Effective Field Theory for Low-Energy Two-Nucleon Systems

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We illustrate how effective field theories work in nuclear physics by using an effective Lagrangian in which all other degrees of freedom than the nucleonic one have been integrated out to calculate the low-energy properties of two-nucleon systems, viz, the deuteron properties, the np 1S0 scattering amplitude and the M1 transition amplitude entering into the radiative np capture process. Exploiting a finite cut-off regularization procedure, we find all the two-nucleon low-energy properties to be accurately described with little cut-off dependence, in consistency with the general philosophy of effective field theories.

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Effective field theories (EFTs) have long proven to be a powerful tool in particle and condensed matter physics [1,2], so it is quite natural that a considerable attention is nowadays paid to the role of EFTs in nuclear physics where phenomenological approaches have traditionally been tremendously successful. Some authors have focused on nucleon-nucleon interactions and two-nucleon systems [3,4] while some [3] on many-body systems including dense matter relevant to relativistic heavy-ion processes and compact stars. One of the most spectacular cases was the recent chiral perturbation calculation of the radiative np capture at thermal energy [4] with an agreement with experiment within 1%. What was calculated in [4] was however the meson-exchange current corrections relative to the single-particle M1 matrix element with the latter borrowed from the accurate Argonne v18 [10] phenomenological two-nucleon wave function. In this respect, one cannot say that it was a complete calculation in the framework of the given EFT, namely, chiral perturbation theory (ChPT) although it was following the strategy of [3] of using ChPT for computing irreducible graphs only.

The purpose of this Letter is to supply the “missing link” that can render Ref. [4] a “first-principle calculation,” that is to obtain the single-particle M1 matrix element within the framework of EFTs [1]. In so doing we will compute the static properties of the bound np state (the deuteron) and the np scattering amplitude in the 1S0 channel. The results come out to be in a surprisingly good agreement with the data, offering a first glimpse of how EFTs work in nuclei.

Since we shall be interested in very low-energy processes with the energy scale $E \ll m_{\pi} \approx 140$ MeV, we will integrate out all massive fields as well as the pion field [6], leaving only the nucleon matter field which can be treated in heavy-fermion formalism (HFF). Since the anti-nucleon field also is integrated out in HFF, there are no “irreducible” loops (there will be, however, “reducible” loops to all orders in solving Lippman-Schwinger equation) and the EFT becomes non-relativistic quantum mechanics where all the interactions appear in the potential. Now the np states we shall study are all very close to the threshold: they are either weakly bound (3S1) or almost bound (1S0). Bound states are not accessible by perturbation expansion and the scattering state with a large scattering length $a$ has a small scale $a^{-1}$, making the convergence of EFTs highly non-trivial. We shall circumvent these difficulties by summing “reducible diagrams” – which amounts to solving Schrödinger (or Lippman-Schwinger) equation and using a cut-off regularization instead of the usual dimensional regularization.

Due to the nonperturbative nature of the Schrödinger equation, unlike perturbative cases, it does matter which regularization scheme one uses in effective theories. Kaplan, Savage and Wise [1] and Luke and Manohar [2] have found that with the dimensional regularization, the EFT breaks down at a very small scale, $p_{\text{cut}} = \frac{1}{\sqrt{\pi e}}$ for large scattering length $a$, where $r_e$ is the effective range and that this problem cannot be ameliorated by introducing the pionic degree of freedom. As pointed out by Beane et al [8] and Lepage [12], the problem can however be resolved if one uses a cut-off regularization. In effective theories, the cut-off has a physical meaning and hence it should not be taken to infinity as one does in renormalizable theories [12]. In fact the strategy of effective field theories is such that one should not pick either too low a cut-off or too high a cut-off: if one chooses too low a cut-off, one risks the danger of throwing away relevant degrees of freedom – and hence correct physics – while if one chooses too high a cut-off, one introduces irrelevant degrees of freedom and hence makes the theory unnecessarily complicated. The astute in doing EFTs is in choosing the proper cut-off. Thus with our effective Lagrangian in which the lightest degree of freedom integrated out is the pion, the natural cut-off scale is set by the pion mass. We find that the optimal cut-off in our case is $\Lambda \sim 200$ MeV as one can see from the results in Table 1 and Figures 1 and 2.

We shall do the calculation to the next-to-leading order
The potential of the EFT is local and hence of zero range in coordinate space, requiring regularization. In order to do the calculation algebraically, we choose the following form of regularization appropriate to a separable potential given by the local Lagrangian:

$$\langle p'|V|p \rangle = S_\Lambda(p')V(p' - p)S_\Lambda(p)$$  \hspace{1cm} (1)

where $S_\Lambda(p) = S(p^2/\Lambda^2)$ is the regulator which suppresses the contributions from $|p| \gtrsim \Lambda$, limit $x \to 0 \; S(x) = 0$, and $V(q) = 0$ is a finite-order polynomial in $q$. Up to the NLO, the most general form of $V(q)$ is

$$V(q) = \frac{4\pi}{M} \left( C_0 + (C_2 \delta^{ij} + D_2 \sigma^{ij})q^i q^j \right),$$  \hspace{1cm} (2)

where $M$ is the nucleon mass and $\sigma^{ij}$ is the rank-two tensor that is effective only in the spin-triplet channel. Thus we have five parameters; two in $^1S_0$ and three in $^3S_1$ channel. In principle, these parameters are calculable from a fundamental Lagrangian (i.e., QCD) but nobody knows how to do this. So in the spirit of EFTs, we shall fix them from experiments. Since the explicit form of the regulator should not matter, we shall choose the Gaussian form,

$$S_\Lambda(p) = \exp \left( -\frac{p^2}{2\Lambda^2} \right)$$  \hspace{1cm} (4)

where $\Lambda$ is the cut-off. (This form of the cut-off functions, strictly speaking, upsets the chiral counting on which we will have more to say later.) The Lippmann-Schwinger (LS) equation for the wavefunction $|\psi\rangle$, $|\psi\rangle = |\varphi\rangle + G^0 \tilde{V}|\psi\rangle$ where $|\varphi\rangle$ is the free wavefunction and $G^0$ is the free two-nucleon propagator depending on the total energy $E$, $(p'|G^0|p) = \frac{E + \not{p}' + \not{p}}{E + \not{p}' + \not{p} + i\delta}$ leads to the $S$-wave function (for the potential (3)) of the form

$$\psi(r) = \varphi(r) + \frac{S(ME) C_E}{1 - \Gamma_E C_E} \left[ 1 - \sqrt{\frac{C_2}{C_E}} (\nabla^2 + ME) \right] - \sqrt{\frac{C_E}{C_E}} S_{12}(\hat{r}) \frac{1}{\sqrt{8}} \frac{\partial}{\partial r} \frac{1}{\partial r} \hat{r} \tilde{\Gamma}_\Lambda(r)$$  \hspace{1cm} (5)

where

$$\Gamma_E = 4\pi \int \frac{d^3p}{(2\pi)^3} \frac{S_2^2(p)}{ME - p^2 + i\delta^+},$$  \hspace{1cm} (6)

$$\hat{\Gamma}_\Lambda(r) = 4\pi \int \frac{d^3p}{(2\pi)^3} \frac{S_\Lambda(p)}{ME - p^2 + i\delta^+} e^{ip \cdot r},$$  \hspace{1cm} (7)

$$Z = (1 - C_2 I_2)^{-2},$$  \hspace{1cm} (8)

$$C_E = a_\Lambda \left( 1 + \frac{1}{2} a_\Lambda r_\Lambda ME \right) + (\sqrt{1 + 2})^2 \tilde{\Gamma}_E,$$  \hspace{1cm} (9)

with

$$a_\Lambda \equiv Z \left[ C_0 + (C_2^2 + \delta_{S1} D_2^2) I_4 \right],$$  \hspace{1cm} (10)

$$r_\Lambda \equiv \frac{2Z}{a_\Lambda} \left[ 2C_2 - (C_2^2 - \delta_{S1} D_2^2) I_2 \right]$$  \hspace{1cm} (11)

where $I_n (n = 2, 4)$ are defined by

$$I_n = -\frac{\Lambda^{n+1}}{(n+1)!} \int_{-\infty}^{\infty} dx x^n S^2(x^2).$$  \hspace{1cm} (12)

With the regulator $\Box$, the integrals come out to be $I_2 = -\frac{1}{4\Lambda^4}$ and $I_4 = -\frac{1}{4\Lambda^4} \Lambda^3$.

The phase shifts can be calculated by looking at the large-$r$ behavior of the wavefunction. To do this, it is convenient to separate the pole contributions of the integrals Eqs.(6, 7) as

$$\tilde{\Gamma}_\Lambda(r) = -\frac{S(ME)}{r} \left[ e^{\sqrt{ME}r} - H(\Lambda r, ME) \right],$$  \hspace{1cm} (14)

where $H(0, \varepsilon) = 1$ which makes $\tilde{\Gamma}_\Lambda(0)$ finite, and that $\lim_{r \to \infty} H(\varepsilon, x) = 0$. The phase shift $\delta$ takes the form

$$p \cot \delta = -\frac{1}{S(\frac{ME}{\Lambda^2})} \left[ I_\Lambda(E) - \frac{1 - \eta^2(E)}{a_\Lambda(1 + \frac{1}{2}a_\Lambda r_\Lambda ME)} \right],$$  \hspace{1cm} (15)

where the $\eta(E)$ is the D/S ratio to be given below (see (E)), which vanishes for the $^1S_0$ channel.

In order to fix the two coefficients $C_0, 2$, we compare (E) to the effective-range expansion

$$p \cot \delta = -\frac{1}{a} + \frac{1}{2}er^2 + \cdots.$$  \hspace{1cm} (16)

We obtain

$$\frac{1}{a_\Lambda} = \frac{1}{a} + \Lambda I(0) = \frac{1}{a} - \frac{\Lambda}{\sqrt{\pi}},$$  \hspace{1cm} (17)

$$r_\Lambda = r_e - \frac{2I(0)}{a} - \frac{4S'(0)}{a^2} = r_e - \frac{4}{\sqrt{\pi}a} + \frac{2}{a\Lambda^2}. $$  \hspace{1cm} (18)

These are essentially the “renormalization conditions” in the standard renormalization procedure. Two important observations to make here: (a) We note that there is an upper bound of $\Lambda, \Lambda_{\text{Max}}$, if one requires that $Z$ be positive and that $C_2$ be real. That is, for $\Lambda > \Lambda_{\text{Max}}$, the potential of the EFT becomes non-Hermitian. With $a = -23.732$ fm and $r_e = 2.697$ fm for the $^1S_0$ channel taken from the Argonne $v_{18}$ potential (which we take to be “experimental”), we find that $\Lambda_{\text{Max}} \approx 348.0$ MeV; (b) the value $\Lambda_{\text{Max}}$ defined such that $Z = 1$ when $\Lambda = \Lambda_{\text{Max}} \approx 172.2$ MeV is quite special. At this point, we have $r_\Lambda = 0$ and $C_2 = 0$, that is, the NLO contribution is identically
This corresponds to the leading-order calculation with the $\Lambda$ chosen to fit the experimental value of the effective range $r_\varepsilon$. A similar observation was made by Beane et al [8] using a square-well potential in coordinate space with a radius $R$, with $R^{-1}$ playing the role of $\Lambda$.

The resulting phase shift with $\Lambda = \Lambda_{Z=1}$ is plotted in Fig. 1. We see that the agreement with the result taken from the Argonne $v_{18}$ potential [10] is perfect up to $p \sim 70$ MeV. Beyond that, we should expect corrections from the next-to-next-order and higher-order terms. In Fig. 2, we show how the phase-shift for a fixed center-of-mass momentum, $p = 68.5$ MeV varies as the cut-off is changed. The solid curve is our NLO result, the dotted one the LO result (with $C_2 = 0$), and the horizontal dashed line the result taken from the $v_{18}$ potential (“experimental”). We confirm that our NLO result is remarkably insensitive to the value of $\Lambda$ for $\Lambda \gtrsim m_\pi$.

It is instructive to compare our result (15) with that obtained with the dimensional regularization [6],

$$p\cot \delta|_{\text{Dim.}} = -a^{-1}(1 + \frac{1}{2}ar_eME)^{-1}. \quad (19)$$

Expanding $p\cot \delta$ of (13) in $ME$, we find that the coefficient of the $n$-th order term is order of $a^{n-1}r_n$. This increases rapidly with $n$ when $a$ is large, disagreeing strongly with the fact that the low-energy scattering is well described by just two terms of the effective range expansion in (14). This observation led the authors of [6] to conclude that the critical momentum scale at which the EFT expansion breaks down is very small for a very large $a$:

$$p_{\text{crit}}|_{\text{Dim.}} \sim \sqrt{2}/(ar_e). \quad (20)$$

We arrive at a different conclusion. With the cut-off regularization, the scattering length $a$ is replaced by an effective one, $a_\Lambda$, that is order of $\frac{1}{\Lambda}$ for large $a$. This agrees with the findings of Beane et al [8] and Lepage [12]. Counting $r_\varepsilon$ to be order of $\Lambda^{-1}$, the $n$-th order coefficient now is $\Lambda^{1-2n}$, as one would expect on a general ground.

The next quantity to consider is the transition $M1$ amplitude for $n+p \to d+\gamma$ and the deuteron structure. For the $np$ capture, we need both the $^1S_0$ scattering wavefunction and the deuteron wave function. The initial state wave function can be written as

$$\psi(r) = \frac{e^{i\delta} \sin \delta}{pr} u_0(r) |^1S_0\rangle,$$

$$u_0(r) = \sin (pr + \delta) - H(\Lambda r, ME/\Lambda^2) + \beta_\Lambda D(\Lambda r) \quad (22)$$

where $p = \sqrt{ME}$ is the center-of-mass momentum and

$$D(\Lambda r) = \frac{4\pi r}{\Lambda^2} \int \frac{d^3 p}{(2\pi)^3} S(\frac{p^2}{\Lambda^2}) e^{ipr}, \quad (23)$$

$$\beta_\Lambda = \frac{(\sqrt{\gamma} - 1)\Lambda^2}{a_\Lambda(1 + \frac{1}{2}ar_\Lambda ME)I_2(S(\frac{ME}{\Lambda^2})))}. \quad (24)$$

As for the $^3S_1$ coupled channel relevant for the final state of the $np$ capture, we use the eigenphase parametrization [13] with the $\eta(E)$ given by $\eta(E) \equiv -\tan \epsilon_1$ where $\epsilon_1$ is the mixing angle,

$$\eta(E) = \frac{\sqrt{\eta_2}}{1 - \eta_2} \frac{1}{1 - \eta_2 ME} = a_\Lambda \frac{\sqrt{\Lambda D_2}}{\Lambda^2} \frac{1}{a_\Lambda} \frac{1}{1 - \frac{1}{2}ar_\Lambda MB_d}. \quad (25)$$

The $D_2$ so far undetermined can be fixed by the deuteron $D/S$ ratio $\eta_d \approx 0.025$ [10] at $E = -B_d$ with $B_d$ the binding energy of the deuteron,

$$\sqrt{\Lambda D_2} = \frac{\eta_d}{1 - \eta_2} \frac{a_\Lambda}{1 - \frac{1}{2}ar_\Lambda MB_d} \left[1 - \frac{1}{2}ar_\Lambda MB_d \right]. \quad (26)$$

Given $C_0$, $C_2$ and $D_2$ for a given $\Lambda$, all other quantities are predictions. The binding energy of the deuteron is determined by the pole position,

$$\gamma S^2(\frac{r_\varepsilon^2}{\Lambda^2}) + I_\Lambda(\frac{r_\varepsilon^2}{\Lambda^2}) = -\frac{1}{a_\Lambda} \frac{1}{1 - \frac{1}{2}ar_\Lambda \gamma^2}. \quad (27)$$

with $\gamma \equiv \sqrt{MB_d}$. The renormalization procedure is the same as for the $^1S_0$ channel. The only difference is that
As mentioned, the Gaussian cut-off brings in terms higher order than NLO which to be consistent, would require corresponding “counter terms” in the potential although our results indicate that the latter cannot be significant. The next task is to incorporate pions into the picture and go up in energy. This would enable us to explore the interplay between the breakdown of EFT and the emergence of a “new physics”, an important and generic issue currently relevant in particle physics where going beyond Standard Model is the Holy Grail. These issues will be addressed in a forthcoming publication.

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**TABLE I.** Deuteron properties and the $M1$ transition amplitude entering into the $np$ capture for various values of $\Lambda$.

| $\Lambda$ (MeV) | 150 | 198.8 | 216.1 | 250 | Exp. \[10\] | $v_{18}$ | \[10\] |
|----------------|-----|-------|-------|-----|---------|--------|-----|
| $B_d$ (MeV)    | 1.799 | 2.114 | 2.211 | 2.389 | 2.225 | 2.225 |
| $A_s$ (fm$^2$) | 0.860 | 0.877 | 0.878 | 0.878 | 0.8846(8) | 0.885 |
| $r_d$ (fm)     | 1.951 | 1.960 | 1.963 | 1.969 | 1.966(7) | 1.967 |
| $Q_d$ (fm$^3$) | 0.231 | 0.277 | 0.288 | 0.305 | 0.286 | 0.270 |
| $P_D$ (%)      | 2.11 | 4.61 | 5.89 | 9.09 | - | 5.76 |
| $\mu_d$        | 0.886 | 0.854 | 0.846 | 0.828 | 0.8574 | 0.847 |
| $M_{1B}$ (fm)  | 4.06 | 4.01 | 3.99 | 3.96 | - | 3.98 |

the value of $\Lambda_{Z=1}$ that makes $Z = 1$ does not coincide with $\Lambda_{A=0}$ that makes $r_A = 0$. Using $a = 5.419$ fm and $r_e = 1.753$ fm \[10\] for the $^3S_1$ channel, we find that $\Lambda_{max} = 304.0$ MeV, $\Lambda_{Z=1} = 198.8$ MeV and $\Lambda_{A=0} = 216.1$ MeV. The resulting (S-wave and $D$-wave) radial wavefunctions of the deuteron are

$$u(r) = e^{-\gamma r} - H(\Lambda r, -\frac{\gamma^2}{\Lambda^2}) + \beta_A D(\Lambda r),$$

$$\omega(r) = \frac{\mu_S}{\gamma^2} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} e^{-\gamma r} - H(\Lambda r, -\frac{\gamma^2}{\Lambda^2}) \right).$$

We now have all the machinery to calculate the deuteron properties: the wavefunction normalization factor, $A_s$, the radius, $r_d$, the quadrupole moment, $Q_d$ and the $D$-state probability, $P_D$. The magnetic moment of the deuteron $\mu_d$ is related to the $P_D$ through

$$\mu_d = \mu_S - \frac{3}{2} \left( \mu_S - \frac{1}{2} \right) P_D$$

where $\mu_S \approx 0.8798$ is the isoscalar nucleon magnetic moment. Finally the one-body isovector $M1$ transition amplitude relevant for $n + p \to d + \gamma$ at threshold \[10\] is

$$M_{1B} = \int_0^\infty dr u(r) u_0(r).$$

The (parameter-free) numerical results are listed in Table 1 for various values of the cut-off $\Lambda$. We see that the agreement with the experiments (particularly for $\Lambda = 216.1$ MeV) is excellent with very little dependence on the precise value of $\Lambda$. It may be coincidental but highly remarkable that even the quadrupole moment which as the authors of \[10\] stressed, the $v_{18}$ potential fails to reproduce, comes out correctly.

We believe to have demonstrated the power of EFTs in low-energy nuclear physics, allowing us to be as close as one can hope to the fundamental theory in the sense put forward in Ref. \[10\]. In particular, it is satisfying that the classic $np$ capture process can be completely understood from a “first-principle” approach. Here the cut-off regularization was found to be highly efficient: With the dimensional regularization the $M1$ matrix element was found to be in total disagreement with the result of the Argonne $v_{18}$ potential.

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