Analytical perturbation theory and Nucleon structure function in infrared region

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(Dated: September 8, 2021)

We employ analytic QCD (anQCD) approach to analyze the unpolarized nucleon structure function (NSF) in deep inelastic scattering (DIS) processes at the next-to-leading order (NLO) accuracy. Considering the unreliable results of underlying perturbative QCD (pQCD) at energy scale $Q^2 \sim \Lambda^2$ and even less we modify the calculations at these scales using anQCD approach and compare them with results from underlying pQCD and also with the available experimental data. In these progresses the massive perturbation theory (MPT) model is also used where an effective mass is attributed to gluons. We finally use the Jacobi polynomials formalism to transfer the calculations from Mellin moment space to Bjorken-$x$ space. To confirm the validity of anQCD approach the Gottfried sum rule is also investigated. The achieved numerical results at low energy scales are compatible with what is expected and corresponding to an admissible behaviour of parton densities.

I. INTRODUCTION

An observable should be an analytic (holomorphic) function in the complex $Q^2$ plane where $|Q^2| > 0$. At high energy scales, i.e. $|Q^2| \gg 1$, we can have a good theoretical description and achieve reliable results which are confirming experimental data, using underlying pQCD. But at the energy scale near to QCD cutoff parameter, i.e., $Q^2 \sim \Lambda^2$ and less, the coupling constant of QCD starts to be growing rapidly and as a result, facing Landau IR-singularities. On the other hand the spacelike QCD observable, such as the nucleon structure functions, do not have such singularities. Accordingly one cannot obtain any reliable results from underlying pQCD, thence we need an efficient approach which eliminates these singularities in order to achieve suitable results. There are various approaches to attain this goal such as Brodsky coupling constant achievement by Ads/CFT [1], the dispersive approach of Dokshitzer [2, 3] and finally Analytic Perturbation Theory (APT) [4-9]. We use the last one to shift and even eliminates the mentioned singularities in calculations of physical quantities such as unpolarized nucleon structure function (NSF) and also the Gottfried sum rule, and thus modify their theoretical predictions.

We refer to [10] for recent related work. In this approach, the running QCD coupling constant $\alpha_s(Q^2) = \frac{\alpha_s(Q^2)}{\pi}$ is transformed in an analytic function of $Q^2$ (analytic for $Q^2 \not= 0$) which is called analytic QCD coupling constant $[A_1(Q^2)]$ that does not have any Landau singularities and we are able to calculate the results for the quantities without any such singularities at the low energies, using the analytic coupling constant.

Among the approaches which eliminate the Landau singularities, we can refer to Fractional Analytic Perturbation Theory [FAPT] [11-14]; 2anQCD [14, 15] and 3anQCD [16] which are based on parameterising the spectral function at low energies by two or three Dirac delta functions, respectively; and finally massive perturbation theory (MPT) [14, 17] which is based on removing the Landau singularities by shifting them into the timelike region. As an attribute that the last method possesses, it considers an effective mass for the gluon. Since we are working on nucleon structure function which contains the singlet and gluon sectors, we decide to apply it so that we can achieve better computational results.

The organization of this paper is as follows. In next section a brief description of essential concepts of APT is reviewed. In Sec.III evolution of parton densities and nucleon structure function, using Jacobi transformation are discussed. Sec.IV is devoted to illustration of the structure function in MPT model. Based on this model the Gottfried sum rule is considered in Sec.V. Finally summary and conclusion is presented in Sec.VI.

II. BASIC CONCEPTS IN ANALYTIC PERTURBATION THEORY

As we mentioned, underlying pQCD coupling suffers from unphysical Landau singularities at $Q^2 \sim \Lambda^2$. Therefore we can not apply it in the low momentum and that is a motivation to use other approaches, especially analytic QCD (anQCD) to achieve fairly accurate results for physical quantities. In this approach we have analytic couplings $A_\nu$ which are free from aforementioned problems. In the following content we will represent the main elements of APT. Application of Cauchy theorem to the running coupling $a(Q^2) = \frac{\alpha_s(Q^2)}{\pi}$, gives us the following spectral relation in general anQCD [14, 18]:

$$A_1(Q^2) = \frac{1}{\pi} \int_{\sigma = \Lambda^2}^{\infty} \frac{d \sigma \rho_1(\sigma)}{(\sigma + Q^2)}, \quad (1)$$

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where
\[ \rho_1(\sigma) \equiv \Im A_1(-\sigma - i\epsilon). \] (2)

Different approaches to consider the discontinuity function \( \rho_1(\sigma) \) and the coupling function \( A_1(Q^2) \) will lead to the various anQCD models. As we pointed out before \( A_1(Q^2) \) is the anQCD-analog of the underlying pQCD coupling \( a(Q^2) = \alpha_s(Q^2)/\pi \), i.e., at \( \sigma \gg \Lambda \) we have \( \equiv \Im a(-\sigma - i\epsilon) = \Im A_1(-\sigma - i\epsilon) \). Let us denote by \( A_\nu \) the anQCD-analog of the pQCD power \( a^\nu \) (where \( \nu \) is not necessarily integer). As an important point we should notice that there is not standard algebra for \( A_\nu \), i.e., \( A_\nu A_\mu \neq A_{\nu+\mu} \) or \( A_{\nu+\mu} \neq (A_\nu)^\mu \). For the construction of \( A_\nu \) in a general anQCD, we follow Ref. [19].

Correct analogs \( A_n(Q^2) \) of the powers \( a^n(Q^2) \) will be achieved, using the logarithmic derivatives of \( A_1 \) [14]:
\[ \bar{A}_{n+1}(Q^2) = \frac{(-1)^n}{\beta_0^n n!} \frac{\partial}{\partial \ln Q^2} n A_1(Q^2). \] (3)

It is obvious that with \( n=0 \) we get \( \bar{A}_1 \equiv A_1 \). Here \( \beta_0 = \frac{1}{4}(11 - \frac{2}{3} N_f) \) is the first coefficient of QCD-\( \beta \) function which is scheme independent where this function is governing by Renormalization Group Equation (RGE) for QCD running coupling constant. Substituting \( A_1 \) in Eq.(1) into Eq.(3) will lead to
\[ \bar{A}_{n+1}(Q^2) = \frac{1}{\pi \beta_0 \Gamma(n+1)} \int_0^\infty \frac{d\sigma}{\sigma} \rho_1(\sigma)L_{-n}(-\sigma/Q^2). \] (4)

In this equation \( n \) is an integer number and it can be extended to noninteger index \( \nu \) as follows [19]:
\[ \bar{A}_{\nu+1}(Q^2) = \frac{1}{\pi \beta_0 \Gamma(\nu+1)} \int_0^\infty \frac{d\sigma}{\sigma} \rho_1(\sigma)L_{-\nu}(-\sigma/Q^2). \] (5)

Here \( L_{-\nu}(\frac{-\sigma}{Q^2}) \) is polylogarithm function. It should be noted that the integral in Eq.(5) is converging at low \( \sigma \) for \( \nu > -1 \) where polylogarithm function is approximated by \( L_{-\nu}(-z) \sim \ln^{-\nu} z \). The analytic analogs \( A_\nu(Q^2) \) can be constructed as linear combinations of \( \bar{A}_{\nu+m} \)s:
\[ A_\nu(Q^2) = \bar{A}_\nu + \sum_{m \geq 1} \bar{k}_m(\nu) \bar{A}_{\nu+m}. \] (6)

The coefficients \( \bar{k}_m(\nu) \) in Eq.(6) have been determined in [19]. Using the analytic coupling constant we can do the required calculations for quantities which contain noninteger power expansion of the coupling constant.

The specific anQCD model that has been used in this paper, is named Massive Perturbation Theory (MPT) in which to achieve a holomorphic coupling, an effective mass is attributed to gluon such as [14, 17]
\[ A_1^{(MPT)}(Q^2, N_f) = a(Q^2 + m_{gl}^2, N_f). \] (7)

The mass scale \( m_{gl} \sim 0.5 - 1 \text{ GeV} \) refers to gluon mass where \( m_{gl}^2 = 0.7\text{GeV}^2 \) is considered here. Since \( m_{gl}^2 > \Lambda^2 \), we get a coupling that is analytic even to scales less than \( \Lambda \). At high energy \( A_1^{(MPT)}(Q^2) \) tends to the pQCD coupling \( a(Q^2) \). It can be seen that the difference of MPT with respect to the underlying pQCD coupling would be given by [14]:
\[ A_1^{(MPT)}(Q^2, N_f) - a(Q^2, N_f) \sim \frac{m_{gl}^2}{Q^2 \ln^2(Q^2/\Lambda^2)}. \] (8)

In the following sections we can observe the benefits of MPT model in comparing with pQCD approach. As an adjunct to this issue we plot in Fig.1 the running coupling constant in two MPT and FAPT models and compare it with the underlying pQCD coupling.

III. JACOBI TRANSFORMATION AND PARTON DENSITIES EVOLUTIONS

To extract the unpolarized NSF in terms of \( Q^2 \) energy scale we need to do the energy evolution for both the singlet and nonsinglet sectors of SF. Here we start by singlet densities where splitting functions are governing their evolution. The singlet quark distribution of hadron is defined by
\[ \Sigma(x, Q^2) = \sum_{i=1}^{N_f} [q_i(x, Q^2) + \bar{q}_i(x, Q^2)]. \] (9)
Here \( q_i(x, Q^2) \) and \( \bar{q}_i(x, Q^2) \) represent the respective number densities of quarks and antiquarks as a function of the carried momentum fraction \( x \). The subscript \( i \) indicates the flavor of the (anti)quark and \( n_f \) stands for the number of effectively massless flavours. Suppressing the fractional dependencies, the coupled evolution equations for the singlet patron and gluon distributions read

\[
\frac{d}{d \ln Q^2} \left( \Sigma \right) = \left( \frac{P_{qq}}{P_{qg}} \frac{P_{gg}}{P_{gg}} \right) \otimes \left( \Sigma_0 \left( g_0 \right) \right),
\]

where \( \otimes \) stands for convolution integral in the momentum variable,

\[
[a \otimes b](x) = \int_x^1 dy a(y)b \left( \frac{x}{y} \right).
\]

The corresponding gluon distribution, \( g(x, Q^2) \), is denoted briefly here by \( g \).

The quark-quark splitting function \( P_{qq} \) [20] in Eq.(10) can be expressed as [21]

\[
P_{qq} = P_{ns}^+ + N_f(P_{qg}^+ + P_{qs}^+) \equiv P_{ns}^+ + P_{ps}^+.
\]

Here \( P_{ns}^+ \) is the non-singlet splitting function. The quantities \( P_{qg}^+ \) and \( P_{qs}^+ \) are the flavour independent sea contributions to the quark-quark and quark-antiquark splitting functions respectively. The gluon-quark entries in Eq.(10) are given by

\[
P_{qg} = N_f P_{qg}, \quad P_{gg} = P_{gg}.
\]

In terms of the flavour independent splitting functions one can write \( P_{qg} = P_{qg}^+ \) and \( P_{gg} = P_{gg}^+ \).

The required calculations can now be continued in Mellin-N space, using Mellin transformation:

\[
a(N) = \int_0^1 dx x^{N-1} a(x).
\]

Then by transforming all needed quantities to Mellin (moment) space, the solution of Eq.(10) at NLO accuracy is rendered by:

\[
\left( \Sigma \right)_g = \left\{ \left( \frac{a_s}{a_0} \right)^{-r-} \left[ e_- + (a_0 - a_s) e_- R_1 e_+ \right] - \left( a_0 - a_s \right) \left( \frac{a_s}{a_0} \right)^{r_-} \left( \frac{e_- - R_1 e_+}{r_+ - r_- - 1} \right) \right\} \left( \Sigma_0 \left( g_0 \right) \right)
\]

(15)

Here we explicitly define \( a_s = a_s / (4 \pi) = a / 4 \). In the last line the following recursive abbreviations have been used [21]

\[
R_0 \equiv \frac{1}{\beta_0} P^{(0)},
\]

\[
R_k \equiv \frac{1}{\beta_0} P^{(k)} - \sum_{i=1}^{k} b_i R_{k-i},
\]

with \( b_k \equiv \beta_k / \beta_0 \). Furthermore for the \( r \) one can write

\[
r = \frac{1}{2 \beta_0} \left[ \begin{array}{c} P^0_{qq} + P^0_{gg} \pm \sqrt{(P^0_{qq} - P^0_{gg})^2 - 4 P^0_{qq} P^0_{gg}} \end{array} \right],
\]

where the following relation for \( e_\perp \) is defined:

\[
e_\perp = \frac{1}{r_\perp - r_0} \left[ R_0 - r_\perp I \right],
\]

(18)

in which \( I \) is representing a unique \( 2 \times 2 \) matrix.

In non-singlet case in order to decouple the combination, it is needed to use the general structure of (anti-)quark (anti-)quark splitting functions as it follows[21]

\[
P_{q_i q_j} = P_{q_i q_j} = \delta_{i k} P_{qq}^{V} + P_{qs}^{s},
\]

(19)

\[
P_{q_i \bar{q}_k} = P_{q_i \bar{q}_k} = \delta_{i k} P_{qg}^{V} + P_{qs}^{s}.
\]

The flavour asymmetries \( q^\pm \) and the total valence distribution \( q^V \) and their corresponding splitting functions are given by [21],

\[
q_{ns,ik} = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k),
\]

\[
q^V = \sum_{r=1}^{N_f} (q_r - \bar{q}_r),
\]

(20)

\[
P_{ns} = P_{qg}^V + P_{qq}^V,\quad P_{ns}^V = P_{qg}^V + P_{qq}^V + n_f(p_{qs}^s - p_{qs}^s) \equiv P_{ns}^V + P_{ps}^s.
\]

For non-singlet quark distributions evolution a similar process exists like the singlet case, but with the obvious simplification that no spurious complexity occurs. Consequently the non-singlet evolution can be written as it follows [21] :

\[
q_{NLO}^V (a_s) = \exp \left[ R_{1}^V (a_0 - a_s) \right] \left( \frac{a_s}{a_0} \right)^{-R_{0}^V} \times q^V (a_0),
\]

(21)

In which \( R_{1}^V = \frac{1}{\beta_0} P_{0}^{(1)} \) and \( R_{1}^V = \frac{1}{\beta_0} P_{1}^{(1)} \) are defined based on non-singlet splitting functions where \( \beta_0 \) and \( \beta_1 \) are the first two universal coefficient of QCD \( \beta \)-function. Accordingly Eq.(21) at next-leading-order (NLO) accuracy can be written:

\[
q_{NLO}^V (a_s) = \left[ 1 + R_{1}^V (a_0 - a_s) \right] \left( \frac{a_s}{a_0} \right)^{-R_{0}^V} \times q^V (a_0),
\]

(22)

Finally, using Eq.(15) and Eq.(22) we can obtain the nucleon structure function at the NLO accuracy in Mellin (moment) N-space as it follows

\[
F_2(N, Q^2) = \sum_{i=u,d,s} e_i^2 q_i(N, Q^2) + a_s(Q^2) C_{2q}^{(1)}(N) \frac{1}{f} \sum_{i=u,d,s} e_i^2 g(N, Q^2).
\]

(23)
Here \( C^k(N) \) s are Wilson coefficient functions which have been calculated in \([22]\). Using Jacobi transformation, as was mentioned before, is an adequate method to convert the calculated results from moment N-space to Bjorken x-space. Details of this method has been described in \([23]\). According to this method we can define the NSF, based on the following relation:

\[
F_2(x, Q^2) = x^\beta (1 - x)^\alpha \sum_{x=0}^{\infty} a_n^{\alpha, \beta} (Q^2) \Theta_n^{(\alpha, \beta)}(x) . \tag{24}
\]

Here \( a_n^{\alpha, \beta} (Q^2) \) is an expansion coefficient and \( \Theta_n^{(\alpha, \beta)}(x) \) is denoting to the Jacobi polynomials and they are related to each other by:

\[
a_n^{\alpha, \beta} (Q^2) = \int_0^1 F_2(x, Q^2) \Theta_n^{(\alpha, \beta)}(x) dx . \tag{25}
\]

By substituting \( \Theta_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n C_{k,n}^{(\alpha, \beta)} x^k \) in Eq.(25) one can get:

\[
a_n^{\alpha, \beta} (Q^2) = \int_0^1 F_2(x, Q^2) \sum_{j=0}^n C_{j,n}^{(\alpha, \beta)} x^j dx \tag{26}
\]

Putting Eq.(26) into Eq.(24) and using the SF in moment-N space by\( M_{F_2}(N, Q^2) = \int_0^1 x^{j-2} F_2(x, Q^2) dx \) we can achieve to SF in Bjorken x-space as it follows \([24]\):

\[
F_2(x, Q^2) = x^\beta (1 - x)^\alpha \sum_{n=0}^{N_{max}} \Theta_n^{(\alpha, \beta)}(x) \sum_{j=0}^n C_{j,n}^{(\alpha, \beta)} M_{F_2}(j + 2, Q^2) \tag{27}
\]

IV. UNPOLARIZED NUCLEON STRUCTURE FUNCTION AND THE MPT MODEL

To do the required computations to extract the nucleon structure function we need first the parton distribution functions (PDFs) at initial energy scale, \( Q_0 \), as the inputs. For this purpose the following parameterized functions are suggested \([25]\).

\[
xu(x, Q_0^2) = N_u x^{a_u} (1 - x)^{b_u} (1 + A_u \sqrt{x} + B_u x + C_u x^2),
\]

\[
xd(x, Q_0^2) = N_d x^{a_d} (1 - x)^{b_d} (1 + A_d \sqrt{x} + B_d x + C_d x^2),
\]

\[
xg(x, Q_0^2) = N_g x^{a_g} (1 - x)^{b_g} (1 + B_g x^{1/2} (1 - x)^{1/2}),
\]

\[
x\Sigma(x, Q_0^2) = N_{\Sigma} x^{a_{\Sigma}} (1 - x)^{b_{\Sigma}} (1 + A_{\Sigma} \sqrt{x} + B_{\Sigma} x),
\]

\[
x\Delta(x, Q_0^2) = N_{\Delta} x^{a_{\Delta}} (1 - x)^{b_{\Delta}} (1 + A_{\Delta} \sqrt{x} + B_{\Delta} x),
\]

\[
xs(x, Q_0^2) = N_s x^{a_s} (1 - x)^{b_s} (1 + A_s \sqrt{x} + B_s x) \tag{28}
\]

Table I: Numerical values of the free parameters in Eq. (28) at \( Q_0^2 = 2 \) GeV\(^2\) in the NLO accuracy \([25]\).

| NLO   | \( N_u \)   | \( N_d \)   | \( N_g \)   | \( N_{\Sigma} \) | \( N_{\Delta} \) | \( N_s \)   |
|-------|-------------|-------------|-------------|-----------------|-----------------|-------------|
| \( a_u \) | 0.55        | 0.92        | 0.047       |                 |                 |             |
| \( b_u \) | 3.61        | 4.6         | 6.1         |                 |                 |             |
| \( A_u \) | 0.8         | -2.8        | 0           |                 |                 |             |
| \( B_u \) | 4.7         | 4.5         | \( \alpha_0 \) | 0              |                 |             |
| \( C_u \) | -0.1        | \( C_d \)   | -2 \( \beta_0 \) | 0              |                 |             |
| \( N_S \) | 0.164       | 57 \( N_u \) | 0.03        |                 |                 |             |
| \( a_{\Sigma} \) | -0.19       | 2.29        | -0.28       |                 |                 |             |
| \( b_{\Sigma} \) | 8.42        | 18.6        | 8.42        |                 |                 |             |
| \( A_{\Sigma} \) | 1.9         | \( A_{\Delta} \) | 1 \( A_s \) | 1.9            |                 |             |
| \( B_{\Sigma} \) | 10          | \( B_{\Delta} \) | 0 \( B_s \) | 10             |                 |             |

In Eq.(28) these definitions are used: \( u_u = u - \bar{u} \), \( d_v = d - \bar{d} \), \( \Sigma = \bar{u} + \bar{d} \) and \( \Delta = d - \bar{u} \). All unknown parameters, including the normalization factors are obtained via the fitting over the related data \([25]\). The results are listed in Table.II.

The computations of this paper is done in mathematica environment, using anQCD.m package \([14]\) and we are going to calculate analytic coupling constant corresponding to the underlying pQCD coupling, re-alising the presented powers in Eq.(23). The relevant mathematica command of MPT coupling constant is \( AMP_{N\nu}[N_f, \nu, Q^2, m^2_{gl}, \Lambda^2] \) which returns the N-loop \((N = 1, 2, 3, 4)\) analytic MPT coupling \( A_{\nu}^{\text{MPT,N}}(Q^2, m^2_{gl}, N_f) \), including the fractional index \( \nu \) at fixed number of active quark flavours \( N_f \), with \( Q^2 \) in the Euclidean domain \((Q^2 > 0)\). In as much we do calculations at NLO approximation, the \( N \)-loop is fixed at 2 (i.e., we use 2-loop MPT). The other commands of various anQCD models, for analytic coupling constant achievement, have been described in \([14]\). Interested reader is encouraged to read also \([19]\).

For simplicity we use the the notation \( a_\nu^\nu \equiv \frac{a_\nu}{4\nu} \) and subsequently \( A_{\nu, \nu} \equiv \frac{A_\nu}{4\nu} \), so the mentioned command becomes \( AMP_{2\nu,3\nu,4\nu}|Q^2, m^2_{gl}, \Lambda^2| \). Here \( N_f = 3 \), \( m^2_{gl} = 0.7 \) GeV\(^2\) and \( \Lambda^2 = 0.35 \) GeV\(^2\).

To employ the MPT model to extract the nucleon structure function, one may do it by applying the model to the evolution equations for singlet and non-singlet sectors of parton densities, given by Eq.(15) and Eq.(22) and finally using the MPT model separately to Eq.(23), containing the Wilson coefficients. This is not admissible since Wilson coefficients and parton densities are not directly observable and it is the nucleon structure function that should be analyzed, and not the factors separately. On this basis we need to resort to Eq.(23) and employ the MPT model entirely on it. Hence each analytical coupling constant is utilized where part of its total exponent number is coming from the evolved parton densities and the rest is back to exponent of coupling constant behind Wilson factors. This procedure is completely correspond-
ing to the property of analytical couplings, presented before by $A_{\nu}A_{\mu} \neq A_{\nu+\mu}$ or $A_{\nu+\mu} \neq (A_{\nu})^\mu$. In fact what we need lastly to calculate can be given summarily by:

$$F_2(N, A_{\nu+1}(Q^2)) = \left[ C_{2g}^{(0)}(N) + A_1(Q^2)C_{2g}^{(1)}(N) \right]$$

$$+ \sum_{i=u,d,s} e_i^2 g_i(N, A_{\nu}(Q^2)) + \frac{1}{f} \sum_{i=u,d,s} e_i^2 g(N, A_{\nu}(Q^2)),$$

(29)

where we replace $A_1 A_{\nu} \rightarrow A_{\nu+1} (\neq A_1 A_{\nu})$.

Considering the numerical values for the required parameters in analytical coupling, the utilized Mathematica command for $A_\nu(Q^2)$ coupling would be $AMPT2|\nu, Q^2, 0.7, 0.35|$ where $\nu$ index is determined via the evolution processes for singlet and gluon densities and also non-singlet density. In practical calculations this index takes the following qualities: $\nu = -R_0, 1 - R_0, 2 - R_0, -r_-, 1 - r_-, 2 - r_-, r_+, 1 - r_+, 2 - r_+$.

Using available data at different energy scales, makes us the possibility to present the $Q$ dependence of $\alpha$ and $\beta$ Jacobi parameters in Eq.(27) as are following:

$$\alpha = -465.737 + 553.088 \exp(Q^2) + 336.376$$

$$\beta = 11.158 + \frac{7.670Q^2}{\log(Q^2)} + 14.308 \sqrt{Q^2} \log(Q^2)$$

(30)

We depict in Fig.2 the $F_2^0(x, Q^2)$ structure function verses $x$ Bjorken variable at different energy scales $Q^2 = 0.15, 0.21, 0.27$ and $0.313 \text{ GeV}^2$, using MPT mode and compare them with E665 experimental data [26]. To indicate the adequate applicability of MPT model at low energy scales we also add to this the results of underlying pQCD for the $F_2^0(x, Q^2)$ structure function. To achieve more precise results, the computation of underlying pQCD are done at the two loops approximation of coupling constant [27–30].

There is a few experimental data in mentioned energies but as it is shown in Fig.2, an appropriate agreement is standing between anQCD results and the available experimental data.

V. GOTTFRIED SUM RULE IN ANQCD APPROACH

Since the advent of quark-parton model, sum rules for nucleon structure functions play an important role for establishing the model. One of the important sum rule is called Gottfried sum rule (GSR). Considering the isospin symmetry for parton densities in proton and neutron, the numerical value for GSR would be different from the reported value by NMC group [31] where they measured the electromagnetic structure function of nucleon through the deep inelastic scattering of muons from proton and deuterons. Here we take into account the GSR such as to include its numerical values at low energy scales. It is then admissible to follow the related calculation, using MPT model. As we referred above, this sum rule provides determination of light flavour asymmetry of the nucleon sea and is given by [32]:

$$S_G = \int_0^1 \frac{1}{x} [F_2^0(x, Q^2) - F_{2n}^0(x, Q^2)]$$

$$= \int_0^1 \frac{1}{3} (u(x) - d(x) + \bar{u}(x) - \bar{d}(x)) dx$$

(31)

The $S_G$ deviates from the expectation of simple quark model. In other words if the nucleon sea were flavour symmetric, i.e. $\bar{u}(x) = \bar{d}(x)$, we then obtain the GSR to be $S_G = \frac{1}{3}$, but it is in contrast with NMC collaboration data in lepton-nucleon DIS [31, 33, 34]. Accordingly the following $S_G$ numerical value has been reported [32, 35]:

$$S_G(Q^2 = 4 \text{ GeV}^2) = 0.235 \pm 0.026$$

(32)
This discrepancy can be associated with existence of perturbative effects in the nucleon sea, which generate light-quark flavour asymmetry $\bar{u}(x, Q^2) < \bar{d}(x, Q^2)$ over significant range of Bjorken variable $x$ [32]. For numerical values of GSR at some specific energies, we can refer to [35].

We apply the MPT model to calculate GSR at low energy scale less than QCD cutoff, $\Lambda$, that is about 0.35 $GeV^2$. The arisen numerical results are listed in Table II. To avoid from numerical difficulty, we take the low limit of integration of $S_G$ in Eq. (31) to be $10^{-7}$. Due to nonexistence of the gluon radiation at low energy scale, the probability of sea quark appearance is very low and it is expected that the $S_G$ value approaches to $\frac{1}{2}$. Consideration of $S_G$ values at low energy scales, as listed in Table II confirms this reality

Table II: Theoretical GSR values, using MPT model at various $Q^2$

| $Q^2$ GeV$^2$ | $S_G$  |
|----------------|--------|
| 0.15           | 0.325  |
| 0.21           | 0.312  |
| 0.27           | 0.301  |
| 0.313          | 0.294  |
| 4              | 0.196  |

VI. SUMMARY AND CONCLUSION

Considering the nonexistence of the pQCD coupling at low spacelike momenta $0 < Q^2 \lesssim \Lambda$, we employed an approach, called anQCD for the purpose of reforming and modifying the calculations at energy scale $Q^2 < \Lambda^2$ to evaluate unpolarized nucleon structure function at the mentioned momenta. In this way considering the importance of gluon density in the singlet sector of nucleon structure function computations, we applied anQCD approach, based on MPT model, which contains a gluon mass parameter. Using this approach, specifically the MPT model, NSF is calculable at all energy scales $Q^2 > 0$ where at moderate and high energies MPT results for NSF are matched to those of the underlying pQCD. It is seen that at low energies $F_2(x, Q^2)$ behaviour is smoother than in the underlying pQCD. We encounter these facts in Fig. 2 at $Q^2 = 0.15, 0.21, 0.27$, and $0.313$ GeV$^2$ respectively. Consequently with due attention to the acceptable conformity between MPT results and available data, we conclude that results of anQCD approach, using MPT model, are more reliable than those of the (underlying) pQCD at low energies.

Also, we evaluated Gottfried sum rule while a nucleon sea flavour asymmetry $(\bar{u}(x, Q^2) < \bar{d}(x, Q^2))$ is considered. The naive GSR indicates a difference in the value with respect to the experimental data, because according to the naive parton model for GSR, $S_G = 1/3$. But experimental data shows a deviation from $\frac{1}{3}$. By applying anQCD approach, specifically MPT model, we achieved a result closer to the experimental data. In addition to experimental energy scale, $Q^2 = 4$ GeV$^2$, we employed this model at energy scales $Q^2 = 0.15, 0.21, 0.27$ and $0.313$ GeV$^2$ due to the applicability of this approach at low energies. Numerical results for $S_G$ at low energy scales, based on the MPT model, gives the results in agreement with the behaviour of the parton densities, which is the correct behaviour at these scales.

The anQCD approach can be employed to calculate the nuclear structure function like $^3He$ and $^7Li$ with data multiplicity for them at low energies. We hope to report on this issue as our further research task.

ACKNOWLEDGMENTS

S. A. T. is grateful to the School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM) to make the required facilities to do this project. The rest of authors are thankful the Yazd university to provide the warm hospitality in connection to this research project.

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