THE CONVERGENCE OF DISCRETE UNIFORMIZATIONS FOR CLOSED SURFACES

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ABSTRACT. The notions of discrete conformality on triangle meshes have rich mathematical theories and wide applications. The related notions of discrete uniformizations on triangle meshes, suggest efficient methods for computing the uniformizations of surfaces. This paper proves that the discrete uniformizations approximate the continuous uniformization for closed surfaces of genus $\geq 1$, when the approximating triangle meshes are reasonably good. To the best of the authors’ knowledge, this is the first convergence result on computing uniformizations for surfaces of genus $> 1$.

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1. INTRODUCTION

The celebrated Poincaré-Koebe uniformization theorem states that any simply connected Riemannian surface $(M, g)$ is conformally equivalent to the unit sphere $S^2$, or the complex plane $\mathbb{C}$, or the open unit disk $\mathbb{D}$. As a consequence, any smooth Riemannian metric $g$ on a closed orientable surface $M$ is conformally equivalent to a Riemannian metric $\tilde{g}$ of constant curvature...
Computing such a uniformization map to $S^2$ or $C$ or $D$, or such a uniformization metric $\tilde{g}$, has a wide range of applications in surface parameterizations, classifications, and matchings and so on. See [1][2][3][4][5] for examples.

Most of the existing methods of computing uniformizations are only for topological disks or topological spheres. Luo [6] and Bobenko-Pinkall-Springborn [7] et al. developed the theory of discrete conformality on triangle meshes, based on the notion of vertex scalings. Here, discrete uniformizations for triangle meshes can be naturally defined, and computed efficiently by minimizing an explicit convex functional. So for surfaces approximated by triangle meshes, we can compute the (approximated) uniformizations efficiently.

This paper proves the convergence of this computing method, for smooth closed surfaces of genus $\geq 1$. Briefly speaking, we proved that for a smooth Riemannian surface $M$ of genus $\geq 1$ and a regular dense geodesic triangulation $T$ of $M$, the discrete uniformization of $T$ approximates the uniformization of $M$, with an error bounded linearly by the maximum size of the triangles in $T$. Our result is a strengthening and a generalization of Gu-Luo-Wu’s convergence theorem (Theorem 6.1 in [8]), which is only for the case of genus 1. To the best of the authors’ knowledge, this is the first convergence result on computing uniformizations for surfaces of genus $> 1$.

Aside from other technicalities, the main ingredients of the proofs are

- a cubic estimate (Lemma 4.3) first given by Gu-Luo-Wu [8], showing that the discrete conformal change approximates the continuous conformal change, and
- explicit formulae (Proposition 3.1 and 3.2) given by Bobenko-Pinkall-Springborn [7], showing that the differential of the discrete curvature map is indeed a discrete elliptic operator, and
- a technical elliptic estimate on graphs (Lemma 2.3), based on discrete isoperimetric inequalities on triangle meshes.

1.1. Related Works. Vertex scaling is a notion of discrete conformality on triangle meshes, and apparently first appeared in Roček-Williams’s work [9] in physics. Luo [6] discovered a variational principle for this notion, and proved the local rigidity of the discrete curvature map, and introduced the discrete Yamabe flow to compute discrete uniformizations. Bobenko-Pinkall-Springborn [7] related the notion of vertex scalings with the volume of hyperbolic tetrahedrons, and wrote down an explicit formula for the variational principle, and proved the global rigidity of the discrete curvature map. They also generalized the theory to the hyperbolic triangle meshes, from the usual Euclidean ones. Gu et al. [10][11] proposed improved versions of vertex scalings, and perfectly solved the prescribed curvature problem for closed surfaces. Springborn [12] proposed a constructive variational proof of the discrete uniformization theorem for spheres, which is equivalent to a theorem of Rivin [13] on a hyperbolic Weyl-type problem. Glickenstein [14][15] extended the notion of vertex scalings to 3-dimensional triangle meshes. In complex analysis we know that the conformal diffeomorphisms between two complex planes are similar transformations. Analogous rigidity results for vertex scalings can be found in [16][17][18]. Springborn et al. [19] and Sun et al. [20] have successfully computed conformal maps using vertex scalings. The convergence of the methods is a fundamental problem, from the viewpoints of both mathematical theories and practical applications. Gu-Luo-Wu
proved a convergence result for the cases of topological disks and tori, and Luo-Sun-Wu [17] and Bücking [21][22] proved convergence results for planar Riemann mappings.

There are several other notions of discrete conformality on triangle meshes. Thurston [23] proposed the idea of using tangential circle packings to approach the discrete conformality. Rodin-Sullivan [24] proved Thurston’s conjecture that the Riemann mapping can be approximated by circle packings. He-Schramm [25][26] extended Rodin-Sullivan’s result to more general triangulations and higher order convergence. Analogous results have been developed for intersecting circle patterns in [27][28][29][30]. By generalizing Colin de Verdière’s variational principle for circle packings [31], Chow-Luo [32] introduced the discrete Ricci flow for circle patterns. Since not all the triangle meshes can be realized as circle patterns, Bowers-Stephenson [33] introduced inversive circle patterns where neighbored circles do not necessarily intersect. The inversive circle patterns have been used to conformally flatten human brains [34][35]. Guo [36] established a convex variational principle for inversive circle patterns and proved the local rigidity of the discrete curvature map. The global rigidity was proved by Luo [37]. Glickenstein [38] unified all the above notions of discrete conformality.

There have been a lot of other methods in computing conformal maps. Pinkall-Polthier [39] proposed a method of conformal parameterization by computing a pair of conjugate harmonic functions. Gu-Yau [40][41] developed the method of computing conformal structures of surfaces by computing the discrete holomorphic one-forms. Dym-Slutsky-Lipman [42] computed the Riemann mapping by minimizing the Dirichlet energy, and proved the convergence in $H^1$-norm. Lévy et al. [43] and Lipman [44] and Lui et al. [45] computed conformal or quasiconformal maps by minimizing or controlling the conformal distortion. Other related works can be found in [46][47][48][49][50][51][52][53].

1.2. Set Up and Main Theorems. Assume $M$ is a two dimensional closed orientable surface, and $T$ is a triangulation of $M$, which is a simplicial complex. Denote $V(T), E(T), F(T)$ as the set of vertices, edges, and triangles of $T$ respectively. Further for $M$ equipped with a Riemannian metric, $T$ is called a geodesic triangulation if any edge in $T$ is a geodesic segment. In this paper, a Riemannian surface $(M, g)$ is always assumed to be smooth, i.e., $C^\infty$.

Given a triangulation $T$, if an edge length $l \in \mathbb{R}_{\geq 0}^{E(T)}$ satisfies the triangle inequalities, then we can determine a Euclidean triangle mesh $(T, l)_E$ by assuming that each triangle in $F(T)$ is a Euclidean triangle with the edge lengths given by $l$. We can construct an analogous hyperbolic triangle mesh $(T, l)_H$, by using hyperbolic triangles instead of the Euclidean triangles. Notice that a Euclidean (resp. hyperbolic) triangle mesh has a piecewise Euclidean (resp. hyperbolic) metric, and all the singular points (or cone points) are in $V(T)$. Given $(T, l)_E$ or $(T, l)_H$, $\theta^i_{jk}$ denotes the inner angle at $i$ in the triangle $\triangle ijk \in F(T)$, and the discrete curvature $K_i$ at a vertex $i \in V(T)$ is defined to be the angle defect

$$K_i = 2\pi - \sum_{jk: ijk \in F(T)} \theta^i_{jk}.$$  

It is easy to see that $(T, l)_E$ (resp. $(T, l)_H$) is globally flat (resp. globally hyperbolic) if and only if $K_i = 0$ for any $i \in V$. 

Definition 1.1 ([6][7]). The Euclidean triangle meshes \((T, l)_E\) and \((T, l')_E\) are called discrete conformal if there exists some \(u \in \mathbb{R}^V(T)\), such that
\[l'_{ij} = e^{\frac{1}{2}(u_i + u_j)}l_{ij}\]
for any \(ij \in E(T)\). The hyperbolic triangle meshes \((T, l)_H\) and \((T, l')_H\) are called discrete conformal if there exists some \(u \in \mathbb{R}^V(T)\), such that
\[\sinh \frac{l'_{ij}}{2} = e^{\frac{1}{2}(u_i + u_j)} \sinh \frac{l_{ij}}{2}\]
for any \(ij \in E(T)\).

Such a vector \(u \in \mathbb{R}^V\) is called a discrete conformal factor, and we denote \(l' = u \ast l\) if (1) holds, and \(l' = u \ast_h l\) if (2) holds. Given \((T, l)_E\) or \((T, l)_H\), \(\theta^i_{jk}(u)\) and \(K_i(u)\) denote the corresponding inner angle and the discrete curvature respectively, in \((T, u \ast l)_E\) or \((T, u \ast_h l)_H\). Let \(K(u) = [K_i(u)]_{i \in V} \in \mathbb{R}^V\). Given \((T, l)_E\) or \((T, l)_H\), \(u \in \mathbb{R}^V\) is called a discrete uniformization conformal factor if \(K(u) = 0\), i.e., \((T, u \ast l)_E\) is globally flat or \((T, u \ast_h l)_H\) is globally hyperbolic.

Luo [6] and Bobenko-Pinkall-Springborn [7] developed the variational principles, saying that
\[\mathcal{F}(u) = \int_u^\ast K(\tilde{u})d\tilde{u}\]
is well defined and locally convex. Here \(\mathcal{F}(u)\) is only defined on a domain of \(\mathbb{R}^V\) where \(u \ast l\) or \(u \ast_h l\) satisfies the triangle inequalities. Bobenko-Pinkall-Sprinborn [7] gave simple and explicit formulae extending \(\mathcal{F}\) to a globally convex functional \(\tilde{\mathcal{F}}\) on \(\mathbb{R}^V\). So if a discrete uniformization conformal factor exists, it can be computed efficiently by minimizing an explicit globally convex functional \(\tilde{\mathcal{F}}\) on \(\mathbb{R}^V\).

Given a smooth Riemannian surface \((M, g)\) of genus 1 (resp. genus > 1), and a geodesic triangulation \(T\) with the edge length \(l \in \mathbb{R}^{E(T)}_{>0}\) measured in \((M, g)\), we view \((T, l)_E\) (resp. \((T, l)_H\)) as a triangle mesh approximation of \((M, g)\) and prove that the discrete uniformization approximates the true uniformization of \((M, g)\) if the triangle mesh is sufficiently regular and dense. For the regularity of a triangle mesh, we use the following definition.

Definition 1.2. \((T, l)_E\) or \((T, l)_H\) is called \((\epsilon_1, \epsilon_2)\)-regular if
(a) \(\theta^i_{jk} \geq \epsilon_1\) for any inner angle, and
(b) \(\theta^i_{ij} + \theta^i_{jk} \leq \pi - \epsilon_2\) for any pair of adjacent triangles \(\triangle ijk\) and \(\triangle ijk'\).

Here part (b) is a uniformly strict Delaunay condition, and could be satisfied by, for example, uniformly acute triangle meshes. Our main convergence results are the following Theorem 1.3 and 1.4. In this paper, if \(x \in \mathbb{R}^A\) is a vector for some finite set \(A\), we use \(|x|\) to denote the infinite norm of \(x\), i.e., \(|x| = |x|_\infty = \max_{i \in A} |x_i|\).

Theorem 1.3. Suppose \((M, g)\) is a closed orientable smooth Riemannian surface of genus 1, and \(\tilde{u} = \tilde{u}_{M, g} \in C^\infty(M)\) is the unique uniformization conformal factor such that \(e^{2\tilde{u}}g\) is flat and \(\text{Area}(M, e^{2\tilde{u}}g) = 1\). Assume \(T\) is a geodesic triangulation, and \(l \in \mathbb{R}^{E(T)}_{>0}\) denotes the edge length in \((M, g)\). Then for any \(\epsilon_1, \epsilon_2 > 0\), there exists a constant \(\delta = \delta(M, g, \epsilon_1, \epsilon_2) > 0\) such that if \((T, l)_E\) is \((\epsilon_1, \epsilon_2)\)-regular and \(|l| < \delta\), then
(a) there exists a unique discrete conformal factor $u \in \mathbb{R}^{V(T)}$, such that $(T, u \ast l)_E$ is globally flat and $\text{Area}(T, u \ast l)_E = 1$, and

(b) $|u - \bar{u}|_{V(T)} \leq C|l|$ for some constant $C = C(M, g, \varepsilon_1, \varepsilon_2)$.

**Theorem 1.4.** Suppose $(M, g)$ is a closed orientable smooth Riemannian surface with genus $> 1$, and $\bar{u} = \bar{u}_{M,g} \in C^\infty(M)$ is the unique uniformization conformal factor such that $e^{2\bar{u}}g$ is hyperbolic. Assume $T$ is a geodesic triangulation, and $l \in \mathbb{R}_{>0}^{E(T)}$ denotes the edge length in $(M, g)$. Then for any $\varepsilon_1, \varepsilon_2 > 0$, there exists a constant $\delta = \delta(M, g, \varepsilon_1, \varepsilon_2) > 0$ such that if $(T, l)_H$ is $(\varepsilon_1, \varepsilon_2)$-regular and $|l| < \delta$, then

(a) there exists a unique discrete conformal factor $u \in V(T)$, such that $(T, u \ast_h l)_H$ is globally hyperbolic, and

(b) $|u - \bar{u}|_{V(T)} \leq C|l|$ for some $C = C(M, g, \varepsilon_1, \varepsilon_2) > 0$.

1.3. Outline of the Proofs. For simplicity we use $\bar{u}$ to denote $\bar{u}|_{V(T)}$. For both Theorem 1.3 and 1.4 we first show that $\bar{u}$ is already a good candidate for the discrete uniformization conformal factor, in the sense that $K(\bar{u})$ is "very small", or equivalently, $(T, \bar{u} \ast l)_E$ (resp. $(T, \bar{u} \ast_h l)_H$) is very close to be globally flat (resp. globally hyperbolic). Then by constructing a flow on the triangle mesh, we perturb $\bar{u}$ to $\bar{u} + \delta u$, such that $K(\bar{u} + \delta u)$ is exactly 0, i.e., $\bar{u} + \delta u$ is a discrete uniformization conformal factor. Based on the fact that $\partial K/\partial u$ is indeed a discrete elliptic operator, $\delta u$ is shown to be bounded by $C|l|$ via a technical discrete elliptic estimate.

1.4. Other Notations. In the remaining part of this paper, if a geodesic triangle $\triangle ABC$ is given, we always denote

(a) $a, b, c$ as the lengths of the edges opposite to $A, B, C$ respectively, and

(b) $A, B, C$ as the inner angles, and

(c) $|\triangle ABC|$ as the area of $\triangle ABC$.

For a domain $U$ in a 2-dimensional surface, the area of $U$ is also denoted as $|U|$, or $|U|_g$ if it is induced from a Riemannian metric $g$. For a finite union $\gamma$ of curves, we denote its length as $s(\gamma)$, or $s_g(\gamma)$ if it is induced from a Riemannian metric $g$. For a point $x$ on the Riemannian surface $(M, g)$ and a radius $r > 0$, denote

$$B(x, r) = B^g(x, r) = \{y \in M : d_g(x, y) < r\}.$$ 

A triangle mesh $(T, l)_E$ or $(T, l)_H$ is simply called $\epsilon$-regular if it is $(\epsilon, \epsilon)$-regular.

1.5. Organization of the Paper. In Section 2 we set up basic notions and properties for discrete calculus on graphs. Section 3 reviews explicit formulae for $\partial K/\partial u$, and introduces elementary estimates on triangles. Theorem 1.3 and 1.4 are proved in Section 4, assuming three elementary geometric lemmas and a discrete elliptic estimate. Section 5 proves the auxiliary lemmas and Section 6 proves the discrete elliptic estimate.

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2. Calculus on Graphs

Assume \( G = (V, E) \) is an undirected connected simple graph, on which we will frequently consider vectors in \( \mathbb{R}^V, \mathbb{R}^E \) and \( \mathbb{R}^E_A \). Here \( \mathbb{R}^E \) and \( \mathbb{R}^E_A \) are both vector spaces of dimension \(|E|\) such that

(a) a vector \( x \in \mathbb{R}^E \) is represented symmetrically, i.e., \( x_{ij} = x_{ji} \), and

(b) a vector \( x \in \mathbb{R}^E_A \) is represented anti-symmetrically, i.e., \( x_{ij} = -x_{ji} \).

A vector in \( \mathbb{R}^E_A \) is also called a flow on \( G \). An edge weight \( \eta \) on \( G \) is a vector in \( \mathbb{R}^E \). Given an edge weight \( \eta \), the gradient \( \nabla x = \nabla_\eta x \) of a vector \( x \in \mathbb{R}^V \) is a flow in \( \mathbb{R}^E_A \) such that

\[
(\nabla x)_{ij} = \eta_{ij}(x_j - x_i).
\]

Given a flow \( x \in \mathbb{R}^E_A \), its divergence \( \text{div}(x) \) is a vector in \( \mathbb{R}^V \) such that

\[
\text{div}(x)_i = \sum_{j \sim i} x_{ij}.
\]

Given an edge weight \( \eta \), the associated Laplacian \( \Delta = \Delta_\eta : \mathbb{R}^V \to \mathbb{R}^V \) is defined as \( \Delta x = \text{div}(\nabla_\eta x) \), i.e.,

\[
(\Delta x)_i = \sum_{j \sim i} \eta_{ij}(x_j - x_i).
\]

There is a discrete Green’s identity on graphs.

**Proposition 2.1** (Green’s identity). Given \( x, y \in \mathbb{R}^V \),

\[
\sum_{i \in V} x_i(\Delta y)_i = \sum_{i \in V} y_i(\Delta x)_i.
\]

*Proof.*

\[
\sum_{i \in V} x_i(\Delta y)_i = \sum_{i \in V} x_i \sum_{j \sim i} \eta_{ij}(y_j - y_i) = \sum_{ij \in E} \eta_{ij}x_iy_j - \sum_{i \in V} x_i y_i \sum_{j \sim i} \eta_{ij}.
\]

Then by symmetry the Green’s identity holds. \( \square \)

A Laplacian is a linear transformation on \( \mathbb{R}^V \), and could be identified as a \(|V| \times |V|\) symmetric matrix. By the definition, \( \Delta 1 = 0 \) where \( 1 = (1, 1, ..., 1) \in \mathbb{R}^V \). Also, it is well known that if \( \eta \in \mathbb{R}^E_{>0} \), then \( \ker(\Delta) = \mathbb{R}1 \) by the connectedness of the graph \( G \).

In the rest of this section, we always assume \( \eta \in \mathbb{R}^E_{>0} \). So \( \Delta \) is invertible on the subspace \( 1^\perp = \{x \in \mathbb{R}^V : \sum_{i \in V} x_i = 0\} \). Denote \( \Delta^{-1} \) as the inverse of \( \Delta \) on \( 1^\perp \). The following regularity property will be needed in the proof of our main theorem.

**Lemma 2.2.** \( (\eta, y) \mapsto \Delta^{-1}_\eta y \) is a smooth map from \( \mathbb{R}^V_{>0} \times 1^\perp \) to \( 1^\perp \).

*Proof.* It is equivalent to show that \( \Phi : (\eta, y) \mapsto (\eta, \Delta^{-1}_\eta y) \) is a smooth mapping from \( \mathbb{R}^V_{>0} \times 1^\perp \) to itself. By the inverse function theorem, it suffices to show that \( \Phi^{-1}(\eta, x) = (\eta, \Delta_\eta x) \) is smooth and \( D(\Phi^{-1}) \) is non-degenerate. The smoothness is obvious, and

\[
D(\Phi^{-1}) = \begin{pmatrix}
\text{id} & \partial \eta / \partial x \\
\partial (\Delta_\eta x) / \partial \eta & \partial (\Delta_\eta x) / \partial x \\
\end{pmatrix} = \begin{pmatrix}
\text{id} & 0 \\
\partial (\Delta_\eta x) / \partial \eta & \Delta_\eta \\
\end{pmatrix}
\]
is indeed nondegenerate, since $\Delta_\eta$ is invertible on $1^\perp$.

Now we introduce the notion of $C$-isoperimetry for a graph $G = (V, E)$ associated with a positive vector $l \in \mathbb{R}^E_{>0}$. Given any $V_0 \subset V$, denote

$$\partial V_0 = \{ij \in E : i \in V_0, j \notin V_0\},$$

and then define the $l$-perimeter of $V_0$ and the $l$-area of $V_0$ as

$$|\partial V_0|_l = \sum_{ij \in \partial V_0} l_{ij} \quad \text{and} \quad |V_0|_l = \sum_{i,j \in V_0, ij \in E} l_{ij}^2$$

respectively.

For a constant $C > 0$, such a pair $(G, l)$ is called $C$-isoperimetric if for any $V_0 \subset V$

$$\min\{ |V_0|_l, |V|_l - |V_0|_l \} \leq C \cdot |\partial V_0|_l^2.$$

We will see, from part (b) of Lemma 4.4, that a uniform $C$-isoperimetric condition is satisfied by regular triangle meshes approximating a closed smooth surface. The following discrete elliptic estimate plays an important role in proving our main theorems. The technical proof is postponed to Section 6.

**Lemma 2.3.** Assume $(G, l)$ is $C_1$-isoperimetric, and $x \in \mathbb{R}^E_A, \eta \in \mathbb{R}^E_{>0}, C_2 > 0, C_3 > 0$ are such that

(i) $|x_{ij}| \leq C_2 l_{ij}^2$ for any $ij \in E$, and

(ii) $\eta_{ij} \geq C_3$ for any $ij \in E$.

Then

$$|\Delta^{-1} \circ \text{div}(x)| \leq \frac{4C_2 \sqrt{C_1 + 1}}{C_3} |l| \cdot |V|^{1/2}.$$

Further if $y \in \mathbb{R}^V$ and $C_4 > 0$ and $D \in \mathbb{R}^{V \times V}$ is a diagonal matrix such that

$$|y_i| < C_4 D_{ii} |l| \cdot |V|^{1/2}$$

for any $i \in V$, then

$$|(D - \Delta_\eta)^{-1}(\text{div}(x) + y)| \leq \left( C_4 + \frac{8C_2 \sqrt{C_1 + 1}}{C_3} \right) |l| \cdot |V|^{1/2}.$$

### 3. Differential of the Discrete Curvatures and Angles

Bobenko-Pinkall-Springborn [7] gave explicit formulae for the infinitesimal changes of the discrete curvature, as the mesh deform in its discrete conformal equivalence class. Here we reformulate their formulae as follows.

**Proposition 3.1** (Proposition 4.1.6 in [7]). Given $(T, l)_E$ and $u \in \mathbb{R}^{V(T)}$ such that $u \ast l$ satisfies the triangle inequalities, define the cotangent weight $\eta \in \mathbb{R}^E$ as

$$\eta_{ij}(u) = \frac{1}{2} \cot \theta_{ij}^k(u) + \frac{1}{2} \cot \theta_{ij}^{k'}(u)$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles in $F(T)$. Then

$$\frac{\partial K}{\partial u}(u) = -\Delta_\eta(u).$$
Proposition 3.2 (Proposition 6.1.7 in [7]). Given \((T, l)_H\) and \(u \in \mathbb{R}^V(T)\) such that \(u \ast_h l\) satisfies the triangle inequalities, denote

\[ \tilde{\theta}^i_{jk}(u) = \frac{1}{2}(\pi + \theta^i_{jk}(u) - \theta^j_{ik}(u) - \theta^k_{ij}(u)) \]

and

\[ w_{ij}(u) = \frac{1}{2} \cot \tilde{\theta}^k_{ij}(u) + \frac{1}{2} \cot \tilde{\theta}^k_{ij}(u) \]

where \(\triangle ijk\) and \(\triangle ijk'\) are adjacent triangles in \(F(T)\). Then

\[ \frac{\partial K}{\partial u}(u) = D(u) - \Delta \eta(u) \]

where

\[ \eta_{ij}(u) = w_{ij}(u)(1 - \tanh^2 \frac{(u \ast_h l)_{ij}}{2}) \]

and \(D = D(u)\) is a diagonal matrix such that

\[ D_{ii}(u) = 2 \sum_{j : ij \in E} w_{ij}(u) \tanh \frac{(u \ast_h l)_{ij}}{2}. \]

To derive Propositions 3.1 and 3.2, one only needs to compute for a single triangle as in Lemma 3.3 and 3.4, and then properly add up the following equation (3) for the Euclidean case, and (4)(5) for the hyperbolic case. We omit the proofs of Propositions 3.1 and 3.2, and postpone the elementary calculations for Lemma 3.3 and 3.4 to Appendix.

Lemma 3.3. Given a Euclidean triangle \(\triangle ABC\), if we view \(A, B, C\) as functions of the edge lengths \(a, b, c\), then

\[ \frac{\partial A}{\partial b} = -\cot C, \quad \frac{\partial A}{\partial a} = \cot B + \cot C \quad \frac{1}{a b \sin C}. \]

Further if \((u_A, u_B, u_C) \in \mathbb{R}^3\) is a discrete conformal factor, and

\[ a = e^{\frac{1}{2}(u_B + u_C)} a_0, \quad b = e^{\frac{1}{2}(u_A + u_C)} b_0, \quad c = e^{\frac{1}{2}(u_A + u_B)} c_0 \]

for some constants \(a_0, b_0, c_0 \in \mathbb{R}_{>0}\), then

\[ \frac{\partial A}{\partial u_B} = \frac{1}{2} \cot C, \quad \frac{\partial A}{\partial u_A} = -\frac{1}{2}(\cot B + \cot C). \]

Lemma 3.4. Given a hyperbolic triangle \(\triangle ABC\), if we view \(A, B, C\) as functions of the edge lengths \(a, b, c\), then

\[ \frac{\partial A}{\partial b} = -\frac{1}{\sinh b}, \quad \frac{\partial A}{\partial a} = \frac{1}{\sinh b \sin C}. \]

Further if \((u_A, u_B, u_C) \in \mathbb{R}^3\) is a discrete conformal factor, and

\[ \sinh \frac{a}{2} = e^{\frac{1}{2}(u_B + u_C)} \sinh \frac{a_0}{2}, \quad \sinh \frac{b}{2} = e^{\frac{1}{2}(u_A + u_C)} \sinh \frac{b_0}{2}, \quad \sinh \frac{c}{2} = e^{\frac{1}{2}(u_A + u_B)} \sinh \frac{c_0}{2} \]

for some constants \(a_0, b_0, c_0 \in \mathbb{R}_{>0}\), then

\[ \frac{\partial A}{\partial u_B} = \frac{1}{2} \cot \tilde{C}(1 - \tanh^2 \frac{c}{2}), \]
and

$$\frac{\partial A}{\partial u_A} = -\frac{1}{2} \cot \tilde{B}(1 + \tanh^2 \frac{b}{2}) - \frac{1}{2} \cot \tilde{C}(1 + \tanh^2 \frac{c}{2}),$$

where $\tilde{B} = \frac{1}{2}(\pi + B - A - C)$ and $\tilde{C} = \frac{1}{2}(\pi + C - A - B)$.

By the differential formulae in Lemma 3.3 and 3.4, it is not difficult to prove the following estimates Lemma 3.5 and 3.6 for perturbations of a single triangle. The proofs of Lemma 3.5 and 3.6 are also in the Appendix.

Lemma 3.5. Given a Euclidean triangle $\triangle ABC$, if all the angles in $\triangle ABC$ are at least $\epsilon > 0$, and $\delta < \epsilon^2/48$, and

$$|a' - a| \leq \delta a, \quad |b' - b| \leq \delta a, \quad |c' - c| \leq \delta c,$$

then $a', b', c'$ form a Euclidean triangle with opposite inner angles $A', B', C'$ respectively, and

$$|A' - A| \leq \frac{24}{\epsilon \delta},$$

and

$$\left| |\triangle A'B'C'| - |\triangle ABC| \right| \leq \frac{576}{\epsilon^2 \delta} \cdot |\triangle ABC|.$$

Lemma 3.6. Given a hyperbolic triangle $\triangle ABC$, if all the angles in $\triangle ABC$ are at least $\epsilon > 0$, and $\delta < \epsilon^3/60$, and

$$a \leq 0.1, \quad b \leq 0.1, \quad c \leq 0.1,$$

and

$$|a' - a| \leq \delta a, \quad |b' - b| \leq \delta a, \quad |c' - c| \leq \delta c,$$

then $a', b', c'$ form a hyperbolic triangle with opposite inner angles $A', B', C'$ respectively, and

$$|A' - A| \leq \frac{30}{\epsilon^2 \delta},$$

and

$$\left| |\triangle A'B'C'| - |\triangle ABC| \right| \leq \frac{120}{\epsilon^2 \delta} \cdot |\triangle ABC|.$$

4. PROOF OF THE MAIN THEOREMS

In this section we first introduce three geometric lemmas, whose proofs are given in Section 5, and then prove Theorem 1.3 and 1.4 in Subsections 4.2 and 4.3 respectively.

4.1. Geometric Lemmas.

Lemma 4.1. Suppose $\triangle_E ABC$, $\triangle_H ABC$ and $\triangle_S ABC$ are Euclidean and hyperbolic and spherical triangles respectively, with the same edge lengths $a, b, c < 0.1$.

(a) If all the inner angles in $\triangle_E ABC$ are at least $\epsilon > 0$, then for any $P \in \{E, H, S\}$,

$$\frac{\epsilon}{8} a^2 \leq |\triangle_P ABC| \leq \frac{1}{\epsilon} a^2.$$
(b) Assume $M_a$ is the middle point of $BC$, and $M_b$ is the middle point of $AC$, and $\triangle P CM_a M_b$ is the geodesic triangle in $\triangle P ABC$ with vertices $C, M_a, M_b$, where $P \in \{E, H, S\}$. Then

$$|\triangle P CM_a M_b| \geq \frac{1}{5} |\triangle P ABC|$$

for any $P \in \{E, H, S\}$.

Remark 4.2. By the well-known Toponogov comparison theorem (see Lemma 5.2), the assumption in part (a) of Lemma 4.1 can be replaced by that all the inner angles in $\triangle H ABC$ are at least $\epsilon > 0$.

The following Lemma 4.3 was first proved by Gu-Luo-Wu (see Proposition 5.2 in [8]). It indicates that our discrete conformal formula $l'_{ij} = e^{(u_i + u_j)/2} l_{ij}$ accurately approximates the local distance after a continuous conformal change.

Lemma 4.3. Suppose $(M, g)$ is a closed Riemannian surface, and $u \in C^\infty(M)$ is a conformal factor. Then there exists $C = C(M, g, u) > 0$ such that for any $x, y \in M$,

$$|e^{u(x)+u(y)} d_g(x, y) - e^{(u(x)+u(y))/2} d_g(x, y)| \leq C d_g(x, y)^3.$$
relative to $V(T) = V(T')$. Let $\bar{l} \in \mathbb{R}^{E(T)} \cong \mathbb{R}^{E(T')}$ denote the geodesic lengths of the edges of $T'$ in $(M, e^{2u}g)$, and then $(T, \bar{l})_E$ is isometric to $(M, e^{2u}g)$ and globally flat.

For simplicity, we will frequently use the notion $a = O(b)$ to denote that if $\delta = \delta(M, g, \epsilon_1, \epsilon_2)$ is sufficiently small, then $|a| \leq C \cdot b$ for some constant $C = C(M, g, \epsilon_1, \epsilon_2)$. For example, $l_{ij} = O(l_{jk})$ for any $\triangle ijk \in F(T)$, and $(\bar{u} \ast l)_{ij} = O(l_{ij})$, and $\bar{l}_{ij} = O(l_{ij})$. The remaining of the proof is divided into three steps.

1. Firstly we show that $(T, \bar{u} \ast l)_E$ is very close to the globally flat triangle mesh $(T, \bar{l})_E$, in the sense that

$$ (\bar{u} \ast l)_{ij} - \bar{l}_{ij} = O(l_{ij}^3) $$

and

$$ K(\bar{u}) = div(x) $$

for some flow $x \in \mathbb{R}^E_A$ satisfying $x_{ij} = O(l_{ij}^3)$.

2. Secondly, we construct a flow $u(t) : [0, 1] \to \mathbb{R}^V$, starting at $u(0) = \bar{u}$, to linearly eliminate the curvature $K(\bar{u})$, i.e., to let

$$ K(u(t)) = (1 - t)K(\bar{u}). $$

Further we show that $|u'(t)| = O(|l|)$, and then $(T, u(1) \ast l)_E$ is globally flat and $u(1) - \bar{u} = O(|l|)$.

3. Lastly we show that $Area((T, u(1) \ast l)_E) - 1 = O(|l|)$, so the area normalization condition can be satisfied by slightly scaling $(T, u(1) \ast l)_E$.

The uniqueness of the discrete uniformization conformal factor is proved by Bobenko-Pinkall-Springborn (see Theorem 3.1.4 in [7]), so we omit its proof here.

4.2.1. Step 1. By lemma [4.4] $(T, \bar{l})_E$ is $\frac{1}{2}\epsilon$-regular if $\delta$ is sufficiently small. For simplicity we denote $\bar{u}|_{V(T)}$ as $\bar{u}$. By lemma [4.3]

$$ \bar{l}_{ij} - (\bar{u} \ast l)_{ij} = O(l_{ij}^3), $$

and then by Lemma [3.5]

$$ \alpha_{jk}^i := \bar{\theta}_{jk}^i - \theta_{jk}^i(\bar{u}) = O(l_{ij}^3) $$

where $\bar{\theta}_{jk}^i$ denotes the inner angle in $(T, \bar{l})_E$. So $(T, \bar{u} \ast l)_E$ is $\frac{1}{2}\epsilon$-regular if $\delta$ is sufficiently small. Let $x \in \mathbb{R}^E_A$ be such that

$$ x_{ij} = \frac{\alpha_{jk}^i - \alpha_{ik}^j}{3} + \frac{\alpha_{jk}^i - \alpha_{ik}^j}{3} $$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles. Then $\alpha_{jk}^i + \alpha_{ik}^j + \alpha_{ij}^k = 0$ and

$$ div(x)_i = \sum_{j: j \sim i} x_{ij} = \sum_{jk: \Delta ijk \in F(T)} \left( \frac{\alpha_{jk}^i - \alpha_{ik}^j}{3} + \frac{\alpha_{jk}^i - \alpha_{ik}^j}{3} \right) = \sum_{jk: \Delta ijk \in F(T)} \alpha_{ij}^i = K_i(\bar{u}), $$

and

$$ x_{ij} = O(l_{ij}^3). $$
4.2.2. Step 2. Let

\[ \tilde{\Omega} = \{ u \in 1^+ : u \ast l \text{ satisfies the triangle inequalities and } (T, u \ast l)_E \text{ is } \frac{\epsilon}{5}\text{-regular} \} \]

and

\[ \Omega = \{ u \in \tilde{\Omega} : |u - \tilde{u}| \leq 1, (T, u \ast l)_E \text{ is } \frac{\epsilon}{4}\text{-regular} \}. \]

Since \((T, \tilde{u} \ast l)_E \) is \( \frac{1}{3}\epsilon\text{-regular} \), \( \tilde{u} \) is in the interior of \( \Omega \). Now consider the following ODE on \( \text{int}(\tilde{\Omega}) \),

\[
\begin{align*}
\begin{cases}
  u'(t) = \Delta_{\eta(u)}^{-1} K(\tilde{u}) = \Delta_{\eta(u)}^{-1} \circ \text{div}(x), \\
  u(0) = \tilde{u}
\end{cases}
\end{align*}
\]

where

\[
\eta_{ij}(u) = \frac{\cot \theta_{ij}^k(u) + \cot \theta_{ij}^{k'}(u)}{2} = \frac{\sin(\theta_{ij}^k(u) + \theta_{ij}^{k'}(u))}{2 \sin \theta_{ij}^k(u) \sin \theta_{ij}^{k'}(u)} \geq \frac{1}{2} \sin \left( \theta_{ij}^k(u) + \theta_{ij}^{k'}(u) \right) \geq \frac{1}{2} \sin \frac{\epsilon}{5},
\]

where \( \triangle ijk, \triangle ijk' \) are adjacent triangles. By Lemma 2.2, the right-hand side of (7) is a smooth function of \( u \), so the ODE (7) has a unique solution \( u(t) \) and

\[ K(u(t)) = (1 - t)K(\tilde{u}) \]

by Proposition 3.1. Assume the maximum existing open interval of \( u(t) \) in \( \Omega \) is \([0, T_0]\) where \( T_0 \in (0, \infty) \). By Lemma 4.4 (\( T, l \)) is \( C \)-isoperimetric for some constant \( C = C(M, g, \epsilon_1, \epsilon_2) \). Then for any \( u \in \Omega \), \((T, u \ast l)_E \) is \((e^{4(|l|+1)}C)\)-isoperimetric by the fact that \( |u| \leq |\tilde{u}| + 1 \). Then by Lemma 2.3 and equations (5)(7)(8), for any \( t \in [0, T_0] \)

\[
|u'(t)| = O(|l| \cdot |V|_l^{1/2}).
\]

By Lemma 4.1 and the fact that \((T, \tilde{l})_E \) is \( \frac{1}{2}\epsilon\text{-regular} \),

\[
|V|_l = \sum_{ij \in E} l_{ij}^2 = O \left( \sum_{ij \in E} \tilde{l}_{ij}^2 \right) = O \left( \sum_{ijk \in F} (\tilde{t}_{ij}^2 + \tilde{t}_{jk}^2 + \tilde{t}_{ik}^2) \right)
\]

\[
= O \left( \sum_{ijk \in F} |(\triangle ijk, \tilde{l})_E| \right) = O(|(T, \tilde{l})_E|) = O(1).
\]

Here recall that \( |(\triangle ijk, \tilde{l})_E| \) denotes the area of the Euclidean triangle, and \( |(T, \tilde{l})_E| \) denotes the area of the piecewise flat surface.

Combining the estimates (9) and (10), we have that for any \( t \in [0, T_0] \)

\[ |u'(t)| = O(|l|). \]

If \( T_0 < 1 \), by Lemma 3.5

\[ |u(T_0) - \tilde{u}| = O(|l|) \quad \text{and} \quad \theta_{jk}^i(u(T_0)) - \theta_{jk}^i(\tilde{u}) = O(|l|), \]

and thus \( u(T_0) \in \text{int}(\Omega) \) if \( \delta \) is sufficiently small. But this contradicts with the maximality of \( T_0 \). So \( T_0 \geq 1 \) and \((T, u(1))_E \) is globally flat and \(|u(1) - \tilde{u}| = O(|l|)\).
4.2.3. **Step 3.** To prove part (a) of the theorem, we only need to scale the mesh \((T, u(1) \ast l)_E\) to make its area equal to 1. To get the estimate in part (b), it remains to show

\[
\log |(T, u(1) \ast l)_E| = O(\|l\|).
\]

Since

\[
|(u(1) \ast l)_{ij} - \bar{l}_{ij}| = |(u(1) \ast l)_{ij} - (\bar{u} \ast l)_{ij}| + |(\bar{u} \ast l)_{ij} - \bar{l}_{ij}|
\]

\[
\leq e^{(u(1)-\bar{u})} - 1)(\bar{u} \ast l)_{ij} + O(l^3_{ij}) = O(|l| \cdot \bar{l}_{ij}).
\]

Since \((T, \bar{l})_E \) is \(\frac{1}{2}\)-regular, by Lemma 3.5 if \(\delta\) is sufficiently small then for any \(\triangle ijk \in F\)

\[
\log \frac{|(\triangle ijk, u(1) \ast l)_E|}{|(\triangle ijk, \bar{l})_E|} = O(|l|)
\]

and

\[
\log |(T, u(1) \ast l)_E| = \log \frac{\sum_{\triangle ijk \in F} |(\triangle ijk, u(1) \ast l)_E|}{\sum_{\triangle ijk \in F} |(\triangle ijk, \bar{l})_E|} = O(|l|).
\]  

4.3. **Proof of Theorem 1.4** For fix two constants \(\epsilon_1, \epsilon_2 > 0\), we assume that \((T, l)_H\) is \((\epsilon_1, \epsilon_2)\)-regular and \(|l| \leq \delta\) where \(\delta = \delta(M, g, \epsilon_1, \epsilon_2) < 1\) is a sufficiently small constant to be determined. Simply note that \((T, l)_H\) is \(\epsilon\)-regular where \(\epsilon = \min\{\epsilon_1, \epsilon_2\}\). By Lemma 4.4, if \(\delta\) is sufficiently small there exists a geodesic triangulation \(T'\) of \((M, e^{2u}g)\) homotopic to \(T\) relative to \(V(T) = V(T')\). Let \(\bar{l} \in \mathbb{R}^{E(T)} \cong R^{E(T')}\) denote the geodesic lengths of edges of \(T'\) in \((M, e^{2u}g)\), and then \((T, \bar{l})_H\) is isometric to \((M, e^{2u}g)\) and globally hyperbolic.

For simplicity, we will frequently use the notion \(a = O(b)\) to denote that if \(\delta = \delta(M, g, \epsilon_1, \epsilon_2)\) is sufficiently small, then \(|a| \leq C \cdot b\) for some constant \(C = C(M, g, \epsilon_1, \epsilon_2)\). For example, we have that

(a) \(l_{ij} = O(l_{jk})\) for any \(\triangle ijk \in F(T)\), and

(b) \((\bar{u} \ast h)_{ij} = O(l_{ij}),\) and

(c) \(\bar{l}_{ij} = O(l_{ij}),\) and

(d) \(\sinh(l_{ij}/2) = O(l_{ij}).\)

The remaining of the proof is divided into two steps.

1. Firstly we show that \((T, \bar{u} \ast h l)_H\) is very close to the globally hyperbolic triangle mesh \((T, \bar{l})_H\), in the sense that

\[
(\bar{u} \ast h l)_{ij} - \bar{l}_{ij} = O(l^3_{ij})
\]

and

\[
K(\bar{u}) = div(x) + y
\]

for some \(x \in \mathbb{R}^E_A\) and \(y \in \mathbb{R}^V\) such that \(x_{ij} = O(l^2_{ij})\) and \(y_i = O(l^3_{ij})\).

2. Secondly, we construct a flow \(u(t) : [0, 1] \to \mathbb{R}^V\), starting at \(u(0) = \bar{u}\), to linearly eliminate the curvature \(K(\bar{u})\), i.e., to let

\[
K(u(t)) = (1 - t)K(\bar{u}).
\]

Further we show that \(|u'(t)| = O(|l|)\), and then \((T, u(1) \ast h l)_H\) is globally hyperbolic and \(u(1) - \bar{u} = O(|l|)\).
The uniqueness of the discrete uniformization conformal factor is also proved by Bobenko-Pinkall-Springborn (see Theorem 6.1.6 in [7]), so we omit its proof here.

4.3.1. Part 1. By lemma \[4.4\] \((T, \bar{l})_H\) is \(\frac{1}{2}\epsilon\)-regular if \(\delta\) is sufficiently small. For simplicity we denote \(\bar{u}|_{V(T)}\) as \(\bar{u}\). By lemma \[4.3\] we get

\[\bar{I}_{ij} - (\bar{u} * l)_{ij} = O(I^3_{ij}).\]

Using the fact that \(|2 \sinh(\frac{x}{2}) - x| \leq |x|^3\) for \(|x| \leq 1\), we have

\[\bar{I}_{ij} - (\bar{u} * h)_{ij} = O(I^3_{ij}).\]

Denote \(\bar{\theta}_{jk}\) as the inner angle in \((T, \bar{l})_H\), and then by Lemma \[5.6\] and \[4.1\] and Remark \[4.2\]

\[\alpha^i_{jk} := \bar{\theta}_{jk} - \theta^i_{jk}(\bar{u}) = O(I^3_{ij})\]

and

\[\alpha^i_{jk} + \alpha^j_{ik} + \alpha^k_{ij} = |(\triangle ijk, \bar{u} * h l)_H| - |(\triangle ijk, \bar{l})_H| = O(I^2_{ij}) \cdot |(\triangle ijk, \bar{l})_H| = O(I^4_{ij}).\]

So \((T, \bar{u} * h l)_H\) is \(\frac{1}{3}\epsilon\)-regular if \(\delta\) is sufficiently small. Let \(x \in \mathbb{R}^E\) and \(y \in \mathbb{R}^V\) be such that

\[x_{ij} = \frac{\alpha^j_{ik} - \alpha^j_{ik} + \alpha^k_{ij}}{3} + \frac{\alpha^j_{ik} - \alpha^j_{ik} + \alpha^k_{ij}}{3} \quad \text{and}\quad y_i = \frac{1}{3} \sum_{j: \triangle ijk \in F(T)} (\alpha^j_{ik} + \alpha^j_{ik} + \alpha^k_{ij})\]

where \(\triangle ijk\) and \(\triangle ijk'\) are adjacent triangles. Then

\[\text{div}(x)_i + y_i = K_i(\bar{u}),\]

and

\[x_{ij} = O(I^2_{ij}),\]

and

\[y_i = O(I^4_{ij})\]

by the fact that any vertex \(i \in V(T')\) has at most \([2\pi/(\epsilon/2)] = O(1)\) neighbors.

4.3.2. Part 2. Let

\[\tilde{\Omega}_H = \{u \in 1^+ : u * h l \text{ satisfies the triangle inequalities and } (T, u * h l)_H \text{ is } \frac{1}{5}\epsilon\text{-regular}\}\]

and

\[\Omega_H = \{u \in \tilde{\Omega} : |u - \bar{u}| \leq 1, (T, u * h l)_H \text{ is } \frac{1}{4}\epsilon\text{-regular}\}.\]

Since \((T, \bar{u} * h l)_H\) is \(\frac{1}{3}\epsilon\)-regular, \(\bar{u}\) is in the interior of \(\Omega_H\). Now consider the following ODE on \(\text{int}(\Omega_H)\),

\[u'(t) = (D(u) - \Delta_{\eta(u)})^{-1} K(\bar{u}) = (D(u) - \Delta_{\eta(u)})^{-1} (\text{div}(x) + y),\]

where \(D(u)\) and \(\eta(u)\) are defined as in Proposition \[3.2\] For any triangle \(\triangle ijk\) and \(u \in \tilde{\Omega}_H\), by Lemma \[4.1\] and Remark \[4.2\] we have

\[|(\triangle ijk, u * h l)_H| = O(I^2_{ij})\]
and
\[ \frac{1}{2}(\pi + \theta_{ij}^k(u) - \theta_{ikj}^l(u) - \theta_{ijk}^l(u)) = \theta_{ij}^k(u) + \frac{1}{2}(\pi - \theta_{ij}^k(u) - \theta_{ikj}^l(u) - \theta_{ijk}^l(u)) = \theta_{ij}^k(u) + O(t_{ij}^l). \]

Now let \( w(u) \) be defined as in Proposition 3.2, and then by the formula
\[ \cot A + \cot B = \frac{\sin(A + B)}{\sin A \sin B} \geq \sin(A + B) \quad \text{for any} \ A, B \in (0, \pi), \]
we have that if \( \delta \) is sufficiently small and \( u \in \tilde{\Omega}_H \)
\[ w_{ij}(u) \geq \frac{1}{2} \sin(\theta_{ij}^k + \theta_{ij}^{k'}) + O(t_{ij}^l) \geq \frac{1}{2} \sin \frac{\epsilon}{5} + O(t_{ij}^l) \geq \frac{1}{4} \sin \frac{\epsilon}{5}, \]
and
\[ D_{ji}(u) \geq 2w_{ij} \tanh^2 \frac{(u * h l)_{ij}}{2} \geq \epsilon' t_{ij}^l, \quad \text{and} \quad \eta_{ij}(u) \geq \frac{1}{8} \sin \frac{\epsilon}{5} \]
for some constant \( \epsilon' = \epsilon'(M, g, \epsilon_1, \epsilon_2) > 0. \)

The right-hand side of equation (13) is a smooth function of \( u \), so the ODE (13) has a unique solution \( u(t) \) and
\[ K(u(t)) = (1 - t)K(\bar{u}) \]
by Proposition 3.2. Assume the maximum existing open interval of \( u(t) \in \Omega_H \) is \( [0, T_0) \) where \( T_0 \in [0, \infty) \). By Lemma 4.4 when \( \delta \) is sufficiently small, \( (T, l) \) is \( C \)-isoperimetric for some constant \( C = C(M, g, \epsilon_1, \epsilon_2) \). Then for any \( u \in \Omega_H \), \( (T, u * h l) \) is \( (e^{A(|u|+1)}C) \)-isoperimetric by the fact that \( |u| \leq |\bar{u}| + 1 \) and
\[ \frac{\sinh a}{a} \geq \frac{\sinh b}{b} \]
for any \( a \geq b > 0. \) By Lemma 4.1 and Remark 4.2 it is not difficult to see
\[ |V|_1 = O(|V|_1) = O(|(T, \bar{L})_H|) = O(1) \quad \text{and} \quad 1 = O(|(T, \bar{L})_H|) = O(|V|_1) = O(|V|_1). \]
Then by Lemma 2.2 and equation (11)(12)(14), for any \( t \in [0, T_0) \)
\[ |u'(t)| = O(|l| \cdot |V|_1^{1/2}) = O(|l|). \]
By Lemma 3.6 if \( T_0 < 1, \)
\[ |u(T_0) - \bar{u}| = O(|l|) \quad \text{and} \quad \theta_{jk}^i(u(T_0)) = \theta_{jk}^i(\bar{u}) = O(|l|), \]
and then \( u(T_0) \in int(\Omega_H) \) if \( \delta \) is sufficiently small. But this contradicts with the maximality of \( T_0 \). So \( T_0 \geq 1 \) and \( (T, u(1))_H \) is hyperbolic and \( |u(1) - \bar{u}| = O(|l|). \)

5. Proof of the Geometric Lemmas

We prove Lemma 4.1 and 4.3 in Subsection 5.1 and 5.2 respectively, and introduce more lemmas in Subsection 5.3, and then prove part (a) and part (b) of Lemma 4.4 in Subsection 5.4 and 5.5 respectively.
5.1. **Proof of Lemma 4.1.** Recall that

**Lemma 4.1.** Suppose \( \triangle EABC, \triangle HABC \) and \( \triangle SABC \) are Euclidean and hyperbolic and spherical triangles respectively, with the same edge lengths \( a, b, c < 0.1 \).

(a) If all the inner angles in \( \triangle EABC \) are at least \( \epsilon > 0 \), then for any \( P \in \{E, H, S\} \),

\[
\frac{\epsilon}{8} a^2 \leq |\triangle_P ABC| \leq \frac{1}{\epsilon} a^2.
\]

(b) Assume \( M_a \) is the middle point of \( BC \), and \( M_b \) is the middle point of \( AC \), and \( \triangle_{P}CM_{a}M_{b} \) is the geodesic triangle in \( \triangle_{P}ABC \) with vertices \( C, M_{a}, M_{b} \), where \( P \in \{E, H, S\} \). Then

\[ |\triangle_{P}CM_{a}M_{b}| \geq \frac{1}{5} |\triangle_{P}ABC| \]

for any \( P \in \{E, H, S\} \).

**Proof of (a).** We begin with three well known Heron’s formulae for Euclidean, hyperbolic and spherical triangles.

\[
|\triangle_{E}ABC|^2 = s(s-a)(s-b)(s-c),
\]

(17)

\[
\tan^2 \frac{|\triangle_{H}ABC|}{4} = \tanh \frac{s}{2} \tanh \frac{s-a}{2} \tanh \frac{s-b}{2} \tanh \frac{s-c}{2},
\]

(18)

\[
\tan^2 \frac{|\triangle_{S}ABC|}{4} = \tan \frac{s}{2} \tan \frac{s-a}{2} \tan \frac{s-b}{2} \tan \frac{s-c}{2},
\]

where \( s = \frac{a+b+c}{2} \).

The hyperbolic Heron’s formula can be found in Theorem 1.1 in [54], and the spherical one is also called L’Huiliier’s Theorem and can be found in Section 4.19.2 in [55].

Notice that \( |\triangle_{E}ABC| \leq a^2 + b^2 + c^2 \leq 0.03 \), and for \( x \in [0, 0.1] \),

\[
\frac{\tanh x}{x} \in (0.99, 1) \quad \text{and} \quad \frac{\tan x}{x} \in (1, 1.01).
\]

So by the three parallel Heron’s formulae and simple approximation estimates, we only need to show the following stronger estimates \((19)\) and \((20)\) for the Euclidean case. By the law of sines in the Euclidean triangle \( \triangle_{E}ABC \),

\[
b = \frac{a \sin \angle E B}{\sin \angle E A} \leq \frac{a}{\sin \epsilon} \leq \frac{\pi}{2} \frac{a}{\epsilon}.
\]

So

(19)

\[
|\triangle_{E}ABC| = \frac{1}{2} ab \sin C \leq \frac{1}{2} a \cdot \frac{\pi}{2} \frac{a}{\epsilon} = \frac{\pi}{4} \frac{a^2}{\epsilon}.
\]

By the triangle inequality, we may assume \( b \geq a/2 \) without loss of generality, and then

(20)

\[
|\triangle_{E}ABC| = \frac{1}{2} ab \sin C \geq \frac{1}{2} a \cdot \frac{a}{2} \cdot \sin \epsilon \geq \frac{\epsilon}{2} \frac{a^2}{\pi}.
\]

\(\square\)
Proof of (b). The Euclidean case is obvious. To prove the hyperbolic and spherical cases, we use the following two formulae

\begin{align}
\cot \frac{\triangle_H ABC}{2} &= \frac{\csc \frac{a}{2} \csc \frac{b}{2} - \cos \angle_H C}{\sin \angle_H C}, \\
\cot \frac{\triangle_S ABC}{2} &= \frac{\csc \frac{a}{2} \csc \frac{b}{2} + \cos \angle_S C}{\sin \angle_S C},
\end{align}

where equation (21) was developed in Theorem 6 of [56]. The equation (22) can be obtained by

\[
\cot \frac{\triangle_S ABC}{2} = \cot \left( \frac{\angle_S A + \angle_S B + \angle_S C - \pi}{2} \right) = -\tan \left( \frac{\angle_S A + \angle_S B}{2} + \frac{\angle_S C}{2} \right)
\]

and the well-known Napier's analogies

\[
\tan \frac{\angle_S A + \angle_S B}{2} = \cot \frac{\angle_S C}{2} \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}}.
\]

Here we only prove the hyperbolic case using equation (11) and the proof for the spherical case is very similar. Firstly we apply the formula (21) to $\triangle_H CM_a M_b$ and get

\[
\cot \frac{\triangle_H CM_a M_b}{2} = \frac{\csc \frac{a}{4} \csc \frac{b}{4} - \cos \angle_H C}{\sin \angle_H C}.
\]

Then

\[
\frac{\tan \frac{\triangle_H CM_a M_b}{2}}{\tan \frac{\triangle_H ABC}{2}} = \frac{\csc \frac{a}{4} \csc \frac{b}{4} - \cos \angle_H C}{\csc \frac{a}{4} \csc \frac{b}{4} - \cos \angle_H C} \geq \frac{(2/a)(2/b) - 1}{(4/a)(4/b)/0.99^2 + 1} = \frac{4 - ab}{16/0.99^2 + ab} \geq \frac{1}{5}.
\]

Since $|\triangle_H CM_a M_b| \leq |\triangle_H ABC|$ and $\tan x$ is increasing on $(0, \infty)$,

\[
\frac{|\triangle_H CM_a M_b|}{|\triangle_H ABC|} \geq \frac{\tan \frac{\triangle_H CM_a M_b}{2}}{\tan \frac{\triangle_H ABC}{2}} \geq \frac{1}{5}.
\]

\[\square\]

5.2. Proof of Lemma 4.3 Recall that

**Lemma 4.2.** Suppose $(M, g)$ is a closed Riemannian surface, and $u \in C^\infty(M)$ is a conformal factor. Then there exists $C = C(M, g, u) > 0$ such that for any $x, y \in M$,

\[
|d_{e^{2u}g}(x, y) - e^\frac{1}{4}(u(x) + u(y))d_g(x, y)| \leq Cd_g(x, y)^3.
\]

It suffices to prove one direction of the inequality, i.e., the following Lemma 5.1. Once we have Lemma 5.1, let $C_1 = C(M, g, u)$ and $C_2 = C(M, e^{2u}g, -u)$ such that for any $x, y \in M$,

\[
d_{e^{2u}g}(x, y) \leq e^\frac{1}{4}(u(x) + u(y))d_g(x, y) + C_1d_g(x, y)^3.
\]

\[
d_g(x, y) \leq e^\frac{1}{4}(-u(x) - u(y))d_{e^{2u}g}(x, y) + C_2d_{e^{2u}g}(x, y)^3.
\]
Then
\[ |d_{e^{2u}}(x, y) - e^{\frac{1}{2}((u(x) + u(y)))} d_g(x, y)| \leq C_1 d_g(x, y)^3 + C_2 \|u\|_{\infty} d_{e^{2u}}(x, y)^3 \leq (C_1 + C_2 e^{4\|u\|_{\infty}}) d_g(x, y)^3. \]

**Lemma 5.1.** Suppose \((M, g)\) is a closed Riemannian surface, and \(u \in C^\infty(M)\) is a conformal factor. Then there exists \(C = C(M, g, u) > 0\) such that for any \(x, y \in M\),
\[ d_{e^{2u}}(x, y) \leq e^{\frac{1}{2}((u(x) + u(y)))} d_g(x, y) + C d_g(x, y)^3. \]

**Proof.** We will use the following two estimates. Assume \(l > 0\) and \(f \in C^2[0, l]\), then
\[ \left| \frac{1}{2} [f(0) + f(l)] - f\left(\frac{l}{2}\right) \right| \leq \frac{l^2}{4} \max_{0 \leq t \leq l} |f''(t)|, \]
and
\[ \left| \int_0^l f(x)dx - l \cdot f\left(\frac{l}{2}\right) \right| \leq \frac{l^3}{24} \max_{0 \leq t \leq l} |f''(t)|. \]

These two estimates can be proved easily by Taylor’s expansions at point \(x_0 = l/2\), and the second estimate is the so-called mid-point rule approximation of definite integrals. Now let \(l = d_g(x, y)\) and \(\gamma : [0, l] \to (M, g)\) be a shortest geodesic connecting \(x, y\), and it suffices to prove
\[ d_{e^{2u}}(x, y) - l \cdot e^{\frac{1}{2}((u(x) + u(y)))} \leq C l^3 \]
for some constant \(C = C(M, g, u)\). Let \(h(t) = u(\gamma(t))\) and then by the two estimates above,
\[ d_{e^{2u}}(x, y) - l \cdot e^{\frac{1}{2}((u(x) + u(y)))} \leq 8_{\gamma} = \frac{1}{2} \cdot e^{\frac{1}{2}((u(x) + u(y)))} \]
\[ = \left[ \int_0^l e^{h(t)} dt - l \cdot e^{h(l/2)} \right] + \left[ l \cdot e^{h(l/2)} - l \cdot e^{h(0) + h(l)} \right] \leq \frac{l^3}{24} \max_{0 \leq t \leq l} |(e^{h(t)})''| + l \cdot e^{\xi} \cdot \left| h\left(\frac{l}{2}\right) - \frac{1}{2} [h(0) + h(l)] \right| \]
where \(\xi\) is between \(h(l/2)\) and \(1/2[h(0) + h(l)]\), and
\[ \left| h\left(\frac{l}{2}\right) - \frac{1}{2} [h(0) + h(l)] \right| \leq \frac{l^2}{4} \max_{0 \leq t \leq l} |h''(t)|. \]

By the compactness of \(M\), and the fact that \(h(t), h'(t), h''(t)\) can be expressed in terms of \(\{u, \nabla u, Hess(u), \gamma, \gamma', \gamma''\}\) under local coordinates \((v^1, v^2)\), we only need to show that on a small domain \(U\) whose closure is a compact subset of a coordinate domain, \(\|u\|_{\infty}, \|\nabla u\|_{\infty}, \|Hess(u)\|_{\infty}, \|\gamma\|_{\infty}, \|\gamma'\|_{\infty}, \|\gamma''\|_{\infty}\) are all bounded by a constant \(C(M, g, u, U)\). It is obvious that \(\|u\|_{\infty}, \|\nabla u\|_{\infty}, \|Hess(u)\|_{\infty}, \|\gamma\|_{\infty}\) are bounded by a constant, by the compactness of \(U\). \(\|\gamma'\|_{\infty}\) is bounded by a constant since \((\gamma'(t), \gamma''(t))_g = 1\), and then \(\|\gamma''\|_{\infty}\) is also bounded by a constant, by the geodesic equation
\[ \frac{d^2}{dt^2} \gamma^i + \sum_{j,k} \Gamma^i_{jk} \left( \frac{d}{dt} \gamma^j \right) \left( \frac{d}{dt} \gamma^k \right) = 0. \]
5.3. Lemmas for the Proof of Lemma 4.4.

**Lemma 5.2.** Assume \(\triangle ABC, \triangle A'B'C', \triangle A''B'C''\) are geodesic Riemannian triangles with the same edge lengths \(a, b, c\), and \(\triangle A'B'C'\) has constant curvature \(-K < 0\), and \(\triangle A''B'C''\) has constant curvature \(K > 0\), and the curvature of \(\triangle ABC\) is always in \([-K, K]\), and

\[
\max\{a, b, c\} < \frac{\pi}{2\sqrt{K}}.
\]

Then we have

\(A' \leq A \leq A''\).

This is a combination of the well-known Toponogov comparison theorem and the CAT(K) Theorem. See Theorem 79 on page 339 in [57] for the Toponogov comparison theorem, and Characterization Theorem on page 704 in [58] or Theorem 1A.6 on page 173 in [59] for the CAT(K) Theorem. We omit the proof here.

**Lemma 5.3.** Assume \(\triangle ABC\) and \(\triangle A'B'C'\) are two geodesic Riemannian triangles with the same edge lengths \(a, b, c\), and the Gaussian curvature on \(\triangle ABC\) and \(\triangle A'B'C'\) are both bounded in \((-K, K)\), and \(\max\{a, b, c\} < \frac{\pi}{3\sqrt{K}}\). Then

\(|A' - A| \leq 2(a + b + c)^2 K|\).

**Proof.** By Lemma 5.2, without loss of generality, we may assume that \(\triangle ABC\) has constant curvature \(-K\) and \(\triangle A'B'C'\) has constant curvature \(K\). Then

\(A' - A > 0, \quad B' - B > 0, \quad C' - C > 0,\)

and by the Gauss-Bonnet theorem

\(0 < A' - A \leq (A' + B' + C') - (A + B + C) = K \cdot (|\triangle A'B'C'| + |\triangle ABC|).\)

By a simple scaling, the Heron’s formulae (17) and (18) can be generalized to the following

\[
\tan^2 \frac{|\triangle ABC| \cdot K}{4} = \tanh \frac{s \sqrt{K}}{2} \tan \frac{(s - a)\sqrt{K}}{2} \tan \frac{(s - b)\sqrt{K}}{2} \tanh \frac{(s - c)\sqrt{K}}{2},
\]

\[
\tan^2 \frac{|\triangle A'B'C'| \cdot K}{4} = \tan \frac{s \sqrt{K}}{2} \tan \frac{(s - a)\sqrt{K}}{2} \tan \frac{(s - b)\sqrt{K}}{2} \tan \frac{(s - c)\sqrt{K}}{2} \leq s^4 K^2,
\]

where \(s = (a + b + c)/2\). So

\(|\triangle ABC| \leq |\triangle A'B'C'| \leq \frac{4}{K} \tan \frac{|\triangle A'B'C'| \cdot K}{4} \leq \frac{4}{K} \cdot s^2 K = (a + b + c)^2\)

and we are done.

**Lemma 5.4.** Suppose \((M, g)\) is a closed Riemannian surface, and \(u \in C^\infty(M)\) is a conformal factor, then for any \(\epsilon > 0\), there exists \(\delta = \delta(M, g, u) > 0\) such that for any \(x, y \in M\) with \(d_g(x, y) < \delta\),

(a) there exists a unique shortest geodesic segment \(l\) in \((M, g)\), and \(l'\) in \((M, e^{2u}g)\), connecting \(x\) and \(y\), and

(b) the angle between \(l\) and \(l'\) at \(x\), measured in \((M, g)\), is less or equal to \(\epsilon\).
Proof. Assume $K(x)$ is the Gaussian curvature of $(M, g)$ at $x$, and $\|K\|_\infty = \max_{x \in M} |K(x)|$. It is easy to find a sufficiently small $\delta$ such that (a) is satisfied, and for any $x \in M$,

$$|B_g(x, \delta)|_g \cdot \|K\|_\infty < \epsilon/2.$$ 

Consider the unit circle bundle $$A = \{(x, \vec{a}) \in TM : x \in M, \|\vec{a}\|_{e^{2u}g} = 1\},$$

and assume we are in local coordinates $(v_1, v_2)$, and $\Gamma^i_{jk}$ are Christoffel symbols for $g$, and $\tilde{\Gamma}^i_{jk}$ are Christoffel symbols for $e^{2u}g$. Then for any geodesic $\gamma(t) = (v_1(t), v_2(t))$ in $(M, e^{2u}g)$ with $\gamma(0) = x$ and $\gamma'(0) = \vec{a}$, the geodesic curvature $k_g$ of $\gamma$ in $(M, g)$ at point $x$ is

$$-\sqrt{g_{11}g_{22} - g_{12}^2}(-\Gamma^1_{11} \dot{v}_1^3 + \Gamma^1_{22} \dot{v}_1^2 - (2\Gamma^1_{12} - \Gamma^0_{11}) \dot{v}_2^2 + (2\Gamma^1_{12} - \Gamma^0_{22}) \dot{v}_1 \dot{v}_2 + \ddot{v}_1 \dot{v}_2 - \ddot{v}_2 \dot{v}_1)$$

(see Theorem 17.19 in [60] for a proof). Here $(\dot{v}_1, \dot{v}_2) = \vec{a}$, and $\dot{v}_1, \dot{v}_2$ are determined by $(\dot{v}_1, \dot{v}_2)$ through the geodesic equations

$$\dot{v}_i + \sum_{j,k} \tilde{\Gamma}^i_{jk} \dot{v}_j \dot{v}_k = 0.$$ 

By this way $k_g$ can be viewed as a smooth function of $(x, \vec{a})$ defined on the compact manifold $A$, and thus is bounded by $[-C, C]$ for some constant $C = C(M, g, u)$.

As shown in Figure 2, assume $l$ and $l'$ start at $x$ and first meet at a point $z$. Let $l_0$ (resp. $l'_0$) be the part of $l$ (resp. $l'$) between $x$ and $z$, and then by the Jordan-Schoenflies theorem $l_0 \cup l'_0$ bounds a small closed disk $D$. Let $\theta_x$ (resp. $\theta_z$) be the intersecting angle of $l_0$ and $l'_0$ at $x$ (resp. at $z$). Then by the Gauss-Bonnet theorem

$$\int_D K \, dA_g + \int_{l_0} k_g ds_g + \int_{l'_0} k_g ds_g + (\pi - \theta_x) + (\pi - \theta_z) = 2\pi$$

and

$$\theta_x \leq \theta_x + \theta_z = \int_D K \, dA_g + \int_{l'_0} k_g ds_g \leq \|K\|_\infty \cdot |D|_g + C \cdot s_g(l') \leq \frac{\epsilon}{2} + C \cdot s_g(l')$$

where

$$s_g(l') \leq e^{\|u\|_{\infty}} \cdot s_{e^{2u}g}(l') \leq e^{\|u\|_{\infty}} \cdot s_{e^{2u}g}(l) \leq e^{2\|u\|_{\infty}} \cdot s_g(l) \leq e^{2\|u\|_{\infty}} \cdot \delta.$$ 

So $\theta_x \leq \epsilon$ if $\delta \leq \epsilon/(2C e^{2\|u\|_{\infty}})$. \qed
5.4. Proof of Part (a) of Lemma 4.4. Recall that

Part (a) of Lemma 4.4. Suppose $(M, g)$ is a closed Riemannian surface, and $T$ is a geodesic triangulation of $(M, g)$, and $l \in \mathbb{R}^{E(T)}_{>0}$ denotes the geodesic lengths of the edges, and $(T, l)_E$ or $(T, l)_H$ is $\epsilon$-regular.

(a) Given a conformal factor $u \in C^\infty(M)$, there exists a constant $\delta = \delta(M, g, u, \epsilon) > 0$ such that if $|l| < \delta$ then there exists a geodesic triangulation $T'$ in $(M, e^{2u}g)$ such that $V(T') = V(T)$, and $T'$ is homotopic to $T$ relative to $V(T)$. Further $(T, l)_E$ and $(T, l)_H$ are $\frac{\epsilon}{2}$-regular where $\bar{l} \in \mathbb{R}^{E(T)} \cong \mathbb{R}^{E(T')}$ denotes the geodesic lengths of the edges of $T'$ in $(M, e^{2u}g)$.

Proof of Part (a) of Lemma 4.4 Denote

1. $\theta^i_{jk}(M)$ as the inner angle of the geodesic triangle in $F(T)$ in $(M, g)$, and
2. $\theta^i_{jk}(E)$ as the inner angle in $(T, l)_E$, and
3. $\theta^i_{jk}(H)$ as the inner angle in $(T, l)_H$, and
4. $\bar{\theta}^i_{jk}(M)$ as the inner angle of the geodesic triangle in $F(T')$ in $(M, e^{2u}g)$, and
5. $\bar{\theta}^i_{jk}(E)$ as the inner angle in $(T, \bar{l})_E$, and
6. $\bar{\theta}^i_{jk}(H)$ as the inner angle in $(T, \bar{l})_H$.

By Lemma 5.3 and 5.4 if $\delta(M, g, u, \epsilon)$ is sufficiently small, then

(a) $|\theta^i_{jk}(M) - \bar{\theta}^i_{jk}(E)| \leq \epsilon/12$ and $|\theta^i_{jk}(M) - \bar{\theta}^i_{jk}(H)| \leq \epsilon/12$, and

(b) for any $ij \in E(T)$ there exists a unique shortest geodesic $e_{ij}$ in $(M, e^{2u}g)$ connecting $i, j$, and

(c) for each $\triangle ijk \in F(T), e_{ij}, e_{ik}, e_{jk}$ bounds a geodesic triangle $F_{ijk}$ in $(M, e^{2u}g)$, and

(d) $|\bar{\theta}^i_{jk}(M) - \bar{\theta}^i_{jk}(E)| \leq \epsilon/12, |\bar{\theta}^i_{jk}(E) - \bar{\theta}^i_{jk}(M)| \leq \epsilon/12, |\bar{\theta}^i_{jk}(H) - \bar{\theta}^i_{jk}(M)| \leq \epsilon/12$, and thus $(T, \bar{l})_E$ and $(T, \bar{l})_H$ are $\epsilon/2$-regular, and

(e) for any vertex $i \in V(T) = V(T')$, its adjacent edges $\{ij\}_{j \sim i}$ in $T$ are placed in the same order as the adjacent edges $\{e_{ij}\}_{j \sim i}$ in $T'$, i.e., there are no folding triangles in $T'$.

We can define a continuous map $f : M \to M$ such that

1. $f(i) = i$ for any $i \in V$, and
2. for any edge $ij \in E(T)$, $f$ is a homeomorphism from $ij$ to $e_{ij}$, and
3. for any $\triangle ijk \in F(T)$, $f$ is a homeomorphism from $\triangle ijk$ to $F_{ijk}$.

Then by above property (e), $f$ is locally a homeomorphism. Further if $\delta$ is sufficiently small, $f$ is homotopic to the identity. Therefore $f$ is a global homeomorphism and $T' = (V, \{e_{ij}\}, \{F_{ijk}\})$ is a triangulation of $M$. □
5.5. **Proof of Part (b) of Lemma 4.4.** Recall that

**Part (b) of Lemma 4.4.** Suppose \((M, g)\) is a closed Riemannian surface, and \(T\) is a geodesic triangulation of \((M, g)\), and \(l \in \mathbb{R}^E(T)\) denotes the geodesic lengths of the edges, and \((T, l)_E\) or \((T, l)_H\) is \(\epsilon\)-regular.

(b) There exists a constant \(\delta = \delta(M, g, \epsilon)\) such that if \(|l| < \delta\), \((T, l)\) is \(C\)-isoperimetric for some constant \(C = C(M, g, \epsilon)\).

We first prove a continuous version.

**Lemma 5.5.** Suppose \((M, g)\) is a closed Riemannian surface, and \(\Omega \subset M\) is an open domain with \(\partial \Omega\) being a finite disjoint union of piecewise smooth Jordan curves, then there exists a constant \(C = C(M, g) > 0\) such that

\[
\min\{|\Omega|, |M - \Omega|\} \leq CL^2
\]

where \(L = s(\partial \Omega)\) denotes the length of \(\partial \Omega\) in \((M, g)\).

**Proof.** If \(\Omega\) is simply connected, then it is well known (See Theorem 4.3 in [61]) that

\[
L^2 \geq |\Omega|(4\pi - 2 \int_\Omega K^+)
\]

where \(K^+(p) = \max\{0, K(p)\}\). Pick \(r = r(M, g) > 0\) smaller than the injectivity radius of \((M, g)\), such that

\[
|B(p, r)| \cdot \|K\|_\infty \leq \pi
\]

for any \(p \in M\).

Now we pick our constant

\[
C(M, g) = \max\{\frac{2\pi}{\pi}, \frac{|M|}{r^2}\}.
\]

If \(L \geq r\), then \(CL^2 \geq |M|\) and we are done. If \(\Omega \subset B(p, r)\) for some \(p \in M\) and is connected, then without loss of generality we may assume \(\Omega\) is simply connected by filling up the holes, and then

(23) \[
CL^2 \geq \frac{2}{\pi} |\Omega|(4\pi - 2 \int_\Omega K^+) \geq |\Omega|(8 - \frac{4\pi}{\pi} |B(p, r)| \cdot \|K\|_\infty) \geq 4|\Omega|
\]

and we are done.

If \(\Omega\) has multiple connected components \(\Omega_1, \ldots, \Omega_n\) with the boundary lengths \(L_1, \ldots, L_n\) respectively, such that each \(\Omega_i\) is in some Riemannian disk \(B(p, r)\), then \(L \geq (L_1 + \ldots + L_n)/2\) since any component of \(\partial \Omega\) is on at most two \(\partial \Omega_i\)’s. So by equation (23)

(24) \[
CL^2 \geq \frac{1}{4} \sum_{i=1}^n CL_i^2 \geq \sum_{i=1}^n |\Omega_i| = |\Omega|
\]

and we are done.

Now we assume \(L < r\) and \(\partial \Omega\) contains Jordan curves \(\gamma_1, \ldots, \gamma_n\) with lengths \(L_1, \ldots, L_n\) respectively. Since \(L_i \leq r\), \(\gamma_i\) is in some Riemannian disk \(B(p, r)\). By the Jordan-Schoenflies theorem, \(\gamma_i\) separates \(M\) into a smaller domain \(U_i \subset B(p, r)\) and a larger domain \(V_i = M - \bar{U}_i\).
and \( \bar{U}_i \) is a topological closed disk. For any \( i \neq j \), since \( \gamma_i \) and \( \gamma_j \) are disjoint, \( \bar{U}_i \subset \bar{U}_j \) or \( \bar{U}_j \subset \bar{U}_i \) or \( \bar{U}_i \cap \bar{U}_j = \emptyset \). So \( \bigcup_{i=1}^{n} \bar{U}_i \) is a finite disjoint union of topological disks, and thus \( M - \bigcup_{i=1}^{n} \bar{U}_i \) is connected. If \( \Omega \subset \bigcup_{i=1}^{n} U_i \), then by equation (24) we are done. Otherwise, \( M - \bigcup_{i=1}^{n} \bar{U}_i \subset \Omega \) and \( M - \bar{\Omega} \subset \bigcup_{i=1}^{n} U_i \), and again by equation (24) \( CL^2 \geq |M - \bar{\Omega}| \) and we are done.

Now we prove part (b) of Lemma 4.4 for the special cases that \((M, g)\) has constant curvature 0 or \(\pm 1\).

**Proof of Part (b) of Lemma 4.4 for the cases of constant curvature 0 or \(\pm 1\).** In this proof each triangle \( \triangle ijk \in F(T) \) is identified as a geodesic triangle in \((M, g)\). Assume \( \delta < 0.1 \), \( V_0 \subset V \), and \( V_1 = V - V_0 \). Let

\[
E_0 = \{ij \in E : i, j \in V_0\}, \quad E_1 = \{ij \in E : i, j \in V_1\}.
\]

Notice that \( \partial V_0 = \partial V_1 \) and \( E = E_0 \cup E_1 \cup \partial V_0 \) is a disjoint union.

For any triangle \( \triangle ijk \in F(T) \), 0 or 2 of its edges are in \( \partial V_0 \). So \( F(T) = F_0 \cup F_2 \) where

\[
F_0 = \{\triangle ijk \in F(T) : \triangle ijk \text{ has 0 edges in } \partial V_0\}, \quad \text{and} \quad F_2 = \{\triangle ijk \in F(T) : \triangle ijk \text{ has 2 edges in } \partial V_0\}.
\]

If \( \triangle ijk \in F_2 \) and \( ij, ik \in \partial V_0 \), let \( \gamma_{ijk} \) be the geodesic segment in \( \triangle ijk \) connecting the middle points \( m_{ij} \) of \( ij \), and \( m_{ik} \) of \( ik \). Then by the triangle inequality \( \frac{1}{2} (l_{ij} + l_{ik}) \geq s(\gamma_{ijk}) \). \( \gamma_{ijk} \) cut \( \triangle ijk \) into two relative open domains \( P^0_{ijk} \) and \( P^1_{ijk} \) such that \( P^0_{ijk} \cap V_0 \neq \emptyset \) and \( P^1_{ijk} \cap V_1 \neq \emptyset \). Given \( \triangle ijk \in F_0 \),

1. if \( i, j, k \in V_0 \), denote \( P^0_{ijk} = \triangle ijk \) and \( P^1_{ijk} = \emptyset \), and
2. if \( i, j, k \in V_1 \), denote \( P^1_{ijk} = \triangle ijk \) and \( P^0_{ijk} = \emptyset \).

The union

\[
\Gamma = \bigcup_{\triangle ijk \in F(T)} \gamma_{ijk}
\]

is a finite disjoint union of piecewise smooth Jordan curves in \((M, g)\), and

\[
P^0 = \bigcup_{\triangle ijk \in F(T)} P^0_{ijk}, \quad \text{and} \quad P^1 = \bigcup_{\triangle ijk \in F(T)} P^1_{ijk}.
\]
are two open domains of \((M, g)\) such that \(\partial P^0 = \Gamma\) and \(P^1 = M - \overline{P^0}\). The above notations are shown in Figure 3. By Lemma 5.5, it suffices to prove that if \(\delta < 0.1\),

\[
|P^1| \geq \frac{\epsilon}{60}(|V|_l - |V_0|_l),
\]
and

\[
|P^0| \geq \frac{\epsilon}{60}|V_0|_l,
\]
and

\[
|\partial V_0| \leq |\partial V_0|_l.
\]

By part (b) of Lemma 4.4 and Remark 4.2 we have that

\[
|V|_l - |V_0|_l = \sum_{ij \in E_0} l_{ij}^2 \leq \frac{4}{\epsilon} \sum_{ij \in E_1 \cup \partial V_0} (|\triangle ijk| + |\triangle ijk'|) \leq \frac{12}{\epsilon} \sum_{\triangle ijk \in F : \partial ijk \cap P^1 \neq \emptyset} |\triangle ijk| \leq \frac{60}{\epsilon} \sum_{\triangle ijk \in F} |P^1_{ijk}| = \frac{60}{\epsilon} |P^1|,
\]
and

\[
|V_0|_l = \sum_{ij \in E_0} l_{ij}^2 \leq \frac{60}{\epsilon} |P^0|,
\]
and

\[
|\partial V_0| = \sum_{ij \in \partial V_0} l_{ij} = \frac{1}{2}(l_{ij} + l_{ik}) \geq \sum_{\triangle ijk \in F_2} s(\gamma_{ijk}) = s(\Gamma).
\]

Now let us prove part (b) of Lemma 4.4 for general smooth surfaces.

**Proof of Part (b) of Lemma 4.4** By the Uniformization theorem, there exists \(u = u_M, g \in C^\infty(M)\) such that \(e^{2u}g\) has constant curvature \(\pm 1\) or 0. By part (a) of Lemma 4.4 if \(\delta\) is sufficiently small, we can find a geodesic triangulation \(T'\) in \((M, e^{2u}g)\) such that \(V(T) = V(T')\), and \(T, T'\) are homotopic relative to \(V\). Further by the inequalities in (a) and (d) in the proof of Lemma 4.4 (a), if \(\delta\) is sufficiently small, any inner angle of \(T'\) in \((M, e^{2u}g)\) is at least \(\epsilon/2\). Let \(\tilde{1} \in \mathbb{R}^{E(T)} \simeq \mathbb{R}^{E(T')}\) denote the geodesic edge lengths in \((M, e^{2u}g)\). Then by our result on surfaces of constant curvature \(\pm 1\) or 0, if \(\delta = \delta(M, e^{2u}g)\) is sufficiently small, \((T, \tilde{1})\) is \(C\)-isoperimetric for some constant \(C = C(M, e^{2u}g) > 0\). Since \(e^{-\|u\|_\infty} \leq \tilde{l}_{ij}/l_{ij} \leq e^{\|u\|_\infty}\), \((T, \tilde{1})\) is \((e^{4\|u\|_\infty} C)\)-isoperimetric.

6. PROOF OF THE DISCRETE ELLIPTIC ESTIMATE

Recall that

**Lemma 2.3.** Assume \((G, l)\) is \(C_1\)-isoperimetric, and \(x \in \mathbb{R}_A, \eta \in \mathbb{R}_E, C_2 > 0, C_3 > 0\) are such that

(i) \(|x_{ij}| \leq C_3 l_{ij}^2\) for any \(i, j \in E\), and

(ii) \(|\eta_{ij}| \geq C_3\) for any \(i, j \in E\).
Then
\[ |\Delta_{\eta}^{-1} \circ \text{div}(x)| \leq \frac{4C_2 \sqrt{C_1 + 1}}{C_3} |l| \cdot |V|^{1/2}. \]

Further if \( y \in \mathbb{R}^V \) and \( C_4 > 0 \) and \( D \in \mathbb{R}^{V \times V} \) is a diagonal matrix such that
\[ |y_i| < C_4 D_{ii} |l| \cdot |V|^{1/2} \]
for any \( i \in V \), then
\[ |(D - \Delta_{\eta})^{-1}(\text{div}(x) + y)| \leq \left( C_4 + \frac{8C_2 \sqrt{C_1 + 1}}{C_3} \right) |l| \cdot |V|^{1/2}. \]

We will first prove Lemma 2.3 assuming Lemma 6.1, and then prove Lemma 6.1.

Proof. Assume
\begin{enumerate}
  \item \( z = \Delta^{-1}(\text{div}(x)) \), and
  \item \( a, b \in V \) are such that \( z_a = \max_i z_i \geq 0 \) and \( z_b = \min_i z_i \leq 0 \), and \( a \neq b \), and
  \item \( u \in \mathbb{R}^V \) is such that \( (\Delta u)_a = 1 \), and \( (\Delta u)_b = -1 \), and \( (\Delta u)_i = 0 \ \forall i \neq a, b \).
\end{enumerate}

By the Green’s identity Lemma 2.1 and Lemma 6.1
\[ |z| \leq z_a - z_b = \sum_i z_i (\Delta u)_i = \sum_i u_i (\Delta z)_i = \sum_i u_i \cdot \text{div}(x)_i \]
\[ = \sum_i u_i \sum_{j, j \sim i} x_{ij} = \sum_{ij \in E} (u_i - u_j) \cdot x_{ij} \leq C_2 \sum_{ij \in E} |u_i - u_j| \cdot l_{ij}^2 \leq \frac{4C_2 \sqrt{C_1 + 1}}{C_3} |l| \cdot |V|^{1/2}. \]

Let
\[ w = (D - \Delta)^{-1}(\text{div}(x) + y) + z, \]
and then
\[ (D - \Delta)w = \text{div}(x) + y + (D - \Delta)z = y + Dz. \]

Assume \( w_i = \max_k w_k \), and then by comparing the \( i \)-th component in (28) we have
\[ D_{ii} w_i \leq ((D - \Delta)w)_i = y_i + D_{ii} z_i \leq y_i + D_{ii} |z|. \]
So
\[ \max_k w_k = w_i \leq |z| + y_i / D_{ii} \leq |z| + \max_k (y_k / D_{kk}) \]
and similarly we also have that
\[ \min_k w_k \geq -|z| + \min_k (y_k / D_{kk}). \]
So
\[ |(D - \Delta)^{-1}(\text{div}(x) + y)| \leq |w| + |z| \leq 2|z| + \max_k (|y_k| / D_{kk}) \]
and we are done.
Lemma 6.1. Assume \((G, l)\) is \(C_1\)-isoperimetric, and the weight \(\eta \in \mathbb{R}_{>0}^E\) satisfies that \(\eta_{ij} \geq C_2\) for some constant \(C_2 > 0\), and \(u \in \mathbb{R}^V\) satisfies that

\[
(\Delta u)_a = 1, \quad \text{and} \quad (\Delta u)_b = -1, \quad \text{and} \quad (\Delta u)_i = 0 \quad \forall i \neq a, b.
\]

then

\[
\sum_{ij \in E} l_{ij}^2 |u_i - u_j| \leq \frac{4\sqrt{C_1 + 1}}{C_2} |l| \cdot |V|^{1/2}.
\]

Proof. We consider the 1-skeleton \(X\) of the graph \(G\) with edge length \(l\). More specifically \(X\) can be constructed as follows. Let \(\bar{X}\) be a disjoint union of line segments \(\{e_{ij} : ij \in E\}\) where each \(e_{ij}\) has length \(l_{ij}\) and two endpoints \(v_{ij}^0, v_{ij}^1\). Then we obtain a connected quotient space \(X\) by identifying the points in \(v_i := \{v_{ij}^0 : ij \in E\}\) for any \(i \in V\).

Assume \(\mu\) is the standard 1-dimensional Lebesgue measure on \(X\) such that \(\mu(e_{ij}) = l_{ij}\). Let \(\nu\) be another measure on \(X\) such that \(d\nu/d\mu \equiv l_{ij}\) on edge \(e_{ij}\). Then we have that \(\nu(e_{ij}) = l_{ij}^2\) and \(\nu(X) = |V| l\).

Assume \(u : V \to \mathbb{R}\) is linearly extended to the 1-skeleton \(X\), and then by maximum principle, \(u_a = \min(u)\) and \(u_b = \max(u)\). Let \(\bar{u} \in (u_a, u_b)\) be such that

\[
\nu(x \in X : u(x) < \bar{u}) \leq |V| l/2, \quad \nu(x \in X : u(x) > \bar{u}) \leq |V| l/2.
\]

Let \(f(x) = l_{ij}|u_i - u_j|\) for \(x \in e_{ij}\), and then \(f\) is well-defined almost everywhere on \(X\), and

\[
\sum_{ij \in E} l_{ij}^2 |u_i - u_j| = \int_X f(x) d\mu \leq \int_{u_a \leq u(x) \leq \bar{u}} f(x) d\mu + \int_{\bar{u} \leq u(x) \leq u_b} f(x) d\mu.
\]

We will prove

\[
\int_{u_a \leq u(x) \leq \bar{u}} f(x) d\mu \leq \frac{2\sqrt{C_1 + 1}}{C_2} |l| \cdot \sqrt{\nu(u(x) < \bar{u})} \leq \frac{2\sqrt{C_1 + 1}}{C_2} |l| \cdot |V|^{1/2}
\]

and then by the symmetry \(\int_{\bar{u} \leq u(x) \leq u_b} f(x) d\mu\) has the same upper bound and we are done.

Let \(u_a = p_0 < p_1 < \cdots < p_s = \bar{u}\) such that \(\{p_0 \cdots p_{s-1}\} = \{u_i : i \in V, u_i < \bar{u}\}\). Noticing that \(\int_{u(x)=p} f(x) d\mu = 0\) for any \(p \in \mathbb{R}\), it suffices to prove that for any \(k \in \{1, \ldots, s\}\)

\[
\int_{p_{k-1} < u(x) < p_k} f(x) d\mu \leq \frac{2\sqrt{C_1 + 1}}{C_2} |l| \cdot \left(\sqrt{\nu(u(x) < p_k)} - \sqrt{\nu(u(x) < p_{k-1})}\right).
\]
In the remaining of the proof, we fix a \( k \in \{1, \ldots, s\} \) and let \( V_k = \{ i \in V : u(i) \leq p_{k-1} \} \). Then for any \( i \in V_k \) and \( ij \in \partial V_k \), \( u_j \geq u_i \) and then

\[
0 = \sum_{i \in V_k} (\Delta u)_i = \sum_{i \in V_k} \sum_{j \sim i} \eta_{ij} (u_j - u_i) = \sum_{i \sim a} \eta_{ia} (u_a - u_i) + \sum_{ij \in \partial V_k : i \in V_k} \eta_{ij} (u_j - u_i) \geq - (\Delta u)_a + C_2 \sum_{ij \in \partial V_k} |u_j - u_i|.
\]

Therefore,

\[
(29) \quad C_2 \sum_{ij \in \partial V_k} |u_j - u_i| \leq (\Delta u)_a = 1.
\]

Let \( e'_{ij} = \{ x : p_{k-1} < u(x) < p_k \} \cap e_{ij} \), and \( l'_{ij} = \mu(e'_{ij}) \). Then \( l'_{ij} = 0 \) if \( ij \notin \partial V_k \). If \( ij \in \partial V_k \), let \( i' \) and \( j' \) be the two endpoints of \( e'_{ij} \). Then \( \{ u(i'), u(j') \} = \{ p_{k-1}, p_k \} \), and

\[
\frac{l'_{ij}}{l_{ij}} = \frac{|u(j') - u(i')|}{|u_j - u_i|} = \frac{p_k - p_{k-1}}{|u_j - u_i|}.
\]

So

\[
(30) \quad \int_{p_{k-1} < u(x) < p_k} f(x) d\mu = \sum_{ij \in \partial V_k} l'_{ij} l_{ij} |u_i - u_j| = (p_k - p_{k-1}) \sum_{ij \in \partial V_k} l^2_{ij} \leq (p_k - p_{k-1}) |l| \cdot |\partial V_k|.
\]

On the other hand, by inequality \((29)\) and Cauchy’s inequality,

\[
\nu(p_{k-1} < u(x) < p_k) = \sum_{ij \in \partial V_k} l'_{ij} l_{ij} = (p_k - p_{k-1}) \sum_{ij \in \partial V_k} l^2_{ij} \leq (p_k - p_{k-1}) \left( \sum_{ij \in \partial V_k} \frac{l^2_{ij}}{|u_j - u_i|} \right) \cdot \left( \sum_{ij \in \partial V_k} |u_j - u_i| \right) \cdot C_2 \geq C_2 (p_k - p_{k-1}) \left( \sum_{ij \in \partial V_k} l_{ij} \right)^2 = C_2 (p_k - p_{k-1}) |\partial V_k|_l^2.
\]

Since \((G, l)\) is \( C_1 \)-isoperimetric, we have that

\[
(32) \quad \nu(u(x) < p_k) \leq |V_k| \cdot l + \sum_{ij \in \partial V_k} l^2_{ij} \leq C_1 |\partial V_k|_l^2 + |\partial V_k|_l^2 = (C_1 + 1) |\partial V_k|_l^2.
\]

Dividing \((31)\) by \( \sqrt{(32)} \) and then we have

\[
(33) \quad \frac{\nu(p_{k-1} < u(x) < p_k)}{\sqrt{\nu(u(x) < p_k)}} \geq \frac{C_2}{\sqrt{C_1 + 1}} (p_k - p_{k-1}) |\partial V_k|_l.
\]
Combining equations \(30\) and \(33\) and then
\[
\int_{p_{k-1} < u(x) < p_k} f(x) \, d\mu \leq \frac{\sqrt{C_1 + 1}}{C_2} \cdot |l| \cdot \frac{\nu(p_{k-1} < u(x) < p_k)}{\sqrt{\nu(u(x) < p_k)}}
\]
\[
\leq \frac{\sqrt{C_1 + 1}}{C_2} \cdot |l| \cdot \frac{\nu(u(x) < p_k) - \nu(u(x) < p_{k-1})}{\sqrt{\nu(u(x) < p_k)}}
\]
\[
\leq 2\frac{\sqrt{C_1 + 1}}{C_2} \cdot |l| \cdot \frac{\nu(u(x) < p_k) - \nu(u(x) < p_{k-1})}{\sqrt{\nu(u(x) < p_k)} + \sqrt{\nu(u(x) < p_{k-1})}}
\]
\[
= 2\frac{\sqrt{C_1 + 1}}{C_2} \cdot |l| \cdot \left( \sqrt{\nu(u(x) < p_k)} - \sqrt{\nu(u(x) < p_{k-1})} \right)
\]
and we are done. \(\square\)

**Appendices**

In this appendix we prove Lemma 3.3 and 3.4 and 3.5 and 3.6.

**Lemma 3.3.** Given a Euclidean triangle \(\triangle ABC\), if we view \(A, B, C\) as functions of the edge lengths \(a, b, c\), then
\[
\frac{\partial A}{\partial b} = -\cot C, \quad \frac{\partial A}{\partial a} = \frac{\cot B + \cot C}{a} = \frac{1}{b \sin C}.
\]

Further if \((u_A, u_B, u_C) \in \mathbb{R}^3\) is a discrete conformal factor, and
\[
a = e^{\frac{1}{2}(u_B+u_C)}a_0, \quad b = e^{\frac{1}{2}(u_A+u_C)}b_0, \quad c = e^{\frac{1}{2}(u_A+u_B)}c_0
\]
for some constants \(a_0, b_0, c_0 \in \mathbb{R}_{>0}\), then
\[
(34) \quad \frac{\partial A}{\partial u_B} = \frac{1}{2} \cot C, \quad \frac{\partial A}{\partial u_A} = -\frac{1}{2}(\cot B + \cot C).
\]

**Proof.** Take the partial derivative on
\[
\cos A = \frac{b^2 + c^2 - a^2}{2bc}
\]
and we have
\[
-\sin A \frac{\partial A}{\partial b} = \frac{2b}{2bc} - \frac{b^2 + c^2 - a^2}{2b^2c} = \frac{b^2 + a^2 - c^2}{2b^2c} = \frac{a \cos C}{bc},
\]
and
\[
\frac{\partial A}{\partial b} = -\frac{a \cos C}{bc \sin A} = -\frac{\cos C}{b \sin C} = -\frac{\cot C}{b}.
\]
Similarly
\[
-\sin A \frac{\partial A}{\partial a} = -\frac{a}{bc}
\]
and
\[
\frac{\partial A}{\partial a} = \frac{a}{bc \sin A} = \frac{1}{b \sin C} \sin A = \frac{\sin B \cos C + \sin C \cos B}{a \sin B \sin C} = \frac{\cot B + \cot C}{a}.
\]
Then equation (34) can be computed easily. □

**Lemma 3.4.** Given a hyperbolic triangle $\triangle ABC$, if we view $A, B, C$ as functions of the edge lengths $a, b, c$, then

$$\frac{\partial A}{\partial b} = -\frac{\cot C}{\sinh b}, \quad \frac{\partial A}{\partial a} = \frac{1}{\sinh b \sin C}.$$ 

Further if $(u_A, u_B, u_C) \in \mathbb{R}^3$ is a discrete conformal factor, and

$$\sinh \frac{a}{2} = e^{\frac{1}{2}(u_B+u_C)} \sinh \frac{a_0}{2}, \quad \sinh \frac{b}{2} = e^{\frac{1}{2}(u_A+u_C)} \sinh \frac{b_0}{2}, \quad \sinh \frac{c}{2} = e^{\frac{1}{2}(u_A+u_B)} \sinh \frac{c_0}{2}$$

for some constants $a_0, b_0, c_0 \in \mathbb{R}_{>0}$, then

(35) $$\frac{\partial A}{\partial u_B} = \frac{1}{2} \cot \tilde{C}(1 - \tanh^2 \frac{c}{2}),$$

and

(36) $$\frac{\partial A}{\partial u_A} = -\frac{1}{2} \cot \tilde{B}(1 + \tanh^2 \frac{b}{2}) - \frac{1}{2} \cot \tilde{C}(1 + \tanh^2 \frac{c}{2}),$$

where $\tilde{B} = \frac{1}{2}(\pi + B - A - C)$ and $\tilde{C} = \frac{1}{2}(\pi + C - A - B)/2$.

**Proof.** Take the partial derivative on

$$\cos A = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$$

and we have

$$-\sin A \frac{\partial A}{\partial b} = \frac{\sinh b \cosh c}{\sinh b \sinh c} - \frac{\cosh^2 b \cosh c - \cosh a \cosh b}{\sinh^2 b \sinh c} = \frac{\cosh a \cosh b - \cosh c}{\sinh^2 b \sinh c} \frac{\sinh a}{\sinh b \sinh c} \cos C,$$

and then by the hyperbolic law of sines,

$$\frac{\partial A}{\partial b} = -\frac{\cos C}{\sinh b \sin C} = -\frac{\cot C}{\sinh b}.$$ 

Similarly

$$-\sin A \frac{\partial A}{\partial a} = -\frac{\sinh a}{\sinh b \sinh c}$$

and then again by the hyperbolic law of sines

$$\frac{\partial A}{\partial a} = \frac{1}{\sinh b \sin C}.$$ 

To prove (35) and (36) we need to compute

$$\frac{\partial c}{\partial u_A} = \frac{\partial \sinh(c/2)}{\partial u_A} \left/ \frac{\partial \sinh(c/2)}{\partial c} \right| = \frac{1}{2} \sinh \frac{c}{2} \left/ \left( \frac{1}{2} \cosh \frac{c}{2} \right) \right| = \tanh \frac{c}{2},$$

and other similar formulae hold.

Since

$$\tanh \frac{x}{2} = \frac{\sinh \frac{x}{2}}{\cosh \frac{x}{2}} = \frac{\sinh \frac{x}{2} \cosh \frac{x}{2}}{\cosh^2 \frac{x}{2}} = \frac{\sinh x}{\cosh x + 1}.$$
and
\[ \text{cosh } b + 1 = \frac{\cos A \cos C + \cos B}{\sin A \sin C} + 1 = \frac{\cos(A - C) + \cos B}{\sin A \sin C}, \]
we have
\[ \frac{\tanh \frac{b}{2}}{\tanh \frac{c}{2}} = \frac{\sin b \cdot \cosh c + 1}{\sin h \cdot \cosh b + 1} \]
\[ = \frac{\sin B \cdot \cos(A - B) + \cos C}{\sin A \sin B \cdot \cos(A - C) + \cos B} \frac{\sin C \cdot \cos(A - C) + \cos C}{\cos A \sin B}, \]
and then
\[ -\cos A + \frac{\tan \frac{b}{2}}{\tanh \frac{c}{2}} = \frac{\cos(A - B) + \cos C - \cos(A - C) \cos A - \cos B \cos A}{\cos(A - C) + \cos B} \]
\[ = \frac{(\cos(A - B) - \cos B \cos A) + (\cos C - \cos(A - C) \cos A)}{\cos(A - C) + \cos B} \]
\[ = \sin A \sin B + \sin A \sin(A - C) \]
\[ = \sin A \cdot \frac{\sin B + \sin(A - C)}{\cos(A - C) + \cos B} \]
\[ = \sin A \cdot \frac{\sin \frac{B + A - C}{2} \cos \frac{B - A - C}{2}}{\cos \frac{A + B - C}{2} \cos \frac{A - B - C}{2}} \]
\[ = \sin A \cdot \tan \frac{A + B - C}{2} \]
\[ = \sin A \cot \tilde{C}, \]
and then
\[ \frac{\partial B}{\partial u_A} = \frac{\partial B}{\partial c} \frac{\partial c}{\partial u_A} + \frac{\partial B}{\partial b} \frac{\partial b}{\partial u_A} \]
\[ = -\frac{\cot A}{\sinh c} \tan \frac{c}{2} + \frac{1}{\sin A \sinh c} \tan \frac{b}{2} \]
\[ = \frac{1}{2\cosh^2 \frac{c}{2}} \left( -\frac{\cos A}{\sin A} + \frac{1}{\sin A \tan \frac{b}{2}} \right) \]
\[ = \frac{1}{2} \left( 1 - \tanh^2 \frac{c}{2} \right) \frac{1}{\sin A} \left( -\cos A + \frac{\tanh \frac{b}{2}}{\tanh \frac{b}{2}} \right) \]
\[ = \frac{1}{2} \left( 1 - \tanh^2 \frac{c}{2} \right) \cot \tilde{C}. \]
By the symmetry equation (35) is true, and for the equation (36), we have that
\[ \frac{\partial A}{\partial u_A} = \frac{\partial A}{\partial c} \frac{\partial c}{\partial u_A} + \frac{\partial A}{\partial b} \frac{\partial b}{\partial u_A} = -\frac{\cot B}{\sinh c} \tan \frac{c}{2} - \frac{\cot C}{\sinh b} \tan \frac{b}{2}. \]
So we need to show

\[- \frac{\cot B}{\sinh c} \tanh \frac{c}{2} - \frac{\cot C}{\sinh b} \tanh \frac{b}{2} = -\frac{1}{2} \cot \tilde{C}(1 + \tanh^2 \frac{c}{2}) - \frac{1}{2} \cot \tilde{B}(1 + \tanh^2 \frac{b}{2}) .\]

Since

\[\frac{\tanh \frac{x}{2}}{\sinh x} = \frac{\sinh \frac{x}{2}}{2 \sinh \frac{x}{2} \cosh^2 \frac{x}{2}} = \frac{1}{\cosh^2 \frac{x}{2}} = \frac{2}{\cosh x + 1}\]

and

\[1 + \tanh^2 \frac{x}{2} = \frac{\cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2}}{\cosh^2 \frac{x}{2}} = \frac{2 \cosh x}{\cosh x + 1},\]

we only need to show

\[\frac{\cot B}{\cosh c + 1} + \frac{\cot C}{\cosh b + 1} = \cot \tilde{C} \frac{\cosh c}{\cosh c + 1} + \cot \tilde{B} \frac{\cosh b}{\cosh b + 1}.\]

We will show that

\[\cot \tilde{B} \frac{\cosh b}{\cosh b + 1} - \frac{\cot B}{\cosh c + 1}\]

is anti-symmetric with respect to \(B\) and \(C\). Recall equation (37) and we have that

\[\cosh b + 1 = \frac{\cos(A - C) + \cos B}{\sin A \sin C} = 2 \cos \frac{A + B - C}{2} \cos \frac{B + C - A}{2},\]

and

\[\frac{\cosh b}{\cosh b + 1} = \frac{\cos A \cos C + \cos B}{\cos(A - C) + \cos B} = \frac{\cos A \cos C + \cos B}{2 \cos \frac{A + B - C}{2} \cos \frac{B + C - A}{2}},\]

and

\[\cot \tilde{B} = \tan \left(\frac{\pi}{2} - \tilde{B}\right) = \tan \frac{A + C - B}{2} .\]

So

\[\cot \tilde{B} \frac{\cosh b}{\cosh b + 1} - \frac{\cot B}{\cosh c + 1} = \tan \frac{A + C - B}{2} \cdot \frac{\cos A \cos C + \cos B}{2 \cos \frac{A + B - C}{2} \cos \frac{B + C - A}{2}} - \cot \tilde{B} \frac{\sin A \sin B}{2 \cos \frac{A + C - B}{2} \cos \frac{B + C - A}{2}} \]

\[= \frac{\sin \frac{A + C - B}{2} (\cos A \cos C + \cos B) - \sin A \cos B \cos \frac{A + B - C}{2}}{2 \cos \frac{A + C - B}{2} \cos \frac{B + C - A}{2} \cos \frac{A + B - C}{2}} .\]
The denominator in the above fraction is symmetric, so we only need to show the numerator is anti-symmetric with respect to \( B, C \).

\[
\sin \frac{A + C - B}{2} (\cos A \cos C + \cos B) - \sin A \cos B \cos \frac{A + B - C}{2}
\]

\[
= (\sin \frac{A}{2} \cos \frac{C - B}{2} + \sin \frac{C - B}{2} \cos \frac{A}{2}) (\cos A \cos C + \cos B) - \sin A \cos B (\cos \frac{A}{2} \cos \frac{C - B}{2} + \sin \frac{A}{2} \sin \frac{C - B}{2})
\]

\[
= \sin \frac{C - B}{2} (\cos \frac{A}{2} \cos A \cos C + \cos \frac{A}{2} \cos B - \sin A \cos B \sin \frac{A}{2})
\]

\[
+ \cos \frac{C - B}{2} (\sin \frac{A}{2} \cos A \cos C + \sin \frac{A}{2} \cos B - \sin A \cos B \cos \frac{A}{2})
\]

\[
= \sin \frac{C - B}{2} (\cos \frac{A}{2} \cos A \cos C + \cos A \cos B \cos \frac{A}{2})
\]

\[
+ \cos \frac{C - B}{2} (\sin \frac{A}{2} \cos A \cos C - \cos A \cos B \sin \frac{A}{2})
\]

\[
= \sin \frac{C - B}{2} \cos A \cos \frac{A}{2} (\cos C + \cos B) + \cos \frac{C - B}{2} \sin \frac{A}{2} \cos A (\cos C - \cos B)
\]

is indeed anti-symmetric with respect to \( B, C \). \( \square \)

**Lemma 3.5.** Given a Euclidean triangle \( \triangle ABC \), if all the angles in \( \triangle ABC \) are at least \( \epsilon > 0 \), and \( \delta < \epsilon^2 / 48 \), and \( \| a' - a \| \leq \delta a, \| b' - b \| \leq \delta a, \| c' - c \| \leq \delta c, \)

then \( a', b', c' \) form a Euclidean triangle with opposite inner angles \( A', B', C' \) respectively, and

\[
\| A' - A \| \leq \frac{24}{\epsilon} \delta,
\]

and

\[
\| \triangle A'B'C' - \triangle ABC \| \leq \frac{576}{\epsilon^2} \delta \cdot |\triangle ABC|.
\]

**Proof.** Let

\[
u_A(t) = t \cdot (\log \frac{b'}{b} + \log \frac{c'}{c} - \log \frac{a'}{a})
\]

and \( u_B(t), u_C(t) \) be defined similarly. Then \( |u'| \leq -3 \log (1 - \delta) \leq 6 \delta \), since

\[
\delta \leq \frac{\epsilon^2}{48} \leq \frac{(\pi/3)^2}{48} \leq 0.1.
\]

Assume

\[
a(t) = e^{\frac{1}{2}(u_B(t)+u_C(t))} a, \quad b(t) = e^{\frac{1}{2}(u_A(t)+u_C(t))} b, \quad c(t) = e^{\frac{1}{2}(u_A(t)+u_B(t))} c,
\]

and then \( a(1) = a', b(1) = b', c(1) = c' \). Let \( A(t), B(t), C(t) \) be the inner angles of the triangle with edge lengths \( a(t), b(t), c(t) \), if well-defined.
Let \( T_0 \in [0, \infty) \) be the maximum real number such that for any \( t \in [0, T_0) \), all \( A(t), B(t), C(t) > \epsilon/2 \). Then \( T_0 > 0 \) and for any \( t \in [0, T_0) \), by Lemma 3.3,

\[
|A'(t)| = \left| \frac{\partial A}{\partial u_A} u_A' + \frac{\partial A}{\partial u_B} u_B' + \frac{\partial A}{\partial u_C} u_C' \right| \leq 2 \cot \frac{\epsilon}{2} \cdot |u'| \leq 12 \delta \cot \frac{\epsilon}{2} \leq \frac{24}{\epsilon},
\]
and similarly \( |B'(t)|, |C'(t)| \leq 24 \delta / \epsilon \). So \( T_0 \geq (\epsilon/2)/(24 \delta / \epsilon) = \epsilon^2/(48 \delta) > 1 \), and \( |A' - A| \leq 24 \delta / \epsilon \).

By Lemma 3.3 for \( t \in (0, 1) \)

\[
\frac{\partial |\triangle ABC|}{\partial a} = \frac{\partial}{\partial a} \left( \frac{1}{2} bc \sin A \right) = \frac{1}{2} bc \cos A \cdot \frac{a}{bc \sin A} = \frac{a(t)}{2 \tan A(t)}
\]

and then by the chain rule

\[
\left| \frac{d|\triangle ABC(t)|}{dt} \right| \leq |u'| \cdot \left( \left| \frac{a^2}{2 \tan A} \right| + \left| \frac{b^2}{2 \tan B} \right| + \left| \frac{c^2}{2 \tan C} \right| \right) \leq 6 \delta \cdot \frac{a(t)^2 + b(t)^2 + c(t)^2}{\epsilon},
\]

where \( a(t) \leq e^{u(t)} a \leq e^{6 \delta t} a \leq 2a \) and \( b(t) \leq 2b \) and \( c(t) \leq 2c \).

Then by Lemma 4.1,

\[
||\triangle A'B'C' - \triangle ABC|| \leq \frac{24 \delta}{\epsilon} (a^2 + b^2 + c^2) \leq \frac{24 \delta}{\epsilon} \cdot 3 \cdot \frac{8}{\epsilon} \cdot |\triangle ABC| = \frac{576}{\epsilon^2} \delta |\triangle ABC|.
\]

**Lemma 3.6.** Given a hyperbolic triangle \( \triangle ABC \), if all the angles in \( \triangle ABC \) are at least \( \epsilon > 0 \), and \( \delta < \epsilon^3 / 60 \), and

\[
a \leq 0.1, \quad b \leq 0.1, \quad c \leq 0.1,
\]

and

\[
|a' - a| \leq \delta a, \quad |b' - b| \leq \delta a, \quad |c' - c| \leq \delta c,
\]

then \( a', b', c' \) form a hyperbolic triangle with opposite inner angles \( A', B', C' \) respectively, and

\[
|A' - A| \leq \frac{30}{\epsilon^2} \delta,
\]

and

\[
|\triangle A'B'C' - \triangle ABC| \leq \frac{120}{\epsilon^2} \delta \cdot |\triangle ABC|.
\]

**Proof.** Let

\[
a(t) = ta' + (1 - t)a, \quad b(t) = tb' + (1 - t)b, \quad c(t) = tc' + (1 - t)c,
\]

and \( A(t), B(t), C(t) \) be the inner angles of the triangle with edge lengths \( a(t), b(t), c(t) \), if well-defined.

Let \( T_0 \in [0, \infty) \) be the maximum real number such that for any \( t \in [0, T_0) \), all \( A(t), B(t), C(t) > \epsilon/2 \). Notice that \( \delta < \epsilon^3 / 60 < 0.1 \) and then for any \( t \in [0, T_0) \), \( \sinh a(t) \in [a, 2a] \) and so on.
By Lemma 3.4

\[ |\dot{A}(t)| = \left| \frac{\partial A}{\partial a} \frac{\partial}{\partial t} + \frac{\partial A}{\partial b} b + \frac{\partial A}{\partial c} c \right| \leq \frac{|a' - a|}{\sinh b(t) \sinh (\epsilon/2)} + \frac{\cot(\epsilon/2)|b' - b|}{\sinh b(t)} + \frac{\cot(\epsilon/2)|c' - c|}{\sinh c(t)} \]

\[ \leq 2 \left( \frac{1}{\sin^2(\epsilon/2)} + 2 \cot(\epsilon/2) \right) \leq 2(\frac{\pi^2}{\epsilon^2} + \frac{4}{\epsilon}) \delta \leq \frac{30}{\epsilon^2} \delta. \]

and similarly \( |\dot{B}(t)|, |\dot{C}(t)| \leq 30\delta/\epsilon^2 \). So \( T \geq (\epsilon/2)/(30\delta/\epsilon^2) = \epsilon^3/(60\delta) > 1 \), and

\[ |A' - A| \leq 30\delta/\epsilon^2, \quad |B' - B| \leq 30\delta/\epsilon^2, \quad |C' - C| \leq 30\delta/\epsilon^2. \]

For \( t \in [0, 1] \), by Lemma 3.4

\[ \left| \frac{\partial(A + B + C)}{\partial a} \right| = \left| \frac{1}{\sinh b \sin C} - \frac{\cot C}{\sinh a} - \frac{\cot B}{\sinh a} \right| = \left| \frac{1}{\sinh a} \frac{\sin A - \sin(B + C)}{\sin B \sin C} \right| \leq \frac{\pi^2(\pi - A(t) - B(t) - C(t))}{\epsilon^2 \sinh a(t)} \leq \frac{2\pi^2}{\epsilon^2} \left| \Delta ABC(t) \right|, \]

and then

\[ \left| \frac{d|\Delta ABC(t)|}{dt} \right| = |\dot{A} + \dot{B} + \dot{C}| \leq \frac{2\pi^2}{\epsilon^2} \left| \Delta ABC(t) \right| \left( \frac{|a' - a|}{a} + \frac{|b' - b|}{b} + \frac{|c' - c|}{c} \right) \leq \frac{6\pi^2\delta}{\epsilon^2} \left| \Delta ABC(t) \right|. \]

So

\[ \frac{|\Delta A'BC'|}{|\Delta ABC|} \in \left[ e^{-6\pi^2\delta/\epsilon^2}, e^{6\pi^2\delta/\epsilon^2} \right] \in \left[ 1 - 6\pi^2\delta/\epsilon^2, 1 + 120\delta/\epsilon^2 \right] \]

and

\[ \left| |\Delta A'BC'| - |\Delta ABC| \right| \leq \frac{120}{\epsilon^2} \delta \cdot |\Delta ABC|. \]

\[ \square \]

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