HAUSDORFF DIMENSION OF PARTICULARLY NON-NORMAL NUMBERS IN DYNAMICAL SYSTEMS FULFILLING THE SPECIFICATION PROPERTY

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Abstract. In this paper, we consider non-normal numbers occurring in dynamical systems fulfilling the specification property. It has been shown that in this case the set of non-normal numbers has measure zero. In the present paper we show that a smaller set, namely the set of particularly non-normal numbers, has full Hausdorff dimension. A particularly non-normal number is a number \( x \) such that there exist two digits, one whose limiting frequency in \( x \) exists and another one whose limiting frequency in \( x \) does not exist.

1. Introduction

Let \( N \geq 2 \) be an integer and \( A := \{0, 1, \ldots, N - 1\} \). Then for every \( x \in [0, 1) \) we denote by

\[
x = \sum_{k=1}^{\infty} d_k(x)N^{-k},
\]

where \( d_k(x) \in A \) for all \( k \geq 1 \), the unique non-terminating \( N \)-ary expansion of \( x \). For every positive integer \( n \) and a block of digits \( b = b_1 \ldots b_k \in A^k \) we write

\[
P_b(x, n) := \frac{|\{0 \leq i < n : d_{i+1}(x) = b_1, \ldots, d_{i+k}(x) = b_k\}|}{n}
\]

for the frequency of the block \( b \) among the first \( n \) digits of the \( N \)-ary expansion of \( x \).

We call a number \( k \)-normal if for every block \( b \in A^k \) of digits of length \( k \), the limit of the frequency \( \lim_{n \to \infty} P_b(x, n) \) exists and equals \( N^{-k} \). A number is called normal with respect to base \( N \) if it is \( k \)-normal for all \( k \geq 1 \). Furthermore, a number is called essentially non-normal if for all \( i \in A \), the limit \( \lim_{n \to \infty} P_i(x, n) \) does not exist, and particularly non-normal if there exists \( i, j \in A \) such that \( \lim_{n \to \infty} P_i(x, n) \) exists, but \( \lim_{n \to \infty} P_j(x, n) \) does not exist.

Already in 1909 Borel [4] showed that the set of absolutely normal numbers has Lebesgue measure 1. Recently the fractal properties of different variants of non-normal numbers have been of interest, see for example [2,14–16]. In particular, Albeverio, Prats’ovyi and Torbin [1] showed in 2005 that the set of particularly non-normal numbers is superfractal, meaning that its Lebesgue measure is 0, whereas its Hausdorff dimension is 1.

Instead of considering \( N \)-ary expansions for an integer \( N \) we might take any \( \beta > 1 \) and represent reals in \( [0, 1] \) in a similar way. These representations are called \( \beta \)-expansions and were introduced by Rényi [19]. For \( \beta > 1 \) let \( S_{\beta} : [0, 1] \to [0, 1] \) be the transformation given by

\[
S_{\beta}(x) = \beta x \mod 1
\]

and \( D_{\beta} = \{0, \ldots, [\beta]\} \) be the corresponding set of digits. Then every \( x \in [0, 1] \) admits a unique representation of the form

\[
x = \sum_{n=1}^{\infty} \frac{d_n}{\beta^n},
\]

with \( d_n = \lfloor \beta S_{\beta}^{n-1}x \rfloor \in D_{\beta} \). For \( \beta > 1 \) and \( x \in [0, 1] \) we denote by \( d_\beta(x) = d_1d_2d_3 \ldots \in D_{\beta}^\infty \) the \( \beta \)-expansion of \( x \). We note that for \( \beta = N \in \mathbb{Z} \) this yields the \( N \)-adic expansion described above.

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Let $\mathcal{L}_\beta$ denote the set of all $\beta$-expansions of elements in $[0, 1]$. A finite word (resp. a sequence) is called $\beta$-admissible if it is a factor of an element (resp. an element) of $\mathcal{L}_\beta$.

Every $x \in [0, 1]$ has a $\beta$-expansion, however, not every sequence $u \in D^\mathbb{N}_\beta$ occurs in $\mathcal{L}_\beta$. In connection with a characterization Parry [17] observed that the expansion of 1 plays a crucial role. In particular, a sequence $u \in D^\mathbb{N}_\beta$ occurs as $\beta$-expansion if and only if

1. the $\beta$-expansion of 1 is infinite, then $S^k(u) \prec_{\text{lex}} d_\beta(1)$ for all $k \geq 0$,
2. the $\beta$-expansion of 1 is finite, i.e. $d_\beta(1) = d_1d_2\ldots d_k0000\ldots$, then $S^k(u) \prec_{\text{lex}} \overline{d_1\ldots d_k}$ for all $k \geq 0$,

where $S$ denotes the shift and $\prec_{\text{lex}}$ denotes the lexicographic ordering. According to this one calls $\beta > 1$ a Parry number if $d_\beta(1)$ is eventually periodic. In this case $\beta$-expansions fulfills the so-called specification property which we will defined in the following section.

The transformation $S_\beta$ is ergodic, and the invariant measure for $S_\beta$ is given by

$$\nu_\beta(B) = \int_B \frac{1}{F(\beta)} \sum_{n \geq 0} \frac{1}{\beta^n} \, dx,$$

where

$$F(\beta) = \int_0^1 \left( \sum_{n \geq 0} \frac{1}{\beta^n} \right) \, dx$$

is the normalising constant (cf. Dajani and Kraaikamp [6]). Using this measure one can easily carry over the definitions of normal and non-normal numbers to $\beta$-expansions. It has been shown in [14] for full shifts and more generally in [13] for systems fulfilling the specification property (under some mild technical conditions) that the set of essentially non-normal numbers is residual or comeagre (a countable intersection of everywhere dense sets).

In the present paper, we calculate the Hausdorff dimension of sets of particularly non-normal numbers in dynamical systems fulfilling the specification property. To this end we will introduce the necessary notation for the statement of the results in the following section. Then we construct a subset of the set of numbers, show that they are all particularly non-normal numbers and calculate their Hausdorff dimension in the last section.

2. Notation and Results

Our notation follows Lind and Marcus [11]. Let $M$ be a compact metric space and let $\varphi: M \to M$ be a continuous mapping. Then $(M, \varphi)$ is called a topological dynamical system.

For $x \in M$ we denote by $\delta_x$ the Dirac measure, which is concentrated on $x$. The empirical measure (of order $n$) is defined as

$$T_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\varphi^k(x)}.$$ 

Note that for a given subset $U \subset M$, $T_n(x)[U]$ is the probability of $\varphi^k(x)$ being in $U$ for $0 \leq k < n$, i.e.

$$T_n(x)[U] = \frac{1}{n} \# \{0 \leq k < n : \varphi^k(x) \in U \}.$$ 

If $M$ is the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $U \subset \mathbb{T}$ is a subinterval, then this corresponds to questions on uniform distribution of the sequence $\{\varphi^k(x)\}_k$. We refer the interested reader to Kuipers and Niederreiter [10], Drmota and Tichy [8] or Bugeaud [2] for more details on uniform distribution.

In the present paper we consider normal numbers and therefore we need a “digital” structure on the set $M$. To this end let $\mathcal{P} = \{P_0, \ldots, P_{N-1}\}$ be a collection of disjoint open subsets of $M$. Then we call $\mathcal{P}$ a topological partition of $M$ if it is the union of the closures of the $P_i$, i.e.

$$M = \overline{P_0} \cup \cdots \cup \overline{P_{N-1}}.$$ 

For the rest of the paper we fix a topological dynamical system $(M, \varphi)$ together with a topological partition $\mathcal{P}$. We note that fixing $\mathcal{P}$ and $\varphi$ suffices.
Now we construct the associated symbolic dynamical system. Let $A = \{0, 1, \ldots, N-1\}$ be a finite set, the alphabet. We denote by $A^\mathbb{N}$ the set of infinite words equipped with the product discrete topology. Furthermore let $S: A^\mathbb{N} \to A^\mathbb{N}$ be the shift operator, i.e. for $\omega = a_1a_2a_3\ldots \in A^\mathbb{N}$

$$S(\omega) = a_2a_3a_4\ldots .$$

Let $\gamma \in A^k$ be a finite word, then we write $|\gamma| = k$ for its length. Furthermore for every $\omega = a_1a_2a_3\ldots \in A^\mathbb{N}$ and $n \geq 1$ a positive integer, we denote by $\omega|_n = a_1a_2\ldots a_n$ the truncation of $\omega$ to the first $n$ letters. For a finite word $\omega = a_1a_2\ldots a_k \in A^k$ we denote by $[\omega]$ the corresponding cylinder set of order $k$, i.e. the set of all infinite words starting with the same letters as $\omega$,

$$[\omega] = \{ \gamma = b_1b_2b_3\ldots \in A^\mathbb{N} : a_i = b_i \text{ for } 1 \leq i \leq k \}.$$ 

Let $A^* = \bigcup_{k=1}^\infty A^k$ be the set of all finite words. A subset $\mathcal{L} \subset A^*$ is called a language. We call a word $\omega = a_1\ldots a_n$ allowed for the pair $(\mathcal{P}, \varphi)$ if the set $\bigcap_{j=1}^n \varphi^{-j}P_{a_j}$ is not empty. Then we denote by $\mathcal{L} := \mathcal{L}_{\mathcal{P}, \varphi}$ the set of all allowed words for the pair $(\mathcal{P}, \varphi)$. There is a unique shift space $X := X_{\mathcal{P}, \varphi}$ whose language is $\mathcal{L}_{\mathcal{P}, \varphi}$ and we call $X$ the symbolic dynamical system corresponding to $(\mathcal{P}, \varphi)$.

We denote by $M_S$ the set of all shift invariant probability measures. A measure $\mu$ is called shift invariant if for all measurable sets $A$ we have that $\mu(A) = \mu(S^{-1}A)$. Furthermore we denote by $\mathfrak{B}$ the set of Borel sets that is generated by the cylinder sets. Then for any $\mu \in M_S$ the tuple $(X, \mathfrak{B}, S, \mu)$ describes a measure theoretical dynamical system.

By abuse of notation we denote by $T_n$ also the empirical measure in $X$:

$$T_n(\omega) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^k(\omega)}.$$ 

Then for each block $b \in A^k$, $T_n(\omega)([b])$ counts the number of occurrences of the block $b$ among the first $n$ digits. This yields the identity

$$P_b(\omega, n) = T_n(\omega)([b]).$$

Now we can pull over the definition of particularly non-normal numbers. We call an infinite word $\omega \in X$ particularly non-normal if there exist two letters $i, j \in A$ such that

$$\lim_{n \to \infty} T_n(\omega)([i]) \text{ exists and } \lim_{n \to \infty} T_n(\omega)([j]) \text{ does not exist.}$$

We denote by $\mathcal{L}_n$ the subset of words of length at most $n$, i.e. $\mathcal{L}_n := \{ w \in A^* : |w| \leq n \}$. Then the Shannon entropy of $\alpha \in M_S$ is defined as

$$h_{\mathcal{B}_n}(\alpha) := \lim_{n \to \infty} -\frac{1}{n} \sum_{w \in \mathcal{L}_n} \alpha(|w|) \log(\alpha(|w|)).$$

A measure $\nu \in M_S$ is called maximal (or an equilibrium state) if it maximises this entropy. For a symbolic dynamical system fulfilling the specification property there need not be a unique maximal ergodic measure, however, the set of ergodic measures is a dense $G_\delta$ subset of $M$ (cf. Proposition 21.9 of Denker et al. [7] or Sigmund [20]). For the case of $\beta$-expansions Bertrand-Mathis [5] showed that the Champernowne word is generic for the unique maximal measure.

For showing that an element $\omega \in A^\mathbb{N}$ is in fact particularly non-normal we look at the limit-points of the frequency vectors $\tilde{P}_k(\omega, n)$. On the other hand in order to calculate the Hausdorff-dimension we consider measures of cylinder sets. In particular, we need information on the limit-points of the empirical measure of the constructed words. Let $f: A^\mathbb{N} \to \mathbb{R}$. Then we call $f$ local if there exist indices $1 \leq i \leq j \leq k < \infty$ such that the function values coincide whenever the sub word on position $i$ up to $j$ coincides, i.e. $f(\omega) = f(\eta)$ whenever $\omega_k = \eta_k$ for $i \leq k \leq j$. Let $\mathcal{B}_n$ be the $\sigma$-algebra generated by $\mathcal{L}_n$ and let $\mathcal{F}_n$ be the set of local functions which are $\mathcal{B}_n$-measurable. Now our topology on $M$, which coincides with the weak* topology, is given by the norm

$$||\rho|| := \sum_{n=1}^{\infty} 2^{-n} ||\rho||_{TV_n}, \text{ where } ||\rho||_{TV_n} := \sup_{f \in \mathcal{F}_n, \|f\| \leq 1} |\langle f, \rho \rangle|.$$
For each $\omega \in \mathcal{A}^\mathbb{N}$ the sequence $\{T_n(\omega)\}_n$ has limit-points, which are clearly $S$-invariant probability measures. If for an $\omega \in \mathcal{A}^\mathbb{N}$ the sequence $\{T_n(\omega)\}_n$ has only one limit point $\alpha$, then we call $\omega$ normal with respect to the measure $\alpha$ (or generic for the measure $\alpha$). Furthermore we call $\omega \in \mathcal{A}^\mathbb{N}$ normal if it is normal with respect to the maximal measure.

Before we start with our construction, we present some hypothesis which we suppose for the rest of this paper. All of them are fulfilled by our motivating example – the $\beta$-expansion.

**H1** Different measures: Let $\nu \in \mathcal{M}_S$ be the maximal measure and $\mu \in \mathcal{M}_S$ be any measure different from the maximal one such that there exists $i, j \in \{0, \ldots, N - 1\}$ with

$$\mu([i]) > \nu([i]) \quad \text{and} \quad \mu([j]) = \nu([j]).$$

Without loss of generality, we assume that $i = 0$ and $j = 1$.

**H2** Specification property: There exists $C \in \mathbb{N}$ such that for any pair $a, b \in \mathcal{L}$ there exists $v \in \mathcal{L}$ with $|v| \leq C$ such that $avb \in \mathcal{L}$. We will write $a \circ b := avb$ for short.

**H3** Approximation property: The maximum measure $\nu \in \mathcal{M}_S$ is non-atomic. There exists a continuous non-negative function $e_\nu$ on $\mathcal{L}$ such that

$$\limsup_n \sup_{\omega \in \mathcal{L}} \left| \frac{1}{n} \log \nu[\omega|_n] + \langle e_\nu, T_n(\omega) \rangle \right| = 0$$

and

$$\exists C_\nu > 0 \quad \text{such that} \quad \langle e_\nu, \rho \rangle \geq C_\nu \quad \forall \rho \in \mathcal{M}_S.$$

**Comments:**

- The hypothesis **H1** guarantees that we have words which tend to the different measures. It is clear from that, that there have to be at least 3 letters since otherwise $\nu([0]) + \nu([1]) = 1$ and if $P_f(\omega, n)$ has a limit, then also $P_l(\omega, n)$.
- We note that we may have relaxed **H2** by letting the length of the word $v$ depend on the length of the left and right word, respectively. However, this makes notation uglier and so we omitted it. For a different definition of the specification property, we refer the interested reader to Chapter 22 of [3].

**Theorem 2.1.** Let $(M, \varphi)$ be a topological dynamical system. Suppose that $\mathcal{P}$ is a finite topological partition of $M$ such that the associated symbolic dynamical system $(X, S)$ satisfies **H1**, **H2** and **H3**. Then the set of particularly non-normal numbers has full Hausdorff dimension.

As an example of a dynamical system satisfying **H1**, **H2** and **H3** we consider the $\beta$-expansions from the introduction.

**Theorem 2.2.** Let $\beta > 2$ be a Parry number. Let $(X, S)$ be the symbolic dynamical system corresponding to the $\beta$-expansion. Then the set of particularly non-normal numbers has full Hausdorff dimension.

**Proof.** Let $\nu \in \mathcal{M}_S$ be the maximal measure. Without loss of generality, assume that $\nu([0]) \neq 0$ and $\nu([1]) \neq 0$. Define $\mu$ as follows. If 1 is a factor of $\omega \in \mathcal{L}$, then $\mu([\omega]) = 0$. Otherwise if 0 is not a factor of $\omega$, then $\mu([\omega]) = \nu([\omega])$. Finally, if 0 is a factor and 1 is not a factor of $\omega$, then consider the map $F : \mathcal{D}_\beta \rightarrow \mathcal{D}_\beta \setminus \{1\}$ which maps $1 \mapsto 0$ and $d \mapsto d$ for $d \neq 1$, and let $\mu([\omega]) = \sum_{a \in F^{-1}(\omega) \cap \mathcal{L}} \nu([a])$. The constructed measure $\mu \in \mathcal{M}_S$ satisfies **H1**. Moreover, if $\beta$ is a Parry number, then the system is sofic by definition. By [3] Section 5, the sofic shifts satisfy **H2** and **H3**. The result then follows by Theorem 2.1. □

3. The construction

The aim of this section is to construct for every positive integer $p$ a set $G_p$ consisting of particularly non-normal numbers and having Hausdorff dimension $\frac{p}{p + 1}$. In particular, the elements of the set $G_p$ will consist of two parts:

1. A fixed part constructed by concatenating parts of the sequences $\omega$ whose empirical measure tends to $\mu$ and
2. A changing part constructed by concatenating words whose empirical measure tends to $\nu$. 

We start by defining sets of finite words, whose empirical measure tends to a given measure $\alpha$.

**Proposition 3.1** ([13] Theorem 2.1, [10] Lemma 2.4). Let $\alpha \in \mathcal{M}_S$. Then there exists a sequence

\[ \Gamma(\alpha, n) \subset \mathcal{A}^n : n \in \mathbb{N}^\bullet \] such that

\[ \lim_{n \to \infty} \frac{1}{n} \log |\Gamma(\alpha, n)| = h_{\text{Sh}}(\alpha), \]

and for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

\[ |P_j(\omega) - \alpha([j])| \leq \varepsilon \]

for $j \in \{0, 1\}$ and all $\omega \in \Gamma(\alpha, n)$.

Corollary 21.15 of Denker et al. [7] tells us that every measure has a generic word. Therefore we choose $\omega \in X$ such that $P_i(\omega, n) \to \mu([i])$ as $n \to \infty$ for $i \in \{0, 1\}$ (the fixed part) and let $(\Gamma(\nu, n))_n$ be given as in Proposition 3.1 (the changing part). Since we need to show that the different parts of the sequences in the set $G_p$ tend to different measures, we have to keep track of their speed of convergence. Thus on the one hand we define for each $n \in \mathbb{N}$ the discrepancy $\delta_n = \max_{\epsilon \in \{0, 1\}} |P_i(\omega, n) - \mu([i])|$.

On the other hand, for each $n \in \mathbb{N}$ we define the parameters $\eta_n$ and $\epsilon_n$ by $\frac{1}{n} \log |\Gamma(\alpha, n)| - h_{\text{Sh}}(\alpha) = \eta_n$ and $\epsilon_n = \max_{\epsilon \in \{0, 1\}} \max_{\omega \in \Gamma(\nu, n)} |P_i(\omega) - \nu([i])|$, respectively. Finally we combine these distances by setting $\varrho_n = \max\{\epsilon_n, \delta_n\}$. Note that $\varrho_n \to 0$ as $n \to \infty$.

Let $B_0^{(1)}$ consist only of the empty word $\epsilon$. We set the length of the initial block as 1 and choose $n_1$ such that for $\omega_1 = \omega|_1$ (the first letter of the word $\omega$) we have

\[ |P_k(\omega_i^{(n_1)} - \mu([i])| \leq 2\varrho_1, \]

for $k \in \{0, 1\}$. For $1 \leq i \leq n_1$, let $B_i^{(1)} = \{\omega_i^{(n_1)}\}$. Denote by $\ell_i^{(1)}$ the length of the word in the set $B_i^{(1)} = \{\omega_i^{(n_1)}\}$. Then for $n_1 + 1 \leq i \leq (p + 1)n_{j-1}$ let

\[ \widetilde{B}_i^{(1)} = B_i^{(1)} \circ \Gamma(\nu, 1) \]

and we choose $B_i^{(1)} \subseteq \widetilde{B}_i^{(1)}$ of maximal cardinality such that each element has the same length $\ell_i^{(1)}$.

Now we continue recursively. To this end we suppose that the sets $B_i^{(j-1)}$ with $0 \leq i \leq (p + 1)n_{j-1} - 1$ have already been constructed. We set

\[ \widetilde{B}_0^{(j)} = B_0^{(j-1)} \circ \Gamma(\nu, j-1), \]

and choose $B_0^{(j)} \subseteq \widetilde{B}_0^{(j)}$ of maximal cardinality such that each element has the same length denoted by $\ell_0^{(j)}$. Let $\omega_j = \omega|_j$. Choose $n_j > n_{j-1}$ such that

\[ |P_i(\tilde{\omega} \circ \omega_j^{(n_j)}) - \mu([i])| \leq 2\varrho_j, \]

for $i \in \{0, 1\}$ for all $\tilde{\omega} \in B_0^{(j)}$, and

\[ \ell_0^{(j)} \to 0 \text{ as } j \to \infty. \]

For $1 \leq i \leq n_j$, let

\[ B_i^{(j)} = B_0^{(j)} \circ \omega_j^{(n_j)} \]

Finally, for $n_j + 1 \leq i \leq (p + 1)n_j$ let

\[ \widetilde{B}_i^{(j)} = B_i^{(j-1)} \circ \Gamma(\nu, j) \]

and again we choose $B_i^{(j)} \subseteq \widetilde{B}_i^{(j)}$ of maximal cardinality such that each element has the same length $\ell_i^{(j)}$.

We may now define our set $G_p$ as the limit of the sets $B_i^{(j)}$, i.e.

\[ G_p = \bigcap_{j \geq 1} \bigcap_i \bigcup_{b \in B_i^{(j)}} [b]. \]
By the construction we obtain that each element in $G_p$ has a unique representation of the form
\[
\underbrace{\omega_1 \odot \omega_1 \odot \cdots \odot \omega_1}_{n_1 \text{ times}} \odot a^{(1)}_1 \odot \cdots \odot a^{(1)}_{\nu_1} \odot \underbrace{\omega_2 \odot \omega_2 \odot \cdots \odot \omega_2}_{n_2 \text{ times}} \odot a^{(2)}_1 \odot \cdots \odot a^{(2)}_{\nu_2} \odot \cdots
\]
with $\omega_j = \omega|_j$ and $a^{(j)}_k \in \Gamma(\nu, j)$ for $1 \leq k \leq \nu_j$.

We finish this section by providing a lower bound for the number of elements in $B_i^{(j)}$.

Lemma 3.2. We have the following:
\[
|B^{(j)}_i| = |B^{(j)}_0| \quad \text{for all } j \text{ and } 1 \leq i \leq n_j; \]
\[
|B^{(j)}_i| = \exp \left( p \sum_{k=1}^{j} n_k k \left( h_{\nu_k} - \eta_k - \frac{ \log C_i }{ k } \right) \right) \quad \text{for all } j \text{ and } n_j + 1 \leq i < \nu_{n_j}; \]
\[
|B^{(j)}_0| \geq \exp \left( p \sum_{k=1}^{j} n_k k \left( h_{\nu_k} - \eta_k - \frac{ \log C_i }{ k } \right) \right) \quad \text{for all } j \geq 1.
\]

Proof. It follows by induction from our construction and Proposition 3.1.

\[\Box\]

4. Particularly non-normal

In this section we want to show that each element of $G_p$ is particularly non-normal, i.e. there exist two digits such that there is no limit frequency for the first but certainly there is one for the second one. Let $N_i(\omega)$ denote the number of occurrences of a digit $i$ in a word $\omega$, and if $\omega$ is finite write $P_i(\omega)$ to denote $P_i(\omega, |\omega|)$ the frequency of occurrences of the digit $i$ in the word $\omega$.

We start by showing that for all $\omega \in G_p$ we have that $P_i(\omega, m) \to \mu([1]) = \mu([1])$ as $m \to \infty$. Let $\omega \in G_p$ and $m \in \mathbb{N}$. Choose $j$ such that $\ell^{(j)}_0 < m \leq \ell^{(j+1)}_0$. First assume that $\ell^{(j)}_{n_j} < m$. In this case,
\[
\omega|_m = \underbrace{\omega_1 \odot \cdots \odot \omega_j \odot a^{(j)}_1 \odot \cdots \odot a^{(j)}_{k-1} \odot \tilde{a}^{(j)}_k}_m,
\]
with $0 \leq k \leq \nu_{n_j}$. Let $v_i$ be the connecting block inserted before the word $a^{(j)}_i$. Then we can write $m = \ell^{(j)}_{n_j} + (k-1)j + \sum_{i=1}^{k} |v_i| + |\tilde{a}^{(j)}_k|$ and get that
\[
P_i(\omega, m) = P_i(\omega, \ell^{(j)}_{n_j}) \frac{\ell^{(j)}_{n_j}}{m} + \frac{j}{m} \sum_{i=1}^{k} P_i(a^{(j)}_i, j) + \frac{1}{m} \sum_{i=1}^{k} N_i(v_i) + \frac{N_i(\tilde{a}^{(j)}_k)}{m}.
\]

Note that
\[
0 \leq \frac{1}{m} \sum_{i=1}^{k} N_i(v_i) \leq \frac{\sum_{i=1}^{k} |v_i|}{\ell^{(j)}_{n_j} + (k-1)j + \sum_{i=1}^{k} |v_i| + |\tilde{a}^{(j)}_k|} \leq \frac{kC}{(n_j + k - 1)j} \quad \xrightarrow{m \to \infty} 0
\]
and
\[
0 \leq \frac{N_i(\tilde{a}^{(j)}_k)}{m} \leq \frac{|\tilde{a}^{(j)}_k|}{\ell^{(j)}_{n_j} + (k-1)j + \sum_{i=1}^{k} |v_i| + |\tilde{a}^{(j)}_k|} \leq \frac{j}{\ell^{(j)}_{n_j}} \leq \frac{1}{n_j} \quad \xrightarrow{m \to \infty} 0.
\]
Finally, by the construction, we have
\[ |P_1(\omega, \ell^{(j)}_{n_j}) - \nu([1])| \leq 2g_j, \]
and for all \( i \)
\[ |P_1(a^{(j)}_i, j) - \nu([1])| \leq g_j. \]
Therefore, we conclude that
\[ P_1(\omega, m) \to \nu([1]) \text{ as } m \to \infty, \ell^{(j)}_{n_j} < m. \]

Suppose now \( m \leq \ell^{(j)}_{n_j} \). In this case, there exists a \( k \) with \( 0 \leq k < n_j \) such that
\[ \omega|_m = \omega_1 \circ \cdots \circ \omega_{j-1} \circ \cdots \circ a^{(j-1)}_{m_j} \circ \omega_j \circ \cdots \circ \omega_j \circ \tilde{\omega}_j, \]
where \( \omega \) is repeated \( k \) times. Let \( v_i \) be the connecting block inserted before the \( i \)th \( \omega_j \) word. We have
\[ m = \ell^{(j)}_{0} + k j + \sum_{i=1}^{k+1} |v_i| + |\tilde{\omega}_j|. \]
With this notation, we have
\[ P_1(\omega, m) = P_1(\omega, \ell^{(j-1)}_{n_{j-1}}) \frac{\ell^{(j-1)}_{n_{j-1}}}{m} + \frac{1}{m} \sum_{i=1}^{m} P_1(a^{(j-1)}_i, j - 1) + \frac{1}{m} \sum_{i=1}^{k+1} N_1(v_i) + \frac{N_1(\tilde{\omega}_j)}{m}. \]
As before, we have
\[ \frac{1}{m} \sum_{i=1}^{k+1} N_1(v_i) \to 0 \text{ and } \frac{N_1(\tilde{\omega}_j)}{m} \to 0. \]
Again by construction
\[ |P_1(\omega, \ell^{(j-1)}_{n_{j-1}}) - \nu([1])| \leq 2g_{j-1}; \]
\[ |P_1(a^{(j-1)}_i, j - 1) - \nu([1])| \leq g_{j-1} \text{ for all } i; \]
\[ |P_1(\omega_j, j) - \nu([1])| \leq g_j. \]
Therefore, we conclude that
\[ P_1(\omega, m) \to \nu([1]) \text{ as } m \to \infty, m \leq \ell^{(j)}_{n_j}. \]

From (4.1) and (4.2), we conclude that \( P_1(\omega, m) \to \nu([1]) \) as \( m \to \infty \) and therefore the limiting frequency of the digit 1 exists.

Now we show that the limiting frequency of digit 0 does not exist. On the one hand, we note that by our construction we have that
\[ P_0(\omega, \ell^{(j)}_{n_j}) \to \mu([0]) \text{ as } j \to \infty. \]
On the other hand, let \( v_i \) be the connecting block inserted before the word \( a^{(j)}_i \). Then we have
\[ P_0(\omega, \ell^{(j+1)}_{0}) = P_0(\omega, \ell^{(j)}_{n_j}) \frac{\ell^{(j)}_{n_j}}{\ell^{(j+1)}_{0}} \sum_{i=1}^{m_j} P_0(a^{(j)}_i, j) + \frac{1}{\ell^{(j+1)}_{0}} \sum_{i=1}^{m_j} N_1(v_i). \]
As above we get that
\[ \frac{1}{\ell^{(j+1)}_{0}} \sum_{i=1}^{m_j} N_1(v_i) \to 0, \quad P_0(\omega, \ell^{(j)}_{n_j}) \to \mu([0]) \quad \text{and} \quad P_0(a^{(j)}_i, j) \to \nu([0]) \]
as \( j \to \infty \). Finally, by (3.2), we have
\[ \frac{\ell^{(j)}_{n_j}}{\ell^{(j+1)}_{0}} \to \frac{1}{p} \quad \text{and} \quad \frac{m_j j}{\ell^{(j+1)}_{0}} \to \frac{p}{p+1}. \]
We conclude that
\[ P_0(\omega, \ell^{(j+1)}_{0}) \to \frac{1}{p+1} \mu([0]) + \frac{p}{p+1} \nu([0]) < \mu([0]) \]
as \( j \to \infty \). It follows from (4.3) and (4.4) that the limiting frequency of the digit 0 in \( \omega \) does not exist, proving that \( \omega \) is indeed a particularly non-normal number.

5. Haussdorff Dimension

For the estimation of the Hausdorff dimension it suffices to consider the cylinder sets. The set \( B_i^{(j)} \) belongs to the \( i \)th word of the \( j \)th block. In the present section we want to ease notation by saying the \( n \)th word is the \( i \)th word of the \( j \)th block. In particular, for each \( n \) there exist unique \( i = i(n) \) and \( j = j(n) \) such that

\[
    n = (p + 1) \sum_{k=1}^{j-1} n_k + i
\]

with \( 1 \leq i \leq (p + 1)n_j \). Similarly we set \( B_n = B_i^{(j)} \),

\[
    y_n = y_i^{(j)} := \min \{ \nu([x]) : x \in B_i^{(j)} \}
\]

and

\[
    E_n = E_i^{(j)} := \exp \left( p \sum_{k=1}^{j-1} n_k k \left( h_{Sh} - \eta_k - \frac{\log C}{k} \right) + \sum_{k=n_j+1}^{i} j \left( h_{Sh} - \eta_j - \frac{\log C}{j} \right) \right).
\]

Then our first tool is the following lemma.

Lemma 5.1. Let \( 0 < s \leq 1 \) and let \( D \) be a cover of \( G_p \) such that \( \nu([w]) < y_N \) for all \( w \in D \). Then there exists \( n \geq N \) such that

\[
    \sum_{w \in D} \nu([w])^s \geq E_n y_n^s.
\]

Proof. Since \( \nu([w]) < y_N \) for all \( w \in D \), each \( w \in D \) has a prefix in \( B_N \), i.e. \( w = vu \) with \( v \in B_N \). Moreover, since \( D \) is a cover, every \( v \in B_N \) has to appear as prefix. Thus we may write

\[
    \sum_{w \in D} \nu([w])^s = \sum_{v \in B_N} \sum_{u \in D} \nu([vu])^s.
\]

Now let \( \tilde{v} \in B_N \) be such that for all \( v \in B_N \)

\[
    \sum_{u : \tilde{v}u \in D} \nu([vu])^s \geq \sum_{u : vu \in D} \nu([vu])^s.
\]

We have

\[
    \sum_{w \in D} \nu([w])^s \geq |B_N| \sum_{u : \tilde{v}u \in D} \nu([\tilde{v}u])^s.
\]

Now, either

\[
    \sum_{u : \tilde{v}u \in D} \nu([\tilde{v}u])^s \geq (y_{N+1})^s, \quad \text{or} \quad \sum_{u : \tilde{v}u \in D} \nu([\tilde{v}u])^s < (y_{N+1})^s.
\]

If the first case holds true, then an application of Lemma 3.2 yields

\[
    \sum_{w \in D} \nu([w])^s \geq \exp \left( p \sum_{k=1}^{j-1} n_k k \left( h_{Sh} - \eta_k - \frac{\log C}{k} \right) + \sum_{k=n_j+1}^{i} j \left( h_{Sh} - \eta_j - \frac{\log C}{j} \right) \right) y_{N+1}^s.
\]

On the other hand, if the latter case is true, then let \( P_{N+1} := \{ v \in B_{N+1} : \tilde{v} \text{ is a prefix of } v \} \). Now every \( u \) with \( \tilde{v}u \in D \) can be written as \( \tilde{v}u = v'u' \), where \( v' \in P_{N+1} \). Since \( D \) is a cover of \( G_p \), all prefixes \( v' \in P_{N+1} \) occur in the decomposition of \( \tilde{v}u = v'u' \). Thus by construction of \( B_{N+1} \) we either have that

\[
    |P_{N+1}| \geq \exp(nj(n+1) \left( h_{Sh} - \eta_{j(n+1)} \right))
\]

if \( i(N+1) > n_j(N+1) \) or \( |P_{N+1}| = 1 \) otheriwse. Similar to above we may choose \( \tilde{v}' \in P_{N+1} \) such that for all \( v' \in P_{N+1} \),

\[
    \sum_{v' : v'u' \in D} \nu([v'u'])^s \geq \sum_{v' : \tilde{v}'u' \in D} \nu([\tilde{v}'u'])^s.
\]
Thus combining this with the estimate above yields
\[ \sum_{w \in \mathcal{D}} \nu([w])^s \geq \mathcal{E}_n \sum_{v \in \mathcal{D}} \nu([v])^s \]
\[ \geq \mathcal{F}_n+1 \sum_{w \in \mathcal{D}} \nu([v])^s. \]
Iterating this argument yields (after finitely many steps since \( \mathcal{D} \) is finite) that
\[ \sum_{w \in \mathcal{D}} \nu([w])^s \geq \mathcal{F}_n y_{n+1}^s. \]

The second ingredient is the following lemma of Pfister and Sullivan [18].

**Lemma 5.2 ([18] Lemma 3.1 and Corollary 3.1).** Let \( \nu \) be a probability measure which possesses a continuous strictly positive regular conditional probability \( \nu(\omega_1|\omega_2, \omega_3, \ldots) \). Then,
\[ e_\nu(\omega) := -\log \nu(\omega_1|\omega_2, \omega_3, \ldots) \]
satisfies (2.1). If \( \nu \) satisfies (2.1), then, for each \( \delta > 0 \), there exist \( m_\delta \) and \( N_\delta \in \mathbb{N} \) such that for all \( n \geq N_\delta \) and for all \( \omega \in \Sigma^\omega \)
\[ \left| e_\nu, T_n(\omega) \right| + \frac{1}{n} \log \nu(\omega_1^n) < \delta. \]

Furthermore, we have
\[ h_{bh}(\nu) = (e_\nu, \nu). \]

Now we are able to calculate the Hausdorff dimension.

**Proof of Theorem 2.1** We suppose that \( s < \frac{1}{p+1} \). Then by our construction
\[ \ell_n T_n(\omega) = \sum_{k=1}^{j-1} n_k kT_k(\omega_k) + \sum_{k=1}^{j-1} \sum_{m=1}^{p,m_k} kT_k(a_m^{(k)}) \]
\[ + ijT_j(\omega_j) + \sum_{m=1}^{j-1} jT_j(a_m^{(j)}) + \mathcal{O} \left( \sum_{k=1}^{j-1} n_k(p+1) + i \right). \]

Similar to the proof of \( \omega \) being particularly non-normal we need to distinguish two cases according to whether \( 0 \leq i < n_j \) or \( n_j \leq i < n_j(p+1) \). For the first case we have
\[ \left\| T_n(\omega) - \sum_{k=1}^{j-1} n_k kT_k(\omega_k) \right\| \leq \frac{1}{\ell_n} \sum_{k=1}^{j-1} n_k kT_k(a_m^{(k)}) - \nu \right\| \leq \sum_{k=1}^{j-1} \frac{n_k k}{\ell_n} \rho_k \leq \frac{p}{p+1} \sum_{k=1}^{j-1} \rho_k. \]

Using the lower bound of Lemma 5.1 together with an application of Lemma 5.2 yields that for each \( \delta > 0 \) there exists \( N_\delta \) such that for \( n \geq N_\delta \) we have
\[ p \sum_{k=1}^{j-1} n_k k h_{bh}(\omega) - \eta_k + s log y_{n+1} \]
\[ \geq \sum_{k=1}^{j-1} n_k k h_{bh}(\omega) (p - (p+1)s) - p \sum_{k=1}^{j-1} n_k k\eta_k - s(i+1)j (e_\nu, T_j(\omega_j)) - s6\ell_{n+1}. \]

For the second case, \( n_j \leq i < n_j(p+1) \), we get by similar means that
\[ \left\| T_n(\omega) - \sum_{k=1}^{j-1} n_k kT_k(\omega_k) \right\| \leq \frac{p}{\ell_n} \sum_{k=1}^{j-1} n_k k\rho_k + (i-n_j)j \eta_j. \]
Again using the lower bound of Lemma 5.1 together with Lemma 5.2 we get that for each $\delta > 0$ there exists $N_\delta$ such that for $n \geq N_\delta$ we have
\[
p \sum_{k=1}^{j-1} n_k k (h_{\text{Sh}}(\nu) - \eta_k) + s \log y_{n+1}\]
\[
\geq \sum_{k=1}^{j-1} n_k k h_{\text{Sh}}(\nu) (p - (p + 1)s) - p \sum_{k=1}^{j-1} n_k k \eta_k - s n_j j (\langle e_\nu, T_j(\omega_j) \rangle - s(i + 1 - n_j) j h_{\text{Sh}}(\nu) - s \delta \ell_{n+1}.
\]

Thus in both cases by taking $\delta > 0$ sufficiently small and noting that $\sum_{k=1}^{j-1} n_k k / \ell_n = \frac{p}{p+1}$ we get that
\[
\liminf_n E_n y_{n+1}^s = \infty,
\]
provided that $s < \frac{p}{p+1}$. Thus $\dim_H(G_p) \geq \frac{p}{p+1}$. Letting $p \to \infty$ shows that the Hausdorff dimension of the set of particularly non-normal numbers is greater or equal to 1. Since it cannot exceed 1, it completes the proof of Theorem 2.1. \hfill \Box

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