Forced MHD turbulence in three dimensions using Taylor-Green symmetries

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We examine the scaling laws of MHD turbulence for three different types of forcing functions and imposing at all times the four-fold symmetries of the Taylor-Green (TG) vortex generalized to MHD; no uniform magnetic field is present and the magnetic Prandtl number is equal to unity. We also include a forcing in the induction equation, and we take the three configurations studied in the decaying case in [E. Lee et al. Phys. Rev.E 81, 016318 (2010)]. To that effect, we employ direct numerical simulations up to an equivalent resolution of 2048\textsuperscript{3} grid points. We find that, similarly to the case when the forcing is absent, different spectral indices for the total energy spectrum emerge, corresponding to either a Kolmogorov law, an Iroshnikov-Kraichnan law that arises from the interactions of turbulent eddies and Alfvén waves, or to weak turbulence when the large-scale magnetic field is strong. We also examine the inertial range dynamics in terms of the ratios of kinetic to magnetic energy, and of the turn-over time to the Alfvén time, and analyze the temporal variations of these quasi-equilibria.

\textbf{I. INTRODUCTION}

Turbulence is a common feature of a variety of flows, from engineering to geophysics and astrophysics. It remains unsolved, due in part to a lack of statistical theory on how to deal with a very large number of modes interacting nonlinearly, and competing with waves. At the moderate Reynolds numbers that are achievable today numerically in three space dimensions on uniform grids of at most 4096\textsuperscript{3} points, one follows accurately the temporal evolution of in excess of 64 billion modes, leading to the creation of myriads of vortex filaments. When coupling to a magnetic field, in the magnetohydrodynamic (MHD) approximation regime for velocities small compared to the speed of light, one observes current and vorticity sheets, that are found to roll-up for sufficiently high Reynolds numbers \textsuperscript{1}.

One question concerns the universality or not of the scaling laws of turbulent flows. There has been much debate concerning this point, in particular in the MHD community as well as when dealing with the dynamics of the atmosphere and the oceans: one way to phrase the question is to ask wether the presence of waves will affect the energy distribution among modes, inertial waves in the rotating case with solid body rotation, gravity waves for stratified turbulence, Alfvén waves in MHD, acoustic waves when the condition of incompressibility is removed, as is necessary in the interstellar medium where supersonic flows are routinely observed. The answer is unambiguous in the regime of weak turbulence when the ratio of characteristic times (wave period and eddy turnover time) is small; this small parameter allows for a natural closure to the statistical problem and constant-flux (as well as zero flux) solutions can be found in terms of power laws as a function of anisotropic wave numbers, the anisotropy arising from the imposition of an external agent (uniform rotation, gravity, or magnetic field), and to the anisotropic dispersion relations \textsuperscript{2}. But this weak-turbulence regime is non-uniform in scale, simply because the variation with scale of the wave period $\tau_W$ and of the eddy turn-over time $\tau_{NL}$ are different; hence, there exists a scale at which these two timescales are equal and the weak turbulence regime breaks down. For stratified flows, this is called the Ozmidov length scale, and for the rotating case, the Zeman scale. Note that for MHD, the situation is different: for stratified and rotating flows, at scales smaller than the Ozmidov or Zeman scales, isotropy and a classical Kolmogorv scaling is likely to recover \textsuperscript{3}, whereas it does not in MHD. In fact, one could argue the opposite: isotropy can prevail at large scales where the effect of the large-scale magnetic field is purely local and its amplitude is comparable to that of the modes it is interacting with, whereas the anisotropic effect due to the imposed large-scale magnetic field is strong at small scale unless reconnection processes are numerous and random enough that isotropy again is recovered. This point is still in debate.

In MHD, another hypothesis has been put forward to understand the dynamical exchanges in a phenomenological way, that of an equality between the two characteristic time scales, an equality that would hold throughout the inertial range \textsuperscript{4}. This hypothesis leads to a Kolmogorov spectrum $E(k) \sim k^{-5/3}$ (hereafter Kp41), expressed in terms of $k$, where the direction refers to that of the external agent, here a uniform magnetic field. It was found in \textsuperscript{5} that the same hypothesis can also lead to an
Iroshnikov-Kraichnan spectrum (IK hereafter) or a weak turbulence spectrum, $E(k_{1}, k_{2}) \sim k_{1}^{-2} f(k_{1})$ (WT hereafter), on the simple basis that $\tau_{W}(k)/\tau_{NL}(k) = r_{\nu}(k)$ can be constant but not necessarily equal to unity: the different regimes appear in that light as emerging from a different rate at which energy is exchanged between its kinetic and magnetic modes. All these spectra have been observed in direct numerical simulations (DNS) of three-dimensional (3D) MHD turbulence. In one particular case, identical velocity fields are used as initial conditions, with comparable invariants (total energy $E_{\text{tot}} = 0.25$ with $E_{\nu}(t = 0) = E_{b}(t = 0)$ where $E_{\nu,b}$ are the kinetic and magnetic energy respectively, with strictly zero magnetic helicity and negligible cross-correlation between the velocity and the magnetic field) [6]. It is the purpose of this paper to pursue the work done in [6], extending it to the statistically steady case in the presence of forcing.

In the next section, we write equations, initial conditions and forcing; the results are given for a high-resolution run in §III and in §IV we compare the evolution of three magnetic configurations, at resolutions of $1024^3$ grid points. Finally, §V is the conclusion.

II. THE EQUATIONS AND THE NUMERICAL SET-UP

The MHD equations for an incompressible fluid with $v$ and $b$ respectively the velocity and magnetic fields in Alfvénic units are:

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho_0} \nabla P + \mathbf{j} \times \mathbf{b} + \nu \nabla^2 v + F_{V}, \quad (1)$$

$$\frac{\partial b}{\partial t} = \nabla \times (v \times b) + \eta \nabla^2 b + F_{M} ; \quad (2)$$

$\rho_0 = 1$ is the (uniform) density, and $b$ is dimensionally a velocity as well, the Alfvén velocity; $P$ is the total pressure, $\nabla \cdot v = \nabla \cdot b = 0$, and $\nu$ and $\eta$ are respectively the kinematic viscosity and magnetic diffusivity: we take $\nu = \eta$ (unit magnetic Prandtl number). Finally, $F_{V}, F_{M}$ are forcing terms introduced both in the momentum and in the induction equations. In principle, above a critical Reynolds number $R_{M}^{C}$, a dynamo mechanism sets in whereby sufficient magnetic excitation is produced at all scales. For the Taylor-Green flow defined below it was shown in reference [7] that $R_{M}^{C}$ depends very strongly on the imposed symmetries. In addition, when imposing all symmetries at all times, it was shown in reference [8] that $R_{M}^{C}$ is very high (of the order of 1000). In this work, like in reference [6], we focus of the fully symmetric problem in order to maximize the available Reynolds number, and thus the maximum resolution (see discussion following Eq. (14) below). Hence, we chose to force the induction equation as a way to mimic the dynamo itself. We note that, by simply breaking the symmetry of the initial conditions and using a general code, this critical parameter is lowered by more than one order of magnitude, but with a substantially costlier computation, by a factor of 32 [8].

The forcing in the induction equation, $F_{M}$, is not a common choice. It is included in order to compensate for the fact that, in the presence of symmetries, the generation of a magnetic field by fluid turbulence (or dynamo effect) occurs above a threshold in magnetic Reynolds number of $R_{M} \approx 1000$, and is slow in this vicinity of $R_{M}$. Another justification for $F_{M} \neq 0$ comes from the dynamics of the Solar Wind [9, 10], with Alfvén wave forcing stemming from coronal mass ejections.

The energy $E_{\text{tot}}$, the cross helicity $H_{C}$ and the magnetic helicity $H_{M}$ are defined as

$$E_{\text{tot}} = E_{\nu} + E_{b} = \langle v^2 + b^2 \rangle / 2 \quad (3)$$

$$H_{C} = \langle v \cdot b \rangle \quad , \quad H_{M} = \langle A \cdot b \rangle \quad (4)$$

where $b = \nabla \times A$ and $A$ is the magnetic potential. In the ideal case ($\nu = \eta = 0$) and without forcing ($F_{V} = F_{M} = 0$) note that these quantities ($E_{\text{tot}}, H_{C}$ and $H_{M}$) are all conserved. Relative helicities measure the relative alignment of vectors, independently of their amplitudes, $\rho_{V} = \cos \langle v, \omega \rangle$, $\rho_{C} = \cos \langle b, \omega \rangle$, $\rho_{M} = \cos \langle A, b \rangle$, with $\omega = \nabla \times v$ the vorticity (the kinetic helicity $H_{V} = \langle v \cdot \omega \rangle$ is an invariant when $b \equiv 0$).

Considering a flow which is $2\pi$-periodic in all spatial dimensions, the kinematic Reynolds number $R_{\nu}$ and the magnetic Reynolds number $R_{m}$ are defined as

$$Re = \frac{L v_{\text{rms}}}{\nu}, \quad R_{m} = \frac{L v_{\text{rms}}}{\eta} \quad (5)$$

where the root-mean square velocity is $v_{\text{rms}} = \sqrt{2E_{\nu}/3}$ and the characteristic length $L$ is defined by

$$L = 2\pi \sum_{k} k^{-1} E_{\nu}(k,t) / \sum_{k} E_{\nu}(k,t) dk \quad (6)$$

where the kinetic energy spectrum $E_{\nu}(k,t)$ (such that $E_{\nu}(t) = \sum_{k} E_{\nu}(k,t)$) is obtained by summing $\frac{1}{2} \langle \hat{u}(k',t) \rangle^2$ on the spherical shells $k-1/2 < |k'| < k+1/2$ ($\hat{u}(k)$ is the Fourier transform of the velocity). Analogously, the magnetic energy spectrum is denoted by $E_{b}(k,t)$ and verifies $E_{b}(t) = \sum_{k} E_{b}(k,t)$.

We now turn to the definition of the external driving volumic forces $F_{V,M}$ in [11, 12] which balance the total energy dissipation and allow to reach a statistically stationary state. Following reference [8], we force the system by setting in [1]

$$F_{V} = f_{v} v^{TG}, \quad (7)$$

where $v^{TG}$ is the Taylor-Green vortex [11] given by

$$v^{TG} = (\sin(x) \cos(y) \cos(z), -\cos(x) \sin(y) \cos(z), 0), \quad (8)$$

and $f_{v}$ is always set to the value 1/16. The force $F_{M}$ is determined in a similar way but using instead of the TG velocity mode, the three magnetic field modes studied
in the decaying MHD runs of reference [6]. These three modes were:

\[
b^I = b_0^I \begin{pmatrix} \cos x \sin y \sin z \\ \sin x \cos y \sin z \\ -2 \sin x \sin y \cos z \end{pmatrix}, \quad b^A = b_0^A \begin{pmatrix} \cos 2x \sin 2y \sin 2z \\ -\sin 2x \cos 2y \sin 2z \\ 0 \end{pmatrix}, \quad b^C = b_0^C \begin{pmatrix} \sin 2x \cos 2y \cos 2z \\ \cos 2x \sin 2y \cos 2z \\ -2 \cos 2x \cos 2y \sin 2z \end{pmatrix}. \tag{9, 10, 11}
\]

The labels \(I, A, C\) stand respectively for insulating, alternate insulating and conducting boundary conditions for the current when considering its orientation with respect to the wall of the so-called fundamental box (see [12]). The coefficients \(b_0^I, b_0^A\) and \(b_0^C\) are such that, for all cases, \(E_b = 1/16\). Thus, all computations have equal initial kinetic and magnetic energy at \(t=0\), with \(E_{\text{tot}} = E_v + E_b = 1/8\).

Like in Eq. (7), the amplitudes of the forcing are chosen as

\[
F_M = f_b b^{\text{LA.C}}, \tag{12}
\]

and the values of \(f_b\) are given below in Table I.

We shall also examine the behavior of the spectral ratios of time-scales \(R_r\) and of modal energies \(R_E\) defined respectively as

\[
R_r(k) = \tau_{NL}(k)/\tau_{A}(k), \quad R_E(k) = E_b(k)/E_v(k), \tag{13}
\]

with \(\tau_{NL}(k) = [k \hat{u}(k)]^{-1}\) and \(\tau_{A}(k) = [kB_0]^{-1}\), \(B_0\) being defined here as the magnetic field in the gravest mode (the first non-zero mode).

Because of the symmetries of the TG vortex extended to MHD, all fields can be represented in Fourier space as:

\[
v_x(r,t) = \sum_{m,n,p=0}^{\infty} u_x(m,n,p,t) \sin mx \cos ny \cos pz, \tag{14}
\]

\[
v_y(r,t) = \sum_{m,n,p=0}^{\infty} u_y(m,n,p,t) \cos mx \sin ny \cos pz, \tag{15}
\]

\[
v_z(r,t) = \sum_{m,n,p=0}^{\infty} u_z(m,n,p,t) \cos mx \cos ny \sin pz, \tag{16}
\]

where \(u(m,n,p,t)\) is equal to zero unless the integers \(m,n,p\) are either all even or all odd. Thus, in spectral space, the following symmetry is fulfilled:

\[
u_x(m,n,p,t) = (-1)^r u_y(n,m,p,t), \tag{17}
\]

\[
u_z(m,n,p,t) = (-1)^r u_z(n,m,p,t), \tag{18}
\]

with \(r = 1\) if \(m,n,p\) are all odd, and \(r = 2\) if \(m,n,p\) are all even.

### III. RESULTS FOR THE C RUNS

The first run on which we report is the one at the highest Reynolds number (and the highest resolution). We give in Fig. Ia, top, the temporal evolution of the kinetic, magnetic and total energy. The thick lines are for Run C1 and the dashed lines for run C2 at lower resolution (see Table I). The ratio of magnetic to kinetic energy

| RUN | \(N\) | \(T_f\) | \(dt\) | \(\nu = \eta\) | \(f_b\) | \(Re\) |
|-----|-----|-----|-----|-----|-----|-----|
| C1  | 2048 | 22.5 | 3.125 \times 10^{-5} | 6.25 \times 10^{-9} | 6.25 \times 10^{-2} | 8700 |
| C2  | 1024 | 68   | 6.25 \times 10^{-4}  | 1.25 \times 10^{-4} | 6.25 \times 10^{-2} | 4470 |
| I   | 1024 | 52   | 6.25 \times 10^{-4}  | 6.25 \times 10^{-4} | 10^{-4} | 1360 |
| A   | 1024 | 52   | 6.25 \times 10^{-4}  | 2.5 \times 10^{-4}  | 6.25 \times 10^{-2} | 2270 |

TABLE I: List of runs and parameters. Resolution \(N\), final time \(T_f\) and time step \(dt\), viscosity and magnetic diffusivity \(\nu = \eta\), forcing parameter \(f_b\) and Reynolds number \(Re\) (see Eqs. 12 and 13).

Relations (17) and (18) allow one to only compute \(v_x\) and \(v_z\). Moreover, the decomposition [14] on either even or odd integers leads to a gain of a factor of 32 in memory and CPU time compared with the general case of Fourier transforms with the same scale separation, or \(k_{\text{max}}/k_{\text{min}}\), with respectively \(k_{\text{min}} = 1\) for a box of length \(2\pi\) and \(k_{\text{max}} = N/3\) with \(N\) the number of grid points per dimension, using a standard 2/3 de-aliasing rule. The code is pseudo-spectral, with a fourth-order Runge-Kutta temporal scheme and with periodic boundary conditions. All previous symmetry relations are implemented to speed up calculations. The code is parallelized up to \(\sim 98,000\) processors on grids of up to \(8196^3\) points, using a hybrid (MPI-Open-MP) algorithm which becomes advantageous at high resolution [13]. Grids used in this work have the equivalent resolution of \(1024^3\) and \(2048^3\) runs. Point parameters are summarized in Table I. To correctly resolve the MHD equations spectrally, a fast decay at large \(k\) (faster than algebraic) of the energy spectrum is required. This condition (called spectral convergence) is quantitatively determined by fitting the exponential decay of the energy spectra by a law of the form \(Ce^{-2\delta k}\) that amounts to a simple Lin-Log linear regression. The value of \(\delta k_{\text{max}}\) furnishes a measure of spectral convergence. We obtain values of \(\delta k_{\text{max}}\) of 5.4, 3.37, 2.3 and 5.2 for the runs C1, C2, I and A respectively, showing that all simulations all well resolved.

We would like to remark that if symmetries are not enforced, due to round noise a symmetry breaking can take place, as studied in [14]. Therefore, during the statistical stationary regime reached after a very long time, systems with and without imposed symmetries are not equivalent. The advantage of imposing the TG symmetries is not purely numerical, it also provide a way to mimic more realistic boundary condition for both, velocity and magnetic field (for a long discussion see for instance [7]).
$E_t = \frac{E_{\text{kin}}}{(2\pi)^2} = k^{\frac{5}{2}}$ and averaged in the same temporal interval as the energy spectrum. As can be seen from Fig. 2b, there is a systematic increase of this ratio in the inertial range, contrary to the hypothesis of critical balance advocated in [4] (see also [5, 15]). Rather, it is the magnetic to kinetic spectra ratio which remains remarkably constant throughout the inertial range, as displayed in Fig. 2b, with as often, a slight excess of magnetic energy, except at the gravest mode that dominates the global energetics; this confirms the earlier findings of the decay case [6], as well as those in numerous other numerical simulations (see e.g. [16]).

When examining the behavior of the C-flow at lower resolution on a grid of $1024^3$ points, we observe that the results are in agreement with these conclusions; the lower resolution simply allows us to compute for longer times, leading to a better temporal averaging. Nevertheless, there may be a trend toward the energetic ratio to inertia, contrary to the hypothesis of critical balance advocated in [4] (see also [5, 15]).

The most striking result of the computations performed in the decaying case presented in [6] is that different initial conditions for the magnetic field only, but with the same global invariants, led to different spectral inertial indices. Will the same occur in the presence of [13] and averaged in the same temporal interval as the energy spectrum.

IV. COMPARATIVE RESULTS FOR THE THREE FORCING FUNCTIONS

The most striking result of the computations performed in the decaying case presented in [6] is that different initial conditions for the magnetic field only, but with the same global invariants, led to different spectral inertial indices. Will the same occur in the presence of [13] and averaged in the same temporal interval as the energy spectrum.
forcing? This is what we are now investigating. We thus address now the question of the scaling of the two other configurations studied in [6] in the decaying case, namely the so-called A- and I- magnetic configurations.

A. The A run

The forcing with the A configuration leads the system to reach a statistically stationary state, both globally for $E_{\text{tot}}$ (shown in Fig. 3.a) and in its kinetic to magnetic energy ratio (as displayed in Fig. 3.b), with a value of that ratio slightly above unity, as often observed in the Solar Wind. Figure 3.b also displays the temporal evolution of $\omega^2 + j^2$, a clear stationary state is observed for $t \gtrsim 30$. The total energy spectrum for this run is rather steep, with the best fit corresponding to a $k^{-2}$ law (see Fig. 4.a), i.e. a law corresponding to weak turbulence (See table II for details). This is in contrast to [6]: in the decaying case, this flow had a spectral index close to $-5/3$. Note that the wave-turbulence behavior in the present case is consistent with the ratio of energy spectra and the ratio of time scales displayed on Fig. 3.b. Note also that this power law can be attributed to the presence of a (quasi)-discontinuity in the magnetic field, as recently found for the decaying case for the I-flow, in the absence of imposed symmetries [17].

In Fig. 4.b below the temporal evolution of the spectral indices of all the runs are compared. Note that the spectral index of the total energy spectrum of the A-flow varies substantially over time. At this point, it is difficult to decide which power law is best followed. Initial behaviour seems consistent with the unforced $-5/3$ value although the inertial law is steepest at later times, and is thus more in favor of a weak turbulence spectrum. However note that steep structures, such as sharp and isolated current and vorticity sheets, can also lead to a “shock” like spectrum. In this context, see the visualizations presented below in Fig. 8 at the end of the present section.

B. The I runs

Let us now examine the dynamics of the magnetic I configuration (see definition in [9]). The temporal evolutions of the energies is displayed in Fig 5.a. The total energy and $\omega^2 + j^2$ seem to reach a quasi steady state although an increase of the ratio of magnetic to kinetic energy is observed in Fig.5. Thus, a stationary state is not reached for that ratio, with marked global oscillations.

The total energy spectrum (compensated by $k^{5/3}$) and the time-scales ratio and energy spectra ratio are shown in Fig 6. A correlation between the oscillatory growth
of the ratio of magnetic to kinetic energy displayed in Fig. 5b and the spectral index of run I displayed as the middle curve in Fig. 7b is apparent. Indeed upward trends in the spectral index, moving towards weak turbulence, correspond to growth of magnetic energy at the expense of the kinetic energy. Observe in Fig. 5b that magnetic energy dominates over the kinetic one and at the same time the ratio of time scales is almost constant in the inertial regime.

As already observed in [6] in the decaying case, it may be that spectral indices vary with time, as the ratio of kinetic to magnetic energy varies as well. Also, it is quite difficult to distinguish between spectral indices that are quite close and this flow seems more undecided than for the other two flows we study in this paper.

To summarize, in Fig. 7a we present the (uncompensated) spectra for the three runs (averaged over time). We also show in Fig. 7b (bottom) the temporal variation of spectral indices for the three runs. We would like to emphasize that the actual values of the exponents shown in Table II, that were obtained after a time average, are not as significant as is their overall behavior displayed in Fig. 7b. Indeed, fluctuations are observed because

FIG. 5: (Color online) a) Temporal evolution of kinetic (blue), magnetic (red) and total (black) energy for Run I (see Table I). b) Temporal evolution of the ratio $E_b/E_v$ for the same run. c) Temporal evolution of $\omega^2 + j^2$ where $\omega = \nabla \times \mathbf{u}$ is the vorticity and $j = \nabla \times \mathbf{b}$ the magnetic current.

FIG. 6: (Color online) Run I (see Table I); all spectra are averaged in time in the interval $t = 30-52$. a) $k^{5/3}$-compensated total energy spectrum. b) Ratios of energy spectra (blue) and time-scales (red).

FIG. 7: (Color online) a) Total energy spectrum for the runs C2, A and I. Black dashed lines display the different fits. b) Temporal evolution of spectral indices for all runs (see inset). Note that the differentiation between the three magnetic configurations is clear at all times, even if the exact value of the indices vary with time.
the scale separation between the inertial and dissipative
ranges is not large enough and finite resolution effects
step in. It is important to remark here that, after an
initial transient, the exponents have each a well defined
behavior and that in particular they do not overlap at
any time for the three runs, revealing the non-universal
character of MHD turbulence.
For completeness and in order to illustrate the
physical-space distribution of magnetic energies in the
the respective runs, we have performed some visualiza-
tions using the VAPOR software [18]. It is apparent on
Fig. 8 that large-scale coherent structures are present in
the I and A flow. In contrast, the C-flow seems more
isotropic. As discussed above it is a possibility that steep
structures, such as sharp and isolated current and vor-
ticity sheets, can also lead to scaling in energy spectrum
as it was reported in [17]. This certainly looks as a pos-
sibility in the cases of I and A flows. Further studies of
these structures are left for future work.

V. CONCLUSIONS
In this paper, we have extended the analysis of non-
universality of MHD spectra from the decaying case per-
formed in [9] to the forced case. We confirm the previ-
ous results, with either IK, K41 or WT spectra emerging
on average, when using Taylor-Green forcing, including
in the induction equation, although temporal variations
may be occurring. Note that lack of universality in MHD
has already been found by other authors, in the context
of heating the solar corona [19–21], or in the presence
of strong correlations between the velocity and the mag-
netic field [22–24]. We also confirm that these different
scalings are linked to the magnetic energy content in the
gravest mode and that, at least for these flows in which
the four-fold symmetries of the TG vortex are imposed
at all time, quasi-equipartition between the kinetic and
magnetic energy obtains in the inertial range, with a vari-
ation of the ratio of the turn-over time to the Alfvén time
consistent with the inertial index of the energy spectrum,
as in the decaying case and contrary to the hypothesis
made in [4]. The influence of strong localized structures
such as quasi-discontinuities can also alter energy scaling
[17].
It should be noted that, even though there is no im-
posed magnetic field in these computations and thus no
imposed anisotropy, an extension of this work could be
to analyze the data in terms of anisotropic scaling with
respect to a locally-defined quasi-uniform field, averag-
ing the induction in a sphere of diameter the integral
scale, as done for example in [25] (see [26] and refer-
cences therein for anisotropic scaling in MHD). It is also well
known that, in the atmosphere and the oceans, different
spectra may emerge according to the relative strength
of the stratification, the rotation and the forcing, as for
example the Garrett-Munk versus Phillips spectra [27].

There may be periods of evolution when the flow tends
to one regime, and at other times to another regime. It
was already observed in [6] that for late times, the dist-
inction between the K41 and IK regimes became diffi-
cult to make but of course the Reynolds number by then
had decreased substantially. It is known that there are
long-time fluctuations in most turbulent flows ([28] and
references therein) and this could lead to alternate ex-
changes of behavior, as already observed in [29] in two-
dimensional MHD, with turbulent periods of the order of
one hundred turn-over times. These long-time fluctua-
tions could lead to long-time fluctuations in spectral in-
dices as well. Such behavior can be attributed to the last-
ing effects of nonlocal interactions between widely sepa-
rated scales [30, 31] as observed in high-resolutions DNS
of MHD turbulence.

It is not known whether these results will stand out
in the limit of infinite Reynolds numbers, and higher
Reynolds number computations will have to be per-
formed in order to confirm the results presented in this
paper. There are other venues that can be taken as well: it
is well-known that magnetic helicity and cross-helicity
play essential roles in the dynamics oh MHD turbulence,
and yet they are quenched by the symmetries in the
present approach where symmetries are enforced at all
times. It is already documented that, for the case of
long time dynamics and perturbing the three initial con-
ditions studied in this paper, vastly different regimes can
be reached, with in particular the ratio of kinetic to mag-
netic energy varying in a large range [16]. This behavior
can be understood in terms of minimization of energy
subjected to the constraints of the invariance of $H_C$ and
$H_M$ [32, 33]. This points out to the possibly essential
role played by the imposed four-fold symmetries. Such
symmetries could be broken in part, as performed in [7],
leading to different modes growing in the dynamo regime.
A similar approach could be taken for the present prob-
lem of lack of universality, still allowing for some sav-
ings in computer resources compared with full-fledged
MHD computations which might otherwise have to be
performed. These issues will need more investigations.

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