Fractional Supersymmetry and Infinite Dimensional Lie Algebras

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In an earlier work extensions of supersymmetry and super Lie algebras were constructed consistently starting from any representation \( D \) of any Lie algebra \( g \). Here it is shown how infinite dimensional Lie algebras appear naturally within the framework of fractional supersymmetry. Using a differential realization of \( g \) this infinite dimensional Lie algebra, containing the Lie algebra \( g \) as a sub-algebra, is explicitly constructed.

1. Introduction

It is a pleasure to write this contribution in memory of the 75th anniversary of D. V. Volkov. Among other things he was a pioneer both in supersymmetry and alternative statistics [1]. This paper can be seen as an attempt to combine those two concepts in a sense specified later on. Symmetry is always a powerful tool to understand properties of systems. Space-time symmetry is central in particle physics. When studying possible symmetries of space-time, two theorems put severe constraints on allowed space-time symmetries: the Coleman and Mandula [2] and Haag, Lopuszanski and Sohnius [3] theorems. They lead to gauge theories and/or supersymmetry. However, for those types of theories bosonic or fermionic representations of the Lorentz group are introduced. So, if one imagine symmetries acting on representations which are neither bosonic nor fermionic, one is, in principle able to define symmetries generalizing supersymmetry. Several possibilities have been considered in the literature [4]–[16], the intuitive idea being that the generators of the Poincaré algebra are obtained as an appropriate product of more fundamental additional symmetries. These new generators are in a representation of the Lorentz algebra which can be neither bosonic nor fermionic (bosonic charges close under commutators and generate a Lie algebra, whilst fermionic charges close under anticommutators and induce super-Lie algebras). Here, we are interested in one possible extension of supersymmetry named fractional supersymmetry.

2. \( F \)–Lie algebras

The natural mathematical structure, generalizing the concept of super-Lie algebras and relevant for the algebraic description of fractional supersymmetry has been introduced in [15] and was called fractional super-Lie algebra (\( F \)–Lie algebra for short). Here, we do not want to go into the detailed definition of this structure but only recall the basic points, useful for our purpose. More details can be obtained in [15]. Basically, an \( F \)–Lie algebra is a \( \mathbb{Z}_F \)–graded vector space,

\[
S = B \oplus A_1 \oplus \cdots \oplus A_{F-1},
\]

which endows the following structure:

1. \( B \) is the graded zero part of \( S \) and is a Lie algebra.

2. All the graded part of \( S, A_i, i = 1, \cdots, F-1 \) (of grade \( i \)) are appropriate representations of \( B \) (in general of infinite dimension \( i.e. \) a Verma module).

3. Denoting \( S^F(D) \) the \( F \)–fold symmetric product of \( D \), there are multilinear, \( B \)–equivariant \( i.e. \) which respect the action of \( B \) maps \( \{ \cdots \} \) from \( S^F(A_k) \) into \( B \). In other words, we assume that
some of the elements of the Lie algebra \( B \) can be expressed as \( F \)–th order symmetric products of “more fundamental generators”.

Of course, to ensure the coherence of this algebraic structure more is needed (Jacobi identities, unitarity, representations, etc.) as can be seen in [13]. From the above definition of \( F \)–Lie algebras, firstly, we observe that no relation between different graded sectors is postulated. Secondly, the sub-space \( B \oplus A_1 \subset S \) satisfies all the conditions of \( F \)–Lie algebras.

3. Fractional supersymmetry

Along the lines of \( F \)–Lie algebras it has then been shown that it is possible to define fractional supersymmetry (FSUSY) starting from any Lie algebra \( \mathfrak{g} \) and any representation \( D \). The basic idea is the following. Let \( \mathfrak{g} \) be a rank \( r \) semi-simple Lie algebra and denote \( \alpha_i \) the set of primitive roots and \( \mu_i \) the set of fundamental weight vectors (2 \( n_i \alpha_i \beta_i = \delta_{ij} \)). If one considers \( B = \mathfrak{g} \oplus D_\mu \) (\( D_\mu \) being the representation associated to the primitive vector \( |\mu \rangle \) with weight \( \mu = \sum_{i=1}^{r} n_i \mu_i, n_i \in \mathbb{N} \)–highest weight representation) as the bosonic part of the \( F \)–Lie algebra (as a semi-direct product)

\[
[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}, \quad [\mathfrak{g}, D_\mu] \subset D_\mu, \quad [D_\mu, D_\mu] = 0,
\]

(2)

one possible solution for the grade one sector is to consider the infinite dimensional representation \( D_{\mu/F} \), with highest weight of weight \( \mu/F = \sum_{i=1}^{r} n_i \mu_i, n_i \in \mathbb{N} \). The action of \( \mathfrak{g} \) on \( D_{\mu/F} \) is extended as follow

\[
[\mathfrak{g}, D_{\mu/F}] \subset D_{\mu/F}, \quad [D_{\mu/F}, D_{\mu/F}] = 0.
\]

(3)

The representation \( D_{\mu/F} \) is in general infinite dimensional and non-unitary and is called a Verma module [17]. In an earlier paper an FSUSY associated to \( \mathfrak{g}, D_\mu \) and \( D_{\mu/F} \) was constructed in an abstract way [13]. It was then noticed that for any Lie algebras it is possible to construct a differential realization and to obtain any representation in terms of appropriate homogeneous monomials [16]. In such a realization FSUSY can be explicitly obtained by action on appropriate homogeneous monomials. Roughly speaking, in FSUSY the representations \( D_\mu \) and \( D_{\mu/F} \) can be related

\[
\mathcal{S}^F (D_{\mu/F}) \sim D_\mu,
\]

(4)

but the main difficulty, in such a construction, is connected to the requirement of relating an infinite dimensional representation \( D_{\mu/F} \) to a finite dimensional one \( D_\mu \) in an equivariant way, i.e. respecting the action of \( \mathfrak{g} \). One possible way of solving this contradiction is to embed \( D_\mu \) into an infinite dimensional (reducible but indecomposable) representation [13]. Another possibility is to embed \( \mathfrak{g} \) into a infinite dimensional algebra dubbed \( V(\mathfrak{g}) \) [15].

4. Fractional supersymmetry and \( \mathfrak{sl}(3, \mathbb{R}) \)

Here, our purpose is to analyze these two points in more details and in particular to construct explicitly the infinite dimensional algebra \( V(\mathfrak{g}) \supset \mathfrak{g} \). The first point was already analyzed in [16] and the differential realizations was the main issue in [16]. Indeed, one additional advantage of the differential realizations given in [16] is to provide the main tool to obtain \( V(\mathfrak{g}) \) from \( \mathfrak{g} \). Up to now, the solution has been obtained only for \( \mathfrak{g} = \mathfrak{sl}(n, \mathbb{R}) \) (the other series are under investigation). Furthermore, the formulas are generic for all \( \mathfrak{sl}(n, \mathbb{R}) \) so we may just focus here on \( \mathfrak{g} = \mathfrak{sl}(3, \mathbb{R}) \).

4.1. Fractional supersymmetry and finite dimensional algebras

Let \( \alpha, \beta \) be the simple roots of \( \mathfrak{sl}(3, \mathbb{R}) \), \( \gamma = \alpha + \beta \) the third positive root and \( \mu_1 \) and \( \mu_2 \) the fundamental weights. Introduce, \( x_1, x_2, x_3 > 0 \) of weight \( |\mu_1| = x_1, |\mu_1 - \alpha| = x_2, |\mu_1 - \alpha - \beta| = x_3 \). This means that \( D_{\mu_1} =< x_1, x_2, x_3 > \) is just the three dimensional fundamental representation. From the definition of this representation and from its explicit construction it follows that the \( \mathfrak{sl}(3, \mathbb{R}) \) generators take the form (the normalization taken here is not conventional, but is most useful for the sequel).
$$E^α = x_1 \partial x_1, \quad E^{-α} = x_2 \partial x_1,$$
$$E^β = x_2 \partial x_2, \quad E^{-β} = x_3 \partial x_2,$$
$$E^γ = x_3 \partial x_3, \quad E^{-γ} = x_3 \partial x_1,$$
$$h_1 = x_1 \partial x_1 - x_2 \partial x_2,$$
$$h_2 = x_2 \partial x_2 - x_3 \partial x_3. \quad (5)$$

Following the results established in [16] all representations of $\mathfrak{sl}(3, \mathbb{R})$ can then be obtained from the fundamental representation and consequently we only need one differential realization to obtain all the representations of $\mathfrak{sl}(3, \mathbb{R})$. Indeed, the highest weight of the representation $D_{p_1 μ_1 + p_2 μ_2}$ is simply $|p_1 μ_1 + p_2 μ_2 > = (x_1)^{p_1} (x_1 \land x_2)^{p_2}$ because (i) $E^{α, β}|p_1 μ_1 + p_2 μ_2 > = 0$ and (ii) $h_1|p_1 μ_1 + p_2 μ_2 > = p_1 |p_1 μ_1 + p_2 μ_2 >$. The representation $D_{p_1 μ_1 + p_2 μ_2}$ is then obtained from the action on $|p_1 μ_1 + p_2 μ_2 > = (x_1)^{p_1} (x_1 ∙ x_2)^{p_2}$ of $E^α, E^β$, $E^γ$ given in (6). A direct calculation shows that when $p_1, p_2$ are integer the representation is finite dimensional, and the operators $E^{±α}, E^{±β}, E^{±γ}$ are nilpotent. However, when at least one of the $p_i$ is not an integer the representation is infinite dimensional. As shown in [16] this procedure can be equally applied for any Lie algebras.

The aim is now to construct an $F$–Lie algebra associated to the three dimensional representation $D_{μ_1}$ (this could have been done for any representation of $\mathfrak{sl}(3, \mathbb{R})$). In the realization (5) the vectorial representation writes

$$D_{μ_1} = \left\{ \begin{array}{l}
x_1 = |μ_1 >, \\
x_2 = |μ_1 - α > = E^{-α}|μ_1 >, \\
x_3 = |μ_1 - α - β > = E^{-β}E^{-α}|μ_1 >.
\end{array} \right. \quad (6)$$

Hence,

$$B = \mathfrak{sl}(3, \mathbb{R}) ⊕ D_{μ_1} \quad (7)$$

is considered for the bosonic (graded zero part) of the $F$–Lie algebra. The natural representation to define the “$F^{th}$–root” of $D_{μ_1}$ is $D_{μ_1/F}$. So, the graded one part is taken as

$$A_1 = D_{μ_1}. \quad (8)$$

The primitive vector of this representation is given by $x_1^{1/F}$. In the differential realization (5), this representation, acting explicitly with the operators $E^α, E^β$ on $x_1^{1/F} = |\frac{1}{F} >$ is easily obtained

$$D_{\frac{1}{F}} = \left\{ \frac{p_1}{F} - nα - pβ > =
\right.$$

$$(x_1)^{1/F} \left( \frac{x_2}{x_1} \right)^n \left( \frac{x_3}{x_2} \right)^p \quad (9)$$

$$\sim (E^{-β})^p (E^{-α})^n \left( \frac{p_1}{F} >, n ∈ \mathbb{N}, 0 ≤ p ≤ n \right),$$

leading to the weight diagram given in figure 1.

Next to define an $F$–Lie algebra associated to $\mathfrak{sl}(3, \mathbb{R}) ⊕ D_{μ_1} ⊕ D_{μ_1/F}$ the following representation (reducible) is considered

$$S^F \left( D_{μ_1/F} \right) = \left\{ \bigotimes^F_{i=1} (x_1)^{1/F} \left( \frac{x_2}{x_1} \right)^n \bigotimes \left( \frac{x_3}{x_2} \right)^{p_i}, \ n_i ∈ \mathbb{N}, 0 ≤ p_i ≤ n_i \right\}, \quad (10)$$

with $\bigotimes$ the symmetric tensorial product.

Now, comparing $x_1$, the primitive vector of $D_{μ_1}$, with $\bigotimes^F (x_1)^{1/F} ∈ S^F \left( D_{μ_1/F} \right)$ it is seen that these two vectors, as primitive vectors, satisfy the same properties

$$h_1(x_1) = x_1, \quad h_1 \left( \bigotimes^F (x_1)^{\frac{1}{F}} \right) = \bigotimes^F (x_1)^{\frac{1}{F}},$$
$$h_2(x_1) = 0, \quad h_2 \left( \bigotimes^F (x_1)^{\frac{1}{F}} \right) = 0,$$
$$E^{α, β, γ}(x_1) = 0, \quad E^{α, β, γ} \left( \bigotimes^F (x_1)^{\frac{1}{F}} \right) = 0. \quad (11)$$

But now, constructing the representation from these primitive vectors, one obtains, on the one hand $D_{μ_1}$, and on the other hand, the infinite dimensional representation

$$\left\{ \bigotimes^F (x_1)^{1/F} \right\} = \left\{ |μ_1 - nα - pβ > = \right.$$

$$(E^{-β})^p (E^{-α})^n \left( \bigotimes^F (x_1)^{1/F} \right), n ∈ \mathbb{N}, 0 ≤ p ≤ n \right\}. \quad (12)$$
However, a direct calculation shows that the following relations hold
\[ E^\alpha |\mu_1 - 2\alpha > 0 \]
\[ E^\alpha |\mu_1 - 2\alpha - \beta > 0 \]
\[ E^\gamma |\mu_1 - 2\alpha - 2\beta > 0. \] (13)

It means that \( Y_{\mu_1} = \langle \otimes^F (x_1)^{1/F} \rangle \) is a Verma module (the operator \( E^\alpha \) is not nilpotent). The relation with the finite dimensional representation \( D_\mu \) is the following isomorphism
\[ D_\mu \cong \langle \otimes^F (x_1)^{1/F} \rangle / M_\mu \] (14)

with \( M_\mu = \{ (E^{-\beta})^p (E^{-\alpha})^n (\otimes^F (x_1)^{1/F}), n \in \mathbb{N}, 0 \leq p \leq n, (n, p) \neq (0, 0), (1, 0), (1, 1) \} \)
the maximal proper sub-representation of \( Y_{\mu_1} \), \( \forall t \in \mathfrak{sl}(3, \mathbb{R}), \forall m \in M_\mu, t(m) \in M_\mu \) (see [14] for more details).

Thus, to obtain an \( F \)-Lie algebra some constraints have to be introduced. Following [15],

\( F \) is defined as the vector space of functions on \( x_1, x_2, x_3 > 0 \). The multiplication map \( m_n : F \times \cdots \times F \rightarrow F \) given by \( m_n(f_1, \cdots, f_n) = f_1 \cdots f_n \) is multilinear and totally symmetric. Hence, it induces a map \( \mu_F \) from \( S^F (\mathcal{F}) \) into \( \mathcal{F} \). Restricting to \( S^F (\mathcal{D}_\mu) \) one gets \( \mu_F : S^F (\mathcal{D}_\mu) \rightarrow S^F (\mathcal{D}_\mu) \)

\[ \mu_F \left( \otimes_{i=1}^F (x_1)^{1/F} \left( \frac{x_2}{x_1} \right)^{n_i} \left( \frac{x_3}{x_2} \right)^{p_i} \right) \]
\[ = x_1 \left( \frac{x_2}{x_1} \right)^{\sum_{i=1}^n n_i} \left( \frac{x_3}{x_2} \right)^{\sum_{i=1}^p p_i}. \] (15)

It is easily seen that \( S^F_{\text{red}} (\mathcal{D}_\mu) = \{ x_1 (\frac{x_2}{x_1})^n (\frac{x_3}{x_2})^p, n \in \mathbb{N}, 0 \leq p \leq n \} \supset D_\mu \). This representation is reducible but indecomposable. Namely, a complement of \( D_\mu \) in \( S^F (\mathcal{D}_\mu) \), stable under \( \mathfrak{sl}(3, \mathbb{R}) \) cannot be
fulfilling the commutation relations \[ \{ E^\alpha, E^\beta \} = \pm E^{\alpha + \beta}, \quad [E^\alpha, E^{-\alpha}] = 2h_\alpha. \]

\( (sl(3, \mathbb{R}) \oplus S^F \{ D_{\mu} \})_{\text{red}} \oplus D_{\mu} \)

\( \supset (sl(3, \mathbb{R}) \oplus D_{\mu}) \oplus D_{\mu} \) (16)

has a structure of \( F \)-Lie algebras. See [15] and [16] for more details.

4.2. Fractional supersymmetry and infinite dimensional algebras

As mentioned there is another way to define an \( F \)-Lie algebra, extending \( g \) into an infinite dimensional Lie algebra. To define this algebraic structure, the key observations are as follow:

(i) For any Lie algebra \( g \) (complexified) and to any positive roots of \( g \) there is a natural \( sl(2, \mathbb{R}) \) sub-algebra generated by \( \{ E^{\pm\alpha}, h_\alpha \} \) and fulfilling the commutation relations \( [h_\alpha, E^{\pm\alpha}] = \pm E^{\pm\alpha}, \quad [E^\alpha, E^{-\alpha}] = 2h_\alpha. \)

(ii) The centerless Virasoro algebra (generated by \( L_n, n \in \mathbb{N} \) with \( [L_n, L_m] = (n-m)L_{n+m} \)) contains an \( sl(2, \mathbb{R}) \) sub-algebra, generated by \( \{ L_{\pm 1}, L_0 \}. \)

(iii) Moreover, for all \( n > 0 \) the generators \( \{ L_{\pm n}, L_0 \} \) of the Virasoro algebra generate a \( sl(2, \mathbb{R}) \) sub-algebra.

Putting (i) and (ii-iii) together, the definition of \( V(g) \) is given, as a possible generalization of the centerless Virasoro algebra (de Witt algebra). This construction, proceeds in two steps. Firstly a partial Lie algebra is defined. Secondly, calculating all the commutators of all the generators defined in the first step, the algebra is closed leading to \( V(g) \).

4.2.1. The partial Lie algebra

Having (i-iii) in mind, the following is assumed, in order to obtain a possible extension of the Virasoro (centerless) algebra

- For any positive roots \( \alpha \) there is a Virasoro algebra, namely generators of weight \( n\alpha, n \in \mathbb{Z} \).

- For all \( n > 0 \) they are generators of weights \( \pm n\alpha, \pm n\beta, \pm n\gamma \) closing, with \( h_1, h_2 \) through an \( sl(3, \mathbb{R}) \) algebra.

Such an algebra was defined in a formal and universal way in [15]. But now, the relevance of the variables \( x_1, x_2, x_3 \) is clear, in the sense that \( \left( \frac{x_i}{x_3} \right)^n \) is of weight \( n\alpha \) and \( \left( \frac{x_i}{x_3} \right)^n \) of weight \( n\beta \) and contain all the informations to construct generators of appropriate weight. It implies that the only generators of weight \( n\alpha \) are \( \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i} \), of weight \( n\beta \) are \( \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i} \), and of weight \( n\gamma \) are \( \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i} \), with \( i = 1, 2, 3 \). (Strictly speaking \( (x_1, x_2, x_3) \) being of weight zero, more generators can be defined.) Finally, introducing (for \( n > 0 \))

\[ E^{\alpha} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad E^{-\alpha} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad E^{\beta} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad E^{-\beta} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad E^{\gamma} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad E^{-\gamma} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \]

\[ \tilde{E}^{\alpha} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad \tilde{E}^{-\alpha} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad \tilde{E}^{\beta} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad \tilde{E}^{-\beta} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad \tilde{E}^{\gamma} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \quad \tilde{E}^{-\gamma} = \left( \frac{x_i}{x_3} \right)^n x_i \partial_{x_i}, \]

a direct and simple calculation shows that these generators (with \( h_1, h_2 \) (see [15])) and \( T = x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} \), endow the following structure:

1. For all \( n > 0 \), \( \{ E^{\pm n\alpha}, E^{\pm n\beta}, E^{\pm n\gamma}, h_1, h_2 \} \) and \( \{ \tilde{E}^{\pm n\alpha}, \tilde{E}^{\pm n\beta}, \tilde{E}^{\pm n\gamma}, h_1, h_2 \} \) generate an \( sl(3, \mathbb{R}) \) algebra.
2. For all positive root \( \varphi = \alpha, \beta, \gamma, \{E_{-n\varphi}, \tilde{E}_{n\varphi}, n \in \mathbb{N}, \frac{1}{2}[\tilde{E}_\varphi, E_{-\varphi}] \} \) and \( \{\tilde{E}_{-n\varphi}, E_{n\varphi}, n \in \mathbb{N}, \frac{1}{2}[E_{-\varphi}, \tilde{E}_\varphi] \} \) generate a Virasoro algebra.

It means that the partial Lie algebra so-constructed admits a concentric and a radial symmetry. Concentric symmetries are related to \( \mathfrak{sl}(3, \mathbb{R}) \)'s sub-algebras and radial ones to Virasoro's sub-algebras. All this is summarized in Figure 2.

### 4.2.2. The algebra \( V[\mathfrak{sl}(3, \mathbb{R})] \)

With the above introduced generators of the partial Lie algebra given in (17) and upon calculating the commutators that are neither on the same circle, nor on the same radial, the generators of \( V[\mathfrak{sl}(3, \mathbb{R})] \) are obtained

\[
X_1^{n,m} = \left( \frac{x_1}{x_2} \right)^n \left( \frac{x_2}{x_3} \right)^m x_1 \partial_{x_1},
\]
\[
X_2^{n,m} = \left( \frac{x_1}{x_2} \right)^n \left( \frac{x_2}{x_3} \right)^m x_2 \partial_{x_2},
\]
\[
X_3^{n,m} = \left( \frac{x_1}{x_2} \right)^n \left( \frac{x_2}{x_3} \right)^m x_3 \partial_{x_3}.
\]

Denoting the generators in a generic way \( X_i^{n,m} = (x_1)^{p_i} (x_2)^{p_2} (x_3)^{p_3} \partial_{x_i}, \) with \( p_1 = n, p_2 = m - n, p_3 = -m \) \( (p_1 + p_2 + p_3 = 0), \) the commutation relations take the form

\[
[X_1^{n,m}, X_2^{n',m'}] = p'_1 X_3^{n+n',m+m'} - p_2 X_1^{n+n',m+m'},
\]
\[
(19)
\]

It can be noticed that this algebra admits a Lie algebra automorphism. Indeed, a direct calculation shows that if \( X_i^{n,m} \rightarrow \rho(X_i^{n,m}) \) such that \( \rho(X_1^{n,m} + X_2^{n,m} + X_3^{n,m}) = 0, \) the commutation relations (19) remains unchanged. The interpretation of the obtained algebra, as well as of the automorphism \( \rho \) is left for a future publication. Notice only that the algebra \( V[\mathfrak{sl}(2, \mathbb{R})]/\text{Ker} \rho \) is just the Virasoro algebra and that the weight diagram of \( V[\mathfrak{sl}(3, \mathbb{R})] \) covers all points \( n\alpha + m\beta, n, m \in \mathbb{Z} \) of the weight lattice with a degeneracy of three.

From the definition of the Lie algebra \( V[\mathfrak{sl}(3, \mathbb{R})] \) it is not difficult to extend the representations (18) and starting with (18) and (19), one gets

\[
\hat{D}_\mu = \left\{ |\mu_1 + r\alpha + s\beta > = \right\}
\]
\[
\hat{D}_\mu = \left\{ \frac{\mu_1}{F} + r\alpha + s\beta > = \right\}
\]
\[
(20)
\]

and

\[
X_1^{n,m} |\mu_1 + r\alpha + s\beta > = (1 + r)|\mu_1 + (r + n)\alpha + (s + m)\beta >
\]
\[
X_2^{n,m} |\mu_1 + r\alpha + s\beta > = (s - r)|\mu_1 + (r + n)\alpha + (s + m)\beta >
\]
\[
X_3^{n,m} |\mu_1 + r\alpha + s\beta > = -s|\mu_1 + (r + n)\alpha + (s + m)\beta >
\]
\[
(21)
\]

It may be seen that the representations given in (18) and (19) are included in their corresponding representations given in (20). It should also be noted that these two representations are not obtained from a primitive vector \( i.e. \) they are not bounded from bellow or above.

Furthermore, in each case the action of \( V[\mathfrak{sl}(3, \mathbb{R})] \) extends the action of \( \mathfrak{sl}(3, \mathbb{R}) \). The
fundamental property of these representations lies in a $V[\mathfrak{sl}(3, \mathbb{R})]$-equivariant map from $\mathcal{S}^E(\hat{\mathcal{D}}_{\alpha/2})$ into $\mathcal{D}_\mu$. This is just the multiplication map $\mu_F$ (see (15)). A direct calculation shows that

$$
\mathcal{S}^E_{\text{red}} \left( \hat{\mathcal{D}}_{\alpha/2} \right) \overset{\text{def}}{=} \mu_F \left( \mathcal{S}^E \left( \hat{\mathcal{D}}_{\alpha/2} \right) \right) \tag{22}
$$

and then $\mathcal{S}^E_{\text{red}} \left( \hat{\mathcal{D}}_{\alpha/2} \right)$ and $\mathcal{D}_\mu$ are isomorphic. Hence,

$$
\left( V[\mathfrak{sl}(3, \mathbb{R})] \oplus \mathcal{D}_\mu \right) \oplus \hat{\mathcal{D}}_{\alpha/2} \tag{23}
$$

is an $F$-Lie algebra.

5. Conclusion

To conclude, considering infinite dimensional representations, one is able to construct a theory generalizing supersymmetry and super-Lie algebras. Indeed, starting from a Lie algebra $\mathfrak{g}$ and a representation $\mathcal{D}_\mu$, the basic point is to consider the infinite dimensional representation $\mathcal{D}_{\mu/F}$. As shown this construction leads naturally to an infinite dimensional algebra $V(\mathfrak{g})$ containing $\mathfrak{g}$ as a sub-algebra. When $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ this algebra reduces to the centerless Virasoro algebra. This construction was done explicitly for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$, but such a construction works equally well for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. It should then be interesting to study this algebra per se: geometrical interpretation, central extension etc.. Indeed, it was then realized that the algebra $V(\mathfrak{sl}(3, \mathbb{R}))/\text{Ker} \rho$ is included into the algebra of vector fields on the Torus $T^2$. Let us mention that a general study is under investigation. Following the same simple principle, we are looking for the possibility to realize some Lie algebras (including Kac-Moody and hyperbolic algebras) as a sub-algebra of the vector fields on a torus. Finally, let us mention that for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ unitary representa-
tions of $(1 + 2)D$ FSUSY have been constructed \[13\]. As observed earlier in this case FSUSY is a symmetry which acts on relativistic anyons \[20\]. The interpretation of FSUSY in higher dimensional space-time is still an open question.

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