M-theory/type IIA duality and K3 in the Gibbons-Hawking approximation

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Abstract

We review the geometry of K3 surfaces and then describe this geometry from the point of view of an approximate metric of Gibbons-Hawking form. This metric arises from the M-theory lift of the tree-level supergravity description of type IIA string theory on the $T^3/\mathbb{Z}_2$ orientifold, the D6/O6 orientifold T-dual to type I on $T^3$. At large base, it provides a good approximation to the exact K3 metric everywhere except in regions that can be made arbitrarily small. The metric is hyperkähler, and we give explicit expressions for the hyperkähler forms as well as harmonic representatives of all cohomology classes. Finally we compute the metric on the moduli space of approximate metrics in two ways, first by projecting to transverse traceless deformations (using compensators), and then by computing the naive moduli space metric from dimensional reduction. In either case, we find agreement with the exact coset moduli space of K3 metrics. The $T^3/\mathbb{Z}_2$ orientifold provides a simple example of a warped compactification. In a companion paper, the results of this paper will be applied to study the procedure for warped Kaluza-Klein reduction on $T^3/\mathbb{Z}_2$.

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1 Introduction

K3 surfaces are ubiquitous in string theory compactifications [2]. As the unique Calabi-Yau manifolds of complex dimension 2, they are the simplest compact special holonomy manifolds, and the simplest manifolds other than tori admitting the covariantly constant spinors required for low energy supersymmetry [34]. K3 surfaces also feature prominently in string duality [66, 42, 43, 2, 3]. For example, in a chain of dualities related by circle reduction, F-theory on K3 is dual to the $E_8 \times E_8$ heterotic string on $T^2$, M-theory on K3 is dual to the heterotic or type I string on $T^3$, and type IIA string theory on K3 is dual to the heterotic or type I string on $T^4$. Fiberwise application of this duality relates M-theory on manifolds of $G_2$ holonomy to the heterotic or type I string on Calabi-Yau 3-folds [1].

The motivation for the present work, and the duality of interest here, is the lift from type IIA string theory on the $T^3/\mathbb{Z}_2$ orientifold to M-theory on K3. Here $T^3/\mathbb{Z}_2$ is the orientifold obtained from type I on $T^3$ after T-duality with respect to all three torus isometries. This duality relates the conventional compactification of M-theory on K3 to a simple IIA warped compactification, where the warping is due to the 16 D6-branes and 8 O6-planes of $T^3/\mathbb{Z}_2$.

This paper is part of a larger investigation, whose overall goal is to elucidate the procedure for Kaluza-Klein reduction of warped compactifications via duality to standard compactifications. A secondary goal along the way, is to learn about compactification on manifolds of SU(2) structure. Virtually all phenomenologically relevant string theory compactifications are of warped type, in which the overall scale factor of 4D spacetime varies over the internal dimensions [29, 11]. This feature modifies 4D mass scales and couplings, and naturally realizes the Randall-Sundrum approach to the hierarchy problem [46]. A long standing obstacle to quantitative prediction has been our incomplete understanding of the analog of standard Kaluza-Klein reduction for warped compactification. Warped Kaluza-Klein reduction was first studied in Refs. [16, 28] and our understanding was greatly enhanced in Refs. [56, 18, 22, 23, 63]. Our investigation takes a complementary route, and will be useful in probing the formalism of Refs. [56, 18, 22].

With this goal in mind, this paper provides a review of the geometry of K3 surfaces emphasizing the tools that will be useful for the duality, of which, a novel feature is the role of an approximate K3 metric exactly dual to the classical supergravity description of $T^3/\mathbb{Z}_2$. At the level of the tree-level type IIA supergravity description of $T^3/\mathbb{Z}_2$, the lift to M-theory gives a “first-order” metric on K3 of Gibbons-Hawking [26] form, through which it is possible to study the differential geometry of K3 more explicitly than is generally the case for Calabi-Yau manifolds.\footnote{An approximate metric of this form has a history of local application to the geometry of K3 in the math literature. See, for example, Ref. [30].} For physicists, differential geometry, the language of general relativity, is more intuitive than algebro-geometric description, in which the quantities appearing in the local supergravity equations—frame, metric, deformations, harmonic forms, Kähler form—are analyzed abstractly rather than explicitly. In contrast, K3 in the Gibbons-Hawking approximation provides a model in which
one can manipulate all of the moving parts of K3 at the level of explicitness that one would really like. There is an explicit metric whose components are simply defined functions of quantities $G_{\alpha \beta}, \beta_{\alpha \beta}, x^I_{\alpha}$ parametrizing an $(\text{SO}(3) \times \text{SO}(19)) \backslash \text{SO}(3, 3 + 16)$ coset, in addition to overall scaling from the volume modulus $V_{K3}$. One can write down the harmonic forms in this metric, and see explicitly how they vary with the moduli. One can write down a frame and a triple of hyperkähler 2-forms and see that all three forms are closed, thus showing that there is no torsion, and that the metric connection has SU(2) holonomy. Finally, one can study the metric on the metric moduli space, and here too, there is an explicit example of a novel phenomenon. The diffeomorphism invariant moduli space metric obtained using the formalism of compensators agrees with the naive moduli space metric from direct dimensional reduction.

We restrict the scope of this paper to the geometry of K3 and the duality between a family of M-theory vacua on K3 and family of IIA vacua on $T^3/\mathbb{Z}_2$. We steer clear of 7D effective field theory questions, in which the moduli promoted from parameters labeling the families to fields depending on the noncompact dimensions of spacetime. The effective field theory analysis will appear in Ref. [52], in which the warped compactification ansatz for type IIA on $T^3/\mathbb{Z}_2$ is derived from the standard compactification ansatz for M-theory on K3, at the level of the tree level IIA supergravity description. All of this is a warm-up for Ref. [53], in which we treat a similar duality [51] relating the $T^6/\mathbb{Z}_2$ type IIB orientifold with $\mathcal{N} = 2$ flux to a class of purely geometry type IIA Calabi-Yau compactifications. The latter will allow us to deduce the warped dimensional reduction procedure for $T^6/\mathbb{Z}_2$ from that for the conventional Calabi-Yau dual.

As an interesting vista along the way to this goal, we observe in Ref. [54] that the class of Calabi-Yau manifolds arising in the duality of the previous paragraph are not only manifolds of SU(3) holonomy, but also manifolds of SU(2) structure. That is, there exists a connection with torsion whose holonomy group is further restricted from SU(3) to SU(2). Just as the dual choice of flux in the type IIB $T^6/\mathbb{Z}_2$ dual spontaneously breaks an $\mathcal{N} = 4$ low energy field theory to $\mathcal{N} = 2$ [35, 49, 50, 51], the SU(2) structure gives the IIA Calabi-Yau compactification an $\mathcal{N} = 4$ effective field theory in which the topology spontaneously breaks the supersymmetry to $\mathcal{N} = 2$. This will be discussed in Ref. [54] based on a first order description very similar to that of the present paper, together with a few exact results. For related work on the SU(2) structure of the Enriques Calabi-Yau 3-fold [21] and the resulting effective field theory, see Refs. [62, 36]. For earlier work on SU(2) structure compactifications, see Refs. [25, 10, 47, 61, 39, 14, 59, 13].

An outline of the paper is as follows:

In Sec. 2 we review the geometry of K3 surfaces, beginning with their definition as compact complex surfaces of trivial canonical bundle, and going on to review their holonomy, Kähler and hyperkähler structure, and (co)homology. On general grounds, the integer (co)homology lattice $H_2(K3, \mathbb{Z})$ splits as $(-E_8) \oplus (-E_8) \oplus (U_{1,1})^{\otimes 3}$ and $(-\text{Spin}(32)/\mathbb{Z}_2) \oplus (U_{1,1})^{\otimes 3}$ where $(-E_8)$ denotes the weight lattice of $E_8$ with the sign of the inner product reversed, and similarly for Spin(32)/$\mathbb{Z}_2$. These splittings are not obvious in the homology basis from the Kummer construction of K3 as the resolution of $T^4/\mathbb{Z}_2$. Therefore,
with future applications in mind, we derive the explicit relations between the Kummer, \((\mathbf{-E_8} \oplus (\mathbf{-E_8}) \oplus (U_{1,1})^{\oplus 3})\) and \((\mathbf{-Spin(32)/Z_2} \oplus (U_{1,1})^{\oplus 3})\) homology bases. These relations, while certainly not new, do not seem to be written down in the literature, and we expect that a clear discussion will be useful to others. Sec. 2 concludes with a discussion of the moduli space of hyperkähler structure and its relation to the moduli space of K3 metrics.

In Sec. 3, we describe the approximate K3 metric of Gibbons-Hawking form, obtained from the lift of the tree-level type IIA supergravity description of \(T^3/\mathbb{Z}_2\) to M-theory. After an overview of the results of this section, we review the Gibbons-Hawking multicenter metrics and the standard supergravity identifications between type IIA string theory and M-theory. Then, we consider in succession the M-theory lift of a collection of \(N\) D6-branes on \(\mathbb{R}^3\), of a collection of \(N\) D6-branes near an O6-plane on \(\mathbb{R}^3\), and finally of the \(T^3/\mathbb{Z}_2\) orientifold with 16 D6 branes and 8 O6-planes. In Sec. 3.4 we give the frame, hyperkähler forms, and a basis of harmonic forms in the approximate metric, identifying the cohomology classes of the latter with those in the exact treatment of Sec. 2.5. Finally, Sec. 3.5 is devoted to the moduli space of the approximate K3 metric, treated from two points of view. First, focusing on the case of the \(16 \times 3\) exceptional deformations, we show that metric deformations due to the explicit dependence of the approximate metric on moduli \(G_{\alpha \beta}, \beta^{\alpha \beta}, \text{ and } x^{I\alpha}\) agree with the transverse traceless deformations generated by harmonic forms, up to compensating diffeomorphisms. The exact coset moduli space follows. Next we consider the naive moduli space metric from dimensional reduction. This differs from the previous moduli space metric in two ways: the metric deformations are not projected to transverse traceless components (i.e., no compensators), and instead, there exists an additional term in the metric. We find that naive moduli space metric precisely agrees with the previous one. Finally, in Sec. 4, we conclude.

The Appendices contain additional background and technical details. App. A describes hyperkähler structures on \(T^4\). App. B treats the homology lattice of K3, deriving intersection numbers and the details of \((\mathbf{-E_8} \oplus (\mathbf{-E_8}) \oplus (U_{1,1})^{\oplus 3})\) and \((\mathbf{-Spin(32)/Z_2} \oplus (U_{1,1})^{\oplus 3})\) splittings. App. C reviews the Lichnerowicz operator, which relates deformations of the Ricci tensor to metric deformations. App. D provides background on the relation between metric deformations and harmonic forms. Finally, App. E treats the metric deformations and compensating vector fields of the Gibbons-Hawking multicenter metric, and evaluates a class of integrals used elsewhere in the paper.

## 2 Review of K3 surfaces

### 2.1 Definition

A K3 surface is a compact complex surface of trivial canonical bundle and \(h^{1,0} = 0\) [7]. The latter condition is necessary only to distinguish a K3 surface from \(T^4\) (an abelian surface). Every K3 surface is Kähler [25], so K3 surfaces are the unique Calabi-Yau manifolds of complex dimension 2. As such, they are ubiquitous in compactifications of string theory. In contrast to Calabi-Yau 3-folds, all K3 surfaces are deformations of one another, so they are all
diffeomorphic \cite{7}. The name K3 was coined in 1958 by André Weil to honor the achievements of geometers Kummer, Kähler, and Kodaira \cite{2}, a short time after mountain climbers first ascended K2 in northern Kashmir, the world’s second highest peak \cite{2}. The two simplest constructions of a K3 surface are the quartic hypersurface in \(\mathbb{P}^4\) and the Kummer surface, defined as the surface obtained by resolving the sixteen orbifold singularities of \(T^4/\mathbb{Z}_2\).

In the next three sections we review the holonomy, hyperkähler structure, and homology of K3 surfaces. Harmonic forms and the moduli space of hyperkähler structure are discussed next, in Sec. 2.2. We conclude our overview of K3 surfaces with a discussion of the metric on K3 metric moduli space in Sec. 2.7. For a more extensive review of the geometry of K3 surfaces, we refer the reader to Refs.\cite{41,2,34,7}.

2.2 Holonomy

K3 surfaces are the unique Calabi-Yau 2-folds. Recall that a Calabi-Yau \(n\)-fold \(X\) is defined to be a compact Kähler manifold of complex dimension \(n\) and trivial canonical bundle. This definition, together with Yau’s theorem, implies an equivalent definition in terms of global SU(\(n\)) holonomy. For simply connected \(X\), the reasoning is as follows:\footnote{The quartic hypersurface is an example of an algebraic K3 surface—the vanishing locus of a set of polynomial equations in a complex projective space. Not all K3 surfaces are algebraic. In particular, the generic Kummer surface is not algebraic.}

(i) The trivial canonical bundle implies vanishing first Chern class \(c_1 \in H^2(X, \mathbb{Z})\), which in turn implies the weaker condition that \(c_1\) vanishes in \(H^2(X, \mathbb{R})\). (ii) Since \(X\) is Kähler, we can then apply Yau’s theorem: \textit{Let \(X\) be a compact complex manifold, of dimension at least 2, with vanishing real first Chern class. Consider a fixed complex structure on \(X\). Given a real class in \(H^{1,1}(X, \mathbb{C})\) of positive norm, there is a unique Ricci flat metric on \(X\) with Kähler form in this cohomology class.} (iii) Finally, the Riemannian holonomy group \(\text{Hol}(g)\) of the Levi-Civita connection of a Ricci flat metric \(g\) follows from the classification of Berger \cite{8}.

Berger proved that for a simply connected Riemannian manifold \(M\) that is not a reducible or symmetric space, the Riemannian holonomy must be \(\text{SO}(m), \text{U}(m), \text{SU}(m), \text{Sp}(m), (\text{Sp}(m) \times \text{Sp}(1))/\mathbb{Z}_2\text{center}, G_2,\) or \(\text{Spin}(7)\). (See, for example, Ref. \cite{9,34}.) Let us focus on the subset of Kähler special holonomy groups \(\text{U}(m), \text{SU}(m),\) and \(\text{Sp}(m),\) and let \(M\) be real \(d\)-dimensional. As cited in Ref. \cite{2}, Berger’s classification states that

1. \(\text{Hol}(M) \subset \text{U}(\frac{d}{2})\) if and only if \(X\) is a Kähler manifold;
2. \(\text{Hol}(M) \subset \text{SU}(\frac{d}{2})\) if and only if \(X\) is a Ricci-flat Kähler manifold;
3. \(\text{Hol}(M) \subset \text{Sp}(\frac{d}{2})\) if and only if \(X\) is a hyperkähler manifold.

\footnote{When \(X\) is not simply connected, the existence of a metric of global SU(\(n\)) holonomy is still an equivalent definition of a Calabi-Yau manifold, even though the reasoning given here must be modified. However, in this case it is a weaker condition to say that \(X\) is Ricci-flat Kähler manifold than to say it is Calabi-Yau. For example, an Enriques surface \(K3/\mathbb{Z}_2\) is a Ricci-flat Kähler manifold, but it is not Calabi-Yau: The canonical bundle is nontrivial and \(c_1\) vanishes in \(H^2(X, \mathbb{R})\) but gives a \(\mathbb{Z}_2\) torsion class in \(H^2(X, \mathbb{Z})\). The restricted holonomy group \(\text{Hol}_0(g)\) is SU(2), but the global holonomy group \(\text{Hol}(g)\) is disconnected.}
Each line contains those that follow, since $U(\frac{d}{2}) \supset SU(\frac{d}{2}) \supset Sp(\frac{d}{2})$. Thus, a Calabi-Yau $n$-fold is a manifold of $SU(n)$ holonomy. In the special case $n = 2$, we have $SU(2) \cong Sp(1)$, and the last two categories collapse to one. Therefore, a K3 surface with a Ricci-flat metric has $SU(2) \cong Sp(1)$ holonomy and is hyperkähler.

Ricci-flat metrics on hyperkähler 4-manifolds are solutions to 4D Euclidean Einstein equations that can be shown to have (anti)selfdual curvature 2-form. For this reason, they are often referred to as gravitational instantons, by analogy to (anti)selfdual Yang-Mills instantons in 4D. The only compact hyperkähler 4-manifolds are K3 and $T^4$. Noncompact hyperkähler 4-manifolds asymptotic to $\mathbb{H}/\Gamma$, where $\Gamma$ is a finite subgroup of $Sp(1)$, are known as asymptotically locally Euclidean (ALE) spaces. As long as $\Gamma$ is not too large, they can be viewed as local models for the resolution of orbifold singularities of K3 surfaces. The prime examples for which explicit metrics can be written down are Taub-NUT spaces [60, 44], multi-Taub-NUT spaces [32], Eguchi-Hanson spaces [20], and finally Gibbons-Hawking multicenter spaces [20, 20], which include the previous ones as special cases. The Gibbons-Hawking ansatz is discussed in Sec. 3.1. An important example in which the metric is known only more implicitly is the Atiyah-Hitchin space [4, 5, 6], the moduli space of two SU(2) ’t Hooft-Polyakov monopoles in four dimensions. Additional constructions beyond the Gibbons-Hawking ansatz include twistor theory [33] and the hyperkähler quotient construction of Kronheimer [37, 38, 27].

2.3 Hyperkähler structure

The definition of a hyperkähler manifold as a 4$m$-dimensional Riemannian manifold of $Sp(m)$ holonomy implies the existence of a triple of integrable almost complex structures satisfying the quaternionic algebra. We now review the structure of Kähler and hyperkähler manifolds and following Chapter 7 of Ref. [34] closely.

2.3.1 Kähler manifolds

It is useful to begin with a review of Kähler structure. First, recall that the complex number field $\mathbb{C}$ is defined as an extension of the reals by writing $z \in \mathbb{C}$ as $z = a + ib$, where $a, b \in \mathbb{R}$ and $i^2 = -1$, and comes with the involution of complex conjugation $\bar{z} = z^* = a - ib$. A Kähler manifold has the local structure of $\mathbb{C}^n$ together with its standard complex structure, Hermitian metric, and Kähler 2-form. For $\mathbb{C}^n$ with holomorphic coordinates $z^j = x^{2j-1} + i x^{2j}$
and antiholomorphic coordinates $z^j = x^{2j-1} - ix^{2j}$, these tensors are

$$ds^2 = \sum_{m=1}^{2n} dx^m \otimes dx^m = \frac{1}{2} \sum_{j=1}^n (dz^j \otimes dz^j + dz^\bar{j} \otimes dz^\bar{j}),$$  \hspace{0.5cm} (1)$$

$$J = \sum_{j=1}^n dx^{2j-1} \otimes \partial_{2j} - dx^{2j} \otimes \partial_{2j-1} = \sum_{j=1}^n dz^j \otimes \partial_j - dz^\bar{j} \otimes \partial_j,$$  \hspace{0.5cm} (2)$$

$$J = \sum_{j=1}^n dx^{2j-1} \wedge dx^{2j} = \frac{1}{2} \sum_{j=1}^n dz^j \wedge dz^\bar{j}.$$  \hspace{0.5cm} (3)$$

On a general Kähler manifold $X$, these expressions hold when coordinate vectors and 1-forms are replaced by the frame and coframe (i.e., vielbein basis). A $2n$ dimensional Kähler manifold has three successive layers of structure:

1. Complex structure. Recall that an almost complex structure (ACS) $J$ on an even-dimensional oriented manifold $X$ is a smooth tensor field of rank $(1,1)$ that maps the (co)tangent bundle to itself and that squares to one. In physics conventions, it has index structure $J_{ab}$ and naturally acts on the cotangent bundle, mapping a 1-form $\omega_p$ to $J^b_p \omega_q$. Its transpose naturally acts on the tangent bundle. An ACS gives a canonical isomorphism between each tangent space $T_p M \cong \mathbb{R}^{2n}$ and $\mathbb{C}^n$. When the Nijenhuis tensor vanishes, the almost complex structure is integrable, and there exist complex coordinates on $X$ with holomorphic transition functions $z'^j = f^j(z)$, independent of $z^\bar{k}$. In this case and we drop the word almost, and refer to $J$ as the complex structure. In the language of $G$-structures and intrinsic torsion, a complex manifold is a manifold of torsion free $GL(n, \mathbb{C})$ structure: the structure group of the frame bundle is reduced from $GL(2n, \mathbb{R})$ to $GL(n, \mathbb{C})$, and the intrinsic torsion vanishes precisely when the Nijenhuis tensor vanishes.

2. Hermitian metric. $X$ is endowed with a metric $g$ such that $g = JgJ^T$, or in components, $g_{j\bar{k}} = (g_{j\bar{k}})^*$ and $g_{j\bar{k}} = g_{\bar{k}j} = 0$. Define the fundamental form $J$ by $J_{pq} = J^r_p g_{rq}$, or equivalently $J = ig_{j\bar{k}} dz^j \wedge dz^\bar{k}$. The volume form on a Hermitian manifold is given by

$$\text{Vol}_M = \frac{1}{n!} J^n.$$  \hspace{0.5cm} (4)$$

3. Kähler structure. $X$ is said to be Kähler, and $J$ called the Kähler form, when $dJ = 0$. Given an almost complex structure and Hermitian metric, the condition $dJ = 0$ is equivalent to $\nabla J = 0$, to $\nabla J = 0$, and to the condition that $X$ have a torsion free $U(n)$ structure. The group $U(n)$ is the subgroup of $GL(n, \mathbb{C})$ preserving $J$, and the intrinsic torsion vanishes precisely when $J$ is covariantly constant and the structure group of the frame bundle is realized as the Riemannian holonomy group. Note that $J_{jk} = \frac{1}{2} \delta_{jk}$ in the holomorphic (and conjugate) coframe basis $\theta^j$ (and $\bar{\theta}^j$), and $U \in U(n)$ acts on this basis as $\theta^j \mapsto U^j_k \theta^k$ (and $\bar{\theta}^j \mapsto (U^*)^j_k \theta^k$).
2.3.2 Hyperkähler manifolds

Hyperkähler manifolds are the natural generalizations of Kähler manifolds when the complex numbers are generalized to the quaternions. Recall that the quaternionic number field $\mathbb{H}$ is defined by writing $q \in \mathbb{H}$ as $q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternionic group, with multiplication defined by $i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j$. (5)

As in the complex case, we define $\bar{q} = q^* = a - bi - cj - dk$, $\text{Re} \ q = a$, and $\text{Im} \ q = bi + cj + dk$. A hyperkähler manifold has the local structure of $\mathbb{H}^{m}$ together with its standard hypercomplex structure, Hermitian metric, and hyperkähler 2-forms. For $\mathbb{H}^{m}$ with quaternionic coordinates $q^j = -x^j + ix^j_1 + jx^j_2 + kx^j_3$, these tensors are $i^j = \partial_{x^j}$, $j^j = \partial_{x^j_1}$, $k^j = \partial_{x^j_2}$, and $l^j = \partial_{x^j_3}$.

$$J^1 = dx^{j_2} \wedge \partial_{j_3} + dx^{j_1} \wedge \partial_{j_4}, \quad J^1 = dx^{j_2} \wedge dx^{j_3} + dx^{j_1} \wedge dx^{j_4},$$
$$J^2 = dx^{j_3} \wedge \partial_{j_1} + dx^{j_2} \wedge \partial_{j_4}, \quad J^2 = dx^{j_3} \wedge dx^{j_1} + dx^{j_2} \wedge dx^{j_4},$$
$$J^3 = dx^{j_1} \wedge \partial_{j_2} + dx^{j_3} \wedge \partial_{j_4}, \quad J^3 = dx^{j_1} \wedge dx^{j_2} + dx^{j_3} \wedge dx^{j_4},$$
$$ds^2 = \sum_{j=1}^{m} \sum_{\alpha=0}^{3} dx^{j \alpha} \otimes dx^{j \alpha} = \text{Re} \left( \sum_{j=1}^{m} dq^j \otimes dq^j \right).$$ (6)

Here, the action of $J^1, J^2, J^3$ on the $dx^{j \alpha}$ induces the same action on $dq^j$ as multiplication by $i, j, k$. The expressions for $J^\alpha$ and the metric can be written concisely as

$$\sum_{j=1}^{m} dq^j \otimes dq^j = g - J^1 i - J^2 j - J^3 k. \quad (7)$$

Note that the metric is Kähler with respect to an $S^2$ worth of complex structures: if $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$, then $n_\alpha J^\alpha$ is a complex structure with corresponding Kähler form $n_\alpha J^\alpha$.

With respect to the complex structure $J^1$, the 2-form

$$J = -J^2 + iJ^3 = \sum_{j=1}^{2m} dz^{2j-1} \wedge dz^{2j} \quad (8)$$

is a complex symplectic form, with similar definitions for $J$ in the complex structure $n_\alpha J^\alpha$. On a general hyperkähler manifold $X$, these expressions hold when coordinate vectors and 1-forms are replaced by the frame and coframe (i.e., vielbein basis). A hyperkähler manifold has three successive layers of structure:

1. Hypercomplex structure. For an oriented 4m-dimensional manifold $X$, we define an almost hypercomplex structure (AHS) to be a triple of almost complex structures $J^\alpha$,

\footnote{The last three relations can also be written more concisely as $ijk = -1$.}

\footnote{Here, to simplify notation, we write $a \wedge b = a \otimes b - b \otimes a$, even when $b$ is a vector rather than a 1-form.}
\(\alpha = 1, 2, 3\), whose action on the tangent bundle satisfies the quaternionic algebra. In physics conventions, this means that \((\mathcal{J}^1)^T, (\mathcal{J}^2)^T, (\mathcal{J}^3)^T\) satisfy the algebra of \(i, j, k\), and \(\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3\) satisfy the algebra of \(-i, -j, -k\). An AHS gives a canonical isomorphism between each tangent space \(T_q M \cong \mathbb{R}^{4m}\) and \(\mathbb{H}^m\). When all three complex structures are integrable, we drop the word almost and refer to the triple as a hypercomplex structure. As above, this means that there is an \(S^2\) worth of complex structures on \(X\). It does not, however, mean that there is a coordinate integrability analogous to that in the complex case. A hypercomplex structure does not imply the existence quaternionic coordinates with transition functions \(q^3 = f^3(q)\), independent of \(q^3\).

2. Hyperhermitian metric. \(X\) is endowed with a metric \(g\) that is hermitian with respect to each complex structure \(\mathcal{J}^\alpha, \alpha = 1, 2, 3\). Define the corresponding fundamental forms \(J^\alpha\) by lowering the vector index of \(\mathcal{J}^\alpha\). Then, the volume form can be written

\[
\text{Vol}_M = \frac{1}{m!} (n_\alpha J^\alpha)^{2m} \quad \text{for any } n_\alpha \in S^2.
\]

and the complex volume form can be written

\[
\Omega_M = \frac{1}{m!} J^{2m}, \quad \text{with} \quad \text{Vol}_M = \frac{1}{4m!} \Omega_M \wedge \bar{\Omega}_M.
\]

Here, \(J = (-n'_\alpha + in''_\alpha)J^\alpha\), where \(n, n', n''\) is a right handed orthonormal basis for \(\mathbb{R}^3\).

3. Hyperkähler structure. \(X\) is said to be hyperkähler, with hyperkähler 2-forms \(J^\alpha, \alpha = 1, 2, 3\), when all three 2-forms are closed, \(dJ^\alpha = 0\). Given a hypercomplex structure and a hyperhermitian metric, this is equivalent to the condition \(\nabla J^\alpha\) for \(\alpha = 1, 2, 3\), and to the condition that \(X\) have torsion free \(\text{Sp}(n)\) structure. The group \(\text{Sp}(m)\) is the subgroup of \(\text{GL}(4m, \mathbb{R})\) preserving \(g, J^1, J^2, J^3\). It can also be viewed as the subgroup of \(\text{GL}(m, \mathbb{H})\) such that \(U^T U = 1\), where \(U\) refers to conjugation in \(\mathbb{H}^3\). It acts as on the quaternionic coframe \(\theta^j\) as \(\theta^j \mapsto U\bar{\theta}^j\) (and the conjugate basis as \(\bar{\theta}^j \mapsto (U)^{-1} k \bar{\theta}^k\)).

The reader can easily check that this preserves the left hand side of

\[
\sum_{j=1}^m \theta^j \otimes \bar{\theta}^j = g - J^1 i - J^2 j - J^3 k.
\]

The intrinsic torsion vanishes precisely when the \(J^\alpha\) are covariantly constant and the structure group of the frame bundle is realized as the Riemannian holonomy group.

Since \(\text{Sp}(m) \supset \text{SU}(2m)\), hyperkähler manifolds are necessarily Calabi-Yau, and their metrics are necessarily Ricci flat. We will discuss the hyperkähler moduli space of K3 and its identification with the metric moduli space in Secs. 2.5 and 2.7.

---

6There are two groups commonly referred to as Sp. The group \(\text{Sp}(m)\) relevant here is compact and has fundamental representation of real dimension \(4m\); it is the straightforward generalization of the special unitary groups when \(\mathbb{C}\) is replaced by \(\mathbb{H}\), and is sometimes denoted \(\text{USp}(m)\). The other group, \(\text{Sp}(2m, \mathbb{R})\) preserves a skew symmetric form with \(\pm I_n\) off diagonal; it is noncompact and has fundamental representation of real dimension \(2m\). The complexification of either is \(\text{Sp}(2m, \mathbb{C})\), and we have \(\text{USp}(2n) = U(2m) \cap \text{SL}(2m, \mathbb{C})\).
2.4 Homology and cohomology

From the definition of K3 as a manifold of trivial canonical bundle and $h^{1,0} = 0$, the interesting part of the homology ring of K3 is the second homology lattice $H_2(K3, \mathbb{Z})$. As we explain below, this lattice is 22 dimensional, and is a selfdual, even unimodular lattice, of signature $(3,19)$. For a proof of this result on general grounds, see the discussion in Sec. 2.3 of Ref. [2].

Even selfdual lattices exist only in signature $(p,q)$ such that $p - q$ is divisible by 8. For example, the signature $(16,0)$ case arises in the construction of the heterotic string. In this case, there are exactly two such lattices: the root lattice of $E_8 \times E_8$ and the weight lattice of $\text{Spin}(32)/\mathbb{Z}_2$. For $p,q > 0$ the solution is unique up to lattice automorphism, and for signature $(3,19)$ the lattice is

$$H_2(K3, \mathbb{Z}) \cong (-E_8) \oplus (-E_8) \oplus (U_{1,1})^3 \cong (-\text{Spin}(32)/\mathbb{Z}_2) \oplus (U_{1,1})^3. \quad (12)$$

Here, $U_{1,1}$ denotes the unique even selfdual lattice of signature $(1,1)$, with inner product $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$.

The goal of this section is to see the result (12) explicitly, starting from the simplest possible description of K3. In the discussion that follows, we first describe the homology of $H_2$ in terms of the 2-tori inherited from $T^4/\mathbb{Z}_2$ in Sec. 2.4.1. This gives a natural basis of $H_2$ from the point of view of the resolution of $T^4/\mathbb{Z}_2$ in Sec. 2.4.1. This gives a natural basis of $H_2$ in terms of the 2-tori inherited from $T^4$ and the 16 exceptional cycles obtained by resolving the $2^4 A_1$ orbifold singularities. This basis, however, does not generate $H_2(K3, \mathbb{Z})$ with integer coefficients. We refer to this basis as the $(A_1)^{16}$ basis, and compute the intersection pairing in this basis in Sec. 2.4.2 and App. B.1. The lattice splittings (12) arise from the resolution of limits in which K3 develops two $E_8$ singularities or a $D_{16}$ singularity. The remaining sections and App. B.2 relate the $(A_1)^{16}$ basis to $(E_8)^2$ and $D_{16}$ bases, realizing the splittings of Eq. (12).

2.4.1 $(A_1)^{16}$ basis

We now specialize to K3 realized as a Kummer surface. Let $S$ denote the smooth resolution $S$ of the orbifold $T^4/\mathbb{Z}_2$ obtained by blowing up its sixteen $A_1$ orbifold singularities, and write $\pi : S \to T^4/\mathbb{Z}_2$. Choose coordinates $x^m, m = 1, 2, 3, 4$ on $T^4$ with periodicities $x^m \cong x^m + 1$, and consider the quotient by the $\mathbb{Z}_2$ involution

$$\sigma : (x^1, x^2, x^3, x^4) \mapsto (-x^1, -x^2, -x^3, -x^4). \quad (13)$$

The involution has $2^4 = 16$ fixed points labeled by the elements of $(\mathbb{F}_2)^4$, i.e., by coordinates such that $2x^m \in \mathbb{F}_2 = \{0, 1\}$ for $m = 1, 2, 3, 4$.

For $E_8$, the root and weight lattices are the same. $\text{Spin}(32)$ is the universal cover of $SO(32)$. Its weight lattice is the root lattice of $SO(32)$ together with the weights of the vector representation and the spinor representations of each chirality. The center of $\text{Spin}(32)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, of which there are three nontrivial elements, each generating a $\mathbb{Z}_2$. Quotienting by one $\mathbb{Z}_2$ eliminates the spinor weights and gives $SO(32)$. Quotienting by the second $\mathbb{Z}_2$ eliminates the vector and negative chirality spinor representations and leaves the group we have denoted $\text{Spin}(32)/\mathbb{Z}_2$. Quotienting by the third $\mathbb{Z}_2$ eliminates the vector and positive chirality spinor representation.
A part of the (co)homology of $S$ is inherited directly from the $T^4$. Let us focus on $H_2(S, \mathbb{Z})$. The subgroup inherited from $H_2(T^4, \mathbb{Z})$ is generated by “sliding cycles” made up of $\mathbb{Z}_2$ invariant pairs of 4-tori on $T^4$. We label these as follows:

\begin{align*}
  f^1 &= T^2_{x^2 x^3} \times \{ p \cup p' \subset T^2_{x^1 x^4} \}, \\
  f^2 &= T^2_{x^2 x^4} \times \{ p \cup p' \subset T^2_{x^3 x^1} \}, \\
  f^3 &= T^2_{x^3 x^2} \times \{ p \cup p' \subset T^2_{x^1 x^3} \}, \\
  f_1 &= T^2_{x^1 x^4} \times \{ p \cup p' \subset T^2_{x^2 x^3} \}, \\
  f_2 &= T^2_{x^2 x^4} \times \{ p \cup p' \subset T^2_{x^3 x^1} \}, \\
  f_3 &= T^2_{x^3 x^2} \times \{ p \cup p' \subset T^2_{x^1 x^3} \}.
\end{align*}

(14)

Here, $f^1$ is the union of a $T^2$ at fixed $x^1, x^4$ (spanning all possible $x^2, x^3$ values), together with its $\mathbb{Z}_2$ image. The point $p$ is any non fixed point, and $p' = -p$. Poincaré duality\(^8\) identifies $H_2(T^4, \mathbb{Z})$ with $H^2(T^4, \mathbb{Z})$. In cohomology, the same classes $[f^\alpha], [f_\alpha], \alpha = 1, 2, 3$ can be represented by

\begin{align*}
  f^1 &= 2dx^1 \wedge dx^4, \\
  f^2 &= 2dx^2 \wedge dx^4, \\
  f^3 &= 2dx^3 \wedge dx^4, \\
  f_1 &= 2dx^2 \wedge dx^3, \\
  f_2 &= 2dx^3 \wedge dx^1, \\
  f_3 &= 2dx^1 \wedge dx^2.
\end{align*}

(15)

Since $H_2(T^4, \mathbb{Z}) \cong H^2(T^4, \mathbb{Z})$, we use the same notation in either case.\(^9\) In addition, there are 16 exceptional cycles from resolving each of the 16 orbifold singularities of $T^2/\mathbb{Z}_2$.

$$E_I, \quad \text{where} \quad I = 1 \ldots 16. \quad (16)$$

As explained in App. [B.1] the basis $[f^\alpha], [f_\alpha], [E_I]$ generates $H_2(S, \mathbb{R})$ over the reals, but only an order 2 sublattice of $H_2(S, \mathbb{Z})$ over the integers. It misses elements of $H_2(S, \mathbb{R})$ inherited from $\mathbb{Z}_2$ invariant 2-tori in $T^4$, of which there are 24:

\begin{align*}
  D^1_s &= T^2_{x^2 x^3} \times \{ p_s \subset T^2_{x^1 x^4} \}, \\
  D_1s &= T^2_{x^1 x^4} \times \{ p_s \subset T^2_{x^2 x^3} \}, \\
  D^2_s &= T^2_{x^2 x^4} \times \{ p_s \subset T^2_{x^3 x^1} \}, \\
  D_2s &= T^2_{x^3 x^1} \times \{ p_s \subset T^2_{x^2 x^4} \}, \\
  D^3_s &= T^2_{x^3 x^2} \times \{ p_s \subset T^2_{x^1 x^3} \}, \\
  D_3s &= T^2_{x^1 x^3} \times \{ p_s \subset T^2_{x^3 x^2} \},
\end{align*}

(17)

where $p_s$, for $s = 1, 2, 3, 4$, runs over the four $\mathbb{Z}_2$ fixed points $(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}) \in T^2$, respectively. Under $T^4 \rightarrow T^4/\sigma$, each of these twenty four 2-tori maps to a $T^2/\mathbb{Z}_2 \cong S^2$. The four fixed points of the latter coincide with four of the sixteen fixed points of $T^4/\mathbb{Z}^2$. Therefore, after resolution of the 16 orbifold singularities, each intersects 4 exceptional cycles. We show in App. [B.1] that the $D^\alpha_s, D_\alpha s$ have cohomology classes

$$[D^\alpha_s] = \frac{1}{2} [f^\alpha] - \frac{1}{2} \sum_{\text{four } I} E_I \quad \text{and} \quad [D_\alpha s] = \frac{1}{2} [f_\alpha] - \frac{1}{2} \sum_{\text{four } I} E_I, \quad (18)$$

where the four $E_I$ that appear in each case can be found in App. [B.1] The complete integer homology lattice $H_2(S, \mathbb{Z})$ is generated by the integer span of $[f^\alpha], [f_\alpha], [E_I], [D^\alpha_s], [D_\alpha s]$.\(^8\) On a $d$-dimensional manifold $X$, Poincaré duality identifies $[B] \in H_p(X, \mathbb{Z})$ with $[\beta] \in H^{d-p}(X, \mathbb{Z})$, where $\int_B \gamma = \int \gamma \wedge \beta$.\(^9\) For notational simplicity, we also leave the various pullbacks and images implicit and do not distinguish between, for example, $f^1$ as a cycle in $T^4, T^4/\mathbb{Z}_2$ and $S$.\(^9\)
In what follows, we will refer to
\[ \chi^{(A)}_a = (f^\alpha, f_\alpha, E_I) \]  
(19)
as the \((A_1)^{16}\) basis, since the \(\chi^{(A)}_I\), for \(I = 1, \ldots, 16\) have zero volume at the \((A_1)^{16}\) orbifold locus in moduli space. Equivalently, the triple of hyperkähler classes \([J^1], [J^2], [J^3]\) are linear combination of only the \([\chi^{(A)}_\alpha]\) and \([\chi^{(A)}_a]\) at the \((A_1)^{16}\) orbifold point. In Secs. 2.4.3 and 2.4.4 we will define bases \(\chi^{(D)}_a\) and \(\chi^{(E)}_a\) having the analogous property at the \(D_{16}\) and \(E_8 \times E_8\) orbifold loci of the moduli space of K3.

It will be helpful to define also a half-integer basis
\[ \xi^{(A)}_a = (f^a, f_a, e^{(A)}_I), \]  
(20)
where the roots \(\chi^{(A)}_I\) of \((A_1)^{16}\) are related to the orthonormal basis \(e^{(A)}_I\) via
\[ \chi_{2i-1}^{(A)} = e_{2i}^{(A)} - e_{2i-1}^{(A)}, \quad \chi_{2i}^{(A)} = e_{2i}^{(A)} + e_{2i-1}^{(A)}, \quad \text{for } i = 1, \ldots, 16, \]  
(21)
as a consequence of which the cohomology classes \([e^A_I]\) are \(\frac{1}{2}\mathbb{Z}\)-valued. The terms root and orthonormal will become meaningful once we define an inner product.

### 2.4.2 Intersection inner product

The inner product on the (co)homology lattice is the intersection number in homology or, equivalently, the cup product in cohomology. If \([B], [C] \in H_2(S, \mathbb{Z})\) are Poincaré dual to \([\beta], [\gamma] \in H^2(S, \mathbb{Z})\), respectively, then we have
\[ \text{#} \alpha \cap \beta = [\alpha] \cup [\beta] = \int_S \alpha \wedge \beta. \]  
(22)
which we denote by \(\alpha \cdot \beta\). Thus we have
\[ f^\alpha \wedge f_\beta = \frac{1}{2} \int_{T^4/\mathbb{Z}_2} f^\alpha \wedge f_\beta = 2 \delta^\alpha_\beta, \]  
(23)
while \(f^\alpha \cdot f_\beta = f_\alpha \cdot f_\beta = 0\).\(^{10}\) The \(E_I\) give
\[ E_I \cdot E_J = -2, \]  
(24)
since they are 2-spheres.\(^{11}\) Finally, \(E_I \cdot f^\alpha = E_I \cdot f_\alpha = 0\), since the sliding cycles \(f\) of can be moved away from the fixed points of \(T^4/\mathbb{Z}_2\). Thus, in the basis \(\chi^{(A)}_a = (f^\alpha, f_\alpha, E_\alpha)\), the lattice inner product is
\[ \chi^{(A)}_a \cdot \chi^{(A)}_b = \eta^{(A)}_{ab}, \quad \text{with } \eta^{(A)}_{ab} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -(A_1)^{16} \end{pmatrix}, \]  
(25)
\(^{10}\)Alternatively two 2-tori on \(T^4\) intersecting transversely have 1 point of intersection. Each \(f\) is a pair of 2-tori. Therefore, two \(f\)'s intersect in 4 points on \(T^4\), or equivalently, 2 points on \(T^4/\mathbb{Z}_2\).
\(^{11}\)As discussed in Ref. 23, using \(c_1(S) = 0\), the self intersection of a genus \(g\) Riemann surface on \(S\) can be shown to be \(2g - 2\).
where \((- (A_1)^{16})_{IJ} = -2\delta_{IJ}\) is minus the Cartan matrix of \((A_1)^{16}\). If we use the orthonormal basis instead of the root basis, then this becomes

\[
\xi^{(A)}_a \cdot \xi^{(A)}_b = \eta_{ab}, \quad \text{with} \quad \eta_{ab} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\] (26)

As already noted, the basis \(\chi^{(A)}_a = (f^\alpha, f_\alpha, E_I)\) does not generate the (co)homology lattice with integer coefficients, since the \(D^a_s\) have half-integer coefficients in this basis. Equivalently, the integer homology lattice \(H_2(K3, \mathbb{Z})\) does not split as sum of the integer sublattices \(\langle f^\alpha, f_\alpha \rangle\) and \(\langle f^\alpha, f_\alpha \rangle^\perp = (-A_1)^{\oplus 16}\). Here angle brackets denote “span of” and \(\perp\) denotes the orthogonal complement with respect to the inner product \((22)\). Secs. 2.4.3 and 2.4.4, we will see that the \(D_{16}\) and \((E_8)^2\) bases \(\chi^{(D)}_a, \chi^{(E)}_a\) are better behaved in this regard. The integer homology lattice \(H_2(K3, \mathbb{Z})\) does split as \((-\text{Spin}(32)/\mathbb{Z}_2) \oplus U_{3,3}\) and \((-E_8) \oplus (-E_8) \oplus U_{3,3}'\).

**2.4.3 \(D_{16}\) basis**

To relate the \((A_1)^{16}\) (co)homology basis \([\chi^{(A)}_a]\) to a \(D_{16}\) basis \([\chi^{(D)}_a]\) we need to find a linear transformation

\[
[\chi^{(A)}_a] = V^b_a [\chi^{(D)}_b]
\] (27)
such that the \([\chi^{(D)}_a]\) are integer (co)homology classes with inner product

\[
\chi^{(D)}_a \cdot \chi^{(D)}_b = \eta^{(D)}_{ab}, \quad \text{with} \quad \eta^{(A)}_{ab} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -D_{16} \end{pmatrix},
\] (28)

where \((-D_{16})_{IJ}\) is minus the Cartan matrix of \(\text{SO}(32)\).

A convenient way to solve this problem is to first express the \(D_{16}\) roots \(\chi^{(D)}_I\) in terms of an orthonormal basis \(e^{(D)}_I\) in the standard way \([24]\),

\[
\chi^{(D)}_1 = e^{(D)}_1 + e^{(D)}_2, \quad \text{and} \quad \chi^{(D)}_I = e^{(D)}_I - e^{(D)}_{I-1} \quad \text{for} \quad I = 2, \ldots, 16
\] (29)

where \((e^{(D)}_I, e^{(D)}_J) = -\delta_{IJ}\), and then define \(\xi^{(D)}_a\) analogously to \(\xi^{(A)}_a\),

\[
\xi^{(D)}_a = (\chi^{(D)}_a, \chi^{(D)}_a, e^{(D)}_I),
\] (30)

so that

\[
\xi^{(D)}_a \cdot \xi^{(D)}_b = \eta_{ab}.
\] (31)

Here, \(\eta_{ab}\) is as defined in Eq. \([26]\). Then, \([\xi^{(A)}_a] = V^b_a [\xi^{(D)}_b]\), where \(V^b_a\) expressed in the new bases must be an \(\text{SO}(3,19)\) matrix preserving \(\eta_{ab}\),

\[
V\eta V^T = \eta.
\] (32)

The task is to find a \(V\) such that the \([\chi^{(D)}_a]\) are integer classes in \(H_2(K3, \mathbb{Z})\).
This problem is easily solved, with a little bit of inspiration from the type IIA D6/O6 orientifold dual to M-theory on K3. The solution is

\[ V_a^b = V_a^b(x) = \begin{pmatrix} 1 & x^T x & 2x^T \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}, \] (33)

where the \( x^{Ia} \), for \( I = 1, \ldots, 16 \) are given by

\[
\begin{align*}
    x^1 & = (0, 0, 0), & x^9 & = (0, 0, \frac{1}{2}), \\
    x^2 & = (0, 0, 0), & x^{10} & = (0, 0, \frac{1}{2}), \\
    x^3 & = (\frac{1}{2}, 0, 0), & x^{11} & = (\frac{1}{2}, 0, \frac{1}{2}), \\
    x^4 & = (\frac{1}{2}, 0, 0), & x^{12} & = (\frac{1}{2}, 0, \frac{1}{2}), \\
    x^5 & = (0, \frac{1}{2}, 0), & x^{13} & = (0, \frac{1}{2}, \frac{1}{2}), \\
    x^6 & = (0, \frac{1}{2}, 0), & x^{14} & = (0, \frac{1}{2}, \frac{1}{2}), \\
    x^7 & = (\frac{1}{2}, \frac{1}{2}, 0), & x^{15} & = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\
    x^8 & = (\frac{1}{2}, \frac{1}{2}, 0), & x^{16} & = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\
\end{align*}
\]

(34)

in the notation \( x^I = (x^{I1}, x^{I2}, x^{I3}) \). A total of 8 solutions are obtained by replacing these \( x^I \) by \( x^I - x_P \), where \( x_P \) is one of the \( 2^3 = 8 \) \( \mathbb{Z}^2 \) fixed points of \( T^3 \) with all coordinates 0 or 1/2. The proof that the resulting \( \chi^{(D)}_I \) are integral is given in App. 11.2. There, it is also shown that in the \( x_P = 0 \) case the integer (co)homology lattice of K3 splits as \((-\text{Spin}(32)/\mathbb{Z}_2) \oplus U_{3,3}\). Here, \( U_{3,3} = \langle D^a_1, f_a \rangle \) and the weight lattice \((-\text{Spin}(32)/\mathbb{Z}_2) = \langle D^a_1, f_a \rangle ^\perp \) is the integer span of the \( D_{16} \) roots \( \chi^{(D)}_I \) together with the chiral spinor weights differing from \( \frac{1}{2} \sum_{I=1}^{16} \xi^{(D)}_I \) by an even number of sign changes.\(^{12}\)

In the type IIA dual, the \( x^I \) become the locations of the 16 D6-branes on \( T^3/\mathbb{Z}_2 \). There are 8 equivalent \( \text{Spin}(32)/\mathbb{Z}_2 \) loci in moduli space in which all D6 branes coincide with an O6-plane at one of the \( 2^3 = 8 \) \( \mathbb{Z}_2 \) fixed points \( x_P \), and the dual K3 surface develops a \( D_{16} \) singularity. To move to the \( (A_1)^{16} \) locus, one needs to move two D6 branes to each O6-plane, giving “\( D_2 \) = (A_1)^2” gauge symmetry at each. The \( x^I \) in Eq. 34 are the D6-brane locations in this configuration.

### 2.4.4 \((E_8)^2\) basis

To relate the \( (A_1)^{16} \) (co)homology basis \( [\chi^{(A)}_a] \) to the \( E_8 \times E_8 \) basis \( [\chi^{(E)}_a] \) we proceed as in the previous section. We seek a linear transformation

\[
[\chi^{(A)}_a] = W^{b|}_a [\chi^{(E)}_b],
\] (35)

\(^{12}\)An equivalent statement holds for other choices of \( x_P \), where shifting one of the three \( x_P^a \) from 0 to 1/2 replaces \( D^a_1 \) by \( D^{a-1}_1 \). (Replacing \( D^a_1 \) by \( D^{a-1}_1 \) or \( D^{a+1}_1 \) by \( D^{a-1}_1 \) is less interesting and just reverses the sign of \( e^{(A)}_I \) and \( e^{(D)}_I \) for \( I \) odd. It is a Weyl reflection of \( (A_1)^{16} \) and \( \text{Spin}(32)/\mathbb{Z}_2 \).)
such that the $[\chi^{(E)}_a]$ are integer (co)homology classes with inner product

$$\chi^{(E)}_a \cdot \chi^{(E)}_b = \eta^{(E)}_{ab}, \quad \text{with} \quad \eta^{(E)}_{ab} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & (-E_8) \oplus (-E_8) \end{pmatrix},$$  

(36)

where $(-E)_{IJ}$ is minus the Cartan matrix of $E_8$.

Again, a convenient way to solve this problem is to first express the $(E_8)^2$ roots $\chi^{(D)}_I$ in terms of an orthonormal basis $e^{(D)}_I$ in the standard way [24],

$$\chi^{(E)}_1 = e^{(E)}_1 + e^{(E)}_2, \quad \chi^{(E)}_I = e^{(E)}_I - e^{(E)}_{I-1} \quad (I = 2, \ldots, 7), \quad \chi^{(E)}_8 = \frac{1}{2} (e^{(E)}_8 + e^{(E)}_1 - \sum_{I=2}^7 e^{(E)}_I),$$

$$\chi^{(E)}_9 = e^{(E)}_9 + e^{(E)}_{10}, \quad \chi^{(E)}_I = e^{(E)}_I - e^{(E)}_{I-1} \quad (I = 10, \ldots, 15), \quad \chi^{(E)}_{16} = \frac{1}{2} (e^{(E)}_{16} + e^{(E)}_9 - \sum_{I=10}^{15} e^{(E)}_I),$$

(37)

where $e^{(E)}_I \cdot e^{(E)}_J = -\delta_{IJ}$, and then define $\xi^{(E)}_a$ analogously to $\xi^{(A)}_a$,

$$\xi^{(E)}_a = (\chi^{(E)}_a, \chi^{(E)}_a, e^{(E)}_I),$$

(38)

so that

$$\xi^{(E)}_a \cdot \xi^{(E)}_b = \eta_{ab}.$$  \hspace{1cm} (39)

Here, $\eta_{ab}$ is again as defined in Eq. (26). Then, $[\xi^{(A)}_a] = W_a^b [\xi^{(E)}_b]$, where $W_a^b$ expressed in the new bases must be an SO(3, 19) matrix preserving $\eta_{ab}$,

$$W \eta W^T = \eta.$$  \hspace{1cm} (40)

The task is to find a $W$ such that the $[\chi^{(E)}_a]$ are integer classes in $H_2(K3, \mathbb{Z})$.

Since the $E_8 \times E_8$ point in the dual IIA D6/O6 orientifold cannot be obtained solely by displacement of D6-branes, we do not expect a transformation of the form [25]. The solution is as follows. For each $I$, let $x^I_\alpha$ denote coordinates on the $T^3$ dual to that of $x^I\alpha$, and define an SO(3, 3 + 16) matrix $\tilde{V}(\tilde{x})$ by

$$\tilde{V}_a^b(\tilde{x}) = \begin{pmatrix} 1 & 0 & 0 \\ x^T \tilde{x} & 1 & 2 \tilde{x}^T \\ \tilde{x} & 0 & 1 \end{pmatrix}. \hspace{1cm} (41)$$

As shown in App. B.2, a (co)homology basis for the $(D_8)^2$ locus in moduli space, at which the K3 surface develops two $D_8$ singularities, can be obtained two equivalent ways. Starting from the $D_{16}$ locus, we can act on the $D_{16}$ basis $\chi^{(D)}_b$ with $V_a^b(y)$, where

$$y^{I_3} = (0^8, \frac{18}{2}), \quad y^{I_1} = y^{I_2} = 0. \hspace{1cm} (42)$$
This has the dual IIA D6/O6 interpretation of displacing 8 of the 16 D6-branes from the fixed point at (0, 0, 0) to the fixed point at (0, 0, 1/2).

Alternatively, starting from the \((E_8)^2\) locus in moduli space, at which the K3 surface develops two \(E_8\) orbifold singularities, we can act on the \((E_8)^2\) basis \(\chi^{(E)}_b\) with \(\bar{V}_a^b(y)\), where

\[
\bar{y}'_3 = (0^7, 1; 0^7, 1), \quad \bar{y}'_1 = \bar{y}'_2 = 0.
\]

Combining these results, the \((D_8)^2\) basis is given by

\[
\xi_a = V_a^b(y)\xi^{(D)}_b = \bar{V}_a^b(-y)\xi^{(E)}_b.
\]

This agrees with the discussion of Wilson lines in Eqs. (11.6.18) and (11.6.19) of Ref. [45].

We deduce that

\[
\xi^{(E)}_b = W_a^b\xi^{(D)}_b, \quad \text{for} \quad W_a^b = \bar{V}_a^c(-y)V_c^b(y)
\]

where we have used the fact that the inverse of \(\bar{V}(y)\) is \(\bar{V}(-y)\).

The \((E_8)^2\) basis has a simple interpretation along lines analogous to those at the end of Sec. 2.4.3. The integer cohomology lattice of K3 splits as \((-E_8) \oplus (-E_8) \oplus U_{3,3}'\), where

\[
U_{3,3}' = \langle D_4^1, D_4^2, D_4^3, D_{34}, f_1, f_2, f_3 \rangle \quad \text{and} \quad (-E_8) \oplus (-E_8) = \langle D_4^1, D_4^2, D_{34}, f_1, f_2, f_3 \rangle, \quad (46)
\]

i.e., by reversing the role of “upper 3” and “lower 3” in the \((-\text{Spin}(32)/\mathbb{Z}_2) \oplus U_{3,3}\) splitting described at the end of the previous section. Other realizations of the \((E_8) \oplus (-E_8) \oplus U_{3,3}'\) splitting are obtained by reversing the role of upper/lower \(\alpha = 1\) or 2 instead of the 3, or by trading a \(D_4^\alpha\) or \(D_{a4}\) for a \(D_4^\alpha\) or \(D_{a2}\).

### 2.5 Moduli space of hyperkähler structure

The cohomology classes \([\xi^{(D)}] = (\xi^{(D)\alpha}, \xi^{(D)\alpha}, \xi^{(D)}_I) \in \Omega_2(\text{K3}, \mathbb{Z}_2)\) form a basis for \(H^2(\text{K3}, \mathbb{R})\). From Eq. (26), the pairing

\[
\alpha \cdot \beta = \int_{\text{K3}} a \wedge \beta
\]

(47)

gives a signature \((3, 19)\) inner product on \(H^2(\text{K3}, \mathbb{R})\). A choice of hyperkähler structure on K3 is equivalent to a choice of positive signature 3-plane in \(H^2(\text{K3}, \mathbb{R})\). This is the choice of three orthogonal kähler classes \([J^a]\), \(\alpha = 1, 2, 3\) of positive norm, modulo rescalings and SO(3) rotations \([J^a] \mapsto O^{\alpha\beta}[J^\beta]\). The choice can be parametrized as follows.

Let

\[
[\omega_a] = V_a^b[\chi_b],
\]

(48)

---

13 The context in Ref. [45] is the T-duality between the \(\text{Spin}(32)/\mathbb{Z}_2\) and \(E_8 \times E_8\) heterotic strings compactified on a circle. The fact that the \(\text{Spin}(32)/\mathbb{Z}_2\) and \(E_8 \times E_8\) Wilson lines are defined on T-dual circles is the same reason that transformations (33) and (41) depend on dual moduli \(y'^3\) and \(\bar{y}'_3\).
where $V_{ab} \in \text{SO}(3, 19)$ satisfies $V \eta V^T = \eta$. Up to a left $\text{SO}(3) \times \text{SO}(19)$ rotation, an arbitrary $\text{SO}(3, 19)$ matrix can be written in the form

$$
V(E, \beta, x) = V(E)V(\beta)V(x) = 
\begin{pmatrix}
E & 0 & 0 \\
0 & E^{-1T} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\beta & -0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^T & 2x^T \\
0 & 1 & 0 \\
x & 1
\end{pmatrix}
$$

(49)

Here, $E_{\alpha\beta} \in \text{GL}(3, \mathbb{R})$, $\beta_{\alpha\beta}$ is antisymmetric $3 \times 3$ matrix, $x^{I\alpha}$ is an arbitrary $16 \times 3$ matrix, and $C_{\alpha\beta} = \beta_{\alpha\beta} - \delta_{IJ}x^{I\alpha}x^{J\beta}$. In components,

$$
[\omega^\alpha] = [E_{\alpha\beta}](\xi^{(D)}_{\alpha} - C_{\alpha\beta}[\xi^{(D)}_{\beta}] + 2x^{J\alpha}[\xi^{(D)}_J]),
$$

$$
[\omega_\alpha] = (E^{-1T})_{\beta\alpha}[\xi^{(D)}_{\beta}],
$$

$$
[\omega_I] = [\xi_I] + x^{I\beta}[\xi^{(D)}_{\beta}].
$$

(50)

A corresponding triple of Kähler classes is then

$$
[J^\alpha] = \sqrt{\frac{V_{K3}}{2}}[\omega_\alpha + \omega^\alpha], \quad \alpha = 1, 2, 3,
$$

(51)

where we have chosen the prefactor so that

$$
V_{K3} = \frac{1}{2} \int_{K3} J^\alpha \wedge J^\alpha \quad \text{(no sum)}
$$

(52)

is the volume of K3, for $\alpha = 1, 2, 3$. Note that the $\omega_\alpha$ have the same intersections as the $\xi^{(D)}_a$,

$$
\int_{K3} \omega_\alpha \wedge \omega_b = \eta_{ab},
$$

(53)

with $\eta$ defined in Eq. (26). So far, we have defined the classes $[\omega_\alpha]$, but not the differential forms themselves. For later convenience, we define the $\omega_\alpha$ to be the harmonic representatives of these (co)homology classes. Whenever we refer to the $\omega_\alpha$ below, we will have in mind this definition.

The identification of the triple $[J^\alpha]$ under $\text{SO}(3)$ rotation is equivalent to the identification of $E_{\alpha\beta}$ under $\text{SO}(3)$ left multiplication. Therefore, the moduli space of hyperkähler structure of $K3$ is the space of volumes $V_{K3}$ and of $\text{SO}(3, 3 + 16)$ matrices $V(\alpha, \beta, E)$ modulo $\text{SO}(3)$ rotations of $E$. Interpreting $E$ as a vielbein for the metric $G_{\alpha\beta} = (E^T E)_{\alpha\beta}$, the hyperkähler moduli space is that of $V_{K3}$, $G_{\alpha\beta}$, $\beta_{\alpha\beta}$, and $x^{I\alpha}$. We can view $V(E, \beta, x)$ as a vielbein for an arbitrary symmetric $\text{SO}(3, 3 + 16)$ matrix $M$ satisfying

$$
M^T \eta M = \eta^{-1}.
$$

(54)

Note that $O^{-1T} = O$, for $O \in \text{SO}(3, \mathbb{R})$, so that $\omega^\alpha$ and $\omega_\alpha$ transform identically under $\text{SO}(3)$. 

by writing
\[ M = V^T K^{-1} V, \] (55)
where \( K_{ab} \) is defined below in Eq. (58). Explicitly, we have
\[
M^{ab} = \begin{pmatrix}
\frac{1}{2} G & \frac{1}{2} G C - \frac{1}{2} C T x & G x^T \\
-\frac{1}{2} C T G + \frac{1}{2} C T x & -x G C + x & x^T x \\
x G & -2 x G x^T + 1
\end{pmatrix}. \] (56)

The moduli space of hyperkähler structure is
\[
\mathcal{M}_{HK} = \mathbb{R}_{>0} \times (\text{SO}(3) \times \text{SO}(19)) \backslash \text{SO}(3, 19)/\Gamma_{3,19}, \] (57)
where the first factor is the overall volume \( V_{K3} \) and the second factor can be interpreted as the moduli space of hypercomplex structure. Here, \( \Gamma_{3,19} \) is the group of lattice automorphisms of the K3 integer (co)homology lattice. The natural coset metric is
\[
ds^2 = -\frac{1}{8} \text{Tr}(dM d(M^{-1})) = \frac{1}{8} \text{Tr}(M^{-1} dM M^{-1} dM), \] (58)
where we have chosen the normalization factor for later convenience. In terms of \( G, \beta, \) and \( x, \) this becomes
\[
ds^2 = \frac{1}{4} G_{\alpha\gamma} G_{\beta\delta} \left( dG^{\alpha\beta} dG^{\gamma\delta} + \tilde{d}_{\alpha\beta} \delta_{\gamma\delta} \right) + 2 \delta_{1,1} G_{\alpha\beta} x^{I\alpha} dx^{I\beta}, \] (59)
where
\[
\tilde{d}_{\alpha\beta} = d\beta_{\alpha\beta} - x^{I\alpha} dx^{I\beta} + x^{I\beta} dx^{I\alpha}. \] (60)
From lattice isomorphisms, the \( x^{I\alpha} \) and \( \beta^{I\alpha} \) can be shown to be periodic with period 1. The space \( \mathcal{M}_{HK} \) can be viewed as a “fibration over a fibration.” The moduli space of 3D metrics \( G \) is the base manifold \( \text{SO}(3) \backslash \text{GL}(3) \). The \( 3 \times 16 = 48 \) periodic \( x^{I\alpha} \) parametrize a \( U(1)^{48} \) fibration over this base. And finally, the \( 3 \) periodic \( \beta^{I\alpha} \) parametrize a \( U(1)^{3} \) fibration over the result. The quantities \( dG^{\alpha\beta}, dx^{I\alpha}, \) and \( \tilde{d}_{\alpha\beta} \) are global 1-forms on \( \mathcal{M}_{HK} \), but the \( d\beta_{\alpha\beta} \) are not, since they shift under \( x^{I\alpha} \rightarrow x^{I\alpha} + 1 \). The \( \tilde{d}_{\alpha\beta} \) are the global 1-form in the \( U(1)^{3} \) fiber directions, with connections
\[
A^{\alpha\beta} I_{\gamma} dx^{I\gamma} = -x^{I\alpha} dx^{I\beta} + x^{I\beta} dx^{I\alpha}. \] (61)
These connections are dual to abelian part the Chern-Simons term of
\[
\tilde{H}_{\text{Het}} = dB - \frac{(2\pi \ell)^2}{4} \text{Tr}_f \left( A \wedge dA - \frac{2}{3} A \wedge A \wedge A \right), \] (62)
in the duality between M-theory on \( K3 \) and the heterotic string on \( T^3 \). Here \( \ell = \sqrt{\alpha'} \). In Sec. 2.7.2 we will see that Eq. (59) agrees with the metric on the metric moduli space of K3.
2.6 Hodge duality and harmonic forms

Given a metric on $K3$, we can define the Hodge star operator, and a positive definite inner product on $H^2(K3, \mathbb{R})$,

$$(\lambda_1, \lambda_2) = \int_{K3} \lambda_1 \wedge \ast \lambda_2.$$  \hfill (63)

If a 2-form is self-dual or anti-self-dual, $\ast \lambda = \pm \lambda$, then $\lambda \cdot \lambda = \pm (\lambda, \lambda)$ respectively. Consequently, there is a decomposition

$$H^2(K3, \mathbb{R}) = H^+ \oplus H^-,$$  \hfill (64)

where $H^+$ ($H^-$) denotes the (anti)-selfdual subspace of $H^2(K3, \mathbb{R})$. By acting with $\pi_{\pm} = \frac{1}{2}(1 \pm \ast)$, we can project any 2-form to its selfdual or anti-selfdual component. However, $\ast$ need not map closed forms to closed forms, so that the result of applying $\pi_{\pm}$ to an arbitrary representative of a cohomology class is not necessarily closed. Implicit in Eq. (64), is that we must use harmonic representatives. The projector $\pi_{\pm}$ indeed maps harmonic forms to harmonic forms, and with this restriction maps closed forms to closed forms.

It is straightforward to show that the triple of Kähler forms satisfy $\ast J^\alpha = J^\alpha$.\footnote{We have $J = \frac{1}{2}J_{ab}dx^a \wedge dx^b$, where $J_{ab} = J_{(a}^c g_{c'b)}$. In complex coordinates, $J^{\hat{k}} = i \delta^{\hat{k}} \hat{i}$ and $J_{\hat{i}} = -i \delta_{\hat{i}}^{\hat{k}}$. For a Hermitian metric, $ds^2 = g_{ij}dz^i \wedge dz^j + g_{ij}dz^i \wedge dz^j$, with $g_{ij} = (g_{ij})^*$. Thus, $J = ig_{ij}dz^i \wedge d\bar{z}^j$. Applying the standard definition of Hodge duality, $\ast J = J$ follows.} Thus,

$$H^+ = \langle \frac{1}{2}[\omega_\alpha + \omega^\alpha] \rangle, \quad \text{and} \quad H^- = \langle \frac{1}{2}[\omega_\alpha + \omega^\alpha] \rangle^\perp = \langle \frac{1}{2}[\omega_\alpha - \omega^\alpha], [\omega_I] \rangle,$$  \hfill (65)

where angle brackets denote “span of” and $\perp$ denotes the orthogonal complement with respect to either of the two inner products. It is useful to introduce the notation

$$\omega^+_\alpha = \frac{1}{2}(\omega_\alpha + \omega^\alpha), \quad \omega^-_\alpha = \frac{1}{2}(\omega_\alpha - \omega^\alpha), \quad \text{with} \quad \ast \omega^\pm_\alpha = \pm \omega^\pm_\alpha.$$  \hfill (66)

Equivalently,

$$\ast[\omega^\alpha] = [\omega_\alpha], \quad \ast[\omega_\alpha] = [\omega^\alpha], \quad \ast[\omega_I] = -[\omega_I],$$  \hfill (67)

from which

$$\int_{K3} [\omega_\alpha] \wedge \ast [\omega_\beta] = K_{ab}, \quad \text{with} \quad K_{ab} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hfill (68)

When expressed in terms of the cohomology classes $[\xi^{(D)}_a]$, the last equation becomes

$$\int_{K3} [\xi^{(D)}_a] \wedge \ast [\xi^{(D)}_b] = M_{ab},$$  \hfill (69)

where $M_{ab}$ is the inverse of the moduli matrix $M^{ab}$ of Eq. (56).
2.7 K3 metric moduli space

In this final section of our review of the geometry of K3, we describe the metric on the moduli space of K3 metrics. We require that this metric be invariant under diffeomorphisms \( K3 \rightarrow K3 \). To achieve the diffeomorphism invariance, compensating vector fields are introduced in Sec. 2.7.1 to project generic metric deformations to transverse traceless gauge. Sec. 2.7.2 relates the transverse traceless deformations to harmonic forms, using the hypercomplex structure to convert differential forms to symmetric tensors.

Background material accompanying this chapter can be found in the Appendices. App. C reviews the Lichnerowicz operator, which acts on a metric deformation to give the deformation of the Ricci tensor. As explained in App. C for a Kähler manifold, the complex structure relates the action of the Laplace-de Rham operator on \((1,1)\)-forms to the action of the Lichnerowicz operator on symmetric tensors. For a Calabi-Yau \( n \)-fold, the holomorphic \((n,0)\)-form achieves the same map for \((1,2)\)-forms. App. D applies this observation to the correspondence between harmonic forms and transverse traceless metric deformations preserving the Ricci-flatness of K3.

2.7.1 Diffeomorphism invariant metric on moduli space

Given a \( d \)-dimensional manifold \( X \) and a family of metrics,

\[
ds^2 = g_{mn}(x; \mu) dx^m dx^n,
\]

parametrized by a moduli space \( \mathcal{M} \) with coordinates \( \mu^i \), a natural guess for the metric on moduli space is

\[
ds^2_\mathcal{M} = \frac{1}{4} \int_X d^d x \sqrt{g} g^{mp} g^{nq} \delta g_{mn} \delta g_{pq} = K_{ij} \delta \mu^i \delta \mu^j,
\]

where \( K_{ij} = \frac{1}{4} \int_X d^d x \sqrt{g} g^{mp} g^{nq} \partial_i g_{mn} \partial_j g_{pq} \) and \( \partial_i g_{mn} = \partial g_{mn} / \partial \mu^i \).

However, this guess is not correct, since we would like the moduli space metric to be purely horizontal in the space of gauge orbits, i.e., diffeomorphism invariant. The expression (71) is not diffeomorphism invariant.

For simplicity, let us restrict to metrics on \( X \) of fixed overall volume \( V_X \). Then solution is as follows. One finds that the transverse traceless gauge of metric deformations is the distinguished purely horizontal gauge \([18, 57, 40]\). For a coordinate chart on \( X \) such that the \( \partial_i g_{mn} \) defined above are transverse and traceless,

\[
\nabla^m \partial_i g_{mn} = 0, \quad g^{mn} \partial_i g_{mn} = 0,
\]

Eq. (71) indeed gives the correct metric on moduli space. More generally, a metric deformation \( \delta g_{mn} = \partial_i g_{mn}(x; \mu) \delta \mu^i \) must be combined with a moduli-dependent diffeomorphism

\[
y^m = x^m - N^m_i(x; \mu) \delta \mu^i
\]
to ensure that this is the case. The quantity $N^m_i \delta \mu^i$ is known as the *compensating vector field* on $X$ (or simply, the *compensator*) corresponding to a change $\delta \mu^i$ in moduli. If we define the $\partial_m g_{mn}$ in the privileged coordinates $y^m$, and then pullback to the original coordinates $x^n$ via

$$g'_{mn}(x, \mu) = g_{pq}(y(x), \mu) \frac{\partial y^p}{\partial x^m} \frac{\partial y^q}{\partial x^n},$$

we find that the corresponding transverse traceless metric deformation $\delta^\perp g_{mn} = \partial_i g_{mn}(x; \mu)$ is

$$\delta^\perp g_{mn} = \partial_i g_{mn} - \mathcal{L}_{N_i} \delta \mu^i g_{mn} = \partial_i g_{mn} - \delta \mu^i(\nabla_m N_n - \nabla_n N_m).$$

(75)

Here, in the first equality, $\mathcal{L}_{N_i}$ denotes the Lie derivative with respect to the compensating vector field. In the second equality, $N_{im} = g_{mn} N^m_i$. The diffeomorphism invariant metric on moduli space is

$$ds^2_{\mathcal{M}} = \frac{1}{4} \int_X d^d x \sqrt{g} g^{mp} g^{nq} \delta^\perp g_{mn} \delta^\perp g_{pq} = K_{ij} \delta \mu^i \delta \mu^j,$$

(76)

where

$$K_{ij} = \frac{1}{4} \int_X d^d x \sqrt{g} g^{mp} g^{nq} \delta^\perp g_{mn} \delta^\perp g_{pq}.$$

Compensators are a necessary feature in all but the simplest string theory compactifications. They modify the naive product metric ansatz in such a way that the moduli kinetic terms come with the correct horizontal moduli space metric \[16\] \[18\] \[57\] \[40\].

### 2.7.2 K3 metric deformations and harmonic forms

The relation between harmonic forms and metric deformations for a Calabi-Yau $n$-fold is reviewed in App. D. The K3 case $n = 2$ is special in that $(1, 1)$-forms generate both Kähler and complex structure deformations. As we have already noted, $H^2(K3, \mathbb{R})$ has signature $(3, 19)$ with respect to the inner product $(\alpha, \beta) = \int \alpha \wedge * \beta$. The hyperkähler 2-forms $J^a$,

$$J^1 = -\text{Re} \Omega, \quad J^2 = \text{Im} \Omega, \quad \text{and} \quad J^3 = J \quad (77)$$

span the selfdual subspace $\mathcal{H}^+$ of signature $(3, 0)$. Their orthogonal complement in $H^2(K3, \mathbb{R})$, of signature $(0, 19)$ is the anti-selfdual subspace $\mathcal{H}^-$ of primitive $(1, 1)$-forms. We can think of the 58 metric deformations of a K3 surface in at least two ways:

#### Complex plus Kähler deformations

There are 20 (real) Kähler deformations and 19 (complex) complex structure deformations generated by $J$ together with the 19 primitive $(1, 1)$-forms. Since the complex structure deformation generated by $J$ leads to vanishing metric deformation, we have a total of $20 + 2 \times 19 = 58$ real metric deformations from

$$h_{mn}(J, \omega) = -\frac{1}{2} (J_m^p \omega_{pm} + J_n^p \omega_{pm}) \quad (\text{Kähler deformations}) \quad (78)$$

\[^{16}\text{Recall that a cohomology class } [\omega] \text{ is said to be } \text{primitive} \text{ when it is not of the form } [J] \wedge [\omega'] \text{ for some } \omega'. \text{ The class } [J] \text{ itself is nonprimitive, since it is of the form } [J] \wedge [1].\]
with \( \omega \) a \((1,1)\)-form and

\[
h_{mn}(\Omega, \omega) = -\frac{1}{2} (\Omega_m^p \omega_{pn} + \Omega_n^p \omega_{pm}) \quad \text{(complex structure deformations)}
\]  

(79)

with \( \omega \) a primitive \((1,1)\)-form.

Note that from \( h_{ij} = -i \omega_{ij} \) in the Kähler case, we obtain a metric deformation \( \delta g_{ij} \propto g_{ij} \) for \( \omega = J \). Thus the Kähler deformation associated to \( J \) simply scales the overall volume of the K3 surface. As a consequence of the discussion in App. [D] all of the deformations (78) and (79) are transverse, and all but the overall volume deformation are traceless.

**Hypercomplex plus volume deformations**

There are \( 3 \times 19 \) (real) hypercomplex metric deformations generated by the 19 primitive \((1,1)\)-forms \( \omega \),

\[
h_{mn}(J^\alpha, \omega) = -\frac{1}{2} ((J^\alpha)_m^p \omega_{pn} + (J^\alpha)_n^p \omega_{pm}) ,
\]  

(80)

plus 1 overall volume deformation, for a total of \( 1 + 3 \times 19 = 58 \) metric deformations.

**Moduli space metric from harmonic forms**

Let \( \omega_A \), for \( A = 1, \ldots, 19 \), denote a basis for the space \( H^- \) of anti-selfdual harmonic forms on K3. Then, a general volume-preserving metric deformation in transverse traceless gauge can be written

\[
\delta(ds^2) = \delta \mu^A \alpha h(J^\alpha, \omega_A), \quad \text{where} \quad h = h_{mn} dx^m dx^n,
\]  

(81)

in terms of \( 3 \times 19 \) deformation parameters \( \delta \mu^A \alpha \). It is possible to show that

\[
\int_{K3} d^4 x \sqrt{g} h_{mn}(J^\alpha, \omega_A) h^{mn}(J^\beta, \omega_B) = \delta^{\alpha\beta} \int_{K3} \omega_A \wedge * \omega_B.
\]  

(82)

Therefore, in this parametrization, the moduli space metric is

\[
ds^2_{\mathcal{M}} = \delta^{\alpha\beta} K_{AB} \delta \mu^A \alpha \delta \mu^B \beta, \quad \text{where} \quad K_{AB} = \frac{1}{4} \int_{K3} \omega_A \wedge * \omega_B.
\]  

(83)

In the notation of Secs. [2.5] and [2.6] the space of anti-selfdual harmonic forms \( H^- \) is spanned by \( \omega^-_\alpha \) and \( \omega_I \). Therefore, a basis of transverse traceless metric deformations is

\[
(h^C)_{\alpha\beta} = \frac{1}{4} (E^{-1T})_{\alpha}^{\alpha'} (E^{-1T})_{\beta}^{\beta'} (h(J_{\alpha'}, \omega^-_{\beta'}) + h(J_{\beta'}, \omega^-_{\alpha'})),
\]  

(84a)

\[
(h^\beta)_{\alpha\beta} = \frac{1}{4} (E^{-1T})_{\alpha}^{\alpha'} (E^{-1T})_{\beta}^{\beta'} (h(J_{\alpha'}, \omega^-_{\beta'}) - h(J_{\beta'}, \omega^-_{\alpha'})),
\]  

(84b)

\[
(h^x)_{I\alpha} = (E^{-1T})_{\alpha}^{\alpha'} h(J_{\alpha'}, \omega_I),
\]  

(84c)

\[23\]
where \( J_\alpha = \delta_{\alpha\beta} J^\beta = J^\alpha \), and where for convenience below, the factors of \( E^{-1} T \) have been included to convert 3D frame indices to coordinate indices. Then, an arbitrary transverse traceless metric deformation of K3 can be parametrized as

\[
\delta(ds^2) = \delta G^{\alpha\beta} (h^G)_{\alpha\beta} + \tilde{\delta} \beta^{\alpha\beta} (h^\beta)_{a\beta} + \delta x^I \alpha (h^x)_{I\alpha},
\]

where we define \( \tilde{\delta} \beta^{\alpha\beta} \) as in Eq. (60). In this parametrization, results (82) and (83) give moduli space metric

\[
ds^2_M = \frac{1}{4} G_{\alpha\alpha'} G_{\beta\beta'} \left( \delta G^{\alpha\beta} \delta G^{\alpha'\beta'} + \tilde{\delta} \beta^{\alpha\beta} \tilde{\delta} \beta^{\alpha'\beta'} \right) + 2 \delta_{IJ} G_{\alpha\beta} \delta x^I \alpha \delta x^J \beta,
\]

which is the SO(3, 19) coset metric (59).

**Metric deformations generated by nonharmonic forms**

As a final generalization, let \( \omega \) denote an anti-selfdual harmonic 2-form on K3 and consider another representative \( \omega' = \omega + d\lambda \) of the same cohomology class in \( H^2(K3, \mathbb{R}) \). Then, it is possible to show that

\[
h_{mn}(J^\alpha, \omega') = h_{mn}(J^\alpha, \omega) - \nabla_m N_n - \nabla_n N_m, \quad \text{where} \quad N_m = -(J^\alpha)_m n \lambda_n,
\]

so that the exact piece of \( \omega' \) generates a diffeomorphism. Thus, general metric deformations,

\[
\delta(ds^2) = \delta G^{\alpha\beta} (h^G)_{\alpha\beta} + \tilde{\delta} \beta^{\alpha\beta} (h^\beta)_{a\beta} + \delta x^I \alpha (h^x)_{I\alpha} \quad \text{diffeomorphisms},
\]

not necessarily transverse, are generated by cohomology representatives that are not necessarily harmonic.

### 3 K3 metric in the Gibbons-Hawking approximation

Having reviewed the geometry of K3, we now turn the explicit, but approximate description of this geometry in terms of the \( \mathbb{Z}_2 \) quotient of a metric of Gibbons-Hawking form. The latter describes a \( U(1) \) principal bundle over \( T^3 \). It is hyperkähler and Calabi-Yau, and positive definite away from neighborhoods of the \( 2^3 \ \mathbb{Z}_2 \) fixed points on \( T^3 \). These neighborhoods becomes arbitrarily small in the large hypercomplex structure limit of small \( U(1) \) fiber and large base. The exact K3 metric differs from the approximate one by replacing these neighborhoods with Atiyah-Hitchin spaces, which smoothly excise the regions in which the metric becomes negative. The results of this section are as follows.

Let us write the metric on K3 as

\[
ds^2_{K3} = (2V_{K3})^{1/2} G_{mn} dx^m dx^n,
\]

where \( G_{mn} \) is the metric for a “unit” K3 of volume \( \frac{1}{2} \), obtained from the resolution of \( T^4/\mathbb{Z}_2 \) for \( T^4 \) of volume 1.
From the duality between M-theory on K3 and type IIA on the $T^3/\mathbb{Z}_2$ orientifold, truncated to the tree-level type IIA supergravity description, we obtain a first-order K3 metric (89) of Gibbons-Hawking form

$$G_{mn}dx^m dx^n = \Delta^{-1} Z \epsilon_{\alpha\beta} dx^\alpha dx^\beta + \Delta Z^{-1} (dx^4 + A)^2.$$  \tag{90}

This metric is derived in Sec. 3.3, after a review of Gibbons-Hawking multicenter metrics in Sec. 3.1 and M-theory/type IIA duality in Sec. 3.2. Here, coordinates $x^m$, for $m = 1, 2, 3, 4$, have periodicity 1, and are subject to an additional $\mathbb{Z}_2$ identification under the involution

$$I_4: x^m \mapsto -x^m.$$  \tag{91}

This metric is that of a circle bundle over $T^3$, quotiented by $\mathbb{Z}_2$.

The quantity $G_{\alpha\beta}$, for $\alpha, \beta = 1, 2, 3$, is a constant metric on $T^3$ of volume $\Delta = \det^{1/2} G$. The function $Z$ on $T^3$ satisfies the Poisson equation

$$-\nabla^2 Z = \sum_{\text{sources } s} Q_s \delta^3(x - x^s),$$  \tag{92}

where the index $s$ runs over (i) 16 points $x^I$ on $T^3$ with $Q_I = 1$, (ii) 16 image points $x^{I'} = -x^I$ with $Q_{I'} = 1$, and (iii) the $2^3 = 8$ fixed points $x^{O_i}$ of $I_3$: $x^a \to x^a$ on $T^3$ with $Q_{O_i} = -4$. The additive constant in $Z$ is chosen so that when two $x^I$ (and their two images $x^{I'}$) coincide with each $x^{O_i}$, we have $Z = 1$, and the metric (90) becomes the orbifold metric on $T^4/\mathbb{Z}_2$.

The connection $A$ is defined by

$$dA = \star_G dZ,$$  \tag{93}

where we choose a gauge condition $G^{\alpha\beta} \partial_\alpha A_\beta = 0$. This determines $A$ only up to a closed 1-form. We fix this ambiguity by setting $A$ equal to an arbitrary constant 1-form $\beta_\alpha dx^\alpha$ at the $T^4/\mathbb{Z}_2$ orbifold locus $x^I = 0$, and

$$\delta A = \delta x^{I\beta} \left( (F_I - F_{I'})_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} x^{I\gamma} \right) dx^\alpha + \delta \beta_\alpha dx^\alpha$$

away from this locus. Here, $F_I = dA_I$ and $F_{I'} = dA_{I'}$ are defined in Eqs. (114) and (122) of Sec. 3.3.

As $x$ approaches $x^I$, the $S^1$ fiber shrinks and the metric is locally that of a smooth Taub-NUT space. On the other hand, near the fixed points $x^{O_i}$, the metric (90) is locally that of Taub-NUT space of *negative* mass parameter. This space is not itself well behaved, and is the large distance approximation to a smooth Atiyah-Hitchin space [55]. The latter is obtained locally in the full nonperturbative lift of type IIA string theory to M-theory [55]. For $\Delta \ll 1$, the metric (90) closely approximates the exact K3 metric everywhere except in a small neighborhood of each $x^{O_i}$, as explained in more detail in Secs. 3.3.2 and 3.3.3 below.

In Sec. 3.4 we write down the hyperkähler forms and a basis of harmonic forms in this metric, showing that the basis approximates that of Sec. 2.5 and has the same inner product. Therefore, the moduli space of hyperkähler structure is identical to that of Sec. 2.5.

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Finally we turn to the metric moduli space in Sec. 3.5. To describe this space, it is useful to write $\beta^{\alpha\beta} = \epsilon^{\alpha\beta\gamma} \beta_{\gamma}$. Then, the approximate metric depends on parameters $G_{\alpha\beta}$, $\beta^{\alpha\beta}$, $x^I$, and the overall volume modulus $V_{K3}$. It is clear that these parameters should determine the moduli as defined in Secs. 2.5 and 2.7.2 and we have suggestively given them the same names. In Sec. 3.5.1, we show that the metric deformations from small changes in the quantities $G_{\alpha\beta}$, $\beta^{\alpha\beta}$, $x^I$ parametrizing the approximate metric precisely agree with the metric deformations generated by the harmonic forms of Sec. 3.3 in the manner described in Sec. 2.7.2. It follows from this and the results of the previous paragraph that the moduli space metric of the approximate K3 metric is the same $R^0 \times (SO(3) \times SO(19)) \backslash SO(3, 19)$ coset metric of the exact discussion in Sec. 2.

As discussed in Sec. 2.7.1, we require that metric on metric moduli space be invariant under diffeomorphisms $K3 \to K3$. To achieve the diffeomorphism invariance, compensating vector fields were introduced in Sec. 2.7.1 to project generic metric deformations to transverse traceless gauge. In Sec. 3.5.2, we consider the moduli space metric from naive dimensional reduction of the $D$-dimensional Einstein-Hilbert action on a $d$-dimensional manifold $X$ ask when this gives the correct result, without compensators. This metric differs from the previous one in two ways: there is no projection of metric deformations to their transverse traceless part, and there are additional terms in the metric obtained by integrating $(\bar{G}^{mn} \delta G_{mn})^2$. We find that Gibbons-Hawking metrics are special in that their naive moduli space metrics precisely agree with the diffeomorphism invariant moduli space metrics of Sec. 2.7.1. Therefore compensators are not necessary, and one can equivalently use the naive metrics from dimensional reduction. We show explicitly that the naive moduli space metric for K3 in the Gibbons-Hawking approximation exactly reproduces the $R^0 \times (SO(3) \times SO(19)) \backslash SO(3, 19)$ coset metric of Sec. 2.

### 3.1 Gibbons-Hawking multicenter metrics

Gibbons-Hawking multicenter metrics [26, 20] are gravitational multi-instanton solutions to general relativity discovered in the late 1970s. Here, the word *instanton* indicates that they are solutions to the 4D Euclidean (rather than Lorentzian) vacuum Einstein equations $R_{mn} = 0$. They have selfdual curvature 2-forms, and in this sense are analogs of Yang-Mills instantons. As discussed in Sec. 2.3.2, they are noncompact hyperkähler 4-manifolds, and they are distinguished by a $U(1)$ isometry. The Gibbons-Hawking metric take the form

$$ds^2 = Z^{-1}(d\phi + \omega \cdot dx)^2 + Z dx \cdot dx,$$

with $\phi \cong \phi + 8\pi M$, (94)

where

$$\nabla Z = \nabla \times \omega \quad \text{and} \quad Z = \epsilon + 2M \sum_{I=1}^{N} \frac{1}{|x - x^I|}.$$ (95)

Here, $\epsilon$ takes the values 0 or 1, and we will refer to the quantity $M > 0$ as the mass parameter.\[17] The choice of metric defines Hodge duality and harmonicity of differential forms.
For $\epsilon = 0$, the metric describes flat $\mathbb{R}^4/\mathbb{Z}_N$ at large $x$ and the space is asymptotically locally Euclidean (ALE). As such, it is a model for the resolution/deformation of the singular space $\mathbb{R}^4/\mathbb{Z}_N \cong \mathbb{H}^4/\mathbb{Z}_N$. Generalizations exist for the other ADE discrete subgroups $\Gamma \subset \text{Sp}(1)$. The case $N = 2$ gives the Eguchi-Hanson space \cite{19}. Since $\mathbb{Z}_1$ is trivial, the case $N = 1$ should be asymptotic to $\mathbb{R}^4$, and is in fact globally $\mathbb{R}^4$.

For $\epsilon = 1$, we have the multi-Taub-NUT metric \cite{32}. The metric describes flat $\mathbb{R}^3 \times S^1$ at large $x$ and the space is asymptotically locally flat (ALF). This is the case relevant to the applications below.

It is useful to give two other presentations of the Gibbons-Hawking metric. Let $\mathbb{R}^4 = 8\pi M$ denote the length of the $\psi$ circle, and define $x^4 = \psi/R^4$, so that $x \approx x + 1$ has unit period. Then, the connection 1-form becomes $A = \omega \cdot dx/R^4$, and we have

$$ds^2 = R^2 Z^{-1} (dx^4 + A)^2 + Z g_{\alpha \beta} dx^\alpha dx^\beta,$$

where

$$dA = R^4 * g dZ, \quad \text{and} \quad Z = \epsilon + \frac{R^4}{4\pi} \sum_{I=1}^N \frac{1}{|x - x^I|}. \quad (96)$$

Here $* g$ denotes the Hodge star operator in the metric $g_{\alpha \beta}$. Although we have derived this expression for the special case $g_{\alpha \beta} = \delta_{\alpha \beta}$, it remains valid for arbitrary constant $\mathbb{R}^3$ metric, provided we understand $|x - x^I|$ to be the length in this metric.

Finally, define a 4D volume modulus $V = \sqrt{g} R_4$ and a "unit" 4D metric $G_{\alpha \beta} = (R_4/\sqrt{g}) g_{\alpha \beta}$. In terms of these variables,

$$ds^2 = V^{1/2} (\Delta Z^{-1} (dx^4 + A)^2 + \Delta^{-1} Z G_{\alpha \beta} dx^\alpha dx^\beta), \quad \Delta = \sqrt{G}, \quad (97)$$

where

$$dA = * G dZ, \quad Z = \epsilon + \frac{\Delta}{4\pi} \sum_{I=1}^N \frac{1}{|x - x^I|}, \quad (98)$$

and $|x - x^I|$ is now the length in the metric $G_{\alpha \beta}$.

### 3.2 M-theory/type IIA duality

The duality between M-theory and type IIA string theory is an exact equivalence, a truncation of which relates the classical 11D and 10D type IIA supergravity actions. In our conventions, the bosonic terms in these supergravity actions are given by

$$\frac{(2\pi \ell)^9}{2\pi} S_{11} = \int d^{11}x \sqrt{-G_{(11)}} \left[ R_{(11)}^{(11)} - \frac{1}{2} \frac{1}{4!} F_{(4)}^2 \right] - \frac{1}{6} \int A_{(3)} \wedge F_{(4)} \wedge F_{(4)}, \quad (99)$$

$$\frac{(2\pi \ell)^8}{2\pi} S_{10} = \int d^{10}x \sqrt{-G_{(10)}} \left[ e^{-2\Phi} \left( R_{(10)}^{(10)} + 4(\partial \Phi)^2 - \frac{1}{2} \frac{1}{3!} H^2 \right) - \frac{1}{2} \frac{1}{2!} F_{(2)}^2 - \frac{1}{2} \frac{1}{4!} F_{(4)}^2 \right] - \frac{1}{2} \int B \wedge F_{(4)} \wedge F_{(4)}. \quad (100)$$
In these conventions, $\ell$ is the 11D Planck length and the 10D string length $\sqrt{\alpha'}$, and the square of a differential form is defined by contracting all indices with the metric. Here, field strengths are related to $p$-form potentials via

$$F_{(4)} = dA_{(3)} \quad \text{(in 11D)},$$

$$H = dB, \quad F_{(2)} = dC_{(1)}, \quad F_{(4)} = dC_{(3)} - C_{(1)} \wedge H \quad \text{(in 10D)}.$$ (101)

If we assume a U(1) isometry in 11D, the 11D supergravity action reduces to the 10D type IIA supergravity action. The identification between 11D and 10D fields is as follows:

$$ds_{11}^2 = e^{-2\Phi/3} ds_{IIA}^2 + (2\pi\ell)^2 e^{4\Phi/3} (dy + A)^2, \quad \text{with periodicity } y \cong y + 1,$$

$$A = \frac{1}{2\pi\ell} C_{(1)}, \quad A_{(3)} = C_{(3)} + (2\pi\ell) dy \wedge B.$$ (102)

In the presence of a $Dp$-brane, the 10D type IIA supergravity action has an additional source term,

$$S_{\text{source}} = \frac{2\pi}{(2\pi\ell)^{p+1}} \int C_{(p+1)}.$$ (103)

### 3.3 The lift of the type IIA $T^3/\mathbb{Z}_2$ orientifold to M-theory on K3

In this section, we derive an approximate K3 metric (90) of Gibbons-Hawking form from the M-theory lift of the tree-level supergravity description of the type IIA $T^3/\mathbb{Z}_2$ orientifold. We do this in three steps, considering the M-theory lift of a collection of $N$ D6-branes transverse to $\mathbb{R}^3$ in Sec. 3.3.1, then adding an O6-plane in Sec. 3.3.2, then finally compactifying the transverse $\mathbb{R}^3$ to $T^3$ and lifting the 16 D6-branes and 8 O6-planes of $T^3/\mathbb{Z}_2$ in Sec. 3.3.3. The discussion closely follows that in Ref. [51] by one of the authors, which in turn relies heavily on Ref. [55].

#### 3.3.1 M-theory lift of a collection of $N$ D6-branes

The type IIA supergravity solution corresponding to $N$ D6 branes located at transverse locations $x^i$ on a space of topology $\mathbb{R}^{6,1} \times \mathbb{R}^3$ with arbitrary constant product metric

$$ds^2 = g_{(\gamma)\mu \nu} dy^\mu dy^\nu + g_{(3)\alpha \beta} dx^\alpha dx^\beta$$ (104)

is

$$ds_{10}^2 = Z^{-1/2} g_{(\gamma)\mu \nu} dy^\mu dy^\nu + Z^{1/2} g_{(3)\alpha \beta} dx^\alpha dx^\beta,$$

$$e^\Phi = e^{4\phi} \left( \sqrt{g_{(3)}} \right)^{3/2} (2\pi\ell)^{3/4} Z^{-3/4}, \quad F_{(2)} = *_3 dZ.$$ (105a)
where $\mu, \nu = 0, \ldots, 6$ and $\alpha, \beta = 1, 2, 3$. Here, $\phi$ is an integration constant\footnote{In the compact setting of Sec. 3.3.3, $\phi$ can be identified with the 7D dilaton. The effective field theory is discussed in Ref. [32].} and $Z(x)$ satisfies the 3D Poisson equation

$$-
abla_{(3)} \cdot Z = 2\pi \ell \sum_{I=1}^{N} \frac{\delta^3(x - x^I)}{\sqrt{g_{(3)}}}. \quad (106)$$

Carrying out the lift to an 11D solution via the identification $104$, and defining

$$G_{\alpha\beta} = e^{\Phi}(2\pi \ell)^2 g_{(3)}^{1/2} Z^{1/2} \sqrt{g_{(3)}} \quad \text{and} \quad \Delta = \det^{1/2} G, \quad (107)$$

we obtain a product metric

$$ds^2_{11} = ds^2_7 + V^{1/2} ds^2_4, \quad V = (2\pi \ell)^4 e^{-4\phi/3}, \quad (108)$$

where $ds^2_7$ is a constant metric on $\mathbb{R}^{6,1}$ and $ds^2_4$ is a multicenter metric

$$ds^2_4 = \bar{G}_{mn} dx^m dx^n = \Delta^{-1} Z G_{\alpha\beta} dx^a dx^b + \Delta^{-1} (dx^4 + A)^2 \quad (109)$$

of Gibbons-Hawking form. Note that $\phi$ gives the overall 4D volume modulus and $\det \bar{G} = Z$. In the metric $G_{\alpha\beta}$, we have

$$dA = *_{G} dZ \quad \text{and} \quad -\nabla^2 Z = \sum_{I=1}^{N} \delta^3(x - x^I), \quad (110)$$

with solution

$$Z = 1 + \sum_{I=1}^{N} Z_I, \quad \text{where} \quad Z_I = \frac{\Delta}{4\pi \|x - x^I\|}, \quad (111)$$

and where $\|x - x^I\|$ is the distance computed in the metric $G_{\alpha\beta}$. Here, we have chosen an integration constant of 1 in the definition of $Z$, for agreement between the metrics $104$ and $105$ at large $x$, far from the D6-branes. Therefore, the metric is a multi-Taub-NUT metric $32$.

The $L^2$ harmonic forms in this metric are known $48$. They are

$$\omega_I = \left( \frac{Z_I}{Z} \right)_{,\alpha} \left( dx^\alpha \wedge (dx^4 + A) - \frac{Z}{2} G^{\alpha\alpha'} \epsilon_{\alpha'\beta\gamma} dx^\beta \wedge dx^\gamma \right)$$

$$= -d \left( A_I - \frac{Z_I}{Z} (dx^4 + A) \right), \quad (112)$$

satisfying

$$\int \omega_I \wedge \omega_J = -\delta_{IJ}, \quad (113)$$

\footnote{In the compact setting of Sec. 3.3.3, $\phi$ can be identified with the 7D dilaton. The effective field theory is discussed in Ref. [32].}
as reviewed in App. E.4. Here, $A_I$ is the magnetic dual of $Z_I$ in the metric $G_{\alpha\beta}$,

\[ dA_I = \ast dZ_I. \]  

The compact homology of this space is generated by 2-spheres $S_{IJ}$, where $S_{IJ}$ is swept out by the expanding and shrinking $x^4$-circle fibration over the line segment from $x^I$ to $x^J$. Since $Z^{-1} \rightarrow 0$ at the source locations $x^I$, the sphere smoothly caps off at the endpoints. In the lift from type IIA to M-theory, a fundamental string stretched between the $I$th and $J$th D6-brane becomes an M-theory membrane wrapped on this sphere. The intersection pairing on $H_2$ is minus the Cartan matrix of $A_{N-1}$. To identify the compact homology lattice with the root lattice of $A_{N-1}$ in terms of its standard orthonormal basis, we can associate the $I$th D6-brane, or $I$th center in M-theory, with a unit vector $e_I$, and the string stretched from $I$th to $J$th D6-brane, or its M-theory lift $S_{IJ}$, with the root $e_J - e_I$.

### 3.3.2 M-theory lift of an O6-plane and a collection of $N$ D6-branes

If we instead begin with $N$ D6-branes and an O6-plane in type IIA, the story is very similar. A new feature is the $\mathbb{Z}_2$ orientifold involution $\Omega(-1)^{F_L}I_3$, where $\Omega$ is worldsheet orientation reversal, $(-1)^{F_L}$ is left-moving fermion parity\(^{19}\) and $I_3$: $x^\alpha \mapsto -x^\alpha$, for $\alpha = 1, 2, 3$, is inversion of $\mathbb{R}^3$. This truncates the type IIA supergravity action to the fields

\[ G_{\mu\nu}, G_{\alpha\beta}, B_{\alpha\mu}, \Phi, C^{(1)}_\alpha, C^{(3)}_{\alpha\beta\mu}. \]  

In the lift to 11D supergravity, the orientifold $\mathbb{Z}_2$ becomes an orbifold $\mathbb{Z}_2$ acting as $I_4$: $x^m \mapsto -x^m$ for $m = 1, 2, 3, 4$. Correspondingly, we define the modulus $V$ of Eq. (108) with a factor of 2,

\[ ds^2_{11} = ds^2_7 + (2V)^{1/2}d\bar{s}^2_4, \quad 2V = (2\pi\ell)^4 e^{-4\phi/3}, \]  

Aside from the $\mathbb{Z}_2$ identification of coordinates, the only aspect of the previous discussion that is modified is the definition of $Z$. The Poisson equation for $Z$ becomes

\[ -\nabla^2 Z = \sum_{\text{sources } s} Q_s \delta^3(x - x^s), \]  

where $s$ runs over $I = 1, \ldots, N, I' = 1', \ldots, N$, and $O$, with (i) $Q_I = 1$ from a D6-brane source at $x^I$ in type IIA; (ii) $Q_{I'} = 1$ from an image D6-brane source at $x^{I'} = -x^I$ in type IIA; (iii) $Q_O = -4$ from an O6-plane at the $I_3$ fixed point at the origin of $\mathbb{R}^3$ in type IIA.\(^{20}\) The solution is

\[ Z = 1 + Z_O + \sum_{I=1}^N (Z_I + Z_{I'}), \]  

where

\[ Z_I = \frac{\Delta}{4\pi|x - x^I|}, \quad Z_{I'} = \frac{\Delta}{4\pi|x + x^I|}, \quad \text{and} \quad Z_O = -4\frac{\Delta}{4\pi|x|}. \]  

\(^{19}\)The factor $(-1)^{F_L}$ is needed to ensure a supersymmetric spectrum. For the supergravity fields, it acts as $-1$ on left-moving Ramond sector states and $+1$ on left-moving Neveu-Schwarz sector states.

\(^{20}\)An O6-plane exactly cancels the RR charge of 2 D6-branes and 2 image D6-branes.
The $L^2$ harmonic forms are
\[
\omega_I = \left(\frac{Z_I - Z_{I'}}{Z}\right)_{,\alpha} \left(dx^\alpha \wedge (dx^4 + A) - \frac{Z}{2} G_{\alpha'\beta'} dx^\beta \wedge dx^\gamma\right)
\]
\[
= -d\left((A_I - A_{I'}) - \frac{(Z_I - Z_{I'})}{Z}(dx^4 + A)\right),
\]
(120)
satisfying
\[
\int \omega_I \wedge \omega_J = -\frac{1}{2} (\delta_{IJ} + \delta_{I'J'}) = -\delta_{IJ},
\]
(121)
where the $\frac{1}{2}$ is due to the $\mathbb{Z}_2$ quotient. It is convenient to define
\[
F_I = dA_I \quad \text{and} \quad F_{I'} = dA_{I'}.
\]
(122)

Then, in terms of cohomology classes with compact support, $[\omega_I] = -[F_I - F_{I'}]$. 

Whereas the lift of a collection of D6 branes gives a smooth positive definite multi-Taub-NUT metric, the lift of the tree-level type IIA supergravity description of an O6-plane gives a singular metric that changes sign near $x = 0$. The interpretation is as follows. The field identification of Eq. (112) assume a U(1) isometry of the M-theory circle. That is because higher Fourier modes around the M-theory circle correspond to states with D0-charge in type IIA, and these are not included in the classical supergravity action [51].

Recall that the mass of a D0-brane is proportional to $e^{-\Phi}$, and for the tree-level D6/O6 solution we have just described, $e^\Phi \propto Z^{-3/4}$. Since a D6 brane has positive Taub-NUT mass parameter, we see that $Z \to \infty$ and D0-branes become infinitely heavy near a D6 brane. Thus, they do not affect the local description of a D6 brane, and the lift of a D6-brane to M-theory truly has an isometry around the M-theory circle.

On the other hand, the tree-level supergravity description of an O6-plane gives -4 times the Taub-NUT mass parameter of a D6-brane in the lift to M-theory. Thus, as we approach an O6-plane, $Z$ decreases to 0 at finite distance, and then becomes negative in a region around the O6 plane. Correspondingly, D0-branes and their bound states become light as they approach this region, and need to be included in the low energy type IIA supergravity theory [51]. The exact M-theory lift does not have an isometry around the M-theory circle, a fact suggested already by the presence of the $\mathbb{Z}_2$ identification, which breaks this isometry at fixed points. The negative mass Taub-NUT space, together with the $\mathbb{Z}_2$ identification $(x, x^4) \approx -(x, x^4)$, defines the large distance approximation to an Atiyah-Hitchin space [55, 44, 45, 6]. It is singular at small $x$, however this just reflects the fact that we have discarded all higher Fourier modes around the M-theory circle. The complete Atiyah-Hitchin geometry is smooth, and excises the $Z < 0$ regions.

The compact homology of the space obtained from the exact M-theory lift is similar to that of the previous section. We again have 2-spheres $S_{IJ}$ arising from the M-theory lift of a string stretched from the $I$th to $J$th D6-brane. Equivalently, these are the $\mathbb{Z}_2$-invariant combinations $S_{IJ} - S_{I'J'}$ on the $\mathbb{Z}_2$ covering space. However, we also obtain new cycles from $S_{II'} + S_{I'J'}$ on the covering space. The latter arise from pairs of strings connected between D6-branes and their $\mathbb{Z}_2$ images on the covering space, or equivalently, between pairs of D6-branes and an O6-plane on the quotient space. In the compact homology lattice, the new
cycles add roots $e_I + e_J$ to the previous lattice spanned by $e_J - e_I$, enhancing the $A_{N-1}$ lattice to $D_N$. The intersection pairing on $H_2$ is minus the Cartan matrix of $D_N$.\[55\]

### 3.3.3 M-theory lift of $T^3/\mathbb{Z}_2$

The $T^3/\mathbb{Z}_2$ orientifold is the analog of the previous section with $y^a$ valued on $T^3$ instead of $\mathbb{R}^3$. It is the background T-dual to type I on $T^3$, via dualization of all three $T^3$ directions. This gives $N = 16$ pairs of D6-branes and image D6-branes and $2^3 = 8$ O6-planes located at the fixed points of $\mathbb{I}_3$ on $T^3$. Choosing coordinate periodicity $x^\alpha \cong x^\alpha + 1$, the fixed points are $x^{O_i}$, for $i = 1, \ldots, 8$, are the $2^3$ points on $T^3$ where $x^\alpha = 0$ or $\frac{1}{2}$ for each $\alpha = 1, 2, 3$. This gives the metric described at the beginning of Sec. 3:

$$ds^2_{K3} = (2V_{K3})^{1/2} \tilde{G}_{mn}dx^m dx^n,$$

where $\tilde{G}_{mn}$ is an approximate metric of Gibbons-Hawking form for a “unit” K3 of volume $\frac{1}{2}$, obtained from the resolution of $T^4/\mathbb{Z}_2$ for $T^4$ of volume $1$,

$$\tilde{G}_{mn}dx^m dx^n = \Delta^{-1} ZG_{\alpha\beta}dx^\alpha dx^\beta + \Delta^{-1}(dx^4 + A)^2.$$  

The solution to the Poisson equation (92) for $Z$ can be expressed formally as

$$Z = 1 + \sum_{i=1}^8 Z_{O_i} + \sum_{I=1}^{16} (Z_I + Z'_I) \quad \text{where} \quad Z_s = Q_s K(x, x^a),$$

and where the Green’s function $K(x, x')$ satisfies

$$-\nabla^2 K(x, x') = \delta^3(x - x') - 1, \quad \text{with} \quad \int_{T^3} d^3x K(x, x') = 0. (124)$$

Here, the delta function is a periodic delta function on $T^3$. The leading constant of unity in Eq. (123) ensures that $Z$ drops out of the metric in the limit that two $x^I, x'^I$ pairs coincide with each $x^{O_i}$ and the sources locally cancel. This is the $T^4/\mathbb{Z}_2$ orbifold limit of K3. From $\int_{T^3} d^3x Z = 1$, the metric $\tilde{G}_{mn}$ indeed gives a “unit” K3 metric of volume $\int_{T^3} d^4x Z = \frac{1}{2} \int_{T^4} d^4x Z = \frac{1}{2}$, whose double cover is a $T^4$ of volume $1$.

In this compact setting, it is meaningful to define a lower dimensional dilaton after reducing to 7D. The parameter $\phi$ of Eq. (105b) can be identified with the 7D dilaton from the IIA point of view. From Eq. (116), this is the same as the overall K3 volume in 11D, $e^{-4\phi/3} = \frac{2V_{K3}}{(2\pi\ell)^4}. (125)$

This approximate K3 metric suffers from the same pathology described at the end of the previous section. Near $x = x'$, the local geometry is that of a smooth Taub-NUT space. However, in a region around the $\mathbb{Z}_2$ fixed points $x^{O_i}$, the metric gives a negative mass Taub-NUT space which is the large distance approximation to the Atiyah-Hitchin space.\[55\][4][5][6],

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obtained in the exact M-theory lift. In the limit of small $x^4$-circle, $\Delta = \sqrt{G} \to 0$, the pathological regions become smaller and smaller, and the metric (90) well approximates the exact K3 metric except in an arbitrarily small neighborhood of each fixed point $x^{O_i}$.

The homology lattice is spanned by the cycles inherited from $T^4/\mathbb{Z}_2$, i.e., the 2-tori $f_\alpha, f^\alpha$ and 2-spheres $D_{as}, D^a$ of Sec. 2.4, together with the $D_{16}$ lattice described at the end of the previous section, from the lift of the D6-branes and any one of the $O_i$. Focusing on the $O_i$ at the origin, the arguments of Sec. 2.4.3 and App. B.2 show that the $D_{16}$ lattice of the previous section is enlarged to the weight lattice of Spin($32$)/$\mathbb{Z}_2$ and that the homology lattice splits as the sum of this weight lattice and a signature $(3, 3)$ lattice $\langle D_\alpha^4, f_\alpha \rangle$.

### 3.4 Harmonic forms in the approximate K3 metric

In this section, we describe the coframe, hyperkähler forms, and a basis of harmonic forms of K3 in the approximate metric (89). We show that this basis of harmonic forms approximates the basis of Sec. 2.5 in the sense that the cohomology classes agree, even though the harmonic representatives of these classes obviously differ in the approximate and exact metrics. As already noted, in the large hypercomplex structure limit $\Delta \ll 1$, the metric (90) closely approximates the exact K3 metric everywhere except in a small neighborhood of each $x^{O_i}$. In the same limit, we expect the harmonic forms of this section to closely approximate the exact harmonic forms.

**Frame**

Let $E^\alpha_\beta$ be a vielbein for the 3D metric $G_{\alpha\beta}$ appearing in the approximate K3 metric (90). Then,

$$ds^2 = \delta_{mn} \bar{\theta}^m \bar{\theta}^n,$$

with volume form $\bar{\theta}^1 \wedge \bar{\theta}^2 \wedge \bar{\theta}^3 \wedge \bar{\theta}^4$. (126)

where $\bar{\theta}^m$ is the coframe

$$\bar{\theta}^\alpha = \Delta^{-1/2} Z^{1/2}(E^{-1})^\alpha_\beta dx^\beta, \quad \bar{\theta}^4 = \Delta^{1/2} Z^{-1/2} (dy^4 + A).$$

(127)

**Hyperkähler forms**

In terms of this coframe, the metric (89) admits a triple of selfdual harmonic forms

$$J^\alpha = \sqrt{\frac{V_{K3}}{2}} \left( 2 \bar{\theta}^\alpha \wedge \bar{\theta}^4 + \delta^{\alpha\alpha'} \epsilon_{\alpha'\beta\gamma} \bar{\theta}^\beta \wedge \bar{\theta}^\gamma \right),$$

(128)

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric tensor with $\epsilon_{123} = 1$. The selfduality is manifest from the form of $J^\alpha$, and the closure $dJ^\alpha = 0$ follows from $dA = *dZ$. In the coordinate basis, this becomes

$$J^\alpha = \sqrt{\frac{V_{K3}}{2}} \left( 2 dx^\alpha \wedge (dx^4 + A) + Z G^{\alpha\alpha'} \epsilon_{\alpha'\beta\gamma} dx^\beta \wedge dx^\gamma \right).$$

(129)

These are the hyperkähler forms of K3 in the approximate metric (89).
Basis of harmonic forms

Next, observe that $J^\alpha$ can be written as the sum of two closed 2-forms

$$J^\alpha = \sqrt{\frac{V_K}{2}}(\omega^\alpha + \omega_\alpha), \quad (130)$$

where

$$\omega^\alpha = 2\bar{\theta}^\alpha \wedge \bar{\theta}^4 + \left(\frac{Z-1}{Z}\right)\epsilon_{\alpha\beta\gamma}\bar{\theta}^\beta \wedge \bar{\theta}^\gamma + d\bar{\lambda}_\alpha, \quad (131)$$

or equivalently, in the coordinate basis,

$$\omega^\alpha = E^\alpha_{\beta\gamma} \zeta^\beta, \quad \omega_\alpha = (E^{-1T})_{\alpha}^\beta \zeta_\beta, \quad (132)$$

where

$$\zeta^\alpha = 2dx^\alpha \wedge (dx^4 + A) + G^{\alpha\alpha'}\epsilon_{\alpha'\beta\gamma}(Z-1)dx^\beta \wedge dx^\gamma + G^{\alpha\alpha'} d\lambda_{\alpha'}, \quad \zeta_\alpha = \epsilon_{\alpha\beta\gamma}dx^\beta \wedge dx^\gamma - d\lambda_\alpha. \quad (133)$$

Here, we have included an exact term $d\bar{\lambda}_\alpha = (E^{-1T})_{\alpha}^\beta d\lambda_\beta$ in the definition of $\omega^\alpha$ and $\omega_\alpha$. We define $\bar{\lambda}_\alpha$ so that $\omega^\alpha$ and $\omega_\alpha$ are harmonic. Then $\bar{\lambda}_\alpha$ satisfies

$$\sqrt{\frac{V_K}{2}}(d\bar{\lambda}_\alpha + \star d\bar{\lambda}_\alpha) = \frac{1-Z}{Z} J^\alpha. \quad (134)$$

Finally, as in Sec. 3.3.2, we obtain anti-selfdual harmonic forms

$$\omega_I = \left(\frac{Z_I - Z_{I'}}{Z}\right)_{\alpha, \alpha'} \left(dx^\alpha \wedge (dx^4 + A) - \frac{Z}{2} G^{\alpha\alpha'} \epsilon_{\alpha'\beta\gamma} dx^\beta \wedge dx^\gamma\right)$$

$$= -d \left((A_I - A_{I'}) - \frac{(Z_I - Z_{I'})}{Z}(dx^4 + A)\right), \quad \text{for } I = 1, \ldots, 16. \quad (135)$$

In summary, a basis of harmonic 2-forms is

$$\omega_a = (\omega^\alpha, \omega_\alpha, \omega_I), \quad (136)$$

and it is straightforward to show that

$$\int \omega_\alpha \wedge \omega_b = \eta_{ab}, \quad (137)$$

with $\eta_{ab}$ defined in Eq. (26).
Identification with the basis of Sec. 2.5

Working in the approximate metric, it is natural to seek to identify the forms (136) with those of Sec. 2.5. At the $T^4/\mathbb{Z}_2$ orbifold locus with two $x^I$ at each fixed point (i.e., $x^I$ given by Eq. (34)), we have $Z = 1$ and $A$ equal to a constant 1-form $\beta_\alpha dx^\alpha$, and we find $\lambda_\alpha = 0$.

In this case, comparing to Sec. 2.4.1, we see that the 2-forms (136) precisely coincide with those of Sec. 2.5 for all $E_{\alpha\beta}$ and $\beta_{\alpha\beta}$. Working in the approximate metric, we would like to identify the forms for all $x'^I_\alpha$ as well. To do so, it suffices to show that the expression for $\delta[\omega_\alpha]$ under a small change $\delta x^I_\alpha$ takes the same form here as in Eq. (50).

From Eq. (50), we should have

$$
\begin{align*}
\delta[\zeta_\alpha] &= -\delta\beta_{\alpha\beta}[z_\beta] + 2\delta x^I_\alpha[\omega_I], \\
\delta[\zeta_\alpha] &= 0, \\
\delta[\omega_I] &= \delta x^I_\beta[\zeta_\beta],
\end{align*}
$$

where

$$
\delta\beta_{\alpha\beta} = \delta\beta_{\alpha\beta} - x'^I_\alpha \delta x^I_\beta + x'^I_\beta \delta x^I_\alpha.
$$

Since $Z_I$ and $Z'_{I'}$ depend on $x'^I_\beta$ only through the combinations $x^a \pm x'^I_\alpha$, under a small change $\delta x^I_\beta$, we have

$$
\delta Z = \delta x^I_\beta \frac{\partial}{\partial x^I_\beta}(Z_I + Z_{I'}) = -\delta x^I_\beta \partial_\beta(Z_I - Z_{I'}). 
$$

Moreover, since $A$ satisfies $dA = \star_G dZ$ in the 3D metric $G_{\alpha\beta}$, we have

$$
\delta A = \delta x^I_\alpha \delta(A_I - A'_{I'}), \quad \text{where} \quad d(\delta A_I) = \star_G d(\delta Z_I) \quad \text{and} \quad d(\delta A'_{I'}) = \star_G d(\delta Z_{I'}). 
$$

Given the form of $\delta Z$ above, an obvious solution is $\delta A_\alpha = -\delta x^I_\beta \partial_\beta(A_I - A'_{I'})_\alpha$. However, a gauge equivalent and more convenient choice is

$$
\delta A_\alpha = \delta x^I_\beta(F_I - F'_{I'})_{\alpha\beta} + c_\alpha, 
$$

where

$$
F_I = dA_I \quad \text{and} \quad F'_{I'} = dA'_{I'}.
$$

Here, $c$ is a moduli-dependent constant 1-form on $T^3$. We choose $c$ so that

$$
\delta A = \left(\delta x^I_\beta(F_I - F'_{I'})_{\alpha\beta} + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \delta\beta^{\beta\gamma}\right) dx^\alpha.
$$

Then, Eq. (133) gives, for example

$$
\delta \zeta^3 = \left(2\delta x^I_\beta(F_I - F'_{I'})_{\delta\beta} + \epsilon_{\delta\beta\gamma} \delta\beta^{\beta\gamma}\right) dx^3 \wedge dx^\delta - \delta x^I_\beta G^{\alpha\beta'} \epsilon_{\alpha'\gamma\delta} \partial_\delta(Z_I - Z'_{I'}) dx^\gamma \wedge dx^\delta. 
$$
In cohomology, it is possible to show that the last equation simplifies as follows:

\[
\delta \zeta^3 = \left[ 2 \delta x^I (F_I - F_{I'}) \delta_\beta + \epsilon_\beta \epsilon_\gamma \epsilon_\delta \delta \beta \gamma \delta \delta \beta \gamma \delta \right] dx^3 \wedge dx^\delta - \delta x^I G^{\beta \alpha} \partial_\alpha (Z_I - Z_{I'}) dx^\gamma \wedge dx^\delta
\]

\[
= \left[ -2 \delta x^I (F_I - F_{I'}) + \epsilon_\beta \epsilon_\gamma \epsilon_\delta \delta \beta \gamma \delta \right] dx^3 \wedge dx^\delta
\]

\[
= -2 \delta x^I (F_I - F_{I'}) - 3 \delta \beta \alpha \delta \beta \delta \beta \gamma \delta \beta \gamma \delta \beta \gamma \delta \beta
\]

(146)

Noting that \( [\omega_I] = -[F_I - F_{I'}] \) and generalizing from \( [\delta \zeta^3] \) to \( [\delta \zeta^\alpha] \), we obtain

\[
[d \zeta^\alpha] = 2 \delta x^I [\omega_I] - 3 \delta \beta \alpha \delta \beta \delta \beta \gamma \delta \beta \gamma \delta \beta
\]

as desired.

### 3.5 Metric moduli space in the Gibbons-Hawking approximation

In this section we describe the metric moduli space of the approximate K3 metric from two points of view.

First, in Sec. 3.5.1, we determine the diffeomorphism invariant moduli space metric of Sec. 2.7.1 applied to the approximate metric. We find agreement with the exact coset moduli space metric of Sec. 2. Then, in Sec. 3.5.2, we consider the naive moduli space metric from dimensional reduction, and show that it agrees with the diffeomorphism invariant metric. This special property is due to the Gibbons-Hawking form of the approximate metric.

The approximate metric depends on parameters \( G_{\alpha \beta}, \beta^{\alpha \beta}, x^I, \) and the overall volume modulus \( V_{K3} \). We have suggestively given these parameters the same names as the moduli as defined in Secs. 2.5 and 2.7.2. To identify the two, we show in Sec. 3.5.1, that the metric deformations due to small changes in the quantities \( x^I \) parametrizing the approximate metric precisely agree with the metric deformations generated by the harmonic forms of Sec. 3.4, in the manner described in Sec. 2.7.2. We expect the identification to hold for \( G_{\alpha \beta} \) and \( \beta^{\alpha \beta} \) as well. Since the inner products of the harmonic forms (136) in the approximate metric are the same as those (50) in the exact metric, it follows that the moduli space metric of the approximate K3 metric is the same \( \mathbb{R}_{>0} \times (SO(3) \times SO(19)) \setminus SO(3, 19) \) coset metric of the exact discussion in Sec. 2.

To achieve the diffeomorphism invariance of the metric on metric moduli space, compensating vector fields were introduced in Sec. 2.7.1 to project generic metric deformations to transverse traceless gauge. In Sec. 3.5.2, we consider the moduli space metric from naive dimensional reduction of the D-dimensional Einstein-Hilbert action on a d-dimensional manifold \( X \). This metric differs from the previous one in two ways: there is no projection of metric deformations to their transverse traceless part, and there are additional terms in the metric obtained by integrating \( (\bar{G}^{mn} \delta G_{mn})^2 \). For the approximate K3 metric (89), we find that the naive moduli space metrics precisely agrees with the diffeomorphism invariant moduli space metrics of Sec. 2.7.1. Thus, the subtlety of compensators and projection to transverse traceless gauge is not necessary for the approximate K3 metric, and one can equivalently use the naive moduli space metric from dimensional reduction.

36
3.5.1 Method 1: moduli space metric using compensators

The unit K3 metric in the Gibbon-Hawking approximation (90)

\[ \bar{G}_{mn}(x; G, \beta, x)dx^m dx^n \]

depends on moduli \( G_{\alpha\beta}, \beta^{\alpha\beta}, \) and \( x^{I\alpha} \), which we would like to identify with the like-named hyperkähler structure moduli of Sec. 2.3. To make this identification, it is necessary to show that metric deformations

\[ \delta(ds^2) = \delta G^{\alpha\beta} \frac{\partial}{\partial G^{\alpha\beta}} \bar{G}_{mn} dx^m dx^n + \tilde{\delta}^{\alpha\beta} \frac{\partial}{\partial \tilde{\beta}^{\alpha\beta}} \bar{G}_{mn} dx^m dx^n + \delta x^{I\alpha} \frac{\partial}{\partial x^{I\alpha}} \bar{G}_{mn} dx^m dx^n \]  

(147)

g Agree with those of Eq. (84) up to compensating diffeomorphism (73). That is,

\[ \frac{\partial}{\partial G^{\alpha\beta}} \bar{G}_{mn} = (h^{G})^{\alpha\beta}_{mn} + \nabla_m ((N^G)_{\alpha\beta})_n + \nabla_n ((N^G)_{\alpha\beta})_m, \]  

(148a)

\[ \frac{\partial}{\partial \tilde{\beta}^{\alpha\beta}} \bar{G}_{mn} = (h^{\tilde{\beta}})_{\alpha\beta}^{\alpha\beta}_m + \nabla_m ((N^{\tilde{\beta}})_{\alpha\beta})_n + \nabla_n ((N^{\tilde{\beta}})_{\alpha\beta})_m, \]  

(148b)

\[ \frac{\tilde{\partial}}{\partial x^{I\alpha}} \bar{G}_{mn} = ((h^x)_{I\alpha})_{mn} + \nabla_m ((N^x)_{I\alpha})_n + \nabla_n ((N^x)_{I\alpha})_m, \]  

(148c)

for the appropriate choice of compensating vector fields

\[ (\delta N^G)^n = \delta G^{\alpha\beta} ((N^G)_{\alpha\beta})^n, \quad (\delta N^{\tilde{\beta}})^n = \tilde{\delta}^{\alpha\beta} ((N^{\tilde{\beta}})_{\alpha\beta})^n, \quad (\delta N^x)^n = \delta x^{I\alpha} ((N^x)_{I\alpha})^n. \]  

(149)

Here,

\[ \frac{\tilde{\partial}}{\partial x^{I\alpha}} = \frac{\partial}{\partial x^{I\alpha}} + (x^{I\beta})^{I\beta}_{\alpha\gamma} - x^{I\gamma} \delta ^{I\beta}_{\alpha\gamma} \]  

(150)

where the quantity appearing in parentheses is \( -A^{\beta\gamma} I\alpha \), with \( A \) the connection defined in Eq. (61).

For the deformations \( \delta x^{I\alpha} \), Eq. (148c) can be shown to hold with the choice

\[ (\delta N^x)^a = -\frac{Z_I}{Z} \delta x^{I\alpha}, \quad (\delta N^x)^4 = 0. \]  

(151)

This is proven in App. E in the simpler context of the Gibbons-Hawking multicenter space from the lift of \( N \) D6 branes. The computation is analogous for K3 in the Gibbons-Hawking approximation.

Equivalently, the metric deformations agree, provided that the harmonic forms \( \omega_I \) generating the deformations are shifted by an exact 2-form. Since

\[ (\delta N^x)_\beta dx^\beta = \Delta^{-1} Z G_{\alpha\beta} (\delta N^\alpha) dx^\beta, \quad \text{with} \quad \Delta^{-1} Z dx^\beta = (E^{-1} T)_{\beta\gamma} J^\beta(dx^4 + A), \]  

(152)

we see from Eq. (87) and the definition (84) of \((h^x)_{I\alpha}\) that the appropriate shift is

\[ \omega_I \mapsto \omega_I - d\left( \frac{Z_I}{Z}(dx^4 + A) \right) = -(F_I - F_I'), \]  

(153)
where in the last equality, we have used Eqs. (122). We expect that appropriate diffeomorphisms can also be found for the deformations $\delta G^{mn}$ and $\tilde{\delta} \beta^{mn}$, but we leave this as an exercise for the future.

As a consequence of Eq. (148), the moduli space metric (76) from transverse traceless deformations of the K3 metric in the Gibbons-Hawking approximation agrees with the exact moduli space metric (86) discussed in Secs. 2.5 and 2.7.

### 3.5.2 Method 2: moduli space metric without compensators

#### Generalities

We now ask the question: under what conditions are the compensators unnecessary? In the case of $D$-dimensional pure gravity, if we ignore the subtlety of compensators and dimensionally reduce on a $d$-dimensional Ricci flat manifold $X$ using the metric ansatz

$$ds^2 = (V_X)^{-\frac{2}{D-4}} g_{\mu\nu}(y)dy^\mu dy^\nu + (2\pi\ell)^2 (nV_X)^{\frac{2}{2}} \tilde{G}_{mn}(\mu(y), x)dx^m dx^n,$$

we find that that $D$ dimensional Einstein-Hilbert action

$$S_{(D)}^{EH} = \frac{2\pi}{(2\pi\ell)^{D-2}} \int d^D x \sqrt{-G^{(D)}} R^{(D)},$$

reduces to a $(D - d)$-dimensional action

$$S^{(D-d)} = S_{(D)}^{EH} + S_{G}^{(D-d)} + S_{V}^{(D-d)},$$

where, writing

$$S = \frac{2\pi}{(2\pi\ell)^{D-d-2}} \int d^{D-d} y \sqrt{-G_E} L,$$

we have

$$L_{(D-d)}^{EH} = R_E,$$

$$L_{G}^{(D-d)} = n \int_X d^d x \sqrt{G} \left( -\frac{1}{4} G^{mp} G^{aq} \partial_p \tilde{G}_{mn} \partial_q \tilde{G}_{pq} + \frac{1}{4} (\partial_E \log G)^2 \right),$$

$$L_{V}^{(D-d)} = -\frac{D-2}{(D-d-2)d} \partial_\mu (\log V) \partial^\mu (\log V).$$

Here, $\tilde{G}_{mn}$ denotes a “unit” metric, satisfying

$$n \int_X d^d x \sqrt{G} = 1,$$

for some normalization constant $n$, and $(2\pi\ell)V_X$ is the volume of $X$. We have introduced the parameter $n$ so that for $X$ an orbifold $Y/\Gamma$, we can choose the convention $\int_Y d^d x \sqrt{G} = 1$ on the covering space $Y$ by setting $n$ equal to the dimension of the orbifold group $\Gamma$. For example, for $K3 \cong T^4/\mathbb{Z}_2$, we would set $n = 2$. 38
From the kinetic terms (158b) and (158c), we can read off the naive metric on moduli space, ignoring the subtlety of compensators. Writing $\delta \log \bar{G} = \bar{G}_{mn} \delta \bar{G}_{mn}$ in Eq. (158b), the naive moduli space metric for constant volume deformations is

$$ds_{\mathcal{M}, \text{naive}}^2 = n \int_X d^d x \sqrt{\bar{G}} \left( \frac{1}{4} \bar{G}^{mp} \bar{G}^{nq} \delta \bar{G}_{mn} \delta \bar{G}_{pq} - \frac{1}{4} (\bar{G}^{mn} \delta \bar{G}_{mn})^2 \right),$$

(160)

and that for the overall volume modulus is

$$ds_V^2 = \frac{D - 2}{(D - d - 2) d} \left( \frac{\delta V_X}{V_X} \right)^2.$$

(161)

The former can be compared to the actual moduli space metric (76) for constant volume deformations from Sec. 2.7.1, which in the present notation becomes

$$ds_{\mathcal{M}}^2 = n \int_X d^d x \sqrt{\bar{G}} \frac{1}{4} \bar{G}^{mp} \bar{G}^{nq} \delta \perp \bar{G}_{mn} \delta \perp \bar{G}_{pq}.$$

(162)

If the moduli space metrics (160) and (162) agree, then compensators are unnecessary. For example, whenever the volume form

$$\text{Vol}_X = n V_X \sqrt{\bar{G}} d x^1 \wedge \ldots \wedge d x^d,$$

(163)

is constant, the two moduli space metrics agree. This is the case for constant metrics on tori, whose volume preserving deformations are automatically transverse and traceless.

**Application to a metric of Gibbons-Hawking form**

A 4D metric of Gibbons-Hawking form,

$$\bar{G}_{mn} d x^m d x^n = \Delta^{-1} Z G_{\alpha \beta} d x^\alpha d x^\beta + \Delta Z^{-1} (d x^4 + A)^2,$$

with $d A = *_G d Z$, $\Delta = \sqrt{\bar{G}}$, (164)

is another special case in which the two moduli space metrics agree and compensators are unnecessary. For a metric of the form (164), we have

$$\sqrt{\bar{G}} = Z, \quad -\frac{1}{4} (\bar{G}^{mn} \delta \bar{G}_{mn}) = -(\delta \log Z)^2,$$

(165)

and the net effect of the projection from $\delta \bar{G}_{mn}$ to $\delta \perp \bar{G}_{mn}$ after integration in Eq. (162) is to “ignore” variations in $Z$ when moduli are varied. Correspondingly, in the naive moduli space metric (160), all $Z$ derivatives cancel between the first and second terms, leaving the same result (up to multiplicative factors of $Z$) as if the metric were

$$G_{mn} d x^m d x^n = \Delta^{-1} G_{\alpha \beta} d x^\alpha d x^\beta + \Delta (d x^4 + A)^2,$$

A further convenient simplification is that the resulting $\bar{G}^{mp} \bar{G}^{nq} \delta \bar{G}_{mn} \delta \bar{G}_{pq}$ terms in the naive moduli space metric are just right to cancel all explicit $\Delta$ derivatives, leaving the same result (again, up to multiplicative factors of $Z$) as if the metric were

$$\bar{G}_{mn} d x^m d x^n = G_{\alpha \beta} d x^\alpha d x^\beta + (d x^4 + A)^2.$$

What remains is

$$ds_{\mathcal{M}, \text{naive}}^2 = n \int_X d^d x \left( \frac{1}{4} Z G^{\alpha \gamma} G^{\beta \delta} \delta G_{\alpha \beta} \delta G_{\gamma \delta} + Z^{-1} G^{\alpha \beta} A_{\alpha} \delta A_{\beta} \right).$$

(166)
Application to K3 in the Gibbons-Hawking approximation

We are now in a position to evaluate the naive moduli space metric of K3 in the Gibbons-Hawking approximation (90). Specializing to $X = K3 \cong T^4/\mathbb{Z}_2$, we set $n = 2$ and write $2 \int_{T^4/\mathbb{Z}_2} = \int_{T^4}$, so that

$$\int_{T^4} d^4x Z = 1,$$

(167)

and the naive moduli space metric becomes

$$ds^2_{M, \text{naive}} = \int_{T^4} d^4x \left( \frac{1}{4} Z G^{\alpha\gamma} G^{\beta\delta} \delta G_{\alpha\beta} \delta G_{\gamma\delta} + Z^{-1} G^{\alpha\beta} \delta A_{\alpha} \delta A_{\beta} \right),$$

(168)

for volume-preserving deformations, and

$$ds^2_V = \frac{9}{20} \left( \frac{\delta V_{K3}}{V_{K3}} \right)^2$$

(169)

for the overall K3 volume modulus (from $D = 11$ and $d = 4$ in Eq. (161)).

In evaluating the first integral in Eq. (168), the quantity $\int_{T^4} d^4x Z = 1$ trivially factors out, leaving

$$\int_{T^4} d^4x Z G^{\alpha\gamma} G^{\beta\delta} \delta G_{\alpha\beta} \delta G_{\gamma\delta} = \frac{1}{4} G^{\alpha\gamma} G^{\beta\delta} \delta G_{\alpha\beta} \delta G_{\gamma\delta}.$$

To evaluate the second integral, we recall from Eq. (144) that $\delta A$ has two terms, one proportional to $\delta x^{I\alpha}$ and the other proportional to $\delta x^{I\alpha} = \delta \beta^{\alpha\beta} = \delta \beta^{\alpha\beta} - x^{I\alpha} dx^{I\beta} + x^{I\beta} dx^{I\alpha}$:

$$\delta A = \left( \delta x^{I\alpha} (F_I - F'_I)_{\gamma\alpha} + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \delta \beta^{\alpha\beta} \right) dx^{\gamma}.$$

The two terms contribute orthogonally to the naive moduli space metric (168). The contribution from $\delta x^{I\alpha}$ term can be evaluated using

$$\int_{T^4} d^4x Z^{-1} (F_I - F'_I)_{\gamma\alpha} (F_J - F'_J)_{\gamma\beta} = (\delta_{IJ} + \delta_{I'J'}) G_{\alpha\beta} = 2 \delta_{IJ} G_{\alpha\beta},$$

(170)

which follows from Eq. (253) of App. E.4. The contribution from the $\delta \beta^{\alpha\beta}$ term gives an integral in which $\int_{T^4} d^4x Z = 1$ again trivially factors out.

Collecting all terms, we find

$$ds^2_{M, \text{naive}} = \frac{1}{4} G_{\alpha\gamma} G_{\beta\delta} \left( \delta G^{\alpha\beta} \delta G^{\gamma\delta} + \delta \beta^{\alpha\beta} \delta \beta^{\gamma\delta} \right) + 2 G_{\alpha\beta} \delta x^{I\alpha} \delta x^{I\beta}.$$

(171)

This naive moduli space metric precisely matches the diffeomorphism invariant moduli space metric $ds^2_M$ obtained in the previous section. So, indeed, compensators are an unnecessary machinery for computing the K3 moduli space metric in the Gibbons-Hawking approximation, and one can equivalently use the naive moduli space metric (160) from dimensional reduction.
4 Conclusions

We have reviewed the geometry of K3 surfaces and have developed this geometry through a novel approximate K3 metric of Gibbons-Hawking form. Our review in Sec. 2 described the holonomy, hyperkähler structure, and homology of K3 surfaces, including an explicit description of the $(-E_8) \oplus (-E_8) \oplus (U_{1,1})^3$ and $\text{Spin}(32)/\mathbb{Z}_2 \oplus (U_{1,1})^3$ splittings of the K3 homology lattice, in terms of six 2-tori inherited from $T^4$ and 16 exceptional divisors of the Kummer resolution of $T^4/\mathbb{Z}_2$. We turned next to harmonic forms and the moduli space of hyperkähler structure, as well as the relation between metric deformations and harmonic forms. Finally, we described the need for compensators and projection to transverse traceless gauge to ensure diffeomorphism invariance of the metric on moduli space, and derived the coset form of the diffeomorphism invariant metric on the moduli space of K3 metrics.

Sec. 3 focused on the approximate K3 metric (90), which we obtained from the M-theory lift of the tree-level type IIA supergravity description of the $T^3/\mathbb{Z}_2$ orientifold. In the large hypercomplex structure limit, the metric closely approximates the exact K3 metric everywhere except in a small neighborhood of 8 points corresponding to the lifts of the orientifold planes. The full nonperturbative lift replaces these regions with smooth Atiyah-Hitchin spaces and gives the exact K3 metric. We described the coframe, hyperkähler forms, and basis of harmonic forms in the approximate metric, identifying the cohomology classes of this basis with known classes in the exact description. The metric components are simple functions of the parameters $G^{\alpha\beta}, \beta^{\alpha\beta}$, and $x^I$, defining the $(\text{SO}(3) \times \text{SO}(19))/\text{SO}(3,19)$ transformation from an integer cohomology basis to the harmonic basis, i.e., parametrizing the hyperkähler moduli space. Finally, we studied the diffeomorphism invariant metric on the metric moduli space, as well as the naive moduli space metric from dimensional reduction. We found that the Gibbons-Hawking form of the approximate metric leads to the novel property that these moduli space metrics agree. Moreover, both coincide with the exact $\mathbb{R}_{>0} \times (\text{SO}(3) \times \text{SO}(19))/\text{SO}(3,19)$ coset metric of K3.

As described in the Introduction, this project forms a part of a larger investigation with the goal of providing a duality derivation of the procedure for warped Kaluza-Klein reduction [52, 53] through duality to conventional compactifications, and a secondary goal of shedding light on compactification of type II string theory on 6D manifolds of $SU(2)$ structure [54], of which the abelian surface fibered Calabi-Yau 3-folds of Refs. [51, 17] are examples. While not themselves realistic, the warped backgrounds of interest here are simplified analogs of the type IIA intersecting D6-brane models and type IIB flux compactifications that feature prominently in model building. The $T^3/\mathbb{Z}_2$ orientifold, upon further compactification on $T^3$, the can be thought of as a baby version of a type IIA intersecting D6-brane model: it based on a $T^6$ rather than Calabi-Yau 3-fold; the orientifold involution preserves a $T^3$ of $T^6$ instead of special Lagrangian 3-cycles of the Calabi-Yau; and all D6-branes and O6-planes are parallel instead of intersecting. Similarly, the type IIB $T^6/\mathbb{Z}_2$ orientifold with $\mathcal{N} = 2$ flux shares much in common with $\mathcal{N} = 1$ or 0 flux compactifications based on F-theory compactifications or Calabi-Yau orientifolds; the main difference here is that there are solely D3-branes and O3-planes, rather than 7-branes wrapping holomorphic cycles. Studying the warped Kaluzak-Klein reduction of these simple models [52, 53] should offer lessons for the
more realistic compactifications of phenomenological interest, and provide useful examples for probing the formalism developed in Refs. [56, 18, 22, 23, 63].

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A Hyperkähler structure on $T^4$

A choice of hyperkähler structure on $T^4$ is analogous to a choice of complex structure on $T^2$. Let us first review the latter in a way that makes the generalization natural, and then go on to discuss hyperkähler structure on $T^4$. This review first appeared as App. A of Ref. [12], but was written with the application to the duality between M-theory on K3 and IIA on $T^3/\mathbb{Z}_2$ in mind. Minor changes have been made to conform to the notation of the present paper.

On $T^2$, we can express the metric as

$$ds^2_{T^2} = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2,$$

(172)

where $\theta^m = \theta^m_n dx^n$ in terms of a vielbein $\theta^m_n$. The complex structure is defined by a tensor $J_m^n$ which we view as a map $J: T^* \rightarrow T^*$, such that

$$J: \theta^2 \rightarrow \theta^1, \quad \theta^1 \rightarrow -\theta^2.$$  

(173)

Lowering the upper index of $J_m^n$ gives the Kähler form $J$ on $T^2$. By $SL(2,\mathbb{Z})$ change of lattice basis for the lattice $\Lambda$ of $T^2 = \mathbb{R}^2/\Lambda$, we can always write

$$\theta^1 = R^1(dx^1 + a^1_2 dx^2),$$

$$\theta^2 = R^2 dx^2, \quad \text{where} \quad x^m \cong x^m + 1.$$  

(174)

The holomorphic 1-form is

$$\theta^z = \theta^1 + i\theta^2 = R^1(dx^1 + \tau_1 dx^2),$$  

(175)

The usual convention in the math literature is the transpose of this; $J$ has index structure $J^m_n$, so that $J$ acts from the left on the tangent space, and $J^T$ acts from the left on the cotangent space. However, if we require that (i) $J$ with holomorphic (antiholomorphic) indices be $+i$ ($-i$), as is customary in both the math and physics conventions, and (ii) the Kähler form $J$ be obtained by lowering one index of the tensor $J$, with no sign change, then we are uniquely led to the conventions used in this paper.
where $\tau_1 = a_1^2 + i \frac{R_2}{R_4}$ is the complex structure modulus.

Likewise, we can express the metric on $T^4$ as
\[
\left. ds^2_{T^4}\right|_{\mathbb{T}^4} = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3 + \theta^4 \otimes \theta^4,
\]
where, again, $\theta^\alpha = \theta^{\alpha}_m dx^m$ in terms of a vielbein $\theta^{\alpha}_m$. The hyperkähler structure is defined by a triple of tensors $(\mathcal{J}^\alpha)_m^n$, $\alpha = 1, 2, 3$, which we view as maps $\mathcal{J}^\alpha: T^* \to T^*$, such that
\[
\begin{align*}
\mathcal{J}^1: & \quad \theta^4 \to \theta^1, \quad \theta^3 \to \theta^2, \quad \theta^1 \to -\theta^4, \quad \theta^2 \to -\theta^3, \\
\mathcal{J}^2: & \quad \theta^4 \to \theta^2, \quad \theta^3 \to \theta^1, \quad \theta^2 \to -\theta^4, \quad \theta^3 \to -\theta^1, \\
\mathcal{J}^3: & \quad \theta^4 \to \theta^3, \quad \theta^2 \to \theta^1, \quad \theta^3 \to -\theta^4, \quad \theta^1 \to -\theta^2.
\end{align*}
\]

The $(\mathcal{J}^\alpha)_m^n$ satisfy
\[
\mathcal{J}^1 \mathcal{J}^2 = -\mathcal{J}^2 \mathcal{J}^1 = -\mathcal{J}^3, \quad (\mathcal{J}^1)^2 = -1,
\]
plus cyclic permutations. Lowering the upper index on $(\mathcal{J}^\alpha)_m^n$ gives a triple of Kähler forms $J^\alpha_{mn}$. The quaternionic 1-form is
\[
\theta^\alpha = \theta^4 - i \theta^1 - j \theta^2 - k \theta^3,
\]
where the quaternions $i, j, k$ satisfy the same algebra as $-\mathcal{J}^\alpha$.

A choice of complex structure on $T^4$ is then a choice of $i$ on the $i, j, k$ unit sphere. By a $\text{SL}(4, \mathbb{Z})$ change of lattice basis for the $T^4$, we can write, in addition to Eq. (174),
\[
\begin{align*}
\theta^3 &= R^3(dx^3 + a_3^1 dx^1 + a_3^2 dx^2), \\
\theta^4 &= R^4(dx^4 + a_4^1 dx^1 + a_4^2 dx^2 + a_4^3 dx^3), \quad \text{where} \quad x^m \cong x^m + 1.
\end{align*}
\]
So, for example, if we choose complex structure $i = k$, then the complex pairing that follows from $\mathcal{J} = \mathcal{J}^3$ is
\[
\begin{align*}
\theta^{z_1} &= R_1 \theta^1 + i R_2 \theta^2 = R^1(dx^1 + \tau_1 dx^2), \\
\theta^{z^2} &= R_4 \theta^4 - i R_3 \theta^3 = R^4(dx^4 + \tau_2^{-1} dx^3 + \ldots),
\end{align*}
\]
where $\tau_2^{-1} = a_4^3 - i \frac{R_3}{R_4}$ and the “...” is a 1-form on $T^2(x^1, x^2)$, which can be interpreted as the connection for a trivial fibration of $T^2(x^3, x^4)$ over $T^2(x^1, x^2)$. The holomorphic (2, 0) form in this case is
\[
\Omega_{(2,0)} = \theta^{z_1} \wedge \theta^{z^2} = -J^1 + i J^2.
\]

If we write the metric on $T^4$ as
\[
ds^2_{T^4} = R_4^2(dx^4 + a^4_\alpha dx^\alpha)^2 + \epsilon_{\alpha\beta} dx^\alpha dx^\beta, \quad i = 1, 2, 3,
\]
\footnote{Here, $-\mathcal{J}^\alpha$ rather than $\mathcal{J}^\alpha$ satisfies the quaternion algebra for the reason discussed in the previous footnote. Note that the tangent space map $(\mathcal{J}^\alpha)^T$ satisfies the quaternion algebra with no minus sign.}

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then the choice of hyperkähler structure is the choice of $T^4$ volume $V_{T^4} = \sqrt{g} R_4$ together with the choice of hypercomplex structure. The latter is the choice of $\beta_\alpha = a^4_{\alpha}$ together with the dimensionless metric $G_{\alpha\beta} = (R_4/\sqrt{g}) g_{\alpha\beta}$. In terms of $V_{T^4}$, $G_{\alpha\beta}$, and $\beta_\alpha$, we can write the $T^4$ metric as

$$ds^2 = \sqrt{V_{T^4}} \left( \Delta(dx^4 + \beta_\alpha dx^\alpha)^2 + \Delta^{-1} G_{\alpha\beta} dx^\alpha dx^\beta \right).$$

(184)

Let us define $\beta^{\alpha\beta} = \epsilon^{\alpha\beta\gamma} \beta_\gamma$, where $\epsilon^{123} = 1$, and let $E^\alpha_\beta$ be a vielbein for the metric $G_{\alpha\beta}$. Then, the choice of $E^\alpha_\beta$ and $\beta^{\alpha\beta}$ parametrizes the $\left( \text{SO}(3) \times \text{SO}(3) \right) \backslash \text{SO}(3,3)/\Gamma_{3,3}$ truncation of the coset $[\mathrm{19}]$, with vielbein

$$V = \begin{pmatrix} E & -E^\beta \\ 0 & E^{-1T} \end{pmatrix}.$$  

(185)

This coset can be interpreted as the choice of positive signature 3-plane spanned by $J^1, J^2, J^3$ in $H^2(T^4, \mathbb{R}) = \mathbb{R}^{3,3}$, modulo lattice isomorphisms of $H^2(T^4, \mathbb{Z})$.

**B The homology lattice of K3**

For completeness, we review the integer homology lattice of K3 via its interpretation as the resolution of the orbifold $T^4/\mathbb{Z}_2$. App. [B.1] is based primarily on Refs. [15] and [65], and has been reproduced from App. B of Ref. [12], with minor notational changes. App. [B.2] derives the $(-E_8) \oplus (-E_8) \oplus U_{1,1}^{3}$ and $(-\text{Spin}(32)/\mathbb{Z}_2) \oplus U_{1,1}^{3}$ splittings of this lattice.

**B.1 Resolution of $T^4/\mathbb{Z}_2$**

Let us view $T^4$ as $T^2(1)(x^1, x^2) \times T^2(2)(x^3, x^4)$ with complex pairing $dz_1 = dx^1 + \tau_1 dx^2$ and $dz_2 = dx^3 + \tau_2 dx^4$. Now consider $T^4/\mathbb{Z}_2$. There are 24 = 16 points of local geometry $\mathbb{C}^2/\mathbb{Z}_2$ (16 $A_1$ singularities), located at the fixed points where each of the four coordinates is equal to 0 or 1/2. There are also 4 + 4 = 8 fixed lines $\mathbb{P}^1$ with a simple description in this complex structure: let $D_{3s}$, $s = 1, 2, 3, 4$ label the divisors $\mathbb{P}^1 = T^2(2)/\mathbb{Z}_2$ located at each of the four fixed points in $(x^1, x^2)$ and $D^3_t$, $t = 1, 2, 3, 4$ denote the divisors $\mathbb{P}^1 = T^2(1)/\mathbb{Z}_2$ located at the four fixed point in $(x^3, x^4)$. The intersections of these $\mathbb{P}^1$s in the singular geometry is illustrated schematically in Fig. 4 (a).

The homology classes of the $D_{3s}$ and $D^3_t$ in the singular geometry are

$$D_{3s} = \frac{1}{2} f_3, \quad D^3_t = \frac{1}{2} f^3,$$

(186)

independent of $s, t$, where $f_3$ is the class of $T^2(2)$ and $f^3$ is the class of $T^2(1)$. Let us focus on the singularity at the “half point”$^{23}$ $p_{st} = D_{3s} \cap D^3_t$, and consider the local model $\mathbb{C}^2/\mathbb{Z}_2$ at this point.

Fig.2 (a) gives the fan for the toric description of $\mathbb{C}^2/\mathbb{Z}_2$. There is a single two dimensional fan of volume 2 generated by the lattice vectors $D_{3s} = (0, 1)$ and $D^3_j = (2, 1)$, each of which

$^{23}$This “half point” is the interpretation of $\int_{K3} (dx^1 \wedge dx^2) \wedge (dx^3 \wedge dx^4) = \frac{1}{2} \int_{T^4} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = 1/2$. 

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Divisors can always be represented in patches as the vanishing loci of local meromorphic functions. For the resolved fan, we include the lattice vector $p_{41}$ to an exceptional divisor $E_{st}$. After resolution, $D_{3s}$ and $D_t^3$ no longer intersect, but each intersects $E_{st}$ in a point. In the figures above, only $p_{41}$ and its blow up $E_{41}$ are labeled explicitly.

Figure 1: (a) In the singular $T^4/{\mathbb{Z}}_2$ (left), each of the sixteen $A_1$ singularities is the “half point” of intersection, $p_{st}$, of two fixed $\mathbb{P}^1$s, $D_{3s}$ and $D_t^3$. (b) In the resolved $K3$ (right), each $p_{st}$ is blown up to an exceptional divisor $E_{st}$. After resolution, $D_{3s}$ and $D_t^3$ no longer intersect, but each intersects $E_{st}$ in a point. In the figures above, only $p_{41}$ and its blow up $E_{41}$ are labeled explicitly.

Figure 2: (a) The fan for the local model $C^2/{\mathbb{Z}}_2$ at the singular point $p_{st}$ in $T^4/{\mathbb{Z}}_2$ (left), and (b) the fan for the resolution (right), with the point $p_{st}$ blown up to the exceptional divisor $E_{st}$.

represents a divisor of $T^4/{\mathbb{Z}}_2$. If we take $p_{st}$ to be the origin of $C^2/{\mathbb{Z}}_2$, then these divisors are $D_{3s} = \{z_1 = 0\}$ and $D_t^3 = \{z_2 = 0\}$. In the toric description, to resolve the singularity, we subdivide the original singular cone into two cones of volume 1 by introducing a new divisor $E_{st}$. $E_{st}$ is the exceptional divisor obtained by blowing up the origin of $C^2/{\mathbb{Z}}_2$. This notation differs slightly from the notation $E_1$ given in the body of the paper, however, in either notation a unique $(s, t)$ or $I$ characterizes each of the 16 singular points of $T^4/{\mathbb{Z}}_2$.

Let us make this more explicit. To each of the lattice components $r$, we associate a monomial $U_r = \prod_{i=1}^2 z_i^r(V_i)^r$, where $(V_i)_r$ is the $r$th component of the lattice vector $V_i$ in the fan. The toric variety is then given by the set of all $(z_1, z_2)$ not in the excluded set $F$, modulo rescalings that leave the $U_r$ invariant. The excluded set $F$ consists of all points that have simultaneous zeros of coordinates whose corresponding $V_i$ do not lie in the same cone. For the unresolved fan of Fig. 3 (a), there is just a single two dimensional cone, so $F = \emptyset$. The only rescaling that leaves $U_1, U_2$ invariant is $\mathbb{Z}_2$: $(z_1, z_2) \rightarrow (-z_1, -z_2)$. So, the toric variety is indeed $\{(z_1, z_2)\}/\mathbb{Z}_2 = C^2/\mathbb{Z}_2$.

For the resolved fan, we include the lattice vector $E_{st} = (1, 1)$ as well, as shown in Fig. 2(b). In this case, $U_1 = z_2 w$ and $U_2 = z_1 z_2 w$, where $w$ is the new coordinate associated to $E_{st}$. The excluded set is $F = \{z_1 = z_2 = 0\}$. The rescaling symmetry of $U_1, U_2$ is $\mathbb{C}^*$: $(z_1, z_2, w) \rightarrow (\lambda z_1, \lambda z_2, \lambda^{-2} w)$. Away from $w = 0$, this gives $(z_1, z_2, 1)/\mathbb{Z}_2 = C^2/\mathbb{Z}_2$ with the $z$-origin deleted. At $w = 0$, we obtain the exceptional $\mathbb{P}^1$, $E_{st} = \{(z_1, z_2, 0) \setminus (0, 0, 0)\}/\mathbb{C}^*$.
functions. However, divisors that globally have such a representation are homologically trivial and have trivial intersection with other divisors. (See, for example, Ref. [31].)

In our toric model for the resolution of $\mathbb{C}^2/\mathbb{Z}_2$, a basis of such global meromorphic functions is $U_1, U_2$. The corresponding homologically trivial divisors are $2D_{3s} + E_{st}$ (from $U_2^2/U_1 = 0$) and $2D_3^3 + E_{st}$ (from $U_1 = 0$). In the compact $K3$ (as explained for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$) in Ref. [15]), these relations become

$$f_3 = 2D_{3s} + \sum_{t=1}^{4} E_{st} \quad \text{independent of } s,$$

$$f^3 = 2D_3^3 + \sum_{s=1}^{4} E_{st} \quad \text{independent of } t,$$

where the divisors $f_3$ and $f^3$ are not homologically trivial, but instead correspond to “sliding divisors” that can be moved away from the (resolved) singularities. They have trivial intersection with the exceptional divisors $E_{st}$ and represent the tori $f_3 = \{z_1 = c_1\} \cup \{z_1 = -c_1\}$ and $f^3 = \{z_2 = c_2\} \cup \{z_2 = -c_2\}$ on the $T^4$ covering space, where $c_1, c_2$ are non fixed points. The corresponding Poincaré dual cohomology classes are $f_3 = 2dx^1 \wedge dx^2$ and $f^3 = 2dx^3 \wedge dx^4$, respectively.

The cycles in $K3$ described so far are those that are particularly simple in the complex structure $J^3$. In the same way, in the complex structure $J^1$ we obtain homology classes $f_1$ and $f^1$ from elliptic curves $T_2^{(2)}$ and $T_2^{(2)}$, located at non fixed points in $(x^2, x^3)$ and $(x^1, x^4)$, respectively. In the complex structure $J^2$ we obtain homology classes $f_2$ and $f^2$ from elliptic curves $T_2^{(2)}$ and $T_2^{(2)}$, located at non fixed points in $(x^3, x^1)$ and $(x^2, x^4)$. Likewise, we obtain divisors $D_{1s}, D_{1t}$ and $D_{2s}, D_{2t}$ by setting the corresponding pairs of coordinates equal to their $\mathbb{Z}_2$ fixed values before the resolution. The homology lattice of $K3$ is the integer span of the overcomplete basis given by the $6f, 24D$ and $16E$ divisors.

### B.2 Splittings of the homology lattice

Returning to the notation of the body of the paper, let us label the 16 exceptional divisors $E_I$ and their corresponding fixed points $(x^1, x^2, x^3, x^4) \in \frac{1}{2} \mathbb{P}_2^4 \subset T^4/\mathbb{Z}_2$ as follows:

| $E_I$ | $(x^1, x^2, x^3, x^4)$ |
|-------|------------------------|
| $E_1$ | $(0, 0, 0, 0)$          |
| $E_2$ | $(\frac{1}{2}, 0, 0, 0)$|
| $E_3$ | $(0, \frac{1}{2}, 0, 0)$|
| $E_4$ | $(\frac{1}{2}, \frac{1}{2}, 0, 0)$|
| $E_5$ | $(0, 0, \frac{1}{2}, 0)$|
| $E_6$ | $(\frac{1}{2}, 0, \frac{1}{2}, 0)$|
| $E_7$ | $(0, \frac{1}{2}, \frac{1}{2}, 0)$|
| $E_8$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$|
| $E_9$ | $(0, 0, 0, \frac{1}{2})$ |
| $E_{10}$ | $(\frac{1}{2}, 0, 0, \frac{1}{2})$ |
| $E_{11}$ | $(0, \frac{1}{2}, 0, \frac{1}{2})$ |
| $E_{12}$ | $(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})$ |
| $E_{13}$ | $(0, 0, \frac{1}{2}, \frac{1}{2})$ |
| $E_{14}$ | $(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$ |
| $E_{15}$ | $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ |
| $E_{16}$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ |
Again, recall that we label the divisors $D$ so that $D_t^1$, for $t = 1, 2, 3, 4$, denote the four $D$ at fixed $(x^1, x^4)$, and $D_s^1$, for $s = 1, 2, 3, 4$, denote the four $D$ at fixed $(x^2, x^3)$, with analogous definitions obtained by cyclic permutation of $1, 2, 3$. Then, from Eq. (187), the 24 divisors $D$ and their locations in $T^4/Z_2$ are

\[
\begin{align*}
(x^1, x^4) &= (0, 0) & D_1^1 &= \frac{1}{2} f_1 - \frac{1}{2} (E_1 + E_5 + E_9 + E_{13}), \\
(x^1, x^4) &= (0, \frac{1}{2}) & D_2^1 &= \frac{1}{2} f_1 - \frac{1}{2} (E_2 + E_6 + E_{10} + E_{14}), \\
(x^1, x^4) &= (\frac{1}{2}, 0) & D_3^1 &= \frac{1}{2} f_1 - \frac{1}{2} (E_3 + E_7 + E_{11} + E_{15}), \\
(x^1, x^4) &= (\frac{1}{2}, \frac{1}{2}) & D_4^1 &= \frac{1}{2} f_1 - \frac{1}{2} (E_4 + E_8 + E_{12} + E_{16}), \\
(x^2, x^4) &= (0, 0) & D_1^2 &= \frac{1}{2} f_1^2 - \frac{1}{2} (E_1 + E_3 + E_9 + E_{11}), \\
(x^2, x^4) &= (0, \frac{1}{2}) & D_2^2 &= \frac{1}{2} f_1^2 - \frac{1}{2} (E_2 + E_4 + E_{10} + E_{12}), \\
(x^2, x^4) &= (\frac{1}{2}, 0) & D_3^2 &= \frac{1}{2} f_1^2 - \frac{1}{2} (E_5 + E_7 + E_{13} + E_{15}), \\
(x^2, x^4) &= (\frac{1}{2}, \frac{1}{2}) & D_4^2 &= \frac{1}{2} f_1^2 - \frac{1}{2} (E_6 + E_8 + E_{14} + E_{16}), \\
(x^3, x^4) &= (0, 0) & D_1^3 &= \frac{1}{2} f_1^3 - \frac{1}{2} (E_1 + E_3 + E_5 + E_7), \\
(x^3, x^4) &= (0, \frac{1}{2}) & D_2^3 &= \frac{1}{2} f_1^3 - \frac{1}{2} (E_2 + E_4 + E_6 + E_8), \\
(x^3, x^4) &= (\frac{1}{2}, 0) & D_3^3 &= \frac{1}{2} f_1^3 - \frac{1}{2} (E_9 + E_{11} + E_{13} + E_{15}), \\
(x^3, x^4) &= (\frac{1}{2}, \frac{1}{2}) & D_4^3 &= \frac{1}{2} f_1^3 - \frac{1}{2} (E_{10} + E_{12} + E_{14} + E_{16}), \\
(x^2, x^3) &= (0, 0) & D_{11} &= \frac{1}{2} f_1 - \frac{1}{2} (E_1 + E_2 + E_3 + E_4), \\
(x^2, x^3) &= (0, \frac{1}{2}) & D_{12} &= \frac{1}{2} f_1 - \frac{1}{2} (E_9 + E_{10} + E_{11} + E_{12}), \\
(x^2, x^3) &= (\frac{1}{2}, 0) & D_{13} &= \frac{1}{2} f_1 - \frac{1}{2} (E_5 + E_6 + E_7 + E_8), \\
(x^2, x^3) &= (\frac{1}{2}, \frac{1}{2}) & D_{14} &= \frac{1}{2} f_1 - \frac{1}{2} (E_{13} + E_{14} + E_{15} + E_{16}), \\
(x^3, x^1) &= (0, 0) & D_{21} &= \frac{1}{2} f_2 - \frac{1}{2} (E_1 + E_2 + E_5 + E_6), \\
(x^3, x^1) &= (0, \frac{1}{2}) & D_{22} &= \frac{1}{2} f_2 - \frac{1}{2} (E_3 + E_4 + E_7 + E_8), \\
(x^3, x^1) &= (\frac{1}{2}, 0) & D_{23} &= \frac{1}{2} f_2 - \frac{1}{2} (E_9 + E_{10} + E_{13} + E_{14}), \\
(x^3, x^1) &= (\frac{1}{2}, \frac{1}{2}) & D_{24} &= \frac{1}{2} f_2 - \frac{1}{2} (E_{11} + E_{12} + E_{15} + E_{16}), \\
(x^1, x^2) &= (0, 0) & D_{31} &= \frac{1}{2} f_3 - \frac{1}{2} (E_1 + E_2 + E_9 + E_{10}), \\
(x^1, x^2) &= (0, \frac{1}{2}) & D_{32} &= \frac{1}{2} f_3 - \frac{1}{2} (E_5 + E_6 + E_{13} + E_{14}), \\
(x^1, x^2) &= (\frac{1}{2}, 0) & D_{33} &= \frac{1}{2} f_3 - \frac{1}{2} (E_3 + E_4 + E_{11} + E_{12}), \\
(x^1, x^2) &= (\frac{1}{2}, \frac{1}{2}) & D_{34} &= \frac{1}{2} f_3 - \frac{1}{2} (E_7 + E_8 + E_{15} + E_{16}).
\end{align*}
\]
In terms of the orthonormal basis $e_I$, defined by
\[ e_1 = \frac{1}{2}(E_2-E_1), \quad e_2 = \frac{1}{2}(E_2+E_1), \quad \ldots \quad e_{15} = \frac{1}{2}(E_{16}-E_{15}), \quad e_{16} = \frac{1}{2}(E_{16}+E_{15}), \]
this becomes
\[
\begin{align*}
D_1^1 &= \frac{1}{2}f^1 - \frac{1}{2}(-e_1 + e_2 - e_5 + e_6 - e_9 + e_{10} - e_{13} + e_{14}), \\
D_2^1 &= \frac{1}{2}f^1 - \frac{1}{2}(e_1 + e_2 + e_5 + e_6 + e_9 + e_{10} + e_{13} + e_{14}), \\
D_3^1 &= \frac{1}{2}f^1 - \frac{1}{2}(-e_3 + e_4 - e_7 + e_8 - e_{11} + e_{12} - e_{15} + e_{16}), \\
D_4^1 &= \frac{1}{2}f^1 - \frac{1}{2}(e_3 + e_4 + e_7 + e_8 + e_{11} + e_{12} + e_{15} + e_{16}), \\
D_5^2 &= \frac{1}{2}f^2 - \frac{1}{2}(-e_1 + e_2 - e_3 + e_4 - e_9 + e_{10} - e_{11} - e_{12}), \\
D_6^2 &= \frac{1}{2}f^2 - \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_9 + e_{10} + e_{11} + e_{12}), \\
D_7^2 &= \frac{1}{2}f^2 - \frac{1}{2}(-e_5 + e_6 - e_7 + e_8 - e_{13} + e_{14} - e_{15} + e_{16}), \\
D_8^2 &= \frac{1}{2}f^2 - \frac{1}{2}(e_5 + e_6 + e_7 + e_8 + e_{13} + e_{14} + e_{15} + e_{16}), \\
D_9^3 &= \frac{1}{2}f^3 - \frac{1}{2}(-e_1 + e_2 - e_3 + e_4 - e_5 + e_6 - e_7 + e_8), \\
D_{10}^3 &= \frac{1}{2}f^3 - \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8), \\
D_{11}^3 &= \frac{1}{2}f^3 - \frac{1}{2}(-e_9 + e_{10} - e_{11} + e_{12} - e_{13} + e_{14} - e_{15} + e_{16}), \\
D_{12}^3 &= \frac{1}{2}f^3 - \frac{1}{2}(e_9 + e_{10} + e_{11} + e_{12} + e_{13} + e_{14} + e_{15} + e_{16}),
\end{align*}
\]

Here, the $E_I$ and $e_I$ coincide with $\chi^{(A)}_I$ and $e^{(A)}_I$ of Sec. 2.4.3 and we have dropped the superscripts in this Appendix for notational simplicity.

The K3 (co)homology lattice is the integer span of $f^\alpha, f_\alpha, D^\alpha, D_{\alpha}, E_I$. We now obtain lattice bases realizing the splittings
\[
H_2(K3, \mathbb{Z}) \cong (-\text{Spin}(32)/\mathbb{Z}_2) \oplus (U_{1,1})^{\oplus 3} \cong (-E_8) \oplus (-E_8) \oplus (U_{1,1})^{\oplus 3},
\]

where $(-\text{Spin}(32)/\mathbb{Z}_2)$ denotes the weight lattice of Spin(32)/Z_2 with opposite sign inner product, $(-E_8)$ denotes the $E_8$ root lattice with opposite sign inner product, and $U_{1,1}$ denotes the even selfdual lattice of signature $(1, 1)$ with inner product \((0 \ 1) \ 24\)

\[\text{B.2.1} \quad (-\text{Spin}(32)/\mathbb{Z}_2) \oplus (U_{1,1})^{\oplus 3} \text{ splitting}\]

To realize a $D_{16} \oplus U_{3,3}$ splitting of $H_2(K3, \mathbb{Z})$, we define
\[
U_{3,3} = \langle f_\alpha, D^\alpha_4 \rangle,
\]
and ask what the orthogonal complement $U_{3,3}^\perp$ is. The map from the (co)homology lattice to $U_{3,3}^\perp$ takes $v$ to $v^\perp = v - a^\alpha f_\alpha - b_\alpha D^\alpha_4$ with $a^\alpha$ and $b_\alpha$ chosen so that $v^\perp \cdot f_\alpha = v^\perp \cdot D^\alpha_4 = 0$.

\footnote{For $E_8$, the weight lattice is the same as the root lattice. For Spin(32), the ratio of the two is $\mathbb{Z}_2 \times \mathbb{Z}_2$. See Footnote [7] for the relation between the weights of Spin(32), Spin(32)/Z_2 and SO(32).}
With this definition, we find

\[
\begin{align*}
    e_1^+ &= e_1, & e_9^+ &= e_9 - \frac{1}{2} f_3, \\
    e_2^+ &= e_2, & e_{10}^+ &= e_{10} - \frac{1}{2} f_3, \\
    e_3^+ &= e_3 - \frac{1}{2} f_1, & e_{11}^+ &= e_{11} - \frac{1}{2} f_1 - \frac{1}{2} f_3, \\
    e_4^+ &= e_4 - \frac{1}{2} f_1, & e_{12}^+ &= e_{12} - \frac{1}{2} f_1 - \frac{1}{2} f_3, \\
    e_5^+ &= e_5 - \frac{1}{2} f_2, & e_{13}^+ &= e_{13} - \frac{1}{2} f_2 - \frac{1}{2} f_3, \\
    e_6^+ &= e_6 - \frac{1}{2} f_2, & e_{14}^+ &= e_{14} - \frac{1}{2} f_2 - \frac{1}{2} f_3, \\
    e_7^+ &= e_7 - \frac{1}{2} f_1 - \frac{1}{2} f_2, & e_{15}^+ &= e_{15} - \frac{1}{2} f_1 - \frac{1}{2} f_2 - \frac{1}{2} f_3, \\
    e_8^+ &= e_8 - \frac{1}{2} f_1 - \frac{1}{2} f_2, & e_{16}^+ &= e_{16} - \frac{1}{2} f_1 - \frac{1}{2} f_2 - \frac{1}{2} f_3,
\end{align*}
\]  

(193)

which can be summarized as

\[
e_I^+ = e_I - x^{I\alpha} f_\alpha,
\]  

(194)

with \( x^{I\alpha} \) given by Eq. (33). Identifying the \( e_I^+ \) of this section with \( e^{(D)}_I \) of Sec. 2.4.3 this extends to the transformation (33) of the full basis from \( e^{(A)}_\alpha \) to \( e^{(D)}_\alpha \). It is straightforward to check that \( e^+_I \pm e^+_J \) is an integer linear combination of \( f^\alpha, f_\alpha, D_\alpha, D_\alpha \) for all \( e^+_I \) and \( e^+_J \), and that the span of the \( e^+_I \pm e^+_J \) is contained in \( U_{3,3}^{\perp} \). Comparing to Eq. (29), we see that \( U_{3,3}^{\perp} \) contains the lattice \((-D_{16})\), where \((-D_{16})\) denotes the \(D_{16}\) root lattice with the sign of the inner product reversed. In fact, \( U_{3,3}^{\perp} \) is even larger than this. Summing the \( e^+_I \), we find

\[
\frac{1}{2} \sum_{I=1}^{16} e^+_I = \frac{1}{2} \left( \sum_{I=1}^{16} e_I \right) - 2(f_1 + f_2 + f_3) = f_3^3 - D_2^3 + D_1^3 - 2(f_1 + f_2 + f_3),
\]  

(195)

so that \( \frac{1}{2} \sum_{I=1}^{16} e^+_I \) is also an integer lattice vector in \( U_{3,3}^{\perp} \subset H_2(K3, \mathbb{Z}) \). By subtracting \(D_{16}\) roots of the form \( e^+_I + e^+_J \) from this lattice vector, we see that \( U_{3,3} \) contains not only the \(D_{16}\) roots, but also the lattice vectors differing from \( \frac{1}{2} \sum_{I=1}^{16} e^+_I \) by an even number of sign flips. The latter are the weights of the chiral spinor representation of \(\text{Spin}(32)\). Together, the roots and chiral spinor weights span \( U_{3,3}^{\perp} \) and form the weight lattice of \(\text{Spin}(32)/\mathbb{Z}_2 \)\(^{25}\)

Therefore,

\[
U_{3,3}^{\perp} \cong (-\text{Spin}(32)/\mathbb{Z}_2),
\]  

(196)

where \((-\text{Spin}(32)/\mathbb{Z}_2)\) denotes the weight lattice of \(\text{Spin}(32)/\mathbb{Z}_2 \) with opposite sign inner product. One can check that \(U_{3,3} \oplus U_{3,3}^{\perp} \) is indeed the whole homology lattice\(^{26}\)

Therefore,

\[
H_2(K3, \mathbb{Z}) \cong (-\text{Spin}(32)/\mathbb{Z}_2) \oplus U_{3,3}.
\]  

(197)

\(^{25}\)See Footnote\(^1\)

\(^{26}\)Over the reals, this is guaranteed. Over the integers, this simply requires checking that no integer (co)homology class decomposes into half integer classes in \(U_{3,3}\) and \(U_{3,3}^{\perp}\). In contrast, for the \((A_1)^{16}\) basis, note that the sum of \(\langle f_\alpha, f^\alpha \rangle \) and \(\langle f_\alpha, f^\alpha \rangle^{\perp} = \langle E_I \rangle \) is a sublattice of the (co)homology lattice of order 2 that misses the \(D_2^\alpha\) and \(D_\alpha, s\), which have half integer coefficients.
If we exchange one or more of the $D^\alpha_4$ for $D^\alpha_2$ in the definition of $U_{3,3}$, then Eq. (194) becomes
\[ e^\perp_I = e_I - (x^{I\alpha} - x_P)f_\alpha, \] (198)
where $x_P$ is the $\mathbb{Z}_2$ fixed point with $x^\alpha$ equal to 0 for each $D^\alpha_4$ and $\frac{1}{2}$ for each $D^\alpha_2$. Exchanging a $D^\alpha_4$ for a $D^\alpha_3$ or a $D^\alpha_2$ for a $D^\alpha_1$ is less interesting. It reverses the sign of $f_\alpha$ in Eq. (198) for odd $I$, which can be undone by the automorphism (Weyl reflection) $e_I, e^\perp_I \to -e_I, -e^\perp_I$ for odd $I$.

Finally, we note that the lattice $U_{3,3}$ splits as
\[ U_{3,3} = U_{1,1} \oplus U_{1,1} \oplus U_{1,1}. \] (199)
For definiteness, consider choice $U_{3,3} = \langle f_\alpha, D^\alpha_4 \rangle$. The basis $D^\alpha_4, f_\alpha$ does not realize this splitting, since $D^\alpha_4 \cdot D^\beta_4 = -1$ for $\alpha \neq \beta$. However, the basis
\[ D^1_4 + f_1 + f_2 + f_3, f_1; \quad D^2_4 + f_2 + f_3, f_2; \quad D^3_4, f_3, \] (200)
indeed has intersection form $\left( \begin{array}{c} 0 & 1 \\ 1 & 0 \end{array} \right)^{\otimes 3}$.

**B.2.2 \((−E_8) \oplus (−E_8) \oplus (U_{1,1})^{\otimes 3}\) splitting**

To realize a \((−E_8) \oplus (−E_8) \oplus U_{3,3}\) splitting of $H_2(K3, \mathbb{Z})$, we observe that the expressions for $D^1_3, D^2_3, D^3_3$ in Eq. (190) do not mix $e_1, \ldots, e_8$ with $e_9, \ldots, e_{16}$. Therefore, let us exchange the roles of $D^3_3, f_3$ and $D^3_3, f^3$ in the Spin(32)/$\mathbb{Z}_2$ discussion, and define
\[ U_{3,3}' = \langle f_1, f_2, f^3, D^1_4, D^2_4, D^3_4 \rangle. \] (201)
What is the orthogonal complement $U_{3,3}'^\perp$? We find
\[
\begin{align*}
e_1^\perp &= e_1, & e_9^\perp &= e_9, \\
e_2^\perp &= e_2, & e_{10}^\perp &= e_{10}, \\
e_3^\perp &= e_3 - \frac{1}{2}f_3, & e_{11}^\perp &= e_{11} - \frac{1}{2}f_1, \\
e_4^\perp &= e_4 - \frac{1}{2}f_1, & e_{12}^\perp &= e_{12} - \frac{1}{2}f_1, \\
e_5^\perp &= e_5 - \frac{1}{2}f_2, & e_{13}^\perp &= e_{13} - \frac{1}{2}f_2, \\
e_6^\perp &= e_6 - \frac{1}{2}f_2, & e_{14}^\perp &= e_{14} - \frac{1}{2}f_2, \\
e_7^\perp &= e_7 - \frac{1}{2}f_1 - \frac{1}{2}f_2, & e_{15}^\perp &= e_{15} - \frac{1}{2}f_1, \\
e_8^\perp &= e_8 - \frac{1}{2}f_1 - \frac{1}{2}f_2 - f^3, & e_{16}^\perp &= e_{16} - \frac{1}{2}f_1 - f^3. 
\end{align*}
\] (202)
Using \((D_1^\perp)^\perp = (D_2^\perp)^\perp = (D_3^\perp)^\perp = 0\) to determine \((f_1^\perp)^\perp, (f_2^\perp)^\perp, (f_3^\perp)^\perp\), we find from Eq. [190] that in the \(e_i^\perp\) basis,

\[
\begin{align*}
D_{11}^\perp &= \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \\
D_{12}^\perp &= \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \\
D_{21}^\perp &= \left(0, 0, 0, 0, 1, 0, 0, 0, 0, 0 \right) \\
D_{22}^\perp &= 0 \\
D_3^\perp &= \left(0, 0, 0, 0, 1, 0, 0, 0, 0, 0 \right) \\
D_4^\perp &= \left(0, 0, 0, 0, 1, 0, 0, 0, 0, 0 \right)
\end{align*}
\]

\[
\begin{align*}
D_{11}^\perp &= \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right); 0^8 \\
D_{12}^\perp &= \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right); 0^8 \\
D_2^\perp &= \left(0^8, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \\
D_4^\perp &= \left(0^8, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)
\end{align*}
\]

\[
\begin{align*}
D_{11}^\perp &= \left(0, -1, 0, -1, 0, 0, 0, 0; 0^8 \right) \\
D_{12}^\perp &= \left(0^8; 0, -1, 0, -1, 0, 0, 0 \right) \\
D_{13}^\perp &= \left(0, 0, 0, 0, 0; 0^8 \right) \\
D_{14}^\perp &= \left(0^8; 0, 0, 0, 0, 0, -1, 0, -1 \right) \\
D_{21}^\perp &= \left(0, -1, 0, 0, 0, -1, 0, 0; 0^8 \right) \\
D_{22}^\perp &= \left(0, 0, 0, -1, 0, 0, 0; 0^8 \right) \\
D_{31}^\perp &= \left(0^8; 0, -1, 0, 0, 0, 0, -1, 0, 0 \right) \\
D_{32}^\perp &= \left(0^8; 0, 0, 0, 0, -1, 0, 0, 0 \right) \\
D_{33}^\perp &= \left(0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0 \right) \\
D_{34}^\perp &= \left(0^8; 0, 0, 0, -1, 0, 0, 0, 0, -1 \right) \\
D_{31}^\perp &= \left(0, -1, 0, 0, 0, 0, 0, 0, 1; 0, -1, 0, 0, 0, 0, 0, 0, 1 \right) \\
D_{32}^\perp &= \left(0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, -1, 0, 0, 0, -1 \right) \\
D_{33}^\perp &= \left(0, 0, 0, -1, 0, 0, 0, 1; 0, 0, 0, -1, 0, 0, 0, 1 \right) \\
D_{34}^\perp &= \left(0^8; 0, 0, 0, -1, 0, 0, 0, 0, -1 \right)
\end{align*}
\]

51
These \( D^\perp \), together with the \( \chi_{2i-1} = e_{2i} - e_{2i-1} \) and \( \chi_{2i} = e_{2i} + e_{2i-1} \) for \( i = 1, \ldots, 8 \), span the lattice \((-E_8) \oplus (-E_8)\). Therefore, we identify the \( e_I^\perp \) here with \( e_I^{(E)} \) of Sec. 2.4.3. Recall that the roots of the lattice \((-E_8)\) are given by all permutations of

\[
(1, 1, 0^6), \quad (1, -1, 0^6), \quad (-1, -1, 0^6),
\]

together with all roots obtained from

\[
(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})
\]

by an even number of sign flips.\(^{27}\)

Finally, on general grounds, the even selfdual lattice \( U_{3,3}' \) should split as \( U_{3,3}' = U_{1,1}^{\oplus 3} \). This was demonstrated in the last section for the \((-\text{Spin}(32)/\mathbb{Z}_2)\oplus U_{3,3}' \) splitting of \( H_2(\text{K3}, \mathbb{Z}) \). For the \((-E_8) \oplus (-E_8) \oplus U_{3,3}' \) splitting now under discussion, the lattice \( U_{3,3}' \) decomposes in the same way, with \( D_3^3, f_3 \) replaced by \( D_{34}, f^3 \).

**B.2.3 Relation to \((-D_{16})^{\oplus 2}\)**

Let \( e_I^{(D)} \) denote the \( e_I^\perp \) of Eq. 193 and \( e_I^{(E)} \) denote the \( e_I^\perp \) of Eq. 202. Define

\[
U_{3,3}'' = \langle f_1, f_2 f_3, D_1^1, D_2^2, f^3 \rangle.
\]

What is the orthogonal complement of \( U_{3,3}'' \)? We find

\[
\begin{align*}
e_1^\perp &= e_1, & e_9^\perp &= e_9, \\
e_2^\perp &= e_2, & e_{10}^\perp &= e_{10}, \\
e_3^\perp &= e_3 - \frac{1}{2} f_1, & e_{11}^\perp &= e_{11} - \frac{1}{2} f_1, \\
e_4^\perp &= e_4 - \frac{1}{2} f_1, & e_{12}^\perp &= e_{12} - \frac{1}{2} f_1, \\
e_5^\perp &= e_5 - \frac{1}{2} f_2, & e_{13}^\perp &= e_{13} - \frac{1}{2} f_2, \\
e_6^\perp &= e_6 - \frac{1}{2} f_2, & e_{14}^\perp &= e_{14} - \frac{1}{2} f_2, \\
e_7^\perp &= e_7 - \frac{1}{2} f_1 - \frac{1}{2} f_2, & e_{15}^\perp &= e_{15} - \frac{1}{2} f_1, \\
e_8^\perp &= e_8 - \frac{1}{2} f_1 - \frac{1}{2} f_2, & e_{16}^\perp &= e_{16} - \frac{1}{2} f_1.
\end{align*}
\]

The \( e_I^\perp \pm e_J^\perp \), for \( I, J = 1, \ldots, 8 \), and for \( I, J = 9, \ldots, 16 \) span two distinct \( D_8 \) root lattices. We have

\[
U_{3,3}'' \perp = -(D_8) \oplus (-D_8),
\]

where \((-D_8)\) denotes the \( D_8 \) root lattice of with opposite sign inner product. Compared to the \( e_I^{(D)} \) basis 193, the \( e_I^\perp \) here differ by a shift of \( e_9^{(D)} \) though \( e_{13}^{(D)} \) by \( \frac{1}{2} f_3 \). Compared to the \( e_I^{(E)} \) basis 202, they differ by a shift of \( e_8^{(E)} \) and \( e_{16}^{(E)} \) by \( f^3 \). Identifying the \( e_I^\perp \) of this section with the \( \xi_I \) of Sec. 2.4.4, these relations extend to the transformations (44) of the full basis from \( \xi^{(D)}_a \) to \( \xi_a \) and from \( \xi^{(E)}_a \) to \( \xi_a \).

\(^{27}\)The former gives the root lattice of SO(16) and the latter the weights of the spinor of SO(16), which is indeed how the root lattice of \( E_8 \) decomposes.
C  The Lichnerowicz operator

Given a metric $g_{mn}$ on a manifold $X$, the change in the Ricci tensor $R_{mn}$ under an infinitesimal metric deformation $\delta g_{mn}$ is

$$\delta R_{mn} = \frac{1}{2}(\Delta_L \delta g)_{mn},$$

(208)

where $\Delta_L$ is the Lichnerowicz operator,

$$-(\Delta_L \delta g)_{mn} = \nabla^p \nabla_p \delta g_{mn} - 2 R^p_{
abla q} \delta g_{pq} - R_m^p \delta g_{pn} - R_n^p \delta g_{mp}.$$  \hspace{1cm} (209)

Here $\nabla_p$ is the metric connection.

The Lichnerowicz operator can be thought of as a symmetric tensor version of the Laplace-de Rham operator, $\Delta = d^\dagger + d\dagger$. The latter acts on differential forms (i.e., antisymmetric tensors). Here, $d\dagger$ is the codifferential, defined by $\ast d\ast$ (with $\ast$ the Hodge star) up to a convention-dependent sign. Harmonic forms are the zero eigenfunctions of the Laplace-de Rham operator. The harmonic forms on $X$ are in one-to-one correspondence with the cohomology of $X$, since there is a unique harmonic representative of each cohomology class.

Explicitly, the Laplace-de Rham operator acts on 2-forms and 3-forms as

$$-(\Delta \omega)_{mn} = \nabla^p \nabla_p \omega_{mn} - 2 R^p_{\nabla q} \omega_{pq} - R_m^p \omega_{pn} - R_n^p \omega_{mp},$$ \hspace{1cm} (210)

and

$$-(\Delta \omega)_{mnp} = \nabla^p \nabla_p \omega_{mnp} - 2 R^p_{\nabla qr} \omega_{mqr} - 2 R^q_{\nabla mp} \omega_{mqr} - 2 R^r_{\nabla np} \omega_{mrq} - R_m^q \omega_{mpq} - R_n^q \omega_{mqp} - R_p^q \omega_{mnq}.$$ \hspace{1cm} (211)

On a Kähler manifold, the complex structure $J_{m^n}$ is covariantly constant $\nabla_p J_{m^n} = 0$ (as is any other version of the same tensor with raised or lowered indices, such as the Kähler form $J_{mn} = J_{mp} g_{pn}$, since $\nabla$ is metric compatible). This leads to the result that

$$(\Delta_L h)_{mn} = \frac{1}{2}(J_{mp}^n (\Delta \omega)_{pq} + J_{np}^m (\Delta \omega)_{qm})$$ \hspace{1cm} (212)

where $h_{mn} = \frac{1}{2}(J_{m^n} \omega_{pn} + J_{n^m} \omega_{pm})$, and $\omega_{mn}$ is any $(1, 1)$-form. The proof uses the symmetry relations

$$R_{mnpq} = -R_{mnpq} = -R_{mnqp}, \quad R_{mnpq} = R_{pqmn}, \quad R_{[mnp]q} = 0,$$ \hspace{1cm} (213)

and is most easily performed in complex coordinates. Recall that the only nonvanishing components of the Riemann tensor on a Kähler manifold are $R_{ij\ell}$ or those related to these by symmetries. Likewise, the Ricci tensor has components $R_{ij} = R_{ji}$.

If in addition, the manifold is Calabi-Yau, then $R_{ij} = 0$ and there exists a covariantly constant $(3, 0)$-form $\Omega_{ijk}$. In this case, we similarly have

$$(\Delta_L h)_{mn} = \frac{1}{2}(\Omega_{m^p}^n (\Delta \chi)_{pq} + \Omega_{n^p}^m (\Delta \chi)_{qm}),$$ \hspace{1cm} (214)

where $h_{mn} = \frac{1}{2}(\Omega_{m^p}^q \chi_{pq} + \Omega_{n^p}^q \chi_{pq})$ and $\chi_{mnp}$ is any $(1, 2)$-form. However, note that primitive $(1, 2)$-forms $J \wedge v$ lead to vanishing $h_{mn}$.
D Metric deformations and harmonic forms

Given a Ricci flat metric $g_{mn}$ on $X$, the metric deformations that preserve Ricci flatness are

$$\delta g_{mn} = \delta \mu^i h_{i(mn)},$$

(215)

where the $\delta \mu^i$ are small parameters and $h_{i(mn)}$ are a complete set of symmetric tensor fields annihilated by the Lichnerowicz operator on $X$. On manifolds of special holonomy, these are closely related to harmonic forms.

D.1 Kähler manifolds

From the results of App. C on a complex Kähler manifold ($dJ = 0$), the covariant constancy of $J^m$ implies that every harmonic $(1, 1)$-form $\omega$ leads to a metric deformation

$$h_{mn} = -\frac{1}{2} (J^m \omega_{pn} + J^n \omega_{pm}),$$

(216)

annihilated by the Lichnerowicz operator. This metric deformation is transverse $\nabla^m h_{mn} = 0$ as a consequence of the harmonicity of $\omega$ and the covariant constancy of $J$. The deformation is traceless $g_{mn} h_{mn} = 0$ when $\omega$ is primitive. Recall that a harmonic form is said to be primitive when it cannot be written as $\omega = J \wedge \omega'$ for some $\omega'$. This is equivalent to the condition that $J^m \omega_{mn...p} = 0$.

D.2 Calabi-Yau $n$-folds

Similarly, on a Calabi-Yau 3-fold, the existence of a holomorphic (3, 0) form $\Omega$ allows us to associate a metric deformation

$$h_{mn} = -\frac{1}{2!} (\Omega^m p q \chi_{pqm} + \Omega^n p q \chi_{pqm}),$$

(217)

annihilated by the Lichnerowicz operator, to each harmonic $(1, 2)$-form $\chi$. In complex coordinates, Eqs. (216) and (217) become, respectively,

$$h_{ij} = -i \omega_{ij} \quad \text{and} \quad h_{ij} = -\frac{1}{2!} \Omega^{k\bar{l}} \chi_{k\bar{l}ij}. \quad (218)$$

The story is very similar for a Calabi-Yau $n$-fold, $n \geq 3$ with the holomorphic $(n, 0)$ form replacing the (3, 0)-form, and $(1, n-1)$-forms replacing (1, 2)-forms:

$$h_{ij} = -i \omega_{ij} \quad \text{and} \quad h_{ij} = -\frac{1}{(n-1)!} \Omega^{k_1...k_{n-1}} \chi_{k_1...k_{n-1}ij}. \quad (219)$$

---

28See App. C for a review of the Lichnerowicz operator. Compared to App. C, this section focuses only on the zero modes of the Lichnerowicz and Laplace-de Rham operators.
The first class of deformations are the Kähler deformations $\delta g_{ij}$ with indices of mixed type. The second class of deformations are the complex structure deformations $\delta g_{ij}$, with indices of the same type. In the latter case, the metric deformation must be combined with a change in the definition of the complex coordinates in order to preserve the hermiticity of the metric. For now, we leave this coordinate redefinition implicit. This metric deformation is transverse as a consequence of the harmonicity of $\omega$ and the covariant constancy of $\Omega$. It is traceless as a consequence of the difference in Hodge type between $\chi$ and $\Omega$.

For $n = 2, 1$, we have a torus or K3 surface. We treat the torus case below. The K3 case is treated in Sec. 2.7.2.

D.3 Tori

For $T^{2n}$, there are $\frac{1}{2}(2n)(2n+1) = n(2n+1)$ real metric moduli. On the other hand, naively, there are $n^2$ real Kähler deformations (the number of real degrees of freedom in $g_{ij}$), and $n^2$ complex complex-structure deformations (the number of complex degrees of freedom in $\tau_{ij}$ of $dz^i = dx^i + \tau_{ij} dy^j$), for a total of $3n^2$ real moduli. The apparent conflict is resolved by observing that some of the complex structure deformations lead to vanishing metric deformation. In particular, $\delta h_{mn} = 0$ for nonprimitive $(1, n-1)$-forms $\chi = J \wedge \omega$, where $\omega$ is a $(0, n-2)$-form.

The reader is invited to check this explicitly for $T^6$, with 21 real metric moduli. In this case, there are 9 Kähler deformations and 9 complex structure deformations. The 3 complex structure deformations generated by $(1, 2)$-forms $J \wedge dz^i$, for $i = 1, \ldots, 3$ vanish, leaving 6 nontrivial metric deformations from complex structure for a total of 9 (Kähler) + $2 \times 6$ (complex) = 21 real metric moduli, as desired.

E Gibbons-Hawking multicenter metric deformations and harmonic forms

The metric deformations and harmonic forms of the approximate K3 metric (90) are closely related to those of the Gibbons-Hawking multicenter metric, so it is helpful to review the latter. Let us write the multicenter metric (97) as

$$ds^2 = \hat{G}_{mn} dx^m dx^n = \Delta^{-1} ZG_{\alpha \beta} dx^\alpha dx^\beta + \Delta Z^{-1} (dx^4 + A)^2 = \delta_{\hat{m}\hat{n}} \theta^{\hat{m}} \theta^{\hat{n}}$$

(220)

where the $\theta^{\hat{m}}$ are a coframe

$$\theta^{\hat{a}} = \Delta^{-1/2} Z^{1/2} E^{\hat{a}}_{\beta} dx^\beta, \quad \theta^4 = \Delta^{1/2} Z^{-1/2} (dx^4 + A),$$

(221)

and $E^{\hat{a}}_{\beta}$ is a vielbein for $G_{\alpha \beta}$. In this Appendix, hats denote frame indices. In the body of the paper, this is clear from context, so we suppress the hats to simplify notation. The quantities $Z$ and $A$ are defined in Eq. (98).
E.1 Metric deformations from explicit moduli dependence

The metric depends on $G_{\alpha \beta}$, the source locations $x^I$, and shifts of $A$ by a constant 1-form $\beta_\alpha dx^\alpha$. As in Sec. 3, we define a covariantized deformation

$$\tilde{\delta} \beta^{\alpha \beta} = \epsilon^{\alpha \beta \gamma} \beta_\gamma - x^{I \alpha} \delta x^{I \beta} + x^{I \beta} \delta x^{I \alpha}.$$

Let focus on the moduli $x^I$, and consider a small change $\delta x^I$ at $\delta G^{\alpha \beta} = \tilde{\delta} \beta^{\alpha \beta} = 0$. Since $Z$ depends on $x^I$ only through $Z_I$ and only through the combination $x - x^I$, we have

$$\delta Z = \delta x^I \cdot \frac{\partial}{\partial x^I} Z = -\delta x^I \cdot \nabla Z_I,$$

where $\nabla$ is the 3D gradient operator. The corresponding metric deformation is

$$\delta (ds^2) = \frac{\delta Z}{Z} \left[ \Delta^{-1} Z g_{\alpha \beta} dx^\alpha dx^\beta - \Delta Z^{-1} R^2 (dx^4 + A)^2 \right] + 2\Delta Z^{-1} \delta A_\alpha dx^\alpha$$

$$= \frac{\delta Z}{Z} \left[ (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2 \right] + 2\theta^4 \delta A^4 \theta^{\hat{4}},$$

where $\delta A^4 = \Delta^{1/2} Z^{-1/2} \delta A$. Equivalently,

$$\delta \theta^{\hat{\alpha}} = \frac{1}{2} \frac{\delta Z}{Z} \theta^{\hat{\alpha}}, \quad \delta \theta^{4} = -\frac{1}{2} \frac{\delta Z}{Z} \theta^{4} + \delta A^4_\alpha \theta^{\hat{\alpha}}. \quad (224)$$

Since $A$ satisfies $dA = *_3 dZ$, we have

$$\delta A = \delta x^{I \alpha} \delta A_{I \alpha}, \quad \text{where} \quad d(\delta A_I) = *_3 d(\delta Z_I).$$

Given the form of $\delta Z$ above, an obvious solution is $\delta A_\alpha = -\delta x^{I \beta} \delta A_{I \alpha}$. However, a gauge equivalent and more convenient choice is

$$\delta A_\alpha = \delta x^{I \beta} F_{I \alpha \beta}, \quad \text{where} \quad F_{I \alpha \beta} = \partial_\alpha A_{I \beta} - \partial_\beta A_{I \alpha}.$$

With this choice, the metric deformation becomes

$$\delta (ds^2) = -\delta x^{I \alpha} \frac{\partial_\beta Z_I}{Z} \left[ \Delta^{-1} Z G_{\alpha \beta} dx^\alpha dx^\beta - \Delta Z^{-1} (dx^4 + A)^2 \right] + 2\delta x^{I \beta} \Delta Z^{-1} F_{I \alpha \beta} dx^\alpha$$

$$= -\frac{\delta x^{I \alpha}}{Z} \frac{\partial_\beta Z_I}{Z} \left[ (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2 \right] + 2\delta x^{I \beta} \theta^{\hat{4}} F_{I \alpha \beta} \theta^{\hat{\alpha}},$$

where $F^4_I = \Delta^{1/2} Z^{-1/2} F_I$ and $F_I = dA_I$. Equivalently,

$$\delta \theta^{\hat{\alpha}} = -\frac{1}{2} \frac{\delta x^{I \beta} \partial_\beta Z_I}{Z} \theta^{\hat{\alpha}}, \quad \delta \theta^{4} = \frac{1}{2} \frac{\delta x^{I \beta} \partial_\beta Z_I}{Z} \theta^{4} + \delta x^{I \beta} \theta^{\hat{4}} F_{I \alpha \beta} \theta^{\hat{\alpha}}. \quad (228)$$
E.2 Metric deformations generated by harmonic forms

For each \( I \), we have the anti-selfdual harmonic 2-form

\[
\omega_I = \left( \frac{Z_I}{Z} \right)_{\dot{\alpha}} \left( \theta^{\dot{\alpha}} \wedge \theta^4 - \delta^{\dot{\alpha} \dot{\beta}} \epsilon_{\dot{\alpha} \dot{\beta} \dot{\gamma}} \theta^\dot{\beta} \wedge \theta^\dot{\gamma} \right) = -d \left( A_I - \frac{Z_I}{Z} (dx^4 + A) \right). \tag{229}
\]

The remaining six harmonic 2-forms analogous to \( \omega_\alpha \) and \( \omega^\alpha \) of Sec. 3.4 are not square integrable on the multicenter space. The triple of hyperkähler 2-forms is

\[
J_{\dot{\alpha}} = \theta^{\dot{\alpha}} \wedge \theta^4 + \delta^{\dot{\alpha} \dot{\beta}} \epsilon_{\dot{\alpha} \dot{\beta} \dot{\gamma}} \theta^\dot{\beta} \wedge \theta^\dot{\gamma}, \quad \text{for} \quad \dot{\alpha} = \dot{1}, \dot{2}, \dot{3}. \tag{230}
\]

By raising the second 2-form index of \( J_{\dot{\alpha}} \), we obtain a triple of complex structures \( (J^{\dot{\alpha}})_m^n \) satisfying

\[
J_{\dot{\alpha}} J_{\dot{\beta}} = -\delta_{\dot{\alpha} \dot{\beta}} - \delta^{\dot{\alpha} \dot{\beta}} \epsilon_{\dot{\alpha} \dot{\beta} \dot{\gamma}} J_{\dot{\gamma}}. \tag{231}
\]

The hyperkähler metric deformations generated by the \( \omega_I \) and deformations parameters \( \delta x^{I \dot{\alpha}} \) are

\[
\delta (ds^2) = \delta x^{I \dot{\alpha}} (h_{I \dot{\alpha}})_{mn} dx^m dx^n, \tag{232}
\]

where

\[
(h_{I \dot{\alpha}})_{mn} = -\frac{1}{2} \left( (J^{\dot{\alpha}})_m^p \omega_{Ipn} + (J^{\dot{\alpha}})_n^p \omega_{Ipm} \right). \tag{233}
\]

After some algebra, we find

\[
(h_{I \dot{1}})_{\dot{m} \dot{n}} \theta^\dot{m} \theta^\dot{n} = \left( \frac{Z_I}{Z} \right)_{\dot{1}} \left[ (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2 \right] + 2 F_{I \dot{1} \dot{4}}^I \theta^\dot{4} \theta^4 + 2 \theta^4 d \left( \frac{Z_I}{Z} \right), \tag{234}
\]

with expressions for \( (h_{I \dot{2}})_{\dot{m} \dot{n}} \) and \( (h_{I \dot{3}})_{\dot{m} \dot{n}} \) obtained by cyclic permutation of \( \dot{1}, \dot{2}, \) and \( \dot{3} \).

E.3 Equivalence

With the identification

\[
\delta x^{I \dot{\alpha}} = \theta^{\dot{\alpha}} \delta x^{I \beta}, \tag{235}
\]

the metric deformations of the last two sections agree, provided the latter is supplemented by a diffeomorphism

\[
x^m \mapsto x^m = x^m - \delta N^m. \tag{236}
\]

It is convenient to lower the index on \( \delta N^m \) and describe the 1-form \( \delta N = \delta N_m dx^m \), which appears in the metric transformation

\[
d s^2 \mapsto ds'^2 = ds^2 - (\nabla_m \delta N_n + \nabla_n \delta N_m) dx^m dx^n. \tag{237}
\]

In the remainder of this section, we prove that

\[
\delta N = -\delta x^{I \dot{\alpha}} \left( \frac{Z_I}{Z} \right) \theta_{\dot{\alpha}}, \tag{238}
\]
where \( \theta^\alpha = \delta^\alpha_{\beta\gamma} \theta^\beta \), or equivalently, in the coordinate basis,

\[
\delta N^\alpha = -\delta x^I \left( \frac{Z_I}{Z} \right), \quad \delta N^4 = 0.
\]  

(238)

For simplicity, consider the case that \( \delta x^I \) is nonzero only for \( \alpha = 1 \). The generalization is straightforward. The difference between the metric deformaton of the previous two sections is

\[
\delta (ds^2) = \delta x^I \left( h_I^{\alpha} \right) \hat{n} \hat{m} \hat{n} \hat{\theta} = -\delta x^I \left[ \frac{Z_I}{Z} \left( \frac{Z_I}{Z} \right) \left( \theta^1 \right)^2 + \left( \theta^2 \right)^2 + \left( \theta^3 \right)^2 - \left( \theta^4 \right)^2 \right] 
\]

\[
-2 \frac{Z_I}{Z} F^i_{I \hat{a}\hat{b}} \theta^\alpha \theta^4 + 2 \partial_{\hat{a}} \left( \frac{Z_I}{Z} \right) \theta^\alpha \theta^1 .
\]  

(239)

To relate this to a diffeomorphism, we need an explicit expression for the covariant derivative operator.

**Maurer-Cartan equations**

It is convenient to work in the basis \( \theta^{\hat{\alpha}} \), \( \theta^4 \) and deduce the connection from the first Maurer-Cartan equations

\[
d\theta^\alpha = \omega^\beta \beta \wedge \theta^\beta + \omega^4 \beta \wedge \theta^4, \quad d\theta^4 = \omega^4 \beta \wedge \theta^\beta,
\]  

(240)

where \( \omega_{\hat{m} \hat{n}} = \omega_{\hat{m}} \theta^\beta \delta_{\hat{n}} \) is antisymmetric. Then,

\[
\nabla_{\hat{m}} B_{\hat{n}} = \partial_{\hat{m}} B_{\hat{n}} - (\omega_{\hat{m}})^{\hat{p}}_{\hat{n}} B_{\hat{p}}.
\]  

(241)

**Connection 1-form**

The coframe was given in \( (221) \). Taking the exterior derivative gives

\[
d\theta^{\hat{\alpha}} = \frac{1}{2} d(\log Z) \wedge \theta^{\hat{\alpha}}, \\
d\theta^4 = -\frac{1}{2} d(\log Z) \wedge \theta^4 + F^4,
\]  

(242)

where \( F^4 = \Delta^{1/2} Z^{-1/2} F \) and \( F = dA \). In components, we write \( F^4 = \frac{1}{2} F^4_{\alpha\beta} dx^\alpha \wedge dx^\beta = \frac{1}{2} F^4_{\hat{\alpha}\hat{\beta}} \theta^{\hat{\alpha}} \wedge \theta^{\hat{\beta}} \).

Comparing to the Maurer Cartan equation, we have

\[
\omega^\beta \beta \wedge \theta^{\hat{\alpha}} + \omega^4 \beta \wedge \theta^4 = \frac{1}{2} \partial_{\hat{\beta}} \log Z \theta^{\hat{\alpha}} \wedge \theta^{\hat{\beta}}, \\
\omega^4 \beta \wedge \theta^{\hat{\beta}} = -\frac{1}{2} \partial_{\hat{\beta}} \log Z \theta^4 - \frac{1}{2} F^4_{\hat{\alpha}\hat{\beta}} \theta^{\hat{\alpha}} .
\]

From the antisymmetry of \( \omega_{\alpha\beta} \) we deduce that

\[
\omega^\beta \beta = \frac{1}{2} \left( \partial_{\hat{\beta}} \log Z \theta^{\hat{\alpha}} - \partial_{\hat{\alpha}} \log Z \theta^{\hat{\beta}} \right) + \frac{1}{2} F^4_{\hat{\alpha} \hat{\beta}} \theta^{\hat{\alpha}}, \\
\omega^4 \beta = -\frac{1}{2} \partial_{\hat{\beta}} \log Z \theta^4 - \frac{1}{2} F^4_{\hat{\alpha}\hat{\beta}} \theta^{\hat{\alpha}}, \\
\omega^{\hat{\beta}} \beta = \frac{1}{2} \partial_{\hat{\beta}} \log Z \theta^4 + \frac{1}{2} F^4_{\hat{\alpha}\hat{\beta}} \theta^{\hat{\alpha}} .
\]  

(243)
Diffeomorphism

Therefore,
\[\nabla_\alpha \delta N_{\beta} + \nabla_\beta \delta N_{\alpha} = \partial_\alpha \delta N_{\beta} + \partial_\beta \delta N_{\alpha} - \left[(\omega_\alpha)^\gamma_{\ \beta} + (\omega_\beta)^\gamma_{\ \alpha}\right] \delta N_{\gamma} - \left[(\omega_\alpha)^4_{\ \beta} + (\omega_\beta)^4_{\ \alpha}\right] \delta N_{4}
\]
\[
= \partial_\alpha \delta N_{\beta} + \partial_\beta \delta N_{\alpha} - \frac{1}{2} \left[ \partial_\beta \log Z \delta^\gamma_{\ \alpha} + \partial_\alpha \log Z \delta^\gamma_{\ \beta} - 2 \partial_1 \log Z \delta^\gamma_{\ \alpha}\right] \delta N_{\gamma},
\]
\[\nabla_\alpha \delta N_4 + \nabla_4 \delta N_\alpha = \partial_\alpha \delta N_4 + \partial_4 \delta N_\alpha - \left[(\omega_\alpha)^\gamma_{\ 4} + (\omega_4)^\gamma_{\ \alpha}\right] \delta N_{\gamma}
\]
\[
= \partial_{\alpha} \delta N_4 + \partial_4 \delta N_{\alpha} - F_{4_{\alpha}} \delta N_{\gamma},
\]
\[\nabla_4 \delta N_4 + \nabla_4 \delta N_4 = \partial_4 \delta N_4 + \partial_4 \delta N_4 - 2(\omega_4)^\gamma_{\ 4} \delta N_{\gamma}
\]
\[
= 2 \partial_4 \delta N_4 - \partial^\gamma \log Z \delta N_{\gamma}.
\]

Writing
\[\delta N = -\delta x^{I_1} \left(\frac{Z_I}{Z}\right) \theta_1,\]
and remembering that \(\delta x^{I_1} = \theta_1^I \delta \theta_1^I\) with \(\theta_1^I\) proportional to \(Z^{1/2}\) (so that \(\partial_{\alpha} \theta_1^I = 1/2 (\partial_{\alpha} \log Z) \theta_1^I\), we find
\[\nabla_1 \delta N_1 + \nabla_1 \delta N_1 = -\delta x^{I_1} \left[2 \partial_1 \left(\frac{Z_I}{Z}\right) + \frac{Z_I}{Z} \partial_1 \log Z\right].\]
\[\nabla_2 \delta N_2 + \nabla_2 \delta N_2 = -\delta x^{I_1} \frac{Z_I}{Z} \partial_1 \log Z,
\]
\[\nabla_3 \delta N_3 + \nabla_3 \delta N_3 = -\delta x^{I_1} \frac{Z_I}{Z} \partial_1 \log Z,
\]
\[\nabla_4 \delta N_4 + \nabla_4 \delta N_4 = \delta x^{I_1} \frac{Z_I}{Z} \partial_1 \log Z,
\]
\[\nabla_1 \delta N_2 + \nabla_2 \delta N_1 = -\delta x^{I_1} \partial_2 \left(\frac{Z_I}{Z}\right),
\]
\[\nabla_1 \delta N_3 + \nabla_3 \delta N_1 = -\delta x^{I_1} \partial_3 \left(\frac{Z_I}{Z}\right),
\]
\[\nabla_1 \delta N_4 + \nabla_4 \delta N_1 = 0,
\]
\[\nabla_2 \delta N_3 + \nabla_3 \delta N_2 = 0,
\]
\[\nabla_2 \delta N_4 + \nabla_4 \delta N_2 = \delta x^{I_1} F_{21}^4 \left(\frac{Z_I}{Z}\right),
\]
\[\nabla_3 \delta N_4 + \nabla_4 \delta N_3 = \delta x^{I_1} F_{31}^4 \left(\frac{Z_I}{Z}\right).
\]

Therefore, \(- (\nabla_m \delta N_n + \nabla_n \delta N_m) dx^m dx^n\) indeed agrees with the difference between the two metric deformations \(\text{(239)}\), as desired. The analogous results for \(\delta x^{I_2}\) and \(\delta x^{I_3}\) are obtained by cyclic permutation of 1, 2, and 3. Thus, we obtain Eq. \(\text{(237)}\) for general \(\delta x^{I_\alpha}\).
In this Appendix we evaluate \( \int \omega_I \wedge \omega_J \) for the anti-selfdual harmonic 2-forms \( \omega_I \) in the Gibbons-Hawking multicenter metric (97), as well as a few related integrals. Let

\[
I_{IJ} = - \int_{X^4} \omega_I \wedge \omega_J. \tag{244}
\]

From Eq. (120), we have

\[
I_{IJ} = 2 \int_{\mathbb{R}^3} d^3x \, Z G^{\alpha \beta} \left( \frac{Z_I}{Z}, \partial_\alpha \left( \frac{Z_J}{Z} \right) \right), \tag{245}
\]

where \( Z = 1 + \sum_I Z_I \) and

\[
-\nabla^2 Z_I = \delta^3(x - x'). \tag{246}
\]

If Eq. (245) is integrated by parts, we find a cancellation between two Laplacians (using \( Z_I/Z = 1 \) at \( x = x' \)), leaving

\[
I_{IJ} = J_{IJ}, \quad \text{where} \quad J_{IJ} \equiv -2 \int_{\mathbb{R}^3} d^3x \, \frac{Z_I}{Z} G^{\alpha \beta} (\partial_\alpha Z) \partial_\beta \left( \frac{Z_J}{Z} \right). \tag{247}
\]

At this point we can apply the following trick:

1. Observe that \( J_{IJ} \) must be symmetric in \( IJ \) as a consequence of the equality \( I_{IJ} = J_{IJ} \) and the definition of \( I_{IJ} \). This is not obvious from the definition of \( J_{IJ} \) in Eq. (247).

2. Integrate \( J_{IJ} \) by parts and use the \( \nabla^2 Z \) expression together with the fact that \( Z_I/Z = \delta_{IJ} \) at \( x = x' \). This relates \( J_{IJ} \) to \( J_{JI} \) plus a term proportional to \( \delta_{IJ} \).

Steps 1 and 2 together allow us to solve for \( I_{IJ} \). The result is

\[
I_{IJ} = \delta_{IJ}. \tag{248}
\]

In a similar manner, the following integrals can be evaluated:

\[
K_{IJ} = 2 \int_{\mathbb{R}^3} d^3x \, \frac{1}{Z} G^{\alpha \beta} (\partial_\alpha Z_I)(\partial_\beta Z_J) = 3\delta_{IJ}, \tag{249}
\]

\[
L_{IJ} = 2 \int_{\mathbb{R}^3} d^3x \, \frac{Z_I Z_J}{Z^3} G^{\alpha \beta} (\partial_\alpha Z)(\partial_\beta Z) = 0, \tag{250}
\]

\[
M_{IJ} = -2 \int_{\mathbb{R}^3} d^3x \, \frac{Z_I}{Z^2} G^{\alpha \beta} (\partial_\alpha Z)(\partial_\beta Z_J) = -\delta_{IJ}. \tag{251}
\]

In particular, for \( F_I = dA_I = \star dZ_I \), where \( \star \) is defined in the 3D metric \( G_{\alpha \beta} \), we have

\[
\int_{X^4} d^4x \, \frac{1}{Z} (F_I)_{\alpha \gamma} (F_I)^{\alpha \gamma} = 2 \int_{X^4} F_I \wedge \star F_J = K_{IJ} = 3\delta_{IJ}. \tag{252}
\]
A closely related integral is
\[ \int_{X^4} d^4x \frac{1}{Z} (F_I)_{\alpha \gamma} (F_I)_{\beta \gamma}, \]
which, by symmetry, is equal to \( \frac{1}{3} G_{\alpha \beta} \) times the previous one:
\[ \int_{X^4} d^4x \frac{1}{Z} (F_I)_{\alpha \gamma} (F_I)_{\beta \gamma} = G_{\alpha \beta} \delta_{IJ}. \] (253)

This integral (or rather its analog with \( F_I \) replaced by \( F_I - F_I' \) and \( \int_{X^4} \) replaced by \( \frac{1}{2} \int_{T^4} \)) appears in Sec. 3.5.2.

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