Abstract

Providing efficient emulations of atomic read/write objects in asynchronous, crash-prone, message-passing systems is an important problem in distributed computing. Communication latency is a factor that typically dominates the performance of message-passing systems, consequently the efficiency of algorithms implementing atomic objects is measured in terms of the number of communication exchanges involved in each read and write operation. The seminal result of Attiya, Bar-Noy, and Dolev established that two pairs of communication exchanges, or equivalently two round-trip communications, are sufficient. Subsequent research examined the possibility of implementations that involve less than four exchanges. The work of Dutta et al. showed that for single-writer/multiple-reader (SWMR) settings two exchanges are sufficient, provided that the number of readers is severely constrained with respect to the number of object replicas in the system and the number of replica failures, and also showed that no two-exchange implementations of multiple-writer/multiple-reader (MWMR) objects are possible. Later research focused on providing implementations that remove the constraint on the number of readers, while having read and write operations that use variable number of communication exchanges, specifically two, three, or four exchanges.

This work presents two advances in the state-of-the-art in this area. Specifically, for SWMR and MWMR systems algorithms are given in which read operations take two or three exchanges. This improves on prior works where read operations took either (a) three exchanges, or (b) two or four exchanges. The number of readers in the new algorithms is unconstrained, and write operations take the same number of exchanges as in prior work (two for SWMR and four for MWMR settings). The correctness of algorithms is rigorously argued. The paper presents an empirical study using the NS3 simulator that compares the performance of relevant algorithms, demonstrates the practicality of the new algorithms, and identifies settings in which their performance is clearly superior.
1 Introduction

Emulating atomic [10] (or linearizable [9]) read/write objects in asynchronous, message-passing environments with crash-prone processors is a fundamental problem in distributed computing. To cope with processor failures, distributed object implementations use redundancy by replicating the object at multiple network locations. Replication masks failures, however, it introduces the problem of consistency because operations may access different object replicas possibly containing obsolete values. Atomicity is the most intuitive consistency semantic as it provides the illusion of a single-copy object that serializes all accesses such that each read operation returns the value of the latest preceding write operation.

Background and Prior Work. The seminal work of Attiya, Bar-Noy, and Dolev [2] provided an algorithm, colloquially referred to as ABD, that implements SWMR (Single Writer, Multiple Reader) atomic objects in message-passing crash-prone asynchronous environments. Operations are ordered using logical timestamps associated with each value. Operations terminate provided some majority of replica servers does not crash. Writes involve a single communication round-trip involving two communication exchanges. Each read operation takes two rounds involving four communication exchanges. Subsequently, Lynch et al. [12] showed how to implement MWMR (Multi-Writer, Multi-Reader) atomic memory where both read and write operations take two communication round trips, for a total of four exchanges.

Dutta et al. [3] introduced a fast SWMR implementation where both reads and writes involve two exchanges (such operations are called ‘fast’). It was shown that this is possible only when the number of readers \( r \) is constrained with respect to the number of servers \( s \) and the number of server failures \( f \), viz. \( r < \frac{s}{2} - 1 \). Other works focused on relaxing the bound on the number of readers in the service by proposing hybrid approaches where some operations complete in one and others in two rounds, e.g., [4].

Georgiou et al. [6] introduced Quorum Views, client-side tools that examine the distribution of the latest value among the replicas in order to enable fast read operations (two exchanges). A SWMR algorithm, called SliQ, was given that requires at least one single slow read per write operation, and where all writes are fast. A later work [5] generalized the client-side decision tools and presented a MWMR algorithm, called CwFr, that allows fast read operations.

Previous works considered only client-server communication round-trips. Recently, Hadjistasi et al. [8] showed that atomic operations do not necessarily require complete communication round trips, by introducing server-to-server communication. They presented a SWMR algorithm, called OhSAM, where reads take three exchanges: two of these are between clients and servers, and one is among servers; their MWMR algorithm, called OhMAM, uses a similar approach. These algorithms do not impose constrains on reader participation and perform a modest amount of local computation, resulting in negligible computation overhead.

Contributions. We focus on the gap between one-round and two-round algorithms by presenting atomic memory algorithms where read operations can take at most “one and a half rounds,” i.e., complete in either two or three exchanges. Complexity results are shown in Table 1; additional details are as follows.

1. We present Erato, Efficient Reads for ATomic Objects, a SWMR algorithm for atomic objects in the asynchronous message-passing model with processor crashes. We improve the three-exchange read protocol of OhSAM [8] to allow reads to terminate in either two or three exchanges

\[ 1 \text{ Ερατώ is a Greek Muse, and the authors thank the lovely muse for her inspiration.} \]
## Models and Definitions

We now present the model, definitions, and notations used in the paper. The system is a collection of crash-prone, asynchronous processors with unique identifiers (ids). The ids are from a totally-ordered set $\mathcal{I}$ that is composed of three disjoint sets, set $\mathcal{W}$ of writer ids, set $\mathcal{R}$ of reader ids, and set $\mathcal{S}$ of replica server ids. Each server maintains a copy of the object. Processors communicate by exchanging messages via asynchronous point-to-point reliable channels; messages may be reordered. We use the term broadcast as a shorthand denoting sending point-to-point messages to multiple destinations.

A quorum system over a set is a collection of subsets, called quorums, such that every pair of quorums intersects. We define a quorum system $Q$ over the set of server ids $\mathcal{S}$ as $Q = \{Q_i : Q_i \subseteq \mathcal{S}\}$; it follows that for any $Q_i, Q_j \in Q$ we have $Q_i \cap Q_j \neq \emptyset$. We assume that every process in the system is aware of $Q$.

**Executions.** An algorithm $A$ is a collection of processes, where process $A_p$ is assigned to processor $p \in \mathcal{I}$. The state of processor $p$ is determined over a set of state variables, and the state of $A$ is a vector that contains the state of each process. Algorithm $A$ performs a step, when some process $p$ (i) receives a message, (ii) performs local computation, (iii) sends a message. Each such action

| Model | Algorithm | Read Exch. | Write Exch. | Read Comm. | Write Comm. |
|-------|-----------|------------|-------------|------------|-------------|
| SWMR  | ABD [2]   | 4          | 2           | $4|\mathcal{S}|$ | $2|\mathcal{S}|$ |
| SWMR  | OhSam [8] | 3          | 2           | $|\mathcal{S}|^2 + 2|\mathcal{S}|$ | $2|\mathcal{S}|$ |
| SWMR  | Sliq [6]  | 2 or 4     | 2           | $4|\mathcal{S}|$ | $2|\mathcal{S}|$ |
| SWMR  | Erato     | 2 or 3     | 2           | $|\mathcal{S}|^2 + 3|\mathcal{S}|$ | $2|\mathcal{S}|$ |
| MWMR  | ABD-MW [2,12] | 4          | 4           | $4|\mathcal{S}|$ | $4|\mathcal{S}|$ |
| MWMR  | OhMam [8] | 3          | 4           | $|\mathcal{S}|^2 + 2|\mathcal{S}|$ | $4|\mathcal{S}|$ |
| MWMR  | CwFr [5]  | 2 or 4     | 4           | $4|\mathcal{S}|$ | $4|\mathcal{S}|$ |
| MWMR  | Erato-MW  | 2 or 3     | 4           | $|\mathcal{S}|^2 + 3|\mathcal{S}|$ | $4|\mathcal{S}|$ |

Table 1: Summary of communication exchanges and communication complexities.

Using client-side tools, Quorum Views, from algorithm Sliq [6]. During the second exchange, based on the distribution of the timestamps, the reader may be able to complete the read. If not, it awaits for “enough” messages from the third exchange to complete. A key idea is that when the reader is “slow” it returns the value associated with the minimum timestamp, i.e., the value of the previous write that is guaranteed to be complete (cf. [8] and [3]). Read operations are optimal in terms of exchanges in light of [7]. Similarly to ABD, writes take two exchanges. (Section 3.)

2. Using the SWMR algorithm as the basis, we develop a MWMR algorithm, Erato-MW. The algorithm supports three-exchange read protocol based on [8] in combination with the iterative technique using quorum views as in [5]. Reads take either two or three exchanges. Writes are similar to ABD-MW and take four communication exchanges (cf. [12]). (Section 4.)

3. We simulate the algorithms using the NS3 simulator and assess their performance under practical considerations by varying the number of participants, frequency of operations, and network topologies. (Section 5.)

Improvements in latency are obtained in a trade-off for communication complexity. Simulation results suggest that in practical settings, such as data centers with well-connected servers, the communication overhead is not prohibitive.
causes the state at \( p \) to change. An execution is an alternating sequence of states and actions of \( A \) starting with the initial state and ending in a state.

**Failure Model.** A process may crash at any point in an execution. If it crashes, then it stops taking steps; otherwise we call the process correct. Any subset of readers and writers may crash. A quorum \( Q \subseteq \mathcal{Q} \) is non-faulty if \( \forall p \in Q, p \) is correct. Otherwise, we say \( Q \) is faulty. We allow for any server to crash as long one quorum is non-faulty.

**Efficiency and Message Exchanges.** Efficiency of implementations is assessed in terms of operation latency and message complexity. Latency of an operation is determined by computation time and the communication delays. Computation time accounts for all local computation within an operation. Communication delays are measured in terms of communication exchanges. The protocol implementing each operation involves a collection of sends (or broadcasts) of typed messages and the corresponding receives. As defined in [8], a communication exchange within an execution of an operation is the set of sends and receives for the specific message type. Traditional implementations in the style of ABD [2] are structured in terms of rounds, each consisting of two exchanges, the first, a broadcast, is initiated by the process executing an operation, and the second, a convergecast, consists of responses to the initiator. The number of messages that a process expects during a convergecast depends on the implementation. Message complexity measures the worst-case total number of messages exchanged.

**Atomicity.** An implementation of a read or a write operation contains an invocation action and a response action. An operation \( \pi \) is complete in an execution, if it contains both the invocation and the matching response actions for \( \pi \); otherwise \( \pi \) is incomplete. An execution is well formed if any process invokes one operation at a time. We say that an operation \( \pi \) precedes an operation \( \pi' \) in an execution \( \xi \), denoted by \( \pi \rightarrow \pi' \), if the response step of \( \pi \) appears before the invocation step in \( \pi' \) in \( \xi \). Two operations are concurrent if neither precedes the other. The correctness of an atomic read/write object implementation is defined in terms of atomicity (safety) and termination (liveness) properties. Termination requires that any operation invoked by a correct process eventually completes. Atomicity is defined following [11]. For any execution \( \xi \), if \( \Pi \) is the set of all completed read and write operations in \( \xi \), then there exists a partial order \( \prec \) on the operations in \( \Pi \), s.t. the following properties are satisfied:

- **A1** For any \( \pi_1, \pi_2 \in \Pi \) such that \( \pi_1 \rightarrow \pi_2 \), it cannot be that \( \pi_2 \prec \pi_1 \)
- **A2** For any write \( \omega \in \Pi \) and any operation \( \pi \in \Pi \), then either \( \omega \prec \pi \) or \( \pi \prec \omega \).
- **A3** Every read operation returns the value of the last write preceding it according to \( \prec \) (or the initial value if there is no such write)

**Timestamps and Quorum Views.** Atomic object implementations typically use logical timestamps (or tags) associated with the written values to impose a partial order on operations that satisfies the properties A1, A2, and A3.

A quorum view refers to the distribution of the highest timestamps that a read operation witnesses during an exchange. Fig. [1] illustrates four different scenarios. Here small circles represent timestamps received from servers, and dark circles represent the highest timestamp, and the light ones represent older timestamps. The quorum system consists of three quorums, \( Q_1, Q_2, \) and \( Q_3 \).

Suppose a read \( \rho \) strictly receives values and timestamps from quorum \( Q_1 \) during an exchange. As presented in [5], \( \rho \) can distinguish three different cases: \( QV(1) \), \( QV(2) \), or \( QV(3) \). Each case can help \( \rho \) derive conclusions about the state of the latest write (complete, incomplete, unknown). If \( QV(1) \) is detected, Fig. [1](a), it means that only one timestamp is received. This means that the write associated with this timestamp is complete. If \( QV(2) \) is detected, Fig. [1](b), this indicates that
the write associated with the highest timestamp is still in progress (because older timestamps are detected in the intersections of quorums). Lastly, if $QV(3)$ is detected, the distribution of timestamps does not provide sufficient information regarding the state of the write. This is because there are two possibilities as shown in Fig. 1(c) and 1(d). In the former the write is incomplete (no quorum has the highest detected timestamp) and in the latter the write is complete in quorum $Q_z$, but the read has no way of knowing this. We will use quorum views as a design element in our algorithms.

3 SWMR Algorithm Erato

We now present and analyze the SWMR algorithm Erato.

3.1 Algorithm Description

In Erato reads take either two or three exchanges. This is achieved by combining the three exchange read protocol of [8] with the use of Quorum Views of [6]. The read protocol design aims to return the value associated with the timestamp of the last complete write operation. We refer to the three exchanges of the read protocol as $e_1$, $e_2$, and $e_3$. Exchange $e_1$ is initiated by the reader, and exchanges $e_2$ and $e_3$ are conducted by the servers. When the reader receive messages during $e_2$, it analyses the timestamps to determine whether to terminate or wait for the conclusion of $e_3$. Due to asynchrony it is possible for the message from $e_3$ to arrive at the reader before messages from $e_2$. In this case the reader still terminates in three exchanges. Similarly to ABD, writes take two exchanges. The code is given in Algorithm 1. We now give the details of the protocols; in referring to the numbered lines of code we use the prefix “L” to stand for “line”.

Reader Protocol. Each reader $r$ maintains several temporary variables. Key variable include $minTS$ and $maxTS$ hold the minimum and the maximum timestamps discovered during the read operation. Sets $RR$ and $RA$ hold the received readRelay and readAck messages respectively. The ids of servers that sent these messages are stored in sets $RRsrv$ and $RAsrv$ respectively. The set $maxTSrv$ keeps the ids of the servers that sent a readRelay message with the timestamp equal to the maximum timestamp $maxTS$.

Reader $r$ starts its operation by broadcasting a readRequest message to the servers (exchange $e_1$). It then collects readRelay messages (from exchange $e_2$) and readAck messages (from exchange $e_3$). The reader uses counter $read_op$ to distinguish fresh message from stale message from prior operations. The messages are collected until messages of the same type are received from some quorum $Q$ of servers (L7-10). If readRelay messages are received from quorum $Q$ then the reader...
Algorithm 1 Reader, Writer, and Server Protocols for SWMR algorithm Erato

At each reader \( r \)

**Variables:**

\[ \text{minTS}, \text{maxTS} \in \mathbb{N}; \text{read_op} \in \mathbb{N} \text{ init } 0 \]

\[ \text{RR}, \text{RA}, \text{maxACK} \subseteq \mathcal{S} \times M \]

\[ v \in \mathbb{V}; \text{RRsrv}, \text{RAsrv}, \text{maxTSrv} \subseteq \mathcal{S} \]

**function** \( \text{Read} \)

\[ \text{read_op} \leftarrow \text{read_op} + 1 \]

\[ (\text{RR}, \text{RA}, \text{RRsrv}, \text{RAsrv}) \leftarrow (0, 0, 0, 0) \]

**bcast** ((readRequest, \( r \), \( \text{read_op} \)) to \( \mathcal{S} \))

**wait until** \( \exists Q \in \mathcal{Q} : (Q \subseteq \text{RRsrv}) \lor (Q \subseteq \text{RAsrv}) \)

if \( \exists Q \subseteq \mathcal{Q} : Q \subseteq \text{RRsrv} \) then

\[ \text{minTS} \leftarrow \min\{\{m.ts\} : (s, m) \in \text{RA} \land s \in Q\} \]

**return** \( (m.v \text{ s.t. } (s, m) \in \text{maxACK}) \)

else if \( \exists Q \subseteq \mathcal{Q} : Q \subseteq \text{RAsrv} \) then

\[ \text{maxTS} \leftarrow \max\{\{m.ts\} : \]

\[ (s, m) \in \text{RR} \land s \in Q\} \]

\[ \text{maxACK} \leftarrow \{s, m) \in \text{RR} : \]

\[ s \in Q \land m.ts = \text{maxTS} \}

\[ \text{maxTSrv} \leftarrow \{s, m) \in \text{maxACK} \}

if \( Q \subseteq \text{maxTSrv} \) then

**\text{Qview1}**

\[ \text{return} (m.v \text{ s.t. } (s, m) \in \text{maxACK}) \]

**\text{Qview2}**

**\text{Qview3}**

**wait until** \( \exists Q \subseteq \mathcal{Q} : Q \subseteq \text{RAsrv} \)

\[ \text{minTS} \leftarrow \min\{\{m.ts\} : \]

\[ (s, m) \in \text{RA} \land s \in Q\} \]

**return** \( (m.v \text{ s.t. } (s, m) \in \text{RA} \land s \in Q) \}

**else** \( \text{Qview3} \)

\[ \text{return} (m.v \text{ s.t. } (s, m) \in \text{RA} \land s \in Q) \}

**Upon receive** \( m \) from \( s \)

if \( m.\text{read_op} = \text{read_op} \) then

if \( m.\text{type} = \text{readRelay} \) then

\[ \text{RR} \leftarrow \text{RR} \cup \{(s, m)\} \]

\[ \text{RRsrv} \leftarrow \text{RRsrv} \cup \{s\} \]

else \( \text{readAck} \)

\[ \text{RA} \leftarrow \text{RA} \cup \{(s, m)\} \]

\[ \text{RAsrv} \leftarrow \text{RAsrv} \cup \{s\} \]

At writer \( w \)

**Variables:**

\( ts \in \mathbb{N}^+, v \in \mathbb{V}, \text{wAck} \subseteq \mathcal{S} \)

Initialization:

\( ts \leftarrow 0, v \leftarrow \perp, \text{wAck} \leftarrow \emptyset \)

**function** \( \text{Write}(\text{val} : \text{input}) \)

\[ (ts, v) \leftarrow (ts + 1, \text{val}) \]

\[ \text{wAck} \leftarrow \emptyset \]

**bcast** ((\( \text{writeRequest}, ts, v, w) \)) to \( \mathcal{S} \)

**wait until** \( \exists Q \in \mathcal{Q} : Q \subseteq \text{wAck} \)

**return**

Upon receive \( m \) from \( s \)

if \( m.ts = ts \) then

\[ \text{wAck} \leftarrow \text{wAck} \cup \{s\} \]

At server \( s \)

**Variables:**

\( ts \in \mathbb{N} \text{ init } 0 ; v \in \mathbb{V} \text{ init } \perp \)

\( D \subseteq \mathcal{S} \text{ init } \{s' : Q \in \mathcal{Q} \land (s, s' \in Q)\} \)

**operations :** \( \mathcal{R} \rightarrow \mathbb{N} \text{ init } 0[\mathcal{R}] \)

**relays :** \( \mathcal{R} \rightarrow 2^S \text{ init } 0[\mathcal{R}] \)

Upon receive((\( \text{readRelay}, ts, v, r, \text{read_op}, s) \)) to \( D \cup r \)

Upon receive((\( \text{writeRequest}, ts', v', w) \))

if \( ts < ts' \) then

\[ (ts, v) \leftarrow (ts', v') \]

**send** ((\( \text{writeAck}, ts, s) \)) to \( w \)

Upon receive((\( \text{readRelay}, ts', v', r, \text{read_op}, s) \))

if \( ts < ts' \) then

\[ (ts, v, up) \leftarrow (ts', v') \]

if \( \text{operations}[r] < \text{read_op} \) then

\[ \text{operations}[r] \leftarrow \text{read_op} \]

\[ \text{relays}[r] \leftarrow \emptyset. \]

if \( \text{operations}[r] = \text{read_op} \) then

\[ \text{relays}[r] \leftarrow \text{relays}[r] \cup \{s\} \]

if \( \exists Q \subseteq \mathcal{Q} : Q \subseteq \text{relays}[r] \) then

**send** ((\( \text{writeAck}, ts, v, \text{read_op}, s) \)) to \( r \)

Examines the timestamps to determine what quorum view is observed (recall Section 2). If \( \text{Qview1} \) is observed, then all timestamps are the same, meaning that the write operation associated with the timestamp is complete, and it is safe to return the value associated with it without exchanging \( \text{r3}. \) (1,20,22). If \( \text{Qview2} \) is observed, then the write associated with the maximum timestamp \( \text{maxTS} \) is
not complete. But because there is a sole writer, it is safe to return the value associated with timestamp $\max TS\text{-}1$, i.e., the value of the preceding complete write, again without exchange $E3$ (L30-33). If $\text{qv}(3)$ is observed, then the write associated with the maximum timestamp $\max TS$ is in progress or complete. Since the reader is unable to decide which case applies, it waits for the exchange $E3$ $\text{readAck}$ messages from some quorum $Q$. The reader here returns the value associated with the minimum timestamp observed (L23-29). It is possible, due to asynchrony, that messages from $E3$ arrive from a quorum before enough messages from $E2$ are gathered. Here the reader decides as above for $E3$ (L11-13).

**Writer Protocol.** Writer $w$ increments its local timestamp $ts$ and broadcasts a $\text{writeRequest}$ message to all servers. It completes once $\text{writeAck}$ messages are received from some quorum $Q$ (L48-52).

**Server Protocol.** Server $s$ stores the value of the replica $v$ and its associated timestamp $ts$. The $\text{relays}$ array is used to store sets of processes that relayed to $s$ regarding a read operation. Destinations set $D$ is initialized to set containing all servers from every quorum that contains $s$. It is used for sending relay messages during exchange $E2$.

In exchange $E1$ of a read, upon receiving message $\langle \text{readRequest},r,\text{read}\_\text{op} \rangle$, the server creates a $\text{readRelay}$ message, containing its $ts$, $v$, and $s$, and broadcasts it in exchange $E2$ to destinations in $D$ and reader $r$ (L62-63).

In exchange $E2$, upon receiving message $\langle \text{readRelay},ts',v',r,\text{read}\_\text{op} \rangle$ $s$ compares its local timestamp $ts$ with $ts'$. If $ts < ts'$, then $s$ sets its local value and timestamp to those enclosed in the message (L69-70). Next, $s$ checks if the received $\text{readRelay}$ marks a new read by $r$, i.e., $\text{read}\_\text{op} > \text{operations}[r]$. If so, then $s$: (a) sets its local counter for $r$ to the enclosed one, $\text{operations}[r] = \text{read}\_\text{op}$; and (b) re-initializes the relay set for $r$ to an empty set, $\text{relays}[r] = \emptyset$ (L71-73). It then adds the sender of the $\text{readRelay}$ message to the set of servers that informed it regarding the read invoked by $r$ (L74-75). Once $\text{readRelay}$ messages are received from a quorum $Q$, $s$ creates a $\text{readAck}$ message and sends it to $r$ in exchange $E3$ of the read (L76-77).

Within a write operation, upon receiving message $\langle \text{writeRequest},ts',v',w \rangle$, $s$ compares its $ts$ to the received one. If $ts < ts'$, then $s$ sets its local timestamp and value to those received, and sends acknowledgment to the writer (L64-67).

### 3.2 Correctness

To prove correctness of algorithm ERATO we reason about its liveness (termination) and atomicity (safety).

**Liveness.** Termination is satisfied with respect to our failure model: at least one quorum $Q$ is non-faulty and each operation waits for messages from a quorum $Q$ of servers. Let us consider this in more detail.

**Write Operation.** Showing liveness is straightforward. Per algorithm ERATO, writer $w$ creates a line:erato:writerequest message and then it broadcasts it to all servers. Writer $w$ then waits for $\text{writeAck}$ messages from a full quorum of servers (L48-52). Since in our failure model at least one quorum is non-faulty, then writer $w$ collects $\text{writeAck}$ messages from a full quorum of live servers and write operation $\omega$ terminates.

**Read Operation.** The reader $r$ begins by broadcasting a readRequest message all servers and waiting for responses. A read operation of the algorithm ERATO terminates when the reader $r$
either (i) collects \texttt{readAck} messages from full quorum of servers or (ii) collects \texttt{readRelay} messages from a full quorum and notices \texttt{QV(1)} or \texttt{QV(2)} \cite{7, 10}. Let’s analyze case (i). Since a full quorum \( Q \) is non-faulty then at least a full quorum of servers receives the \texttt{readRequest} message. Any server \( s \) that receives this message broadcasts \texttt{readRelay} message to every server that belongs to the same quorum with, and the invoker \( r_i \). That is its destinations set \( D \cup \{ r_i \} \) \cite{62, 63}. In addition, no server ever discards any incoming \texttt{readRelay} messages. Any server, whether it is aware or not of the \texttt{readRequest}, always keeps a record of the incoming \texttt{readRelay} messages and takes action as if it is aware of the \texttt{readRequest}. The only difference between server \( s_i \) that received a \texttt{readRequest} message and server \( s_k \) that does not, is that \( s_i \) is able to broadcast \texttt{readRelay} messages, and \( s_k \) broadcasts \texttt{readRelay} messages when \( s_k \) receives the \texttt{readRequest} message. Each non-failed server receives \texttt{readRelay} messages from a full quorum of servers and sends a \texttt{readAck} message to reader \( r \). Therefore, reader \( r \) can always collect \texttt{readAck} messages from a full quorum of servers, decide on a value to return, and terminate \cite{11, 13}. In case where case (ii) never holds then the algorithm will always terminate from case (i). Thus, since any read or write operation will collect a sufficient number of messages and terminate then \textit{liveness} is satisfied.

Based on the above, it is always the case that acknowledgment messages \texttt{readAck} and \texttt{writeAck} are collected from a full quorum of servers in any read and write operation, thus ensuring \textit{liveness}.

\textbf{Atomicity.} To prove atomicity we order the operations with respect to the timestamps associated with the written values. For each execution of the algorithm there must exist a partial order \( \prec \) on the operations that satisfy conditions A1, A2, and A3 given in Section 2. Let \( ts_\pi \) be the the timestamp at the completion of \( \pi \) when \( \pi \) is a write, and the timestamp associated with the returned value when \( \pi \) is a read. We now define the partial order as follows. For two operations \( \pi_1 \) and \( \pi_2 \), when \( \pi_1 \) is any operation and \( \pi_2 \) is a write, we let \( \pi_1 \prec \pi_2 \) if \( ts_\pi_1 < ts_\pi_2 \). For two operations \( \pi_1 \) and \( \pi_2 \), when \( \pi_1 \) is a write and \( \pi_2 \) is a read we let \( \pi_1 \prec \pi_2 \) if \( ts_\pi_1 \leq ts_\pi_2 \). The rest of the order is established by transitivity, without ordering the reads with the same timestamps. We now state the following lemmas.

It is easy to see that the \( ts \) variable in each server \( s \) is monotonically increasing. This leads to the following lemma.

\textbf{Lemma 1} In any execution \( \xi \) of Erato, the variable \( ts \) maintained by any server \( s \) in the system is non-negative and monotonically increasing.

\textbf{Proof.} Upon receiving a timestamp \( ts \), a server \( s \) updates its local timestamp \( ts_s \) iff \( ts > ts_s \), \cite{65, 66, 69, 70}, and the lemma follows. \( \square \)

Next, we show that any read operation that follows a write operation, it receives \texttt{readAck} messages the servers where each included timestamp is at least as the one returned by the complete write operation.

\textbf{Lemma 2} In any execution \( \xi \) of Erato, if a read operation \( \rho \) succeeds a write operation \( \omega \) that writes \( ts \) and \( v \), i.e., \( \omega \to \rho \), and receives \texttt{readAck} messages from a quorum \( Q \) of servers, set \( RA \), then each \( s \in RA \) sends a \texttt{readAck} message to \( \rho \) with a timestamp \( ts_s \) s.t. \( ts_s \geq ts \).

\textbf{Proof.} Let \( wAck \) be the set of servers from a quorum \( Q_a \) that send a \texttt{writeAck} message to \( \omega \), let \( RelaySet \) be the set of servers from a quorum \( Q_b \) that sent \texttt{readRelay} messages to server \( s \), and let
$RA$ be the set of servers from a quorum $Q_e$ that send a readAck message to $\rho$. Notice that it is not necessary that $a \neq b \neq c$ holds.

Write operation $\omega$ is completed. By Lemma 1 if a server $s$ receives a timestamp $ts$ from a process $p$ at some time $T$, then $s$ attaches a timestamp $ts'$ s.t. $ts' \geq ts$ in any message sent at any time $T' \geq T$. Thus, every server in $wAck$, sent a writeAck message to $\omega$ with a timestamp greater or equal to $ts$. Hence, every server $s_x \in wAck$ has a timestamp $ts_{s_x} \geq ts$. Let us now examine a timestamp $ts_s$ that server $s$ sends to read operation $\rho$.

Before server $s$ sends a readAck message to $\rho$, it must receive readRelay messages from a full quorum $Q_b$ of servers, RelaySet ([76]-[77]). Since both $wAck$ and RelaySet contain messages from a full quorum of servers, and by definition, any two quorums have a non-empty intersection, then $wAck \cap RelaySet \neq \emptyset$. By Lemma 1, any server $s_x \in aAck \cap RelaySet$ has a timestamp $ts_{s_x}$ s.t. $ts_{s_x} \geq ts$. Since server $s_x \in RelaySet$ and from the algorithm, server’s $s$ timestamp is always updated to the highest timestamp it noticed ([69]-[69]), then when server $s$ receives the message from $s_x$, it will update its timestamp $ts_s$ s.t. $ts_s \geq ts_{s_x}$. Server $s$ creates a readAck message where it encloses its local timestamp and its local value, $(ts_s, v_s)$ ([77]). Each $s \in RA$ sends a readAck to $\rho$ with a timestamp $ts_s$ s.t. $ts_s \geq ts_{s_x} \geq ts$. Thus, $ts_s \geq ts$, and the lemma follows.

Now, we show that if a read operation succeeds a write operation, then it returns a value at least as recent as the one written.

**Lemma 3** In any execution $\xi$ of Erato, if a read $\rho$ succeeds a write operation $\omega$ that writes timestamp $ts$, i.e. $\omega \rightarrow \rho$, and returns a timestamp $ts'$, then $ts' \geq ts$.

**Proof.** A read operation $\rho$ terminates when it either receives (a) readRelay messages from a full quorum $Q$ or (b) readAck messages from a full quorum $Q$ ([77]).

We first examine case (b). Let’s suppose that $\rho$ receives readAck messages from a full quorum $Q$ of servers, $RA$. By lines [11]-[13], it follows that $\rho$ decides on the minimum timestamp, $ts' = minTS$, among all the timestamps in the readAck messages of the set $RA$. From Lemma 2, $minTS \geq ts$ holds, where $ts$ is the timestamp written by the last complete write operation $\omega$. Then $ts' = minTS \geq ts$ also holds. Thus, $ts' \geq ts$.

Now we examine case (a). In particular, case (a) terminates when the reader process notices either (i) $qv(1)$ or (ii) $qv(2)$ or (iii) $qv(3)$. Let $wAck$ be the set of servers from a quorum $Q_e$ that send a writeAck message to $\omega$. Since the write operation $\omega$, that wrote value $v$ associated with timestamp $ts$ is complete, and by monotonicity of timestamps in servers (Lemma 1), then at least a quorum $Q_a$ of servers has a timestamp $ts_a$ s.t. $ts_a \geq ts$. In other words, every server in $wAck$, sent a writeAck message to $\omega$ with a timestamp $ts_a$ greater or equal to $ts$.

Let’s suppose that $\rho$ receives readRelay messages from a full quorum $Q_b$ of servers, $RR$. Since both $wAck$ and $RR$ contain messages from a full quorum of servers, quorums $Q_a$ and $Q_b$, and by definition any two quorums have a non-empty intersection, then $wAck \cap RR \neq \emptyset$. Since every server in $wAck$ has a timestamp $ts_a \geq ts$ then any server $s_x \in wAck \cap RR$ has a timestamp $ts_{s_x}$ s.t. $ts_{s_x} \geq ts_a \geq ts$.

If $\rho$ noticed $qv(1)$ in $RR$, then the distribution of the timestamps indicates the existence of one and only timestamp in $RR$, $ts'$. Hence, $ts' \geq ts_{s_x} \geq ts_a \geq ts$. Based on the algorithm ([20]-[22]), the read operation $\rho$ returns value $v$ associated with $ts'$ and $ts' \geq ts$ holds.

Based on the definition of $qv(2)$, if it is noticed in $RR$, then there must exist at least two servers in $wAck \cap RR$ with different timestamps and one of them holds the maximum timestamp. Let $s_k$ be the one that holds the maximum timestamp $ts_{s_k}$ (or $maxTS$) and $s_m$ the server that
Lemma 5 In any execution $\xi$ of Erato, if $\rho_1$ and $\rho_2$ are two fast read operations, take 2 exchanges to complete, such that $\rho_1$ precedes $\rho_2$, i.e., $\rho_1 \rightarrow \rho_2$, and $\rho_1$ returns the value for timestamp $t_1$, then $\rho_2$ returns the value for timestamp $t_2 \geq t_1$.

Proof. Let the two operations $\rho_1$ and $\rho_2$ be invoked by processes with identifiers $r_1$ and $r_2$ respectively (not necessarily different). Also, let $RA_1$ and $RA_2$ be the sets of servers from quorums $Q_a$ and $Q_b$ (not necessarily different) that sent a readAck message to $r_1$ and $r_2$ during $\rho_1$ and $\rho_2$.

Assume by contradiction that read operations $\rho_1$ and $\rho_2$ exist such that $\rho_2$ succeeds $\rho_1$, i.e., $\rho_1 \rightarrow \rho_2$, and the operation $\rho_2$ returns a timestamp $t_2$ that is smaller than the $t_1$ returned by $\rho_1$, i.e., $t_2 < t_1$. Based on the algorithm, $\rho_2$ returns a timestamp $t_2$ that is smaller than the minimum timestamp received by $\rho_1$, i.e., $t_1$, if $\rho_2$ obtains $t_2$ and $v$ in the readAck message of some server $s_x \in RA_2$, and $t_2$ is the minimum timestamp received by $\rho_2$.

Let us examine if $s_x$ sends a readAck message to $\rho_1$ with timestamp $t_2$, i.e., $s_x \in RA_1$. By Lemma 4 and since $\rho_1 \rightarrow \rho_2$, then it must be the case that $t_2 \leq t_1$. According to our assumption $t_1 > t_2$, and since $t_1$ is the smallest timestamp sent to $\rho_1$ by any server in $RA_1$, then it follows that $r_1$ does not receive the readAck message from $s_x$, and hence $s_x \notin RA_1$.

Now let us examine the actions of the server $s_x$. From the algorithm, server $s_x$ collects readRelay messages from a full quorum $Q_c$ of servers before sending a readAck message to $\rho_2$ (L76-76). Let $RRSet_{s_x}$ denote the set of servers from the full quorum $Q_c$ that sent readRelay to $s_x$. Since, both $RRSet_{s_x}$ and $RA_1$ contain messages from full quorums, $Q_c$ and $Q_a$, and since any two quorums have a non-empty intersection, then it follows that $RRSet_{s_x} \cap RA_1 \neq \emptyset$.

Thus there exists a server $s_i \in RRSet_{s_x} \cap RA_1$, that sent (i) a readAck to for $\rho_1$, and (ii) a readRelay to $s_x$ during $\rho_2$. Note that $s_i$ sends a readRelay for $\rho_2$ only after it receives a read request from $\rho_2$. Since $\rho_1 \rightarrow \rho_2$, then it follows that $s_i$ sent the readAck to $\rho_1$ before sending the readRelay to $s_x$. By Lemma 4 if $s_i$ attaches a timestamp $t_{s_i}$ in the readAck to $\rho_1$, then $s_i$ attaches a timestamp $t_{s_i}'$ in the readRelay message to $s_x$, such that $t_{s_i}' \geq t_{s_i}$. Since $t_1$ is the minimum timestamp received by $\rho_1$, then $t_{s_i} \geq t_1$, and hence $t_{s_i}' \geq t_1$ as well. By Lemma 4 and since $s_x$ receives the readRelay message from $s_i$ before sending a readAck to $\rho_2$, it follows that $s_x$ sends a timestamp $t_2$ s.t. $t_2 \geq t_{s_i}' \geq t_1$. Thus, $t_2 \geq t_1$ and this contradicts our initial assumption. \qed

Lemma 5 In any execution $\xi$ of Erato, if $\rho_1$ and $\rho_2$ are two fast read operations, take 2 exchanges to complete, such that $\rho_1$ precedes $\rho_2$, i.e., $\rho_1 \rightarrow \rho_2$, and $\rho_1$ returns the value for timestamp $t_1$, then $\rho_2$ returns the value for timestamp $t_2 \geq t_1$. 

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Proof. Let the two operations $\rho_1$ and $\rho_2$ be invoked by processes with identifiers $r_1$ and $r_2$ respectively (not necessarily different). Also, let $RR_1$ and $RR_2$ be the sets of servers from quorums $Q_a$ and $Q_b$ (not necessarily different) that sent a readRelay message to $r_1$ and $r_2$ during $\rho_1$ and $\rho_2$.

The algorithm terminates in two communication exchanges when a read operation $\rho$ receives readRelay messages from a full quorum $Q$ and based on the distribution of the timestamp it either notices (a) QV(1) or (b) QV(2). We now examine the four cases.

Case (i), $\rho_1 \to \rho_2$ and both $\rho_1$ and $\rho_2$ notice QV(1). It is known that all the servers in $RR_1$ replied to $\rho_1$ with timestamp $ts_1$. Since by definition, any two quorums have a non-empty intersection it follows that $RR_1 \cap RR_2 \neq \emptyset$. From that and by Lemma 1 then every server $s_x \in RR_1 \cap RR_2$ has a timestamp $ts'$ such that $ts' \geq ts_1$. Since $\rho_2$ notices QV(1) in $RR_2$, then the distribution of the timestamps indicates the existence of one and only timestamp in $RR_2$, $ts_2$. Thus, $ts_2 \geq ts' \geq ts_1$.

Case (ii), $\rho_1 \to \rho_2$ and $\rho_1$ notices QV(1) and $\rho_2$ notices QV(2). It is known that all the servers in $RR_1$ replied to $\rho_1$ with timestamp $ts_1$. Since by definition, any two quorums have a non-empty intersection it follows that $RR_1 \cap RR_2 \neq \emptyset$. From that and by Lemma 1 then every server $s_x \in RR_1 \cap RR_2$ has a timestamp $ts'$ such that $ts' \geq ts_1$. Since $\rho_2$ notices QV(2) in $RR_2$, then there must exist at least two servers in $RR_1 \cap RR_2$ with different timestamps and one of them holds the maximum timestamp. Let $s_m$ be the one that holds the maximum timestamp $ts_m$ s.t. $\text{max}TS = ts_m > ts_1$. Since (a) any server $s_x \in RR_1 \cap RR_2$ has a timestamp $ts'$ s.t. $ts' \geq ts_1$, and (b) $s_k \in RR_1 \cap RR_2$ holds the maximum timestamp $ts_k$ (or maxTS), and (c) $s_m \in RR_1 \cap RR_2$ and (d) $\text{max}TS = ts_k > ts_m$ then it follows that $\text{max}TS = ts_k > ts_m \geq ts_1$. Thus, $ts_k$ (or maxTS) must be strictly greater from $ts_1$, $\text{max}TS = ts_k > ts_1$. Based on the algorithm, when $\rho$ notices QV(2) in $RR_2$ then it returns the value associated with the previous maximum timestamp, that is the value associated with $\text{max}TS-1$ (110-113). Since $\text{max}TS = ts_k > ts_1$, then for the previous maximum timestamp, denoted by $ts_2$, which is only one unit less than $\text{max}TS$, then the following holds, $\text{max}TS > \text{max}TS-1 = ts_2 \geq ts_1$, thus $ts_2 \geq ts_1$.

Case (iii), $\rho_1 \to \rho_2$ and $\rho_1$ notices QV(2) and $\rho_2$ notices QV(1). Since $\rho_1$ notices QV(2) in $RR_1$ then there exist a subset of servers $Smax$, $Smax \subset RR_1$, that hold the maximum timestamp $\text{max}TS$ and a subset of servers $Spre$, $Spre \subset RR_1$, that hold timestamp $\text{max}TS-1$. Based on the algorithm, $\rho_1$ returns $ts_1$ s.t. $ts_1 = \text{max}TS - 1$ from the set of servers in $Spre$. Since $RR_1 \cap RR_2 \neq \emptyset$, and QV(1) indicates the existence of one and only timestamp, then $\rho_1$ can notice QV(1) in two cases; (a) all the servers in $RR_1 \cap RR_2 \subseteq Spre$ or (b) all the servers in $RR_1 \cap RR_2 \subseteq Smax$. By Lemma 1 and if (a) holds then $\rho_2$ returns $ts_2$ s.t. $ts_2 \geq ts_1$; else, if (b) holds then $\rho_2$ returns $ts_2$ s.t. $ts_2 > ts_1$.

Case (iv), $\rho_1 \to \rho_2$ and both $\rho_1$ and $\rho_2$ notice QV(2). The distribution of the timestamps that $\rho_1$ notices, indicates that the write operation associated with the maximum timestamp, $\text{max}TS$, is ongoing, i.e., not completed. By the property of well formedness and the existence of a sole writer in the system then we know that $ts_1$ corresponds to the latest complete write operation, $ts_1 = \text{max}TS-1$. By Lemma 3 $\rho_2$ will not be able to return a timestamp $ts_2$ s.t. $ts_2 < \text{max}TS - 1$. Thus $ts_2 \geq ts_1$ holds and the lemma follows.

Lemma 6 In any execution $\xi$ of ERATO, if $\rho_1$ and $\rho_2$ are two read operations such that $\rho_1$ precedes $\rho_2$, i.e., $\rho_1 \to \rho_2$, and $\rho_1$ returns timestamp $ts_1$, then $\rho_2$ returns a timestamp $ts_2$, s.t. $ts_2 \geq ts_1$.

Proof. We are interested to examine the cases where one of the read operation is fast and the other is semifast. In particular, cases (i) $\rho_1 \to \rho_2$ and $\rho_1$ is semifast and $\rho_2$ is fast and (ii) $\rho_1 \to \rho_2$ and $\rho_1$ is fast and $\rho_2$ is semifast.
Let the two operations $\rho_1$ and $\rho_2$ be invoked by processes with identifiers $r_1$ and $r_2$ respectively (not necessarily different). Also, let $RR_1, RA_1$ and $RR_2, RA_2$ be the sets of servers from full quorums (not necessarily different) that sent a readRelay and readAck message to $\rho_1$ and $\rho_2$ respectively.

We start with case (i). Since read operation $\rho_1$ is semifast, then based on the algorithm, the timestamp $ts_1$ that is returned it is also the minimum timestamp noticed in $RA_1$. Before a server $s$ sends readAck messages to $\rho_1$ (that form RA), it must receive readRelay messages from a full quorum of servers. Thus, by Lemma 1 monotonicity of the timestamps at the servers we know that the minimum timestamp that a full quorum has by the end of $\rho_1$ is $ts_1$. Read operation $\rho_2$ receives readRelay messages from a full quorum of servers, $RR_2$. By definition of quorums, since both $RA_1$ and $RR_2$ are from a full quorum of servers then it follows that $RA_1 \cap RR_2 \neq \emptyset$. Thus every server $s_x \in RA_1 \cap RR_2$ holds a timestamp $ts'$ s.t. $ts' \geq ts_1$.

If $\rho_2$ notices $QV(1)$ in $RR_2$ then the distribution of the timestamps in $RR_2$ indicates the existence of one and only timestamp, maxTS. From the above, it follows that for the timestamp $ts_2$ that $\rho_2$ returns $maxTS = ts_2 \geq ts' \geq ts_1$ holds.

On the other hand, if $\rho_2$ notices $QV(2)$ in $RR_2$, then based on the distributions of the timestamps in $QV(2)$ there must exist at least two servers in $RA_1 \cap RR_2$ with different timestamps and the one must be the maximum. Since every server $s_x \in RA_1 \cap RR_2$ holds a timestamp $ts'$ s.t. $ts' \geq ts_1$ then the maximum timestamp maxTS cannot be equal to $ts_1$. If that was the case, $\rho_2$ would have noticed $QV(1)$. In particular, now $maxTS > ts_1$ holds. Based on the algorithm, when $\rho$ notices $QV(2)$ in $RR_2$ then it returns the value $v$ associated with the previous maximum timestamp, that is the value associated with $maxTS-1$. Since $maxTS > ts_1$, then for the previous maximum timestamp, denoted by $ts_2$, which is only one unit less than $maxTS$, then the following holds, $maxTS > maxTS - 1 = ts_2 \geq ts_1$, thus $ts_2 \geq ts_1$.

We now examine case (ii). Since $\rho_1$ is fast, it follows that it has either noticed $QV(1)$ or $QV(2)$ in $RR_1$. If $QV(1)$ was noticed, and $\rho_1$ returned a value associated with maximum timestamp $ts_1$, then by the completion of $\rho_1$ a full quorum has a timestamp $ts'$ s.t. $ts' \geq ts_1$. Now, since read operation $\rho_2$ is semifast, then based on the algorithm, the timestamp $ts_2$ that is returned it is the minimum timestamp noticed in $RA_2$. Before a server $s$ sends readAck messages to $\rho_2$ (that form RA), it must receive readRelay messages from a full quorum of servers, RelaySet. By Lemma 1 monotonicity of the timestamps at the servers and $RR_1 \cap RelaySet \neq \emptyset$, then every server in $RA_2$ has a timestamp $ts_2$ s.t. $ts_2 \geq ts' \geq ts_1$.

If $QV(2)$ was noticed in $RR_1$, based on the algorithm, $\rho_1$ returned a value associated with previous maximum timestamp, that is $ts_1$. By the completion of $\rho_1$ a full quorum has a timestamp $ts'$ s.t. $ts' \geq ts_1$. Read operation $\rho_2$ is semifast, and the returned timestamp $ts_2$ is the minimum timestamp noticed in $RA_2$. A server $s$ sends readAck messages to $\rho_2$ (that form RA), when receives readRelay messages from a full quorum of servers, RelaySet. By Lemma 1 and since $RR_1 \cap RelaySet \neq \emptyset$, then every server in $RA_2$ has a timestamp $ts_2$ s.t. $ts_2 \geq ts' \geq ts_1$.

The rest of the cases are proved in Lemmas 4 and 5 and the lemma follows. \[\square\]

**Theorem 7** Algorithm Erato implements an atomic SWMR object.

**Proof.** We now use the lemmas above and the partial order definition to reason about each of the three conditions A1, A2 and A3.

A1 For any $\pi_1, \pi_2 \in \Pi$ such that $\pi_1 \rightarrow \pi_2$, it cannot be that $\pi_2 < \pi_1$.

When the two operations $\pi_1$ and $\pi_2$ are reads and $\pi_1 \rightarrow \pi_2$ holds, then from Lemma 4 it follows that the timestamp of $\pi_2$ is no less than the one of $\pi_1$, $ts_{\pi_2} \geq ts_{\pi_1}$. If $ts_{\pi_2} > ts_{\pi_1}$ then by the ordering
definition $\pi_1 < \pi_2$ is satisfied. When $ts_{\pi_2} = ts_{\pi_1}$ then the ordering is not defined, thus it cannot be the case that $\pi_2 < \pi_1$. If $\pi_2$ is a write, the sole writer generates a new timestamp by incrementing the largest timestamp in the system. By well-formedness (see Section 2), any timestamp generated in any write operation that precedes $\pi_2$ must be smaller than $ts_{\pi_2}$. Since $\pi_1 \rightarrow \pi_2$, then it holds that $ts_{\pi_1} < ts_{\pi_2}$. Hence, by the ordering definition it cannot be the case that $\pi_2 < \pi_1$. Lastly, when $\pi_2$ is a read and $\pi_1$ a write and $\pi_1 \rightarrow \pi_2$ holds, then from Lemma 3 it follows that $ts_{\pi_2} \geq ts_{\pi_1}$. By the ordering definition, it cannot hold that $\pi_2 < \pi_1$ in this case either.

**A2** For any write $\omega \in \Pi$ and any operation $\pi \in \Pi$, then either $\omega < \pi$ or $\pi < \omega$.

If the timestamp returned from $\omega$ is greater than the one returned from $\pi$, i.e. $ts_\omega > ts_\pi$, then $\pi < \omega$ follows directly. Similarly, if $ts_\omega < ts_\pi$ holds, then $\omega < \pi$ follows. If $ts_\omega = ts_\pi$, then it must be that $\pi$ is a read and $\pi$ discovered $ts_\omega$ in a quorum view QV(1) or QV(3). Thus, $\omega < \pi$ follows.

**A3** Every read operation returns the value of the last write preceding it according to $\prec$ (or the initial value if there is no such write).

Let $\omega$ be the last write preceding read $\rho$. From our definition it follows that $ts_{\rho} \geq ts_\omega$. If $ts_{\rho} = ts_\omega$, then $\rho$ returns the value conveyed by $\omega$ to some servers in a quorum $Q_i$ satisfying either QV(1) or QV(3). If $ts_{\rho} > ts_\omega$, then $\rho$ obtains a larger timestamp, but such a timestamp can only be created by a write that succeeds $\omega$, thus $\omega$ does not precede the read and this cannot be the case. Lastly, if $ts_{\rho} = 0$, no preceding writes exist, and $\rho$ returns the initial value.

Having shown liveness and atomicity of algorithm Erato the result follows.

### 3.3 Performance

We now assess the performance of Erato in terms of (i) latency of read and write operations as measured by the number of communication exchanges, and (ii) the message complexity of read and write operations.

**Communication and Message Complexity.** By inspection of the code, write operations take 2 exchanges and read operations take either 2 or 3 exchanges. The (worst case) message complexity of write operations is $2|S|$ and of read operations is $|S|^2 + 2|S|$, as follows from the structure of the algorithm. We now give additional details.

**Operation Latency.** Write operation latency: According to algorithm Erato, writer $w$ sends line:eratowriterequest messages to all servers during exchange $E1$ and waits for writeAck messages from a full quorum of servers during $E2$. Once the writeAck messages are received, no further communication is required and the write operation terminates. Therefore, any write operation consists of 2 communication exchanges.

Read operation latency: A reader sends a readRequest message to all the servers in the first communication exchange $E1$. Once the servers receive the readRequest message they broadcast a readRelay message to all servers and the reader in exchange $E2$. The reader can terminate at the end of the $E2$ if it receives readRelay messages and based on the distribution of the timestamp it notices QV(1) or QV(2). If this is not the case, the operation goes into the third exchange $E3$. Thus read operations terminate after either 2 or 3 communication exchanges.

**Message Complexity.** We measure operation message complexity as the worst case number of exchanged messages in each read and write operation. The worst case number of messages corresponds to failure-free executions where all participants send messages to all destinations according to the protocols.
Write operation: A single write operation in algorithm Erato takes 2 communication exchanges. In the first exchange $e_1$, the writer sends a writeRequest message to all the servers in $S$. The second exchange $e_2$, occurs when all servers in $S$ send a writeAck message to the writer. Thus, at most $2|S|$ messages are exchanged in a write operation.

Read operation: Read operations in the worst case take 3 communication exchanges. Exchange $e_1$ occurs when a reader sends a readRequest message to all servers in $S$. The second exchange $e_2$ occurs when servers in $S$ send readRelay messages to all servers in $S$ and to the requesting reader. The final exchange $e_3$ occurs when servers in $S$ send a readAck message to the reader. Summing together $|S| + (|S|^2 + |S|) + |S|$, shows that in the worst case, $|S|^2 + 3|S|$ messages are exchanged during a read operation.

4 MWMR Algorithm Erato-mw

We now aim for a MWMR algorithm that involves two or three communications exchanges per read operation and four exchanges per write operation. The read protocol of algorithm Erato relies on the fact of the sole writer in the system: based on the distribution of the timestamp in a quorum $Q$, if the reader knows that the write operation is not complete, then any previous write is complete (by well-formedness). In the MWMR setting this does not hold due to the possibility of concurrent writes. Consequently, algorithm Erato-mw, in order to allow operations to terminate in either two or three communication exchanges, adapts the read protocol from algorithm OhMam in combination with the iterative technique using quorum views of CwFr. The latter approach not only predicts the completion status of a write operation, but also detects the last potentially complete write operation. The code is given in in Algorithm 2.

4.1 Detailed Algorithm Description

To impose an ordering on the values written by the writers we associate each value with a tag $tg$ defined as the pair $(ts, id)$, where $ts$ is a timestamp and $id$ the identifier of a writer. Tags are ordered lexicographically (cf. [12]).

Reader Protocol. Readers use state variables similarly to algorithm Erato. Reader $r$ broadcasts a readRequest message to all servers, then awaits either (a) readRelay messages from some quorum, or (b) readAck messages from some quorum (L10-14). The key departure here is in how the reader handles case (a) when $qV(2)$ is detected, which indicates that the write associated with the maximum tag is not complete. Here the reader considers past history and discovers the tag associated with the last complete write. This is accomplished in an iterative manner, by removing the servers that respond with the maximum tag in the responding quorum $Q$ and repeating the analysis (L21-39). During the iterative process, if $r$ detects $qV(1)$ it returns the value associated with the maximum tag discovered during the current iteration. If no iteration yields $qV(1)$, then eventually $r$ observes $qV(3)$. In the last case, $qV(3)$ is detected when a single server remains in some intersection of $Q$. If so, the reader waits readAck messages to arrive from some quorum, and returns the value associated with the minimum tag. If case (b) happens before case (a), then $r$ proceeds identically as in the case where $qV(3)$ is detected (L15-19).

Writer Protocol. Similarly to the four-exchange implementation [12], a writer broadcasts a writeDiscover message to all servers, and awaits “fresh” discoverAck messages from some quorum $Q$ (L55-58). Among these responses the writer finds the maximum timestamp, $maxTS$, increments
Algorithm 2 Reader, Writer and Server Protocols for MWMR algorithm ERATO-MW

1. At each reader $r$
2. Variables:
3. $v \in V$; $read_{op} \in N$; $minTAG, maxTAG \in T$
4. $RR, RA, maxACK \subseteq S \times M$ init $\emptyset$
5. $RRsrv, RAsrv, maxTGrsv \subseteq S$ init $\emptyset$
6. Initialization:
7. $minTAG \leftarrow (0, 0)$, $maxTAG \leftarrow (0, 0)$
8. $v \leftarrow \bot$, $read_{op} \leftarrow 0$
9. function Read
10. $read_{op} \leftarrow read_{op} + 1$
11. $(RR, RA, maxACK) \leftarrow (\emptyset, \emptyset, \emptyset)$
12. $(RRsrv, RAsrv, maxTGrsv) \leftarrow (\emptyset, \emptyset, \emptyset)$
13. $bcast((readRequest, r, read_{op}))$ to $S$
14. wait until $(\exists Q \in Q : Q \subseteq RRsrv \lor Q \subseteq RAsrv)$
15. if $(\exists Q \in Q : Q \subseteq RAsrv)$ then
16. $minTAG \leftarrow \min((m.ts, m.id) :$
17. $(s, m) \in RA \land s \in Q)$
18. return $(m.v \ s.t. (s, m) \in RA \land s \in Q$
19. $\land (m.ts, m.id) = minTAG)$
20. else if $(\exists Q \in Q : Q \subseteq RRsrv)$ then
21. while $(Q \neq \emptyset)$ do
22. $maxTAG \leftarrow \max((m.ts, m.id) :$
23. $(s, m) \in RA \land s \in Q)$
24. $maxACK \leftarrow \{s, m) \in RR : s \in Q \land$
25. $(m.ts, m.id) = maxTAG)$
26. $maxTGrsv \leftarrow \{s \in Q :$
27. $(s, m) \in maxACK)$
28. if $Q \subseteq maxTGrsv$ then
29. /* Qview1**/ return $(m.v \ s.t. (s, m) \in maxACK)$
30. if $(\exists Q' \in Q, Q' \neq Q : Q' \cap Q \subseteq maxTGrsv)$
31. then
32. /* Qview2**/ $Q \leftarrow Q - maxTGrsv$
33. Upon receive $m$ from $s$
34. if $m.read_{op} = read_{op}$ then
35. $RR \leftarrow RR \cup \{s, m\}$
36. $RRsrv \leftarrow RRsrv \cup \{s\}$
37. else // readAck
38. $RA \leftarrow RA \cup \{s, m\}$
39. $RAsrv \leftarrow RAsrv \cup \{s\}$
40. At each writer $w$
41. Variables:
42. $ts \in N$, $v \in V$, $write_{op} \in N$, $maxTS \in N$
43. $Acks \subseteq S \times M$ init $\emptyset$, $AcksSrv \subseteq S$ init $\emptyset$
44. Initialization:
45. $ts \leftarrow 0$, $v \leftarrow \bot$, $write_{op} \leftarrow 0$, $maxTS \leftarrow 0$
46. function Write(val : input)
47. $write_{op} \leftarrow write_{op} + 1$
48. $(Acks, AcksSrv) \leftarrow (\emptyset, \emptyset)$
49. $bcast((writeDiscover, write_{op}, w))$ to $S$
50. wait until $(\exists Q \in Q : Q \subseteq AcksSrv)$
51. $maxTS \leftarrow \max((m.ts) :$
52. $(s, m) \in Acks \land s \in Q)$
53. $(ts, id, v) \leftarrow (maxTS + 1, i, val)$
54. $write_{op} \leftarrow write_{op} + 1$
55. $(Acks, AcksSrv) \leftarrow (\emptyset, \emptyset)$
56. $bcast((writeRequest, ts, v, write_{op}))$ to $S$
57. wait until $(\exists Q \in Q : Q \subseteq AcksSrv)$
58. return()
59. Upon receive $m$ from $s$
60. if $m.write_{op} = write_{op}$ then
61. $Acks \leftarrow Acks \cup \{s, m\}$
62. $AcksSrv \leftarrow AcksSrv \cup \{s\}$
63. At server $s$
64. Variables and Initialization:
65. $ts \in N$ init $0$, $id \in W$ init $\bot$, $v \in V$ init $\bot$
66. operations : $R \rightarrow N$ init $0^{\infty}$
67. write_ops : $W \rightarrow N$ init $0^{\infty}$
68. relays : $R \rightarrow 2^S$ init $0^{\infty}$
69. $D \subseteq S$ init $\{s : (\exists Q \in Q, (s, s_i \in Q)\}
70. Upon receive((writeDiscover, write_{op}, w))
71. send ((discoverAck, ts, id, s_i)) to $w$
72. Upon receive
73. $((writeRequest, ts', v', id', write_{op}, w))$
74. if write_ops[w] < write_{op} then
75. write_ops[w] \leftarrow write_{op}
76. if $(ts < ts') \lor (ts = ts' \land id < id')$ then
77. $(ts, id, v) \leftarrow (ts', id', v')$
78. send((writeAck, write_{op}, s)) to $w$
79. Upon receive((readRequest, r, read_{op}))
80. $bcast(readRelay, ts, id, v, r, read_{op}, s)$ to $D \cup r$
81. Upon receive((readRelay, ts', id', v', r, read_{op}, s))
82. if $(ts < ts') \lor (ts = ts' \land id < id')$ then
83. $(ts, id, v) \leftarrow (ts', id', v')$
84. if operations[r] < read_{op} then
85. operations[r] \leftarrow read_{op}; relays[r] \leftarrow \emptyset.
86. if operations[r] = read_{op} then
87. relays[r] \leftarrow relays[r] \cup \{s\}
88. if $(\exists Q \in Q : Q \subseteq relays[r])$ then
89. send ((readAck, ts, id, v, read_{op}, s)) to $r$
it, and associates it and its own id with the new value by broadcasting the new timestamp, its id, and the new value in a writeRequest message to all servers. The write completes when writeAck messages are received from some quorum Q (L60-65).

**Server Protocol.** Servers react to messages from readers exactly as in Algorithm 1. We now describe how the messages from writers are handled.

1. Upon receiving message \langle writeDiscover, write_op, w \rangle, server s replies with a discoverAck message containing its local tag and value. (L78-79).

2. Upon receiving message \langle writeRequest, ts', id', v', write_op, w \rangle, server s compares lexicographically its local tag with the received one. If (ts, id) < (ts', id'), then s updates its local information and replies using writeAck message (L81-86).

### 4.2 Correctness

To prove correctness of algorithm ERATO-MW we reason about its liveness (termination) and atomicity (safety) as in Section 3.2 except using tags instead of timestamps.

**Liveness.** Termination is satisfied with respect to our failure model: at least one quorum Q is non-faulty and each operation waits for messages from a quorum Q of servers. Let us consider this in more detail.

**Write Operation.** Writer w finds the maximum tag by broadcasting a discover message to all servers and waits to collect discoverAck replies from a full quorum of servers (L55-58). Since a full quorum of servers is non-faulty, then at least a full quorum of live servers will collect the discover messages and reply to writer w. Once the maximum timestamp is determined, then writer w updates its local tag and broadcasts a writeRequest message to all servers. Writer w then waits to collect writeAck replies from a full quorum of servers before it terminates. Again, at a full quorum of servers will collect the writeRequest messages and will reply to writer w (L60-65).

**Read Operation.** A read operation of the algorithm ERATO-MW terminates when the reader r either (i) collects readAck messages from full quorum of servers or (ii) collects readRelay messages from a full quorum and throughout the iterative procedure it notices QV(1) or QV(3) (L21-39). Case (i) is identical as in Algorithm ERATO and liveness is ensured as reasoned in section 3.2. For case (ii), in the worst case, during the iterative analysis QV(3) will be noticed once one server remains in the replying quorum. This is identical to case (i) and the case follows.

Based on the above, any read or write operation collect a sufficient number of messages to terminate, guaranteeing liveness.

**Atomicity.** As given in Section 3.2, atomicity can be reasoned about in terms of timestamps. In the MWMR setting we use tags instead of timestamps, and here we show how algorithm ERATO-MW satisfies atomicity using tags. It is easy to see that the tg variable in each server s is monotonically increasing. This leads to the following lemma.

**Lemma 8** In any execution ξ of ERATO-MW, the variable tg maintained by any server s in the system is non-negative and monotonically increasing.

**Proof.** When server s receives a tag tg then s updates its local tag tg_s if and only if tg > tg_s (L84-85).

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Lemma 9 In any execution $\xi$ of Erato-MW, if a write $\omega$ writes tag $tg'$ and succeeds a read operation $\rho$ that returns a tag $tg$, i.e., $\rho \rightarrow \omega$, then $tg' > tg$.

Proof. Let $RR$ be the set of servers that belong to quorum $Q_a$ and sent readRelay messages to $\rho$. Let $dAck$ be the set of servers from a quorum $Q_b$ that sent discoverAck messages to $\omega$. Let $wAck$ be the set of servers from a quorum $Q_c$ that sent writeAck messages to $\omega$ and let $RA$ be the set of servers from a quorum $Q_d$ that sent readAck messages to $\rho$. It is not necessary that $a \neq b \neq c \neq d$ holds.

Based on the read protocol, the read operation $\rho$ terminates when it either receives (a) readRelay messages from a full quorum $Q$ or (b) readAck messages from a full quorum $Q$ (L10-14).

Case (a), based on the algorithm, during the iterative analysis $\rho$ terminates once it notices $QV(1)$ or $QV(3)$ in the messages received from $RR$. If $QV(1)$ is noticed then the distribution of the timestamps indicates the existence of one and only tag in $Q_a$ and that is, the maximum tag in $Q_a$ at the current iteration. Read $\rho$ returns the value associated with the current maximum tag, $tg$ and terminates. The following writer $\omega$, initially it broadcasts a writeDiscover message to all servers, and it then awaits for “fresh” discoverAck messages from a full quorum $Q_b$, that is, set $dAck$ (155-58). Observe that each of $RR$ and $dAck$ sets are from a full quorum of servers, $Q_a$ and $Q_b$ respectively, and so $RR \cap dAck \neq \emptyset$. By Lemma 8 any server $s_k \in RR \cap dAck$ has a tag $tg_{sk}$ s.t. $tg_{sk} \geq tg$. Since $\omega$ generates a new local tag-value $(tg', v)$ pair in which it assigns the timestamp to be one higher than the one discovered in the maximum tag from set $dAck$, it follows that $tg' > tg$. Write operation $\omega$ broadcasts the value to be written associated with $tg'$ in a writeRequest message to all servers and it awaits for writeAck messages from a full quorum $Q_c$ before completion, set $wAck$ (160-65). Observe that each of $dAck$ and $wAck$ sets are from a full quorum of servers, $Q_b$ and $Q_c$ respectively, and so $dAck \cap wAck \neq \emptyset$. By Lemma 8 any server $s_k \in dAck \cap wAck$ has a tag $tg_{sk}$ s.t. $tg_{sk} \geq tg' > tg$ and the result for this case follows.

Now we examine if $QV(3)$ is noticed. When that holds, based on the algorithm, the reader awaits readAck messages from a full quorum $Q$ of servers, set $RA$. By lines 15 - 19 it follows that $\rho$ decides on the minimum tag, $tg = \text{min} TG$, among all the tags in the readAck messages of the set $RA$ and terminates. Again, $\omega$, initially it broadcasts a writeDiscover message to all servers, and it then awaits for “fresh” discoverAck messages from a full quorum $Q_b$, that is, set $dAck$. Each of $RA$ and $dAck$ sets are from a full quorum of servers, $Q_d$ and $Q_b$ respectively, and so $RA \cap dAck \neq \emptyset$. By Lemma 8 any server $s_k \in RA \cap dAck$ has a tag $tg_{sk}$ s.t. $tg_{sk} \geq tg$. Since $\omega$ generates a new local tag-value $(tg', v)$ pair in which it assigns the timestamp to be one higher than the one discovered in the maximum tag from set $dAck$, it follows that $tg' > tg$. Furthermore, $\omega$ broadcasts the value to be written associated with $tg'$ in a writeRequest message to all servers and it awaits for writeAck messages from a full quorum $Q_c$ before completion, set $wAck$ (160-65). Observe that each of $dAck$ and $wAck$ sets are from a full quorum of servers, $Q_b$ and $Q_c$ respectively, and so $dAck \cap wAck \neq \emptyset$. By Lemma 8 any server $s_k \in dAck \cap wAck$ has a tag $tg_{sk}$ s.t. $tg_{sk} \geq tg' > tg$ and the result for this case follows.

Lastly, case (b) where read $\rho$ terminates because it received readAck messages from a full quorum of servers $Q$, it is the same as in case (a) when reader observers $QV(3)$ and the lemma follows. □

Lemma 10 In any execution $\xi$ of Erato-MW, if a write $\omega_1$ writes tag $tg_1$ and precedes a write $\omega_2$ that writes tag $tg_2$, i.e., $\omega_1 \rightarrow \omega_2$, then $tg_2 > tg_1$.  

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Proof. Let $w\text{Ack}_1$ be the set of servers from a full quorum $Q_a$ that send a writeAck message within write operation $\omega_1$. Let $d\text{Ack}_2$ be the set of servers from a full quorum $Q_b$ (not necessarily different from $Q_a$) that send a discoverAck message within write operation $\omega_2$.

Lemma assumes that $\omega_1$ is complete. By Lemma 8, we know that if a server $s$ receives a tag $tg$ from a process $p$, then $s$ includes tag $tg'$ s.t. $tg' \geq tg$ in any subsequent message. Thus, servers in $w\text{Ack}_1$ send a writeAck message within $\omega_1$ with tag at least tag $tg_1$.

Once $\omega_2$ is invoked, it collects discoverAck messages from a full quorum of servers in the set, $d\text{Ack}_2$ (L55-58). Since $Q_a \subseteq w\text{Ack}_1$ and $Q_b \subseteq d\text{Ack}_2$ then $w\text{Ack}_1 \cap d\text{Ack}_2 \neq \emptyset$. By Lemma 8, any server $s_k \in w\text{Ack}_1 \cap d\text{Ack}_2$ has a tag $tg_{s_k}$ s.t. $tg_{s_k} \geq tg_1$. Thus, the invoker of $\omega_2$ discovers the maximum tag, $maxTG$, from the tags found in $d\text{Ack}_2$ s.t. $maxTG \geq tg_{s_k} \geq tg_1$ (L60). It then increases the timestamp from in the maximum tag discovered by one, sets it’s local tag to that and associates it with its id $i$ and the value $val$ to be written, $local\_tag = (maxTS + 1, i, val)$ (L61). We know that, $local\_tag > maxTG \geq tg_1$, hence $local\_tag > tg_1$.

Lastly, $\omega_2$ attaches its local tag $local\_tag$ in a writeRequest message which it broadcasts to all the servers, and terminates upon receiving writeAck messages from a full quorum of servers. By Lemma 8, $\omega_2$ receives writeAck messages with a tag $tg_2$ s.t. $tg_2 \geq local\_tag > tg_1$ hence $tg_2 > tg_1$. This completes the Proof of the lemma.

We now show that any read operation that follows a write operation, and it receives readAck messages the servers where each included tag is at least as the one returned by the complete write operation.

Lemma 11 In any execution $\xi$ of Erato-mw, if a read operation $\rho$ succeeds a write operation $\omega$ that writes $tg$ and $v$, i.e., $\omega \rightarrow \rho$, and receives readAck messages from a quorum $Q$ of servers, set RA, then each $s \in RA$ sends a readAck message to $\rho$ with a tag $tg_s$ s.t. $tg_s \geq tg$.

Proof. Let $w\text{Ack}$ be the set of servers from a quorum $Q_a$ that send a writeAck message to $\omega$, let RelaySet be the set of servers from a quorum $Q_b$ that sent readRelay messages to server $s$, and let RA be the set of servers from a quorum $Q_c$ that send a readAck message to $\rho$. Notice that it is not necessary that $a \neq b \\neq c$ holds.

Write operation $\omega$ is completed. By Lemma 8, if a server $s$ receives a tag $tg$ from a process $p$ at some time $t$, then $s$ attaches a tag $tg'$ s.t. $tg' \geq ts$ in any message sent at any time $t' \geq t$. Thus, every server in $w\text{Ack}$, sent a writeAck message to $\omega$ with a tag greater or equal to $tg$. Hence, every server $s \in w\text{Ack}$ has a tag $tg_s \geq tg$. Let us now examine a tag $tg_s$ that server $s$ sends to read operation $\rho$.

Before server $s$ sends a readAck message to $\rho$, it must receive readRelay messages from a full quorum $Q_b$ of servers, RelaySet (L96-97). Since both $w\text{Ack}$ and RelaySet contain messages from a full quorum of servers, and by definition, any two quorums have a non-empty intersection, then $w\text{Ack} \cap RelaySet \neq \emptyset$. By Lemma 8, any server $s_x \in a\text{Ack} \cap RelaySet$ has a tag $tg_{s_x}$ s.t. $tg_{s_x} \geq tg$. Since server $s_x \in RelaySet$ and from the algorithm, server’s $s$ tag is always updated to the highest tag it noticed (L90-91), then when server $s$ receives the message from $s_x$, it will update its tag $tg_s$ s.t. $tg_s \geq tg_{s_x}$. Server $s$ creates a readAck message where it encloses its local tag and its local value, $(tg_s, v_s)$ (L97). Each $s \in RA$ sends a readAck to $\rho$ with a tag $tg_s$ s.t. $tg_s \geq tg_{s_x} \geq tg$. Thus, $tg_s \geq tg$, and the lemma follows.

Now, we show that if a read operation succeeds a write operation, then it returns a value at least as recent as the one written.
Lemma 12 In any execution $\xi$ of ERATO-MW, if a read $\rho$ succeeds a write operation $\omega$ that writes tag $tg$, i.e. $\omega \rightarrow \rho$, and returns a tag $tg'$, then $tg' \geq tg$.

Proof. A read operation $\rho$ terminates when it either receives (a) readRelay messages from a full quorum $Q$ or (b) readAck messages from a full quorum $Q$ (L10-L14).

We first examine case (b). Let’s suppose that $\rho$ receives readAck messages from a full quorum $Q$ of servers, $RA$. By lines 15 - 19, it follows that $\rho$ decides on the minimum tag, $tg' = \min TG$, among all the tags in the readAck messages of the set $RA$. From Lemma 11, $\min TG \geq tg$ holds, where $tg$ is the tag written by the last complete write operation $\omega$. Then $tg' = \min TG \geq tg$ also holds. Thus, $tg' \geq tg$.

Now we examine case (a). Case (a) is an iterative procedure that terminates when the reader notices either (i) QV(1) or (iii) QV(3). When QV(2) is observed then it is the case where the write associated with the maximum tag is not yet complete, thus we proceed to the next iteration to discover the latest potentially complete write. This, by removing all the servers with the maximum tag from $Q$ and repeating the analysis. If no iteration was interrupted because of QV(1) then eventually QV(3) will be noticed, when a single server remains in some intersection of $Q$ (L21-L39).

Let $wAck$ be the set of servers from a quorum $Q_a$ that send a writeAck message to $\omega$. Since the write operation $\omega$, that wrote value $v$ associated with tag $tg$ is complete, and by monotonicity of tags in servers (Lemma 3), then at least a quorum $Q_a$ of servers has a tag $tg_a$ s.t. $tg_a \geq tg$.

Let’s suppose that $\rho$ receives readRelay messages from a full quorum $Q_b$ of servers, $RR$. Since both $wAck$ and $RR$ contain messages from a full quorum of servers, quorums $Q_a$ and $Q_b$, and by definition any two quorums have a non-empty intersection, then $wAck \cap RR \neq \emptyset$. Since every server in $wAck$ has a tag $tg_a \geq tg$ then any server $s_x \in wAck \cap RR$ has a tag $tg_{s_x}$ s.t. $tg_{s_x} \geq tg_a \geq tg$.

Assume by contradiction that at the $i^{th}$ iteration $\rho$ noticed QV(1) in $RR$ and returned a tag $tg'$ s.t. $tg' < tg$. Since every server $s_x \in wAck \cap RR$ has a tag $tg_{s_x}$ s.t. $tg_{s_x} \geq tg$ and since QV(1) returned a tag $tg'$ s.t. $tg' < tg$, then it must be the case that none of the servers in $wAck \cap RR$ were participating in QV(1). Therefore, it must be the case that all servers in $wAck \cap RR$ were removed during the analysis at a previous iteration $k$, s.t. $k < i$. However, we know that the iterative procedure, in the worst case, it will notice QV(3) once a single server remains in an intersection of the quorum we examine. This contradicts the fact that all servers in $wAck \cap RR$ were removed from $Q_a$ during the analysis. Thus, if QV(1) is noticed, then the distribution of the tags yielded the existence of one and only tag, the current maximum tag. At least one server $s_x$ from $\in wAck \cap RR$ will participate in QV(1), hence $\rho$ will return a tag $tg'$ s.t. $tg' = tg_{s_x} \geq tg$.

Lastly, when QV(3) is noticed during the iterative procedure then $\rho$ waits for readAck messages from a full quorum $Q$ before termination, (L31-L37), proceeds identically as in case (b) and the lemma follows.

Lemma 13 In any execution $\xi$ of ERATO-MW, if $\rho_1$ and $\rho_2$ are two semi-fast read operations, take 3 exchanges to complete, such that $\rho_1$ precedes $\rho_2$, i.e., $\rho_1 \rightarrow \rho_2$, and $\rho_1$ returns the value for tag $tg_1$, then $\rho_2$ returns the value for tag $tg_2 \geq tg_1$.

Proof. Let the two operations $\rho_1$ and $\rho_2$ be invoked by processes with identifiers $r_1$ and $r_2$ respectively (not necessarily different). Also, let $RA_1$ and $RA_2$ be the sets of servers from quorums $Q_a$ and $Q_b$ (not necessarily different) that sent a readAck message to $r_1$ and $r_2$ during $\rho_1$ and $\rho_2$.

Assume by contradiction that read operations $\rho_1$ and $\rho_2$ exist such that $\rho_2$ succeeds $\rho_1$, i.e., $\rho_1 \rightarrow \rho_2$, and the operation $\rho_2$ returns a tag $tg_2$ that is smaller than the $tg_1$ returned by $\rho_1$, i.e.,
\[\text{Lemma 14} \text{ In any execution } \xi \text{ of ERATO-MW, if } \rho_1 \text{ and } \rho_2 \text{ are two fast read operations, take } 2 \text{ exchanges to complete, such that } \rho_1 \text{ precedes } \rho_2, \text{ i.e., } \rho_1 \rightarrow \rho_2, \text{ and } \rho_1 \text{ returns the value for tag } tg_1, \text{ then } \rho_2 \text{ returns the value for tag } tg_2 \geq tg_1.\]

\textbf{Proof.} \ Let the two operations \(\rho_1\) and \(\rho_2\) be invoked by processes with identifiers \(r_1\) and \(r_2\) respectively (not necessarily different). Let \(RR_1\) and \(RR_2\) be the quorums (not necessarily different) that sent a \texttt{readRelay} message to \(r_1\) and \(r_2\) during \(\rho_1\) and \(\rho_2\) respectively.

The algorithm terminates in \textit{two} communication exchanges when a read operation \(\rho\) receives \texttt{readRelay} messages from a full quorum \(Q\) and based on the distribution of the tags during the \(i^{th}\) iteration of the analysis, \(i \geq 1\), it notices \texttt{QV}(1).

Observe that if there exists a server \(s_k \in RR_1\), that replies with a tag \(tg_{s_k}\) s.t. \(tg_{s_k} < tg_1\) then \(\rho_1\) wouldn’t be able to notice \texttt{QV}(1) and return \(tg_1\). Thus, since \texttt{QV}(1) is noticed during the \(i^{th}\) iteration then it is known that all the servers in \(RR_1\) replied to \(\rho_1\) with a tag \(tg_s\) s.t. \(tg_s \geq tg_1\).

This is clear since every server \(s_x\) that was removed during the iterative analysis at iteration \(j\) s.t. \(j > i\), server \(s_x\) holds a tag \(tg_{s_x} \geq tg_s\).

Since by definition, any two quorums have a non-empty intersection it follows that \(RR_1 \cap RR_2 \neq \emptyset\). From that and by Lemma 8 then every server \(s_x \in RR_1 \cap RR_2\) has a tag \(tg'\) such that \(tg' \geq tg_1\). When \(\rho_2\) notices \texttt{QV}(1) in \(RR_2\) at the \(m^{th}\) iteration of the analysis, \(m \geq 1\), we know that \texttt{QV}(1) consists tags that come from the set \(RR_1 \cap RR_2\). Notice that if \(RR_1 \cap RR_2 = \emptyset\) holds at iteration \(m\), then it means that the algorithm would have stopped at an earlier iteration when either \(RR_1 \cap RR_2 \neq \emptyset\) or \(|RR_1 \cap RR_2| = 1\) holds.

Since the distribution of the tags during \(m^{th}\) iteration indicates the existence of one and only tag and since servers from \(RR_1 \cap RR_2\) participate then \(\rho\) returns a value associated with \(tg_2\) s.t. \(tg_2 \geq tg' \geq tg_1\) and the lemma follows. \(\square\)
Lemma 15 In any execution $\xi$ of Erato-MW, if $\rho_1$ and $\rho_2$ are two read operations s.t. $\rho_1$ precedes $\rho_2$, i.e., $\rho_1 \rightarrow \rho_2$, and $\rho_1$ returns tag $tg_1$, then $\rho_2$ returns a tag $tg_2$, s.t. $tg_2 \geq tg_1$.

Proof. We are interested to examine the cases where one of the read operation is fast and the other is semifast. In particular, cases (i) $\rho_1 \rightarrow \rho_2$ and $\rho_1$ is semifast and $\rho_2$ is fast and (ii) $\rho_1 \rightarrow \rho_2$ and $\rho_1$ is fast and $\rho_2$ is semifast.

Let the two operations $\rho_1$ and $\rho_2$ be invoked by processes with identifiers $r_1$ and $r_2$ respectively (not necessarily different). Also, let $RR_1, RA_1$ and $RR_2, RA_2$ be the sets of servers from full quorums (not necessarily different) that sent a readRelay and readAck message to $\rho_1$ and $\rho_2$ respectively.

We start with case (i). Since read operation $\rho_1$ is semifast, then based on the algorithm, the tag $tg_1$ that is returned is also the minimum tag noticed in $RA_1$. Before a server $s$ sends readAck messages to $\rho_1$ (that form $RA_1$), it must receive readRelay messages from a full quorum of servers. Thus, by Lemma 8 monotonicity of the tags at the servers we know that the minimum tag that a full quorum has by the end of $\rho_1$ is $tg_1$. Read operation $\rho_2$ receives readRelay messages from a full quorum of servers, $RR_2$. By definition of quorums, since both $RA_1$ and $RR_2$ are from a full quorum of servers then it follows that $RA_1 \cap RR_2 \neq \emptyset$. Thus every server $s_x \in RA_1 \cap RR_2$ holds a tag $tg'$ s.t. $tg' \geq tg_1$.

For $\rho_2$ to notice $qv(1)$ in $RR_2$ at the $m^{th}$ iteration of the analysis, $m \geq 1$, it means that in $qv(1)$ participate servers that belong to $RR_1 \cap RR_2$. Notice that if $RR_1 \cap RR_2 = \emptyset$ holds at iteration $m$, then it means that the algorithm would have stopped at an earlier iteration when either $RR_1 \cap RR_2 \neq \emptyset$ or $|RR_1 \cap RR_2| = 1$ holds. Since the distribution of the tags during $m^{th}$ iteration indicates the existence of one and only tag and since servers from $RR_1 \cap RR_2$ participate then $\rho$ returns a value associated with $tg_2$ s.t. $tg_2 \geq tg' \geq tg_1$ and the case follows.

We now examine case (ii). Since $\rho_1$ is fast, it follows that it has noticed $qv(1)$ in $RR_1$. If $qv(1)$ was noticed at the $m^{th}$ iteration of the analysis, $m \geq 1$, and $\rho_1$ returned a value associated with maximum tag during $m^{th}$ iteration, $tg_1$, then by the completion of $\rho_1$ a full quorum has a tag $tg'$ s.t. $tg' \geq tg_1$. Now, since read operation $\rho_2$ is semifast, then based on the algorithm, the tag $tg_2$ that is returned it is the minimum tag noticed in $RA_2$. Before a server $s$ sends readAck messages to $\rho_2$ (that form $RA_2$), it must receive readRelay messages from a full quorum of servers, RelaySet. By Lemma 8 monotonicity of the tags at the servers and $RR_1 \cap RelaySet \neq \emptyset$, then every server in $RA_2$ has a tag $tg_2$ s.t. $tg_2 \geq tg' \geq tg_1$ and the case follows.

The rest of the cases are proved in Lemmas 13 and 14 and the lemma follows.

Theorem 16 Algorithm Erato-MW implements an atomic MWMR object.

Proof. We use the above lemmas and the operations order definition to reason about each of the three atomicity conditions A1, A2 and A3.

A1 For any $\pi_1, \pi_2 \in II$ such that $\pi_1 \rightarrow \pi_2$, it cannot be that $\pi_2 < \pi_1$.

If both $\pi_1$ and $\pi_2$ are writes and $\pi_1 \rightarrow \pi_2$ holds, then from Lemma 10 it follows that $tg_{\pi_2} > tg_{\pi_1}$. From the definition of order $<$ we have $\pi_1 < \pi_2$. If $\pi_1$ is a write and $\pi_2$ a read and $\pi_1 \rightarrow \pi_2$ holds, then from Lemma 12 it follows that $tg_{\pi_2} \geq tg_{\pi_1}$. By definition $\pi_1 < \pi_2$ holds. If $\pi_1$ is a read, $\pi_2$ is a write and $\pi_1 \rightarrow \pi_2$ holds, then from Lemma 9 it follows that $\pi_2$ returns a tag $tg_{\pi_2}$ s.t. $tg_{\pi_2} > tg_{\pi_1}$. By the order definition $\pi_1 < \pi_2$ is satisfied. If both $\pi_1$ and $\pi_2$ are reads and $\pi_1 \rightarrow \pi_2$ holds, then from Lemma 13 it follows that $tg_{\pi_2} \geq tg_{\pi_1}$. If $tg_{\pi_2} > tg_{\pi_1}$, then by the ordering definition $\pi_1 < \pi_2$ holds. When $tg_{\pi_2} = tg_{\pi_1}$ then the ordering is not defined, thus it cannot be that $\pi_2 < \pi_1$.
A2 For any write $\omega \in \Pi$ and any operation $\pi \in \Pi$, then either $\omega \prec \pi$ or $\pi \prec \omega$.
If $tg_\omega > tg_\pi$, then $\pi \prec \omega$ follows directly. Similarly, if $tg_\omega < tg_\pi$ holds, then it follows that $\omega \prec \pi$.
When $ts_\omega = ts_\pi$ holds, then because all writer tags are unique (each server increments timestamps monotonically, and the server ids disambiguate among servers) $\pi$ can only be a read. Since $\pi$ is a read and the distribution of the tag written by $\omega$ satisfies either $qv(1)$ or $qv(3)$, it follows that $\omega \prec \pi$.

A3 Every read operation returns the value of the last write preceding it according to $\prec$ (or the initial value if there is no such write).
Let $\omega$ be the last write preceding read $\rho$. From our definition it follows that $tg_\rho \geq tg_\omega$. If $tg_\rho = tg_\omega$, then $\rho$ returned a value written by $\omega$ in some servers in a quorum $Q$. Read $\rho$ either was fast and during the iterative analysis it noticed a distribution of the tags in $Q$ that satisfied $qv(1)$ or $\rho$ was slow and waited for $readAck$ messages from a full quorum $Q$. In the latter, the intersection properties of quorums ensure that $\omega$ was the last complete write. If $tg_\rho > tg_\omega$ holds, it must be the case that there is a write $\omega'$ that wrote $tg_\rho$ and by definition it must hold that $\omega \prec \omega' \prec \rho$. Thus, $\omega$ is not the preceding write and this cannot be the case. Lastly, if $tg_\rho = 0$, no preceding writes exist, and $\rho$ returns the initial value. □

4.3 Performance
By inspection of the code, write operations take 2 exchanges and read operations take either 2 or 3 exchanges. The (worst case) message complexity of write operations is $2|S|$ and of read operations is $|S|^2 + 2|S|$, as follows from the structure of the algorithm. We now provide additional details.

Operation Latency. Write operation latency: According to algorithm Erato-mw, writer $w$ sends $discover$ messages to all servers in exchange $e_1$ and waits for $discoverAck$ messages from a full quorum of servers in $e_2$. Once the $discoverAck$ messages are received from $e_2$, then writer $w$ broadcasts a $writeRequest$ message to all servers in $e_3$. It then waits for $writeAck$ messages from a full quorum of servers from $e_4$. No further communication is required and the write operation terminates. Thus a write operation consists of 4 communication exchanges.

Read operation latency: A reader sends a $readRequest$ message to all the servers in the first communication exchange $e_1$. Once the servers receive the $readRequest$ message they broadcast a $readRelay$ message to all servers and the reader in exchange $e_2$. The reader can terminate at the end of the $e_2$ if it receives $readRelay$ messages and based on the distribution of the tags through the iterative procedure it notices $qv(1)$ or $qv(3)$. If not, the operation goes into the third exchange $e_3$. Thus read operations terminate after either 2 or 3 communication exchanges.

Message Complexity. Write operation: Write operations in algorithm Erato-mw take 4 communication exchanges. The first and the third exchanges, $e_1$ and $e_3$, occur when a writer sends $discover$ and $writeRequest$ messages respectively to all servers in $S$. The second and fourth exchanges, $e_2$ and $e_4$, occur when servers in $S$ send $discoverAck$ and $writeAck$ messages respectively back to the writer. Thus $4|S|$ messages are exchanged in any write operation.

Read operation: The structure of the read protocol of Erato-mw is similar as in Erato, thus, as reasoned in Section 3.3 $|S|^2 + 3|S|$ messages are exchanged during a read operation.
5 Empirical Evaluations

We now compare the algorithms using the NS3 discrete event simulator [1]. The following SWMR algorithms Erato, ABD [2], OHSAM [8], and SLIQ [6], and the corresponding MWMR algorithms: Erato-mw, ABD-mw [12], OhMAM [8], and CwFr [5] were simulated. For comparison, we implemented benchmark LB that mimics the minimum message requirements: LB does two exchanges for reads and writes, and neither performs any computation nor ensures consistency.

We developed two topologies that use the same array of routers, but differ in the deployment of server and client nodes. Clients are connected to routers over 5Mbps links with 4ms delay and the routers over 10Mpbs links with 6ms delay. In Series topology, Fig.2(a), a server is connected to each router over 10Mbps bandwidth with 2ms delay, modeling a network where servers are separated and appear to be in different networks. In Star topology, Fig.2(b), servers are connected to a single router over 50Mbps links with 2ms delay, modeling a network where servers are in close proximity and well-connected, e.g., a datacenter. Clients are located uniformly with respect to the routers.

Performance. We assess algorithms in terms of operation latency that depends on communication delays and local computation time. For operation latency we combine two clocks: the simulation clock to measure communication delays, and a real time clock for computation delays. The sum of the two yields latency.

Experimentation Setup. To subject the system to high communication traffic, no failures are assumed (ironically, crashes reduce network traffic). Communication is via point-to-point bidirectional links implemented with a DropTail queue.

Scenarios. The scenarios are designed to test (i) the scalability of the algorithms as the number of readers, writers, and servers increases; (ii) the contention effect on efficiency, and (iii) the effects of chosen topologies on the efficiency. For scalability we test with the number of readers $|R|$ from the set $\{10, 20, 40, 80\}$ and the number of servers $|S|$ from the set $\{9, 16, 25, 36\}$. Algorithms are evaluated with matrix quorums (unions of rows and columns). For the MWMR setting we range the number of writers $|W|$ over the set $\{10, 20, 40\}$. We issue reads (and writes) every $rInt$ (and $wInt$ respectively) from the set of $\{2, 4\}$ seconds. To test contention we define two invocation schemes: fixed and stochastic. In the fixed scheme all operations are scheduled periodically at a constant interval. In the stochastic scheme reads are scheduled randomly from...
the intervals \([1..rInt]\) and write operations from the intervals \([1..wInt]\).

**Results.** We note that generally the new algorithms outperform the competition. A closer examination yields the following observations.

*Scalability:* Increased number of readers and servers increases latency in both settings. Observe Fig. 5(a),(b) for SWMR and Fig. 5(e),(f) for MWMR algorithms. Not surprisingly, latency is better for smaller numbers of readers, writers, and servers. However, the relative performance of the algorithms remains the same.

*Contention:* The efficiency of the algorithms is examined under different concurrency schemes. We notice that in the *stochastic* scheme reads complete faster than in the *fixed* scheme – Fig. 5(b) and 5(c) for the SWMR and Fig. 5(f) and 5(g) for the MWMR setting. This outcome is expected as the *fixed* scheme causes congestion. For the *stochastic* scheme the invocation time intervals are distributed uniformly (randomness prevents the operations from being invoked simultaneously), and this reduces congestion in the network and improves latency.

*Topology:* Topology substantially impacts performance and the behavior of the algorithms. This can be seen in Figures 5(b) and 5(d) for the SWMR setting, and Figures 5(f) and 5(h) for the MWMR setting. The results show clearly that the proposed algorithms outperform the competition in the *Star* topology, where servers are well-connected using high bandwidth links.

6 Conclusions

We focused on the problem of emulating atomic read/write shared objects in the asynchronous, crash-prone, message-passing settings with the goal of synthesizing algorithms where read operations can *always* terminate in *less* than two communication round-trips. We presented such algorithms for the SWMR and MWMR models. We rigorously reasoned about the correctness of our algorithms. The algorithms impose no constraints on the number of readers, and no constraints on the number of writers (in the MWMR model). The algorithms are shown to be optimal in terms of *communication exchanges* with unconstrained participation. The empirical study demonstrates the practicality of the new algorithms, and identifies settings in which their performance is clearly superior.

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Figure 3: Simulation results for SWMR (a-d) and MWMR (e-g). Horizontal axis is the number of readers. Vertical axis is latency.
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