CONTINUOUS ACTION OF LIE GROUPS ON $\mathbb{R}^n$ AND FRAMES

G. ÓLAFSSON

Abstract. Wavelet and frames have become a widely used tool in mathematics, physics, and applied science during the last decade. In this article we discuss the construction of frames for $L^2(\mathbb{R}^n)$ using the action of closed subgroups $H \subset \text{GL}(n, \mathbb{R})$ such that $H$ has an open orbit $O$ in $\mathbb{R}^n$ under the action $(h, \omega) \mapsto (h^{-1})^T(\omega)$. If $H$ has the form $ANR$, where $A$ is simply connected and abelian, $N$ contains a co-compact discrete subgroup and $R$ is compact containing the stabilizer group of $\omega \in O$ then we construct a frame for the space $L^2_O(\mathbb{R}^n)$ of $L^2$-functions whose Fourier transform is supported in $O$. We apply this to the case where $H^T = H$ and the stabilizer is a symmetric subgroup, a case discussed for the continuous wavelet transform in [5].

Introduction

The wavelet transform, and more generally time frequency analysis, has become a widely used and studied tool in mathematics, physics, engineering, and applied science during the last decade. One of the interesting aspects is the role played by abstract harmonic analysis and representation theory of locally compact groups. In wavelet theory one studies square integrable representations of semidirect products $G = \mathbb{R}^n \rtimes_a H$, and in time frequency analysis representations of the Heisenberg group are used to understand Gabor frames built from a lattice $\Gamma \subset \mathbb{R}^{2d}$. In this article we will discuss frames built from the continuous wavelet transform and discrete subsets $\Gamma$ of $G$.

In the language of representation theory the continuous wavelet transform on the line is given by taking the matrix coefficients of the natural representation $\pi$ of the $(ax + b)$-group, i.e., the group of dilations and translations on the line, on the Hilbert space $L^2(\mathbb{R})$. Thus

$$\pi(a, b)\psi(x) = |a|^{-1/2}\psi\left(\frac{x - b}{a}\right) = T_b D_a \psi(x)$$

and

$$W\psi(f)(a, b) = (f | \pi(a, b)\psi) = |a|^{-1/2} \int_{\mathbb{R}} f(x)\psi\left(\frac{x - b}{a}\right) \, dx.$$  \hfill (0.1)

1991 Mathematics Subject Classification. 42C40,43A85.

Key words and phrases. Wavelet transform, frames, Lie groups, square integrable representations, reductive groups.

Research supported by NSF grants DMS-0070607 and DMS-0139783.
Here $T_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ stands for the unitary isomorphism corresponding to translation $T_b f(x) = f(x - b)$ and $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unitary map corresponding to dilation $D_a f(x) = |a|^{-1/2}f(x/a)$, $a \neq 0$.

The discrete wavelet transform is obtained by sampling the wavelet transform, given by a suitable wavelet $\psi$, of a function $f$ at points gotten by replacing the full $(ax + b)$-group by a discrete subset generated by translation by integers and dilatations of the form $a = 2^n$:

$$W^d_{\psi}(f)(2^{-n}, -2^{-n}m) = \left( f \mid \pi((2^n, m)^{-1})\psi \right) = 2^{n/2} \int_{\mathbb{R}} f(x)\overline{\psi(2^n x + m)} \, dx.$$ 

Hence, the corresponding frame is

$$(0.2) \quad \{ \pi((2^n, m)^{-1})\psi \mid n, m \in \mathbb{Z} \}.$$ 

The inverse refers here to the inverse in the $(ax + b)$-group.

This observation, in particular (0.1) is the basis for the generalization of the continuous and discrete wavelet transform to higher dimensions and more general settings. For the continuous wavelet transform the relation to representation theory of the $(ax + b)$-group was already pointed out by Grossmann, Morlet, and Paul in 1985 [16, 17]. Since then several people have worked on wavelets related to actions of topological groups acting on $\mathbb{R}^n$. Without trying to be complete we would like to name the work of Ali, Antoine, and Gazeau, [1, 2], Bernier and Taylor [4], Führ and Führ and Mayer [11, 12, 13, 14], and finally Laugesen, Weaver, Weiss, and Wilson [19]. In most of these cases the group generalizing the $(ax + b)$-group is a semidirect product $\mathbb{R}^n \times_s H$, where $H$ is a closed subgroup of $\text{GL}(n, \mathbb{R})$. For the continuous wavelet transform one often assumes that the group $H$ has open orbits $O_1, \ldots, O_r$ such that the complement of their union has measure zero. As a further condition for the existence of wavelet functions, or admissible functions, one needs that for $\omega$ in an open orbit the stabilizer $H^\omega = \{ h \in H \mid (h^{-1})^T(\omega) = \omega \}$ is compact. In [8] the case where this condition is not satisfied were discussed. Instead it was assumed that $H$ is reductive, $H^T = H$, and that the stabilizer group $H^\omega$ is a symmetric subgroup, i.e., there exists an involutive automorphism $\tau : H \to H$ such that $H_\tau^\omega \subset H^\omega \subset H^\tau$. For any given open orbit $O$ we then able to construct a group $Q = ANR$ such that the following holds:

1. There exists points $\omega_0, \ldots, \omega_k \in O$ such that the $Q$-orbits $O_j = Q^T(\omega_j)$, $0 \leq j \leq k$, are open and $O \setminus (O_1 \cup \ldots \cup O_k)$ has measure zero.
2. The group $Q$ has the form $ANR$ where $A$ is simply connected and abelian, $N$ is simply connected and unipotent, and $R$ is compact and containing the stabilizer of $\omega_j$, $0 \leq j \leq k$. In particular the stabilizer of $\omega_j$ in $Q$ is compact.

Our aim in this article is to use the special structure of the group $Q = ANR$ listed above, to construct frames for $L^2(\mathbb{R}^n)$ generalizing (0.2). Our ideas are based on the article [4] by Bernier and Taylor, but it should be pointed out that several of the ideas
in [2] are based on previous work of others. We would like to mention the article by Daubechies, Grossmann, and Meyer [4], the work of Feichtinger and Gröchenig [10, 15], Al, Antoine, and Gazeau [2, 4], and finally the work of Heil and Walnut [18].

In [4] the authors considered a subgroup \( H \subset \text{GL}(n, \mathbb{R}) \) as above and assume that \( H \) acts freely on \( \mathbb{R}^n \), i.e., the stabilizer group is trivial. Define an action of \( H \) on \( \mathbb{R}^n \) by \( a \cdot x = (a^{-1})^T(x) \). The authors introduced the notion of separated sets and frame generators. A separated set \( \Gamma \) is a subset of \( H \) such that there exist a compact set \( B \subset \mathcal{O} \), \( B^o \neq \emptyset \), such that \( a \cdot B \cap b \cdot B \neq \emptyset \) implies that \( a = b \). In particular all the translates \( a^T B \), \( a \in \Gamma \), of \( B \) are disjoint. A frame generator is a pair \((\Gamma, \mathbb{F})\) where \( \Gamma \subset H \) is separated and \( \mathbb{F} \) is a compact subset of \( \mathcal{O} \) such that \( \bigcup_{a \in \Gamma} a \cdot \mathbb{F} = \mathcal{O} \). In [4] the authors show, that if \((\Gamma, \mathbb{F})\) is a frame generator, then there exist a function \( \psi \) and a discrete set \( \{v(m) \in \mathbb{R}^n \mid m \in \mathbb{Z}^n\} \) such that the set \( \{\pi((a,v(m))^{-1}) \psi \mid a \in \Gamma, m \in \mathbb{Z}^n\} \) is a frame for the Hilbert space of \( L^2 \)-functions whose Fourier transform is supported in \( \mathcal{O} \).

In the first part of this article we show that the same construction can be carried out if the action is not free, but the stabilizer group is compact. Motivated by the construction in [4] we apply this to groups of the form ANR as in (2) above except we do not need to assume that \( N \) is simply connected. The assumption needed is, that \( N \) contains a discrete subgroup \( \Gamma \) such that \( \Gamma \setminus N \) is compact. In this case we can carry out the construction by Bernier and Taylor to get a frame related to an open ANR-orbit, c.f. Theorem 4.1 and Theorem 4.2. We recall in section 4 the construction from [8] and explain it using the action of \( \text{GL}(n, \mathbb{R}) \) on the space of symmetric matrices.

Then the author would like to thank C. Heil and G. Weiss for helpful comments and corrections.

1. Separated sets

Let \( H \) be a closed subgroup of \( \text{GL}(n, \mathbb{R}) \). Then \( H \) acts in a natural way on \( \mathbb{R}^n \). We will also consider the action

\[
(h, v) \mapsto h \cdot v := (h^{-1})^T(v)
\]

of \( H \) on \( \mathbb{R}^n \). Here \( a^T \) denotes the transpose of the matrix \( a \in \text{GL}(n, \mathbb{R}) \). We denote by \( \theta : \mathbb{R}^n \to \mathbb{R}^n \) the homomorphism \( \theta(h) = (h^{-1})^T \). For simplicity we will also write \( h^\theta = \theta(h) \). We assume that there exist an open orbit \( \mathcal{O} \subset \mathbb{R}^n \) under the twisted action \( (h, v) \mapsto h^\theta(v) \). For \( \omega \in \mathcal{O} \) let

\[
H^\omega := \{h \in H \mid h \cdot \omega = \omega\}
\]

be the stabilizer of \( \omega \) in \( H \). Notice that \( H^\omega = \{h \in H \mid h^T(v) = v\} \) as \( H^\omega \) is a subgroup of \( H \). Because of the applications that we have in mind, we assume from now on that \( H^\omega \) is compact. The following definition is from [4]:

**Definition 1.1.** Let \( H \) be a locally compact Hausdorff topological group acting on the locally compact Hausdorff topological space \( X \). A subset \( \Gamma \subset H \) is called separated if there exist a compact set \( B \subset X \) such that \( B^o \neq \emptyset \) and \( h \cdot B \cap k \cdot B = \emptyset \) for all \( h, k \in \Gamma, h \neq k \). We then say that \( \Gamma \) is separated by \( B \).
Example 1.2. Let $H = \mathbb{R}^+ \text{SO}(n)$. Let $A \subset \text{SO}(n)$ be a non-empty subset and let $\lambda > 1$.

$$\Gamma := \{ \lambda^k a \mid k \in \mathbb{Z}, a \in A \}.$$ 

Let $0 < \alpha < \beta$ be such that $\lambda \alpha > \beta$ and define

$$B = \{ v \in \mathbb{R}^n \mid \alpha \leq \|v\| \leq \beta \}.$$ 

Then $B$ is compact with non-empty interior. If $b = \lambda^k a \in \Gamma$ then

$$b \cdot B = \{ v \in \mathbb{R}^n \mid \lambda^{-k} \alpha \leq \|v\| \leq \lambda^{-k} \beta \}.$$ 

Suppose that $k \leq m$ and that $\lambda^k \text{SO}(n) \cdot B \cap \lambda^m \text{SO}(n) \cdot B$. Then $\lambda^{-k} \alpha \leq \lambda^{-m} \beta$ and, hence,

$$\lambda^{m-k} \alpha \leq \beta$$

which is only possible if $m - k = 0$. It follows that $\Gamma$ is separated by $B$.

Fix from now on $\omega_0 \in \mathcal{O}$ and recall that we are assuming that $L := \{ h \in H \mid h^T(\omega_0) = \omega_0 \}$ is compact. We can always assume that $\omega_0 \in B^o$. Otherwise take $b \in H$ such that $b \cdot \omega_0 \in B^o$. Thus $\omega_0 = b^{-1} \cdot B$. Let $\Gamma' := \Gamma b$ and $B' := b^{-1} \cdot B$. Then for $h, k \in \Gamma$, $h \neq k$ we have

$$(hb) \cdot (b^{-1} \cdot B) \cap (kb) \cdot (b^{-1} \cdot B) = h \cdot B \cap k \cdot B = \emptyset$$

so that $\Gamma'$ is separated by $B'$.

Lemma 1.3. Let $L = \{ h \in H \mid h^\theta(\omega_0) = \omega_0 \}$ and let $\kappa : H \to \mathcal{O}$ be the map $h \mapsto h^\theta(\omega_0)$ that defines an $H$-isomorphism $H/L \simeq \mathcal{O}$. Let $B \subset \mathcal{O}$ be compact. Then $\tilde{B} := \kappa^{-1}(B) \subset H$ is a right $L$-invariant compact subset of $H$ such that $\kappa(\tilde{B}) = B$. Furthermore the following holds:

1. $\omega_0 \in B$ if and only if $e \in \tilde{B}$;
2. $B^o \neq \emptyset$ if and only if $\tilde{B}^o \neq \emptyset$;
3. If $\tilde{B}^o \neq \emptyset$ then $\kappa(\tilde{B}^o) = B^o$ and $\tilde{B}^o$ is right $L$-invariant.

Proof. All of this is well known, but let us go over the argument here. That $\tilde{B}L = \tilde{B}$ follows from the fact that $\kappa(ab) = a^\theta(\kappa(b)) = (ab) \cdot \omega_0$.

(a) follows by $\kappa(e) = \omega_0$.

(b) and (c) follows from the fact that $\kappa$ is open and continuous.

We can assume that $e \in \tilde{B}$. Then $L \subset \tilde{B}$. Let $V \subset H$ be an open neighborhood of $e$ such that $V$ is compact. Then $\kappa(V) \subset \mathcal{O}$ is open and

$$B \subset \bigcup_{g \in H} g \cdot \kappa(V).$$ 

Hence, there are finitely many $g_1, \ldots, g_n$ such that

$$B \subset \bigcup_{j=1}^n g_j \cdot \kappa(V).$$
Let $S := \bigcup_{j=1}^{n} g_j V$. Then $S$ is compact and $B \subset \kappa(S)$. It follows that $\tilde{B} \subset SL$. But $SL$ is compact as the continuous image of the compact set $S \times L \subset H \times H$ under the continuous map $H \times H \to H$, $(a,b) \mapsto ab$. As $\tilde{B}$ is closed it follows that $\tilde{B}$ is compact.

**Lemma 1.4.** Let $\Gamma \subset H$ be a separated set. Let $D \subset O$ be compact. Let $S = \kappa^{-1}(D)$. Then for each $a \in \Gamma$:
\[
\{b \in \Gamma \mid a \cdot D \cap b \cdot D \neq \emptyset\} = \{b \in \Gamma \mid aS \cap bS \neq \emptyset\}.
\]

**Proof.** We have
\[
\kappa(aS \cap bS) = a^\theta(D) \cap b^\theta(D).
\]
Hence, if $aS \cap bS \neq \emptyset$ then $a^\theta(D) \cap b^\theta(D) \neq \emptyset$ and it follows that the right hand side is greater or equal to the left hand side. Assume now that $x \in a^\theta(D) \cap b^\theta(D)$. Then there exists $s,t \in S$ such that $\kappa(as) = \kappa(bt)$. Hence, there exist $h \in L$ such that $as = bth$. As $SL = S$ it follows that $aS \cap bS \neq \emptyset$ and, hence,
\[
\{b \in \Gamma \mid a \cdot D \cap b \cdot D \neq \emptyset\} \leq \{b \in \Gamma \mid aS \cap bS \neq \emptyset\}
\]
finishing the proof.

We have now the necessary tools to prove the main results of this section.

**Theorem 1.5.** Suppose that $\Gamma \subset H$ and that $B \subset O$. Let $\tilde{B} = \kappa^{-1}(B)$. Then $\Gamma$ is separated by $B$ in $O$ if and only of $\Gamma$ is separated by $\tilde{B}$ in $H$ under the natural action of $H$ on $H$ given by left multiplication.

**Proof.** We have already seen that $\tilde{B}$ is compact with $\tilde{B}^o$ non-empty. Assume that $a \tilde{B} \cap b \tilde{B} \neq \emptyset$ for some $a,b \in \Gamma$ then it follows by Lemma 1.4 that $a \cdot B \cap b \cdot B \neq \emptyset$. Hence, $a = b$ as $\Gamma$ is separated by $B$.

**Theorem 1.6.** Suppose that $\Gamma \subset H$ is a separated subset of $H$. Let $D \subset O$ be compact. Then
\[
\sup_{k \in \Gamma} \#\{h \in \Gamma \mid h \cdot D \cap k \cdot D \neq \emptyset\} < \infty.
\]

**Proof.** This has been proved in [4] for the case where the action of $H$ is free. Using Lemma 1.4 and Theorem 1.5 the general statement is reduced to that case and, hence, the claim.

---

2. THE CONTINUOUS WAVELET TRANSFORM

In this section we review some basic facts about the continuous wavelet transform on $\mathbb{R}^n$ with respect to a group action, see [8, 16, 17] for more information and references. Denote by $\text{Aff}(\mathbb{R}^n)$ the group of invertible affine linear transformations on $\mathbb{R}^n$. Then $\text{Aff}(\mathbb{R}^n)$ consists of pairs $(x,h)$ such that $h \in \text{GL}(n,\mathbb{R})$ and $x \in \mathbb{R}^n$. The action of $(x,h) \in \text{Aff}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is given by
\[
(x,h)(v) = h(v) + x.
\]
The product is the composition of maps. Thus
\[(x, a)(y, b) = (a(y) + x, ab)\]
and the inverse of \((x, a) \in \text{Aff}(\mathbb{R}^n)\) is given by
\[(x, a)^{-1} = (-a^{-1}(x), a^{-1}).\]
Thus \(\text{Aff}(\mathbb{R}^n)\) is the semidirect product of the abelian group \(\mathbb{R}^n\) and the group \(\text{GL}(n, \mathbb{R})\);
\[\text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \times_s \text{GL}(n, \mathbb{R}).\]
Define a unitary representation of \(\text{Aff}(\mathbb{R}^n)\) on \(L^2(\mathbb{R}^n)\) by
\[(2.1) \quad [\pi(x, a)f](v) = |\det(a)|^{-1/2} f((x, a)^{-1}(v)) = |\det(a)|^{-1/2} f(a^{-1}(v - x)).\]
For \(f \in L^2(\mathbb{R}^n)\) denote by \(\hat{f}\) the Fourier transform of \(f\)
\[\hat{f}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-i(x|\omega)} \, dx, \quad f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).\]
We denote by \(\hat{\pi}(x, a)\) the unitary action on \(L^2(\mathbb{R}^n)\) given by
\[\hat{\pi}(x, a)f(v) = \sqrt{|\det(a)|} e^{-i(x|v)} f(a^T(v)) = \sqrt{|\det(a)|} e^{-i(x|v)} f(a^{-1} \cdot v).\]
The Fourier transform intertwines the representations \(\pi\) and \(\hat{\pi}\).

**Lemma 2.1.** Let \(f \in L^2(\mathbb{R}^n)\) and \((x, a) \in \text{Aff}(\mathbb{R}^n)\). Then
\[\pi(x, a)f(\omega) = \hat{\pi}(x, a)\hat{f}(\omega).\]

Let \(H \subset \text{GL}(n, \mathbb{R})\) be a closed subgroup. Denote by \(G := \mathbb{R}^n \times_s H\) the subgroup of \(\text{Aff}(\mathbb{R}^n)\) given by
\[G = \{(x, a) \in \text{Aff}(\mathbb{R}^n) \mid a \in H\}.
We assume that there exists open sets \(\{O_j\}_{j \in J}\), where \(J\) is a finite or countably infinite index set, such that

- (W1) Each \(O_j\) is invariant and homogeneous under the action of \(H\) given by \((a, v) \mapsto a \cdot v = a^0(v)\);
- (W2) We have \(O_i \cap O_j = \emptyset\) if \(i \neq j\);
- (W3) The complement of \(\bigcup_{j \in J} O_j\) has measure zero with respect to the Lebesgue measure on \(\mathbb{R}^n\).

For a measurable function \(f\) denote by \(\text{Supp}(f)\) the complement of the maximal open set \(U \subset \mathbb{R}^n\) such that \(f(x) = 0\) for almost all \(x \in U\). For \(\emptyset \neq U\) open in \(\mathbb{R}^n\) denote by \(L^2_U(\mathbb{R}^n)\) the closed subspace of \(L^2(\mathbb{R}^n)\) given by
\[(2.2) \quad L^2_U(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid \text{Supp}(\hat{f}) \subset \overline{U}\}.
Suppose that \(U\) is \(H\)-invariant under the twisted action \(h \cdot v = h^0(v)\), then by \(2.1\) it follows that \(L^2_U(\mathbb{R}^n)\) is \(G\)-invariant. Furthermore, by Theorem 3.4 in [8], \(L^2_U(\mathbb{R}^n)\) is irreducible if
and only if $U$ is homogeneous. Hence, the decomposition of $L^2(\mathbb{R}^n)$ into irreducible parts is given by

\begin{equation}
L^2(\mathbb{R}^n) \simeq \bigoplus_{j \in \mathcal{J}} L^2_{\mathcal{O}_j}(\mathbb{R}^n).
\end{equation}

Denote by $dh$ a left invariant Haar measure on $H$. Then a left invariant Haar measure on $G$ is given by $dg = (2\pi)^{-n} |\det(a)|^{-1} dadv$.

**Definition 2.2.** Suppose that $\emptyset \neq U$ is an open subset of $\mathbb{R}^n$. Then a nonzero function $f \in L^2_{\mathcal{O}_j}(\mathbb{R}^n)$ is called **admissible** if $\pi_{g,f}(h) := (g \mid \pi(x,a)f)$ is in $L^2(G)$ for all $g \in L^2_{\mathcal{O}_j}(\mathbb{R}^n)$.

A simple calculation, see [8], shows that

\begin{equation}
\int_G |(g \mid \pi(x,a)f)|^2 \frac{dadx}{|\det(a)|} = (2\pi)^n \int_U |\hat{g}(\omega)|^2 \int_H |\hat{f}(h^T(\omega))|^2 dhd\omega
\end{equation}

In particular, if $U$ is homogeneous, then $C_f = \int_H |\hat{f}(h^T(\omega))|^2 dh$ is independent of $\omega \in U$ and, hence,

\[ \int_G |(g \mid \pi(x,a)f)|^2 \frac{dadx}{|\det(a)|} = C_f \|g\|^2. \]

In particular $f$ is admissible if and only if $H \ni h \mapsto f(h^T(\omega)) \in \mathbb{C}$ is in $L^2(H)$, which in particular implies that the condition

(W4) For all $\omega \in U$ we have that $H^\omega = \{h \in H \mid h^T(\omega) = \omega\}$ is compact

has to be satisfied.

### 3. Separated sets and Frames

In this section we recall some basic facts from [4] on how to construct frames from the continuous wavelet transform using separating sets. Let us also recall that we are assuming that $H \subset \text{GL}(n, \mathbb{R})$ is closed and that $\mathcal{O}$ is a homogeneous open subset of $\mathbb{R}^n$ such that the condition (W4) is satisfied.

Let us start with the well known definition:

**Definition 3.1.** Let $H$ be a Hilbert space. A sequence $\{v_n\}$ in $H$ is called a **frame** if there exits numbers $A, B > 0$ such that for all $v \in H$ we have

\[ A\|v\|^2 \leq \sum_n |(v \mid v_n)|^2 \leq B\|v\|^2. \]

The numbers $A$ and $B$ are called **frame bounds**.
Definition 3.2. Let $H$ be a locally compact Hausdorff topological group acting transitively on the locally compact Hausdorff topological space $X$. A frame generator is a pair $(\Gamma, F)$ where $\Gamma$ is a countable separated subset of $H$ and $F$ is a compact subset of $X$ such that

\[(3.1) \quad X = \bigcup_{a \in \Gamma} a \cdot F.\]

In our case we will take $X = O$ or $X = H$. Notice (3.1) implies in this case that $F^o \neq \emptyset$ and that for each $a \in \Gamma$ we have $\#\{b \in \Gamma \mid a \cdot F \cap b \cdot F \neq \emptyset\} < \infty$. Notice also that $(\Gamma, F)$ is a frame generator for the action on $O$ if and only if $(\Gamma, \kappa^{-1}(F))$ is a frame generator for the action of $H$ on $H$ by multiplication.

Let us now go back to the situation considered in the previous sections. Let $(\Gamma, F)$ be a frame generator, let $D \subset O$ be a compact subset of $O$ such that $F \subset D^o$. Let $R \subset \mathbb{R}^n$ be a parallelepiped such that $D \subset R$. Choose $a_j < b_j$ ($j = 1, \ldots, n$) and a basis $v_j \in \mathbb{R}^n$ ($j = 1, \ldots, n$) such that

\[R = \left\{ \sum_{j=1}^n x_j v_j \mid a_j \leq x_j \leq b_j \right\}.\]

Let $w_1, \ldots, w_n$ be the dual base to $v_1, \ldots, v_n$, i.e., $(v_i \mid w_j) = \delta_{ij}$. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ define $w(m) \in \mathbb{R}^n$ by

\[w(m) := \frac{1}{b_j - a_j} \sum_{j=1}^n m_j w_j.\]

Finally we define $e_m : \mathbb{R}^n \to \mathbb{C}$ by

\[e_m(v) = \frac{1}{\sqrt{\text{Vol}(R)}} \exp\left( \sum_{j=1}^n 2\pi i (v \mid w(m)) \right) \chi_R(v)\]

where $\chi_F$ denotes the indicator function of a set $F \subset \mathbb{R}^n$. We identify $e_m$ with its restriction to $R$. Then $\{e_m\}_{m \in \mathbb{Z}^n}$ is an orthonormal basis for $L^2(R)$. Let

\[\alpha := \sup_{a \in \Gamma} \#\{b \in \Gamma \mid a \cdot D \cap b \cdot D \neq \emptyset\}.\]

Then $\alpha$ is finite by Lemma 1.6. Let $\varphi \in L^2_O(\mathbb{R}^n)$ be such that:

(F1) $\text{Supp}(\hat{\varphi}) \subset D$;
(F2) $a(\varphi) := \inf_{\omega \in F} |\hat{\varphi}(\omega)| > 0$;
(F3) $b(\varphi) := \sup_{\omega \in D} |\hat{\varphi}(\omega)| < \infty$.

In particular we could take $\varphi$ such that $\hat{\varphi} = \chi_F$.

Theorem 3.3 (Bernier-Taylor). Assume that $(\Gamma, F)$ is a frame generator and that $\varphi \in L^2_O(\mathbb{R}^n)$ satisfies the conditions (F1), (F2), and (F3). Then, with the above notation, the sequence

\[\{\pi((a, w(m))^{-1})\varphi\}_{(a, m) \in \Gamma \times \mathbb{Z}^n}\]

is a frame for $L^2_O(\mathbb{R}^n)$ with frame bounds $A = \text{Vol}(R)a(\varphi)^2$ and $B = \text{Vol}(R)ab(\varphi)^2$. 
Proof. See [4], Theorem 3. □

**Example 3.4.** Let \( H = \mathbb{R}^+ \text{SO}(n) \) as in Example 1.2. Then \( H \) has two orbits, \( \{0\} \) and \( \mathcal{O} := \mathbb{R}^n \setminus \{0\} \) in \( \mathbb{R}^n \). Notice that \( L_2^2(\mathbb{R}^n) = L^2(\mathbb{R}^n) \) in this case. If \( u \in \mathcal{O} \) and \( r \neq 1 \) then \( ru \neq u \) and, hence, \( H^\omega \) is a closed subgroup of \( \text{SO}(n) \) and therefore compact. In fact it is easy to see that \( H^\omega \) is isomorphic to \( \text{SO}(n-1) \) for all \( \omega \in \mathcal{O} \). Thus all the conditions (W1) – (W4) are fulfilled.

Let \( \lambda > 1 \) and let \( \Gamma = \{ \lambda^n \mid n \in \mathbb{Z} \} \). Then \( \Gamma \) is separated by example 1.2. Choose \( \rho < \sigma \) such that \( \lambda \rho \leq \sigma \) and define \( \mathcal{F} = \{ v \in \mathbb{R}^n \mid \rho \leq \|v\| \leq \sigma \} \). Let \( v \in \mathcal{O} \). Then there exist an \( n \in \mathbb{Z} \) such that \( \lambda^n \rho \leq \|v\| < \lambda^{n+1} \rho \leq \lambda^n \sigma \). Hence, \( v \in \lambda^{-n} \mathcal{F} = \Gamma \cdot \mathcal{F} \). Thus \((\Gamma, \mathcal{F})\) is a frame generator.

**Example 3.5.** In this example we consider a case of a subgroup \( H \subset \mathbb{R}^+ \text{SO}(1, n) \) acting on \( \mathbb{R}^{n+1} \) which is more complicated than the example 3.4. But our construction relies on the fact that the action on each of the open orbits is free, i.e., the stabilizer is trivial.

For \( \lambda \in \mathbb{R}^+ = \{ r \in \mathbb{R} \mid r \neq 0 \} \), \( t \in \mathbb{R} \), and \( x \in \mathbb{R}^{n-1} \) define \( a(\lambda, t), n(x) \in \text{GL}(n+1, \mathbb{R}) \) by

\[
a(\lambda, t) = \lambda \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix}
\]

and

\[
n(x) = \begin{pmatrix} 1 + \frac{1}{2} \|x\|^2 & x^T & \frac{1}{2} \|x\|^2 \\ x & I_{n-1} & x \\ -\frac{1}{2} \|x\|^2 & -x^T & 1 - \frac{1}{2} \|x\|^2 \end{pmatrix}
\]

and let

\[
A := \{ a(\lambda, t) \mid \lambda > 0, t \in \mathbb{R} \} \quad \text{and} \quad N = \{ n(x) \mid x \in \mathbb{R}^{n-1} \}.
\]

Then \( A \) and \( N \) are abelian groups. But calculations are in fact easier using the corresponding Lie algebras, which are abelian and isomorphic to \( \mathbb{R}^2 \), respectively \( \mathbb{R}^{n-1} \). For that let

\[
H(s, t) = s I_{n+1} + s (E_{1,n+1} + E_{n+1}) \quad \text{and} \quad X(x) = \begin{pmatrix} 0 & x^T & 0 \\ x & 0 & x \\ 0 & -x^T & 0 \end{pmatrix},
\]

where \( x \in \mathbb{R}^{n-1} \) and \( E_{\nu\mu} = (\delta_{\nu \delta \rho})_{ij} \). Define

\[
a = \{ H(s, t) \mid s, t \in \mathbb{R} \} \simeq \mathbb{R}^2 \quad \text{and} \quad n = \{ X(x) \mid x \in \mathbb{R}^{n-1} \} \simeq \mathbb{R}^{n-1}.
\]

Denote by

\[
X \mapsto \exp(X) = e^X = \sum_{j=0}^{\infty} \frac{X^j}{j!}
\]

the matrix exponential function. Then

\[
e^{H(s, t)} = a(e^s, t) \quad \text{and} \quad e^{X(x)} = n(x).
\]
Furthermore exp : \( \mathfrak{a} \to A \) and exp : \( \mathfrak{n} \to N \) is a group homomorphism, i.e., in both cases we have \( e^{X+Y} = e^X e^Y \) (\( X, Y \in \mathfrak{a} \) or \( X, Y \in \mathfrak{n} \)), hence, the multiplication in \( A \), respectively, \( N \) can be reduced to the usual addition in \( \mathbb{R}^2 \), respectively \( \mathbb{R}^{n-1} \).

A simple calculation shows that

\[
a(\lambda, t)n(x)a(\lambda, t)^{-1} = n(e^{-t}x).
\]

In particular it follows that \( H = AN = NA \) is a closed subgroup of GL\((n+1, \mathbb{R})\) with \( N \) a normal subgroup. Next we notice that \( a(\lambda, t)\theta = a(\lambda, t)^{-1} = a(\lambda^{-1}, -t) \), \( n(x)^{-1} = n(-x) \), and

\[
n(x)^{\theta} = n(-x)^T = \begin{pmatrix} 1 + \frac{1}{2}||x||^2 & -x^T & -\frac{1}{2}||x||^2 \\ -x & I_{n-1} & x \\ \frac{1}{2}||x||^2 & -x^T & 1 - \frac{1}{2}||x||^2 \end{pmatrix}.
\]

Hence, the twisted action of \( a(\lambda, t) \) and \( n(x) \) is given by

\[
a(\lambda, t) \cdot v = \lambda^{-1}(\cosh(t)v_1 - \sinh(t)v_{n+1}, v_2, \ldots, v_n, -\sinh(t)v_1 + \cosh(t)v_{n+1})^T
\]

and

\[
n(x) \cdot v = v + (v_1 - v_{n+1})(\frac{1}{2}||x||^2, -x^T, \frac{1}{2}||x||^2)^T - \left( \sum_{j=1}^{n-1} x_j v_{j+1} \right) (1, 0, \ldots, 0, 1)^T.
\]

In particular, if we take \( v = e_1 \) and \( v = e_{n+1} \), where \( \{e_j\}_{j=1}^{n+1} \) is the standard basis of \( \mathbb{R}^n \), we get

\[
n(x)a(\lambda, t) \cdot e_1 = \lambda^{-1}(\cosh(t) + \frac{e^t}{2}||x||^2, -e^tx^T, -\sinh(t) + \frac{e^t}{2}||x||^2)^T
\]

and

\[
n(x)a(\lambda, t) \cdot e_{n+1} = \lambda^{-1}(-\sinh(t) - \frac{e^{-t}}{2}||x||^2, e^{-t}x^T, \cosh(t) - \frac{e^{-t}}{2}||x||^2)^T.
\]

Notice that the stabilizer of \( e_1 \) and \( e_{n+1} \) is trivial. Let \( \beta \) be the bilinear form

\[
\beta(v, w) = v_1 w_1 - \sum_{j=2}^{n+1} v_j w_j.
\]

We can now describe the four open \( H \)-orbits:

\[
\mathcal{O}_1 = \{ v \in \mathbb{R}^{n+1} \mid \beta(v, v) > 0, v_1 > 0 \} = H \cdot e_1
\]

\[
\mathcal{O}_2 = \{ v \in \mathbb{R}^{n+1} \mid \beta(v, v) > 0, v_1 < 0 \} = H \cdot (-e_1)
\]

\[
\mathcal{O}_3 = \{ v \in \mathbb{R}^{n+1} \mid \beta(v, v) < 0, v_1 < v_{n+1} \} = H \cdot e_{n+1}
\]

\[
\mathcal{O}_4 = \{ v \in \mathbb{R}^{n+1} \mid \beta(v, v) < 0, v_1 > v_{n+1} \} = H \cdot (-e_{n+1})
\]

The complement of \( \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_4 \) is given by

\[
\mathbb{R}^{n+1} \setminus (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_4) = \{ v \in \mathbb{R}^{n+1} \mid \beta(v, v) = 0 \} \cup \{ v \in \mathbb{R}^{n+1} \mid \beta(v, v) < 0, v_1 = v_{n+1} \}
\]
which obviously has measure zero in \( \mathbb{R}^n \). Thus (W1) – (W4) holds. Notice also, that if we allow \( \lambda \) to take positive and negative values, i.e., replace \( H \) by the non-connected group \( \{ a(\lambda, t) \mid \lambda \in \mathbb{R}^*, t \in \mathbb{R} \} N \); then there are only two open orbits, \( \mathcal{O}_1 \cup \mathcal{O}_2 \) and \( \mathcal{O}_3 \cup \mathcal{O}_4 \).

Define now
\[
\Gamma_A = \{ \exp(H(n,m)) \mid n, m \in \mathbb{Z} \} \quad \text{and} \quad \Gamma_N = \{ \exp X(x) \mid x \in \mathbb{Z}^n \} .
\]
Then \( \Gamma_A \) and \( \Gamma_N \) are discrete subgroups of \( A \) respectively \( N \) and \( N/\Gamma_N \) is compact. Let (3.2)
\[
\Gamma := \Gamma_A \Gamma_N \subset H .
\]
Then \( \Gamma \) is a discrete subset of \( H \), but notice that \( \Gamma \) is not a group. For \( \epsilon > 0 \) denote by
\[
B_\epsilon = \{ x \in \mathbb{R}^{n-1} \mid \|x\| \leq \epsilon \}.
\]
Choose \( 0 < \delta < 1/4 \) such that \( e^t B_{1/2} \subset B_{3/4} \) for all \( t \in [-\delta, \delta] \). Then \( e^t \mathbb{Z}^{n-1} \cap B_{1/2} = \{0\} \) for all \( |t| \leq \delta \). Let
\[
B = \{ a(e^s, t)n(x) \mid |s| \leq 1/4, |t| \leq \delta, x \in B_{1/2} \} .
\]
Then \( B \subset H \) is compact and \( B^0 \neq \emptyset \). Let \( a, b \in \Gamma \) and assume that \( aB \cap bB \neq \emptyset \). Then there exists \( a(e^r, t)n(x), a(e^s, u)n(y) \in B \), such that \( aa(e^r, t)n(x) = ba(e^s, u)n(y) \). Write \( a = a(e^{n_1}, m_1)n(m_1) \) and \( b = a(e^{n_2}, m_2)n(m_2) \) with \( n_j, m_j \in \mathbb{Z} \), and \( m_j \in \mathbb{Z}^{n-1} \). Then we have
\[
aa(e^r,t)n(x) = a(e^{n_1+r}, m_1 + t)n(e^s m_1 + x) = ba(e^s, u)n(y) = a(e^{n_2+s}, m_2 + u)n(e^s m_2 + y).
\]
But this is only possible if
\[
n_1 + r = n_2 + s, \quad m_1 + t = m_2 + s \quad \text{and} \quad e^s m_1 + x = e^s m_2 + y .
\]
But then \( n_1 - n_2 = s - r \in \mathbb{Z} \cap [-1/2, 1/2] = \{0\} \) and, hence, \( n_1 = n_2 \) and \( s = r \). Similarly it follows that \( m_1 = m_2 \) and \( t = u \). Thus
\[
e^s m_1 + x = e^s m_2 + y
\]
or
\[
e^s(m_1 - m_2) = y - x \in e^s \mathbb{Z}^{n-1} \cap B_{1/2} = \{0\}
\]
which implies that \( m_1 = m_2 \) and \( y = x \). In particular it follows that \( \Gamma \) is separated by \( B \) in \( H \). Let \( \omega_1 = e_1, \omega_2 = -e_2, \omega_3 = e_{n+1} \) and \( \omega_4 = -e_{n+1} \). For \( j = 1, 2, 3, 4 \) the map
\[
H \ni h \mapsto \kappa_j(g) := h^\theta(\omega_j) = h \cdot \omega_j \in \mathcal{O}_j
\]
is a diffeomorphism such that \( \kappa_j(ab) = a \cdot \kappa_j(b) \). It follows that \( \Gamma \) is separated by \( B_j := \kappa_j(B) \). For \( j = 1, \ldots, 4 \) let
\[
\mathbb{F}_j = \kappa_j(\{ a(e^s, t)n(x^T) \mid |s| \leq 1, |t| \leq 1, |x_k| \leq 1 (k = 1, \ldots, n-1) \}) .
\]
Then a simple calculation shows that \( (\Gamma, \mathbb{F}_j) \) is a frame generator.
Example 3.6. Let $H$ be a locally compact Hausdorff topological group, and assume that there exist a countable discrete subgroup $\Gamma \subset H$ such that $\Gamma \setminus H$ is compact. Then there exist a compact subset $K \subset H$ such that $e \in K^c$, $K^{-1} = \{k^{-1} \mid k \in K\} = K$ and $K^2 = \{ab \mid a, b \in K\} \cap \Gamma = \{e\}$. Assume that $a, b \in \Gamma$ and $aK \cap bK \neq \emptyset$. Then $b^{-1}a \in K^2 \cap \Gamma = \{e\}$ and therefore $a = b$. It follows that $\Gamma$ is separated by $K$. As $\Gamma \setminus H$ is compact there exist $F \subset H$ such that $\Gamma F = H$. Thus $(\Gamma,F)$ is a frame generator.

Assume now that $H \subset \text{GL}(n, \mathbb{R})$ is closed and that $O \subset \mathbb{R}^n$ is an open orbit. We assume that there exist $\omega \in O$ such that $\Gamma \cap H\omega = \{e\}$. Let, as usual, $\kappa : H \to O$ be the canonical map $\kappa(a) = a^0(\omega) = (a^{-1})^T(\omega)$. Then $\Gamma$ is separated by $B = \kappa(K)$ and $(\Gamma,\kappa(F))$ is a frame generator. We would expect that by modifying the proof of Theorem 3 in [4] one can remove the condition that $\Gamma \cap H\omega = \{e\}$, which would give several examples of reductive groups acting on $\mathbb{R}^n$. The question is also, if one can remove the condition that $H\omega$ is compact, by $\Gamma \cap H\omega$ is finite and one assumes that $\Gamma \setminus G/H\omega$ is compact.

4. Action of some special groups and frame generators

There are natural examples where $\mathbb{R}^n$ contains finitely many open orbits satisfying (W1) – (W3) but some of which do not have compact stabilizers. In [8] it was shown that in the case where $H$ is reductive, or more simply stated, $H^T = H$ and the stabilizer $L = H^\omega$, $\omega \in O$, is a symmetric subgroup of $H$, then one can always find a subgroup $Q = \text{RAN} = \text{ANR}$ such that $O$ decomposes – up to a set of measure zero – into finitely many open orbits such that (W1) – (W4) holds. More importantly, the structure of the group $Q$ is relatively simple and well understood. In particular we have the following:

1. The map $A \times N \times R \ni (a,n,r) \mapsto anr \in Q$ is a diffeomorphism;
2. $R$ is a compact group and the stabilizer of $\omega$ is contained in $R$;
3. $A$ is abelian and $A$ and $R$ commutes;
4. Both $R$ and $A$ normalize the group $N$;
5. Let $a = \{X \in M(n, \mathbb{R}) \mid \forall t \in \mathbb{R} : e^{tX} \in A\}$

be the Lie algebra of $A$. Then $\exp : a \to A$ is an isomorphism of abelian groups, i.e., $e^{X+Y} = e^X e^Y$ for $X, Y \in a$;
6. Let $n = \{X \in M(n, \mathbb{R}) \mid \forall t \in \mathbb{R} : e^{tX} \in N\}$

be the Lie algebra of $N$. Then $\exp : n \to N$ is a diffeomorphism.

We refer to [8] for the exact construction, but for completeness we recall some of the main constructions in the next section, but first we will show how those facts can be used to construct a separated set and a frame generator. The construction is very much in the spirit of Example 3.5. In fact we do not need the last conditions, so from now on we will assume that $Q = \text{ANR} \subset \text{GL}(n, \mathbb{R})$ is a closed subgroup satisfying the conditions (1) – (5).
Let $H_1, \ldots, H_r$ be a basis for $\mathfrak{a}$. For $t \in \mathbb{R}^r$ let $a(t) := \exp(\sum_{j=1}^r t_j H_j)$. Let
\begin{equation}
\Gamma_A = \{a(m) \in A \mid m \in \mathbb{Z}^r\}.
\end{equation}
Then $\Gamma_A$ is a discrete subgroup of $A$. If $\Gamma_N$ is a discrete subgroup of $N$ let $\Gamma = \Gamma_A \Gamma_N$. Then $\Gamma$ is a discrete subset of $AN$, but in general $\Gamma$ is not a subgroup.

**Theorem 4.1.** Assume that $Q = ANR$ is a closed subgroup of $GL(n, \mathbb{R})$ such that (1) – (5) above holds. Suppose that $\Gamma_N$ is a discrete subgroup of $N$. Set $\Gamma = \Gamma_A \Gamma_N$. Assume that $\mathcal{O} \subset \mathbb{R}^n$ is an open $Q$-orbit such that $\overline{\mathcal{O}} \setminus \mathcal{O}$ has measure zero. Let $\omega \in \mathcal{O}$ and assume that $Q \omega = L \subset R$. Then $\Gamma$ is a separated set.

**Proof.** For $\epsilon > 0$ denote by $B_\epsilon$ the closed ball in $\mathfrak{n}$ with center zero and radius $\epsilon$, $B_\epsilon = \{X \in \mathfrak{n} \mid \text{Tr}(XX^T) \leq \epsilon\}$. Let $K_\epsilon = \exp(B_\epsilon)$. Choose $\epsilon > 0$ such that $\exp : B_{2\epsilon}^0 \rightarrow \exp(B_{2\epsilon}^0)$ is a diffeomorphism, and
\begin{equation}
K_\epsilon^4 \cap \Gamma = \{e\}.
\end{equation}
Choose $0 < \delta \leq 1/4$ such that for $|t_j| \leq \delta$, $j = 1, \ldots, r$, we have
\begin{equation}
\exp(a(t)B_\epsilon a(t)^{-1}) = a(t)K_\epsilon a(t)^{-1} \subset K_\epsilon^2.
\end{equation}
This is possible because the action $A \times \mathfrak{n} \ni (a, X) \mapsto aXA^{-1} \in \mathfrak{n}$ is continuous. For $X \in \mathfrak{n}$ let $n(X) = \exp(X)$. Define (using the obvious notation)
$$B(Q) = \{a(r)n(X)b \mid |r_j| \leq \delta, X \in B_\epsilon, b \in B\} \subset Q.$$ 
Then $B(Q)$ is compact with $B(Q)^0 \neq \emptyset$.

Assume that we have $g_1 = \gamma_1 \eta_1, g_2 = \gamma_2 \eta_2 \in \Gamma$, $\gamma_j \in \Gamma_A$ and $\eta_j \in \Gamma_N$ ($j = 1, 2$), such that $g_1B(Q) \cap g_2B(Q) \neq \emptyset$. Then we can find $a_j = a(r_j) \in \{a(r) \mid |r_j| \leq \delta\} \subset A$, $n_j = n(X_j) \in \{n(X) \mid X \in B_\epsilon\}$, and $b_j \in B$ ($j = 1, 2$), such that
$$\gamma_1 \eta_1 a_1 n_1 b_1 = \gamma_2 \eta_2 a_2 n_2 b_2.$$ 
But then
\begin{equation}
\gamma_1 a_1 ((a_1^{-1} \eta_1 a_1)n_1)b_1 = \gamma_2 a_1 (a_2^{-1} \eta_2 a_2)n_2 b_2.
\end{equation}
As the map $Q \simeq A \times N \times R$ (cf. condition (a)) we must have $\gamma_1 a_1 = \gamma_2 a_2$, $(a_1^{-1} \eta_1 a_1)n_1 = (a_2^{-1} \eta_2 a_2)n_2$, and $b_1 = b_2$. Write $\gamma_j = a(m_j)$ with $m_j \in \mathbb{Z}^r$ ($j = 1, 2$). Then
$$a(m_1 + r_1) = a(m_2 + r_2).$$ 
As the exponential map $\exp : \mathfrak{a} \rightarrow A$ is an isomorphism of groups (cf. (5)) it follows that $m_1 - m_2 = r_2 - r_1 \in \mathbb{Z}^r \cap \{r \in \mathbb{Z}^r \mid |r| \leq \delta\} = \{0\}$. Hence, $m_1 = m_2$ and $r_1 = r_2$. It follows that $\gamma_1 = \gamma_2$ and $a_1 = a_2$. Let $a = a_1 = a_2$. Then we have
$$a^{-1} \eta_1 a(X_1) = a^{-1} \eta_2 a(X_2)$$ 
or (by conjugating by $a$):
$$\eta_1 n(aX_1a^{-1}) = \eta_2 n(aX_2a^{-1}).$$
Write \( \eta = \eta_2^{-1}\eta_1 \in \Gamma_N \). Then we get
\[
\eta = n(aX_2a^{-1})n(-aX_2a^{-1}) \in \Gamma \cap K^4 = \{e\}
\]
by (1.2) and (1.3). But then \( \eta_1 = \eta_2 \) and \( n_1 = n_2 \) showing that \( B(Q) \) separate \( \Gamma \) in \( Q \).

Let \( \kappa : Q \to \mathcal{O}, p \mapsto p \cdot \omega = (p^{-1})^T(\omega) \). Then \( \kappa(pq) = p \cdot \kappa(q) \). Then \( B := \kappa(B(Q)) \) is compact and \( B^o = \kappa(B(Q)^o) \neq \emptyset \). We claim that \( B \) separate \( \Gamma \) in \( \mathcal{O} \). For that assume that there are \( g_1, g_2 \in \Gamma \) such that \( g_1 \cdot B \cap g_2 \cdot B \neq \emptyset \). Then there exists \( b_1, b_2 \in B(Q) \) such that \( g_1 b_1 = g_2 b_2 \). Then it follows that \( g_1 = g_2 \) and, hence, the claim.

Assume now that \( \Gamma_N \) is a discrete subgroup of \( N \) such that \( \Gamma_N \setminus N \) is compact. Choose \( F_N \subset N \) compact such that \( \Gamma_N F_N = N \). Then \( F_N^o \neq \emptyset \). Define \( F_A = \{a(t) \mid \forall j \in \{1, \ldots, r\} : |t_j| \leq 1 \} \) and
\[
F_Q := F_A F_N B \subset Q.
\]
Then \( F_Q \) is compact and \( F_Q^o \neq \emptyset \). Furthermore \( \Gamma F_Q = Q \) as \( \Gamma_A F_A = A, \Gamma_N \subset \{a \gamma a^{-1} \mid a \in F_A, \gamma \in \Gamma_N \} \), and \( \Gamma_N F_N = N \).

**Theorem 4.2.** Assume that \( Q = ANR \) is a closed subgroup of \( GL(n, \mathbb{R}) \) such that (1) – (5) above holds. Suppose that \( \Gamma_N \) is a discrete subgroup of \( N \) such that \( \Gamma_N \setminus N \) is compact. Set \( \Gamma = \Gamma_A \Gamma_N \) and define \( F_Q = F_A F_N B \) as above. Assume further that \( \mathcal{O} \subset \mathbb{R}^n \) is an open \( Q \)-orbit such that \( \mathcal{O} \setminus \mathcal{O} \) has measure zero. Let \( \omega \in \mathcal{O} \) and assume that \( Q^\omega = L \subset R \). Let \( F = F_Q \cdot \omega \). Then \( (\Gamma, F) \) is a frame generator.

**Proof.** This follows from the discussion just before the statement of the Theorem.

By condition (6) we see that the group \( N \) as constructed in [8] is a simply connected nilpotent Lie group. We recall here the most general statement about the existence of co-compact subgroups of nilpotent Lie group, but first let us recall the following definition. Let \( g \) be a Lie algebra and let \( \{X_1, \ldots, X_r\} \) be a basis for \( g \). Then we can write
\[
[X_i, X_j] = \sum_{k=1}^{r} c_{ijk} X_k.
\]
The constants \( c_{ijk} \) \( (1 \leq i, j, k \leq r) \) are called the *structure constants* of \( g \) relative to the basis \( \{X_1, \ldots, X_r\} \).

**Theorem 4.3** (Malcev, 1949). Suppose that \( N \) is a simply connected nilpotent Lie group with Lie algebra \( n \). Then \( N \) contains a co-compact discrete subgroup if and only if \( n \) has a basis with rational structure constants.

The following is also well known and follows from Theorem 4.3. See also [6], p. 511, for proof.

**Theorem 4.4.** Assume that \( N \subset GL(n, \mathbb{R}) \) is unipotent (i.e., the Lie algebra is nilpotent) and defined over \( \mathbb{Q} \). Let \( N_{\mathbb{Z}} := N \cap GL(n, \mathbb{Z}) \). Then \( N_{\mathbb{Z}} \setminus N \) is compact.

We notice also the following simple application of the ideas in the proof of Theorem 4.3 and Theorem 4.2.
Lemma 4.5. Let $H \subset \text{GL}(n, \mathbb{R})$ be a closed subgroup. Assume that $O$ is an open $H$-orbit under the twisted action such that $H^\omega$ is compact for $\omega \in O$. Assume that $\overline{O \setminus O}$ has measure zero and that there exist a co-compact discrete subgroup $\Gamma \subset H$ such that $\Gamma \cap H^\omega = \{e\}$. Then there exits a compact subset $F \subset O$ such that $(\Gamma, F)$ is a frame generator.

Proof. Let $F \subset H$ be a compact subset such that such that $\Gamma F = G$. □

5. Action of reductive groups

In this section we recall the construction from [8] of the group $Q$. This in particular gives us a simple criteria for the existence of a co-compact discrete subgroup $\Gamma_N$ in $N$. We refer to [8] for proofs.

Definition 5.1. A closed subgroup $H \subset \text{GL}(n, \mathbb{R})$ is called reductive if there exist a $x \in \text{GL}(n, \mathbb{R})$ such that $xHx^{-1}$ is invariant under transposition, $a \mapsto a^T$.

We assume from now on that $H$ is reductive. For simplicity we can then assume that $H^T = H$. In order to handle cases where $O = H \cdot \omega$ do not necessarily have compact stabilizers, we will assume that there exist an involution $\tau : H \rightarrow H$ such that $H^\tau \subset H^\omega = \{h \in H \mid h^T(\omega) = \omega\} \subset H^T$ where

$$H^\tau = \{h \in H \mid \tau(h) = h\}$$

and the subscript "o" indicates the connected component containing the identity element. Notice that the involution $\tau$ may depend on the open orbit. We can assume that $L$ is also invariant under transposition. Then $\theta : h \mapsto (h^T)^{-1}$ and $\tau$ commute. Let $K = O(n) \cap H = \{k \in H \mid k^T = k^{-1}\}$. Then $K$ is a maximal compact subgroup of $H$ and $L \cap K$ is a maximal compact subgroup of $L$. Denote by $\mathfrak{h}$ the Lie algebra of $H$, i.e.,

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{s} = \mathfrak{l} \oplus \mathfrak{q} = \mathfrak{k} \cap \mathfrak{l} \oplus \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{s} \cap \mathfrak{l} \oplus \mathfrak{s} \cap \mathfrak{q}$$

where

$$\mathfrak{s} = \{X \in \mathfrak{h} \mid X^T = X\}$$

is the subspace of symmetric matrices, and

$$\mathfrak{q} = \{X \in \mathfrak{h} \mid \tau(X) = -X\}.$$

Notice that

$$[\mathfrak{l}, \mathfrak{q}] \subset \mathfrak{q} \quad \text{and} \quad [\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}.$$
Let \( \Delta = \{ \alpha \in h^a | h^a \neq \{0\} \} \setminus \{0\} \). Notice that the set \( \Delta \) is finite. Hence, there is a \( X, \in a \) such that \( \alpha(X,)=0 \) for all \( \alpha \in \Delta \). Let \( \Delta^+ = \{ \alpha | \alpha(X,)>0 \} \) and Let \( n = \bigoplus_{\alpha \in \Delta^+} h^\alpha \). Let \( m_1 = \{ X \in h | [a, X] = \{0\} \} \), and \( m = \{ X \in m_1 | \forall Y \in a : (X | Y) = 0 \} \). Then \( m_1 = m \oplus a \). Furthermore \( m, n, \) and \( p = m \oplus a \oplus n \) are subalgebras of \( h \) and
\[
\begin{align*}
h &= \mathfrak{k} + p \\
&= l + p.
\end{align*}
\]
Notice that this is not a direct sum in general because \( \mathfrak{k} \cap p = \mathfrak{k} \cap m \) and \( l \cap p = l \cap m \). Let \( m_2 \) be the algebra generated by \( m \cap s \), i.e., \( m_2 = [m \cap s, m \cap s] \oplus m \cap s \). Then \( m_2 \) is an ideal in \( m \) and contained in \( m \cap l \). Let
\[
\mathfrak{r} := \{ X \in m | \forall Y \in m_2 : (X, Y) = 0 \} = m_2^\perp.
\]
Then \( \mathfrak{r} \) is an ideal in \( m \) and \( m = \mathfrak{r} \oplus m_2 \). Let
\[
N_K(a) = \{ k \in K | \forall X \in a : \text{Ad}(a)X \in a \}.
\]

Even if we are not going to use it, we would also like to recall the isomorphism \( q \simeq T_\omega(\mathcal{O}) \) given in the following way. For \( X \in q \) define a derivation \( D_X \) by
\[
D_X(f) := \frac{d}{dt}f(e^{tX} \cdot \omega)|_{t=0}.
\]
Then \( q \ni X \mapsto D_X \in T_\omega(\mathcal{O}) \) is a linear isomorphism. As \( L \) fixes \( \omega \) it follows that \( L \) acts on \( T_\omega(\mathcal{O}) \). Denote by \( \ell_h \) the map \( \ell_h(\eta) = h \cdot \eta \). Then for \( h \in L \) we have
\[
(d\ell_h)_\omega(D_X)f = \frac{d}{dt}f(h e^{tX} \cdot \omega)|_{t=0} = \frac{d}{dt}f(e^{tAd(h)(X)} \cdot \omega)|_{t=0} = D_{Ad(h)(X)}(f).
\]
Hence, the action of \( L \) corresponds to the natural action of \( L \) on \( q \). In particular it follows then that the tangent bundle \( T(\mathcal{O}) \) can be described as the vector bundle \( T(\mathcal{O}) = H \times_L q \).
and
\[ N_{L \cap K}(a) = N_K(a) \cap L. \]

Finally let
\[ M_K = Z_K(a) = \{ k \in K \mid \forall X \in a : aXa^{-1} = X \} \]
and
\[ M_L = Z_{L \cap K} = L \cap M_K(a) \]
Then
\[ W = N_K(a)/M_K \quad \text{and} \quad W_0 = N_{K \cap L}(a)/M_L \subset W. \]
are finite groups For 0 \leq j \leq k = \#W/W_0 choose s_j \in N_K such that s_0 = e and by obvious abuse of notation
\[ W = \bigcup_{j=0}^k s_j W_0 \quad \text{(disjoint union).} \]

Let \( P = \{ a \in H \mid \text{Ad}(a)p = p \} \), \( A = \{ e^X \mid X \in a \} \) and \( N = \{ e^X \mid X \in n \} \). Then \( P, A, \) and \( N \) are closed subgroups of \( H \), and \( A, N \subset P \). Let \( M_2 \) be the group generated by \( \exp(m_2) \), and \( R_o = \exp(r) \). Then \( F = \exp(ia) \cap K \subset M_K \) is finite and such that \( R = FR_o \) is a group. Furthermore
\[ R \times M_2 \times A \ni (r, m, a) \mapsto rma \in Z_H(A) \]
is a diffeomorphism. Notice that by definition \( F \) is central in \( Z_H(A) \) and \( mFm^{-1} = F \) for all \( m \in N_K(a) \). Let \( M = RM_2 \). Then \( P = MAN \). Furthermore each element \( p \in P \) has a unique expression \( p = man \) with \( m \in M, a \in A, n \in N \). The final step is now to define
\begin{equation}
(5.1) \quad Q := RAN = ANR \subset P.
\end{equation}

Then \( Q \) is a closed subgroup of \( H \) with Lie algebra \( q = r \oplus a \oplus n \). Notice that \( h = l + (r \oplus a \oplus n) \) and \( l \cap (r \oplus a \oplus n) = l \cap r \).

By Theorem 4.3 in \[8\] we have the following:

**Theorem 5.2.** Choose \( e = s_0, s_1, \ldots, s_k \in W \) such that \( W \) is the disjoint union of the cosets \( s_j W_0 \). Then
\[ \bigcup_{j=0}^k Qs_j L \subset H \]
is open and dense. Furthermore there exist an analytic function \( \psi : H \to \mathbb{C} \) such that
\[ H \setminus \bigcup_{j=0}^k Qs_j L = \{ h \in H \mid \psi(h) = 0 \}. \]

In particular \( H \setminus \bigcup_{j=0}^k Qs_j L \) has measure zero.
Theorem 5.3. Let the notation be as above. Let $O \subset \mathbb{R}^n$ be an open orbit such that $L = H\omega$ is a symmetric subgroup. Let $O_j = Q \cdot (s_j \cdot \omega)$. Then

$$\bigcup_{j=0}^k O_j \subset O$$

is open and

$$O \setminus \bigcup_{j=0}^k O_j \subset O$$

has measure zero. Furthermore the stabilizer in $Q$ of $\tilde{\omega}$ in $O_j$ ($1 \leq j \leq k$) is compact.

Example 5.4 (GL($n$, $\mathbb{R}$) acting on symmetric matrices). Let $V$ be the space Sym($n$, $\mathbb{R}$) of symmetric $n \times n$ matrices. Then $V \simeq \mathbb{R}^{n(n+1)/2}$ Under this identification the standard inner product on $\mathbb{R}^{n(n+1)/2}$ corresponds to the inner product $(X, Y) := \text{Tr}(XY) = \text{Tr}(XY^T)$ on $V$. Define an action of $H = \text{GL}(n, \mathbb{R})$ on $V$ by

$$a(X) = (a^{-1})^T X a^{-1}.$$

Then

$$a \cdot X = a^\theta(X) = a X a^T.$$

Each symmetric matrix is up to conjugation determined by the signature and rank. The set $V_{\text{reg}} = \{X \in V \mid \det X \neq 0\}$ is open and dense in $V$ and has measure zero. Furthermore each matrix in $V_{\text{reg}}$ is conjugate to one of the matrices $I(0) = I_n$, $I(n) = -I_n$ or

$$I(p) = \begin{pmatrix} I_{n-p} & 0 \\ 0 & I_{n-p} \end{pmatrix}$$

where $1 \leq p \leq n - 1$. Denote the corresponding orbit by $O_p$. Notice that $X \mapsto -X$ defines a $\text{GL}(n, \mathbb{R})$ isomorphism $O_p \simeq O_{n-p}$. The group $O(p, n-p)$ is by definition given by

$$O(p, n-p) = \{g \in \text{GL}(n, \mathbb{R}) \mid gI(p)g^T = I(p)\}$$

where we use the notation $O(n) = O(n, 0) = O(0, n)$. Notice that $O(p, n-p)$ is compact if and only if $p = n$. It follows that only the orbits $O_0$ and $O_n$ satisfy the condition (W4). As before we let $\theta : \text{GL}(n, \mathbb{R}) \to \text{GL}(n, \mathbb{R})$ be the involution $\theta(g) = (g^{-1})^T$. For $j = 0, \ldots, n$ we define $\tau_p : \text{GL}(n, \mathbb{R}) \to \text{GL}(n, \mathbb{R})$ by

$$\tau_p(g) = I(p)\theta(g)I(p).$$

Then

$$\tau_p(g) = g \iff gI(p)g^T = I(p)$$

and, hence,

$$H^{\tau_p} = \{g \in \text{GL}(n, \mathbb{R}) \mid \tau_p(g) = g\} = O(p, n-p) = H^{I(p)}$$

and hence the orbit $O_p \simeq \text{GL}(n, \mathbb{R})/O(p, n-p)$ is a symmetric space.

By abuse of notation we denote the derived involutions on $M(n, \mathbb{R})$ by the same letters, i.e.,

$$\theta(X) = -X^T, \quad \tau_p(X) = -I(p)X^TI(p).$$
Then $\theta(\exp X) = \exp(\theta(X))$ and $\tau_p(\exp X) = \exp(\tau_p(X))$. Define as in the last section:

\[
\begin{align*}
\mathfrak{k} &= \{X \in M(n, \mathbb{R}) \mid \theta(X) = X\} = \mathfrak{o}(n), \\
\mathfrak{s} &= \{X \in M(n, \mathbb{R}) \mid \theta(X) = -X\} = \text{Sym}(n, \mathbb{R}), \\
\mathfrak{h} &= \{X \in M(n, \mathbb{R}) \mid \tau_p(X) = X\} = \mathfrak{o}(p, n-p), \\
\mathfrak{q} &= \{X \in M(n, \mathbb{R}) \mid \tau_p(X) = -X\},
\end{align*}
\]

where we leave out the dependence of $\mathfrak{h}$ and $\mathfrak{q}$ on $p$ as that should be clear in each case. Then

\[
\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{s} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{s} \cap \mathfrak{h} \oplus \mathfrak{s} \cap \mathfrak{q}.
\]

In this case the abelian subalgebra $\mathfrak{a}$ is given by:

\[
\mathfrak{a} = \{d(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in \mathbb{R}\}
\]

where $d(d_1, \ldots, t_n) = d(\mathfrak{k})$ stands for the diagonal matrix with diagonal entries $t_1, \ldots, t_n$. In this case $\mathfrak{a}$ is maximal abelian in $\mathfrak{s}$ and in fact maximal abelian in $\mathfrak{h}$. Hence, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\mathfrak{l}$ is trivial. Thus we will have $P = Q$ in this case.

A simple calculation shows that

\[
[d(\mathfrak{k}), E_{ij}] = (t_i - t_j)E_{ij},
\]

i.e., the matrices $E_{ij}$ are the joint eigenvectors of $\{\text{ad}(d(\mathfrak{k})) \mid d(\mathfrak{k}) \in \mathfrak{a}\}$, with eigenvalues $t_i - t_j$. Define $\alpha_{ij} : \mathfrak{a} \to \mathbb{R}$ by $\alpha_{ij}(d(\mathfrak{k})) = t_i - t_j$. Then $\Delta = \{\alpha_{ij} \mid 1 \leq i, j \leq n, i \neq 0\}$. Let $\Delta^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$. Then $\mathfrak{n}$ is the Lie algebra of upper triangular matrices with zero on the main diagonal:

\[
\mathfrak{n} = \bigoplus_{1 \leq i < j \leq n} \mathbb{R}E_{ij}.
\]

Furthermore $A = \exp(\mathfrak{a}) = \{d(e^{t_1}, \ldots, e^{t_n}) \mid d(\mathfrak{k}) \in \mathfrak{a}\}$ and $N = \exp(\mathfrak{n})$ is the group of upper triangular matrices with one on the main diagonal. So in particular $AN$ is the group of upper triangular matrices with positive diagonal elements. In this case the group $M$ is given by $M = \{d(\epsilon) \mid \epsilon_j = \pm\} \simeq \{-1, 1\}^n$ and $P = MAN$ is the group of regular upper triangular matrices. The group $N_K(A)$ is the finite group of elements with only one coefficient non-zero in each column and row, and that non-zero element is either 1 or $-1$. Let $\mathfrak{S}_n$ be the group of permutations of $\{1, \ldots, n\}$. Then $W$ acts on $\mathfrak{a}$ by

\[
\sigma \cdot d(t_1, \ldots, t_n) = d(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n)}).
\]

Notice that this action can be realized as the conjugation by the orthogonal matrix

\[
s_{ij} = (I_n - E_{ii} - E_{jj}) + E_{ij} - E_{ji} \in M'.
\]

It follows that $W \simeq \mathfrak{S}_n$. The group $L \cap K = \text{O}(p, n-p) \cap \text{O}(n)$ is given by

\[
L \cap K \simeq \text{O}(p) \times \text{O}(n-p).
\]
where the isomorphism is given by

$$(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$ 

Hence, $W_0 \simeq \mathcal{G}_p \times \mathcal{G}_{n-p}$. In particular each of the open $\text{GL}(n, \mathbb{R})$-orbits is decomposed into $\frac{n!}{p!(n-p)!}$ $P$-orbits. Finally we remark that in this example we can take

$$\Gamma_N = N \cap \text{GL}(n, \mathbb{Z})$$

the group of upper triangular matrices with integer coefficients and one on the diagonal.

**Remark 5.5.** Examples of pairs $(H, \mathbb{R}^n)$ such that $H$ is reductive and has finitely many open orbits of full measure is given by the *pre-homogeneous vector spaces of parabolic type* (see [5]). But there are simple examples where the stabilizer is not symmetric. For that let $H = \text{SL}(2, \mathbb{R})$. Then $H$ acts on $\mathbb{R}^2$ in a natural way and the orbits are $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. Hence there is only one open orbit, and that orbit has full measure. The stabilizer of $e_1$ is the group

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

which is not symmetric in $\text{SL}(2, \mathbb{R})$. Let

$$Q = \left\{ \begin{pmatrix} a & 0 \\ y & 1/a \end{pmatrix} \mid a \neq 0, y \in \mathbb{R} \right\}$$

Then $Q$ has three orbits $\{0\}$, $\{(0, y)^T \in \mathbb{R}^2 \mid y \neq 0\}$ and the open orbit $\mathcal{O} = \{(x, y)^T \in \mathbb{R}^2 \mid x \neq 0\}$. The stabilizer of $e_1 \in \mathcal{O}$ is trivial and, hence, we can replace $\text{SL}(2, \mathbb{R})$ by $Q$ to construct frames. Notice that $Q$ is isomorphic to the $(ax + b)$-group.

**Remark 5.6.** Assume that $(H, \mathbb{R}^n)$ is a pre-homogeneous vector space of parabolic type. Then one can show that the same group $Q$ works for all the open orbits [5]. Furthermore the group $Q$ is admissible in the sense of Laugesen, Weaver, Weiss, and Wilson [19] and contains an expansive matrix.

**References**

[1] S.T. Al, J.-P. Antoine, and J.-P. Gazeau: Square integrability of group representations on homogeneous space, I. Reproducing triples and frames. *Ann. Inst. Henri Poincaré* 55 (1991), 829–856
[2] S.T. Al, J.-P. Antoine, and J.-P. Gazeau: *Coherent states, wavelets and their generalizations*. New York, Springer, 2000
[3] P. Aniello, G. Cassinelli, E. De Vito, and A. Levrero: Wavelet transforms and discrete frames associated to semidirect products. *J. Math. Phys.* 39 (1998), 3965–3973
[4] D. Bernier, K. F. Taylor: Wavelets from square-integrable representations. *SIAM J. Math. Anal.* 27 (1996), 594–608
[5] N. Bopp, H. Rubenthaler, *Local Zeta functions attached to the minimal spherical series for a class of symmetric spaces*. Preprint, Strasbourg, 2002
[6] A. Borel, Harish-Chandra: Arithmetic subgroups of algebraic groups. *Ann. Math.* 75 (1962) 485–535
[7] I. Daubechies, A. Grossmann, and Y. Meyer: Painless nonorthogonal expansions. *J. Math. Phys.* 27 (1986), 1271–1283
[8] R. Fabec, G. Ólafsson: The Continuous Wavelet Transform and Symmetric Spaces. To appear in Acta Applicandae Math.

[9] H.G. Feichtinger, and K. Gröchenig: Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view. In C. K. Chui (ed): Wavelets: A tutorial in theory and applications, 359–398. Academic Press, Boston, 1992

[10] H.G. Feichtinger, and K. Gröchenig: Banach spaces related to integrable group representations and their atomic decomposition. J. Funct. Anal. 86 (1989), 307–340

[11] H. Führ: Wavelet frames and admissibility in higher dimensions. J. Math. Phys. 37 (1996), 6353–6366

[12] H. Führ: Continuous wavelet transforms with Abelian dilation groups. J. Math. Phys. 39 (1998), 3974–3986

[13] H. Führ: Admissible vectors for the regular representation. To appear in Proceedings of the AMS

[14] H. Führ, M. Mayer: Continuous wavelet transforms from semidirect products: Cyclic representations and Plancherel measure. To appear in J. Fourier Anal. Appl.

[15] K. Gröchenig: Aspects of Gabor analysis on locally compact abelian groups. In H. G. Feichtinger, and T. Strohmer (ed): Gabor Analysis and Algorithms. Theory and Applications. Birkhäuser 1998

[16] A. Grossmann, J. Morlet, and T. Paul: Transforms associated to square integrable group representations I. General results. J. Math. Phys. 26 (1985), 2473–2479

[17] A. Grossmann, J. Morlet, and T. Paul: Transforms associated to square integrable group representations II. Examples. Ann. Inst. Henri Poincaré: Phys. Theor. 45 (1986), 293–309

[18] C.E. Heil, and D.F. Walnut: Continuous and discrete wavelet transform. SIAM Rev. 31 (1989), 628–666

[19] R.S. Laugesen, N. Weaver, G.L. Weiss, E.N. Wilson: A characterization of the higher dimensional groups associated with continuous wavelets. J. Geom. Anal. 12 (2002), 89–102.

[20] Macev: On a class of homogeneous spaces. Izvestiya Akademii Nauk SSSR Seriya Mathematischeskaia 13 (1949), 9–32

[21] G. Ólafsson: Fourier and Poisson transform associated to semisimple symmetric spaces. Invent. Math. 90 (1987), 605-629

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail address: olafsson@math.lsu.edu