Geometric global quantum discord of two-qubit X states

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Xu [Jianwei Xu, J. Phys. A: Math. Theor. 45 405304 (2012)] generalized geometric quantum discord [B. Dakic, V. Vedral, and Č. Brukner, Phys. Rev. Lett. 105 190502 (2010)] to multipartite states and proposed a geometric global quantum discord. Almost at the same time, Hassan and Joag [J. Phys. A: Math. Theor. 45 345301 (2012)] introduced total quantum correlations in a general N-partite quantum state and obtained exact computable formulas for the total quantum correlations in a N-qubit quantum state. In this paper, we pointed out that the geometric global quantum discord and the total quantum correlations are identical. We derive the analytical formulas of the geometric global quantum discord and geometric quantum discord for two-qubit X states, respectively, and give five concrete examples to demonstrate the use of our formulas. Finally, we prove that the geometric quantum discord is a tight lower bound of the geometric global quantum discord.

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I. INTRODUCTION

Quantum correlations, which are a fundamental character of a multiparticle quantum system and an essential resource for quantum information processing [1], initially studied in the entanglement-versus-separability scenario [2–4]. While entanglement has attracted much effort, however, it has been found that the entanglement is not the only characteristic of a quantum system, and it has no advantage for some quantum information tasks. In some cases [5–7], although there is no entanglement, certain quantum information processing tasks can still be done efficiently by using quantum discord [8–10], which is believed more workable than the entanglement. The quantum discord (QD), first introduced by Olivier and Zurek [8] and by Henderson and Vedral [9], is a measure of quantum correlations, which extends beyond entanglement, and a quantum-versus-classical paradigm for correlations [11–13].

In spite of its merit, because the calculation of quantum discord involves a difficult optimization procedure, it is sometimes hard to obtain analytical results except for a few families of two-qubit states [14–22]. Huang have proved that computing quantum discord is NP-complete, the running time of any algorithm for computing quantum discord is believed to grow exponentially with the dimension of the Hilbert space. Therefore, computing quantum discord in a quantum system even with moderate size is not possible in practice [23]. So recently, some authors restrict their research to two-qubit X states, which was frequently encountered in condensed matter systems, quantum dynamics, etc. [24–26] with an interest in the dynamics of quantum discord [27]. M. Ali first studied the quantum discord for two-qubit X states and derived an explicit expression for X state [14–15], but Lu immediately gave a counterexample to Ali’s results [28]. Then Chen analyzed Ali’s results and pointed out that Ali’s algorithm is only valid for a class of X states; there is a family of X states, for which Ali’s algorithm is not correct because the inequivalence between the minimization over positive operator-valued measures and that over von Neumann measurements [29]. Soon after, Rau, one of the authors of Ref. [14, 15], and his co-worker extend the procedure in Ref. [14] for calculating discord of two-qubit X-states to so called extended X-states, which contain N qubits. They also give a formula to calculate the geometric measure of quantum discord for qubit-qudit systems [30]. In this aspect, Huang also gave a counterexample to the analytical formula derived in [24], and proposed an analytical formula with very small worst-case error [31].

Considering the difficulty to calculate the quantum discord, Dakic et al. [32] introduced a geometric measure of quantum discord, which also was named geometric discord (GD), and obtained the analytic formula for two qubit states. After Dakic’s paper published soon, Luo and Fu generalized GD to an arbitrary bipartite system and derived an explicit tight lower bound on GD [33]. Rana et al. and Hassan et al. also independently obtained the rigorous lower bound to GD [34–35]. D. Girolami et al. gave another expression of GD for qubit-qudit states [36, 37]. Tufarelli et al. proposed an algorithm to calculate GD for any 2 × d systems, which is valid for d → ∞ case [38].

Because the original definitions of both QD and GD consider a set of local measurements only on one subsystem. It is not symmetrical for two subsystems in the two-partie case. Rulli et al. suggested a symmetric extension of GD named global quantum discord (GQD) [39], which has been extended to q-global quantum discord [40]. Some analytical expressions of GQD for some special quantum states also have been found [41]. On the other hand, inspired by Rulli’s work, Xu generalized the geometric quantum discord to multipartite states and proposed a geometric global quantum discord (GGQD) [42], which is alternatively called as symmetric or two-side geometric measure of quantum discord for two-qubit system [43–44]. Almost meanwhile, Hassan and Joag proposed total quantum correlations (TQC) and presented an algorithm to calculate TQC for a N-partite quantum state [45]. Compared with QD and GD, obviously, the study of GQD and GGQD as well as TQC is not yet enough. So, in this paper,
we first prove that GGQD and TQC are identical, then we restrict ourselves to the study of GGQD. We devote to derive an explicit analytical expression of GGQD for two-qubit X states.

This paper is organized as follows. In the next section, we give a brief review of QGD and GGQD as well as TQC and prove that GGQD and TQC are identical. We derive the analytical formulas of GGQD and QGD of two-qubit X states in Sec. III. Some demonstrating examples are given in Sec. VI. A related discussion is presented in Sec. VII and we give concluding remarks in the last section.

II. BRIEF REVIEW OF GEOMETRIC MEASURE OF QUANTUM DISCORD AND GEOMETRIC GLOBAL QUANTUM DISCORD

For convenience of later use, we present a brief review of QD, GD and GGQD as well as TQC. The GD of a bipartite state $\rho$ on a system $H^a \otimes H^b$ with marginals $\rho^a$ and $\rho^b$ can be expressed as

$$Q(\rho) = \min_{\Pi^a} \{I(\rho) - I(\Pi^a(\rho))\}. $$

(1)

Here the minimum is over von Neumann measurements (one-dimensional orthogonal projectors summing up to the identity) $\Pi^a = \{\Pi^a_k\}$ on subsystem $a$, and

$$\Pi^a(\rho) = \sum_k (\Pi^a_k \otimes I^b)\rho(\Pi^a_k \otimes I^b),$$

(2)

is the resulting state after the measurement. $I(\rho) = S(\rho^a) + S(\rho^b) - S(\rho)$ is the quantum mutual information, $S(\rho) = -\text{tr} \rho \ln \rho$ is the von Neumann entropy, and $I^b$ is the identity operator on $H^b$. The GD for a state $\rho$ is defined as [32]:

$$D(\rho) = \min_\chi \|\rho - \chi\|_1^2,$$

(3)

where the minimum is over the set of zero-discord states (i.e., $Q(\chi) = 0$) and $\|\rho - \chi\|_1^2 := \text{tr}(\rho - \chi)^2$ is the square of Hilbert-Schmidt norm of Hermitian operators. The GD of any two-qubit state is evaluated as

$$D(\rho) = \frac{1}{4}(\|x\|^2 + \|T\|^2 - k_{\text{max}}),$$

(4)

where $x := (x_1, x_2, x_3)^T$ is a column vector, $\|x\|^2 := \sum_i x_i^2$, $x_i = \text{tr}(\rho(\sigma_i \otimes 1^b))$, $T := (t_{ij})$ is a matrix and $t_{ij} = \text{tr}(\rho(\sigma_i \otimes \sigma_j))$, $k_{\text{max}}$ is the largest eigenvalue of matrix $xx^T + TT^T$.

Since Dakić et al. proposed the GD, many authors extended Dakić’s results to the general bipartite states. Luo and Fu evaluated the GD for an arbitrary state $\rho$ and obtained an explicit formula

$$D(\rho) = \text{tr}(C^2) - \max_A \text{tr}(ACC^A),$$

(5)

where $C = (c_{ij})$ is an $m^2 \times n^2$ matrix, given by the expansion $\rho = \sum c_{ij} X_i \otimes Y_j$ in terms of orthonormal operators $X_i \in L(H^a)$, $Y_j \in L(H^b)$ and $A = (a_{ij})$ is an $m \times m$ matrix given by $a_{ij} = \text{tr}(|k\rangle \langle k| X_i) = \langle k| X_i |k\rangle$ for any orthonormal basis $|k\rangle$ of $H^a$. They also gave a tight lower bound for GD of arbitrary bipartite states [33]. Recently, a different tight lower bound for GD of arbitrary bipartite states was given by S. Rana et al. [34], and Ali Saif M. Hassan et al. [35] independently. Other explicit expressions of GD for two-qubit system and $2 \otimes d$ systems are also found [36–38].

Though QD and GD have been revealed as useful measurements, they are originally not symmetric for its all subsystems. As an extension of GD, Rulli proposed a global quantum discord (GQD) for an arbitrary multipartite state $\rho_{A_1\cdots A_N}$ as [39]:

$$D(\rho_{A_1\cdots A_N}) = \min_{\{\Pi_A\}} \{S(\rho_{A_1\cdots A_N}) \|\Phi(\rho_{A_1\cdots A_N})\}$$

$$- \sum_{j=1}^N S(\rho_{A_j} \|\Phi_j(\rho_{A_j})\}).$$

(6)

To calculate $D(\rho_{A_1\cdots A_N})$ conveniently, Xu has given an equivalent expression of Eq. (6) [41]

$$D(\rho_{A_1\cdots A_N}) = \max_{\Pi} \sum_{k=1}^N \{S(\rho_{A_k}) - S(\rho_{A_1 A_2\cdots A_N})$$

$$- \sum_{k=1}^N \rho_{A_k} \|\Phi_k(\rho_{A_k})\}).$$

(7)

where $\Pi = \Pi_{A_1 A_2\cdots A_N}$ is any locally projective measurement performed on $A_1 A_2\cdots A_N$. Inspired by Rulli’s work, Xu generalized the GD to multipartite states and proposed a geometric global quantum discord (GGQD) [42]. GGQD is also called as a symmetric or two-sided geometric measure of quantum discord for two-qubit system [43–44]. The definition of GGQD for state $\rho_{A_1 A_2\cdots A_N}$ is

$$D^{G}(\rho_{A_1 A_2\cdots A_N}) = \min_{\sigma_{A_1 A_2\cdots A_N}} \{\text{tr}[\rho_{A_1 A_2\cdots A_N} - \sigma_{A_1 A_2\cdots A_N}]^2$$

$$: D(\sigma_{A_1 A_2}) = 0\},$$

(8)

where $D(\sigma_{A_1 A_2})$ is defined by Eq. (6). To simplify calculation of Eq. (8), Xu derived two equivalent formulas of GGQD. The first one is:

$$D^{G}(\rho_{A_1 A_2\cdots A_N}) = \min_{\Pi} \{\text{tr}[\rho_{A_1 A_2\cdots A_N} - \Pi(\rho_{A_1 A_2})]^2$$

$$: \text{tr}[\rho_{A_1 A_2\cdots A_N}]^2 - \max_{\Pi} \{\text{tr}[\Pi(\rho_{A_1 A_2\cdots A_N})]^2\},$$

(9)

where $\Pi$ is the same as the one in Eq. (7). The second formula of GGQD can be expressed as
where \(C_{\alpha_1\alpha_2\cdots\alpha_N}\) and \(A_{\alpha_1i_k}\) are determined by

\[
\rho_{A_{1}A_{2} \cdots A_{N}} = \sum_{\alpha_1\alpha_2\cdots\alpha_N} C_{\alpha_1\alpha_2\cdots\alpha_N} X_{\alpha_1} \otimes X_{\alpha_2} \otimes \cdots \otimes X_{\alpha_N},
\]

and \(\{X_{\alpha_k}\}_{n_k=1}^{\infty}\) are orthonormal bases of \(L(H_k)\), which were constituted by all Hermitian operators on \(H_k\).

For any two-qubit state \(\rho_{AB}\), Eqs. (10,12) are reduced to:

\[
D^G(\rho_{AB}) = \sum_{\alpha_1\beta_1, \alpha_2\beta_2} \langle \alpha_1\beta_1 | \sum_{i,j} A_{\alpha_1i} B_{\beta_2j} C_{\alpha_2\beta} \rangle^2,
\]

(13)

\[
C_{\alpha\beta} = \text{tr}(\rho_{AB} X_{\alpha} Y_{\beta}),
\]

(14)

\[
A_{\alpha i} = \langle i | X_{\alpha} | i \rangle, \quad B_{\beta j} = \langle j | Y_{\beta} | j \rangle,
\]

(15)

here, for consistency with other literature, such as [33, 35], we have exchanged the indexes of \(A\) and \(B\) in Eq. (15), which do not affect the latter results. On the other hand, in Eqs. (14) and (15), \(X_0 = I^A/\sqrt{2}, \quad X_i = \sigma_i^A/\sqrt{2}, \quad Y_0 = I^B/\sqrt{2}, \quad Y_i = \sigma_i^B/\sqrt{2}, \quad i = 1, 2, 3\). where \(I^A\) and \(\sigma_i^A\) are 3 x 3 unitary matrix and Pauli matrix for qubit \(A\), \(I^B\) and \(\sigma_i^B\) are the same for qubit \(B\), respectively. We can further express Eq. (13) in the matrix form,

\[
D^G(\rho_{AB}) = \text{tr}(CC^t) - \text{max}_{AB} \text{tr}(ACB^tB^tA^t),
\]

(16)

In the following, we shall see that the definitions of GGQD and TQC are different in form, but they are identical to each other. For this end, we recall that, obviously, Eq. (9) is also valid for GGQD with \(\Pi = \Pi_k\), \(k = 1, 2, \cdots, N\), which only performed on \(k\)th part of the system. Keeping this in mind, we can rewrite Eq. (17) for \(N = 2\) as

\[
Q(\rho) = D_1(\rho) + D_2(\Pi^{(1)}(\rho))
\]

\[
= \text{tr}[\rho^2] - \text{max}_{\Pi} \text{tr}[\Pi^{(1)}(\rho)^2]
\]

\[
+ \text{tr}[\Pi^{(2)}(\rho)^2] - \text{max}_{\Pi} \text{tr}[\Pi^{(2)}(\Pi^{(1)}(\rho))^2]
\]

\[
= \text{tr}[\rho^2] - \text{max}_{\Pi} \text{tr}[\Pi^{(1)}(\rho)^2]
\]

\[
= \text{tr}[\rho^2] - \text{max}_{\Pi} \text{tr}[\Pi(\rho)^2] = D^G(\rho)
\]

(18)

In the above equation, \(\Pi = \Pi^{(2)}\Pi^{(1)}\), the terms \(\text{tr}[\Pi^{(1)}(\rho)^2]\) and \(\text{tr}[\Pi^{(2)}(\Pi^{(1)}(\rho))^2]\) after the second equal sign are canceled because \(\Pi^{(1)}\) maximize \(\text{tr}[\Pi^{(1)}(\rho)^2]\). The proof of \(Q(\rho) = D^G(\rho)\) for \(N \geq 3\) cases is similar and straightforward. The identity of GGQD with TQC is not surprising, because these two measurements both use the original definition of the geometric measure of quantum discord to every individuals of the system. Due to this identity, therefore, hereafter we use the name ‘geometric global quantum discord (GGQD)’, which also stand for ‘total quantum correlations (TQC)’. In the next section, we use Eq. (16) to calculate the GGQD of X state.

### III. GGQD of Two-Qubit X State

The two-qubit X state usually arises as the two-particle reduced density matrix in many physical systems. In the computational basis \(\{00, 01, 10, 11\}\), the visual appearance of its density matrix resembles the letter X leading it to be called as X state. The density matrix of a two-qubit X state

\[
\rho_{AB} = \begin{pmatrix}
\varrho_{00} & 0 & 0 & \varrho_{03} \\
0 & \varrho_{11} & \varrho_{12} & 0 \\
0 & \varrho_{12} & \varrho_{22} & 0 \\
\varrho_{03} & 0 & 0 & \varrho_{33}
\end{pmatrix}
\]

(19)

has nonzero elements only on the diagonal and the antidiagonal, where \(\varrho_{00}, \varrho_{11}, \varrho_{22}, \varrho_{33} \geq 0\) satisfy \(\varrho_{00} + \varrho_{11} + \varrho_{22} + \varrho_{33} = 1\). The antidiagonal elements \(\varrho_{03}, \varrho_{12}\) are generally complex numbers, but can be made real and nonnegative by the local unitary transformation \(e^{-i\theta_1\sigma_z} \otimes e^{-i\theta_2\sigma_z} = \begin{pmatrix} e^{-i\theta_1} & 0 & 0 & 0 \\
0 & e^{i\theta_1} & 0 & 0 \\
0 & 0 & e^{i\theta_2} & 0 \\
0 & 0 & 0 & e^{-i\theta_2}
\end{pmatrix}\) with \(\theta_1 = -(\arg \varrho_{03} + \arg \varrho_{12})/4\), \(\theta_2 = -(\arg \varrho_{03} - \arg \varrho_{12})/4\), where
\[ \sigma \text{ is the Pauli matrix. Hereafter we assume } \theta_{03}, \theta_{12} \geq 0. \text{ Recall that the matrix } C \text{ in Eq. (15) can be written as} \]
\[ C = (C_{ij}) = \frac{1}{2} \begin{pmatrix} 1 & y' \\ x & T \end{pmatrix}, \quad (20) \]
Matrix A and B in Eq. (16) can be expressed as \[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & a \\ 1 & -a \end{pmatrix}, \quad (21) \]
\[ B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & b \\ 1 & -b \end{pmatrix}, \quad (22) \]
where \[ a = \{a_1, a_2, a_3\} = \sqrt{2}(A_{11}, A_{12}, A_{13}), \quad b = \{b_1, b_2, b_3\} = \sqrt{2}(B_{11}, B_{12}, B_{13}) \text{ and } ||a|| = ||b|| = 1. \] Using Eqs. (20, 22), we can easily get the first term in Eq. (16).
\[ \text{tr}(CC') = \frac{1}{4}(1 + ||x||^2 + ||y||^2 + ||T||^2) \quad (23) \]

\[ D^G(\rho_{AB}) = \frac{1}{4}(||x||^2 + ||y||^2 + ||T||^2 - \frac{1}{2} \max_b ||x||^2 + ||bT'||^2 + \sqrt{(||x||^2 - ||bT'||^2)^2 + 4(||y||^2)^2}). \quad (26) \]

The second step to maximize \[ \text{tr}(AC'B'C'A') \] in Eq. (16) is reduced to maximize \[ ||x||^2 + ||bT'||^2 + \sqrt{(||x||^2 - ||bT'||^2)^2 + 4(||y||^2)^2} \] in above equation on \[ b = \{b_1, b_2, b_3\}. \] For X state (19),
\[ x' = \{x_1, x_2, x_3\} = \{0, 0, \theta_{00} + \theta_{11} - \theta_{22} - \theta_{33}\}, \quad (27) \]
\[ y' = \{y_1, y_2, y_3\} = \{0, 0, \theta_{00} - \theta_{11} + \theta_{22} - \theta_{33}\}, \quad (28) \]
\[ T = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix} \]
\[ = \begin{pmatrix} \theta_{12} + \theta_{03} & 0 & 0 \\ 0 & \theta_{12} - \theta_{03} & 0 \\ 0 & 0 & \theta_{00} - \theta_{11} + \theta_{22} - \theta_{33} \end{pmatrix}. \]
\[ ||x||^2 + ||bT'||^2 + \sqrt{(||x||^2 - ||bT'||^2)^2 + 4(||y||^2)^2} = x_3^2 + V + 2b_2^2b_3^2 + \sqrt{(x_3^2 - V^2)^2 - 4Wx_3^2}, \quad (30) \]
where \[ W = b_2^2T_{11}^2 + b_2^2T_{22}^2, \quad V = b_3^2T_{33}^2 + W. \quad (31) \]
To further maximize Eq. (30) we let
\[ b_1 = \sin \theta \cos \phi, \quad b_2 = \sin \theta \sin \phi, \quad b_3 = \cos \theta, \]
and the second term
\[ \text{tr}(AC'B'C'A') = \frac{1}{4}[1 + y'b'by + a(xx' + Tb'bT')a']. \quad (24) \]
The maximum over matrixes A and B in Eq. (16) can be done by two steps. First, we maximize \[ a(xx' + Tb'bT')a' \] on matrix A. The maximum of this term is the largest eigenvalue \[ \lambda_{\max-A} \text{ of matrix } xx' + Tb'bT'. \] According to the Lemma 1 of Ref. [44], which states that for any two vectors \([a]|a\rangle \text{ and } |b\rangle \text{ (not necessarily normalized)} \) in \( \mathbb{R}^3 \), the largest eigenvalue of the matrix \([a]|a\rangle \langle a| + |b\rangle \langle b| \) is \[ \lambda = |a|^2 + |b|^2 + \sqrt{(a^2 - b^2)^2 + 4(a|b|^2)} \] with \[ a^2 = \langle a|a \rangle \text{ and } b^2 = \langle b|b \rangle, \]
we get
\[ \lambda_{\max-A} = \frac{1}{2}||x||^2 + ||bT'||^2 + \sqrt{(||x||^2 - ||bT'||^2)^2 + 4(||y||^2)^2}. \quad (25) \]
Substituting Eqs. (23, 25) into Eq. (16), we obtain the GGQD of any two-qubit systems
\[ f(\theta, \phi) = \frac{1}{2} \begin{pmatrix} x_3^2 + 2y_2^2 \cos^2 \theta + \gamma(\theta, \phi) \\ \sqrt{\gamma(\theta, \phi) - x_3^2} + 4T_{33}^2x_3^2 \cos^2 \theta \end{pmatrix}, \]
\[ \gamma(\theta, \phi) = T_{33}^2 \cos^2 \theta + \mu(\phi) \sin^2 \theta, \quad \mu(\phi) = T_{11}^2 \cos^2 \phi + T_{22}^2 \sin^2 \phi. \quad (32) \]
Ignoring the relative maximum \( x_3^2 \) of \( f(\theta, \phi) \), we find
\[ \frac{\partial f(\theta, \phi)}{\partial \theta} = 0, \quad \frac{\partial f(\theta, \phi)}{\partial \phi} = 0. \]
Eq. (30) becomes
\[ f(\theta, \phi) = \frac{1}{2} \begin{pmatrix} x_3^2 + 2y_2^2 \cos^2 \theta \gamma(\theta, \phi) \\ \sqrt{\gamma(\theta, \phi) - x_3^2} + 4T_{33}^2x_3^2 \cos^2 \theta \end{pmatrix}, \]
\[ \gamma(\theta, \phi) = T_{33}^2 \cos^2 \theta + \mu(\phi) \sin^2 \theta, \quad \mu(\phi) = T_{11}^2 \cos^2 \phi + T_{22}^2 \sin^2 \phi. \quad (32) \]
Finally, we obtain the maximum value of Eq. (30)
\[ \text{max} [x_0^2 + \theta_{11}^2 + \theta_{22}^2 + \theta_{33}^2 - 1/4, (\theta_{12} + \theta_{03})^2] \]
and the GGQD of X states
\[ D^G(\rho_X) = \theta_{00}^2 + \theta_{11}^2 + \theta_{22}^2 + \theta_{33}^2 - \frac{1}{4} + 2(\theta_{12}^2 + \theta_{03}^2) \]
\[ - \text{max} [\theta_{00}^2 + \theta_{11}^2 + \theta_{22}^2 + \theta_{33}^2 - \frac{1}{4}, (\theta_{12} + \theta_{03})^2]. \quad (34) \]
For comparing GGQD with GD for some X states in next section, we also calculated the GD of X state according to
Ref. [33] and got the following formula:
\[
D(\rho_X) = \frac{1}{2}(\rho_{00} + \rho_{11} + \rho_{22} + \rho_{33}) \\
- \rho_{00}\rho_{22} - \rho_{11}\rho_{33} + 2(\rho_{12} + \rho_{03}) \\
- \max\left\{ \frac{1}{2}(\rho_{00} + \rho_{11} + \rho_{22} + \rho_{33}) \right\} \\
- \rho_{00}\rho_{22} - \rho_{11}\rho_{33} - (\rho_{12} + \rho_{03})^2. \tag{35}
\]
In simplifying Eqs. (34) and [35], the condition \( \rho_{00} + \rho_{11} + \rho_{22} + \rho_{33} = 1 \) was repeatedly used. This formula can also be derived by Eq.(23) of Ref. [30].

IV. ILLUSTRATIVE EXAMPLES

In this section, we give some concrete examples to demonstrate the use of formulas obtained in above section.

(1) As the first example, we consider the initial state \( \rho = a|\psi^+\rangle\langle \psi^+| + (1 - a)|1_A, 1_B\rangle\langle 1_A, 1_B| + |0\rangle\langle 0|, 0 < a \leq 1 \), where \( |\psi^+\rangle = (|0_A, 0_B\rangle + |1_A, 1_B\rangle)/\sqrt{2} \) is a maximally entangled state [14]. The density matrix of this state is:
\[
\rho_X = \begin{pmatrix}
\frac{a}{2} & 0 & 0 & \frac{a}{2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{a}{2} & 0 & 0 & 1 - \frac{a}{2}
\end{pmatrix}.
\tag{36}
\]
The corresponding GGQD and GD are
\[
D^G(\rho_X) = D(\rho_X) = \frac{a^2}{2}. \tag{37}
\]
We plot \( D^G(\rho_X) \) and \( D(\rho_X) \) in Fig. 1. We noticed that GGQD and GD are completely coincident in this state.

(2) We take the class of states defined as \( \rho = a|\psi^+\rangle\langle \psi^+| + (1 - a)|1_A, 1_B\rangle\langle 1_A, 1_B| + |0\rangle\langle 0|, 0 \leq a \leq 1 \), where \( |\psi^+\rangle = (|0_A, 0_B\rangle + |1_A, 1_B\rangle)/\sqrt{2} \) is a maximally entangled state [14]. The density matrix of this state is:
\[
\rho_X = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & a & 0 & a \\
0 & 0 & a & 0 \\
0 & 0 & 0 & 1 - a
\end{pmatrix}.
\tag{38}
\]
The corresponding GGQD and GD are
\[
D^G(\rho_X) = \begin{cases}
\frac{a^2}{4}(3 - 8a + 7a^2), & 0 \leq a < \frac{1}{2} \\
\frac{a^2}{2}, & \frac{1}{2} < a \leq 1.
\end{cases} \tag{39}
\]
\[
D(\rho_X) = \begin{cases}
\frac{a^2}{2}(1 - 3a + 3a^2), & 0 \leq a < \frac{1}{2} \\
\frac{a^2}{2}, & \frac{1}{2} < a \leq 1.
\end{cases} \tag{40}
\]
We plot \( D^G(\rho_X) \) and \( D(\rho_X) \) for the state (38) in Fig. 2. We see that \( D^G(\rho_X) = D(\rho_X) \), for \( 0 \leq a \leq \frac{1}{2} \) and \( D^G(\rho_X) \geq D(\rho_X), \) for \( \frac{1}{2} < a \leq 1 \). Finally \( D^G(\rho_X) = D(\rho_X) \) when \( a = 1 \).

(3) We take the class of states defined as \( \rho = \frac{1}{3}\{(1 - a)|0_A, 0_B\rangle\langle 0_A, 0_B| + 2|\psi^+\rangle\langle \psi^+| + a|1_A, 1_B\rangle\langle 1_A, 1_B|, 0 \leq a \leq 1 \), where \( |\psi^+\rangle \) is the same as in example (2) [14]. The density matrix of this state is:
\[
\rho_X = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2a}{3}
\end{pmatrix}.
\tag{41}
\]
The corresponding GGQD and GD are
\[
D^G(\rho_X) = \frac{1}{36}(7 - 8a + 8a^2), \tag{42}
\]
\[
D(\rho_X) = \frac{1}{18}(3 - 2a + 2a^2). \tag{43}
\]
We plot \( D^G(\rho_X) \) and \( D(\rho_X) \) for the state (41) in Fig. 3. We see that \( D^G(\rho_X) \) and \( D(\rho_X) \) have the same minimum values \( a = \frac{1}{2} \). The two curves are symmetrical about \( a = \frac{1}{2} \).

(4) Two atoms in the Tavis-Cumming model [46]. We consider two atoms (A and B), each of which interacting resonantly with a single quantized cavity field (system C) in a Fock state. This system is described by the two-atom Tavis-Cumming (TC) Hamiltonian: \( H = \hbar g[(\sigma_+ + \sigma^-)_C + (\sigma_+^A + \sigma^-_B)_A], \) where \( \sigma_j \) and \( \sigma_J \) denote the Pauli ladder operators for the \( j \)th atom, \( a(a^\dagger) \) stands for the annihilation (creation) operator of photons in cavity \( C \), and \( g \) is the coupling constant.
We consider that the system is initially in the state $|\psi(0)\rangle = (\alpha|0_40_n\rangle + \beta|1_41_n\rangle)|n_c\rangle$. Because the total number of excitations is conserved by TC Hamiltonian, the cavity mode will develop within a five-dimensional Hilbert space spanned by $\{|(n-2)_C\rangle, |(n-1)_C\rangle, |n_c\rangle, |(n+1)_C\rangle, |(n+2)_C\rangle\}$ for $n \geq 2$. When $n = 0, 1$ the dimension will be 3 and 4, respectively. On the other hand, since the atomic system evolves within the subspace $\{|0_40_n\rangle, |+_C\rangle, |1_41_n\rangle\}$ with $+_C = (|1_40_n\rangle + |0_41_n\rangle)/\sqrt{2}$ independently of $n$, for our purpose, we only need to consider $n = 0$ case. Solving the Schrödinger equation, we obtain the state of the system at time $t$.

$$|\psi(t)\rangle = c_1(t)|0_40_n\rangle|2_C\rangle + c_2(t)|+_C\rangle + c_3(t)|1_41_n\rangle|0_C\rangle + c_4(t)|0_40_n\rangle|0_C\rangle$$

(44)

with the following probability amplitudes

$$c_1(t) = -\frac{\sqrt{2}}{3}\beta[1 - \cos(\sqrt{6}gt)],$$
$$c_2(t) = -\frac{i\beta}{\sqrt{3}}\sin(\sqrt{6}gt),$$
$$c_3(t) = \beta\left(1 + \frac{1}{3}[\cos(\sqrt{6}gt) - 1]\right),$$
$$c_4(t) = \alpha.$$  

(45)

Now, we take trace of density operator $\rho = |\psi(t)\rangle\langle\psi(t)|$ over cavity $C$ resulting in the reduced density matrix of the qubit system

$$\rho_{AB} = \begin{pmatrix}
|c_1|^2 + |c_4|^2 & 0 & 0 & |c_AC_4| \\
0 & \frac{|c_3|^2}{2} & |c_3|^2 & 0 \\
0 & |c_3|^2 & \frac{|c_3|^2}{2} & 0 \\
|c_3c_4| & 0 & 0 & |c_3|^2
\end{pmatrix}.$$  

(46)

Using Eqs. (44) and (45), we obtain

$$D^{G}(\rho_{AB}) = (|c_1|^2 + |c_4|^2)^2 + |c_2|^4 + |c_3|^4 + 2|c_3c_4|^2 - \frac{1}{4} - \max\left\{\frac{1}{2}|c_2|^4 + |c_3|^4 + (|c_1|^2 + |c_4|^2)^2 - \frac{1}{4}, (\frac{1}{2}|c_2|^2 + |c_3c_4|^2)\right\}.$$  

(47)

$$D(\rho_{AB}) = \frac{1}{2}(|c_1|^4 + 4|c_2|^4 + |c_3|^4 + |c_4|^4 - |c_2|^2) + (|c_1|^2 + 2|c_3|^2)|c_4|^2 - \max\left\{\frac{1}{2}|c_2|^4 - 1 - |c_3|^2 + (1 - |c_2|^2)|c_3|^2, (\frac{1}{2}|c_2|^2 + |c_3c_4|^2)\right\}.$$  

(48)

In this case, $D^{G}(\rho_{AB})$ and $D(\rho_{AB})$ as functions of dimensionless time $\tau = \sqrt{6}gt/(6\pi)$ are plotted in Fig. 4 which shows that $D^{G}(\rho_{AB})$ and $D(\rho_{AB})$ change periodically with a period $T_\tau = 1$. In addition, they simultaneously arrive their maximums and minimums. Furthermore, the practical calculation shows the results for $n \geq 1$ are the same as Fig. 4 which enhances that the evolution of two atomic system is independent of $n$, as pointed out earlier. (5) As a final example, let us consider two atoms $A$ and $B$ in a common reservoir $C$ [46]. We suppose that the initial state of this system was $|\Psi(0)\rangle = (g_{AB}|g_{AB}\rangle + e_{AB}|e_{AB}\rangle)|0_C\rangle$, where $|0\rangle = \prod_k|0_k\rangle$ is the reservoir vacuum state. The overall state of the system at time $t$ can be written as

$$|\Psi(t)\rangle = \alpha|g_{AB}\rangle|0_C\rangle + c_1(t)|e_{AB}\rangle|0_C\rangle + c_2(t)|+_AB\rangle|1_C\rangle + c_3(t)|g_{AB}\rangle|2_C\rangle.$$  

(49)

where $|+_AB\rangle = (|e_{AB}\rangle + |g_{AB}\rangle)/\sqrt{2}$ and $|k\rangle$ denotes the collective states of the reservoir in $k$ excitations. The probability amplitudes for this case are

$$c_1(t) = e^{-\gamma t},$$
$$c_2(t) = e^{-\gamma t},$$
$$c_3(t) = \sqrt{1 - \alpha^2 - c_1^2(t) - c_2^2(t)}.$$  

(50)

Tracing out the reservoir, we obtain the density matrix of atomic subsystem

$$\rho_{AB} = \begin{pmatrix}
\alpha^2 + c_3^2 & 0 & 0 & \alpha c_1 \\
0 & \frac{c_2^2}{2} & \frac{c_2^2}{2} & 0 \\
0 & \frac{c_2^2}{2} & \frac{c_2^2}{2} & 0 \\
\alpha c_1 & 0 & 0 & c_1^2
\end{pmatrix},$$  

(51)

which is just an $X$ state. The corresponding GGQD and GD...
We have derived analytical formulas of GGQD and GD for two-qubit $X$ states. Here we give some useful remarks. First, it should be pointed out that Eqs. (16,26), from which Eq. (34) was derived, not only applicable to two-qubit $X$ states, but also to any two-qubit states. Second, because of $tr(ACB^*BC^*A^*) = tr(BC^*A^*ACB^*)$, we can alternatively first optimize system $B$, then system $A$. This is equivalent to exchange subsystems $A$ and $B$, and transpose matrix $C$. Of course, the two procedures give the same results. Third, and more important, we find that GGQD are always greater than or equal to GD in five examples we have given. In fact, this is true for any $X$ state. We present a proof as follows.

First, using $tr(\rho_X) = \rho_{00} + \rho_{11} + \rho_{22} + \rho_{33} = 1$ we easily obtain

\[
\begin{align*}
D_G(\rho_{AB}) & = 3/4 - 2[c_1^2(c_2^2 + c_3^2) + c_2^2(c_3^2(\alpha^2 + \alpha^2))] - \max\{c_1^2 + c_2^2/2 + (\alpha^2 + c_3^2)^2 - 1/4, (c_3^2/2 + \alpha c_1)^2\}, \\
D(\rho_{AB}) & = 1/4[2 + 7c_2^2 - 6c_2^2 - 4c_2^2(c_2^2 - \alpha^2)] - \max\{(1/2) - 3c_2^2(1 - c_2^2) - 2c_1^2(c_3^2 + \alpha^2) - c_3^2/2), (c_2^2/2 + c_1\alpha)^2\}.
\end{align*}
\]

In deducing above two equations, $c_1^2 + c_2^2 + c_3^2 + \alpha^2 = 1$ has been used. We plot $D_G(\rho_{AB})$ and $D(\rho_{AB})$ as functions of the dimensionless time $\gamma t$ in Fig. 5. $D_G(\rho_{AB})$ and $D(\rho_{AB})$ have

the same initial values $2\alpha^2(1-\alpha^2)$ and two relative maxima as well as one relative minimum, respectively. Their corresponding relative maximums and relative minimums are close to each other. When $t \to \infty$, $D_G(\rho_{AB})$ and $D(\rho_{AB})$ simultaneously go to zero.

V. DISCUSSION

We have derived analytical formulas of GGQD and GD for two-qubit $X$ states. Here we give some useful remarks. First, it should be pointed out that Eqs. (16,26), from which Eq. (34) was derived, not only applicable to two-qubit $X$ states, but also to any two-qubit states. Second, because of $tr(ACB^*BC^*A^*) = tr(BC^*A^*ACB^*)$, we can alternatively first optimize system $B$, then system $A$. This is equivalent to exchange subsystems $A$ and $B$, and transpose matrix $C$. Of course, the two procedures give the same results. Third, and more important, we find that GGQD are always greater than or equal to GD in five examples we have given. In fact, this is true for any $X$ state. We present a proof as follows.

First, using $tr(\rho_X) = \rho_{00} + \rho_{11} + \rho_{22} + \rho_{33} = 1$ we easily obtain

\[
\begin{align*}
\frac{d}{d\gamma} \rho_{00} + \rho_{11} + \rho_{22} + \rho_{33} & = \frac{1}{2}(\rho_{00} + \rho_{22}) - \frac{1}{2}(\rho_{11} + \rho_{33}) + (\rho_{12} - \rho_{03})^2, \\
D_G(\rho_{AB}) - D(\rho_{AB}) & = \frac{1}{4}[2(\rho_{00} + \rho_{22}) - 1] - \frac{1}{4}[2(\rho_{11} + \rho_{33}) - 1] \geq 0 (57)
\end{align*}
\]

(2) $\rho_{00} + \rho_{11} + \rho_{22} + \rho_{33} - 1/4 \geq (\rho_{12} + \rho_{03})^2 \geq (\rho_{00} + \rho_{11} + \rho_{22} + \rho_{33})/2 - \rho_{00}\rho_{22} - \rho_{11}\rho_{33}.$

\[
\begin{align*}
D_G(\rho_{AB}) & = 2(\rho_{12} + \rho_{03}), \\
D(\rho_{AB}) & = (\rho_{12} - \rho_{03})^2, \\
D_G(\rho_{AB}) - D(\rho_{AB}) & = \frac{1}{4}[(\rho_{00} - 2\rho_{22})^2 + (\rho_{11} - \rho_{33})^2] + (\rho_{12} - \rho_{03})^2, \\
D_G(\rho_{AB}) - D(\rho_{AB}) - \frac{1}{2}[(\rho_{00} - 2\rho_{22})^2 + (\rho_{11} - \rho_{33})^2] & \geq \frac{1}{4}[(\rho_{00} - \rho_{22})^2 + (\rho_{11} - \rho_{33})^2] \geq 0 (60)
\end{align*}
\]
two-qubit $X$ states. In addition, we have further found that GD is the tight lower bound of GGQD, which means that GD is a good approximation at least for $X$ states. There are some interesting opening problems needed to be studied in this respect, such as, are there any analytical expressions of GGQD for qubit-qudit system? Can GD be a tight lower bound of GGQD for any bipartite system? We shall report our research results on these issues later.

VI. SUMMARY

In summary, we have proved GGQD and TQC are the same. Then we obtained analytical formulas of GGQD and GD for

$$\text{(3) } (\hat{g}_{12} + \hat{g}_{33})^2 \leq (\hat{g}_{00}^2 + \hat{g}_{11}^2 + \hat{g}_{22}^2 + \hat{g}_{33}^2)/2 - \hat{g}_{00}\hat{g}_{22} - \hat{g}_{11}\hat{g}_{33} \leq \hat{g}_{00}^2 + \hat{g}_{11}^2 + \hat{g}_{22}^2 + \hat{g}_{33}^2 - 1/4;$$

$$D_G(\rho_{AB}) = D(\rho_{AB}) = 2(\hat{g}_{12}^2 + \hat{g}_{03}^2). \quad (61)$$

We conclude that $D_G(\rho_{AB}) \geq D(\rho_{AB})$ for any $X$ state from Eqs. (54,57,60,61).