An Objective Definition of Damage Spreading - Application to Directed Percolation

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We present a general definition of damage spreading in a pair of models. Using this general framework, one can define damage spreading in an objective manner, that does not depend on the particular dynamic procedure that is being used. The formalism is applied to the Domany-Kinzel cellular automaton in one dimension; the active phase of this model is shown to consist of three sub-phases, characterized by different damage-spreading properties.

PACS numbers: 05.70.Ln, 64.60.Ak, 64.60.Ht, 89.80.+h
Key words: damage spreading, directed percolation

I. INTRODUCTION

The concept of damage spreading was introduced in the context of biologically motivated dynamical systems by Stuart Kauffman [1]. The question posed is whether the phase-space trajectories of two slightly different copies of a dynamic system, subjected to the same thermal noise, will stay close (or even merge) at long times or, alternatively, will they diverge? Damage spreading made its first appearance in the physics literature in the mid-eighties [2–4], and attracted considerable interest and attention. The main reason behind this initial enthusiasm was the hope that damage may spread (indicating chaotic behavior) in some regions of a system’s parameter space and disappear or heal elsewhere. This possibility intrigued researchers, since if indeed realized, it would have indicated the existence of different dynamic phases in various complex systems (such as spin-glasses) [4]. The initial enthusiasm concerning damage spreading has abated during subsequent years; the main reason being an apparent lack of an objective, observer-independent measure of whether damage does or does not spread in a given system. Even for relatively simple models, such as the two-dimensional ferromagnetic Ising model, different results were obtained when heat bath or Metropolis dynamics were used [3–5]. Both these dynamic procedures are phenomenological (since they satisfy detailed balance, they can be used to generate equilibrium ensembles) and the two are equally legitimate to mimic the temporal evolution of a system in contact with a thermal reservoir. If spreading or healing of damage were to indicate some intrinsic property of the system, one would not expect the result to depend on the details of exactly which phenomenological procedure was used to generate its dynamics.

The purpose of this communication is to pose the “right” question; i.e. one which has a well defined objective answer. The essence of the argument is to consider the entire family of dynamic procedures that are consistent with the physically dictated constraints of the problem. For any particular system one of three possibilities may hold:

1. Damage is spreading for every member of the family of dynamic procedures
2. Damage heals for every member of this family
3. Damage spreads for a subset of the possible dynamic procedures, and heals for the complementing subset.

Hence the only question regarding damage spreading that has an unambiguous, observer-independent answer is: to which of these three classes a particular system belongs?

To demonstrate the general concept introduced here we studied the simplest dynamic model in which damage spreading has been observed, the one-dimensional Domany-Kinzel (DK) cellular automaton [7], for which we found the phase diagram presented in Fig. 1.

The DK automaton is a two-parameter model whose temporal evolution contains, as special cases, the bond and site directed percolation problems. The main point made by DK was universality: namely, that the entire family of observed transitions of the one-dimensional cel-
lular automaton is in the directed percolation universality class. DK identified two phases; a “dry” or “frozen” phase, in which all initial conditions evolve to the absorbing state, and an “active” or percolating phase. Some years later Martins et al. discovered that in a certain region of the active phase damage spreads, and it heals elsewhere. More detailed investigations, using simulations as well as analytic (mean field) approximations confirmed the existence of this “chaotic phase”. Its boundary, however, was shown to depend on the manner in which the dynamic procedure of the underlying DK model is carried out, while the evolution of a single replica is completely insensitive to the dynamic procedure. This prompted Grassberger to observe that “it is misleading to speak of different phases in the DK automaton...instead these are different phases for very specific algorithms for simulating pairs of such automata”. This observation is the precise analog of the problematic nature of viewing DS as a manifestation of a dynamic transition in spin models, where, as mentioned above, it was well known that different dynamics that yield identical equilibrium properties can give rise to different results for damage spreading. Thus, again, DS becomes a “subjective” concept, which is devoid of well defined meaning for the DK model, whose phases should be determined by the properties of a single evolving system.

The main purpose of this paper is to point out that if one defines the most general family of dynamic rules that are consistent with the physics of the problem being studied (Sec II), DS has an objective, observer-independent meaning. Past work on DS in the DK model is reviewed in Sec III and in Sec IV the existence of the three well defined distinct phases described in the Introduction is established for the DK model by numerical simulations and analytical arguments.

We also tested and confirmed a recent conjecture of Grassberger, to the effect that the damage spreading transition is in the directed percolation universality class. Analytical support for this conjecture came so far from approximate mean-field arguments and an exact statement first made by Kohring and Schreckenberg, who noted that on the line the dynamics of damage spreading in the DK automaton is precisely identical to the evolution of the DK automaton itself, and hence on this line DS is trivially in the DP universality class. This being a rather special line, it is of interest to try to establish such precise mapping of DS to DP also elsewhere in the plane. In Sec IV A we present such an extension.

**II. DAMAGE SPREADING - GENERAL FORMALISM**

**A. Rules for Legitimate Damage Spreading Procedures**

We turn now to present our arguments for the possibility of defining an observer-independent measurement of damage spreading. By this we do not mean that DS is reflected in the dynamic behavior of a single system, so that Grassberger’s observation still holds; DS is a property of a pair of automata. It is possible, however, to address the lack of objectivity implicit in one’s freedom to choose the precise algorithm that is used for the evolution of the pair of replicas. If every observer can pick his favorite dynamic rule, get results (on DS) that depend on the rule used, while no measurement done on an evolving single system can differentiate between the rules - indeed it appears contradictory to claim that DS reflects “phases” of the model that is being investigated. Nevertheless such phases can be defined in a precise way.

To overcome this apparent paradox we formulate quite general and physically motivated restrictions on the possible dynamic rules that one can use for studying DS. By “physical” we mean that the restrictions are dictated by the dynamics of the single evolving system. The restrictions are as follows:

1. The dynamic rules for the evolution of the pair of replicas are such that the evolution of a single replica is according to its “natural” dynamics.

2. The transition probability matrix for a site i for the pair of replicas can depend only on those sites that affect the evolution of site i under the dynamic rules of a single system.

3. The rules that govern evolution for the pair do not break any of the symmetries of the single-replica dynamics.

The first restriction simply means that the fact that we are watching two systems evolving in parallel should not affect the behavior of any one of them. The second constraint means that if the evolution of site i is affected, say, only by the states of its nearest neighbors, the relative states taken on site i by the two replicas should not feel longer range interactions. For example, if site i and all its neighbors j are in the same state in the two replicas, in this sense DS, as defined for stochastic dynamics, differs from dynamics in deterministic nonlinear systems. In that case one can find signatures of chaotic behavior in following the phase-space trajectory of a single evolving system. Divergence of two initially neighboring trajectories (indicating the existence of a positive Lyapunov exponent) is a computationally feasible tool to ascertain the chaotic nature of a single system’s trajectory, but is not essential to consider two replicas in order to define chaos.
we do not expect damage to be generated at $i$ by a damaged site which is far away (i.e. not one of the neighbors of $i$). The third rule implies, for example, that if there is a left-right symmetry in the evolution of a single system the same must hold for the pair of replicas.

Clearly, the subjectivity in defining the damage spreading procedure that was described above has now been shifted to this point - to selecting the restrictions that define which DS procedure is “legitimate”. We do believe that there is much less arbitrariness, however, in this kind of subjectivity than what was done before - choosing, at random, one out of a continuum of physically equivalent procedures.

B. Transfer matrix formalism

We now introduce a general formalism for damage spreading problems in one-dimensional models with parallel updates. Consider a one-dimensional spin chain with $L$ sites and certain boundary conditions to be specified. At each site $j = 1, \ldots, L$ a local spin variable $\sigma_j = 1, \ldots, s$ is placed which can be in $s$ different states. We denote by $P_t(\sigma_1, \ldots, \sigma_L)$ the probability to find the system at time $t$ in the state \{\(\sigma_1, \ldots, \sigma_L\)\} which is positive and normalized by

$$\sum_{\sigma_1, \ldots, \sigma_L} P_t(\sigma_1, \ldots, \sigma_L) = 1. \quad (1)$$

The time evolution of a system with parallel updates is discrete and can be described by a transfer matrix $T$, whose element $T_{\sigma_1, \ldots, \sigma_L}^{\tau_1, \ldots, \tau_L}$ is the probability of making a transition from state \{\(\sigma_1, \ldots, \sigma_L\)\} at time $t$ to state \{\(\tau_1, \ldots, \tau_L\)\} at time $t+1$. The corresponding discrete time master equation has the form

$$P_{t+1}(\tau_1, \ldots, \tau_L) = \sum_{\sigma_1, \ldots, \sigma_L} T_{\sigma_1, \ldots, \sigma_L}^{\tau_1, \ldots, \tau_L} P_t(\sigma_1, \ldots, \sigma_L) \quad (2)$$

The conservation of probability (1) implies that

$$\sum_{\tau_1, \ldots, \tau_L} T_{\sigma_1, \ldots, \sigma_L}^{\tau_1, \ldots, \tau_L} = 1 \quad (3)$$

for all configurations \{\(\sigma_1, \ldots, \sigma_L\)\}, which means that the elements in each column of the transfer matrix add up to one.

In what follows we assume that each site is updated simultaneously\(^2\) and that an update at some site $j$ depends locally on $n$ sites of the previous configuration, say \{\(\sigma_{j-m}, \ldots, \sigma_{j+m}\)\} where $m = \frac{4n}{L}$. This means that the transfer matrix factorizes:

$$T_{\sigma_{1}, \ldots, \sigma_{L}}^{\tau_{1}, \ldots, \tau_{L}} = \prod_{j=1}^{L} T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}}. \quad (4)$$

Each factor $T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}}$ conserves probability separately:

$$\sum_{\tau_{j}=1}^{S} T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}} = 1. \quad (5)$$

We now consider two replicas $S$ and $S'$ of the same system. Denote by \{\(\sigma_1, \ldots, \sigma_L\)\} and \{\(\tau_1, \ldots, \tau_L\)\} the states of $S$ and by \{\(\sigma'_1, \ldots, \sigma'_L\)\} and \{\(\tau'_1, \ldots, \tau'_L\)\} those of $S'$. In order to define simultaneous temporal evolution of these systems, one has to generalize the transfer matrix:

$$T_{\sigma_{1}, \ldots, \sigma_{L}}^{\tau_{1}, \ldots, \tau_{L}} \rightarrow T_{\sigma_{1}, \ldots, \sigma_{L}}^{\tau_{1}, \ldots, \tau_{L}} T_{\sigma'_{1}, \ldots, \sigma'_{L}}^{\tau'_1, \ldots, \tau'_L} \quad (6)$$

$$T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}} \rightarrow T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}} T_{\sigma'_{j-m}, \ldots, \sigma'_{j+m}}^{\tau'_1, \ldots, \tau'_L}. \quad (7)$$

The transfer matrix for the total system \((S, S')\) is restricted by the requirement that each replica should evolve as before, i.e. integrating out the degrees of freedom of one of the replicas results in the previous transfer matrix of the other replica:

$$\sum_{\tau_{j}=1}^{S} T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}} = T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}} \quad (8)$$

These restrictions already imply probability conservation for the total system:

$$\sum_{\tau_{j}, \tau'_{j}=1}^{S} T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}} T_{\sigma'_{j-m}, \ldots, \sigma'_{j+m}}^{\tau'_1, \ldots, \tau'_L} = 1. \quad (9)$$

In order to study damage spreading, a further restriction is imposed: Once both replicas reach the same state (no damage), their temporal evolution is identical:

$$T_{\sigma_{j-m}, \ldots, \sigma_{j+m}}^{\tau_{j-m}, \ldots, \tau_{j+m}} \delta_{\tau_{j}, \tau'_{j}} \quad (10)$$

We define damage as the Hamming distance between the states \{\(\sigma_1, \ldots, \sigma_L\)\} and \{\(\sigma'_1, \ldots, \sigma'_L\)\} at any given time, that is the fraction of sites for which $\sigma_j \neq \sigma'_j$:

$$\Delta = \frac{1}{L} \sum_{j=1}^{L} \Delta_j = \frac{1}{L} \sum_{j=1}^{L} (1 - \delta_{\sigma_j, \sigma'_j}) \quad (10)$$

The restrictions (8), (9), and (10) impose dependences among the $s^{2n+2}$
matrix elements $T_{\sigma_{j-m},...,\sigma_{j+m}}^{\sigma_j',\sigma_j}$. The number of independent degrees of freedom can be counted as follows. First notice that because of eq. (6) only $2s-1$ of the $2s$ equations in (3) and (4) are independent. Thus, even though for any given initial configuration \( \{\sigma_{j-m},\ldots,\sigma_{j+m};\sigma_{j-m}',\ldots,\sigma_{j+m}'\} \) the number of different final configurations is $s^2$, the number of independent matrix elements is $s^2 - (2s-1) = (s-1)^2$. On the other hand eq. (9) implies that for $s^n$ possible initial configurations \( \{\sigma_{j-m},\ldots,\sigma_{j+m};\sigma_{j-m}',\ldots,\sigma_{j+m}'\} \) the matrix elements are already defined. Furthermore the whole system is symmetric under exchange of the replicas which gives another factor $\frac{1}{2}$. Thus the total number of independent degrees of freedom of the transfer matrix is

$$\frac{1}{2} (s^{2n} - s^n) (s-1)^2$$  \hspace{1cm} (11)

C. Algorithmic implementation

In a numerical simulation the temporal evolution described by the transfer matrix $T$ can be realized as follows. At each time step all sites are updated independently, i.e. $\sigma_j$ is replaced by a new value $\tau_j$ according to probabilistic rules which depend only on the previous configuration $\sigma_{j-m},\ldots,\sigma_{j+m}$. Since in this paper we discuss only two-state models, let us from now on restrict our attention to the case $s=2$, with $\sigma_j = 0,1$ (the generalization to $s>2$ is straightforward).

We introduce a stochastic binary variable $r_{\sigma_{j-m},\ldots,\sigma_{j+m}}$, that denotes the value assigned to site $j$ in one update, given the state of its neighborhood $\sigma_{j-m},\ldots,\sigma_{j+m}$:

$$\tau_j := r_{\sigma_{j-m},\ldots,\sigma_{j+m}}$$  \hspace{1cm} (12)

The numbers $r_{\sigma_{j-m},\ldots,\sigma_{j+m}}$ are generated probabilistically in some procedure, such that updates at different sites or different times are uncorrelated. Furthermore, their expectation value, averaged over many realizations of random numbers, is given by the corresponding matrix element of the transfer matrix:

$$\langle r_{\sigma_{j-m},\ldots,\sigma_{j+m}} \rangle = T_{\sigma_{j-m},\ldots,\sigma_{j+m}}^{\tau_j=1,\ldots,\tau_j=1}$$  \hspace{1cm} (13)

Usually this procedure is implemented by generating a random number $z$ from a uniform distribution in the interval $0<z<1$, and comparing it with the transition probability $T_{\sigma_{j-m},\ldots,\sigma_{j+m}}^{\tau_j=1}$, to get the assigned value of $r_{\sigma_{j-m},\ldots,\sigma_{j+m}}$:

$$r_{\sigma_{j-m},\ldots,\sigma_{j+m}} = \begin{cases} 1 & \text{if } z < T_{\sigma_{j-m},\ldots,\sigma_{j+m}}^{\tau_j=1} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (14)

This prescription is not unique, however, and there are many other possibilities to select $r_{\sigma_{j-m},\ldots,\sigma_{j+m}}$ in a manner that satisfies eq. (13). This constraint specifies the temporal evolution of a single system. The existence of correlations between any two random variables $r_{\sigma_{j-m},\ldots,\sigma_{j+m}}$ and $r_{\sigma_{j-m}',\ldots,\sigma_{j+m}'}$ will play no role in the evolution of a single system, since only one of the two will be used for any given update. The situation is different, however, in damage spreading problems in which the two different replicas of the system are evolving simultaneously. The updated state of the pair of replicas is governed by the joint transition probabilities

$$\text{Prob}(\tau_j = r, \tau_j' = r') = T_{\sigma_{j-m},\ldots,\sigma_{j+m};\sigma_{j-m}',\ldots,\sigma_{j+m}'}^{\tau_j=1,\ldots,\tau_j'=1}$$  \hspace{1cm} (15)

Therefore correlations between the two random variables $r, r'$ will influence the temporal evolution of the combined system and, therewith, the properties of damage spreading. Such correlations are contained in the two-point correlation functions of the random variables, which are given by the elements of the combined transfer matrix for both replicas:

$$\langle r_{\sigma_{j-m},\ldots,\sigma_{j+m}} r_{\sigma_{j-m}',\ldots,\sigma_{j+m}'} \rangle = T_{\sigma_{j-m},\ldots,\sigma_{j+m};\sigma_{j-m}',\ldots,\sigma_{j+m}'}^{\tau_j=1,\ldots,\tau_j'=1}$$  \hspace{1cm} (16)

In this formalism the restriction (3)-(4), stating that each replica separately evolves in the same way as the original system, is satisfied automatically. The same applies to the second restriction (5) which ensures that in case of ‘no damage’ both systems evolve in parallel.

Obviously, the number of degrees of freedom specifying damage spreading is just the number of two-point correlation functions. In the case of two-state models there are $2^n (2^n - 1)$ such correlations, which agrees with the number of degrees of freedom counted in eq. (11).

Three-point correlations do not affect the evolution of a pair of replicas, because in each update only two random variables, $r_{\sigma_{j-m},\ldots,\sigma_{j+m}}$ and $r_{\sigma_{j-m}',\ldots,\sigma_{j+m}'}$, are used. However, they would start to play a role in damage spreading problems with three replicas. Generally, $k$-point correlations will be felt in systems consisting of at least $k$ replicas.

III. DAMAGE SPREADING IN THE DK MODEL:
A BRIEF REVIEW

In this section we review briefly past work on damage spreading in the DK automaton. We emphasize the manner in which DS was calculated by various authors, and the manner in which different ways of defining DS are embedded in the general formal framework of Sec II.

The DK automaton is defined as follows: a binary variable $\sigma_i(t) = 0,1$ characterizes the state of site $i$ at (discrete) time $t$. $\sigma = 1$ means that the site is wet or active, whereas $\sigma = 0$ means that it is dry. The automaton
evolves by a parallel update rule, which can be stated, using the notation of Sec II, as follows:

\[ T^r_{\sigma_{t-1}, \sigma_{t+1}} = T^r_{\sigma_{t-1}, \sigma_{t+1}} \]

\[ T^1_{\sigma_{t-1}, \sigma_{t+1}} = \begin{cases} 
0 & \text{if } \sigma_{t-1} = \sigma_{t+1} = 0 \\
p_1 & \text{if } \sigma_{t-1} \neq \sigma_{t+1} \\
p_2 & \text{if } \sigma_{t-1} = \sigma_{t+1} = 1
\end{cases} \]

That is, the state of site \( i \) at time \( t + 1 \) depends only on the states of its two neighbors at time \( t \); only wet sites can give rise to a wet site, with probabilities \( p_1 \) if one neighbor was wet and \( p_2 \) if both were wet.

Using the notations introduced in Sec II the transition probabilities in the DK model are defined by the one-point expectation values of three stochastic binary variables \[ \langle r_0 \rangle = \langle r_{10} \rangle = p_1, \quad \langle r_{11} \rangle = p_2. \] This model has a dry phase and a wet phase, separated by a transition line which has been determined with high accuracy by various numerical methods. In spite of its simplicity, the model has not been solved exactly, except for the special line \( p_2 = 1 \). At all points on the phase boundary, except the special line, the transition to the active or wet phase is characterized by directed percolation (DP) exponents.

Damage spreading properties between two replicas are controlled by correlations between the random variables:

\[ \langle r_0 r_{11} \rangle = \langle r_{10} r_{11} \rangle = \tilde{\alpha} \]
\[ \langle r_0 r_{10} \rangle = \tilde{\beta} \]
\[ \langle r_0 r_{10} r_{11} \rangle = \tilde{\gamma} \]

According to the arguments discussed in Sec II, only one- and two-point functions enter the transfer matrix, which means that \( \tilde{\gamma} \) is an irrelevant parameter in the present problem.

Martins et al. \[ 3 \] were the first to address the issue of damage spreading in the DK model. Two nearly identical initial configurations were allowed to evolve on two replicas, using the same random numbers for both (the precise initial configurations were allowed to evolve on two replicas). As discussed in Sec II, the evolution of a single replica is completely insensitive to whether one or two random variables are used in the dynamic procedure, which prompted Grassberger \[ 2 \] to make his observation quoted in the Introduction.

Finally we note that Grassberger has formulated recently \[ 12 \] a conjecture, which is a natural extension of previous statements \[ 13 \] regarding universality of directed percolation transitions for models with nonsymmetric absorbing states \[ 21 \]. According to this conjecture, the evolution of a single replica is completely insensitive to whether one or two random variables are used in the dynamic procedure, which prompted Grassberger \[ 2 \] to make his observation quoted in the Introduction.

For reasons that will be evident shortly, we call this dynamic process, that uses a single random number, **maximally correlated**.

Kohring and Schreckenberg recognized the fact that one could, in principle, use two different random numbers to determine \( \tau_i \) and \( \tau'_{i} \), if at least one of the two neighbor sites was damaged at time \( t \). In fact they studied DS using two different random numbers \( z_0 \) and \( z_{11} \), their DS procedure has two fully correlated binary variables \( (r_0 \) and \( r_{10} \) and two uncorrelated ones \( (r_0 \) and \( r_{11} \):

\[ r_0 = r_{10} = \theta(p_1 - z), \quad r_{11} = \theta(p_2 - z) \]

was made, which can be expressed as

\[ \tilde{\beta} = \langle r_0 r_{10} \rangle = p_1 \]
\[ \tilde{\alpha} = \langle r_0 r_{11} \rangle = \min(p_1, p_2) \]

The dynamical process is generated by setting

\[ \sigma_i(t+1) = r_{\sigma_{t-1} \sigma_{t+1}} \]
\[ \sigma'_i(t+1) = r_{\sigma'_{t-1} \sigma'_{t+1}} \]

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\[ r_0 = r_{10} = \theta(p_1 - z), \quad r_{11} = \theta(p_2 - z) \]

the correlations being

\[ \tilde{\beta} = \langle r_0 r_{10} \rangle = p_1 \]
\[ \tilde{\alpha} = \langle r_0 r_{11} \rangle = \langle r_0 \rangle \langle r_{11} \rangle = p_1 p_2 \]

The dynamics generated by using on the first replica \( \tau_i = r_{\sigma_{t-1} \sigma_{t+1}} \) and \( \tau'_i = r_{\sigma'_{t-1} \sigma'_{t+1}} \) on the second gave rise to a shift of the original “phase boundary” (as obtained with a single random number, eq. \[ 21 \]). As discussed in Sec II, the evolution of a single replica is completely insensitive to whether one or two random variables are used in the dynamic procedure, which prompted Grassberger \[ 2 \] to make his observation quoted in the Introduction.

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IV. TRUE PHASES IN THE DK MODEL

As discussed in Sec III, the most general dynamic rule that can be defined for two replicas of the DK automaton, in accordance with these constraints, has two degrees of freedom or parameters, \( \alpha \) and \( \beta \). As it turns out (see Appendix A), the possible values that \( \alpha \) and \( \beta \) can take are restricted by requiring that all transition rates in the transfer matrix have to be positive. For any value of \( p_1, p_2 \), the range of allowed values of the parameters \( \alpha \) and \( \beta \) is given by

\[
\max(0, p_1 + p_2 - 1) \leq \alpha \leq \min(p_1, p_2) \\
\max(0, 2\alpha - p_2, 2p_1 - 1, 2p_1 - 2\alpha + p_2) \leq \beta \leq p_1 
\]

(25)

There are three important special cases, namely those of

- maximal correlations: \( \alpha = \min(p_1, p_2) \), \( \beta = p_1 \)
- no correlations: \( \alpha = p_1 p_2 \), \( \beta = p_1^2 \)
- minimal correlations: \( \alpha = p_1 + p_2 - 1 \), \( \beta = 2p_1 - 1 \)

In the case of minimal correlations, the values listed above hold only in the region \( 2p_1 + p_2 > 2 \) (see Appendix A).

A. Exact results

We turn now to show that for \( p_2/2 \leq p_1 \leq 1 - p_2/2 \) the damage spreading process can be mapped exactly onto a directed percolation process. Kohring and Schreckenberg [3] have shown that such a mapping holds on the line \( p_2 = 0 \). Clearly, their choice of parameters [24] is a particular case of our damage spreading procedure, which is the most general one that satisfies rules 1 - 3 listed above. Therefore we find a wider (two-dimensional) region in the \( p_1, p_2 \) plane in which such a mapping is possible. To see this, let \( \Delta_i = 1 - \delta_{\sigma_i, \sigma_i'} \) be the damage at site \( i \). By \( P_D(\Delta_i = 1 \mid \sigma_{i-1}\sigma_{i+1}, \sigma_{i-1}', \sigma_{i+1}') \) we denote the probability to generate a damaged site for a given initial configuration in a particular update. These probabilities are listed in Table I, in which we introduced for brevity the notation

\[
X = 2p_1 - 2\beta \quad Y = p_1 + p_2 - 2\alpha 
\]

(26)

In general, the probability for generating damage on site \( i \) depends on the previous states of both replicas, i.e. on \( (\sigma_{i-1}\sigma_{i+1}, \sigma_{i-1}', \sigma_{i+1}') \); knowledge of \( \Delta_{i-1} \) and \( \Delta_{i+1} \) does not suffice to determine \( \Delta_i \) at the next time step. Thus damage spreading itself cannot be seen as an independent process. We may, however, pose the following question: under which conditions will damage spread as if it were generated by an independent process? That is, when do we have

\[
P_D(\Delta_i \mid \sigma_{i-1}\sigma_{i+1}, \sigma_{i-1}', \sigma_{i+1}') = P_D(\Delta_i \mid \Delta_{i-1}, \Delta_{i+1})
\]

(27)

In order to satisfy this condition, any two entries in Table I, that correspond to the same initial damage \( \{\Delta_{i-1}, \Delta_{i+1}\} \), should be equal. For example all four initial configurations

\[
\{\sigma_{i-1}\sigma_{i+1} \mid \sigma_{i-1}', \sigma_{i+1}'\} = \{(11; 00), (10; 01), (01; 10), (00; 11)\}
\]

have the same initial damage \( \{\Delta_{i-1}, \Delta_{i+1}\} = \{1, 1\} \). In order to satisfy eq. (27), the four entries \( (p_2, X, X, p_2) \) must have the same value, i.e. we must have \( p_2 = X \). A similar consideration leads to the condition \( Y = p_1 \); that is, we must have

\[
P_D(1|00) = 0 \\
P_D(1|01) = P_D(1|10) = p_1 = Y \\
P_D(1|11) = p_2 = X
\]

(28)

Note that these are precisely the update rules of the DK process. Using the definitions (26), we see that the correlations must satisfy

\[
\alpha = \frac{p_2}{2}, \quad \beta = p_1 - \frac{p_2}{2}.
\]

(29)

Since the correlation parameters are restricted by eq. (23), the allowed range for \( p_1 \) and \( p_2 \) in which these conditions can hold is a triangle in the phase diagram:

\[
p_2/2 \leq p_1 \leq 1 - p_2/2
\]

(30)

To summarize: we have proved that within this triangle we can find correlations \( \alpha, \beta \) such that the damage spreading process follows the dynamical rules of a single DK automaton.
Say we have a line in the \((p_1, p_2)\) plane that lies within this region. For every point \((p_1^*, p_2^*)\) on this line we can find \(\alpha, \beta\) values for which DS evolves precisely like a DK automaton with parameters \((p_1^*, p_2^*)\). Since part of the transition line of the DK model (from dry to wet phase) lies in the triangle \((21)\), on any trajectory that crosses this part of the phase boundary we will observe a damage spreading transition precisely at the DP transition and with DP exponents (provided we chose \(\alpha, \beta\) according to eq. \((22)\)). In particular, this holds for the line \(p_2 = 0\), as discovered in \([13]\); note that for \(p_2 = 0\) their choice of correlations, eq. \((24)\) precisely satisfy eq. \((23)\).

**B. Results from comparing probabilities**

Other useful results can be obtained by comparing probability tables of different pairs of automata. The basic idea is that by increasing (decreasing) all probabilities in Table I, damage spreading will be more (less) likely. More precisely, if a pair of DK automata described by parameters \(p_1^*, p_2^*, \alpha^*, \beta^*\) exhibits damage spreading, we expect that any other pair of automata with parameters \(p_1, p_2, \alpha, \beta\) satisfying

\[
\begin{align*}
p_1^* &\leq p_1, \\
p_2^* &\leq p_2, \\
p_1^* + p_2^* - 2\alpha^* &\leq p_1 + p_2 - 2\alpha, \\
2p_1^* - 2\beta^* &\leq 2p_1 - 2\beta
\end{align*}
\]

exhibits damage spreading as well. Vice versa, if damage heals in a pair of automata described by \(p_1^*, p_2^*, \alpha^*, \beta^*\), then for any other pair with \(p_1, p_2, \alpha, \beta\) obeying

\[
\begin{align*}
p_1^* &\geq p_1, \\
p_2^* &\geq p_2, \\
p_1^* + p_2^* - 2\alpha^* &\geq p_1 + p_2 - 2\alpha, \\
2p_1^* - 2\beta^* &\geq 2p_1 - 2\beta
\end{align*}
\]

we expect damage to heal.

Although these statements are very plausible, we were not able to prove them rigorously. However, we performed various numerical tests which turned out to be consistent with the inequalities stated above.

Because of these inequalities the boundaries between the three regions in the phase diagram correspond to extremal correlations \(\alpha\) and \(\beta\). For example, if at a point \((p_1, p_2)\) damage spreads in a model with maximal correlations \(\alpha^{max} = \min(p_1, p_2)\) and \(\beta^{max} = p_1\), then eq. \((31)\) implies that damage spreads also for every \(\alpha\) and \(\beta\) in the allowed range \((23)\). This, however, means that the point \((p_1, p_2)\) belongs to region 1 in the phase diagram. Therefore the phase boundary of region 1 coincides with the DS transition line for maximal correlations. It turns out (see Fig. 2) that this line lies entirely in the region \(2p_1 + p_2 > 2\) so that minimal correlations are well defined (see Appendix B).

Alternatively one can compare the probabilities for generating damage in a pair of DK automata to the probabilities of generating a wet site in a single DK automaton. To this end one simply has to use the same inequalities setting \(\alpha^* = p_2^*/2\) and \(\beta^* = p_1^* - p_2^*/2\). For example, if \(p_1^*\) and \(p_2^*\) represent a point in the wet phase of the DK phase diagram, then for all pairs of automata parametrized by \(p_1, p_2, \alpha, \beta\) and satisfying

\[
\begin{align*}
p_1^* &\leq p_1, \\
p_2^* &\leq p_2, \\
p_1^* &\leq p_1 + p_2 - 2\alpha \\
p_2^* &\leq 2p_1 - 2\beta
\end{align*}
\]

damage will spread. On the other hand, if \(p_1^*\) and \(p_2^*\) belong to the dry phase of the DK model then in all pairs of automata with

\[
\begin{align*}
p_1^* &\geq p_1, \\
p_2^* &\geq p_2, \\
p_1^* &\geq p_1 + p_2 - 2\alpha \\
p_2^* &\geq 2p_1 - 2\beta
\end{align*}
\]

damage does not spread.

As an illustration of eq. \((34)\) consider the point \(M_1\) in Fig. 2. Setting \(p_1^* = p_1^* \approx 0.809\) and \(p_2^* = 0\) we obtain the conditions

\[
\begin{align*}
p_1^* &\leq p_1, \\
p_1^* &\leq p_1 + p_2 - 2\alpha, \\
0 &\leq 2p_1 - 2\beta
\end{align*}
\]

Using the bounds \((25)\) we find from these inequalities that in the triangle \(p_1 - p_2 \geq p_1^*\) damage spreads with certainty for any \(\alpha\) in the allowed range. In Fig. 2 this triangle is indicated as a shaded region.

**C. Terminal points of the phase boundaries**

We turn now to derive, using the arguments introduced above, a few exact results concerning the phase boundaries for minimal and maximal correlations. As explained above, these boundaries are the transition lines between the damage-spreading phases 1, 2 and 3 described in the Introduction and shown on 1. In order to make our arguments easier to follow, we present in Fig. 2 all the lines and special points that are mentioned.

Let us consider first the case of **maximal correlations**. That is, for every point \((p_1, p_2)\) in the phase diagram we assign \(\alpha = \min(p_1, p_2)\) and \(\beta = p_1\) and look for the boundary \(B_{max}\) between the region in which damage spreads and the one in which it doesn’t. Denote the wet-to-dry transition line of the DK model by \(B_{wet}\) (see Fig. 2).
We now prove that $B_{\text{max}}$ and $B_{\text{wet}}$ can intersect only at the point $M_1 = (p_1^*, 0)$, where $B_{\text{wet}}$ intersects the $p_1$ axis. On this axis maximal correlations correspond to the choice $\alpha = \beta = 0$, which also satisfy the conditions ($29$), i.e. the damage spreading process can be mapped onto a single DK model. Therefore we know that a DS transition will occur precisely at $p_1 = p_1^*$, so that at all points $(p_1 > p_1^*, p_2 = 0)$ we must have DS. On the other hand, as we will now show, for maximal correlations there cannot be DS on any point on $B_{\text{wet}}$; to see this, note that (for the region of interest, $p_1 > 1/2$) maximal correlations imply the inequalities

\begin{align}
\begin{aligned}
 p_1 + p_2 - 2\alpha &= |p_1 - p_2| < p_1 \\
 2p_1 - 2\beta &= 0 < p_2
\end{aligned}
\end{align}

(36)

According to eqs. ($14$) this implies that on the DK transition line in the region $p_1 + p_2/2 > 1$ damage does not spread. Therefore $M_2$ is the terminal point of $B_{\text{min}}$.

Having located the endpoints $M_1$ and $M_2$, we now turn to the opposite end of the lines $B_{\text{max}}$ and $B_{\text{min}}$. Note that for $p_1 = 1$ the bounds ($25$) collapse to $\alpha = p_2, \beta = 1$, i.e. maximal and minimal correlations are identical and hence $B_{\text{max}}$ and $B_{\text{min}}$ meet at some point $M_3$ on the $p_1 = 1$ line. The three special “multicritical” points discussed above determine the topology of the phase diagram for DS. In order to obtain high precision quantitative information about the location of the transition lines we performed numerical studies of damage spreading in the DK model.

D. Numerical results

In order to obtain accurate numerical estimates for the critical parameters of models with absorbing states one usually has to let the system evolve for extremely long times $\mathcal{T}$. Grassberger overcame the difficulty posed by long transients and obtained good statistics by simulating $n$ replicas of the same system in parallel, using simple bit manipulations on computers with unsigned words of length $n$ $[12]$. Using this multi-spin encoding method he measured the decay in damage on a one-million site chain, allowing it to evolve for hundreds of thousands of time steps. Because of the improved statistics he was able to determine the critical exponents for damage spreading at a particular transition point with high accuracy.

Another method to determine the critical point efficiently is the so-called gradient method which was introduced by Zebende and Penna $[9]$. In this method a gradient in $p_1$ and $p_2$ is arranged along the chain. The values of the parameters at the two end-points of the chain are chosen to be in different phases, i.e. on different sides of the transition point. This allows the critical point to be determined by measuring the average location of the boundary of the active (damaged) cluster.

In the present work we used a combination of multi-spin encoding and the gradient method. In combining these methods, a number of problems emerged which we solved as follows:

1. In order to measure the damage spreading transition point, one has to find the first position (approaching from the non-spreading phase) where damage occurs. Simulating $n = 64$ lattices in parallel, this has to be done for each of the $64 \cdot 63/2 = 2016$ pairs of replicas. To do this one has to set
up a 64 × 64 table in order to keep track of damaged pairs. Moreover, one has to scan the words bit by bit which makes it impossible to use parallel bit manipulations. The large amount of CPU time needed for this process usually kills the advantage one gains from the multi-spin encoding. In order to solve this problem, we used a simplified search algorithm which is based on fast bit operations. The price we pay is that only ≈ 75% of all possible pairs are taken into account. We proved that the error of this method does not bias the measurement of the transition point.

2. Zebende and Penna started each run with a single damaged seed located somewhere on the chain. It is not clear whether the choice of the location influences the results. In order to circumvent this problem, we used initial conditions with randomly distributed damage all over the chain.

3. The gradient method is a finite-size simulation and therefore boundary conditions may play an important role. In the work of Zebende and Penna the boundary conditions can be understood as dry walls and it is not clear to what extent they affect the measurements. In order to minimize this effect, we created, on a chain of 2N sites, a gradient with reflection symmetry (p1(i) = p1(2N − i) and p1(i) = p1(2N − i)) and measured the boundary of the damaged cluster on both sides. We expect finite-size effects to be less important for these periodic boundary conditions.

The phase diagram of the DK automaton, obtained using the multiple lattice gradient method, is presented in Fig. 1. First, we verified numerically the prediction that larger correlations correspond to smaller damage and vice versa. This was done by scanning the (α, β) space for various points in the (p1, p2) plane. Next we determined the DS transition lines for minimal and maximal correlations. Typical gradient values of 1.2 · 10−5 were used for lattice sizes L = 8192 and upwards. A transient period of at least 2L was followed by an averaging period of L time steps. For (p1, p2) near the the transition lines longer transient times were used. The terminal points of the phase boundaries were determined with high accuracy. Using a chain with L = 16384 sites, gradients down to 1.22 · 10−6 and transients of 231072, we measured the following critical values at these special points: p1∗ = 0.8087(5) (on the p2 = 0 line); p2∗ = 0.3130(5) (on the p1 = 0 line). The new triple point was located at p1∗ = 0.744(10), p2∗ = 0.526(10). This was done without using our analytic result that identified this point as the crossing of L with B_{crit}; the value of p1∗ + p2∗/2 = 1.007(15) agrees with the predicted value (of 1) for points on L.

Measuring the density of damage along the gradient of the chain, we could estimate the density exponent β. At the terminal points we found β = 0.302(30) for p2 = 0 and β = 0.296(30) on the p1 = 1 line. We also measured the exponent at a point (p1, p2) = (0.85, 0.35) which lies inside phase 3. This was done by crossing the DS phase boundary while varying the correlations α and β, yielding the value β = 0.279(10). All results are in fair agreement with the expected density exponent of directed percolation β = 0.277(1) [21], [22].

V. SUMMARY

We have rules that a most general damage spreading procedure should satisfy. These rules are most natural: they ensure that the evolution of a single replica is not affected by the fact that two replicas are evolving simultaneously; that the range of damage spreading does not exceed the range of interactions in the original single model and that the two evolving replicas respect the symmetries of the model. These rules can be cast in a formal setting, using transfer matrices. Using these formal definitions we were able to parametrize the most general damage spreading procedure for any given model in terms of correlation coefficients between various stochastic binary variables. Thus we are considering all possible damage spreading procedures and identify different damage spreading phases in terms of the manner in which this complete set of procedures behaves. Three possible phases can occur: one in which damage spreads for all allowed procedures, one in which it does not spread for any procedure and the third, in which for some procedures damage spreads while for others it does not.

These ideas were implemented for the Domany-Kinzel automaton, for which the three phases were identified, using a combination of numerical and analytic methods. We have shown that in an extended region of the model’s parameter space damage spreading can be mapped onto the evolution of the DK automaton itself. This observation supports Grassberger’s recent conjecture to the effect that damage spreading is in the directed percolation universality class. This was also confirmed by numerical tests (performed in regions where the above mentioned mapping does not hold).

APPENDIX A: GENERATION OF CORRELATED RANDOM VARIABLES

In this Appendix we explain in detail how correlated random variables r_{01}, r_{10} and r_{11}, that govern the evolu-
tion of the DK-model, can be generated. We also prove the allowed ranges for $\tilde{\alpha}$ and $\tilde{\beta}$ given in eq. (A2). Finally, we explain the manner in which minimal correlations are given by the expression presented in Sec. V.

Since in each update $r_{01}$, $r_{10}$, and $r_{11}$ can be either zero or one, there are eight possible combinations. By $\pi_{r_{01},r_{10},r_{11}}$, we denote the (positive) probability to generate the combination $\{r_{01},r_{10},r_{11}\}$. These probabilities are normalized by:

$$\pi_{000} + \pi_{001} + \pi_{010} + \pi_{011} + \pi_{100} + \pi_{101} + \pi_{110} + \pi_{111} = 1,$$

(A1)

Once the probabilities $\pi_{r_{01},r_{10},r_{11}}$ are known, the numbers $r_{01}$, $r_{10}$, and $r_{11}$ can be generated by taking one uniformly distributed random number $0 < z < 1$ and selecting one of the eight possible outcomes according to the probabilities $\pi_{r_{01},r_{10},r_{11}}$. The three-point correlation functions can be represented in terms of the $\pi$'s:

$$\pi_{111} = \langle r_{01}r_{10}r_{11} \rangle$$
$$\pi_{110} = \langle r_{01}r_{10}(1-r_{11}) \rangle$$

Collecting all identities of this type, we obtain seven equations:

$$\pi_{100} + \pi_{101} + \pi_{110} + \pi_{111} = p_1$$
$$\pi_{010} + \pi_{011} + \pi_{110} + \pi_{111} = p_1$$
$$\pi_{001} + \pi_{011} + \pi_{101} + \pi_{111} = p_2$$
$$\pi_{011} + \pi_{111} = \tilde{\alpha}$$
$$\pi_{011} + \pi_{111} = \tilde{\alpha}$$
$$\pi_{110} + \pi_{111} = \tilde{\beta}$$
$$\pi_{111} = \tilde{\gamma}$$

(A3)

Collecting all identities of this type, we obtain seven equations:

$$\pi_{000} + \pi_{001} + \pi_{010} + \pi_{011} + \pi_{100} + \pi_{101} + \pi_{110} + \pi_{111} = 1.$$
Acknowledgments
H.H. would like to thank the Minerva foundation for financial support. E.D. thanks the Newton Institute at Cambridge University and the Department of Physics at Oxford University for their hospitality and support during 1994, when this work was started. We also thank B. Derrida and D. Dhar for very helpful discussions and comments, D. Stauffer for calling some useful references to our attention and P. Grassberger for sending us preprints of his work prior to publication.

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