Probability Distribution-free General Scenario Programming
Applications in Smart Grid Optimization under Both Exogenous and Endogenous Uncertainty

Qifeng Li

Abstract This paper presents a novel solution paradigm of general optimization under both exogenous and endogenous uncertainties. This solution paradigm consists of a probability distribution (PD)-free method of obtaining deterministic equivalents and an innovative approach of scenario reduction. First, unlike the existing methods that use scenarios sampled from pre-known PD functions, the PD-free method uses historical measurements of uncertain variables as input to convert the logical models into a type of deterministic equivalents called General Scenario Program (GSP). Our contributions to the PD-free deterministic equivalent construction reside in generalization (making it applicable to general optimization under uncertainty rather than just chance-constrained optimization) and extension (enabling it to the problems under endogenous uncertainty via developing an iterative and a non-iterative frameworks). Second, this paper reveals some unknown properties of the PD-free deterministic equivalent construction, such as the characteristics of active scenarios and repeated scenarios. Base on this discoveries, we propose a concept and methods of strategic scenario selection which can effectively reduce the required number of scenarios as demonstrated in both mathematical analysis and numerical experiments. Numerical experiments are conducted on two typical smart grid optimization problems under exogenous and endogenous uncertainties.

Keywords Endogenous uncertainty · general scenario programming · optimization under uncertainty

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1 Introduction

This paper considers the following logical model of optimization under uncertainty (OU):

\[
\begin{align*}
\text{(OU)} & \quad \min_x f_0(x) + \mathbb{E}_w[f_1(x, w)] \quad (1a) \\
\text{s.t.} & \quad \mathbb{P}_w[g(x, w) \leq 0] \geq 1 - \alpha, \quad (1b)
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the vector of (continuous, integer, or mixed continuous and integer) decision variables, \( f_0 : \mathbb{R}^n \to \mathbb{R}, f_1 : \mathbb{R}^{n+r} \to \mathbb{R}, \) and \( g : \mathbb{R}^{n+r} \to \mathbb{R}^m \) are general functions, \( \mathbb{P} \) and \( \mathbb{E} \) represent probability and expectation respectively, and \( \alpha \in \mathbb{R}^m \) represent the allowed probabilities of constraint violation. For simplicity but without loss of generality, only inequality constraints are considered in (1b) since there are explicit and inexplicit methods of equivalently representing equations as inequalities. The vector of uncertainties \( w = [u, v]^T \in \mathbb{R}^r \) consists of exogenous uncertainties (ExU) \( u \in \mathbb{R}^{r_1} \) and endogenous uncertainties (EnU) \( v \in \mathbb{R}^{r_2} \), which follow certain probability distributions (PD):

\[
\begin{align*}
\mathbb{P}[u_i] & = p_i(u_i), \quad u_i \in \mathcal{U}, \; i = 1, \ldots, r_1 \quad (2a) \\
\mathbb{P}[v_j] & = q_j(v_j, x), \quad v_j \in \mathcal{V}, \; j = 1, \ldots, r_2 \quad (2b)
\end{align*}
\]

where \( r_1 + r_2 = r \), \( \mathcal{U} \) and \( \mathcal{V} \) are the uncertain sets. Here, ExU and EnU mean the decision-independent and decision-dependent uncertainties respectively.

Formulation (OU) is a general logical model of optimization under uncertainty. Given that (1b) is a set of chance constraints, (OU) represents the chance-constrained optimization (CCO) if \( f_1 = 0 \). When \( \alpha = 0 \), (OU) reduces to a stochastic optimization (SO) and further reduces to a robust optimization (RO) if \( f_1 = 0 \). However, problem (OU) can not be directly solved by mature optimization algorithms (e.g., interior-point, quasi-Newton, and conjugate gradient) or computer solvers that are based on these algorithms (e.g., Knitro, Gurobi, and Mosek). In the solution process, an inevitable step is converting the logical model (OU) into its deterministic equivalents, i.e. linear, nonlinear, or integer program. Inspired by the widely adopted term “convexification” in the optimization field, which refers to the process of converting or approximating non-convex problems into convex ones, we define a verb “determinisfy” to refer to convert a logical model into its deterministic equivalent, and the term “deterministication” to refer to the process of determinisfyng.

The existing deterministication methods assume that the PD functions are known a prior and determinisfy the logical model (OU) using scenarios of the uncertain variables sampled from the PD functions. However, such PD-based deterministication methods creates an “exponential dilemma”, i.e. the problem sizes of the resulting deterministic equivalents increase exponentially as the number of uncertain variables grows. For instance, \( a \) samples for each uncertain variable out of \( b \) will result in \( a^b \) scenarios. The authors of...
used randomly selected scenarios to convert a convex CCO under only ExU into a uncertain/random convex program which is a deterministic equivalent candidate of the original CCO. They also provided a closed-form relation between the numbers of input scenarios and the degree of “equivalent.” In the past decade, researchers has attempted to extend this PD-free method to continuous nonconvex [9] and mixed-integer convex cases [10][11]. However, we found that the numbers of needed scenarios determined in these literature are unnecessarily high, so that the applicability of this approach to large engineering systems is very limited. Moreover, EnU is not considered in these existing research.

Based on the above-mentioned PD-free deterministication method, this paper develops a novel technical path for effectively solving a more general problem, i.e. (OU). On this research path, we aim at achieving a high equivalence using as small numbers of scenarios as possible rather than finding the exact closed-form formulation of the relation between the numbers of input scenarios and the degree of “equivalent.”. As a result, this deterministication method can be applied to general optimization under uncertainty. More importantly, we will enable this method for determinisfying problems with EnU, which has not been investigated in existing research.

2 PD-Free Deterministication under Exogenous Uncertainty

For simplicity, we start our discussions with the logical model (OU) under only ExU, where \( w \) reduces to only \( u \). This section aims at establishing a methodology of PD-Free deterministication by generalizing the existing findings on scenario optimizations.

2.1 General Scenario Program

Denoting the (OU) under ExU as (OU-ExU), the developed PD-Free Deterministication method determinisfies the logical model (OU-ExU) into the following deterministic equivalent:

\[
\begin{align*}
\text{(GSP-ExU)} & \quad \min_x f_0(x) + \frac{1}{N} \sum_{i=1}^{N} f_1(x, u^{(i)}) \\
\text{s.t.} & \quad g(x, u^{(i)}) \leq 0, \quad (u^{(i)} \in \mathcal{U}, \ i = 1, \ldots, N)
\end{align*}
\]

which is called general scenario program (GSP) in this paper. In the (GSP-ExU) formulation, \( \mathcal{U} \) is a big enough set of historical measurements of \( u \). Since \( u^{(i)} \ (i = 1, \ldots, N) \) are \( N \) scenarios randomly picked up from \( \mathcal{U} \), the transformation from (OU-ExU) to (GSP-ExU) is free from PD functions. Let \( x^*_N \) be the optimal solution of (GSP-ExU) with \( N \) input scenarios, we have the following definition.
**Definition 1:** A confidence factor $\epsilon$ ($0 \leq \epsilon \leq 1$) is defined to capture the probability that $x_N^*$ is optimal to (OU-ExU), namely $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and
\[
\begin{align*}
\epsilon_1 &= \mathbb{P}_N[\mathbb{P}_u[g(x_N^*, u) \leq 0] \geq 1 - \alpha] \\
\epsilon_2 &= \mathbb{P}_N[f_0(x_N^*) + \mathbb{E}_u[f_1(x_N^*, u) \leq f_0(\tilde{x}) + \mathbb{E}_u[f_1(\tilde{x}, u)]]
\end{align*}
\] (4a)
\[
\text{if } \mathbb{P}_u[g(\tilde{x}, u) \leq 0] \geq 1 - \alpha,
\]
where $u \in \mathcal{U}$.

The confidence factor $\epsilon$ can be interpreted as the degree that the deterministic (GSP-ExU) is equivalent to (OU-ExU) or the accuracy of using the deterministic (GSP-ExU) to approximate (OU-ExU). Since $\epsilon$ represents a desired confidence level, it is a given value for example 0.99.

2.2 Existing Findings on Chance-constrained Optimization under ExU

This subsection considers a subset of (OU-ExU):
\[
\begin{align*}
\text{(CCO-ExU)} &\quad \min_x f_0(x) \\
\text{s.t. } &\quad \mathbb{P}_u[g(x, u) \leq 0] \geq 1 - \alpha,
\end{align*}
\] (5a)
\[
\] (5b)

which is a chance-constrained optimization (CCO) under ExU. Following (GSP-ExU), the corresponding deterministic equivalent of (CCO-ExU) in the form of GSP is given as:
\[
\begin{align*}
\text{(GSP-CCO-ExU)} &\quad \min_x f_0(x) \\
\text{s.t. } &\quad g(x, u^{(i)}) \leq 0. \ (u^{(i)} \in \mathcal{U}, \ i = 1, \ldots, N)
\end{align*}
\] (6a)
\[
\] (6b)

The existing research mainly covers three special cases of (GSP-CCO-ExU) as tabulated in Table 1. The findings of the existing research are summarized in the following proposition.

**Table 1** The three special cases of (CCO-ExU) in existing research

| Objective function | Constraints | Variables |
|--------------------|-------------|-----------|
| Case 1             | $f_0(x) = c^T x$ | continuous |
| Case 2             | $g$ is convex on $x$ | mixed-integer |
| Case 3             | $f_0$ is any function | continuous |

**Proposition 1** on (GSP-CCO-ExU): $x_N^*$ is optimal to (CCO-ExU) with a confidence level of $\epsilon$ if,
1. for Case 1:
\[
N \geq \frac{\epsilon}{\epsilon - 1} \frac{1}{\alpha} (m + \ln \frac{1}{1 - \epsilon})
\] (7)
2. For Case 2:

$$N \geq \arg \min_{N_{\text{min}}} \left\{ \sum_{k=0}^{(n_1+1)2^{n_2}} \left[ \frac{N_{\text{min}}^!}{k!(N_{\text{min}} - k)!} \alpha^k(1 - \alpha)^{N_{\text{min}} - k} \right] = 1 - \epsilon \right\}$$  \(8\)

3. For Case 3:

$$N \geq \arg \min_{N_{\text{min}}} \left\{ \sum_{k=0}^{N_{\text{min}}} \left[ \frac{N_{\text{min}}^!}{k!(N_{\text{min}} - k)!} \left(1 - \frac{1}{N_{\text{min}} - k} \log \frac{1}{\beta} - \frac{1}{N_{\text{min}} - k} \log \frac{N_{\text{min}}!}{N_{\text{min}}!} \right) \right] = 1 - \epsilon \right\},$$  \(9\)

where \(\epsilon\) is Euler’s number, \(n_1\) and \(n_2\) are the numbers of continuous and integer variables, and \(N_{\text{min}}\) is the solution of the equation inside the curly brackets.

Proof: Find the proof of Case 1 in [7,8], the proof of Case 2 in [10], and the proof of Case 3 in [9]. □

2.3 A Hypothesis on the PD-free Deterministication of (OU-ExU)

This subsection introduces our hypothesis on the PD-free Deterministication we established in Subsection 2.1.

Hypothesis: Let \(x_N^*\) be the optimal solution of (GSP-ExU) with \(N\) input scenarios, then:

1. There exists a number \(N_{\text{min}}\) that, if \(N \geq N_{\text{min}}\), \(x_N^*\) is also optimal to (OU-ExU) with a confidence factor \(\epsilon\).
2. For a specific case, the relation among \(N_{\text{min}}\), \(\epsilon\), \(\alpha\), \(n\) and \(r\) can be formulated by a closed-form expression \(N_{\text{min}} = h(\epsilon, \alpha, n, r)\) where:

$$\frac{\partial h}{\partial \epsilon}, \frac{\partial h}{\partial n}, \frac{\partial h}{\partial r} > 0 \quad \text{and} \quad \frac{\partial h}{\partial \alpha} < 0.$$  \(10\)

A qualitative explanation: If a scenario \(u^{(j)}\)’s probability is \(P[u^{(j)}]\), it will most likely appear \((N \ast P[u^{(j)}])\) times in the selected scenario set \(\mathcal{N} = \{u^{(i)}, i = 1, \ldots, N\}\) according to the basic theory in statistics. If \((N \ast P[u^{(j)}]) \geq 1\), \(u^{(j)}\) is most likely selected and inputted to \(\mathcal{N}\), such that the optimal solution \(x_N^*\) is feasible to \(u^{(j)}\). Otherwise, \(u^{(j)}\) is most likely not selected and \(x_N^*\) is not necessarily feasible to \(u^{(j)}\). In other words, if \(x_N^*\) is required to be feasible to all scenarios whose probability is equal or bigger than \(\alpha\), one needs to select no less than \(1/\alpha\) scenarios, which provides that \(N_{\text{min}} = g(\epsilon, \alpha, n, r) = 1/\alpha\) without the need of any PD function. Of course, this is just a simple intuitive explanation. In reality, \(N_{\text{min}}\) is also determined by other parameters, such as \(\epsilon\) [7,8]. Observing from proposition 1, we conjecture that the analytical function \(h\) is determined by the problem type (i.e. continuous or mixed-integer, and...
convex or nonconvex). Moreover, we think that $N_{\text{min}}$ grows following $\epsilon$, $n$, and $r$ while it decreases as $\alpha$ increases.

This hypothesis on the PD-free deterministicization of (OU-ExU) is inspired by the existing findings on that of the (CCO-ExU) reviewed in the previous subsection. We provide a hypothesis rather than a proved theorem here since we found that the $N_{\text{min}}$s determined by (7)-(9) are unnecessarily large. It is worth pointing out that the conditions in proposition 1 are sufficient but not necessary for the $\epsilon$-optimality. In reality, engineers are generally interested in how to use as small numbers of scenarios as possible to achieve a high $\epsilon$ more than the exact expressions of $N_{\text{min}}$. The relations in (10) indicate that $N_{\text{min}}$ grows following $\epsilon$, $n$, and $r$ while it decreases as $\alpha$ increases. Moreover, the $N_{\text{min}}$s calculated in proposition 1 (i.e. the existing research) are unacceptably big for large-scale engineering systems, such as smart grids. In next section, we will show that it is possible to achieve a high $\epsilon$ with a much smaller number of input scenarios $N$, i.e. $N \ll N_{\text{min}}$.

3 Strategic Scenario Selection

3.1 Theories of Active Scenarios and Repeated Scenarios

In the theory of constrained optimization, an active constraint means the constraints that cause the limitation on the objective function \cite{12}. In other words, the optimal solution $x^*$ will not be changed by removing inactive constraints. Motivated by this characteristics of constrained optimization, we have the following definition and lemma.

**Definition.** Let $\mathcal{N}$ be the set of $N$ scenarios randomly selected from the historical measurements of uncertain variables $\mathcal{U}$, i.e. $\mathcal{N} = \{u(i) \mid i = 1, \ldots, N\}$. For a given optimal solution $x^*$ of a GSP, $u(i) \in \mathcal{N}$ is a active scenario if $g(\cdot, u(i))$ contain at lease one active constraints.

**Lemma 1** (on active scenarios): For Case 1, the number of active scenarios $\hat{N}$ is less than or equal to the number $n$ of variables $x$ in (CCO-ExU).\[\]

**Proof:** According to Lemma 2.2 of \cite{8}, any finite dimensional (GSP-CCO-ExU) of Case 1 has at most $n$ active constraints if it is feasible. Since $g(\cdot)$ in (5b)/(6b) is a vector of $m$ functions, each scenario $u(i)$ contributes $m$ constraints to the optimization model. Even for the most conservative case, where each active scenario only contain one active constraint, the number of active scenarios $\hat{N}$ is at most $n$. As a result, $\hat{N}$ will not exceed $n$ for all other cases.\[\]

Lemma 1 and proposition 1 together imply $\hat{N} \ll N_{\text{min}} \leq N$ for Case 1 of (CCO-ExU). This relation is also applicable to nonconvex (OU-ExU) with integer decision variables, although the analytical expressions of $N^{\text{min}}$ for these problems are still not exactly known. Moreover, we have the following proposition.
Proposition 2 (on repeated scenarios): There exist repeated scenarios in $\mathcal{N}$ which can be removed without impacting the solutions of GSPs.

Proof: In statistics, if the probability of a scenario $u^{(s)}$ is $P[u^{(s)}]$, it will likely appear $(N \times P[u^{(s)}])$ times in set $\mathcal{N}$. However, in optimization, one $u^{(s)}$ is sufficient instead of $(N \times P[u^{(s)}])$, which means that the rest $(N \times P[u^{(s)}]-1)\ u^{(s)}$’s can be removed without impacting the optimal solution.

3.2 Methods of Strategic Scenario Selection

The purpose of strategic scenario selection (SSS) is to find $\hat{\mathcal{N}} (\hat{\mathcal{N}} \leq \bar{\mathcal{N}} \ll \mathcal{N}^\text{min})$ scenarios that contains all active scenarios and $\hat{\mathcal{N}}$ is as small as possible. Inspired by proposition 2, we use the following dissimilarity logic to avoid repeated scenarios

$$u^{(k+1)} \neq u^{(i)}, \ (i = 1, \ldots, k) \quad (11)$$

which means the new selected scenario $u^{(k+1)} \in \mathcal{N}$ is different from all previous selected scenarios. It is straightforward to know that the logic (11) can effectively eliminate repeated scenarios in $\mathcal{N}$. A numerical experiment showing the effectiveness of (11) is given in Section 5.

In this research, we are interested in removing not only the repeated scenarios but also more inactive scenarios to further reduce the size of set $\mathcal{N}$. For an engineering perspective, we propose a close-loop learning-aided framework of SSS as shown in Figure 1. First, many engineering systems like smart grids [13] are significantly impacted by some physical conditions, such as time span, weather, ambient temperature, and seasons. This physical information will be collected and processed in the first sub-module of the SSS core module. Then, in the second sub-module, properly selected machine learning algorithms which, on one hand, helps figure out the best physical information in sub-module 1 and, on the other hand, select effective scenarios guided by the physical information analyzed in sub-module 1. Here, we only provide introductory information about this advanced SSS framework while we’ll provide a detailed discussion on it in a future engineering paper.

Remark 1. Although the hypothesis does not provide exact mathematical expressions on the needed number of scenarios for a desired confidence level, it offers a unique opportunity of effectively solving optimization under uncertainty for complex engineering systems. First, one can create a large scenario set $\mathcal{N}$. Then, under the help of the SSS methods, the numbers of scenarios that are finally input to (GSP-ExU) will be acceptably small with the same confidence level as inputting the original large set $\mathcal{N}$. 
4 Endogenous Uncertainty-compatible PD-free GSP

This section presents another core contribution of this paper, i.e. the EnU-compatible PD-free GSP. This section will develop two paradigms for deterministicizing (OU) into its deterministic equivalents in the form of GSP.

4.1 A Non-iterative Paradigm of Deterministication

We develop the following PD-free GSP as the deterministic equivalent of (OU):

\[
\begin{align*}
\text{(GSP)} & \quad \min \quad f_0(x) \ + \ \frac{1}{N} \sum_{i=1}^{N} \{f_1(x, w^{(i)}) + \lambda_i (y_i - y^{(i)})\} \quad (12a) \\
\text{s.t.} & \quad g(x, w^{(i)}) \leq 0, \ (i = 1, \ldots, N) \quad (12b) \\
& \quad y_i = q(v^{(i)}, x), \ ((w^{(i)}, y^{(i)}) \in W) \quad (12c)
\end{align*}
\]

where \( \lambda_i \) is a Lagrangian multiplier, \( y_i \) denotes the joint probability of scenario \( w^{(i)} \) which is a function of \( x \), and \( q(v, x) = \prod_{j=1}^{2} q_j(v_j, x) \). The historical data set \( U \) is extended by including the joint probability \( y^{(i)} \) of each scenario \( w^{(i)} \) and denoted as \( W \). If function \( q(\cdot) \) in (12c) is not given, both \( y^{(i)} \) and \( q \) can be obtained by training or fitting the historical data. Let \( x_N^* \) be the optimal solution of (GSP) with \( N \geq N_{\min} \) input scenarios, we have the following theorem.

**Theorem 1** (on PD-free GSP under EnU): If the hypothesis holds true, then:

i. \( x_N^* \) is a lower bound to the solution of (OU) with a confidence level of \( \epsilon \).

ii. \( x_N^* \) is optimal to (OU) with a confidence level of \( \epsilon \) if (OU) is convex and \( q \) is affine on \( x \).

**Proof:** First, we obtain the following re-formulation of (OU), which is in the form of optimization under ExU, by explicitly including the PD functions
(2b) of EnU as a constraint:

\[
\begin{align*}
\min_x & \quad (1a) \\
\text{s.t.} & \quad (1b) \\
& \quad y = q(v, x) = \prod_{j=1}^{r_2} q_j(v_j, x) = \prod_{j=1}^{r_2} y_j(v_j)
\end{align*}
\]  

where the uncertain variables \( u \in \mathbb{R}^{r_1} \) follow PD functions \((2a)\) while the uncertain variables \( v \in \mathbb{R}^{r_2} \) follow the PD functions \( \mathbb{P}[v_j] = y_j(v_j) \) that is in the form of ExU PD functions.

Then, we consider a Lagrangian relaxation of \((13)\):

\[
\begin{align*}
\min_x & \quad (1a) - \lambda (y - q(v, x)) \\
\text{s.t.} & \quad (1b),
\end{align*}
\]  

which is an (OU-ExU). According to (GSP-ExU), model \((12)\) is exactly the deterministic GSP equivalent of \((14)\). Assuming the hypothesis true, the optimal solution \( x_N^* \) of \((12)\) is optimal to \((14)\) with a confidence level of \( \epsilon \). It suffices to show that the optimal solution of \((14)\) is a lower bound of the solution of \((13)\), i.e. the solution of (OU), since \((14)\) is a Lagrangian relaxation of \((13)\). Further, the optimal solution of \((14)\) is also optimal to \((13)\) if \((13)\) is convex according to the zero-gap property of Lagrangian relaxation for convex problems. Model \((13)\) is convex if (OU) is convex with a \( q \) function that is affine on \( x \).

Further, we consider a subset of (OU):

\[
\begin{align*}
\text{(CCO)} \quad & \min_x f_0(x) \\
\text{s.t.} & \quad \mathbb{P}_w[g(x, w) \leq 0] \geq 1 - \alpha, 
\end{align*}
\]  

which is a CCO under both ExU and EnU. Following (GSP), the corresponding deterministic equivalent of (CCO) in the form of GSP is given as:

\[
\begin{align*}
\text{(GSP-CCO)} \quad & \min_x f_0(x) + \frac{1}{N} \sum_{i=1}^{N} \lambda_i (y_i - y^{(i)}) \\
\text{s.t.} & \quad g(x, w^{(i)}) \leq 0, \quad (i = 1, \ldots, N) \\
& \quad y_i = q(v^{(i)}, x), \quad ((w^{(i)}, y^{(i)}) \in W)
\end{align*}
\]  

Let \( x_N^* \) be the optimal solution of (GSP-CCO) with \( N \) input scenarios, we have the following lemma of the above theorem.

**Lemma 2** (on PD-free GSP for (CCO) under both ExU and EnU):

i. For Case 1, \( x_N^* \) is also optimal to (CCO) with a confidence level of \( \epsilon \) if \( N \) satisfies condition \((7)\).

ii. For Case 2, \( x_N^* \) is a lower bound to the solution of (CCO) with a confidence level of \( \epsilon \) if \( N \) satisfies condition \((8)\).
For Case 3, \( x_N^* \) is a lower bound to the solution of \( (CCO) \) with a confidence level of \( \epsilon \) if \( N \) satisfies condition (9).

**Proof:** According to proposition 1, the hypothesis holds true for these cases. It suffices to prove this lemma by applying the conclusions in the theorem.

**Remark 2.** Lemma 2 indicates that, using \( (GSP) \) as the deterministic equivalent to solve an \( (OU) \) problem, the type of uncertain variables does not bring any difference to the needed number of scenarios.

### 4.2 An Iterative Paradigm

For determinisfying the optimization under \( \text{EnU} \), the above non-iterative paradigm \( (GSP) \) preserves the beauty of \( (GSP-\text{ExU}) \). While many optimization problems in engineering systems are nonconvex, the optimality only holds for the convex cases under the above non-iterative framework. Moreover, the function \( q \) in (12c) may be either unavailable or very expensive/time-consuming to obtain for some cases. As a remedy to the situation that the non-iterative paradigm can not work effectively, we propose an iterative paradigm as shown in Figure 2. For setting up this algorithm, \( K \ast N \) scenarios are selected to produce a data set \( \mathcal{N} = \{ \mathcal{N}^{(1)}, \ldots, \mathcal{N}^{(K)} \} \) where the \( i \)th data point in the \( k \)th subset is \( (x^{(k)}, w^{(k,i)}) \). The advantage of this iterative paradigm resides in that it leverages the deterministic equivalent \( (GSP-\text{ExU}) \) for solving the optimization \( (OU) \).

\[ k = 0 \]

- Let \( k = k + 1 \) and search for a subset \( \mathcal{N}^{(k)} \) in \( \mathcal{N} \) where \( x^{(k)} \approx x^*(k) \).

- Solve problem (2) with \( u^{(k,i)} \in \mathcal{N}^{(k)} \) \((i = 1, \ldots, N)\) as scenario input and obtain an optimal solution \( x^*(k+1) \).

- \( \|x^*(k+1) - x^*(k)\| < \sigma \)?
  - No
  - Yes

- Calculation ends and return the optimal solution of \( (5) \).

Fig. 2 The iterative algorithm of endogenous uncertainty-compatible PD-free GSP.
5 Applications and Numerical Experiments

This section presents two applications of the PD-free GSP solution paradigm in solving smart grid optimization (SGO) [14] problems. One is under exogenous uncertainty while the other is under endogenous uncertainty.

5.1 Optimal Power Flow with High Renewable Energy

This subsection considers a classical optimal power flow (OPF) [15] problem with high level of renewable energy as an example of SGO under exogenous uncertainty. Under the chance-constrained logical model, the OPF formulation is given as

\[
\begin{align*}
\min_{P^G, Q^G, \beta} & \quad E_u \left[ \sum_k f_{1,k}(P^G_k + \beta_k \sum_l u_l) \right] \\
\text{s.t.} & \quad P_u \left[ g(P^G, Q^G, \beta, u) \right] \leq 0 \quad \text{or} = 0 \\
& \quad 1 - \alpha \geq 1 - \alpha
\end{align*}
\]  

(17a)

(17b)

where the cost \( c^i_k \) is a quadratic function of the real part of the real-time generation of bus \( k \in \text{bus set } B \), decision variables \( x = [P^G, Q^G, \beta] \), and uncertain variables \( u \), that follow exogenous PD functions (2a), are the difference between the real-time (actual) renewable generation and the forecasted ones \( P^R \). The detailed expressions of the vector function \( g \) in (17b) are given as

\[
\begin{align*}
V^T H^P_k V + P^L_k - P^R_k - u_k & = (P^G_k + \beta_k \sum_l u_l) \\
V^T H^Q_k V + \eta (P^L_k - P^R_k - u_k) & = (Q^G_k + \eta \beta_k \sum_l u_l) \\
V^T H_{ij} V & \leq S^\text{Thermal}_{ij} \\
P^G_k, Q^G_k & \leq P^G_k, Q^G_k \leq \bar{P}^G_k, \bar{Q}^G_k, V_k & \leq \|V_k\|_2 \leq \bar{V}_k
\end{align*}
\]

(18a)

(18b)

(18c)

(18d)

where \( P^G_k \) and \( Q^G_k \) denote the bases of active and reactive power generation respectively, \( P^L_k \) represents the active load, \( (P^R_k - u_k) \) is the real-time active power of renewable generation, and \( \beta_k \) is the participation factor of the generator at bus \( k \) [16]. For simplicity, we do not consider the load uncertainty in this application. Vector \( V \) of the real and imaginary parts of bus voltages are state variables in this problem, while \( V_k \) is a two-dimension sub-vector of \( V \). Note that the state variables vary following the decision and control variables. The parameter \( \eta \) implies that the loads and renewable generators are operated in the constant power factor mode. Constraint (18c) represents the thermal limits of power lines.

A deterministic equivalent of (17) based on the PD-free GSP model (3) is established and given as

\[
\begin{align*}
\min_{P^G, Q^G, \beta} & \quad \frac{1}{N} \sum_{i} \sum_k f_{1,k}(P^G_k + \beta_k \sum_l u^{(i)}_l) \\
\text{s.t.} & \quad g(P^G, Q^G, \beta, u^{(i)}) \leq 0, \quad (i = 1, \ldots, \tilde{N})
\end{align*}
\]

(19a)

(19b)
where \( u^{(i)} \in \tilde{N} \) which is the set of scenarios selected by the SSS algorithm, and \( \tilde{N} = [\tilde{N}] \).

5.2 OPF in Distribution Systems under Market-induced Consumer Demand Uncertainty

Market-enabling is one of the key features of smart grids, where the locational marginal prices (LMP) of distribution nodes are determined by the optimal solution of distribution OPF \cite{17,18} while customers decide their electricity consumption based on the real-time LMPs. On one hand, the forecasted loads (i.e. consumer demands) are parameters for obtaining the optimal solution of decision variables. On the other hand, the energy prices in turn impacts the real-time loads while one can consider that the energy price is determined by the optimal solution of the DOPF.

A distribution OPF problem under endogenous uncertainty (DOPF-EnU) is considered in this subsection, where the EnU comes from the consumer demands that depend on the energy prices. We use the following chance-constrained optimization formulation (20) to capture the logical model of DOPF-EnU:

\[
\begin{align*}
\min_{P^G, Q^G, \beta} & \quad E_v[\sum_k f_{1,k}(P^G_k + \beta \sum_l v_l)] \\
\text{s.t.} & \quad P^G \geq 0 \quad \text{or} = 0 \quad \geq 1 - \alpha
\end{align*}
\]  

(20a)

(20b)

where uncertain variables \( v \), that follow the endogenous PD functions (2b), are the difference between the real-time (actual) loads and the forecasted loads \( P^L \). The detailed expressions of the vector function \( g \) include (18c), (18d), and

\[
\begin{align*}
V^T H^k_v V + (P^L_k + v_k) - (P^G_k + \beta \sum_l v_l) &= 0 \quad \text{(21a)} \\
V^T H^q_v V + \eta (P^L_k + v_k) - (Q^G_k + \eta \beta \sum_l v_l) &= 0 \quad \text{(21b)}
\end{align*}
\]

where \( (P^L_k + v_k) \) is the real-time active load at bus \( k \). A deterministic equivalent of (20) based on the PD-free GSP model (12) is created and given as follows

\[
\begin{align*}
\min_{S^G, \beta} & \quad \frac{1}{N} \sum_i \left\{ \sum_k f_{1,k}(P^G_k + \beta \sum_l v_l^{(i)}) + \lambda_i(y_i - \hat{y}_i) \right\} \\
\text{s.t.} & \quad g(P^G, Q^G, \beta, v^{(i)}) \leq 0, \quad (i = 1, \ldots, \tilde{N}) \\
& \quad y_i = g(v^{(i)}, P^G, Q^G, \beta)
\end{align*}
\]

(22a)

(22b)

where \( (v, y)^{(i)} \in \tilde{Y} \).
5.3 Numerical Experiments

5.3.1 PD-free v.s. PD-based

This experiment evaluates the performance of the PD-free determinization via comparing it with that of PD-based determinization on problem (17). We uses the convex relations of constraints (18) [19,20,21,22] in this experiment. As a result, the problem (17) is convex so that we can use expression (7) to calculate the needed numbers of scenarios determined by the PD-free methods. The needed numbers of the PD-based and PD-free scenarios on some IEEE test systems are tabulated in the Table 2. In this numerical experiment, we considered a constraint violation probability of $\alpha = 5\%$ and an accuracy of $90\%$ (i.e. $\epsilon = 90\%$). For the PD-based sampling, we assume that all ExU variables follow the normal distributions and 5 samples per normal distribution can provide 90% accuracy. It can be observed from the table that the needed number of PD-based scenarios increases exponentially over the number of uncertain variable. The needed number of PD-free scenarios is much less than that of the PD-based ones for large systems.

5.3.2 With SSS v.s. Without SSS

This numerical experiment solves an optimal multi-period power system operation problem with energy storage under the uncertainty of both loads and

![Fig. 3](image-url) Numerical results of dissimilarity-based scenario selection. Each step-up on the curves means an active scenario has been added.

| IEEE system | n  | r  | Needed of scenarios          |
|-------------|----|----|-------------------------------|
|             |    |    | PD-based | PD-free                  |
| 9-bus       | 22 | 3  | 125      | 768                      |
| 57-bus      | 124| 7  | 78k      | 4k                       |
| 118-bus     | 305| 19 | 1.9*10^{13} | 9.8k                   |
renewable energy resources whose one single period formulation is highly similar to (17). In other words, the model size of this problem is multiple times as (17)’s. The effectiveness of the dissimilarity logic (7) is evaluated through solving such a computationally challenging problem on the IEEE 9-, 57- and 118-bus systems, where an operation horizon of 24 hours is considered. In this experiment, scenarios are added to the PD-free GSP (3) one-by-one of which the simulation results are plotted in Figure 3. For the 9-bus case, after adding around 50 scenarios selected by the dissimilarity-based method, the objective value no longer increases no matter how many scenarios will be added. That means the solution process has reached the optimal solution. These numbers for the 57- and 118-bus cases are around 160 and 340 respectively. It is worth noting that, if we consider the constraints in (18), \( N_{9}^{\text{min}}=44k \), \( N_{57}^{\text{min}}=236k \) and \( N_{118}^{\text{min}}=693k \) according to equation (7). In other words, the dissimilarity-based scenario selection dramatically reduces the needed number of scenarios by effectively removing the repeating scenarios. In our future research, we will seek to develop more advanced SSS methods for more effective scenario reduction as described in Subsection 3.2.

6 Conclusion

This paper establishes a novel solution paradigm—the probability distribution (PD)-free general scenario programming—for general optimization under both exogenous and endogenous uncertainties based on an existing PD-free determinization methods. According to the discussions throughout the paper, we can conclude that our contributions includes: 1) generalizing the PD-free determinization method, which originally works for chance-constrained optimization only, and makes it applicable to more general optimization problems under uncertainty; 2) extending the PD-free determinization method, which originally works for problems under exogenous uncertainty only, and makes it applicable to problems under endogenous uncertainty; 3) revealing more properties of the PD-free determinization, such as the properties on active scenarios and repeated scenarios; and 4) developing strategic scenarios selection methods which can effective reduce the needed scenarios for a desired confidence level. The applications of the proposed approaches on two smart grid optimization problems under exogenous and endogenous uncertainties shows that they have a high capability of solving related optimization problems of complex engineering systems like smart grids.

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