ON CRITICAL POINTS OF BLASCHKE PRODUCTS

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ABSTRACT. We obtain an upper bound for the derivative of a Blaschke product, whose zeros lie in a certain Stolz-type region. We show that the derivative belongs to the space of analytic functions in the unit disk, introduced recently in [6]. As an outcome, we obtain a Blaschke-type condition for critical points of such Blaschke products.

1. INTRODUCTION

Given a sequence \( \{z_n\} \subset \mathbb{D} \) subject to the Blaschke condition
\[
\alpha := \sum_{n=1}^{\infty} (1 - |z_n|) < \infty, \tag{1}
\]
let
\[
B(z) = \prod_{n=1}^{\infty} b_n(z), \quad b_n(z) = \frac{\overline{z}_n}{|z_n|} \frac{z_n - z}{1 - \overline{z}_n z}
\]
be a Blaschke product with the zero set \( Z(B) = \{z_n\} \). With no loss of generality we will assume that \( B(0) \neq 0 \).

One of the central problems with Blaschke products is that of the membership of their derivatives in classical function spaces, \( B' \in X \). There is a vast literature on the problem, starting from investigations of P. Ahern and his collaborators [1, 2, 3, 4] and D. Prota [14] in 1970s, up to quite recent results of the Spanish school [8, 9, 10], see also [12, 15]. The above mentioned spaces \( X \) are primarily the Hardy spaces \( H^p \), the Bergman spaces \( A^p \), the Banach envelopes of the Hardy spaces \( B^p \) etc. Recall the definition of the Bergman spaces \( A^p \), \( p > 0 \):
\[
A^p = \{ f \in \mathcal{A}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p \, dx \, dy < \infty \}, \quad z = x + iy.
\]

In this paper we will add to the list some new spaces \( X = \mathcal{A}(E, \rho) \) of analytic functions in the unit disk, introduced recently in [6]. Given a closed set \( E = \overline{E} \subset \mathbb{T} \) and \( \rho > 0 \), we say that an analytic function \( f \) belongs to \( \mathcal{A}(E, \rho) \) if
\[
|f(z)| \leq C_1 \exp \left( \frac{C_2}{d^p(z, E)} \right), \quad d(z, E) = \text{dist}(z, E)
\]

Date: November 16, 2010.

1991 Mathematics Subject Classification. Primary: 30D50; Secondary: 31A05, 47B10.
is the distance from \( z \in \mathbb{D} \) to \( E \), \( C_{1,2} \) are positive constants.

The simplest and most general result drops out immediately from the Schwarz–Pick lemma for functions \( g \) from the unit ball of \( H^\infty \):

\[
|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2},
\]

and states, that \( g' \in A^p \) for all \( 0 < p < 1 \). The result is sharp: there exists a Blaschke product \( B \) such that \( B'/A^1 \) (W. Rudin).

To proceed further, one should impose some additional restrictions either on absolute values \( |z_n| \), stronger than (1), or on location, distribution of arguments of zeros \( z_n \) etc. For instance, a typical result of the first type is due to Protas [14]:

\[
\sum_{n=1}^{\infty} (1 - |z_n|)^r < \infty, \quad 0 < r < \frac{1}{2} \Rightarrow B' \in H^{1-r},
\]

and \( \frac{1}{2} \) is sharp.

We are more interested in the second direction, related to the location of zeros. A typical assumption here is that \( Z(B) \) belongs to certain regions inside the unit disk.

Let \( t \in \mathbb{T}, \gamma \geq 1 \). Following [5, 3, 10] we introduce regions

\[
R(t, \gamma, K) := \{ \lambda \in \mathbb{D} : |t - \lambda|^\gamma \leq K(1 - |\lambda|) \}, \quad K \geq 1.
\]

For \( \gamma = 1, K > 1 \) this is the standard Stolz angle. When \( \gamma > 1 \) the region touches the circle \( \mathbb{T} \) at the vertex \( t \) with the power degree of tangency. The following result claims that \( B' \) belongs to \( H^p \) or \( A^p \) as soon as \( Z(B) \subset R(t, 1, K) \).

**Theorem A.** Let \( Z(B) \subset R(t, 1, K) \). Then

1. \( B' \in H^p, \quad p < \frac{1}{2}, \) and \( \frac{1}{2} \) is sharp;
2. \( B' \in A^p, \quad p < \frac{3}{2}, \) and \( \frac{3}{2} \) is sharp.

The first statement is proved in [9, Theorem 2.3], for the second one see [9, 8]. For related results in the case \( Z(B) \subset R(t, \gamma, K) \) with \( \gamma > 1 \), see [10, Section 3].

We study the same problem for more general Stolz-type regions.

A function \( \phi \) on the right half-line will be called a model function, if it is nonnegative, continuous and increasing, and

\[
\phi(x) \leq Cx, \quad x \geq 0, \quad C = C(\phi) > 0.
\]

We define a Stolz angle associated with a model function \( \phi \) with the vertex at \( t \in \mathbb{T} \) as

\[
S_\phi(t, K) = S(t, K) := \{ \lambda \in \mathbb{D} : \phi(|t - \lambda|) \leq K(1 - |\lambda|) \}, \quad K > 0.
\]

Since \( |t - \lambda| \leq 2 \) for \( t, \lambda \in \mathbb{D} \), it is clear that regions (2) are of the form (4) for an appropriate \( \phi \). Precisely, one can put \( \phi(x) = x^\gamma \) for \( 0 \leq x \leq 2 \), and...
φ(x) = 2^x − 1 for x ≥ 2. Next, given a closed set \( E = \overline{E} \subset \mathbb{T} \) we define a Stolz region, associated with a model function φ and the set E, as

\[
S(E, K) := \{ \lambda \in \mathbb{D} : \phi(d(\lambda, E)) \leq K(1 - |\lambda|) \} = \bigcup_{t \in E} S(t, K).
\]

Here is our main result.

**Theorem 1.** Let \( B \) be a Blaschke product such that \( Z(B) \subset S(E, K) \). Then

\[
|B'(z)| \leq 2(2C + K)^2 \sum_{n=1}^{\infty} (1 - |z_n|) \phi^{-2} \left( \frac{d(z, E)}{6} \right).
\]

(5)

For the standard Stolz angle and \( E = \{t\} \) we take \( \phi(x) = x \), so

\[
|B'(z)| \leq \frac{C_3}{|t - z|^2},
\]

and part (1) in Theorem A follows. Similarly, for the region \( R(t, \gamma, K) \), \( \gamma > 1 \), (5) implies

\[
|B'(z)| \leq \frac{C_4}{|t - z|^{2\gamma}},
\]

and we come to the following result (cf. [10, Remark 1]).

**Corollary 2.** If \( Z(B) \subset R(t, \gamma, K) \), \( \gamma > 1 \), then \( B' \in H^p \) for all \( p < 1/2\gamma \).

We are particularly interested in the model function \( \phi(x) = \exp\{-x^{-\rho}\} \), \( \rho > 0 \). In this case (5) says that \( B' \in \mathcal{A}(E, \rho) \).

Denote \( Z(B') = \{z'_n\} \) the zero set of \( B' \). Each result of the form \( B' \in X \) provides some information about the critical points of \( B \) (zeros of \( B' \)), as long as the information about zero sets of functions from \( X \) is available. The most general condition applied to an arbitrary Blaschke product arises from the fact that \( B' \in A^p \), \( p < 1 \), so (cf. [11, Theorem 4.7])

\[
\sum_{n=1}^{\infty} \frac{1 - |z'_n|}{(\log \frac{1}{1-|z'_n|})^{1+\varepsilon}} < \infty, \quad \forall \varepsilon > 0.
\]

On the other hand there are Blaschke products \( B \) such that \( \sum 1 - |z'_n| = \infty \) (see, e.g., [13]).

A Blaschke–type condition for zeros of functions from \( \mathcal{A}(E, \rho) \) is given in a recent paper [6]. To present its main result we define, following P. Ahern and D. Clark [4, p.113], the type \( \beta(E) \) of a closed subset \( E \) of the unit circle as

\[
\beta(E) := \sup\{\beta \in \mathbb{R} : |E_x| = O(x^\beta), \ x \to 0\},
\]

where \( E_x := \{t \in \mathbb{T} : d(t, E) < x\} \), \( x > 0 \), is an \( x \)-neighborhood of \( E \), \( |E_x| \) its normalized Lebesgue measure. For the equivalent definition and properties of the type see also [6, 7].
Theorem 3. Given a closed set \( E \subset \mathbb{T} \) and a Blaschke product \( B \), assume that \( Z(B) \subset S_\phi(E, K) \), \( \phi(x) = \exp\{-x^{-\rho}\}, \rho > 0 \). Then

\[
\sum_{n=1}^{\infty} (1 - |z_n'|) d(\rho - \beta(E) + \epsilon)_+(z_n', E) < \infty, \quad \forall \epsilon > 0,
\]

\( \beta(E) \) is the type of \( E \), \( (a)_+ = \max(a, 0) \).

It is clear that \( \beta(E) = 1 \) for each finite set \( E \).

Corollary 4. Let \( Z(B) \subset S_\phi(t, K) \) with the same \( \phi \). Then

\[
\sum_{n=1}^{\infty} (1 - |z_n'|) |t - z_n'|^{(\rho-1+\epsilon)_+} < \infty, \quad \forall \epsilon > 0.
\]

The authors thank the referee for a number of comments that improved the paper write-up.

2. Main results

A model function \( \phi \) is nonnegative and increasing, so for all \( x, y, u \geq 0 \)

\[
\phi \left( \frac{x + y + u}{3} \right) \leq \phi(x) + \phi(y) + \phi(u).
\]

(6)

We begin with the following result, which is similar to Vinogradov’s lemma from [16].

Lemma 5. Let \( z \in \mathcal{D}, t \in \mathbb{T} \) and \( \lambda \in S(t, K) \). Then

\[
\frac{1}{|1 - \lambda z|} \phi \left( \frac{|t - z|\lambda|}{3} \right) \leq 2C + K.
\]

(7)

Proof. With no loss of generality we assume that \( t = 1 \). Since

\[
|1 - z|\lambda| = |1 - \lambda z + z(\lambda - |\lambda|)| \leq |1 - \lambda z| + |\lambda - |\lambda||
\]

\[
\leq |1 - \lambda z| + (1 - |\lambda|) + |1 - \lambda|,
\]

then by (6)

\[
\phi \left( \frac{|1 - z|\lambda|}{3} \right) \leq \phi(|1 - \lambda z|) + \phi(1 - |\lambda|) + \phi(|1 - \lambda|),
\]

so

\[
\frac{1}{|1 - \lambda z|} \phi \left( \frac{|1 - z|\lambda|}{3} \right) \leq A_1 + A_2 + A_3.
\]

By (3), for the first two terms we have

\[
A_1 = \frac{\phi(|1 - \lambda z|)}{|1 - \lambda z|} \leq C, \quad A_2 \leq \frac{\phi(1 - |\lambda|)}{1 - |\lambda|} \leq C.
\]

As for the third one,

\[
A_3 \leq \frac{\phi(|1 - \lambda|)}{1 - |\lambda|} \leq K
\]

according to the assumption \( \lambda \in S(1, K) \). The proof is complete.
Our main result gives a bound for the derivative $B'$ in the case when the zero set $Z = Z(B) \subset S(E, K)$.

Proof of Theorem 1. Denote $Z(t) := Z \cap S(t, K), t \in E$. Then there is an at most countable set $\{t_k\}_{k=1}^{\omega}, \omega \leq \infty, t_k \in E$, so that $z_k \in Z(t_k)$, and $Z = \bigcup_k Z(t_k)$. It is clear that there is a disjoint decomposition

$$Z = \bigcup_k Z_k, \quad Z_k \neq \emptyset, \quad Z_k \subset Z(t_k), \quad Z_j \cap Z_k = \emptyset, \quad j \neq k.$$

Let us label the set $Z$ in such a way that

$$Z = \{z_{kj}\}, \quad k = 1, 2, \ldots, \omega, \quad j = 1, 2, \ldots, \omega_k, \quad \{z_{kj}\}_{j=1}^{\omega_k} \subset Z_k.$$

We proceed with the expression

$$B'(z) = \sum_{j=1}^{\infty} b'_n(z)B_n(z), \quad B_n(z) = \frac{B(z)}{b_n(z)},$$

so

$$|B'(z)| \leq \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - z_n z|^2} |B_n(z)| \leq \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - z_n z|^2} = \sum_{k=1}^{\omega} \sum_{j=1}^{\omega_k} \frac{1 - |z_{kj}|^2}{|1 - z_{kj} z|^2}.$$

Note that $|t - z| \leq 2|t - z|\lambda|$ for all $z, \lambda \in \mathbb{D}$ and $t \in \mathbb{T}$. Indeed,

$$|t - z|\lambda| \geq 1 - |z\lambda| \geq 1 - |\lambda|$$

and

$$|t - z| \leq |t - z|\lambda| + |z|(1 - |\lambda|) \leq 2|t - z|\lambda|,$$

as claimed. Hence

$$\phi \left( \frac{|t - z|}{6} \right) \leq \phi \left( \frac{|t - z|\lambda|}{3} \right),$$

and by (7)

$$\phi^2 \left( \frac{d(z, E)}{6} \right) |B'(z)| \leq \sum_{k=1}^{\omega} \sum_{j=1}^{\omega_k} \frac{1 - |z_{kj}|^2}{|1 - z_{kj} z|^2} \phi^2 \left( \frac{|t_k - z|}{6} \right)
\leq \sum_{k=1}^{\omega} \sum_{j=1}^{\omega_k} \frac{1 - |z_{kj}|^2}{|1 - z_{kj} z|^2} \phi^2 \left( \frac{|t_k - z| |z_{kj}|}{3} \right)
\leq 2(2C + K)^2 \sum_{n=1}^{\infty} (1 - |z_n|),$$

which is (5). The proof is complete.

Theorem 3 is a direct consequence of Theorem 1 and [6, Theorem 3].
References

[1] P. Ahern, The mean modulus of the derivative of an inner function, Indiana Univ. Math. J., 28 (1979), 311–347.
[2] P. Ahern, The Poisson integral of a singular measure, Can. J. Math. 35 (1983), 735–749.
[3] P. Ahern and D. Clark, On inner functions with $H^p$ derivatives, Michigan Math. J., 21 (1974), 115–127.
[4] P. Ahern and D. Clark, On inner functions with $B^p$ derivatives, Michigan Math. J., 23 (1976), 107–118.
[5] G. Cargo, Angular and tangential limits of Blaschke products and their successive derivatives, Canad. J. Math., 14 (1962), 334–348.
[6] S. Favorov and L. Golinskii, A Blaschke-type condition for analytic and subharmonic functions and application to contraction operators, Amer. Math. Soc. Transl., 226 (2) (2009), 37–47.
[7] S. Favorov and L. Golinskii, Blaschke-type conditions for analytic functions in the unit disk: inverse problems and local analogs, preprint arXive:1007.3020v1 [math.CV] 18 Jul 2010.
[8] D. Girela and J. Peláes, On the membership in Bergman spaces of the derivative of a Blaschke product with zeros in a Stolz domain, Canad. Math. Bull., 49 (3) (2006), 381–388.
[9] D. Girela, J. Peláes, and D. Vucotić, Integrability of the derivative of a Blaschke product, Proc. Edinburgh Math. Soc., 50 (2007), 673–687.
[10] D. Girela, J. Peláes, and D. Vucotić, Interpolating Blaschke products: Stolz and tangential approach regions, Constr. Approx., 27 (2008), 203–216.
[11] H. Hedenmalm, B. Korenblum, K. Zhu, Theory of Bergman spaces. Graduate Texts in Mathematics, vol. 199. Springer-Verlag, New York, 2000.
[12] J. Mashreghi and M. Shabankhah, Integral means of the logarithmic derivative of Blaschke products, Comp. Meth. Func. Theory, 9 (2) (2009), 421–433.
[13] Ch. Pommerenke, On the Green’s function of Fuchsian groups, Ann. Acad. Sci. Fenn., 2 (1976), 409–427.
[14] D. Protas, Blaschke products with derivative in $H^p$ and $B^p$, Michigan Math. J., 20 (1973), 393–396.
[15] D. Protas, Blaschke products with derivatives in function spaces, preprint arXive:1001.5098v2 [math.CV] 10 Jun 2010.
[16] S. A. Vinogradov, Multiplication and division in the space of analytic functions with area integrable derivative, and some related spaces, Zap. Nauchn. Semin. POMI, 222 (1994), 45–77. (Russian)

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