CLOSED MINIMAL SURFACES IN CUSPED HYPERBOLIC THREE-MANIFOLDS

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Abstract. Motivated by classical theorems on minimal surface theory in compact hyperbolic 3-manifolds, we investigate the questions of existence and deformations for least area minimal surfaces in complete noncompact hyperbolic 3-manifold of finite volume. We prove any closed immersed incompressible surface can be deformed to a closed immersed least area surface within its homotopy class in any cusped hyperbolic 3-manifold. Our techniques highlight how special structures of these cusped hyperbolic 3-manifolds prevent any least area minimal surface going too deep into the cusped region.

1. Introduction

1.1. Minimal surfaces in hyperbolic 3-manifolds. Minimal surfaces are fundamental objects in geometry. In 3-manifold theory, the existence and multiplicity of minimal surfaces often offer important geometrical insight into the structure of the ambient 3-manifold (see for instance [Rub05, Mee06]), they also have important applications in Teichmüller theory, Lorentzian geometry and many other mathematical fields (see for example [Rub07, KS07]). By Thurston’s geometrization theory, the most common geometry in a 3-manifold is hyperbolic ([Thu80]), and this paper is a part of a larger goal of studying closed incompressible minimal surfaces in hyperbolic 3-manifolds.

Before we state our main result, we briefly motivate our effort by making some historic notes on minimal surface theory in three different types of hyperbolic 3-manifolds, namely, compact hyperbolic 3-manifolds, quasi-Fuchsian manifolds, and cusped hyperbolic 3-manifolds (complete, noncompact, and of finite volume).

Let $M^3$ be a complete Riemannian 3-manifold (with or without boundary), and let $\Sigma$ be a closed surface which is immersed or embedded in $M^3$, then $\Sigma$ is called a minimal surface if its mean curvature vanishes identically, further we call it least area if the area of $\Sigma$ with respect to the induced metric from $M^3$ is no greater than that of any other surface which is homotopic or isotopic to $\Sigma$ in $M^3$.

A closed surface is called incompressible in $M^3$ if the induced map between the fundamental groups is injective, where we don’t require that an incompressible map to be
an embedding. Throughout this paper, we always assume that a closed incompressible surface is of genus at least two and is oriented.

In the case when $M^3$ is a closed Riemannian 3-manifold, Schoen and Yau ([SY79]) and Sacks and Uhlenbeck ([SU82]) showed that if $S \subset M^3$ is a closed incompressible surface, then $S$ is homotopic to an immersed least area minimal surface $\Sigma$ in $M^3$. The techniques of [SY79, SU82] extend to the case $M^3$ is a compact (negatively curved) 3-manifold with mean convex boundary (i.e. $\partial M^3$ has non-negative mean curvature with respect to the inward normal vector), then there still exists an immersed least area minimal surface $\Sigma$ in any homotopy class of incompressible surfaces (see [MSY82, HS88]). Note that the existence of immersed closed surfaces in closed hyperbolic 3-manifolds (to start the minimization process) follows from recent remarkable resolution of the surface subgroup conjecture by Kahn-Markovic ([KM12]).

Recall that a quasi-Fuchsian manifold is a complete (of infinite volume) hyperbolic 3-manifold diffeomorphic to the product of a closed surface and $\mathbb{R}$. Since the convex core of any geometrically finite quasi-Fuchsian manifold is compact with mean convex boundary, one finds the existence of closed incompressible surface of least area in this class of hyperbolic 3-manifolds. In [Uhl83], Uhlenbeck initiated a systematic study of the moduli theory of minimal surfaces in hyperbolic 3-manifolds, where she also studied a subclass of quasi-Fuchsian manifolds which we call almost Fuchsian. $M^3$ is called almost Fuchsian if it admits a closed minimal surface of principal curvatures less than one in magnitude. Such a minimal surface is unique and embedded in the almost Fuchsian manifold (see also [FHS83]), and therefore one can study the parameterization of the moduli of almost Fuchsian manifolds via data on the minimal surface (see for instance [GHW10, HW13, San13]). For the uniqueness and multiplicity questions of minimal surfaces in quasi-Fuchsian manifolds, or in general hyperbolic 3-manifolds, one can refer to [And83, Wan12, HL12, HW15] and references within.

This paper will address the existence question for immersed closed incompressible minimal surfaces in another important class of hyperbolic 3-manifolds: cusped hyperbolic 3-manifolds. $M^3$ is called a cusped hyperbolic 3-manifold if it is a complete non-compact hyperbolic 3-manifold of finite volume. There are many examples of this type, frequently the complements of knots and links in the 3-sphere $S^3$. Mostow rigidity theorem ([Mos73]) extends to this class of hyperbolic 3-manifolds by Prasad ([Pra73]), however the techniques used in [SY79, SU82] to find incompressible minimal surfaces do not. It is well-known that any cusped hyperbolic 3-manifold admits infinitely many immersed closed minimal surfaces ([Rub05]), however, they may not be embedded, nor incompressible. Using min-max theory, very recently, Collin, Hauswirth, Mazet and Rosenberg in [CHMR14, Theorem A] proved the existence of an embedded (not necessarily incompressible) compact minimal surface in $M^3$. It has been a challenge to show the existence of immersed (or embedded) closed incompressible minimal surface in hyperbolic 3-manifolds.

For the rest of the paper, we always assume $M^3$ is an oriented cusped hyperbolic 3-manifold.
1.2. Main result. In 3-manifold theory, it is a question of basic interest to ask if one can deform an immersed surface in its homotopy class to some area minimizing surface. Instead of looking for the existence of an oriented, immersed, closed, incompressible minimal surface in a cusped hyperbolic 3-manifold $M^3$, we aim to prove that one can deform any immersed closed incompressible surface into a least area minimal surface in its homotopy class. More specifically, we show:

**Theorem 1.1.** Let $S$ be a closed orientable surface of genus at least two, which is immersed in a cusped hyperbolic 3-manifold $M^3$. If $S$ is incompressible, then $S$ is homotopic to an immersed least area minimal surface in $M^3$.

We use relatively elementary tools, taking advantage special structure of the cusps. Given the cusped hyperbolic 3-manifold $M^3$ and an immersed incompressible surface $S$, we make one truncation to obtain a compact 3-manifold $M^3(\tau_4)$ of negative curvature and totally geodesic boundary. The location where this truncation takes place is determined by $M^3$ and $S$ (see Remark 2.3). We obtain quantitative estimates on how deep this least area minimal surface can reach into the cusped region of $M^3$ (see Remark 2.3 and Corollary 5.7). The geometric structures both in the upper-half space $\mathbb{H}^3$ and the cusped hyperbolic 3-manifold $M^3$ play crucial role in our arguments to keep the least area minimal surface in the region not arbitrarily far into the cusp. We observe that any cusped region is a topologically solid torus with the core curve removed, and an area minimizing closed incompressible surface can only have certain ways to intersect the boundary of a cusped region.

Our techniques easily apply to the case when an embedded incompressible surface is in presence in $M^3$, namely, we prove the following statement, which was originally shown by Collin, Hauswirth, Mazet and Rosenberg:

**Corollary 1.2 ([CHMR14, Theorem B]).** Let $S$ be a closed orientable embedded surface in a cusped hyperbolic 3-manifold $M^3$ which is not a 2-sphere or a torus. If $S$ is incompressible and non-separating, then $S$ is isotopic to an embedded least area minimal surface.

Their original argument for this result is to cut further and further into the cusp(s), and apply the results [HS88] each time to obtain a sequence of least area minimal surfaces, then show there is at least one such minimal surface in the hyperbolic region by applying two forms of the maximum principle.

Note that a general existence theorem for an immersed closed essential surface in any cusped hyperbolic 3-manifold was established in [CLR97]. It is very special that there exist some cusped hyperbolic 3-manifolds which do NOT admit any embedded closed essential surfaces ([Hat82]).

1.3. Outline of the proof. We actually prove the embedded case first, namely, Corollary 1.2. There are essentially two parts for it. First we modify the hyperbolic metric in $\mathbb{H}^3$ to obtain a submanifold of $M^3$ in the quotient with sufficiently long cusped regions, and the modified metric around all boundaries so that the submanifold is a
compact negatively curved manifold with totally geodesic boundaries. By results of [MSY82, HS88], there is a least area minimal surface $\Sigma$ (with respect to the new metric, not the hyperbolic metric) in the homotopy class of a closed incompressible surface $S$ in this compact submanifold. The heart of the argument is then to guarantee it does not drift into infinity of $M^3$. We deploy a co-area formula (see Lemma 5.8) as our main tool for this. We can then show that $\Sigma$ is actually contained in the subregion of the submanifold which is still equipped with the hyperbolic metric. Hence $\Sigma$ is a least area minimal surface with respect to the hyperbolic metric. It is oriented as well since the surface $S$ is non-separating. To prove our main theorem, we lift an immersed essential surface to an embedded non-separating and incompressible surface in a finite cover of $M^3$, where we can apply prior arguments and take advantage of the hyperbolic geometry of a cusped hyperbolic 3-manifold to show the existence of a least area minimal surface in the homotopy class of any closed immersed incompressible surface.

1.4. Organization. The organization of the paper is as follows: in §2, we cover necessary background material and fix some notations; in §3, we modify the upper-half space model of $\mathbb{H}^3$ to set up hemispheres as barriers for minimal surfaces in $\mathbb{H}^3$; in §4, we move down to the cusped hyperbolic 3-manifold $M^3$ and its maximal cusped regions. Using the modification in previous section we obtain a truncated Riemannian 3-manifold of negative curvature. Finally in §5, we prove our main result.

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2. Preliminary

2.1. Kleinian groups and cusped hyperbolic 3-manifolds. We will work with the upper-half space model of the hyperbolic space $\mathbb{H}^3$, i.e.

$$\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\},$$

equipped with metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}. \quad (2.1)$$

The hyperbolic space $\mathbb{H}^3$ has a natural compactification: $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The orientation preserving isometry group of the upper-half space $\mathbb{H}^3$ is given by $\text{PSL}_2(\mathbb{C})$, which consists of linear fractional transformations that preserve the upper-half space.
A (torsion free) discrete subgroup $\Gamma$ of $\text{PSL}_2(\mathbb{C})$ is called a Kleinian group, and the quotient space $M^3 = \mathbb{H}^3 / \Gamma$ is a complete hyperbolic 3-manifold whose fundamental group $\pi_1(M^3)$ is isomorphic to $\Gamma$. Conversely, if $M^3$ is a complete hyperbolic 3-manifold, then there exists a holonomy $\rho : \pi_1(M^3) \to \text{PSL}_2(\mathbb{C})$ such that $\Gamma = \rho(\pi_1(M^3))$ is a (torsion free) Kleinian group and $M^3 = \mathbb{H}^3 / \rho(\pi_1(M^3))$.

Mostow-Prasad’s Rigidity Theorems imply that hyperbolic volume is a topological invariant for hyperbolic 3-manifolds of finite volume, that is to say, these hyperbolic 3-manifolds are completely determined by their fundamental groups. Jørgensen and Thurston (see [Thu80, Chapter 5–6]) proved that the set of volumes of orientable hyperbolic 3-manifolds is well ordered and of order type $\omega^\omega$. Since any non-orientable hyperbolic 3-manifold is double-covered by an orientable hyperbolic 3-manifold, then the set of volumes of all hyperbolic 3-manifolds is also well ordered.

Many examples of the cusped hyperbolic 3-manifold come from the complements of hyperbolic knots [Thu82, Corollary 2.5] on $\mathbb{S}^3$. In general cusped hyperbolic 3-manifolds can be described as follows (see [Thu80, Theorem 5.11.1]):

**Theorem 2.1.** A cusped hyperbolic 3-manifold is the union of a compact submanifold which is bounded by tori and a finite collection of horoballs modulo $\mathbb{Z} \oplus \mathbb{Z}$ actions.

By the works of Marden, Thurston, Bonahon ([Mar74, Thu80, Bon86]), any closed incompressible surface of genus at least two in a cusped hyperbolic 3-manifold is always geometrically finite, i.e. it’s either quasi-Fuchsian or essential with accidental parabolics (see also the proof of Theorem 5.3 in [Wu04]). It is well-known that some cusped hyperbolic 3-manifolds do not contain any embedded closed incompressible surfaces ([Hat82]). A fundamental fact about any cusped hyperbolic 3-manifold is the following property which can be found in for instance survey [AFW15]:

**Theorem 2.2.** The fundamental group of a cusped hyperbolic three-manifold is LERF, i.e. locally extended residually finite.

As a corollary of Theorem 2.2, if $S$ is a closed incompressible surface (with genus $\geq 2$) immersed in a cusped hyperbolic 3-manifold $M^3$, then $S$ can be lifted to an embedded nonseparating closed incompressible surface, in a finite cover of $M^3$ (see [Sco78, Sco85, Lon88, Mat02]).

### 2.2. Maximal cusps and maximal cusped regions.

In this subsection, we briefly describe the maximal cusps and maximal cusped regions of the cusped hyperbolic 3-manifold $M^3$, and they will play important roles in our construction. For more details, one can go to for instance [Ada05, Mar07].

Suppose that $M^3$ has been decomposed into a compact component (which is called the compact core of $M^3$) and a finite set of cusps (or ends), each homeomorphic to $T^2 \times [0, \infty)$, where $T^2$ represents a torus. Each cusp can be realized geometrically as the image of some horoball $\mathcal{H}$ in $\mathbb{H}^3$ under the covering map from $\mathbb{H}^3$ to $M^3$. If we lift any such cusp to the upper-half space model $\mathbb{H}^3$ of the hyperbolic space, we obtain a parameter family of disjoint horoballs.
Assume first that \( M^3 \) has exactly one cusp, and we lift it to the corresponding set of disjoint horoballs, each of which is the image of any other by some group element. Expand the horoballs equivariantly until two first become tangent. The projection of these expanded horoballs back to \( M^3 \) is called the maximal cusped region of \( M^3 \), denoted by \( C \).

Assume that one such horoball \( \mathcal{H} \) is centered about \( \infty \). We may normalize the horoball \( \mathcal{H} \) so that \( \partial \mathcal{H} \) is a horizontal plane with Euclidean height \( \text{one above the xy-plane} \). Thus \( \mathcal{H} = \{(x, y, t) \mid t \geq 1 \} \). Let \( \rho : \pi_1(M^3) \to \text{PSL}_2(\mathbb{C}) \) be the holonomy of \( M^3 \). Then \( \Gamma = \rho(\pi_1(M^3)) \) is a (torsion free) Kleinian group with parabolic elements. Let \( \Gamma_\infty \) be the parabolic subgroup of \( \Gamma \) which fixes \( \infty \), it’s then well-known that \( \Gamma_\infty \) is generated by two elements \( z \mapsto z + \mu \) and \( z \mapsto z + \nu \), where \( \mu \) and \( \nu \) are non-trivial complex numbers which are not real multiples of each other. Obviously \( \mathcal{H} \) is invariant under \( \Gamma_\infty \), and the quotient \( \mathcal{H}/\Gamma_\infty \) is just the maximal cusped region \( C \) of \( M^3 \) described above. Also \( T^2 = \partial \mathcal{H}/\Gamma_\infty \) is a torus.

The fundamental domain of the parabolic group \( \Gamma_\infty \) in the horoball \( \mathcal{H} \) is denoted by \( A \times [1, \infty) \), where \( A \subset \partial \mathcal{H} \) is a parallelogram spanned by the complex numbers \( \mu \) and \( \nu \). It is not hard to see that the Euclidean area of \( A \), which is given by \( \text{Im}(\mu \nu) \), is the same as that of the torus \( T^2 \).

We may equip the horoball \( \mathcal{H} \) with the warped product metric \( ds^2 = e^{-2\tau}(dx^2 + dy^2) + d\tau^2 \), by letting \( \tau = \log t \) for \( t \geq 1 \). Then the metric on the maximal cusped region \( C = T^2 \times [0, \infty) \) can be written in the form

\[
(2.2) \quad ds^2 = e^{-2\tau} ds_{\text{eucl}}^2 + d\tau^2, \quad \tau \geq 0,
\]

where \( ds_{\text{eucl}}^2 \) is the standard flat metric on the torus \( T^2 \) induced from that of \( \partial \mathcal{H} \).

If \( M^3 \) has more than one cusp, we define the maximal cusped region for each cusp exactly as above. It’s possible that the maximal cusped regions in a cusped hyperbolic 3-manifold can intersect.

Now suppose that the cusped hyperbolic 3-manifold \( M^3 \) has \( k \) cusps, whose maximal cusped regions are denoted by \( C_i = T^2 \times [0, \infty), i = 1, \ldots, k \). Let \( \tau_0 > 0 \) be the smallest number such that each maximal cusped region \( T^2 \times (\tau_0, \infty), i = 1, 2, \ldots, k \), is disjoint from any other maximal cusped regions of \( M^3 \).

For any constant \( \tau \geq \tau_0 \), let \( M^3(\tau) \) be the compact subdomain of \( M^3 \) which is defined as follows:

\[
(2.3) \quad M^3(\tau) = M^3 - \bigcup_{i=1}^{k} (T^2 \times (\tau, \infty)) \ .
\]

By this construction, \( M^3(\tau) \) is a compact submanifold of \( M^3 \) with concave boundary components with respect to the inward normal vectors.

For each \( i \) with \( 1 \leq i \leq k \), we lift \( M^3 \) to the upper-half space model of the hyperbolic space \( \mathbb{H}^3 \) such that one horoball \( \mathcal{H}_i \) corresponding to the maximal cusped region \( C_i \) is centered at \( \infty \) and \( \partial \mathcal{H}_i \) passes through the point \( (0, 0, 1) \). Suppose that \( \Gamma_\infty^i \) is the
subgroup of $\Gamma$, which is generated by two elements $z \mapsto z + \mu_i$ and $z \mapsto z + \nu_i$, where $\mu_i$ and $\nu_i$ are non-trivial complex numbers that are not real multiples of each other.

Now we may define a constant as follows:

(2.4) \[ L_0 = \max \left\{ e^{\tau_0}, |\mu_1| + |\nu_1|, \ldots, |\mu_k| + |\nu_k| \right\} > 0. \]

Remark 2.3. Note that this constant is independent of $S$. If $S$ is embedded in $M^3$, we will prove that the closed incompressible least area minimal surface $\Sigma$ in Corollary 1.2 is contained in $M^3(\tau_3)$, where $\tau_3 = \log(3L_0)$. If $S$ is only assumed to be immersed in $M^3$, then by Theorem 2.2, we may lift $S$ to an embedded incompressible surface in a finite cover $N^3$ of $M^3$, which is also a cusped hyperbolic 3-manifold. In this case, we will show that immersed minimal surface in Theorem 1.1 is contained in $M^3(\tilde{\tau}_3)$ for $\tilde{\tau}_3 = \log(3\tilde{L}_0)$, where $\tilde{L}_0$ is defined similarly according to the information of the cusped regions of $N^3$.

3. Constructing Barriers in Hyperbolic Three-space

In this section we work entirely in the hyperbolic space $\mathbb{H}^3$ instead of the quotient cusped hyperbolic 3-manifold $M^3$. Our goal will be to construct hemispheres in $\mathbb{H}^3$ which can be used as barriers for minimal surfaces. To do this, we will first modify the standard hyperbolic metric on $\mathbb{H}^3$ to get a new metric which is non-positively curved. This procedure gives us the flexibility we need to obtain barriers.

3.1. Modifying the hyperbolic space. For fixed constants $L_2 > L_1 > 0$, we define a smooth cut-off function $\varphi : (0, \infty) \to [0, \infty)$ as follows (see Figure 1):

(i) $\varphi(t) = \frac{1}{t}$, if $0 < t \leq L_1$;
(ii) $\varphi(t)$ is is strictly decreasing on $[L_1, L_2)$, with $\varphi(L_1) = \frac{1}{L_1}$ and $\varphi(L_2) = 0$;
(iii) $\varphi(t) \equiv 0$ if $t \geq L_2$;
(iv) We also require $\varphi$ to satisfy the following inequality:

(3.1) \[ 0 \leq \varphi(t) \leq \frac{1}{t}, \quad \text{for all } t > 0. \]

Figure 1. A graph of $\varphi(t)$

Figure 2. A graph of $f(t)$
We now define another smooth function $f(t) : (0, \infty) \rightarrow (0, \infty)$ by solving the following equation:

\[(3.2) \quad \frac{f'(t)}{f(t)} = \varphi(t), \quad \text{for all } t > 0.\]

And we may require $f(t)$ to satisfy the following (see Figure 1) properties:

(i) $f(t) = t$, if $0 < t \leq L_1$;
(ii) $f(t)$ is strictly increasing on the interval $(L_1, L_2)$;
(iii) $f(t)$ is a constant, if $t \geq L_2$.

Now we consider an upper-half space model of the modified hyperbolic space $(\mathbb{U}^3, \bar{g})$, constructed as follows:

(i) $\mathbb{U}^3 = \mathbb{R}^3_+ = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$;
(ii) the new metric is given by

\[(3.3) \quad \bar{g}(x, y, t) = \frac{dx^2 + dy^2 + dt^2}{(f(t))^2}.\]

Comparing with the standard hyperbolic metric (2.1) on $\mathbb{H}^3$, one sees that $\bar{g}$ is just the hyperbolic metric for $t \in (0, L_1]$, and flat beyond $t = L_2$. In fact, we have the following result, which was not explicitly listed but can be derived from the proof of [Zho99, Theorem 4.1]. We include a proof here for the sake of completeness.

**Proposition 3.1.** [Zho99] The upper-half space $(\mathbb{U}^3, \bar{g})$ is non-positively curved.

**Proof.** Recalling from (3.3), we may choose a local coordinate system such that $\bar{g}_{ij} = \frac{\delta_{ij}}{f(t)^2}$, for $\{i, j\} = \{1, 2, 3\}$. We can then workout the Christoffel symbols $\{\bar{\Gamma}^k_{ij}\}$ with respect to this metric $\bar{g}$ according to the formula:

\[\bar{\Gamma}^k_{ij} = \frac{1}{2} \bar{g}^{km}(\bar{g}_{mi,j} + \bar{g}_{mj,i} - \bar{g}_{ij,m}).\]

We find these Christoffel symbols are:

(i) $\bar{\Gamma}^1_{13} = \bar{\Gamma}^1_{31} = \bar{\Gamma}^2_{23} = \bar{\Gamma}^2_{32} = \bar{\Gamma}^3_{33} = -\frac{f'(t)}{f(t)}$,
(ii) $\bar{\Gamma}^3_{11} = \bar{\Gamma}^3_{22} = \frac{f'(t)}{f(t)}$, and
(iii) all others are equal to 0.

One can then verify the sectional curvatures of the space $(\mathbb{U}^3, \bar{g})$ at a point $(x, y, t)$ are given by

\[(3.4) \quad K_{12} = -(f'(t))^2, \quad \text{and} \quad K_{13} = K_{23} = f''(t)f(t) - (f'(t))^2.\]

Note that, by (3.2), we have

\[
\frac{f''(t)f(t) - (f'(t))^2}{f^2(t)} = \left(\frac{f'(t)}{f(t)}\right)' = \varphi'(t) \leq 0, \quad \text{for all } t > 0.
\]

Therefore the space $(\mathbb{U}^3, \bar{g})$ is non-positively curved. \qed
In order to show a convexity statement in Theorem 3.3, we need to calculate the principal curvatures of some surfaces immersed in \((\mathbb{U}^3, \bar{g})\), if these surfaces are special with respect to a metric that is conformal to \(\bar{g}\) in \(\mathbb{U}^3\). The tool can be found in the following more general lemma:

**Lemma 3.2 ([Lóp13]).** For \(m \geq 3\), let \((\mathcal{M}, g)\) be an \(m\)-dimensional Riemannian manifold and let \(\sigma : \mathcal{M} \to \mathbb{R}^+\) be a smooth positive function on \(\mathcal{M}\). Define the metric \(\bar{g} = \sigma^2 g\). Let \(\iota : S \to \mathcal{M}\) be an immersion of an orientable hypersurface. If \(\kappa\) is a principal curvature of \((S, \iota^*g)\) with respect to the unit normal vector field \(N\), and then

\[
\bar{\kappa} = \frac{\kappa}{\sigma} - \frac{1}{\sigma^2} d\sigma(N)
\]

is a principal curvature of \((S, \iota^*\bar{g})\) with respect to the unit normal vector field \(\bar{N} = N/\sigma\), and \(d\sigma(N)\) is the differential of \(\sigma\) along \(N\).

By Proposition 3.1, we know that the space \((\mathbb{U}^3, \bar{g})\) is non-positively curved. We now want to understand the structure of some special figures in \((\mathbb{U}^3, \bar{g})\). This will become important in Theorems 4.1 and 4.2: we need to construct a submanifold in \(M^3\) is of negative curvature and it is a quotient from a subregion in \(\mathbb{H}^3\) by the same Kleinian group.

**Theorem 3.3.** The subspace \(\{(x, y, t) \in \mathbb{U}^3 \mid 0 < t < L_2\}\) is a negatively curved space (with respect to the metric \(\bar{g}\)), with a totally geodesic boundary \(\{(x, y, t) \in \mathbb{U}^3 \mid t = L_2\}\). Furthermore, any horizontal plane in \((\mathbb{U}^3, \bar{g})\) is either convex with respect to the upward normal vector \(N = (0, 0, 1)\), or totally geodesic.

**Proof.** In order to apply Lemma 3.2, on the space \(\mathbb{U}^3\), the metric \(g\) will be designated as the Euclidean metric, and the conformal factor \(\sigma(x, y, t) = \frac{1}{f(t)}\), where \(f(t)\) is defined previously, and \(\bar{g} = \frac{g}{f^2(t)}\) is the modified metric on \(\mathbb{U}^3\) which is nonpositively curved by Proposition 3.1, and negatively curved in the subspace \(\{(x, y, t) \in \mathbb{U}^3 \mid 0 < t < L_2\}\) of \(\mathbb{U}^3\).

For any horizontal plane that passes through \((0, 0, t)\), its unit normal vector at the point \((x, y, t)\) with respect to the Euclidean metric \(g\) is given by \(N = \frac{\partial}{\partial t}\).

Since

\[
d\sigma(N) = \text{grad}(1/f(t)) \cdot N = -\frac{f'(t)}{f^2(t)},
\]

where grad is the gradient with respect to the Euclidean metric \(g\) and \(\cdot\) denotes the Euclidean inner product of vectors, then by (3.5), we find the principal curvatures of the plane with respect to the new metric \(\bar{g}\)

\[
\bar{\kappa}_i(x, y, t) = 0 - f^2(t)(-\frac{f'(t)}{f^2(t)}) = f'(t), \quad i = 1, 2.
\]

By the construction of the function \(f(t)\), we have

- \(f'(t) > 0\) if \(0 < t < L_2\), and
- \(f'(t) \equiv 0\) if \(t \geq L_2\).
Therefore any horizontal plane through the \((0,0,t)\) is either convex with respect to the normal vector \(N = (0,0,1)\) if \(0 < t < L_2\), or totally geodesic if \(t \geq L_2\).

**Remark 3.4.** Similarly one can show that any vertical plane is totally geodesic, and any vertical straight line is a geodesic with respect to the new metric \(\bar{g}\).

### 3.2. Barriers.

The following result guarantees that hemispheres in \((\mathbb{U}^3, \bar{g})\) can be used as the barrier surfaces to prevent the least area minimal surface \(\Sigma\) from entering into each cusped region of \(M^3\) too far.

**Theorem 3.5.** For any positive constant \(r\), let

\[ S^2_+(r) = \{(x,y,t) \mid x^2 + y^2 + t^2 = r^2, \ t > 0\} \]

be a hemisphere in \((\mathbb{U}^3, \bar{g})\) with radius \(r\). Then \(S^2_+(r)\) is non-concave with respect to the inward normal vector field, i.e. the principal curvatures of \(S^2_+(r)\) are nonnegative with respect to the inward normal vector field.

**Proof.** Let \(g\) again denote the standard Euclidean metric on \(\mathbb{R}^3\). At a point \(p = (x,y,\sqrt{r^2 - x^2 - y^2})\) on \(S^2_+(r)\), the inward normal vector field on the hemisphere \(S^2_+(r)\) with respect to the Euclidean metric \(g\) is given by

\[ N(p) = \left(\frac{-x}{r}, \frac{-y}{r}, -\frac{\sqrt{r^2 - x^2 - y^2}}{r}\right). \]

The principal curvatures \(\kappa_1\) and \(\kappa_2\) of \(S^2_+(r) \subset (\mathbb{R}^3_+, g)\) with respect to the normal vector \(N\) are identically equal to \(\frac{1}{r}\).

As in the proof of Theorem 3.3, we set \(\sigma(x,y,t) = \frac{1}{f(t)}\), where the positive function \(f(t)\) is defined by solving the equation (3.2). Let \(\bar{\kappa}_i\) \((i = 1, 2)\) be the principal curvatures of \(S^2_+(r) \subset (\mathbb{U}^3, \bar{g})\) at \(p\) with respect to an orientation \(\bar{N}(p) = f \left(\sqrt{r^2 - x^2 - y^2}\right) N(p)\).

Now we apply (3.5), the principal curvatures \(\bar{\kappa}_i\) \((i = 1, 2)\) at \(p\) are then given by:

\[
\bar{\kappa}_i(p) = f \left(\sqrt{r^2 - x^2 - y^2}\right) \cdot \frac{1}{r} - f' \left(\sqrt{r^2 - x^2 - y^2}\right) \cdot \frac{\sqrt{r^2 - x^2 - y^2}}{r} \cdot \frac{1}{f(t)} \cdot \frac{1}{f'(t)} \cdot \frac{1}{f'(t)} \\
= \frac{f \left(\sqrt{r^2 - x^2 - y^2}\right)}{r} \left(1 - \varphi \left(\sqrt{r^2 - x^2 - y^2}\right) \sqrt{r^2 - x^2 - y^2}\right) \geq 0,
\]

where we use the property (3.1). This completes the proof. □

### 4. Truncating Cusped Hyperbolic Three-manifold

We want to construct a submanifold in a cusped hyperbolic 3-manifold \(M^3\) whose boundary components are concave with respect to the inward normal vectors. The idea is to remove some horoballs of certain sizes from \(\mathbb{H}^3\) in §4.1, then modify the hyperbolic metric in the remaining regions according to previous section, and we have to of course verify, in §4.2, that the Kleinian group \(\Gamma\) of \(M^3\) preserves the new metric (otherwise we get a different hyperbolic 3-manifold in the quotient).
4.1. **Truncated hyperbolic space.** As before we assume that the cusped hyperbolic 3-manifold $M^3$ has $k$ cusps, whose maximal cusped regions are denoted by $C_i = T^2_i \times [0, \infty)$, $i = 1, \ldots, k$. We also denote $\rho : \pi_1(M^3) \to PSL_2(\mathbb{C})$ as the holonomy so that $\Gamma = \rho(\pi_1(M^3))$ is a Kleinian group.

For the $i$-th cusped region $T^2_i \times [\tau, \infty)$, let $H^i(\tau)$ be the corresponding horoball centered at $\infty$, whose boundary is a horizontal plane passing through the point $(0, 0, e^{\tau})$, i.e.

\begin{equation}
H^i(\tau) = \{(x, y, t) \in \mathbb{H}^3 \mid t \geq e^{\tau}\}.
\end{equation}

In particular, $H^i(0)$ is the corresponding (maximal) horoball $H_i$ centered at $\infty$. We also denote $H^i_0(\tau)$ as the interior of (4.1).

Recall that $\tau_0 > 0$ is the smallest number such that each maximal cusped region $T^2_i \times (\tau_0, \infty)$, $i = 1, 2, \ldots, k$, is disjoint from any other maximal cusped regions of $M^3$. When $\tau \geq \tau_0$, the subset $\Omega(\tau)$ of $\mathbb{H}^3$ is obtained by removing a disjoint collection of open horoballs, namely,

\begin{equation}
\Omega(\tau) = \mathbb{H}^3 - \bigcup_{i=1}^{k} \bigcup_{\gamma \in \Gamma} \gamma(H^i_0(\tau))
\end{equation}

is called a truncated hyperbolic 3-space (see [BH99, p.362]).

It is clear that $\Omega(\tau)$ is invariant under $\Gamma$, so

\begin{equation}
\Omega(\tau)/\Gamma = M^3(\tau).
\end{equation}

We define four constants

\begin{equation}
\tau_j = \log(j \cdot L_0), \quad \text{for } j = 1, 2, 3, 4,
\end{equation}

where the constant $L_0$ is defined by (2.4). Note that by this definition (4.4) and by (2.4), we have $\tau_4 > \tau_3 > \tau_2 > \tau_1 \geq \tau_0 > 0$.

We are particularly interested in the subregion $\Omega(\tau_4)$, and we define a new metric on it as follows:

(i) We equip the subregion $\Omega(\tau_3)$ with the standard hyperbolic metric.

(ii) The subregion $\Omega(\tau_4)\setminus\Omega^0(\tau_3)$ (where $\Omega^0(\tau_3)$ is the interior of $\Omega(\tau_3)$) consists of countably infinitely many disjoint subregions which can be divided into $k$ families $\mathcal{H}_1, \ldots, \mathcal{H}_k$, such that each family $\mathcal{H}_i$ is the lift of the cusped subregion $T^2_i \times [\tau_3, \tau_4]$.

For an element $U_i \in \mathcal{H}_i$, we may assume that it can be described as

\begin{equation}
U_i = \{(x, y, t) \in \mathbb{H}^3 \mid 3L_0 \leq t \leq 4L_0\}.
\end{equation}

We equip the region $U_i$ with the new metric

\begin{equation}
\bar{ds}^2 = \frac{dx^2 + dy^2 + dt^2}{(f(t))^2},
\end{equation}
where the function $f$ is defined on $[3L_0, 4L_0]$ just as in §2 (i.e. $L_1 = 3L_0$ and $L_2 = 4L_0$). Similarly we may define the same new metric on the other elements in $\mathcal{H}$, and so on the elements from the other families.

We denote $\tilde{g}$ the new metric on the space $\Omega(\tau_4)$. Now we apply Theorem 3.3 to arrive at the following:

**Theorem 4.1.** The compact space $(\Omega(\tau_4), \tilde{g})$ is a negatively curved space with (countably infinitely many) totally geodesic boundary components.

4.2. **The Kleinian group.** The Kleinian group $\Gamma$ preserves the hyperbolic metric, but we need to show it also preserves the new metric $\tilde{g}$ on $\Omega(\tau_4)$. More precisely,

**Theorem 4.2.** The group $\Gamma$ is a subgroup of $\text{Isom}(\Omega(\tau_4), \tilde{g})$, the isometry group of $\Omega(\tau_4)$ with respect to the negatively curved metric $\tilde{g}$.

**Proof.** In order not to introduce a different cut-off process, we proceed here with a straightforward (but lengthy) argument.

Let $p$ and $q$ be two points in $\Omega(\tau_4)$, and we need to show that $d(p, q) = d(\gamma(p), \gamma(q))$ for any element $\gamma \in \Gamma$, where $d(\cdot, \cdot)$ denotes the distance function with respect to the new metric $\tilde{g}$. More precisely, let $c$ be the (unique) geodesic from $p$ to $q$, we shall prove that $\gamma \circ c$ is the (unique) geodesic from $\gamma(p)$ to $\gamma(q)$ for any $\gamma \in \Gamma$. Moreover we shall prove that $c$ and $\gamma \circ c$ have the same length with respect to the metric $\tilde{g}$, so $\gamma$ is an isometry of $\Omega(\tau_4)$ with respect to the metric $\tilde{g}$ for any $\gamma \in \Gamma$. Therefore $\Gamma$ is a subgroup of $\text{Isom}(\Omega(\tau_4), \tilde{g})$.

By Theorem 4.1, the manifold $(\Omega(\tau_4), \tilde{g})$ is negatively curved. Then there is a unique geodesic $c : [0, L] \rightarrow (\Omega(\tau_4), \tilde{g})$ parameterized by arc length, such that $c(0) = p$ and $c(L) = q$. If the geodesic $c([0, L])$ is totally contained in $\Omega(\tau_3)$, we are done by the definition of the function $f(t)$ (note that $f(t) = t$ for $t \in (0, 3L_0)$). If $c([0, L])$ is entirely contained in any component of $\Omega(\tau_4) - \Omega^c(\tau_3)$, then $f(t)$ is a strictly increasing function and $\gamma$ preserves the distance.

In general, similar to Corollary 11.34 in [BH99] on page 364, the geodesic $c$ is expressed as a chain of non-trivial paths $c_1, \ldots, c_n$, each parameterized by arc length, such that

(i) each of the paths $c_i$ is either a hyperbolic geodesic or else its image is contained in one component of $\Omega(\tau_4) - \Omega^c(\tau_3)$;

(ii) if $c_i$ is a hyperbolic geodesic then the image of $c_{i+1}$ is contained in one component of $\Omega(\tau_4) - \Omega^c(\tau_3)$, and vice versa.

Suppose that each geodesic segment $c_i$ is parameterized by $c_i(s) = c(s)$ for $s \in [s_{i-1}, s_i]$, where $0 = s_0 < s_1 < \cdots < s_n = L$ is a partition of the interval $[0, L]$. Then we write $c = c_1 * c_2 * \cdots * c_n$ in the sense that $c(s) = c_i(s)$ if $s \in [s_{i-1}, s_i]$. By the above argument, we have that each curve $\gamma \circ c : [s_{i-1}, s_i] \rightarrow (\Omega(\tau_4), \tilde{g})$ is a geodesic for $i = 1, \ldots, n$.

We need to show that the curve $\gamma \circ c = (\gamma \circ c_1) * \cdots * (\gamma \circ c_n)$ is a geodesic from $\gamma(p)$ to $\gamma(q)$. We will proceed by induction. To start, $(\gamma \circ c_1)$ is a geodesic segment.
Now suppose that \((\gamma \circ c_1) \ast \cdots \ast (\gamma \circ c_{j-1})\) is a geodesic segment, and \((\gamma \circ c_1) \ast \cdots \ast (\gamma \circ c_j)\) is not a geodesic segment, then there exists a (unique) geodesic segment \(c' : [0, s_j] \to (\Omega(\tau_4), \bar{g})\) such that \(c'(0) = \gamma(p)\) and \(c'(s_j) = \gamma(c(s_j))\), and furthermore the \(\bar{g}\)-length of \(c'([0, s_j]) < s_j\). However, \(\Gamma\) is a subgroup of \(\text{PSL}(2, \mathbb{C})\), whose elements are conformal, therefore they preserve the angle. Now three geodesic segments \((\gamma \circ c_1) \ast \cdots \ast (\gamma \circ c_{j-1})([s_0, s_{j-1}]), \gamma \circ c_j([s_{j-1}, s_j])\) and \(c'([0, s_j])\) would form a geodesic triangle whose sum of its inner angles is \(\geq \pi\). This is a contradiction.

Therefore \(\gamma \circ c = (\gamma \circ c_1) \ast \cdots \ast (\gamma \circ c_n)\) is a geodesic segment from \(\gamma(p)\) to \(\gamma(q)\), and then \(d(\gamma(p), \gamma(q)) = L = d(p, q)\).

As a corollary, we consider the resulting quotient manifold:

**Corollary 4.3.** The manifold \(M^3(\tau_4) = \Omega(\tau_4)/\Gamma\) can be equipped with a new metric induced from the covering space, still denoted by \(\bar{g}\), such that \((M^3(\tau_4), \bar{g})\) is a compact negatively curved 3-manifold with totally geodesic boundary components.

We now make special remarks here on \(M^3(\tau_4)\) and its submanifolds \(M^3(\tau_3)\) before we move to the proofs.

**Remark 4.4.** According to the construction of the submanifold \(M^3(\tau_4)\), it is homeomorphic to \(M^3\), so its fundamental group \(\pi_1(M^3(\tau_1))\) is also LERF.

By the definition of \(f(t)\) in §3.1 and the definition of four constants \((4.4)\), the modified metric \(\bar{g}\) restricted to \(M^3(\tau_3)\) is the hyperbolic metric. The submanifold \((M^3(\tau_3), \bar{g})\) is a compact hyperbolic 3-manifold whose boundary components are concave with respect to the inward normal vectors.

**5. Proof of Main results**

In §4 we constructed a submanifold \(M^3(\tau_4) = \Omega(\tau_4)/\Gamma\) in any cusped hyperbolic 3-manifold \(M^3 = \mathbb{H}^3/\Gamma\) with a modified metric \(\bar{g}\) such that \((M^3(\tau_4), \bar{g})\) is a compact negatively curved 3-manifold with mean convex boundary components with respect to the inward normal vectors.

In this section we may assume that \(S\) is an embedded closed incompressible surface with genus \(\geq 2\) contained in \(M^3(\tau_1) \subset M^3\) until we begin to prove Theorem 1.1 on page 19. Now by the argument in [MSY82, HS88], there exists an embedded closed incompressible least area minimal surface \(\Sigma\) isotopic to \(S\) in \(M^3(\tau_4)\) with respect to the modified metric \(\bar{g}\). In this case, we will say that \(\Sigma\) is an embedded least area minimal surface isotopic to \(S\) in \((M^3(\tau_4), \bar{g})\).

**Remark 5.1.** If \(S\) is only guaranteed to be immersed, fortunately we then use an additional fact that \(\pi_2(M^3) = 0\), and apply [SU82, HS88] to find the existence of an immersed least area surface \(\Sigma\) homotopic to \(S\) in \((M^3(\tau_4), \bar{g})\).

The key will be showing that the embedded minimal surface \(\Sigma\) is contained in \((M^3(\tau_3), \bar{g})\), a hyperbolic subregion of \((M^3(\tau_4), \bar{g})\). A key ingredient of the rest of
the argument is that a cusped region has very simple geometry, and an embedded closed incompressible surface of the least area can only intersect the region (if at all) in a predictable way (see Proposition 5.3).

5.1. Minimal surface intersecting toric region. As before, we assume that the oriented cusped hyperbolic 3-manifold $M^3$ has $k$ cusps, such that each maximal cusped region is parametrized by $\mathcal{C}_i = T^2_i \times (0, \infty)$ for $i = 1, \ldots, k$. Suppose that $\rho: \pi_1(M^3) \to \text{PSL}_2(\mathbb{C})$ is the holonomy so that $\Gamma = \rho(\pi_1(M^3))$.

Let $S_{g,n}$ be a surface of genus $g$ with $n$ boundary components (i.e. a closed genus $g$ surface with $n$ disjoint open disk-type subdomains removed). If $S_{g,n}$ has negative Euler characteristic, i.e. $\chi(S_{g,n}) < 0$, then $S_{g,n}$ must satisfy one of the following conditions:

- If $g \geq 2$, then $n \geq 0$.
- If $g = 1$, then $n \geq 1$.
- If $g = 0$, then $n \geq 3$.

It’s easy to verify that $\pi_1(S_{g,n})$ is non-abelian in the above three cases.

For a simple closed curve $\alpha \subset S_{g,n}$, it is said to be essential if no component of $S_{g,n} \setminus \alpha$ is a disk, and it is said to be non-peripheral if no component of $S_{g,n} \setminus \alpha$ is an annulus. We denote $T'$ as a solid torus with a core curve removed and we state a simple lemma to be used later:

**Lemma 5.2.** Let $S_{g,n}$ be a surface of negative Euler characteristic embedded in $T'$ such that $\partial S_{g,n} \subset \partial T'$ if $n \geq 1$. Then there exists at least one essential simple closed curve $\alpha \subset S_{g,n}$ such that $\alpha$ bounds a disk $D \subset T'$.

**Proof.** We prove that the homomorphism $\pi_1(S_{g,n}) \to \pi_1(T')$ induced by the embedding $S_{g,n} \to T'$ can’t be injective. In fact, if any essential simple closed curve $\alpha$ in $S_{g,n}$ is non-contractible in $T'$, then the embedding induces an injection between fundamental groups $\pi_1(S_{g,n}) \to \pi_1(T') = \mathbb{Z} \oplus \mathbb{Z}$. Thus $\pi_1(S_{g,n})$ is isomorphic to a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$. But this is impossible, since $\pi_1(S_{g,n})$ is non-abelian whereas $\mathbb{Z} \oplus \mathbb{Z}$ is abelian. Therefore the kernel of the homomorphism $\pi_1(S_{g,n}) \to \pi_1(T')$ is nontrivial.

Next we shall prove the existence of an essential simple closed curve in $S_{g,n}$, which represents an element in the kernel of the homomorphism $\pi_1(S_{g,n}) \to \pi_1(T')$. Since $\chi(S_{g,n}) < 0$, it contains a separating simple closed curve $\alpha$, in other words homologically trivial, so it is a commutator, and must be contained in the kernel of the homomorphism $\pi_1(S_{g,n}) \to \pi_1(T')$.

Recall that there are four constants only depending on $M^3$: $\tau_j = \log(jL_0)$, for $j = 1, 2, 3, 4$. Suppose that the embedded closed incompressible surface $S$ is contained in $M^3(\tau_1)$.

We consider a compact submanifold $M^3(\tau_4)$ of $M^3$, equipped with the new metric $\tilde{g}$ (see Corollary 4.3), so that $(M^3(\tau_4), \tilde{g})$ is a compact negatively curved 3-manifold whose boundary components are all totally geodesic. By above arguments, we have an embedded closed incompressible least area minimal surface $\Sigma$ isotopic to $S$ in $(M^3(\tau_4), \tilde{g})$. 

Furthermore, $\Sigma$ is disjoint from $T_i^2 \times \{\tau_i\}$ for $i = 1, \ldots, k$, since each boundary component of $(M^3(\tau_i), \bar{g})$ is totally geodesic.

We need to show the least area minimal surface $\Sigma$ is contained in $(M^3(\tau_j), \bar{g})$, that is to say, $\Sigma$ is a minimal surface with respect to the hyperbolic metric. If $\Sigma$ does not intersect with $T_i^2 \times \{\tau_2\}$, then we are done (since it can not be contained entirely in the cusped region). Therefore we can just assume that $\Sigma \cap (T_i^2 \times [0, \tau_2])$ is non-empty. We are interested in how $\Sigma$ intersects with the region $T_i^2 \times [0, \tau_4)$.

**Proposition 5.3.** Each component of $\Sigma \cap (T_i^2 \times [0, \tau_4))$ is either a minimal disk whose boundary is a null-homotopic Jordan curve in $T_i^2 \times \{0\}$, or a minimal annulus whose boundary consists of essential Jordan curves in $T_i^2 \times \{0\}$.

Moreover each component of $\Sigma \cap (T_i^2 \times [0, \tau_4))$ is boundary compressible, i.e. each component can be isotoped into $T_i^2 \times \{0\}$ such that the isotopy fixes the boundary of the component.

**Proof.** Let $\Sigma'$ be a component of $\Sigma \cap (T_i^2 \times [0, \tau_4))$. Since $(M^3(\tau_4), \bar{g})$ is a compact negatively curved 3-manifold with totally geodesic boundary components, so the least area minimal surface $\Sigma$ is disjoint from its boundary. Therefore the boundary of $\Sigma'$ is contained in $T_i^2 \times \{0\}$. Since $\Sigma$ is incompressible while $T_i^2$ is a torus, we have very few cases to consider:

**Case I:** Suppose that $\Sigma'$ is a surface of negative Euler characteristic. We suppose that $\Sigma'$ is homeomorphic to the surface $S_{g,n}$ with $\chi(S_{g,n}) = 2 - 2g - n < 0$.

Firstly, no non-peripheral essential curves in $\Sigma'$ are null homotopic in the region $T_i^2 \times [0, \tau_4)$. Otherwise such a curve is also a non-peripheral essential curve in $\Sigma$. This is impossible since the surface $\Sigma$ is incompressible.

Secondly no peripheral essential curves in $\Sigma'$ are null homotopic in the region $T_i^2 \times [0, \tau_4)$ either. Otherwise, let $\alpha$ be a boundary component of $\Sigma'$ which is null homotopic in $T_i^2 \times \{0\}$. Now since $\Sigma$ is incompressible, $\alpha$ must bound a disk $D$ in $\Sigma$, which is also a minimal surface embedded in $(M^3(\tau_4), \bar{g})$.

We claim that the minimal disk $D$ with $\partial D = \alpha$ must be contained in $T_i^2 \times [0, \tau_4)$. Assume that $D$ is not entirely contained in $T_i^2 \times [0, \tau_4)$, then $\partial D$ is mean convex with respect to both the inward normal vector and the modified metric $\bar{g}$, according to the argument in [MY82, pp. 155–156], $\alpha$ bounds an embedded least area minimal disk $D' \subset T_i^2 \times [0, \tau_4)$ (recall that $T_i^2 \times \{\tau_4\}$ is totally geodesic with respect to the metric $\bar{g}$) and all of these kinds of least area minimal disks with the same boundary $\alpha$ must be contained in $T_i^2 \times [0, \tau_4)$.

Recall that $D$ is assumed not to be entirely contained in $T_i^2 \times (0, \tau_4)$, so $\text{Area}(D) > \text{Area}(D')$, where the area $\text{Area}(\cdot)$ is with respect to the modified metric $\bar{g}$ on $M^3(\tau_4)$. Since $\pi_2(M^3(\tau_4)) = \pi_2(M^3) = 0$, $D$ is isotopic to $D'$ with boundary fixed in $M^3(\tau_4)$. Let $\Pi$ be the surface defined by $\Pi = (\Sigma - D) \cup D'$, then $\Pi$ is isotopic to $\Sigma$ and $\text{Area}(\Pi) < \text{Area}(\Sigma)$, where the area $\text{Area}(\cdot)$ is also with respect to the metric $\bar{g}$. But this contradicts the assumption that
\(\Sigma\) is a least area minimal surface isotopic to \(S\) in \((M^3(\tau_4), \bar{g})\). Therefore \(D\) must be itself contained in \(T^2_1 \times [0, \tau_4]\).

Next we choose \(\varepsilon\) sufficiently small such that \(\alpha \times [0, \varepsilon] \subset \Sigma'\), therefore each simple closed curve \(\alpha_t = \Sigma' \cap (T^2_1 \times \{t\})\) is null homotopic in \(T^2_1 \times \{t\}\) for \(0 \leq t \leq \varepsilon\), then similarly we have a minimal disk \(D_t\) such that \(D_t \subset \Sigma \cap (T^2_1 \times [t, \tau_4])\) for \(0 \leq t \leq \varepsilon\). But this is impossible since otherwise \(\Sigma\) would self-intersect (uncountably) infinitely many times at \(\alpha_t\) for \(0 < t \leq \varepsilon\).

Therefore, if \(\Sigma'\) is a surface of negative Euler characteristic in \(T^2_1 \times [0, \tau_4]\), then no closed essential curves in \(\Sigma'\) is null homotopic in \(T^2_1 \times [0, \tau_4]\). But this is impossible by applying Lemma 5.2.

**Case II:** Suppose that \(\Sigma'\) is an annulus such that at least one of its boundary components, say \(\alpha\), is null-homotopic in \(T^2 \times \{0\}\). Then apply the similar argument as in Case I (in this case, \(\alpha\) is also a peripheral essential curve in \(\Sigma'\)), we know that this is impossible.

Thus each component of \(\Sigma \cap (T^2_1 \times [0, \tau_4])\) is either a minimal disk whose boundary is a null-homotopic Jordan curve in \(T^2_1 \times \{0\}\), or a minimal annulus whose boundary consists of two essential Jordan curves in \(T^2_1 \times \{0\}\). It’s easy to see that each component of \(\Sigma \cap (T^2_1 \times [0, \tau_4])\) is boundary compressible. \(\square\)

### 5.2. Good positioned Jordan curves on tori.

We start by making a definition of Jordan curves being in good position on a torus. This will be important for what follows.

**Definition 5.4.** Let \(M^3\) be a cusped hyperbolic 3-manifold and \(C = T^2 \times [0, \infty)\) be a maximal cusped region of \(M^3\). A Jordan curve (i.e., simple closed) \(\alpha \subset T^2 \times \{\tau\}\) is said to be in “good position” if one of the lifts of \(\alpha\) to \(\mathbb{H}^3\) is contained in \(A \times \{e^{\tau}\}\), where \(A\) is the fundamental domain of the parabolic group \(\Gamma_{\infty} = (z \mapsto z + \mu, z \mapsto z + \nu)\) in the horosphere \(\{(x, y, 1) \mid (x, y) \in \mathbb{R}^2\}\).

From the above definition, we have the following statement:

**Proposition 5.5.** A Jordan curve \(\alpha \subset T^2 \times \{\tau\}\) is in good position if the Euclidean length of \(\alpha\) is less than \(\min\{2|\mu|, 2|\nu|, 2|\mu \pm \nu|\}\), while if it is not in good position then the Euclidean length of \(\alpha\) is at least \(\min\{2|\mu|, 2|\nu|, 2|\mu \pm \nu|\}\).

If \(\alpha \subset T^2 \times \{\tau\}\) is an essential Jordan curve, then \(\alpha\) is not in good position.

Recall from (4.4) that we have 4 constants: \(\tau_j = \log(j \cdot L_0)\) for \(j = 1, 2, 3, 4\), where the constant \(L_0\) is defined in (2.4). And these constants are ordered: \(\tau_4 > \tau_3 > \tau_2 > \tau_1 > 0\). As in the previous subsection, we assume \(\Sigma \cap (T^2_1 \times [0, \tau_2])\) is non-empty. We first observe the following fact:

**Proposition 5.6.** Let \(\Sigma'\) be a component of \(\Sigma \cap (T^2_1 \times [0, \tau_4])\). If there exists some \(\tau \in [0, \tau_2]\), such that \(\Sigma' \cap (T^2_1 \times \{\tau\})\) consists of Jordan curves in good position, then each component of \(\Sigma' \cap (T^2_1 \times \{\tau'\})\) is also in good position for all \(\tau' \in [\tau, \tau_2]\).
Proof. By Theorem 4.2, we can lift \((M_3(\tau_4), \bar{g})\) to the truncated negatively curved space \((\Omega(\tau_4), \bar{g})\) such that \(T_i^2 \times \{0\}\) is lifted to the horizontal plane passing through the point \((0, 0, 1)\). Suppose that the barycenter of the fundamental domain \(A_i\) of the parabolic group generated by \(z \mapsto z + \mu_i\) and \(z \mapsto z + \nu_i\) is the point \((0, 0, 1)\).

Suppose \(D\) is a component of \(\Sigma' \cap (T_i^2 \times [\tau, \tau_4])\) such that \(\partial D \subset T_i^2 \times \{\tau\}\) is in good position, then by the arguments in Proposition 5.3, and Proposition 5.5, \(D\) must be a disk and \(\partial D\) must be a null-homotopic Jordan curve in \(T_i^2 \times \{\tau\}\). Let \(\bar{D}\) be a lift of \(D\) such that \(\partial \bar{D} \subset A_i \times \{e^\tau\}\).

We define the following:

\[
B_i = A_i \times [e^\tau, 4L_0].
\]

We want to show that \(\bar{D}\) must be contained in \(B_i\). In fact, it is a minimal disk such that \(\partial \bar{D} \subset A_i \times \{e^\tau\}\) is null-homotopic. Then we are left with very few cases:

(i) The minimal disk \(\bar{D}\) doesn’t have any subdisk below the horizontal plane through the point \((0, 0, e^\tau)\), since \(D \subset T_i^2 \times [\tau, \tau_4]\) by the assumption.

(ii) Since all vertical planes are totally geodesic (see Theorem 3.3 and Remark 3.4), the minimal disk \(\bar{D}\) does not have any subdisk outside \(B_i\) by Hopf’s maximum principle.

Thus \(\bar{D}\) must be contained in the domain \(B_i\). This is certainly true for the other lifts of \(D\) which are given by \(\gamma(\bar{D})\) for \(\gamma \in \Gamma\). By definition, for \(\tau' \geq \tau\), each component of \(\Sigma' \cap (T_i^2 \times \{\tau'\})\) is in good position. \(\square\)

As a corollary, and taking advantage of Theorem 3.5 that we can use hemispheres as barriers, we find:

Corollary 5.7. If there exists some \(\tau \in [0, \tau_2]\) such that \(\Sigma' \cap (T_i^2 \times \{\tau\})\) consists of Jordan curves in good position, then \(\Sigma'\) is contained in \(T_i^2 \times [0, \tau_3]\), i.e. \(\Sigma'\) is a least area disk or annulus with respect to the hyperbolic metric.

Proof. Recall from (4.5) and (4.6), the modified metric \(\bar{g}\) is flat for \(t > 4L_0\), and hyperbolic when \(t < 3L_0\). For convenience, we denote two new constants: \(L_3 = \sqrt{e^{2\tau} + (\frac{L_0}{2})^2}\) and \(L_4 = \sqrt{\frac{17}{2}}L_0\). Since \(\tau \leq \tau_2 = \log(2L_0)\), so we have

\[
L_3 \leq \frac{\sqrt{17}}{2}L_0 < 3L_0 < 4L_0 < L_4.
\]

Therefore \(A_i \times \{L_4\}\) is totally geodesic with respect to the metric \(\bar{g}\).

We consider the subregion \(\mathcal{B}_i'\) of \(\mathcal{B}_i\), which is defined by

\[
\mathcal{B}_i' = B_i \cap \left\{ \bigcup_{L_3 \leq \tau \leq L_4} S_{L_i}(\tau) \right\}.
\]

By Theorem 3.5, the subregion \(\mathcal{B}_i'\) is foliated by the non-concave spherical caps with respect to the downward normal vectors. By the definition of \(L_0\) in (2.4), the spherical cap \(\mathcal{B}_i \cap S_{L_3}(L_3)\) lies above \(A_i \times \{e^\tau\}\).
Recall from the proof of Proposition 5.6 that $D$ is a component of $\Sigma' \cap (T_i^2 \times [\tau, \tau_4])$ such that $\partial D \subset T_i^2 \times \{\tau\}$ is in good position, and $\tilde{D}$ is a lift of $D$ such that $\partial \tilde{D} \subset A_i \times \{e^\tau\}$. Therefore by the maximum principle, $\tilde{D}$ is contained in $B_i$ and below the spherical cap $B \cap S^2_i(L_3)$. In other words, the Euclidean height of $\tilde{D}$ is at most $L_3$.

By (5.2), we have $\tilde{D} \subset A_i \times [e^\tau, 3L_0]$. This is true for other lifts of $D$ which are given by $\gamma(\tilde{D})$, for all $\gamma \in \Gamma$. Since the Kleinian group preserves the metric $\bar{g}$ (Theorem 4.2), we have $D \subset T_i^2 \times [\tau, \tau_3]$, and therefore

$$
\Sigma' \subset (T_i^2 \times [0, \tau]) \cup (T_i^2 \times [\tau, \tau_3]) = T_i^2 \times [0, \tau_3].
$$

The proof of the Corollary is complete. \qed

5.3. Completing the proof. First we need a version of the co-area formula modified from that in [CG06, p.399]. The proof of (5.3) in the following Lemma 5.8 can be found in [Wan12].

**Lemma 5.8.** If $M^3$ is a Riemannian 3-manifold with nonempty boundary $\partial M^3$, and $F$ is a component of $\partial M^3$ such that its $s$-neighborhood $\mathcal{N}_s(F) \subset M^3$ is a trivial normal bundle over itself. If $\Sigma_1 \subset M^3$ is a surface such that $\Sigma_1 \cap \mathcal{N}_s(F) \neq \emptyset$, then

$$
\text{Area}(\Sigma_1 \cap \mathcal{N}_s(F)) = \int_0^s \int_{\Sigma_1 \cap \partial \mathcal{N}_s(F)} \frac{1}{\cos \theta} \, dl d\tau,
$$

where the angle $\theta$ is defined as follows: For any point $q \in \Sigma_1$, set $\theta(q)$ to be the angle between the tangent space to $\Sigma_1$ at $q$, and the radial geodesic which is through $q$ (emanating from $q$) and is perpendicular to $F$.

To complete the proof of Theorem 1.1, we just need to find one $\tau \in [0, \tau_2]$ satisfying the assumption in Proposition 5.6. And we show this $\tau$ may be chosen as just $\tau_2$:

**Theorem 5.9.** Let $\Sigma'$ be a component of $\Sigma \cap (T_i^2 \times [0, \tau_1])$, then any component of $\Sigma' \cap (T_i^2 \times \{\tau_2\})$ is a Jordan curve in good position.

**Proof.** Assume that $\Sigma'$ is a component of $\Sigma \cap (T_i^2 \times [0, \tau_1])$ such that at least one component of $\Sigma' \cap (T_i^2 \times \{\tau_2\})$ is not in good position, then by Proposition 5.6, for each $\tau \in [0, \tau_2]$, $\Sigma' \cap (T_i^2 \times \{\tau\})$ has at least one component that is not in good position.

By Proposition 5.5, for all $\tau \in [0, \tau_2]$, we have:

$$
\text{Length} \left( \Sigma' \cap (T_i^2 \times \{\tau\}) \right) \geq \min\{2|\mu_i|, 2|\nu_i|, 2|\mu_i \pm \nu_i|\} e^{-\tau}.
$$

To apply the co-area formula (5.3), we choose $F = T_i^2 \times \{0\}$, and for each $\tau \in [0, \tau_2]$, we set

$$
\mathcal{N}_s(F) = \{p \in T_i^2 \times [0, \tau_2] \mid \text{dist}(p, F) \leq \tau\},
$$

where dist$(\cdot, \cdot)$ is the hyperbolic distance function. Now we apply the co-area formula (5.3) to find:

$$
\text{Area} \left( \Sigma' \cap (T_i^2 \times [\tau_1, \tau_2]) \right) = \int_{\tau_1}^{\tau_2} \int_{\Sigma' \cap \partial \mathcal{N}_s(F)} \frac{1}{\cos \theta} \, dl d\tau.
$$
Here we used the fact that $L_0 \geq |\mu_i| + |\nu_i|$ (Equation 2.4) and $\tau_j = \log(jL_0)$ for $j = 1, 2$.

By Proposition 5.3, $\Sigma'$ is either a least area disk or a least area annulus (that is boundary compressible), so we may isotope $\Sigma' \cap (T_i^2 \times [\tau_1, \tau_4])$ to a disk or an annulus $A$ contained in $T_i^2 \times \{\tau_1\}$ such that $\partial A = \Sigma' \cap (T_i^2 \times \{\tau_1\})$. Let $\Sigma''$ be a new surface defined by

$$\Sigma'' = (\Sigma' \cap (T_i^2 \times [0, \tau_1])) \cup A .$$

Then $\partial \Sigma'' = \partial \Sigma'$ and $\Sigma''$ is isotopic to $\Sigma'$ with boundary fixed in $\Sigma' \cap (T_i^2 \times [0, \tau_4])$. By the above inequality, we have $\text{Area}(\Sigma'') < \text{Area}(\Sigma')$, but this contradicts the fact that $\Sigma'$ is a least area minimal surface in the region $T_i^2 \times [0, \tau_4]$. Therefore any component of $\Sigma' \cap (T_i^2 \times \{\tau_2\})$ is a Jordan curve in good position, and then any component of $\Sigma \cap (T_i^2 \times \{\tau_2\})$ is also in good position. □

We may now complete the proof:

**Proof of Theorem 1.1.** We consider two cases:

**Case I:** $S$ is assumed to be embedded in $M^3$. This will complete the proof for Corollary 1.2.

According to both Theorem 5.1 and the remarks before Theorem 6.12 in [HS88], there is an embedded incompressible least area minimal surface $\Sigma$ isotopic to $S$ in $(M^3(\tau_4), \tilde{g})$. By Theorem 5.9, all components of $\Sigma \cap (T_i^2 \times \{\tau_2\})$ are in good position, then by Corollary 5.7, each component of $\Sigma \cap (T_i^2 \times [0, \tau_4])$ is disjoint from $T_i^2 \times (\tau_3, \tau_4]$. Therefore we have

$$\Sigma \cap (T_i^2 \times [0, \tau_4]) \subset T_i^2 \times [0, \tau_3] , \quad \text{for } i = 1, \ldots, k ,$$

which implies that $\Sigma$ is a minimal surface with respect to the hyperbolic metric.

Next we claim that the minimal surface $\Sigma \subset M^3(\tau_3)$ is a least area minimal surface isotopic to $S$ in the cusped hyperbolic 3-manifold $M^3$.

In fact, let $L_0'$ be an arbitrary real number such that $L_0' \geq L_0$, and let $\tau_j' = \log(j \cdot L_0')$ for $j = 1, 2, 3, 4$. Obviously $\tau_j' \geq \tau_j$, and so $M^3(\tau_j') \subset M^3(\tau_j)$ for $j = 1, 2, 3, 4$. We can construct the truncated 3-manifold $M^3(\tau_j')$ with a modified metric $\tilde{g}'$ as in §4, i.e.

- $\tilde{g}'|M^3(\tau_j')$ is hyperbolic, and
and similarly any least area minimal surface $\Sigma'$ isotopic to $S$ in $(M^3(\tau_4'), \bar{g}')$ must be contained in $M^3(\tau_3)$, so it can be considered as a minimal surface isotopic to $S$ in $(M^3(\tau_4), \bar{g})$. But we know that $\Sigma$ is the least area minimal surface isotopic to $S$ in $(M^3(\tau_4), \bar{g})$, so we must have $\text{Area}(\Sigma') \geq \text{Area}(\Sigma)$ with respect to the hyperbolic metric on $M^3$ (since both $\Sigma$ and $\Sigma'$ are contained in $M^3(\tau_3)$ and $\bar{g}|M^3(\tau_3) = \bar{g}'|M^3(\tau_3)$ are both hyperbolic). Let $L'_0 \to \infty$, we know that $\Sigma$ is a least area minimal surface isotopic to $S$ in $M^3$.

**Case II:** $S$ is only assumed to be immersed in $M^3$.

By Theorem 2.2, we may lift $S$ to an embedded nonseparating closed incompressible surface in a finite cover $\tilde{M}^3$ of $M^3$. It’s easy to see that $\tilde{M}^3$ is also a cusped hyperbolic 3-manifold. Suppose that $\tilde{M}^3$ has $\ell$ maximal cusped regions $\tilde{T}_i^2 \times [0, \infty)$, $\ell \geq 0$, such that each parabolic group corresponding to the horosphere $\tilde{T}_i^2 \times \{0\}$ is generated by $\langle z \mapsto z + \mu_i, \bar{z} \mapsto z + \bar{\mu}_i \rangle$ for $i = 1, \ldots, \ell$. Note that $\ell \geq k$, where $k$ is the number of the maximal cusped regions of $M^3$. We define

$$
\tilde{L}_0 = \max \left\{ e^{\tau_0}, |\bar{\mu}_1| + |\bar{\nu}_1|, \ldots, |\bar{\mu}_\ell| + |\bar{\nu}_\ell| \right\} > 0,
$$

where $\tau_0 > 0$ is the same number as in §2.2, i.e. $\tau_0$ is the smallest number such that each maximal cusped region is disjoint from any other maximal cusped regions of $M^3$ and $\tilde{M}^3$ respectively. Obviously $\tilde{L}_0 \geq L_0$, where $L_0$ is defined by (2.4). Similarly, we also define

$$
\tilde{\tau}_j = \log(j \cdot \tilde{L}_0), \quad j = 1, 2, 3, 4.
$$

Then $\tilde{\tau}_j \geq \tau_j$ for $j = 1, 2, 3, 4$. We still need some notations. It’s easy to verify that the submanifold

$$
\tilde{M}^3(0) = \tilde{M}^3 - \bigcup_{i=1}^\ell (\tilde{T}_i^2 \times [0, \infty))
$$

of $\tilde{M}^3$ is a finite cover of the submanifold $M^3(0)$ of $M^3$ defined by (4.3) for $\tau = 0$. Just as we did in §4, we can define the truncated manifold $\tilde{M}^3(\tilde{\tau}_4)$ and the modified metric $\bar{g}$ on $\tilde{M}^3(\tilde{\tau}_4)$ such that

- $\bar{g}|\tilde{M}^3(\tilde{\tau}_3)$ is hyperbolic, and
- $\bar{g}|(M^3(\tau_3) - \tilde{M}^3(\tilde{\tau}_3))$ is defined as in §4.

By the construction, $\tilde{M}^3(\tilde{\tau}_4) \to M^3(\tau_4)$ is a finite cover. Let $\tilde{g}$ be the modified metric on $M^3(\tau_4)$ defined by the finite cover map such that

- $\tilde{g}|M^3(\tau_3)$ is hyperbolic,
- $\tilde{g}|(M^3(\tau_4) - M^3(\tau_3))$ is defined as in §4, and
- the finite cover $(M^3(\tau_4), \tilde{g}) \to (M^3(\tau_3), \bar{g})$ is a local isometry.

According to the definition the metric $\tilde{g}$ on $M^3(\tilde{\tau}_4)$, particularly we know that $\tilde{g}|M^3(\tau_3)$ is hyperbolic since $\tau_3 \leq \tilde{\tau}_3$. 

Since $S$ is assumed to be immersed in $M^3(\tau_4)$, according to both Theorem 5.3 and the remarks before Theorem 6.12 in [HS88], there exists an immersed least area minimal surface $\Sigma$ homotopic to $S$ in $M^3(\tau_4)$ with respect to the modified metric $\bar{g}$. It is also incompressible in $M^3(\tau_4)$. Note that it might not be minimal with respect to the hyperbolic metric. On the other hand, since $\pi_1(M^3(\tau_4)) = \pi_1(M^3)$ is LERF by Theorem 2.2, we may lift $S$ to an embedded closed incompressible surface $\bar{S}$ in $\bar{M}^3(\bar{\tau}_4)$, and lift $\Sigma$ to a (possibly only immersed) minimal surface $\bar{\Sigma}$ homotopic to $\bar{S}$ in $\bar{M}^3(\bar{\tau}_4)$ with respect to the modified metric $\bar{g}$. Since $\Sigma$ is a least area minimal surface homotopic to $S$ in $M^3(\tau_4)$ with respect to the modified metric $\bar{g}$, the minimal surface $\bar{\Sigma}$ is a least area minimal surface homotopic to $S$ in $(\bar{M}^3(\bar{\tau}_4), \bar{g})$. Then using both Theorem 5.1 and the remarks before Theorem 7.1 in [FHS83], the minimal surface $\bar{\Sigma}$ is an embedded least area minimal surface isotopic to $\bar{S}$ in $(\bar{M}^3(\bar{\tau}_4), \bar{g})$.

Now just as we did in Case I, we apply both Corollary 5.7 and Theorem 5.9 to $\bar{\Sigma}$ in the non-positively curved manifold $(\bar{M}^3(\bar{\tau}_4), \bar{g})$, then we have

$$\bar{\Sigma} \cap (\bar{T}_i^2 \times [0, \bar{\tau}_4]) \subset \bar{T}_i^2 \times [0, \bar{\tau}_3], \quad \text{for } i = 1, \ldots, \ell.$$ 

This means $\bar{\Sigma} \subset (\bar{M}^3(\bar{\tau}_3), \bar{g})$, so it is a least area minimal surface with respect to the hyperbolic metric. Therefore we have $\Sigma \subset (M^3(\tau_3), \bar{g})$, i.e., $\Sigma$ is an immersed closed least area minimal surface homotopic to $S$ in $M^3(\tau_3)$ with respect to the hyperbolic metric, and furthermore it is also a least area minimal surface homotopic to $S$ in the cusped hyperbolic 3-manifold $M^3$ as we did in Case I.

\[\square\]

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