Covariant Hamiltonian formalism for $F(R)$-gravity

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Abstract
In this short note we apply Weyl–De Donder formalism, also known as covariant Hamiltonian formalism, for $F(R)$-gravity. We derive covariant Hamiltonian and derive corresponding equations of motion.

Keywords Covariant Hamiltonian formalism · $F(R)$ gravity · Alternative theories of gravity

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1 Introduction and summary

The fundamental concept of modern field theories is their formulation with the help of action and corresponding Lagrangian density. This formulation is manifestly covariant which reflects basic concept of general relativity. Dynamics of these fields is then governed by Lagrange equations of motion which are again manifestly covariant. In other words, in the Lagrangian formulation all space-time coordinates are treated on equal footing.

The situation is different when we switch to the Hamiltonian formalism when one direction known as time direction is selected and Hamiltonian equations of motion
basically determine evolution of the system along this direction. Clearly such a splitting of space-time into space and time breaks manifest covariance of the theory. On the other hand there exists covariant Hamiltonian formulation of field theory known as Weyl–De Donder theory [1,2]. Let us demonstrate its principle on the simple example of the Lagrangian for scalar field \( \phi \) that clearly depend on derivative \( \partial_a \phi \). In conventional canonical formalism we consider time derivative as special one and define conjugate momentum as derivative of Lagrangian density with respect to \( \partial_t \phi \). In case of Weyl–De Donder theory we treat all partial derivatives on the equal footing which clearly preserves diffeomorphism invariance. Then covariant canonical Hamiltonian density depends on conjugate momenta \( p^a_M \) which are variables conjugate to \( \partial_a \phi^M \). This approach is also known as multisymplectic field theory, see for example [3–5], for review, see [6] and for recent interesting application of this formalism in string theory, see [7,8].

Intuitively it is clear that such a covariant Hamiltonian approach could be very convenient for all covariant theories and especially it would be very appropriate in case of general relativity. In fact, covariant Hamiltonian formulation of gravity was presented by P. Horava in his paper [13]. It was shown there that the analysis simplifies considerably when instead of conventional canonical variable \( g_{ab} \) we use \( f_{ab} \equiv \sqrt{-g} g^{ab} \). In fact, an importance of this variable in the holographic formulation of gravity was stressed recently by T. Padmanabhan in his works, for very clear and detailed discussion of properties of gravity formulated in \( f_{ab} \) variables we recommend his paper [11].

It is natural to ask the question whether covariant Hamiltonian analysis can be performed with more general theories of gravity, as for example \( F(R)–\)gravity, for review and extensive list of references see [14–16]. These theories can be considered as simplest modification of general relativity that have a potential to explain some aspects of inflation or dark energy. On the other hand considering \( F(R)–\)gravity as generalization of general relativity, it is natural to study whether it is possible to find its covariant Hamiltonian formulation. In fact, this is the goal of this paper 1. As the first step we introduce auxiliary fields in order to be able to deal with \( F(R)–\)gravity and then we carefully split Lagrangian density for \( F(R)–\)gravity into bulk and boundary part. We find that there is an additional contribution to the bulk part with respect to ordinary gravity that is proportional to the partial derivative of the auxiliary field. Then we can straightforwardly proceed to the covariant Hamiltonian formulation when we firstly determine momenta conjugate to \( g_{ab} \). We find that they are rather complicated which has been previously stressed in [11]. Then, following discussion presented in that paper, we proceed to the canonical variable \( f_{ab} = \sqrt{-g} g^{ab} \) that are related to \( g_{ab} \) through point transformations. We determine momenta \( N^c_{ab} \) conjugate to \( f_{ab} \) and \( p^c_B \) conjugate to auxiliary field \( B \). Finally we invert relations between conjugate momenta \( N^c_{ab} \) and Christoffel symbols and also between \( p^c_B \) and derivative of \( B \) and obtain corresponding Hamiltonian. We also determine canonical equations of motion.

Let us outline our results. We find covariant Hamiltonian for \( F(R)–\)gravity which is the first step in the application of this formalism to more general form of gravity. We also determine corresponding equations of motion. It would be certainly nice to

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1 For related works, see for example [23,24].
extend this result along directions that are related to Padmanabhan’s work. It would be also nice to apply covariant Hamiltonian analysis in case of more general theories of gravity, as for example curvature-squared gravity [17]. Finally, it would be interesting to find covariant Hamiltonian formalism for $F(R)$—gravity formulated in Jordan frame and compare it with the result derived in this paper and perform similar analysis as in [18–20]. We hope to return to these problems in future.

This paper is organized as follows. In the next section (2) we introduce Lagrangian for $F(R)$—gravity and split it into bulk and boundary parts. Then in section (3) we present canonical formalism in $g_{ab}$ and $M^{abc}$ variables. Finally in section (4) we proceed to the Hamiltonian analysis in $f_{ab}$, $N_{ab}^c$ variables and determine corresponding Hamiltonian density and canonical equations of motion.

### 2 Introduction of $F(R)$-gravity

In this section we introduce an action for $F(R)$—gravity. This theory is generalization of Einstein-Hilbert action when term linear in curvature $R$ is replaced by arbitrary function $F(R)$. Explicitly, the action for $F(R)$—gravity has the form

$$S = \int d^4x \mathcal{L}^F, \quad \mathcal{L}^F = \frac{1}{16\pi} \sqrt{-g} F(R), \quad (1)$$

where we set gravitational constant and speed of light to be equal one and where $F$ is an arbitrary function of Ricci scalar $R$ that is defined as

$$R = g^{mn} R_{mn} = g^{mn} R^k_{mkn},$$

$$R^k_{lmn} = \partial_m \Gamma^k_{nl} - \partial_n \Gamma^k_{ml} + \Gamma^k_{np} \Gamma^p_{ml} - \Gamma^k_{np} \Gamma^p_{ml}. \quad (2)$$

In order to find Hamiltonian formulation of this theory we proceed in the similar way as in case of ordinary canonical formalism for $F(R)$—gravity, for very nice discussion we recommend the paper [21]. Explicitly, we introduce two scalar fields $A$ and $B$ and write the Lagrangian density in the form [22]

$$\mathcal{L}^F = \frac{1}{16\pi} \sqrt{-g} \left[ F(B) + A(R - B) \right]. \quad (3)$$

where the equations of motion for $A$ implies $R = B$ while equation of motion for $B$ implies $F'(B) - A = 0$ where $F'(B) \equiv \frac{dF}{dB}$. Inserting the first result into (3) we obtain original action. Instead it is useful to use second equation to express $A = F'(B)$ and write the Lagrangian density (3) in the form

$$\mathcal{L}^F = \frac{1}{16\pi} \sqrt{-g} \left[ F(B) + F'(B)(R - B) \right]. \quad (4)$$

---

2. We work with the metric $g_{ab}$ with signature $(-1, 1, 1, 1)$ where $a, b, c, \cdots = 0, \ldots, 3$.

3. Let us briefly explain why it is necessary to introduce auxiliary fields instead of working with the original action (1) directly when we want to find canonical formulation of $F(R)$—gravity, either classical or covariant. The point is that when we started with the original action with the definition of canonical momenta we would not be able to invert the relation between derivatives of metric and conjugate momenta due to the presence of general function of $F(R)$ that explicitly depends on derivatives of metric. For that reason it is convenient to introduce two auxiliary fields where now $F(B)$ is ordinary function of scalar field.
In order to check consistency of this action let us solve equations of motion for $B$ that follow from (4)

$$F'(B) + F''(B)(R - B) - F'(B) = F''(B)(R - B) = 0 \quad (5)$$

which again implies $R = B$ on condition $F''(B) \neq 0$, where $F''(B) \equiv \frac{d^2 F}{d^2 B}$. Inserting this result into (4) we obtain original Lagrangian density (1). In fact, for $F''(B) = 0$ we find that $F'(B) = K$ where $K = \text{const}$ and hence $F(B) = KB + C$ where $C$ is another constant. Inserting this result into (4) we obtain Einstein-Hilbert action with cosmological constant proportional to $C$ after appropriate rescaling. This can be already seen from the action (4) when inserting $F''(B) = 0$ into it we obtain Einstein-Hilbert action that does not depend on $B$ at all. In other words, the equation (5) has sense on condition $F''(B) \neq 0$. In what follows we will presume condition $F''(B) \neq 0$ that leads to $F(R)$--gravity. In fact, we should impose more stronger condition that $F''(B) \neq 0$ during the whole dynamical evolution of the system. For example, in case of $F(B) = Bk$, $k > 2$ we should impose the condition that $B \neq 0$ during the whole evolution of the system.

We will also work with (4) rather than with the action (3) since all fields that are presented in (4) contain their partial derivatives. Clearly analysis performed above shows that these two actions are equivalent.

Finally we determine equations of motion for $g_{mn}$ from the action (4) in order to see full equivalence between action (1) and (4). Performing variation of (4) with respect to $\delta g_{mn}$ we get following equation

$$F'(B)R_{mn} - \frac{1}{2}g_{mn}[F(B) + F'(B)(R - B)]$$

$$- \nabla_m \nabla_n F'(B) + g_{mn} \frac{1}{\sqrt{-g}} \partial_k [\sqrt{-g}g^{kl} \partial_l F'(B)] = 0. \quad (6)$$

Then using (5) we find that equations of motion (5) reduces to the standard equations of motion of $F(R)$ gravity that are derived from the action (1), see for example [14]. These results fully demonstrate equivalence of actions (1) and (4).

Let us now proceed to the covariant canonical formalism of $F(R)$--gravity. Following [11] we write Ricci scalar as

$$R = Q^{mnl}_k R^k_{mnl}, \quad (7)$$

where

$$Q^{mnl}_k = \frac{1}{2}(g^{ml} \delta^m_n - g^{mn} \delta^l_k), \quad Q^m_{knl} = -Q^m_{kln}. \quad (8)$$

As the next step we split Lagrangian density (4) into bulk and boundary part [9,10], for recent careful discussion, see also [11]4. This procedure is trivial in case of terms

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4 It is important to stress that the variation of the bulk term alone leads to the equations of motion while the surface term, when integrated over a horizon, is related to the entropy of the horizon, as was shown for example in [12].
quadratic in Christoffel symbols in Riemann tensor however we should be more careful
with terms that contain partial derivatives of Christoffel tensor since for example
\[
\sqrt{-g} F'(B) Q_{k}^{mnl} \partial_m \Gamma^k_{nl} = \partial_m [\sqrt{-g} F'(B) Q_{k}^{mnl} \Gamma^k_{nl}]
- \partial_m [\sqrt{-g} Q_{k}^{mnl}] F'(B) \Gamma^k_{nl} - \sqrt{-g} Q_{k}^{mnl} F''(B) \partial_m B \Gamma^k_{nl}.
\] (9)

The second term on the second line gives bulk contribution corresponding to the kinetic
term for auxiliary field $B$. Further, the first term gives also bulk contribution as follows
from the fact that
\[
\partial_c (\sqrt{-g} Q^{bcd}_a) = -\sqrt{-g} \Gamma^b_{cm} Q^m_a Q^{bcd} - \Gamma^m_{ca} Q^{bcd} \sqrt{-g}
\] (10)

which is consequence of the condition $\nabla_c (\sqrt{-g} Q^{bcd}_a) = 0$. Collecting these terms
together we obtain that surface term has explicit form
\[
16 \pi \mathcal{L}_{surf}^F = 2 \partial_c [\sqrt{-g} F'(B) Q_{a}^{bcd} \Gamma^a_{bd}]
= 2 \sqrt{-g} F''(B) \partial_c B Q_{a}^{bcd} \Gamma^a_{bd} - 4 F'(B) \sqrt{-g} Q_{a}^{bcd} \Gamma^a_{dm} \Gamma^m_{bc}
+ 2 F'(B) \sqrt{-g} Q_{a}^{bcd} \partial_c \Gamma^a_{bd}.
\] (11)

and hence we find that decomposition of the Lagrangian density of $F(R)-$gravity into
surface and bulk terms has the form
\[
16 \pi \mathcal{L}^F = 16 \pi \mathcal{L}_{quadr}^F + 16 \pi \mathcal{L}_{surf}^F,
16 \pi \mathcal{L}_{quadr}^F = 2 \sqrt{-g} F'(B) Q_{a}^{bcd} \Gamma^a_{dm} \Gamma^m_{bc} - 2 \sqrt{-g} F''(B) \partial_c B Q_{a}^{bcd} \Gamma^a_{bd}
+ \sqrt{-g} [F(B) - F'(B) B],
16 \pi \mathcal{L}_{surf}^F = 2 \partial_c [\sqrt{-g} F'(B) Q_{a}^{bcd} \Gamma^a_{bd}].
\] (12)

The Lagrangian density (12) is our starting point for the covariant Hamiltonian
formulation of $F(R)-$gravity which will be analysed in the next section.$^5$

$^5$ Let us also say few words about covariant canonical analysis of the Lagrangian (3). Even in this case
we can perform separation of the Lagrangian into bulk and boundary term where however now $B$
would be variable without partial derivative in the quadratic action. As a result we would get that its conjugate
momenta would be zero which would be primary constraint in terms of the terminology of constraint
systems. Certainly we could derive covariant Hamiltonian density for dynamical variables $g_{mn}$, $A$ and field
$B$. However then equation of motion for $B$ would be algebraic equation that could be solved for $B$
at least in principle. Then the result should be inserted back to the remaining equations of motion. In case
of standard canonical analysis this procedure is equivalent to the requirement of the preservation of the
primary constraint which leads to the secondary constraints that together with primary constraints form two
second class constraints. On the other hand it is not exactly clear how such a formalism could be applied
in case of covariant canonical formalism however some progress can be found in [27]. For that reason we
mean that it is useful to eliminate auxiliary field $A$ by its equation of motion before we proceed to the
covariant canonical formalism.
3 Covariant Hamiltonian formalism with \((g_{ab}, M^{abc})\) as canonical variables

In this section we proceed to the covariant Hamiltonian formalism of \(F(R)\)–gravity when we consider \(g_{bc}\) as basic dynamical variable. Then, according to the basic principle of covariant field theory, the conjugate momentum \(M^{abc}\) is defined as variation of the bulk part of the action with respect to \(\partial_a g_{bc}\). Explicitly, we know that quadratic part of Lagrangian density for \(F(R)\)–gravity has the form (12), or alternatively, using explicit form of \(Q_{bcd}^a\) we have

\[
16\pi L_{quad}^F = \sqrt{-g} F'(B) [\Gamma^{h}_{dk} \Gamma^{k}_{gh} g^{gd} - \Gamma^{f}_{fk} \Gamma^{k}_{gh} g^{gh}] - F''(B) \sqrt{-g} \partial_c B[\Gamma^{c}_{bd} g^{bd} - g^{cb} \Gamma^{d}_{bd}] + \sqrt{-g} [F(B) - F'(B)B].
\]

(13)

In order to define conjugate momenta \(M^{abc}\) we need following expression

\[
\frac{\delta \Gamma^{k}_{bc}}{\delta \partial_p g_{rs}} = \frac{1}{4} \delta^p_b (g^{kr} \delta^s_c + g^{ks} \delta^r_c) + \frac{1}{4} \delta^p_c (g^{kr} \delta^s_b + g^{ks} \delta^r_b) - \frac{1}{4} g^{kp} (\delta^r_b \delta^s_c + \delta^r_c \delta^s_b).
\]

(14)

Using this result we get

\[
16\pi M^{abc} = \frac{\partial L_{quad}^F}{\partial (\partial_a g_{bc})} = \frac{1}{2} F'(B) \sqrt{-g} [g^{bd} \Gamma^{a}_{dk} g^{kc} + g^{cd} \Gamma^{a}_{dk} g^{kb}] - \frac{1}{2} F'(B) \sqrt{-g} \Gamma^{k}_{fk} (g^{f b} g^{ac} + g^{f c} g^{ab}) - \frac{1}{2} F'(B) \sqrt{-g} \Gamma^{a}_{gh} g^{gh} + \frac{1}{2} F''(B) \sqrt{-g} \partial_f B [g^{f b} g^{ac} + g^{f c} g^{ab}] + F''(B) \sqrt{-g} \partial_f B g^{fa} \]

(15)

and momentum conjugate to \(B\) as

\[
16\pi p^c_B = \frac{\partial L_{quad}^F}{\partial (\partial_c B)} = -F''(B) \sqrt{-g} [\Gamma^{a}_{bd} g^{cd} - g^{cb} \Gamma^{d}_{bd}].
\]

(16)

To proceed further let us introduce \(V^a = -g_{bc} M^{abc}\) that, using (15) has the form

\[
16\pi V^a = \sqrt{-g} F'(B) \Gamma^{a}_{gh} g^{gh} - F'(B) \sqrt{-g} \Gamma^{k}_{fk} g^{fa} - 3 F''(B) \sqrt{-g} \partial_f B g^{fa}.
\]

(17)
From $V^a$ we can express $\partial_f B$ as

$$
\partial_f B = -\frac{16\pi g_f a}{3F'(B)\sqrt{-g}} \left( -g_{bc} M^{abc} + \frac{F'(B)}{F''(B)} p_B^a \right).
$$

Then we can insert this result into definition of $M^{abc}$ and find relation between $M^{abc}$ and $\Gamma^a_{bc}$. However resulting expression is rather complicated and corresponding Hamiltonian as well. For that reason we rather focus on set of variables that were used in [13].

**4 Hamiltonian analysis in new variables ($f^{ab}$, $N^c_{ab}$)**

We define new metric variable as

$$
f^{ab} = \sqrt{-g} g^{ab}.
$$

Even if we call it as a new one we should stress that it was used in classical literature long time ago [9,10] and it was also used by P. Horava for covariant Hamiltonian formulation of gravity [13]. An importance of these variables was also stressed many times in works by Padmanabhan, see for example [11]. Explicitly, as was shown in [11] these variables are very useful for the simplification of the variation of the action and also the variation of these variables on horizon has direct thermodynamical interpretation in case of ordinary general relativity. For all these reasons it is reasonable to use these variables in case of covariant canonical formalism for $F(R)$—gravity.

From definition of $f^{ab}$ given in (18) we find that

$$
f = \det f^{ab} = \det g
$$

and also we have following relation between variation of $g^{ab}$ and $f^{ab}$

$$
\delta g^{ab} = \frac{\delta f^{ab}}{\sqrt{-f}} - \frac{1}{2\sqrt{-f}} f^{ab} f_{mn} \delta f^{mn} \equiv \frac{1}{\sqrt{-f}} B^{ab}_{mn} s f^{mn},
$$

where

$$
B^{ab}_{mn} = \frac{1}{2} (\delta^a_m \delta^b_n + \delta^b_m \delta^a_n) - \frac{1}{2} f^{ab} f_{mn} = \frac{1}{2} (\delta^a_m \delta^b_n + \delta^b_m \delta^a_n - g^{ab} g_{mn}),
$$

where we defined $f_{ab}$ as inverse to $f^{ab}$ and it has explicit form $f_{ab} = \frac{1}{\sqrt{-g}} g_{ab}$.

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6 We should stress that it is still possible to find covariant canonical formulation of $F(R)$—gravity with the variables $(g_{ab}, M^{abc})$ in principle. However the canonical analysis simplifies considerably when we use new variables that will be introduced in the next section so that we prefer them over original set of variables $(g_{ab}, M^{abc})$.  

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Now let us consider $f^{ab}$ as canonical variable and introduce corresponding conjugate momenta $N^c_{ab}$ as variation of the action with respect to $\partial_c f^{ab}$. Since boundary term does not contribute to this definition we have

$$N^c_{ab} = \frac{\partial L^F_{\text{quadr}}}{\partial (\partial_c f^{ab})} = \frac{\partial L^F_{\text{quadr}}}{\partial (\partial_d g_{mn})} \frac{\partial (\partial_d g_{mn})}{\partial (\partial_c f^{ab})}. \quad (22)$$

Now $f^{ab}$ and $g_{mn}$ are related by point transformations so that $f^{ab} = f^{ab}(g_{mn})$ or inverse $g_{mn} = g_{mn}(f^{ab})$. Then

$$\partial_d g_{mn} = \frac{\delta g_{mn}}{\delta f^{ab}} \partial_d f^{ab} \quad (23)$$

so that $\partial_d g_{mn} = \partial_d g_{mn}(f^{ab}, \partial_c f^{ab})$. Then clearly we have

$$\frac{\delta (\partial_d g_{mn})}{\delta (\partial_c f^{ab})} = \frac{\delta g_{mn}}{\delta f^{ab}} \delta^c_d. \quad (24)$$

Returning to $N^c_{ab}$ we obtain

$$N^c_{ab} = \frac{\partial L^F_{\text{quadr}}}{\partial (\partial_c g_{mn})} (-g_{mk} \frac{1}{\sqrt{-f}} B^{kli}_{ab} g_{ln}) = -M^{cmn} \frac{1}{\sqrt{-f}} B_{mn,ab}, \quad (25)$$

where we used

$$\frac{\delta g_{mn}}{\delta f^{ab}} = -\frac{1}{\sqrt{-f}} g_{mk} B^{kli}_{ab} g_{ln} \quad (26)$$

and also we defined $B_{mn,ab}$ as

$$B_{mn,ab} = g_{mk} B^{kli}_{ab} g_{ln} = \frac{1}{2} (g_{ma} g_{nb} + g_{mb} g_{na} - g_{mn} g_{ab}). \quad (27)$$

Since

$$16\pi M^{abc} = \frac{1}{2} F' \sqrt{-g} \left[ g^{bd} \Gamma^a_{dk} g^{kc} + g^{cd} \Gamma^a_{dk} g^{kb} \right]$$

$$-\frac{1}{2} F' \sqrt{-g} \Gamma^{k}_{fk} (g^{fb} g^{ac} + g^{fc} g^{ab}) - \frac{1}{2} F'(B) \sqrt{-g} g^{bc} \Gamma^a_{gh} g_{gh}$$

$$+ \frac{1}{2} F'(B) \sqrt{-g} \Gamma^{k}_{fk} g^{ab} g^{bc}$$

$$- \frac{1}{2} F''(B) \sqrt{-g} \partial_d B g^{fbc} + g^{fc} g^{ab}) + F''(B) \sqrt{-g} \partial_d B g^{fbc} \quad (28)$$
we obtain from (25) following conjugate momenta $N^c_{ab}$

\[
16\pi N^c_{ab} = -F'(B)\Gamma^c_{ab} + \frac{1}{2} F'(B)(\Gamma^k_{ak} \delta^c_b + \Gamma^k_{bk} \delta^c_a)
+ \frac{1}{2} F''(B)(\partial_a B \delta^c_b + \partial_b B \delta^c_a) + \frac{1}{2} F''(B)\partial_g B f^{gc} f_{ab}
\]  
(29)

and

\[
16\pi p^c_B = -F''(B)[\Gamma^c_{bd} f^{bd} - f^{cb} \Gamma^d_{bd}].
\]  
(30)

It is crucial that for $F(B) = B$, $F'(B) = 1$, $F''(B) = 0$ we get that $N^c_{ab}$ has the same form as in case of pure gravity and that $p^c_B = 0$. Further, if we take the trace $f^{ab} N^c_{ab}$ we obtain

\[
16\pi f^{ab} N^c_{ab} = -F'(B)\Gamma^c_{ab} f^{ab} + F'(B)\Gamma^k_{kb} f^{bc} + 3 F''(B)\partial_g B f^{gc}.
\]  
(31)

In case when $F'(B) = 1$ we get the same result as in pure general relativity case while for $F''(B) \neq 0$ we can use previous equation and also definition of canonical momenta $p^c_B$ given in (30) to express $\partial_g B$ as

\[
16\pi f^{ab} N^c_{ab} = 16\pi \frac{F'(B)}{F''(B)} p^c_B + 3 F''(B)\partial_g B f^{gc}
\]  
(32)

so that

\[
\partial_g B = \frac{16\pi}{3 F''(B)} f^{gc} \left( f^{ab} N^c_{ab} - \frac{F'(B)}{F''(B)} p^c_B \right).
\]  
(33)

Inserting this result into (29) we obtain

\[
16\pi N^c_{ab} = -F'(B)\Gamma^c_{ab} + \frac{1}{2} F'(B)(\Gamma^k_{ak} \delta^c_b + \Gamma^k_{bk} \delta^c_a)
+ \frac{16\pi}{6} (f_{ad} f^{mn} N^d_{mn} \delta^c_b + f_{bd} f^{mn} N^d_{mn} \delta^c_a + f^{mn} N^c_{mn} f_{ab})
- \frac{16\pi}{6} \frac{F'(B)}{F''(B)} (f_{ad} p^d_B \delta_b + f_{bd} p^d_B \delta_a + p^c_B f_{ab})
\]  
(34)

that allows us to write

\[
16\pi \tilde{N}^c_{ab} = -F'(B)\Gamma^c_{ab} + \frac{1}{2} F'(B)(\Gamma^k_{ak} \delta^c_b + \Gamma^k_{bk} \delta^c_a).
\]
where
\[
\tilde{N}_{ab}^c = N_{ab}^c - \frac{1}{6} (f_{ab} f^{mn} N_{mn}^d \delta_b^c + f_{bd} f^{mn} N_{mn}^d \delta_a^c + f^{mn} N_{mn}^c f_{ab})
\]
+ \frac{1}{6} \frac{F'(B)}{F''(B)} (f_{ab} p_B^d \delta_b^c + f_{bd} p_B^d \delta_a^c + p_B^c f_{ab}). \tag{35}
\]

As the next step we have to find inverse relation between $\Gamma_{ab}^c$ and $\tilde{N}_{ab}^c$. Following [11] we presume that it has the form
\[
\Gamma_{ab}^c = a \tilde{N}_{ab}^c + b (\tilde{N}_{ad}^d \delta_b^c + \tilde{N}_{bd}^d \delta_a^c), \tag{36}
\]
where $a, b$ are unknown coefficients which we determine after inserting this ansatz into (35). Explicitly we get
\[
\frac{16\pi}{F'(B)} \tilde{N}_{ab}^c = -(a \tilde{N}_{ab}^c + b (\tilde{N}_{ad}^d \delta_b^c + \tilde{N}_{bd}^d \delta_a^c))
\]
+ \frac{1}{2} [(a \tilde{N}_{ad}^d + b (4 \tilde{N}_{ak}^k + \tilde{N}_{ak}^k)) \delta_b^c + (a \tilde{N}_{bd}^d + b (\tilde{N}_{bk}^k + 4 \tilde{N}_{bk}^k) \delta_a^c)]. \tag{37}
\]
Comparing left and right side of this expression we obtain following equations
\[
a = -\frac{16\pi}{F'(B)}, \quad -b + \frac{(a + 5b)}{2} = 0 \tag{38}
\]
that has solution
\[
b = -\frac{a}{3}. \tag{39}
\]
Using these values of $a$ and $b$ we obtain inverse relation
\[
\Gamma_{ba}^c = -\frac{16\pi}{F'(B)} \tilde{N}_{ab}^c + \frac{16\pi}{3 F'(B)} (\tilde{N}_{ad}^d \delta_b^c + \tilde{N}_{bd}^d \delta_a^c). \tag{40}
\]
Now the Hamiltonian density has the form
\[
\mathcal{H}^F = \partial_c f_{ab}^c N_{ab}^c + p_B^c \partial_c B - \mathcal{L}_{\text{quad}}^F = (\Gamma_{dc}^d f_{ab}^c - \Gamma_{cd}^a f_{ab}^d - \Gamma_{dc}^b f_{da}^a) N_{ab}^c
\]
+ $p_B^c \partial_c B - \mathcal{L}_{\text{quad}}^F = \frac{1}{16\pi} \frac{F'(B) [\Gamma_{dk}^h \Gamma_{gh}^k f_{gd} - \Gamma_{fk}^h \Gamma_{gh}^k f_{gh}]}{f_{ab}^c (\Gamma_{ab}^b - \Gamma_{ab}^f f_{ab})} - \frac{1}{16\pi} \sqrt{-f} [F(B) - F'(B) B]. \tag{41}
\]
using
\[ \partial_c f^{ab} = \partial_c \sqrt{-g} g^{ab} + \sqrt{-g} \partial_c g^{ab} = \Gamma^d_{dc} f^{ab} - \Gamma^a_{cd} f^{db} - \Gamma^b_{dc} f^{da}, \]

(42)

where we used the fact that \( \nabla_c \sqrt{-g} = 0 \) implies
\[ \partial_c \sqrt{-g} = \Gamma^d_{dc} \sqrt{-g}. \]

(43)

In the same way the condition \( \nabla_c g^{ab} = 0 \) implies
\[ \partial_c g^{ab} = -(\Gamma^a_{cd} g^{db} + \Gamma^b_{cd} g^{da}). \]

(44)

Then inserting (40) into (41) we obtain Hamiltonian density in the form
\[ H_f = \frac{16 \pi}{F'(B)} N^h_{dk} f^{gd} N^k_{gh} - \frac{16 \pi}{3 F'(B)} N^m_{mk} f^{kg} N^d_{dg} \]
\[ + \frac{16 \pi}{3 F''(B)} P_B f_{gc}(f^{ab} N^c_{ab} - \frac{F'(B)}{F''(B)} P_B C^c) - \frac{1}{16 \pi} \sqrt{-f}[F(B) - F'(B) B]. \]

(45)

Finally in order to complete Hamiltonian analysis we have to insert explicit form of \( N^c_{ab} \) given in (35) to (45). In fact, after some tedious calculations we find that the Hamiltonian density of \( F(R) \) gravity has the form
\[ H^E = \frac{16 \pi}{F'} \left[ N^h_{dk} f^{dg} N^k_{gh} - \frac{1}{3} N^m_{mk} f^{kg} N^n_{ng} \right] \]
\[ - \frac{16 \pi}{6} F'[\frac{1}{F''} P^a - \frac{1}{F'} N^a_{mn} f^{mn}] f_{ab} \left( \frac{1}{F''} P^b - \frac{1}{F'} f^{rs} N^b_{rs} \right) \]
\[ - \frac{1}{16 \pi} \sqrt{-f}[F(B) - F'(B) B]. \]

(46)

This is the final form of the covariant Hamiltonian density for \( F(R) \) gravity. Observe that the expression on the first line corresponds to the Hamiltonian density for ordinary Einstein-Hilbert action when we set \( F' = 1 \). Further, it is instructive to compare this density with the standard Hamiltonian treatment of \( F(R) \) gravity that is based on \( 3 + 1 \) splitting of target space-time, see for example [21]. In \( 3 + 1 \) decomposition metric decomposes into \( N, N_i, g_{ij} \) components where \( N \) and \( N_i \) are non-dynamical variables while the dynamical ones are \( g_{ij} \) only. As a consequence of non-dynamical nature of \( N, N_i \) that Hamiltonian is sum of four first class constraints. Of course, the field \( B \) is dynamical as well. However in case of covariant canonical formalism there is no split of space-time metric and we also showed that all partial derivatives of \( f^{ab} \) and \( B \) can be inverted and expressed as functions of conjugate covariant momenta.

As a result there are no primary constraints in the covariant Hamiltonian treatment of \( F(R) \) gravity and hence which is very nice due to the limited understanding of the treatment of constraints in covariant Hamiltonian formalism, for more details, see footnote (6).
In order to find corresponding equations of motion we begin with the canonical form of the action

\[ S = \int d^4x \left( N^{ac} \delta_c f^{ab} + p^c_B \delta_c B - \mathcal{H}^F \right) \]  

so that its variation has the form

\[ \delta S = \int d^4x \left( \delta N^{ac} \delta_c f^{ab} + N^{ac} \delta_c \delta f^{ab} + \delta p^c_B \delta_c B + p^c_B \delta_c \delta B \right) \]

that implies following equations of motion

\[ \partial_c f^{ab} = \frac{\delta \mathcal{H}^F}{\delta N^{ac}} = \frac{16\pi}{F'} \left[ f^{bg} N^{a}_{gc} + f^{ag} N^{b}_{gc} \right] \]

\[ \partial_c N^{ac} = \frac{\delta \mathcal{H}^F}{\delta f^{ab}} = -8\pi F' \left[ N^{h}_{ka} N^{k}_{bh} + N^{h}_{kb} N^{k}_{ah} \right] + \frac{16\pi}{3} f^{ab} f^{ch} W^h, \]

\[ \partial_a B = \frac{\delta \mathcal{H}^F}{\delta p^a_B} = -16\pi F' \left[ f^{ab} W^a \right] + \frac{16\pi}{3} f^{ab} \left( N^{cd} f^{cd} - F^{c} F^{c}_{ab} \right) \]

that agrees with (33). Then we determine equations of motion for \( p^c_B \)

\[ \partial_c p^c_B = -\frac{\delta \mathcal{H}^F}{\delta B} = \frac{16\pi}{F'^2} \left[ N^{h}_{kd} f^{d} f^{k}_{gh} - \frac{1}{3} N^{mn}_{mk} f^{kg} N^{n}_{gk} \right] + \frac{16\pi}{6} F'' W^a f^{ab} W^b + \frac{16\pi}{3} F' \left[ -\frac{F'''}{F'^2} p^a + \frac{F''}{F'^2} N^{ia}_{mn} f^{mn} \right] \]

where we defined \( W^a \) as

\[ W^a = \frac{1}{F''} p^a - \frac{1}{F'} N^{ad}_{c} f^{cd} \]

We would like to show that these equations of motion are equivalent to Lagrangian equations of motion of \( F(R) \)-gravity. Note that \( N^{ac}_{ab} \) and \( f^{ab} \) are independent variables. To proceed further we introduce metric}
\( g_{ab} = \sqrt{-f} f_{ab} \). We can also introduce new independent variable \( \Gamma^c_{ab} \) through the relation

\[
16\pi N^c_{ab} = -F' \Gamma^c_{ab} + \frac{1}{2} F'(\Gamma^k_{ak} \delta^c_b + \Gamma^k_{bk} \delta^c_a) \\
\quad - \frac{16\pi}{6} F' f_{ad} W^d \delta^c_b - \frac{16\pi}{6} F' f_{bd} W^d \delta^c_a - \frac{16\pi}{6} F' f_{ab} W^c.
\]

(54)

Further we contract (49) with \( g_{ab} \) and we obtain

\[
\sqrt{-f} f_{ab} \partial_c f^{ab} = \left( \frac{64\pi}{3F'} N^a_{ac} + \frac{64\pi}{3} f_{ch} W^h \right) \sqrt{-f}
\]

(55)

that can be written as

\[
\partial_c \sqrt{-g} = \left( \frac{32\pi}{3F'} N^a_{ac} + \frac{32\pi}{3} f_{ch} W^h \right) \sqrt{-g} = \Gamma^b_{bc} \sqrt{-g}
\]

(56)

using (54). Note that this is the same result as in case of ordinary general relativity action that was found in [11]. Then, we would like to calculate \( \sqrt{-g} \nabla_c g^{ab} \) where covariant derivative is defined with the help of the Christoffel symbols \( \Gamma^c_{ab} \) that still do not depend on \( g^{ab} \). Explicitly, we have

\[
\sqrt{-g} \nabla_c g^{ab} = \partial_c f^{ab} - \partial_c \sqrt{-g} g^{ab} + \Gamma^a_{cd} g^{db} \sqrt{-g} + \Gamma^b_{cd} g^{da} \sqrt{-g}.
\]

(57)

As the next we use inverse relation between \( \Gamma^c_{ab} \) and \( N^c_{ab} \) in the form

\[
\Gamma^c_{ba} = -\frac{16\pi}{F'} \left( N^c_{ab} + \frac{F'}{6} (f_{ad} W^d \delta^c_b + f_{bd} W^d \delta^c_a + f_{ab} W^c) \right) \\
\quad + \frac{16\pi}{3F'} \left( N^d_{ad} \delta^c_b + N^d_{bd} \delta^c_a + \frac{F'}{6} (f_{bd} W^d \delta^c_a + f_{ad} W^d \delta^c_b) \right)
\]

(58)

using (35) and (40). Inserting this result into (57) we obtain that

\[
\sqrt{-g} \nabla_c g^{ab} = 0
\]

(59)

as a consequence of (49). This is well known relation that allow us to express \( \Gamma^a_{bc} \) as functions of metric and their partial derivatives.

Further, from (51) we express \( p^c_B \) as functions of \( N^c_{ab} \) and \( \partial_c B \) that, using (29), means that \( p^c_B \) is function of \( \partial_c B \) and \( \Gamma^a_{bc} \). Then inserting this relation into (52) we obtain the equation

\[
2 \partial_c [F''(B) \hat{Q}^b_{a cd} \Gamma^a_{bd}] + 2 F''(B) \hat{Q}^b_{a cd} \Gamma^a_{am} \Gamma^a_{bc} \\
- 2 F''' \partial_c B \hat{Q}^b_{a cd} \Gamma^a_{bd} - \sqrt{-f} F''(B) B \\
= 2 F''(B) \partial_c [\hat{Q}^b_{a cd} \Gamma^a_{bd}] + 2 F''(B) \hat{Q}^b_{a cd} \Gamma^a_{am} \Gamma^a_{bc} - \sqrt{-f} F''(B) B = 0.
\]

(60)
where

\[ \tilde{\mathcal{Q}}_{n}^{mnl} \equiv \frac{1}{2}(f^{ml} \delta_{n}^{k} - f^{mn} \delta_{l}^{k}). \]  

(61)

Note that the equation (60) agrees with the equation of motion that arises from the variation of \( \mathcal{L}_{\text{quadr}}^{F} \) given in (12) with respect to \( B \). Further, we can explicitly calculate the partial derivative in the first term on the second line and using (49) we can express the equation (60) in the form

\[ \sqrt{-f} F''(B)(R - B) = 0 \]  

(62)

that has exactly the same form as the equation (5) that implies \( B = R \).

Finally we return to the equations of motion for \( N_{ab}^{c} \) that are given in (50). We can express \( N_{ab}^{c} \) using \( \Gamma_{bc}^{a} \) and insert it into equations (50) where we also use (51) in order to replace \( W_{b}^{c} \) with \( \partial_{c} B \). Then after some manipulations we obtain equations of motion that follow from the variation of (12) with respect to \( g_{ab} \) that are equivalent to equations of motion (6).

Further, the boundary term for \( F(R) \)–gravity has the form

\[ \mathcal{L}_{\text{surf}}^{F} = -\partial_{c} \left[ \frac{F'}{F_{n}^{n}} p_{B}^{c} \right] \]  

(63)

which shows an importance of the field \( B \) and corresponding conjugate momenta \( p_{B}^{c} \).

On the other hand if we presume that the solution of the equation of motion for \( B \) is \( B = \text{const} \) we find that \( p_{B}^{b} \) is equal \( -\frac{F''}{F_{n}^{n}} N_{mn}^{b} f_{mn} \). Inserting this result into (63) we find that it is equal to

\[ \mathcal{L}_{\text{surf}}^{F} = -\partial_{c} [N_{mn}^{c} f_{mn}] \]  

(64)

which is the same as in case of pure gravity. Certainly it would be nice to study consequence of this result for thermodynamics aspects of \( F(R) \)–gravity.

This paper was devoted to the covariant canonical formulation of \( F(R) \)–gravity. As far as we know this is first time when such an analysis was performed. In some way it is related to the study of Lanczos-Lovelock gravity that was given in [23,24] however \( F(R) \)–gravity is simpler since the theory is defined with general function of curvature \( R \) while Lanczos-Lovelock Lagrangians are more complicated even if they lead to the equations of motions with derivatives up to second order. We also mean that this analysis is the first step in the analysis of relation between \( F(R) \)–gravities and non-relativistic thermodynamics of space-time as was suggested in [25,26]. This relation is currently under investigation.

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