Abelian Magnetic Monopoles and Topologically Massive Vector Bosons in Scalar-Tensor Gravity with Torsion Potential

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Abstract

A Lagrangian formulation describing the electromagnetic interaction - mediated by topologically massive vector bosons - between charged, spin-$\frac{1}{2}$ fermions with an abelian magnetic monopole in a curved spacetime with non-minimal coupling and torsion potential is presented. The covariant field equations are obtained. The issue of coexistence of massive photons and magnetic monopoles is addressed in the present framework. It is found that despite the topological nature of photon mass generation in curved spacetime with isotropic dilaton field, the classical field theory describing the nonrelativistic electromagnetic interaction between a point-like electric charge and magnetic monopole is inconsistent.

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1 Introduction

Extended Theories of Gravity have become a sort of paradigm in the study of gravitational interaction since several motivations push for enlarging the traditional scheme of Einstein’s General Relativity (GR) \cite{1}. Such issues come, essentially, from cosmology and quantum field theory. In the first case, it is well known that higher-order derivative theories and scalar-tensor theories give rise to inflationary cosmological solutions capable, in principle, of solving the shortcomings of the Standard Cosmological Model. Besides, they have relevant features also from the quantum cosmology viewpoint. In the second case, every unification scheme as Superstrings, Supergravity or Grand Unified Theories, takes into account effective actions where nonminimal couplings to the geometry or higher-order terms in the curvature invariants come out. Such contributions are due to one-loop or higher-loop corrections in the high-curvature regimes near the full (not yet available) quantum gravity regime. In the weak-limit approximation, all these classes of theories should be expected to reproduce Einstein’s GR which, in any case, is experimentally tested only in this limit. This issue is debatable however, since several relativistic theories do not reproduce those of GR in the Newtonian approximation.

Magnetic monopoles were first proposed by Dirac in the framework of classical electrodynamics in his classic works \cite{2}. The main purpose for the introduction of monopoles was to provide a physical explanation for the quantization of electric charge. This is known as the Dirac quantization rule. Antisymmetric tensor gauge fields analogous to the torsion potential employed here were proposed some time ago in the literature. The first description of such an antisymmetric field was due to Ogievetskii and Polubarinov \cite{3}. In 1973, Kalb and Raymond \cite{4} described classical
string interactions by means of an antisymmetric field the interpretation of which is that of a potential generated by the string. Scherk and Schwarz [5] showed that torsion could be viewed as the product of the antisymmetric field of string theory multiplied by a scalar field. In their work, the spacetime metric was not covariantly constant. In [6], Fradkin and Tseytlin derived an effective Lagrangian density in the low energy limit of string theory, describing not only gravity but also a scalar (dilaton) and antisymmetric field.

In the present work we implement a scalar-tensor generalization of gravity in the sense of Brans-Dicke [7] with non-vanishing curvature and torsion, whereby the gravitational coupling constant becomes a scalar field. This scalar field is identified as the dilaton. Being a new dynamical variable of the theory, we include a kinetic term for the dilaton in the total system Lagrangian density. Gravitational theories with torsion such as Einstein-Cartan theory [9] or Poincaré gauge theory [10, 11, 12, 13] describe the torsion as being the anti-symmetric part of a generalized affine connection or as the Cartan structure equation for the dual frame field taken as a gauge potential. By contrast, we assume in this paper that torsion is derived from a (anti-symmetric) second-rank tensor potential [14] which could be further generalized by considering bi-vectors [15]. We admit topological interaction between torsion and electromagnetic gauge potentials and consider the electromagnetic interaction between charged, nonrelativistic spin-1/2 fermions with an abelian magnetic monopole. As a consequence of the electromagnetic and torsion gauge field coupling, the fermion-monopole interaction is mediated by topologically massive vector bosons. The material Lagrangian density is taken to be that of the Dirac minimally coupled type.

It is known that massive photons and magnetic monopoles of the Dirac type cannot coexist within the same theory defined over flat Minkowski spacetime [16, 17]. In such scenarios, the photon mass is usually introduced in an ad hoc manner by explicitly breaking the gauge symmetry of the theory. In this work we consider whether such an incompatibility emerges from the a priori inclusion of photon mass or from the specific mechanism for gauge boson mass generation. Moreover, due attention is given to the role of the curved spacetime geometry and isotropic dilation field with regard to this incompatibility.

The paper is organized as follows: In Section 2, the Lagrangian density representing the matter background is specified. In Section 3, the total system Lagrangian density including gravity, gauge, matter and interaction terms is obtained. The electromagnetic and torsion gauge field, Einstein, Klein-Gordon, (nonlinear) Dirac equations of Heisenberg-Pauli type and Bianchi identities in the electromagnetic and torsion sectors are obtained in Section 4. In Section 5, we investigate the possibility of coexistence of magnetic monopoles with topologically massive vector bosons within the framework of Scalar-Tensor Gravity with Torsion Potential. Our conclusions are presented in Section 6.

2 The Matter Background

To make a distinction between the coordinate (or holonomic indices) and local Lorentz (or non-holonomic) coordinates, we use Greek indices (μ, ν = 0, 1, 2, 3) for the former and Latin indices (j, k = 0, 1, 2, 3) for the latter. Latin indices are raised and lowered with the Minkowski metric η_{ij}. Greek indices are raised and lowered with the spacetime metric g_{αβ}. We use geometrized units (ℏ = c = 1) throughout this work.

In the present Section, we are concerned with constructing a Lagrangian formulation of the dynamics of spinor valued fields ψ(x) defined over a curved manifold endowed with torsion. The equations of motion are given as the Euler-Lagrange equations for the corresponding action-integral $I(Ω) = \int_{Ω} d^{4}x L(ψ(x), \partialψ(x); x)$ defined over a spacetime volume Ω. In order to introduce spinor fields in the Riemann-Cartan geometry considered here, it is convenient to choose an orthonormal (Lorentz) basis vectors $e_{i} = e_{i}^{α}(x) e_{α}$ for the tangent space satisfying $e_{i} \cdot e_{j} = η_{ij}$, where $e_{α} \equiv \partial_{α} = \frac{∂}{∂x^α}$ represents a coordinate basis in the tangent space $T_{p}$ at point $P$ in the spacetime manifold,
\[ \eta_{ij} = \text{diag}(-1, +1, +1, +1) \] 

is the Minkowski metric and \( g_{\alpha \beta} = e_{\alpha} \cdot e_{\beta} \) is the metric of curved spacetime. The metrics \( \eta_{ij} \) and \( g_{\alpha \beta} \) are related via

\[
g_{\alpha \beta} := e_{\alpha}^i(x) e_{\beta}^j(x) \eta_{ij}. \tag{1} \]

The quantities \( e_{\alpha}^i(x) \), called tetrads, are coefficients of the dual (1-form) basis non-holonomic co-vectors \( \theta^\mu(x) = e_{\alpha}^i(x) dx^\gamma \) that satisfy the orthogonality relations \( e_{\alpha}^i e_{\alpha}^\mu = \delta^\mu_i \). In particular, the tetrads constitute transformation matrices that map from local Lorentz (with non-holonomic coordinates \( x^a \)) to coordinate (with holonomic coordinates \( x^\mu \)) bases, i.e., \( v^a = e_{\alpha}^a v^\alpha \) with \( v^i = e_{\alpha}^a e_{\alpha}^i \).

The components \( e_{\alpha}^i(x) \) and \( e_{\alpha}^a(x) \) transform as covariant and contravariant vectors (under the Poincaré group) of the frame \( x^\mu \), if and only if the rotations \( \partial_{[\mu} e_{\alpha]}^a \) vanish at all points. The equations \( \partial_{[\mu} e_{\alpha]}^a = 0 \) are the so-called integrability conditions \([10]\), implying \( e_{\alpha}^i(x) = \partial_i x^\alpha \). If the integrability condition is not satisfied, the reference frame formed by \( e_{\alpha}^i(x) \) and \( e_{\alpha}^a(x) \) is said to be non-holonomic. The quantities

\[
\Omega_{\alpha \beta} := e_{\nu c}(x) \left[ e_{\alpha}^\mu(x) \partial_\mu e_{\beta}^\nu(x) - e_{\beta}^\mu(x) \partial_\mu e_{\alpha}^\nu(x) \right], \tag{2}
\]

are called the objects of non-holonomicity. They measure the non-commutativity of the tetrad basis \([9]\). We may readily define tensors and various algebraic operations with tensors at a given point in the spacetime manifold. Comparison of tensors at different points however, requires introduction of a linear connection via the process of parallel transport. The linear connection defines a covariant derivative operator \( \hat{D} \). In non-holonomic coordinates, the parallel transport of an orthonormal basis \( e_i \) is given by \([9]\) \( \delta e_i = -\omega_{ij}^k e_k dx^j = -\omega_{ia}^k e_k dx^a \). The associated covariant derivative is given by \( D_\mu \psi = \left( \partial_\mu + \omega_{\mu \beta}^a \gamma_{ab} \right) \psi \), while for the contravariant components of a non-holonomic vector we have \( D_\mu v^i := \partial_\mu v^i + \omega_{\mu j}^i v^j \). Note that \( D_\mu \) is a coordinate representation of the operator \( \hat{D} \). The coefficients \( \omega_{\alpha \beta}^a \) are known as the spin-connection and the matrix \( \gamma_{ik} \) is an irreducible spinorial representations of the Lorentz group defined by

\[
\gamma_{ik} = \frac{1}{2} (\gamma_i \gamma_k - \gamma_k \gamma_i). \tag{3}
\]

Under local Lorentz transformation (LT) the covariant derivative itself should transform as a scalar since it does not carry a Lorentz (Latin) index. Thus \( D_\mu v^i \) LT \( D'_\mu v'^i = \Lambda^i_\mu D_\mu v^i \) where \( \Lambda^i_\mu := \frac{\partial e^i_\mu}{\partial x^\alpha} \) is a non-holonomic transformation matrix. Making use of the equation for \( D_\mu v^i \), \( D'_\mu v'^i \) and the fact that \( \partial_\mu \gamma_{ab} = 0 \) (since the Minkowski metric is constant) we obtain the transformation property of the spin connection

\[
\omega_{\alpha \beta}^a \rightarrow \omega'_{\alpha \beta}^a = \Lambda^a_\alpha \Lambda^b_\beta \omega_{\alpha \beta}^i - \left( \partial_\mu \Lambda^a_\alpha \right) \Lambda^b_\beta. \tag{4}
\]

Parallel transport is a unique geometric operation that is independent of the choice of frame. The relative rotation of a coordinate (holonomic) basis vector \( e_\alpha \) is given by \( dx^\alpha \left( \partial_\alpha e_i^\gamma + \Gamma^\gamma_{\alpha \beta} e_k^\beta \right) e_\alpha = dx^\alpha \left( \nabla_\alpha e_i^\beta \right) e_j^\beta e_j \) with the affine connection \( \Gamma^\gamma_{\mu \nu} = e_i^\rho(x) D_\nu e_i^\mu(x) = -e_i^\rho(x) D_\nu e^\rho_i(x) \) defining the covariant derivative \( \nabla_\alpha := \partial_\alpha + \Gamma^\beta_{\alpha \gamma} \Xi_{\beta \gamma} \). The matrices \( \Xi_{\alpha \beta} = -\Xi_{\beta \alpha} \) are generators of the Lorentz group satisfying the Lie algebra

\[
[\Xi_{ij}, \Xi_{kl}] = \eta_{ik} \Xi_{jl} + \eta_{jl} \Xi_{ik} - \eta_{jk} \Xi_{il} - \eta_{il} \Xi_{jk}, \tag{5}
\]

with \( \Xi_{ij} = e_i^\alpha e_j^\beta \Xi_{\alpha \beta} \). At this juncture we emphasize that there is only one linear connection. It may be expressed in either holonomic or non-holonomic frames of reference. As will be shown, these two representations of the linear connection are related by \([17]\). Moreover, the linear connection (expressed in either reference frame) is not a priori torsion free. Indeed, it will be shown that the linear connection does contain torsion, the latter being equivalently defined by either \([13]\) or \([18]\).
The covariant derivative of a quantity $v^\lambda$ ($v_\lambda$) which behaves like a contravariant (covariant) vector under the local Poincaré transformation is given by

$$\nabla_\nu v^\lambda = \partial_\nu v^\lambda + \Gamma^\lambda_{\mu\nu} v^\mu, \quad \nabla_\nu v^\mu = \partial_\nu v^\mu - \Gamma^\lambda_{\mu\nu} v^\lambda. \quad (6)$$

In analogy to (4), the transformation property for the affine connection coefficients $\Gamma^\rho_{\mu\nu}$ is given by

$$\Gamma^\lambda_{\mu\nu} \to \Gamma'^\lambda_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \Gamma^\gamma_{\alpha\beta} + \Lambda^\alpha_\mu \Lambda^\rho_\lambda \Lambda^\rho_{\alpha\nu}, \quad (7)$$

where $\Lambda^\alpha_\mu := \frac{\partial \alpha_\mu}{\partial x^\rho}$ is the holonomic transformation matrix and $\Lambda^\rho_{\alpha\nu} \equiv \partial_\alpha \partial_\nu x^\rho$. In view of the inhomogenous term $\Lambda^\alpha_\mu \Lambda^\rho_\lambda \Lambda^\rho_{\alpha\nu}$ in (7), the linear connection is not a tensor.

The parallel transport of a vector around an infinitesimal closed path is proportional to the curvature of the manifold which may be calculated as [21]

$$[D_k, D_l] \psi (x) = \frac{1}{2} R^{ij}_{kl} \gamma_{ij} \psi (x) + C_{ijkl} D_i \psi (x), \quad (8)$$

with $D_k := e^\mu_k D_\mu$. The central charge $R^{ij}_{kl}$ and structure functions $C^i_{jk}$ of the deformed algebra [8] are given (in non-holonomic coordinates) by the Cartan structure equations

$$R^{ij}_{kl} (\omega) := e^i_\lambda e^j_\rho R^\rho_{\mu\nu} = \partial_\mu \omega^i_{\nu j} - \partial_\nu \omega^i_{\mu j} + \omega^i_{\kappa \mu} \omega^\kappa_{\nu j} - \omega^i_{\kappa \nu} \omega^\kappa_{\mu j}, \quad C_{ijk} = e^{i} e_{j} e_{k} \left( e^{\rho} e^{\nu} - e^{\mu} e^{\nu} \right) D_{\rho} e_{\mu} (x). \quad (9)$$

The curvature tensor $R^\lambda_{\rho\mu\nu}$ (expressed in holonomic coordinates) is defined by,

$$R^\alpha_{\gamma\rho\lambda} (\Gamma) = \partial_\gamma \Gamma^\alpha_{\rho\lambda} - \partial_\rho \Gamma^\alpha_{\gamma\lambda} + \Gamma^\gamma_{\rho\sigma} \Gamma^\sigma_{\gamma\lambda} - \Gamma^\gamma_{\rho\lambda} \Gamma^\sigma_{\gamma\sigma}. \quad (10)$$

It is interesting to observe the similarity in structure of the curvature tensors in (10) and the first equation in (9). Indeed, there is only one curvature tensor since these two quantities can be transformed into each other via appropriate tetrad index saturation, $R^{ij}_{kl} (\omega) = e^i_\lambda e^j_\rho R^\rho_{\mu\nu} (\Gamma)$. We can therefore view $R^\alpha_{\gamma\rho\lambda} (\Gamma)$ in (10) and $R^{ij}_{kl} (\omega)$ in (9) as holonomic and non-holonomic representations, respectively, of the same spacetime curvature.

Since the basis vectors (in either holonomic or non-holonomic frames) change from one point in the spacetime manifold to another, the derivative of a vector must be given by [22] $\partial_\mu v^i = \partial_\mu \left( v^i e_i \right) = \left( \partial_\mu v^i \right) e_i + v^i \left( \partial_\mu e_i \right) \equiv \left( \nabla_\mu v^i \right) e_i$. This implies that $\partial_\mu e_j = \omega^i_{\mu j} e_i$. For similar reasons, we conclude $\partial_\mu e^i = \Gamma^i_{\rho\mu} e^\rho$. Thus, if we choose a transformation in (4) which leads from a non-holonomic to a holonomic frame, then we find [9] [22]

$$\partial_\mu e^\lambda_i - \omega^k_{ji} e^\lambda_k + \Gamma^\lambda_{\mu\nu} e^\mu_i \equiv \nabla_\mu e^\lambda_i = 0, \quad \partial_\nu e^\lambda_i + \omega^\lambda_{k\nu} e^k_i - \Gamma^\lambda_{\mu\nu} e^\mu = D_\nu e^\lambda_i = 0, \quad (11)$$

since $\partial_\mu e_j = \partial_\mu \left( e_j \cdot e_\nu \right) = \omega^i_{\mu j} e_i \cdot e_\nu + \Gamma^\rho_{\nu\mu} e_j \cdot e_\rho = \omega^i_{\mu j} e_i + \Gamma^\rho_{\nu\mu} e_j$. Observe that $\nabla_\nu = \nabla_{\gamma} (\Gamma + \omega)$. Recalling (11) and using (11), we may derive the so-called metricity condition $\nabla_\lambda (\nabla_\mu g_{\nu\sigma}) = \nabla_\mu (\nabla_\nu g_{\rho\sigma}) = \nabla_\lambda (\nabla_{\gamma} (\Gamma + \omega)) \left( e^\lambda_i (x) e^\nu_j (x) \eta_{ij} \right) = 0$. This metricity condition enables the definition of the linear connection $\Gamma^\sigma_{\rho\mu} = \tilde{\Gamma}^\sigma_{\rho\mu} + T^\sigma_{\rho\mu}$, where the quantity $\tilde{\Gamma}^\sigma_{\rho\mu}$ can be identified as the Christoffel connection coefficient

$$\tilde{\Gamma}^\sigma_{\rho\mu} := \frac{1}{2} g^{\kappa\sigma} \left( \partial_\kappa g_{\rho\mu} + \partial_\rho g_{\kappa\mu} - \partial_\mu g_{\kappa\rho} \right) \quad (12)$$

and $T^\sigma_{\rho\mu}$ is the torsion tensor defined as the asymmetric part of the affine connection,

$$T^\alpha_{\beta\gamma} := \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}. \quad (13)$$
With Eqs. (12) and (13) in hand, the quantity $R_{\alpha}^{\beta} = R_{\alpha \mu}^{\beta}$ in (9) can be expressed in terms of its torsion-free $\hat{R}_{\alpha}^{\beta}$ and torsion dependant contributions as [21]

$$R_{\alpha \mu}^{\beta} = e_{\gamma}^{\beta} (x) e_{\kappa}^{\gamma} \left( \hat{R}_{\alpha \mu}^{\gamma} + 2\hat{\nabla}_{\beta} T_{\alpha \mu}^{\gamma} + 2T_{\alpha \beta}^{\gamma} T_{\mu \lambda}^{\beta} \right),$$

where $\hat{\nabla}_{\beta} A_{\alpha} := \partial_{\beta} A_{\alpha} + \Gamma_{\alpha \beta}^{\gamma} A_{\gamma}$, $\hat{\nabla}_{\beta} A_{\alpha} := \partial_{\beta} A_{\alpha} - \hat{\Gamma}_{\beta \mu} A_{\alpha}$, the square brackets in $T_{\alpha \beta}^{\gamma}$ represent anti-symmetrization with respect to $i, j$, $\beta$ being fixed and $\hat{R}_{\alpha}^{\beta} = R_{\alpha}^{\beta} \left( \Gamma \rightarrow \hat{\Gamma} \right)$. We note that the torsion tensor $T_{\alpha \beta} = T_{\alpha \beta \gamma}$ takes the form

$$T_{\alpha \beta} = \hat{T}_{\alpha \beta} \left( \hat{\Gamma} \right) + \hat{\nabla}_{\alpha} T_{\beta \mu} - \hat{\nabla}_{\beta} T_{\alpha \mu} + T_{\alpha \beta} T_{\mu \lambda} - T_{\mu \beta} T_{\alpha \lambda},$$

(15)

where the torsion-free contribution $\hat{T}_{\alpha \beta}$ is defined as,

$$\hat{T}_{\alpha \beta} = \hat{R}_{\alpha \beta} \left( \hat{\Gamma} \right) \hat{\Gamma}_{\alpha \beta}, (16)$$

From (11) we can deduce a relation that allows to compute the affine connection in terms of the spin connection (and tetrad) or vice-versa, namely [22]

$$\Gamma_{\alpha \beta \gamma} = e_{\alpha}^{\alpha} \left( \partial_{\beta} e_{\mu} - \omega_{\beta \mu} e_{\nu} \right).$$

(17)

It is interesting to observe that substituting $\Gamma = \Gamma (\omega)$ from (17) into (13) leads to

$$T_{\alpha \beta} e_{\alpha}^{\beta} = e_{\beta}^{\beta} \left( D_{\beta} e_{\gamma}^{\gamma} - D_{\gamma} e_{\beta}^{\beta} \right) \equiv C_{\alpha \beta},$$

(18)

which establishes a means to transform between the holonomic torsion tensor $T_{\alpha \beta}$ in (13) and the non-holonomic structure functions $C_{\beta \gamma}$ in (9) (and vice-versa) in terms of appropriate tetrad index saturation. This situation is entirely analogous to the transformation from $R_{\alpha \beta \gamma} (\omega)$ to $R_{\alpha \beta \gamma} (\Gamma)$ (and vice-versa) via tetrad index saturation. From (12) or (9), the torsion tensor can be viewed as a sort of field strength associated with the tetrad coefficients that describes a twist of the tetrad under parallel transport (relative to a given basis) that is independent of the effect of curvature (i.e., a twist in a plane perpendicular to the plane of parallel transport). This is to be compared with the interpretation of torsion as the asymmetric part of the affine connection according to (13). Equation (18) can be solved for the spin connection, yielding (12)

$$\omega_{\alpha \beta} := \frac{1}{2} \left( \Omega_{\alpha \beta} + \Omega_{\beta \alpha} - \Omega_{\alpha \beta \gamma} \right) e_{\mu}^{\gamma} (x) + T_{\alpha \beta \gamma}.$$  

(19)

The quantities $T_{\alpha \beta}$ are related to the spacetime torsion tensor $T_{\alpha \beta}$ according to $T_{\alpha \beta} := e_{\beta}^{\alpha} (x) e_{\alpha}^{\beta} T_{\alpha \beta \gamma}$. We assume in this work that the torsion is totally antisymmetric and of potential type, that is, we employ the ansatz that $T_{\alpha \beta} \psi$ is derived from a second-rank, tensor potential $H_{\mu \nu} = -H_{\nu \mu}$ according to [13]

$$T_{\alpha \beta} \psi = \partial_{\mu} H_{\beta \gamma}.$$  

(20)

The Lagrangian density for a fermion field $\psi (x)$ in curved spacetime [13, 19] with torsion is given by

$$L_{\text{matter}} = \frac{i}{2} \left[ \left(D_{\mu} \psi \right) \gamma_{\mu} \psi - \bar{\psi} \gamma_{\mu} D_{\mu} \psi \right] - m \bar{\psi} \psi - e_{\lambda} A_{\lambda} \psi_{\mu}, j_{\mu}^{\lambda} := i \bar{\psi} \gamma_{\mu} \psi, \psi,$$

(21)

where $\bar{\psi}$ is the Pauli conjugate of the Dirac field $\psi$ defined by $\bar{\psi} (x) = i \psi^{\dagger} (x) \gamma_{0}$. (\dagger) is the Hermitian conjugate and $\gamma$ represents the appropriate Dirac $\gamma$-matrix with $\gamma_{\mu} := e_{\mu}^{\gamma} (x) \gamma^{\gamma}$. $A_{\mu}$ is the
electromagnetic 4-vector potential, \( e \) is the electric charge of the fermion and \( j_{(e)}^\mu \) is the fermion current. The Lagrangian density (21) can be re-written as

\[
\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{matter}} - \frac{1}{8} T_{\mu \alpha \beta} \bar{\psi} \left\{ \gamma^\mu, \gamma^{\alpha \beta} \right\} \psi - e A_{\mu} j_{(e)}^\mu, \quad j_{(e)}^\mu := i \bar{\psi} \gamma^\mu \psi,
\]

where

\[
\mathcal{L}_{\text{matter}} = \frac{i}{2} \left[ \left( \hat{D}_\alpha \bar{\psi} \right) \gamma^\alpha \psi - \bar{\psi} \gamma^\alpha \tilde{D}_\alpha \psi \right] - m \bar{\psi} \psi,
\]

with \( \tilde{D}_\alpha \psi := \partial_\alpha \psi - \frac{1}{2} \tilde{\omega}_{\alpha ij} \gamma^{ij} \psi \) and \( \hat{D}_\alpha \bar{\psi} := \partial_\alpha \bar{\psi} + \frac{1}{2} \tilde{\omega}_{\alpha ij} \gamma^{ij} \bar{\psi} \), \( \tilde{\omega}_{\alpha ij} = \frac{1}{2} e_a (x) (\Omega_{\alpha ij} + \Omega_{jci} - \Omega_{ijc}) \) being the torsion-free spin connection. Using the following relations

\[
\begin{align*}
-\frac{1}{2} T_{\mu \alpha \beta} \bar{\psi} \left\{ \gamma^\mu, \gamma^{\alpha \beta} \right\} \psi &= \frac{1}{2} T_{\mu \alpha \beta} \bar{\psi} \left( \gamma^{\beta \alpha} \gamma^\mu - \gamma^{\mu \alpha} \gamma^\beta \right), \\
\gamma^\mu \gamma^\nu \gamma^\lambda \varepsilon_{\mu \nu \lambda \sigma} &= \left\{ \gamma^\mu, \gamma^{\nu \lambda} \right\} \varepsilon_{\mu \nu \lambda \sigma} = 3 \left( \gamma_5 \gamma_5 \right), \\
\gamma^\mu \gamma^\nu \gamma^\lambda &= \gamma^{[\mu \nu \lambda]},
\end{align*}
\]

we obtain

\[
T_{\mu \alpha \beta} \bar{\psi} \left\{ \gamma^\mu, \gamma^{\alpha \beta} \right\} \psi = \frac{1}{2} \varepsilon_{\alpha \beta \mu \nu} j_{5 \nu} \psi,
\]

where \( j_{5 \nu} \) is the fermion pseudo-current. Defining the torsion axial-vector (also referred to as the torsion dual in what follows)

\[
T^\nu := \frac{1}{3!} \varepsilon^{\alpha \beta \mu \nu} T_{\alpha \beta \mu}.
\]

The interaction between the Dirac field and torsion has been reduced to a coupling of the fermion axial current to a torsion axial-vector \( T_\mu \). Thus, the matter Lagrangian density in curved space with torsion [20] and electromagnetic fields reads

\[
\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{matter}} + \frac{3i}{8} T_\mu j_{5 \mu} - e A_{\mu} j_{(e)}^\mu.
\]

### 3 The Total System Lagrangian Density

We now consider the geometrical setting in which the matter content - represented by Lagrangian density [28] - is immersed. The Einstein-Hilbert Lagrangian density is given by

\[
\mathcal{L}_{\text{geom}} = \sqrt{-g} \frac{R}{k_0},
\]

where \( k_0 = \frac{16 \pi G}{c^4} \) and \( R = g^{ij} R_{ij} \) is the scalar curvature. Note that \( l_p = (G)^{1/2} \) is the Planck constant (in geometrized units). In the Brans-Dicke generalization of gravity, one introduces a scalar field \( \Phi \) via the replacement \( G \rightarrow e^{2\Phi} G \) (i.e. \( k_0 \rightarrow e^{2\Phi} k_0 \)). For simplicity, let \( \alpha = \frac{e^{-2\Phi}}{k_0} \).

With this generalization and the transformation \( k_0 \rightarrow e^{2\Phi} k_0 \), the geometrical Lagrangian density becomes

\[
\mathcal{L}_{\text{geom}} = \sqrt{-g} \alpha R = \sqrt{-g} \frac{e^{-2\Phi}}{k_0} \left( \hat{R} + \partial_\gamma T_\alpha \gamma^\alpha + T_\alpha \beta^\lambda T_\lambda^\beta \right).
\]
so-called Einstein frame [24, 23]. For this reason we perform a conformal transformation on the metric tensor
\[ g_{\mu\nu} \rightarrow g'_{\mu\nu} = e^{2\Phi} g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow g'^{\mu\nu} = e^{-2\Phi} g^{\mu\nu}, \] (31)
which leads to
\[ \sqrt{-g'} = e^{4\Phi} \sqrt{-g}, \] (32)
where \( g = \text{det} g_{\mu\nu} \) and \( g' = e^{8\Phi} g \). Under the conformal transformation (31) the Christoffel symbols transform according to,
\[ \hat{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + g^\alpha{}_{\alpha\beta} \left[ (\partial_\mu \Phi) g_{\nu\beta} + (\partial_\nu \Phi) g_{\beta\mu} - (\partial_\beta \Phi) g_{\mu\nu} \right]. \] (33)
With the conformally transformed Christoffel symbols (33), the correspondingly transformed scalar curvature is given by
\[ \hat{R} \rightarrow \hat{R}' = g'^{\mu\nu} \left( \partial_\mu \tilde{\Gamma}^\nu_{\beta\mu} - \partial_\nu \tilde{\Gamma}^\beta_{\beta\mu} + \tilde{\Gamma}^\nu_{\nu\lambda} \tilde{\Gamma}^\lambda_{\beta\mu} - \tilde{\Gamma}^\nu_{\nu\lambda} \tilde{\Gamma}^\lambda_{\beta\mu} \right). \] (34)
By direct calculation, we obtain
\[ \hat{R}' = e^{2\Phi} \hat{R} + 6e^{2\Phi} g'^{\mu\nu} \left[ (\partial_\mu \Phi) (\partial_\nu \Phi) + \frac{1}{2} \partial_\mu \partial_\nu \Phi \right]. \] (35)
Letting \( \phi = 2\Phi \), the geometrical Lagrangian density becomes
\[ \mathcal{L}'_{\text{geom}} = \frac{1}{\kappa_0} \left[ \hat{R} - e^{-2\phi} T_{\mu\nu\sigma} T^{\mu\nu\sigma} + \frac{3}{2} (\partial^\mu \phi) (\partial_\mu \phi) + \frac{3}{2} \Box \phi \right], \] (36)
where \( \Box := g^{\mu\nu} \partial_\mu \partial_\nu \). It is worth observing that the dilaton kinetic term in is generated by the conformal transformation (31) acting on the curvature scalar taking \( \hat{R} \) to \( \hat{R}' \). Moreover, we note that the Lagrangian density (36) is true up to a total divergence that is proportional to \( \partial_\gamma (e^{-2\Phi} T_{\gamma\alpha} \gamma^\alpha} \).

As a working hypothesis we assume the dilaton \( \phi \) possess an isotropic field configuration (i.e. \( \phi (\hat{r}) = \phi (|\hat{r}|^2) \)).

It is straightforward to verify that under conformal transformation (31) \( \mathcal{L}_{\text{matter}} \) is invariant. Using the conformal transformation on spinor fields [25]
\[ \psi \rightarrow \psi' = e^{-\frac{1}{2} \phi(|\hat{r}|)} \psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = e^{\frac{1}{2} \phi(|\hat{r}|)} \bar{\psi}, \] (37)
we determine
\[ \hat{\mathcal{L}}_{\text{matter}} \rightarrow \hat{\mathcal{L}}'_{\text{matter}} = \hat{\mathcal{L}}_{\text{matter}} + 3 \partial_\mu \left( \phi j^\mu_{(e)} \right) - 3 \phi \partial_\mu j^\mu_{(e)}. \] (38)
If the fermion current is conserved, then we expect \( \partial_\mu j^\mu_{(e)} = 0 \). Since \( \partial_\mu \left( \phi j^\mu_{(e)} \right) \) is a total divergence it does not contribute to the equations of motion so it may be ignored. The interaction term \( \frac{3}{4} T_{\mu\nu\sigma} \bar{\psi} \gamma^{[\mu\nu\gamma\delta]} \psi \) is invariant under the conformal transformations (31) since \( T_{\mu\nu\sigma} \) is postulated to be so, and the spin energy potential \( \tau_{\mu\nu\sigma} := \bar{\psi} \gamma^{[\mu\nu\gamma\delta]} \psi \) is trivially invariant under scale transformations. It is obvious that the mass term is invariant under scale transformations.

Having introduced the geometrical setting and matter content (electrically charged, nonrelativistic spin-\( \mathbf{1} \) particles) of the model, we now consider electromagnetic interaction between such prototype matter and abelian magnetic monopoles, where the photons mediating this interaction are topologically coupled to the anti-symmetric torsion potential. We are concerned with investigating whether the incompatibility of massive photons and magnetic monopoles within a classical theory is a consequence of the \( \text{a priori} \) inclusion of photon mass or is related to the specific mechanism for gauge boson mass generation. We include the monopole in a non-dynamical manner. The Lagrangian density describing the gauge sector of this scenario is given by
\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \mu_0 \varepsilon_{\alpha\beta\rho\sigma} A^\alpha \partial_\beta H^{\rho\sigma}, \quad F_{\mu\nu} := F_{\mu\nu} + *G_{\mu\nu}, \] (39)
where the Hodge dual (\ast) of $\mathcal{G}^{\mu\nu}$ is defined by $\ast\mathcal{G}^{\mu\nu} := \frac{1}{2\pi}e^{\mu\nu\rho\sigma}\mathcal{G}_{\rho\sigma}$. The electromagnetic field strength has the usual form $F_{\nu\sigma} := \partial_{\nu}A_{\sigma} - \partial_{\sigma}A_{\nu}$ while the monopole contribution is given by

$$G^{\rho\sigma}(\vec{r}) := 4\pi c(m) \int dx^{\rho} \wedge dx^{\sigma} \delta^{(4)}(\vec{r} - \vec{r}_{\text{monopole}}), \quad (40)$$

where $c(m)$ is the magnetic charge. Observe that (40) is antisymmetric and is responsible for breaking the Bianchi identity (56) in the $A_{\mu}$ sector. The total Lagrangian density $L_{\text{total}}(\phi, H_{\mu\nu}, A_{\mu}, e_{\mu}^i, \psi) = L_{\text{geom}}(\phi, e_{\mu}) + L_{\text{gauge}}(H_{\mu\nu}, A_{\mu}) + L_{\text{matter}}(\psi, \phi)$ is given by,

$$L_{\text{total}} = \frac{1}{k_{0}} \left( \dot{R} - e^{-2\phi(|\vec{r}|)}T_{\mu\nu\sigma}T^{\mu\nu\sigma} + \frac{3}{2} (\partial_{\mu}\phi)(\partial_{\mu}\phi) + \frac{3}{2} \Box \phi \right) - \frac{1}{4} F_{\mu\nu}^{\rho\sigma} F_{\mu\nu}^{\rho\sigma} + \mu_0 \epsilon_{\alpha\beta\rho\sigma} A^{\alpha} \partial^{\beta} H^{\rho\sigma} +$$

$$\frac{i}{2} \left( [\dot{D}_{\mu}\psi] \gamma^{\mu} \psi - \bar{\psi} \gamma^{\mu} \dot{D}_{\mu} \psi \right) + \frac{3i}{8} T_{\mu\nu} j_{5}^{\mu} - e A_{\mu} j_{(e)}^{\mu} - m \bar{\psi} \psi. \quad (41)$$

We remark that the coupling term proportional to $\mu_0\epsilon$ describes a topological interaction between gauge fields $A^{\alpha}$ and $H^{\rho\sigma}$. This may be understood from the lack of $\mu_0\epsilon$-dependent terms in the canonical energy-momentum tensor $\Sigma_{\mu\nu}$ appearing in (55). This fact reflects the lack of energy associated with the interaction. Such interaction has no local propagating degrees of freedom, hence being topological in nature [26].

### 4 Field Equations

By variation of the action $I = \int \sqrt{-g} d^{4}x L_{\text{total}}(\phi, H_{\mu\nu}, A_{\mu}, e_{\mu}^i, \psi)$ with respect to $\phi, H_{\mu\nu}, A_{\mu}$ and $\bar{\psi}$, and requiring the coefficients of each variation independently vanish, we obtain the equations of motion

$$\begin{align*}
\frac{\partial L}{\partial \phi} - \partial_{\mu} \left( \frac{\partial L}{\partial \partial_{\mu} \phi} \right) &= 0, \\
\frac{\partial L}{\partial H_{\mu\nu}} - \partial_{\sigma} \left( \frac{\partial L}{\partial \partial_{\sigma} H_{\mu\nu}} \right) &= 0, \\
\frac{\partial L}{\partial \psi} - \partial_{\mu} \left( \frac{\partial L}{\partial \partial_{\mu} \psi} \right) &= 0, \\
\frac{\partial L}{\partial \bar{\psi}} - \partial_{\sigma} \left( \frac{\partial L}{\partial \partial_{\sigma} \bar{\psi}} \right) &= 0.
\end{align*} \quad (42)$$

To obtain the explicit form of the dynamical equations for the fermions we recall that the Dirac $\gamma$-matrices are covariantly constant,

$$\nabla_{\kappa} \gamma_{i} = \partial_{\kappa} \gamma_{i} - \Gamma_{\kappa}^{\alpha} \gamma_{\alpha} + \left[ \gamma_{i}, \hat{\Gamma}_{\kappa} \right] = 0 \text{ with } \hat{\Gamma}_{\kappa} = \frac{1}{8} \left[ (\partial_{\kappa} \gamma_{i}) \gamma^{i} - \Gamma_{\kappa}^{\alpha} \gamma_{\alpha} \gamma^{i} \right] \quad (43)$$

The $4 \times 4$ matrices $\hat{\Gamma}_{\kappa}$ are real matrices used to induce similarity transformations on quantities with spinor transformation properties [27], that is $\gamma_{i} \rightarrow \gamma^{i} = \hat{\Gamma}^{-1} \gamma_{i} \hat{\Gamma}$. Varying $\hat{\Gamma}_{\kappa}$ leads to $\delta \hat{\Gamma}_{\kappa} = \frac{1}{8} \left[ (\partial_{\kappa} \delta \gamma_{i}) \gamma^{i} - (\delta \Gamma_{\kappa}^{\alpha}) \gamma_{\alpha} \gamma^{i} \right]$. Since we require the anticommutator condition on the gamma matrices $\gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = g_{\mu\nu} 1$ (Dirac algebra) to hold, the variation of the metric gives

$$2 \delta g^{\mu\nu} = \{\delta \gamma^{\mu}, \gamma^{\nu}\} + \{\gamma^{\mu}, \delta \gamma^{\nu}\}. \quad (44)$$

One solution to this equation is $\delta \gamma^{\nu} = \frac{1}{2} \gamma_{\sigma} \delta \gamma^{\sigma\nu}$. With the aid of this result, we can write $(\partial_{\kappa} \delta \gamma_{i}) \gamma^{i} = \frac{1}{2} \partial_{\kappa} \left( \gamma^{i} \delta g_{\nu i} \right) \gamma^{i}$. Finally, exploiting the anti-symmetry in $\gamma_{\mu\nu}$ we obtain

$$\delta \hat{\Gamma}_{\kappa} = \frac{1}{8} \left( g_{\nu\sigma} \delta \Gamma_{\kappa}^{\nu\sigma} - g_{\mu\sigma} \delta \Gamma_{\kappa}^{\mu\sigma} \right) \gamma^{\mu\nu}. \quad (45)$$

With the above variational relations, it is straightforward to show that the dynamical equation for the fermions is a nonlinear Dirac equation [28] of Heisenberg-Pauli type,

$$\left[ \gamma^{\mu} \left( \hat{D}_{\mu} - i e A_{\mu} \right) + \frac{3}{8} T_{\mu\nu\sigma} \gamma^{[\mu} \gamma_{\nu] \gamma^{\sigma]} - m \right] \psi = 0. \quad (46)$$
For the scalar field $\phi$ we obtain the Klein-Gordon equation

$$
\Box \phi((\vec{r})) - \frac{4}{3} e^{-2\phi(\vec{r})} T_{\mu\nu} T^{\mu\nu} = 0.
$$

(47)

To obtain the analogue of the Einstein equations the following calculations involving the metric tensor $g_{\mu\nu}$ and its determinant $g = \det (g_{\mu\nu})$ are useful. Recall $gg^{\mu\nu} = \frac{\delta g}{\delta g_{\rho\sigma}}$ and $gg_{\mu\nu} = -\frac{\delta g}{\delta g^{\rho\sigma}}$.

Now since $\delta \sqrt{-g} = \frac{\delta g}{\delta g^{\mu\nu}} g_{\mu\nu} = -\frac{\delta g}{2\sqrt{-g}}$, we can write $\delta g = gg^{\mu\nu} \delta g_{\mu\nu}$. Thus, we obtain

$$
\delta \sqrt{-g} = -\frac{gg^{\mu\nu} \delta g_{\mu\nu}}{2\sqrt{-g}}.
$$

(48)

Writing the metric in terms of the tetrads $g^{\mu\nu} = e^i_i e^\nu_i$, we observe $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} (\delta e^i_i e^\nu_i + e^{i\nu}_i \delta e^\nu_i)$. By using $\delta e^\nu_i = \delta (\eta^i_i e^\nu_i) = \eta^i_i \delta e^\nu_i$, we are able to deduce

$$
\delta \sqrt{-g} = -\sqrt{-g} e^\mu_i \delta e^\nu_i.
$$

(49)

To compute the variation of the scalar curvature $R$ we must consider the variation of the ordinary Ricci tensor $\tilde{R}_{\mu\nu} = e^i_i \tilde{R}_{\mu\nu}$ which is given by $\delta \tilde{R}_{\mu\nu} = \delta e^i_i \tilde{R}_{\rho\sigma} + e^i_i \delta \tilde{R}_{\mu\nu}$. In an inertial frame the Ricci tensor reduces to $\tilde{R}_{\mu\nu} = \eta^i_i \tilde{R}_{\rho\sigma} + e^i_i \delta \tilde{R}_{\mu\nu}$, so that $\delta \tilde{R}_{\mu\nu} = \delta e^i_i \tilde{R}_{\mu\nu} + e^i_i \left( \partial_{\nu} \delta \Gamma^\beta_{\mu\rho} - \partial_{\mu} \delta \Gamma^\beta_{\nu\rho} \right)$. The second term can be converted into a surface term and does not contribute to the field equations, so it may be ignored leading to conclude

$$
\delta \tilde{R}_{\mu\nu} = \delta e^i_i \tilde{R}_{\mu\nu}.
$$

(50)

With the aid of $\delta \tilde{R}_{\mu\nu}$ we may write the variation $\delta R$ as

$$
\delta R = \tilde{R}_{\mu\nu} \delta g_{\mu\nu} + g^{\mu\nu} \left( \nabla_{\lambda} \delta \Gamma^\lambda_{\mu\nu} - \nabla_{\nu} \delta \Gamma^\lambda_{\mu\lambda} \right) - T^\alpha_{\beta\gamma} \delta T^\beta_{\gamma\alpha}.
$$

(51)

With the above variational calculations involving the metric and Ricci tensor it is not difficult to deduce the Einstein-like equations

$$
G^\mu_{\nu} + \Theta^\mu_{\nu} = k_0 \Sigma^\mu_{\nu},
$$

(52)

with

$$
\Theta^\mu_{\nu} = -\left( 2T_{\nu\rho\sigma} T^{\rho\mu\sigma} + \nabla_{\nu} T^{\mu\rho}_{\sigma} - \frac{3}{8} P^\mu_{\nu} + e^{-\phi(\vec{r})} Q^\mu_{\nu} + \left( 1 - e^{-2\phi(\vec{r})} \right) S^\mu_{\nu} \right),
$$

(53)

where

$$
\left\{ \begin{array}{l}
\hspace{1cm} P^\mu_{\nu} = \delta^\mu_{\rho} (\partial_{\sigma} \phi) (\partial_{\nu} \phi) - (\partial^\rho \phi) (\partial_{\nu} \phi) + (\delta^\mu_{\rho} \phi) (\partial_{\nu} \phi), \\
S^\mu_{\nu} = \delta^\mu_{\rho} T_{\lambda\sigma\tau} T^{\lambda\sigma\tau} + 3 T_{\nu\rho\sigma} T^{\rho\mu\sigma} - 3 T_{\nu\rho\sigma} T^{\rho\mu\sigma}, \quad Q^\mu_{\nu} = \delta^\mu_{\rho} F_{\lambda\sigma} F^{\rho\lambda} - F^{\mu\rho} F_{\nu\rho}
\end{array} \right.
$$

(54)

The Einstein tensor $G^\mu_{\nu}$ is given by the standard form $G^\mu_{\nu} = R^\mu_{\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R}$, while the energy momentum tensor $\Sigma_{\mu\nu}$ reads

$$
\Sigma_{\mu\nu} = \psi \gamma_{(\mu} \tilde{D}_{\nu)} \psi - \tilde{D}_{\mu} (\bar{\psi} \gamma_{(\nu)} \psi) + A_{(\mu} j^e_{(\nu)} - A j^e_{(\nu)} + T_{(\mu} j_{\nu)} - T j^2_{\mu\nu}.
$$

(55)

The Bianchi identities for the $A_{\mu}$ and $H_{\mu\nu}$-sectors read

$$
\partial_{\mu} F^{\mu\nu} = -\partial_{\mu} G^{\mu\nu} = -j^e_{(\mu)} \text{ and } \tilde{\nabla}_{\sigma} \left( e^{-2\phi(\vec{r})} T^{\sigma} \right) = 0,
$$

(56)
where \( j^\nu_{(m)} \equiv (\rho_{(m)}, \tilde{j}_{(m)}) \), \( \rho_{(m)} = e_{(m)} \delta^{(3)}(\vec{r} - \vec{r}_{\text{monopole}}) \) and \( \tilde{j}_{(m)} = 0 \) since we work in the monopole rest frame. For the gauge fields \( A_\beta \) and \( H_{\rho\sigma} \) we obtain the equations of motion

\[
\partial_\mu F^{\mu\nu} + 2\mu_0 T^\nu = e j^\nu_{(e)},
\]

\[
\frac{1}{k_0} \nabla_\sigma \left( e^{-2\phi(|\vec{r}|)} T^{\mu\nu\sigma} \right) - \mu_0 \, \ast F^{\mu\nu} = \epsilon^{\mu\nu\sigma\rho} \nabla_\sigma j^5_{\rho},
\]

By integrating (58) we find the solution

\[
T^{\mu\nu\sigma} = e^{2\phi(|\vec{r}|)} \left[ k_0 \epsilon^{\mu\nu\sigma\rho} (j^5_{\rho} + \mu_0 A_\rho) + \Lambda^{\mu\nu\sigma} \right], \tag{59}
\]

where \( \Lambda^{\mu\nu\sigma} \) arises from the process of integration and must satisfy \( \nabla_\sigma \Lambda^{\mu\nu\sigma} = 0 \) and \( \Lambda^{(\mu\nu\sigma)} = 0 \). The general form of \( \Lambda^{\mu\nu\sigma} \) that fulfills both conditions is \( \Lambda^{\mu\nu\sigma} = \epsilon^{\mu\nu\sigma\gamma} \partial_\gamma f \), where \( f \) is a scalar function. From the solutions (59) it is clear that the sources of torsion are spinors, dilatons and the electromagnetic gauge fields. Furthermore, equations (57), (58) and (59) describe a system of interacting charged fermions and abelian magnetic monopoles where the interaction is mediated by topologically massive vector boson with mass \( m_5^2 = 2\mu_0 \).

5 Generalized Maxwell’s Equations, Classical Hamiltonian Formulation and Magnetic Field Symmetry

It is known that the classical non-relativistic theory describing the massless electromagnetic scattering of an electric charge from a fixed magnetic monopole has a well defined Hamiltonian formulation [29]. Alternatively however, it is equally well known that one cannot construct a self-consistent quantum field theory describing the nonrelativistic electromagnetic interaction mediated by massive photons between a point-like electric charge and a magnetic monopole [16]. In one of our previous work [17], we showed that this inconsistency arises in the classical theory itself.

In this Section of the paper, we explore the possibility of constructing a self-consistent nonrelativistic classical theory where magnetic monopoles and topologically massive vector bosons coexist in the framework of scalar-tensor gravity with torsion potential.

5.1 Generalized Maxwell’s Equations

In this subsection, we begin by decomposing, for convenience, the electromagnetic field strength and torsion tensors into their boost and spatial components according to

\[
F_{\mu\nu} = \begin{cases} F_{0i} & = (\vec{E}_i)^i, \\ F_{ij} & = -\epsilon_{ijk} (\vec{B}_k)^j, \end{cases}, \quad T_{\mu\nu\rho} = \begin{cases} T_{0ij} & = -\epsilon_{ijk} (\vec{E}_k)^j, \\ T_{ijk} & = \epsilon_{ijk} \vec{B}, \end{cases} \text{ and } T^\mu = (\vec{B}, \vec{E})
\]

(60)

and using the Bianchi identities (56) and the field equations (57) and (58) we obtain the Maxwell-torsion equations in standard vector notation

\[
\vec{\partial} \cdot \vec{E} (\vec{r}) = \rho_{(e)} - 2\mu_0 \vec{B} (\vec{r}), \quad \vec{\nabla} \cdot \left[ e^{-2\phi(|\vec{r}|)} \vec{E} (\vec{r}) \right] = 0, \quad \vec{\partial} \cdot \vec{B} (\vec{r}) = \rho_{(m)},
\]

(61)

\[
\vec{\partial} \times \vec{E} (\vec{r}) = -\partial_i \vec{B} (\vec{r}), \quad \vec{\nabla} \times \left[ e^{-2\phi(|\vec{r}|)} \vec{E} (\vec{r}) \right] = k_0 \mu_0 \vec{B} (\vec{r}) + k_0 \vec{\nabla} \times \vec{j}_5,
\]

(62)

\[
\vec{\partial} \times \vec{B} (\vec{r}) = \vec{j}_{(e)} + \partial_i \vec{E} (\vec{r}) - 2\mu_0 \vec{E} (\vec{r}), \quad \vec{\nabla} \left[ e^{-2\phi(|\vec{r}|)} \vec{B} (\vec{r}) \right] = k_0 \vec{\nabla} \rho_5 - k_0 \mu_0 \vec{E} (\vec{r}).
\]

(63)

where \( \vec{\partial} \) represents the ordinary nabla differential operator of flat space, \( j^{\mu\nu} = \nabla_\sigma \tau^{\mu\sigma} = \epsilon^{\mu\nu\alpha\beta} \nabla_\alpha j_\beta^5 \) and \( j_5^\beta = (\rho_5, \vec{j}_5) \). The pseudo-current \( j_5^\beta \) arising from the spin energy potential contributes to
the diffusive magnetic potential. Observe that the magnetic current $\vec{j}_m$ is absent from the first equation in (62) since we are working in the rest frame of the monopole. In absence of electric fields, charges and currents, as well as the absence of magnetic current and the zeroth component $\vec{B}$ of the torsion dual $T^\mu$, the maxwell-torsion equations become:

$$\vec{\partial} \cdot \vec{E}(\vec{r}) = 0, \vec{\nabla} \cdot \left[e^{-2\phi(|\vec{r}|)}\vec{E}(\vec{r})\right] = 0, \vec{\partial} \cdot \vec{B}(\vec{r}) = \rho(m), \quad (64)$$

$$\vec{\partial} \times \vec{E}(\vec{r}) = 0, \vec{\nabla} \times \left[e^{-2\phi(|\vec{r}|)}\vec{E}(\vec{r})\right] = k_0\mu_0\vec{B}(\vec{r}) + k_0\vec{\nabla} \times \vec{j}_5, \quad (65)$$

$$\vec{\partial} \times \vec{B}(\vec{r}) = -2\mu_0\vec{E}(\vec{r}). \quad (66)$$

The total static magnetic field of the system is comprised of the point-like magnetic charge, string, diffuse magnetic field (arising from the spatial components $\vec{E}$ of the torsion dual $T^\mu$) and spin-magnetic (arising from $\vec{j}_5$) contributions

$$\vec{B}(\vec{r}) = \vec{B}_{\text{monopole}}(\vec{r}) + \vec{B}'(\vec{r}) = \left[\vec{\partial} \times \vec{A}_{\text{monopole}} + e_{(m)}\vec{h}(\vec{r})\right] + \vec{\partial} \times \left[e^{-2\phi(|\vec{r}|)}\vec{E}(\vec{r}) + k_0\vec{j}_5\right]$$

$$= \vec{\partial} \times \vec{A} + e_{(m)}\vec{h}(\vec{r}), \quad (67)$$

where $\vec{A} = \vec{A}_{\text{monopole}} + \vec{A}'$ with $\vec{A}' = e^{-2\phi(|\vec{r}|)}\vec{E} + k_0\vec{j}_5$ and the vector $\vec{A}_{\text{monopole}}$ is a singular vector potential representing the field of the fixed monopole $\vec{A}_{\text{monopole}}(\vec{r}) = \frac{e_{(m)}\vec{r}}{r^2}$. (72).

The quantity $\vec{h}(\vec{r})$ is the magnetic string function

$$|\vec{h}(\vec{r})| = \frac{4\pi}{r^2} \frac{\delta(\theta)\delta(\phi)}{\sin \theta}(\hat{n} \times \vec{r}), \quad \theta \neq \pi, \quad (69)$$

with semi-infinite singularity line oriented along the negative z-axis. The quantity $\vec{h}(\vec{r})$ is encountered in monopole theory. The magnetic field $\vec{B}_{\text{monopole}}$ in (67) generated by the point-like magnetic charge is given by

$$\vec{B}_{\text{monopole}}(\vec{r}) = \frac{e_{(m)}\vec{r}}{r^3}, \quad (70)$$

whereas $\vec{B}'(\vec{r})$ in (67) has the form

$$\vec{B}'(\vec{r}) = b^{(1)}(r, \hat{n} \cdot \vec{r})\vec{r} + b^{(2)}(r, \hat{n} \cdot \vec{r})\hat{n}, \quad (71)$$

with $b^{(1)}$ and $b^{(2)}$ being general scalar field functions and $\hat{n}$ denoting a unitary vector along the monopole string. Combining equations (70) and (71), equation (67) becomes

$$\vec{B}(\vec{r}) = \frac{e_{(m)}\vec{r}}{r^3} + b^{(1)}(r, \hat{n} \cdot \vec{r})\vec{r} + b^{(2)}(r, \hat{n} \cdot \vec{r})\hat{n}. \quad (72)$$

It is clear from equation (72) that no spherically magnetic solutions are allowed within Maxwell’s generalized equations. Moreover, the magnetic fields $\vec{B}_{\text{monopole}}(\vec{r})$ and $\vec{B}'(\vec{r})$ satisfy

$$\vec{\partial} \cdot \vec{B}_{\text{monopole}}(\vec{r}) = e_{(m)}\delta^{(3)}(\vec{r}), \quad \vec{\partial} \times \vec{B}_{\text{monopole}}(\vec{r}) = 0 \quad (73)$$

and

$$\vec{\partial} \cdot \vec{B}'(\vec{r}) = 0, \quad \vec{\partial} \times \vec{B}'(\vec{r}) = -m_2^2(\vec{A}_{\text{monopole}} + e^{-2\phi(|\vec{r}|)}\vec{E}), \quad m_2^2 = 2\mu_0 \quad (74)$$

respectively. Notice that because of the second equation in (74) is consistent with the non spherical symmetry of the total magnetic field in equation (72).
5.2 Classical Hamiltonian Formulation and Magnetic Field Symmetry

In this subsection, given the magnetic field solutions obtained in the previous subsection, we consider the possibility of constructing a classical non-relativistic Hamiltonian formulation of a theory describing a point-like electric particle with charge $e$ and mass $m$ moving in the field of a fixed monopole of charge $e_{(m)}$. We require that the constructed Poisson algebra of such Hamiltonian formulation be consistent with the symmetry of the total magnetic vector field obtained in the above subsection.

The Hamiltonian that describes to the above system is given by,

$$H_{\text{total}} \overset{\text{def}}{=} \left( \frac{\nabla - e\vec{A}}{2m} \right)^2 + H_{\text{string}}, \quad H_{\text{string}} = -ee_{(m)} \int \left( \frac{d\vec{r}}{dt} \times \vec{h}(\vec{r}) \right) \cdot d\vec{r}. \quad (75)$$

The classical equation of motion arising from (75), becomes

$$m \frac{\nabla}{\nabla t} \vec{v} - e \frac{d\vec{r}}{dt} \times \left( \vec{\partial} \times \vec{A} \right) - ee_{(m)} \frac{d\vec{r}}{dt} \times \vec{h}(\vec{r}) = 0, \quad \vec{v} = \frac{d\vec{r}}{dt}, \quad \vec{A} = \vec{A}_{\text{monopole}} + e^{-2\phi(|\vec{r}|)} \vec{E}$$

where in component form $\frac{\nabla \vec{v}^k}{\nabla t} \overset{\text{def}}{=} \frac{dv^k}{dt} + \dot{\Gamma}^k_j v^i v^j$ with $v^k = \frac{dr^k}{dt}$, $k = 1, 2, 3$. For simplicity it is assumed that $|\vec{r}_5| \ll \left| \frac{1}{k_5} e^{-2\phi(|\vec{r}|)} \vec{E}(\vec{r}) \right|$ so we may neglect $\vec{\partial} \times \vec{j}_5$ in the following analysis. Under this hypothesis, the total magnetic field $\vec{B}(\vec{r})$ reduces to

$$\vec{B}(\vec{r}) = \left[ \vec{\partial} \times \vec{A}_{\text{monopole}} + e_{(m)} \vec{h}(\vec{r}) \right] + \vec{\partial} \times \left[ e^{-2\phi(|\vec{r}|)} \vec{E}(\vec{r}) \right] = \vec{\partial} \times \vec{A} + e_{(m)} \vec{h}(\vec{r}).$$

Since we require that $\vec{B}(\vec{r})$ be a vector field, we must verify that the quantity $e^{-2\phi(|\vec{r}|)} \vec{E}$ transforms appropriately under spatial rotations. Given that we are in a curved space, we must define the spatial rotation generator associated to the Hamiltonian (75) such that it satisfies a proper Poisson bracket with $\vec{J}$ space rotation generator associated to the Hamiltonian (75) such that it satisfies a proper Poisson bracket with $\vec{J}$.

Finally $\vec{s}$ is defined as $\vec{s} = \vec{r} \times \vec{B}$.

We make use of a result proved in [31], namely that in a curved spacetime the fundamental Poisson brackets are always conserved. Thus, in the curved spacetime that we consider, the Poisson brackets between two generic functions $u(\vec{p}, \vec{r}, t)$ and $g(\vec{p}, \vec{r}, t)$ of the dynamical variables $\vec{p}$ and $\vec{r}$, are defined in usual manner as

$$\{u(\vec{p}, \vec{r}, t), g(\vec{p}, \vec{r}, t)\} \overset{\text{def}}{=} \sum_i \left( \partial_{p^i}u\partial_{r^i}g - \partial_{r^i}u\partial_{p^i}g \right). \quad (80)$$

In what follows, we employ the basic canonical Poisson bracket structure for the conjugate variables,

$$\{r_i, r_j\} = 0, \quad \{r_i, p^j_{\text{flat}}\} = -\delta_{ij}, \quad \{p^i_{\text{flat}}, p^j_{\text{flat}}\} = 0. \quad (81)$$
Finally, the fourth bracket on the rhs of (82) is given by

\[
\{ J_l, J_l \} = \{ \varepsilon_{ijk} r_j (p_k - A_k) + s_i, \varepsilon_{lmn} r_m (p_n - A_n) + s_l \} 
\]

\[
= \{ \varepsilon_{ijk} r_j p_k - \varepsilon_{ijk} r_j A_k + s_i, \varepsilon_{lmn} r_m p_n - \varepsilon_{lmn} r_m A_n + s_l \} 
+ \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m p_n \} 
- \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m p_n \} 
+ \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m A_n \} + \{ s_i, s_l \} .
\]

Note that in the previous Section and for the remainder of this subsection, we set the electric charge \( e = 1 \) for convenience. The first bracket on the right hand side (rhs) of (82) becomes

\[
\{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m p_n \} = \left\{ \varepsilon_{ijk} r_j \left( P_k^\text{flat} - \tilde{\Gamma}_k \right), \varepsilon_{lmn} r_m \left( P_n^\text{flat} - \tilde{\Gamma}_n \right) \right\} 
\]

\[
= \left\{ \varepsilon_{ijk} r_j P_k^\text{flat}, \varepsilon_{lmn} r_m P_n^\text{flat} \right\} 
- \left\{ \varepsilon_{ijk} r_j r_i P_i^\text{flat}, \varepsilon_{lmn} r_m r_i P_i^\text{flat} \right\} 
+ \left\{ \varepsilon_{ijk} r_j \tilde{\Gamma}_k, \varepsilon_{lmn} r_m \tilde{\Gamma}_n \right\} 
- \left\{ \varepsilon_{ijk} r_j \tilde{\Gamma}_k, \varepsilon_{lmn} r_m \tilde{\Gamma}_n \right\} .
\]

where

\[
\left\{ \varepsilon_{ijk} r_j P_k^\text{flat}, \varepsilon_{lmn} r_m P_n^\text{flat} \right\} = r_i P_i^\text{flat} - r_i P_i^\text{flat} ,
\]

\[
- \left\{ \varepsilon_{ijk} r_j P_k^\text{flat}, \varepsilon_{lmn} r_m r_i P_i^\text{flat} \right\} = \delta_i r_n A_i - r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_m P_k^\text{flat} \{ A_n, r_j \} ,
\]

\[
- \left\{ \varepsilon_{ijk} r_j \tilde{\Gamma}_k, \varepsilon_{lmn} r_m \tilde{\Gamma}_n \right\} = -\delta_i r_n A_i + r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_m \tilde{\Gamma}_n \{ r_i, \tilde{\Gamma}_k \} 
- \varepsilon_{ijk} \varepsilon_{lmn} r_m \tilde{\Gamma}_n \{ r_i, \tilde{\Gamma}_k \} .
\]

Similarly, the second bracket on the rhs of (82) reduces to

\[
- \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m A_n \} = -\left\{ \varepsilon_{ijk} r_j \left( P_k^\text{flat} - \tilde{\Gamma}_k \right), \varepsilon_{lmn} r_m A_n \right\} 
\]

\[
= -\left\{ \varepsilon_{ijk} r_j P_k^\text{flat}, \varepsilon_{lmn} r_m A_n \right\} 
+ \left\{ \varepsilon_{ijk} r_j \tilde{\Gamma}_k, \varepsilon_{lmn} r_m A_n \right\} 
\]

where

\[
- \left\{ \varepsilon_{ijk} r_j P_k^\text{flat}, \varepsilon_{lmn} r_m A_n \right\} = \delta_i r_n A_i - r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_m P_k^\text{flat} \{ A_n, r_j \} ,
\]

\[
+ \left\{ \varepsilon_{ijk} r_j \tilde{\Gamma}_k, \varepsilon_{lmn} r_m A_n \right\} = -\varepsilon_{ijk} \varepsilon_{lmn} r_m A_n \{ r_i, \tilde{\Gamma}_k \} 
- \varepsilon_{ijk} \varepsilon_{lmn} r_m \tilde{\Gamma}_k \{ A_n, r_j \} .
\]

The third bracket on the rhs of (82) is similar to the second with \( \tilde{A} \) and \( \tilde{p} \) being interchanged such that,

\[
- \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m p_n \} = -\delta_i r_n A_i + r_i A_i - \varepsilon_{ijk} \varepsilon_{lmn} r_m P_k^\text{flat} \{ r_m, A_k \} 
+ \varepsilon_{ijk} \varepsilon_{lmn} r_m A_k \{ \tilde{\Gamma}_j, r_n \} 
+ \varepsilon_{ijk} \varepsilon_{lmn} r_m A_k \{ \tilde{\Gamma}_j, r_n \} .
\]

Finally, the fourth bracket on the rhs of (82) is given by

\[
\{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m A_n \} = -\varepsilon_{ijk} \varepsilon_{lmn} r_j A_n \{ r_m, A_k \} 
- \varepsilon_{ijk} \varepsilon_{lmn} r_m A_k \{ A_n, r_j \} .
\]
Combining these results, we obtain

\[ \{ J_i, J_j \} = r p_i^{\text{flat}} - r_i p_j^{\text{flat}} + \delta_{il} r_j \hat{\Gamma}_l - \delta_{jl} r_i \hat{\Gamma}_l + r_i \hat{\Gamma}_l - r_l \hat{\Gamma}_i + + \delta_{il} r_i A_n - \delta_{jl} r_j A_n + r_i A_l - r_l A_l + + \varepsilon_{ijk \ell m n} (r_{m} p_k^{\text{flat}} \{ A_n, r_j \} - r_{j} p_{m}^{\text{flat}} \{ r_m, A_k \}) + + \varepsilon_{ijk \ell m n} (r_{m} p_k^{\text{flat}} \{ A_k, r_j \} - r_{j} p_{m}^{\text{flat}} \{ r_m, A_k \}) + + \varepsilon_{ijk \ell m n} (r_{m} p_k^{\text{flat}} \{ \hat{\Gamma}_n, r_j \} - r_{j} p_{m}^{\text{flat}} \{ r_m, \hat{\Gamma}_k \}) + + \varepsilon_{ijk \ell m n} (r_{j} \hat{\Gamma}_n \{ A_k, r_m \} - r_{m} \hat{\Gamma}_k \{ A_n, r_j \}) + + \varepsilon_{ijk \ell m n} (r_{j} \hat{\Gamma}_n \{ A_k, r_m \} - r_{m} \hat{\Gamma}_k \{ A_n, r_j \}) + + \varepsilon_{ijk \ell m n} (r_{j} \hat{\Gamma}_n \{ A_k, r_m \} - r_{m} \hat{\Gamma}_k \{ A_n, r_j \}) - \varepsilon_{\ell k s} \delta_{l} \hat{\Gamma}_i. \] (93)

The full antisymmetry of the Levi-Civita tensor leads to

\[ \varepsilon_{ijk \ell m n} r_{m} p_k^{\text{flat}} \{ A_n, r_j \} - \varepsilon_{ijk \ell m n} r_{j} p_{n}^{\text{flat}} \{ A_n, r_m \} = (\varepsilon_{ijk \ell m n} - \varepsilon_{imn} \varepsilon_{ijk}) r_{m} p_k^{\text{flat}} \{ A_n, r_j \} = 0 \] (94)

and

\[ \varepsilon_{ijk \ell m n} r_{m} p_k^{\text{flat}} \{ \hat{\Gamma}_n, r_j \} - \varepsilon_{ijk \ell m n} r_{j} p_{n}^{\text{flat}} \{ \hat{\Gamma}_n, r_m \} = (\varepsilon_{ijk \ell m n} - \varepsilon_{imn} \varepsilon_{ijk}) r_{m} p_k^{\text{flat}} \{ \hat{\Gamma}_n, r_j \} = 0. \] (95)

Thus,

\[ \{ J_i, J_j \} = -\varepsilon_{ijk} J_k \] (96)

proving that \( \bar{J} \) is the generator of spatial rotations. Using \( \bar{J} \) we can now show that \( e^{-2\phi(|\vec{r}|)} \vec{E}(\vec{r}) \) transforms as a vector,

\[ \{ J_i, e^{-2\phi(|\vec{r}|)} \vec{E}_i \} = \{ J_i, e^{-2\phi(|\vec{r}|)} \} \vec{E}_i + e^{-2\phi(|\vec{r}|)} \{ J_i, \vec{E}_i \} \] (97)

\[ = \varepsilon_{\ell k} e^{-2\phi(|\vec{r}|)} \vec{E}_k \text{ since } \{ J_i, e^{-2\phi(|\vec{r}|)} \} = 0. \]

The vector \( e^{-2\phi(|\vec{r}|)} \vec{E}(\vec{r}) \) can be shown \[16\] to have a general functional form

\[ e^{-2\phi(|\vec{r}|)} \vec{E}(\vec{r}) = k_0 m^2 \xi (m_\varepsilon \vec{r}, m_\varepsilon \vec{r} \cdot \hat{n}) (\hat{n} \times \vec{r}), \] (98)

where \( \xi \) is a generic scalar field function. We emphasize again that form the \( \vec{\partial} \times \vec{B}'(\vec{r}) \) equation in \[74\], it is evident that no spherically symmetric magnetic \( \vec{B}'(\vec{r}) \) field solution exists, i.e. \( \vec{B}'(\vec{r}) \neq \vec{B}'(\vec{r}) \).

We now study whether the symmetry properties of the magnetic field obtained above is compatible with the Poisson algebra of the system. This is accomplished by determining if the magnetic field transforms as a vector under spatial rotations by computing the Poisson bracket \( \{ J_i, B_j \} \). It is convenient to begin this analysis by calculating the curved space canonical momentum \( \vec{p} \) and the curved space kinetic momentum vector \( \vec{P} \) Poisson brackets,

\[ \{ p_i, p_j \} = \{ p_i^{\text{flat}}, \hat{\Gamma}_i, p_j^{\text{flat}}, \hat{\Gamma}_j \} \]

\[ = \{ p_i^{\text{flat}}, p_j^{\text{flat}} \} - \{ p_i^{\text{flat}}, \hat{\Gamma}_j \} - \{ \hat{\Gamma}_i, p_j^{\text{flat}} \} + \{ \hat{\Gamma}_i, \hat{\Gamma}_j \} \]

\[ = \{ \hat{\Gamma}_j, p_i^{\text{flat}} \} - \{ \hat{\Gamma}_i, p_j^{\text{flat}} \} = -\math{R}_{ij} \text{ where } \math{R}_{ij} = \partial_i \hat{\Gamma}_j + \partial_j \hat{\Gamma}_i \] (99)
and

\[ \{P_i, P_j\} = \{ p_i - A_i, p_j - A_j \} \]
\[ = \{ p_i, p_j \} - \{ p_i, A_j \} - \{ A_i, p_j \} + \{ A_i, A_j \} \]
\[ = \{ A_j, p_i \} - \{ A_i, p_j \} - \mathcal{R}_{ij} \]
\[ = \{ A_j, p_i^{\text{flat}} - \hat{\Gamma}_i \} - \{ A_i, p_j^{\text{flat}} - \hat{\Gamma}_j \} - \mathcal{R}_{ij} \]
\[ = \{ A_j, p_i^{\text{flat}} \} - \{ A_i, \hat{\Gamma}_i \} - \{ A_i, p_j^{\text{flat}} \} + \{ A_i, \hat{\Gamma}_j \} - \mathcal{R}_{ij} \]
\[ = -\partial_i A_j + \partial_j A_i - \{ A_j, \hat{\Gamma}_i \} + \{ A_i, \hat{\Gamma}_j \} - \mathcal{R}_{ij} \]
\[ = - (\partial_i A_j - \partial_j A_i) - \mathcal{R}_{ij} = -\varepsilon_{ijk} B_k - \mathcal{R}_{ij}. \]

Note that we employed the Dirac-veto \( B_k = (\varepsilon_{klm} \partial_l A_m + e(m) h_k) \) in obtaining (100), that is, we impose that the electrically charged particle must never pass through the string \([32]\) and therefore the electric charge does not "feel" the magnetic field contribution originating from the string function \( \vec{h}(\vec{r}) \). From (100) we conclude

\[ B_k = -\frac{1}{2} \varepsilon_{ijk} \left( \{P_i, P_j\} + \mathcal{R}_{ij} \right). \] (101)

We can now calculate the Poisson bracket \( \{J_i, B_j\} \)

\[ \{J_i, B_j\} = -\frac{1}{2} \varepsilon_{lmj} \{J_i, \mathcal{R}_{lm} + \{P_l, P_m\}\} \]
\[ = -\frac{1}{2} \varepsilon_{lmj} \{J_i, \mathcal{R}_{lm}\} - \frac{1}{2} \varepsilon_{lmj} \{J_i, \{P_l, P_m\}\} \]
\[ = \frac{1}{2} \varepsilon_{lmj} \{J_i, \{P_l, P_m\}\} - \frac{1}{2} \varepsilon_{lmj} \{J_i, \{P_l, P_m\}\} \]
by using the Jacobi identities

\[ \{J_i, \{P_l, P_m\}\} + \{P_m, \{J_i, P_l\}\} + \{P_l, \{J_i, P_m\}\} = 0, \] (103)
\[ \{J_i, \{p_l, p_m\}\} + \{p_m, \{J_i, p_l\}\} + \{p_l, \{J_i, p_m\}\} = 0, \] (104)

and

\[ \{P_m, \{J_i, P_l\}\} = -\varepsilon_{ilk} \{P_m, P_k\} = -\varepsilon_{ilk} [-\varepsilon_{mkn} B_n - \mathcal{R}_{mk}], \] (105)
\[ \{P_l, \{P_m, J_l\}\} = -\{P_l, \{J_l, P_m\}\} = \varepsilon_{imk} \{P_l, P_k\} = \varepsilon_{imk} [-\varepsilon_{lkn} B_n - \mathcal{R}_{lk}], \] (106)
\[ \{p_m, \{J_i, p_l\}\} = -\varepsilon_{ilk} \{p_m, p_k\} = -\varepsilon_{ilk} (-\mathcal{R}_{mk}), \] (107)
\[ \{p_l, \{p_m, J_l\}\} = -\{p_l, \{J_l, p_m\}\} = \varepsilon_{imk} \{p_l, p_k\} = \varepsilon_{imk} (-\mathcal{R}_{lk}). \] (108)

Using (105), (106), (107), (108) together with the Jacobi identities (103) and (104), we obtain

\[ \{J_i, \{P_l, P_m\}\} = -(\varepsilon_{ilk} [-\varepsilon_{mkn} B_n - \mathcal{R}_{mk}] + \varepsilon_{imk} [-\varepsilon_{lkn} B_n - \mathcal{R}_{lk}]) \]
\[ = -\varepsilon_{ilk} \varepsilon_{mkn} B_n + \varepsilon_{imk} \varepsilon_{lkn} B_n - \varepsilon_{ilk} \mathcal{R}_{mk} + \varepsilon_{imk} \mathcal{R}_{lk} \]
\[ = -\delta_d B_m + \delta_{im} B_l - \varepsilon_{ilk} \mathcal{R}_{mk} + \varepsilon_{imk} \mathcal{R}_{lk} \]

and

\[ \{J_i, \{p_l, p_m\}\} = -[\varepsilon_{ilk} (-\mathcal{R}_{mk}) + \varepsilon_{imk} (-\mathcal{R}_{lk})] \]
\[ = -\varepsilon_{ilk} \mathcal{R}_{mk} + \varepsilon_{imk} \mathcal{R}_{lk}. \] (110)
Substituting (109) and (110) into (102) leads to

\[
\{ J_i, B_j \} = \frac{1}{2} \varepsilon_{lmj} \left[ -\varepsilon_{ilk} R_{mk} + \varepsilon_{imk} R_{lk} \right] - \frac{1}{2} \varepsilon_{lmj} \left[ -\delta_{il} B_m + \delta_{im} B_l - \varepsilon_{ilk} R_{mk} + \varepsilon_{imk} R_{lk} \right] + \frac{1}{2} \varepsilon_{lmj} \varepsilon_{ilk} \left( R_{mk} - R_{ml} \right) + \frac{1}{2} \varepsilon_{lmj} \varepsilon_{ilk} \left( R_{mk} - R_{ml} \right)
\]

\[
= -\varepsilon_{mi} B_m.
\]

It is known from (70) that \( \vec{B}_{\text{monopole}} \) is spherically symmetric and following [17], it can be shown that the diffuse magnetic field (with vector potential of form (98)) must exhibit spherical symmetry

\[
\vec{B}'(\vec{r}) = B'(r) \hat{r}.
\]

in order to satisfy (111). Such spherically symmetric solutions however, are incompatible with the second equation in (74). This result implies it is not possible to formulate a consistent classical theory describing nonrelativistic point-like charged particles interacting with magnetic monopoles without a "visible" string via topologically massive vector bosons in curved spacetime with isotropic dilation since there is no way to construct a consistent Lie algebra.

6 Conclusion

In this article we considered a Brans-Dicke generalization of gravity with non-vanishing curvature and torsion of potential type. An action describing electromagnetic interaction between charged, nonrelativistic fermions with an abelian magnetic monopole, where the interaction is mediated by topologically massive vector bosons, was proposed. The gauge field mass is a direct consequence of the (topological) coupling - characterized by \( \mu_0 \) - between the electromagnetic 4-vector and the second-rank torsion potential. This coupling is said to be topological due to the lack of \( \mu_0 \)-dependent terms in the canonical energy-momentum tensor. The field equations for the theory as well as the Bianchi identities in the electromagnetic and torsion sectors were obtained. From the solutions to the torsion field equation (59) we observe that the sources of torsion are spinors, dilatons and photons. The dilatonic contribution arises from the non-minimal torsion-dilaton coupling while the electromagnetic contribution is due to the aforementioned topological interaction.

Assuming an isotropic dilaton field configuration, the quantity \( e^{-2\phi(|\vec{r}|)} \tilde{E} \) plays the role of a massive photon-like term with mass \( m_2^2 = 2\mu_0 \). This term together with pseudo-current \( \vec{j}_5 \) arising from the spin energy potential constitute the total diffusive magnetic potential \( \vec{A}' \). It was demonstrated that the Poisson bracket \( \{ J, B \} \) in curved spacetime is not only well defined but identical in structure to the flat spacetime dilaton free case. It can be shown following [17] that under the isotropic dilaton and Dirac veto ansatz, together with the limit \( \left| \vec{j}_5 \right| \ll \left| \frac{1}{\kappa_0} e^{-2\phi(|\vec{r}|)} \tilde{E}(\vec{r}) \right| \), spherically symmetric magnetic field solutions are required in order to satisfy the Poisson bracket \( \{ J, B \} \). Although \( \vec{B}_{\text{monopole}} \) is spherically symmetric, spherical solutions for the diffuse magnetic field \( \vec{B}' \) are inconsistent with the nonvanishing of \( \partial \times \vec{B}'(\vec{r}) \) in (74) despite the topological nature of photon mass effectively generated by \( m_2^2 e^{-2\phi(|\vec{r}|)} \tilde{E}(\vec{r}) \). For this reason we conclude that the incompatibility between massive photons and magnetic monopoles (without visible string) in the present classical framework is not a consequence of the specific nature of photon mass generation. What is more, the incompatibility survives the transition from flat to curved spacetime and persists even in presence of (isotropic) dilaton fields. With regard to the matter content of the theory, it is interesting to observe that depending on the sign of the fermion electric charge, the pseudo-current \( \vec{j}_5 \) could serve to either enhance or degrade the massive photon-like term. A measurable consequence of this would be an associated increase or decrease of the diffuse magnetic field intensity arising from the diffuse vector potential \( \vec{A}' \).
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