SPECTRAL PROPERTIES OF ORDINARY DIFFERENTIAL OPERATORS ADMITTING SPECIAL DECOMPOSITIONS

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Abstract. We investigate spectral properties of ordinary differential operators related to expressions of the form \( D^\epsilon + a \). Here \( a \in \mathbb{R} \) and \( D^\epsilon \) denotes a composition of \( \delta \) and \( \delta^+ \) according to the signs in the multi-index \( \epsilon \), where \( \delta \) is a first order linear differential expression, called delta-derivative, and \( \delta^+ \) is its formal adjoint in an appropriate \( L^2 \) space. In particular, Sturm-Liouville operators that admit the decomposition of the type \( \delta^+ \delta + a \) are considered.

We propose an approach, based on weak delta-derivatives and delta-Sobolev spaces, which is particularly useful in the study of the operators \( D^\epsilon + a \). Finally we examine a number of examples of operators, which are of the relevant form, naturally arising in analysis of classical orthogonal expansions.

1. Introduction. The Sturm-Liouville theory constitutes an important part of analysis. The theory encompasses the general theory of Sturm-Liouville differential equations and the theory of Sturm-Liouville operators included in the general theory of unbounded operators on a Hilbert space.

Given an open interval \( I \subset \mathbb{R} \) and a set of Sturm-Liouville coefficients \( \{w, r, s\} \), that is a triple of real valued functions on \( I \) satisfying some natural smoothness and positivity assumptions, the associated Sturm-Liouville differential expression is

\[
\mathcal{L}_{\{w,r,s\}} = \frac{1}{w(x)} \left( - \frac{d}{dx} \left( r(x) \frac{d}{dx} \right) + s(x) \right).
\]

With this expression one can associate a boundary value problem or an unbounded operator acting on \( L^2(I, w(x) \, dx) \).

In [22] a unified approach to the theory of Riesz transforms and conjugacy in the setting of multi-dimensional orthogonal expansions was proposed. Specified to dimension one this included investigation of second order linear differential operators of the form \( L = \delta^+ \delta + a \), where \( a \) was a non-negative constant, \( \delta = p(x) \frac{d}{dx} + q(x) \) was an appropriate 'derivative' acting on functions on an interval \( I \), and \( \delta^+ \) denoted its formal adjoint in \( L^2(I, w(x) \, dx) \). An essential assumption consisted in existence of an orthonormal basis \( \{\varphi_j\}_{j \in \mathbb{N}} \) in \( L^2(I, w(x) \, dx) \) being eigenfunctions of \( L \) (together with some minor technical hypotheses).

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The purpose of this paper is to investigate spectral properties of differential operators related to the expression of the form $D^\epsilon + a$, where $a \in \mathbb{R}$ and $D^\epsilon$ denotes a composition of $\delta$ and $\delta^+$ according to the signs in the multi-index $\epsilon$. This is then applied to Sturm-Liouville operators that admit the decomposition $\delta^+ \delta + a$. As an archetype of such decomposition can serve the decomposition of the one-dimensional harmonic oscillator

$$-\frac{d^2}{dx^2} + x^2 = \delta^+ \delta + 1,$$

where $\delta = \frac{d}{dx} + x$ and $\delta^+ = -\frac{d}{dx} + x$ are commonly called the creation and annihilation operators (more symmetric form of the above is $-\frac{d^2}{dx^2} + x^2 = \frac{1}{2}(\delta \delta^+ + \delta^+ \delta)$). The basic tools we introduce and use in our investigation are weak delta-derivatives and delta-Sobolev spaces. These tools prove to be highly adequate in describing explicitly objects associated to the investigated operators. In particular, this is visible in construction of natural self-adjoint extensions, including the Friedrichs extension, of $D^\epsilon + a$ with domain $C^\infty_c(I)$ in case when $\epsilon = \epsilon'$, i.e. when $D^\epsilon$ is symmetric (see Section 3 for the definition of $\epsilon'$).

It is worth noting that the idea of using factorization of the type $A^*A$, where $A^*$ is formally adjoint to $A$, is not new and was broadly used in analysis of bounded or unbounded operators. This includes, for instance, the case of some second order partial differential operators of divergence form. We point out, however, that adding a constant $a$ is important from the point of view of developed theory; see comments in the example at the end of Section 2. We also emphasize that the developed theory applies to higher order differential operators and in the case of second order differential operators Sturm-Liouville operators in divergent form are included. See Section 2.

The Friedrichs extensions of differential (or quasi-differential) operators were widely investigated in the literature. See [27] and references therein. In particular, for the Sturm-Liouville differential expressions with minimal assumptions on the Sturm-Liouville coefficients the Friedrichs extension of the associated minimal operators were analysed. In this paper we consider the Sturm-Liouville coefficients to be $C^\infty$ functions and the role of the corresponding minimal operator is played by the associated operator with domain $C^\infty_c(I)$. The Friedrichs extension of this operator, more precisely, of $D^\epsilon + a$, is described in Theorem 5.2.

The concept of Sobolev-type (or potential-type) spaces associated to second order differential operators related to different discrete orthogonal expansions was used by a number of authors. For instance, Graczyk et al. [10] investigated Sobolev-potential spaces in the (multidimensional) context of Laguerre polynomial expansions. Bongioanni and Torrea [6, 7] investigated Sobolev-type spaces in the context of Hermite and Laguerre expansions, in particular with respect to the Laguerre function system $\{L_n^\alpha\}$, together with the relations of these spaces to potential spaces. Betancor et al. [5] and Langowski [17] did analogous research in the context of ultraspherical and Jacobi polynomial expansions, respectively.

We mention that we do not pretend to develop our theory in full generality and assume the involved coefficients, in particular the Sturm-Liouville coefficients, to be $C^\infty$ functions. This is enough in all classic applications and avoids entering in technical details. It is worth pointing out that, although main applications are directed to the second-order differential operators, the theory we develop treats higher order differential operators as well. An example of source where self-adjoint extensions of higher order ordinary differential operators were studied is [24].
We now comment briefly on the literature that can be helpful in following this paper (needless to say, we offer a limited number of sources). A comprehensive catalogue of Sturm-Liouville differential equations together with summary of Sturm-Liouville theory is included in [9]. The monograph [28] encompasses detailed presentation of the theory. The survey [13] contains a discussion of Hilbert space theory of Sturm-Liouville operators with some emphasis on the minimal and maximal operators; also a number of illustrative examples is examined there. For a short (but instructive) treatment of this theory for operators of the type \(-\frac{d^2}{dx^2} \pm s(x)\) or, more generally, of the type \(-\frac{d}{dx}(r(x)\frac{d}{dx}) + s(x)\), the reader is referred to [23]. Another source that can be consulted with respect to spectral theory of Sturm-Liouville operators is [26].

The paper is organized as follows. In Section 2 preliminary facts are gathered. Section 3 contains basic definitions and is mainly concerned with the proofs of auxiliary results related to weak delta-derivatives. Section 4 is a reminder on the theory of unbounded operators with some emphasis on the construction of the Friedrichs extension of a symmetric lower semibounded operator. Section 5 contains definition and basic results on delta-Sobolev spaces. Section 6 has an accessory character and includes a discussion of a Hajmirzaahmad idea and a comment on the Liouville form of a Sturm-Liouville operator. A substantial part of the paper, Section 7, is devoted to discussion of examples, mainly coming out from harmonic analysis of orthogonal expansions, that illustrate our theory. Finally, in Section 8 we gather explanations of auxiliary facts needed earlier.

**Notation.** We use fairly standard notation for (complex-valued) function or distribution spaces. For instance, \(AC_{\text{loc}}(I)\) denotes the space of all functions \(f\) on \(I\) such that \(f \in AC[\alpha, \beta]\), for every bounded interval \([\alpha, \beta] \subset I\). Usual derivatives or weak derivatives of a function \(f\) will be denoted by using the same symbols, \(f', f''\) or \(f^{(n)}\). It will be clear from the context which types of derivatives are considered. Recall that given \(f \in L^1_{\text{loc}}(I)\) we say that **weak derivative** of \(f\) exists provided there is \(g \in L^1_{\text{loc}}(I)\) such that

\[
\int_I -\varphi'(x)f(x) \, dx = \int_I \varphi(x)g(x) \, dx, \quad \varphi \in C_c^\infty(I).
\]

Then we set \(f' := g\) and call \(g\) the weak derivative of \(f\). At some places we will be a bit loose when it comes to distinguishing between differential expressions and operators generated by them. Finally, in several places we shall tacitly use the fact that for a continuous and positive function \(w\) on \(I\) one has \(L^1_{\text{loc}}(I, w(x) \, dx) = L^1_{\text{loc}}(I, dx)\); we shall write \(L^1_{\text{loc}}(w)\) for short.

**2. Preliminaries.** Let the interval \(I = (b, c), \ -\infty \leq b < c \leq \infty\), be given and let \(w\) be a weight function on \(I\), by which we mean a real-valued positive \(C^\infty\) function. For real-valued \(p, q \in C^\infty(I), \ p(x) \neq 0\) for \(x \in I\), consider the first order linear differential expression

\[
\mathcal{D} = \mathcal{D}_{\{w, p, q\}} = p(x) \frac{d}{dx} + q(x),
\]

treated as an operator on the Hilbert space \(L^2(w) := L^2(I, w(x) \, dx)\). We shall call \(\mathcal{D}\) the **delta-derivative associated with the triple** \(\{w, p, q\}\). The formal adjoint to \(\mathcal{D}\), in the sense that

\[
\langle \mathcal{D}\varphi, \psi \rangle_{L^2(w)} = \langle \varphi, \mathcal{D}^*\psi \rangle_{L^2(w)}, \quad \varphi, \psi \in C_c^\infty(I),
\]

(2.1)
is
\[ \mathfrak{d}^+ = -p(x) \frac{d}{dx} + q^+(x), \]
where
\[ q^+(x) = q(x) - p(x) \frac{w'(x)}{w(x)} - p'(x). \]
Note that \( \mathfrak{d}^+ \) is the delta-derivative associated with the dual triple \( \{ w, -p, q^+ \} \) and thus any definition or statement formulated for \( \mathfrak{d} \), in the context of general triple, has its counterpart for \( \mathfrak{d}^+ \), which need not to be restated explicitly. Also, notice that \((q^+)^+ = q\) and \((\mathfrak{d}^+)^+ = \mathfrak{d}\) and, in general, skew-symmetry does not hold, \( \mathfrak{d}^+ \neq -\mathfrak{d} \). This lack of skew-symmetry has a serious impact in several places of developed theory.

Each operator of the form \( \mathfrak{d}^+ \mathfrak{d} + a \), where \( \mathfrak{d} \) is associated with a triple \( \{ w, p, q \} \) and \( a \in \mathbb{R} \), is a Sturm-Liouville operator corresponding to the Sturm-Liouville triple \( \{ w, r, s \} \) with \( r := wp^2 \) and \( s := w(q^+ q - pq' + a) \). Moreover, \( \mathfrak{d}^+ \mathfrak{d} + a \) is symmetric and lower semibounded (with a lower bound \( a \)) when considered on \( C_c^\infty (I) \). But not every Sturm-Liouville operator \( L_{\{w, r, s\}} \), with \( r > 0 \), admits decomposition of the form
\[ L_{\{w, r, s\}} = \mathfrak{d}^+ \mathfrak{d} + a, \tag{2.2} \]
for some \( \mathfrak{d} = \mathfrak{d}_{\{w, p, q\}} \) and \( a \in \mathbb{R} \). For instance, for \( w = r \equiv 1 \) and \( s(x) = -x \), the Schrödinger operator \( -\frac{d^2}{dx^2} - x \) on \( L^2(\mathbb{R}) \) is not lower semibounded (fix \( \varphi \in C_c^\infty (\mathbb{R}) \) such that \( \text{supp} \varphi \subset [0, 1] \), \( \varphi(x) > 0 \) for \( x \in (0, 1) \) and \( \int_0^1 \varphi^2 \, dx = 1 \) and consider the sequence \( f_n(x) = \varphi(x - n), n \in \mathbb{N} \) and hence decomposition (2.2) is not possible.

It is easily seen that (2.2) holds for the Sturm-Liouville operators in divergent form, namely, \( L_{\{w, w, 0\}} = \mathfrak{d}^+ \mathfrak{d} \) with
\[ \mathfrak{d} = \frac{d}{dx}, \quad \mathfrak{d}^+ = -\frac{d}{dx} \frac{w'}{w}, \]
and, more generally, \( L_{\{w, r, 0\}} = \mathfrak{d}^+ \mathfrak{d} \) with
\[ \mathfrak{d} = \left( \frac{r}{w} \right)^{1/2} \frac{d}{dx}, \quad \mathfrak{d}^+ = - \left( \frac{r}{w} \right)^{1/2} \frac{d}{dx} - \left( \frac{1}{2} \frac{w'}{w} \left( \frac{r}{w} \right)^{1/2} \right)' + \left( \frac{r}{w} \right)^{1/2} \frac{w'}{w}. \]
However, for general \( L_{\{w, r, s\}} \) with an arbitrary \( s \) things are not so easy. For simplicity consider the case of \( w = r = 1 \) and assume for a moment that \( \tilde{s} \) is a given real-valued continuous function on \( I \). It is immediately seen that the factorization
\[ -\frac{d^2}{dx^2} + \tilde{s}(x) = \left( -\frac{d}{dx} + q(x) \right) \left( \frac{d}{dx} + q(x) \right) \]
holds on \( I \) with a real-valued \( C^1 \) function \( q \) if and only if the Riccati equation
\[ q' - q^2 = -\tilde{s} \tag{2.3} \]
has a real-valued global solution on \( I \). On the other hand, it is well-known (see, for instance, [15, Chapter 5]) that (2.3) has a real-valued global solution if and only if the linear equation
\[ y'' - \tilde{s}(x)y = 0 \tag{2.4} \]
has a positive solution on \( I \). Indeed, if \( y \) is such a solution, then \( q := -y'/y \) solves (2.3). Conversely, if \( q \) solves (2.3), then \( y := \exp(-\int q) > 0 \) solves (2.4).
Thus we conclude that, in the simplified case \( w = r = 1 \), given real-valued \( s \in C^1(I) \) the decomposition

\[
- \frac{d^2}{dx^2} + s(x) = \left( - \frac{d}{dx} + q(x) \right) \left( \frac{d}{dx} + q(x) \right) + a
\]

exists with some real-valued \( q \) and \( a \in \mathbb{R} \) if and only if there exists \( a \in \mathbb{R} \) such that the linear equation \( y'' - (s(x) - a)y = 0 \), possesses a positive solution.

It is illustrative to consider the case \( s(x) = -x \) and show by a direct calculation that for any \( a \in \mathbb{R} \), \( y'' + (x + a)y = 0 \) fails to possess a positive solution. In fact, due to the shift in the independent variable, it suffices to verify this for \( a = 0 \). But \( u \) solves \( y'' + xy = 0 \) if and only if \( v(x) := u(-x) \) solves \( y'' - xy = 0 \). The analysis of the latter equation is contained, for instance, in [18, Section 5.17]. It is known that the Airy functions \( \text{Ai}(x) \) and \( \text{Bi}(x) \) form the system of fundamental solutions for this equation. Moreover, the following asymptotics hold (see [18, 5.17.11])

\[
\begin{bmatrix} \text{Ai} \\ \text{Bi} \end{bmatrix}(-x) = \begin{bmatrix} 1 \ & \ 1 \\ -1 & -1 \end{bmatrix} \pi^{-1/2} x^{-1/4} \begin{bmatrix} \cos \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) \\ \sin \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) \end{bmatrix} + O(x^{-7/4}), \quad x \to \infty.
\]

(A comment is necessary at this point. Namely, there is a discrepancy in proper understanding of the symbol \( \approx \) in [18, 5.17.11]. This cannot be understood as in the top of [18, p.9] since the functions on the right hand sides of [18, 5.17.11] have zeroes. Also, this cannot be understood as in [18, (1.4.1)] since this would mean the coincidence of zeroes of functions on both sides of [18, 5.17.11] which is not the case. The proper understanding is presented above and the exponent in the remainder is calculated by means of the asymptotic representation [18, 5.11.6]. Also, we point out that the factor \( 1/\sqrt{3} \) is missing in the right hand side of the second line of [18, 5.17.11].)

In particular, this shows that the Airy functions have an oscillatory character for large negative values. Therefore, to check that any solution, i.e. \( C_1 \text{Ai}(x) + C_2 \text{Bi}(x) \), \( C_1, C_2 \in \mathbb{R} \), fails to be a positive function on \( \mathbb{R} \), we assume that both \( C_1 \) and \( C_2 \) are non-zero. Then, let \( x_k > 0 \), \( k \in \mathbb{N} \), be defined by the equation \( \frac{2}{3} x_k^{3/2} - \frac{\pi}{4} = k\pi \), so that \( \text{Bi}(-x_k) = O(x_k^{-7/4}) \) and \( \text{Ai}(-x_k) = \pi^{-1/2} x_k^{-1/4} (-1)^k + O(x_k^{-7/4}) \), \( k \to \infty \). This shows that \( C_1 \text{Ai}(-x_k) + C_2 \text{Bi}(-x_k) \) oscillates for large \( k \).

In fact, given a general Sturm-Liouville operator \( L_{w,r,s} \), search for its relevance the decomposition relies on an attempt of solving a first order Ricatti differential equation; see [22, (8.5)] and comments on [22, p.708].

Finally, it is interesting to compare the decomposition (2.2) with Mammana’s factorization of a linear ordinary differential operator; see [15, Chapter 5]. Mammana’s result specified to second order linear differential operators says that

\[
L_{A,B} = - \frac{d^2}{dx^2} + A(x) \frac{d}{dx} + B(x),
\]

where \( A \) and \( B \) are real-valued continuous functions on a given interval \( I \), admits the decomposition

\[
L_{A,B} = \left( - \frac{d}{dx} + \alpha(x) \right) \left( \frac{d}{dx} + \beta(x) \right) \tag{2.6}
\]

where \( \alpha \) and \( \beta \) are, in general, complex-valued continuous functions on \( I \). A local factorization of the form (2.6) with real-valued continuous \( \alpha \) and \( \beta \) had been known for some time. The point in the quoted result is that (2.6) is global, i.e. is valid on the whole interval \( I \) if one allows \( \alpha \) and \( \beta \) to be complex-valued.
We now examine the following simple example that shows, in particular, that the relevant decompositions, if exist, are not unique.

Example. Consider \( L_\pm = -\frac{d^2}{dx^2} \pm 1 \) on \( I = \mathbb{R} \), so that \( w = r \equiv 1 \) and \( s \equiv \pm 1 \). Then for every \( \tau \geq 0 \) we have
\[
L_\pm = \left( -\frac{d}{dx} + \sqrt{\tau} \right) \left( \frac{d}{dx} + \sqrt{\tau} \right) + (\pm 1 - \tau),
\]
globally on \( \mathbb{R} \). In particular, for \( L_+ \) and \( \tau = 1 \) the decomposition (2.2) holds with \( a = 0 \). Observe, however, that for \( L_- \) we cannot claim that for some \( \tau \geq 0 \) (2.2) holds with \( a = 0 \). This strongly indicates that allowing an additive constant in (2.2) is crucial for further development of the theory. On the other hand, for every \( \tau < 1 \) we also have
\[
L_\pm = \left( -\frac{d}{dx} + \sqrt{-\tau} \cot(\sqrt{-\tau}x) \right) \left( \frac{d}{dx} + \sqrt{-\tau} \cot(\sqrt{-\tau}x) \right) + (\pm 1 - \tau),
\]
but this time (2.2) is limited to the interval \((0, \pi/\sqrt{-\tau})\), i.e., it holds locally.

3. Weak delta-derivatives. From now on till the end of Section 5, if not specified otherwise, \( d \) is assumed to be fixed; the associated triple is \( \{w, p, q\} \).

We begin with notion of weak delta-derivatives.

Definition 3.1. Let \( f \in L^1_{loc}(w) \). We say that weak \( d \)-derivative of \( f \) exists provided there is \( g \in L^1_{loc}(w) \) such that
\[
\int_I \varphi(x) \overline{f(x)} w(x) \, dx = \int_I \varphi(x) \overline{g(x)} w(x) \, dx, \quad \varphi \in C^\infty_c(I).
\]
Then we set \( d_{weak} f := g \) and call \( g \) the weak \( d \)-derivative of \( f \).

Clearly, if \( w = p \equiv 1, q \equiv 0 \), then \( d = \frac{d}{dx}, d^+ = -d \), and the notion of weak \( d \)-derivative coincides with the notion of usual (classic) weak derivative; we then write \( f' \) rather than \( d_{weak} f \). Observe that using in the above definition the complex rather than the real 'pairing' is immaterial for the mentioned coincidence.

To define higher order weak delta-derivatives and to avoid clumsy symbols it is convenient to introduce the following notation :
\[
d^\varepsilon = \begin{cases} \bar{d}, & \varepsilon = 1, \\ d^+, & \varepsilon = -1, \end{cases} \quad p^\varepsilon(x) = \begin{cases} q(x), & \varepsilon = 1, \\ q^+(x), & \varepsilon = -1, \end{cases}
\]
and \( p^\varepsilon(x) = \varepsilon p(x) \) for \( \varepsilon \in \{-1,1\} \). Moreover, the set of all multi-indices \( \epsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n \), \( n \in \mathbb{N} \), will be denoted by \( \mathcal{E} \) and \( |\epsilon| = n \) will be called the length of \( \epsilon \). In addition, given \( \epsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) we shall write
\[
\epsilon' = (-\varepsilon_n, \ldots, -\varepsilon_1) \quad \text{and} \quad \bar{\epsilon} = (\varepsilon_2, \ldots, \varepsilon_n).
\]
Let \( D^\epsilon = d^{\varepsilon_1} \circ \ldots \circ d^{\varepsilon_n} \). Here and later on we consider \( D^\epsilon \) as an operator on \( L^2(w) \) with domain \( \text{Dom}(D^\epsilon) = C^\infty_c(I) \). Note that \( D^{\epsilon'} \) is the formal adjoint to \( D^\epsilon \), that is
\[
\langle D^{\epsilon'} \varphi, \psi \rangle_{L^2(w)} = \overline{\langle \varphi, D^\epsilon \psi \rangle}_{L^2(w)}, \quad \varphi, \psi \in C^\infty_c(I).
\]
In particular, if \( \epsilon \in \mathcal{E} \) is anti-symmetric, \( \epsilon' = \epsilon \), then \( D^\epsilon \) is symmetric. Note also, that the condition \( \epsilon' = \epsilon \) forces the length of \( \epsilon \) to be even, \( |\epsilon| = 2n \), and, moreover, the shape of \( \epsilon \) to be of the form \( \epsilon = (\tau, \tau) \), for some \( \tau \in \mathcal{E}, |\tau| = n \). Then, for \( \varphi \in C^\infty_c(I) \),
\[
\langle D^\epsilon \varphi, \varphi \rangle_{L^2(w)} = \langle D^{\epsilon'} \varphi, D^\epsilon \varphi \rangle_{L^2(w)} \geq 0,
\]
which means that \( D^\epsilon \) is non-negative. In addition, if \( \epsilon' = \epsilon \) and \( a \in \mathbb{R} \), then \( D^\epsilon + a \) is symmetric and lower semibounded with \( a \) as a lower bound.
Now, for any $n \geq 1$ and any multi-index $\epsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{E}$ we say $g \in L^1_{\text{loc}}(w)$ is the weak delta-derivative of $f \in L^1_{\text{loc}}(w)$ of order $n$ corresponding to $\epsilon$, provided
\[
\int_I D^n_{\epsilon} \varphi(x) f(x) w(x) \, dx = \int_I \varphi(x) g(x) w(x) \, dx, \quad \varphi \in C^\infty_c(I).
\]
We set $D_{\epsilon}^n f := g$. This agrees with previous Definition 3.1.

It is also clear that if $f \in C^n(I)$, then the weak delta-derivatives of $f$ up to order $n$ exist and $D_{\epsilon}^n f = D^n f$ for any multi-index $\epsilon$ of length $\leq n$. Finally we add that for any $\epsilon$ the weak delta-derivative $D_{\epsilon}^n f$, if exists, is uniquely determined.

To define the action of $\partial$, $\partial^+$, or more generally, $D^r$, on distributions, note that $\partial$, and similarly $\partial^+$, maps continuously $D(I)$ into itself ($D(I) := C^\infty_c(I)$). Therefore, for $T \in D'(I)$ we define $D^r T \in D'(I)$ by
\[
D^r T := T \circ D^r.
\]
In particular, $\partial T := T \circ \partial^+$, $\partial^+ T := T \circ \partial$. This definition agrees with definition of just introduced weak delta-derivatives, provided we identify $f \in L^1_{\text{loc}}(w)$ with the distribution
\[
D(I) \ni \varphi \mapsto \int_I \varphi(x) f(x) w(x) \, dx.
\]

More specifically, $D^r f = D_{\epsilon}^n f$ for $f \in L^1_{\text{loc}}(w)$, provided $D_{\epsilon}^n f$ exists.

Since the (formal) composition $D^r = \partial^1 \circ \ldots \circ \partial^{n-1}$ gives the linear differential expression of order $n$ with real $C^\infty$ coefficients, we denote them by $r_j^r$, $0 \leq j \leq n$, so that
\[
D^r = r_n^r \frac{d^n}{dx^n} + \ldots + r_1^r \frac{d}{dx} + r_0^r.
\]
Thus, for $n = 1$ we have $r_1^r(x) = p^r(x)$, $r_0^r(x) = q^r(x)$, and, more generally, for $n \geq 1$, $r_n^r(x) = \prod_{j=1}^n p_j^{r_j}(x)$. Moreover, for $n \geq 2$ and $0 \leq k \leq n-1$, as a consequence of $D^r = \partial^{n-1} \circ D^1$, we have the relations
\[
r_k^r(x) = p_k^{r_k}(x) \left( (r_{k-1}^{r_k})'(x) + r_{k-1}^{r_k}(x) \right) + q_k^{r_k}(x) r_k^{r_k}(x),
\]
where, for $k = 0$, $r_{-1}^r$ is understood as 0.

We begin to investigate intrinsic properties of functions having weak delta-derivatives. Observe that, in particular, the results of Lemma 3.2 and Proposition 1 include the case of usual weak derivatives.

**Lemma 3.2.** Let $T \in D'(I)$ and $\epsilon \in \mathcal{E}$. If $D^r T = 0$, then $T$ is a $C^\infty$ function on $I$.

**Proof.** Since the leading term of the differential operator $D^r$ is not vanishing on $I$, $D^r$ is elliptic on $I$ and hence hypoelliptic. Consequently, $T \in C^\infty(I)$. \[\square\]

In the proof of the following proposition we shall use the operator
\[
R_\partial g(x) = \exp \left( - \int_\xi^x \frac{q(t)}{p(t)} \, dt \right) \cdot \int_\xi^x \frac{g(t)}{p(t)} \exp \left( \int_\xi^t \frac{q(s)}{p(s)} \, ds \right) \, dt,
\]
and its analogue for $\partial^+$, where $\xi \in I$ is arbitrarily fixed. $R_\partial$ is smoothing in the sense that for any $g \in L^1_{\text{loc}}(w)$ one has $R_\partial g \in C(I)$, $R_\partial g \in AC_{\text{loc}}(I)$, and for $g \in C^n(I)$, $R_\partial g \in C^{n+1}(I)$. Moreover, $R_\partial$ is the right inverse to $\partial$, that is for $g \in L^1_{\text{loc}}(w)$, $(R_\partial g)'$ exists a.e. on $I$, $R_\partial g \in L^1_{\text{loc}}(w)$, and $\partial (R_\partial g) = g$, i.e., $R_\partial g$ solves the equation
\[
p(x) h' + q(x) h = g, \quad \text{a.e. on } I.
\]
Given $\sigma, \tau \in \mathcal{E}, |\sigma| = k$, $|\tau| = m$, we define $\sigma \oplus \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m)$. 
Proposition 1. Let \( f \in L^1_{\text{loc}}(w) \), \( \epsilon \in \mathcal{E} \), \( |\epsilon| = n \geq 2 \), and assume \( D^\epsilon f \) exists. Then, for any \( 1 \leq m \leq n-1 \) and \( \tau \in \mathcal{E} \), \( |\tau| = m \), also \( D^\tau f \) exists. Moreover, if \( \sigma \in \mathcal{E} \) is such that \( 1 \leq |\sigma| \leq k - 1 \) and \( |\sigma| + |\tau| \leq n \), then \( D^\sigma D^\tau f \) exists and equals \( D^\sigma D^\tau f \).

Proof. By assumption, \( h := D^\epsilon f \in L^1_{\text{loc}}(w) \). Consider \( R_\epsilon h \), where \( R_\epsilon = R_{\epsilon_n} \circ \ldots \circ R_{\epsilon_1} \), with the convention that \( R_\epsilon = R_0 \) for \( \epsilon = 1 \) and \( R_\epsilon = R_{\epsilon_0} \) for \( \epsilon = -1 \). Clearly \( R_\epsilon h \in C^{n-1}(I) \), \( (R_\epsilon h)^{(n-1)} \in AC_{\text{loc}}(I) \), and \( D^\epsilon(R_\epsilon h) = D^\epsilon f \). By Lemma 3.2, \( f - R_\epsilon h \) is a \( C^\infty \) function on \( I \), hence \( f \) is a \( C^{n-1} \) function and the first claim follows. It is now clear that \( D^\tau f, D^\sigma D^\tau f \in C^{n-1-m}(I) \) and \( D^\tau f = D^\sigma D^\tau f \). Hence \( D^\sigma D^\tau f \) exists and equals \( D^\sigma f \).

We shall need the following technical result (that certainly is known and belongs to the mathematical folklore). Notice, that existence of the weak derivative of order \( n \) implies, by Proposition 1, existence of the weak derivatives of lower orders, \( j = 1, \ldots, n-1 \), and hence the right hand sides of (3.1) and (3.2) below indeed make sense.

Lemma 3.3. Let \( f \in L^1_{\text{loc}}(w) \), \( n \geq 1 \), and assume the weak derivative \( f^{(n)} \) exists. Then, for any \( h \in C^\infty(I) \) also \( (hf)^{(n)} \) exists and

\[
(hf)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} h^{(n-j)} f^{(j)}. \tag{3.1}
\]

Moreover, if \( f \in L^1_{\text{loc}}(w) \), \( b \in C^\infty(I) \), \( b > 0 \), and we assume that \( (bf)^{(n)} \) exists, then also \( f^{(n)} \) exists and

\[
f^{(n)} = \sum_{j=0}^{n} \binom{n}{j} \left( \frac{1}{b} \right)^{(n-j)} (bf)^{(j)}. \tag{3.2}
\]

Proof. The proof of the first part of the lemma goes by induction and, since the procedure is natural, we omit it. The second part is an immediate consequence of the first.

We conclude this section with result that shows a close connection of weak-delta derivatives and classic weak derivatives. Recall that existence of the weak derivative \( f^{(n)} \) implies existence of all lower order weak derivatives.

Theorem 3.4. Let \( f \in L^1_{\text{loc}}(w) \) and \( \epsilon \in \mathcal{E} \) be of length \( n \). If \( D^\epsilon f \) exists, then the weak derivative \( f^{(n)} \) exists and

\[
D^\epsilon f = r^n_0(x)f^{(n)} + \ldots + r^n_1(x)f' + r^n_0(x)f. \tag{3.3}
\]

Conversely, if the weak derivative \( f^{(n)} \) exists, then \( D^\epsilon f \) exists and (3.3) is satisfied.

Proof. For the first claim we proceed by induction and begin with \( n = 1 \) and \( \mathcal{D} \) (there is no need to consider \( \mathcal{D}^\pm \) separately) so \( \epsilon = 1 \). Fix \( f \in L^1_{\text{loc}}(w) \). Existence of \( \mathcal{D} f \) means that there is \( g \in L^1_{\text{loc}}(w) \) such that for every \( \varphi \in C^\infty_c(I) \) it holds

\[
\int_I \left( -p(x)\varphi''(x) + q^+(x)\varphi(x) \right) f(x) w(x) \, dx = \int_I \varphi(x) g(x) w(x) \, dx,
\]

or, equivalently,

\[
\int_I \varphi'(x)p(x)w(x)f(x) \, dx = -\int_I \varphi(x) \left( g(x) - q^+(x) f(x) \right) w(x) \, dx.
\]
This means that \( p(x)w(x)f(x) \) has weak derivative equal to \( (g(x) - q^+(x)f(x))w(x) \), and, consequently, by Lemma 3.3, also \( f \) has weak derivative.

It remains to check that (3.3) holds for the considered case. This means justification of the equality

\[
\int_I \partial^+ \varphi(x)f(x)w(x)\,dx = \int_I \varphi(x)(p(x)f'(x) + q(x)f(x))\,w(x)\,dx,
\]

for every \( \varphi \in C_c^\infty(I) \). But this is just a matter of a direct calculation with the aid of the fact that if \( f \) has weak derivative and \( h \in C_c^\infty(I) \), then also \( hf \) has weak derivative, see Lemma 3.3.

We now proceed the induction step. Let \( n \geq 2 \). Since \( D_{\text{weak}}^f \) exists, by Proposition 3.3, \( D_{\text{weak}}^f \) also exists, and thus, by induction hypothesis, the weak derivatives \( f', f'', \ldots, f^{(n-1)} \), exist. Existence of \( D_{\text{weak}}^f \) also means that there is \( g \in L^1_{\text{loc}}(w) \) such that for every \( \varphi \in C_c^\infty(I) \) it holds

\[
\int_I D' \varphi(x)f(x)w(x)\,dx = \int_I \left( r_n'(x)\varphi^{(n)}(x) + \ldots + r_1'(x)\varphi'(x) + r_0'(x)\varphi(x) \right)\overline{f(x)}w(x)\,dx = \int_I \varphi g w\,dx.
\]

From this one has

\[
\int_I \varphi^{(n)}(x)r_n'(x)w(x)f(x)\,dx = \int_I \varphi(x)g(x)w(x)\,dx - \sum_{k=0}^{n-1} \int_I \varphi^{(k)}(x)r_k'(x)w(x)f(x)\,dx.
\]

But for each \( k = 1, \ldots, n-1 \), \( r_k'(x)w(x)f(x) \) has weak derivative of order \( k \) and hence,

\[
\int_I \varphi^{(k)}(x)r_k'(x)w(x)f(x)\,dx = (-1)^k \int_I \varphi(x)(r_k'(x)w(x)f(x))^{(k)}\,dx.
\]

Putting things together it follows that \( r_n'(x)w(x)f(x) \) has weak derivative of order \( n \). Consequently, \( f \) has weak derivative of order \( n \).

Assume now that (3.3) holds for every multi-index of length \( n-1 \). Since \( D_{\text{weak}}^f \) exists, it holds

\[
\forall \varphi \in C_c^\infty(I) \int_I D' \varphi(x)f(x)w(x)\,dx = \int_I \varphi(x) D_{\text{weak}}^f f(x)w(x)\,dx.
\]

Our aim is now to show that

\[
\forall \varphi \in C_c^\infty(I) \int_I D' \varphi(x)\overline{f(x)}w(x)\,dx = \int_I \varphi(x) R_{\text{weak}}^f f(x)w(x)\,dx,
\]

where, for a given \( \epsilon \), \( R_{\text{weak}}^f \) denotes the right-hand side of (3.3). Then, combining the above will give (3.3).

To verify (3.4) recall that for \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) we set \( \epsilon' = (-\epsilon_n, \ldots, -\epsilon_1) \), and \( \tilde{\epsilon} = (\epsilon_2, \ldots, \epsilon_n) = (-\epsilon_n, \ldots, -\epsilon_2) \). We write the sequence of equalities

\[
\int_I D' \varphi f \, w \, dx = \int_I D^{(-\epsilon_n, \ldots, -\epsilon_1)}(\overline{\varphi^{(-\epsilon_1)}}) f \, w \, dx = \int_I \overline{\varphi^{(-\epsilon_1)} D_{\text{weak}}^f} f \, w \, dx \]

\[
= \int_I \overline{\varphi^{(-\epsilon_1)}} D_{\text{weak}}^f f \, w \, dx = \int_I \overline{\varphi^{(-\epsilon_1)} R_{\text{weak}}^f} f \, w \, dx \]

The second and third equality follow from Proposition 1, the fourth is due to induction hypothesis. Finally, the fifth equality is obtained by using the result from
the first step of the induction procedure and combining Proposition 1 and Lemma 3.3 together with relations for the coefficients involved in \( R^{\epsilon}_{\text{weak}} f \).

This finishes the induction step and hence the proof of the first claim. The proof of the converse claim, also inductive, relies on appropriate reversing of just used arguments and thus it is omitted. The proof of the theorem is therefore completed. \( \square \)

Theorem 3.4 permits to enhance Proposition 1.

**Corollary 1.** Let \( f \in L^{1}_{\text{loc}}(w) \), \( \epsilon \in \mathcal{E} \), \( |\epsilon| = n \geq 1 \), and assume \( D^{\epsilon}_{\text{weak}} f \) exists. Then, for any and \( \tau \in \mathcal{E} \), \( |\tau| = n \), also \( D^{\tau}_{\text{weak}} f \) exists.

Indeed, the assumption implies existence of the weak derivative \( f^{(n)} \). The argument used in the proof of the converse claim in Theorem 3.4 shows that the right hand side of (3.3) with \( \epsilon \) replaced by \( \tau \) is the weak delta-derivative \( D^{\tau}_{\text{weak}} f \).

What is more interesting, analogous argument with the aid of Theorem 3.4 also shows that the theory of weak delta-derivatives is, in some sense, independent on the choice of the initial \( \delta \).

**Corollary 2.** Let \( \delta = \delta_{\{w,p,q\}} \) be given. Let \( f \in L^{1}_{\text{loc}}(w) \), \( \epsilon \in \mathcal{E} \), and assume \( D^{\epsilon}_{\text{weak}} f \) exists. Then, for any other delta-derivative \( \delta_{\epsilon} \) associated with a triple \( \{w,p_{\epsilon},q_{\epsilon}\} \) also \( D^{\epsilon}_{\text{weak}} f \) exists, where \( D^{\epsilon}_{\delta} \) denotes the composition \( \delta_{\epsilon} \circ \cdots \circ \delta_{p} \).

Finally, the following corollary is an immediate consequence of Proposition 1, Theorem 3.4 and the fact (see [19, Corollary, p.3]) that if \( f \in L^{1}_{\text{loc}}(I) \) and the weak derivative \( f^{(n)} \) exists and belongs to \( L^{2}(I) \), then for every \( k = 0,1,\ldots,n-1 \), we have \( f^{(k)} \in L^{2}_{\text{loc}}(I) \).

**Corollary 3.** Let \( f \in L^{1}_{\text{loc}}(w) \). If \( D^{\epsilon}_{\text{weak}} f \) exists and belongs to \( L^{2}(w) \), then \( f \in L^{2}_{\text{loc}}(w) \) and for every \( 1 \leq m \leq |\epsilon|-1 \) and every \( \tau \in \mathcal{E} \), \( |\tau| = m \), we have \( D^{\epsilon}_{\text{weak}} f \in L^{2}_{\text{loc}}(w) \).

4. **A remainder on the theory of unbounded operators.** We commence with serving a minimal portion of well-known facts from the theory of unbounded operators, including the notions of the Friedrichs extension, deficiency indices and essential self-adjointness, that will be needed subsequently.

Let \( S \) be a densely defined symmetric operator on a complex Hilbert space \((H,\langle \cdot,\cdot \rangle_{H})\). Then \( S \) is closable and its closure \( \overline{S} \) is also symmetric. Moreover, \( (\overline{S})^{*} = S^{*} \). The deficiency indices of \( S \) are defined by \( d_{\pm}(S) = \dim \ker(S^{\pm} \pm i) \in \mathbb{N} \cup \{\infty\} \). \( S \) possesses a self-adjoint extension if and only if \( d_{-}(S) = d_{+}(S) \). This happens, for instance, if \( S \) is nonnegative or, more generally, lower semibounded. The latter means that \( \langle Sf,f \rangle_{H} \geq c\langle f,f \rangle_{H} \) for \( f \in \text{Dom}(S) \), where \( c \in \mathbb{R} \). The largest \( c \) satisfying the above condition is called the lower bound for \( S \).

It is well known that any densely defined symmetric and lower semibounded operator \( S \) has a self-adjoint extension which is also lower semibounded and preserves the lower bound. The construction of this operator, denoted \( S_{F} \), was given by Friedrichs in 1933. \( S_{F} \) is nowadays called the Friedrichs extension of \( S \).

The construction of \( S_{F} \) is based on the theory of (sesquilinear) forms. An important result in this theory says that for a given \((H,\langle \cdot,\cdot \rangle_{H})\) there is a one-to-one correspondence between the set of all densely defined Hermitian lower semibounded closed forms and the set of all self-adjoint lower semibounded operators on \( H \). If \( S \) is such a form, then in this correspondence \( A_{S} \) denotes the relevant operator; if \( A \)
is such an operator, then \( s_A \) denotes the relevant form. Moreover, for every \( A \) we have \( A_{(s_A)} = A \) and for every \( s \) we have \( s_{(A_s)} = s \).

More precisely, the associated operator \( A_s \) is defined by first determining its domain,

\[
\text{Dom}(A_s) = \{ h \in \text{Dom}(s): \exists u_h \in H \ \forall h' \in \text{Dom}(s) \ s[h,h'] = \langle u_h, h' \rangle \},
\]

and then by setting its action on \( h \in \text{Dom}(A_s) \) by \( A_s h = u_h \). See \([23, \text{Chapter 10}} \) and Section 3 of Chapter 12\]. Also recall, that closedness of a non-negative form \( s \) means that the norm \( \|x\|_s := (s[x,x] + \langle x, x \rangle)^{1/2} \) defined on \( \text{Dom}(s) \) is complete.

The construction of the Friedrichs extension now goes as follows. Let \( S \) be as above and let \( S[x,y] = \langle Sx, y \rangle_H \), \( x, y \in \text{Dom}(S) \), be the form associated to \( S \). It is immediately seen that \( s_S \) is densely defined Hermitian and lower semibounded. But more importantly, \( s_S \) is closable, see \([23, \text{Lemma 10.16}] \). Let \( \overline{s_S} \) be the closure of \( s_S \). Although the completion procedure in the definition of \( \overline{s_S} \) is abstract from its nature it can be shown that \( \overline{s_S} \) may be realized in \( H \), which means, in particular, that \( \text{Dom}(\overline{s_S}) \subset H \). Then, \( S_F \) is just \( A_{\overline{s_S}} \), the operator associated to \( \overline{s_S} \). See, for instance, \([23, \text{Definition 10.6}] \). Finally we add that for \( a \in \mathbb{R} \) we have \( (S+a)_F = S_{F+a} \).

Let \( d_-(S) = d_+(S) \). Then \( S \) is essentially self-adjoint, that is \( \overline{s} \) is the unique self-adjoint extension of \( S \), if and only if \( d_+(S) = 0 \). Moreover, if \( d_+(S) = n < \infty \), then the family of all self-adjoint extensions of \( S \) can be explicitly parametrized by the unitary group \( U(n) \).

Crucial for the study of deficiency indices and self-adjointness of Sturm-Liouville operators is nowadays the classical theory founded by H. Weyl. The reader is referred to \([23, \text{Chapter 15}] \), where the theory is presented with some limitations or, for instance, to \([26] \), where the theory is outlined in full generality. Recall that basic in this theory is the property of regularity and singularity of endpoints of the given interval for the considered Sturm-Liouville operator \( \mathcal{L} \), and this property remains unchanged for \( \mathcal{L} + a \), \( a \in \mathbb{R} \). Equally important is classification of endpoints into limit point (LP) case or limit circle (LC) case. Again, the property of being LP or LC case remains unchanged for \( \mathcal{L} + a \). Recall that regular points fall into the LC case.

The proposition that follows is a well known consequence of famous Weyl’s alternative for Sturm-Liouville operators so that the case of \( \mathcal{D}^\dagger \mathcal{D} + a \) is also covered. See, for instance, \([26, \text{pp.79 - 80}] \) or \([23, \text{Theorem 15.10}] \) (where the case of the Sturm-Liouville operator of the form \( -\frac{d^2}{dx^2} + s(x) \) is discussed).

**Proposition 2.** Let \( a \in \mathbb{R} \). The symmetric and lower semibounded operator \( \mathcal{D}^\dagger \mathcal{D} + a \) has deficiency indices:

\begin{itemize}
  \item[(2, 2)] if it is in the LC case at both endpoints;
  \item[(1, 1)] if it is in the LC case at one endpoint and in the LP case at the other endpoint;
  \item[(0, 0)] if it is in the LP case at both endpoints.
\end{itemize}

We shall make use of the following general result. It describes a natural self-adjoint extension of a symmetric operator on a separable Hilbert space for which there exists an orthonormal basis included in its domain and consisting of eigenfunctions of this operator.

**Lemma 4.1.** \([8, \text{Lemma 1.2.2}] \) Let \( S \) be a symmetric operator with domain \( \text{Dom}(S) \) on a separable Hilbert space \( H \). Assume \( \{f_n\} \) is an orthonormal basis in \( H \) contained in \( \text{Dom}(S) \) which consists of eigenfunctions of \( S \), \( Sf_n = \lambda_n f_n \), \( \lambda_n \in \mathbb{R} \). Then the
is self-adjoint and extends $S$. Moreover, $S$ is essentially self-adjoint.

Clearly, for any subspace $E$ of $\text{Dom}(S)$, $S$ extends $S|_{E}$. In the next section we shall use Lemma 4.1 for $E = C_{c}^{\infty}(I)$ with $I = (0, 1)$ or $I = (0, \infty)$, with $S$ being one of the considered differential operators satisfying the assumptions of Lemma 4.1 and such that $C_{c}^{\infty}(I) \subset \text{Dom}(S)$. All this will happen in an appropriate $L^{2}$ space as a Hilbert space.

5. Delta-Sobolev spaces. We now come to introducing and discussing the delta-Sobolev spaces. These spaces can be regarded in the broader context of weighted Sobolev spaces.

Let $\epsilon \in \mathcal{E}$ be arbitrary.

**Definition 5.1.** The delta-Sobolev space $W^{\epsilon}(I, w)$ is the space

$$W^{\epsilon}(I, w) = \{f \in L^{2}(w): D_{\text{weak}}^{\epsilon}f \text{ exists and is in } L^{2}(w)\}$$

equipped with the inner product

$$\langle f, g \rangle_{W^{\epsilon}(I, w)} = \langle f, g \rangle_{L^{2}(w)} + \langle D_{\text{weak}}^{\epsilon}f, D_{\text{weak}}^{\epsilon}g \rangle_{L^{2}(w)}.$$ 

The closure of $C_{c}^{\infty}(I)$ in $W^{\epsilon}(I, w)$ with respect to the norm generated by $\langle \cdot, \cdot \rangle_{W^{\epsilon}(I, w)}$ is denoted $W_{0}^{\epsilon}(I, w)$.

Specified to $\epsilon = (1)$, the $\partial$-Sobolev space is (we write $W^{\partial}$ rather than $W^{(1)}$)

$$W^{\partial}(I, w) = \{f \in L^{2}(w): D_{\text{weak}}^{\partial}f \text{ exists and is in } L^{2}(w)\}$$

and the associated norm is $\|f\|_{W^{\partial}(I, w)} = (\|f\|_{L^{2}(w)}^{2} + \|D_{\text{weak}}^{\partial}f\|_{L^{2}(w)}^{2})^{1/2}$. Since, $C_{c}^{\infty}(I) \subset W_{0}^{\partial}(I, w) \subset W^{\partial}(I, w)$, both $W_{0}^{\partial}(I, w)$ and $W^{\partial}(I, w)$ are dense in $L^{2}(w)$.

The following proposition has a relatively standard proof hence we omit it.

**Proposition 3.** The delta-Sobolev space $W^{\epsilon}(I, w)$ is a Hilbert space.

Consequently, also $W_{0}^{\epsilon}(I, w)$ is a Hilbert space. In the classic case of $\partial = \frac{d}{dx}$ ($w = p = 1, q = 0$), we have $D_{\text{weak}}^{\partial}f = -\frac{d_{w}^{2}}{dx}$, and the space

$$W^{(-1,1)}(I) = \{f \in L^{2}(I, dx): f'' \in L^{2}(I, dx)\}$$

is denoted in [19, Section 1.1.4] as $W_{2}^{2}(I)$. In the proposition below, as in [23], the symbol $H^{n}(J), n \geq 1$, means the classic Sobolev space on an open interval $J \subset \mathbb{R}$.

**Proposition 4.** Let $|\epsilon| = n$. For any bounded interval $J \subset I$ and every $f \in W^{\epsilon}(I, w)$ one has $f|_{J} \in H^{n}(J)$. In particular, for any bounded $I$ and $w \equiv 1$, $W^{\epsilon}(I) \subset H^{n}(I)$.

**Proof.** Let $f \in W^{\epsilon}(I, w)$ so that $g := D_{\text{weak}}^{\epsilon}f$ exists and is in $L^{2}(w)$. Then, the argument from the proof of Proposition 1 shows that $f = R_{\epsilon}g + G$, where $G \in C^{\infty}(I)$, $R_{\epsilon}g \in C^{n-1}(I)$, $(R_{\epsilon}g)(n-1) \in AC_{\text{loc}}(I)$, and $D_{\text{weak}}^{\epsilon}(R_{\epsilon}g) = g$. Hence, for any bounded $J \subset I$, $f|_{J} \in H^{n}(J)$. \hfill $\square$

The delta-Sobolev spaces are useful tools in describing explicitly domains of operators related to compositions of $\partial$ and $\partial^{\pm}$, and domains of the corresponding maximal operators. We begin with description of the conjugate operator $(D')^{\ast}$. Recall that $D' = \partial^{+} \circ \ldots \circ \partial^{n}$ and $\text{Dom}(D') = C_{c}^{\infty}(I)$.
Proposition 5. We have \( \text{Dom}((D^*)^*) = W^*(I, w) \) and
\[
(D^*)^* f = D_{\text{weak}}^* f \quad \text{for} \quad f \in W^*(I, w).
\]

**Proof.** It suffices to compare definition of \( \text{Dom}((D^*)^*) \) (based on the general theory),
\[
\text{Dom}((D^*)^*) = \{ f \in L^2(w) : \exists g \in L^2(w) \forall \varphi \in C_c^\infty(I) \langle D^* \varphi, f \rangle_{L^2(w)} = \langle \varphi, g \rangle_{L^2(w)} \},
\]
with definition of \( W^*(I, w) \). Equality \( (D^*)^* f = D_{\text{weak}}^* f \) is then a consequence of this comparison. \( \square \)

We now define the minimal and the maximal operators related to \( D^* \). These two operators are important because in case when \( \epsilon' = \epsilon \), i.e. when \( D^* \) is symmetric, self-adjoint extensions of \( D^* \) lie in between \( \{ D^* \geq 0 \} \) for \( \epsilon' = \epsilon \), so such extensions do exist.

Define \( D_{\text{min}}^* := \overline{D^*} \), the closure of \( D^* \) (we point out that \( D^* \) is closable since \( \text{Dom}((D^*)^*) \) is dense in \( L^2(w) \)); moreover, \( (D^*)^* = (\overline{D^*})^* \), and \( D_{\text{max}}^* \) as the operator defined on the domain \( \text{Dom}(D_{\text{max}}^*) = W^*(I, w) \) by
\[
D_{\text{max}}^* f = D_{\text{weak}}^* f.
\]

An easy argument shows that \( D_{\text{max}}^* \) is closed and hence \( D_{\text{min}}^* \subset D_{\text{max}}^* \). Indeed, assume that \( f_n \in W^*(I, w) \) and for some \( f, g \in L^2(w) \) we have \( f_n \to f \) and \( D_{\text{weak}}^* f_n \to g \). This means that \( \{ f_n \} \) and \( D_{\text{weak}}^* f_n \) are Cauchy sequences in \( L^2(w) \) and hence \( \{ f_n \} \) is a Cauchy sequence in \( W^*(I, w) \). By Proposition 3, there is \( F \in W^*(I, w) \) such that \( f_n \) converges to \( F \) in \( W^*(I, w) \). Consequently, \( f_n \to F \) in \( L^2(w) \) and hence \( f = F \in W^*(I, w) \). Since also \( D_{\text{weak}}^* f_n \to g \) in \( L^2(w) \), a simple argument then shows that \( g = D_{\text{weak}}^* f \).

**Proposition 6.** We have \( (D_{\text{min}}^*)^* = D_{\text{max}}^* \) and \( (D_{\text{max}}^*)^* = D_{\text{min}}^* \). Moreover, for \( \epsilon' = \epsilon \) every self-adjoint extension \( T \) of \( D^* \) satisfies \( D_{\text{min}}^* \subset T \subset D_{\text{max}}^* \).

**Proof.** We begin with the first equality and prove the inclusion \( (D_{\text{min}}^*)^* \subset D_{\text{max}}^* \).
Let \( f \in \text{Dom}(D_{\text{max}}^*) \). This means that \( D_{\text{weak}}^* f \) exists and belongs to \( L^2(w) \). In other words, it holds
\[
\forall \varphi \in C_c^\infty(I) \langle D^* \varphi, f \rangle_{L^2(w)} = \langle \varphi, D_{\text{max}}^* f \rangle_{L^2(w)},
\]
which means that \( f \in \text{Dom}((D^*)^*) = \text{Dom}((\overline{D^*})^*) \) and \( D_{\text{max}}^* f = (D^*)^* f = (\overline{D^*})^* f \).

To prove the opposite inclusion, \( (D_{\text{min}}^*)^* \subset D_{\text{max}}^* \), let \( f \in \text{Dom}((D_{\text{min}}^*)^*) \) and set \( g = (D_{\text{min}}^*)^* f \). This means, in particular, that
\[
\forall \varphi \in C_c^\infty(I) \langle D^* \varphi, f \rangle_{L^2(w)} = \langle \varphi, g \rangle_{L^2(w)}.
\]
Consequently, \( D_{\text{weak}}^* f \) exists and equals \( g \). Hence \( f \in \text{Dom}(D_{\text{max}}^*) \) and \( g = D_{\text{max}}^* f \).

Since \( D_{\text{min}}^* \) is closed, we have \( (D_{\text{min}}^*)^* = D_{\text{min}}^* \) and thus the second equality is a consequence of the first one. The last claim of the proposition is obvious. \( \square \)

Let \( \epsilon \in \mathcal{E} \) be anti-symmetric, \( \epsilon' = \epsilon \), that is \( \epsilon = (\tau', \tau) \) for some \( \tau \in \mathcal{E} \), \( |\tau| = n \), \( |\epsilon| = 2n \). Define the form \( t' \) on the domain \( W^*_{\tau}(I, w) \) by
\[
t'[f, g] = \int_I D_{\text{weak}}^* f(x) \overline{D_{\text{weak}}^* g(x)} w(x) \, dx.
\]
The form \( t' \) restricted to \( W^*_{\tau}(I, w) \) will be denoted as \( t'_{\tau} \). Thus \( \text{Dom}(t'_{\tau}) = W^*_{\tau}(I, w) \) and \( \text{Dom}(t') = W^*(I, w) \).
In what follows, for \( c' = c \), \( \mathbb{D}^{c'} \) and \( \mathbb{D}_0^{c} \) will mean the operators associated with the forms \( t^{c} \) and \( t_0^{c} \), respectively.

**Theorem 5.2.** Let \( c' = c \) and \( a \in \mathbb{R} \). The operators \( \mathbb{D}^{c'} + a \) and \( \mathbb{D}_0^{c} + a \) are self-adjoint and lower semibounded extensions of \( D^{c'} + a \). Moreover, \( \mathbb{D}_0^{c} + a \) is the Friedrichs extension of \( D^{c'} + a \).

**Proof.** Obviously, it suffices to consider the case \( a = 0 \). The form \( t^{c} \) is Hermitian and nonnegative. Moreover, it is closed and this fact is just a consequence of the completeness of the norm \( \| \cdot \|_{W_0^{c'}(I,w)} \). The same is valid for \( t_0^{c} \). By the general theory, \( \mathbb{D}^{c} \) and \( \mathbb{D}_0^{c} \) are self-adjoint and non-negative. We now check that they extend \( D^{c} \). It suffices to consider \( \mathbb{D}_0^{c} \) only. From the general definition

\[
\text{Dom}(\mathbb{D}_0^{c}) = \{ f \in W_0^{c'}(I,w) : \exists u_f \in L^2(w) \ \forall g \in W_0^{c'}(I,w) \ | t_0^{c}(f,g) = (u_f,g)_{L^2(w)} \},
\]

and \( \mathbb{D}_0^{c} f = u_f \). We claim that \( C_0^\infty(I) \subset \text{Dom}(\mathbb{D}_0^{c}) \) and \( \mathbb{D}_0^{c} \varphi = D^{c} \varphi \) for \( \varphi \in C_0^\infty(I) \). For this purpose we need to check that given \( \varphi \in C_0^\infty(I) \) for every \( g \in W_0^{c'}(I,w) \) it holds

\[
\int_I D^{c'} \varphi(x)\overline{D^{c}g(x)} w(x) \, dx = (D^{c'} \varphi,g)_{L^2(w)}.
\]

(5.1)

The proof of (5.1) goes by induction and is based on the integration by parts formula for absolutely continuous functions on a closed finite interval. Let \( J \subset I \) be such an interval with support of \( \varphi \) included in \( J \). We prove (5.1) with \( I \) replaced by \( J \).

For \( n = 1 \), assuming \( c = (-1,1) \) (there is no need to consider separately the case \( c = (1,-1) \)) and \( g \in W_0^{c'}(I,w) \) we must justify that

\[
\int_J \partial^{n+1} \varphi(x)\partial_{\text{weak}} g(x) w(x) \, dx = \int_J \partial^{n} \varphi(x)\overline{g(x)} w(x) \, dx.
\]

This is indeed the case, since by Theorem 3.4 the weak derivative \( g' \) exists on \( I \) and consequently \( g \) is absolutely continuous on \( J \). Moreover, \( \partial_{\text{weak}} g(x) = p(x)g'(x) + q(x)g(x) \) and application of the integration by parts formula plus a small calculation shows the required equality. Performing the induction step goes along the same lines (additionally we need to invoke here Proposition 1) so we skip the details.

It remains to prove that \( \mathbb{D}_0^{c} = (D^{c'})_F \). We take the form \( s^{c'} = (D^{c'} f,g)_{L^2(w)} \) on the domain \( \text{Dom}(s^{c'}) = C_0^\infty(I) \) and consider its closure \( \overline{s^{c'}} \). We now claim that

\[
\overline{s^{c'}} = t_0^{c}.
\]

(5.2)

This is enough for our purpose since then, with the notation of Section 4, we have

\[
\text{Dom}((D^{c'})_F) = \text{Dom}(A_{\overline{s^{c'}}}) = \text{Dom}(A_{t_0^{c}}),
\]

and as one immediately sees, the latter space coincides with \( \text{Dom}(\mathbb{D}_0^{c}) \). Moreover, it follows that \( (D^{c'})_F f = \mathbb{D}_0^{c} f \) for \( f \) from these joint domains. Returning to (5.2), we note that it is a consequence of the fact that \( C_0^\infty(I) \) lies densely in \( \text{Dom}(t_0^{c}) = W_0^{c'}(I,w) \) and \( t_0^{c} \) is closed. Here are details. Clearly \( t_0^{c} \) extends \( s^{c} \) and hence the inclusion \( \subset \) follows. For the opposite inclusion let \( f \in \text{Dom}(t_0^{c}) = W_0^{c'}(I,w) \) and take \( \{ \varphi_n \} \subset C_0^\infty(I) \) such that \( \varphi_n \to f \) in \( W_0^{c'}(I,w) \). In particular, this means that \( \varphi_n \to f \) and \( D^{c'} \varphi_n \to D^{c'} \varphi \) in \( L^2(w) \). We want to show that \( f \in \text{Dom}(\overline{s^{c'}}) \).

For this purpose it suffices (see [23, p.224]) to ensure existence of \( \{ \varphi_n \} \subset C_0^\infty(I) \) convergent to \( f \) in \( L^2(w) \) and such that \( s^{c'}(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0 \) as \( n, m \to \infty \). But

\[
s^{c'}(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = (D^{c'}(\varphi_n - \varphi_m), (\varphi_n - \varphi_m)_{L^2(w)}) = (D^{c}(\varphi_n - \varphi_m), D^{c'}(\varphi_n - \varphi_m))_{L^2(w)}
\]
and the latter required convergence to 0 follows since \( D^\tau \varphi_n \) being convergent in \( L^2(w) \) is a Cauchy sequence there. \( \square \)

In the classic case of \( \mathfrak{d} = \frac{d}{dx} \) \((w = p \equiv 1, q \equiv 0)\), we have \( W^\varnothing(I) = H^1(I) \) and \( W_0^\varnothing(I) = H^1_0(I) \), and for \( \epsilon = (\epsilon_1, 1) \), \( D^\epsilon \) and \( D_0^\epsilon \), being self-adjoint extensions of \( -\frac{d^2}{dx^2} = \mathfrak{d}^+ \circ \mathfrak{d} \), are the Neumann and Dirichlet Laplacians on \( I \), respectively. Up to this point we assumed that the delta-derivative \( \mathfrak{d} \) is fixed. Then, given \( \epsilon \in \mathcal{E} \), we can consider \( W^\epsilon(I, w) \) as to be associated to the concrete operator, namely \( L := D^\epsilon \). It is interesting to observe that in case when \( \epsilon = \epsilon' \), definition of \( W^\varnothing(I, w) \), now assumed to be associated to \( L \), is independent on the realization of \( L \) as a composition of deltas of the same length. Precisely, if \( L = D^\epsilon_j = \mathfrak{d}^+ \circ \cdots \circ \mathfrak{d}^+ \), \( j = 1, 2 \), and \( \mathfrak{d} = \mathfrak{d}_{\{w, p_j, q_j\}} \), then, for \( \epsilon = \epsilon' \), \( D^\epsilon_{\text{weak}} \) and thus \( W^\epsilon_j(I, w) \), coincide for \( j = 1, 2 \), and the same is for \( W_{0,j}^\epsilon(I, w) \), \( j = 1, 2 \). Consequently, in the same situation, the self-adjoint extensions \( D^\epsilon_j \) and \( D_{0,\epsilon} \) of \( D^\epsilon_j \), coincide for \( j = 1, 2 \).

6. **Two auxiliaries.** Throughout the discussion of examples in Section 7 we shall use a device discovered by Hajmirzaahmad [11, 12]. She noted that the constraint \( \alpha > -1 \) on the type parameter \( \alpha \), usually assumed in the harmonic analysis of Laguerre polynomial expansions, may be removed. The same was done with the constraints \( \alpha > -1, \beta > -1 \) in the case of Jacobi expansions. The idea is clever and natural. It relies (in the Laguerre case), roughly, on reflecting the setting corresponding to \( \alpha > -1 \) by a use of an appropriate intertwining operator to cover the case \( \alpha < 1 \). This idea may be easily adapted in other settings of orthogonal expansions, see [21, 1], and we do this in our discussion. Moreover, the idea may be also adapted in continuous orthogonal settings; we comment this in Subsection 6.1.

To make the discussion of examples in Section 7 relatively close, in the next two subsections we first put the Hajmirzaahmad idea in an abstract framework (we believe, with some profit to propagate this idea) and then comment on “Liouville form” of a Sturm-Liouville operator.

6.1. **Abstract framework of Hajmirzaahmad idea.** Hajmirzaahmad idea is based on the following simple observation. Suppose \( L_j, j = 1, 2 \), are two symmetric operators on Hilbert spaces \( H_j \). Suppose also, that there exists a unitary isomorphism \( V : H_1 \to H_2 \) such that \( L_1 \circ V = V \circ (L_2 - c) \), for some real constant \( c \). Then spectral properties of \( L_1 \) and \( L_2 - c \), equivalently \( L_1 \) and \( L_2 \), coincide.

In fact in our examples we shall work with families of operators. Therefore we examine specific examples of such one-parameter or two-parameter families where the required unitary isomorphisms naturally exist. Assume that we have a one-parameter family \( \{L_\tau\}, \tau \in \mathbb{R} \), of Sturm-Liouville operators in divergent form

\[
L_\tau f = \frac{1}{v^{2\tau+1}} \left( (-v^{2\tau+1} f')' + sf \right),
\]

where \( v \) and \( s \) are given and \( I \) is fixed. Denote \( d\mu_\tau(x) = v(x)^{2\tau+1} dx \). (Writing the exponent of \( v \) in the form \( 2\tau + 1 \) we adjust to the setting of examples considered below.) In this notation the mapping \( V_\tau : L^2(\mu_\tau) \to L^2(\mu_\tau) \), \( V_\tau f = v^{-2\tau} f \), is a unitary isomorphism. It is easily checked that

\[
L_\tau (V_\tau f) = V_\tau (L_{-\tau} f - 2\tau \frac{v''}{v} f),
\]
Therefore, if for some $a \succ a$ for some $\tau$ type parameter and for $L_c$ plicity, that $L \in a > a$ bases apply in the following sense: if for some $\tau, \rho = 0$, and for every $\tau > -a$ bases as above exist in $L^2(\mu_\tau)$ and for $L_\tau$, then such bases also exist for every $\tau < a$ and hence the whole range of type parameter $\tau$ is covered with this respect.

Analogous observation can be done in the continuous setting. Assume, for simplicity, that $c_\nu = 0$, that is $V_\tau$ intertwines $L_\tau$ and $L_{-\tau}$ for every $\tau$. Assume also that for some $a > 0$ and for every $\tau > -a$ we have at our disposal a unitary isomorphism $H_\tau: L^2(\mu_\tau) \to L^2(\mu_\tau)$ such that $H_\tau(L_\tau f) = \lambda(\tau)H_\tau f$ for $f \in C_c^\infty(I)$, where $\lambda$ is a function of the parameter $\tau > -a$. It follows that for every $\tau < a$ the mapping

$$H_\tau^{\text{exo}} := V_\tau \circ H_\tau \circ V_{-\tau}$$

is a unitary isomorphism on $L^2(\mu_\tau)$ and, moreover, $H_\tau^{\text{exo}}(L_\tau f) = \lambda(-\tau)H_\tau^{\text{exo}} f$ for $f \in C_c^\infty(I)$.

A similar situation occurs in the two-parameter case of operators

$$L_{\tau, \rho} f = \frac{1}{v_1^{2\tau+1}v_2^{2\rho+1}} \left( - (v_1^{2\tau+1}v_2^{2\rho+1}f')' + sf \right),$$

$\tau, \rho \in \mathbb{R}$, where $v_1, v_2$ and $s$ are given. Then denoting $d\mu_{\tau, \rho}(x) = v_1(x)^{2\tau+1}v_2(x)^{2\rho+1} dx$ we have the unitary isomorphism

$$V_{\tau, \rho}: L^2(\mu_{-\tau, -\rho}) \to L^2(\mu_{\tau, \rho}), \quad V_{\tau, \rho} f = v_1^{-2\tau}v_2^{-2\rho} f,$$

and assuming $v_1, v_2$ satisfy

$$\frac{v_1''}{v_1} \equiv c_{v_1}, \quad \frac{v_2''}{v_2} \equiv c_{v_2},$$

on $I$, we have

$$L_{\tau, \rho} \circ V_{\tau, \rho} = V_{\tau, \rho} \circ (L_{-\tau, -\rho} - c_{\tau, \rho, v_1, v_2}), \quad c_{\tau, \rho, v_1, v_2} = 2\tau c_{v_1} + 2\rho c_{v_2}. \quad (6.4)$$

Analogously, using the unitary isomorphism

$$V^{(1)}_{\tau}: L^2(\mu_{-\tau, -\rho}) \to L^2(\mu_{\tau, \rho}), \quad V^{(1)}_{\tau} f = v_1^{-2\tau} f,$$

we have $L_{\tau, \rho} \circ V^{(1)}_{\tau} = V^{(1)}_{\tau} \circ (L_{-\tau, -\rho} - c_{\tau, 0, v_1, v_2})$, and similarly when the roles of $\tau$ and $\rho$ are changed and $V^{(2)}_{\rho}$ is considered. Consequently, remarks concerning orthonormal bases apply in the following sense: if for some $a > 0$ and $b > 0$ and every $\tau > -a$ and $\rho > -b$ there exists an orthonormal basis in $L^2(\mu_{\tau, \rho})$ consisting of eigenfunctions of $L_{\tau, \rho}$, then bases with analogous properties also exist for every $\tau$ and $\rho$.

Coming back to condition (6.1) we note that the general solution of (6.1) is: $ax + b$ for $c_v = 0$, $a \sinh(\sqrt{c_v}x) + b \cosh(\sqrt{c_v}x)$ for $c_v > 0$, and $a \sin(\sqrt{-c_v}x) + b \cos(\sqrt{-c_v}x)$ for $c_v < 0$; since $v > 0$ on $I$, we additionally assume that $a, b$ are such that the corresponding expressions are positive on $I$ (in particular, $a^2 + b^2 > 0$). Two out of these three types of weights will appear in our examples. (The cases of $v = \cos x/2$
or, more generally, \( v_1 = \cos x/2 \) and \( v_2 = \sin x/2 \), appear in analysis of operators associated to ultraspherical or Jacobi polynomial expansions, respectively.)

6.2. Liouville form. Although the term ‘Liouville (normal) form’ seems to be reserved for differential equations (see, for instance, [9, Sections 7 and 12]) we adopt it here in the operator theory context.

It frequently happens that in the case of a weight \( w \neq 1 \) on \( I \), together with the operator \( L_{1,\varphi} \), acting in \( L^2(w) \), an associated operator, which is its replique in the \( L^2(dx) \) setting through the unitary isomorphism \( U : L^2(w) \rightarrow L^2(dx) \), \( Uf = \sqrt{w}f \), is considered. We have in mind the operator

\[
L_{w,r,s}^0 := U \circ L_{w,r,s} \circ U^{-1}.
\]

The above equality says that \( L_{w,r,s}^0 \) and \( L_{w,r,s} \) are intertwined by \( U \). A computation then shows that

\[
L_{w,r,s}^0 = L_{\hat{r},\hat{s}},
\]

with \( \hat{r} = \frac{r}{w} \) and

\[
\hat{s} = \frac{s}{w} + \frac{1}{2} \frac{w''}{w} - \frac{3}{4} \frac{r(w')^2}{w^3} + \frac{1}{2} \frac{r'w'}{w^2}.
\]

In particular, if \( r = w \), then \( \hat{r} = 1 \) and

\[
\hat{s} = \frac{s}{w} + \frac{1}{2} \frac{w''}{w} - \frac{1}{4} \frac{(w')^2}{w^2},
\]

so that \( L_{w,w,s}^0 \) becomes \( L_{1,1,s} \), a Schrödinger operator with potential \( \hat{s} \).

Clearly, if \( L_{w,r,s} = \mathfrak{d} + a \), then \( L_{w,r,s}^0 = \mathfrak{d}^0 + \mathfrak{d}^0 + a \) with

\[
\mathfrak{d}^0 = U \circ \mathfrak{d} \circ U^{-1}, \quad \mathfrak{d}^0 + a = U \circ \mathfrak{d} + U^{-1}.
\]

(6.5)

For an operator in divergent form, \( L_{w,w,s} \), these recipes transform to

\[
\mathfrak{d}^0 = \frac{d}{dx} - \frac{w'}{2w}, \quad \mathfrak{d}^{0,+} = -\frac{d}{dx} - \frac{w'}{2w}.
\]

We pause for a moment to remark that an insight into the relevant definitions immediately shows the following relation between the delta-Sobolev and delta\(^0\)-Sobolev spaces: for \( \epsilon \in \mathcal{E} \) we have \( W^{\epsilon,\epsilon}(I, dx) = U(W^\epsilon(I, w)) \). Consequently, the operators \( \mathfrak{D}^\epsilon \) and \( \mathfrak{D}^{0,\epsilon} \) are conjugated by \( U \), \( \mathfrak{D}^{0,\epsilon} = U \circ \mathfrak{D}^\epsilon \circ U^{-1} \).

Also spectral properties of \( L_{w,r,s} \) and \( L_{w,r,s}^0 \) coincide. In particular, if \( \{ \phi_n \} \) is an orthonormal basis in \( L^2(w) \) consisting of eigenfunctions of \( L_{w,r,s} \), then \( \{ U\phi_n \} \) is an orthonormal basis in \( L^2(dx) \) consisting of eigenfunctions of \( L_{w,r,s}^0 \) and the eigenvalues in both cases are the same. Moreover, if \( L_{w,r,s} \) and \( L_{w,r,s}^0 \) denote the differential operators given by the corresponding differential expressions and with domains \( C_c^\infty(I) \) (so that \( U(\text{Dom}(L_{w,r,s})) = \text{Dom}(L_{w,r,s}^0) \)), then spectral characteristics of both operators (for instance, deficiency indices and hence essential self-adjointness) are identical and self-adjoint extensions of both operators are in a 1–1 correspondence through the conjugation with \( U \). The latter statement precisely means that \( L \) with domain \( \text{Dom}(L) \) is a self-adjoint extension of \( L_{w,r,s} \) if and only if \( U \circ L \circ U^{-1} \) with domain \( U(\text{Dom}(L)) \) is a self-adjoint extension of \( L_{w,r,s}^0 \). Finally we observe that also the self-adjoint extensions of \( L_{w,r,s} \) and \( L_{w,r,s}^0 \) that result from Theorem 5.2 are conjugated: \( \mathfrak{D}^0_{w,r,s} = U \circ \mathfrak{D}_{w,r,s} \circ U^{-1} \) and \( \text{Dom}(L_{w,r,s}^0) = U(\text{Dom}(L_{w,r,s})) \) and similarly for \( L_{w,r,s}^0 \) and \( L_{w,r,s}^0 \).

The association \( L_{w,r,s} \rightarrow L_{w,r,s}^0 \) will appear in the discussed examples.
7. Examples. We now illustrate the theory by considering a number of examples of ordinary second order differential operators that appear in harmonic analysis of (continuous) Bessel transforms and Fourier-Jacobi transforms or (discrete) Fourier-Bessel and Laguerre expansions. For every such operator $L$ acting within a natural $L^2$ space, we furnish the decomposition (well known and widely used in a number of papers concerning the beforementioned settings)

$$L = \mathfrak{d} + a,$$

(7.1)

with appropriate delta-derivative and $a \in \mathbb{R}$. Each of these operators is therefore of the form $D^2 + a$ with $\epsilon = (-1, 1)$ hence Theorem 5.2 applies and results in obtaining the corresponding self-adjoint extensions of $L$ denoted in the spirit of notation from Theorem 5.2, here and later on, by $L$ and $L_0$. In each of the considered examples, and operators labeled by $\alpha \in \mathbb{R}$ in the Laguerre case, $\nu \in \mathbb{R}$ in the Bessel case or by $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ in the Jacobi function case (sometimes $\mathbb{R}$ will be naturally reduced to $[0, \infty)$), we characterize the set of these involved type parameters for which the corresponding operator is essentially self-adjoint. Moreover, in each case when it is not so, we furnish two different concrete self-adjoint extensions (of course there are infinitely many of them).

Relevant decompositions of operators considered below have a long history and it is impossible to cite here full list of references. The reader is referred to [22, Section 7] and the references therein. Here for the divergent form operators we apply the principles described after (2.2) and for operators in Liouville form we use (6.5). We also mention that we borrow the concept of notation from [21] and use the labels cls and exo to distinguish between classic and exotic self-adjoint extensions (and other objects). In addition we shall use the mathbb font to denote self-adjoint extensions resulting from Theorem 5.2. Thus the operators corresponding to $B_\nu$, $B_\nu^\circ$, $L_\alpha$, $L_\alpha^\circ$, $J_{\alpha, \beta}$, etc. are $\mathbb{B}_\nu$, $\mathbb{B}_\nu^\circ$, $\mathbb{L}_\alpha$, $\mathbb{L}_\alpha^\circ$, $\mathbb{J}_{\alpha, \beta}$, etc., and similarly with the additional 0 subscript.

7.1. Bessel operators $B_\nu$ and $B_\nu^\circ$. The first of these two operators is

$$B_\nu = -\frac{d^2}{dx^2} - \frac{2\nu + 1}{x} \frac{d}{dx} = -\frac{1}{x^{2\nu+1}} \frac{d}{dx} \left( x^{2\nu+1} \frac{d}{dx} \right), \quad \nu \in \mathbb{R},$$

(7.2)

with $\text{Dom}(B_\nu) = C^\infty_0(I)$, discussed in [9, Section 11], that can be considered on $L^2(d\mu_\nu) := L^2(I, x^{2\nu+1} dx)$, where $I = (0, \infty)$ or $I = (0, 1)$. In both cases of $I$ decomposition (7.1) holds for $L = B_\nu$ with $a = 0$ and the associated delta-derivative $\mathfrak{d}_\nu$ and its formal adjoint $\mathfrak{d}^+_\nu$ given by

$$\mathfrak{d}_\nu = \frac{d}{dx}, \quad \mathfrak{d}^+_\nu = -\frac{d}{dx} + \frac{2\nu + 1}{x}.$$

The considered case corresponds to $v(x) = x$, see Subsection 6.1, and hence $c_\nu = 0$. Therefore the unitary isomorphism

$$V_\nu : L^2(d\mu_{-\nu}) \rightarrow L^2(d\mu_\nu), \quad V_\nu f = (\cdot)^{-2\nu} f,$$

(7.3)

intertwines $B_\nu$ and $B_{-\nu}$. Consequently, spectral properties of $B_{-\nu}$ and $B_\nu$ coincide. This is indeed confirmed by a direct analysis. In case of $I = (0, \infty)$, according to endpoint classification in $L^2(d\mu_\nu)$, see [9, Section 11], the endpoint $\infty$ is LP for any $\nu \in \mathbb{R}$, while the endpoint 0 is: LP for $|\nu| \geq 1$, and is LC for $|\nu| < 1$. Thus, $B_\nu$ has deficiency indices $(0, 0)$ for $|\nu| \geq 1$ and $(1, 1)$ otherwise. In case of $I = (0, 1)$ endpoint classification in $L^2(d\mu_\nu)$ is not furnished in [9], however it is clear that also in this case properties of the endpoint 0 coincide with these presented above while
the endpoint 1 is regular. In particular, in both cases, $B_\nu$ is essentially self-adjoint if and only if $|\nu| \geq 1$ and then the extension $\mathbb{B}_\nu = \mathbb{B}_{\nu,0}$ that results from Theorem 5.2 is the unique self-adjoint extension of $B_\nu$.

We now furnish constructions of self-adjoint extensions of $B_\nu$ in terms of spectral resolutions based on either related integral transforms or orthogonal expansions. We discuss the cases $I = (0, \infty)$ and $I = (0,1)$ separately.

A) $I = (0, \infty)$. Recall that for any $\nu \in \mathbb{R}$, the functions $\varphi_\nu'(x) = J_\nu(xy)/(xy)^\nu$, $y \in \mathbb{R}_+$, are eigenfunctions of $B_\nu$, $B_\nu \varphi_\nu' = y^2 \varphi_\nu'$. Here $J_\nu$ denotes the Bessel function of the first kind of order $\nu \in \mathbb{R}$; see also [21, Section 4]. Clearly, $B_\nu^\text{cls}$ and $B_\nu^\text{exo}$ coincide with $\mathbb{B}_\nu = \mathbb{B}_{\nu,0}$ for $\nu \geq 1$ and $\nu \leq -1$, respectively.

In the overlapping case $-1 < \nu < 1$, $B_\nu^\text{cls}$ and $B_\nu^\text{exo}$ are two different self-adjoint extensions of $B_\nu$, unless $\nu = 0$. In the latter case we have $B_0^\text{cls} = B_0^\text{exo}$ but an additional argument allows to construct $\tilde{B}_0^\text{cls}$, a self-adjoint extension of $B_0$, different from $B_0^\text{cls}$; see the appendix, Section 8.

B) $I = (0,1)$. Given $\nu > -1$ let $\lambda_\nu := \lambda_n, n \in \mathbb{N}_+$, denote the sequence of successive positive zeros of $J_\nu(x)$. Then the functions

$$\phi_n'(x) = c_{n,\nu} J_\nu(\lambda_n x)/(\lambda_n x)^\nu, \quad c_{n,\nu} = \sqrt{2} |J_{\nu+1}(\lambda_n)|^{-1} \lambda_n,$$

$n \in \mathbb{N}_+$, form a complete orthonormal system in $L^2(d\mu_\nu)$ and are eigenfunctions of $B_\nu$, $B_\nu \phi_n' = \lambda_n^2 \phi_n'$. Thus, in the case $\nu > -1$, Lemma 4.1 applied to $S$ given by the differential expression in (7.2) with domain

$$\text{Dom}(S) = \{ f \in C^2(0, \infty) \cap L^2(d\mu_\nu) : - f'' - \frac{2\nu + 1}{x} f' \in L^2(d\mu_\nu) \},$$

and to the orthonormal basis $\{ \phi_n' \}_{n \in \mathbb{N}_+}$, with the choice of $E := C_c^\infty(0,1)$, results in the self-adjoint extension $S$ of $B_\nu$ now denoted $B_\nu^\text{cls}$.

In the case $\nu < 1$, using properties of $V_\nu$ it follows that for $\phi_{n,\nu}^\text{exo} := V_\nu \phi_{n,\nu}$, one has $B_\nu(\phi_{n,\nu}^\text{exo}) = \lambda_n^2 \phi_{n,\nu}^\text{exo}$ and $\{ \phi_{n,\nu}^\text{exo} \}_{n \in \mathbb{N}_+}$ is an orthonormal basis in $L^2(d\mu_\nu)$. Applying Lemma 4.1 to $S$ and $E$ as above, and to the orthonormal basis
\{\phi^{v, \text{exo}}_n\}_{n \in \mathbb{N}_+}, \) results in the self-adjoint extension \( S \) of \( B_\nu \) denoted \( B^{exo}_\nu \). Clearly, \( B^{\text{cls}}_\nu \) and \( B^{exo}_\nu \) coincide with \( B_\nu \) for \( \nu \geq 1 \) and \( \nu \leq -1 \), respectively.

In the overlapping case \(-1 < \nu < 1\), \( B^{\text{cls}}_\nu \) and \( B^{exo}_\nu \) are two different self-adjoint extensions of \( B_\nu \), unless \( \nu = 0 \). In the latter case we have \( B^{\text{cls}}_0 = B^{exo}_0 \) but, similarly to the case \( A \), an additional argument allows to construct \( \hat{B}^{\text{cls}}_0 \), a self-adjoint extension of \( B_0 \), different from \( B^{\text{cls}}_0 \), see the Appendix.

We now pass to the operator \( B'_\nu \). According to the procedure from Subsection 6.2 applied to \( w = w_\nu = x^{2\nu+1} \) and \( U = U_\nu \), this operator is

\[
B'_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad \nu \in \mathbb{R}, \tag{7.4}
\]

with \( \text{Dom}(B'_\nu) = C^\infty_c(I) \). It can be considered on \( L^2(dx) := L^2(I, dx) \), where \( I = (0, \infty) \) or \( I = (0, 1) \). The operator was discussed in [22, Section 7.8] (in the context of \( L^2((0, 1), dx) \)) and in [9, Section 12]. Since \( B^2_\nu = B'_\nu \), we can restrict \( \nu \) to \( \nu \geq 0 \). (For a connection, with respect to the Liouville normal form, between the left-hand sides of (7.4) and (7.2) see [9, p. 285].)

A direct computation based on (6.5) shows that

\[
\vartheta^\nu = \frac{d}{dx} - \frac{\nu + 1/2}{x}, \quad \vartheta^{\nu,+} = -\frac{d}{dx} - \frac{\nu + 1/2}{x}.
\]

Note that for \( \nu \geq 0 \) we have \( B^2_\nu = U_\nu \circ B^{\text{cls}}_\nu \circ U_\nu \) as well, and hence decomposition (7.1) for \( B^0_\nu \) also holds with \( \vartheta^{\nu,-} \) and \( \vartheta^{\nu,+} \), and \( a = 2(-\nu + 1) \).

For \( \nu \geq 0 \) endpoint classification and deficiency indices of \( B^0_\nu \) coincide with those of \( B_\nu \). See also [23, Example 15.2], where the case of \( I = (0, \infty) \) is discussed in this matter. In particular, \( B^0_\nu \) is essentially self-adjoint if and only if \( \nu \geq 1 \) and, in this case, \( \mathbb{B}^0_\nu = \mathbb{B}^0_{\nu,0} \) is the unique self-adjoint extension of \( B^0_\nu \). Also realizations of self-adjoint extensions of \( B^0_\nu \) are derived from those for \( B_\nu \) through the conjugation with \( U_\nu \), that is \( B^{\nu, \text{cls}/\text{exo}}_\nu := U_\nu \circ B^{\text{cls}/\text{exo}}_\nu \circ U_\nu \). Clearly, for \( 0 < \nu \leq 1 \) we have \( B^{\nu, \text{exo}}_\nu \neq B^{\nu, \text{cls}}_\nu \) and for \( \nu > 1 \), \( B^{\nu, \text{exo}}_\nu \) and \( B^{\nu, \text{cls}}_\nu \) coincide with \( \mathbb{B}^0_\nu \). Moreover, for \( \nu = 0 \), \( B^{\nu, \text{cls}}_0 := U_0 \circ B^{\nu, \text{exo}}_0 \circ U_0 \) is a self-adjoint extension of \( B^0_0 \) different from \( B^{\text{cls}}_0 \).

Finally we remark that constructions of self-adjoint extensions of \( B^0_\nu \) may be also obtained in a 'direct' way in terms of spectral resolutions based on either related integral transforms or orthogonal expansions. We have in mind the (non-modified) Hankel transform \( \mathcal{H}_\nu := U_\nu \circ H_\nu \circ U_{-\nu} \) (see [3, (1.3)]) and its exotic version \( \mathcal{H}^{exo}_\nu := U_\nu \circ H^{exo}_\nu \circ U_{-\nu} \) in the case of \( I = (0, \infty) \), or the orthonormal bases \( \{\psi^\nu_k\}_{k \in \mathbb{N}_+} \), where \( \psi^\nu_k := U_\nu \phi^\nu_k \) (see [22, Section 7.8]) and \( \{\psi^{\nu, \text{exo}}_k\}_{k \in \mathbb{N}_+} \), where \( \psi^{\nu, \text{exo}}_k := U_\nu \phi^{\nu, \text{exo}}_k \).

### 7.2. Laguerre operators \( L_\alpha \) and \( L^\alpha_0 \)

The first of these two operators is

\[
L_\alpha = -\frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx} + x^2 = \frac{1}{x^{2\alpha+1}} \left( -\frac{d}{dx} \left( x^{2\alpha+1} \frac{d}{dx} \right) + x^{2\alpha+3} \right), \quad \alpha \in \mathbb{R}, \tag{7.5}
\]

with \( \text{Dom}(L_\alpha) = C^\infty_c(0, \infty) \), that acts on \( L^2(dx) := L^2((0, \infty), x^{2\alpha+1} dx) \), and was discussed in [22, Section 7.6], under the restriction \( \alpha > -1 \), in connection with harmonic analysis of Laguerre expansions of convolution type. Decomposition (7.1) for \( L = L_\alpha \) holds with \( a = 2(\alpha + 1) \) and

\[
\vartheta_\alpha = \frac{d}{dx} + x, \quad \vartheta^{\alpha,+} = -\frac{d}{dx} - \frac{2\alpha + 1}{x} + x.
\]
Although endpoint classification of $L_\alpha$ in $L^2(d\mu_\alpha)$ is not discussed in [9], it may be checked that the endpoint $\infty$ is LP for any $\alpha$, while the endpoint 0 is: LP for $|\alpha| \geq 1$, regular for $-1 < \alpha < 0$ and is LC for $0 \leq \alpha < 1$. Thus, $L_\alpha$ has deficiency indices $(0,0)$ for $|\alpha| \geq 1$ and $(1,1)$ otherwise. In particular, $L_\alpha$ is essentially self-adjoint if and only if $|\alpha| \geq 1$ and, in this case, $L_\alpha$ is its unique self-adjoint extension.

When it comes to constructing self-adjoint extensions of $L_\alpha$ in terms of spectral resolutions based on an orthonormal basis of eigenfunctions, it is well known that for $\alpha > -1$ the Laguerre functions $\{l_\alpha^n\}_{n \in \mathbb{N}}$ (see [22, Section 7.6]), form an orthonormal basis in $L^2(d\mu_\alpha)$ and $L_\alpha l_\alpha^n = \lambda_{n,\alpha} l_\alpha^n$, $\lambda_{n,\alpha} = 2(2n+\alpha+1)$. Thus, Lemma 4.1 applied to $S$ given by the differential expression in (7.5) with

$$\text{Dom}(S) = \{f \in C^2(0, \infty) \cap L^2(d\mu_\alpha): -f'' - \frac{2\alpha + 1}{x} f' + x^2 f \in L^2(d\mu_\alpha)\},$$

and to the orthonormal basis $\{l_\alpha^n\}_{n \in \mathbb{N}}$, with the choice of $E := C_c^\infty(0, \infty)$, results in the self-adjoint extension $\hat{S}$ of $L_\alpha$ now denoted $L_\alpha^{\text{cls}}$.

For the case $\alpha < 1$ we first note that the unitary isomorphism $V_\alpha$ ((7.3) with $\nu$ replaced by $\alpha$) intertwines $L_\alpha$ and $L_{-\alpha}$, $L_{-\alpha} \circ V_\alpha = V_\alpha \circ L_{-\alpha}$. Thus for $\alpha < 1$ the system $l_\alpha^{\alpha,\text{exo}} = V_\alpha l_{-\alpha}^\alpha$, $n \in \mathbb{N}$, is an orthonormal basis in $L^2(d\mu_\alpha)$ and one has $L_\alpha l_\alpha^{\alpha,\text{exo}} = \lambda_{n,\alpha} l_\alpha^{\alpha,\text{exo}}$. By applying again Lemma 4.1 to $S$ and $E$ as above, and the orthonormal basis $\{l_\alpha^{\alpha,\text{exo}}\}_{n \in \mathbb{N}}$, results in a self-adjoint extension of $L_\alpha$ denoted $L_\alpha^{\alpha,\text{exo}}$.

Clearly, $L_\alpha^{\text{cls}}$ and $L_\alpha^{\alpha,\text{exo}}$ coincide with $L_{\alpha,0} = L_{\alpha,0}^{\alpha,\text{exo}}$ for $\alpha \geq 1$ and $\alpha \leq -1$, respectively.

In the overlapping case $-1 < \alpha < 1$, $L_\alpha^{\text{cls}}$ and $L_\alpha^{\alpha,\text{exo}}$ are two different self-adjoint extensions of $L_\alpha$, unless $\alpha = 0$. In the latter case we have $L_0^{\text{cls}} = L_0^{\alpha,\text{exo}}$, but $L_0^{\text{cls}} \neq L_0^{\alpha,\text{exo}}$.

We now pass to the associated operator $L_\alpha^\circ$. According to the procedure from Subsection 6.2 applied to $w = w_\alpha = x^{2\alpha+1}$ and $U = U_\alpha = x^{\alpha+1/2}$, this operator is

$$L_\alpha^\circ = -\frac{d^2}{dx^2} + \frac{\alpha^2 - 1/4}{x^2} + 1^2, \quad \alpha \in \mathbb{R},$$

(7.6)

with $\text{Dom}(L_\alpha^\circ) = C^\infty_c(0, \infty)$, that acts on $L^2(dx) = L^2((0, \infty), dx)$ and was discussed, under the restriction $\alpha > -1$, in [22, Section 7.5] in connection with harmonic analysis of *Laguerre expansions of Hermite type*. Since $L_{-\alpha} = L_\alpha^\circ$, we can restrict $\alpha$ to $\alpha \geq 0$.

A direct computation based on (6.5) shows that

$$\mathcal{L}_\alpha = \frac{d}{dx} + x - \frac{1}{x}(\alpha + \frac{1}{2}), \quad \mathcal{L}_{\alpha}^{\alpha, +} = -\frac{d}{dx} + x - \frac{1}{x}(\alpha + \frac{1}{2}).$$

Note, however, that for $\alpha \geq 0$ we have $L_\alpha = U_{-\alpha} \circ L_{-\alpha} \circ U_\alpha$ as well, and hence decomposition (7.1) for $L_\alpha^\circ$ also holds with $\mathcal{L}_\alpha$ and $\mathcal{L}_\alpha^{\alpha, +}$, and $a = 2(-\alpha + 1)$.

According to the discussion in Subsection 6.2 endpoint classification and deficiency indices of $L_\alpha^\circ$ and $L_\alpha$ coincide in the range $\alpha \geq 0$. In particular, for $\alpha \geq 0$, $L_\alpha^\circ$ is essentially self-adjoint if and only if $\alpha \geq 1$ and, in this case, $\mathcal{L}_\alpha^\circ = \mathcal{L}_{\alpha,0}^\circ$ is the unique self-adjoint extension of $L_\alpha^\circ$. Also realizations of self-adjoint extensions of $L_\alpha^\circ$ are derived from those for $L_\alpha$ through the conjugation with $U_\alpha$, that is $L_{\alpha,0}^{\text{cls/exo}} := U_\alpha \circ L_\alpha^{\text{cls/exo}} \circ U_{-\alpha}$. 'Direct' constructions of self-adjoint extensions of $L_\alpha^\circ$ are obtained in terms of spectral resolutions based on orthonormal bases $\{\varphi_\alpha^{\alpha,\text{exo}}\}_{n \in \mathbb{N}}$, where $\varphi_\alpha^{\alpha} := U_\alpha l_\alpha^n$ (see [22, Section 7.5]) and $\{\varphi_\alpha^{\alpha,\text{exo}}\}_{n \in \mathbb{N}}$, where $\varphi_\alpha^{\alpha,\text{exo}} := U_\alpha l_\alpha^{\alpha,\text{exo}}$.

7.3. Laguerre operators $L_\alpha^\circ$ and $L_\alpha^{\alpha,\circ}$. The first of these two operators is

$$L_\alpha^\circ = -x \frac{d^2}{dx^2} - (\alpha + 1 - \frac{1}{x}) \frac{d}{dx} = -\frac{1}{x^{\alpha+1} e^{-x} \frac{d}{dx}} \left(x^{\alpha+1} e^{-x} \frac{d}{dx}\right), \quad \alpha \in \mathbb{R},$$

(7.7)
with $\text{Dom}(L_\alpha^o) = C_c^\infty(0, \infty)$, that acts on $L^2(d\mu_\alpha^o) := L^2((0, \infty), x^\alpha e^{-x} \, dx)$. Spectral properties of this operator were discussed, among others, in [12], [22, Section 7.2] (under the restriction $\alpha > -1$), [9, Section 27], and in the introductory section of [21]. Decomposition (7.1) holds for $L = L_\alpha^o$ with $a = 0$ and

$$\delta_\alpha^o = \sqrt{x} \frac{d}{dx}, \quad \delta_\alpha^{o,+} = -\sqrt{x} \frac{d}{dx} - \frac{\alpha + 1/2}{\sqrt{x}} + \sqrt{x}.$$  

According to endpoint classification in $L^2(d\mu_\alpha^o)$, see [9, Section 27], the endpoint $\infty$ is LP for any $\alpha$, while the endpoint 0 is: LP for $\alpha \leq -1$ and for $\alpha \geq 1$, regular for $-1 < \alpha < 0$ and is LC for $0 \leq \alpha < 1$. Thus, $L_\alpha^o$ has deficiency indices $(0, 0)$ for $|\alpha| \geq 1$ and $(1, 1)$ otherwise. In particular, $L_\alpha^o$ is essentially self-adjoint if and only if $|\alpha| \geq 1$ and, in this case, $L_\alpha^o$ is the unique self-adjoint extension of $L_\alpha^o$.

When it comes to constructing self-adjoint extensions of $L_\alpha^o$ in terms of spectral resolutions based on an orthonormal basis of eigenfunctions, recall that it is well known that the classical Laguerre polynomials $L_n^\alpha$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, are eigenfunctions of $L_\alpha^o$. $L_\alpha^o L_n^\alpha = n L_n^\alpha$. Moreover, for $\alpha > -1$ they form an orthogonal basis in $L^2(d\mu_\alpha^o)$. Let $\tilde{L}_n^\alpha = L_n^\alpha/\|L_n^\alpha\|_{L^2(d\mu_\alpha^o)}$ denote the Laguerre polynomials normalized in $L^2(d\mu_\alpha^o)$. Thus, in case $\alpha > -1$, Lemma 4.1 applied to $S$ given by the differential expression in (7.7) with

$$\text{Dom}(S) = \{ f \in C^2(0, \infty) \cap L^2(d\mu_\alpha^o) : -xf''' - (\alpha + 1 - x)f' \in L^2(d\mu_\alpha^o) \},$$

and to the orthonormal basis $\{ \tilde{L}_n^\alpha \}_{n \in \mathbb{N}}$, with the choice of $E := C_c^\infty(0, \infty)$, results in the self-adjoint extension of $L_\alpha^o$ denoted $L_\alpha^{o, \text{ex}}$.

A simple calculation shows (note that this case does not fit into the general framework of Subsection 6.1) that the unitary isomorphism

$$V_\alpha : L^2(d\mu_\alpha^o) \rightarrow L^2(d\mu_\alpha^o), \quad V_\alpha f(x) = x^\alpha f(x),$$

satisfies $L_\alpha^o \circ V_\alpha = V_\alpha \circ (L_\alpha^o - \alpha)$. Therefore, it follows that for $\alpha < 1$ the system $\{x^{-\alpha} \tilde{L}_{n-1}^\alpha \}_{n \in \mathbb{N}}$, is an orthonormal basis in $L^2(d\mu_\alpha^o)$ and one has $L_\alpha(x^{-\alpha} \tilde{L}_{n}^\alpha) = (n - \alpha)x^{-\alpha} \tilde{L}_{n}^\alpha$. Again by applying Lemma 4.1 to $S$ and $E$ as above, and the new orthonormal basis $\{x^{-\alpha} \tilde{L}_{n}^\alpha \}_{n \in \mathbb{N}}$, results in the self-adjoint extension of $L_\alpha^o$ denoted $L_\alpha^{o, \text{exo}}$. See [21, Section 1 and 3]. Obviously, $L_\alpha^{o, \text{cls}}$ and $L_\alpha^{o, \text{exo}}$ coincide with $L_\alpha^o = L_\alpha^o, 0$ for $\alpha \geq 1$ and $\alpha < -1$, respectively.

It remains to consider $-1 < \alpha < 1$. Clearly, $L_\alpha^{o, \text{cls}}$ and $L_\alpha^{o, \text{exo}}$ are then two different self-adjoint extensions of $L_\alpha^o$, unless $\alpha = 0$. In the latter case we have $L_0^{o, \text{cls}} = L_0^{o, \text{exo}}$, but $L_0^{o, \text{cls}} \neq L_0^o$.

Passing to the associated operator $L_\alpha^{o, \circ}$ observe that according to the procedure from Subsection 6.2 applied to $w = w_\alpha = e^{-x} x^\alpha$ and $U = U_\alpha$, it follows that

$$L_\alpha^{o, \circ} = -x \frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x^2}{4} - \frac{\alpha + 1}{2},$$

and that decomposition (7.1) holds for $L = L_\alpha^{o, \circ}$ with $a = 0$ and

$$\delta_\alpha^{o, \circ} = \sqrt{x} \frac{d}{dx} + \frac{1}{2} \left( \sqrt{x} - \frac{\alpha}{\sqrt{x}} \right), \quad \delta_\alpha^{o, \circ,+} = -\sqrt{x} \frac{d}{dx} + \frac{1}{2} \left( \sqrt{x} + \frac{\alpha + 1}{\sqrt{x}} \right),$$

(see also [7] but note that the relevant differential operators differ by the constant $(\alpha + 1)/2$). Moreover, for $\alpha > -1$, $U_\alpha L_\alpha^o$ are the (standard) Laguerre functions

$$L_\alpha^o(x) = \left( \Gamma(n + 1)/\Gamma(n + \alpha + 1) \right)^{1/2} e^{-x/2} x^{\alpha/2} L_n^\alpha(x), \quad n \in \mathbb{N},$$
that form an orthonormal basis in $L^2((0,\infty),dx)$ and are eigenfunctions, with eigenvalues $\lambda_n = n$, of $L^\alpha$. Comments concerning deficiency indices and essential self-adjointness of $L^\alpha$, the exotic system $\{\mathcal{L}_n^{\alpha,\text{exo}}\}$ and classic and exotic extensions $L^\alpha,\text{ch/exo}$, are similar to those from previous subsections and hence we skip them.

7.4. Jacobi function operators $J_{\alpha,\beta}$ and $J^\circ_{\alpha,\beta}$. The first of these two operators is

$$J_{\alpha,\beta} = -\frac{d^2}{dx^2} - \left((2\alpha + 1) \coth x + (2\beta + 1) \tanh x\right) \frac{d}{dx},$$

where

$$A_{\alpha,\beta}(x) = 2^{2(\alpha + \beta + 1)}(\sinh x)^{2\alpha + 1}(\cosh x)^{2\beta + 1},$$

considered on $L^2(d\mu_{\alpha,\beta}) := L^2((0,\infty),A_{\alpha,\beta}(x)dx)$ and with domain $\text{Dom}(J_{\alpha,\beta}) = C^\infty_c(I)$. The operator was investigated in [16] (see also [2]). Decomposition (7.1) holds for $L = J_{\alpha,\beta}$ with $a = 0$ and

$$\mathfrak{d}_{\alpha,\beta} = \frac{d}{dx}, \quad \mathfrak{d}^+_{\alpha,\beta} = -\frac{d}{dx} - (2\alpha + 1) \coth x - (2\beta + 1) \tanh x.$$

Following the procedure from Subsection 6.2 shows that

$$J^\circ_{\alpha,\beta} = -\frac{d^2}{dx^2} + \left(\alpha^2 - \frac{1}{4}\right) (\coth x)^2 + \left(\beta^2 - \frac{1}{4}\right) (\tanh x)^2 + c_{\alpha,\beta},$$

where $c_{\alpha,\beta} = (2\alpha + 1)(2\beta + 1) + \alpha + \beta + 1$. Decomposition (7.1) holds for $L = J^\circ_{\alpha,\beta}$ with $a = 0$ and

$$\mathfrak{d}^\circ_{\alpha,\beta} = \frac{d}{dx} - \frac{A_{\alpha,\beta}(x)'}{2A_{\alpha,\beta}(x)}, \quad \mathfrak{d}^{\circ,+}_{\alpha,\beta} = \frac{d}{dx} - \frac{A_{\alpha,\beta}(x)'}{2A_{\alpha,\beta}(x)},$$

where

$$\frac{A_{\alpha,\beta}(x)'}{2A_{\alpha,\beta}(x)} = (\alpha + 1/2) \coth x + (\beta + 1/2) \tanh x.$$

We now focus on $J_{\alpha,\beta}$, the corresponding results for $J^\circ_{\alpha,\beta}$ follow respectively. From the considerations in Subsection 6.1 we know that the mapping

$$V_{\alpha,\beta}: L^2(A_{-\alpha,-\beta}(x)dx) \to L^2(A_{\alpha,\beta}(x)dx), \quad V_{\alpha,\beta} f = (\sinh x)^{-2\alpha}(\cosh x)^{-2\beta} f,$$

is a unitary isomorphism such that

$$J_{\alpha,\beta} \circ V_{\alpha,\beta} = V_{\alpha,\beta} \circ (J_{-\alpha,-\beta} - 2(\alpha + \beta)).$$

Analogously, using the unitary isomorphism

$$V^{(1)}_{\alpha}: L^2(A_{-\alpha,\beta}(x)dx) \to L^2(A_{\alpha,\beta}(x)dx), \quad V^{(1)}_{\alpha} f = w_{1-2\alpha} f,$$

we have $J_{\alpha,\beta} \circ V^{(1)}_{\alpha} = V^{(1)}_{\alpha} \circ (J_{-\alpha,-\beta} - 2\alpha)$, and similarly when the roles of $\alpha$ and $\beta$ are changed and $V^{(2)}_{\beta}$ is considered.

Endpoint classification in $L^2(d\mu_{\alpha,\beta})$, furnished in [9, Section 25] for the case $\alpha \geq -1/2, \beta \geq -1/2$, extends, due to the unitary isomorphisms $V_{\alpha,\beta}, V^{(1)}_{\alpha}$ and $V^{(2)}_{\beta}$, to the whole range of type parameters $\alpha, \beta$. Consequently we have: the endpoint $\infty$ is LP for any $(\alpha,\beta) \in \mathbb{R} \times \mathbb{R}$, while the endpoint 0 is: LP for $|\alpha| \geq 1$, and any $\beta \in \mathbb{R}$, and is LC for $|\alpha| < 1$ and any $\beta \in \mathbb{R}$. Therefore, $J_{\alpha,\beta}$ has deficiency indices $(0,0)$ for $|\alpha| \geq 1$ and any $\beta \in \mathbb{R}$, and $(1,1)$ otherwise. In particular, $J_{\alpha,\beta}$ is essentially self-adjoint if and only if $|\alpha| \geq 1$ and $\beta \in \mathbb{R}$, and, in this case, $J_{\alpha,\beta} = J_{\alpha,\beta;0}$ is the unique self-adjoint extension of $J_{\alpha,\beta}$. 

SPECTRAL PROPERTIES OF ORDINARY DIFFERENTIAL OPERATORS 1983
When it comes to constructing concrete self-adjoint extensions of $J_{\alpha,\beta}$, similarly to the case of $B_\nu$ on $L^2((0,\infty),d\mu_\nu)$, this may be done in terms of an integral transform, called Fourier-Jacobi transform, and denoted $\mathcal{F}_{\alpha,\beta}$. An important property of this transform is that for sufficiently smooth functions $f$ on $(0,\infty)$ it holds

$$\mathcal{F}_{\alpha,\beta}(J_{\alpha,\beta}f)(y) = (y^2 + (\alpha + \beta + 1)^2)\mathcal{F}_{\alpha,\beta}(f)(y), \quad y \in \mathbb{C}.$$  

Moreover, for $\alpha > -1$ and $\beta \in \mathbb{R}$, $\mathcal{F}_{\alpha,\beta}$ extends to a unitary isomorphism between $L^2((0,\infty),d\mu_\alpha)$ and $L^2((0,\infty) \cup D_{\alpha,\beta},d\nu_\beta)$, where $\nu_{\alpha,\beta}$ is a Borel measure on $(0,\infty) \cup D_{\alpha,\beta}$ with density on $(0,\infty)$ given in terms of a Harish-Chandra function and a purely atomic measure on the finite set $D_{\alpha,\beta}$ (in case $\alpha > -1$ and $|\beta| \leq \alpha + 1$, $D_{\alpha,\beta}$ is empty). For all these facts we refer the reader to [16] (see also [2]). This gives rise to defining a self-adjoint extension of $J_{\alpha,\beta}$, denoted $J_{\alpha,\beta}^{\text{cls}}$, in a similar way as the extension $B_\nu^{\text{cls}}$ of $B_\nu$ was obtained, and then to define exotic extensions $J_{\alpha,\beta}^{\text{exo}}$ for the whole range of parameters $(\alpha,\beta)$. We skip the details. Clearly, for $|\alpha| \geq 1$ and $\beta \in \mathbb{R}$, the extensions $J_{\alpha,\beta}^{\text{cls/exo}}$ coincide with $J_{\alpha,\beta} = J_{\alpha,\beta,0}$.

8. Appendix. We now outline reasoning that leads to a construction of a self-adjoint extension of $B_0$ different from $B_0^{\text{cls}}$ or, equivalently, a self-adjoint extension of $B_0^{\text{cls}}$ different from $B_0^{\text{cls}}$.

Let $Y_\nu$ denote the Bessel function of the second kind of order $\nu \in \mathbb{R}$. Recall that $J_\nu$ and $Y_\nu$ are linearly independent solutions of the second order linear differential equation $u'' + \frac{1}{\nu^2}u' + (1-\nu^2/x^2)u = 0$ and the recurrence relations satisfied by $J_\nu$ and $Y_\nu$ are the same. Also asymptotic estimate for large $x$ is for $Y_\nu$, the same as for $J_\nu$, namely $Y_\nu(x) = \mathcal{O}(x^{-1/2})$, $x \to \infty$. Similarly for small $x$, except the case $\nu = 0$, where there is an essential difference in behavior of $Y_0$ when compared with $J_0$, that is, $Y_0(x) \approx \log x$, $x \to 0^+$. For all these facts we refer the reader to [18, Chapter 5].

For a moment we now consider any $\nu > -1$. Given $\nu > -1$ let $\tau_n := \tau_{n,\nu}$, $n \in \mathbb{N}_+$, denote the sequence of successive positive zeros of $Y_\nu(x)$. This sequence has properties analogous to that of $\{\lambda_{n,\nu}\}_{n \in \mathbb{N}_+}$. See [20, Chapter 10]. In particular it is known that $\lim_{n \to \infty} \tau_n = \infty$.

We begin with the case $I = (0,1)$. Following an argument from [18, pp. 128-129] it becomes clear that the functions $Y_\nu(\tau_n x)$, $n \in \mathbb{N}_+$, are pairwise orthogonal in $L^2((0,1),x\,dx)$. Then

$$\psi_n^\nu(x) = d_{n,\nu}(\tau_n x)^{1/2}Y_\nu(\tau_n x), \quad n \in \mathbb{N}_+,$$

where $d_{n,\nu}$ denotes an appropriate positive $L^2$ normalization constant, is a complete orthonormal system in $L^2(dx)$ consisting of eigenfunctions of $B_\nu$, $B_\nu \psi_n^\nu = \tau_n^2 \psi_n^\nu$. Completeness is a (nontrivial) separate issue but it may be proved as in the case of $\{\psi_n^\alpha\}_{n \in \mathbb{N}_+}$; see Hochstadt [14]. Thus we end up in a place where we apply again Lemma 4.1 to $S$ and $E$ as in Subsection 7.1, case B), but with $\{\psi_n^0\}_{n \in \mathbb{N}_+}$ as the orthonormal basis. The resulting self-adjoint extension denoted $\hat{B}_0^{\text{cls}}$ is different from $B_0^{\text{cls}}$.

Passing to the case $I = (0,\infty)$ we now limit consideration to $\nu = 0$ and define the Hankel-type-transform

$$\hat{H}_0 f(y) = \int_0^\infty f(x)(xy)^{1/2}Y_0(xy)\,dx, \quad y > 0.$$  

It is not difficult to see that repeating the reasoning from the proof of [4, Lemma 2.7] allows to show that $\hat{H}_0$ is an isometry on $C_{c}^\infty(0,\infty)$. (Observe that completeness of
\{\hat{\varphi}^0_n\}_{n\in\mathbb{N}^+} \text{ is crucial in this reasoning.) Thus } \hat{H}_0 \text{ extends to a unitary isomorphism of } L^2(dx); \text{ we use the same symbol to denote this (unique) extension. To be fair, there are points in the proof of [4, Lemma 2.7] that must be appropriately modified. We now pause for a moment to discuss the differences.}

Firstly, the pointwise estimate [4, (2.6)] remains valid if we put \( \psi^0_n \) on the left-hand side and \((n\pi)^{1/2}\log(n\pi) \) in the upper line on the right-hand side. This is sufficient to claim that the result analogous to [4, Lemma 2.5] holds true (with \( \nu = \mu = 0 \) and \( \psi^0_n \) in place of \( \psi^\nu_n \)) and justifies a selection, in the beginning of the proof of [4, Lemma 2.7], of a subsequence of appropriate partial sums converging almost everywhere. Secondly, the estimates [4, (2.7) and (2.8)] remain valid with replacement of \( \mathcal{H}_\nu \) by \( \hat{H}_0 \) and \( y^{\nu+1/2} \) by \( y^{1/2}\log y \). Then, using appropriately modified notation from the proof of [4, Lemma 2.7] it may be checked that for \( h_\nu(y) = (xy)^{1/2}Y_0(xy)\hat{H}_0 f(y), \text{ the asymptotic estimates } h_\nu(y) = O(y), \text{ } y \to 0^+, \text{ and } h_\nu(y) = O(y^{-2}), \text{ } y \to \infty, \text{ are true. This is sufficient to claim that estimates analogous to those on [4, pp.116-117] can be carried out and thus to conclude that } \hat{H}_0 \text{ is an isometry on } C^\infty_c(0, \infty). \text{ Finally, we define } \hat{B}_c^{0, \text{cls}} \text{ in terms of } \hat{H}_0 \text{ in the same way as } B_0^{0, \text{cls}} \text{ can be defined in terms of } \mathcal{H}_0 \text{ (see the comments at the end of Section 7.1). It is clear that } \hat{B}_c^{0, \text{cls}} \neq B_0^{0, \text{cls}}. \text{ }

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