Cosmology with non-minimal coupled gravity: inflation and perturbation analysis

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Abstract
We study a scalar-tensor cosmological model where the Einstein tensor is non-minimally coupled to the free scalar-field dynamics. Using the Friedmann–Robertson–Walker metric, we investigate the behavior of the scale factor for vacuum, matter, and dark energy-dominated eras. In particular, we focus on the inflationary behavior of the early Universe. Moreover, we study the perturbation analysis of this model in order to confront the inflation under consideration with observational results.

Keywords: inflation, kinetic-coupled gravity, perturbations

1. Introduction

It is known that scalar fields play an especially important role in cosmology. For example, one may mention the numerous inflationary models in which inflation is typically driven by a fundamental scalar field, the so-called inflation. The general form of the action for a scalar-tensor theory with a single scalar field minimally coupled to gravity is given by\textsuperscript{1}[1]

\[ S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi} + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \]  

(1.1)

where \( g_{\mu\nu} \) is the metric tensor, \( g = \text{det}(g_{\mu\nu}) \), \( R \) is the scalar-curvature, and \( V(\phi) \) is the scalar-field potential. Also, there are scalar-tensor theories with non-minimal kinetic coupling to the Ricci tensor [2], the scalar curvature \( R \), and the f(R) and f(T) theories of modified gravity [3, 4].

In the application of the above gravity theories to the cosmology of the early Universe, the role of the scalar-field potential to establish an inflation is unavoidable. However, which

\textsuperscript{1} We have used the units \( G = 1 \).
potential can exactly describe the correct inflation at the early Universe is an open problem. In
general, the slowly varying potentials should behave like a large effective cosmological
constant suitable for driving an inflation; however, the appropriate choice of \( V(\phi) \) satisfying
the requirements of inflation results in the known problem of fine tuning of the cosmological
constant. One way to get rid of the controversial role of the scalar field potential in inflationary models was introduced by Linde as a model: so-called chaotic inflation [5], where extremely simple potentials can lead to inflation. Also, some models have been introduced to represent natural inflation [6] and inflation with non-minimal derivative coupling [7, 8]. One may also work with non-minimal coupling of the scalar-field dynamics with an Einstein tensor with a vanishing or constant potential to establish the inflationary scenarios [9, 10]. This is a minimal model, because it is economic to consider a free scalar-field rather than a scalar-field subject to a potential term. In other words, it seems more reasonable to think that the Universe requires an action to trigger the inflation by the least factors: geometry and free scalar field.

In the present paper, we study a scalar-tensor cosmological model where the Einstein tensor is non-minimally coupled to the free scalar-field dynamics. We intend to bring this cosmological model in the framework of an inflation model having suitable slow-roll conditions, an exit mechanism from inflation, creation of baryonic matter after inflation, and so on. More importantly, we study the perturbation analysis of this inflation model in order to confront the inflation under consideration with Planck and BICEP2 results. In section 2, we study the cosmology with non-minimal kinetic-coupled gravity and introduce our inflation model, which is followed by matter-dominant and dark energy-dominant eras. In section 3, we study the cosmic perturbations inside and outside the horizon. In section 4, we study the vacuum fluctuation of the inflation field to obtain the scalar spectral index and tensor-to-scalar ratio. The paper is ended with a conclusion.

2. Cosmology with non-minimal kinetic-coupled gravity

Let us consider a free (without potential term) scalar field whose kinetic term is coupled both
with the metric tensor \( g_{\mu\nu} \) and the Einstein tensor \( G_{\mu\nu} \). We write the action as

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{8\pi} - (g^{\mu\nu} + \alpha G^{\mu\nu}) \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - 2\Lambda \right] + S_m, \tag{2.1}
\]

where \( R \) is the Ricci scalar, \( \alpha \) is a coupling parameter with dimension of \((\text{length})^2\), \( \Lambda \) is a positive cosmological constant, and \( S_m \) is the matter action. In fact, if one takes the most general action, having coupling functions of the metric with the scalar-field derivatives, namely, non-minimal coupling of the geometry with the kinetic term of the scalar field, then it turns out that in order for the field equations to be second-order, one is required to take the action (2.1) [2]. Equations of motion for the scalar field and the metric field are obtained by varying (2.1) with respect to \( \phi \) and \( g_{\mu\nu} \), respectively, as

\[
\left( g^{\mu\nu} + \alpha G^{\mu\nu} \right) \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} = 0, \tag{2.2}
\]

\[
G_{\mu\nu} = 8\pi \left[ T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\phi)} + \alpha \Theta_{\mu\nu} \right], \tag{2.3}
\]

where the total energy-momentum tensor is divided into three parts, as follows

\[
T_{\mu\nu}^{(m)} = \left( \rho_m + p_m \right) u_\mu u_\nu + p_m g_{\mu\nu}, \tag{2.4}
\]
\[ T^{(\phi)}_{\mu\nu} = V_\mu \phi V_\nu \phi - \frac{1}{2} g_{\mu\nu} V_\rho \phi V^\rho \phi - \Lambda g_{\mu\nu}, \]  \hspace{1cm} (2.5)

\[ \Theta_{\mu\nu} = -\frac{1}{2} V_\mu \phi V_\nu \phi R + 2 V_\mu \phi V_\nu \phi R^{\alpha}_{\mu\alpha\beta} \\
+ V^\alpha \phi V^\beta \phi R_{\mu\alpha\beta} + V_\mu \phi V^\alpha \phi V_\nu \phi V_\alpha \phi \\
- V_\mu \phi V_\nu \phi \Delta \phi - \frac{1}{2} (V\phi)^2 G_{\mu\nu} \\
+ g_{\mu\nu} \left[ -\frac{1}{2} V^\alpha \phi V^\beta \phi V_\alpha \phi V_\beta \phi + \frac{1}{2} (\Box \phi)^2 \\
- V_\alpha \phi V_\beta \phi R^{\alpha\beta} \right]. \]  \hspace{1cm} (2.6)

where \( T^{(m)}_{\mu\nu} \) and \( (T^{(\phi)}_{\mu\nu} + \Theta_{\mu\nu}) \) are independently conserved. Using the Friedmann–Robertson–Walker (FRW) flat \((k = 0)\) background metric with \(a(t)\) being the scale factor

\[ ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2 \right), \]  \hspace{1cm} (2.7)

the field equations are obtained

\[ 3H^2 = 4\pi \phi^2 \left( 1 - 9aH^2 \right) + \Lambda + 8\pi \rho_m, \]  \hspace{1cm} (2.8)

\[ 2H + 3H^2 = -4\pi \phi^3 \left[ 1 + \alpha \left( 2H + 3H^2 + 4H\dot{\phi}\dot{\phi}^{-1} \right) \right] \\
+ \Lambda - 8\pi \rho_m, \]  \hspace{1cm} (2.9)

\[ (\ddot{\phi} + 3H\dot{\phi}) - 3\alpha \left( H^2 \dot{\phi} + 2HH\dot{\phi} + 3H\dot{\phi} \right) = 0. \]  \hspace{1cm} (2.10)

Therefore, one obtains the total density and pressure, respectively, as

\[ \rho_t = \rho_m + \frac{\Lambda}{8\pi} + \frac{\dot{\phi}^2}{2} \left( 1 - 9aH^2 \right), \]  \hspace{1cm} (2.11)

\[ p_t = p_m - \frac{\Lambda}{8\pi} + \frac{\dot{\phi}^2}{2} \left[ 1 + \alpha \left( 2H + 3H^2 + 4H\dot{\phi}\dot{\phi}^{-1} \right) \right], \]  \hspace{1cm} (2.12)

where a dot denotes the derivative with respect to \(t\). Equation (2.10) can be easily integrated to

\[ \dot{\phi} = \frac{\sqrt{2\lambda}}{a^3 \left( 1 - 3aH^2 \right)}, \]  \hspace{1cm} (2.13)

where \( \lambda \) is a positive constant of integration.

In the three following subsections, we will set up a cosmological model in a systematic way, which includes an inflation era with suitable slow-roll conditions, an exit mechanism from inflation, a deceleration era, and an acceleration era of the Universe.

2.1. Inflationary Universe

By differentiating (2.13) with respect to time and substituting \( \dot{\phi} \) and \( \phi \) in (2.14) and (2.15), we obtain

\[ \phi = \frac{\sqrt{2\lambda}}{a^3 \left( 1 - 3aH^2 \right)}, \]  \hspace{1cm} (2.13)
In the very early Universe, where there is no baryonic matter, namely $\rho_{m} = p_{m} = 0$, we define $\rho_{v} = \rho_{0}$ and $p_{v} = p_{0}$ as the density and pressure of the vacuum state. Then, we combine equations (2.14) and (2.15) to obtain the following equation of state

\begin{equation}
\rho_{v} = \rho_{0} + \frac{\Lambda}{8\pi} + \frac{\lambda}{a^{6}(1 - 3aH^{2})^{2}}\left(1 - 9aH^{2}\right),
\end{equation}

\begin{equation}
p_{v} = \rho_{0} - \frac{\Lambda}{8\pi}
+ \frac{\lambda}{a^{6}(1 - 3aH^{2})^{2}}\left[1 + a\left(2\dot{H} + 3H^{2} + 4H\dot{\phi}^{2}\right)\right].
\end{equation}

Although not all inflationary models require an equation of state $p_{v} = -\rho_{v}$, and even most of them require an equation of state near it, here we regularly assume that the inflation model requires the vacuum equation of state as $p_{v} = -\rho_{v}$. So, in order to achieve this necessary condition for inflation, according to (2.16), we need to satisfy the following differential equation for $\lambda \neq 0$

\begin{equation}
\left[1 - 9aH^{2}\right] + aH\left(1 + \frac{12aH^{2}}{1 - 3aH^{2}}\right) = 0.
\end{equation}
One may solve this differential equation to obtain the plot $H(t)$, as depicted in figure 1. As is seen in figure 1, considering a typical small value of $\alpha$, the behavior for $H(t)$ is so favored to model an inflationary cosmology with an almost constant $H \approx 0$ and large $H$ for $t > 10^{-35}$ s. Actually, the Hubble parameter $H$ may suddenly (typically within $10^{-35}$ s) approach the large and almost constant asymptotic value

$$H \equiv H_\alpha \approx \sqrt{\frac{1}{9\alpha}} \approx 1.2 \times 10^{35} \text{sec}^{-2},$$

(provided that $\alpha \sim 10^{-71/2}$, namely, $\alpha$ is assumed to be an infinitesimally small but non-vanishing parameter\(^2\). The subscript $\alpha$ denotes the Hubble parameter at the regime where the contribution of kinetic-coupled gravity is dominant. Therefore, we realize that imposing the vacuum equation of state results in the typically large and almost constant Hubble parameter (2.18). Now, we aim to establish an inflation model for the behaviors of scale factor and scalar-field dynamics. The cosmological evolution of the Universe at the vacuum-dominated state is described by

$$p_v = -\rho, \quad (2.19)$$

$$a(t) \propto \exp(H_\alpha t), \quad (2.20)$$

and (see (2.13))

$$\dot{\phi}(t) \propto \exp(-3H_\alpha t). \quad (2.21)$$

We know the vacuum era $p_v = -\rho$ corresponds to the following approximate equations

$$3H^2 \approx \frac{1}{3a}, \quad (2.22)$$

$$\dot{H} \approx 0. \quad (2.23)$$

If we define the Hubble slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv \frac{\dot{\epsilon}}{2He}, \quad (2.24)$$

then, as long as $|\epsilon| \ll 1, |\eta| \ll 1$, the inflationary stage is guaranteed.

2.1.1. Exit mechanism. In order to study the exit mechanism from the inflationary era, we investigate the condition under which the vacuum equation of state is violated. To this end, we combine (2.14) and (2.15) as

\(^2\) The non-vanishing requirement of $\alpha$, which prevents some expressions like (2.18) or (2.31) (see below) from diverging, arises from the following argument. If one lets $\alpha \rightarrow 0$, then by using (2.13) the equation (2.16) becomes $p_v = -\rho + \dot{\phi}^2$. Now, if we assume a large kinetic energy $\dot{\phi}^2 \gg 1$ in the present inflation model, instead of a large potential energy $V(\phi) \gg 1$ in the standard inflation models, then $p_v \neq -\rho$, and we lose the vacuum equation of state $p_v = -\rho$, which is necessary for our inflation model. Therefore, the limit of $\alpha \rightarrow 0$ together with the base assumption of $\dot{\phi}^2 \gg 1$, will damage the inflation model; hence, we require the non-vanishing assumption of $\alpha$. In fact, if one is interested in the general relativistic limit of the present non-minimal kinetic-coupled inflation model, then the limit $\alpha \rightarrow 0$ should be accompanied by $\dot{\phi}^2 \rightarrow 0$, which is nothing but the so-called slow-roll approximation $\dot{\phi}^2 \ll V(\phi)$. In this way, using (2.14) and (2.15), we obtain the standard expressions $p_v = \frac{\lambda}{8\pi}$ and $p_v = -\frac{\rho}{8\pi}$ in a de Sitter space with vacuum equation of state $p_v = -\rho$.\]
\[ p_\epsilon + \rho_\epsilon = \frac{2\lambda}{a^6 \left( 1 - 3aH^2 \right)^2} \left[ \left( 1 - 9aH^2 \right) + aH + \frac{12a^2H^2\dot{H}}{1 - 3aH^2} \right], \]  

(2.25)

where, contrary to (2.17), it is assumed that the RHS will be no longer vanishing when the inflation reaches to its end, namely, \( p_\epsilon \neq -\rho_\epsilon \). On the other hand, combining (2.8) and (2.9) and using (2.25) results in

\[ -\dot{H} = \frac{8\pi\lambda \left( 1 - 9aH^2 \right)}{a^6 \left( 1 - 3aH^2 \right)^2} + \frac{8\pi\lambda \alpha}{3a^3 \left( 1 - 9aH^2 \right)}. \]  

(2.26)

Since during the inflation the asymptotic (maximum) value of \( H \) is \( 1/3\sqrt{a} \) (see (2.18)), when we approach the end of inflation, it is strongly expected that the value of \( H \) is below this upper bound, so \( \left( 1 - 9aH^2 \right) \) is positive. Moreover, \( \lambda > 0, \alpha > 0 \). Therefore, one may expect that when the inflation reaches its end, according to (2.26), the quantity \( \dot{H} \) is expected to become negative, and so \( H \) starts decreasing.

On the other hand, substituting (2.13) in (2.8), with \( \rho_m = 0 \), we obtain

\[ H^2 = \frac{8\pi\lambda \left( 1 - 9aH^2 \right)}{3a^6 \left( 1 - 3aH^2 \right)^2} + \frac{8\pi\lambda}{3a^3}. \]  

(2.27)

Now, by equating (2.26) and (2.27) we can find the condition under which \( \epsilon \to 1 \). This results in the following equation

\[ \frac{2\lambda \left( 1 - 9aH^2 \right)}{3a^6 \left( 1 - 3aH^2 \right)^2} = \Lambda, \]  

(2.28)

which is obtained by ignoring the second term in the denominator of (2.26) due to the small value of the coupling \( \alpha \). Equation (2.28) is the condition under which the slow-roll approximations \( |\epsilon| \ll 1, |\eta| \ll 1 \) are violated, and the inflation reaches to its end. Therefore, using (2.28) we realize that the inflation is ended at \( t_f \) when the scale factor is inflated to the order of magnitude

\[ a \left( t_f \right) = \left[ \frac{2\lambda \left( 1 - 9aH^2 \right)}{3\Lambda \left( 1 - 3aH^2 \right)^2} \right]^{1/6}. \]  

(2.29)

Note that, although according to (2.20) the inflationary expansion of the scale factor is determined just by the coupling parameter \( \alpha \), however, according to (2.29) for a given value of \( \lambda \), the end of inflation, namely \( t_f \), is determined essentially by the cosmological constant \( \Lambda \). Actually, this is a reasonable result, because it tells us that both the essential parameters \( \alpha \) and \( \Lambda \) should be involved in the beginning and ending of the inflation. On the other hand, comparing (2.13) with (2.29) we find

\[ \phi^2 \left( t_f \right) = \frac{3\Lambda}{\left( 1 - 9aH^2 \right)}. \]  

(2.30)

It is seen that, similar to the scale factor, the kinetic energy of the scalar field at the end of inflation is also determined by both of the parameters \( \alpha \) and \( \Lambda \). Although, according to (2.21),

3 Note that \( \rho_m \) should become nonzero just after the end of inflation; hence, before the inflation is ended, namely, when the inflation is going to its end, we can ignore the matter density.
the kinetic energy of the scalar field is decreasing with time during inflation, however, from (2.30) it turns out that for a given value of positive cosmological constant, and considering the very small positive value of \((1 - 9\alpha H^2)\), one can obtain a large kinetic energy for the scalar field, even at the end of inflation, which can play the role of a large energy source for reheating the Universe after inflation.

2.1.2. e-folding. One may obtain the number of e-folding during the inflation as

\[
N = \int_{t_i}^{t_f} \frac{H dt}{\sqrt{9\alpha}} = \sqrt{\frac{1}{9\alpha}} (t_f - t_i).
\]

Within the typical short period of time \((t_i = 10^{-35}) < t < (t_f = 10^{-33})\) required by particle physics, and using \(\sqrt{\frac{1}{9\alpha}} \approx 1.2 \times 10^{35}\), we obtain \(N \sim 120\), well above \(N \sim 60\), which is at least needed to overcome the problems of standard cosmology. If we assume that the initial size of the Universe before inflation was about the Planck length \(10^{-34}\) m, then the 120 number of the e-folding results in a final size of the Universe at the end of inflation as large as \(a(t_f) \sim 10^{22}\) m. This large size will remove all the problems of standard cosmology.

The last important point is to investigate ghost instability. From quantum field theory we know that a ghost is a degree of freedom whose propagator has the wrong sign, giving rise to a negative norm state on quantization. A gravity theory with fourth-order derivatives in the kinetic term inevitably has ghosts [11]. However, in the case of the kinetic term coupled to the metric and Einstein tensors, the equations of motion for the scalar field are reduced to the second order. Therefore, from the physical point of view this theory can be interpreted as a good theory [12]. To prevent ghosts in our model we insist that the effective kinetic energy of the scalar field is non-negative. According to equation (2.8), the effective kinetic energy of the scalar field is \(\frac{H^4}{4} \beta_1^2 (1 - 9\alpha H^2)\). Since \((1 - 9\alpha H^2) > 0\), the effective kinetic energy of the scalar field is positive, and so the presence of ghosts will be prevented.

2.2. Radiation- and matter-dominated Universe

After the inflationary era the scale factor \(a\) becomes exponentially large (see (2.20)); hence, the last terms of (2.14) and (2.15) containing \(a^{-6}\) become ignorable. At this time, namely \(t_f\), the very fast decrease in the kinetic energy of the scalar field (see (2.21)) starts and can be balanced by the creation of baryonic matter, with the density and pressure related by the equation of state \(p_b = \omega_2 \rho_b\). At this stage, we have just two components left, as follows

\[
\rho_m = \rho_m + \frac{\Lambda}{8\pi}, \quad (2.32)
\]

\[
p_m = p_m - \frac{\Lambda}{8\pi}. \quad (2.33)
\]

Assuming a small cosmological constant in comparison with the sufficiently large values of matter density and pressure, the cosmological evolution of the Universe at this stage with an ignorable cosmological constant is well-known, as follows [13, 17]

(1) for \(\omega_m = \frac{1}{3}\) we have the radiation-dominant era with the scaling behavior \(\rho_m \propto a^{-4}\) and time evolution

\[
a(t) \propto t^{1/2};
\]
(II) for \(\omega_m = 0\) we have the matter-dominant era with the scaling behavior \(\rho_m \propto a^{-3}\) and time evolution
\[
a(t) \propto t^{2/3}.
\]

### 2.3. Dark energy-dominated Universe

At a late time and an old Universe, the scale factor becomes so large that \(\rho_m \propto a^{-3} \ll \Lambda/8\pi\). This stage of evolution is governed by the cosmological constant
\[
\rho_\Lambda = \frac{\Lambda}{8\pi},
\]
\[
p_\Lambda = -\frac{\Lambda}{8\pi},
\]
which represents the new phase of the vacuum state as
\[
p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi},
\]
where the cosmological constant plays the role of dark energy. The cosmological evolution of this dark energy-dominated Universe is well known as the de Sitter expansion [13, 17]
\[
a(t) \propto \exp(H_At),
\]
where
\[
H_A = \sqrt{\frac{\Lambda}{3}}.
\]

### 3. Cosmic perturbation

#### 3.1. Background Universe

In this section, we focus on the inflationary Universe and study the cosmological perturbations. Let us consider a flat FRW background Universe with the metric
\[
d\sigma^2 = a^2(\eta)\left(-d\eta^2 + d^2x + d^2y + d^2z\right),
\]
where \(\eta\) is the conformal time defined as \(d\eta = dt/a(t)\). In the background Universe the scalar field is homogeneous, namely,
\[
\phi = \phi(\eta).
\]
The background scalar-field equation in terms of the conformal time \(\eta\) becomes
\[
\begin{align*}
-1 + 3\frac{\dot{H}}{a^2} \frac{\ddot{\phi}}{a^2} - 2\ddot{\phi} - 1 + 3\frac{\dot{H}^2}{a^2} \\
+ 3\dot{H} \left(-1 + \frac{\alpha}{a^2} \left(2\dot{H} + \dddot{H}\right)\right) = 0,
\end{align*}
\]
where $\bar{H} = \frac{\dot{a}}{a}$ is the Hubble function with respect to the conformal time in the background Universe, and $\dot{}$ denotes a derivative with respect to the conformal time. Also, the background energy-momentum tensor is given by the following components

$$T^0_0 = -\rho = -\frac{1}{2}a^{-2}\left(-1 + 9\alpha \frac{H^2}{a^2}\right)\dot{\phi}^2,$$

$$T^0_i = 0,$$

$$T^j_i = \rho \delta^j_i = \frac{a^{-2}}{2}\left(-1 - \frac{\alpha}{a^2}\left(2\bar{H} - 3\bar{H}^2\right)\dot{\phi}^2 - 4\alpha \frac{H\dot{\phi}\phi^*}{a^2}\right).$$  \hspace{1cm} (3.4)

The Friedman equations are

$$3\dot{H}^2 = -8\pi\left(\frac{1}{2}\left(-1 + 9\alpha \frac{H^2}{a^2}\right)\dot{\phi}^2\right).$$  \hspace{1cm} (3.5)

$$2\dot{H} + \bar{H}^2 = -8\pi\left(\frac{1}{2}\left(-1 + \frac{\alpha}{a^2}\left(2\bar{H} - 3\bar{H}^2\right)\dot{\phi}^2 - 2\alpha \frac{H\dot{\phi}\phi^*}{a^2}\right)\right).$$  \hspace{1cm} (3.6)

$$\dot{H} = -\frac{8\pi}{3}\left(-1 - \frac{3\alpha}{2a^2}\left(\bar{H} - 3\bar{H}^2\right)\dot{\phi}^2 - 3aH\dot{\phi}\phi^*\right).$$  \hspace{1cm} (3.7)

### 3.2. Perturbed Universe in the Newtonian gauge

The metric of the perturbed Universe in the Newtonian gauge is

$$g_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu},$$  \hspace{1cm} (3.8)

and

$$ds^2 = a^2(\eta)(-(1 + 2\Psi)d^2\eta + (1 - 2\Psi)\delta_{ij}dx^i dx^j).$$  \hspace{1cm} (3.9)

where $\Psi$ is the gauge-invariant Newtonian potential, which characterizes the metric perturbations. The determinant of the perturbed metric is obtained

$$g = -a^8(1 - 4\Psi).$$  \hspace{1cm} (3.10)

In this perturbed metric, the scalar-field equation takes the form

$$g^0_0\left(1 + \alpha G^0_0\right)\left(\phi^* - \Gamma^0_0 \phi' - \Gamma^0_0 \phi_0\right) + g^0_i\left(1 + \alpha G^0_i\right)\left(V^2\phi - \Gamma^0_i \phi' - \Gamma^0_i \phi_0\right) + \frac{2\alpha}{a^2} G^0_i \left(\phi_j' - \Gamma^0_i \phi_{i,j}\right) = 0.$$  \hspace{1cm} (3.11)

Now, we divide the scalar field into a background and perturbed parts

$$\phi = \phi(\eta) + \delta \phi(\eta, \vec{x}).$$  \hspace{1cm} (3.12)
Also, the required perturbed geometrical quantities in the Newtonian gauge are [15],
\[ \Gamma_{00}^{0} = \dot{H} + \Psi, \]
\[ \Gamma_{ii}^{0} = \dot{\Psi}, \]
\[ \Gamma_{ij}^{0} = \dot{H}(1 - 4\dot{\Psi})\delta_{ij} - \delta_{ij}\dot{\Psi}, \]
\[ \Gamma_{00}^{i} = -\ddot{\Psi}, \]
\[ \Gamma_{ij}^{i} = \dot{H}\delta_{ij} - \delta_{ij}\dot{\Psi}, \]
\[ \Gamma_{jk}^{i} = -\delta_{i}^{j}\Psi_{,k} - \delta_{i}^{k}\Psi_{,j} + \delta_{jk}\Psi_{,i}, \]
\[ G_{0}^{0} = -3a^{-2}\dot{H}^{2} + a^{-2}\left\{ -2\nabla^{2}\Psi + 6\dot{H}\Psi + 6\dot{H}^{2}\Psi \right\}, \]
\[ G_{0}^{i} = -G_{i}^{0} = a^{-2}\left\{ -2\Psi_{,i} - 2\ddot{H}\Psi_{,i} \right\}, \]
\[ G_{i}^{j} = a^{-2}\left( -2\dot{H}' - \ddot{H}' \right)\delta_{ij}, \]
\[ + a^{-2}\left( 2\Psi'' + 6\dot{H}\Psi + \left( 4\dot{H}' + 2\dot{H}^{2} \right)\Psi \right). \] (3.13)

Substituting equations (3.13) and (3.12) into (3.11), we get the field perturbation equation as follows

\[ \left\{ -1 + \frac{3\alpha}{a^{2}}\dot{H}^{2} \right\}(\delta \phi''') + 2\dot{H}\left( -1 + \frac{3\alpha}{a^{2}}\dot{H} \right)\delta \phi'' = \left\{ -1 + \frac{\alpha}{a^{2}}(2\dot{H}' + \dot{H}^{2}) \right\}V^{2}(\delta \phi) = -la^{2}, \] (3.14)

where

\[ I = \frac{1}{a^{2}}\left\{ -1 + \frac{3\alpha}{a^{2}} \right\}(\Psi')' \]
\[ + \frac{1}{a^{2}}\left\{ -\frac{\alpha}{a^{2}} \right\} \left( -2\nabla^{2}\Psi + 6\dot{H}\Psi' + 6\dot{H}'\dot{H}\Psi \right) \]
\[ + 2\Psi - \frac{6\alpha\Psi'}{a^{2}}(\dot{\phi}' - \dot{H}\dot{\phi}) \]
\[ - \frac{3\dot{H}}{a^{2}} \left\{ \frac{\alpha}{a^{2}} \right\} \left\{ -2\Psi' + 6\dot{H}\Psi' + \left( 4\dot{H}' + 2\dot{H}^{2} \right)\Psi \right\} \]
\[ + 2\Psi + \frac{2\alpha\Psi'}{a^{2}} \left( -2\dot{H}' - \dot{H}^{2} \right) \dot{\phi}' \]
\[ - \frac{3(4\Psi'\dot{H} + \Psi')}{a^{2}} \left\{ -1 + \frac{\alpha}{a^{2}}(2\dot{H}' + \dot{H}^{2}) \right\} \dot{\phi}'. \] (3.15)

Using the slow-roll approximation \( \dot{H} \approx 0 \), and equations (3.14) and (3.15) become

\[ \left\{ -1 + \frac{3\alpha}{a^{2}}\dot{H}^{2} \right\}(\delta \phi''') + 2\dot{H}(\delta \phi'' - V^{2}(\delta \phi)) = -la^{2}, \] (3.16)
where

\[ I' = \frac{1}{a^2} \left( -1 + \frac{3\alpha}{a^2} \right) (-\Psi' \Phi') \]
\[ + \frac{1}{a^2} \left( -\alpha \frac{\Psi''}{a^2} \right) (-2V^2 \Psi + 6\dot{H} \Psi' + 6\dot{H}^2 \Psi) + 2\Psi \]
\[ - \frac{6\alpha\Psi}{a^2} \hat{H} \right) \left( \Phi' - \dot{H} \Phi' \right) \]
\[ - \frac{3\dot{H}}{a^2} \left( \frac{\alpha}{a^3} \left( 2\Psi'' + 6\dot{H} \Psi' + 6\dot{H}^2 \Psi \right) \right) \]
\[ + 2\Psi - \frac{6\alpha \Psi}{a^2} \hat{H} \right) \left( \Phi' \right) \]
\[ - \frac{3}{a^2} \left( 4\Psi \hat{H} + \Psi' \right) \left( -1 + \frac{3\alpha}{a^2} \hat{H}^2 \right) \Phi'. \] (3.17)

In order to obtain the equations of perturbations, we have to linearize the Einstein equations

\[ G^\mu_\nu = R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = 8\pi T^\mu_\nu \] (3.18)

for small inhomogeneities about the FRW background Universe. The components of the Einstein tensor for the background metric (2.7) are obtained easily and result in [13]

(3.19)

\[ \tilde{G}_0^0 = \frac{\dot{H}^2}{a^2}, \tilde{G}_i^0 = 0, \]
\[ \tilde{G}_i^j = \frac{1}{a^2} \left( 2\dot{H} + \hat{H} \right) \delta_{ij}. \] (3.20)

The linearized equations for perturbations are

\[ \delta G^\mu_\nu = 8\pi \delta T^\mu_\nu. \] (3.21)

The direct calculation of (3.21) for the metric (3.9) gives the equations [13]

\[ V^2 \Psi - 3\dot{H} \left( \Psi + \dot{H} \Psi \right) = 4\pi a^2 \delta T^0_0, \] (3.22)
\[ \left( \Psi + \dot{H} \Psi \right)_j = 4\pi a^2 \delta T^0_j, \] (3.23)
\[ \left( \Psi'' + 3\dot{H} \Psi + \left( 2\dot{H}^2 + \hat{H} \right) \Psi \right) \delta_{ij} = -4\pi a^2 \delta T^j_j. \] (3.24)

Equation (3.14) contains two unknown variables, \( \delta \phi \) and \( \Psi \), and should be supplemented by one of the equations (3.23)–(3.24). It is convenient to use (3.23). Substituting \( \phi = \phi + \delta \phi(\eta, \vec{x}) \) in (2.6), using equation (3.13), and after a tedious but straightforward calculation, we obtain

\[ \delta T^0_i = a^{-2} \left( 1 + 6\alpha a^{-2} \left( 2\dot{H}^2 + \hat{H}^2 \right) \right) \phi \delta \phi + 4\alpha a^{-4} \Psi \phi^2. \] (3.25)

Hence, equation (3.22) becomes

\[ \Psi + \dot{H} \Psi = 4\pi \left\{ \left( 1 + 6\alpha a^{-2} \left( 2\dot{H}^2 + \hat{H}^2 \right) \right) \phi \delta \phi + 4\alpha a^{-4} \Psi \phi^2 \right\}. \] (3.26)
Now, we solve (3.14) and (3.26) in two limiting cases: i) for short-wavelength perturbations, where the physical wavelength $\lambda_{ph}$ is much smaller than the curvature scale $H^{-1}$, and ii) for long-wavelength perturbations where $\lambda_{ph}$ is much larger than the curvature scale $H^{-1}$. Since the curvature scale does not change very much during the inflation, and the physical scale of perturbations grows like $\lambda \sim a/k$, hence we are interested in the short-wavelength perturbation, where the physical wavelength starts smaller than the Hubble length but eventually exceeds it. We may fix the amplitude of these modes by vacuum fluctuations through the uncertainty principle. Then, we study how the amplitude of these perturbations evolves after it crosses the Hubble length.

3.3. Inside the Hubble scale

For the short-wavelength perturbations we have $\lambda_{ph} \ll H^{-1}$ or, equivalently, $k \gg Ha \sim |\eta|^{-1}$. Moreover, for a very large $k\eta$ the spatial derivative term dominates in (3.14), and its solution becomes as $\exp(\pm ik\eta)$ to leading order. On the other hand, the gravitational field oscillates ($\Psi \sim k\Psi'$) and can be estimated from (3.26) as

$$\Psi = \frac{4\pi}{k} \left( 1 + 6\alpha a^{-2} \left( 2H' + \bar{H}^2 \right) \right) \delta \phi,$$

where $\dot{\phi} = a\dot{\phi}$. Taking into account that during the inflation $H \approx 0$, we will get

$$\Psi = \frac{4\pi}{k} \left( 1 + 18\alpha a^{-2} \bar{H}^2 \right) \delta \phi.$$

Now, using this equation and also $k \gg Ha$, after a tedious but straightforward calculation we find that equation (3.16) reduces to the following equation in terms of physical time

$$(\ddot{\delta \phi} + 3H \dot{\delta \phi} + \frac{k^2}{a^2}(\delta \phi) \approx 0,$$

where use has been made of the slow-roll approximation. This equation is easily solved with $H$ being constant. Figure 2 shows the behavior of $\delta \phi$ as a function of Hubble time ($Ht$).

3.4. Evolution through horizon exit

To take advantage of the slow-roll approximation for the perturbations, we need to recast (3.14) and (3.26) in terms of the physical time $t$ as

$$\left( -1 + 3aH^2 \right) (\ddot{\delta \phi}) + 3H \left( -1 + \alpha \left( 2H + 3H^2 \right) \right) (\dot{\delta \phi}) - \left( -1 + \alpha \left( 2H + 3H^2 \right) \right) V^2 (\delta \phi) = -I,$$

$$\Psi + H\Psi = 4\pi \left( \left( -1 + 6\alpha \left( 2H + 3H^2 \right) \right) \delta \phi + 4\alpha \Psi \delta \phi \right).$$

where $I$ is the quantity defined in equation (3.15), which is expressed here in terms of the physical time $t$. The spatial derivative term $V^2 \phi \approx k^2 \phi$ can be neglected for long-wavelength inhomogeneities (i.e., $k \ll Ha$). To find the non-decaying slow-roll mode we next omit terms proportional to $\Psi$ [13]. The equations for the perturbations in the slow-roll regime result in
where we have removed the overbars on the background quantities for simplicity. But, in the slow-roll approximation the scalar-field equation is

\[ \frac{\ddot{\alpha} \delta \phi}{\delta \phi} + \frac{\ddot{\alpha}}{\alpha} \delta \phi \approx 0. \] (3.34)

Then, we can ignore the RHS of equation (3.32) and rewrite (3.32) as

\[ \left( -1 + 3aH^2 \right) \left( \delta \dot{\phi} + 3H \delta \phi \right) \approx 0. \] (3.35)

This equation is also easily solved with $H$ being constant. The solution of $\delta \phi$ can be set to depict a behavior, as shown in figure 3, where $\delta \phi$ is almost constant for $Ht \gtrsim 2$. In conclusion, for $a \lesssim a_k \sim \frac{1}{H_0}$ the perturbations are inside the horizon, and their amplitudes decrease with time, as is seen in figure 2. After several Hubble time, for example $Ht \gtrsim 2$, the perturbations cross the horizon, for $a \gtrsim a_k$, and their amplitudes freeze out at their last values crossing the horizon, as is seen in figure 3. Combining two perturbations inside and outside the horizon, we may obtain the desired behavior of the perturbations, as is depicted in figure 4.
4. Vacuum fluctuation of the inflation field

The scalar-field perturbation equation, in the absence of metric perturbation, is obtained as

\[ (-1 + 3aH^2) \left( \delta \dot{\phi} + 3H \delta \phi - \nabla^2 \delta \phi \right) = 0, \tag{4.1} \]

where use has been made of (3.12), (2.10) and the slow-roll approximation \( H \approx \text{constant} \). Using (2.22), which results in \( (-1 + 3aH^2) \neq 0 \), equation (4.1) in the conformal time \( \eta \) is rewritten as

\[ \delta \phi'' + 2H \delta \phi' - \nabla^2 \delta \phi = 0. \tag{4.2} \]

For a given Fourier component having wave number \( k \), and by defining \( u \equiv a \delta \phi_k \), we obtain \(^4\)

\[ u'' + \left\{ k^2 - \frac{1}{\eta^2} (3\varepsilon + 2) \right\} u = 0, \tag{4.3} \]

\(^4\) Note that the perturbation analysis of the scalar-field equation, described by (4.3), is almost indistinguishable from that studied in the usual inflationary models, up to terms including slow-roll parameters in the RHS (see 13). This agreement may be a good point, because we obtain the same perturbative analysis of common inflation models (including a scalar-field potential) in our inflation model (not including a scalar-field potential). Of course, if we would consider the metric perturbations too, then we could obtain highly nonlinear and complicate perturbative differential equations with possibly a different perturbative analysis. To avoid this complexity, and just for simplicity, we have considered the scalar-field perturbations over a fixed background.
where use has been made of the slow-roll regime to keep terms up to the first order in the slow-roll parameter \( \varepsilon \), namely, \( \varepsilon^2 \approx 0 \). By defining a new function \( S \) so that \( u S = \left( k^2 \eta^2 - \nu^2 \right) \), we obtain

\[
\eta^2 S^\prime + \eta S^\prime + \left( k^2 \eta^2 - \nu^2 \right) u = 0, \quad \left( \nu \approx \frac{3}{2} + \varepsilon \right)
\]

whose solutions are the Hankel functions. For early time \((-k\eta \to \infty)\) and late time \((-k\eta \to 0)\) we have, respectively [15]

\[
\delta \phi_k = C_k \left( \frac{2}{\pi} \right)^{\frac{1}{4}} a^{\frac{5}{2}} e^{-ik\eta},
\]

\[
\delta \phi_k = C_k a^{-1} \left( \frac{2}{\pi} \right)^{\frac{1}{4}} 2^{-\frac{1}{2}} \Gamma \left( \nu \right) \Gamma \left( \frac{3}{2} \right) (-k\eta)^{-\nu} \propto (-\eta)^{\frac{3}{2} + \varepsilon - \nu}.
\]

Therefore, using \( \nu \approx \frac{3}{2} + \varepsilon \), the perturbation \( \delta \phi_k \) at late time becomes almost independent of the conformal time. The power spectrum of these vacuum fluctuations is defined as [14]

\[
P_{\delta \phi}(k) = L^3 \left( \frac{k^3}{2\pi^2} \right) \langle (\delta \phi)^2 \rangle.
\]

Well below the horizon exit, \( k \gg \dot{H} \), the field operator \( \delta \phi_k (\eta) \) becomes the Minkowski space field operator, and we have standard, typical vacuum fluctuations in \( \phi \). Well above the horizon exit, the scale dependence of the power spectrum of \( \delta \phi \) fluctuations is obtained as [15]
which leads to the scalar spectral index
\[ n_s = 1 - 2\varepsilon. \] (4.9)

The tensor-to-scalar ratio is also obtained as [16]
\[ r = \frac{P_T}{P_R} = \left[ \frac{\dot{\phi}^2}{\pi H^2} \right]_{\eta = aH} = \left[ \frac{9\phi^2}{\pi} \right]_{\eta = aH}, \] (4.10)

where use has been made of equations (2.21) and (2.31) together with \( \frac{H}{\dot{\phi}} = \frac{dW}{d\phi}, H \approx 0 \).

Comparing with the upper bounds obtained by the recent Planck and BICEP2 measurements [19–21]
\[ n_s = 0.9603 \pm 0.0073, \quad r \approx \begin{cases} 
0.11 & \text{Planck} \\
0.2 & \text{BICEP2}, 
\end{cases} \] (4.11)

we find \( \varepsilon \approx 2 \times 10^{-5} \) and
\[ [\phi]_{\eta = aH} \approx \begin{cases} 
0.19 M_p & \text{Planck} \\
0.26 M_p & \text{BICEP2}, 
\end{cases} \] (4.12)

which predicts the vacuum expectation value of the scalar field, at the time of leaving the horizon, in terms of the Planck mass.

5. Conclusion and discussion

Motivated by the fact that the common inflationary scenarios usually need a scalar-field potential to trigger the inflation, and that taking the proper inflation potential without fine tuning and cosmological constant problems is still an unsolved issue, we have studied the cosmological implications of a kinetic-coupled scalar-tensor gravity to establish a systematic inflation model that is capable of transition to the matter-dominant and dark energy-dominant eras. Moreover, we have studied the perturbation analysis of this inflation model in order to confront the inflation under discussion with the recent observational results. Note that the simplified perturbation analysis followed here, leads to no constraints on the parameter \( \alpha \).

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