Magnetic moment of the $\rho$-meson in QCD sum rules

A. Samsonov*

Institute of Theoretical and Experimental Physics,
Bol’shaya Cheremushkinskaya, 25, Moscow, 117259 Russia

Abstract

The magnetic moment $\mu$ of the $\rho$-meson is calculated in the framework of QCD sum rules in external fields. Bare loop calculations (parton model) give: $\mu_{\text{part}} = 2.0$. The contribution of operators of dimension 6 reduces this value: $\mu = 1.5 \pm 0.3$.

*e-mail:sams@heron.itep.ru
1 Introduction

Investigation of the static properties of vector mesons provides an important information about strong interaction of hadrons. In particular, the vector dominance hypothesis (VDM) supposes that the interaction of real or virtual photon with hadrons proceeds in such a way that the photon first transforms into vector mesons $\rho$, $\omega$, $\phi$, which then undergo interaction with hadrons. In the consistent lagrangian formulation of VDM it is assumed [1] (for review, see [2]) that $\rho$-mesons are Yang-Mills vector bosons. In the framework of this hypothesis the $\rho$-meson magnetic moment is equal to 2 (in units $e\hbar/(2m_\rho c)$), at least if strong interaction is neglected.

The goal of this paper is to calculate the $\rho$-meson magnetic moment in QCD, using the method of QCD sum rules in external fields [3],[4].

In paper [5] the $\rho$-meson formfactors were found at intermediate momentum transfer by QCD sum rules. By extrapolation of the $\rho$-meson magnetic formfactor to the point $Q^2 = 0$ (outside the applicability domain of the technique) it was found that the $\rho$-meson magnetic moment $\mu$ is close to 2. However, this result can not be considered as conclusive; the direct calculation of $\mu$ in QCD in model independent way is still absent. The $\rho$-meson magnetic moment was calculated in the Dyson-Schwinger equation based models [6],[7] and in the framework of relativistic quantum mechanics [8].

Here we work in the limit of zero quark masses, $\alpha_s$-corrections are neglected.

2 Phenomenological part of the sum rule

We consider the correlator of two vector currents in the external electromagnetic field:

$$\Pi_{\mu\nu}(p) = i \int d^4x \, e^{ipx} \langle T(j_\mu(x)j^+_{\nu}(0)) \rangle_F .$$

Here subscript $F$ denotes the presence of the external electromagnetic field with strength $F_{\rho\lambda}$ and $j_\mu$ is the vector current with $\rho$-meson quantum numbers: $j_\mu = \overline{u} \gamma_\mu d$. Its matrix element is

$$\langle \rho^+ | j_\mu | 0 \rangle = \left( m_\rho^2 / g_\rho \right) e_\mu ,$$

where $m_\rho$ is the $\rho$-meson mass, $g_\rho$ is the $\rho$-gamma coupling constant, $g_\rho^2/(4\pi) = 1.27$, and $e_\mu$ is the $\rho$-meson polarization vector.

In the limit of weak external field we consider only linear in $F_{\rho\lambda}$ terms in the correlator $\Pi_{\mu\nu}$ (1):

$$\Pi_{\mu\nu} = \Pi^0_{\mu\nu} + i \sqrt{4\pi \alpha} \Pi_{\mu\nu\chi\sigma} F_{\chi\sigma} .$$

We find magnetic moment from sum rule for the invariant function $\Pi(p^2)$ at certain kinematical structure of $\Pi_{\mu\nu\chi\sigma}$ (3). To obtain this sum rule, we calculate $\Pi$ at $p^2 < 0$ as the operator product expansion series. On the other hand, we saturate dispersion relation...
for $\Pi$ by the contributions of physical states. After equating of these representations the required sum rule appears.

Therefore, first of all one should choose kinematical structure.

The electromagnetic vertex of the $\rho$-meson has the following general form [5]:

$$\langle \rho(p + q, e')|j_\chi^{el}|\rho(p, e')\rangle = -e'_{\sigma} e'_{\rho} \left( \left( (2p + q)_{\chi} g_{\rho\sigma} - (p + q)_{\rho} g_{\chi\sigma} - p_{\sigma} g_{\rho\chi} \right) F_{1}(-q^2) + 
\right. \\
\left. + \left( g_{\chi\rho} q_{\sigma} - g_{\chi\sigma} q_{\rho} \right) F_{2}(-q^2) + \frac{1}{m_{\rho}^2} (p + q)_{\rho} p_{\sigma} (2p + q)_{\chi} F_{3}(-q^2) \right). \quad (4)$$

In (4) $j_\chi^{el} = e_u \vec{\gamma}_\chi u + e_d \vec{\gamma}_\chi d$ is electromagnetic current, $e_u, e_d$ are $u$- and $d$-quark charges and $F_{1}, F_{2}, F_{3}$ are electric, magnetic and quadrupole formfactors correspondingly,

$$F_{1}(0) = 1, \quad \mu = 1 + F_{2}(0), \quad (5)$$

$\mu$ is the $\rho$-meson magnetic moment.

Using (2) and (4), we obtain for the $\langle 0|j_\mu|\rho\rangle \langle \rho|j_\chi|\rho\rangle \langle \rho|j_{\nu}|0\rangle \epsilon_{\chi}$ transition:

$$-i \sum_{r,r'} \langle 0|j_\mu|\rho^{r'}\rangle \langle \rho^{r'}|j_\chi^{el}|\rho^{r'}\rangle \langle \rho^{r'}|j_{\nu}|0\rangle \epsilon_{\chi} = \quad (6)$$

$$= i \frac{m_{\rho}^2}{g_{\rho}^2} \sum_{r,r'} e_{\mu}^{r'} e_{\nu}^{r'} e_{\rho}^{r'} e_{\chi}^{r'} \left( \left( (2p + q)_{\chi} g_{\rho\sigma} - (p + q)_{\rho} g_{\chi\sigma} - p_{\sigma} g_{\rho\chi} \right) F_{1}(-q^2) + 
\right. \\
\left. + \left( g_{\chi\rho} q_{\sigma} - g_{\chi\sigma} q_{\rho} \right) F_{2}(-q^2) + \frac{1}{m_{\rho}^2} (p + q)_{\rho} p_{\sigma} (2p + q)_{\chi} F_{3}(-q^2) \right).$$

Here $\epsilon_{\chi}$—photon polarization, $r$, $r'$ are the $\rho$-meson polarization indices. Let us consider in this expression linear in $q_{\sigma}$ terms. We sum over $\rho$-meson polarizations, retain the antisymmetric over $\chi, \sigma$ part, introduce $F_{\chi\sigma} = i(\epsilon_{\chi} q_{\sigma} - \epsilon_{\sigma} q_{\chi})$ and obtain for (6):

$$-i \frac{m_{\rho}^4}{2g_{\rho}^2} F_{\chi\sigma} \left( (F_{2} + \frac{1}{2} F_{1}) \frac{1}{p_{\rho}} \left( p_{\nu}(p_{\chi} g_{\mu\sigma} - p_{\sigma} g_{\mu\chi}) - p_{\mu}(p_{\chi} g_{\nu\sigma} - p_{\sigma} g_{\nu\chi}) \right) + 
\right. \\
\left. + \frac{1}{2} F_{1} \frac{1}{p_{\rho}} \left( p_{\nu}(p_{\chi} g_{\mu\sigma} - p_{\sigma} g_{\mu\chi}) + p_{\mu}(p_{\chi} g_{\nu\sigma} - p_{\sigma} g_{\nu\chi}) \right) + (F_{2} + F_{1})(g_{\mu\chi} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\chi}) \right).$$

Formfactor $F_{3}$ does not give linear in $q_{\sigma}$ contribution.

Thus, we choose the structure

$$p_{\nu}(p_{\chi} g_{\mu\sigma} - p_{\sigma} g_{\mu\chi}) - p_{\mu}(p_{\chi} g_{\nu\sigma} - p_{\sigma} g_{\nu\chi}). \quad (7)$$

In comparison with another possible structure, $g_{\mu\chi} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\chi}$, (7) contains two additional powers of momentum in the numerator, which result in better convergence of the operator expansion series.
It should be noted here that, as follows from the vector current conservation, the antisymmetric over field indices $\chi, \sigma$ structure in $\Pi_{\mu\nu\chi\sigma}^{(3)}$ is antisymmetric over $\rho$-meson indices $\mu, \nu$ too.

From (5) one can see that $F_2(0) + 1/2F_1(0) = \mu - 1/2$.

Thus, one should calculate the invariant function $\Pi(p^2)$ at the structure (7) in $\Pi_{\mu\nu\chi\sigma}$. In the dispersion relation for $\Pi$ we use the simplest model of physical spectrum, which contains the lowest resonance and continuum. Phenomenological representation of $\Pi$ has the form:

$$\Pi(p^2) = \int ds \frac{\rho_L(s)}{(s - p^2)^2} + \ldots,$$

$$\rho_L(s) = -\frac{m_\rho^4}{2g_\rho^2 s}(\mu - \frac{1}{2})\delta(s - m_\rho^2) + f(s)\theta(s - s_\rho).$$

Here dots mean the contributions of non-diagonal transitions (for example, $\langle 0 | j_\mu | \rho^* \rangle \langle \rho^* | j_\chi \epsilon_\chi | \rho \rangle \langle \rho | j_\nu | 0 \rangle$, where $\rho^*$ is the excited state with the same quantum numbers as $\rho$), function $f$ represent continuum contribution and $s_\rho$ is the continuum threshold for the $\rho$-meson.

Retaining only the terms, which do not vanish after Borel transformation, we obtain:

$$\Pi(p^2) = -\frac{m_\rho^2}{2g_\rho^2 (m_\rho^2 - p^2)^2} + \frac{\tilde{C}}{m_\rho^2 - p^2} + \int_{s_\rho}^{\infty} ds \frac{f(s)}{(s - p^2)^2}, \tag{8}$$

where $\tilde{C}$ appears due to non-diagonal transitions.

3 Calculation of the vector current correlator

Now let us calculate $\Pi(p^2)$, basing on the operator product expansion in QCD.

The quark propagator in the external electromagnetic field $F_{\mu\nu}$ in the fixed-point gauge $x_\mu A_\mu = 0$, $A_\mu = -1/2F_{\mu\nu}x_\nu$ can be found in [3]:

$$\langle T q^a_\alpha(x)\bar{q}^b_\beta(0) \rangle_F = \frac{i\delta^{ab}(\hat{x})_{\alpha\beta}}{2\pi^2 x^4} - \frac{\delta^{ab}g_{\alpha\beta}}{12}(\bar{q}q) - \frac{i\delta^{ab}(\bar{q}\sigma_\rho\lambda)q_F}{48}(\gamma_\rho\gamma_\lambda - \gamma_\lambda\gamma_\rho)_{\alpha\beta} - \delta^{ab}\sqrt{4\pi\alpha}\epsilon_\rho q_F^{\mu\lambda}(\hat{x}\gamma_\rho\gamma_\lambda + \gamma_\rho\gamma_\lambda\hat{x})_{\alpha\beta}.$$ 

Here $e_\rho$ is the quark charge, $\alpha, \beta$ are spinor indices, $a, b$ - color indices and (see [3])

$$\langle \bar{q}\sigma_\rho\lambda q \rangle_F = \sqrt{4\pi\alpha}\epsilon_\rho \chi F_{\rho\lambda}(\bar{q}q), \chi is the quark condensate magnetic susceptibility.

The expression for the quark propagator in the external electromagnetic and soft gluon fields has the following form in momentum representation [3]:

$$\hat{S}_{FG} = -\frac{ige_\rho\sqrt{4\pi\alpha}F_{\rho\lambda}C_{\sigma\tau}^{m}t^{n}}{2p^6}(\gamma_\lambda\gamma_\tau\gamma_\rho\gamma_\sigma\hat{p} - 2p_\lambda\gamma_\tau\gamma_\sigma\gamma_\rho - 2p_\tau\gamma_\lambda\gamma_\rho\gamma_\sigma - \frac{8p_\rho p_\tau g_{\lambda\sigma}}{p^2}\hat{p} + 2g_{\lambda\tau}\gamma_\sigma\gamma_\rho\hat{p} - 2g_{\lambda\tau}g_{\rho\sigma}\hat{p}).$$
Here $G_{\sigma\tau}^n$ is the gluon field strength and $t^n$ are the color matrices.

The contribution of the loop diagrams to $\Pi(p^2)$ is equal to

$$-\frac{3}{16\pi^2} \int_0^\infty \frac{ds}{(s-p^2)^2}. \tag{9}$$

According to the quark-hadron duality, the continuum contribution in the interval of $P^2 = -p^2$ from $s_p$ to infinity is determined by the bare loop in this interval. Therefore, function $f$ in (8) is constant: $f = -3/(16\pi^2)$.

The loop diagrams correspond to the operator of the lowest dimension $F_{\rho\lambda}$. Operators of dimension 4 are absent. As was shown in [3], operator $\bar{q}(D_{\mu}\gamma_\nu - D_\nu\gamma_\mu)q$ has opposite with respect to the electromagnetic field C-parity and can not be induced by them, while operator $\epsilon_{\mu\nu\rho\lambda}\gamma_\mu D_\lambda q$ vanishes due to equation of motion for massless quarks.

There are a number of vacuum expectation values of operators of dimension 6: $\langle \bar{q}\sigma_{\rho\lambda}q \rangle_{F} \langle \bar{q}g \rangle$, $\langle G_{\sigma\tau}^n G_{\sigma\tau}^n \rangle_{F\mu\nu}$ and

$$g \langle (G_{\mu\lambda}^a D_\nu - \hat{D}_\nu G_{\mu\lambda}^a) - (G_{\nu\lambda}^a D_\mu - \hat{D}_\mu G_{\nu\lambda}^a) \rangle_{F} \gamma_\lambda t^n q, \tag{10}$$

$$\epsilon_{\mu\nu\rho\lambda} g \langle (G_{\rho\xi}^a D_\lambda + \hat{D}_\lambda G_{\rho\xi}^a) \gamma_\xi t^n q \rangle_{F},$$

$$d^{ikl} \langle (G_{\mu\lambda}^i G_{\rho\mu}^j G_{\nu}^l D_\lambda - C_{\nu\lambda} G_{\xi}^k G_{\rho\mu}^l) \rangle_{F},$$

where $D_\mu$ is the covariant derivative and $d^{ikl}$ are SU(3) structure constants.

The diagrams, corresponding to the operator $\langle G_{\sigma\tau}^n G_{\sigma\tau}^n \rangle_{F\mu\nu}$, have infrared divergence. We introduce the cut-off over transversal momenta $\lambda$ and obtain their contribution into $\Pi(p^2)$:

$$-\frac{1}{36} \left(\frac{\alpha_s}{\pi} G^2\right) \left(\frac{1}{2\lambda^4 p^2} - \frac{1}{6\lambda^2 p^4} + \frac{3}{p^6}\right). \tag{11}$$

This divergence is probably cancelled by the contribution of field induced vacuum expectation values (10) (for example, this was shown in [3] for dimension 4 operators and symmetric tensor field). Usually such vacuum expectation values can be calculated by constructing corresponding sum rule. But for operators (10) this approach is inapplicable because of their high dimension.

The dominating contribution appears from no loop diagrams with hard gluon exchange. In our case such diagrams contain the operator $\langle \bar{q}\sigma_{\rho\lambda}q \rangle_{F} \langle \bar{q}q \rangle$. They give:

$$\frac{2}{9} \frac{g^2 \langle \bar{q}q \rangle^2 \chi}{p^6}. \tag{12}$$

It should be noted here that the quark condensate magnetic susceptibility $\chi$ is negative.

Collecting the expressions (9), (11), (12), one can find the operator product expansion part of the sum rule:

$$\Pi(p^2) = -\frac{3}{16\pi^2} \int_0^\infty \frac{ds}{(s-p^2)^2} + \frac{2}{9} \frac{g^2 \langle \bar{q}q \rangle^2 \chi}{p^6} - \frac{1}{36} \left(\frac{\alpha_s}{\pi} G^2\right) \left(\frac{1}{2\lambda^4 p^2} - \frac{1}{6\lambda^2 p^4} + \frac{3}{p^6}\right). \tag{13}$$

The uncertainty in the contribution of the vacuum expectation values (10) is included into the errors.
4 Results and discussion

After Borel transformation

\[
\hat{B}(M^2) = \lim_{p^2/n \to \infty} \frac{(P^2)^{n+1}}{n!} \left( - \frac{d}{dp^2} \right)^n, \quad P^2 = -p^2 > 0
\]

we equate the phenomenological (8) and operator product expansion (13) parts of sum rule and obtain:

\[
\mu - \frac{1}{2} + CM^2 = \frac{3g^2_p M^2}{8\pi^2 m^2_p} \left( 1 - e^{-s_p/M^2} \right) e^{m^2_p/M^2} - \frac{g^2_p}{m^2_p} e^{m^2_p/M^2} \left( - \frac{2g^2_fq^2}{9M^2} + \frac{1}{36} \frac{\alpha_s G^2}{\pi} \left( \frac{M^2}{\lambda^2} + \frac{1}{3\lambda^2} + \frac{3}{M^2} \right) \right). \quad (14)
\]

\(C\) appears due to nondiagonal transitions.

We use the following values of parameters:

- \(m_\rho = 0.77 \text{ GeV}\) — the \(\rho\)-meson mass,
- \(g^2_p/(4\pi) = 1.27\) — the \(\rho-\gamma\) coupling constant,
- \(s_\rho = 1.5 \text{ GeV}^2\) — the continuum threshold for \(\rho\)-meson,
- \(\langle (\alpha_s/\pi) G^2 \rangle = 0.009 \pm 0.007 \text{ GeV}^4\) — the gluon condensate [10],
- \(g^2_fq^2 = (0.28 \pm 0.09) \times 10^{-2} \text{ GeV}^6\) — the quark condensate [10],
- \(\chi = -(5.7 \pm 0.6) \text{ GeV}^{-2}\) — the quark condensate magnetic susceptibility [11],
- \(\lambda^2 = 0.8 \text{ GeV}^2\) — the cut-off over transversal momenta.

First of all, let us consider the contribution of the bare loop (and continuum). It is given by the first term in (14). In [12] the following relation for \(g_\rho\) can be found:

\[
\frac{g^2_p M^2}{4\pi^2 m^2_p} \left( 1 - e^{-s_p/M^2} \right) e^{m^2_p/M^2} = 1. \quad (15)
\]

Substituting (15) into (14) and omitting for a while the terms with quark and gluon condensates, one can obtain very simple answer:

\[
\mu - \frac{1}{2} = \frac{3}{2}. \quad (16)
\]

We see that in the parton model approximation the \(\rho\)-meson magnetic moment is equal to 2. This result agrees with prediction of the vector dominance hypothesis.

Now let us analyze the whole equation (14). In order to find the value of magnetic moment, we approximate the right-hand side of (14) (see fig.1) by a straight line in the interval \(0.9 \text{ GeV}^2 \leq M^2 \leq 1.3 \text{ GeV}^2\) and find it ordinate at zero Borel mass.

Thus we obtain:

\[
\mu = 1.5. \quad (16)
\]

The contribution of the operators of dimension 6 to this value does not exceed 20%.

The contribution of the terms, which contain \(\lambda^2\), is not more than 20% of the total
contribution of dimension 6 operators. That is why variation of $\lambda^2$ within the interval $0.6 \text{GeV}^2 \leq \lambda^2 \leq 1.0 \text{GeV}^2$ does not change the value of magnetic moment.

The variations of the values of the quark and gluon condensates within the given limits change the value of magnetic moment by $\lesssim 10\%$ each.

The uncertainty in the value of the quark condensate magnetic susceptibility results in the error about few percent in the value of the magnetic moment. Variation of the continuum threshold for the $\rho$-meson $s_{\rho}$ in the reasonable limits gives the same effect.

Supposing that the contribution of vacuum expectation values (10) does not exceed 50% of that from diagrams with hard gluon exchange (12), we obtain after collecting all uncertainties:

$$\mu = 1.5 \pm 0.3.$$  

This is our final result.

In [13] it was shown that the approximation procedure is correct (nonlinear terms can be safely neglected), when $\mu \gg CM^2$. In our case $CM^2/\mu \approx 0.2 \div 0.3$.

Thus we find that in parton model (bare loop) approximation the $\rho$-meson magnetic moment $\mu_{\text{part}} = 2$, whereas the nonperturbative interactions decrease this quantity by a quarter: $\mu = 1.5 \pm 0.3$. It is important to mention that all accounted operator product expansion corrections are negative, i.e. results in decrease of $\mu$ in comparison with $\mu_{\text{part}} = 2$. Since the effective values of the Borel parameter $M^2$ are about 1 GeV$^2$, one may expect that perturbative corrections are remarkable and can reach $\sim 20\%$.

The value of the $\rho$-meson magnetic moment was calculated in a number of papers within the Dyson-Schwinger equation based models. In [6] the value $\mu = 2.69$ was found. In [7] several results are compared, and the values of $\mu$ lie between 2.5 and 3.0. Relativistic quantum mechanics model gives $\mu = 2.23 \pm 0.13$. Unfortunately, while $\alpha_s$-corrections are not calculated (we plan to do it in the next paper), it is hard to say with certainty if this discrepancy is real or not.

The author is grateful to B.L. Ioffe for posing the problem and valuable discussions and to A.G. Oganesian for helpful discussions.

The work is supported in part by grants CRDF RP2-2247, INTAS 2000 Project 587 and RFFI 00-02-17808.

References

[1] N. Kroll, T. Lee and B. Zumino, Phys.Rev. 157 (1967) 1376.

[2] B. Ioffe, V. Khoze and L. Lipatov, Hard Processes, North Holland, 1984, ch.5.

[3] B. Ioffe and A. Smilga, Nucl.Phys. B232 (1984) 109.

[4] I. Balitsky and A. Yung, Phys.Lett. B129 (1983) 328.

[5] B. Ioffe and A. Smilga, Nucl.Phys. B216 (1983) 373.

[6] F. Hawes and M. Pichowsky, Phys.Rev. C59 (1999) 1743.
[7] M. Hecht and B.H.J. McKellar, Phys.Rev. C57 (1998) 2638.
[8] J.P.B.C de Melo and T. Frederico, Phys.Rev. C55 (1997) 2043.
[9] A. Oganesian and A. Samsonov, JHEP 0109 (2001) 002.
[10] B. Ioffe, hep-ph/0207191, Phys.Atom.Nucl., in press.
[11] V. Belyaev and Ya. Kogan, Yad.Fiz. 40 (1984) 1035 (Sov.J.Nucl.Phys. 40 (1984) 659).
[12] M. Shifman, A. Vainshtein and V. Zakharov, Nucl.Phys. B147 (1979) 385, 448.
[13] B. Ioffe, Yad.Fiz. 58 (1995) 1492.

Figure 1: The right-hand side of equation (14) $R(M^2)$ as the function of $M^2$. 