On the level set version of partial uniform ellipticity and applications

Ri-Rong Yuan*

Abstract

We derive level set version of partial uniform ellipticity for symmetric concave functions. This suggests an effective approach to investigate second order fully nonlinear equations of elliptic and parabolic type.

1 Introduction

Let $f$ be a smooth symmetric function defined in an open symmetric convex cone $\Gamma \subset \mathbb{R}^n$ containing the positive cone

$$\Gamma_n = \{ \lambda \in \mathbb{R}^n : \text{each } \lambda_i > 0 \} \subseteq \Gamma$$

with vertex at the origin and with nonempty boundary $\partial \Gamma \neq \emptyset$. The study of fully nonlinear equations of the form

$$F(D^2 u) := f(\lambda(D^2 u)) = \psi \text{ in } \Omega \subset \mathbb{R}^n$$

starts from the pioneering work [1] of Caffarelli-Nirenberg-Spruck. Since then the equations of this type have been extensively studied in real and complex variables. The following two basic hypotheses are imposed in the literature:

$$f \text{ is a concave function in } \Gamma,$$  

\* School of Mathematics, South China University of Technology, Guangzhou 510641, China  
Email address: yuanrr@scut.edu.cn
\[ f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n. \]  

(1.3)

In some cases one may replace (1.3) by a weaker condition

\[ f_i(\lambda) \geq 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n. \]  

(1.4)

As is well known, the typical examples satisfying (1.2)-(1.3) are as follows:

\[ f(\lambda) = \sigma_k^{1/k}(\lambda) \text{ or } (\sigma_k/\sigma_l)^{1/(k-l)}(\lambda), \quad 1 \leq l < k \leq n, \quad \Gamma = \Gamma_k \]

where \( \sigma_k \) is the \( k \)-th elementary symmetric function. Here

\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \quad \forall 1 \leq j \leq k \}. \]

The linearized operator of (1.1) at \( u \) is given by

\[ L_u w = \frac{\partial F(D^2 u)}{\partial u_{ij}} \cdot w_{ij}. \]

One can check that the eigenvalues of \( \left( \frac{\partial F}{\partial u_{ij}}(D^2 u) \right) \) are precisely given by

\[ f_1(\lambda), \ldots, f_n(\lambda) \text{ for } \lambda = \lambda(D^2 u). \]

In particular, for the Poisson equation corresponding to \( f(\lambda) = \sum_{i=1}^n \lambda_i, \)

\[ f_i(\lambda) \equiv 1, \quad \forall \lambda \in \mathbb{R}^n, \quad \forall 1 \leq i \leq n. \]

This means that (1.1) is uniformly elliptic. However, the fully nonlinear equations analogous to (1.1) fail to be uniformly elliptic in general, which causes various hard difficulties in the investigation, especially in proof of a priori (interior) estimates. Consequently, it is important to compare \( f_i(\lambda) \) with \( \sum_{j=1}^n f_j(\lambda) \). This leads to the notion of partial uniform ellipticity.

**Definition 1.1** (Partial uniform ellipticity). Let \( H \) be a symmetric nonempty subset of \( \Gamma \). We say that \( f \) is of \( m \)-uniform ellipticity in \( H \), if (1.4) holds and there exists a uniform positive constant \( \vartheta \) such that for any \( \lambda \in H \) with \( \lambda_1 \leq \cdots \leq \lambda_n, \)

\[ f_i(\lambda) \geq \vartheta \sum_{j=1}^n f_j(\lambda) > 0, \quad \forall 1 \leq i \leq m. \]  

(1.5)

In particular, \( n \)-uniform ellipticity is also called fully uniform ellipticity.
The author [25] introduced an integer $\kappa$ for $\Gamma$

$$\kappa_{\Gamma} = \max \left\{ k : (-\alpha_1, \cdots, -\alpha_k, \alpha_{k+1}, \cdots, \alpha_n) \in \Gamma, \text{ where } \alpha_j > 0, \forall 1 \leq j \leq n \right\} \quad (1.6)$$

and proved that the concave symmetric functions satisfying

$$\lim_{t \to +\infty} f(t\lambda) > f(\mu) \text{ for any } \lambda, \mu \in \Gamma \quad (1.7)$$

is exactly of $(\kappa_{\Gamma} + 1)$-uniform ellipticity in $\Gamma$. More precisely, there exists a uniform positive constant depending only on $\Gamma$ such that for any $\lambda \in \Gamma$ with $\lambda_1 \leq \cdots \leq \lambda_n$,

$$f_i(\lambda) \geq \vartheta_{\Gamma} \sum_{j=1}^{n} f_j(\lambda), \quad \forall 1 \leq i \leq 1 + \kappa_{\Gamma}. \quad (1.8)$$

In the case $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, (1.8) was proved by Lin-Trudinger [16]. Such a partial uniform ellipticity is relevant to various partial differential equations of elliptic and parabolic type. A surprising consequence of the conclusion (1.8) is that a type 2 cone means in some sense that the corresponding equations are uniformly elliptic.

However, the condition (1.7) is not fulfilled in some situations. For instance, it does not allow

$$\sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq -K \sum_{i=1}^{n} f_i(\lambda), \text{ for some } K \geq 0, \quad \forall A \leq f(\lambda) \leq \overline{A}. \quad (1.9)$$

Such a condition includes

$$\sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq \delta > 0, \quad \forall A \leq f(\lambda) \leq \overline{A} \quad (1.10)$$

as a special case. These two conditions appeared in the study of certain fully nonlinear equations from differential geometry, see e.g. [2, 11, 17, 20, 8, 9] and the references therein. We shall remark that assumptions (1.10) and (1.9) hold only on the range of the given function.

Motivated by this and related topics, it would be necessary to derive the level set version of partial uniform ellipticity.

*The paper [25] is essentially extracted from [arXiv:2011.08580] and [arXiv:2101.04947].
Before stating results we introduce some notions and impose appropriate assumptions. For $\sigma$, we denote the level set by
\[
\partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \}.
\]
Conditions (1.2) and (1.4) imply that (see Lemma A.1)
\[
\sum_{i=1}^{n} f_i(\lambda) > 0 \text{ for } f(\lambda) < \sup_{\Gamma} f.
\]
(1.11)
As a result, the level set $\partial \Gamma^\sigma$ (when $\partial \Gamma^\sigma \neq \emptyset$) is a smooth complete noncompact convex hypersurface. Throughout this paper we assume
\[
\sigma < \sup_{\Gamma} f \text{ and } \partial \Gamma^\sigma \neq \emptyset.
\]
Let’s denote
\[
t_\lambda = \frac{\sum_{i=1}^{n} f_i(\lambda) \lambda_i}{\sum_{j=1}^{n} f_j(\lambda)}, \quad \vec{1} = (1, \cdots, 1) \in \mathbb{R}^n.
\]
Geometrically, the tangent plane $T_\lambda \partial \Gamma^\sigma$ of $\partial \Gamma^\sigma$ at $\lambda \in \partial \Gamma^\sigma$, intersects the diagonal at $t_\lambda \vec{1}$, i.e.
\[
T_\lambda \partial \Gamma^\sigma \cap \{ t \vec{1} : t \in \mathbb{R} \} = \{ t_\lambda \vec{1} \}.
\]
We assume $t_\lambda$ has lower bound
\[
\liminf_{|\lambda| \to +\infty, \lambda \in \partial \Gamma^\sigma} t_\lambda > -\infty.
\]
(1.12)
And then we denote
\[
\tau_\sigma = \inf_{\lambda \in \partial \Gamma^\sigma} t_\lambda.
\]
Let $c_\sigma$ be the positive constant with $f(c_\sigma \vec{1}) = \sigma$. By (1.2), we know
\[
\tau_\sigma \leq c_\sigma
\]
with equality holding if and only if
\[
f_1(\lambda) = f_2(\lambda) = \cdots = f_n(\lambda), \quad \forall \lambda \in \partial \Gamma^\sigma.
\]
Consequently, we assume throughout this paper that
\[
\tau_\sigma < c_\sigma.
\]
Definition 1.2. Let $\Gamma_{\sigma,f}$ denote the cone

$$
\Gamma_{\sigma,f} = \{ t(\lambda - \tau) : \lambda \in \partial \Gamma^r, t > 0 \}.
$$

Furthermore $\Gamma_{\sigma,f}(1) := \partial \Gamma^r - \tau \vec{1}$ simply denotes a slice of $\Gamma_{\sigma,f}$. For such $\Gamma_{\sigma,f}$, we define

$$
\kappa_{\Gamma_{\sigma,f}} = \max \{ k : (\alpha_1, \cdots, \alpha_k, \alpha_{k+1}, \cdots, \alpha_n) \in \Gamma_{\sigma,f}, \alpha_i > 0 \}.
$$

Below we state the results on partial uniform ellipticity.

Theorem 1.3. Assume (1.2), (1.4) and (1.12) hold. Then there exists a uniform positive constant $\theta_{\Gamma_{\sigma,f}}$ depending only on $\Gamma_{\sigma,f}$ such that for each $\lambda \in \partial \Gamma^r$ with $\lambda_1 \leq \cdots \leq \lambda_n$,

$$
f_i(\lambda) \geq \theta_{\Gamma_{\sigma,f}} \sum_{j=1}^{n} f_j(\lambda), \quad \forall i \leq 1 + \kappa_{\Gamma_{\sigma,f}}.
$$

Remark 1.4. When replacing (1.4) and (1.12) by (1.7), Theorem 1.3 gives back (1.8).

As a consequence of Theorem 1.3, we can confirm an important inequality.

Theorem 1.5. Suppose, in addition to (1.2) and (1.4), that (1.9) holds for $\sup_{\vec{A}} f < \underbar{A} < \overline{A} < \sup_{\Gamma} f$. Then there is a positive constant $\theta$ depending on $K, \overline{A}, \underbar{A}$ such that

$$
f_i(\lambda) \geq \theta \left( 1 + \sum_{j=1}^{n} f_j(\lambda) \right) \text{ if } \lambda_i \leq -K, \forall \overline{A} \leq f(\lambda) \leq \underbar{A}.
$$

(1.13)

In particular, if $K = 0$, i.e. $\sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq 0$ for $\underbar{A} \leq f(\lambda) \leq \overline{A}$, then

$$
f_i(\lambda) \geq \theta \left( 1 + \sum_{j=1}^{n} f_j(\lambda) \right) \text{ if } \lambda_i \leq 0.
$$

(1.14)

Remark 1.6. The inequality (1.14) was imposed as a vital assumption by Li [15] and later by many experts to study certain geometric PDEs from classical differential geometry and conformal geometry, see e.g. [20, 10, 17, 3, 4, 6, 21, 8]. Our results can improve related results obtained there.

The paper is organized as follows. The level set version of partial uniform ellipticity is derived in Section 2. As applications, we derive the interior estimates for first and second order derivatives for complex fully nonlinear equations with Laplacian terms in Section 3, and briefly discuss real Hessian fully nonlinear equations in Section 4. In Appendixes, we summarize and prove some lemmas.
2 Partial uniform ellipticity: Level set version

2.1 Proof of Theorem 1.3

The concavity assumption (1.2) gives
\[ \sum_{i=1}^{n} f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda), \quad \forall \lambda, \mu \in \Gamma. \quad (2.1) \]

First we prove the following lemma.

**Lemma 2.1.** Suppose (1.2), (1.4) and (1.12) hold. Then
\[ \sum_{i=1}^{n} f_i(\lambda)\mu_i \geq 0, \quad \forall \lambda \in \partial \Gamma^\sigma, \forall \mu \in \Gamma_{\sigma,f}. \]

**Proof.** Given \( \lambda \in \partial \Gamma^\sigma \). Without loss of generality, we choose \( \mu \in \Gamma_{\sigma,f}(1) \). So \( \mu + \tau_{\sigma,1} \in \partial \Gamma^\sigma \). The inequality (2.1) simply yields
\[ \sum_{i=1}^{n} f_i(\lambda)(\tau_{\sigma} + \mu_i - \lambda_i) \geq 0. \]

Thus
\[ \sum_{i=1}^{n} f_i(\lambda)\mu_i \geq \sum_{i=1}^{n} f_i(\lambda)\lambda_i - \tau_{\sigma} \sum_{i=1}^{n} f_i(\lambda) \geq 0. \]

\[ \square \]

Let \( \lambda_1 \leq \cdots \leq \lambda_n \), the concavity and symmetry of \( f \) imply
\[ f_1(\lambda) \geq \cdots \geq f_n(\lambda) \] and \[ f_1(\lambda) \geq \frac{1}{n} \sum_{i=1}^{n} f_i(\lambda). \]

If \( \kappa_{\Gamma^\sigma,f} = 0 \) then Theorem 1.3 clearly follows. For \( \kappa_{\Gamma^\sigma,f} \geq 1 \), it is a consequence of the following proposition.

**Proposition 2.2.** In addition to (1.2), (1.4) and (1.12), we assume \( \kappa_{\Gamma^\sigma,f} \geq 1 \). Let \( \alpha_1, \cdots, \alpha_n \) be \( n \) positive constants with
\[ (-\alpha_1, \cdots, -\alpha_{\kappa_{\Gamma^\sigma,f}}, \alpha_{1+\kappa_{\Gamma^\sigma,f}}, \cdots, \alpha_n) \in \Gamma_{\sigma,f}. \]
Assume in addition that \( \alpha_1 \geq \cdots \geq \alpha_{\kappa_{\Gamma^\sigma,f}} \). Then for \( \lambda \in \partial \Gamma^\sigma \) with \( \lambda_1 \leq \cdots \leq \lambda_n \),
\[ f_{1+\kappa_{\Gamma^\sigma,f}}(\lambda) \geq \frac{\alpha_1}{\sum_{i=1+\kappa_{\Gamma^\sigma,f}}^{n} \alpha_i - \sum_{i=2}^{n} \alpha_i} f_1(\lambda). \quad (2.2) \]
Proof. According to Lemma 2.1, we have

$$- \sum_{i=1}^{\kappa_{\sigma,f}} \alpha_i f_i(\lambda) + \sum_{i=1+\kappa_{\sigma,f}}^n \alpha_i f_i(\lambda) \geq 0.$$  

This simply yields $f_{1+\kappa_{\sigma,f}}(\lambda) \geq \frac{\alpha_1}{\sum_{i=1+\kappa_{\sigma,f}}^n \alpha_i} f_i(\lambda)$. In addition, one derives (2.2) by using iteration.  

□

Remark 2.3. In the case $\kappa_{\Gamma,\sigma,f} \geq 1$, the constant $\vartheta_{\Gamma,\sigma,f}$ in Theorem 1.3 can be achieved as

$$\vartheta_{\Gamma,\sigma,f} = \sup_{\alpha(-\alpha_1,\ldots,-\alpha_n,\alpha_1+\kappa_{\sigma,f},\ldots,\alpha_n) \in \Gamma_{\sigma,f}} \frac{\alpha_1/n}{\sum_{i=1+\kappa_{\sigma,f}}^n \alpha_i - \sum_{i=2}^{\kappa_{\sigma,f}} \alpha_i}.$$

2.2 A new criterion for $f$ satisfying (1.7)

Building on Lemma B.1, we can deduce a new criterion for (1.7).

Lemma 2.4. In the presence of (1.2), (1.4) and

$$\sum_{i=1}^n f_i(\lambda) > 0 \text{ in } \Gamma,$$  

condition (1.7) is equivalent to

$$\Gamma \subseteq \Gamma_{\sigma,f}, \quad \forall \sup_{\partial \Gamma} f < \sigma < \sup_{\Gamma} f.$$  

Proof. $\Leftarrow$ Fix $\lambda, \mu \in \Gamma$, let $\sigma = f(\lambda)$. Since $\Gamma \subseteq \Gamma_{\sigma,f}$, we have $\sum_{i=1}^n f_i(\lambda)\mu_i \geq 0$ by Lemma 2.1. Thus (1.7) holds by Lemma B.5.

$\Rightarrow$ For $\lambda \in \Gamma$ and $\sup_{\partial \Gamma} f < \sigma < \sup_{\Gamma} f$, one has $f(t\lambda) > \sigma$ for some $t > 0$. By Lemma B.1, $\tau_{\sigma} \geq 0$. There is $0 < t_0 < t$ such that $f(t_0\lambda + \tau_{\sigma} \vec{1}) = \sigma$. This yields

$$\Gamma \subseteq \Gamma_{\sigma,f}.$$  

□

The $(\kappa_{\Gamma} + 1)$-uniform ellipticity as asserted in (1.8) follows as a consequence of Theorem 1.3, Lemma 2.4 and Corollary B.2.
2.3 Confirming an inequality

**Proposition 2.5.** Suppose (1.2), (1.4) and (1.12) hold. Then for \( \lambda \in \partial \Gamma^\sigma \),

\[
f_i(\lambda) \geq \vartheta_{\Gamma^\sigma} + \sum_{j=1}^{n} f_j(\lambda) \text{ whenever } \lambda_i \leq \tau_{\sigma}.
\]  

(2.5)

In particular, replacing (1.12) by

\[
\sum_{i=1}^{n} f_i(\lambda) \lambda_i \geq 0 \text{ in } \partial \Gamma^\sigma,
\]  

(2.6)

then for any \( \lambda \in \partial \Gamma^\sigma \) we get

\[
f_i(\lambda) \geq \vartheta_{\Gamma^\sigma} + \sum_{j=1}^{n} f_j(\lambda) \text{ if } \lambda_i \leq 0.
\]  

(2.7)

**Proof.** Given \( \lambda \in \partial \Gamma^\sigma \) with \( \lambda_1 \leq \cdots \leq \lambda_n \). Let \( \mu_i = \lambda_i - \tau_{\sigma} \), then \( \mu = (\mu_1, \cdots, \mu_n) \in \Gamma_{\sigma,f} \). By the definition of \( \kappa_{\Gamma_{\sigma,f}, \mu_1+\kappa_{\Gamma_{\sigma,f}}} \geq 0 \), i.e. \( \lambda_1+\kappa_{\Gamma_{\sigma,f}} \geq \tau_{\sigma} \). For each \( \lambda_i \leq \tau_{\sigma} \), we have

\[
f_i(\lambda) \geq f_{1+\kappa_{\Gamma_{\sigma,f}}}(\lambda).
\]

Consequently, (2.5) follows from Theorem 1.3. \( \square \)

It follows from (2.1) that if \( f \) satisfies (1.12) then there is a positive constant \( \theta = \vartheta(\sigma) \) depending only on \( \sigma \) such that

\[
\sum_{i=1}^{n} f_i(\lambda) \geq \theta(\sigma) \text{ in } \partial \Gamma^\sigma.
\]  

(2.8)

Theorem 1.5 then follows from (2.8) and Proposition 2.5.

Below we consider two special cases.

**Lemma 2.6.** If \((f, \Gamma)\) satisfies (1.2), (1.3) and

\[
\lim_{t \to 0^+} f(t\vec{1}) > -\infty,
\]  

(2.9)

then we have \( \sum_{i=1}^{n} f_i(\lambda) \lambda_i > 0 \) and (1.14).

**Proof.** According to Lemmas B.1 and B.6, we have \( \sum_{i=1}^{n} f_i(\lambda) \lambda_i > 0 \). Theorem 1.5 then gives (1.14). \( \square \)
Lemma 2.7. Let \((f, \Gamma)\) satisfy (1.2) and

\[ f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial \Gamma \tag{2.10} \]

then we have \( \sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq 0, (1.4) \) and \((1.14)\).

Proof. The conclusions \((1.4)\) and \( \sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq 0 \) are deduced from Lemma B.3. Again, by Theorem 1.5 we have \((1.14)\). □

Remark 2.8. These two lemmas allows one to improve some results on Weingarten equations obtained by Li [15] and Trudinger [20] respectively; we decide to omit the details here.

3 Applications to complex fully nonlinear equations

Let \((M, \omega)\) be a compact Hermitian manifold of complex dimension \(n \geq 2\) possibly with boundary. Let \(\chi\) be a smooth real \((1, 1)\)-form, \(\psi\) a \(C^2\)-smooth function, \(\Delta\) the Laplacian operator,

\[ Z = \frac{1}{(n - 1)!} \ast \Re(\sqrt{-1}\partial u \wedge \bar{\partial} \omega^n), \]

where \(\ast\) is the Hodge star operator with respect to \(\omega\). Recently, Székelyhidi-Tosatti-Weinkove [18] proved Gauduchon’s conjecture, by solving the Monge-Ampère equation for \((n - 1)\)-PSH functions on a closed Hermitian manifold

\[ \left( \omega_0 + \frac{1}{n - 1}(\Delta u\omega - \sqrt{-1}\partial \bar{\partial} u) + Z \right)^n = e^{(n-1)\phi} \omega^n, \quad \omega_0 > 0. \tag{3.1} \]

When \(M\) admits a balanced metric and an astheno-Kähler metric, it closely connects to the Form-type Calabi-Yau equation [5]. Subsequently, the author [24]† solved the Dirichlet problem for \((3.1)\), thereby extending Székelyhidi-Tosatti-Weinkove’s results to complex manifolds with boundary. In addition, the author [24] has investigated the Dirichlet problem for equations of the form

\[ f \left( \lambda \left( \chi + \frac{1}{n - 1}(\Delta u\omega - \sqrt{-1}\partial \bar{\partial} u) + Z \right) \right) = \psi \tag{3.2} \]

†It is essentially extracted from the second parts of [arXiv:2001.09238] and [arXiv:2106.14837].
where \( \chi \) is a smooth real \((1,1)\)-form, \( \lambda(A) \) are the eigenvalues of \( A \) with respect to \( \omega \). More precisely, when imposing (1.7), \( \Gamma = \Gamma_n \) and the unbound condition

\[
\lim_{t \to +\infty} f(\lambda_1 + t, \cdots, \lambda_{n-1} + t, \lambda_n) = \sup_{\Gamma} f, \quad \forall \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma,
\]

the Dirichlet problem for (3.2) was solved by the author in [24, Section 7]. The assumption (3.3) allows

\[
f = \left( \frac{\sigma_n}{\sigma_k} \right)^{1/(n-k)}, \quad 0 \leq k \leq n - 2.
\]

While \( \Gamma \neq \Gamma_n \), even without assuming (3.3) as well with degenerate right-hand side, the Dirichlet problem for (3.2) was completely solved there. These results reveal that there are significant differences between \( \Gamma = \Gamma_n \) and \( \Gamma \neq \Gamma_n \). Based on the partial uniform ellipticity, we are able to figure out those differences.

**Proposition 3.1.** Given \((f, \Gamma)\), we define

\[
\tilde{f}(\lambda) = f(\mu), \quad \mu_i = \sum_{j \neq i} \lambda_j, \quad \tilde{\Gamma} = \{ \lambda \in \mathbb{R}^n : \mu \in \Gamma \}.
\]

In the presence of (1.2) and (1.7), we have the following:

(1) If \( \Gamma \neq \Gamma_n \) holds, then \( \tilde{\Gamma} \) is of type 2 cone and \( \tilde{f} \) is of fully uniform ellipticity in \( \tilde{\Gamma} \).

(2) If \( \Gamma = \Gamma_n \), then \( \tilde{f} \) is of \((n-1)\)-uniform ellipticity in \( \tilde{\Gamma} \).

The proposition shows that in the presence of (1.2) and (1.7), equation (3.2) is uniformly elliptic in the case \( \Gamma \neq \Gamma_n \), while it is only of \((n-1)\)-uniform ellipticity when \( \Gamma = \Gamma_n \). Such differences motivate us to derive more delicate results. We apply level set version of partial uniform ellipticity to derive interior estimates for (3.2), when imposing proper restrictions:

\[
\Gamma \neq \Gamma_n,
\]

\[
\lim_{t \to +\infty} f(t, \cdots, t, 0) > \sup_M \psi,
\]

\[
\sum_{i=1}^n f_i(\lambda)\lambda_i \geq 0 \text{ in } \Gamma^{\psi, \phi}
\]

where

\[
\Gamma^{\psi, \phi} = \left\{ \lambda : \inf_M \psi \leq f(\lambda) \leq \sup_M \psi \right\}.
\]
Theorem 3.2. Let $B_r$ be a geodesic ball in $(M, \omega)$. Suppose (1.2), (1.4), (3.5), (3.6) and (3.7) hold. Then for any solution $u \in C^4(B_r)$ to equation (3.2) satisfying

$$\lambda \left( \chi + \frac{1}{n-1} \left( \Delta u \omega - \sqrt{-1} \partial \overline{\partial} u \right) + Z \right) \in \Gamma \text{ in } B_r,$$

there is a uniform positive constant $C$ depending only on $|u|_{C^0(B_r)}$, $|\psi|_{C^2(B_r)}$ and geometric quantities on $B_r$, such that

$$\sup_{B_{r/2}} (|\nabla u|^2 + |\partial \overline{\partial} u|) \leq \frac{C}{r^2}.$$  

According to Lemma B.1 and Corollary B.2 below, (1.4) and (3.7) are simultaneously satisfied when $f$ satisfies (1.2) and (1.7). As a consequence, we obtain

Corollary 3.3. Theorem 3.2 holds when $(f, \Gamma)$ satisfies (1.2), (1.7) and (3.5).

In fact interior estimate (3.10) still holds for equations

$$f \left( \lambda \left( \chi + \Delta u \omega - \sqrt{-1} \partial \overline{\partial} u + \gamma Z \right) \right) = \psi$$

provided that $(f, \Gamma)$ satisfies (1.2) and (1.7), $\gamma$ is a $C^2$-smooth function, and $\varrho$ is a $C^2$-smooth function satisfying

$$\varrho < \frac{1}{1 - \kappa_{\Gamma} \theta_{\Gamma}} \text{ and } \varrho \neq 0.$$  

(3.12)

Here $\kappa_{\Gamma}$ and $\theta_{\Gamma}$ are the constants in (1.6) and (1.8), respectively.

Theorem 3.4. Let $B_r$ be a geodesic ball in $(M, \omega)$, and let $u \in C^4(B_r)$ be a solution to (3.11) in $B_r$ with

$$\lambda \left( \chi + \Delta u \omega - \sqrt{-1} \partial \overline{\partial} u + \gamma Z \right) \in \Gamma \text{ in } B_r.$$  

Suppose in addition that (1.2), (1.7) and (3.12) hold. Then

$$\sup_{B_{r/2}} (|\nabla u|^2 + |\partial \overline{\partial} u|) \leq \frac{C}{r^2}$$

where $C$ depends only on $|u|_{C^0(B_r)}$, $|\psi|_{C^2(B_r)}$ and geometric quantities on $B_r$.

The restriction (3.12) to parameter $\varrho$ was imposed by the author [25] to study conformal deformation of modified Schouten tensors. When $\Gamma = \Gamma_{\nu}$ it reduces to

$$\varrho < 1 \text{ and } \varrho \neq 0.$$  

(3.13)

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Hence it does not cover the Monge-Ampère equation for \((n - 1)\)-PSH function. The equations analogous to (3.11) were also studied in [12] under the assumption (3.13). Notice (3.12) allows the critical case \(\varrho = 1\) when \(\Gamma \neq \Gamma_n\). Our result in Theorem 3.4 is new.

**Remark 3.5.** The main difficulty in Székelyhidi-Tosatti-Weinkove’s proof of second estimate is to deal with the bad terms due to the gradient terms \(\partial u, \overline{\partial u}\) from \(Z\). Their proof depends heavily on delicate structures of \(Z\), which cannot be extended to more general cases. In contrast with Székelyhidi-Tosatti-Weinkove’s estimates, our results assert the interior estimates for second and first order derivatives when \(\Gamma \neq \Gamma_n\). In fact, such interior estimates are not true for general complex fully nonlinear equations.

### 3.1 Interior estimates for equations of fully uniform ellipticity

In this subsection we are concerned with an equation of the form

\[
F(g_{ij}) := f(\lambda(g_{ij})) = \psi
\]  \hspace{1cm} (3.14)

where \(g_{ij} = u_{ij} + \chi_{ij} + S^k_{ij}u_k + S^k_{ji}u_k, \lambda(g_{ij})\) denote the eigenvalues of \(g_{ij}\) with respect to \(g_{ij}\). We call \(u\) an admissible function if \(\lambda(g_{ij}) \in \Gamma\). In addition, we assume that there exists a positive constant \(\theta\) such that

\[
f_i(\lambda) \geq \theta \sum_{j=1}^{n} f_j(\lambda) \text{ in } \Gamma \supseteq \overline{\psi}, \quad \forall 1 \leq i \leq n.
\]  \hspace{1cm} (3.15)

We prove the following interior estimates.

**Theorem 3.6.** Let \(B_r\) be a geodesic ball in \((M, \omega)\). Suppose (1.2), (1.4), (3.6) and (3.15) hold. Then for any admissible solution \(u \in C^4(B_r)\) to (3.14) in \(B_r\), we have

\[
\sup_{B_{r/2}} (|\partial \overline{\partial u}| + |\nabla u|^2) \leq \frac{C}{r^2}
\]

where \(C\) is a uniform constant depending only on \(\theta^{-1}, |u|_{C^0(B_r)}, |\psi|_{C^2(B_r)}\) and geometric quantities on \(B_r\).
3.1.1 Preliminaries

The linearized operator of equation (3.14), say \( \mathcal{L} \), at solution \( u \) is locally given by

\[
\mathcal{L}v = F^i v_i + F^i S^k_{ij} v_k + F^i S^k_{ji} v_k
\]

(3.16)

where \( F^i = \frac{\partial F}{\partial \bar{g}^i} \). One knows that the eigenvalues of \( F^i \) (w.r.t. \( g^i \)) are precisely

\[
f_1(\lambda), \ldots, f_n(\lambda), \quad \text{where} \quad \lambda = \lambda(g).
\]

Moreover

\[
F^i g_{ij} = \sum_{i=1}^n f_i(\lambda) \lambda_i, \quad \sum_{i=1}^n F^i g_{ij} = \sum_{i=1}^n f_i(\lambda).
\]

From condition (3.6), the right-hand side must satisfy

\[
\sup_M \psi < \sup_{\Gamma} f.
\]

Combining with Lemma A.1 below,

\[
\sum_{i,j=1}^n F^i g_{ij} = \sum_{i=1}^n f_i(\lambda) > 0.
\]

(3.17)

Remark 3.7. The condition (3.15) yields that equation (3.14) is in effect uniformly elliptic at admissible solution \( u \). That is automatically satisfied when imposed the same conditions as that of Theorem 3.2 or Corollary 3.3.

The following formulas are standard

\[
\begin{align*}
  u_{1jk} - u_{kj1} &= T^l_{ik} u_{lj}, \\
  u_{1i\bar{i}} - u_{\bar{i}i1} &= R^l_{i1\bar{i}} u_{lp} - R_{1i\bar{i}p} u_{\bar{i}l} + 2 \Re \{ \bar{T}^l_{ij} u_{ij} \} + T^p_{il} \bar{T}^q_{lp} u_{pq}.
\end{align*}
\]

(3.18)

Denote

\[
w = |\nabla u|^2 \text{ and } Q = |\partial \bar{\partial} u|^2 + |\partial \bar{\partial} u|^2.
\]

Under local coordinates \( z = (z_1, \ldots, z_n) \) around \( z_0 \), with \( g_{ij}(z_0) = \delta_{ij} \), we have by straightforward computations

\[
w_j = u_k u_{ki} + u_k u_{jk},
\]

\[
w_{ij} = u_k u_{kj} + u_{kji} + u_{kij} + u_{kij} + R_{ijkl} u_{kl} - T^l_{ik} u_{ij} u_k - T^l_{jk} u_{ij} u_k.
\]

One then obtains

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Lemma 3.8. We have
\[ F_{ij} w_i w_j \leq 2wQ \sum F^{ii}, \]
and there exists $C > 0$ such that
\[ L(w) \geq \frac{3\theta Q}{4} \sum F^{ii} - Cw \sum F^{ii} - C|\nabla \psi| \sqrt{w}. \]

3.1.2 Interior estimate for first derivative

Let’s consider the quantity
\[ m_0 = \max_{\mathcal{M}} \eta |\nabla u|^2 e^\phi \]
where $\eta \geq 0$ and $\phi = \phi(z, u)$ are functions to be determined. Suppose that $m_0$ is attained at an interior point $z_0 \in M$. We choose local coordinates $(z_1, \ldots, z_n)$ such that $g_{ij} = \delta_{ij}$ at $z_0$. As above we denote $w = |\nabla u|^2$. Without loss of generality, $w \geq 1$ at $z_0$. From above, the function $\log \eta + \log w + \phi$ achieves a maximum at $z_0$ and therefore,
\[ \frac{\eta_i}{\eta} + \frac{w_i}{w} + \phi_i = 0, \quad \frac{\eta_j}{\eta} + \frac{w_j}{w} + \phi_j = 0, \quad (3.19) \]

\[ L(\log \eta + \log w + \phi) \leq 0. \quad (3.20) \]

Combining (3.19) with Cauchy-Schwarz inequality, we derive
\[ \frac{1}{w^2} F_{ij} w_i w_j \leq \frac{1 + \epsilon}{\epsilon \eta^2} F_{ij} \eta_i \eta_j + (1 + \epsilon) F_{ij} \phi_i \phi_j. \quad (3.21) \]

As a result, combining with Lemma 3.8 and let $8\epsilon \leq \theta$, we derive at $z_0$
\[ L \log w \geq \frac{(3\theta - 8\epsilon)Q}{4w} \sum F^{ii} - C \sum F^{ii} - C|\nabla \psi| \sqrt{w} - \frac{1 - \epsilon^2}{\epsilon \eta^2} F_{ij} \eta_i \eta_j - (1 - \epsilon^2) F_{ij} \phi_i \phi_j \quad (3.22) \]

On the other hand,
\[ L \log \eta = \frac{F_{ij} \eta_i \eta_j}{\eta} + F_{ij} S^{\kappa \lambda} \eta_i \eta_k \eta_j \frac{\eta_\lambda}{\eta} + F_{ij} S^{\kappa \lambda} \eta_i \eta_k \eta_j \frac{\eta_\lambda}{\eta^2} - F_{ij} \eta_i \eta_j \eta_j \frac{\eta_0}{\eta^2}. \quad (3.23) \]
To derive the interior estimate, following [13] (see also [12]) we take \( \eta \) to be a smooth function with compact support in \( B_r \subset M \) satisfying
\[
0 \leq \eta \leq 1, \quad \eta|_{B_{\frac{r}{2}}} \equiv 1, \quad |\nabla \eta| \leq \frac{C \sqrt{\eta}}{r}, \quad |\nabla^2 \eta| \leq \frac{C}{r^2}.
\] (3.24)

Thus
\[
\frac{1 + \epsilon}{\epsilon \eta^2} F^{ij} \eta_i \eta_j + F^{ij} S^k_{ij} \eta_k + F^{ij} S^{kj}_{ij} \eta_k - \frac{1}{\eta} F^{ij} \eta_i \eta_j \leq \frac{C}{\epsilon \eta^2 \eta} \sum F^{\tilde{\alpha}}.
\] (3.25)

As in [7], let \( \phi = v^{-N} \) where \( v = u - \inf_{B_r} u + 2 \) (\( v \geq 2 \) in \( B_r \)) and \( N \geq 1 \) is an integer that is chosen later. By direct computation
\[
\phi_i = -N v^{-N-1} u_i, \quad \phi_i = -N v^{-N-1} u_i, \\
\phi_{ij} = N(N + 1) v^{-N-2} u_i u_j - N v^{-N-1} u_{ij}.
\]

So
\[
F^{\tilde{\alpha}} \phi_i \phi_j = N^2 v^{-2N-2} F^{ij} u_i u_j
\] (3.26)

and
\[
\mathcal{L} \phi = N(N + 1) v^{-N-2} F^{ij} u_i u_j - N v^{-N-1} F^{ij} u_i u_j - N v^{-N-1} F^{ij} (F^{ij} S^k_{ij} u_k + F^{ij} S^{kj}_{ij} u_k) \\
= N(N + 1) v^{-N-2} F^{ij} u_i u_j - N v^{-N-1} F^{ij} (g_{ij} - \chi_{ij}).
\] (3.27)

The concavity of \( f \) implies that there is a positive constant \( C_1 \) such that
\[
F^{ij} (g_{ij} - \chi_{ij}) = \sum_{i=1}^{n} f_i \lambda_i \leq C_1 \sum_{i=1}^{n} f_i.
\] (3.28)

We choose \( N \gg 1 \) so that \( N v^{-N} < 1, \) then
\[
N(N + 1) v^{-N-2} - N^2 v^{-2N-2} \geq N^2 v^{-N-2}.
\] (3.29)

Plugging (3.22)-(3.23) and (3.25)-(3.29) into (3.20), we obtain
\[
\theta w N^2 v^{-N-2} \sum F^{\tilde{\alpha}} + \frac{\theta Q}{2w} \sum F^{\tilde{\alpha}} \leq C N v^{-N-1} \sum F^{\tilde{\alpha}} + \frac{C}{r^2 \eta} \sum F^{\tilde{\alpha}} + \frac{C}{\sqrt{w}}
\]

Note
\[
\frac{\theta w N^2 v^{-N-2}}{2} \sum F^{\tilde{\alpha}} + \frac{\theta Q}{2w} \sum F^{\tilde{\alpha}} \geq \theta N v^{-N-1} \sqrt{Q} \sum F^{\tilde{\alpha}}
\]

and that there exists \( R_0 > 0 \) such that for any \( \lambda \) with \( |\lambda| \geq R_0 \)
\[
|\lambda| \sum_{i=1}^{n} f_i(\lambda) \geq \frac{f(|\lambda|) - f(\lambda)}{2} \geq \frac{f(R_0 \lambda) - \psi}{2} > 0.
\]

As a result, we obtain Theorem 3.6.
3.1.3 Interior estimate for second order derivatives

We derive second order interior estimate.

**Proof.** As in [12] we consider the following quantity

\[ P := \sup_{z \in M} \max_{\xi \in T_z^0 M} e^{2\phi} g_{\bar{p}q} \xi p \bar{S}_q \sqrt{|g_{\bar{q}r} g_{\bar{q}k} \xi \bar{S}_k |} / |\xi|^3 \]

where \( \phi \) is a function depending on \( z \) and \( |\nabla u| \). Assume that it is achieved at an interior point \( p_0 \in M \) for some \( \xi \in T_{p_0}^0 M \). The quantity \( P \) is inspired by [19]. We choose local coordinates \( z = (z_1, \cdots, z_n) \) around \( p_0 \), such that at \( p_0 \)

\[ g_{ij} = \delta_{ij}, \quad g_{i\bar{j}} = \delta_{i\bar{j}} \lambda_j \quad \text{and} \quad F_{ij} = \delta_{ij} f_i. \]

The maximum \( P \) is achieved for \( \xi = \partial_1 \) at \( p_0 \). We assume \( g_{1\bar{1}} \geq 1 \); otherwise we are done.

In what follows the computations are given at \( p_0 \). Similar to the computations in [12] one has

\[ g_{1\bar{i}} + g_{1\bar{i}} \phi_i = 0, \quad g_{1\bar{i}} + g_{1\bar{i}} \phi_i = 0, \quad (3.30) \]

\[ 0 \geq \frac{F^{\bar{i}} g_{1\bar{i}}}{g_{1\bar{i}}} + F^{\bar{i}} (\phi_{\bar{i}} - \phi_{\bar{i}}) + \frac{1}{8 g_{1\bar{1}}} \sum_{k>1} F^{\bar{i}} g_{1\bar{k}} g_{k\bar{1}} - C \sum F^{\bar{i}}. \quad (3.31) \]

Combining with the standard formula (3.18), together with straightforward computation, we can derive

\[ g_{1\bar{i}} \geq g_{\bar{1}i} + 2 \Re (\bar{T}_{i\bar{j}} g_{1\bar{j}}) + 2 \Re (S_{1\bar{i}} g_{\bar{1}l} - S_{1\bar{i}} g_{\bar{1}l}) - C \sqrt{Q} \quad (3.32) \]

where as above one denotes

\[ Q = |\partial \bar{\partial} u|^2 + |\partial \bar{\partial} u|^2. \]

Differentiating equation (3.14) twice (using covariant derivative), we obtain

\[ F^{\bar{i}} g_{\bar{i}l} = \psi_1, \quad (3.33) \]

\[ F^{\bar{i}} g_{\bar{i}1\bar{l}} = \psi_{1\bar{l}} - F^{i\bar{j}h} g_{i\bar{j}1\bar{h}l}. \quad (3.34) \]
Then we have
\[ F^{\bar{u}}_{g_{11}} \geq 2 \Re(\bar{F}^{\bar{u}} T_{i j}^{1} g_{1 j}) - 2 \Re F^{\bar{u}}_{\bar{S}}^{1} g_{11} - C \sqrt{Q} \sum F^{\bar{u}}. \]  
(3.35)

Putting the above inequalities into (3.31) we get
\[ 0 \geq g_{11} F^{\bar{u}}(\phi_{\bar{i}} - \phi_{i}) - 2 \Re(F^{\bar{u}} S_{\bar{g}_{11}}) + 2 \Re(F^{\bar{u}} T_{i j}^{1} g_{1 j}) - C \sqrt{Q} \sum F^{\bar{u}}. \]

Let \( \phi = \log \eta + \varphi(w) \), where as above \( w = |\nabla u|^{2} \), and \( \eta \) is the cutoff function given by (3.24). Then
\[ \mathcal{L}\phi = \frac{\mathcal{L}\eta}{\eta} - F^{\bar{u}} \frac{|\eta|^{2}}{\eta^{2}} + \varphi' \mathcal{L}w + \varphi'' F^{\bar{u}}|w_{i}|^{2}, \]  
(3.36)

\[ F^{\bar{u}}|\phi_{i}|^{2} + 2 \Re F^{\bar{u}} \bar{T}_{i j}^{1} \phi_{j} \leq \frac{4}{3} F^{\bar{u}}|\phi_{i}|^{2} + C \sum F^{\bar{u}} \]  
(3.37)

and
\[ F^{\bar{u}}|\phi_{i}|^{2} \leq \frac{3}{2} F^{\bar{u}}|\varphi_{i}|^{2} + 3 F^{\bar{u}} \frac{|\eta|^{2}}{\eta^{2}}. \]

As in [7] we set
\[ \varphi = \varphi(w) = \left(1 - \frac{w^{2}}{2N}\right)^{-\frac{1}{2}} \text{ where } N = \sup_{w > 0} |\nabla u|^{2}. \]

One can check \( \varphi' = \frac{\varphi^{3}}{4N}, \varphi'' = \frac{\varphi^{5}}{16N} \) and \( 1 \leq \varphi \leq \sqrt{2}. \) And so
\[ \varphi'' - 2 \varphi' = \frac{\varphi^{5}}{16N^{2}} \left(3 - 2\varphi\right) > \frac{\varphi^{5}}{96N^{2}}. \]  
(3.38)

By Lemma 3.8 we have
\[ \mathcal{L}(w) \geq \frac{3\theta Q}{4} \sum F^{\bar{u}} - C(1 + \sum F^{\bar{u}}). \]  
(3.39)

By (3.24) we obtain
\[ 0 \leq \eta \leq 1, \quad \eta|_{B_{r}} \equiv 1, \quad F^{\bar{u}} \frac{|\eta|^{2}}{\eta^{2}} \leq \frac{C}{r^{2}}, \quad \frac{\mathcal{L}\eta}{\eta} \leq \frac{C}{r^{2}} \sum F^{\bar{u}}. \]  
(4.40)

In conclusion we finally derive
\[ 0 \geq \frac{9\theta Q}{16N} \sum F^{\bar{u}} - \frac{C}{r^{2}} \sum F^{\bar{u}} - C \frac{\sqrt{Q}}{g_{11}} \sum F^{\bar{u}}. \]  
(3.41)

This gives
\[ g_{11} \leq \frac{C}{r^{2}}. \]
3.2 Completion the proof of Theorem 3.2

First, Theorem 1.3 implies the following result.

**Theorem 3.9.** Assume (1.2), (1.4) and (2.6) hold. Then for fixed \( \sigma \),

1. If \( \partial \Gamma^\nu \backslash \tilde{\Gamma}_n \neq \emptyset \), then \( \kappa_{\Gamma, f} \geq 1 \).
2. If \( \Gamma \) is of type 2 cone and
   \[
   f(0, \cdots, 0, t) > \sigma \text{ for some } t > 0, \tag{3.42}
   \]
   then \( \kappa_{\Gamma, f} = n - 1 \) and \( f \) is of fully uniform ellipticity when restricted to \( \partial \Gamma^\nu \).

**Proof.** Prove (1): By (2.6), \( \tau_\sigma \geq 0 \). Let \( \lambda \in \partial \Gamma^\nu \backslash \tilde{\Gamma}_n \) and we assume \( \lambda_n < 0 \). Note that \( \lambda - \tau_\sigma \tilde{1} \in \Gamma_{\sigma, f} \) and \( \lambda_n - \tau_\sigma \leq \lambda_n < 0 \). Thus \( \kappa_{\Gamma, f} \geq 1 \).

Prove (2): From (3.42), there are some positive constants \( \epsilon_0, t_0 \) such that \( f(-\epsilon_0, \cdots, -\epsilon_0, t_0) > \sigma \). And then there is \( 0 < \beta < 1 \) such that \( f(-\beta \epsilon_0, \cdots, -\beta \epsilon_0, \beta t_0) = \sigma \). Thus \( (-\beta \epsilon_0 - \tau_\sigma, \cdots, -\beta \epsilon_0 - \tau_\sigma, \beta t_0 - \tau_\sigma) \in \Gamma_{\sigma, f} \) as required. \( \square \)

Next we present functions being of fully uniform ellipticity.

**Corollary 3.10.** Let \( (\bar{f}, \tilde{\Gamma}) \) be as defined in (3.4). Let \( \sup_{\partial \Gamma^\nu} f < \sigma < \sup_{\Gamma} f \). Suppose that \( (f, \Gamma) \) satisfies (1.2), (1.4), (2.6), \( \Gamma \neq \Gamma_n \) and

\[
 f(t, \cdots, t, 0) > \sigma \text{ for some } t > 0. \tag{3.43}
\]

Then \( \bar{f} \) is of fully uniform ellipticity when restricted to \( \{ \lambda \in \tilde{\Gamma} : \bar{f}(\lambda) = \sigma \} \).

Corollary 3.10 immediately implies the following:

**Proposition 3.11.** If (1.2), (1.4), (3.5), (3.6) and (3.7) hold, then equation (3.2) is of fully uniform ellipticity at any solution satisfying (3.9).

This proposition confirms all the assumptions imposed in Theorem 3.6, thereby obtaining Theorem 3.2.
4 Applications to Hessian equations

Let \((M, g)\) be a \(n\)-dimensional compact Riemannian manifold, possibly with boundary \(\partial M, \tilde{M} = M \cup \partial M\). We consider the Hessian equations

\[
f(\lambda(\nabla^2 u + A)) = \psi.
\]  

(4.1)

where \(\psi\) is a smooth function and \(A\) is a smooth symmetric \((0, 2)\)-type tensor. The second order estimate for (4.1) on curved Riemannian manifolds was studied by Guan [8, Section 3], extending previous results in literature, see e.g. [14, 21]. The gradient estimate is the remaining task to the study of Hessian equations. However, it is rather hard to prove gradient estimate for Hessian equations on curved Riemannian manifolds. The gradient estimate was obtained in [14] under assumptions (2.9), \[\lim_{|\lambda| \to +\infty} \sum_{i=1}^{n} f_i(\lambda) = +\infty\] and that the Riemannian manifold admits nonnegative sectional curvature, and later extended by Urbas [21] with replacing such restrictions by (1.14) and (2.10). One may use Lemmas 2.6 and 2.7 to improve their results.

In fact, Theorem 1.5 and Proposition 2.5 allow us to derive the gradient estimate for more general equations when

\[
\sum_{i=1}^{n} f_i(\lambda) \lambda_i \geq -K_0 \sum_{i=1}^{n} f_i(\lambda) \text{ for some } K_0 \geq 0.
\]  

(4.2)

If (4.2) holds for any \(\lambda \in \Gamma^\psi_{\tilde{M}}\), then according to Proposition 2.5 and (2.8), we obtain a more general inequality than (1.14)

\[
f_i(\lambda) \geq \theta + \theta \sum_{j=1}^{n} f_j(\lambda) \quad \text{if } \lambda_i \leq -K_0, \quad \forall \lambda \in \Gamma^\psi_{\tilde{M}}.
\]  

(4.3)

As an application, one can follow a strategy, analogous to that used in [21], to derive gradient bound for solutions to (4.1) under the C-subsolution assumption

\[
\lim_{t \to +\infty} f(\lambda(\nabla^2 u + A) + te_i) > \psi \text{ in } \tilde{M}, \quad \forall 1 \leq i \leq n
\]  

(4.4)

where \(e_i\) is the \(i\)-th standard basis vector of \(\mathbb{R}^n\).

**Proposition 4.1.** In addition to (1.2), (1.4) and (4.2), we assume there is a \(C^2\)-smooth C-subsolution \(\underline{u}\). Let \(u \in C^2(M) \cap C^1(\tilde{M})\) be a solution to (4.1) with \(\lambda(\nabla^2 u + A) \in \Gamma\), then

\[
\sup_{M} |\nabla u| \leq C(1 + \sup_{\partial M} |\nabla u|),
\]

where \(C\) depends on \(|\psi|_{\Gamma^\psi_{\tilde{M}}}, |\underline{u}|_{C^2(\tilde{M})}\) and other known data.
With gradient estimate at hand, as in [8, 23], we can prove the following:

**Theorem 4.2.** Let \((M^n, g)\) be a compact Riemannian manifold with smooth smooth boundary. Let \(\varphi \in C^\infty(\partial M)\) and \(\psi \in C^\infty(\bar{M})\) be a function satisfying \(\inf_M \psi > \sup_{\partial M} f\). Suppose that there is an admissible function \(u \in C^{3,1}(\bar{M})\) satisfying

\[
f(\lambda(\nabla^2 u + A)) \geq \psi, \quad u|_{\partial M} = \varphi.
\]

In addition to (1.3), (1.2), we assume (4.2) holds for

\[
\inf_M \psi \leq f(\lambda) \leq \sup_M f(\lambda(\nabla^2 u + A)).
\]

Then there exists a unique smooth admissible function \(u\) solving (4.1) with \(u|_{\partial M} = \varphi\).

**Remark 4.3.** Condition (4.2) for \(K_0 = 0\) was also used by [8] to derive boundary estimate for Dirichlet problem of (4.1), and later by [9] (for \(K_0 \geq 0\)) to study first initial boundary problems. Our results indicate that the technique assumptions (i)-(iii) imposed in [8, Theorem 1.10] as well as assumptions (i)-(iv) in [9, Theorem 1.9] can be removed.

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**A Some standard lemmas**

In this appendix we summarize some standard lemmas.

**Lemma A.1.** Let \(f\) satisfy (1.2) and (1.4), then (1.11) holds.

The above lemma has been used in [12]. Below we present another one.

**Lemma A.2.** Suppose \(f\) satisfies (1.2) and (1.4). Then for any \(\sigma\) with \(\sigma < \sup_f f\) and \(\partial \Gamma^\sigma \neq \emptyset\), there exists \(c_\sigma \bar{1} \in \partial \Gamma^\sigma\).

**Proof.** For \(\sigma < \sup_f f\), the level set \(\partial \Gamma^\sigma\) (if \(\partial \Gamma^\sigma \neq \emptyset\)) is a convex noncompact hypersurface contained in \(\Gamma\). Moreover, \(\partial \Gamma^\sigma\) is symmetric with respect to the diagonal.
Let \( \lambda^0 \in \partial \Gamma^\gamma \) be the closest point to the origin. (Such a point exists, since \( \partial \Gamma^\gamma \) is a closed set). The idea is to prove \( \lambda^0 \) is the one we look for.

Assume \( \lambda^0 \) is not in the diagonal. Then by the Implicit Function Theorem, and the convexity and symmetry of \( \partial \Gamma^\gamma \), one can choose \( \lambda \in \partial \Gamma^\gamma \) which has strictly less distance than that of \( \lambda^0 \). It is a contradiction.

\[ \square \]

B  Criterion for \( f \) satisfying (1.7)

We summarize characterizations of \( f \) when it satisfies (1.2) and (1.7). The following lemma was first proposed by [22]\(^\dagger\) and further reformulated in [25].

**Lemma B.1** ([22, 25]). For \( f \) satisfying (1.2), the following statements are equivalent.

- \( f \) satisfies (1.7).
- \( \sum_{i=1}^n f(\lambda)\mu_i > 0, \ \forall \lambda, \mu \in \Gamma \).
- \( f(\lambda + \mu) > f(\lambda), \ \forall \lambda, \mu \in \Gamma \).

**Corollary B.2** ([25]). Assume (1.2) and (1.7) hold. Then we have (1.4) and \( \sum_{i=1}^n f(\lambda) > 0 \).

Inspired by the following key observation derived from (2.1)

For any \( \lambda, \mu \in \Gamma \), \( \sum_{i=1}^n f_i(\lambda)\mu_i \geq \limsup_{t \to +\infty} f(t\mu)/t \)

the author [22] introduced the following two conditions:

For any \( \lambda \in \Gamma \), \( \lim_{t \to +\infty} f(t\lambda) > -\infty \),

\begin{equation}
(\text{B.1})
\end{equation}

For any \( \lambda \in \Gamma \), \( \limsup_{t \to +\infty} f(t\lambda)/t \geq 0 \).

\begin{equation}
(\text{B.2})
\end{equation}

Obviously, it leads to

\(^\dagger\)The paper is extracted from [arXiv:2203.03439] and the first parts of [arXiv:2001.09238; arXiv:2106.14837].
Lemma B.3 ([22]). Suppose \( f \) satisfies (1.2) and (B.2). Then
\[
\sum_{i=1}^{n} f_i(\lambda)\mu_i \geq 0 \text{ for any } \lambda, \mu \in \Gamma.
\]  
(B.3)

In addition, \( f_i(\lambda) \geq 0 \) in \( \Gamma \) for all \( 1 \leq i \leq n \). In particular it satisfies
\[
\sum_{i=1}^{n} f_i(\lambda)\lambda_i \geq 0, \quad \forall \lambda \in \Gamma.
\]  
(B.4)

We have criteria for concave symmetric functions.

Lemma B.4. In the presence of (1.2), the following statements are equivalent.

- \( f \) satisfies (B.1).
- \( f \) satisfies (B.2).
- \( f \) satisfies (B.3).
- \( f \) satisfies (B.4).

We can deduce the following lemma when (2.3) holds.

Lemma B.5. Suppose that (1.2) and (2.3) hold. Then the following statements are equivalent to each other.

- \( f \) satisfies (1.7).
- \( f \) satisfies (B.1).
- \( f \) satisfies (B.2).
- \( f \) satisfies (B.3).
- \( f \) satisfies (B.4).

Proof. From Lemmas B.1 and B.4, it requires only to prove
\[(B.2) \Rightarrow (1.7).\]

Fix \( \lambda, \mu \in \Gamma \). Note that \( t\mu - \lambda - \bar{1} \in \Gamma \) for \( t > t_{\mu,(\lambda+\bar{1})} > 0 \). Together with (2.1) and (2.3), we derive
\[f(t\mu) \geq f(\lambda) + \sum_{i=1}^{n} f_i(t\mu) > f(\lambda) \text{ for } t > t_{\mu,(\lambda+\bar{1})}.\]

\( \square \)
Lemma B.6 ([22]). If $f$ satisfies (1.2), (1.3) and (2.9), then it obeys (1.7).

Remark B.7. Lemma B.5 was proved in [22] when $f$ satisfies (1.2)-(1.3).

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