Research Article

X-ranks for embedded varieties and extensions of fields

Edoardo Ballico

Department of Mathematics, University of Trento, Trento, Italy

Abstract

Let $X \subset \mathbb{P}^r$ be a projective embedded variety defined over a field $K$. Results relating maximum and generic $X$-rank of points of $\mathbb{P}^r(K)$ and $\mathbb{P}^r(L)$ are given, where $L$ is a field containing $K$. Some of these results are algebraically closed for $K$ and $L$. In other results (e.g. on the cactus rank), $L$ is a finite extension of $K$.

Keywords: cactus rank; Veronese variety; $X$-rank.

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1. Introduction

In this paper we fix an extension of fields, say $K \subset L$, a projective variety $X$ defined over $K$ and an embedding of $X$ into a projective space $\mathbb{P}^r$ defined over $K$. Thus $\mathbb{P}^r(K) \subset \mathbb{P}^r(L)$. For each $a \in \mathbb{P}^r(K)$ there are several different notions of ranks with respect to $X(K)$ and $X(L)$. In Section 3 we consider the case in which $K$ is not algebraically closed and $L$ is a finite extension of $K$ (see Theorem 3.1), and in the rest of the paper we consider the case in which both $K$ and $L$ are algebraically closed.

Fix algebraically closed fields $K \subset L$. Take $F \in \{K,L\}$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over $K$. We also assume that the embedding of $X$ in $\mathbb{P}^r$ is defined over $K$ and that $X$ is non-degenerate, i.e. $X(K)$ spans $\mathbb{P}^k(K)$. Thus $X(L)$ spans $\mathbb{P}^r(L)$. Set $n := \dim X$. For any scheme or algebraic subset $Z \subset \mathbb{P}^r(K)$ (respectively, $Z \subset \mathbb{P}^r(L)$) defined over $K$ (respectively, over $L$) let $(Z)_K \subset \mathbb{P}^r(K)$ (respectively, $(Z)_L \subset \mathbb{P}^r(L)$) denote the linear span of $Z$ over $K$ (respectively, over $L$). Note that $(\langle Z \rangle)_L = \langle Z \rangle_L$ for any $Z \subset \mathbb{P}^r(K)$. For all positive integers $t$ let $S(X(F),t)$ denote the set of all subsets of $X(F)$ with cardinality $t$. The set $S(X(F),t)$ is an irreducible quasi-projective variety of dimension $n$. For any $o \in \mathbb{P}^r(F)$ the $X(F)$-rank $r_{X(F)}(o)$ of $o$ is the minimal cardinality of a subset of $X(F)$ containing $o$ in its linear span. For any positive integer $t$ let $S(X(F),o,t)$ denote the set of all $S \subset S(X(F),t)$ such that $o \in \langle S \rangle_F$ and $o \notin \langle S' \rangle_F$ for any $S' \subset S$. Each set $S(X(F),o,t)$ is constructible by a theorem of Chevalley (see [8, Ex. II.3.18]). Note that $r_{X(F)}(o)$ is the minimal integer $t$ such that $S \subset S(X(F),t) \neq \emptyset$. Now assume $o \in \mathbb{P}^r(K)$. Since $K$ is algebraically closed, it is easy to check (and well-known) that $r_{X(L)}(o) = r_{X(K)}(o)$ and that $S(X(L),o,t)$ is the constructible $L$-set associated to $S(X(K),o,t)$ (see Remark 2.1 for more details). In particular $S(X(L),o,t)$ and $S(X(L),o,t)$ have the same number of irreducible components and the bijection between their irreducible components preserves the dimension of the components. In particular $S(X(L),o,t) = S(X(K),o,t)$ if and only if either $S(X(K),o,t) = \emptyset$ or $S(X(K),o,t)$ is finite. For any positive integer $t$ let $\sigma_t(X(F)) \subset \mathbb{P}^n(F)$ denote the closure in $\mathbb{P}^n(F)$ of the union of all $(S)_F$, $S \in S(X(F),t)$. Each $\sigma_t(X(F))$ is irreducible and $\sigma_t(X(K))$ is the $L$-variety associated to the $K$-variety $\sigma_t(X(K))$. The first integer $a$ such that $\sigma_a(X(F)) = \mathbb{P}^n(F)$ is the same for $F = K$ and $F = L$. It is often call the generic $X(K)$-rank (respectively, generic $X(L)$), because it is the $X(K)$-rank (respectively, $X(L)$-rank) of a non-empty open subset of $\mathbb{P}^n(K)$ (respectively, $\mathbb{P}^n(L)$). For any positive integer $t$ let $R(X(F),t)$ denote the set of all $o \in \mathbb{P}^n(F)$ such that $r_{X(F)}(o) = t$. Each $R(X(F),t)$ is constructible (Lemma 2.1). See Remark 2.1 for the definition and construction of the $L$-associated set of any constructible subset of $\mathbb{P}^n(K)$.

It is easy to prove the following result (its proof is given after Lemma 2.1).

Theorem 1.1. For each positive integer $t$ the constructible $L$-set $R(X(L),t)$ is the $L$-set associated to $R(X(K),t)$.

The maximum among all $X(F)$-rank is the largest integer $a$ such that $R(X(F),a) \neq \emptyset$. Thus Theorem 1.1 has the following corollary.

Corollary 1.1. The maxima of the $X(K)$-ranks and of the $X(L)$-ranks are the same.
Take any \( o \in \mathbb{P}^r(F) \). The open \( X(F) \text{-rank} \) \( o_{X(F)}(o) \) of \( o \) is the minimal integer \( t > 0 \) such that for all closed sets \( T \subseteq X(F) \) there is \( S \in S(X(F), t) \) such that \( o \in (S)_F \) and \( S \cap T = \emptyset \) (see [1, 9]). Obviously \( o_{X(F)}(o) \geq r_{X(F)}(o) \), but very often the strict inequality holds. For instance, \( o_{X(F)}(o) > 1 \) for all \( o \in \mathbb{P}^r(F) \). Since \( X(K) \) is Zariski dense in \( X(L) \), \( o_{X(L)}(o) \leq o_{X(K)}(o) \) for all \( o \in \mathbb{P}^r(K) \). Let \( O(X(F), t) \) denote the set of all \( o \in \mathbb{P}^r(F) \) such that \( o_{X(F)}(t) \). We also prove the following results.

**Theorem 1.2.** We have \( o_{X(L)}(o) = o_{X(K)}(o) \) for all \( o \in \mathbb{P}^r(K) \).

**Theorem 1.3.** The following properties are true:

1. The generic open \( X(F) \text{-rank} \) is the same for \( F = K \) and \( F = L \).

2. The maximum open \( X(F) \text{-rank} \) is the same for \( F = K \) and \( F = L \).

3. Each set \( O(X(F), t) \) is constructible and \( O(X(L), t) \) is the L-constructible set associated to \( O(X(K), t) \).

### 2. Proofs of Theorems 1.1, 1.2, and 1.3

**Remark 2.1.** Let \( Y(K) \) be a projective variety defined over \( K \) and let \( E \subseteq Y(K) \) be a constructible subset. We define the L-constructible set \( E(L) \subseteq Y(L) \) associated to \( E \) in the following way. If \( E \) is a finite set, then \( E(L) := E \). Thus we may assume \( \dim E > 0 \) and use induction on the integer \( \dim E \). Let \( \overline{E} \) be the closure of \( E \) in \( Y(K) \). Let \( \overline{E} = A_1 \cup \cdots \cup A_r \) be the irreducible components of \( \overline{E} \). Set \( E_i := A_i \cap E \). Each set \( E_i \) is constructible. Note that each \( E_i \) contains a non-empty open subset \( U_i \) of \( A_i \). Thus the set \( U_i \setminus E_i \) is a constructible set of dimension \( < \dim E \). By the inductive assumption we have defined the constructible sets \( (E_i \setminus U_i)(L) \). Set \( E_i(L) := U_i(L) \cup (E_i \setminus U_i)(L) \) and \( E(L) := E_1(L) \cup \cdots \cup E_r(L) \). It is easy to check that the definition of \( E(L) \) does not depend on the choice of \( Y(K) \), we only need a \( K \)-variety containing \( \overline{E} \). Note that there is a bijection between the irreducible component of \( \overline{E}(L) \) and \( \overline{E} \) (respectively, \( \overline{E}(L) \setminus E(L) \) and \( \overline{E} \setminus E \) ) which preserves the dimension.

**Observation 2.1.** Note that \( E(L) = E \) if and only if \( E \) is finite. In all other cases \( E(L) \) (respectively, \( E \)) has the cardinality of \( L \) (respectively, \( K \)) and hence \( E(L) \setminus E(K) \) is infinite and its Zariski closure contains all non-isolated points of \( E(L) \).

Observation 2.1 is applied to \( R(X(K), t) \) and \( R(X(L), t) \) by Theorem 1.1 and to all constructible sets used in the proofs of the results stated in the introduction. By [6, Theorem 3.1] each \( R(X(F), t) \) has positive dimension, except at most when \( t \) is the maximal \( X(F) \text{-rank} \).

**Lemma 2.1.** Each set \( R(X(F), t) \) is constructible.

**Proof:** Since \( R(X(F), 1) = X(F) \), we may assume \( t > 1 \) and use induction on the integer \( t \). Since \( R(X(F), t) \cap R(X(F), x) = \emptyset \) for all \( x < t \), it is sufficient to prove that \( A := \bigcup_{1 \leq x \leq t} R(X(F), x) \) is constructible. The set \( E \) is the image of \( S(X(F), t) \) by the evaluation map.

**Proof of Theorem 1.1.** Lemma 2.1 says that \( R(X(K), t) \) and \( R(X(L), t) \) are constructible. Since \( X(L) \) is the \( L \)-set of \( X(K) \), we may use induction on \( t \) to prove the theorem. It is sufficient to mimic the proof of Lemma 2.1.

**Proof of Theorem 1.2.** For any positive integer \( t \) set

\[
X(F)(t) := \bigcup_{S \in S(X(F), t)} S \subseteq X(F).
\]

Since \( S(X(F), t) \) is constructible, a theorem of Chevalley gives that \( X(F)(t) \) is constructible and that for each constructible set \( \Sigma \subseteq S(X(F), t) \) the set \( ev(\Sigma) := \bigcup_{S \in S(X(F), t)} S \subseteq X(F) \) is constructible (see [8, Ex. II.3.18, II.3.19]). Fix any \( o \in \mathbb{P}^r(F) \).

**Observation:** The open \( X(F) \text{-rank} \) \( o_{X(F)}(o) \) of \( o \) is the first positive integer \( t \) such that \( ev(S(X(F), o, t)) \) is Zariski dense in \( X(F) \).

Now assume \( o \in \mathbb{P}^r(K) \). Since \( K \) is algebraically closed, the Observation gives \( o_{X(L)}(o) = o_{X(K)}(o) \).

**Proof of Theorem 1.3.** It is sufficient to prove Part (3). The observation in the proof of Theorem 1.2 and a theorem of Chevalley (see [8, Ex. II.3.18, II.3.19]) first gives that each \( O(X(F), t) \) is constructible and then that \( O(X(L), t) \) is the L-constructible set associated to \( O(X(K), t) \).
3. When $K$ is not algebraically closed

Let $K$ be a field which is not algebraically closed. We fix an inclusion $K \subset \overline{K}$. Let $X \subset \mathbb{P}^r$ be an embedding (defined over $K$) of the integral projective variety $X$ defined over $K$. We assume that $X(\overline{K})$ is non-degenerate, but we do not assume that $X(K)$ spans $\mathbb{P}^r(K)$ (we allow the case $X(K) = \emptyset$). For each $a \in \overline{K}$ let $\deg(a)$ be the degree of the minimum polynomial of $a$ over $K$, i.e. the dimension of the $K$-vector space $K(a)$. We fix a system of homogeneous coordinates $x_0, \ldots, x_r$ of $\mathbb{P}^r(K)$.

For each $a = (a_0 : \cdots : a_r) \in \mathbb{P}^r(\overline{K})$ with, say, $a_i \neq 0$ the degree $\deg_1(a)$ of $a$ is the maximum of all integers $\deg(a_j/a_i)$, $0 \leq j \leq r$, and let $\deg_2(a)$ be the degree of the extension $K(a_0/a_1, \ldots, a_r/a_i)$ of $K$. The integer $\deg_3(a)$ is the degree of the normal closure of $K(a_0/a_1, \ldots, a_r/a_i)$ as an extension of $K$. The integers $\deg_1(a)$, $\deg_2(a)$ and $\deg_3(a)$ are well-defined, i.e. they do not depend upon the choice of the index $i$ such that $a_i \neq 0$. If $K$ is a finite field $\mathbb{F}_q$, then $\deg_2(a) = \deg_3(a)$ for all $a$, because all finite extensions of $\mathbb{F}_q$ are Galois extensions. However, even for a finite field we may have $\deg_1(a) < \deg_2(a)$ if $r \geq 2$ (Example 3.1). If $K$ is real closed ([4]), then $\overline{K} = K(i)$ and any $a$ has $\deg_1(a) = \deg_2(a) = \deg_3(a) \in \{1, 2\}$. For $K = \mathbb{Q}$ and any $r \geq 2$ there are easy examples with $\deg_1(a) < \deg_2(a) < \deg_3(a)$. For any finite set $S \subset \mathbb{P}^r(\overline{K})$, $S \neq \emptyset$ let $\deg_1(S)$ be the maximum of all $\deg_1(a)$, $a \in S$. Let $\deg_2(S)$ (respectively, $\deg_3(S)$) be the degree of the extension (respectively, normal extension) of $K$ generated by the ratios of the homogeneous coordinates of all $a \in S$.

Take $o \in \mathbb{P}^r(\overline{K})$ and fix $i \in \{1, 2, 3\}$. Set $t := r_{X(\overline{K})}(o)$. Let $DR_i(X, K, o)$ denote the minimum of all $\deg_i(S)$ for some $S \subset S(X(\overline{K}), t)$. We say that $a = (a_0 : \cdots : a_r) \in \mathbb{P}^r(\overline{K})$ is separable over $K$ if all ratios $a_j/a_i$ with $a_i \neq 0$ are separable over $K$. Obviously if $a_i \neq 0$ it is sufficient to test all $a_j/a_i$. If $K$ is perfect, then every $a \in \mathbb{P}^r(\overline{K})$ is separable over $K$. The field $K$ is perfect if either $K$ is a finite field or $\text{char}(K) = 0$.

Example 3.1. Take $r = 2$, $K = \mathbb{F}_q$ and $a = (1 : u : v)$ with $u \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$ and $v \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. We have $\deg_1(a) = 3$ and $\deg_2(a) = 6$.

The fact that all finite extensions of a finite field are Galois extensions has the following byproduct.

**Proposition 3.1.** Take $K = \mathbb{F}_q$. Fix $o \in \mathbb{P}^r(\overline{K})$ and set $t := r_{X(\overline{K})}(o)$. Assume $\#S(X(\overline{K}), o, t) = 1$. Then

\[ DR_2(X, K, o) \leq t \deg_2(o). \]

**Proof.** Set $x := \deg_2(o)$ and $y := DR_2(X, K, o)$. Write $\{S\} = S(X(\overline{K}), o, t)$. Consider $X$ over $\mathbb{F}_{q^r}$. Since $o \in \mathbb{P}^r(\mathbb{F}_{q^r})$ and $S$ is the unique element of $S(X(\overline{K}), t)$ computing the $X(\overline{K})$-rank of $o$, $S$ is invariant for the Galois group of the extension $\mathbb{F}_{q^r}/\mathbb{F}_{q^t}$. Thus $y \leq (\#S)x$. We have $t = \#S$. \( \square \)

For any field $E \supset K$ let $\rho(X(E))$ denote the maximal integer $t$ such that any subset of $X(F)$ with cardinality $t$ is linearly independent. Of course, if $E \subset E'$, then $X(E) \subset X(E')$ and hence $\rho(X(E')) \leq \rho(X(E))$. If $E$ is algebraically closed, it is easy to check that $\rho(X(E')) = \rho(X(E))$ for any field $E' \supset E$.

**Remark 3.1.** Fix $o \in \mathbb{P}^r(\overline{K})$ and assume $2r_{X(\overline{K})} \leq \rho(X(\overline{K}))$. Then we have

\[ \#S(X(\overline{K}), o, r_{X(\overline{K})}) = 1. \]

**Remark 3.2.** Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, be the $d$-Veronese embedding of $\mathbb{P}^n$, i.e. the embedding induced The cohomology of a projective space easily gives that $\rho(\nu_d(\mathbb{P}^n)(E)) = d + 1$ for any field $E$. In particular we may apply Remark 3.1 to any $o \in \mathbb{P}^r(\overline{K})$ such that

\[ r_{X(\overline{K})}(o) \leq \left\lfloor \frac{d+1}{2} \right\rfloor. \]

The proof of Proposition 3.1 gives the following result.

**Proposition 3.2.** Fix a separable $o \in \mathbb{P}^r(\overline{K})$ and assume $\#S(X(\overline{K}), o, r_{X(\overline{K})}(o)) = 1$. Then $DR_3(X, K, o) \leq r_{X(\overline{K})}(o) \deg_3(o)$.

By Remark 3.2, Proposition 3.2 may be applied to the $d$-Veronese embedding of any projective space, but just in a very restricted range of ranks.

Other notions of ranks for homogeneous polynomials are the slice rank and the Schmidt rank (often called strength). The recent preprint [10] by Lempert and Ziegler proves stronger versions of all our attempts related to this notion over a non-algebraically closed field with characteristic 0.

**Remark 3.3.** Take any field $K$ such that $\text{char}(K) = 0$ and let $X \subset \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, be the image of the the $d$-Veronese embedding of $\mathbb{P}^n$. Fix $a \in X(K)$ and $o \in \mathbb{P}^r(K)$.
If we do not search for a small degree extension of $K$ on which it is defined all points (or the set) defining the $X(K)$-rank, then we may get far better bounds. We recall that the cactus $X(K)$-rank of $a \in \mathbb{P}^r(K)$ is the minimal degree of a zero-dimensional scheme $Z \subset X(K)$ whose linear span contains $a$ (see [2,3,5,7]). Fix a finite extension $L$ of $K$ such that $a$ is defined over $K$. We call cactus $L$-rank the minimal degree of a zero-dimensional scheme $Z \subset X(K)$ defined over $L$ and whose linear span contains $a$. We call strong cactus $L$-rank the minimal degree of a zero-dimensional scheme $Z \subset X(K)$ defined over $L$ and whose linear span contains $a$. Obviously every connected $Z$ defined over $L$ may be use to test the strong cactus rank.

**Theorem 3.1.** Assume $\text{char}(K) = 0$. Let $L$ be any finite extension of $K$. Fix an integer $d \geq 3$. Let $X(K) \subset \mathbb{P}^r$, $r = (n+d) - 1$, be the image of the $d$-Veronese embedding of $\mathbb{P}^n$. If $d = 2k + 1$ is odd, set $N := 2\binom{n+k}{n}$. If $d = 2k + 2$ is even, set $$N := \binom{n+k}{n} + \binom{n+k+1}{n}.$$ Then every $a \in \mathbb{P}^r(L)$ has strong cactus $L$-rank $\leq N$.

**Proof.** Fix $b \in X(L)$. The proof of [3, Theorem 3] gives the existence of a zero-dimensional scheme $Z \subset X(K)$ defined over $L$, spanning $a$ and with $Z_{\text{red}} = \{b\}$. Since $Z$ is connected, it gives an upper bound for the strict cactus $L$-rank of $a$. \hfill $\square$

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