Abstract
Under the quenched approximation, we perform a lattice calculation for the mass of the ground $4^{++}$ glueball state in $E^{++}$ channel on a $D = 3 + 1$ lattice. Our calculation shows that the mass of this state is $M_G(4^{++}) = 3.65(6)(18)\text{GeV}$, which rules out the $4^{++}$ or mainly $4^{++}$ glueball state interpretation for $\xi(2230)$.
1 Introduction

As the experiments reported\(^1\), there maybe exist one candidate for glueball state, namely, \(\xi(2230)\), which has been seen mainly in \(J/\Psi\) radiative decay with a number of decay channels. But its spin has not been determined experimentally:\(^2\) we don’t know whether its spin is 2\(^{++}\) or 4\(^{++}\). Therefore, one hopes to calculate the masses of 2\(^{++}\) and 4\(^{++}\) glueball states on lattice for the spin interpretation of \(\xi(2230)\) since lattice QCD is regarded as the most reliable theory to study hadrons.

Meanwhile, lattice simulation has achieved tremendous success in the study of glueball states. For instance, some authors show mass spectra\(^3,4\) for glueball states. But, so far we do not know how to calculate the mass of 4\(^{++}\) glueball state assuredly on \(D = 3 + 1\) lattice. On a \(D = 2 + 1\) cubic lattice, in Ref. \(^5\), Johnson and Teper performed a calculation for these states. Nevertheless, there are some disputes about the interpretation of the calculation, i.e., one does not know whether the mass calculated in Ref. \(^5\) is really the mass of 4\(^{++}\) glueball states. In addition, as authors consider, one can hardly extent the construction of the operator in Ref. \(^5\) into the \(D = 3 + 1\) lattice.

Therefore, from the point of views of both experiment and theory, it is important to calculate the 4\(^{++}\) glueball mass on lattice. Basing on the connection between the continuum limits of the asymptotic expansion of the chosen operator and the construction of the operator with definite \(J^{PC}\), we have developed a new procedure\(^6\) which can calculate masses of glueball states with arbitrary \(J^{PC}\). According to this procedure, a lattice calculation for the mass of the ground 4\(^{++}\) glueball states is performed in this paper by the construction of operators, corresponding states of which are in \(E^{++}\) channel in the simulation. The leading term in expansion of this operator belongs to 4\(^{++}\) representation of \(SO(3)^{PC}\) group. Therefore, from the asymptotic expansion of this constructed operators, we assure that, in continuum case, the spin of corresponding states is 4\(^{++}\). Then, the mass of 4\(^{++}\) glueball state was obtained. We found that the calculated mass is prominently different with that of 2\(^{++}\) glueball states\(^7\) and it is about 1.5 and 2 times of that of tensor and scalar glueball state respectively. We should argued that our results rule out the possible interpretation of the ground 4\(^{++}\) glueball states for \(\xi(2230)\).

This paper is organized as follows. In section 2 we show how to construct 4\(^{++}\) glueball operators in \(E^{++}\) channel. We should also present our results and conclusions in section 3 and section 4.

2 The Construction of the Operator Corresponding to 4\(^{++}\) Glueball state

As we know, a definite state \(|\psi>\) is generated by a current \(o\) acting on vacuum \(|0>\):

\[
|\psi> = o|0>.
\] (1)
and the character of $|\psi>$ can be described by the current. Therefore, as mentioned in [7], to calculate the mass of glueball state with spin $J^{PC} = 4^{++}$, we should carry out three steps. First, we should write out nonet currents which transform as the representation $4^{++}$ under the $SO(3)^{PC}$ group and decompose them into irreducible representations $A_1^{++}, E^{++}, T_1^{++}$ and $T_2^{++}$ of the group $O^{PC}$ (the cubic group with parity and charge conjugation transform) in the subduced representation $^1$. Then we construct corresponding operators on lattice. Its continuum limits should be just those currents mentioned above. Next we should also use the variational procedure to extract masses of the ground $4^{++}$ states and excited $4^{++}$ states.

We shall calculate the mass of the ground $4^{++}$ glueball states in this paper. Among nonet currents, there is only one type of such nonet current, the mass dimension of which is the lowest one. This nonet current is of the form $Tr(B\bar{D}\bar{D}B)$ and we expect that it should give the most contribution to the ground states with spin $J^{PC} = 4^{++}$ due to dimensional analysis. Therefore, we will only consider such nonet current and the construction of corresponding operators on lattice in this paper. Apparently such disposal will bring up some errors and its affectation will be considered in the error estimate in section 3.

The Nonet Current and Its Decomposition

To simplify, we denote magnetic field $B_{x(y, z)} = B_{x(y, z)}^a \frac{x^a}{2}$ by $B_{1(2, 3)}$ and covariant derivative $D_{x(y, z)}$ by $D_{1(2, 3)}$ ($D_i = \partial_i - i[A_i, \cdot]$) respectively here. Obviously one can treat $B_i$'s as bases of the spin $J = 1$, and $B_\pm = \mp \frac{1}{\sqrt{2}}(B_1 \pm iB_2)$ and $B_0 = B_3$ as three standard bases $^2$. It is similar for covariant derivatives $D_i$'s, which can also be combined into three standard bases: $D_\pm = \mp \frac{1}{\sqrt{2}}(D_1 \pm iD_2)$ and $D_0 = D_3$.

Since spins of magnetic field $\vec{B}$ and covariant derivative operator $\vec{D}$ are both 1, the nonzero gauge invariant nonet current of the mass dimension 6 is sole, so that according to C-G coefficients, one can write these 9 gauge invariant currents unambiguously in the standard form up to full covariant derivatives (The following $c_i$'s $(i = 4, \cdots, -4)$ are eigen-currents of operator $\hat{J}_3$, i.e., $\hat{J}_3(c_i) = i \cdot c_i$ $^3$):

$$c_4 = \text{Tr}[B_+D_+D_+B_+ + DG],$$

$$c_3 = \text{Tr}[B_+D_+D_0B_+ + B_+D_+D_+B_0 + DG],$$

$$c_2 = \frac{1}{\sqrt{7}}\text{Tr}[(B_+D_+D_+B_+ + B_+D_0B_+ + B_+D_+D_+B_0 + 2B_+D_+D_0B_0 + 2B_+D_0D_+B_0)\text{D}],$$

$$c_1 = \frac{1}{\sqrt{7}}\text{Tr}[B_+D_0D_-B_+ + B_0D_+D_-B_+ + B_+D_+D_-B_0].$$

---

$^1$ The subduced representation used here is the representation obtained by trivially embedding the $O^{PC}$ group into the $SO(3)^{PC}$ group.

$^2$ Here $B_\pm$ and $B_0$ is the eigen-current of the operator $\hat{J}_3$, i.e., $\hat{J}_3(B_\pm) = \pm B_\pm$ and $\hat{J}_3(B_0) = 0$.

$^3$ Phenomenologically, we can regard this nonet current as $^5D_4$ multi-current and the corresponding state as $^5D_4$ multi-state.
decomposition into of which are 1, 2, 3, 3 respectively. For our aim we only show the result of the 1DG representation A form the bases of subduced representation with nonet current mentioned above. As 2O

Then we turn to the According to the procedure introduced in Ref. [10], we can get two bases of the These two currents are combinations of following three linear dependent ones: 31

Here DG is some full covariant derivative terms. In fact, one can get \( c_{-i} (i = 1, 2, 3, 4) \) easily by the swap \( + \leftrightarrow - \) in \( c_i \).

Then we turn to the \( O^{PC} \) group, the finite subgroup of \( SO(3)^{PC} \) group, to form the bases of subduced representation with nonet current mentioned above. As shown in Ref. [1], the subduced representation is reducible and one can reduce this representation into irreducible ones \( A_{1}^{1+}, \ E^{++}, \ T_{1}^{++} \) and \( T_{2}^{++} \), the dimension of which are 1, 2, 3, 3 respectively. For our aim we only show the result of the decomposition into \( E^{++} \) here.

According to the procedure introduced in Ref. [10], we can get two bases of the representation \( E^{++} \):

\[
\begin{align*}
\text{e}_1 & = -\frac{1}{\sqrt{2}} (c_2 + c_{-2}) \\
& \propto \text{Tr} [B_1(D_1^2 - D_2^2)B_1 - B_2(D_2^2 - D_3^2)B_2 + B_3(D_3^2 - D_1^2)B_3 \\
& + 4B_2D_3D_2B_3 - 4B_1D_3D_1B_3 + DG]; \\
\text{e}_2 & = \frac{1}{2} \left[ \sqrt{\frac{7}{6}} c_4 + \sqrt{\frac{7}{6}} c_{-4} - \sqrt{\frac{5}{3}} c_0 \right] \\
& \propto \text{Tr} [B_1(D_1^2 + D_3^2 - 2D_2^2)B_1 + B_2(D_2^2 + D_3^2 - 2D_1^2)B_2 \\
& + B_3(D_1^2 + D_2^2 - 2D_3^2)B_3 + 4B_1D_3D_1B_3 \\
& + 4B_2D_3D_2B_3 - 8B_1D_2D_1B_2 + DG].
\end{align*}
\]  

(11)

These two currents are combinations of following three linear dependent ones:
\[ e^{(1)} = \text{Tr}[B_2(D_2^2 - D_1^2)B_2 - B_3(D_3^2 - D_1^2)B_3 + B_1(D_3^2 - D_1^2)B_1 \\
+ 4B_3D_1D_3B_1 - 4B_2D_1D_2B_1 + DG]; \]
\[ e^{(2)} = \text{Tr}[B_3(D_3^2 - D_2^2)B_3 - B_1(D_1^2 - D_2^2)B_1 + B_2(D_1^2 - D_3^2)B_2 \\
+ 4B_1D_2D_3B_2 - 4B_3D_2D_3B_2 + DG]; \]
\[ e^{(3)} = \text{Tr}[B_1(D_1^2 - D_3^2)B_1 - B_2(D_2^2 - D_3^2)B_2 + B_3(D_2^2 - D_1^2)B_3 \\
+ 4B_2D_3D_2B_3 - 4B_1D_3D_1B_3 + DG]. \] (12)

These three currents are of equivalence up to a cyclic permutation \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \) of superscripts and subscripts in Eq. (12).

Then we consider how to construct operators on lattice, which are in the representation \( E^{++} \) and their continuum limits are just \( e^{(i)}(i = 1, 2, 3) \).

**The Construction of Operators on Lattice**

In this paragraph, we assign that \((i, j, k)\) is even permutation of \((1, 2, 3)\), i.e., \((i, j, k) = (1, 2, 3), (2, 3, 1)\) or \((3, 1, 2)\).

Using the link variable as defined in the discussion of improved action\[9, 8\]:

\[ U(n, i) = T \exp(\int_0^\infty dt A_i(an + \hat{t}) \), \] (13)
we denote Wilson operators by figures \[ ]\[1\]. For instance, we denote the plaquette operator

\[ O_{ij} = \sum_n O_{ij}(n) = Re \sum_n \text{Tr}[U(n, i)U(n + \hat{i}, j)U^{-1}(n + \hat{j}, i)U^{-1}(n, j)] \] (14)
by the figure \( \sum_n \vcenter{\hbox{\includegraphics[width=1cm]{sumfig}}} \) for simplification. Here the summation is only over all the spatial links in the same time slice.

Let’s consider the following five types of operators \( d_k^{(1)} - d_k^{(5)} \):

1. The first type of operators are \( d_k^{(1)}(k = 1, 2, 3) \). They are from the following operators:

\[ b_k^{(1)} = \epsilon_k^{(1)} + \epsilon_k^{(1)}; \] (15)

where

\[ \epsilon_k^{(1)} = \sum_n \vcenter{\hbox{\includegraphics[width=1.5cm]{epsilonfig}}} }. \] (16)

\[ ^{4}\text{We use a notation, where } A_i \text{ is an anti-hermitian, traceless } N \times N \text{-matrix.} \]
The expansion of the operator $b_k^{(1)}$ according to lattice spacing $a$ is

$$b_k^{(1)} = \sum_n \text{Tr}[4 + 5a^4(B_j^2 + B_i^2) + \frac{a^6}{12}B_i(17D_j^2 + 5D_k^2 + 12D_i^2)B_i$$
$$+ \frac{a^6}{12}B_j(17D_i^2 + 5D_k^2 + 12D_j^2)B_j - 2a^6B_iD_jD_kB_j$$
$$+ DG] + O(a^8). \quad (17)$$

Here $B_i = -\frac{1}{2} \sum_{jk} \epsilon_{ijk} F_{jk}$ and $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. Then the operator $d_k^{(1)}$ is:

$$d_k^{(1)} = b_i^{(1)} - b_j^{(1)}$$

$$= \sum_n \text{Tr}[7a^4(B_{ij}^2 - B_{ii}^2) + \frac{a^6}{12}B_i(19D_j^2 + 19D_k^2 + 24D_i^2)B_i$$
$$- \frac{a^6}{12}B_j(19D_i^2 + 19D_k^2 + 24D_j^2)B_j + a^6B_k(4D_i^2 - D_k^2)B_k$$
$$- 2a^6(B_jD_kD_iB_k - B_iD_kD_iB_k) + DG] + O(a^8). \quad (18)$$

2. Next, we consider operators $d_k^{(2)}(k = 1, 2, 3)$. They are from operators

$$b_k^{(2)} = \epsilon_{ijk}^{(2)}$$

where

$$\epsilon_{ijk}^{(2)} = \sum_x (\text{rectangle} + \text{diagram}). \quad (19)$$

The definition and the expansion according to lattice spacing $a$ of operators $d_k^{(2)}$ are

$$d_k^{(2)} = b_i^{(2)} - b_j^{(2)}$$

$$= \sum_n \text{Tr}[7a^4(B_{ij}^2 - B_{ii}^2) + \frac{a^6}{12}B_i(19D_j^2 + 19D_k^2 + 24D_i^2)B_i$$
$$- \frac{a^6}{12}B_j(19D_i^2 + 19D_k^2 + 24D_j^2)B_j + a^6(B_jD_kD_iB_k - B_iD_kD_iB_k)$$
$$+ DG] + O(a^8). \quad (20)$$

3. Then we study operators $d_k^{(3)}$, which are from operators

$$b_k^{(3)} = \epsilon_{ijk}^{(3)}$$

$$= \sum_x (\text{rectangle} + \text{diagram} + \text{diagram} + \text{diagram}). \quad (21)$$
For $d_k^{(3)}$ we get
\[
d_k^{(3)} = b_i^{(3)} - b_j^{(3)} = \sum_n \text{Tr}[2a^4(B_i^2 - B_j^2) + \frac{a^6}{6} B_i(D_i^2 + D_k^2)B_j - \frac{a^6}{6} B_i(D_j^2 + D_k^2)B_i + a^6(B_jD_kD_2B_k - B_iD_kD_jB_k) + DG] + O(a^8). \tag{23}
\]

4. Operators $d_k^{(4)}$ are from
\[
\begin{align*}
v b_k^{(4)} &= \epsilon_{ij}^{(4)} = \sum_x (\begin{array}{c}
\text{a} \\
\text{i} \\
\text{n}
\end{array}) + \sum_x (\begin{array}{c}
\text{j} \\
\text{a} \\
\text{n}
\end{array}) + \sum_x (\begin{array}{c}
\text{j} \\
\text{a} \\
\text{n}
\end{array}) + \sum_x (\begin{array}{c}
\text{j} \\
\text{a} \\
\text{n}
\end{array}) \text{).} \\
\end{align*}
\tag{24}
\]

And again
\[
d_k^{(4)} = b_i^{(4)} - b_j^{(4)} = \sum_n \text{Tr}[\frac{a^6}{2} (B_j(D_i^2 + D_k^2)B_j - B_i(D_j^2 + D_k^2)B_i + DG] + O(a^8). \tag{25}
\]

5. At last we study the operator $d_k^{(5)}$ which are constructed by
\[
b_k^{(5)} = \epsilon_{ijk}^{(5)} = \sum_n (\begin{array}{c}
\text{a} \\
\text{i} \\
\text{n}
\end{array}) \text{).} \tag{26}
\]

The construction is
\[
d_k^{(5)} = b_i^{(5)} - b_j^{(5)} = \sum_n \text{Tr}[2a^4(B_i^2 - B_j^2) + \frac{a^6}{6} B_i(D_j^2 + D_k^2)B_i + a^6B_jD_kD_2B_k - B_iD_kD_jB_k] + DG] + O(a^8). \tag{27}
\]

Therefore we let
\[
e_k = d_k^{(1)} + d_k^{(2)} + 5d_k^{(3)} + 2d_k^{(4)} + 4d_k^{(5)} = a^6 \sum_n \text{Tr}[B_i(D_i^2 - D_k^2)B_i - B_j(D_j^2 - D_k^2)B_j + B_k(D_j^2 - D_i^2)B_k + 4B_jD_kD_2B_k - 4B_iD_kD_jB_k] + DG] + O(a^8). \tag{28}
\]

While comparing Eq. (28) and Eq. (12), we find that operators $e_k(k = 1, 2, 3)$ are just our aimed operators. Noticing that these operators are all 8-link, so that we need not consider the "tadpole" renormalization due to the mean field theory.\cite{11}

We should utilize operators $e_k$ to calculate the mass of $4^{++}$ glueball states in this paper.

Results and some discussions are shown in the forthcoming section.
3 Calculation Results

Under the quench approximation, we perform our calculation on an anisotropic $8^3 \times 40$ lattice with improved action introduced in Ref. \cite{12}:

\[ S_{II} = \beta \left\{ \frac{5\Omega_{sp}}{3\xi u_s^3} + \frac{4\xi\Omega_{tp}}{3u_s^2u_t^2} - \frac{\Omega_{sr}}{12\xi u_s^6} - \frac{\xi\Omega_{str}}{12u_s^4u_t^2} \right\}, \tag{29} \]

where $\beta = 6/g^2$, $g$ is the QCD couple constant, $u_s$ and $u_t$ are mean link renormalization parameters (we set $u_t = 1$), $\xi = a_s/a_t = 5.0$ is the aspect ratio, and $\Omega_{sp}$ includes the summation over all spatial plaquettes on the lattice, $\Omega_{tp}$ indicates the temporal plaquettes, $\Omega_{sr}$ denotes the planar $2 \times 1$ spatial rectangular loops and $\Omega_{str}$ refers to the short temporal rectangles (one temporal and two spatial links). More detail is given in Ref. \cite{12}. In each $\beta$ calculation, we set 2880 sweeps to make configurations reach to equilibrium and then make total 8100 measurements in 90 bins (we perform measurement one time every four sweeps). The other calculation parameters, such as $u_s^4$, $r_s/r_0$ and $a_s$, are as the same as those in Ref. \cite{4}.

Our results in different $\beta$ are shown in Table 1:

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$\beta$ & 1.7 & 1.9 & 2.2 & 2.4 & 2.5 \\
\hline
$4^{++}$ & 1.61(1) & 1.35(7) & 0.97(4) & 0.79(3) & 0.70(5) \\
\hline
$2^{++}$ & 1.019(3) & 0.95(1) & 0.71(2) & 0.548(6) & 0.519(4) \\
\hline
\end{tabular}
\end{center}

\textbf{Table. 1} $4^{++}$ glueball energy $m_{GL}$ for each $\beta$. The numerals in the brackets are error estimates (The data for $2^{++}$ glueball is from Ref. \cite{4}).

Now we discuss a little about the error estimate.

As discussed in many papers, one may regard that the chosen improved action breaks the rotation symmetry up to $O(a^4)$ in error estimate. But since the mass dimension of terms which break the rotation symmetry up to $O(a^4)$ is eight and the lowest mass dimension of our chosen operators in Eq. (28) is six, the upper limit of precision here is $O(a^2)$.

On the other hand, among the currents with mass dimension 7 or 9, there is no current with $P = +$. Therefore, the precision calculated here is actually $O(a^2)$ and the systematic error can be written out as the form $c_2a_s^2 + c_4a_s^4 + \cdots$, which should be used to fit our simulation data. Simulation results and fitting curve are shown in Fig. 1. The bottom curve and data\footnote{Note: The data for $2^{++}$ glueball is from Ref. \cite{4}} in Fig. 1 are for the $2^{++}$ glueball state which also belong to $E^{++}$ channel.

From the data and fitting curve, the statistical error of the mass measurement is 0.044GeV. As argued in ref. \cite{4}, there are 1% systematic error due to aspect ratio. In addition, our approach leads into about 0.5% systematic error. Therefore, the total systematic error is about 1.1%(0.041GeV). So the mass of $4^{++}$ glueball states is 3.65(6)GeV. Including the uncertainty in $r_0^{-1} = 410(20)MeV$, our final results are: $M_G(4^{++}) = 3.65(6)(18)GeV$ which is about 1.5 times and twice of masses of the tensor and scalar glueball states respectively.
4 Conclusion

Through the construction of operators in $E^{++}$ channel which are complicated to some degree, we perform a calculation of mass of the ground $4^{++}$ glueball state under the quenched approximation. Due to the expansion of the chosen operator, we confirm that, in continuum, the mass calculated here is the mass of the ground $4^{++}$ glueball state and it is $M_G(4^{++}) = 3.65(6)(18) GeV$.

Apparently, our result rules out the $J^{PC} = 4^{++}$ glueball interpretation for $\xi(2230)$ even by taking into account the opinion that the mass will be shifted by as much as 20% in going to full QCD\textsuperscript{[13]}. While noticing that the common calculated mass for tensor glueball state is about $2.4 GeV$ under the quench approximation, we claim that our calculation supports the $2^{++}$ glueball interpretation for $\xi(2230)$ if this state exists and it is confirmed as a glueball.
References

[1] Baltrusaitis, R. M., et al., Phys. Rev. Lett. 56(1986), 107; Caso, C., et al., 1998, Eur. Phys. J. C3, 1.

[2] Alde, D., et al., GAMS Collaboration, Phys. Lett. B177(1986), 120; Aston, D., et al., LASS Collaboration, Phys. Lett. 201B(1988), 169; Blundell, H. and S. Godfrey Phys. Rev. D53(1996), 3700; Bai, J.Z., et al., BES Collaboration, Phys. Rev. Lett. 76(1996), 3502.

[3] C. Michele and M. Teper, Nucl. Phys. B314(1989) 347; UKQCD collagenation et al, Phys. Lett. B309(1993) 378; H. Chen et al, Nucl. Phys. B(Suppl)34(1994) 357.

[4] C. J. Morningstar and M. Peardon, Phys. Rev. D60(1999) 034509.

[5] R. Johnson and M. Teper, Nucl. Phys. B(Suppl) 73(1999) 267.

[6] B. Berg and A. Billoire, Nucl. Phys. B221(1983) 109.

[7] D. Q. Liu et al, hep-lat/0103018.

[8] O. Nachtmann: "High Energy Collision and Nonperturbative QCD" in the lecture notes at the workshop "Topics in Field Theories" held on Oct. 1993, Kloster Banz, Germany.

[9] D. Q. Liu et al, hep-lat/0104007.

[10] D. Q. Liu et al, hep-lat/0011087.

[11] G. P. Lepage and P. B. Mackenzie, Phys. Rev. D48(1993) 2250.

[12] C. J. Morningstar and M. Peardon Phys. Rev. D56(1997) 4043-4061.

[13] H. Chen, et al, Nucl. Phys. B(Suppl.) 34(1994) 357.
Figure caption

Figure 1  The mass of $4^{++}$ glueball against the lattice spacing $(a_s/r_0)^2$. Our fitting curve for the data of $4^{++}$ glueball state is $m(4^{++}, a_s) = 3.65 - 1.22(a_s/r_0)^2 + 2.74(a_s/r_0)^4 (unit: GeV)$. In continuum case, the mass of $4^{++}$ glueball state is 3.65(4)GeV if we only consider the statistical error. The bottom data and curve: $m(2^{++}, a_s) = 2.417 + 0.783(a_s/r_0)^2 - 0.787(a_s/r_0)^4 (unit: GeV)$ are for the mass of $2^{++}$ glueball state which is shown in Ref. [7].
