Galois lines for space elliptic curve with $j = 12^3$

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Abstract The $V_4$-lines for each linearly normal space elliptic curve form the edges of a tetrahedron, in addition the elliptic curve with $j = 12^3$ has $Z_4$-lines. We show the arrangement of $V_4$ and $Z_4$-lines concretly for the curve. As a corollary we obtain that each irreducible quartic curve with genus one has at most two Galois points, which is a correction of the previous paper (Yoshihara, Algebra Colloq 19(no. spec 01):867–876, 2012).

Keywords Galois line · Space elliptic curve · Galois group

Mathematics Subject Classification Primary 14H50; Secondary 14H20

1 Introduction

We have been studying Galois embedding of algebraic varieties Yoshihara (2007), in particular, of elliptic curves $E$. In this case, by Lemma 8 in Yoshihara (2012) we can assume the embedding is associated with the complete linear system $|nP_0|$ for some $n \geq 3$, where $P_0 \in E$. Let $f_n : E \hookrightarrow \mathbb{P}^{n-1}$ be the embedding and put $C_n = f_n(E)$. Then we consider the Galois subspaces, Galois group, the arrangement of Galois subspaces and etc. for $C_n$ in $\mathbb{P}^{n-1}$. In the previous papers (Duyaguit and Yoshihara 2005; Yoshihara 2012) we have treated in the case where $n = 4$ and settled almost all

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questions. However, the arrangement of $V_4$ and $Z_4$-lines has not been determined in sufficient detail for $j(E) = 12^3$, i.e., the curve with an automorphism of order four with a fixed point.

The purpose of this article is as follows:

(1) In the paper (Yoshihara 2012), Corollary 2 and (2) of Lemma 12 contain errors. We make the corrections of them.

(2) We show the arrangement of Galois lines for $j = 12^3$ concretely.

The constitution of this article is as follows: In Sect. 2 we state the main theorem and mention some other results. In Sect. 3 we make the corrections of Corollary 2 and Lemma 12 in Yoshihara (2012). In Sect. 4 we make the proof of the main theorem. In Sect. 5 we explain the method of computations in Sect. 4. Finally in Sect. 6 we mention a remark. Note that the proof of this article depends on neither Corollary 2 nor (2) of Lemma 12 of Yoshihara (2012).

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2 Statement of result

**Theorem 1** The arrangement of all the Galois lines for $C_4$, where $j(C_4) = 12^3$, is illustrated by the union of the following two figures:
In these figures, • denotes the intersection of $V_4$-lines and ◦ denotes the intersection of a $V_4$ and a $Z_4$-line. Four points $Q_0$, $Q_1$, $Q_2$ and $Q_3$ are not coplanar. These points form vertexes of a tetrahedron. Let $L_{ij}$ be the line passing through $Q_i$ and $Q_j$ ($0 \leq i < j \leq 3$). Then, all the $V_4$-lines are $\ell_{01}$, $\ell_{02}$, $\ell_{03}$, $\ell_{12}$, $\ell_{13}$ and $\ell_{23}$. Except these lines, each line is a $Z_4$-line. For each vertex there exist two $Z_4$-lines passing through it. Two $Z_4$-lines which do not pass through the same vertex are disjoint. A $Z_4$-line meets $V_4$-lines at two points as is shown above. If the one is the vertex $Q_i$, then we let the other be $R_{ij}$, where $(i, j)$ is a $Z_4$-line and $Q_i$ is a $Z_4$-point. Let $L_{ij}$ be the line passing through $Q_i$ and $Q_j$, where $(i, j)$ is a $Z_4$-line. For each vertex there exist two $Z_4$-lines passing though $Q_i$. A $Z_4$-point is at most two $V_4$-points form vertexes of a tetrahedron. Let $\pi : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ be the projection with the center $R_{ij}$.

In Corollary 2 in Yoshihara (2012) we must assume $j(E) \neq 12^3$. So we correct the corollary as follows:

**Corollary 2** Let $\Gamma$ be an irreducible quartic curve in $\mathbb{P}^2$ and $E$ the normalization of it. Assume the genus of $E$ is one. If $j(E) = 12^3$ (resp. $\neq 12^3$), then the number of Galois points is at most two (resp. one).

In fact, Takahashi found the curve defined by: $s^4 + s^2 u^2 + t^4 = 0$. It is easy to see that the genus of the normalization is one and $(s : t : u) = (0 : 1 : 0)$ is a $Z_4$-point and $(1 : 0 : 0)$ is a $V_4$-point. By using Theorem 1, we can find many such examples as follows:

**Example 3** Let $L_{ij}$ and $\ell_{pq}$ be the $Z_4$ and $V_4$-lines passing though $R_{ij}$, where $0 \leq i \leq 3$, $j = 1, 2$ and if $i = 0$ or 3 (resp. 1 or 2), then $(p, q) = (1, 2)$ (resp. $(0, 3)$). Let $\pi_{ij} : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ be the projection with the center $R_{ij}$. Then, $\pi_{ij}(C_4) = \Gamma_{ij}$ is an irreducible quartic curve and the points $\pi_{ij}(L_{ij})$ and $\pi_{ij}(\ell_{pq})$ are $Z_4$ and $V_4$-points, respectively. For example, take the point $R = (0 : 0 : 1 : 0)$ as the projection center. Then, $\pi_R(X : Y : Z : W) = (X : Y : W)$. The $Z_4$-line $L : X = Y = 0$ and $V_4$-line $\ell : X + 4Y = W = 0$ pass through $R$. The defining equation of $\pi_R(C_4)$ is $W^4 = XY(X - 4Y)^2$, $\pi_R(L) = (0 : 0 : 1)$ and $\pi_R(\ell) = (4 : 1 : 0)$. By the projective change of coordinates

$$X = X' - iY', \quad Y = -(X' + iY')/4$$

we get the example of Takahashi.

We have an interest in the group generated by the Galois groups belonging to Galois points (Kanazawa et al. 2001; Miura and Ohbuchi 2015). Let $\wp(z)$ be the Weierstrass $\wp$-function associated with the lattice $\mathcal{L}$ such that $\wp_2(\mathcal{L}) = 1$ and $\wp_3(\mathcal{L}) = 0$. Then $\mathcal{L}$ is given as $\mathbb{Z}c + \mathbb{Z}ci$, where $c$ is a positive number and $i = \sqrt{-1}$. Indeed we have $c/2 = 1.8540746\ldots$ by Abramowitz and Stegun (2017), [p. 658] (in the previous
paper Yoshihara (2012) we had to use this lattice). Let $G_0$ (resp. $G$) be the group generated by the Galois group belonging to $V_4$-lines (resp. $V_4$ or $Z_4$-line) for $C$. Then we have the following.

**Corollary 4** (1) In case $j \neq 12^3$, we have $G = G_0 = \langle \rho_0, \rho_1, \rho_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. An example of the curve with this group is given in Kanazawa and Yoshihara (2011)

$$(4y^4 + 5xy^2 - 1)^2 = xy^2(x + 8y^2)^2.$$ 

(2) In case $j = 12^3$ we can show $G = \langle \sigma_0, \sigma_2, \sigma_6 \rangle$. Putting

$$\alpha(z) = z + \frac{1}{2}c, \quad \beta(z) = z + \frac{3 + i}{4}c,$$

where $c$ is the positive number given above, we have $\langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and

$$G \cong \langle \alpha, \beta \rangle \rtimes \langle \sigma_0 \rangle.$$

It is easy to see that $G_0$ is a normal subgroup of $G$. In particular $|G| = 32$ and $G$ is called an elliptic exceptional group $E(2, 2, 4)$ in Kanazawa and Yoshihara (2011). Furthermore this group appears as the group by the embedding of degree 32 of the elliptic curve $j(E) = 12^3$.

**3 Correction**

In this section we work on only the correction of Yoshihara (2012) and use the same notation. We had to assume $j(C) \neq 1$ in the first sentence of Corollary 2, so it should be revised as follows:

**Correction 1**

**Corollary 2** If a plane quartic curve $\Gamma$ with genus one has an outer Galois point, then the Galois group $G$ is isomorphic to $V_4$ or $Z_4$. In the latter case $G \cong Z_4$, the $j$-invariant of the normalization of $\Gamma$ is $12^3$. Further, $\Gamma$ is given as $\pi_Q(C)$, where

(1) $\pi_Q$ is the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from the point $Q$, and

(2) $C$ is an elliptic curve in $\mathbb{P}^3$, and

(3) $Q$ is not a vertex of the tetrahedron and $Q \in \Sigma$ (resp. $Q \in \Sigma'$) if $G \cong V_4$ (resp. $Z_4$).

Therefore, the number of Galois points is at most one if $j(C) \neq 12^3$. ($j(C) \neq 1$ in the notation of Yoshihara (2012)). The error comes from the assertion (2) of Lemma 12.
Correction 2
Delete the assertion (2) from Lemma 12.

The assertion (1) of Lemma 12 is left. The assertion (2) is used only in the statement “If $\ell_1$ and $\ell_2$ meet, then we have $\tau_1^2 = \tau_2^2$ by Lemma 12” before Claim 1. We revise the proof of the last part of Theorem 2 as follows:

Delete the sentences “If $\ell_1$ and $\ell_2$ meet, then...” we have $(1+i)(\alpha_1 - \alpha_2) \in \mathcal{C}$.

Then, we insert the following sentence between Claims 1 and 2.

In case $G_{ij} \cap G_{pq} \neq \{id\}$ in Claim 1, two $\mathcal{Z}_4$-lines $\ell(ij)$ and $\ell(pq)$ meet by (1) of Lemma 12.

After Claim 2 we complete the proof of Theorem 2 as follows. First we recall the last part of the proof of Lemma 20.

Remark The possibility of $\alpha$ is as follows, where $\alpha = (m+n)i/4$ is the translational part of the complex representation $\sigma(z) = iz + \alpha$.

$$(m, n) = (0, 0), (2, 2), (2, 0), (0, 2), (3, 1), (1, 3), (1, 1), (3, 3)$$

Suppose two $\mathcal{Z}_4$-lines $\ell$ and $\ell'$ meet except at the vertex. Then, we will get a contradiction. Let $\ell$ and $\ell'$ pass through vertexes $Q$ and $Q'$ respectively. Let $\ell''$ be the $V_4$-line connecting two vertexes $Q$ and $Q'$, and let $H$ the plane containing $\ell$, $\ell'$ and $\ell''$. The following three cases take place:

1. $H \cap C$ consists of one point.
2. $H \cap C$ consists of two points.
3. $H \cap C$ consists of four points.

Take a point $P$ in $H \cap C$, which is corresponded to $a \in \mathbb{C}$, i.e., $P = \phi \bar{\phi} \pi(a)$, where the notation is given in Yoshihara (2012) below Lemma 9. Let the generators of the Galois groups associated with $\ell$, $\ell'$ and $\ell''$ be $\sigma, \sigma'$ and $\{\tau, \tau'\}$ respectively. Then the following three sets are equal (mod $\mathcal{L}$).

$\{a, \sigma(a), \sigma^2(a), \sigma^3(a)\} \equiv \{a, \sigma'(a), \sigma'^2(a), \sigma'^3(a)\} \equiv \{a, \tau(a), \tau'(a), \tau \tau'(a)\}$

We conclude from this that $\ell$ and $\ell'$ meet at a vertex.

Let $\sigma(z) = iz + \alpha, \sigma'(z) = iz + \alpha'$ and $\tau(z) = z + \beta, \tau'(z) = -z + \beta'$ be the complex representations.

In the case (1) we have $\sigma(a) \equiv \sigma'(a)$, which means $\alpha \equiv \alpha'(\text{mod } \mathcal{L})$, hence $\sigma = \sigma'$ on $E$. By Lemma 10 we have $\ell = \ell'$. This is a contradiction.

In the case (2) we have $\{a, \sigma(a)\} \equiv \{a, \sigma'(a)\}$. Since this set consists of two points, we have also $\sigma = \sigma'$. This is a contradiction.

In the case (3), $\sigma'(a)$ is equal to (i) $\sigma(a)$, (ii) $\sigma^2(a)$ or (iii) $\sigma^3(a)$. From the case (i) we get $\alpha \equiv \alpha'(\text{mod } \mathcal{L})$. Similarly this is a contradiction. From the case (ii) we get $ia + \alpha' \equiv -a + (i+1)\alpha$ (mod $\mathcal{L}$), this means $(i+1)a \equiv (i+1)\alpha - \alpha'$ (mod $\mathcal{L}$). In addition we have another relation (ii-1) $\sigma^2(a) = \sigma(a)$ or (ii-2) $\sigma'^2(a) = \sigma^3(a)$. From the sub-case (ii-1) we get $(i+1)a \equiv (i+1)\alpha' - \alpha$ (mod $\mathcal{L}$). Combining the above two relations we have $5(\alpha - \alpha') \equiv (2r + s) + (-r + 2s)i$ (mod $\mathcal{L}$), where $r, s \in \mathbb{Z}$. Recall that $\alpha$ and $\alpha'$ can be expressed as $(m+ni)c/4$ and $(m'+n'i)c/4$, respectively. Suppose
Hereafter we treat only the case \( m \neq m' \) or \( n \neq n' \). Then we have \( c = 4(r + 2s)/5(m - m') \) or \( c = 4(-r + 2s)/5(n - n') \), respectively. Since \( c/2 \) has the value 1.8540746 . . . , it can not be expressed as above, this is a contradiction. Thus this case holds only if \( \alpha \equiv \alpha' (\text{mod} \, \mathcal{L}) \). This is a contradiction. From the sub-case (ii-2), we get two relations \( a + (i + 1)a' \equiv -ia + ia\alpha (\text{mod} \, \mathcal{L}) \) and \((i + 1)a \equiv (i + 1)\alpha - \alpha' (\text{mod} \, \mathcal{L}) \), we infer from these that \((i + 1)(\alpha - \alpha') \equiv 0 \) mod(\mathcal{L}), i.e., \( \alpha - \alpha' \) can be expressed as \( 2(\alpha - \alpha') \equiv (r + s) + (-r + s)i \), where \( r, s \in \mathbb{Z} \). This implies that \( \ell \) and \( \ell' \) meet at a vertex by Claim 2. This is a contradiction. Now we treat the last case (iii). We have \( \sigma'(a) = \sigma^3\alpha = \tau(a) \) or \( \tau'(a) \). From these relations we get \( ia + \alpha' \equiv -ia + i\alpha \equiv a + \beta \) or \( -a + \beta' (\text{mod} \, \mathcal{L}) \), where \( 2\beta \equiv 2\beta' \equiv 0 \) (mod \( \mathcal{L} \)). In the former case we have \( 2ia \equiv i\alpha - \alpha' \) and \( 2ia + 2\alpha' \equiv 2a \), since \( 2\beta \equiv 0 \). Thus we get \( 2a \equiv i\alpha - \alpha' + 2\alpha' = i\alpha + \alpha' \), so that \( 2ai \equiv -\alpha + \alpha' \). Hence we have \((i + 1)(\alpha - \alpha') \equiv 0 \). Similarly we have \((1 - i)(\alpha + \alpha') \equiv 0 \) in the latter case. By the same reason as above, we have a contradiction. This completes the proof.

4 Proof

Hereafter we treat only the case \( j(E) = 12^3 \). We use the following notation and convention as in Yoshihara (2012). Let us recall briefly:

- \( \pi : \mathbb{C} \longrightarrow E = \mathbb{C}/\mathcal{L}, \mathcal{L} = \mathbb{Z}c + \mathbb{Z}ci \)
- \( x = \varphi(z), \, y = \varphi'(z), \varphi\)-functions with respect to \( \mathcal{L} \).
- \( \varphi : \mathbb{C} \longrightarrow \mathbb{C}/\mathcal{L} \sim \mathbb{C}, \, y^2 = 4x^3 - x \)
- \( P_\alpha := \varphi(\alpha) \in C, \, (\alpha \in \mathbb{C}) \), in particular, \( P_0 = \varphi(0) \)
- \( + \) denotes the sum of complex numbers \( \alpha + \beta \) in \( \mathbb{C} \) and at the same time the sum of divisors \( P_\alpha + P_\beta \) on \( E \)
- \( \sim : \) linear equivalence
- \( \) Note that \( P_\alpha + P_\beta \sim P_{\alpha + \beta} + P_0 \) holds true.
- \( V_4 : \) Klein’s four group
- \( Z_n : \) cyclic group of order \( n \)
- \( \langle \cdots \rangle : \) the group generated by \( \cdots \)

Since the embedding is associated with \(|4P_0|\), we can assume it is given by

\[ f = f_4 : E \longrightarrow \mathbb{P}^3, \, f(x, y) = (1 : x^2 : x : y) \]

Put \( C_4 = f(E) \). The \( V_4 \)-lines have been determined in Yoshihara (2012). Recall that the Galois group associated with \( V_4 \)-line is \( \langle \rho_i, \rho_j \rangle \) for some \( i, \, j \) where \( 0 \leq i < j \leq 3 \). Let \( \sigma \) be a complex representation of a generator of the group associated with \( Z_4 \)-line. As we see in the proof of Lemma 20 in Yoshihara (2012), \( \sigma \) can be expressed as \( \sigma(z) = iz + (m + ni)e/4 \), where \( (m, n) = (0, 0), \, (2, 2), \, (3, 1), \, (1, 3), \, (1, 1), \, (3, 3), \, (2, 0) \) or \( (0, 2) \). So we put as follows:
(0) $\sigma_0(z) = iz$
(1) $\sigma_1(z) = iz + \frac{1 + i}{2} c$
(2) $\sigma_2(z) = iz + \frac{3 + i}{2} c$
(3) $\sigma_3(z) = iz + \frac{1 + 3i}{2} c$
(4) $\sigma_4(z) = iz + \frac{1 + 4i}{4} c$
(5) $\sigma_5(z) = iz + \frac{4 + 3i}{4} c$
(6) $\sigma_6(z) = iz + \frac{1 + c}{2}$
(7) $\sigma_7(z) = iz + \frac{c}{2}$

Furthermore we put

$\rho_0(z) = -z, \quad \rho_1(z) = -z + \frac{1}{2} c, \quad \rho_2(z) = -z + \frac{i}{2} c, \quad \rho_3(z) = -z + \frac{1 + i}{2} c. $

Note that

$\rho_0 \equiv \sigma_0^2 \equiv \sigma_1^2 (\mod \mathcal{L}), \quad \rho_1 \equiv \sigma_2^2 \equiv \sigma_3^2 (\mod \mathcal{L}),$
$\rho_2 \equiv \sigma_4^2 \equiv \sigma_5^2 (\mod \mathcal{L}), \quad \rho_3 \equiv \sigma_6^2 \equiv \sigma_7^2 (\mod \mathcal{L}).$

Let $V$ be the vector space spanned by $\{1, x^2, x, y\}$ over $\mathbb{C}$. If $\sigma$ is an element of the Galois group associated with a Galois line $\ell$, then it induces a linear transformation $M(\sigma)$ of $V$. The $M(\sigma)$ defines a projective transformation, we denote it by the same letter. It has the following properties:

(1) Some eigenvalue belongs to at least two independent eigenvectors.
(2) We have $M(\sigma)(\ell) = \ell$, i.e., $M(\sigma)$ induces an automorphism of $\ell \cong \mathbb{P}^1$.

There are two characterizations for the vertexes, one is the following Lemma 17 in Yoshihara (2012):

Lemma 5 There exist exactly four irreducible quadratic surfaces $S_i$ ($0 \leq i \leq 3$) such that each $S_i$ has a singular point and contains C. Let $Q_i$ be the unique singular point of $S_i$. Then the four points are not coplanar.

The other one is as follows:

Lemma 6 The $M(\rho_i)$ ($0 \leq i \leq 3$) has two eigenvalues $\lambda_{i1}$ and $\lambda_{i2}$ which belong to one and three independent eigenvectors, respectively. Let $Q_i$ be the point in $\mathbb{P}^3$ defined by the eigenvector having the eigenvalue $\lambda_{i1}$. Then, these points coincide with the ones in Theorem 1. The line passing through $Q_i$ and $Q_j$ ($0 \leq i < j \leq 3$) is a $V_4$-line. Four points $\{Q_1, Q_2, Q_3, Q_4\}$ are not coplanar, so they form a vertex of a tetrahedron.

Proof These are checked by direct computations. To find the action of $\rho_i$ on the vector space $V$, we can use the action on $x = \wp(z)$ and $y = \wp'(z)$. We make use of the addition formulas of $\wp$ and $\wp'$, see in the proof of Lemma 15 in Yoshihara (2012).

$$
\rho_0^* (1, x^2, x, y) = (1, x^2, x, -y),
$$
$$
\rho_1^* (1, x^2, x, y) = (4x^2 - 4x + 1, x^2 + x + \frac{1}{4}, 2x^2 - \frac{1}{2}, 2y),
$$
$$
\rho_2^* (1, x^2, x, y) = (4x^2 + 4x + 1, x^2 - x + \frac{1}{4}, -2x^2 + \frac{1}{2}, 2y),
$$
$$
\rho_3^* (1, x^2, x, y) = (4x^2, \frac{1}{4}, -x, -y).
$$
We obtain the following representation matrices:

\[
M(\rho_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M(\rho_1) = \begin{pmatrix} 1 & 4 & -4 \\ 1/4 & 1 & 1 \\ -1/2 & 2 & 0 \end{pmatrix}, \\
M(\rho_2) = \begin{pmatrix} 1 & 4 & 4 \\ 1/4 & 1 & -1 \\ 1/2 & -2 & 0 \end{pmatrix}, \quad M(\rho_3) = \begin{pmatrix} 0 & 4 & 0 \\ 1/4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Therefore, the eigenvalues \(\lambda\) and eigenvectors (mod constant multiplications) of \(M(\rho)\) can be computed as follows:

\[
M(\rho_0) \lambda = -1 : (0, 0, 0, 1) \quad \lambda = 1 : (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) \\
M(\rho_1) \lambda = -2 : (4, -1, 2, 0) \quad \lambda = 2 : (1, 0, -1/4, 0), (0, 1, 1, 0), (0, 0, 0, 1) \\
M(\rho_2) \lambda = -2 : (4, -1, -2, 0) \quad \lambda = 2 : (4, 0, 1, 0), (0, 1, -1, 0), (0, 0, 0, 1) \\
M(\rho_3) \lambda = 4 : (4, 1, 0, 0) \quad \lambda = -4 : (4, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)
\]

\[ \square \]

Similarly, we can find \(Z_4\)-lines by the following results. For the sake of completeness we will give the outlines of the computations in the next section.

\[
\sigma_0^* \left( 1, x^2, x, y \right) = \left( 1, x^2, -x, iy \right),
\]

\[
\sigma_1^* \left( 1, x^2, x, y \right) = \left( 4x^2, \frac{1}{4}, x, ix \right),
\]

\[
\sigma_2^* \left( 1, x^2, x, y \right) = \left( -2y + \sqrt{2} \left( i - 1 \right) x^2 - \sqrt{2} \left( 1 + i \right) x - \frac{\sqrt{2} \left( i - 1 \right)}{4},
\]

\[
- \frac{1}{2} y - \frac{\sqrt{2} \left( i - 1 \right)}{4} x^2 + \frac{\sqrt{2} \left( i + 1 \right)}{4} x
\]

\[
+ \frac{\sqrt{2} \left( i - 1 \right)}{16}, \quad \frac{\sqrt{2} \left( i + 1 \right)}{2} x^2
\]

\[
+ \frac{\sqrt{2} \left( i - 1 \right)}{2} x - \frac{\sqrt{2} \left( 1 + i \right)}{8}, \quad 2x^2 + \frac{1}{2} \right),
\]

\[
\sigma_3^* \left( 1, x^2, x, y \right) = \left( 4\sqrt{2} iy - \left( 1 + i \right) \left( 4x^2
\right.
\]

\[
+ 4ix - 1 \right), \quad \frac{1}{4} \left( 4\sqrt{2} iy + \left( 1 + i \right) \left( 4x^2 + 4ix - 1 \right) \right),
\]

\[
i - \frac{1}{2} \left( 4x^2 - 4ix - 1 \right),
\]

\[
- \sqrt{2} i \left( 4x^2 + 1 \right) \right).
\]

\[ \square \]
\( \sigma_4^* (1, x^2, x, y) = \left( -2\sqrt{2}(1+i)y - 4ix^2 - 4x + i, -\frac{1-i}{\sqrt{2}}y + ix^2 + x - \frac{i}{4}, 2x^2 + 2ix - \frac{1}{2} \right), \)

\( \sigma_5^* (1, x^2, x, y) = \left( 2\sqrt{2}(1+i)y - 4ix^2 - 4x + i, \frac{1+i}{\sqrt{2}} + ix^2 + x - \frac{i}{4}, 2x^2 + 2ix - \frac{1}{2} \right), \)

\( \sigma_6^* (1, x^2, x, y) = \left( 4x^2 + 4x + 1, x^2 - x + \frac{1}{4}, 2x^2 - \frac{1}{2}, -2iy \right), \)

\( \sigma_7^* (1, x^2, x, y) = \left( 4x^2 - 4x + 1, x^2 + x + \frac{1}{4}, -2x^2 + \frac{1}{2}, -2iy \right). \)

\[
M(\sigma_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad M(\sigma_1) = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},
\]

\[
M(\sigma_2) = -\sqrt{2}i \begin{pmatrix} (i+1)/4 & i-1 & -i+1 & -\sqrt{2}i \\ -(1+i)/16 & (1+i)/4 & -(1+i)/4 & -i/2\sqrt{2} \\ (1-i)/8 & -(1+i)/2 & (1+i)/2 & 0 \\ i/2\sqrt{2} & \sqrt{2}i & 0 & 0 \end{pmatrix},
\]

\[
M(\sigma_3) = \begin{pmatrix} 1+i & -4(1+i) & 4(1-i) & -4\sqrt{2}i \\ -(1+i)/4 & 1+i & -1+i & \sqrt{2}i \\ (1-i)/2 & -2(1-i) & 2(1+i) & 0 \\ -\sqrt{2}i & -4\sqrt{2}i & 0 & 0 \end{pmatrix},
\]

\[
M(\sigma_4) = \begin{pmatrix} i & -4i & -4 -2\sqrt{2}(1+i) \\ -i/4 & i & 1 & -(1+i)\sqrt{2} \\ -1/2 & 2 & 2i & 0 \\ -(1+i)/\sqrt{2} & 2\sqrt{2}(1+i) & 0 & 0 \end{pmatrix},
\]

\[
M(\sigma_5) = \begin{pmatrix} i & -4i & -4 2\sqrt{2}(1+i) \\ -i/4 & i & 1 & (1+i)\sqrt{2} \\ -1/2 & 2 & 2i & 0 \\ (1+i)/\sqrt{2} & 2\sqrt{2}(1+i) & 0 & 0 \end{pmatrix},
\]

\[
M(\sigma_6) = \begin{pmatrix} 1 & 4 & 4 & 0 \\ 1/4 & 1 & 1 & 0 \\ -1/2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}, \quad M(\sigma_7) = \begin{pmatrix} 1 & 4 & -4 & 0 \\ 1/4 & 1 & 1 & 0 \\ 1/2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}. \]

Eigenvalues \( \lambda \) and eigenvectors (mod constant multiplications) of \( M(\sigma) \) are as follows:
\[
\begin{align*}
M(\sigma_0) & \quad \lambda = -1 : (0, 0, 1, 0) \\
& \quad \lambda = 1 : (1, 0, 0, 0), (0, 1, 0, 0) \\
& \quad \lambda = i : (0, 0, 0, 1) \\
M(\sigma_1) & \quad \lambda = -1 : (4, -1, 0, 0) \\
& \quad \lambda = 1 : (4, 1, 0, 0), (0, 1, 0, 0) \\
& \quad \lambda = i : (0, 0, 0, 1) \\
M(\sigma_2) & \quad \lambda = \sqrt{2} : (4, -1, -2, 0) \\
& \quad \lambda = -\sqrt{2}i : (4, 0, 1, \sqrt{2}i), (0, 1, -1, \sqrt{2}i) \\
& \quad \lambda = \sqrt{2}i : (4, 1, 0, -2\sqrt{2}i) \\
M(\sigma_3) & \quad \lambda = 4i : (4, -1, -2, 0) \\
& \quad \lambda = 4 : (4, 0, 1, -2\sqrt{2}i), (0, 1, -1, -\sqrt{2}i) \\
& \quad \lambda = -4 : (4, 1, 0, 2\sqrt{2}i) \\
M(\sigma_4) & \quad \lambda = -2 - 2i : (4, 1, 0, 2\sqrt{2}) \\
& \quad \lambda = 2 + 2i : (4, 0, -1, -\sqrt{2}), (0, 1, 1, \sqrt{2}) \\
& \quad \lambda = -2 + 2i : (4, -1, 2, 0) \\
M(\sigma_5) & \quad \lambda = -2 - 2i : (4, 1, 0, -2\sqrt{2}) \\
& \quad \lambda = 2 + 2i : (4, 0, -1, \sqrt{2}), (0, 1, 1, \sqrt{2}) \\
& \quad \lambda = -2 + 2i : (4, -1, 2, 0) \\
M(\sigma_6) & \quad \lambda = 2i : (4, -1, 2i, 0) \\
& \quad \lambda = -2i : (4, -1, -2i, 0), (0, 0, 0, 1) \\
& \quad \lambda = 2 : (4, 1, 0, 0) \\
M(\sigma_7) & \quad \lambda = 2i : (4, -1, -2i, 0) \\
& \quad \lambda = -2i : (4, -1, 2i, 0), (0, 0, 0, 1) \\
& \quad \lambda = 2 : (4, 1, 0, 0)
\end{align*}
\]

The proof of Corollary 2 is the same as Corollary 2 in Yoshihara (2012). It is sufficient to note the intersection points of Galois lines. In the case where \( j(E) = 12^3 \), there exist points which are not the vertexes \( Q_i \) \((0 \leq i \leq 3)\) but the intersection of \( V_4 \) and \( Z_4 \)-lines. The projection from such points yield the curve with two Galois points.

## 5 Computation of \( \sigma_i^* \)

In Sect. 4 we have calculated several values by using the software “maxima” step by step.

First note that both \( E = \mathbb{C}/\mathcal{L} \) and the curve \( y^2 = 4x^3 - x \) have additions. The addition on the curve is given as follows.

\[
(a, b) + (c, d) = (e, f),
\]

\[
e = \frac{(4ac - 1)(a + c) - 2bd}{4(a - c)^2},
\]

\[
f = \frac{(4a^3 + 12a^2c - 3a - c)d - (4c^3 + 12ac^2 - 3c - a)b}{4(a - c)^3}.
\]
At the double point we have

\[
2(a, b) = \left(\frac{(4a^2 + 1)^2}{16b^2}, \frac{(4a^2 + 1)(4a^2 - 4a - 1)(4a^2 + 4a - 1)}{32b^3}\right).
\]

Since the points of order two lie on \( y = 0 \), we have \((0,0), (\pm \frac{i}{2}, 0)\). The point \((0,0)\) corresponds to \(1 + \frac{i}{2}c\). Moreover the coordinates of the points of order four are \((-\frac{i}{2}, \pm \frac{1+i}{\sqrt{2}}), (\frac{i}{2}, \pm \frac{1-i}{\sqrt{2}})\). These points are corresponded to \(\frac{3+i}{4}c, \frac{1+3i}{4}c, \frac{1+i}{4}c, \frac{3+3i}{4}c\).

We decompose \(\sigma_4(z) = iz + \frac{3+i}{4}c\) into a rotation \(z \rightarrow iz\) and a translation \(z \rightarrow z + \frac{3+i}{4}c\).

The rotation on \(E\) : \(z \rightarrow iz\) is corresponded to \((x, y) \rightarrow (-x, iy)\) on \(C\). Translations are more complicated. We show the method of computation by taking the example of \(\sigma_4^*\).

First we assume that the addition of \(\frac{3+i}{4}c\) on \(E\) corresponds to the addition of the point \((\frac{i}{2}, \frac{1-i}{\sqrt{2}})\) on \(C\). Using the addition formula, we have the representation of translation \(\tau\).

\[
\tau(x, y) = (x, y) + \left(\frac{i}{2}, \frac{1-i}{\sqrt{2}}\right)
= \left(\frac{i(2x + i)^2 - 2\sqrt{2}(1-i)y}{2(2x - i)^2}, -\sqrt{2}(1 + i) \cdot \frac{2x + i}{2x - i} \cdot \frac{i(2x + i)^2 - 2\sqrt{2}(1-i)y}{2(2x - i)^2}\right),
\]

\[
\sigma_4(x, y) = \tau(-x, iy).
\]

Then we have

\[
\sigma_4^*(x) = \frac{i(2x - i)^2 - 2\sqrt{2}(1 + i)y}{2(2x + i)^2},
\]
\[
\sigma_4^*(y) = -\sqrt{2}(1 + i) \cdot \frac{2x - i}{2x + i} \cdot \frac{i(2x - i)^2 - 2\sqrt{2}(1 + i)y}{2(2x + i)^2}.
\]

Moreover we have

\[
\sigma_4^*(x^2) = \frac{(i(2x - i)^2 - 2\sqrt{2}(1 + i)y)^2}{4(2x + i)^4}.
\]

Put

\[
f(x, y) = -2\sqrt{2}(1 + i)y + i(2x - i)^2,
\]
\[
g(x, y) = -2\sqrt{2}(1 + i)y - i(2x - i)^2.
\]
Since
\[(2x + i)^4 = f(x, y) \cdot g(x, y),\]
we have
\[
\sigma_4^*(1 : x^2 : x : y) = \left(1 : \frac{f(x, y)^2}{4(2x + i)^4} : \frac{f(x, y)}{2(2x + i)^2} : -\sqrt{2}(1 + i) \cdot \frac{2x - i}{2x + i} \cdot \frac{f(x, y)}{2} \right)
\]
\[
= \left(g(x, y) : \frac{f(x, y)}{4} : \frac{(2x + i)^2}{2} : -\sqrt{2}(1 + i) \cdot \frac{(2x - i)(2x + i)}{2} \right),
\]
\[
\sigma_4^*(1) = -2\sqrt{2}(1 + i)y - 4ix^2 - 4x + i,
\]
\[
\sigma_4^*(x^2) = -\frac{1 + i}{\sqrt{2}}y + ix^2 + x - \frac{i}{4},
\]
\[
\sigma_4^*(x) = 2x^2 + 2ix - \frac{1}{2},
\]
\[
\sigma_4^*(y) = -2\sqrt{2}(1 + i)x^2 - \frac{1 + i}{\sqrt{2}}.
\]
Therefore we obtain
\[
M(\sigma_4) = \begin{pmatrix}
i & -4i & -4 & -2\sqrt{2}(1 + i) \\
-\frac{i}{4} & i & 1 & -\frac{1+i}{\sqrt{2}} \\
-\frac{1}{2} & 2 & 2i & 0 \\
-\frac{1+i}{\sqrt{2}} & -2\sqrt{2}(1 + i) & 0 & 0
\end{pmatrix}.
\]

5.1 About maxima
In the computations we have used maxima 5.37.3 latest version, and front end wnMaxima 15.08.1+git.
Choose output form “none” otherwise do not display parenthesis ().

5.2 Computation of $\tau$ by maxima
Define
\[
f(a,b,c,d):=((4ac-1)(a+c)-2bd)/(4(a-c)^2).
g(a,b,c,d):=((4a^3+12a^2c-3a-c)d-(4c^3+12ac^2-3c-a)b)/(4(a-c)^3).
ft(x,y):=f(x,y,%i/2,(1-%i)/sqrt(2)).
gt(x,y):=g(x,y,%i/2,(1-%i)/sqrt(2)).
We get
ft(x,y)=((x+%i/2)*(2*%i*x-1)-sqrt(2)*(1-%i)*y)/(4*(x-%i/2)^2).

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We have studied the group generated by the Galois group belonging to Galois points (Kanazawa et al. 2001; Miura and Ohbuchi 2015). In the case of Galois embedding of elliptic curves, we have the following.

**Remark 7** For each Galois embedding let $G$ be the group generated by the Galois groups belonging to the Galois subspaces. Then $G$ can be realized as the Galois group for some Galois embedding of the elliptic curve.

**Proof** We infer readily the theorem from Theorems 7.4 and 7.7 in Kanazawa and Yoshihara (2011).  

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