg}_{NLO, B} = U^{(2,0,0)}(n_0; Q, \mu_1) g^2 \bar{u}_{n_2} \left[ \frac{\bar{q}_1}{\bar{q}_0} \sqrt{n_1 \cdot n'_1} v_1^\alpha + \frac{\bar{k}_1}{\bar{q}_1} \sqrt{n_2 \cdot n'_2} \frac{\gamma_1^\alpha}{2} \gamma_{n'_2}^\beta \right]$$

$$+ \frac{\bar{k}_1}{\bar{q}_0} \sqrt{n_1 \cdot n'_1} \frac{\gamma_1^\alpha}{2} \gamma_{n'_2}^\beta + \frac{\bar{q}_2}{\bar{q}_1} \frac{\bar{k}_2}{(q_2 + k_1)} \sqrt{n_2 \cdot n'_2} \frac{\gamma_2^\beta}{2}$$

$$A^{qg}_{NLO, C} = U^{(2,0,0)}(n_0; Q, \mu_1) g^2 \bar{u}_{n_2} \left[ \frac{\bar{q}_1}{\bar{q}_0} \sqrt{n_1 \cdot n'_1} v_1^\alpha + \frac{\bar{k}_1}{\bar{q}_1} \sqrt{n_2 \cdot n'_2} \frac{\gamma_2^\beta}{2} \gamma_{n'_2}^\alpha \right]$$

$$+ \frac{\bar{q}_2}{(q_2 + k_1)\bar{q}_0} \sqrt{n_2 \cdot n'_2} \frac{\gamma_2^\beta}{2}$$

and $n_1 \cdot n'_1$ is defined in terms of $n_2$, $n'_2$ and $n'_2$ in Eqs. (B18).

The vectors $v_1$ and $v_2$ are defined in Eqs. (B12), (B17) and (B18). The values of $q_0^2$, $q_1^2$
and $(q_1 + k_2)^2$ are given in Eqs. (B19) and (B20). As with the one-gluon emission, we can
neglect the terms with $n_1^\alpha$ and $n_2^\beta$ as they are orthogonal to the $B_{n'_2}^\alpha$ and $B_{n'_2}^\beta$ fields. The
SCET$_2$ amplitude for $\langle 0|\mathcal{O}_2^{(2)}(n_2, n'_2)|q_{n_2} g_{n'_2} g_{n_2} g_{n'_2}\rangle$ is:

$$\langle 0|\bar{\chi}_{n_2} g B_{n_2}^\alpha g B_{n'_2}^\beta |q_{n_2} g_{n'_2} g_{n_2} g_{n'_2}\rangle = g^2 \bar{u}_{n_2} \gamma_{n'_2}^\alpha \epsilon_{n'_2}^\beta v_{n_2}.$$ (D36)

In Eq. (D36) we have explicitly written the polarization vectors of the external gluons. For
the jet-structure corrections, we get:

$$C_{2,NLO}^{(2)}(n_2, n'_2) = U^{(2,0,0)}(n_0; Q, \mu_1) d_1^{J \alpha \beta}(n_2, n'_1, n'_2) \Theta_{\delta_2}[n_2 \cdot n'_2] \Theta_{\delta_2}[n_2 \cdot n'_2] \Theta_{\delta_2}[n_2 \cdot n'_1],$$ (D37)

where

$$d_1^{J \alpha \beta}(n_2, n'_1, n'_2) = d_{1,A}^{J \alpha \beta}(n_2, n'_1, n'_2) + d_{1,B}^{J \alpha \beta}(n_2, n'_1, n'_2) + d_{1,C}^{J \alpha \beta}(n_2, n'_1, n'_2).$$ (D38)
with
\[
\begin{align*}
\mathcal{O}^{J,\alpha\beta}(n_2, n_1', n_2') &= \left[ \frac{q_2}{q_1} \sqrt{n_2 \cdot n_2' \nu_2} + \frac{k_2}{q_1} \sqrt{n_2 \cdot n_2' \frac{\psi_2}{2} \gamma_{n_2'}^\beta} \right] \\
\times \left[ \frac{q_1}{q_0} \sqrt{n_1 \cdot n_1' \nu_1} + \frac{k_1}{q_1} \sqrt{n_1 \cdot n_1' \frac{\psi_1}{2} \gamma_{n_1'}^\alpha} \right] \frac{q_1 q_0}{q_1^2} \gamma_{n_0}^\mu \, , \\
\mathcal{O}^{J,\alpha\beta}(n_2, n_1', n_2') &= \left[ \frac{q_1}{q_0} \sqrt{n_1 \cdot n_1' \nu_1} + \frac{k_2}{q_1} \sqrt{n_2 \cdot n_2' \frac{\psi_2}{2} \gamma_{n_2'}^\alpha} \right] \\
&+ \frac{k_1}{q_0} \sqrt{n_1 \cdot n_1' \frac{\psi_1}{2} \gamma_{n_1'}^\alpha} + \frac{q_2}{q_1} \frac{k_2}{q_1 (q_2 + k_1)} \sqrt{n_2 \cdot n_2' \gamma_{n_2'}^\alpha} \\
&- \frac{k_1 k_2}{q_0 (q_2 + k_1)} \sqrt{n_1 \cdot n_1' \gamma_{n_1'}^\alpha \frac{\psi_1}{2}} \right] \\
\times \left[ \frac{q_2}{q_1} \sqrt{n_2 \cdot n_2' \nu_2} - \frac{k_1}{q_0} \sqrt{n_1 \cdot n_1' \nu_1} \right] \\
&+ \frac{q_2}{q_1} \frac{k_2}{q_1 (q_2 + k_1) q_1} \sqrt{n_2 \cdot n_2' \frac{\psi_2}{2} \gamma_{n_2'}^\beta} - \frac{k_2}{q_1} \frac{k_1}{q_0} \sqrt{n_1 \cdot n_1' \frac{\psi_1}{2} \gamma_{n_1'}^\beta} \right] \\
\times \frac{q_2 + k_1}{q_1} \frac{q_0}{q_0} \gamma_{n_0}^\mu , \\
\mathcal{O}^{J,\alpha\beta}(n_2, n_1', n_2') &= \left[ \frac{1}{q_2 + k_1} \gamma_{n_1'}^\alpha \gamma_{n_2'}^\beta + \frac{1}{q_2 + k_2} \gamma_{n_2'}^\alpha \gamma_{n_1'}^\beta \right] \frac{q_0}{q_0} \gamma_{n_0}^\mu .
\end{align*}
\]
\[
\text{The } \Theta \text{ functions in Eq. (D37) show that } C_{2, \text{NLO}}^{(2)J}(n_2, n_1', n_2') \text{ comes from the two-jet SCET}_1 \text{ operators. To examine the power counting of } C_{2, \text{NLO}}^{(2)J}(n_2, n_1', n_2'), \text{ we have to consider that this coefficient comes from matching SCET}_1 \text{ to SCET}_2 \text{ in the region where } n_2 \cdot n_1' \sim n_2 \cdot n_2' \sim n_1' \cdot n_2' \sim \lambda^2/\eta^4, \text{ thus we have}
\]
\[
C_{2, \text{NNLO}}^{(2)J}(n_2, n_1', n_2') \sim \lambda^{-2},
\]
\[
\text{and since this multiplies } \mathcal{O}^{(2)}_2 \sim \lambda^8, \text{ by comparison with Eq. (D7) we see that we get an NLO}(\lambda) \text{ contribution.}
\]

We proceed similarly to calculate the coefficient $C_{2, \text{NNLO}}^{(2)J,H,\alpha}(n_2, n_1', n_2')$ and show that it is $\mathcal{O}(\lambda^{-1})$. We decompose the SCET$_1$ amplitude:
\[
A_{\text{NNLO}}^{qgqg} = C_{1, \text{NLO}}^{(1)}(n_0, n_0) \int dx \langle 0 | T \{ \mathcal{L}_{\text{SCET}_1}(x) \mathcal{O}_1^{(1)} \} | q_{n_2} g_{n_1'} g_{n_2'} q_{n_3} \rangle_2 ,
\]
in
\[
A_{\text{NNLO}}^{qgqg} = A_{\text{NNLO}, A}^{qgqg} + A_{\text{NNLO}, B}^{qgqg} ,
\]
where $A, B$ correspond to the two graphs in Fig. 21. We have:
From Eqs. (D36) and (D43) we can see that the Wilson coefficient $C^{(2)}_{H,a, \text{NNLO}}(n_2, n_1, n_2')$ is

$$C^{(2)}_{H,a, \text{NNLO}}(n_2, n_1, n_2') = U^{(2,1,0)}(n_0, n_0; Q, \mu_1) d_1^{H, \alpha \beta}(n_2, n_1, n_2') \times \Theta_{\delta_2[n_2-n_1]} \Theta_{\delta_2[n_2'-n_1]},$$

where

$$d_1^{H, \alpha \beta}(n_2, n_1, n_2') = d_{1,A}^{H, \alpha \beta}(n_2, n_1, n_2') + d_{1,B}^{H, \alpha \beta}(n_2, n_1, n_2')$$

with

$$d_{1,A}^{H, \alpha \beta}(n_2, n_1, n_2') = \left[ \frac{\vec{q}_2}{\vec{q}_2 + \vec{k}_2} \sqrt{n_2 \cdot n_2'} v_2^\beta \right.$$

$$+ \frac{k_2}{q_2 + k_2} \sqrt{n_2 \cdot n_2'} \frac{\phi_2}{2} \frac{\gamma_{n_2', \parallel}^\alpha}{Q} \left. \gamma_{n_1', \parallel}^\alpha \frac{\bar{q}_1}{\bar{q}_1} n_1'^\mu - \bar{n}_\mu \right]$$

$$d_{1,B}^{H, \alpha \beta}(n_2, n_1, n_2') = \left[ \frac{\bar{q}_1}{q_0 (q_2 + k_1)} \sqrt{n_1 \cdot n_1'} v_1^\alpha + \frac{\bar{k}_1 \bar{k}_2}{q_0 (q_2 + k_1)} \sqrt{n_1 \cdot n_1'} \frac{\phi_1}{2} \frac{\gamma_{n_1', \parallel}^\alpha}{Q} \right.$$

$$+ \frac{k_1}{q_0 (q_2 + k_1)} \sqrt{n_1 \cdot n_1'} \gamma_{n_1', \parallel}^\alpha \frac{\bar{q}_1}{\bar{q}_1} n_1'^\mu - \bar{n}_\mu \left. \right].$$
To get the power counting of $C_{2,\text{NNLO}}^{(2)H,a}(n_2, n_1', n_1')$, as in the previous case, we have to consider that the matching is done in a region where $n_2 \cdot n_1' \sim n_2 \cdot n_1' \sim n_1 \cdot n_2' \sim \lambda^2/\eta^4$. This implies:

$$C_{2,\text{NNLO}}^{(2)H,a}(n_2, n_1', n_2') \sim \lambda^{-1},$$

which justifies its labeling as NNLO($\lambda$).

We now turn to calculate the coefficient $C_{2}^{(2)}(n_1, n_1', n_1')$. We will proceed as above. We decompose $C_{2}^{(2)}(n_1, n_1', n_1')$ as:

$$C_{2}^{(2)}(n_2, n_1', n_1') = C_{2,\text{NLO}}^{(2)J}(n_2, n_1', n_1') + C_{2}^{(2)H}(n_2, n_1', n_1'),$$

where

$$C_{2,\text{NLO}}^{(2)J}(n_2, n_1', n_1') = C_{1,\text{LO}}^{(0)}(n_0) \int dx_1 dx_2 \langle 0 | \mathcal{O}_2^{(2)}(n_2, n_1', n_1') | q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}} \rangle_2$$

and

$$C_{2}^{(2)H}(n_2, n_1', n_1') = C_{1,\text{NLO}}^{(1)}(n_0, n_0) \int dx \langle 0 | \mathcal{L}_{\text{SCET1}}(x) \mathcal{L}_{\text{SCET1}}(x_2) \mathcal{O}_1^{(0)}(n_0) | q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}} \rangle_2 + C_{1,\text{T}}^{(1)}(n_0, n_0) \int dx \langle 0 | \mathcal{L}_{\text{SCET1}}(x) \mathcal{T}_1^{(1)} | q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}} \rangle_2.$$  

We further set:

$$C_{2}^{(2)H}(n_2, n_1', n_1') = C_{2,\text{NNLO}}^{(2)H}(n_2, n_1', n_1') + C_{2,\text{NLO}}^{(2)H}(n_2, n_1', n_1'),$$

where $C_{2,\text{NLO}}^{(2)H}(n_2, n_1', n_1')$ is the coefficient of the contribution that reproduces the second line in Eq. (D50), etc. We will only calculate $C_{2,\text{NLO}}^{(2)J}(n_2, n_1', n_1')$ and show that it scales as $\lambda^{-2}$. This is the only operator of this form that we need to calculate the amplitude squared at NLO($\lambda$).

To calculate the amplitude on the RHS of Eq. (D49), we can use Eqs. (D35), which are written in terms of $n_2, n_1'$ and $n_1'$ that are parallel to the external particles, and take the limit $n_2' \cdot n_1' \rightarrow \lambda^2/\eta^4$. In this case the two gluons are collinear in SCET$_2$. Thus, we can define $C_{2,\text{NLO}}^{(2)J}(n_2, n_1', n_1')$ as

$$C_{2,\text{NLO}}^{(2)J}(n_2, n_1', n_1') = U^{(2,0,0)}(n; Q, \mu_1) d_2^{J \alpha \beta}(n_2, n_1', n_1') \Theta_2[n_2 \cdot n_1'],$$
where

\[ d^{\alpha \beta}_{2}(n_{2}, n'_{1}, n'_{2}) = \lim_{n'_{2} \rightarrow n'_{1}, n'_{2} \rightarrow n_{2}} q^{\alpha \beta}(n_{2}, n'_{1}, n'_{2}) \]  \(\text{(D53)}\)

\[ = \left( \frac{\bar{q}_{2}}{q_{1}} \sqrt{n_{2} \cdot n_{2}^{\prime} v_{2}^{\beta}} + \frac{\bar{k}_{2}}{q_{1}} \sqrt{n_{2} \cdot n_{2}^{\prime} \frac{\gamma_{n_{2}^{\prime}}}{2}} \right) \times \left[ \frac{q_{1}}{\bar{q}_{0}} \sqrt{n_{1} \cdot n_{1}^{\prime} v_{1}^{\alpha}} + \frac{\bar{k}_{1}}{q_{0}} \sqrt{n_{1} \cdot n_{1}^{\prime} \frac{\gamma_{n_{1}^{\prime}}}{2}} \right] \frac{2\bar{q}_{0}}{q_{1}^{2} \bar{q}_{2} k_{1}(n_{2}, n_{1}^{\prime}) + \bar{q}_{2} k_{2}(n_{2}, n_{2}^{\prime})} + \frac{q_{0}}{\bar{q}_{0}} \sqrt{n_{1} \cdot n_{1}^{\prime} v_{1}^{\alpha}} + \frac{\bar{k}_{1}}{q_{1}} \sqrt{n_{1} \cdot n_{1}^{\prime} \frac{\gamma_{n_{1}^{\prime}}}{2}} \right]

\[ + \frac{\bar{k}_{1}}{q_{0}} \sqrt{n_{1} \cdot n_{1}^{\prime} \frac{\gamma_{n_{1}^{\prime}}}{2}} \frac{2\bar{q}_{0}}{q_{1}^{2} \bar{q}_{2} k_{1}(n_{2}, n_{1}^{\prime}) + \bar{q}_{2} k_{2}(n_{2}, n_{2}^{\prime})} \times \gamma_{n_{0}^{\prime}} \bigg|_{n_{1}^{\prime}=n_{2}^{\prime}} \]  \(\text{(D53)}\)

In Eqs. (D53) there is a difference in the notation between the LHS and RHS. On the LHS, we have labeled the quark with \(n_{2}\) and the two gluons with \(n'_{1}\) because the coefficient \(\Theta\) is for the operator \(C_{2}^{(2)}(n_{2}, n'_{1}, n'_{2})\), where the gluons are collinear. On the RHS, \(n_{2}, n'_{1}\) and \(n'_{2}\) are the directions parallel to the quarks and the two gluons as defined in Eqs. (B10) and (B15). We encode that the two gluons are collinear using the \(\Theta\) function on the RHS of Eq. (D53) with \(\delta_{3} = \lambda^{3}/\eta^{4}\). It restricts that \(n'_{1} \cdot n'_{2} \lesssim \lambda^{4}/\eta^{4}\). On the RHS of Eq. (D53) we could decompose \(n'_{2}\) in terms of \(n'_{1}\) and avoid inserting the \(\Theta\), but it is convenient to leave \(n'_{2}\) explicit because it will make the matching easier to SCET_{3}. We notice that the RHS of Eq. (D53) is just equal to the coefficient \(C_{2,\text{NLO}}^{(2)}(n_{2}, n'_{1}, n'_{2})\) defined in Eq. (D37) with the substitution \(q_{0}^{2} \rightarrow \bar{q}_{2} k_{1}(n_{2}, n'_{1})/4 + \bar{q}_{2} k_{2}(n_{2}, n'_{2})/4\). Knowing that \(n'_{1} \cdot n'_{2} \sim \lambda^{4}/\eta^{4}\), \(n_{1} \cdot n'_{2} \sim \lambda^{2}/\eta^{4}\) and \(n_{1} \cdot n'_{1} \sim \lambda^{2}/\eta^{4}\), it is easy to check that Eq. (D53) scales as \(\lambda^{-2}\). The information that \(C_{2,\text{NLO}}^{(2)}(n_{2}, n'_{1}, n'_{2})\) comes from a two-jet SCET_{1} operator, is encoded in the \(\Theta\)-functions of Eq. (D52).

For the coefficient \(C_{2}^{(2)}(n_{2}, n'_{1}, n_{2})\), we decompose it as:

\[ C_{2}^{(2)}(n_{2}, n'_{1}, n_{2}) = C_{2,\text{NLO}}^{(2)}(n_{2}, n'_{1}, n_{2}) + C_{2}^{(2)H}(n_{2}, n'_{1}, n_{2}), \]  \(\text{(D54)}\)
where

\[ C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \langle 0 | O_2^{(2)} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 \]

\[ = C^{(0)}_{1,\text{LO}}(n_0) \int dx \langle 0 | T \{ L_{\text{SCET}_1}(x_1) L_{\text{SCET}_1}(x_2) O_1^{(0)} \} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 \]

\[ - C^{(1)}_{2,\text{LO}}(n_1, n'_1) \int dx \langle 0 | T \{ L_{\text{SCET}_2}(x) O_2^{(1)} \} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 , \]

and

\[ C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \langle 0 | O_2^{(2)} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 \]

\[ = C^{(1)}_{1,\text{LO}}(n_0, n_0) \int dx \langle 0 | T \{ L_{\text{SCET}_1}(x) O_1^{(1)} \} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 \]

\[ - C^{(2)}_{2,\text{NLO}}(n_2, n'_1) \int dx \langle 0 | T \{ L_{\text{SCET}_2}(x) O_2^{(1)} \} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 \]

\[ + C^{(1)}_{1,\text{LO}}(n_0, n_0) \int dx \langle 0 | T \{ L_{\text{SCET}_1}(x) T^{(1)}_1 \} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 \]

\[ - C^{(2)}_{2,\text{NLO}}(n_2, n'_1) \int dx \langle 0 | T \{ L_{\text{SCET}_2}(x) O_2^{(1)} \} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 , \]

We write \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) as

\[ C^{(2)}_{2}(n_2, n'_1, n_2) = C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) + C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) , \]

where \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) is the coefficient of the contribution that reproduces the second and third line in the Eq. \( \text{(D55)} \), and \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) as \( \lambda^{-2} \), \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) as \( \lambda^{-1} \) and \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) as \( \lambda^0 \). Since \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) interferes with the LO operator, \( C^{(1)}_{2,\text{NLO}}(n_1, n'_1) \), to have the amplitude squared up to NLO(\( \lambda \)) we need both \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) and \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \). We start with \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \). To calculate the amplitude in the second line in Eq. \( \text{(D55)} \), we use Eq. \( \text{(D35)} \) and take the limit \( n_2 \cdot n'_2 \rightarrow \lambda^4/\eta^4 \) with \( n_2 \cdot n'_1 \sim n'_2 \cdot n'_2 \sim \lambda^2/\eta^4 \). (We could alternatively take the limit \( n_2 \cdot n'_1 \rightarrow \lambda^4/\eta^4 \) with \( n_2 \cdot n'_2 \sim n'_2 \cdot n'_1 \sim \lambda^2/\eta^4 \).) It is easy to check that

\[ \lim_{n_2 \rightarrow n'_2 \rightarrow \lambda^4/\eta^4} d_{1,A}^{J \alpha \beta}(n_2, n'_1, n'_2) \langle 0 | O_2^{(2)} | q_{n_2} g_{n'_1} q_{m_2} \rangle \]

\[ = C^{(1)}_{2,\text{LO}}(n_2, n'_1) \int dx \langle 0 | T \{ L_{\text{SCET}_2}(x) O_2^{(1)} \} | q_{n_2} g_{n'_1} q_{m_2} \rangle_2 . \]  \( \text{(D58)} \)

With Eq. \( \text{(D58)} \), we can write \( C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) \) as

\[ C^{(2)}_{2,\text{NLO}}(n_2, n'_1, n_2) = U^{(2,0,0)}(n_0; Q, \mu_1) d_{3}^{J \alpha \beta}(n_2, n'_1, n_2) \Theta \delta_{2} [n_2 \cdot n'_1] \]  \( \text{(D59)} \)
where

\[
d_3^{\alpha\beta}(n_2, n_1', n_2') = \lim_{n_2, n_2' \to \lambda^4/\eta^4} \left( d_{1B}^{\alpha\beta}(n_2, n_1', n_2') + d_{1C}^{\alpha\beta}(n_2, n_1', n_2') \right)
\]

\[
= \left( \frac{q_1}{\eta_0} \sqrt{n_1 \cdot n_1'} v_1^\alpha + \frac{k_1}{\eta_0} \sqrt{n_1 \cdot n_1'} \frac{g_1^\alpha}{2} n_1'^\perp \right.
\]

\[
- \frac{k_1 k_2}{\eta_0 (q_2 + k_1)} \sqrt{n_1 \cdot n_1'} \gamma_0^\alpha n_1'^\perp \frac{g_1}{2} 
\]

\[
\times \left[ - \frac{k_1}{\eta_0} \sqrt{n_1 \cdot n_1'} v_1^\beta - \frac{k_2 k_1}{(q_2 + k_1) \eta_0} \sqrt{n_1 \cdot n_1'} \frac{g_1^\beta}{2} n_1'^\perp \right.
\]

\[
+ \left[ \frac{1}{q_2 + k_1} \gamma_0^\alpha n_1'^\perp \gamma_0^\beta n_1'^\perp + \frac{1}{q_2 + k_2} \gamma_0^\beta n_1'^\perp \gamma_0^\alpha n_1'^\perp \right] \gamma_n^\perp 
\]

\[
\times \frac{2 \eta_0}{k_1 (k_2 (n_2 \cdot n_1') + q_2 (n_2 \cdot n_1'))} \bigg|_{n_2 = n_2'}.
\]

The scaling of the dot products of \( n \)'s in this configuration make the coefficient \( C_{2,\text{NNLO}}^{(2)J}(n_2, n_2, n_1') \sim \lambda^{-2} \). As previously for \( C_{2,\text{NNLO}}^{(2)J}(n_1, n_1', n_1') \), we prefer leaving \( (D59) \) in terms of \( n_2, n_1' \) and \( n_2' \). To calculate \( C_{2,\text{NNLO}}^{(2)H}(n_2, n_2, n_1') \) we proceed in the same way. We have

\[
C_{2,\text{NNLO}}^{(2)H}(n_2, n_2, n_1') = U^{(2,1,0)}(n_0, n_0; Q, \mu_1) a_3^{H\alpha\beta}(n_2, n_1', n_2) \Theta_{\delta_2}[n_2 \cdot n_1'],
\]

where

\[
a_3^{H\alpha\beta}(n_2, n_1', n_2') = \lim_{n_2, n_2' \to \lambda^4/\eta^4} d_{1B}^{H\alpha\beta}(n_2, n_1', n_2')
\]

\[
= \left[ \frac{q_1}{\eta_0} \sqrt{n_1 \cdot n_1'} v_1^\alpha + \frac{k_1}{\eta_0} \sqrt{n_1 \cdot n_1'} \frac{g_1^\alpha}{2} n_1'^\perp \right.
\]

\[
- \frac{k_1 k_2}{\eta_0 (q_2 + k_1)} \sqrt{n_1 \cdot n_1'} \gamma_0^\alpha n_1'^\perp \frac{g_1}{2} 
\]

\[
\times \left[ - \frac{k_1}{\eta_0} \sqrt{n_1 \cdot n_1'} v_1^\beta - \frac{k_2 k_1}{(q_2 + k_1) \eta_0} \sqrt{n_1 \cdot n_1'} \frac{g_1^\beta}{2} n_1'^\perp \right.
\]

\[
+ \left[ \frac{1}{q_2 + k_1} \gamma_0^\alpha n_1'^\perp \gamma_0^\beta n_1'^\perp + \frac{1}{q_2 + k_2} \gamma_0^\beta n_1'^\perp \gamma_0^\alpha n_1'^\perp \right] \gamma_n^\perp 
\]

\[
\times \frac{2 \eta_0}{k_1 (k_2 (n_2 \cdot n_1') + q_2 (n_2 \cdot n_1'))} \bigg|_{n_2 = n_2'}.
\]

In Eq. \( (D61) \) we use the fact that,

\[
\lim_{n_2, n_2' \to \lambda^4/\eta^4} d_{1, \alpha/\eta}^{H\alpha\beta}(n_2, n_1', n_2') \langle 0 | \mathcal{O}_2^{(2)}(2) | q_{n_2} g_{n_2} g_{n_2} q_{n_2} \rangle_2
\]

\[
= C_{2,\text{NNLO}}^{(1)} \int dx \langle 0 | \mathcal{L}_{\text{SCET}_2}(x) \mathcal{O}_2^{(1)}(1) | q_{n_2} g_{n_2} g_{n_2} q_{n_2} \rangle_2.
\]

In Eqs. \( (D60), (D62) \), there is again a difference in the notation between the LHS and RHS similar to Eq. \( (D59) \). Since \( C_{2,\text{NNLO}}^{(2)H}(n_2, n_1', n_2) \) and \( C_{2,\text{NNLO}}^{(2)H}(n_2, n_1', n_2) \) come from SCET, two-jet operators, we include the appropriate \( \Theta \)-functions in Eqs. \( (D59), (D61) \).

We have that all the NLO(\( \lambda \)) terms for two gluon matching come from the SCET\( \text{I} \) operator, \( \mathcal{O}_{1}^{(0)}(n_0) \), and are jet-structure corrections. At NNLO(\( \lambda \)) we have only hard corrections. Before matching SCET\( \text{II} \) to SCET\( \text{III} \), we have to insert in the coefficients the SCET\( \text{II} \) running factors. Below we list all the needed SCET\( \text{II} \) coefficients to NNLO(\( \lambda \)) that we have calculated.
with the appropriate RG kernels. From the matching of one-gluon emission, we have:

\[ C^{(1)}_{2,\text{LO}}(n_1, n'_1) = U^{(2,1,0)}(n_1, n'_1; \mu_1, \mu) U^{(2,0,0)}(n_0; Q, \mu_1) c_{\text{LO}}(n_0) \frac{\bar{q}_0}{q_0} \gamma_{n_0,1}, \tag{D64} \]

\[ C^{(1)H,a}_{2,\text{NLO}}(n_1, n'_1) = U^{(2,1,0)}(n_1, n'_1; \mu_1, \mu) U^{(2,0,0)}(n_0, n_0; Q, \mu_1) \otimes c^{H,a}_{2,\text{NLO}}(n_0, n_0), \]

\[ C^{(1)H}_{2,\text{NNLO}}(n_1, n'_1) = U^{(2,1,0)}(n_1, n'_1; \mu_1, \mu) U^{(2,0,0)}(n_0, n_0; Q, \mu_1) \otimes c^H_{2,\text{NNLO}}(n_0, n_0), \]

\[ C^{(1)H,b}_{2,\text{NLO}}(n_1, n'_1) = U^{(2,1,0)}(n_1, n'_1; \mu_1, \mu) C^{(1)}(n_1, n'_1) \tilde{\Theta}_{\delta_2}[n_1 \cdot n'_1], \]

where the coefficient in [D64] without the SCET\(_2\) RG-kernel is defined in Eq. (D11), the second and third in Eqs. (D17), and the last in (D19). From the matching of two-gluon emission we have the coefficients:

\[ C^{(2)J}_{2,\text{NLO}}(n_2, n'_1, n'_2) = U^{(2,1,0)}(n_2, n'_1, n'_2; \mu_1, \mu) U^{(2,0,0)}(n_0; Q, \mu_1) \]

\[ \times d^J_2(n_2, n'_1, n'_2) \Theta_{\delta_2}[n_2 \cdot n'_1] \Theta_{\delta_2}[n_2 \cdot n'_2], \tag{D65} \]

\[ C^{(2)J}_{2,\text{NNLO}}(n_2, n'_1, n'_1) = U^{(2,1,0)}(n_2, n'_1, n'_1; \mu_1, \mu) U^{(2,0,0)}(n_0; Q, \mu_1) \]

\[ \times d^J_2(n_2, n'_1, n'_1) \Theta_{\delta_2}[n_2 \cdot n'_1] \Theta_{\delta_2}[n_2 \cdot n'_2], \]

\[ C^{(2)J}_{2,\text{NNLO}}(n_2, n'_1, n'_2) = U^{(2,1,0)}(n_2, n'_1, n'_2; \mu_1, \mu) U^{(2,0,0)}(n_0, n_0; Q, \mu_1) \]

\[ \times d^J_2(n_2, n'_1, n'_2) \Theta_{\delta_2}[n_2 \cdot n'_1] \Theta_{\delta_2}[n_2 \cdot n'_2], \]

\[ C^{(2)H}_{2,\text{NNLO}}(n_2, n'_1, n'_2) = U^{(2,1,0)}(n_2, n'_1, n'_2; \mu_1, \mu) U^{(2,0,0)}(n_0, n_0; Q, \mu_1) \]

\[ \otimes d^H_3(n_2, n'_1, n'_2) \Theta_{\delta_2}[n_2 \cdot n'_1] \Theta_{\delta_2}[n_2 \cdot n'_2], \]

where the coefficients without SCET\(_2\) running are defined in Eqs. (D37, D52, D59, D61). The RG kernels are given in Eqs. (D18, D19, and D20). As discussed below Eq. (89), we have a convolution because SCET fields collinear to the same direction can exchange longitudinal momentum during the running.

**Appendix E: Matching SCET\(_2\) to SCET\(_3\), SCET\(_N\)**

We match SCET\(_2\) to SCET\(_3\) before proceeding to the general case and listing a set of master operators for SCET\(_N\). The SCET\(_3\) operators necessary for matching up to two-gluon emission are: \(O_3^{(0)}(n_0)\), \(O_3^{(1)}(n_0, n_0)\), \(O_3^{(1)}(n_1, n'_1)\), \(O_3^{(2)}(n_2, n'_1, n'_2)\), \(O_3^{(2)}(n_2, n'_2, n'_1)\), \(O_3^{(2)}(n_2, n'_1, n'_1)\). We have seen that to describe the parton shower for one emission, we only need the coefficient of the SCET\(_2\) operator, \(O_3^{(1)}(n_1, n'_1)\). Similarly, in SCET\(_3\) we need the coefficient of the operator \(O_3^{(2)}(n_2, n'_1, n'_2)\). We can follow the same steps from App. [D] to calculate the Wilson coefficients \(C_3^{(2)}(n_2, n'_1, n'_2)\). In this way, it is not difficult to show that

\[ C_3^{(2)}(n_2, n'_1, n'_2) = C_3^{(2)}(n_2, n'_1, n'_2) \]

\[ + C_3^{(2)H,a} + C_3^{(2)H,b} + C_3^{(2)J,a} + C_3^{(2)J,b} + C_3^{(2)J,c}, \tag{E1} \]

\[ + C_3^{(2)H,a} + C_3^{(2)H,b}, \]

where

\[ C_3^{(2)J}(n_2, n'_1, n'_2) = C^{(1)}_{3,\text{LO}}(n_2, n'_1) C_2^{(1)}(n_1, n'_1), \tag{E2} \]
\[ C^{(2)H,a}_{3,NLO}(n_2, n'_1, n'_2) = C^{(1)}_{3,LO}(n_2, n'_2) C^{(1)H,a}_{2,NLO}(n_1, n'_1), \]
\[ C^{(2)H,b}_{3,NLO}(n_2, n'_1, n'_2) = C^{(1)}_{3,LO}(n_2, n'_2) C^{(1)H,b}_{2,NLO}(n_1, n'_1), \]
\[ C^{(2)J,1}_{3,NLO}(n_2, n'_1, n'_2) = C^{(2)J}_{2,NLO}(n_2, n'_1, n'_2) \tilde{\Theta}_{\delta_3} [n_2 \cdot n'_2] \tilde{\Theta}_{\delta_3} [n_2 \cdot n'_1] \tilde{\Theta}_{\delta_3} [n'_2 \cdot n'_1], \]
\[ C^{(2)J,2}_{3,NLO}(n_2, n'_1, n'_2) = C^{(2)J}_{2,NLO}(n_2, n'_1, n'_2) \tilde{\Theta}_{\delta_3} [n_2 \cdot n'_2] \tilde{\Theta}_{\delta_3} [n_2 \cdot n'_1] \tilde{\Theta}_{\delta_3} [n'_2 \cdot n'_1], \]
\[ C^{(2)J,3}_{3,NLO}(n_2, n'_1, n'_2) = C^{(2)J}_{2,NLO}(n_2, n'_1, n'_2) \Theta_{\delta_3} [n_2 \cdot n'_2] \tilde{\Theta}_{\delta_3} [n_2 \cdot n'_1] \tilde{\Theta}_{\delta_3} [n'_2 \cdot n'_1], \]
\[ C^{(2)H,a}_{3,NNLO}(n_2, n'_1, n'_2) = C^{(1)}_{3,LO}(n_2, n'_2) C^{(1)H,a}_{2,NNLO}(n_1, n'_1), \]
\[ C^{(2)H,b}_{3,NNLO}(n_2, n'_1, n'_2) = C^{(2)H}_{2,NNLO}(n_2, n'_1, n'_2) \Theta_{\delta_3} [n_2 \cdot n'_2] \tilde{\Theta}_{\delta_3} [n_2 \cdot n'_1] \tilde{\Theta}_{\delta_3} [n'_2 \cdot n'_1], \]
and
\[ C^{(1)}_{3,LO}(n_2, n'_2) = \left( \frac{2}{k_2} \left( \frac{q_2}{k_2} n_{1\perp} \right) + \frac{\left( \frac{\hat{q}_2}{q_2} n_{1\perp} \right) \gamma_{n_{1\perp}}}{q_2} \right) \frac{\hat{q}_1 \hat{p}_1}{4} \Theta_{\delta_3} [n_2 \cdot n'_2]. \]  

On the LHS of the equations in the first, second and third line of (E2) we can write \( n_1 \) in terms of \( n_2, n'_2 \) and \( n'_1 \) using the formulas in (B16). The SCET_2 coefficients \( C^{(2)J}_{2,NLO} \) and \( C^{(2)H}_{2,NNLO} \) are defined in Eqs. (D65). \( C^{(1)H,a}_{2,NLO}(n_1, n'_1), C^{(1)H,b}_{2,NLO}(n_1, n'_1), \) and \( C^{(1)H}_{2,NNLO}(n_1, n'_1) \) are given in Eq. (D64), and \( C^{(1)}_{2,LO}(n_1, n'_1) \) in Eq. (52). As with any SCET_i \( \rightarrow \) SCET_{i+1} matching, we encode the definition of collinearity from the higher scale theory in the lower one by \( \Theta \) functions (cf. discussion in Sec. IIIA). Some of the SCET_2 coefficients above already contained such factors as a result of matching to SCET_1. In Eq. (E2), we write out the new ones that appear with \( \Theta_{\delta_3} \), with \( \delta_3 = \lambda^3/\eta^4 \) according to our usual convention. Since all the coefficients above multiply \( O_3^{(2)} \), the scaling of contributions comes from them alone, with \( C^{(2)}_{3,LO} \sim \lambda^{-3} \), the NLO terms \( \sim \lambda^{-2} \), and NNLO going as \( \lambda^{-1} \).

At LO, the contribution in SCET_3 is given by the replacement procedure on the LO contribution in SCET_2, \( C^{(1)}_{2,LO}(n_1, n'_1) \tilde{O}_2^{(1)} \). We multiply it by the running function \( U^{(1)}(n_1, n'_1; \mu_1, \mu) \) and apply the replacement:

\[ \langle \bar{\chi}_{n_2} \rangle_i \rightarrow \left( c^{(i)}_{LO}(n_1) \right)_{ji} \langle \bar{\chi}_{n_1} \rangle_j g B^{n'_2}_{\alpha} \]  

where \( c^{(i)}_{LO}(n_1) \) is

\[ c^{(i)}_{LO}(n_1) = \left( \frac{2}{k_2} \left( \frac{q_2}{k_2} n_{1\perp} \right) + \frac{\left( \frac{\hat{q}_2}{q_2} n_{1\perp} \right) \gamma_{n_{1\perp}}}{q_2} \right) \frac{\hat{q}_1 \hat{p}_1}{4} \Theta_{\delta_3} [n_2 \cdot n'_2]. \]  

Eq. (E4) has the same structure as Eq. (D15). If we go on with the matching down to SCET_N, we find that the LO result would be given by applying the above replacement \( N-1 \) times. At SCET_N we could match everything to the operator \( O_3^{(N-1)}(n_{N-1}, n'_1, \ldots, n'_{N-1}) \), and the LO coefficient is

\[ C^{(N-1)}_{N,LO} = \prod_{k=1}^{N-1} U^{(2,k-1,0)}(n_{k-1}, n'_1, \ldots, n'_{k-1}; \mu_{k-1}, \mu_k) c^{(k)}_{LO}(n_{k-1}) \Gamma^\mu, \]  

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with \( \mu_k \sim (k^2_n)_{n-1} \) given in Eq. (64) and
\[
c^\alpha_{\text{LO}}(n_k) = \left( 2 \frac{(q_{k+1})^{\alpha}}{k^2_{k+1}} + \frac{(q_{k+1})_{n+1} \gamma_{n+1}^{\alpha}}{q_{k+1}} \right) \frac{\phi_{k+1}}{4} \Theta_k \left[ n_{k+1}, n_{k+1}^\prime \right],
\]
where \( \delta_k = \lambda^{2k-3} / \eta^4 \).

At NLO(\( \lambda \)), we have two kinds of corrections: hard-scattering and jet-structure. We notice that the NLO(\( \lambda \)) hard-scattering terms in SCET\(_3\) are just given by those in SCET\(_2\) with the application of the replacement rule (E4). If we go on with the matching down to SCET\(_N\), we find that we get NLO(\( \lambda \)) hard-scattering by applying the above replacement rule \( N - 2 \) times to the SCET\(_2\) hard-scattering operators. Thus, we can consider this as a correction to the matrix elements that we pass to a LL shower:
\[
C^{(N-1)H}_{N,NLO} = \left( C^{(1)H,a}_{2,\text{NLO}}(n_1, n_1^\prime) + C^{(1)H,b}_{2,\text{NLO}}(n_1, n_1^\prime) \right) \\
\times \left( \prod_{k=2}^{N-1} U^{(2,k-1,0)}(n_{k-1}, n_{k-1}^\prime, \ldots, n_{k-1}^\prime; \mu_{k-1}, \mu_k) c^\alpha_{\text{LO}}(n_{k-1}) \right).
\]

This approach also works for hard-scattering at NNLO(\( \lambda \)). Since we did not get \( C^{(2)H,a}_{3,\text{NNLO}} \) from a replacement rule, it contains one less factor of \( c^\alpha_{\text{LO}} \).
\[
C^{(N-1)H}_{N,\text{NNLO}} = C^{(1)H,a}_{2,\text{NNLO}}(n_1, n_1^\prime) \left( \prod_{k=2}^{N-2} U^{(2,k-1,0)}(\mu_k, n_{k-1}, n_{k-1}^\prime; \mu_k) c^\alpha_{\text{LO}}(n_{k-1}) \right) \\
+ C^{(1)H,b}_{2,\text{NNLO}}(n_2, n_1^\prime, n_2^\prime) \left( \prod_{k=3}^{N-3} U^{(2,k-1,0)}(\mu_k, n_{k-1}, n_{k-1}^\prime; \mu_k) c^\alpha_{\text{LO}}(n_{k-1}) \right),
\]
where the coefficients \( C^{(1)H,a}_{2,\text{NNLO}}(n_1, n_1^\prime) \) and \( C^{(1)H,b}_{2,\text{NNLO}}(n_2, n_1^\prime, n_2^\prime) \) are defined in Eqs. (D38, D53, D60).

The NLO(\( \lambda \)) jet-structure corrections in SCET\(_3\) are given by \( C^{(2)I,1}_{3,\text{NLO}}(n_2, n_1^\prime, n_2^\prime) \mathcal{O}_3(2) \), where \( I = \{1, 2, 3\} \), are given by the LO SCET\(_1\) operator \( \bar{\chi}_{n_0} \gamma^\mu \chi_{n_0} \) in three steps: First, we multiply it by the running factor \( U^{(1)}(n_0; Q, \mu_1) \), second, we apply the replacements
\[
(\bar{\chi}_{n_2}); \rightarrow (h^\alpha_i)_{ji}(n_2, n_1^\prime, n_2^\prime)(\bar{\chi}_{n_1}_j) g B_{\alpha}^{n^\prime_1 \rightarrow} g B_{\beta}^{n^\prime_2 \rightarrow},
\]
where
\[
h^\alpha_i(n_2, n_1^\prime, n_2^\prime) = d^\alpha_i(n_2, n_1^\prime, n_2^\prime) \Theta_3[n_2 \cdot n_1^\prime] \Theta_3[n_2 \cdot n_1^\prime] \Theta_3[n_2^\prime \cdot n_1^\prime],
\]
\[
= d^\alpha_i(n_2, n_1^\prime, n_2^\prime) \Theta_3[n_2 \cdot n_1^\prime] \Theta_3[n_2 \cdot n_1^\prime] \Theta_3[n_2^\prime \cdot n_1^\prime],
\]
\[
= d^\alpha_i(n_2, n_1^\prime, n_2^\prime) \Theta_3[n_2 \cdot n_1^\prime] \Theta_3[n_2 \cdot n_1^\prime] \Theta_3[n_2^\prime \cdot n_1^\prime].
\]
at any matching SCET$_i$ to SCET$_{i+2}$. Thus, the NLO($\lambda$) jet-structure coefficients for the SCET$_N$ operator are:

$$C^{(N-1)J}_{N,NLO} = \sum_{l=1}^{N-2} C^{(N)J}_{N,NLO}(l),$$  \hspace{1cm} (E12)

where

$$C^{(N-1)J}_{N,NLO}(l) = \sum_{l=1}^{3} \left[ \left( \prod_{k=1}^{l-1} U^{(2,k-1,0)}(n_{k-1}, n'_1, \ldots, n'_k; \mu_k, \mu_k) c_{LO}^{\alpha_k}(n_{k-1}) \right) \right] \times \left[ U^{(l+1)}_I(\mu_l, \mu_{l+1}) \otimes h_1^{\alpha\beta}(n_{l+1}, n'_1, n'_{l+1}) \times \left( \prod_{k=l+1}^{N-1} U^{(2,k-1,0)}(n_{k-1}, n'_1, \ldots, n'_{k-1}; \mu_k, \mu_k) c_{LO}^{\alpha_k}(n_{k-1}) \right) \right] \Gamma^\mu,$$  \hspace{1cm} (E13)

with

$$U^{(l+1)}_I(\mu_l, \mu_{l+1}) = U^{(2,l+1,0)}(n_{l+1}, n'_1, \ldots, n'_l, n'_{l+1}; \mu_l, \mu_{l+1}),$$  \hspace{1cm} (E14)

$$U^{(l+1)}_I(\mu_l, \mu_{l+1}) = U^{(2,l+1,0)}(n_{l+1}, n'_1, \ldots, n'_l, n'_{l+1}; \mu_l, \mu_{l+1}),$$  \hspace{1cm} (E15)

and

$$h_1^{\alpha\beta}(n_{l+1}, n'_1, n'_{l+1}) = d_1^{\alpha\beta}(n_{l+1}, n'_1, n'_{l+1}) \tilde{\Gamma}_{\delta_{l+1}}[n_{l+1} \cdot n'_1] \tilde{\Gamma}_{\delta_{l+1}}[n_{l+1} \cdot n'_1],$$

$$h_1^{\alpha\beta}(n_{l+1}, n'_1, n'_{l+1}) = d_2^{\alpha\beta}(n_{l+1}, n'_1, n'_{l+1}) \tilde{\Gamma}_{\delta_{l+1}}[n_{l+1} \cdot n'_1] \tilde{\Gamma}_{\delta_{l+1}}[n_{l+1} \cdot n'_1],$$

$$h_1^{\alpha\beta}(n_{l+1}, n'_1, n'_{l+1}) = d_3^{\alpha\beta}(n_{l+1}, n'_1, n'_{l+1}) \tilde{\Gamma}_{\delta_{l+1}}[n_{l+1} \cdot n'_1] \tilde{\Gamma}_{\delta_{l+1}}[n_{l+1} \cdot n'_1].$$  \hspace{1cm} (E16)

The coefficients $d_I^{\alpha\beta}$ here are equal to the coefficients $d_I^{\alpha\beta}$ defined in Eqs. (D38, D53, D60) upon the substitution $(n_2, n'_1, n'_2) \rightarrow (n_{l+1}, n'_1, n'_{l+1})$ and $\delta_3 \rightarrow \delta_{l+1}$.

**Appendix F: $O(\alpha_s^2)$ Correction to Splitting Function**

One of the cross-checks on our results is the rederivation of (the abelian part of) the $O(\alpha_s^2)$ correction to the $q \rightarrow qq$ splitting function, $P_{qq}^{(1)}$. This follows from obtaining the NLO($\lambda$) correction to two-gluon emission. For comparison, we have chosen the classic result of Curci et al. [62]. The full expression for $P_{qq}^{(1)}$ involves many real and virtual contributions. Here we will only explicitly calculate the $\sim C_F^2$ component of $P_{qq}^{(1)}$ and show it agrees. (Obtaining the full result requires additional non-abelian diagrams.) Ref. [62] splits the abelian, two-gluon, real emission contributions to $P_{qq}^{(1)}$ into two topologically inequivalent diagrams, the box and crossed graphs, Fig. 22. We calculated each of these individually.

The SCET$_1$ amplitude contains three graphs for two-gluon emission. These are shown in Fig. 20 and we give the corresponding amplitudes in Eqs. (D34). In order to obtain $P_{qq}^{(1)}$, we will need to square the amplitudes and partially integrate over phase space. Thus, we need to choose an explicit kinematics. We redraw, in Fig. (23), our vector labels for
two-gluon emission. We choose a somewhat nonstandard assignment for our variables. This is to aid in the comparison with [62]. The final state parton shower occurs for timelike virtual particles, and momentum fractions decrease the farther we are from the initial hard scattering. By contrast, [62] considered a DIS-type process where the shower is spacelike. Since the radiation in that case comes from initial states, the momentum fractions decrease toward the hard interaction. Only at LO in $\alpha_s$ are the spacelike and timelike splitting functions equal, by the Gribov-Lipatov relation [77]. At higher orders, this gets violated, but there is a straightforward conversion procedure, detailed in [62, 78]. We, however, choose our kinematics such that our variable relations are equivalent to those for a spacelike process. For example, $P_{q\bar{q}}^{(1)}$ is a function of $x \equiv \bar{q}_0/\bar{q}_2$. In a spacelike process, $x \in [0, 1]$. Rather than convert our answer, we will also define $x$ as above, even though this means for us $x \in [1, \infty)$. Other integration variables will have their ranges shifted so that they have the same relation with $x$ as in DIS, and thus they enter into our expression in the same way. Lastly, we do not do the phase space integration for $q_2$. While this is necessary for the timelike splitting function, the analogous particle for a spacelike process is a fixed initial state. Thus, for comparison purposes, we can leave it undone. Our vectors are as follows (note that this is
a different frame from the one used previously for matching):

\[ q_2 = \{ p, 0, 0, p \} \]

\[ k_1 = \left\{ -z_1 p - \frac{k_{1 \perp}^2}{4p z_1}, k_{1 \perp} \cos(\phi_1), k_{1 \perp} \sin(\phi_1), -z_1 p + \frac{k_{1 \perp}^2}{4p z_1} \right\} \]

\[ k_2 = \left\{ -z_2 p - \frac{k_{2 \perp}^2}{4p z_2}, k_{2 \perp}, 0, -z_2 p + \frac{k_{2 \perp}^2}{4p z_2} \right\} \]

\[ q_0 = \left\{ x p + \frac{q_0^2 + |\vec{k}_{1 \perp} + \vec{k}_{2 \perp}|^2}{4px}, \vec{k}_{1 \perp} + \vec{k}_{2 \perp}, x p - \frac{q_0^2 + |\vec{k}_{1 \perp} + \vec{k}_{2 \perp}|^2}{4px} \right\}. \quad (F1) \]

Before proceeding, we wish to note some things about our assignment. First of all, while it is redundant to include \( q_0 = k_1 + k_2 + q_2 \), we will integrate over \( d^4 q_0 \) and wanted to present our parametrization. We see that \( x = 1 - z_1 - z_2 \). This is consistent with the spacelike case, but here, \( z_1, z_2 \in (-\infty, 0] \), hence the minus signs in \( k_1 \) and \( k_2 \). Additionally, only the relative azimuthal angle between \( k_1 \) and \( k_2 \) is physical. Thus, to simplify our formulas, we fix \( k_2 \) in the \( x - z \) plane.

As a last step before squaring and integrating, we will introduce our measure and integral parametrization. While one could integrate the full final state phase space including the antiquark, we instead exploit the factorization of the the cross section into a hard interaction \( \mathcal{H} \), a radiation-function \( \mathcal{K} \), and fragmentation functions \( q_{B,F}(x) \) which determine how the partons arrange themselves into hadrons. Schematically, \( \sigma = \mathcal{H} \otimes (\mathcal{K}_{LO}(x, q^2) + \mathcal{K}_{J,NLO}(x, q^2) + \ldots) \otimes \Pi q_{B,F}(x) = \mathcal{H} \otimes (\mathcal{R}_{LO} + \mathcal{R}_{NLO} + \ldots) \). For our computations we need only integrate the phase space for \( \mathcal{R} \), and it will remain independent of the details of \( \mathcal{H} \).

Taking \( d \equiv 4 + \epsilon \):

\[ \mathcal{R}_{LO} = \sum_i \frac{2}{q_i} \int \prod_{j=1}^{i-1} \frac{d^{d-1}k_j}{z_j} d^d q_0 \text{PP} \left[ |C^{(i-1)}_{i,LO} \langle 0 | O^{(i-1)}_i | q(i - 1) g \bar{q} \rangle|^2 \right] \]

\[ \times \delta(x - \bar{q}_0/\bar{q}_i) \delta(q^2 - (q_i + \sum_{j=1}^i k_j)^2) (2\pi)^d \delta^{(d)}(q_0 - \sum_{j=1}^i k_j)), \]

\[ \mathcal{R}_{J,NLO} = \sum_i \frac{2}{q_i} \int \prod_{j=1}^{i-1} \frac{d^{d-1}k_j}{z_j} d^d q_0 d^2 \text{PP} \left[ |C^{(i-1)}_{i,NLO} \langle 0 | O^{(i-1)}_i | q(i - 1) g \bar{q} \rangle|^2 \right] \]

\[ \times \delta(x - \bar{q}_0/\bar{q}_i) \delta(q^2 - (q_i + \sum_{j=1}^i k_j)^2) (2\pi)^d \delta^{(d)}(q_0 - \sum_{j=1}^i k_j)), \quad (F2) \]

and the \( q_i \) phase space and spin-sum are moved into \( \mathcal{H} \). We define \( z_j \) analogously to Eqs. (F1).

The setup we describe in the body of the paper uses Wilsonian cutoffs in phase space, both to keep the contributions of different operators distinct via \( \Theta \)'s and to cutoff soft and collinear divergences via some shower resolution parameter which keeps configurations outside of nonperturbative regimes. In the shower language the \( \Pi q_{B,F}(x) \) term in \( \mathcal{R} \) signifies the hadronization model and may depend on more than just \( x \) variables, and the \( \mathcal{K} \) term signifies the infrared finite fully differential shower computations. In Eq. (F2) we are integrating over \( \perp \)-momenta to carry out the perturbative comparison with Curci et al. Here we are
implicitly in the $\overline{\text{MS}}$ scheme, and it is the perturbative IR divergences in $\mathcal{R}$ that get absorbed by $q_B, \gamma(x)$. The $\mathcal{R}$ terms that we need consist of only the $1/\epsilon^2$ and $1/\epsilon$ portions of the corresponding operator expectation values. The non-pole contributions from $C_{i,\text{LO}}^{(i-1)}$ and $C_{i,\overline{\text{MS}}}^{(i-1)}$ (Eqs. E6 and E13), along with higher-order corrections are in higher order terms in the $\mathcal{R}$ functions. The hard-scattering corrections are in $\mathcal{H}$. The reason we extract only the pole terms is that these are precisely what give the expression for $P_{qq}^{(0)}$ and $P_{q\bar{q}}^{(1)}$. In addition to selecting the pole part, we also define PP to remove those portions of the matrix element which enter into $\mathcal{H}$ such as the final quark spin-sum, current $\Gamma$, and antiquark quantities.

In this $\overline{\text{MS}}$ factorization scheme, we need to define our correction operator differently than in Apps. D and E. Since $P_{qq}^{(1)}$ requires the calculation of two-gluon emission, we find it simplest here to calculate in SCET$_3$ where only $C_{3,\text{NLO}}^{(2),J,1}$ in Eq. (E2) contributes. This corresponds to taking limits such that only its $\Theta$-function equals one, while the other jet-structure coefficients are zero. Since we integrate it over all of phase space, which includes the strongly-ordered limit, we need to subtract the LO contribution. This just comes from $C_{3,\text{LO}}(n_2, n'_1, n'_2)\mathcal{O}_{3}^{(2)}$, but we take care to only remove the pole parts consistent with $\overline{\text{MS}}$. We can thus write the subtraction as:

$$
\mathcal{R}_{J,\text{NLO}}^{q\to qgq} = \int d\Pi_{k_1, k_2, q_0} \text{PP} \left[ \left| C_{3,\text{NLO}}^{(2),J,1}(n_2, n'_1, n'_2) \langle 0 | \mathcal{O}_3^{(2)} | qgq \rangle \right|^2 - \left. \left| C_{3,\text{LO}}^{(2),J,1}(n_2, n'_1, n'_2) \langle 0 | \mathcal{O}_3^{(2)} \Gamma^\mu | qgq \rangle \right|_{\overline{\text{MS}}} \right],
$$

where $C_{3,\text{NLO}}^{(2),J,1}$ is evaluated such that $\Theta = 1$ over all of phase space. We will describe the subtraction portion in detail below, but first we concentrate on the correction term.

By fixing the virtuality of $q_0^2 = q^2$, we can obtain an expression without having to know its exact limits, which will depend on the details of the hard scattering. For $P_{qq}^{(1)}$, one only needs to calculate one-loop corrections to single emission and tree-level double emission, and we now specialize to the latter case. We perform the $d$-dimensional integration over $d^d q_0$ and rewrite the integral in terms of $k_{1\perp}$ and $k_{2\perp}$ dependent functions with $z_{1,2}$-dependent coefficients. Using the same parametrization as Ref. [73], we can write:

$$
\mathcal{R}_{J,\text{NLO}}^{q\to qgq} = \frac{1}{(16\pi^2)^2} \int \frac{dq^2}{z_1} \frac{dz_2}{z_2} \frac{d^d k_{1\perp}}{\pi} \frac{d^d k_{2\perp}}{\pi} \delta(1 - x - z_1 - z_2) \delta(q^2 - (a_1 k_{1\perp}^2 + a_2 k_{2\perp}^2 - k_{1\perp} \cdot k_{2\perp}))
\times \frac{1}{q^4} \left( A(z_1, z_2) + B(z_1, z_2) \frac{k_{1\perp} \cdot k_{2\perp}}{k_{2\perp}^2} + C(z_1, z_2) \frac{k_{1\perp} \cdot k_{2\perp}}{k_{2\perp}^2} \frac{k_{1\perp}^2}{k_{2\perp}^2} \right.
\left. + D(z_1, z_2) \frac{k_{1\perp} \cdot k_{2\perp}^2}{k_{1\perp}^2 k_{2\perp}^2} + E(z_1, z_2) \frac{k_{1\perp}^2}{k_{2\perp}^2} + F(z_1, z_2) \frac{k_{2\perp}^2}{k_{1\perp}^2} \right) - [\text{LO}],
$$

where $a_1 = -(1 - z_2)/z_1$ and $a_2 = -(1 - z_1)/z_2$. The functions $A, B, C, D$ are defined in [73], and their corresponding $k_{1\perp}$ integrals are finite. We can check the intermediate step of their integration with [73]. The terms in our $q^2$ $\delta$-function have a relative sign compared to theirs, as our $q^2 > 0$. As a computational aside, we found it easiest to pass to a change of variables: $(u \equiv k_{1\perp} k_{2\perp}, w \equiv k_{1\perp}/k_{2\perp})$. Then the $\delta$-function just enforces:

$$
u = u_0 \equiv \frac{q^2 w}{a_1 w^2 + a_2 - 2 w \cos(\phi_1)}.
$$
Performing all but the $dz_i$ integrals in $\mathcal{R}$, we get Table III, which corresponds to [79]'s Table 5.

| Function of $\mathbf{k}_i \perp$ in integrand of equation (F4) | Contribution to $\mathcal{R}$ multiplying $\frac{q^2}{(16\pi^2)^2} \int dz_1 \, dz_2 \, \delta(1 - z_1 - z_2 - x)$ |
|---------------------------------------------------------------|------------------------------------------------------------------------------------------------------|
| $1$                                                            | $A(z_1, z_2)$                                                                                       |
| $\frac{k_{1\perp} \cdot k_2}{k_{1\perp}}$                     | $- \frac{z_2}{1 - z_1} B(z_1, z_2)$                                                                  |
| $\frac{k_{1\perp} \cdot k_2 \perp}{k_{2\perp}}$                | $- \frac{z_1}{1 - z_2} C(z_1, z_2)$                                                                  |
| $(k_{1\perp} \cdot k_{2\perp})^2$                             | $(1 + \frac{x}{2 z_1 z_2} \ln \left[ \frac{x}{(1 - z_1)(1 - z_2)} \right]) D(z_1, z_2)$            |

TABLE III: Purely finite contributions to $\mathcal{R}$

We thus reproduce the earlier result.

The $E$, $F$ functions multiply integrals that lead to single $\epsilon$ poles after the $dk_i \perp$ integrals (and double poles after integrating $q^2$), and so we must be more careful in treating them. These double poles correspond to the LO contribution, which we are explicitly subtracting as it does not contribute to $P_{qq}^{(1)}$. We discuss the subtraction in detail below Eq. (F12). For now we concentrate on the divergent integrals multiplying $E$ and $F$. When we did our computations for Table (III), we were helped by the finiteness of the expressions under the $dk_i \perp$ integration. We could thus take $\epsilon \to 0$ for these terms, which greatly simplifies their integrals. By contrast, we will need to keep the $\epsilon$-dependence of the $E$, $F$ terms, which results in an intractable computation. To get around this, one can introduce subtraction functions, which simply reproduce the $\epsilon$ poles (these are merely a computational aid and are not related to the subtraction of LO). We will need to take care that they do not remove any finite pieces. Secondly, since their full contribution to $\mathcal{R}$ is $\propto 1/\epsilon^2$, we will need to include for $E$ and $F$ any terms $\propto \epsilon$ that multiply $\frac{k_{1\perp}^2}{k_{2\perp}}$ or $\frac{k_{1\perp}^2}{k_{2\perp}^2}$. These arise from doing Dirac algebra in $d$-dimensions.

To do the integrals in $\mathcal{R}$ which multiply $E$ and $F$, we will change variables to $u$, $w$, and perform the $u$ integration as well as the trivial $\phi_2$ azimuthal one. We get for this contribution to $\mathcal{R}$:

$$
\mathcal{R}|_{E,F} = \frac{1}{(16\pi^2)^2} \frac{2}{\pi} \int dq^2 \, dz_1 \, dz_2 \, z_1 \, z_2 \, d\phi_1 \, dw \, \delta(1 - x - z_1 - z_2) \times \left( \frac{wu_0^{2+\epsilon}}{2q^2} E(z_1, z_2) + \frac{u_0^{2+\epsilon}}{2wq^2} F(z_1, z_2) \right) \frac{1}{q^4},
$$

where $u_0$ is defined by equation (F6). We only need the leading poles in $\epsilon$, and so rather than performing the $w$ and $\phi_1$ integrals for the functions multiplying $E$, $F$, we will define subtraction functions to reproduce the poles of $\frac{wu_0^{2+\epsilon}}{2q^2}$, $\frac{u_0^{2+\epsilon}}{2wq^2}$, respectively:

$$
S_E = \frac{q^2}{2a_1^2} \frac{w^{-\epsilon}}{(w + 1)},
$$

$$
S_F = \frac{q^2}{2a_2^2} \frac{w^\epsilon}{(w + w^2)}.
$$

(F8)


| Function of $k_{\perp}$ in integrand of equation (F4) | Contribution to $\mathcal{R}$ multiplying |
|-----------------------------------------------------|--------------------------------------------|
| $k_{1\perp}^2$/$k_{2\perp}^2$ | $\frac{g^2}{(16\pi^2)^2} \int dz_1 dz_2 \delta(1 - z_1 - z_2 - x)$ |
| $k_{2\perp}^2$/$k_{1\perp}^2$ | $\left[ \frac{2 x_1}{(1-x_1^2)^2} \epsilon \left(1 - \epsilon \ln \left[ \frac{1-x_1}{x_1} \right] \right) + \frac{z_1}{2z_1(1-z_1)^2} (z_1 z_2 + x (\ln \frac{z_2(1-z_2)}{z_2 x} - 1)) \right] E(z_1, z_2)$ |

| $A_E \equiv \frac{w u^{2+\epsilon}}{2wq^2} E,F | \equiv \frac{q^2 w^3}{2(a_2 + a_1 w^2 - 2w \cos(\phi_1))^2} \left( \frac{w q^2}{a_2 + a_1 w^2 - 2w \cos(\phi_1)} \right)^\epsilon |
| $A_F \equiv \frac{u^{2+\epsilon}}{2wq^2} F | \equiv \frac{q^2}{2w(a_2 + a_1 w^2 - 2w \cos(\phi_1))^2} \left( \frac{w q^2}{a_2 + a_1 w^2 - 2w \cos(\phi_1)} \right)^\epsilon $ (F9)

Integrating these in $w$ gives us a pure $1/\epsilon$ term. Subtracting them from the functions in equation (F7):

$A_E \equiv \frac{w u^{2+\epsilon}}{2wq^2} E,F \equiv \frac{q^2 w^3}{2(a_2 + a_1 w^2 - 2w \cos(\phi_1))^2} \left( \frac{w q^2}{a_2 + a_1 w^2 - 2w \cos(\phi_1)} \right)^\epsilon$,

$A_F \equiv \frac{u^{2+\epsilon}}{2wq^2} F \equiv \frac{q^2}{2w(a_2 + a_1 w^2 - 2w \cos(\phi_1))^2} \left( \frac{w q^2}{a_2 + a_1 w^2 - 2w \cos(\phi_1)} \right)^\epsilon$ (F9)

leads to finite integrals, allowing us to pass to the $\epsilon \to 0$ limit prior to integration, making the calculation tractable. After integrating $w$ and $\phi_1$, we want the $\epsilon^{-1,0}$ pieces as these turn into the single and double poles upon doing the $q^2$ integral and contribute to $\mathcal{R}_{J,NLO}$. The $\epsilon^0$ piece has one contribution besides that from $(A_{E,F} - S_{E,F})|_{\epsilon=0}$ ($S_{E,F}$ contributes a pure $1/\epsilon$ pole). Our $w$ integration goes from 0 to $\infty$, and we obtained $S_{E,F}$ by expanding $A_{E,F}$ in the appropriate $w \to 0, \infty$ limit to pick up the pole, while carefully regulating the other integration limit so as not to contribute its own spurious divergence or any subleading terms. However, we see that in equation (F9), taking these limits actually results in factors $(a_1 w)^{-\epsilon}$ and $(w/a_2)^\epsilon$. Expanding the $a_i^{\pm \epsilon}$ to LO in $\epsilon$ does not affect $S_{E,F}$. Nonetheless, since the subtraction functions have $1/\epsilon$ poles, including the NLO part of the $\epsilon$-expansion will yield an $\epsilon^0$ contribution. This $O(\epsilon^0)$ term is not in $A_{E,F}|_{\epsilon=0}$ since they send $u_0^{\epsilon} \to 1$. Thus, we have the following addition to the contributions from the integration of $\mathcal{R}_{E,F}$:

$B_E = -\epsilon \ln(a_1) \frac{q^2}{2a_1^2} \frac{w^{-\epsilon}}{w+1},$

$B_F = -\epsilon \ln(a_2) \frac{q^2}{2a_2^2} \frac{w^\epsilon}{w+w^2}$. (F10)

In the end, our $\epsilon^{-1,0}$ contributions after $w$ and $\phi_1$ integration come from: $S_{E,F} + B_{E,F} + (A_{E,F} - S_{E,F})|_{\epsilon=0}$. For integrating the first two terms, we leave the full $\epsilon$ dependence as this was tractable. Collecting everything, we can obtain the counterpart to Table [III] for $E$, $F$, (Table [IV]).

Having set up this much of the integration, we can take the amplitude squared from the process of interest and decompose it in terms of the $A(z_1, z_2)$, $B(z_1, z_2)$, etc. basis. We then simply have to read off the results from Tables [III] and [IV] and perform the $z_{1,2}$ integrals. One of these is made trivial by the remaining $x$-dependent $\delta$-function. As mentioned at the beginning of this Appendix, [62] recognizes two topologically distinct contributions, which
we shall refer to as box and crossed (cf. Fig. 22), because of their appearance as cut two-loop diagrams. We can identify them in our calculation by their color structures ($C_F^2$ and $C_F^2 - \frac{1}{2}C_F C_A$, respectively). In fact, we can already calculate the entire crossed contribution as it only involves terms from Table III having no double pole contribution to $R$ and thus requiring no subtraction of LO. Determining the box graph, however, involves treating the LO subtraction properly.

As this subtraction is one of the more subtle points of the computation, we will present it in some detail. Its handling is tied up with what one means precisely by a “subleading splitting function.” At LO in $\alpha_s$, the definition is clear. The same splitting function that gives us the probability for a $1 \rightarrow 2$ radiation also determines the running of parton densities:

$$Q^2 \frac{\partial}{\partial Q^2} f(x, Q^2) = \int_x^1 \frac{dz}{z} P_{qq} \left( \frac{x}{z}, \alpha_s(Q^2) \right) f(z, Q^2),$$

where the $O(\alpha_s)$ part of $P_{qq}$, $P_{qq}^{(0)}$, is given by Eq. (3). To determine $P_{qq}^{(1)}$, we have had to calculate a $1 \rightarrow 3$ splitting, thus the probabilistic interpretation in terms of radiation is nontrivial as it involves a mix of $1 \rightarrow 2$ and $1 \rightarrow 3$ processes. At the level of Eq. (F11), though, we see that we are just correcting PDF evolution. In addition to the real-emission calculation that we are pursuing, one can alternatively determine $P_{qq}$ from the anomalous dimension of certain twist-2 operators [80, 81]. Ref. [62] made a comparison to this approach and found agreement to $O(\alpha_s^2)$. Since $P_{qq}^{(1)}$ is thus a two-loop object, it has the scheme dependence one would expect at this order, and so we need to make sure that we compute in the same scheme, which is why we do our LO subtraction in MS. In SCET, one could attempt the same cross-check from a straightforward two-loop calculation after fixing to one’s renormalization scheme of choice.

We will now show how to subtract the LO portion in the calculation of $R_{q\rightarrow qggJ, NLO}$. We get a double collinear pole associated with the strongly-ordered emission of two gluons. We want to write this as removing the emission coming from our LO operator, $C_{3,LO}^{(2)}$. As with any subtraction scheme, while the pole is unambiguous, we need to make sure to remove the appropriate finite pieces. We note that $c_{3,LO}^{(2)}$ defined by Eq. (57) contains NLO($\lambda$) pieces (in SCET$_3$ power counting) which come from the offshellness of the intermediate quark. It is true that the LO replacement rule, Eq. (56), gives only the splitting function times the logarithmic, collinear divergence. Nonetheless, the Wilson coefficients given by Eq. (63) for offshell quarks have additional terms. From the point of view of amplitude matching, this poses no problem. However, if we want to copy [62]’s scheme, then we can only subtract poles associated with the pure LO result after integration. As an operator subtraction in SCET$_3$, this means we need to change $C_{3,LO}^{(2)}$. In order to recover the correct splitting function with no NLO contribution, we will need to project the offshell quark momentum to an onshell one with the same $\bar{p}$-fraction. This alone, though, does not specify the spatial orientation of the vector and will not necessarily kill the subleading terms. To do that, we write the replacement rule, but in the limit that the offshell quark’s daughters are exactly collinear with it. Equivalently, if we are in the frame determined by $\bar{n} = \{1, 0, 0, -1\}$, we can project the quark momentum along $\bar{n} = \{1, 0, 0, 1\}$, i.e. $q_i \rightarrow \bar{\bar{n}} = q_i'. \bar{n}$. Since the replacement rule also makes reference to the quark’s parent’s momentum, we also need to project it to what it would be if it had emitted an onshell quark with $q_i'$. Thus, $q_{i-1} \rightarrow k_i + q_i' = q_{i-1}'$. In the
end, this changes our replacement rule coefficient for the \( j \)th quark to:

\[
\frac{c'_{\text{LO}}}{\alpha_{j+1}} = \frac{\bar{q}_j}{q_j^2} \left( n_j^\alpha + \frac{(q'_{j+1})^\nu n'_{j+1}}{4} \right),
\]

where \( q'_j = \bar{q}_j n_j^\nu/2 \). Thus \( c'_{\text{LO}} \) has the same form as \( c_{\text{LO}} \) but with a different orientation for its momenta. This changes the expression for \( C_{3,\text{LO}}^{(2)} \) to involve \( c'_{\text{LO}} \) instead of \( c_{\text{LO}} \) (cf. Eq. (63)).

After the \( dq^2 \) integration, the 1/\( \epsilon \) term in \( R_{J,\text{NLO}}^{q\to qgg} \) will allow us to read off \( P_{qq}^{(1)} \). As a reminder, we need this subtraction operator because our NLO(\( \lambda \)) term, \( C_{3,\text{NLO}}^{(2),J} (n_2, n'_1, n'_2) \), is supported over all of phase space, and thus contains LO portions. We therefore have

\[
R_{J,\text{NLO}}^{q\to qgg} = \int d\Pi_{k_1, k_2, q_0} \text{PP} \left[ |C_{3,\text{LO}}^{(2),1,J} (n_2, n'_1, n'_2) \rangle \langle 0 | C_{3}^{(2)} | qggq \rangle |^2 \right. \\
- \left. \left( |c'_{\text{LO}}^{\alpha_1} c'_{\text{LO}}^{\alpha_2} \langle 0 | C_{3}^{(2)} | qggq \rangle |^2 \right)_{\overline{\text{MS}}} \right].
\]  

(13)

The \( \overline{\text{MS}} \) indicates that we are only subtracting pole parts of the LO contribution with no finite pieces. However, there is still an ambiguity over which pole parts we subtract, since the LO contribution has a double pole from its two collinear divergences, but we are at some liberty to decide which single pole parts we remove as well. As we expect, this subtraction operator squared takes the form of a convolution of two splitting functions:

\[
\int d\Pi \left( |c'_{\text{LO}}^{\alpha_1} c'_{\text{LO}}^{\alpha_2} \langle 0 | C_{3}^{(2)} | qggq \rangle |^2 \right)_{\overline{\text{MS}}} = 2 \int \left. d^4 q_2 \delta(q_2^2) dq_2 dq_2 dq_2 dq_2 (1 - y)^{\frac{d}{2} - 1 + \epsilon/2} \frac{\alpha^2}{2 \pi^2} \right. \\
\times \frac{1}{y} P_{d,qq}^{(0)}(y) \frac{P_{4,qq}^{(0)}(x/y)}{\epsilon} \text{Tr} \left[ \bar{q}_2 \gamma_4 \Omega \right],
\]

(14)

where the trace contains those terms that get passed to the hard function, \( \mathcal{H} \), along with the \( q_2 \) phase space by the projector PP. This includes the final quark spin-sum and phase space, the current \( \Gamma \) which is a spectator for both LO and jet-structure corrections, and quantities related to the antiquark (cf. Eq. (27)). What may seem surprising is that the two splitting functions live in different dimensions. The reason for this particular scheme for regulating phase space has to do with the alternate, two-loop method for calculating \( P_{qq}^{(1)} \), which was the original approach. For that result, in \( \overline{\text{MS}} \) we would subtract a pure pole counterterm, regulate the loop integral in \( d \)-dimensions, and leave external particles in 4d. Since the phase space integrals are related to loops by cuts, we see above that our \( y \)-integral is, in fact, in \( d \)-dimensions, but the splitting involving two external particles is left simply in four.

Looking at the SCET\(_1\) diagrams for the process (Fig. 20), the amplitude \( c'_{\text{LO}}^{\alpha_1} c'_{\text{LO}}^{\alpha_2} \langle 0 | C_{3}^{(2)} | qggq \rangle \) comes from a subset of diagrams \( A^2 \) and \( B^2 \). The expression for subtraction is thus:

\[
\text{PP} \left[ \int d\Pi \left( |c'_{\text{LO}}^{\alpha_1} c'_{\text{LO}}^{\alpha_2} \langle 0 | C_{3}^{(2)} | qggq \rangle |^2 \right)_{\overline{\text{MS}}} \right] = \int dq_2^2 d\frac{x}{z_1} \left. \frac{z_1}{x + z_1} \right)^{\frac{d}{2}} \\
\times (q_2^2)^{-1 + \epsilon/2} \frac{\alpha^2}{2 \pi^2} \frac{1}{x + z_1} \left[ \frac{1 + \frac{x}{x + z_1}}{x + z_1} + \frac{\epsilon}{2} \left( 1 - \frac{x}{x + z_1} \right) \right] \left( \frac{1 + (x + z_1)^2}{x + z_1 - 1} \right) \left( \frac{1 + (x + z_1)^2}{x + z_1 - 1} \right) \\
+ z_1 \leftrightarrow z_2,
\]

(15)
where we now act with PP, dropping the trace from Eq. (F14) and keeping only those terms needed for the computation of $R$ and $P_{qq}^{(1)}$. We can note several things about this expression. For concreteness, we discuss the $z_1$-dependent term corresponding to graph $A^2$, Fig. [20]. The $\hat{p}$ fraction of $q_0$ relative to $q_1$ is $x/(x + z_1)$, and that of $q_1$ to $q_2$ is $x + z_1$, in terms of the variables in Eq. (F14), $y' = x/(x + z_1)$. Performing the integrals leads to double and single poles. For later use, we write down the result of doing the $dq^2$, $dz_i$ integrals, where one of latter is trivial since we have $\delta(1 - x - z_1 - z_2)$ sitting inside the phase space integral (cf. Eq. [F4]).

\[
\int d\Pi PP \left[ \left( |c'_{\text{LO}}|^{\alpha_1} c'_{\text{LO}}^{\alpha_2} \langle 0 | \mathcal{O}_3^{(2)} \Gamma_{qg}^\mu | q\bar{q}gg \rangle \right)^2 \right]_{\overline{\text{MS}}} =
2 x p \frac{\alpha^2}{2\pi^2} \left( \frac{1}{\epsilon^2} \frac{2}{x - 1} \left[ (-2 \left( x^2 + 1 \right) \log(\lambda) + (x - 1)^2) + 4 \left( x^2 + 1 \right) \log(\lambda) - \left( x^2 - 1 \right) \log(\lambda) \right] + 2 \frac{1}{\epsilon (x - 1)^2 x} \left[ 2x(x - 1) \left( 2 \left( x^2 + 1 \right) \left( \text{Li}_2(1 - x) - \text{Li}_2 \left( \frac{1}{x} \right) \right) + \left( x^2 - 1 \right) \text{Li}_2 \left( \frac{x - 1}{x} \right) \right) + x \left( -2 \left( x^2 + 1 \right) (x - 1) \log(\lambda) + 4 \left( x^2 + 1 \right) \left( \log \left( \frac{x - 1}{x} \right) + \log \left( \frac{x}{x - 1} \right) \right) + 6x \left( x^2 + 1 \right) (x - 1) \log^2(x - 1) - 2x(x - 1)^3 \right) \right) \right)
\] (F16)

where we have done the $dz_i$ integrals between $1 - x + \lambda$ and $-\lambda$ to regulate soft divergences. All $\lambda$-dependence cancels out of the final answer, which gives us a consistency check on the scheme.

Before comparing $P_{qq}^{(1)}$, we can check our setup with $P_{qq}^{(0)}$, by looking at the $\mathcal{O}(\alpha_s)$ contribution to $R_{\text{LO}}$. We see that [62] gets the following contribution:

\[
P_{qq}^{(0)} = \left( \frac{\alpha_s}{2\pi} \right) \frac{21 + x^2}{\epsilon - 1 - x}.
\] (F17)

Calculating in SCET$_1$, we get the following amplitude squared:

\[
A_{q\rightarrow qg} = \frac{\bar{q}_0 q_2}{q_0^2} \left( \frac{2n \cdot k_1}{k_1} + \frac{2k_{1,\perp} \cdot q_{1,\perp}}{q_{1,\perp} k_{1,\perp}} - \frac{q_{1,\perp}^2}{q_1^2} \right) \text{Tr}[\bar{q}_0 \gamma_0 \Omega \Omega^\dagger]
\] (F18)

With our definition of $R_{\text{LO}}$ in Eq. (F14), we get:

\[
P_{qq}^{(0)} = \left( \frac{\alpha_s}{2\pi} \right) \frac{21 + x^2}{\epsilon - 1 - x}.
\] (F19)

The overall minus sign between Eqs. (F17) and (F19) is due to the difference between the spacelike and timelike processes. It arises in the $dz_i$ integral. Even though the $z_i$ dependence is the same in the two calculations, and the integration limits are the same, 0 and $1 - x$. For us, $1 - x < 0$, but in [62], it is positive.
We will compare the different contributions to double emission separately. In SCET\textsubscript{1}, the C graph in Fig. 20 will give box and crossed terms when interfered with itself and the A and B ones. We identify the crossed contribution by inserting the color structure and taking those terms proportional to $C_F^2 - \frac{1}{3} C_F C_A$. As mentioned above, it only contains the integrals in Table III. In terms of its notation, we have:

| Function defined in Eq. (F4) | Value in crossed diagram |
|-----------------------------|-------------------------|
| $A(z_1, z_2)$               | $16x(z^2 + xz_1 + (z_1 - 1)z_1 + 1)$ |
| $B(z_1, z_2)$               | $8(x^2(z_1 - 1) - xz_1 + z_1 - 1)$ |
| $C(z_1, z_2)$               | $8(x^2 + (x - 1)z_1 + 2 + z_1)$ |
| $D(z_1, z_2)$               | $16(x^2 + 1)$ |

**TABLE V: Contributions to crossed amplitude squared diagram**

The box contribution additionally contains the functions in Table IV, though we are only interested in the finite parts. Their $z_i$ dependence is:

| Function defined in Eq. (F4) | Value in box diagram |
|-----------------------------|---------------------|
| $A(z_1, z_2)$               | $12x^2 + 8xz_1 + 8(z_1 - 1)z_1 + 12$ |
| $B(z_1, z_2)$               | $8(z_1 - 1)(x^2 + (z_1 - 2)z_1 + 2)$ |
| $C(z_1, z_2)$               | $8(x + z_1)(2x^2 + 2xz_1 + z_1^2 + 1)$ |
| $D(z_1, z_2)$               | $0$ |
| $E(z_1, z_2)$               | $4\left[\frac{(2x^4 + 6x^3z_1 + x^2(7z_1^2 + 2) + 2x(2z_1^2 + 2 + 2z_1 + 1) + z_1^2 + z_1^2)}{z_1^2}\right]$ |
| $F(z_1, z_2)$               | $4\left[\frac{(2x^2(z_1 - 2) + 2x^2 + (z_1 - 1)^2)}{(x + z_1 - 1)^2}\right]$ |

**TABLE VI: Contributions to box amplitude squared diagram**

For the crossed contribution, we perform the multiplication in Table III with the functions defined in Table IV and integrate $dz_1$, having already done the trivial $dz_2$ integral. We again use a cutoff to avoid soft divergences, thus its range is between $1 - x + \lambda$ and $-\lambda$. In the end, we obtain:

$$P_{qq}^{(1)} = \left(\frac{\alpha_s}{2\pi}\right)^2 \left[\frac{1 + x^2}{x - 1}\right] \left(4\ln(x - 1) - \ln^2(x) - \ln(\lambda) - 2(x + 1)\ln(x)\right).$$

(F20)
The $\lambda$-dependent pieces will cancel against those from the box contribution. The other terms agree with [62] up to the previously discussed minus sign, and wherever $\ln(1-x)$ appears in the spacelike calculation, we get $\ln(x-1)$. Since our integrand and integration region are real, the imaginary pieces generated by $\ln(1-x)$ when making $x>1$ all must cancel.

The box calculation proceeds similarly using the functions defined in Tables III and VI. We also include the terms proportional to $E(z_1,z_2)$ and $F(z_1,z_2)$ and we have subtracted the appropriate contribution, Eq. (F16) from that given by $C^{(2)\text{I}}_3$. Doing all this, we get:

$$P^{(1)}_{qq}\text{box} = \left(\frac{\alpha_s}{2\pi}\right)^2 \left[ \left(\frac{1 + x^2}{x-1}\right) (\ln(\lambda) - \ln(x-1)) + 2(2x-1) \ln(x) \right].$$

(F21)

The soft divergent pieces cancel against the crossed contribution, and once again we agree with [62] up to an overall sign, and the continuation $\ln(1-x) \to \ln(x-1)$.

In addition to these real emission contributions to the $C^2_F$ portion of $P^{(1)}_{qq}$, there are also single-emission, one-loop diagrams, shown in Fig. 24. We can account for their contributions in SCET easily. We have already derived the tree-level expression for single emission (Eqs. F17 and F19). Furthermore, both the quark wavefunction renormalization and the vertex renormalization are the same in SCET as in QCD [21]. Thus, we recover the entire, gauge-invariant, $\propto \alpha_s^2 C^2_F$ contribution to the splitting function, in agreement with Ref. [62],

$$P^{(1)}_{qq\text{abelian}} = C^2_F \frac{\alpha_s^2}{2\pi} \left[ (1-x) \ln(x) - \frac{3}{2} \frac{1 + x^2}{1-x} \ln(x) - 2 \frac{1 + x^2}{1-x} \ln(x) \ln(1-x) \right. \left. - \frac{1}{2} (1+x) \ln^2(x) - 5(1-x) - \frac{5}{2} (1+x) \ln(x) \right].$$

(F22)

Here we have written $P^{(1)}_{qq}$ with its usual sign conventions for spacelike evolution.

FIG. 24: Single emission, one-loop contributions to $P^{(1)}_{qq}$.

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