Dynamics of Connected Rigid Bodies in a Perfect Fluid

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Abstract—This paper presents an analytical model and a geometric numerical integrator for a system of rigid bodies connected by ball joints, immersed in an irrotational and incompressible fluid. The rigid bodies can translate and rotate in three-dimensional space, and each joint has three rotational degrees of freedom. This model characterizes the qualitative behavior of three-dimensional fish locomotion. A geometric numerical integrator, referred to as a Lie group variational integrator, preserves Hamiltonian structures of the presented model and its Lie group configuration manifold. These properties are illustrated by a numerical simulation for a system of three connected rigid bodies.

I. INTRODUCTION

Fish locomotion has been investigated in the fields of biomechanics and engineering (see [1] and references therein). This is a challenging problem as it involves interaction of a deformable fish body with an unsteady fluid, through which an internal muscular force of the fish is translated into an external propulsive force exerted on the fluid.

Various mathematical models of fish locomotion have been formulated. A quasi-static model based on a steady state flow theory is developed in [2], and an elastic plate model that treats a fish as an elongated slender body is studied in [3], [4], [5]. The effects of body thickness for the slender body model are considered in [6]. Numerical models involving computational fluid dynamics techniques appear in [7], [8]. The body of a fish is modeled as a planar articulated rigid body in [9], [10], [11].

The planar articulated rigid body model has become popular in engineering area, as it depicts underwater robotic vehicles that move and steer by changing their shape [12], [13]. Furthermore, if it is assumed that the ambient fluid is incompressible and irrotational, then equations of motion of the articulated rigid body can be derived without explicitly incorporating fluid variables [9]. The effect of the fluid is accounted by added inertia terms of the rigid body. This model is known to characterize the qualitative behavior of fish swimming properly [9]. Based on this assumption, optimal shape changes of a planar articulated body to achieve a desired locomotion has been studied in [14], [15].

By following [9], [10], [11], we consider a system of connected rigid bodies immersed in an incompressible and irrotational fluid, and we first develop an analytical model of it. The contribution of this paper is that the connected rigid bodies can freely translate and rotate in three-dimensional space, and each joint has three rotational degrees of freedom. This is important for understanding the locomotion of a fish with a blunt body and a large caudal fin.

The second part of this paper deals with a geometric numerical integrator of connected rigid bodies in a perfect fluid. Geometric numerical integration is concerned with developing numerical integrators that preserve geometric features of a system, such as invariants, symmetry, and reversibility [16]. It is critical for a numerical simulation of Hamiltonian systems on a Lie group to preserve both the symplectic property of Hamiltonian flows and the Lie group structure [17]. A geometric numerical integrator, referred to as a Lie group variational integrator, has been developed for a Hamiltonian system on an arbitrary Lie group in [18].

A system of connected rigid bodies is a Hamiltonian system, and its configuration manifold is expressed as a product of the special Euclidean group and copies of the special orthogonal group. This paper develops a Lie group variational integrator for the connected rigid bodies in a perfect fluid based on the results presented in [18]. The proposed geometric numerical integrator preserves symplecticity and momentum maps, and exhibits desirable energy properties. It also respects the Lie group structure of the configuration manifold, and avoids the singularities and complexities associated with local coordinates.

In summary, this paper develops an analytical model and a geometric numerical integrator for a system of connected rigid bodies in a perfect fluid. These provide a three-dimensional mathematical model and a reliable numerical simulation tool that characterizes the qualitative properties of fish locomotion.

This paper is organized as follows. A system of connected rigid bodies immersed in a perfect fluid is described in Section III. An analytical model and a Lie group variational integrator are developed in Section III and in Section IV respectively, followed by a numerical example in Section V.

II. CONNECTED RIGID BODIES IMMERSED IN A PERFECT FLUID

Consider three connected rigid bodies immersed in a perfect fluid. We assume that these rigid bodies are connected by a ball joint that has three rotational degrees of freedom, and the fluid is incompressible and irrotational. We also assume each body has neutral buoyancy: the mass of the
body equals the mass of the fluid it displaces. This model is illustrated by Fig. II.

We choose a reference frame and three body-fixed frames. The origin of each body-fixed frame is located at the mass center of the rigid body and it is aligned along the principal axes. Define

\[ R_i \in \text{SO}(3) \] Rotation matrix from the \( i \)-th body-fixed frame to the reference frame

\[ \Omega_i \in \mathbb{R}^3 \] Angular velocity of the \( i \)-th body, represented in the \( i \)-th body-fixed frame

\[ x \in \mathbb{R}^3 \] Vector from the origin of the reference frame to the mass center of the 0-th body, represented in the reference frame

\[ d_{ij} \in \mathbb{R} \] Vector from the mass center of the \( i \)-th body to the ball joint connecting the \( i \)-th body with the \( j \)-th body, represented in the \( i \)-th body-fixed frame

\[ m_i^b \in \mathbb{R} \] Mass of the \( i \)-th body

\[ J_i^b \in \mathbb{R}^{3\times3} \] Inertia matrix of the \( i \)-th body

for \( i, j \in \{0, 1, 2\} \).

A configuration of this system can be described by the location of the mass center of the central body, and the attitude of each rigid body with respect to the reference frame. So, the configuration manifold is \( G = \text{SE}(3) \times \text{SO}(3) \times \text{SO}(3) \), where \( \text{SO}(3) = \{ R \in \mathbb{R}^{3\times3} | R^T R = I, \det R = 1 \} \), and \( \text{SE}(3) = \text{SO}(3) \odot \mathbb{R}^3 \).

The attitude kinematics equation is given by

\[ \dot{R}_i = R_i \hat{\Omega}_i \]

for \( i \in \{0, 1, 2\} \), where the hat map \( \hat{\cdot} : \mathbb{R}^3 \rightarrow \text{so}(3) \) is defined such that \( \hat{xy} = x \times y \) for any \( x, y \in \mathbb{R}^3 \).

III. CONTINUOUS-TIME ANALYTICAL MODEL

In this section, we develop continuous-time equations of motion for a system of connected rigid bodies in a perfect fluid. As the fluid is irrotational, equations of motion can be expressed without explicitly incorporating fluid variables, and the effects of the ambient fluid is encountered by added inertia terms [9]. To simplify expressions for the added inertia terms, we assume each body is an ellipsoid.

We first find an expression for the Lagrangian of the system, and substitute it into Euler-Lagrange equations.

A. Lagrangian

The total kinetic energy of connected rigid bodies immersed in a fluid can be written as the sum of the kinetic energy of the rigid bodies \( T_{B_i} \) and the kinetic energy of the fluid \( T_f \):

\[ T = \sum_{i=0}^{2} T_{B_i} + T_f. \]

Kinetic energy of rigid bodies: Let \( V_i \in \mathbb{R}^3 \) be the velocity of the mass center of the \( i \)-th body represented in the \( i \)-th body-fixed frame for \( i \in \{0, 1, 2\} \). Since \( \hat{x} \) represents the velocity of the 0-th rigid body in the reference frame, we obtain

\[ V_0 = R_0^T \hat{x}. \] (1)

The location of the mass center of the first rigid body can be written as \( x + R_0d_{01} - R_1d_{10} \) with respect to the reference frame. Therefore, \( V_1 \) is given by

\[ V_1 = R_1^T (x + R_0\hat{\Omega}_0d_{01} - R_1\hat{\Omega}_1d_{10}) = R_1^T \hat{x} - R_1^T R_0\hat{\Omega}_0d_{01} + d_{10}\hat{\Omega}_1. \] (2)

Similarly,

\[ V_2 = R_2^T \hat{x} - R_2^T R_0d_{02}\hat{\Omega}_0 + d_{20}\hat{\Omega}_2. \] (3)

The kinetic energy of rigid bodies is given by

\[ T_{B_i} = \sum_{i=0}^{2} \frac{1}{2} m_i^b V_i \cdot V_i + \frac{1}{2} \Omega_i \cdot J_i^b \Omega_i. \] (4)

Kinetic energy of fluid: The kinetic energy of the fluid is given by

\[ T_f = \frac{1}{2} \int_{\mathcal{F}} \rho_f ||u||^2 \, dv, \]

where \( \rho_f \) is the density of the fluid, \( u \) is the velocity field of the fluid and \( dv \) is the standard volume element in \( \mathbb{R}^3 \). Since the flow is irrotational, the velocity field can be expressed as a gradient of a potential. Under these conditions, the kinetic energy of the fluid can be written as

\[ T_f = \sum_{i,j=0}^{2} \frac{1}{2} M_{ij}^f V_i \cdot V_j + \frac{1}{2} \Omega_i \cdot J_{ij}^f \Omega_j + D_{ij}^f V_i \cdot \Omega_j, \]

where \( M_{ij}^f, J_{ij}^f, D_{ij}^f \in \mathbb{R}^{3\times3} \) are referred to as added inertia matrices [19]. Here we assume that the flow near one rigid body is not affected by other rigid bodies: the added inertia matrices \( M_{ij}^f, J_{ij}^f, D_{ij}^f \) are equal to zero when \( i \neq j \). The resulting model captures the qualitative properties of the
interaction between rigid body dynamics and fluid dynamics correctly [9], [14].

Expressions for added inertia matrices for an ellipsoidal body are derived in [20]. Let \( l_q \in \mathbb{R} \) be the length of the \( q \)-th principal axis of an ellipsoid for \( q \in \{1, 2, 3\} \). Define constants
\[
\gamma_q = l_1 l_2 l_3 \int_0^\infty \frac{d\nu}{(l_q^2 + \nu)(l_q^2 + \nu)(l_q^2 + \nu)}
\]
for \( q \in \{1, 2, 3\} \) and
\[
\lambda_1 = \frac{1}{5} m^b \left( \frac{(l_2^2 - l_3^2)^2 (\gamma_3 - \gamma_2)}{2(l_2^2 - l_3^2) + (l_2^2 + l_3^2)(\gamma_2 - \gamma_3)} \right).
\]
Constants \( \lambda_2 \) and \( \lambda_3 \) are given by cyclic permutations of this expression. Then, the added inertia matrices of the ellipsoid are given by
\[
M^f = m^b \text{diag} \left[ \frac{\gamma_1}{2 - \gamma_1}, \frac{\gamma_2}{2 - \gamma_2}, \frac{\gamma_3}{2 - \gamma_3} \right],
\]
\[
J^f = \text{diag} [\lambda_1, \lambda_2, \lambda_3],
\]
\[
D^f = 0.
\]
Using these expressions, we find added inertia matrices \( M^f, J^f \) for each rigid body.

In summary, the kinetic energy of the fluid surrounding ellipsoidal rigid bodies is given by
\[
T_f = \sum_{i=1}^2 \frac{1}{2} M^f_i V_i \cdot V_i + \frac{1}{2} \Omega^i \cdot J^f_i \Omega^i.
\]

**Total kinetic energy:** Define total inertia matrices
\[
M_i = m^b I_{3 \times 3} + M^f_i,
\]
\[
J_i = J^b_i + J^f_i
\]
for \( i \in \{0, 1, 2\} \). From (4) and (9), the total kinetic energy is given by
\[
T = \sum_{i=0}^2 \frac{1}{2} M_i V_i \cdot V_i + \frac{1}{2} \Omega_i \cdot J_i \Omega_i.
\]

Substituting (10) and (11), this can be written as
\[
T = \frac{1}{2} \xi^T \Pi (R_0, R_1, R_2) \xi,
\]
where \( \xi = [\Omega_0; \hat{x}; \Omega_1; \Omega_2] \in \mathbb{R}^{12} \) and the matrix \( \Pi(R_0, R_1, R_2) \in \mathbb{R}^{12 \times 12} \) is given by (5). Since there is no potential field, this is equal to the Lagrangian of the connected rigid bodies immersed in a perfect fluid.

**B. Euler-Lagrange Equations**

Euler-Lagrange equations for a mechanical system that evolves on an arbitrary Lie group are given by
\[
\frac{d}{dt} D_\xi L(g, \xi) - ad^*_g \cdot D_\xi L(g, \xi) - T_g^* L_g \cdot D_g L(g, \xi) = 0,
\]
(14)
\[
\dot{g} = g \xi,
\]
(15)
where \( L : TG \simeq G \times \mathfrak{g} \to \mathbb{R} \) is the Lagrangian of the system [18]. Here \( D_\xi L(g, \xi) \in \mathfrak{g}^* \) denotes the derivative of the Lagrangian with respect to \( \xi \in \mathfrak{g} \), \( ad^*_g : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^* \) is the co-adjoint operator, and \( T_g^* L_g : T^* \mathbb{G} \to \mathfrak{g}^* \) denotes the cotangent lift of the left translation map \( L_g : \mathbb{G} \to \mathbb{G} \) (see [21] for the detailed definitions).

Using this result, we develop Euler-Lagrange equations of a system of connected rigid bodies in a perfect fluid. To simplify the derivation, we consider the configuration manifold given by \( G = \text{SO}(3) \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3 \), left-trivialize \( TG \) to yield \( \mathbb{G} \times \mathfrak{g} \), and identify its Lie algebra \( \mathfrak{g} \) with \( \mathbb{R}^{12} \) by the hat map. For \( \xi = [\Omega_0; \hat{x}; \Omega_1; \Omega_2] \in \mathfrak{g} \) and \( p = [p_0; p_x; p_1; p_2] \in \mathfrak{g}^* \), the co-adjoint operator is given by
\[
ad^*_g p = [-\Omega_0 p_0; -p_x; -\Omega_1 p_1; -\Omega_2 p_2].
\]

**Derivatives of the Lagrangian:** The derivative of the Lagrangian with respect to \( \xi \) is given by
\[
D_\xi L(g, \xi) = \Pi(R_0, R_1, R_2) \xi.
\]
(16)
The derivative of the Lagrangian with respect to \( g = (R_0, x, R_1, R_2) \in G \) can be written as
\[
T_g^* L_g \cdot D_g L(g, \xi) = \left[ T_1^* L_{R_0} \cdot D_{R_0} L; D_x L; T_1^* L_{R_1} \cdot D_{R_1} L; T_1^* L_{R_2} \cdot D_{R_2} L \right].
\]
(17)
An expression for the first term of this can be found as follows. For any \( \eta_0 \in \mathbb{R}^3 \), let \( g_0 = [R_0 \exp \epsilon \eta_0; x, R_1, R_2] \in G \). Then, we have
\[
(T_1^* L_{R_0} \cdot D_{R_0} L) \cdot \eta_0 = \left. \frac{d}{de} \right|_{e=0} \Pi(R_0, R_1, R_2) \xi = \hat{x}^T R_0 M_0 \eta_0 \hat{R}_0^T \hat{x} + \sum_{i=1}^2 \left( -\Omega_0^T \hat{d}_{0i} R_0^T R_i M_i \hat{R}_i \Omega_0 \right)
\]
\[
- \hat{x}^T R_i M_i \hat{R}_i \Omega_0 \hat{d}_{0i} R_0 \Omega_0 - \hat{x}^T R_i M_i \hat{R}_i \hat{d}_{0i} \Omega_0 R_0 \hat{R}_0 \Omega_0 - \hat{x}^T \hat{R}_0 \hat{d}_{0i} \Omega_0 R_0 \hat{R}_0 \Omega_0 R_i M_i \hat{R}_i \hat{d}_{0i} \Omega_0
\]
where we use identities: \( x \cdot y = x^T y = y^T x \), \( \hat{y} = -\hat{x} \) for any \( x, y \in \mathbb{R}^3 \). Since this is satisfied for any \( \eta_0 \in \mathbb{R}^3 \), we
The continuous-time Euler-Lagrange equations and Hamilton’s equations developed in the previous section provide analytical models of the connected rigid bodies in a perfect fluid. However, they are not suitable for a numerical study since a direct numerical integration of those equations using a general purpose numerical integrator, such as an explicit Runge Kutta method, may not preserve the geometric properties of the system accurately [16].

Variational integrators provide a systematic method of developing geometric numerical integrators for Lagrangian/Hamiltonian systems [22]. As it is derived from a discrete analogue of Hamilton’s principle, it preserves symplecticity and the momentum map, and it exhibits good total energy behavior. Lie group methods conserve the structure of a Lie group configuration manifold as it updates a group element using the group operation [23].

These two methods have been unified to obtain a Lie group variational integrator for Lagrangian/Hamiltonian systems evolving on a Lie group [18]. This preserves symplecticity and group structure of those systems concurrently. It has been shown that this property is critical for accurate and efficient simulations of rigid body dynamics [17].

In this section, we develop a Lie group variational integrator for the connected rigid bodies in a perfect fluid. We first obtain an expression for a discrete Lagrangian and substitute it into the discrete-time Euler-Lagrange equations.

### A. Discrete Lagrangian

Let $h > 0$ be a fixed integration step size, and let a subscript $k$ denote the value of a variable at the $k$-th time step. We define a discrete-time kinematics equation as follows. Define $f_k = (F_{0k}, F_{1k}, F_{2k}) \in \mathbb{G}$ for $\Delta x_k \in \mathbb{R}^3$, $F_{0k}, F_{1k}, F_{2k} \in \text{SO}(3)$ such that $g_{k+1} = g_k f_k$:

$$
r_{0k+1} = r_{0k} + \Delta x_k, \quad r_{1k+1} = r_{1k} + F_{1k}, \quad r_{2k+1} = r_{2k} + F_{2k}.
$$

Therefore, $f_k$ represents the relative update between two integration steps. This ensures that the structure of the Lie group configuration manifold is numerically preserved.

A discrete Lagrangian $L_d(g_k, f_k) : \mathbb{G} \times \mathbb{R} \rightarrow \mathbb{R}$ is an approximation of the Jacobi solution of the Hamilton–Jacobi equation, which is given by the integral of the Lagrangian along the exact solution of the Euler-Lagrange equations over a single time step:

$$
L_d(g_k, f_k) \approx \int_0^h L(\hat{g}(t), \hat{g}^{-1}(t)\dot{\hat{g}}(t)) \, dt,
$$

where $\hat{g}(t) : [0, h] \rightarrow \mathbb{G}$ satisfies Euler-Lagrange equations with boundary conditions $\hat{g}(0) = g_k$, $\hat{g}(h) = g_k f_k$. The resulting discrete-time Lagrangian system, referred to as a variational integrator, approximates the Euler-Lagrange equations to the same order of accuracy as the discrete Lagrangian approximates the Jacobi solution.

The kinetic energy given by (13) can be rewritten as

$$
T = \frac{1}{2} x^T R_0 M_0 R_0^T \dot{x} + \frac{1}{2} \Omega^T J_0 \Omega.
$$
From this, we choose the discrete Lagrangian as
\[
L_{d_k} = \frac{1}{2h} \Delta x^T R_k M_k \Delta x_k + \frac{1}{h} \text{tr}[(I - F_k) J_{d_k}] \\
+ \frac{1}{2h} \delta_{d_0}^T (F_k^T - I) R_k^T M_k R_k (F_k - I) d_{0i} \\
+ \frac{1}{h} \Delta x^T R_k M_k \Delta x_k R_k (F_k - I) d_{0i} \\
- \frac{1}{h} \delta_{d_0}^T (F_k^T - I) R_k^T M_k (F_k - I) d_{0i},
\]

where nonstandard inertia matrices are defined as
\[
J_{d_0} = \frac{1}{2} \text{tr}[J_0] I - J_0, \tag{31}
\]
\[
J_{d_i} = \frac{1}{2} \text{tr}[J_i I - J_i], \quad J_i = I - \delta_{i0} M_i \delta_{i0}, \tag{32}
\]
for \(i \in \{1, 2\}.

**B. Discrete-time Euler-Lagrange Equations**

For a discrete Lagrangian on \(G \times \mathbb{G}\), the following discrete-time Euler-Lagrange equations, referred to as a Lie group variational integrator, were developed in [18].

\[
\begin{align*}
\mathcal{T}_x L_{f_k} : \mathbf{D}_{f_k} L_{d_k} - \text{Ad}^{*}_{f_{k+1}} \cdot (\mathcal{T}_x L_{f_{k+1}} : \mathbf{D}_{f_{k+1}} L_{d_{k+1}}) \\
+ \mathcal{T}_x L_{g_{k+1}} : \mathbf{D}_{g_{k+1}} L_{d_{k+1}} = 0, \tag{33}
\end{align*}
\]

where \(\text{Ad}^* : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*\) is co-Ad operator [21].

Using this result, we develop a Lie group variational integrator for connected rigid bodies in a perfect fluid. For \(f = (F_0, \Delta x, F_1, F_2) \in G\) and \(p = [p_0; p_2; p_1; p_2] \in \mathfrak{g}^* \cong \mathbb{R}^{12}\), the co-Ad operator is given by \(\text{Ad}^*_{f} p = [F_0 p_0; p_2; F_1 p_1; F_2 p_2] = [(F_0 p_0 F_0^T)^\vee; p_2; (F_1 p_1 F_1^T)^\vee; (F_2 p_2 F_2^T)^\vee]\), where the vee map \(\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^{3}\) denotes the inverse of the hat map.

**Derivatives of the discrete Lagrangian:** We find expressions for the derivatives of the discrete Lagrangian. The derivative of the discrete Lagrangian with respect to \(F_{0k}\) is given by
\[
\mathbf{D} F_{0k} L_{d_k} : \delta F_{0k} = \frac{1}{h} \text{tr}[-\delta F_{0k} J_{d_0}] + \frac{1}{h} \sum_{i=1}^2 A^T_{ik} R_{0k} \delta F_{0k} d_{0i},
\]
where we define, for \(i \in \{1, 2\},
\[
A_{ik} = R_{0k} M_i \left( R^T_{ik} B_{ik} - (F_{ik} - I) d_{0i} \right), \tag{35}
\]
\[
B_{ik} = \Delta x_k + R_{0k} (F_{0k} - I) d_{0i}. \tag{36}
\]

The variation of \(F_{0k}\) can be written as \(\delta F_{0k} = F_{0k} \hat{\delta}_{0k}\) for \(\hat{\delta}_{0k} \in \mathbb{R}^3\). Therefore, we have
\[
\mathbf{D} F_{0k} L_{d_k} : (F_{0k} \hat{\delta}_{0k}) = \left( \mathcal{T}_x L_{F_{0k}} \cdot \mathbf{D} F_{0k} L_{d_k} \right) : \hat{\delta}_{0k} = \frac{1}{h} \text{tr}[-F_{0k} \hat{\delta}_{0k} J_{d_0}] + \frac{1}{h} \sum_{i=1}^2 A^T_{ik} R_{0k} \hat{\delta}_{0k} d_{0i}.
\]

By repeatedly applying a property of the trace operator, \(\text{tr}[AB] = \text{tr}[BA] = \text{tr}[A^T B^T]\) for any \(A, B \in \mathbb{R}^{3 \times 3}\), the first term can be written as \(\text{tr}\{-F_{0k} \hat{\delta}_{0k} J_{d_0}\} = \text{tr}\{-\hat{\delta}_{0k} J_{d_0} F_{0k}\} = \text{tr}\{\hat{\delta}_{0k} (J_{d_0} F_{0k} - F_{0k} J_{d_0})\}\). Using a property of the hat map, \(x^T y = \frac{1}{2} \text{tr}[\hat{x} \hat{y}]\) for any \(x, y \in \mathbb{R}^3\), this can be further written as \((J_{d_0} F_{0k} - F_{0k} J_{d_0})\hat{\delta}_{0k}\). As \(\hat{x} y = -\hat{y} x\) for any \(x, y \in \mathbb{R}^3\), the second term can be written as \(A^T_{ik} R_{0k} \hat{\delta}_{0k} d_{0i} = -A^T_{ik} R_{0k} \hat{\delta}_{0k} \hat{\delta}_{0k} \hat{\delta}_{0k}\). Using these, we obtain
\[
\mathcal{T}_x L_{F_{0k}} : \mathbf{D} F_{0k} L_{d_k} = \frac{1}{h} (J_{d_0} F_{0k} - F_{0k} J_{d_0})^\vee + \frac{1}{h} \sum_{i=1}^2 \delta_{0i} R^T_{0k+1} A_{ik} \hat{\delta}_{0k} \hat{\delta}_{0k}. \tag{37}
\]

Similarly, we can derive the derivatives of the discrete Lagrangian as follows.
\[
\mathcal{T}_x L_{f_{ik}} : \mathbf{D} f_{ik} L_{d_k} = \frac{1}{h} (J_{d_0} F_{0k} - F_{0k} J_{d_0})^{\vee\vee} - \frac{1}{h} \delta_{0i} F^T_{ik} M_i R^T_{ik} B_{ik}, \tag{38}
\]
\[
\mathbf{D} \Delta x_k L_{d_k} = \frac{1}{h} R_{0k} M_0 R^T_{0k} \Delta x_k + \frac{1}{h} A_{ik} + \frac{1}{h} F_{0k}, \tag{39}
\]
\[
\mathbf{D} \Delta \hat{x}_k L_{d_k} = \frac{1}{h} (M_0 R^T_{0k} \Delta x_k) \hat{\delta}_{0k} R^T_{0k} \Delta x_k + \frac{1}{h} \sum_{i=1}^2 ((F_{0k} - I) d_{0i})^\vee R^T_{0k} A_{ik}. \tag{40}
\]

**Discrete-time Euler-Lagrange Equations:** Substituting \(37\) into \(33\), and rearranging, discrete-time Euler-Lagrange equations for the connected rigid bodies immersed in a perfect fluid are given by
\[
(J_{d_0} F_{0k} - F_{0k} J_{d_0})^\vee - (F_{0k+1} J_{d_0} - J_{d_0} F^T_{0k+1})^\vee \\
+ (M_0 R^T_{0k+1} \Delta x_{k+1} + A_{ik} + A_{ik})^\vee \\
+ \sum_{i=1}^2 \hat{\delta}_{0i} R^T_{0k+1} M_i R^T_{ik} B_{ik} \\
+ (F^T_{ik} \delta_{0i} R^T_{ik} M_i R^T_{ik} B_{ik}) B_{ik} + 0, \tag{42}
\]
\[
(J_{d_0} F_{0k} - F_{0k} J_{d_0})^\vee - (F_{0k+1} J_{d_0} - J_{d_0} F^T_{0k+1})^\vee \\
- \hat{\delta}_{0i} R^T_{0k+1} M_i R^T_{ik} B_{ik} \\
+ (F^T_{ik} \delta_{0i} R^T_{ik} M_i R^T_{ik} B_{ik}) B_{ik} + 0, \tag{43}
\]
\[
R_{0k} M_0 R^T_{0k} \Delta x_k + A_{ik} + A_{ik} = 0, \tag{44}
\]
\[
- R_{0k+1} M_0 R^T_{0k+1} \Delta x_{k+1} + A_{ik+1} + A_{ik+1} = 0, \tag{45}
\]
\[
R_{0k+1} = R_{0k} F_{0k}, \tag{46}
\]
\[
R_{ik+1} = R_{ik} F_{ik}, \tag{46}
\]
\[
x_{ik+1} = x_k + \Delta x_k. \tag{47}
\]
where inertia matrices are given by (31), (32), and $A_{ik}, B_{ik} \in \mathbb{R}^3$ are given by (35), (36) for $i \in \{1, 2\}$. For given $(g_0, f_0) \in \mathbb{G} \times \mathbb{G}$, $g_1 \in \mathbb{G}$ is obtained by (45)–(47), and $f_1 \in \mathbb{G}$ is obtained by solving (42)–(44). This yields a discrete-time Lagrangian flow map $(g_0, f_0) \rightarrow (g_1, f_1)$, and this process is repeated.

**Discrete-time Hamilton’s Equations:** Discrete-time Legendre transformation is given by

$$
\mu_k = -T^T e^T g_k \cdot D_{g_k} L_{d_k} + Ad_{f_k}^* \cdot (T^T e^T f_k \cdot D_{f_k} L_{d_k}).
$$

Substituting this into discrete-time Euler-Lagrange equations, we obtain discrete-time Hamilton’s equations for the connected rigid bodies immersed in a perfect fluid as follows.

$$
h_{p_{0k}} = (F_{0f} J_{0d} - J_{0d} F_{0f}^T)^\vee - (M_0 R_{0k}^T \Delta x_k)^\wedge R_{0k}^T \Delta x_k + \sum_{i=1}^2 d_{0i} R_{0k}^T A_{ik},
$$

(48)

$$
h_{p_{ik}} = (F_{ik} J_{id}^T - J_{id}^T F_{ik}^T)^\vee - \frac{1}{h} F_{ik} d_{0i} M_{i} R_{ik}^T B_{ik} - \frac{1}{h} R_{ik}^T A_{ik} B_{ik},
$$

(49)

$$
h_{p_{xk}} = R_{0k} M_{0} R_{0k}^T \Delta x_k + A_{1k} + A_{2k},
$$

(50)

$$
R_{0k+1} = R_{0k} F_{0k},
$$

(51)

$$
R_{ik+1} = R_{ik} F_{ik},
$$

(52)

$$
x_{k+1} = x_k + \Delta x_k,
$$

(53)

$$
h_{p_{0k+1}} = (J_{0f} F_{0k} - F_{0k} J_{0f})^\vee + \sum_{i=1}^2 d_{0i} R_{0k+1}^T A_{ik},
$$

(54)

$$
h_{p_{i+1}} = (J_{id}^\vee F_{ik} - F_{ik} J_{id})^\vee - d_{0i} F_{ik} M_{i} R_{ik}^T B_{ik},
$$

(55)

$$
p_{x_{k+1}} = p_{x_k},
$$

(56)

where inertia matrices are given by (31), (32), and $A_{ik}, B_{ik} \in \mathbb{R}^3$ are given by (35), (36) for $i \in \{1, 2\}$. Given $(g_0, \mu_0) \in \mathbb{G} \times \mathbb{G}^*$, $f_1 \in \mathbb{G}$ is obtained by solving (48)–(50), and $g_1 \in \mathbb{G}$ is obtained by (51)–(53). The momenta at the next step is obtained by (54)–(56). This yields a discrete-time Hamiltonian flow map $(g_0, \mu_0) \rightarrow (g_1, \mu_1)$, and this process is repeated.

**V. Numerical Example**

We show computational properties of the Lie group variational integrator developed in the previous section. The principal axes of each ellipsoid are given by

- **Body 0:** $l_1 = 8$, $l_2 = 1.5$, $l_3 = 2$ (m),
- **Body 1, 2:** $l_1 = 5$, $l_2 = 0.8$, $l_3 = 1.5$ (m).

We assume the density of fluid is $\rho = 1$ kg/m$^3$. The corresponding inertia matrices are given by

$$
M_0 = \text{diag}[1.0659, 2.1696, 1.6641], \quad (\text{kg})
$$

$$
M_1 = M_2 = \text{diag}[0.2664, 0.6551, 0.3677] \quad (\text{kg}),
$$

$$
J_0 = \text{diag}[1.3480, 20.1500, 25.3276] \quad (\text{kgm}^2),
$$

$$
J_1 = J_2 = \text{diag}[0.1961, 1.7889, 2.9210] \quad (\text{kgm}^2).
$$

The location of the ball joints with respect to the mass center of each body are chosen as

$$
d_{01} = -d_{02} = [8.8, 0, 0], \quad d_{10} = -d_{20} = [5.5, 0, 0] \quad (\text{m}).
$$

The initial conditions are as follows:

$$
R_{00} = I, \quad \Omega_{00} = [0.2, 0.1, 0.5] \quad (\text{rad/s}),
$$

$$
R_{10} = I, \quad \Omega_{10} = [0.1, -0.3, -0.2] \quad (\text{rad/s}),
$$

$$
R_{20} = I, \quad \Omega_{20} = [-0.1, 0.4, -0.6] \quad (\text{rad/s}),
$$

$$
x_0 = [0, 0, 0] \quad (\text{m}), \quad \dot{x}_0 = [0, -0.4142, -0.5900] \quad (\text{m/s}).
$$

We compute discrete-time Hamiltonian flow according to (48)–(56), and as comparison, we numerically integrate the continuous-time Hamilton’s equations (26)–(28) using an explicit, variable step size, Runge-Kutta method. The timestep of the Lie group variational integrator is $h = 0.001$ and the maneuver time is 100 seconds.

**VI. Conclusions**

We have developed continuous-time equations of motion and a geometric numerical integrator, referred to as a Lie group variational integrator, for a system of connected rigid bodies immersed in a perfect fluid. The rigid bodies are modeled as three-dimensional ellipsoids, and each joint has three rotational degrees of freedom. This model characterizes qualitative behaviors of three-dimensional fish locomotion.
The continuous-time equations of motion provide an analytical model that is defined globally on the Lie group configuration manifold, and the Lie group variational integrator preserves the geometric features of the system, thereby yielding a reliable numerical simulation tool.

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