A simplified BBGKY hierarchy for correlated fermionic systems from a Stochastic Mean-Field approach

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The stochastic mean-field (SMF) approach allows to treat correlations beyond mean-field using a set of independent mean-field trajectories with appropriate choice of fluctuating initial conditions. We show here, that this approach is equivalent to a simplified version of the Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy between one-, two-, ..., N-body degrees of freedom. In this simplified version, one-body degrees of freedom are coupled to fluctuations to all orders while retaining only specific terms of the general BBGKY hierarchy. The use of the simplified BBGKY is illustrated with the Lipkin-Meshkov-Glick (LMG) model. We show that a truncated version of this hierarchy can be useful, as an alternative to the SMF, especially in the weak coupling regime to get physical insight in the effect beyond mean-field. In particular, it leads to approximate analytical expressions for the quantum fluctuations both in the weak and strong coupling regime. In the strong coupling regime, it can only be used for short time evolution. In that case, it gives information on the evolution time-scale close to a saddle point associated to a quantum phase-transition. For long time evolution and strong coupling, we observed that the simplified BBGKY hierarchy cannot be truncated and only the full SMF with initial sampling leads to reasonable results.

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I. INTRODUCTION

The BBGKY hierarchy [1–3] is an exact reformulation of the problem of interacting fermions where one-body degrees of freedom (DOFs) are coupled to two-body DOFs that are themselves coupled to three-body DOFs and so on and so forth. One of the advantages of this hierarchy is that it illustrates how correlations can affect the evolution of the one-body density matrix. For this reason, the set of equations between one-body evolution and many-body correlations are often used as a starting point to develop approximations beyond mean-field (see for instance [4]). However, due to the increasing complexity occurring when complex correlations are included, a truncation scheme of the hierarchy is necessary.

The proposal of an appropriate truncation of the BBGKY hierarchy has been the subject of extensive work in the past [5–7] and leads to the so-called Time-Dependent Density Matrix (TDDM) approach to interacting systems where not only the one-body density is followed in time but also eventually the two-body and eventually three-body density matrices. The flexibility of choosing the proper equations of motion is still the subject of intensive work [8]. In particular the absence of a definite strategy for truncating the BBGKY can lead to uncontrolled results with varying quality as illustrated in Ref. [9].

It has been shown recently that the stochastic mean-field approach can provide rather accurate description of many-body effects beyond the Time-Dependent Hartree-Fock (TDHF) or Time-Dependent Hartree-Fock Bogolyubov (TDHFB) approach in several test cases [10–12] where the approximate treatment can be confronted to the exact solution. In this approach, the system is described by a set of initial conditions with fluctuating one-body density, followed by deterministic TDHF or TDHF-B trajectories [13]. Each trajectory evolves through its own self-consistent mean-field for a given initial condition and is independent from the others. Among the interesting aspects of the SMF approach one can mention that beyond mean-field effects are incorporated although only mean-field type evolution is needed. One of the surprising results turns out to be that the SMF technique can provide a better approximation compared to an approach where two-body degrees of freedom are explicitly introduced in the description [12].

One of the main objectives of the present work is to clarify how a theory that only involves mean-field evolution can incorporate properly many-body effects. Below, we make explicit connection between the SMF approach and the BBGKY hierarchy. We show that the SMF theory is equivalent to solve a set of coupled equations between one-body DOFs and higher-order correlations without truncation. This set of equations can be seen as a simplified version of the standard BBGKY hierarchy. We finally explore if this simplified BBGKY hierarchy can be used either as a numerical tool or to get physical insight in correlated systems if a truncation is assumed.
II. THE SMF APPROACH AND ITS CONNECTION TO A NON-TRUNCATED BBGKY HIERARCHY

A. Basic aspects of the stochastic mean-field approach

In the SMF theory, the N-body problem is replaced by a set of deterministic time-dependent mean-field trajectories \[13\],

\[
ith \frac{d\rho(t)}{dt} = \left[ h(\rho(t)), \rho(t) \right],
\]

(1)

where \((n)\) labels a given trajectory, \(\rho^{(n)}\) is the one-body density along this trajectory and \(h(\rho^{(n)})\) is the associated self-consistent mean-field. For a system interacting through a two-body interaction \(v_{12}\), \(h\) is given by:

\[
h[\rho] = t + \text{Tr}_{\{\hat{v}_{12}\rho_2\}},
\]

(2)

where \(t\) is the kinetic term and \(\hat{v}\) denotes antisymmetric effective interaction. Here, we use the convention of Refs. \[13\], i.e. the index \(^{\gamma}\hat{v}^{\delta}\) in \(\rho_i\) means that the density acts on the \(i^{th}\) particle, similarly \(\text{Tr}_{\{\}}\) means that the partial trace is made on the \(i^{th}\) particle. Starting from the above expression and using this convention, we recover the standard mean-field expression

\[
h_{ij}[\rho] = t_{ij} + \sum_{mn} \hat{v}_{ik,jm}\rho_{mk}.
\]

In the SMF approach, each trajectory is deterministic and fluctuations stem only from the initial conditions \(\rho^{(n)}(t_0)\). The statistical properties of the density matrices are usually chosen in such a way that they reproduce at least in an approximate way the initial quantum phase-space of the problem. This is usually done by imposing specific properties of different moments of the initial density fluctuations \(\delta \rho^{(n)}(t_0) = \rho^{(n)}(t_0) - \bar{\rho}^{(n)}(t_0)\). \(\bar{\rho}^{(n)}\) denotes here the statistical average over the initial conditions. Most often, the moments are chosen to reproduce quantum fluctuations of the initial many-body states. In its original form \[13\], the SMF theory has been formulated assuming (i) that the initial state is either a pure Slater determinant or a statistical ensemble of independent particles. In both cases, the one-body density is associated with occupation numbers denoted by \(n_i\) in the canonical basis. (ii) The initial quantum phase-space has been approximated by a Gaussian statistical ensemble. Then, initial phase-space sampling is completely specified by the first and second moments of \(\rho^{(n)}(t_0)\) that are conveniently chosen as:

\[
\bar{\rho}^{(n)}_{ij}(t_0) = \delta_{ij}n_i,
\]

\[
\delta \rho^{(n)}_{ij}(t_0)\delta \rho^{(n)}_{kl}(t_0) = \frac{1}{2} \delta_{ij}\delta_{lk} [n_i(1 - n_j) + n_i(1 - n_j)].
\]

Starting from this original formulation, it has been realized that the SMF approach can be extended to superfluid systems by replacing the N-body problem by a set of Time-Dependent Hartree-Fock Bogolyubov (TDHFB) with initial fluctuations on both the normal and anomalous density matrices \[10\]. In addition, we have shown recently that an improved treatment of the initial phase-space \[15\] without the Gaussian approximation can further ameliorate the many-body dynamics beyond the mean-field and allows to describe initially correlated many-body states.

More surprisingly, a recent study on the Hubbard model \[12\] has shown that the SMF approach can lead to a better than state of the art Green function techniques including explicitly two-body effects \[16\]. This actually might appear as a surprising result since in the SMF approach only mean-field trajectories are required. It is one of the goals of the present article to show how correlations are incorporated in our theory. In particular, we show that the SMF approach is equivalent to a simplified BBGKY hierarchy where 2-body, 3-body, .... N-body correlations are approximately propagated in time.

B. Many-body evolution in SMF

In the SMF theory, the quantum expectation value of any \(k\)-body operator is replaced by a statistical average over the different trajectories. Considering a set of \(k\) one-body operators, denoted by \(A(1), \cdots, A(k)\), their expectation values within SMF are given by

\[
A(1)^{(n)} \cdots A(k)^{(n)} = \sum_{\alpha_1, \beta_1} A_{\alpha_1, \beta_1}(1) \cdots A_{\alpha_k, \beta_k}(k) \rho^{(n)}_{\alpha_1, \beta_1} \cdots \rho^{(n)}_{\beta_k, \alpha_k}.
\]

Therefore, the knowledge of any many-body observable is equivalent to the knowledge of the time evolution of the set of moments \(M_1, M_2, \cdots, M_{1-k}\) defined through:

\[
\overline{M_{1-k}} = \rho^{(n)}_{\alpha_1, \beta_1} \cdots \rho^{(n)}_{\beta_k, \alpha_k}.
\]

We show in appendix A starting from the equation \[4\] that the different moments evolve according to the following coupled equations:

\[
ith \frac{d}{dt} M_{1, \cdots, k} = \sum_{\alpha_1} \left[ t_{\alpha_1} M_{1, \cdots, k} \right] + \sum_{\alpha_1} \text{Tr}_{\{\} (\overline{\tilde{v}_{\alpha_k+1}, M_{1, \cdots, k+1}})}.
\]

(3)

These equations are formally very close to the BBGKY hierarchy of density matrices in many-body systems \[4\] except that here, the many-body densities are replaced by different moments of the density \(\rho^{(n)}\). Another important difference is that part of the fermionic aspects are lost in this description. Indeed, the moments \(M_{1, \cdots, k}\) here are symmetric with respect to the exchange of two indices. Therefore, part of the quantum correlations induced by Fermionic statistic are lost in the theory.
This is one of the differences with the TDDM approach where many-body densities truly associated to fermions are solved in time.

One can equivalently rewrite the above set of equations as coupled equations between the average one-body density denoted for simplicity below as $\bar{\rho}$ and the centered moments $C_{1,k}$ defined through:

$$C_{1...k} = \delta \rho_1^{(n)} \cdots \delta \rho_k^{(n)}. \quad (4)$$

The resulting coupled equation are (see appendix A):

$$i \hbar \frac{d}{dt} \bar{\rho}(t) = [h(\bar{\rho}(t)), \bar{\rho}(t)] + \text{Tr}_2 \{\tilde{\nu}_{12}, C_{12}\}, \quad (5)$$

together with (for $k \geq 2$):

$$i \hbar \frac{d}{dt} C_{1...k} = \left[ \sum_{\alpha \leq k} t_{\alpha}, C_{1...k} \right] + \sum_{\alpha = 1}^k \text{Tr}_{k+1} \{\tilde{\nu}_{\alpha k+1}, C_{1...k} P_{k+1}\}$$

$$+ \sum_{\alpha = 1}^k \text{Tr}_{k+1} \{\tilde{\nu}_{\alpha k+1}, C_{1...k} - (\alpha-1)(\alpha+1)...(k+1) P_{\alpha}\}$$

$$+ \sum_{\alpha = 1}^k \text{Tr}_{k+1} \{\tilde{\nu}_{\alpha k+1}, C_{1...k} C_{\alpha k+1}\}$$

$$+ \sum_{\alpha = 1}^k \text{Tr}_{k+1} \{\tilde{\nu}_{\alpha k+1}, C_{1...k+1}\}. \quad (6)$$

Equation (5) is similar to the first equation of the BBGKY hierarchy where usually $C_{12}$ stands for the two-body correlation matrix. This equation clearly points out that effects beyond the standard mean-field are accounted for in the SMF approach. A more precise discussion on what many-body effects are included is given below.

### C. Beyond mean-field effects in SMF theory

The fact that the SMF approach is equivalent to solve an unrestricted set of coupled equations between the one-body density and higher-order moments is a clear advantage of this technique. As a direct conclusion, it includes not only two-body effects but also higher order correlation effects. This might explain why, in previous applications, it has given better results compared to the calculation using BBGKY hierarchy truncated at second order.

Still, because of the replacement of quantum average by classical average, it is not anticipated that all many-body effects are properly included with the SMF framework. Let us write explicitly the equation on $C_{12}$. Starting from Eq. (5), after some straightforward manipulations, we obtain:

$$i \hbar \frac{d}{dt} C_{12} = \{h_1(\bar{\rho}), \bar{\rho}\} + \text{Tr}_2 \{\tilde{\nu}_{12} + \tilde{\nu}_{23}, C_{12}\}.$$  

$$+ \text{Tr}_3 \{\tilde{\nu}_{13} + \tilde{\nu}_{23}, C_{13}\}, \quad (7)$$

where $h_i$ is the mean-field Hamiltonian acting on the particle $i$. This equation corresponds to a simplified version of the second BBGKY equation for correlation (see Eq. (55-58) of Ref. [14]). On the negative side, we first observe that the term $B_{12}$ (Eq. (57) of [14]) that is responsible for both direct in-medium collisions and pairing effects is missing. Regarding in-medium collisions, it is anticipated that the SMF approach only valid at low internal excitation of the system where the in-medium collisions are strongly hindered due to the Pauli exclusion principle. It is worth mentioning that the inclusion of in-medium collisions can be made eventually by considering a Langevin process where fluctuations are introduced continuously in time [17]. Regarding the pairing term, although the complete proof is out of the scope of the present work, it is anticipated that this term is partially included when the superfluid version of SMF is used [10].

On the positive side of SMF we can remark:

- The first line of Eq. (7) shows that the SMF approach properly propagate the initial fluctuations in the average self-consistent mean-field.

- The second line of Eq. (7) shows that the term $P_{12}$ usually appearing in the second BBGKY equation (Eq. (58) of [14]) is approximately accounted for. We note in particular, that the Pauli principle is not fully respected in the SMF approach. This is indeed not surprising due to the simple classical assumption made on the average moments.

- The last line of Eq. (7) shows that a great advantage of the resulting equation is that 3-body and higher correlation effects are included.

In order to better understand what type of correlations are retained in Eq. (7), it is interesting to note that the average density evolution can be rewritten as

$$i \hbar \frac{d}{dt} \bar{\rho}(t) = [h(\bar{\rho}(t)), \bar{\rho}(t)] + \{\delta h[\rho^{(n)}], \delta \rho^{(n)}\]. \quad (8)$$

This alternative form evidences that the effect beyond mean-field stems from the correlation between the density fluctuation and the resulting mean-field fluctuation. Such correlations and their effects on one-body dynamics have been studied in Ref. [18] for small amplitude vibrations around the ground state of atomic nuclei. In that case, they lead to a coupling of the collective motion to surface vibrations (the so-called particle-phonon coupling). This coupling is known as a dominant source of fragmentation of the nuclear collective response at low temperature [4].
D. Discussion

The recent success of the SMF approach and more particularly the good agreement between the approximate treatment and exact solutions in some model cases [10–12] indicate that this approach retains some of the most important correlation effects and properly incorporate their effects on one-body degrees of freedom. The equivalent formulation given above in terms of a simplified BBGKY hierarchy is helpful to clearly pinpoint the retained terms. As a side product of the present work, we note that the present development leads to a new approximate hierarchy of equations of motion that is much less involved than the standard BBGKY one. In particular, it is commonly accepted that the main difficulty in many-body theories based on the truncation of the BBGKY hierarchy is the absence of a systematic prescription to truncate the coupled equations at a given order [9]. The coupled equations derived above can be seen as an alternative way to incorporate many-body effects.

Let us make a few remarks on the advantages of the new equations of motion: (i) first, the equation of motion are much simpler than the standard BBGKY hierarchy, as we will see below, in some cases, third-order or even fourth order equation can be obtained without difficulty. (ii) When possible, it is clear that it is preferable to directly perform the full SMF theory, i.e. sample a set of mean-field trajectories. However, in some cases, this might become rather cumbersome due to number of trajectories required to get small statistical error-bars. Then, the corresponding BBGKY hierarchy can be an alternative approximate method to solve SMF. (iii) As we pointed out in Ref. [15], a proper account for the initial quantum phase-space might require to go beyond the Gaussian assumption of the initial sampling. In simple models, it is possible to directly get the initial phase-space with some semi-classical techniques as in [15]. In more complex systems, this seems more complicated and the sampling beyond the Gaussian approximation might not be possible. Alternatively, one can use the simplified BBGKY hierarchy imposing that different moments at initial time exactly equals the initial quantum moments. (iv) Last, as we will see below, the simplified form of the BBGKY equation can lead to interesting physical information. This is illustrated below, where approximate analytical expressions are obtained for the evolution including beyond mean-field effects.

III. APPLICATION

The goal of the present section is to illustrate the interesting aspects associated to the hierarchy of coupled equations motivated by the SMF approach. For this purpose, we took as an example the Lipkin-Meshkov-Glick (LMG) Model [20–23] that has already been used to benchmark the SMF theory using phase-space initial sampling both in the Gaussian [11] and non-Gaussian assumptions. In the two previous applications, a set of mean-field trajectories have been explicitly followed in time using the equations of motion:

\[
\begin{aligned}
\frac{d}{dt} j_x^{(n)} &= -j_y^{(n)} + 2\chi j_y^{(n)} j_z^{(n)}, \\
\frac{d}{dt} j_y^{(n)} &= j_x^{(n)} + 2\chi j_x^{(n)} j_z^{(n)}, \\
\frac{d}{dt} j_z^{(n)} &= -4\chi j_y^{(n)} j_z^{(n)},
\end{aligned}
\]

with fluctuating initial conditions. Here, \( j_x^{(n)}, j_y^{(n)}, j_z^{(n)} \) are the three reduced quasi-spin components and \( \chi \) is a constant that measures the strength of the two-body interaction. All technical details associated to the LMG model, its SMF solution, as well as the possibility to get an exact solution have been extensively discussed in our previous work [11–15].

To get the closed set of equations, we follow the general strategy depicted above and separate the average quasi-spin from its fluctuation as \( \langle j \rangle + \delta j \). Starting from Eq. 9, we obtain

\[
\begin{aligned}
\frac{d}{dt} \langle j_x \rangle &= (-1 + 2\chi j_y \rangle j_y + 2\chi \Sigma_{yz}, \\
\frac{d}{dt} \langle j_y \rangle &= (1 + 2\chi j_z \rangle j_z + 2\chi \Sigma_{zx}, \\
\frac{d}{dt} \langle j_z \rangle &= -4\chi \langle j_y \rangle j_y - 4\chi \Sigma_{xy},
\end{aligned}
\]

where \( \Sigma_{ij} \equiv \langle \delta j_i^{(n)} \delta j_j^{(n)} \rangle \) are the average fluctuations of the reduced quasi-spin. This evolution corresponds to the first coupled equation of the simplified BBGKY hierarchy. Higher order equation can be deduced from the equations of motion of the fluctuating part, that are given by:

\[
\begin{aligned}
\frac{d}{dt} \langle \delta j_x \rangle &= (-1 + 2\chi \langle j_y \rangle \delta j_y + 2\chi \delta j_y \delta j_z \\
&+ 2\chi [\delta j_y \delta j_z - \Sigma_{yz}], \\
\frac{d}{dt} \langle \delta j_y \rangle &= (1 + 2\chi \langle j_z \rangle \delta j_z + 2\chi \delta j_x \delta j_z \\
&+ 2\chi [\delta j_x \delta j_z - \Sigma_{zx}], \\
\frac{d}{dt} \langle \delta j_z \rangle &= -4\chi \langle j_y \rangle \delta j_y + 4\chi \delta j_y \delta j_z - 4\chi [\delta j_x \delta j_y - \Sigma_{xy}].
\end{aligned}
\]

From these equations, any equation of motion of the average quasi-spin centered moments can be derived rather easily. An illustration is given in appendix [13] for the evolution of second, third and fourth centered moments, denoted respectively by \( \Sigma_{ij}, \Sigma_{ijk} \) and \( \Sigma_{ijkl} \). The average equation of motion greatly simplifies due to the fact that the Hamiltonian is invariant with respect to the rotation \( R_z = e^{i\pi j_z} \). Accordingly, if the initial condition is also invariant with respect to this symmetry, all moments containing an odd number of components of \( \delta j_x \) and \( \delta j_y \) are equal to zero at all times. Here, we consider initial
Fluctuations
0 4 8
time
0 5 10 15 20 25 30
012345678

A. Weak coupling regime

In many physical situations, we are mainly interested in the average evolution of one-body observables as well as their quantum fluctuations. Standard time-dependent Hartree-Fock is usually convenient for the former one but provides a rather poor approximation for the fluctuations. We already know that the SMF approach can greatly improve the description of two-body degrees of freedom from previous applications [11, 15]. We show here, that the QC-TDDM2 approach where only the average quasi-spin and their fluctuations are followed in time, already provides a great improvement compared to the original TDHF.

In Fig. 1, the exact results obtained for \( \chi = 0.5 \) are compared with the SMF and QC-TDDM2 calculations. We see that the QC-TDDM2 approach does also provide the correct evolution of the second moments \( \Sigma^2_{zz} \) and \( \Sigma^2_{yy} \) even if higher order effects are neglected. Only at large time, deviation with the exact solutions starts to be visible in Fig. 1. We see however that \( \Sigma^2_{zz} \) remains equal to zero in QC-TDDM2. This could have been anticipated from Eq. (11) where we see that the only way to have \( \Sigma^2_{zz} \) evolving in time is to include the effects of third order moments.

The good agreement between QC-TDDM2 and the exact evolution in the weak coupling regime, despite the fact that it corresponds to a simplified BBGKY hierarchy truncated at order 2, is already an interesting result. This indirectly proves that SMF is able to grasp some important physical effects. Actually, because of the simplicity of the QC-TDDM2 approach, we can even obtain

\[
\frac{d}{dt} \Sigma_{zz} = -4\chi \Sigma^2_{xy},
\]

while the relevant second moments evolutions are given by:

\[
\begin{align*}
\frac{d}{dt} \Sigma^2_{xx} &= 2(-1 + 2\chi \Sigma^2_{xy}) + 4\chi \Sigma^3_{xxy}, \\
\frac{d}{dt} \Sigma^2_{yy} &= 2(1 + 2\chi \Sigma^2_{xy}) + 4\chi \Sigma^3_{yxy}, \\
\frac{d}{dt} \Sigma^2_{xy} &= (-1 + 2\chi \Sigma^2_{xy}) + (1 + 2\chi \Sigma^2_{xy}) \Sigma^2_{xx} + 2\chi \Sigma^3_{yxy} + 2\chi \Sigma^3_{xxy}, \\
\frac{d}{dt} \Sigma^2_{zz} &= -8\chi \Sigma^3_{xzy}.
\end{align*}
\]

We see that the second order moments are coupled to the third moments. Similarly, the third moments evolution induces a coupling to the 4th moments and so on and so forth. The explicit form of the third and fourth moments evolution are given in the appendix A. It should be noted that these equations are obtained in a rather straightforward way starting from the SMF framework. In the following, the evolution obtained by truncating the set of equation to the second moments, third moments or fourth moments will be respectively referred to as Quasi-Classical TDDM2 (QC-TDDM2), QC-TDDM3 and QC-TDDM4, while the original technique that samples the fourth moments will be respectively referred to as Quasi-Classical TDDM4, while the original technique that samples the third moments evolution are given in the appendix B. It should be noted that these equations are obtained in a rather straightforward way starting from the SMF framework. In the following, the evolution obtained by truncating the set of equation to the second moments, third moments or fourth moments will be respectively referred to as Quasi-Classical TDDM2 (QC-TDDM2), QC-TDDM3 and QC-TDDM4, while the original technique that samples the third moments evolution are given in the appendix B. It should be noted that these equations are obtained in a rather straightforward way starting from the SMF framework.
approximate analytical solutions for $\Sigma_{xx}^2$ and $\Sigma_{yy}^2$. Indeed, let us assume that the average evolution of $\bar{j}_z$ can be approximated by its mean field solution in the evolution of second moments. Then $\bar{j}_z \simeq j_0 = -1/2$ for all time. The equation of motion can be written as:

\[
\begin{align*}
\frac{d}{dt} \Sigma_{xx}^2 &= -2\Omega_+ \Sigma_{xx}^2, \\
\frac{d}{dt} \Sigma_{yy}^2 &= +2\Omega_+ \Sigma_{yy}^2, \\
\frac{d}{dt} \Sigma_{xy}^2 &= -\Omega_- \Sigma_{xy}^2 + \Omega_+ \Sigma_{xx}^2,
\end{align*}
\]

where $\Omega_{\pm} = |2\chi j_0 \pm 1|$. Introducing the frequency $\omega = 2\sqrt{\Omega_+ \Omega_-} = 2\omega_0$, we immediately deduce that:

\[
\frac{d^2}{dt^2} \Sigma_{xy}^2 = -\omega^2 \Sigma_{xy}^2,
\]

that could be easily integrated to finally give (using the fact that $\Sigma_{xy}^2(0) = 0$):

\[
\begin{align*}
\Sigma_{xx}^2(t) &= \Sigma_{xx}^2(0) + \left(\frac{\Omega_-^2 \Sigma_{xx}^2(0) - \omega_0^2 \Sigma_{xx}^2(0)}{\omega_0^2}\right) \sin^2(\omega_0 t), \\
\Sigma_{yy}^2(t) &= \Sigma_{yy}^2(0) + \left(\frac{\Omega_-^2 \Sigma_{yy}^2(0) - \omega_0^2 \Sigma_{yy}^2(0)}{\omega_0^2}\right) \sin^2(\omega_0 t).
\end{align*}
\]

(12)

In Fig. 1 we see that the results of these analytical expressions are very close to the QC-TDDM2 case and henceforth to the exact case. This gives an example where the simple formula obtained with QC-TDDM2 and the approximate treatment of the average evolution of $\bar{j}_z$ provides interesting information in the many-body correlated dynamics. In particular, it directly gives the expression of the relevant frequency $\omega_0$ that drives the two-body evolution.

In order to describe also the fluctuation of the $z$ component of the quasi-spin, one needs to include higher order moments effects. The equations of motion for third and fourth moments are given in appendix B. In Fig. 2 we show the evolution of $\Sigma_{zz}^2$ for different order of truncation of the hierarchy up to QC-TDDM4. In this figure, we see that including third order moments and fourth order moments improves gradually the description of $\Sigma_{zz}^2$. In the latter case, we see that the result is almost on top of the exact result.

In the bottom part of Fig. 2 we also show the result of SMF assuming a Gaussian sampling of the initial phase-space. We see in particular that the Gaussian approximation has an impact on the quality of the result. For comparison, we also show the results of QC-TDDM3 and QC-TDDM4 imposing that all third moments are initially zero as it should be for a Gaussian distribution. We observe again that the QC-TDDM4 results tend to the complete SMF case assuming initially Gaussian phase-space. These comparisons illustrate one of the point raised above. In more complicated situations of interacting many-body systems, initial sampling beyond the Gaussian approximation might be impossible. Then, using a truncated version of the simplified BBGKY hierarchy with non-zero third and/or fourth order moments might be a useful alternative. As a side remark, we note that the truncation of the hierarchy and/or a wrong assumption on the initial conditions can lead to unphysical values of the observables, here negative values of the fluctuations. This is a phenomenon that also happens when the standard BBGKY hierarchy is truncated. This problem does not occur when the full SMF is used.

B. Strong coupling regime

1. Second order truncation

In Fig. 3 the results of QC-TDDM2 are compared to the exact and SMF case for $\chi = 1.8$. It is known
that the SMF result is expected to be worse as the two-body interaction increases. Still, we see that SMF results remain comparable with the exact results over the period displayed in Fig. 3. We observe that the QC-TDDM2 results are able to describe the very short time evolution but then rather fast deviate from the correct evolution. In that case, the truncation to second order does not seem to be appropriate.

Similarly to the previous case, to understand the very short time dynamics, one can assume that the average evolution of \( j_z \) identifies with the mean-field one. Then, for \( \chi > 1 \), we end-up with the set of equations:

\[
\begin{align*}
\frac{d}{dt} \Sigma_{xx}^2 &= -2\Omega_0 - \Sigma_{xy}^2, \\
\frac{d}{dt} \Sigma_{yy}^2 &= -2\Omega_0 + \Sigma_{xy}^2, \\
\frac{d}{dt} \Sigma_{xy}^2 &= -\Omega - \Sigma_{xx}^2 - \Omega - \Sigma_{yy}^2,
\end{align*}
\]

that could be solved to give:

\[
\begin{align*}
\Sigma_{xx}^2(t) &= \Sigma_{xx}^2(0) + \left( \frac{\Omega^2 \Sigma_{xy}^2(0) + \omega^2 \Sigma_{xx}^2(0)}{\omega^2} \right) \sinh^2(\omega_0 t), \\
\Sigma_{yy}^2(t) &= \Sigma_{yy}^2(0) + \left( \frac{\Omega^2 \Sigma_{xy}^2(0) + \omega^2 \Sigma_{yy}^2(0)}{\omega^2} \right) \sinh^2(\omega_0 t).
\end{align*}
\]

This divergent behavior reflects the fact that the initial state is located at a saddle point associated to a spontaneous symmetry breaking. The above formula can grasp the short time behavior and give access to the typical instability time-scale associated to this symmetry breaking (see Fig. 3). However, non-linear effects and higher order effects are needed to describe longer time dynamics. The complete SMF approach that includes correlations to all order not gives access not only to the instability time but also to longer time evolution.

2. Third and fourth order truncation

In Fig. 4, the evolution obtained with higher order truncations (QC-TDDM3 and QC-TDDM4) are compared with the exact solution. We see in this figure that the QC-TDDM4 follows the exact result for longer time compared to QC-TDDM2. However, in both QC-TDDM3 and QC-TDDM4, the trajectory diverges after some rather short time. We note that truncated dynam-

ics diverges while the full SMF calculation with initial fluctuations doesn’t. This is a clear indication that either a simplified BBGKY truncated at some order or the sharp truncation strategy that is used here, i.e. assuming directly fluctuations to be zero above a certain order are not appropriate in the strong coupling regime and/or close to a spontaneous symmetry breaking case. In that case, in the absence of a clear prescription for truncation, there is no alternative than performing explicitly the initial sampling.

IV. CONCLUSION

In the present work, we show that the recently proposed Stochastic Mean-Field approach that consists in sampling initial quantum fluctuations associated to a system of interacting fermions and then follow each individual trajectory using the associated time-dependent mean-field is equivalent to a simplified BBGKY hierarchy of equations of motion. Similarly to the standard BBGKY approach, in this hierarchy, average evolution of one-body degrees of freedom are coupled to second order fluctuations that are themselves coupled to third order fluctuation, and so on.

We show that the simplified hierarchy is retaining spe-
specific terms of the standard BBGKY one. The quality of the SMF approach in many recent applications [10–12, 15, 25] indicates that the retained terms are among the important ones beyond the mean-field approximation at least for short time evolution. This finding is helpful in particular to understand why the SMF technique can be more accurate than an approach based on the standard BBGKY assuming a truncation at some order, like the TDDM approach. Indeed, while some correlations are lost in SMF, it propagates correlations to any orders.

Using the LMG model as an illustrative example, we show that the simplified BBGKY approach can be truncated and used as an alternative to the complete SMF theory in the weak coupling regime. In this regime, it is shown that it can even give better results that the complete SMF theory where a Gaussian assumption for the initial phase-space is made. It addition, in some cases, it gives interesting physical insight beyond the independent particle approximation. In the strong coupling regime, a sharp truncation seems to be adequate only for very short time evolution, unstable behavior is observed in the LMG model for longer times. Still it provides the timescale associated to the spontaneous symmetry breaking occurring in the LGM model. For strong coupling, the complete SMF implementation remains more adequate and leads to better description of correlations over longer time, without unstable evolution.

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Appendix A: connection between SMF approach and BBGKY hierarchy

In the present appendix, the intermediate steps to obtain the set of coupled equations (3) and (6) are given. The main assumption of the SMF theory is that all trajectories identify with a mean-field evolution given by Eq. (1). Using the expression of the mean-field hamiltonian, we obtain:

\[
\frac{i\hbar}{dt} M_1^{(n)} = \sum_{\alpha} t_{\alpha} \rho_{\alpha}^{(n)} - \rho_{\alpha}^{(n)} t_{\gamma} \beta + \sum_{\gamma \lambda \lambda'} \tilde{\gamma}_{\lambda \lambda'} \left( \rho_{\lambda \lambda'}^{(n)} \rho_{\gamma \beta}^{(n)} - \sum_{\gamma \lambda \lambda'} \rho_{\lambda \lambda'}^{(n)} \rho_{\gamma \beta}^{(n)} \right) \tilde{\gamma}_{\lambda \lambda',\beta}.
\]

This equation could be rewritten formally as:

\[
\frac{i\hbar}{dt} d \rho_1^{(n)} = \left[ t_1, \rho_1^{(n)} \right] + \text{Tr}_2 \left[ \tilde{\gamma}_{12}, \rho_1^{(n)} \rho_2^{(n)} \right].
\]  

Note that here we used the fact that \( \text{Tr}_2(\tilde{\gamma}_{12} \rho_2^{(n)}) = \text{Tr}_2(\rho_2^{(n)} \tilde{\gamma}_{12}) \). Taking the average, we directly obtain the first equation of the hierarchy (3). Higher order equations are immediately obtained by using the property:

\[
\frac{i\hbar}{dt} \left[ \rho_1^{(n)} \cdots \rho_k^{(n)} \right] = \sum_{\alpha=1}^k \rho_1^{(n)} \cdots \left[ \frac{i\hbar}{dt} \rho_\alpha^{(n)} \right] \cdots \rho_k^{(n)}
\]

\[
= \sum_{\alpha=1}^k \rho_1^{(n)} \cdots \left[ t_{\alpha}, \rho_\alpha \right] \cdots \rho_k^{(n)} + \sum_{\alpha=1}^k \rho_1^{(n)} \cdots \text{Tr}_{(k+1)} \left[ \tilde{\gamma}_{\rho_{k+1}, \rho_{\alpha}^{(n)}} \right] \cdots \rho_k^{(n)}
\]

Introducing the notation \( M_{1\cdots k}^{(n)} = [\rho_1^{(n)} \cdots \rho_k^{(n)}] \), we then end-up with the fact that the equation of motion of \( M_{1\cdots k}^{(n)} \) is coupled to \( M_{1\cdots (k+1)}^{(n)} \):

\[
\frac{i\hbar}{dt} M_{1\cdots k}^{(n)} = \sum_{\alpha=1}^k \left[ t_{\alpha}, M_{1\cdots k}^{(n)} \right] + \sum_{\alpha=1}^k \text{Tr}_{k+1} \left[ \tilde{\gamma}_{\rho_{k+1}, \rho_{1\cdots (k+1)}} \right].
\]

The set of equations (3) then correspond to the average version of the above coupled equations.

1. Average evolution of centered moments

In order to get the equations of motion for the centered moments defined in Eq. (4), it is first convenient to obtain the evolution of the fluctuations \( \delta \rho^{(n)} \) with respect to the average. Subtracting the average evolution of \( \rho^{(n)} \) obtained above to the Eq. (A1) gives:

\[
\frac{i\hbar}{dt} \delta \rho_1^{(n)} = \left[ t_1, \delta \rho_1^{(n)} \right] + \text{Tr}_2 \left[ \tilde{\gamma}_{12}, \delta \rho_1^{(n)} \rho_2^{(n)} \right] + \text{Tr}_2 \left[ \tilde{\gamma}_{12}, \rho_1^{(n)} \delta \rho_2^{(n)} \right] + \text{Tr}_2 \left[ \tilde{\gamma}_{12}, \delta \rho_1^{(n)} \delta \rho_2^{(n)} - \overline{\delta \rho_1^{(n)} \delta \rho_2^{(n)}} \right].
\]
Similarly as in the previous section, we can then use the fact that:

\[ i\hbar \frac{d}{dt} \left( \delta \rho_1^{(n)} \cdots \delta \rho_k^{(n)} \right) = \hbar \sum_{\alpha=1}^k \delta \rho_1^{(n)} \cdots \left( \frac{d\delta \rho_\alpha^{(n)}}{dt} \right) \cdots \delta \rho_k^{(n)}, \]

to obtain the general equation of motion (valid for \( k \geq 2 \)):

\[
\begin{align*}
&i\hbar \frac{d}{dt} \left( \delta \rho_1^{(n)} \cdots \delta \rho_k^{(n)} \right) = \left[ \sum_{\alpha \leq k} t_{\alpha}, \delta \rho_1^{(n)} \cdots \delta \rho_k^{(n)} \right] \\
&\quad + \sum_{\alpha=1}^k \text{Tr}_{k+1} \left[ \tilde{v}_{\alpha k+1}, \left( \delta \rho_1^{(n)} \cdots \delta \rho_k^{(n)} \right) \tilde{\rho}_{k+1} \right] + \sum_{\alpha=1}^k \text{Tr}_{k+1} \left[ \tilde{v}_{\alpha k+1}, \left( \delta \rho_1^{(n)} \cdots \tilde{\rho}_{\alpha} \cdots \delta \rho_k^{(n)} \right) \delta \rho_{k+1}^{(n)} \right] \\
&\quad + \sum_{\alpha=1}^k \text{Tr}_{k+1} \left[ \tilde{v}_{\alpha k+1}, \left( \delta \rho_1^{(n)} \cdots \delta \rho_{k+1}^{(n)} \right) \right] \sum_{\alpha=1}^k \text{Tr}_{k+1} \left[ \tilde{v}_{\alpha k+1}, \left( \delta \rho_1^{(n)} \cdots \sum_{\alpha(k+1)} \cdots \delta \rho_k^{(n)} \right) \right], \quad (A2)
\end{align*}
\]

Taking the average, we deduce Eq. [6].

**Appendix B: Simplified BBGKY hierarchy for the LGM model**

Starting from the dynamical evolution of the fluctuations given in the main text, one can deduce the evolution of the second moments through the use of

\[
\frac{d\Sigma_{ij}^{(n)}}{dt} = \delta j_i \frac{d}{dt} \delta j_j + \delta j_i \frac{d}{dt} \delta j_j. \quad (B1)
\]

The resulting equation are given by:

\[
\begin{align*}
\frac{d}{dt} \Sigma_{xx} &= 2(-1 + 2\chi_{\overline{z}z})\Sigma_{xx}^2 + 4\chi_{\overline{z}x}\Sigma_{x}^2, \\
\frac{d}{dt} \Sigma_{yy} &= 2(1 + 2\chi_{\overline{z}z})\Sigma_{xx}^2 + 4\chi_{\overline{z}y}\Sigma_{x}^2, \\
\frac{d}{dt} \Sigma_{zz} &= -8\chi_{\overline{z}x}\Sigma_{x}^2 + \chi_{\overline{z}y}\Sigma_{y}^2, \\
\frac{d}{dt} \Sigma_{xy} &= (-1 + 2\chi_{\overline{z}j})\Sigma_{xy}^2 + (1 + 2\chi_{\overline{z}j})\Sigma_{zz}^2 \\
&\quad + 2\chi_{\overline{z}x}\Sigma_{y}^2 + 2\chi_{\overline{z}y}\Sigma_{x}^2, \\
\frac{d}{dt} \Sigma_{xz} &= (-1 + 2\chi_{\overline{z}j})\Sigma_{yz}^2 + 2\chi_{\overline{z}y}\Sigma_{x}^2 \\
&\quad - 4\chi_{\overline{z}x}\Sigma_{y}^2 - 4\chi_{\overline{z}y}\Sigma_{x}^2, \\
\frac{d}{dt} \Sigma_{yz} &= (1 + 2\chi_{\overline{z}j})\Sigma_{yz}^2 + 2\chi_{\overline{z}y}\Sigma_{z}^2 \\
&\quad - 4\chi_{\overline{z}x}\Sigma_{y}^2 - 4\chi_{\overline{z}y}\Sigma_{x}^2.
\end{align*}
\]

It is worth mentioning that in the exact case, since the Hamiltonian is invariant under the rotation \( R_z = e^{i\pi J_z} \), most of the moments appearing above will be zero during the evolution. The same property holds if the initial phase-space is invariant under this symmetry. Then, any moments that contains an odd number of \( x \) or \( y \) are equal to zero. The resulting equation of motion for the average fluctuations is given in Eq. [11].

Using a similar strategy, the evolution of the third order moments is given by:

\[
\begin{align*}
\frac{d}{dt} \Sigma_{xx}^3 &= 2(-1 + 2\chi_{\overline{z}z})\Sigma_{xx}^3 + 4\chi_{\overline{z}x}\Sigma_{x}^3, \\
\frac{d}{dt} \Sigma_{yy}^3 &= 2(1 + 2\chi_{\overline{z}z})\Sigma_{xx}^3 + 4\chi_{\overline{z}y}\Sigma_{x}^3, \\
\frac{d}{dt} \Sigma_{zz}^3 &= -8\chi_{\overline{z}x}\Sigma_{x}^3 + \chi_{\overline{z}y}\Sigma_{y}^3, \\
\frac{d}{dt} \Sigma_{xy}^3 &= (-1 + 2\chi_{\overline{z}j})\Sigma_{xy}^3 + (1 + 2\chi_{\overline{z}j})\Sigma_{zz}^3 \\
&\quad + 2\chi_{\overline{z}x}\Sigma_{y}^3 + 2\chi_{\overline{z}y}\Sigma_{x}^3, \\
\frac{d}{dt} \Sigma_{xz}^3 &= (-1 + 2\chi_{\overline{z}j})\Sigma_{yz}^3 + 2\chi_{\overline{z}y}\Sigma_{x}^3 \\
&\quad - 4\chi_{\overline{z}x}\Sigma_{y}^3 - 4\chi_{\overline{z}y}\Sigma_{x}^3, \\
\frac{d}{dt} \Sigma_{yz}^3 &= (1 + 2\chi_{\overline{z}j})\Sigma_{yz}^3 + 2\chi_{\overline{z}y}\Sigma_{z}^3 \\
&\quad - 4\chi_{\overline{z}x}\Sigma_{y}^3 - 4\chi_{\overline{z}y}\Sigma_{x}^3.
\end{align*}
\]

while the \( 4^{th} \) moments evolution reads as (neglecting higher orders terms):

\[
\begin{align*}
\frac{d}{dt} \Sigma_{xx}^4 &= 4(-1 + 2\chi_{\overline{z}j})\Sigma_{xx}^4, \\
\frac{d}{dt} \Sigma_{yy}^4 &= 4(1 + 2\chi_{\overline{z}j})\Sigma_{yy}^4, \\
\frac{d}{dt} \Sigma_{zz}^4 &= 16\chi_{\overline{z}x}\Sigma_{x}^4, \\
\frac{d}{dt} \Sigma_{xy}^4 &= 3(-1 + 2\chi_{\overline{z}j})\Sigma_{xy}^4 + (1 + 2\chi_{\overline{z}j})\Sigma_{xx}^4, \\
\frac{d}{dt} \Sigma_{xz}^4 &= 3(1 + 2\chi_{\overline{z}j})\Sigma_{xz}^4 + (-1 + 2\chi_{\overline{z}j})\Sigma_{yy}^4, \\
\frac{d}{dt} \Sigma_{yz}^4 &= 2(1 + 2\chi_{\overline{z}j})\Sigma_{yz}^4 + 2(-1 + 2\chi_{\overline{z}j})\Sigma_{yz}^4, \\
\frac{d}{dt} \Sigma_{xy}^3 &= 2(1 + 2\chi_{\overline{z}j})\Sigma_{xy}^3 + 8\chi_{\overline{z}x}\Sigma_{y}^3, \\
\frac{d}{dt} \Sigma_{xz}^3 &= 2(-1 + 2\chi_{\overline{z}j})\Sigma_{xz}^3 + 8\chi_{\overline{z}x}\Sigma_{x}^3, \\
\frac{d}{dt} \Sigma_{yz}^3 &= (1 + 2\chi_{\overline{z}j})\Sigma_{yz}^3 + (-1 + 2\chi_{\overline{z}j})\Sigma_{yz}^3 \\
&\quad + 8\chi_{\overline{z}x}\Sigma_{y}^3.
\end{align*}
\]
It should be noted that the above equations are rather straightforward to obtain from the SMF approach. In comparison, using standard BBGKY approach would be more cumbersome and would lead to more complicated expressions.

[1] N.N. Bogolyubov, J. Phys. (URSS) 10, 256 (1946).
[2] H. Born, H.S. Green, Proc. Roy. Soc. A 188, 10 (1946).
[3] J.G. Kirwood, J. Chem. Phys. 14, 180 (1946).
[4] D. Lacroix, S. Ayik, and Ph. Chomaz, Prog. Part. Nucl. Phys. 52, 497 (2004).
[5] W. Cassing, S.J. Wang, Z. Phys. A 337, 1 (1990).
[6] M. Gong, M. Tohyama, Z. Phys. A 335, 153 (1990).
[7] K.-J. Schmitt, P.-G. Reinhard, and C. Toepffer, Z. Phys. A 336, 123 (1990).
[8] Mitsuru Tohyama and Peter Schuck, Eur. Phys. J. A 50, 77 (2014).
[9] A. Akbari, M.J. Hashemi, A. Rubio, R.M. Nieminen, and R. van Leeuwen, Phys. Rev. B 85, 235121 (2012).
[10] D. Lacroix, D. Gambacurta and S. Ayik, Phys. Rev. C 87, 061302(R) (2013).
[11] D. Lacroix, S. Ayik, and B. Yilmaz, Phys. Rev. C 85, 041602(R) (2012).
[12] Denis Lacroix, S. Hermanns, C. M. Hinz, and M. Bonitz Phys. Rev. B 90, 125112 (2014).
[13] S. Ayik, Phys. Lett. B 658, 174 (2008).
[14] D. Lacroix and S. Ayik, Eur. Phys. J. A50, 95 (2014).
[15] Bulent Yilmaz, Denis Lacroix, and Resul Curebal, Phys. Rev. C 90, 054617 (2014).
[16] Sebastian Hermanns, Niclas Schlünzen, and Michael Bonitz Phys. Rev. B 90, 125111 (2014).
[17] D. Lacroix, Phys. Rev. C 73, 044311 (2006).
[18] S. Ayik and Y. Abe, Phys. Rev. C 64, 024609 (1996).
[19] G. F. Bertsch, P. F. Bortignon, and R. A. Broglia, Rev. Mod. Phys. 55, 287 (1983).
[20] H. J. Lipkin, N. Meshkov, and A. J. Glick, Nucl. Phys. A 62, 188 (1965).
[21] N. M. D. Agassi and H. J. Lipkin, Nucl. Phys. A86, 321 (1966).
[22] A. P. Severyukhin, M. Bender, and P.-H. Heenen, Phys. Rev. C74, 024311 (2006).
[23] P. Ring and P. Schuck, The Nuclear Many-Body Problem (Springer-Verlag, New York, 1980).
[24] P. Bouche and H. Flocard, Nucl. Phys. A 437, 189 (1985).
[25] B. Yilmaz, S. Ayik, D. Lacroix, and O. Yilmaz, Phys. Rev. C 90, 024613 (2014).
[26] Wei-Min Zhang, Da Hsuan Feng, and Robert Gilmore Rev. Mod. Phys. 62, 867 (1990).
[27] K.-K. Kan, P. C. Lichtner, M. Dworzecka, and J. J. Griffin, Phys. Rev. C 21, 1098 (1980).