NEW DERIVED AUTOEQUIVALENCES OF HILBERT SCHEMES AND GENERALIZED KUMMER VARIETIES

ANDREAS KRUG

Abstract. We show that for every smooth projective surface $X$ and every $n \geq 2$ the push-forward along the diagonal embedding gives a $\mathbb{P}^{n-1}$-functor into the $\mathfrak{S}_n$-equivariant derived category of $X^n$. Using the Bridgeland–King–Reid–Haiman equivalence this yields a new autoequivalence of the derived category of $n$ points on $X$. In the case that the canonical bundle of $X$ is trivial and $n = 2$ this autoequivalence coincides with the known EZ-spherical twist induced by the boundary of the Hilbert scheme. We also generalise the 16 spherical objects on the Kummer surface given by the exceptional curves to $n^4$ orthogonal $\mathbb{P}^{n-1}$-Objects on the generalised Kummer variety.

1. Introduction

For every smooth projective surface $X$ over $\mathbb{C}$ and every $n \in \mathbb{N}$ there is the Bridgeland–King–Reid–Haiman equivalence (see [BKR01] and [Hai01])

$$\Phi: \mathcal{D}^b(X^{[n]}) \xrightarrow{\sim} \mathcal{D}^b_{\mathfrak{S}_n}(X^n)$$

between the bounded derived category of the Hilbert scheme of $n$ points on $X$ and the $\mathfrak{S}_n$-equivariant derived category of the cartesian product of $X$. In [Plo07] Ploog used this to give a general construction which associates to every autoequivalence $\Psi \in \text{Aut}(\mathcal{D}^b(X))$ an autoequivalence $\alpha(\Psi) \in \text{Aut}(\mathcal{D}^b(X^{[n]}))$ on the Hilbert scheme. Recently, Ploog and Sosna [PS12] gave a construction that produces out of spherical objects (see [ST01]) on the surface $\mathbb{P}^n$-objects (see [HT06]) on $X^{[n]}$ which in turn induce further derived autoequivalences. On the other hand, there are only very few autoequivalences of $\mathcal{D}^b(X^{[n]})$ known to exist independently of $\mathcal{D}^b(X)$:

- There is always an involution given by tensoring with the alternating representation in $\mathcal{D}^b_{\mathfrak{S}_n}(X^n)$, i.e with the one-dimensional representation on which $\sigma \in \mathfrak{S}_n$ acts via multiplication by $\text{sgn}(\sigma)$.
- Addington introduced in [Add11] the notion of a $\mathbb{P}^n$-functor generalising the $\mathbb{P}^n$-objects of Huybrechts and Thomas. He showed that for $X$ a K3-surface and $n \geq 2$ the Fourier–Mukai transform $F_\mu: \mathcal{D}^b(X) \to \mathcal{D}^b(X^{[n]})$ induced by the universal sheaf is a $\mathbb{P}^{n-1}$-functor. This yields an autoequivalence of $\mathcal{D}^b(X^{[n]})$ for every K3-surface $X$ and every $n \geq 2$.
- For $X = A$ an abelian surface the pull-back along the summation map $\Sigma: A^{[n]} \to A$ is a $\mathbb{P}^{n-1}$-functor and thus induces a derived autoequivalence (see [Mea12]).
- The boundary of the Hilbert scheme $\partial X^{[n]}$ is the codimension 1 subvariety of points representing non-reduced subschemes of $X$. For $n = 2$ it equals $X^{[2]}_\Delta := \mu^{-1}(\Delta)$ where $\mu: X^{[2]} \to S^2X$ denotes the Hilbert-Chow morphism. For $n = 2$ and $X$ a surface with trivial canonical bundle it is known (see [Huy06, examples 8.49 (iv)]) that every line bundle on the boundary of the Hilbert scheme is an EZ-spherical object (see [Hor05]).
and thus also induces an autoequivalence. We will see in remark 4.6 that the induced automorphisms given by different choices of line bundles on $X^{[2]}_{\Delta}$ only differ by twists with line bundles on $X^{[2]}$. Thus, we will just speak of the autoequivalence induced by the boundary referring to the automorphism induced by the EZ-spherical object $O_{\mu}|_{X^{[2]}_{\Delta}}(-1)$.

In this article we generalise this last example to surfaces with arbitrary canonical bundle and to arbitrary $n \geq 2$. More precisely, we consider the functor $F: \text{D}^b(X) \to \text{D}^b_{S_n}(X^n)$ which is defined as the composition of the functor $\text{triv}: \text{D}^b(X) \to \text{D}^b_{S_n}(X^n)$ given by equipping every object with the trivial $S_n$-linearisation and the push-forward $\delta_*: \text{D}^b_{S_n}(X) \to \text{D}^b_{S_n}(X^n)$ along the diagonal embedding. Then we show in section 3 the following.

\textbf{Theorem 1.1.} For every $n \in \mathbb{N}$ with $n \geq 2$ and every smooth projective surface $X$ the functor $F: \text{D}^b(X) \to \text{D}^b_{S_n}(X^n)$ is a $\mathbb{P}^{n-1}$-functor.

In section 4 we show that for $n = 2$ the induced autoequivalence coincides under $\Phi$ with the autoequivalence induced by the boundary. In section 5 we compare the autoequivalence induced by $F$ to some other derived autoequivalences of $X^{[n]}$ showing that it differs essentially from the standard autoequivalences and the autoequivalence induced by $F_n$. In particular, the Hilbert scheme always has non-standard autoequivalences even if $X$ is a Fano surface. In the last section we consider the case that $X = A$ is an abelian surface. We show that after restricting our $\mathbb{P}^{n-1}$-functor from $A^{[n]}$ to the generalised Kummer variety $K_{n-1}A$ it splits into $n^4$ pairwise orthogonal $\mathbb{P}^{n-1}$-objects. They generalise the 16 spherical objects on the Kummer surface given by the line bundles $O_{C}(-1)$ on the exceptional curves.

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\section{2. $\mathbb{P}^n$-functors}

A $\mathbb{P}^n$-\textit{functor} is defined in [Add11] as a functor $F: \mathcal{A} \to \mathcal{B}$ of triangulated categories admitting left and right adjoints $L$ and $R$ such that

(i) There is an autoequivalence $H$ of $\mathcal{A}$ such that

\[ RF \simeq \text{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n. \]

(ii) The map

\[ HRF \hookrightarrow RFRF \xrightarrow{ReF} RF \]

with $\varepsilon$ being the counit of the adjunction is, when written in the components

\[ H \oplus H^2 \oplus \cdots \oplus H^n \oplus H^{n+1} \to \text{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n, \]

of the form

\begin{pmatrix}
\ast & \ast & \cdots & \ast & \ast \\
1 & \ast & \cdots & \ast & \ast \\
0 & 1 & \cdots & \ast & \ast \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \ast
\end{pmatrix}.
(iii) $R \simeq H^n L$. If $\mathcal{A}$ and $\mathcal{B}$ have Serre functors, this is equivalent to $S_B F H^n \simeq F S_A$.

In the following we always consider the case that $\mathcal{A}$ and $\mathcal{B}$ are (equivariant) derived categories of smooth projective varieties and $F$ is a Fourier–Mukai transform. The $\mathbb{P}^n$-twist associated to a $\mathbb{P}^n$-functor $F$ is defined as the double cone

$$P_F := \text{cone} (\text{cone}(FHR \to FR) \to \text{id}).$$

The map defining the inner cone is given by the composition

$$FHR \xrightarrow{FjR} FRFR \xrightarrow{\varepsilon FR - FR\varepsilon} FR$$

where $j$ is the inclusion given by the decomposition in (i). The map defining the outer cone is induced by the counit $\varepsilon: FR \to \text{id}$ (for details see [Add11]). Taking the cones of the Fourier–Mukai transforms indeed makes sense, since all the occurring maps are induced by maps between the integral kernels (see [AL12]). We set $\ker R := \{ B \in \mathcal{B} \mid RB = 0 \}$. By the adjoint property it equals the right-orthogonal complement $(\text{im} F)\perp$.

**Proposition 2.1** ([Add11 section 3]). Let $F: \mathcal{A} \to \mathcal{B}$ be a $\mathbb{P}^n$-functor.

(i) We have $P_F(B) = B$ for $B \in \ker R$.

(ii) $P_F \circ F \simeq H^{n+1}[2]$.

(iii) The objects in $\text{im} F \cup \ker R$ form a spanning class of $\mathcal{B}$.

(iv) $P_F$ is an autoequivalence.

**Example 2.2.**

(i) Let $\mathcal{B} = D^b(X)$ for a smooth projective variety $X$. A $\mathbb{P}^n$-object (see [HT06]) is an object $E \in \mathcal{B}$ such that $E \otimes \omega_X \simeq E$ and $\text{Ext}^*(E, E) \cong H^*(\mathbb{P}^n, \mathbb{C})$ as $\mathbb{C}$-algebras (the ring structure on the left-hand side is the Yoneda product and on the right-hand side the cup product). A $\mathbb{P}^n$-object can be identified with the $\mathbb{P}^n$-functor

$$F: \mathbb{D}^b(\text{pt}) \to \mathcal{B} , \quad \mathbb{C} \mapsto E$$

with $H = [-2]$. Note that the right adjoint is indeed given by $R = \text{Ext}^*(E, \underline{-})$. The $\mathbb{P}^n$-twist associated to the functor $F$ is the same as the $\mathbb{P}^n$-twist associated to the object $E$ as defined in [HT06].

(ii) A $\mathbb{P}^1$-functor is the same as a spherical functor (see [Ann07]) where the unit

$$\text{id} \xrightarrow{\eta} RF \to H$$

splits. In this case there is also the spherical twist given by

$$T_F := \text{cone} \left( FR \xrightarrow{\varepsilon} \text{id} \right).$$

It is again an autoequivalence with $T_F^2 = P_F$ (see [Add11 p. 33]).

**Lemma 2.3.**

(i) Let $\Psi \in \text{Aut}(\mathcal{A})$ such that $\Psi \circ H \simeq H \circ \Psi$. Then $F \circ \Psi$ is again a $\mathbb{P}^n$-functor with the property

$$P_{F \circ \Psi} \simeq P_F.$$

(ii) Let $\Phi: \mathcal{B} \to \mathcal{C}$ be an equivalence of triangulated categories. Then $\Phi \circ F$ is again a $\mathbb{P}^n$-functor with the property that

$$P_{\Phi \circ F} \circ \Phi \simeq \Phi \circ P_F.$$

**Proof.** The proof is analogous to the proof of the corresponding statement for spherical functors [Ann07 proposition 2].
Corollary 2.4. Let $E_1, \ldots, E_n \in \mathcal{B}$ be a collection of pairwise orthogonal (that means $\text{Hom}^*(E_i, E_j) = 0 = \text{Hom}^*(E_j, E_i)$ for $i \neq j$) $\mathbb{P}^n$-objects with associated twists $p_i := P_{E_i}$. Then

$$\mathbb{Z}^n \to \text{Aut}(A), \quad (\lambda_1, \ldots, \lambda_n) \mapsto p_1^{\lambda_1} \circ \cdots \circ p_n^{\lambda_n}$$

defines a group isomorphism $\mathbb{Z}^n \cong \langle p_1, \ldots, p_n \rangle \subset \text{Aut}(B)$.

Proof. By part (ii) of the previous lemma the $p_i$ commute which means that the map is indeed a group homomorphism onto the subgroup generated by the $p_i$. Let $g = p_1^{\lambda_1} \circ \cdots \circ p_n^{\lambda_n}$. Then $g(E_i) = E_i[2n\lambda_i]$ by proposition 2.1. Thus, $g = \text{id}$ implies $\lambda_1 = \cdots = \lambda_n = 0$.

Lemma 2.5. Let $X$ be a smooth variety, $T \in \text{Aut}(D^b(\mathbb{P}^n))$, and $A, B \in D^b(\mathbb{P}^n)$ objects such that $TA = A[i]$ and $TB = B[j]$ for some $i \neq j \in \mathbb{Z}$. Then $A \perp B$ and $B \perp A$.

Proof. See [Add11, p. 11].

Remark 2.6. This shows together with proposition 2.1 that for a $\mathbb{P}^n$-functor $F$ with $H = [-\ell]$ for some $\ell \in \mathbb{Z}$ there does not exist a non-zero-object $A$ with $TF(A) = A[\ell]$ for any values of $i$ besides $0$ and $-n\ell + 2$ because such an object would be orthogonal to the spanning class $\text{im } F \cup \ker R$.

3. The diagonal embedding

Let $X$ be a smooth projective surface over $\mathbb{C}$ and $2 \leq n \in \mathbb{N}$. We denote by $\delta : X \to X^n$ the diagonal embedding. We want to show that $F : D^b(\mathbb{P}^n) \to D^b(\mathbb{P}^n_X)$ given as the composition

$$D^b(\mathbb{P}^n) \xrightarrow{\text{triv}} D_{\mathcal{S}_n}(\mathbb{P}^n) \xrightarrow{\delta} D_{\mathcal{S}_n}(X^n)$$

defines a group isomorphism $\mathbb{Z}^n \cong \langle p_1, \ldots, p_n \rangle \subset \text{Aut}(\mathcal{S}_n)$ as the composition

$$D^b(\mathbb{P}^n_X) \xrightarrow{\text{id}} D_{\mathcal{S}_n}(\mathbb{P}^n_X) \xrightarrow{\lambda_n} D^b(X^n)$$

of the usual right adjoint (see [WH99] for equivariant Grothendieck duality) and the functor of taking invariants. We consider the standard representation $\rho$ of $\mathcal{S}_n$ as the quotient of the regular representation $\mathbb{C}^n$ by the one dimensional invariant subspace. The normal bundle sequence

$$0 \to TX \to T_{X^n/X} \to N \to 0$$

where $N := N_\delta = N_{X/X}$ is of the form

$$0 \to TX \to T^\oplus_{X^n} \to N \to 0$$

where the map $T_X \to T^\oplus_{X^n}$ is the diagonal embedding. When considering $T_{X^n/X}$ as a $\mathcal{S}_n$-sheaf equipped with the natural linearisation it is given by $T_X \otimes \mathbb{C}^n$ where $\mathbb{C}^n$ is the regular representation. Thus, as a $\mathcal{S}_n$-sheaf, the normal bundle equals $TX \otimes \rho$. We also see that the normal bundle sequence splits using e.g. the splitting

$$TX \otimes \mathbb{C}^n \to TX, \quad (v_1, \ldots, v_n) \mapsto \frac{1}{n}(v_1 + \cdots + v_n).$$

Theorem 3.1 ([ACT12]). Let $\iota : Z \hookrightarrow M$ be a regular embedding of codimension $c$ such that the normal bundle sequence splits. Then there is an isomorphism

$$\iota^* \iota_* (\underline{\_}) \cong (\bigoplus_{i=0}^c \wedge^i N_{Z/M}[i])$$

(1)
of endofunctors of $D^b(Z)$.

**Corollary 3.2.** Under the same assumptions, there is an isomorphism
\[(2) \quad i^! t_* (\_ ) \simeq (\_ ) \otimes ( \bigoplus_{i=0}^c \Lambda^i N_{Z/M}[-i]) \]

**Proof.** Tensorise both sides of (1) by $i^! O_M \simeq \Lambda^i N_{Z/M}[-c]$.

**Lemma 3.3.** The monad multiplication $i^! e t_* : i^! t_* i^! t_* \to i^! t_*$ is given under the above isomorphism (if chosen correctly) by the wedge pairing
\[
\left( \bigoplus_{i=0}^c \Lambda^i N_{Z/M}[-i] \right) \otimes \left( \bigoplus_{j=0}^c \Lambda^j N_{Z/M}[-j] \right) \to \bigoplus_{k=0}^c \Lambda^k N_{Z/M}[-k].
\]

**Proof.** For $E \in D^b(M)$ the object $i^! E$ can be identified with $\mathcal{H}om(i_* O_Z, E)$ considered as an object in $D^b(Z)$. Under this identification the counit map $\mathcal{H}om(i_* O_Z, E) \to E$ is given by $\varphi \mapsto \varphi(1)$ (see [Har66, section III.6]). Now we get for $F \in D^b(Z)$ the identifications
\[
i^! t_* i^! t_* F \simeq \mathcal{H}om(i_* O_Z, i_* O_Z) \otimes_{O_Z} \mathcal{H}om(i_* O_Z, i_* O_Z) \otimes_{O_Z} F
\]
and $i^! t_* F \simeq \mathcal{H}om(i_* O_Z, i_* O_Z) \otimes_{O_Z} F$ under which the monad multiplication equals the Yoneda product. It is known (see [LH09, p. 442]) that the Yoneda product corresponds to the wedge product when choosing the right isomorphism.

In the case that $i = \delta$ from above this yields the isomorphism of monads
\[(3) \quad \delta^! \delta_*(\_ ) \simeq (\_ ) \otimes ( \bigoplus_{i=0}^{2(n-1)} \Lambda^i (T_X \otimes \varrho)[-i]).\]

**Lemma 3.4** ([Sca09a, Appendix B]). Let $V$ be a two-dimensional vector space with a basis consisting of vectors $u$ and $v$. Then the space of invariants $[\Lambda^i (V \otimes \varrho)]^{\mathbb{G}_m}$ is one-dimensional if $0 \leq i \leq 2(n-1)$ is even and zero if it is odd. In the even case $i = 2\ell$ the space of invariants is spanned by the image of the vector $\omega^\ell$, where
\[
\omega = \sum_{i=1}^{k} u e_i \wedge v e_i \in \Lambda^2 (V \otimes \mathbb{C}^n),
\]
under the projection induced by the projection $\mathbb{C}^n \to \varrho$.

**Corollary 3.5.** For a vector bundle $E$ on $X$ of rank two and $0 \leq \ell \leq n-1$ there is an isomorphism
\[
[\Lambda^{2\ell} (E \otimes \varrho)]^{\mathbb{G}_m} \simeq (\Lambda^2 E)^{\otimes \ell}.
\]

**Proof.** The isomorphism is given by composing the morphism
\[
(\Lambda^2 E)^{\otimes \ell} \to \Lambda^\ell (E \otimes \mathbb{C}^n), \quad x_1 \otimes \cdots \otimes x_\ell \mapsto \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} x_{i_1} e_{i_1} \wedge \cdots \wedge x_{i_\ell} e_{i_\ell}
\]
with the projection induced by the projection $\mathbb{C}^n \to \varrho$.

We set $H := \Lambda^2 T_X[-2] = \omega_X^\vee [-2] = S_X^{-1}$.

**Corollary 3.6.** There is the isomorphism of functors
\[
RF \simeq \text{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^{n-1}.
\]
Proof. This follows by formula (3) and corollary 3.5.

**Lemma 3.7.** The map $HRF \rightarrow RF$ defined in condition (ii) for $\mathbb{P}^n$-functors is for this pair $F \rightleftharpoons R$ given by the matrix

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}.
$$

Proof. The generators $\omega^f$ from lemma 3.4 are mapped to each other by wedge product. By lemma 3.3 the monad multiplication is given by wedge product. □

**Lemma 3.8.** There is the isomorphism $S_{X^n}FH^{n-1} \simeq FS_X$.

Proof. For $E \in D^b(X)$ there are natural isomorphisms

$$
S_{X^n}FH^{n-1}(E) = \omega_{X^n}[2n] \otimes \delta_s(E \otimes \omega_X^{-(n-1)}[-2(n-1)]) \simeq \omega_{X} \otimes \delta_s(E \otimes \omega_X^{-(n-1)})[2] \\
\simeq \delta_s(E \otimes \omega_X[2]) = FS_X(E).
$$

□

All this together shows theorem 1.1, i.e. that $F = \delta_s \circ \text{triv}$ is indeed a $\mathbb{P}^{n-1}$-functor.

4. Composition with the Bridgeland–King–Reid–Haiman equivalence

The isospectral Hilbert scheme $I^nX \subset X^n \times X^n$ is defined as the reduced fibre product $I^nX := (X^n \times X^n)_{\text{red}}$ with the defining morphisms being the Hilbert–Chow morphism $\mu: X^n \rightarrow S^nX$ and the quotient morphism $\pi: X^n \rightarrow S^nX$. Thus, there is the commutative diagram

$$
\begin{array}{ccc}
I^nX & \overset{q}{\longrightarrow} & X^n \\
\downarrow p & & \downarrow \pi \\
X^n & \overset{\mu}{\longrightarrow} & S^nX.
\end{array}
$$

The Bridgeland–King–Reid–Haiman equivalence is the functor

$$
\Phi := FM_{O_{X^n}} \circ \text{triv} = p_* \circ q^* \circ \text{triv}: D^b(X^n) \rightarrow D^b_{S^n}(X^n).
$$

By the results in [BKR01] and [Hai01] it is indeed an equivalence. The isospectral Hilbert scheme can be identified with the blow-up of $X^n$ along the union of all the pairwise diagonals $\Delta_{ij} = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j\}$ (see [Hai01]). By lemma 2.4 the functor composition $\Phi^{-1} \circ F: D^b(X) \rightarrow D^b(X[n])$ is again a $\mathbb{P}^n$-functor and thus yields an autoequivalence of the derived category of the Hilbert scheme.

**Lemma 4.1.** Let $G: \mathcal{A} \rightarrow \mathcal{B}$ be a right-exact functor between abelian categories such that $\mathcal{A}$ has a $G$-adapted class and let $X^\bullet \in D^-(\mathcal{A})$ be a complex such that $\mathcal{H}^n(X^\bullet)$ is $G$-acyclic for every $n \in \mathbb{Z}$. Then $\mathcal{H}^n(LG(X^\bullet)) = GH^n(X^\bullet)$ holds for every $n \in \mathbb{Z}$.

Proof. This follows from the spectral sequence

$$
E_2^{p,q} = L^pG\mathcal{H}^q(X^\bullet) \implies E^n = \mathcal{H}^n(LG(X^\bullet)).
$$

□
Lemma 4.3. For every $E$ over generalisation of [Huy06, Proposition 11.12] for a blow-up

Proof. The functor $\Phi$ Here, the line bundle $O$ of a smooth projective variety $X$ is known to be an EZ-spherical object (see [Huy06, examples 8.49 (iv)]). That means that the functor

$$\tilde{F}_L : D^b(X) \to D^b(X^{[2]}), \quad E \mapsto j_*(L \otimes \mu_E^* E)$$

is a spherical functor where the maps $j$ and $\mu$ come from the fibre diagram

$$X^{[2]} \xrightarrow{j} X^{[2]} \quad \mu \downarrow \quad \mu \downarrow$$

$$X \xrightarrow{d} S^2 X$$

with $d$ being the diagonal embedding. The map $\mu$ is a $\mathbb{P}^1$-bundle.

Proposition 4.2. Let $X$ be a smooth projective surface (with arbitrary canonical bundle). Then there is an isomorphism of functors $\Phi^{-1} \circ F \simeq \tilde{F}_{\mathcal{O}_X(-1)}$, where $\Phi^{-1}$ is the inverse of the BKRH-equivalence and $F = \delta_* \circ \text{triv} : D^b(X) \to D^b_{\mathbb{G}_m}(X^{[2]})$ from the previous section.

Proof. The functor $\Phi^{-1}$ is given by the composition $\bigoplus \delta \circ \text{FM}_{\mathcal{Q}}^{X_2 \to X^{[2]}}$ with Fourier–Mukai kernel $\mathcal{Q} = \mathcal{O}_{I^2 X} \otimes q^* \omega_{X^{[2]}}[4]$. The isospectral Hilbert scheme $I^2 X$ is the blow-up of $X^2$ along the diagonal. In particular, it is smooth. Let $E = p^{-1}(\Delta)$ be the exceptional divisor of the blow up. The $\mathbb{P}^1$-bundles $p_\Delta : E \to X$ and $\mu_\Delta : X^{[2]} \to X$ are isomorphic via $q : I^2 X \to X^{[2]}$. The canonical bundle of the blow-up is given by $\omega_{I^2 X} \cong p^* \omega_{X^2} \otimes \mathcal{O}(E)$. Let $N$ be the normal bundle of the codimension 4 regular embedding $I^2 X \to X^{[2]} \times X^2$. By adjunction formula

$$\wedge^4 N \cong \omega_X^{[2]} \otimes \omega_{I^2 X} \cong q^* \omega_{X^{[2]}}[4] \otimes \mathcal{O}(E).$$

It follows by Grothendieck-Verdier duality for regular embeddings that

$$\mathcal{Q} = \mathcal{O}_{I^2 X} \otimes q^* \omega_{X^{[2]}}[4] \cong \wedge^4 N[-4] \otimes q^* \omega_{X^{[2]}}[4] \simeq \mathcal{O}(E).$$

Here, the line bundle $\mathcal{O}(E)$ is equipped with the natural $\mathfrak{S}_2$-linearisation which is trivial over $E$, i.e. $\mathcal{O}_E(E) = \mathcal{O}_{p_\Delta}(-1)$ carries the trivial $\mathfrak{S}_2$-action. We need the following slight generalisation of [Huy06, Proposition 11.12] for a blow-up

$$E \xrightarrow{i} \tilde{X} \quad \pi \downarrow \quad \pi \downarrow$$

$$Y \xrightarrow{j} X$$

of a smooth projective variety $X$ along a smooth subvariety $Y$ of codimension $c$.

Lemma 4.3. For every $\mathcal{F} \in \text{Coh}(Y)$ and every $k \in \mathbb{Z}$ there is an isomorphism

$$\mathcal{H}^k(p^* j_* \mathcal{F}) \cong i_* \left( \pi^* \mathcal{F} \otimes \wedge^{-k} \Omega_{\pi} \otimes \mathcal{O}_{\pi}(-k) \right).$$

Proof. This can be proven locally. Hence, we may assume that $Y = Z(s)$ is the zero locus of a global section of a vector bundle $\mathcal{E}$ of rank $c$. Thus, the blow-up diagram can be enlarged
to

\[
E \xrightarrow{\iota} \tilde{X} \xrightarrow{\iota} \mathbb{P}(\mathcal{E})
\]

\[
\pi \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow p \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow g
\]

\[
Y \xrightarrow{j} X \xrightarrow{id} X
\]

where \(\iota\) is a closed embedding of codimension \(c - 1\) such that the normal bundle \(M\) has the property \(\land^k M_{|E} = \land^{-k}\Omega_\pi \otimes \mathcal{O}_\pi(-k)\) (see [Huy06, p. 252]). The outer square is a flat base change. It follows that

\[
(4) \quad \iota_* p^* j_* \mathcal{F} \simeq \iota_* \iota^* g^* j_* \mathcal{F} \simeq \iota_* \iota^* p^* i_* \pi^* \mathcal{F} \simeq \iota_* i_\pi^* (\pi^* \mathcal{F} \otimes i^* \iota^* \mathcal{O}_\tilde{X})
\]

where the last isomorphism is given by applying the projection formula two times. Now \(\mathcal{H}^k (\pi^* \mathcal{F} \otimes i^* \iota_* \mathcal{O}_\tilde{X}) \equiv \pi^* \mathcal{F} \otimes \land^{-k} \mathcal{M} \equiv \pi^* \mathcal{F} \otimes \land^{-k} \Omega_\pi \otimes \mathcal{O}_\pi(-k)\).

By \([1]\) it follows that

\[
\mathcal{H}^k (\pi^* \mathcal{F} \otimes i^* \iota_* \mathcal{O}_\tilde{X}) \cong \pi^* \mathcal{F} \otimes \land^{-k} \mathcal{M} \otimes \mathcal{O}_\pi(-k)
\]

which proves the assertion since \(\iota_* : \text{Coh}(\tilde{X}) \to \text{Coh}(\mathbb{P}(\mathcal{E}))\) is fully faithful.

**Remark 4.4.** If \(X\) carries an action by a finite group \(G\) and \(Y\) is invariant under this action, \(G\) also acts on the blow-up \(\tilde{X}\). The bundle \(M_{|E}\) of the proof is a quotient of the normal bundle \(N_{E/\mathbb{P}(\mathcal{E})} \equiv \pi^* \eta Y/X\). In the case that there is a group action this quotient is \(G\)-equivariant. Thus, the formula of the lemma is in this case also true for \(\mathcal{F} \in \text{Coh}_G(X)\) with the action on the right hand side induced by the linearization of the wedge powers of \(M\) respectively \(\eta Y/X\).

In the case of the blow-up \(p: I^2 X \to X^2\) the center \(\Delta\) of the blow-up has codimension 2. Thus, \(p^* \delta_* \mathcal{F}\) is cohomologically concentrated in degree 0 and \(-1\). Since \(p^* \delta_* \mathcal{F}\) is concentrated on \(E\) where \(G_2\) is acting trivially, one can take the invariants even before applying the push-forward along \(q: I^2 X \to X^2\). The group \(G_2\) acts on \(\land^0 N_{\Delta/X^2}\) trivially and on \(N_{\Delta/X^2}\) alternating. Hence, by the previous remark we have for \(\mathcal{F} \in \text{Coh}(X)\) equipped with the trivial group action that \(\mathcal{H}^0 (p^* \delta_* \mathcal{F})^{G_2} = p^*_\Delta \mathcal{F}\) and \(\mathcal{H}^1 (p^* \delta_* \mathcal{F})^{G_2} = 0\). In particular, every \(\mathcal{F} \in \text{Coh}(X)\) is acyclic under the functor \(\land^0 \mathcal{F} \otimes \delta_\ast\mathcal{O}_\mathcal{F}\) with \(\land^0 \mathcal{F} \otimes \delta_\ast\mathcal{O}_\mathcal{F}\). This implies that \(\land^0 \mathcal{F} \otimes \delta_\ast\mathcal{O}_\mathcal{F}\) is concentrated for every \(\mathcal{F} \in \text{D}^b(X)\). Together with \(\mathcal{O}_{|E} \cong \mathcal{O}_E (\mathcal{E}) \cong \mathcal{O}_{|\Delta} (-1) \cong \mathcal{O}_{|\Delta} (-1)\) this proves proposition \([1,2]\).□

**Remark 4.5.** The proposition says in particular that \(\tilde{F}_{\mathcal{O}_{|\Delta} (-1)}\) is also a spherical functor in the case that \(\omega_X\) is not trivial. One can also prove this directly and for general \(L\) instead of \(\mathcal{O}_{|\Delta} (-1)\).

**Remark 4.6.** Since \(X_{[2]}^\Delta\) is a \(\mathbb{P}^1\)-bundle over \(X\), every line bundle on it is of the form \(L \cong \mu^\Delta K \otimes \mathcal{O}_{|\Delta} (i)\) for some \(K \in \text{Pic} \ X\) and \(i \in \mathbb{Z}\). The canonical bundle of \(X_{[2]}^\Delta\) is given by \(\mu^\Delta \omega_X \otimes \mathcal{O}_{|\Delta} (-2)\). The Hilbert-Chow morphism \(\mu\) is a crepant resolution, i.e. \(\omega_{[2]} X \equiv \mu^\Delta \omega_{2X}\). Thus,

\[
\omega_{[2]} X \cong \mu^\Delta (\omega_{2X}) \cong \mu^\Delta \omega_X^2.
\]

Let \(N = \mathcal{O}_{X_{[2]}^\Delta} (X_{[2]}^\Delta)\) be the normal bundle of \(X_{[2]}^\Delta\) in \(X_{2}[2]\). By adjunction formula it is given by \(\mu^\Delta \omega_X \otimes \mathcal{O}_{|\Delta} (-2)\). There is a line bundle \(D \in \text{Pic} \ X_{[2]}\) (namely the determinant
of the tautological sheaf $O_X^{[2]}$ such that $-2c_1(D) = [X^{[2]}_\Delta] = c_1(O_X^{[2]})$ (see [Leh99 lemma 3.8]). Its restriction $D|_{X^{[2]}_\Delta}$ is of the form $\mu_\Delta M \otimes O_{X^{[2]}_\Delta}(1)$ for some $M \in \Pic X$. Using this, we can rewrite for a general $L = \mu_\Delta K \otimes O_{\mu_\Delta} i \in \Pic X^{[2]}_\Delta$ the spherical functor $\tilde{F}_L$ as $\tilde{F}_L = M_D^{i+1} \circ \tilde{F}_{O_{\mu_\Delta} (-1)} \circ M_Q$ for some $Q \in \Pic X$ where $M_Q$ is the autoequivalence given by tensor product with $Q$. The analogous of lemma 2.3 for spherical functors thus yields $t_L = M_D^{i+1} \circ t_{O_{\mu_\Delta} (-1)} \circ M_D^{(-i+1)}$ where $t_L$ is the spherical twist associated to $\tilde{F}_L$.

**Remark 4.7.** For general $n \geq 2$ every object in the image of $\Phi^{-1} \circ F$ still is supported on $X^{[n]}_\Delta = \mu^{-1}(\Delta)$.

5. Comparison with other autoequivalences

In the following we will denote the $\mathbb{P}^n$-twist associated to $F$ respectively $\Phi^{-1} \circ F$ by $b \in \Aut(D^{b}_{\mathbb{G}_m}(X^n)) \cong \Aut(D^{b}(X^{[n]}))$. In the case that $n = 2$ the functor $F$ is spherical (see example 2.2 (ii)). We denote the associated spherical twist by $\sqrt{b}$.

**Proposition 5.1.** The automorphism $b$ is not contained in the group of standard automorphisms

$$\Aut(D^{b}(X^{[n]})) \supset \Aut_{\mathcal{X}}(X^{[n]}) \cong \mathbb{Z} \times \left( \Aut(X^{[n]} \times \Pic(X^{[n]})) \right)$$

generated by shifts, push-forwards along automorphisms and taking tensor products by line bundles. The same holds in the case $n = 2$ for $\sqrt{b}$.

**Proof.** Let $[\xi] \in X^{[n]} \setminus X^{[n]}_\Delta$, i.e. $\supp \xi \geq 2$. Then by remark 1.7 and proposition 2.1 (i), we have $b(C([\xi])) = C([\xi])$. Let $g = [\xi] \circ \varphi \circ M_L \in \Aut_{\mathcal{X}}(X^{[n]})$ where $M_L$ is the functor $E \mapsto E \otimes L$ for an $L \in \Pic X^{[n]}$. Then $g(C([\xi])) = C(\varphi([\xi]))[\ell]$. Thus, the assumption $b = g$ implies $\ell = 0$ and also $\varphi = \id$, since $X^{[n]} \setminus X^{[n]}_\Delta$ is open in $X^{[n]}$. Thus, the only possibility left for $b \in \Aut_{\mathcal{X}}(X^{[n]})$ is $b = M_L$ for some line bundle $L$ which cannot hold by proposition 2.1 (ii). The proof that $\sqrt{b} \notin \Aut_{\mathcal{X}}(X^{[n]})$ is the same. \[\square\]

In [Plo07] Ploog gave a general construction which associates to derived autoequivalences of the surface $X$ derived autoequivalences of the Hilbert scheme $X^{[n]}$. Let $\Psi \in \Aut(D^{b}(X))$ with Fourier–Mukai kernel $\mathcal{P} \in D^{b}(X \times X)$. The object $\mathcal{P} \in D^{b}(X \times X)$ carries a natural $\mathcal{G}_n$-linearisation given by permutation of the box factors. Thus, it induces a $\mathcal{G}_n$-equivariant derived autoequivalence $\alpha(\Psi) := \op_{\mathcal{P}}^{\mathcal{E}n}$ of $X^{[n]}$. This gives the following.

**Theorem 5.2.** ([Plo07]). The above construction gives an injective group homomorphism

$$\alpha: \Aut(D^{b}(X)) \rightarrow \Aut(D^{b}_{\mathbb{G}_m}(X^{[n]})) \cong \Aut(D^{b}(X^{[n]}))$$

**Remark 5.3.** For every $\varphi \in \Aut(X)$ we have $\alpha(\varphi) = (\varphi^n)_*$ where $\varphi^n$ is the $\mathcal{G}_n$-equivariant automorphism of $X^n$ given by $\varphi(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n))$. Furthermore, $\varphi$ acts on $X^{[n]}$ by the morphism $\varphi^n$, which is given by $\varphi^n([\xi]) = [\varphi(\xi)]$, and on $X^n$ by the morphism $\varphi^n$. Since the Bridgeland–King–Reid–Haiman equivalence is the Fourier–Mukai transform with kernel the structural sheaf of $P^n X$, it is $\op(X)$-equivariant, i.e. $\Phi \circ (\varphi^n)_* \cong (\varphi^n)_* \circ \Phi$. Thus, $\alpha(\varphi) \in \Aut(D^{b}_{\mathbb{G}_m}(X^{[n]}))$ corresponds to $\varphi^n \in \Aut(D^{b}(X^{[n]}))$. For $L \in \Pic X$ we have $\alpha(M_L) = M_L^{[\mathcal{E}n]}$ where $L^{[\mathcal{E}n]}$ is considered as a $\mathcal{G}_n$-equivariant line bundle with the natural linearization. Under $\Phi$ the autoequivalence $M_{L^{[\mathcal{E}n]}}$ corresponds to $M_{D_L} \in \Aut(D^{b}(X^{[n]}))$ where $D_L \in \Pic X^{[n]}$ is the line bundle $D_L := \mu^{\ast}((L^{[\mathcal{E}n]})_{\mathcal{G}_n})$ (see [Knu12 lemma 9.2]).
Lemma 5.4.  \(\text{(i) For every automorphism } \varphi \in \text{Aut}(X) \text{ we have } b \circ \alpha(\varphi) = \alpha(\varphi) \circ b \text{ and for } n = 2 \text{ also } \sqrt{b} \circ \alpha(\varphi) = \alpha(\varphi) \circ \sqrt{b}.\)  
(ii) For every line bundle \(L \in \text{Pic}(X)\) we have \(b \circ \alpha(M_L) = \alpha(M_L) \circ b\) and for \(n = 2\) also \(\sqrt{b} \circ \alpha(M_L) = \alpha(M_L) \circ \sqrt{b}\).

Proof. We have \(\alpha(\varphi) \circ F \simeq F \circ \varphi\) and \(\alpha(M_L) \circ F \simeq F \circ M_L^n\). The assertions now follow by lemma 2.3 (for \(\sqrt{b}\) one has to use the analogous result \[Ann07\] proposition 2) for spherical twists.

Let \(G \subset \text{Aut}(D^b(X^{[n]}))\) be the subgroup generated by \(b\), shifts, and \(\alpha(\text{DAut}_{st}(X))\).

Proposition 5.5. The map  
\[S : \mathbb{Z} \times \mathbb{Z} \times (\text{Aut}(X) \times \text{Pic}(X)) \rightarrow \text{Aut}(D^b_{\text{FN}}(X^n))\] defines a group isomorphism onto \(G\).

Proof. By the previous lemma, \(b\) indeed commutes with \(\alpha(\Psi)\) for \(\Psi \in \text{DAut}_{st}(X)\). Together with theorem 5.2 and the fact that shifts commute with every derived automorphism, this shows that \(S\) is indeed a well-defined group homomorphism with image \(G\). Now consider \(g = b^k \circ [\ell] \circ \alpha(\varphi) \circ \alpha(M_L)\) and assume \(g = \text{id}\). For every point \([\xi] \in X^{[n]} \setminus X^n\) we have \(g(\mathcal{C}([\xi])) = \mathcal{C}([\varphi([\xi])][\ell]]\) which shows \(\ell = 0\) and \(\varphi = \text{id}\), i.e. \(g = b^k \circ M_{[\xi]}\). Hence, for \(A \in D^b(X)\) its image under \(F\) gets mapped to \(g(FA) = F(A \otimes [k(2n - 2)])\) for some line bundle \(N\) on \(X\), which shows that \(k = 0\). Finally, \(g = M_{[\xi]}\) is trivial only if \(L = O_X\).

Remark 5.6. Again, the analogous statement with \(b\) replaced by \(\sqrt{a}\) holds.

Let now \(X\) be a K3-surface. In this case Addington has shown in \[Add11\] that the Fourier–Mukai transform \(F_{X_\Xi} : D^b(X) \rightarrow D^b(X^{[n]})\) with kernel the universal sheaf \(\mathcal{I}_\Xi\) is a \(\mathbb{P}^{n-1}\) functor with \(H = [-2]\). Here, \(\Xi \subset X \times X^{[n]}\) is the universal family of length \(n\) subschemes. We denote the associated \(\mathbb{P}^{n-1}\)-twist by \(a\) and in case \(n = 2\) the spherical twist by \(\sqrt{a}\).

Lemma 5.7. For every point \([\xi] \in X^{[n]} \setminus \partial X^{[n]}\), i.e. \(\xi = \{x_1, \ldots, x_n\}\) red with pairwise distinct \(x_i\), the object \(\alpha(\mathcal{C}([\xi]))\) is supported on the whole \(X^{[n]}\). In case \(n = 2\) the same holds for the object \(\sqrt{\alpha(\mathcal{C}([\xi]))}\).

Proof. We set for short \(A = \mathcal{C}([\xi])\). We will use the exact triangle of Fourier–Mukai transforms \(F \rightarrow F' \rightarrow F''\) with kernels \(\mathcal{P}' = O_{X \times X^{[n]}}\) and \(\mathcal{P}' = O_{\Xi}\). The right adjoints form the exact triangle \(R'' \rightarrow R' \rightarrow R\) with kernels \(\mathcal{Q}' = O_{\Xi}^{[2]}\) and \(\mathcal{Q}' = O_{X \times X^{[n]}}[2]\). Over the open subset \(X^{[n]} \setminus \partial X^{[n]}\), the universal family \(\Xi\) is smooth and thus on \(\Xi \times X^{[n]}\) the object \(O_{\Xi}^{[2]}\) is a line bundle concentrated in degree \(2\). This yields
\[R''(A) = O_\xi[0], \quad R'(A) = H^*(X^{[n]}, A) \otimes O_X[2] = O_X[2].\]

Setting \(H^i = \mathcal{H}^i(R(A))\) the long exact cohomology sequence gives \(H^{-2} = O_X, H^{-1} = O_\xi, \text{ and } H^0 = 0\) for all other values of \(i\). The only functor in the composition \(F = \text{pr}_{X^{[n]}}(\text{pr}_{X^{[n]}}(\mathcal{I}_\Xi))\) that needs to be derived is the push-forward along \(\text{pr}_{X^{[n]}}\). The reason is that the non-derived functors \(\text{pr}_{X^{[n]}}\) as well as \(\text{pr}_{X^{[n]}} \otimes O_\Xi\) are exact (see \[Sca09\] proposition 2.3) for the latter. Thus, there is the spectral sequence
\[E_2^{p,q} = \mathcal{H}^p(F(H^q)) \Rightarrow E_\infty^n = \mathcal{H}^n(FR(A))\]
associated to the derived functor \(\text{pr}_{X^{[n]}}\). It is zero outside of the \(-1\) and \(-2\) row. Now \(F'(O_\xi) = H^*(X, O_\xi) \otimes O_X^{[n]} = O_X^{[n]}[0]\) and \(F''(O_\xi)\) is also concentrated in degree zero since
\( \Xi \) is finite over \( X[n] \). By the long exact sequence we see that all terms in the \(-1\) row except for \( E^{-1}_2 \) and \( E^{-1}_2 \) must vanish. Furthermore,

\[
F^*(H^{-2}) = H^*(X, O_X) \otimes O_{X[n]} = O_{X[n]}[0] \oplus O_{X[n]}[-2]
\]

and \( F^*(H^{-2}) \) is a locally free sheaf of rank \( n \) concentrated in degree zero since \( \Xi \) is flat of degree \( n \) over \( X[n] \). This shows that the \(-2\) row of \( E_2 \) is zero outside of degree 0, 1, and 2 and that \( E^{-1}_2 \) is of positive rank. By the positioning of the non-zero terms it follows that \( E^{-1}_2 = E^{-2}_0 \) and thus also \( E^{-1} = \mathcal{H}^{-1}(FR(A)) \) is of positive rank. Furthermore, we can read off the spectral sequence that the cohomology of \( FR(A) \) is concentrated in the degrees \(-2, -1, \) and 0. Now, by the long exact sequences associated to the cones occurring in the definition of the spherical respectively the \( \mathbb{P}^n \)-twist it follows that \( \mathcal{H}^{-2}(\sqrt{a}(A)) \) as well as \( \mathcal{H}^{-2}(a(A)) \) are of positive rank.

**Proposition 5.8.**
(i) The subgroup \( H \) generated by \( a \) and push-forwards along natural automorphisms, i.e. autoequivalences of the form \( \varphi^*_{\alpha} = \alpha(\varphi_{\alpha}) \), is isomorphic to \( \mathbb{Z} \times \text{Aut}(X) \).
(ii) \( b \notin H = \langle a, \{ \varphi^{[n]}_{\alpha} \}_{\varphi \in \text{Aut}(X)} \rangle \).
(iii) \( a \notin G = \langle b, [\ell], \alpha \{ \text{DAut}_{st}(X) \} \rangle \).

The same results hold for \( a \) replaced by \( \sqrt{a} \) and \( b \) replaced by \( \sqrt{b} \).

**Proof.** We have for \( \varphi \in \text{Aut}(X) \) that \( \varphi^{[n]}_{\alpha} \circ F_a = F_a \circ \varphi_{\alpha} \) which by lemma 2.3 shows that \( a \) commutes with \( \varphi^{[n]}_{\alpha} \). The reason is that the subvariety \( \Xi \subset X \times X[n] \) is invariant under the morphism \( \varphi \times \varphi^{[n]} \). Because of \( a^k \circ \varphi^{[n]}_{\alpha}(F_a(A)) = F_a(\varphi^{[n]}_{\alpha}A)[k(2(n-1))] \) for \( A \in \mathcal{D}^b(X) \), there are no further relations in the group \( H \) which shows (i). The autoequivalence \( g = a^k \circ \varphi^{[n]}_{\alpha} \in H \) has \( g(F(\mathcal{O}_X)) = F(\mathcal{O}_X)[k(2(n-1))] \). Thus, by remark 2.6 the equality \( b = g \) implies \( k = 1 \). But also \( b = a \circ \varphi^{[n]}_{\alpha} \) can not hold comparing the values of both sides on \( \mathcal{O}(\{ \xi \}) \) for \( \{ \xi \} \in X[n] \setminus \partial X[n] \). The assertion (iii) also is shown by comparing the values of the autoequivalences on \( \mathcal{O}(\{ \xi \}) \).

Using the same arguments as in [Add11, p.11-12 and p.39-40] one can also show that \( b \) does not equal a shift of an autoequivalence induced by a \( \mathbb{P}^n \)-object on \( X[n] \) or of an autoequivalence of the form \( \alpha(T_E) \) for a spherical twist \( T_E \) on the surface. In particular, \( b \) is an exotic autoequivalence in the sense of [PST12].

### 6. \( \mathbb{P}^n \)-objects on generalised Kummer varieties

Let \( A \) be an abelian surface. There is the summation map

\[
\Sigma : A^n \rightarrow A \quad , \quad (a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i.
\]

We set \( N_{n-1}A := \Sigma^{-1}(0) \). It is isomorphic to \( A^{n-1} \) via e.g. the morphism

\[
A^{n-1} \rightarrow N_{n-1}A \quad , \quad (a_1, \ldots, a_{n-1}) \mapsto (a_1, \ldots, a_{n-1}, -\sum_{i=1}^{n-1} a_i).
\]

The subvariety \( N_{n-1}A \subset A^n \) is \( G_n \)-invariant. Thus, we have \( N_{n-1}A/G_n \subset S^nA \). The *generalized Kummer variety* is defined as \( K_{n-1}A := \mu^{-1}(N_{n-1}A/G_n) \), i.e. it is the subvariety of the Hilbert scheme \( A^{[n]} \) consisting of all points representing subschemes whose weighted support
Proposition 6.3. For every 2-torsion point \( a \) of \( A \) the skyscraper sheaf \( \mathcal{C}(\delta(a)) \) is a \( \mathbb{P}^n \)-object in \( \text{D}^b_\mathcal{S}_n(N_{n-1}A) \).

Proof. Indeed, using the results for the invariants of \( ^\wedge N_{\Delta/A^n} \) of section 3.

\[
\text{Hom}^*_{\text{D}^b_\mathcal{S}_n}(\mathcal{C}(\delta(a)), \mathcal{C}(\delta(a))) \cong \text{Ext}^*(\mathcal{C}(\delta(a)), \mathcal{C}(\delta(a)))^{\mathcal{S}_n} \cong ^\wedge N_{\Delta/A^n}(\delta(a))^{^\wedge N_{\Delta/A^n}(\mathcal{S}_n)} \\
\cong ^\wedge N_{\Delta/A^n}(\delta(a))^{^\wedge (\mathcal{S}_n)} \\
\cong \mathbb{C} \oplus \mathbb{C}[-2] \oplus \cdots \oplus \mathbb{C}[-2n].
\]

Remark 6.2. For two different \( n \)-torsion points the skyscraper sheaves are orthogonal which makes the associated twists commute. Thus, we have an inclusion (see corollary 2.4)

\[
\mathbb{Z}^{n^2} \subset \text{Aut}(\text{D}^b_\mathcal{S}_n(N_{n-1}A)) \cong \text{Aut}(\text{D}^b(K_{n-1}A)).
\]

In the case \( n = 2 \) the generalised Kummer variety \( K_{n-1}A = K_1A \) is just the Kummer surface \( K(A) \). Moreover, there is an isomorphism of commutative diagrams

\[
\begin{array}{ccc}
\mathbb{P}^nA & \xrightarrow{p} & N_1A & \xrightarrow{\bar{\pi}} & A \\
\downarrow{\cong} & & \downarrow{\pi} & & \downarrow{\pi} \\
K_1A & \xrightarrow{\mu} & N_1A/\mathcal{S}_2 & K(A) & \xrightarrow{\mu} A/\mathcal{S}_2
\end{array}
\]

where \( p \) and \( \mu \) in the right-hand diagram are the blow-ups of the 16 different 2-torsion points respectively of their image under the quotient under the involution \( \iota = (-1) \). For a 2-torsion point \( a \in A_2 \) we denote by \( E(a) \) the exceptional divisor over the point \([a] \in A/\mathcal{S}_2 \). Since \( E(a) \) is a rational curve in the \( K3 \)-surface \( K(A) \), every line bundle on it is a spherical object in \( \text{D}^b(K(A)) \).

Proposition 6.3. For every 2-torsion point \( \Phi^{-1}(\mathcal{C}(\delta(a))) = \mathcal{O}_{E(a)}(-1) \) holds.

Proof. Using the isomorphism of the commutative diagrams above the proof is nearly the same as the proof of proposition 4.2.

There is no known homomorphism \( \text{Aut}(\text{D}^b(A)) \to \text{Aut}(\text{D}^b(K_{n-1}A)) \) analogous to Ploog’s map \( \alpha \). But at least one can lift line bundles \( L \in \text{Pic} A \) (by restricting \( D_L \)) and group automorphisms \( \varphi \in \text{Aut}(A) \) (by restricting \( \varphi^{[n]} \)) to the generalised Kummer variety. Recently,
Meachan has shown in [Mea12] that the restriction of Addington’s functor to the generalised Kummer variety $K_n(A)$ for $n \geq 2$ (i.e. the Fourier–Mukai transform with kernel the universal sheaf) is still a $\mathbb{P}^{n-1}$ functor and thus yields an autoequivalence $\bar{a}$. Comparing these autoequivalences with those induced by the above $\mathbb{P}^n$-objects one gets results similar to the results of section 5.

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Universität Bonn
E-mail address: akrug@math.uni-bonn.de