Mark and Marshak boundary conditions in surface harmonics method

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Abstract. In the surface harmonics method (SHM) as a boundary condition it can be used a link of even angular momentum vector of the neutron distribution with odd momentum vector to obtain a finite difference equations. On the border with the vacuum such boundary conditions are called Mark or Marshak boundary conditions. Instead of odd angular momenta the SHM uses their linear combinations (so called neutron levels), so in this paper it is obtained the matrix of connection between odd angular momentum vector and the neutron levels vector (such as Marshak boundary conditions) and matrices of connection between odd angular momentum vector and the neutron levels vector(such as Mark boundary conditions). In present article connection matrix is used in SHM and for computation of the reference neutron distributions in PN - approximation (up to N=97).

1. Introduction

To solve the neutron transport equation in the PN-approximation with vacuum boundary conditions, the Mark and Marshak boundary conditions [1] can be used at the reactor boundary. The authors of the paper turned to the boundary conditions of Mark and Marshak because in the surface harmonics method (SHM, e.g. [1]), in the boundary conditions at the outer boundary of the system (here and in the subsequent formulas, the left one), we use the coupling matrix \( \Gamma \) of odd angular momenta vector \( \Phi \) [1], \( \Phi = -\Gamma I \).

In this paper we obtain matrices \( \Gamma \) for the Mark boundary conditions, including for even PN-approximations. Using the example of test tasks used in [1], the work of Mark and Marshak boundary conditions in even and odd PN-approximations (up to N = 97) of the spherical harmonics method, as well as in different approximations (even and odd, up to \( L^{SHM} = 9 \ )) of the surface harmonics method. In addition, the development of computer technology allows us to consider and compare the work of these boundary conditions in sufficiently high PN-approximations of the method of spherical harmonics.

2. Basic formulas

The Marshak conditions in the one-group flat case are written in the form of integral equalities:
where \( P_i(\mu) \) is the Legendre polynomial, \( \mu = \Omega_x \), \( x_L \) and \( x_R \) the left and right boundary of the plate, \( i = 1, 3, 5, ..., N \) (as a rule, odd \( N \) are used, therefore \((N + 1) / 2\) conditions).

Mark’s conditions are written in the form of zero neutron flux density for some selected directions:

\[
\Phi(x_L, \mu_i) = 0 \quad \mu_i > 0 \quad \Phi(x_R, \mu_i) = 0 \quad \mu_i < 0
\] (2)

In odd \( P_N \)-approximations, it is recommended [1] to take the zeros of the Legendre polynomial \((N + 1)\)th (even) order as the distinguished directions. Since this polynomial has \((N + 1)\) roots, this ensures by \((N + 1)/2\) conditions on each boundary (which is required for closure of the system of equations).

In the following formulas, the following notation will be used for the vectors of odd \( I \) and even \( \Phi \) angular momenta at the boundary of the system:

\[
I = \begin{pmatrix}
\Phi_1(x_L) \\
\vdots \\
\Phi_{2j-1}(x_L)
\end{pmatrix} \\
\Phi = \begin{pmatrix}
\Phi_{1}(x_L) \\
\vdots \\
\Phi_{2j-1}(x_L)
\end{pmatrix}
\]

where \( j \) - the line number, \( J = N / 2 + 1 \) for even and \( J = (N + 1) / 2 \) for odd \( N \).

In addition, in even approximations we have to use combinations of two consecutive even moments. In [2,1] these combinations are called levels. The level vector \( \Phi \) is obtained by multiplying the vector of even moments from the left by the rectangular matrix \( Y \):

\[
\tilde{\Phi} = Y\Phi, \text{ where } Y = \begin{pmatrix}
1 & 2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 2j-1 & 2j & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2J-1 & 2J
\end{pmatrix}
\]

2.1. The boundary conditions of Marshak

The matrices \( \Gamma \) for the boundary conditions of Marshak were obtained in [1]; nevertheless, for completeness and demonstration of the approach, basic formulas and results are given here.

2.1.1. Odd \( P_N \)-approximations (Marshak)

Substituting in (1) the expansion of the neutron flux density on the left boundary along the Legendre polynomials (on the right-hand side the calculations are similar):

\[
\Phi(x_L, \mu) = \sum_{n=0}^{N} \frac{2n+1}{4\pi} \Phi_n(x_L) P_n(\mu)
\]

writing separately summation over even and odd indices \((i = 1, 2, ..., (N + 1)/2)\), using, where possible, the normalization of the Legendre polynomials, we obtain:

\[
\sum_{j=1}^{(N+1)/2} \left[ (4j - 3)p_j \Phi_{2j-1}(x_L) + \Phi_{2j-1}(x_L) \right] = 0
\]

where \( p_j = \int_0^1 P_{2j-1}(\mu) P_{2j-1}(\mu) d\mu \).
Using the previously introduced vectors $I$ of odd and $\Phi$ even moments of dimension $(N+1)/2$ from the last relation, we have $A_{\text{odd}} \Phi + I = 0$.

We introduce a matrix $A_{\text{odd}}$ of dimension $(N+1)/2 \times (N+1)/2$ with elements $(A_{\text{odd}})_{ij} = (4j - 3)p_{ij}$. Then the required relation is written in the form:

$$\Phi = -\Gamma_{\text{odd}} I,$$

where $\Gamma_{\text{odd}} = A_{\text{odd}}^{-1}$ (3)

Generally speaking, the coupling matrix of the even and odd angular momenta on the boundary with the vacuum can be obtained by multiplying (1) by even Legendre polynomials $i = 0, 2, 4, ..., N-1$. In this case we get:

$$\Phi = -\Gamma^{(\text{even})}_{\text{odd}} I,$$ (3.1)

and the matrix $\Gamma^{(\text{even})}_{\text{odd}}$ will be filled with elements $(\Gamma^{(\text{even})}_{\text{odd}})_{ij} = (4j - 1)p_{ij}$ (see the next section).

It should be noted that in this case we do not obtain a limiting transition in the $P_{1}$-approximation to the diffusion approximation, that is, instead of the one used $\Phi_0 = -2\Phi_1$ (from the expressions for the unilateral neutron currents in the diffusion approximation), we get $\Phi_0 = -1.5\Phi_1$, because of which it is difficult to hope for good conditions performance (3.1). Indeed, trial calculations using the boundary conditions (3.1) in the lower $P_N$-approximations strongly lose (in the test tasks described below, up to 2% in the multiplication factor in the $P_1$ and $L_{15}^{\text{SHM}}$ -approximation) similar to the calculations using conditions (3). Therefore, in the following text, conditions (3.1) do not appear.

Note that when using odd $P_N$-approximations, the use of levels is not necessary, because due to the fact that for the last even moment there is no pair to form the level, the cross-linking of the levels reduces to a cross-linking of even angular moments.

2.1.2. Even $P_N$-approximations (Marshak)

In even approximations it is appropriate to multiply and integrate the expression (1) with the Legendre polynomial of even order. We get:

$$\sum_{j=1}^{N/2} \phi_{2j-1}(x_L) \delta_{ij} + \sum_{j=1}^{N/2} (4j - 1)p_{ij} \Phi_{2j-1}(x_L) = 0.$$ (4)

Using the vector of odd angular momenta $I$ of dimension $N/2$, the vector $\Phi$ of even moments of dimension $N/2 + 1$ and a rectangular matrix $A_{\text{even}}$ of dimension $(N/2 + 1) \times N/2$ with elements $(A_{\text{even}})_{ij} = (4j - 1)p_{ij}$, from (4) we have $\Phi = -A_{\text{even}} I$. Multiplying the last relation from the left to the rectangular matrix $Y$ (see above), we obtain the desired relation:

$$\Phi = -\Gamma_{\text{even}} I,$$ where $\Gamma_{\text{even}} = YA_{\text{even}}$

For even $N$ multiplication and integration (1) with a polynomial of odd order does not lead to the desired result because of the impossibility of moving from the even-moment vector to the level vector (since the matrix $Y$ is rectangular and does not have an inverse).

We mention that in the $P_2$-approximation we have $\Phi_0 = -2.25\Phi_1$. 

2.2. Boundary conditions of Mark

In this section, the use of Mark's boundary conditions in even $P_N$-approximations of the spherical harmonics method is new.

2.2.1. Odd $P_N$-approximations (Mark)

When Mark's boundary conditions are formulated, odd $P_N$-approximations (at each boundary $(N + 1)/2$ of conditions) use the fact that a polynomial of even order $P_{N+1}(\mu)$ has $(N + 1)/2$ “neg-
ative" zeros and \((N + 1)/2\) zeros for positive \(\mu\). If we denote these positive roots (the zeros of the polynomial) \(\mu_i^{N+1}\), then the boundary conditions, using (2), can be written (on the left boundary) as follows:

\[
\frac{2n+1}{4\pi} \Phi_n(x_L)P_n(\mu_i^{N+1}) = 0 \quad i = 1, 2, \ldots, (N + 1)/2
\]

Writing separately summation over even and odd indices \(n\), we have:

\[
\sum_{j=1}^{(N+1)/2}[4(j-1)+1]P_{2(j-1)}^{(N+1)}(\mu_i^{N+1})\Phi_{2(j-1)}(x_L) + \sum_{j=1}^{(N+1)/2}[2(2j-1)+1]P_{2j-1}^{(N+1)}(\mu_i^{N+1})\Phi_{2j-1}(x_L) = 0 \quad (5)
\]

Using the previously introduced vectors of odd \(I\) and even \(\Phi\) moments of dimension \((N + 1)/2\), relation (5) is written in the matrix-vector form:

\[
O_{odd} \Phi + T_{odd} I = 0
\]

or

\[
\Phi = -\Gamma_{odd} I, \quad \text{where} \quad \Gamma_{odd} = O_{odd}^{-1}T_{odd} \quad (6)
\]

and the elements of the square matrices \(O_{odd}^{-1}\) and \(T_{odd}\) are calculated by the formulas:

\[
(O_{odd})_{ij} = (4j-3)P_{2(j-1)}^{(N+1)}(\mu_i^{N+1}) \quad (T_{odd})_{ij} = (4j-1)P_{2j-1}^{(N+1)}(\mu_i^{N+1})
\]

When using odd \(P_N\)-approximations, the use of levels is also not necessary, since due to the fact that for the last even moment there is no pair to form the level, the cross-linking of the levels reduces to a cross-linking of even angular moments.

In the \(P_1\)-approximation, we also do not have a limiting transition to the diffusion approximation: instead of the traditional \(\Phi_0 = -2\Phi_1\), we have \(\Phi_0 = -1.73\Phi_1\), we can therefore expect that in the lowest approximations of the spherical harmonics method (and also of the surface harmonics method), calculations using the Mark conditions will underestimate the eigenvalue by compared with the conditions of Marshall.

### 2.2.2. Even \(P_N\)-approximations (Mark)

In an even \(P_N\)-approximation there are as many linearly independent solutions (and unknown coefficients in front of them) as in the previous odd one - therefore, in the boundary conditions, combinations of moments of distribution (levels) of neutrons are used. Therefore, at each boundary it is necessary to put \(N/2\) conditions. At first glance, the same zeros \(\mu_i^N\) of the polynomial \(P_N(x_\mu)\) can be chosen as \(\mu_i\), as in the previous odd approximation. By analogy with (5), we write \((i = 1, 2, \ldots, N/2)\):

\[
\sum_{j=1}^{N/2}[4(j-1)+1]P_{2(j-1)}^{N}(\mu_i^{N})\Phi_{2(j-1)}(x_L) + \sum_{j=1}^{N/2}[2(2j-1)+1]P_{2j-1}^{N}(\mu_i^{N})\Phi_{2j-1}(x_L) = 0 \quad (7)
\]

In the first sum of expression (7) there are \(N/2 + 1\) terms, but the last term contains \(P_N(\mu_i^N)\), which is zero. Therefore, in fact conditions (7) coincide with conditions (5).

This conclusion also says that in this case (for an even \(N\)) we can not form a matrix of coupling of odd moments and levels. To form a level vector, we need to write \(N/2+1\) equations to use the matrix \(Y\) to go to the level vector of dimension \(N/2\). You can consider two options:

- a) to write down the conditions for the distribution of neutrons to be zero \(\mu_i^{N+1}\) for the zeros of the next odd polynomial, but their odd number (one of the roots is necessarily zero), so the equation \(\mu_i^{N+1} = 0\) must be used both on the left and on the right boundary;

- b) as distinguished \(\mu_i\), use \(\mu_i^{N+2}\) - zeros of polynomial \(P_{N+2}(\mu)\), that is, the next even polynomial. These zeros are \(N+2\), therefore at each boundary we can write down the \(\frac{N + 2}{2} = \frac{N}{2} + 1\) equations.
In any case, we can write, for example, on the left boundary \((N + 2)/2\) equations of the type (7):

\[
\sum_{j=1}^{N/2} [4(j-1)+1]P_{2j-1}(\mu^{N+2})\Phi_{2j-1}(x_L) + \sum_{j=1}^{N/2} [2(2j-1)+1]P_{2j-1}(\mu^{N+2})\Phi_{2j-1}(x_L) = 0,
\]

or, in the matrix form:

\[
O_{\text{even}} \Phi + T_{\text{even}} I = 0 \tag{8}
\]

It is important to note that the elements of the matrices \(O_{\text{even}}\) and \(T_{\text{even}}\) are calculated by the same formulas of paragraph 1.2.1 as for matrices \(O_{\text{odd}}\) and \(T_{\text{odd}}\), but the matrix \(O_{\text{even}}\) is square with dimension \(N/2(N+1)/2\), and \(T_{\text{even}}\) is a rectangular \((N/2 \times N\) - row \((N+2)/2, and the columns \(N/2)\).

From (7) we can write \(\Phi = O_{\text{even}}^{-1}T_{\text{even}} I\), and after multiplying by \(Y\) on the left we have the desired relation:

\[
\Phi = -\Gamma_{\text{even}} I, \text{ where } \Gamma_{\text{even}} = YO_{\text{even}}^{-1}T_{\text{even}} \tag{9}
\]

We note that in the \(P_2\)-approximation we have \(\Phi_0 = -2.32\Phi_1\) for the case (8) with a choice of zeros in a), and \(\Phi_0 = -2.23\Phi_1\) for case b). Knowing from the trial calculations that even approximations overestimate the multiplication factor (and the odd ones underestimate), we conclude that it is preferable to use case \(\Phi_0 = -2.23\Phi_1\) (b), and we will not mention case (a) later. We can note that in the lowest even approximations of the method of spherical harmonics Mark's conditions will be somewhat preferable to the conditions of Marshak.

Thus, in the even and odd approximations of the spherical harmonics method (and SHM) we have single-valued boundary conditions for Marshak and Mark. It should be said that when using the Marshak conditions it is necessary to count the elements \(p_{ij}\) once by numerical integration (for example, for the \(P_{97}\)-approximation approximately 2500 numbers), and then this matrix can be used for all lower approximations. In the case of Marc conditions, it is necessary to find the zeros of all even Legendre polynomials (approximately 1250 values using approximations up to \(N=97\)), calculate the polynomial values for the required selected directions (about 2500 values), and use these values to construct individual matrices \(\Gamma\) for each approximation by matrix transformations. In this sense, although the initial number of numbers for obtaining the matrix \(\Gamma\) is the same, but the acquisition of this matrix is more cumbersome in the case of using Mark's conditions. If, on the other hand, the matrix for Mark's conditions is received once "stored" then the number of stored numbers will be approximately 100 times larger for our case, when \(N \leq 97\).

Next, we will compare for each variant of the Mark and Marshak boundary conditions.

3. Test tasks

In solving one-velocity one-dimensional (planar, multilayer) problems by the spherical harmonics method, one can numerically find the roots of the characteristic equation, analytically write the solution in each layer of the system as a linear combination of independent solutions \((N+1\) for odd or \(N/2\) for even approximations). The coefficients of the linear combination (and the multiplication factor-the eigenvalue of the problem) are found from the solution of the system of linear equations of the cross-linking of the distribution of neutrons (angular momenta) on the boundaries of layers and boundary conditions (such as Marshak or Mark). All these procedures are written in VBA.

In the surface harmonics method, as in the homogenization method, layers (periodically repeating) are combined into cells, the neutron distribution in the cell is constructed as a linear combination of trial functions, each of which is calculated in the same approximation as the reference solution, and satisfies certain inhomogeneous boundary conditions. The coefficients of the linear combination (and the multiplication factor - the eigenvalue of the problem) are found from the solution of the system of linear equations of the cross-linking of the distribution of neutrons (angular momenta) at cell bounda-
and boundary conditions (such as Marshak or Mark). It is important that only the lowest angular momenta are cross-linked at the cell boundaries, up to $L_{SHM}$, and including ($L_{SHM} \leq 9$), so it is recommended that cell boundaries be chosen where the angular distribution of neutrons can be described by a small number of angular moments [7].

3.1. Results of the 1st test task

In the first test task from [4], a set of successively located plates of $U$ and $U$-Pu separated by Na layers is considered in planar geometry (figure 1). In the present work, calculations were made for a system with 5 layers $U$ (the number of $U$-Pu plates per unit is less than - 4). In [4], the reference solution was calculated in the S32-approximation, in the present paper we take the result obtained in the P97 approximation for the reference solution. Note that the results of P96-approximation differ by 0.001 in own meaning (see table 1), but for the purposes of this paper it is not significant.

![Figure 1. Geometry and material properties in the test task](image)

We note that even SHM-approximations are introduced in the table (in the same way as in [5, 6]) in order to show (by analogy with the method of spherical harmonics) the differences between the results obtained in the cross-linking at the cell boundaries of “pure” even moments (odd-even approximations), and the “levels” obtained by cross-linking (even approximations).
Table 1. Deviation of the multiplication factor from the reference value in the first test task (in $10^{-5}$)

| № | Option with 5 U-plates | Option with 11 U-plates |
|---|------------------------|------------------------|
|   | SpHM       | SHM       | SpHM       | SHM       | SpHM       | SHM       | SpHM       | SHM       |
| 1 | -4014.7    | -1850.6   | -4993.8    | -2772.9   | -2351.3    | -797.5    | -2939.2    | -1366.4   |
| 2 | 13288.3    | 2100.3    | 13153.1    | 2054.7    | 19863.3    | 521.0     | 19839.0    | 488.9     |
| 3 | -1913.0    | -338.1    | -2233.0    | -639.6    | -1314.3    | -72.6     | -1412.0    | -169.4    |
| 4 | 12182.4    | 462.8     | 12146.9    | 420.1     | 10395.2    | 89.4      | 10378.3    | 69.9      |
| 5 | -1425.2    | -94.6     | -1537.4    | -201.7    | -1142.6    | -19.7     | -1173.7    | -50.5     |
| 6 | 7779.3     | 142.5     | 7753.2     | 113.1     | 6756.8     | 32.1      | 6747.1     | 21.5      |
| 7 | -1213.4    | -34.5     | -1260.4    | -79.8     | -1028.5    | -8.8      | -1042.3    | -22.5     |
| 8 | 5534.8     | 55.8      | 5517.3     | 36.7      | 4859.7     | 15.8      | 4853.9     | 9.5       |
| 9 | -1071.7    | -15.1     | -1095.0    | -37.7     | -928.0     | -4.7      | -928.0     | -12.1     |
| 10| 4192.4     | -        | 4180.5     | -         | 3707.0     | -         | 3703.2     | -         |
| 30| 840.7      | -        | 839.9      | -         | 763.0      | -         | 763.0      | -         |
| 31| -322.5     | -        | -321.8     | -         | -287.8     | -         | -287.8     | -         |
| 45| -159.1     | -        | -159.3     | -         | -142.8     | -         | -142.8     | -         |
| 46| 402.1      | -        | 401.8      | -         | 368.2      | -         | 368.2      | -         |
| 56| 284.4      | -        | 284.2      | -         | 261.6      | -         | 261.6      | -         |
| 57| -87.9      | -        | -88.2      | -         | -79.3      | -         | -79.3      | -         |
| 72| 184.9      | -        | 184.8      | -         | 171.0      | -         | 170.9      | -         |
| 73| -36.5      | -        | -36.8      | -         | -33.1      | -         | -33.1      | -         |
| 96| 118.2      | -        | 118.1      | -         | 109.8      | -         | 109.8      | -         |
| 97| 0.0        | -        | 0.1        | -         | 0.0        | -         | -0.1       | -         |

From the data in table 2, we see the expected results in terms of the spherical harmonics method, namely: in odd approximations Mark's conditions somewhat lose to the Marshak conditions (up to 1% in the multiplication factor in the $P_1$-approximation), but with the increase in the approximation number, this loss becomes completely insignificant - about 0.02% in the $P_9$-approximation, and the angular distribution of neutrons at the boundary with the vacuum is described almost identically - see figure 2.

In even-sided spherical harmonics method approximations, Mark's conditions are slightly more preferable (improvement is less than 0.02%), but the difference from the frame by modulus is significantly (several times) greater than in the previous odd approximations (and convergence seems slower).
Figure 2. Angular distributions of neutrons at the boundary with vacuum under Mark and Marshak conditions for the P_{97}-approximation for a variant with 5 plates.

As for the surface harmonics method, in an odd-to-close approximation, the picture is similar to an spherical harmonics method (Marshak's conditions are preferable), but in even approximations, according to the data in table 2, Mark's conditions work better than the conditions of Marshak, while the multiplication factor is different from frame by modulus is less than in the previous odd SHM-approximation. But even approximations of SHM (and spherical harmonics method) are usually not used, since in even approximations there is a discontinuity in the neutron flux density (integrated in the directions of neutron flight), and this, as a rule, does not like users (not physically). For illustration, figure 3 shows the differences in the total neutron flux density in $L_3^{\text{SHM}}$ - and $L_4^{\text{SHM}}$-approximations from the reference (for a single average normalization).

Figure 3. Differences from the reference neutron flux density in the $L_3^{\text{SHM}}$ - and $L_4^{\text{SHM}}$-approximations for the version with 5 plates.
Indeed, with the best value of the eigenvalue, in figure 3, the discontinuities with a large scale ±0.06 $\Gamma_4^{\text{SHM}}$ - approximation (in the $\Gamma_4^{\text{SHM}}$ - approximation ±0.02) are conspicuous. It is because of them that a greater preference is given to odd approximations (matching of pure even moments).

3.2. The results of the 2nd test task
If the single-group cross sections in the first test task of [4] correspond to a fast reactor, then in the second test task, plates with single-section cross sections corresponding to water and MOX fuel in a reactor on thermal neutrons (5, 10, 20, 40 plates with MOX fuel). The most difficult to calculate by the method of homogenization (and the method of surface harmonics method) are variants with a small number of plates (cells). Therefore, in this paper we present the results of calculating the eigenvalue (the neutron multiplication factor) in variants with the number of plates 5 and 10. The solution is the solution of the problem in $P_{81}$-approximation. The results of calculating these variants (the deviation of the multiplication factor from the reference value) in different approximations of the spherical and surface harmonics method with different types of boundary conditions are given in table 2.

| №   | Option with 5 MOX-plates | Option with 10 MOX-plates |
|-----|--------------------------|--------------------------|
|     | Marshak | Mark | Marshak | Mark |
| SpHM | SHM | SpHM | SHM | SpHM | SHM | SpHM | SHM |
| 1   | -1861.3 | -1001.5 | -2934.3 | -1980.2 | -393.6 | -130.9 | -578.6 | -307.2 |
| 2   | 2316.3 | 774.4 | 2253.9 | 706.1 | 590.2 | 98.7 | 578.0 | 85.8 |
| 3   | -510.8 | -122.7 | -676.2 | -293.7 | -139.7 | -16.4 | -164.3 | -41.9 |
| 4   | 764.2 | 226.2 | 731.2 | 191.8 | 194.7 | 31.1 | 189.3 | 25.5 |
| 5   | -246.6 | -47 | -314.1 | -117.3 | -66.6 | -6.6 | -76.5 | -16.9 |
| 6   | 352.6 | 108.9 | 333.5 | 89.1 | 88.3 | 15.3 | 85.2 | 12.1 |
| 7   | -123.6 | -25.8 | -159.5 | -63.7 | -32.9 | -3.6 | -38.1 | -9.1 |
| 8   | 191.8 | 63 | 179.2 | 50.0 | 47.5 | 9.1 | 45.5 | 7.1 |
| 9   | -67.3 | -16.4 | -89.3 | -62.1 | -17.8 | -2.3 | -21.0 | -5.7 |
| 30  | 13.3 | - | 12.1 | - | 3.3 | - | 3.1 | - |
| 45  | -1.6 | - | -2.2 | - | -0.4 | - | -0.5 | - |
| 56  | 4.4 | - | 4.0 | - | 1.1 | - | 1.0 | - |
| 73  | -0.2 | - | -0.8 | - | -0.1 | - | -0.2 | - |
| 80  | 2.5 | - | 2.3 | - | 0.6 | - | 0.6 | - |

The second test task is easier for calculating both the spherical harmonics method and the surface harmonics method (homogenization method). We see that according to the results of the calculation of the second test task, we can draw the same conclusions as in Section 2.1.

4. Conclusion
The numerical values necessary for the formulation of the Mark and Marshak boundary conditions in the form of a coupling matrix for the even and odd angular momenta of the neutron distribution $\Phi = -\Gamma I$ at the boundary of the system are obtained. It is shown that the Marshak conditions are generally preferable in the spherical and surface harmonics methods. However, when calculating the multiplication factor in the surface harmonics method, which, as a rule, will be applied in low approximations, Mark's conditions are quite competitive with the conditions of Marshak.
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