EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE ANISOTROPIC QUASI-GEOSTROPHIC EQUATIONS IN THE SOBOLEV SPACE

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Abstract. In this paper, we focus on the two-dimensional surface quasi-geostrophic equation with fractional horizontal dissipation and vertical thermal diffusion which represents a general case of the classical surface quasi-geostrophic equation. On the one hand, we will show the local existence and uniqueness of the solution in Sobolev space $H^{2-2\alpha}(\mathbb{R}^2) \cap H^{2-2\beta}(\mathbb{R}^2)$, which is the critical space in the classical case. Furthermore, we will demonstrate that the solution is global even when the initial data is very small. Finally, we will study the asymptotic representation of our global solution in infinity.

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1. Introduction

We consider the two-dimensional anisotropic quasi-geostrophic equation:

\[
\begin{align*}
\partial_t \theta + u_\theta \nabla \theta + \mu |\partial_1|^{2\alpha} \theta + \nu |\partial_2|^{2\beta} \theta &= 0, & t > 0, x \in \mathbb{R}^2, \\
u = R^\perp \theta &= t > 0, x \in \mathbb{R}^2, \\
\theta(x, 0) &= \theta_0(x), & x \in \mathbb{R}^2,
\end{align*}
\]

(AQG)

\* $\theta(x, t)$ and $u_\theta = u_\theta(x, t) = (u_\theta^1(x, t), u_\theta^2(x, t))$ are the unknown potential temperature and the velocity field of the fluid, respectively.

\* $\theta_0$ is the given initial potential temperature.

\* $\mu, \nu > 0$ and $\alpha, \beta \in (0, 1)$.

\* $R^\perp = (\overline{-R}_2, \overline{R}_1)$, where $R_1$ and $R_2$ are the Riesz transforms defined by

\[
R_j \theta = \mathcal{F}^{-1} \left( \frac{i\xi_j}{|\xi|} \mathcal{F}(\theta)(\xi) \right), \quad j \in \{1, 2\}.
\]
The equation arises from geophysics and has a strong physical background. If $\mu = \nu = 0$, then it reduces to the inviscid case, which shares some important features with the 3D Euler equations such as the vortex stretching mechanism. This inviscid case equation is an important model of geophysical fluid dynamics, which describes the evolution of the surface temperature field in the rotating stratified fluid. The first mathematical studies of this equation was carried out in 1994s by Constantin, Majda and Tabak. For more details and mathematical and physical explanations of this model we can consult [3-6].

In this manuscript, we show the local existence and uniqueness of the solution of system (AQQ). More precisely, for the given initial data $\theta^0$ in $H^{\max(2-2\alpha,2-2\beta)}(\mathbb{R}^2) := H^{2-2\alpha}(\mathbb{R}^2) \cap H^{2-2\beta}(\mathbb{R}^2)$, there exists a positive constant $T_0$ determined by $\alpha, \beta$ and $\theta^0$ such that (AQQ) possesses a unique local classical solution on $C([0,T_0], H^{\max(2-2\alpha,2-2\beta)}(\mathbb{R}^2))$. Next, we prove the global existence for a small initial data in the same space, that is to say, a positive constant $\varepsilon$ exists such, if $\|\theta^0\|_{H^{\max(2-2\alpha,2-2\beta)}} < \varepsilon$, then, the local solution is global one. Finally we prove some optimal decay results of the global solution of (AQQ) with small initial data.

Before we state the main result, we recall some known results for the existence theorem of the classical quasi-geostrophic equation:

\[
\begin{align*}
\begin{cases}
\partial_t \theta + u_\theta \nabla \theta + \kappa(-\Delta)\gamma \theta &= 0, & t > 0, x \in \mathbb{R}^2, \\
u_\theta &= \mathcal{R}^{1,2} \theta, & t > 0, x \in \mathbb{R}^2, \\
\theta(x,0) &= \theta^0(x), & x \in \mathbb{R}^2.
\end{cases}
\end{align*}
\]

(SQG)

Easy to see that when $\gamma = \alpha = \beta$ and $\kappa = \mu = \nu$, we have (AQQ) becomes the classical dissipation (SQG) equation. This last equation was studied in three different cases: subcritical case when $\gamma \in (\frac{1}{2},1)$, critical case $\gamma = \frac{1}{2}$ and last case when $\gamma \in (0,\frac{1}{2})$ called supercritical case. Constantin and Wu [7] established the existence of a unique global solution and decay estimates with regard to the $L^2$ norm for the initial data $\theta^0 \in L^2(\mathbb{R}^2)$ in the subcritical case. For the critical case, there are many results for this case, for example, Constantin and Vicol [8] showed the global in time existence of the smooth solution $\theta \in \mathcal{S}(\mathbb{R}^2)$. See also the works [9-11] where same type of results have been obtained.

However, the supercritical cases, for an initial data, whether it remains globally regular or not, is an interesting open problem. Although the global well-posedness is still open for the this case for (SQG) equation. Some interesting result, showed by Miura [12] who proved the unique local existence of the solution in the critical space $H^{2-2\gamma}(\mathbb{R}^2)$. See also the works in [13] that have produced similar results.

To our knowledge, the first time the system of equations in (AQQ) has been studied, is by Zhuan in [14], who established the global regularity when the dissipation powers are restricted to a suitable range. More specifically

**Theorem 1.1** (see [14]). Let $\theta^0 \in H^s(\mathbb{R}^2)$ with $s \geq 2$ and $\alpha, \beta > 0$. Then there exists a positive $T(\|\theta^0\|_{H^s}) > 0$ such that for the system (AQQ) admits a unique solution

\[
\theta \in C([0,T], H^s(\mathbb{R}^2)), \quad |\partial_1|^\alpha \theta, |\partial_2|^\beta \theta \in L^2([0,T], H^s(\mathbb{R}^2)).
\]

Moreover, if $\alpha, \beta \in (0,1)$ satisfy

\[
\beta > \begin{cases}
\frac{1}{2\alpha+1}, & 0 < \alpha \leq \frac{1}{2} \\
\frac{1-\alpha}{2\alpha}, & \frac{1}{2} < \alpha < 1,
\end{cases}
\]

then, the system (AQQ) admits a unique global solution $\theta$ such that for any $T > 0$ we have

\[
\theta \in C([0,T], H^s(\mathbb{R}^2)), \quad |\partial_1|^\alpha \theta, |\partial_2|^\beta \theta \in L^2([0,T], H^s(\mathbb{R}^2)).
\]
Motivated by these previous studies, we first show the local solution existence to (AQG) for the initial data \( \theta^0 \in H^s(\mathbb{R}^2) \), with \( s = \max\{2 - 2\alpha, 2 - 2\beta\} \). Our main result is as follows:

**Theorem 1.2.** Let \( \alpha, \beta \in (0, 1) \) such that

\[
\min\{\alpha, \beta\} < \frac{1}{2}.
\]

We assume \( s = \max\{2 - 2\alpha, 2 - 2\beta\} \). Let \( \theta^0 \in H^s(\mathbb{R}^2) \), then, there exists a positive time \( T_0 \) such that (AQG) admits a unique solution

\[
\theta \in C([0, T_0]; H^s(\mathbb{R}^2)), \quad |\partial_1|^\alpha \theta, |\partial_2|^\beta \theta \in L^2([0, T_0]; H^s(\mathbb{R}^2)).
\]

Moreover, there exists a constant \( c > 0 \) such that, if

\[
\|\theta^0\|_{H^s} < c,
\]

then

\[
\theta \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)), \quad |\partial_1|^\alpha \theta, |\partial_2|^\beta \theta \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^2)).
\]

In addition the following estimate holds

\[
\|\theta(t)\|_{H^s} + \int_0^t \|\partial_1|^\alpha \theta(\tau)\|_{H^s}^2 d\tau + \int_0^t \|\partial_2|^\beta \theta(\tau)\|_{H^s}^2 d\tau \leq \|\theta^0\|_{H^s}.
\] (1.2)

**Remark 1.**

(i) We assume that \( \mu = \nu = 1 \) to simplify the calculus and some procedures in the proofs of our results.

(ii) We can select without loss the generality \( \alpha = \min\{\alpha, \beta\} \), the proof will be in the same way if we have the opposite.

(iii) Very recently in [15], Zhuan a proved the global well-posedness existence of (AQG) in the Sobolev space

\[
H^{\max\{s, \frac{2 - 2\alpha}{1 - \alpha}\}, \frac{2 - 2\alpha}{1 - \alpha}} := H^s(\mathbb{R}^2) \cap \dot{H}^{2 - \frac{2\alpha}{1 - \alpha}}(\mathbb{R}^2), \quad s \geq 0,
\]

where \( \varepsilon > 0 \) is very small. This result not true only if \( (\alpha, \beta) \in \mathbb{F} \), where

\[
\mathbb{F} := (0, 1]^2 \setminus \{ (\alpha, \beta) \in (0, 1]^2 \text{ satisfy (1.1)} \}.
\]

In the same paper, the author showed the decay estimate for the solution with additionally that

\[
\theta^0 \in H^{\frac{2\alpha}{1 - \alpha}; \frac{2 - 2\beta}{1 - \beta}}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2), \quad p \in [1, 2).
\]

In the next theorem for the decays as time goes to infinity of the solution:

**Theorem 1.3.** Let \( \theta \in C(\mathbb{R}^+, H^s(\mathbb{R}^2)) \), \( s = \max\{2 - 2\alpha, 2 - 2\beta\} \), a global solution of (AQG) equation such that

\[
\|\theta^0\|_{H^s} < c,
\]

then

\[
\lim_{t \to +\infty} \|\theta(t)\|_{H^s} = 0.
\]

The following is a breakdown of the paper’s structure. In Section 2, we review some of the notations and estimations that will be used in the next parts. In Section 3, we prove the existence theorem for any initial data \( \theta^0 \in H^{2 - 2\alpha}(\mathbb{R}^2) \). Finally, in Section 4, we prove **Theorem 1.3**.

Throughout this paper, the constant, which may differ in each line, is denoted by \( C \) throughout this work. \( C = C(a_1, ..., a_n) \) specifies that \( C \) is entirely dependent on \( a_1, ..., a_n \).
2. Notations and Preliminary Results

In this short section, we collect some notations and definitions that will be used later, and we give some technical lemmas.

2.1. Notations. Let $\mathcal{S}(\mathbb{R}^2)$ denote the space of Schwartz class functions defined on $\mathbb{R}^2$ and $\mathcal{S}'(\mathbb{R}^2)$ denote the space of tempered distributions. For $f \in \mathcal{S}(\mathbb{R}^2)$, we denote by $\hat{f}$ or $\mathcal{F}(f)$, the Fourier transform of $f$, defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx, \quad \forall \xi \in \mathbb{R}^2,$$

and the inverse Fourier transform of $f$ defined by

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} f(\xi) d\xi, \quad \forall x \in \mathbb{R}^2.$$  

Recall that $\mathcal{F}$ is an isometry on $L^2$ and satisfies

$$(f, g)_{L^2} = (\hat{f}, \hat{g})_{L^2}. \tag{2.3}$$

- The convolution product of a suitable pair of function $f = (f_1, f_2)$ and $g = (g_1, g_2)$ on $\mathbb{R}^2$ is given by

$$f \ast g(x) = \int_{\mathbb{R}^2} f(x - y) g(y) dy.$$  

Also, we set

$$f \otimes g := (g_1 f, g_2 f) \quad \text{and} \quad \text{div}(f \otimes g) := (\text{div}(g_1 f), \text{div}(g_2 f)).$$

- The fractional operators are defined through the Fourier transform, namely, for any $s \in \mathbb{R}$

$$|\partial_1|^s f = \mathcal{F}^{-1} \left( \xi \mapsto |\xi_1|^s \hat{f}(\xi) \right), \quad |\partial_2|^s f = \mathcal{F}^{-1} \left( \xi \mapsto |\xi_2|^s \hat{f}(\xi) \right) \quad \text{and} \quad |\nabla|^s f = \mathcal{F}^{-1} \left( \xi \mapsto |\xi|^s \hat{f}(\xi) \right). \tag{2.4}$$

- For $s \in \mathbb{R}$, let $H^s(\mathbb{R}^2)$ denote the Sobolev non-homogeneous space of order $s$, which is a Hilbert space with the inner product

$$(f, g)_{H^s} = ((1 + |\nabla|^2)^{\frac{s}{2}} f, (1 + |\nabla|^2)^{\frac{s}{2}} g)_{L^2} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$$

and the norm $\|f\|_{H^s} = \sqrt{(f, f)_{H^s}}$. We also denote $\dot{H}^s(\mathbb{R}^2)$ the Sobolev homogeneous space of order $s$ with the inner product

$$(f, g)_{\dot{H}^s} = ((|\nabla|^s f, |\nabla|^s g)_{L^2} = \int_{\mathbb{R}^2} |\xi|^{2s} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$$

and the norm $\|f\|_{\dot{H}^s} = \sqrt{(f, f)_{\dot{H}^s}}$. If $s > 0$, then it is well known that the norm equivalence

$$\|f\|_{H^s} \sim \|f\|_{L^2} + \|f\|_{\dot{H}^s}.$$

- If $(X, \|\cdot\|_X)$, be a Banach space, $p \in [0, +\infty]$ and $T > 0$. We define

$$L^p_T(X) := L^p([0, T], X)$$

the space of all measurable functions $t \in [0, T] \mapsto f(t) \in X$ such that

$$\|f(t)\|_X \in L^p([0, T]).$$

- We present some features for the mollifier in $\mathbb{R}^2$ of the Friedrichs type, as specified by

$$J_{N}(f) := \mathcal{F}^{-1} \left( \xi \mapsto \mathcal{X}_{B(0,N)}(\xi) \hat{f}(\xi) \right),$$
where $N \in \mathbb{N}$, $f \in L^2(\mathbb{R}^2)$, $B(0, N) = \{\xi \in \mathbb{R}^2; |\xi| < N\}$ and

$\mathcal{X}_{B(0,N)} : \mathbb{R}^2 \to \{0,1\}$

$\xi \mapsto \mathcal{X}_{B(0,N)}(\xi) = \begin{cases} 1 & \text{if } |\xi| < 1, \\ 0 & \text{else.} \end{cases}$

2.2. Preliminary Results. Next, we introduce some Lemma that will be used in the proof of our results.

**Lemma 2.1** (see [1]). Let $H$ be Hilbert space and $(x_n)$ be a bounded sequence of elements in $H$ such that

$x_n \to x$ in $H$ and

$\lim_{n \to +\infty} \|x_n\| \leq \|x\|.

Therefore $\lim_{n \to +\infty} \|x_n - x\| = 0$.

**Lemma 2.2** (see [2]). Let $p \in [2, +\infty)$ and $\sigma \in [0,1)$ such that

$$\frac{1}{p} + \frac{\sigma}{2} = \frac{1}{2}.$$ Then, there is a constant $C > 0$ such that

$$\|f\|_{L^p(\mathbb{R}^4)} \leq C\|\nabla|^\sigma f\|_{L^2(\mathbb{R}^4)}.$$**Lemma 2.3** (see [2]). Let $s_1, s_2$ be two real numbers such that $s_1 < 1$ and $s_1 + s_2 > 0$. Then, there exists a positive constant $C = C(s_1, s_2)$ such that for all $f, g \in H^{s_1}(\mathbb{R}^2) \cap H^{s_2}(\mathbb{R}^2)$:

$$\|fg\|_{H^{s_1+s_2-1}} \leq C (\|f\|_{H^{s_1}} \|g\|_{H^{s_2}} + \|f\|_{H^{s_2}} \|g\|_{H^{s_1}}).$$

Moreover, in addition $s_2 < 1$, there exists a positive constant $C = C(s_1, s_2)$ such that, for all $f \in H^{s_1}(\mathbb{R}^2)$ and $g \in H^{s_2}(\mathbb{R}^2)$:

$$\|fg\|_{H^{s_1+s_2-1}} \leq C \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}.$$**Lemma 2.4.** For $s_1, s_2 \in \mathbb{R}$ and $t \in [0,1]$, the interpolation inequalities, respectively, in the homogeneous and non-homogeneous Sobolev spaces

$$\|f\|_{H^{s_1+(1-t)s_2}} \leq \|f\|_{H^{s_1}} \|f\|_{H^{1-t}s_2}^{1-t},$$

$$\|f\|_{H^{s_1+(1-t)s_2}} \leq \|f\|_{H^{s_1}} \|f\|_{H^{1-t}s_2}^{1-t}.$$**Lemma 2.5.** Let $s, s' \in \mathbb{R}$ and $\alpha, \beta \in (0,1)$, such that $s' < s + \alpha$ and $\alpha \leq \beta$ then for any $f \in S(\mathbb{R}^2)$

$$\|\nabla|^\alpha f\|_{H^{s'}} \leq \|f\|_{H^{s'}} + \|\partial_1|^\alpha f\|_{H^{s'}} + \|\partial_2|^\alpha f\|_{H^{s'}}.$$**Proof.** For $f \in S(\mathbb{R}^2)$, we have

$$\|\nabla|^\alpha f\|_{H^{s'}}^2 = \int_{\mathbb{R}^2} (|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha}) |\mathcal{F}(|\nabla|^\alpha f)(\xi)|^2 d\xi \leq 2 \int_{\mathbb{R}^2} |\xi_1|^{2\alpha} |\mathcal{F}(|\nabla|^\alpha f)(\xi)|^2 d\xi + 2 \int_{\mathbb{R}^2} |\xi_2|^{2\alpha} |\mathcal{F}(|\nabla|^\alpha f)(\xi)|^2 d\xi = \|\partial_1|^\alpha f\|_{H^{s'}}^2 + \|\partial_2|^\alpha f\|_{H^{s'}}^2.$$ But $s' - s < \alpha \leq \beta$ then their exist $z \in (0,1]$ such that $\alpha = z \times \beta + (1-z) \times (s' - s)$ and by interpolation inequality we get

$$\|\partial_1|^\alpha f\|_{H^{s'}} \leq \|f\|_{H^{s'}}^z \|\partial_2|^\beta f\|_{H^{s'}}^{1-z} \leq (1-z)\|f\|_{H^{s'}} + z\|\partial_2|^\beta f\|_{H^{s'}} \leq \|f\|_{H^{s'}} + \|\partial_2|^\beta f\|_{H^{s'}}.$$
Lemma 2.6 (see [13]). For any $p \in (1, +\infty)$, there is a constant $C(p) > 0$ such that
\[
\|R^\perp \theta\|_{L^p} \leq C(p) \|\theta\|_{L^p}.
\] (2.9)

We recall the following important commutator and product estimates:

Lemma 2.7. For $s > 1$, if $f, g \in S(\mathbb{R}^2)$ then for any $\alpha \in (0, 1)$
\[
\|\nabla \|^s (fg) - f|\nabla \|^s g\|_{L^2} \leq s^2 C(\alpha) \left( \|\nabla \|^{s+\alpha} f\|_{L^2} \|\nabla \|^{1-\alpha} g\|_{L^2} + \|\nabla \|^{s-1+\alpha} g\|_{L^2} \|\nabla \|^{2-\alpha} f\|_{L^2} \right).
\] (2.10)

Proof. We have
\[
\|\nabla \|^{s}(fg) - f|\nabla \|^s g\|_{L^2}^2 = \int_{\mathbb{R}^2} |\mathcal{F}(|\nabla \|^s (fg) - f|\nabla \|^s g)(\xi)|^2 \, d\xi
\]
\[
\leq \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \|\xi|^s - |\eta|^s|\tilde{f}(\xi - \eta)\|\tilde{g}(\eta)\|d\eta \right)^2 \, d\xi,
\]
By using the elementary inequality
\[
\|\xi|^s - |\eta|^s| \leq s^2 s^{-1} (|\xi - \eta|^s + |\eta|^{s-1}|\xi - \eta|)
\]
we get
\[
\|\nabla \|^{s}(fg) - f|\nabla \|^s g\|_{L^2}^2 \leq s^{22^{(s-1)}} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \|\xi - \eta\|^s|\tilde{f}(\xi - \eta)|\tilde{g}(\eta)\| + |\xi - \eta||\tilde{f}(\xi - \eta)||\eta|^{s-1}|\tilde{g}(\eta)|\|d\eta \right)^2 d\xi
\]
\[
\leq s^{22^s}\|f_1g_1\|_{L^2}^2 + s^{22^s}\|f_2g_2\|_{L^2}^2,
\]
where
\[
\mathcal{F}(f_1)(\xi) = |\xi|^s F(f)(\xi) F(g)(\xi) = \|F(g)(\xi)\|,
\]
\[
\mathcal{F}(f_2)(\xi) = |\xi|^s F(f)(\xi) F(g)(\xi) = |\xi|^{s-1}/F(g)(\xi)\|
\]
Using Hölder inequality with
\[
\frac{\alpha}{2} + \frac{1-\alpha}{2} = \frac{1}{2}
\]
and by Lemma 2.2, we get
\[
\|\nabla \|^{s}(fg) - f|\nabla \|^s g\|_{L^2} \leq s^2 \left( \|f_1\|_{L^2} \|g_1\|_{L^2} + \|f_2\|_{L^2} \|g_1\|_{L^2} \right)
\]
\[
\leq s^2 C(\alpha) \left( \|\nabla \|^{s+\alpha} f\|_{L^2} \|\nabla \|^{1-\alpha} g\|_{L^2} + \|\nabla \|^{s-1+\alpha} g\|_{L^2} \|\nabla \|^{2-\alpha} f\|_{L^2} \right),
\]
which finished the proof. ■

Lemma 2.8. Let $\alpha, \beta \in (0, 1)$ and we pose $A(\xi) = |\xi|^{2\alpha} + |\xi|^{2\beta}$, then
\[
|\xi| \leq C(\alpha, \beta) \left( A(\xi) \frac{1}{2^\alpha} + A(\xi) \frac{1}{2^\beta} \right), \quad \forall \xi \in \mathbb{R}^2.
\] (2.11)

Proof. Let $\alpha, \beta \in (0, 1)$,

- If $|\xi_1| \leq |\xi_2|$, then we have
\[
|\xi|^{2\beta} \leq (|\xi_1|^{2\beta} + |\xi_2|^{2\beta}) \leq 2|\xi_2|^{2\beta} \leq 2A(\xi).
\]
Therefore
\[
|\xi| \leq 2^{\frac{1}{2^\beta}} A(\xi) \frac{1}{2^\beta} \leq 2^{\frac{1}{2^\beta}} \left( A(\xi) \frac{1}{2^\alpha} + A(\xi) \frac{1}{2^\beta} \right).
\] (2.12)
* If \(|\xi_2| \leq |\xi_1|\) then we have
\[
|\xi|^{2\alpha} \leq 2(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha}) \leq 2|\xi_1|^{2\alpha} \leq 2A(\xi).
\]
Therefore
\[
|\xi| \leq 2^{1\over 2\alpha} A(\xi)^{1\over \alpha} \leq 2^{1\over \alpha} \left( A(\xi)^{1\over \alpha} + A(\xi)^{2\over \alpha} \right). \tag{2.13}
\]
Collecting (2.12) and (2.13), we get the result.

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. In the first part, we show the local existence and uniqueness of the solution. The second part is devoted to prove that this solution is continuous. The global existence of solution will showed in the last part.

3.1. Local existence and uniqueness: We start by proving the existence of local solution of (AQG) equation in the following space
\[
\mathcal{X}_{T_0} = \left\{ f \in L^\infty_{T_0}(H^{2-2\alpha}(\mathbb{R}^2)) \cap C_{T_0}(L^2(\mathbb{R}^2)) : |\partial_t|^\alpha f, |\partial_x|^\beta f \in L^2_{T_0}(H^{2-2\alpha}(\mathbb{R}^2)) \right\}.
\]

• Existence: The classical study of the existence of the solution necessitates a very small initial date, therefore we will write the initial condition as a sum of higher and lower frequencies.

For that, for any \(\varepsilon > 0\), their exist \(N \in \mathbb{N}\) such that \(\theta_0 = a^0 + b^0\), where
\[
\begin{cases}
* a^0 := \mathcal{J}_N(\theta^0) \in \bigcap_{r \in \mathbb{R}} H^r(\mathbb{R}^2), \\
* b^0 := \theta^0 - a^0, \\
* \|b^0\|_{H^{2-2\alpha}} < \varepsilon.
\end{cases} \tag{3.1}
\]
According to Theorem 1.1, we get that the following system
\[
\begin{cases}
\partial_t a + |\partial_t|^{2\alpha} a + |\partial_x|^{2\beta} a + u_x.\nabla a = 0, \\
a(0) = a^0 \in H^5(\mathbb{R}^2),
\end{cases}
\]
admit a unique solution \(a \in C_{T_0}(H^5(\mathbb{R}^2)) \subset C_{T_0}(H^{2-2\alpha}(\mathbb{R}^2))\), where \(T_0 = {\frac{C_N}{\|\theta^0\|_{H^{2-\alpha}}^4}} > 0\), moreover,
\[
\|a\|_{L^\infty_{T_0}(H^5)} \leq M_N.
\]
To show the existence of solution of system (AQG), it suffices to show that the following system
\[
\begin{cases}
\partial_t b + |\partial_t|^{2\alpha} b + |\partial_x|^{2\beta} b + u_x.\nabla b + u_y.\nabla a + u_b.\nabla b = 0, & (x, t) \in \mathbb{R}^2 \times [0, T_0], \\
b(x, 0) = b^0(x),
\end{cases} \tag{AQG}'
\]
adopts a solution in \(\mathcal{X}_{T_0}\), where \(T_0 > 0\) is a variable that will be decided later.

In order to do this, we may for instance make use of the Friedrichs method. The first step is to consider the following approximate system of (AQG)',
\[
\begin{cases}
\partial_t b + |\partial_t|^{2\alpha} \mathcal{J}_n b + |\partial_x|^{2\beta} \mathcal{J}_n b + \mathcal{J}_n (u_b.\nabla \mathcal{J}_n b) + \mathcal{J}_n (u_x u_b.\nabla a) + \mathcal{J}_n (u_y u_b.\nabla \mathcal{J}_n b) = 0, \\
b(x, 0) = \mathcal{J}_n b^0(x).
\end{cases} \tag{AQG}_n'
\]
Using the Cauchy-Lipschitz theorem, for any fixed $n \in \mathbb{N}$, there exist $T_n = T(n, \|b^0\|_{H^{2-2\alpha}}) \in (0, T^0)$, such that, the system $(AQG)_n$ admit a unique local solution $b_n$ on $[0, T_n]$. Moreover, $\mathcal{J}_n b_n$ is also a solution of $(AQG)_n$ with the same initial data, the fact that $\mathcal{J}_n^2 = \mathcal{J}_n$. According to the uniqueness, we have

$$\mathcal{J}_n b_n = b_n$$

Consequently, for any $n \in \mathbb{N}$, $b_n$ is solution of the following equation

$$\partial_t b_n + |\partial_1|^{2\alpha} b_n + |\partial_2|^{2\beta} b_n + \mathcal{J}_n (u_n \nabla v_n) + \mathcal{J}_n (u_n \nabla a) + \mathcal{J}_n (u_b \nabla b_n) = 0 \quad (3.2)$$

Taking the inner product of (3.2) with $b_n$ we get

$$\frac{1}{2} \frac{d}{dt} \|b_n\|_{L^2}^2 + \|\partial_1|^{\alpha} b_n\|_{L^2}^2 + \|\partial_2|^{\beta} b_n\|_{L^2}^2 \leq \|\mathcal{J}_n (u_b \nabla a), b_n\|_{L^2} \leq \|u_b \nabla a\|_{L^2} \|b_n\|_{L^2} \leq M_N \|b_n\|_{L^2}^2 \quad (3.3)$$

Gronwall’s Lemma implies that

$$\|b_n(t)\|_{L^2}^2 + \int_0^t \|\partial_1|^{\alpha} b_n(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\partial_2|^{\beta} b_n(\tau)\|_{L^2}^2 d\tau \leq \|b^0\|_{L^2}^2 + 2M_N \int_0^t \|b_n(\tau)\|_{L^2}^2 d\tau$$

Now, we get

$$\mathcal{J}_n (u_b \nabla a), \nabla|b_n| \leq \mathcal{J}_n (u_b \nabla a), \nabla|b_n| = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$$

where

$$\mathcal{H}_1 = - \|\nabla|^{2-2\alpha}(u_b \nabla a), \nabla|^{\alpha} b_n\|_{L^2}$$

$$\mathcal{H}_2 = - \|\nabla|^{2-2\alpha}(u_b \nabla a), \nabla|^{\alpha} b_n\|_{L^2}$$

$$\mathcal{H}_3 = - \|\nabla|^{2-2\alpha}(u_b \nabla a), \nabla|^{\alpha} b_n\|_{L^2}$$

In what follows, we shall estimate the terms at the right hand side of (3.4) one by one. To estimate the first term, we use $\text{div} u_n = 0$ and the first inequality in Lemma 2.7 to conclude

$$|\mathcal{H}_1| = \|\nabla|^{2-2\alpha}(u_b \nabla b_n), \nabla|^{2-2\alpha} b_n\|_{L^2} \leq C \|\nabla|^{2-2\alpha}(u_b \nabla b_n), \nabla|^{2-2\alpha} b_n\|_{L^2} \leq C \|\nabla|^{2-2\alpha}(u_b \nabla b_n), \nabla|^{2-2\alpha} b_n\|_{L^2} \leq C \|\nabla|^{\alpha} b_n\|_{L^2} \|b_n\|_{H^{2-2\alpha}}$$

$$\leq C \|\nabla|^{2-2\alpha}(u_b \nabla b_n), \nabla|^{\alpha} b_n\|_{L^2} \leq C \|\nabla|^{2-2\alpha}(u_b \nabla b_n), \nabla|^{\alpha} b_n\|_{L^2} \leq C \|\nabla|^{2-2\alpha}(u_b \nabla b_n), \nabla|^{\alpha} b_n\|_{L^2} \leq C \|\nabla|^{\alpha} b_n\|_{H^{2-2\alpha}}"
The fact that \( \alpha \leq \beta \), then by Lemma 2.5, we have

\[
|||\nabla^\alpha b_n|||_{H^2-2\alpha} \leq \|b_n\|_{L^2} + |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha} + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha} \\
\leq \|b^0\|_{L^2e^{MN T^0}} + |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha} + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha},
\]

which implies

\[
|H_1| \leq C_N \left(1 + |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha} + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}\right) \|b_n\|_{H^2-2\alpha} \\
\leq \frac{1}{2} |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + \frac{1}{2} |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2 + C_N (1 + \|b_n\|_{H^2-2\alpha}^2).
\]

(3.5)

Move now to estimate the term \( H_2 \), the fact that \( 2 - 2\alpha > 1 \) we have

\[
|H_2| \leq \|u_{b_n} \nabla u\|_{H^2-2\alpha} \|b_n\|_{H^2-2\alpha} \\
\leq \|u_{b_n}\|_{H^2-2\alpha} \|u\|_{H^2-2\alpha} \|b_n\|_{H^2-2\alpha} \\
\leq C (1 + \|b_n\|_{H^2-2\alpha}^2).
\]

(3.6)

Finally, following the estimate of \( H_1 \), with \( \text{div} u_{b_n} = 0 \), we get

\[
|H_3| = ||(\nabla|^{2-2\alpha}(u_{b_n} \nabla b_n) - u_{b_n} \nabla|^{2-2\alpha} b_n, |\nabla|^{2-2\alpha} b_n)_{L^2}| \\
\leq C \||\nabla|^{\alpha} b_n|||_{H^2-2\alpha}^2 \|b_n\|_{H^2-2\alpha} \\
\leq C \left(\|b^0\|_{L^2e^{2MN T^0}}^2 + |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2\right) \|b_n\|_{H^2-2\alpha} \\
\leq C \left(|||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2 \right) \|b_n\|_{H^2-2\alpha} + C_N (1 + \|b_n\|_{H^2-2\alpha}^2)
\]

(3.7)

Collecting the estimates (3.4), (3.5), (3.6) and (3.7), we obtain

\[
\frac{d}{dt} |||b_n|||_{H^2-2\alpha}^2 + |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2 \leq C_N (1 + \|b_n\|_{H^2-2\alpha}^2) \\
+ C \|b_n\|_{H^2-2\alpha} \left(|||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2\right).
\]

Let \( T'_n = \sup \left\{T \in (0,T^0); \|b_n\|_{L^\infty(H^2-2\alpha)} \leq 2\varepsilon \right\} \). Since the function \( (t \mapsto \|b_n\|_{H^2-2\alpha}) \) is continuous in \([0,T^0]\), that implies the existence of \( T'_n \), moreover we have for any \( t \in [0,T'_n) \)

\[
\frac{d}{dt} |||b_n|||_{H^2-2\alpha} + |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2 \leq C_N (1 + 4\varepsilon^2) + 2C\varepsilon \left(|||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2\right).
\]

We can choose \( \varepsilon < \frac{1}{4C} \), so we get:

\[
2C\varepsilon \left(|||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2\right) \leq \frac{1}{2} |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + \frac{1}{2} |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2.
\]

Therefore

\[
\frac{d}{dt} |||b_n|||_{H^2-2\alpha} + \frac{1}{2} |||\partial_1|^{\alpha} b_n|||_{H^2-2\alpha}^2 + \frac{1}{2} |||\partial_2|^{\beta} b_n|||_{H^2-2\alpha}^2 \leq C_N (1 + 4\varepsilon^2).
\]

We integrate on \([0,t]\), for any \( t \in (0,T'_n) \), we get

\[
\|b_n(t)|||_{H^2-2\alpha}^2 + \int_0^t |||\partial_1|^{\alpha} b_n(\tau)|||_{H^2-2\alpha}^2 d\tau + \int_0^t |||\partial_2|^{\beta} b_n(\tau)|||_{H^2-2\alpha}^2 d\tau \leq \|b^0|||_{H^2-2\alpha}^2 + C_N t (1 + 4\varepsilon^2).
\]

Taking

\[
T_0 = \frac{\varepsilon^2}{2C_N (1 + 4\varepsilon^2)}.
\]
We can show that $T_0 < T_n$, for any $n \in \mathbb{N}$, moreover, for any $t \in [0, T_0]$

$$\|b_n(t)\|^2_{H^{2-2\alpha}} + \int_0^t \|\partial_1^{\alpha} b_n(\tau)\|^2_{H^{2-2\alpha}} d\tau + \int_0^t \|\partial_2^{\alpha} b_n(\tau)\|^2_{H^{2-2\alpha}} d\tau \leq 2\varepsilon. \quad (3.8)$$

So we can conclude that $(b_n)_{n \in \mathbb{N}}$ is uniformly bounded in $C_{T_0}(H^{2-2\alpha}(\mathbb{R}^2))$ and

$$\{\partial_1^{\alpha} b_n\}, \{\partial_2^{\alpha} b_n\} \subset L^2_{T_0}(H^{2-2\alpha}(\mathbb{R}^2)).$$

Thanks to these bounds, there remains the passage to the limit of this suite of solutions. This passage to the limit due to the classical argument by combining Ascoli’s theorem and Cantor’s diagonal process. Finally we get

that the system (AQQ) admits a solution $b \in X_{T_0}$.

Therefore $\theta = a + b$ is solution of system (AQQ) in $X_{T_0}$.

- **Uniqueness:** It remains to show that this solution is unique in the space $X_{T_0}$.

For that let $\theta_1$ and $\theta_2$ be two solutions of $(AQQ)$, $u_{g_1} = \mathcal{R}^{-1} \theta_1$ and $u_{g_2} = \mathcal{R}^{-1} \theta_2$. We assume $\omega = \theta_1 - \theta_2$, then $\omega$ is solution of the following equation

$$\partial_t \omega + |\partial_1^{2\alpha} \omega + \partial_2^{3\alpha} \omega + u_1 \nabla \omega + u_2 \nabla \theta^2 = 0, \quad (3.9)$$

where $u_\omega = \mathcal{R}^{-1} \omega = u_{g_1} - u_{g_2}$. Taken the inner product of (3.9) with $\omega$, we get

$$\frac{1}{2} \frac{d}{dt} \|\omega(t)\|^2_{L^2} + \|\partial_1^{\alpha} \omega\|^2_{L^2} + \|\partial_2^{\alpha} \omega\|^2_{L^2} \leq \|v \nabla \theta^2\|_{L^2} \|\omega\|_{L^2}.$$

Using Lemma 2.3 for $s_1 = \alpha < 1$, $s_2 = 1 - \alpha < 1$, we get

$$\|v \nabla \theta^2\|_{L^2} \leq C\|\nabla \omega\|_{L^2} \|\nabla \theta^2\|_{H^{2-2\alpha}},$$

and by Lemma 2.5, we have

$$\|v \nabla \theta^2\|_{L^2} \leq C (\|\omega\|_{L^2} + \|\partial_1^{\alpha} \omega\|_{L^2} + \|\partial_2^{\alpha} \omega\|_{L^2}) \|\nabla \omega\|_{H^{2-2\alpha}} \|\omega\|_{L^2} \leq \|\partial_1^{\alpha} \omega\|_{L^2} + \|\partial_2^{\alpha} \omega\|_{L^2} + C (1 + \|\nabla \theta^2\|_{H^{2-2\alpha}}^2) \|\omega\|_{L^2}.$$

Therefore

$$\frac{d}{dt} \|\omega(t)\|^2_{L^2} \leq C (1 + \|\nabla \theta^2\|_{H^{2-2\alpha}}^2) \|\omega\|^2_{L^2}.$$

Integrate on $[0, t]$, $t \in (0, T_0]$ we get

$$\|\omega(t)\|^2_{L^2} \leq C \int_0^t (1 + \|\nabla \theta^2(\tau)\|_{H^{2-2\alpha}}^2) \|\omega\|^2_{L^2} d\tau$$

Using Gronwall’s Lemma and the fact $\left(t \mapsto 1 + \|\nabla \theta^2(\tau)\|_{H^{2-2\alpha}}^2\right) \in L^1([0, T_0])$ we can deduce that $\omega = 0$ in $[0, T_0]$ which gives the uniqueness.
3.2. Continuity. Finally, we will show that the solution is continuous in $H^{2-2\alpha}(\mathbb{R}^2)$.

First of all, we have $\theta \in C^t_t (H^s(\mathbb{R}^2))$, $s < 2 - 2\alpha$. Therefore, if $(s_k)_{k \in \mathbb{N}}$ a positive real sequence such that $1 < s_k < s_{k+1} < 2 - 2\alpha$ and $s_k \to 2 - 2\alpha$,

then, we have

$$\frac{d}{dt} \| \theta(t) \|^2_{H^{s_k}} + 2 \| \partial_1 |^{s_k} \theta \|^2_{H^{s_k}} + 2 \| \partial_2 |^{s_k} \theta \|^2_{H^{s_k}} + 2 (|\nabla|^{s_k} (u_\theta, \nabla \theta), |\nabla|^{s_k} \theta)_{L^2} = 0.$$  \hspace{1cm} (3.10)

Now, let $0 \leq t < t' \leq T_0$, we integer (3.10) in $[t, t']$, we get

$$\| \theta(t') \|^2_{H^{s_k}} \leq \| \theta(t) \|^2_{H^{s_k}} + 2 \int_t^{t'} \left( \| \partial_1 |^{s_k} \theta \|^2_{H^{s_k}} d\tau + \| \partial_2 |^{s_k} \theta \|^2_{H^{s_k}} \right) d\tau + 2 \int_t^{t'} (|\nabla|^{s_k} (u_\theta, \nabla \theta), |\nabla|^{s_k} \theta)_{L^2} d\tau, \hspace{1cm} (3.11)$$

and

$$\| \theta(t') \|^2_{H^{s_k}} \leq \| \theta(t) \|^2_{H^{s_k}} + 2 \int_t^{t'} \left( \| \partial_1 |^{s_k} \theta \|^2_{H^{s_k}} d\tau + \| \partial_2 |^{s_k} \theta \|^2_{H^{s_k}} \right) d\tau + 2 \int_t^{t'} (|\nabla|^{s_k} (u_\theta, \nabla \theta), |\nabla|^{s_k} \theta)_{L^2} d\tau. \hspace{1cm} (3.12)$$

Using Lemma 2.7 and Lemma 2.5 and the fact that $u_\theta$ has divergence free and $1 < s_k < 2 - 2\alpha$, we get

$$\int_t^{t'} (|\nabla|^{s_k} (u_\theta, \nabla \theta), |\nabla|^{s_k} \theta)_{L^2} d\tau \leq \int_t^{t'} \| |\nabla|^{s_k} (u_\theta, \nabla \theta) - u_\theta, |\nabla|^{s_k} \nabla \theta\|_{L^2} \| |\nabla|^{s_k} \theta\|_{L^2} d\tau$$

$$\leq s_k 2^{s_k} C \int_t^{t'} \| |\nabla|^{s_k + \alpha} \theta \|_{L^2} \| |\nabla|^{2-\alpha} \theta \|_{L^2} \| \theta \|^2_{H^{2-2\alpha}} d\tau$$

$$\leq C(\alpha) \| \theta \|^2_{L^\infty_t (H^{2-2\alpha})} \int_t^{t'} \| |\nabla|^{s_k} \theta \|^2_{H^{2-2\alpha}} d\tau$$

$$\leq C(\alpha) \int_t^{t'} (\| \partial_1 |^{s_k} \theta \|^2_{H^{2-2\alpha}} + \| \partial_2 |^{s_k} \theta \|^2_{H^{2-2\alpha}}) d\tau + C(\alpha) (t' - t). \hspace{1cm} (3.13)$$

Then, we collect (3.11) and (3.12) with the inequality (3.13) to get

$$\| \theta(t') \|^2_{H^{s_k}} \leq \| \theta(t) \|^2_{H^{s_k}} + C(\alpha) \int_t^{t'} \left( \| \partial_1 |^{s_k} \theta \|^2_{H^{2-2\alpha}} d\tau + \| \partial_2 |^{s_k} \theta \|^2_{H^{2-2\alpha}} \right) d\tau + C(\alpha) (t' - t), \hspace{1cm} (3.14)$$

and

$$\| \theta(t') \|^2_{H^{s_k}} \leq \| \theta(t) \|^2_{H^{s_k}} + C(\alpha) \int_t^{t'} \left( \| \partial_1 |^{s_k} \theta \|^2_{H^{2-2\alpha}} d\tau + \| \partial_2 |^{s_k} \theta \|^2_{H^{2-2\alpha}} \right) d\tau + C(\alpha) (t' - t). \hspace{1cm} (3.15)$$

We have $\lim_{k \to +\infty} \left( 1 + |\xi|^{2s_k} \right) \widehat{\theta}(\xi, t)^2 = \left( 1 + |\xi|^{2(2-2\alpha)} \right) |\widehat{\theta}(\xi, t)|^2$, moreover

$$\| \theta(t) \|^2_{H^{s_k}} = \int_{|\xi| \leq 1} \left( 1 + |\xi|^{2s_k} \right) |\widehat{\theta}(\xi, t)|^2 d\xi + \int_{|\xi| > 1} \left( 1 + |\xi|^{2s_k} \right) |\widehat{\theta}(\xi, t)|^2 d\xi.$$ 

The fact that

$$\int_{|\xi| \leq 1} \left( 1 + |\xi|^{2s_k} \right) |\widehat{\theta}(\xi, t)|^2 d\xi \leq 2 \| \theta \|^2_{L^2},$$

and by Lebesgue’s dominated convergence theorem, we get

$$\lim_{k \to +\infty} \int_{|\xi| \leq 1} \left( 1 + |\xi|^{2s_k} \right) |\widehat{\theta}(\xi, t)|^2 d\xi = \int_{|\xi| \leq 1} \left( 1 + |\xi|^{2(2-2\alpha)} \right) |\widehat{\theta}(\xi, t)|^2 d\xi.$$ 

Moreover, we have

$$\left( 1 + |\xi|^{2s_k} \right) |\widehat{\theta}(\xi, t)|^2 \leq \left( 1 + |\xi|^{2s_{k+1}} \right) |\widehat{\theta}(\xi, t)|^2, \hspace{1cm} \forall |\xi| > 1.$$
therefore, by the Monotonic Convergence Theorem, we obtain
\[ \lim_{k \to +\infty} \int_{|\xi| > 1} (1 + |\xi|^{2\alpha}) \tilde{\theta}(\xi, t)^2 d\xi = \int_{|\xi| > 1} (1 + |\xi|^{2(2-\alpha)}) \tilde{\theta}(\xi, t)^2 d\xi. \] (3.16)
Finally, we have for any \( t \in [0, T_0] \), \( \lim_{k \to +\infty} \|\theta(t)\|_{H^{2-\alpha}} = \|\theta(t)\|_{H^{2-\alpha}} \), which implies
\[ \|\theta(t)\|_{H^{2-\alpha}}^2 \leq \|\theta(t')\|_{H^{2-\alpha}}^2 + C(\alpha) \int_{t}^{t'} \left( \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 d\tau + \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 d\tau \right) + C(\alpha)(t' - t), \] (3.17)
and
\[ \|\theta(t')\|_{H^{2-\alpha}}^2 \leq \|\theta(t)\|_{H^{2-\alpha}}^2 + C(\alpha) \int_{t}^{t'} \left( \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 d\tau + \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 d\tau \right) + C(\alpha)(t' - t). \] (3.18)
After that, we pass \( \limsup \) and \( \liminf \) for the last equations, then, we get
\[ \lim_{t \to t'} \sup_{t' \to t} \|\theta(t')\|_{H^{2-\alpha}} \leq \|\theta(t)\|_{H^{2-\alpha}} \quad \text{and} \quad \lim_{t \to t'} \inf_{t' \to t} \|\theta(t')\|_{H^{2-\alpha}} \leq \|\theta(t)\|_{H^{2-\alpha}}, \] (3.19)
moreover \( \theta(t) \to \theta(t') \) if \( t \to t' \) and \( \theta(t) \to \theta(t') \) if \( t' \to t \). Therefore by Lemma 2.1 we have
\[ \lim_{t \to t'} \|\theta(t) - \theta(t')\|_{H^{2-\alpha}} = 0 \quad \text{and} \quad \lim_{t' \to t} \|\theta(t') - \theta(t)\|_{H^{2-\alpha}} = 0, \]
which implies that \((t \to \theta(t))\) is continue in \([0, T_0]\) in \( H^{2-\alpha}(\mathbb{R}^2) \).

So we can conclude that
\[ \theta \in C([0, T_0], H^{2-\alpha}(\mathbb{R}^2)). \]

3.3. Global solution. In this section, we prove if \( \|\theta^0\|_{H^{2-\alpha}} < c \), we get a global solution in \( C(\mathbb{R}^+, H^{2-\alpha}(\mathbb{R}^2)) \) satisfying (1.2).

Let \( \theta \in C([0, T^*), H^{2-\alpha}(\mathbb{R}^2)) \) be a maximal solution of the system (AQG), using Lemma 2.7 and the fact \( \text{div}\, \theta = 0 \), we infer that
\[ \frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{2-\alpha}}^2 + \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 + \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 \leq \left( \|\nabla\|_{2-\alpha}^2 (u_\theta, \nabla \theta), |\nabla |_{2-\alpha}^2 \theta \right)_{L^2} \leq C \|\nabla\|_{2-\alpha}^2 \theta \|_{L^2} \|\nabla |_{2-\alpha}^2 \theta \|_{L^2}, \] (3.20)
By interpolation, we get
\[ \|\nabla |_{2-\alpha}^2 \theta \|_{L^2} \leq C \left( \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 + \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 \right), \]
which implies
\[ \frac{d}{dt} \|\theta(t)\|_{H^{2-\alpha}}^2 + 2 \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 + 2 \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 \leq C \left( \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 + \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 \right) \|\theta\|_{H^{2-\alpha}} \] (3.21)
Let \( T_1 = \sup \{ T \in (0, T^*) : \|\theta\|_{L^p(H^{2-\alpha})} \leq 2 \|\theta^0\|_{H^{2-\alpha}} \} \), then for any \( t \in [0, T_1] \), we have
\[ \frac{d}{dt} \|\theta(t)\|_{H^{2-\alpha}}^2 + 2 \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 + 2 \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 \leq 2C \left( \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 + \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 \right) \|\theta^0\|_{H^{2-\alpha}}. \] (3.22)
Taking \( c = \frac{1}{2C} \), then, if we have \( \|\theta^0\|_{H^{2-\alpha}} < c \), so for any \( t \in [0, T_1) \)
\[ \frac{d}{dt} \|\theta(t)\|_{H^{2-\alpha}}^2 + \|\partial_1^\alpha \theta\|_{H^{2-\alpha}}^2 + \|\partial_2^\alpha \theta\|_{H^{2-\alpha}}^2 \leq 0, \] (3.23)
which implies
\[ \|\theta(t)\|_{H^{2,\infty}}^2 + \int_0^t \|\partial_1^0 \theta(t)\|_{H^{2,\infty}}^2 \, dt + \int_0^t \|\partial_1^1 \theta(t)\|_{H^{2,\infty}}^2 \, dt \leq \|\theta_0\|_{H^{2,\infty}}^2. \]

Finally, we get \( T = T^* \). Hence \( T^* = +\infty \), which complete the proof.

4. Proof of Theorem 1.3

The proof is done in two steps. In the first step, we prove
\[ \lim_{t \to +\infty} \|\theta(t)\|_{H^{2,\infty}} = 0, \] (4.1)
where we use
\[ \|\theta(t)\|_{H^{2,\infty}} + \int_0^t \|\partial_1^0 \theta(t)\|_{H^{2,\infty}}^2 \, dt + \int_0^t \|\partial_1^1 \theta(t)\|_{H^{2,\infty}}^2 \, dt \leq \|\theta_0\|_{H^{2,\infty}}^2, \quad \forall t \geq 0. \]

In the second step, we prove
\[ \lim_{t \to +\infty} \|\theta(t)\|_{L^2} = 0. \] (4.2)

**First step:** By interpolation theorem we have
\[ \|\theta(t)\|_{H^{2,\infty}} \leq C \left( \|\partial_1^0 \theta(t)\|_{H^{2,\infty}}^2 + \|\partial_2^1 \theta(t)\|_{H^{2,\infty}}^2 \right). \]

We integrate over \( \mathbb{R}^+ \) we get
\[ \int_0^{+\infty} \|\theta(t)\|_{H^{2,\infty}}^2 \, dt \leq C \int_0^{+\infty} \|\partial_1^0 \theta(t)\|_{H^{2,\infty}}^2 \, dt + C \int_0^{+\infty} \|\partial_2^1 \theta(t)\|_{H^{2,\infty}}^2 \, dt \leq C \|\theta_0\|_{H^{2,\infty}}^2. \] (4.3)

Let \( \varepsilon > 0 \) and \( E(\varepsilon) = \{ t \geq 0; \|\theta(t)\|_{H^{2,\infty}} > \varepsilon \} \), then
\[ \varepsilon^2 \lambda(E) = \int_E \|\theta(t)\|_{H^{2,\infty}}^2 \, dt \leq \int_0^{+\infty} \|\theta(t)\|_{H^{2,\infty}}^2 \, dt \leq C \|\theta_0\|_{H^{2,\infty}}^2. \] (4.4)

where \( \lambda(E) \) is the Lebesgue measure of \( E \), then \( \lambda(E) < +\infty \). So for \( r > 0 \), there exists \( t_0 \in [0, \lambda(E) + r] \) such that \( t_0 \notin E \) and
\[ \|\theta(t_0)\|_{H^{2,\infty}} \leq \varepsilon. \]

Thus, \( \lim_{t \to +\infty} \|\theta(t)\|_{H^{2,\infty}} = 0. \)

**Second step:** Let \( \delta \) a strictly positive real number strictly less than 1, we define the operators \( A_\delta(D) \) and \( B_\delta(D) \), respectively, by the following:
\[ A_\delta(D)f = F^{-1} \left( \xi \mapsto \chi_{\mathcal{P}(0,\delta)}(\xi) F(f)(\xi) \right), \]
\[ B_\delta(D)f = F^{-1} \left( \xi \mapsto (1 - \chi_{\mathcal{P}(0,\delta)})(\xi) F(f)(\xi) \right), \] (4.5)

where
\[ \mathcal{P}(0,\delta) = \{ \xi \in \mathbb{R}^2 : A(\xi) \leq \delta \}. \]

We define \( w_\delta = A_\delta(D)\theta \) and \( v_\delta = B_\delta(D)\theta \); \( F(\theta) = F(w_\delta) + F(v_\delta) \). Then,
\[ \partial_t w_\delta + |\partial_1|^2 w_\delta + |\partial_2|^2 v_\delta + A_\delta(D)(u_\theta, \nabla \theta) = 0 \] (4.6)
and
\[ \partial_t v_\delta + |\partial_1|^2 v_\delta + |\partial_2|^2 v_\delta + B_\delta(D)(u_\theta, \nabla \theta) = 0 \] (4.7)
Taking the scalar product of (4.6) equation with \( w_\delta \), we get
\[
\frac{1}{2} \frac{d}{dt} \| w_\delta(t) \|_{L^2}^2 \leq \| (A_\delta(D)(u_\theta, \nabla \theta), w_\delta)_{L^2} \|
\]
\[
\leq \int_{A(\xi) \leq \delta} |\xi| |u_\theta \otimes \theta(\xi)| |\tilde{w}_\delta(\xi)| d\xi
\]
\[
\leq \int_{A(\xi) \leq \delta} |\xi|^{2-2\beta} |\xi|^{2\beta-1} |u_\theta \otimes \theta(\xi)| |\tilde{w}_\delta(\xi)| d\xi
\]
\[
\leq C(\alpha, \beta) \int_{A(\xi) \leq \delta} \left( A(\xi) \frac{dR}{dt} + A(\xi) \frac{d\alpha}{dt} \right)^{2-2\beta} |\xi|^{2\beta-1} |u_\theta \otimes \theta(\xi)| |\tilde{w}_\delta(\xi)| d\xi
\]
\[
\leq C(\alpha, \beta) \delta^{1-\beta} \int_{\mathbb{R}^2} |\xi|^{2\beta-1} |u_\theta \otimes \theta(\xi)| |\tilde{w}_\delta(\xi)| d\xi
\]
\[
\leq C(\alpha, \beta) \delta^{1-\beta} \| u_\theta \otimes \theta \|_{H^{2\beta-1}} \| w_\delta \|_{L^2}
\]
\[
\leq C(\alpha, \beta) \delta^{1-\beta} \| \nabla^B \theta \|_{L^2} \| w_\delta \|_{L^2}.
\]

By interpolation Theorem, \( \alpha \leq \beta < 2 - \alpha \) we have
\[
\| \nabla^B \theta \|_{L^2} \leq \| \partial_1^B \theta \|_{L^2} + \| \partial_2^B \theta \|_{L^2}
\]
\[
\leq \| \partial_1^B \theta \|_{L^2} + \| \partial_2^B \theta \|_{H^{2\beta-2\alpha}} + \| \partial_2^B \theta \|_{L^2}
\]

Using \( \| w_\delta \|_{L^2} \leq \| \theta \|_{L^2} \leq \| \theta^0 \|_{L^2} \), and integer in \([0, t]\) we get
\[
\| w_\delta(t) \|_{L^2}^2 \leq \| w_\delta^0 \|_{L^2}^2 + C(\alpha, \beta) \delta^{1-\beta} \| \theta^0 \|_{L^2} \int_0^t \left( \| \partial_1^B \theta \|_{H^{2\beta-2\alpha}} + \| \partial_2^B \theta \|_{L^2} \right)^2 d\tau
\]
\[
\leq \| w_\delta^0 \|_{L^2}^2 + C(\alpha, \beta) \delta^{1-\beta} \| \theta^0 \|_{L^2}^2 \int_0^t \delta \to 0^+ \to 0.
\]

Therefore
\[
\lim_{\delta \to 0^+} \| w_\delta(t) \|_{L^2(\mathbb{R}^+, L^2)} = 0.
\]

So for any \( \varepsilon > 0 \), there exists \( \delta_0 > 0 \) such that
\[
\| w_{\delta_0}(t) \|_{L^2(\mathbb{R}^+, L^2)} < \frac{\varepsilon}{2}.
\]

(4.8)

On the other hand, we deduce that
\[
\| v_\delta \|_{L^2}^2 \leq \int_{A(\xi) > \delta} \frac{A(\xi)}{A(\xi)} |\tilde{\theta}(\xi)|^2 d\xi
\]
\[
\leq \frac{1}{\delta} \int_{\mathbb{R}^2} |\xi|^{2\beta} d\xi
\]
\[
\leq \frac{1}{\delta} (\| \partial_1^B \theta \|^2 + \| \partial_2^B \theta \|^2),
\]

which implies
\[
\int_0^{+\infty} \| v_\delta \|_{L^2}^2 d\tau \leq \frac{1}{\delta} \int_0^{+\infty} (\| \partial_1^B \theta \|^2 + \| \partial_2^B \theta \|^2) d\tau \leq \frac{1}{\delta} \| \theta^0 \|^2.
\]
Let $S_\varepsilon(\delta_0) = \{ t \geq 0; \| v_{\delta_0} \|_{L^2} > \frac{\varepsilon}{2} \}$, then

$$
\left( \frac{\varepsilon}{2} \right)^2 \lambda(S_\varepsilon(\delta_0)) \leq \int_{S_\varepsilon(\delta_0)} \| v_{\delta_0} \|_{L^2}^2 dt \\
\leq \int_0^t \| v_{\delta_0} \|_{L^2}^2 dt \\
\leq \frac{1}{\delta_0} \| \theta_0 \|_{L^2}^2,
$$

We pose

$$T_\varepsilon = \left( \frac{2}{\varepsilon} \right)^2 \frac{1}{\delta_0} \| \theta_0 \|_{L^2}^2 < +\infty,$$

then $\lambda(S_\varepsilon(\delta_0)) \leq T_\varepsilon$. So there exists $t_2 \in [0, T_\varepsilon + 1] \setminus S_\varepsilon(\delta_0)$ such

$$\| v_{\delta_0}(t_2) \|_{L^2} \leq \frac{\varepsilon}{2}. \quad (4.9)$$

By the equation (4.8) and (4.9), we get

$$\| \theta(t_2) \|_{L^2} \leq \varepsilon.$$

Thus, $\lim_{t \to +\infty} \| \theta(t) \|_{L^2} = 0$, and this finishes the proof in this case.

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