Melvin Matrix Models

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Abstract: In this short note we construct the DLCQ description of the flux seven-branes in type IIA string theory and discuss its basic properties. The matrix model involves dipole fields. We explain the relation of this nonlocal matrix model to various orbifolds. We also give a spacetime interpretation of the Seiberg-Witten-like map, proposed in a different context first by Bergman and Ganor, that converts this matrix model to a local, highly nonlinear theory.

Keywords: M-Theory, p-branes, Superstring Vacua
1. Introduction

The Melvin solution of Einstein-Maxwell theory [8] has recently attracted a new wave of attention of string theorists; see for instance [2], [7], [9]. The axially symmetric universe with magnetic field parallel to the axis (a flux-string in 4 dimensions) admits a description in terms of Kaluza-Klein theory. The corresponding solution of the five-dimensional gravity turns out to be flat but has nontrivial identifications. Because we want to study physics of string theory, the corresponding configuration has six more dimensions and represents a flux seven-brane of type IIA string theory and can be rewritten as a flat metric in eleven-dimensional M-theory

$$ds^2 = -dt^2 + dy_m dy^m + dz^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_{11}^2$$ (1.1)

with nontrivial identifications

$$(t, y_m, z, \rho, \varphi, x_{11}) \equiv (t, y_m, z, \rho, \varphi + 2\pi n_2 + n_1 \beta, x_{11} + 2\pi n_1 R), \quad n_1, n_2 \in \mathbb{Z}. \quad (1.2)$$

Here $t$ and $z$ denote the coordinates parallel to the flux-brane and $y_m = (y_1, y_2, \ldots, y_6)$ are the extra six stringy dimensions; $x_{11}$ denotes the standard eleventh coordinate compactified on a circle to give us type IIA string theory (later, when we construct the matrix model, we will call it $x_9$) and $(\rho, \varphi)$ are the polar coordinates of the two-plane transverse to the flux-brane. Because we deal with a theory containing fermions, the periodicity of $\varphi$ that leads to a completely identical state should perhaps be written as $4\pi$. Also $x_{11}$ is a periodic variable but $x_{11} \rightarrow x_{11} + 2\pi R$ must be accompanied$^1$ by

$^1$Our variable $\beta$ is related to $B$ of [8] by $\beta = 2\pi RB$. 

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1. Introduction

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a twist $\varphi \to \varphi + \beta$. Clearly $\beta$ is a periodic variable with period $4\pi$: $\beta + 4\pi$ induces an identification equivalent to the identification computed from $\beta$: we will always consider $\beta$ from the fundamental interval $(-2\pi, 2\pi)$.

Furthermore for $\beta = 2\pi$ the operation $\varphi \to \varphi + 2\pi$ accounts for a change of sign of the fermions and therefore leads to the Scherk-Schwarz theory with antiperiodic fermions $\mathbb{T}$ which was studied in the context of string theory by Rohm $\mathbb{4}$. Bergman and Gaberdiel conjectured that such a compactification of M-theory is dual to type 0A string theory $\mathbb{5}$ in the same way as M-theory on a “periodic” circle is a dual of type IIA string theory. A piece of evidence for this claim was also shown in $\mathbb{6}$ where the matrix description of the Scherk-Schwarz compactification of M-theory was constructed: the matrix string limit $\mathbb{12}$-$\mathbb{14}$ of this gauge theory was shown to describe type 0A closed strings in the Green-Schwarz light-cone variables. Because the matrix model constructed in the present paper represents a generalization of the matrix model in $\mathbb{3}$, we will discuss the type 0A matrix string limit, too.

We will see in the following subsection that $\beta$ can be interpreted as the Ramond-Ramond magnetic field on the axis of the flux-brane and that values $\beta + 4\pi k$ are dual to each other for every $k \in \mathbb{Z}$. Furthermore Costa and Gutperle $\mathbb{7}$ argued that such a configuration is also dual to type 0A theory with magnetic field $\beta + 2\pi + 4\pi k'$. We will also present a DLCQ argument for this conjecture.

### 1.1 The language of the Ramond-Ramond flux-branes

The flat eleven-dimensional solution (1.1) with the identification indicated by (1.2) admits a ten-dimensional type IIA interpretation. We must first replace $\varphi$ by a new coordinate $\tilde{\varphi} = \varphi - 2\pi R\beta x_{11}$ so that the identification of $x_{11}$ and $x_{11} + 2\pi R$ does not affect $\tilde{\varphi}$. This construction thus involves a dimensional reduction along a helix and $\tilde{\varphi}$ will play the role of the polar coordinate in the resulting type IIA picture. (Note that $\beta$ and $\beta + 4\pi k$ lead to different type IIA descriptions.)

The standard identification of the eleven-dimensional Einstein frame metric $ds_{11}$ and the ten-dimensional type IIA string frame metric $ds_{10}$, the dilaton field $\phi$ and the Ramond-Ramond one-form potential $A_\mu$ reads

$$ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3}(dx_{11} + A_\mu dx^\mu)^2. \quad (1.3)$$

For our particular solution this gives us

$$ds_{10}^2 = \Lambda^{1/2} \left(-dt^2 + dy_m dy^n + dz^2 + d\rho^2\right) + \Lambda^{-1/2} \rho^2 d\tilde{\varphi}^2 \quad (1.4)$$

and

$$e^{4\phi/3} = \Lambda = 1 + (2\pi R\beta \rho)^2, \quad A_\tilde{\varphi} = \frac{2\pi R\beta \rho^2}{\Lambda}. \quad (1.5)$$

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2. Construction of the matrix model

M(atrix) theory \cite{11} can be used to describe physics of string/M-theory in some spacetimes with a simple enough asymptotic structure and with a sufficient number of large dimensions, using the so-called Discrete Light Cone Quantization (DLCQ).

Our model is a $\mathbb{Z}$ orbifold of the original BFSS matrix model where the generator of $\mathbb{Z}$ acts as

$$X_9 \rightarrow X_9 + 2\pi R, \quad X_1 + iX_2 \rightarrow (X_1 + iX_2)e^{i\beta}. \quad (2.1)$$

We will derive the matrix model using the classical orbifolding procedure. First, we enhance the gauge group $U(N)$ to $U(N \times \infty)$ where $\infty$ indicates points on the circle $\sigma \in [0, 4\pi]$: matrices $X^i, \Pi^i, \theta^i$ also become operators on the space of complex functions supported on the circle. Then we impose the restriction that guarantees that the group generated by (2.1) is identified with a subgroup of the gauge group, generated by $e^{i\sigma/2}$. For matrices such as $X_3, \ldots X_8$ which are left unaffected by (2.1) this simply means that

$$e^{i\sigma/2} \cdot X_i \cdot e^{-i\sigma/2} = X_i, \quad i = 3, 4, \ldots 8 \quad (2.2)$$

because $X_i$’s transform as adjoint of $U(N \times \infty)$. This equation implies that $X_i$ commutes with any function of $\sigma$, i.e. is itself equal to a function of $\sigma$. The matrix elements are proportional to $\delta(\sigma_m - \sigma_n)$ where $\sigma_m, \sigma_n$ stand for the two (continuous) indices. Therefore we can forget one of two $\sigma$’s. Similarly

$$e^{i\sigma/2} \cdot X_9 \cdot e^{-i\sigma/2} = X_9 + 2\pi R \quad (2.3)$$

is solved by

$$X_9 = x_9(\sigma) - 4\pi iR \frac{\partial}{\partial \sigma} \quad (2.4)$$

i.e. $X_9$ becomes the covariant derivative of the resulting 1+1-dimensional maximally supersymmetric Yang-Mills theory. Apart from the general $N \times N$ matrix-valued function of $\sigma$, $X_9$ contains the term with matrix elements $-4\pi i R \delta'(\sigma_m - \sigma_n)$. And what about $Z = X_1 + iX_2$? The condition reads

$$e^{i\sigma/2} \cdot Z \cdot e^{-i\sigma/2} = Ze^{i\beta} \quad (2.5)$$

which is solved by $Z$ proportional to the factor $\delta(\sigma_m - \sigma_n + 2\beta)$. In the language of the 1+1-dimensional Yang-Mills theory, the scalar field $Z$ is not local anymore: it transforms as $(\mathbf{N}, \bar{\mathbf{N}})$ under two $U(N)$ groups located at different points $\sigma_m, \sigma_n$, separated by the fixed interval $2\beta$. More generally the fields in the Yang-Mills theory become dipoles whose length is equal to $2\beta J_{12}$ where $J_{12}$ is a component of the angular momentum. Namely the spinors $\theta$ become dipoles of length $\pm \beta$. 


2.1 Type 0A matrix string limit

For $\beta = 2\pi$ all the fields $X_i$ become local fields again (the length of the dipoles is a multiple of $4\pi$) but all $\theta$'s behave as dipoles of length $\pm 2\pi$, stretched between the opposite points of the circle. One can make this theory local by identifying points $\sigma$ and $\sigma + 2\pi$. The gauge group is then $U(N) \times U(N)$ – each factor comes from one of the opposite points and the two factors get interchanged if one goes around the reduced circle $\sigma \in [0, 2\pi]$. The fields $X_i$ still transform in the adjoint representation of the gauge group while the fermions $\theta$ transform in the bifundamental representation $(N, \bar{N})$.

One can derive the matrix string limit [12]-[14] of this model. For long strings the dipole character of the fields becomes irrelevant and the main difference from the type IIA matrix string theory is the existence of the gauge transformations satisfying (in the nonlocal, dipole language) $U(\sigma + 2\pi) = -U(\sigma)$, for example $U(\sigma) = \exp(i\sigma/2)$ under which the bifundamental representation is odd while the bosonic fields in adjoint do not transform at all. The Green-Schwarz fermions $\theta$ are therefore also allowed to be antiperiodic on the long string. Together with the corresponding GSO-like projection (identifed with the requirement of the gauge invariance under $U$) such a construction gives the correct description of type 0A string theory using the Green-Schwarz fermions. Although such a derivation is purely classical and at quantum level the spacetime interpretation breaks down [6], one can still consider this type 0A matrix string limit to be formal evidence for the conjecture of Bergman and Gaberdiel [5] relating type 0A string theory and the Scherk-Schwarz compactification of M-theory.

In a similar fashion, one can also take the limit of M-theory on an infinitely small two-torus with periodic conditions on one circle and the Scherk-Schwarz conditions on the other circle. Perturbatively, such a model can be understood as an orbifold of type IIB matrix string theory [12], [13], [23] and leads to type 0B matrix string theory written in Green-Schwarz variables [6]. The duality symmetry $SL(2, \mathbb{Z})$ of type IIB string theory is reduced down to its subgroup $\Gamma(2)$ that preserves the boundary conditions.

Quantum mechanically, such nonsupersymmetric configurations are unstable. The spacetime is not static and we are not justified to quantize the theory in DLCQ because the spacetime does not contain two null Killing vectors. At the level of the matrix model, this problem manifests itself as a two-loop divergence that destroys the spacetime interpretation of the matrix model [6].

However, we can construct supersymmetric versions of the fluxbranes, too. The simplest case is to add an extra twist to (2.1), namely

$$X_3 + iX_4 \rightarrow (X_3 + iX_4)e^{-i\beta}. \quad (2.6)$$

The complex scalar field $Z' = X_3 + iX_4$ will then transform as a dipole oriented in the opposite direction than $Z = X_1 + iX_2$. Such a “supersymmetric F5-brane” has been
discussed in [2]. It can be understood as a pair of intersecting F7-branes: each twist represents a single F7-brane and we try to describe their superposition. Our matrix model for such an intersection has two complex scalar dipole fields of length $2\beta$. One half of the fermions transform as dipoles of length $2\beta$, too. The other half transform as local fields in adjoint of $U(N)$ and represent the unbroken supersymmetries. We believe that such a matrix model for the supersymmetric F5-brane should have the usual spacetime interpretation.

2.2 The case of the rational twists

One can of course formulate the matrix model for the flux seven-brane in the local “quiver” language [24] whenever $\beta$ is a rational multiple of $4\pi$, $\beta = 4\pi \cdot p/q$. The circumference of the new $\sigma$-circle is then $4\pi/q$ and the dipole fields transform in the representations derived from a loop quiver diagram (the extended Dynkin diagram of $A_{q-1}$, i.e. in the bifundamental representations $(N_i, \tilde{N}_{i+p})$ where $i = 1, \ldots q$ of the gauge group $U(N)^q$. For $p = 1$ and $q \to \infty$ the dipole character of the fields could be neglected and the theory would be very similar in spirit to the theory of the artificial dimensions recently constructed by Arkani-Hamed, Cohen and Georgi [20] but we will not study the details of this correspondence here.

We can also note some properties of the matrix model for the supersymmetric F5-brane (a pair of intersecting F7-branes). For a rational twist $\beta = 4\pi \cdot p/q$ we obtain a matrix model that differs from the matrix model with the same gauge group $U(N)^q$ for the $A_{q-1}$ singularity by the permutation of the $U(N)$ factors only. This permutation of the factors associated with $\sigma \to \sigma + 4\pi/q$ is the matrix description of the Ramond-Ramond Wilson line that distinguishes the F5-brane from the $A_{q-1}$ singularity [2].

2.3 Seiberg-Witten map and its interpretation

The transformation rule of the dipole fields (of length $2\beta$, such as $Z$, stretched between $\sigma - \beta$ and $\sigma + \beta$) under the gauge transformations is identical to that of the open (untraced) Wilson lines

$$W_{\sigma - \beta, \sigma + \beta} = P \exp \left( i \int_{\sigma - \beta}^{\sigma + \beta} A_{\sigma}(\sigma')d\sigma' \right).$$

(2.7)

It is therefore easy to redefine the dipoles in the following way:

$$Z(\sigma - \beta, \sigma + \beta) = P \exp \left( i \int_{\sigma - \beta}^{\sigma} A_{\sigma}(\sigma')d\sigma' \right) \tilde{Z}(\sigma) P \exp \left( i \int_{\sigma}^{\sigma + \beta} A_{\sigma}(\sigma')d\sigma' \right)$$

(2.8)

We chose a symmetric convention where the new $\tilde{Z}$ sits in the middle of the Wilson line. We could have defined the map in a less symmetric way but all such choices lead to a theory with local fields and infinitely many higher derivative terms. An
important thing to note is that while $\beta$ and $\beta + 4\pi k$, $k \in \mathbb{Z}$ lead to the same physics (as seen in 11 dimensions), the redefinitions (2.7) associated with them are different: this is the matrix realization of the “different” type IIA descriptions where $\beta$ differs by multiples of $4\pi$. Our notation $\tilde{Z}$ coincides with the angular variable $\tilde{\varphi} = \varphi - 2\pi R \beta x_{11}$ defined earlier: $\tilde{Z} = \tilde{\rho} e^{i\tilde{\varphi}}$.

This redefinition of the variables is a dipole counterpart of the Seiberg-Witten map [21] which was written using the open Wilson lines by Liu [22].

$$F_{\mu\nu}(k) = \int d^D k \, L_s \left[ \sqrt{\det(1 - \theta \hat{F})} \left( \frac{1}{1 - \theta \hat{F}} \right)_{\mu\nu} W(x, C) \right] e^{ik \cdot x} \quad (2.9)$$

In this formula, the commutative $U(1)$ field strength $F_{\mu\nu}$ is expressed in terms of a straight Wilson line $W(x, C)$. The determinant and the rational function of the noncommutative $\hat{F}$ are understood as a power series expansion. The length of the Wilson line is proportional to the momentum, $\Delta x^\mu = \theta_{\mu\nu} k^\nu$. The symbol $L_s$ guarantees that all the operators are inserted in a path-ordered fashion.

In fact, Liu’s prescription for the Seiberg-Witten map is a natural generalization of Ganor and Bergman’s form of the corresponding map removing the dipoles. In both cases, the fields are redefined by an open Wilson line. In the case of dipoles, the Feynman vertices acquire a phase linear in momentum, and consequently the open Wilson line has a fixed length. In the noncommutative case, the Feynman vertices include phases bilinear in momenta, and therefore the length of the Wilson line is proportional to the momentum.

3. Conclusions and open questions

In this short note, we constructed the matrix model for the flux seven-branes and their intersections. The matrix model involves dipole fields. Using open Wilson lines, such dipole fields can be converted to the local fields. The Wilson lines wrapped around the circle $n$ times lead to different type IIA interpretations of the configuration where the Ramond-Ramond field strength on the axis differs by $n$ times its period. The Wilson lines that make all the bosonic fields local while the fermionic fields transform as dipoles stretched between the opposite points represent different type IIA interpretations of the theory.

Matrix models with dipoles could be useful to learn something about physics of strings in backgrounds with a Ramond-Ramond field strength. Dipole fields can also serve as a simple toy model for noncommutative geometry. In both cases the Feynman vertices acquire an extra phase; in the case of the dipoles, the phase is linear in momenta, while in the case of noncommutative geometry it is bilinear. In both cases, a Seiberg-Witten map involving open Wilson lines can be used to transform the action into a very nonlinear dynamics of local fields.
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