The Spectral Slope in the Cold Dark Matter Cosmogony

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Abstract

In a recent paper, we suggested that the density fluctuation spectra arising from power-law (or extended) inflation, which are tilted with respect to the Harrison–Zel’dovich spectrum, may provide an explanation for the excess large scale clustering seen in galaxy surveys such as the APM survey. In the light of the new results from COBE, we examine in detail here cold dark matter cosmogonies based on inflationary models predicting power-law spectra. Along with power-law and extended inflation, this class includes natural inflation. The latter is of interest because, unlike the first two, it produces a power-law spectrum without significant gravitational wave production. We examine a range of phenomena, including large angle microwave background fluctuations, clustering in the galaxy distribution, bulk peculiar velocity flows, the formation of high redshift quasars and the epoch of structure formation.

Of the three models, only natural inflation seems capable of explaining the large scale clustering of optical galaxies. Such a model, though at best marginal even at present, has some advantages over standard CDM and on most grounds appears to perform at least as well. Power-law inflation’s primary interest may ultimately only be in permitting a larger bias parameter than standard CDM; it appears unable to explain excess clustering. Most models of extended inflation are ruled out at a high confidence level.

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1 Introduction

Although recently it has been running into trouble from several sources, the standard Cold Dark Matter (CDM) model (White et al 1987; Frenk et al 1988; Efstathiou 1990) has shown some considerable success in explaining the clustering of galaxies we see around us. This model is based upon the idea that there exists a primordial spectrum of density fluctuations which takes on a particular form known as the Harrison–Zel’dovich or flat spectrum. Most frequently, this choice is motivated as being a prediction of the inflationary universe model. Nevertheless, it has long been known, at least within the inflation community, that the predicted spectrum is only approximately flat, and that within any given model there are readily calculable deviations from scale invariance. An example is provided by the standard ‘chaotic’ inflation scenario (Linde 1983, 1987, 1990), wherein a logarithmic correction (see eg Salopek, Bond & Bardeen 1989; Mukhanov, Feldman & Brandenberger 1992; Schaefer & Shafi 1992) adds some power at large scales, giving a density contrast at horizon crossing of 5–10% greater at $100h^{-1}$ Mpc than at $10h^{-1}$ Mpc. Hitherto, when our knowledge of the primeval spectrum was restricted to studies of local clustering up to scales of perhaps $100h^{-1}$ Mpc, these corrections where rightly regarded as insignif-

A certain class of inflationary models predict an even more dramatic change from the Harrison–Zel’dovich case, wherein the primeval spectrum generated exhibits power-law deviations from flatness. There are several such inflationary scenarios. The simplest is power-law inflation (Abbott & Wise 1984b; Lucchin & Matarrese 1985), which can be realised via a scalar field evolving in a potential of exponential form and which leads to an expansion of the universe which is not the conventional nearly exponential growth, but rather has the scale factor growing as a rapid power law in time. A second option is provided by the so-called extended inflation scenarios (La & Steinhardt 1989; Kolb, Salopek & Turner 1990; Kolb 1991), wherein modifications to Einsteinian gravity allow inflation to proceed via a first-order phase transition. Yet another option is provided by ‘natural inflation’ (Freese, Frieman & Olinto 1990; Adams et al 1992), a specific case of the more general situation of a scalar field evolving near the top of an inverted harmonic oscillator potential which also gives a power-law spectrum.

While power-law primeval spectra are often written down, the consequences of such a spectrum in the complete CDM cosmogony have not been fully investigated. The first detailed discussion of such models was made by Vittorio, Matarrese and Lucchin (1988), where they also allowed the density parameter $\Omega$ to be less than one.
Subsequent investigations by these authors and collaborators (Tormen, Lucchin & Matarrese 1992; Tormen et al 1992) have concentrated primarily on analysis of bulk velocity flows. Interesting constraints in the $n$–$\Omega$ plane were also obtained by Suto and Fujita (1990) and by Suto, Gouda and Sugiyama (1990) based on predictions for the cosmic Mach number combined with the then current microwave anisotropy limits. A pre–COBE analysis much less specific than that given here has been given by Kashlinsky (1992). In a recent paper with Will Sutherland (Liddle, Lyth & Sutherland 1992, henceforth LLS), we examined the consequences of such spectra for galaxy clustering. The motivation lay in recent observations such as the APM survey (Mad- dox et al 1990, 1991), which have shown that on large scales there appears to be stronger clustering than the standard CDM model can provide. These power-law spectra do indeed possess extra power on large scales. We found that for a certain range of values of the slope, one could reproduce the clustering seen in the APM survey, while withstanding the then current microwave background bounds.

Given this success, the emergence of the COBE data makes this an excellent time to examine this scenario in more detail, especially as in a certain sense the COBE data too show a rather larger amplitude of fluctuations than one would expect in the standard biassed CDM model. We have also taken this opportunity to analyse several other aspects of the scenario, though we have not gone so far as to carry out $N$-body simulations. Instead, we have restricted ourselves to calculations which can be done analytically or semi-analytically.

Recently there have been several papers analysing various aspects of power-law spectra (Salopek 1992; Davis et al 1992; Liddle & Lyth 1992; Lucchin, Matarrese & Mollerach 1992; Adams et al 1992; Lidsey & Coles 1992), which have however tended to deal with specific issues relating to particular models. With the exception of Adams et al, these papers have also failed to come up with detailed constraints on the scenario. Our aim here is to provide a more unified treatment, covering all the inflationary possibilities for generating power-law spectra, with the ultimate aim of constraining the slope of the spectrum and the bias parameter for the different models. One has already seen limits on the slope quoted by COBE (Smoot et al 1992) of $n = 1.1 \pm 0.5$, these numbers being simply derived from the variation of microwave anisotropies with scale between around $10^0$ and the quadrupole, and thus being independent of the specific choice of dark matter. [It has been suggested (D. T. Wilkinson, conference talk, Oxford, June 1992) that a more detailed error analysis (recognising that the error bars on the data are not independent) will increase the size of the errors on $n$.] It is natural for one to expect that if one makes the additional requirement, within a specific cosmogony, that the appropriate galaxy clustering is obtained at scales a factor of 100 smaller than those COBE samples, then the limits on the spectral slope become yet tighter. On the other hand, one should bear in mind that the COBE error bars on $n$ are presumably, like the rest of the data, 1-sigma error bars and so a stronger level of exclusion would allow further deviations from scale invariance.
Our general aim is to constrain the spectral index and bias parameter in the cold dark matter cosmogony by finding limits on the spectral amplitude at different length scales. Of most interest for this purpose will be the amplitude on large scales (> 1000 \( h^{-1} \) Mpc) from the COBE experiment, and on intermediate scales using the velocity data (which see the mass spectrum itself rather than the galaxy spectrum). In particular we shall use results from the QDOT survey (Saunders et al 1991; Saunders, Rowan-Robinson & Lawrence 1992) which fix the amplitude at around 20 \( h^{-1} \) Mpc. It is important to note that both these scales are expected to be well into the linear regime and thus linear calculational techniques can be used. To aid comparison with other work, we shall normally give the inferred normalisation in terms of the optical galaxy bias parameter \( b \) defined later, which is assumed to be the reciprocal of the mass dispersion in 8 \( h^{-1} \) Mpc spheres using the CDM spectrum. Note that this use should not be taken to imply that the results have anything to do with optical galaxies or scales which are close to entering the nonlinear regime.

Having obtained limits in this way on the mass spectrum, reference to optical galaxies provides limits on the bias parameter, enabling us to present our constraints in the form of allowed regions in the \( n-b \) plane. In order to clarify why particular choices of \( n \) appear in these sections, let us presage here the ultimate result that \( n \) is constrained in any inflation model to be no less than 0.7, at very high confidence. With the allowed values in mind, we then go on to investigate whether any further constraints can be imposed by an examination of aspects of the formation of nonlinear structure, wherein however the calculations are technically more tricky and comparison with observation much more ambiguous.

The layout of this paper is thus as follows, and generally works from large to small scales. Firstly, section 2 discusses the inflationary models giving rise to power-law spectra. In section 3, we discuss our processed CDM spectrum and its normalisation. Our aim here is to be as explicit as possible in our definitions and in where the uncertainties lie. Section 4 discusses large angle microwave background anisotropies, a discussion dominated of course by the observations of the COBE satellite. These calculations refer to the largest scales inside our horizon. In section 5, we review the results of LLS concerning the application of power-law spectra to the excess large-scale galaxy clustering. In this section we also discuss the bulk flows, which serve as another indicator of the power spectrum amplitude on these scales. Section 6 deals with the issue of shorter scales, corresponding to typical galaxy and cluster masses. There we investigate a range of topics concerning the entry into the nonlinear regime. In this region of the spectrum, our power-law spectra offer less power than standard CDM, and hence one expects the structure formation constraint to be more stringent here. Finally, we conclude and discuss our results.

The universe is taken to have critical density, with present Hubble parameter \( H_0 = 100h \text{ km s}^{-1} \text{ Mpc} \) where \( h = 0.5 \). For ease of comparison with other work we often take \( h^{-1} \text{ Mpc} \) for the distance unit. The units are taken to be such that \( c = 1 \) and (in section 2) \( h = 1 \), and the Planck scale is defined by \( m_{Pl} = G^{-1/2} \).
2 Inflationary Models giving Power-Law Spectra

As discussed in the introduction, there are at least three different inflationary models giving rise to power-law spectra. While there are certainly relations between these as we shall see, each has its own features and we shall discuss each in turn.

Let us first briefly sketch the derivation of the spectra, following Liddle and Lyth (1992). A scalar field \( \phi \) moving in a potential \( V \) is usually taken to satisfy slow-roll conditions (\( \dot{\epsilon} \equiv d/d\phi \))

\[
\dot{\phi} = -\frac{1}{3H} V' \\
\epsilon \equiv \frac{m_{Pl}^2}{16\pi} \left( \frac{V'}{V} \right)^2 \ll 1 \\
\eta \equiv \frac{m_{Pl}^2 V''}{8\pi V} \ll 1
\]

\( H \equiv \dot{a}/a \) is the Hubble parameter during inflation, where \( a \) is the scale factor normalised to \( a = 1 \) at present. Denoting the epoch at the end of inflation by a subscript 2, and the epoch \( k = aH \) when a scale \( k \) leaves the horizon with a subscript \( k \), the number of Hubble times \( N(t_k) \) between a scale \( k \) leaving the horizon and the end of inflation is

\[
N(t_k) = \frac{8\pi}{m_{Pl}^2} \int_{\phi_k}^{\phi_2} \frac{V}{V'} d\phi = 60 + \ln \frac{V^{1/4}}{10^{16}\text{GeV}} + 2 \ln \frac{V^{1/4}}{V_2^{1/4}} - \ln(k/H_0) \tag{2.4}
\]

In the slow-roll approximation, the spectrum of density fluctuations is given by

\[
\delta_H^2(k) = \frac{32}{75} \frac{V_k}{m_{Pl}^4 \epsilon_k^{-1}} \tag{2.5}
\]

where \( \delta_H \) is roughly the density contrast at horizon entry and is defined precisely in the following section. The effective spectral index is given by

\[
1 - n \equiv -\frac{d \ln[\delta_H^2(k)]}{d \ln k} = (6\epsilon_k - 2\eta_k) \tag{2.6}
\]

We see immediately that in order to have significant deviations from the flat \( n = 1 \) spectrum one must come close to violating at least one of the slow-roll conditions. Corrections to these slow-roll results have been calculated exactly for power-law inflation (Lyth & Stewart 1992a), and to first order in the departure from slow-roll for the general case (Lyth & Stewart 1992b). As discussed in our earlier paper (Liddle & Lyth 1992) they are not important in practice.

The models giving rise to power-law spectra are as follows.
1. **Power-Law Inflation.**

Power-law inflation $a \propto t^p$ arises (Abbott & Wise 1984b; Lucchin & Matarrese 1985; Barrow 1987; Liddle 1989) when a scalar field $\phi$ evolves down an exponential potential of the form

$$V(\phi) = V_0 \exp \left( \frac{16\pi}{p m_{Pl}^2} \phi \right)$$

(2.7)

It is of particular interest to researchers in inflation because although the scalar field comes close to violating the slow-roll conditions, exact analytic solutions exist both for the dynamics of inflation and for the density perturbations generated. The exact formula for the spectral index is (Lyth & Stewart 1992a)

$$n = 1 - 2/(p - 1)$$

(2.8)

In the limit of large $p$ the spectrum tends rapidly towards the flat $n = 1$ spectrum.

2. **Extended Inflation.**

Extended inflation (La & Steinhardt 1989; Kolb 1991) looks very different from power-law inflation, but is in fact related during inflation by a conformal transformation which allows the usual power-law inflation machinery to be utilised in its study. It is based on modifications to the gravitational sector of the theory, which allow a first-order inflationary phase transition to complete satisfactorily. The original model (La & Steinhardt 1989) was based on a Brans-Dicke theory with parameter $\omega$, and although this proved insufficient to allow present day tests of general relativity to be satisfied, it has remained the paradigm around which more complicated working models are based. The spectral slope in such a model is given by (Kolb, Salopek & Turner 1990; Guth & Jain 1992; Lyth & Stewart 1992a)

$$n = \frac{2\omega - 9}{2\omega - 1}$$

(2.9)

with extended inflation being conformally equivalent to power-law inflation with $2p = \omega + 3/2$.

A crucial difference between power-law and extended inflation is that extended inflation suffers an additional constraint, as one must avoid the large bubbles generated as inflation ends from being so profuse as to unacceptably distort the microwave background. This constrains $\omega$ as a function of the inflaton energy scale $M$ as (Liddle & Wands 1991; Liddle & Lyth 1992)

$$\omega < 20 + 0.7 \log_{10} (M/m_{Pl})$$

(2.10)
Bounding $M$ using the microwave background limits on the fluctuation amplitude gives $\omega \lesssim 17$, corresponding to $n \lesssim 0.75$. We have discussed the significance of this bound in an earlier paper (Liddle & Lyth 1992); the prognosis for the extended inflation model is not good, as we shall recap in this paper.

3. **Natural Inflation.**

The natural inflation model (Freese, Frieman & Olinto 1990; Adams *et al* 1992) is based on a pseudo-Nambu-Goldstone boson evolving in a potential

$$V(\phi) = \Lambda^4 (1 \pm \cos(\phi/f))$$

(2.11)

where $\Lambda$ and $f$ are mass scales. Evolution near the top of this potential (as is required for sufficient inflation with parameter choices significantly tilting the spectrum), gives rise to a power-law spectrum. In fact, this is a realisation of the more general case of a scalar field evolving near the top of an inverted harmonic oscillator potential

$$V(\phi) = V_0 - \frac{1}{2}m^2\phi^2$$

(2.12)

which gives a spectrum of slope

$$n = 1 - \frac{m^2m_{Pl}^2}{4\pi V_0}$$

(2.13)

For natural inflation one has $n = 1 - m^2_{Pl}/8\pi f^2$.

That natural inflation gives the same spectrum as power-law inflation may seem surprising in the light of claims that one can reconstruct inflationary potentials from a given spectrum (Hodges & Blumenthal 1990). In fact, these two models can be regarded as different regimes of an all-encompassing potential (which one can calculate in the manner of Hodges and Blumenthal) which is essentially $1/\cosh^2 \phi$ with various factors thrown in. In the $\phi \simeq 0$ region we have the inverted harmonic oscillator, while at large $\phi$ we have the exponential region.

In a sense, this is the unique potential from which all inflationary models giving power-law spectra arise. With this potential, the scalar field can roll from the top down to the exponential region, while in a slow-roll approximation generating an exact power-law spectrum. However, one must also note that as discussed above, the fact that the spectrum has been tilted implies that the slow-roll approximations are at best only just satisfied, so there will be corrections to the slow-roll spectrum in all regions of this potential. In the exponential region these corrections are known to affect only the amplitude and not the slope; one hopes for the same near the top of the potential.

Each of the above models can produce a power-law spectrum of the appropriate amplitude. However, as discussed in many recent papers (Krauss and White 1992;
Salopek 1992; Davis et al. 1992; Liddle & Lyth 1992; Lucchin, Matarrese & Mollerach 1992; Souradeep & Sahni 1992), one must pay attention to the predicted amplitude of gravitational waves (this was noted for power-law inflation long ago by Fabbri, Lucchin and Matarrese (1986)). In regimes where the slow-roll approximation is invalid or close to violation, one expects that these can become large, and the above work has confirmed that for power-law inflation (and by inference for extended inflation), the microwave contributions from gravitational waves may not only be large, but can dominate those from the scalar density fluctuations. To a good enough approximation for our purposes (see Lucchin, Matarrese & Mollerach (1992) for more precise results), the ratio $R$ of the tensor to the scalar contribution to the squared microwave multipoles is independent of the multipole and given by (Liddle & Lyth 1992)

$$R \approx 12.4 \frac{m^2_{pl}}{16\pi} \left(\frac{V''}{V}\right)^2$$

(2.14)

where the potential and its derivative are to be evaluated at the point corresponding to a given scale passing out of the horizon. In general $R$ is scale-dependent, but for power-law and extended inflation one has the scale-independent ratio

$$R \approx \frac{12.4}{p}$$

(2.15)

and thus for values of $p$ less than around 12 ($n \lesssim 0.82$) the tensor modes dominate the observed microwave background anisotropies.

On the other hand, for natural inflation the ratio $R$ is extremely small as the relevant scales leave the horizon, and hence there are no gravitational wave corrections of note in that case. Thus natural inflation (and similar models) provide an inflationary mechanism for generating power-law spectra without the large gravitational wave accompaniment. We shall return to the role of gravitational waves in section 4 for their microwave background implications, but for now we shall turn to the spectrum of density fluctuations.

3 The Spectrum

3.1 Definitions

The matter density contrast is $\delta(\mathbf{x}, t) \equiv (\rho(\mathbf{x}) - \bar{\rho})/\bar{\rho}$, where $\rho(\mathbf{x})$ is the matter density, $\bar{\rho}$ is its spatial average, $t$ is time and $\mathbf{x} \equiv (x^1, x^2, x^3)$ are comoving Cartesian coordinates. There is some freedom in the definition of $t$ and $\mathbf{x}$, but this ‘gauge freedom’ is not significant during the matter dominated era which concerns us (Lyth & Stewart 1990). As long as they are small, $\delta$ and related quantities evolve according to the linear equations of cosmological perturbation theory, with each Fourier mode evolving independently. At each comoving point during the matter dominated era, $\delta$ is proportional to the scale factor $a$, or equivalently to $(aH)^{-2}$. 

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According to the inflationary models described in the last section the primeval density contrast is a Gaussian random field at very early times, and therefore as long as linear evolution holds. As such its stochastic properties are completely determined by its spectrum. Different definitions of the spectrum exist in the literature. Letting $\delta_k$ denote the Fourier coefficient of the density contrast in a box of coordinate volume $V$, with the normalisation $\delta_k = V^{-1} \int e^{-ik.x} \delta(x) d^3x$, we work with the spectrum $P$ defined by

$$P = V(k^3/2\pi^2)\langle|\delta_k|^2\rangle$$

(3.1)

where the bracket denotes the average over a small region of $k$-space. With this definition the dispersion $\sigma$ of the density contrast is given by

$$\sigma^2 = \int_0^\infty P(k)\frac{dk}{k}$$

(3.2)

In the literature, our $P$ is denoted variously by $P_\rho$ (Salopek, Bond and Bardeen 1989), $P_\rho$ (Lyth & Stewart 1990), $\Delta^2$ (Kolb & Turner 1990), $\delta\rho_k/\rho$ (Linde 1990) and $d\sigma^2_\rho/d\ln k$ (Bond & Efstathiou 1991). The quantity usually denoted by $P$ in the literature is equal to $Nk^{-3}P$, where $N$ is an author-dependent normalisation factor which is often left undefined.

The standard assumption for the $k$-dependence of the primordial spectrum is $P \propto k$, corresponding to $P \propto k^4$. This is variously referred to as a Harrison–Zel’dovich spectrum, a flat spectrum, or a scale-free spectrum, and it is generated by the vacuum fluctuation of the inflaton field if the slow-roll conditions are well satisfied. Here we are exploring a power-law primordial spectrum $P \propto k^n$, corresponding to $P \propto k^{3+n}$, with $n < 1$.

During the matter dominated era, the spectrum may be written

$$P(k) = \left(\frac{k}{aH}\right)^4 T^2(k)\delta^2_H(k)$$

(3.3)

Up to a gauge dependent numerical factor (Lyth & Stewart 1990), $\delta^2_H$ is the primeval spectrum at the epoch $k = aH$ when the scale $k$ leaves the horizon. (In the literature $\delta_H(k)$ is often written as $\delta\rho/\rho|_{\text{hor.}}$.) Its scale dependence is $\delta^2_H \propto k^{n-1}$, which is why the standard choice $n = 1$ is referred to as a flat or scale–invariant spectrum.

The quantity $T(k)$ is known as the transfer function. It measures distortions to the primeval spectrum generated once a given scale comes within the horizon. On those very large scales which enter the horizon long after matter domination ($k^{-1} \gg 100\ \text{Mpc}$) it is equal to 1. To calculate it on smaller scales one has to know the matter content of the universe (see Efstathiou (1990) for a discussion covering various physical situations). For hot dark matter, free streaming erases perturbations on scales up to around the horizon scale at matter–radiation equality, but for cold dark matter the predominant effect is simply that the growth of modes which enter the horizon during radiation domination is suppressed.
Taking the dark matter to be cold, the standard picture of the early universe determines a practically unique transfer function once the baryon density \( \Omega_B \) is fixed. There are however considerable discrepancies between the parametrisations which appear in the literature. For \( \Omega_B = 0 \) there are two widely quoted parametrisations, one due to Bond and Efstathiou (1984) and the other to Bardeen et al (1986). We use the former, as given by Efstathiou (1990)

\[
T(k) = \left[1 + \left(ak + (bk)^{3/2} + (ck)^2\right)^\nu\right]^{-1/\nu}
\]

(3.4)

where

\[
a = 6.4 \left(\Omega h^2\right)^{-1} \text{Mpc} ; \quad b = 3 \left(\Omega h^2\right)^{-1} \text{Mpc} ; \quad c = 1.7 \left(\Omega h^2\right)^{-1} \text{Mpc} ; \quad \nu = 1.13
\]

(3.5)

In figure 1 this parametrisation is compared with that of Bardeen et al (1986), and with two others. At scales around \( k = 2\pi/10h^{-1} \text{Mpc}^{-1} \), the discrepancies can be of order 10%. In particular, the parametrisation of Davis et al (1985), which has the fewest fitting parameters, appears rather less accurate than the others.

Adams et al (1992) claim that Bardeen et al’s parametrisation is extremely accurate for \( \Omega_B = 0 \), and that increasing \( \Omega_B \) to the value \( \Omega_B \simeq 0.06 \) favoured by nucleosynthesis decreases it by about 15% at \( k \simeq 2\pi/10h^{-1} \text{Mpc} \). This would bring it 5% below the parametrisation that we have adopted, forcing the normalisation up by roughly that amount.

### 3.2 The filtered density contrast

The density contrast \( \delta(x) \) will evolve linearly as long as its dispersion satisfies \( \sigma \ll 1 \), except in those rare regions where it becomes \( \gtrsim 1 \) and gravitational collapse takes place. Following the usual practice, we assume when necessary that the linear evolution is at least roughly valid right up to the epoch \( \sigma \approx 1 \). Soon after that epoch, a large fraction of the matter collapses into gravitationally bound objects, and linear evolution becomes completely invalid. The epoch of non-linearity, defined as the epoch when \( \sigma = 1 \), corresponds to a redshift \( z_{nl} \) given by \( (1 + z_{nl}) = 1/\sigma_0 \), where \( \sigma_0 \) is the linearly evolved quantity evaluated at the present time.

A ‘filtered’ density contrast \( \delta(R_f, x) \), which has a smaller dispersion \( \sigma(R_f) \), can be constructed by cutting off the Fourier expansion of \( \delta(x) \) above some minimum wavenumber \( \approx 1/R_f \), or equivalently by smearing it over a region with size \( \approx R_f \). The filtered quantity will evolve linearly until the later epoch \( \sigma(R_f) = 1 \).

A precise definition of the filtered density contrast is made by means of a ‘window function’ \( W(r) \), which is equal to 1 at \( r = 0 \) and which falls off rapidly beyond some radius \( R_f \) (Peebles 1980, Kolb & Turner 1990). The filtered density contrast is

\[
\delta(R_f, x) = \int W(R_f, |x' - x|)\delta(x')d^3x'
\]

(3.6)
and its spectrum is
\[ \mathcal{P}(R_f, k) = \left[ \tilde{W}(R_f, k)/V_f \right]^2 \mathcal{P}(k) \]  
(3.7)
where
\[ \tilde{W}(R_f, k) = \int e^{-ik \cdot x} W(R_f, r) d^3 x \]  
(3.8)
and
\[ V_f = \int W(R_f, r) d^3 x \]  
(3.9)
The filtered dispersion is
\[ \sigma^2(R_f) = \int_0^\infty \left[ \tilde{W}(R_f, k)/V_f \right]^2 \mathcal{P}(k) \frac{dk}{k} \]  
(3.10)
The quantity \( V_f \) is the volume ‘enclosed’ by the filter. It is convenient to define the associated mass \( M = \rho_0 V_f \), where \( \rho_0 \) is the mean comoving density which in an \( h = 0.5 \) universe is given by \( \rho_0 = 3H_0^2/8\pi G = 6.94 \times 10^{10} M_\odot \text{ Mpc}^{-3} \) (\( M_\odot \) being the solar mass). One normally uses \( M \) instead of \( R_f \) to specify the scale, writing \( \delta(M, x) \) and \( \sigma(M) \).

The two popular choices are the Gaussian filter
\[ W(R_f, r) = \exp(-r^2/2R_f^2) \]  
(3.11)
\[ V_f = (2\pi)^{3/2} R_f^3 \]  
(3.12)
\[ \tilde{W}(R_f, k)/V_f = \exp(-kR_f) \]  
(3.13)
\[ M = 4.36 \times 10^{12} h^2 (R_f/1 \text{ Mpc})^3 M_\odot \]  
(3.14)
and the top hat filter which smears uniformly over a sphere of radius \( R_f \)
\[ W(R_f, r) = \theta(r - R_f) \]  
(3.15)
\[ V_f = 4\pi R_f^3/3 \]  
(3.16)
\[ \tilde{W}(R_f, k)/V_f = 3 \left( \frac{\sin(kR_f)}{(kR_f)^3} - \frac{\cos(kR_f)}{(kR_f)^2} \right) \]  
(3.17)
\[ M = 1.16 \times 10^{12} h^2 (R_f/1 \text{ Mpc})^3 M_\odot \]  
(3.18)
The Gaussian filter is the most convenient for the theoretical calculations described in Section 6, but the top hat filter is widely used to as a means of presenting data as in Section 5.

### 3.3 The conventional normalisation

The final ingredient required to fix the CDM spectrum is the overall normalisation. In the past this has usually been done by comparing the linearly evolved theory with observation, for the galaxy correlation function on a scale of order 10\( h^{-1} \) Mpc. Two
standard prescriptions exist (Efstathiou 1990). The first requires that the dispersion \( \sigma_g(r) \) of galaxy counts in spheres of radius \( r \) has the observed value unity at \( r = 8h^{-1}\text{Mpc} \) (Davis & Peebles 1983). The second requires that \( J_3(r) \), the integral of the second moment of the galaxy correlation function up to distance \( r \), has the observed value (Davis & Peebles 1983) \( 270 h^{-3} \text{Mpc}^{-3} \) at \( r = 10h^{-1}\text{Mpc} \).

These normalisation schemes both refer to the statistics of the clustering of luminous galaxies, not to the clustering of mass as described by the spectrum \( \mathcal{P} \). It is an important ingredient of the CDM cosmogony that these are not necessarily taken to be the same — so-called biased galaxy formation (Bardeen et al. 1986; Efstathiou 1990). To be specific the mass density correlation function \( \xi \), related to the spectrum by

\[
\xi(r) = \int_0^\infty \mathcal{P}(k) \frac{\sin kr}{kr} \frac{dk}{k}
\]  

is assumed to be related to the correlation function \( \xi_{gg} \) of luminous galaxies by

\[
\xi_{gg}(r) \simeq b^2 \xi(r)
\]

with a bias parameter \( b \) which is roughly independent of the scale \( r \). Including the bias parameter, the normalisations implied by the two schemes are

\[
\sigma_g^2(r)/b^2 = 9 \int_0^\infty \mathcal{P}_0(k) \left[ \frac{\sin kr}{kr^3} - \frac{\cos kr}{(kr)^2} \right]^2 \frac{dk}{k}
\]

\[
= (1/b)^2 \quad \text{at } r = 8h^{-1}\text{Mpc}
\]  

and

\[
2r^{-3}J_3(r)/b^2 = \int_0^\infty 2\mathcal{P}_0(k) \left( \frac{1}{kr} \right)^2 \left[ \frac{\sin kr}{kr} - \cos kr \right] \frac{dk}{k}
\]

\[
= (.73/b)^2 \quad \text{at } r = 10h^{-1}\text{Mpc}
\]

In each case, \( \mathcal{P}_0 \) is the linearly evolved spectrum, evaluated at the present epoch.

One sees that \( \sigma_g(r)/b \) is the dispersion of the linearly evolved density contrast with top hat filtering on the scale \( r \), whereas \( (2r^{-3}J_3(r))^{1/2}/b \) is its dispersion on the same scale with a different choice of filter. As noted earlier, the dispersion of a filtered quantity evolves linearly only as long as it is \( \lesssim 1 \), and this condition is only marginally satisfied by the quantities encountered in the two conventional normalisation schemes. The non-linear correction will be worse if \( b \simeq 1 \) than if \( b \simeq 2 \), and Couchman and Carlberg (1992) have claimed on the basis of numerical simulations that it is indeed significant in the former case. Evaluating the integrals, one finds that the \( J_3 \) normalisation is typically 10% lower than the normalisation using \( \sigma_g \), across the range of \( n \) in which we shall be interested. These are shown in figure 2. As remarked above, different choices of transfer function parametrisation can give a correction of around 10% as well.
In what follows we always adopt the $\sigma_g$ normalisation scheme, in the sense that normalisation of the spectrum is specified by quoting the quantity $b$ defined by Eq. (3.21). In words, $b^{-1}$ is defined as the present value of the dispersion of the linearly evolved matter density contrast, with top hat filtering on the scale $R_f = 8h^{-1}\text{Mpc}$. Only occasionally do we invoke the much stronger assumptions, that the galaxy correlation function is given by $\xi_{gg}(r) \simeq b^2 \xi(r)$ and that $\xi(r)$ is equal to the linearly evolved quantity on scales $r \simeq 10h^{-1}\text{Mpc}$. Our most significant results, summarised in figure 11 later in the paper, do not involve these assumptions.

Figure 3 shows $bP_{0}^{1/2}(k)$ for a selection of $n$ values, where as always $P$ is the linearly evolved quantity and the subscript zero indicates evaluation at the present epoch. The spectra for different $n$ typically cross at a scale of around $k^{-1} = 2\pi/32h^{-1}\text{Mpc}$. Figure 4 shows the corresponding dispersion $b\sigma(M)$ for $n = 1$ and $n = 0.7$, for both Gaussian and top hat filtering. The mass scale $M$ runs from $M = 10^6M_\odot$, the Jeans mass at decoupling (Peebles 1980, Kolb & Turner 1990) to $M = 10^{18}M_\odot$, the mass of the universe on the scale of the CfA survey. The comoving wavenumber $k^{-1}$ runs over the corresponding range of filtering radius $R_f$.

For $n = 1$ the spectrum $P(k)$ is practically flat on small scales. For $n < 1$ the power on small scales is reduced, though even for $n = 0$ not by as much as in the standard hot dark matter model (White, Frenk & Davis 1983) or with string-seeded hot dark matter (Albrecht & Stebbins 1992)). As a result the spectrum starts to fall on scales $k^{-1} \lesssim 1\text{Mpc}$. The implication of this change for small scale structure is discussed in Section 6.

4 Large Scales: Large Angle Microwave Background Anisotropies

Measurements of the cosmic microwave background anisotropy at large angular scales provide the only evidence we possess as to the amplitude of the spectrum on scales $\gtrsim 100h^{-1}\text{Mpc}$. Ignoring the possibility of reionisation, the surface of last scattering for the cosmic microwave background lies some $190h^{-1}\text{Mpc}$ inside the particle horizon (Hogan, Kaiser & Rees 1982), whose present distance is $2H_0^{-1} = 6000h^{-1}\text{Mpc}$. This means that an angular scale $\theta_0$ corresponds to a linear scale $r = 101\theta h^{-1}\text{Mpc}$.

For angular scales $\gg 1^\circ$, the linear scale is much bigger than the horizon size at last scattering. This means that the microwave anisotropy does not involve such awkward physical processes as Thomson scattering from moving electrons and the like, which vastly complicate matters when one reaches scales of a degree or less (Bond & Efstathiou 1987). It is in fact given by the well-known and understood Sachs-Wolfe effect (Sachs & Wolfe 1967), which corresponds simply to fluctuations in the gravitational potential across the microwave sky caused by the density fluctuations.

If re-ionisation occurs the distance of the surface of last scattering will be multiplied by some factor $\epsilon < 1$, and the Sachs-Wolfe formula will be valid only on scales...
Re-ionisation is not thought to be likely in the standard CDM model, due to the relatively late onset of nonlinear behaviour which is further exacerbated with power-law spectra. From now on the possibility of re-ionisation is ignored.

The only data we consider here are those from the COBE experiment (Smoot et al 1992), which to date provides the only positive detection of anisotropies. It corresponds to an angular resolution of order $10^0$, where one may safely ignore the possibility of re-ionisation. The corresponding linear resolution is of order $10^3h^{-1}$ Mpc. Experiments with smaller angular resolution have so far reported only upper limits, which are compatible with the COBE detection for the CDM model with $n < 1$.

Let us first review the pertinent results from COBE (Smoot et al 1992). The most useful and significant measurement made is that of the variance of the temperature fluctuations smoothed on a scale of $10^0$. The value quoted is $(1.1 \pm 0.2) \times 10^{-5}$. Like all the COBE results, the error limits quoted are 1-sigma. Of less theoretical importance is the measurement of the quadrupole anisotropy $Q_2$, which is also susceptible to experimental contamination from the galaxy in particular. This is quoted as $Q_2 = (5 \pm 1.5) \times 10^{-6}$.

COBE also have results on all intermediate scales, readily obtained by the appropriate smoothing of their basic data set. These are the data used to calculate the 1-sigma limits on the best fit primeval spectral slope $n = 1.1 \pm 0.5$. Unfortunately, this intermediate data has not been given in the paper, and so consequently we must work solely with the information given above and the slope limits.

Now we proceed to the formalism, which follows closely that of Scaramella and Vittorio (1990). [See also Abbott & Wise 1984a; Bond & Efstathiou 1987; Scaramella & Vittorio 1988; Efstathiou 1990, 1991.] For studies at large angles, the most convenient approach is the expansion of the temperature fluctuation field in spherical harmonics. Dropping the dipole term which is dominated by our peculiar velocity relative to the comoving rest frame, one writes

$$\frac{\Delta T}{T}(x, \theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{+l} a_l^m(x) Y_l^m(\theta, \phi)$$ (4.1)

where $\theta$ and $\phi$ are angles on the sky and $x$ is the observer position. With gaussian statistics for the original density field, the coefficients $a_l^m(x)$ are gaussian distributed random variables of position, with zero mean and with a rotationally invariant variance which depends only on $l$,

$$\langle a_l^m(x) \rangle = 0 \quad ; \quad \langle |a_l^m(x)|^2 \rangle = \Sigma_l^2$$ (4.2)

where the angle brackets are averages over all observer positions. It is important to appreciate however that what one can actually measure is only the temperature anisotropies at a single observer point, our own. Thus in practice one only gets a single realisation of the probability distribution functions given above. This leads to statistical uncertainties, as an individual realisation may not reflect the properties of
the ensemble averaged system. This effect, normally called the cosmic variance, is significant for observables which depend only on a limited number of the \( a_i^m \), and in particular for the quadrupole which only depends on the five \( a_2^m \) (two of which are redundant rotational information). This is the fundamental reason why the quadrupole measurement is not particularly useful for constraining theories. On the other hand, the variance at \( 10^0 \) depends on a significant number of the \( a_i^m \) and is far less susceptible to statistical vagaries.

The multipoles \( Q_l \) as seen from a particular observer point are defined from the variances of the individual \( a_i^m \) as

\[
Q_l^2 = \frac{1}{4\pi} \sum_{m=-l}^{+l} |a_i^m|^2
\]

In fact, the temperature autocorrelation function measured by a single observer

\[
C(\alpha) \equiv \langle \frac{\Delta T}{T}(\theta_1, \phi_1) \frac{\Delta T}{T}(\theta_2, \phi_2) \rangle_\alpha
\]

where the angle brackets indicate an average over all directions separated by an angle \( \alpha \), is given exactly in terms of the \( Q_l^2 \) via

\[
C(\alpha) = \sum_{l=2}^{\infty} Q_l^2 P_l(\cos \alpha)
\]

Note that our \( Q_l^2 \) have a different normalisation to that of Scaramella and Vittorio (1990). It is defined here so that \( \sqrt{Q_l^2} \) is exactly the quantity COBE measures.

Since the \( Q_l^2 \) are the sums of squared gaussian random variables, they are described by chi-squared distributions with the appropriate number of degrees of freedom; for the \( l \)-th multipole there are \( 2l+1 \) degrees of freedom, and in particular the quadrupole is described by a \( \chi^2_5 \) distribution.

The expectation of \( Q_l^2 \), again averaged over observer positions, is directly related to the variance \( \Sigma_l^2 \) of the \( a_i^m \), via

\[
\langle Q_l^2 \rangle = \frac{1}{4\pi} (2l + 1) \Sigma_l^2
\]

The values of the \( \Sigma_l^2 \) are readily calculated. On these large angular scales, the temperature anisotropies from the density perturbations are dominated by the Sachs-Wolfe effect (Sachs & Wolfe 1967). The appropriate formula for this curvature contribution is (Peebles 1982)

\[
\Sigma_l^2(\text{scalar}) = \pi \int_0^\infty \frac{dk}{k} j_l^2 (2k/aH) \delta_H^2
\]

where we write ‘scalar’ to distinguish from other contributions to the variances. Here \( j_l \) is the spherical Bessel function. With \( \delta_H^2(k) \propto k^{-1/n} \) this becomes

\[
\Sigma_l^2(\text{scalar}) = \frac{1}{8} [\frac{\sqrt{\pi}}{2} l(l+1) \frac{\Gamma((3-n)/2)}{\Gamma((4-n)/2)} \frac{\Gamma(l + (n-1)/2)}{\Gamma(l + (5-n)/2)}] \frac{2l+1}{l(l+1)} \delta_H^2(H_0/2)
\]
As discussed in section 2, there is in power-law and extended inflation models an additional contribution to the multipoles from gravitational waves. The contribution to the squared multipoles from gravitational waves is adequately given (for the range of $l$ values we will use) from the scalar contribution above just by

$$\Sigma_i^2(\text{grav}) = \frac{12}{p} \Sigma_i^2(\text{scalar})$$

so that

$$\Sigma_i^2(\text{total}) = \frac{15 - 13n}{3 - n} \Sigma_i^2(\text{scalar})$$

In contrast, natural inflation has a negligible gravitational wave contribution, even though it can give a tilted spectrum just like the power-law case.

### 4.1 The quadrupole anisotropy

As we shall see below, the quadrupole is not particularly useful for constraining the models, and we shall ultimately drop it in favour of the $10^0$ result. Nevertheless, it is worth examining the predictions of the model to see why we come to this conclusion.

The above equations readily allow us to calculate the mean of the predicted distribution function for $Q_2^2$. It is given in figure 5, again as a function of spectral slope, for both types of inflationary model. Because the spectral normalisation is uncertain by the bias factor, we have plotted the quantity $b^2 \langle Q_2^2 \rangle$. The COBE result of $2.5 \times 10^{-11}$ corresponds almost exactly to the mean prediction for the flat spectrum when the bias is one, whereas at $n = 0.6$ the predicted mean is nearly 12 times the COBE result even for the natural inflation scenario, seemingly requiring a bias of over three. However, that conclusion neglects the statistical nature of the quadrupole prediction.

As discussed above, the quadrupole prediction is not for a unique value but for a $\chi^2$ probability distribution function (pdf) with the given mean. For a given observer, the prediction is for a given realisation from that distribution. Let us discuss the predictions in terms of a quantity $q \equiv 10^{10} b^2 Q_2^2$. Then the mean value $\bar{q} = 10^{10} b^2 \langle Q_2^2 \rangle$, and the $\chi^2(q)$ pdf with that mean is

$$\chi^2(q) = \frac{1}{3 \sqrt{2\pi} (\bar{q}/5)^{5/2}} q^{3/2} \exp \left( -\frac{5q}{2\bar{q}} \right)$$

This distribution has a very broad spread (the variance being $2/5$ of the mean squared), and thus realisations of it may differ considerably from the mean. The spread is indicated in figure 5 by the vertical bars through the mean predictions.

Let us for the time being assume the COBE measurement to be perfect (ie not subject to observational errors). Then one can only exclude a given theory on the basis of some exclusion level, where the experimental result is far along the tail of the distribution. A conventional choice would be that 95% of the distribution should predict values above (or below) the experimental measurement. Thus one can exclude
at 95% confidence only those theories which predict a mean sufficiently high that 95% of the distribution is above the COBE value of $q_{\text{exp}} = 0.25 b^2$. This one can readily calculate via error functions, to discover that one can only say with 95% confidence that $\bar{q} < 1.1 b^2$, corresponding to $\sqrt{\langle Q^2 \rangle} < 1.05 \times 10^{-5} b$.

One can loosen this constraint even further by incorporating the fact that the COBE results possess experimental errors. One can model the COBE data with a probability distribution function describing the expected values were the experiment to be repeated — a sensible choice might be to assume $Q^2_{\text{exp}}$ to be gaussian distributed with mean $5 \times 10^{-6}$ and width $1.5 \times 10^{-6}$ — and use this to construct a pdf $p_{\text{exp}}(q_{\text{exp}})$ for the experimental measurement $q_{\text{exp}}$. One then tests each member of the experimental distribution for exclusion and takes a weighted mean, thus constructing a rejection functional on the predicted theoretical pdfs as

$$R = \int_0^{\infty} p_{\text{exp}}(q_{\text{exp}}) \left| 1 - 2 \int_0^{q_{\text{exp}}} \chi_5^2(q) dq \right| dq_{\text{exp}}$$

(4.12)

where $| \cdot |$ signals the modulus. Defined on one side of the probability distribution like this, a value $R > 0.9$ signals a 95% rejection of the theoretical pdf in the light of the modelled experimental data. With the experimental modelling as suggested above, then predicted mean values of $\bar{q}$ up to $2.3 b^2$ are allowed, corresponding to 95% confidence that $\sqrt{\langle Q^2 \rangle} < 1.5 \times 10^{-5} b$.

Hence one sees that the statistical uncertainties in the theoretical quadrupole prediction make it of little use in constraining theories. For example, with this last constraint $n = 0.6$ is allowed with a mild bias of 1.15.

4.2 The variance at $10^0$

Following the filter function formalism of Bond and Efstathiou (1987) (see also Efstathiou 1991), the mean (over all observer points) of the anisotropy seen by a given experiment is given by

$$\left< \left( \frac{\Delta T}{T} \right)^2 \right> = \frac{1}{2\pi} \sum_l (2l + 1) \Sigma_l^2 F_l$$

(4.13)

where $F_l$ is a filter function appropriate to a given experimental configuration.

For ground-based experiments which typically feature two and three beam configurations, this filter function can be rather complex. For the COBE experiment it is much simpler, as COBE reconstruct the smoothed fluctuation field across the entire sky and calculate the variance of that. The original COBE beam is well approximated by a gaussian with Full Width Half Maximum of $7^0$. However, before calculating the variance they smooth again by convolving with a further $7^0$ FWHM gaussian, a procedure equivalent to an original smoothing by a $10^0$ FWHM gaussian. Such a gaussian has a variance $\sigma^2 = (4.25^0)^2$. Thus the appropriate form of the filter
function is
\[ F_l = \frac{1}{2} \exp \left( - (4.25\pi l/180)^2 \right) \] (4.14)

It is now trivial to calculate the predicted $10^9$ variance as a function of $n$, and in figure 6 we plot its square root, henceforth denoted $\Delta T |_{10^9}$, multiplied by the bias, for both power-law and natural inflation. In the latter case, one sees that the results are essentially exactly linear. We have been unable to show why this should be analytically. The appropriate fitting function is
\[ \frac{\Delta T}{T} |_{10^9}(n) = \exp (2.62(1 - n)) \cdot \frac{\Delta T}{T} |_{10^9}(n = 1) \] (4.15)

and so for power-law inflation one has
\[ \frac{\Delta T}{T} |_{10^9}(n) = \sqrt{\frac{15 - 13n}{3 - n}} \exp (2.62(1 - n)) \cdot \frac{\Delta T}{T} |_{10^9}(n = 1) \] (4.16)

One can also readily calculate the cosmic variance, giving the spread in values about these means that would be measured by differently positioned observers. The variance of the $Q^2_l$ is $2\langle Q^2_l \rangle / (2l + 1)$. For the $10^9$ result, this is 10% at $n = 1$, rising to 12% at $n = 0.6$ (this result remains true whether or not there is a gravitational wave contribution to the anisotropies). This is a negligible correction to the larger COBE error in the present observations, but is ultimately a limiting obstacle to the conclusions one can draw on this large angular scale.

4.3 Microwave background constraints

In establishing constraints on the bias at fixed $n$, it is the $10^9$ data which are of primary interest. Some particular values worthy of note are that for the flat spectrum of $\Delta T |_{10^9} = 1.05 \times 10^{-5} / b$, and that for $n = 0.6$ of $\Delta T |_{10^9} = 5.17 \times 10^{-5} / b$ for power-law inflation and $\Delta T |_{10^9} = 2.99 \times 10^{-5} / b$ for natural inflation. These of course are to be compared with the COBE observations of $\Delta T |_{10^9} = (1.1 \pm 0.2) \times 10^{-5}$.

The mean quadrupole prediction is also very similar to the flat spectrum prediction with bias one, but the statistical uncertainties make this comparison rather less relevant. It is worth recalling that an extrapolation of the quadrupole from the COBE data at smaller angles, assuming a power-law spectrum, gives a somewhat larger prediction for the mean quadrupole of $16 \pm 4 \mu K$, corresponding to $(6 \pm 1.5) \times 10^{-6}$ (Smoot et al 1992). Note that this extrapolation estimates the mean quadrupole, not a specific realisation. Hence this again favours values of the bias not much exceeding 1, unless one inserts large scale power to boost the theoretical prediction for the mean.

Lacking full access to the COBE data, one must make an operational choice as to what to take as the observational limits. The COBE team provide fits of their data to power-law spectra, where both the slope and amplitude are treated as fitting
parameters. The two pertinent pieces of information are that the observed (root of the) variance at $10^0$ is $(1.1 \pm 0.2) \times 10^{-5}$, and the allowed slopes in the fit carried out by the COBE team are $n = 1.1 \pm 0.5$, where both errors are 1-sigma. These are of course not independent, as $10^0$ is at the lower end of the fitting range. Nevertheless, if one takes the upper 2-sigma value of the extrapolated quadrupole ($24 \mu K$), then this actually gives a prediction for the $10^0$ variance which is well above the 2-sigma limit on the $10^0$ data (for $n = 1$, the root of the mean $10^0$ variance is about 2.03 times the mean quadrupole, the relative factor falling as $n$ decreases). This conclusion (also implicit in Efstathiou, Bond and White (1992)), argues that the fits are not rigid enough to be of much use in constraining theories. Because of the cosmic variance and also experimental uncertainties, the quadrupole itself also appears of little use in this type of analysis. Consequently, we choose to adopt simply the $10^0$ result as it stands, without further incorporation of the COBE data. We relax their error bars to 2-sigma, and for many purposes we are only interested in the upper limit thus given of $1.5 \times 10^{-5}$. This is a particularly useful way of utilising the results, as it seems likely that this number can only go downwards if it is to avoid conflict with other experiments and so any limits quoted on its basis are likely only to become stronger with improved observations.

The full limits on the bias as a function of $n$ from this criterion, for both styles of inflationary model, are plotted in figure 11 at the end of section 5 along with other constraints derived in that section. Some sample results are that for $n = 1$ we have the obvious $0.7 < b < 1.5$, while for $n = 0.6$ we require $3.4 < b < 7.4$ (power-law inflation) and $2.0 < b < 4.3$ (natural inflation). Let us finally recall the uncertainties of normalisation. Were one to have used the $J_3$ spectral normalisation rather than $\sigma_8$, these bias values would be lowered by about 10%; the use of the Bardeen et al (1986) transfer function rather than the one we use would reduce the required bias by a further 10%, and a higher baryonic content raise it by a similar amount.

5 Intermediate Scales: Galaxy Clustering and Bulk Velocity Flows

Now we consider scales which are big enough that the filtered density contrast is still evolving linearly, yet small enough that there is information about the galaxy correlation function and about the peculiar velocity field. Thus we are discussing scales from around $10h^{-1}$ Mpc up to perhaps $100h^{-1}$ Mpc. The galaxy correlation function is sensitive to the combination $b\delta(M, \mathbf{x})$, whereas the peculiar velocity field is is sensitive to $\delta(M, \mathbf{x})$ itself.
5.1 The galaxy correlation function

There is ever increasing evidence that the amount of large-scale galaxy clustering is greater than can be accommodated in the standard CDM cosmogony. The most striking piece of evidence is provided by the measurement of the galaxy angular correlation function \( w(\theta) \) in the APM survey (Maddox et al. 1990, 1991), based on a sample of over two million galaxies. Further evidence pointing to the same conclusion has been provided by the ‘counts in cells’ of the QDOT survey (Saunders et al. 1991), and more recently by a survey of redshifts of APM galaxies (Dalton et al. 1992), and in the power spectrum inferred from the CfA survey (Vogeley et al. 1992) and that inferred from the Southern Sky Redshift Survey (Park, Gott & da Costa 1992).

Our earlier work with Will Sutherland (LLS) on structure from the power-law inflation model was primarily concerned with an investigation of whether or not this excess clustering could be intrinsic to the primeval spectrum, were the spectrum to be of power-law form. In this subsection we recap briefly on that analysis and its conclusions. The remaining results of that paper, concerning microwave anisotropies and the formation of nonlinear structure, are superceded by the other sections of this paper.

On the large angular scales where the discrepancy arises, the angular correlation function \( w(\theta) \) can readily be calculated in linear theory. It is given (Peebles 1980) by Limber’s equation as an integral over the galaxy correlation function \( \xi_{gg}(r) = b^2 \xi(r) \), where the mass correlation function \( \xi(r) \) is defined in section 3. A linear calculation is expected to be reliable on scales above about 2°. The results appear in figure 7, reproduced from LLS. The flat spectrum \( n = 1 \) falls well below the observational data, but one can see that as the spectrum is tilted, the extra power does indeed make itself evident in the clustering statistics. It is suspected that the APM survey may contain small residual systematics which bias the observational estimates upwards, so it seems reasonable to regard values of \( n \) from 0.3 to 0.6 as good fits to the excess clustering data. (This is actually rather more restrictive than the range allowed by Efstathiou, Bond and White (1992), as we see below.) We emphasise once more that this result is independent of the value of the bias parameter.

It has become a common reference point to consider spectra with low effective \( \Omega_h \) when considering clustering data. Figure 8 compares the processed spectra for \( n = 1 \), 0.6 and 0.5 with that of a CDM model with a flat spectrum but with \( \Omega_h \sim 0.2 \). Such a model has been much touted, especially by Efstathiou and collaborators (Efstathiou, Sutherland & Maddox 1991; Efstathiou 1991; Efstathiou, Bond & White 1992), as providing a good fit to all of the excess clustering data. One sees that the tilted power-law spectra resemble this spectrum much more closely than does standard CDM, though with some additional reduction in short scale power. The \( \Omega_h \sim 0.2 \) model has also been claimed as a good fit to the distribution of rich clusters (Lilje 1992; Scaramella 1992), to the CfA survey (Vogeley et al. 1992) and to the Southern Sky redshift survey (Park, Gott & da Costa 1992), all of which appear inconsistent with standard CDM. We thus expect tilted spectra to also do well on these criteria,
though we have not attempted a direct comparison here as we lack suitable data.

One can quantify this connection somewhat by utilising a quantity introduced by Wright et al (1992) called the excess power, which is a functional of the power spectrum defined as

\[ E[P] = 3.4 \frac{\sigma(25h^{-1}\text{Mpc})}{\sigma(8h^{-1}\text{Mpc})} \]  

For standard CDM and our transfer function, \( E = 0.95 \) (there is a very weak dependence on the choice of transfer function, which still plays a role on these scales). We find a reasonable fit is given simply by

\[ E[n] = 1.44 - n/2 \quad ; \quad 0.3 \leq n \leq 1 \]  

This can be related to the \( \Gamma \) parameter of Efstathiou, Bond and White (1992) as

\[ \Gamma = \frac{1}{2} \left( \frac{1.88}{2.88 - n} \right)^{10/3} \]  

The range of \( \Gamma \) these authors consider a reasonable fit to the APM data is \( 0.15 < \Gamma < 0.30 \), corresponding to \( 0.15 < n < 0.67 \), which we see is actually rather looser than the range we took above.

The relation between \( n \) and \( \Gamma \) provides an excellent means to compare results of different authors on intermediate length scales. It is worth remembering though that well away from the scales \( 8h^{-1} \) and \( 25h^{-1} \) Mpc used to define excess power the spectra do show significant differences, and so this relation is not as useful for considering the microwave background anisotropies, and even less so on short scales where the power-law spectra show considerably less power than their \( \Gamma \)-equivalents.

It would be invidious of us to close this subsection without reference to a huge body of literature which explains the excess clustering by a variety of means other than by adding intrinsic power to the primeval spectrum. One way in which this can be done is to modify the transfer function. Introducing a cosmological constant with \( \Omega_{\Lambda} \sim 0.8 \) provides a realisation of the \( \Gamma \sim 0.2 \) standard CDM model discussed above (Efstathiou, Sutherland & Maddox 1990; Efstathiou, Bond & White 1992), which remains viable though under threat from observations of quasar lensing (see eg Fukugita et al 1992). A 17 keV neutrino with a lifetime of the order of 10 years can also provide an appropriate modification of the transfer function (Bond & Efstathiou 1991), though the evidence for such an object from particle physics seems to be evaporating. Hybrid models combining both hot and cold dark matter in carefully chosen combination are also finding increasing favour (Schaefer & Shafi 1992; Davis, Summers & Schlegel 1992; Taylor & Rowan-Robinson 1992).

An alternative strategy is to modify the galaxy formation process itself. A particularly elegant proposal is to allow quasars to suppress local galaxy formation (Babul & White 1991), though this must now contend also with the COBE measurements implying a bias of around 1. Cooperative galaxy formation, a phenomenological (at
present) model in which galaxy formation is favoured in the neighbourhood of other galaxies, has also been proposed (Bower et al 1992), which can have the effect of a scale-dependent bias. Finally, it has been suggested that a detailed analysis of nonlinear evolution assuming a bias of around 1 (made more attractive by COBE) coupled with a velocity antibias shows that there is no conflict between the predictions of this model and the observed galaxy clustering (Couchman & Carlberg 1992).

5.2 Bulk peculiar velocities

There are many controversial aspects to the measurements of bulk velocity flows, but nonetheless they provide a useful measure of the absolute magnitude of the power spectrum on intermediate scales (for a recent review, see Kashlinsky & Jones (1991)). Quite detailed work on the bulk velocity flows has been carried out for the power-law spectrum (and also allowing $\Omega < 1$) by Tormen, Lucchin and Matarrese (1992) and by Tormen et al (1992), though our analysis will take a somewhat different and simplified approach.

Just as one can filter the density contrast, one can also filter the peculiar velocity field to obtain a bulk flow quantity, on any desired scale. As long the density contrast on that scale is in the linear regime, the bulk flow can be constructed from the linearly evolved Fourier components $v_k$, given by (Peebles 1980; Kolb & Turner 1990)

$$v_k = i\hat{k}\frac{aH}{k}\delta_k$$

where $\hat{k}$ is the unit vector in the $k$-direction.

Through the galaxy redshifts, the peculiar velocity field at the present epoch is observable within a sphere around us whose radius is a few hundred Mpc. Smearing over a scale $\sim 10h^{-1}$ Mpc, one obtains a bulk velocity field which is still evolving linearly. Taking the mass density contrast as being $b^{-1}$ times the galaxy number density it too can be observed, and inserting both quantities into Eq. (5.4) the value of $b$ can be estimated. This method of estimating $b$ has been extensively explored for the standard case $n = 1$. Until recently, applied for instance to the region around the Virgo supercluster, it seemed definitely to require a bias factor $b$ significantly bigger than 1 (Peebles 1980; Kolb & Turner 1990). With the advent of better measurements this is no longer the case, and a value $b = 1$ is apparently possible (Dekel 1991). Without direct access to the data it seems difficult to know how this result is affected if $n < 1$. The possibility of a velocity bias (Carlberg, Couchman & Thomas 1990; Couchman & Carlberg 1992) further complicates the analysis.

We base our comments here around the recent results concerning IRAS galaxies form the QDOT survey. Our analysis essentially mimics that of Efstathiou, Bond and White (1992). One gets two useful statistics from the QDOT survey. The first is that, in combination with bulk flow data, one can estimate directly the bias of IRAS galaxies. In general, the bias of the infra-red selected (and thus typically young)
IRAS galaxies will not be the same as that of the optically selected galaxies discussed thus far, and so we denote this bias by $b_I$. It is often stated that IRAS galaxies are somewhat less clustered than their optical counterparts; for example Saunders, Rowan-Robinson and Lawrence (1992) suggest $b_I = (0.69 \pm 0.09)b$ at the 1-sigma level. According to Taylor and Rowan-Robinson (1992), there are three independent dynamical estimates of $b_I$, giving respectively $b_I = 1.23 \pm 0.23, 1.16 \pm 0.21$ and $1.2 \pm 0.1$ where all errors are 1-sigma. One cannot statistically combine errors from the same data set, so we adopt as a result (which we assume has something like 2-sigma errors) that $b_I = 1.2 \pm 0.3$.

The second useful statistic is the dispersion of counts in cells in $30h^{-1}$ Mpc cubes, given as $0.46 \pm 0.07$ (Saunders, Rowan-Robinson & Lawrence 1992) where we have doubled the quoted error bars. Because we have a good estimate of the bias, we can immediately calculate the mass variance in these cubes, and then use the spectrum to calculate the variance at any other scale. In particular, one can scale down to $8h^{-1}$ Mpc and thus determine limits on the bias parameter $b$. Note that the use of the optical bias parameter is just a way of representing the results, without implying any reference either to optically selected galaxies or to scales on the point on nonlinearity. We note also that even if one did not know the variance of the QDOT counts in cells, one would already have strong limits from the knowledge that optical galaxies are not too much more clustered than IRAS ones.

The comparison between the bias limits obtained by this method and those obtained from the COBE measurements represent a particularly strong combination. In our earlier paper (Liddle & Lyth 1992) we pointed out that because of the strong gravitational contribution to the microwave anisotropies, this comparison rules out at least the simpler models of extended inflation because of the $n < 0.75$ bubbles constraint on these models. In figure 11 at the end of this section, these constraints appear together and highlight the allowed regions of the $n-b$ parameter plane.

Before finishing this section, we mention one other simple comparison which can be made — to the bulk flows in spheres as measured by POTENT. This approach considers the dispersion of $v = |v|$, given on each scale through the spectrum of $v$,

$$\mathcal{P}_v(k) = \left( \frac{aH}{k} \right)^2 \mathcal{P}(k)$$

(5.5)

Using this approach it is easy to consider the effect of $n$. Figure 9 illustrates the scaling of the dispersion with $R_f$ for a top hat filter. Smaller values of $n$ give significantly higher predictions, but the scaling with $R_f$ is roughly the same.

To compare this result with observation we use the velocity field reconstruction results from the POTENT method, pioneered by Bertschinger, Dekel and collaborators (Bertschinger & Dekel 1989; Dekel, Bertschinger & Faber 1990; Bertschinger et al 1990; Dekel 1991). It is well known (Kolb & Turner 1990) that typical theories, including standard hot and biased cold dark matter models and also models seeded by topological defects, tend to predict bulk velocities rather lower than those observed, particularly for standard CDM with high bias. Nevertheless, it is not trivial
to compare theory with observations, because one predicts only the \textit{rms} velocity averaged over the entire universe and the observed distribution may not be a fair sample. In particular, the possible contaminating effects of the great attractor (if backside infall can be unambiguously identified (Mathewson, Ford & Buchhorn 1992)), may distort observations. In the Monte Carlo work analysing bulk flows of Tormen \textit{et al} (1992), special criteria for choosing the observer points are employed \textit{before} statistical comparisons are made.

The POTENT results are obtained via a two-stage smoothing (Dekel 1991). First, the original data is smoothed with a gaussian of radius $12h^{-1}$ Mpc, and then a top hat of radius $R_f$ is used to provide the published data. In figure 10, we apply this two-stage smoothing to our spectra, which reduces the predictions, especially at short scales. These are compared with the POTENT data at different biases. It is seen that standard CDM at bias 1 performs rather well, significantly better than $n = 0.6$ at bias 2. Nonetheless, it appears that either model is still tenable given the statistical nature of the quantity being observed. In particular, the influence of large scales filters down significantly to small scales with this statistic, so an unexpectedly large fluctuation (and the great attractor would certainly be one in these CDM models) can easily move the whole observed data set to well above the mean, as predicted from the spectrum, that would be observed in a fair sample.

Some numerical analysis (based on Monte Carlo simulations) of bulk velocity flows with power-law spectra has been made by Tormen \textit{et al} (1992), who consider 18 models combining all possible combinations of $b = 1, 1.5, 2; n = 1, 0.5, 0; \Omega = 0.4, 1$. Of most interest to us is the $\Omega = 1, n = 0.5, b = 1.5$ model, which most closely represents the criteria we are establishing in this paper. They employ a maximum likelyhood test, and find that this model is twice as likely as the model with flat spectrum and bias 1.5, and is four times more likely than the flat spectrum with bias 1.

\section*{5.3 Summary of the intermediate/large scale results}

One of the key aims of this paper is to investigate the constraints in the $n$–$b$ parameter plane. The most significant constraints from large and intermediate scales are plotted in figure 11, and all the lines on this figure are to be taken as representing the 2-sigma range on either side of the (unplotted) mean value. The constraints plotted are

- The upper limit on $n$ (independent of $b$) obtained from requiring a fit to the APM data (dotted line). We have taken our constraint to be $n < 0.6$ (see figure 7), though we note that Efstathiou, Bond and White (1992) allow a range for their $\Gamma$ which is roughly equivalent to letting $n$ be as large as 0.67, which is very conservative.

- The limits on bias as a function of $n$ from the QDOT survey (dot–dashed lines), as discussed in section 5.2. In a sense these are the weak link in the diagram,
as although we have been very conservative in the errors on this observation it is not inconceivable that changes will be seen in the future. Nonetheless, we believe that these limits should be taken very seriously and may well strengthen.

- The COBE limits on bias as a function of $n$ for power-law and extended inflation (solid lines) and for natural inflation (dashed line), as discussed in section 4.3. These indicate the upper and lower 2-sigma range of the COBE $10^0$ data. These are very strong results, as it seems inevitable that if any change to the observations occurs it will be to lower the fluctuations and thus increase the required bias.

By themselves, the QDOT and COBE data allow a region of the $n$-$b$ plane which is roughly a triangle with corners \( \{n, b\} = \{0.7, 1.6\}, \{1, 1.6\} \) and \( \{1, 1\} \). Leaving aside for the moment the problematical galaxy clustering represented by the APM data, we ask in the next section whether data on smaller scales can discriminate between points within this triangle.

### 6 Small Scales: Galaxies and Galaxy Clusters

Now we study small scales $k^{-1} \lesssim 10h^{-1}$ Mpc, corresponding to $M \lesssim 10^{15} M_\odot$. This is the range on which gravitationally bound objects exist, in particular galaxies with mass $10^6 M_\odot \lesssim M \lesssim 10^{13} M_\odot$, and galaxy clusters (including the small ones known as groups) with mass $10^{13} M_\odot \lesssim M \lesssim 10^{15} M_\odot$.

In contrast with the treatment so far, we shall not attempt in this section to work at the 2-sigma level. The uncertainties in observational data, where they are considered at all, will be taken simply as those quoted in the literature.

For small scales, there are many quantities which one might hope to calculate and compare with observation. A partial list is the following.

- The redshift of formation of a given class of objects, such as bright galaxies or rich galaxy clusters.
- The number density $n(z, > M)$ of all gravitationally bound systems with mass bigger than $M$, which exist at redshift $z$.
- The mean virial velocity of a given class of objects.
- The number density $n(z, > v)$ of all gravitationally bound systems with virial velocity bigger than $v$, which exist at redshift $z$.
- The correlation function $\xi_{obj}(r)$ of a given class of objects. If it has the same shape as the correlation function $\xi(r)$ of the mass over scales of interest, it is useful to define a bias parameter by $\xi_{obj} = b^2_{obj} \xi$. 

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• The angular correlation \( w(\theta) \) of bright galaxies, on the angular scales \( \theta \lesssim 2^\circ \) which correspond to linear scales \( r \lesssim 10h^{-1}\) Mpc.

• The dispersion \( \sigma_v(r) \) of the relative velocity of a pair of galaxies separated by distance \( r \).

As galaxies and clusters are gravitationally bound objects, these quantities cannot be calculated using only the linear cosmological perturbation theory which was adequate for intermediate and large scales. Considerable insight can, however, still be gained from the linear theory if it is combined with order of magnitude estimates inspired by a spherical model of gravitational collapse. The general conclusion from this approach is that the \( n = 1 \) case is viable, with \( b \) somewhere in the range \( 1 \lesssim b \lesssim 2 \).

In the context of the present paper, we wish to ask whether the linear approach will also permit other values of \( n \) and \( b \), within the triangle allowed by the QDOT and COBE data.

In the \( n = 1 \) case, small scales have also been studied through non-linear calculations, namely \( N \)-body simulations. Most such calculations agree that the \( n = 1 \) case is viable on small scales (Davis et al 1985; Gott et al 1986; White et al 1987; Carlberg \& Couchman 1989; Frenk et al 1990; Carlberg et al 1990; Bertschinger \& Gelb 1991; Couchman \& Carlberg 1992), though some find that it has problems (Suto et al 1992, and other references sited there). A useful summary of the situation for \( n = 1 \) has been given by Davis et al (1992). The corresponding calculations for \( n < 1 \) are so far in their infancy (Vogeley et al 1992; Park et al 1992; Cen et al 1992), but so far they seem to suggest that a value \( n < 1 \) is preferred. We will not consider numerical simulations further in the present paper.

### 6.1 Gravitational collapse

The linear approach makes essential use of the filtered density contrast \( \delta(M, x) \). Following the usual practice for theoretical calculations, we use the Gaussian filter, defined by Eqs. (3.11)–(3.14). the dispersion of the filtered density contrast is thus

\[
\sigma(M) = \int_0^\infty P(k) \exp(-R_j^2k^2) \frac{dk}{k} \tag{6.1}
\]

At this point we need to distinguish between the density contrast of the cold dark matter, and that of the baryonic matter. They are the same on filtering scales \( M \) well in excess of \( 10^6M_\odot \). As \( M \) is reduced below this figure, the density contrast of the baryonic matter falls off sharply, because pressure forces prevent its growth until long after decoupling (Peebles 1980; Kolb \& Turner 1990). In contrast the cold dark matter has negligible pressure, and its density contrast grows like \( (1 + z)^{-1} \) just as it does on larger scales. This difference is presumed to account for the absence of luminous galaxies with mass below \( 10^6M_\odot \). In what follows, \( \delta(M, x) \) will always
denote the density contrast of the cold dark matter, though we are usually interested only in scales bigger than $10^6 M_\odot$.

As long as it is less than 1, the filtered density contrast evolves linearly except in those rare regions of space where it exceeds 1 in magnitude. In the 50% of these regions where it is positive, gravitational collapse takes place. At least initially, the collapsing regions have mass $\gtrsim M$, because the filtered density contrast does not ‘see’ structure on smaller scales.\footnote{As discussed later, this statement needs modifying when the collapsing regions are very rare.} In the approach that we are considering, gravitational collapse is modelled by taking the collapsing region be spherically symmetric. One then finds (Peebles 1980, Eqs. 19.50 and 19.53) that when a given mass shell stops expanding, the mean density inside it is a factor $9\pi^2/16$ times bigger than the mean density of the universe. If the matter inside the shell had evolved linearly (density contrast $\propto (1 + z)^{-1}$), its density contrast at that time would have been $\delta = (3/5)(3\pi/4)^{2/3} = 1.06$. After it has stopped expanding, the shell collapses. If spherical symmetry continues to hold, and one neglects pressure forces, the shell has collapsed to a point by the time that the age of the universe has increased by precisely a factor 2, corresponding to a mean density contrast within the linearly evolved shell of $\delta = (3/5)(3\pi/2)^{2/3} = 1.69$. Numerical studies (e.g. Peebles 1970) indicate that by about this time, pressure forces will in fact have virialised the random motion of the constituents of the object. After this initial virialisation, the object can lose energy (dissipation) which further increases its virial velocity.

The use of the spherical collapse model in the present context is somewhat problematical. First, filtering on a mass scale $M$ will distort the profile of a peak with mass of order $M$, making it lower and broader. Thus the linearly evolved filtered density contrast will not have precisely the behaviour described above, even for a spherical peak. The departure from sphericity of a linearly evolved collapsing region ‘seen’ by the filtered density contrast can be estimated from the linear theory. As we shall see it is quite significant, and that of the true (unfiltered) density contrast will be more so. Finally, any initial departure from sphericity is amplified during the collapse (Peebles 1980). Nevertheless, the fact that the peaks of the density contrast are at least roughly spherical (during the linear regime), as well as numerical simulations that have been done for the $n = 1$ case, indicate that two features of the spherical collapse model translate into reality.

- The regions with mass $\gtrsim M$ which have undergone gravitational collapse can be at least approximately identified with the regions where the linearly evolved density contrast $\delta(M, x)$ exceeds some threshold $\delta_c$.

- A collapsing region does not fragment into a large number of separate objects, which means that the mass of the resulting gravitationally bound systems is also $\gtrsim M$.

The optimal choice of the threshold $\delta_c$ is a matter of debate. Many authors take
the value 1.69 inspired by the above spherical collapse model. On the other hand, comparison of the linear estimate of $n(M)$ (described below) with the estimate from numerical simulations suggests a smaller value, Carlberg and Couchman (1989) advocating $\delta_c = 1.44$ and Efstathiou and Rees (1988) advocating $\delta_c = 1.33$ (but see also Brainerd and Villumsen (1992)). When making numerical estimates we set $\delta_c = 1.33$, while keeping in mind the effect of taking a bigger value.

As the collapsed regions $\delta(M, x) > \delta_c$ represent exceptionally large fluctuations of a Gaussian random field, there are powerful mathematical results concerning their stochastic properties. We shall use some of them in what follows, drawing extensively on the work of Bardeen et al (1986, henceforth BBKS).

### 6.2 The epoch of structure formation

On each mass scale, the linear regime ends at the epoch $z_{nl}(M)$ when the filtered linearly evolved dispersion $\sigma(z, M)$ of the density contrast is equal to 1. Since this quantity is proportional to $(1 + z)^{-1}$, the epoch is given by

$$1 + z_{nl}(M) = \sigma_0(M)$$

where the subscript 0 denotes the present value. At this epoch, the formerly rare regions where gravitational collapse is taking place on scales $\gtrsim M$ become common.

Before studying the linear regime which is our main concern, let us ask how this gravitational collapse process proceeds. In particular we want to know whether the bottom-up picture of structure formation, known to occur if $n = 1$, will also occur if $n < 1$.

At the epoch $z_{nl}(M)$ the collapsing regions, which have mass $\gtrsim M$, give rise to gravitationally bound structures with mass $\gtrsim M$ provided that they do not fragment into a large number of pieces. If $\sigma(M)$ is increasing significantly as the mass decreases, then at this same epoch the density contrast filtered on mass scales substantially bigger than $M$ is still evolving linearly in most parts of the universe. In that case, only a small fraction of the mass of the universe is bound into objects with mass much bigger than $M$. Moreover, $N$–body simulations indicate that fragmentation does not then occur, so that the objects formed at the epoch $z_{nl}(M)$ do not have mass much less than $M$ either. The result is a bottom-up scenario, in which gravitationally bound systems of successively bigger mass $M$ form at the successively later epochs $z_{nl}(M)$. After a system has formed, it may become part of a bigger gravitationally bound system, remaining a discrete object, but it may also merge with other systems.

The scale dependence of $\sigma(M)$ depends on that of the spectrum $P(k)$, on the corresponding scale $k^{-1} \sim R_f$. In the cosmologically interesting range $M > 10^6 M_\odot$, these scale dependences were already shown in figures 3 and 4 both for the standard choice $n = 1$ and for some smaller values. One sees that for $n = 1$, $\sigma(M)$ increases steadily as $M$ decreases, corresponding to the fact that $P(k)$ does not turn over as $k^{-1}$ decreases. In contrast, for $n < 1$ the spectrum $P(k)$ turns over below some
mass scale, so that $\sigma(M)$ levels out. This difference in the small scale behaviour arises from the small scale behaviour $\mathcal{P}(k) \propto T^2(k) k^{13+n}$. According to the parametrisation Eq. (3.4), $T^2(k)$ is proportional to $k^{-4}$ which makes $\mathcal{P}(k) \propto k^{n-1}$. This means that $\sigma(M)$ increases logarithmically as $M$ decreases if $n = 1$, whereas if $n < 1$ it approaches the unfiltered value $\sigma$, the difference going to zero like $k^{n-1}$. (Strictly speaking the above statements are modified by logarithmic factors because $T^2$ goes like $k^{-4} \log^2 k$ (BBKS), but they do not change the qualitative behaviour.)

The small scale behaviour described in the last paragraph is expected to persist down to some very small scale, below which the spectrum cuts off sharply and $\sigma(M)$ becomes practically equal to the unfiltered quantity $\sigma$. This scale is called the coherence scale and depends on the nature of the cold dark matter. If the cold dark matter consists of subnuclear particles, the coherence scale is determined by their interactions. For instance, if it consists of the lightest supersymmetric particle the coherence scale is the Hubble scale at the epoch when the particle becomes non-relativistic. If it consists of the axion, the coherence scale is the Hubble scale at the epoch when the axion acquires mass through QCD effects.

On the basis of these considerations, we arrive at the following picture of structure formation, which covers the case $n < 1$ as well as the familiar case $n = 1$. The first structure forms at some epoch given by

$$1 + z_{nl} = \sigma_0$$

(6.3)

where $\sigma_0$ is the unfiltered quantity. For $n = 1$, this epoch is very early and depends on the nature of the cold dark matter. Decreasing $n$ makes the epoch later, and to an increasingly good approximation allows it to be calculated without knowing the nature of the cold dark matter. After this epoch, structure forms according to the bottom-up picture.

Let us estimate the maximum mass $M_{\text{max}}$, above which structures collapsing at a given epoch $z_{nl}(M)$ are rare. It should be such that $\sigma_0(M_{\text{max}})$ is significantly less than 1, so that the density contrast filtered on the scale $M_{\text{max}}$ is still evolving linearly almost everywhere in the universe. To be definite, let us require that it is still evolving linearly in 90% of the volume of the universe. From the Gaussian distribution, this corresponds to $\sigma_0(M_{\text{max}}) = 1/1.63 = .61$. For $M = 10^6 M_\odot$ this criterion gives $M_{\text{max}} = 10^9 M_\odot$ ($10^{10} M_\odot$) for $n = 1 (.7)$. For this value of $M$, the epoch $z_{nl}(M)$ is a slowly varying function of $M$, so the conclusion is that a broad band of masses downwards of $M = 10^9 M_\odot$ to $10^{10} M_\odot$ collapse at the same time, around the epoch $z_{nl}(10^6 M_\odot)$. For $M = 10^{12} M_\odot$, the criterion gives $M_{\text{max}} = 2 \times 10^{13} M_\odot$, more or less independent of $n$. For this value of $M$, the epoch $z_{nl}(M)$ is a fairly rapidly varying function, so the conclusion is that a fairly narrow band of masses $M = 10^{12} M_\odot$ to about $10^{13} M_\odot$ collapse around the epoch $z_{nl}(10^{12} M_\odot)$.

We conclude that the bottom-up picture is more or less the same for any $n$ in the range $.7 < n < 1$. A broad range of masses below about $10^9 M_\odot$ collapses at about the same time, but subsequent collapse takes place in an increasingly narrow mass
The epoch $z_{nl}(M)$ can be read off figure 4, for $n = 1$ and $n = .7$. It becomes more recent as $M$ or $n$ are reduced. Let us consider three representative values of $M$. First take $M = 10^6 M_\odot$. Then $z_{nl}(M)$ is quite early, $1 + z_{nl}(M) = 18/b (9/b)$ for $n = 1 (.7)$. A broad range of mass scales downwards of about $10^9 M_\odot$ collapse at around this epoch. Second, take $M = 10^{12} M_\odot$. Then $1 + z_{nl}(M) = 4.5/b (3.7/b)$ for $n = 1 (.7)$. This is the epoch when a significant fraction of the mass of the universe collapses into objects with mass of order $10^{12} M_\odot$, the mass of large galaxies. However, the favoured explanation of a bias factor $b > 1$ is that luminous galaxies originate from exceptionally high peaks of the evolved density contrast, and therefore form well before the epoch $z_{nl}(M)$ (BBKS; Efstathiou 1990). Finally consider $M = 10^{15} M_\odot$. Then, $1 + z_{nl}(M) = .82/b$, almost independent of $n$. This means that the epoch $z_{nl}(M)$, at which a significant fraction of the mass collapses into large galaxy clusters, lies in the future. The prediction is therefore that the observed large clusters originated from exceptionally high peaks of the density contrast, which again implies a bias factor for these objects (Kaiser 1984). The bias factor for galaxies and galaxy clusters will be discussed quantitatively below.

Finally, we consider a different criterion which is sometimes proposed for the epoch of structure formation, namely that it is the epoch when the linearly evolved spectrum $P(k)$ is equal to 1 on the corresponding scale $k^{-1} = R_f$. This is practically equivalent to the criterion $\sigma(M) = 1$ if $P(k)$ is significantly increasing, say like $k^m$ with $m \sim 1$, because then the filtered spectrum is $P(k) \exp(-k^2 R_f^2) \simeq P(R_f^{-1}) \delta(\log(kR_f))$, where $\delta$ denotes the Dirac delta function. The condition for this to be the case is more or less the same as the condition for bottom-up structure formation to occur. It fails, in particular, when $k$ corresponds to the maximum of $P$ which occurs when $n < 1$, so it is not true that in that case the first structure forms at the epoch when $1 + z$ is equal to the maximum value of $P(k)$, as was incorrectly stated in LLS.

### 6.3 The statistics of the collapsed regions

Now we study the stochastic properties of the collapsed regions, defined as regions where the linearly evolved density contrast $\delta(M,x)$ exceeds a threshold $\delta_c$. In them, $\delta(M,x)$ is more than $\nu$ standard deviations above zero, where

$$\nu(z, M) = \frac{\delta_c}{\sigma(z, M)} \stackrel{(6.4)}{=} \frac{\delta_c(1 + z)}{\sigma_0(M)} \stackrel{(6.5)}{=} \frac{1 + z}{1 + z_{nl}(M)} \stackrel{(6.6)}{=} \frac{1 + z}{1 + \frac{1}{b}}$$

We are working in the linear regime, corresponding to $\nu > \delta_c > 1$.

The collapsed regions occupy a volume fraction $V$ given by the Gaussian distri-
distribution, 
\[ \frac{dV}{d\nu} = \frac{1}{\sqrt{2\pi}} e^{-\nu^2/2} \]  
(6.7)
leading to
\[ V(\nu) = \text{erfc}(\nu/\sqrt{2})/2 \] 
(6.8)
\[ = (2\pi)^{-1/2} \nu^{-1} e^{-\nu^2/2} (1 - \nu^{-2} + O(\nu^{-4})) \]  
(6.9)
For \( \nu = 1, 2, 3, 4 \) the volume fraction is \( V = .16, .023, .0013, .000031 \). In practice one is not interested in values \( \nu \gtrsim 4 \), because the collapsed regions are then too rare to be physically significant. The corresponding mass fraction is about \( (1 + \delta_c) = 2 \) to \( 3 \) times bigger than the volume fraction.

To say more one needs to know the shape of the spectrum. We shall list the relevant results given by BBKS. They involve only two moments of the spectrum, defined by
\[ \langle k^2(M) \rangle = \sigma^{-2} \int_0^\infty k^2 \exp(-k^2 R_f^2) \mathcal{P}(k) \frac{dk}{k} \]  
(6.10)
\[ \langle k^4(M) \rangle = \sigma^{-2} \int_0^\infty k^4 \exp(-k^2 R_f^2) \mathcal{P}(k) \frac{dk}{k} \]  
(6.11)
The quantity \( \langle k^2 \rangle \) is the mean of the \( \nabla^2 \) operator, i.e., of the quantity \( \delta^{-1} \nabla^2 \delta \). Similarly, \( \langle k^4 \rangle \) is the mean of \( \nabla^4 \).

A relevant length scale is defined by \( R_*^2 = 3 \langle k^2 \rangle / \langle k^4 \rangle \). For any spectral index \( n > -1 \), it is easy to show that in the limit of small filtering scale \( R_f \),
\[ \frac{R_*}{R_f} = \left( \frac{6}{1+n} \right)^{1/2} \]  
(6.12)
For the case of CDM with \( .7 < n < 1 \), the ratio is in the range 1 to 3 for the entire range of cosmologically interesting masses.

Another relevant length scale is \( \langle k^2 \rangle^{-1/2} \). On large filtering scales, such that \( \mathcal{P}(k) \) is increasing fairly strongly at \( k^{-1} \simeq R_f \), the ratio \( \langle k^2 \rangle^{-1/2} / R_f \) is close to 1. As the scale is reduced it increases, but is \( \lesssim 10 \) in the cosmologically interesting range \( M > 10^6 M_\odot \).

Finally, it is convenient to define the dimensionless parameter
\[ \gamma(M) = \langle k^2 \rangle / \langle k^4 \rangle^{1/2} \]  
(6.13)
It falls from about \( .7 \) to about \( .3 \) as \( M \) decreases from \( 10^{15} M_\odot \) to \( 10^6 M_\odot \), for \( .7 < n < 1 \).

For sufficiently large \( \nu \), each collapsed region is a sphere surrounding a single peak of \( \delta \). However, the departure from sphericity is considerable in the cosmologically
interesting regime. BBKS show that a quantity $x^{-1}$, which is roughly the fractional departure from sphericity, is well approximated by

$$x = \gamma \nu + \theta(\gamma, \gamma \nu)$$  \hspace{1cm} (6.14)

where

$$\theta(\gamma, \gamma \nu) = \frac{3(1 - \gamma^2) + (1.216 - .9\gamma^4) \exp[-\gamma/2(\gamma \nu/2)^2]}{[3(1 - \gamma^2) + .45 + (\gamma \nu/2)^2]^1/2 + \gamma \nu/2}$$  \hspace{1cm} (6.15)

The asphericity is plotted in figure 12 for $M = 10^8 M_\odot$ and $10^{14} M_\odot$, for both $n = 1$ and $n = .7$, and is seen to be $\gtrsim .3$ even at $\nu = 4$ and $10^{14} M_\odot$. We emphasise that this is the asphericity seen in the linearly evolved, filtered density contrast. The asphericity in the true, unfiltered density contrast will be bigger, and will increase during collapse.

Three useful number densities are given by BBKS. First, the density $n_\chi$ of the Euler number of the surfaces bounding the collapsed regions is

$$\frac{1}{2} n_\chi(\nu, <k^2>) = \frac{\langle (k^2)/3 \rangle^{3/2}}{(2\pi)^2} (\nu^2 - 1) e^{-\nu^2/2}$$  \hspace{1cm} (6.16)

Second, the number density of upcrossing points on these surfaces is

$$n_{up}(\nu, <k^2>, \gamma) = \frac{\langle (k^2)/3 \rangle^{3/2}}{(2\pi)^2} \left[\nu^2 - 1 + \frac{4\sqrt{3}}{5\gamma^2(1 - 5\gamma^2/9)^{1/2}} \exp(-5\gamma^2 \nu^2/18)\right] e^{-\nu^2/2}$$  \hspace{1cm} (6.17)

An upcrossing point on a surface of constant $\delta$ is defined as one where $\nabla \delta$ points along some chosen direction.

The third number density is $n_{peak}$, the number density of peaks which are more than $\nu$ standard deviations high. BBKS give expressions for $n_{peak}$, but they point out also that in the cosmologically interesting regime it is quite well approximated by $n_{up}$. We shall use this approximation in what follows. It suggests that if a collapsed region contains several peaks, they are not buried deep inside it; rather, the boundary of the region is presumably corrugated, wrapping itself partially around each peak.

In the limit $\nu \gamma \gg 1$, $n_{peak} = n_{up} = \frac{1}{2} n_\chi$. This is in accordance with the fact that in this regime, each surface is a sphere surrounding a single peak. It contributes +1 to the number of upcrossing points, and +2 to the Euler number. As $\nu$ decreases the surface becomes deformed, but at first its contributions to $n_{up}$ and $n_\chi$ are not affected. Eventually though, it may become so corrugated that it has more than one upcrossing point, and may become a torus so that its Euler number is less than 2 (of course it then has more than one upcrossing point whatever its shape). As a result, its contribution to $n_\chi/2$ becomes less than its contribution to $n_{up}$. The ratio $2n_{peak}/n_\chi$ (approximated as $2n_{up}/n_\chi$) is plotted in figure 13. (We are not interested in the regime $\nu < 1$, but for the record $n_\chi$ is negative there, indicating that the surfaces $\delta(M, x) = \nu \sigma$ definitely do not have spherical topology; in fact they percolate leading to a sponge–like topology for the regions $\delta(M, x) > \delta_c$ (Melott 1990).)
One would like to know the number density $n_{\text{coll}}$ of the collapsed regions. It satisfies the inequality $n_{\chi}/2 < n_{\text{coll}} < n_{\text{peak}}$, and so is equal to $n_{\chi}/2$ in the limit $\nu\gamma \gg 1$. According to figure 13, this defines it within a factor 2 for $\nu > 3$. The average number $N$ of peaks per collapsed region satisfies $1 < N < 2n_{\text{peak}}/n_{\chi}$, so according to figure 13 it is no bigger than 2 for $\nu \gtrsim 3$.

### 6.4 The mass of the collapsed regions

As the filtered density contrast does not contain structure on scales much less than the filtering scale $R_f$, we expect the average radius of a peak to be $\gtrsim R_f$. It can be estimated from the number density of peaks with arbitrary height $n_{\text{peak}}(-\infty)$, which is equal to $0.016R_\ast^{-3}$ where the length $R_\ast$ was defined at the beginning of the last subsection. This is also the number density of troughs, so a rough estimate of the average radius of a peak or trough at half height is $(2n_{\text{peak}}(-\infty))^{-1/3}/4 = 0.8R_\ast$. As we discussed earlier, $R_\ast/R_f \simeq 1$ to 3 for all cases of interest, so the average peak size is roughly of order $R_f$. Since there are few peaks with radius less than $R_f$, this suggests that the probability distribution of peak sizes is fairly narrow, most peaks having a radius around the average.

We would like to compare the average peak radius $R_f$ with the average radius of a collapsed region. Equivalently, we would like to compare the filtering volume $V_f$ with the average volume of a collapsed region. The latter is equal to the volume fraction occupied by the collapsed regions, Eq. (6.9), divided by their number density $n_{\text{coll}}$. In general we do not know $n_{\text{coll}}$, but we do know the peak number density $n_{\text{peak}}$, so we can calculated the average volume $V_{\text{peak}}$ per peak of a collapsed region. It is useful to give the result as a fraction to the filtering volume $V_f$,

$$V_{\text{peak}}/V_f = \frac{1}{2}\text{erfc}(\nu/\sqrt{2})/(n_{\text{peak}}V_f) \quad (6.18)$$

The corresponding mass fraction is roughly $M_{\text{peak}}/M = (1 + \delta_{c})V_{\text{peak}}/V_f$. The volume fraction is plotted in figure 14, and one sees that except for large $\nu$ it is bigger than 1.

At large $\nu$, the ratio may be calculated using the approximation $n_{\text{up}} = n_{\chi}/2$ together with Eq. (6.9),

$$V_{\text{peak}}/V_f \simeq (2\pi)^{3/2}((k^2)/3)^{-3/2}\nu^{-3}/V_f \quad (6.19)$$

$$= \left(\frac{3}{R_f^2(k^2)^{1/2}}\right)^{3/2}\nu^{-3} \quad (6.20)$$

The fact that it is small indicates that a typical collapsed region is sitting on top of a peak, as opposed to the situation for smaller $\nu$ where it encompasses most of the peak. This is natural, because for large $\nu$ a collapsed region is exceptionally high.
6.5 The number density $n(> M)$

The main application of these results is to estimate the number density $n(> M)$ of gravitationally bound systems with mass bigger than $M$, at a given epoch before $z_{nl}(M)$. The systems are supposed to be identifiable by looking at the linearly evolved density contrast $\delta(M, \mathbf{x})$. Each collapsed region, defined as one in which $\delta(M, \mathbf{x}) > \delta_c$, is supposed to contain one or more systems with mass bigger than $M$. As we have just seen this cannot be correct for large $\nu$, but let us accept it for the moment.

If each collapsed region is identified with a single system, then $n(> M) = n_{\text{coll}}$. In general this recipe is useless for lack of an expression for $n_{\text{coll}}$. A different prescription, which does lead to a calculable expression, is to identify each peak within a collapsed region with a different collapsed object,

$$n(> M) = n_{\text{peak}} \approx n_{\text{up}}$$

This estimate (usually without the simplifying second equality) is widely used in the literature. It is certainly the same as the estimate $n(> M) = n_{\text{coll}}$ for large $\nu$, where we know that there is just one peak per collapsed region. To what extent the prescriptions are the same for lower $\nu$ is not known, because the number of peaks per collapsed region is not known.

If at some epoch the linearly evolved density contrast does have many peaks within a collapsed region, an interesting situation arises. At a somewhat earlier epoch, $\delta(M, \mathbf{x})$ was smaller, and a separate contour $\delta(M, \mathbf{x}) = \delta_c$ was wrapped around each peak. In other words, each peak of the linearly evolved density contrast, filtered on scale $M$, was inside a single collapsed region, and presumably represented a separate gravitationally bound system. At the later epoch when the collapsed region encompasses many peaks of the linearly evolved density contrast, we have a bigger gravitationally bound system. If the original systems survive, the identification of each peak with a separate system is correct, but it misses the larger system which contains the original systems. Of course, missing this one system does not affect the total count much, so if this case is typical of collapsed regions containing many peaks the estimate $n(> M) = n_{\text{peak}}$ is better than the estimate $n(> M) = n_{\text{coll}}$. If, on the other hand, the original systems have merged, that identification is wrong, and the whole of the collapsed region should be identified with just one gravitationally bound system. If this case is typical, the estimate $n(> M) = n_{\text{coll}}$ would be better, if only we had a formula for it. Which case is the more likely? A clue is provided by the observation made earlier, that if there are several peaks in a collapsed region they typically seem to lie near the surface of the region, a part of the surface wrapping itself around each peak. This picture would suggest that the estimate $n(> M) = n_{\text{peak}}$ is the more reasonable, the peaks of the linearly evolved density contrast in a typical collapsed region representing structures which have not existed long enough to merge.

A different prescription was used by Press and Schechter (1974), to derive a widely
used alternative formula. They worked with the differential number density,

\[
dn/dM = d/dM n(> M)
\]

(6.22)

At a given epoch, if the filtering mass \( M \) is increased by an amount \( dM \) then \( \nu \) is increased by an amount \( d\nu \), and the volume fraction occupied by the collapsed regions is reduced by an amount \( dV \) given by Eq. (6.7). Press and Schechter suppose that the eliminated volume consists of objects with mass between \( M \) and \( M + dM \), corresponding to the idealisation that filtering the density contrast on any mass scale \( M \) cuts out precisely those objects with mass less than \( M \) while leaving unaffected objects with mass bigger than \( M \). Ignoring the overdensity \( \approx (1 + \delta_c) \) of the collapsed regions this implies that the number density \( dn \) of such objects is given by

\[
M dn/dM = \left[ M d(R_f^2) / dM \right] \left[ d(\sigma^2(M)) / d\sigma^2(M) \right] \nu dV / dV dn
\]

(6.23)

\[
= \left[ -\sigma^2(M) \langle k^2 \rangle \right] \left[ -\nu / 2\sigma^2(M) \right] \left[ 1 / \sqrt{2\pi} e^{-\nu^2/2} \right] \left[ 1 / V_f \right]
\]

(6.24)

\[
= R_f^2 \langle k^2 \rangle \left( 2R_f^2 / 3 \right) \left( 1 / 4\pi^2 R_f^2 \right) \nu e^{-\nu^2/2}
\]

(6.25)

Press and Schechter multiplied this formula by a factor 2, so that when integrated over all masses it would give the total mass density of the universe, rather than just the half corresponding to the regions of space where the linearly evolved density contrast is positive. They thus arrived at the estimate

\[
n(> M) \approx n_{ps} \equiv \int_M^\infty \langle k^2 \rangle' R_f' \nu' e^{-\nu'^2/2} dM' / M'
\]

(6.26)

In this equation, \( R_f' = R_f(M') \), and similarly for \( < k^2 >' \) and \( \nu' \). The factor 2 inserted by Press and Schechter is not justified by their argument, because the linearly evolved density contrast has nothing to do with reality in the non-linear regime \( \sigma(M) > 1 \). On the other hand, the neglected overdensity gives a factor \( \approx (1 + \delta_c) = 2 \) to 3. Thus the factor 2 goes in the right direction, and the Press-Schechter formula is reasonably well founded theoretically.

From Eq. (6.4),

\[
\nu' / \nu = \sigma(M) / \sigma(M')
\]

(6.27)

The right hand side is independent of \( b, \delta_c \) and \( z \), so it follows that \( n_{ps} \), like \( n_{peak} \), depends on these quantities through \( \nu \), which involves the combination \( b\delta_c(1 + z) \). In figure 15 is plotted the ratio of the two alternative estimates \( n(> M) = n_{ps} \) and \( n(> M) = n_{peak} \), for \( M = 10^{10} M_\odot \) and for \( M = 10^{15} M_\odot \). One sees that the estimates agree to better than a factor 2 for \( \nu \lesssim 2 \). Presumably, this indicates that in this regime
the assumptions underlying the two estimates are compatible, in that increasing $M$ by a small amount cuts out portions of the collapsed regions which have mass of order $M$ and are centred on peaks with height of order $\nu(M)$.

For large $\nu$, the Press-Schechter estimate falls below $n_{\text{peak}}$. This can be understood analytically, from the expression

$$\frac{d n_{\text{peak}}}{dM} \sim \frac{d n_{\text{up}}}{dM} = \frac{\partial n_{\text{up}}}{\partial \nu} \frac{d \nu}{dM} + \frac{\partial n_{\text{up}}}{\partial \langle k^2 \rangle} \frac{d \langle k^2 \rangle}{dM} + \frac{\partial n_{\text{up}}}{\partial \gamma} \frac{d \gamma}{dM}$$

(6.28)

The first term dominates for large $\nu$, leading to the ratio

$$\frac{d n_{\text{ps}}/dM}{d n_{\text{peak}}/dM} = 2 \left( \frac{3}{R^2 \langle k^2 \rangle} \right)^{3/2} \nu^{-3}$$

(6.29)

Apart from the factor 2, this is just the filter volume divided by the average volume of a collapsed region (Eq. (6.20)).

Both estimates are too big when $\nu$ is big. The estimate $n_{\text{peak}}(\simeq n_{\text{coll}})$ is too big because it counts many ‘collapsed regions’ which have mass less than the filtering mass and do not therefore contribute to $n(>M)$. To obtain an accurate estimate, one would have to excise such regions before evaluating $n_{\text{peak}}$. The Press-Schechter estimate is too big because it assumes that all of the reduction in volume of the collapsed regions in going from $M$ to $M + \delta M$ corresponds to structures with mass in this range, whereas some of the eliminated volume will belong to collapsed regions with mass less than $M$. To obtain an accurate estimate, one would again have to excise these regions before calculating the eliminated volume and hence obtaining the Press-Schechter estimate. Neither of these effects can be quantified analytically, because relevant statistical results are not known. The fact that the ratio Eq. (6.29) is 2 times the ratio of the filtering volume to the average volume of a collapsed region presumably indicates that the regions which ought to be excised dominate both $n_{\text{peak}}$ and the reduction in volume leading to the Press-Schechter estimate. The implication is that even the latter is substantially too high at large $\nu$.

In what follows we focus the estimate $n(>M) = n_{\text{peak}}(\simeq n_{\text{up}})$ because of its simplicity, but give some results also for $n_{\text{ps}}$.

### 6.6 Bias factors

Before comparing the number densities with observation, we need one more result from the analysis of BBKS, namely an estimate of the galaxy and galaxy cluster bias factors at the present epoch. According to the CDM cosmogony such a bias factor will be present for any class of objects, if they form at an epoch when $\nu$ is significantly bigger than 1 and therefore originate as exceptionally high peaks of the density contrast. In general the bias occurs partly through the excess clustering of
these peaks, and partly through additional non-linear clustering after the objects have formed. An estimate including both effects is

\[ b_{\text{obj}} = 1 + \frac{\tilde{\nu}}{\sigma(M)} \]  

(6.30)

where

\[ \tilde{\nu} = \nu - \frac{\gamma \theta(\gamma, \gamma \nu)}{1 - \gamma^2} \]  

(6.31)

In these expressions, everything on the right hand side is to be evaluated at the epoch when the objects form.

6.7 Comparison with observation

The prediction \( n(> M) = n_{\text{peak}} \) is plotted against redshift in figure 16, for masses \( M = 10^{15} M_\odot, 10^{12} M_\odot, 10^{10} M_\odot \) and \( 10^8 M_\odot \). For each case, three curves are given corresponding to the parameter choices \( \{ n, b \} = \{ 1, 1 \}, \{ 1, 1.6 \} \) and \( \{ .7, 1.6 \} \). Each curve ends at the epoch \( z_{\text{nl}}(M) \), when the linear approach ceases to be valid.

Let us consider first the case \( M = 10^{15} M_\odot \), which corresponds to very large galaxy clusters. Since this mass corresponds to the normalisation scale \( R_f \sim 10 h^{-1} \text{Mpc} \) there is little dependence on \( n \) in the range \( .7 < n < 1 \). There is however strong dependence on \( b \), and several authors (Kaiser 1984; Bardeen et al 1986; Bardeen et al 1987; Dalton et al 1992; Nichol et al 1992; Efstathiou, Bond and White 1992; Adams et al 1992) have claimed that \( b \sim 1.6 \) is preferred over \( b = 1 \). Let us first reproduce this result, then comment on its uncertainty.

Since \( \sigma(10^{15} M_\odot) = .8/b \), we assume that the filtered density contrast on this scale is still evolving linearly at the present epoch. The quantities of interest are the number density \( n(> M) \) and the ratio \( b_c/b \) of the cluster bias factor to the galaxy bias factor. The prediction for \( n(> M) \) is \( 5.0 \times 10^{-6} \text{Mpc}^{-3} \) if \( b = 1 \), and \( 1.0 \times 10^{-6} \text{Mpc}^{-3} \) if \( b = 1.6 \). Following for instance BBKS and Bardeen et al (1987), we identify clusters in this mass range with Abell clusters of richness class \( > 1 \), and hence observed number density \( 7.5 \times 10^{-7} \text{Mpc}^{-3} \) (Bahcall & Soniera 1983). Thus, the prediction for \( n(> M) \) is about right if \( b = 1.6 \), but too big if \( b = 1 \). Coming to \( b_c/b \), one finds from Eq. (6.30) that if \( b = 1.6 \), then \( b_c = 4.0 \) giving \( b_c/b = 2.5 \). If, on the other hand, \( b = 1.0 \), then \( b_c = 1.5 \) which gives \( b_c/b = 1.5 \). Recent estimates (Dalton et al 1992; Nichol et al 1992) give \( \xi_{cc}(r) = (r_0/r)^{1.9 \pm .3} \) with \( r_0 = (13 \pm 5) h^{-1} \text{Mpc} \). Dividing by the galaxy correlation function \( \xi_{gg} = (5 h^{-1} \text{Mpc}/r)^{1.8} \) gives therefore \( b_c/b_g = 2.4 \pm .9 \) as an observational estimate. Again, \( b = 1.6 \) is preferred.

So far so good, but what about the uncertainty? Unfortunately it is big. First, suppose that we take \( \delta_c \) to be equal to 1.69, instead of the 1.33 used in the above estimates. For \( b = 1 \) (1.6) this multiplies \( n(> M) \) by a factor .56 (.16), and multiplies \( b_c \) by a factor 1.4 (1.4). Second, suppose instead that we multiply \( M \) by a factor 2, on the ground that there may be this amount of uncertainty in the observational
value of the masses of galaxy clusters (and remembering also that filtering the density contrast on mass scale $M$ does not completely eliminate all structure with mass less than $M$). This multiplies $n(> M)$ by a factor .30 (.09) and multiplies $b_c$ by a factor 1.6 (1.7). Third, do neither of these things but replace $n_{\text{peak}}$ by $n_{\text{ps}}$ which is lower (though not low enough remember) for large $\nu$. This multiplies $n(> M)$ by a factor .78 (.35) without changing $b_c$. Each of these not unreasonable changes has a significant effect on the predictions. Finally, implement them all simultaneously. This multiplies $n(> M)$ by a factor .03 (.0004), and multiplies $b_c$ by a factor 2.3 (2.4). The effect is to make $b = 1$ strongly preferred over $b = 1.6$!

Our conclusion is that one cannot yet reliably constrain $b$ by considering the present number density and bias factor of rich clusters. On the other hand, these quantities are certainly very sensitive to $b$, and would constrain it well if the uncertainties could be removed. Similar remarks probably apply to earlier epochs, but as yet the data are too sparse to say much.

For galaxies, the linear epoch ends before the present. The number density $n(> M)$ at the end of the non-linear epoch cannot be estimated reliably, and it will also evolve with time because of merging and other non-linear phenomena. As a result, no reliable prediction is possible for the quantity $n(> M)$ at the present epoch.

However, in the biased theory of galaxy formation, the formation of luminous galaxies is supposed to stop during the linear regime, at some epoch given roughly by Eq. (6.30). By demanding that this value reproduces the value of $b$ used to normalise the amplitude one obtains the epoch of luminous galaxy formation, and hence the observed number density, as a function of $b$. This approach has been implemented for $n = 1$ by BBKS and by Bardeen et al (1987). Here we extend the calculation to $n = .7$, and comment on the uncertainty.

The stars in figure 16b for $b = 1.6$ indicate the epoch of formation of luminous galaxies which is needed to reproduce this bias factor, according to Eq. (6.30). This epoch is $z = 4.7$ (2.8) for $n = 1$ (.7), and the corresponding value of $n(> M)$ is $4.8 \times 10^{-4}$ Mpc$^{-3}$ $(7.9 \times 10^{-4}$ Mpc$^{-3})$. Ignoring merging etc., this comoving number density should be equal to the presently observed number density $n_g(> M)$ of luminous galaxies for $M = 10^{12} M_\odot$. An observational estimate of $n_g(> M)$ is the Schechter parametrisation, which taking the ratio of luminosity $L$ to mass $M$ to be independent of $M$ is (Ellis et al 1988)

$$M \frac{dn_g}{dM} = \phi_* \left( \frac{M}{M_*} \right)^{-0.7} e^{-M/M_*}$$

(6.32)

In this formula

$$\phi_* = 1.56 \times 10^{-2} h^3 \text{Mpc}$$

(6.33)

Integrating $Mdn_g/dM$ over all masses gives $M_* = 1.6 \Omega_{\text{gal}} \times 10^{13} h^{-1} M_\odot$, where $\Omega_{\text{gal}}$ is the contribution to $\Omega$ of luminous galaxies (including the dark halos) and we take $\Omega_{\text{gal}} = .1$. For $M = 10^{12} M_\odot$, this gives an observational estimate $n_g(> M) = 1.75 \times 10^{-3}$ Mpc$^{-3}$, which is marked by an arrow on the y-axis of figure 16a.
According to this calculation, the predicted number density of luminous galaxies is somewhat too small for $b = 1.6$, and in fact one needs a value $b \simeq 1.2$ to 1.3 to reproduce it. But now set $M = 5 \times 10^{11} M_\odot$ in the theoretical calculation, on the ground that the observational value of $M$ could be a factor 2 too high. With $b = 1.6$ this gives a somewhat earlier epoch of formation $z = 5.9 (3.6)$ for $n = 1 (.7)$. The corresponding number densities are $n_g(>M) = 5.8 \times 10^{-4} (1.2 \times 10^{-3})$, which look much more healthy, and the actual value of $b$ needed for consistency is now around 1.4. Even without looking at any other sources of uncertainty (such as galaxy merging, which BBKS emphasise), it seems clear that it is very difficult to pin down $b$ within the range $1 < b < 1.6$ from these considerations.

One conclusion, though, does seem fairly robust, which is that going from $n = 1$ to $n = .7$ makes the redshift of bright galaxy formation later by around 2 units. As evidence accumulates from $N$-body simulations and from observation, this difference may eventually be able to pin down the required value of $n$.

Finally, consider the results for $M = 10^{10} M_\odot$ and $10^8 M_\odot$, as shown in figures 16c and d. The observed number densities $n_g(>M)$ according to the Schechter parametrisation are again shown by arrows (though they become increasingly uncertain as the mass is reduced). Within the biased galaxy formation theory one expects luminous galaxies with these masses to form before the epoch $z_{nl}(M)$, but the observational bias factor for them is not known, and there is no reason why it should be equal to $b$ which refers to bright galaxies. However, the epoch of formation would need to be a lot earlier than $z_{nl}(M)$ to give agreement with the observed $n_g(>M)$. It is presumably more reasonable to suppose that merging has taken place, reducing $n_g(>M)$ from its original value. Again, there does not seem to be a useful constraint on the parameters $n$ and $b$.

The quasar density

So far we have focussed on observations at small redshift. One can also ask about observation at $z \gtrsim 1$. In particular, quasars have now been seen out to a redshift of about 5, and are the oldest observed objects in the universe. In order to produce the observed luminosity, some galaxies must have evolved to contain a sufficient concentration of mass-energy. One then needs to estimate the minimum mass required and the number density, in order to discover if the CDM cosmogony can explain the observations. Such a comparison was made for $n = 1$ by Efstathiou and Rees (1988). Using both the $n_{peak}$ and Press-Schechter prescriptions they calculated $n(>M)$ for various masses as a function of redshift. Our results are far higher than theirs, but the difference can be explained if they chose $b = 2.5$ and defined $b$ with respect to $J_3$ normalisation instead of $\sigma_g$ normalisation (they are not explicit about these choices, but our calculation essentially agrees with theirs if these changes are made in it). As in the case of large clusters at the present epoch, the extreme sensitivity to normalisation is caused by the fact that $\nu$ is substantially bigger than 1.
The number density of quasars assumed by Efstathiou and Rees was $10^{-6}h^3$ Mpc$^{-3}$, out to a redshift of around 4. Since then, observations have become more stringent, and to be consistent with present data one requires this number density out to a redshift of 5 (Martin Rees, private communication). This value also assumes that the quasar lifetimes are not too short, in which case one needs multiple generations. One also needs to know the mass which is required to be evolving nonlinearly to harbour the quasar, and they estimated at the time that $10^{12}M_\odot$ was the smallest that one could safely get away with. However, recent work (Martin Rees, private communication) has suggested that only $10^{10}M_\odot$ is required, which corresponds to a large loosening of the constraint. As one sees from figures 16b or 16c, the upshot of all this is that enough quasars may form at high redshift, even if $n = .7$ and $b = 1.6$.

Another cosmological requirement on high redshifts is the Gunn-Peterson constraint on the amount of neutral hydrogen in the inter-galactic medium. In the CDM cosmogony this implies that some structure has formed before $z = 5$, in order to re-ionise the hydrogen (Schneider, Schmidt & Gunn 1989), but it is not clear how much or of what kind. Even with $n = .7$, structure with $M \leq 10^{10}M_\odot$ forms at the epoch $1 + z \simeq 9/b$ so a value for $b$ as high as 1.6 need not be problematical.

6.8 The virial velocity in a galaxy or cluster

The linear approach has been pushed further by some authors, to try to predict the virial velocity $v(M)$ of structures of mass $M$ (Blumenthal et al 1984; Bardeen et al 1987; Evrard 1989; Henry & Arnaud 1991; Evrard & Henry 1991; Adams et al 1992). The virial velocity of a gravitationally bound system is defined as the rms velocity of its constituents in the centre of mass frame. According to the virial theorem it is given by

$$v^2 = \frac{GM}{R_g} \tag{6.34}$$

where $M$ the mass of the system and $R_g$ is its gravitational radius, defined by the requirement that its potential energy is $-GM^2/R_g$. The idea is to relate $R_g$ to the comoving size $R_{com}$ of the object in the early universe, defined by

$$M = (4\pi/3)R_{com}^3 \rho_0 \tag{6.35}$$

where $\rho_0 = 3H_0^2/(8\pi G)$ is the present mass density. To obtain such a relation one can use the spherical collapse model of Section 6.2, in the following manner. First, assume that the system has virialised, with no energy loss, before the epoch $z_{form}$ when in the absence of pressure forces the system would have contracted to a point. Next, assume that the system collapsed from an initial configuration at rest, with a density profile of the same shape as the profile after virialisation and without energy loss (dissipation). It then follows from the virial theorem, Eq. (6.34), that $R_g$ is equal to one half of the initial gravitational radius. Finally, set the initial gravitational radius
equal to the initial radius of the edge of the object, defined as the sphere containing mass $M$. Using the results already quoted in section 6.2 this gives

$$R_g^{-3} = 8 \frac{9\pi^2}{16} R_{\text{com}}^{-3} (1 + z_{\text{max}})^{-3}$$
(6.36)

$$= 32 \frac{9\pi^2}{16} R_{\text{com}}^{-3} (1 + z_{\text{form}})^{-3}$$
(6.37)

$$= 178 R_{\text{com}}^{-3} (1 + z_{\text{form}})^{-3}$$
(6.38)

Then Eqs. (6.34) and (6.35) give

$$\left(\frac{v}{126 \text{ km s}^{-1}}\right)^2 = \left(\frac{M}{10^{12} M_\odot}\right)^{2/3} (1 + z_{\text{form}})$$
(6.39)

This is the expression quoted by several of the authors mentioned above (Evrard 1989; Henry & Arnaud 1991; Evrard & Henry 1991), and the others presumably used a similar expression though they are less explicit. Obviously, the assumptions leading to this relation are at best extremely crude approximations. It seems clear, in fact, that any unique relation between the mass, virial velocity and formation epoch of gravitationally bound systems can only be a rough approximation.

In figure 16, the arrows on the horizontal axes show $z_{\text{form}}(v)$, calculated from Eq. (6.39), corresponding to estimates of the upper and lower observational limits on observational values of $v$ for objects of mass $M$. These estimates of $v$ are extremely crude. For galaxies, they correspond roughly to those given by Blumenthal et al (1984) without any attempt to update that analysis in the light of more recent observations. For clusters, the upper limit corresponds roughly to X-ray observations (Henry & Arnaud 1991); the observational lower limit of about 1000 km s$^{-1}$ would in that case correspond to negative $z$, a value of about 1300 km s$^{-1}$ corresponding to $z = 0$.

The most direct way of utilising the relation $z_{\text{form}}(M, v)$ is to set $z_{\text{form}}$ equal to $z_{nl}(M)$. This gives the virial velocity $v(M)$ of systems formed when the scale $M$ goes nonlinear. Blumenthal et al (1984) identified such systems with luminous galaxies and with galaxy clusters, but according to the biased galaxy formation theory that identification is wrong in the former case. Alternatively, the relation can be combined with a theoretical estimate of the number density $dn(M, z)$ of objects with mass between $M$ and $M + dM$, which form at epochs between $z$ and $z + dz$, to calculate the number density $n(z, > v)$ of objects with virial velocity bigger than $v$, which exist at redshift $z$. This is essentially the approach taken by the other authors mentioned above.

We saw in Section 6.7 that even an estimate of $n(z, > M)$ is subject to large uncertainties, with the result that it cannot be used to reliably constrain the parameters $n$ and $b$ within the range allowed by the QDOT and COBE data. An estimate of $n(z, > v)$ inherits uncertainties of the same type, plus additional ones coming from
the use of the relation $z_{\text{form}}(M,v)$. It is unclear how to quantifying the total uncertainty, but it seems likely that it will again prevent one from constraining $n$ and $b$ within the allowed range.

### 6.9 Summary

In this long section we have discussed galaxies and clusters, making the maximum use of linear theory despite the fact that these objects are gravitationally bound. Our discussion has extended earlier $n = 1$ calculations to the case $0.7 < n < 1$. It has treated some points of principle not considered in earlier discussions, and in some cases has made for the first time a crude estimate of the uncertainty in the calculations. Despite, or perhaps because of, these improvements the conclusion is that they do not provide significant additional constraints on the pair of parameters $n$ and $b$, beyond those already provided by the QDOT and COBE data.

### 7 Discussion and Conclusions

As small scale structure does not seem to provide any additional constraints over the large scale results of sections 4 and 5, our final constraints are those illustrated in figure 11. From this, we conclude that for natural inflation one must have $n > 0.70$ (which is a separate 2-sigma exclusion on two pieces of data). If one is willing to push all the observations to their limits and fold in extra uncertainties such as allowing $h < 0.5$, it is just conceivable that such a model allows a fit to the clustering data such as APM. The most important requirement for this is that the microwave fluctuation level should be at the top of its permitted range. One sees from the spread of the dashed lines on figure 11 that if the COBE result is exactly correct then the limit on $n$ will immediately be pushed well above 0.8.

For power-law and extended inflation, the limit is already $n > 0.84$ at high confidence. This clearly excludes the possibility that these models can fit the clustering data, and indeed we have already remarked (Liddle & Lyth 1992) that in fact this limit rules out simple forms of extended inflation, as they give a big bubble constraint (Liddle & Wands 1991) $n < 0.75$. (Changing from a cold dark matter cosmogony to a different choice also does not seem to salvage the situation (Liddle & Lyth 1992).) The strength of this constraint is due in the main to the large contribution of gravitational waves to the microwave anisotropies at low $n$; $n = 0.84$ is in fact coincidentally close to the break-even point where gravitational and scalar modes contribute the same amount. Note once again that the limit will strengthen dramatically if the microwave result comes down. If the COBE result is exact, then that will push $n$ well above 0.9.

Assuming one takes the QDOT result seriously, then the bias is also strongly constrained with a 2-sigma maximum of around 1.6. For both types of inflationary
model, the lowest allowed $n$ has this maximum bias. Hence the most optimistic power-law model, which may just fit the clustering data, appears to be a natural inflation style model with values of $n \sim 0.70$ and $b \sim 1.6$. In addition to an improvement on clustering issues, such a model also seems to deal adequately with the formation of structure (contrary to our more pessimistic assessment in LLS).

A preference has been expressed for models with $n$ in the range 0.8 to 0.9 (see eg Salopek 1992; Davis et al 1992; Lucchin, Matarrese & Mollerach 1992), because these models are claimed as allowing a higher bias than the $n = 1$ model. However, with the current error range on the COBE data our figure 11 shows little benefit in this at present, because even at $n = 1$ the COBE error bars allow quite substantial biases at $n = 1$ (and remember this is without including that the spectrum from say a chaotic inflation model has a slight but not insignificant slope and also mildly significant gravitational wave contributions). Even at $n = 1$ it appears that QDOT rather than COBE provides the upper limit to the bias. However, as soon as the upper limit to the fluctuations comes down, either by tightened error bars or independent measurement, this effect may become more relevant.

To conclude, we have found strong constraints on the slope of the primeval spectrum when generated by various inflationary models. Most extended inflation models appear to be ruled out completely. Power-law inflation is viable, but only for values of $n \gtrsim 0.84$, too high to allow an explanation of the clustering data. Natural inflation (and related) models are the most promising candidates for generating useful power-law spectra — provided the true level of fluctuations is close to the top of the COBE range they seem marginally able to explain the excess large scale clustering as now seen in many optical surveys. In most aspects, such a model does at least as well as a bias one standard CDM model, with the advantage of a more plausible cluster abundance as well as helping with galaxy clustering statistics. We must note however that even without including corrections to the flat spectrum and a gravitational wave contribution, even standard CDM appears compatible with biases up to 1.4 or 1.5. Such a model may however have difficulties with the clustering data.

Final Note: As we were completing this paper, we received preprints by Cen et al (1992) which discusses various astrophysical consequences of power-law spectrum including $N$-body results (but without considering gravitational waves), and by Adams et al (1992) discussing the astrophysics with particular emphasis on the natural inflation scenario. Where we cover the same ground as these papers, our results appear to be in good agreement.

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Figure Captions

Figure 1
A selection of transfer functions, from Efstathiou (1990) (used in this paper), Bardeen et al (1986), Davis et al (1985) and Holtzmann (1989), given as appropriate to CDM universes with \( \Omega = 1, \ h = 0.5 \), low baryonic content and three neutrino species. They feature 4, 5, 3 and 4 fitting parameters respectively.

Figure 2
The normalisation \( b\delta_H(k = 1 \text{ Mpc}^{-1}) \) as a function of \( n \). The solid line is from the \( \sigma_8 \) normalisation (used in this paper), while the dashed line shows the rival \( J_3 \) normalisation.

Figure 3
The present day spectra, as calculated in the linear approximation, for a selection of values of \( n \). One sees the additional large scale power and the deficit on short scales when one compares \( n < 1 \) to the standard CDM spectrum. Note \( k \) is in Mpc\(^{-1}\), without a factor of \( h \).

Figure 4
The dispersion \( b\sigma(M) \) as a function of mass, for \( n = 1 \) and \( n = 0.7 \), and with both choices of filter. The top hat filtered spectra are unity at \( M \sim 10^{15} M_\odot \), as required by the normalisation. The gaussian filtering gives significantly smaller answers than does the top hat, as its smearing gives a higher contribution to the larger scales.

Figure 5
The triangles indicate the prediction for the mean quadrupole as a function of \( n \) for power-law inflation, and the stars for natural inflation. The dotted lines plot the COBE observation for two choices of bias (note the bias implicit in the y-axis). The CDM prediction for the mean at bias 1 is very close to the COBE result. The vertical bars on the starred data indicate the spread of the pdf for the quadrupole (for clarity the bars have been omitted for the triangles — they are the same size); 95% of the pdf is above the bottom of the bars, while 95% is below the top. The \( \chi^2 \) distribution is not symmetric, so the bars are skewed to higher values (somewhat concealed by the log plot). A value of \( n \) is allowed at 95% exclusion if the observations cut through its vertical bar. Modelling the observational errors (see text) gives an even looser criterion.

Figure 6
The predictions for \( b \frac{\Delta T}{T} \bigg|_{10^9} \), as defined in text, along with the COBE limits (at 1-sigma) for bias 1 and 2. The upper line represents the power-law inflation predictions, the lower those for natural inflation. As for the quadrupole, the \( n = 1 \) prediction at bias 1 is very close to the COBE result.
Figure 7
The predicted angular correlation functions for a choice of $n$ are plotted alongside the observational data from the APM survey (Maddox et al 1990). With the anticipated residual systematics, values of $n$ between about 0.3 and 0.6 provide reasonable fits, while the standard CDM curve falls well below the data.

Figure 8
A comparison of models with different $n$ against a standard CDM model but with $\Omega h = 0.2$.

Figure 9
The predicted rms velocity flows, when smoothed with a top hat of radius $R_f$, for different choices of $n$.

Figure 10
The predicted rms velocity flows in a configuration mimicking the POTENT observational data. The velocity field is first smoothed with a $12h^{-1}$ Mpc gaussian, reducing the short scale power, and then smoothed with top hat filters of radius $R_f$, giving predictions smaller than in figure 9. The solid lines indicate the predictions for $n = 1$ and $n = 0.7$. The stars indicate the POTENT observational data at bias 1 (read from figure 4 of Dekel (1991)), and the triangles the same at bias 1.6. The error bars on the data (the last ones just overlap) are 1-sigma. Finally, we emphasise that the theoretical curves are averages over all observer points, whereas the observations are a single realisation, with correlated errors due to long wavelength domination of bulk flows.

Figure 11
The limits on the bias $b$ as a function of $n$. The dotted line shows an upper limit on $n$ to fit the APM data. All other lines are 2-sigma limits. The dot-dashed lines indicate the range allowed by QDOT, and apply to all models. The solid and dashed lines indicate the limits from COBE for power-law (and extended) inflation and natural inflation respectively. For power-law inflation, one has $n > 0.84$ and $b < 1.5$, which rules out extended inflation which needs $n < 0.75$ to satisfy the bubbles constraint. For natural inflation, one has $n > 0.70$ and $b < 1.6$. This is well outside our APM fit range (though it is close to the range allowed by Efstathiou, Bond & White (1992)). If one does not attempt to fit optical galaxy clustering, then the allowed region is that marked with the star, bounded by the microwave limits appropriate to a given scenario.

Figure 12
The asphericity parameter $x^{-1}$ plotted against $\nu$.

Figure 13
The ratio $2n_{up}/n_\chi$ plotted against $\nu$. 

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Figure 14
The ratio $V_{\text{peak}}/V_f$ plotted against $\nu$.

Figure 15
The ratio $n_{\text{up}}/n_{\text{P-S}}$ plotted versus $\nu$. The ratio is not very sensitive to $n$, and only the case $n = 1$ is plotted.

Figure 16
The number density $n(>M)$ as a function of redshift. It is calculated theoretically by equating it with the number density of peaks of the linearly evolved density contrast, filtered on the mass scale $M$. Figures 16a–d refer respectively to $M = 10^{15} M_\odot$, $10^{12} M_\odot$, $10^{10} M_\odot$ and $10^8 M_\odot$. For each case there are three curves. They correspond to the three choices $\{n, b\} = \{.7, 1.6\}$, $\{1, 1.6\}$ and $\{1, 1\}$ which mark the corners of the more or less triangular region allowed by the QDOT and COBE data (figure 11). The arrows on the vertical axes give the observed galaxy number densities at the present epoch. The arrows on the horizontal axes indicate very roughly the ‘observed’ range of formation epochs, deduced from the indicated range of virial velocities. Each curve ends at the epoch when $\sigma(M) = 1$, signalling the end of linear evolution. According to the theory of biassed galaxy formation, luminous galaxies form significantly before that epoch. In the case of bright galaxies, the epoch can be calculated by demanding that the resulting bias factor is equal to $b$, and it is indicated by a star in figure 16b. The theoretical and observational uncertainties in are discussed in the text.