Structure and properties of the algebra of partially transposed permutation operators

Marek Mozrzymas¹, Michal Horodecki²,³ and Michal Studziński²,³

¹Institute for Theoretical Physics, University of Wroclaw, 50-204 Wroclaw, Poland
²Institute for Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland
³National Quantum Information Centre of Gdańsk, 81-824 Sopot, Poland

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We consider the structure of algebra of operators, acting in n-fold tensor product space, which are partially transposed on the last term. Using purely algebraical methods we show that this algebra is semi-simple and then, considering its regular representation, we derive basic properties of the algebra. In particular, we describe all irreducible representations of the algebra of partially transposed operators and derive expressions for matrix elements of the representations. It appears that there are two types of irreducible representations of the algebra. The first one is strictly connected with the representations of the group S(n-1) induced by irreducible representations of the group S(n). The second type is structurally connected with irreducible representations of the group S(n-1).

Keywords: partial transposition, Peres-Horodecki criterion, symmetric group, irreducible representation

I. INTRODUCTION

In this manuscript we present strong mathematical tool to study PPT property for certain class of states. Namely we consider case when some operator is invariant under $U^\otimes (n-k) \otimes (U^*)^\otimes k$ transformations (where $*$ denotes complex conjugation) and then it can be decomposed in terms of partially transposed permutation operators [11]. Thanks to this we see that to full analysis it is enough to investigate properties algebra of partially transposed permutation operators on last k subsystems. We present full solution for the simplest but nontrivial case $k=1$. Namely we present how to construct irreducible representations of the algebra of partially transposed permutation operators for an arbitrary number of subsystems n and an arbitrary dimension $d$ of local Hilbert space $\mathcal{H}$.

The concept of partial transposition plays an important role in the theory of quantum information. The most known example is the PPT-criterion (Positive Partial Transpose) or Peres-Horodecki criterion [8, 10] which gives us necessary and sufficient condition for separability in bipartite case when dimensions of subsystems are $2 \times 2$ and $2 \times 3$. From these works we know that whenever spectrum of partially transposed bipartite density operator is positive our state is separable. For higher dimensions and multipartite case Peres-Horodecki criterion gives only necessary condition, but still concept of partial transposition is worth to investigate. When we are dealing with states having some symmetries problem of separability becomes much more easier to analysis. In particular Eggeling and Werner in [9] present result on separability properties for tripartite states which are $U^\otimes 3$ invariant using PPT property and tools from group theory for an arbitrary dimension of subsystem space. In [12] authors present solution on open problem of existence of four-qubit entangled symmetric states with positive partial transposition and generalize them to systems consisting an arbitrary number of qubits. Namely authors provide criteria for separability of such states formulated in terms of their ranks. PPT property turned out also relevant for a problems in computer science: it is relaxation of some complexity problem, which can be written in terms of separability [14, 15].

A more concrete application of the results of this paper is given in [7]. Namely in this paper the authors present description of universal quantum cloning machines using techniques presented here and in [6]. They connect quantity which describes quality of the clones (so called fidelity) with matrix representations of above mentioned algebra. This result gives new insight into mathematical structure of quantum cloning machines and quantum cloning in general. One more concrete motivation behind the present study is the following: the future generalization of this paper, i.e. extension main results to $U^\otimes (n-k) \otimes (U^*)^\otimes k$ case, for an arbitrary $k$ might allow us to find analytical expressions for output Rényi entropy for two copies of channel coming from some subspaces suggested by Schur-Weyl [5] duality for some fixed number of subsystems n and an arbitrary dimension of Hilbert space $d$, which is relevant for violation of additivity of minimum output entropy.

The algebra of partially transposed permutation operators, which will be denoted $A_n^d(d)$, is the algebra of operators representing the elements of the symmetric group $S(n)$ acting permutationally on a basis of tensor product space $(C^d)^\otimes n$, which are transposed on the last term of the tensor product. From the definition of this algebra it follows that it is generated, in the natural way, by partially transposed operators representing permutations of the group $S(n)$, which will be denoted $V_d(\sigma)^n : \sigma \in S(n)$. The important feature of the algebra $A_n^d(d)$ is the fact that it contains a subalgebra $A_{n-1}(d)$, generated by operators representing the subgroup $S(n-1) \subset S(n)$, which are not
changed by the partial transposition and which act on the basis of the space \((\mathbb{C}^d)^\otimes n\) in the standard permutational way (these operators will be denoted \(V_d(\sigma) : \sigma \in S(n-1)\)). From this it follows that the algebra \(A_n^d(d)\) is a sum of two subspaces

\[ A_n^d(d) = M + A_{n-1}(d), \]

where the subspace \(M\) is an ideal generated by operators \(V_d(\sigma)^{d_n}\) representing permutations \(\sigma \in S(n)\), which permute the number \(n\). The operators generating the ideal \(M\) are non-invertible, which shows that the partial transposition changes strongly the properties of the generating operators. This ideal \(M\) is the most important in our studies (therefore we call it \(M\) as main) because, as we will show, it contains all non-trivial irreducible representations of the algebra \(A_n^d(d)\), where by non-trivial representation we mean a representation in which the non-invertible generators of \(A_n^d(d)\) are represented by operators not equal to zero. The natural generators of the algebra \(A_n^d(d)\) may be linearly dependent or linearly independent, which depends on the relation between \(n\) and \(d\).

The non-invertible generators of the ideal \(M\) such a representation of the first type of the group \(S\) representation of \(d\) complicated formulas. When \(d\) the representations of the group \(A\) this paper coincide with the results obtained by different method in our previous paper [6].

\[ A_n^d(d) = \{ \text{operators generating the ideal } M \}. \]

The structure the algebra \(A_n^d(d)\) and in particular, the structure of its irreducible representations, also depends on the relation between \(n\) and \(d\). The properties of the algebra \(A_n^d(d)\) were considered in our previous paper [6], where we have described the irreducible representations of the algebra \(A_n^d(d)\) in the case when the natural generators of \(A_n^d(d)\) are linearly independent. We have described also the irreducible representations of the algebra \(A_n^d(d)\) in some particular cases when the generators of \(A_n^d(d)\) were linearly dependent. These results were obtained using the representation approach, in which we considered the action of the operators of the algebra \(A_n^d(d)\) on a basis of the natural representation space \((\mathbb{C}^d)^\otimes n\).

In this paper we apply a different, purely algebraical methods. First we derive the formula for the multiplication rule for the natural generators of the algebra \(A_n^d(d)\) and then, treating the algebra in an abstract way, we consider its properties. In particular we show that the algebra \(A_n^d(d)\) is semi-simple. From the semi-simplicity of the algebra \(A_n^d(d)\), it follows that it is a direct sum of matrix ideals, which are in turns, a direct sums of all left minimal ideals of \(A_n^d(d)\), or equivalently, of all irreducible representations of \(A_n^d(d)\). Considering the left regular representation of the algebra \(A_n^d(d)\) we construct all matrix ideals of \(A_n^d(d)\) and next we describe all irreducible representations of this algebra. From our results it follows that the irreducible representations of the algebra \(A_n^d(d)\) are of two types, which differ structurally. In the first type, included in the ideal \(M\), the irreducible representations of \(A_n^d(d)\) are indexed by irreducible representations of the group \(S(n-1)\) and they are strictly connected with the representations of the group \(S(n-1)\) induced by these irreducible representations of \(S(n-1)\). In these representations, when the condition \(d > n - 2\) is satisfied, the elements \(V_d(\sigma) : \sigma \in S(n-1)\) of the algebra \(A_n^d(d)\) are represented as in the representations of the group \(S(n-1)\) induced by irreducible representations of \(S(n-2)\) and the dimension of such a representation of the first type of \(A_n^d(d)\) is equal to the dimension the induced representation of \(S(n-1)\). The non-invertible generators of the ideal \(M\) are represented in these representations by non-trivial and rather complicated formulas. When \(d \leq n - 2\), the situation is more complicated, because in this case some of the irreducible representations of the first type may be defined on some subspace of the representation space of induced representation of \(S(n-1)\). From the semi-simplicity of the algebra \(A_n^d(d)\) it follows that there exists in \(A_n^d(d)\) an ideal \(S\) such that

\[ A_n^d(d) = M \oplus S. \]

The ideal \(S\) contains irreducible representations of the algebra \(A_n^d(d)\) of different structure then the representations of the first type included in the ideal \(M\). These representations of the second type are indexed by some irreducible representations of the group \(S(n-1)\). The generators \(V(\sigma) : \sigma \in S(n-1)\) of the algebra \(A_n^d(d)\) are represented naturally by operators of irreducible representations of \(S(n-1)\), whereas the generators of the ideal \(M\) in \(A_n^d(d)\) e.i. partially transposed operators, are represented trivially by zero operators. So in the representations of the second type only the subalgebra \(A_{n-1}(d)\) of \(A_n^d(d)\) is represented non-trivially and the ideal \(S\) contains irreducible representations of the algebra \(A_n^d(d)\) that may be called semi-trivial.

In the case when the natural generators of the algebra \(A_n^d(d)\) are linearly independent the results obtained in this paper coincide with the results obtained by different method in our previous paper [6].

At the end of this section we note connection between algebra \(A_n^d(d)\) and Walled Brauer Algebra [19, 20, 22, 23] (subalgebra of Brauer Algebra [16–18]). Namely algebra of partially transposed permutation operators is a representation of Walled Brauer Algebra [11]. From the paper [21] we know that whenever \(d > n - 1\) the dimension of Walled Brauer Algebra is equal to \(n!\) and dimension of \(A_n^d(d)\) is also equal to \(n!\). So in this case this two algebras
are isomorphic. When condition \( d > n - 1 \) is not fulfilled we have \( \dim A_n^t(d) < n! \) while dimension of Walled Brauer Algebra is still equal to \( n! \) - we do not have isomorphic between this two algebras.

One important implication of lack of isomorphism is the issue of semisimplicity. Translating necessary and sufficient condition from \([23]\) into language of number of systems and local dimensions of the Hilbert space we obtain that Walled Brauer Algebra is semisimple iff \( d > n - 2 \) and also from the same work we know how to label irreducible components. For \( d = n - 1 \) both the algebra \( A_n^t(d) \) and Walled Brauer Algebra are not isomorphic anymore, but they are still semisimple. When condition \( d > n - 2 \) is not satisfy, then algebra of partially transposed permutation operators \( A_n^t(d) \) is still semisimple while Walled Brauer Algebra is not.

Summarizing, from previous works \([19, 20, 22, 23]\) we know conditions when Walled Brauer Algebra is simply reducible and we know how to label their irreducible components, but we do not know how to construct matrix elements of irreducible representations in our case (algebra of partially transposed permutation operators). This work gives solution of this problem for the simplest case, namely when partial transposition is taken over last subsystem. We have solved this problem also for the case when algebra \( A_n^t(d) \) is not isomorphic to Walled Brauer Algebra. Namely in our paper it is shown that, the irreducible representations of the algebra \( A_n^t(d) \) are labelled by representations of the group \( S(n-1) \) induced by irreducible representations of \( S(n-2) \) in the first kind and by irreducible representations of \( S(n-1) \) in the case of the second kind of irreducible representations of the algebra \( A_n^t(d) \). The matrix forms of these irreducible representations of the algebra \( A_n^t(d) \) are expressed by matrices of irreducible representations of the groups \( S(n-1) \) and \( S(n-2) \).

Our paper is organized in the following way. In section \( \text{II} \) we remind briefly the basic concepts and results in theory of groups, complex finite-dimensional algebras and their representations. In section \( \text{III} \) we introduce the algebra of partially transformed operators \( A_n^t(d) \) and derive some its properties. The main results of this paper is presented in section \( \text{IV} \) in the remaining sections and appendices of the paper contain the derivation of the main results. In particular in section 5 we give the construction of the matrix ideals and minimal left ideals (irreducible representations) of the algebra \( A_n^t(d) \).

\section*{II. PRELIMINARIA}

In this paper we will have to deal with the group \( S(n) \), finite-dimensional complex algebras, in particular the group algebra \( \mathbb{C}[S(n)] \), and their representations, therefore in this section we remind briefly basic properties of these structures that will be applied later on. These results my be found in \([14]\). In the following we will use terminology of modules and representations, which in fact, are equivalent, but in case of algebras we will use rather concept of of modules and in case of group we will use the terminology of representations. The representations of finite groups and left modules of their group algebras are strictly connected, namely we have

\textbf{Theorem 1.} There is a one-to-one correspondence between finite-dimensional complex representations of a finite-dimensional algebras and the left modules of such algebras. In particular it holds for any group algebra of a finite group and any matrix representation of the group \( G \) defines, in a unique way, a matrix representation of the its group algebra \( \mathbb{C}[G] \).

If the representation of the group is unitary, then the corresponding representation of the group algebra, as an algebra of operators, is a \( \mathbb{C}^* \) – algebra. The algebra of partially transformed operators, that we will have to deal with is in fact, as we will see, a semi-simple algebra. Therefore we remind here the definition and the basic properties of semisimple algebras.

\textbf{Definition 2.} \([4]\) The algebra \( A \) is semisimple if it not possesses properly nilpotent \(^1\) elements other then zero. An element \( a \in A \) is properly nilpotent if \( \forall x \in A \) the elements \( xa \) and \( ax \) are a nilpotent elements. Note however, that if \( xa \) is nilpotent then \( ax \) is also nilpotent.

We have the following criterion for the semisimplicity.

\textbf{Theorem 3.} A finite-dimensional complex algebra is semi-simple iff it is a direct sum of ideals, such that each one is isomorphic with a matrix algebra. We will call such ideals matrix ideals.

If the algebra is a \( \mathbb{C}^* \) – algebra then semi-simplicity follows from its structure , and we have

\(^1\) \( x \in A \) is a nilpotent element of algebra \( A \) if \( x^n = 0 \) for some \( n \in \mathbb{N} \).
Theorem 4. [2] Every finite-dimensional $C^*$-algebra is a direct sum of matrix algebras and consequently is a semisimple algebra.

The semi-simplicity determines the structure and properties of irreducible representations of the algebra in the following way.

Theorem 5. [3] If an associative algebra $A$ over $\mathbb{C}$ is semi-simple then every irreducible $A$-module (i.e., every irreducible representation of $A$) is isomorphic to some left minimal (i.e., irreducible) ideal of $A$. Moreover any left minimal ideal $I$ of $A$ is of the form

$$I = Ae = \{ae : a \in A\},$$

where $e \in A$ is a primitive idempotent i.e. $e^2 = e$ and $e$ is not the sum if two idempotents $a, b \in A$ such that $ab = 0$.

Example 6. The matrix algebra

$$M(n, \mathbb{C}) = \text{span}_\mathbb{C}\{e_{ij} : e_{ij}e_{kl} = \delta_{jk}e_{il}, \; i, j = 1, \ldots, n\}$$

is a semi-simple algebra of dimension $n^2$ and the irreducible $M(n, \mathbb{C})$ left-modules (representations) are generated by primitive idempotents $e_{ij}, i = 1, \ldots, n$ and are isomorphic to the space $\mathbb{C}^n$. The algebra $M(n, \mathbb{C})$ endowed with hermitian conjugation is a $C^*$-algebra.

Proposition 7. The structure of the matrix ideals (i.e., the ideals that are generated by elements $E_{ij} : i, j = 1, \ldots, n$ which satisfy matrix multiplication) which appear in the decomposition of a semisimple algebra in the statement of Th. 5 is the same as the structure of the matrix algebras described in the above Ex. 6. They are direct sums of left minimal ideals of the algebra generated by the primitive idempotents (the “diagonal” elements) $E_{ii}, i = 1, \ldots, n$ of the ideal.

The construction of matrix ideals of the algebra representing a group algebra (such an algebra is always semisimple) is described in the Appendix C, where the statement of Th. 8 is an example for of the Proposition 7.

In our studies of irreducible representations of the algebra of partially transposed operators the concept of any induced representation of a given group $G$ will play an important role. In the matrix form the induced representation $\Phi$ of a group $G$ induced by a representation $\varphi$ of the subgroup $H$ is defined in the following way.

Definition 8. [4] Let $\varphi : H \to M(n, \mathbb{C})$ be a matrix representation of a subgroup $H$ of the group $G$. Then the matrix form of the induced representation $\pi = \text{ind}_H^G(\varphi)$ of a group $G$ induced by an irreps. $\varphi$ of the subgroup $H \subset G$ has the following block matrix form

$$\forall g \in G \quad \pi_{ab, bj}(g) = (\Phi_{ij}(g_a^{-1}ggb}),$$

where $g_a, a = 1, \ldots, |G : H|$ are representatives of the left cosets $G/H$ and

$$\Phi_{ij}(g_a^{-1}ggb) = \begin{cases} \Phi_{ij}(g_a^{-1}ggb) & \text{if } g_a^{-1}ggb \in H, \\ 0 & \text{if } g_a^{-1}ggb \notin H. \end{cases}$$

Any induced representation of a given group $G$ may be extended to the representation of its group algebra $\mathbb{C}[G]$. In next section we will use the following:

Notation 9. Any permutation $\sigma \in S(n)$ defines, in a natural and unique way, two natural numbers $a, b \in \{1, 2, \ldots, n\}$

$$n = \sigma(a), \quad b = \sigma(n)$$

Thus we may characterize any permutation by these two numbers in the following way

$$\sigma \equiv \sigma_{(a, b)} \equiv \sigma_{ab}.$$  

Note that in general $a, b$ may be different except the case, when one of them is equal to $n$, because in this case we have

$$a = n \iff b = n.$$  

When $a = n = b$, then $\sigma(n) = n$ and we will use abbreviation $\sigma = \sigma_{(n, n)} \equiv \sigma_n \in S(n - 1) \subset S(n)$.  

Example 10.

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
\end{pmatrix}^{(2,1)} = 
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
\end{pmatrix}^{(3,3)}.
\]

The irreducible representations of the symmetric group \(S(m)\), which are uniquely characterized by partitions \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) or equivalently by Young diagrams \(Y(\lambda)\) will be denoted by greek symbols \(\varphi^\lambda, \psi^V\) etc.

III. THE ALGEBRA OF PARTIALLY TRANSPOSED OPERATORS.

Let us consider a representation \(V\) of the group \(S(n)\) in the space \(H \equiv (\mathbb{C}^d)^{\otimes n}\) defined in the following way

**Definition 11.** \(V : S(n) \to \text{Hom}( (\mathbb{C}^d)^{\otimes n} )\) and

\[
\forall \sigma \in S(n) \quad V(\sigma) e_1 \otimes e_2 \otimes \ldots \otimes e_n = e_{i_\sigma^{-1}(1)} \otimes e_{i_\sigma^{-1}(2)} \otimes \ldots \otimes e_{i_\sigma^{-1}(n)},
\]

where \(d \in \mathbb{N}\) and \(\{e_i\}_{i=1}^d\) is an orthonormal basis of the space \(\mathbb{C}^d\).

The representation \(V : S(n) \to \text{Hom}( (\mathbb{C}^d)^{\otimes n} )\) is defined in a given basis \(\{e_i\}_{i=1}^d\) of the space \(\mathbb{C}^d\) (and consequently in a given basis of \(H\)), so in fact it is a matrix representation.

**Remark 12.** The representation \(V : S(n) \to \text{Hom}( (\mathbb{C}^d)^{\otimes n} )\) depends explicitly on the dimension \(d\), so in fact we should write \(V \equiv V_d\) but for simplicity we will omit the index \(d\), unless it will be necessary.

The representation \(V\) of the group \(S(n)\) is unitary and we have

\[
\forall \sigma \in S(n) \quad V(\sigma)^\dagger = V(\sigma^{-1}),
\]

where \(\dagger\) denotes usual hermitian conjugation with respect to the scalar product in \((\mathbb{C}^d)^{\otimes n}\).

For \(d > 1\) the representation \(V\) is always reducible and we have

**Proposition 13.** \(\square\) The irreducible representation \(\varphi^\alpha\) of \(S(n)\), indexed by the partition \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)\), is contained in \(V\) if \(d \geq k \equiv h(\alpha)\). In particular if \(d \geq n\) then all irreducible representations of \(S(n)\) are included in the representation \(V\) of \(S(n)\). When \(d \geq k \equiv h(\alpha)\) then the multiplicity of the irreducible representation \(\varphi^\alpha\) of \(S(n)\) is equal to

\[
\frac{1}{n!} \sum_{\sigma \in S(n)} \chi^\alpha(\sigma^{-1}) d^l(\sigma),
\]

where \(\chi^\alpha\) is the character of \(\varphi^\alpha\), \(l(\sigma)\) is the number of cycles in the permutation \(\sigma\) and \(\chi^V(\sigma) = d^l(\sigma)\) is the character of the representation \(V : S(n) \to \text{Hom}( (\mathbb{C}^d)^{\otimes n} )\).

The representation \(V\) of \(S(n)\) extends in a natural way to the representation of the group algebra \(\mathbb{C}[S(n)]\) and in this way we get the algebra

\[
A_n(d) = \text{span}_\mathbb{C}\{V(\sigma) : \sigma \in S(n)\} \subset \text{Hom}( (\mathbb{C}^d)^{\otimes n} )
\]

of operators representing the elements of the group algebra \(\mathbb{C}[S(n)]\). Obviously the dimension of the algebra \(A_n(d)\) depends on the dimension \(d\) and in general we have \(n! = \dim \mathbb{C}[S(n)] \geq \dim A_n(d)\) and therefore the algebra \(A_n(d)\) is only homomorphic to \(\mathbb{C}[S(n)]\). From Prop. 13 and Th. B4 in App. C it follows

**Proposition 14.** If \(d \geq n\) then the operators \(V(\sigma) : \sigma \in S(n)\) are linearly independent and \(\dim A_n(d) = n!\), if \(d < n\) then the operators are linearly dependent and the dimension of the algebra \(A_n(d)\) is smaller then \(n!\).

The algebra \(A_n(d)\) contain a natural subalgebra

\[
A_{n-1}(d) = \text{span}_\mathbb{C}\{V(\sigma_n) : \sigma_n \in S(n-1)\}.
\]

Extending anti-linearly the hermitian conjugation \(\dagger\) of the representation \(V[S(n)]\), we endow the algebra \(A_n(d)\) with a \(C^*\)-algebra structure getting a \(C^*\)-algebra. The semi-simplicity of the algebra \(A_n(d)\) follows from the complete reducibility of the representation \(V : S(n) \to \text{Hom}( (\mathbb{C}^d)^{\otimes n} )\) or from the \(C^*\)-algebra structure of \(A_n(d)\).

Our main task in this paper is to study the properties of the following algebra of partially transposed operators
Definition 15. For $A_n(d) = \text{span}_C \{ V(\sigma) : \sigma \in S(n) \}$ we define a new complex algebra

$$A_{tn}^n(d) = \text{span}_C \{ V(\sigma)^{tn} : \sigma \in S(n) \} \subset \text{Hom}((C^d)^{\otimes n}),$$

where the symbol $t_n$ describes the partial transpose in the last place in the space $\text{Hom}((C^d)^{\otimes n})$. The elements $V(\sigma)^{tn} : \sigma \in S(n)$ will be called natural generators of the algebra $A_{tn}^n(d)$.

Directly from this definition it follows

Proposition 16. a) if $\sigma = \sigma_n \in S(n-1) \subset S(n)$ then $V(\sigma_n)^{tn} = V(\sigma_n)$ i.e. $V(S(n-1)) \subset A_{tn}^n(d)$, where $S(n-1) \subset S(n)$ means the natural embedding: $\sigma = \sigma_n \in S(n-1) \subset S(n)$ then $\sigma(n) = n$,

b) dim$_C A_{tn}^n(d) = \text{dim}_C A_n^d(d)$

The second statement in this Proposition follows from the invertibility of the partial transpose and from the Propositions 14, 15 we get

Corollary 17. When $d < n$ then the generating elements $V(\sigma)^{tn} : \sigma \in S(n)$ of the algebra $A_{tn}^n(d)$ are linearly dependent.

Remark 18. Because the partial transpose does not change the elements $V(\sigma_n) \in V(S(n-1)) \subset A_{tn}^n(d)$ therefore, in the following, we will write simply $V(\sigma_n)$ instead $V(\sigma_n)^{tn}$ when $\sigma_n \in S(n-1)$.

The essential for our studies of the properties of the algebra $A_{tn}^n(d)$ is to describe the composition law of this algebra. A rather laborious direct calculation gives the following result

Theorem 19. The elements $V(\sigma)^{tn} : \sigma \in S(n)$ which span the algebra $A_{tn}^n(d)$ have the following composition rule

$$V(\sigma_n)^{tn} V(\rho_n)^{tn} = V(\sigma_n) V(\rho_n) = V(\sigma_n \rho_n),$$

$$V(\sigma_n) V(\rho_{(a,b)})^{tn} = V(\sigma_n \rho_{(a,b)})^{tn}, \quad V(\sigma_n^{(a,b)})^{tn} V(\rho_n) = V(\sigma_n^{(a,b)} \rho_n)^{tn},$$

$$V(\sigma_{(a,b)})^{tn} V(\rho_{(p,q)})^{tn} = d^{tn} V[(\sigma(q)n)\sigma_{(a,b)}\rho_{(p,q)}(pn)]^{tn},$$

where $a,b \neq n$ and $c,d \neq n$ and the multiplication rule depends explicitly on $d$.

According to these formulas the composition of operators $V(\sigma_{(a,b)})^{tn}$ is expressed by composition of standard permutations.

Remark 20. Note that, contrary to the case of the standard algebra $A_n(d)$, the composition rule in the algebra $A_{tn}^n(d)$ depends explicitly on the dimension $d$. Therefore for different values of $d$ we have to deal with different algebras.

Remark 21. The explicit formulas for the multiplication in the algebra $A_{tn}^n(d)$ allows to consider this algebra, in purely abstract way, as an algebra generated by the elements $V(\sigma)^{tn} : \sigma \in S(n)$, which satisfies the multiplication rules stated in the Th. 19. In the following we will treat the algebra in this way.

Many particular cases follow almost immediately from the above composition law.

Example 22.

$$V(kn)^{tn} V(jn)^{tn} = V([jn](kn))^{tn}, \quad k \neq j.$$

$$V(kn)^{tn} V(kn)^{tn} = d V((kn))^{tn},$$

$$V(ijn)^{tn} V(ijn)^{tn} = V(ijn)^{tn}$$

so in particular $V(kn)^{tn}$ and $V(ijn)^{tn}$ are (essential) projectors.
Example 23. In the table below we present composition properties for algebra \( A_{n}^{d}(d) \).

| \( \circ \) | \( \mathbb{I} \) | \( (132)^{1} \) | \( (123)^{1} \) | \( (12)^{1} \) | \( (13)^{1} \) | \( (23)^{1} \) |
|---|---|---|---|---|---|---|
| \( \mathbb{I} \) | \( \mathbb{I} \) | \( (132)^{1} \) | \( (123)^{1} \) | \( (12)^{1} \) | \( (13)^{1} \) | \( (23)^{1} \) |
| \( (132)^{1} \) | \( (132)^{1} \) | \( (132)^{1} \) | \( d(23)^{1} \) | \( (23)^{1} \) | \( d(132)^{1} \) | \( (23)^{1} \) |
| \( (123)^{1} \) | \( (123)^{1} \) | \( d(13)^{1} \) | \( (123)^{1} \) | \( (13)^{1} \) | \( (23)^{1} \) | \( d(123)^{1} \) |
| \( (12)^{1} \) | \( (12)^{1} \) | \( (13)^{1} \) | \( (23)^{1} \) | \( \mathbb{I} \) | \( (132)^{1} \) | \( (123)^{1} \) |
| \( (13)^{1} \) | \( (13)^{1} \) | \( (13)^{1} \) | \( d(13)^{1} \) | \( (123)^{1} \) | \( d(13)^{1} \) | \( (123)^{1} \) |
| \( (23)^{1} \) | \( (32)^{1} \) | \( d(132)^{1} \) | \( (23)^{1} \) | \( (132)^{1} \) | \( (132)^{1} \) | \( d(23)^{1} \) |

TABLE I: For simplicity here we have \( t = t_{3} \). From this table it follows that \( (132)^{1}, (123)^{1}, (13)^{1}, (23)^{1} \) of the algebra \( A_{n}^{d}(d) \) are idempotents or essential idempotents. The operator \( (12) \), as well \( \mathbb{I} \), remains unchanged under the transposition on the third position \( t_{3} \).

The hermitian conjugation in the space \( \text{Hom}((\mathbb{C}^{d})^{\otimes n}) \) commutes with the partial transpose i.e. we have

\[
\forall \sigma \in S(n) \quad (V(\sigma)^{I_{n}})^{\dagger} = ((V(\sigma)^{\dagger})^{I_{n}} = V(\sigma^{-1})^{I_{n}}
\]

so the algebra \( A_{n}^{d}(d) \) is invariant under the hermitian conjugation \( \dagger \) and moreover we have

Moreover directly form the definition of the operators \( V(\sigma)^{I_{n}} \) we have:

**Proposition 24.** The algebra \( A_{n}^{d}(d) \) with the hermitian conjugation is a \( C^{*} \)-algebra i.e.

\[
\forall a, b \in A_{n}^{d}(d) \quad (ab)^{\dagger} = b^{\dagger}a^{\dagger}.
\]

From the Theorem [4] we get

**Corollary 25.** The algebra \( A_{n}^{d}(d) \) is a semi-simple algebra for any value of \( d \) and \( n \).

We have also one more consequence of the Theorem [19]

**Corollary 26.** For \( a, b \neq n \) the elements \( V(\sigma_{(a,b)})^{I_{n}} \) are not invertible and the set \( M = \text{span}_{\mathbb{C}}\{V(\sigma_{(a,b)})^{I_{n}} : a, b \neq n\} \) is an ideal of the algebra \( A_{n}^{d}(d) \). We have the following decomposition of the algebra \( A_{n}^{d}(d) \)

\[
A_{n}^{d}(d) = M + \text{span}_{\mathbb{C}}\{V(\sigma_{n})^{I_{n}} = V(\sigma_{n}) : \sigma_{n} \in S(n-1)\}.
\]

Note that the second component in the simple sum, which in general, is not simple, is not an ideal of the algebra \( A_{n}^{d}(d) \). The ideal \( M \), being semi-simple, has a unit element \( e \), which in fact an idempotent of the algebra \( A_{n}^{d}(d) \) and \( M = eA_{n}^{d}(d)e \). Using the basic properties of the idempotents \( e, \mathbb{I} - e \), one can construct another ideal

\[
S = (\mathbb{I} - e)A_{n}^{d}(d)(\mathbb{I} - e) \subset A_{n}^{d}(d)
\]

which is, as we will see in next section, closely related to the group algebra \( \mathbb{C}[S(n-1)] \) such that

**Proposition 27.** The algebra \( A_{n}^{d}(d) \) has the following decomposition into a direct sum of two ideals

\[
A_{n}^{d}(d) = eA_{n}^{d}(d)e \oplus (\mathbb{I} - e)A_{n}^{d}(d)(\mathbb{I} - e) \equiv M \oplus S,
\]

where

\[
MS = 0.
\]

This Proposition is the first step in the decomposition of the algebra \( A_{n}^{d}(d) \) into a direct sum of matrix algebras. In the Section 5 we will show that the ideals \( M \) and \( S \) are in fact, the direct sums of matrix ideals of the algebra \( A_{n}^{d}(d) \), such that each one is a direct sum of minimal left ideals (irreducible representations) of the algebra \( A_{n}^{d}(d) \). The construction however, is rather complicated and laborious and it is the content of the remaining part of this paper, therefore in the following section we formulate the theorems describing the structure and irreducible representations of the algebra \( A_{n}^{d}(d) \), which is in fact the main result of the paper.
IV. MAIN RESULT: THE STRUCTURE AND IRREDUCIBLE REPRESENTATIONS OF THE ALGEBRA $A^d_n(d)$.

It appears that the structure of irreducible representations of the algebra $A^d_n(d)$ is closely related to the structure of the representation $\text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha)$ of the group $S(n-1)$ induced by irreducible representations $\varphi^\alpha$ of the group $S(n-2)$ and the properties of irreducible representations of $A^d_n(d)$ depends strongly on the relation between $d$ and $n$. Before presenting the main results of this paper we have to describe briefly some object appearing in the structure of the algebra $A^d_n(d)$, in particular the properties of the induced representation $\text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha)$. The matrix form of such a representations given in Def. 8. The irreducible representations of the group $S(n-2)$ are characterized by the partitions $\alpha = (\alpha_1, ..., \alpha_k)$ of $n-2$, which describe also the corresponding Young diagram $Y(\alpha)$. The representation $\text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha)$ is completely and simply reducible i.e. we have $\mathds{1}$.

Proposition 28.

$$\text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) = \bigoplus_{\nu} \varphi^{\nu},$$

where the sum is over all partitions $\nu = (\nu_1, ..., \nu_k)$ of $n-1$, such that their Young diagrams $Y(\nu)$ are obtained from $Y(\alpha)$ by adding, in a proper way, one box.

From this Proposition it follows that the induced representation $\text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha)$ may be described in two bases. The first one, is the basis of the matrix form of the induced representation given in Def. 8 and it is of the form

$$\{e^\alpha_i(a) : a = 1, ..., n-1, \quad i = 1, ..., \text{dim} \varphi^\alpha\},$$

where the index $a = 1, ..., n-1$ describes the the costes $S(n-1)/S(n-2)$ and the the index $i = 1, ..., \text{dim} \varphi^\alpha$ is the index of a matrix form of $\varphi^\alpha$. The second one is a basis of the reduced form of $\text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha)$, given Prop. 28 which is of the form

$$\{f^{\nu}_{j\nu} : \psi^{\nu} \in \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha), \quad j\nu = 1, ..., \text{dim} \psi^{\nu}\}.$$

The next important objects are the following matrices

Definition 29. For any irreducible representation $\varphi^\alpha$ of the group $S(n-2)$ we define the block matrix

$$Q^d_{n-1}(\alpha) = Q(\alpha) = (d^a_{ij} \varphi^\alpha_{ji}((a n-1)(ab)(b n-1))) = (Q^d_{ij}(\alpha)),$$

where $a, b = 1, ..., n-1$, $i, j = 1, ..., \text{dim} \varphi^\alpha$ and the blocs of the matrix $Q(\alpha)$ are labelled by indices $(a, b)$ whereas the elements of the blocks are labelled by the indices of the irreducible representation $\varphi^\alpha = (\varphi^\alpha_{ij})$ of the group $S(n-2)$ and $Q(\alpha) \in M((n-1)w^\alpha, \mathbb{C})$. Note that if $a, b \neq n-1 \Rightarrow (a n-1)(ab)(b n-1) = (ab)$, but if $a = n-1$ and $b \neq n-1$ then $(a n-1)(ab)(b n-1) = id$, where id denotes identity permutation.

The matrices $Q(\alpha)$ are hermitian and their structure and properties are described in the Appendix A where it has been shown, that the eigenvalues $\lambda_V$ of the matrix $Q(\alpha)$ are labelled by the irreducible representations $\psi^\nu \in \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha)$ and the multiplicity of $\lambda_V$ is equal to $\text{dim} \psi^\nu$. The essential for properties of the irreducible representations of the algebra $A^d_n(d)$ is the fact, that at most one (up to the multiplicity) eigenvalue $\lambda_V$ of the matrix $Q(\alpha)$ may be equal to zero (Cor. 68 in App. A).

The structure of the algebra $A^d_n(d)$ is the following

Theorem 30. The algebra $A^d_n(d)$ is a direct sum of two ideals

$$A = M \oplus S$$

and the ideals $M$ and $S$ has different structures.
a) The ideal $M$ is of the form

$$M = \bigoplus_a U(a),$$

where $U(a)$ are ideals of the algebra $A_n^1(d)$ characterized by the irreducible representations $\varphi^\lambda$ of the group $S(n - 2)$, such that $\varphi^\lambda \in V_d[S(n - 2)]$ and

$$U(a) = \text{span}_\mathbb{C}\{u_{ij}^{ab}(\alpha) : a, b = 1, \ldots, n - 1, i, j = 1, \ldots, w^a\}$$

with

$$u_{ij}^{ab}(\alpha)u_{kl}^{pq}(\beta) = \delta_{a\beta}Q_{ik}^{bp}(\alpha)u_{jl}^{aq}(\alpha).$$

The ideals $U(a)$ are matrix ideals such that

$$U(a) \simeq M(\text{rank} \, Q(\alpha), \mathbb{C}),$$

in particular when $\det Q(\alpha) \neq 0$ we have

$$U(a) \simeq M((n - 1) \dim \varphi^\lambda, \mathbb{C}).$$

b) The ideal $S$ has the following structure:

1) if $d \geq n$

$$S \simeq \bigoplus_\nu M(\dim \varphi^\nu, \mathbb{C})$$

where $\nu$ runs over all irreducible representations of the group $S(n - 1)$

2) if $d < n$ then

$$S \simeq \bigoplus_\nu M(\dim \varphi^\nu, \mathbb{C})$$

where now $\nu$ runs over all irreducible representations of the group $S(n - 1)$ such that $d > h(\nu)$ ($h(\nu)$ is defined in Prop.13).

The matrix ideals contained in the ideals $M$ and $S$ contains all minimal left ideals i.e. all irreducible representations of the algebra $A_n^1(d)$. The next theorems describes all these representations.

The structure of the irreducible representations of the algebra $A_n^1(d)$, included in the ideal $M$, is completely determined by irreducible representations $\varphi^\lambda$ of the group $S(n - 2)$, therefore we will denote them $\Phi^\lambda_A$.

**Theorem 31.** The irreducible representations $\Phi^\lambda_A$ of the algebra $A_n^1(d)$ contained in the ideal $U(\alpha) \subset M$ (Th.30) are indexed by the irreducible representations $\varphi^\alpha$ of the group $S(n - 2)$, such that $\varphi^\alpha \in V_d[S(n - 2)]$ and if $\{f^\nu_j : \nu \in \text{ind}_{S(n-2)}(\varphi^\lambda), \quad j = 1, \ldots, \dim \varphi^\nu\}$ is the reduced basis of the induced representation $\text{ind}_{S(n-2)}(\varphi^\lambda)$, then the vectors $\{f^\nu_j : \nu \neq 0\}$ form the basis of the irreducible representation of the algebra $A_n^1(d)$ and the natural generators of $A_n^1(d)$ act on it in the following way

$$V(\alpha)^{\rho^\nu}f^\nu_j(\alpha) = \sum_{\rho, \lambda, k} \sqrt{\lambda_\rho z^+(\alpha)^{\rho\alpha}z(\alpha)^{\lambda\nu}k_j} \lambda_\nu f^\rho_j(\alpha)$$

(1)

where the summation is over $\rho$ such that $\lambda_\rho \neq 0$. Due to the condition $\varphi^\alpha \in V_d[S(n - 2)]$ the eigen-values $\lambda_\nu$ of $Q(\alpha)$ are non-negative. The unitary matrix $Z(\alpha) = (z(\alpha)^{\nu\nu}_{kj})$ has the form

$$z(\alpha)^{\nu\nu}_{kj} = \frac{\dim \varphi^\nu}{\sqrt{N^\nu_j(n - 1)!}} \sum_{\sigma \in S(n - 1)} \psi_j^\nu_{kj}(\sigma^{-1})\delta_{a\nu, \nu(\sigma)}\varphi^\nu_{kr}[(a n - 1)\sigma(q n - 1)],$$

(2)
with
\[ N_{j\nu}^\nu = \dim \psi_{j\nu}^\nu \frac{n!}{(n-1)!} \sum_{\sigma \in S(n-1)} \psi_{j\nu}^\nu (\sigma^{-1}) \delta_q \sigma(q) \psi_{n\nu}^\nu [(q n - 1) \sigma(q n - 1)], \tag{3} \]
where the indices \( q = 1, \ldots, n-1, \quad r = 1, \ldots, \dim \varphi^\alpha \) are fixed and such that \( N_{j\nu}^\nu > 0 \) (see Th.59, 85 and Cor. 86). For \( \sigma_n \in S(n-1) \) we have
\[ V(\sigma_n) f_{j\nu}^\nu (\alpha) = \sum_{\rho \neq j} \psi_{j\nu}^\nu (\sigma_n) f_{\rho\nu}^\nu (\alpha). \]

In particular when \( \det Q(\alpha) \neq 0, \) (i.e. when all \( \lambda_v \neq 0 \)) then the representation \( \Phi_A^\alpha \) is the induced representation \( \text{ind}_{S(n-2)}^{S(n-1)} (\varphi^\alpha) \) (in the reduced form) for the subalgebra \( V_d[S(n-1)] \subset A_{n\nu}^\nu (d) \). In this case the dimension of the irreducible representation is equal to
\[ \dim \Phi_A^\alpha = (n-1) \dim \varphi^\alpha = \dim(\text{ind}_{S(n-2)}^{S(n-1)} (\varphi^\alpha)). \]

When \( \det Q(\alpha) = 0, \) (i.e. when one, up to the multiplicity, eigen-value \( \lambda_\theta \) of \( Q(\alpha) \) is equal to 0), then the irreducible representation of \( A_{n\nu}^{\nu} (d) \) is defined on a subspace \( \{ y_{j\nu}^\nu : \lambda_v \neq \lambda_\theta \} \) of the representation space \( \text{ind}_{S(n-2)}^{S(n-1)} (\varphi^\alpha) \) and the representation has dimension is equal to
\[ \dim \Phi_A^\alpha \text{ are } = \dim(\text{ind}_{S(n-2)}^{S(n-1)} (\varphi^\alpha)) - \dim \psi^\nu = \text{rank } Q(\alpha). \]

This case takes the place when
\[ d = i - a_i - 1 \]
for some \( a_i \) in the partition \( \alpha = (\alpha_1, \ldots, a_i, \ldots, \alpha_k) \) characterizing the irreducible representation \( \varphi^\alpha \), under condition that \( \nu = (\alpha_1, \ldots, a_i + 1, \ldots, \alpha_k) \) characterizes the representation \( \psi^\nu \) of \( S(n-1) \).

The ideal \( U(\alpha) \) is a direct sum of \( \text{dim } \Phi_A^\alpha \) of irreducible representations \( \Phi_A^\alpha \).

The formula for the eigenvalues \( \lambda_v \) of matrices \( Q(\alpha) \) are derived in the Appendix A (Th. 60).

Remark 32. Note that even if \( \dim \varphi^\alpha = 1 \), we have \( \dim \Phi^\alpha = n - 1 \).

The matrix forms of these representations are the following

**Proposition 33.** In the reduced matrix basis \( \{ f_{j\nu}^\nu : \nu = \theta \} \) of the ideal \( U(\alpha) \) the natural generators \( V(\sigma_a^\nu)_{\nu=1}^n \) and \( V(\sigma_n) \) of \( A_{n\nu}^{\nu} (d) \) are represented by the following matrices
\[ M_{j\nu}^\nu (V(\sigma_a^\nu)_{\nu=1}^n) f_{j\nu}^\nu = \sum_{k=1, \ldots, \dim \varphi^\alpha} \sqrt{\lambda_j} z_k^\nu (\alpha) \otimes z_k^\nu (\sigma_n) \sqrt{\lambda_j} \nu, \nu \neq \theta, \]
\[ M_{j\nu}^\nu (V(\sigma_n)_{\nu=1}^n) f_{j\nu}^\nu = \delta_{\nu, \nu} \psi_{j\nu}^\nu (\sigma_n). \]

From the properties of the matrix \( Q(\alpha) \) (Cor. 69) one gets

**Proposition 34.** If \( d > n - 2 \), then \( \det Q(\alpha) \neq 0 \) and the irreducible representations \( \Phi_A^\alpha \) described in Th. 31 are induced representation \( \text{ind}_{S(n-2)}^{S(n-1)} (\varphi^\alpha) \) for the subalgebra \( V_d[S(n-1)] \subset A_{n\nu}^{\nu} (d) \), so their dimension is equal to \( (n-1) \dim \varphi^\alpha \). When \( d \leq n - 2 \), then for some \( \varphi^\alpha \) it may appear that \( \det Q(\alpha) = 0 \) and consequently the irreducible representation \( \Phi^\alpha \) of \( A_{n\nu}^{\nu} (d) \) is define on a subspace of the irreducible representation \( \text{ind}_{S(n-2)}^{S(n-1)} (\varphi^\alpha) \).

When \( \det Q(\alpha) \neq 0 \), the equivalent form of the irreducible representation \( \Phi_A^\alpha \) form Th. 31 in the basis \( \{ e_i^\alpha (\alpha) : i = 1, \ldots, \dim \varphi^\alpha \} \) is given in the Prop. 30, 32

The representations of the algebra \( A_{n\nu}^{\nu} (d) \) included in the ideal \( S \) are much simpler.
Theorem 35. Each irreducible representation \( \psi^\nu \) of the group \( S(n-1) \), which appears in the decomposition of the ideal \( S \) given in the Th. 31 b) defines irreducible representations \( \Psi^\nu \) of the algebra \( A_{S}^{n}(d) \) in the following way

\[
\Psi^\nu(a) = \begin{cases} 
0 & \text{if } a \in M, \\
\psi^\nu(\sigma_n) & \text{if } a = \sigma_n \in S(n-1).
\end{cases}
\]

So in this representation the non-invertible element of the ideal \( M \) are represented trivially by zero and therefore we call these representation of the algebra \( A_{S}^{n}(d) \) semi-trivial. The matrix forms of these representations are simply matrix forms of the irreducible representations of the group algebra \( \mathbb{C}[S(n-1)] \subset A_{S}^{n}(d) \) and zero matrices for the elements of the ideal \( M \).

Corollary 36. All irreducible representations of the algebra \( A_{S}^{n}(d) \) of dimension one are included in the ideal \( S \). In particular, because for the identity representation \( \psi^{\text{id}} \) of \( S(n-1) \) we have \( h(\psi^{\text{id}}) = 1 < d : d \geq 2 \), the algebra \( A_{S}^{n}(d) \) has a trivial representation \( \Psi^{\text{id}} \), in which the elements of the ideal \( M \) are represented by zero and the elements \( V_\nu(\sigma) : \sigma \in S(n-1) \) are represented by number 1.

Let us consider some examples.

Example 37. The simplest case is when \( n = 2 \) and \( d \geq 2 \). The corresponding algebra \( A_{2}^{2}(d) = \text{span}_{\mathbb{C}}\{1, V(12)^{12} : (V(12)^{12})^{2} = dV(12)^{12}\} \) is two-dimensional and commutative. In this simplest case we cannot apply Th. 31 because the group \( S(n-2) = S(2-2) = S(0) \) does not exist. However due to the simplicity, in particular commutativity of \( A_{2}^{2}(d) \) it is easy to derive the structure of the algebra using elementary calculations, which give

\[
A_{2}^{2}(d) = M \oplus S : M = \mathbb{C}V(12)^{12}, \quad S = \mathbb{C}(1 - \frac{1}{d}V(12)^{12}) \Rightarrow A_{2}^{2}(d) \simeq \mathbb{C} \oplus \mathbb{C}
\]

and

\[
V(12)^{12})^{2} = dV(12)^{12}, \quad V(12)^{12}(1 - \frac{1}{d}V(12)^{12}) = 0.
\]

In this simplest all irreducible representations of the algebra \( A_{2}^{2}(d) \) are one-dimensional and they are generated by one-dimensional ideals \( M \) and \( S \). In the second representation the generator \( V(12)^{12} \) is represented by the zero operator.

Example 38. I) \( n = 3 \) and \( d \geq 3 \) all generating elements \( V_{\alpha}^{3}(\sigma) : \sigma \in S(3) \) are linearly independent (Th.87, Prop.16) and we have \( S(3-2) = S(1) = \{\text{id}\} \) and this group has only trivial irreducible representation \( \psi^{\text{id}} \). The group \( S(3-1) = S(2) \) has two one-dimensional irreducible representations, trivial \( \psi^{\text{id}} \) and \( \text{sgn} \) denoted \( \psi^{s} \) and the induced representation \( \text{ind}_{S(1)}^{S(2)}(\psi^{\text{id}}) \) has the following decomposition

\[
\text{ind}_{S(1)}^{S(2)}(\psi^{\text{id}}) = \psi^{\text{id}} \oplus \text{sgn}
\]

and is a regular representation of the group \( S(2) \). In this case the matrices \( Q(\alpha) \) and \( Z(\alpha) \) have the form

\[
Q(\text{id}) = Q(\text{sgn}) = \begin{pmatrix} d & 1 \\ 1 & d \end{pmatrix}, \quad Z(\text{id}) = \frac{\sqrt{5}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

and, for \( d \geq 2 \), \( Q(\text{id}) \) is always invertible, because we have \( d > n-2 = 1 \) and therefore in the ideal \( M \) we have to deal with the induced presentations of \( A_{2}^{3}(d) \subset A_{3}^{3}(d) \) only. The irreducible representation \( \Phi^{\text{id}}_{A} \) of the algebra \( A_{3}^{3}(d) \), included in the ideal \( M \), has the following matrix form

\[
M_{\sigma}^{\text{id}}(V_{13}(13)) = \frac{1}{2} \begin{pmatrix} d + 1 & -\sqrt{d^2 - 1} \\ -\sqrt{d^2 - 1} & d - 1 \end{pmatrix}, \quad M_{\sigma}^{\text{id}}(V_{13}(23)) = \frac{1}{2} \begin{pmatrix} d + 1 & \sqrt{d^2 - 1} \\ \sqrt{d^2 - 1} & d - 1 \end{pmatrix}
\]

and

\[
M_{\sigma}^{\text{id}}(V_{13}(123)) = \frac{1}{2} \begin{pmatrix} d + 1 & \sqrt{d^2 - 1} \\ -\sqrt{d^2 - 1} & 1 - d \end{pmatrix}, \quad M_{\sigma}^{\text{id}}(V(\sigma_3)) = \begin{pmatrix} \text{id}(\sigma_3) & 0 \\ 0 & \text{sgn}(\sigma_3) \end{pmatrix}
\]
where $c_3 \in S(2)$. The ideal $S$ is in this case two-dimensional and is a direct sum of two non-isomorphic one-dimensional semi-trivial representations of algebra $A^4_3(d)$, generated by all irreducible representations of $S(2)$ (because we have $d \geq n - 1 = 2$), which are the following

$$
\Psi^{id}(id) = \Psi^{id}(12) = 1, \quad \Psi^{id}((123)^{t_3}) = \Psi^{id}((132)^{t_3}) = \Psi^{id}((13)^{t_3}) = \Psi^{id}((23)^{t_3}) = 0
$$

and

$$
\Psi^{s}(id) = \Psi^{s}(12) = -1, \quad \Psi^{s}((123)^{t_3}) = \Psi^{s}((132)^{t_3}) = \Psi^{s}((13)^{t_3}) = \Psi^{s}((23)^{t_3}) = 0,
$$

The algebra $A^4_3(d)$ has the following structure

$$A^4_3(d) \simeq M(2, \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C},$$

so it is isomorphic to algebra $\mathbb{C}[V_d(S(3))]$. The representation $\Phi^{id}_A$ from the the ideal $M \simeq M(2, \mathbb{C})$, may be written, using Prop. 52 in an equivalent but simpler form

$$\Phi^{id}(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi^{id}((123)^{t_3}) = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}, \quad \Phi^{id}((132)^{t_3}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Phi^{id}((23)^{t_3}) = \begin{pmatrix} d & 1 \\ 0 & 0 \end{pmatrix},
$$

For the group $S(2)$ this representation is the induced representation $\text{ind}_{S(1)}^{S(2)}(\Phi^{id})$ in a non-reduced form.

II) When $n = 3, d = 2$. In this case the irreducible representation $\text{sgn}$ of the group $S(3)$ is not included in $A_3(2)$, because $h(\text{sgn}) = 3 > d = 2$, therefore the generating elements $V_{d}^{id}(\sigma) : \sigma \in S(n)$ are linearly dependent (Th.87, Prop.16). The matrix $Q(id)$ is still invertible because $d = 2 \geq n - 1 = 2$ so the ideal $M$ is the same as in case I). The ideal $S$ contains only one irreducible representation $\Psi^{id}$, because $h(\Psi^{id}) = 1 < d = 2$ and $h(\Psi^{\text{sgn}}) = 2 \not\geq d = 2$. So $S \simeq \mathbb{C}$ and

$$A^4_3(2) \simeq M(2, \mathbb{C}) \oplus \mathbb{C} \simeq \mathbb{C}[V_2(S(3))].$$

The next example is more interesting.

**Example 39.** For $n = 4$ we have $S(4 - 2) = S(2)$ and we have two one-dimensional representations of the group $S(2)$ $\Phi^{id}$, $\Phi^{\text{sgn}}$ and the group $S(4 - 1) = S(3)$ has three non-trivial irreducible representations, characterized by partitions $v_{id} = (3)$, $v_2 = (2, 1)$ and $v_3 = (1, 1, 1)$. The induced representations $\text{ind}_{S(2)}^{S(3)}(\Phi^{\alpha})$, where $\alpha = \text{id}, \text{sgn}$ are irreducible representations of $S(2)$ have the following structure

$$\text{ind}_{S(2)}^{S(3)}(\Phi^{id}) = \psi^{v_2} \oplus \psi^{id}, \quad \text{ind}_{S(2)}^{S(3)}(\Phi^{\text{sgn}}) = \psi^{v_2} \oplus \psi^{\text{sgn}}$$

and we chose the following matrix representation for $\psi^{v_2}$

$$
\psi^{v_2}(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi^{v_2}(13) = \begin{pmatrix} 0 & \epsilon \\ \epsilon^{-1} & 0 \end{pmatrix}, \quad \psi^{v_2}(23) = \begin{pmatrix} 0 & \epsilon^{-1} \\ \epsilon & 0 \end{pmatrix},
$$

$$
\psi^{v_2}(123) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \psi^{v_2}(132) = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix},
$$

where $\epsilon^3 = 1$. In case of the algebra $A^4_4(d)$, its structure and the structure of its irreducible representations depends on the value of the dimension $d$. We have two cases.
1) When \( n = 4, d \geq 4 \), then the condition \( d > n - 2 = 2 \) is satisfied and the irreducible representations \( \Phi^\alpha \) of the ideal \( M \), are induced representations \( \text{ind}_{\text{id}}^{S(3)}(\psi^\alpha) \) for the subalgebra \( A_3(d) \subset A_4(d) \), where \( \alpha = \text{id}, \text{sgn} \) are irreducible representations of \( S(2) \). Using the explicit form of the representation \( \psi^{\alpha_2} \) of \( S(3) \) we get that the matrices \( Q(\text{id}) \) and \( Z(\text{id}) \) are of the form

\[
Q(\text{id}) = \begin{pmatrix} d & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{pmatrix}, \quad Z(\text{id}) = \frac{1}{\sqrt{3}} \begin{pmatrix} \varepsilon & \varepsilon^2 & 1 \\ \varepsilon^2 & \varepsilon & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

The irreducible representation \( \Phi^\text{id}_A \) of the algebra \( A_4(d) \), included in the ideal \( M \), has the following matrix form

\[
M^\text{id}_f(V^{\alpha}(14)) = \frac{1}{3} D^\text{id} \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & 1 \end{pmatrix} D^\text{id}, \quad M^\text{id}_f(V^{\alpha}(24)) = \frac{1}{3} D^\text{id} \begin{pmatrix} 1 & \varepsilon^2 & \varepsilon \\ \varepsilon & 1 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix} D^\text{id},
\]

\[
M^\text{id}_f(V^{\alpha}(34)) = \frac{1}{3} D^\text{id} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} D^\text{id}, \quad M^\text{id}_f(V(\sigma_4)) = \begin{pmatrix} \psi^{2}(\sigma_4) & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_4 \in S(3),
\]

where

\[
D^\text{id} = \begin{pmatrix} \sqrt{d-1} & 0 & 0 \\ 0 & \sqrt{d-1} & 0 \\ 0 & 0 & \sqrt{d+2} \end{pmatrix}.
\]

In the basis from Prop. 52 this irreducible representation of \( A_4(d) \) looks simpler

\[
M^\text{id}_f(V^{\alpha}(14)) = \begin{pmatrix} d & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M^\text{id}_f(V^{\alpha}(24)) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & d & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
M^\text{id}_f(V^{\alpha}(34)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & d \end{pmatrix}, \quad M^\text{id}_f(V(\sigma_4)) = \delta_{\text{ir}(j)}, \quad \sigma_4 \in S(3)
\]

So for \( A_3(d) \subset A_4(d) \) this a natural representation. Similarly, for the irreducible representation \( \Phi^\text{sgn}_A \) we get that

\[
Q(\text{sgn}) = \begin{pmatrix} d & 1 & 1 \\ -1 & d & 1 \\ -1 & -1 & d \end{pmatrix}, \quad Z(\text{sgn}) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\varepsilon & -\varepsilon^{-1} & -1 \\ -\varepsilon^{-1} & -\varepsilon & -1 \\ 1 & 1 & 1 \end{pmatrix},
\]

The irreducible representation \( \Phi^\text{sgn}_A \) of the algebra \( A_4(d) \), included in the ideal \( M \), has the following matrix form

\[
M^\text{sgn}_f(V^{\alpha}(14)) = \frac{1}{3} D^s \begin{pmatrix} 1 & \varepsilon^2 & \varepsilon \\ \varepsilon & 1 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix} D^s, \quad M^\text{sgn}_f(V^{\alpha}(24)) = \frac{1}{3} D^s \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & 1 \end{pmatrix} D^s,
\]

\[
M^\text{sgn}_f(V^{\alpha}(34)) = \frac{1}{3} D^s \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} D^s, \quad M^\text{sgn}_f(V(\sigma_4)) = \begin{pmatrix} \psi^{2}(\sigma_4) & 0 \\ 0 & \text{sgn}(\sigma_4) \end{pmatrix},
\]
where
\( D^i = \begin{pmatrix} \sqrt{d+1} & 0 & 0 \\ 0 & \sqrt{d+1} & 0 \\ 0 & 0 & \sqrt{d-2} \end{pmatrix} \)

Again, in the basis from Prop. 52 the irreducible representation \( \Phi^{\text{sgn}}_A \) of \( A_4^{(d)} \) looks simpler

\[
M_{i}^{\text{sgn}}(V^{i_1}(14)) = \begin{pmatrix} d & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{i}^{\text{sgn}}(V^{i_1}(24)) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & d & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
M_{j}^{\text{sgn}}(V^{i_2}(34)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & d \end{pmatrix}, \quad M_{j}^{\text{sgn}}(V(\sigma_4)_{ij}) = \delta_{ir(j)} \text{sgn}((i3)e_4(j3)),
\]

The ideal \( M \) has the following structure

\[
M = M(3, \mathbb{C}) \oplus M(3, \mathbb{C}),
\]

where each induced representation \( \text{ind}^{S(3)}_{S(2)}(\psi^n) \) is included with the multiplicities equal to their dimensions. The ideal \( S \), is a direct sum of three non-isomorphic semi-trivial representations \( \Psi^v \) of the algebra \( A_4^{(d)} \) (also with multiplicities equal to their dimensions), generated by all irreducible representations \( \psi^v \) of \( S(3) \) \( (v_4 = (3), v_2 = (2, 1) \text{ and } v_3 = (1, 1, 1)) \), because in this case we have \( d \geq n = 4 \) (see Theorem 30). It means that, in these semi-trivial representations the natural generators \( V(\sigma_n) : \sigma_n \in S(n-1) \) are represented by operators \( \psi^v(\sigma_n) \), whereas the elements of the ideal \( M \) are represented by zero operator. Therefore the ideal \( S \) has the following structure

\[
S = M(2, \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}.
\]

II) \( n = 4, d = 3 \). In this case the irreducible representation \( \text{sgn} \) of the group \( S(4) \) is not included in \( A_4(3) \), because \( h(\text{sgn}) = 4 > d = 3 \), so from (Th.87, Prop.16) we get that the generating elements \( V^{i_1}_d(\sigma) : \sigma \in S(4) \) are linearly dependent. The matrices \( Q(\text{id}) \) and \( Q(\text{sgn}) \) are invertible, because \( d = 3 \geq n = 3 \), therefore the ideal \( M \) has the same form as in previous case I). The ideal \( S \) contains the irreducible representations \( \Psi^{\text{id}} \) and \( \Psi^{v_2}, h(\Psi^{\text{id}}), h(\Psi^{v_2}) < 3 \), so \( S \simeq M(2, \mathbb{C}) \oplus \mathbb{C} \).

III) In the case \( d = 2 \), the irreducible representations \( \text{sgn} \) and \( (2, 1, 1) \) of the group \( S(4) \) are not included in \( A_4(2) \), because \( h(\text{sgn}) = 4, h(2, 1, 1) = 3 > d = 2 \), therefore, again from Th.87, Prop.16 we get that the generating elements \( V^{i_1}_d(\sigma) : \sigma \in S(4) \) are linearly dependent. The ideal \( M \) in algebra \( A_4^{(d)} \) has another structure of its irreducible representations, because we have \( d = n-2 = 2 \) and from Prop.34 and Cor.67 it follows that, in this case the irreducible representation \( \Phi^{\text{id}}_A \) is the induced representation \( \text{ind}^{S(3)}_{S(2)}(\psi^{\text{id}}) \) for the subalgebra \( A_3(d) \) but the irreducible representation \( \Phi^{\text{sgn}}_A \) is defined only on two-dimensional subspace of the representation \( \text{ind}^{S(3)}_{S(2)}(\psi^{\text{sgn}}) \) (because \( \det Q(\text{sgn}) = 0 \)). The representation \( \Phi^{\text{sgn}}_A \) has, in this case the same form as for the case I) but we have to set \( d = 2 \) and the irreducible representation \( \Phi^{\text{sgn}}_A \) of \( A_4^{(d)} \) has the following matrix form

\[
M_{f}^{\text{sgn}}(V^{i_1}(14)) = \begin{pmatrix} 1 & \varepsilon^2 \\ \varepsilon & 1 \end{pmatrix}, \quad M_{f}^{\text{sgn}}(V^{i_1}(24)) = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon^2 & 1 \end{pmatrix},
\]

\[
M_{f}^{\text{sgn}}(V^{i_2}(34)) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_{f}^{\text{sgn}}(V(\sigma_4)) = \psi^2(\sigma_4), \quad \sigma_4 \in S(3)
\]

In this case we can not use the Prop.49 because \( \det Q(\text{sgn}) = 0 \) for \( d = 2 \). We have also

\[
M = M(3, \mathbb{C}) \oplus M(2, \mathbb{C})
\]
Similarly, from Th.30b and 35 it follows that, in this case the representations $\Psi^{sgn}$ and $\Psi^{v_2}$ of the group $S(3)$ are not included in the ideal $S$, because $h(sgn) = 3$, $h(v_2) = 2 \neq d = 2$. Therefore the ideal $S$ contains only one irreducible representation $\Psi^{id}$ and has the following structure

$$S \cong \mathbb{C}.$$ 

The results obtained in these examples were obtained, in an equivalent form, in [6] were different methods had been used.

V. CONSTRUCTION OF THE IRREDUCIBLE REPRESENTATIONS OF THE ALGEBRA $A_{n}^m(d)$.

In this section we will prove our main results stated in Theorems 30, 31 and 35 of previous section. In the proof we will construct all representations of the algebra $A_{n}^m(d)$. Because the algebra $A_{n}^m(d)$ is semi-simple then, from the Theorem 5 and Proposition 7 we get that all irreducible representations of the algebra $A_{n}^m(d)$ are included, as a left minimal ideals, in the left regular representation of this algebra. More precisely these left minimal ideals are included in the matrix ideals of $A_{n}^m(d)$, so in fact we have to look for the matrix ideals of the algebra $A_{n}^m(d)$. This a purely algebraical approach, alternative to representation approach described in [6], is based on the properties of the multiplication in the algebra $A_{n}^m(d)$ described in Th.19. First we will construct the matrix ideals of $M$ and all irreducible representations of the algebra $A_{n}^m(d)$ included in the ideal $M$, next all irreducible representations included in the complementary ideal $S$, which is closely related with the algebra $\mathbb{C}[S(n - 1)]$.

A. Matrix ideals and left minimal ideals of the ideal $M$ of the algebra $A_{n}^m(d)$.

We define new generating elements of the ideal $M$ in the following way

**Definition 40.** Let $\varphi^a$, $a = 1, \ldots, k$ be all inequivalent irreps of the group $S(n - 2)$ which are supposed to be unitary and $w^a = \dim \varphi^a$. For any $\varphi^a$, $a = 1, \ldots, k$ we define

$$ u_{ij}^{ab}(\alpha) = \frac{w^a}{(n - 2)!} V(an)^{ia} \sum_{\sigma \in S(n-2)} \varphi^a_{n}(\sigma^{-1}) V[(a n - 1)\sigma(b n - 1)]$$

$$= V(an)^{ia} V[(a n - 1)] V[E_{ij}^{b}] V[(b - 1)]$$

where $a, b = 1, \ldots, n - 1$ and $E_{ij}^{b}$ are matrix operators of the representation $\varphi^a$ defined in the Def. 69, Appendix C and $i, j = 1, \ldots, w^a$ are indices of the matrix form of $\varphi^a$.

**Remark 41.** Here, a natural question arises, why the group $S(n - 2)$ and its representations appears in this Definition. The group $S(n - 2)$ arises in a more natural way in the studies of irreducible representations of the algebra $A_{n}^m(d)$ in the representation approach where one consider the properties of the operators of the algebra $A_{n}^m(d)$ acting on a given basis of its natural representation space $(\mathbb{C}^d)^{\otimes n} [6]$. Using however, the results of this section, one can show that the ideal $M$ is a direct sum of subalgebras isomorphic to the group algebras $\mathbb{C}[S(n - 2)]$ (but we omit this derivation) which allows to construct left ideals in $M$ (i.e. representations of the algebra $A_{n}^m(d)$) using irreducible representations of the group $S(n - 2)$.

For a given irreducible representation $\varphi^a$ of the group $S(n - 2)$ we get a set of $((n - 1) \dim \varphi^a)^2$ elements $\{u_{ij}^{ab}(\alpha)\}$ which are either all non-zero vectors or all of them are zero. In fact we have

**Proposition 42.** a) For a given irreducible representation $\varphi^a$ of the group $S(n - 2)$ the operators $u_{ij}^{ab}(\alpha)$: $a, b = 1, \ldots, n - 1$, $i, j = 1, \ldots, w^a$ are different from zero iff the the irreducible representation $\varphi^a$ of the group $S(n - 2)$ appears in the permutation representation $V_{a}[S(n - 2)]$ of the group $S(n - 2)$. 
b) The vectors \( \{ u_{ij}^{ab}(\alpha) \} \) span the ideal \( M \) i.e. we have

\[
M = \text{span}_\mathbb{C}\{ u_{ij}^{ab}(\alpha) : a, b = 1, ..., n - 1, \quad \varphi^a \in V_d[S(n - 2)], \quad i, j = 1, ..., w^a \}
\]

**Proof.** a) The representation \( V : S(n) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \) of the group \( S(n) \) is also, in a natural way, the representation of its subgroup \( S(n - 2) \) i.e. we have also \( V_d : S(n - 2) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \). From the statements of the Appendix C we know that the matrix operators

\[
E_{ij}^a = \frac{w^a}{(n - 2)!} \sum_{\sigma \in S(n-2)} \varphi_{ij}^a(\sigma^{-1})V_d[\sigma] \in \text{Hom}((\mathbb{C}^d)^{\otimes n})
\]

of the representation \( \varphi^a \) of the subgroup \( S(n - 2) \) are nonzero iff the irreducible representation \( \varphi^a \) belongs to the representation \( V : S(n - 2) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \). From the invertibility of the operators \( V(an) \) and \( V[(an - 1)] \) (in the representation \( V : S(n) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \)) we get that for any \( a, b = 1, ..., n - 1 \) the elements

\[
\tilde{w}_{ij}^a(\alpha) = V(an)V[(a n - 1)]V[E_{ij}^a]V[(b n - 1)] \in A_n(d)
\]

are nonzero iff \( \varphi^a \) belongs to the representation \( V : S(n - 2) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \). Now the statement of the Proposition follows from the invertibility of the partial transpose \( t_n \) (which is a linear transformation) and from properties of multiplication in the algebra \( A_n^d(d) \) Th. 19.

b) The algebra \( A_n(d) \), which homomorphic with the group algebra \( \mathbb{C}[S(n)] \), is a sum of two subspaces

\[
A_n(d) = \tilde{M} + V[\mathbb{C}[S(n - 1)]
\]

where

\[
\tilde{M} = \text{span}_\mathbb{C}\{ V(\sigma_{ab}) : a, b \neq n \}.
\]

The subspace \( \tilde{M} \) has the following form

\[
\tilde{M} = \sum_{a, b = 1, ..., n - 1} S_{ab} : S_{ab} = \text{span}_\mathbb{C}\{ V(\sigma) : \sigma \in S(n) \land \sigma = \sigma_{ab} \}.
\]

It is easy to check that for a given pair \( (a, b) \), \( a, b = 1, ..., n - 1 \) we have

\[
S_{ab} = V(an)V[(a n - 1)]V_d[S(n - 2)]V[(b n - 1)],
\]

so taking into account (see Appendix C) that

\[
V_d[S(n - 2)] = \bigoplus_{\varphi^a \in V_d[S(n - 2)]} \text{span}_\mathbb{C}\{ E_{ij}^a \}
\]

we get that

\[
\tilde{M} = \text{span}_\mathbb{C}\{ \tilde{w}_{ij}^a(\alpha) : a, b = 1, ..., n - 1, \quad \varphi^a \in V_d[S(n - 2)], \quad i, j = 1, ..., w^a \}
\]

and again from the invertibility of the partial transpose we get the statement b) of the Proposition. \( \square \)

Similarly as the generating elements \( V(\sigma)^{t_n} : \sigma \in S(n) \) the vectors \( u_{ij}^{ab}(\alpha) \), in general, need not to be linearly independent which depends on the the dimension parameter \( d \). Arguing however, in a similar way as in the above proof and using Prop. 14 one gets
Proposition 43. a) The vectors \( \{u_{ij}^{ab}(\alpha)\} \), in general, need not to be linearly independent but for a given pair \((a, b), a, b = 1, \ldots, n - 1 \) and given irreducible representation \( \varphi^a \in V[\alpha(S(n - 2))] \) the set of \((w^a)^2 \) nonzero vectors \( u_{ij}^{ab}(\alpha), \) \( i, j = 1, .., w^a \) is linearly independent.

b) When \( d \geq n \) then the both sets of vectors \( \{V(\sigma)^{\alpha} : \sigma \in S(n)\} \) and \( \{u_{ij}^{ab}(\alpha) : a, b = 1, \ldots, n - 1, \ \alpha = 1, \ldots, k, \ i, j = 1, \ldots, w^a \} \) are linearly independent and they form a bases of the ideal \( M \).

Using multiplication rule for the algebra \( A_n^k(d) \) (Th. 19), one can derive the following properties of the elements \( \{u_{ij}^{ab}(\alpha)\} \) of \( M \subset A_n^k(d) \)

Proposition 44. Suppose that \( \varphi^a \in V[S(n - 2)] \), i.e. \( u_{ij}^{pq}(\alpha) \neq 0 \), then for any \( V(\sigma_{ab})^{\alpha} \in M \) we have

\[
V(\sigma_{ab})^{\alpha} u_{ij}^{pq}(\alpha) = d^{\delta_{pq}} \sum_{k=1}^{\alpha^n} q_k^{\alpha_k} [(b n - 1)q_{ab}(ap)(p n - 1)] u_{kj}^{pq}(\alpha),
\]

(4)

where \( \sigma_{ab} = (bn)_{ab} = \sigma_{ab}(an) \in S(n - 1), a, b \neq n, \) in particular

\[
V(an)^{\alpha} u_{ij}^{pq}(\alpha) = d^{(\delta_{pq})} \sum_{k=1}^{\alpha^n} q_k^{\alpha_k} [(a n - 1)(ap)(p n - 1)] u_{kj}^{pq}(\alpha),
\]

and for any \( \sigma_n \in S(n - 1) \)

\[
V(\sigma_n) u_{ij}^{pq}(\alpha) = \sum_{k=1}^{\alpha^n} q_k^{\alpha_k} [(\sigma_n(p) n - 1)\sigma_n(p n - 1)] u_{kj}^{pq}(\alpha),
\]

(5)

the multiplication rule for these vectors is the following

\[
u_{ij}^{ab}(\alpha) u_{kj}^{pq}(\beta) = \delta_{ab} d^{\delta_{pq}} \sum_{k=1}^{\alpha^n} q_k^{\alpha_k} [(b n - 1)(bp)(p n - 1)] u_{ij}^{pq}(\alpha).
\]

(6)

In particular one has

\[
u_{ab}^{aa}(\alpha) u_{ii}^{aa}(\beta) = \delta_{ab} \delta_{ii} u_{ii}^{aa}(\alpha),
\]

i.e. the vectors \( u_{ii}^{aa}(\alpha) \) are essential projectors.

From the transformation law of the natural generators (Eq.1, Eq.2) we see that the left action of an arbitrary element \( V(\sigma)^{\alpha} \in A \) on the elements \( u_{ij}^{pq}(\alpha) \) changes only the first upper and first lower indices in \( u_{ij}^{pq}(\alpha) \), whereas the second indices remain unchanged. Therefore we may formulate

Corollary 45. For a given partition \( \alpha \) such that \( \varphi^a \in V[S(n - 2)] \) and for arbitrary fixed values of \( b \in \{1, \ldots, n - 1\} \) and \( j \in \{1, \ldots, w^a\} \) a linear space

\[
U_{ij}^a(\alpha) \equiv \text{span}_C \{u_{ij}^{ab}(\alpha) : a = 1, \ldots, n - 1, \ i = 1, \ldots, w^a\} \subset M
\]

is a left submodule of the algebra \( A_n^k(d) \) in \( M \) i.e. it is a representation of the algebra \( A_n^k(d) \) which appears in the regular representation of the algebra \( A_n^k(d) \) with the multiplicity \( (n - 1) \dim \varphi^a \) in \( M \). It is clear also that for \( \alpha \) such that \( \varphi^a \in V[S(n - 2)] \) the subspace

\[
U(\alpha) = \bigoplus_{a=1}^{n-1, \dim \varphi^a} U_{ij}^a(\alpha) = \text{span}_C \{u_{ij}^{ab}(\alpha) : a, b = 1, \ldots, n - 1, \ i, j = 1, \ldots, w^a\}
\]

is an ideal of \( M \subset A_n^k(d) \), generated by \(((n - 1) \dim \varphi^a)^2 \) elements. We have also the following decomposition of the ideal \( M \) into a direct sum of ideals

\[
M = \bigoplus_a U(\alpha),
\]

where the sum is over \( a : \varphi^a \in V[S(n - 2)] \).
In fact the representation $U^p_n(a)$ of the algebra $A^p_n(d)$ is generated by the irreducible representation $\varphi^a$ of the group $S(n-2)$. We see also that each irreducible representation $\varphi^a$, $a = 1, \ldots, k$ of the group $S(n-2)$, such that $\varphi^a \in V[S(n-2)]$, generates $(n-1) \dim \varphi^a$ isomorphic representations of the algebra $A^p_n(d)$, indexed by $a \in \{1, \ldots, n-1\}$ and $i \in \{1, \ldots, w^a\}$. This isomorphism follows from the fact that the transformation laws (Eq. (4), Eq. (5)) in Prop. 44 does not depend on the indices $q$ and $j$. We will show that these representations are in fact, all irreducible representations of the algebra $A^p_n(d)$.

Obviously the representation $U^p_j(a)$ of the algebra $A^p_n(d)$ is also a representation of its subalgebra $\mathbb{C}[S(n-1)]$ and consequently of the group $S(n-1)$. From the representation rule (Eq. (5)), for the elements $V(\sigma_n) \in V[S(n-1)]$ one deduce

**Theorem 46.** Suppose that the vectors $\{u^b_{ij}(a)\}$ are linearly independent (so they form a basis of $U(a)$), then for any $b = 1, \ldots, n-1$ and $j = 1, \ldots, \dim \varphi^a$ the representation $U^b_j(a)$ of the algebra $A^p_n(d)$ from Cor. 45 as a reducible representation of the group $S(n-1) \subset A^p_n(d)$, is in fact the representation of $S(n-1)$ induced by the irrep $\varphi^a$ of the subgroup $S(n-2) \subset S(n-1)$, i.e. $U^b_j(a)$ is the representation space of $\text{ind}_{S(n-2)}^{S(n-1)}(\varphi^a)$, for any $b = 1, \ldots, n-1$, $j = 1, \ldots, \dim \varphi^a$.

**Proof.** In our case $G = S(n-1) \supset S(n-2) = H$ and we chose the transpositions $(an-1, a = 1, \ldots, n-1)$ as the natural representatives of the left cosets $S(n-1)/S(n-2)$. Then taking the representation $\varphi^a$ of the subgroup $S(n-2)$, using the Definition 8 we get

$$\forall \sigma \in S(n-1) \quad \pi_{ai,bj}(\sigma) = \delta_{ai}(\sigma) \varphi^a(\sigma(b \ n-1) \sigma(b \ n-1))$$

which is exactly the block matrix representing the elements $\sigma = \sigma_n \in S(n-1) \subset A^p_n(d)$ in the irreducible representation $U^a_n(\alpha)$ of the algebra $A^p_n(d)$ given in the Proposition 44.

The multiplication rule for the elements $\{u^b_{ij}(a)\}$ is similar to the matrix multiplication and in fact, it is the matrix multiplication when the representation $\varphi^a$ is the identity representation of the group $S(n-2)$. When the representation $\varphi^a$ is not the trivial representation then the properties of the multiplication rule for the elements $\{u^b_{ij}(a)\}$ depends on the properties of the matrix $Q(\alpha)$ (defined in Def. 29), which appears in the multiplication law of the elements $\{u^b_{ij}(a)\}$ (Prop. 44, Eq. (5)).

The properties and the structure of the matrices $Q(\alpha)$ are described in the Appendix A. The multiplication rule for the elements $\{u^b_{ij}(a)\}$ in the ideal $U(a)$ (eq. (6)), may be written now

$$u^b_{ij}(a)u^p_{kl}(b) = \delta_{i\beta}Q_{jk}(b)u^{aq}_{il}(a).$$

In the Appendix B we derived a basic properties of the algebras with multiplication of this type. The algebra

$$U(a) = \text{span}_\mathbb{C}\{u^b_{ij}(a) : a, b = 1, \ldots, n-1, \ i, j = 1, \ldots, w^a\}$$

is semisimple (because it is $C^*$-algebra, Th.4) and its properties depends strongly on the properties of the matrix $Q(\alpha)$. In particular the dimension of the algebra $U(a)$ depend on the rank of the matrix $Q(\alpha)$.

From the results derived in the Appendices A and B it follows that if $\det Q(\alpha) \neq 0$, then all vectors $\{u^b_{ij}(a)\}$ are linearly independent (so they form a basis of $U(a)$) and the algebra $U(a)$ is isomorphic with the matrix algebra $M((n-1) \dim \varphi^a, \mathbb{C})$ (see Th. 75 and Th. 78 in Appendix B). The case when $\det Q(\alpha) = 0$ is more complicated, because then the vectors $\{u^b_{ij}(a)\}$ are linearly dependent and we have to construct a basis of the algebra $U(a)$.

The universal construction of a reduced basis, which applies to the both cases is described in the Appendix B (Th. 78). The basis constructed in this theorem we will call the reduced basis of the ideal $U(a)$ because firstly it is constructed using the matrix $Z(\alpha)$, that reduces the induced representation $\text{ind}_{S(n-2)}^{S(n-1)}(\varphi^a)$ to the direct sum of irreducible representations and secondly, in case when $\det Q(\alpha) = 0$ in its construction we reduce the set of linearly dependent generators to linearly independent one.

Now, using Th. 78 from the Appendix B we will construct the reduced basis in the algebra $U(a)$ when $\varphi^a \in V(\sigma_n) \subset V(\sigma_n) \subset V[S(n-2)]$. The construction of the basis is based on the diagonalization of the matrix $Q(\alpha)$ appearing in the
The multiplication law of the algebra $U(\alpha)$ (Eq. 6), which is realized by the following similarity transformation (see Th. 60, Eq. (A1), Appendix A)

$$\sum_{ak} \sum_{bl} z^a(\alpha)^{ba}_{\mu \nu} Q(\alpha)^{b}_{kl} z(\alpha)^{\mu}_{ij} = \delta^{\mu \nu} \delta_{ij} \lambda_{\mu \nu},$$

where the unitary matrix $Z(\alpha) = (z(\alpha))_{ij}^{\mu \nu}$ reduces the induced representation $\text{ind}_{S(n-1)}^{S(n-2)}(\varphi^\alpha)$ to the sum $\bigoplus \lambda \psi^\lambda$ of irreducible representations of $S(n-1)$ (Prop. 25).

Now, using the matrix $Z(\alpha)$, according to the Th. 78 in the Appendix B, we define new elements in the ideal $U(\alpha)$ in the following way

$$y_{\mu \nu}^{\rho \sigma}(\alpha) = \sum_{ai,\theta} (z^{-1})_{ij}^{\mu \rho} h_{a i}^{\rho \sigma}(\alpha) z_{k j}^{\nu \sigma},$$

where the indices $\mu, \nu$ labels the irreducible representations $\psi^\mu, \psi^\nu$ of the group $S(n-1)$, which appears in the reduction of the induced representation $\Phi^\alpha = \text{ind}_{S(n-1)}^{S(n-2)}(\varphi^\alpha)$ and $j_i$ are matrix indices of $\psi^\nu$ (see Appendix A). From the Th. 78 in the Appendix B it follows, that if $\det Q(\alpha) = 0$, which means that for one (and only one), up to the multiplicity, Cor. 68, irreducible representation $\psi^\nu$ of $S(n-1)$ appearing in $\text{ind}_{S(n-1)}^{S(n-2)}(\varphi^\alpha) = \bigoplus \lambda \psi^\lambda$ the corresponding eigenvalue $\lambda_{\mu \nu}$ of the matrix $Q(\alpha)$ is zero, then we have

$$y_{\mu \nu}^{\rho \sigma}(\alpha) = y_{\mu \nu}^{\rho \sigma}(\alpha) = 0 : \forall \mu, j, \nu$$

and the all remaining non-zero vectors $\{y_{\mu \nu}^{\rho \sigma}(\alpha)\}$ form the reduced basis of the ideal $U(\alpha)$ such that

$$y_{\mu \nu}^{\rho \sigma}(\alpha)y_{\mu \nu}^{\rho \sigma}(\beta) = \delta^{\rho \theta} \delta^{\mu \sigma} \lambda_{\mu \nu} y_{\mu \nu}^{\rho \sigma}(\alpha), \quad \mu, \nu, \rho, \sigma \neq \theta,$$

where $\mu, \nu, \rho, \sigma$ label the irreducible representations of $S(n-1)$ that appears in the reduction of the induced representation $\Phi^\alpha = \text{ind}_{S(n-1)}^{S(n-2)}(\varphi^\alpha)$, such that the corresponding eigenvalues $\lambda_{\mu \nu}, \lambda_{\nu \rho}, \lambda_{\rho \sigma}$ of the matrix $Q(\alpha)$ are non-zero. In particular from this multiplication rule we get

$$(y_{\mu \nu}^{\rho \sigma}(\alpha))^2 = \lambda_{\nu} y_{\mu \nu}^{\rho \sigma}(\alpha)$$

e.i $y_{\mu \nu}^{\rho \sigma}(\alpha)$ is an essential projector. Again from the last theorem in the Appendix B it follows that the ideal $U(\alpha)$ is isomorphic with the matrix algebra $M(\text{rank } Q(\alpha), \mathbb{C})$ and the dimension of the ideal $U(\alpha)$ is

$$\dim U(\alpha) = (\text{rank } Q(\alpha))^2 = ((n-1) \dim \varphi^\alpha - \dim \psi^\varphi)^2.$$  

Note that in this case the vectors $\{y_{\mu \nu}^{\rho \sigma}(\alpha)\}$, as linearly dependent do not form a basis of $U(\alpha)$.

If all eigenvalues of the the matrix $Q(\alpha)$ are non zero (e.i when $\det Q(\alpha) \neq 0$), then from the Th. 78 in the Appendix B we get that, all the reduced vectors $\{y_{\mu \nu}^{\rho \sigma}(\alpha)\}$ defined in the (Eq. 7) are non-zero and they form a new basis of the algebra $U(\alpha)$, because in this case the vectors $\{y_{\mu \nu}^{\rho \sigma}(\alpha)\}$ also form a basis of the algebra $U(\alpha)$ and the ideal $U(\alpha)$ is isomorphic to the matrix algebra $M((n-1) \dim \varphi^\alpha, \mathbb{C})$. In both cases, when $\det Q(\alpha) \neq 0$, then from the Th. 78 in the Appendix B we may recast the vectors of the basis $\{y_{\mu \nu}^{\rho \sigma}(\alpha) : \mu, \nu \neq \theta\}$, according the formula given in Th. 78 in the Appendix B the following way

$$y_{\mu \nu}^{\rho \sigma}(\alpha) \rightarrow f_{\mu \nu}^{\rho \sigma}(\alpha) = \frac{1}{\sqrt{\lambda_{\mu \nu}}},$$

and a new basis $\{f_{\mu \nu}^{\rho \sigma}(\alpha)\}$ satisfies the matrix multiplication rule

$$f_{\mu \nu}^{\rho \sigma}(\alpha)f_{\mu \nu}^{\rho \sigma}(\beta) = \delta^{\rho \theta} \delta^{\mu \sigma} f_{\mu \nu}^{\rho \sigma}(\alpha), \quad \mu, \nu, \rho, \sigma \neq \theta$$

and therefore we will call this basis reduced matrix basis.

From this matrix multiplication rule it follows that, for a fixed values of $\mu \neq \theta, j_i$, the basis vectors $\{f_{\mu \nu}^{\rho \sigma}(\alpha) : \nu \neq \theta, j_i = 1, \ldots, \dim \psi^\nu\}$ span the irreducible left modules of the algebra $A^u_n(d)$ (Th. 5 and Prop. 7). Now we have to derive the transformation rule for the natural generators $V(\sigma)^{\lambda}_{\mu \nu} \in A^u_n(d), \sigma \in S(n)$ in these irreducible representations. A laborious but purely technical calculation shows that, they are the following
Proposition 47. Let \( \{ f^{\mu}_{j_\nu j_\rho} : \mu, \nu, \neq \theta \} \) be the reduced matrix basis of the ideal \( U(\alpha) \), then for a fixed values of \( \mu, j_\nu \) the vectors \( \{ f^{\mu}_{j_\nu j_\rho} \} \) form a basis of an irreducible module \( U^{\mu}_{j_\nu}(\alpha) = \text{span}_C \{ f^{\mu}_{j_\nu} : \nu \neq \theta, j_\nu = 1, \ldots, \dim \psi^{\nu} \} \) and we have

\[
V(\alpha)^n f^{\nu}_{j_\nu j_\rho} (\alpha) = \sum_{\rho \neq \theta, j_\rho} (\sum_k \sqrt{\lambda_{\rho z^*(\alpha)}^{\rho a}} z(\alpha)_{j_\rho k} \sqrt{\lambda_{\nu z^*(\alpha)}^{\nu a}} f^{\mu}_{j_\nu j_\rho} (\alpha),
\]

where the unitary matrix \( Z(\alpha) = (z(\alpha)_{j_\rho k}^{\alpha}) \) has the form

\[
z(\alpha)_{j_\rho k}^{\alpha} = \frac{\dim \psi^{\nu}}{\sqrt{N^\nu_{j_\nu}(n-1)!}} \sum_{\sigma \in S(n-1)} \psi^{\nu}_{j_\nu j_\rho} (\sigma^{-1}) \delta_{\sigma(c)} \phi_{\nu a}^a ([a n-1] \sigma(q n-1)],
\]

with

\[N^\nu_{j_\nu} = \frac{\dim \psi^{\nu}}{(n-1)!} \sum_{\sigma \in S(n-1)} \psi^{\nu}_{j_\nu j_\rho} (\sigma^{-1}) \delta_{\sigma(c)} \phi_{\nu a}^a ([a n-1] \sigma(q n-1)],
\]

where the indices \( q = 1, \ldots, n-1, r = 1, \ldots, \dim \phi^a \) are fixed and such that \( N^\nu_{j_\nu} > 0 \) (see Th.59, 85 and Cor. 86) and for \( \sigma_n \in S(n-1) \) we have

\[
V(\sigma_n) f^{\nu}_{j_\nu j_\rho} (\alpha) = \sum_{j_\rho} \psi^{\nu}_{j_\rho j_\nu} (\sigma_n) f^{\nu}_{j_\rho j_\nu} (\alpha),
\]

so for the elements \( V(\sigma_n) \in V(S(n-1)) \) the irreducible left modul \( U^{\mu}_{j_\nu}(\alpha) \) is a direct sum of the irreducible representations \( \psi^{\nu} \) of the group \( S(n-1) \), appearing in the induced representation \( \text{ind}_{S(n-2)}^{S(n-1)}(\phi^a) \), such that \( \mu \neq \theta \).

Remark 48. If the matrix \( Q(\alpha) \) is invertible then the above transformation rule for the elements \( V(\sigma_n)^{\mu} = V(\sigma_n) \) is the transformation law for the induced representation \( \text{ind}_{S(n-2)}^{S(n-1)}(\phi^a) \) in the form reduced to the direct sum of irreducible components (Th. 60, Eq. (4)).

From transformation rules for the natural generators of the algebra \( A_\infty^\mu (d) \) in the reduced matrix bases one can easily deduce the the matrix forms of the irreducible representations of the algebra \( A_\infty^\mu (d) \), which are given in Th. 34 and Prop. 33 in Sec. IV.

Unfortunately the transformation rules of the natural generators \( V(\tau)^{\nu} \in A_\infty^\mu (d) \) in the reduced basis of the ideal \( U(\alpha) \), given in the Prop. 47, are complicated because the matrix \( Z(\alpha) \), which appears in the transformation rules has complicated structure. However, in case when \( \det Q(\alpha) \neq 0 \), then transformation rules for the irreducible left modules included in the ideal \( U(\alpha) \) can be written equivalently in simpler form given in the Prop. 44 because in this case, the vectors \( \{ u^{\alpha}_{i j} (\alpha) : a, b = 1, \ldots, n-1, \ i, j = 1, \ldots, \dim \phi^a \} \) also form a basis of the ideal \( U(\alpha) \). The transformation rules for the natural generators of the algebra \( A_\infty^\mu (d) \) are much simpler in this basis but this is not a matrix basis. Using the fact that \( \det Q(\alpha) \neq 0 \), one can construct a new matrix basis without using the matrix \( Z(\alpha) \), in which the transformation rules remains as simple as in Prop. 44.

The new basis is defined as follows

Definition 49. If the matrix \( Q(\alpha) \) of the ideal \( U(\alpha) \) is invertible then we define the new elements of \( U(\alpha) \) in the following way

\[
e^{\alpha}_{i j} (\alpha) = \sum_{\gamma \delta} (Q^{-1})^{\gamma s}_{i j} (\alpha) u^{\delta}_{k l} (\alpha) : a, b, s = 1, \ldots, n-1, \ i, j, k, l = 1, \ldots, \dim \phi^a \]

where \( \alpha : \phi^a \in V_d[S(n-2)] \).

From the invertibility of the matrix \( Q(\alpha) \) it follows that the elements \( \{ e^{\alpha}_{i j} (\alpha) \} \) form a new basis of the ideal \( M \). For this basis of \( U(\alpha) \) we have
Proposition 50.

\[ e_{ij}^{ab}(\alpha) e_{kl}^{cd}(\beta) = \delta_{ib} \delta_{kc} \delta_{jk} e_{kl}^{ad}(\alpha), \]

\[ V(\sigma_{ab}) e_{ij}^{q}(\alpha) = d^{\sigma_{ab}} \sum_{k=1}^{n^a} q_{k}^{\alpha}((b n - 1) \sigma_{ab}(ac)(c n - 1)) e_{kj}^{q}(\alpha), \quad a, b \neq n, \]

where \( \sigma_{ab} = (bn) \sigma_{ab} = (an) \in S(n - 1) \) and \((b n - 1) \sigma_{ab}(ad)(d n - 1) \in S(n - 2) \) and

\[ V(\sigma_n) e_{ij}^{q}(\alpha) = \sum_{k=1}^{n^a} q_{k}^{\alpha}((\sigma_n(c) n^{-1}) \sigma_n(c n - 1)) e_{kj}^{q}(\alpha). \]

So the transformation rule of the natural generators of the algebra \( A_n^l(d) \) in the basis \( \{ e_{ij}^{q}(\alpha) \} \) are expressed only by matrices of the irreducible representations \( \varphi^a \) of \( S(n - 2) \) and they are much simpler then analogous formulas for the reduced matrix basis \( \{ f_{ij}^{q}(\alpha) \} \).

The multiplication rule of the basis elements \( e_{ij}^{ab}(\alpha) \) is in fact the matrix multiplication rule. To see this we introduce the following notation

\[ e_{ij}^{ab}(\alpha) = e_{AB}(\alpha) : A = (a - 1)w^a + i, \quad B = (b - 1)w^b + j, \]

where \( w^a = \dim \varphi^a \), then

\[ e_{AB}(\alpha) e_{CD}(\beta) = \delta_{AB} \delta_{CD} e_{AD}, \]

and for fixed irreducible representation \( \alpha \) the elements \( \{ e_{AB}(\alpha) : A, B = 1, \ldots, (n - 1)w^a \} \) span an ideal in \( M((n - 1)w^a, \mathbb{C}) \). The elements \( \{ e_{ij}^{ab}(\alpha) : \alpha = 1, \ldots, k, \quad a = 1, \ldots, n - 1, \quad i = 1, \ldots, \dim \varphi^a \} \) are primitive (irreducible) idempotents of the algebra \( A_n^l(d) \).

From the above theorem easily follows

**Corollary 51.** For any \( x = 1, \ldots, n - 1, \quad s = 1, \ldots, \dim \varphi^a \) the subspace

\[ E^x_s(\alpha) \equiv \text{span}_{\mathbb{C}} \{ e_{js}^{ax}(\alpha) : a = 1, \ldots, n - 1, \quad j = 1, \ldots, w^a \} \subset U(\alpha) \subset M \]

is a left minimal ideal (i.e. is an irreducible representation) of the algebra \( A_n^l(d) \) such that

\[ U(\alpha) = \bigoplus_{x,s} E^x_s(\alpha) \simeq M((n - 1)w^a, \mathbb{C}) \]

and all the irreducible representations \( E^x_s(\alpha) \) for any \( x = 1, \ldots, n - 1, \quad s = 1, \ldots, \dim \varphi^a \) are isomorphic, so a fixed irreducible representation \( \varphi^a \) of \( S(n - 2) \) gives us \( (n - 1)w^a \) isomorphic irreducible representations.

The fact that the subspace \( E^x_s(\alpha) \) is the irreducible representation space follows from the relation

\[ E^x_s(\alpha) = \{ a e_{js}^{ax}(\alpha) : a \in A \}, \]

where \( e_{js}^{ax}(\alpha) \) is a primitive (irreducible) idempotent which follows immediately from the matrix multiplication rule of the elements \( \{ e_{js}^{ax}(\alpha) \} \) (Th. 5).

The matrix form of irreducible representations of the algebra \( A_n^l(d) \) for the bases \( \{ e_{ij}^{ab}(\alpha) \} \) are the following

**Proposition 52.** When \( \det Q(\alpha) \neq 0 \), the natural generators \( V(\sigma_{ab}) e_{ij}^{q} \) and \( V(\sigma_n) \) of \( A_n^l(d) \) are represented in the irreducible representation space \( E^x_s(\alpha) = \text{span}_{\mathbb{C}} \{ e_{js}^{ax}(\alpha) : a = 1, \ldots, n - 1, \quad j = 1, \ldots, w^a \} \subset U(\alpha) \) by the matrices of the form

\[ M^x_{\sigma_{ab}}(V(\sigma_{ab})) e_{pq}^{a}(\alpha) = \delta_{bq} d^\delta a \sigma_{ab}(aq)(q n - 1)), \quad a, b \neq n \]

and

\[ M^x_{\sigma_n}(V(\sigma_n)) e_{pq}^{a}(\alpha) = \delta_{pq} a \sigma_{n}(a n - 1)), \quad a, b \neq n \]

and we see that that these matrices do not depend on the indices \( x, s \) so in each isomorphic irreducible representation space \( E^x_s(\alpha) \) \( x = 1, \ldots, n - 1, \quad s = 1, \ldots, \dim \varphi^a \) the operators \( V(\sigma_{ab}) e_{ij}^{q} \) are represented by the same matrices.
One can check that the matrices $M^\alpha(\sigma^tn)\rho^tn$ that represent the elements $V(\sigma^tn)$ of the algebra $A^t_n(d)$ satisfy the multiplication rule of the algebra $A^t_n(d)$ (see Th. 19 above) i.e. we have

$$M^\alpha(\sigma^tn)M^\beta(\rho^tn) = d^{\delta\sigma}(\sigma(d)n)\sigma_{ab}(dn)\rho^tn,$$

so we have to deal with the irreducible matrix representations of the algebra $A^t_n(d)$.

We may conclude this subsection formulating the following

**Corollary 53.** The ideal $M$ is a direct sum of irreducible representations (left minimal ideals), indexed by irreducible representations $\phi^\alpha \in V_d[S(n-2)]$ of $S(n-2)$ and each irreducible representations has multiplicity in $M$ equal to its dimension.

**B. Matrix ideals and left minimal ideals of the ideal $S$ of the algebra $A^t_n(d)$.**

In this section we will describe the structure of ideal $S$ of the algebra $A^t_n(d)$, complementary to the ideal $M$ constructed in Prop. 28. From the general properties of the semi-simple algebras it follows that each matrix ideal $U(\alpha)$ in $M$, being semisimple, has a unit element $e_\alpha$, which is an idempotent of the algebra $A^t_n(d)$. These units of the ideal $U(\alpha)$ are of the form

$$e_\alpha = \sum_{v \neq \theta, j_i} f_{ij}^uv,$$

in the reduced matrix basis, where $\theta : \lambda_\theta = 0$ and when $\det Q(\alpha) \neq 0$

$$e_\alpha = \sum_{a=1,...,n-1} \sum_{j_i} e_i^a(a)$$

in the basis $\{e_i^a(a)\}$ of $U(\alpha)$. The sum of all these units is a unit $e$ of the ideal $M$ and we have

**Proposition 54.** The element

$$e = \sum_{\alpha: \phi^\alpha \in V_d[S(n-2)]} e_\alpha$$

is a unit of the ideal $M$ i.e. we have

$$\forall m \in M \quad me = em = m$$

and

$$M = eA^t_n(d)e.$$
Theorem 55. The ideal is of the form

$$S = \text{span}_C \{ V(\sigma_n)(1 - e) : \sigma_n \in S(n - 1) \} = A_{n-1}(d)(1 - e),$$

where $A_{n-1}(d) \subset A_n^\mu(d)$ the the elements $\{ V(\sigma_n)(1 - e) : \sigma_n \in S(n - 1) \}$ are natural generators of the ideal $S$. The ideal $S$ (as a left $A_n^\mu(d)$ module) is a representation of the algebra $A_n^\mu(d)$. The natural generators of algebra $A_n^\mu(d)$ acts on the basis elements of $S$ in the following way

$$\forall m \in M \quad m(V(\sigma_n)(1 - e)) = 0,$$

$$\forall V(\rho_n) \in V_d[S(n - 1)] \quad V_d(\rho_n)(V_d(\sigma_n)(1 - e)) = V_d(\rho_n\sigma_n)(1 - e)$$

i.e. the elements of the ideal $M$ acts on $S$ trivially as zero operators (the elements of $M$ are not invertible). We have also

$$\forall V_d(\rho_n), V_d(\sigma_n) \in V_d[S(n - 1)] \quad V_d(\rho_n)(1 - e)(V_d(\sigma_n)(1 - e)) = V_d(\rho_n\sigma_n)(1 - e)$$

Thus the ideal $S$ is a representation of the algebra $\mathbb{C}[S(n - 1)]$ i.e. is homomorphic with this algebra. The elements $V(\sigma) : \sigma \in S(n)$ are, in general linearly depended, but we have: if $d \geq n$ then

$$S = \bigoplus_v \text{span}_C \{ E_{ij}^{S(n-1)}(\psi^\nu)(1 - e) \},$$

where

$$E_{ij}^{S(n-1)}(\psi^\nu) = \frac{\dim \psi^\nu}{(n - 1)!} \sum_{\sigma_n \in S(n-1)} \psi^\nu_{ij}(\sigma_n^{-1}) V(\sigma_n)$$

and $v$ runs over all irreducible representations of $S(n - 1)$ and if $d < n$, then

$$S = \bigoplus_v \text{span}_C \{ E_{ij}^{S(n-1)}(\psi^\nu)(1 - e) \},$$

where $v : h(\psi^\nu) < d$ and the elements $E_{ij}^{S(n-1)}(\psi^\nu)(1 - e)$ in both cases are linearly independent.

From this it follows

Corollary 56. Let $\psi^\nu \nu = 1, ..., q$ be all irreducible representations of the group $S(n - 1)$. Then

$$S \simeq \bigoplus_v M(\dim \psi^\nu, \mathbb{C})$$

where, if $d \geq n$, the direct sum is over all irreducible representations $\psi^\nu$ of the group $S(n - 1)$, and when $d < n$ then the direct sum is over $v : h(\psi^\nu) < d$. Each irreducible representation $\psi^\nu$ of the group $S(n - 1)$ defines an irreducible representation $\Psi^\nu$, of the algebra $A_n^\mu(d)$ in the following way

$$\Psi^\nu(a) = \begin{cases} 0 : a \in M, \\ \psi^\nu(\sigma_n) : a = \sigma_n \in S(n - 1). \end{cases}$$

So in this representation the non-invertible elements of the ideal $M$ are represented trivially by zero and therefore we call this representation of the algebra $A_n^\mu(d)$ semi-trivial. The matrix forms of these irreducible representations are simply matrix forms of the irreducible representations of the group algebra $\mathbb{C}[S(n - 1)] \subset A_n^\mu(d)$ and zero matrices for the elements of the ideal $M \subset A_n^\mu(d)$.

Thus from the above results we get that the algebra $A_n^\mu(d)$ admits irreducible representations of two kinds. The irreducible representations from the ideal $M$ are generated by the irreducible representations of the group $S(n - 2)$ and in these representations the algebra $A_n^\mu(d)$ is represented nontrivially. The second kind of the irreducible representations of $A_n^\mu(d)$ are included in the ideal $S$ and they are generated by irreducible representations of the group $S(n - 1)$. These representations are in fact, the irreducible representations of the subalgebra $\mathbb{C}[V(S(n - 1))] \subset A_n^\mu(d)$ in which all the partially transposed operators $V(\sigma_{ab})' \in M$ are represented trivially by zero operators.
***VI. ACKNOWLEDGMENT***

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**Appendix A: The properties of the matrices \( Q(\alpha) \).**

In this appendix we derive the basic properties of the matrices \( Q(\alpha) \) were defined in Def 29.

Directly from the definition it follows that the matrix \( Q(\alpha) \) are hermitian. In fact we have

**Proposition 57.** For any unitary representation \( q^\alpha \) of the group \( S(n - 2) \) the matrix \( Q(\alpha) \) is hermitian. This result holds even when the representation \( q^\alpha \) is not reducible.

The structure of the matrix \( Q(\alpha) \) is strictly connected with the representation of the group \( S(n - 1) \) induced by the irreducible representation \( q^\alpha \) of its subgroup \( S(n - 2) \), namely we have

**Proposition 58.** The matrix \( Q(\alpha) \) has the following form

\[
Q(\alpha) \equiv Q_{n-1}(\alpha) = \sum_{1 \leq a < b \leq n - 1} \Phi^\alpha(ab) + (d - F) \Phi^\alpha(\text{id}),
\]

where \( F = \frac{(n-2)(n-3)}{2} \frac{\chi^\alpha(12)}{\dim q^\alpha} \), \( \chi^\alpha \) is the character of the representation \( q^\alpha \) and \( \Phi^\alpha = \text{ind}_{S(n-2)}^{S(n-1)}(q^\alpha) \). Thus we see that the matrix \( Q_{n-1}(\alpha) \) is the sum of matrices representing, in the induced representation of \( S(n-1) \), all transposition in the group \( S(n - 1) \) plus the unit matrix multiplied by the number \( d - F \) depending only on \( d \) and the parameters of the representation \( (q^\alpha, V^\alpha) \) of the group \( S(n - 2) \).

In the proof of this Proposition, as well in the of the main theorem of this Appendix below, we will need the following

**Theorem 59.** Let \( (q^\alpha, V^\alpha) \) be an irreducible representation of the group \( S(m) \) with the character \( \chi^\alpha \). Then

\[
\sum_{\sigma \in K} q^\alpha(\sigma) = \frac{n_K}{\dim q^\alpha} \chi^\alpha(\sigma^{\text{inv}}) \text{id}_V^\alpha,
\]

where \( K \) is a class of conjugated elements, \( n_K \) is the number of permutations in the class \( K \) and \( \sigma^{\text{inv}} \) is any representant of the class \( K \). In particular for the class of the transpositions we have

\[
\sum_{(ab) \in S(m)} q^\alpha(ab) = \frac{m(m-1)}{2} \frac{\chi^\alpha(12)}{\dim q^\alpha} \text{id}_V^\alpha.
\]

The statement of the theorem follows from the fact that \( \sum_{\sigma \in K} q^\alpha(\sigma) \) belongs to the center of the algebra \( q^\alpha(S(m)) \), so from the irreducibility of the representation \( (q^\alpha, V^\alpha) \) and Schur Lemma we get the result. Now we go back to the proof of the Proposition 58.

**Proof.** Let \( (q^\alpha, V^\alpha) \) be an irreducible representation of the group \( S(n - 2) \) characterized by the partition(or the Young diagram) \( \alpha \). Consider the representation \( \Phi^\alpha \) of the group \( S(n - 1) \) induced by the representation \( (q^\alpha, V^\alpha) \) of the subgroup \( S(n - 2) \subset S(n - 1) \). The block matrices \( \Phi^\alpha(\sigma), \sigma \in S(n - 1) \) have dimension \( N = (n - 1) \dim q^\alpha \). If we chose the representatives of the cosets \( S(n - 1) \backslash S(n - 2) \) in the standard way i.e. as \( (1 n - 1), (2 n - 1), \ldots, (n -
2n - 1) and use the standard formula for induced matrix representation, then we have

$$
\Phi^a(ab) = \begin{pmatrix}
\varphi^a(ab) & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \varphi^a(ab)
\end{pmatrix},
$$

for $a, b \leq n - 2$ and

$$
\Phi^a(n - 1b) = \begin{pmatrix}
\varphi^a(b - 1b) & 0 & 0 \\
0 & \varphi^a(2b) & 0 \\
0 & 0 & \varphi^a(n - 2b)
\end{pmatrix},
$$

where $b \leq n - 2$. Comparing these matrices with the matrix $Q(a) \equiv Q_{n-1}(a)$ given in its definition we get the statement of the Proposition

$$
Q(a) \equiv Q_{n-1}(a) = \sum_{1 \leq a < b \leq n-1} \Phi^a(ab) + (d - f)\Phi^a(id),
$$

where $f = \frac{(n-2)(n-3)}{2} \frac{\chi^\alpha(12)}{\dim \varphi^\alpha}$ and we see that the matrix $Q_{n-1}(a)$ belongs to the algebra $\Phi^a[S(n - 1)]$.

The solution of the most important problem of eigenvalues and eigenvectors of the matrix $Q(a)$ is given in the following main theorem of this Appendix

**Theorem 60.** a) Let $\varphi^\alpha$ any irreducible representation of the group $S(n - 2)$, $\alpha = (\alpha_1, \ldots, \alpha_k)$ its partition and $Y^\alpha$ the corresponding Young diagram. The rank $r = r(\alpha)$ (or $r(Y^\alpha)$) of the partition $\alpha$ is the length of diagonal of its Young diagram. Suppose that for some $i$ the sequence $\nu = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_k)$ is a partition of $n - 1$, so it defines an irreducible representation $\psi^\nu$ of the group $S(n - 1)$ and for the Young diagrams it means that, the Young diagram $Y^\nu$ is obtained from the Young diagram $Y^\alpha$ by adding, in the $i$-th row, one box. Then the corresponding matrix $Q_{n-1}(a)$ has the following eigenvalue

i) if $r(Y^\nu) = r(Y^\nu)$, then

$$
\lambda_\nu = d + \alpha_i + 1 - i, \quad i = 1, \ldots, k + 1,
$$

and if $i = k + 1$ we set $\alpha_{k+1} = 0$.

ii) if $r(Y^\alpha) + 1 = r(Y^\nu)$ which may occur only if $i = r + 1$, then

$$
\lambda_\nu = d.
$$

The case ii) describes the situation when adding, in a proper way one box to Young diagram $Y^\alpha$ we extend its diagonal. The multiplicity of the eigenvalue $\lambda_\nu$ is equal to $\dim \psi^\nu$ and the number of pairwise distinct eigenvalues of the matrix $Q_{n-1}(a)$ is equal to Young diagrams $Y^\nu$ that one can obtain from the Young diagram $Y^\alpha$ by adding, in a proper way, one box.

b) The unitary matrix $Z(\alpha) = (z(\alpha)_{kj})$ which reduce the induced representation $\Phi^\alpha = \text{ind}_{S(n-2)}^{S(n-1)} \varphi^\alpha$ into the irreducible components has the form

$$
z(\alpha)_{kj} = \frac{1}{\sqrt{N_{\nu}}} (E_{\nu, \nu})_{kj} = \frac{\dim \psi^\nu}{\sqrt{N_{\nu}}} \sum_{\sigma \in S(n-1)} \psi^\nu_{kj} (\sigma^{-1}) \delta_{\sigma(a)} \varphi^\nu_{kj} ([a n - 1] \sigma(q n - 1)], \equiv
$$
Remark 61. The part a) of this theorem gives an explicit and remarkable simple dependence on eigenvalues of the matrix $Q^{\psi}$ and let $\psi$ be all irreducible representations of the group $S(n-1)$ whose Young diagrams are obtained from the Young diagram $\alpha$ of $\varphi^a$ by adding, in a proper way, one box and $(\psi^{\psi^a}_{\psi^a}(\sigma))$ is a matrix form of $\sigma \in S(n-1)$ in the representation $\psi^a$, $E^{\psi^a}_{\psi^a}$, is a hermitian projector of rank one in the representation space $\Phi^a$ defined by $\psi^a$ (see Def. 79 in App. C) and the double index $(q,r)$ is fixed and chosen in such a way that $\text{dim}(\psi^{\psi^a}) > 0$, which according Th. 84 and Cor. 85, is always possible. The double index $(q,r)$ shows which non-zero column from the projector $(E^{\psi^a}_{\psi^a})^{al}_{kr}$ we chose to construct the matrix $z(\alpha)_{jk}^{\psi^a}$. We have also

$$
\sum_{ak} \sum_{bl} z^a(\alpha)_{jk}^{\psi^a} \Phi^a(\sigma)_k^l \psi(\alpha)_l^m = \delta^a \phi^a_{jk}^{\psi^a}(\sigma).
$$

In particular

$$
\sum_{ak} \sum_{bl} z^a(\alpha)_{jk}^{\psi^a} Q(\alpha)_k^l \psi(\alpha)_l^m = \delta^a \phi^a_{jk}^{\psi^a} \lambda^a,
$$

(A1)

and the columns of the matrix $Z(\alpha) = (z(\alpha)_{jk}^{\psi^a})$ are eigenvectors of the matrix $Q(\alpha)$.

Remark 62. The part a) of this theorem gives an explicit and remarkable simple dependence on eigenvalues of the matrix $Q_{n-1}(\alpha)$ on the partition $\alpha = (\alpha_1, \ldots, \alpha_k)$ which defines the the irreducible representation $\varphi^a$ and consequently the matrix $Q_{n-1}(\alpha)$.

The proof of this theorem uses for lemmas which are succeeding steps of this proof. The first is the following

Lemma 62. Let $\varphi^a$ any irreducible representation of the group $S(n-2)$, $\alpha$ its partition (or Young diagram), $\chi^\alpha$ its character and let $\psi^a$ be all irreducible representations of the group $S(n-1)$ whose Young diagrams are obtained from the Young diagram $\alpha$ of $\varphi^a$ by adding, in a proper way, one box. By $\chi^\psi$ we denote their characters, where $\psi$ is the partition of $n-1$ which labels the representation $\psi^a$. Then the distinct eigenvalues of the matrix $Q(\alpha)$ generated by the irreducible representation $\varphi^a$ of $S(n-2)$ are labelled by the partitions $\nu$ and are of the form

$$
\lambda = d + \frac{(n-1)(n-2)}{2} \frac{\chi^\psi(ab)}{\dim \psi^a} - \frac{(n-2)(n-3)}{2} \frac{\chi^{\psi^a(ab)}}{\dim \varphi^a},
$$

where $(ab), a, b < n-2$ is an arbitrary transposition in $S(n-2)$, the eigenvalue $\lambda$ has multiplicity $\dim \psi^a$.

Proof. The induced representation $\Phi^a$ of the group $S(n-1)$ is reducible, in fact we have

$$
\Phi^a = \bigoplus_{\lambda} \psi^\lambda,
$$

where $\psi^\lambda$ are irreducible representations of the group $S(n-1)$ whose Young diagrams are obtained from the Young diagram $\alpha$ of $\varphi^a$ by adding, in a proper way, one box. This means that one can transform the matrix $Q_{n-1}(\alpha)$ to the reduced block diagonal $Q_{n-1}^R(\alpha)$ form by a similarity transformation generated by a block matrix $Z = (z_{ij}^\lambda)$, $j_\lambda = 1, \ldots, \dim \psi^\lambda$ such that

$$
Q_{n-1}^R(\alpha) = Z^{-1} Q_{n-1}(\alpha) Z = \bigoplus_{\lambda} (\sum_{1 \leq a < b \leq n-1} \psi^\lambda(ab)) + (d - f) \mathbb{1},
$$

where the representations $\psi^\lambda$ of $S(n-1)$ are irreducible so from the Theorem we get

$$
\sum_{1 \leq a < b \leq n-1} \psi^\lambda(ab) = \frac{(n-1)(n-2)}{2} \frac{\chi^\psi(12)}{\dim \psi^a} \mathbb{1},
$$

and it is clear that the similarity transformation simply diagonalizes the matrix $Q_{n-1}(\alpha)$ giving all eigenvalues with their multiplicities. The matrix $Q(\alpha)$, as a block matrix of an induced representation, has in natural way, double
indices \( a, j : a = 1, \ldots, n - 1, \quad j = 1, \ldots, \dim \varphi^a \), where \( \varphi^a \) is an irreducible representation of the group \( S(n - 2) \). After the reduction of the induced representation \( \Phi^a \) of the group \( S(n - 1) \) to the form

\[
\Phi^a = \bigoplus_{\lambda} \psi^\lambda,
\]

where \( \psi^\lambda \) are irreducible representations of the group \( S(n - 1) \), the matrix \( Q(\alpha) \) is transformed to the reduced form \( Q^R_{n-1}(\alpha) \) which has indices naturally related to its block diagonal structure

\[
Q^R_{n-1}(\alpha) = \left( (Q^R_{n-1})^{\nu\mu}_{jk}(\alpha) \right)
\]

and

\[
(Q^R_{n-1})^{\nu\mu}_{jk}(\alpha) = \sum_{i, b, k} (z^{-1})^{\nu a}_{i, j} Q^{ab}_{jk}(\alpha) z^{b\mu}_{k j}
\]

The formula for the eigenvalue of the matrix \( Q_{n-1}(\alpha) \), given in the Lemma 62 is not entirely analytical because there is no analytical formula for irreducible characters of the group \( S(n) \). For arbitrary \( n \in \mathbb{N} \). However for a given \( n - 2 \in \mathbb{N} \) and given partition \( \alpha \) of \( n - 2 \) and can always calculate the value of the corresponding character on the class of transpositions as well the value of the irreducible character \( \chi^\psi \) on the class of transpositions.

The formula for the eigenvalues of the matrix \( Q_{n-1}(\alpha) \) derived in the Lemma 62 express the eigenvalues by characters if irreducible representations of \( S(n - 2) \) and \( S(n - 1) \). The next step necessary for the proof of the Th. 60 is to derive the formula for the eigenvalues of the matrix \( Q_{n-1}(\alpha) \) which allows to write down (practically without calculations) these eigenvalues using some characteristic of Young diagram of the irreducible representation \( \varphi^a \) of the group \( S(n - 2) \). For this we will need the following

**Definition 63.** Let \( \alpha = (a_1, \ldots, a_k) \) be a partition of \( n - 2 \) and \( Y^a \) corresponding Young diagram. We define the rank \( r = r(\alpha) \) (or \( r(\ Y^a) \) ) of the partition \( \alpha \) is the length of diagonal of its Young diagram. Let \( a_i, b_i, i = 1, \ldots, r \) be respectively, the number of the boxes below and to the right of the \( i \)-th box in the diagonal, reading from the upper left to the lower right. So we have

\[
a_1 > a_2 > \ldots > a_r, \quad b_1 > b_2 > \ldots > b_r,
\]

and

\[
\begin{pmatrix}
a_1 & a_2 & \ldots & a_r \\
b_1 & b_2 & \ldots & b_r
\end{pmatrix}
\]

is called the characteristic of the partition \( \alpha \) or equivalently of its Young diagram.

**Example 64.** \( \alpha = (4, 2, 2) \) so

\[
Y^a =
\]

then the rank is \( r = 2 \) and the characteristic is

\[
\begin{pmatrix}
2 & 1 \\
3 & 0
\end{pmatrix}
\]

The important point in the proof of the part a) of the main theorem is the following result proved by Frobenius.

**Theorem 65.** Let \( \varphi^a \) the irreducible representation of the group \( S(m) \) indexed by the partition \( \alpha \) or rank \( r \) and characteristic

\[
\begin{pmatrix}
a_1 & a_2 & \ldots & a_r \\
b_1 & b_2 & \ldots & b_r
\end{pmatrix}
\]

then the value of its character on the class of transpositions is given by very simple formula

\[
\chi^a(12) = \frac{\dim V^a}{m(m - 1)} \sum_{i=1}^r (b_i(b_i + 1) - a_i(a_i + 1)).
\]
A substitution of this formula into the formula for eigenvalues of the matrix $Q_{n-1}(\alpha)$ derived in the Lemma 62 gives us

**Lemma 66.** Let $\phi^\alpha$ any irreducible representation of the group $S(n-2)$, $\alpha = (a_1, \ldots, a_k)$ its partition of rank $r = r(\alpha)$ with characteristic

$$
\begin{pmatrix}
    a_1 & a_2 & \ldots & a_r \\
    b_1 & b_2 & \ldots & b_r
\end{pmatrix}
$$

and the Young diagram $Y^\alpha$. Suppose that for some $i$ the sequence $\nu = (a_1, \ldots, a_i + 1, a_k)$ is a partition of $n - 1$, so it defines an irreducible representation $\psi^\nu$ of the group $S(n-1)$ and for the Young diagrams it means that the Young diagram $Y^\nu$ is obtained from the Young diagram $Y^\alpha$ by adding, in the $i$-th row, one box. Then the corresponding matrix $Q_{n-1}(\alpha)$ has the following eigenvalue

a) $$\lambda_\nu = d + b_i + 1 \quad \text{if} \quad i \leq r,$$

b) $$\lambda_\nu = d - (a_j + 1) : j = \lambda_i + 1 \quad \text{if} \quad i > r \quad \text{and} \quad r(\nu) = r(Y^\nu),$$

c) $$\lambda_\nu = d \quad \text{if} \quad i = r + 1 \quad \text{and} \quad r(\nu) + 1 = r(Y^\nu).$$

The case c) describes the situation when adding, in a proper way one box to Young diagram $Y^\alpha$ we extend its diagonal. The multiplicity of the eigenvalue $\lambda_\nu$ is equal to $\dim \psi^\nu$ and the number of pairwise distinct eigenvalues of the matrix $Q_{n-1}(\alpha)$ is equal to Young diagrams $Y^\nu$ that one can obtain from the Young diagram $Y^\alpha$ by adding, in a proper way, one box.

Thus derivation of the eigenvalues of the matrix $Q_{n-1}(\alpha)$ using this Lemma need not any calculations. It is enough to look what Young diagrams $Y^\nu$ one can obtain from the Young diagram $Y^\alpha$ by adding, in a proper way, one box and then the place of this extra box in $Y^\nu$ determines what number from the characteristic of the Young diagram $Y^\alpha$ one should add to $d \pm 1$ or to $d$ in order to get an eigenvalue of the matrix $Q_{n-1}(\alpha)$.

The last step of the proof of the part a) of the Th. 60 is a simple analysis of dependence between the numbers of the partition $\alpha = (a_1, \ldots, a_k)$ of the irreducible representation $\phi^\alpha$ on the numbers of its characteristic

$$
\begin{pmatrix}
    a_1 & a_2 & \ldots & a_r \\
    b_1 & b_2 & \ldots & b_r
\end{pmatrix}
$$

which leads to the remarkable simply formula for the eigenvalues of the matrix $Q_{n-1}(\alpha)$ given in the part a) of the main theorem in this Appendix.

The proof of the part b) of the main theorem consists of a rather laborious but purely technical calculation where, in order to show the statements of the theorem, one uses the orthogonality relations for irreducible representations of the symmetric group which are of the form

$$
\frac{1}{m!} \sum_{\sigma \in S(m)} \phi^\alpha_{ij}(\sigma^{-1}) \phi^\beta_{kl}(\sigma) = \frac{1}{\dim \phi^\alpha} \delta^{\alpha \beta} \delta_{ij} \delta_{kl},
$$

where $\phi^\alpha$ and $\phi^\beta$ are irreducible representations of the symmetric group $S(m)$.

The explicit formulas for the eigenvalues and eigenvectors of the matrix $Q_{n-1}(\alpha)$ derived in the Th. 60 leads to several conclusions.

For any Young diagram $Y^\alpha$ we may always add one box at the end of first line as well below the last line of $Y^\alpha$, therefore from the main theorem one deduce

**Corollary 67.** Under assumptions of the theorem 60 we have: for $\alpha = (a_1, \ldots, a_k)$ the numbers

$$
\lambda_\nu = d + a_1, \quad \lambda_\nu = d - k
$$

are always the eigenvalues of the matrix $Q_{n-1}(\alpha)$ and

$$
d = k \Rightarrow \lambda_k = 0.
$$
The existence of zero eigenvalues of the matrices $Q_{n-1}(\alpha)$ is essential for the inversibility of these matrices. The following conclusion describes when the zero eigenvalues in matrices $Q_{n-1}(\alpha)$ do appear.

**Corollary 68.** The matrix $Q_{n-1}(\alpha)$ has an eigenvalue equal to zero iff for some $i$ the sequence $\nu = (\alpha_1, \alpha_i + 1, \ldots, \alpha_k)$ is a partition of $n - 1$, so it defines an irreducible representation $\psi^\nu$ of the group $S(n - 1)$ and

$$d = i - \alpha_i - 1.$$  

The multiplicity of this zero eigenvalue is equal to $\dim \psi^\nu$ and such a zero eigenvalue may appear only for one irreducible representation $\psi^\nu$.

Directly from the formula for the eigenvalues of the matrix $Q_{n-1}(\alpha)$, given in Th. 59 one gets also

**Corollary 69.** If $d > n - 2$ then the matrix $Q(\alpha)$ is (strictly) positive i.e. $Q(\alpha) > 0$ and consequently if $d > n - 2$ the matrix $Q(\alpha)$ is invertible for any irreducible representation $\psi^\alpha$.

An alternative proof of this statement is given in [5].

Using the formula of this theorem one can easily calculate the eigenvalues for particular matrices $Q_{n-1}(\alpha)$.

The simplest cases are the following

**Example 70.** When the irreducible representation $\psi^\alpha$ of the group $S(n - 2)$ is the identity representation then from the theorem we get

$$\lambda_0 = d + n - 2, \quad \lambda_1 = d - 1,$$

with the multiplicities $1$ and $n - 2$ respectively.

Similarly

**Example 71.** When the irreducible representation $\psi^\alpha$ of the group $S(n - 2)$ is the $\text{sgn}$ representation then from the theorem we get

$$\lambda_0 = d - (n - 2), \quad \lambda_1 = d + 1,$$

with the multiplicities $1$ and $n - 2$ respectively.

These results one can obtain independently, by a direct calculation of eigenvalues of the matrices $Q(\alpha)$ using standard procedures, which are easily applicable in these simplest cases. The next examples are less simple

**Example 72.** Let $\alpha = (2, 1, \ldots, 1)$ be a partition of $n - 2$. So $r(\alpha) = 1$. Now by adding to the Young diagram $Y^\alpha$ one box one can obtain three young diagrams $Y^\nu$ describing the irreducible representations of the group $S(n - 1)$.

I) $\nu = (3, 1, 1, \ldots, 1)$ and this the case a) i) in the main theorem, so the eigenvalue takes the form

$$\lambda_{\nu} = d + 2.$$

II) $\nu = (2, 2, 1, \ldots, 1)$ so and this is the case a) ii), where the diagonal of the Young diagram $Y^\alpha$ is extended therefore

$$\lambda_{\nu} = d.$$

III) $\nu = (2, 1, 1, \ldots, 1)$ then we have to deal again with the case a) i) of the theorem so we get

$$\lambda_{\nu} = d - n + 3.$$

**Example 73.** Let $\alpha = (n - 3, 1, \ldots, 0)$ be a partition of $n - 2$. So $r(\alpha) = 1$ and by adding to the Young diagram $Y^\alpha$ one box one obtain three Young diagrams $Y^\nu$ describing the irreducible representations of the group $S(n - 1)$.

I) $\nu = (n - 2, 1, 0, \ldots, 0)$ and this the case a) i) in the main theorem, so the eigenvalue takes the form

$$\lambda_{\nu} = d + n - 3.$$

II) $\nu = (n - 3, 2, 0, \ldots, 0)$ and this the case a) ii), where the diagonal of the Young diagram $Y^\alpha$ is extended therefore

$$\lambda_{\nu} = d.$$

III) $\nu = (n - 3, 1, 1, \ldots, 0)$ now it is the case a) i) in the Theorem so we get

$$\lambda_{\nu} = d - 2.$$
Appendix B

In this Appendix we will consider the properties of the following algebra

**Definition 74.** Let \( A = (a_{ij}) \in M(m, \mathbb{C}) \), then the algebra \( X_A \) is defined as
\[
X_A = \text{span}_\mathbb{C} \{ x_{ij} : i, j = 1, \ldots, m \},
\]
where
\[
x_{ij}x_{kl} = a_{jk}x_{il}.
\]
and we do not assume that the elements \( \{ x_{ij} \} \) are linearly independent, thus the algebra \( X_A \) is a complex finite-dimensional algebra of the dimension at most \( m^2 \).

Obviously the properties of the algebra \( X_A \) depends on the properties of the matrix \( A \). In fact we have

**Theorem 75.** Suppose that the matrix \( A \) in the algebra \( X_A \) is invertible then we have two possibilities
a) \( X_A = \{ 0 \} \) such that the algebra \( X_A \) a zero algebra,
b) If \( X_A \neq \{ 0 \} \) then algebra \( X_A \) is isomorphic to the matrix algebra \( M(m, \mathbb{C}) \) and the elements \( \{ x_{ij} : i, j = 1, \ldots, m \} \) are linearly independent, in particular \( x_{ij} \neq 0 \) for some indices \( i, j = 1, \ldots, m \), and form the basis of \( X_A \), and in this case the unit of the algebra \( X_A \) is of the form
\[
1 = \sum_{i,j=1,...,m} (a_{ij}^{-1})x_{ij},
\]
where \( A^{-1} = (a_{ij}^{-1}) \).

**Proof.** Suppose that \( x_{ij} = 0 \) for some indices \( i, j = 1, \ldots, m \). Then from multiplication law for the algebra \( X_A \) we get
\[
\forall k, l = 1, \ldots, m, \quad x_{ij}x_{kl} = a_{jk}x_{il} = 0
\]
and because the matrix \( A = (a_{ij}) \) is invertible (so it has no zero columns or zero rows) then we get \( x_{ij} = 0 \Rightarrow \forall k, l = 1, \ldots, m \quad x_{ik} = x_{il} = 0 \) and consequently \( \forall k, l = 1, \ldots, m \quad x_{kl} = 0 \).

If \( X_A \neq \{ 0 \} \) then defining new basis \( y_{ij} = \sum_{i=1,...,m} (a_{ik}^{-1})x_{ij} \), we get
\[
y_{ij}y_{kl} = \delta_{jk}y_{il}.
\]

So we see that the invertibility of the matrix \( A \) strongly determines the properties of the vectors \( \{ x_{ij} : i, j = 1, \ldots, m \} \) which span the algebra \( X_A \). We have also

**Theorem 76.** Suppose the the vectors \( \{ x_{ij} : i, j = 1, \ldots, m \} \) which span the algebra \( X_A \) are linearly independent and \( \det(A) = 0 \) then there exist in the algebra \( X_A \) a nonzero properly nilpotent element and consequently the algebra \( X_A \) is not semisimple.

**Proof.** From the assumption \( \det(A) = 0 \) it follows that there exist s nonzero vector \( u = (u_1, u_2, \ldots, u_m) \in \mathbb{C}^m \) such that
\[
Au = 0 \iff \sum_{k=1,...,m} (a_{ik})u_k = 0 : i = 1, \ldots, m.
\]
Consider now, for arbitrary \( l = 1, \ldots, m, \) an element \( w_l = \sum_{k=1,...,m} u_kx_{kl} \in X_A \) which from the first assumption is nonzero. From the multiplication law of the algebra \( X_A \) we get
\[
x_{ij}w_l = \sum_{k=1,...,m} u_kx_{ij}x_{kl} = \sum_{k=1,...,m} u_ka_{jk}x_{il} = 0 \quad \forall i, j = 1, \ldots, m.
\]
which means that the nonzero elements \( w_l \) are properly nilpotent and therefore from the Def. the algebra \( X_A \) is not semisimple.
From this theorem it follows

**Corollary 77.** If the algebra \( X_A \) is semisimple and \( \det(A) = 0 \), then the vectors \( \{x_{ij} : i, j = 1, \ldots, m\} \) which span the algebra \( X_A \) are linearly dependent.

In this case the question naturally arises, how to reduce the set of linearly dependent vectors \( \{x_{ij} : i, j = 1, \ldots, m\} \) to the set of linearly independent ones. If the algebra \( X_A \) is semisimple then we have the following method of constructing basis of \( X_A \).

**Theorem 78.** Let the algebra \( X_A \) be a semisimple algebra such that

\[
X_A = \text{span}_\mathbb{C}\{x_{ij} : i, j = 1, \ldots, m\},
\]

where

\[
x_{ij}x_{kl} = a_{jk}x_{il}.
\]

and the matrix \( A = (a_{ij}) \) is diagonalizable i.e.

\[
Z^{-1}AZ = \text{diag}(\lambda_1 \neq 0, \ldots, \lambda_p \neq 0, 0, \ldots, 0) \iff \sum_{jk} z_{ij}^{-1} a_{jk} z_{kl} = \lambda_i \delta_{il}, \quad Z \in M(m, \mathbb{C}).
\]

So if \( p = m \), then the matrix \( A \) is invertible. Define a new elements of the algebra \( X_A \)

\[
y_{sr} = \sum_{jk} z_{ij}^{-1} x_{ij} z_{is}.
\]

These elements have the following properties

\[
y_{sr} = y_{rs} = 0 \quad \forall s = 1, \ldots, m, \quad r > p,
\]

the remaining non-zero vectors \( \{y_{ij} : i, j = 1, \ldots, p\} \) form the basis of the algebra \( X_A \), which we call a reduced basis, and they satisfy the following multiplication rule

\[
y_{ij}y_{kl} = \lambda_j \delta_{jl} y_{il}, \quad i, j, k, l = 1, \ldots, p.
\]

A simple rescaling of the basis vectors \( \{y_{ij} : i, j = 1, \ldots, p\} \) of the form

\[
y_{ij} \rightarrow f_{ij} = \frac{1}{\sqrt{\lambda_i \lambda_j}} y_{ij}
\]

gives a new basis of the algebra \( X_A \), which satisfies the matrix multiplication rule

\[
f_{ij}f_{kl} = \delta_{jl} f_{il}, \quad i, j, k, l = 1, \ldots, p
\]

and this proves that the algebra \( X_A \) is isomorphic with the matrix algebra \( M(p, \mathbb{C}) \).

**Appendix C**

In this Appendix we describe the method of reducing the algebra generated by operators representing a finite group in a given representation, to the direct sum of matrix algebras.

**Notation 79.** Let \( G \) be a finite group of order \( |G| = n \) which has \( r \) classes of conjugated elements. Then \( G \) has exactly \( r \) inequivalent, irreducible representations, in particular \( G \) has exactly \( r \) inequivalent, irreducible matrix representations. Let

\[
\phi^\alpha : G \rightarrow \text{Hom}(V^\alpha), \quad \alpha = 1, 2, \ldots, r, \quad \dim V^\alpha = w^\alpha
\]

be all inequivalent, irreducible representations of \( G \) and let chose these representations to be all unitary (always possible) i.e.

\[
\phi^\alpha(g) = (\phi^\alpha_{ij}(g)), \quad i, j = 1, 2, \ldots, w^\alpha, \quad (\phi^\alpha_{ij}(g))^\dagger = (\phi^\alpha_{ij}(g))^{-1}
\]

and \( V^\alpha \) are corresponding representation spaces.
The matrix elements \( \varphi^\alpha_{ij}(g) \) will play a crucial role in the following.

Any complex finite-dimensional representation \( D : G \to \text{Hom}(V) \) of the finite group \( G \), where \( V \) is a complex linear space (dim \( V = w \)), generates a algebra \( A_V[G] \subset \text{Hom}(V) \) which homomorphic to the group algebra \( \mathbb{C}[G] \) and in particular isomorphic if the representation \( D \) is faithful. Obviously

\[
A_V[G] = \text{span}_\mathbb{C}\{D(g), \ g \in G\}.
\]

We have also the following decomposition of the representation \( D : G \to \text{Hom}(V) \)

\[D = \bigoplus_{a=1}^r k_a \varphi^a, \quad V = \bigoplus_{a=1}^r k_a V^a, \quad k_a \in \mathbb{N} \cup \{0\},\]

where \( k_a \) is the multiplicity of the irreducible representation \( D^a \) in \( D \). If the operators \( D(g) \) are linearly independent, then they form a basis of the algebra \( A_V[G] \) and in this case dim \( A_V[G] = |G| \). It is also possible, using matrix irreducible representations, to construct a new basis which has a remarkable properties, very useful in applications of representation theory. Below we define the new basis and we use it in the study of the linear independence of the group operators \( D(g), g \in G \).

**Definition 80.** Let \( D : G \to \text{Hom}(V) \) be an unitary representation of a finite group \( G \) and let \( \varphi^a : G \to \text{Hom}(V^a) \) be all inequivalent, irreducible representations of \( G \). Define a matrix operators in the following way

\[
E^a_{ij} = \frac{w^a}{n} \sum_{g \in G} \varphi^a_{ji}(g^{-1})D(g), \quad a = 1, 2, ..., r, \quad i, j = 1, 2, ..., w^a, \quad E^a_{ij} \in A_V[G] \subset \text{Hom}(V).
\]

**Remark 81.** In this definition we do not assume that the operators \( D(g), g \in G \) are linearly independent.

The matrix operators have noticeable properties listed in the

**Theorem 82.**

I) There are exactly \( |G| = n \) operators \( E^a_{ij} \) but in general they need not to be distinct and we have

\[
D(g) = \sum_{ij\alpha} \varphi^a_{ij}(g)E^a_{ij}
\]

II) the operators \( E^a_{ij} \) are orthogonal with respect to the Hilbert-Schmidt scalar product in the space \( \text{Hom}(V) \).

\[
(E^a_{ij}, E^\beta_{kl}) = \text{tr}((E^a_{ij})^{\dagger}E^\beta_{kl}) = k_\alpha \delta^a_\beta \delta_{ik}\delta_{jl},
\]

where \( k_\alpha \) is the multiplicity of the irreducible representation \( \varphi^a \) in \( D \) and it does not depend on \( i, j = 1, 2, ..., w^a \).

III) the operators \( E^a_{ij} \) satisfy the following composition rule

\[
E^a_{ij}E^\beta_{kl} = \delta^a_\beta \delta_{ij}E^\beta_{il},
\]

in particular \( E^a_{ij} \) are orthogonal projections.

IV) If the irreducible representation \( \varphi^a \) is included in the representation \( D \), then for a fixed \( j = 1, 2, ..., w^a \) the operators \( E^a_{ij} : i = 1, 2, ..., w^a \) are the basis of the representation \( \varphi^a \) i.e. we have

\[
D(h)E^a_{ij} = \sum_{k=1}^{w^a} \varphi^a_{kj}(h)E^a_{ij}.
\]

**Remark 83.** Note that from point II) of the theorem it follows that if \( k_\alpha = 0 \) for some \( \alpha \in \{1, ..., r\} \) (that is irreducible representation \( \varphi^a \) is not included in \( D \)), then necessarily \( E^a_{ij} = 0 \) for all \( i, j = 1, 2, ..., w^a \). This important property will be considered in details below. From this part of the theorem it follows also that the equations

\[
E^a_{ij} = \frac{w^a}{n} \sum_{g \in G} \varphi^a_{ji}(g^{-1})D(g)
\]

describe transformation of orthogonalization of operators \( D(g), g \in G \) in the space \( \text{Hom}(V) \) with the Hilbert-Schmidt scalar product.
Using elementary methods or the famous Peter-Weyl theorem one can prove

**Theorem 84.** The operators \( \{ E_{ij}^\alpha : \alpha = 1, 2, ..., r, \ i, j = 1, 2, ..., w^\alpha \} \) are linearly independent if and only if the operators \( \{ D(g), \ g \in G \} \) are linearly independent. Obviously in this case both sets form a bases of the algebra \( A_V[G] \).

From Th.81 it follows, in particular, that if the irreducible representation \( \phi^\alpha \) of \( G \) has multiplicity one in the representation \( D \) then the operators \( E_{jj}^\alpha, \ j = 1, 2, ..., w^\alpha \) are a hermitian projectors of rank one. Such a projectors have the following properties

**Theorem 85.** Let \( P = (p_{ij}) \in M(n, \mathbb{C}) \) and \( P^2 = P = P^+, \ \text{rank}(P) = 1 \) and let \( P_q = (p_{iq}) : i = 1, ..., n \) be a non-zero column of \( P \), then

\[
0 < p_{qq} \leq 1.
\]

**Corollary 86.** If the irreducible representation \( \phi^\alpha \) of \( G \) has multiplicity one in the representation \( D \), then for any \( j = 1, 2, ..., w^\alpha \), there exist a non-zero column \( q \) in the projector \( E_{jj}^\alpha \), such that \((E_{jj}^\alpha)_{qq} > 0\).

The structure of the algebra \( A_V[G] \), in particular its dimension, is the following.

**Theorem 87.**

\[
A_V[G] = \bigoplus_\alpha \text{span}_{\mathbb{C}} \{ E_{ij}^\alpha : \phi^\alpha \in D, \}
\]

in particular

\[
\dim A_V[G] = \sum_{\alpha : \phi^\alpha \in D} (\dim \phi^\alpha)^2.
\]

The operators \( D(g) : g \in G \) are linearly independent, in this case \( \dim A_V[G] = |G| \) iff each irreducible representations \( \phi^\alpha, \alpha = 1, 2, ..., r \), appears in the decomposition \( D = \bigoplus_{\alpha=1}^r k_\alpha \phi^\alpha \) e.i. \( k_\alpha \geq 1 \), \( \ \forall \alpha = 1, 2, ..., r \).

From this theorem it follows that in order to check the linear independence of the operators \( D(g), \ g \in G \) in the representation algebra algebra \( A_V[G] \) it is enough to know the multiplicities \( k_\alpha \) of the irreducible representations \( \phi^\alpha, \alpha = 1, 2, ..., r \) in the representation \( D \).
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