Completeness of Trajectories of Relativistic Particles Under Stationary Magnetic Fields

A.M. Candela‡ A. Romero† and M. Sánchez‡

‡Dipartimento di Matematica, Università degli Studi di Bari “A. Moro”,
Via E. Orabona 4, 70125 Bari, Italy
\texttt{candela@dm.uniba.it}

†,‡Departamento de Geometría y Topología,
Facultad de Ciencias, Universidad de Granada,
18071 Granada, Spain
\texttt{aromero@ugr.es, 2sanchezm@ugr.es}

\textbf{Abstract}

The second order differential equation \( \frac{D^2\gamma(t)}{dt^2} = F_\gamma(t)(\dot{\gamma}(t)) - \nabla V(\gamma(t)) \) on a Lorentzian manifold describes, in particular, the dynamics of particles under the action of a electromagnetic field \( F \) and a conservative force \(-\nabla V\). We provide a first study on the extendability of its solutions, by imposing some natural assumptions.

\textit{Key words and phrases:} Second order differential equation; Lorentzian manifold; completeness of inextensible trajectories; electromagnetic field; stationary and conformal vector fields.

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1 Introduction

Let \((M, g)\) be a (connected, finite–dimensional) Lorentzian manifold and denote by \(\pi : M \times \mathbb{R} \rightarrow M\) the natural projection. Giving a \((1,1)\) smooth tensor field \(F\) along \(\pi\) and a smooth vector field \(X\) along \(\pi\), let us consider the second order differential equation

\[
\frac{D\dot{\gamma}}{dt}(t) = F_{(\gamma(t), t)}(\dot{\gamma}(t)) + X_{(\gamma(t), t)}, \tag{E}
\]

where \(D/dt\) denotes the covariant derivative along \(\gamma\) induced by the Levi–Civita connection of \(g\) and \(\dot{\gamma}\) represents the velocity field along \(\gamma\).

Taking \(p \in M\) and \(v \in T_p M\), there exists a unique inextensible smooth curve \(\gamma : I \rightarrow M, 0 \in I\), solution of \((E)\) which satisfies the initial conditions

\[
\gamma(0) = p, \quad \dot{\gamma}(0) = v.
\]

Such a curve is called complete if \(I = \mathbb{R}\) and forward (resp. backward) complete when \(I = (a, b)\) with \(b = +\infty\) (resp. with \(a = -\infty\)).

In the recent article [1], we have investigated the completeness of the inextensible solutions of \((E)\) when \((M, g)\) is a Riemannian manifold both, in the autonomous and in the non–autonomous case, in particular when \(X\) derives from a potential. Furthermore, such results have been applied for studying a special class of Lorentzian manifolds, which generalize the parallelly propagated waves (briefly, \(pp–waves\)), whose geodesic completeness follows from the completeness of the trajectories of a suitable version of \((E)\) stated in a Riemannian manifold (see [1, 2]).

Here, our aim is investigating directly the completeness of the inextensible solutions of \((E)\) in a Lorentzian manifold. Nevertheless, such a problem is much more complicated in this case. In fact, it includes, for example, the geodesic completeness of \((M, g)\) (i.e., the case \(F = 0\) and \(X = 0\)). This problem is handled in the Riemannian case by means of the classical Hopf–Rinow theorem, but nothing similar holds in the Lorentzian one (see the survey [3]). So, in order to consider a manageable Lorentzian problem, some additional assumptions will be made. This will allow to introduce a new type of results, which may be extended in further works.

So, as a physically meaningful framework, we will assume that \(X\) derives from an (autonomous) potential \((X = -\nabla V)\) and \(F\) is skew–adjoint. In particular, when timelike trajectories are taken into account, \((E)\) will describe
the dynamics of relativistic particles subject to an electromagnetic field $F$ (i.e., being accelerated through a term which corresponds to the Lorentz force law) plus an exterior potential $V$ (see, e.g., [4, p. 88]).

This framework still includes the problem of geodesic completeness. Hence, in order to prove the completeness of the solutions of $(E)$, we will select a representative case were the problem has been solved for geodesics, namely, the case when a timelike conformal vector field $K$ exists, the so–called *conformastationary spacetimes* (see [5]). In this ambient, the hypothesis $F(K) = 0$ means that the *conformastationary observers* (those moving along the integral curves of $K$) do not feel any electric force, but only magnetic ones.

Finally, as a simplifying hypothesis, we will assume that $M$ is compact. About this hypothesis, recall that, on one hand, the techniques will be extensible to the non–compact case (in the spirit of [5]) and, on the other hand, the compact case is not by any means trivial, even for geodesic completeness (see [6, 7, 8]).

**Theorem 1.** Let $(M, g)$ be a compact Lorentzian manifold, $F$ a smooth $(1, 1)$ skew–adjoint tensor field on $M$ and $V : M \to \mathbb{R}$ a smooth potential. If a timelike conformal Killing vector field $K$ exists such that $F(K) = 0$, then each inextensible solution of

$$\frac{D\dot{\gamma}}{dt}(t) = F_{\gamma(t)}(\dot{\gamma}(t)) - \nabla V(\gamma(t)) \quad (E_0)$$

must be complete.

This paper is organized in the following way: the main results in the Riemannian case are recalled in Section 2, the specific Lorentzian tools are introduced in Section 3 and the proof of Theorem 1 is developed in Section 4.

## 2 Background about the Riemannian Case

In this section we outline the main results obtained in the Riemannian case (for their proofs, see [7]), even though only some of the tools will be applicable to the Lorentzian one. To this aim, we need some definitions.

Firstly, we recall that the $(1, 1)$ tensor field $F$ can be decomposed as

$$F = S + H,$$
where $S$ is the self-adjoint part of $F$ with respect to $g$, and $H$ is the skew-adjoint one.

From now till the end of this section, let $(M, g)$ be a Riemannian manifold. Taking any $t \in \mathbb{R}$ and considering the slice $M \times \{t\}$, we denote

$$S_{\text{sup}}(t) := \sup_{p \in M, v \in T_pM} \{ g(v, S(p, t)v) \},$$
$$S_{\text{inf}}(t) := \inf_{p \in M, v \in T_pM} \{ g(v, S(p, t)v) \},$$
$$\|S(t)\| := \max\{ |S_{\text{sup}}(t)|, |S_{\text{inf}}(t)| \}.$$

We say that $S$ is bounded (resp. upper bounded; lower bounded) along finite times when, for each $T > 0$ there exists a constant $N_T$ such that

$$\|S(t)\| < N_T \quad \text{(resp. $S_{\text{sup}}(t) < N_T$; $S_{\text{inf}}(t) > -N_T$)}$$
for all $t \in [-T, T]$. (1)

Moreover, if $X$ is a vector field along $\pi$ and $d$ denotes the distance canonically associated to the Riemannian metric $g$, we say that $X$ grows at most linearly in $M$ along finite times if for each $T > 0$ there exists $p_0 \in M$ and some constants $A_T, C_T > 0$ such that

$$\sqrt{g(X(p, t), X(p, t))} \leq A_T d(p, p_0) + C_T \quad \text{for all} \quad (p, t) \in M \times [-T, T].$$
(2)

Obviously, conditions (1), (2) are independent of the chosen point $p_0$.

**Theorem 2.** Let $(M, g)$ be a complete Riemannian manifold and consider a $(1, 1)$ tensor field $F$ and a vector field $X$ both time-dependent and smooth.

If $X$ grows at most linearly in $M$ along finite times and the self-adjoint part $S$ of $F$ is bounded (resp. upper bounded; lower bounded) along finite times, then each inextensible solution of (E) must be complete (resp. forward complete; backward complete).

In particular, if $M$ is compact then any inextensible solution of (E) is complete for any $X$ and $F$.

Let us point out that the hypotheses in Theorem 2 are optimal (see [1, Example 1]) and do not require that $X$ is conservative, i.e., it depends on a potential. Now, let $V : M \times \mathbb{R} \to \mathbb{R}$ be a smooth time-dependent potential, and emphasize as $\nabla^M V$ the gradient of the function $p \in M \mapsto V(p, t) \in \mathbb{R}$, for each fixed $t \in \mathbb{R}$. In this setting, Theorem 2 reduces to the following result.
Corollary 3. Let \((M, g)\) be a complete Riemannian manifold, consider a \((1,1)\) tensor field \(F\), eventually time–dependent, with self–adjoint component \(S\), and let \(V : M \times \mathbb{R} \to \mathbb{R}\) be a smooth potential. If \(S\) is bounded along finite times and \(\nabla^M V(p, t)\) grows at most linearly in \(M\) along finite times, then each inextensible solution of \((E)\) must be complete.

The proof of Theorem 2 is based on the similar result proved when both \(F\) and \(X\) are time–independent as the non–autonomous case \((E)\) can be reduced to the autonomous one by working on the manifold \(M \times \mathbb{R}\) (see [1, Proposition 1]). On the contrary, when \(X\) is a time–dependent conservative vector field, another result can be stated but with a direct proof in the non–autonomous case. In order to describe such a result, we need a further notion.

A function \(U : M \times \mathbb{R} \to \mathbb{R}\) grows at most quadratically along finite times if for each \(T > 0\) there exist \(p_0 \in M\) and some constants \(A_T, C_T > 0\) such that

\[
U(p, t) \leq A_T d^2(p, p_0) + C_T \quad \text{for all} \quad (p, t) \in M \times [-T, T]
\]

(again, this property is independent of the chosen \(p_0\)).

Theorem 4. Let \((M, g)\) be a complete Riemannian manifold, \(F\) a smooth time–dependent \((1,1)\) tensor field on \(M\) with self–adjoint component \(S\) and \(V : M \times \mathbb{R} \to \mathbb{R}\) a smooth time–dependent potential.

Assume that \(S\) is bounded (resp. upper bounded; lower bounded) along finite times and \(-V\) grows at most quadratically along finite times.

If also \(\partial V / \partial t : M \times \mathbb{R} \to \mathbb{R}\) (resp. \(\partial V / \partial t; -\partial V / \partial t\)) grows at most quadratically along finite times, then each inextensible solution of

\[
\frac{D\gamma}{dt}(t) = F(\gamma(t), t)(\dot{\gamma}(t)) - \nabla^M V(\gamma(t), t) \quad (E^*)
\]

must be complete (resp. forward complete; backward complete).

When particularized to autonomous systems, Theorem 4 extends the results by Weinstein and Marsden in [9] and in [10, Theorem 3.7.15]. Furthermore, in the non–autonomous case, it generalizes widely the results by Gordon in [11].

3 The Lorentzian Setting

From now till the end of this paper, let \((M, g)\) be a Lorentzian manifold and assume that \(F\) is a time–independent smooth \((1,1)\) tensor field on \(M\) and
$X$ is a time–independent smooth vector field on $M$, so that we consider the autonomous problem

$$
\frac{D\dot{\gamma}}{dt}(t) = F_{\gamma(t)}(\dot{\gamma}(t)) + X_{\gamma(t)}.
$$

(\tilde{E})

First of all, recall the following result which is a direct consequence of the existence and uniqueness of solutions to second order differential equations (see Lemma 4 and Remark 6 in [1], that apply also in the autonomous Lorentzian case).

**Lemma 5.** There exists a unique vector field $G$ on the tangent bundle $TM$ such that the curves $t \mapsto (\gamma(t), \dot{\gamma}(t))$ are the integral curves of $G$ for any solution $\gamma$ of equation (\tilde{E}).

Recall that an integral curve $\rho$ of a vector field defined on some bounded interval $[a, b)$, where $b < +\infty$, can be extended to $b$ (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$, such that $\{\rho(t_n)\}_n$ converges (see [12, Lemma 1.56]). The following technical result follows directly from this fact and Lemma 5.

**Lemma 6.** Let $\gamma : [0, b) \to M$ be a solution of equation (\tilde{E}) with $0 < b < +\infty$. The curve $\gamma$ can be extended to $b$ as a solution of (\tilde{E}) if and only if there exists a sequence $\{t_n\}_n \subset [0, b)$ such that $t_n \to b^-$ and the sequence $\{\gamma(t_n), \dot{\gamma}(t_n)\}_n$ is convergent in $TM$.

Assume that the vector field $X$ derives from a smooth potential $V : M \to \mathbb{R}$, i.e., $X = -\nabla V$, and, hence, (\tilde{E}) reduces to $(E_0)$. Furthermore, suppose that $F$ is skew–adjoint, so it results $g(Y, F(Y)) = 0$ for any vector field $Y$. In this setting, if $\gamma : (a, b) \to \mathbb{R}$ is a solution of $(E_0)$, then

$$
\frac{d}{dt}(g(\dot{\gamma}(t), \dot{\gamma}(t)) + 2V(\gamma(t))) = 2g(\dot{\gamma}(t), F(\dot{\gamma}(t)) - \nabla V(\gamma(t)))
$$

$$
+ 2g(\nabla V(\gamma(t)), \dot{\gamma}(t))
$$

$$
= 0 \quad \text{for all } t \in (a, b)
$$

and a constant $c \in \mathbb{R}$ exists such that

$$
g(\dot{\gamma}(t), \dot{\gamma}(t)) + 2V(\gamma(t)) = c \quad \text{for all } t \in (a, b).
$$

(3)

Let us point out that, if $M$ is a compact Lorentzian manifold, the conservation law (3) implies that $g(\dot{\gamma}(t), \dot{\gamma}(t))$ is bounded. Anyway, unlike the Riemannian case, this is not enough for applying Lemma 6 and so proving the completeness of all the inextendible solutions of $(E_0)$. 

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Example 7. (1) There are examples of compact Lorentzian manifolds which have incomplete inextensible geodesics, i.e. solutions of $(E_0)$ in the simplest case $F = 0, V = 0$. In fact, if $(M, g)$ is a Clifton–Pohl torus, then it is a compact Lorentzian manifold but it is not geodesically complete (see [12, Example 7.16]).

(2) There are examples of geodesically complete Lorentzian manifolds $(M, g)$ with bounded $\|F\|^2 = |\sum \epsilon^{\mu\nu} F_{\mu\nu}|$ and $\|X\|^2 = |\sum \epsilon^{\mu\nu} X_{\mu\nu}|$ such that they admit incomplete inextensible solutions of $(\tilde{E})$. Indeed, it is enough to consider $M = \mathbb{R}^2$ and $g = dx \otimes dy + dy \otimes dx$ with $F = 0$ and $X = 2x^3 \frac{\partial}{\partial x}$. Direct computations show that the corresponding problem $(\tilde{E})$ has incomplete inextensible solutions.

It is a relevant fact that a compact Lorentzian manifold is geodesically complete if it admits a timelike conformal vector field (see [5]). Thus, it is natural to assume the existence of such infinitesimal conformal symmetry to deal with the extendibility of the solutions of $(E_0)$.

Definition 8. A vector field $K$ is called conformal Killing, or simply conformal, if the Lie derivative with respect to $K$, $\mathcal{L}_K$, satisfies

$$ \mathcal{L}_K g = 2\sigma g $$

(the local flows of $K$ are conformal maps) for some $\sigma \in C^\infty(M)$. In the case $\sigma = 0$, $K$ is called Killing.

In particular, if $K$ is a conformal vector field and $\gamma$ is a geodesic, we have

$$ \frac{d}{dt} g(K, \dot{\gamma}) = \sigma(\gamma) g(\dot{\gamma}, \dot{\gamma}), \quad \text{with } g(\dot{\gamma}, \dot{\gamma}) \text{ constant.} $$

Hence, if $K$ is Killing then $g(K, \dot{\gamma})$ is a constant.

More in general, if $\gamma : I \rightarrow M$ is any curve, from (4) it follows

$$ g(\nabla_{\dot{\gamma}} K, \dot{\gamma}) = \sigma(\gamma) \ g(\dot{\gamma}, \dot{\gamma}) $$

which implies

$$ \frac{d}{dt} g(K, \dot{\gamma}) = g(K, D_{\dot{\gamma}}) + \sigma(\gamma) \ g(\dot{\gamma}, \dot{\gamma}). \quad (5) $$

As already remarked, the compactness of $M$ and the boundedness of $g(\dot{\gamma}, \dot{\gamma})$ are not enough to assure that the image of $\dot{\gamma}$ is contained in a compact subset of the tangent bundle $TM$. So, in order to prove our main result, some extra Lorentzian tools are needed.
Lemma 9. Let \((M, g)\) be a (time-orientable) compact Lorentzian manifold with a timelike vector field \(Z\) such that \(g(Z, Z) = -1\). Assume that \(F\) is a skew–adjoint \((1,1)\) tensor field and \(V\) is a smooth potential on \(M\). If \(\gamma : I \to M\) is a solution of \((E_0)\) such that \(g(Z, \dot{\gamma})\) is bounded in \(I\), then there exists a compact subset \(C\) of \(TM\) which contains the image of \(\dot{\gamma}\).

Proof. As \(\gamma\) is a solution of \((E_0)\) and \(M\) is compact, from (3) it follows that \(g(\dot{\gamma}, \dot{\gamma})\) is bounded. On the other hand, consider the 1–form \(\omega\) equivalent to \(Z\) with respect to \(g\), i.e. \(\omega(X) = g(Z, X)\) for any vector field \(X\), and the related tensor field \(g_R = g + 2\omega \otimes \omega\) which is a Riemannian metric on \(M\). By definition, it results
\[
g_R(\dot{\gamma}, \dot{\gamma}) = g(\dot{\gamma}, \dot{\gamma}) + 2g(Z, \dot{\gamma})^2;
\]
whence, in our hypotheses, \(g_R(\dot{\gamma}, \dot{\gamma})\) is bounded on \(I\). Thus, a constant \(c > 0\) exists such that
\[
(\gamma(I), \dot{\gamma}(I)) \subset C, \quad C := \{(p, v) \in TM : p \in M, g_R(v, v) \leq c\},
\]
with \(C\) compact in \(TM\).

4 Proof of the Main Result

Now, we are ready to prove our main result.

Proof of Theorem 1. Without loss of generality, let \(I = [0, b), 0 < b < +\infty\), be the domain of a forward–inextensible solution \(\gamma\) of \((E_0)\). As \(F\) is skew–adjoint and null on \(K\), it results
\[
g(K, F_\gamma(\dot{\gamma})) = -g(F_\gamma(K), \dot{\gamma}) = 0,
\]
then from (4) it follows
\[
\frac{d}{dt}g(K, \dot{\gamma}) = g(K, F_\gamma(\dot{\gamma})) - g(K, \nabla V(\gamma)) + \sigma(\gamma) g(\dot{\gamma}, \dot{\gamma})
\]
\[
= -g(K, \nabla V(\gamma)) + \sigma(\gamma) g(\dot{\gamma}, \dot{\gamma}).
\]

From the compactness of \(M\) and (3) we have that both \(g(K, \nabla V(\gamma))\) and \(\sigma(\gamma) g(\dot{\gamma}, \dot{\gamma})\) are bounded on \(I\), then \(c_1 > 0\) exists such that
\[
\left| \frac{d}{dt}g(K, \dot{\gamma}) \right| \leq c_1 \text{ on } I;
\]
whence, as $I$ is a bounded interval, a constant $c_2 > 0$ exists such that

$$|g(K, \dot{\gamma})| \leq c_2 \text{ on } I. \quad (6)$$

Now, let us define $Z = \frac{K}{\|K\|}$, with $\|K\|^2 = -g(K, K) > 0$ as $K$ is timelike. So, $Z$ is a timelike vector field with $g(Z, Z) = -1$, and (6) implies

$$|g(Z, \dot{\gamma})| \leq mc_2 \text{ on } I,$$

where $m = \max\limits_{M} \|K\|^{-1} > 0$ (which exists as $M$ is compact).

Then, by applying Lemmas 9 and 6 we yield a contradiction. \hfill \Box

**Remark 10.** In particular, we may consider $F = 0$ and $V = 0$ in Theorem 1 and, therefore, this result extends the theorem on completeness in [5].

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