Cremmer-Gervais Quantum Lie Algebra

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Abstract

We describe a quantum Lie algebra based on the Cremmer-Gervais R-matrix. The algebra arises upon a restriction of an infinite-dimensional quantum Lie algebra.

1 Introduction

The notion of a quantum Lie algebra is a modification of the notion of a Lie algebra. Quantum Lie algebras arise as the algebras generated by the quantum analogs of vector fields in the framework of the bicovariant differential calculus on quantum groups [1] (for an introduction see e.g. [2]). Many constructions from the theory of Lie algebras can be generalized for quantum Lie algebras (for example, the standard complex, BRST operator etc. [3,4,5]).
In this Note we outline the quantum Lie algebra having the so called Cremmer-Gervais R-matrix \(^7\) as the braid matrix. This can be seen as a first step in constructing the BRST operator for the bicovariant differential calculus based on the Cremmer-Gervais R-matrix.

### 2 Quantum Lie Algebra

The bicovariant differential calculus is characterized by functionals \(\chi_i\) and \(f^i_j\) on a Hopf algebra \(\mathcal{A}\) ("the algebra of functions on a quantum group") satisfying the relations

\[
\begin{align*}
\chi_i \chi_j - \sigma^{kl}_{ij} \chi_k \chi_l & = C^{jk}_i \chi_k, \quad \sigma^{kl}_{ij} f^a_k f^b_l = f^k_i f^j_l \sigma^{ab}_{kl}, \\
\sigma^{kl}_{ij} \chi_k f^a_l + C^{ij}_k f^a_l & = f^k_i f^j_l C^{a}_{kl} + f^a_i \chi_j, \quad \chi_i f^a_j = \sigma^{kl}_{ij} f^a_k \chi_l.
\end{align*}
\]

The relations for \(C^{ij}_k\) and the braid matrix \(\sigma^{ij}_{kl}\) (\(C^{ij}_k\) and \(\sigma^{ij}_{kl}\) are subject to certain conditions, see below) are such that \(C^{ij}_k = \chi_k (M^i_j)\) and \(\sigma^{ij}_{kl} = f^j_l (M^i_k)\), where the matrix \(M \in \mathcal{A}\) is given by the right coaction on the space of left-invariant forms

\[\Delta_R(\omega^j) = \omega^j \otimes M^i_j, \quad M^i_j \in \mathcal{A}, \quad \omega^i \in \Gamma.\]

The algebra (1) endowed with the comultiplication \(\Delta\), counit \(\epsilon\) and antipode \(S\),

\[
\begin{align*}
\Delta f^i_j &= f^i_k \otimes f^k_j, \quad \epsilon(f^i_j) = \delta^i_j, \quad S(f^i_j) f^k_j = \delta^i_j = f^k_i S(f^j_k), \\
\Delta \chi_i &= 1 \otimes \chi_i + \chi_j \otimes f^j_i, \quad \epsilon(\chi_i) = 0, \quad S(\chi_i) = -\chi_j S(f^j_i),
\end{align*}
\]

becomes a Hopf algebra which we will be denote by \(\mathcal{L}\). The subalgebra generated by \(\chi_i\) is called quantum Lie algebra.

The relations for \(\mathcal{L}\) can be written in a concise way with the help of a single R-matrix \(^6\). Let us make a convention that the small indices \(i, j, \ldots, k\) run over a set \(\mathcal{I}\) and the capital indices \(I, J, \ldots, K\) run over the set \(\mathcal{I}_0 := 0 \cup \mathcal{I}\). Denote by \(\hat{R}\) and \(T\) the following matrices

\[
\hat{R}^{ij}_{kL} = \begin{pmatrix} \delta^j_k & C^{ij}_{kl} \\ 0 & \sigma^{ij}_{kl} \end{pmatrix}, \quad \hat{R}^{0i}_{0L} = \delta^i_L, \quad T^i_j = \begin{pmatrix} 1 & \chi_j \\ 0 & f^j_i \end{pmatrix},
\]

i.e., \(\hat{R}^{ij}_{kl} = \sigma^{ij}_{kl}, \hat{R}^{0i}_{kl} = C^{ij}_{kl}, \hat{R}^{0A}_{0B} = \delta^A_B, \hat{R}^{0A}_{0B} = \delta^A_B\) and \(T^i_j = f^j_i, T^0 = 1\) and all others entries are equal to zero. Suppose now that \(R\) is a solution of the Yang-Baxter equation

\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}.
\]
Then the Hopf algebra relations (1) and (2) are equivalent to

\[ \hat{R}_{IJK}^{AB} T_{KL}^{AB} = T_{IL}^{JK} \hat{R}_{KL}^{AB}, \quad \Delta T_I^J = T^K_J \otimes T_I^K, \]

\[ S(T_I^K) T_J^K = \delta_I^J = T_I^K S(T_J^K), \quad \epsilon(T_I^J) = \delta_I^J. \]

The Yang-Baxter relation for \( \hat{R} \) implies, for the components \( \sigma_{ij}^{kl} \) and \( C_{ij}^k \),

\[
\begin{align*}
C_{im}^k C_{sj}^m - \sigma_{ij}^{kl} C_{nk}^i C_{sl}^m &= C_{ij}^k C_{nk}^m, \\
\sigma_{ij}^{kl} \sigma_{nk}^a \sigma_{sl}^b &= \sigma_{mi}^k \sigma_{sj}^a \sigma_{kl}^b, \\
\sigma_{ij}^{kl} C_{nk}^s \sigma_{sl}^a &+ C_{ij}^l \sigma_{nl}^{am} = \sigma_{ni}^k \sigma_{sj}^a C_{kl}^m + \sigma_{ni}^m \sigma_{sj}^a C_{kl}^m, \\
C_{nk}^s \sigma_{sj}^{am} &= \sigma_{ij}^{kl} \sigma_{nk}^a C_{sl}^m.
\end{align*}
\] (4)

Here the first relation is the “braided” Jacobi identity and the second one is simply the braid relation for \( \sigma \), \( \sigma_{23} \sigma_{12} \sigma_{23} = \sigma_{12} \sigma_{23} \sigma_{12} \).

Given a braid matrix \( \sigma \) it is natural to ask if non-zero structure constants \( C_{ij}^k \) consistent with \( \sigma \) exist or is there a non-trivial quantum Lie algebra structure compatible with \( \sigma \). As we have seen this question is equivalent to finding a suitable extension (3) of the R-matrix \( \sigma \).

In this Note we obtain an infinite-dimensional R-matrix which upon restrictions yields finite-dimensional quantum Lie algebras compatible with the Cremmer–Gervais R-matrix [7].

3 Cremmer-Gervais extended

We apply the elegant method used in [8] and then in [9] where the Yang-Baxter operators are realized as operators in a certain space of functions. Finite-dimensional R-matrices arise upon a restriction of the operator domain to an appropriate invariant finite-dimensional subspace, such as the space of polynomials of bounded degree.

For a ring \( K \), let \( K(x) \) be the ring of rational functions in \( x \) with coefficients in \( K \). An endomorphism of \( K \) extends to an endomorphism of \( K(x) \) (which acts only on the coefficients of a rational function). Having an endomorphism \( \phi \in \text{End} \ C(x, y) \), introduce \( \phi_{12} \in \text{End} \ C(x, y, z) \) considering \( C(x, y, z) \) as \( C(x, y)(z) \). In the same vein, \( \phi_{13} \in \text{End} \ C(x, z)(y) \) and \( \phi_{23} \in \text{End} \ C(y, z)(x) \) and the functional Yang-Baxter equation reads \( \phi_{12} \phi_{13} \phi_{23} = \phi_{23} \phi_{13} \phi_{12} \).

Given a rational function \( F(x, y) \) with series expansion (around 0) \( F(x, y) = \sum_{i,j \in \mathbb{Z}} F_{i,j} x^i y^j \), define the operation \( \text{reg}_{x,y} \) which maps \( F(x, y) \) to the non-singu-
lar part \( f(x, y) \) of its expansion,
\[
f(x, y) = \text{reg}_{x,y} F(x, y) := \sum_{i,j \geq 0} F_{i,j} x^i y^j.
\]

**Theorem 1.** Let \( \hat{R} \) be the following linear operator in \( \text{End} \mathbb{C}(x, y) \)
\[
\hat{R} = P + \beta \frac{y}{x-y} (P-I) \text{reg}_{x,y} + C \frac{\text{eval}_{x=0} (P-I) \text{reg}_{x,y}}{x},
\]
\( \beta = 1 - q^{-2} \) and \( C \) are arbitrary constants. Here \( I \) stands for the identity operator, \( P \) for the permutation \( (PF)(x, y) = F(y, x) \) and \( \text{eval}_{x=0} \) is the evaluation at \( x = 0 \); in other words, for an arbitrary \( F(x, y) \in \mathbb{C}(x, y) \) the result of the action of the operator \( \hat{R} \) reads
\[
(\hat{R}F)(x, y) = F(y, x) + \beta y \frac{f(y, x) - f(x, y)}{x-y} + C \frac{f(y, 0) - f(0, y)}{x}.
\]

The operator \( \hat{R} \) satisfies the braid equation
\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}.
\]

**Proof.** The braid equation for the operator \( \hat{R} \) is equivalent to the Yang-Baxter equation
\[
[\hat{R}_{12}, \hat{R}_{13}] + [\hat{R}_{12}, \hat{R}_{23}] + [\hat{R}_{13}, \hat{R}_{23}] = 0.
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\]

\(4\)
Proof of the lemma. The operator in the left hand side of (8) depends only on the regular part of a function $F(x, y, z) \in \mathbb{C}(x, y, z)$ therefore it is enough to check the assertion on an arbitrary polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$. Since the coefficients $\beta$ and $C$ are arbitrary, the classical Yang-Baxter equation for $r$ splits into three components. The component proportional to $\beta^2$ is the classical Yang-Baxter equation for $\rho$; it is satisfied: $\rho$ is the classical Cremmer-Gervais $r$-matrix \[9, 10\]. Next, a straightforward verification shows that

$$
\rho_{13}\rho_{23} = 0, \quad \rho_{23}\rho_{13} = 0, \quad \rho_{23}\rho_{12} = 0, \quad [\rho_{12}, \rho_{13}] + \rho_{12}\rho_{23} = 0,
$$

on polynomials. The sum (with corresponding signs) of these equalities is the component proportional to $\beta C$. Finally, a straightforward verification shows that

$$
\rho_{23}\rho_{12} = 0, \quad \rho_{23}\rho_{13} = 0, \quad \rho_{12}\rho_{23} = 0, \quad \rho_{12}\rho_{13} + \rho_{12}\rho_{23} = 0
$$

on polynomials and the classical Yang-Baxter equation for the operator $s$ (the component, proportional to $C^2$) follows. □

The Yang-Baxter equation for $R = I + r$ holds true if the operator $r$ satisfies the classical Yang-Baxter equation and the Yang-Baxter equation

$$
r_{12}r_{13}r_{23} = r_{23}r_{13}r_{12}.
$$

(9)

To check this identity on an arbitrary function $F(x, y, z) \in \mathbb{C}(x, y, z)$ it is again enough to check it on an arbitrary polynomial function $f(x, y, z) \in \mathbb{C}[x, y, z]$. Now (9) splits into four components. The component proportional to $\beta^3$ vanishes ($\rho$ satisfies a stronger equation, see \[10\]). Next, a direct verification shows that

$$
\rho_{12}\rho_{13}\rho_{23} = 0, \quad \rho_{12}\rho_{13}\rho_{23} = 0, \quad \rho_{23}\rho_{13}\rho_{12} = 0,
$$

$$
\rho_{23}\rho_{13}\rho_{12} = 0, \quad \rho_{12}\rho_{13}\rho_{23} = 0, \quad \rho_{23}\rho_{13}\rho_{12} = 0
$$

on polynomials; the vanishing of the component proportional to $\beta^2 C$ follows. Finally, each term in the components, proportional to $\beta C^2$,

$$
\rho_{12}\rho_{13}\rho_{23} = 0, \quad \rho_{12}\rho_{13}\rho_{23} = 0, \quad \rho_{12}\rho_{13}\rho_{23} = 0,
$$

$$
\rho_{23}\rho_{13}\rho_{12} = 0, \quad \rho_{23}\rho_{13}\rho_{12} = 0, \quad \rho_{23}\rho_{13}\rho_{12} = 0,
$$

5
and $C^3$, 
\[ s_{12}s_{13}s_{23} = 0, \quad s_{23}s_{13}s_{12} = 0, \]
vanishes separately, which ends the proof of the theorem. □

Let $V = \bigoplus_{i=0}^{n} C e_i$ be a finite-dimensional vector space of functions $\frac{p(x)}{x}$ where $p(x)$ is a polynomial of degree not higher than $n$. Identify $V \otimes V$ with the space of functions $\frac{p(x,y)}{xy}$ where $p(x,y)$ is a polynomial of degree not higher than $n$ in $x$ and not higher than $n$ in $y$. The space $V \otimes V$ is stable under the action of the operator $\hat{R}$. The matrix of the restricted operator $\hat{R}(e_K \otimes e_L) = \sum_{I,J=0}^{n} e_I \otimes e_J \hat{R}_{KL}^{IJ}$ (which we denote again by $\hat{R}$) is given by
\[
\hat{R}(x^{K-1}y^{L-1}) = \sum_{I,J=0}^{n} \hat{R}_{KL}^{IJ} x^{I-1}y^{J-1}; \quad I, J, K, L = 0, \ldots, n.
\]
The non-vanishing entries of the matrix $\hat{R}_{KL}^{IJ}$ read as follows
\[
\hat{R}_{KL}^{0J} = \hat{R}_{0K}^{0j} = \delta_K^J, \quad \hat{R}_{kl}^{ij} = C_{kl}^{ij} = C(\delta_l^i \delta_k^j - \delta_k^i \delta_l^j),
\]
\[ i, j, k, l = 1, \ldots, n. \]

The latter submatrix $\hat{R}_{kl}^{ij}$ is the member (with $p = 1$) of the Cremmer-Gervais family of non-unitary R-matrices
\[
(\hat{R}_{CG,1})_{kl}^{ij} = p^{k-l} \delta_l^i \delta_k^j + (1 - q^{-2}) \left( \sum_{k \leq s < l} - \sum_{l \leq s < k} \right) \delta_s^i \delta_{k+l-s}^j.
\]

We sum up these results in the following corollary.

**Corollary 3.** The above finite-dimensional restriction of the operator $\hat{R}$, defined by (5), gives rise to a quantum Lie algebra associated with the $p = 1$ member of the family of non-unitary Cremmer-Gervais R-matrices. The non-zero structure constants $C_{kl}^{ij}$ are all equal to $\pm C$ (the constant $C$ can be set to 1 by rescalings),
\[
C_{j1}^j = -C_{1j}^j = C, \quad j = 1 \ldots n.
\]
Remark 1. Our treatment is an extension of the construction of [9] in which the finite-dimensional Cremmer-Gervais matrices arise upon restrictions of infinite-dimensional functional R-matrices to the spaces of polynomials. The boundary (unitary) Cremmer-Gervais solution of the Yang-Baxter equation can be treated along the same lines [9]. The boundary Cremmer-Gervais R-matrix as well gives rise to a quantum Lie algebra which will be described elsewhere.

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