Toughness and the existence of tree-connected \(\{f, f+k\}\)-factors

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Abstract

Let \(G\) be a graph and let \(f\) be a positive integer-valued function on \(V(G)\) satisfying \(2m \leq f \leq b\), where \(b\) and \(m\) are two positive integers with \(b \geq 4m^2\). In this paper, we show that if \(G\) is \(b^2\)-tough and \(|V(G)| \geq b^2\), then it has an \(m\)-tree-connected factor \(H\) such that for each vertex \(v\),

\[d_H(v) \in \{f(v), f(v) + 1\}.\]

Next, we generalize this result by giving sufficient conditions for a tough graph to have a tree-connected factors \(H\) such that for each vertex \(v\), \(d_H(v) \in \{f(v), f(v) + k\}\). As an application, we prove that every \(64b(b-a)^2\)-tough graph \(G\) of order at least \(b+1\) with \(ab|V(G)|\) even admits a connected factor whose degrees lie in the set \(\{a, b\}\), where \(a\) and \(b\) are two integers with \(2 \leq a < b < \frac{6}{5}a\). Moreover, we prove that every \(16\)-tough graph \(G\) of order at least three admits a \(2\)-connected factor whose degrees lie in the set \(\{2, 3\}\), provided that \(G\) has a \(2\)-factor with girth at least five. This result confirms a weaker version of a long-standing conjecture due to Chvátal (1973).

Keywords:
Spanning tree; tree-connectivity; spanning Eulerian; spanning closed trail; connected factor; toughness.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed and a simple graph is a graph without multiple edges. Let \(G\) be a graph. The vertex set, the edge set, the minimum degree, the maximum degree, and the number of components of \(G\) are denoted by \(V(G)\), \(E(G)\), \(\delta(G)\), \(\Delta(G)\), and \(\omega(G)\), respectively. The degree \(d_G(v)\) of a vertex \(v\) is the number of edges of \(G\) incident to \(v\). The set of edges of \(G\) that are incident to \(v\) is denoted by \(E_G(v)\). We denote by \(d_G(C)\) the number of edges of \(G\) with exactly one end in \(V(C)\), where \(C\) is a subgraph of \(G\). For a set \(X \subseteq V(G)\), we denote by \(G[X]\) the induced subgraph of \(G\) with the vertex set \(X\) containing precisely those edges of \(G\) whose ends lie in \(X\). For a set \(A\) of integers, an \(A\)-factor is a spanning subgraph with vertex degrees in \(A\). If for each vertex \(v\), \(A(v)\) is a set of integers, an \(A\)-factor is a spanning subgraph \(F\) such that for each vertex \(v\), \(d_F(v) \in A(v)\). Let \(F\) be a spanning
subgraph of \( G \). For an edge set \( E \), we denote by \( F - E \) the graph obtained from \( F \) by removing the edges of \( E \) from \( F \). Likewise, we denote by \( F + E \) the graph obtained from \( F \) by inserting the edges of \( E \) into \( F \). For convenience, we use \( e \) instead of \( E \) when \( E = \{e\} \). For two edge sets \( E_1 \) and \( E_2 \), we also use the notation \( E_1 + E_2 \) for the union of them. The graph obtained from \( G \) by contracting any component of \( F \) is denoted by \( G/F \). The graph \( F \) is said to be trivial, if it has no edge. Let \( S \subseteq V(G) \). The graph obtained from \( G \) by removing all vertices of \( S \) is denoted by \( G \setminus S \). Denote by \( G \setminus [S,F] \) the graph obtained from \( G \) by removing all edges incident to the vertices of \( S \) except the edges of \( F \). Note that while the vertices of \( S \) are deleted in \( G \setminus S \), no vertices are removed in \( G \setminus [S,F] \). Denote by \( e_G(S) \) the number of edges of \( G \) with both ends in \( S \). Let \( P \) be a partition of \( V(G) \). Denote by \( e_G(P) \) the number of edges of \( G \) whose ends lie in different parts of \( P \). A graph \( G \) is called \( m \)-tree-connected, if it has \( m \) edge-disjoint spanning trees. In addition, an \( m \)-tree-connected graph \( G \) is called minimally \( m \)-tree-connected, if \(|E(G)| = m(|V(G)| - 1) \). In other words, for any edge \( e \) of \( G \), the graph \( G \setminus e \) is not \( m \)-tree-connected. The vertex set of any graph \( G \) can be expressed uniquely as a disjoint union of vertex sets of some induced \( m \)-tree-connected subgraphs. These subgraphs are called the \( m \)-tree-connected components of \( G \). For a graph \( G \), we define the parameter \( \Omega_m(G) = m|P| - e_G(P) \) to measure tree-connectivity, where \( P \) is the unique partition of \( V(G) \) obtained from the \( m \)-tree-connected components of \( G \). Note that \( \Omega_1(G) \) is the same number of components of \( G \), while \( \Omega_m(G) \) is less or equal than the number of \( m \)-tree-connected components of \( G \). The definition implies that the null graph \( K_0 \) with no vertices is not \( m \)-tree-connected and \( \Omega_m(K_0) = 0 \). In this paper, we assume that all graphs are nonnull, except for the graphs that obtained by removing vertices. We say that a graph \( F \) is \( m \)-sparse, if \( e_F(S) \leq m|S| - m \), for all nonempty subsets \( S \subseteq V(G) \). It is not hard to check that \( \Omega_m(F) < \Omega_m(F \setminus e) \), for any edge \( e \) of \( F \). Clearly, \( 1 \)-sparse graphs are forests. A graph \( G \) is \( m \)-sparse if and only if its \( m \)-tree-connected components are minimally \( m \)-tree-connected. Note that all maximal \( m \)-sparse factors of \( G \) form the bases of a matroid, see [2]. Note also that several basic tools in this paper for working with sparse and tree-connected graphs can be obtained using matroid theory. Let \( t \) be a positive real number, a graph \( G \) is said to be \( t \)-tough, if \( \omega(G \setminus S) \leq \max\{1, \frac{1}{t}|S|\} \) for all \( S \subseteq V(G) \). The bipartite index \( bi(G) \) of a graph \( G \) is the smallest number of all \(|E(G) \setminus E(H)| \) taken over all bipartite factors \( H \). Throughout this article, all variables \( k \) and \( m \) are positive integers.

Recently, the present author [6] investigated connected factors with small degrees and established the following theorem. This result is an improvement several results due to Win (1989) [14], Ellingham and Zha (2000) [4], and Ellingham, Nam, and Voss (2002) [3].

**Theorem 1.1.**([6]) Let \( G \) be a graph with a factor \( F \) of which every component contains at least \( c \) vertices with \( c \geq 2 \). Let \( h \) be a nonnegative integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),

\[
\omega(G \setminus S) \leq \sum_{v \in S} \left( \frac{c}{2c - 2} h(v) - \frac{1}{c - 1} \right) + 2 + \frac{1}{c - 1} \omega(G[S]),
\]

then \( G \) has a connected factor \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq h(v) + d_F(v) \).

In this paper, we generalize Theorem 1.1 to the following tree-connected version with more complicated arguments. The special case \( c = 2 \) of this theorem was former studied in [7].
Theorem 1.2. Let $G$ be a graph with a factor $F$. Assume that for every $m$-tree-connected component $C$ of $F$, $|V(C)| + \frac{m}{2c-2}d_F(C) \geq c$ in which $c \geq 2$. Let $h$ be an integer-valued function on $V(G)$. If for all $S \subseteq V(G)$, 
\[ \Omega_m(G \setminus S) < \sum_{v \in S} \left( \frac{c}{2c-2}h(v) - \frac{m}{c-1} \right) + m + 1 + \frac{1}{c-1}\Omega_m(G[S]), \]
then $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v)$.

In 1973 Chvátal [1] conjectured that there exists a positive real number $t_0$ such that every $t_0$-tough graph of order at least three admits a Hamiltonian cycle (connected 2-factor). In 1989 Win [14] gave the first step forward to confirm this conjecture by proving that every 1-tough graph of order at least three admits a connected $\{1, 2, 3\}$-factor. In 2000 Ellingham and Zha [4] improved Win’s result by proving that every 4-tough graph of order at least three admits a connected $\{2, 3\}$-factor. In Subsection 4.3, we prove the following stronger assertion for graphs with higher toughness.

Theorem 1.3. Let $G$ be a graph of order at least three. If $G$ is 16-tough, then it admits a 2-connected $\{2, 3\}$-factor, provided that $G$ admits a 2-factor with girth at least five.

In 1990 Katerinis [11] formulated the following sufficient toughness condition for the existence of $f$-factors. Next, some sufficient toughness-type conditions for the existence of connected $\{f, f + 1\}$-factors were investigated in [3, 4, 8].

Theorem 1.4.([11], see [8]) Let $G$ be a graph, let $b$ be a positive integer with $b \geq 2$, and let $f$ be a positive integer-valued function on $V(G)$ with $f \leq b$. If $G$ is $b^2$-tough and $|V(G)| \geq b^2$, then it has a factor $F$ such that for all vertices $v$, $d_H(v) = f(v)$, except possibly for a vertex $u$ satisfying $d_H(u) = f(u) + 1$.

In Section 4.1, we apply Katerinis’s result to give a sufficient toughness condition for the existence of $m$-tree-connected $\{f, f + 1\}$-factors as mentioned in the abstract. Moreover, we apply it together with the following recent result to give a sufficient toughness condition for a graph to have a tree-connected $\{f, f + k\}$-factor.

Theorem 1.5.([10]) Let $G$ be a $(2m + 2m_0 + 6k)$-tree-connected graph satisfying $bi(G) \geq k - 1$ and $k > m + m_0 \geq 0$, and let $f$ be a positive integer-valued function on $V(G)$. If for each vertex $v$, $f(v) + m_0 \leq \frac{1}{2}d_G(v) \leq f(v) + k - m$, then $G$ has an $m$-tree-connected factor $H$ such that its complement is $m_0$-tree-connected and for each vertex $v$, 
\[ d_H(v) \in \{f(v), f(v) + k\}, \]
provided that $(k - 1)\sum_{v \in V(G)} f(v)$ is even.
2 Preliminary results

Here, we state a fundamental theorem for finding tree-connected factors with small degrees which is similar to Theorem 3.1 in [7] which provides a tree-connected version for Theorem 1 in [4]. Before doing so, let us state some basic tools for working with sparse graphs and tree-connected graphs which are as well-known in terms of matroid theory, see [7].

Lemma 2.1. Let $G$ be a graph with an $m$-sparse factor $F$. If $e \in E(G) \setminus E(F)$ joins different tree-connected components of $F$, then $F + e$ is still $m$-sparse.

Lemma 2.2. Let $G$ be a graph with an $m$-sparse factor $F$. If $xy \in E(G) \setminus E(F)$ and $Q$ is a minimal $m$-tree-connected subgraph of $F$ including $x$ and $y$, then for every $e \in E(Q)$, the graph $F - e + xy$ remains $m$-sparse.

Lemma 2.3. Let $G$ be a graph with two $m$-sparse factors $F$ and $F_0$. If for a vertex set $X$, $F[X]$ is $m$-tree-connected, then $F - E(F[X]) + E(F_0[X])$ remains $m$-sparse.

Proof. Set $H = F - E(F[X]) + E(F_0[X])$ and let $A$ be a nonempty subset of $V(H)$. If $X \cap A$ is empty, then $e_{H}(A) = e_{F}(A) \leq m(|A| - 1)$. So, suppose that $X \cap A$ is not empty. Obviously, $e_{F}(A) - e_{F}(X \cap A) \leq e_{F}(X \cup A) - e_{F}(X)$. Since $e_{F}(X) = m(|X| - 1)$ and $e_{F}(X \cup A) \leq m(|X| + |A| - |X \cap A| - 1)$, we must have $e_{F}(A) - e_{F}(X \cap A) \leq m(|A| - |X \cap A|)$. Since $e_{F_0}(X \cap A) \leq m(|A \cap X| - 1)$, we therefore have $e_{H}(A) = e_{F}(A) - e_{F}(X \cap A) + e_{F_0}(X \cap A) \leq m(|A| - 1)$. Hence $H$ is $m$-sparse and the proof is completed. \[\square\]

Now, we are ready to prove the main result of this section.

Theorem 2.4. Let $G$ be a graph with an $m$-sparse factor $F$ and let $h$ be a nonnegative integer-valued function on $V(G)$. If $H$ is an $m$-sparse factor of $G$ with $te(H, h + d_{F}) = 0$ and with the minimum $\Omega_{m}(H)$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\Omega_{m}(G \setminus [S, F]) = \Omega_{m}(H \setminus [S, F])$.
2. For each vertex $v$ of $S$, $d_{H}(v) = h(v) + d_{F}(v)$.

Proof. Define $V_{0} = \emptyset$. For any $S \subseteq V(G)$ and $u \in V(G) \setminus S$, let $A(S, u)$ be the set of all $m$-sparse factors $H'$ of $G$ containing $F$ with $te(H', h + d_{F}) = 0$ such that $\Omega_{m}(H') = \Omega_{m}(H)$, $H'[X]$ is $m$-tree-connected, and $H'$ and $H$ have the same edges, except for some of the edges of $G$ whose ends are in $X$, where $H[X]$ is the $m$-tree-connected component of $H \setminus [S, F]$ containing $u$. Now, for each integer $n$ with $n \geq 2$, recursively define $V_{n}$ as follows:

$V_{n} = V_{n-1} \cup \{ v \in V(G) \setminus V_{n-1} : d_{H'}(v) = h(v) + d_{F}(v) \text{ for all } H' \in A(V_{n-1}, v) \}.$
Now, we prove the following claim.

**Claim.** Let $x$ and $y$ be two vertices in different $m$-tree-connected components of $H \setminus [V_{n-1}, F]$. If $xy \in E(G) \setminus E(H)$, then $x \in V_n$ or $y \in V_n$.

**Proof of Claim.** By induction on $n$. Suppose, to the contrary, that vertices $x$ and $y$ are in different $m$-tree-connected components of $H \setminus [V_{n-1}, F]$, $xy \in E(G) \setminus E(H)$, and $x, y \notin V_n$. Let $X$ and $Y$ be the vertex sets of the $m$-tree-connected components of $H \setminus [V_{n-1}, F]$ containing $x$ and $y$, respectively. Since $x, y \notin V_n$, there exist $H_x \in \mathcal{A}(V_{n-1}, x)$ and $H_y \in \mathcal{A}(V_{n-1}, y)$ with $d_{H_x}(x) < h(x) + d_F(x)$ and $d_{H_y}(y) < h(y) + d_F(y)$.

For $n = 1$, define $H'$ to be the factor of $G$ containing $F$ with

$$E(H') = E(H) + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]).$$

Since $xy$ joins different $m$-tree-connected components of $H$, by Lemma 2.1, $H'$ is still $m$-sparse and so $\Omega_m(H') < \Omega_m(H)$. Since $te(H', h + d_F) = 0$, we arrive at a contradiction. Now, suppose $n \geq 2$. By the induction hypothesis, $x$ and $y$ are in the same $m$-tree-connected component of $H \setminus [V_{n-2}, F]$ with the vertex set $Z$ so that $X \cup Y \subseteq Z$. Let $Q$ be a minimal $m$-tree-connected subgraph of $H$ including $x$ and $y$. Notice that the vertices of $Q$ lie in $Z$ and also $Q$ includes at least a vertex $z$ of $Z \cap V_{n-1}$ with $zz' \in E(Q) \setminus E(F)$.

By Lemma 2.2, the graph $H - zz' + xy$ is still $m$-sparse. Now, let $H'$ be the factor of $G$ containing $F$ with

$$E(H') = E(H) - zz' + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]).$$

According to Lemma 2.3, one can easily conclude that $H'$ is also $m$-sparse. Since $|E(H')| = |E(H)|$, we must have $\Omega_m(H') = \Omega_m(H)$. For each $v \in V(H')$, we have

$$d_{H'}(v) \leq \begin{cases} d_{H_x}(v) + 1, & \text{if } v \in \{x, y\}; \\ d_H(v), & \text{if } v = z', \\
\end{cases} \quad \text{and} \quad d_{H'}(v) = \begin{cases} d_{H_x}(v), & \text{if } v \in X \setminus \{x, z'\}; \\ d_{H_y}(v), & \text{if } v \in Y \setminus \{y, z'\}; \\ d_H(v), & \text{if } v \notin X \cup Y \cup \{z, z'\}. \end{cases}$$

It is not hard to check that $d_{H'}(z) < d_H(z) \leq h(z) + d_F(z)$ and $H'$ lies in $\mathcal{A}(V_{n-2}, z)$. Since $z \in V_{n-1} \setminus V_{n-2}$, we arrive at a contradiction. Hence the claim holds.

Obviously, there exists a positive integer $n$ such that and $V_1 \subseteq \cdots \subseteq V_{n-1} = V_n$. Put $S = V_n$. For each $v \in V_i \setminus V_{i-1}$, we have $H \in \mathcal{A}(V_{i-1}, v)$ and so $d_H(v) = h(v) + d_F(v)$. This establishes Condition 2. Because $S = V_n$, the previous claim implies Condition 1 and completes the proof.

\[\square\]

### 3 Highly tree-connected factors with bounded degrees

Our aim in this section is to prove Theorem 1.2 with a more stronger version. Before formulating the main result, we shall begin with some lemmas that allows us to make the proof shorter.
3.1 Comparing tree-connectivity of $G \setminus [S, F]$ and $G \setminus S$

In this subsection, we shall compare two parameters $\Omega_m(G \setminus S)$ and $\Omega_m(G \setminus [S, F])$. Before stating the main comparison, let us make the following simpler version. Note that $\Omega_m(H \setminus [S, F]) = \Omega_m(H \setminus S) + m|S|$ when $F$ is the trivial factor.

**Lemma 3.1.** Let $G$ be a graph with a factor $F$. If $S \subseteq V(G)$, then
\[
\Omega_m(G \setminus [S, F]) \leq \Omega_m(G \setminus S) + mt_s - i_s.
\]
where $t_s$ is the number of $m$-tree-connected components of $G \setminus [S, F]$ whose vertices entirely lie in $S$, and $i_s$ is the number of edges of $F$ incident to vertices in $S$ joining different $m$-tree-connected components of $G \setminus [S, F]$.

**Proof.** Define $P$ to be the partition of $V(G)$ obtained from the $m$-tree-connected components of $G \setminus [S, F]$. Let $P_s = \{A \in P : A \subseteq S\}$ and $P'_s = \{A \in P : A \not\subseteq P_s\}$, and also $P_0 = \{A \setminus S : A \in P'\}$. For every $A \in P$, define $P_A$ to be the partition of $A \setminus S$ obtained from the $m$-tree-connected components of $G[A] \setminus S$. Also, define $e_A$ to be the number of edges with both ends in $A \setminus S$ joining different $m$-tree-connected components of $G[A] \setminus S$. Since for every $A \in P'$, $m(|P_A| - 1) \geq e_A$, we must have
\[
\Omega_m(G \setminus [S, F]) = m|P_s| + m|P'_s| - e_{G\setminus [S, F]}(P) \leq m|P_s| + \sum_{A \in P'} (m|P_A| - e_A) - e_{G\setminus [S, F]}(P),
\]
which implies that
\[
\Omega_m(G \setminus [S, F]) \leq m|P_s| + m \sum_{A \in P'} |P_A| - e_{G\setminus S}(P_0) - i_s \leq mt_s + \Omega_m(G \setminus S) - i_s.
\]
Hence the assertion holds. \(\square\)

The following theorem is a generalization of Lemma 5.1 in [6] and provides a useful relationship between two parameters $\Omega_m(G \setminus S)$ and $\Omega_m(G \setminus [S, F])$.

**Lemma 3.2.** Let $G$ be a graph with a factor $F$. Let $c \in (1, \infty)$ be a real number and let $\xi : V(G) \to [0, m]$ be a real function in which for every $m$-tree-connected component $C$ of $F$, $\sum_{v \in V(C)} \xi(v) \geq m - \frac{m}{c-1}(|V(C) - 1| - \frac{1}{2}d_F(C))$. If $S \subseteq V(G)$, then
\[
\Omega_m(G \setminus [S, F]) \leq \Omega_m(G \setminus S) + \frac{1}{(c-1)}e_F(S) + \sum_{v \in S} \xi(v).
\]
In addition, $\Omega_m(G \setminus [S, F]) \leq \Omega_m(G \setminus S) + e_F(S) + \sum_{v \in S} \max\{0, m - d_F(v)\}$.

**Proof.** We denote by $P_s$ be the set of vertex sets of all $m$-tree-connected components of $G \setminus [S, F]$ whose vertices entirely lie in $S$. In the first statement, since $e_F(A) \geq m(|A| - 1)$ for every $A \in P_s$, we must have
\[
m - \frac{1}{2}d_F(A) \leq \frac{m(|A|-1)}{(c-1)} + \sum_{v \in A} \xi(v) \leq \frac{1}{(c-1)}e_F(A) + \sum_{v \in A} \xi(v),
\]
which implies that
\[ m|P_s| - i_s \leq \sum_{A \in P_s} (m - \frac{1}{2}d_F(A)) \leq \frac{1}{(c-1)} \sum_{A \in P_s} e_F(A) + \sum_{v \in S} \xi(v) \leq \frac{1}{(c-1)}e_F(S) + \sum_{v \in S} \xi(v). \]

Now, let us define \( P_s' \) to be the set of all \( A \in P_s \) with \(|A| \geq 2\) and define \( S' \) to be the set of all \( v \in S \) with \( \{v\} \in P_s \). In the second statement, since \( e_F(A) \geq m(|A| - 1) \geq m \) for every \( A \in P_s' \), we must have
\[ m|P_s'| + m|S'| - i_s \leq \sum_{A \in P_s'} e_F(A) + m|S'| - \sum_{v \in S'} d_F(v) + e_F(S') \leq \sum_{v \in S'} (m - d_F(v)) + e_F(S), \]
which implies that
\[ m|P_s| - i_s \leq \sum_{v \in S} \max\{0, m - d_F(v)\} + e_F(S). \]

Hence the two assertions follow from Lemma 3.1.

\[ \square \]

3.2 Extending \( m \)-sparse factors

The following lemma is a common generalization of Lemma 3.1 in [6] and Lemma 4.2 in [7].

**Lemma 3.3.** Let \( H \) be an \( m \)-sparse graph with a factor \( F \). If \( S \subseteq V(H) \) and \( \mathcal{F} = H \setminus E(F) \), then
\[ \sum_{v \in S} d_F(v) = \Omega_m(H \setminus [S,F]) - \Omega_m(H) + e_F(S). \]

**Proof.** By induction on the number of edges of \( \mathcal{F} \) which are incident to the vertices in \( S \). If there is no edge of \( \mathcal{F} \) incident to a vertex in \( S \), then the proof is clear. Now, suppose that there exists an edge \( e = uu' \in E(\mathcal{F}) \) with \(|S \cap \{u,u'\}| \geq 1\). Hence

1. \( \Omega_m(H) = \Omega_m(H \setminus e) - 1 \),
2. \( \Omega_m(H \setminus [S,F]) = \Omega_m((H \setminus e) \setminus [S,F]) \),
3. \( e_F(S) = e_{\mathcal{F}\setminus e}(S) + |S \cap \{u,u'\}| - 1 \),
4. \( \sum_{v \in S} d_F(v) = \sum_{v \in S} d_{\mathcal{F}\setminus e}(v) + |S \cap \{u,u'\}|. \)

Therefore, by the induction hypothesis on \( H \setminus e \) with the factor \( F \) the lemma holds.

\[ \square \]

The following theorem is essential in this section.

**Theorem 3.4.** Let \( G \) be a graph with a factor \( F \) and let \( h \) be a nonnegative integer-valued function on \( V(G) \). Define \( \mathcal{A} \) to be the set of all \( m \)-sparse factors \( \mathcal{F} \) of \( G \) whose edges join different \( m \)-tree-connected components of \( F \) and for each vertex \( v \), \( d_F(v) \leq h(v) \). If for all \( S \subseteq V(G) \),
\[ \Omega_m(G \setminus [S,F]) \leq \sum_{v \in S} h(v) + m - \max_{\mathcal{F} \in \mathcal{A}} e_F(S), \]
then \( G \) has an \( m \)-tree-connected factor \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq h(v) + d_F(v) \)
Proof. First, suppose that $F$ is $m$-sparse. Let $H$ be an $m$-sparse factor of $G$ containing $F$ with $te(H, h + d_F) = 0$ and with the minimum $\Omega_m(H)$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 2.4. Put $\mathcal{F} = H \setminus E(F)$ so that $\mathcal{F} \in \mathcal{A}$. By Lemma 3.3 and Theorem 2.4,

$$\sum_{v \in S} h(v) = \sum_{v \in S} d_{\mathcal{F}}(v) = \Omega_m(H \setminus [S, F]) - \Omega_m(H) + e_{\mathcal{F}}(S),$$

and so

$$\Omega_m(H) = \Omega_m(G \setminus [S, F]) + e_{\mathcal{F}}(S) - \sum_{v \in S} h(v) \leq m.$$ 

Hence $\Omega_m(H) = m$ and the theorem holds. Now, suppose that $F$ is not $m$-sparse. Remove some of the edges of the $m$-tree-connected components of $F$ until the resulting graph $F'$ becomes $m$-sparse such that their $m$-tree-connected components have the same vertices. It is enough, now, to apply the theorem on $F'$ and finally add the edges of $E(F) \setminus E(F')$ to that explored $m$-tree-connected factor. \qed

The following corollary shows an interesting application of Theorem 3.4. The special case $m = 1$ of this result says that every linear forest of every 1-tough graph can be extend to a spanning tree with maximum degree at most 3.

**Corollary 3.5.** Let $G$ be a graph with an $m$-sparse factor $F$ satisfying $\Delta(F) \leq 2m$. Then $F$ can be extended to an $m$-tree-connected factor $H$ satisfying $\Delta(H) \leq 2m + 1$, if for all $S \subseteq V(G)$,

$$\Omega_m(G \setminus S) \leq \frac{1}{2}|S| + m.$$ 

Proof. For each vertex $v$, define $f(v) = 2m + 1 - d_F(v)$ so that $f(v) \geq 1$. Let $S$ be a subset of $V(G)$ and let $P$ be the set of vertex sets of $m$-tree-connected components of $G \setminus [S, F]$ whose vertices entirely lie in $S$. By the assumption and Lemma 3.1,

$$\Omega_m(G \setminus [S, F]) \leq \Omega_m(G \setminus S) + m|P| - \frac{1}{2} \sum_{A \in P} d_F(A) \leq \frac{1}{2}|S| + m - \sum_{A \in P} \left(\frac{1}{2}d_F(A) - m\right).$$

Since $F$ is $m$-sparse, for every $A \in P$, we must have

$$0 \leq m(|A| - 1) - e_F(A) = \sum_{v \in A} \left(m - \frac{1}{2}d_F(v)\right) + \frac{1}{2}d_F(A) - m = \sum_{v \in A} \left(\frac{1}{2}f(v) - \frac{1}{2}\right) + \frac{1}{2}d_F(A) - m,$$

which implies that

$$\frac{1}{2}|S| - \sum_{A \in P} \left(\frac{1}{2}d_F(A) - m\right) \leq \frac{1}{2}|S| + \sum_{v \in A \in P} \left(\frac{1}{2}f(v) - \frac{1}{2}\right) \leq \sum_{v \in S} \frac{1}{2}f(v).$$

Let $\mathcal{F}$ be an arbitrary $m$-sparse factor of $G$ whose edges join different $m$-tree-connected components of $F$ and for each vertex $v$, $d_{\mathcal{F}}(v) \leq f(v)$. Obviously, $e_{\mathcal{F}}(S) \leq \sum_{v \in S} \frac{1}{2}d_{\mathcal{F}}(v) \leq \sum_{v \in S} \frac{1}{2}f(v)$. Therefore,

$$\Omega_m(G \setminus [S, F]) \leq \sum_{v \in S} \frac{1}{2}f(v) + m \leq \sum_{v \in S} f(v) + m - e_{\mathcal{F}}(S).$$

Hence the assertion follows from Theorem 3.4. \qed
3.3 Extending factors with large tree-connected components

The following theorem provides a stronger version for Theorem 1.2 which is motivated by the main result in [6].

**Theorem 3.6.** Let $G$ be a graph with a factor $F$ and let $h$ be a nonnegative integer-valued function on $V(G)$. Let $c \in [2, \infty)$ be a real number and let $\xi : V(G) \to [0, m]$ be a real function in which for every $m$-tree-connected subgraph $C$ of $F$, $\sum_{v \in V(C)} \xi(v) \geq m - \frac{m}{c-1}(\lfloor |V(C)| - 1 \rfloor) - \frac{1}{2}d_F(C)$. If for all $S \subseteq V(G)$, $\Omega_m(G \setminus S) < \sum_{v \in S} (h(v) - \frac{m}{c} - \xi(v)) + m+1 + \frac{1}{c-1}\Omega_m(G[S]) - \frac{c-2}{c-1}\min\{\lfloor \sum_{v \in S} \frac{h(v)}{2}\rfloor, m|S| - \Omega_m(G[S])\}$, then $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v)$

**Proof.** First, suppose that $F$ is $m$-sparse. Let $H$ be an $m$-sparse factor of $G$ containing $F$ with $te(H, h + d_F) = 0$ and with the minimum $\Omega_m(H)$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 2.4. Put $F = H \setminus E(F)$. By Lemma 3.3 and Theorem 2.4,

$$\sum_{v \in S} h(v) = \sum_{v \in S} d_F(v) = \Omega_m(H \setminus [S, F]) - \Omega_m(H) + e_F(S),$$

and so

$$\Omega_m(H) = \Omega_m(G \setminus [S, F]) - \sum_{v \in S} h(v) + e_F(S). \tag{1}$$

Also, by Lemma 3.2,

$$\Omega_m(G \setminus [S, F]) \leq \Omega_m(G \setminus S) + \frac{1}{c-1}e_F(S) + \sum_{v \in S} \xi(v). \tag{2}$$

Since $e_F(S) + e_F(S) = e_H(S) \leq m|S| - \Omega_m(G[S]),$

$$e_F(S) + \frac{1}{c-1}e_F(S) \leq \frac{c-2}{c-1}e_F(S) + \frac{1}{c-1}(m|S| - \Omega_m(G[S])).$$

In addition, since $e_F(S) \leq \frac{1}{2} \sum_{v \in S} d_F(v) = \lfloor \frac{1}{2} \sum_{v \in S} h(v) \rfloor$, we must have

$$e_F(S) + \frac{1}{c-1}e_F(S) \leq \frac{c-2}{c-1}\min\{\lfloor \frac{1}{2} \sum_{v \in S} h(v) \rfloor, m|S| - \Omega_m(G[S])\} + \frac{1}{c-1}(m|S| - \Omega_m(G[S])). \tag{3}$$

Therefore, Relations (1), (2), and (3) can conclude that

$$\Omega_m(H) \leq \Omega_m(G \setminus S) - \sum_{v \in S} (h(v) - \frac{m}{c} - \xi(v)) - \frac{1}{c-1}\Omega_m(G[S]) + \frac{c-2}{c-1}\min\{\lfloor \sum_{v \in S} \frac{h(v)}{2}\rfloor, |S| - \Omega_m(G[S])\} < 2.$$ 

Hence $\Omega_m(H) = m$ and the theorem holds. Now, suppose that $F$ is not $m$-sparse. Remove some of the edges of the $m$-tree-connected components of $F$ until the resulting graph $F'$ becomes $m$-sparse such that their $m$-tree-connected components have the same vertices. For every $m$-tree-connected component $F[A]$ of $F'$, we still have $d_{F'}(A) = d_F(A)$. It is enough, now, to apply the theorem on $F'$ and finally add the edges of $E(F) \setminus E(F')$ to that explored $m$-tree-connected factor. \qed
Remark 3.7. In Theorem 3.6 we could select $c$ and $\xi$ depending on $S$. In addition, we could replace the estimation $\frac{c}{2m}\min\{\lfloor\sum_{v \in S} \frac{h(v)}{2}\rfloor, m|S| - \Omega_m(G[S]) - e_F(S)\}$ when $F$ is $m$-sparse.

When $\xi = 0$, the theorem becomes much simpler as the next corollary.

Corollary 3.8. Let $G$ be a graph with a factor $F$ of which every $m$-tree-connected component $C$ satisfies $|V(C)| + \frac{c}{2m}d_F(C) \geq c$ and $c \geq 2m + 1$. Let $h$ be a nonnegative integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$\Omega_m(G \setminus S) < \sum_{v \in S} \left(\frac{c}{2c - 2}h(v) - \frac{m}{c - 1}\right) + m + 1 + \frac{1}{c - 1}\Omega_m(G[S]),$$

then $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v)$.

Proof. By the assumption, for every $m$-tree-connected component $C$ of $F$, we must have $0 \geq m - \frac{m}{c-1}(|V(C)| - 1) - \frac{1}{2}d_F(C)$. Thus it is enough to apply Theorem 3.6 with $\xi(v) = 0$. \hfill $\Box$

The following corollary is an improvement of Corollary 3.5 for graphs with higher toughness.

Corollary 3.9. Let $G$ be a simple graph with a factor $F$ satisfying $\Delta(F) \leq 2m$. Then $F$ can be extended to an $m$-tree-connected factor $H$ satisfying $\Delta(H) \leq 2m + 1$, if for all $S \subseteq V(G)$,

$$\Omega_m(G \setminus S) \leq \frac{1}{4m}|S| + m.$$

Proof. For each vertex $v$, define $h(v) = 2m + 1 - d_F(v)$ and $\xi(v) = \frac{1}{|V(C_v)|}\max\{0, m - \frac{1}{2}(|V(C_v)| - 1) - \frac{1}{2}d_F(v, C_v)\}$, where $C_v$ is the $m$-tree-connected component of $F$ including $v$, and also $d_F(v, C_v)$ denotes the number of edge of $F$ incident to $v$ having exactly one vertex in $C$. According to this definition, it is easy to see that $\sum_{v \in V(C)} \xi(v) \geq m - \frac{1}{2}(|V(C)| - 1) - \frac{1}{2}d_F(C)$, where $C$ is a $m$-tree-connected component of $F$. Since $F$ has no multiple edges, we must have $d_F(v, C) \geq d_F(v) - (|V(C_v)| - 1)$. Thus

$$\xi(v) \leq \frac{1}{|V(C_v)|}\max\{0, m - \frac{1}{2}d_F(v)\} = \frac{1}{|V(C_v)|}(m - \frac{1}{2}d_F(v)).$$

This implies that

$$\frac{2m + 1}{4m}h(v) - \frac{1}{2} - \xi(v) \geq \frac{2m + 1}{4m}(2m + 1 - d_F(v)) - \frac{1}{|V(C_v)|}(m - \frac{1}{2}d_F(v)) \geq \frac{1}{4m}.$$ 

Therefore, by applying Theorem 3.6 with $c = 2m + 1$, the factor $F$ can be extended to an $m$-tree-connected factor $H$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v) = 2m + 1$. Hence the assertion holds. \hfill $\Box$

4 Highly tree-connected $\{f, f+1\}$-factors including given $f$-factors

4.1 Tree-connected $\{f, f+1\}$-factors

When we consider the special cases $h(v) \leq 1$, Corollary 3.8 becomes simpler as the following result. This theorem plays an essential role of the results of this section.
Theorem 4.1. Let $G$ be a graph with a factor $F$ of which every $m$-tree-connected component $C$ satisfies $|V(C)| + \frac{a}{2m}d_F(C) \geq c$ and $c \geq 2m + 1$. If for all $S \subseteq V(G)$,
\[ \omega(G \setminus S) \leq \frac{c - 2m}{2m(c - 1)}|S| + 1 \]
then $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \in \{d_F(v), d_F(v) + 1\}$, and $d_H(u) = d_F(u)$ for an arbitrary given vertex $u$.

Proof. Let $G'$ be the union of $m$ copies of $G$ with the same vertex set. It is easy to check that $\frac{1}{m}\Omega_m(G' \setminus S) = \omega(G \setminus S)$ for every $S \subseteq V(G)$. Define $h(u) = 0$ and $h(v) = 1$ for each vertex $v$ with $v \neq u$. By Corollary 3.8, the graph $G'$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq h(v) + d_F(v)$. According to the construction, the graph $H$ must have no multiple edges of $E(G') \setminus E(F)$. Hence $H$ itself is a factor of $G$ and the proof is completed. $\square$

The following corollary shows an application of Theorem 4.1.

Corollary 4.2. Let $G$ be a simple graph and let $f$ be a positive integer-valued function on $V(G)$ with $f \geq a \geq 2m$, where $a$ is a positive integer. Assume that $G$ contains a factor $F$ such that for all vertices $v$, $d_H(v) = f(v)$, except possibly for a vertex $u$ with $d_H(u) = f(u) + 1$. If for all $S \subseteq V(G)$,
\[ \omega(G \setminus S) \leq \frac{a+1-2m}{2ma}|S| + 1, \]
then $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \in \{f(v), f(v)+1\}$.

Proof. Since $F$ is simple, it is easy to check that $d_F(C) \geq \delta(F) - (|V(C)| - 1)$, where $C$ is an $m$-tree-connected subgraph. Since $a \geq 2m$, we must have $|V(C)| + \frac{a}{2m}d_F(C) \geq |V(C)| + (\delta(F) + 1 - |V(C)|) \geq a + 1$. Thus by applying Theorem 4.1 with $c = a + 1$, the graph $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq d_F(v) + 1$, and also $d_H(u) = d_F(u)$. This implies that $H$ is an $m$-tree-connected $\{f, f + 1\}$-factor. $\square$

Corollary 4.3. Let $G$ be a graph, let $b$ be a positive integer, and let $f$ be a positive integer-valued function on $V(G)$ with $2m \leq f \leq b$. If $G$ is $b^2$-tough and $|V(G)| \geq b^2$, then it has an $m$-tree-connected $\{f, f + 1\}$-factor.

Proof. We may assume that $G$ is a simple graph, by deleting multiple edges from $G$ (if necessary). By Theorem 1.4, the graph $G$ has a factor $F$ such that for each vertex $v$, $d_F(v) = f(v)$, except possibly for a vertex $u$ with $d_F(u) = f(u) + 1$. By Corollary 4.2, the graph $G$ has an $m$-tree-connected $\{f, f + 1\}$-factor. $\square$

4.2 Tree-connected $\{a, a + 1\}$-factors in $a$-tough graphs

Enomoto, Jackson, Katerinis, and Saito (1985) [5] showed that every $r$-tough graph $G$ of order at least $r + 1$ with $r|V(G)|$ even admits an $r$-factor. For the case that $r|V(G)|$ is odd, the same arguments can imply that
the graph $G$ admits a factor whose degrees are $r$, except for a vertex with degree $r + 1$. A combination of Theorem 4.1 and this result can conclude the next results.

**Corollary 4.4.** Every $rt$-tough graph $G$ of order at least $r + 1$ has an $m$-tree-connected $\{r, r + 1\}$-factor, where $r \geq 2m$ and $t = \max\{1, \frac{2m}{r + 1 - 2m}\}$.

**Proof.** We may assume that $G$ is a simple graph, by deleting multiple edges from $G$ (if necessary). Let $F$ be a factor of $G$ such that each of whose vertices has degree $r$, except for at most one vertex $u$ with degree $r + 1$ [5]. By applying Corollary 4.2, the graph $G$ has an $m$-tree-connected $\{r, r + 1\}$-factor. □

**Corollary 4.5.** Every $4m^2$-tough graph of order at least $2m + 1$ has an $m$-tree-connected $\{2m, 2m + 1\}$-factor.

**Proof.** Apply Corollary 4.4 with $r = 2m$. □

**Corollary 4.6.** ([3, 4], see [6]) Every $\max\{r, 4\}$-tough graph of order at least $r + 1$ admits a connected $\{r, r + 1\}$-factor, where $r \geq 2$.

**Proof.** Apply Corollary 4.4 with $m = 1$. □

### 4.3 2-edge-connected $\{2f, 2f + 1\}$-factors

The following theorem gives a sufficient toughness condition for extending 2-factors with girth at least five to 2-connected $\{2, 3\}$-factors. Ellingham and Zha [4] proved that 2-factors with girth at least three of 4-tough graphs can be extended to connected $\{2, 3\}$-factors.

**Theorem 4.7.** Let $G$ be a graph with a factor $F$ of which every component is 2-edge-connected and contains at least $c$ vertices with $c \geq 5$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \frac{c - 4}{4c - 4} |S| + 1,$$

then $G$ has a 2-edge-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \in \{d_F(v), d_F(v) + 1\}$.

**Proof.** Duplicate the edges of $F$ in $G$ and call the resulting graphs $F'$ and $G'$. Obviously, every 2-tree-connected component of $F'$ contains at least $c$ vertices. By applying Theorem 4.1 with $m = 2$, the graph $G'$ has a 2-tree-connected factor $H'$ containing $F'$ such that for each vertex $v$, $d_{H'}(v) \leq d_{F'}(v) + 1$. Remove a copy of $F$ from $H'$ and call the resulting graph $H$. It is easy to check that $H/F$ is still 2-tree-connected and for each vertex $v$, $d_H(v) \leq d_{F'}(v) + 1$. By the assumption, every component of $F$ is 2-edge-connected. Hence $H$ itself is 2-edge-connected and the proof is completed. □
**Corollary 4.8.** Let $G$ be a simple graph and let $f$ be an integer-valued function on $V(G)$ with $f \geq 2$. If $G$ has a $2f$-factor and for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \frac{1}{16}|S|+1$, then $G$ has a 2-edge-connected $(2f, 2f+1)$-factor including a $2f$-factor.

**Proof.** Let $F$ be a $2f$-factor of $G$. Note that every component of $F$ must be Eulerian which is 2-edge-connected. Since $F$ is simple and has minimum degree at least four, every component of it contains at least five vertices. Thus by applying Theorem 4.7 with $c = 5$, the graph $F$ can be extended to a 2-edge-connected $(2f, 2f+1)$-factor $H$. \hfill $\Box$

**Corollary 4.9.** Let $G$ be a simple graph having a $2$-factor $F$ with girth at least five. Then $G$ has a 2-connected $(2, 3)$-factor containing $F$. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \frac{1}{16}|S|+1$.

**Proof.** By applying Theorem 4.7 with $c = 5$, the graph $F$ can be extended to a 2-edge-connected $(2, 3)$-factor $H$. Since $\Delta(H) \leq 3$, the graph $H$ has no cut vertices, which can complete the proof. \hfill $\Box$

**Corollary 4.10.** Every 16-tough graph $G$ of girth at least five has a 2-connected $(2r, 2r+1)$-factor.

**Proof.** We may assume that $G$ is a simple graph, by deleting multiple edges from $G$ (if necessary). First, we choose a $2r$-factor $F$ of $G$ [5], and next we choose a 2-factor $F_0$ of $F$. Since $F_0$ is simple, it has girth at least five. Thus by Corollary 4.9, the factor $F_0$ can be extended to a 2-connected $(2, 3)$-factor $H$. It is easy to check that $H \cup F$ is the desired factor we are looking for. \hfill $\Box$

## 5 Highly tree-connected $\{f, f+k\}$-factors

Our aim in this section is to give a sufficient toughness condition for the existence of tree-connected $\{f, f+k\}$-factors. For this purpose, we need to apply the second lemma in our proof, which shows that tough enough graphs can have sufficiently large bipartite index.

**Lemma 5.1.**[9] Every $2m$-tree-connected graph has an $m$-tree-connected bipartite factor.

**Lemma 5.2.** Let $G$ be a graph with a $(2k-2)$-tree-connected factor $F$ in which $|V(G)| \geq 4k-2$. If $G$ is $(4k-3)$-tough, then it has a matching $M$ of size $k-1$ such that $bi(F \cup M) \geq k-1$.

**Proof.** Let $k_0 = k-1$ and $t = 4k_0 + 1$. By Lemma 5.1, there is a bipartition $X, Y$ of $V(G)$ such that $F[X, Y]$ is $k_0$-tree-connected. We may assume that $|X| \geq |V(G)|/2$. Let $M$ be a matching of $G[X]$ with the maximum size so that $G[X] \setminus V(M)$ consists of isolated vertices. Define $S = Y \cup V(M)$. Since $t$-iso-tough, we must have

$$|X| - 2|E(M)| = \omega(G \setminus S) \leq \frac{1}{t}|S| + 1 = \frac{1}{t}(|Y| + 2|E(M)|) \leq \frac{1}{t}(|X| + 2|E(M)|),$$
which implies that $|E(M)| \geq \frac{b-1}{2b+2}|X| \geq \frac{b-1}{4b+4}|V(G)| \geq k_0$. Let $e_1, \ldots, e_{k_0}$ be $k_0$ edges of $M$ and decompose $F[X, Y]$ into $k_0$ disjoint spanning trees $T_1, \ldots, T_{k_0}$. Obviously, $T_i + e_i$ is not bipartite and so contains an odd cycle. This implies that $F \cup M$ has $k_0$ odd disjoint cycles, and hence it has the bipartite index at least $k_0$. Hence the proof is completed. \qed

Now, are in a position to prove the main result of this section.

**Theorem 5.3.** Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$ satisfying $3m + 2m_0 + 6k < f + k \leq b$ and $m + m_0 < k$, where $k$, $b$, $m$, and $m_0$ are four nonnegative integers. If $G$ is $4b^2$-tough and $|V(G)| \geq 4b^2$, then $G$ has an $m$-tree-connected factor $H$ such that its complement is $m_0$-tree-connected and for each vertex $v$,

$$d_H(v) \in \{f(v), f(v) + k\},$$

provided that $(k-1) \sum_{v \in V(G)} f(v)$ is even.

**Proof.** For each vertex $v$, define $h(v) = f(v) + k - m - 1$. Since $f + k > 3m + 2m_0 + 6k$, we must have $2m' \leq 2h(v) \leq b'$, where $m' = 2m + 2m_0 + 6k$ and $b' = 2b - 2m - 2$. By the assumption, $|V(G)| \geq 4b^2 \geq (b')^2$. Thus by Corollary 4.3, the graph $G$ has an $m'$-tree-connected $\{2h, 2h + 1\}$-factor $G'$. Since $|V(G)| \geq 4b^2 \geq 4k - 2$ and $m' \geq 2k - 2$, by Lemma 5.2, there is a matching $M$ of size $k - 1$ such that $bi(G' \cup M) \geq k - 1$. Let $G_0 = G' \cup M$. Note that for each vertex $v$, $2h(v) \leq d_{G_0}(v) \leq 2h(v) + 2$. Since $m + m_0 < k$, we must have $f(v) + m_0 \leq h(v) \leq \frac{1}{2}d_{G_0}(v) \leq h(v) + 1 = f(v) + k - m$. Therefore, by Theorem 1.5, the graph $G_0$ has an $m$-tree-connected $\{f, f + k\}$-factor $H$ such that its complement is $m_0$-tree-connected and so does $G$. \qed

### 6 Highly tree-connected $\{a, b\}$-factors

When $f$ is a constant function, Theorem 5.3 becomes simpler as the following version. By replacing Corollary 4.4 in the proof, we also improve the needed toughness by a linear bound.

**Theorem 6.1.** Let $G$ be a graph and let $a$ and $b$ be two positive integers with $ab|V(G)|$ even. Let $m$ and $m_0$ be two nonnegative integers with

$$a + m + m_0 < b < \frac{1}{5}(6a - 3m - 2m_0).$$

If $G$ is max$\{2b, 256(b-a)^2\}$-tough and $|V(G)| \geq 2b$, then $G$ admits an $m$-tree-connected $\{a, b\}$-factor such that its complement is $m_0$-tree-connected.

**Proof.** Let $k = b - a$ and $h = b - m - 1$. Since $b > 3m + 2m_0 + 6k$, we must have $m' \leq h$, where $m' = 2m + 2m_0 + 6k \geq 8k$. By the assumption, $|V(G)| \geq 2b \geq 2h + 1$. Since $G$ is max$\{2b, 4(m')^2\}$-tough, by applying Corollary 4.4 with $r = 2h$, the graph $G$ has an $m'$-tree-connected $\{2h, 2h + 1\}$-factor $G'$. Since $|V(G)| \geq 2b \geq 4k - 2$ and $m' \geq 2k - 2$, by Lemma 5.2, there is a matching $M$ of size $k - 1$ such that
Proof. By Theorem 1.5, the graph $G_0$ has an $m$-tree-connected $\{a, b\}$-factor $H$ such that its complement is $m_0$-tree-connected and so does $G$. \hfill \square

Corollary 6.2. Let $G$ be a graph and let $a$ and $b$ be two positive integers with $ab|V(G)|$ even and $a < b \leq \frac{1}{6}(6a + 1)$. If $G$ is max$\{2b, 64(b - a)^2\}$-tough and $|V(G)| \geq b + 1$, then $G$ admits a connected $\{a, b\}$-factor.

Proof. If $|V(G)| \leq 2b$, then $G$ must be the complete graph and so it is easy to see that it has a connected $r$-factor, where $r \in \{a, b\}$ and $r|V(G)|$ is even. We may assume that $|V(G)| > 2b$. For the case $b = a + 1$, the assertion follows from Corollary 4.4. For the case $b \geq a + 2$, the assertion follows Theorem 6.1 by setting $m = 1$ and $m_0 = 0$. \hfill \square

In the following, we are going to refine the special case $b = a + 2$ of Corollary 6.2. For this purpose, we need to replace the following lemma in the proof.

Lemma 6.3. ([10]) Let $G$ be a 4-tree-connected graph and let $f$ be a positive integer-valued function on $V(G)$ satisfying $\sum_{v \in V(G)} f(v) \geq 2$. If for each vertex $v$, $2f(v) \leq d_G(v) \leq 2f(v) + 2$, then $G$ has a connected factor $H$ such that for each vertex $v$, $d_H(v) \in \{f(v), f(v) + 2\}$.

The following theorem provides a natural generalization for Corollary 4.6.

Theorem 6.4. Every max$\{2a, 64\}$-tough graph $G$ of order at least $a + 1$ with $a|V(G)|$ even admits a connected $\{a, a + 2\}$-factor, where $a \geq 2$.

Proof. The special case $a = 2$ was proved in [7]. First suppose that $a \geq 4$. If $|V(G)| \leq 2a$, then $G$ must be complete and so it admits a connected $a$-factor. We may therefore assume that $|V(G)| > 2a$. Thus by Corollary 4.4, the graph $G$ has a 4-tree-connected $\{2a, 2a + 1\}$-factor $G'$. Now, by Lemma 6.3, the graph $G'$ has a connected $\{a, a + 2\}$-factor and so does $G$. Now, suppose that $a = 3$. Again, by Corollary 4.4, the graph $G$ has a 3-tree-connected $\{6, 7\}$-factor $G'$. By counting the number of edges of $G'$, one can conclude that there is an edge $e$ of $G'$ such that $G' - e$ is 3-tree-connected. This can imply that $G'$ is $(2, 1)$-partition-connected which means that it can be decomposed into a 2-tree-connected factor $T$ and a factor $F$ having an orientation with minimum out-degree at least 1. Thus by Corollary 5.4 in [10], the graph $G'$ has a connected $\{3, 5\}$-factor and so does $G$. \hfill \square

Corollary 6.5. Every max$\{4a, 64\}$-tough graph $G$ of order at least $2a + 1$ admits a spanning closed trail meeting each vertex $a$ or $a + 1$ times, where $a \geq 1$.

Proof. By Theorem 6.4, the graph $G$ has a connected $\{2a, 2a + 2\}$-factor and so it admits a spanning closed trail meeting each vertex $a$ or $a + 1$ times. \hfill \square
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