Julia set describes quantum tunneling in the presence of chaos

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Abstract.
We find that characteristics of quantum tunneling in the presence of chaos can be regarded as a manifestation of the Julia set of the complex dynamical system. Several numerical evidences for the standard map together with a rigorous statement for the Hénon map are presented demonstrating that the complex classical paths which contribute to the semiclassical propagator are dense in the Julia set. Chaotic tunneling can thus be characterized by the transitivity of the dynamics and high density of the trajectories on the Julia set.

Recent studies on tunneling in multidimensions have revealed that the existence of chaos affects the signature of quantum tunneling. The observation of purely quantum mechanical calculation in chaotic systems shows that tunneling can become chaotic or chaos seems to assist tunneling [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The idea capturing such a novel aspect of tunneling looks very attractive, but a direct connection between chaos and tunneling can only be accomplished by interpreting the quantum phenomenon by the trajectory of the classical dynamics [6, 7, 8, 10, 11]. When one is particularly interested in the tunneling process, the use of complex trajectories is essential since the transition due to tunneling occurs where the real-valued classical trajectories cannot reach.

The most well-known technique using the complex space is so-called instanton method in which the tunneling penetration is evaluated mainly by a single classical path moving on the reversed potential [12, 13]. On the other hand, in chaotic systems, it has been found in the time-domain semiclassical analysis that a bunch of complex paths almost equally contribute to the tunneling transition between classically forbidden regions [10, 11]. They typically form a tree-like fractal structure in the complex initial value plane, and its outstanding appearance compels us to prepare some concept which controls dominating complex paths in the semiclassical sum of contributing candidates [10, 11]. All the characteristic structures appearing in the tunneling wavefunction in chaotic systems originate from it.
However, the *Laputa chain*, which was so introduced in [10, 11], has been still phenomenological so far, and remains even mysterious if no link to some concept compatible with the dynamical system theory is made. One may thus naturally ask why such a structure play a special role in the complex trajectory description of chaotic tunneling, and what sort of mechanism underlies such conspicuous objects in the complex plane. The purpose of this Letter is to provide a clear answer to these questions. Our final claim is simple and would be rather natural: *Julia set is the origin of chaotic tunneling.*

Let us begin with introducing the model system we are concerned with. The system we study here is a family of two-dimensional area preserving maps, in which the mixed phase space being realized in a certain range of the parameter space. The time evolution of the phase point \((p, \theta)\) is given as the mapping rule as

\[
(p_{n+1}, \theta_{n+1}) = F(p_n, \theta_n) \equiv \left( H'(p_n) - V'(\theta_n), \theta_n + H'(p_n) - V'(\theta_n) \right).
\]

(1)

Here, \(H(p) = p^2/2\) and \(V(\theta) = K \sin \theta\) are the most standard choice, but suitable modification or replacement of the kinetic or the potential term is sometimes helpful and will be made according to the target of the analysis.

Since the map model does not have the energy as the Hamiltonian flow problem does so, one cannot consider the tunneling through the energetic barrier, which may be a normal setting of the tunneling problem. Instead, dynamical confinement due to classically disconnected components such as KAM tori and chaotic components in the phase space plays the role of barriers, and the quantum transition between such invariant regions is regarded as tunneling [14].

At least in the first setting of the problem, it is not at all obvious that several different situations, such as the tunneling transition out of the quasiperiodic region into some chaotic component, or its reverse process, or that between different chaotic components, could be treated on the same footing. However, as will be described below and also become one of the most important point in the present report, the choice of initial and final states does not matter to the whole story.

Typical quantum mechanical wavefunctions in the mixed phase space are displayed in Fig. 1. In both models, the tails of the wavefunctions do not monotonically decay even in the tunneling regime, rather there appear several unexpected structures; the crossovers of the slope, the plateau regions and irregular interference patterns on it. All these characteristics are only qualitatively featured [10, 11], but they are commonly observed not only in the dynamical tunneling problem, but also in the energetic barrier tunneling [15].

The semiclassical approach, which is extensively developed in recent studies of quantum chaos or quantum chaology [16], works quite well even when one employs it as a tool describing the tunneling process. Apart from an added technical (but sometimes crucial) problem originating from the Stokes phenomenon [17], which we do not enter into details here, our task in the semiclassical analysis is essentially the same as the real
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one, that is, to evaluate the Van-Vleck propagator:

\[ \Psi_n(p_0, p_n) \approx \sum_{p_0=\alpha \atop p_n=\beta} A_n(p_0, \theta_0) \exp \left\{ -\frac{i}{\hbar} S_n(p_0, \theta_0) \right\}, \]  

(2)

where the summation is taken for all \((p_0, \theta_0)\) which satisfy the boundary conditions for the initial momentum \(p_0 = \alpha\) and the final momentum \(p_n = \beta\). Here, \(S_n(p_0, \theta_0) = \sum_{j=1}^n [H(\theta_j) - V(\theta_j) + \theta_j (p_j - p_{j+1})]\) is the action along a classical trajectory, and \(A_n(p_0, \theta_0) = \left[ 2\pi \hbar (\partial p_n / \partial \theta_0)_{p_0} \right]^{-\frac{1}{2}}\) represents the amplitude factor associated with its stability.

Figure 1. Quantum (bold line) and semiclassical wavefunction (dashed line) for (a) the model with \(H_0(p) = \frac{p^2}{2} \left( \frac{p}{p_d} \right)^6 + \omega p\) and \(V(\theta) = K \sin \theta\), where \(p_d = 5\), \(\omega = 2\) and \(K = 1.2\), and (b) the model with \(H_0(p) = \frac{p^2}{2}\) and \(V(\theta) = K \sin \theta\), where \(K = 1.5\). In both cases, the initial wavepacket is set as \(\Psi(p_0) = \delta(p)\). The real-valued classical orbits cannot reach the region outside the dashed lines within the time step taken here \((n = 6\) for (a) and \(n = 5\) for (b)). The semiclassical wavefunction is shifted in order to clarify the structure.

Since we here take the \(p\)-representation, \(p_0\) should be a real quantity. So, the canonical partner \(\theta_0\) may be used to identify the (complexified) trajectories contributing to the sum (2), and it is then allowed to be complex as \(\theta_0 = \xi + i\eta\) (\(\xi, \eta\) real). We visualize the contributing complex paths by displaying the set \([10, 11]\):

\[ \mathcal{M}_n \equiv \bigcup_{\beta \in \mathbb{R}} \mathcal{M}_n^{\alpha, \beta} = \bigcup_{\beta \in \mathbb{R}} \{ (p, \theta) \in \mathbb{C}^2 \mid p_n = \beta \} \]  

(3)

on the \(\theta_0\)-plane of the slice \(\{p_0 = \alpha\}\) for some initial condition \(\alpha \in \mathbb{R}\). The set \(\mathcal{M}_n\) on the \(\theta_0\)-plane, which usually looks like clouds or wisteria trellis on a macroscopic scale, is decomposed into finer and finer structures as it is magnified \([10, 11]\). One can see that its basic element is a string with various scales. Each string represents a trajectory appearing in the semiclassical sum (2). We note that in the integrable limit only the
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branches connected with the real plane (i.e., \( \eta = 0 \)) survive and all other complicated objects disappear \([10, 11]\).

A huge amount of candidate paths may discourage us since it appears to be no more possible to establish a simple view of tunneling in the presence of chaos. However, among all possible candidates the complex paths forming a sequential structure, which runs in the vertical direction at the center of Fig. 2(a) and clearly discernible from the other aggregated strings, exceed any other candidate paths in amplitude. We have called such a characteristic structure the \textit{Laputa chain} \([10, 11]\). As shown in Fig. 1, one finds that semiclassical sum including only such complex paths contained in the Laputa chains has reproduced almost all details of tunneling into chaotic regions. Our task is, therefore, reduced to clarifying what this marked structure appearing in the initial value plane represents.

\textbf{Figure 2.} (a) A magnification of the initial value representation \( \mathcal{M}_n \) on the \( \theta_0 \)-plane in case of the standard map with \( K = 1.5 \) and \( n = 40 \). The range shown above is given as \( 4.197487346 \times 10^{-2} \leq \xi \leq 4.197487362 \times 10^{-2} \), and \( 6.693592 \times 10^{-4} \leq \eta \leq 6.693601 \times 10^{-4} \). The initial momentum is set as \( p_0 = 0 \). \( \mathcal{M}_n \)-set consists of a bunch of self-similar objects, whose basic element is a string. Except for the central part, such strings are so densely aggregated that individual string cannot be resolved in this scale. However, magnifying the black area, one can find a similar structure in the scale shown here, that is, the black area is also composed of a bunch of string objects. Each string represents an individual component of the semiclassical sum (2). The strings running in the vertical direction look as if they cross with each other, but actually they avoid with very narrow gaps, that is, the strings form a serial chain-like structure connected via narrow gaps. (b) The slice of \( K^+ \) by \( \{ p_0 = \alpha \} \). This is numerically obtained by plotting the initial points whose trajectories remain a ball in \( \mathbb{C}^2 \) with a certain finite radius, \( r = 10^3 \), in this case.

The reason why some complex paths dominate the others in the semiclassical sum (4) is, in general, that the imaginary part of their action, \( \text{Im} \, S_n(p_0, \theta_0) \), are relatively
small. This is because the absolute value of each term in (2) is mainly governed by
$\text{Im} S_n(p_0, \theta_0)$, rather than the amplitude factor $A_n(p_0, \theta_0)$. This in turn means that
the complex paths forming the Laputa chain should gain small imaginary action as
compared to the other paths not forming the chain structure. Indeed, as shown later,
the trajectories initially placed on the Laputa chain approach the real $(p, \theta)$-plane
exponentially, which provide minimal or relatively small imaginary action.

Conversely, one can say that this property characterizes the Laputa chain and
makes them distinguishable from the others. Furthermore, they are specific in that
those trajectories stay in bounded regions because after coming close to the real plane
they almost follow the behavior of the trajectories on the real plane and the real orbits
are all bounded in the present situation.

This is a hint to link the Laputa chain to a proper object compatible with the
theory of dynamical systems, since the Julia set, which plays a central role in the
complex dynamical systems, is specified as the set satisfying such a property. More
precisely, the forward Julia set $J^+$ is defined as the boundary of the set $K^+$ of points
whose forward orbits remain in a finite region [18]:

$$K^+ = \{ (p, \theta) \mid \{ F^n(p, \theta) \}_{n=0} \text{is bounded} \}$$

and

$$J^+ = \partial K^+.$$

The polynomial diffeomorphism like Hénon map $f$, which is defined on $\mathbb{C}^2$, has
a polynomial inverse, so both the forward and the backward iterations can be considered.
In such a case we define $K^+$ (resp. $K^-$) as the set of points in $\mathbb{C}^2$ whose forward (resp. backward)
orbits are bounded, and $J^+$ (resp. $J^-$) to be the boundary of $K^+$ (resp. $K^-$)
which we call the forward (resp. backward) Julia set. The set $J \equiv J^+ \cap J^-$ is called the
Julia set of $f$. The forward (or backward) Julia set $K^\pm$ is where the orbits have sensitive
dependence on initial conditions, which means that the chaotic motion is realized on it.

Remarkably enough, such a purely mathematical object enters into physics as
quantum tunneling in chaos. Indeed, as shown in Fig. 2(b), the similarity between
the chain-shaped structure demonstrated in Fig. 2(a) and the slice of $K^+$ by the same
plane is obvious. The slice of $K^+$ shows a typical dendrite-like structure which often
appears in the one-dimensional complex dynamical systems [19]. The number of strings
constituting the Laputa chain increases in an exponential rate, so coincidence between
them becomes better as the time proceeds [11, 23]. Note that the orbits put on the
highly aggregated branches surrounding the Laputa chain do not stay in a finite phase
space domain but rapidly escape to infinity.

It is possible to provide a rigorous statement if one focuses on the cubic potential
model given by putting $H(p) = p^2/2$ and $V(\theta) = c\theta - \theta^3/3$. The map (I)
is transformed to a standard form of the Hénon map:

$$f : (x, y) \mapsto (y, y^2 + (1 - c) - x),$$

\[ (5) \]
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by the affine change of coordinate \((p, \theta) = (y - x, y - 1)\). The Hénon map is known to be one of the simplest nonlinear systems in the two-dimensional space, and its dynamics is extensively studied by several authors. Among them, investigation from the complex dynamical point of view has been developed in the last decade (see, for example, [20, 21, 22] and the references therein) by using the pluripotential theory, the theory of currents, etc.

As in the case of the standard map, it is reasonable to focus on the \(\text{Im} S_n(p_0, \theta_0)\) of each trajectory, but to be compatible with the invariant set of the dynamical system one should consider the set of trajectories having the property described above in the limit of \(n\) going to infinity. The most natural condition would be to select the complex or orbits whose \(\text{Im} S_n(p_0, \theta_0)\) has a finite limit even when \(n\) goes to infinity. Such a filtering only serves as a necessary condition for semiclassically contributing orbits, but it is at least true that the trajectories whose \(\text{Im} S_n(p_0, \theta_0)\) are divergent cannot contribute to the semiclassical summation since those orbits either tends to zero in their magnitude or will be removed by the Stokes phenomenon [17]. Therefore we define the \textit{Laputa chains} as,

\[
\mathcal{C}_{\text{Laputa}} \equiv \{(p, \theta) \in \mathcal{M}_\infty | \text{Im} S_n(p, \theta) \text{ converges absolutely at } (p, \theta)\} \quad (7)
\]

In this definition \(\mathcal{M}_\infty\) is an object introduced to represent the limit of \(\mathcal{M}_n\)-set when \(n\) goes to infinity. More precisely,

\[
\mathcal{M}_\infty \equiv \bigcup_{\beta \in \mathbb{R}} \mathcal{M}_\infty^\beta,
\]

where \(\mathcal{M}_\infty^\beta\) is given as the Hausdorff limit of \(\mathcal{M}_n^{*, \beta} \equiv \{(p, \theta) \in \mathbb{C}^2 | p_n = \beta\}\) (compare eq. (3)). Thus, the set \(\mathcal{M}_\infty\) corresponds to \(\mathcal{M}_n\) for the time step \('n = \infty'). It is possible to prove that this Hausdorff limit itself contains the forward Julia set \(J^+\) [23, 11], which in itself is a partial verification of our numerical observation. The following assertion concerning the relation between \(\mathcal{C}_{\text{Laputa}}\) thus defined and \(J^+\) is proved by the second-named author:

**Theorem.** Let \(F\) be the time-one map on \(\mathbb{C}^2\) associate to the kicked rotor [7] with \(H(p) = p^2/2\) and \(V(\theta) = c\theta - \theta^3/3\), and \(h_{\text{top}}(F)\) be the topological entropy of \(F\),

(i) If \(h_{\text{top}}(F|_{\mathbb{R}^2}) > 0\), then \(\mathcal{C}_{\text{Laputa}} \supset J^+\).

(ii) If \(F\) is hyperbolic on \(J\) and \(h_{\text{top}}(F|_{\mathbb{R}^2}) > 0\), then \(\overline{\mathcal{C}_{\text{Laputa}}} = J^+\).

(iii) If \(F\) is hyperbolic on \(J\) and \(h_{\text{top}}(F|_{\mathbb{R}^2}) = \log 2\), then \(\mathcal{C}_{\text{Laputa}} = J^+\).

Here \(\overline{X}\) indicates the closure of the set \(X\). The rough sketch of the proof is as follows [23, 11]: that \(h_{\text{top}}(F|_{\mathbb{R}^2}) > 0\) implies the existence of a saddle periodic point \(X\) in the real phase space. A principal tool we will employ is the following result which is established by Bedford and Smillie [20, 21, 22]: For a complex one-dimensional locally closed submanifold \(M\) in either \(J^+\) or an algebraic variety, there is a constant \(c > 0\) so that

\[
\lim_{n \to +\infty} \frac{1}{2^n} [f^{-n} M] = c \cdot dd^c G^+ \quad (9)
\]
in the sense of current, where \([M]\) is the current of integration of \(M\), i.e. \([M](\phi) \equiv \int_M \phi|_M\), and \(dd^c\) is the complex Laplacian. In this statement, \(G^+\) represents the Green function for \(K^+\) given by

\[
G^+(x, y) \equiv \lim_{n \to +\infty} \frac{1}{2^n} \log^+ \|f^n(x, y)\| .
\] (10)

It is easily shown that the support of \(dd^c G^+\) coincides with \(J^+\). From this theorem we see that the stable manifold of any periodic saddle \(p\) is dense in \(J^+\), that is, \((W^s(p)) = J^+\). Using this result, together with the fact that the Hausdorff limit \(\mathcal{M}_\infty\) contains \(J^+\), we obtain the desired claim.

This claim gives a mathematical verification to the observation numerically found. Indeed, as shown in Fig. 3(a), the trajectories leaving the Laputa chains approach exponentially to the real \((p, \theta)\)-plane. This makes \(\text{Im } S_n(p_0, \theta_0)\) converge absolutely. As a demonstration of the theorem, we show in Figs. 3(b) and (c) the set \(\bigcup_{\beta<\beta_0} \mathcal{M}^*_n(\beta)\) for a fixed \(\beta_0\) and the slice of \(J^+\) by \(\{p_0 = \alpha\}\) for the Hénon map. One can see how \(\mathcal{M}_n\)-set shrinks to the \(J^+\) as a function of \(n\).

![Figure 3](image.png)

**Figure 3.** (a) The distance of from the real plane as a function of time is displayed for the orbits whose initial conditions are put on the Laputa chains. The solid line denote the case of the standard map. The dotted and broken lines are the ones for the Hénon map. When the Julia set \(J\) exists only on the real plane, the orbits always approach the real plane directly(dotted line), but otherwise the orbits first move around in \(\mathbb{C}^2\) space, and then approach the real plane (broken line). (b)-(c) The set \(\bigcup_{\beta<\beta_0} \mathcal{M}^*_n(\beta)\) \((\beta_0 = 10^{10})\) for the Hénon map is shown as the solid curves in case of (b) \(n = 9\) and (c) \(n = 10\). The slice of \(J^+\) by \(\{p = \alpha\}\) is shown as the dots in each figure.

Notice that the assumption \(h_{\text{top}}(F|_{\mathbb{R}^2}) > 0\) in the above theorem is a mathematical expression which corresponds to the fact that the underlying classical dynamics \(F|_{\mathbb{R}^2}\) is chaotic. We also note that the slice of the forward Julia set \(J^+\) by \(\{p = \alpha\}\) can be shown to have positive capacity for any initial condition \(\alpha \in \mathbb{R}\). Thus, the theorem above suggests that, unlike the instanton solutions in the integrable case, a bunch of
paths in $\mathbb{C}^2$ contributes to the tunneling phenomena if the underlying classical mechanics is chaotic.

It should be noted that the assumption in (i) covers the system with mixed phase space, which is the most generic situation in physics. In addition, the physical implication or interpretation of another theorem of Bedford and Smillie on the transitivity of the dynamics $[20, 21, 22]$ is suggestive in our problem. It states that for any $\mathbb{C}^2$-neighborhoods of any two points in the chaotic regions there is an orbit in $\mathbb{C}^2$ connecting them, even in the case where the chaotic regions in the real plane are mutually disjointed by KAM tori. This property exactly guarantees the transition between any disconnected regions on the real-valued classical dynamics, and non-zero tunneling amplitude of the wavefunction in arbitrary regions is always realized due to the transitivity on the Julia set.

In this way, with the help of strong mathematical claims, which could be established only by extending the dynamics to the complex space, we can clearly understand the reason why chaos seems to assist tunneling and can become chaotic; these can be attributed to such high density of the tunneling paths in $J^+$ and the transitivity of the complexified dynamics. So far, the structure of the Julia set has been an object which mainly attracts the interest of mathematicians. But the present result implies that the Julia set is really observable as chaotic tunneling in various physical and chemical phenomena.

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