Notes for Miscellaneous Lectures

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Abstract

Here I share a few notes I used in various course lectures, talks, etc. Some may be just calculations that in the textbooks are more complicated, scattered, or less specific; others may be simple observations I found useful or curious.

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1 Nemirovski Estimate of Common Mean of Arbitrary Distributions with Bounded Variance

The popular Chernoff bound\(^1\) assume severe restrictions on distribution: it must be cut-off, or vanish exponentially, etc. In [Nemirovsky Yudin\(^2\)] an equally simple bound uses no conditions at all beyond independence and known bound on variance. It is not widely used because it is not explained anywhere with an explicit tight computation. I offer this version:

Assume independent variables \(X_i(\omega)\) with the same unknown mean \(m\) and known lower bounds \(B_i^2\) on inverses \(1/v_i\) of their variance. We estimate \(m\) as \(M(\omega)\) with probability \(P(\pm(M-m) \geq \varepsilon) \overset{\Delta}{=} p^+ < 2^{-k}\) for \(k\) close to \(\sum_i(B_i\varepsilon)^2/12\). First, we normalize \(X_i\) to set \(\varepsilon = 1\), spread them into \(n\) groups, and take in each group \(j\) its \(B_i^2\)-weighted mean \(x_j(\omega)\).

The inverse variance bounds \(b_j^2\) for \(x_j\) are additive; we grow groups to assure \(b_j > 2\) and to increase the sum \(k\) of heights \(h_j \overset{\Delta}{=} \log_2 b_j^2 + b_j^{-4}\). (The best \(h/b^2 > 1/12\) comes with \(b^2 \approx 6\).\(^3\)

For \(s \subset [1, n]\), let \(b_s \overset{\Delta}{=} \prod_{j \in s} b_j\). Let \(L \overset{\Delta}{=} \cup_t L_t\) consist of light \(s\): those whose largest superset \(s'\) with \(b_{s'} < b_{[1,n]}\) has \(\|s\| + t\) elements. As \(s \in L_t\) do not include each other, \(\|L_t\| \leq (\binom{n}{\lfloor n/2 \rfloor})^2 \sqrt{2/(\pi n)},\) by Sperner’s theorem, and since \(n! = (n/e)^n \sqrt{\pi(2n+1)/3} + \varepsilon/n, \varepsilon \in [0, 1].\)

Our \(M\) is the \((\log b_j)\)-weighted median of \(x_j\). Let \(S^\pm(\omega) \overset{\Delta}{=} \{j : \pm(x_j-m) < 1\}.\) Then \(\pm(M(\omega)-m) \geq 1\) means \(S^\pm(\omega) \in L\). By Chebyshev’s inequality, \(p^\pm = P(j \notin S^\pm) \leq 1/(b_j^2+1).\)

We assume \(p^+_j = 1/(b_j^2+1):\) the general case follows by so modifying the distribution without changing \(m, b_j\), or decreasing \(p^+\) (respectively \(p^-\)). If \(s \in L_t, S^\pm(\omega) = s\) has probability

\[
p^+_s = b_s^2 / \prod_{j \leq n} (b_j^2 + 1) < 4^{-t} \prod_{j \leq n} b_j / (b_j^2 + 1) = 4^{-t} 2^{-(k+n)}.
\]

So,

\[
p^+ + p^- \leq \sum_{t \geq 0} \sum_{s \in L_t} (p^+_s + p^-_s) \leq 2(\sum_{t \geq 0} 4^{-t}) 2^{-(k+n)} 2^n \sqrt{2/(\pi n)} < 2^{-k} \sqrt{5/n}.
\]

\(^1\)First studied by S.N. Bernstein: *Theory of Probability.*, Moscow, 1927. Tightened by Wassily Hoeffding in: Probability inequalities for sums of bounded random variables, J.Am.Stat.Assoc. 58(301):13-30, 1963.

\(^2\)A.S.Nemirovsky, D.B.Yudin. *Problem Complexity and Method Efficiency in Optimization.* Wiley, 1983.

\(^3\)Giving up tightness, the rest may be simplified: assure \(b_j \geq b \sqrt{2} + 1\) and replace \(b, h_j\) with \(b, 1/2\).
2 Leftover Hash Lemma

The following Lemma is often useful to convert a stream of symbols with absolutely unknown (except for a lower bound on its entropy) distribution into a source of perfectly uniform random bits $b \in \{0, 1\}$.

The version I give is close to that in [HILL], though some aspects are closer to that from [GL]. Unlike [GL], I do not restrict hash functions to be linear and do not guarantee polynomial reductions, i.e. I forfeit the case when the unpredictability of the source has computational, rather than truly random, nature. However, like [GL], I restrict hash functions only in probability of collisions, not requiring pairwise uniform distribution.

Let $G$ be a probability distribution on $Z_2^n$ with Renyi entropy $-\log \sum_x G^2(x) \geq m$. Let $f_h(x) \in Z_2^{2t}$, $h \in Z_2^t$, $x \in Z_2^n$ be a hash function family in the sense that for each $x, y \neq x$ the fraction of $h$ with $f_h(x) = f_h(y)$ is $\leq 2^{-k} + 2^{-m}$. Let $U^t$ be the uniform probability distribution on $Z_2^t$ and $s = m - k - 1$. Consider a distribution $P(h, a) = 2^{-t}G(f_h^{-1}(a))$ generated by identity and $f$ from $U^t \otimes G$. Let $L_1(P, Q) = \sum_z |P(z) - Q(z)|$ be the $L_1$ distance between distributions $P$ and $Q = U^t$, $i = t + k$. It never exceeds their $L_2$ distance

$$L_2(P, Q) = \sqrt{2^i \sum_z (P(z) - Q(z))^2}.$$ 

Lemma 1 (Leftover Hash Lemma).

$$L_1(P, U^i) \leq L_2(P, U^i) < 2^{-s/2}.$$

Note that $h$ must be uniformly distributed but can be reused for many different $x$. These $x$ need to be independent only of $h$, not of each other as long as they have $\geq m$ entropy in the distribution conditional on all their predecessors.

Proof.

$$\left( L_2(P, U) \right)^2 = 2^i \sum_{h, a} P(h, a) + 2^i \sum_z (2^{-2i} - 2P(z)2^{-i}) = 2^i \sum_{h, a} P(h, a)^2 - 1$$

$$= -1 + 2^i \sum_{x, y} G(x)G(y)2^{-2t} \sum_a \|\{h: f_h(x) = f_h(y) = a\}\|$$

$$= -1 + 2^{k-t} \sum_{x, y} G(x)G(y)\|\{h: f_h(x) = f_h(y)\}\|$$

$$= -1 + 2^{k-t} \left( \sum_x G(x)^22^t + \sum_{x, y \neq x} G(x)G(y)\|\{h: f_h(x) = f_h(y)\}\| \right)$$

$$\leq -1 + 2k2^{-m} + 2^{k-t}(1 - 2^{-m})2^t(2^{-k} + 2^{-m}) < 2^{-s}.$$
3 Disputed Ballots and Poll Instabilities

Here is another curious example of advantages of quadratic norms.

The ever-vigilant struggle of major parties for the heart of the median voter makes many elections quite tight. Add the Electoral College system of the US Presidential elections and the history may hang on a small number of ballots in one state. The problem is not in the randomness of the outcome. In fact, chance brings a sort of fair power sharing unplagued with indecision: either party wins sometimes, but the country always has only one leader. If a close race must be settled by dice, so be it. But the dice must be trusty and immune to manipulation!

Alas, this is not what our systems assure. Of course, old democratic traditions help avoiding outrages endangering younger democracies, such as Ukraine. Yet, we do not want parties to compete on tricks that may decide the elections: appointing partisan election officials or judges, easing voter access in sympathetic districts, etc. Better to make the randomness of the outcome explicit, giving each candidate a chance depending on his/her share of the vote. It is easy to implement the lottery in an infallible way, the issue is how its chance should depend on the share of votes.

In contrast to the present one, the system should avoid any big jump from a small change in the number of votes. Yet, chance should not be proportional to the share of votes. Otherwise each voter may vote for himself, rendering election of a random person. The present system encourages voters to consolidate around candidates acceptable to many others. The ‘jumpless’ system should preserve this feature. This can be done by using a non-linear function: say the chance in the post-poll lottery be proportional to the squared number of votes. In other words, a voter has one vote per each person he agrees with. Consider for instance an 8-way race where the percents of votes are 60, 25, 10, 1, 1, 1, 1, 1. The leader’s chance will be 5/6, his main rival’s 1/7, the third party candidate’s 1/43 and the combined chance of the five ‘protest’ runners 1/866.

This system would force major parties to determine the most popular candidate via some sort of primaries, and will almost exclude marginal runners. However it would have no discontinuity rendering any small change in the vote distribution irrelevant. The system would preserve an element of chance, but would be resistant to manipulation.

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6 The dependence of lottery odds on the share of votes may be sharper. Yet, it must be smooth to minimize the effects of manipulation. Even (trusty) noise alone, e.g., discarding a randomly chosen half of the votes, can “smooth” the system a little.
Universal Heuristics: How do humans solve “unsolvable” problems?

Lots of crucial problems defeat current computer arts but yield to our brains. Great many of them can be stated in the form of inverting easily computable functions. Still other problems, such as extrapolation, are related to this form. We have no idea which difficulties are intrinsic to these problems and which just reflect our ignorance. We will remain puzzled pending major foundational advances such as, e.g., on $P=\neq NP$. And yet, traveling salesmen do get to their destinations, mathematicians do find proofs of their theorems, and physicists do find patterns in transformations of their elementary particles! How is this done, and how could computers emulate their success?

Brains of insects solve problems of such complexity and with such efficiency, as we cannot dream of. Yet, few of us would be flattered with a comparison to the brain of an insect :-). What advantage do we, humans, have? One is the ability to solve new problems, those on which evolution did not train generations of our ancestors. We must have some pretty universal methods, not restricted to the specifics of focused problems. Of course, it is hard to tell how, say, the mathematicians search for their proofs. Yet, the diversity and dynamism of math achievements suggest that some pretty universal methods must be at work.

In fact, whatever the difficulty of inverting functions $x = f(y)$ is, we know a “theoretically” optimal algorithm for all such problems, one that cannot be sped-up by more than a constant factor, even on a subset of instances $x$. It searches for solutions $y$, but in order of increasing complexity $Kt$, not increasing length: short solutions may be much harder to find than long ones. $Kt(y|x)$ can be defined as the minimal sum of (1) the bit-length of a prefixless program $p$ transforming $x$ into $y$ and (2) the log of the running time of $p$.

Extrapolations could be done by double-use of this concept. The likelihood of a given extrapolation consistent with known data decreases exponentially with the length of its shortest description. This principle, Occam Razor, was clarified in papers by Ray Solomonoff and his followers (see also https://arxiv.org/abs/1403.4539 and its references).

Decoding short descriptions should not take more time than the complexity of the process that generated the data. The major hurdle in implementing Occam Razor is finding short descriptions: it may be exponentially hard. Yet, this is an inversion problem, and the above optimal search applies. Such approaches contrast with the methods employed currently by CS - universal algorithms are used heavily, but mostly for negative results.

The point of this note is to emphasize the following problem: The above methods are optimal only up to constant factors. Nothing is known about these factors, and simplistic attempts make them completely unreasonable. Current theory cannot even answer straight questions, such as, e.g., is it true that some such optimal algorithm cannot be sped-up 10-fold on infinitely many instances? Yet humans do seem to use such generic methods successfully, raising hopes for a reasonable approach to these factors.

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7 The speed is defined to include the time for running $f$ on the solution $y$ to check it.

8 Realistically, $p$ runs on data which specify the instance, but also encompass other available relevant information, possibly including access to a huge database, such as a library, or even the Internet.
5 A Magic Trick

A book “Mathematics for Computer Science” by Eric Lehman, F Thomson Leighton, and Albert R Meyer has a very nice magic trick with cards. I used in my class some variation of it described below (with book authors permission).

The trick is performed by a Wizard (W) and his assistant (A) for the viewers (V).

In W’s absence, V choose and give A four cards out of 52 deck. A places them in a row with one of them (H) hidden (turned back up) and exits. W then enters and guesses H.

However, placing H in the middle of the 3 open cards hints that the cards order is informative, spoiling the surprise. I would instead place the chosen cards so that, 3 contiguous cards are open and 1 hidden, or all are hidden (sometimes stellar patterns are so favorable to magic that wizards need no information at all! :-).

First, some terms: Senior (S), Junior (J), Middle (M) below refer to the order of ranks or rank-suit pairs. Kings (K) are special. If chosen cards include King of spades (K0), all cards are hidden; K1 always is J, K2 is M, K3 is S. A 4-set is a set of 4 cards with no K0.

A string is an ordered 4-set with the first or last card replaced by a symbol H (hidden). G is a bipartite graph of 4-sets connected to four strings obtained by hiding one card and ordering the rest to reflect the rank of H. A hidden K is treated as a duplicate of the respective (J, M, or S) non-K open card. The Wizard only needs to figure the suit of H.

G breaks into small connected components distinguished by their sets R of non-K ranks of the 4 chosen cards and ranks’ multiplicity (including K as duplicates). With a uniform degree 4, G has a perfect matching, described below, for A,W to use.

In a 4-set, let α be the $\mathbb{Z}_4$ sum of all suits in single-suit ranks. Multiple suits in a rank are viewed in a circle ($\mathbb{Z}_7$ if $|R|=1$, else $\mathbb{Z}_5$) including respective Kings (but not K2 for $|R|=2$).

Let β (and $\beta'$ if 2 such ranks) be 0 if the suits are consecutive, else 1. Notations like $j, j'=j+1+\beta \pmod{5}$. Let γ be 2 if $|R|=2$ with K2 present, else γ=0. Below is a simple matching, blind to $\mathbb{Z}_5, \mathbb{Z}_7$ rotations. (I omit cases with just $j, m, s$ permuted):

| | | |
|---|---|---|---|
| $|R|=1$ | $H$ is the suit in a row (in $\mathbb{Z}_7$) adjacent to 1-suit-shorter gap (left is preferred). |
| $|R|=2$ | suits $j, j', s, s'$: $H=j$ if $\beta=\beta'$, else $H=s$. |
| $|R|=2$ | suits $j, c=K2, s, s'$ or $j, s, c'=s', s''$: $H=j$ if $\alpha=\beta+\gamma$; $H=c$ if $\alpha+\beta=1$; else $H=s$. |
| $|R|=3$ | suits $j, j', m, s$: $H=s$ if $x=(\alpha+\beta \pmod{4})$ is 0; $H=m$ if $x=1$; else $H=j$. |
| $|R|=4$ | The seniority of $H$ reflects $\alpha$. |

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9Problem 15.48 in a preprint: https://courses.csail.mit.edu/6.042/fall17/mcs.pdf

10In Russia, the special one would be Queen, not King: Queen of Spades is attributed a special malice. :-)
