HBT Correlators – 
Current Formalism vs. 
Wigner Function Formulation

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Abstract: We clarify the relationship between the current formalism developed by Gyulassy, Kaufmann and Wilson and the Wigner function formulation suggested by Pratt for the 2-particle correlator in Hanbury-Brown Twiss interferometry. When applied to a hydrodynamical description of the source with a sharp freeze-out hypersurface, our results remove a slight error in the prescription given by Makhlin and Sinyukov which has led to confusion in the literature.

It is widely accepted that if the nuclear matter created in ultra-relativistic heavy-ion collisions attains a high enough energy density, it will undergo a phase transition into a quark-gluon plasma. For this reason, it is of great interest to determine the energy densities actually attained in these collisions. The total interaction energy of a given reaction can be directly measured by particle calorimeters and spectrometers. Although there is no analogous direct measurement for the size of the reaction region, Hanbury-Brown Twiss interferometry [1] provides an indirect measurement in terms of the correlations between produced particles.

Ten years ago, Pratt [2] used the covariant current formulation of Gyulassy, Kaufmann and Wilson [3] to show that the correlations between two particles could be expressed in terms of one-particle pseudo-Wigner functions. Although Pratt’s derivation was non-relativistic, it provided a valuable link between the experimental data and many semi-classical event generators whose output came in the form of one-particle

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distributions. Since that time, different methods have been used to relativistically generalize Pratt’s result \cite{4, 5, 6}, but to our knowledge, the simplest generalization (using the covariant current formalism covariantly) has never been published. The aim of this letter is twofold: (1) to fill the above void in the literature, and (2) to show that by applying the final result to hydrodynamical models with 3-dimensional freezeout hypersurfaces, a dispute in the literature about the correct form of the 2-particle correlator in these models can be resolved.

The covariant single- and two-particle distributions for bosons are defined by

\[ P_1(p) = E \frac{dN}{d^3p} = E \langle \hat{a}^+(p)\hat{a}(p) \rangle, \]  

\[ P_2(p_a, p_b) = E_a E_b \frac{dN}{d^3p_a d^3p_b} = E_a E_b \langle \hat{a}^+(p_a)\hat{a}^+(p_b)\hat{a}(p_b)\hat{a}(p_a) \rangle, \]  

where \( \hat{a}^+(p) \) (\( \hat{a}(p) \)) creates (destroys) a particle with momentum \( p \). The two particle correlation function is then given by \cite{3}

\[ C(p_a, p_b) = \frac{\langle N \rangle^2}{\langle N(N-1) \rangle} \frac{P_2(p_a, p_b)}{P_1(p_a)P_1(p_b)}. \]

Using the classical covariant current formalism of \cite{3, 7} we will show that for a general class of chaotic current ensembles the two particle distribution for bosons obeys a Wick theorem:

\[ P_2(p_a, p_b) = \frac{\langle N(N-1) \rangle}{\langle N \rangle^2} \left( P_1(p_a)P_1(p_b) + |\bar{S}(p_a, p_b)|^2 \right), \]

where we define the following covariant quantity

\[ \bar{S}(p_a, p_b) = \sqrt{E_a E_b} \langle \hat{a}^+(p_a)\hat{a}(p_b) \rangle. \]

We will then show that \( \bar{S} \) is equal to the Fourier transform of a kind of Wigner function:

\[ \bar{S}(p_a, p_b) = \tilde{S}(q, K) = \int d^4x e^{-iq \cdot x} S(x, K), \]

where the off-shell 4-vector \( K = \frac{1}{2}(p_a + p_b) \) is the average of two on-shell \( (p_i^0 = E_i) \) 4-momenta, and \( q = p_a - p_b \) is the off-shell difference of the same two momenta so
that their scalar product vanishes, \( K^\mu q_\mu = 0 \). For the special case of \( p_a = p_b \), \( K = p_a \) becomes on-shell and

\[
\tilde{S}(p_a, p_a) = \tilde{S}(0, K) = P_1(p_a)
\]

(7)

It should be noted that eqn.(3) involves a 4-dimensional Wigner transform, in contrast to the 3-dimensional expression suggested by Pratt [2] which neglects retardation effects.

In [3] it was shown that a classical source current \( J(x) \) generates free outgoing pions in a state which satisfies

\[
\hat{a}(p)|J\rangle = i\tilde{J}(p)|J\rangle,
\]

(8)

where

\[
\tilde{J}(p) = \int \frac{d^4 x}{\sqrt{(2\pi)^3 2E_p}} \exp[i(E_p t - p \cdot x)] J(x)
\]

(9)

is the on-shell Fourier transform of the source \( J(x) \), and \( \langle J|J\rangle = \int d^3 p |\tilde{J}(p)|^2 = 1 \). For classical currents, the ensemble expectation values in eqns. [1], [2], and [3] can then be defined in terms of a density operator \( \hat{\rho} \) involving the state \( |J\rangle \) such that \( \langle \hat{O} \rangle = \text{tr}(\hat{\rho} \hat{O}) \).

Generalizing the result of [7] in order to allow for arbitrary \( x - p \) correlations, we consider an ensemble of chaotic source currents at positions \( x_i \) with momenta \( p_i \),

\[
J(x) = \sum_{i=1}^{N} e^{i\phi_i} e^{-ip_i \cdot (x-x_i)} J_0(x - x_i),
\]

(10)

where \( \phi_i \) is a random phase. The momenta \( p_i \) of the sources can, but need not be on the boson mass-shell; for example, the source could be a decaying \( \Delta \)-resonance with 3-momentum \( p_i \). The on-shell Fourier transform of (10) is

\[
\tilde{J}(p) = \sum_{i=1}^{N} e^{i\phi_i} e^{ip \cdot x_i} \tilde{J}_0(p - p_i),
\]

(11)

where

\[
\tilde{J}_0(p - p_i) = \int \frac{d^4 x}{\sqrt{(2\pi)^3 2E_p}} e^{i(p-p_i) \cdot x} J_0(x)
\]

(12)
is the Fourier transform of $J_0(x)$, and $p$ is on-shell while $p_i$ may be off-shell.

We then choose a density operator such that

$$\text{tr}(\hat{\rho} \hat{O}) = \sum_{N=0}^{\infty} P_N \prod_{i=1}^{N} \int d^4x_i d^4p_i \rho(x_i, p_i) \int_0^{2\pi} \frac{d\phi_i}{2\pi} \langle J_i \mid \hat{O} \mid J \rangle$$

(13)

where $\rho(x_i, p_i)$ is the covariant probability density of the source points $(x_i, p_i)$ in phase space, and $P_N$ is the probability distribution for the number of sources in the reaction. These probabilities are normalized as follows:

$$\int d^4x d^4p \rho(x, p) = 1, \quad \sum_{N=0}^{\infty} P_N = 1.$$  \hspace{1cm} (14)

Using (8) and the above definitions, it is straightforward to show that

$$P_1(p) = E_p \langle |\tilde{J}(p)|^2 \rangle = \langle N \rangle E_p \int d^4x_1 d^4p_1 \rho(x_1, p_1) \langle \tilde{J}_0(p - p_1) \rangle^2$$

$$= \langle N \rangle E_p \int d^4p_1 \tilde{\rho}(p_1) \langle \tilde{J}_0(p - p_1) \rangle^2.$$  \hspace{1cm} (15)

The single particle spectrum is thus obtained by folding the momentum spectrum $|\tilde{J}_0(p)|^2$ of the individual source currents $J_0$ with the 4-momentum distribution of the sources, $\tilde{\rho}(p) = \int d^4x \rho(x, p)$.

Similarly, if one neglects cases in which two particles are emitted from exactly the same point \[3\], one finds:

$$P_2(p_a, p_b) = \frac{\langle N(N - 1) \rangle}{\langle N \rangle^2} E_a E_b \left[ \langle |\tilde{J}(p_a)|^2 \rangle \langle |\tilde{J}(p_b)|^2 \rangle + \langle \tilde{J}^*(p_a) \tilde{J}(p_b) \rangle \langle \tilde{J}^*(p_b) \tilde{J}(p_a) \rangle \right]$$

(16)

which proves eqn.(4) by way of (8).

Using eqn.(9), we find the following relationship:

$$\tilde{J}^*(p_a) \tilde{J}(p_b) = \int \frac{d^4x_1 d^4x_2}{(2\pi)^3 2 \sqrt{E_a E_b}} \exp(-ip_a \cdot x_1 + ip_b \cdot x_2) J^*(x_1) J(x_2)$$

$$= \int \frac{d^4x d^4y}{(2\pi)^3 2 \sqrt{E_a E_b}} \exp(-iq \cdot x - iK \cdot y) J^*(x + \frac{1}{2}y) J(x - \frac{1}{2}y),$$

(17)
where \( x = \frac{1}{2}(x_1 + x_2) \) and \( y = x_1 - x_2 \). The above relation proves eqn.(\[3\]) as long as the following expression for the Wigner function is used:

\[
S(x, K) = \int \frac{d^4 y}{2(2\pi)^3} e^{-iK\cdot y} \langle J^*(x + \frac{1}{2}y)J(x - \frac{1}{2}y) \rangle .
\] (18)

The average on the r. h. s. is defined in the sense of eqn.(13) and can be evaluated with the help of the definition (10) to yield

\[
S(x, K) = \langle N \rangle \int d^4 z d^4 q \rho(x - z, q) S_0(z, K - q) ,
\] (19)

where

\[
S_0(x, p) = \int \frac{d^4 y}{2(2\pi)^3} e^{-ip\cdot y} J_0^*(x + \frac{1}{2}y)J_0(x - \frac{1}{2}y) \]

(20)

is the Wigner function associated with an individual source \( J_0 \). Thus the one- and two-particle spectra can be constructed from a Wigner function which is obtained by folding the Wigner function for an individual boson source \( J_0 \) with the Wigner distribution \( \rho \) of the sources. Eqn.(19) is useful for the calculation of quantum statistical correlations from classical Monte Carlo event generators for heavy-ion collisions: \( \langle N \rangle \rho(x, p) \) can be considered as the distribution of the classical phase-space coordinates of the boson emitters (decaying resonances or 2-body collision systems), and \( S_0(x, p) \) as the Wigner function of the free bosons emitted at these points. Replacing the former by a sum of \( \delta \)-functions describing the space-time locations of the last interactions and the boson momenta just afterwards, and the latter by a product of two Gaussians with momentum spread \( \Delta p \) and coordinate spread \( \Delta x \) such that \( \Delta x \Delta p \geq \hbar/2 \), we recover the expressions derived in [6].

Using eqns. (3) to (6), our final result is then

\[
C(p_a, p_b) = 1 + R(q, K)
\] (21)

where the “correlator” \( R \) is given by

\[
R(q, K) = \frac{|\tilde{S}(q, K)|^2}{\tilde{S}(0, p_a) \tilde{S}(0, p_b)} .
\] (22)

Equation (3) is the starting point for a practical evaluation of the above correlator. It should be noted that due to the on-shell condition of (9), it is impossible to
reconstruct \( S(x, K) \) from the correlator in a model independent way. Thus any analysis of data on \( R(q, K) \) necessarily involves suitable model assumptions for \( S(x, K) \), in particular for the \( x - K \) correlations in the source distribution. In most practical applications one takes for \( S(x, K) \) a classical (on-shell) phase-space distribution. In hydrodynamical models, for example, this phase-space distribution is taken as a local equilibrium Bose-Einstein distribution localized on a 3-dimensional freeze-out hypersurface \( \Sigma(x) \) which separates the thermalized interior of an expanding fireball from the free-streaming particles on its exterior [8]:

\[
S_\alpha(x, K) = \frac{2s_\alpha + 1}{(2\pi)^3} \int_\Sigma K^\mu d^3\sigma_\mu(x') \delta^{(4)}(x - x') \exp\left\{\beta(x')[K \cdot u(x') - \mu_\alpha(x')]\right\} - 1 .
\] (23)

Here \( s_\alpha \) and \( \mu_\alpha \) denote the spin and chemical potential of the emitted particle species \( \alpha \), while \( u_\nu(x) \), \( \beta(x) \), and \( d^3\sigma_\mu(x) \) denote the local hydrodynamic flow velocity, inverse temperature, and normal-pointing freeze-out hypersurface element. Inserting this equation into (7), one obtains the Cooper-Frye formula [9]

\[
\tilde{S}(0, p) = P_1(p) = \int_\Sigma p^\mu d^3\sigma_\mu(x) f(x, p)
\] (24)

where we define the distribution function (for clarity we drop the index \( \alpha \) for the particle species)

\[
f(x, p) = \frac{2s + 1}{(2\pi)^3} \frac{1}{\exp\{\beta(x)[p \cdot u(x) - \mu(x)]\} - 1} .
\] (25)

For the numerator of the correlator,

\[
|\tilde{S}(q, K)|^2 = \int_\Sigma K^\mu d^3\sigma_\mu(x) K^\nu d^3\sigma_\nu(y) f(x, K) f(y, K) \exp[iq \cdot (x - y)] ,
\] (26)

we find an expression which is very similar to the one given in [10]. There, however, each of the two distribution functions under the integral featured on-shell arguments \( p_a \) and \( p_b \), respectively, instead of the common (off-shell) average argument \( K \) as in (26). This error in [10] can be traced back to an inaccurate transition from finite discrete volumes along the freeze-out surface \( \Sigma \) to the continuum limit [11]. Taking over this inaccuracy produces (in particular for very rapidly expanding sources) unphysical [12] oscillations of the correlator around zero at large values of \( q \) [13, 14]
which are inconsistent with the manifestly positive definite nature of the correlator \((22)\).

The symmetric form \((26)\) (in contrast to the asymmetric one given in \([10]\)) allows one to replace the exponential by the cosine and to split the expression into two real 3-dimensional integrals:

\[
|\tilde{S}(\mathbf{q}, \mathbf{K})|^2 = (\tilde{S}_1(\mathbf{q}, \mathbf{K}))^2 + (\tilde{S}_2(\mathbf{q}, \mathbf{K}))^2,
\]

where

\[
\tilde{S}_{1,2}(\mathbf{q}, \mathbf{K}) = \int_{\Sigma} K^\mu d^3\sigma(x) f(x, K) \left\{ \cos(q \cdot x) \right\}. \tag{28}
\]

This facilitates the numerical evaluation of the correlator.

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