Shadow cast by a rotating and nonlinear magnetic-charged black hole in perfect fluid dark matter

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Abstract

We derived an exact solution of the spherically symmetric Hayward black hole surrounded by perfect fluid dark matter (PFDM). By applying the Newman-Janis algorithm, we generalized it to the corresponding rotating black hole. Then, we studied the shadows of rotating Hayward black hole in PFDM. The apparent shape of the shadow depends upon the black hole spin $a$, the magnetic charge $Q$ and the PFDM intensity parameter $k$ ($k < 0$). The shadow is a perfect circle in the non-rotating case ($a = 0$) and a deformed one in the rotating case ($a \neq 0$). For a fixed value of $a$, the size of the shadow increases with the increasing $|k|$, but decreases with the increasing $Q$. We further investigated the black hole emission rate. We found that the emission rate decreases with the increasing $|k|$ (or $Q$) and the peak of the emission shifts to lower frequency. Finally, we discussed the observational prospects corresponding to the supermassive black hole Sgr A* at the center of the Milky Way.
I. INTRODUCTION

In recent years, various kinds of astronomical observations strongly reveal that black holes do exist in our universe. So far, the strongest evidence are the recent experimental announcements the detection of gravitational waves (GWs) [1] by the LIGO and VIRGO observatories and the captured image of the black hole shadow of a supermassive M87 black hole by the Event Horizon Telescope (EHT) based on the very-long baseline interferometry (VLBI) [2, 3]. Among the different methods used to determine the nature of the black hole, observing the shadow of the black hole remains probably the most interesting one. A black hole shadow is the optical appearance that occurs when there is a bright distant light source behind the black hole. To a distant observer, it appears as a two-dimensional dark zone. The observation of the shadow provides a tentative way to find out the parameters of the black hole. Synge [4] studied the shadow of the Schwarzschild black hole, he pointed out that the edge of the shadow is rounded. Bardeen [5] was the first who studied the shadow of the Kerr black hole. It can be seen that for Kerr black hole, the shadow is no longer circular. Recent works also considered extended Kerr black holes, such as Kerr-de Sitter black holes [6–8], deformed black holes [9], accelerated Kerr black holes [10], Kerr black holes in the presence of extra dimensions [11, 12]. See Refs. [13–34] for more recent research.

One of mysterious properties of the black hole is that it itself has a singularity at the origin of the spacetime, at which the curvatures, densities become infinite and the predictive power of physical laws is completely broken down. It is widely believed that the spacetime singularities are the reflection of the incompleteness of General Relativity. Thus, many efforts have been devoted to introduce the black hole without having a spacetime singularity. Surprisingly, Bardeen obtained a black hole solution without a singularity in 1968 [35]. After that, more regular (non-singular) black holes
such as Ayón-Beato and García black hole [36], Berej-Matyjasek-Trynieki-Wornowicz black hole [37], and Hayward black hole [38], were proposed. The spherically symmetric Hayward black hole is described by the metric

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2, \quad f(r) = 1 - \frac{2Mr^2}{r^3 + Q^3}, \quad (1)$$

where $Q$ and $M$ are the magnetic charge and mass, respectively. Hayward black hole behaves like the Schwarzschild black hole at the large distances and at the short distances like the de-Sitter geometry.

According to the standard model of cosmology, our current universe contains around 68% dark energy, around 28% dark matter and less than 4% baryonic matter [39]. Here, dark matter is non-baryonic and non-luminous. Therefore, it is extremely important to study the black hole physics in the presence of dark matter. In recent years, the black hole surrounded by quintessence dark energy have attracted much attention. For example, Kiselev [40] considered the Schwarzschild black hole surrounded by the quintessential energy and then Toshmatov and Stuchlík [41] extended it to the Kerr-like black hole; the Hayward black holes surrounded by quintessence have been studied in Ref. [42], etc [43–47]. On the other hand, as a dark matter candidate, the perfect fluid dark matter has been proposed by Kiselev [48]. In this work, following Refs. [41, 42], we generalize the Schwarzschild black hole surrounded by PFDM to the spherically symmetric Hayward black hole. In addition, using Newman-Janis algorithm, we obtain the rotating Hayward black hole in PFDM.

The paper is organized as follows. The next section is the derivation of the spherically symmetric Hayward black hole surrounded by PFDM. In Sec. III, by applying the Newman-Janis algorithm we obtain the rotating Hayward black hole surrounded by PFDM. In Sec. IV, we study the photon motion around the rotating Hayward black hole with PFDM. Black hole shadow of the rotating Hayward black hole with
nonvanishing PFDM intensity parameter is considered in Sec. V. In Sec. VI, we investigate the energy emission rate of the rotating black hole in PFDM. Conclusions and discussions are presented in Sec. VII. Planck units $\hbar = G = c = k_B = 1$ are used throughout the paper.

II. STATIC AND SPHERICALLY SYMMETRIC NONLINEAR MAGNETIC-CHARGED BLACK HOLE IN PERFECT FLUID DARK MATTER

Let us consider Einstein gravity coupled to a nonlinear electromagnetic field in the presence of the perfect fluid dark matter. It is described by the following equations:

$$G_{\mu\nu} = 2 \left( \frac{\partial \mathcal{L}(F)}{\partial F} F_{\mu\lambda} F^{\nu\lambda} - \delta^{\nu}_{\mu} \mathcal{L} \right) + 8\pi T_{\mu}^{\nu} \text{(PFDM)},$$

$$\nabla_{\mu} \left( \frac{\partial \mathcal{L}(F)}{\partial F} F^{\nu\mu} \right) = 0.$$  \hspace{1cm} (2)

Here, $F_{\mu\nu} = 2 \nabla_{[\mu} A_{\nu]}$ and $\mathcal{L}$ is a function of $F \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ given by [49]

$$\mathcal{L}(F) = 3M |Q|^3 \frac{(2Q^2 F)^{\frac{3}{2}}}{\left(1 + (2Q^2 F)^{\frac{3}{4}}\right)^2},$$

where $Q$ and $M$ are the parameters associated with magnetic charge and mass, respectively.

In this work, we consider the black holes surrounded by the perfect fluid dark matter, following Kiselev [40, 48] and Li and Yang [50], the energy density of PFDM is given by

$$T^t_t = T^r_r = \frac{1}{8\pi r^3} k,$$

with $k$ denoting the intensity of the PFDM. The value of $k$ can be both positive and negative [51]. Here we only consider the theoretical negative values of $k$. The case of positive $k$ can be studied in a similar way as presented in this work.
To obtain a metric satisfies Eqs. (2) and (3), let us commence with a spherically symmetric spacetime
\[
d s^2 = -f(r) \, dt^2 + f(r)^{-1} \, dr^2 + r^2 d\Omega^2, \quad f(r) = 1 - \frac{2m(r)}{r},
\]
and use the ansatz for Maxwell field [52]
\[
F_{\mu\nu} = \left( \delta^\theta_{\mu} \delta^\varphi_{\nu} - \delta^\theta_{\nu} \delta^\varphi_{\mu} \right) B(r, \theta).
\]
With these choices, Eqs. (3) are easily integrated,
\[
F_{\mu\nu} = \left( \delta^\theta_{\mu} \delta^\varphi_{\nu} - \delta^\theta_{\nu} \delta^\varphi_{\mu} \right) c(r) \sin \theta.
\]
Using the Bianchi identities
\[
0 = dF = c'(r) \sin(\theta) dr \wedge d\theta \wedge d\varphi,
\]
one can find that \( c(r) = \text{const.} = Q \), where the integration constant has been chosen as \( Q \). Further, one can get \( F = Q^2/2r^4 \). In order to give a direct physical interpretation to \( Q \), we consider the following integral
\[
\frac{1}{4\pi} \int_{S^2_\infty} F = \frac{Q}{4\pi} \int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\varphi = Q,
\]
where \( S^2_\infty \) is a two-sphere at the infinity. From Eq. (10), one can confirm that \( Q \) is the magnetic monopole charge. For simplicity, without loss of generality, we consider \( Q > 0 \).

Now, with the help of the above equations, the time component of Eq. (2) reduces to
\[
- \frac{2}{r^2} \frac{dm(r)}{dr} = - \frac{6MQ^3}{(r^3 + Q^3)^2} + \frac{k}{r^3}.
\]
Integrating Eq. (11) from \( r \) to \( \infty \) and using that \( M = \lim_{r \to \infty} \left( m(r) + \frac{k}{2} \ln \frac{r}{|k|} \right) \), one finally gets
\[
f(r) = 1 - \frac{2Mr^2}{r^3 + Q^3} + \frac{k}{r} \ln \frac{r}{|k|}.
\]
Thus the metric of exact spherically symmetric solutions for the Einstein equations describing the nonlinear magnetic-charged black holes surrounded by perfect fluid dark matter is given by

\[
    ds^2 = -\left(1 - \frac{2Mr^2}{r^3 + Q^3} + \frac{k}{r} \ln \frac{r}{|k|}\right) dt^2 + \frac{dr^2}{1 - \frac{2Mr^2}{r^3 + Q^3} + \frac{k}{r} \ln \frac{r}{|k|}} + r^2 d\Omega^2.
\]

In the absence of PFDM, i.e. \( k = 0 \), we can obtain the non-linear magnetic-charged black hole in the flat background or the Hayward-like black hole \[38]\.

**III. ROTATING AND NONLINEAR MAGNETIC-CHARGED BLACK HOLE IN PERFECT FLUID DARK MATTER**

Many methods have been developed in theory to compute rotating solutions from static ones and the most widely known method is the Newman-Janis algorithm (NJA) and its generalizations. The NJA was first proposed by Newman and Janis in 1965 \[53]\ and widely used in many articles \[41, 42, 54–60\]. In this work, we will use the NJA modified by Azreg-Aïnou \[61]\ to obtain the Rotating nonlinear magnetic-charged black hole surrounded by perfect fluid dark matter.

Consider the general static and spherically symmetric metric:

\[
    ds^2 = -f(r) dt^2 + g(r)^{-1} dr^2 + h(r) d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2.
\]

At the first step of NJA, we transform the spherically symmetric space-time metric (14) from the Boyer-Lindquist (BL) coordinates \((t,r,\theta,\varphi)\) to the Eddington-Finkelstein (EF) coordinates \((u,r,\theta,\varphi)\). After introducing the coordinate transformation defined by

\[
    du = dt - \frac{dr}{\sqrt{fg}},
\]

6
the line element (14) takes the form

\[ ds^2 = -f(r) \, du^2 - 2\sqrt{\frac{f(r)}{g(r)}} \, du \, dr + h(r) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right). \]  

(16)

In terms of the null tetrads satisfy the relations

\[ l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = l_\mu m^\mu = n_\mu m^\mu = 0, \]
\[ l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \]
the nonzero components of the inverse metric associated with the line element (16) can be expressed as

\[ g^{\mu \nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu, \]

(17)

where

\[ l^\mu = \delta^\mu_r, \]
\[ n^\mu = \sqrt{\frac{g(r)}{f(r)}} \delta^\mu_u - \frac{f(r)}{2} \delta^\mu_r, \]
\[ m^\mu = \frac{1}{\sqrt{2h(r)}} \delta^\mu_\theta + \frac{i}{\sqrt{2h(r) \sin \theta}} \delta^\mu_\varphi, \]
\[ \bar{m}^\mu = \frac{1}{\sqrt{2h(r)}} \delta^\mu_\theta - \frac{i}{\sqrt{2h(r) \sin \theta}} \delta^\mu_\varphi. \]  

(18)

Next, we take the critical step of the NJA, which is to perform complex coordinate transformations in the \( u - r \) plane

\[ u \rightarrow u - ia \cos \theta, \]
\[ r \rightarrow r + ia \cos \theta, \]

(19)

by which \( \delta^\mu_\nu \) transform as vectors:

\[ \delta^\mu_r \rightarrow \delta^\mu_r, \quad \delta^\mu_u \rightarrow \delta^\mu_u, \]
\[ \delta^\mu_\theta \rightarrow \delta^\mu_\theta + ia \sin \theta (\delta^\mu_u - \delta^\mu_r), \quad \delta^\mu_\varphi \rightarrow \delta^\mu_\varphi. \]

(20)

At the same time, we assume that the functions \( f(r), g(r) \) and \( h(r) \) in Eq. (18) also turn into a new form: \( f(r) \rightarrow F(r, a, \theta), \ g(r) \rightarrow G(r, a, \theta), \) and \( h(r) \rightarrow \Sigma = \)
Thus, the effect of the transformation (19) on Eq. (18) is

\[ l^\mu = \delta^\mu_r, \quad n^\mu = \sqrt{G/F} \delta^\mu_u - \frac{1}{2} F \delta^\mu_r, \]
\[ m^\mu = \frac{1}{\sqrt{2\Sigma}} \left( \delta^\mu_\theta + ia \sin \theta \left( \delta^\mu_u - \delta^\mu_r \right) + \frac{i}{\sin \theta} \delta^\mu_\varphi \right), \]
\[ \bar{m}^\mu = \frac{1}{\sqrt{2\Sigma}} \left( \delta^\mu_\theta - ia \sin \theta \left( \delta^\mu_u - \delta^\mu_r \right) - \frac{i}{\sin \theta} \delta^\mu_\varphi \right). \]

Then by means of Eq. (17), we obtain the spacetime metric tensor \( g^{\mu\nu} \) as

\[ g^{uu} = \frac{a^2 \sin^2 \theta}{\Sigma}, \quad g^{rr} = G + \frac{a^2 \sin^2 \theta}{\Sigma}, \]
\[ g^{\theta\theta} = \frac{1}{\Sigma}, \quad g^{\varphi\varphi} = \frac{1}{\Sigma \sin^2 \theta}, \]
\[ g^{ur} = g^{ru} = -\sqrt{G/F} - \frac{a^2 \sin^2 \theta}{\Sigma}, \]
\[ g^{u\varphi} = g^{\varphi u} = \frac{a}{\Sigma}, \quad g^{r\varphi} = g^{\varphi r} = -\frac{a}{\Sigma}. \]

Accordingly, the covariant metric in the EF coordinates \((u, r, \theta, \phi)\) reads

\[ ds^2 = -Fdu^2 - 2\sqrt{\frac{F}{G}} dudr + 2a \left( F - \frac{F}{G} \right) \sin^2 \theta dud\varphi + \Sigma d\theta^2 \]
\[ + 2a \sin^2 \theta \sqrt{\frac{F}{G}} d\varphi + \sin^2 \theta \left[ \Sigma + a^2 \left( 2\sqrt{\frac{F}{G}} - F \right) \sin^2 \right] d\varphi^2. \]

The final step of NJA is to bring (23) to the BL coordinates by a coordinate transformations:

\[ du = dt + \lambda (r) dr, \quad d\varphi = d\phi + \chi (r) dr, \]

where the functions \( \lambda (r) \) and \( \chi (r) \) can be found using the requirement that all nondiagonal components of the metric tensor (except for the coefficient \( g_{t\phi} \) \( g_{\phi t} \)) are equal to zero, Thus

\[ \lambda (r) = -\frac{p (r) + a^2}{g (r) h (r) + a^2}, \quad \chi (r) = -\frac{a}{g (r) h (r) + a^2}, \]

8
with
\[ p(r) = \sqrt{\frac{g(r)}{f(r)}}h(r) \] (26)
and
\[ F(r, a, \theta) = \frac{(gh + a^2 \cos^2 \theta) \Sigma}{(p^2 + a^2 \cos^2 \theta)^2}, \quad G(r, a, \theta) = \frac{gh + a^2 \cos^2 \theta}{\Sigma}. \] (27)

Finally, the rotating solution corresponding to the spherically symmetric metric (14) can therefore be obtained as
\[
d s^2 = -\left(\frac{gh + a^2 \cos^2 \theta}{p + a^2 \cos^2 \theta}\right) \Sigma dt^2 + \frac{\Sigma}{gh + a^2} dr^2 - 2a \sin^2 \theta \left(\frac{p - gh}{(p + a^2 \cos^2 \theta)^2}\right) \Sigma d\phi dt
+ \Sigma d\theta^2 + \Sigma \sin^2 \theta \left(1 + a^2 \sin^2 \theta \frac{2p - gh + a^2 \cos^2 \theta}{(p + a^2 \cos^2 \theta)^2}\right) d\phi^2.
\] (28)

In the case of Hayward black holes in PFDM, comparing the line elements (6) with (14), one can find
\[ g(r) = f(r), \quad p(r) = h(r) = r^2. \] (29)

Substituting the above expressions into (28), we obtain the metric of rotating Hayward black holes in perfect fluid dark matter in the form
\[
d s^2 = -\left(1 - \frac{r^2 - f(r) r^2}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta_r} dr^2 - \frac{2a \sin^2 \theta \left(r^2 - f(r) r^2\right)}{\Sigma} dtd\phi
+ \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} \left((r^2 + a^2)^2 - a^2 \Delta_r \sin^2 \theta\right) d\phi^2,
\] (30)
with
\[ \Sigma = r^2 + a^2 \cos^2 \theta, \]
\[ \Delta_r = r^2 f(r) + a^2, \]
\[ f(r) = 1 - \frac{2Mr^2}{r^3 + Q^3} + \frac{k}{r} \ln \frac{r}{|k|}. \] (31)
IV. PHOTON ORBITS

For obtaining the geodesic equations, we use the Hamilton-Jacobi equation and Carter constant separable method [62]. The Hamilton-Jacobi equation takes the form as

\[
\frac{\partial S}{\partial \tau} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \tag{32}
\]

where \(S\) is the Jacobi action and \(\tau\) is an affine parameter. We select the corresponding action as

\[
S = \frac{1}{2} m^2 \tau - Et + S_r(r) + S_\theta(\theta), \tag{33}
\]

where \(m\) is proportional to the rest mass of the particle. Energy \(E\) and angular momentum \(L\) are constant of motion related to the associated Killing vectors \(\partial/\partial t\) and \(\partial/\partial \phi\). \(S_r(r)\) and \(S_\theta(\theta)\) are functions of \(r\) and \(\theta\), respectively.

For rotating Hayward black hole in PFDM, the Hamilton-Jacobi equation yields:

\[
\frac{1}{2} g^{tt} \frac{\partial S}{\partial t} \frac{\partial S}{\partial t} + \frac{1}{2} g^{rr} \frac{\partial S}{\partial r} \frac{\partial S}{\partial r} + \frac{1}{2} g^{\theta\theta} \frac{\partial S}{\partial \theta} \frac{\partial S}{\partial \theta} + \frac{1}{2} g^{\phi\phi} \frac{\partial S}{\partial \phi} \frac{\partial S}{\partial \phi} + g^{\phi t} \frac{\partial S}{\partial \phi} \frac{\partial S}{\partial t} = -\frac{\partial S}{\partial \tau}. \tag{34}
\]

Then, we solve it for \(S_r(r)\) and \(S_\theta(\theta)\) as follows [63]:

\[
\frac{dS_r}{dr} = \frac{\sqrt{R(r)}}{\Delta_r}, \tag{35}
\]

\[
\frac{dS_\theta}{d\theta} = \sqrt{\Theta(\theta)},
\]

where

\[
R(r) = (aL - (a^2 + r^2)E)^2 - (K + (aE - L)^2 + m^2 r^2) \Delta_r, \tag{36}
\]

\[
\Theta(\theta) = K - a^2 m^2 \cos^2(\theta) - \left(\frac{L^2}{\sin^2\theta} - a^2 E^2\right) \cos^2 \theta. \tag{37}
\]
with $K$ as a constant of separation. We can replace $S_r(r)$ and $S_\theta(\theta)$ in formula (33) with

$$S_r = \int^r \frac{\sqrt{R(r)}}{\Delta_r} \, dr \quad (38)$$

By following [64], we can write the equations of the geodesic motion in the form

$$\Sigma \dot{t} = \frac{(r^2 + a^2)E - aL(r^2 + a^2)}{\Delta_r} - a(aE \sin^2(\theta) - L), \quad (39)$$

$$\Sigma \dot{r} = \sqrt{R(r)}, \quad (40)$$

$$\Sigma \dot{\theta} = \sqrt{\Theta(\theta)}, \quad (41)$$

$$\Sigma \dot{\phi} = \frac{a((r^2 + a^2)E - aL)}{\Delta_r} - \frac{aE \sin^2(\theta) - L}{\sin^2(\theta)}, \quad (42)$$

where overdot (’) stands for the derivative with respect to the proper time $\tau$.

As we are interested in black hole shadows, we consider only null geodesics, for which $m = 0$. The motion of photon is determined by two impact parameters $\xi = L/E$ and $\eta = K/E^2$. We define $R_p(r) = R(r)/E^2$ and $\Theta_p(\theta) = \Theta(\theta)/E^2$.

Then, for photon, we rewrite Eqs. (36) and (37) as:

$$R_p(r) = (a\xi - (a^2 + r^2))^2 - (\eta + (a - \xi)^2)\Delta_r, \quad (43)$$

$$\Theta_p(\theta) = \eta - \left(\frac{\xi^2}{\sin^2 \theta} - a^2\right) \cos^2 \theta. \quad (44)$$

For spherical orbit, a test photon has zero radial velocity ($\dot{r} = 0$) and zero radial acceleration ($\ddot{r} = 0$), By (40), this requires that

$$R_p(r) = 0, \quad \frac{dR_p(r)}{dr} = 0. \quad (45)$$

Combining Eqs. (43) and (45), we get the expressions of $\xi$ and $\eta$ as

$$\xi = \frac{-4r\Delta_r + a^2\Delta'_r + r^2\Delta'_r}{a\Delta'_r},$$

$$\eta = \frac{r^2 (16a^2\Delta_r - 16\Delta'_r + 8r\Delta_r\Delta'_r - r^2\Delta'_r^2)}{a^2\Delta'^2_r}. \quad (46)$$
Inserting (46) into (44), using the fact that $\Theta_p(\theta)$ is non negative, one can get an inequality that determines the photon region [6]

$$K : 16r^2a^2\Delta_r \sin^2(\theta) - (4r\Delta_r - \Sigma\Delta'_r)^2 \geq 0.$$  \hspace{1cm} (47)

When $a = 0$, Eq. (47) degenerates into an equality,

$$(4r\Delta_r - \Sigma\Delta'_r)^2 = 0.$$ \hspace{1cm} (48)

This means that the photon regions degenerate into photon spheres.

A spherical lightlike geodesic at $r = r_p$ is stable with respect to radial perturbations if $\frac{d^2R_p(r)}{dr^2}|_{r=r_p} < 0$, and unstable if $\frac{d^2R_p(r)}{dr^2}|_{r=r_p} > 0$. With the help of (43) and (46), one can get

$$\frac{d^2R_p(r)}{dr^2} = \frac{8E^2(2r\Delta_r\Delta'_r + r^2\Delta'^2 - 2r^2\Delta_r\Delta''_r)}{\Delta'}.$$ \hspace{1cm} (49)

where ($'$) denotes the derivative with respect to $r$.

Fig. 1 shows the photon region $K$ in the $(r, \theta)$ plane with $a = 1$, $Q = 0.01$ and $k = -0.01$, where stable and unstable spherical light rays are distinguished. The unstable spherical light rays determine the contour of the shadow.

V. BLACK HOLE SHADOW

Now we investigate the shadow of a rotating Hayward black hole in PFDM. Here, we assume that the light sources exist at infinity and are distributed uniformly in all directions. To determine the shape of the black hole shadow, we introduce the celestial coordinates $\alpha$ and $\beta$ as

$$\alpha = \lim_{r_o \to \infty} \left(-r_o^2 \sin \theta \frac{d\phi}{dr}|_{\theta \to \iota}\right),$$

$$\beta = \pm \lim_{r_o \to \infty} \left(r_o^2 \frac{d\theta}{dr}|_{\theta \to \iota}\right),$$ \hspace{1cm} (50)
Figure 1: Photon region of rotating nonlinear magnetic-charged black hole surrounded by PFDM with $a = 1$, $Q = 0.01$ and $k = -0.01$.

where $r_0$ is the distance from the black hole to observer, $i$ is the angle between the rotation axis of the black hole and the line of sight of the observer. Here we assume the observer is at infinity.

Using the geodesic equations (39)-(42), the celestial coordinates, as a function of $\xi$ and $\eta$, take the form

$$
\alpha = -\frac{\xi}{\sin i},
$$

$$
\beta = \pm \sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i}.
$$

In the equatorial plane ($i = \pi/2$), $\alpha$ and $\beta$ reduce to

$$
\alpha = -\xi,
$$

$$
\beta = \pm \sqrt{\eta}.
$$
Furthermore, using Eqs. (31), (46) and (52), we have
\[ \alpha^2 + \beta^2 = \xi^2 + \eta = 2r^2 + a^2 + \frac{8\Delta_r[2 - (rf'(r) + 2f(r))]}{(rf'(r) + 2f(r))^2}, \]  
(53)
where
\[ f'(r) = \frac{2Mr^4 - 4MQ^3r}{r(Q^3 + r^3)^2} + \frac{k}{r^2} \left(1 - \ln \frac{r}{|k|}\right). \]  
(54)

Different shapes of the shadow can be obtained by plotting \( \beta \) against \( \alpha \). Fig. 2 shows the plots of the shadows of the rotating nonlinear magnetic-charged black hole surrounded by PFDM for different values of the parameters \( a \) and \( k \). We found the size of the shadow increases with the increasing \( |k| \) and the shadow gets more and more distorted with the increasing \( a \). Interestingly, as shown in the last picture of Fig. 2, we found that when \( |k| \) increases, the most left position (marked by \( L \) in Fig. 5) of the shadow will first move to the left and then to the right. As shown in Fig. 3, we studied the dependence of the horizontal coordinate \( \alpha_l \) of the most left position of the Shadow on the spin parameter \( a \) and the PFDM intensity parameter \( k \) with \( Q = 0.5 \). In the case of \( a = 0.8 \), we found that when \( k \approx -1.23 \), the value of \( \alpha_l \) is the smallest.

Fig. 4 shows the plots of the shadows of the rotating nonlinear magnetic-charged black hole surrounded by PFDM for different values of the parameters \( a \) and \( Q \). We found the size of the shadow decreases with the increasing \( Q \), but it is not obvious.

In order to extract detailed information from the shadow and connect to astronomical observations, it is recommended to introduce two observables the radius \( R_s \) and the distortion parameter \( \delta_s \) [65]. As shown in Fig. 5, \( R_s \) is the radius of the reference circle passing through three points: the top one \( T (\alpha_t, \beta_t) \), the bottom one \( B (\alpha_b, \beta_b) \) and the most right one \( R (\alpha_r, 0) \). The distortion parameter \( \delta_s \) is defined as
\[ \delta_s = \frac{D_s}{R_s} = \frac{|\alpha_l - \alpha_a|}{R_s}, \]  
(55)
Figure 2: Examples of shadows of the rotating nonlinear magnetic-charged black hole surrounded by PFDM for different values of the parameters $a$ and $k$ with the magnetic charge $Q = 0.5$. 
Figure 3: Dependence of the horizontal coordinate ($\alpha_l$) of the most left position (marked by $L$ in figure 5 ) of the shadow on the spin parameter $a$ and the PFDM intensity parameter $k$ with $Q = 0.5$.

where $D_s$ is the distance from the most left position ($L$) of the shadow to the reference circle ($A$). The observable $R_s$ is defined as

$$R_s = \frac{(\alpha_l - \alpha_r)^2 + \beta_t^2}{2|\alpha_r - \alpha_t|}.$$  \hspace{1cm} (56)

We show in Fig. 6 the evolution of the radius $R_s$ and the distortion parameter $\delta_s$ with the parameters $a$ and $k$. We found the radius $R_s$ increases with the increasing $|k|$, this result is consistent with the result in Fig. 2. $\delta_s$ is not monotonic with respect to $k$. There exists a $k_0 (\approx -0.56)$. When $k < k_0$, $\delta_s$ increases with the increasing $|k|$. When $k > k_0$, $\delta_s$ decreases with the increasing $|k|$. Fig. 7 describes the behaviour of observables $R_s$ and $\delta_s$ with respect to the parameters $a$ and $Q$. We found that magnetic charge $Q$ diminishing the shadow radius $R_s$, while increases the distortion parameter $\delta_s$. From Figs. 6 and 7, we also find that both $R_s$ and $\delta_s$ increase with the increasing spin parameter $a$.  

16
Figure 4: Examples of shadows of the rotating nonlinear magnetic-charged black hole surrounded by PFDM for different values of the parameters $a$ and $Q$ with the PFDM intensity parameter $k = -0.5$.
Figure 5: Schematic illustration of the black hole shadow and the reference circle.

Figure 6: The radius $R_s$ (left) and the distortion parameter $\delta_s$ (right) of the black hole shadow against the parameter $k$ for different values of the spin parameter $a$ with the magnetic charge $Q = 0.5$. 
VI. ENERGY EMISSION RATE

The high energy absorption cross section of a black hole oscillates around a limiting constant value $\sigma_{lim}$. For an observer at an infinite distance, the black hole shadow corresponds to the high energy absorption cross section. Here, we compute the energy emission rate [66] using

$$\frac{d^2 E(\omega)}{d\omega dt} = \frac{2\pi^2 \sigma_{lim}}{e^{\omega/T} - 1} \omega^3,$$

the limiting constant value can be expressed as

$$\sigma_{lim} \approx \pi R_s^2,$$

$\omega$ represent the frequency of photon and $T$ is the Hawking temperature for the outer event horizon ($r_+$) which is defined by

$$T = \lim_{\theta=0, r \to r_+} \frac{\partial_r \sqrt{-g_{tt}}}{2\pi \sqrt{g_{rr}}}.$$

In our case, for the rotating Hayward black hole in PFDM
\[ g_{tt} = - \left( 1 - \frac{r^2 - f(r)r^2}{\Sigma} \right), \quad g_{rr} = \frac{\Sigma}{\Delta r}. \]  

Thus, we obtain the Hawking temperature as

\[ T = \frac{r^2_+ f'(r_+) \left( r^2_+ + a^2 \right) + 2a^2 r_+ (f(r_+) - 1)}{4\pi \left( r^2_+ + a^2 \right)^2}. \]

We show energy emission rate in Fig. 8, against the frequency \( \omega \) for different values of the parameter \( a \) and \( k \). We can see that the peak of the emission decreases with increasing \( |k| \) and shifts to left, i.e., to lower frequency. In Fig. 9, we describe the energy emission rate against the frequency \( \omega \) for different values of the parameter \( a \) and \( Q \). It is clear that the peak of the emission decreases with the increasing \( Q \) and also shifts to lower frequency.

![Figure 8: Evolution of the emission rate with the frequency \( \omega \) for different values of the parameters \( a \) and \( k \) with the magnetic charge \( Q = 0.5 \).](image)

VII. CONCLUSIONS AND DISCUSSIONS

In this paper, we have obtained the exact solution of the static spherically symmetric Hayward black hole surrounded by perfect fluid dark matter and generalized it to the corresponding rotating metric with the Newman-Janis algorithm. We show
that the photon regions of rotating Hayward black hole in PFDM can be divided into the stable regions and the unstable regions. Then we studied the black hole shadow. The shadow is a perfect circle in the non-rotating case \((a = 0)\) and a deformed one in the rotating case \((a \neq 0)\). We adopted two observables, the radius \(R_s\) and the distortion parameter \(\delta_s\), characterizing the apparent shape of the black hole shadow. For a fixed value of \(a\), \(R_s\) increases with the increasing \(|k|\), but decreases with the increasing \(Q\). There exists a \(k_0\). When \(k < k_0\), \(\delta_s\) increases with the increasing \(|k|\). When \(k > k_0\), \(\delta_s\) decreases with the increasing \(|k|\). The black hole emission rate is also investigated, we find that the emission rate decreases with the increasing \(|k|\) (or \(Q\)) and the peak of the emission shifts to lower frequency.

We can also use the shadow radius \(R_s\) to determine the angular size of the black hole. The angular size is given by \(\theta_s = R_s M / D_o\), with \(M\) the black hole mass and \(D_o\) the distance from the observer to the black hole \([67]\). The angular size can be further expressed as \(\theta_s = 9.87098 \times 10^{-6} R_s (M/M_\odot) (1\text{kpc}/D_o)\) \(\mu\text{as}\) with \(M_\odot\) the solar mass. For the supermassive black hole Sgr A* at the center of the Milky Way, \(M = 4.3 \times 10^6 M_\odot\) and \(D_o = 8.3\) kpc, we show the calculation result in table I. From the table, we find that to extract the information of the black hole spin \(a\) or magnetic charge \(Q\), the required angular resolution must be less than 1 \(\mu\text{as}\).
\[ a=0.1 \]

| \( k \) | \( Q=0.3 \) | \( Q=0.6 \) | \( Q=0.9 \) |
|-------|--------|--------|--------|
| \( R_s \) | 7.2316 | 8.3992 | 9.2450 |
| \( \delta_s (\%) \) | 0.0790 | 0.0758 | 0.0812 |
| \( \theta_s \) | 36.981 | 42.953 | 47.278 |

\[ a=0.8 \]

| \( k \) | \( Q=0.3 \) | \( Q=0.6 \) | \( Q=0.9 \) |
|-------|--------|--------|--------|
| \( R_s \) | 7.2329 | 8.4007 | 9.2468 |
| \( \delta_s (\%) \) | 6.0122 | 5.6908 | 6.1653 |
| \( \theta_s \) | 36.988 | 42.960 | 47.287 |

Table I: The radius \( R_s \), the distortion parameter \( \delta_s \) and the angular size \( \theta_s \) for the supermassive black hole Sgr A* at the center of the Milky Way, for different values of black hole spin \( a \), magnetic charge \( Q \) and PFDM parameter \( k \).

However, for PFDM parameter \( k \), a resolution of 1 µas is sufficient. Currently, EHT has a resolution of \( \sim 60 \) µas at 230 GHz. In the future, the spacebased VLBI RadioAstron [68] will obtain a resolution of \( \sim 1-10 \) µas. We expect that future observations can distinguish the influence of PFDM on the shadow of the black hole Sgr A*.  

22
CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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