Hidden Symmetry of the CKM and Neutrino Mapping Matrices

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Abstract

We propose that the smallness of the light quark masses is related to the smallness of the $T$ (i.e. $CP$) violation in hadronic weak interactions. Accordingly, for each of the two quark sectors ("upper" and "lower") we construct a $3 \times 3$ mass matrix in a bases of unobserved quark states, such that the "upper" and "lower" basis states correspond exactly via the $W^\pm$ transitions in the weak interaction. In the zeroth approximation of our formulation, we assume $T$ conservation by making all matrix elements real. In addition, we impose a "hidden symmetry" (invariance under simultaneous translations of all three basis quark states in each sector), which ensures a zero mass eigenstate in each sector.

Next, we simultaneously break the hidden symmetry and $T$ invariance by introducing a phase factor $e^{i\chi}$ in the interaction for each sector. The Jarlskog invariant $J_{CKM}$, as well as the light quark masses are evaluated in terms of the parameters of the model. Comparing formulas, we find that most unknown factors drop out, resulting in a simple relation with $J_{CKM} = (m_d m_s/m_b^2)^{1/2} A \lambda^3 \cos \frac{1}{2} \chi$, to leading order in $\chi$ and $m_s/m_b$, with $A$, $\lambda$ the Wolfenstein parameters. (Because of the large top quark mass, the contribution from upper quark sector can be neglected.) Setting $J_{CKM} = 3.08 \times 10^{-5}$, $m_b = 4.7 GeV$ (1s mass), $m_s = 95 MeV$, $A = 0.818$ and $\lambda = 0.227$, we find $m_d \cos^2 \frac{1}{2} \chi \approx 2.4 MeV$, consistent with the accepted value $m_d = 3 - 7 MeV$.

We make a parallel proposal for the lepton sectors. With the hidden symmetry and in the approximation of $T$ invariance, both the masses of $e$ and $\nu_1$ are zero. The neutrino mapping matrix $V_\nu$ is shown to be of the same Harrison-Scott form which is in agreement with experiments. We also examine the correction due to $T$ violation, and evaluate the corresponding Jarlskog invariant $J_\nu$.

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1. Introduction

In a recent paper[1], we postulate a new symmetry of the neutrino mass matrix in terms of the field operators $\nu_e$, $\nu_\mu$ and $\nu_\tau$. This symmetry enables us to derive the Harrison-Scott form[2,3] of the neutrino mapping matrix $V_\nu$. However, the formalism has a built-in asymmetry between the charged leptons and the neutral ones. In this paper, we modify the symmetry introduced in [1], as that $e$, $\mu$, $\tau$ and $\nu_1$, $\nu_2$, $\nu_3$ are now set on a similar basis. Furthermore, the new symmetry can also be extended to quarks $d$, $s$, $b$ and $u$, $c$, $t$. For clarity, we first discuss how the new symmetry, called hidden symmetry, can be realized in the quark sectors leading to the CKM matrix $U_{CKM}$. Next, we discuss its application to leptons, resulting again in the Harrison-Scott form of the neutrino mapping matrix $V_\nu$.

In the quark sector, let $q_i(\downarrow)$ and $q_i(\uparrow)$ be the quark states ”diagonal” in $W^\pm$ transitions:

$$q_i(\downarrow) \Leftrightarrow q_i(\uparrow) + W^-$$

and

$$q_i(\uparrow) \Leftrightarrow q_i(\downarrow) + W^+$$

with $i = 1$, $2$, $3$. Their electric charges in units of $e$ are $-\frac{1}{3}$ for $q_i(\downarrow)$, and $+\frac{2}{3}$ for $q_i(\uparrow)$. However, these are not the observed mass eigenstates $d$, $s$, $b$ and $u$, $c$, $t$. Likewise, let $l_i(\downarrow)$ and $l_i(\uparrow)$ be the lepton states ”diagonal” in the corresponding $W^\pm$ transitions:

$$l_i(\downarrow) \Leftrightarrow l_i(\uparrow) + W^-$$

and

$$l_i(\uparrow) \Leftrightarrow l_i(\downarrow) + W^+$$

with their electric charge unit $-1$ for $l_i(\downarrow)$, and $0$ for $l_i(\uparrow)$ and $i = 1$, $2$, $3$. Again, neither $l_i(\downarrow)$ nor $l_i(\uparrow)$ are the mass eigenstates $e$, $\mu$, $\tau$ and $\nu_1$, $\nu_2$, $\nu_3$. Thus, for each of these four triplets

$$\{q_1(\downarrow), q_2(\downarrow), q_3(\downarrow)\}, \quad \{q_1(\uparrow), q_2(\uparrow), q_3(\uparrow)\} \quad \{l_1(\downarrow), l_2(\downarrow), l_3(\downarrow)\} \quad \text{and} \quad \{l_1(\uparrow), l_2(\uparrow), l_3(\uparrow)\}$$

(1.3)
there exists a separate $3 \times 3$ mass matrix, denoted by

$$M(q_1), M(q_\uparrow), M(l_\downarrow) \text{ and } M(l_\uparrow)$$ (1.4)

respectively. As we shall discuss, these mass matrices satisfy a common set of rules due to hidden symmetry, leading to a unifying formalism of both the CKM matrix $U_{CKM}$ and the neutrino mapping matrix $V_\nu$.

In what follows, we begin our discussion in the approximation assuming time reversal invariance $T$. Thus, the mass matrix $M(q_1), M(q_\uparrow), M(l_\downarrow)$ and $M(l_\uparrow)$ are all $3 \times 3$ real symmetric matrices. The corresponding mass operators are

$$\mathcal{M}(q_1) = \left(\bar{q}_1(\downarrow), \bar{q}_2(\downarrow), \bar{q}_3(\downarrow)\right)M(q_1) \begin{pmatrix} q_1(\downarrow) \\ q_2(\downarrow) \\ q_3(\downarrow) \end{pmatrix},$$ (1.5)

$$\mathcal{M}(q_\uparrow) = \left(\bar{q}_1(\uparrow), \bar{q}_2(\uparrow), \bar{q}_3(\uparrow)\right)M(q_\uparrow) \begin{pmatrix} q_1(\uparrow) \\ q_2(\uparrow) \\ q_3(\uparrow) \end{pmatrix},$$ (1.6)

$$\mathcal{M}(l_\downarrow) = \left(\bar{l}_1(\downarrow), \bar{l}_2(\downarrow), \bar{l}_3(\downarrow)\right)M(l_\downarrow) \begin{pmatrix} l_1(\downarrow) \\ l_2(\downarrow) \\ l_3(\downarrow) \end{pmatrix},$$ (1.7)

and

$$\mathcal{M}(l_\uparrow) = \left(\bar{l}_1(\uparrow), \bar{l}_2(\uparrow), \bar{l}_3(\uparrow)\right)M(l_\uparrow) \begin{pmatrix} l_1(\uparrow) \\ l_2(\uparrow) \\ l_3(\uparrow) \end{pmatrix}.$$ (1.8)

In (1.5), $q_i(\downarrow)$ and $\bar{q}_i(\downarrow)$ are related to the corresponding Dirac field operators $\psi(q_i(\downarrow))$ and its Hermitian conjugate $\psi^\dagger(q_i(\downarrow))$ by

$$q_i(\downarrow) = \psi(q_i(\downarrow)) \text{ and } \bar{q}_i(\downarrow) = \psi^\dagger(q_i(\downarrow))\gamma_4.$$ (1.9)

Likewise, in (1.6)-(1.8) $q_i(\uparrow), \bar{q}_i(\uparrow), l_i(\downarrow)$, etc. are similarly related to their corresponding Dirac field operators. We assume that each of these four mass operators (1.5)-(1.8) satisfies a hidden symmetry with $\mathcal{M}(q_1)$ invariant under the transformation

$$q_1(\downarrow) \rightarrow q_1(\downarrow) + z, \quad q_2(\downarrow) \rightarrow q_2(\downarrow) + \eta_1 z \text{ and } q_3(\downarrow) \rightarrow q_3(\downarrow) + \xi_1\eta_1 z.$$ (1.10)
where $z$ is a space-time independent constant element of the Grassmann algebra anticommuting with the Dirac field operators, and $\xi$, $\eta$ are $c$-numbers. It will be shown in the next section that (1.10) implies a zero down-quark mass in the absence of $T$ violation. Similar symmetries are also assumed for other triplets $\{q_i(\uparrow)\}$, $\{l_i(\downarrow)\}$ and $\{l_i(\uparrow)\}$. Thus, we correlate the nearly zero masses of $d$, $u$, $e$ and $\nu_1$ with $T$ invariance and the new symmetry.

In Section 3, we derive the form of CKM matrix in the same zeroth approximation of $T$ invariance. The violation of $T$ invariance will be discussed in Section 4. As will be shown, to the first approximation of small $T$ violation, we derive an interesting formula relating $T$ violating Jarlskog invariant $J$ with quark masses:

$$
J = \left( \frac{m_d m_s}{m_b^2} \right)^\frac{1}{2} A \lambda^3 \cos[\frac{1}{2} \chi_T(\downarrow)] + O\left( \frac{m_u m_c}{m_t^2} \right)^\frac{1}{2}
$$

(1.11)

where $A \approx 0.818$, $\lambda \approx 0.2272$ are the Wolfenstein parameters. Using the experimental values $J \approx 3.08 \cdot 10^{-5}$, $m_s \approx 95\text{MeV}$ and $m_b \approx 4.7\text{GeV}$ (1s mass), we find

$$
m_d \cos^2[\frac{1}{2} \chi_T(\downarrow)] \approx 2.4\text{MeV},
$$

(1.12)

where $\chi_T(\downarrow)$ is the $T$-violating phase in the $\downarrow$ quark sector. Since $\cos\frac{1}{2} \chi_T(\downarrow) \leq 1$, we have

$$
m_d \geq 2.4\text{MeV}
$$

(1.13)

consistent with the range $m_d \approx 3$ to $7\text{MeV}$ quoted by the particle data group.

In section 5 and 6, we discuss lepton sectors. As in Ref.[1], we show how the new hidden symmetry can also lead to the Harrison-Scott form of the neutrino-mapping matrix $V_\nu$ in agreement with experiments. The Jarlskog invariant in the lepton sector is also calculated to the lowest order of the $T$ violating interaction.
2. Hidden Symmetry

Consider first the \( \{q_i(\downarrow)\} \) sector. In the approximation of \( T \) invariance, the \( 3 \times 3 \) matrix in (1.5) becomes a real symmetric matrix \( M_0(q_i) \) characterized by six real parameters: three diagonal and three off-diagonal elements. We propose to represent the corresponding mass operator \( M(q_i) \) by

\[
M_0(q_i) = \alpha_i |q_3(\downarrow) - \xi_i q_2(\downarrow)|^2 + \beta_i |q_2(\downarrow) - \eta_i q_1(\downarrow)|^2 + \gamma_i |q_1(\downarrow) - \zeta_i q_3(\downarrow)|^2
\]

with also six real parameters \( \alpha_i, \beta_i, \gamma_i, \xi_i, \eta_i \) and \( \zeta_i \). Their relation with the six diagonal and off-diagonal elements of an arbitrary symmetric matrix \( M(q_i) \) is given in the Appendix. We impose the hidden symmetry requirement that \( M_0(q_i) \) be invariant under the transformation (1.10). Substituting (1.10) into (2.1) and requiring the symmetry, we see that these three parameters \( \xi_i, \eta_i \) and \( \zeta_i \) must satisfy

\[
\xi_i \eta_i \zeta_i = 1. \tag{2.2}
\]

The corresponding mass matrix \( M(q_i) \) defined by (1.5) is

\[
M_0(q_i) = \begin{pmatrix}
\gamma + \beta \eta^2 & -\beta \eta & -\gamma \zeta \\
-\beta \eta & \beta + \alpha \xi^2 & -\alpha \xi \\
-\gamma \zeta & -\alpha \xi & \alpha + \gamma \zeta^2
\end{pmatrix}_{\downarrow}
\]

where the suffix \( \downarrow \) on the right hand side indicates that the parameters \( \alpha, \beta, \gamma, \xi, \eta, \zeta \) refer to \( \alpha_i, \beta_i, \gamma_i, \xi_i, \eta_i \) and \( \zeta_i \) respectively. From (2.3), we see that the determinant of \( M_0(q_i) \) is given by

\[
|M_0(q_i)| = \left[ \alpha \beta \gamma (\xi \eta \zeta - 1)^2 \right]_{\downarrow}. \tag{2.4}
\]

Thus, for

\[
(\xi \eta \zeta)_{\downarrow} \equiv \xi_i \eta_i \zeta_i = 1. \tag{2.5}
\]

we have

\[
|M_0(q_i)| = 0. \tag{2.6}
\]

Choose \( \alpha_i, \beta_i \) and \( \gamma_i \) to be all positive. The operator \( M_0(q_i) \) is then positive; condition (2.5) implies the smallest eigenvalue of \( M(q_i) \), the down quark mass, to be zero; i.e.,

\[
m_d = 0 \tag{2.7}
\]
on account of the hidden symmetry requirement (1.10) and the approximation of $T$ invariance.

This result can also be seen directly from the symmetry requirement (1.10). In the three-dimensional space of coordinate axes $q_1(\downarrow)$, $q_2(\downarrow)$ and $q_3(\downarrow)$, the transformation (1.10) represents a translation along the direction parallel to the three dimensional unit vector

$$
\epsilon_\downarrow \propto \begin{pmatrix} 1 \\ \eta_\downarrow \\ \xi_\downarrow \eta_\downarrow \end{pmatrix}.
$$

The assumed invariance under (1.10) is identical to the invariance of $M_0(q_\downarrow)$ under a translation along the vector $\epsilon_\downarrow$; thus, $\epsilon_\downarrow$ is an eigenvector of the corresponding mass matrix $M_0(q_\downarrow)$, with zero eigenvalue (i.e., zero mass).

Likewise, under the transformation $\downarrow \rightarrow \uparrow$,

$$(\alpha_\downarrow, \beta_\downarrow, \gamma_\downarrow) \rightarrow (\alpha_\uparrow, \beta_\uparrow, \gamma_\uparrow)$$

and

$$(\xi_\downarrow, \eta_\downarrow, \zeta_\downarrow) \rightarrow (\xi_\uparrow, \eta_\uparrow, \zeta_\uparrow)$$

we have

$$M_0(q_\downarrow) \rightarrow M_0(q_\uparrow) \text{ and } M_0(q_\downarrow) \rightarrow M_0(q_\uparrow).$$

As in (1.10), the hidden symmetry

$$q_1(\uparrow) \rightarrow q_1(\uparrow) + z, \quad q_2(\uparrow) \rightarrow q_2(\uparrow) + \eta_\uparrow z \quad \text{and} \quad q_3(\uparrow) \rightarrow q_3(\uparrow) + \xi_\uparrow \eta_\uparrow z$$

implies the corresponding invariance of the mass operator $M_0(q_\uparrow)$, in the approximation of $T$ invariance. Thus, (2.11) implies

$$\xi_\uparrow \eta_\uparrow \zeta_\uparrow = 1$$

and the up quark $u$ to be of zero mass; i.e., with hidden symmetry and $T$ invariance,

$$m_u = 0.$$
3. CKM Matrix (neglecting $T$ violation)

In this section, we discuss the CKM matrix in the same zeroth approximation by neglecting $T$ violation. Let $(U_↓)_0$ and $(U_↑)_0$ be the unitary matrices that diagonalize $M_0(q_↓)$ and $M_0(q_↑)$:

$$(U_↓)_0 M_0(q_↓)(U_↓)_0 = \begin{pmatrix} m_0(d) & 0 & 0 \\ 0 & m_0(s) & 0 \\ 0 & 0 & m_0(b) \end{pmatrix}$$ (3.1)

and

$$(U_↑)_0 M_0(q_↑)(U_↑)_0 = \begin{pmatrix} m_0(u) & 0 & 0 \\ 0 & m_0(c) & 0 \\ 0 & 0 & m_0(t) \end{pmatrix}.$$ (3.2)

The corresponding CKM matrix is given by

$$(U_{CKM})_0 = (U_↑)_0(U_↓)_0.$$ (3.3)

In accordance with (2.7) and (2.13), we have in the notation of (3.1) and (3.2)

$$m_0(d) = m_0(u) = 0.$$ (3.4)

Without $T$ violation, $(U_↓)_0$, $(U_↑)_0$ and $(U_{CKM})_0$ are each a $3 \times 3$ real orthogonal matrix characterized by three real parameters.

In (2.3), the mass matrix $M_0(q_↓)$ has six parameters $\alpha_↓$, $\beta_↓$, $\gamma_↓$, $\xi_↓$, $\eta_↓$ and $\zeta_↓$. With the constraint $\xi_↓ \eta_↓ \zeta_↓ = 1$ in accordance with (2.5), there are still five independent parameters in $M_0(q_↓)$. Together with $M_0(q_↑)$, we have $5 + 5 = 10$ parameters. Assuming that the only observables are the quark masses and the CKM matrix. Since $m_0(d) = m_0(u) = 0$ in this approximation, there are only four nonzero masses $m_0(s)$, $m_0(b)$, $m_0(c)$ and $m_0(t)$. In addition, the CKM matrix with $T$ invariance is characterized by three real parameters; together, there are

$$4 + 3 = 7$$

observables in this approximation. That means among the 10 parameters, there are

$$10 - 7 = 3$$ (3.5)
parameters which are "unphysical". The elimination of these three unphysical
parameters is analogous to the gauge condition in a vector field theory. As we
shall see, a convenient choice is to eliminate two of these three by requiring

\[
\frac{\beta_\downarrow}{\gamma_\downarrow} = \zeta_\downarrow^2 \quad \text{and} \quad \frac{\beta_\uparrow}{\gamma_\uparrow} = \zeta_\uparrow^2. \tag{3.6}
\]

Define four real angular variables \(\theta_\downarrow, \phi_\downarrow\) and \(\theta_\uparrow, \phi_\uparrow\) by

\[
\xi_\downarrow = \tan \phi_\downarrow, \quad \xi_\uparrow = \tan \phi_\uparrow
\]

\[
\eta_\downarrow = \tan \theta_\downarrow \cos \phi_\downarrow \quad \text{and} \quad \eta_\uparrow = \tan \theta_\uparrow \cos \phi_\uparrow. \tag{3.7}
\]

It can be readily verified that with (3.6) the eigenstates of \(M_0(q_\downarrow)\) become
quite simple, given by

\[
\epsilon_\downarrow = \begin{pmatrix} \cos \theta_\downarrow \\ \sin \theta_\downarrow \cos \phi_\downarrow \\ \sin \theta_\downarrow \sin \phi_\downarrow \end{pmatrix} \quad \text{with eigenvalue } \lambda(\epsilon_\downarrow), \tag{3.8}
\]

\[
p_\downarrow = \begin{pmatrix} -\sin \theta_\downarrow \\ \cos \theta_\downarrow \cos \phi_\downarrow \\ \cos \theta_\downarrow \sin \phi_\downarrow \end{pmatrix} \quad \text{with eigenvalue } \lambda(p_\downarrow), \tag{3.9}
\]

and

\[
P_\downarrow = \begin{pmatrix} 0 \\ -\sin \phi_\downarrow \\ \cos \phi_\downarrow \end{pmatrix} \quad \text{with eigenvalue } \lambda(P_\downarrow). \tag{3.10}
\]

Here \(\lambda(\epsilon_\downarrow), \lambda(p_\downarrow)\) and \(\lambda(P_\downarrow)\) are the same 0th order approximation \(m_0(d), m_0(s)\)
and \(m_0(b)\) in (3.1), with

\[
m_0(d) = \lambda(\epsilon_\downarrow) = 0, \tag{3.11}
\]

\[
m_0(s) = \lambda(p_\downarrow) = \beta_\downarrow [1 + \eta_\downarrow^2(1 + \xi_\downarrow^2)] \tag{3.12}
\]

and

\[
m_0(b) = \lambda(P_\downarrow) = \alpha_\downarrow (1 + \xi_\downarrow^2) + \beta_\downarrow. \tag{3.13}
\]

In terms of \(\xi_\downarrow\) and \(\eta_\downarrow\), the statevector \(\epsilon_\downarrow\) satisfies (2.8).
Likewise, the eigenstates of $M_0(q)$ are

$$
\epsilon = \begin{pmatrix}
\cos \theta \\
\sin \theta \cos \phi \\
\sin \theta \sin \phi
\end{pmatrix}
$$

with eigenvalue $\lambda(\epsilon)$, \hspace{1cm} (3.14)

and

$$
p = \begin{pmatrix}
-\sin \theta \\
\cos \theta \cos \phi \\
\cos \theta \sin \phi
\end{pmatrix}
$$

with eigenvalue $\lambda(p)$, \hspace{1cm} (3.15)

where

$$m_0(u) = \lambda(\epsilon) = 0,
$$

$$m_0(c) = \lambda(p) = \beta [1 + \eta^2 (1 + \xi^2)]]
$$

and

$$m_0(t) = \lambda(p) = \alpha(1 + \xi^2) + \beta.
$$

Correspondingly, the $3 \times 3$ unitary matrices $(U_\downarrow)_0$ and $(U_\uparrow)_0$ of (3.1) and (3.2) are given by

$$(U_\downarrow)_0 = (\epsilon, p, P)$$

and

$$(U_\uparrow)_0 = (\epsilon, p, P).$$

Thus, in accordance with (3.3), the corresponding CKM matrix in the same approximation is given by

$$
(U_{CKM})_0 =\begin{pmatrix}
\cos \theta \downarrow \cos \theta \uparrow & -\sin \theta \downarrow \cos \theta \uparrow & \sin \theta \downarrow \sin \phi \\
+ \sin \theta \downarrow \sin \theta \uparrow \cos \phi & + \cos \theta \downarrow \sin \theta \uparrow \cos \phi \\
- \cos \theta \downarrow \sin \theta \uparrow & \sin \theta \downarrow \sin \theta \uparrow & \cos \theta \downarrow \sin \phi \\
+ \sin \theta \downarrow \cos \theta \uparrow \cos \phi & + \cos \theta \downarrow \cos \theta \uparrow \cos \phi \\
- \sin \theta \downarrow \sin \phi & - \cos \theta \downarrow \sin \phi & \cos \phi
\end{pmatrix},
$$

(3.22)
in which
\[ \phi = \phi_\uparrow - \phi_\downarrow. \] (3.23)
Equations (3.6) and (3.23) eliminate the three unphysical variables, as we shall see. Upon comparison with experimental values, we find from (3.22)
\[ \theta_\uparrow - \theta_\downarrow = \text{Cabibbo angle} \] (3.24)
with
\[ \sin(\theta_\uparrow - \theta_\downarrow) \approx \lambda = 0.227. \] (3.25)
By taking the ratio of (1,3) and (2,3) matrix elements of \((U_{CKM})_0\), we estimate \(\theta_\uparrow = O(\lambda)\); likewise, from the corresponding (3,1) and (3,2) matrix elements, \(\theta_\downarrow = O(\lambda)\). Using the (2,3) matrix element, we derive
\[ \sin \phi \approx A\lambda^2 \] (3.26)
with \(A = 0.818\).

We observe that the dependence of \((U_{CKM})_0\) on \(\phi_\downarrow\) and \(\phi_\uparrow\) is only through \(\phi = \phi_\uparrow - \phi_\downarrow\). Thus, \((U_{CKM})_0\) is independent of \(\phi_\uparrow + \phi_\downarrow\), which together with the two conditions given by (3.6) eliminate the 3 unphysical parameters mentioned in (3.5).
4. $T$-Violation

In the approximation of $T$ invariance, by using (2.1) and constraints

$$\xi_1 \eta_1 \zeta_1 = 1 \quad \text{and} \quad \frac{\beta_{\perp}}{\gamma_1} = \zeta_1^2 \quad (4.1)$$

in accordance with (2.5) and (3.6), we find that the mass operator (2.1) can also be written as

$$M_0(q_\downarrow) = \alpha_{\perp} |q_3(\downarrow) - \xi_1 q_2(\downarrow)|^2 + \beta_{\perp} |q_2(\downarrow) - \eta_1 q_1(\downarrow)|^2 + \beta_1 |q_3(\downarrow) - \xi_1 \eta_1 q_1(\downarrow)|^2 \quad (4.2)$$

With $T$ violation, we replace $M_0(q_\downarrow)$ by

$$M(q_\downarrow) = \alpha_{\perp} |q_3(\downarrow) - \xi_1 e^{i \chi_T} q_2(\downarrow)|^2 + \beta_{\perp} |q_2(\downarrow) - \eta_1 q_1(\downarrow)|^2 + \beta_1 |q_3(\downarrow) - \xi_1 \eta_1 q_1(\downarrow)|^2 \quad (4.3)$$

in which

$$\chi_T = \chi_T(\downarrow) \quad (4.4)$$

is the $T$-violating phase factor for the $\downarrow$ quark sector. The corresponding mass matrix defined by (1.5) is given by

$$M(q_\downarrow) = M_0(q_\downarrow) + M_1(q_\downarrow) \quad (4.5)$$

with $M_0(q_\downarrow)$ given by (2.3). Because of (2.5) and the first equation in (3.6), $M_0(q_\downarrow)$ can also be written as

$$M_0(q_\downarrow) = \left( \begin{array}{ccc} \beta_\eta^2 (1 + \xi^2) & -\beta_\eta & -\beta \xi \eta \\ -\beta_\eta & \beta + \alpha \xi^2 & -\alpha \xi \\ -\beta \xi \eta & -\alpha \xi & \alpha + \beta \end{array} \right) \quad (4.6)$$

The $T$ violating term in (4.5) is

$$M_1(q_\downarrow) = \alpha_{\perp} \xi_1 \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-i \chi_T(\downarrow)} \\ 0 & 1 - e^{i \chi_T(\downarrow)} & 0 \end{array} \right) \quad (4.7)$$

Because of $T$ violation, the mass of $d$ quark is not zero and the CKM matrix is unitary but not real.
4.1 $d$ quark mass

In accordance with (3.11)-(3.13), the eigenvalues of $M_0(q_\downarrow)$ are $\lambda(\epsilon_\downarrow) = 0$, $\lambda(p_\downarrow)$ and $\lambda(P_\downarrow)$, whereas those of $M(q_\downarrow)$ are the observed quark masses $m = m_d$, $m_s$ and $m_b$ determined by

$$|M(q_\downarrow) - m| = 0. \quad (4.8)$$

By using (3.11)-(3.13) and (4.5)-(4.7), we find (4.8) to be the cubic equation,

$$m(m - \lambda(p_\downarrow))(m - \lambda(P_\downarrow)) = |M(q_\downarrow)| \quad (4.9)$$

with

$$|M(q_\downarrow)| = 2\alpha_\downarrow\beta_\downarrow^2\xi^2_\downarrow \eta^2_\downarrow (1 - \cos \chi_T(\downarrow)). \quad (4.10)$$

Since in accordance with (3.12)-(3.13), $\lambda(p_\downarrow) = m_0(s)$ and $\lambda(P_\downarrow) = m_0(b)$ are the zeroth order values of $m_s$ and $m_b$, both are $\gg m_d$. From (3.7) and (3.12)-(3.13),

$$m_0(s) = \lambda(p_\downarrow) = \beta_\downarrow \sec^2 \theta_\downarrow$$

and

$$m_0(b) = \lambda(P_\downarrow) = \alpha_\downarrow \sec^2 \phi_\downarrow + \beta_\downarrow. \quad (4.11)$$

By setting $m = m_d$, $m_s$ and $m_b$ respectively in (4.9), we have

$$m_d = [(m_0(s) - m_d)(m_0(b) - m_d)]^{-1} |M(q_\downarrow)|, \quad (4.12)$$

$$m_s - m_0(s) = -[m_s(m_0(b) - m_s)]^{-1} |M(q_\downarrow)| \quad (4.13)$$

and

$$m_b - m_0(b) = [m_b(m_b - m_0(s))]^{-1} |M(q_\downarrow)|. \quad (4.14)$$

Thus, neglecting corrections $O(m_d/m_s)$ and $O(m_d/m_b)$, we find from (4.12)-(4.14)

$$m_d \cong [m_sm_b]^{-1} |M(q_\downarrow)|, \quad (4.15)$$

$$m_s - m_0(s) \cong -[m_s(m_b - m_s)]^{-1} |M(q_\downarrow)| \quad (4.16)$$

and

$$m_b - m_0(b) \cong [m_b(m_b - m_s)]^{-1} |M(q_\downarrow)|. \quad (4.17)$$

Likewise, (4.11) leads to

$$\beta_\downarrow \cong m_s \cos^2 \theta_\downarrow, \quad (4.18)$$
\[
\alpha_\downarrow \cong (m_b - m_s \cos^2 \theta_\downarrow) \cos^2 \phi_\downarrow
\]  
(4.19)

and
\[
m_d \cong 2m_s(1 - \frac{m_s}{m_b} \cos^2 \theta_\downarrow) \sin^2 \theta_\downarrow \cos^2 \theta_\downarrow \sin^2 \phi_\downarrow \cos^2 \phi_\downarrow (1 - \cos \chi_T(\downarrow)).
\]  
(4.20)

In (4.20), we may further neglect \(m_s/m_b\) as compared to 1; this yields
\[
m_d \cong 2m_s \sin^2 \theta_\downarrow \cos^2 \theta_\downarrow \sin^2 \phi_\downarrow \cos^2 \phi_\downarrow (1 - \cos \chi_T(\downarrow)).
\]  
(4.21)

When \(\chi_T(\downarrow) = 0\), we have \(m_d = 0\).

4.2 Eigenstates of \(M(q_\downarrow) = M_0(q_\downarrow) + M_1(q_\downarrow)\)

Let \(d, s, b\) be the normalized eigenstates of \(M(q_\downarrow)\), with
\[
M(q_\downarrow)|d\rangle = m_d|d\rangle
\]
\[
M(q_\downarrow)|s\rangle = m_s|s\rangle
\]  
(4.22)
\[
M(q_\downarrow)|b\rangle = m_b|b\rangle.
\]

Throughout the paper, the \(3 \times 1\) normalized state vectors \(|d\rangle, |s\rangle, |b\rangle\) may be denoted simply by \(d, s, b\) as well. Likewise, the states \(\epsilon_\downarrow, p_\downarrow\) and \(P_\downarrow\) of (3.8)-(3.10) may also be denoted by \(|\epsilon_\downarrow\rangle, |p_\downarrow\rangle\) and \(|P_\downarrow\rangle\). Introduce the perturbation matrix
\[
g_\downarrow \cong (U_\downarrow)_0^\dagger M_1(q_\downarrow)(U_\downarrow)_0
\]  
(4.23)

by using \(\epsilon_\downarrow, p_\downarrow\) and \(P_\downarrow\) as base vectors, with \((U_\downarrow)_0\) given by (3.20). To the lowest order in \(\sin \chi_T(\downarrow)\),
\[
g \equiv g_\downarrow = i \alpha_\downarrow \xi_\downarrow \sin \chi_T(\downarrow) \begin{pmatrix} 0 & 0 & \sin \theta_\downarrow \\ 0 & 0 & \cos \theta_\downarrow \\ -\sin \theta_\downarrow & \cos \theta_\downarrow & 0 \end{pmatrix} + O(\chi_T^2). \quad (4.24)
\]

The corresponding eigenstates \(d, s, b\) to the first order in \(\sin \chi_T(\downarrow)\) are given by
\[
|d\rangle = |\epsilon_\downarrow\rangle - \frac{1}{\lambda(P_\downarrow)} g_{pe}|P_\downarrow\rangle
\]
\[
|s\rangle = |p_\downarrow\rangle + \frac{1}{m_s - \lambda(P_\downarrow)} g_{pp}|P_\downarrow\rangle
\]  
(4.25)
and

\[ |b\rangle = |P_\uparrow\rangle + \frac{1}{m_b} g_{\epsilon \rho} |\epsilon_\downarrow\rangle + \frac{1}{m_b - \lambda (p_\downarrow)} g_{p \rho} |p_\downarrow\rangle \]

where

\[ g_{\epsilon \rho} = g^*_{\epsilon \rho} = i\alpha \xi_\downarrow \sin \theta \sin \chi_T(\downarrow) \]

and

\[ g_{p \rho} = g^*_{p \rho} = i\alpha \xi_\downarrow \cos \theta \sin \chi_T(\downarrow) \]

are the first order nonzero matrix elements of \( g_\downarrow \) in accordance with (4.24).

### 4.3 CKM Matrix and Jarlskog Invariant

Anticipating that the Jarlskog Invariant \( J \) in this model is dominated by the \( \downarrow \) quark sector because of

\[ \frac{m_u m_c}{m_t^2} \ll \frac{m_d m_s}{m_b^2} \]

in accordance with (1.11), our discussions can be much simplified by setting the \( T \)-violating phase

\[ \chi_T(\uparrow) = 0 \]

as an approximation. In this case,

\[ |u\rangle \cong |\epsilon_\uparrow\rangle, \quad |c\rangle \cong |p_\uparrow\rangle \quad \text{and} \quad |t\rangle \cong |P_\uparrow\rangle \]

and

\[ U_\uparrow \cong (U_\uparrow)_0 = (\epsilon_\uparrow, \ p_\uparrow, \ P_\uparrow). \]

The corresponding CKM matrix is given by

\[ U_{CKM} = U_\uparrow^\dagger U_\downarrow, \]

with its matrix elements in this approximation given by

\[
\begin{align*}
U_{ud} &= u^\dagger d = \epsilon^\dagger_\downarrow \epsilon_\uparrow + i(\alpha \xi \sin \theta \sin \chi_T)_\downarrow \frac{\epsilon^\dagger_\downarrow P_\downarrow}{m_b}, \\
U_{cd} &= c^\dagger d = p^\dagger_\downarrow \epsilon_\uparrow + i(\alpha \xi \sin \theta \sin \chi_T)_\downarrow \frac{p^\dagger_\downarrow P_\downarrow}{m_b}, \\
U_{td} &= t^\dagger d = P^\dagger_\downarrow \epsilon_\uparrow + i(\alpha \xi \sin \theta \sin \chi_T)_\downarrow \frac{P^\dagger_\downarrow P_\downarrow}{m_b}, \\
U_{us} &= u^\dagger s = \epsilon^\dagger_\downarrow p_\uparrow + i(\alpha \xi \cos \theta \sin \chi_T)_\downarrow \frac{\epsilon^\dagger_\downarrow P_\downarrow}{m_b - m_s}, \\
U_{cs} &= c^\dagger s = p^\dagger_\downarrow p_\uparrow + i(\alpha \xi \cos \theta \sin \chi_T)_\downarrow \frac{p^\dagger_\downarrow P_\downarrow}{m_b - m_s}, \\
U_{ts} &= t^\dagger s = P^\dagger_\downarrow p_\uparrow + i(\alpha \xi \cos \theta \sin \chi_T)_\downarrow \frac{P^\dagger_\downarrow P_\downarrow}{m_b - m_s},
\end{align*}
\]
etc., in which $\epsilon^\dagger \epsilon$, $p^\dagger \epsilon$, etc. are given by the approximate matrix elements in $(U_{CKM})_0$ of (3.22).

Define

$$S_1 \equiv U^*_{ud} U_{us}$$
$$S_2 \equiv U^*_{cd} U_{cs}$$
and

$$S_3 \equiv U^*_{td} U_{ts}.$$  

We have

$$S_1 + S_2 + S_3 = 0$$

and the Jarlskog Invariant

$$J = ImS_1^* S_2 = ImS_2^* S_3 = ImS_3^* S_1.$$  

Assume $\theta^\dagger$ and $\theta_\downarrow$ are all small and $O(\lambda)$, with $\lambda$ given by (3.25). To the lowest order in powers of $\lambda = 0.227$ and of $m_s/m_b$, we find

$$J \simeq \frac{m_s}{m_b} \sin(\theta^\dagger - \theta_\downarrow) \sin \theta_\downarrow \cos \theta^\dagger \sin \phi^\dagger \cos \phi \sin \phi \sin \chi_T$$

where

$$\phi = \phi^\dagger - \phi_\downarrow.$$  

Combining the square root of (4.21) with (4.35), we derive, to the accuracy of the calculated order,

$$J = \left( \frac{m_d m_s}{m_b^2} \right)^{1/2} \sin(\theta^\dagger - \theta_\downarrow) \sin \phi \cos[\frac{1}{2} \chi_T(\downarrow)].$$

By using (3.25)-(3.26), we derive (1.11) for $J$, which is consistent with all available data. As the time reversal violating phase $\chi_T(\downarrow) \to 0$, both $J$ and $(m_d m_s/m_b^2)^{1/2}$ approach zero; their ratio remains fixed by the $T$-conserving elements of the CKM matrix:

$$\sin(\theta^\dagger - \theta_\downarrow) \cdot \sin \phi = A\lambda^2 \cdot \lambda.$$  

It is satisfying that this limiting value is consistent with available experimental data, as shown by (1.11)-(1.13).
5. Lepton Sectors (neglecting $T$ violations)

The application of hidden symmetry to the lepton sectors will be examined in this and the following sections.

5.1 General Discussion

The lepton mass operators are given by (1.7)-(1.8). In the zeroth approximation of $T$ invariance, these operators can be written as

$$M_0(l_\downarrow) = a_\downarrow |l_3(\downarrow) - \kappa_\downarrow l_2(\downarrow)|^2 + b_\downarrow |l_2(\downarrow) - \rho_\downarrow l_1(\downarrow)|^2 + c_\downarrow |l_1(\downarrow) - \sigma_\downarrow l_3(\downarrow)|^2$$  (5.1)

and

$$M_0(l_\uparrow) = a_\uparrow |l_3(\uparrow) - \kappa_\uparrow l_2(\uparrow)|^2 + b_\uparrow |l_2(\uparrow) - \rho_\uparrow l_1(\uparrow)|^2 + c_\uparrow |l_1(\uparrow) - \sigma_\uparrow l_3(\uparrow)|^2$$  (5.2)

where the twelve parameters $a_\uparrow, a_\downarrow, \kappa_\uparrow, \kappa_\downarrow, b_\uparrow, b_\downarrow, \cdots$ are all real, with at least six of them $a_\uparrow, a_\downarrow, b_\uparrow, b_\downarrow, c_\downarrow, c_\uparrow$ positive. As in (2.2) and (3.6), we impose

$$\kappa_\downarrow \rho_\downarrow \sigma_\downarrow = \kappa_\uparrow \rho_\uparrow \sigma_\uparrow = 1,$$

$$\frac{b_\downarrow}{c_\downarrow} = \sigma_\downarrow^2 \text{ and } \frac{b_\uparrow}{c_\uparrow} = \sigma_\uparrow^2.$$  (5.3, 5.4)

Hence, as in (4.2) these mass operators become

$$M_0(l_\downarrow) = a_\downarrow |l_3(\downarrow) - \kappa_\downarrow l_2(\downarrow)|^2 + b_\downarrow |l_2(\downarrow) - \rho_\downarrow l_1(\downarrow)|^2 + b_\downarrow |l_3(\downarrow) - \kappa_\downarrow \rho_\downarrow l_1(\downarrow)|^2$$  (5.5)

and

$$M_0(l_\uparrow) = a_\uparrow |l_3(\uparrow) - \kappa_\uparrow l_2(\uparrow)|^2 + b_\uparrow |l_2(\uparrow) - \rho_\uparrow l_1(\uparrow)|^2 + b_\uparrow |l_3(\uparrow) - \kappa_\uparrow \rho_\uparrow l_1(\uparrow)|^2.$$  (5.6)

Correspondingly, the mass matrices $M(l_\downarrow)$ and $M(l_\uparrow)$ defined by (1.7)-(1.8) are, similar to (4.6),

$$[M_0(l)]_\downarrow \text{ or } \uparrow = \begin{pmatrix} b \rho^2 (1 + \kappa^2) & -b \rho & -b \kappa \rho \\ -b \rho & b + a \kappa^2 & -a \kappa \\ -b \kappa \rho & -a \kappa & a + b \end{pmatrix}_\downarrow \text{ or } \uparrow.$$  (5.7)
Since their determinants satisfy
\[ |M_0(l_{\downarrow})| = |M_0(l_{\uparrow})| = 0, \] (5.8)
each matrix has an eigenvector of zero eigenvalue:
\[ M_0(l_{\downarrow})\delta_{\downarrow} = 0 \] (5.9)
and
\[ M_0(l_{\uparrow})\delta_{\uparrow} = 0. \] (5.10)
As in (3.1)-(3.2), let \((V_{\downarrow})_0\) and \((V_{\uparrow})_0\) be the real unitary matrices that diagonalize \(M_0(l_{\downarrow})\) and \(M_0(l_{\uparrow})\):
\[ (V_{\downarrow})_0^\dagger M_0(l_{\downarrow})(V_{\downarrow})_0 = \begin{pmatrix} m_0(e) & 0 & 0 \\ 0 & m_0(\mu) & 0 \\ 0 & 0 & m_0(\tau) \end{pmatrix} \] (5.11)
and
\[ (V_{\uparrow})_0^\dagger M_0(l_{\uparrow})(V_{\uparrow})_0 = \begin{pmatrix} m_0(\nu_1) & 0 & 0 \\ 0 & m_0(\nu_2) & 0 \\ 0 & 0 & m_0(\nu_3) \end{pmatrix} \]. (5.12)
The corresponding zeroth order neutrino mapping matrix is
\[ (V_\nu)_0 = (V_{\downarrow})_0(V_{\uparrow})_0. \] (5.13)
Note that the roles of \(\uparrow\) and \(\downarrow\) in (5.13) are switched in comparison between those in (3.3) because of our accustomed definitions of \(V_\nu\) and \(U_{CKM}\).

We will further simplify the lepton mass operators by assuming
\[ \kappa_{\downarrow} = -1, \rho_{\downarrow} = y, \sigma_{\downarrow} = -\frac{1}{y} \] (5.14)
\[ \kappa_{\uparrow} = x, \rho_{\uparrow} = -\frac{1}{\sqrt{2}} \text{ and } \sigma_{\uparrow} = -\frac{\sqrt{2}}{x} \]
with \(x, y\) two small real parameters; i.e.,
\[ |x| << 1 \text{ and } |y| << 1. \] (5.15)
Thus, (5.1) and (5.2) become

\[ M_0(l_\downarrow) = a_\downarrow |l_3(\downarrow) + l_2(\downarrow)|^2 + b_\downarrow |l_2(\downarrow) - yl_1(\downarrow)|^2 + b_\downarrow |l_3(\downarrow) + yl_1(\downarrow)|^2 \]  

(5.16)

and

\[ M_0(l_\uparrow) = a_\uparrow |l_3(\uparrow) - xl_2(\uparrow)|^2 + b_\uparrow |l_2(\uparrow) + \sqrt{\frac{2}{3}} l_1(\uparrow)|^2 + b_\uparrow |l_3(\uparrow) - \sqrt{\frac{2}{3}} xl_1(\uparrow)|^2. \]  

(5.17)

Correspondingly (5.7) can be written as

\[ M_0(l_\downarrow) = \begin{pmatrix} 2b_\downarrow y^2 & -b_\downarrow y & b_\downarrow y \\ -b_\downarrow y & a_\downarrow + b_\downarrow & a_\downarrow \\ b_\downarrow y & a_\downarrow & a_\downarrow + b_\downarrow \end{pmatrix} \]  

(5.18)

and

\[ M_0(l_\uparrow) = \begin{pmatrix} \frac{1}{2} b_\uparrow (1 + x^2) & \sqrt{\frac{2}{3}} b_\uparrow & \sqrt{\frac{2}{3}} b_\uparrow x \\ \sqrt{\frac{2}{3}} b_\uparrow & b_\uparrow + a_\uparrow x^2 & -a_\uparrow x \\ \sqrt{\frac{2}{3}} b_\uparrow x & -a_\uparrow x & a_\uparrow + b_\uparrow \end{pmatrix}. \]  

(5.19)

Since the matrices (5.18) and (5.19) are special cases of the matrix (2.3) with the constraints (2.5) and (3.6), their eigenvectors and eigenvalues can be readily obtained. Arrange the three eigenvalues \( \lambda_e, \lambda_m \) and \( \lambda_t \) of \( M_0(l_\downarrow) \) in the same order as those in (3.11)-(3.13) and (5.11); we have

\[ m_0(e) = \lambda_e = 0, \]  

(5.20)

\[ m_0(\mu) = \lambda_m = b_\downarrow (1 + 2y^2) \]  

(5.21)

and

\[ m_0(\tau) = \lambda_t = 2a_\downarrow + b_\downarrow. \]  

(5.22)

Likewise, the eigenvalues of \( M_0(l_\uparrow) \), in the same ascending order as (3.17)-(3.19) and (5.12), are

\[ m_0(\nu_1) = \lambda_n = 0, \]  

(5.23)

\[ m_0(\nu_2) = \lambda_l = \frac{1}{2} b_\uparrow (3 + x^2) \]  

(5.24)

and

\[ m_0(\nu_3) = \lambda_L = a_\uparrow (1 + x^2) + b_\uparrow. \]  

(5.25)
Similarly, by using (3.8)-(3.10) and (3.14)-(3.16) the corresponding eigenvectors of \( M_0(l_u) \) and \( M_0(l_u) \) can be readily written down.

### 5.2 A limiting Case

In the limit

\[
x \to 0 \quad \text{and} \quad y \to 0,
\]

the mass matrices (5.18) and (5.19) become

\[
[M_0(l_u)]_\downarrow \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_\downarrow + b_\downarrow & a_\downarrow \\ 0 & a_\downarrow & a_\downarrow + b_\downarrow \end{pmatrix}
\]

and

\[
[M_0(l_u)]_\uparrow \rightarrow \begin{pmatrix} \frac{1}{2} b_\uparrow & \sqrt{\frac{1}{2}} b_\uparrow & 0 \\ \sqrt{\frac{1}{2}} b_\uparrow & b_\uparrow & 0 \\ 0 & 0 & a_\uparrow + b_\uparrow \end{pmatrix}.
\]

In accordance with (5.11)-(5.12), the matrices \((V_\downarrow)_0\) and \((V_\uparrow)_0\) becomes

\[
(V_\downarrow)_0 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}
\]

and

\[
(V_\uparrow)_0 \rightarrow \begin{pmatrix} -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

with the corresponding neutrino-mapping matrix \((V_\nu)_0\) given by the Harrison-Scott form

\[
(V_\nu)_0 \equiv (V_\downarrow)_0^\dagger (V_\uparrow)_0 \rightarrow \begin{pmatrix} -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}.
\]

Thus, the matrices (5.18) and (5.19) agree with all existing experimental data provided \( x \) and \( y \) are small.
6. Lepton Sectors (with $T$ violation)

With $T$ violations, we generalize (5.16)-(5.17) and write in accordance with (1.7)-(1.8):

$$\mathcal{M}(l_{\downarrow}) = \begin{pmatrix} l_1(\downarrow), & l_2(\downarrow), & l_3(\downarrow) \end{pmatrix} \begin{pmatrix} l_1(\downarrow) \\ l_2(\downarrow) \\ l_3(\downarrow) \end{pmatrix}$$

$$= a_{\downarrow}|l_3(\downarrow) + e^{i\phi_1}l_2(\downarrow)|^2 + b_{\downarrow}|l_2(\downarrow) - y l_1(\downarrow)|^2 + b_{\downarrow}|l_3(\downarrow) + y l_1(\downarrow)|^2, \quad (6.1)$$

and

$$\mathcal{M}(l_{\uparrow}) = \begin{pmatrix} l_1(\uparrow), & l_2(\uparrow), & l_3(\uparrow) \end{pmatrix} \begin{pmatrix} l_1(\uparrow) \\ l_2(\uparrow) \\ l_3(\uparrow) \end{pmatrix}$$

$$= a_{\uparrow}|l_3(\uparrow) - x l_2(\uparrow)|^2 + b_{\uparrow}|l_2(\uparrow)|^2 + e^{i\phi_1}|l_1(\uparrow)|^2 + b_{\uparrow}|l_3(\uparrow) + \sqrt{2} x l_1(\uparrow)|^2. \quad (6.2)$$

The matrices $M(l_{\downarrow})$ and $M(l_{\uparrow})$ can also be written as

$$M(l_{\downarrow}) = M_0(l_{\downarrow}) + M_1(l_{\downarrow}) \quad (6.3)$$

and

$$M(l_{\uparrow}) = M_0(l_{\uparrow}) + M_1(l_{\uparrow}) \quad (6.4)$$

with $M_0(l_{\downarrow})$ and $M_0(l_{\uparrow})$ given by (5.18) and (5.19),

$$M_1(l_{\downarrow}) = a_{\downarrow} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{-i\phi_1} - 1 \\ 0 & e^{i\phi_1} - 1 & 0 \end{pmatrix} \quad (6.5)$$

and

$$M_1(l_{\uparrow}) = \frac{1}{\sqrt{2}} b_{\uparrow} \begin{pmatrix} 0 & e^{-i\phi_1} - 1 & 0 \\ e^{i\phi_1} - 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6.6)$$

in which $\phi_{\downarrow}$ and $\phi_{\uparrow}$ are the $T$-violating phases.

The determinants of $M(l_{\downarrow})$ and $M(l_{\uparrow})$ are

$$|M(l_{\downarrow})| = 2 a_{\downarrow} b_{\downarrow}^2 y^2 (1 - \cos \phi_{\downarrow}) \quad (6.7)$$
\[ |M(l_\uparrow)| = a_\uparrow b_\uparrow^2 x^2 (1 - \cos \phi_\uparrow). \] (6.8)

The three eigenvalues of \( M(l_\downarrow) \) are the physical masses of charged leptons \( e, \mu \) and \( \tau \):

\[ m = m_e, \ m_\mu, \ m_\tau, \] (6.9)
determined by the cubic equation

\[ m(m - \lambda_m)(m - \lambda_\ell) = |M(l_\downarrow)| \] (6.10)

with \( \lambda_m, \ \lambda_\ell \) given by (5.21) and (5.22). Likewise, the eigenvalues of \( M(l_\uparrow) \) are the three neutrino masses

\[ m = m_1, \ m_2, \ m_3, \] (6.11)
determined by

\[ m(m - \lambda_\ell)(m - \lambda_L) = |M(l_\uparrow)| \] (6.12)

with \( \lambda_\ell, \ \lambda_L \) given by (5.24)-(5.25). As in (4.9), Eqs. (6.10) and (6.12), together with (5.20)-(5.25) give a simple way to derive the physical masses of leptons with the inclusion of \( T \) violation effects.

On the other hand, for evaluation of eigenvectors it is more convenient to transform

\[ l_1(\downarrow) = l'_1(\downarrow), \ l_2(\downarrow) = e^{-i\phi_\downarrow/2} l'_2(\downarrow) \quad \text{and} \quad l_3(\downarrow) = e^{i\phi_\downarrow/2} l'_3(\downarrow). \] (6.13)

Thus,

\[ \mathcal{M}(l_\downarrow) = \left( \begin{array}{c} l'_1(\downarrow), \ l'_2(\downarrow), \ l'_3(\downarrow) \end{array} \right) \left( H_0(l_\downarrow) + H_1(l_\downarrow) \right) \left( \begin{array}{c} l'_1(\downarrow) \\ l'_2(\downarrow) \\ l'_3(\downarrow) \end{array} \right) \] (6.14)

with

\[ H_0(l_\downarrow) = \begin{pmatrix} 2b_\downarrow y^2 & 0 & 0 \\ 0 & a_\downarrow + b_\downarrow & a_\downarrow \\ 0 & a_\downarrow & a_\downarrow + b_\downarrow \end{pmatrix} \] (6.15)

and

\[ H_1(l_\downarrow) = b_\downarrow y \begin{pmatrix} 0 & -z_\downarrow^* & z_\downarrow \\ -z_\downarrow & 0 & 0 \\ z_\downarrow^* & 0 & 0 \end{pmatrix}, \] (6.16)
with 
\[ z_\downarrow = e^{i\frac{1}{2} \phi_1}. \]  
(6.17)

Likewise, introduce \( l_1'(\uparrow) \) and \( l_2'(\uparrow) \) through 
\[ l_1(\uparrow) = e^{-i\phi_1} l_1'(\uparrow), \quad l_2(\uparrow) = e^{i\phi_1} l_2'(\uparrow) \quad \text{and} \quad l_3(\uparrow) = l_3'(\uparrow). \]  
(6.18)

Correspondingly, 
\[ \mathcal{M}(l_\uparrow) = \left( \bar{p}_1'(\uparrow), \bar{p}_2'(\uparrow), \bar{p}_3'(\uparrow) \right) \left( H_0(l_\downarrow) + H_1(l_\downarrow) \right) \begin{pmatrix} l'_1(\uparrow) \\ l'_2(\uparrow) \\ l'_3(\uparrow) \end{pmatrix} \]  
(6.19)

with 
\[ H_0(l_\downarrow) = \begin{pmatrix} \frac{b_1}{2}(1 + x^2) & \sqrt{\frac{T}{2}} b_\uparrow & 0 \\ \sqrt{\frac{T}{2}} b_\downarrow & b_\uparrow + a_\downarrow x^2 & 0 \\ 0 & 0 & a_\downarrow + b_\downarrow \end{pmatrix} \]  
(6.20)

and 
\[ H_1(l_\downarrow) = x \begin{pmatrix} 0 & 0 & \sqrt{\frac{T}{2}} b_\uparrow & 0 \\ 0 & 0 & -a_\downarrow z_\downarrow^* & 0 \\ \sqrt{\frac{T}{2}} b_\downarrow z_\downarrow^* & -a_\downarrow z_\downarrow & 0 \end{pmatrix}, \]  
(6.21)

with 
\[ z_\uparrow = e^{i\frac{1}{2} \phi_1}. \]  
(6.22)

The smallness of \( x \) and \( y \) makes it possible to use \( H_1(\downarrow) \) and \( H_1(\uparrow) \) as perturbations.

Let \( V_\downarrow \) and \( V_\uparrow \) be the unitary matrices that diagonalize \( M(l_\downarrow) \) and \( M(l_\uparrow) \) of (6.1) and (6.2). The corresponding neutrino mapping matrix is 
\[ V_\nu = V_\downarrow^\dagger V_\uparrow. \]  
(6.23)

Define
\[ T_1 = (V_\nu)_21 (V_\nu)_31, \]
\[ T_2 = (V_\nu)_22 (V_\nu)_32 \]  
(6.24)

and 
\[ T_3 = (V_\nu)_23 (V_\nu)_33. \]
Their sum satisfies

\[ T_1 + T_2 + T_3 = 0. \tag{6.25} \]

The corresponding Jarlskog invariant for leptons is

\[ \mathcal{J}_\nu = \text{Im} T_1^* T_2 = \text{Im} T_2^* T_3 = \text{Im} T_3^* T_1. \tag{6.26} \]

To first order in \( x \) and \( y \), we find

\[ \mathcal{J}_\nu = -\frac{y}{3\sqrt{2}} \left[ \cos \frac{\phi_\downarrow}{2} \sin \left( \frac{\phi_\downarrow}{2} + \phi_\uparrow \right) + \frac{b_\downarrow}{2a_\downarrow + b_\downarrow} \sin \phi_\downarrow \cos \left( \frac{\phi_\downarrow}{2} + \phi_\uparrow \right) \right] \]
\[ + \frac{x}{3} \frac{a_\downarrow b_\downarrow}{(a_\downarrow + b_\downarrow)(2a_\downarrow - b_\downarrow)} \left[ -\sin \phi_\downarrow + \sin \left( \phi_\downarrow + \phi_\uparrow \right) \right]. \tag{6.27} \]

As in (4.37), there is an interesting relation between \( \mathcal{J}_\nu \) and lepton masses, which will be discussed in a separate paper.

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Appendix

In (2.3), the $3 \times 3$ symmetric matrix is written in the form

$$M = \begin{pmatrix} \gamma + \beta \eta^2 & -\beta \eta & -\gamma \\ -\beta \eta & \beta + \alpha \xi^2 & -\alpha \xi \\ -\gamma & -\alpha \xi & \alpha + \gamma \zeta^2 \end{pmatrix} \quad (A.1)$$

with six parameters $\alpha$, $\beta$, $\gamma$, $\xi$, $\eta$ and $\zeta$. On the other hand, any such symmetric $M$ can also be expressed by

$$M = \begin{pmatrix} c & -e & -f \\ -e & b & -d \\ -f & -d & a \end{pmatrix} \quad (A.2)$$

with

$$a = \alpha + \gamma \zeta^2, \quad (A.3)$$
$$b = \beta + \alpha \xi^2, \quad (A.4)$$
$$c = \gamma + \beta \eta^2, \quad (A.5)$$
$$d = \alpha \xi, \quad (A.6)$$
$$e = \beta \eta \quad (A.7)$$

and

$$f = \gamma \zeta. \quad (A.8)$$

In this Appendix, we give the answer to the inverse problem: Given $a$, $b$, $\cdots$, $f$ what are the corresponding $\alpha$, $\beta$, $\cdots$, $\zeta$?

Introduce

$$A = bc - e^2 \quad (A.9)$$
$$B = ae^2 + bf^2 - cd^2 - abc \quad (A.10)$$

and

$$C = (ac - f^2)d^2. \quad (A.11)$$

One can readily verify that $\alpha$ satisfies

$$A \alpha^2 + B \alpha + C = 0. \quad (A.12)$$

Knowing $\alpha$, (A.6) gives $\xi$. From $\alpha \xi^2$, $\beta$ can be determined. Thus, $\eta$ is known from (A.7). Likewise, $\gamma$ can be deduced from (A.5) and $\zeta$ from (A.3).

For $\alpha$, $\beta$, $\gamma$, $\xi$, $\eta$, $\zeta$ real, so are $a$, $b$, $c$, $d$, $e$, $f$. However, the converse is not always true, as can be readily studied by examining the solutions of the quadratic equation (A.12).