The \((2, m, 2, \infty, r)\)-Wizard of Houses

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Abstract. The \((n, m, l, t, r)\)-Wizard of houses is a game played by \(n\) players \(p_1, p_2, p_3, \ldots, p_n\) on a board with \(m\) houses. There are \(l\) colors of token, and \(t\) tokens for each color. Each player chooses one color from the \(l\) colors, so that each has a different color. Player \(p_1\) starts the game by choosing one token of any color and putting it into any house. Players \(p_2, p_3, p_4, \ldots, p_n\) follow suit, until one round is completed. The game is over when the \(r^{th}\) round ends or after the tokens are all played, whichever comes first. In each house, each player who has the highest number of his own colored tokens receives one point. The winner is the player who has the highest score. In this paper, we determine the \((2, m, 2, \infty, r)\)-Wizard of houses. We provide non-losing strategies for both players.

Keyword: Wizard of Houses, Board Game, Thailand Board Game.

1. Introduction

Game theory is one of the important branches of Mathematics. It concerns strategic decision making. Game theory can be used to solve problems in fields such as Economics, Computer Science, Psychology, and Political Science. We first give some definitions of game theory which are used in this paper. An agent who makes decisions in a game is called a player. A strategy is one of the given possible actions of a player. For notation and terminologies which are not defined here, we follow Reny et al. [1] and Straffin [2].

Board games have been played all over the world for many thousands of years. They have been separately invented in many countries. For example, Tic-Tac-Toe originated in ancient Egypt of around 1300 BCE. It has been widespread to many countries and has been studied widely. Beck [3] studied Tic-Tac-Toe in the form of a combinatorial game. Later, Beeler [4] considered this game in the form of a graph game. Go was invented in China around 500 BCE. Since this game is a highly competitive game, researchers would like to find strategies to play this game e.g. Huang et al. [5]. Furthermore, some researchers use computers to develop move patterns such as Coulom [6]. Hex was invented in Denmark in 1952. Some researchers provided solutions of this game of some sizes such as Yang et al. [7] and Henderson et al. [8]. In 2014, Huneke et al. [9] revised this game from a square board to a cylinder board. For more detailed history of board games, see Walker [10].

In this paper, we introduce a board game called “The Wizard of Houses”, inspired by “The Wizard of Five Rings”, which was introduced by Piamdamrong sak and Sanongyad. This game was first launched in the Thailand Board Games of the Year 2016.

Let \(n, m, l, t\) and \(r\) be positive integers such that \(n \leq l\). The \((n, m, l, t, r)\)-Wizard of Houses is a board game, where \(n\) is the number of players, \(m\) is the number of houses, \(l\) is the number of colors, \(t\) is the number of tokens of each color, and \(r\) is the number of rounds. Let \(P = \{p_1, p_2, p_3, \ldots, p_n\}\) be the set of players, \(H = \{h_1, h_2, h_3, \ldots, h_m\}\) the set of houses, and \(C = \{c_1, c_2, c_3, \ldots, c_l\}\) the set of
colors of tokens. At the beginning of the game, each player chooses one color from the \( l \) available, and all the players get different colors. Without loss of generality, we assume that player \( p_i \) chooses the color \( c_i \) for each \( i \in \{1, 2, 3, \ldots, n\} \). The sequence of play for one round is \( p_1, p_2, p_3, \ldots, p_n \). It is played for \( r \) rounds or until all the tokens are used, whichever is first. On each turn, a player chooses one token of any color and puts it into any house. In any house, a player who has the maximum tokens gains one point. If two players have the same number of tokens, each receives one point. If there are no tokens in a particular house, no points are awarded. The goal of each player is to get the highest score. In this paper, we consider a game which has two players, \( m \) houses, two colors of token, unlimited tokens of each color, and \( r \) rounds. This game is denoted as the \((2, m, 2, \infty, r)\)-Wizard of Houses. Note that this game is over only when the \( r^{th} \) round ends.

The following is an example of the \((2, m, 2, \infty, r)\)-Wizard of Houses.

**Example 1** In the \((2,4, 2, \infty, 3)\)-Wizard of Houses, player \( p_1 \) first puts a \( c_1 \)-token into house \( h_3 \), and player \( p_2 \) puts a \( c_2 \)-token into house \( h_3 \), as shown in Figure 1. In the second round, \( p_1 \) puts a \( c_1 \)-token into \( h_2 \), and \( p_2 \) puts a \( c_2 \)-token, into \( h_2 \) as shown in Figure 2. In the final round, \( p_1 \) puts a \( c_2 \)-token into \( h_2 \), and \( p_2 \) a \( c_1 \)-token as shown in Figure 3. The game is now over. Both players get two points, so this game ends in a draw.

\[ \text{Figure 1. First round of the (2,4, 2, \infty, 3) - Wizard of Houses.} \]

\[ \text{Figure 2. Second round of the (2,4, 2, \infty, 3) - Wizard of Houses.} \]
Figure 3. Final round of the (2,4,∞,3)-Wizard of Houses.

To prove the main theorems of the (2,m,2,∞,r)-Wizard of Houses, we use the following lemma.

**Lemma 2** In the (n,m,l,t,r)-Wizard of Houses, removing one token of each color from the same house at the end of the game does not affect the player ranking.

**Proof** Suppose that there is a house containing tokens of all colors. Let s_i be the score of the player p_i for each \( i \in \{ 1,2,3,\ldots,n \} \). When we remove a token of each color from that house, we have the following two cases.

- **Case 1** The house is empty. Thus, the new score of \( p_i \) is \( s_i - 1 \) for all \( i \). Hence, the player ranking does not change.

- **Case 2** The house is non-empty. Thus, \( p_i \) still has \( s_i \) points for all \( i \). Hence, the player ranking does not change.

Lemma 2 can also be used in a general version of this game. The process of removing one token of each color from the same house is called a *simplification* of the game. The game is in *simple form* if it cannot be simplified further.

2. The First Player’s Non-Losing Strategy and The Second Player’s Drawing Strategy

For the (2,m,2,∞,r)-Wizard of Houses, we notice that, if a house has the same number of \( c_1 \)-tokens and \( c_2 \)-tokens, then both players get one point. To get a draw, the second player may try to make every house have the same number of \( c_1 \) and \( c_2 \)-tokens, as shown in the following strategy.

**Strategy 3** (The second player’s drawing strategy)

\[ p_1 \text{ puts a token into one house, } p_2 \text{ then puts a token of different color into the same house.} \]

We next show that, if the second player uses Strategy 3, then the game ends in a draw, as shown by the following theorem.

**Theorem 4** The second player has a drawing strategy for the (2,m,2,∞,r)-Wizard of Houses.

**Proof** Assume \( p_2 \) uses Strategy 3. It is easy to derive that both players have zero score in the simple form. By Lemma 2, this game ends in a draw.

We can then extend the second player’s drawing strategy to demonstrate the first player’s non-losing strategy, follows:
Strategy 5 (The first player’s non-losing strategy)

First, $p_1$ puts a $c_1$-token into any house. When $p_2$ puts a token into one house, $p_1$ then puts a token of different color into the same house.

Theorem 6 The first player has a non-losing strategy for the $(2, m, 2, \infty, r)$-Wizard of Houses.

Proof Assume that $p_1$ uses Strategy 5. Apart from the first round of $p_1$ and the last round of $p_2$, the number of $c_1$ and $c_2$-tokens are the same in every house. By Lemma 2, it is enough to consider only the first round of $p_2$ and the last round of $p_2$. Note that $p_1$ places a $c_1$-token in the first round. If $p_2$ puts a $c_1$-token in any house in the last round, then $p_1$ is the winner. Otherwise, $p_2$ places a $c_2$-token in the last round. Then both players have one point in the simple form of this game, so the game ends in a draw.

Theorems 4 and 6 produce Theorem 7.

Theorem 7 Neither player in the $(2, m, 2, \infty, r)$-Wizard of Houses has any winning strategy.

3. Non-Losing Strategies for Both Players

In this section, we give another non-losing strategy for the first player, and extend the second player’s drawing strategy to a non-losing strategy.

Strategy 8 (The first player's non-losing strategy)

First, $p_1$ puts a $c_1$-token into any house. When $p_2$ puts a token into one house, $p_1$ then puts a $c_1$-token into the same house.

It is obvious that Strategy 8 is at least as good as Strategy 5. Thus, Strategy 8 is a non-losing strategy by Theorem 7.

Strategy 9 (The second player's non-losing strategy)

When $p_1$ puts a token into one house, $p_2$ then puts a $c_2$-token into the same house.

The following is an example in which $p_2$ uses Strategy 9.

Example 10 In the $(2, 4, 2, \infty, 3)$-Wizard of Houses, player $p_1$ first puts a $c_1$-token into house $h_3$, and player $p_2$ puts a $c_2$-token into house $h_3$. In the second round, $p_1$ puts a $c_1$-token into $h_2$ and $p_2$ puts a $c_2$-token into $h_2$. In the last round, $p_1$ puts a $c_2$-token into $h_2$ and $p_2$ puts a $c_2$-token into $h_2$, as shown in Figure 4. The game is now over. As $p_1$ gets one point and $p_2$ gets two points, $p_2$ is the winner.

Figure 4. The last round of the $(2,4,2,\infty,3)$-Wizard of Houses.
We can see that $p_1$ plays the same strategy as in Examples 1 and 10; however, $p_2$ uses Strategy 3 in Example 1 and Strategy 9 in Example 10. It is clear that the result for $p_2$ in Example 10 is strictly better than in Example 1. Using Strategy 9, $p_2$ may win the game or at least get a draw. Hence, Strategy 9 is a non-losing strategy.

4. Conclusions
In the $(2, m, 2, \infty, r)$-Wizard of Houses, we provide two non-losing strategies for the first player. We also give one drawing strategy and one non-losing strategy for the second player. Furthermore, we show that neither player has a definitely winning strategy.

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