ALGEBRAIC SOLUTIONS OF THE
MULTICOMPONENT KP HIERARCHY

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Abstract. It is shown that it is possible to write down tau functions for the $n$-component KP hierarchy ($n$-KP) in terms of non-abelian theta functions. This is a generalization of the rank 1 situation; that is, the relation of theta functions of Jacobians and tau functions for the KP hierarchy.

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1. Introduction

This paper is concerned with the generalization for higher rank of the relation of theta functions of Jacobians and the KP hierarchy. More precisely, the well known relation between the theta function of a Jacobian and the tau function of a certain point of the infinite Grassmannian $Gr^0(k((z)))$ ([Kr2, SW, Sh], see also [MP, Pl, PW]) is generalized for non-abelian theta functions and $Gr^0(k((z))^{[m]}^* )$. From this point of view, the main result is the Theorem 5.1 that shows that it is possible to write down tau functions for the $n$-component KP hierarchy ($n$-KP) in terms of non-abelian thetas. However, further research must be made to obtain explicit expressions.
It is worth remarking that this result is twofold; on the one hand, it shows that non-abelian thetas give solutions for \( n \)-KP; on the other, it suggests the possibility of characterize non-abelian thetas in terms of differential equations of the \( n \)-KP (see [Sh, MP] for the rank 1 case).

In a certain sense, §2 is the core of the paper. It contains three new results; namely, Lemma 2.5 (Addition formula) and Theorems 2.6 and 2.7 (Residue Bilinear Identity). These results are the key for proving that there exists a 1-1 correspondence between wave functions for the \( n \)-KP and Baker-Akhiezer (BA) functions of points of the infinite Grassmannian \( \text{Gr}^0(k((z))^{\oplus n^2}) \) (see §3). Then, our strategy is rather simple (see §5): a generalization of the well-known Krichever map ([Kr2, Mm, Ml]) is used to show that there is a subscheme:

\[
i : \hat{U}(r,d) \hookrightarrow \text{Gr}^0(k((z))^{\oplus n^2})
\]

together with a projection onto the moduli space of vector bundles, \( \pi : \hat{U}(r,d) \to U(r,d) \) (some data must be previously fixed) such that:

\[
i^*_F \Omega_+ = \pi^* \theta_F
\]

\[
i^*_F \text{Det} \cong \pi^* \mathcal{O}(\Theta_{\{F\}})
\]

(see §5 for notations and precise statements). This proves that non-abelian thetas give rise to tau functions for the \( n \)-KP.

Before explaining how the paper is organized, let us point out an intermediate result that deserves special mention; namely, Theorem 4.3 that computes the equations defining the subscheme of \( \text{Gr}(k((z))^{\oplus n}) \times \text{Gr}(k((z))^{\oplus n \cdot r}) \) whose set of rational points is:

\[
\{(C, p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n, M, \beta)\}
\]

where \( C \) is a curve, \( p_i \in C \), \( \alpha_i : \hat{O}_{C, p_i} \simeq k[[z]] \), \( M \) is a rank \( r \) torsion free sheaf, and \( \beta : \hat{M}_{\{p_i\}} \cong k[[z]]^{\oplus n \cdot r} \).

In §2 the approach of [MP] to the KP hierarchy, which is based in the “geometry of formal curves”, is developed for the \( n \)-KP; that is, for the infinite Grassmannian of \( E((z)) \) where \( E \) is a finite dimensional \( k \)-vector space. This enables us to define tau functions for the \( n \)-KP in terms of global sections of the determinant bundle and to generalize the Addition formula for this situation (see Lemma 2.5). Similarly to the rank 1 case, the BA function of an arbitrary point of \( \text{Gr}(E((z))) \) is defined as a certain deformation of the tau function. However, from the Theorem 2.7 it follows that our definition agrees with the standard one ([Kr1, Kr2, KrN], see [Kr4, P1, PW] for overviews on the subject) for those points coming from algebro-geometric data. This section finishes with the generalization of the Residue Bilinear Identity (Theorem 2.7).
Section §3 recalls the definition of the $n$-KP and shows that there exists a 1-1 correspondence between wave functions for the $n$-KP and BA functions of points of the infinite Grassmannian $Gr^0(k((z))^{\oplus n^2})$.

Moduli spaces are studied in section §4 in terms the Krichever map and infinite Grassmannians. Although it is not needed for our main purpose, we have considered convenient to include here the equations for these moduli spaces (Theorem [1,3]).

The last section unveils the deep relation among tau functions of algebro-geometric points and non-abelian thetas, which is the “expected” generalization of the rank 1 situation.

Finally, I hope that these results will help in the study of some related problems; particularly, those related with higher rank vector bundles over curves and infinite Grassmannians and with differential operators (e.g. higher rank commutative subrings of differential operators, Darboux-Backlund transform, etc).

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2. INFINITE GRASSMANNIANS

2.A. Background. Let us address the reader to [AMP, MP] for the scheme-theoretic approach to infinite Grassmannians. However, it is convenient to recall some basic facts in order to fix notations and to point out the statements which we shall need.

Since we are concerned with the multicomponent KP hierarchy, we shall not deal with general infinite Grassmannians, but only with that of $V = E((z))$ where $E$ is a $n$-dimensional $k$-vector space. Denote $E[[z]]$ by $V^+$ and consider the linear topology in $V$ given by $\{z^m V^+ | m \in \mathbb{Z}\}$ as a basis of neighborhoods of $(0)$.

Then, we know that there exists a $k$-scheme $Gr(V)$ locally covered by the open subschemes:

$$F_A(S) := \{\text{sub-$\mathcal{O}_S$-modules $\mathcal{L} \subset \hat{V}_S$ such that $\mathcal{L} \oplus \hat{A}_S = \hat{V}_S$}\}$$

where $S$ is a $k$-scheme and:

- $A \subset V$ is a subspace such that $A \sim V^+$; that is, $\dim(A + V^+)/A \cap V^+ < \infty$;
- $\hat{A}_S$ is defined by $\lim_{B \sim V^+} ((A/A \cap B) \otimes \mathcal{O}_S)$ for a subspace $A \subseteq V$.

(For instance, when $E = k$ one has $\hat{V}_S^+ = \lim_m \mathcal{O}_S[z]/z^m =: \mathcal{O}_S[[z]]$,

and $\hat{V}_S = \lim_m z^{-m} \mathcal{O}_S[[z]] =: \mathcal{O}_S((z))$.)
The generalization of some “good” properties of the 1-dimensional case requires a choice of a chain of strict inclusions $V_0^+ = V^+ \subset V_1^+ \subset \ldots \subset V_n^+ = z \cdot V^+$ (note that it follows that the inclusions are of codimension 1). Some remarkable facts are:

- the function:
  $$\text{Gr}(V) \longrightarrow \mathbb{Z}$$
  $$L \mapsto \dim(L \cap V^+) - \dim(V/L + V^+)$$
  gives the decomposition of $\text{Gr}(V)$ in connected components, which will be denoted by $\text{Gr}^n(V) \ (n \in \mathbb{Z})$;
- there is a natural line bundle on $\text{Gr}(V)$ defined (on the connected component $\text{Gr}^n(V)$) by the determinant of the perfect complex:
  $$\mathcal{L} \oplus (\hat{V}_m^+)_{\mathcal{O}_{\text{Gr}}(V)} \longrightarrow (\hat{V}_m)_{\mathcal{O}_{\text{Gr}}(V)}$$
  where $\mathcal{L}$ is the submodule of $\hat{V}_{\mathcal{O}_{\text{Gr}}(V)}$ corresponding to the universal object, and $V_m$ is defined as $z^m \cdot V_r^+$ with $m = q \cdot n + r$ and $0 \leq r < n$;
- the addition morphism in the above complex gives canonically a global section of the dual of that bundle, $\Omega_+ \in H^0(\text{Gr}(V), \text{Det}^*)$.

Remark 1. Although as abstract scheme $\text{Gr}(V)$ is independent of the dimension of $E$, it is straightforward that the groups acting (naturally) on the Grassmannians do depend on it. The standard procedure to introduce these Grassmannians consists of a “re–labelling” the indexes of $k((z))$ and taking $V_m^+ = T^m(V^+)$, where:

$$\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
z & 0 & \ldots & 0 & 0
\end{pmatrix}$$

but in this case the induced $k((z))$-module structures are different.

For simplicity’s sake, we will assume that:

- $E = k^\oplus n$; that is, we consider a basis $\{e_1, \ldots, e_n\}$ of $E$, so that the elements of $E((z))$ may be thought as $E$-valued series in $z$ or as $n$-tuples of series;
- $k$ is an algebraically closed field of characteristic 0.

2.B. $\tau$ Functions. Analogously to the one dimensional case, the formal geometry language ([AMP],[MP]) will be the base of our approach to the definition $\tau$-function. From now on, a pair $(E, \mathcal{T})$ consisting of
a \( n \)-dimensional \( k \)-vector space and a semisimple commutative subalgebra \( \mathcal{T} \subseteq \text{End}(E) \) will be fixed.

Although the constructions and results below hold in greater generality (e.g. \( \dim \mathcal{T} \leq \dim E \)), we will assume that \( \mathcal{T} \) is \( n \)-dimensional and that there exist a basis \( \{T_1, \ldots, T_n\} \) such that \( T_i(e_j) = \delta_{ij} e_i \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \). (The existence follows from [Bou] Chp. 8, §9 n.3).

Motivated by the fact that the choice of a basis of \( \mathcal{T} \) induces an isomorphism of the completion of the symmetric algebra generated by \( \mathcal{T} \), and \( k[[t_1, \ldots, t_n]] \), define the \( n \)-dimensional formal variety by:

\[
\hat{C}^n := \text{Spf}(k[[t_1, \ldots, t_n]])
\]

And let \( \Gamma^n_- \) be the direct limit of the symmetric products of \( \hat{C}^n \), \( \lim_{\rightarrow r} S^r \hat{C}^n \).

Now, we will study an action of \( \Gamma^n_- \) that, roughly said, is induced by that of \( \mathcal{T} \) on \( V \) at an infinitesimal level.

Analogously to the case \( n = 1 \) (see Theorem 3.6 of [AMP]), one has:

**Theorem 2.1.** \( \Gamma^n_- \), is a formal group scheme whose \( S \)-valued points are \( n \)-tuples of series on \( z^{-1} \):

\[
(1 + \sum_{i>0} a_{i1} z^{-i}, \ldots, 1 + \sum_{i>0} a_{in} z^{-i})
\]

(where \( a_{ij} \in H^0(S, \mathcal{O}_S) \) are nilpotents) with componentwise multiplication as composition law.

Let \( \prod^r \hat{C}^n \) be the formal spectrum of \( k[[\{s_{ij}\}_{1 \leq i \leq n}]] \). Then, the ring of \( \Gamma^n_- \) is \( \lim_{\rightarrow r} k[[\{s_{ij}\}_{1 \leq i \leq n}]] \) (where \( s_{ij} \) is to be understood as the \( i \)-th symmetric function on \( t_{1j}, t_{2j}, \ldots \)). Since \( \text{char}(k) = 0 \), the exponential map gives an isomorphism of \( \Gamma^n_- \) with an additive group scheme, the universal element might be written as \( (\exp(\sum_{i>0} s_{1i} z^{-i}), \ldots, \exp(\sum_{i>0} s_{in} z^{-i})) \).

Consider the action \( \mu : \Gamma^n_- \times \text{Gr}(V) \to \text{Gr}(V) \) induced by that of \( \Gamma^n_- \) on \( V \) given by componentwise multiplication; or, equivalently by:

\[
(\exp(\sum_{i>0} s_{i1} z^{-i}), \ldots, \exp(\sum_{i>0} s_{in} z^{-i})) \cdot v := \exp(\sum_{j=1}^n \sum_{i>0} s_{ij} z^{-i} T_j)(v)
\]

Note that \( \mu \) preserves the determinant bundle; that \( (\mu_U^*)^* \text{Det}^* \) is trivial; and, that the group structure of \( \Gamma^n_- \) gives a trivialization of it.

**Definition 2.2.** Then \( \tau \)-function of a rational point \( U \in \text{Gr}(V) \), \( \tau_U \), is the inverse image of the global section of the determinant bundle \( \Omega_+ \) by the morphism \( \mu_U : \Gamma^n_- \times \{U\} \to \text{Gr}(V) \).
2.C. **Baker-Akhiezer Functions.** Recall that the Abel morphism:

\[ \text{Spf } k[[\bar{z}]] \times \Gamma_{\bar{z}} \rightarrow \Gamma_{\bar{z}} \]

is the associated with the series \((1 - \bar{z})^{-1} \cdot (1 + \sum_{i>0} a_i \bar{z}^{-i})\). Let \(\phi_j : \text{Spf}(k[[\bar{z}]] \times \Gamma_n \rightarrow \Gamma_n\) be the morphism given by the Abel morphism in the \(j\)-th entry and by the identity on the others.

**Definition 2.3.** The Baker-Akhiezer function of a rational point \(U \in \text{Gr}^r(V)\) (with \(\Omega_+(U) \neq 0\)), \(\psi_U(z, s)\), is the vector valued function:

\[ \psi_U(z, s) := \exp\left(- \sum_{i,j} T_j \frac{s_{ij}}{z^i}\right) \cdot \frac{1}{\tau_U(s)} \cdot (\phi_1^*(\tau_U), \ldots, \phi_n^*(\tau_U)) \]

**Remark 2.** In our definitions of tau and BA functions, the commutativity of \(T\) is extremely important. It implies that expressions like \(\prod_{1 \leq i \leq N} (1 - T_j \frac{t_{ij}}{z})\), \(\exp(-\sum_{i,j} T_j \frac{s_{ij}}{z^i})\) are well defined and, further, that the Abel morphisms \(\phi_j\) and the morphism \(\mu_U\) are compatible.

In order to introduce the adjoint BA function, one must assume that there is a non-degenerate symmetric pairing:

\[ T_2 : E \times E \rightarrow k \]

Then, \(E((z))\) carries a natural non-degenerate pairing, \(\text{Res}\), given by:

\[ \text{Res}\left(\sum f_i z^i\right) \cdot \left(\sum g_j z^j\right) := \text{Res}_{z=0} \sum_{i,j} T_2(f_i, g_j) z^{i+j} dz = \sum_i T_2(f_i, g_{-i-1}) \]

When \(E\) is the algebra of matrices, \(M_{r \times s}(k)\) (including the case \(r = 1\)), we shall consider the pairing given by:

\[ (A, B) \mapsto \text{Tr}(A \cdot B^t) \]

**Definition 2.4.** The adjoint BA function of a rational point \(U \in \text{Gr}(V)\) is:

\[ \psi_U^*(z, s) := \psi_{U^*}(z, -s) \]

2.D. **Addition formula.** Let \(N > 0\) be an integer. Let \(U \in \text{Gr}(V)\) be a rational point. Consider the morphism:

\[ \tilde{\mu}_U^N : \prod_{1 \leq j \leq N} \mathcal{C}^n \rightarrow \text{Gr}(V) \]

given by:

\[ \left( \prod_{1 \leq i \leq N} (1 - T_j \frac{t_{ij}}{z}) \right)^{-1}(U) \]

(the ring of the \(j\)-th copy of \(\prod \mathcal{C}^n\) in \(\prod \mathcal{C}^n\) is \(k[[t_{1j}, \ldots, t_{Nj}]]\)).
Lemma 2.5. Let $U \in \text{Gr}^0(V)$ satisfy $\Omega_+(U) \neq 0$. Then, for all $N \gg 0$ the inverse image of $\Omega_+$ by $\bar{\mu}_U^N$ is given (up to a non-zero scalar) by the following expression:

$$
(\prod_{j=1}^n \Delta_j)^{-1} \cdot \det \begin{pmatrix}
 f_1^1(t_{11}) & \cdots & f_1^1(t_{N1}) & \cdots & f_1^n(t_{1n}) & \cdots & f_1^n(t_{Nn}) \\
 \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 f_1^M(t_{11}) & \cdots & f_1^M(t_{N1}) & \cdots & f_n^M(t_{1n}) & \cdots & f_n^M(t_{Nn})
\end{pmatrix}
$$

where $M := N \cdot n$ and $\{f^i = (f^i_1, \ldots, f^i_n)\}_{1 \leq i \leq M}$ is a basis of $V^+ \cap z^N U$.

Proof. We have to compute the determinant of the inverse image of the complex:

$$
\mathcal{L} \rightarrow V/V^+
$$

by the morphism $\bar{\mu}_U^N$, which is $[1 - (T_j t_{ij})]^{-1} \cdot U \rightarrow V/V^+$. It is straightforward that its determinant coincides with that of the following complex:

$$
\text{Im}(h) \rightarrow V/z^N U
$$

where $h$ is the homothety:

$$
h : V^+ \rightarrow V^+
$$

$$(f_1, \ldots, f_n) \mapsto \left( \prod_{i=1}^N (z - t_{i1}) f_1, \ldots, \prod_{i=1}^N (z - t_{in}) f_n \right)
$$

Recall that $V^+ = E[[z]] = k[[z]] \oplus \ldots \oplus k[[z]]$. Consider now the following evaluation map:

$$
k[[z]] \oplus \ldots \oplus k[[z]] \rightarrow \mathcal{M}_{Nn} := \bigoplus_{1 \leq i \leq N, 1 \leq j \leq n} k[[t_{ij}]]
$$

$$(f_1(z), \ldots, f_n(z)) \mapsto (f_1(t_{11}), \ldots, f_1(t_{N1}), \ldots, f_n(t_{1n}), \ldots, f_n(t_{Nn}))
$$

and observe that the cokernel of $h$ is isomorphic to:

$$
k[[z]]/\prod_{i} (z - t_{i1}) \oplus \ldots \oplus k[[z]]/\prod_{i} (z - t_{in})
$$

Now, we construct the following exact sequence of complexes (written vertically):

$$
0 \rightarrow \text{Im}(h) \rightarrow V^+ \rightarrow \text{Coker}(h) \rightarrow 0
$$

$$
\pi \downarrow \quad \quad (\pi, v^N) \downarrow \quad \quad v^N \downarrow
$$

$$
0 \rightarrow V/z^N U \rightarrow (V/z^N U) \oplus \mathcal{M}_{Nn} \rightarrow \mathcal{M}_{Nn} \rightarrow 0
$$

Since the determinant of the morphism in the complex of the left hand side is precisely $(\bar{\mu}_U^N)^* \Omega_+$, it is sufficient to calculate the others.
First, observe that:
\[
\det(\bar{v}_N) = \prod_{j=1}^{n} \Delta_j \quad \text{where} \quad \Delta_j := \prod_{i<k} (t_{ij} - t_{kj})
\]

To compute the determinant of the complex in the middle, note that in the diagram:
\[
\begin{array}{ccccccc}
0 & \rightarrow & V^+ \cap z^NU & \rightarrow & V^+ & \rightarrow & V^+/(V^+ \cap z^NU) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{M}_{Nn} & \rightarrow & \mathcal{M}_{Nn} \oplus V/z^NU & \rightarrow & V/z^NU & \rightarrow & 0
\end{array}
\]

the determinant of the morphism of the right hand side is a non-zero constant for \(N \gg 0\) (since \(U\) lies in \(\text{Gr}_0(V)\) and \(\Omega_+(U) \neq 0\)). We conclude now since in the determinant of the morphism of the complex of the left hand side is:
\[
\begin{array}{ccccccc}
\det \left( f^1_1(t_{11}) & \ldots & f^1_1(t_{1N}) & \ldots & f^n_1(t_{1n}) & \ldots & f^n_1(t_{Nn}) \\
\vdots & & \vdots & & \vdots & & \\
M \cdot f^1_1(t_{11}) & \ldots & f^M_1(t_{1N}) & \ldots & f^M_n(t_{1n}) & \ldots & f^M_n(t_{Nn})
\end{array}
\]

where \(M := N \cdot n\) and \(\{f^i = (f^i_1, \ldots, f^i_n)|1 \leq i \leq M\}\) is a basis of \(V^+ \cap z^NU\).

**Remark 3.** Since the previous Lemma shows that \((\prod_j \Delta_j)((\mu^0_U)^*\Omega_+)(U)\) is a determinant for all \(U\), it follows that one can generalize the “Addition Formula” of \([SS]\) to this case.

**Theorem 2.6.** Let \(U_0 \in \text{Gr}^m(V)\) be a rational point. Then, there exists polynomials \(p_{ij} \in k[[t_{ij}]]\) and \(\zeta_m \in V_m^+\) generating \(V_m^+/V_{m-1}^+\) such that for every rational point, \(U\), lying on a (Zarisky) open neighbourhood of \(U_0\) the following formula holds:
\[
\psi_U(z, s) = z \cdot \zeta_m^{-1} \cdot \sum_{i>0} (f^i_1(z)p_{i1}(s), \ldots, f^i_n(z)p_{in}(s))
\]

where \(\{f^i(z) = (f^i_1(z), \ldots, f^i_n(z))\}_{i>0}\) is a basis of \(U\).

**Remark 4.** Lemma 2.5 and Theorem 2.6 have straightforward generalizations for the case \(\dim T \leq \dim E\).

**Proof.** Once we have proved the Lemma 2.5, this proof is a generalization of that of Theorem 4.8 of \([MP]\).

Let us consider a rational point \(U \in \text{Gr}^m(V)\). In order to compute the \(j\)-th entry of its BA function, we will calculate first the expression \(\phi^*_j N^\Omega_+ \phi^*_N(U)\) for all \(N \gg 0\), where \(\phi_N : \prod^N \hat{C}^m \times \{U\} \rightarrow \text{Gr}(V)\) and
\( \phi_{j,N} \) consist of composing with the Abel morphism in the \( j \)-th entry. However, we shall use the fact that \( \phi_{j,N} \) and the morphism \( \prod^N \hat{C}^n \times \{ U_j \} \to \text{Gr}(V) \) coincide \((U_j := (1, \ldots, (1 - \frac{z}{t_i})^{-1}, \ldots, 1) \) in the \( j \) place).

Observe that \( U_j \in \text{Gr}^{m+1}(V) \) and choose \( \zeta_m \) such that \( \zeta_m^{-1} \cdot U_0 \in \text{Gr}^0(V) \) lies on the complementary of the zero locus of the section \( \Omega_+ \).

Then, it is clear that the points \( U \in \text{Gr}^m(V) \) such that \( \Omega_+ (\zeta_m^{-1} \cdot U) \neq 0 \) define a (Zariski open) neighborhood of \( U_0 \), and that Lemma 2.5 might be applied to \( \zeta_m^{-1} \cdot U \).

Let \( \{ f^1, \ldots, f^{M+1} \} \) be a basis of \( (1, \ldots, z^{-1}, \ldots, 1) z^{-N} V^+ \cap \zeta_m^{-1} U \) such that \( \{ f^1, \ldots, f^M \} \) is a basis of \( z^{-N} V^+ \cap \zeta_m^{-1} U \). Let \( \tilde{f}^i \) be \( z^N \cdot f^i \).

Applying the Lemma 2.5, one gets:

\[ \phi_{j,N}^* \Omega_+ (U) = \prod_{l=1}^{N} (\tilde{z} - t_{lj})^{-1} \cdot \tilde{z} \cdot \left( \tilde{f}_j^{M+1} \prod_l t_{lj} + \sum_{l=1}^{M} \tilde{f}_j^l \cdot \tilde{p}_{lj}(t) \right) \]

(up to a scalar). Taking inverse limit in \( N \), replacing \( \tilde{z} \) by \( z \), and recalling that the \( j \)-th entry of the BA function is \( \prod (1 - \frac{u_{lj}}{z})^{-1} \cdot \phi_{j,N}^* \Omega_+^0 (U) \), it follows that the \( j \)-th entry is:

\[ z \cdot \sum_{l>0} f^l_j (z) p_{lj}(t) \]

where \( \{ f^l = (f^l_1, \ldots, f^l_n) | l > 0 \} \) is a basis of \( \zeta_m^{-1} U \). Finally, the very construction of the polynomials \( p_{lj}(t) \) implies that they are symmetric in the \( t \), so that they can be expressed in terms of their symmetric functions \( s \). The claim is proved.

2.E. Bilinear Identity. The previous Theorem together with the definition of the pairing Res imply easily the following:

**Theorem 2.7.** Let \( U, U' \in \text{Gr}(V) \) be two rational points, then the Residue Bilinear Identity holds:

\[ \text{Res} \left( \frac{1}{z} \psi_U(z, s) \right) \cdot \left( \frac{1}{z} \psi_{U'}^*(z, s') \right) = 0 \]

if and only if \( U \subseteq U' \) (the equality holds precisely when \( U \) and \( U' \) are in the same connected component).

3. \( n \)-component KP hierarchy

Here, we introduce the \( n \)-component KP hierarchy (\( n \)-KP) in a very concise way. We will define it as a system of Lax equations and follow closely \([3]\) (see also \([1]\)). Another common approach is based on representation theory (see, for instance, \([4,5]\)). The last
approach might be “included” in ours by studying the action of the linear group of $E((z))$ on the space of global sections of the determinant bundle (see [P2]).

3.A. Pseudodifferential Operators. Let us begin this section summarizing some standard definitions and properties of pseudo-differential operators (pdo).

For a $\mathbb{C}$-algebra $A$ and a $\mathbb{C}$-derivation $\partial : A \to A$, one considers the $A$-module of pdo:

$$\mathcal{P} := \left\{ \sum_{i \leq n} a_i \partial^i \mid a_i \in A, n \in \mathbb{Z} \right\}$$

The following generalization of the Leibnitz rule:

$$\left( \sum_{i} a_i \partial^i \right) \left( \sum_{j} b_j \partial^j \right) := \sum_{i,j} \sum_{k \geq 0} \binom{i}{k} a_i (\partial^k b_j) \partial^{i+j-k}$$

endows $\mathcal{P}$ with a $\mathbb{C}$-algebra structure. Moreover, $\mathcal{P}$ contains a distinguished $\mathbb{C}$-algebra; namely, the algebra $\mathcal{D}$ of differential operators (those elements $\sum_{i \leq n} a_i \partial^i$ such that $a_i = 0$ for all $i \leq 0$).

A pdo $\sum_{i \leq n} a_i \partial^i$ is called of order $n$ iff $a_n \neq 0$. The subspace of the operators of order less or equal than $n \in \mathbb{Z}$ will be denoted by $\mathcal{P}(n)$. Since $\mathcal{P} = \mathcal{D} \oplus \mathcal{P}(-1)$, every operator $P$ decomposes as a sum $P_+ + P_-$. Finally, define the adjoint of $P = \sum_{i \leq n} a_i \partial^i$ to be $P^* = \sum_{i \leq n} (-\partial)^i a_i$.

Observe that:

- $\mathcal{P}(n)\mathcal{P}(m) \subseteq \mathcal{P}(n+m)$;
- the Leibnitz rule induces a composition law in the affine subspace $1 + \mathcal{P}(-1) \subset \mathcal{P}$;
- $1 + \mathcal{P}(-1)$ acts transitively on $\partial + \mathcal{P}(-1)$ by conjugation;
- the stabilizer of $\partial$ consists of those pdo with constant coefficients. (Here, $a$ is constant iff $\partial a = 0$).

In the particular case of matrix valued functions, we impose one constraint; namely, the leading coefficient, $a_n$, is the identity matrix, $I$.

**Definition 3.1.** A $n \times n$-matrix-valued oscillating function is a formal expression of the type:

$$\left( I + \sum_{i < 0} M_i(s) \bar{z}^i \right) \cdot e^{\xi(s, \bar{z})}$$

where $M_i(s)$ are $n \times n$-matrices, and:

$$\xi(s, \bar{z}) := \sum_{i > 0} \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix} \bar{z}^i$$
From now on, we will consider oscillating functions and pdo over the ring \( M_{n \times n}(C[[s]]) \) \( (s = \{ s_i \}_{i \geq 0}) \). Define \( \partial_{ij} := \frac{d}{ds_{ij}} \) and \( \partial = \sum_{j=1}^{n} \partial_{ij} \).

3.B. \textit{n-component KP}. Let \( L, C^{(1)}, \ldots, C^{(n)} \) be pdo with \( n \times n \) matrix coefficients of the form:

\[
L = 1 \partial + L_1(s) \partial^{-1} + L_2(s) \partial^{-2} + \ldots \\
C^{(i)} = E_i + C_1^{(i)}(s) \partial^{-1} + C_2^{(i)}(s) \partial^{-2} + \ldots
\]

where \( E_i \) is a matrix whose only non-zero entry is 1 in the \((i, i)\) place.

Then, the \( n \)-component KP hierarchy is the following set of Lax equations:

\[
\begin{cases}
\partial_{ij} L = \left[ (L^i C^{(j)})_+, L \right] & 1 \leq j, k \leq n, \ i > 0 \\
\partial_{ij} C^{(k)} = \left[ (L^i C^{(j)})_+, C^{(k)} \right]
\end{cases}
\]  

(3.2)

where \( LC^{(j)} = C^{(j)} L \), \( C^{(j)} C^{(k)} = \delta_{jk} C^{(j)} \) and \( \sum_{j=1}^{n} C^{(j)} = I \).

The above system might be regarded as the compatibility condition of the following system of differential equations:

\[
Lw = \bar{z} \cdot w \quad , \quad C^{(j)} w = w \cdot E_j \quad , \quad \partial_{ij} w = B^{(j)}_i w
\]  

(3.3)

for a formal oscillating matrix function \( w(\bar{z}, s) = (I + \sum_{l<0} w_l(s) \bar{z}^l) e^{\xi(s, \bar{z})} \) and \( B^{(j)}_i := ((LC^{(j)})^i)_+ = (L^i C^{(j)})_+ \).

Theorem 3.4. There are 1-1 correspondence between the set of solutions of the \( n \)-component KP hierarchy and the set of rational points of the open subscheme of \( \text{Gr}^0(V) \) given by \( \Omega_+ \neq 0 \) \( (V = M_{n \times n}(k((z))) \) and \( z = \frac{1}{t} \).

For proving the Theorem we shall need the following generalization of the Lemma of [DJKM] proved in [KVL]:

Lemma 3.5. Let \( P, Q \) be two pdo with matrix coefficients. If:

\[
\text{Res}_{z=0} P(s, \partial) e^{\xi(s, z)} \cdot Q'(s', \partial') e^{\xi(s', z)} dz = 0
\]

then \( (P \cdot Q^*)^- = 0 \). (Here, the superscript \( t \) denotes the transpose).

Proof. It is well known that the system [3.2] has a solution \( w \) if and only if there exists a pdo \( P = I + \sum_{l<0} P_l(s) \partial^l \) satisfying [3.3] with \( w \) replaced by \( P \). In that case, it follows that \( w_i(s) = P_i(s) \).

Further, if there exists \( P \) as above solving [3.3], then \( L = P \partial P^{-1}, \ldots \)

C^{(i)} = P E_i P^{-1}.

Let \( P = I + \sum_{l<0} P_l(s) \partial^l \) be a pdo. If it is a solution, we have:

\[
\partial_{ij} P(s, \partial) = \bar{z}^i P(s, \partial) E_j - (L^i C^{(j)})_+ P(s, \partial)
\]
thus, for \( w(s, \bar{z}) = P(s, \bar{z})e^{\xi(s,\bar{z})} \):
\[
\partial_{ij} w(s, \bar{z}) = \bar{z}^j P(s, \bar{z}) E_j e^{\xi(s,\bar{z})} - (L^i C^{(j)})_j w(s, \bar{z})
\]
and:
\[
\partial_{ij} w(s, \bar{z})|_{s=0} = \bar{z}^j (0, \ldots, 1, \ldots, 0) + \sum_{k<i} \bar{z}^k w_k
\]
where \( w_k \) are certain vector valued functions. Summing up, given a solution \( P \) we have shown that the vector space, \( U \), generated by \( w(s, \bar{z}) \) (as the variables \( s \) vary) belongs to \( \text{Gr}^0(V) \). It is easy to check that \( \Omega^+(U) \neq 0 \).

Conversely, given a point \( U \in \text{Gr}^0(V) \) let \( P \) be a pdo such that \( \psi_U(z, s) = P(s, \partial)e^{\xi(s,\bar{z})} \). Theorems 2.6 and 2.7 imply that its BA functions satisfy the following relation:
\[
\text{Res} \left( \frac{1}{z}(\partial_{ij} - B_{ij}) \psi_U(z, s) \right) \cdot \left( \frac{1}{z} \psi_U^*(z, s') \right) = 0 \quad \forall 1 \leq j \leq n, i \geq 1
\]
so, the Lemma implies \( (\partial_{ij} P - B_{ij}^{(j)} P) \cdot P^* = 0 \).

Observe that \( (\partial_{ij} - B_{ij}^{(j)}) P \) is of negative order since:
\[
(\partial_{ij} - B_{ij}^{(j)}) \psi_U(z, s) = (\partial_{ij} P + z^i P E_j - B_{ij}^{(j)} P)e^{\xi(s,\bar{z})} = (\partial_{ij} P + P \partial^i E_j - B_{ij}^{(j)} P)e^{\xi(s,\bar{z})} = (\partial_{ij} P + L^i C^{(j)} P - B_{ij}^{(j)} P)e^{\xi(s,\bar{z})}
\]

Then, it follows that \( \partial_{ij} P - B_{ij}^{(j)} P \) must be identically zero; that is, \( \psi_U \) is a solution of the \( n \)-KP.

**Remark 5.** Observe that the equations 3.2 determine \( P \) up to right multiplication by an operator \( 1 + \sum_{i<0} A_i \partial^i \) with constant coefficients. ([74]).

Let us finish this section with a brief comment on Wronskian solutions. This “Wronskian Method” permits us to construct solutions of the \( n \)-KP starting with \( n \) solutions of the KP; equivalently (in terms of Grassmannians), it is a procedure to construct a point of \( \text{Gr}(k((z))^{\oplus n^2}) \) from a point of \( \text{Gr}(k((z))^{\oplus n}) \).

Consider the “Wronskian embedding”:
\[
\mathcal{W} : \text{Gr}(k((z))^{\oplus n}) \hookrightarrow \text{Gr}(k((z))^{\oplus n^2})
\]
\[
U \mapsto U \oplus U^{(1)} \oplus \ldots \oplus U^{(n-1)}
\]
where:
\[
U^{(i)} := \{ (\frac{d}{dz})^i f(z) | f(z) \in U \} 
\]
Now, an easy calculation shows the following relation between the BA functions:

\[ \psi_{W(U)} = \text{Wronskian}(\psi_U) \]

that is, if \( \psi_U \) is the vector valued function \( (\psi_U^{(1)}, \ldots, \psi_U^{(n)}) \) then \( \psi_{W(U)} \) is the determinant of the matrix \( \{(d/dz)^{i-1} \psi_U^{(j)} \}_{1 \leq i, j \leq n} \).

4. Moduli Spaces

Now, the results of [MP] may be generalized in order to give equations for the moduli space of algebraic data:

\[ \tilde{M}_{g,n}^r = \{(C, p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n, M, \beta) \} \]

where \( C \) is a curve, \( p_i \in C, \alpha_i : \tilde{O}_{C,p_i} \simeq k[[z]], \) \( M \) is a rank \( r \) torsion free sheaf, and \( \beta : \tilde{M}(p_i) \to k[[z]]^{\oplus n \cdot r}. \) (Here, \( \tilde{\cdot} \) denotes the completion of a sheaf along a divisor). But, let us be more precise.

Given a flat curve \( \pi : C \to S \) and a Cartier divisor \( D \) denote:

\[ \tilde{\mathcal{O}}_{C,D} = \varprojlim_n \mathcal{O}_C / \mathcal{O}_C(-n) \]

where \( \mathcal{O}_C(-1) \) is the ideal sheaf of \( D \). Assume that \( D \) is of finite degree, flat and smooth over \( S \). (Here smooth over \( S \) means that for every closed point \( x \in D \) there exists an open neighborhood \( U \) of \( x \) in \( C \) such that the morphism \( U \to S \) is smooth). Then, \( \tilde{\mathcal{O}}_{C,D} \) is a sheaf of \( \mathcal{O}_S \)-algebras. We also define the following sheaf of \( \mathcal{O}_S \)-algebras:

\[ \tilde{\Sigma}_{C,D} = \varprojlim_m \tilde{\mathcal{O}}_{C,D}(m) \]

**Definition 4.1.** Let \( S \) be a \( k \)-scheme. Define the functor \( \tilde{M}_{g,n}^r \) over the category of \( k \)-schemes by:

\[ S \rightsquigarrow \tilde{M}_{g,n}^r(S) = \{ \text{families} (C, p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n, M, \beta) \text{ over } S \} \]

where these families satisfy:

1. \( \pi : C \to S \) is a proper flat morphism, whose geometric fibres are reduced curves of arithmetic genus \( g \);
2. \( p_i : S \to C (1 \leq i \leq n) \) is a section of \( \pi, \) such that (when considered as a Cartier Divisor over \( C \), also denoted by \( p_i \)) is of relative degree 1, flat and smooth over \( S \).
3. for each irreducible component of \( C \) there is at least one divisor \( p_i \) lying on it;
4. \( \alpha_i (1 \leq i \leq n) \) is an isomorphism of \( \mathcal{O}_S \)-algebras \( \tilde{\Sigma}_{C,p_i} \to \mathcal{O}_S((z)). \)
5. \( M \) is a torsion free rank \( r \) bundle on \( C \).
6. $\beta$ is a direct sum of $n$ isomorphisms of $\mathcal{O}_S$-modules $\tilde{M}_{p_i} \sim \mathcal{O}_S[[z]]^{\oplus r}$ \((1 \leq i \leq n)\).

On the set $\tilde{M}_{g,n}^r(S)$ one can define an equivalence relation, $\sim$: $(C, \{p_i, \alpha_i\}, M, \beta)$ and $(C', \{p'_i, \alpha'_i\}, M', \beta')$ are said to be equivalent, if there exists an isomorphism $C \to C'$ (over $S$) such that the first family goes to the second under the induced morphisms.

Definition 4.2. The moduli functor of $\tilde{M}_{g,n}^r$, is the functor over the category of $k$-schemes defined by the sheafification of the functor:

$$S \rightsquigarrow \tilde{M}_{g,n}^r(S)/\sim$$

Theorem 4.3. There is an injective map of functors:

$$\tilde{M}_{g,n}^r \rightarrow \text{Gr}(k((z))^{\oplus n}) \times \text{Gr}(k((z))^{\oplus n-r})$$

$$(C, \{p_i, \alpha_i\}, M, \beta) \mapsto (H^0(C - \{p_1, \ldots, p_n\}, \mathcal{O}_C), H^0(C - \{p_1, \ldots, p_n\}, M))$$

Moreover, $\tilde{M}_{g,n}^r$ is representable by a closed subscheme of $\text{Gr}^{1-g}(k((z))^{\oplus n}) \times \text{Gr}(k((z))^{\oplus n-r})$.

Proof. Let $(C, \{p_i, \alpha_i\}, M, \beta)$ be a point of $\tilde{M}_{g,n}^r$. Define the divisor $D$ by $p_1 + \ldots + p_n$ and the sheaf $M(m)$ by $M \otimes \mathcal{O}_C(mD)$. Proceeding as in Proposition 6.3 of [MP], one shows that the sheaf $\varinjlim_m \pi_* M(m)$ is an $S$-valued point of $\text{Gr}(\Sigma_{C,\{p_i\}}^{\oplus n-r})$. Moreover, it lies on the component of index $\text{deg}(M) + r(1 - g)$.

Now, the morphism of the statement maps $(C, \{p_i, \alpha_i\}, M, \beta) \in \tilde{M}_{g,n}^r(S)$ to the pair:

$$\left(\varinjlim_m \pi_* \mathcal{O}_C(m), \varinjlim_m \pi_* M(m)\right)$$

which are understood as submodules of $\mathcal{O}_S((z))^{\oplus n}$ and of $\mathcal{O}_S((z))^{\oplus n-r}$ via $\alpha_1 \oplus \ldots \alpha_n$ and $\beta$, respectively.

It is clear that the image lies on the subset of $\text{Gr}^{1-g}(k((z))^{\oplus n}) \times \text{Gr}(k((z))^{\oplus n-r})$ defined by:

$$\{(A, B) \text{ such that } A \cdot A = A \text{ and } A \cdot B = B\} \quad (4.4)$$

which is actually a closed subscheme (by the same arguments of the proof of Theorem 6.5 of [MP]). Here, the composition laws are given as follows: an element $(f_1, \ldots, f_n)$ of $k((z))^{\oplus n}$ is represented as the diagonal matrix $\begin{pmatrix} f_1 & 0 \\ \vdots & \ddots \\ 0 & \cdots & f_n \end{pmatrix}$; an element of $k((z))^{\oplus n-r}$ is thought as a $n \times r$ matrix; then, the composition laws are those induced by the multiplication of matrices).
Starting with a pair \((A, B)\) of that subscheme, a well known procedure (see, for instance, Theorem 6.4 in \([MP]\)) enables us to construct algebro-geometric data \((C, \{p_i, \alpha_i\}, M, \beta) \in \hat{M}_{g,n}^r\) whose image is \((A, B)\). Further, both constructions are inverse of each other.

Finally, it is worth observing that our construction, although analogous, does not follow from the \(n = 1\) case, since \(H^0(C - \{p_1, \ldots, p_n\})\) is a subalgebra of \(k((z))^{\otimes n}\), while \(H^0(C - \{p_1\}) \oplus \ldots \oplus H^0(C - \{p_n\})\) does not.

**Remark 6.** A direct consequence of Theorem 2.6 is that (with a suitable normalization) the restriction of our BA function to a rational point corresponding to \((C, \{p_i, \alpha_i\}, M, \beta)\) gives the standard notions of A-hkiezer functions in all its flavours: vector valued, matrix valued, multi-punctured (\([Kr1, Kr3, KrN]\), see also \([Kr4, Pr, PW, SW]\)).

In this case, the Bilinear Residue Identity is geometrically meaningful; it is equivalent to the fact that the sum of the residues must vanish.

**Theorem 4.5.** Let \((A, B)\) be a rational point of \(\text{Gr}^{1-g}(k((z))^{\otimes n}) \times \text{Gr}(k((z)))^{\otimes n \cdot r}\) and let \(\zeta_A \in V\) such that \(\Omega_+(\frac{1}{z} A) \neq 0\). Then \((A, B)\) lies in \(\hat{M}_{g,n}^r\) if and only if their BA functions satisfy:

\[
\begin{align*}
\text{Res} \left( (1, \ldots, 1) \cdot \left( \frac{1}{z} \psi_A^*(z, s) \right) \right) &= 0 \\
\text{Res} \left( \frac{1}{z} \psi_A(z, s) \cdot \frac{1}{z} \psi_A(z, s') \cdot \left( \frac{1}{z} \psi_A^*(z, s'') \right) \right) &= 0 \\
\text{Res} \left( \frac{1}{z} \psi_A(z, s) \cdot \frac{1}{z} \psi_B(z, s') \cdot \left( \frac{1}{z} \psi_B^*(z, s'') \right) \right) &= 0
\end{align*}
\]

**Proof.** Firstly, recall that the pairing considered in \(k^{\otimes n \cdot r}\) is \((A, B) \mapsto \text{Tr}(A \cdot B^t)\). Using Theorems 2.7 and 2.9, it is easy to check that the equations defining the subscheme 4.4 are those given in the statement.

Note that the equations given in this Theorem can be transformed into a set of differential equations for tau functions that might be used to characterize certain sections of bundles over \(\hat{M}_{g,n}^r\). (See \([MP]\) for how this transformation is made). On the other hand, these equations can also be written as algebraic relations among sections of the determinant bundle.

5. Solutions to the n-KP and non-abelian theta functions

Fix two positive integers \(r, d\) such that \((r, d) = 1\). Let us fix a smooth curve \(C\) and a vector bundle \(F\) such that \(\text{rank}(F) = r\) and \(\text{deg}(F) = -d + r(g-1)\).
Denote by $\mathcal{U}_s(r,d)$ the moduli space of rank $r$ degree $d$ stable vector bundles on $C$. Recall that $\mathcal{U}_s(r,d)$ carries a natural line bundle associated to the Weil divisor given by:

$$\{ M \text{ such that } h^0(M \otimes F) > 0 \}$$

The most important properties of this bundle follows from its construction as a determinant. Let us recall the construction following [DN, Po].

Let $\mathcal{P}$ be a Poincaré bundle on $C \times \mathcal{U}_s(r,d)$ and $p_i (i = 1, 2)$ the projection onto the $i$-th factor. Let us consider a resolution of $Q := \mathcal{P} \otimes p_1^*F$ by locally free sheaves:

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$$

such that $p_2^*(V_0) = 0$. Since the vertical arrow is a quasi-isomorphism, it induces an isomorphism:

$$\text{Det}(Rp_2^*Q) \simeq \text{Det}(Rp_2^*V)$$

Note that $\text{Det}(Rp_2^*V) \simeq \wedge(R^1p_2^*(V_0)) \otimes \wedge(R^1p_2^*(V_1))^{-1}$ (where $\wedge$ denotes the top exterior algebra). It is now easy to check that the morphism $\alpha$ induces a section:

$$\theta_F := \text{det}(\alpha) \in H^0(\mathcal{U}_s(r,d), \text{Det}(Rp_2^*Q)^{-1})$$

Lemma 2.1 of [Po] proves that: a) the above construction does not depend on the resolution; b) the line bundle $\text{Det}(Rp_2^*Q)$ does depend only on the equivalence class of $F$, $[F]$, in the Grothendieck group of algebraic coherent sheaves over $C$, $K(C)$ (see also [DN]); and, c) the zero locus of the section $\text{det}(\alpha)$ is the subscheme whose closed points are those bundles $M$ such that $h^0(C, M \otimes F) \neq 0$. Moreover, Lemma 1.2 of [Po] shows that $\text{Det}(Rp_2^*Q)$ does not depend on the choice of a universal bundle (up to isomorphisms) provided that $d \text{rank}(F) + r \chi(F) = 0$; or, equivalently, $\chi(M \otimes F) = 0$ for all $M \in \mathcal{U}_s(r,d)$ (this is why we have taken $\text{deg}(F) = -d + r(g - 1)$).

Summing up, given a class $[F] \in K(C)$ (such that $d \text{rank}(F) + r \chi(F) = 0$) there is an associated line bundle on $\mathcal{U}_s(r,d)$, $\mathcal{O}(\Theta[F]) := \text{Det}(Rp_2^*Q)$. For each element $F' \in [F]$, there is a section of the dual of this bundle, $\theta_{F'}$, whose zero locus is $\{ M \text{ s.t. } h^0(C, M \otimes F') \neq 0 \}$.

Let us now relate the above picture with infinite Grassmannians. As before, some data must be fixed.

Let $r, d$ be two positive integers such that $(r, d) = 1$. Let us fix $(C, p, \alpha, F, \gamma) \in \mathcal{M}_{g,1}^r$ such that $C$ is smooth and $\text{deg}(F) = -d + r(g - 1)$. 

From §4 we know that:

\[ \hat{U}(r, d) := \{ (M, \beta) \text{ s.t. } (C, p, \alpha, M, \beta) \in \hat{M}_{g,1}^r \text{ and } \deg(M) = d \} \]

is a scheme. Let \( \hat{U}_s(r, d) \) be the open subscheme consisting of the points of \( \hat{U}(r, d) \) such that \( M \) is stable.

If \( \mathcal{M} \) is the universal bundle of \( \hat{U}_s(r, d) \) and \( V = M_{r \times r}(k((z))) \), then the submodule:

\[ \lim_{\substack{m \to \infty \n}} \pi_*(\mathcal{M}(m) \otimes F) \hookrightarrow V \otimes \mathcal{O}_{\hat{U}(r, d)} \]

(via \( \beta \otimes \gamma \) and \( \alpha \)) corresponds to the morphism of schemes:

\[ \iota_F : \hat{U}_s(r, d) \hookrightarrow \text{Gr}^0(V) \]

which for rational points is given by:

\[ (M, \beta) \mapsto H^0(C - p, M \otimes F) \]

The above discussion and the exactness of the following sequence:

\[ 0 \to \pi_*(\mathcal{M} \otimes F) \to \lim_{\substack{m \to \infty \n}} \pi_*(\mathcal{M}(m) \otimes F) \to V/V^+ \to R^1\pi_*(\mathcal{M} \otimes F) \to 0 \]

show the following:

**Theorem 5.1.** Let \( \pi : \hat{U}_s(r, d) \to U_s(r, d) \) be the canonical projection. It holds that:

- \( (\iota_F^*\Omega_+)(M, \beta) \neq 0 \iff h^0(C - p, M \otimes F) \neq 0; \)
- \( \iota_F^*\Omega_+ = \pi^*\theta_F \) (up to a non zero constant);
- \( \iota_F^*\text{Det} \sim \pi^*\mathcal{O}(\Theta_{[F]}). \)

Roughly said, the Theorem states that, analogously the the rank 1 case (see [Kr2, Sh]), it is theoretically possible to give solutions for the \( n \)-KP in terms of certain non-abelian theta functions. However, further research must be made in this direction to obtain explicit expressions.

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