Faster Width-dependent Algorithm for Mixed Packing and Covering LPs

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Abstract

In this paper, we give a faster width-dependent algorithm for mixed packing-covering LPs. Mixed packing-covering LPs are fundamental to combinatorial optimization in computer science and operations research. Our algorithm finds a $1 + \varepsilon$ approximate solution in time $O(Nw/\varepsilon)$, where $N$ is number of nonzero entries in the constraint matrix, and $w$ is the maximum number of nonzeros in any constraint. This run-time is better than Nesterov’s smoothing algorithm which requires $O(N\sqrt{nw}/\varepsilon)$ where $n$ is the dimension of the problem. Our work utilizes the framework of area convexity introduced in [Sherman-FOCS’17] to obtain the best dependence on $\varepsilon$ while breaking the infamous $\ell_8$ barrier to eliminate the factor of $\sqrt{n}$. The current best width-independent algorithm for this problem runs in time $O(N/\varepsilon^2)$ [Young-arXiv-14] and hence has worse running time dependence on $\varepsilon$. Many real life instances of the mixed packing-covering problems exhibit small width and for such cases, our algorithm can report higher precision results when compared to width-independent algorithms. As a special case of our result, we report a $1 + \varepsilon$ approximation algorithm for the densest subgraph problem which runs in time $O(md/\varepsilon)$, where $m$ is the number of edges in the graph and $d$ is the maximum graph degree.

1 Introduction

Mixed packing and covering linear programs (LPs) are a natural class of LPs where coefficients, variables, and constraints are non-negative. They model a wide range of important problems in combinatorial optimization and operations research. In general, they model any problem which contains a limited set of available resources (packing constraints) and a set of demands to fulfill (covering constraints).

Two special cases of the problem have been widely studied in literature: pure packing, formulated as $\max_x \{b^T x \mid Px \leq p\}$; and pure covering, formulated as $\min_x \{b^T x \mid Cx \geq c\}$ where $P, p, C, c, b$ are all non-negative. These are known to model fundamental problems such as maximum bipartite graph matching, minimum set cover, etc [9]. Algorithms to solve packing and covering LPs have also been applied to great effect in designing flow control systems [3], scheduling problems [14], zero-sum matrix games [11] and in mechanism design [19]. In this paper, we study the mixed packing and covering (MPC) problem, formulated as checking the feasibility of the set: $\{x \midPx \leq p, Cx \geq c\}$, where $P, C, p, c$ are non-negative. We say that $x$ is an $\varepsilon$-approximate solution to MPC if it belongs to the relaxed set $\{x \mid P(x-(1+\varepsilon)p, Cx \geq (1-\varepsilon)c\}$. MPC is a generalization of pure packing and pure covering, hence it is applicable to a wider range of problems such as multi-commodity flow on graphs [17, 15], non-negative linear systems and X-ray tomography [17].

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General LP solving techniques such as the interior point method can approximate solutions to MPC in as few as \(O(\log(1/\epsilon))\) iterations - however, they incur a large per-iteration cost. In contrast, iterative approximation algorithms based on first-order optimization methods require poly\((1/\epsilon)\) iterations, but the iterations are fast and in most cases are conducive to efficient parallelization. This property is of utmost importance in the context of ever-growing datasets and the availability of powerful parallel computers, resulting in much faster algorithms in relatively low-precision regimes.

### 1.1 Previous work

In literature, algorithms for the MPC problem can be grouped into two broad categories: width-dependent and width-independent. Here, width is an intrinsic property of a linear program which typically depends on the dimensions and the largest entry of the constraint matrix, and is an indication of the range of values any constraint can take. In the context of this paper and the MPC problem, we define \(w_p\) and \(w_c\) as the maximum number of non-zeros in any constraint in \(P\) and \(C\) respectively. We define the width of the LP as \(w_{\text{max}} = \max(w_p, w_c)\).

One of the first approaches used to solve LPs was Langrangian-relaxation: replacing hard constraints with loss functions which enforce the same constraints indirectly. Using this approach, Plotkin, Schmoys and Tardos \[14\] and Grigoriadis and Khachiyan \[8\] obtained width-dependent polynomial-time approximation algorithms for MPC. Luby and Nisan \[9\] gave the first width-dependent parallelizable algorithm for pure packing and pure covering, which ran in \(\tilde{O}(\varepsilon^{-4})\) parallel time, and \(\tilde{O}(N\varepsilon^{-4})\) total work. Here, parallel time (sometimes termed as depth) refers to the longest chain of dependent operations, and work refers to the total number of operations in the algorithm.

Young \[17\] extended this technique to give the first width-independent parallel algorithm for MPC in \(\tilde{O}(\varepsilon^{-4})\) parallel time, and \(\tilde{O}(md\varepsilon^{-2})\) total work\[11\]. Young \[18\] later improved his algorithm to run using total work \(O(N\varepsilon^{-2})\). Mahoney \textit{et al.} \[10\] later gave an algorithm with a faster parallel run-time of \(\tilde{O}(\varepsilon^{-3})\).

The other most prominent approach in literature towards solving an LP is by converting it into a smooth function \[11\], and then applying general first-order optimization techniques \[11\]. Although the dependence on \(\epsilon\) from using first-order techniques is much improved, it usually comes at the cost of sub-optimal dependence on the input size and width. For the MPC problem, Nesterov’s accelerated method \[13\], as well as Bienstock and Iyengar’s adaptation \[5\] of Nesterov’s smoothing \[11\], give rise to algorithms with runtime linearly depending on \(\varepsilon^{-1}\), but with far from optimal dependence on input size and width. For pure packing and pure covering problems, however, Allen-Zhu and Orrechia \[1\] were the first to incorporate Nesterov-like acceleration while still being able to obtain near-linear width-independent runtimes, giving a \(\tilde{O}(N\varepsilon^{-1})\) time algorithm for the packing problem. For the covering problem, they gave a \(\tilde{O}(N\varepsilon^{-1.5})\) time algorithm, which was then improved to \(\tilde{O}(N\varepsilon^{-1})\) by \[16\]. Importantly, however, the above algorithms do not generalize to MPC.

### 1.2 Our contributions

We give the best parallel width-dependent algorithm for MPC, while only incurring a linear dependence on \(\varepsilon^{-1}\) in the parallel runtime and total work. Additionally, the total work has near-linear dependence on the input-size. Formally, we state our main theorem as follows.

**Theorem 1.1.** There exists a parallel \(\varepsilon\)-approximation algorithm for the mixed packing covering problem, which runs in \(\tilde{O}(w \cdot \varepsilon^{-1})\) parallel time, while performing \(\tilde{O}(w \cdot N \cdot \varepsilon^{-1})\) total work, where \(N\) is the total number of non-zeros in the constraint matrices, and \(w\) is the width of the given LP.

Table\[1\] compares the running time of our algorithm to previous works solving this problem (or its special cases).

Sacrificing width independence for faster convergence with respect to precision proves to be a valuable trade-off for several combinatorial optimization problems which naturally have a low width. Prominent examples of such problems which are not pure packing or covering problems include

\[d\] here is the maximum number of constraints that any variable appears in.
Table 1: Comparison of runtimes of ε-approximation algorithms for the mixed packing covering problem.

|                   | Parallel Runtime | Total Work | Comments      |
|-------------------|------------------|------------|---------------|
| Young [17]        | \( \tilde{O}(\varepsilon^{-4}) \) | \( \tilde{O}(md\varepsilon^{-2}) \) | \( d \) is column-width |
| Bienstock and Iyengar [5] | \( \tilde{O}(n^{2.5}w_{1.5}^{}w_{-1}^{}) \) | \( \tilde{O}(w\cdot N\sqrt{n}\varepsilon^{-1}) \) | width-dependent |
| Nesterov [13]     | \( \tilde{O}(w\sqrt{n}\varepsilon^{-1}) \) | \( \tilde{O}(w\cdot N\sqrt{n}\varepsilon^{-1}) \) | width-dependent |
| Young [18]        | \( \tilde{O}(\varepsilon^{-4}) \) | \( \tilde{O}(N\varepsilon^{-2}) \) |                     |
| Mahoney et al. [10]| \( \tilde{O}(\varepsilon^{-3}) \) | \( \tilde{O}(N\varepsilon^{-3}) \) |                     |
| This paper        | \( \tilde{O}(w\varepsilon^{-1}) \) | \( \tilde{O}(w\cdot N\varepsilon^{-1}) \) | width-dependent |

multicommodity flow and densest subgraph, where the width is bounded by the degree of a vertex. In a large number of real-world graphs, the maximum vertex degree is usually small, hence our algorithm proves to be much faster when we want high-precision solutions. We explicitly show that this result directly gives the fastest algorithm for the densest subgraph problem on low-degree graphs in Appendix C.

2 Notation and Definitions

For any integer \( q \), we represent using \( \| \cdot \|_q \) the \( q \)-norm of any vector. We represent the infinity-norm as \( \| \cdot \|_\infty \). We denote the infinity-norm ball (sometimes called the \( \ell_\infty \) ball) as the set \( B^n_\infty(r) \) defined as \( \{ x \in \mathbb{R}^n : \| x \|_\infty \leq r \} \). The nonnegative part of this ball is denoted as \( B^n_{+\infty}(r) \) for radius \( r = 1 \), we drop the radius specification and use a short notation \( B^n_\infty \) and \( B^n_{+\infty} \). We denote the extended simplex of dimension \( k \) as \( \Delta_k^+ \) defined as \( \{ x \in \mathbb{R}^k : \sum_{i=1}^k x_i \leq 1 \} \). For any \( y \geq 0_k \), \( \text{proj}_{\Delta_k^+}(y) = y/\| y \|_1 \) if \( \| y \|_1 \geq 1 \). Further, for any set \( K \), we represent its interior, relative interior and closure as \( \text{int}(K) \), \( \text{relint}(K) \) and \( \text{cl}(K) \), respectively. Function \( \exp \) is applied to a vector element wise. Division of two vectors of same dimension is also performed element wise.

For any matrix \( A \), we use \( \text{nnz}(A) \) to denote the number of nonzero entries in it. We use \( A_{i,:} \) and \( A_{:,j} \) to refer to the \( i \)-th row and \( j \)-th column of \( A \) respectively. We use notation \( A_{ji} \) or \( A_{j,i} \) alternatively to denote element in \( j \)-th row and \( i \)-th column of matrix \( A \). \( \| A \|_\infty \) denotes the operator norm \( \| A \|_{\infty \rightarrow \infty} \) defined as \( \sup_{x \neq 0} \frac{\| Ax \|_\infty}{\| x \|_\infty} \). For a symmetric matrix \( A \) and an antisymmetric matrix \( B \), we define an operator \( \geq_1 A \geq_1 B \Leftrightarrow \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \) is positive semi-definite. We formally define an \( \varepsilon \)-approximate solution to the mixed packing-covering (MPC) problem as follows.

**Definition 2.1.** We say that \( x \) is an \( \varepsilon \)-approximate solution of the mixed packing-covering problem if \( x \) satisfies \( x \in B^n_{+\infty}(1) \), \( P \cdot x \leq (1+\varepsilon)1_p \) and \( C \cdot x \geq (1-\varepsilon)1_c \).

Here, \( 1_k \) denotes a vectors of 1’s of dimension \( k \) for any integer \( k \). The saddle point problem on two sets \( x \in X \) and \( y \in Y \) can be defined as follows:

\[
\min_{x \in X} \max_{y \in Y} L(x, y)
\]

where \( L(x, y) \) is some bilinear form between \( x \) and \( y \). For this problem, we define the primal-dual gap function as \( \sup_{(x,y) \in X \times Y} L(x, y) - L(\bar{x}, \bar{y}) \). This gap function can be used as measure of accuracy of the above saddle point solution.

**Definition 2.2.** We say that \((x, y) \in X \times Y\) is an \( \varepsilon \)-optimal solution for (1) if \( \sup_{(x,y) \in X \times Y} L(x, y) - L(\bar{x}, \bar{y}) \leq \varepsilon \).

3 Technical overview

The mixed packing-covering (MPC) problem is formally defined as follows.
Given two nonnegative matrices $P \in \mathbb{R}^{p \times n}$, $C \in \mathbb{R}^{c \times n}$, find an $x \in \mathbb{R}^n$, $x \geq 0$, $\|x\|_\infty \leq 1$ such that $Px \leq 1_p$ and $Cx \geq 1_c$ if it exists, otherwise report infeasibility.

Note that the vector of 1’s on the right-hand side of packing and covering constraints can be obtained by simply scaling each constraint appropriately. We also assume that each entry in the matrices $P$ and $C$ is at most one. This assumption, and subsequently the $\ell_\infty$ constraints on $x$ also cause no loss of generality.

We reformulate MPC as a saddle point problem, as defined in Section 2:

\[
\lambda^* \overset{\text{def}}{=} \min_{x \in \mathbb{R}_+^n} \max_{y \in \Delta_p^+, z \in \Delta_c^+} L(x, y, z),
\]

where $L(x, y, z) := [y^T z^T] \begin{bmatrix} P & -1_p \\ -C & 1_c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$. The relation between the two formulations is shown in Section 4. For the rest of the paper, we focus on the saddle point formulation 2.

$\eta(x) \overset{\text{def}}{=} \max_{y \in \Delta_p^+, z \in \Delta_c^+} L(x, y, z)$ is a piecewise linear convex function. Assuming oracle access to this “inner” maximization problem, the “outer” problem of minimizing $\eta(x)$ can be performed using first order methods like mirror descent, which are suitable when the underlying problem space is the unit $\ell_\infty$ ball. One drawback of this class of methods is that their rate of convergence, which is standard for non-accelerated first order methods on non-differentiable objectives, is $O(1/k)$ to obtain an $\varepsilon$-approximate minimizer $x$ of $\eta$ which satisfies $\eta(x) \leq \eta^* + \varepsilon$, where $\eta^*$ is the optimal value. This means that the algorithm needs to access the inner maximization oracle $O(1/\varepsilon^2)$ times, which can become prohibitively large in the high precision regime.

Note that even though $\eta$ is a piecewise linear non-differentiable function, it is not a black box function, but a maximization linear functions in $x$. This structure can be exploited using Nesterov’s smoothing technique [11]. In particular, $\eta(x)$ can be approximated by choosing a strongly convex function $\phi : \Delta_p^+ \times \Delta_c^+ \to \mathbb{R}$ and considering

\[
\tilde{\eta}(x) = \max_{y \in \Delta_p^+, z \in \Delta_c^+} L(x, y, z) - \phi(y, z).
\]

This strongly convex regularization yields $\tilde{\eta}$ is a Lipschitz-smooth convex function. If $L$ is the constant of Lipschitz smoothness of $\tilde{\eta}$ then application of any of the accelerated gradient methods in literature will converge in $O(\sqrt{\varepsilon/L})$ iterations. Moreover, it can also be shown that in order to construct a smooth $\varepsilon$-approximation $\tilde{\eta}$ of $\eta$, the Lipschitz smoothness constant $L$ can be chosen to be of the order $O(1/\varepsilon)$, which in turn implies an overall convergence rate of $O(1/\varepsilon)$.

In particular, Nesterov’s smoothing achieves an oracle complexity of $O(\|P\|_\infty + \|C\|_\infty) D_x \max\{D_y, D_z\} \varepsilon^{-1}$, where $D_x, D_y$ and $D_z$ denote the sizes of the ranges of their respective regularizers which are strongly convex functions. $D_y$ and $D_z$ can be made of the order of $\log p$ and $\log c$, respectively. However, $D_x$ can be problematic since $x$ belongs to an $\ell_\infty$ ball. More on this will soon follow.

Nesterov’s dual extrapolation algorithm [12] gives a very similar complexity but is a different algorithm in that it directly addresses the saddle point formulation 2 rather than viewing the problem as optimizing a non-smooth function $\eta$. The final convergence for the dual extrapolation algorithm is given in terms of the primal-dual gap function of the saddle point problem 2. This algorithms views the saddle point problem as solving variational inequality for an appropriate monotone operator in joint domain $(x, y, z)$. Moreover, as opposed to smoothing techniques which only regularize the dual, this algorithm regularizes both primal and dual parts, hence is a different scheme altogether.

Note that for both schemes mentioned above, the maximization oracle itself has an analytical expression which involves matrix-vector multiplication. Hence each call to the oracle incurs a sequential run-time of $\min\|P\|_\infty + \min\|C\|_\infty$. Then, overall complexity for both schemes is of order $O((\min\|P\|_\infty + \min\|C\|_\infty)(\|P\|_\infty + \|C\|_\infty) D_x \max\{D_y, D_z\} \varepsilon^{-1})$.

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3This transformation can be achieved by adapting techniques from [13] while increasing dimension of the problem up to a logarithmic factor. Details of this fact are in the Appendix B in the full paper (supplementary file). For the purpose of the main text, we work with this assumption.

4Definitions of Lipschitz-smoothness and strong convexity can be found in many texts in nonlinear programming and machine learning, e.g. [13]. Intuitively, $f$ is Lipschitz-smooth if the rate of change of $\nabla f$ can be bounded by a quantity known as the “constant of Lipschitz smoothness”.
The $\ell_\infty$ barrier

Note that the both methods, i.e., Nesterov’s smoothing and dual extrapolation, involves a $D_x$ term, which denotes the range of a convex function over the domain of $x$. The following lemma states a lower bound for this range in case of $\ell_\infty$ balls.

**Lemma 3.1.** Any strongly convex function has a range of at least $\Omega(\sqrt{n})$ on any $\ell_\infty$ ball.

Since $D_x \geq \sqrt{\frac{n}{2}}\sqrt{7}$ for each member function of this wide class, there is no hope of eliminating this $\sqrt{n}$ factor using techniques involving explicit use of strong convexity.

So, the goal now is to find a function with a small range over $\ell_\infty$ balls, but still act as good enough regularizers to enable accelerated convergence of the descent algorithm. In pursuit of breaking this $\ell_\infty$ barrier, we draw inspiration from the notion of area convexity introduced by Sherman [15]. Area convexity is a weaker notion than strong convexity, however, it is still strong enough to ensure that accelerated first order methods still go through when using area convex regularizers. Since this is a weaker notion than strong convexity, we can construct area convex functions which have range of $O(n^{o(1)})$ on $\ell_\infty$ ball.

First, we define area convexity, and then go on to mention its relevance to the saddle point problem (2). Area convexity is a notion defined in context of a matrix $A \in \mathbb{R}^{a \times b}$ and a convex set $K \subseteq \mathbb{R}^{a+b}$.

Let $M_A \overset{\text{def}}{=} \begin{bmatrix} 0_{b \times b} & -A^T \\ A & 0_{a \times a} \end{bmatrix}$.

**Definition 3.2.** A function $\phi$ is area convex with respect to a matrix $A$ on a convex set $K$ iff for any $t, u, v \in K$, $\phi$ satisfies $\phi\left(\frac{t+u+v}{3}\right) \leq \frac{1}{3}(\phi(t) + \phi(u) + \phi(v)) - \frac{1}{3\sqrt{3}}(v-u)^T M_A (u-t)$

To understand the definition above, first lets look at the notion of strong convexity. $\phi$ is strongly convex if for any two points $t, u, \frac{1}{2}(\phi(t) + \phi(u)) \leq \phi(\frac{1}{2}(t + u))$ by an amount proportional to $\|t-u\|^2$. Definition 3.2 generalizes this notion in context of matrix $A$ for any three points $x, y, z$. $\phi$ is area-convex on set $K$ if for any three points $t, u, v \in K$, we have $\frac{1}{4}(\phi(t) + \phi(u) + \phi(v)) \leq \phi(\frac{1}{4}(t + u + v))$ by an amount proportional to the area of the triangle defined by the convex hull of $t, u, v$.

Consider the case that points $t, u, v$ are collinear. For this case, the area term (i.e., the term involving $M_A$ in Definition 3.2) is 0 since matrix $M_A$ is antisymmetric. In this sense, area convexity is even weaker than strict convexity. Moreover, the notion of area is parameterized by matrix $A$. To see a specific example of this notion of area, consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $t, u, v \in \mathbb{R}^2$. Then, for all possible permutations of $t, u, v$, the area term takes a value equal to $\pm (t_1(u_2 - v_2) + u_1(v_2 - t_2) + v_1(t_2 - u_2))$. Since the condition holds irrespective of the permutation so we must have that $\phi(\frac{t+u+v}{3}) \leq \frac{1}{4}(\phi(t) + \phi(u) + \phi(v)) - \frac{1}{3\sqrt{3}}|t_1(u_2 - v_2) + u_1(v_2 - t_2) + v_1(t_2 - u_2)|$. But note that area of triangle formed by points $t, u, v$ is equal to $\frac{1}{2}|t_1(u_2 - v_2) + u_1(v_2 - t_2) + v_1(t_2 - u_2)|$. Hence the area term is just a high dimensional matrix based generalization of the area of a triangle.

Coming back to the saddle point problem (2), we need to pick a suitable area convex function $\phi$ on the set $B_{\infty,x}^0 \times \Delta_+^r \times \Delta_+^r$. Since $\phi$ is defined on the joint space, it has the property of joint regularization vis a vis (2). However, we need an additional parameter: a suitable matrix $M_A$. The choice of this matrix is related to the bilinear form of the primal-dual gap function of (2). We delve into the technical details of this in Section 4 however, we state that the matrix is composed of $P, C$ and some additional constants. The algorithm we state exactly follows Nesterov’s dual extrapolation method described earlier. One notable difference is that in [12], they consider joint regularization by a strongly convex function which does not depend on the problem matrices $P, C$ but only on the constraint set $B_{\infty,x}^0 \times \Delta_+^r \times \Delta_+^r$. Our area convex regularizer, on the other hand, is tailor made for the particular problem matrices $P, C$ as well as the constraint set.

### 4 Area Convexity for Mixed Packing Covering LPs

In this section, we present our technical results and algorithm for the MPC problem, with the end goal of proving Theorem 1.7. First, we relate an $(1 + \varepsilon)$-approximate solution to the saddle point...
Thus, the gap function in Definition 4.1 can be written in the bilinear form (Definition 3.2). which is generated using a simple “gadget” function. We show that this area convex function satisfies all key requirements and hence achieves the desired accelerated rate of convergence. This section closely follows [15], in which the author chooses an area convex function specific to the undirected multicommodity flow problem. Due to space constraints, we relegate almost all proofs to Appendix A and simply include pointers to proofs in [15] when it is directly applicable.

4.1 Saddle Point Formulation for MPC

Consider the saddle point formulation in (2) for MPC problem. Given a feasible primal-dual feasible solution pair \((x, y, z)\) and \((\bar{x}, \bar{y}, \bar{z})\) for (2), we denote \(w = (x, u, y, z)\) and \(\bar{w} = (\bar{x}, \bar{u}, \bar{y}, \bar{z})\) where \(u, \bar{u} \in \mathbb{R}\). Then, we define a function \(Q : \mathbb{R}^{n+1+p+c} \times \mathbb{R}^{n+1+p+c} \rightarrow \mathbb{R}\) as

\[
Q(w, \bar{w}) \overset{\text{def}}{=} [y^T \, z^T] \begin{bmatrix} P & -1_p \\ -C & 1_c \end{bmatrix} x - [y^T \, z^T] \begin{bmatrix} P & -1_p \\ -C & 1_c \end{bmatrix} \bar{x}.
\]

Note that if \(u = \bar{u} = 1\), then

\[
\sup_{w \in \mathbb{W}} Q(w, \bar{w}) = \sup_{\bar{x} \in \mathbb{B}_n^+, \bar{y} \geq \Delta_p^+, \bar{z} \in \Delta_c^+} L(x, \bar{y}, \bar{z}) - L(\bar{x}, y, z)
\]

is precisely the primal-dual gap function defined in Section 2. Notice that if \((x^*, y^*, z^*)\) is a saddle point of (2), then we have

\[
L(x^*, y, z) \leq L(x^*, y^*, z^*) \leq L(x, y^*, z^*)
\]

for all \(x \in \mathbb{B}_n^+, y \in \Delta_p^+, z \in \Delta_c^+\). From above equation, it is clear that \(Q(w, w^*) \geq 0\) for all \(w \in \mathbb{W}\) where \(\mathbb{W} \overset{\text{def}}{=} \mathbb{B}_n^+ \times \{1\} \times \Delta_p^+ \times \Delta_c^+\) and \(w^* = (x^*, 1, y^*, z^*) \in \mathbb{W}\). Moreover, \(Q(w^*, w^*) = 0\). This motivates the following accuracy measure of the candidate approximate solution \(w\).

**Definition 4.1.** We say that \(w \in \mathbb{W}\) is an \(\varepsilon\)-optimal solution of (2) iff

\[
\sup_{w \in \mathbb{W}} Q(w, \bar{w}) \leq \varepsilon.
\]

**Remark 4.2.** Recall the definition of \(M_A\) for a matrix \(A\) in Section 2. We can rewrite \(Q(w, \bar{w}) = \bar{w}^T J w\) where \(J = M_{H_H}\) and

\[
H = \begin{bmatrix} P & -1_p \\ -C & 1_c \end{bmatrix} \quad \Rightarrow \quad J := \begin{bmatrix} 0_{n \times n} & 0_{n \times 1} & -P^T & C^T \\ 0_{1 \times n} & 0_{1 \times 1} & 1_p & -1_c \\ P & -1_p & 0_{p \times p} & 0_{p \times c} \\ -C & 1_c & 0_{c \times p} & 0_{c \times c} \end{bmatrix}.
\]

Thus, the gap function in Definition 4.1 can be written in the bilinear form \(\sup_{w \in \mathbb{W}} \bar{w}^T J w\).

**Lemma 4.3.** Relates the \(\varepsilon\)-optimal solution of (2) to the \(\varepsilon\)-approximate solution to MPC.

**Lemma 4.3.** Let \((x, y, z)\) satisfy \(\sup_{(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{B}_n^+ \times \Delta_p^+ \times \Delta_c^+} L(x, \bar{y}, \bar{z}) - L(\bar{x}, y, z) \leq \varepsilon\). Then either

1. \(x\) is an \(\varepsilon\)-approximate solution of MPC, or
2. \(y, z\) satisfy \(y^T (P \bar{x} - 1_p) + z^T (-C \bar{x} + 1_c) \geq 0\) for all \(\bar{x} \in \mathbb{B}_n^+\).

This lemma states that in order to find an \(\varepsilon\)-approximate solution of MPC, it suffices to find \(\varepsilon\)-optimal solution of (2). Henceforth, we will focus on \(\varepsilon\)-optimality of the saddle point formulation (2).

4.2 Area Convexity with Saddle Point Framework

Here we state some useful lemmas which help in determining whether a differentiable function is area convex. We start with the following remark which follows from the definition of area convexity (Definition 3.2).
Remark 4.4. If \( \phi \) is area convex with respect to \( A \) on a convex set \( K \), and \( \hat{K} \subseteq K \) is a convex set, then \( \phi \) is area convex with respect to \( A \) on \( \hat{K} \).

The following two lemmas from [15] provide the key characterization of area convexity.

**Lemma 4.5.** Let \( A \in \mathbb{R}^{2 \times 2} \) symmetric matrix. \( A \geq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \iff A \succeq 0 \) and \( \det(A) \geq 1 \).

**Lemma 4.6.** Let \( \phi \) be twice differentiable on the interior of convex set \( K \), i.e., \( \text{int}(K) \).

1. If \( \phi \) is area convex with respect to \( A \) on \( \text{int}(K) \), then \( d^2 \phi(x) \succeq_M A \) for all \( x \in \text{int}(K) \).

2. If \( d^2 \phi(x) \succeq_M A \) for all \( x \in \text{int}(K) \), then \( \phi \) is area convex with respect to \( \frac{1}{2}A \) on \( \text{int}(K) \).

Moreover, if \( \phi \) is continuous on \( \text{cl}(K) \), then \( \phi \) is area convex with respect to \( \frac{1}{2}A \) on \( \text{cl}(K) \).

In order to handle the operator \( \geq_1 \) (recall from Section 2), we state some basic but important properties of this operator, which will come in handy in later proofs.

**Remark 4.7.** For symmetric matrices \( A \) and \( C \) and antisymmetric matrices \( B \) and \( D \),

1. If \( A \succeq B \) then \( A \succeq_1 (-B) \).
2. If \( A \succeq B \) and \( \lambda \geq 0 \) then \( \lambda A \succeq \lambda B \).
3. If \( A \succeq B \) and \( C \succeq D \) then \( A + C \succeq_1 (B + D) \).

Having laid a basic foundation for area convexity, we now focus on its relevance to solving the saddle point problem \[3\]. Considering Remark 4.4, we can write the gap function criterion of optimality in terms of bilinear form of the matrix \( \Lambda \). Suppose we have a function \( \phi \) which is area convex with respect to \( H \) on set \( W \). Then, consider the following jointly-regularized version of the bilinear form:

\[
\tilde{\eta}(w) := \sup_{w \in W} w^T Jw - \phi(w).
\]  

(3)

Similar to Nesterov’s dual extrapolation, one can attain \( O(1/\varepsilon) \) convergence of accelerated gradient descent for function \( \tilde{\eta}(w) \) in \[3\] over variable \( w \). In order to obtain gradients of \( \tilde{\eta}(w) \), we need access to \( \arg\max_{w \in W} \tilde{\eta}(w) \). However, it may not be possible to find an exact maximizer in all cases. Again, one can get around this difficulty by instead using an approximate optimization oracle of the problem in \[3\].

**Definition 4.8.** A \( \delta \)-solution oracle (OSO) for \( \phi : W \rightarrow \mathbb{R} \) takes input \( a \) and outputs \( w \in W \) such that

\[
a^T w - \phi(w) \geq \sup_{\bar{w} \in W} a^T \bar{w} - \phi(\bar{w}) - \delta.
\]

Given \( \Phi \) as a \( \delta \)-OSO for a function \( \phi \), consider the following algorithm (Algorithm 4.2):

**Algorithm 1** Area Convex Mixed Packing Covering (AC-MPC)

Initialize \( w_0 = (0_n, 1, 0_{p+c}) \)

for \( t = 0, \ldots, T \) do

\[ w_{t+1} \leftarrow w_t + \Phi(Jw_t + 2J\Phi(Jw_t)) \]

end for

For Algorithm 4.2 [15] shows the following:

**Lemma 4.9.** Let \( \phi : W \rightarrow [-\rho, 0] \). Suppose \( \phi \) is area convex with respect to \( 2\sqrt{\rho}H \) on \( W \). Then for \( J = M_H \) and for all \( t \geq 1 \) we have \( w_t/t \in W \) and,

\[
\sup_{w \in W} \frac{\tilde{\eta}(w)}{t} \leq \delta + \frac{\rho}{t}.
\]

In particular, in \( \frac{1}{\rho} \) iterations, Algorithm 4.2 obtain \( (\delta + \varepsilon) \)-solution of the saddle point problem \[2\].

The analysis of this lemma closely follows the analysis of Nesterov’s dual extrapolation.
Note that, each iteration consists of $O(1)$ matrix-vector multiplications, $O(1)$ vector additions, and $O(1)$ calls to the approximate oracle. Since the former two are parallelizable to $O(\log n)$ depth, the same remains to be shown for the oracle computation to complete the proof of the run-time in Theorem 1.1.

Recall from the discussion in Section 3 that the critical bottleneck of Nesterov’s method is that diameter of the $\ell_p$ ball is $\Omega(\sqrt n)$, which is achieved even in the Euclidean $\ell_2$ norm. This makes $p$ in Lemma 4.9 to also be $\Omega(\sqrt n)$, which can be a major bottleneck for high dimensional LPs, which are commonplace among real-world applications.

Although, on the face of it, area convexity applied to the saddle point formulation (2) has a similar framework to Nesterov’s dual extrapolation, the challenge is to construct a $\phi$ for which we can overcome the above bottleneck. Particularly, there are three key challenges to tackle:

1. We need to show that existence of a function $\phi$ that is area convex with respect to $H$ on $\mathcal{W}$.
2. $\phi : \mathcal{W} \to [-\rho, 0]$ should be such that $\rho$ is not too large.
3. There should exist an efficient $\delta$-OSO for $\phi$.

In the next subsection, we focus on these three aspects in order to complete our analysis.

4.3 Choosing an area convex function

First, we consider a simple 2-D gadget function and prove a “nice” property of this gadget. Using this gadget, we construct a function which can be shown to be area convex using the aforementioned property of the gadget.

Let $\gamma_\beta : \mathbb{R}^2 \to \mathbb{R}$ be a function parameterized by $\beta$ defined as

$$
\gamma_\beta(a, b) = ba \log a + \beta b \log b.
$$

**Lemma 4.10.** Suppose $\beta \geq 2$. Then $d^2 \gamma_\beta(a, b) \geq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ for all $a \in (0, 1]$ and $b > 0$.

Now, using the function $\gamma_\beta$, we construct a function $\phi$ and use the sufficiency criterion provided in Lemma 4.6 to show that $\phi$ is area convex with respect to $J$ on $\mathcal{W}$. Note that our set of interest $\mathcal{W}$ is not full-dimensional, whereas Lemma 4.6 is only stated for int and not for relint. To get around this difficulty, we consider a larger set $\overline{\mathcal{W}} \supset \mathcal{W}$ such that $\overline{\mathcal{W}}$ is full dimensional and $\phi$ is area convex on $\overline{\mathcal{W}}$. Then we use Remark 4.3 to obtain the final result, i.e., area convexity of $\phi$.

**Theorem 4.11.** Let $w = (x, u, y, z)$ and define

$$
\phi(w) = \frac{1}{\beta} \sum_{i=1}^{p} \sum_{j=1}^{n} P_{ij} \gamma_{p_i}(x_j, y_i) + \sum_{i=1}^{p} \sum_{j=1}^{n} C_{ij} \gamma_{c_i}(x_j, z_i) + \sum_{i=1}^{c} \gamma_2(u, z_i),
$$

where $p_i = 2 \cdot \frac{\|P_{ij}\|}{\|C_{ij}\|}$ and $c_i = 2 \cdot \frac{\|C_{ij}\|}{\|P_{ij}\|}$, then $\phi$ is area convex with respect to $\frac{1}{3} \begin{bmatrix} P & -1_p \\ -C & 1_c \end{bmatrix}$ on set $\overline{\mathcal{W}} := \mathcal{B}_{n, \ell_p}^{p+1}(1) \times \Delta_u^+ \times \Delta_c^+$. In particular, it also implies $6\sqrt{3} \phi$ is area convex with respect to $2\sqrt{3} \begin{bmatrix} P & -1_p \\ -C & 1_c \end{bmatrix}$ on set $\overline{\mathcal{W}}$.

Theorem 4.11 addresses the first part of the key three challenges. Next, Lemma 4.12 shows an upper bound on the range of $\phi$.

**Lemma 4.12.** Function $\phi : \mathcal{W} \to [-\rho, 0]$ then $\rho = O(\|P\|_{\ell_p} \log p + \|C\|_{\ell_p} \log c)$.

Finally, we need an efficient $\delta$-OSO. Consider the following alternating minimization algorithm.

**Algorithm 2** $\delta$-OSO for $\phi$

Input $a \in \mathbb{R}^{n+1}, a^1 \in \mathbb{R}^p, a^2 \in \mathbb{R}^c, \delta > 0$
Initialize $(x^0, u^0) \in \mathcal{B}_{n, \ell_p}^{p+1} \times \{1\}$ arbitrarily.

for $k = 1, \ldots, K$

$$
(y^k, z^k) \leftarrow \arg\max_{y \in \Delta_u^+ \times \Delta_p^+} \ y^T a^1 + z^T a^2 - \phi(x^{k-1}, u^{k-1}, y, z)
$$

$$
(x^k, u^k) \leftarrow \arg\max_{(x, u) \in \mathcal{B}_{n, \ell_p}^{p+1} \times \{1\}} [x^T u] a - \phi(x, u, y^k, z^k)
$$

end for
We thank Richard Peng for many important pointers and discussions.

Lemma 4.13. For $\delta > 0$, Algorithm 2 is a $\delta$-OSO for $\phi$ which converges in $O(\log \frac{1}{\delta})$ iterations.

We show that for our chosen $\phi$, we can compute the two argmax in each iteration of Algorithm 2 analytically with computation time $O(\text{nnz}(P) + \text{nnz}(C))$ and hence we obtain a $\delta$-OSO running in $O(\text{nnz}(P) + \text{nnz}(C) \log \frac{1}{\delta})$ total work. Parallelizing matrix-vector multiplications, eliminates the dependence on $\text{nnz}(P)$ and $\text{nnz}(C)$, at the cost of another $\log(N)$ term.

Lemma 4.14. Each argmax in Algorithm 2 can be computed as follows:
\[
\begin{align*}
x^k &= \min \{ \exp \left( \frac{2}{\text{nnz}(P)_x + \text{nnz}(C)_x} a_1 - 1 \right), 1_n \} \text{ for all } j \in [n], \\
y^k &= \text{proj}_{A^+_P} \left( \exp \left( \frac{1}{2[\text{nnz}(P)_x + 1]} a_1 - P x^{k-1} \log x^{k-1} \right) \right) \\
z^k &= \text{proj}_{A^+_C} \left( \exp \left( \frac{1}{2[\text{nnz}(C)_x + 1]} a_2 - C x^{k-1} \log x^{k-1} \right) \right) \\
\end{align*}
\]
In particular, we can compute $x^k, y^k, z^k$ in $O(\text{nnz}(P) + \text{nnz}(C))$ work and $O(\log(N))$ parallel time.

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A Proof of auxiliary results

In this section, we include proofs of lemmas from the main paper. In some cases, the lemmas are direct restatements of results from other papers, for which we provide appropriate pointers.

Proof of Lemma 3.1 Consider an arbitrary strongly convex function \( d \). Assume WLOG that \( d(0) = 0 \). (otherwise, we can shift it accordingly). We will show that \( \max_{x \in B^r_+(r)} d(x) \geq \frac{nr^2}{2} \) by induction on \( n \) for set \( B^r_+(r) \). This suffices because \( B^r_+(r, 1) \) is isomorphic to \( B^r_+(\frac{1}{2}) \). The claim holds for \( n = 1 \) by the definition of strong convexity. Now, suppose it is true for \( n - 1 \). Then there exists \( \bar{x} \in B^{n-1}_+(r) \) such that \( d(\bar{x}) \geq \frac{(n-1)r^2}{2} \). Moving \( r \) units in the last coordinate from \( \bar{x} \) in the direction of nonnegative slope, suppose we reach \( \tilde{x} \in B^n_+(r) \). Then, due to strong convexity of \( d \), we have
\[
d(\tilde{x}) \geq d(\bar{x}) + \frac{1}{2} \| \tilde{x} - \bar{x} \|^2 \geq \left( \frac{1}{2} (n-1)r^2 + \frac{1}{2} r^2 \right) = \frac{nr^2}{2}.
\]

Proof of Lemma 4.3 Suppose we are given \( (x, y, z) \) such that \( \sup_{(x,y,z) \in B^r_+(1)} L(x, y, z) \leq \varepsilon \). Then, we have
\[
d_{x-y-z} \leq \varepsilon.
\]
where implication follows by optimality over extended simplices \( \Delta^+_p \). We obtain that \( \max\{\Delta^+_p \} \) is isomorphic to \( \Delta^+_p \). Theorem 1.6 in [15]. The proof follows from

Proof of Lemma 4.5. Consider an arbitrary strongly convex function \( d \). Assume WLOG that \( d(0) = 0 \) (otherwise, we can shift it accordingly). We will show that \( \max_{x \in B^r_+(r)} d(x) \geq \frac{nr^2}{2} \) by induction on \( n \) for set \( B^r_+(r) \). This suffices because \( B^r_+(r, 1) \) is isomorphic to \( B^r_+(\frac{1}{2}) \). The claim holds for \( n = 1 \) by the definition of strong convexity. Now, suppose it is true for \( n - 1 \). Then there exists \( \bar{x} \in B^{n-1}_+(r) \) such that \( d(\bar{x}) \geq \frac{(n-1)r^2}{2} \). Moving \( r \) units in the last coordinate from \( \bar{x} \) in the direction of nonnegative slope, suppose we reach \( \tilde{x} \in B^n_+(r) \). Then, due to strong convexity of \( d \), we have
\[
d(\tilde{x}) \geq d(\bar{x}) + \frac{1}{2} \| \tilde{x} - \bar{x} \|^2 \geq \left( \frac{1}{2} (n-1)r^2 + \frac{1}{2} r^2 \right) = \frac{nr^2}{2}.
\]

Proof of Lemma 4.6 This lemma appears exactly as Theorem 1.6 in [15]. The proof follows from the same.

Proof of Lemma 4.7 Let \( B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) and \( T := \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \). Then \( A \geq B \) iff \( T \geq 0 \) iff all principle minors of \( T \) are nonnegative. Now, \( T \geq 0 \) implies \( A \geq 0 \). It is easy to verify that third principle minor is nonnegative iff \( \det(A) \geq 1 \). So \( T \geq 0 \) implies \( A \) must be invertible. Then, applying Schur complement lemma, we obtain that \( T \geq 0 \iff A + BA^{-1}B \geq 0 \). Now let \( A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \) then \( A^{-1} = \frac{1}{ad - b^2} \begin{bmatrix} d & -b \\ -b & a \end{bmatrix} \). It is easy to verify that \( A + BA^{-1}B = A(1 - \frac{1}{\det(A)}) \). This implies \( T \geq 0 \iff A \geq 0 \) and \( \det(A) \geq 1 \). Hence we conclude the proof.

Proof of Lemma 4.8. This lemma appears exactly as Theorem 1.6 in [15]. The proof follows from the same.






































































































































































































































































































































































































































































































Proof of Proposition 4.7

1. $A \succeq_1 B \iff \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \succeq 0 \\
\iff x^T Ax + y^T Ay + y^T Bx - x^T By \geq 0, \quad \forall x, y \\
\iff x^T Ax + y^T Ay - y^T Bx + x^T By \geq 0, \quad \forall x, y \\
\iff \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \succeq 0 \iff A \succeq_1 (-B)$

Here, the third equivalence follows after replacing $y$ by $-y$. Hence we conclude the proof of part 1.

2. $A \succeq_1 B \iff \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \succeq 0 \iff \begin{bmatrix} \lambda A & -\lambda B \\ \lambda B & \lambda A \end{bmatrix} \succeq 0 \iff \lambda A \succeq \lambda B$

3. $A \succeq_1 B$ implies $\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \succeq 0$. Similarly $C \succeq_1 D$ implies $\begin{bmatrix} C & -D \\ D & C \end{bmatrix} \succeq 0$. Hence

$$\begin{bmatrix} A + C & -(B + D) \\ (B + D) & (A + C) \end{bmatrix} \succeq 0.$$

So we obtain $A + C \succeq_1 (B + D)$.

Proof of Lemma 4.5

This lemma appears as Theorem 1.3 in [15], and the proof follows from the same.

Proof of Lemma 4.10

We use equivalent characterization proved in Lemma 4.5. We need to show that $d^2 \gamma_\beta(a, b) \geq 0$ and $\det(d^2 \gamma_\beta(a, b)) \geq 1$ for all $a \in (0, 1]$ and $b > 0$. First of all, note that $d^2 \gamma_\beta$ is well-defined on this domain. In particular, we can write

$$d^2 \gamma_\beta(a, b) = \begin{bmatrix} \frac{\beta}{a} & 1 + \log a \\ 1 + \log a & \frac{\beta}{a} \end{bmatrix}.$$

Note that a $2 \times 2$ matrix is PSD if and only if its diagonal entries and determinant are nonnegative. Clearly diagonal entries of $d^2 \gamma_\beta(a, b)$ are nonnegative for the given values of $\beta, a$, and $b$. Hence, in order to prove the lemma, it suffices to show that $\det(d^2 \gamma_\beta(a, b)) \geq 1$. $\det(d^2 \gamma_\beta(a, b)) = \frac{\beta}{a^2} - (1 + \log a)^2$ is only a function of $a$ for any fixed value of $\beta \geq 2$. Moreover, it can be shown that $\det(d^2 \gamma_\beta)$ is a decreasing function of $a$ on set $(0, 1]$. Clearly, the minimum occurs at $a = 1$. However, $\det(d^2 \gamma_\beta(1, b)) = \beta - 1 \geq 1$ for all $b > 0$. Hence we have that $\det(d^2 \gamma_\beta(a, b)) \geq 1$ for all $a \in (0, 1], b > 0$ and $\beta \geq 2$.

Finally to see the claim that $\det(d^2 \gamma_\beta)$ is a decreasing function of $a \in (0, 1]$ for any $\beta \geq 2$, consider

$$\frac{d}{da}(\det(d^2 \gamma_\beta(a, b))) = \frac{\beta}{a^2} - \frac{2(1 + \log a)}{a} \leq \frac{2(1 + a(1 + \log a))}{a^2} < 0$$

where the last inequality follows from the observation that $1 + a + a \log a > 0$ for all $a \in (0, 1]$. Hence we conclude the proof.

Proof of Theorem 4.11

Note that $\gamma_{c, i}, \gamma_{p, i}$ are twice differentiable in the int$(\tilde{W})$. So by Lemma 4.6 part 2, it is sufficient to prove that $d^2 \phi(w) \succeq_1 0$ for all $w \in \text{int}(\tilde{W})$.

By definition, we have $\gamma_{c, i} \geq 2$ for all $i \in [c]$ and $\gamma_{p, i} \geq 2$ for all $i \in [p]$. Moreover $x_j \in (0, 1)$ and
We have \( y_i > 0, z_i > 0 \) for any \( w = (x, u, y, z) \in \text{int}(W) \). Then by Lemma 4.10 and Proposition 4.7 we have

\[
d^2 \phi(w) = \sum_{i=1}^{p} \sum_{j=1}^{n} P_{ij} d^2 \gamma_{p_i}(x_j, y_i) + \sum_{i=1}^{p} d^2 \gamma_2(u, y_i) + \sum_{i=1}^{c} \sum_{j=1}^{n} C_{ij} d^2 \gamma_{c_i}(x_j, z_i) + \sum_{i=1}^{c} d^2 \gamma_2(u, z_i)
\]

\[
\geq \left( \sum_{i=1}^{p} \sum_{j=1}^{n} -P_{ij} e_j \otimes e_{n+1+i} + \sum_{i=1}^{p} e_{n+1} \otimes e_{n+1+i} \right.
\]

\[
+ \sum_{i=1}^{c} \sum_{j=1}^{n} C_{ij} e_j \otimes e_{n+p+i} + \sum_{i=1}^{c} (-1)e_{n+1} \otimes e_{n+1+p+i}\right),
\]

where \( e_k \otimes e_1 = e_k e_1^T - e_1 e_k^T \). Here we used \( P_{ij} d^2 \gamma_{p_i}(x_j, y_i) \geq -P_{ij} e_j \otimes e_{n+1+i} \) using Lemma 4.10 Proposition 4.7 part 1, part 2 and \( C_{ij} d^2 \gamma_{c_i}(x_j, z_i) \geq C_{ij} e_j \otimes e_{n+1+p+i} \) using Lemma 4.10 Proposition 4.7 part 2. Similar arguments can be made about terms inside the other two summations. Finally we used Proposition 4.7 part 3 to obtain (4). Note matrix in the last sum term is in fact \( \gamma \).

It is clear that since \( d^2 \phi \geq 1 \) hence using Proposition 4.7 part 2, we have \( 6\sqrt{3}\phi \geq 6\sqrt{3}J \). Then by Lemma 4.6 part 2, we obtain \( 6\sqrt{3}\phi \) is area convex with respect to \( 2\sqrt{3} \left[ \begin{array}{cc} P & -\frac{1}{p} \\ -C & \frac{1}{e} \end{array} \right] \) on set \( W \).

Note that the set of interest \( W \subset \bar{W} \). Moreover, \( W \) is a convex subset. By Remark 4.4 one can see that \( 6\sqrt{3}\phi \) is area convex with respect to \( 2\sqrt{3} \left[ \begin{array}{cc} P & -\frac{1}{p} \\ -C & \frac{1}{e} \end{array} \right] \) on set \( W \). Hence we conclude the proof.

**Proof of Lemma 4.12** Note that \( \gamma_\beta(a, b) \leq 0 \) for any \( a \in [0, 1], b \in [0, 1], \beta \geq 0 \). Since \( P_{ij} \geq 0, C_{kj} \geq 0 \) for all possible values of \( i, j, k \) hence we clearly have \( \phi(w) \leq 0 \) for all \( w \in W \). Now we prove that lower bound is not too small.

We have

\[
\sum_{i=1}^{p} \sum_{j=1}^{n} P_{ij} \gamma_{p_i}(x_j, y_i) = \sum_{i=1}^{p} \sum_{j=1}^{n} P_{ij} (y_i x_j \log x_j + p_i y_i \log y_i)
\]

\[
\geq -\sum_{i=1}^{p} \sum_{j=1}^{n} P_{ij} y_i \frac{1}{e} + \sum_{i=1}^{p} p_i y_i \log y_i \sum_{j=1}^{n} P_{ij}
\]

\[
= -\sum_{i=1}^{p} \sum_{j=1}^{n} P_{ij} y_i \frac{1}{e} + \sum_{i=1}^{p} 2\|P\|_{x_i} y_i \log y_i
\]

\[
\geq -\sum_{i=1}^{p} \|P\|_{x_i} y_i \frac{1}{e} + \sum_{i=1}^{p} 2\|P\|_{x_i} y_i \log y_i
\]

\[
\geq -\|P\|_{x_i} \frac{1}{e} - 2\|P\|_{x_i} \log p
\]

Note that if \( w \in W \) implies \( u = 1 \). So

\[
\sum_{i=1}^{p} \gamma_2(u, y_i) = \sum_{i=1}^{p} 2y_i \log(y_i) \geq -2 \log p
\]

Similarly, we have

\[
\sum_{i=1}^{c} \sum_{j=1}^{n} C_{ij} \gamma_{c_i}(x_j, z_i) \geq -\|C\|_{x_i} \frac{1}{e} - 2\|C\|_{x_i} \log c
\]

\[
\sum_{i=1}^{c} \gamma_2(u, z_i) \geq -2 \log c
\]

Taking sum of all four terms, we conclude the proof.
Proof of Lemma 4.14 Note that maximization with respect to $u$ is trivial since $u = 1$ is a fixed variable. We first look at maximization with respect to $x \in B^{n}_{+, \infty}(1)$. Writing the first order necessary condition of Lagrange multipliers, we have

$$a_j - \sum_{i=1}^{p} P_{ij} \left. \frac{\partial}{\partial t} \gamma_{p_i}(t,v) \right|_{(t,v) = (x_j, y_i)} - \sum_{i=1}^{c} C_{ij} \left. \frac{\partial}{\partial t} \gamma_{c_i}(t,v) \right|_{(t,v) = (x_j, z_i)} - \lambda_j = 0$$

$$\Rightarrow a_j - \left\{ \sum_{i=1}^{p} P_{ij} y_i + \sum_{i=1}^{c} C_{ij} z_i \right\} (1 + \log x_j) - \lambda_j = 0.$$

Here $\lambda_j$ is the Lagrange multiplier corresponding to the case that $x_j = 1$. By complimentary slackness, we have $\lambda_j > 0$ iff $x_j = 1$.

This implies $x_j = \min \left\{ \exp \left( \sum_{i=1}^{n} \frac{a_i}{P_{ij} y_i + \sum_{i=1}^{c} C_{ij} z_i} - 1 \right), 1 \right\}$ for all $j \in [n]$.

Now we consider maximization with respect to $y, z$. Note that there are no cross-terms of $y_i$ and $z_i$, i.e., $\frac{\partial}{\partial y_i} \gamma_{p_i}$ is independent of $z$ variable and vice-versa. So we can optimize them separately. From first order necessary condition of Lagrange multipliers for $y$, we have

$$a_j - \sum_{j=1}^{n} P_{ij} \left. \frac{\partial}{\partial u} \gamma_{p_i}(t,v) \right|_{(t,v) = (x_j, y_i)} - \left. \frac{\partial}{\partial u} \gamma_{2}(t,v) \right|_{(t,v) = (u, y_i)} - \lambda = 0$$

$$\Rightarrow a_j - \sum_{j=1}^{n} P_{ij} (x_j \log x_j + p_i (1 + \log y_i)) - u \log u|_{u=1} - 2(1 + \log y_i) - \lambda = 0$$

$$\Rightarrow a_j - \sum_{j=1}^{n} P_{ij} x_j \log x_j - 2\|P\|_{\infty} (1 + \log y_i) - \lambda = 0$$

where last relation follows due to definition of $p_i$ and $\lambda$ is Lagrange multiplier corresponding to the constraint $\sum_{i=1}^{p} y_i \leq 1$. By complimentary slackness, we have $\lambda > 0$ iff $\sum_{i=1}^{p} y_i = 1$.

Eliminating $\lambda$ from above equations, we obtain $y = \text{proj}_{\Delta}^{+} \left( \exp \left( \frac{1}{2\|P\|_{\infty} + 1} \left( a^1 - Px \log x \right) \right) \right)$.

Similarly, we obtain $z = \text{proj}_{\Delta}^{+} \left( \exp \left( \frac{1}{2\|C\|_{\infty} + 1} \left( a^2 - Cx \log x \right) \right) \right)$.

It is clear from the analytical expressions that for each iteration of Algorithm 2 we need $O(\text{nnz}(P) + \text{nnz}(C))$ time. Hence total runtime of Algorithm 2 is $O((\text{nnz}(P) + \text{nnz}(C)) \log \frac{1}{\delta})$.

B Proof of width reduction for the MPC problem

In Section 3 we made the assumption that all entries

This assumption follows from the results in [16]. We outline this proof in this section for completeness.

For the purpose of this proof, we introduce notation $[k] := \{1, \ldots, k\}$.

Suppose we are given an instance of mixed packing covering of the form

$$P x \leq 1_p, C x \geq 1_c, x \geq 0_n.$$

Case 1: For each column $P_{i,j}$ associated with variable $x_i$, let $P_{j,i} \overset{\text{def}}{=} \max_{j \in [p]} P_{j,i} > 0$. Then we consider the following updates to MPC in order to reduce diameter.

Suppose, without loss of generality, $C_{1,i} = \max_{j \in [c]} C_{j,i}$ and $C_{c,i} = \min_{j \in [c]} C_{j,i}$. If $C_{1,i} \leq P_{j,i}$, then we can update $P_{j,i} = \frac{1}{P_{j,i}} P_{j,i}, C_{c,i} = \frac{1}{P_{j,i}} C_{j,i}$ and $x_i = P_{j,i} x_i$. Then we observe that each element in $P_{i,j}, C_{i,j}$ is at most 1. Moreover, due to the packing constraint $\bar{P}_{i,j} \bar{x} \leq 1$, we note that for any feasible $\bar{x}, \bar{P}_{j,i} \bar{x} \leq 1$. Finally, since $\bar{P}_{j,i} = 1$, we have that $\bar{x}_i \leq 1$ lies in the support of constraint set. So we replaced the $i$-th column and corresponding $i$-th variable of the system by an equivalent system.
Similarly, if $C_{c,i} \geq P_{j,i}$ then consider $x^{sol}$ defined as

$$x_k^{sol} := \begin{cases} \frac{1}{P_{j,i}} & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$$

Then $x^{sol}$ is already a feasible solution of MPC. So we may assume that $C_{c,i} < P_{j,i} < C_{1j}$. In this case, define $r_i = \frac{C_{c,i}}{P_{j,i}}$ and $n_i = \lceil \log r_i \rceil$. We make $n_i$ copies of the column $C_{c,i}$ and denote by the tuple $(i,l)$ the columns of a new matrix $\hat{C}_{(i,l)}$ where $l \in [n_i]$. Similarly, we add $n_i$ copies of variable $x_i$ denoted as $\hat{x}_{(i,l)}$. We make similar changes to $P_{j,i}$. Note that this system is equivalent to earlier system in the sense that any solution $\hat{x}_{(i,l)}, l \in [n_i]$ can be converted into a solution of the earlier system since $x_i = \sum_{l \in [n_i]} \hat{x}_{(i,l)}$. However, this allows us to reduce the elements of $\hat{C}$ along with certain box constraints on $\hat{x}_i$, which was our original goal. For each $j \in [c], l \in [n_i]$, redefine

$$\hat{C}_{j,(i,l)} = \min\{C_{j,i}, 2^l P_{j,i}\}$$

and for variable $\hat{x}_{(i,l)}$, add the constraint

$$\hat{x}_{(i,l)} \leq \frac{2}{2^l P_{j,i}}.$$

Claim B.1. MPC \(\hat{C}, \hat{P}\) and the new system defined by matrices $\hat{C}, \hat{P}$ and variable $\hat{x}$ are equivalent.

Proof. For this proof, let us focus on $i$-th column and $i$-th variable.

For any feasible solution $\hat{x}$, consider $x_i = \sum_{l \in [n_i]} \hat{x}_{(i,l)}$. This $x_i$ does not violate any covering constraint since $\hat{C}_{j,(i,l)} \leq C_{j,i}$. The packing constraints also follow because we have not made any changes to the elements corresponding to the packing constraints $\hat{P}_{j,(i,l)}$.

For the other direction, the key fact to note is that any feasible $x$ satisfies $x_i \leq \frac{1}{2^l P_{j,i}}$ due to packing constraint $P_{j,i} x \leq 1$. Let $l_i$ be the largest index such that

$$x_i \leq \frac{2}{2^l P_{j,i}},$$

and then let

$$\hat{x}_{(i,l)} = \begin{cases} x_i & \text{if } l = l_i \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\hat{x}_{(i,l)}$ satisfies the constraint in (6) for all $l \in [n_i]$. Moreover, for constraint $j$, we must have $\hat{C}_{j,i} \hat{x} \geq 1$. Note that if $\hat{C}_{j,(i,l)} = C_{j,i}$ then there is nothing to prove. So we assume that $C_{j,i} > \hat{C}_{j,(i,l)} = 2^l P_{j,i}$. Then we must have that $l_i < n_i$ in this case, by definition of $n_i$. This then gives $\hat{x}_{(i,l_i)} = x_i \geq \frac{1}{2^l P_{j,i}}$ by our choice of $l_i$ being the largest possible. Then we know that $\hat{C}_{j,(i,l_i)} = 2^l P_{j,i}$, and hence the $j$-th covering constraint is satisfied.

Packing constraints are satisfied trivially since there is no change in elements of $\hat{P}_{(i,l)}$ for all $l \in [n_i]$. Hence the claim follows.

Finally the proof follows by change of variables as $\bar{x}_{(i,l)} = 2^{l-1} P_{j,i}$ and $\bar{C}_{(i,l)} = \frac{1}{2^{l-1} P_{j,i}} \hat{C}_{(i,l)}$.

Further, note that all elements of $\hat{P}_{(i,l)}$ are at most 1 for all $l \in [n_i]$, and all elements of $\hat{C}_{(i,l)}$ are at most 2 for all $l \in [n_i]$ and $\bar{x}_{(i,l)} \leq 1$ for all $l \in [n_i]$.

Case 2: Suppose $P_{j,i} = 0$. This implies that in variable $x_i$, this is a purely covering problem. So we can increase $x_i$ to satisfy the $j$-th covering constraint such that $C_{j,i} > 0$ independent of the packing constraints and problem reduces to smaller packing covering problem in remaining variables and covering constraints $j$ such that $C_{j,i} = 0$. For this smaller packing covering problem, we can apply the method in Case 1 again.
C Application to the Densest Subgraph problem

In this section, we apply the result in Theorem 1.1 to the densest subgraph problem. We define the density of a graph \( G = \langle V, E \rangle \) as \( |V|/|E| \) (half the average degree of \( G \)). Hence, the densest subgraph of \( G \) is induced on a subset of vertices \( U \subseteq V \) such that

\[
U \overset{\text{def}}{=} \arg\max_{S \subseteq V} \frac{|E(S)|}{|S|},
\]

where \( E(S) \) denotes the set of edges in the subgraph of \( G \) induced by \( S \).

The following is a well-known LP formulation of the densest subgraph problem, introduced in [7], which we denote using PRIMAL\((G)\). The optimal objective value is known to be \( \rho^*_G \).

\[
\begin{align*}
\text{maximize} & \quad \sum_{e \in E} y_e \\
\text{subject to} & \quad y_e \leq x_u, x_v, \quad \forall e = uv \in E \\
& \quad \sum_{v \in V} x_v \leq 1, \\
& \quad y_e \geq 0, x_v \geq 0, \quad \forall e \in E, \forall v \in V
\end{align*}
\]

We then construct the dual LP for the above problem. Let \( f_e(u) \) be the dual variable associated with the first \( 2m \) constraints of the form \( y_e \leq x_u \), and let \( D \) be associated with the last constraint. We get the following LP, which we denote by DUAL\((G)\), and whose optimum is also \( \rho^*_G \).

\[
\begin{align*}
\text{minimize} & \quad D \\
\text{subject to} & \quad f_e(u) + f_e(v) \geq 1, \quad \forall e = uv \in E \\
& \quad \sum_{e \ni v} f_e(v) \leq D, \quad \forall v \in V \\
& \quad f_e(u) \geq 0, f_e(v) \geq 0, \quad \forall e = uv \in E
\end{align*}
\]

Parametrizing with respect to \( D \), this becomes a mixed packing covering LP. The solution to the densest subgraph problem is simply the smallest value of \( D \) for which the LP is feasible. Since \( D \) can take at most \( O(|V|^3) \) values in total, the densest subgraph problem can be reduced to solving \( O(|E|) \) instances of MPC, where the number of nonzeros \( N \) in the matrix is \( O(|E|) \) and the width \( w \) is simply the maximum degree in \( G \). This gives the following corollary.

**Corollary C.1.** Given a graph \( G = \langle V, E \rangle \) with maximum degree \( \Delta \), we can find the \((1 + \varepsilon)\)-approximation to the maximum subgraph density of \( G \), \( \rho^*_G \), in parallel time \( \widetilde{O}(\Delta \varepsilon^{-1}) \) and total work \( \widetilde{O}(\Delta |E| \varepsilon^{-1}) \).

The previous fastest algorithms for densest subgraph do not depend on \( \Delta \) - however, their dependence on \( 1/\varepsilon \) is quadratic [2]. Corollary C.1 gives the fastest algorithm for this problem in the high precision regime \((\varepsilon < 1/\Delta)\), since its dependence on \( \varepsilon^{-1} \) is only linear.