Discrete breathers in classical spin lattices

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Discrete breathers (nonlinear localised modes) have been shown to exist in various nonlinear Hamiltonian lattice systems. In the present paper we study the dynamics of classical spins interacting via Heisenberg exchange on spatial $d$-dimensional lattices (with and without the presence of single-ion anisotropy). We show that discrete breathers exist for cases when the continuum theory does not allow for their presence (easy-axis ferromagnets with anisotropic exchange and easy-plane ferromagnets). We prove the existence of localised excitations using the implicit function theorem and obtain necessary conditions for their existence. The most interesting case is the easy-plane one which yields excitations with locally tilted magnetisation. There is no continuum analogue for such a solution and there exists an energy threshold for it, which we have estimated analytically. We support our analytical results with numerical high-precision computations, including also a stability analysis for the excitations.

I. INTRODUCTION

The phenomenon of dynamical localisation has been a subject of intense theoretical research. It is well known that classical Hamiltonian lattices possess periodic in time and localised in space solutions called discrete breathers or intrinsic localised modes. A recent explosion of interest to discrete breathers has happened due to the fact that they may exist in lattice models of interacting identical particles. Breathers in continuum models (like, for example, the well known sine-Gordon equation) exist only due to the high symmetry of the system and are therefore structurally unstable. Discrete breathers are generic solutions of nonlinear lattice equations. Their existence is based on the fact that all possible resonances of multiples of their frequency with the bounded band of small amplitude plane waves (BSAPW) above the classical ground state can be avoided. So far discrete breathers have been proven to be generic solutions in both Hamiltonian and dissipative systems. Several cases of experimental observation of discrete breathers have been reported (in Josephson junction arrays, arrays of weakly coupled waveguides, low dimensional crystals, and biological systems (myoglobin)).

Due to its spatial periodicity, lattices of interacting spins are ideal systems to observe discrete breathers as well. Here we will concentrate on large spins which may be described classically. Nonlinear waves in magnetic systems have been extensively studied during the last three decades. The results of these studies provide a lot of information about the properties of solitary waves (particularly, breathers) in magnets, since it is possible in many cases to obtain explicit solutions to them. However, neglecting discreteness effects may lead to losing important features of the nonlinear wave dynamics. For instance, since only high symmetry continuous systems possess breather solutions, the area of the potentially interesting models is artificially reduced. Another drawback of continuous systems is that the consideration of nontopological localised excitations is typically restricted to space dimension one.

In the last decade, a number of papers has appeared where localised modes in magnets were treated as essentially discrete objects (also an attempt of experimental observation of discrete breathers in antiferromagnets has been made). However, no rigorous existence proofs have been given, and only the most simplest cases (from the point of view of symmetries) have been considered. Preserving these symmetries one can continue those solutions to the space continuous limit.

The aim of this work is to present breather excitations for spin lattices in cases where the symmetries will not allow for a similar mode construction in space continuous cases. Also we will not restrict the consideration to one-dimensional systems. Below we present a rigorous existence proof for discrete breathers in magnetic systems using the anti-continuum limit. With the help of this proof and of a Newton iteration method, we show the existence of discrete breathers in ferromagnetic lattices with anisotropic exchange interaction. We also consider an easy-plane ferromagnet and find a new type of discrete breathers, with several spins precessing around the hard (single ion anisotropy) axis, while all the others precess around an axis which lies in the easy plane. Note that only the simplest case of monochromatic in time breathers has been investigated in most of previ-
ous papers. Such a situation simplifies considerably the treatment of the system, and important families of solutions can be lost. Our studies do not depend in any way on the number of higher harmonics in the time evolution of a breather.

This paper is organised as follows. The next section presents the model Hamiltonian and the equations of motion. In Sec. III, we consider an easy-axis ferromagnet, discuss the implementation of the anti-continuum limit, and give a rigorous existence proof for discrete breathers. In Sec. IV, we study an easy-plane ferromagnet. Then Sec. V presents a study of a two-dimensional lattice with easy-plane anisotropy. Discussions and conclusions are given in Sec. VI.

II. HAMILTONIAN AND EQUATIONS OF MOTION

We consider a lattice of classical spins described by the Hamiltonian with Heisenberg XYZ exchange interaction

\[ H = -\frac{1}{2} \sum_{\alpha=x,y,z} \sum_{\mathbf{n} \neq \mathbf{n'}} J^n_{\alpha\alpha'} S_n^\alpha S_{n'}^\alpha - D \sum_n S_n^2. \]  

(1)

Here \( S_n^x, S_n^y, S_n^z \) are the \( \mathbf{n} \)th spin components (\( \mathbf{n} \) labels a lattice site) which satisfy the normalisation condition

\[ S_n^x^2 + S_n^y^2 + S_n^z^2 = S^2. \]  

(2)

For simplicity, the total spin magnitude can be normalised to unity: \( S = 1 \). The constants \( J_x, J_y, J_z > 0 \) are the exchange integrals and \( D \) is the on-site anisotropy constant.

The equations of motion for the spin components in the one-dimensional spin chain with nearest-neighbour interactions are the well known Landau-Lifshitz equations:

\[
\dot{S}_n^x = \frac{1}{2} \left[ J_y S_n^y (S_{n-1}^x + S_{n+1}^x) - J_z S_n^z (S_{n-1}^x + S_{n+1}^x) \right] \\
- 2DS_n^x S_n^z,
\]

(3)

\[
\dot{S}_n^y = \frac{1}{2} \left[ J_z S_n^z (S_{n-1}^y + S_{n+1}^y) - J_x S_n^x (S_{n-1}^y + S_{n+1}^y) \right] \\
+ 2DS_n^y S_n^z,
\]

\[
\dot{S}_n^z = \frac{1}{2} \left[ J_x S_n^x (S_{n-1}^z + S_{n+1}^z) - J_y S_n^y (S_{n-1}^z + S_{n+1}^z) \right].
\]

Generalisation to higher lattice dimensions is straightforward.

III. EASY-AXIS FERROMAGNET

First, we consider spin lattices with a ground state that corresponds to all spins directed along a given axis (we assume this axis to be the Z-axis). This can be achieved either by introducing a strong exchange anisotropy \( J_x, J_y << J_z \), or by introducing an on-site anisotropy term \( D > 0 \). Before we study breather solutions of Eqs. (3), let us study the dispersion laws for linear spin waves.

A. Dispersion laws

First, we consider the easiest case, a ferromagnetic chain without ion anisotropy \( (D = 0) \), but with a strong exchange anisotropy: \( 0 \leq J_x, J_y << J_z \). In this case, the ground state is \( S_n^x = \pm 1, S_n^y = S_n^z = 0 \). Linearising the equations of motion around one of these ground states, e.g., \( S_n^x = \delta_x \sin (qn - \omega t), S_n^y = \delta_y \cos (qn - \omega t), S_n^z = \text{const} \approx 1 \), we obtain

\[ \omega_n^2(q) = (J_z - J_x \cos q)(J_z - J_y \cos q). \]  

(4)

This dispersion law is shown in Fig. 1, with the edges of the linear band \( \omega_L(q) \) being given by

\[ \omega_0^2 = (J_z - J_x)(J_z - J_y), \omega_\pi^2 = (J_x + J_z)(J_y + J_z). \]  

(5)

![Fig. 1. Dispersion law for the ferromagnetic chain with strong exchange anisotropy.](image)

Let us explain how we can use the information about dispersion relations in order to formulate expectations about existence or nonexistence of breathers. It has been shown in [2] that breather solutions of the full nonlinear equations bifurcate from certain nonlinear plane wave solutions. These specific plane wave solutions are time periodic and reduce to band edge plane waves (BEPW) in the limit of small amplitudes, i.e., to linear waves with \( \omega_L(q) \) being an extremum of \( \omega_L(q) \). A necessary prerequisite of the existence of breathers is that the frequency of these BEPW’s is detuning away from the linear band \( \omega_L(q) \) with increasing amplitude or energy density of the wave.

Let us consider an excitation which corresponds to the value of the wave number \( q = 0 \) from above. This is a space-homogeneous excitation \( S_n^x = S^x \) and the Landau-Lifshitz equations yield...
\[ S^x = (J_y - J_z) S^y S^z, \]
\[ S^y = (J_z - J_x) S^z S^x, \]
\[ S^z = (J_x - J_y) S^x S^y. \]

As can be seen from these equations, even for \( J_x = J_y \) we have softening in the dispersion law at \( q = 0 \). Then \( S^z = 0 \), and for \( S^x = A \cos \omega t \) the spin precession frequency

\[ \omega^2 = (1 - A^2)(J_z - J_x)^2 < (J_z - J_x)^2 = \omega_0^2, \]

is below the lower edge of the linear band, which indicates an occurrence of discrete breathers in the band gap. Note that for the completely isotropic model \( J_x = J_y = J_z = J \), there is no gap and consequently no breather solutions are to be expected.

It is easy to check that a similar analysis of the upper band edge yields lowering of the BEPW frequency with increasing amplitude, i.e. its frequency is attracted by \( \omega_L(q) \) instead of being repelled. Consequently we do not expect breathers to bifurcate from the upper band edge.

For \( D \neq 0 \) there are no qualitative changes. The ground state of the chain remains to be the same, whereas the gap in the dispersion law widens:

\[ \omega_L^2(q) = (J_z - J_x \cos q)(J_z - J_y \cos q) + 4D \left( J_z - \frac{J_x + J_y}{2} \cos q \right) + 4D^2, \]

with

\[ \omega_0^2 = (J_z - J_x)(J_z - J_y) + 4D \left( J_z - \frac{J_x + J_y}{2} \right) + 4D^2, \]

\[ \omega_2^2 = (J_x + J_z)(J_y + J_z) + 4D \left( J_z + \frac{J_x + J_y}{2} \right) + 4D^2. \]

Note that the band gap exists even in the case of isotropic exchange (all \( J_\alpha \)'s are equal). Therefore, easy axis anisotropy increases the chances of breather existence.

### B. Implementation of the anti-continuum limit

Now, following MacKay and Aubry\cite{13}, we apply the anti-continuum (AC) limit to our system. The principle of the AC limit consists in decoupling the lattice sites and exciting only one or a small number of them, keeping all the other in the ground state. Then, upon switching on the interaction, the persistence of the localised solution is shown. As a prerequisite for the successful existence proof and continuation of the breather solution, the initial ‘decoupled’ periodic orbit must be anharmonic\cite{14} and the breather frequency and all its multiples should not resonate with the linear magnon band. In the case of strongly anisotropic exchange \( J_x, J_y \ll J_z \), the particular case of the AC limit means simply setting \( J_x = J_y = 0 \).

In this case the \( Z \)-component of each spin is conserved. The solution of (\ref{1}) reduces to the precession of decoupled spins around the \( Z \) axis with frequencies which depend on the realised value of the \( Z \)-components of nearest neighbours (due to nonzero \( J_z \)):

\[ S_n + i S_n^y = A_n e^{i(\omega_n t + \varphi_n)}, \quad \dot{S}_n^z = 0, \]

where the precession frequency of the \( n \)th spin is given by

\[ |\omega_n| = J_z \left( S_{n-1}^z + S_{n+1}^z \right) + 2DS_n^z, \quad A_n^2 = 1 - S_n^2. \]

1. Vanishing single-ion anisotropy \((D = 0)\)

The initial choice of one precessing spin and all the others being at rest

\[ S_n^z = (\ldots, 1, 1, 1, S_0, 1, 1, 1, \ldots) \quad (12) \]

with \( S_0 < 1 \) cannot be used to generate breathers because the frequency of this solution \( \omega = J_z \) resonates with the linear band [more precisely, with the lower edge of the linear band \( \omega_0 \), see Eq. (\ref{3})]. Therefore it cannot be continued to the region of non-zero \( J_x \) and \( J_y \). A way out is simply to excite three neighbouring spins:

\[ S_n^z = (\ldots, 1, 1, 1, S_1, S_0, S_1, 1, 1, 1, \ldots). \]

Here \( 0 < S_0 < S_1 < 1 \) and, since the precession frequency of all the three central spins must be the same, \( S_0 = 2S_1 - 1 \). In this case, the precession frequency \( \omega = J_z S_1 \), allowing for the absence of resonances with the linear spectrum frequency \( \omega_0 = J_z \), must satisfy the condition \( k \omega \neq \omega_0 \).

2. Additional single-ion anisotropy \((D > 0)\)

In this case we may use the AC limit with the ansatz (\ref{13}). The initial distribution of \( Z \)-spin components are chosen as follows

\[ S_n^z = (\ldots, 1, 1, 1, S_0, 1, 1, 1, \ldots) \quad (14) \]

Here one central spin is precessing with frequency \( \omega = J_z + 2DS_0 \). All other spins are at rest. Small deviations from their equilibrium states yield precession with frequencies

\[ \omega_1 = \omega_0 + \frac{1}{2}(S_0 - 1), \quad \omega_0 = J_z(1 + 2D), \]

distributed along the lattice in the following way

\[ \omega_n = (\omega_0, \omega_1, \omega_0, \omega_1, \omega_0, \omega_1, \omega_0, \omega_1, \ldots). \]

3
Discrete breathers can be continued from the AC limit if the following non-resonance conditions are satisfied:

\[ k\omega \neq \omega_0, \quad k\omega \neq \omega_1, \quad k \in \mathbb{Z}. \tag{17} \]

Taking into account that \( S_0 = (\omega - J_z)/2D \) and substituting it into the non-resonance condition \( \omega \neq \omega_1 \), we get

\[ k\omega \neq \frac{J_z}{2} + 2D - \frac{J_z^2}{4D} + \frac{J_z}{4D}\omega \]

\[ \equiv -4D \left( \frac{J_z}{4D} - 1 \right) \left( \frac{J_z}{4D} + 1 \right) + \frac{J_z}{4D}\omega. \tag{18} \]

Note that for \( k = 1 \) the resonance will occur for any breather frequency if \( J_z = 4D \). For this set of parameters, a breather continuation from the AC limit is not possible for any frequency.

For this particular case we try another ansatz, namely the even-parity case:

\[ S_n^x = (\ldots, 1, 1, 0, S_0, S_0, 1, 1, 1, \ldots), \]

\[ \omega_n = (\ldots, \omega_0, \omega_0, \omega_1, \omega, \omega, \omega_1, \omega_0, \omega_0, \ldots) \tag{19} \]

with \( \omega_0 \) and \( \omega_1 \) being the same as in Eq. (15), and

\[ \omega = \frac{J_z}{2} (1 + S_0) + 2DS_0. \tag{20} \]

Then, using the non-resonant condition (17) which is valid for this ansatz as well, we obtain

\[ k\omega \neq \omega_1 = \frac{(J_z + 4D)^2 - J_z^2}{2(J_z + 4D)} + \frac{J_z}{J_z + 4D}\omega. \tag{21} \]

It follows from this expression that the even-parity AC limit allows for a continuation of the breather solution for all values of \( J_z \) and \( D \).

3. Isotropic exchange \( J_x = J_y = J_z \equiv J \)

Here the ansatz (12) can be used and the frequencies in the AC limit will be distributed as \( \omega_n = 2DS_n^\pm \). The eigenfrequencies of the non-excited spins do not depend on the values of the adjacent spins and equal \( \omega_0 \). Thus, the only non-resonance condition to be fulfilled is \( k\omega \neq \omega_0 \).

C. Existence proof for magnetic breathers

Here we present a rigorous proof of the existence of discrete breathers for the particular case of strongly anisotropic exchange and \( D > 0 \).

**Theorem 1** If a periodic orbit of the Hamiltonian (4) with a frequency \( \omega \) is non-resonant (\( k\omega \neq \omega_0,1, \quad k \in \mathbb{Z} \) and \( \omega_L(q) \)) and anharmonic, then the periodic orbit of the equations of motion (1) at \( \alpha = \{J_x, J_y\} = 0 \) given by the spin precession frequency (4) has a locally unique continuation as a periodic orbit of the equations (3) with the same period \( T = 2\pi/\omega \) for a sufficiently small \( \alpha \).

**Proof.** Let \( SL_T \) be the space of bounded infinite sequences \( z = \{z_n\}_{n \in \mathbb{Z}} \) of triplets \( z_n = (S_n^x, S_n^y, S_n^z) \) of continuously differentiable functions of a period \( T = 2\pi/\omega \) with symmetry properties:

\[ S_n^x(t) = S_n^x(-t), \quad S_n^y(t) = -S_n^y(-t), \quad S_n^z = S_n^z(-t). \tag{22} \]

Then the size of oscillations on the \( n \)th site will be measured by the following norm:

\[ |z_n| = \sup \{ |S_n^x(t)|, |S_n^y(t)|, |\dot{S}_n^x(t)|, |\dot{S}_n^y(t)|; t \in R \}. \tag{23} \]

Next, the size of \( z \in SL_T \) is given by

\[ |z| = \sup \{ |z_n|; n \in \mathbb{Z} \}. \tag{24} \]

and therefore \( SL_T \) is a Banach space.

Consider now another Banach space \( SM_T \) of bounded infinite sequences \( w = \{w_n\}_{n \in \mathbb{Z}} \) of triplets \( w_n = (M_n^x, M_n^y, M_n^z) \) of continuous functions of a period \( T \) with the following symmetry properties:

\[ M_n^x(t) = -M_n^x(-t), \quad M_n^y(t) = M_n^y(-t), \quad M_n^z = -M_n^z(-t), \tag{25} \]

and the norms

\[ |w_n| = \sup \{ |M_n^x(t)|, |M_n^y(t)|, |M_n^z(t) - t|; t \in R \}, \]

\[ |w| = \sup \{ |w_n|; n \in \mathbb{Z} \}. \tag{26} \]

Define the mapping \( F: SL_T \to SM_T \)

\[ F(z, \alpha) = w \]

with \( \alpha = \{J_x, J_y\}, \quad z = \{S_n^x, S_n^y, S_n^z\}_{n \in \mathbb{Z}}, \quad w = \{M_n^x, M_n^y, M_n^z\}_{n \in \mathbb{Z}} \), and

\[ M_n^x = \frac{1}{2} \left[ J_y S_n^y \left( S_n^y - 1 + S_n^y + 1 \right) - J_z S_n^z \left( S_n^z - 1 + S_n^z + 1 \right) \right] \]

\[ - 2D S_n^y S_n^z - \dot{S}_n^x, \]

\[ M_n^y = \frac{1}{2} \left[ J_x S_n^x \left( S_n^x - 1 + S_n^x + 1 \right) - J_z S_n^z \left( S_n^z - 1 + S_n^z + 1 \right) \right] \]

\[ + 2D S_n^x S_n^z - \dot{S}_n^y, \tag{27} \]

\[ M_n^z = \frac{1}{2} \left[ J_x S_n^x \left( S_n^x - 1 + S_n^x + 1 \right) - J_z S_n^z \left( S_n^z - 1 + S_n^z + 1 \right) \right] \]

\[ - \dot{S}_n^z. \]

The symmetric solutions of the equations of motion (4) are in one-to-one correspondence with zeroes of \( F \), i.e., \( F(z, \alpha) = 0 \), especially in the case when \( \alpha = 0 \). Using the implicit function theorem, we prove this solution to have a locally unique continuation \( z(\alpha) \) for sufficiently small \( \alpha \), such that \( F(z(\alpha), \alpha) = 0 \) provided \( F \in C^1 \) and its derivative with respect to \( z, D \) is invertible at \( \alpha = 0 \).
To show the invertibility of $DF$, we linearise the map around the periodic orbit at $\alpha = 0$
\[ \delta M = DF \delta S \] (28)
and show that it is invertible simultaneously in the following three parts of the lattice: the central site $n = 0$, its adjacent sites $n = \pm 1$, and the remainder of the lattice. Invertibility of $DF$ is equivalent to the invertibility of the corresponding matrix in (29).

(i) For $n \neq 0, \pm 1$ we have
\[ \delta M_n^x = -\delta S_n^y \omega_0 - \delta S_n^z , \]
\[ \delta M_n^y = \delta S_n^x \omega_0 - \delta S_n^w , \]
\[ \delta M_n^z = -\delta S_n^w . \] (29)

Since the functions $\delta M_n^a$ and $\delta S_n^a$ are periodic in time, we can expand them into Fourier series
\[ \delta \ldots = \sum_{k=-\infty}^{+\infty} \delta \ldots(k) e^{i k \omega t} . \] (30)

The time-symmetry requirements (22) ensures $\delta \ldots(k \geq 0)$ to be either purely real or imaginary, so that
\[ \delta M_n^a(-k) = -\delta M_n^a(k) , \quad \delta S_n^a(-k) = \delta S_n^a(k) , \]
\[ \delta M_n^a(0) = \delta M_n^a(0) = \delta S_n^a(0) = 0. \]

As a result, for $k > 0$ we obtain
\[ \delta M_n^a(k) = -\omega \delta S_n^a(0) - i k \omega \delta S_n^a(k) , \]
\[ \delta M_n^a(0) = \omega_0 \delta S_n^a(0) - i k \omega \delta S_n^a(0) , \] (32)
\[ \delta M_n^a(0) = -i k \omega \delta S_n^a(0) . \]

These equations appear to be decoupled with respect to $k$ and they can be inverted if $k^2 \omega^2 \neq \omega_0^2$. The third equation can be inverted for $k \neq 0$ if $\omega \neq 0$. For $k = 0$ the inversion is impossible, but this degeneracy can be lifted by imposing the normalisation condition (3). In fact, the third equation can be dropped because $S_n^a$ is defined by $S_n^a$ and $S_n^b$ with accuracy up to a sign. All that we have to check is the inequality $S_n^{a x} + S_n^{a y} \leq 1$.

(ii) For $n = \pm 1$ we act similarly to the previous case:
\[ \delta M_1^x = -\delta S_1^y \omega_1 - \delta S_1^z , \]
\[ \delta M_1^y = \delta S_1^x \omega_1 - \delta S_1^w , \]
\[ \delta M_1^z = -\delta S_1^w . \] (33)

A similar condition for invertibility can be found: $k^2 \omega^2 \neq \omega_1^2$.

(iii) Case $n = 0$. Using the canonical coordinate $y$ and momentum $p$
\[ S_0^x = \sqrt{1 - y^2} \cos p, \quad S_0^y = \sqrt{1 - y^2} \sin p, \quad S_0^z = y. \] (34)

we obtain the following Hamilton equations:
\[ \dot{y} = \frac{\partial H}{\partial p} , \quad \dot{p} = -\frac{\partial H}{\partial y} . \] (35)

We define the pair $(u, v)$ instead of the set $(M_0^x, M_0^y, M_0^z)$, with
\[ u = \frac{\partial H}{\partial p} - \dot{y} , \quad v = -\dot{p} - \frac{\partial H}{\partial y} . \] (36)

The new function satisfies the following symmetries:
\[ y(t) = y(-t) , \quad p = \omega t + h(t) , \quad h(t) = -h(-t) , \]
\[ y(t + T) = y(t) , \quad h(t) = h(t + T) . \] (37)

Here $h(t)$ is a periodic function in time. Keeping in mind that $S_{\pm 1}$ and $S_1^a$ are fixed by variations of $S_{\pm 1}$, we obtain
\[ \delta u = 2D \delta y - \delta \dot{y} , \quad \delta v = -\delta \dot{y} . \] (38)

After considering the corresponding Fourier series with respect to time, we find the following inversion conditions: $\omega \neq 0$ for $k \neq 0$ and $D \neq 0$ for $k = 0$.

Once the invertibility is shown, by the implicit function theorem, the initial solution $z(0)$ has a locally unique continuation $z(\alpha)$ in $SL_T$, and thus the theorem has been proved. $\square$

This result can be easily extended to lattices in higher dimensions, antiferromagnets, and systems with larger interaction radius.

D. A method of computation of discrete breathers

and linear stability analysis

For numerical simulations it is convenient to use stereographic coordinates. The new coordinates incorporate the normalisation condition and reduce the problem with three unknown real functions per site to the problem with one unknown complex function per site:
\[ \xi_n = \frac{S_n^a}{1 + |\xi_n|^2} , \]
\[ \xi_n = \frac{\xi_n + \xi_n^*}{1 + |\xi_n|^2} . \] (39)

The inverse transform is given by
\[ S_n^a = \frac{\xi_n + \xi_n^*}{1 + |\xi_n|^2} , \quad S_n^y = \frac{1}{1 + |\xi_n|^2} , \quad S_n^z = \frac{1 - |\xi_n|^2}{1 + |\xi_n|^2} . \] (40)

In these new coordinates the Landau-Lifshitz equations take the form
\[ \dot{\xi}_n = \frac{1}{4} \left[ (J_x + J_y) \left( \xi_n - \xi_n^* \frac{\xi_{n-1} - \xi_{n-1}^*}{1 + |\xi_{n-1}|^2} + \xi_{n+1} - \xi_{n+1}^* \frac{\xi_{n+1} - \xi_{n+1}^*}{1 + |\xi_{n+1}|^2} \right) + (J_x - J_y) \frac{\xi_n - \xi_n^*}{1 + |\xi_{n-1}|^2} + \frac{\xi_{n+1} - \xi_{n+1}^*}{1 + |\xi_{n+1}|^2} \right) \]
\[ + 2J_2 \xi_n \left( \frac{1 - |\xi_{n-1}|^2}{1 + |\xi_{n-1}|^2} + \frac{1 - |\xi_{n+1}|^2}{1 + |\xi_{n+1}|^2} \right) - 8D \xi_n \frac{1 - |\xi_n|^2}{1 + |\xi_n|^2} . \] (41)
The computation of discrete breathers is done using the Newton map.

The linear stability analysis of discrete breathers is performed by linearising Eqs. (11): \( \xi_n(t) = \xi_n^{(0)}(t) + \xi_n(t) \) around the breather periodic orbit, and solving afterwards the eigenvalue problem

\[
\begin{pmatrix}
\text{Re} \, \epsilon_n(T) \\
\text{Im} \, \epsilon_n(T)
\end{pmatrix} = \mathcal{M} \begin{pmatrix}
\text{Re} \, \epsilon_n(0) \\
\text{Im} \, \epsilon_n(0)
\end{pmatrix}.
\]  

(42)

If the eigenvalues \( \Lambda \) of the Floquet matrix \( \mathcal{M} \) are found to be located on the unit circle of the complex plane, then according to the Floquet theorem, the periodic orbit is stable, otherwise it is unstable.

**E. Breather solutions in an easy-axis ferromagnet**

Breathers in magnetic lattices with an easy-axis anisotropy can be viewed as localised spin excitations with the spins precessing around one of the ground states of the system (which we have chosen \( S^z = 1 \) at the beginning of the paper), so that the effective radius of this precession decreases to zero at \( n \to \pm \infty \). The case of an isotropic exchange in \( XY \) (\( J_x = J_y \)) is the simplest one because the \( S^z \) component is conserved in the solution, and therefore the separation of the time and the space variables

\[
S_n^+ = S_n^z + i S_n^y = A_n e^{i \omega t}
\]

(43)
is possible in the Landau-Lifshitz equation [this can easily be seen also from Eq. (11)]. The precession amplitudes \( A_n \) do not depend on time and the Landau-Lifshitz equations (2) are reduced to a set of algebraic equations which can be solved by a simple iteration procedure. This is true both in the case of strong exchange anisotropy \( J_x = J_y \ll J_z \) and also when the exchange is isotropic and an on-site anisotropy \( D > 0 \) is present.

The breather existence in the \( J_{x,y} - D \) plane is governed by the non-resonance conditions given in Subsec. 11 for one single harmonics \( k = 1 \). With the growth of \( J \) the breather frequency may hit the linear spectrum which marks the boundary of breather existence on this plane. The nature of the other non-resonance condition is different, e.g., we cannot continue a breather solution for small \( J = J_{x,y} \) when \( J_x = 4D \), however breathers exist for larger values of \( J \). Note, that in the case of \( J_x = J_y \) discrete breathers have a continuum equivalent which is a breather solution of the integrable nonlinear Schrödinger equation. The reason is that the \( XY \) exchange symmetry allows one to find solutions which are monochromatic in time (13). As long as the linear band provides a gap and the nonlinearity allows for pushing the breather frequency into the gap, localised excitations may be found regardless of the degree of discreteness of the system, which can be characterised by the ratio of the gap to the bandwidth of the spin wave spectrum.

Now we consider a rhombic chain with \( J_x \neq J_y < J_z \). Breaking isotropy in the \( XY \) plane implies that \( S^z \) is not conserved in the solution anymore, and according to the Landau-Lifshitz equations, it is impossible to represent the breather solution in the form (13). This implies that breathers will have an infinite number of harmonics in time, and consequently the spin dynamics is more complicated. Each spin now draws an ‘elliptic’ trajectory on the unit sphere, elongated towards the larger component of \( J_x \) or \( J_y \). Fig. 2.

![FIG. 2. Dynamics of the central spin with site number \( n_0 = 11 \) in the chain of \( N = 21 \) spins with \( J_x = 0.1, J_y = 0.23, J_z = 1, D = 0 \). shows such a dynamics of the central spin \( n_0 = 11 \) of the breather in a chain consisting of \( N = 21 \) spins. The breather profile at some instant of time is shown in Fig. 2.](image-url)
Due to the broken symmetry in the XY plane we are not able to find breathers in the corresponding continuum problem. The reason simply is that the linear band of a continuum equation may still have a gap but will be unbounded from above. Consequently there will be unavoidable resonances of higher harmonics of a breather with the linear band causing in general nonexistence of the solution itself. Here we have a nontrivial case where the discreteness of the lattice provides the necessary support for breathers which is missing in the continuum case. The computation of breather periodic orbits in this case cannot be reduced to solving a system of algebraic equations and we have to work in the full phase space by using, e.g., a generalised Newton map.

Let us briefly discuss the stability of the obtained breather solutions. Previous stability studies have shown that the stability depends on the breather parity (i.e., its spatial symmetry). Our Floquet analysis of the eigenvalues of the stability matrix confirms these findings. We obtain that the site-centred breathers (continued from the one-site breather) are stable in the limit of small exchange (see Fig. 4a)

while the bond-centred breathers [continued from the two-site breathers, see ansatz Eq. (19)] are unstable arbitrarily close to the AC limit with the unstable eigenvalue being located on the positive half or the real axis outside of the unit circle (see Fig. 4b).

IV. EASY-PLANE FERROMAGNET

In the case of an easy-plane anisotropy we choose $D < 0$ and $J_x = J_y = J_z = J$. The ground state of the system, without loss of generality, can be assumed to be
\[ S_n^x = 1, \quad S_n^y = S_n^z = 0. \] (44)

Note that the ground state is degenerate, so that the spins can be oriented arbitrarily in the \( XY \) plane, but they must stay parallel to each other.

**A. Linear dispersion law and the anti-continuum limit**

Linearising the equations of motion in the vicinity of the ground-state (44) we obtain the following dispersion law:

\[ \omega^2(q) = J^2(1 - \cos q)^2 + 2J|D|(1 - \cos q). \] (45)

This is an ‘acoustic’-type dispersion law with

\[ \omega_0^2 = \omega^2(0) = 0, \quad \omega_\pi^2 = \omega^2(\pi) = 4J(J + |D|), \] (46)

and therefore the breather frequencies in this case should lie above the linear band.

The implementation of the AC limit can be achieved by setting \( J = 0 \) and exciting one or several spins, so that they would start to precess around the hard axis with a frequency \( \omega = 2|D|S_0 \), where \( S_0 \) is the \( z \) projection of the spin. If the non-resonance condition \( \omega \neq \omega_\pi = 0 \) is satisfied, breather solutions can be continued.

Breathers do not exist in the continuum limit with easy-plane anisotropy. The reason is again that the corresponding linear band is gapless and unbounded, so that it covers the whole real axis. Correspondingly, there is no place for a frequency of a localised excitation on the real axis which does not resonate with the linear band.

An essentially discrete model has been studied only in the case of a strong magnetic field directed along the hard axis. In this case the hard axis effectively becomes an easy axis and, as a result, the spins precess around the \( Z \) axis with a constant \( S^z \) component. In this case, a separation of variables (43) is possible which simplifies the treatment of the system.

**B. Breather solutions of the easy-plane ferromagnet**

As stated above, we do not apply an external magnetic field, and therefore we do not change the ground state. As in the previous case of easy-axis, we compute breather periodic orbits from the AC limit using the generalised Newton method. As a result we obtain solutions, schematically described in Fig. 5 (for one ‘out-of-plane’ spin) or for two parallel precessing ‘out-of-plane’ spins, as shown in Fig. 6.

For non-zero \( J \) previously non-excited spins start to precess with small amplitudes around the \( X \)-axis while the plane of precession of the ‘out-of-plane’ spin is no longer parallel to the easy plane, but is slightly tilted. Breathers with more than two precessing spins can be also created.

Depending on its frequency, the breather width changes. When the frequency approaches the upper edge of the linear band, the breather becomes more delocalised. However, this does not qualitatively influence its core structure, i.e., the effective precessing axis of the central spin is not continuously tilted towards the \( X \) axis upon lowering the breather frequency down to the linear band edge. The central spin dynamics can be viewed as a periodic (closed) orbit of a point confined to the unit sphere. Let the \( XY \) plane be the equatorial one. Then for large breather frequencies the point performs small circles around the north (or south) pole. Lowering the breather frequency does not change the fact that the loop still encircles the \( Z \)-axis. Thus the breather solution can not be deformed into a slightly perturbed and weakly localised BEPW. This makes clear that the easy-plane ferromagnet lattice supports breather solutions with a local magnetisation tilt which have no analog in a continuum theory.

The situation is illustrated by Fig. 7.
where the profiles of two breathers are represented: one corresponds to the frequency $\omega_{\text{wide}} = 0.6649$, which is very close to the upper edge of the linear band $\omega_\pi = 0.6633$ and the other one has the frequency $\omega_{\text{narrow}} = 1.1967$, which is far above the linear band (see the inset to Fig. 7).

The first solution is more delocalised, which can be seen from Fig. 8. However, the central spin still precesses in a way similar to the ‘narrow’ breather, i.e. it encircles the north pole on some lower latitude as compared to the narrow breather (see Fig. 8, two curves on the unit sphere correspond to two breather periodic orbits, discussed above).

FIG. 8. Dynamics of the central spin with site number $n_0 = 15$ in the chain, described in Fig. 7.

Hence, even when very close to the linear band, our breathers have a structure, which has no analogue in the continuum case. Moreover, we have investigated the dependence of the breather energy on the breather frequency (see Fig. 9). We observe that there exists an energy threshold since the breather energy attains a non-zero minimum when its frequency is still not equal to the edge of the linear spin wave spectrum. Note that for lattices of interacting scalar degrees of freedom discrete breathers have typically zero lower energy bounds in spatial dimension $d = 1$ and become nonzero only for $d = 2, 3$. The reason for the appearance of a lower nonzero bound in the present case is due to the already mentioned fact that the breather of the easy plane ferromagnet system does not deform into a perturbed band edge magnon wave. Instead the central spin(s) is precessing around the Z axis. This topological difference is the reason for the appearance of nonzero lower energy bounds. Such energy thresholds may be very important as they show up in contributions to thermodynamic quantities which depend exponentially on temperature. To eliminate possible size effects, we repeated calculations for Fig. 9 for a chain with $N = 50$ spins. The difference between the curves was negligibly small.
Energy thresholds can be estimated analytically in the limit of small exchange $J$. Ignoring displacements of all in-plane spins we obtain the threshold energy for a breather with $M$ out-of-plane precessing spins, normalised to the ground state:

$$E(\omega) \approx 2M|D|S_0^{z^2},$$

where $S_0^z = \omega/2|D|$ is $Z$-component of the precessing spin in the

AC limit [see Eq. (3)]. The factor 2 comes from the fact that we should take into account the contribution of the breather tails.

Equation (47) can be obtained using the following argumentation. For large values of $\nu$ in Fig. the main contribution to the breather energy comes from the $M$ 'out-of-plane' precessing spins, because the tail amplitudes of the breather are small (see, for example, Fig. 7). For $\nu \to 0$ the energy contribution from the tails is actually diverging. Thus the height of the minima of the curves in Fig. can be estimated as two times the contribution coming from the central spins. Substitution into the above formula for the band edge frequency $\omega_\pi$ yields $E = E(\omega_\pi) \approx 2MJ + O(J^2)$, For the case, considered in Fig. for $M = 1$ (one precessing spin) the analytic result yields $\bar{E} \approx 0.2$ while numerics give the value 0.33. In the case of two precessing spins ($M = 2$) the numerical result yields $\bar{E} \approx 0.44$ while analytical estimate predicts the value 0.4.

Increase of the frequency leads to a decrease of the precession radius of the central spin. In the AC limit, an upper bound for the breather frequency is defined by $\omega = 2D$, which corresponds to the central (precessing) spin being parallel to the $Z$ axis. This bound continues to exist when exchange is switched on. After reaching this frequency threshold, the breather becomes a stationary (time-independent) solution. The existence of such a solution has been verified numerically by solving the time-independent Landau-Lifshitz equations.

C. Stability of breather solutions and their asymptotic properties

We have investigated the stability of our solutions with the help of the Floquet analysis (for details see Subsec. IIID) and with direct Runge-Kutta simulations. It appears that for small $J$'s breathers with one precessing spin (see Fig. 5) are unstable, while a stable configuration which corresponds to two parallel precessing spins (see Fig. 6) is stable. Note that similar results have been obtained for the FPU-type lattice (1). Stability tests also included the following numerical experiment. A periodic breather orbit $\{S_n^{x}(t), S_n^{y}(t), S_n^{z}(t)\}$ is perturbed by deviating one of the central spins $S_n^{z}(0)$ from $S_n^{z}(0)$ and simultaneously the equations of motion (9). The error function

$$\Delta(t) = \min_{\tau \in [0,T]} \left[ \sum_{n=1}^{N} \sum_{\alpha=(x,y,z)} \left| S_n^{\alpha}(t) - S_n^{\alpha}(0) \right|^2 \right] (48)$$

was calculated on each breather oscillation period $T$. In Fig. 10, such a function (with $\varepsilon = -0.0025$, in a chain consisting of $N = 1000$ spins with periodic boundary conditions) is shown for the breather solution with the two in-phase precessing spins (see Fig. 6).

![Graph showing the error function](image)

FIG. 10. Time dependence of the effective error $\Delta$ for the breather solution with $J = 0.1, D = -1$ and oscillation period $T = 4.4248$.

The error function is bounded during a significant period of time (more than 10000 breather oscillation periods) and the breather structure remains preserved. Similar experiments have been performed for other types of breathers. They yield similar results. This demonstrates the stability of the discussed excitation.

For a better understanding of the internal breather dynamics, the data will be represented using the Fourier
expansion of the breather periodic orbit

\[ S_\alpha^n(t) = C_\alpha^n(\omega;n) + \sum_{k=1}^{\infty} [A_\alpha^k(\omega;n) \cos k\omega t + B_\alpha^k(\omega;n) \sin k\omega t], \quad \alpha = x, y, z. \]  

(49)

We plot the space configuration of \( C_0 \) and \( C_n = \sqrt{A_n^2 + B_n^2} \).

The 'logarithmic' profile of such a solution is given in Fig. 11. We have plotted the space dependence of its Fourier harmonics (from the zeroth to the fifth one) for one particular stable solution.

Let us analyse the behaviour and the exponential spatial decay of these harmonics. As can be seen from Fig. 11, the zeroth (static) component is present. According to this figure, the zeroth component decays exponentially in space. This seems to be surprising, since the corresponding zero frequency resonates with the bottom of the acoustic-type linear band [see Eqs. (45)-(46)].

To understand the results in Fig. 11 we linearise the equations of motion (3) around the ground state (44) in the breather tails. We obtain the following equations for \( S_y \) and \( S_z \) components (\( S_x \) is assumed to be equal 1 with higher than linear corrections):

\[ -\frac{J}{2|D|} (S_{n+1}^z - 2S_n^z + S_{n-1}^z) + 2S_n^z = 0, \]

\[ S_{n+1}^y - 2S_n^y + S_{n-1}^y = 0. \]  

(50)

The numerical results suggest that the static \( S_n^y \) component is zero. This satisfies the second equation in (50). The first equation in (50) allows for an exponential decay of the static \( S_n^z \) component. Its decay can be characterised by the value \( \lambda_0^z \) if \( C_0^z(\omega;n) \sim \exp \left(-\lambda_0^z|n|\right), \quad |n| \to \infty \).
\( \lambda_0 > 0 \). The substitution of this ansatz into (45) yields

\[
\lambda_0^z = \ln \left[ 1 + \frac{2|D|}{J} + \sqrt{\left( 1 + \frac{2|D|}{J} \right)^2 - 1} \right].
\] (51)

The spatial decay of all the other (non-zero) harmonics of the breather solution can be obtained from the dispersion law (45) by substituting \( q = \pi - i\lambda^z \) and solving this equation with respect to \( \lambda \) with frequencies \( \Omega_k = k\omega \). As a result, we get

\[
\lambda_k^z = \ln \left[ \zeta + \sqrt{\zeta^2 - 1} \right] = \frac{\sqrt{D^2 + \Omega_k^2} - |D|}{J} - 1, \quad k = 1, 2, \ldots
\] (52)

Since the Fourier components for \( S^y_n \) decay in space as \( S^z_n \) (except for the static one), we have omitted them. The spatial decay of the Fourier components of \( S^x_n \) can be obtained using the normalisation condition (3), and therefore, for small deviations from the ground state (44), the following expansion is true:

\[
S^x_n \approx 1 - \frac{S^y_n^2}{2} - \frac{S^z_n^2}{2} + O \left( S^{(y,z)}_n^4 \right).
\]

Substituting here the Fourier expansion for \( S^y_n \) and \( S^z_n \), we see that only the product of terms containing harmonics \( k\omega \) and \( (m \pm k)\omega \) of \( S^{y,z} \) will contribute to the decay of the \( m \)th harmonic of \( S^x_n \). We have to choose the smallest exponent of all possible ones to obtain the leading order decay rate:

\[
\lambda_m^x = \min_{k=0,1,\ldots,m} \left[ \lambda_k^z + \lambda_{m \pm k}^z \right]
\] (53)

As a result, the following relations have been obtained for the first five harmonics of the \( S^x \) component: \( \lambda_0^x = 2\lambda_1^x \), \( \lambda_1^x = \lambda_1^y + \lambda_2^z \), \( \lambda_2^x = 2\lambda_1^x \), \( \lambda_3^x = \lambda_1^y + \lambda_3^z \), \( \lambda_4^x = \lambda_1^z + \lambda_3^z \). The comparison of these theoretical results with the values of \( \lambda \) extracted from the numerical data is given in Table I.

### Table I. Numerically and analytically computed values of the decay exponents \( \lambda_k \) of Fourier \( k = 0, 1, \ldots, 4 \) harmonics for different spin components. Parameter values are the same as in Fig. 7.

| Order, \( k \) | \( \lambda_k^x \) numerical | \( \lambda_k^y \) numerical | \( \lambda_k^x \) analytical | \( \lambda_k^y \) analytical |
|-------------|-----------------|-----------------|-----------------|-----------------|
| 0           | 3.5914          | 3.5903          | 3.1846          | 3.1856          |
| 1           | 4.9463          | 4.8055          | 1.8067          | 1.7951          |
| 2           | 3.5859          | 3.5903          | 2.9979          | 3.0103          |
| 3           | 4.8499          | 4.8055          | 3.5597          | 3.5690          |
| 4           | 5.4399          | 5.3641          | 3.9195          | 3.9309          |

As a result, the following relations have been obtained for the first five harmonics of the \( S^x \) component: \( \lambda_0^x = 2\lambda_1^x \), \( \lambda_1^x = \lambda_1^y + \lambda_2^z \), \( \lambda_2^x = 2\lambda_1^x \), \( \lambda_3^x = \lambda_1^y + \lambda_3^z \), \( \lambda_4^x = \lambda_1^z + \lambda_3^z \). The comparison of these theoretical results with the values of \( \lambda \) extracted from the numerical data is given in Table I.
As can be seen from this Table, the agreement between the numerical and analytical values of $\lambda$ decreases with the order of the Fourier components, which we think is due to the smallness of the higher order components.

V. TWO-DIMENSIONAL LATTICE WITH EASY-PLANE ANISOTROPY

Finally, we briefly consider a two-dimensional system, namely an easy-plane ferromagnet with nearest-neighbour exchange interactions. We have numerically simulated the Landau-Lifshitz equations for this system:

$$\dot{S}_{mn}^x = J_y S_{mn}^z \left( S_{m-1,n}^y + S_{m+1,n}^y + S_{m,n-1}^y + S_{m,n+1}^y \right)$$

$$- J_z S_{mn}^y \left( S_{m-1,n}^z + S_{m+1,n}^z + S_{m,n-1}^z + S_{m,n+1}^z \right)$$

$$- 2D S_{mn}^y S_{mn}^z$$

$$\dot{S}_{mn}^y = J_z S_{mn}^x \left( S_{m-1,n}^z + S_{m+1,n}^z + S_{m,n-1}^z + S_{m,n+1}^z \right)$$

$$- J_x S_{mn}^z \left( S_{m-1,n}^x + S_{m+1,n}^x + S_{m,n-1}^x + S_{m,n+1}^x \right)$$

$$+ 2D S_{mn}^x S_{mn}^z$$

$$\dot{S}_{mn}^z = J_x S_{mn}^y \left( S_{m-1,n}^x + S_{m+1,n}^x + S_{m,n-1}^x + S_{m,n+1}^x \right)$$

$$- J_y S_{mn}^x \left( S_{m-1,n}^y + S_{m+1,n}^y + S_{m,n-1}^y + S_{m,n+1}^y \right)$$

using again a fourth-order Runge-Kutta scheme with various initial spin configurations. The results of our simulations to some extent are similar to the one-dimensional problem. In Fig. 11, we show the simplest possible configurations of breathers which involve four ‘out-of-plane’ precessing spins. No stable breathers with one precessing spin are possible, similarly to the one-dimensional model. Also, there are no stable breathers with two or three precessing spins (at least in the limit of small $J$). Among the three possible stable configurations shown in Fig. 12, the first one [see panel (a)] corresponds to four spins precessing parallel and in phase, similar to its one-dimensional counterpart. The second two configurations do not have analogues in the one-dimensional case, but have also similar parity properties. The cases shown in Fig. 12(b,c) represent breathers with two spins precessing around the $Z$ axis in the positive direction (marked by dots) and two spins precessing around the negative direction (marked by crosses).

Simulations have been performed on a lattice of $150 \times 150$ spins with $J = 0.11$ and $D = -1$.

VI. SUMMARY AND DISCUSSIONS

Summarising, we have considered discrete breathers in different (easy-axis and easy-plane) classical ferromagnetic spin lattices as essentially discrete objects. We have shown systematically how to implement the anti-
continuum limit for different types of magnetic lattices (different types of anisotropy). Depending on the type of anisotropy, discrete breathers have properties similar to breather solutions for other nonlinear lattices. In the case of an easy-axis anisotropy the breather solution appears in the gap of the linear magnon band as do the breathers of the Klein-Gordon-type models. In the easy-plane case, there is no gap in the linear band and the breather frequency lies above the band; these breathers resemble the breathers of the FPU-type chains (also known as the Sievers-Takeno modes).

The concept of the anti-continuum limit helps us, first, to show rigorously the existence of discrete breathers, and, second, to compute the breather solutions numerically. The existence proof has been performed for the one-dimensional $XYZ$ Heisenberg ferromagnetic chain with a strong exchange ($J_{x,y} \ll J_z$). The proof can be easily generalised to the presence of an easy-axis anisotropy and to larger lattice dimensions. The numerical continuation of the solutions from the AC limit has been done with the help of a Newton iteration scheme. Note that so far only breathers with one nonzero Fourier component in time have been studied, due to the fact that it is much easier to treat them both numerically and analytically. The solutions we have studied allow the infinite number of harmonics, as in $XYZ$ model, for example.

Why is it important to study discrete systems if the continuum approximation can give an analytical solution? First of all, it is known that the breathers are non-generic for most continuous models. Therefore many systems may be incorrectly referred to as those which do not possess breathers. We demonstrated this circumstance for the easy axis ferromagnet, where the smallest exchange anisotropy in the hard plane leads to a loss of breathers in the continuum model, but not in the case of a spatial lattice. In addition we have obtained breather solutions for easy plane ferromagnets which have simply no continuum analog. This is due to the fact that the spins in the center of the excitation precess around a tilted axis leading to a local tilt of the magnetisation.

Finally, we would like to address some important unanswered questions in this area. The first problem is how to treat quantum spin lattices (e.g., when the total spin is too small to treat the lattices classically) and what is the quantum analogue of the spin breather. Another important question is the breather’s mobility. So far, there is no rigorous existence proof for moving breathers; however, Lai and Sievers have obtained some numerical results with highly mobile spin breathers. Since their results are concerned only with breathers with one Fourier component in time, it is still questionable whether breathers with an infinite number of harmonics can freely propagate along the lattice.

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