On the Importance of Hubs in Hopfield Complex Neuronal Networks under Attack

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The organizational principles behind the connectivity of a complex network are known to influence its behavior. In this work we investigate, using the Hopfield model, the influence of the network architecture on the performance for associative recall while the network is under hub and edge attack. We show, by using four different attack strategies, that although the importance of hubs is more definite for Barabási-Albert neuronal networks, the random removal of the same amount of edges as in a hub may imply a greater reduction of memory recall.

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I. INTRODUCTION

A great deal of the current interest on networks, neuronal or complex, stems from the fact that the functionality of network systems is highly influenced, and even determined, by their respective connectivity (e.g. [8]). While most neuronal network models were initially fully connected, increasing interest has been focused more recently on systems with diluted (e.g. [13, 14, 22, 23]) and/or structured (e.g. [4, 7, 12, 17, 20, 25, 27, 28]) connectivity. A series of interesting points and questions are implied by such investigations, especially regarding structured networks of connections, i.e. models departing from uniformly random connections. For instance, given that metabolic and spatial biological pressures constraining the number of dendrites and axons in a neuronal cell, it is virtually impossible to have fully-connected biological networks (e.g. [10]). What are the functional implications of such unavoidable partial connections? Which partially connected structure (e.g. random, small world, scale free) allows the best functionalities? Are hubs particularly important for scale-free neuronal networks? In which sense they affect the recall of patterns in associative networks such as the Hopfield model?

Because they concentrate connections, hubs have been found to play a fundamental role in defining the connectivity of scale free complex networks (e.g. [11, 24]). For instance, as they connected to many nodes, hubs provide bypasses between a large number of pair of nodes, contributing to the overall reduction of the mean shortest path in the network. It is intuitive to expect that hubs would also be particularly important for the dynamics of systems with scale-free connectivity, such as the Hopfield complex neuronal networks considered in [12, 25]. Such a question provided the main motivation for the present article. In order to try to answer it, we performed node and edge attack on random and Hopfield complex networks and monitor the effect over the recall performance. Among the reported experiments, we successively remove hubs from scale free Hopfield models, and then remove the same number of edges by node and edge attack on random models. Although hubs are confirmed to have special importance for the memory capacity of the network, it was remarkably found that their removal is not so impacting as the random elimination of the same number of edges from complex or random networks. One explanation for such a phenomenon is the fact that a hub is most directly and strongly associated to a single bit of the patterns to be trained and recovered.

The article starts by reviewing the Hopfield model and complex networks, and follows by describing the simulation framework and the obtained results. It concludes by interpreting and discussing the results and their implications for further researches.

II. METHODOLOGY

The Hopfield model [16] is a model of autoassociative memory which has been studied and generalized thoroughly since its first appearance. Today, the value of the original model is mostly epistemologic, as practical implementations are constrained by several issues such as limited efficiency for storing several patterns. Nevertheless, being a simple and intuitive model, it provides an effective framework for performance studies under varying conditions while allowing the quantification of the impact of the choice of specific influencing elements.

In this paper we follow closely the implementation described by [15]. A set of $M$ binary string patterns are represented by vectors as $P_i = [p_1, p_2, ..., p_n]^T$, whose $N$ elements are in either of the possible states $p_j = \pm 1$, randomly chosen. This information is stored in a weight matrix $W$, according to the follow expression

$$W = \frac{1}{N} \sum_{i=1}^{M} P_i P_i^T - MI,$$  \hspace{1cm} (1)

In order to retrieve a pattern stored initially, say $Q_n = [q_1, q_2, ..., q_n]^T$, it is necessary to iterate the following expression

$$Q_{n+1} = \text{sgn} \left[ W Q_n \right],$$ \hspace{1cm} (2)
where \( sng[.] \) is a hard limiting function. In this implementation, whenever the value of the state of a neuron becomes exactly zero, the previous state of that neuron is used instead, so that the neurons are always either firing (+1) or quiescent (-1). The element \( w_{ij} \) of \( W \) is the weight of the connection of neuron \( i \) with neuron \( j \) with the \( W \) matrix representing a fully connected network, except for loops (i.e. \( w_{ii} = 0 \)).

A diluted version of the Hopfield model still retaining pattern recognition properties, see [23], can be obtained by multiplying, in elementwise fashion, the weight matrix by a sparse weight connection matrix, build upon some assumed heuristics.

We consider in this paper two models of complex networks, namely the random and Barabási-Albert networks, as connection matrices underlying Hopfield models. A network with \( N \) nodes can be represented by a symmetric adjacency matrix \( C \) of rank \( N \), with each element \( c_{ij} \) assuming unit value for connected nodes and null values for disconnected ones. We set \( c_{ii} = 0 \) to prevent self connections. The random network is obtained simply by choosing connections \( c_{ij} = 1 \) with probability \( \gamma \). The parameter \( \gamma \) therefore determines the density of connections.

In the Barabási-Albert model [1, 3], we start with a small random network \( \lambda \) with \( m_0 \) nodes. For each node \( a_i \) the probability \( p_i \) of a new connection is given as \( k(a_i)/\sum_k(k(a_i)) \), where \( k(a_i) \) is the node degree. Consider a new node \( b \) with \( m \) edges. A node \( a_i \) of the initial network \( \lambda \) is allowed to connect to one of the \( m \) edges of the new node \( b \) with the probability \( p_i \). In other words, the degree of node \( a_i \) defines its chance of receiving new connections, suggesting the paradigm ‘the rich gets richer’.

### III. SIMULATIONS

In this section we consider both kinds of network architectures (i.e. random and scale free), characterized by their respective connection matrix \( C \), as defined in the previous section. The training matrix \( W \) of the Hopfield model, calculated for a initial set of random patterns \( P_0 \), is diluted by direct multiplication with the connection matrix \( C \) of the network under analysis. The pattern \( P_1 \) will stand for the reference pattern for these experiments. We add 10% of noise to \( P_1 \) before trying to recover \( P_1 \) through the iterative recall process, see Equation 2. After fifty interactions, an overlap index is calculated, measuring how much of the recovered pattern \( P_r \) matches the original pattern \( P_1 \). This process is repeated for an increasing number of patterns \( P_i \), \( i = 1, 2, ..., M \). The calculation of the overlap index is performed by

\[
\text{Overlap}(i) = \frac{\langle P_r \cdot P_i \rangle(i)}{N},
\]

where \( N \) is the total number of neurons in the network and \( i = 1, ..., M \). This procedure produces an overlap curve as a function of the number of patterns stored \( i \), quantifying the degradation of performance as the number of patterns grows. Figure 2 shows the overlap curves for several experiments, each averaged over a hundred realizations, exploring both Barabási-Albert and Random network architectures with varying \( \langle k \rangle \), implemented by changing \( m \).

To implement hub attack, in each experiment one hub is identified and eliminated from \( W \) by setting to zero all the elements of the connection matrix along the respective column and row. We eliminate hubs either sequentially, i.e., from the more connected to the less connected hub, or at random, with uniform probability. Figure 2 shows four experiments in which 30 hubs, belonging to two types of networks (random or Barabási-Albert), were successively attacked in two manners (orderly or not). The Barabási-Albert networks were grown with parameters \( m_0 = m = 100 \). The random network counterparts were obtained with equivalent \( \langle k \rangle \). Both types of networks included \( N = 800 \) neurons. As expected, the recall performance decreased with the number of trained patterns and as the hubs were eliminated. The mean node degree \( \langle k \rangle \) was also directly affected. Marked differences appear then in these cases, discriminating the impact of hub withdrawals, orderly or not, for each architecture. Note, for instance, that the Barabási networks are more sensitive to orderly hub removal than a random network with equivalent degree. Note also that both networks respond essentially in the same manner to random hub deletion.

The similar effect of random hub withdrawal from both network models can be better illustrated by defining a recall index as corresponding to the number of patterns at which the respective overlap value reaches 0.8. We also define the attack rate as the number of eliminated hubs divided by the total number of neurons. Figure 3 shows the degradation of the recall index for each experiment as a function of the attack rate, showing clearly the different behavior of a Barabási network under orderly hub elimination. Yet, Figure 4 shows the stability of such behavior under variations of network parameters \( N \) and \( \langle k \rangle \).

These results support the intuitive expectation about hub importance for the inner workings of the Hopfield memory model. Note nevertheless that the comparisons between Random and Barabási network were carried out regarding only hub attack and, in this manner, it is not guaranteed that the same amount of links or edges is eliminated from both networks. In what follows, we carry out further experiments in order to evaluate the relative importance of edges and hub removal.

We consider three experiments ensuring that the number of edges eliminated from each networks is always the same. These experiments are illustrated in Figure 5. In the first step, the hubs in a Barabási network are attacked, as indicated in Figure 5A. The black circle represents a hub and the dashed line its eliminated edges. From an identical Barabási network we then eliminate
FIG. 1: Overlap as a function of the number of patterns for Barabási-Albert and Random Network with different \( \langle k \rangle \).

FIG. 2: Overlap as a function of the number of patterns for Barabási-Albert and Random networks under both random and orderly hub attack.

\( e = 6 \) edges at random, as shown in Figure 3. In our experiments with large networks this procedure is repeated until 300 hubs are eliminated. The overlap curve for both networks obtained at each step, after another 30 hubs were eliminated, is shown in Figure 3 (A and B). For an equivalent Random network, the \( e \) edges are eliminated either at random or by removing nodes at random until the total amount of eliminated edges equals \( e \), as shown in Figure 3 C and D, respectively. In the actual experiments, overlaps curves are again produced for each set of 30 eliminated hubs. The results for this experiments are showed in the Figure 3 C and D, respectively. Figure 7 shows the performance of the networks under attack, quantified by the recall index, in terms of the attack ratio.
FIG. 3: Recall index as a function of the attack rate for $N = 800$ and $\langle k \rangle = 160.7$

FIG. 4: The slope, defined in Figure 3, as a function of two important network parameters.

IV. DISCUSSION AND CONCLUSION

Although largely overlooked by initial studies in artificial neuronal networks, the connectivity between neurons plays a decisive role in constraining and even defining the dynamical properties of the neuronal systems, especially the potential of memory recall. The results obtained in our investigation substantiate further such a phenomenon, with special attention focused on the importance of hubs to the overall memory recall, as quantified by the overlap index. The main interesting conclusions identified from our experiments are listed and discussed in the following:

- **Effect of connectivity dilution:** As expected, all types of edge and node dilutions act in order to reduce the recall in the Hopfield model, for both Barabási and Random networks. See Figure 1.

- **Ordered hub attack:** The removal of hubs, starting with the most connected ones, from Barabási networks tends to cause more severe loss of recall performance than when removing hubs from random networks. This conclusion is implied by the fact that the curve for ordered attack to hubs in Barabási networks, see Figure 3, correspond to the strongest recall deterioration. This reflects the relative importance of the hubs in Barabási-Albert neuronal networks, as a consequence of the fact that hubs in such nets tend to have higher node degree (and therefore more edges) than hubs in random networks with similar average degree.

- **Random node attack:** Random attack to both types of networks did not tend to produce different performance reduction, as indicated by the two entangled curves representing random attack to both types of networks in Figure 7.

- **Localized effect of hub removal:** A particular interesting point in the obtained results, for both types of network models, regards the fact that the
FIG. 5: A - hub attack to Barabási network (black circle), eliminating \( e = 13 \) edges (dashed line), B - edge attack to the same Barabási network eliminating \( e = 13 \) edges at random, C - edge attack to Random network eliminating 13 edges at random, D - node attack to Random network eliminating nodes until the amount of eliminated edges reaches \( e = 13 \), that gives a total of 5 nodes.

FIG. 6: The corresponding overlap curves for the situations described in Figure 5.
FIG. 7: Recall index as a function of the attack rate for edge attack to both types of networks
random removal of the same quantity of edges as in a large hub from the same network has a stronger effect on the recall performance than removing that hub. This property is supported by the fact that the curves, in Figure 7, for hub (node) attack are higher than the respective edge attack curves. This is a direct consequence of the fact that the removal of the hub implies more localized connectivity deterioration, affecting predominantly the single bit of the recovered pattern which is directly related to the hub.

All in all, it has become clear that hub and edge attack, even when involving the same number of links, have quite distinct effects on the memory recall of Hopfield networks running on random and Barabási-Albert networks.

Future works related to the currently reported investigation should consider the effect of edge and hub attack on other network topologies such as small-world and Sznajd. It would also be interesting to extend such experimental analyses to artificial neuronal network models such as the perceptron and self-organizing maps.

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