GALOIS THEORY FOR BIALGEBROIDS, DEPTH TWO AND NORMAL HOPF SUBALGEBRAS

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Abstract. We reduce certain proofs in [16, 11, 12] to depth two quasibases from one side only, a minimalistic approach which leads to a characterization of Galois extensions for finite projective bialgebroids without the Frobenius extension property. We prove that a proper algebra extension is a left $T$-Galois extension for some right finite projective left bialgebroid over some algebra $R$ if and only if it is a left depth two and left balanced extension. Exchanging left and right in this statement, we have a characterization of right Galois extensions for left finite projective right bialgebroids. Looking to examples of depth two, we establish that a Hopf subalgebra is normal if and only if it is a Hopf-Galois extension. We characterize finite weak Hopf-Galois extensions using an alternate Galois canonical mapping with several corollaries: that these are depth two and that surjectivity of the Galois mapping implies its bijectivity.

1. Introduction and Preliminaries

Hopf algebroids arise as the endomorphisms of fiber functors from certain tensor categories to a bimodule category over a base algebra. For example, Hopf algebroids over a one-dimensional base algebra are Hopf algebras while Hopf algebroids over a separable $K$-algebra base are weak Hopf algebras. Galois theory for right or left bialgebroids were recently introduced in [11, 12, 13] based on the theory of Galois corings [3] and ordinary definitions of Galois extensions [19, 7] with applications to depth two extensions. In particular, Frobenius extensions that are right Galois over a left finite projective right bialgebroid are characterized in [12] as being of depth two and right balanced. Then a Galois theory for Hopf algebroids, especially of Frobenius type, was introduced in [1, 11] with applications to Frobenius extensions of depth two and weak Hopf-Galois extensions over finite dimensional quantum groupoids. Although they break with the tradition of defining Galois extensions over bialgebras and have a more complex definition, Galois extensions over Hopf algebroids have more properties in common with Hopf-Galois extensions. However, several of these properties will follow from any Galois theory for bialgebroids which is in possession of two Galois mappings equivalent due to a bijective antipode, sometimes denoted by $\beta$ and $\beta'$, as is the case for finite Hopf-Galois theory [19, ch. 8], finite weak Hopf-Galois theory (see the last section in this paper), possibly some future, useful weakening of Hopf-Galois theory to Hopf algebroids over a symmetric algebra, a Frobenius algebra or some other type of base algebra.

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In [1] a characterization similar to that in [12] for depth two Frobenius one-sided balanced extensions is given in terms of Galois extensions over Hopf algebroids with integrals. This shows in a way that the main theorem in [12] makes no essential use of the hypothesis of Frobenius extension (only that a Frobenius extension is of left depth two if and only if it is of right depth two), and it would be desirable to remove the Frobenius hypothesis. This is then the objective of section 2 of this paper: to show that Galois extensions over one-sided finite projective bialgebroids are characterized by one-sided depth two and balance conditions on the extension (Theorem 2.1). This requires among other things some care in re-doing the two-sided arguments in [16] to show that the structure $T := (A \otimes_B A)_B$ on a one-sided depth two extension $A \mid B$ with centralizer $R$ is still a one-sided finite projective right bialgebroid (proposition 1.1). This provides the objective of the rest of this section; in the appendix in section 5, we include some related results for the $R$-dual bialgebroid $S := \text{End}_B A_B$ of a one-sided depth two extension $A \mid B$. These two sections may be read as an introduction to depth two theory.

A depth two complex subalgebra is a generalization of normal subgroup [14]. The question was then raised whether depth two Hopf subalgebras are precisely the normal Hopf subalgebras ($\supseteq$ in [14]). In a very special case, this is true when the notion of depth two is narrowed to $H$-separability [11], an exercise in going up and down with ideals as in commutative algebra. We study in section 3 the special case of depth two represented by finite Hopf-Galois extensions: we show that a Hopf-Galois Hopf subalgebra is normal using a certain algebra epimorphism from the Hopf overalgebra to the Hopf algebra which is coacting Galois, and comparing dimensions of the kernel with the associated Schneider coalgebras.

A special case of Galois theory for bialgebroids is weak Hopf-Galois theory [3, 7, 11, 12], (where Hopf-Galois theory is in turn a special case): for depth two extensions, each type of Galois extension occurs as we move from any centralizer to separable centralizers to one-dimensional centralizers. Conversely, each type of Galois extension, so long as it is finitely generated, is of depth two [16, 11, 12]. In section 4 we complete the proof that a weak Hopf-Galois extension is left depth two by studying the alternative Galois mapping $\beta' : A \otimes_B A \to A \otimes H$ where $\beta'(a \otimes a') = a_{(0)} a' \otimes a_{(1)}$. As a corollary we find an interesting factorization of the Galois isomorphism of a weak Hopf algebra over its target subalgebra. In a second corollary, a direct proof is given that a surjective Galois mapping for an $H$-extension is automatically bijective, if $H$ is a finite dimensional weak Hopf algebra. Finally, it is shown by somewhat different means than in [3] that a weak bialgebra in Galois extension of its target subalgebra has an antipode reconstructible from the Galois mapping. We provide some evidence for more generally a weak bialgebra, which coacts Galois on an algebra over a field, having an antipode, something which is true for bialgebroids by a result of Schauenburg [21].

Let $K$ be any commutative ground ring in this paper. All algebras are unital associative $K$-algebras and modules over these are symmetric unital $K$-modules. We say that $A \mid B$ is an extension (of algebras) if there is an algebra homomorphism $B \to A$, proper if this is monic. This homomorphism induces the natural bimodule structure $B A_B$ which is most important to our set-up. The extension $A \mid B$ is left depth two (left D2) if the tensor-square $A \otimes_B A$ is centrally projective w.r.t. $A$ as natural $B$-$A$-bimodules: i.e.,

$$B A \otimes_B A_A \oplus \cong \otimes^n B A_A.$$
This last statement postulates the existence then of a split \( B \)-\( A \)-epimorphism from a direct sum of \( A \) with itself \( n \) times to \( A \otimes_B A \).

Making the clear-cut identifications \( \text{Hom} \left( (B \otimes_B A) \oplus (B \otimes_B A), B \otimes_B A \right) \cong \text{End}_B A_R \) and \( \text{Hom} \left( (B \otimes_B A) \oplus (B \otimes_B A), (A \otimes_B A)^B \right) \), we see that left D2 is characterized by there being a left D2 quasibase \( t_i \in (A \otimes_B A)^B \) and \( \beta_i \in \text{End}_B A_R \) such that for all \( a, a' \in A \)

\[
    a \otimes_B a' = \sum_{i=1}^{n} t_i \beta_i(a) a'.
\]

The algebras \( \text{End}_B A_B \) and \( (A \otimes_B A)^B \) (note that the latter is isomorphic to \( \text{End}_A A \otimes_B A \) and thus receives an algebra structure) are so important in depth two theory that we fix (though not unbendingly) brief notations for these:

\[
    S := \text{End}_B A_B \quad T := (A \otimes_B A)^B.
\]

Similarly, a right depth two extension \( A \mid B \) is defined by switching from the natural \( B \)-\( A \)-bimodules in the definition above to the natural \( A \)-\( B \)-bimodules on the same structures. Thus an extension \( A \mid B \) is right D2 if \( A \otimes_B A_B \oplus \ast \cong \otimes^m A_B \).

Equivalently, if there are \( m \) paired elements \( u_j \in T, \gamma_j \in S \) such that

\[
    a \otimes a' = \sum_{j=1}^{m} a \gamma_j(a') u_j
\]

for all \( a, a' \in A \).

A depth two extension is one that is both left and right D2. These have been studied in \[15, 11, 12\] among others, but without a focus on left or right D2 extensions. Note that a left D2 extension \( A \mid B \) has right D2 extension \( A^{\text{op}} \mid B^{\text{op}} \) when we pass to opposite algebras. This gives in fact a natural one-to-one correspondence between left D2 extensions and right D2 extensions.

Let \( t, t' \) be elements in \( T \), where we write \( t \) in terms of its components using a notation that suppresses a possible summation in \( A \otimes_B A \): \( t = t^1 \otimes t^2 \). Then the algebra structure on \( T \) is simply

\[
    tt' = t'^1 t^1 \otimes t'^2 t^2, \quad 1_T = 1_A \otimes 1_A
\]

There is a standard “groupoid” way to produce right and left bialgebroids, which we proceed to do for \( T \). There are two commuting embeddings of \( R \) and its opposite algebra in \( T \). A “source” mapping \( s_R : R \to T \) given by \( s_R(r) = 1_A \otimes r \), which is an algebra homomorphism. And a “target” mapping \( t_R : R \to T \) given by \( t_R(r) = r \otimes 1_A \) which is an algebra anti-homomorphism and clearly commutes with the image of \( s_R \). Thus it makes sense to give \( T \) an \( R \)-\( R \)-bimodule structure via \( s_R \), \( t_R \) from the right: \( r \cdot t \cdot r' = t s_R(r') t_R(r) = t(r \otimes r') = rt^1 \otimes t^2 r' \), i.e., \( rT_r \) is given by

\[
    r \cdot t^1 \otimes t^2 \otimes r' = rt^1 \otimes t^2 r'.
\]

**Proposition 1.1.** Suppose \( A \mid B \) is either a right or a left D2 extension. Then \( T \) is a right \( R \)-bialgebroid, which is either left f.g. \( R \)-projective or right f.g. \( R \)-projective respectively.

**Proof.** First we suppose \( A \mid B \) is left D2 with quasibases \( t_i \in T \), \( \beta_i \in S \). The proof that \( T \) is a right \( R \)-bialgebroid in \[15, 5.1\] carries through verbatim except in one place where a right D2 quasibase made a brief appearance, where coassociativity of
the coproduct needs to be established through the introduction of an isomorphism. Thus we need to see that

\[ T \otimes_R T \otimes_R T \xrightarrow{\cong} (A \otimes_B A \otimes_B A \otimes_B A)^B \]

via \( t \otimes t' \otimes t'' \mapsto t^1 \otimes t^2 t'^1 \otimes t'^2 t''^1 \otimes t''^2 \). The inverse is given by

\[ a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto \sum_{i,j} b_i \otimes_R a_j (\beta_i(a_1)a_2) a_3 \otimes_B a_4. \]

for all \( a_i \in A \) (\( i = 1, 2, 3, 4 \)).

In the case that we only use a right D2 quasibase, this inverse is given by

\[ a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto \sum_{j,k} a_1 \otimes a_2 \gamma_k(a_3 a_j(a_4)) \otimes_R u_k \otimes_R u_j. \] (5)

Both claimed inverses are easily verified as such by using the right and left D2 quasibase equations repeatedly.

The module \( T_R \) is finite projective since eq. (6) implies a dual bases equation

\[ t = \sum_i t_i f_i(t), \] for each \( t \in T \subseteq A \otimes_B A \), where \( f_i(t) := \beta_i(t^i) t^2 \) define \( n \) maps in \( \text{Hom}(T_R, R_R) \).

Suppose \( A \mid B \) is right D2 with quasibase \( u_j \in T, \gamma_j \in S \). The algebra structure on \( T \) is given in the introduction above as is the \( R \)-\( R \)-bimodule structure. What remains is specifying the \( R \)-coring structure on \( T \) and checking the five axioms of a right bialgebroid. The coproduct \( \Delta : T \to T \otimes_R T \) is given by

\[ \Delta(t) := \sum_j (t^1 \otimes_B \gamma_j(t^2)) \otimes_R u_j, \] (6)

which is clearly left \( R \)-linear, and right \( R \)-linear as well since

\[ \Delta(tr) = \sum_j t^1 \otimes \gamma_j(t^2 r) \otimes u_j \xrightarrow{\cong} t^1 \otimes 1 \otimes t^2 r \]

under the isomorphism \( T \otimes_R T \cong (A \otimes_B A \otimes_B A)^B \) given by \( t \otimes t' \mapsto t^1 \otimes t^2 t'^1 \otimes t'^2 \), which is identical to the image of

\[ \Delta(t)r = \sum_j t^1 \otimes \gamma_j(t^2) \otimes u_j r \mapsto t^1 \otimes 1 \otimes t^2 r. \]

Coassociativity \( (\Delta \otimes \text{id}_T)\Delta = (\text{id}_T \otimes \Delta)\Delta \) follows from applying the isomorphism

\[ T \otimes_R T \otimes_R T \cong (A \otimes_B A \otimes_B A \otimes_B A)^B \]

given above in this proof to the left-hand and right-hand sides applied to a \( t \in T \):

\[ \sum_j \Delta(t^1 \otimes \gamma_j(t^2)) \otimes u_j = \sum_{j,k} t^1 \otimes \gamma_k(\gamma_j(t^2)) \otimes_R u_k \otimes_R u_j \xrightarrow{\cong} t^1 \otimes_B 1_A \otimes_B 1_A \otimes_B t^2. \]

\[ \sum_j (t^1 \otimes \gamma_j(t^2)) \otimes_R \Delta(u_j) = \sum_{j,k} (t^1 \otimes \gamma_j(t^2)) \otimes_R (u_j \otimes \gamma_k(u_j^2)) \otimes_R u_k \]

which also maps into \( t^1 \otimes_B 1_A \otimes_B 1_A \otimes_B t^2 \) under the same isomorphism.

The counit \( \varepsilon : T \to R \) of the \( R \)-coring \( T \) is given by

\[ \varepsilon(t) := t^1 t^2 \] (7)

i.e., the multiplication mapping \( A \otimes_B A \to A \) restricted to \( T \) (and taking values in \( R \) since \( bt = tb \) for all \( b \in B \)). Clearly, \( \varepsilon(rtr') = r \varepsilon(t) r' \) for \( r, r' \in R, t \in T \), and that
Definition 1.2. In detail, the definition is equivalent to the following.

The subalgebra of coinvariants is $\Delta R_{\delta}$.

We next verify the five axioms of a right bialgebroid [10, 2.1].

1. $\Delta(1_T) = 1_T \otimes 1_T$ since $\gamma_j(1_A) \in R$ and $1_T = \sum_j \gamma_j(1_A)u_j$.

2. $\varepsilon(1_T) = 1_A$ since $1_T = 1_A \otimes 1_A$.

3. $\varepsilon(tt') = \varepsilon(t_R\varepsilon(t)t') = \varepsilon(s_R\varepsilon(t)t')$ $(t, t' \in T)$ since $\varepsilon(tt') = t^1t_1t_2t_2$, $t_R\varepsilon(t) = t^1t^2 \otimes_B 1_A$ and $s_R\varepsilon(t) = 1_A \otimes_B t^1t^2$.

4. $(s_R r \otimes 1_T)\Delta(t) = (1_T \otimes t_R(r))\Delta(t)$ for all $r \in R, t \in T$ since the left-hand side is

$$\sum_j (t^1 \otimes_B r \gamma_j(t^2)) \otimes_R u_j \mapsto t^1 \otimes_B r \otimes_B t^2$$

under the isomorphism $T \otimes_T T \cong (A \otimes_B A \otimes_B A)^R$ given by $t \otimes_T t' \mapsto t^1 \otimes_B t^1t^2t_2$ and the right-hand side is equal to

$$\sum_j (t^1 \otimes_B \gamma_j(t^2)) \otimes_R (u^1_jr \otimes_B u^2_j) \mapsto t^1 \otimes_B r \otimes_B t^2$$

with the same image element.

5. $\Delta(tt') = \Delta(t)\Delta(t')$ for all $t, t' \in T$ in the tensor subalgebra (denoted by $T \times_T T$ with the straightforward tensor multiplication) of $T \otimes_T T$ (which makes sense thanks to the previous axiom). This follows from both sides having the image element $t^1t^1t^2t_2t^2t_2$ under the isomorphism $T \otimes_T T \cong (A \otimes_B A \otimes_B A)^R$, which is clear for the left-hand side of the equation and for the right-hand side we note it equals

$$\sum_{j, k} (t^1t^1_1 \otimes_B \gamma_j(t^2)\gamma_k(t^2)) \otimes_R (u^1_ju^2_j \otimes_B u^2_ju^2_k).$$

Now apply $t \otimes t' \mapsto t^1 \otimes_B t^2t_1 \otimes_B t^2$ and the right D2 quasibase equation twice.

This completes the proof that $(T, R, s_R, t_R, \Delta, \varepsilon)$ is a right bialgebroid.

Finally $rT$ is finite projective via an application of the right D2 quasibase eq. [8].

A right comodule algebra is an algebra in the tensor category of right $R$-comodules $\mathcal{H}$. In detail, the definition is equivalent to the following.

**Definition 1.2.** Let $T$ be any right bialgebroid $(T, R, \hat{s}, \hat{t}, \Delta, \varepsilon)$ over any base algebra $R$. A right $T$-comodule algebra $A$ is an algebra $A$ with algebra homomorphism $R \rightarrow A$ (providing the $R$-$R$-bimodule structure on $A$) together with a coaction $\delta : A \rightarrow A \otimes_T T$, where values $\delta(a)$ are denoted by the Sweedler notation $a_{(0)} \otimes a_{(1)}$, such that $A$ is a right $T$-comodule over the $R$-coring $T$ [8, 18.1], $\delta(1_A) = 1_A \otimes 1_T$, $r a_{(0)} \otimes a_{(1)} = a_{(0)} \otimes \hat{t}(r)a_{(1)}$ for all $r \in R$, and $\delta(aa') = \delta(a)\delta(a')$ for all $a, a' \in A$.

The subalgebra of coinvariants is $A^{\alpha_T} := \{a \in A | \delta(a) = a \otimes 1_T\}$. We call $A$ a right $T$-extension of $A^{\alpha_T}$.

**Lemma 1.3.** For the right $T$-comodule $A$ introduced just above, $R$ and $A^{\alpha_T}$ commute in $A$.

**Proof.** We note that

$$\delta(rb) = b \otimes_R \hat{s}(r) = \delta(br)$$
for \( r \in R, \ b \in A^{\text{co} T} \). But \( \delta \) is injective by the counitality of comodules, so \( rb = br \) in \( A \) (suppressing the morphism \( R \to A \)).

**Definition 1.4.** Let \( T \) be any right bialgebroid over any algebra \( R \). A \( T \)-comodule algebra \( A \) is a right \( T \)-Galois extension of its coinvariants \( B \) if the (Galois) mapping \( \beta : A \otimes B \to A \otimes_R T \) defined by \( \beta(a \otimes a') = aa'(0) \otimes a'(1) \) is bijective.

Left comodule algebras over left bialgebroids and their left Galois extensions are defined similarly, the details of which are in [3]. The values of the coaction is in extension. Let \( R \) be any right bialgebroid over any algebra \( A \). In particular, \( A \) must be faithful.

**Theorem 2.1.** Let \( A \mid B \) be a proper algebra extension. Then

1. \( A \mid B \) is a right \( T \)-Galois extension for some left finite projective right bialgebroid \( T \) over some algebra \( R \) if and only if \( A \mid B \) is right \( D2 \) and right balanced.

2. \( A \mid B \) is a left \( T \)-Galois extension for some right finite projective left bialgebroid \( T \) over some algebra \( R \) if and only if \( A \mid B \) is left \( D2 \) and left balanced.

**Proof.** (\( \Rightarrow \)) Suppose \( T \) is a left finite projective right bialgebroid over some algebra \( R \). Since \( R T \oplus \ast \cong \ast R R \) for some positive integer \( t \), we apply to this the functor \( A \otimes_R - \) from left \( R \)-modules into \( \ast \)-modules which results in \( A A \otimes_B \cong \ast A A \), after using the Galois \( A-B \)-isomorphism \( A \otimes_B \cong A \otimes R T \). Hence, \( A \mid B \) is right \( D2 \).

Let \( E := \text{End}_B A \). We show \( A \mid B \) is balanced by the following device. Let \( R \) be an algebra, \( M_R \) and \( R V \) be \( B \)-modules with \( R V \) finite projective. If \( \sum_j m_j \phi(v_j) = 0 \) for all \( \phi \) in the left \( R \)-dual \( \ast V \), then \( \sum_j m_j \otimes_R v_j = 0 \). Thus, this follows immediately by using dual bases \( f_i \in \ast V, w_i \in V \).

Given \( F \in \text{End}_E A \), it suffices to show that \( F = \rho_b \) for some \( b \in B \). Since \( \lambda_a \in E \), \( F \circ \lambda_a = \lambda_a \circ F \) for all \( a \in A \), whence \( F = \rho_{F(1)} \). Designate \( F(1) = x \). If we show that \( x(0) \otimes x(1) = x \otimes 1 \) after applying the right \( T \)-valued coaction on \( A \), then \( x \in A^{\text{co} T} = B \). For each \( \alpha \in \text{Hom}(R T, R R) \), define \( \overline{\alpha} \in \text{End} A \) by \( \overline{\alpha}(y) = y(0)\alpha(y(1)). \) Since \( \rho_r \in E \) for each \( r \in R \) by lemma,

\[
x\alpha(1_T) = F(\overline{\alpha}(1_A)) = \overline{\alpha}(F(1_A)) = x(0)\alpha(x(1))
\]

for all \( \alpha \in \ast T \). Hence \( x(0) \otimes_R x(1) = x \otimes 1_T \).

(\( \Leftarrow \)) It follows from the proposition that a right \( D2 \) extension \( A \mid B \) has a left finite projective right bialgebroid \( T := (A \otimes_B A)^B \) over the centralizer \( R \) of the extension. Let \( R \to A \) be the inclusion mapping. We check that \( A \) is a right \( T \)-comodule algebra via the coaction \( \rho_R : A \to A \otimes_R T \) on \( A \) given by

\[
(8) \quad \rho_R(a) = a(0) \otimes a(1) := \sum_j \gamma_j(a) \otimes u_j.
\]
First, we demonstrate several properties by using the isomorphism \( \beta^{-1} : A \otimes_R T \xrightarrow{\sim} A \otimes_B A \) given by \( \beta^{-1}(a \otimes t) = at = at^1 \otimes t^2 \) [12, 3.12(iii)] with inverse \( \beta(a \otimes a') = \sum_j a_{(0)j} \otimes a'_{(1j)} \) (cf. right D2 quasibase eq. (2)). This shows straightaway that the Galois mapping \( \beta : A \otimes_B A \rightarrow A \otimes_R T \) is bijective. Then \( A \otimes_R T \otimes_R T \cong A \otimes_B A \otimes_B A \) via \( \Phi := (id_A \otimes \beta^{-1})(\beta^{-1} \otimes id_T) \), so coassociativity \((id_A \otimes \Delta) \rho_R = (\rho_R \otimes id_T) \rho_R \) follows from

\[
\Phi(id \otimes \Delta_T) \circ \rho_R = \sum_{j,k} \gamma_j(a)u^1_j \otimes_B \gamma_k(u^2_j)u^1_k \otimes_B u^2_k = \sum_k 1 \otimes \gamma_k(a)u^1_k \otimes u^2_k = 1 \otimes 1 \otimes a
\]

\[
= \sum \gamma_k(\gamma_j(a))u^1_k \otimes u^2_k = \phi((\rho_R \otimes id) \rho_R(a)).
\]

We note that \( \rho_R \) is right \( R \)-linear, since

\[
\rho_R(ar) = \sum \gamma_j(ar) \otimes u \beta^{-1}(1 \otimes B ar = \beta^{-1}(\rho_R(a)r)
\]

since \( \rho_R(a)r = \sum \gamma_j(a) \otimes u_j r \). Also, \( a_{(0)} \epsilon_T(a_{(1)}) = \sum \gamma_j(a)u^1_j u^2_j = a \) for all \( a \in A \).

Next,

\[
\beta^{-1}(r \otimes a_{(0)} \otimes a_{(1)}) = \sum_r r \gamma_j(a)u_j = r \otimes_B a = \sum \gamma_j(a)u^1_j r \otimes u^2_j = \beta^{-1}(a_{(0)} \otimes T_R(r)a_{(1)}).
\]

Whence the statement \( \rho_R(aa') = \rho_R(a) \rho_R(a') \) makes sense for all \( a, a' \in A \). We check the statement:

\[
\beta^{-1}(\rho_R(a) \rho_R(a')) = \sum \gamma_j(a) \gamma_k(a')u_j = \sum \gamma_j(a) \gamma_k(a')u_j^1 u_j^2 u_k = 1 \otimes a a' = \sum \gamma_j(aa')u_j = \beta^{-1}(\rho_R(aa')).
\]

Also \( \rho_R(1_A) = 1_A \otimes_R 1_T \) since \( \gamma_j(1_A) \in R \). Finally we note that for each \( b \in B \)

\[
\rho_R(b) = \sum \gamma_j(b) \otimes_R u_j = b \otimes \sum \gamma_j(u_j) = b \otimes 1_T
\]

so \( B \subseteq A^{op, mun} \). Conversely, if \( \rho_R(x) = x \otimes 1_T = \sum \gamma_j(x) \otimes u_j \) applying \( \beta^{-1} \) we obtain \( x \otimes B 1 = 1 \otimes_B x \). Let \( f \in \text{End}_A \). Then applying \( \mu(f \lambda(a) \otimes id) \) to this we obtain \( f(ax) = f(a)x \) since \( \lambda(a) \in \text{End}_A \) for each \( a \in A \). It follows that \( f \rho(x) = \rho(x)f \) so \( \rho(x) \in \text{End}_A \). Since \( A_B \) is balanced, \( \rho(x) = \rho(b) \) for some \( b \in B \), whence \( x = b \in B \).

The second part of the theorem is proven similarly (or alternatively, apply the first part with the opposite algebra technique mentioned in the introduction). In the \( \Leftarrow \) direction, we convert the right \( R \)-bialgebroid \( T \) to a left \( R \)-bialgebroid \( T^{op} \) with \( s_L = t_R, t_L = s_L \), the same \( R \)-comodule structure and opposite multiplication, which leads to the left \( R \)-\( R \)-bimodule structure coinciding with the usual \( R \)-\( R \)-bimodule structure on \( T \) in eq. (11). We then define a left \( T^{op} \)-comodule algebra structure on \( A \) via \( \rho_L : A \rightarrow T \otimes_R A \) defined via left D2 quasibases by

\[
\rho_L(a) := \sum_i t_i \otimes \beta_i(a).
\]

The isomorphism \( T \otimes_R A \cong A \otimes_B A \) given by \( t \otimes a \mapsto t^1 \otimes t^2 a \) is inverse to the Galois mapping \( \beta_L(a \otimes a') = a_{(-1)} \otimes a_{(0)}a' \) by the left D2 quasibase eq. (11). One
needs the opposite multiplication of $T$ when showing $\rho_L(aa') = \rho_L(a)\rho_L(a')$ for $a, a' \in A$. 

Let $T$ be a left finite projective right bialgebroid over some algebra $R$ in the next corollary.

**Corollary 2.2.** Suppose $A \mid B$ is a right $T$-extension. If the Galois mapping $\beta$ is a split $A$-$B$-monomorphism, then $A \mid B$ is a right $(A \otimes_B A)^B$-Galois extension.

**Proof.** This follows from $AA \otimes_B AB \otimes \ast \cong AA \otimes_B T$ and the arguments in the first few paragraphs of the proof above (the balance argument makes only use of $A \mid B$ being a right $T$-extension). Hence, $A \mid B$ is right D2 and right balanced. Whence $A \mid B$ is right Galois extension w.r.t. the bialgebroid $(A \otimes_B A)^B$. 

Notice that $T$ is possibly not isomorphic to $(A \otimes_B A)^B$. For example, one might start with a Hopf algebra Frobenius extension with split monic Galois map and conclude it is a weak Hopf-Galois extension (if the centralizer is separable, the antipode being constructible from the Frobenius structure).

3. **Galois extended Hopf subalgebras are normal**

There is a question of whether depth two Hopf subalgebras are normal [14, 3.4]. In this section we answer this question in an almost unavoidable special case, namely, when the Hopf subalgebra forms a Hopf-Galois extension with respect to the action of a third Hopf algebra. Since a depth two extension with one extra condition is a Galois extension for actions of bialgebroids or weak bialgebras [12], the situation of ordinary Hopf-Galois extension would seem to be a critical step.

Let $k$ be a field. All Hopf algebras in this section are finite dimensional algebras over $k$. Recall that a Hopf subalgebra $K \subseteq H$ is a Hopf algebra $K$ w.r.t. the algebra and coalgebra structure of $H$ (with counit denoted by $\varepsilon$) as well as stable under the antipode $\tau$ of $H$. Recall the Nichols-Zoeller result that the natural modules $H_K$ and $K_H$ are free. $K$ is normal in $H$ if $\tau(a_{(1)})xa_{(2)} \in K$ and $a_{(1)}x\tau(a_{(2)}) \in K$ for all $x \in K, a \in H$. Equivalently, if $K^+$ denotes the kernel of the counit $\varepsilon$, $K$ is a normal Hopf subalgebra of $H$ if the left algebra ideal and coideal $HK^+$ is equal to the right ideal and coideal $K^+H$ [19, 3.4.4].

In considering another special case of D2 Hopf subalgebras, we showed in [11] that $H$-separable Hopf subalgebras are normal using favorable properties for $H$-separable extensions of going down and going up for ideals. However, we noted that such subalgebras are not proper if $H$ is semisimple, e.g., $H$ is a complex group algebra. In [14, 3.1] we showed that depth two subgroups are normal subgroups using character theory (for $k = \mathbb{C}$). We also noted the more general converse that normal Hopf $k$-subalgebras are Hopf-Galois extensions and therefore D2. Next we extend this to the characterization of normal Hopf subalgebras below, one that we believe is not altogether unexpected but unnoted or not adequately exposed in the literature.

**Theorem 3.1.** Let $K \subseteq H$ be a Hopf subalgebra. Then $K$ is normal in $H$ if and only if $H \mid K$ is a Hopf-Galois extension.

**Proof.** $(\Rightarrow)$ This is more or less implicit in [19, 3.4.4], where it is also shown [19] chs. 7,8 that $H$ is a crossed product by a counital 2-cocycle of $K$ with the quotient Hopf algebra $\overline{H}$ (a cleft $\overline{H}$-extension or Galois extension with normal basis). Since
\( HK^+ = K^+ H \) under normality of \( K \), it becomes a Hopf ideal, so we form the Hopf algebra \( \overline{H} := H/\overline{KK}^+ \), which coacts naturally on \( H \) via the comultiplication and quotient projection. The coinvariants are precisely \( K \) since \( H_K \) is faithfully flat. The Galois mapping \( \beta : H \otimes_K H \to H \otimes \overline{H} \) given by \( \beta(a \otimes a') = aa'_{(1)} \otimes \overline{a'}_{(2)} \) is an isomorphism with inverse given by \( x \otimes \overline{y} \mapsto x \tau(y_{(1)}) \otimes y_{(2)} \).

\( (\Leftarrow) \) Suppose \( H \) is a \( W \)-Galois extension of \( K \) where \( W \) is a Hopf algebra with right coaction \( \rho : H \to H \otimes W \) on \( H \). We define a mapping \( \Phi : H \to W \) by \( \Phi(h) = \varepsilon_H(h_{(0)})h_{(1)} \), i.e., \( \Phi = (\varepsilon_H \otimes \text{id}_W) \circ \rho \). We note that \( \Phi \) is an algebra homomorphism since \( \rho \) and \( \varepsilon_H \) are (and augmented since \( \varepsilon_W \circ \Phi = \varepsilon_H \)). Also, \( \Phi : H \to W \) is a right \( W \)-comodule morphism since \( H \) is a right \( W \)-comodule with \( \rho \) and \( \Delta_W \) obeying a coassociativity rule. Next we note that \( \Phi \) is an epi since given \( w \in W \), there is \( \sum_i h_i \otimes h_i' \in H \otimes_K H \) such that \( 1 \otimes w = \sum_i h_i h_i'_{(0)} \otimes h_i'_{(1)} \). Applying \( \varepsilon_H \otimes \text{id}_W \) to this, we obtain

\[
\Phi(\sum_i \varepsilon_H(h_i)h_i') = w.
\]

We note that \( \ker \Phi \) contains \( K^+ \) since \( K = H^{co_W} = \{ h \in H \mid \rho(h) = h \otimes 1_W \} \). Consider the coalgebra and right quotient \( H \)-module \( H/\overline{K}^+ H := \overline{H} \) as well as the coalgebra and left quotient \( H \)-module \( H/\overline{K}^+ H := \overline{H} \). In this case, \( \Phi \) induces \( \overline{\Phi} : \overline{H} \to W \) and \( \overline{\Phi} : \overline{H} \to W \). (They are respectively right and left \( H \)-module morphisms w.r.t. the modules \( W_K \) and \( W \).) By Schneider \cite[2.3]{Schneider}, the Galois quotient mapping \( \overline{\beta} : H \otimes_K H \to H \otimes \overline{H} \) given by \( \overline{\beta}(x \otimes y) = xy_{(1)} \otimes \overline{y_{(2)}} \) is bijective (since \( K \) is a left coideal subalgebra of \( H \)). But the Hopf subalgebra \( K \) is also a right coideal subalgebra satisfying a right-handed version of Schneider’s lemma recorded in \cite[2.4]{Schneider}; whence the Galois mapping \( \overline{\beta} : H \otimes_K H \to \overline{H} \otimes \overline{H} \) given by \( \overline{\beta}(x \otimes y) = x_{(1)} \otimes x_{(2)} \) is bijective as well.

Observe now that \( H_K \) is free of rank \( n \), let’s say, so \( \overline{\beta} \) bijective implies that \( \dim_k W = n \). Similarly, \( \overline{\beta} \) bijective implies \( \dim_k \overline{H} = n \) and \( \overline{\beta} \) bijective implies \( \dim_k \overline{H} = n \). It follows that the vector space epimorphisms \( \overline{\Phi} : \overline{H} \to W \) and \( \overline{\Phi} : \overline{H} \to W \) are isomorphisms. But \( \overline{\Phi} \) factors through \( \overline{H} \to H/\overline{H} \) induced by \( K^+ H \subseteq H K^+ H \); similarly, \( \overline{\Phi} \) factors through \( \overline{H} \to H/\overline{H} K^+ H \), so both these canonical mappings are monic. It follows that \( H K^+ = H K K^+ H \), whence \( K \) is a normal Hopf subalgebra in \( H \).

In the proof of \( \Leftarrow \) above, we can go further to conclude that \( \overline{H} \) is a Hopf algebra isomorphic to \( W \) as augmented algebras. However, the theory of deforming the comultiplication of a Hopf algebra by a 2-cocycle \cite[2.3.4]{Schneider} shows that there are pairs of Hopf algebras isomorphic as augmented algebras yet non-isomorphic as Hopf algebras. Additionally, there are examples of algebra extensions which are Hopf-Galois w.r.t. two different Hopf algebras. We therefore do not know a priori if \( \overline{H} \) and \( W \) are isomorphic as Hopf algebras.

4. Weak Hopf-Galois extensions are depth two

In this section we study right Galois extensions of special bialgebroids - the weak Hopf-Galois extensions, cf. \cite{Bliedtner, Bucataru, Paterson, Raczkiewicz}. By exploiting the antipode in weak Hopf algebras, we find an alternative Galois mapping which characterizes weak Hopf-Galois extensions. This leads to several corollaries that finite weak Hopf-Galois
extensions are right as well as left depth two extensions, that they may be defined by only a surjective Galois map, and that a weak Hopf algebra over its target separable subalgebra is an example of such. We propose a number of problems for further study in the young subject of weak Hopf-Galois extensions.

Weak Hopf algebras are a special case of Hopf algebroids - those with separable base algebra [8, 10]: the separable algebra has an index-one Frobenius system which one uses to convert mappings to the base and tensors over the base to linear functionals and tensors over a ground field. There is an example of one step in how to conversely view a weak Hopf algebra \( H \) as a Hopf algebroid over its left coideal subalgebra \( H^L \) in the proof of corollary 4.4 below.

Let \( k \) be a field. A weak Hopf algebra \( H \) is first a weak bialgebra, i.e., a \( k \)-algebra and \( k \)-coalgebra \( (H, \Delta, \varepsilon) \) such that the comultiplication \( \Delta : H \to H \otimes_k H \) is linear and multiplicative, \( \Delta(ab) = \Delta(a)\Delta(b) \), and the counit is linear just as for bialgebras; however, the change (or weakening of the axioms) is that \( \Delta \) and \( \varepsilon \) may not be unital, \( \Delta(1) \neq 1 \otimes 1 \) and \( \varepsilon(1_H) \neq 1_k \), but must satisfy
\[
\varepsilon(ab) = \varepsilon(ab_1)\varepsilon(b_2) = \varepsilon(ab_2)\varepsilon(b_1) \quad (\forall a, b, c \in H)
\]

There are several important projections that result from these axioms:
\[
\Pi^L(x) := \varepsilon(1(1)x)1(2) \\
\Pi^R(x) := 1(1)\varepsilon(x1(2)) \\
\Pi^L(x) := 1(1)\varepsilon(1(2)x) \\
\Pi^R(x) := \varepsilon(x1(1))1(2) \quad (\forall x \in H)
\]

We denote \( H^L := \text{Im} \Pi^L = \text{Im} \Pi^R \) and \( H^R := \text{Im} \Pi^R = \text{Im} \Pi^L \). These subalgebras are separable \( k \)-algebras [6].

In addition to being a weak bialgebra, a weak Hopf algebra has an antipode \( S : H \to H \) satisfying the axioms
\[
S(x(1))x(2) = \Pi^R(x) \\
x(1)S(x(2)) = \Pi^L(x) \\
S(x(1))x(2)S(x(3)) = S(x) \quad (\forall x \in H)
\]

The antipode turns out to be bijective for finite dimensional weak Hopf algebras (which we will assume for the rest of this section), an anti-isomorphism of algebras with inverse denoted by \( S^{-1} \).

The reader will note from the axioms above that a Hopf algebra is automatically a weak Hopf algebra. For a weak Hopf algebra that is not a Hopf algebra, consider a typical groupoid algebra such as \( H = M_n(k) \), the \( n \times n \)-matrices over \( k \) (the groupoid here being a category with \( n \) objects where each Hom-group has a single invertible arrow). Let \( e_{ij} \) denote the \((i,j)\)-matrix unit. For example, \( M_n(k) \) is a weak Hopf algebra with the counit given by \( \varepsilon(e_{ij}) = 1 \), comultiplication by \( \Delta(e_{ij}) = e_{ij} \otimes e_{ij} \) and antipode given by \( S(e_{ij}) = e_{ji} \) for each \( i, j = 1, \ldots, n \) (extending the Hopf algebra structure of group algebras). In this case, \( H^L = H^R \) and is equal to the diagonal matrices. The corresponding projections are given by \( \Pi^L(e_{ij}) = e_{ii} \).
= \Pi^L(e_{ij}) \text{ and } \Pi^R(e_{ij}) = e_{jj} = \Pi^R(e_{ij}). \text{ Note that } \varepsilon(1_H) = n1_k \text{ which is zero if the characteristic of } k \text{ divides } n.

There are a number of equations in the subject that we will need later (cf. [28, 2.9, 2.24]):

\begin{align}
(19) & \quad \Pi^L = S \circ \Pi^L \\
(20) & \quad \Pi^R = S \circ \Pi^R \\
(21) & \quad S(a_{(2)})a_{(1)} = \Pi^L(a) \\
(22) & \quad a_{(2)}S(a_{(1)}) = \Pi^L(a) \\
(23) & \quad a_{(1)} \Pi^L(a_{(2)}) = 1_{(1)}a \otimes 1_{(2)} \\
(24) & \quad \Pi^R(a_{(1)})a_{(2)} = 1_{(1)} \otimes a_{(2)} \\
(25) & \quad \Pi^R(ab) = b_{(1)} \varepsilon(ab_{(2)}) \\
(26) & \quad a\Pi^L(b) = \varepsilon(a_{(1)}b)a_{(2)} \quad (\forall a, b \in H)
\end{align}

where e.g. eq. (21) follows from applying the inverse-antipode to eqs. (20) and (16).

We recall the definition of a right $H$-comodule algebra $A$, its subalgebra of coinvariants, and Galois coaction for $H$ a weak bialgebra (e.g. in [27]):

**Definition 4.1.** Let $H$ be a weak bialgebra with $A, H$ both $k$-algebras. $A$ is a right $H$-comodule algebra if there is a right $H$-comodule structure $\rho : A \to A \otimes_k H$ such that $\rho(ab) = \rho(a)\rho(b)$ for each $a, b \in A$ and any of the equivalent conditions for $\rho(a) := a_{(0)} \otimes a_{(1)}$ are satisfied:

\begin{align}
(27) & \quad 1_{(0)} \otimes 1_{(1)} \in A \otimes H^L \\
(28) & \quad a_{(0)} \otimes \Pi^L(a_{(1)}) = 1_{(0)}a \otimes 1_{(1)} \\
(29) & \quad a_{(0)} \otimes \Pi^R(a_{(1)}) = a1_{(0)} \otimes 1_{(1)} \quad (\forall a \in H) \\
(30) & \quad 1_{(0)} \otimes 1_{(1)} \otimes 1_{(2)} = (\rho(1_A) \otimes 1_H)(1_A \otimes \Delta(1_H))
\end{align}

The coinvariants are defined by:

\[ B := \{ b \in A | b_{(0)} \otimes b_{(1)} = 1_{(0)}b \otimes 1_{(1)} = b1_{(0)} \otimes 1_{(1)} \}, \]

the second equation following from equations directly above. We say $A$ is a weak $H$-Galois extension of $B$ if the mapping $\beta : A \otimes_B A \to A \otimes_k H$ given by $\beta(a \otimes a') = aa'_{(0)} \otimes a'_{(1)}$ is bijective onto:

\[ \overline{A \otimes H} = (A \otimes H)\rho(1) = \{ a1_{(0)} \otimes h1_{(1)} | a \in A, h \in H \}. \]

For finite dimensional weak Hopf algebras and their actions, we only need require $\beta$ be surjective in the definition of weak Hopf-Galois extension, as $\beta$ is automatically injective by [11, 2] or corollary 4.16 below. Note that Im $\rho \subseteq \overline{A \otimes H}$, an $A$-$B$-subbimodule and that $\beta$ is an $A$-$B$-bimodule morphism w.r.t. the structure $\alpha' \cdot (a \otimes h) \cdot b = a'ab \otimes h$ on $A \otimes H$. These definitions correspond to the case of a separable base algebra in the definitions of right comodule algebras, Galois coring and Galois coactions for bialgebroids given in [11, 12].

We now establish the Hopf algebra analogue of an alternate Galois mapping characterizing Galois extension. This would correspond to working with a lefthanded version of the Galois coring considered in [17].
Proposition 4.2. Suppose $H$ is a weak Hopf algebra and $A$ a right $H$-module algebra with notation introduced above. Let $\beta': A \otimes_B A \to A \otimes H$ be defined by
\begin{equation}
\beta'(a \otimes a') = a(0)a' \otimes a(1)
\end{equation}
and $\eta: A \otimes H \to A \otimes H$ be the map defined by
\begin{equation}
\eta(a \otimes h) = a(0) \otimes a(1)S(h).
\end{equation}
Then $\beta' = \eta \circ \beta$ and $\beta : A \otimes_B A \to \overline{A \otimes H}$ is respectively injective, surjective or bijective iff $\beta'$ is injective, surjective or bijective onto
\[\overline{A \otimes H} := \rho(1)(A \otimes H).\]
In particular, $A|B$ is a weak $H$-Galois extension iff $\beta': A \otimes_B A \to \overline{A \otimes H}$ is bijective.

Proof. Notice that $\overline{A \otimes H}$ is a $B$-$A$-sub-bimodule of $A \otimes H$, and that $\text{Im} \ \eta$ and $\text{Im} \ \beta' \subseteq \overline{A \otimes H}$. Next note that an application of eq. (28) gives
\[\eta \beta(a \otimes a') = \eta(aa'(0) \otimes a'(1)) = a(0)a'(0) \otimes a(1)a'(1)S(a'(2)) = a(0)a'(0) \otimes a(1)\Pi^L(a'(1)) = a(0)1(0)a' \otimes a(1)1(1) \not= a(0)a' \otimes a(1) = \beta'(a \otimes a').\]
We define another linear self-mapping of $A \otimes H$ given by $\overline{\eta}(a \otimes h) = a(0) \otimes \overline{S}(h)a(1)$. Note that $\text{Im} \ \overline{\eta}$ and $\text{Im} \ \beta \subseteq \overline{A \otimes H}$.

Let $p: A \otimes H \to \overline{A \otimes H}$, $\overline{\eta}: A \otimes H \to \overline{A \otimes H}$ be the straightforward projections given by $p(a \otimes h) = a1(0) \otimes h1(1)$, and $\overline{\eta}(a \otimes h) = 1(0)a \otimes 1(1)h$. We show below that $\eta \circ p = \eta$, $\overline{\eta} \circ \overline{\eta} = \overline{\eta}$, $\eta \circ \overline{\eta} = p$ and $\overline{\eta} \circ \eta = \rho$, from which it follows that the restrictions of $\eta$, $\overline{\eta}$ to $\overline{A \otimes H}$, $\overline{A \otimes H}$ are inverses to one another, so that there is a commutative triangle connecting $\beta$, $\beta'$ via $\eta$.

\[
\begin{array}{ccc}
A \otimes_B A & \xrightarrow{\beta} & \overline{A \otimes H} \\
\downarrow & & \downarrow \\
A \otimes H & \xrightarrow{\eta} & A \otimes H
\end{array}
\]

First, we note that $\eta \circ p = \eta$ since
\[\eta(a1(0) \otimes h1(1)) = a(0)1(0) \otimes a(1)1(1)S(h1(2)) = a(0)1(0) \otimes a(1)\Pi^L(1(1))S(h) = a(0) \otimes a(1)S(h) = \eta(a \otimes h)
\]
by eqs. (14) and (27).

Secondly, we note that $\overline{\eta} \circ \overline{\eta} = \overline{\eta}$ since
\[\overline{\eta}(1(0)a \otimes 1(1)h) = 1(0)a(0) \otimes \overline{S}(h)\overline{S}(1(2))1(1)a(1) = 1(0)a(0) \otimes \overline{S}(h)\Pi^R(1(1))a(1) = 1(0)a(0) \otimes \overline{S}(h)1(1)a(1) = \overline{\eta}(a \otimes h)
\]
by eqs. (21) and (29).

Next we note that \( \eta \circ \eta = p \) since

\[
\eta(a_0(0) \otimes a_1(1)S(h)) = a_0(0) \otimes S(a_2(2)S(h))a_1(1)
\]

\[
= a_0(0) \otimes h\Pi^R(a_1(1))
\]

\[
= a_0(0) \otimes h_1(1) = p(a \otimes h)
\]

by eqs. (21) and (29).

Finally we note \( \eta \circ \eta = \eta \) since

\[
\eta(a_0(0) \otimes S(h)a_1(1)) = a_0(0) \otimes a_1(1)S(a_2(2))h
\]

\[
= a_0(0) \otimes \Pi^L(a_1(1))h = \eta(a \otimes h)
\]

by eq. (20).

Again let \( H \) be a finite dimensional weak Hopf algebra. Recall that the \( k \)-dual \( H^* \) is also a weak Hopf algebra, by the self-duality of the axioms, and acts on \( H \) by the usual right action \( x \looparrowright \psi = \psi(x_{(1)})x_{(2)} \) and a similarly defined left action. In addition, a right \( H \)-comodule algebra \( A \) corresponds to a left \( H^* \)-module algebra \( A \) via \( \psi \cdot a := \eta(a_0(1)) \psi(a_1) \). Following Kreimer-Takeuchi and Schneider, there are two proofs that surjectivity of \( \beta \) is all that is needed in the definition of a weak Hopf \( H \)-Galois extension \( \#2 \). As a corollary of the proposition, we offer a third and direct proof.

**Corollary 4.3.** \( \#4 \) \( \#2 \) Let \( A \) be a right \( H^* \)-module algebra and \( B \) its subalgebra of coinvariants \( A^{co H^*} \). If \( \beta : A \otimes_B A \to A \otimes H^* \) is surjective, then the natural module \( A_B \) is f.g. projective and \( A|B \) is a weak \( H^* \)-Galois extension.

**Proof.** We know from \( \#2 \) that \( H \) and \( H^* \) are both Frobenius algebras with non-degenerate left integral \( t \in H \) satisfying \( ht = \Pi^L(h)t \) for all \( h \in H \) as well as \( t \looparrowright T = 1_H \) for some \( T \in H^* \). Since \( \beta \) is surjective, there are finitely many paired elements \( a_i, b_i \in A \) such that

\[
1(0) \otimes T1(1) = \sum_i a_i b_i(0) \otimes b_i(1).
\]

Let \( \phi_i(a) := t \cdot (b_i a) \) for every \( a \in A \). Then \( \{a_i\}, \phi_i \) are dual bases for the module \( A_B \) by a computation that \( \sum_i a_i \phi_i(a) = a \) for all \( a \in A \), almost identical with \( \#19 \) p. 132] for Hopf algebra actions (using the identity \( 1(0)a_0(0) \otimes 1(1)a_1(1) = a_0(0) \otimes a_1(1) \) at one point).

Finally, one shows that \( \beta' \) is injective, for if \( \sum_j u_j \otimes v_j \in \ker \beta' \), we compute

\[
\sum_j u_j \otimes v_j = \sum_{i,j} a_i \otimes \phi_i(u_j)v_j = \sum_{i,j} a_i \otimes (t_1(1) \cdot b_i)u_j(0)v_j(u_j(1), t_2(2)) = 0
\]

as in \( \#19 \) p. 132]. By the proposition, \( \beta \) is then injective, whence a bijection of \( A \otimes_B A \) onto \( A \otimes H \).

We next offer an example of weak Hopf-Galois extension with an alternative proof. For example, if \( H = M_n(k) \) considered above, the Galois map \( \beta = (\mu \otimes \id) \circ (\id \otimes \Delta) \) given by \( \beta(e_{ij} \otimes e_{jk}) = e_{ik} \otimes e_{jk} \) with coinvariants \( H^L \) the diagonal matrices and \( 1(1) \otimes 1(2) = \sum_i e_{ii} \otimes e_{ii} \), is an isomorphism by a dimension count. The general picture is the following:
Corollary 4.4. [7, 2.7] Define a coaction on \( H \) by \( a_{(0)} \otimes a_{(1)} = a_{(1)} \otimes a_{(2)} \) for \( a \in H \). Then \( H \) is a weak Hopf \( H \)-Galois extension of its coinvariants \( H^L \).

Proof. We note that \( H^L \subseteq H^{coH} \) since \( \Delta(x^L) = 1_{(1)}x^L \otimes 1_{(2)} \) for \( x^L \in H^L \) [6, 2.7a]. The converse follows from \( x \in H^{coH} \) implies
\[
x = \varepsilon(x_{(1)})x_{(2)} = \varepsilon(1_{(0)}x)1_{(1)} \in H^L.
\]

Next we note that \( \beta' \) factors into isomorphisms in the following commutative diagram, where \( \sigma : H \otimes H \rightarrow H \otimes H \) is the standard twist involution:

\[
\begin{array}{ccc}
H \otimes_{H^L} H & \xrightarrow{\beta'} & H \otimes H \\
q \cong & & \cong \\
H \otimes H & \xrightarrow{\eta} & H \otimes H
\end{array}
\]

where \( q(x \otimes y) := \overline{\theta(\overline{x} \otimes_k y)} \) is well-defined since \( S(1_{(1)}) \otimes 1_{(2)} \) is a separability element for the separable \( k \)-algebra \( H^L \) [4 prop. 2.11]. Its inverse is given by \( q^{-1}(\overline{\theta(x \otimes y)}) = S(x) \otimes y \). The mapping \( \sigma \circ (S \otimes S) \) has an obvious inverse and is well-defined since \( S(1) = 1 \) and \( S \) is an anti-coalgebra homomorphism. \( \square \)

We provide the complete proof that a weak Hopf \( H \)-Galois extension is depth two [12 3.2]:

Corollary 4.5. A weak \( H \)-Galois extension \( A \mid B \) is right and left depth two.

Proof. The algebra extension \( A \mid B \) is right D2 since the Galois mapping \( \beta : A \otimes_B A \rightarrow A \otimes H \) and the projection \( p : A \otimes H \rightarrow A \otimes H \) are \( A \)-\( B \)-bimodule morphisms [11 3.1]. Whence \( A \otimes_B A \) is \( A \)-\( B \)-isomorphic to a direct summand \( \text{Im } p \) within \( \oplus^n A \) where \( n = \dim H \).

Similarly \( A \mid B \) is left D2 since the alternate Galois isomorphism \( \beta' \) and projection \( \overline{\eta} \) onto \( A \otimes H \) are both \( B \)-\( A \)-bimodule morphisms. \( \square \)

The proof of the corollary sidesteps the problem of showing \( A \mid B \) is a Frobenius extension, which then implies that left D2 \( \Leftrightarrow \) right D2. It is likely that a direct proof using \( \beta \) and \( \beta' \) that a weak \( H \)-Galois extension is Frobenius may be given since there are nondegenerate integrals in \( H^* \) which would define a Frobenius homomorphism via the dual action of \( H^* \) on \( A \) (with invariants \( B \)). In addition we have avoided starting only with a weak bialgebra having Galois action on \( A \) and showing the existence of an antipode on \( H \) in extension of [21] for Galois actions of bialgebras.

If we denote
\[
\sum_i \ell_i(h) \otimes_B r_i(h) := \beta^{-1}(1_{(0)} \otimes h1_{(1)}),
\]
we note that
\[
\begin{align*}
1_{(0)} \otimes 1_{(1)}S(h) &= \eta(1 \otimes h) \\
&= \beta'(\beta^{-1}(1_{(0)} \otimes h1_{(1)})) \\
&= \sum_i \ell_i(h)_{(0)}r_i(h) \otimes \ell_i(h)_{(1)},
\end{align*}
\]
which can conceivably be made to descend to a formula for the antipode of \( H \) in terms of just the isomorphism \( \beta \).

We then propose two problems and provide some evidence for each.

**Problem 4.6.** If \( H \) is a finite dimensional weak Hopf algebra and \( A \mid B \) is \( H \)-Galois, provide a direct proof that \( A \mid B \) is a Frobenius extension (cf. [11 3.7]).

For example, if \( H \) is a Galois extension of \( H^L \) as in corollary 4.4 we expect such a Frobenius extension based on Pareigis’s theorem that a Frobenius subalgebra \( B \) of a Frobenius algebra \( A \), where the natural module \( A_B \) is f.g. projective and the Nakayama automorphism of \( A \) stabilizes \( B \), yields a \( \beta \)-Frobenius extension \( A \mid B \) where \( \beta \) is the relative Nakayama automorphism of \( A \) and \( B \) (restrict one and compose with the inverse of the other). For example, if \( H \) has an \( S \)-invariant non-degenerate integral, the Nakayama automorphism is \( S^2 \) [5 3.20], also the Nakayama automorphism of \( H^L \), so \( \beta = \text{id} \) and \( H \mid H^L \) is an ordinary Frobenius extension.

**Problem 4.7.** If \( H \) is a weak bialgebra and \( A \mid B \) is \( H \)-Galois, is \( H \) necessarily a weak Hopf algebra?

Again this is true in the special case of the weak Hopf-Galois extension in corollary 4.4, a result in [3 Brzezinski-Wisbauer]; we give another proof which may extend to the general problem. Note that the definition of weak Hopf-Galois extension does not make use of an antipode nor does \( H^{coH} = H^L \) in corollary 4.4.

**Theorem 4.8.** [3 36.14] Let \( H \) be a weak bialgebra. If the right \( H \)-coalgebra \( H \) with coaction \( \varrho = \Delta_H \) is Galois over \( H^L \), then \( H \) is a weak Hopf algebra.

**Proof.** In terms of the notation in eq. (33) we define an antipode \( S : H \to H \) by

\[
S(h) = \sum_i \varepsilon(\ell_i(h)(1) r_i(h)) \ell_i(h)(2)
\]

(37)

Note that by eq. (26), \( S(h) = \sum_i \ell_i(h) \Pi^L(r_i(h)) \) for \( h \in H \). In order to prove that \( S \) satisfies the three eqs. (16), (17) and (18), we note the three equations below for a general right \( H \)-comodule algebra \( A \) over a weak bialgebra \( H \) where \( A \) is \( H \)-Galois over its coinvariants \( B \); the proofs are quite similar to those in [21].

\[
\sum_i \ell_i(h) \otimes r_i(h)(0) \otimes r_i(h)(1) = \sum_i \ell_i(h(1)) \otimes r_i(h(1)) \otimes h(2)
\]

(38)

\[
\sum_i a_{(0)} \ell_i(a(1)) \otimes_B r_i(a(1)) = 1 \otimes_B a
\]

(39)

\[
\sum_i \ell_i(h)r_i(h) = 1(0)\varepsilon(h1(1)) \quad (\forall a \in A, h \in H)
\]

(40)

Next we note three equations in \( A \otimes H \), two of which we need here (and all three might play a role in an answer to problem 4.7).

\[
\sum_i \ell_i(h(1))_{(0)} r_i(h(1)) \otimes \ell_i(h(1))_{(1)} h(2) = 1(0) \otimes 1(1) \Pi^R(h)
\]

(41)

\[
\sum_i \ell_i(h(2))_{(0)} r_i(h(2)) \otimes h(1) \ell_i(h(2))_{(1)} = 1(0) \otimes \Pi^L(h1(1))
\]

(42)

\[
\sum_i \ell_i(h(1))_{(0)} r_i(h(3)) \ell_i(h(3))_{(0)} r_i(h(3)) \otimes \ell_i(h(1))_{(1)} h(2) \ell_i(h(3))_{(1)} =
\]

(43)
\[
\sum_i \ell_i(h)_{(0)} r_i(h) \otimes \ell_i(h)_{(1)}.
\]
They are established somewhat similarly to [21] and left as exercises.

Applying eq. (41) with \( A = H \) and \( a_{(0)} \otimes a_{(1)} = a_{(1)} \otimes a_{(2)} \), we obtain one of the antipode axioms:
\[
S(h_{(1)}) h_{(2)} = \sum_i \varepsilon(\ell_i(h_{(1)}) r_i(h_{(2)})) \ell_i(h_{(1)}) h_{(2)}
\]
\[
= \varepsilon(1_{(1)} 1_{(2)} \Pi^R(h)) = \Pi^R(h). \quad (\forall h \in H)
\]

Applying eq. (42), we obtain
\[
h_{(1)} S(h_{(2)}) = \sum_i \varepsilon(\ell_i(h_{(2)}) r_i(h_{(2)})) h_{(1)} \ell_i(h_{(2)})_{(2)}
\]
\[
= \varepsilon(1_{(1)} \Pi^L(h_{1(2)})) = \Pi^L(h) \quad (\forall h \in H)
\]

Finally we see \( S \) is an antipode from the just established eq. (16) and applying eq. (24):
\[
\Pi^R(h_{(1)}), S(h_{(2)}) = \sum_i \Pi^R(h_{(1)}) \Pi^L(r_i(h_{(2)})) = \sum_i \ell_i(h) \Pi^L(r_i(h)) = S(h)
\]

where we use the general fact that \( \beta \) is left \( A \)-linear, so \( \sum_i 1_{(0)} \ell_i(h_{1(1)}) \otimes r_i(h_{1(1)}) = \beta^{-1}(1'_{(0)} 1_{(0)} \otimes h 1'_{(1)} 1_{(1)}) = \sum_i \ell_i(h) \otimes r_i(h) \).

5. Appendix

In this section we answer some natural questions about the theory of one-sided depth two extensions. One of the apparent questions after a reading of proposition 1.1 would be if the endomorphism algebra \( S \) is also a bialgebroid over the centralizer, to which the next proposition provides an answer in the affirmative.

**Proposition 5.1.** Suppose \( A \mid B \) is either a right or a left D2 extension with centralizer \( R \). Then \( S \) is a left \( R \)-bialgebroid, which is either right f.g. \( R \)-projective or left f.g. \( R \)-projective respectively.

**Proof.** The algebra structure comes from the usual composition of endomorphisms in \( S = \text{End}_B A_B \). The source and target mappings are \( s_L(r) = \lambda(r) \) and \( t_L(r) = \rho(r) \), whence the structure \( R S_R \) is given by
\[
r \cdot \alpha \cdot r' = \lambda(r) \rho(r') \alpha = r \alpha(-) r'.
\]

Suppose now we are given a right D2 structure on \( A \mid B \) by quasibases \( u_j \in T, \gamma_j \in S \). The \( R \)-coring structure on \( R S_R \) is given by a coproduct \( \Delta : S \to S \otimes_R S \) defined by
\[
\Delta(\alpha) = \sum_j \gamma_j \otimes u_j^2 \alpha(u_j^2 -),
\]
and a counit \( \varepsilon : S \to R \) given by
\[
\varepsilon(\alpha) = \alpha(1_A)
\]
Clearly $\varepsilon$ is an $R$-$R$-bimodule mapping with $\varepsilon(1_S) = 1_A$, satisfying the counitality equations and
\[ \varepsilon(\alpha\beta) = \varepsilon(\alpha s_L(\varepsilon(\beta))) = \varepsilon(\alpha t_L(\varepsilon(\beta))). \]
Also $\Delta$ is right $R$-linear and $\Delta(1_S) = 1_S \otimes_R 1_S$. By making the identification
\[ S \otimes_R S \cong \text{Hom}(B A \otimes B A_B, B A_B), \quad \alpha \otimes \beta \mapsto (a \otimes a' \mapsto \alpha(a)\beta(a')) \]
with inverse $F \mapsto \sum \gamma_j \otimes u_j^1 F(u_j^2 \otimes -)$, we see that the coproduct is left $R$-linear, satisfies $\alpha(1) t_L(r) \otimes \alpha(2) = \alpha(1) \otimes \alpha(2) s_L(r)$ for all $r \in R$, and $\Delta(\alpha \beta) = \Delta(\alpha) \Delta(\beta)$ for all $\alpha, \beta \in S$. For with the independent variables $x, x' \in A, \alpha, \beta \in S$ and $r \in R$, each of these expressions becomes equal to $ra(xx'), \alpha(xrx')$, and $\alpha(\beta(xx'))$ respectively.

The coproduct $\Delta$ is coassociative since
\[ S \otimes_R S \otimes_R S \xrightarrow{\cong} \text{Hom}(B A \otimes B A B A_B, B A_B), \quad \alpha \otimes \beta \otimes \gamma \mapsto (x \otimes y \otimes z \mapsto \alpha(x)\beta(y)\gamma(z)) \]
with inverse given by
\[ F \mapsto \sum_{i,j,k} \gamma_i \otimes u_i^1 \gamma_j (u_j^2 \gamma_k(-)) \otimes u_j^1 F(u_j^2 u_k^1 \otimes u_k^2 \otimes -) \]
Applying this identification to $(\Delta \otimes \text{id}_S)\Delta(\alpha)$ and to $(\text{id}_S \otimes \Delta)\Delta(\alpha)$ then to $x \otimes_B y \otimes_B z$ both expressions equal $\alpha(xy)z$.

$S_R$ is f.g. projective since for each $\alpha \in S$, we have $\alpha = \sum_j \gamma_j h_j(\alpha)$ where $h_j \in \text{Hom}(S_R, R_R)$ is defined by $h_j(\alpha) = u_j^1 \alpha(u_j^2)$.

The proof that given left D2 quasibases $t_i \in T$, $\beta_i \in S$, we have left f.g. projective left bialgebroid $S$ with identical bialgebroid structure is similar and therefore omitted.

Suppose $A | B$ is right D2. Then we have seen that $S$ is a right finite projective left bialgebroid while $T$ is a left finite projective right bialgebroid. There is a nondegenerate pairing between $S$ and $T$ with values in the centralizer $R$ given by $(t | \alpha) := t^1 \alpha(t^2)$, since
\[ (47) \quad \eta : R T \xrightarrow{\cong} \text{Hom}(S_R, R_R) \]
via $\eta(t) = (t | -)$ with inverse $\phi \mapsto \sum_j \phi(\gamma_j) u_j$. By proposition [16, 2.5] a right f.g. projective left bialgebroid $S$ has a right $R$-bialgebroid $R$-dual $S^*$. The question is then if the bialgebroid $S^*$ is isomorphic to the bialgebroid $T$ via $\eta$? The question is partly answered in the affirmative by corollary [16, 5.3], where it is shown without using left D2 quasibases that $T$ and $S^*$ are isomorphic via the pairing above as algebras and $R$-$R$-bimodules.

**Corollary 5.2.** Suppose $A | B$ is right D2. Then $T$ is isomorphic as right bialgebroids over $R$ to the right $R$-dual of $S$ via $\eta$. If $A | B$ is left D2, then $T$ is isomorphic to the bialgebroid left $R$-dual of $S$.

**Proof.** What remains to check in the first statement is that $\eta$ is a homomorphism of $R$-corings using right D2 quasibases. We compute:
\[ (t(1) \cdot t(2) | \alpha') | \alpha) = \sum_j (t^1 \otimes_B \gamma_j(t^2) u_j^1 \alpha'(u_j^2) | \alpha) = t^1 \alpha(\alpha'(t^2)) = (t | \alpha \circ \alpha'), \]
Whence $\Delta(\eta(t)) = \eta(t(1)) \otimes \eta(t(2))$ by uniqueness [16, 2.5 (41)].
The proof of the last statement is similar to the first in using the pairing $[\alpha | t] := \alpha(t^1)t^2$ and the right bialgebroid of the left dual of a left bialgebroid in [16, 2.6]. The details are left to the reader. □

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