Maximum size intersecting families of bounded minimum positive co-degree

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Abstract

Let $H$ be an $r$-uniform hypergraph. The minimum positive co-degree of $H$, denoted $\delta_{r-1}^+(H)$, is the minimum $k$ such that if $S$ is an $(r-1)$-set contained in a hyperedge of $H$, then $S$ is contained in at least $k$ distinct hyperedges of $H$. We determine the maximum possible size of an intersecting $r$-uniform $n$-vertex hypergraph with minimum positive co-degree $\delta_{r-1}^+(H) \geq k$ and characterize the unique hypergraph attaining this maximum, for $n$ sufficiently large. Our proof is based on the delta-system method.

1 Introduction

A hypergraph $\mathcal{H}$ is intersecting if for every pair of hyperedges $h, h' \in E(\mathcal{H})$ we have $h \cap h' \neq \emptyset$. The celebrated theorem of Erdős, Ko and Rado [3] gives that for $n \geq 2r$, the maximum size of an intersecting $r$-uniform $n$-vertex hypergraph is $\binom{n-1}{r-1}$. The Erdős-Ko-Rado theorem is a cornerstone of extremal combinatorics and has many proofs, extensions and generalizations. See the excellent survey of Frankl and Tokushige [10] for a history of extremal problems for intersecting hypergraphs.

The degree of a set of vertices $S$ in a hypergraph $\mathcal{H}$ is the number of hyperedges containing $S$, i.e., $|\{h \in E(\mathcal{H}) \mid S \subseteq h\}|$. Denote by $\delta_S(\mathcal{H})$ the minimum degree of an $s$-element subset of the vertices of $\mathcal{H}$. In this way, $\delta_1(\mathcal{H})$ is the standard minimum degree of a vertex in $\mathcal{H}$.

Huang and Zhao [14] considered a minimum degree version of the Erdős-Ko-Rado theorem. In particular, they proved that for $n \geq 2r + 1$, if $\mathcal{H}$ is an intersecting $r$-uniform $n$-vertex hypergraph, then $\mathcal{H}$ has minimum degree $\delta_1(\mathcal{H}) \leq \binom{n-2}{r-2}$. The Huang-Zhao proof uses the linear algebra method and later a combinatorial proof was given by Frankl and Tokushige [9] for $n \geq 3k$. Kupavskii [16] gave an extension of this result and showed that
for \( t < r \) and \( n \geq 2k + 2t/(1 - t/k) \), an intersecting \( r \)-uniform \( n \)-vertex hypergraph \( \mathcal{H} \) satisfies \( \delta_{t}(\mathcal{H}) \leq \binom{n-t-1}{r-1} \).

In the more general hypergraph setting, Mubayi and Zhao \[17\] introduced the notion of co-degree Turán numbers, i.e., the maximum possible value of \( \delta_{r-1}(\mathcal{H}) \) among all \( r \)-uniform \( n \)-vertex hypergraphs \( \mathcal{H} \) not containing a specified subhypergraph \( \mathcal{F} \). In their paper they give several results that show that the co-degree extremal problem behaves differently from the classical Turán problem.

Motivated by these degree versions of Erdős-Ko-Rado and co-degree Turán numbers we propose the following hypergraph degree condition.

**Definition 1.** Let \( \mathcal{H} \) be an \( r \)-uniform hypergraph. The minimum positive co-degree of \( \mathcal{H} \), denoted \( \delta^{+}_{r-1}(\mathcal{H}) \), is the minimum \( k \) such that if \( S \) is an \((r-1)\)-set contained in a hyperedge of \( \mathcal{H} \), then \( S \) is contained in at least \( k \) distinct hyperedges of \( \mathcal{H} \).

As an example, let us examine hypergraphs that contain no \( F_5 = \{abc, abd, cde\} \) to compare the co-degree and positive co-degree settings. Frankl and Füredi \[8\] (see \[15\] for a strengthening) showed that the complete balanced tripartite 3-uniform hypergraph has the maximum number of hyperedges among all 3-uniform \( n \)-vertex \( F_5 \)-free hypergraphs, for \( n \) sufficiently large. This construction has minimum co-degree 0 and it is easy to see that minimum co-degree at least 2 guarantees the existence of an \( F_5 \). On the other hand, the balanced tripartite hypergraph has minimum positive co-degree \( n/3 \) and it can be shown that minimum positive co-degree greater than \( n/3 \) implies the existence of a \( F_5 \).

Note that in problems where we can suppose that our graph does not contain isolated vertices, the positive co-degree in a graph is equal to the minimum degree of a vertex. This suggests positive co-degree as a reasonable notion of “minimum degree” in a hypergraph.

In this paper we are interested in determining the maximum size of an intersecting \( r \)-uniform \( n \)-vertex hypergraph with positive co-degree at least \( k \). The condition \( k \geq 1 \) is always satisfied, so in this case the maximum is \( \binom{n-1}{r-1} \) as given by the Erdős-Ko-Rado theorem. An easy argument shows that in an intersecting hypergraph, the uniformity is always at least the minimum positive co-degree, i.e., \( r \geq k \); see Proposition \[4\].

We will prove that the maximum-size intersecting hypergraph with minimum positive co-degree \( k \) and \( n \) sufficiently large is the following hypergraph.

**Definition 2.** Fix integers \( r \geq k \) and a set \( X \) of \( 2k-1 \) vertices. The \( r \)-uniform hypergraph consisting of all hyperedges containing at least \( k \) vertices of \( X \) is a \( k \)-kernel system.

Clearly a \( k \)-kernel system is intersecting. Observe that the number of hyperedges in an \( r \)-uniform \( n \)-vertex \( k \)-kernel system \( \mathcal{H} \) is

\[
|E(\mathcal{H})| = \sum_{i=k}^{\max(r,2k-1)} \binom{2k-1}{i} \binom{n-2k+1}{r-i} \geq \binom{2k-1}{k} \binom{n-2k+1}{r-k} = \Omega(n^{r-k}).
\]

Note that a 1-kernel system is the hypergraph consisting of all hyperedges containing a fixed vertex \( x \), i.e., the maximal hypergraph in the Erdős-Ko-Rado theorem. Interestingly, \( k \)-kernel systems appear as solutions to maximum degree versions of the Erdős-Ko-Rado theorem. Let us give two examples. First, as a special case of a more general theorem of Frankl \[7\] implies that if \( \mathcal{H} \) is a maximum-size intersecting \( r \)-uniform \( n \)-vertex hypergraph with maximum degree at most \( 2\binom{n-3}{r-2} + \binom{n-3}{r-3} \), then \( \mathcal{H} \) is a 2-kernel system, provided \( n \)
is large enough. Second, Erdős, Rothschild and Szemerédi (see [2]) posed the following question: determine the maximum size of an intersecting \( r \)-uniform \( n \)-vertex hypergraph \( H \) such that each vertex contained in at most \( c|E(H)| \) hyperedges for \( r \geq 3 \) and \( 0 < c < 1 \). They proved when \( c = 2/3 \) (and \( n \) large), that a 2-kernel system is the unique hypergraph attaining this maximum. Frankl [5] showed that for \( 2/3 \leq c < 1 \) and \( n \) large enough, \( H \) has no more hyperedges than a 2-kernel system. For \( 3/5 < c < 2/3 \) and \( n \) large enough, Füredi [5] showed that the 3-kernel system is one of six non-isomorphic hypergraphs attaining this maximum. In the case when \( 1/2 < c \leq 3/5 \) and \( n \) large enough, Frankl [5] showed that \( H \) has no more hyperedges than a 3-kernel system, although the unique hypergraph attaining this maximum is not isomorphic to a 3-kernel system.

The main result of this paper is as follows:

**Theorem 3.** Let \( H \) be an intersecting \( r \)-uniform \( n \)-vertex hypergraph with minimum positive co-degree \( \delta^+_{r-1}(H) \geq k \) where \( 1 \leq k \leq r \). If \( H \) has the maximum number of hyperedges, then for \( n \) large enough \( H \) is a \( k \)-kernel system.

Theorem [3] holds when \( n \) is at least \( \Omega(r^k) \). In Section [5] we give two results that suggest that Theorem [3] holds for \( n \) at least \( \Omega(r^{k+1}) \).

As an open question, it would be interesting to determine the range of \( n \) as a function of \( r \) and \( k \) where our results hold. Also, we only considered the positive co-degree of \( (r-1) \)-sets. Similarly, we can define \( \delta^+_k(H) \) to be the minimum \( k \) such that if \( S \) is an \( s \)-set contained in a hyperedge of \( H \), then \( S \) is contained in at least \( k \) distinct hyperedges. There may be interesting questions to be answered under this more general condition.

## 2 Proof of Theorem 3

First, let us observe that the uniformity of an intersecting hypergraph is always at least the minimum positive co-degree.

**Proposition 4.** If \( H \) is an intersecting \( r \)-uniform \( n \)-vertex hypergraph with minimum positive co-degree \( \delta^+_{r-1}(H) \geq k \), then \( r \geq k \).

**Proof.** Assume, for the sake of a contradiction, that \( k > r \). Let \( h = \{x_1, x_2, \ldots, x_r\} \) be a hyperedge of \( H \). The \((r-1)\)-set \( h \setminus x_1 \) has co-degree at least \( k \), so there is a vertex \( y_1 \notin h \) such that \( h \setminus x_1 \cup \{y_1\} \) is a hyperedge of \( H \). Similarly, the \((r-1)\)-set \( h \setminus \{x_1, x_2\} \cup \{y_1\} \) has co-degree at least \( k \), so there is a vertex \( y_2 \notin h \) such that \( h \setminus \{x_1, x_2\} \cup \{y_1, y_2\} \) is a hyperedge of \( H \). As long as \( k > r \) we can repeat this process to obtain a hyperedge \( h \setminus \{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_r\} = \{y_1, \ldots, y_r\} \) that is in \( H \). Now as \( H \) and \( \{y_1, \ldots, y_r\} \) are disjoint we have a contradiction. \( \square \)

An \( r \)-uniform hypergraph \( S \) is a **sunflower** if every pairwise intersection of the hyperedges is the same set \( Y \) called the core of the sunflower. We call the sets \( h \setminus Y \) for \( h \in E(S) \) the petals of the sunflower \( S \). Note that the petals are pairwise disjoint. For a sunflower \( S \) let \( c(S) \) denote the size of the core of \( S \).

**Lemma 5** (Sunflower Lemma, Erdős and Rado [4]). **Fix positive integers** \( r \geq 3 \) and \( C \). If \( G \) is an \( r \)-uniform hypergraph with

\[ |E(G)| \geq r!(C-1)^r, \]

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then \( \mathcal{G} \) contains a sunflower with \( C \) petals.

Let \( f(r, C) \) denote the minimum integer such that an \( r \)-uniform hypergraph with \( f(r, C) \) hyperedges contains a sunflower with \( C \) petals. The determination of \( f(r, C) \) is a well-known open problem in combinatorics. A recent breakthrough in [1] gave a bound on \( f(r, C) \) of about \( (\log r)^{r^{1+o(1)}} \).

In general we cannot force a sunflower to have a core of a specified size unless we increase the number of hyperedges in the host hypergraph.

**Lemma 6.** Fix integers \( r \geq 3 \) and \( C \geq 1 \) and let \( n \) be large enough. If \( \mathcal{G} \) is an \( r \)-uniform \( n \)-vertex hypergraph with

\[
|E(\mathcal{G})| \geq 2^{r^{r-k}} \binom{n-k-1}{r-k-1} f(r, Cr^{r-k}),
\]

then \( \mathcal{G} \) contains a sunflower with \( C \) petals and core of size at most \( k \).

Observe that Lemma 6 is sharp in the order of magnitude of \( n \). Indeed, the \( r \)-uniform \( n \)-vertex hypergraph consisting of all hyperedges containing a fixed set \( Y \) of \( k+1 \) vertices contains \( \binom{n-k-1}{r-k-1} \) hyperedges, but no sunflower with a core of size at most \( k \) as any two hyperedges intersect in at least \( k+1 \) vertices.

**Proof.** For the sake of a contradiction, suppose that \( \mathcal{G} \) contains no sunflower with \( C \) petals and core of size at most \( k \).

Iteratively remove from \( \mathcal{G} \) a sunflower \( \mathcal{S} \) with exactly \( Cr^{c(S)-k} \) petals such that at each step we choose a sunflower with minimum available core size \( c(S) \). Let \( p \) be the number of steps in this sunflower removal procedure. Note that \( p \) grows with \( n \) as at each step we remove at most \( Cr^{r-k} \) hyperedges from \( \mathcal{G} \) and we only need constant number of hyperedges to guarantee the existence of a sunflower with \( Cr^{c(S)-k} \) petals. In particular, we have

\[
p \geq \frac{|E(\mathcal{G})| - f(r, Cr^{r-k})}{Cr^{r-k}} \geq \frac{|E(\mathcal{G})|}{2Cr^{r-k}}
\]

for \( n \) large enough.

The core of each removed sunflower is of size at least \( k+1 \) and at most \( r-1 \). Therefore, there is some integer \( s \) such that there are at least \( p/r \) cores of size \( s \) among the removed sunflowers. Some of these cores may be identical. Let us compute the maximum multiplicity of a core \( Y \). There are at most \( \binom{n-|Y|}{r-|Y|} \) hyperedges containing \( Y \) and each removed sunflower with core \( Y \) has exactly \( Cr^{|Y|-k} \) hyperedges. Therefore, the maximum multiplicity of a core \( Y \) is at most

\[
\frac{1}{Cr^{|Y|-k}} \binom{n-|Y|}{r-|Y|} \leq \frac{1}{Cr} \binom{n-k-1}{r-k-1}
\]

for \( n \) large enough. Therefore, there is a collection of at least

\[
(p/r) \cdot Cr \binom{n-k-1}{r-k-1}^{-1} \geq C \cdot \frac{|E(\mathcal{G})|}{2Cr^{r-k}} \binom{n-k-1}{r-k-1}^{-1} \geq f(r, Cr^{r-k})
\]

distinct cores of size \( s \). Let \( Y_1, Y_2, \ldots, Y_q \) be these cores and let \( \mathcal{S}_i \) be the sunflower with core \( Y_i \) for \( i = 1, 2, \ldots, q \). Note that each of these sunflowers has exactly \( Cr^s^{-k} \) petals.
Let \( t \) be the first step in the sunflower removal procedure in which a sunflower with core of size \( s \) is chosen to be removed. This implies that all later cores are of size at least \( s \). Now we will show that there is a sunflower \( B \) with core of size less than \( s \) and \( C^{r-\ell(B)-k} \) petals among the hyperedges in the sunflowers \( S_1, S_2, \ldots, S_q \). Before removing the sunflower in step \( t \), all hyperedges of the sunflowers \( S_1, S_2, \ldots, S_q \) are still in \( H \). Therefore, the sunflower \( B \) with core of size less than \( s \) could be chosen in step \( t \), this will contradict the choice of \( t \).

We may think of the \( s \)-sets \( Y_1, \ldots, Y_q \) as an \( s \)-uniform hypergraph on the vertex set of \( H \). As \( q \geq f(r, C^{r-k}) \geq f(s, C^{s-k}) \geq f(s, C^{s-r}) \), the \( s \)-sets \( Y_1, \ldots, Y_q \) contain an \( s \)-uniform sunflower \( A \) with \( C^{s-k} \) petals and core \( Y^* \) of size less than \( s \). By relabelling, we may suppose that \( Y_i \) is a member of \( A \) for \( i = 1, 2, \ldots, q \). Note that the petals \( Y_i \setminus Y^* \) of \( A \) are pairwise disjoint by definition. The sunflower \( A \) is not in the hypergraph \( H \) as it is \( s \)-uniform. However, each hyperedge of \( A \) is the core of some sunflower \( S_i \) in \( H \). Therefore, we will use the members of \( A \) to identify an \( r \)-uniform sunflower \( B \) with core \( Y^* \) in \( H \). The main idea will be carefully choose a petal from each sunflower \( S_i \) whose core is a member of \( A \). To this end, define \( B \) as follows:

First pick any hyperedge of \( S_1 \); denote it by \( h_1 \). Now suppose we have chosen \( \ell \) hyperedges \( h_1, h_2, \ldots, h_\ell \) that form a sunflower with core \( Y^* \). The union of these hyperedges contains \( \ell(r - |Y^*|) \) vertices outside of \( Y^* \). Therefore, as long as

\[
C^{s-k} \geq \ell(r - |Y^*|),
\]

(1)

there is a petal \( Y_i \setminus Y^* \) of \( A \) that is disjoint from each of the hyperedges \( h_1, h_2, \ldots, h_\ell \). The corresponding sunflower \( S_i \) with core \( Y_i \) has

\[
C^{s-k} > \ell(r - |Y^*|)
\]

petals by (1). Therefore, there is a petal \( P \) of \( S_i \) that is also disjoint from the hyperedges in \( h_1, h_2, \ldots, h_\ell \). Let \( h_{\ell+1} \) be the hyperedge \( P \cup Y_i \). Now we have a sunflower with \( \ell + 1 \) petals and core \( Y^* \). We may repeat this procedure as long as \( \ell \) satisfies (1), i.e., until \( \ell = C^{s-k-1} \). This implies that the number of petals in sunflower \( B \) is at least

\[
C^{s-k-1}.
\]

As \( B \) has core \( Y^* \) of size \( c(B) < s \) we have a contradiction to the choice of sunflower in step \( t \). \( \square \)

We can also give an upper-bound on the size of a core of a sunflower in an intersecting hypergraph.

**Lemma 7.** If \( S \) is a sunflower with at least \( r + 1 \) petals in an intersecting \( r \)-uniform hypergraph \( \mathcal{G} \) with \( \delta^{+}_{r+1}(\mathcal{G}) \geq k \), then the core \( Y \) of \( S \) satisfies \( |Y| \geq k \).

**Proof.** For the sake of contradiction, assume that the core \( Y \) of \( S \) is small, i.e., \( |Y| < k \). Observe that \( Y \) is a transversal of \( \mathcal{G} \), i.e., every hyperedge of \( \mathcal{G} \) intersects \( Y \). Indeed, as the petals of the sunflower \( S \) are pairwise vertex-disjoint, each hyperedge of \( \mathcal{G} \) must intersect the core \( Y \) in order to intersect each of the at least \( r + 1 \) hyperedges associated with the petals of the sunflower.

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Now let $Y'$ be a minimum transversal in $G$. Thus $|Y'| \leq |Y| < k$ and the minimality of $Y'$ guarantees the existence of a hyperedge $h$ that intersects $Y'$ in exactly one element. The $(r - 1)$-set $h \setminus Y'$ is contained in at most $k - 1$ hyperedges of $G$; one for each element of $Y'$. This contradicts the positive co-degree condition on $G$. □

**Proof of Theorem 3** We have observed that a $k$-kernel system has minimum positive co-degree at least $k$. Therefore, we may assume that

$$|E(h)| \geq \binom{2k - 1}{k} \binom{n - 2k + 1}{r - k} = \Omega(n^{r-k}).$$

Therefore, for $n$ large enough, Lemmas 6 and 7 guarantees the existence of a sunflower $S$ with $C = (r + 1) \cdot r^{k-1}$ petals and core of size $k$. Denote the core of $S$ by $Y = \{y_1, y_2, \ldots, y_k\}$.

**Claim 8.** There is a set of vertices $Z = \{z_1, z_2, \ldots, z_{k-1}\}$ such that $Z \cap Y = \emptyset$ and $Z \cup \{y_k\}$ is the core of a sunflower with $r + 1$ petals.

**Proof.** We will prove the following stronger claim: For $0 \leq i \leq k - 1$, there is a set of vertices $Z_i = \{z_1, z_2, \ldots, z_i\}$ such that $Y \cap Z_i = \emptyset$ and $Z_i \cup \{y_k, y_{k-1}, \ldots, y_{i+1}\}$ is the core of a sunflower $S_i$ with $(r + 1) \cdot r^{k-1-i}$ petals. The claim follows from the case $i = k - 1$.

We proceed by induction on $i$. The base case $i = 0$ is immediate as $Z_0 = \emptyset$ and $S_0 = S$ is a sunflower with core $Z_0 \cup \{y_k, y_{k-1}, \ldots, y_1\} = Y$ with $(r + 1) \cdot r^{k-1}$ petals.

Now suppose $i > 0$ and the statement holds for $i - 1$. Let $S_{i-1}$ be a sunflower given by the inductive hypothesis.

For each petal $P$ in $S_{i-1}$ consider the $(r - 1)$-set $P \cup Z_{i-1} \cup \{y_k, \ldots, y_i\} \setminus y_i$. By the positive co-degree condition on $H$, the set $P \cup Z_{i-1} \cup \{y_k, \ldots, y_i\} \setminus y_i$ is contained in $k$ hyperedges of $H$. Therefore, as $i \leq k - 1$, there is a vertex $x(P)$ such that $x(P) \not\in \{y_1, y_2, \ldots, y_i\}$ and $\{x(P)\} \cup P \cup Z_{i-1} \cup \{y_k, \ldots, y_i\} \setminus y_i$ is a hyperedge of $H$.

Now suppose there are distinct vertices $x_1, x_2, \ldots, x_{r+1}$ among the vertices in $\{x(P) \mid P$ is a petal in $S\}$. Let $P_1, P_2, \ldots, P_{r+1}$ be the petals corresponding to these vertices, i.e., $\{x_j\} \cup P_j \cup Z_{i-1} \cup \{y_k, \ldots, y_i\} \setminus y_i \in E(H)$ for $j = 1, 2, \ldots, r+1$. Then $Z_{i-1} \cup \{y_k, \ldots, y_i\} \setminus y_i$ is the core of size $k - 1$ of a sunflower with petals $P_j \cup \{x_j\}$ for $j = 1, 2, \ldots, r + 1$ in $H$. This contradicts Lemma 7. Therefore, there are at most $r$ distinct vertices among the vertices in $\{x(P) \mid P$ is a petal in $S\}$. This implies that there is a vertex $x$ that is the vertex $x(P)$ for at least $\frac{1}{r}|E(S_{i-1})| \geq (r + 1) \cdot r^{k-2-(i-1)}$ petals $P$ in $S_{i-1}$. Put $z_i = x$ and $Z_i = \{z_1, z_2, \ldots, z_i\}$ and let $S_i$ be the sunflower consisting of $(r + 1) \cdot r^{k-1-i}$ hyperedges of $S_{i-1}$ containing $x = z_i$. Observe that $Z_i \cup \{y_k, \ldots, y_{i+1}\}$ is the core of sunflower $S_i$ with $(r + 1) \cdot r^{k-1-i}$ petals. □

Let $S_Z$ be a sunflower with $r + 1$ petals and core $Z \cup \{y_k\}$ given by Claim 8. There are at most $r(r + 1)$ vertices in $S_Z$, so we may choose $r + 1$ petals of $S$ that are each vertex-disjoint from the vertices of $S_Z$. Call the resulting sunflower $S_Y$. Note that $S_Y$ has $r + 1$ petals and core $Y$.

**Claim 9.** For every petal $P$ in $S_Z$ and every $y \in Y$ we have that $P \cup Z \cup \{y\}$ is a hyperedge in $H$.

\[\text{This argument will appear again in the proof of Lemma 14.}\]
implies that each hyperedge containing \( Q \) for each petal \( k \) condition guarantees it is contained in \( Y \) intersect the \( k \) hyperedges of \( S_Y \). As \( S_Y \) has more than \( r \) petals, each of the \( k \) hyperedges containing \( P \) must contain a distinct vertex of \( Y \).

We now continue with a technical claim that will imply the theorem.

**Claim 10.** For every \( k \)-set \( T \subset Y \cup Z \) we have:

1. \( Q \cup T \in E(\mathcal{H}) \) for every petal \( Q \) of \( S_Y \),
2. \( ((Y \cup Z) \setminus T) \cup \{s\} \cup P \in E(\mathcal{H}) \) for every \( s \in T \) and petal \( P \) of \( S_Z \).

**Proof.** We proceed by induction on \( t = |T \cap Z| \). Note that \( t \leq k - 1 \). When \( t = 0 \) we have that \( T = Y \), then (1) is immediate as \( Q \cup Y \in E(S_Y) \subset \mathcal{H} \) and (2) follows from Claim [9].

So let \( t > 0 \) and suppose the statement of the claim holds for smaller values. As \( t > 0 \), there exists a \( z \in Z \cap T \) and a \( y \in Y \setminus T \). Fix an arbitrary petal \( Q \) of \( S_Y \). Put \( T' = \{y\} \cup T \setminus z \) and note that \( |T' \cap Z| = t - 1 \). Therefore, by the inductive hypothesis we have \( Q \cup T' \in E(\mathcal{H}) \) and \( ((Y \cup Z) \setminus T') \cup \{s\} \cup P \in E(\mathcal{H}) \) for every \( s \in T' \) and petal \( P \) of \( S_Z \).

By the positive co-degree condition, the \( (r - 1) \)-set \( Q \cup T' \setminus y \) is contained in \( k \) hyperedges. Moreover, \( Q \cup T' \setminus y \) disjoint from the \( k \) hyperedges \( ((Y \cup Z) \setminus T') \cup \{y\} \cup P \) for each petal \( P \) of \( S_Z \). Thus, each of the \( k \) hyperedges containing \( Q \cup T' \setminus y \) must intersect the \( k \) hyperedges \( ((Y \cup Z) \setminus T') \cup \{y\} \cup P \) for each petal \( P \) of \( S_Z \). As \( S_Z \) has \( r + 1 \) petals, this implies that each hyperedge containing \( Q \cup T' \setminus y \) intersects the \( (r + 1) \)-set \( ((Y \cup Z) \setminus T) \cup \{y\} \). In particular, \( (Q \cup T' \setminus y) \cup \{s\} = Q \cup T \) is a hyperedge of \( \mathcal{H} \). This proves (1).

In order to prove (2) let us fix an arbitrary petal \( P \) of \( S_Z \). By (1), the \( (r - 1) \)-set \( ((Y \cup Z) \setminus T) \cup P \) is contained in a hyperedge of \( \mathcal{H} \) and therefore the positive co-degree condition guarantees it is contained in \( k \) hyperedges. In order for these hyperedges to intersect the \( r + 1 \) hyperedges \( Q \cup T \) for each petal \( Q \) of \( S_Y \) we have that each set of the form \( ((Y \cup Z) \setminus T) \cup \{s\} \cup P \) for \( s \in T \) must be a hyperedge of \( \mathcal{H} \).

We are now ready to complete the proof of Theorem [8]. Suppose that there is a hyperedge \( h \in E(\mathcal{H}) \) such that \( |h \cap (Y \cup Z)| \leq k - 1 \). Then there exists a \( k \)-set \( T \subset Y \cup Z \) such that \( T \) is disjoint from \( h \). Moreover, there is a petal \( Q \) in \( S_Y \) that is disjoint from \( h \). By Claim [10] we have that \( T \cup Q \in E(\mathcal{H}) \) which is disjoint from \( h \in E(\mathcal{H}) \). This violates the intersecting property of \( \mathcal{H} \), a contradiction.

Therefore, every hyperedge \( h \in E(\mathcal{H}) \) intersects \( Y \cup Z \) in at least \( k \) vertices. This implies that \( \mathcal{H} \) is a subhypergraph of a \( k \)-kernel system, i.e., as \( \mathcal{H} \) is edge-maximal it is exactly a \( k \)-kernel system.

**Remark.** Observe that the proof of Theorem [8] gives a stability result. In particular, if \( \mathcal{H} \) has enough edges to apply Lemma [6] then we have that \( \mathcal{H} \) is a subhypergraph of a \( k \)-kernel system.
3 Improved thresholds on $n$

We now show that in the case $k \leq 3$, Theorem 3 holds for $n$ only about $r^{k+1}$. In Theorem 3 we need $n$ to be about $r^k$. Recall that two hypergraphs $A$ and $B$ are cross-intersecting if for every pair of hyperedges $A \in E(H)$ and $B \in E(H)$ we have $A \cap B \neq \emptyset$. Also, a transversal for a hypergraph $H$ is a set of vertices $T$ such that $X \cap h \neq \emptyset$ for every hyperedge $h \in E(H)$. The transversal number $\tau(H)$ is the minimum $t$ such that there is a transversal $T$ of $H$ of size $t$.

We begin with a result due to Frankl [6] (see also [11]) on the size of an intersecting hypergraph with given minimum transversal size.

**Lemma 11** (Frankl, [6]). Let $H$ be an intersecting $r$-uniform $n$-vertex hypergraph with minimal transversal size $\tau(H) \geq t$, then

$$|E(H)| \leq (r^{t-1} + o(1)) \left( \frac{n-t}{r-t} \right).$$

**Proposition 12.** Let $H$ be an intersecting $r$-uniform $n$-vertex hypergraph with minimum positive co-degree $\delta^+_{r-1}(H) \geq 2$. If $H$ has the maximum number of hyperedges, then for $n$ large enough $H$ is a 3-kernel system.

**Proof.** We distinguish three cases based on the minimum transversal size $\tau(H)$ of $H$.

**Case 1:** $\tau(H) = 1$.

Then there is a vertex $x$ in each hyperedge of $H$. Fix a hyperedge $h \in E(H)$ and observe that the $(r-1)$-set $h \setminus x$ is contained in exactly one hyperedge which violates the positive co-degree condition.

**Case 2:** $\tau(H) \geq 3$.

Then Lemma 11 gives

$$|E(H)| \leq (r^2 + o(1)) \left( \frac{n-3}{r-3} \right),$$

which for $n = \Omega(r^3)$ is smaller than $3\left( \frac{n-3}{r-2} \right)$, a contradiction.

**Case 3:** $\tau(H) = 2$.

Let $\{x, y\}$ be a minimum transversal of $H$. Consider the $(r-1)$-uniform hypergraphs $H_x = \{h \setminus x \mid h \in E(H)\}$ and $h \cap \{x, y\} = \{x\}$ and $H_y = \{h \setminus y \mid h \in E(H)\}$ and $h \cap \{x, y\} = \{y\}$. First observe that this pair of hypergraphs is cross-intersecting as $H$ is intersecting. Now observe that any hyperedge $h \in E(H_x)$ is a set of size $r-1$ that is contained in a hyperedge of $H$. Thus, $h$ has co-degree at least 2 and, therefore must be a member of $H_y$. This implies that $H_x$ and $H_y$ are identical. Therefore $H_x$ is intersecting.

A simple calculation shows that if $H_x$ is not a maximal star, then $H$ has too few hyperedges. Thus, every hyperedge of $H_x$ contains a fixed vertex $z$. Therefore, every hyperedge of $H$ contains at least two of $\{x, y, z\}$, i.e., maximality implies that $H$ is a 3-kernel system.

We now turn to the case when $k = 3$. Here we are not able to show uniqueness of the extremal construction. However, we do give a matching upper-bound that holds for a larger range of values on $n$ than in Theorem 3. We will need two lemmas. The first is due to Frankl [7].
Lemma 13 (Frankl, [7]). Let $A$ and $B$ be cross-intersecting hypergraphs on vertex set $[N]$ such that $A$ is $a$-uniform and $B$ is $(a + 1)$-uniform and intersecting. If $N > 2a + 1$, then

$$|A| + |B| \leq \binom{N}{a}.$$

Part of our next lemma was proved in Lemma 7. We include a full argument here to keep this section is self-contained.

Lemma 14. Let $H$ be an intersecting $r$-uniform $n$-vertex hypergraph with minimum positive co-degree $\delta^+_{r-1}(H) \geq k$. If $H$ is edge-maximal and $n$ is large enough, then $H$ has minimal transversal size $\tau(H) = k$.

Proof. First suppose that $\tau(H) < k$. Let $X$ be a minimal transversal for $H$ and consider a hyperedge $h$ that intersects $X$ in exactly one element. Such a hyperedge exists as otherwise $X$ is not minimal. The $(r-1)$-set $h \setminus X$ is contained in at most $k-1$ hyperedges of $H$; one for each element of $X$. This contradicts the co-degree condition on $H$.

Now suppose that $\tau(H) > k$. Lemma 11 gives $|E(H)| = (r^k + o(1))(n^{r-k-1})$. On the other hand, our construction has at least $\binom{2k-1}{k} \binom{n-2k+1}{r-k}$ hyperedges. Therefore, for $n = \Omega(r^{k+1})$ we have a contradiction, thus, $\tau(H) = k$.

Finally, we need a technical definition to construct auxillary hypergraphs from $H$.

Definition 15. Let $H$ be an $r$-uniform hypergraph and let $T$ be a fixed set of vertices in $H$. For a subset $S \subset T$ define

$$H^T_S = \{ h - S \mid h \in E(H) \text{ and } h \cap T = S \},$$

i.e., $H_S$ is the $(r-|S|)$-uniform hypergraph constructed by removing $S$ from each hyperedge of $H$ that intersects $T$ in exactly $S$.

For ease of notation we will often denote $H^T_S$ by $H^T_{x_1x_2 \ldots x_s}$ when $S = \{x_1, x_2, \ldots, x_s\}$.

Theorem 16. Let $H$ be an intersecting $r$-uniform $n$-vertex hypergraph with minimum positive co-degree $\delta^+_{r-1}(H) \geq 3$. If $H$ has the maximum number of hyperedges, then for $n$ large enough,

$$|E(H)| = 10 \binom{n-5}{r-3} + 5 \binom{n-5}{r-4} + \binom{n-5}{r-5}.$$  

Note that Theorem 16 holds with a smaller threshold on $n$ than in Theorem 3 when $k = 3$, but we do not prove uniqueness of the extremal construction.

Proof. By Lemma 14 we may assume the minimum transversal size of $H$ is $\tau(H) = 3$. Let $X = \{x, y, z\}$ be a minimum transversal of $H$.

Consider the three $(r-1)$-uniform hypergraphs $H^X_x$, $H^X_y$ and $H^X_z$. First observe that any pair of these hypergraphs is cross-intersecting as $H$ is intersecting. Now observe that any hyperedge $h \in H^X_x$ is a set of size $r - 1$ that is contained in a hyperedge of $H$. Therefore, $h$ has co-degree at least 3. This implies that $h$ is also a member of $H^X_y$ and $H^X_z$. Thus, all three hypergraphs $H^X_x, H^X_y, H^X_z$ are identical. Moreover, this implies that $H^X$ is intersecting.

Recall that the shadow of an $r$-uniform hypergraph $G$ is the collection of all $(r-1)$-sets contained in a hyperedge of $G$. We denote the shadow of $G$ by $\Delta(G)$.  

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Claim 17. For each hyperedge $h \in \mathcal{H}^X_{xy}$ there is some hyperedge $g \in \mathcal{H}^X_x$ that contains $h$. Thus, 
\[ |E(\mathcal{H}^X_{xy})| \leq |\Delta(\mathcal{H}^X_x)|. \]

Proof. Let $h$ be an arbitrary hyperedge of $\mathcal{H}^X_{xy}$. Consider the $(r - 1)$-set $A = h \cup \{y\}$. The set $A$ has co-degree at least 3, so it is contained in three hyperedges of $\mathcal{H}$; one such edge is $A \cup \{z\}$, another could be $A \cup \{x\}$, so there exists at least one hyperedge of the form $A \cup \{w\}$ where $w \not\in \{x, y, z\}$. However, $A \cap \{x, y, z\} = \{y\}$, so $(A \cup \{w\}) \setminus y \in E(\mathcal{H}^X_y) = E(\mathcal{H}^X_x)$. \hfill $\square$

We distinguish three cases based on $\tau(\mathcal{H}_x^X)$.

Case 1: $\tau(\mathcal{H}_x^X) = 1$.

Let $u$ be a minimal transversal of $\mathcal{H}_x^X$. Every hyperedge of $\mathcal{H}_x^X, \mathcal{H}_y^X, \mathcal{H}_z^X$ contains $u$, therefore, every hyperedge of $\mathcal{H}$ contains at least two vertices from $\{x, y, z, u\}$. Put $T = X \cup \{u\} = \{x, y, z, u\}$.

Claim 18. All six hypergraphs $\mathcal{H}_{ij}^T$ for $i, j \in T = \{x, y, z, u\}$ are identical.

Proof. It is enough to show that $\mathcal{H}_{xy}^T \subseteq \mathcal{H}_{xz}^T$ as the choice of the three vertices $x, y, z$ from $T$ is arbitrary. Let $h \in E(\mathcal{H}_{xy}^T)$ and consider the $(r - 1)$-set $A = h \cup \{x\}$. By the co-degree condition on $\mathcal{H}$ we have that $A$ is contained in at least three hyperedges. Each of these hyperedges includes at least two vertices from $\{x, y, z, u\}$, so $A$ is in $A \cup \{y\}, A \cup \{z\}$ and $A \cup \{u\}$. Therefore, $h \in E(\mathcal{H}_{xz}^T)$.

Now as $\mathcal{H}_{xy}^T$ and $\mathcal{H}_{zu}^T$ are cross-intersecting we have that $\mathcal{H}_{xy}^T$ is intersecting. Thus, 
\[ |E(\mathcal{H})| \leq 6 \binom{n - 5}{r - 3} + 4 \binom{n - 4}{r - 3} + \binom{n - 4}{r - 4}. \]

Applying Pascal's identity gives 
\[ |E(\mathcal{H})| \leq 10 \binom{n - 5}{r - 3} + 5 \binom{n - 5}{r - 4} + \binom{n - 5}{r - 5}. \]

Case 2: $\tau(\mathcal{H}_x^X) = 2$.

Let $u, v$ be a minimal transversal of $\mathcal{H}_x^X$, i.e., every hyperedge of $\mathcal{H}_x^X$ contains at least one of $u, v$. As $\mathcal{H}_x^X = \mathcal{H}_y^X = \mathcal{H}_z^X$ we have that every hyperedge of $\mathcal{H}$ contains at least two vertices from $T = \{x, y, z, u, v\}$. Note that there is no hyperedge that intersects $T$ in exactly $u$ and $v$, so $\mathcal{H}_{uv}^T$ is empty. For simplicity, we consider the empty hypergraph as intersecting.

Claim 19. The hypergraph $\mathcal{H}_{ij}^T$ is intersecting for any $i, j \in T$.

Proof. Suppose not. Then there are hyperedges $A, B \in \mathcal{H}_{ij}^T$ such that $A \cap B = \emptyset$. By the co-degree condition, the $(r - 1)$-set $A \cup i$ is contained in at least three hyperedges of $\mathcal{H}$. Since each hyperedge of $\mathcal{H}$ contains at least two elements from $T$, there is a hyperedge $A \cup \{i, s\}$ where $s \in T \setminus \{i, j\}$. Similarly, the $(r - 1)$-set $B \cup \{j\}$ is contained in some hyperedge $B \cup \{j, t\}$ where $t \in T \setminus \{s, i, j\}$. However, the hyperedges $A \cup \{i, s\}$ and $B \cup \{j, t\}$ are disjoint which violates the intersecting property of $\mathcal{H}$.

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Now any $H^T_{ij}$ and $H^T_{T \setminus \{i,j\}}$ are cross-intersecting and $(r - 2)$ and $(r - 3)$-uniform, respectively. Therefore, by Lemma 13 we have

$$|E(H^T_{ij})| + |E(H^T_{T \setminus \{i,j\}})| \leq \binom{n - 5}{r - 3}.$$ 

Thus

$$|E(H)| = \sum_{S \subseteq T} |E(H^T_S)| \leq 10 \binom{n - 5}{r - 3} + 5 \binom{n - 5}{r - 4} + \binom{n - 5}{r - 5}.$$

Case 3: $\tau(H^X_{i}) \geq 3$.

Then Lemma 11 gives

$$|E(H^X_{i})| \leq ((r - 1)^2 + o(1)) \left( \binom{n - 1 - 3}{r - 1 - 3} \right) \leq (r^2 + o(1)) \binom{n - 4}{r - 4}.$$ 

The remaining hyperedges of $H$ are counted by $H^X_{xyz}$ and $H^X_{ij}$ for $i, j \in \{x, y, z\}$. By Claim 17 we have

$$|E(H^X_{xyz})| \leq |\Delta(H^X_{i})| \leq (r - 1) |E(H^X_{x})| \leq (r^3 + o(1)) \binom{n - 4}{r - 4}.$$ 

Finally, $|E(H^X_{xyz})| \leq \binom{n - 3}{r - 3}$. Thus,

$$|E(H)| \leq \binom{n - 3}{r - 3} + 6(r^3 + r^2 + o(1)) \binom{n - 6}{r - 4}$$

which is smaller than $10 \binom{n - 5}{r - 3}$ for $n = \Omega(r^4)$.

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