RESEARCH ARTICLE

Tooth bending strength of gears with a progressive curved path of contact

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Abstract

The article presents a comprehensive study on the tooth bending strength of spur gears with a progressive curved path of contact, or so-called S-gears. Systematic gear meshing simulations were conducted to study the effects of S-gear geometry parameters on tooth bending strength. Different S-gear geometries were analysed in a systematically organized manner, and a comparison was made against a standard 20° pressure angle involute shape. Furthermore, different material combinations, e.g. polymer/polymer, steel/polymer, and steel/steel, of both drive and driven gear were analysed within a meaningful range of loads. The gear profile shape, material combination of the drive and the driven gear, and the transmitted load were found as the main parameters affecting gear tooth bending stress. Complex, non-linear relations between the recognized effects and the corresponding root stress were observed. Based on the numerical results, a shape factor, which considers the above-mentioned effects, was introduced, and a model for root strength control of S-gears was proposed and verified employing the finite element method (FEM).

Keywords: S-gears; stress; bending strength; polymers; gears; FEM

Nomenclature

- \(\sigma_F\) MPa: Root stress
- \(\sigma_{F0}\) MPa: Nominal root stress
- \(K_A\): Application factor
- \(K_V\): Dynamic factor
- \(K_{FL}\): Face load factor for tooth root stress
- \(K_{FL0}\): Transverse load factor for tooth root stress
- \(K_F\): Factor for tooth root load
- \(Y_{fa}\): Form factor
- \(Y_{sa}\): Stress correction factor (notch effect)
- \(Y_c\): Contact ratio factor for root stress
- \(Y_\beta\): Helix angle factor for root stress
- \(Y_2\): S-gears' shape factor for the root stress
- \(F_t\) N: Nominal tangential load, the transverse load tangential to the reference cylinder
- \(b\) mm: Face width
- \(m_n\) mm: Normal module
- \(h_a\) mm: Addendum height
- \(h_f\) mm: Dedendum height
- \(\rho_f\) mm: Root fillet radius
- \(d_i\) mm: Reference diameter
- \(d_{ti}\) mm: Root diameter
- \(n\): Curvature parameter
- \(\alpha_0\): Pressure angle in the pitch point C
- \(s_{Fn}\) mm: Critical section length (tooth root normal chord)
- \(z_1\): Number of teeth for the pinion
- \(z_2\): Number of teeth for the gear
- \(E\) MPa: Young's modulus
- \(\mu\): Coefficient of friction
- \(\varepsilon_s\): Transverse contact ratio

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1. Introduction

Involute gears are the most commonly used type of gears in practice due to their unique properties. The cutting tools have a simple shape; the rack profile for cutting involute gears has a circular arc for the addendum and the dedendum consists of two involutes. The improved Wildhaber–Novikov–Nagata gears showed less sensitivity to centre distance errors in the tests by Ariga and Nagata (1985) performed.

Peng et al. (2017) studied a new gear geometry, where the tooth addendum has the shape of a circular arc, and the dedendum the form of a cycloid. In an analytical study, it was found that such gears have a constant transmission ratio and more favourable stress conditions than involute gears. Chen et al. (2016) introduced a computer program for the geometric generation and meshing simulation of hyperboloid gears with a circular tooth form. The authors analysed the sensitivity of such gears to shaft misalignments. Wang et al. (2019) presented a novel design of a high contact ratio internal gear drive. Following the gear meshing theory, the tooth addendum profiles were deduced from a circular arc-shaped path of contact connecting the intersection point of addendum diameters and the intersection point of the pitch diameters. The dedendum profiles were then derived as the envelope to the family of the addendum profiles on the mating gear. A contact ratio as high as 7.8 was achieved in the case studies presented by authors, leading to a significant reduction in contact and bending stress when compared to an internal involute gear drive of the same size. A new type of transmission with a face gear split-flow system was introduced by Shuai et al. (2020a). The system exhibits stable and reliable transmission and strong carrying capacity, making it of high potential for helicopter transmission systems.

To increase the load-carrying capacity of involute gears, Wildhaber proposed asymmetric gears in 1930 (Fuentes et al., 2013). In asymmetric gears, both tooth flanks are still of involute shape, just that different pressure angles are used on the tooth’s drive and coast side. Asymmetric gears were widely studied by Kapelevich (2013), who introduced conversion methods for them. Shuai et al. (2019a) formulated a formula for calculating the tooth bending stress of asymmetric gears while considering the effect of friction between the meshing teeth. The application of an asymmetric form of teeth into the design of internal helical gears was presented by Shuai et al. (2019b). The effect of root cracks on the meshing of asymmetric gears was researched by Doğan and Karpat (2019) utilizing numerical modelling. Mohan and Senthivelan (2014) investigated the tooth bending strength of polymer asymmetric gears. The load-carrying capacity of the gear drive can also be increased by using herringbone gears, which exhibit compact structure and are widely used in the aviation and marine industry (Shuai et al., 2020b). In some cases, non-circular gears are also used, usually to gain variable transmission ratios and minimize transmission error (Karpov et al., 2017); they are further used in gear pumps (Liu et al., 2019). Trigonometric functions like a sine (Luo et al., 2008) and cosine (Koide et al., 2017) have also been used to define gear tooth profiles. More favourable sliding conditions and lower contact pressure were observed for such gears than for involute ones.

Regardless of the reported advantages, no other gear type has prevailed in practice. A lot of research has been done on involute gears. Today, advanced models and optimization procedures are being developed to design gears that are optimal from the perspective of load bearing, efficiency, and manufacturing (Marjanovic et al., 2012; Amani et al., 2017; Artoni, 2019). With profile shifts, the centre distance can be adjusted for involute gear pairs, and a gear’s load-bearing capacity can be increased (Miler et al., 2017). Profile shifts affect the contact sliding conditions and, therefore, a gear pair’s efficiency (Miler et al., 2018). Low specific sliding is especially important for polymer gears, where it is desirable that the least amount of heat should be generated as possible. Lower specific sliding also correlates with the lower wear rate of polymer gears.

The tooth bending stress can be reduced by optimizing the root fillet and lowering the fillet stress concentration. This can be done in two ways, i.e. by optimizing the tooth root directly on the gear (Kapelevich & Shekhtman, 2003) or by optimizing the tool shape. When gears are manufactured by cutting the root, the fillet shape depends on the shape of the cutting tool. The tool shape is standardized for involute gears (ISO 53:1998), where four different tool tip radii are possible. The root fillets of gears cut with standard tools are of trochoidal shape. Gears with circular fillets and enlarged trochoidal fillets were compared in several studies (Spitas et al., 2005; Spitas & Spitas, 2007). The bending strength of gears with fewer than 17 teeth was found to be higher when using circular fillets than for gears with trochoidal fillets, irrelevant of the tool tip radius used. For the case

Indices

1 pinion
2 gear
of higher numbers of teeth, however, no significant differences were observed. The standard tool tip shape was optimized by Pedersen (2009) to improve the root fillet shape and reduce the stress concentration. Reductions in tooth bending stress from 10% to 12% were reported. In the subsequent study by Pedersen (2015), the root fillet was optimized on the final gear, and then the tool shape was found based on the optimized gear shape. Reductions in bending stress from 8% to 31% were reported, depending on the number of teeth, where the gears with lower numbers of teeth exhibited higher root stress reduction. Such methods of reducing the root stress concentration can also be employed for non-involute gears. For polymer gears, which are mainly produced by injection molding, there is a lot more design freedom for root shape optimization. Senthivelan and Gnanamoorthy (2006) analysed the effect of root fillet radius on the performance of injection-molded PA 66 gears. Gears with root radii of 0.38 times the module and 0.125 times the module were tested at three different torque levels. A higher load carrying capacity was reported for the gears with the larger root fillet radius.

The use of polymer gears is evermore increasing due to several advantages they have when compared to metal gears. The mass production of polymer gears is cheaper, and they can operate without external lubrication; furthermore, polymer gear drives weigh less, and their NVH behaviour is better. One of the main problems with polymer gears is the lack of reliable methods for their design. There are currently no international standards that would provide design calculation methods for polymer gears. However, some national standards are available, e.g. BS 6168:1987, and the Japanese standard JIS B 1759:2013, which is based on the ISO 6336:2006 with some modifications discussed in the work of Moriwaki et al. (2014). Some guidelines from various engineering associations are also available, e.g. VDI 2736:2014, which was published in 2014 and is the most up-to-date guideline for designing polymer gears, and the design guidelines issued by AGMA (ANSI/AGMA 1106-A97:1997; AGMA 920-A01:2001), which only address the gear geometry and fail to propose conversion models for polymer gears. Various researchers, such as Bravo et al. (2015) and Mao (2007), have also presented models for designing polymer gears. These models are limited to particular cases and are relatively complex to be used in practice by gear designers. Tavčar et al. (2021) presented a multicriteria function for integrated polymer gear design optimization. To support the polymer gear design process also accelerated testing methods are available (Pogačnik & Tavčar, 2015; Tavčar et al., 2018). However, all existing design methods and conversion models apply only for involute gears.

The present study was focused on the gears with a progressive curved path of contact (S-gears), which under certain conditions show better performance than involute gears. This drives the opportunity for a significant competitive advantage (Zorko et al., 2017). The majority of mass-produced polymer gears are manufactured through injection molding. The gear profile shape has no significant effect on the cost of the injection-molding tool, leading to the possibility of economically justified use of particular, non-involute gear geometry (S-gears). In addition to economically justified manufacturing, relatively simple conversion models are also required for the broader use of S-gears in practice. The effect of S-gear profiles on the tooth bending strength has never been systematically investigated, not even for metal gears, let alone for polymer gears, where power transmission conditions are even more complex. Therefore, this study aimed to research this effect and propose a shape factor that would take into account the specific S-gear tooth shape and could be used in a modified standard model for root strength control (Zorko et al., 2019a).

2. S-gears

The S-gear profile shape was invented by Hlebanja and patented in 2000 (Hlebanja & Hlebanja, 2000). The new gear geometry was named after its characteristic S-shaped path of contact (Fig. 1). Due to the S shape, the pressure angle of S-gears is not constant along the path of contact. The involute tooth profile presented in Fig. 1a has a standard 20° pressure angle, whereas the S-gear profile presented in Fig. 1b has a 20° pressure angle only at the pitch point C. The addendum part of the S-gear rack profile is defined by

\[
y_1 = a_P \cdot m \cdot \left(1 - \left(1 - \frac{x_1}{m}\right)^n\right), \tag{1}
\]

where \((x_1, y_1)\) are Cartesian coordinates with their origin at kinematic point C, while \(a_p\) is the size factor and \(n\) is the curvature parameter (Hlebanja & Hlebanja, 2008). The dedendum part of the rack is defined by its semisymmetrical counterpart. A detailed procedure for the S-gear profile generation can be found in work of Kulovec and Duhovnik (2013). Both the \(a_p\) and \(n\) parameters affect the shape of the S-gear tooth profile shape and thus the gear’s expected load capacity. The size factor is dependent on the initial pressure angle \(a_{P0}\), i.e. the pressure angle at kinematic point C (equation 2).

\[
a_p = \tan(90° - a_{P0}) \quad \frac{n}{1000} \quad \text{as}n \quad \text{as} \quad \text{as} \quad \text{as} \quad \text{as} \quad \text{as}\tag{2}
\]

In several studies (Hlebanja & Okorn, 1999; Hlebanja & Hlebanja, 2008; Hlebanja, 2011), the S-gear geometry was compared
with the involute one. The primary reported advantages of S-gears over involute gears included convex/concave contact at the beginning and end of the meshing, and consequently lower contact pressure in that part of the meshing cycle, lower sliding velocities, less sliding in contact, and consequently more minor losses. Greater tooth root thickness was reported for S-gears than for standard involute gears with the same module, which in theory leads to a higher tooth bending strength (Hlebanja & Okorn, 1999). The above-mentioned studies were conducted for empirically chosen fixed values of parameters \( a_p \) and \( n \). The influence of the size factor \( a_p \) and the curvature parameter \( n \) on the shape of the S-gear profile was analysed by Kulovec and Duhovnik (2013). The effect on root thickness, curvature radius, sliding speeds, and theoretical contact ratio was studied on a rigid gear pair consisting of two gears of the same size (module 1 mm and 20 teeth).

The analytical and experimental research on S-gears conducted to date (Hlebanja & Okorn, 1999; Hlebanja, 2011; Kulovec & Duhovnik, 2013; Duhovnik et al., 2016; Zorko et al., 2017) deals with S-gear designs where fixed values of parameters \( a_p \) and \( n \), which define the S-gear profile geometry, were empirically chosen. No systematic study on applying S-gear geometry and its defining parameters to plastic gears has been presented yet. Experimental research would be pretty challenging since for every different set of parameters, a new molding insert or cutting tool would be needed. Testing the effect of different parameters on the S-gear performance is therefore inappropriate since there are too many design variants and impact factors. A systematic approach by employing numerical methods as presented in this study is therefore more convenient.

The S-gear geometry inventors state that one of the S-gear advantages is larger root thickness, leading to increased root strength. However, root thickness can also be increased for involute gears by using larger pressure angles, positive profile shifts, or both. In principle, a larger critical section length should lead to lower root stress; however, this is not always the case. In addition to root thickness and the size of the critical cross-section, the load-induced teeth deflection and the corresponding contact ratio increase have a significant effect on the root stress (Hasl et al., 2017, 2018).

Tooth bending strength is not the only criterion that has to be taken into account in polymer gear design (Tavčar et al., 2021), since wear (Zorko et al., 2019b) and thermal failures (Zorko et al., 2017) can occur, as well as pitting in oil-lubricated cases (Illenberger et al., 2019). Operating temperature is often a decisive factor for root strength since fatigue strength is temperature-dependent for polymers. Therefore, when comparing different gear geometries, other aspects must also be taken into account. Regarding the operational temperature, thermal failure, and wear, gear designs with low specific sliding are advantageous.

The above-mentioned advantages of S-gears could contribute to a higher performance of polymer gears, where usually the challenge is to transmit as much power as possible through as small volume of the transmission as possible. S-gears exhibit a large potential for use in high-performance polymer gear applications (Zorko et al., 2017).

This paper presents a comprehensive study on the S-gear profile and its effect on tooth bending strength. The impact of the curvature parameter \( n \) and the number of teeth \( z \) will be systematically analysed for different material combinations of the drive and driven gear and within a meaningful range of loads. Thus, the effect of the load-induced tooth deflection and the actual contact ratio will be taken into account.

3. The Load-Carrying Capacity of Polymer Gears

Polymer gears must be appropriately designed to operate well. Polymer gear failure types include temperature-induced failure, wear, and fatigue (Singh et al., 2017). The prevailing failure type in a given application with polymer gears depends on operating conditions, i.e. on material pairing, lubrication, and loads. For example, when a polymer gear pair, where an identical material is used for the drive and the driven gear, operates at a relatively high load, the gears will fail due to excessive temperature load. The thermal failure is considered to occur when during operation gears heat up over a limit temperature that is permissible for a continuous operation under load. When the polymer material is heated over the permissible temperature softens and severe plastic deformation occurs.

At a lower load, the most likely failure type will be wear, and in the event of lubrication, the wear rate is reduced, and the gears might fail due to fatigue. A combination of several failure types can often be observed, e.g. material melting and tooth breakage, or wear and tooth breakage resulting from reduced tooth thickness (Zorko et al., 2019b). Severe wear can, in most cases, be avoided by adequately selecting materials for the drive and driven gear. It was reported in the work of Mao et al. (2009) that the wear rate of a polyoxymethylene (POM)/polyamide (PA) gear pair is significantly lower than the wear rate of a POM/POM gear pair. For steel/polymer gear pairs, the wear rate can be reduced by altering the microgeometry of gears and making surface modifications to the steel gear (Zorko et al., 2019b).

When a gear pair with polymer gears is properly designed, it will operate at a steady-state temperature that is below the permissible temperature for continuous operation and no thermal failure should occur. Thermal failures are usually the result of overload, where the heat generation is higher than heat dissipation and no steady-state operational temperature is reached. Attempts to formulate an analytical model for the polymer S-gears operating temperature calculation have already been proposed (Hlebanja et al., 2017) and are still being upgraded. To study the operating temperature of individual S-gear designs also numerical models can be used (Fernandes et al., 2018; Cerne et al., 2019; Roda-Casanova & Sanchez-Marín, 2019). This study was therefore focused on the tooth bending strength of S-gears and on developing a calculation model for root strength control (Zorko et al., 2019a). According to DIN 3990:1987 (as well as ISO 6336:2006), the actual root stress in the steel gear is calculated by the equation

$$\sigma_F = K_A \cdot K_Y \cdot K_{F\beta} \cdot K_{F\alpha} \cdot \sigma_{F0}.$$  \( \text{(3)} \)

where \( K_A \) is the application factor, which takes into account the externally influenced variations of input or output torque, \( K_Y \) is the dynamic factor, which takes into account the effect of internal dynamic effects, and \( K_{F\beta} \) and \( K_{F\alpha} \) are the face load and transverse load factors for the tooth root stress, respectively, which take into account the uneven distribution of load over the face width and in the transverse direction. Using DIN 3990:1987 Method C, these factors are not profile-shape dependent and can also be determined for the S-gears using the prescribed methods in the DIN standard. The nominal root stress \( \sigma_{F0} \) is calculated by the equation

$$\sigma_{F0} = Y_{F0} \cdot Y_{sa} \cdot Y_{k} \cdot Y_{p} \cdot \frac{F_t}{B \cdot m}.$$  \( \text{(4)} \)

Limiting the study on spur gears, the helix angle factor value is \( Y_{k} = 1 \). The effect of the tooth shape on the nominal root
stress can be expressed as
\[ Y_{fa} \cdot Y_{sa} \cdot Y_r = \sigma_{f0} \cdot \frac{b \cdot m_n}{F_t}. \] (5)

A new factor can be introduced, which takes into account the effects of the tooth profile shape \( Y_{fa} \), stress concentration in the tooth root \( Y_{sa} \), and transverse contact ratio \( Y_r \):
\[ Y_Z = Y_{fa} \cdot Y_{sa} \cdot Y_r. \] (6)

To characterize the factor \( Y_{fa} \), the value of the nominal root stress \( \sigma_{f0} \) must be evaluated for a specific case. If the geometry of the gear pair and the transmitted load are known, the nominal root stress \( \sigma_{f0} \) can be calculated using an FE model, and the factor \( Y_{fa} \) can be evaluated using equation (5). If the shape factor value is known, then equation (7) can be used for root strength control of S-gears:
\[ \sigma_F = K_A \cdot K_V \cdot K_{F\beta} \cdot K_{F_a} \cdot Y_Z - \frac{F_t}{b \cdot m_n} \leq \sigma_{FP}. \] (7)

The calculated actual root stress must be smaller than the permissible root stress \( \sigma_{FP} \), which for steel gears should be determined according to ISO 6336:2006 or DIN 3990:1987, in the same way as for involute gears. Factors that consider the effect of gear accuracy, shaft misalignments, housing tolerances, and shaft deflections have not yet been defined for polymer gears. Due to the elastic properties of thermoplastics, the recommendation is to use \( K_F = (K_A \cdot K_V \cdot K_{F\beta} \cdot K_{F_a}) \approx K_A \) when the condition \( b / m_n \leq 12 \) is met (VDI 2736:2014). The permissible root stress \( \sigma_{FP} \) for polymer S-gears can be determined from VDI 2736:2014, taking into account the findings of Haal et al. (2017), and, for materials where there are no data in the guideline, through testing (Pogačnik & Tavčar, 2015; Duhovnik et al., 2016; Zorko et al., 2017, 2020; Tavčar et al., 2018). Values of the S-gear shape factor for root stress \( Y_Z \) were determined in this study, as presented in the following pages.

4. Methodology for the Characterization of S-gears’ Shape Factor for Root Stress

The values of the factor \( Y_{fa} \) were determined through a systematic run of numerical simulations. The methodology used to determine the factor values is presented in a flowchart in Fig. 2. In-house developed, custom Python software was used to generate the S-gear profile geometry. S-gears with an initial pressure angle \( \alpha_{20} = 20° \) were analysed, along with the values of the curvature parameter \( n \) in the range of \( 1.0 \leq n \leq 2.2 \). When the curvature parameter is set to \( n = 1.0 \), equation (1) becomes a linear function, the rack profile has straight teeth, and the involute gear profile is obtained. In Fig. 3, the effect of the curvature parameter \( n \) on the tooth profile shape is presented. Increasing the parameter \( n \) leads to an increase in the critical section length \( s_{cr} \). In all the simulated meshing gear pairs, the normal module and the face width were kept constant, namely at \( m_n = 1 \text{ mm} \) and \( b = 6 \text{ mm} \). To determine the effect of the S-gear shape on the root stress, all of the gear’s other geometric parameters had to be kept constant. The tip and the root diameters were kept constant for all analysed values of the parameter \( n \). The addendum height was \( h_a = 1 \cdot m \) and the dedendum height was \( h_f = 1.2 \cdot m \). A root fillet radius \( r_F = 0.3 \text{ mm} \) was chosen since this was the largest possible root radius that could be applied to all of the analysed gear geometries. Using a larger fillet radius than \( r_F = 0.3 \text{ mm} \) proved geometrically impossible for S-gears with higher curvature parameter \( n \). For comparison also a maximum possible radius for each design was used, resulting in a full round circular root fillet (FRCF). The tooth profile shape also depends on the number of teeth (Fig. 4), wherefore five different combinations of the drive and driven gear were analysed. The number of teeth of the drive gear was \( z_1 = 20 \) in all cases, and the number of teeth of the driven gear was \( z_2 = [20, 30, 50, 75, 100] \).

To determine the \( Y_Z \) values for polymer S-gear pairs, simulations were carried out for four different tangential forces on the teeth \( F_T \): 40 N, 80 N, 120 N, and 160 N. The factor values were first evaluated for the material combination POM/PA66 of the drive and driven gear since this is the combination tested in the previous experimental study of S-gears (Zorko et al., 2017). The POM/PA combination was also reported in other studies (Mao et al., 2009) as beneficial for gear applications. For comparison, the shape factor was also evaluated for the steel gear pairs with S-form teeth, where tangential forces of 160 N and 500 N were modelled. During the study, it was also found appropriate to investigate the steel/polymer gear pairs, which was done for the combination steel/POM and steel/PA66, both of which are commonly used combinations in practical applications. Tangential forces of magnitudes 40 N, 80 N, and 160 N were simulated for the cases of a steel/polymer gear pair. The goal of the selected criteria was to consider a wide range of possible geometric variations, meaningful loads, and most often used polymer materials for gear applications.

For polymer gear pairs, it was expected that the load-induced teeth deflection would increase the contact ratio, leading to a change in load sharing. The tangential load \( F_T = 40 \text{ N} \) was chosen based on previous experience, where such a load led to a tooth bending fatigue failure (Duhovnik et al., 2016; Zorko et al., 2017). Based on the initial magnitude, other force magnitudes were chosen as multipliers of the initial 40 N. The force magnitude 160 N was the limit case, for which the root stress of the smaller drive gear was 80 MPa. This is the highest stress that is meaningful for analysis since larger stress values would lead to the immediate failure of a plastic gear. Therefore, the gears were analysed in the range of usual operational loads. In all the simulated cases, the gear face width was set to be 6 mm, so the shape factor values are referred to a normalized force on the face width \( F_T / b \). That way, the factor values can also be used for gears with different face widths.

For a rigid material (e.g. steel), the load-induced increase of contact ratio in most cases is negligible, wherefore the force magnitude was not essential. A large enough force to obtain reasonable convergence is required for a successful simulation, as at smaller loads, contact discontinuities can occur. The most commonly used models for the root strength control of steel gears (ISO, DIN, AGMA) do not consider the teeth deflection. The force magnitude 160 N was chosen so as not to induce any significant teeth deflection but to obtain reasonable convergence. The magnitude 500 N was then analysed, as this is a realistic load for steel gears. A small amount of load-induced teeth deflection was expected at this load, even for the steel gears.

The nominal root stress \( \sigma_{f0} \), on whose basis the \( Y_Z \) values for a specific design of S-gear were evaluated, was defined based on the calculated von Mises stress in the root area. The region of interest (ROI) where the von Mises stress was considered was on the tooth’s drive side, as presented in Fig. 5. In the work of Wen et al. (2018), it was reported that the 30° tangent method (ISO, DIN) or the Lewis parabola (AGMA) method sometimes gives unreliable results when determining the critical section location (CSL) for involute gears. The CSL defined by the 30° tangent method is irrelevant to the meshing point, where the CSL characterized by Lewis parabola changes as the meshing point changes. A
process of locating the critical section was never really defined for the S-gears. As presented in Fig. 5b, the selected ROI considers the whole tooth’s root area. Therefore, by choosing a larger ROI in the root area, the stress at the exact critical section was calculated, and an unreliable determination of the critical section was avoided.

The von Mises stress is a scalar value, which does not give information about the type of stress (tensile, compressive, or...
Figure 3: The effect of the curvature parameter $n$ on the S-gear tooth shape and the critical section length $s_{Fn}$ (gear parameters: $m_n = 1$ mm and $z = 75$).

Figure 4: The effect of the number of teeth on the tooth profile shape: (a) involute gear ($n = 1.0$) and (b) S-gear ($n = 2.0$).

shear). The root cracks and the tooth root fracture in the final stage occur due to tensile stress in the tooth root, which is a consequence of the tangential load $F_T$ acting on the tooth. Therefore, the shape factor values were determined for the von Mises stress calculated in the meshing part, where it was clear that the stress in the root was tensile. The radial load on the tooth $F_R$ and the frictional force $F_{fr}$ imposed compressive stress on the active flank side (drive side), which was taken into account when using the von Mises stress and which is neglected in the ISO and DIN standards. During the meshing cycle, after the LP-STC of the gear, there was a rise in the von Mises stress observed in some cases, which is a consequence of the next pair of teeth in the mesh. This causes compressive stress on the non-active flank side extending to the ROI. This effect was most prominent for the S-gears with curvature parameter values $n = 2.0$ and $2.2$ and a comparatively larger number of teeth. The shape factor
values were determined based on the von Mises stress in the meshing part between the initial point of the contact and the LPSTC of the gear.

5. Numerical Model

The nominal root stress \( \sigma_{F0} \) was calculated by simulating a gear meshing process with an FEM model. The numerical models were constructed and solved using the ANSYS Workbench 17.2 software. The geometrical model of the drive and the driven gear had five teeth, and the entire post-processing was done on the third tooth, as it passes all characteristic meshing points (cf. Fig. 6). The material properties of all materials, including polymers, were modelled as linear elastic, taking account of material parameters \( E_{\text{Steel}} = 210 \, \text{GPa}, \nu_{\text{Steel}} = 0.30, E_{\text{POM}} = 2800 \, \text{MPa}, \nu_{\text{POM}} = 0.35, \) and \( E_{\text{PA66}} = 3500 \, \text{MPa}, \nu_{\text{PA66}} = 0.40. \) Frictional contact was modelled between the meshing tooth flanks, where \( \mu = 0.18 \) (VDI 2736:2014) was used for the unlubricated POM/PA66 gear pair, \( \mu = 0.1 \) (Linke & Börner, 2010) was used for the case of oil lubricated steel gear pairs, and \( \mu = 0.2 \) (VDI 2736:2014) was used for the unlubricated steel/polymer gear pairs.

Simulating a single meshing cycle under the considered loads, the non-linear material properties of polymers were not expected to influence the material response noticeably. The load on a single tooth acts in a short time period and thus viscous properties do not become evident. The assumption of linear elastic behaviour is generally considered adequate when conducting stress calculations in polymer gear design, bearing in mind that the calculated strains are below the polymer material’s yield point (Hasl et al., 2017; Černe et al., 2019). A comparison with a viscoplastic model was made by Černe et al. (2020), who confirmed that the presumption of linear elastic mechanical behaviour yields a sufficiently accurate approximation of the material’s behaviour for practical thermo-mechanical modelling purposes in gear design applications.

Simulations were conducted in 2D, considering a plane stress state. Quadratic order finite elements (PLANE183) were used for domain discretization. Contact conditions were simulated using CONTA172 elements on the drive gear and TARGE169 elements on the driven one. To achieve an accurate stress calculation, mesh refinement was applied in the contact areas between the meshing flanks and the tooth root area (Fig. 6). The average composite quality of the finite element mesh was 0.96. This metric is defined as the volume ratio to the sum of the square of the edge lengths for 2D quad/tri elements. A value of 1 indicates a perfect square/triangle, while 0 indicates that the ele-
ment has a zero or negative volume. A mesh of very high quality was accordingly used, enabling good accuracy in calculating the stress.

The geometry of the numerical model consisted of two gear bodies each with five teeth. The post-processing was done on the third teeth, which meshed through all characteristic meshing points. An ROI was selected in the root area on the drive (ROI1) and the driven (ROI2) tooth. The hole of the drive gear 1 was constrained to the fixed point located in the origin of the coordinate system x1,y1. Translations in directions x1 and y1 were constrained and only rotation around the origin of coordinate system x1,y1 was allowed. In the same manner, the side edges and the inner edge of the driven gear’s rim were constrained to a fixed point located in the origin of coordinate system x2,y2. This way the rest of the driven gear’s body was considered as rigid. Rotation ω was prescribed around the origin of coordinate system x1,y1 on the drive gear, and torque M was prescribed on the driven gear in the direction opposite to rotation.

When modelling frictional contact and considering large deflections, simulations become non-linear, leading to a significant increase in the problem-solving time. In this study, the focus was on accurately calculating the bending stress, wherefore a traditional approach of mesh refinement was selected. To reduce the computational costs, the simulations were run in 2D. Since over 500 simulations had to be performed to obtain the results presented further on, the problems were solved in batch mode on an HPC, where several cases could be run in parallel.

A convergence test was conducted, employing the h-refinement method. The results obtained with several element sizes in the ROI are presented in Fig. 7. The nominal root stress calculated according to the ISO 6336 Method B was taken as a reference for the steel gear pairs, and the VDI 2736:2014 calculated stress for the POM/PA66 gear pairs. The load F1 = 160 N was prescribed in both cases. Involute (ISO 53:1998 profile C) gear geometry and a 20/20 gear pair with module m = 1 mm and face width b = 6 mm were considered. For the steel gear pair, the stress calculated with the element size 0.016 mm was 3.5% higher than that calculated according to the standard ISO 6336:2006 Method B. There was no discrepancy between the root stress calculated with element sizes of 0.025 and 0.016 mm, so this was considered as the converged result, and the element size 0.016 mm was chosen for the following simulations. The VDI 2736:2014 is derived from DIN 3990:1987 Method C, where the material stiffness and the load-induced contact ratio increase are not considered when calculating the root stress (Hasl et al., 2018; Lu et al., 2020). Therefore, the FEM-calculated root stress is lower for the POM/PA66 gear pair. When analysing the POM/PA66 gear pair, the result was converged for all the element sizes, so the same element size as for steel gears was chosen.

6. Results

6.1. Shape factor for root stress in polymer S-gears

The stress state in the tooth root during the entire path of contact was calculated using the presented numerical model. The root stress for the 20/75 gear pairs is shown in Fig. 8. To determine the factor Y2, the maximum value of the drive gear’s nominal root stress σF for was used as calculated during the meshing between point A and the roll angle 43°–45° before the von Mises stress in the ROI started to increase again. The values of other terms in equation (5) were defined as b = 6 mm, mb = 1 mm, and F1 = 40 N, 80 N, 120 N, and 160 N. The values of the factor Y2 could then be determined using equations (5) and (6). The factor Y2 values were determined for different loads per mm width of the tooth, i.e. 6.67, 13.33, 20, and 26.67 N/mm (Fig. 9).

It was observed that in the case of lower loads and a higher number of teeth, involute gears and S-gears with a lower value of parameter n (smaller critical section length) showed better load-carrying characteristics, as can be seen in Fig. 9a and b. The reason for the jump in the parameter Y2 for gears with n = 1.0, n = 1.2, and 75 teeth is that the maximum tensile root stress for those two geometries occurs in the initial part of the meshing, as can be seen in Fig. 8. It would be intuitively expected that gears with a larger critical section length sFn would exhibit lower root stress, but this was not the case (Fig. 9). For the 20 N/mm load, minor differences between the analysed cases were observed. Based on the results, it could be concluded that, in the case of polymer gears, the load-induced teeth deflection and thus the increased contact ratio have considerable influence on the root stress. According to the values of Y2, the contact ratio increase has a more significant impact on the tooth bending strength than the greater critical section length. The single pair tooth contact area was significantly reduced for involute gears and S-gears with a smaller critical section length (lower n values). This is a consequence of a larger contact ratio due to gear geometry and an additional increase in the contact ratio due to load-induced teeth deflection. The theoretical contact ratio of S-gears is reduced as the curvature parameter n increases; further on, the teeth of S-gears with larger n values deflect less due to

Figure 7: Convergence test for steel/steel (left) and POM/PA66 (right) combination and comparison with standard-calculated root stress.
Figure 8: The calculated nominal root stress for a POM/PA66 gear pair during meshing from A to E, load \( F_T/b = 20 \text{ N/mm} \); red dotted lines correspond to involute gears \( n = 1.0 \) and the black dotted lines to S-gears with \( n = 2.2 \).

Figure 9: S-gears’ shape factor for the root stress \( Y_z \) (for S-gears with \( \alpha_{10} = 20^\circ \) and fillet root radius ratio \( \rho_F/m = 0.3 \)).
a larger critical section length. Therefore, increasing the value of \( n \) reduces the increase of the contact ratio. It turns out that, from the perspective of load bearing, it is more favourable if the teeth deflect more. However, large teeth deflections are a source of increased transmission error (Meuleman et al., 2007).

The effect of employing an FRCF was studied for the boundary designs, i.e. involute gears \((n = 1.0 \text{ FRCF})\) and S-gears with the largest critical section length \((n = 2.2 \text{ FRCF})\). The shape factor values for FRCF gears are included in Fig. 9, where they can be compared with designs with a fixed root fillet radius of \( \rho_f = 0.3 \text{ mm} \). In most cases, the design with an FRCF resulted in a lower tooth bending stress and a lower value of the shape factor \( Y_z \). However, it can be seen that there is no unique solution for which one design would be the best in all analysed cases.

At the lowest analysed load of 6.67 N/mm, the \( n = 2.2 \text{ FRCF} \) S-gears show the lowest shape factor value for gears with up to 25 teeth. For a higher number of teeth, the lowest shape factor value was found for \( n = 1.0 \text{ FRCF involute gears} \). At the load 13.33 N/mm, FRCF involute gears exhibit the lowest root stress, i.e. the lowest shape factor in the entire analysed range. At 20 N/mm load, the FRCF involute gears show the lowest shape factor for gears up to 65 teeth, while for a higher number of teeth, the lowest value was observed for \( n = 2.2 \text{ S-gears with } \rho_f = 0.3 \text{ mm} \). At the highest analysed load of 26.67 N/mm, the lowest shape factor value can be observed for FRCF involute gears up to 45 teeth. After this point, the lowest shape factor value was again observed for \( n = 2.2 \text{ S-gears, with } \rho_f = 0.3 \text{ mm} \). It can be concluded that for lower loads (6.67 and 13.33 N/mm), the involute gears with FRCFs exhibit the best tooth bending strength. Conversely, at higher loads, a change of trend is observed, where \( n = 2.2 \text{ S-gears with root fillet } \rho_f = 0.3 \text{ mm} \) show the highest tooth bending strength.

6.2. Shape factor for root stress in polymer S-gears meshing with steel pinions

Combinations of POM and PA66 gears meshing with a steel pinion were analysed for three different load cases. The results are presented in Fig. 10. In a steel/polymer gear pair, only the polymer gear is expected to deform, leading to a contact ratio somewhere in between that of a steel gear pair and a POM/PA66 gear pair. Values of the S-gears’ shape factor correspond with this statement since the values are higher than those for the POM/PA66 gears and lower than those for the steel gear pairs. In all the analysed load cases of S-gear geometry with \( n = 2.2 \), the one with the largest critical section length exhibited the best root load-carrying capacity.

When analysing the FRCF designs, a substantial reduction of root stress was also found for steel/polymer gear pairs; however, different trends were observed from those of polymer/polymer gear pairs. A more uniform trend can be observed; however, the S-gear design \((n = 2.2 \text{ FRCF})\) proved the best in most cases. For the loads of 6.67 and 13.33 N/mm, the \( n = 2.2 \text{ FRCF S-gears} \) exhibited the lowest shape factor for up to 85 teeth. However, for gears with between 85 and 100 teeth, a lower shape factor was observed for FRBC involute gears. A more complex trend was observed at higher loads of 26.66 N/mm, where for POM gears with up to 50 teeth and PA66 gears with up to 40 teeth, the lowest shape factor was found for FRCF involute gears. Conversely, for a higher number of teeth, a change of trend was observed, where the \( n = 2.2 \text{ FRCF S-gears} \) exhibited the lowest shape factor. At the upper end of the analysed range of teeth, another trend change was observed since lower tooth bending stress was calculated in FRCF involute gears.

6.3. Shape factor for root stress in steel S-gears

The shape factor \( Y_z \) for the analysed steel S-gear pairs is presented in Fig. 11. For steel gears, the values of the shape factor for S-gear root stress \( Y_z \) are as intuitively expected. Increasing the parameter \( n \) increases the critical section length \( s_{fn} \), and the tooth’s load-bearing cross-section is larger, thus leading to lower root stress. However, at a higher load and a number of teeth in the range between 75 and 100, a reduction in the root stress for gears with \( n = 1.0 \) was observed (Fig. 11b). This indicates that steel gears are also subject to a load-induced contact ratio increase and the corresponding reduction of stress (Xie et al., 2018).

The load-induced contact ratio increase will be analysed in the discussion section.

With smaller loads, where teeth deflections were negligible, the \( n = 2.2 \text{ FRCF S-gears} \) show the lowest shape factor value. The shape factor curve for FRBC involute gears intersects the curve of \( n = 2.2 \text{ S-gears with root fillet } \rho_f = 0.3 \text{ mm} \), where lower shape factor values were found for \( n = 2.2 \text{ S-gears with more than 50 teeth} \). Similar trends can be observed for higher and more realistic loads; hence, \( n = 2.2 \text{ FRCF S-gears} \) exhibited the lowest shape factor. The shape factor curves of FRBC involute gears and \( n = 2.2 \text{ S-gears with } \rho_f = 0.3 \text{ mm} \) coincide in the region between 50 and 75 teeth, while for other numbers of teeth, the FRBC involute gears showed a lower shape factor value.

7. Discussion

7.1. Model verification and limitations

The proposed model for the root strength control of S-gears is based on the verified numerical model and formulated in the same way as standard models for calculating load root carrying capacity (ISO 6336:2006; VDI 2736:2014). From the presented results, it can be concluded that the nominal root stress \( \sigma_{n0} \) is dependent on the critical section length \( s_{fn} \), the actual contact ratio, the material of gears in meshing, and the nominal load. All these effects were taken into account when determining a new shape factor for S-gear root stress. The factor \( Y_z \) takes into account the impact of the specific shape of the S-gear tooth profile, the stress concentration in the root, and the impact of the actual contact ratio on the S-gear root stress. The factor’s values were determined by systematic implementation of numerical simulations using a verified numerical model (Fig. 7).

Thus, it can be stated that the values of factor \( Y_z \) were correctly determined and that the proposed model is eligible to use for S-gear design. It should be pointed out that for polymer gears, the \( Y_z \) factor values take into account the actual contact ratio. The load-induced deflection was not investigated as much for steel gears as for polymer gears since only two different loads were analysed. A maximum contact ratio increase of 8% was observed when simulating the higher load \( F_T = 500 \text{ N} \). In real-life applications, steel gears exhibit load-induced teeth deflection as well, so the rise in the actual contact ratio can be considered an additional level of safety for the root strength.

The proposed model takes into account the load-induced increase of the contact ratio. Polymer gears heat up during operation, leading to the deterioration of the polymer material’s mechanical properties. A constant elastic modulus was considered when determining the S-gear shape factor values. For polymer materials, the elastic modulus is temperature dependent; however, it is impossible to predict in advance at which temperature the newly designed gears will operate. Therefore, using the
elast modulus at room temperature (23°C) seemed the most adequate since such an approach is on the safe side when calculating the tooth bending stress. With a reduction in the elastic modulus, some additional teeth deflection is expected, leading to increased contact ratio and lower tooth bending stress. However, gear designers need to be careful when controlling tooth bending strength since with temperature increase, the polymer material’s bending fatigue strength is also reduced. The effect of
reduced Young's modulus will be discussed further in this section.

Shape factors for S-gears were determined based on numerical simulations, where the meshing of gear pairs with module $m = 1$ mm was simulated. The accuracy of the proposed S-gears conversion model was tested for other modules. The model-calculated root stress was compared against the root stress calculated with an FEM model. Two gear geometries were investigated, i.e., involute gears ($n = 1.0$) and S-gears with $n = 2.2$. The model's accuracy was tested for the module range from 0.5 to 5 mm. The magnitude of the tangential force on the tooth $F_T$ was 160 N in all steel gear cases and 40 N in all POM/PA66 gear pair cases. The model-calculated root stress was compared with FEM calculated for steel/steel and POM/PA66 gear pairs. For polymer gear pairs, the effect of temperature was also studied. Young's modulus at 50°C was used for POM and PA66. The values were used as reported in the VDI 2736:2014 guideline $E_{\text{POM-50°C}} = 2500$ MPa, $E_{\text{PA66-50°C}} = 1800$ MPa. Since the S-gear shape factors were determined for the root fillet ratio $\rho_F/m = 0.3$, the same ratio was used for the gear geometry in FEM simulations. The comparison of model-calculated and FEM-calculated stresses for steel gear pairs is presented in Fig. 12. For the steel S-gears, the highest discrepancy was observed for gear pairs with a module size $m = 0.5$ mm, where FEM-calculated root stress was 8% lower than model-calculated root stress (Fig. 12a). The highest difference for involute steel gears was in the case of gears with module $m = 5$ mm where FEM-calculated stress was 9.5% lower; however, in the mentioned case, the absolute value of the discrepancy was only 2 MPa (Fig. 12b).

The comparison of model-calculated and FEM-calculated stresses for POM/PA66 gear pairs is presented in Fig. 13. The most significant discrepancy for S-gears was observed for the polymer gear pairs with a module size $m = 0.5$ mm and taking into account Young's modulus reduction at a temperature of 50°C. The difference between the model-calculated and FEM-calculated root stresses was 10%, where the model-calculated root stress was higher (Fig. 13a). In all other cases, the discrepancy between the model-calculated and FEM results was a maximum of 2%. A more considerable difference of 32% was observed when calculating the root stress for the involute polymer gear
pairs with module $m = 0.5 \text{ mm}$ and taking into account operation at $50^\circ \text{ C}$ (Fig. 13b). Comparing the model-calculated and FEM-calculated root stresses, it was observed that the model is conservative for gears with modules smaller than 1 mm. For modules larger than 1 mm, slightly lower root stress was calculated since the shape factor includes the effect of load-induced deflection of gears with a module size of 1 mm. The discrepancy between the calculated root stresses is in the range of 2% to 5% for modules larger than 1 mm. Considering this, the error incurred is not too big for practical use since the FEM-based results take hours of work to obtain, while the model-calculated root stress can be obtained in a matter of minutes. Nevertheless, for optimal final design, S-gears should be analysed with FEM if possible. Using the proposed model, the design procedure is quicker since there is no need to construct an FEM model for each design iteration. Since the profile shape does not change with the gear module, the tooth only gets scaled. Good agreement between the simulation-calculated and model-calculated root stresses was also obtained for other modules. However, changing the pressure angle, root fillet radius, and the addendum/deaddendum coefficient, or applying other profile modifications would result in a different gear profile shape. In that case, the S-gear shape factor values presented in this study are not valid.

Standards ISO 6336 and DIN 3990 advocate FEM as one of the proposed conversion procedures in Method A for the most accurate calculation of the nominal root stress. Based on the comparison of the proposed model- and FEM-calculated results, it can be concluded that the model-calculated root stress is also accurate enough for gears with different modules. There is a need to further test the model, which will be done during future projects. The model will be tested for S-gear pairs with different transmission ratios, different loads, and different material combinations. Such model verification is time consuming since for every FEM case, both the gear geometry and numerical model must be prepared.

### 7.2. Effect of the actual contact ratio

The kinematics of polymer gears significantly differs from the kinematics of metal gears. Due to the polymer material’s lower stiffness, the teeth deflections are so large that they lead to an extension of the path of contact. This results in a changed load sharing ratio and transmission error. Besides the teeth deflections, the other major factor that affects the meshing is deviations in the gear geometry. However, the main reason for the transmission error is found in the large teeth deflections (Meuleman et al., 2007). The values of the factor $Y_2$ were determined based on gear pairs whose drive gear had 20 teeth. Such a drive gear geometry was chosen because the test gears used in previous S-gear studies had 20 teeth (Duhovnik et al., 2016; Zorko et al., 2017). In practical applications, almost certainly, a different number of teeth will be used for a drive gear. If the drive gear’s number of teeth is reduced down to 17, the contact ratio does not change much. The difference between a gear pair where the drive gear has 17 teeth and a gear pair where the drive gear has 20 teeth is 2%. An even lower number of teeth can be used, but the difference in contact ratio gets larger. For a 14-tooth drive gear, the contact ratio is reduced between 10% and 20%, depending on the number of teeth of the driven gear. In that case, it makes sense to consider the discrepancy in contact ratio since the change will affect the root stress. This significant reduction happens because the involute teeth undergo undercutting when the number of teeth is lower than 17, leading to a shorter flank length. Since S-gears do not exhibit undercutting, a minor change in the contact ratio is expected when lowering the number of teeth under 17.

Figure 14a presents the calculated contact ratio for analysed steel gear pairs loaded with $F_T = 500 \text{ N}$ ($F_T/b = 83.33 \text{ N/mm}$). Involute gears ($n = 1.0$) exhibit the largest contact ratio between the analysed gear geometries, and the lowest contact ratio was calculated for S-gears with curvature parameter $n = 2.2$. The transverse contact ratio increase when the tangential force $F_T$ is increased from $160 \text{ N}$ ($F_T/b = 26.67 \text{ N}$) to $500 \text{ N}$ ($F_T/b = 83.33 \text{ N}$) is presented in Fig. 14b. The increase in the transverse contact ratio for steel gears ranges from 2% to 8%. The smallest relative increase is for the involute gear pair 20/20 and the most significant relative increase is for the S-gear pair 20/50, with $n = 2.2$. As the contact ratio depends on the material combination and the load transmitted, the proposed model is only valid for the analysed material combinations (and those materials with a similar elastic modulus) and in the range of the analysed loads.

The calculated transverse contact ratios for the analysed polymer gear pairs are presented in Fig. 15. By increasing the
number of the driven gear's teeth, the contact ratio increases, and the increase in the contact ratio is even more pronounced due to large teeth deflections at higher loads (Fig. 15b).

Depending on the load and the gear pair geometry, the relative increase in the contact ratio for POM/PA66 gear pairs is from 13% to 60% compared to steel gear pairs loaded with 160 N (Fig. 16).

7.3. Effect of the material combination

The S-gear shape factor (\(Y_2\)) values were determined for the polymer/polymer, steel/steel, and steel/polymer material combinations of the drive and driven gear. The mechanical properties of POM (\(E_{\text{POM}} = 2800\) MPa) and PA66 (\(E_{\text{PA66}} = 3500\) MPa), which are most often used in polymer gear applications, were
Figure 17: Difference between the $Y_z$ values due to the elastic modulus of the material: (a–c) POM vs. PA66 and (d) POM vs. steel.

Figure 18: (a) Comparison between the $Y_z$ values due to the elastic modulus of the material ($F_T / b = 26.67 \, \text{N/mm}$). (b) Relative difference between the factor values depending on the material and load.

taken into account. Comparing the shape factor values for the polymer/polymer and steel/polymer combinations to the values for the steel gear pair, it was found that a smaller elastic modulus of the polymer material has the effect of increasing the contact ratio and, consequently, lowering the value of the factor $Y_z$. Based on the calculated values of $Y_z$, the discrepancies of the factor values determined for the steel/POM and steel/PA66 pairs were analysed at all load levels (Fig. 17a–c).
No significant differences in the shape factor values were observed between the two analysed steel/polymer combinations. The most considerable difference was 6%, when, for most of the geometries and loads analysed, the differences were less than 3%. More significant differences were observed for gears with the curvature parameters from $n = 1.0$ to $1.6$, which have a smaller critical section length and exhibit more significant teeth deflection. For S-gears with curvature parameter $n = 2.2$, which showed the best tooth bending strength, the difference was in the range of 1%, which is negligible. Based on the small discrepancy, it can be concluded that the determined factor values can also be applied to other polymer materials since the elastic modules of the majority of engineering polymers used for gears are in the same range.

Significant differences were observed when comparing the shape factor values determined for a steel/POM combination against that determined for steel gear pairs (Fig. 17d). Differences from 7% to 25% were observed, and the values for the steel/POM combination were lower. Thus, when using the proposed model, the shape factor values determined for each material combination analysed, i.e. steel/steel, polymer/polymer, and steel/polymer, should be used. A wide range of material combinations occurring in practice have been considered within a meaningful load range, which enables widespread use of the proposed model.

For higher loads, a combination of a fibre-reinforced polymer composite gear in pair with a steel or polymer gear is often used in practical applications. The fibre-reinforced polymer composite gears generally do not exhibit isotropic mechanical properties. This is the case, especially when manufacturing them with injection molding, where the fibres are oriented in the direction of melt flow. The mechanical properties in the particular directions can therefore be quite different. To obtain an accurate result, this non-isotropic material behaviour must be considered within the numerical simulation. For a rough estimation also an average elastic modulus can be used, modelling the composite material on a macroscale (Barbero, 2008). An analysis was made to study the deviation of the value of $Y_z$ in the case of a steel/composite gear pair. An average modulus of elasticity $E = 10\, 000\, \text{MPa}$ was taken into account when modelling the composite gear. This elastic modulus represents the mechanical behaviour of PA66 reinforced with 30% glass fibres (PA-GF) and other reinforced polymers. The calculation was made for the boundary cases of the considered gear geometry areas, i.e. for curvature parameters of $n = 1.0$ and $2.2$. Simulations were run for all analysed load levels. Differences between $Y_z$ values for steel, polymer, and composite gears were observed. The values of $Y_z$ for involute gears ($n = 1.0$) and a small number of teeth (20–30) were closer to those for steel gears, while for a higher number of teeth (50–100), the values were closer to those determined for the polymer material (Fig. 18a). For S-gears with the curvature parameter $n = 2.2$, the values of the factor $Y_z$ were always closer to those determined for the polymer gears at each analysed load (Figs 18a and 19). The maximum observed difference for this geometry was 6.6% (Fig. 18b). Based on this analysis, it can be concluded that the values of the root stress shape factor determined for the material combination steel/polymer can also be used for the initial design calculations of S-gears made of polymer composite materials. To calculate stress in the composite gear more accurately, a numerical model is needed that adequately considers the composite material’s mechanical properties. This exceeds the scope of this study and indicates the need for further research.

7.4. Effect of the root fillet radius

The root fillet radius affects the stress concentration in the tooth’s root. Thus, to compare different S-gear profiles properly, the root fillet radius was fixed. Not using a constant root fillet radius in all analysed cases would affect the result; hence, it would not be possible to distinguish the effect of different S-gear profiles properly. The largest possible root fillet radius, which could be applied to all studied gear geometries, was selected for the study.

The effect of the root fillet radius in the standard ISO 6336:2006 is taken into account with the stress concentration factor $Y_s$, which was studied on the geometry of external spur gears with a 20° pressure angle, utilizing measurements and calculations using finite elements and integral equation methods. In this study, a root fillet was modelled directly on the gear; that is, the root fillet was not dependent on the tip shape of the tool. For polymer gears, such an approach is more adequate due to the specific manufacturing technology. Variation in fillet radii would additionally complicate the study. However, the effect of
Table 1: Comparison of the critical section length and specific sliding for involute and S gears. Note: For S-gears with a 25° initial pressure angle, the maximum value of the curvature parameter is $n = 1.8$ due to the peaking limit.

| Gear geometry | No. | $z_1$ | $z_2$ | Pressure angle $\alpha_n$ | Profile shift $x_1$ | Profile shift $x_2$ | $S_{Fn1}$ | $S_{Fn2}$ | Specific sliding in point A | Specific sliding in point E |
|---------------|-----|-------|-------|--------------------------|-------------------|------------------|-------------|-------------|--------------------------|--------------------------|
| Involute      | 1   | 20    | 20    | 20                       | 0.0               | 0.0               | 1.906       | 1.906       | −3.869                   | −3.814                   |
|               | 2   | 20    | 20    | 20                       | +0.5              | −0.5              | 2.153       | 1.575       | −1.110                   | −12.508                  |
|               | 3   | 20    | 20    | 25                       | 0.0               | 0.0               | 2.102       | 2.102       | −1.752                   | −1.761                   |
|               | 4   | 20    | 20    | 25                       | +0.5              | −0.5              | 2.383       | 1.759       | −0.668                   | −4.032                   |
|               | 5   | 50    | 20    | 20                       | 0.0               | 0.0               | 1.906       | 2.129       | −4.066                   | −1.264                   |
|               | 6   | 50    | 20    | 20                       | +0.5              | −0.5              | 2.153       | 1.985       | −0.873                   | −2.074                   |
|               | 7   | 50    | 20    | 25                       | 0.0               | 0.0               | 2.102       | 2.366       | −1.451                   | −0.809                   |
|               | 8   | 50    | 20    | 25                       | +0.5              | −0.5              | 2.383       | 2.195       | 0.561                    | −1.277                   |
| S-gears       | 11  | 20    | 20    | 20                       | 2.2               | /                 | 2.278       | 2.278       | −0.800                   | −0.800                   |
|               | 12  | 20    | 20    | 20                       | 1.6               | /                 | 2.010       | 2.010       | −1.754                   | −1.765                   |
|               | 13  | 20    | 20    | 25                       | 1.8               | /                 | 2.544       | 2.544       | −0.671                   | −0.677                   |
|               | 14  | 20    | 20    | 25                       | 1.6               | /                 | 2.373       | 2.373       | −0.797                   | −0.797                   |
|               | 15  | 50    | 20    | 20                       | 2.2               | /                 | 2.296       | 2.520       | −0.560                   | −0.452                   |
|               | 16  | 50    | 20    | 20                       | 1.6               | /                 | 2.010       | 2.281       | −1.311                   | −0.818                   |
|               | 17  | 50    | 20    | 25                       | 1.8               | /                 | 2.561       | 2.808       | −0.512                   | −0.447                   |
|               | 18  | 50    | 20    | 25                       | 1.6               | /                 | 2.425       | 2.626       | −0.594                   | −0.462                   |

Employing FRCFs was analysed since significant root stress reduction was reported for such designs (Spitas et al., 2005; Spitas & Spitas, 2007).

Two boundary designs, i.e. involute ($n = 1.0$) and S-gears with curvature parameter $n = 2.2$, were analysed employing an FRCF. A substantial root stress reduction was observed in the vast majority of analysed cases; however, in some cases, the designs with the fixed fillet radius $\rho_f = 0.3\, \text{mm}$ proved better in terms of lower root stress. A larger root fillet radius decreases stress concentration in the root area and increases the tooth's foundation stiffness. The teeth with larger root fillet rounding and therefore increased foundation stiffness deflect less during meshing; hence, the load-induced contact ratio increase is lower. In the cases where lower nominal root stress was calculated for gears with a fixed fillet radius $\rho_f = 0.3$, the effect of increased contact ratio was higher than the effect of reducing the stress concentration and consequently increasing the foundation stiffness. However, as already pointed out, in most cases, the effect of reducing the stress concentration was dominant. Such a design improvement is particularly appropriate when manufacturing polymer gears with injection molding, where full round circular fillet root shapes can be made without any additional cost. However, for steel gears produced by cutting, custom tools are required for manufacturing, which usually results in higher prices.

Employing the FRCF did not yield the lowest root stress in all analysed cases. For steel gears, where teeth deflections are smaller, the effect was as intuitively expected. Nonetheless, the results show no unique solution where one gear design could be outlined as the best. The best design is case dependent since the S-gears’ shape factor value varies with the curvature parameter $n$, the number of teeth, material combination, load, root fillet radius, and some other effects that were not included in the study (pressure angle, addendum/dedendum height, root or tip relief, and reduced elastic modulus at elevated temperature).

7.5. Comparison of the involute and S-gear geometry

There has already been some analytical and experimental research conducted regarding the comparison of involute and S-gear geometry (Hlebanja & Okorn, 1999; Hlebanja, 2011; Kulovec & Duhovnik, 2013; Duhovnik et al., 2016; Zorko et al., 2017). Experimental research is quite challenging since for every different set of parameters, a new molding insert or cutting tool is needed. Testing the effect of different parameters on the S-gear performance is therefore inappropriate since there are too many design variants and impact factors. No systematic study on the application of S-gear geometry to plastic gears has been presented yet.

One of the potential advantages of the S-gear geometry is larger root thickness, leading to increased root strength. However, root thickness can also be increased for involute gears by employing various tooth profile modifications, e.g. positive profile shift, larger pressure angle, increased root fillet rounding, or a combination of the latter. It is expected that a larger critical section length should lead to lower root stress, but as was observed when analysing POM/PA66 gear pairs, this is not always the case. The teeth of polymer gears with a larger root thickness exhibit higher stiffness leading to lower load-induced teeth deflection and lower contact ratio increase that has a significant effect on the root stress (Hasi et al., 2017, 2018).

In polymer gear design, the tooth bending strength is not the only criterion that has to be considered since wear (Zorko et al., 2019b) and thermal failures (Zorko et al., 2017) are common failure modes. Also, pitting failure mode was observed when gears were tested in oil-lubricated cases (Illeberger et al., 2019). The multicriteria approach is often most appropriate for the integrated design of polymer gears (Tavčar et al., 2021). For polymer gears, the operating temperature is an important decisive factor for root strength since polymer’s fatigue strength is temperature dependent. Hence, when comparing different gear geometries the other aspects have to be considered. Gear designs that exhibit low specific sliding are advantageous in terms of operational temperature, thermal failure, and wear. The specific sliding and critical section lengths were analysed and compared for selected involute and S-gear designs (Table 1). The critical section length $S_{Fn}$ was determined with the 30° tangent method.

From the cases analysed, it was observed that, with a proper geometry of S-gears, a larger critical section length for the
pinion and gear and smaller specific sliding could be obtained. Parameters highlighted in Table 1 define the gear geometry with the largest critical section length $s_{Fn}$ and lowest value of specific sliding. The 25° pressure angle geometry exhibited the largest critical section length and the lowest specific sliding among the analysed involute gears. The best geometry of the analysed S-gears showed a 7% larger $s_{Fn}$ for a 20/20 gear pair and a 20% larger $s_{Fn}$ for a 20/50 gear pair. Also, the specific sliding in both cases was smaller for S-gears. With smaller specific sliding, less heat generation due to frictional losses is also expected. Specific sliding changes during the path of contact; for better presentation, the specific sliding for some of the analysed geometries is presented in Fig. 20.

When applying the profile shift, the condition was to preserve the same centre distance and same transmission ratio. With using a positive profile shift on one gear a negative profile shift had to be applied to the other gear. When doing this, usually the specific sliding increases since the gear’s active tooth flank of the gear with a negative profile shift is much shorter than that of the gear with a positive profile shift.

8. Conclusions

The effect of the S-gear profile geometry on the tooth bending strength of steel and polymer spur gears was systematically studied, and the following conclusions can be drawn:

1. Complex non-linear relations between the gear profile shape, material combination of drive and driven gear, the transmitted load, and the corresponding root stress were observed. It was found that, for polymer gears, a greater critical section length does not necessarily mean a higher tooth bending strength, as was the case for steel gears.
2. The standard model for the tooth bending strength control of involute gears was modified to be eligible for S-gears. In the modified model, the specific shape of the S-gear tooth profile is taken into account with introducing a new shape factor $r_{YZ}$. The load-induced contact ratio increase was found to have a significant effect on reducing the tooth bending stress. The proposed S-gear shape factors therefore include this effect and enable a more accurate calculation of root stress than the standard model, where this effect is not considered.
3. It was observed that teeth deflection and the corresponding contact ratio increase are dependent on the material stiffness of the pinion and the gear, gear geometry (curvature parameter and number of teeth), and the transmitted load. The tooth bending strength of gears with a progressive curved path of contact was analysed for steel/steel, POM/PA66, steel/POM, and steel/PA66 gear pairs. For steel/steel gear pairs, a 2%–8% contact ratio increase was observed, for the POM/PA66 gear pairs, a 30%–60% increase, for the steel/POM gear pairs, an 8%–37% increase, and for steel/PA66 gear pairs, a 7%–33% increase.
4. The proposed modified model and S-gear shape factor values are based on the analysed material combinations and can be used for materials with a similar elastic modulus. The complex non-linear interrelations between the parameters affecting the contact ratio increase make it challenging to develop a universal conversion model.
5. Penalty factors used in the standard root stress calculation model for involute gears (ISO 6336, DIN 3990, VDI 2736) are based on the critical section determined by the 30° tangent method. This method is not always accurate for involute gears and was never investigated and confirmed for S-gears. A wider ROI, considering the entire root area, was therefore used to determine the maximum stress at the exact critical section on which the S-gear shape factor is based. Shape factors determined by the selected approach are more accurate than assuming the critical section defined by the 30° tangent method.
6. The modified model’s adequacy was tested for different gear modules, and good agreement between the FEM-calculated and model-calculated root stresses was observed. A 10% difference between the model-calculated and FEM-calculated root stresses was observed for steel/steel gear pairs with a smaller module ($m = 0.5$ mm), and a maximum 2% difference was observed in the case of larger modules ($m = 2$ mm, $m = 5$ mm). For a polymer/polymer gear pair, a maximum 2% difference was found in all analysed cases.

9. Future Scope

By studying the effect of S-gear geometry on the tooth bending stress, a step towards the S-gear design guideline was made.
Tooth bending strength was studied as one of the most critical criteria; however, methods for temperature, wear, and contact stress control would need to be investigated.

Several modifications, like enlarged root rounding, increased pressure angle, and positive profile shift (for involute gears), can be applied to increase the critical section length. Further studying the effect of these parameters is necessary.

An extensive set of data was obtained in this study. These data could be processed by various machine learning algorithms, which could be used to develop a machine learning-based model for predicting tooth bending stress.

The high-cycle tooth bending fatigue was never systematically investigated for polymer S-gears. Experimental testing of S-gears would be necessary to study the thermo-mechanical behaviour of polymer S-gears and their effect on tooth bending fatigue.

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Data Availability

The raw and processed data required to reproduce these findings were made available as supplementary data to this article.

Conflict of interest statement

None declared.

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