“BOOTSTRAP DOMAIN OF DEPENDENCE”: BOUNDS AND TIME DECAY OF SOLUTIONS OF THE WAVE EQUATION

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Abstract. This article introduces a novel “bootstrap domain-of-dependence” concept, according to which, for all time following a given illumination period of arbitrary duration, the wave field scattered by an obstacle is encoded in the history of boundary scattering events for a time-length equal to the diameter of the obstacle, measured in time units. Resulting solution bounds provide estimates on the solution values in terms of a short-time history record, and they establish super-algebraically fast decay (i.e., decay faster than any negative power of time) for a wide range of scattering obstacles—including certain types of “trapping” obstacles whose periodic trapped orbits span a set of positive volumetric measure, and for which no previous fast-decay theory was available. The results, which do not rely on consideration of the Lax-Phillips complex-variables scattering framework and associated resonance-free regions in the complex plane, utilize only real-valued frequencies, and follow from use of Green functions and boundary integral equation representations in the frequency and time domains, together with a certain $q$-growth condition on the frequency-domain operator resolvent.

1. Introduction

We present a novel “bootstrap domain-of-dependence” concept (bootstrap DoD) and associated bounds on solutions of the scattering problem for the wave equation on an exterior region $\Omega^\circ$. The classical domain-of-dependence concept [26] for a space-time point $(r_0, T_0)$ concerns the initial and boundary values that determine the free-space solution at that space-time point. The bootstrap DoD ending at a given time $T_0$, in contrast, involves field values on the scattering boundary $\Gamma = \partial \Omega = \partial \Omega^\circ$ over the length of time $T_\ast$ preceding $T_0$, where $T_\ast$ equals the amount of time that is sufficient for a wave in free space to traverse a distance equal to the diameter of the obstacle. As discussed in this paper, in absence of additional illumination after time $t = T_0 - T_\ast$, the bootstrap DoD completely determines the scattered field on $\Gamma$ for all times $t \geq T_0$. Further, solution bounds that result from the bootstrap DoD approach provide estimates on the solution values in terms of a short-time history record, and they establish super-algebraically fast decay (i.e., decay faster than any negative power of time) for a wide range of scattering obstacles—including certain types of trapping obstacles for which no previous

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fast-decay theory was available. These results, whose proofs do not rely on consideration of the Lax-Phillips complex-variables scattering framework and associated resonance-free regions in the complex plane, utilize only real-valued frequencies, and follow from use of Green functions and boundary integral equation representations in the frequency and time domains, together with a certain $q$-growth condition on the frequency-domain operator resolvent. As discussed in Remark 4.3, on the basis of some of the materials in Section 3 and most of those in Section 4, related but less informative super-algebraically decaying solution bounds can also be obtained by relying on the complete time-history of the incident field instead of the bootstrap DoD.

(Estimates based on the classical concept of domain of dependence have previously been provided, including, in particular, a “domain-of-dependence inequality” for the problem of scattering by obstacles [44, Thm. 5.2]. We note however, that that is not a decay result and, in fact, it plays an important but very different role: by establishing that at most exponential growth can occur, it provides the necessary stability elements in a proof of existence and uniqueness based on energy considerations.)

In addition to the decay-problem application, the bootstrap DoD estimates introduced in this paper provide a valuable tool in connection with the numerical analysis of certain frequency-time “hybrid” numerical methods introduced recently [1] for the time-domain wave equation, which proceed by Fourier transformation from the time domain into the frequency domain. These hybrid solvers evaluate solutions of the wave equation by partitioning incident wave fields into a sequence of smooth compactly-supported wave packets followed by Fourier transformation in time for each packet, and they incorporate a number of novel approaches designed to accurately capture all high-frequency behavior—both in time, for high frequencies, and in frequency, for large time. The overall solution is then reconstructed as a sum of many wave equation solutions, each with a distinct center in time. Unlike the aforementioned complete time-history bounds, the bootstrap DoD bounds introduced in this paper provide a natural theoretical basis for efficient truncation of this sum while meeting a prescribed accuracy tolerance.

Returning to the decay problem we note that, in contrast with previous approaches, that typically rely on energy arguments and/or on analytic continuation of the frequency-domain resolvent, the method proposed in this paper is based on use of boundary integral equations for the frequency-domain and time-domain problems (along with the Fourier transform that relates them), and it characterizes the multiple-scattering effects and trapping character of domain boundaries (which are described by means of billiard-ball trajectories, see Section 1.2) in terms of the growth of the norm of the resolvent for the frequency-domain problem as the real frequency $\omega$ grows without bound. In detail, without recourse to complex-analytic methods, the new approach to the decay problem proceeds on the basis of the bootstrap DoD formulation introduced in Section 3 which, as mentioned above, captures the impact of the complete time history up to a given time $t$ in terms of the history over the time interval, immediately preceding $t$, of time-length $T^*$, required for free-space propagation across a distance equal to the diameter of the obstacle. The resulting bounds provide super-algebraically-fast time-decay energy estimates (decaying faster than any negative power of time) for a wide range of (both trapping and non-trapping) obstacles satisfying a certain $q$-growth condition.
on the resolvent operator as a function of the real frequency $\omega$. In particular, this theory establishes the first rapid decay estimates for specific types of trapping obstacles, such as those depicted in Figure 1 which are not equal to unions of convex obstacles—and whose periodic trapped orbits, in fact, span a set of positive volumetric measure.

1.1. OVERVIEW. This paper is organized as follows. A brief but somewhat detailed overview of previous decay results for the obstacle scattering problem is presented in Section 1.2. Then, after preliminaries presented in Section 2, Section 3 introduces the bootstrap domain-of-dependence formulation and it presents Theorem 3.2, which shows that for obstacles satisfying the $q$-growth condition, if the Neumann trace is “small” on a (slightly extended) bootstrap DoD time interval, then, absent additional illumination during and after that interval, it must remain “permanently small”, that is, small for all times subsequent to that interval. Section 4 then extends the results of Section 3, establishing, in Theorem 4.1 and Corollaries 4.2 and 4.3, various super-algebraically fast-decay results, including super-algebraic decay of the local energy (eq. 1.2 below) for all obstacles satisfying the $q$-growth condition.

1.2. ADDITIONAL BACKGROUND ON DECAY THEORY. In order to provide relevant background concerning decay estimates we briefly review the literature on this long-standing problem, and we note the significant role played in this context by the shape of the obstacle $\Omega$. Previous studies of the decay problem establish exponential decay of solutions for certain classes of domain shapes—including star-shaped domains [34], domains that are “non-trapping” with respect to rays [33, 36], and unions of strictly convex domains that satisfy certain spacing criteria [23, 25]. (A domain is non-trapping if each billiard ball traveling in the exterior of $\Omega$, which bounces off the boundary $\Gamma = \partial \Omega$ in accordance with the law of specular reflection, and which starts within any given ball $B_R$ of radius $R$ containing $\Omega$, eventually escapes $B_R$ with bounded trajectory length [33].)

Early results [29, 31, 35], obtained on the basis of energy estimates in the domains of time and frequency, establish exponential decay for “star-shaped” obstacles, that is, obstacles $\Omega$ which, for a certain $r_0 \in \Omega$, contain the line segment connecting $r_0$ and any other point $r \in \Omega$. As noted in [29, 30], exponential decay generally implies analyticity of the resolvent operator in a strip around the real $\omega$ frequency axis. A significant generalization of these results was achieved in [36] using a hypothesis somewhat more restrictive than the non-trapping condition, while [33] established exponential local-energy decay for all non-trapping obstacles. All of these results establish exponential decay

$$E(u, D, t) \leq C e^{-\alpha t} E(u, \infty, 0), \quad \alpha > 0,$$

for the local energy

$$E(u, D, t) = \int_D |\nabla u(r, t)|^2 + |u_t(r, t)|^2 \, dV(r)$$

contained in a compact region $D \subset \Omega^c$ in terms of the energy $E(u, \infty, 0)$ contained in all of $\Omega^c$—the latter one of which is, of course, conserved. As reviewed in Remark 4.4, a uniform decay estimate of the form (1.1) cannot hold for trapping obstacles.
In this connection it is tangentially relevant to consider reference [14], which establishes sub-exponential decay for the problem of scattering by a globally defined smooth potential. The method utilized in that work relies on use of simple resolvent manipulations for the differential scattering operator \( P = -\frac{1}{\omega^2}\Delta + V(r) \), where \( V \in C_0^\infty(\mathbb{R}^n) \) denotes the globally defined potential. Such manipulations are not applicable in the context of the Green function-based operators for the impenetrable-scattering problem, for which, in particular, the frequency \( \omega \) is featured in the Green function exponent of the integral scattering operator instead of a linear factor \( \frac{1}{\omega} \) in the corresponding differential operator.

As mentioned above, a decay estimate of the form (1.1) cannot hold for trapping obstacles. However, by relying on analytic continuation of the frequency-domain resolvent into a strip surrounding the real-axis in the complex frequency domain \( \omega \), exponential decay (of a different character than expressed in (1.1); see Remark 4.4) has been established for certain trapping geometries [18, 23, 25]. In view of previous work leading to results of exponential decay, even for trapping geometries, the question may arise as to whether the results of super-algebraically fast convergence presented in this paper could actually be improved to imply, for example, exponentially fast decay for trapping geometries which merely satisfy the \( q \)-growth condition. A definite answer in the negative to such a question is provided in [24, Thm. 1]. This contribution exhibits an example for which, consistent with earlier general suggestions [30, p. 158], the sequence of imaginary parts of the pole-locations of the scattering matrix in the complex plane tends to zero as the corresponding sequence of real parts tends to infinity; clearly, such a domain cannot exhibit exponential decay in view of the Paley-Wiener theorem [38, Thm. I]. In detail, the example presented in [24] concerns a domain \( \Omega \) consisting of a union of two disjoint convex obstacles, where the principal curvature of each connected component vanishes at the closest point between the obstacles. Reference [5] provides a general-obstacle inverse polylogarithmic decay estimate that is the only previous decay result applicable to such a trapping domain. In view of this background it may be suggested that the real-\( \omega \) decay analysis presented in this paper provides significant progress, as it establishes super-algebraic decay for a wide range of obstacles not previously treated by classical scattering theory, including the aforementioned vanishing-curvature example [24] and the connected and significantly more strongly trapping structures depicted in Figure 1, for which the trapped rays form a set of positive volumetric measure.

The aforementioned references [18, 23, 25] establish exponential decay for wave scattering for certain trapping structures consisting of unions of disjoint convex obstacles (but see Remark 4.4), and thereby answer in the negative a conjecture by Lax and Phillips [30, p. 158] according to which exponential decay could not occur for any trapping structure (in view of the Lax/Phillips conjectured existence, for all trapping obstacles, of a sequence of resonances \( \lambda_j \) for which \( \text{Im} \lambda_j \rightarrow 0^- \) as \( j \rightarrow \infty \)). The trapping structures with exponential decay are taken to equal a disjoint union of two smooth strictly convex obstacles in [23], and otherwise unions of disjoint convex obstacles in [25]; in all cases the geometries considered give rise to sets of trapping rays spanning three-dimensional point sets of zero volumetric measure: single rays in [18,23], and countable sets of primitive trapped rays in [25] as implied by Assumption (H.2) in that reference. To the authors’ knowledge, these are the only known results on fast decay of solutions in trapping geometries. Despite these
known exceptions to the Lax-Phillips conjecture, it has been surmised 24 that “it seems very sure that the conjecture remains to be correct for a great part of trapping obstacles.”—which would disallow exponential decay for most trapping obstacles—thus providing an interesting perspective on the main results presented in this paper—which, in particular, establish super-algebraic decay for certain obstacles for which the set of trapped rays span a set of positive measure.

2. Preliminaries

2.1. Dirichlet problems for the three-dimensional wave-equation. We consider the problem of scattering of an incident field \( b \) by an (open) bounded obstacle \( \Omega \subset \mathbb{R}^3 \) with Lipschitz boundary. More precisely, letting \( \Box = \frac{\partial^2}{\partial t^2} - c^2 \Delta \) denote the d’Alembertian operator with wave-speed \( c > 0 \) and given an open set \( \Omega^{inc} \) containing the closure \( \overline{\Omega} \) of \( \Omega \), \( \Omega \subset \Omega^{inc} \), we study the solution \( u \) of the initial and boundary value problem

\[
\begin{align*}
\Box u(r,t) &= 0, \quad \text{for} \quad (r,t) \in \overline{\Omega}^c \times (0, \infty), \\
u(r,0) &= \frac{\partial u}{\partial t}(r,0) = 0, \quad \text{for} \quad r \in \Omega^c, \\
u(r,t) &= -\gamma^+ b(r,t), \quad \text{for} \quad (r,t) \in \Gamma \times (0, \infty),
\end{align*}
\]

on the complement \( \Omega^c \) of \( \Omega \), for a given incident-field function

\[
b : \Omega^{inc} \times \mathbb{R} \to \mathbb{R} \quad \text{satisfying} \quad b \in C^2(\Omega^{inc} \times \mathbb{R}) \quad \text{and} \quad \Box b = 0 \quad \text{in} \quad \Omega^{inc} \times \mathbb{R},
\]

(cf. Remark 2.3 below), where \( \Gamma = \partial \Omega^c = \partial \Omega \) denotes the boundary of the obstacle \( \Omega \), and where \( \gamma^+ \) (Definition 6 in Appendix A) denotes the exterior trace operator. Compatibility of the boundary values with the initial condition requires \( b(r,0) = \frac{\partial b}{\partial t}(r,0) = 0 \) for \( r \in \Gamma \). In fact, for compatibility with the integral equation formulation (2.6) below, letting \( \mathbb{R}^- = \{ t \in \mathbb{R} : t \leq 0 \} \), throughout this paper we assume that

\[
b \in C^2(\Omega^{inc} \times \mathbb{R}) \cap L^2(\Omega^{inc} \times \mathbb{R}) \quad \text{and} \quad b(r,t) = 0 \quad \text{for} \quad (r,t) \in \overline{\Omega} \times \mathbb{R}^-.
\]

The solution \( u \) is the “scattered” component of the total field \( u^{tot} = u + b \); with these conventions, we clearly have \( \gamma^+ u^{tot}(r,t) = 0 \) for \( r \in \Gamma \).

**Remark 2.1.** In the case \( \Omega^{inc} = \mathbb{R}^3 \), the function \( v = u^{tot} = u + b \), satisfies the initial-value problem

\[
\begin{align*}
\Box v(r,t) &= 0, \quad \text{for} \quad (r,t) \in \overline{\Omega}^c \times (0, \infty), \\
v(r,0) &= g_0(r), \quad \frac{\partial v}{\partial t}(r,0) = g_1(r), \quad \text{for} \quad r \in \Omega^c, \\
v(r,t) &= 0, \quad \text{for} \quad (r,t) \in \Gamma \times (0, \infty),
\end{align*}
\]

in that domain, where \( g_0(r) = u^{tot}(r,0) \) and \( g_1(r) = \frac{\partial}{\partial t} u^{tot}(r,0) \). Conversely, given smooth data \( g_0 \) and \( g_1 \) in all of \( \Omega^c \cap \Omega^{inc} \), an incident field \( b \) can be obtained by first extending \( g_0 \) and \( g_1 \) to all of space \( \mathbb{R}^3 \) [37, Thm. 3.10] and using the extended data as initial data for a free-space wave equation with solution \( b \). In other words, the problems (2.1) and (2.4) are equivalent. □

**Remark 2.2.** The classical literature on decay rates for the wave equation [30] concerns problem (2.4) with the additional assumption that \( g_0 \) and \( g_1 \) are compactly supported. In view of the strong Huygens principle [26], which is valid in the present
three-dimensional context, the procedure described in Remark 2.1 translates the spatial compact-support condition on \( g_0 \) and \( g_1 \) into a temporal compact-support condition for the function \( b \) in \( (2.8) \), i.e. that the incident field vanishes on the boundary \( \Gamma = \partial \Omega \) after a certain initial “illumination time period” has elapsed. □

**Remark 2.3.** Even though the main problem considered in this paper, problem (2.1), is solely driven by the Dirichlet data \( (2.1c) \), our analysis relies on assumptions that are imposed not only on the function \( \gamma^+ b \), but also on the values \( \gamma^+ \partial_n b \) of the normal derivative of \( b \) on \( \Gamma \). Specifically, various results presented in this paper assume that, for an integer \( s \geq 0 \) the incident field \( b \) satisfies the “\( s \)-regularity conditions”

\[
\gamma^+ b \in H^{s+1}(\mathbb{R}; L^2(\Gamma)) \quad \text{and} \quad \gamma^+ \partial_n b \in H^s(\mathbb{R}; L^2(\Gamma)),
\]

where, for an integer \( p \), \( H^p(\mathbb{R}; L^2(\Gamma)) \) denotes the Sobolev-Bochner space of order \( p \) with values in \( L^2(\Gamma) \), as defined in Appendix A. Clearly, for the most relevant incident fields \( b \), such as those mentioned in Remark 2.1, namely, solutions of the wave equation that are smooth and compactly supported in time over any compact subset of \( \Omega^{inc} \), the \( s \)-regularity conditions hold for all non-negative integers \( s \). □

**2.2. Time-domain single layer potential.** The boundary value problem (2.1) can be reduced \(^2\) to an equivalent time-domain integral equation formulation, in which a boundary integral density \( \psi \) (defined to vanish for all \( t < 0 \) ) is sought, that satisfies the spatio-temporal boundary integral equation

\[
(S\psi)(\mathbf{r}, t) = \gamma^+ b(\mathbf{r}, t) \quad \text{for} \quad (\mathbf{r}, t) \in \Gamma \times \mathbb{R}.
\]

Here, calling

\[
G(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta((t - t') - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|}
\]

the Green function for the three-dimensional wave equation, \( S = \gamma^+ \mathcal{S} \) denotes the trace of the time-domain single-layer potential

\[
(\mathcal{S}\mu)(\mathbf{r}, t) = \int_{-\infty}^{t} \int_{\Gamma} G(\mathbf{r}, t; \mathbf{r}', t') \mu(\mathbf{r}', t') \, d\sigma(\mathbf{r}') \, dt', \quad (\mathbf{r}, t) \in \mathbb{R}^3 \times \mathbb{R}.
\]

Note that the single-layer potential and its restriction to the boundary may be expressed without recourse to distributions in the forms

\[
(\mathcal{S}\mu)(\mathbf{r}, t) = \int_{\Gamma} \frac{\mu(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} \, d\sigma(\mathbf{r}'), \quad \mathbf{r} \in \mathbb{R}^3,
\]

and

\[
(S\mu)(\mathbf{r}, t) = \int_{\Gamma} \frac{\mu(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} \, d\sigma(\mathbf{r}'), \quad \mathbf{r} \in \Gamma,
\]

respectively. Clearly, these operators are well-defined, for example, for densities \( \mu \in \mathcal{S}(\mathbb{R}; L^2(\Gamma)) \), where \( \mathcal{S}(\mathbb{R}; L^2(\Gamma)) \) denotes \(^{22}\) Def. 2.4.21\) the Schwartz space of smooth and rapidly decaying functions of \( t \in \mathbb{R} \) with values in \( L^2(\Gamma) \). The potential \( (2.9) \) is also well-defined for functions \( \mu \) in the space \( L^2(\mathbb{R}; L^2(\Gamma)) \), a fact that follows easily for \( \mathbf{r} \not\in \Gamma \), on account of the smoothness of the Green-function kernel in the integrand \( (2.8) \) for such values of \( \mathbf{r} \). As discussed in Section 2.4 the same is true for \( \mathbf{r} \in \Gamma \). More precisely, as shown in Lemma 2.6 the integral operator \( (2.10) \) maps \( L^2(\mathbb{R}; L^2(\Gamma)) \) continuously into itself.
As is well known [2], problem (2.6) admits a unique solution. (In fact, Lemma 3.6 below shows that $\psi \in L^2(\mathbb{R}; L^2(\Gamma))$ for obstacles $\Omega$ satisfying the $q$-growth condition (Definition 2).) Once $\psi$ has been obtained, the solution $u$ of (2.1) is given by
\begin{equation}
(2.11) \quad u(r, t) = (S_\psi)(r, t), \quad r \in \Omega^c.
\end{equation}
As is well known, further, the solution $\psi$ of (2.6) equals the Neumann trace of the solution $u$:
\begin{equation}
(2.12) \quad \psi(r, t) = \gamma^+ \frac{\partial u^{\text{tot}}}{\partial n}(r, t), \quad r \in \Gamma.
\end{equation}
In the functional setting utilized in this paper, the result (2.12) follows directly from Lemma 2.7 below together with the corresponding result [8, Thm. 2.44] for solutions of the frequency-domain single-layer equation (which is uniquely solvable for all frequencies except for the measure-zero set of square roots of Laplace eigenvalues in the domain $\Omega$).

As shown in Corollary 4.3, in view of (2.11), spatio-temporal estimates and temporal decay rates for the density $\psi$ imply corresponding decay properties for the energy $E(u, D, t)$ in (1.2) over any given compact set $D \subset \Omega^c$.

Remark 2.4. Throughout this paper three different kinds of notation are used to denote the application of an operator to a function, namely, e.g. in the case of the operator in (2.6),
\begin{equation}
(2.13) \quad (S_\omega \psi) = S[\psi] = S\psi.
\end{equation}

2.3. The Fourier transform and frequency-domain layer potentials.
As indicated above, this paper presents time-decay estimates on the solutions $\psi$ of the integral equation problem (2.6), including results for certain classes of trapping obstacles. Our analysis is based on consideration of frequency-domain counterparts of problems (2.1) and (2.6). On one hand, the frequency-domain counterpart of (2.1) for a given frequency $\omega$ is the Dirichlet problem for the Helmholtz equation with wavenumber $\kappa = \kappa(\omega) = \omega/c$,
\begin{align}
(2.14a) \quad \Delta U^f(r, \omega) + \kappa^2(\omega) U^f(r, \omega) &= 0, \quad \text{for } r \in \Omega^c, \\
(2.14b) \quad U^f(r, \omega) &= -\gamma^+ B^f(r, \omega), \quad \text{for } r \in \Gamma,
\end{align}
with unknown and incident field given by
\begin{equation}
(2.15) \quad U^f = \mathcal{F}[u] \quad \text{and} \quad B^f = \mathcal{F}[b],
\end{equation}
respectively (see Remark 2.5 below). The frequency-domain counterpart of (2.6), in turn, is the equation $S_\omega \psi^f = \gamma^+ B^f$, where $S_\omega$ denotes the frequency-domain single-layer operator introduced in equation (2.19) below. But the latter equation is not uniquely solvable at some frequencies, and we thus use the uniquely solvable frequency-domain combined-field integral equation
\begin{equation}
(2.16) \quad A_{\omega, \eta} \psi^f = \gamma^+ \partial_n B^f - i\eta \gamma^+ B^f
\end{equation}
for the same unknown, where the operator $A_{\omega, \eta}$ is presented in (2.22). Lemma 3.6 below shows that, indeed, for each $\omega \in \mathbb{R}$, the solution $\psi^f = \psi^f(r, \omega)$ of (2.16) coincides with the temporal Fourier transform of $\psi = \psi(r, t)$: $\psi^f(r, \omega) = \mathcal{F}[\psi](r, \omega), \quad r \in \Gamma$—where, using (2.17) below, the notational convention $\mathcal{F}[\psi](r, \omega) = \mathcal{F}[\psi(r, \cdot)](\omega)$ has been introduced.
Remark 2.5. Throughout this article the superscript $f$ is often used to emphasize the dependence of a given function on the temporal frequency $\omega$. The situation occurs frequently in connection with the use of the Fourier transform. For example, for a function $h(t)$ of the time variable $t$, the Fourier transform of $h$ could be denoted by

$$(2.17) \quad H^f(\omega) = \mathcal{F}[h](\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} \, dt.$$ 

Here $h$ and $H^f = \mathcal{F}[h]$ could denote either complex-valued scalar functions or functions with values on a Banach space $X$ over the complex numbers. In the latter case the integral on the right-hand side of (2.17) indicates integration in the sense of Bochner [15, 21, 22].

Our analysis relates the decay problem under consideration to the growth of norms of associated frequency-domain solution operators as the frequency grows without bound. Motivated in part by work concerning numerical analysis of integral equations, studies of such norm growths, which provide an indicator of the energy-trapping character of obstacles, have been undertaken over the last several decades, and have resulted in frequency-growth estimates for geometries that exhibit a variety of trapping behavior [3, 4, 6, 7, 10, 39]. The relevant frequency-domain operators are introduced in the following definition.

Definition 1 (Frequency-domain operators). Let $\Omega$ denote a Lipschitz domain with boundary $\Gamma$. Then, calling $G_{\omega}$ the Green function for the Helmholtz equation (2.14a) with wavenumber $\kappa(\omega) = \omega/c$, $G_{\omega}(r, r') = \frac{e^{i\omega r - r'}}{4\pi| r - r'|}$, we define the single-layer potential $S_{\omega}$, and the single-layer and adjoint double-layer operators, $S_{\omega}$ and $K_{\omega}^*$, respectively (see e.g. [8]),

$$(2.18) \quad (S_{\omega})_{\mu}(r) = \int_{\Gamma} G_{\omega}(r, r')\mu(r') \, d\sigma(r'), \quad r \in \mathbb{R}^3,$$

$$(2.19) \quad (S_{\omega})_{\mu}(r) = \int_{\Gamma} G_{\omega}(r, r')\mu(r') \, d\sigma(r'), \quad r \in \Gamma, \quad \text{and}$$

$$(2.20) \quad (K_{\omega}^*)_{\mu}(r) = \int_{\Gamma} \frac{\partial G_{\omega}(r, r')}{\partial n(r)}\mu(r') \, d\sigma(r'), \quad r \in \Gamma.$$ 

Well known expressions for the exterior and interior traces of the single layer potential and its normal derivative [32, p. 219] tell us that, for a given $\eta \in \mathbb{R}$,

$$(2.21) \quad (\gamma^+ \partial_n - i\eta \gamma^+) (S_{\omega})_{\mu} = \left( \mp \frac{1}{2} I + K_{\omega}^* - i\eta S_{\omega} \right)_{\mu}.$$ 

In what follows we utilize the interior-trace instance of (2.21) for a real coupling parameter $\eta \neq 0$, and we thus define the combined-field operator

$$(2.22) \quad A_{\omega, \eta} = \frac{1}{2} I + K_{\omega}^* - i\eta S_{\omega}, \quad \eta \neq 0.$$ 

For a given $\omega_0 > 0$ and for $\omega \geq 0$ we also define

$$(2.23) \quad A_{\omega} = A_{\omega, \eta_0(\omega)} \quad \text{where} \quad \eta_0(\omega) = \begin{cases} 1, & \text{if } 0 \leq \omega < \omega_0 \\ \omega, & \text{if } \omega \geq \omega_0, \end{cases}$$ 

where the dependence of $A_{\omega}$ on $\omega_0$ has been suppressed.
2.4. Time- and Frequency-domain Functional Spaces. This paper develops an $L^2(\Gamma)$-theory for time-domain scattering. We rely, in part, on classical results for frequency-domain layer potentials on Lipschitz domains. The associated literature is a storied one, and it contains well known contributions as described in [32, p. 209]; the corresponding results required in this paper are outlined in the following lemma.

**Lemma 2.6.** Let $\Gamma$ denote a Lipschitz boundary. Then, the operators $S_\omega$, $K_\omega^*$ and $A_\omega$ are continuous linear operators on $L^2(\Gamma)$ for each $\omega \geq 0$. Further, the frequency-domain operator $\tilde{S}$ given by

$$
\tilde{S}[\mu](\omega, \mathbf{r}) = S_\omega[\mu](\mathbf{r}), \quad \mathbf{r} \in \Gamma, 
$$

is a continuous operator on the space $L^2(\mathbb{R}; L^2(\Gamma))$ (i.e., the space $H^s(\mathbb{R}; H^s(\mathcal{U})$ in Definition 7 with $r = s = 0$ and $\mathcal{U} = \Gamma)$;

$$
\tilde{S} : L^2(\mathbb{R}; L^2(\Gamma)) \to L^2(\mathbb{R}; L^2(\Gamma)).
$$

Finally, for $\text{Re}(\eta) \neq 0$ the operator $A_{\omega, \eta}$ is invertible for all $\omega \geq 0$.

**Proof.** Proofs for the results concerning the mapping properties of the operators $S_\omega$, $K_\omega^*$ and $A_\omega$ can be found in [13] and [32, Thm. 6.12 and pp. 209]; and the invertibility of $A_\omega$ is established e.g. in [8, Thm. 2.27]. (The continuity result for $S_\omega$ with $\omega = 0$, which easily implies the result for $\omega \neq 0$, was initially established in [43].)

The continuity of the operator $\tilde{S}$, finally, can be obtained from the norm-boundedness relation

$$
\|S_\omega\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C \quad \text{for all } \omega \in \mathbb{R},
$$

which is given e.g. in [7, Thm. 3.3] for $\omega \geq 0$ and which follows also for $\omega < 0$ in view of the Hermitian symmetry relation $S_\omega \mu^f(\mathbf{r}, \omega) = S_{-\omega} \mu^f(\mathbf{r}, \omega)$. Indeed, for $\mu^f \in L^2(\mathbb{R}; L^2(\Gamma))$, equation (2.26) tells us that

$$
\int_{\Gamma} \left| \int_{\Gamma} \frac{e^{i\omega|\mathbf{r}' - \mathbf{r}|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \mu^f(\mathbf{r}', \omega) \, d\sigma(\mathbf{r}') \right|^2 \, d\sigma(\mathbf{r}) \leq C \left\| \mu^f(\cdot, \omega) \right\|^2_{L^2(\Gamma)},
$$

and, thus, integrating with respect to $\omega$, we obtain

$$
\|\tilde{S} \mu^f\|^2_{L^2(\mathbb{R}; L^2(\Gamma))} \leq C \int_{-\infty}^{\infty} \left\| \mu^f(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega = C \left\| \mu^f \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))},
$$

establishing the desired continuity property for the operator $\tilde{S}$. \hfill \Box

As indicated in the following Lemma, the time-domain single-layer operator is also continuous in the space $L^2(\mathbb{R}; L^2(\Gamma))$, and the frequency- and time-domain single-layer operators are related by the Fourier transform.

**Lemma 2.7.** The time-domain single-layer boundary integral operator (2.10) is well defined for $\mu$ in the function-valued Schwartz space $S(\mathbb{R}, L^2(\Gamma))$ of infinitely smooth and rapidly decaying functions on the real line, and it may be extended to a continuous operator

$$
S : L^2(\mathbb{R}; L^2(\Gamma)) \to L^2(\mathbb{R}; L^2(\Gamma)).
$$
Further, for each \( \mu \in L^2(\mathbb{R}; L^2(\Gamma)) \) the temporal Fourier transform of the operator (2.29) applied to \( \mu \) equals the frequency domain single layer operator (2.25) applied to the Fourier transform of \( \mu \):

\[
\mathcal{F} [S[\mu]] = \tilde{S} [\mathcal{F}[\mu]].
\]

Proof. See Appendix B. \( \square \)

2.5. Growth of operators norms. This article utilizes a certain “\( q \)-growth condition” which, for a given \( q \in \mathbb{R} \), may or may not be satisfied by a given obstacle \( \Omega \)—namely, that a constant \( C \) exists such that, for all \( \omega \in \mathbb{R} \), the relation (2.31) below holds. The \( q \)-growth condition is fundamentally a statement on the high-frequency character of the operator \( A^{-1}_\omega \). Further, the \( q \)-growth condition is indeed a condition on the domain \( \Omega \) which is independent on the constant \( \omega_0 \) in (2.23).

This can be verified by noting that, given any compact one-dimensional intervals \( I_\eta \) and \( I_\omega \), where \( I_\eta \) is bounded away from 0, the norm \( \|A^{-1}_\omega\|_{L^2(\Gamma) \to L^2(\Gamma)} \) is uniformly bounded for \((\eta, \omega) \in I_\eta \times I_\omega \). The latter property, in turn, can be established on the basis of the theory [8, 32] for the combined field operator on a Lipschitz domain, together with a compactness argument based on Taylor expansions of the oscillatory exponential factor of the Green function as well as a Neumann series for the resolvent around each pair \((\eta, \omega) \in I_\eta \times I_\omega \).

**Definition 2** (\( q \)-growth condition). For real \( q \) and \( \omega_0 > 0 \), a Lipschitz domain \( \Omega \) and its boundary \( \Gamma \) are said to satisfy the \( q \)-growth condition iff for all \( \omega \geq 0 \) the operator \( A^\omega \) in (2.23) satisfies the bound

\[
\|A^{-1}_\omega\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C(1 + \omega^2)^{q/2}
\]

for some constant \( C > 0 \), or, equivalently, the bounds

\[
\|A^{-1}_\omega\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \begin{cases} 
C_1, & \text{if } \omega \leq \omega_0 \\
C_2\omega^q, & \text{if } \omega > \omega_0
\end{cases}
\]

for certain constants \( C_1 > 0 \) and \( C_2 > 0 \). \( \square \)

Per [10, Lem. 6.2], polynomially growing bounds on the norm of this inverse operator in the high-frequency regime can be obtained on the basis of corresponding polynomially growing bounds as \( \omega \to \infty \) on the norm of the resolvent operator \((\Delta + \omega^2)^{-1}\) outside \( \Omega \), with zero Dirichlet boundary conditions on \( \Gamma \). The results of the present article thus imply super-algebraically-fast local energy decay of solutions to (2.1) for any Lipschitz domain for which such real-axis high-frequency resolvent bounds for the Helmholtz operator can be established.

**Remark 2.8.** It is known that the \( q \)-growth condition is satisfied with a variety of \( q \)-values for various classes of obstacles [3, 7, 9, 10, 42]. For example, reference [3] Thm. 1.13 shows that a smooth non-trapping obstacle satisfies the \( q \)-growth condition with \( q = 0 \). A related \( q = 0 \) result is presented in [9] for merely Lipschitz domains, but under the stronger assumption that the obstacle is star-shaped. Reference [10] shows that for “hyperbolic” trapping regions (in which all periodic billiard ball trajectories are unstable), a merely logarithmic growth in \( \omega \) results, while for certain “parabolic” trapping regions, stronger \((q = 2 \text{ or } q = 3)\) growth takes place. It is also known that much more strongly-trapping obstacles exist, including obstacles for which exponentially-large inverse operator norms \( \|A^{-1}_\omega\|_{L^2(\Gamma) \to L^2(\Gamma)} \) occur [4, 6, 28] (and which, therefore, do not satisfy the \( q \)-growth condition for any value of \( q \)). \( \square \)
Figure 1. Examples of connected trapping obstacles that satisfy the $q$-growth condition ($q = 3$) and for which super-algebraically-fast wave equation time decay rates are established in this article. Left: Visualization of the obstacle mentioned in Remark 2.9: a connected trapping obstacle satisfying the $q$-growth condition of Definition 2 for $q = 3$. Right: A trapping obstacle containing a deep cavity (arbitrary cavity depths are admissible), that satisfies the $q$-growth condition with the same value of $q$.

Remark 2.9. Existing results can be used to obtain examples of connected trapping obstacles containing cavities which satisfy the $q$-growth condition in Definition 2 for some value of $q$. For example, the obstacle

$$\Omega = [-1,1] \times [-1,1] \times [-1,1] \setminus [-1/2,1/2] \times [-1/2,1/2] \times [0,1]$$

(a cube with a smaller cube removed from one of its faces, as displayed in the left panel of Figure 1), is an \((R_0,R_1,a)\) parallel trapping obstacle in the sense of \([10, ~\text{Def. 1.9}]\), for $R_1 > e^{1/4}R_0$, $R_0 \geq \sqrt{3}/2$, and $a = 1$. According to \([10, ~\text{Cor. 1.14 and Rem. 1.16}]\), $\Omega$ satisfies the $q$-growth condition with $q = 3$. Smoothing of the corners of this obstacle results in a connected trapping obstacle that satisfies a $q$-growth condition with $q = 2$. □

3. Uniform “bootstrap domain-of-dependence” boundary density estimates

This section presents uniform-in-time “bootstrap domain-of-dependence” estimates (Theorem 3.2) on the solution $\psi$ of the time-domain boundary integral equation (2.6). In particular, these estimates show that if $\psi$ is “small” throughout $\Gamma$ for any domain-of-dependence time interval of length

(3.1)  
$$T_* := \text{diam}(\Gamma)/c = \max_{r, r' \in \Gamma} |r - r'|/c$$

after the $\Gamma$-values of the incident field have been turned off, then $\psi$ will remain small for all time thereafter. At a fundamental level, the domain-of-dependence analysis presented in this section exploits an interesting property of equation (2.6) (made
precise in Lemmas 3.9 and 3.13, namely, that, after the incident field $b(r,t)$ has been turned off permanently throughout $\Gamma$, the values of the solution $\psi(r,t)$ over any given domain-of-dependence time length $T^*$ determine the solution uniquely for all subsequent times: the boundary integral density over the interval $I_T$ encodes the necessary information to reproduce all future scattering events and, in particular, it encapsulates the effect of all previously imposed boundary values, over time periods of arbitrarily long duration.

For technical reasons we utilize domain-of-dependence intervals $I_T$, as detailed in Definition 3, of a length slightly larger than $T^*$—larger by a small amount $2\tau > 0$. Any positive value of $\tau$ can be used, and the selection only affects the multiplicative constants and integration domains in the main domain-of-dependence estimates presented in this paper. The rest of this section is organized as follows. Definitions 3 and 4 lay down the conventions necessary to subsequently state and prove the main result of the section, namely, Theorem 3.2. After these definitions, the statement of the theorem is introduced. Lemmas 3.6 through 3.18 then establish a series of results required in the proof of the theorem, and then, concluding the section, the theorem’s proof is presented.

**Definition 3** (Domain-of-dependence interval). Using the definition $T^* := \text{diam}(\Gamma)/c$ for a given Lipschitz boundary $\Gamma$, and given real numbers $T$ and $\tau > 0$, the time interval

$$I_T = I_{T,T^*,\tau} = [T - T^* - 2\tau, T),$$

will be referred to as the $\tau$-extended “domain-of-dependence” interval ($\tau$DoD) relative to the “observation” time $T$. (As suggested by the notations in (3.2), the dependence of $I_T$ on the parameters $T^*$ and $\tau$ will not be made explicit in what follows.)

**Remark 3.1.** For future reference we mention here that the interval $I_T$ plays two important roles in our analysis. On one hand, this interval figures prominently, for a given value $T = T_0 > 0$, in the statements of the two main theorems of this paper, namely, Theorems 3.2 and 4.1, for any fixed given time $T_0 > 0$, the estimates provided by these theorems are valid for times past the upper endpoint $T_0$ of the interval $I_{T_0}$, provided the incident field vanishes at all times after the lower endpoint $T_0 - T^* - 2\tau$ of $I_{T_0}$. On the other hand, in order to adequately translate the fast oscillations that take place in inverse Fourier-transform integrands for large times into the decay rates claimed in Theorem 4.1, a time-recentering technique is utilized that is embodied in Remark 4.7 and Lemma 4.10 and which is exercised in the portions of the proof of Theorem 4.1 containing equations (4.80), (4.82) and (4.84). This time-recentering technique is based on consideration of the interval $I_T$, but, this time, with $T = 0 < T_0$. In this manner, the idea of recentering the problem in time around $t = 0$, that is exploited in [1] to reduce oscillations for algorithmic optimization purposes, is manifested in the present contribution in the use of the recentered interval $I_0$ equal to $I_T$ with $T = 0$, to properly account for fast oscillations, as functions of frequency, that are observed for large time in frequency-domain inverse Fourier-transform integrands. From an analytical point of view, the time-recentering of frequency-domain integrals to the interval $I_0$ allows us to usefully exploit the frequency-time isometry inherent in the Plancherel theorem, to produce Sobolev estimates, but without incurring the uncontrollably large derivatives of the Fourier transform integrands with respect to frequency that result
as \( t \) grows, and which, instead of the Sobolev estimates provided in Lemma 4.8, would yield Sobolev estimates with constants \( C \) that grow without bound as \( T_0 \) grows. \( \square \)

**Definition 4** (Time-windowed solutions). For a given solution \( \psi \) of (2.6) and a given \( \tau \text{-DoD} \) interval \( I_T \), using smooth non-negative window functions

\[
 w_-(t) = \begin{cases} 
 1 & \text{for } t < -\tau \\
 0 & \text{for } t \geq 0 \\
 1 & \text{for } t \geq 0,
\end{cases}
\]

(3.3) \( w_+ = \) given \( \tau \) defined over the real line, which satisfy the “Partition-of-unity” relation \( w_- + w_+ = 1 \), we define the auxiliary densities

\[
(3.4) \quad \psi_{-,T}(\mathbf{r},t) = w_-(t-T)\psi(\mathbf{r},t), \quad \psi_{+,T}(\mathbf{r},t) = w_+(t-T)\psi(\mathbf{r},t),
\]

whose temporal supports satisfy supp \( \psi_{-,T} \subset (-\infty, T] \) and supp \( \psi_{+,T} \subset [T-\tau, \infty) \), and for which the relation

\[
(3.5) \quad \psi(\cdot, t) = \psi_{-,T}(\cdot, t) + \psi_{+,T}(\cdot, t)
\]

holds for all real values of the time variables \( t \) and \( T \). We further define

\[
(3.6) \quad \psi_{*,T}(\mathbf{r},t) = w_+(t-T+T_0+\tau)w_-(t-T)\psi(\mathbf{r},t),
\]

which is nonzero only in the interior of \( I_T \). \( \square \)

The smooth temporal decompositions introduced in Definition 1 play central roles in the derivations of the uniform-in-time bounds and decay estimates presented in this paper. Noting that \( \psi \) is identical to \( \psi_{+,T_0} \) for \( t > T_0 \), to produce such bounds and estimates we first relate \( \psi_{+,T_0} \) to \( \psi_{-,T_0} \) and thereby obtain a norm bound for \( \psi_{+,T_0} \) over the slightly larger time interval \( [T_0 - \tau, \infty) \)—which then yields, in particular, the desired domain-of-dependence estimate for \( \psi \) on the interval \( [T_0, \infty) \). The necessary bounds for \( \psi_{+,T_0} \) are produced via Fourier transformation into the frequency domain in conjunction with use of the \( q \)-growth condition introduced in Definition 2. A similar approach is followed for the time-derivatives of the incident field data, leading to bounds in temporal Sobolev norms of arbitrary orders, and, in particular, via Sobolev embeddings, to uniform bounds in time. As suggested above, the approach intertwines the frequency and time domains, and it thus incorporates frequency-domain estimates while also exploiting the time domain Huygens’ principle, cf. for example the relations (3.38) and (3.48).

The main result of this section, Theorem 3.2, which is stated in what follows, relies on the definition of Sobolev-Bochner spaces presented in Appendix A.

**Theorem 3.2.** Let \( p \) and \( q \) denote non-negative integers, let \( T_0 > 0 \) and \( \tau > 0 \) be given, and assume (i) \( \Gamma \) satisfies the \( q \)-growth condition (Definition 2); (ii) The incident field satisfies the \( s \)-regularity conditions (2.5) with \( s = p + 2q + 1 \); and, (iii) The incident field \( \mathbf{b} = \mathbf{b}(\mathbf{r}, t) \) satisfies (2.3) and it vanishes for \( (\mathbf{r}, t) \in \mathbb{R} \times \{T_0 \cup [T_0, \infty)\} \), with \( I_T \) as in Definition 3. Then, the solution \( \psi \) of (2.6) satisfies both the \( H^p \) estimate

\[
(3.7) \quad ||\psi||_{H^p([T_0, \infty); L^2(\Gamma))} \leq C(\Gamma, \tau, p) ||\psi||_{H^{p+q+1}(I_{T_0}; L^2(\Gamma))} < \infty,
\]

and, in the case \( p = 1 \) the time-uniform estimate

\[
(3.8) \quad ||\psi(\cdot, t)||_{L^2(\Gamma)} \leq C(\Gamma, \tau) ||\psi||_{H^{q+2}(I_{T_0}; L^2(\Gamma))} \quad \text{for} \quad t > T_0,
\]

where the constants \( C = C(\Gamma, \tau, p) \) and \( C = C(\Gamma, \tau) \) are independent of \( T_0 > 0 \).
Remark 3.3. Several results in the present contribution, including Theorem 3.2, Lemma 3.19, Theorem 4.1, and Lemma 4.10, utilize the \(q\)-growth condition-based estimate provided by Lemma 3.6 below to conclude that \(\psi \in H^p(\mathbb{R}; L^2(\Gamma))\) for certain values of \(p\), and thus guarantee the less stringent condition \(\psi \in H^p([\xi_1, \xi_2]; L^2(\Gamma))\) (for certain values \(-\infty < \xi_1 < \xi_2 < \infty\)) that is actually needed to make the results meaningful. We note, however, that the regularity assumptions in these theorems and lemmas can be relaxed by using, instead, Laplace-domain bounds for the purpose of establishing the aforementioned less stringent condition. Indeed, in contrast to Lemma 3.6, on the basis of Laplace-domain estimates (in particular [12, Thm. 4.2] in conjunction with [11, Lem. 2], cf. also [2, 31]) such bounds show that if \(b\) is causal and satisfies

\[
\gamma^+ b \in H^{r+1}([0, T]; L^2(\Gamma)) \quad \text{and} \quad \gamma^+ \partial_n b \in H^r([0, T]; L^2(\Gamma)),
\]

then \(\psi \in H^r([0, T]; L^2(\Gamma))\). While we eschewed use of these bounds to achieve a simpler and more self-contained presentation, we note here how the regularity assumptions of the various lemmas and theorems would be relaxed by application of these auxiliary results. The \(s\)-regularity assumptions in Theorem 3.2, Lemma 3.19, Theorem 4.1, and Lemma 4.10 would be relaxed, respectively, to requirements of \(s\) regularity with \(s = p + q + 1\), \(s = p + 1\), \(s = p + (n+1)(q+1)\), and \(s = (n+1)(q+1)\). We additionally note that none of the estimates are quantitatively affected—these assumptions are only used to ensure that the right-hand side quantities in the desired estimates are finite. We finally mention that all of the assumptions mentioned here, including the original and the relaxed assumptions, are immediately satisfied for data \(b\) that is smooth in time—see Remark 2.3.

The proof of Theorem 3.2 is deferred to the end of this section, following a series of eight preparatory Lemmas (the first and last of which are Lemmas 3.6 and 3.19). The first of these lemmas relates the size of the solution of equation (2.6) to the size of the imposed incident fields \(b(r,t)\) in various norms.

Remark 3.4. According to (2.2), and without loss of generality, throughout this paper the incident field \(b(r,t)\) is assumed to be real-valued. It follows that the solution \(\psi(r,t)\) of (2.6) is real-valued, which implies that its Fourier transform \(\psi_f(r,\omega)\) satisfies the Hermitian symmetry relation

\[
\psi_f(r, -\omega) = \psi_f(r, \omega).
\]

Our studies of frequency-domain operator norms are therefore restricted to the range \(\omega \geq 0\), as is common practice in the mathematical frequency-domain scattering literature [3, 4, 10]. Such symmetry relations apply to other quantities defined above and in what follows, including \(\psi_{\pm,T}(r,t)\), \(\psi_{s,T}(r,t)\), \(h_T(r,t)\), etc., that are defined in terms of the real-valued density \(\psi(r,t)\).

Remark 3.5. It will be necessary in what follows (specifically, in the proofs of the \(H^p\) estimates in Theorem 3.2 and Theorem 4.1 for \(p > 0\), and associated preparatory lemmas), to allow for right-hand sides other than the specific incident-field function \(b(r,t)\) present in the Dirichlet problem (2.6), and the integral equation formulation (3.10), and we thus consider scattering problems of an incident field function \(\tilde{b}(r,t)\), similar to \(b\), for which we have a corresponding integral equation formulation

\[
S\tilde{\psi}(r,t) = \gamma^+ \tilde{b}(r,t) \quad \text{for} \quad (r,t) \in \Gamma \times \mathbb{R},
\]
with solution \( \tilde{\psi} \), where \( S \) denotes the single layer integral operator \((2.10)\). Generalizing \((2.3)\), and letting \( \mathbb{R}_{\alpha} = \{ t \in \mathbb{R} : t \leq \alpha \} \), throughout the remainder of this paper we assume that, for some \( \alpha \in \mathbb{R} \),

\[
(3.11) \quad \tilde{b} \in C^2(\Omega^{\text{inc}} \times \mathbb{R}) \cap L^2(\Omega^{\text{inc}} \times \mathbb{R}) \text{ and } \tilde{b}(r, t) = 0 \text{ for } (r, t) \in \bar{\Omega} \times \mathbb{R}_{\alpha} \]

(so that, in particular, \( \tilde{b} \) is square-integrable as a function of time for \(-\infty < t < \infty \)).

We also define \( \tilde{\psi}_{-,T}, \tilde{\psi}_{+,T}, \) and \( \tilde{\psi}_{+,T} \) paralleling the definitions of \( \psi_{-,T}, \psi_{+,T}, \) and \( \psi_{+,T} \), respectively, in Definition \( \[\] \). Similarly, with reference to \((2.15)\), we write

\[
(3.12) \quad \tilde{B}^f = \mathcal{F}[\tilde{b}].
\]

**Lemma 3.6** (Well-posedness of the time-domain integral equation \((3.10)\)). Let \( p \) and \( q \) denote non-negative integers, and assume that the obstacle \( \Omega \) satisfies the \( q \)-growth condition. Let \( \Omega^{\text{inc}} \) denote an open domain containing \( \bar{\Omega} \), and assume the function \( b \) is a solution to the wave equation in \( \Omega^{\text{inc}} \), which satisfies \((3.11)\) as well as the \( s \)-regularity conditions \((2.5)\) with \( s = q \). Then, for each fixed \( \omega \in \mathbb{R} \) and each \( \eta \in \mathbb{C} \) with \( \text{Re}(\eta) \neq 0 \), the solution \( \tilde{\psi}^f \) of the \( \eta \)-dependent uniquely solvable second-kind integral equation

\[
(3.13) \quad A_{\omega, \eta} \tilde{\psi}^f(r, \omega) = \gamma^+ \partial_n \tilde{B}^f(r, \omega) - i\eta \gamma^+ \tilde{B}^f(r, \omega), \quad r \in \Gamma,
\]

is itself independent of \( \eta \), and therefore, using \((2.23)\), the \( \eta \) independent function \( \tilde{\psi}^f \) also satisfies the uniquely solvable second-kind integral equation

\[
(3.14) \quad A_{\omega} \tilde{\psi}^f(r, \omega) = \gamma^+ \partial_n \tilde{B}^f(r, \omega) - i\eta_0(\omega) \gamma^+ \tilde{B}^f(r, \omega), \quad r \in \Gamma.
\]

Further, there exists a unique solution \( \tilde{\psi} \in L^2(\mathbb{R}; L^2(\Gamma)) \) satisfying the integral equation \((3.10)\), which is given by the inverse Fourier transform, with respect to \( \omega \), of the function \( \tilde{\psi}^f = \tilde{\psi}^f(r, \omega) \). If, in addition, the incident field \( b \) satisfies the \( s \)-regularity conditions \((2.5)\) with \( s = p + q \), then \( \tilde{\psi} \in H^p(\mathbb{R}; L^2(\Gamma)) \) and

\[
(3.15) \quad \left\| \tilde{\psi} \right\|_{H^p(\mathbb{R}; L^2(\Gamma))} \leq C(\Gamma) \left( \left\| \gamma^+ \partial_n \tilde{b} \right\|_{H^{p+q}(\mathbb{R}; L^2(\Gamma))} + \left\| \gamma^+ \tilde{b} \right\|_{H^{p+q+1}(\mathbb{R}; L^2(\Gamma))} \right).
\]

Finally, if \( \tilde{b} \) satisfies the \( s \)-regularity conditions \((2.5)\) with \( s = p + q + 1 \), then \( \tilde{\psi} \in C^p(\mathbb{R}; L^2(\Gamma)) \), and the norm of \( \tilde{\psi} \) in \( C^p(\mathbb{R}; L^2(\Gamma)) \) is bounded by the corresponding boundary data, as it follows from \((3.15)\) and the Sobolev lemma (Lemma \( \[\] \)) below.

**Remark 3.7.** With reference to Remark \( \[\] \), we note that, for the particular case \( \tilde{b} = b \), Lemma \( \[\] \) tells us that the Fourier transform \( \psi^f \) of the solution \( \psi \) to \((2.6)\) satisfies the frequency-domain equation

\[
(3.16) \quad A_{\omega} \psi^f(r, \omega) = \gamma^+ \partial_n B^f(r, \omega) - i\eta_0(\omega) \gamma^+ B^f(r, \omega), \quad r \in \Gamma.
\]

**Proof of Lemma \( \[\] \)** As is well known \([8, \text{Thm. 2.27}]\), for any \( \eta \in \mathbb{C} \) satisfying \( \text{Re}(\eta) \neq 0 \), the integral equation \((3.13)\) is uniquely solvable, its solution \( \tilde{\psi}^f \) does not depend on \( \eta \), and \( \tilde{\psi}^f \) additionally satisfies \([8, \text{Thms. 2.44 and 2.46}]\) the single-layer integral equation

\[
(3.17) \quad S_{\omega} \tilde{\psi}^f(r, \omega) = \gamma^+ \tilde{B}^f(r, \omega), \quad (r, \omega) \in \Gamma \times \mathbb{R}.
\]

(\text{The references cited establish the result for } \omega \geq 0; \text{ the statement for } \omega < 0 \text{ then follows directly in view of the aforementioned Hermitian symmetry property.})

Clearly, substituting \( \eta = \eta_0(\omega) \) in \((3.13)\) tells us that \( \tilde{\psi}^f \) is also a solution of \((3.14)\).
To complete the proof of the lemma we now note that, for any given non-negative integer \(r\), (3.14) tells us that
\[
(1 + \omega^2)^{r/2} \tilde{\psi}^f = (1 + \omega^2)^{r/2} A_r^{-1} \left( \gamma^+ \partial_n \tilde{B}^f - i \eta_0 \gamma^+ \tilde{B}^f \right),
\]
and, thus, in view of (2.31) and taking into account the relation \(|\eta_0| \leq (1 + \omega^2)^{1/2}\) that follows from (2.23), we obtain
\[
\int_{-\infty}^{\infty} (1 + \omega^2)^r \left\| \tilde{\psi}^f(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \, d\omega \leq C \int_{-\infty}^{\infty} \left( (1 + \omega^2)^{r+q} \left\| \gamma^+ \partial_n \tilde{B}^f(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 + (1 + \omega^2)^{r+q+1} \left\| \gamma^+ \tilde{B}^f(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \right) \, d\omega.
\]
In particular, equation (3.18) with \(r = 0\) tells us that that \(\tilde{\psi}^f \in L^2(\mathbb{R}; L^2(\Gamma))\), in view of the \(s\)-regularity conditions assumed with \(s = q\). As a result, the inverse time Fourier transform \(\tilde{\psi} = \mathcal{F}^{-1} \tilde{\psi}^f\) (see (2.17)) is well-defined. Since \(\tilde{\psi}^f\) satisfies (3.17) for every \(\omega\), it follows from (2.30) that its inverse Fourier transform \(\tilde{\psi}\) satisfies (3.10), and, thus the existence of a solution to (3.10), given by the inverse Fourier transform of \(\tilde{\psi}^f\), follows. To establish the uniqueness of this solution in \(L^2(\mathbb{R}; L^2(\Gamma))\), we assume \(\lambda \in L^2(\mathbb{R}; L^2(\Gamma))\) satisfies (\(S\lambda\))(\(r, t\)) = 0 for \((r, t) \in \Gamma \times \mathbb{R}\). Using (2.24) and (2.30), we see that the Fourier transform \(\lambda^f = \mathcal{F}[\lambda]\) satisfies the equation \((S_\omega \lambda^f(\cdot, \omega))(\mathbf{r}) = 0\) for each \(\omega \in \mathbb{R}\). Since \(S_\omega\) is invertible for almost every \(\omega\) (or, more precisely, for all \(\omega \in \mathbb{R}\) except for the measure-zero set of \(\omega\) for which \(-\omega^2\) is a Dirichlet eigenvalue for the Laplace operator in the interior of \(\Omega\) [7, Thm. 2.25]), we conclude that \(\lambda^f(\cdot, \omega) \equiv 0\) for almost every \(\omega \in \mathbb{R}\), and thus \(\lambda = \mathcal{F}^{-1}[\lambda^f] \equiv 0\). The existence and uniqueness of solutions of (3.10) thus follows.

The estimate (3.15) is equivalent to (3.18) with \(r = p\), in view of the equivalence of the norms displayed in equations (A.2) and (A.4) in Appendix A below. Under the additional assumption of \(s\)-regularity with \(s = q + 1\), the fact that \(\tilde{\psi} \in C^p(\mathbb{R}; L^2(\Gamma))\), together with an associated norm bound, result from an application of (3.15) with \(p\) substituted by \(p + 1\) and the use of the Sobolev embedding Lemma 3.18. The proof is now complete. \(\square\)

In preparation for Lemma 3.9 we note the relation
\[
\mathcal{F} [\mathcal{F}[\mu]] = \tilde{\mathcal{F}} [\mathcal{F}[\mu]]
\]
that relates the time- and frequency-domain single layer potentials (2.8) and (2.18) (where we have set
\[
\tilde{\mathcal{F}}[\mu](\mathbf{r}, \omega) = \mathcal{F}_{\omega}[\mu(\cdot, \omega)](\mathbf{r}), \quad (\mathbf{r}, \omega) \in \mathbb{R}^3 \times \mathbb{R};
\]
cf. equations (2.24) and (2.30)), and which, as in Appendix B, follows via an application of the change of variables (B.1) to the left-hand term of the equation. Also, letting
\[
\tilde{u}_{\cdot, T}(\mathbf{r}, t) := (\tilde{\mathcal{F}} \tilde{\psi}_{\cdot, T})(\mathbf{r}, t), \quad (\mathbf{r}, t) \in \mathbb{R}^3 \times \mathbb{R},
\]
we define the function
\[
\tilde{b}_T(\mathbf{r}, t) := \tilde{b}(\mathbf{r}, t) - \tilde{u}_{\cdot, T}(\mathbf{r}, t), \quad (\mathbf{r}, t) \in \Omega^{inc} \times \mathbb{R},
\]
and, using (3.19), its Fourier transform

\[ \hat{H}^f_T(r, \omega) = \hat{B}^f_T(r, \omega) - S^f_T(\omega, \omega), \quad r \in \Omega^{inc} \times \mathbb{R}. \]  

Remark 3.8. With reference to Remark 3.5 we mention that Lemma 3.9 and subsequent lemmas apply in particular to the function \( \tilde{b} = b \) in (2.6); when such applications arise we will accordingly use untilded notations such as e.g.

\[ u_{-T}(r, t) := (\mathcal{F} \psi_{-T})(r, t), \quad (r, t) \in \mathbb{R}^3 \times \mathbb{R}, \]

instead of (3.21).

\[ h_T(r, t) := b(r, t) - u_{-T}(r, t), \quad r \in \Omega^{inc} \times \mathbb{R}, \]

instead of (3.22), and

\[ H^f_T(r, \omega) = B^f_T(r, \omega) - S^f_T(\omega, \omega), \quad r \in \Omega^{inc} \times \mathbb{R}, \]

instead of (3.23). \( \square \)

Relying on the \( \tau \text{DoD} \) interval \( I_T \) introduced in Definition 3.3 as well as the windowed time-dependent boundary densities \( \psi_{-T}, \psi_{+T}, \) and \( \psi_{\eta,T} \) introduced in Definition 3.4 and Remark 3.5, and the trace operators \( \gamma^\pm \) on \( \Gamma \) in Definition 6.4 (Appendix A), Lemmas 3.9 to 3.16 below develop estimates on the time-dependent density \( \psi_{+T} \) and bounds on the norms of certain frequency-domain operators, all of which are used in the proof of Theorem 3.2. The subsequent Lemma 3.18 is the Sobolev Lemma for functions with values in a Banach space (i.e. \( L^2(\Gamma) \)), that is used, in particular, to establish the theorem’s uniform-in-time estimate (3.8).

**Lemma 3.9** (Direct second-kind integral equations for \( \tilde{\psi}^f_{+T} \)). Assume that the obstacle \( \Omega \) satisfies the \( q \)-growth condition for some integer \( q \geq 0 \). Let \( \bar{b} \) denote a function defined in an open set \( \Omega^{inc} \) containing \( \bar{\Omega} \), and assume that \( \bar{b} \) is a solution to the wave equation in \( \Omega^{inc} \) that satisfies (3.11) as well as the \( s \)-regularity conditions (2.5) with \( s = q \). Additionally, let \( T \) denote a real number with associated time-windowed solutions \( \bar{\psi}_{+T} \) and \( \bar{\psi}_{-T} \) as in Definition 3.4 and Remark 3.5 where \( \psi \) is the solution to (3.10). Then \( \bar{\psi}_{+T} \) solves the integral equation

\[ (S\bar{\psi}^f_{+T})(r, t) = \gamma^+h_T(r, t), \]

(cf. (2.6)) where \( h_T \) is given by (3.22). Further, for each fixed \( \omega \geq 0 \) the Fourier transform \( \tilde{\psi}^f_{+T} \) of \( \bar{\psi}_{+T} \) satisfies the second-kind integral equation

\[ \left( A_\omega \tilde{\psi}^f_{+T} \right) (r, \omega) = \gamma^-\partial_n \hat{h}_T(r, \omega) + i\eta_0(\omega)\gamma^- \hat{H}^f_T(r, \omega), \quad r \in \Gamma, \]

where \( A_\omega \) and \( \hat{H}^f_T \) are given by equations (2.23) and (3.28), respectively.

Remark 3.10. With reference to Remark 3.5 we note that, for the particular case \( \bar{b} = b \), Lemma 3.9 tells us that the Fourier transform \( \tilde{\psi}^f \) of the solution \( \psi \) to (2.6) satisfies

\[ \left( A_\omega \tilde{\psi}^f_{+T} \right) (r, \omega) = \gamma^-\partial_n \hat{H}^f_T(r, \omega) - i\eta_0(\omega)\gamma^- \hat{H}^f_T(r, \omega), \quad r \in \Gamma, \]

where

\[ H^f_T(r, \omega) = B^f_T(r, \omega) - \mathcal{F} \psi_{-T}(r, \omega). \] \( \square \)
Remark 3.11. It is important to note a certain structural difference between equations (3.14) and (3.28) (both of which concern the frequency-domain operator $A_\omega$), namely, that the right hand side in the first equation is expressed in terms of the exterior trace operator $\gamma^+$ while the one for the second utilizes the interior trace operator $\gamma^-$ instead. In the first case using $\gamma^-$ on the right hand side produces identical results, since, according to (3.11), the incident field is assumed to be sufficiently smooth on the set $\Omega^{inc}$ and, therefore, in a neighborhood of $\Gamma$. In the second case, however, as indicated in the proof of Lemma 3.9, interior traces appear in view of the relation [32, p. 219]

\begin{equation}
\gamma^- \partial_n \mathcal{S}_\omega = \frac{1}{2} I + K^*_\omega
\end{equation}

for the values of the interior normal-derivative of the single-layer potential on $\Gamma$, which happens to reproduce the $\frac{1}{2} I + K^*_\omega$ contribution in the expression (2.22)-[2.23] for the operator $A_\omega$, while use of an exterior normal derivative would yield a different result, on account of the jump conditions for the normal derivatives of the single layer potential across $\Gamma$. In a related interpretation, the presence of $\gamma^-$ reflects the fact that the Green-formula derivations of direct integral equation formulations, like the $A_\omega$-equations we use, rely on consideration of the interior problem for the single layer potential that represents the incident field. \hfill \Box

Proof of Lemma 3.9. Since $\tilde{b}$ satisfies the assumptions in Lemma 3.6, an application of that lemma with $p = 0$ tells us that there exists a unique solution $\tilde{\psi} \in L^2(\mathbb{R}; L^2(\Gamma))$ of equation (3.10). Using the partition of unity decomposition embodied in (3.4), equation (3.10) may be re-expressed in the form (3.27), where $\tilde{h}_T$ and the associated $\tilde{u}_{-T}$ are given by (3.22) and (3.21) respectively.

To establish (3.28) we use Lemma 3.6 again which tells us that $\tilde{\psi}_f = \mathcal{F}[\tilde{\psi}]$ solves (3.14). But using the identity $\tilde{\psi}_f = \tilde{\psi}_{+T} + \tilde{\psi}_{-T}$ (see Definition 4 and Remark 3.5), together with the relations $\gamma^- \partial_n \tilde{B}_f^l = \gamma^+ \partial_n \tilde{B}_f^l$, and $\gamma^- \tilde{B}_f^l = \gamma^+ \tilde{B}_f^l$, which result in view of the assumed smoothness of $\tilde{B}_f^l$ in a neighborhood of $\Gamma$, we thus obtain

\begin{equation}
(A_\omega \tilde{\psi}_{+T}^l)(r, \omega) = - (A_\omega \tilde{\psi}_{-T}^l)(r, \omega) + \gamma^- \partial_n \tilde{B}_f^l(r, \omega) - i \eta_0(\omega) \gamma^- \tilde{B}_f^l(r, \omega), \quad r \in \Gamma.
\end{equation}

In view of the definition (2.23) of $A_\omega$ and the relation (3.31) we find that

\begin{equation}
(A_\omega \tilde{\psi}_{+T}^l)(r, \omega) = \gamma^- \partial_n \left( \tilde{B}_f^l - \mathcal{S}_\omega \tilde{\psi}_{-T}^l \right) - i \eta_0(\omega) \gamma^- \left( \tilde{B}_f^l - \mathcal{S}_\omega \tilde{\psi}_{-T}^l \right), \quad r \in \Gamma.
\end{equation}

Using the definition (3.23) of $\tilde{H}_f^l$ together with (3.20), finally, the desired equation (3.28) follows, and the proof is complete. \hfill \Box

Lemma 3.12 (Frequency $L^2$ bounds on the solution of (3.28)). Let $T \in \mathbb{R}$ and $q \geq 0$, let $\Gamma = \partial \Omega$ satisfy the $q$-growth condition, and assume $b$ satisfies the conditions of Lemma 3.9. Then there exist constants $C_1 = C_1(\Gamma) > 0$ and $C_2 = C_2(\Gamma) > 0$
such that $\tilde{\psi}^f_{+,\gamma}$ satisfies

$$
\left\| \tilde{\psi}^f_{+,\gamma} \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))} \leq C_1 \int_0^\infty \left\| \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right\|^2_{L^2(\Gamma)} d\omega
\left. + C_2 \int\limits_{\omega_0}^{\infty} \omega^{2q} \left\| \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\omega\gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right\|^2_{L^2(\Gamma)} d\omega, \right.
$$

(3.34)

where $\tilde{H}^f_T$ is given by \[3.28\].

Proof. In view of Lemma \[2.6\] (cf. Lemma \[3.6\]), $A_\omega$ is invertible for all $\omega > 0$, and, thus, using \[3.28\] we obtain $\tilde{\psi}^f_{+,\gamma} = A_\omega^{-1} \left( \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\eta_0 \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right)$. It follows that, for all $\omega > 0$,

$$
\left\| \tilde{\psi}^f_{+,\gamma} \right\|^2_{L^2(\Gamma)} \leq \left\| A_\omega^{-1} \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \left\| \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right\|^2_{L^2(\Gamma)}.
$$

Using the $q$-growth condition \[2.32\] we then obtain

$$
\left\| \tilde{\psi}^f_{+,\gamma} \right\|^2_{L^2(\Gamma)} \leq \frac{C_1}{2} \left\| \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right\|^2_{L^2(\Gamma)}, (0 \leq \omega < \omega_0)
$$

and

$$
\left\| \tilde{\psi}^f_{+,\gamma} \right\|^2_{L^2(\Gamma)} \leq \frac{C_2}{2} \omega^{2q} \left\| \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\omega\gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right\|^2_{L^2(\Gamma)}, (\omega > \omega_0)
$$

for certain constants $C_1$ and $C_2$. Noting that by Hermitian symmetry (see \[3.9\]) we have $\left\| \tilde{\psi}^f_{+,\gamma} \right\|_{L^2(\Gamma)} = \left\| \tilde{\psi}^f_{+,\gamma} \right\|_{L^2(\Gamma)}$ for $\omega \in \mathbb{R}$, it follows that

$$
\left\| \tilde{\psi}^f_{+,\gamma} \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))} = \frac{2}{2} \int_0^\infty \left\| \tilde{\psi}^f_{+,\gamma} \right\|^2_{L^2(\Gamma)} d\omega
\leq C_1 \int_0^\infty \left\| \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right\|^2_{L^2(\Gamma)} d\omega
\left. + C_2 \int_{\omega_0}^{\infty} \omega^{2q} \left\| \gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega) - i\omega\gamma^{-\partial_n \tilde{H}^f_T(-\gamma \cdot \omega)} \right\|^2_{L^2(\Gamma)} d\omega, \right.
$$

(3.37)

as desired. \hfill \square

Lemma 3.13 (About $\tilde{h}_T$: Limited temporal history horizon and bounded support).

Let $T > 0$ be given. Further, let $\tilde{b}$ denote a function defined in the open set $\Omega^{inc}$ containing $\Omega$, and assume that $\tilde{b}$ is a solution of the wave equation in $\Omega^{inc}$ that satisfies \[3.11\] for some $\alpha \in \mathbb{R}$ and which vanishes for $(r, t) \in \bar{\Omega} \times \{t \in T \cup \{T, \infty\}\}$ (that is, for $r \in \bar{\Omega}$ and $t \geq T - T_* - 2T_\gamma$; see Definition \[3.4\]). Then, the function $\tilde{h}_T$ defined in \[3.22\] satisfies

$$
\tilde{h}_T(r, t) = \begin{cases} 
-\tilde{u}_{+,\gamma}(r, t), & t \geq T - T_\gamma \\
0, & t < T - T_\gamma 
\end{cases} \quad \text{for } r \in \Omega,
$$

(3.38)

where, with reference to Definition \[4.2\] and Remark \[3.3.4\]

$$
\tilde{u}_{+,\gamma}(r, t) = (\mathcal{F} \tilde{\psi}_{+,\gamma})(r, t).
$$

(3.39)

Further, the function $\tilde{h}_T(r, t)$ has a bounded temporal support:

$$
supp \tilde{h}_T(r, \cdot) \subset [T - T_\gamma, T + T_*], \quad \text{for } r \in \bar{\Omega},
$$

(3.40)
Remark 3.14. The boundedness of the temporal support of the function \( \tilde{h}_T(r,t) = b - \bar{u}_{-T} \) provides an essential element in the proof of Theorem 4.1 and associated lemmas; additional details in this regard are provided in Remark 4.9. Note that, in particular, the right-hand side \( \gamma^+ \bar{h}_T(r,t) \) of (3.27) vanishes at times for which the solution \( \bar{\psi}_{+T} \) generically does not vanish. □

Remark 3.15. In view of (3.6), and taking into account Remark 3.5, Lemma 3.13 tells us that \( \tilde{h}_T \) is determined by the restriction of \( \bar{\psi}(r,t) \) to the interval \( I_T \), and, thus, in view of (3.27), the same is true of \( \bar{\psi}_{+T} \). Since, by (3.4), \( \bar{\psi}(r,t) \) coincides with \( \bar{\psi}_{+T}(r,t) \) for all \( t \geq T \), it follows that, for such values of \( t \) the solution \( \bar{\psi}(r,t) \) is solely determined by the restriction of \( \bar{\psi}(r,t) \) to the interval \( I_T \). In other words, for a given time \( T \) and in absence of illumination for \( t \geq T - T_\tau - 2\tau \), \( \bar{\psi}(r,t) \) is determined by the values of the same solution function \( \bar{\psi}(r,t) \) in the bounded time interval \( I_T \) immediately preceding time \( T \). Thus, Lemma 3.13 implies a bootstrap domain-of-dependence relation for solutions of the wave equation. □

Proof of Lemma 3.13. We first establish (3.38) for \( t \geq T - \tau \) and \( r \in \bar{\Omega} \). Since \( \bar{b} \) vanishes for \( t \geq T - T_\tau - 2\tau \), it suffices to show that \( \bar{u}_{-T}(r,t) = \tilde{u}_{+T}(r,t) \) for \( t \geq T - \tau \), or, equivalently, in view of (3.21) and (3.39), that

\[
(\mathcal{F}\bar{\psi}_{-T})(r,t) = (\mathcal{F}\bar{\psi}_{-T})(r,t) \quad \text{for} \quad t > T - \tau.
\]

Letting \( \xi = t - |r - r'|/c \) denote the second argument of the density factor \( \mu(r', t - |r - r'|/c) \) in the integrand (2.9), the density factors in (\( \mathcal{F}\bar{\psi}_{-T} \)) and (\( \mathcal{F}\bar{\psi}_{+T} \)) are given by \( w_-(\xi - T)\psi(r, \xi) \) and \( w_+(\xi - T + T_\tau + \tau)\psi(r, \xi) \), respectively. But, in view of (3.3), \( w_+(\xi - T + T_\tau + \tau) = 1 \) since, using (3.1), we see that \( \xi - T + T_\tau + \tau \geq \xi - T + \tau + |r - r'|/c = t - T + \tau \), and, thus, \( \xi - T + T_\tau + \tau > 0 \) in the present case \( t > T - \tau \). It follows that for such times the two density factors coincide, showing that (3.41) holds, and, thus, that \( \bar{u}_{-T}(r,t) = \tilde{u}_{+T}(r,t) \) for \( t > T - \tau \)—which establishes (3.38) for \( t > T - \tau \).

To complete the proof of (3.38) it remains to show that \( \tilde{h}_T(r,t) = 0 \) for \( r \in \bar{\Omega} \) and \( t < T - \tau \). But (3.27) tells us that \( \tilde{h}_T = 0 \) for \( r \in \Gamma \) and \( t < T - \tau \), since, by definition, \( \bar{\psi}_{+T}(r,t) \) itself vanishes for \( r \in \Gamma \) and \( t < T - \tau \). In view of (3.22), it follows that \( \tilde{h}_T(r,t) \) is a solution of the wave equation for \( r \in \Omega \) with vanishing boundary values for \( t \in (-\infty, T - \tau) \) which satisfies vanishing initial conditions throughout \( \Omega \) for a sufficiently large negative value of \( t \). Thus, \( \tilde{h}_T(r,t) \) vanishes for \( (r,t) \in \bar{\Omega} \times (-\infty, T - \tau) \), by solution uniqueness, and (3.38) follows.

In order to establish (3.40), finally, it suffices, in view of (3.38), to show that \( \tilde{h}_T(r,t) = 0 \) for all \( r \in \bar{\Omega} \) and \( t > T + T_\tau \). To do this we note from Definition 4.1 that \( \text{supp} \bar{\psi}_+(r, \cdot) \subset I_T = [T - T_\tau - 2\tau, T] \). Thus, equation (3.39), which can be expressed in the form \( \bar{u}_{+T}(r,t) = \int_T^{\infty} \frac{\psi(r', t - |r - r'|/c) \, d\sigma(r')}{{4\pi|r - r'|}} \), tells us that \( \tilde{h}_T(r,t) = 0 \) for \( (r,t) \in \bar{\Omega} \times [T + T_\tau, \infty) \). Equation (3.40) thus follows and the proof is complete. □

Lemma 3.16. Let \( \Gamma \) denote the boundary of a Lipschitz obstacle. Then there exist constants \( C_1 = C_1(\Gamma) > 0 \) and \( C_2 = C_2(\Gamma) > 0 \) such that the operator norm bounds

\[
\| (\gamma^{-1} \partial_n \pm i \omega \gamma^{-1} ) \mathcal{L} \|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C_1(1 + \omega^2)^{1/2}
\]
and
\[(3.43) \quad \left\| (\gamma^{-}\partial_{n} \pm i\gamma^{-}) \mathcal{J}_{\omega} \right\|_{L^{2}(\Gamma) \to L^{2}(\Gamma)} \leq C_{2}(1 + \omega^{2})^{1/2}\]
hold for all \(\omega \geq 0\).

**Proof.** The relations \[(3.42)\] and \[(3.43)\] follow directly from the bounds
\[(3.44) \quad \left\| \gamma^{-}\mathcal{J}_{\omega} \right\|_{L^{2}(\Gamma) \to L^{2}(\Gamma)} \leq D_{1}, \quad \text{and} \quad \left\| K_{\omega}^{*} \right\|_{L^{2}(\Gamma) \to L^{2}(\Gamma)} \leq D_{2} \omega + D_{3},\]
valid for all \(\omega \geq 0\) (where \(D_{1}, D_{2}, D_{3} > 0\) are constants), which are presented in reference [7] Thms. 3.3 and 3.5. Indeed, in view of \[(2.21)\] it follows that there exist constants \(\bar{D}_{1} > 0\) and \(\bar{D}_{2} > 0\) and \(C_{1} > 0\) such that
\[\left\| (\gamma^{-}\partial_{n} \pm i\omega\gamma^{-}) \mathcal{J}_{\omega} \right\|_{L^{2}(\Gamma) \to L^{2}(\Gamma)} = \left\| \frac{1}{2} I + K_{\omega}^{*} \pm i\omega S_{\omega} \right\|_{L^{2}(\Gamma) \to L^{2}(\Gamma)} \leq (\bar{D}_{1} + \bar{D}_{2} \omega) \leq C_{1}(1 + \omega^{2})^{1/2}.\]

Similarly,
\[\left\| (\gamma^{-}\partial_{n} \pm i\gamma^{-}) \mathcal{J}_{\omega} \right\|_{L^{2}(\Gamma) \to L^{2}(\Gamma)} \leq C_{2}(1 + \omega^{2})^{1/2}\]
for some constant \(C_{2} > 0\), and the result follows. □

**Lemma 3.17.** Let \(q\) denote a non-negative integer, let \(\Gamma = \partial\Omega\) satisfy the \(q\)-growth condition (Definition [3]), let \(T_{0}\) denote a given real number such that \(\bar{b}\) vanishes for \((r, t) \in \bar{\Omega} \times \{T_{0} \cup [T_{0}, \infty)\}\), and let \(\bar{H}_{T_{0}}^{f}\) be defined by \[(3.23)\] with \(T = T_{0}\). Then for some constant \(C > 0\) independent of \(T_{0}\) and \(\bar{b}\) we have
\[(3.45) \quad \int_{0}^{\omega_{0}} \left\| (\gamma^{-}\partial_{n} - i\gamma^{-}) \bar{H}_{T_{0}}^{f}(\cdot, \omega) \right\|^{2}_{L^{2}(\Gamma)} d\omega \]
\[+ \int_{\omega_{0}}^{\infty} \omega^{2q} \left\| (\gamma^{-}\partial_{n} - i\omega\gamma^{-}) \bar{H}_{T_{0}}^{f}(\cdot, \omega) \right\|^{2}_{L^{2}(\Gamma)} d\omega \]
\[\leq C \int_{\infty}^{\infty} (1 + \omega^{2})^{q+1} \left\| \bar{J}_{\omega}^{f}(\cdot, \omega) \right\|^{2}_{L^{2}(\Gamma)} d\omega.\]

**Proof.** In order to obtain the desired bound we define the differential operators
\[(3.46) \quad \mathcal{R} = (\gamma^{-}\partial_{n} - i\gamma^{-}) \quad \text{and} \quad \mathcal{T} = \left(-i\frac{\partial}{\partial t}\right)^{q} \left(\gamma^{-}\partial_{n} - \frac{\partial}{\partial t}\gamma^{-}\right),\]
whose Fourier symbols are given by
\[\hat{\mathcal{R}} = (\gamma^{-}\partial_{n} - i\gamma^{-}) \quad \text{and} \quad \hat{\mathcal{T}} = \omega^{q} (\gamma^{-}\partial_{n} - i\omega\gamma^{-}).\]
Then, denoting by \(\mathcal{Q}\) the sum of quantities on the left-hand side of \[(3.45)\] and using \[(3.38)\] together with Plancherel’s theorem we obtain
\[\mathcal{Q} = \int_{0}^{\omega_{0}} \left\| \bar{H}_{T_{0}}^{f}(\cdot, \omega) \right\|^{2}_{L^{2}(\Gamma)} d\omega + \int_{\omega_{0}}^{\infty} \left\| \bar{H}_{T_{0}}^{f}(\cdot, \omega) \right\|^{2}_{L^{2}(\Gamma)} d\omega \]
\[\leq \int_{\Gamma} \int_{-\infty}^{\infty} \left| \bar{H}_{T_{0}}^{f}(r, \omega) \right|^{2} d\omega d\sigma(r) + \int_{\Gamma} \int_{-\infty}^{\infty} \left| \tilde{T}_{T_{0}}(r, \omega) \right|^{2} d\omega d\sigma(r)\]
\[= \int_{\Gamma} \int_{-\infty}^{\infty} \left| \bar{u}_{\omega}(T_{0}, t') \right|^{2} dt' d\sigma(r) + \int_{\Gamma} \int_{-\infty}^{\infty} \left| T_{\omega}(T_{0}, t') \right|^{2} dt' d\sigma(r)\]
\[= \int_{\Gamma} \int_{T_{0} - \tau}^{\infty} \left| \bar{u}_{\omega}(T_{0}, t') \right|^{2} dt' d\sigma(r) + \int_{\Gamma} \int_{T_{0} - \tau}^{\infty} \left| T_{\omega}(T_{0}, t') \right|^{2} dt' d\sigma(r),\]
where the last equality follows from (3.38) by virtue of the temporal locality of the operators \( \mathcal{R} \) and \( \mathcal{T} \). In view of the non-negativity of the integrands on the right-hand side of this estimate, it follows that

\[
Q \leq \int_{-\infty}^{\infty} |\mathcal{R} \tilde{u}_{s,T_0}(r,t')|^2 \, dt' \, d\sigma(r) + \int_{-\infty}^{\infty} |\mathcal{T} \tilde{u}_{s,T_0}(r,t')|^2 \, dt' \, d\sigma(r).
\]

Using once again Plancherel’s theorem, (3.48) becomes

\[
\int_{-\infty}^{\infty} \left\| \mathcal{R} \tilde{u}_{s,T_0}(\cdot,\omega) \right\|_{L^2(\Gamma)}^2 \, \omega \, d\omega + \int_{-\infty}^{\infty} \left\| \mathcal{T} \tilde{u}_{s,T_0}(\cdot,\omega) \right\|_{L^2(\Gamma)}^2 \, \omega \, d\omega,
\]

where \( \tilde{u}_{s,T_0} \) denotes the temporal Fourier transform of the single layer potential \( \tilde{u}_{s,T_0} \),

\[
\tilde{u}_{s,T_0}(r,\omega) = \left( \mathcal{R} \omega \tilde{u}_{s,T_0} \right)(r,\omega).
\]

Using (3.47), (3.42) and (3.43) to bound the integrands on the right-hand side of (3.49), and then using the fact that \( \omega^{2r} \leq (1 + \omega^2)^r \) for nonnegative \( r \), we obtain

\[
Q \leq C_1 \int_{-\infty}^{\infty} (1 + \omega^2) \left\| \tilde{u}_{s,T_0}(\cdot,\omega) \right\|_{L^2(\Gamma)}^2 \, \omega \, d\omega
\]

\[
+ C_2 \int_{-\infty}^{\infty} \omega^{2q} (1 + \omega^2) \left\| \tilde{u}_{s,T_0}(\cdot,\omega) \right\|_{L^2(\Gamma)}^2 \, \omega \, d\omega
\]

\[
\leq C \int_{-\infty}^{\infty} (1 + \omega^2)^{q+1} \left\| \tilde{u}_{s,T_0}(\cdot,\omega) \right\|_{L^2(\Gamma)}^2 \, \omega \, d\omega.
\]

for a suitable constant \( C \), and the result follows.

In order to establish the uniform-in-time estimates (3.8) and (4.2) we utilize a Bochner-space version of the Sobolev lemma (cf. Definition 7) over a physical domain \( \mathcal{U} \). This result, which is presented in what follows, is primarily used in the surface-domain case \( \mathcal{U} = \Gamma \), but it is also used for volumetric domains in Corollary 4.4. The proof results by merely incorporating the Bochner space nomenclature in the classical arguments used for the real-valued case [19, Lem. 6.5].

**Lemma 3.18** (Sobolev embedding in Bochner spaces). Let \( k \) denote a nonnegative integer, let \( s \geq 0 \), and let \( \mathcal{U} \) denote a subset of \( \mathbb{R}^3 \), where either \( \mathcal{U} = \Gamma \) equals the Lipschitz boundary of an open and bounded domain \( \Omega \subset \mathbb{R}^3 \), or \( \mathcal{U} = D \) equals an open and bounded domain \( D \subset \mathbb{R}^3 \) with a Lipschitz boundary. Then for \( r > k + 1/2 \), \( H^r(\mathbb{R}; H^s(\mathcal{U})) \) is continuously embedded in \( C^k(\mathbb{R}; H^s(\mathcal{U})) \): \( H^r(\mathbb{R}; H^s(\mathcal{U})) \hookrightarrow C^k(\mathbb{R}; H^s(\mathcal{U})) \). In particular, there exists a constant \( C \) such that for all functions \( a \in H^r(\mathbb{R}; H^s(\mathcal{U})) \) the bound

\[
\max_{0 \leq t \leq T} \sup_{0 \leq k \leq r} \left\| \frac{\partial^k}{\partial t^k} a(t) \right\|_{H^s(\mathcal{U})} \leq C \| a \|_{H^r(\mathbb{R}; H^s(\mathcal{U}))}
\]

holds, where \( \frac{\partial^k}{\partial t^k} a(t) \) denotes the \( k \)-th classical derivative of \( a \) with respect to \( t \) (see Appendix A).

**Proof.** Let \( a \in H^r(\mathbb{R}; H^s(\mathcal{U})) \). We first show that \( \tilde{a}(\omega) \in L^1(\mathbb{R}; H^s(\mathcal{U})) \) for all nonnegative integers \( \ell \leq k \). Indeed, using the relation \( \tilde{\partial_t} a(\omega) = (i\omega)^{\ell} \tilde{a}(\omega) \) (in
accordance with the Fourier transform convention (2.17)) we obtain
\[
\int_{-\infty}^{\infty} \| \hat{\partial^\ell a}(\omega) \|_{H^r(\mathcal{U})} \, d\omega = \int_{-\infty}^{\infty} \| (i\omega)^{\ell} \hat{a}(\omega) \|_{H^r(\mathcal{U})} \, d\omega \\
\leq \int_{-\infty}^{\infty} (1 + \omega^2)^{k/2} \| \hat{a}(\omega) \|_{H^r(\mathcal{U})} \, d\omega \\
(3.52)
= \int_{-\infty}^{\infty} (1 + \omega^2)^{(k-r)/2} \| \hat{a}(\omega) \|_{H^r(\mathcal{U})} (1 + \omega^2)^{r/2} \, d\omega \\
\leq \left( \int_{-\infty}^{\infty} (1 + \omega^2)^{k-r} \, d\omega \right)^{1/2} \left( \int_{-\infty}^{\infty} (1 + \omega^2)^r \| \hat{a}(\omega) \|_{H^r(\mathcal{U})}^2 \, d\omega \right)^{1/2} \\
\leq C \| a \|_{H^r(\mathbb{R}; H^s(\mathcal{U}))}
\]
where we used the fact that \( r > k + 1/2 \) to bound the integral of \( (1 + \omega^2)^{k-r} \) by a constant independent of \( a \), and where we utilized the equivalent norm of the space \( H^r(\mathbb{R}; H^s(\mathcal{U})) \) displayed in (A.4). It follows that \( \hat{\partial^\ell a} \in L^1(\mathbb{R}; H^s(\mathcal{U})) \), as claimed.

By the Bochner version of the Riemann-Lebesgue lemma [22, Lem. 2.4.3] (which ensures that the Fourier transform of an \( L^1(\mathbb{R}; H^s(\mathcal{U})) \) function is a norm-continuous function of \( \omega \) that tends to zero in norm as \( \omega \to \infty \)), in conjunction with the fact that, as established above, \( \hat{\partial^\ell a} \in L^1(\mathbb{R}; H^s(\mathcal{U})) \), it follows that
\[
(3.53)
\hat{\partial^\ell a} \in C(\mathbb{R}; H^s(\mathcal{U})), \quad 0 \leq \ell \leq k.
\]

Since by hypothesis \( a \in H^r(\mathbb{R}; H^s(\mathcal{U})) \), further, it follows that for \( 0 \leq \ell \leq k \), \( \hat{\partial^\ell a} \in L^2(\mathbb{R}; H^s(\mathcal{U})) \), and thus by the Bochner Plancherel theorem [27, Thm. 2.20] we have, additionally, \( \hat{\partial^\ell a} \in L^1(\mathbb{R}; H^s(\mathcal{U})) \cap L^2(\mathbb{R}; H^s(\mathcal{U})) \). Now, applying the Bochner Fourier inversion theorem [27] (see also [45, §8.4]), we obtain
\[
(3.54) \quad \partial^\ell a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\partial^\ell a}(\omega) e^{i\omega t} \, d\omega, \quad 0 \leq \ell \leq k,
\]
which, in view of (3.53), shows that \( \partial^\ell a(t) = \frac{1}{2\pi} \hat{\partial^\ell a}(-t) \in C(\mathbb{R}; H^s(\mathcal{U})) \) for \( 0 \leq \ell \leq k \) and, therefore, that \( H^r(\mathbb{R}; H^s(\mathcal{U})) \subseteq C^k(\mathbb{R}; H^s(\mathcal{U})) \). Finally, using (3.54) and then (3.52) we see that, for \( 0 \leq \ell \leq k \),
\[
\sup_{t \in \mathbb{R}} \left\| \frac{\partial^\ell}{\partial t} a(t) \right\|_{H^s(\mathcal{U})} = \sup_{t \in \mathbb{R}} \frac{1}{2\pi} \left\| \int_{-\infty}^{\infty} \hat{\partial^\ell a}(\omega) e^{i\omega(-t)} \, d\omega \right\|_{H^s(\mathcal{U})} \\
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| \hat{\partial^\ell a}(\omega) \right\|_{H^s(\mathcal{U})} \, d\omega \leq C \| a \|_{H^r(\mathbb{R}; H^s(\mathcal{U}))}.
\]
This establishes (3.51) and concludes the proof of the lemma.

**Lemma 3.19.** Let \( p, q \) denote non-negative integers, assume \( \Gamma \) satisfies the \( q \)-growth condition, and assume \( \tilde{b} \) satisfies the \( s \)-regularity conditions \([2.5] \) with \( s = p + q + 1 \). Then, letting \( \tilde{\psi} \) denote the solution of (3.10), we have \( \tilde{\psi} \in C^p(\mathbb{R}; L^2(\Gamma)) \), and \( \partial^\ell \tilde{\psi} \) satisfies the integral equation
\[
(3.55) \quad \left( S \partial^\ell \tilde{\psi} \right)(r, t) = \gamma^+ \partial^\ell \tilde{b}(r, t) \quad \text{for} \quad (r, t) \in \Gamma \times \mathbb{R}.
\]

**Proof.** From Lemma 3.6 we have \( \tilde{\psi} \in H^{p+1}(\mathbb{R}; L^2(\Gamma)) \), and, thus, by Lemma 3.18 we obtain \( \tilde{\psi} \in C^p(\mathbb{R}; L^2(\Gamma)) \) as claimed. Similarly, by Lemma 3.18 \( \tilde{b} \) satisfies
\[ \gamma^+ \tilde{b} \in C^{p+q+1}(\mathbb{R}; L^2(\Gamma)) \subset C^p(\mathbb{R}; L^2(\Gamma)). \] The proof is now completed inductively, by differentiation of (3.10) under the integral sign in the expression (2.10) for the operator \( S \).

The proof of Theorem 3.2 is presented in what follows.

**Proof of Theorem 3.2**

The proof of (3.7) follows by first obtaining an \( L^2 \)-in-time estimate for the solution \( \tilde{\psi} \) of (3.10) with a generic right-hand side \( \tilde{b} \) in suitable Sobolev-Bochner spaces, and then applying that estimate to the particular cases \( \tilde{\psi} = \psi \) (for \( \tilde{b} = b \)) and \( \tilde{\psi} = \partial_t^p \psi \) (for \( \tilde{b} = \partial_t^p b \), see Definition 7).

To obtain the necessary estimates for these functions \( \tilde{\psi} \), in turn, we develop \( L^2 \)-in-time estimates for the quantities \( \tilde{\psi}_{+, T_0} \) that are related to the solution \( \tilde{\psi} \) of (3.10) via Definition 4 and Remark 3.5. Since, by hypothesis, both aforementioned selections \( \tilde{\psi} = \psi \) and \( \tilde{\psi} = \partial_t^p b \) satisfy \( \gamma^+ \tilde{b} \in H^{q+1}(\mathbb{R}; H^1(\Gamma)) \) and \( \gamma^+ \partial_n \tilde{b} \in H^q(\mathbb{R}; L^2(\Gamma)) \), it follows that the conditions of Lemma 3.9 are met, and, thus Lemma 3.12 tells us that \( \tilde{\psi}_{+, T_0} \) satisfies the estimate (3.34). Using this bound in conjunction with the relation \( \tilde{\psi} = \tilde{\psi}_{+, T_0} \) for \( t > T_0 \) (see (3.4)) and Plancherel’s identity we obtain

\[
\left\| \tilde{\psi} \right\|_{L^2(T_0, \infty); L^2(\Gamma)}^2 \leq C_1 \int_{T_0}^\infty \left\| \gamma^- \partial_n \tilde{H}_{T_0}^f (\cdot, \omega) - i \gamma^- \tilde{H}_{T_0}^f (\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \omega^{2q} \, \mathrm{d} \omega 
\]

\[
\text{and} \quad C_2 \int_{T_0}^\infty \omega^{2q} \left\| \gamma^- \partial_n \tilde{H}_{T_0}^f (\cdot, \omega) - i \omega \gamma^- \tilde{H}_{T_0}^f (\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \omega \, \mathrm{d} \omega.
\]

Using (3.45) in Lemma 3.17, further, to estimate the integrals on the right-hand side of (3.56) we obtain

\[
\left\| \tilde{\psi} \right\|_{L^2(T_0, \infty); L^2(\Gamma)}^2 \leq \tilde{C} \int_{T_0}^\infty (1 + \omega^2)^q \left\| \tilde{\psi}_{+, T_0} \right\|_{L^2(\Gamma)}^2 \omega \, \mathrm{d} \omega,
\]

and, in view of the equivalence of the norms (A.2) and (A.4), we re-express (3.57) as

\[
\left\| \tilde{\psi} \right\|_{L^2(T_0, \infty); L^2(\Gamma)}^2 \leq \tilde{C} \left( \left\| \tilde{\psi}_{+, T_0} \right\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2 + \left\| \partial_t^{q+1} \tilde{\psi}_{+, T_0} \right\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2 \right).
\]

Applying the Leibniz formula to the expression

\[
\partial_t^{q+1} \tilde{\psi}_{+, T_0} (r, t) = \partial_t^{q+1} \left( w_+ (t - T_0 + T_* + \tau) w_- (t - T_0) \tilde{\psi}(r, t) \right)
\]

and noting that for all \( r \in \Gamma \) and for \( 1 \leq i \leq p \) we have \( \text{supp} \partial_t^i \tilde{\psi}_{+, T_0} (r, \cdot) \subset I_{T_0} \), using straightforward bounds on the functions \( w_- \) and \( w_+ \) (that do not depend on \( T_0 \)—see Definition 4), we obtain

\[
\left\| \tilde{\psi} \right\|_{L^2(T_0, \infty); L^2(\Gamma)}^2 \leq \sum_{i=0}^{q+1} C_i \left\| \partial_t^i \tilde{\psi} \right\|_{L^2(I_{T_0}; L^2(\Gamma))}^2, \quad C_i = C_i(\Gamma, \tau).
\]
Using the continuity of the inclusion map in Sobolev spaces, it follows that
\begin{equation}
\| \tilde{\psi} \|_{L^2([T_0, \infty); L^2(\Gamma))} \leq C \| \tilde{\psi} \|_{H^{s+1}(I_{T_0}; L^2(\Gamma))} < \infty, \quad C = C(\Gamma, \tau),
\end{equation}
where the finiteness of the norm over \( I_{T_0} \) in \( 3.60 \) follows from Lemma 3.6—since that lemma tells us that \( \psi \in H^{q+1}(I_{T_0}; L^2(\Gamma)) \), in view of the assumed hypotheses \( \gamma^+ \tilde{b} \in H^{q+2}(\mathbb{R}; L^2(\Gamma)) \) and \( \gamma^+ \partial_n \tilde{b} \in H^{q+1}(\mathbb{R}; L^2(\Gamma)) \) for each of \( \tilde{b} = b \) and \( \tilde{b} = \partial_t^2 b \).

Applying \( 3.60 \) with \( \tilde{b} = b \) yields \( 3.7 \) in the case \( p = 0 \). Furthermore, Lemma 3.19 with \( \tilde{b} = b \) implies that \( \tilde{\psi} = \psi \) satisfies \( 3.55 \). But this equation can be expressed in the form \( 3.10 \) with \( \tilde{b} = \partial_t^2 b \) and \( \psi = \partial_t^2 \psi \), for which the estimate \( 3.60 \) becomes
\begin{equation}
\| \partial_t^2 \psi \|_{L^2([T_0, \infty); L^2(\Gamma))} \leq C \| \partial_t^2 \psi \|_{H^{q+1}(I_{T_0}; L^2(\Gamma))} < \infty, \quad C = C(\Gamma, \tau),
\end{equation}
which, together with \( 3.60 \) in the case \( \tilde{b} = b \) and \( \tilde{\psi} = \psi \), and using once again the continuity of the inclusion map in Sobolev spaces, implies \( 3.7 \).

Applying \( 3.7 \) with \( p = 1 \) together with Lemma 3.18 yields
\begin{equation}
\sup_{t > T_0} \| \psi(\cdot, t) \|_{L^2(\Gamma)} \leq \tilde{C} \| \psi \|_{H^1([T_0, \infty); L^2(\Gamma))} \\
\leq C(\Gamma, \tau) \| \psi \|_{H^{q+2}(I_{T_0}; L^2(\Gamma))},
\end{equation}
and, thus, \( 3.8 \). The proof is now complete. \( \square \)

4. **Super-algebraic decay of boundary densities and local energies**

This section extends the theoretical results of Section 3: it establishes that not only is it possible to bound the density \( \psi \) in the unbounded time interval \([T_0, \infty)\) by its values on the preceding bounded subinterval \( I_{T_0} \), as shown in Theorem 3.2 but also, in Theorem 4.1 below, the main theorem of this paper, that the temporal Sobolev and maximum norms of the solution \( \psi \) on time intervals of the form \([T_0 + t, \infty), \) \( t > 0 \), each decay rapidly (e.g., super-algebraically fast for temporally smooth incident signals) as \( t \to \infty \); see also Remark 4.5 where a related but somewhat modified decay result and proof strategy are suggested. The statement and proof of Theorem 4.1 rely on the nomenclature introduced in Sections 2 and 3. Two important corollaries to this theorem, namely Corollaries 4.2 and 4.3, relate Theorem 4.1 to rapid decay of solutions of the wave equation. Following the statement and proof of Corollary 4.3 a brief discussion is presented that lays out the main lines of the proof of Theorem 4.1: a detailed proof of this theorem is presented at the end of this section, following a sequence of preparatory lemmas.

**Theorem 4.1.** Let \( p, q \) and \( n \) denote non-negative integers, \( n > 0 \), let \( T_0 > 0 \) and \( \tau > 0 \) be given, and assume (i) \( \Gamma \) satisfies the \( q \)-growth condition (Definition 3); (ii) The incident field \( b(\mathbf{r}, t) \) satisfies the \( s \)-regularity conditions \( 2.5 \) with \( s = p + q + (n + 1)(q + 1) \); as well as, (iii) The incident field \( b = b(\mathbf{r}, t) \) satisfies \( 2.3 \) and it vanishes for \( (\mathbf{r}, t) \in \Omega \times \{ I_{T_0} \cup [T_0, \infty) \} \), with \( I_{T_0} \) as in Definition 3. Then for \( t > T_0 \) the solution \( \psi \) of \( 2.6 \) satisfies the \( t \)-decay estimate
\begin{equation}
\| \psi \|_{H^{p+1}([t, \infty); L^2(\Gamma))} \leq C(\Gamma, \tau, p, n)(t - T_0)^{1/2 - n} \| \psi \|_{H^{p+1}(I_{T_0}; L^2(\Gamma))} < \infty.
\end{equation}
If \( p \geq 1 \) then \( \psi \) additionally satisfies the temporally pointwise \( t \)-decay estimate
\begin{equation}
\| \psi(\cdot, t) \|_{L^2(\Gamma)} \leq C(\Gamma, \tau, n)(t - T_0)^{1/2 - n} \| \psi \|_{H^{n+1}(I_{T_0}; L^2(\Gamma))} < \infty
\end{equation}
for all \( t > T_0 \).
Corollary 4.2 (Spatial-$L^2$/pointwise solution decay). Let $q$ denote a non-negative integer, let $\Gamma$ satisfy the $q$-growth condition (Definition 3), and assume that the data $b$ for the problem (2.1)-(2.2) is such that
\begin{equation}
\gamma^+ b \in C^\infty(\mathbb{R}; L^2(\Gamma)) \quad \text{and} \quad \gamma^+ \partial_n b \in C^\infty(\mathbb{R}; L^2(\Gamma)).
\end{equation}

Further, assume that $b = b(r,t)$ satisfies (2.3), and that, for given $T_0 > 0$ and $\tau > 0$, $b(r,t)$ vanishes for $(r,t) \in \Omega \times [0, t_0 + \tau]$, where $t_0 = T_0, T, \ldots$ is defined in Definition 3 with $T_0 = \text{diam}(\Gamma)/c$ as indicated in eq. (3.1). Then, letting $D = \Omega^c \cap \{|r| < R\}$, for given $R > 0$ such that $\Omega \subset \{|r| < R\}$, and defining $r_{max} = \sup_{r \in D, r' \in \Gamma} |r - r'|$, for each pair of integers $n > 0$ and $p > 0$ there exists a constant $C = C(p, n, \tau, R, \Gamma) > 0$ such that the solution $u$ to (2.1) satisfies the $t$-decay estimate
\begin{equation}
\|\partial_t^p u(t)\|_{L^2(D)} \leq C(t - T_0 - r_{max}/c)^{1/2-n} \|\psi\|_{H^{p+(n+1)(q+1)+1}(I_0; L^2(\Gamma))} < \infty,
\end{equation}
for all $t \in (T_0 + r_{max}/c, \infty)$.

Further, $u$ also decays super-algebraically fast with increasing time $t$ at each point $r$ outside $\Omega$. More precisely, given any compact set $K \subset (\Omega^c)$ and defining $r_{max} = \sup_{r \in K, r' \in \Gamma} |r - r'|$, for each pair of integers $n > 0$ and $p > 0$ there exists a constant $C = C(p, n, \tau, \Gamma) > 0$ such that $u$ satisfies the $t$-decay estimate
\begin{equation}
|\partial_t^p u(r,t)| \leq C(t - T_0 - r_{max}/c)^{1/2-n} \|\psi\|_{H^{p+(n+1)(q+1)+1}(I_0; L^2(\Gamma))} < \infty,
\end{equation}
for all $(r, t) \in K \times (T_0 + r_{max}/c, \infty)$.

Proof. Except for hypothesis (ii) of Theorem 4.1, all conditions of that Theorem follow immediately from hypotheses of the present corollary. Hypothesis (ii), in turn, follows since by hypothesis (4.3) and the assumed compact support of $b$, this function satisfies $\gamma^+ b \in H^p(\mathbb{R}; L^2(\Gamma))$ and $\gamma^+ \partial_n b \in H^p(\mathbb{R}; L^2(\Gamma))$ for every integer $p \geq 0$. Thus, the conditions of Theorem 4.1 are satisfied for every integer $n > 0$ and every integer $p \geq 0$ and we thus have
\begin{equation}
\|\partial_t^p \psi\|_{H^1((t',\infty); L^2(\Gamma))} \leq \|\psi\|_{H^{p+1}((t',\infty); L^2(\Gamma))} \\
\leq C_1(p, n, \Gamma, \tau)(t' - T_0)^{1/2-n} \|\psi\|_{H^{p+(n+1)(q+1)+1}(I_0; L^2(\Gamma))}
\end{equation}
for $t' > T_0$.

The estimates (4.4) and (4.5) are obtained in what follows by relying on a corresponding estimate on $\partial_t^p u$ in the norm of $H^1((T_0 + r_{max}/c, \infty); L^2(\Gamma))$, on one hand, and an estimate on the norm of $\partial_t^p u(r,\cdot)$ in $H^1((t + r_{max}/c, \infty))$ for each $(r, t) \in K \times (T_0 + r_{max}/c, \infty)$, on the other hand. Estimate (4.4) is established by first differentiating $s$ times under the integral sign in the integral representation (2.11) for the solution $u$ (for certain values of the integer $s$) obtaining $\partial_t^s u(r, t) = (\mathcal{S} \partial_t^s \psi)(r, t)$, and then applying the Cauchy-Schwarz inequality to the formula (2.9) for $\mathcal{S}$ to obtain, for all $t'$,
\begin{equation}
\|\partial_t^s u(\cdot, t')\|_{L^2(D)}^2 = \int_D |\partial_t^s u(r, t')|^2 \, dV(r) \\
\leq \frac{1}{16\pi^4} \int_D \left( \int_{\Gamma} |\partial_t^s \psi(r', t' - |r - r'|/c)|^2 \, d\sigma(r') \right) \left( \int_{\Gamma} \frac{d\sigma(r')}{|r - r'|^2} \right) \, dV(r).
\end{equation}
Integrating this bound in $t'$ for $t' \geq t$, for a given $t$, we obtain

\begin{align}
\| \partial_t^s u \|_{L^2([t, \infty); L^2(D))} &\leq \frac{1}{16\pi^2} \int_D \left[ \left( \int_t^{\infty} \int_\Gamma |\partial_t^s \psi(r', t' - |r - r'|/c)|^2 d\sigma(r') dt' \right) \times \right. \\
&\left. \left( \int_\Gamma \frac{d\sigma(r')}{|r - r'|^2} \right) \right] dV(r).
\end{align}

Letting $I_{s,1}(r, t)$ and $I_{s,2}(r)$ denote, respectively, the first and second factors in the integrand of the integral over $D$,

\begin{align}
I_{s,1}(r, t) &= \int_t^{\infty} \int_\Gamma |\partial_t^s \psi(r', t' - |r - r'|/c)|^2 d\sigma(r') dt', \\
I_{s,2}(r) &= \int_\Gamma \frac{d\sigma(r')}{|r - r'|^2},
\end{align}

we seek to bound each of $I_{s,1}(r, t)$ and $I_{s,2}(r)$ by quantities independent of $r$. Considering first $I_{s,1}(r, t)$ for fixed $r \in D$, a change in integration order and a slight extension of the integration domain in the $t'$ variable yields

\begin{align}
I_{s,1}(r, t) &\leq \int_t^{\infty} \int_{1 - r_{\text{max}}/c}^{\infty} |\partial_t^s \psi(r', t')|^2 d\sigma(r') dt' = \| \partial_t^s \psi \|^2_{L^2([t - r_{\text{max}}/c, \infty); L^2(\Gamma))}.
\end{align}

The integral of $I_{s,2}(r)$ over $D$, in turn, is bounded by an $R$- and $\Gamma$-dependent constant since by switching the order of integration we obtain

\begin{align}
\int_D I_{s,2}(r) dV(r) = \int_\Gamma \left( \int_D \frac{1}{|r - r'|^2} dV(r) \right) d\sigma(r'),
\end{align}

where the inner integral is bounded for each $r' \in \Gamma$ by an $R$- and $\Gamma$-dependent constant. (The latter statement can easily be established by using a spherical coordinate system centered at each $r' \in \Gamma$ for integration in $r \in D$.) We have therefore established that

\begin{align}
\| \partial_t^s u \|^2_{L^2([t, \infty); L^2(D))} &\leq C_2 \| \partial_t^s \psi \|^2_{L^2([t - r_{\text{max}}/c, \infty); L^2(\Gamma))}.
\end{align}

Taking separately $s = p$ and $s = p + 1$ in (4.11) and adding the results we obtain

\begin{align}
\| \partial_t^p u \|^2_{H^1([t, \infty); L^2(D))} &\leq C_3 \| \partial_t^p \psi \|^2_{H^1([t - r_{\text{max}}/c, \infty); L^2(\Gamma))}.
\end{align}

Using (4.12) in conjunction with (4.6) and the Sobolev embedding Lemma 3.18 we obtain

\begin{align}
\| \partial_t^p u(\cdot, t) \|^2_{L^2(D)} \leq C_4 \| \partial_t^p u \|^2_{H^1([t, \infty); L^2(D))} \leq \sqrt{C_3} C_4 \| \partial_t^p \psi \|^2_{H^1([t - r_{\text{max}}/c, \infty); L^2(\Gamma))} \leq C_5(\Gamma, \tau, n) (t - T_0 - r_{\text{max}}/c)^{1/2 - n} \| \psi \|^2_{H^{p(n+1)(\rho+1)/4+1}(\Gamma)}.
\end{align}

and, thus, (4.4) follows.

To establish (4.5), in turn, we once again use $s$-times differentiation under the integral sign in the variable $t$ in the representation (2.11) followed by the Cauchy-Schwarz inequality in the formula (2.9) for $\mathcal{S}$ to obtain

\begin{align}
|\partial_t^s u(r, t')|^2 &\leq \left( \int_\Gamma \frac{1}{|r - r'|^2} d\sigma(r') \right) \left( \int_\Gamma |\partial_t^s \psi(r', t' - |r - r'|/c)|^2 d\sigma(r') \right) \\
&\leq C_6^2 \int_\Gamma |\partial_t^s \psi(r', t' - |r - r'|/c)|^2 d\sigma(r'),
\end{align}

where

\begin{align}
\| \partial_t^s \psi \|^2_{H^1([t - r_{\text{max}}/c, \infty); L^2(\Gamma))} &\leq C_3 \| \partial_t^p \psi \|^2_{H^1([t - r_{\text{max}}/c, \infty); L^2(\Gamma))}.
\end{align}
where \( C^2 \) is \( \sup_{\rho \in \mathcal{R}} \int_{\Gamma} \frac{1}{|\rho - \omega|} \, d\sigma(\rho') \). Integrating in \( t' \) and using (4.10) to estimate the resulting quantity \( I_{s,1}(r, t) \), for each \( r \in \mathcal{R} \) we obtain the \( L^2 \) bound
\[
(4.14) \quad \int_{t}^{\infty} |\partial^s_t u(r, t')|^2 \, dt' \leq C_0 \| \partial^s_t \psi \|_{L^2([t-r_{\max}/c, \infty); L^2(\Gamma))}^2.
\]
Taking separately \( s = p \) and \( s = p + 1 \) in (4.14), adding the results, and using (4.6) in conjunction with the Sobolev embedding Lemma 3.18 we obtain, again for \( (r, t) \in \mathcal{R} \times (T_0 + r_{\max}/c, \infty) \),
\[
|\partial^p_t u(r, t)| \leq C_7 \| \partial^p_t u(r, \cdot) \|_{H^1([t, \infty))} \leq C_8 \| \partial^p_t \psi \|_{L^2([t-r_{\max}/c, \infty); L^2(\Gamma))} \leq C_9 (\Gamma, r, n) (t - T_0 - r_{\max}/c)^{1/2-n} \| \psi \|_{H^{p+(n+1)(q+1)/2}(I_{T_0}; L^2(\Gamma))} < \infty,
\]
establishing (4.5). The proof is complete.

While equation (4.5) provides a concrete solution decay estimate at each spatial point \( r \) outside \( \Omega \), the associated constant \( C \) does increase without bound as \( r \) approaches \( \Omega \) (cf. (4.13)). The result presented next, in contrast, which holds for arbitrary bounded subsets of \( \Omega^c \), establishes decay of the local energy expression (1.2) considered in much of the literature—thus enriching the previous estimate (4.4) by incorporating a norm containing spatial derivatives. The proof below relies on the estimates provided in Corollary 4.2 in conjunction with standard elliptic regularity properties.

**Corollary 4.3 (Local energy decay estimates).** Let \( \Gamma \) and \( b \) satisfy the hypotheses of Corollary 4.2 and let \( u \) denote the solution to the problem (2.1). Then, the local energy \( E \) (equation (1.2)) associated with the solution \( u \) decays super-algebraically fast as \( t \to \infty \). More precisely, letting \( R > 0 \) be such that the radius-\( R \) ball \( B_R = \{ |r| \leq R \} \) contains \( \Omega^c \), defining the compact region \( D = \Omega^c \cap B_R \), and letting \( r_{\max} = \sup_{r \in D, t' \in \Gamma} |r - r'| \), for each integer \( n > 0 \) there exists a constant \( C = C(\Gamma, R, \tau, n) > 0 \) such that
\[
(4.15) \quad E(u, D, t) \leq C(t - T_0 - r_{\max}/c)^{1-2n} \| \psi \|_{H^{n+1}(\Gamma)}^2 \| \psi \|_{H^{p+(n+1)(q+1)/2}(I_{T_0}; L^2(\Gamma))} < \infty,
\]
for all \( t \in (T_0 + r_{\max}/c, \infty) \).

**Proof.** In view of the bound on \( \int_{D} |u_t(r, t)|^2 \, dV(r) \) provided by Corollary 4.2 it suffices to establish a corresponding estimate for \( \int_{D} |\nabla u(r, t)|^2 \, dV(r) \) for \( t \in (T_0 + r_{\max}/c, \infty) \). To do this, we use the fact that \( u \) is a solution of the wave equation (2.1a) as well as the hypothesis that \( \gamma^+ b(r', t') = 0 \) for \( (r', t') \in \Gamma \times \{I_{T_0} \cup [T_0, \infty)\} \), which together imply that, for each \( t' \in I_{T_0} \cup [T_0, \infty) \), \( u \) satisfies the exterior Dirichlet problem
\[
(4.16a) \quad \Delta u(r, t') = f(r, t'), \quad \text{for} \quad r \in \Omega^c,
(4.16b) \quad u(r, t') = 0, \quad \text{for} \quad r \in \Gamma = \partial \Omega,
\]
for the Poisson equation (4.16a), where, for \( r \in \Omega^c \), \( f(r, t') = \frac{1}{c^2} \partial^2_t u(r, t') \). Exploiting, in addition, the relation (4.4) with \( p = 2 \), we obtain, for \( t > T_0 + r_{\max}/c \),
\[
(4.17) \quad \| f(\cdot, t) \|_{L^2(D)} \leq C_1 \| (t - T_0 - r_{\max}/c)^{1/2-n} \| \psi \|_{H^{(n+1)(q+1)/2}(I_{T_0}; L^2(\Gamma))} < \infty,
\]
which, in particular, implies that
\[
(4.18) \quad f(\cdot, t) \in L^2(D) \quad \text{for} \quad t \in (T_0 + r_{\max}/c, \infty).
\]
To establish regularity for the Poisson solution we introduce the region $D_\varepsilon = \Omega^c \cap \{|r| \leq R + \varepsilon\}$ for some $\varepsilon > 0$ as well as, for each fixed $t \in (T_0 + r_{\text{max}}/c, \infty)$, a function $\varphi : D_\varepsilon \rightarrow \mathbb{R}$ satisfying $\varphi(r) = u(r, t)$ in a neighborhood of $|r| = R + \varepsilon$ as well as $\varphi = 0$ in both, a neighborhood of $\partial \Omega$ and in the region $|r| > R + 2\varepsilon$—which can be easily constructed by multiplication of $u$ by a smooth cutoff function in the radial variable. Moreover, using the representation (2.11) for $r \in D_\varepsilon$ bounded away from $\partial \Omega$, we see that $\varphi \in H^1(D_\varepsilon)$ for every $t$ since $\psi \in C^\infty(\mathbb{R}; L^2(\Gamma))$ per the assumptions in Corollary 4.2. Applying the regularity result [20, Thm. 8.9] to the radial variable. Moreover, using the representation (2.11) for $\varphi$ can be easily constructed by multiplication of $u$ by a smooth cutoff function in the radial variable. Indeed, differentiating that formula once with respect to the normal $n$-sentation formula (2.11). Thus, differentiating that formula once with respect to the normal $n$-sentation formula (2.11) yields

\begin{equation}
\int_D |\nabla u(r, t)|^2 \, dV(r) = \langle u(r, t), \gamma \partial_n u(r, t) \rangle_{\partial D} - \int_D u(r, t) \Delta u(r, t) \, dV(r),
\end{equation}

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$ and where $\gamma$ denotes the trace operator on $D$. In fact, since $\gamma u(\cdot, t) = 0$ on $\Gamma$, and since $u(\cdot, t)$ is smooth for $r$ in a neighborhood of $\partial B_R$, we have $\langle \gamma u(\cdot, t), \gamma \partial_n u(\cdot, t) \rangle_{\partial D} = \int_{\partial B_R} u(r, t) \partial_n u(r, t) \, d\sigma(r)$.

Using (4.19) we now obtain a bound on $E(u, D, t)$. Since $u(\cdot, t)$ satisfies (4.16), using the Cauchy-Schwarz inequality we obtain

\begin{equation}
\int_D |\nabla u(r, t)|^2 \, dV(r) = \int_{\partial B_R} u(r, t) \partial_n u(r, t) \, d\sigma(r) - \int_D u(r, t) f(r, t) \, dV(r)
\leq \|u(\cdot, t)\|_{L^2(\partial B_R)} \|\partial_n u(\cdot, t)\|_{L^2(\partial B_R)} + \|u(\cdot, t)\|_{L^2(D)} \|f(\cdot, t)\|_{L^2(D)}.
\end{equation}

The first volumetric term on the right-hand side in (4.20) can be estimated using Corollary 4.2 with $p = 0$:

\begin{equation}
\|u(\cdot, t)\|_{L^2(D)} \leq C_2(t - T_0 - r_{\text{max}}/c)^{1/2-n} \|\psi\|_{H^{(n+1)(q+1)+1}(I_{T_0}; L^2(\Gamma))},
\end{equation}

Then, using (4.17) and the continuity of the inclusion map in Sobolev spaces, (4.21) provides a bound for the second summand on the right-hand side of (4.20):

\begin{equation}
\|u(\cdot, t)\|_{L^2(D)} \|f(\cdot, t)\|_{L^2(D)} \leq C_3(t - T_0 - r_{\text{max}}/c)^{1-2n} \|\psi\|_{H^{(n+1)(q+1)+3}(I_{T_0}; L^2(\Gamma))}^2.
\end{equation}

We now to turn to the first summand on the right-hand side of (4.20), and we estimate each one of the corresponding boundary terms by relying on the representation formula (2.11). Indeed, differentiating that formula once with respect to the normal $n$ and $s$-times with respect to time with $s = 0, 1$, and adding the results, for $(r, r') \in \partial B_R \times \mathbb{R}$, we obtain the relation

\begin{equation}
\partial_n \partial_s^r u(r, t') = -\int_{\Gamma} \left( \frac{\hat{\mathbf{r}} \cdot (r - r') \partial_t^s \hat{\mathbf{r}} \psi(r', t' - |r - r'|/c)}{|r - r'|^3}
+ \frac{\hat{\mathbf{r}} \cdot (r - r') \partial_t^{s+1} \psi(r', t' - |r - r'|/c)}{c|r - r'|^2} \right) \, d\sigma(r'),
\end{equation}

where $\hat{\mathbf{r}}$ denotes the trace operator on $\partial B_R \times \mathbb{R}$. In fact, since $\gamma u(\cdot, t) = 0$ on $\Gamma$, and since $u(\cdot, t)$ is smooth for $r$ in a neighborhood of $\partial B_R$, we have $\langle \gamma u(\cdot, t), \gamma \partial_n u(\cdot, t) \rangle_{\partial D} = \int_{\partial B_R} u(r, t) \partial_n u(r, t) \, d\sigma(r)$.
for which the Cauchy-Schwarz inequality implies, for \((r, r') \in \partial B_R \times \mathbb{R}\), that
\[
|\partial_n \partial_t^r u(r, t')|^2 \leq C_4(\Gamma, R, \varepsilon) \int_{\Gamma} |\partial_t^r \psi(r, t' - |r - r'|/c)|^2 \, d\sigma(r') + C_5(\Gamma, R, \varepsilon) \int_{\Gamma} |\partial_t^{r+1} \psi(r', t' - |r - r'|/c)|^2 \, d\sigma(r').
\]
(4.24)

The argument we use to bound \(\partial_n u\) uniformly in time is similar to the one used in the proof of Corollary 4.2: we obtain bounds on \(\partial_n u(r, \cdot)\) in the norm of \(H^1(\mathbb{R})\) by taking \(s = 0\) and \(s = 1\) in (4.24), adding the results, and integrating the resulting inequality over \(t' > t\); the uniform-in-time bound then follows from the Sobolev Lemma. Indeed, applying (4.24) with \(s = 0\) and \(s = 1\) and using the definition (4.9) of \(I_{0,1}(r, t), I_{1,1}(r, t)\) and \(I_{2,1}(r, t)\), we obtain
\[
\|\partial_n u(r, \cdot)\|_{H^1(\mathbb{R})}^2 \leq C_4(\Gamma, R, \varepsilon) I_{0,1}(r, t) + C_5(\Gamma, R, \varepsilon) I_{1,1}(r, t) + (C_4(\Gamma, R, \varepsilon) + C_5(\Gamma, R, \varepsilon)) I_{1,1}(r, t).
\]
(4.25)

Then, using the bound (4.10), \(I_{0,1}(r, t) \leq \|\partial_t^r \psi\|_{L^2(\mathbb{R})}^2\), we obtain the relation
\[
\|\partial_n u(r, \cdot)\|_{H^1(\mathbb{R})} \leq C_6(\Gamma, R, \varepsilon) \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})}.
\]
(4.26)

In conjunction with the Sobolev embedding Lemma 3.18 and the bound (4.1) of Theorem 4.1, the bound (4.26) implies that
\[
|\partial_n u(r, t)| \leq C_7(\Gamma, R, \varepsilon) \|\partial_n u(r, \cdot)\|_{H^1(\mathbb{R})} \leq C_7 C_6(\Gamma, R, \varepsilon) \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})},
\]
(4.27)

for \((r, t) \in \partial B_R \times (T_0 + r_{\text{max}}/c, \infty)\). Taking \(L^2(\partial B_R)\) norms in both (4.25) and in (4.27) yields the desired estimate for the first summand on the right-hand side of (4.20):
\[
\left\|u(\cdot, t)\right\|_{L^2(\partial B_R)} \left\|\partial_n u(\cdot, t)\right\|_{L^2(\partial B_R)} \leq C_9(t - T_0 - r_{\text{max}}/c)^{1 - 2n} \times \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})}.
\]
(4.28)

for \(t \geq T_0 + r_{\text{max}}/c\), where we once again used the continuity of the inclusion map in Sobolev spaces.

As suggested above, to obtain decay estimates for the second term in (4.22) we use the bound (4.4) with \(p = 1\), which tells us that
\[
|\partial_n u(\cdot, t)|_{L^2(D)} \leq C_{10}(t - T_0 - r_{\text{max}}/c)^{1/2 - n} \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})},
\]
(4.29)

for \(t \in (T_0 + r_{\text{max}}/c, \infty)\).

To complete the proof we now utilize (4.20) with right-hand side terms substituted by (4.22) and (4.28), together with the bound (4.29), and obtain the bound
\[
E(u, D, t) \leq \left\|u(\cdot, t)\right\|_{L^2(\partial B_R)} \left\|\partial_n u(\cdot, t)\right\|_{L^2(\partial B_R)} + \left\|u(\cdot, t)\right\|_{L^2(D)} \left\|f(\cdot, t)\right\|_{L^2(D)} + \left\|\partial_n u(\cdot, t)\right\|_{L^2(D)}^2 \leq C_9(t - T_0 - r_{\text{max}}/c)^{1 - 2n} \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})} \cdot \|\psi\|_{H^2(\mathbb{R})},
\]
(4.30)
The desired estimate (4.15) now follows, once again, by virtue of the continuity of the inclusion map in Sobolev spaces.

**Remark 4.4.** The well known contribution [40] shows that, for a trapping obstacle $\Omega$ and for an arbitrarily large time $T$, initial conditions can be selected so that the local energy $E(u,D,t)$ in (1.1) with $D = \Omega^c \cap \{r \leq R\}$ remains arbitrarily close to the initial energy value $E(u,D,0)$ for $0 \leq t \leq T$. In particular, the result [40] implies that a decay bound of the form (1.1) that is uniformly valid for all admissible initial conditions and for all compact sets $D \subset \Omega^c$ cannot hold for a trapping obstacle. References [23, 25] do present, however, uniformly valid decay estimates relative to higher order Sobolev norms over the complete exterior domain, which are valid for trapping structures consisting of certain unions of convex obstacles, for which the trapping rays span spatial regions of zero measure (indeed, a single ray in the case of the structure consisting of two convex connected obstacles in [23]), and more generally, as implied by Assumption (H.2) in [25]. In the same spirit, Theorem 4.1 and Corollaries 4.2 and 4.3 present decay results for trapping obstacles satisfying the $q$-growth condition (including a result of decay for the local energy $E(u,D,t)$) in terms of higher-order surface Sobolev norms, that are uniformly valid for all admissible incident fields. In particular, these results apply to obstacles such as those depicted in Figure 1 (for which the trapping rays span volumetric regions of positive measure), in addition to the examples [23, 25] which, per reference [10], satisfy the $q$-growth condition with $q = 1$.

The overall approach to the proof of Theorem 4.1 relies critically on the time-history domain-of-dependence ideas described in Remark 3.15 (see also Remark 4.5 where a related but somewhat modified decay result and proof strategy are suggested). Technically, the proof of Theorem 4.1 proceeds on the basis of an argument resulting from integration by parts with respect to the temporal frequency $\omega$, which requires $\omega$-differentiation of a certain function $\psi^f(r,\omega)$ closely related to the boundary integral density $\psi^f(r,\omega)$. Certain necessary results concerning differentiability of boundary integral operators and associated integral densities are established in a series of lemmas presented in the Appendix. Some of the main elements of the proof of Theorem 4.1, in turn, are presented in Lemmas 4.6 through 4.10 below. Thus, with reference to the Appendix, Lemma 4.6 provides pointwise bounds on density derivatives. Then, the technical Lemma 4.8 (of a similar character to Lemma 3.17) establishes bounds on the integrals of certain quantities in Lemma 4.6 and Lemma 4.10 provides the primary decay estimate used in the proof of Theorem 4.1.

**Remark 4.5.** Before proceeding with the proof of Theorem 4.1 we note that a related but somewhat less informative decay result and proof strategy, which do not depend on the bootstrap DoD concept, can be contemplated. In the alternative approach the decay proof results once again from an argument based on integration by parts (in frequency-domain) in the Fourier integral that represents the time-domain solution. The proof of such a result proceeds as follows: starting from the given time-domain data $b(r,t)$ defined in the neighborhood $\Omega^{inc}$ of $\Omega$, a Fourier transform is performed to obtain the right-hand side of the integral equation (2.16). Since $\Gamma$ satisfies the $q$-growth condition, it follows that the solution $\psi^f(r,\omega)$ of this equation admits an upper bound that grows polynomially as a function of $\omega$. Using an argument similar to the one presented in Lemma 4.6 below, corresponding
polynomially growing bounds are obtained for the $\omega$ derivatives of $\psi^j(r, \omega)$. Thus, the decay result can be obtained by an integration by parts argument followed by a bound on the integral of the resulting integrands. Such a bound can be obtained by relying on smooth partitioning of the integral together with a Young inequality-based estimate in an argument similar to the one in Lemma 4.10. The resulting time-decay bound resembles the bound (4.1). But, unlike equation (4.1), which expresses decay in terms of the norm of the solution $\psi$ itself over the bootstrap DoD $I_{T_0}$, the alternative bound expresses decay in terms of the norm, over all time, of the right-hand side function $b(r, t)$ and its normal derivative on $\Gamma$. Thus Theorem 4.1 provides a significantly more precise decay estimate, but it does so at the cost of certain added complexity in the proof—which is mainly confined to Lemma 4.8.

Following the aforementioned proof plan for Theorem 4.1 we first present, in Lemma 4.6, estimates on frequency-derivatives of certain frequency-domain density solutions.

**Lemma 4.6.** Let $q$ denote a non-negative integer, assume $\Omega$ satisfies the $q$-growth condition (Definition 2), and let $b$ satisfy the assumptions of Lemma 3.13 (so that, in particular, $b(r, t)$ is $C^2$ and temporally compactly supported in the time interval $[\alpha, T-T, -2\tau]$ for all $r \in \Omega$) as well as the $s$-regularity conditions (2.3) with $s = q$. Further, let $\psi$ and $\mu = \psi^T$ denote the solutions to (3.10) and (3.28), respectively, and, with reference to (3.28), define $\bar{R}_T(r, \omega)$ by

\[
\bar{R}_T(r, \omega) = \gamma^{-\omega} \partial_\omega \bar{H}_T(r, \omega) - \partial_\omega \omega_0 \gamma^{-\omega} \bar{H}_T(r, \omega), \quad r \in \Gamma,
\]

for $\omega \geq 0$, and by Hermitian symmetry for $\omega < 0$: $\bar{R}_T(r, \omega) = \bar{R}_T(r, -\omega)$. Then $\mu = \psi^T \in C^\infty(\mathbb{R}; L^2(\Gamma))$ and $\bar{R}_T \in C^\infty(\mathbb{R} \setminus \pm \omega_0; L^2(\Gamma))$, and for all nonnegative integers $p$ and all $\omega \neq \pm \omega_0$ (cf. Definition 7) we have

\[
\| \partial_\omega^p \mu(\omega) \|^2_{L^2(\Gamma)} \leq \sum_{i=0}^{p} \left( \sum_{j=0}^{(i+1)(q+1)-1} \partial_\omega^p \omega_0^{2j} \left\| \partial_\omega^{p-i} \bar{R}_T(\omega, \omega) \right\|^2_{L^2(\Gamma)} \right),
\]

where $\partial_\omega^p > 0$ denote certain non-negative constants independent of $T$. Additionally, for each nonnegative integer $p$ there exists a constant $C$ (dependent on $p$, $\alpha$, $T$, and on certain norms of $b$), such that

\[
\| \partial_\omega^p \mu(\omega) \|^2_{L^2(\Gamma)} \leq C|\omega|^{(p+1)(q+1)}
\]

for all sufficiently large values of $|\omega|$.

**Proof.** Let $p$ denote a nonnegative integer and let $\omega \neq \pm \omega_0$. Lemma C.3 tells us that $\bar{R}_T \in C^\infty(\mathbb{R} \setminus \pm \omega_0; L^2(\Gamma))$ and $\psi^T \in C^\infty(\mathbb{R} \setminus \pm \omega_0; L^2(\Gamma))$, as claimed. We restrict the remainder of the proof to the case $\omega > 0$; the full result then follows by the property of Hermitian symmetry satisfied by $\mu = \psi^T$ (Remark 3.4).

In order to establish (4.32) we first show that

\[
\| \partial_\omega^p \mu(\omega) \|^2_{L^2(\Gamma)} \leq \sum_{i=0}^{p} \left( \sum_{j=0}^{(i+1)(q+1)-1} \partial_\omega^p \omega_0^{2j} \left\| \partial_\omega^{p-i} \bar{R}_T(\omega, \omega) \right\|^2_{L^2(\Gamma)} \right),
\]
for certain constants $b_{ij}^s \geq 0$. The proof of (4.34) proceeds inductively: assuming that there exist constants $b_{ij}^s \geq 0$ ($0 \leq s \leq p$) such that the relation

$$(4.35) \quad \|\partial^s \omega \mu(\cdot, \omega)\|_{L^2(\Gamma)} \leq \sum_{i=0}^s \left( \sum_{j=0}^{(i+1)(q+1)-1} b_{ij}^s \right) \|\partial^{s-i} \tilde{R}_T(\cdot, \omega)\|_{L^2(\Gamma)}$$

holds for all nonnegative integers $s \leq p$, we show that there exist constants $b_{ij}^{s+1} \geq 0$ such that (4.35) holds for $s = p + 1$. (The base case $s = 0$ follows from (3.28) on account of the $q$-growth condition.) To carry out the inductive step we use (C.21) in Lemma C.3 which tells us that

$$(4.36) \quad (\partial^{p+1} \omega \mu)(\mathbf{r}, \omega) = A_{\omega}^{-1} \left( \partial^{p+1} \tilde{R}_T(\mathbf{r}, \omega) - \sum_{k=1}^{p+1} a_{k}^{p+1} (\partial^k \alpha A_{\omega}) (\partial^{p+1-k} \mu)(\mathbf{r}, \omega) \right)$$

for certain integers $a_{k}^{p+1}; k = 1, \ldots, p + 1$. But the $q$-growth condition tells us that there exist positive $C_1, C_2$ such that $\|A_{\omega}^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C_1 + C_2\omega^q$. Further, the operator-norm bound (C.17) in Lemma C.2 tells us that, for certain constants $\alpha_{0k}$ and $\alpha_{1k}$, we have $\|\partial^k \alpha A_{\omega}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \alpha_{0k} + \alpha_{1k}\omega$ for all $\omega \in \mathbb{R}^+ \setminus \omega_0$. It thus follows from (4.36) that

$$(4.37) \quad \left| \partial^{p+1} \omega \mu(\cdot, \omega) \right|_{L^2(\Gamma)} \leq (C_1 + C_2\omega^q) \left( \left| \partial^{p+1} \tilde{R}_T(\cdot, \omega) \right|_{L^2(\Gamma)} + \sum_{k=1}^{p+1} |a_{k}^{p+1}| (\alpha_{0k} + \alpha_{1k}\omega) \left| \partial^{p+1-k} \mu(\cdot, \omega) \right|_{L^2(\Gamma)} \right).$$

Substituting (4.36) with $s = p + 1 - k$ for $k = 1, \ldots, p + 1$ into (4.37) we obtain

$$\left| \partial^{p+1} \omega \mu(\cdot, \omega) \right|_{L^2(\Gamma)} \leq (C_1 + C_2\omega^q) \left| \partial^{p+1} \tilde{R}_T(\cdot, \omega) \right|_{L^2(\Gamma)}$$

$$+ \sum_{k=1}^{p+1} |a_{k}^{p+1}| (C_1\alpha_{0k} + C_2\alpha_{0k}\omega^q + C_1\alpha_{1k}\omega + C_2\alpha_{1k}\omega^{q+1}) \times$$

$$\times \left[ \sum_{i=0}^{p+1-k} \sum_{j=0}^{(i+1)(q+1)-1} b_{ij}^{p+1-k-i} \left| \partial^{p+1-k-i} \tilde{R}_T(\cdot, \omega) \right|_{L^2(\Gamma)} \right],$$

from which, expanding the products, we obtain

$$(4.38) \quad \left| \partial^{p+1} \omega \mu(\cdot, \omega) \right|_{L^2(\Gamma)} \leq (C_1 + C_2\omega^q) \left| \partial^{p+1} \tilde{R}_T(\cdot, \omega) \right|_{L^2(\Gamma)} + (A),$$

where

$$(A) = \sum_{k=1}^{p+1} \sum_{i=0}^{p+1-k} \sum_{j=0}^{(i+1)(q+1)-1} |a_{k}^{p+1}| b_{ij}^{p+1-k} \left( C_1\alpha_{0k}\omega^j + C_2\alpha_{0k}\omega^{q+j} \right)$$

$$+ C_1\alpha_{1k}\omega^{j+1} + C_2\alpha_{1k}\omega^{q+j+1} \left| \partial^{p+1-k-j} \tilde{R}_T(\cdot, \omega) \right|_{L^2(\Gamma)}.$$

It is easy to check that (4.38) implies that there exist constants $b_{ij}^{p+1} \geq 0$ such that the relation (4.35) with $s = p + 1$ holds. Indeed, the first term on the right-hand side of (4.38) and every term arising from the summations in (A) can be expressed as
a constant multiplied by a term of the form $\omega^{\ell} \left\| \partial_{\omega}^{m} \tilde{R}_{T}(\cdot, \omega) \right\|_{L^{2}(\Gamma)}$ for some $m \leq p$ and for some $\ell \leq (p + 1)(q + 1) + q$—all of which match corresponding terms in (4.35) with $s = p + 1$. This concludes the inductive proof, showing that for each nonnegative integer $p$ there exist constants $b_{ij}^{p} \geq 0$ such that the inequality (4.34) holds. The desired inequality (4.32) follows directly from (4.34) using the relation $\sum_{i=1}^{p} a_{i}^{2} \leq p \sum_{i=1}^{p} \|a_{i}\|^{2}$.

In view of (4.34), in order to establish the bound (4.33) we estimate the expression $\|\partial_{\omega}^{\ell} \tilde{R}_{T}(\cdot, \omega)\|_{L^{2}(\Gamma)}$ in (4.34) (with $\ell = p - i$, $0 \leq i \leq p$). To do this we note that, in view of (4.31) and (2.23), for $\omega \geq \omega_{0}$ we have $\tilde{R}_{T} = (\gamma - \partial_{n} - i\omega\gamma^{-}) \tilde{H}^{T}_{T}$. Thus, using (3.26) and (2.21), we obtain the relation

$$
\partial_{\omega}^{\ell} \tilde{R}_{T} = \partial_{\omega}^{\ell} \gamma^{-} \partial_{n} \tilde{B}^{T} - \partial_{\omega}^{\ell} \left(i\omega\gamma^{-} \tilde{B}^{T}\right) - \partial_{\omega}^{\ell} \left(\frac{1}{2} I + K_{\omega}^{*} - i\omega S_{\omega}\right) \tilde{\psi}_{-T}, \quad (\omega > \omega_{0}),
$$

whose right-hand terms we estimate in what follows. In view of (3.12), the function $\gamma^{-} \tilde{B}^{T}$ equals the temporal Fourier transform of the compactly-supported function $\gamma^{-} \tilde{b} = \gamma^{+} \tilde{b}$. But, in view of the s-regularity hypotheses and other assumptions in force, the function $\gamma^{-} \tilde{b} = \gamma^{-} \tilde{b}(t, r)$ is an element of $L^{2}(\mathbb{R}; L^{2}(\Gamma))$ that is compactly-supported as a function of $t$. We may thus differentiate under the Fourier-transform integral sign in (3.12), which shows that $\partial_{\omega}^{\ell} \gamma^{-} \tilde{B}^{T}$ equals the temporal Fourier transform of the compactly-supported function $(-it)^{\ell} \gamma^{-} \tilde{b} \in L^{2}(\mathbb{R}; L^{2}(\Gamma))$. Using the Cauchy-Schwarz inequality it follows that for a certain constant $\tilde{C}_{0} > 0$ (that depends on $p$, $\alpha$, $T$, and $\|\gamma^{-} \tilde{b}\|_{L^{2}(\mathbb{R}; L^{2}(\Gamma))}$), but which does not depend on $\omega$) we have $\|\partial_{\omega}^{\ell} \gamma^{-} \tilde{B}^{T}\|_{L^{2}(\Gamma)} \leq \tilde{C}_{0}$ for $0 \leq \ell \leq p$ and for all $\omega \in \mathbb{R}$. In view of the triangle inequality we conclude that, for each $0 \leq \ell \leq p$ and for all $\omega \in \mathbb{R}$ we have

$$
\left\| \partial_{\omega}^{\ell} \left(\omega \gamma^{-} \tilde{B}^{T}(\cdot, \omega)\right) \right\|_{L^{2}(\Gamma)} \leq \tilde{C}_{1}(1 + \omega)
$$

where $\tilde{C}_{1}$ denotes a constant which, once again, depends on $p$, $\alpha$, $T$, and on $\|\gamma^{-} \tilde{b}\|_{L^{2}(\mathbb{R}; L^{2}(\Gamma))}$ but does not depend on $\omega$, which provides the needed estimate of the second right-hand term. A similar argument applied to $\partial_{\omega}^{\ell} \gamma^{-} \partial_{n} \tilde{B}^{T}$ leads to the desired estimate

$$
\left\| \partial_{\omega}^{\ell} \gamma^{-} \partial_{n} \tilde{B}^{T}(\cdot, \omega) \right\|_{L^{2}(\Gamma)} \leq \tilde{C}_{2},
$$

for the first right-hand term in (4.39), where $\tilde{C}_{2}$ denotes a constant with dependencies analogous to those found for $\tilde{C}_{1}$.

In order to obtain a bound for the last term on the right-hand side of (4.39), in turn, we note that $\tilde{\psi}_{-T}$ is compactly-supported (cf. (3.4)) and satisfies $\tilde{\psi}_{-T} \in L^{2}(\mathbb{R}; L^{2}(\Gamma))$ (cf. Lemma 3.6), and, thus, an argument similar to the one leading to (4.40) yields the bound

$$
\left\| \partial_{\omega}^{\ell} \tilde{\psi}_{-T}(\cdot, \omega) \right\|_{L^{2}(\Gamma)} \leq \tilde{C}_{3}
$$

for all $\ell$, $0 \leq \ell \leq p$, and all $\omega > 0$, where $\tilde{C}_{3}$ depends on the integer $p$, $\alpha$, $T$, and, via Lemma 3.6 with $p = 0$, on $\|\gamma^{+} \tilde{b}\|_{H^{p+1}(\mathbb{R}; L^{2}(\Gamma))}$ and $\|\gamma^{+} \partial_{n} \tilde{b}\|_{H^{p}(\mathbb{R}; L^{2}(\Gamma))}$. Then, employing the operator norm bounds presented in Lemma C.1 for $\partial_{\omega}^{m} S_{\omega}$ and $\partial_{\omega}^{m} K_{\omega}^{*}$
(\ell = 0, \ldots, p) together with (4.42) we obtain the bound

\[ \| \partial_{\nu}^\ell \left( \frac{1}{2} I + K_{\omega} - i\omega S_{\omega} \right) \tilde{\psi}_{-T}(\cdot, \omega) \|_{L^2(\Gamma)} \leq \tilde{C}_4 (1 + \omega) \]

for all integers \ell, 0 \leq \ell \leq p and for all \omega > 0, where \tilde{C}_4 denotes a constant dependent on \tilde{C}_3 and the constants in Lemma C.1 but which is independent of \omega.

Estimating the norm of (4.39) by means of the triangle inequality, and bounding the right-hand side of the resulting inequality by means of (4.40), (4.41) and (4.43), we obtain

\[ \| \partial_{\nu}^\ell \tilde{R}_T(\cdot, \omega) \|_{L^2(\Gamma)} \leq \tilde{C}_5 \omega \quad (\omega > \omega_0), \]

for all integers \ell, 0 \leq \ell \leq p, where \tilde{C}_5 denotes a constant independent of \omega but dependent on \omega_0, \ p, \ \alpha \ and \ T_0, and on the norm values \|\gamma^+\hat{b}\|_{H^{r+1}(\mathbb{R} \times L^2(\Gamma))} \) and \( \|\gamma^+ \partial_{n} \hat{b}\|_{H^{r}(\mathbb{R} \times L^2(\Gamma))}. \) The estimate (4.33) follows directly from (4.34) and (4.44). □

**Remark 4.7.** In order to obtain the estimates in Theorem 4.1 it will be necessary to perform time-recentering on the data \( \tilde{b} \) and the solution \( \tilde{\psi} \)—see Remarks 3.1 and 3.5. Given a real number \( T_0 \) we define for a given \( \tilde{b} \) the time-shifted “breve” quantities

\[ b(r, t) = \tilde{b}(r, t + T_0), \quad \tilde{\psi}(r, t) = \tilde{\psi}(r, t + T_0). \]

With reference to Remark 3.5, note that \( \tilde{\psi}(r, t) \) equals the solution \( \tilde{\psi}(r, t) \) of (3.10) with \( \tilde{b} \) substituted by \( \tilde{b}. \) Note also that if \( \tilde{b} \) satisfies (3.11) for some \( \alpha \) then \( \tilde{b} \) satisfies (3.11) with \( \alpha \) substituted by \( \alpha - T_0. \) We will consider the \( T_0 \)-dependent density \( \tilde{\psi} \) in conjunction with the interval \( I_0 \) equal to \( I_T \) with \( T = 0 \) (see Remark 3.1), and we thus define \( \tilde{\psi}_{\pm,0} \) and \( \tilde{\psi}_{*,0} \) by analogy with \( \psi_{\pm,T} \) and \( \psi_{*,T} \) in Definition 4, but with \( \tilde{\psi} \) in lieu of \( \psi \) and with \( T = 0. \)

Consistent with the conventions laid out in Definition 4 and Remark 3.5 and letting

\[ \tilde{u}_{\pm,0}(r, t) = (\mathcal{S} \tilde{\psi}_{\pm,0})(r, t) \quad \text{and} \quad \tilde{u}_{*,0}(r, t) = (\mathcal{S} \tilde{\psi}_{*,0})(r, t), \quad (r, t) \in \mathbb{R}^3 \times \mathbb{R}, \]

we define also the function \( \tilde{h}_0 \) (cf. (3.22)),

\[ \tilde{h}_0(r, t) = \tilde{b}(r, t) - \tilde{u}_{-0}(r, t), \quad (r, t) \in \Omega^{inc} \times \mathbb{R}, \]

and, using (3.19), its Fourier transform

\[ \tilde{H}_0^f(r, \omega) = \tilde{B}_f^f(r, \omega) - \mathcal{F}_{\omega} \tilde{\psi}_{-0}^f(r, \omega), \quad r \in \Omega^{inc} \times \mathbb{R}. \]

Similarly, the right-hand side of (3.28) becomes

\[ \tilde{R}_0(r, \omega) = \gamma^+ \partial_{n} \tilde{H}_0^f(r, \omega) - i \hat{\tau}_0(\omega) \gamma^+ \tilde{H}_0^f(r, \omega), \quad (r, \omega) \in \Gamma \times \mathbb{R}. \]

**Lemma 4.8.** Let \( T_0 \) and \( \tau \) denote given real numbers, let \( q \) denote a non-negative integer, let \( \tilde{b} \) be defined as in Remark 4.7, and assume that \( b(r, t) \) vanishes for all \( (r, t) \in \Omega \times \{I_0 \cup [0, \infty)\} \) (where \( I_0 = I_T \) with \( T = 0 \), see Definition 3). Additionally, let \( \Omega \) satisfy the \( q \)-growth condition and assume \( \tilde{b} \) satisfies the \( s \)-regularity conditions (2.5) with \( s = q \). Finally, assume that for a given non-negative integer \( n, \tilde{\psi}_{*,0} \)
satisfies \( \bar{\psi}_{s,0} \in H^{n+1}(I_0; L^2(\Gamma)) \). Then for all integers \( m \geq 0 \) and all integers \( j \) such that \( 0 \leq j \leq n \) we have

\[
\int_0^\infty \omega^{2j} \left\| \partial^m_\omega \bar{R}_0(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \ d\omega \leq C \left\| \bar{\psi}_{s,0} \right\|_{H^{n+1}(I_0; L^2(\Gamma))}^2,
\]

where \( C \) is a constant independent of \( T_0 \) and \( b \).

**Remark 4.9.** In order to obtain an upper bound on integrals containing powers of the temporal variable \( t \), the estimate (4.50) in the proof of Lemma 4.8 relies in an essential manner on the domain-of-dependence relations (3.38) and (3.40) for the function \( \tilde{h}_0 \) (cf. Remark 3.15)—which, limiting the integration of the aforementioned powers of \( t \) to a bounded interval, yields meaningful integral estimates necessary for the proof of the lemma. \( \square \)

**Proof of Lemma 4.8.** Since \( \bar{b} \) satisfies the \( s \)-regularity conditions with \( s = q \), using Lemma 3.6 with reference to Remark 4.7 shows that the quantity \( \bar{\psi}_{s,0} \) in (4.48) (which also enters in the definition (4.49)) satisfies \( \bar{\psi}_{s,0} \in L^2(\mathbb{R}; L^2(\Gamma)) \). In view of (4.49) we have

\[
(4.50) \quad \int_0^\infty \omega^{2j} \left\| \partial^m_\omega \bar{R}_0(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \ d\omega = \int_0^\infty \int_\Gamma \left| \bar{S}_{jm} \bar{H}_0^j (r, \omega) \right|^2 \ d\sigma(r) \ d\omega,
\]

where the operator \( \bar{S}_{jm} \) is defined as

\[
(4.51) \quad \bar{S}_{jm} = \omega^j \partial^m_\omega \left( \gamma^{-} \partial_n - \imath \theta_0 \gamma^{-} \right).
\]

In view of the definition (2.23) of the function \( \eta_0(\omega) \) in (4.51), which depends on whether \( 0 \leq \omega < \omega_0 \) or \( \omega > \omega_0 \), (4.50) is re-expressed in the form

\[
(4.52) \quad \int_0^\infty \omega^{2j} \left\| \partial^m_\omega \bar{R}_0(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \ d\omega = \int_\Gamma \left( \int_0^{\omega_0} + \int_{\omega_0}^\infty \right) \left| \bar{S}_{jm} \bar{H}_0^j (r, \omega) \right|^2 \ d\omega \ d\sigma(r).
\]

Then, defining the operators

\[
(4.53) \quad \bar{S}_{jm}^1 = \omega^j \partial^m_\omega \left( \gamma^{-} \partial_n - \imath \gamma^{-} \right), \quad and \quad \bar{S}_{jm}^2 = \omega^j \partial^m_\omega \left( \gamma^{-} \partial_n - \omega \gamma^{-} \right),
\]

which clearly satisfy \( \bar{S}_{jm} = \bar{S}_{jm}^1 \) for \( \omega < \omega_0 \) and \( \bar{S}_{jm} = \bar{S}_{jm}^2 \) for \( \omega > \omega_0 \), (4.52) yields

\[
\int_0^\infty \omega^{2j} \left\| \partial^m_\omega \bar{R}_0(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \ d\omega
\]

\[
= \int_\Gamma \int_0^{\omega_0} \left| \bar{S}_{jm}^1 \bar{H}_0^j (r, \omega) \right|^2 \ d\omega \ d\sigma(r) + \int_\Gamma \int_{\omega_0}^\infty \left| \bar{S}_{jm}^2 \bar{H}_0^j (r, \omega) \right|^2 \ d\omega \ d\sigma(r)
\]

\[
\leq \int_\Gamma \int_{-\infty}^{\omega_0} \left| \bar{S}_{jm}^1 \bar{H}_0^j (r, \omega) \right|^2 \ d\omega \ d\sigma(r) + \int_\Gamma \int_{\omega_0}^\infty \left| \bar{S}_{jm}^2 \bar{H}_0^j (r, \omega) \right|^2 \ d\omega \ d\sigma(r).
\]

Thus, utilizing the time-domain operators

\[
(4.54) \quad S_{jm}^1 = (\imath \partial / \partial t)^j (\gamma^{-} \partial_n - \imath \gamma^{-}) \quad and \quad S_{jm}^2 = -(\imath \partial / \partial t)^j (\gamma^{-} \partial_n - \partial / \partial t \gamma^{-})
\]
corresponding to (4.53), together with Plancherel’s theorem and equations (4.47)–(4.49), we obtain

\[ \int_0^\infty \omega^{2j} \left\| \partial^m_w \tilde{R}_0(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \, d\omega \]

(4.55) \leq \int_\Gamma \int_{-\infty}^{T_*} \left| S^1_{jm}(\tilde{h}_0(r, t')) \right|^2 \, dt' \, d\sigma(r) + \int_\Gamma \int_{-\infty}^{T_*} \left| S^2_{jm}(\tilde{h}_0(r, t')) \right|^2 \, dt' \, d\sigma(r)

= \int_\Gamma \int_{-\tau}^{T_*} \left| S^1_{jm}(\tilde{u}_{*,0}(r, t')) \right|^2 \, dt' \, d\sigma(r) + \int_\Gamma \int_{-\tau}^{T_*} \left| S^2_{jm}(\tilde{u}_{*,0}(r, t')) \right|^2 \, dt' \, d\sigma(r),

where, since \( \Gamma = \partial \Omega \subseteq \bar{\Omega} \), the last equality follows by using (3.38) and (3.40) with \( T = 0 \) (see Remark 4.7). Letting

\[ v_1 = (\gamma^- \partial_n - i \gamma^-) \tilde{u}_{*,0}, \quad \text{and} \quad v_2 = (\gamma^- \partial_n - \frac{\partial}{\partial t} \gamma^-) \tilde{u}_{*,0}, \]

and calling \( a_\ell = (\frac{1}{t}) \), by Leibniz’s product rule we then obtain

\[ \int_0^\infty \omega^{2j} \left\| \partial^m_w \tilde{R}_0(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \, d\omega \]

\[ \leq \int_\Gamma \int_{-\tau}^{T_*} \left| (-i \frac{\partial}{\partial t'})^j (-it')^m v_1(r, t') \right|^2 \, dt' \, d\sigma(r) \]

\[ + \int_\Gamma \int_{-\tau}^{T_*} \left| (-i \frac{\partial}{\partial t'})^j (-it')^m v_2(r, t') \right|^2 \, dt' \, d\sigma(r) \]

\[ = \int_\Gamma \int_{-\tau}^{T_*} \left| \sum_{\ell=0}^{j} a_\ell \left( \frac{\partial}{\partial t'} \right)^\ell (t')^m \left( \left( \frac{\partial}{\partial t'} \right)^{j-\ell} v_1(r, t') \right) \right|^2 \, dt' \, d\sigma(r) \]

\[ + \int_\Gamma \int_{-\tau}^{T_*} \left| \sum_{\ell=0}^{j} a_\ell \left( \frac{\partial}{\partial t'} \right)^\ell (t')^m \left( \left( \frac{\partial}{\partial t'} \right)^{j-\ell} v_2(r, t') \right) \right|^2 \, dt' \, d\sigma(r). \]

Substituting the exact value of the derivative \( (\frac{\partial}{\partial t'})^\ell (it')^m \) in these expressions, we further obtain

\[ \int_0^\infty \omega^{2j} \left\| \partial^m_w \tilde{R}_0(\cdot, \omega) \right\|_{L^2(\Gamma)}^2 \, d\omega \]

\[ \leq \int_\Gamma \int_{-\tau}^{T_*} \left| \sum_{\ell=0}^{j} \hat{a}_\ell (t')^m-\ell \left( \left( \frac{\partial}{\partial t'} \right)^{j-\ell} v_1(r, t') \right) \right|^2 \, dt' \, d\sigma(r) \]

\[ + \int_\Gamma \int_{-\tau}^{T_*} \left| \sum_{\ell=0}^{j} \hat{a}_\ell (t')^m-\ell \left( \left( \frac{\partial}{\partial t'} \right)^{j-\ell} v_2(r, t') \right) \right|^2 \, dt' \, d\sigma(r), \]

where \( \hat{a}_\ell = \frac{m!}{(m-\ell)!} a_\ell \) for \( m-\ell \geq 0 \) and \( \hat{a}_\ell = 0 \) for \( m-\ell < 0 \). Since the \( t' \)-integration is limited to the bounded region \([-\tau, T_*]\) the quantities \( |t'|^{m-\ell} \) are bounded by a
constant (which, importantly, is independent of $T_0$—see Remark 3.1), and thus
\[
\int_0^\infty \omega^{2j} \left\| \partial_m^n \hat{R}_0(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega
\]
\[
\leq C_1 \sum_{\ell=0}^j \left( \int_\Gamma \int_{-\infty}^{T_\ast} \left( \left( \frac{\partial}{\partial t'} \right)^{\ell} v_1(r, t') \right)^2 \, dt' \, d\sigma(r) + \left( \left( \frac{\partial}{\partial t'} \right)^{\ell} v_2(r, t') \right)^2 \, dt' \, d\sigma(r) \right)
\]
\[
\leq C_1 \sum_{\ell=0}^j \left( \int_\Gamma \int_{-\infty}^\infty \left( \left( \frac{\partial}{\partial t'} \right)^{\ell} v_1(r, t') \right)^2 \, dt' \, d\sigma(r) + \left( \left( \frac{\partial}{\partial t'} \right)^{\ell} v_2(r, t') \right)^2 \, dt' \, d\sigma(r) \right),
\]
where the last inequality simply bounds the space-time norm on the finite temporal region $[-\tau, T_\ast]$ by the full time integral on $\mathbb{R}$. In view of (4.56), using once again the Plancherel theorem, and denoting, per Remark 2.5, $\hat{U}_{\ast,0}^f$ the Fourier transform of $\hat{u}_{\ast,0}$, we estimate
\[
\int_0^\infty \omega^{2j} \left\| \partial_m^n \hat{R}_0(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega
\]
\[
\leq C_1 \sum_{\ell=0}^j \left( \int_\Gamma \int_{-\infty}^{\infty} \left| \omega^{\ell} (\gamma^- \partial_n - i \gamma^-) \hat{U}_{\ast,0}^f(r, \omega) \right|^2 \, d\omega \, d\sigma(r)
\]
\[
+ \int_\Gamma \int_{-\infty}^{\infty} \left| \omega^{\ell} (\gamma^- \partial_n - i \omega \gamma^-) \hat{U}_{\ast,0}^f(r, \omega) \right|^2 \, d\omega \, d\sigma(r) \right)
\]
\[
\leq \tilde{C}_1 \int_\Gamma \int_{-\infty}^{\infty} \left( 1 + \omega^2 \right)^{j/2} \left( \gamma^- \partial_n - i \gamma^- \right) \hat{U}_{\ast,0}^f(r, \omega) \right|^2 \, d\omega \, d\sigma(r)
\]
\[
+ \tilde{C}_1 \int_\Gamma \int_{-\infty}^{\infty} \left( 1 + \omega^2 \right)^{j/2} \left( \gamma^- \partial_n - i \omega \gamma^- \right) \hat{U}_{\ast,0}^f(r, \omega) \right|^2 \, d\omega \, d\sigma(r).
\]
We thus have established that
\[
\int_0^\infty \omega^{2j} \left\| \partial_m^n \hat{R}_0(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega
\]
(4.57)
\[
\leq \tilde{C}_1 \int_{-\infty}^{\infty} \left( 1 + \omega^2 \right)^j \left\| (\gamma^- \partial_n - i \gamma^-) \hat{U}_{\ast,0}^f(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega
\]
\[
+ \tilde{C}_1 \int_{-\infty}^{\infty} \left( 1 + \omega^2 \right)^j \left\| (\gamma^- \partial_n - i \omega \gamma^-) \hat{U}_{\ast,0}^f(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega.
\]
To complete the proof we now use Lemma 3.16 which provides the frequency-wise bounds
\[
\left\| (\gamma^- \partial_n - i \gamma^-) \hat{U}_{\ast,0}^f(\cdot, \omega) \right\|_{L^2(\Gamma)} \leq D(1 + \omega^2)^{1/2} \left\| \hat{U}_{\ast,0}^f(\cdot, \omega) \right\|_{L^2(\Gamma)},
\]
and
\[
\left\| (\gamma^- \partial_n - i \omega \gamma^-) \hat{U}_{\ast,0}^f(\cdot, \omega) \right\|_{L^2(\Gamma)} \leq E(1 + \omega^2)^{1/2} \left\| \hat{U}_{\ast,0}^f(\cdot, \omega) \right\|_{L^2(\Gamma)},
\]
where \( D, E > 0 \) are constants independent of \( \omega, \tilde{b} \) and \( \tilde{\psi} \). Substituting \((4.58)\) and \((4.59)\) in \((4.57)\) we conclude that

\[
\int_0^\infty \omega^{2j} \left\| \partial^m \tilde{R}_0(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega \\
\leq \int_{-\infty}^\infty C_2(1+\omega^2)^j \left\| \tilde{\psi}_{*,0}^{\varphi}(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega \\
\quad + \int_{-\infty}^\infty C_3(1+\omega^2)^j \left\| \tilde{\psi}_{*,0}^{\varphi}(\cdot, \omega) \right\|^2_{L^2(\Gamma)} \, d\omega.
\]

\[
\leq C \left\| \tilde{\psi}_{*,0}^{\varphi} \right\|^2_{H^{j+1}(\mathbb{R}; L^2(\Gamma))} = C \left\| \tilde{\psi}_{*,0}^{\varphi} \right\|^2_{H^{j+1}(l_0; L^2(\Gamma))},
\]

as desired. \( \square \)

Lemma 4.8 is used in what follows to establish the main building block in the proof of Theorem 4.1, namely, Lemma 4.10. The proof of Lemma 4.10, in turn, incorporates a smooth windowing procedure which relies on use of compactly-supported time-domain window functions. For definiteness, we utilize the time-window functions introduced in what follows.

**Definition 5.** Letting \( v(u) = \exp\left(\frac{2e^{-1/n}}{u-1}\right) \), we define

\[
w(s) = \begin{cases} 
1 - v\left(\frac{s+2s_0}{s_0}\right), & -2s_0 \leq s \leq -s_0 \\
1, & -s_0 < s < s_0 \\
v\left(\frac{s-s_0}{s_0}\right), & s_0 \leq s \leq 2s_0 \\
0, & |s| > 2s_0,
\end{cases}
\]

\quad \text{and} \quad w_{\varphi}(s) = w(s - \varphi),

where \( \varphi \in \mathbb{R} \) denotes an important "time-shift" parameter that enacts the time-recentering approach alluded to in Remark 3.1 and where \( s_0 > 0 \) is a fixed parameter that, in the context of this paper, can be selected arbitrarily. \( \square \)

Clearly, the functions \( w \) and \( w_{\varphi} \): (i) Satisfy \( w, w_{\varphi} \in C^\infty(\mathbb{R}) \); (ii) Equal 1 in an interval of length \( 2s_0 \); (iii) Increase (decrease) from 0 to 1 \((1\,\text{to}\,0)\) in intervals of length \( s_0 \); (iv) Satisfy \( 0 \leq w(s) \leq 1 \) and \( 0 \leq w_{\varphi}(s) \leq 1 \) for all \( s \in \mathbb{R} \). It is easy to check that for every \( \varphi \in \mathbb{R} \) we have

\[
w_{\varphi+s_0}(s) + w_{\varphi+4s_0}(s) = 1 \\
w_{\varphi+s_0+3\ell s_0}(s) = 0 \quad (\ell \notin \{0,1\}) \quad \text{for} \quad s \in [\varphi, \varphi+3s_0].
\]

In particular, the functions \( w_{3\ell s_0}(s) \) with \( \ell \in \mathbb{Z} \) form a partition of unity, wherein at most two functions in the family do not vanish at any given \( s \in \mathbb{R} \).

**Lemma 4.10.** Let \( n \) and \( q \) denote non-negative integers, \( n > 0 \), let \( T_0 > 0 \), let \( b \) be defined as in Remark 3.5, and assume that \( b \) vanishes for all \( (r, t) \in \overline{\Omega} \times \{ I_{T_0} \cup [T_0, \infty) \} \), with \( I_{T_0} \) as in Definition 3. Additionally, let \( \mathcal{Q} \) satisfy the \( q \)-growth condition and assume \( b \) satisfies the \( s \)-regularity conditions \((2.5)\) with \( s = (n+1)(q+1) + q \). Then, the functions \( \tilde{\psi}_{*,0}^{\varphi} \) and \( \tilde{\psi}^{\varphi} \) defined in Remark 4.7 satisfy

\[
\left\| w_{\varphi} \tilde{\psi}_{*,0}^{\varphi} \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))} \leq C(\Gamma, \tau, n, s_0) \varphi^{-2n} \left\| \tilde{\psi}^{\varphi} \right\|^2_{H^{(n+1)(q+1)}(l_0; L^2(\Gamma))},
\]

for all \( \varphi > 0 \), where \( C(\Gamma, \tau, n, s_0) \) denotes a constant independent of \( \varphi, T_0, \) and \( \tilde{b} \).
Proof. We first note that since by hypothesis \( \tilde{b}(r, t) \) vanishes for \( (r, t) \in \Omega \times \{I_0 \cup [T_0, \infty)\} \), it follows that \( \tilde{b}(r, t) \) vanishes for all \( (r, t) \in \Omega \times \{I_0 \cup [0, \infty)\} \). Similarly, the \( s \)-regularity condition hypotheses are satisfied for \( \tilde{b} \).

In order to establish the desired decay estimate \((4.62)\) for \( w_s \tilde{\psi}_{+0} \), by Plancherel’s theorem we may instead provide an estimate for the \( L^2 \) norm of its Fourier transform:

\[
(4.63) \quad \left\| w_s \tilde{\psi}_{+0} \right\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2 = \left\| \tilde{w}_s \tilde{\psi}_{+0} - \tilde{w}_s * \tilde{\psi}_{+0} \right\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2.
\]

Using the relation \( \tilde{w}_s(\omega) = e^{-i\omega \varphi} \tilde{w}(\omega) \) we obtain

\[
(4.64) \quad \left( \tilde{w}_s * \tilde{\psi}_{+0} \right)(r, \omega) = \int_{-\infty}^{\infty} e^{-i\omega \varphi} \tilde{w}(\tau) \tilde{\psi}_{+0}(r, \omega - \tau) d\tau,
\]

and we proceed to integrate by parts this integral \( n \) times with respect to \( \tau \). To do this we note that, (i) Both \( \tilde{w}_s(\omega) \) and \( \tilde{\psi}_{+0}(r, \omega) \) are infinitely differentiable functions of \( \omega \) (in view of Definition \( 3 \), Remark \( 4.7 \) and Lemma \( C.3 \)); (ii) The Fourier transform \( \tilde{w}(\tau) \) and all of its derivatives tends to zero faster than any negative power of \( \tau \) as \( \tau \to \pm \infty \), as it befits the Fourier transform of a compactly supported function; and (iii) For each non-negative integer \( m \) there exist an integer \( N_m > 0 \) and a constant \( C_m > 0 \) such that \( \left\| \partial^m \tilde{\psi}_{+0}(\cdot, \omega - \tau) \right\|_{L^2(\Gamma)} \leq C_m |\omega - \tau|^{N_m} \) as \( |\tau| \to \infty \), as it follows directly from \((4.33)\) in Lemma \( 4.6 \). Thus, integrating by parts \((4.64)\), using Leibniz’s product rule, and noting that all boundary terms at \( \tau = \pm \infty \) vanish, \((4.64)\) becomes

\[
(4.65) \quad \left( \tilde{w}_s * \tilde{\psi}_{+0} \right)(r, \omega) = \left( \frac{1}{i \varphi} \right)^n \int_{-\infty}^{\infty} e^{-i\omega \varphi} \left( \sum_{m=0}^{n} a_m \left( \partial^m_{\tau} \tilde{\psi}_{+0}(\cdot, \omega - \tau) \right) \times \left( \partial^m_{\omega} \tilde{\psi}_{+0}(r, \omega - \tau) \right) \right) d\tau,
\]

where \( a_m = \binom{n}{m} \).

In view of \((4.63)\) and \((4.65)\), calling, for \( 0 \leq m \leq n \),

\[
(4.66) \quad F_{nm}^\varphi(\tau) = e^{-i\varphi} \partial^{-m}_{\tau} \tilde{w}(\tau) \quad \text{and} \quad P_m(r, \tau) = \partial^m_{\tau} \tilde{\psi}_{+0}(r, \tau),
\]

and using the relation \( \left\| \sum_{i=1}^{n} a_i \right\|^2 \leq n \sum_{i=1}^{n} ||a_i||^2 \), we obtain

\[
(4.67) \quad \left\| w_s \tilde{\psi}_{+0} \right\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2 \leq (n+1) \varphi^{-2n} \sum_{m=0}^{n} |a_m|^2 \left\| F_{nm}^\varphi \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))}.
\]

In order to obtain a bound on the norms on the right-hand side of \((4.67)\) we rely on the Bochner version \( [22, \text{Lem. 1.2.30}] \) of Young’s convolution inequality, and we thus first establish that the required hypotheses are satisfied, namely, that

\[
(4.68) \quad F_{nm}^\varphi \in L^1(\mathbb{R}) \quad \text{and} \quad P_m \in L^2(\mathbb{R}; L^2(\Gamma)) \quad \text{for} \quad 0 \leq m \leq n.
\]

The first of these relations is easily established: since \( w \) is smooth and compactly supported, it is in the Schwartz class, and thus \( [19] \), its Fourier transform is also in the Schwartz class—and, in particular, \( \tilde{w} \) and all of its derivatives are elements
of $L^1(\mathbb{R})$. To verify the second relation in (4.68), on the other hand, we first note from Lemma 3.9 that, for $\omega \geq 0$, $\tilde{\psi}_{+,0}$ satisfies the integral equation

$$
\left( A_\omega \tilde{\psi}_{+,0} \right) = \tilde{R}_0(r, \omega),
$$

where $\tilde{R}_0(r, \omega)$ is defined by (4.49). It then follows from (4.32) in Lemma 4.6 that, for $m \leq n$, there exist constants $d_{ij}^m > 0$ such that for all $\omega \in \mathbb{R}$, $\omega \neq \pm \omega_0$,

$$
\| P_m(\cdot, \omega) \|^2_{L^2(\Gamma)} \leq \sum_{i=0}^m \sum_{j=0}^{(i+1)(q+1)-1} d_{ij}^m \omega^{2j} \left\| \partial^{m-i}_\omega \tilde{R}_0(\cdot, |\omega|) \right\|^2_{L^2(\Gamma)}. \tag{4.69}
$$

Integrating, we obtain

$$
\int_{-\infty}^{\infty} \| P_m(\cdot, \omega) \|^2_{L^2(\Gamma)} \ d\omega \leq \sum_{i=0}^m \sum_{j=0}^{(i+1)(q+1)-1} 2d_{ij}^m \int_{0}^{\infty} \omega^{2j} \left\| \partial^{m-i}_\omega \tilde{R}_0(\cdot, |\omega|) \right\|^2_{L^2(\Gamma)} \ d\omega. \tag{4.70}
$$

We now use Lemma 4.8 to estimate each term on the right-hand side of (4.70). The main hypothesis of that lemma, which, in the present context amounts to the requirement that $\tilde{\psi}_{+0} \in H^{(n+1)(q+1)}(I_0; L^2(\Gamma))$, follows from Lemma 3.6 since $\tilde{b}$ satisfies the $s$-regularity conditions (2.5) with $s = (n+1)(q+1) + q$ (which hold in view of the $s$-regularity condition hypotheses for $\tilde{b}$ and the definition (4.45) of $\tilde{b}$).

For $m \leq n$ we may therefore write

$$
\int_{-\infty}^{\infty} \| P_m(\cdot, \omega) \|^2_{L^2(\Gamma)} \ d\omega \leq C_1 \left\| \tilde{\psi}_{+0} \right\|^2_{H^{(n+1)(q+1)}(I_0; L^2(\Gamma))} \leq C_2 \left\| \tilde{\psi}_{+0} \right\|^2_{H^{(n+1)(q+1)}(I_0; L^2(\Gamma))}, \tag{4.71}
$$

with $C_1$ and $C_2 > 0$ independent of $\tilde{b}$ and $T_0$, which, in particular, establishes the second relation in (4.68).

Having established (4.68), we may apply Young’s convolution inequality [22, Lemma 1.2.30] to each of the terms in the right-hand sum in (4.67) and obtain

$$
\left\| w_{\varphi} \tilde{\psi}_{+0} \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))} \leq C(n) \varphi^{-2n} \sum_{m=0}^n |a_m|^2 \left\| F_{nm}^\varphi \right\|^2_{L^1} \left\| P_m \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))}, \tag{4.72}
$$

(since, of course, $\left\| F_{nm}^\varphi \right\|_{L^1(\mathbb{R})} = \left\| \partial^{m-n}_\omega \tilde{w} \right\|_{L^1(\mathbb{R})}$), which using (4.71) yields

$$
\left\| w_{\varphi} \tilde{\psi}_{+0} \right\|^2_{L^2(\mathbb{R}; L^2(\Gamma))} \leq C(\Gamma, \tau, n, s_0) \varphi^{-2n} \left\| \tilde{\psi}_{+0} \right\|^2_{H^{(n+1)(q+1)}(I_0; L^2(\Gamma))}. \tag{4.62}
$$

The claimed relation (4.62) follows from this inequality since, according to Remark 4.7 $\tilde{\psi}_{+0}(r', t') = w_-(t' + T_r + \tau) w_+(t') \tilde{\psi}(r', t')$, and since each derivative of each one of the functions $w_-$ and $w_+$ is uniformly bounded. □

On the basis of the preparatory lemmas established above in this section we now present the proof of Theorem 4.1.
Proof of Theorem 4.1. Letting (cf. (4.45))

\[ \tilde{b}(r, t) = b(r, t + T_0) \quad \text{and} \quad \tilde{\psi}(r', \theta) = \psi(r', \theta + T_0), \]

instead of (4.1) we establish the equivalent \( \theta \)-decay \( (\theta = t - T_0 > 0) \) estimate

\[ \left\| \tilde{\psi} \right\|_{H_{p}(\Gamma; L^{2}(\Gamma))} \leq C(\Gamma, \tau, p, n)\theta^{1/2-n} \left\| \tilde{\psi} \right\|_{H_{p+n}(\Gamma; L^{2}(\Gamma))} < \infty, \]

where \( I_0 \) (i.e. \( I_T \) for \( T = 0 \)) was introduced in Definition 3. Clearly, \( \tilde{\psi} \) in (4.73) satisfies the integral equation

\[ (S\tilde{\psi})(r, t) = \gamma + \tilde{b}(r, t) \quad \text{for} \quad (r, t) \in \Gamma \times \mathbb{R}, \]

which coincides with (3.10) for \( \tilde{b}(r, t) = \tilde{b}(r, t) = b(r, t + T_0) \) and \( \tilde{\psi} = \psi \).

To establish (4.74) we first obtain certain decay results for each element of a sequence of bounded time intervals and then produce the final estimate by summing over the sequence. Using (4.61) with \( \varphi = \theta + 3\ell s_0 \) together with the identity \( \tilde{\psi}(\cdot, t') = \tilde{\psi}_{\theta,0}(\cdot, t') \) for \( t' > 0 \) (see Remark 4.7) and the relation \( (A + B)^2 \leq 2A^2 + 2B^2 \) we obtain

\[ \left\| \tilde{\psi} \right\|_{L^{2}(\Gamma; L^{2}(\Gamma))}^2 = \int_{\theta}^{\infty} \int_{\Gamma} \left( \sum_{\ell=0}^{\infty} \left( w_{\theta + 3\ell s_0} + w_{\theta + 3(\ell+1)s_0} \right) \tilde{\psi}_{\theta,0}(r', t') \right)^2 d\sigma(r') dt' \]

\[ \leq 4 \sum_{\ell=0}^{\infty} \left\| w_{\theta + 3\ell s_0} \tilde{\psi}_{\theta,0} \right\|_{L^{2}(\Gamma; L^{2}(\Gamma))}^2. \]

The same argument applied to \( \partial_t^p \tilde{\psi} \) tells us that

\[ \left\| \partial_t^p \tilde{\psi} \right\|_{L^{2}(\Gamma; L^{2}(\Gamma))}^2 \leq 4 \sum_{\ell=0}^{\infty} \left\| w_{\theta + 3\ell s_0} \tilde{\psi}_{\theta,0} \right\|_{L^{2}(\Gamma; L^{2}(\Gamma))}^2, \]

where

\[ \tilde{\psi}_{\theta,0} = w_{\theta}(t) \partial_t^p \tilde{\psi}. \]

Combining (4.76) and (4.77) and using Definition 7 we obtain

\[ \left\| \tilde{\psi} \right\|_{H_{p}(\Gamma; L^{2}(\Gamma))} \leq 4 \sum_{\ell=0}^{\infty} \left( \left\| w_{\theta + 3\ell s_0} \tilde{\psi}_{\theta,0} \right\|_{L^{2}(\Gamma; L^{2}(\Gamma))}^2 \right), \]

(4.79)

To complete the proof we now estimate the \( L^2 \) norms on the right-hand side of (4.79), that is to say, the norms \( \left\| w_{\varphi} \tilde{\psi}_{\theta,0} \right\|_{L^{2}(\Gamma; L^{2}(\Gamma))} \) and \( \left\| w_{\varphi} \tilde{\psi}_{\theta,0} \right\|_{L^{2}(\Gamma; L^{2}(\Gamma))} \) for \( \varphi = \theta + 3\ell s_0 \), \( \ell = 0, 1, \cdots \); the desired result then follows by addition of the resulting estimates.
Decay estimate for $\|w_\varphi \tilde{\psi}_{p,.0}\|_{L^2(\mathbb{R};L^2(\Gamma))}$. Since by assumption $b$ satisfies the $s$-regularity conditions (2.5) with $s = (n+1)(q+1) + q$, and since, by hypothesis, $\tilde{b}(r', t')$ vanishes for $(r', t') \in \Omega \times \{I_0 \cup [0, \infty)\}$, Lemma 4.10 with $\tilde{b} = b$ yields

$$\|w_\varphi \tilde{\psi}_{p,.0}\|_{L^2(\mathbb{R};L^2(\Gamma))} \leq C(\Gamma, \tau, n, s_0)\varphi^{-2n} \|\tilde{\psi}\|_{H^{(n+1)(q+1)}(I_0; L^2(\Gamma))}^2$$

for arbitrary $\varphi > 0$, where $C$ denotes a constant independent of $\varphi$, $T_0$, and $b$.

Decay estimate for $\|w_\varphi \tilde{\psi}_{p,.0}\|_{L^2(\mathbb{R};L^2(\Gamma))}$ $(p > 0)$. Lemma 3.19 tells us that

$$\|S\partial^\ell_b \tilde{\psi}(r, t)\| = \gamma^p \partial^\ell_b \tilde{\psi}(r, t), \quad \text{for} \quad (r, t) \in \Gamma \times \mathbb{R},$$

and we may thus apply Lemma 4.10 to obtain decay estimates for $w_\varphi \tilde{\psi}_{p,.0}$. Certainly, $\partial^\ell_b \tilde{\psi}$ satisfies the hypotheses of that lemma: (i) $\partial^\ell_b \tilde{\psi}$ vanishes for $(r', t') \in \Omega \times \{I_0 \cup [0, \infty)\}$ since, by hypothesis, $b$ vanishes in that set; and (ii) $\partial^\ell_b \tilde{\psi}$ satisfies the $s$-regularity conditions (2.5) with $s = (n+1)(q+1) + q$, as it follows from the present hypotheses on $b$. Applying Lemma 4.10 with $\tilde{b} = \partial^\ell_b \tilde{\psi}$ we then obtain the estimate

$$\|w_\varphi \tilde{\psi}_{p,.0}\|_{L^2(\mathbb{R};L^2(\Gamma))}^2 \leq C(\Gamma, \tau, n, s_0)\varphi^{-2n} \|\partial^\ell_b \tilde{\psi}\|_{H^{(n+1)(q+1)}(I_0; L^2(\Gamma))}^2,$$

for arbitrary $\varphi > 0$, where $C$ is again a constant independent of $\varphi$, $T_0$, and $b$.

Combined decay estimate. Using (4.79), (4.80), and (4.82) we obtain

$$\|\tilde{\psi}\|_{H^p([\theta, \infty); L^2(\Gamma))}^2 \leq 4 \sum_{\ell=0}^\infty \left( \|w_{\theta+3\ell s_0 + s_0} \tilde{\psi}_{p,.0}\|_{L^2(\mathbb{R};L^2(\Gamma))}^2 + \|w_{\theta+3\ell s_0 + s_0} \tilde{\psi}_{,p,.0}\|_{L^2(\mathbb{R};L^2(\Gamma))}^2 \right) \leq C_1 \sum_{\ell=0}^\infty \left( (\theta + 3\ell s_0 + s_0)^{-2n} \|\tilde{\psi}\|_{H^{(n+1)(q+1)}(I_0; L^2(\Gamma))}^2 + (\theta + 3\ell s_0 + s_0)^{-2n} \|\partial^\ell_b \tilde{\psi}\|_{H^{(n+1)(q+1)}(I_0; L^2(\Gamma))}^2 \right),$$

where $C_1$ is a constant dependent only on $\Gamma$, $\tau$, $n$ and $s_0$. It follows that

$$\|\tilde{\psi}\|_{H^p([\theta, \infty); L^2(\Gamma))}^2 \leq C \|\tilde{\psi}\|_{H^{p+(n+1)(q+1)}(I_0; L^2(\Gamma))} \sum_{\ell=0}^\infty (\theta + 3\ell s_0 + s_0)^{-2n},$$

where, again, $C = C(\Gamma, \tau, n, s_0) > 0$. Since $\theta > 0$, the term $(\theta + 3\ell s_0 + s_0)^{-n}$ is a strictly decreasing and positive function of $\ell$, and, thus, estimating the sum by an integral it is easy to check that

$$\sum_{\ell=0}^\infty (\theta + 3\ell s_0 + s_0)^{-2n} \leq \tilde{C}\theta^{-2n+1}, \quad \tilde{C} > 0,$$

which in conjunction with (4.84) establishes (4.74), and, therefore, its equivalent decay estimate (4.1). The result (4.2), finally, follows from use of (4.1) with $p = 1$ together with Lemma 3.18. The proof is now complete. □
Appendix A. Sobolev-Bochner spaces

This appendix introduces the class of Bochner spaces and the associated trace operators utilized in this paper.

Definition 6. We denote by $\gamma^+$ and $\gamma^-$ the well-known trace operators
$$
\gamma^+ : H^s_{\text{loc}}(\mathbb{R}^3 \setminus \Omega) \to H^{s-1/2}(\Gamma),
$$
and
$$
\gamma^- : H^s(\Omega) \to H^{s-1/2}(\Gamma),
$$
with $1/2 < s \leq 1$ [32 Thm. 3.37], that are associated with a Lipschitz obstacle $\Omega$ and its boundary $\Gamma$. \hfill \Box

The Sobolev-Bochner spaces and norms introduced in this section, whose elements are functions defined on the complete real line, are similar to, but different from, the corresponding Sobolev-Bochner spaces and norms on finite and semi-infinite intervals that have been used in the context of parabolic PDEs, cf. [17 §5.9]. The corresponding definitions for bounded and semi-infinite intervals, which are also used in this paper, are defined analogously.

Definition 7 (Sobolev-Bochner spaces). For an integer $r \geq 0$, $\partial_\xi^r u(r, \xi)$ denotes the $r$-th distributional derivative of the function $u$ with respect to the real variable $\xi$. Further, for given integers $r, s \geq 0$ and a given set $\mathcal{U} \subset \mathbb{R}^3$, where either $\mathcal{U} = \Gamma$ equals the Lipschitz boundary of an open and bounded domain $\Omega \subset \mathbb{R}^3$, or $\mathcal{U} = D$ equals an open and bounded domain $D \subset \mathbb{R}^3$ with a Lipschitz boundary, we define the Sobolev-Bochner spaces

(A.1) \hspace{1cm} H^r(\mathbb{R}; H^s(\mathcal{U})) = \left\{ u : \mathbb{R} \to H^s(\mathcal{U}) \mid \|u\|_{H^r(\mathbb{R}; H^s(\mathcal{U}))} < \infty \right\},

with the norm

(A.2) \hspace{1cm} \|u\|_{H^r(\mathbb{R}; H^s(\mathcal{U}))} = \left[ \|u\|^2_{L^2(\mathbb{R}; H^s(\mathcal{U}))} + \|\partial_\xi^r u\|^2_{L^2(\mathbb{R}; H^s(\mathcal{U}))} \right]^{1/2},

where the $L^2(\mathbb{R}; H^s(\mathcal{U}))$-norm of a function $v : \mathbb{R} \to H^s(\mathcal{U})$ is given by

(A.3) \hspace{1cm} \|v\|_{L^2(\mathbb{R}; H^s(\mathcal{U}))} = \left( \int_{-\infty}^{\infty} \|v(\cdot, \xi)\|^2_{H^s(\mathcal{U})} \, d\xi \right)^{1/2}.

Note that the integrals inherent in the Sobolev-Bochner norms may be interpreted in the sense of Bochner [15,21], or, equivalently, and more simply, as double integrals with respect to $r \in \mathcal{U}$ and $\xi \in \mathbb{R}$. \hfill \Box

An equivalent norm for $H^r(\mathbb{R}; H^s(\mathcal{U}))$ is given by

(A.4) \hspace{1cm} \|u\|_{H^r(\mathbb{R}; H^s(\mathcal{U}))} = \left( \int_{-\infty}^{\infty} (1 + \omega^2)^{r/2} \|U^f(\cdot, \omega)\|^2_{H^s(\mathcal{U})} \, d\omega \right)^{1/2},

where $U^f$ denotes the Fourier transform of $u$ with respect to $\xi$. (The equivalence of this norm to (A.2) results from a simple application of the classical Plancherel theorem [19 Thm. 6.1] for real-valued functions together with Fubini’s theorem; cf. [27] where the general Bochner case is considered for $r = 0$.) For integers $k, s \geq 0$ we also utilize the spaces

(A.5) \hspace{1cm} C^k(\mathbb{R}; H^s(\mathcal{U})) = \left\{ u \in H^s_{\text{loc}}(\mathbb{R}; H^s(\mathcal{U})) \mid \partial_\xi^k u \in C(\mathbb{R}; H^s(\mathcal{U})) \right\},
where $C(\mathbb{R}; H^s(\mathcal{U}))$ denotes the set of all strongly continuous functions $u : \mathbb{R} \to H^s(\mathcal{U})$. Note that this notation is consistent with the classical definition of Bochner $C^k$ spaces: any element $u = u(r, \xi)$ in the space of $C^k(\mathbb{R}; H^s(\mathcal{U}))$ introduced in (A.5) is indeed $k$-times (strongly) continuously differentiable as a function of $\xi$—since its weak and strong derivatives coincide almost everywhere [16, Cor. 64.32], and since, by definition, the weak derivative is continuous. Finally we define $C^\infty(\mathbb{R}; H^s(\mathcal{U})) = \cap_{k=1}^\infty C^k(\mathbb{R}; H^s(\mathcal{U}))$.

**Appendix B. Proof of Lemma 2.7**

*Proof.* We first establish the $L^2$-norm continuity of the operator (2.29) restricted to $S(\mathbb{R}, L^2(\Gamma))$. To do this, given $\mu \in S(\mathbb{R}, L^2(\Gamma))$, we use the change of variables

$$
\tau = t - |r - r'|/c
$$

(B.1)

to re-express the norm of $S\mu$ in the form

$$
\|S\mu\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2 = \int_{-\infty}^{\infty} \int_{\Gamma} \left| \int_{\Gamma} \frac{\mu(r', \tau)}{4\pi |r - r'|} \, d\sigma(r') \right|^2 \, d\sigma(r) \, d\tau.
$$

(B.2)

Since $\mu \in S(\mathbb{R}, L^2(\Gamma)) \subset L^2(\mathbb{R}, L^2(\Gamma))$, Tonelli’s theorem tells us that $\mu(r', \tau)$ is an element of $L^2(\Gamma)$ for all fixed $\tau \in \mathbb{R}$ outside a set of measure zero. We can therefore invoke (2.26) with $\omega = 0$ to obtain

$$
\int_{\Gamma} \left| \int_{\Gamma} \frac{\mu(r', \tau)}{4\pi |r - r'|} \, d\sigma(r') \right|^2 \, d\sigma(r) \leq C \|\mu(\cdot, \tau)\|_{L^2(\Gamma)}^2
$$

for all $\tau$ outside a set of measure zero. Integrating with respect to $\tau$ we then obtain

$$
\|S\mu\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2 \leq C \int_{-\infty}^{\infty} \|\mu(\cdot, \tau)\|_{L^2(\Gamma)}^2 \, d\tau = C \|\mu\|_{L^2(\mathbb{R}; L^2(\Gamma))}^2.
$$

(B.3)

The extension of the operator $S$ to an operator as indicated in (2.29), as well as the continuity of that operator, follow from (B.4), by using suitable Cauchy sequences and the Plancherel theorem, in view of the density [22, Prop. 2.4.23] of the space $S(\mathbb{R}; L^2(\Gamma))$ in $L^2(\mathbb{R}; L^2(\Gamma))$.

In view of the continuity of the operators $\tilde{S}$, $S$, and the Fourier transform $\mathcal{F}$ in the space $L^2(\mathbb{R}; L^2(\Gamma))$ (cf., respectively, (2.25), (2.29), and the Bochner-Plancherel theorem [27]), together with the aforementioned density of the space $S(\mathbb{R}; L^2(\Gamma))$ in $L^2(\mathbb{R}; L^2(\Gamma))$, to establish (2.30) it suffices to show that this equality holds for for each $\mu \in S(\mathbb{R}; L^2(\Gamma))$. But a function $\mu \in S(\mathbb{R}; L^2(\Gamma))$, $\mu = \mu(r, t)$, is necessarily integrable for $(r, t) \in \Gamma \times \mathbb{R}$, and we may thus use the change of variables (B.1) followed by the Fubini theorem to obtain

$$
\int_{-\infty}^{\infty} e^{-i\omega t} \int_{\Gamma} \frac{\mu(r', t - |r - r'|/c)}{4\pi |r - r'|} \, d\sigma(r') \, dt = \int_{\Gamma} \frac{e^{i\omega r - |r - r'|}}{4\pi |r - r'|} \, d\sigma(r') \int_{-\infty}^{\infty} e^{-i\omega \tau} \mu(r', \tau) \, d\tau,
$$

(B.5)

establishing (2.30) and thus completing the proof of the lemma. \qed
This appendix presents necessary technical lemmas concerning the differentiability of frequency domain solutions with respect to the temporal frequency $\omega$. In detail, Definition 8 introduces relevant boundary integral operators with frequency-differentiated kernels; Lemmas C.1 and C.2 show that certain standard integral operators in scattering theory are strongly differentiable with respect to frequency, and they present frequency-explicit bounds on the norms of the frequency-derivatives of these operators; and Lemma C.3 establishes that, for temporally compactly supported data $\tilde{b}$, the frequency-domain integral equation solution is infinitely differentiable as a function of frequency, and its derivatives can be produced by means of Leibniz's formula.

**Definition 8.** With reference to Definition 1, for $m \in \mathbb{N}_0$ define the operators $S_{\omega,m}$ and $K_{\omega,m}^*$ by

$$S_{\omega,0} = S_{\omega}, \quad K_{\omega,0}^* = K_{\omega}^*,$$

and, for $m \geq 1$,

$$S_{\omega,m} = \lim_{\Delta \omega \to 0} \frac{S_{\omega,m-1} - S_{\omega,m-1}}{\Delta \omega}, \quad \bar{\omega} = \omega + \Delta \omega,$$

and

$$K_{\omega,m}^* = \lim_{\Delta \omega \to 0} \frac{K_{\omega,m-1}^* - K_{\omega,m-1}^*}{\Delta \omega}, \quad \bar{\omega} = \omega + \Delta \omega,$$

where the limit is understood in the sense of the $L^2(\Gamma)$ operator norm. In other words, using the symbol $\partial_m^\omega$ to denote the $m$-th derivative of an operator with respect to frequency (defined as the limit of quotients of increments, with convergence according to the $L^2(\Gamma)$ operator norm), we have

$$\partial_m^\omega S_{\omega} = S_{\omega,m}$$

and

$$\partial_m^\omega K_{\omega}^* = K_{\omega,m}^*,$$

for all non-negative integers $m$. Further, $\partial_m^\omega S_{\omega}$ and $\partial_m^\omega K_{\omega}^*$ satisfy

$$\|\partial_m^\omega S_{\omega}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C_1, \quad \text{and} \quad \|\partial_m^\omega K_{\omega}^*\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C_2 + C_3|\omega|,$$

for all $\omega \in \mathbb{R}$, for certain $\omega$-independent finite constants $C_j = C_j(\Gamma, m), \ j = 1, 2, 3$. 

---

**Lemma C.1.** The operators $S_{\omega,m}$ and $K_{\omega,m}^*$ are strongly differentiable, and for all positive integers $m$ we have

$$S_{\omega,m} = \lim_{\Delta \omega \to 0} \frac{S_{\omega,m-1} - S_{\omega,m-1}}{\Delta \omega}, \quad \bar{\omega} = \omega + \Delta \omega,$$

and

$$K_{\omega,m}^* = \lim_{\Delta \omega \to 0} \frac{K_{\omega,m-1}^* - K_{\omega,m-1}^*}{\Delta \omega}, \quad \bar{\omega} = \omega + \Delta \omega,$$

where the limit is understood in the sense of the $L^2(\Gamma)$ operator norm. In other words, using the symbol $\partial^m_\omega$ to denote the $m$-th derivative of an operator with respect to frequency (defined as the limit of quotients of increments, with convergence according to the $L^2(\Gamma)$ operator norm), we have

$$\partial^m_\omega S_{\omega} = S_{\omega,m}$$

and

$$\partial^m_\omega K_{\omega}^* = K_{\omega,m}^*,$$

for all non-negative integers $m$. Further, $\partial^m_\omega S_{\omega}$ and $\partial^m_\omega K_{\omega}^*$ satisfy

$$\|\partial^m_\omega S_{\omega}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C_1, \quad \text{and} \quad \|\partial^m_\omega K_{\omega}^*\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C_2 + C_3|\omega|,$$

for all $\omega \in \mathbb{R}$, for certain $\omega$-independent finite constants $C_j = C_j(\Gamma, m), \ j = 1, 2, 3$. 

---

**Appendix C. Frequency-differentiated integral equation solutions**

This appendix presents necessary technical lemmas concerning the differentiability of frequency domain solutions with respect to the temporal frequency $\omega$. In detail, Definition 8 introduces relevant boundary integral operators with frequency-differentiated kernels; Lemmas C.1 and C.2 show that certain standard integral operators in scattering theory are strongly differentiable with respect to frequency, and they present frequency-explicit bounds on the norms of the frequency-derivatives of these operators; and Lemma C.3 establishes that, for temporally compactly supported data $\tilde{b}$, the frequency-domain integral equation solution is infinitely differentiable as a function of frequency, and its derivatives can be produced by means of Leibniz’s formula.
Proof of Lemma C.1. We establish (C.4) and (C.5) by showing that there exist positive constants \( \tilde{C}_1 \) and \( \tilde{C}_2 \) dependent only on \( \Gamma \) and \( m \) such that for all sufficiently small \( |\Delta \omega| > 0 \) we have

\[
(C.7) \quad \left\| S_{\omega,m-1} - S_{\omega,m-1} \frac{\Delta \omega}{\Delta \omega} \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \tilde{C}_1 |\Delta \omega|,
\]

and

\[
(C.8) \quad \left\| K_{\omega,m-1}^* - K_{\omega,m-1}^* \frac{\Delta \omega}{\Delta \omega} \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \tilde{C}_2 |\Delta \omega|.
\]

To do this we rely on the expressions

\[
(C.9) \quad \partial_m^\omega G_{\omega}(r,r') = \left( \frac{|r-r'|}{c} \right)^m \frac{e^{i\frac{\omega}{c} |r-r'|}}{4\pi |r-r'|},
\]

and

\[
(C.10) \quad \frac{\partial_m^\omega \partial G_{\omega}(r,r')}{\partial n(r)} = \frac{(r-r') \cdot n(r)}{4\pi |r-r'|^3} e^{i\frac{\omega}{c} |r-r'|} \left( i \frac{|r-r'|}{c} \right)^m \left( \frac{\omega}{c} \frac{|r-r'|}{c} + m - 1 \right) = \frac{i}{c} \left( \frac{r-r'}{4\pi |r-r'|^2} \right) e^{i\frac{\omega}{c} |r-r'|} \left( i \frac{|r-r'|}{c} \right)^{m-1} \left( \frac{\omega}{c} \frac{|r-r'|}{c} + m - 1 \right),
\]

that are valid for all \( m \in \mathbb{N}_0 \).

In order to establish (C.7) and (C.8), we let \( a \in L^2(\Gamma) \) and we note that the difference between \( S_{\omega,m} \) and the associated quotient of increments, when applied to \( a \), reads

\[
(C.11) \quad \int_{\Gamma} \left[ \frac{1}{\Delta \omega} \left( \partial_m^{\omega-1} G_{\omega} - \partial_m^{\omega-1} G_{\omega} - \partial_m^\omega G_{\omega} \right) \right] a(r') \, d\sigma(r') = \int_{\Gamma} \frac{1}{4\pi |r-r'|} \left[ \frac{1}{\Delta \omega} \left( i \left( \frac{|r-r'|}{c} \right)^m \left( e^{i\frac{\omega}{c} |r-r'|} - e^{i\frac{\omega}{c} |r-r'|} \right) \right) \right] a(r') \, d\sigma(r') = \int_{\Gamma} \frac{1}{4\pi |r-r'|} \left[ Q_{\Delta \omega}^1(r,r') a(r') \right] \, d\sigma(r'),
\]

where \( Q_{\Delta \omega}^1 \) is the term in square brackets in the next to last integral. Similarly, the difference between \( K_{\omega,m}^* \) and the associated quotient of increments, applied to
\[ a, \text{ equals} \]
\[ \int_\Gamma \left[ \frac{1}{|\Delta \omega|} \left( \frac{\partial G_{\omega}}{\partial \Omega(r)} - \frac{\partial G_{\omega}}{\partial \Omega(r)} \right) \right] a(r') \, d\sigma(r') \]
\[ = \int_\Gamma \frac{(r - r') \cdot n(r)}{4\pi |r - r'|^2} \left[ \frac{1}{|\Delta \omega|} \left( \frac{i |r - r'|}{c} \right)^{m-1} \left( \frac{i \omega}{c} |r - r'| - \frac{i \omega}{c} |r - r'| \right) \right] \]
\[ + \frac{i}{c} (m - 2) \left( \frac{i |r - r'|}{c} \right)^{m-2} \left( e^{i \frac{\omega}{c} |r - r'|} - e^{i \frac{\omega}{c} |r - r'|} \right) \]
\[ - \frac{i}{c} e^{i \frac{\omega}{c} |r - r'|} \left( \frac{i |r - r'|}{c} \right)^{m-1} \left( \frac{i \omega}{c} |r - r'| + m - 1 \right) \]
\[ f(r') \, d\sigma(r') \]
\[ = \int_\Gamma \frac{(r - r') \cdot n(r)}{4\pi |r - r'|^2} Q_{\Delta \omega}^2(r, r') f(r') \, d\sigma(r'), \]

where we have used the second expression in (C.10), and where \( Q_{\Delta \omega}^2 \) is the quantity in square brackets in the next to last integral.

Since, for \( m \in \mathbb{N} \) and for all \( r, r' \in \Gamma \), \( Q_{\Delta \omega}^1 \) and \( Q_{\Delta \omega}^2 \) amount to difference quotients of smooth functions of \( \omega \) minus the corresponding derivatives, use of second-order Taylor expansions of \( Q_{\Delta \omega}^1 \) and \( Q_{\Delta \omega}^2 \) in the variable \( \omega \) ensures that, for certain constants \( D_1 \) and \( D_2 \), \( |Q_{\Delta \omega}^1(r, r')| \leq D_1 |\Delta \omega| \) and \( |Q_{\Delta \omega}^2(r, r')| \leq D_2 |\Delta \omega| \) for all \( r, r' \in \Gamma \), and for all \( \Delta \omega \) in a bounded interval around \( \Delta \omega = 0 \).

Using the bound on \( Q_{\Delta \omega}^2 \) together with (C.11) we obtain
\[ (C.13) \, \left| (S_{\omega, m-1} - S_{\omega, m-1})_{\Delta \omega} - S_{\omega, m} \right| |a| \leq D_1 |\Delta \omega| \int_\Gamma \frac{1}{4\pi |r - r'|} |a(r')| \, d\sigma(r'). \]

Similarly, using the bound for \( Q_{\Delta \omega}^2 \) together with (C.12) we obtain
\[ (C.14) \, \left| \left( K_{\omega, m-1}^\ast - K_{\omega, m-1}^\ast \right)_{\Delta \omega} - K_{\omega, m}^\ast \right| |a| \leq D_2 |\Delta \omega| \int_\Gamma \frac{1}{4\pi |r - r'|} |a(r')| \, d\sigma(r'), \]

where the relation \( |(r - r') \cdot n(r)|/|r - r'| \leq 1 \) was used. But the integral expressions on the right-hand sides of (C.13) and (C.14) are Laplace single-layer operators, equal to \( S_{\omega} \) with \( \omega = 0 \), acting on \( |a| \), and, thus, by (2.26) we obtain
\[ \left\| \left( S_{\omega, m-1} - S_{\omega, m-1} \right)_{\Delta \omega} - S_{\omega, m} \right\|_{L^2(\Gamma)} \leq CD_1 |\Delta \omega| \|a\|_{L^2(\Gamma)}, \]

and
\[ \left\| \left( K_{\omega, m-1}^\ast - K_{\omega, m-1}^\ast \right)_{\Delta \omega} - K_{\omega, m}^\ast \right\|_{L^2(\Gamma)} \leq CD_2 |\Delta \omega| \|a\|_{L^2(\Gamma)}, \]

where \( C = C(\Gamma) \) denotes the operator norm \( \|S_0\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \) of the Laplace single-layer operator \( S_0 : L^2(\Gamma) \rightarrow L^2(\Gamma) \). The relations (C.7) and (C.8), and thus (C.4) and (C.5), follow directly.
In order to establish the estimates in (C.6), finally, let \( a \in L^2(\Gamma) \) and \( m \in \mathbb{N} \). Then, using (C.9) and the Cauchy-Schwarz inequality we obtain

\[
\| (\partial^m_\omega S_\omega) a \|_{L^2(\Gamma)}^2 = \int_{\Gamma} \left| \int_{\Gamma} \left( \frac{1}{c} \frac{|r - r'|}{4\pi |r - r'|} \right)^m e^{i \frac{c}{2} |r - r'|} a(r') d\sigma(r') \right|^2 d\sigma(r) \leq \left( \frac{\text{diam}(\Gamma)}{4\pi c^m} \right)^2 \int_{\Gamma} |a(r')|^2 d\sigma(r')
\]

for some constant \( C_1 = C_1(\Gamma, m) \) independent of \( \omega \), from which the first inequality in (C.6) follows. Similarly, using the second expression in (C.10) and the fact that \( |(r - r') \cdot n(r)|/|r - r'| | \leq 1 \) we obtain

\[
\| (\partial^m_\omega K_\omega^*) a \|_{L^2(\Gamma)}^2 = \int_{\Gamma} \left| \int_{\Gamma} \left( \frac{1}{c} \frac{|r - r'|}{4\pi |r - r'|} \right)^{m-1} \left( 1 + \frac{c}{\text{diam}(\Gamma)} \right)^{m-1} e^{i \frac{c}{2} |r - r'|} a(r') d\sigma(r') \right|^2 d\sigma(r) \leq C'_{1}(\Gamma, m) (C'_{2}(\Gamma)|\omega| + C'_{3}(m))^2 \| S_0 \|_{L^2(\Gamma)}^2 \]

where \( C_j = C_j(\Gamma, m), j = 2, 3 \), are positive constants independent of \( \omega \), and where in the last step we used once again the boundedness (2.26) in norm enjoyed by the single-layer operator \( S_0 \). This establishes the second inequality in (C.6) and it completes the proof of the Lemma. \( \square \)

**Lemma C.2.** The operator \( A^{(m)}_\omega \) in (C.3) is strongly infinitely differentiable at all \( \omega \neq \pm \omega_0 \), and for all \( m \in \mathbb{N} \) we have

\[
A^{(m)}_\omega = \lim_{\Delta \omega \to 0} \frac{A^{(m-1)}_{\bar{\omega}} - A^{(m-1)}_{\omega}}{\Delta \omega}, \quad \bar{\omega} = \omega + \Delta \omega,
\]

in the sense of the \( L^2(\Gamma) \) operator norm. Further, for all \( m \in \mathbb{N} \) and \( \omega \neq \pm \omega_0 \) we have

\[
\partial^m_\omega A_\omega = A^{(m)}_\omega, \quad A^{(m)}_\omega = \begin{cases} \partial^m_\omega K^*_\omega - i \partial^m_\omega S_\omega, & \text{for } 0 \leq |\omega| < \omega_0, \\
\partial^m_\omega K^*_\omega - i m \partial^{m-1}_\omega S_\omega - i \omega \partial^m_\omega S_\omega, & \text{for } |\omega| > \omega_0,
\end{cases}
\]

and \( \partial^m_\omega A_\omega \) satisfies

\[
\| \partial^m_\omega A_\omega \|_{L^2(\Gamma)} \leq \alpha_0 + \alpha_1 \omega
\]

for some \( \omega \)-independent positive constants \( \alpha_j = \alpha_j(\Gamma, m), j = 0, 1 \).

**Proof.** Equations (C.15) and (C.16) follow directly from Lemma C.1 and the product differentiation rule. The frequency-explicit operator norm bound (C.17) follows
immediately from the estimates (C.6) and (C.16). Indeed, for $0 \leq |\omega| < \omega_0$, using the first case in (C.16) we obtain
\[
\|\partial^m_\omega A_\omega\|_{L^2(\Gamma)} \leq \|\partial^m_\omega K^\omega\|_{L^2(\Gamma)} + \|\partial^m_\omega S_\omega\|_{L^2(\Gamma)},
\]
while for $|\omega| > \omega_0$, from the second case in (C.16) we have
\[
\|\partial^m_\omega A_\omega\|_{L^2(\Gamma)} \leq \|\partial^m_\omega K^\omega\|_{L^2(\Gamma)} + m\|\partial^m_\omega S_\omega\|_{L^2(\Gamma)} + \omega\|\partial^m_\omega S_\omega\|_{L^2(\Gamma)}.
\]
In either case (C.17) follows for some constants $\alpha_j = \alpha_j(\Gamma, m)$, and the proof is complete.

**Lemma C.3.** Let $\bar{b}$ satisfy the assumptions of Lemmas 3.9 and 3.13 and let $\bar{\psi}$ denote the solution to (3.10). Further, using the expressions for $\eta_0(\omega)$ and $H^f_\Gamma$ given in (2.23) and (3.23), respectively, define $\bar{R}_T(\omega, \tau)$ by
\[
\bar{R}_T(\omega, \tau) = \gamma^{-1} \partial_\mu \tilde{H}^f_\Gamma(\omega, \tau) - \eta_0(\omega)\gamma^{-1} \tilde{H}^f_\Gamma(\omega, \tau),
\]
for $\omega \geq 0$, and by Hermitian symmetry for $\omega < 0$: $\bar{R}_T(\omega, \tau) = \bar{R}_T(\tau, -\omega)$. Then $\bar{R}_T \in C^\infty(\mathbb{R} \setminus \pm \omega_0; L^2(\Gamma))$, the solution $\tilde{\psi}^f_{+T}$ to (3.28) satisfies $\tilde{\psi}^f_{+T} \in C^\infty(\mathbb{R}; L^2(\Gamma))$, and, letting $\mu = \tilde{\psi}^f_{+T}$ and $a_k^p = p_k$ ($k = 1, \ldots, p, p \in \mathbb{N} \cup \{0\}$), for all $\omega \in \mathbb{R}^+ \setminus \{\omega_0\}$ and for all non-negative integers $p$ we have
\[
(A_\omega (\partial^p_\omega \mu))(\omega, \tau) = \partial^p_\mu \bar{R}_T(\omega, \tau) - \sum_{k=1}^p a_k^p \left(\partial^p_\omega A_\omega\right) (\partial^{p-k}_\omega \mu)(\omega, \tau).
\]

**Proof.** According to (3.40), $\bar{h}_T$ is compactly supported as a function of time within the temporal interval $[\alpha, T - \tau - 2\tau]$. Thus, by differentiation under the Fourier-transform integral sign we see that the Fourier transform $\tilde{H}^f_\Gamma$ of $\bar{h}_T$ is infinitely differentiable in the sense introduced at the end of Appendix A, $\tilde{H}^f_\Gamma \in C^\infty(\mathbb{R}; L^2(\Gamma))$. (All necessary differentiations under the integral sign are easily justified using the dominated convergence theorem, since $\bar{h}_T$ is compactly supported.) It follows that the function $R_T$ defined by (C.20) is infinitely differentiable with respect to $\omega$ for each $\omega \in \mathbb{R}^+ \setminus \{\omega_0\}$ (see also Remark 3.4 concerning negative $\omega$).

It remains to show that $\mu = \tilde{\psi}^f_{+T} \in C^\infty(\mathbb{R}; L^2(\Gamma))$ and that (C.21) holds. Suppressing $\tau$-dependence, consider the equation
\[
A_\omega \mu(\omega) = R_T(\omega) \quad \text{or, equivalently} \quad \mu(\omega) = A_\omega^{-1} R_T(\omega).
\]
To establish the differentiability of $\mu$ we first show that the operator $A_\omega^{-1}$ is strongly infinitely differentiable (the quotients of increments converge strongly, as in Lemmas C.1 and C.2) with respect to $\omega$, and that
\[
\partial_\omega A_\omega^{-1} = -A_\omega^{-1} (\partial_\omega A_\omega) A_\omega^{-1}.
\]
To do this we first note that $A_\omega^{-1}$ is strongly continuous with respect to $\omega$ (cf. [41]), as it follows directly from the relations
\[
\|A_\omega^{-1} (A_\omega + \Delta_\omega) - A_\omega^{-1}\|_{L^2(\Gamma)} \leq \|A_\omega^{-1} (A_\omega + \Delta_\omega) - A_\omega^{-1}\|_{L^2(\Gamma)} \leq C \|A_\omega + \Delta_\omega - A_\omega\|_{L^2(\Gamma)} \rightarrow 0 \quad \text{as} \quad \Delta_\omega \rightarrow 0.
\]
Then, using Lemma C.2 together with the identity
\[ A^{-1}_{\omega} + \Delta \omega - \Delta^{-1} \omega = -A^{-1}_{\omega} A_{\omega} + \Delta \omega - A_{\omega} A^{-1}_{\omega} \]
in view of the strong continuity of \( A_{\omega} \) with respect to \( \omega \), it follows that \( A^{-1}_{\omega} \) is strongly differentiable, and that for every \( \omega \in \mathbb{R}^+ \setminus \omega_0 \) equation (C.22) holds. Utilizing the easily established rule for differentiation of an operator product of the form \( B_{\omega} C_{\omega} \) together with (C.22), further, shows that \( A^{-1}_{\omega} \) is infinitely differentiable. The differentiability of \( \mu = \psi_{+,T} \) to all orders and for all real frequencies \( \omega \) then follows directly—including at \( \omega = \omega_0 \), since, by Lemma 3.6, \( \psi_{+,T} \) does not depend on the specific choice of \( \omega_0 \). The expression (C.21), finally, results from an application of Leibniz’s differentiation rule to the quantity \( A_{\omega} \mu_{\omega} \). The proof is complete.

\[ \square \]

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