Assertional Logic: Towards an Extensible Knowledge Model (extended abstract)

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Abstract

We argue that extensibility is a key challenge for knowledge representation. For this purpose, we propose assertional logic - a knowledge model for easier extension with new AI building blocks. In assertional logic, all syntactic objects are categorized as set theoretic constructs including individuals, concepts and operators, and all kinds of knowledge are formalized by equality assertions. When extending with a new building block, one only needs to consider its interactions with the basic form of knowledge (i.e., equality assertions) without going deeper into its interactions with other existing ones. We first present a primitive form of assertional logic that uses minimal assumed knowledge and constructs. Then, we show how to extend it by definitions, which are special kinds of knowledge, i.e., assertions. As a case study, we show how assertional logic can be used to unify logic and probability, and more important AI building blocks including time.

1 Introduction

Knowledge representation (KR) is one of the central focuses of Artificial Intelligence (AI). KR intends to syntactically formalize information in an application domain as knowledge. Then, complex problems in the domain can be solved by reasoning about the knowledge. KR is not only of its own interests but also highly influential to many other subfields in AI, including expert systems, multi-agent systems, planning, uncertainty, cognitive robotics and semantic Web [4-14].

Nevertheless, KR has encountered huge difficulties nowadays. One of the most critical challenges is that there are too many features and building blocks to be considered [14]. For instance, propositions, variables, connectives, rules, actions, probability, preferences, common sense, time/space, fuzzy, mental states and so on. In fact, KR has made huge successes on formalizing these building blocks individually. However, combing them together, even several of them, seems an extremely difficult task. This severely restricts KR’s capability in applications since many domains, e.g. robotics and natural language understanding, need multiple building blocks at the same time.

To address this issue, a lot of effort has been dedicated to extend existing KR formalisms with more building blocks, for instance, to extend classical logic with actions and probabilities, to extend answer set programming with preferences and first-order
quantifiers and so on. Again, some interesting and successful approaches have been proposed along this direction. However, adding one building block normally needs to completely redefine the syntax and semantics thus results in a much more complicated formalism. More critically, there are always a lot more building blocks waiting to be incorporated.

Against this backdrop, we argue that KR needs an extensible knowledge model, in which new building blocks and features can be easily incorporated without redefining the semantics all over again and those building blocks can be freely assembled without proposing a totally different formalism. This is much closer to our natural language in the sense that obtaining new knowledge incrementally does not need to redefine a completely new system.

In fact, the recent development of description logics provides an excellent starting point [2]. By interpreting individuals, concepts and roles as domain elements, unary predicates and binary predicates respectively, one can easily extend the basic description logics with more building blocks (e.g., nominal, number restrictions, inverse and transitive roles etc.) based on the same foundational semantics. Also, one can freely assemble those building blocks into different fragments of description logics such as ALC, SHIQ, SHION and so on. However, the extensibility of description logics is still quite limited. Many important AI building blocks, e.g., actions, probability, time, rules, etc. are difficult to be incorporated within.

For this purpose, we propose assertional logic, a new knowledge model that allows us to easily extend a current system with new building blocks, i.e., syntactic objects. In assertional logic, all syntactic objects are categorized as set theoretic constructs including individuals, concepts and operators, and all kinds of knowledge are uniformly formalized by equality assertions of the form \( a = b \), where \( a \) and \( b \) are either atomic individuals or compound individuals. Semantically, individuals, concepts and operators are interpreted as elements, sets and functions respectively in set theory and knowledge of the form \( a = b \) means that the two individuals \( a \) and \( b \) are referring to the same element. For addressing extensibility in assertional logic, the key idea is that, when extending with a new building block, syntactically we categorize it as syntactic objects (i.e., individuals, concepts or operators), and semantically we only need to consider its interactions with the basic form of knowledge (i.e., equality assertions) without going deeper into its interactions with existing building blocks.

We first present the primitive form of assertional logic that uses minimal assumed knowledge and primitive constructs. Then, we show how to extend it with more building blocks by definitions, which are special kinds of knowledge, i.e., assertions used to define new syntactic objects including individuals, concepts and operators. Once these new objects are defined, they can be used as a basis to define more. For instance, we show how to define multi-assertions by using Cartesian product, and nested assertions by using multi-assertions etc.

As a case study, we show how to extend assertional logic for capturing logic and probability, and more important AI building blocks including time. Note that our intention is not to reinvent the wheel of these building blocks but to borrow existing excellent work on formalizing these building blocks separately and assemble them within one framework (i.e., assertional logic) so that they can live happily together.
2 Meta Language and Prior Knowledge

One cannot build something from nothing. Hence, in order to establish assertional logic, we need some basic knowledge. Of course, we need an informal meta language for the purpose of explanation. The syntax and semantics of the meta language are pre-assumed. As usual, we use a natural language such as English. Nevertheless, this meta language is used merely for explanation and it should not affect the syntax as well as the semantics of anything defined formally.

Only a meta level explanation language is not enough. Other than this, we also need some core objects and knowledge, whose syntax and semantics are pre-assumed. These are called prior objects and prior knowledge. For instance, when defining real numbers, we need some prior knowledge about natural numbers; when defining probability, we need some prior knowledge about real numbers.

In assertional logic, we always treat the equality symbol “=” as a prior object. There are some prior knowledge associated with the equality symbol. For instance, “=” is an equivalence relation satisfying reflexivity, symmetricity, and transitivity. Also, “=” satisfies the general substitution property, that is, if \( a = b \), then \( a \) can be used to replace \( b \) anywhere. In this extended abstract, we also assume some prior objects and knowledge in set theory \([9]\) and classical logic \([6]\).

3 Assertional Logic: the Primitive Form

In this section, we present the primitive form of assertional logic. As the goal of assertional logic is to syntactically represent knowledge in application domains, there are two essential tasks, i.e., how to capture the syntax of the domain and how to represent knowledge in it. Of course, as a logic, we need to provide a formal semantics.

3.1 Capturing the syntax

Given an application domain, a syntactic structure (structure for short if clear from the context) of the domain is a triple \( \langle I, C, O \rangle \), where \( I \) is a collection of individuals, representing objects in the domain, \( C \) a collection of concepts, representing groups of objects sharing something in common and \( O \) a collection of operators, representing relationships and connections among individuals and concepts. Concepts and operators can be nested and considered as individuals as well. If needed, we can have concepts of concepts, concepts of operators, concepts of concepts of operators and so on.

An operator could be multi-ary, that is, it maps a tuple of individuals into a single individual. Each multi-ary operator \( O \) is associated with a domain of the form \((C_1, \ldots, C_n)\), representing all possible values that the operator \( O \) can operate on, where \( C_i, 1 \leq i \leq n, \) is a concept. We call \( n \) the arity of \( O \). For a tuple \((a_1, \ldots, a_n)\) matching the domain of an operator \( O \), i.e., \( a_i \in C_i, 1 \leq i \leq n, \) \( O \) maps \((a_1, \ldots, a_n)\) into an individual, denoted by \( O(a_1, \ldots, a_n) \). We also use \( O(C_1, \ldots, C_n) \) to denote the set \( \{ O(a_1, \ldots, a_n) \mid a_i \in C_i \} \), called the range of the operator \( O \).

\footnote{Note that in set theory, a tuple of sets is a Cartesian product of some sets, which itself is a set as well. Therefore, multi-ary operators can essentially be viewed as unary.}
Operators are similar to functions in first-order logic but differ in two essential ways. First, operators are many-sorted as $C_1, \ldots, C_n$ could be different concepts. More importantly, $C_1, \ldots, C_n$ could be high-order constructs, e.g., concepts of concepts, concepts of operators and so on.

For instance, consider a family relationship domain, in which Alice and Bob are individuals, Human, Woman and Female are concepts and Father, Mother and Aunt are operators etc.

3.2 Representing knowledge

Let $\langle I, C, O \rangle$ be a syntactic structure. A term is an individual, either an atomic individual $a \in I$ or the result $O(a_1, \ldots, a_n)$ of an operator $O$ operating on some individuals $a_1, \ldots, a_n$. We also call the latter compound individuals.

An assertion is of the form

$$a = b,$$  \hspace{1cm} (1)

where $a$ and $b$ are two terms. Intuitively, an assertion of the form (1) is a piece of knowledge in the application domain, claiming that the left and right side are referring to the same object. Here, $=$ is the equality binary operator among individuals, and it is always considered as a prior knowledge. In this sense, $a = b$ can be understood in alternative way that $=(a, b)$ is true. A knowledge base is a set of assertions. Terms and assertions can be considered as individuals as well. For instance, in the family relationship domain, $\text{Father}(Alice) = Bob, \text{Father}(Alice) = \text{Uncle}(Michael)$ are assertions.

Similar to concepts that group individuals, we use schemas to group terms and assertions. A schema term is either an atomic concept $C \in C$ or the collection of results $O(C_1, \ldots, C_n)$, where $C_i, 1 \leq i \leq n$ are concepts. Essentially, a schema term represents a set of terms, in which every concept is grounded by a corresponding individual. Then, a schema assertion is of the same form as form (1) except that terms can be replaced by schema terms. Similarly, a schema assertion represents a set of assertions.

We say that a schema term/assertion mentions a set $\{C_1, \ldots, C_n\}$ of concepts if $C_1, \ldots, C_n$ occur in it, and only mentions if $\{C_1, \ldots, C_n\}$ contains the set of all concepts mentioned in it. Note that it could be the case that two or more different individuals are referring to the same concept $C$ in schema terms and assertions. In this case, we need to use different copies of $C$, denoted by $C^1, C^2, \ldots$, to distinguish potentially different individuals. For instance, all assertions $x = y$, where $x$ and $y$ are human, are captured by the schema assertion $\text{Human}^1 = \text{Human}^2$. On the other side, in a schema, the same copy of a concept $C$ can only refer to the same individual. For instance, $\text{Human} = \text{Human}$ is the set of all assertions of the form $x = x$, where $x \in \text{Human}$.

3.3 The semantics

We introduce a set theoretic semantics for assertional logic. Since we assume some prior knowledge including set theory, in the semantics, we freely use those individuals
(e.g., the empty set), concepts (e.g., the set of all natural numbers) and operators (e.g.,
the set union operator) in the semantics without any explanation because the meanings
of these syntactic objects are pre-fixed.

An interpretation (also called a possible world) is a pair \( \langle \Delta, \mathcal{I} \rangle \), where \( \Delta \) is a
domain of elements, and \( \mathcal{I} \) is a mapping function that admits all prior knowledge, and
maps each individual into a domain element in \( \Delta \), each concept into a set in \( \Delta \) and
each \( n \)-ary operator into an \( n \)-ary function in \( \Delta \). The mapping functions \( \mathcal{I} \) can be
generalized into mapping from terms to elements. Similar to terms and assertions,
interpretations can also be considered as individuals to be studied as well.

It is important to emphasize that an interpretation has to admit all the prior knowl-
edge. For instance, since we assume set theory, suppose that one interprets two indi-
viduals \( x \) and \( y \) as an element \( a \) in the domain, then the concepts \( \{x\} \) and \( \{y\} \) must be
interpreted as \( \{a\} \), and \( x = y \) must be interpreted as \( a = a \).

Let \( \mathcal{I} \) be an interpretation and \( a = b \) an assertion. We say that \( \mathcal{I} \) is a model of
\( a = b \), denoted by \( \mathcal{I} \models a = b \) iff \( \mathcal{I}(a) = \mathcal{I}(b) \), also written \( a^\mathcal{I} = b^\mathcal{I} \). Let \( KB \) be a
knowledge base. We say that \( \mathcal{I} \) is a model of \( KB \), denoted by \( \mathcal{I} \models KB \), iff \( \mathcal{I} \) is a model
of all assertions in \( KB \). We say that an assertion \( A \) is a property of \( KB \), denoted by
\( KB \models A \), iff for models of \( KB \) are also models of \( A \). In particular, we say that an
assertion \( A \) is a tautology iff it is modeled by all interpretations.

Since we assume set theory as our prior knowledge, we directly borrow some set
theoretic constructs on individuals, concepts and operators. For instance, we can use
\( \cup(C_1, C_2) \) (also written as \( C_1 \cup C_2 \)) to denote a new concept that unions two concepts
\( C_1 \) and \( C_2 \). Applying this to assertions, we can see that assertions of the primitive
form (1) can indeed represent many important features in knowledge representation.
For instance, the membership assertion, stating that an individual \( a \) is an instance of
a concept \( C \) is the following assertion \( \in(a, C) = \top \) (also written as \( a \in C \) ). The containment assertion, stating that a concept \( C_1 \) is contained by another concept \( C_2 \),
is the following assertion \( \subseteq(C_1, C_2) = \top \) (also written as \( C_1 \subseteq C_2 \) ). The range declaration, stating that the range of an operator \( O \) operating on some concept \( C_1 \)
equals to another concept \( C_2 \) is the following assertion \( O(C_1) = C_2 \).

4 Extending New Syntactic Objects by Definitions

As argued in the introduction section, extensibility is a critical issue for knowledge
representation and modeling. In assertional logic, we use definitions for this purpose.
Definitions are (schema) assertions used to define new syntactic objects (including
individuals, concepts and operators) based on existing ones. Once these new syntactic
objects are defined, they can be used to define more. Note that definitions are nothing
extra but special kinds of knowledge (i.e. assertions).

We start with defining new individuals. An individual definition is an assertion of
the form
\[
a = t,
\]
where \( a \) is an atomic individual and \( t \) is a term. Here, \( a \) is the individual to be defined.
This assertion claims that the left side \( a \) is defined as the right side \( t \). For instance,
\( 0 = \emptyset \) means that the individual \( 0 \) is defined as the empty set.
Defining new operators is similar to defining new individuals except that we use schema assertions instead. Let \( O \) be an operator to be defined and \((C_1, \ldots, C_n)\) its domain. An operator definition is a schema assertion of the form

\[
O(C_1, \ldots, C_n) = T,
\]

(3)

where \( T \) is a schema term that mentions concepts only from \( C_1, \ldots, C_n \). It could be the case that \( T \) only mentions some of \( C_1, \ldots, C_n \). Note that if \( C_1, \ldots, C_n \) refer to the same concept, we need to use different copies.

Since a schema assertion represents a set of assertions, essentially, an operator definition of the form (3) defines the operator \( O \) by defining the value of \( O(a_1, \ldots, a_n) \) one-by-one, where \( a_i \in C_i \), \( 1 \leq i \leq n \). For instance, for defining the successor operator \( \text{Succ} \), we can use the schema assertion

\[
\text{Succ}(N) = \{ N, \{ N \} \}
\]

meaning that, for every natural number \( n \), the successor of \( n \), is defined as \( \{ n, \{ n \} \} \), i.e., \( \text{Succ}(n) = \{ n, \{ n \} \} \).

Defining new concepts is somewhat different. As concepts are essentially sets, we directly borrow set theory notations to define concepts. There are four ways to define a new concept.

**Enumeration** Let \( a_1, \ldots, a_n \) be \( n \) individuals. Then, the collection \( \{ a_1, \ldots, a_n \} \) is a concept, written as

\[
C = \{ a_1, \ldots, a_n \}.
\]

(4)

For instance, we can define the concept \( \text{Digits} \) by \( \text{Digits} = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \} \).

**Operation** Let \( C_1 \) and \( C_2 \) be two concepts. Then, \( C_1 \cup C_2 \) (the union of \( C_1 \) and \( C_2 \)), \( C_1 \cap C_2 \) (the intersection of \( C_1 \) and \( C_2 \)), \( C_1 \setminus C_2 \) (the difference of \( C_1 \) and \( C_2 \)), \( C_1 \times C_2 \) (the Cartesian product of \( C_1 \) and \( C_2 \)), \( 2^{C_1} \) (the power set of \( C_1 \)) are concepts. Operation can be written by assertions as well. For instance, the following assertion

\[
C = C_1 \cup C_2
\]

(5)

states that the concept \( C \) is defined as the union of \( C_1 \) and \( C_2 \). As an example, one can define the concept \( \text{Man} \) by \( \text{Man} = \text{Human} \cap \text{Male} \).

**Comprehension** Let \( C \) be a concept and \( A(C) \) a schema assertion that only mentions concept \( C \). Then, individuals in \( C \) satisfying \( A \), denoted by \( \{ x \in C \mid A(x) \} \) (or simply \( C \mid A(C) \)), form a concept, written as

\[
C' = C \mid A(C).
\]

(6)

For instance, we can define the concept \( \text{Male} \) by \( \text{Male} = \{ \text{Animal} \mid \text{Sex(Animal)} = \text{male} \} \), meaning that \( \text{Male} \) consists of all animals whose sex are male.

**Replacement** Let \( O \) be an operator and \( C \) a concept on which \( O \) is well defined. Then, the individuals mapped from \( C \) by \( O \), denoted by \( \{ O(x) \mid x \in C \} \) (or simply \( O(C) \)), form a concept, written as

\[
C' = O(C).
\]

(7)

For instance, we can define the concept \( \text{Parents} \) by \( \text{Parents} = \text{ParentOf} \circ \text{Human} \), meaning that it consists of all individuals who is a \( \text{ParentOf} \) some human.
Definitions can be incremental. We may define some syntactic objects first. Once defined, they can be used to define more. One can always continue with this incremental process. For instance, in arithmetic, we define the successor operator first. Once defined, it can be used to define the add operator, which is further served as a basis to define more useful syntactic objects.

For clarity, we use the symbol “::=” to replace “=” for definitions. We force uniqueness of definitions. That is, each syntactic object can only be defined at most once.

Since terms and assertions can be considered as individuals as well, we can define new type of terms and assertions by definitions. As an example, we can simply extend assertions of the form \((1)\) into multi-assertions by using Cartesian product. We first define multi-assertions of a fixed number of assertions. Given a number \(n\), we define a new operator \(M_n\) for multi-assertions with arity \(n\) by the following schema assertion:

\[
M_n(C_1 = D_1, \ldots, C_n = D_n) ::= (C_1, \ldots, C_n) = (D_1, \ldots, D_n),
\]

where \(C_i, D_i, 1 \leq i \leq n\), are concepts of terms. Notice that, \((C_1, \ldots, C_n) = (D_1, \ldots, D_n)\) is a single assertion of the form \((1)\). In this sense, an \(n\)-ary multi-assertion is just a syntax sugar. Then, we define the concept of multi-assertions:

\[
Multi\text{-}Assertion ::= \bigcup_{1 \leq i \leq \infty} M_i(A_1, \ldots, A^i),
\]

where \(A_1, \ldots, A^i\) are \(i\) copies of standard assertions. For convenience, we use \(Assertion_1, \ldots, Assertion_n\) to denote a \(n\)-ary multi-assertion. The important thing is, once multi-assertion is defined, it can be used to define more syntactic objects.

As another example, we can use multi-assertion to define nested assertions. We first define nested terms as follows:

\[
\begin{align*}
Nested\text{-}Term & ::= Term \cup N\text{-}Term \\
N\text{-}Term & ::= Op(Nested\text{-}Term).
\end{align*}
\]

Then, nested assertions can be defined as

\[
Nested\text{-}Assertion ::= Nested\text{-}Term = Nested\text{-}Term.
\]

Nested assertions can be represented by non-nested multi-assertions by introducing new individuals. Whenever a result of nested term is used, we introduce a new individual to replace it and claim that this new individual is defined as the nested term. That is, for every nested term \(Op(a_1, \ldots, Op'(b_1, \ldots, b_m), \ldots, a_m)\) occurred in a nested assertion, we introduce a new atomic individual \(a'\); replace the above term with \(Op(a_1, \ldots, a', \ldots, a_m)\) and add a new assertion \(a' = Op'(b_1, \ldots, b_m)\). For instance, the nested assertion \(Op(a, Op(b, Op'(c))) = Op'(d)\) is defined as \(Op(a, x) = Op'(d), x = Op(b, y), y = Op'(c)\), where \(x\) and \(y\) are new individuals. In this sense, nested assertion is essentially a multi-assertion, which can be represented as a single assertion. Therefore, nested assertion is a syntactic sugar of the primitive form as well.

Again, once nested assertion is defined, it can be used as basis to define more, so on and so forth. Using nested assertions can simplify the representation task. However,
one cannot overuse nested assertions since, essentially, every use of a nested term introduces a new individual. For instance, one can easily get lost with a nested assertion like \( Op(a, Op(b, Op'(c))) = Op'(d) \).

5 Embedding Classical Logic into Assertional Logic

In the previous section, we show how to extend assertions of the primitive form (1) into multi-assertions and nested assertions. In this section, we continue with this task to show how to define more complex forms of assertions with logic connectives, including not only propositional connectives but also quantifiers.

We start with the propositional case. Let \( A \) be the concept of nested assertions. We introduce a number of operators over \( A \) in assertional logic, including \( \neg (a = a') \) (for negation), \( \land (A^1, A^2) \) (for conjunction), \( \lor (A^1, A^2) \) (for disjunction) and \( \to (A^1, A^2) \) (for implication).

There could be different ways to define these operators in assertional logic. Let \( a = a' \) and \( b = b' \) be two (nested) assertions. The propositional connectives are defined as follows:

\[
\neg(a = a') ::= \{a\} \cap \{a'\} = \emptyset
\]

\[
\land(a = a', b = b') ::= ((a) \cap \{a'\}) \cup ((b) \cap \{b'\}) = \{a, a', b, b'\}
\]

\[
\lor(a = a', b = b') ::= ((a) \cap \{a'\}) \cup ((b) \cap \{b'\}) \neq \emptyset
\]

\[
\to(a = a', b = b') ::= ((a, a') \setminus \{a\} \cap \{a'\}) \cup ((b) \cap \{b'\}) \neq \emptyset.
\]

We also use \( a \neq a' \) to denote \( \neg(a = a') \). One can observe that the ranges of all logic operators are nested assertions. Hence, similar to multi-assertion and nested assertion, propositional logic operators are syntactic sugar as well in assertional logic.

It can be observed that all tautologies in propositional logic (e.g., De-Morgan’s laws) are also a tautology in assertional logic in the sense that each proposition is replaced by an assertion and each propositional connective is replaced by corresponding logic operators in assertional logic.

Now we consider to define operators for quantifiers, including \( \forall \) (for the universal quantifier) and \( \exists \) (for the existential quantifier). The domain of quantifiers is a pair \((C, A(C))\), where \( C \) is a concept and \( A(C) \) is a schema assertion that only mentions \( C \).

The quantifiers are defined as follows:

\[
\forall(C, A(C)) ::= C \setminus A(C) \subseteq C
\]

\[
\exists(C, A(C)) ::= C \setminus A(C) \neq \emptyset
\]

Intuitively, \( \forall(C, A(C)) \) is true if those individuals \( x \) in \( C \) such that \( A(x) \) holds equals to the concept \( C \) itself, that is, for all individuals \( x \) in \( C \), \( A(x) \) holds; \( \exists(C, A(C)) \) is true if those individuals \( x \) in \( C \) such that \( A(x) \) holds does not equal to the empty set, that is, there exists at least one individual \( x \) in \( C \) such that \( A(x) \) holds. We can see that the ranges of quantifiers are nested assertions as well. Thus, quantifiers are also syntactic sugar of the primitive form.
Note that quantifiers defined here are ranging from an arbitrary concept \( C \). If \( C \) is a concept of all atomic individuals and all quantifiers range from the same concept \( C \), then these quantifiers are first-order. Nevertheless, the concepts could be different. In this case, we have many-sorted first-order logic. Moreover, \( C \) could be complex concepts, e.g., a concept of all possible concepts. In this case, we have monadic second-order logic. Yet \( C \) could be many more, e.g., a concept of assertions, a concept of concepts of terms etc. In this sense, the quantifiers become high-order. Finally, the biggest difference is that \( C \) can even be a concept of assertions so that quantifiers in assertional logic can quantify over assertions (corresponding to formulas in classical logics), while this cannot be done in classical logics.

A problem arises whether there is cyclic definition as we assume first-order logic as our prior knowledge. Nevertheless, although playing similar roles, operators (over assertions) defined in assertional logic are considered to be different from logic connectives (over propositions/formulas) since they are on a different layer of definition. The main motivation is for the purpose of extensibility, i.e., by embedding classical logic connectives into operators in assertional logic, we can easily extend it with more components and building blocks including probability.

6 Incorporating Probability and More

Probability is another important building block for knowledge representation and modeling. In the last several decades, with the development of uncertainty in artificial intelligence, a number of influential approaches [3, 7, 8, 10, 11, 13] have been developed, and important applications have been found in machine learning, natural language processing etc.

In this section, we show how logic and probability can be unified through assertions in assertional logic. The basic idea is that, although the interactions between logic and probability are complicated, their interactions with assertions of the form [1] could be relatively easy. As shown in the previous section, the interactions between logic and assertions can be defined by a few lines. In this section, following Gaifman’s idea [7], we show that this is indeed the case for integrating assertions with probability as well. As a result, the interactions between logic and probability will be automatically established via assertions.

6.1 Integrating assertions with probability

Since operations over real numbers are involved in defining probability, we need to assume a theory of real number as our prior knowledge.

Gaifman [7] proposed to define the probability of a logic sentence by the sum of the probabilities of the possible worlds satisfying it. Following this idea, in assertional logic, we introduce an operator \( Pr \) (for probability) over the concept \( A \) of assertions. The range of \( Pr \) is the set of real numbers. For each possible world \( w \), we assign an associated weight \( W_w \), which is a positive real number. Then, for an assertion \( A \), the
probability of $A$, denoted by $Pr(A)$, is defined by the following schema assertion:

$$Pr(A) = \frac{\sum_{w, w \models A} W_w}{\sum_{w} W_w}. \quad (8)$$

This definition defines the interactions between probability and assertions. In case that there are a number of infinite worlds, we need to use measure theory. Nevertheless, this is beyond the scope of our paper, which focuses on how to use assertional logic for extensible knowledge modeling.

Once we have defined the probability $Pr(A)$ of an assertion $A$ as a real number, we can directly use it inside other assertions. In this sense, $Pr(A) = 0$.

We also extend this definition for conditional probability. We again introduce a new operator $Pr$ over pairs of two assertions. Following a similar idea, the conditional probability $Pr(A_1 | A_2)$ of an assertion $A_1$ providing another assertion $A_2$, also denoted by $Pr(A_1 | A_2)$, is defined by the following schema assertion:

$$Pr(A_1 | A_2) = \frac{\sum_{w, w \models A_1, w \models A_2} W_w}{\sum_{w, w \models A_2} W_w}. \quad (9)$$

Again, once conditional probability is defined as a real number, we can use it arbitrarily inside other assertions. Similarly, we can derive some properties about conditional probabilities, including the famous Bayes’ theorem, i.e.,

$$Pr(A_1) \times Pr(A_2 | A_1) = Pr(A_2) \times Pr(A_1 | A_2).$$

for all assertions $A_1$ and $A_2$.

### 6.2 Interactions between logic and probability through assertions

Although we only define probabilities for assertions of the basic form, the interactions between probability and other building blocks, e.g., logic, are automatically established since assertions connected by logic operators can be reduced into the primitive form. In this sense, we can investigate some properties about the interactions between logic and probability. For instance, it can be observed that Kolmogorov’s third probability axiom is a tautology in assertional logic. That is, let $A_1, \ldots, A_n$ be $n$ assertions that are pairwise disjoint. Then, $Pr(A_1 \lor \cdots \lor A_n) = Pr(A_1) + \cdots + Pr(A_n)$.

One can verify that many axioms and properties in the literature regarding the interactions between logic and probability are tautologies in assertional logic as well, for instance, the additivity axiom: $Pr(\phi) = Pr(\phi \land \psi) + Pr(\phi \land \neg \psi)$ and the distributivity axiom: $\phi \equiv \psi$ implies that $Pr(\phi) = Pr(\psi)$, for any two assertions $\phi$ and $\psi$. In

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\(^2\)Note that Kolmogorov’s probability axioms are defined for events instead of assertions. To represent an event in assertional logic, one can use its effects (i.e., postconditions), which is an assertion. Then, that two events are disjoint if and only if their postconditions do not have common models.
this sense, assertional logic can also be used to validate existing properties about the interactions of logic and probability. In addition, it may foster new discoveries, e.g., the interactions between higher-order logic and probability and some properties about nested probabilities.

Note that we do not intend to reinvent the wheel of defining probability nor its interactions with logic. All definitions about (conditional) probability are borrowed from the literature, and all properties are thoroughly discussed. Instead, we take probability as a case study to show how one building block (e.g., logic) and another (e.g., probability) can be interacted through assertions without going deeper into the interactions between themselves.

### 6.3 More building blocks

More critically, there are a lot of important building blocks in knowledge representation and modeling so that it is barely possible to clarify the interactions among them all. Nevertheless, it becomes possible to unify them altogether in assertional logic as one only needs to consider the interactions between these building blocks and the basic form of assertions separately. Consequently, the interactions among these building blocks will be automatically established via assertions, as what we did for unifying logic and probability.

As another case study, we consider how to formalize time in assertional logic. Time itself can be understood in different ways such as time points, time interval, LTL and CTL. Following the same idea of incorporating probability, we first consider the interactions between time and assertions. We start with the simple case of integrating assertions with time points. Let $T_p$ be a concept of time points (natural numbers, real numbers or other special forms). We introduce a binary operator $t$ whose domain is a pair $(I, T_p)$ and whose range is the concept of individuals. Intuitively, $t(i, tp)$ is the value of individual $i$ at time point $tp$. Then, we introduce temporal formulas, a new Boolean operator $T$ whose domain is a pair $(A, T_p)$ be the following schema assertion:

$$T(a = b, tp) ::= t(a, tp) = t(b, tp).$$  \(10\)

Then, the interactions between time points and logic connectives and probability can be automatically established. We are able to investigate some properties. For instance, for all assertions $A$ and $B$, $T(A, tp) = \top$ iff $T(\neg A, tp) = \bot$; $T(A \land B, tp) = \top$ $|$ $T(A, tp) = \top$ etc. Hence, we have an integrated formalism combing logic, probability and time points in assertional logic.

One can further introduce new concepts and operators for other temporal assertions, e.g., time interval, and define relationships among time intervals similar to Allen’s interval algebra. Again, the key point is that for extending with a new building block (e.g., time interval) in assertional logic, one only needs to formalize it by syntactic objects and define its interactions with the basic form of assertions. Then, the interactions between this building block and others (e.g., logic, probability or time points) will be automatically established through assertions. Moreover, one can freely assemble some of the building blocks together as different fragments in assertional logic.
7 Conclusion

In this extended abstract, we argue that extensibility is a critical challenge for knowledge representation. For this purpose, we propose assertional logic, in which the syntax of an application domain is captured by individuals (i.e., objects in the domain), concepts (i.e., groups of objects sharing something in common) and operators (i.e., connections and relationships among objects), and knowledge in the domain is simply captured by equality assertions of the form $a = b$, where $a$ and $b$ are terms.

In assertional logic, without redefining the semantics, one can extend a current system with new syntactic objects by definitions, which are special kinds of knowledge (i.e., assertions). Once defined, these syntactic objects can be used to define more. We extend the primitive form of assertional logic with multi-assertions and nested assertions, and more interestingly, logic connectives and quantifiers. Interestingly, such a simple extension is even more expressive than high-order logic as quantifiers can be used to quantify over assertions (i.e., formulas).

We further consider how to extend assertional logic with other important AI building blocks. The key point is that, when one wants to integrate a new building block in assertional logic, she only needs to formalize it as syntactic objects (including individuals, concepts and operators) and defines its interactions with the basic form of assertions (i.e., $a = b$). The interactions between this building block and others will be automatically established since all complicated assertions can essentially be reduced to the basic form. This enables extensibility for knowledge representation and modeling. As a case study, we briefly discuss how to incorporate probability and time points in this extended abstract.

This extended abstract is only concerned with the representation task and the definition task, and we leave the reasoning task to our future work. Certainly, how to do efficient and effective but not necessarily complete reasoning/inference is a key issue in assertional logic. Nevertheless, as the foundation, representation and definition are worth study because of their own importance. For instance, in assertional logic, how to formalize dynamics, i.e., actions and their effects, is indeed a challenge worth pursuing.

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