Blow-up problems for a parabolic equation coupled with superlinear source and local linear boundary dissipation

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Abstract

In this paper, we consider the finite time blow-up results for a parabolic equation coupled with superlinear source term and local linear boundary dissipation. Using a concavity argument, we derive the sufficient conditions for the solutions to blow up in finite time. In particular, we obtain the existence of finite time blow-up solutions with arbitrary high initial energy. We also derive the upper bound and lower bound of the blow up time.

Keywords: Parabolic equation; Boundary dissipation; Finite time blow-up; Concavity method; Blow up time

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1 Introduction

In this paper, we consider the following parabolic equation coupled with superlinear source term and local linear boundary dissipation:

\[
\begin{aligned}
&u_t - \Delta u = |u|^{p-2}u, \quad x \in \Omega, \quad t > 0, \\
&u(x, t) = 0, \quad x \in \Gamma_0, \quad t > 0, \\
&\frac{\partial u}{\partial \nu} = -|u_t|^{m-2}u_t, \quad x \in \Gamma_1, \quad t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \( p > 2, \) \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) \( (n \geq 1) \) with \( C^1 \) boundary \( \partial \Omega \) and \( \nu \) denotes the unit outward normal vector to \( \partial \Omega. \) Let \( \{\Gamma_0, \Gamma_1\} \) be a partition of the boundary \( \partial \Omega \) such that

\[\partial \Omega = \Gamma_0 \cup \Gamma_1, \quad \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset,\]

where \( \Gamma_0 \) and \( \Gamma_1 \) are measurable over \( \partial \Omega, \) endowed with \((n - 1)\)-dimensional surface measure \( \sigma. \) We assume \( \sigma(\Gamma_0) > 0 \) throughout this paper.

Problem (1.1) can be used to describe a heat reaction-diffusion process which occurs inside a solid body \( \Omega \) surrounded by a fluid, with contact \( \Gamma_1 \) and having an internal cavity with contact boundary \( \Gamma_0. \) The function \( u = u(x, t) \) represents the temperature at point \( x \) and time \( t. \) The quantity of heat produced by the reaction is proportional to a superlinear power of the temperature, i.e. \( |u|^{p-2}u \) with \( p > 2. \) To avoid the internal explosion inside \( \Omega, \) a refrigerating system is introduced in the fluid. The refrigerating system works in such a way that the heat absorbed from the fluid is proportional to a power of the rate of change of the temperature, which can be described by

\[\frac{\partial u}{\partial \nu} = -|u_t|^{m-2}u_t, \quad x \in \Gamma_1,\]

where \( \frac{\partial u}{\partial \nu} \) represents the heat flux from \( \Omega \) to the fluid.

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Evolution equations with dynamical boundary conditions have been studied by many authors. We refer the readers to [1–3, 7, 13–16, 18] and the references therein. In this paper, we mainly focus on the finite time blow-up of solutions for parabolic equations with dynamical boundary conditions. In [9], using a certain concavity technique, Levine and Smith obtained the finite time blow-up result for the following problem:

\[
\begin{align*}
\begin{cases}
  u_t - \Delta u &= 0, &x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial n} &= f(u), &x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), &x \in \Omega,
\end{cases}
\end{align*}
\] (1.2)

provided the initial data \(u_0\) satisfies

\[
\frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 \, dx - \int_{\partial \Omega} \left( \int_0^{u_0(s)} f(z) \, dz \right) \, ds < 0.
\]

In [10], similar results were derived for more general classes of higher order equations. In [11], using the potential well method, Levine and Smith obtained the finite time blow-up result for problem (1.2) provided the initial data \(u_0\) satisfies

\[
\frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 \, dx - \int_{\partial \Omega} \left( \int_0^{u_0(s)} f(z) \, dz \right) \, ds < d, \quad \int_\Omega |\nabla u_0(x)|^2 \, dx - \int_{\partial \Omega} f(u_0(s)) \, u_0(s) \, ds < 0,
\]

where \(d\) is the potential well depth. In [12], Vitillaro obtained the local and global existence for the solutions of the following heat equation with local nonlinear boundary damping and source terms:

\[
\begin{align*}
\begin{cases}
  u_t - \Delta u &= 0, &x \in \Omega, \ t > 0, \\
  u(x, t) &= 0, &x \in \Gamma_0, \ t > 0, \\
  \frac{\partial u}{\partial n} &= -|u|^m - 2u_t + |u|^{p-2}u, &x \in \Gamma_1, \ t > 0, \\
  u(x, 0) &= u_0(x), &x \in \Omega.
\end{cases}
\end{align*}
\]

In [6], Fiscella and Vitillaro considered problem (1.1) with local nonlinear boundary dissipation (i.e. \(m \geq 2\)). Using the monotonicity method of J.L. Lions and a contraction argument, they showed the results of local well-posedness. They also proved that the weak solution blows up in finite time provided

\[
m < m_0(p) := \frac{2(n + 1)p - 4(n - 1)}{n(p - 2) + 4}
\]

and

\[
J(u_0) = \frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 \, dx - \frac{1}{p} \int_\Omega |u_0(x)|^p \, dx < d, \quad K(u_0) = \int_\Omega |\nabla u_0(x)|^2 \, dx - \int_\Omega |u_0(x)|^p \, dx < 0.
\]

It is natural to ask whether or not problem (1.1) admits finite time blow-up solutions with arbitrary high initial energy, especially for the case of \(J(u_0) \geq d\). The main purpose of this paper is to answer this question for the case of linear boundary dissipation (i.e. \(m = 2\)). We find two subsets \(B_1\) and \(B_2\) in the space \(H^1_\Gamma(\Omega)\), which are invariant under the semi-flow associated with problem (1.1). Combining them with a concavity argument, we prove that the weak solution of problem (1.1) blows up in finite time provided the initial data belongs to \(B_1 \cup B_2\). For any \(a \in \mathbb{R}\), we can construct a function \(u_0 \in B_2\) such that \(J(u_0) = a\) (see Corollary 3.5). In particular, we also derive the upper bound and lower bound for the blow up time.

The rest of this paper is organized as follows. In Section 2, we introduce some notations, definitions and conclusions that will be used in the sequel, including the result of local well-posedness. We establish a concavity argument for problem (1.1) in Section 3.1, then give the criteria of finite time blow-up in Section 3.2. Finally, combining with the interpolation inequality of Gagliardo–Nirenberg, we derive the lower bound for the blow up time in Section 3.3.

## 2 Preliminaries

For convenience, we denote \(\| \cdot \|_q = \| \cdot \|_{L^q(\Omega)}\), \(\| \cdot \|_{q, \Gamma_\alpha} = \| \cdot \|_{L^q(\Gamma_\alpha)}\) for \(1 \leq q \leq \infty\), and the Hilbert space

\[
H^1_{\Gamma_\alpha}(\Omega) = \left\{ w \in H^1(\Omega) : w|_{\Gamma_\alpha} = 0 \right\},
\]
where \( w|_{\Gamma_0} \) stands for the restriction of the trace of \( w \) on \( \partial \Omega \) to \( \Gamma_0 \). We also denote \((\cdot, \cdot)\) and \((\cdot, \cdot)_{\Gamma_1}\) as the inner products on the Hilbert spaces \( L^2(\Omega) \) and \( L^2(\Gamma_1) \) respectively. The trace theorem implies the existence of the continuous trace mapping

\[
H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\partial \Omega).
\]

Since \( \sigma(\Gamma_0) > 0 \), a Poincaré-type inequality holds, see [4, Theorem 6.7-5], and consequently \( \|\nabla w\|_2 \) is equivalent to the norm

\[
\|w\|_{H^1_{\Gamma_0}} = (\|w\|_2^2 + \|\nabla w\|_2^2)^{\frac{1}{2}}
\]

in the space \( H^1_{\Gamma_0}(\Omega) \). Based on the above arguments, we can define the following positive optimal constants

\[
S_1 = \sup_{w \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \frac{\|w\|_{2, \Gamma_0}}{\|\nabla w\|_2}, \quad S_2 = \sup_{w \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \frac{\|w\|_{2, \Gamma_0}}{\|\nabla w\|_2}.
\]  (2.1)

We now introduce the definition of weak solution.

**Definition 2.1.** Assume that \( u_0 \in H^1_{\Gamma_0}(\Omega) \),

\[
2 \leq p \leq 1 + \frac{2}{\varepsilon},
\]

where \( 2^* \) is the critical exponent of Sobolev embedding \( H^1(\Omega) \hookrightarrow L^q(\Omega), \) i.e., \( 2^* = \frac{2n}{n-2} \) if \( n \geq 3 \); \( 2^* = \infty \) if \( n = 1, 2 \). The function \( u \) is said to be a weak solution of problem (1.1) in \([0, T] \times \Omega\) if

(a) \( u \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega)); u_t \in L^2((0, T) \times \Omega); \)
(b) the spatial trace of \( u \) on \((0, T) \times \partial \Omega \) (which exists by the trace theorem) has a distributional time derivative \( u_t \) on \((0, T) \times \partial \Omega \), belong to \( L^2((0, T) \times \partial \Omega); \)
(c) for all \( \phi \in H^1_{\Gamma_0}(\Omega) \) and for almost all \( t \in [0, T] \) the distributional identity

\[
(u_t(t), \phi) + (\nabla u(t), \nabla \phi) + (u_t(t), \phi)_{\Gamma_1} = \int_\Omega \int_\Omega |u(t)|^{p-2}u(t)\phi
\]

holds true;
(d) \( u(0) = u_0 \).

**Remark 2.2.** By (a) and the continuous embedding

\[
H^1(0, T; L^2(\Omega)) \hookrightarrow C([0, T]; L^2(\Omega)),
\]

we have \( u \in C([0, T]; L^2(\Omega)) \), therefore \( u(0) \) makes sense in (d).

The following result of local well-posedness for problem (1.1) can be obtained by the monotonicity method of J.L. Lions and a contraction argument.

**Theorem 2.3** ([6, Theorem 2]). Let

\[
2 \leq p \leq 1 + \frac{2^*}{2},
\]  (2.3)

Then, for any \( u_0 \in H^1_{\Gamma_0}(\Omega) \), problem (1.1) has a unique weak maximal solution \( u \) in \([0, T_{\text{max}}) \times \Omega\), where \( T_{\text{max}} \) is the maximal existence time for the weak solution. Moreover \( u \in C([0, T_{\text{max}}]; H^1_{\Gamma_0}(\Omega)) \),

\[
u_t \in L^2((0, T) \times \Gamma_1) \cap L^2((0, T) \times \partial \Omega) \quad \text{for any} \ T \in (0, T_{\text{max}}),
\]

the energy identity

\[
\frac{1}{2}\|\nabla u\|_2^2 |_{s}^{t} + \int_{s}^{t} (\|u(\tau)\|_2^2 + \|u(\tau)\|_{2, \Gamma_1}^2) \, d\tau = \int_{s}^{t} \int_{\Omega} |u(t)|^{p-2}u(t) \, dx \, dt
\]  (2.4)

holds for \( 0 \leq s \leq t < T_{\text{max}} \) and the following alternative holds:

(i) either \( T_{\text{max}} = \infty; \)
(ii) or $T_{\text{max}} < \infty$ and
\[
\lim_{t \to T_{\text{max}}} \|u(t)\|_{H^1_{\Gamma_0}} = +\infty.
\]

**Remark 2.4.** In fact, if $p = 2$, the weak solution $u$ is global, see [6, Appendix B].

When $2 < p \leq 1 + \frac{2s}{2}$, we introduce the following functionals
\[
J(w) = \frac{1}{2}\|\nabla w\|_2^2 - \frac{1}{p}\|w\|_p^p, \quad K(w) = \|\nabla w\|_2^2 - \|w\|_p^p, \quad w \in H^1_{\Gamma_0}(\Omega).
\]

According to Definition 2.1 and Theorem 2.3, we obtain the following lemma.

**Lemma 2.5.** Assume that $u_0 \in H^1_{\Gamma_0}(\Omega)$,
\[2 < p \leq 1 + \frac{2s}{2}\]
and $u$ is the unique maximal weak solution of problem (1.1) in $[0, T_{\text{max}}) \times \Omega$. Then

(i) \[
\frac{d}{dt}\|u(t)\|_p^p = p \int_\Omega |u(t)|^{p-2}u(t)u_t(t)\,dx \quad \text{for a.e. } t \in (0, T_{\text{max}});
\]

(ii) \[
\frac{d}{dt}J(u(t)) = - (\|u_t(t)\|_{2, \Gamma_1}^2 + \|u_t(t)\|_2^2) \leq 0 \quad \text{for a.e. } t \in (0, T_{\text{max}});
\]

(iii) \[
\frac{d}{dt}\rho(t) = (u(t), u_t(t)) + (u(t), u_t(t))_{\Gamma_1} = -K(u(t)) \quad \text{for a.e. } t \in (0, T_{\text{max}}),
\]

where \[
\rho(t) = \frac{1}{2}\|u(t)\|_2^2 + \frac{1}{2}\|u(t)\|_{2, \Gamma_1}^2.
\]

**Proof.** For the proof of statement (i), we refer to [6, Lemma 3].

By statement (i) and (2.4), we have
\[
J(u(t)) - J(u_0) = - \int_0^t (\|u_\tau(\tau)\|_2^2 + \|u_\tau(\tau)\|_{2, \Gamma_1}^2) \,d\tau
\]
for any $t \in [0, T_{\text{max}})$. In view of the regularity of the weak solution, the function $t \mapsto J(u(t))$ is absolutely continuous on $[0, T_{\text{max}})$ and therefore (2.6) holds.

Since $u \in C([0, T_{\text{max}}); H^1_{\Gamma_0}(\Omega))$ and
\[
u_t \in L^2((0, T) \times \Gamma_1) \cap L^2((0, T) \times \Omega)
\]
for any $t \in (0, T_{\text{max}})$, the function $\rho$ is of absolutely continuous on $(0, T_{\text{max}})$ and
\[
\frac{d}{dt}\rho(t) = (u(t), u_t(t)) + (u(t), u_t(t))_{\Gamma_1} \quad \text{for a.e. } t \in (0, T_{\text{max}}).
\]

Taking $\phi = u(t)$ in (2.2), we get
\[
\frac{d}{dt}\rho(t) = (u, u_t) + (u, u_t)_{\Gamma_1} = -K(u(t)) \quad \text{for a.e. } t \in (0, T_{\text{max}}).
\]

\[\boxdot\]

Finally, we introduce the potential well depth by
\[
d = \inf_{w \in N} J(w) = \inf_{w \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \sup_{\lambda > 0} J(\lambda w),
\]
where $N$ is the Nehari manifold
\[
N = \{w \in H^1_{\Gamma_0}(\Omega) \mid K(w) = 0\} \setminus \{0\}.
\]
It is easy to verify (see [6, Lemma 2])

\[ d = \left(\frac{1}{2} - \frac{1}{p}\right) B_1^{-\frac{2p}{p-2}} > 0, \]

where

\[ B_1 = \sup_{w \neq 0 \in H_{1,0}^1(\Omega)} \frac{\|w\|_p}{\|\nabla w\|_2}. \]

**Lemma 2.6.** When \(2 < p \leq 1 + \frac{2^*}{2}\), it holds that

\[ \|w\|_p^p > \frac{2p}{p-2}d \]

for any

\[ w \in N_- = \{w \in H_{1,0}^1(\Omega) \mid K(w) < 0\}. \]

**Proof.** Since

\[ \|\nabla w\|_2^2 - \|w\|_p^p = K(w) < 0, \]

we have \(w \neq 0\) and therefore \(K(\lambda^* w) = 0\) with

\[ \lambda^* = \left(\frac{\|\nabla w\|_2^2}{\|w\|_p^p}\right)^{\frac{1}{p-2}} \in (0, 1), \]

i.e. \(\lambda^* w \in N\). In view of the definition of the potential well depth \(d\), it holds

\[ d = \inf_{w \in N} J(w) \leq J(\lambda^* w) \]

\[ = \frac{p-2}{2p} \|\lambda^* w\|_p^p + \frac{1}{2} K(\lambda^* w) \]

\[ = \frac{p-2}{2p} (\lambda^*)^p \|w\|_p^p \]

\[ < \frac{p-2}{2p} \|w\|_p^p. \]

So we consequently obtain

\[ \|w\|_p^p > \frac{2p}{p-2}d. \]

The set \(N_-\) plays an important role in the study of finite time blow-up. In fact, we have the following proposition for the necessary condition of finite time blow-up.

**Proposition 2.7.** Assume that

\[ 2 < p \leq 1 + \frac{2^*}{2} \]

and the weak solution \(u\) of problem (1.1) blows up in finite time. Then, there exists \(t^* \in [0, T_{\text{max}})\) such that

\[ u(t^*) \in N_- \]

**Proof.** Suppose, by the contrary, that

\[ K(u(t)) \geq 0 \] for all \(t \in [0, T_{\text{max}})\).

Then, by Lemma 2.5(ii), we have

\[ J(u_0) \geq J(u(t)) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{p} \|u(t)\|_p^p \]

\[ = \frac{p-2}{2p} \|\nabla u\|_2^2 + \frac{p}{p} K(u(t)) \]

\[ \geq \frac{p-2}{2p} \|\nabla u(t)\|_2^2 \]

for any \(t \in [0, T_{\text{max}})\), which contradicts Theorem 2.3(ii).
3 The finite time blow-up results

3.1 A concavity argument

We will apply Levine’s concavity method to obtain the finite time blow-up results.

**Lemma 3.1** ([8]). Assume that a positive function $F$ on $[0, T]$ satisfies the following conditions:

(i) $F$ is differentiable on $[0, T]$, and $F'$ is absolutely continuous on $[0, T]$ with $F'(0) > 0$;
(ii) there exists a positive constant $\alpha > 0$ such that

$$F(t)F''(t) - (1 + \alpha) (F'(t))^2 \geq 0 \quad \text{for a.e } t \in [0, T].$$

Then

$$T \leq \frac{F(0)}{\alpha F'(0)}.$$

**Remark 3.2.** The condition (ii) in Lemma 3.1 implies that the function

$$G(t) = (F(t))^{-\alpha}$$

is concave on $[0, T]$.

Suppose that $2 < p \leq 1 + \frac{2^*}{2}$ and $u$ is the unique maximal weak solution of problem (1.1) with initial data $u_0 \in H^1_{1,0}(\Omega)$. Denote

$$\rho(t) = \frac{1}{2} \|u(t)\|^2_2 + \frac{1}{2} \|u(t)\|^2_{2, \Gamma_1}, \quad t \in [0, T_{\text{max}}).$$

According to Lemma 2.5 (iii), it holds that

$$\frac{d}{dt} \rho(t) = (u, u_t) + (u, u_t)_{\Gamma_1} = -K(u(t)) \quad \text{for a.e. } t \in (0, T_{\text{max}}).$$

Choose an arbitrary $T$ such that

$$0 < T < T_{\text{max}}$$

and define the auxiliary function

$$F(t) = \int_0^t \rho(\tau)d\tau + (T - t) \rho(0) + \frac{1}{2} \beta (t + \sigma)^2, \quad t \in [0, T],$$

where $\beta$ and $\sigma$ are the positive parameters to be determined later. In view of (2.5), (2.7) and Lemma 2.5 (iii), we have

$$F'(t) = \rho(t) - \rho(0) + \beta(t + \sigma)$$

$$= \int_0^t \left( \frac{d}{dt}\rho(\tau)d\tau + (t + \sigma) \rho(\tau) \right)$$

$$= \int_0^t (u, u_t) d\tau + \int_0^t (u, u_t)_{\Gamma_1} d\tau + \beta(t + \sigma)$$

for any $t \in [t_0, T]$ and

$$F''(t) = \frac{d}{dt} \rho(t) + \beta$$

$$= -K(u(t)) + \beta$$

$$= -\left( pJ(u(t)) - \frac{p - 2}{2} \|\nabla u(t)\|^2_2 \right) + \beta$$

$$= \frac{p - 2}{2} \|\nabla u(t)\|^2_2 - p \left[ J(u(0)) - \int_0^t (\|u_t(\tau)\|^2_2 + \|u_t(\tau)\|^2_{2, \Gamma_1}) d\tau \right] + \beta$$
\[
\frac{d}{dt} \|u(t)\|_{L^2}^2 + p \int_0^t \|u(\tau)\|_{L^2}^2 \, d\tau + p \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau - pJ(u_0) + \beta \leq \frac{p-2}{2} \|
abla u(t)\|_{L^2}^2 + p \int_0^t \|u(\tau)\|_{L^2}^2 \, d\tau + p \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau - pJ(u_0) + \beta
\]
(3.3)

for a.e. \( t \in (0, T) \). Note that the auxiliary function \( F \) is positive on \([0, T]\) and \( F'(0) = \beta \sigma > 0 \).

Now we derive an estimation for \( FF'' - \lambda (F')^2 \), where \( \lambda > 0 \) is a constant to be determined later.

Using the Cauchy-Schwartz inequality and Young’s inequality, we obtain

\[
\xi(t) = \left[ \int_0^t \|u(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right]
\cdot \left[ \int_0^t \|u_t(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right]
\leq \left[ \int_0^t \|u(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right] \cdot \left( \int_0^t \|u_t(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right)
\geq 0
\]

for any \( t \in [0, T] \). Therefore, in view of (3.1)-(3.3), it holds that

\[
FF'' - \lambda (F')^2 \equiv FF'' - \lambda \left[ \int_0^t (u, u_t) \, d\tau + \int_0^t (u_t, u_{\Gamma_1}) \, d\tau + \beta(t + \sigma)^2 \right]^2
\leq FF'' + \lambda \left[ \xi(t) - \left( \int_0^t \|u(\tau)\|_{L^2}^2 + \int_0^t \|u(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right) \cdot \left( \int_0^t \|u_t(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right) \right]
\leq FF'' - 2\lambda F(t) \cdot \left( \int_0^t \|u_t(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right)
\leq F(t) \cdot \left[ \left( \frac{p-2}{2} \|
abla u(t)\|_{L^2}^2 + p \int_0^t \|u_t(\tau)\|_{L^2}^2 \, d\tau + p \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau - pJ(u_0) + \beta \right) - 2\lambda \left( \int_0^t \|u_t(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau + \beta(t + \sigma)^2 \right) \right]
\leq F(t) \cdot \left[ \left( \frac{p-2}{2} \|
abla u(t)\|_{L^2}^2 + (p - 2\lambda) \int_0^t \|u_t(\tau)\|_{L^2}^2 \, d\tau + (p - 2\lambda) \int_0^t \|u_t(\tau)\|_{L^2,\Gamma_1}^2 \, d\tau - pJ(u_0) + (1 - 2\lambda)\beta \right) \right].
\]

Taking \( \lambda = \frac{p}{2} \), we finally obtain the following estimation:

\[
FF'' - \frac{p}{2} (F')^2 \geq F \cdot \left[ \frac{p-2}{2} \|
abla u(t)\|_{L^2}^2 - pJ(u_0) - (p - 1)\beta \right] \quad \text{for a.e. } t \in [0, T].
\]
(3.4)
Lemma 3.3. Assume that $\rho(0) > 0$ and there exists some positive parameter $\beta$ such that
\[
\frac{p-2}{2} \|\nabla u(t)\|_2^2 - pJ(u_0) - (p-1) \beta \geq 0
\]
holds for any $t \in (0,T]$. Then
\[
0 < T \leq \frac{8\rho(0)}{(p-2)^2 \beta}
\]

Proof. Note that $\frac{p}{2} > 1$. In view of Lemma 3.1 and (3.4), we have
\[
T \leq \frac{F(0)}{\frac{p}{2} \cdot F'(0)} = \frac{T \rho(0) + \frac{1}{2} \beta \sigma^2}{\frac{p}{2} \cdot \beta \sigma} = \frac{2\rho(0)}{(p-2)\beta}T + \frac{\sigma}{p-2}, \tag{3.5}
\]
To guarantee
\[
\frac{2\rho(0)}{(p-2)\beta} \sigma < 1,
\]
we restrict the range of $\sigma$ to be $\left(\frac{2\rho(0)}{(p-2)\beta}, +\infty\right)$. Therefore, by (3.5), we obtain
\[
T \leq \left(1 - \frac{2\rho(0)}{(p-2)\beta} \right)^{-1} \frac{\sigma}{p-2}. \tag{3.6}
\]
A direct calculation shows that the right hand side of (3.6) takes its minimum at
\[
\sigma = \sigma_\beta = \frac{4\rho(0)}{(p-2)\beta} \in \left(\frac{2\rho(0)}{(p-2)\beta}, +\infty\right).
\]
Since $T$ is independent of the parameter $\sigma$, we finally obtain
\[
T \leq \left(1 - \frac{2\rho(0)}{(p-2)\beta} \sigma_\beta \right)^{-1} \frac{\sigma_\beta}{p-2} = \frac{8\rho(0)}{(p-2)^2 \beta}.
\]

3.2 The finite time blow-up criteria

Based on Lemma 3.3, we give the following finite time blow-up criteria.

Theorem 3.4. Assume that
\[
2 < p \leq 1 + \frac{2^*}{2}
\]
and the initial data $u_0$ belongs to one of the following sets:
\[
B_1 = \left\{ w \in H^1_{\text{loc}}(\Omega) : J(w) < d \text{ and } K(w) < 0 \right\},
\]
\[
B_2 = \left\{ w \in H^1_{\text{loc}}(\Omega) : J(w) < \frac{p-2}{2p(S_1 + S_2)} (\|w\|_2^2 + \|w\|_{2, \Gamma_1}^2) \right\}.
\]

Then the weak solution $u$ of problem (1.1) blows up in finite time. Moreover, if $u_0 \in B_1$, we have
\[
T_{\text{max}} \leq \frac{4(p-1)}{p(p-2)^2} \|u_0\|_2^2 + \|u_0\|_{2, \Gamma_1}^2 - d - J(u_0)
\]
if $u_0 \in B_2$, we have
\[
T_{\text{max}} \leq \frac{4(p-1)}{p(p-2)^2} \frac{\|u_0\|_2^2 + \|u_0\|_{2, \Gamma_1}^2}{2p(S_1 + S_2)} - J(u_0)
\]

Proof. Part 1. The case of $u_0 \in B_1$.
First, we show that $u(t) \in B_1$ holds for any $t \in [0,T_{\text{max}})$ provided $u_0 \in B_1$.
By the energy identity (2.7), we have
\[
J(u(t)) = J(u_0) - \int_0^t \left(\|u_0\|_2^2 + \|u(t)\|_{2, \Gamma_1}^2\right) \, d\tau \leq J(u_0) - d \tag{3.7}
\]
for any \( t \in [0, T_{\max}) \). We only need to prove that \( K(u(t)) < 0 \) for any \( t \in [0, T_{\max}) \). Since \( u \in C([0, T_{\max}); H^1_\Gamma(\Omega)) \), the mapping \( t \mapsto K(u(t)) \) is continuous on \([0, T_{\max})\). Suppose, on the contrary, that there exists \( t_1 \in (0, T_{\max}) \) such that
\[
K(u(t)) < 0 \quad \text{for any} \ t \in [0, t_1); \quad K(u(t_1)) = 0.
\]
Considering Lemma 2.6 and the continuity of the mapping \( t \mapsto \|\nabla u(t)\|_2 \), we have
\[
\|\nabla u(t_1)\|_2^2 = \lim_{t \to t_1^-} \|\nabla u(t)\|_2^2 \geq \frac{2p}{p-2} d > 0,
\]
so \( u(t_1) \in N \). While, the definition of the potential well depth \( d \) shows that
\[
d = \inf_{u \in N} J(u) \leq J(u(t_1)),
\]
which contradicts (3.7).

According to the above argument, \( u(t) \in B_1 \) for any \( t \in [0, T_{\max}) \) provided \( u_0 \in B_1 \). It is obvious that \( \rho(0) > 0 \). In view of Lemma 2.6, we have
\[
\frac{p-2}{2} \|\nabla u(t)\|_2^2 - pJ(u(t)) - (p-1)\beta > pd - pJ(u_0) - (p-1)\beta \geq 0
\]
for any \( t \in (0, T_{\max}) \) and any
\[
\beta \in \left(0, \frac{p}{p-1} (d - J(u_0))\right).
\]
According to Lemma 3.3, we obtain
\[
0 < T \leq \frac{8(p-1)\rho(0)}{p(p-2)^2 (d - J(u_0))} < \infty
\]
for any \( T \in (0, T_{\max}) \). So the maximal existence time \( T_{\max} \) of the weak solution \( u \) satisfies that
\[
0 < T_{\max} \leq \frac{4(p-1)}{p(p-2)^2} \frac{\|u_0\|_2^2 + \|u_0\|_{2, \Gamma_1}^2}{d - J(u_0)},
\]
hence \( u \) blows up in finite time.

**Part 2. The case of \( u_0 \in B_2 \).**

By Lemma 2.5 (iii) and (2.1), we have
\[
\frac{d}{dt} \rho(t) = -K(u(t)) = \frac{p-2}{2} \|\nabla u(t)\|_2^2 - pJ(u(t)) \geq \frac{p-2}{2} \frac{2}{S_1 + S_2} \rho(t) - pJ(u(t)) = \frac{p}{A} \left(\rho(t) - AJ(u(t))\right)
\]
for a.e. \( t \in (0, T_{\max}) \), where
\[
A = \frac{p(S_1 + S_2)}{p-2} > 0.
\]
Let \( H(t) = \rho(t) - AJ(u(t)) \). In view of Lemma 2.5 (ii) and (3.8), we obtain
\[
\frac{d}{dt} H(t) = \frac{d}{dt} \rho(t) - A \frac{d}{dt} J(u(t)) \geq \frac{d}{dt} \rho(t) \geq \frac{p}{A} H(t)
\]
for a.e. \( t \in (0, T_{\max}) \). Using Gronwall’s inequality, we have \( H(t) \geq e^{\frac{pt}{A}} H(0) \). The assumption \( u_0 \in B_2 \) implies that
\[
H(0) = \rho(0) - AJ(u_0) = \frac{1}{2} \left(\|u_0\|_2^2 + \|u_0\|_{2, \Gamma_1}^2\right) - \frac{p(S_1 + S_2)}{p-2} J(u_0) > 0,
\]
so
\[
\frac{d}{dt} \rho(t) \geq \frac{p}{A} H(t) \geq \frac{p}{A} e^{\frac{pt}{A}} H(0) > 0
\]
for a.e. \( t \in (0, T_{\max}) \), which means that \( \rho(t) \) is nondecreasing on \([0, T_{\max})\).
In view of (2.1) and the monotonicity of $\rho(t)$, it follows that

$$
\begin{align*}
\frac{p-2}{2}\|\nabla u(t)\|^2 - pJ(u_0) - (p-1)\beta \\
\geq \frac{p-2}{2} \cdot \frac{2}{S_1 + S_2} \rho(t) - pJ(u_0) - (p-1)\beta \\
\geq \frac{p-2}{2} \cdot \frac{2}{S_1 + S_2} \rho(0) - pJ(u_0) - (p-1)\beta \\
= \frac{p}{A} H(0) - (p-1)\beta \\
\geq 0
\end{align*}
$$

for any $t \in (0, T_{\text{max}})$ and any

$$
\beta \in \left(0, \frac{pH(0)}{A(p-1)}\right).
$$

According to Lemma 3.3, we obtain

$$
0 < T \leq \frac{8A(p-1)\rho(0)}{p(p-2)^2 H(0)}
$$

for any $T \in (0, T_{\text{max}})$. So the maximal existence time $T_{\text{max}}$ of the weak solution $u$ satisfies that

$$
0 < T_{\text{max}} \leq \frac{4(p-1)}{p(p-2)^2} \cdot \frac{\|u_0\|^2 + \|u_0\|_2^2}{\|u_0\|^2 + \|u_0\|_2^2} \cdot \max_{0 < t < T_{\text{max}}} \{ J(u_0) \},
$$

hence $u$ blows up in finite time.

It is obvious that both $B_1$ and $B_2$ are none-empty sets. Moreover, the following corollary implies that, for any $a \in \mathbb{R}$, there exists $u_0 \in H^1_\Omega(\Omega)$ with initial energy $J(u_0) = a$ which leads to finite time blow-up solution.

**Corollary 3.5.** For any $a \in \mathbb{R}$, denote the energy level set by

$$
J^a := \{ w \in H^1_\Omega(\Omega) | J(w) = a \}.
$$

Then $J^a \cap B_2 \neq \emptyset$.

**Proof.** Assume that $\Omega_1$ and $\Omega_2$ are two disjoint open subdomains of $\Omega$, and

$$
\text{dist} (\Omega_1, \partial \Omega) > 0, \quad \text{dist} (\Omega_2, \partial \Omega) > 0, \quad \text{dist} (\Omega_1, \Omega_2) > 0.
$$

According to the proof of Theorem 3.7 in [17], there exists a sequence $\{v_k\} \subset H^1_\Omega(\Omega_1)$ such that

$$
\frac{1}{2} \int_{\Omega_1} |\nabla v_k(x)|^2 \, dx - \frac{1}{p} \int_{\Omega_1} |v_k(x)|^p \, dx \to +\infty \quad \text{as } k \to \infty. \tag{3.9}
$$

On the other hand, choosing an arbitrary nonzero function $w \in C^\infty_\Omega(\Omega)$ with the support $\text{supp } w \subset \Omega_2$, then

$$
a - \left( \frac{1}{2} \int_{\Omega_2} |\nabla (rw(x))|^2 \, dx - \frac{1}{p} \int_{\Omega_2} |rw(x)|^p \, dx \right) \to +\infty \quad \text{as } r \to +\infty, \tag{3.10}
$$

and there exists $r_0 > 0$ such that

$$
\frac{p-2}{2p(S_1 + S_2)} \int_{\Omega_2} |w(x)|^2 \, dx = r^2 \cdot \frac{p-2}{2p(S_1 + S_2)} \int_{\Omega_2} |w(x)|^2 \, dx > a \tag{3.11}
$$

for any $r > r_0$. By (3.9) and (3.10), there exist $k_0 \in \mathbb{N}_+$ and $r_1 > r_0$ such that

$$
\frac{1}{2} \int_{\Omega_1} |\nabla v_{k_0}(x)|^2 \, dx - \frac{1}{p} \int_{\Omega_1} |v_{k_0}(x)|^p \, dx = a - \left( \frac{1}{2} \int_{\Omega_2} |\nabla (r_1w(x))|^2 \, dx - \frac{1}{p} \int_{\Omega_2} |r_1w(x)|^p \, dx \right) \tag{3.12}
$$

Let $u_0 = \tilde{v} + r_1w$, where

$$
\tilde{v}(x) = \begin{cases} 0, & x \in \Omega \setminus \Omega_1 \\ v_{k_0}(x), & x \in \Omega_1. \end{cases}
$$
It is easy to verify that \( u_0 \in H^1_{\Gamma_0}(\Omega) \) and \( u_0(x) = 0 \) on \( \Omega \setminus (\Omega_1 \cup \Omega_2) \). In view of (3.11) and (3.12), we finally obtain

\[
J(u_0) = \frac{1}{2} \left( \int_{\Omega_1} + \int_{\Omega_2} \right) |\nabla u_0(x)|^2 \, dx - \frac{1}{p} \left( \int_{\Omega_1} + \int_{\Omega_2} \right) |u_0(x)|^p \, dx \\
= \left( \frac{1}{2} \int_{\Omega_1} |\nabla v_k_0(x)|^2 \, dx - \frac{1}{p} \int_{\Omega_1} |v_k_0(x)|^p \, dx \right) + \left( \frac{1}{2} \int_{\Omega_2} |\nabla (r_1 w(x))|^2 \, dx - \frac{1}{p} \int_{\Omega_2} |r_1 w(x)|^p \, dx \right) \\
\geq \frac{p - 2}{2p(S_1 + S_2)} \int_{\Omega_2} |r_1 w(x)|^2 \, dx \\
< \frac{p - 2}{2p(S_1 + S_2)} \int_{\Omega_2} |u_0(x)|^2 \, dx \\
= \frac{p - 2}{2p(S_1 + S_2)} (\|u_0\|_2^2 + \|u_0\|_{2, \Gamma_1}^2),
\]

i.e., \( u_0 \in J^a \cap B \). This completes the proof. \( \square \)

### 3.3 Lower bound of the blow up time

In view of the proof of Theorem 3.4, the sets \( B_1 \) and \( B_2 \) are invariant under the semi-flow associated with problem (1.1), that is to say, \( u(t) \in B_1 \) for any \( t \in (0, T_{max}) \) provided the initial data \( u_0 \in B_1 \), and \( u(t) \in B_2 \) for any \( t \in (0, T_{max}) \) provided the initial data \( u_0 \in B_2 \). On the other hand, by (2.1), we have

\[
K(w) = \|\nabla w\|^2 - \|w\|^p_p \\
= p J(w) - \frac{p - 2}{2} \|\nabla w\|_2^2 \\
\leq p \left[ J(w) - \frac{p - 2}{2p(S_1 + S_2)} (\|w\|_2^2 + \|w\|_{2, \Gamma_1}^2) \right] \\
< 0
\]

for the case of \( w \in B_2 \). Therefore, it holds that

\[
u(t) \in N_- = \{ w \in H^1_{\Gamma_0}(\Omega) \mid K(w) < 0 \}
\]

for any \( t \in [0, T_{max}) \) provided the initial data \( u_0 \in B_1 \cup B_2 \). Based on the above arguments, it is natural to ask whether or not the condition \( u_0 \in N_- \) is sufficient enough for finite time blow-up. This is not an easy task and we only refer the reader to [5] for similar research.

Now we derive the lower bound of the blow-up time.

**Theorem 3.6.** Assume that

\[
2 < p \leq 1 + \frac{2^*}{2}, \quad p < 2 + \frac{4}{n},
\]

the weak solution \( u \) of problem (1.1) blows up in finite time and \( u(t) \in N_- \) for any \( t \in [0, T_{max}) \). Then

\[
T_{max} \geq \frac{\tilde{C}}{ \left( \|u_0\|_2^2 + \|u_0\|_{2, \Gamma_1}^2 \right)^{\frac{2(p - 2)}{2p - 2}}},
\]

where \( \tilde{C} \) is a positive constant that will be determined in the proof.

**Proof.** Since \( u(t) \in N_- \) for any \( t \in [0, T_{max}) \), considering Lemma 2.6 and the Sobolev embedding theorem, we have \( \|\nabla u(t)\|_2 > 0, \|u(t)\|_p > 0 \) and

\[
\rho(t) = \frac{1}{2} \|u(t)\|_2^2 + \frac{1}{2} \|u(t)\|_{2, \Gamma_1}^2 \geq \frac{1}{2} \|u(t)\|_2^2 > 0
\]

for any \( t \in [0, T_{max}) \). Using the interpolation inequality of Gagliardo–Nirenberg, we have

\[
\|\nabla u(t)\|_2^2 < \|u(t)\|_p^p \leq S_3 \|u(t)\|_{2, \Gamma_1}^{2(p(1 - \sigma))} \|\nabla u(t)\|_2^{p\sigma}
\]

(3.13)
for any \( t \in [0, T_{\text{max}}) \), where \( S_3 \) is a positive constant and \( \sigma = \frac{n(p - 2)}{2p} \). Since \( p < 2 + \frac{4}{n} \), it is easy to verify that \( 2 - p\sigma > 0 \), so

\[
\|\nabla u(t)\|_2 \leq S_3^{\frac{1}{p-\sigma}} \|u(t)\|^{\frac{p-2}{2}}_2 \quad \text{for any } t \in [0, T_{\text{max}}).
\]

From Lemma 2.5 (iii) and (3.13), it follows that

\[
\frac{d}{dt} \rho(t) = -K(u(t)) = \|u(t)\|^p_p - \|\nabla u(t)\|_2^2
\]

\[
< \|u(t)\|^p_p \leq S_3 \|u(t)\|^{p(1-\sigma)}_2 \|\nabla u(t)\|_2^{p\sigma}
\]

\[
\leq S_3 \|u(t)\|^{p(1-\sigma)}_2 S_3^{\frac{p\sigma}{p-\sigma}} \|u(t)\|_2^{\frac{p\sigma(p-2)}{2-p\sigma}}
\]

\[
= S_3^{\frac{2}{p-\sigma}} \|u(t)\|_2^{\frac{2}{p-\sigma}}
\]

\[
\leq S_3^{\frac{2}{p-\sigma}} [2\rho(t)]^{\frac{2}{p-\sigma}}
\]

\[
= S_4 [\rho(t)]^{\frac{2}{p-\sigma}}
\]

for a.e. \( t \in [0, T_{\text{max}}) \), where

\[ S_4 = S_3^{\frac{1}{p-\sigma}} \cdot 2^{\frac{2}{p-\sigma}} > 0 \]

Noting that \( \frac{p-2}{2-p\sigma} > 1 \), integrating the differential inequality

\[
\frac{d}{dt} \rho(t) < S_4 [\rho(t)]^{\frac{2}{p-\sigma}},
\]

we then obtain

\[
\rho(t) - \rho(0) = -S_4 \cdot \frac{p-2}{2-p\sigma} t \tag{3.14}
\]

for any \( t \in (0, T_{\text{max}}) \). Since the weak solution \( u \) blows up in finite time, from Theorem 2.3 (ii), it follows that \( \lim_{t \to (T_{\text{max}}^-)} \rho(t) = +\infty \). Letting \( t \to (T_{\text{max}}^-) \) in (3.14), we then obtain

\[
-\rho(t) - \rho(0) \geq -S_4 \cdot \frac{p-2}{2-p\sigma} T_{\text{max}}.
\]

Finally, we have

\[
T_{\text{max}} \geq \frac{2 - p\sigma}{S_4(p-2)} \rho(t) - \rho(0) = \frac{\tilde{C}}{(\|u_0\|_2^2 + \|u_0\|_{2, \Gamma_1})^{2\rho(t)-2}},
\]

where

\[
\tilde{C} = \frac{2 - p\sigma}{S_4(p-2)} \cdot 2^{\frac{2}{p-\sigma}} > 0.
\]

\[ \square \]

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**References**

[1] I. Bejenaru, J. I. Diaz, and I. I. Vrabie. An abstract approximate controllability result and applications to elliptic and parabolic systems with dynamic boundary conditions. *Electron. J. Differ. Eq.*, 2001(50):1–19, 2001.
[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and I. Lasiecka. Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping–source interaction. *J. Differ. Equ.*, 236(2):407–459, 2007.

[3] I. Chueshov, M. Eller, and I. Lasiecka. On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation. *Commun Part. Diff. Eq.*, 27(9-10):1901–1951, 2002.

[4] P. G. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications*. SIAM-Society for Industrial and Applied Mathematics, 2013.

[5] F. Dickstein, N. Mizoguchi, P. Souplet, and F. Weissler. Transversality of stable and Nehari manifolds for a semilinear heat equation. *Calc. Var. Partial Diff.*, 42(3):547–562, 2011.

[6] A. Fiscella and E. Vitillaro. Local Hadamard well-posedness and blow-up for reaction-diffusion equations with non-linear dynamical boundary conditions. *Discrete Contin. Dyn. Syst.-Ser. A*, 33(11&12):5015–5047, 2013.

[7] T. Hintermann. Evolution equations with dynamic boundary conditions. *P. Roy. Soc. Edinb. A*, 113(1-2):43–60, 1989.

[8] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $P u_t = -Au + F(u)$. *Arch. Ration. Mech. Anal.*, 51(5):371–386, 1973.

[9] H. A. Levine and L. E. Payne. Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time. *J. Differ. Equ.*, 16(2):319–334, 1974.

[10] H. A. Levine and L. E. Payne. Some nonexistence theorems for initial-boundary value problems with nonlinear boundary constraints. *Proc. Am. Math. Soc.*, 46(2):277–284, 1974.

[11] H. A. Levine, R. A. Smith, and L. E. Payne. A potential well theory for the heat equation with a nonlinear boundary condition. *Math. Method Appl. Sci.*, 9(1):127–136, 1987.

[12] E. Vitillaro. Global existence for the heat equation with nonlinear dynamical boundary conditions. *P. Roy. Soc. Edinb. A*, 135(1):175–207, 2005.

[13] E. Vitillaro. On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and source. *Arch. Ration. Mech. Anal.*, 223(3):1183–1237, 2017.

[14] E. Vitillaro. On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources. *J. Differ. Equ.*, 265(10):4873–4941, 2018.

[15] E. Vitillaro. Blow–up for the wave equation with hyperbolic dynamical boundary conditions, interior and boundary nonlinear damping and sources. *Discrete Contin. Dyn. Syst.-Ser. S*, 14(12):4575–4608, 2021.

[16] J. L. Vázquez and E. Vitillaro. Heat equation with dynamical boundary conditions of reactive–diffusive type. *J. Differ. Equ.*, 250(4):2143–2161, 2011.

[17] M. Willem. *Minimax Theorems*. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Boston Inc, Boston, 1996.

[18] X. Yang and Z. Zhou. Blow-up problems for the heat equation with a local nonlinear Neumann boundary condition. *J. Differ. Equ.*, 261(5):2738–2783, 2016.