Nonsignaling as the consistency condition for local quasi-classical probability modeling of a general multipartite correlation scenario

Elena R Loubenets

Applied Mathematics Department, Moscow State Institute of Electronics and Mathematics, Moscow 109028, Russia

Received 6 September 2011, in final form 8 March 2012
Published 23 April 2012
Online at stacks.iop.org/JPhysA/45/185306

Abstract

We specify for a general correlation scenario a particular type of local quasi hidden variable (LqHV) model (Loubenets 2012 J. Math. Phys. 53 022201)—a deterministic LqHV model, where all joint probability distributions of a correlation scenario are simulated via a single measure space with a normalized bounded real-valued measure not being necessarily positive and random variables, each depending only on a setting of the corresponding measurement at the corresponding site. We prove that an arbitrary multipartite correlation scenario admits a deterministic LqHV model if and only if all its joint probability distributions satisfy the consistency condition, constituting the general nonsignaling condition formulated in Loubenets (2008 J. Phys. A: Math. Theor. 41 445303). This mathematical result specifies a new probability model that has a measure-theoretic structure resembling the structure of the classical probability model but incorporates the latter only as a particular case. The local version of this quasi-classical probability model covers the probabilistic description of each nonsignaling correlation scenario, in particular, each correlation scenario on a multipartite quantum state.

PACS numbers: 03.65.Ta, 02.50.Cw, 03.67.–a

1. Introduction

A possibility of the description of quantum measurements in terms of the classical probability model has been a point of intensive discussion ever since the seminal publications of von Neumann [1], Kolmogorov [2], Einstein, Podolsky and Rosen (EPR) [3] and Bell [4, 5].

Although, in the quantum physics literature, one can still find the misleading claims on a peculiarity of ‘quantum probabilities’ and ‘quantum events’, the probabilistic description of each quantum measurement satisfies the Kolmogorov axioms [2] for the theory of probability.

1 On the misleading character of such statements, see also [6].
Namely, each measurement on a quantum system represented initially by a state $\rho$ on a complex separable Hilbert space $H$ is described by the probability space $(\Lambda, F_{\Lambda}, \text{tr}[\rho M(\cdot)])$, where $\Lambda$ is a set of measurement outcomes, $F_{\Lambda}$ is a $\sigma$-algebra of observed events $F \subseteq \Lambda$ and $\text{tr}[\rho M(\cdot)] : F_{\Lambda} \rightarrow [0, 1]$ is the probability measure with values $\text{tr}[\rho M(F)]$, $F \in F_{\Lambda}$, each defining the probability that, under this quantum measurement, an outcome $\lambda$ belongs to a set $F \in F_{\Lambda}$. Here, $M$ is a normalized ($M(\Lambda) = I_H$) measure with values $M(F)$, $F \in F_{\Lambda}$, that are positive operators on $H$—that is, a normalized positive operator-valued (POV) measure on $(\Lambda, F_{\Lambda})$.

The measure-theoretic structure of the Kolmogorov axioms [2] is crucial and the probabilistic description of each measurement in every application field satisfies these probability axioms.

However, the classical probability model, which is also often named after Kolmogorov in the mathematical physics literature and where system observables and states are represented by random variables and probability measures on a single measurable space $(\Omega, F_{\Omega})$, describes correctly randomness in the classical statistical mechanics and many other application fields, but fails either to reproduce noncontextually [10] the statistical properties of all quantum observables on a Hilbert space of a dimension $\text{dim } H \geq 3$ or to simulate via random variables, each depending only on a setting of the corresponding measurement at the corresponding site, the probabilistic description of a quantum correlation scenario on an arbitrary $N$-partite quantum state. For details and references, see section 1.4 in [11] and the introduction in [12].

The probabilistic description of an arbitrary multipartite correlation scenario also cannot be reproduced via the classical probability model.

Note that, in the quantum theory literature, the interpretation of quantum measurements in the classical probability terms is generally referred to as a hidden variable (HV) model.

In [13], we have introduced for a general correlation scenario the notion of a local quasi hidden variable (LqHV) model, where locality and the measure-theoretic structure inherent to a local hidden variable (LHV) model are preserved, but the positivity of a simulation measure is dropped. We have proved [13] that every quantum $S_1 \times \cdots \times S_N$-setting correlation scenario admits LqHV modeling and specified the state parameter determining quantitatively a possibility of an $S_1 \times \cdots \times S_N$-setting LHV description of an $N$-partite quantum state.

In this paper, we further develop the LqHV approach introduced in [13]. The paper is organized as follows.

In section 2, we specify for a general multipartite correlation scenario the notion of a deterministic LqHV model, where all joint probability distributions of a correlation scenario are simulated via a single measure space with a normalized bounded real-valued measure and random variables, each depending only on a setting of the corresponding measurement at the corresponding site. We show that the existence for a general correlation scenario of some LqHV model implies the existence for this scenario of a deterministic LqHV model.

In section 3, we prove that an arbitrary multipartite correlation scenario admits a deterministic LqHV model if and only if all its joint probability distributions satisfy the consistency condition, constituting the general nonsignaling condition formulated by equation (10) in [12].

In section 4, we summarize the main mathematical results of this paper and discuss their conceptual implication.

---

2 In the measure theory, this triple is called a measure space.

3 The description of a quantum measurement via a POV measure was introduced by Davies and Lewis [7, 8].

4 In the probability theory, the term ‘Kolmogorov probability model’ refers to the probabilistic description of a measurement via the Kolmogorov axioms, see, for example, [9].

5 For the general framework on the probabilistic description of multipartite correlation scenarios, see [12].
2. A deterministic LqHV model

Consider an $N$-partite correlation scenario, where each $n$th of $N \geq 2$ parties (players) performs $S_n \geq 1$ measurements with outcomes $\lambda_n \in \Lambda_n$ of an arbitrary type and $\mathcal{F}_{\Lambda_n}$ is a $\sigma$-algebra of events $F_n \subseteq \Lambda_n$ observed at the $n$th site. We label each measurement at the $n$th site by a positive integer $s_n = 1, \ldots, S_n$ and each of $N$-partite joint measurements, induced by this correlation scenario and with outcomes $(\lambda_1, \ldots, \lambda_N) \in \Lambda_1 \times \cdots \times \Lambda_N$, by an $N$-tuple $(s_1, \ldots, s_N)$, where the $n$th component refers to a measurement at the $n$th site.

For concreteness, we further specify an $S_1 \times \cdots \times S_N$ -setting correlation scenario by the symbol $\mathcal{E}_S$, where $S := S_1 \times \cdots \times S_N$, and denote by $p^{(E_3)}_{(s_1, \ldots, s_N)}$ a probability measure, defined on the direct product $^6(\Lambda_1 \times \cdots \times \Lambda_N, \mathcal{F}_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N})$ of measurable spaces $(\Lambda_n, \mathcal{F}_{\Lambda_n})$, $n = 1, \ldots, N$, and describing an $N$-partite joint measurement $(s_1, \ldots, s_N)$ under a scenario $\mathcal{E}_S$.

**Remark 1.** The superscript $E_3$ in notation $p^{(E_3)}_{(s_1, \ldots, s_N)}$ indicates that, in contrast to a correlation scenario represented by the so-called nonsignaling boxes [15, 16] and described by joint probability distributions $p^{(E_3)}_{(s_1, \ldots, s_N)} = p^{(E_3)}(s_1 = 1, \ldots, s_N = 1, \ldots, S_N)$, each depending only on settings of the corresponding measurements at the corresponding sites, for a general correlation scenario $\mathcal{E}_3$, each distribution $p^{(E_3)}_{(s_1, \ldots, s_N)}$ may also depend on settings of all (or some) other measurements. The latter is, for example, the case under a classical correlation scenario with ‘one-sided’ or ‘two-sided’ memory [17].

If under an $N$-partite joint measurement $(s_1, \ldots, s_N)$ of a scenario $\mathcal{E}_S$ only outcomes of $M < N$ parties $1 \leq n_1 < \cdots < n_M \leq N$ are taken into account while outcomes of all other parties are ignored, then the joint probability distribution of outcomes observed at these $M$ sites is described by the marginal probability distribution:

$$p^{(E_3)}_{(s_1, \ldots, s_N)}(\Lambda_1 \times \cdots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \cdots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \cdots \times \Lambda_n).$$

(1)

In particular,

$$p^{(E_3)}_{(s_1, \ldots, s_N)}(\Lambda_1 \times \cdots \times \Lambda_n)$$

(2)

is the probability distribution of outcomes observed at the $n$th site under a joint measurement $(s_1, \ldots, s_N)$ of the scenario $\mathcal{E}_S$.

**Remark 2.** Throughout this paper, for a measure $\tau$ on the direct product $(\Lambda \times \cdots \times \Lambda', \mathcal{F}_{\Lambda} \otimes \cdots \otimes \mathcal{F}_{\Lambda'})$ of some measurable spaces, we often use notation $\tau(d\lambda \times \cdots \times d\lambda')$ outside an integral. This allows us to easily specify the structure of different marginals of $\tau$.

For the probabilistic description of a general correlation scenario, consider the following simulation model introduced in [13].

**Definition 1 ([13]).** An $S_1 \times \cdots \times S_N$-setting correlation scenario $\mathcal{E}_S$, with joint probability distributions $p^{(E_3)}_{(s_1, \ldots, s_N)}$, $s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N$, and outcomes $(\lambda_1, \ldots, \lambda_N) \in \Lambda_1 \times \cdots \times \Lambda_N$ of an arbitrary type admits a local quasi hidden variable (LqHV) model if all of its joint probability distributions admit the representation

$$p^{(E_3)}_{(s_1, \ldots, s_N)}(F_1 \times \cdots \times F_N) = \int_{\Omega} P^{(s_1)}_{\omega}(F_1 | \omega) \cdots P^{(s_N)}_{\omega}(F_N | \omega) v_{\mathcal{E}_S}(d\omega),$$

(3)

$$F_1 \in \mathcal{F}_{\Lambda_1}, \ldots, F_N \in \mathcal{F}_{\Lambda_N},$$

$^6$ Recall that the product $\sigma$-algebra $\mathcal{F}_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}$ on $\Lambda_1 \times \cdots \times \Lambda_N$ is the smallest $\sigma$-algebra generated by the set of all rectangles $F_1 \times \cdots \times F_N \subseteq \Lambda_1 \times \cdots \times \Lambda_N$ with measurable ‘sides’ $F_n \in \mathcal{F}_{\Lambda_n}$, $n = 1, \ldots, N$, see [14].
in terms of a single measure space \((\Omega, \mathcal{F}_\Omega, \nu)\) with a normalized bounded real-valued measure \(\nu_{E_3}\), and conditional probability measures \(P_n^{(s_n)}(\cdot | \omega) : \mathcal{F}_{\Lambda_n} \rightarrow [0, 1]\), defined \(\nu_{E_3}\)-a.e. (almost everywhere) on \(\Omega\) and such that, for each \(s_n \equiv 1, \ldots, S_n\) and every \(n = 1, \ldots, N\), the function \(P_n^{(s_n)}(F_n | \cdot) : \Omega \rightarrow [0, 1]\) is measurable for each \(F_n \in \mathcal{F}_{\Lambda_n}\).

In a triple \((\Omega, \mathcal{F}_\Omega, \nu)\) representing a measure space, \(\Omega\) is a non-empty set, \(\mathcal{F}_\Omega\) is a \(\sigma\)-algebra of subsets of \(\Omega\) and \(\nu\) is a measure on a measurable space \((\Omega, \mathcal{F}_\Omega)\). A real-valued measure \(\nu\) is called normalized if \(\nu(\Omega) = 1\) and bounded [14] if \(|\nu(F)| \leq M < \infty\) for all \(F \in \mathcal{F}_\Omega\). Note that each bounded real-valued measure \(\nu\) admits \([14]\) the Jordan decomposition \(\nu = \nu^+ - \nu^-\) via positive measures

\[
v^+(F) := \sup_{F \in \mathcal{F}_\Omega, F \subseteq F} \nu(F), \quad v^-(F) := -\inf_{F \in \mathcal{F}_\Omega, F \subseteq F} \nu(F), \quad \forall F \in \mathcal{F}_\Omega,
\]

with disjoint supports.

We stress that, in a LqHV model \((3)\), a normalized bounded real-valued measure \(\nu_{E_3}\) has a simulation character and may, in general, depend (via the subscript \(E_3\)) on measurement settings at all or some sites, as an example, see measure \((39)\) in \([13]\).

The structure of each LqHV model is such that though some values of a simulation measure \(\nu_{E_3}\) may be negative, the integral standing on the right-hand side of representation \((3)\) is non-negative for all \(F_1 \in \mathcal{F}_{\Lambda_1}, \ldots, F_n \in \mathcal{F}_{\Lambda_n}\).

If, for a correlation scenario \(E_3\), there exists representation \((3)\), where a normalized bounded real-valued measure \(\nu_{E_3}\) is positive (hence, is a probability measure), then this scenario admits an LHV model formulated for a general case by equation \((26)\) in \([12]\).

As is discussed in detail in \([13]\), the concept of an LqHV model incorporates as particular cases and generalizes in one whole both types of simulation models known in the literature—an LHV model and an affine model \([18]\). Note that the latter model, where all distributions \(P_n^{(s_n)}(\cdot | \omega), s_n = 1, \ldots, S_n, n = 1, \ldots, N\), have the special form:

\[
P_n^{(s_n)}(F_n | \omega) = \chi_{\mathcal{F}_{\Lambda_n}(F_n)}(\omega), \quad \forall F_n \in \mathcal{F}_{\Lambda_n},
\]

\(\nu_{E_3}\)-a.e. on \(\Omega\).

Here, \(f^{-1}(F) = \{\omega \in \Omega \mid f(\omega) \in F\}\) is the pre-image of a set \(F \in \mathcal{F}_{\Lambda}\) under a mapping \(f : \Omega \rightarrow \Lambda\), and \(\chi_D(\cdot)\) is the indicator function of a subset \(D \subseteq \Omega\), that is, \(\chi_D(\omega) = 1\) for \(\omega \in D\) and \(\chi_D(\omega) = 0\) for \(\omega \notin D\).

The notion of a deterministic LqHV model generalizes the concept of a deterministic LHV model formulated for a general multipartite correlation scenario in section 4 of \([12]\).

From \((3)\) and \((5)\) it follows that if an \(S_1 \times \cdots \times S_N\)-setting correlation scenario \(E_3\) admits a deterministic LqHV model, then all its joint probability distributions \(P_n^{(E_3)}(\cdot)\) admit the representation

\[
P_n^{(E_3)}(F_1 \times \cdots \times F_N) = \int_{\Omega} \chi_{f^{-1}_1(F_1)}(\omega) \cdots \chi_{f^{-1}_N(F_N)}(\omega) \nu_{E_3}(d\omega)
\]

\(\nu_{E_3}\)-a.e. on \(\Omega\).

\(\text{Definition 2.}\) An LqHV model \((3)\) is called deterministic if there exist \(\mathcal{F}_{\Lambda_n}/\mathcal{F}_{\Lambda_n}\)-measurable functions (random variables) \(f_{n,s_n} : \Omega \rightarrow \Lambda_n\), such that, in representation \((3)\), all conditional probability measures \(P_n^{(s_n)}(\cdot | \omega), s_n = 1, \ldots, S_n, n = 1, \ldots, N\), have the special form:

\[
P_n^{(s_n)}(F_n | \omega) = \chi_{\mathcal{F}_{\Lambda_n}(F_n)}(\omega), \quad \forall F_n \in \mathcal{F}_{\Lambda_n},
\]

\(\nu_{E_3}\)-a.e. on \(\Omega\).

7 The terms ‘deterministic HV model’ and ‘stochastic HV model’ were first introduced by Fine \([19]\) for a bipartite scenario with two settings and two outcomes per site.
\[ F_1 \in \mathcal{F}_{\Lambda_1}, \ldots, F_N \in \mathcal{F}_{\Lambda_N}, \] 
(6)

via a normalized bounded real-valued measure \( \nu_{\mathcal{E}} \) on some measurable space \((\Omega, \mathcal{F}_\Omega)\) and random variables \(f_{n,s_n} : \Omega \to \Lambda_n\), each depending only on a setting of the \(s_n\)th measurement at the \(n\)th site.

We stress that, in a deterministic LqHV model, the relation between a simulation measure \(\nu_{\mathcal{E}}\) and random variables \(f_{n,s_n}\), \(s_n = 1, \ldots, S_n, n = 1, \ldots, N\), modeling scenario measurements is such that the joint probabilities of scenario events are reproduced due to (6) only via non-negative values of \(\nu_{\mathcal{E}}\).

Representation (6), in turn, implies that for arbitrary bounded measurable real-valued functions \( \phi_n : \Lambda_n \to \mathbb{R}, n = 1, \ldots, N\), the product expectation

\[ \langle \phi_1(\lambda_1) \cdot \ldots \cdot \phi_N(\lambda_N) \rangle_{(\Omega_1, \ldots, \Omega_N)}^{(\mathcal{E}_0)} := \int_\Lambda \phi_1(\lambda_1) \cdot \ldots \cdot \phi_N(\lambda_N) P_{(\Omega_1, \ldots, \Omega_N)}^{(\mathcal{E}_0)}(d\lambda_1 \times \ldots \times d\lambda_N) \] 
(7)

takes the form

\[ \langle \phi_1(\lambda_1) \cdot \ldots \cdot \phi_N(\lambda_N) \rangle_{(\Omega_1, \ldots, \Omega_N)}^{(\mathcal{E}_0)} = \int (\phi_1 \circ f_{1,s_1}) (\omega) \cdot \ldots \cdot (\phi_N \circ f_{N,s_N}) (\omega) \nu_{\mathcal{E}_0}(d\omega), \] 
(8)

which differs from the form of the product expectations in a deterministic LHV model (see equation (31) in [12]) only by the fact that a normalized bounded real-valued measure \(\nu_{\mathcal{E}_0}\) in (8) does not need to be positive.

Recall that, for a given correlation scenario, a deterministic LHV model constitutes the version of the local classical probability model, where only the observed joint probability distributions are reproduced [12].

Therefore, a deterministic LqHV model (6) corresponds to the local quasi-classical probability model, where, in contrast to the local classical probability model, an 'underlying' probability space is replaced by a measure space \((\Omega, \mathcal{F}_\Omega, \nu)\) with a normalized bounded real-valued measure \(\nu\) not necessarily positive and where

(i) observables with a value space \((\Lambda, \mathcal{F}_\Lambda)\) are represented only by such random variables \(f : \Omega \to \Lambda\) for which \(\nu(f^{-1}(F)) \geq 0, \forall F \in \mathcal{F}_\Lambda\);

(ii) a joint measurement of two observables \(f_1\) and \(f_2\), each with a value space \((\Lambda_n, \mathcal{F}_{\Lambda_n})\), is possible if and only if \(\nu(f_1^{-1}(F_1) \cap f_2^{-1}(F_2)) \geq 0\) for all \(F_n \in \mathcal{F}_{\Lambda_n}\).

The following statement is proved in appendix A.

**Proposition 1.** If an \(S_1 \times \cdots \times S_N\)-setting correlation scenario \(\mathcal{E}_S\) admits some LqHV model (3), then it also admits a deterministic LqHV model (6).

This proposition and theorem 1 of [13] imply.

**Proposition 2.** An \(S_1 \times \cdots \times S_N\)-setting correlation scenario \(\mathcal{E}_S\) admits a deterministic LqHV model (6) if and only if, on the direct product space \((\Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}, \mathcal{F}_{\Lambda_1}^{\otimes S_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}^{\otimes S_N})\), there exists a normalized bounded real-valued measure \(\mu_{\mathcal{E}_S}\)

\[ \mu_{\mathcal{E}_S}(d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)}), \]

\[ \lambda_n^{(s_n)} \in \Lambda_n, \quad s_n = 1, \ldots, S_n, \quad n = 1, \ldots, N, \] 
(9)

returning all joint probability distributions \(P_{(\Omega_1, \ldots, \Omega_N)}^{(\mathcal{E}_S)}\) of a scenario \(\mathcal{E}_S\) as the corresponding marginals.

\[ \text{See Remark 2.} \]
3. The general consistency theorem

Let us now analyze under what condition on joint probability distributions an arbitrary multipartite correlation scenario admits a deterministic LqHV model.

Suppose that under an $S_1 \times \cdots \times S_N$-setting correlation scenario $\mathcal{E}_S$, for all joint measurements $(s_1, \ldots, s_n)$, $(s_1', \ldots, s_n')$ with $1 \leq M < N$ common settings $s_{n_1}, \ldots, s_{n_M}$ at arbitrary sites $1 \leq n_1 < \cdots < n_M \leq N$, the marginal probability distributions (1) of outcomes observed at these sites coincide, that is,

$$P^{\mathcal{E}_S}_{(s_1, \ldots, s_n)}(\Lambda_1 \times \cdots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \cdots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \cdots \times \Lambda_{n_F})$$

$$= P^{\mathcal{E}_S}_{(s_1', \ldots, s_n')}^{( \mathcal{E}_S )}(\Lambda_1 \times \cdots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \cdots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \cdots \times \Lambda_{n_F}).$$

As we discuss this in section 3 of [12], for a general correlation scenario (11), theorem 1 implies the general nonsignaling condition and the EPR locality condition, respectively. Moreover, (10) the nonsignaling condition and the EPR locality condition, respectively. This means that condition (10) should be distinguished from the condition

$$P^{\mathcal{E}_S}_{(s_1, \ldots, s_n)}(\Lambda_1 \times \cdots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \cdots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \cdots \times \Lambda_{n_F})$$

$$= P^{\mathcal{E}_S}_{(s_1, \ldots, s_n)}(d\lambda_{n_1} \times \cdots \times d\lambda_{n_M}),$$

$s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N$, $M = 1, \ldots, N$, (11) usually argued in the literature to follow if $M < N$ from condition (10).

Although condition (11) implies condition (10), the converse is not, in general, true; see proposition 1 in [12].

In view of their physical interpretations discussed in detail in [12], we call conditions (10) and (11) the nonsignaling condition and the EPR locality condition, respectively. Moreover, since in the literature on quantum information specifically the joint combination of conditions (10) and (11) is often called nonsignaling, in order to exclude a possible misunderstanding, we further refer to the consistency condition (10) as the general nonsignaling condition.

We stress that the nonsignaling condition in the sense of [15, 16] implies the general nonsignaling condition (10), but the converse of this statement is not, in general, true.

The following theorem is proved in appendix B.

**Theorem 1.** An $S_1 \times \cdots \times S_N$-setting correlation scenario $\mathcal{E}_S$ admits a deterministic LqHV model (6) if and only if all its joint probability distributions $P^{\mathcal{E}_S}_{(s_1, \ldots, s_n)}$, $s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N$, satisfy the consistency condition (10), constituting the general nonsignaling condition formulated in [12].

Consider, in particular, an $S_1 \times \cdots \times S_N$-setting correlation scenario performed on an $N$-partite quantum state $\rho$ on a complex separable Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and described by the joint probability distributions

$$\text{tr} \left[ \rho \left[ M^{(1)}_{n_1}(F_1) \otimes \cdots \otimes M^{(N)}_{n_N}(F_N) \right] \right],$$

$$F_n \in \mathcal{F}_n, \quad s_n = 1, \ldots, S_n, \quad n = 1, \ldots, N,$$

(12) where $M^{(n)}_{s_n}$ is a POVM measure on $(\Lambda_n, \mathcal{F}_{\Lambda_n})$ representing on a Hilbert space $\mathcal{H}_n$ a quantum measurement $s_n$ at the $n$th site.

Since each quantum correlation scenario (12) satisfies condition (10) (as well as condition (11)), theorem 1 implies

---

9. See, for example, [15, 16, 18] and references therein.

10. For this notion, see the introduction.
Corollary 1. For every quantum state $\rho$ on a complex separable Hilbert space $H_1 \otimes \cdots \otimes H_N$ and arbitrary positive integers $S_1, \ldots, S_N \geq 1$, the probabilistic description of each quantum $S_1 \times \cdots \times S_N$-setting correlation scenario (12) admits a deterministic LqHV model.

In view of the above proposition 1, the statement of corollary 1 agrees with the statement of theorem 2 in [13].

4. Conclusions

In this paper, we have introduced (definition 2) the notion of a deterministic LqHV model, where all joint probability distributions of a multipartite correlation scenario are simulated via a single measure space $(\Omega, \mathcal{F}_\Omega, \nu)$, with a normalized bounded real-valued measure $\nu$ not necessarily positive, and random variables which are local in the sense that each of these random variables depends only on a setting of the corresponding measurement at the corresponding site.

We have proved (theorem 1) that a general $S_1 \times \cdots \times S_N$-setting correlation scenario admits a deterministic LqHV model if and only if all its joint probability distributions satisfy the consistency condition (10), constituting the general nonsignaling condition formulated in [12].

This general result, in particular, implies (corollary 1) that the probabilistic description of each $S_1 \times \cdots \times S_N$-setting correlation scenario (12) on an $N$-partite quantum state admits modelling in local quasi-classical probability terms.

From the conceptual point of view, these mathematical results specify a new probability model that has the measure-theoretic structure $(\Omega, \mathcal{F}_\Omega, \nu)$, resembling the structure of the classical probability model but reduces to the latter iff a normalized bounded real-valued measure $\nu$ is positive. In the frame of this quasi-classical probability model:

(i) observables with a value space $(\Lambda, \mathcal{F}_\Lambda)$ are represented only by such random variables $f : \Omega \to \Lambda$ for which $\nu(f^{-1}(F)) \geq 0, \forall F \in \mathcal{F}_\Lambda$;

(ii) a joint measurement of two observables $f_1$ and $f_2$, each with a value space $(\Lambda_n, \mathcal{F}_{\Lambda_n})$, is possible if and only if $\nu(f_1^{-1}(F_1) \cap f_2^{-1}(F_2)) \geq 0$ for all $F_n \in \mathcal{F}_{\Lambda_n}, n = 1, 2$.

In the quasi-classical probability model, the relation between a simulation measure $\nu$ and random variables modeling observables is such that probabilities of the observed events are reproduced only via positive values of a normalized bounded real-valued measure $\nu$.

The local version of the quasi-classical probability model covers (theorem 1) the probabilistic description of each nonsignaling multipartite correlation scenario, in particular, each multipartite correlation scenario (corollary 1) on an $N$-partite quantum state.

Appendix A.

Proof of proposition 1. Let a scenario $\mathcal{E}_S$ admit an LqHV model (3). Introduce the normalized real-valued measure

$$\mu_{\mathcal{E}_S}(d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)})$$

$$: = \int_{\Omega} \left\{ \prod_{n=1}^{N} P_n^{(s_n)}(d\lambda_n^{(s_n)} | \omega) \right\} \nu_{\mathcal{E}_S}(d\omega). \quad (A.1)$$

This measure is bounded (see the proof of theorem 1 in [13]) and returns all distributions $P_{(s_1, \ldots, s_N)}(\mathcal{E}_S)$ of the scenario $\mathcal{E}_S$ as the corresponding marginals. The latter means the factorizable
representation
\[
P_{\epsilon_1, \ldots, \epsilon_N}(F_1 \times \cdots \times F_N) = \int \chi_{F_1}(\lambda_1^{(S_1)}) \cdot \cdots \cdot \chi_{F_N}(\lambda_N^{(S_N)}) \, \mu_{\epsilon_1}(d\lambda_1^{(S_1)}) \cdot \cdots \cdot \mu_{\epsilon_N}(d\lambda_N^{(S_N)}),
\]
\[
F_1 \in \mathcal{F}_{A_1}, \ldots, F_N \in \mathcal{F}_{A_N},
\]
for all \( s_n = 1, \ldots, S_n, \) \( n = 1, \ldots, N. \) Denote
\[
\tilde{\omega} := (\lambda_1^{(S_1)}, \ldots, \lambda_n^{(S_n)}, \ldots, \lambda_N^{(S_N)}),
\]
\[
\Omega := A_1^{S_1} \times \cdots \times A_N^{S_N}, \quad \mathcal{F}_{\Omega} := \mathcal{F}_{A_1}^{\otimes S_1} \otimes \cdots \otimes \mathcal{F}_{A_N}^{\otimes S_N},
\]
\[
\tilde{\nu}_{\epsilon_n}(d\tilde{\omega}) := \mu_{\epsilon_n}(d\lambda_1^{(S_1)} \times \cdots \times d\lambda_n^{(S_n)} \times \cdots \times d\lambda_N^{(S_N)})
\]
and introduce the \( \mathcal{F}_{\Omega}/\mathcal{F}_{A_n} \)-measurable functions \( f_{n, n} : \Omega \to \Lambda_n, \) each defined by the relation
\[
f_{n, n}(\tilde{\omega}) = \lambda_n^{(s_n)}.
\]
Then
\[
\chi_{F_1}(\lambda_1^{(s_1)}) = \chi_{f_{1, n}^{-1}(F_1)}(\tilde{\omega}), \quad \forall F_1 \in \mathcal{F}_{A_1},
\]
and, in view of (A.3), (A.4), representation (A.2) takes the form
\[
P_{\epsilon_1(1), \ldots, \epsilon_N(1)}(F_1 \times \cdots \times F_N) = \int_{\Omega} \chi_{f_{2, n}^{-1}(F_2)}(\tilde{\omega}) \cdots \chi_{f_{N, n}^{-1}(F_N)}(\tilde{\omega}) \tilde{\nu}_{\epsilon_1}(d\tilde{\omega}) = \tilde{\nu}_{\epsilon_1}(f_{1, n}^{-1}(F_1) \cap \cdots \cap f_{N, n}^{-1}(F_N)).
\]
This proves the statement of proposition 1. \( \square \)

Appendix B.

Proof of theorem 1. If an \( N \)-partite correlation scenario \( \mathcal{E}_S \), with a setting \( S = S_1 \times \cdots \times S_N \), admits a deterministic LqHV model (6), then, clearly, the consistency condition (10) is fulfilled.

Conversely, let the scenario \( \mathcal{E}_S \) satisfy the consistency condition (10). Consider first a bipartite \( (N = 2) \) scenario \( \mathcal{E}_S \) with a setting \( S = S_1 \times S_2 \) and joint probability distributions \( P_{\epsilon_1, \epsilon_2}(F_1 \times F_2) \) satisfying condition (10). Since, under condition (10), marginals \( P_{\epsilon_1, \epsilon_2(1)}(F_1 \times A_2), \ldots, P_{\epsilon_1, \epsilon_2(2)}(F_1 \times A_2) \) at site ‘1’ coincide for all \( s_1 = 1, \ldots, S_1 \) for the simplicity of notation, we denote these coinciding marginals as
\[
P_{\epsilon_1}^{(S_1)}(A_1^2 \times A_1^2) = \cdots = P_{\epsilon_1}^{(S_1)}(A_1^1 \times A_1^2) := P_{\epsilon_1}^{(S_1)}(F_1),
\]
for all \( F_1 \in \mathcal{F}_{A_1} \) and each \( s_1 = 1, \ldots, S_1 \) at site ‘1’. The superscript \( \mathcal{E}_S \) in notation \( P_{\epsilon_1}^{(S_1)} \) indicates that, for a general correlation scenario, this marginal does not need to depend only on a setting of measurement \( s_1 \) at site ‘1’ (see remark 1).

Quite similarly,
\[
P_{\epsilon_1}^{(S_1)}(A_1^1 \times F_2) = \cdots = P_{\epsilon_1}^{(S_1)}(A_1^1 \times F_2) := P_{\epsilon_1}^{(S_1)}(F_2),
\]
for all \( F_2 \in \mathcal{F}_{A_2} \) and each \( s_2 = 1, \ldots, S_2 \) at site ‘2’.

Introduce the normalized bounded real-valued bounded measure \( \mu_{\mathcal{E}_S} \) on \( (A_1^{S_1} \times A_2^{S_2}), \)
\[
\mathcal{F}_{A_1}^{\otimes S_1} \otimes \mathcal{F}_{A_2}^{\otimes S_2}
\]
with values
\[
\mu_{\mathcal{E}_S} \left( F_1^{(1)} \times \cdots \times F_1^{(S_1)} \times F_2^{(1)} \times \cdots \times F_2^{(S_2)} \right)
\]
\[
:= \sum_{s_1, s_2} \left\{ P_{\epsilon_1}^{(S_1)}(F_1^{(s_1)}) \times \cdots \times P_{\epsilon_2}^{(S_2)}(F_2^{(s_2)}) \prod_{s_1 \neq s_1} P_{\epsilon_1}^{(S_1)}(F_1^{(s_1)}) \prod_{s_2 \neq s_2} P_{\epsilon_2}^{(S_2)}(F_2^{(s_2)}) \right\}
\]
\[
- (S_1 S_2 - 1) \prod_{s_1} P_{\epsilon_1}^{(S_1)}(F_1^{(s_1)}) \prod_{s_2} P_{\epsilon_2}^{(S_2)}(F_2^{(s_2)}),
\]
\]
for all $F_n(s_n)\in\mathcal{F}_{A_n}$, $s_n = 1, \ldots , S_n$, $n = 1, 2$. It is easy to verify that this measure returns all joint probability distributions $P(\xi_3)$ of a bipartite nonsignaling scenario $\xi_3$ as the corresponding marginals. By proposition 2, this implies that a bipartite correlation scenario satisfying condition (10) admits a deterministic LqHV model.

Let $N = 3$. Consider a tripartite correlation scenario $\xi_3$ with a setting $S = S_1 \times S_2 \times S_3$ and joint probability distributions $P(\xi_3)$ satisfying condition (10). In addition to the one-party marginals denoted similar to notation (B.2), we denote by $P(\xi_3)$, $P(\xi_3)$ and $P(\xi_3)$ the coinciding two-party marginals at the corresponding sites, that is,

\[
P(\xi_3)(F_1 \times F_2 \times \Lambda_3) = \cdots = P(\xi_3)(F_1 \times F_2 \times \Lambda_3) := P(\xi_3)(F_1 \times F_2),
\]

\[
P(\xi_3)(F_1 \times \Lambda_2 \times F_3) = \cdots = P(\xi_3)(F_1 \times \Lambda_2 \times F_3) := P(\xi_3)(F_1 \times F_3),
\]

\[
P(\xi_3)(\Lambda_1 \times F_2 \times F_3) = \cdots = P(\xi_3)(\Lambda_1 \times F_2 \times F_3) := P(\xi_3)(F_2 \times F_3),
\]

for all $F_n \in \mathcal{F}_{A_n}$, $s_n = 1, \ldots , S_n$, $n = 1, 2, 3$.

Similar to our construction of measure (B.3) for a bipartite case, introduce the normalized bounded real-valued measure $\widetilde{\mu}_{\xi_3}$ on $(\Lambda_1^S \times \Lambda_2^S \times \Lambda_3^S \times \mathcal{F}_{A_1} \otimes \mathcal{F}_{A_2} \otimes \mathcal{F}_{A_3})$ with values

\[
\widetilde{\mu}_{\xi_3}(F(1) \times \cdots \times F_{n-1}(s_{n-1}) \times F_n(s_n) \times \cdots \times F_{n+k-1}(s_{n+k-1})
\]

\[
- (S_1 - 1) \prod_{s_1} P(\xi_3)(F_1(s_1)) \sum_{s_2} \prod_{s_3} P(\xi_3)(F_2(s_2) \times F_3(s_3)) \prod_{\tilde{s}_2 \neq s_2} P(\xi_3)(F_{\tilde{s}_2}(\tilde{s}_{\tilde{s}_2})) \prod_{\tilde{s}_3 \neq s_3} P(\xi_3)(F_{\tilde{s}_3}(\tilde{s}_{\tilde{s}_3}))
\]

\[
- (S_2 - 1) \prod_{s_2} P(\xi_3)(F_2(s_2)) \sum_{s_1} \prod_{s_3} P(\xi_3)(F_1(s_1) \times F_3(s_3)) \prod_{\tilde{s}_1 \neq s_1} P(\xi_3)(F_{\tilde{s}_1}(\tilde{s}_{\tilde{s}_1})) \prod_{\tilde{s}_3 \neq s_3} P(\xi_3)(F_{\tilde{s}_3}(\tilde{s}_{\tilde{s}_3}))
\]

\[
- (S_3 - 1) \prod_{s_3} P(\xi_3)(F_3(s_3)) \sum_{s_1} \prod_{s_2} P(\xi_3)(F_1(s_1) \times F_2(s_2)) \prod_{\tilde{s}_1 \neq s_1} P(\xi_3)(F_{\tilde{s}_1}(\tilde{s}_{\tilde{s}_1})) \prod_{\tilde{s}_2 \neq s_2} P(\xi_3)(F_{\tilde{s}_2}(\tilde{s}_{\tilde{s}_2}))
\]

\[
+ (2S_1S_2S_3 - S_1S_2 - S_2S_3 - S_1S_3 + 1) \prod_{s_1} P(\xi_3)(F_1(s_1)) \prod_{s_2} P(\xi_3)(F_2(s_2)) \prod_{s_3} P(\xi_3)(F_3(s_3)),
\]

for all sets $F_n(s_n) \in \mathcal{F}_{A_n}$, $s_n = 1, \ldots , S_n$, $n = 1, 2, 3$. Measure $\widetilde{\mu}_{\xi_3}$ returns all joint probability distributions $P(\xi_3)$ of a tripartite nonsignaling scenario $\xi_3$ as the corresponding marginals. By proposition 2, the latter implies that a correlation scenario $\xi_3$ satisfying condition (10) admits a deterministic LqHV model.

The obvious generalization to an arbitrary $N$-partite case of the measure constructions used in (B.3) and (B.5) proves the sufficiency part of theorem 1.

References

[1] Von Neumann J 1932 Mathematische Grundlagen der Quantenmechanik (Berlin: Springer)

[2] Kolmogorov A N 1933 Grundbegriffe der Wahrscheinlichkeitsrechnung (Berlin: Springer)

[3] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777
[4] Bell J S 1964 Physica 1 195
[5] Bell J S 1966 Rev. Mod. Phys. 38 447
[6] Khrennikov A Y and Loubenets E R 2004 Found. Phys. 34 689
[7] Davies E B and Lewis J T 1970 Commun. Math. Phys. 17 239
[8] Davies E B 1976 Quantum Theory of Open Systems (London: Academic)
[9] Shiraev A N 2011 Probability 5th edn (Moscow: MCNMO)
    Shiraev A N 1996 Probability 2nd edn (Berlin: Springer)
[10] Kochen S and Specker E 1968 J. Math. Mech. 17 59
[11] Holevo A S 2001 Statistical Structure of Quantum Theory (Berlin: Springer)
[12] Loubenets E R 2008 J. Phys. A: Math. Theor. 41 445303
[13] Loubenets E R 2012 J. Math. Phys. 53 022201
[14] Dunford N and Schwartz J T 1957 Linear Operators: Part I. General Theory (New York: Interscience)
[15] Barrett J, Linden N, Massar S, Pironio S, Popescu S and Roberts D 2005 Phys. Rev. A 71 022101
[16] Barrett J 2007 Phys. Rev. A 75 032304
[17] Barrett J, Collins D, Hardy L, Kent A and Popescu S 2002 Phys. Rev. A 66 042111
[18] Degorre J, Kaplan M, Laplante S and Roland J 2009 The communication complexity of non-signaling distributions Mathematical Foundations of Computer Science 2009 34th Int. Symp. MFCS 2009, Novy Smokovec, High Tatras, Slovakia, August 2009 (Lecture Notes in Computer Science vol 5734) ed R Královič and D Niwinski (Berlin: Springer) pp 270–81
[19] Fine A 1982 Phys. Rev. Lett. 48 291