On construction of $k$-regular maps to Grassmannians via algebras of socle dimension two

Joachim Jelisiejew, Hanieh Keneshlou

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Abstract

A continuous map $\mathbb{C}^n \to \text{Gr}(\tau, N)$ is $k$-regular if the $\tau$-dimensional subspaces corresponding to images of any $k$ distinct points span a $\tau k$-dimensional space. For $\tau = 1$ this essentially recovers the classical notion of a $k$-regular map $\mathbb{C}^n \to \mathbb{C}^N$. We provide new examples of $k$-regular maps, both in the classical setting $\tau = 1$ and for $\tau \geq 2$, where these are the first examples known. Our methods come from algebraic geometry, following and generalizing [BJJM19]. The key and highly nontrivial part of the argument is proving that certain loci of the Hilbert scheme of points have expected dimension. As an important side result, we prove the irreducibility of the punctual Hilbert scheme of $k$ points on a threefold, for $k \leq 11$.

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*Faculty of Mathematics, Informatics and Mechanics, Stefana Banacha 2, 02-097 Warszawa, Poland. Supported by Polish NCN grant 2020/39/D/ST1/00132. A revision of this work has partially supported by the Thematic Research Programme “Tensors: geometry, complexity and quantum entanglement”, University of Warsaw, Excellence Initiative – Research University and the Simons Foundation Award No. 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for the years 2021-2023.

†Department of Mathematics and Statistics, Universitätsstraße 10, 78464 Konstanz, Germany.
1 Introduction

A continuous map \( f : \mathbb{C}^n \to \mathbb{C}^N \) is (linearly) \( k \)-regular if the images \( f(x_1), \ldots, f(x_k) \) are linearly independent for any pairwise distinct points \( x_1, \ldots, x_k \). The problem of the existence of such maps for given \( k, n, N \) is researched classically in topology and is also important in applications, for example to approximation theory, see [BJJM19]. This problem decomposes into two distinctly differently flavored halves: providing lower and upper bounds for the minimal \( N = N(n, k) \) for which a \( k \)-regular map exists. The lower bounds are provided using algebro-topological methods [BLZ16, BCLZ16], while the upper bounds follow from constructions in [BJJM19].

A continuous map \( f : \mathbb{C}^n \to \mathbb{P}(\mathbb{C}^N) \) is projectively \( k \)-regular if the images \( f(x_1), \ldots, f(x_k) \) span a \((k - 1)\)-dimensional projective subspace for any pairwise distinct points \( x_1, \ldots, x_k \). This notion is closely related to linear \( k \)-regularity, see [BJJM19, Lemma 2.3].

A natural generalization of projectively \( k \)-regular maps comes from replacing \( \mathbb{P}(\mathbb{C}^N) \) by a Grassmannian \( \text{Gr}(\tau, N) \) of \( \tau \)-dimensional subspaces of \( \mathbb{C}^N \). A continuous map \( f : \mathbb{C}^n \to \text{Gr}(\tau, N) \) is \( k \)-regular if for every pairwise distinct points \( x_1, \ldots, x_k \in \mathbb{C}^n \) the subspaces \( f(x_1), \ldots, f(x_k) \) jointly span a \((k \cdot \tau)\)-dimensional subspace of \( \mathbb{C}^N \), i.e., their basis vectors are jointly linearly independent. For \( \tau = 1 \) this recovers the notion of a projectively \( k \)-regular map. Also in this context one would like to bound the minimal \( N = N(\tau, n, k) \) for which a \( k \)-regular map exists.

In this article, we prove the following result.

**Theorem 1.1.** Consider the function \( \widetilde{N}(\tau, k, n) \) defined by

1. If \( k \leq 8 \), or if \( \tau \leq 2 \) and \( k \leq 11 \), then \( \widetilde{N}(\tau, k, n) := (n - 1)(k - 1) + rk. \)
2. Otherwise \( \widetilde{N}(\tau, k, n) = n(k - 1) - 1 + rk. \)

Then there exists a \( k \)-regular map \( \mathbb{C}^n \to \text{Gr}(\tau, \widetilde{N}(\tau, k, n)). \)

In the classical case \( \tau = 1 \) this extends the bounds from [BJJM19] for \( k = 10, 11 \). The method of construction, which follows the ideas of [BJJM19], is the following.

1. By a direct construction, a \( k \)-regular algebraic map \( f_0 : \mathbb{C}^n \to \text{Gr}(\tau, N_0) \) exists for \( N_0 \gg 0. \)
2. We seek a \( W \subset \mathbb{C}^{N_0} \) such that the composed map \( f_W : \mathbb{C}^n \to \text{Gr}(\tau, N_0) \to \text{Gr}(\tau, N_0 - \dim W) \) induced by \( \mathbb{C}^{N_0} \to \mathbb{C}^{N_0}/W \) is still everywhere defined and \( k \)-regular. In fact, since \( \mathbb{C}^n \) is homeomorphic to any open ball around the origin, it is enough for \( f_W \) to be defined and \( k \)-regular around the origin.
3. A subspace \( W \) satisfies the requirements above if it does not intersect a certain bad locus, given by the spans of all \( f_0(Z) \) for \( Z \subset \mathbb{C}^n \) a zero-dimensional degree \( k \) subscheme supported at the origin. The dimension of this bad locus is bounded from above by the dimension of the family of possible \( Z \), which is the punctual Hilbert scheme \( \text{Hilb}_k(\mathbb{A}^n, 0) \), see below.
4. The bad locus above is covered by spans of $Z$ which have socle dimension $\leq \tau$, so we may restrict to these. This is perhaps geometrically obscure, but allows reducing to bounding the dimension of the locus $\text{Hilb}_1^k(A^n, 0) \subseteq \text{Hilb}_k(A^n, 0)$ parameterizing them.

The final remaining task, to bound $\dim \text{Hilb}_1^k(A^n, 0)$, is completely disconnected from topology, but very far from trivial, especially to get the best estimate in Point 1. It occupies the major part of the current paper and we discuss it at length below. Let us only remark that the above Theorem 1.1 is close to optimal when employing the methods of [BJJM19], as follows from Theorem 1.2 below. Hence radically new approaches are necessary to improve the upper bounds on $N(\tau, n, k)$ further. We also remark that the case $\tau = 1$ is qualitatively easier than the other ones, thanks to the fact that in this case one deals exclusively with Gorenstein algebras, see §4 for details, and a mature theory of classification of those algebras exists.

We work over an algebraically closed field $k$ of characteristic zero. The dimension of the bad locus above is governed by a sublocus of the Hilbert scheme of points $\text{Hilb}_k(A^n)$. The Hilbert scheme is a quasi-projective scheme and its closed points are in bijection with closed subschemes $Z \subseteq A^n$ that are zero-dimensional and of degree $k$; the simplest example of $Z$ is a tuple of $k$ closed points of $A^n$. The topology of the Hilbert scheme is defined functorially, see for example [BJJM19, §2.4] for details. We denote by $[Z] \in \text{Hilb}_k(A^n)$ the point corresponding to $Z \subseteq A^n$.

Let $\text{Hilb}_k(A^n, 0) \rightarrow \text{Hilb}_k(A^n)$ denote the closed locus consisting of $Z \subseteq A^n$ such that $Z$ is supported only at the origin. This is the punctual Hilbert scheme, see §2.4 for details. It is a projective, in particular proper, scheme. (A warning: some authors use the name “punctual” for the whole $\text{Hilb}_k(A^n)$.)

Compared with $\text{Hilb}_k(A^n)$, much less is known about the punctual Hilbert scheme $\text{Hilb}_k^1(A^n, 0)$. This scheme has a distinguished irreducible component, the curvilinear locus which is the locus of curvilinear schemes: schemes isomorphic to $\text{Spec}(k[\epsilon]/\epsilon^2)$, see [Jar83]. This locus has dimension $(n-1)(k-1)$, thus $\dim \text{Hilb}_k^1(A^n, 0) \geq (n-1)(k-1)$. We say that $(n-1)(k-1)$ is the expected dimension of $\text{Hilb}_k^1(A^n, 0)$ and that a locus of $\text{Hilb}_k^1(A^n, 0)$ is negligible if its dimension is at most $(n-1)(k-1)$. Let $\text{Hilb}_k^1(A^n, 0) \subseteq \text{Hilb}_k(A^n, 0)$ denote the open locus that consists of $[Z = \text{Spec}(A)]$ where the socle of $A$ is at most $\tau$-dimensional; for example $\tau = 1$ corresponds to a Gorenstein $Z$. Our first main result concerns $(n, k, \tau)$ for which the dimension of $\text{Hilb}_k^1(A^n, 0)$ is expected.

**Theorem 1.2.** For every $k \leq 8$ and $n$ the scheme $\text{Hilb}_k^1(A^n, 0)$ has expected dimension while for every $k \geq 9$ the dimension of $\text{Hilb}_k^1(A^{k-4}, 0)$ is higher than expected. For every $k \leq 11$ and $n$ the scheme $\text{Hilb}_k^1(A^n, 0)$ has expected dimension, while for $k \geq 13$ the dimension of $\text{Hilb}_k^1(A^{\lceil k/2 \rceil - 1}, 0)$ is higher than expected and for $k \geq 14$ the dimension of $\text{Hilb}_k^1(A^{\lceil k/2 \rceil - 1}, 0)$ is higher than expected. This information can be summarized as follows, where no means no if $n$ is big enough.

| $k \leq 8$ | $9 \leq k \leq 11$ | $k = 12$ | $k = 13$ | $k \geq 14$ |
|------------|-----------------|---------|---------|---------|
| $\text{Hilb}_k^1(A^n, 0)$ | ✓ | no | no | no |
| $\text{Hilb}_k^1(A^n, 0), \tau \geq 3$ | ✓ | no | no | no |
| $\text{Hilb}_k^1(A^n, 0)$ | ✓ | ✓ | ? | no |
| $\text{Hilb}_k^1(A^n, 0)$ | ✓ | ✓ | ? | no |

*Table. Does $\text{Hilb}_k^1(A^n, 0)$ have expected dimension for every $n$?*

Before the current work, the case $\tau = 1$ was investigated in [BJJM19, Theorem A.16] where it was proven that $\dim \text{Hilb}_k^1(A^n, 0) = (k-1)(n-1)$ for $k \leq 9$. Apart from this, to the authors’ knowledge, no results were known for general $n$. For $n \leq 2$ the closure of the curvilinear locus is the only component; the punctual Hilbert scheme is irreducible by a result of Briançon [Bri77], see also [Jar77]. In [ES87, Corollary (1.2)] the authors reprove Briançon’s result via a beautiful topological argument, which relies heavily on that $\text{Hilb}_k^1(A^2)$ is smooth and irreducible. In the current article, we extend the irreducibility result to threefolds, for $k \leq 11$. 3
Theorem 1.3. For \( k \leq 11 \) the scheme \( \text{Hilb}_k(\mathbb{A}^3, 0) \) is equal to the closure of the curvilinear locus, so it is irreducible of dimension \( 2(k - 1) \).

The punctual Hilbert scheme does not depend on the global geometry [Fog68, Proposition 2.2] so we immediately obtain the following generalization.

Theorem 1.4. Let \( x \in X \) be a smooth point on a threefold \( X \). Then for \( k \leq 11 \) the scheme \( \text{Hilb}_k(X, x) \) is irreducible of dimension \( 2(k - 1) \).

It was known that \( \text{Hilb}_k(\mathbb{A}^3) \) is irreducible for \( k \leq 11 \) [DJNT17, HJ18, Šiv12]. However, this by no means implies irreducibility of \( \text{Hilb}_k(\mathbb{A}^3, 0) \). Indeed, the proof of irreducibility of \( \text{Hilb}_k(\mathbb{A}^3) \) as in [DJNT17] proceeds by proving that every scheme \( Z \subset \mathbb{A}^3 \) is a degeneration of \( \Gamma \subset \mathbb{A}^3 \) which is a union of \( k \) points. Obviously, the scheme \( \text{Hilb}_k(\mathbb{A}^3, 0) \) contains no such \( \Gamma \), so the picture is as in Figure 1. None of the known general techniques used to prove irreducibility of Hilbert schemes adapts effectively to analysing irreducibility of \( \text{Hilb}_k(\mathbb{A}^3, 0) \) and here we develop a new method.

![Figure 1: The schemes \( \text{Hilb}_k(\mathbb{A}^3) \) and \( \text{Hilb}_k(\mathbb{A}^3, 0), k \leq 11 \).](image)

1.1 Proof ideas I: general and Białynicki-Birula slicing

The major part of the proof of Theorem 1.3 is to prove an appropriate part of Theorem 1.2: to show that the dimension of \( \text{Hilb}_k(\mathbb{A}^3, 0) \) is at most \( 2(k - 1) \) and additionally that there is at most one irreducible component of dimension \( 2(k - 1) \). Since the dimension of \( \text{Hilb}_k(\mathbb{A}^3, 0) \) at every point is at least \( 2(k - 1) \) by general nonsense, see Proposition 2.7, this implies that this component is the whole \( \text{Hilb}_k(\mathbb{A}^3, 0) \).

However, bounding the dimension of \( \text{Hilb}_k(\mathbb{A}^3, 0) \) is hard. The points of \( \text{Hilb}_k(\mathbb{A}^3, 0) \) are quite impossible to classify for \( k > 6 \) and the number of cases is best described as swarming, see tables below. Moreover, in contrast with \( \text{Hilb}_k(\mathbb{A}^n) \), the scheme \( \text{Hilb}_k(\mathbb{A}^n, 0) \) has not been much investigated. Therefore, the only one readily available estimate is “dimension at a point is at most the dimension of the tangent space”. But even this method has a serious caveat. Namely, it is known classically that for a zero-dimensional \( Z \subset \mathbb{A}^n \) we have

\[
T_{[Z]} \text{Hilb}_k(\mathbb{A}^n) = \text{Hom}_k[\kappa[x_1, \ldots, x_s]](I, \kappa[x_1, \ldots, x_s]/I),
\]

where \( I = I(Z) \subset H^0(O_{\mathbb{A}^n}) = \kappa[x_1, \ldots, x_n] \). However, for \( [Z] \in \text{Hilb}_k(\mathbb{A}^n, 0) \) no such nice description of the subspace \( T_{[Z]} \text{Hilb}(\mathbb{A}^n, 0) \) seems to exist, see [BS17] for the surface case. Given that \( \text{Hilb}_k(\mathbb{A}^3) \) is irreducible for \( k \leq 11 \), we have \( \dim_k \text{Hom}(I, \kappa[x_1, \ldots, x_s]/I) \geq 3k \) which is not a useful estimate. Even worse, the locus \( \text{Hilb}_k(\mathbb{A}^3, 0) \) is the most singular part of \( \text{Hilb}_k(\mathbb{A}^3) \); indeed every zero-dimensional \( Z \subset \mathbb{A}^3 \) can be degenerated to a scheme supported only at the origin by taking the limit at zero of the torus action on \( \mathbb{A}^3 \) by scalar multiplication. This implies that the tangent spaces to the Hilbert scheme at points of \( \text{Hilb}_k(\mathbb{A}^3, 0) \) are frequently of dimension much higher than \( 3k \).

To provide an upper bound on the dimension, we introduce a technique that we baptize informally as \textit{Białynicki-Birula slicing}. It seems applicable much more generally, outside the realm of Hilbert
schemes, whenever one has a singular moduli space with a torus action. We discuss it here for arbitrary \( \mathbb{A}^n \), not just \( n = 3 \).

Fix a one-dimensional torus \( \mathbb{G}_m \) and its action on \( \mathbb{A}^n \) by \( t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n) \). Every point except the origin diverges with \( t \to \infty \). We consider the Białynicki-Birula-decomposition \( \text{Hilb}_k^G(A^n) \) of the Hilbert scheme, which subdivides this scheme by grouping together points that have their limit at infinity in the same connected component of the fixed points [Dri13, JS19]. The precise definition of the Białynicki-Birula decomposition will not be important for us, below we gather all its relevant properties. The Białynicki-Birula decomposition comes with morphisms of schemes

\[
\begin{align*}
\text{Hilb}_k^G(A^n) & \xrightarrow{\theta} \text{Hilb}_k(A^n) \\
\downarrow \pi & \\
\text{Hilb}_k^G(A^n) & \text{where } \theta \text{ is the forgetful map, while } \pi \text{ maps a subscheme to its limit at } t = \infty. \text{ Since schemes supported outside the origin do not admit a limit at infinity, the image of } \theta \text{ is, as a set, } \text{Hilb}_k(A^n, 0) \text{ [Jel19, Proposition 3.3]. The closed subscheme } \text{Hilb}_k(A^n, 0) \text{ is } \mathbb{G}_m \text{-stable, hence inherits a decomposition } \text{Hilb}_k^G(A^n, 0) \text{ and the corresponding maps. This subscheme is also projective, see } \S 2.4, \text{ so the forgetful map } \text{Hilb}_k^G(A^n, 0) \to \text{Hilb}_k(A^n, 0) \text{ is bijective on points and hence }
\end{align*}
\]

\[\dim \text{Hilb}_k^G(A^n, 0) = \dim \text{Hilb}_k(A^n, 0).\]

The map \( \pi \) restricted to \( \text{Hilb}_k(A^n, 0) \) has an explicit description as the associated-graded-algebra map, see \( \S 1.2 \) for details. In particular, all elements of a fiber of \( \pi \) share the same Hilbert function.

The points of \( \text{Hilb}_k^G(A^n) \) correspond to subschemes \( Z \subset \mathbb{A}^n \) whose ideals are homogeneous with respect to the standard grading. Thus, this scheme has at least as many connected components as there are Hilbert functions \( H \):

| number of Hilbert functions \( H \), with arbitrary \( H(1) \): | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---------------------------------------------------------------|---|---|---|---|---|---|---|---|---|----|----|
| possible Hilbert functions \( H \) for any fixed Hilbert function \( H \) with \( \sum H = k \), consider the open-closed locus \( \text{Hilb}_k^G(A^n, 0) \subset \text{Hilb}_k^G(A^n) \) parameterizing points \( [\text{Spec}(A)] \) where \( A \) has Hilbert function \( H \), we call it the graded stratum associated to \( H \). Let \( \text{Hilb}_H(A^n, 0) \subset \text{Hilb}_k^G(A^n, 0) \) denote the locus parameterizing points \( [\text{Spec}(A)] \) where \( A \) is a local algebra with Hilbert function \( H \), this is the stratum associated to the Hilbert function \( H \). By the description of \( \pi \) as the associated-graded-algebra map, we have \( \text{Hilb}_H(A^n, 0) = \pi^{-1}(\text{Hilb}_k^G(A^n, 0)) \).

Moreover, \( (\text{Hilb}_H(A^n, 0))^G_m = \text{Hilb}_H^G(A^n, 0) \) as the notation suggests. In Figure 2 we give a very schematic description of its Białynicki-Birula decomposition, compare Figure 1. We would like to stress that the division of points of \( \text{Hilb}_k(A^n, 0) \) into strata using the Hilbert function is a very classical idea. The contribution of the Białynicki-Birula decomposition is to give this classical idea a functorial interpretation. One important product of this interpretation is a formula for tangent space to the stratum, which we discuss now.

For a \( \mathbb{G}_m \)-fixed point \( [Z] \) corresponding to a homogeneous ideal \( I \subset R = k[x_1, x_2, \ldots, x_n] \) we have

\[T_{[Z]} \text{Hilb}_k^G(A^n, 0) = (T_{[Z]} \text{Hilb}_k(A^n))^G_m = \text{Hom}_R(I, R/I)^G_m.\]

where \( \text{Hom}(I, R/I)^G_m \) consists of homomorphisms \( \varphi : I \to R/I \) such that \( \varphi(\mathfrak{m}) \subset (R/I)^G_m \) for every \( i \geq 0 \), i.e., the homomorphisms that do not lower the degree, see for example [Jel19, (2.2), (2.4)]. This restriction of the tangent space to its nonnegative part is the first key improvement. Indeed, when \( n = 3 \), then \( \dim_k \text{Hom}(I, R/I)^G_m \leq 2(k - 1) \) for almost all homogeneous ideals, see Proposition 3.5.
Figure 2: Schematic description of Białynicki-Birula decomposition of the scheme $\text{Hilb}_k^+(\mathbb{A}^3, 0)$.

and the table below. To bound the dimension of the tangent spaces, we resort again to semicontinuity: choose a Borel subgroup $B \subseteq \text{GL}_n$ that commutes with $\mathbb{G}_m$. It acts on the components of the projective scheme $\text{Hilb}_k^\mathbb{G}_m(\mathbb{A}^n)$ and so by semicontinuity to bound the tangent space dimension (degree-wise) everywhere, it is enough to bound it for Borel-fixed points. Borel-fixed points correspond to monomial ideals, so there are finitely many of them for a fixed $k$. In fact, they correspond to very special monomial ideals [AL19]. We list the numbers below, to provide some context.

| $n = 3$ | $k = 1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|--------|--------|---|---|---|---|---|---|---|---|----|----|
| number of monomial ideals (MacMahon function) | 1 | 3 | 6 | 13 | 24 | 48 | 86 | 160 | 282 | 500 | 859 |
| number of Borel-fixed ideals | 1 | 1 | 2 | 3 | 4 | 1 | 1 | 1 | 1 | 2 | 2 |
| number of Borel-fixed ideals with $\dim T_{Z,0} \geq 2(k-1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 4 |

There remains a small number of special cases for $n = 3$ which have to be resolved by hand. There also remain many more problematic cases for $n > 3$, see below. To tackle those we need to employ other tools, such as the theory of Macaulay’s inverse systems and partial classification results that will be discussed below.

| $n = k$ | $k = 1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|--------|--------|---|---|---|---|---|---|---|---|----|----|
| number of Borel-fixed ideals | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 20 | 32 | 50 | 77 |
| number of Borel-fixed ideals with $\dim T_{Z,0} \geq (n-1)(k-1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 4 | 8 | 16 |

**Dimension estimates.** For a $\mathbb{G}_m$-fixed $[Z]$ given by a homogeneous $I \subset R = \mathbb{k}[x_1, \ldots, x_n]$ its tangent space

$$T_{[Z]} := \text{Hom}_\mathbb{k}(I, R/I)_{\geq 0}$$

decomposes into the space $(T_{[Z],0})_0$ which is the tangent space to $\text{Hilb}_k^\mathbb{G}_m(\mathbb{A}^n)$ and $(T_{[Z],0})_{>0}$ which is the tangent space to the fiber over $[Z]$. Moreover, the fiber dimension of $\pi$ is upper-semicontinuous, because this map is a cone over a projective morphism, see [Sza21, Proof of Theorem 3.1] for details. Hence, we have the following estimates for the dimension of $\text{Hilb}_k^+(\mathbb{A}^n, 0)$ near $[Z]$:

1. **tangent space estimate** $\dim_{[Z]} \text{Hilb}_k^+(\mathbb{A}^n, 0) \leq \dim_k \text{Hom}_\mathbb{k}(I, R/I)_{\geq 0}$,
2. **base-and-tangent-to-fiber estimate** $\dim_{[Z]} \text{Hilb}_k^+(\mathbb{A}^n, 0) \leq \dim_{[Z]} \text{Hilb}_k^\mathbb{G}_m(\mathbb{A}^n) + \dim_k \text{Hom}_\mathbb{k}(I, R/I)_{>0}$,
3. **tangent-to-base-and-fiber estimate** $\dim_{[Z]} \text{Hilb}_k^+(\mathbb{A}^n, 0) \leq \dim_k \text{Hom}_\mathbb{k}(I, R/I)_0 + \dim_{[Z]} \pi^{-1}(\{[Z]\})$,
4. **base-and-fiber estimate** $\dim_{[Z]} \text{Hilb}_k^+(\mathbb{A}^n, 0) \leq \dim_{[Z]} \text{Hilb}_k^\mathbb{G}_m(\mathbb{A}^n) + \dim_{[Z]} \pi^{-1}(\{[Z]\})$.

In most of the remaining cases, these observations are sufficient. In few, we need to resort to the following observation, which we call local constancy. If $[Z]$ is $\mathbb{G}_m$-fixed and a smooth point of $\text{Hilb}_k^+(\mathbb{A}^n, 0)$ then the map $\pi$ is smooth at $[Z]$ and so is $\text{Hilb}_k^\mathbb{G}_m(\mathbb{A}^n)$. In particular $\dim_k \text{Hom}_\mathbb{k}(I, R/I)_{>0}$
and \( \dim_k \text{Hom}_k(I, R/I)_0 \) are constant in a neighbourhood of \([Z]\). This is used to rule out the cases where the tangent space has a surplus of exactly one with respect to the expected dimension: if the dimension of the stratum is indeed higher than expected, then the point has to be smooth and we can apply the above observation and find a witness of non-constancy.

1.2 Proof ideas II: explicit computations

Finally, there are much fewer but still quite a few cases where \( \dim_k \text{Hom}_k(I, R/I)_{\geq 0} \) is higher than \((n - 1)(k - 1) + 1\) and for these we need to peek deeper into the structure of the stratum. We can rephrase the Diagram 1.5 more concretely on the level of closed points as

\[
\bigcup_{H : \sum H = k} \{ I \subset R \mid \text{rad}(I) = (x_1, \ldots, x_n), H_{R/I} = H \} \quad \text{bij} \quad \{ I \subset R \mid \text{rad}(I) = (x_1, \ldots, x_n), \dim_k R/I = k \}
\]

where \( R = k[x_1, \ldots, x_n] \). Using this description, we see that for a homogeneous ideal \( I \), the fiber of \( \pi \) over \([I]\) consists of all ideals having \( I \) as the initial ideal, hence it is possible to describe the fiber explicitly. It is not quite true that such a description is straightforward, and in fact, it is not, especially since in many cases the fiber is very singular. To facilitate it we resort to a number of tricks and classification theorems that are too technical to describe here. We attempt to present only one, which can be called vertex removal.

Typically, the graded stratum \( \text{Hilb}^G_m(A^n, 0) \) contains small substrata consisting of highly singular points. The whole torus \( G_m^n \) acts on \( \text{Hilb}_H(A^n, 0) \) and \( \text{Hilb}^G_m(A^n, 0) \), so we can fix a one-dimensional torus \( T \subset G_m^n \) and consider the induced decompositions. An example of such is the left vertex in Figure 2. The trick is to choose the weights of this other \( T \) appropriately, so that near the vertex the action is divergent, as on the right part of Figure 2. Then the dimension of the non-vertex part can be bounded by using tangent spaces, while the dimension of the vertex part can be bounded directly, since this part becomes "small"; see Proposition 5.16 for an example of such an argument.

Notation for the Hilbert schemes

In the table below, we summarize the various loci appearing in this article. See §2.2 for details.

| object | symbol |
|--------|--------|
| The Hilbert scheme of \( k \) points | \( \text{Hilb}_k(A^n) \) |
| The punctual Hilbert scheme | \( \text{Hilb}_k(A^n, 0) \) |
| The locus of algebras with socle dimension \( \leq \tau \) | \( \text{Hilb}^\tau_k(A^n, 0) \) |
| The locus of homogeneous ideals | \( \text{Hilb}^G_k(A^n, 0) \) |
| The locus of homogeneous ideals with fixed Hilbert function \( H \) | \( \text{Hilb}^G_k(A^n, 0) \) |
| The Białynicki-Birula decomposition of the punctual Hilbert scheme | \( \text{Hilb}^B_k(A^n, 0) \) |
| The locus with fixed Hilbert function \( H \) | \( \text{Hilb}^H_k(A^n, 0) \) |
| The locus of Gorenstein algebras with fixed Hilbert function \( H \) | \( \text{Hilb}^{\text{Gor}}_H(A^n, 0) \) |

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2 Preliminaries

In this section, we collect facts and notations regarding the Hilbert scheme, the Hilbert function and Macaulay’s inverse systems, used throughout this article. As already stated in the introduction, throughout we work over an algebraically closed field $k$ of characteristic zero. All computations below and in the introduction are available as an Macaulay2 ancillary file for the arXiv version of this paper.

2.1 Socle and Hilbert function

Let $\hat{R} = k[[x_1, \ldots, x_n]]$ be the power series ring in $n$ variables with the maximal ideal $m = (x_1, \ldots, x_n)$. Let $A = \hat{R}/I$ be a local Artin ring for an $m$-primary ideal $I \subset \hat{R}$, with the unique maximal ideal $n = m/I$ and residue field $k$. Recall that an $m$-primary ideal is just an ideal $I$ such that $I \supset m^r$ for some $r$. Since for every $r$ we have canonically

$$ (2.1) \quad \frac{\hat{R}}{m^r} \cong \frac{R}{(x_1, \ldots, x_n)^r} $$

we could view $A$ as a quotient of $R = k[x_1, \ldots, x_n]$ by an ideal $I$ satisfying $I \supset (x_1, \ldots, x_n)^r$ for some $r$.

The socle $\text{Soc}(A)$ of $A$ is the annihilator of the maximal ideal in $A$. The socle is a $k$-vector space. The socle dimension of $A$ is defined by $\tau(A) := \dim_k \text{Soc}(A)$. The ring $A$ is Gorenstein if $\tau(A) = 1$, see for example [Eis95, Chapter 21]. The associated graded ring of $A$, denoted $\text{gr}(A)$, is the vector space $\bigoplus_{n \geq 0} n!/n^+1$ with natural multiplication. When $A = k[x_1, \ldots, x_n]/I$, where $I \supset (x_1, \ldots, x_n)^r$, then $\text{gr}(A) = k[x_1, \ldots, x_n]/(n(I))$, where $\text{in}(I)$ is the ideal generated by the smallest degree forms of elements of $I$. We have $\tau(\text{gr} A) \geq \tau(A)$ and typically strict inequality occurs.

**Definition 2.2.** The Hilbert function $H_A : \mathbb{N} \to \mathbb{N}$ of a local ring $A$ with a maximal ideal $n$ is defined by $H_A(n) := \dim_k n!/n^+1$. It is the Hilbert function of its associated graded ring.

Let $s$ denote the largest integer such that $n^s \neq 0$, the so-called socle degree of $A$. The Hilbert function of $A$ can be then represented by a vector $H_A = (1, H_A(1), \ldots, H_A(s))$ or a series $\sum_{i=0}^s H_A(i)n^i$. Moreover, since $n^s \subset \text{Soc}(A)$, we get $H_A(s) \leq \tau(A)$. We define the Hilbert function of an $A$-module similarly.

2.2 The Hilbert scheme of points and its subloci

The Hilbert scheme $\text{Hilb}_k(A^n)$ of $k$ points on $A^n$ is an open subscheme of the projective scheme $\text{Hilb}_k(\mathbb{P}^n)$. Numerous nice introductions to Hilbert scheme of points exist, for example [FGI'05, Chapters 5-6], [Str96, Ber12]. Here we only collect the facts which will be useful in the article.

Let $R = k[x_1, \ldots, x_n]$. The degree of a zero-dimensional subscheme $Z = \text{Spec}(R/I)$ of $A^n$ is $\text{deg}(Z) := \dim_k (R/I)$. Closed points of $\text{Hilb}_k(A^n)$ correspond bijectively to zero-dimensional closed subschemes $Z \subseteq A^n$ of degree $k$, or, in other words, to ideals $I \subseteq R$ with $\dim_k R/I = k$. We write $[Z]$ or $[I]$ for the point corresponding to a subscheme $Z$ or ideal $I$. The tangent space at a closed point $[\text{Spec}(R/I)] \in \text{Hilb}_k(A^n)$ is given by $\text{Hom}_k(I, R/I)$, see for example [Str96, Theorem 10.1]. The locus of $[Z] \in \text{Hilb}_k(A^n)$ with $Z$ smooth is open in the Hilbert scheme. Its closure is called the smoothable component, we denote it by $\text{Hilb}^{\text{sm}}_k(A^n)$.

A zero-dimensional subscheme $Z$ is a finite disjoint union $Z_1 \cup \ldots \cup Z_e$, where the summands are irreducible subschemes, so, as a set, every $Z_i$ is a singleton $p_i \in A^n$. The support of $Z$ is a formal sum $\sum_{i=1}^e \text{deg}(Z_i)p_i$. Let $\text{Sym}_k(A^n) = (A^n)^k / S_k$ be the $k$-fold symmetric product of $A^n$. The Hilbert-Chow
morphism \( \rho_k : \text{Hilb}_k(\mathbb{A}^n) \to \text{Sym}_k(\mathbb{A}^n) \) sends a zero-dimensional scheme \( Z \subset \mathbb{A}^n \) to its support \([\text{Ber12}, \text{Theorem 2.16}]\). It extends to a morphism \( \tilde{\rho}_k : \text{Hilb}_k(\mathbb{P}^n) \to \text{Sym}_k(\mathbb{P}^n) \) and we have a cartesian diagram

\[
\begin{array}{ccc}
\text{Hilb}_k(\mathbb{A}^n) & \xrightarrow{\rho_k} & \text{Sym}_k(\mathbb{A}^n) \\
\downarrow \text{open} & & \downarrow \text{open} \\
\text{Hilb}_k(\mathbb{P}^n) & \xrightarrow{\tilde{\rho}_k} & \text{Sym}_k(\mathbb{P}^n)
\end{array}
\]

The punctual Hilbert scheme is defined as \( \text{Hilb}_k(\mathbb{A}^n, 0) := \rho_k^{-1}(d[0]) = \rho_k^{-1}(d[0]), \) where \( 0 \in \mathbb{A}^n \) is the origin. As a topological space, the punctual Hilbert scheme consists of subschemes supported only at the origin. The description by \( \tilde{\rho}_k \) implies that this scheme is projective. We obtain the corresponding locus

\[
\text{Hilb}_k^{\text{sm}}(\mathbb{A}^n, 0) := \text{Hilb}_k(\mathbb{A}^n, 0) \cap \text{Hilb}_k^{\text{sm}}(\mathbb{A}^n).
\]

The invariants discussed in §2.1 give rise to loci in the punctual Hilbert scheme which we now formally introduce. For \([I] \in \text{Hilb}_k(\mathbb{A}^n, 0)\) the socle dimension of \( R/I \) is the minimal number of generators of the canonical module \( \omega_{R/I} \), so the sublocus \( \text{Hilb}_k^{\text{i}}(\mathbb{A}^n, 0) \subseteq \text{Hilb}_k(\mathbb{A}^n, 0) \) that consists of \([I] \) with \( \tau(R/I) \leq \tau \) is open. For every \( i \), the dimension \( \dim_R R/(I + R_i) \) is upper-semicontinuous as \([I] \) varies in the punctual Hilbert scheme, hence for a given \( H : \mathbb{N} \to \mathbb{N} \) the locus of \([I] \) with \( H_{R/I} = H \) is locally closed. We denote it by \( \overline{\text{Hilb}}_H(\mathbb{A}^n, 0) \).

For generalities on the Białynicki-Birula decomposition \( \text{Hilb}_k^{\text{BB}}(\mathbb{A}^n) \) of the Hilbert scheme, we refer the reader to [Jel19]. The Białynicki-Birula decomposition \( \text{Hilb}_k^{\text{BB}}(\mathbb{A}^n, 0) \) of the punctual Hilbert scheme can be formally defined using the machinery of [JS19]. It follows from [Jel19, Proposition 3.3] that the inclusion \( \text{Hilb}_k^{\text{BB}}(\mathbb{A}^n, 0) \hookrightarrow \text{Hilb}_k(\mathbb{A}^n) \) is an isomorphism, but we will not use this fact in the present article. The restriction of the map \( \theta \) from the introduction is a map \( \text{Hilb}_k^{\text{BB}}(\mathbb{A}^n, 0) \rightarrow \text{Hilb}_k(\mathbb{A}^n, 0) \) which is bijective on points (but not an isomorphism!). We identify the closed points of those two schemes. There is a morphism \( \pi : \text{Hilb}_k^{\text{BB}}(\mathbb{A}^n, 0) \rightarrow \text{Hilb}_k^{\text{BB}}(\mathbb{A}^n, 0) \) which on the level of points sends \([I] \) to the initial ideal \([\text{in}(I)] \). We define \( \text{Hilb}_H(\mathbb{A}^n, 0) \) as \( \pi^{-1}(\text{Hilb}_H^{\text{BB}}(\mathbb{A}^n, 0)) \). The forgetful map \( \text{Hilb}_k(\mathbb{A}^n, 0) \rightarrow \text{Hilb}_k(\mathbb{A}^n, 0) \) restricts to a map \( \text{Hilb}_H(\mathbb{A}^n, 0) \rightarrow \overline{\text{Hilb}}_H(\mathbb{A}^n, 0) \) bijective on points and so

\[
\dim \text{Hilb}_H(\mathbb{A}^n, 0) = \dim \overline{\text{Hilb}}_H(\mathbb{A}^n, 0);
\]

in fact those two schemes can be thought of as two scheme structures on the same set of closed points. We have a much better control on the tangent space of \( \text{Hilb}_H(\mathbb{A}^n, 0) \), so we will mostly consider this one. We define the open sublocus

\[
\text{Hilb}_H^{\text{or}}(\mathbb{A}^n, 0) := \text{Hilb}_H(\mathbb{A}^n, 0) \cap \text{Hilb}_k(\mathbb{A}^n, 0).
\]

The scheme \( \text{Hilb}_H^{\text{or}}(\mathbb{A}^n, 0) \) is a union of connected components of \( \text{Hilb}_k^{\text{or}}(\mathbb{A}^n, 0) \), so the scheme \( \text{Hilb}_H(\mathbb{A}^n, 0) \) is a union of connected components of \( \text{Hilb}_k^{\text{BB}}(\mathbb{A}^n, 0) \). We call this union the stratum corresponding to \( H \), or, when an ideal \( I \) with \( H_{R/I} = H \) is given, the stratum of \( I \). (We believe that \( \text{Hilb}_H^{\text{or}}(\mathbb{A}^n, 0) \) is connected, so the stratum is connected as well. However, this result seems to be not present explicitly in the literature and it is not important for the current article.)

Let \([I] \in \text{Hilb}_k(\mathbb{A}^n)\). Since \( I \) is a homogeneous ideal, the tangent space at \([I] \in \text{Hilb}_k(\mathbb{A}^n)\) has a structure of a graded \( R \)-module \( \text{Hom}_k(I, R/I) = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_k(I, R/I)_d \) with \( \text{Hom}_k(I, R/I)_d \) consisting of all homomorphisms \( \varphi : I \rightarrow R/I \) of \( R \)-modules such that \( \varphi(I) \subseteq (R/I)_{i+d} \) for all \( i \). We set

\[
\text{Hom}_k(I, R/I)_{\geq 0} := \bigoplus_{i \geq 0} \text{Hom}_k(I, R/I)_i \quad \text{and} \quad \text{Hom}_k(I, R/I)_{> 0} := \bigoplus_{i > 0} \text{Hom}_k(I, R/I)_i.
\]
By [Jel19, Theorem 4.2] the tangent space at \([I] \in \text{Hilb}^r_k(\mathbb{A}^n, 0)\) is isomorphic to \(\text{Hom}_R(I, R/I)_{\geq 0}\). By a similar argument, the tangent space at \([I] \to \pi^{-1}([I])\) is \(\text{Hom}_R(I, R/I)_{> 0}\).

Although the tangent space \(\text{Hom}_R(I, R/I)\) can be computed by hand, we usually use the computer algebra system \text{Macaulay2} to compute it quickly in explicit cases, most of them related to monomial ideals. See Proposition 3.1 for an example of such a computation.

### 2.3 Macaulay’s Inverse Systems

Let \(S = \mathbb{k}[y_1, \ldots, y_n]\) be the polynomial ring with \(n\) variables which we view as a vector space. Recall the rings \(R = \mathbb{k}[x_1, \ldots, x_n] \subset \hat{R} = \mathbb{k}[x_1, \ldots, x_n]\). The ring \(\hat{R}\) acts on \(S\) by the partial derivation map \(\partial : \hat{R} \times S \to S\) defined as follows:

\[
\partial^\alpha \circ y^\beta := \begin{cases} 
\binom{n}{\alpha} y^{\beta - \alpha} & \beta \geq \alpha, \\
0 & \text{otherwise},
\end{cases}
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_n)\) and \(\beta = (\beta_1, \ldots, \beta_n)\) are vectors in \(\mathbb{N}^n\), \(\binom{n}{\alpha} = \prod_{i=1}^{n} \binom{n}{\alpha_i}\), and \(\beta \geq \alpha\) if \(\beta_i \geq \alpha_i\) for all \(1 \leq i \leq n\). With this action, \(S\) can be viewed as a \(\hat{R}\)-module. In fact, the partial derivation map is up to scalars equivalent to the action of \(\hat{R}\) on \(S\) by contraction, see for example [IK99, Appendix A], because we are over a field of characteristic zero. We will mostly work with the polynomial ring \(\hat{R}\) rather than the whole \(\hat{R}\), thus below we restrict to the \(R\)-action on \(S\). This is mostly a formal choice.

**Definition 2.3.** Let \(I \subset R\) be an \((x_1, \ldots, x_n)\)-primary ideal of \(R\). The Macaulay inverse system of \(I\) is

\[
I^\perp = \{g \in S : I \circ g = 0\}.
\]

This is an \(R\)-submodule of \(S\). Its generators are called dual generators of \(I\). Given a subset \(E \subset S\), the annihilator of \(E\) is the ideal

\[
\text{Ann}_R(E) = \{f \in R : f \circ E = 0\},
\]

which is called the apolar ideal of \(E\).

Note that \(\text{Ann}_R(E) = \text{Ann}_R(M)\), where \(M = R \circ E\) is the \(R\)-submodule of \(S\) generated by \(E\). If \(I \subseteq R\) is a homogeneous ideal, then \(I^\perp\) is spanned by homogeneous polynomials, and if \(E\) is spanned by homogeneous elements, then \(\text{Ann}_R(E) \subseteq R\) is homogeneous. Also, if \(I \supseteq (x_1, \ldots, x_n)^r\), then \(I^\perp \subset S_{<r}\). We will often abbreviate \(\text{Ann}_R(\cdot)\) to \(\text{Ann}(\cdot)\) when the ring is clear from the context.

The above constructions are justified by the following theorem.

**Theorem 2.4** (Macaulay’s duality). For every \((x_1, \ldots, x_n)\)-primary ideal \(I\) of \(R\) and finitely generated \(R\)-submodule \(M \subset S\) we have \(\text{Ann}_R(I^\perp) = I\) and \((\text{Ann}_R(M))^\perp = M\). In this way, the operations \(\perp\) and \(\text{Ann}_R(\cdot)\) give an inclusion-reversing bijection between finitely generated \(R\)-submodules of \(S\) and \((x_1, \ldots, x_n)\)-primary ideals of \(R\). Moreover, the \(R\)-module \(I^\perp\) is minimally generated by \(r(R/I)\) elements. A zero-dimensional \(A = R/I\) is Gorenstein of socle degree \(s\) if and only if \(I^\perp\) is a principal \(R\)-module generated by a polynomial of degree \(s\).

**Proposition 2.5** (parameterizing by polynomials vs by the Hilbert scheme). For a Hilbert function \(H\) with \(k = \sum H\) consider the set \(\mathcal{L}_H\) of all polynomials \(f\) such that \(R/\text{Ann}_R(f)\) has Hilbert function \(H\); this is a locally closed subset of \(S_{<k}\). Then there is a surjective map \(\mathcal{L}_H \to \text{Hilb}^r_k(\mathbb{A}^n, 0)\) whose fibers are \(k\)-dimensional. In particular, we have

\[
\dim \text{Hilb}^r_k(\mathbb{A}^n, 0) = \dim \mathcal{L}_H - k.
\]

**Proof.** For a polynomial \(f \in S_{<k}\) to fix the Hilbert function of \(R/\text{Ann}_R(f)\) is the same as fixing the dimensions of the spaces \(m_i \circ f\) for every \(i\). This shows that \(\mathcal{L}_H\) is locally closed. The surjectivity and the claim about fiber dimensions are classical, see for example [Ems78, Iar94]. □
2.4 Lower bounds on dimension of components of $\text{Hilb}_k^n(A^n, 0)$

In this section, we prove that every component of $\text{Hilb}_k^n(A^n, 0)$ has at least the excepted dimension $(n - 1)(k - 1)$. We begin with a general observation.

**Proposition 2.6** (weak purity of the total branch locus). Let $f : X \to S$ be a finite flat degree $d$ morphism of schemes locally of finite type over $k$. Assume $f$ is generically étale. Consider the locus $\mathcal{Z} := \{s \in S \mid \text{the geometric fiber } f^{-1}(s) \text{ has only one point}\}$

This is a closed subset of $S$ and for every $z \in \mathcal{Z}$ we have $\dim_z \mathcal{Z} \geq \dim_z S - (d - 1)$.

**Proof.** Closedness of $\mathcal{Z}$ is well-known, for example [Sta21, Tag 0BU1]. The statement on the dimension is local around $z$, so we may replace $S$ by a component through $z$ which has dimension $\dim_z S$. Next, we reduce to the situation when $f : X \to S$ has $d$ sections $s_1, \ldots, s_d$ such that $X = \bigcup s_i(S)$. Assume we have already $p$ sections $s_1, \ldots, s_p$, where $p$ is possibly zero. Suppose $X \neq \bigcup s_i(S)$, take any irreducible component $S'$ of $X$ other than $s_i(S), \ldots, s_p(S)$ and view it with reduced scheme structure. Consider the base change

$$X' := X \times_S S' \longrightarrow X,$$

$$\downarrow f'$$

$$S' \longrightarrow S.$$

The map $f'$ is finite flat degree $d$ by base change. Since $f$ is flat, it is torsion free, so every component of $X$ dominates $S$. Moreover, $f$ is closed as a finite map. Thus $S' \to S$ is onto and so $f'$ is generically étale. This map has $p + 1$ sections: $p$ sections which are pullbacks of $s_i$ and an additional section $i \times \text{id}$ that comes from $i : S' \to X$. The intersections of every two of these sections with a fiber over a general point of $S'$ are disjoint, hence indeed we obtained $p + 1$ disjoint sections (this in particular shows that one could not obtain $d + 1$ sections, so the procedure ends exactly for $p = d$). By replacing $S$ as above, we assume $f : X \to S$ is such that $X = \bigcup_{i=1}^d X_i$, where $f|_{X_i} : X_i \to S$ are isomorphisms. We finally replace $S$ by its normalization (which is a finite surjective map since $S$ was a finite type scheme) and shrink it so that $S = \text{Spec}(A)$ is affine and $X = \text{Spec}(B)$, where $B$ is a free $A$-algebra.

We observe now that $X$ is connected in codimension one, which by definition means that $X \setminus V$ is connected for every closed $V$ with $\dim V \leq \dim S - 2$. Indeed, suppose not, then there exist nontrivial idempotents in $H^0(X \setminus V, \mathcal{O}_X)$. Since $A$ is normal, we have

$$A = \bigcap \{ A_p \mid p \in \text{Spec}(A), \dim(A/p) = \dim(A) - 1 \}.$$

Since $B$ is a free $A$-module, we have

$$B = \bigcap \{ B_p \mid p \in \text{Spec}(A), \dim(A/p) = \dim(A) - 1 \},$$

where $B_p = (A \setminus p)^{-1} B$. Therefore, the idempotents above uniquely extend to global sections $H^0(X, \mathcal{O}_X)$ which are idempotents themselves; so $X$ is disconnected. But this is absurd since the unique preimage of $z$ lies in every irreducible component $X_i$. This concludes the proof that $X \setminus V$ is connected.

The locus where $f : X \to S$ is not étale is, locally near $z$, given by a single equation [Sta21, Tag 0BVH]. This locus coincides with the locus of $s \in S$ such that in $f^{-1}(s)$ two of the components $X_1, \ldots, X_j$ coincide. Consider the graph with vertices $X_i$ and an edge joining $X_i$ and $X_j$ if $\dim(X_i \cap X_j) = d - 1$. If this graph is disconnected, then $X$ is not connected in codimension one. Hence this graph is connected and we may choose a spanning tree $e_1 = (X_{i_1}, X_{j_1}), \ldots, e_{d-1} = (X_{i_{d-1}}, X_{j_{d-1}})$. The image $S_m$ of $X_{i_m} \cap X_{j_m}$ is a codimension one subscheme of $S$ that contains $z$, so the intersection $S_1 \cap S_2 \cap \ldots \cap S_{d-1}$ is near $z$ a
subscheme of codimension at most \( d - 1 \), see for example [Eis95, Chapter 10, 10.2, 10.5, 10.6]. For every point \( z' \) of this intersection consider the fiber \( F = f^{-1}(z') \). By assumption, the intersection \( F \cap X_m \cap X_{jm} \) is nonempty for every \( m \). But \( F \cap X_m = F \cap s_j(S) \) is (as a scheme!) a point, and similarly for \( F \cap X_{jm} \). This shows that \( F \cap X_m = F \cap X_{jm} \) for every \( m \). Since we iterate over a spanning tree of a graph with vertices \( X_1, \ldots, X_d \), we deduce that topologically \( |F| \cap |X| = |F| \cap \bigcup_j |X_j| = \bigcup_j |F| \cap X_j | \) is a point. Therefore, \( z' \in Z \) and so \( S_1 \cap S_2 \cap \ldots \cap S_{d-1} \subset Z \), which concludes the proof.

As a Corollary, we get the following result, whose latter part was proven classically by different methods in [Gaf88, Theorem 3.5].

**Proposition 2.7.** The dimension of \( \text{Hilb}^m_k(\mathbb{A}^n, 0) \) at every its point is at least \((n - 1)(k - 1)\).

**Proof.** By Proposition 2.6 applied to the universal family of \( \text{Hilb}^m_k(\mathbb{A}^n) \) we get that the locus \( Z \subset \text{Hilb}^m_k(\mathbb{A}^n) \) parameterizing schemes supported at a single point has codimension at most \( k - 1 \). This locus is fibered over \( \mathbb{A}^n \) by sending the scheme to its support (formally, this map comes from restricting the Hilbert-Chow morphism). The locus \( \text{Hilb}^m_k(\mathbb{A}^n, 0) \) is a fiber of \( Z \to \mathbb{A}^n \), hence it has codimension at most \( k - 1 + n \).

Iarrobino [Iar87, p.310] asked whether the lower bound \((n - 1)(k - 1)\) from Proposition 2.7 holds for every component of \( \text{Hilb}_k(\mathbb{A}^n, 0) \). Recently, Satriano and Staal [SS21] gave a negative answer already for \( n = 4 \).

## 3 Irreducibility of punctual Hilbert schemes on threefolds for \( k \leq 11 \)

In this subsection, we prove the irreducibility of \( \text{Hilb}_k(\mathbb{A}^3, 0) \) for \( k \leq 11 \). Recall from §2.2 that the stratum \( \text{Hilb}_H(\mathbb{A}^3, 0) \) of an ideal \( I \) is the union of some connected components of \( \text{Hilb}_I^m(\mathbb{A}^3, 0) \); the closed points of this union are ideals \( I' \) with \( H_{R/I'} = H_{R/I} \) and the tangent space to the stratum at a \( \text{G}_m \)-fixed \([I]\) is given by \( \text{Hom}_R(I, R/I)_{\mathbb{Z}_0} \). As explained in the introduction, having Proposition 2.7, the main difficulty is in proving that the dimension of all but one strata is smaller than expected. We do it by bounding the dimension of their tangent spaces.

**Proposition 3.1.** Consider Borel-fixed ideals \( I \subset \mathbb{k}[x_1, x_2, x_3] \) such that \( \dim_k \mathbb{k}[x_1, x_2, x_3]/I \leq 11 \). Then
\[
D(I) := \dim_k \left( T_{[I]} \text{Hilb}_k(\mathbb{A}^3) \right)_{\mathbb{Z}_0} - 2(k - 1)
\]
is negative for all of them with the exception of the following ideals:

(i) \( I = (x_1, x_2, x_3^2) \) where \( D(I) = 0 \),

(ii) \( I = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2) \) where \( D(I) = 0 \) and \( H_{R/I} = (1, 3, 3, 1) \),

(iii) \( I = (x_1^2, x_1x_2, x_2x_3, x_1x_3^2, x_2x_3^2, x_3^3) \) where \( D(I) = 0 \) and \( H_{R/I} = (1, 3, 3, 1, 1) \),

(iv) \( I = (x_1x_3, x_1x_2, x_1^2, x_1x_2x_3, x_2x_3, x_3x_2, x_3^2) \) where \( D(I) = 0 \) and \( H_{R/I} = (1, 3, 3, 2, 1) \),

(v) \( I = (x_1^2, x_1x_2, x_2x_3, x_1x_3^2, x_2x_3^2, x_3^3) \) where \( D(I) = 0 \) and \( H_{R/I} = (1, 3, 3, 1, 1) \),

(vi) \( I = (x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_3x_2, x_3^3) \) where \( D(I) = 2 \) and \( H_{R/I} = (1, 3, 3, 2, 1) \),

(vii) \( I = (x_1x_3, x_1x_2, x_1^2, x_1x_2x_3, x_2x_3, x_3x_2, x_3^2) \) where \( D(I) = 0 \) and \( H_{R/I} = (1, 3, 3, 2, 1, 1) \),

(viii) \( I = (x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_3x_2, x_3^3) \) where \( D(I) = 0 \) and \( H_{R/I} = (1, 3, 3, 1, 1, 1) \),
(ix) \( I = (x_2^3, x_1 x_2, x_1^2, x_1 x_3, x_2 x_3^2, x_3^4) \) where \( D(I) = 2 \) and \( H_{R/I} = (1, 3, 3, 2, 1, 1) \),

(xi) \( I = (x_2^3, x_1 x_2, x_1^2, x_2 x_3^2, x_1 x_3, x_3^4) \) where \( D(I) = 0 \) and \( H_{R/I} = (1, 3, 3, 3, 1) \),

(xii) \( I = (x_1 x_2, x_1^2, x_2 x_3, x_2x_3^2, x_3^4) \) where \( D(I) = 1 \) and \( H_{R/I} = (1, 3, 4, 2, 1) \).

\( \text{Proposition 3.2. Let } H = (1, n, 3, 2, 1). \) The Hilbert scheme \( \text{Hilb}_{H}^S(A^n, 0) \) set-theoretically consists of the following loci:

(1) Ideals dually generated by \( \ell_1, \ell_2, q \), where \( \ell_1, \ell_2 \) are two independent linear forms, and \( q \) is a quadric independent from \( \ell_1, \ell_2 \). This is a locus parameterized by an open subset of \( \text{Gr}(1,n) \times \text{Gr}(1,n) \times \text{Gr}(1,\binom{n+1}{2} - 2) \), whence it has dimension \( 2n + \binom{n+1}{2} - 5 \).

(2) Ideals dually generated by \( Q \) and \( q \), where \( Q \) lies in \( \text{Sym}^4(V) \) for a two-dimensional linear subspace \( V \subset S_1 \) and is such that the space of second-order partial derivatives of \( Q \) is two-dimensional, while \( q \) is a quadric linearly independent from the second order partial derivatives of \( Q \). The parameter space for \( Q \) is an open subset of a divisor in a \( \text{P}(\text{Sym}^4 k^2) \)-bundle over \( \text{Gr}(2,n) \), while the parameter space for \( q \) is an open subset of \( \text{Gr}(1,\binom{n+1}{2} - 2) \). Summing up, the locus is irreducible and has dimension \( 2n + \binom{n+1}{2} - 4 \).

(3) Ideals dually generated by all linear forms and a \( Q \), where \( Q \) is a general quartic in \( \text{Sym}^4(V) \) for a two-dimensional linear subspace \( V \subset S_1 \). This locus is parameterized by an open subset of a \( \text{P}(\text{Sym}^4 k^2) \)-bundle over \( \text{Gr}(2,n) \) and hence has dimension \( 2n \).

(4) Ideals dually generated by a perfect quartic \( \ell^4 \) and a non-perfect cubic \( c \) whose first order partial derivatives together with \( \ell^4 \) span a 3-dimensional subspace. This gives a locus of dimension at most \( 3n - 2 \).

In the case (4) we do not have a parameterization, but the estimate above is enough for our purposes.

\( \text{Proof.} \) Let \( I \) be the homogeneous ideal of an algebra with the given Hilbert function. Since \( I^+ \) is generated (non-minimally) by a quartic, two cubics and three quadrics as an \( R \)-module, the possibilities (1)-(3) and the corresponding dimension counts are obvious. We only prove the bound on the dimension of the last locus. Let \( \ell^4 \) be a perfect quartic and let \( c \) be a cubic whose partial derivatives together with \( \ell^4 \) generate a 3-dimensional vector space. If \( c \) does not depend essentially on three variables, then the upper bound \( n - 1 + \dim \text{Gr}(2, n) + 3 = 3n - 2 \) follows immediately, so we suppose it does. Let \( c \in k[W] \), for \( W \subset S_1 \) three-dimensional. The choice of \( c \) is determined by a choice of a linear form \( \alpha \in W^* \) such that \( \alpha \circ c = \ell^4 \) and a choice of an element in \( \text{Sym}^3 V \), where \( V \) is the two-dimensional space of linear forms in \( W \) annihilated by \( \alpha \). At most, we obtain a \( \dim \text{Gr}(3, n) + \dim \text{Gr}(1,W) + \dim W^* + \dim k[V]_3 = 3n \) dimensional choice. This parameterization is not one-to-one, as adding \( \ell^3 \) to \( c \) and/or multiplying it by a nonzero scalar does not change the system. Therefore, the final estimate is \( 3n - 2 \), as claimed.

\( \text{Remark 3.3.} \) By the very same argument as in Proposition 3.2, for \( H = (1, n, 3, 2, 1, ..., 1) \), with at least two trailing “1”, the Hilbert scheme \( \text{Hilb}_{H}^S(A^n, 0) \) consists of two loci, parameterizing ideals analogous to the ideals in (1) and (4), and with the same dimension estimates, respectively.
Proposition 3.4. Let $H = (1, 3, 4, 2, 1)$. The Hilbert scheme $\text{Hilb}_{H}^{\infty}(\mathbb{A}^{3}, 0)$ set-theoretically consists of the following loci:

1. Ideals dually generated by $t_1^3, t_2^3$ for two independent $t_1, t_2 \in S_1$, and two quadrics $q_1, q_2$ independent from $t_1^3, t_2^3$. This is a locus parametrized by an open subset of $\text{Gr}(1, 3) \times \text{Gr}(1, 3) \times \text{Gr}(2, 4)$ whence it has dimension 8.

2. Ideals dually generated by a perfect quartic $t^4$ and a general cubic. The cubic can be taken modulo the $t^8$ term, so the locus is parameterized by an open subset of $\text{Gr}(1, 3) \times \text{Gr}(1, 9)$, so it has dimension 10.

3. Ideals dually generated by $Q$ and $q_1, q_2$, where $Q$ lies in $\text{Sym}^4(V)$ for a two-dimensional linear subspace $V \subset S_1$ and is such that the space of second-order partial derivatives of $Q$ is two-dimensional, while $q_1, q_2$ are quadrics linearly independent from the second order partial derivatives of $Q$. This locus is an open subset of a divisor in a $\text{P}(\text{Sym}^4 k^2)$-bundle over $\text{Gr}(2, 3) \times \text{Gr}(2, 4)$, whence it has dimension $(2 + 4 + 4) - 1 = 9$.

4. Ideals dually generated by $Q$ and $q$, where $Q$ is an element in $\text{Sym}^4(V)$ for a two-dimensional linear subspace $V \subset S_1$ and is such that the space of second-order partial derivatives of $Q$ is three-dimensional, while $q$ is a quadric linearly independent from the second order partial derivatives of $Q$. This locus is parametrized by an open subset of a $\text{P}(\text{Sym}^4 k^2)$-bundle over $\text{Gr}(2, 3) \times \text{Gr}(1, 3)$ and hence of dimension 8.

5. Ideals dually generated by $t^4, c, q$, where $t$ is a linear form, $c$ is a cubic in $S$ with first-order partial derivatives are linearly dependent with $t^4$, and a quadric $q$ independent from partial derivatives of $c$ and $t^4$. This gives a locus of dimension at most 9.

Proof. We only note that the bound 9 in (5) follows from the bound 7 in Proposition 3.2(4) increased by $2 = \dim \text{Gr}(1, 3)$ to account for the choice of a quadric $q$ modulo the partials of $c$ and $t^4$. 

Proposition 3.5. The stratum of $(x_1, x_2, x_3^k)$ is $2(k - 1)$-dimensional, while the strata for all other exceptional ideals from Proposition 3.1 have dimension strictly less than $2(\deg(I) - 1)$. Moreover, the point $(x_1, x_2, x_3^k)$ is a smooth point of its stratum.

Proof. Consider first the curvilinear ideal $I = (x_1, x_2, x_3^k)$. We directly see that the dimension of the tangent space $\text{Hom}_R(I, R/I)_{\geq 0}$ to its stratum is $2(k - 1)$. Moreover, for every $f_1, f_2 \in (R[I]_{\leq k-1}$ the ideal $(x_1 - f_1(x_3), x_2 - f_2(x_3), x_3^k)$ lies in the stratum of $I$. Hence this stratum has dimension at least $2(k - 1)$ near $[I]$. Together with the tangent space dimension, this implies that $[I]$ is a smooth point of its $2(k - 1)$-dimensional stratum.

Consider now the remaining ideals $I$. First, pick any of them with $D(I) = 0$. Suppose that its stratum is $2(k - 1)$-dimensional. Then $D(I) = 0$ implies that $[I]$ is a smooth point of its stratum. Suppose $I'$ is a homogeneous ideal and suppose there is a one-dimensional torus $T$ action on $\mathbb{A}^3$ such that in the induced action on $\text{Hilb}_k(\mathbb{A}^3)$ the point $[I']$ converges to $[I]$. The curve $[I]$ degenerating $[I']$ to $[I]$ lies in the $G_{m}$-fixed locus and in particular in the stratum of $[I]$. Since $[I]$ is a smooth point of the stratum, the smooth locus of the stratum intersects the curve $[I]$ in an open, $G_{m}$-stable subset that contains the unique $G_{m}$-fixed point of the curve; thus the smooth locus contains the whole curve $[I]$. In particular, the point $[I']$ is smooth in this stratum, hence

$$(\text{Hilb}_k(\mathbb{A}^3))_{\geq 0} = 2(k - 1).$$

By semicontinuity of each graded piece of the tangent space, we even obtain

$$\left(\text{Hilb}_k(\mathbb{A}^3)\right)_i = \left(\text{Hilb}_k(\mathbb{A}^3)\right)_{i-0} \quad \text{for all} \quad i \geq 0.$$
It is enough to exhibit an ideal $I'$ violating this last condition. Below we give a list of those, in each case pointing out the differing degree. In all cases, we fix a $T$-action $t \cdot (y_1, y_2, y_3) = (t^a y_1, t^b y_2, t^c y_3)$ with $a_1 \gg a_2 \gg a_3$ and consider the limit at $t \to 0$, so the initial form of a homogeneous polynomial is its lex-largest monomial in $S$.

(ii), (iii), (v), (viii) consider the apolar ideal of $y_1^2 + y_3^2$, $y_2 y_3$, $y_3^{4+i}$, degree $i$ for $i = 0, 1, 2, 3$, respectively.

(iv), (vii) consider the apolar ideal of $y_1^2 + y_2^2$, $y_2 y_3^2$, $y_3^{4+i}$, degree $i$ for $i = 0, 1$, respectively.

(x) consider the apolar ideal of $y_1 y_3^2 + y_2 y_3^2, y_3^5$, degree 1.

In the case (vi), by Proposition 3.2 we see that the $\mathbb{G}_m$-fixed locus near this point has dimension at most 8, while the degree zero part of the tangent space at $[I]$ has dimension 10. By Proposition 3.1 the strictly positive part of this tangent space also has dimension 10; in particular, the fiber has dimension at most 10. By the base-and-tangent-to-fiber estimate, the stratum of $[I]$ has dimension at most $8 + 10 = 2 \cdot (10 - 1)$. Suppose that the stratum has dimension exactly $2 \cdot (10 - 1)$. By the estimation above, the $\mathbb{G}_m$-fixed locus near $[I]$ has dimension exactly 8. By the bounds from Proposition 3.2 this means that $[I] \in \text{Hilb}^{10}_{\mathbb{G}_m}(\mathbb{A}^3, 0)$ lies in the closure $\mathcal{V}$ of the irreducible locus from Proposition 3.2(2) and that there exists an open neighbourhood of $[I] \in \mathcal{V}$ such that the the fiber of $\pi$ is smooth 10-dimensional over any point of this neighborhood. The strictly positive degree parts of the tangent spaces are constant for this open neighbourhood. Consider the ideal $I'$ apolar to $y_1^2 + y_3^2$, $y_2 y_3^2 + y_3^4$. It lies in $\mathcal{V}$ and its lex-initial ideal is $I$, so the $T$-orbit of $[I']$ is a curve in $\mathcal{V}$ which intersects the above neighbourhood nontrivially. Yet, the tangent spaces to the stratum at $[I']$ and $[I]$ have dimensions differing in degree 1, which gives a contradiction. The very same strategy works in the case (ix) with Remark 3.3, the ideal $I'$ apolar to $y_1^2 + y_3^2$, $y_2 y_3^2$, $y_3^5$ and degree 2.

In the case (xi) the degree zero part of the tangent space is 12-dimensional, while the possible homogeneous inverse systems are at most 10-dimensional by Proposition 3.4, so by the base-and-tangent-to-fiber estimate the stratum has dimension at most 19. This concludes the last case and the whole proof.

\[\Box\]

**Proof of Theorem 1.3.** By [DJNT17] we have $\text{Hilb}^0_k(\mathbb{A}^3) = \text{Hilb}^m_k(\mathbb{A}^3)$. By Proposition 2.7 each of the irreducible components of $\text{Hilb}^m_k(\mathbb{A}^3, 0)$ has dimension at least $2(k - 1)$. Take any such component $Z$. There is a stratum $\text{Hilb}_{\mathbb{G}_m}^1(\mathbb{A}^3, 0)$ which contains an open subset of this component, so the dimension of $Z \cap \text{Hilb}_{\mathbb{G}_m}^1(\mathbb{A}^3, 0)$ is at least $2(k - 1)$. Consider a $\mathbb{G}_m$-fixed point $[I]$ of this intersection. The ideal $I$ is graded and the nonnegative part of its tangent space to the Hilbert scheme has dimension at least $2k - 1$. The generic initial ideal $I_0$ of $I$ is Borel-fixed [Eis95, §15.9]. By semicontinuity, both its stratum and the nonnegative part of its tangent space have dimension at least $2k - 1$. But by Propositions 3.1, 3.5 there is a single such point, corresponding to $I_0 = (x_1, x_2, x_3^d)$. It follows that the generic initial ideal of $I$ is in the curvilinear locus, hence $I$ itself is in this locus. Thus the stratum of $I$ coincides with the stratum of $I_0$ which is equal to the curvilinear locus, so the chosen component lies in this locus. This concludes the proof.

\[\Box\]

### 4 Construction of regular maps to Grassmannian

In this section, we derive some existence statements for regular maps. They generalize the upper bounds from [BJJM19] and are proven using a similar method, although the key Proposition 4.2, which generalizes [BB14, Lemma 2.3] (that actually refers to [BGL13]), has to be proven in a bit subtler way.
4.1 Reduction to small socle

Let $V$ be a $k$-vector space and $\text{Gr}(r, V)$ be the Grassmannian of its $r$-dimensional subspaces. Let $U'$ be the universal subbundle, which is a rank $r$ vector bundle over $\text{Gr}(r, V)$. For a zero-dimensional scheme $Z$ with a morphism $f: Z \to \text{Gr}(r, V)$, let $U_Z$ be the pullback of this bundle to $Z$ so that we have

$$ U_Z \xrightarrow{\text{closed}} Z \times V \xrightarrow{\text{pr}_2} V $$

and let $(Z)_f$ be the linear space spanned in $V$ by $\text{pr}_2(U_Z)$, where $\text{pr}_2$ is the second projection. We stress that $(Z)_f$ depends not only on $Z$ but also on the morphism $f: Z \to \text{Gr}(r, V)$.

**Example 4.1.** The motivation for the symbol $(Z)_f$ is the case when $Z$ is a tuple of $k$ points. Then the map $f: Z \to \text{Gr}(r, V)$ corresponds to choosing $k$ subspaces $W_1, \ldots, W_k \subset V$ and so $(Z)_f = W_1 + \ldots + W_k$.

In particular, for $\tau = 1$ and for $Z \to \text{Gr}(1, V) = P(V)$ we get the usual notion of projective span of $Z$.

For a zero-dimensional irreducible scheme $Z$, let $\tau(Z) := \tau(H^0(Z, \mathcal{O}_Z)) = \dim_k \text{Soc}(H^0(Z, \mathcal{O}_Z))$. For example, $\tau(Z) = 1$ if and only if $Z$ is Gorenstein. The following is the key result that will allow us to reduce from $\text{Hilb}_k(A^0, 0)$ to its sublocus $\text{Hilb}_k^0(A^0, 0)$. It is a generalization of [BGL13, Lemma 3.5 (iii) $\implies$ (i)], which dealt with the case $\tau = 1$.

**Proposition 4.2.** Let $Z$ be a zero-dimensional irreducible scheme with $\tau(Z) > \tau$ together with a morphism $f: Z \to \text{Gr}(r, V)$. Then $(Z)_f = \bigcup_{Z' \subseteq Z} (Z')_{f|_{Z'}}$.

Consequently, we have

$$ (Z)_f = \bigcup \left\{ (Z')_{f|_{Z'}} \mid Z' \subset Z, \tau(Z') \leq \tau \right\}. $$

For the proof, we need the following auxiliary lemma.

**Lemma 4.3.** Let $Z$ be a zero-dimensional irreducible scheme with $\tau(Z) > \tau$. Let $\mathcal{E}$ be a rank $\tau$ locally free sheaf on $Z$ and let $E = H^0(Z, \mathcal{E})$. Finally, let $\varphi: E \to k$ be a $k$-linear map. Then there exists a proper subscheme $Z' \subseteq Z$ and a $k$-linear map $H^0(Z', \mathcal{E}|_{Z'}) \to k$ such that $\varphi$ factors as

$$ H^0(Z, \mathcal{E}) \to H^0(Z', \mathcal{E}|_{Z'}) \to k, $$

where the first map comes from restricting $\mathcal{E}$ to $Z'$.

**Proof.** The sheaf $\mathcal{E}$ is actually free, so we assume $\mathcal{E} = \mathcal{O}_Z^\oplus r$ and consequently $E = H^0(Z, \mathcal{O}_Z)^\oplus r$. Let $i_p: H^0(Z, \mathcal{O}_Z) \to E$ be the injection of the $p$-th factor. The space

$$ \bigcap_{p=1}^r \ker(\varphi \circ i_p) \subseteq H^0(Z, \mathcal{O}_Z) $$

is a corank $\leq \tau$ subspace. By assumption, $\dim_k \text{Soc}(H^0(Z, \mathcal{O}_Z)) > \tau$, so this subspace intersects the socle nontrivially. Let $r$ be a non-zero element of this intersection. The subscheme $Z' = \text{V}(r)$ is the required one. \hfill \square

**Proof of Proposition 4.2.** Let $0 \to U' \to \mathcal{O}_{\text{Gr}(r, V)} \otimes V \to Q \to 0$ be the universal sequence of locally free sheaves on $\text{Gr}(r, V)$, where $U'$ has rank $r$. Dualizing and taking symmetric powers, we obtain the sequence

$$ 0 \to \mathcal{K} \to \mathcal{O}_{\text{Gr}(r, V)} \otimes \text{Sym}(V^*) \to \text{Sym}(U^*) \to 0 $$

where $\mathcal{K} = (Q^*)$ is the ideal sheaf generated by degree one elements and with $\mathcal{K}_1 = Q^*$; it is exactly the ideal sheaf of $U' \hookrightarrow \text{Gr}(r, V) \times V$. Restricting this sequence to $Z$ via $f$ we obtain the sequence

$$ 0 \to \mathcal{K}_Z \to \mathcal{O}_Z \otimes \text{Sym}(V^*) \to \text{Sym}(\mathcal{E}) \to 0 $$

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where $\mathcal{E} = U^n|_Z$ and so $\mathcal{K}_Z$ is the ideal sheaf of $U_Z \rightarrow Z \times V$. The ideal of $\text{pr}_Z(U_Z)$ is $\mathcal{K}_Z \cap \text{Sym}(V^*)$, where $\text{Sym}(V^*) \to \mathcal{O}_Z \otimes \text{Sym}(V^*)$ is the usual inclusion. Note that this ideal is homogeneous. Finally, the linear span of $\text{pr}_Z(U_Z)$ is given by taking only the linear part of this ideal. So we discard the nonlinear parts of the above sequence and take global sections to obtain

$$
\begin{align*}
0 & \longrightarrow H^0(\mathcal{K}_Z)_0 \longrightarrow H^0(Z, \mathcal{O}_Z) \otimes \mathbb{K} V^* \longrightarrow H^0(Z, \mathcal{E}) \longrightarrow 0
\end{align*}
$$

A $\mathbb{K}$-point $x \in V$ lies in $\langle \text{pr}_Z(U_Z) \rangle$ if and only if $m_x$ contains $\varphi$. For such a point we complete the above diagram to

$$
\begin{align*}
0 & \longrightarrow H^0(\mathcal{K}_Z)_0 \longrightarrow H^0(Z, \mathcal{O}_Z) \otimes \mathbb{K} V^* \longrightarrow H^0(Z, \mathcal{E}) \longrightarrow 0
\end{align*}
$$

where $\varphi_x$ is a map making the diagram commutative. By Lemma 4.3, there is a proper closed subscheme $Z' \subset Z$ such that the relevant part of the above diagram completes to a commutative diagram

$$
\begin{align*}
\begin{array}{ccc}
V^* & \xrightarrow{ev_x} & \mathbb{K} \\
\downarrow & & \downarrow \varphi_x \\
H^0(\mathcal{K}_Z)_0 & \longrightarrow & H^0(Z, \mathcal{E})
\end{array}
\end{align*}
$$

Reversing the above argument, we get that $x$ lies in the linear span of $\text{pr}_Z(U_{Z'})$ as claimed.

**Example 4.4.** Let $Z = \text{Spec}(\mathbb{K}[x_1, x_2, x_3]/(x_1, x_2, x_3)^2)$, so that $\tau(Z) = 3$. Let $V$ be a $\mathbb{K}$-vector space. A morphism $f : Z \rightarrow \text{Gr}(2, V)$ corresponds to a rank two locally free subsheaf $\mathcal{F} \subset \mathcal{O}_Z \otimes \mathbb{K} V$ such that $(\mathcal{O}_Z \otimes \mathbb{K} V)/\mathcal{F}$ is locally free. Since $Z$ is zero-dimensional, the sheaf $\mathcal{F}$ is free and determined by its space of global sections $H^0(\mathcal{F})$ which is spanned as an $H^0(\mathcal{O}_Z)$-module by two elements

$$
e_{11} + e_{12}x_1 + e_{13}x_2 + e_{14}x_3, e_{21} + e_{22}x_1 + e_{23}x_2 + e_{24}x_3 \in \mathbb{K}[x_1, x_2, x_3]/(x_1, x_2, x_3)^2 \otimes \mathbb{K} V,$$

where $e_{11}, \ldots, e_{14}, e_{21}, \ldots, e_{24} \in V$. By definition of $f$, we have $U_Z = \mathcal{F}$ as subsheaves of $\mathcal{O}_Z \otimes V$, so that $\mathcal{E}$ from the proof is equal to $\mathcal{F}$ and the map $\varphi : V^* \rightarrow H^0(\mathcal{E}) = H^0(\mathcal{F})^*$ is given by

$$\varphi(v^*) = (v^*(e_{11}), \ldots, v^*(e_{14}), v^*(e_{21}), \ldots, v^*(e_{24})).$$

It follows that $(Z)_f = \langle e_{11}, \ldots, e_{24} \rangle$. Consider a point $p = \sum_{1 \leq i < j \leq 4} a_{ij}e_{ij}$ in this span, where $a_{ij} \in \mathbb{K}$. Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{K}^3$ be a nonzero vector perpendicular to $(a_{12}, a_{13}, a_{14})$ and $(a_{22}, a_{23}, a_{24})$. Reversing the above argument we get that $p$ lies in $(Z')_f$ where $Z' = V(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) \subset Z$.

**4.2 Existence of regular maps and its connection with the Hilbert scheme**

In this section $\mathbb{K} = \mathbb{C}$ and we consider $\mathbb{C}^n$ mostly as a topological space with Euclidean topology. We also consider maps $X \rightarrow \text{Gr}(\tau, V)$ for more general $X$. In fact, we will mostly use the case $X = \mathbb{P}^n$, see Corollary 4.7, but it seems more natural to give the general definitions. Correspondingly, we will consider $\text{Hilb}_k(\mathbb{P}^n, p)$ where $p$ is a fixed closed point, and similar loci instead of $\text{Hilb}_k(\mathbb{A}^n, 0)$.
**Definition 4.5.** Let $X$ be a topological space. A map $f : X \to \text{Gr}(\tau, V)$ is $k$-regular if for every tuple of $k$ distinct points $x_1, \ldots, x_k \in X$ the linear space $(f(x_1), \ldots, f(x_k))$ is $\tau k$-dimensional. Let $X$ be an algebraic variety. A morphism $f : X \to \text{Gr}(\tau, V)$ is strongly $k$-regular if for every zero-dimensional smoothable subscheme $Z \subset X$ of degree $k$ the linear space $(Z)_{f|_Z}$ is $\tau k$-dimensional.

In light of Example 4.1, strongly regular maps are regular. Moreover, the existence of a regular map for a certain $N$ implies the existence of this map for any integer $\geq N$. The definition of a $k$-regular map makes sense for every map of sets $f : X(C) \to \text{Gr}(\tau, V)(C)$.

Let $f : X \to \mathbb{P}^{N-1} = \mathbb{P}(\mathbb{C}^N)$ be a morphism of varieties. Let $\mathbb{C}^N = \bigoplus_{i=1}^r \mathbb{C}N_i$ and define the morphism $f^\tau : X \to \text{Gr}(\tau, \mathbb{C}^N)$ by

$$f^\tau(x) = \left\langle f(x)e_1, f(x)e_2, \ldots, f(x)e_r \right\rangle,$$

where $f(x) \in \mathbb{C}^N$ is any element of the line $f(x) \in \mathbb{P}(\mathbb{C}^N)$. More formally, the morphism $f^\tau$ is defined as follows. Over $\mathbb{P}^{N-1}$ we have the universal line subbundle $U' \hookrightarrow \mathbb{P}^{N-1} \times \mathbb{C}^N$ whose fiber over a point $\mathbb{P}^{N-1}$ is the corresponding line in $\mathbb{C}^N$. In fact, this is a special case of the universal subbundle for a Grassmannian, as in the previous section. Pulling back $U'$ via $f$ we get a subbundle $U_f \hookrightarrow X \times \mathbb{C}^N$. The direct sum $U_f e_1 \oplus \ldots \oplus U_f e_r \hookrightarrow \bigoplus_{i=1}^r \mathbb{C}N_i$ is a bundle over $X$ that is a rank $\tau$ subbundle of $\mathbb{C}^N$ so it induces a morphism $X \to \text{Gr}(\tau, \mathbb{C}^N)$.

When we view $\mathbb{P}^{N-1}$ as $\text{Gr}(1, \mathbb{C}^N)$, Definition 4.5 gives us notions of $k$-regularity for $f$. Explicitly, $f$ is $k$-regular if for every $k$ distinct points of $X$, their images under $f$ span a $(k-1)$-dimensional projective subspace and $f$ is strongly $k$-regular if for every smoothable zero-dimensional subscheme $Z \subset X$ of degree $k$, its image $f(Z)$ spans a $(k-1)$-dimensional projective subspace.

**Lemma 4.6.** If $f$ is $k$-regular then $f^\tau$ is $k$-regular. If $f$ is strongly $k$-regular then $f^\tau$ is strongly $k$-regular.

**Proof.** We first prove the second statement. Take a zero-dimensional smoothable subscheme $Z \subset X$ of degree $k$. By strong $k$-regularity, $W' := (Z)_f \subset \mathbb{C}^N$ is a $k$-dimensional linear space. The vector space $(Z)_f$ contains the subspaces $W_i$ for all $i$, hence it contains the $\tau k$-dimensional vector space $\bigoplus_{i=1}^r W_i$, which proves that $f^\tau$ is strongly $k$-regular. For the proof of $k$-regularity, one may restrict to $Z$ being a tuple of points and repeat the argument (one may also do a much more elementary argument).

**Corollary 4.7.** For every $\tau, n, k$ there exists an $N$ and a strongly $k$-regular morphism $\mathbb{P}^n \to \text{Gr}(\tau, \mathbb{C}^N)$.

**Proof.** Follows from Lemma 4.6 and the existence of strongly $k$-regular morphisms from $\mathbb{P}^n$ to $\mathbb{P}^{N-1}$, see [BJJM19, Lemma 5.10].

Fix a closed point $p \in \mathbb{P}^n$, and consider a strongly $k$-regular morphism $F : \mathbb{P}^n \to \text{Gr}(\tau, \mathbb{C}^N)$. It induces a map

$$F_p : \text{Hilb}^k_{\mathbb{P}^n}(p) \to \text{Gr}(k\tau, \mathbb{C}^N)$$

that sends $[Z]$ to $(Z)_F$. We pull back the universal subbundle of the Grassmannian via $F_p$ to obtain a bundle $U_p$ on $\text{Hilb}^k_{\mathbb{P}^n}(p)$ and projectivise this bundle to obtain

$$\begin{array}{ccc}
\text{P}(U_p) & \hookrightarrow & \text{Hilb}^k_{\mathbb{P}^n}(p) \times \mathbb{P}^{N-1} \\
\downarrow & & \\
\text{Hilb}^k_{\mathbb{P}^n}(p)
\end{array}$$

The areole is the image of $\text{P}(U_p)$ in $\mathbb{P}^{N-1}$ As a set, it is the union of $\text{P}((Z)_F)$ for $Z$ ranging over all zero-dimensional smoothable degree $k$ subschemes $Z \to \mathbb{P}^n$ supported only at the point $p$. We denote the areole by $\alpha_{k,p} = \alpha_{k,p}(F)$. By construction, we have $\dim \alpha_{k,p} \leq \dim \text{P}(U_p) = \tau k - 1 + \dim \text{Hilb}^k_{\mathbb{P}^n}(p)$. The following gives a key improvement of the upper bound on the dimension of $\alpha_{k,p}$.
Proposition 4.8. With the above notations, we have
\[ \dim a_{k, p} \leq \max\{r_i - 1 + \dim \operatorname{Hilb}^i_{a}(P^n, p) \mid 1 \leq i \leq k\}. \]

Proof. Using Proposition 4.2, we have
\[ a_{k, p} = \bigcup_{i \leq k} \left\{ P((Z)_{F_p}) \mid Z \in \operatorname{Hilb}^i_{a}(P^n, p) \right\} \subseteq \bigcup_{i \leq k} \left\{ P((Z')_{F_p}) \mid Z' \in \operatorname{Hilb}^i_{a}(P^n, p) \right\} \]
and so \( \dim a_{k, p} \leq \max\{r_i - 1 + \dim \operatorname{Hilb}^i_{a}(P^n, p) \mid 1 \leq i \leq k\}. \)

Proposition 4.9 (Reduction of N). There exists a \( k \)-regular continuous map \( C^n \to \operatorname{Gr}(r, C^{M+1}) \) for \( M = \dim a_{k, p} \).

It should be stressed that the obtained \( k \)-regular map is usually not algebraic.

Proof. This is a straightforward generalization of [BJJM19, Theorem 5.7]; we sketch the argument below. A general subspace \( W \subset C^N \) of dimension \( N - M - 1 \) satisfies \( P(W) \cap a_{k, p} = \emptyset \), since the dimensions of these two varieties sum up to \( N - M - 2 + M = N - 2 < N - 1 \). Consider the projection \( \pi : C^N \to C^N/W \); the latter space is isomorphic to \( C^{M+1} \). This projection induces a rational map \( \pi : \operatorname{Gr}(r, C^N) \to \operatorname{Gr}(r, C^N/W) \). Since \( P(W) \) is disjoint from the areole, it is, in particular, disjoint from the subspace corresponding to \( F(p) \), so the composed map \( F_W = \pi \circ F : P^n \to \operatorname{Gr}(r, C^N/W) \) is a well-defined morphism on a Zariski-open neighborhood of \( p \). Suppose that for every Euclidean-open ball \( B_{\varepsilon} \) of radius \( \varepsilon \) around \( p \) the map \( F_W|_{B_{\varepsilon}} : B_{\varepsilon} \to \operatorname{Gr}(r, C^N/W) \) is not \( k \)-regular. For every \( \varepsilon \) choose a \( k \)-tuple of points \( x_1(\varepsilon), \ldots, x_k(\varepsilon) \in B_{\varepsilon} \) such that \( x_1(\varepsilon), \ldots, x_k(\varepsilon) \) witnesses the non-regularity, so the corresponding subspaces do not span a \( k \tau \)-dimensional space in \( C^N/W \). Taking a countable sequence \( (\varepsilon_n) \) converging to zero, we obtain a map \( \mathbb{N} \to \operatorname{Hilb}^i_{a}(P^n) \) that sends \( \varepsilon_n \) to \( \{x_1(\varepsilon_n), \ldots, x_k(\varepsilon_n)\} \). Since the points \( x_i(\varepsilon_n) \) converge to \( p \) with \( n \to \infty \), the Zariski closure of the image of \( \mathbb{N} \to \operatorname{Hilb}^i_{a}(P^n) \) contains a point \( [Z] \) corresponding to a subscheme \( Z \subset P^n \) supported only at \( p \). By semicontinuity, the span \( (Z)_{F_p} \subset C^N/W \) is also of dimension less than \( k \tau \), hence \( (Z)_{F_p} \) intersects \( W \) in a nonzero subspace, which is a contradiction with the choice of \( W \). The obtained contradiction shows that for some \( \varepsilon \) the continuous map \( F_W|_{B_{\varepsilon}} \) is \( k \)-regular. Composing with a homeomorphism \( B_{\varepsilon} \cong C^n \) we obtain the required map. 

Putting together Propositions 4.9 and 4.8 we obtain the following statement.

Corollary 4.10. For every \( r, n, k \) there exists a \( k \)-regular map \( C^n \to \operatorname{Gr}(r, C^{M+1}) \) for \( N = \max\{r i - 1 + \dim \operatorname{Hilb}^i_{a}(P^n, p) \mid 1 \leq i \leq k\} \).

5 The scheme \( \operatorname{Hilb}^2_k(A^n, 0) \) has expected dimension for \( k \leq 11 \)

In this section, we prove the upper bound \((n - 1)(k - 1)\) on the dimension of \( \operatorname{Hilb}^2_k(A^n, 0) \) for \( k \leq 11 \). Subdividing the \( \operatorname{Hilb}_k(A^n, 0) \) according to the Hilbert function, as described in (1.6), it is enough to prove this bound on the dimension of
\[ \operatorname{Hilb}^2_H(A^n, 0) := \operatorname{Hilb}_H(A^n, 0) \cap \operatorname{Hilb}^2_k(A^n, 0) \]
for every possible Hilbert function with \( k = \sum H \leq 11 \) and \( H(s) \leq 2 \). Such Hilbert functions are subdivided into three groups as in the following subsections.
5.1 Hilbert functions with $H(3) = 1$

First, we consider the Hilbert functions with $H(3) = 1$. Such a function has the form $(1, H(1), H(2), 1, 1, \ldots, 1)$ and so any minimal dual generating set has, without loss of generality, one “large degree” element $f$ and other elements of degree at most two; informally speaking such an algebra is close to the Gorenstein algebra given by the dual generator $f$. The Hilbert function of the apolar algebra of $f$ is $(1, a, b, 1, 1, \ldots, 1)$ and it follows from the general theory of Iarrobino’s symmetric decomposition (e.g. [CJN15, Remark (2), p. 1532]) that $a \geq b$. We will also assume $a = n$, as this is the only case of interest for us.

Proposition 5.1. Let $\text{Hilb}_{H}^{\text{Gor}}(\mathbb{A}^n, 0)$ be the locus of Gorenstein algebras with the Hilbert function $H = (1, n, b, 1, 1, \ldots, 1)$. Let $s$ be the largest number such that $H(s) \neq 0$. Then the dimension of $\text{Hilb}_{H}^{\text{Gor}}(\mathbb{A}^n, 0)$ is

$$(n - 1)(s - 3) + (n - b)b + \binom{b + 2}{3} - 1 + \binom{n + 2}{2} - (1 + n + b).$$

Proof. Every such Gorenstein algebra is given by an ideal $I = \text{Ann}(f)$ for a single polynomial $f \in S$. By [Jel94, Thm 5.3AB] (see also [CJN15, Proposition 3.3]) there exists a standard form of $f$. This is a form $g \in S$ of the shape

$$(5.2) \quad g = y_1^n + a_{−1}y_1^{n−1} + \ldots + a_ny_1^n + c + q,$$

where $a_i \in \mathbb{k}$, $c \in \mathbb{k}[y_1, \ldots, y_b]$ and $q \in \mathbb{k}[y_1, \ldots, y_b]_{\leq 2} = S_{\leq 2}$ such that there exists an automorphism $\phi : \overline{R} \rightarrow \overline{R}$ satisfying $\phi^*(g) = f$, see [Jel17, §2.2] for the details about the transformation $\phi^* : S \rightarrow S$ dual to $\phi$. This last condition can be rephrased as $I = \varphi(\text{Ann}(g))$. Conversely, for a $g$ as in (5.2) with $c, q$ general and for any $\varphi$, we get a form $\varphi^*(g)$ which is a dual generator of a Gorenstein algebra with the correct Hilbert function. It remains to count the possible $g$ and $\varphi$. By the explicit form of $\varphi^*$ as in [Jel17, §2.2, Equation (5)] it is enough to consider $\varphi \in \text{Aut}(\overline{R}/m^{s+1})$. Let $\text{StrdForms} \subset S_{\leq 1}$ be the affine space of $g$ in the form (5.2). The parameter space for $f$ is the image of the map

$$P : \text{Aut}(\overline{R}/m^{s+1}) \times \text{StrdForms} \rightarrow S_{\leq 1}$$

that sends $(\varphi, g)$ to $\varphi^*(g)$. Take a general $(\varphi, g)$ in the domain of $P$. Since $\mathbb{k}$ has characteristic zero, the tangent map $dP$ is surjective onto $T_{\varphi^*(g)} \text{Im} P$ and its image has dimension equal to $\dim \text{Im} P$. Since $P$ is equivariant with respect to $\text{Aut}(\overline{R}/m^{s+1})$, we can and do assume $\varphi = \text{id}$. By [Jel17, Proposition 2.18] the image of $dP$ is given by

$$\sum_{i=1}^{n} y_i(m \circ g) + \langle y_1^n, \ldots, y_n^n \rangle + \mathbb{k}[y_1, \ldots, y_b]_{3} + \mathbb{k}[y_1, \ldots, y_b]_{\leq 2}$$

$$= \langle y_1^s | s \in S_{1}, 3 \leq i \leq s - 1 \rangle + S_1 y_1^n + \sum_{i=1}^{n} y_i \cdot \langle x_j \circ g | j = 2, 3, \ldots, n \rangle + \mathbb{k}[y_1, \ldots, y_b]_{3} + \mathbb{k}[y_1, \ldots, y_b]_{\leq 2}.$$

This space has dimension $n(s - 3) + (n - b) + (n - b)(b - 1) + \binom{b + 2}{3} + \binom{n + 2}{2}$. Using Proposition 2.5 we deduce that

$$\dim \text{Hilb}_{H}^{\text{Gor}}(\mathbb{A}^n, 0) = n(s - 3) + (n - b)b + \binom{b + 2}{3} + \binom{n + 2}{2} - (1 + n + b) - s - 2$$

$$= (n - 1)(s - 3) + (n - b)b + \binom{b + 2}{3} - 1 + \binom{n + 2}{2} - (1 + n + b).$$

\[\square\]

Remark 5.3. Using the ideas from [Jel17, §2], it is not hard to actually describe the locus of possible $f$. For example, in the case $(1, 3, 2, 1, 1)$ we have $f = t_1^1 + a t_1^2 t_3 + F_5(t_1, t_2) + F_2(t_1, t_2, t_3)$, where
\(\alpha \in \kappa\) and \(F_i\) is homogeneous of degree \(i\) and \(\ell_i\) are linear forms. The above parameterization gives 3 + 1 + (1 + 4) + 10 = 19 parameters and hence we obtain an 11-dimensional family by Proposition 2.5, in concordance with Proposition 5.1. This more explicit argument works also in positive characteristics. We will not use it.

**Corollary 5.4.** Let \(H = (1, n, b, 1, 1, \ldots, 1)\). The dimension of \(\text{Hilb}_H^G(A^n, 0)\) is at most the maximum of

\[
\dim \text{Hilb}_H^G(A^n, 0) + \binom{n + 1}{2} - b - 1 \quad \text{and} \quad \dim \text{Hilb}_H^G(A^n, 0),
\]

where \(H - (0, 0, 1, 0, \ldots)\) means a Hilbert function that differs from \(H\) only at position 2 and is one less at that position.

**Proof.** The inverse system of an element of \(\text{Hilb}_H^G(A^n, 0)\) may have a minimal generator in degree two or not. This subdivides \(\text{Hilb}_H^G(A^n, 0)\) into loci whose dimensions are as in the statement. \(\square\)

**Corollary 5.5.** For every Hilbert function \(H = (1, a, b, 1, 1, \ldots, 1)\) with \(\sum H \leq 12\), the bound from Corollary 5.4 is less than or equal to \((\sum H - 1)(a - 1)\), hence the corresponding locus is negligible.

**Proof.** By Invariance of Codimension [BJJM19, Proposition A.4] we may take \(a = n\). Increasing the socle degree \(s\) by one increases both the bound and the expected dimension by \(n - 1\), so it does not change negligibility, so we may take \(s = 3\). By an argument analogous to [CJN15, Remark (2), p. 1532] we have \(b \leq a + 1 = n + 1\). Then the statement becomes a direct check of all possible cases using Proposition 5.1. \(\square\)

**Remark 5.6.** The bound 12 in Corollary 5.5 is sharp: the inequality from this corollary is false already for \(H = (1, 5, 6, 1)\), as we will compute more explicitly in §5.4.

### 5.2 Hilbert functions with \(H(2) \leq 2\)

In this case we use the tangent space estimate.

**Theorem 5.7.** Let \(H\) be a Hilbert function with \(k = \sum H\). Suppose \(H(2) \leq 2\). Then \(\dim \text{Hilb}_H^G(A^n, 0) < (k - 1)(n - 1)\).

**Proof.** When \(H(2) = 1\), then in fact \(H(i) \leq 1\) for all \(i \geq 2\), see [CJN15, Remark 2.7], and the bound follows directly from the estimate in Proposition 5.1. The case \(H(2) = 2\) follows from Corollary 5.10. \(\square\)

This theorem allows us to greatly reduce the number of cases that appear. It is also a partial strengthening of the Briançon-Iarrobino result that implies a similar claim for \(H(1) \leq 2\). Note that we have no assumptions on \(k\) here, and the bound is general.

To prove the theorem we apply several steps. Using invariance of codimension, we reduce to considering algebras with \(H(1) = n\). We also assume \(H(2) = 2\), then \(H(i) \leq 2\) for all \(i \geq 2\), see [CJN15, Remark 2.7]. The first step is to show that there are only a few isomorphism classes of graded algebras with the required Hilbert function. The next step is to bound the strictly positive part of the tangent space at each such algebra, to control the fibers of the map \(\pi\) sending an algebra to its associated graded. We will now analyze the tangent space to the Hilbert scheme at the points of our stratum. First, we find the Borel-fixed ones.

**Lemma 5.8.** Suppose \(I \subset R = \kappa[x_1, \ldots, x_n]\) is a Borel-fixed ideal whose quotient algebra \(A = R/I\) is zero-dimensional with \(H(2) = 2\). Let \(s\) be the socle degree of \(A\) and let \(t \leq s\) be the largest degree where \(H(t) = 2\). Then the inverse system of \(I\) is generated by the following elements: \(S_1, y_n^s, y_{n-1}y_n^{-1}\).
Proof. The ideal $I$ is Borel-fixed, hence monomial. Since $H_A(2) = 2$, the only possibility for $I_2$ is to contain all monomials except $x_n^2$ and $x_{n-1}x_n$. Then the inverse system $I^x$ satisfies $I_2^x = \langle y_n^2, y_{n-1}y_n \rangle$. This shows that $I_2^x \subseteq k[y_{n-1}, y_n]$ and we reduce to the case $n = 2$, so that $y_{n-1} = y_1$, $y_n = y_2$, and $I_2 = ky_1^2$. Since $x_1^2 \in I$, we have $I_2^x \subseteq \langle y_2^2, y_1y_2^2 \mid a, b \geq 0 \rangle$, so $I^x$ is generated by $S_1$ and $y_2^2$ and $y_1y_2^2$ for some $a, b \geq 0$. Moreover, in all degrees where $I^x$ is non-zero, it is Borel-fixed hence contains the largest monomial, which is $y_2^2$. Therefore, we conclude that $a = s$. It follows from the Hilbert function that $b = t - 1$. 

\[\text{Lemma 5.9.} \] Let $A = R/I$ be a graded algebra whose inverse system is given by $S_1$ together with $y_1^2$ and $y_1^{-1}y_2$ for some $1 \leq t \leq s$. Then the non-negative part of $T_A = \text{Hom}_R(I, R/I)$ has Hilbert series

\[T^s = (n+1) \left(\binom{n+1}{2} - 2 - (n-1) \right) (T^t+T^{s-t}) + (n-1) \sum_{i=2}^s H_A(i)T^{t-i},\]

where $T^a$ is interpreted as zero for $a < 0$.

Proof. The ideal $I$ is generated by $x_1^2$, $x_1x_i$, $x_2x_i$, $x_i$ for $i, j \geq 3$ and $x_1^2x_3, x_3^{t+1}$. For $2 \leq a \leq t$ the space $(R/I)_a$ has a basis $\{x_1^a, x_1^{a-1}x_3\}$, while for $t < a \leq s$ it has a basis $\{x_1^a\}$. Consider a homomorphism $\varphi : I \rightarrow R/I$ of degree $d = a - 2 \geq 0$. The generator $x_1^{t+1}$ is sent to zero by degree reasons. The generator $x_1^2x_3$ is sent to some power of $x_1$, so contributes 1 in degrees $1, 2, \ldots, s-t-1$.

Write $\varphi(x_1x_j) = \lambda_{ij}x_i^a + \mu_{ij}x_1^{a-1}x_3 \mod I$ for any $x_1x_j \in I$ with $i \leq j$; here if $a > s$ or, respectively, $a > t$ we use the convention that $\lambda_{ij}$ or, respectively, $\mu_{ij}$, is zero. For $a = s$ the element $\lambda_{ij}$ can be chosen arbitrarily and for $a = t$ the element $\mu_{ij}$ can be chosen arbitrarily, as these two coefficients stand next to socle elements. Below we assume that those two specific coefficients are zero.

If $i, j \geq 3$ then the syzygies $x_1(x_1x_j) = x_1(x_1x_i)$ and $x_2(x_1x_i) = x_2(x_1x_j)$ force $x_1\varphi(x_1x_i) = 0 = x_2\varphi(x_1x_j)$, so $\lambda_{ij} = \mu_{ij} = 0$. It remains to consider $i = 1$ and $i = 2$. Assume $j \geq 3$. The syzygy $x_1(x_1x_j) = x_2(x_1x_j)$ implies that

\[(\lambda_{ij} - \mu_{ij})x_i^a x_2 + \lambda_{ij}x_1^{a+1} = 0 \mod I.\]

If $a - 1 \geq t$, then $\mu_{ij} = 0$ by the convention above. Otherwise $x_i^a x_2 \notin I$ and so $\mu_{ij} = \lambda_{ij}$. If $a \geq s$, then $\lambda_{ij} = 0$ by convention, while for $a < s$ we get $\lambda_{ij} = 0$ because $x_1^{a+1} \notin I$. It follows that $\varphi(x_2x_i)$ is uniquely determined by $\varphi(x_1x_i)$.

The syzygy $(x_1^2x_3) = (x_1^2)x_3^2$ shows that the coefficient of $x_1^i$ in $\varphi(x_1^2)$ is uniquely determined by $\varphi(x_1^2)$ for $b + i \leq s$ which gives an additional constraint for homomorphisms in degrees $0, 1, \ldots, s-t-2$. Summing up, the homomorphism $\varphi$ is uniquely determined by choosing

1. two arbitrarily coefficients (next to socle elements) for every quadric generator except $x_3^2$ and $x_1x_i$ for $i \geq 3$. This contributes $\left(\binom{n+1}{2} - 2 - (n-1) \right) (T^t+T^{s-2})$,
2. an arbitrary image of $x_1^2x_3$ which, taking into account the restrictions for $\varphi(x_1^2)$, contributes $T^{s-t-1}$,
3. arbitrary images of $x_i^2$ and $x_1x_i$ for $i \geq 3$ which contributes $(n-1) \sum_{i=2}^s H_A(i)T^{t-i}$.

Summing up the three contributions, we obtain the result.

\[\text{Corollary 5.10.} \] For $H(2) = 2$ the stratum Hilb$_H(A^n, 0)$ has dimension at most

\[(k-1)(n-1) - 1.\]
Proof. By the tangent space estimate and semicontinuity, it is enough to bound the non-negative part of the tangent space at Borel-fixed points. By Lemma 5.8 there is only one such point [I] and the bound on its tangent space is given in Lemma 5.9:

\[
\dim_k T_{[I]} \text{Hilb}_H(A^n, 0) \leq 1 + 2 \left( \binom{n+1}{2} - 2 - (n - 1) \right) + (n - 1) \sum_{i=2}^{s} H_A(i) = 1 + (n + 1)n - 2(n + 1) + (n - 1)(k - 1 - n) = (n - 1)(k - 1) - 1. \]

Using Lemma 5.8, we could machine-check the tangent spaces for \( k \leq 11 \). However the argument in Corollary 5.10 is quite clean and works for any \( k \).

5.3 The remaining Hilbert functions

In this subsection, we treat the special cases which are not covered previously. Recall the map \( \pi \) in (1.6) that sends each local algebra to its associated graded algebra. The following is an easy but useful result.

**Proposition 5.11** ([CEV09, Proposition 4.3]). Let \( H = (1, n, a, b) \) and \( t = \binom{n + 1}{2} - a \). Then every fiber of \( \pi \) is irreducible of dimension \( tb \).

**Sketch of proof.** There are \( t \) quadric generators of any graded quotient \( A \) of \( k[x_1, \ldots, x_n] \) with Hilbert function \( H \) and they can be send to arbitrary elements of \( A_3 \) to obtain an element of the fiber: the conditions coming from syzygies are vacuous as \( H(4) = 0 \).

**Proposition 5.12.** Let \( H = (1, n, 3, 2) \). The locus \( \text{Hilb}^2_H(A^n, 0) \) is negligible.

**Proof.** We are to prove that the dimension of the locus is at most \( (n + 5)(n - 1) \). By Proposition 5.11, the fibers of \( \pi \) have dimension \( n(n + 1) - 6 \). For an algebra \( R/I \) as in the statement, its inverse system is generated by two polynomials of degree three \( f_1, f_2 \). Therefore, the leading terms of all degree two elements of this system are partials of the leading terms of \( f_1, f_2 \). Therefore, the inverse system of \( \pi(A) \) will have no minimal quadric generators. It is enough to prove that the locus of graded algebras with the Hilbert function \( H \) and with no minimal quadric generators in the inverse system has dimension at most \( (n + 5)(n - 1) - (n(n + 1) - 6) = 3n + 1 \). To do this we will decompose it into locally closed subloci, according to the properties of the dual generators.

Consider first the inverse systems which contain a cubic \( F \) essentially in three variables. The Hilbert function of the apolar algebra of \( F \) is \( (1, 3, 3, 1) \) and the inverse systems are parameterized by choosing a 3-dimensional subspace \( V \) of linear forms, choosing \( F \in P(\text{Sym}^3 V) \), and choosing a codimension one subspace in the space of minimal cubic generators of \( \text{Ann}(F) \). For example, by Boij-Soederberg theory, it is known that there are at most two minimal cubic generators of \( \text{Ann}(F) \). Therefore, the above parameterization gives an upper bound of \( \dim \text{Gr}(3, n) + 9 + 1 = 3n + 1 \). This concludes this case.

Consider now the inverse systems where every cubic depends essentially on at most two variables. Here, consider first the inverse systems which contain a perfect cube \( \ell^3 \). Such an inverse system contains, as minimal generators, the cube \( \ell^3 \) and a cubic \( F \) which depends essentially on two variables. This locus is parameterized by choosing a linear form \( \ell \), a space \( V \) of linear forms for \( F \) and next choosing \( F \in P(\text{Sym}^3 V) \). This gives \( \dim \text{Gr}(1, n) + \dim \text{Gr}(2, n) + 3 = 3n - 2 < 3n + 1 \).

Finally, consider the inverse systems where every cubic depends essentially on exactly two variables. Since \( H(2) = 3 < 2 \cdot 2 \), the spaces of first-order partials for cubics must intersect. Therefore, we can parameterize the whole locus by choosing a 3-dimensional subspace of linear forms, choosing two of its 2-dimensional subspaces \( V_1, V_2 \), and choosing \( F \in P(\text{Sym}^3 V) \) for \( i = 1, 2 \). This gives a parameterization by \( \dim \text{Gr}(3, n) + 2 + 2 + 3 + 3 = 3n + 1 \) parameters.
Remark 5.13. The locus of algebras with Hilbert function \( H = (1, n, 3, 2) \) without restrictions on \( \tau \) is not negligible: choosing two perfect cubes, an arbitrary quadric, and using Proposition 5.11 gives a locus of dimension \( 2(n - 1) + \binom{n+1}{2} - 2 + n(n + 1) - 6 \). Hence, keeping track of \( \tau \) is essential.

Proposition 5.14. Let \( H = (1, 4, 4, 2) \). The locus \( \text{Hilb}^2_H(\mathbb{A}^4, 0) \) is negligible.

Proof. In general terms, the proof is similar to the one of Proposition 5.12. We only consider the graded algebras and our aim is to show that their locus has dimension at most \( 10 \cdot 3 - 6 \cdot 2 = 18 \).

First, consider the inverse systems which contain a cubic \( F \) essentially depending on 4 variables, so that the corresponding ideal has Hilbert function \((1,4,4,1)\). By [CN11, Proposition 4.6] the ideal \( \text{Ann}(F) \) has at most three-dimensional space of cubic minimal generators. Moreover, if \( \text{Ann}(F) \) has any cubic minimal generator, then \( F \) is a limit of direct sums, by [BBKT15, Theorem 1.7] or by [CN11, Lemma 4.5] together with the form given in equation [BBKT15, (1)]. Such direct sums are parameterized by a locus of dimension \( \dim \text{Gr}(1, 4) + \dim \text{Gr}(3, 4) + \dim (\text{Sym}^3 \mathbb{k}^4) \times (\text{Sym}^3 \mathbb{k}^2) \) which gives in total 15 parameters (choosing a decomposition \( \mathbb{k}^2 \oplus \mathbb{k}^2 \) yields a similar count). Together with the choice of a codimension one space of cubic minimal generators, this gives at most 17 parameters. This concludes the subcase. From now on we assume that every cubic in the inverse system essentially depends on at most 3 variables.

Second, consider the inverse systems in which some cubic depends on at most two variables. They are parameterized in a naive manner by \( \dim \text{Gr}(3, 4) + \dim \text{Gr}(2, 4) + \dim (\text{Sym}^3 \mathbb{k}^1) = 19 \) parameters. Moreover, a general such pair of cubics gives rise to an inverse system with Hilbert function \((1, 4, 3, 2)\). Hence the cubics giving rise to \((1, 4, 4, 2)\) are parameterized by a proper closed (perhaps reducible) subvariety which thus has dimension at most 18. This concludes this case.

Finally, consider the inverse systems in which every cubic depends essentially on exactly three variables. Consider such a system \( F_1, F_2 \) and assume \( x_1 \circ F_1 = 0, x_2 \circ F_2 = 0 \). Since \( H(2) = 4 \), the, a priori 6-dimensional, space \( x_2 \circ F_1, x_3 \circ F_1, x_3 \circ F_1, x_1 \circ F_2, x_3 \circ F_2, x_4 \circ F_2 \) has dimension at most four. Therefore, there are two independent relations

\[ \ell_{1,1} \circ F_1 = \ell_{2,1} \circ F_2 \quad \text{and} \quad \ell_{1,2} \circ F_1 = \ell_{2,2} \circ F_2. \]

Without loss of generality, we may assume that \( \ell_{i,a} \) does not contain \( x_i \) for \( i = 1, 2 \). Note that \( \ell_{1,a} \) are linearly independent, since otherwise some combination of \( \ell_{2,a} \) annihilates \( F_2 \), which is impossible since \( F_2 \) depends essentially on three variables and \( \ell_{2,a} \) do not contain \( x_2 \). We observe that for \( i = 1, 2 \) we have

\[ \ell_{1,j} \circ (x_2 \circ F_1) = x_2 \circ (\ell_{1,j} \circ F_1) = x_2 \circ (\ell_{2,j} \circ F_2) = \ell_{2,j} \circ (x_2 \circ F_2) = 0. \]

We deduce that the quadric \( x_2 \circ F_1 \in \mathbb{k}[y_2, y_3, y_4] \) is annihilated by a two-dimensional space \( \ell_{4,a} \subset \mathbb{k}[x_2, x_3, x_4] \), and thus \( x_2 \circ F_1 \) is a perfect square. Therefore, our locus is parameterized by choosing two three-dimensional spaces of linear forms \( V_1, V_2 \) and choosing \( F_1 \in \mathbb{P}(\text{Sym}^3 V_1) \) such that \( x_{3-i} \circ F_1 \) is a perfect square, where \( x_i = V_i^4 \). The choices of \( V_i \) and \( F_i \) a priori give \( 2 \cdot (3 + 9) = 18 \) parameters, but \( x_{3-i} \circ F_1 \) is a perfect square is a codimension three condition on \( F_i \), so in fact we get \( 2 \cdot (3 + 9 - 3) = 18 \) parameters. This concludes the whole proof.

Remark 5.15. It seems surprisingly hard to properly describe the locus from Proposition 5.14. Devising a general tool to do this would be interesting.

Proposition 5.16. Let \( H = (1, 4, 3, 2, 1) \). The locus \( \dim \text{Hilb}^2_H(\mathbb{A}^4, 0) \) is negligible.

Proof. The locus of homogeneous part of \( \text{Hilb}^2_H(\mathbb{A}^4, 0) \) decomposes into four subloci as in Proposition 5.2. We consider them case by case, but not in order. For each locus, we either bound the dimension...
of each fiber of $\pi$ directly or perform a degeneration argument to reduce to bounding the dimensions of special fibers and then compute the tangent space to the fiber.

Case 3.2(3). Here the parameterization is straightforward. The associated graded system is generated by a quartic $F$ in two variables, say $F \in \mathbb{k}[y_1, y_2]$, and linear forms. The inverse system of the fiber is generated by degree four polynomial $f = F + f_2 + f_3$ and linear forms. The part $f_2$ can be arbitrarily chosen modulo the partials of $F$, which gives a 7-dimensional choice. The cubic part $f_3$ has to satisfy $x_3 \circ f_3, x_4 \circ f_3$ being partials of $F$, hence $x_3 \circ f_3, x_4 \circ f_3 \in \mathbb{k}[y_1, y_2]$ so that $f_3 \in \langle y_1, y_2, y_3, y_4 \rangle \mathbb{k}[y_1, y_2]$. Now lies in a 10-dimensional space of choices. Together, we see that the fiber is at most 17-dimensional, so the whole locus — using the conclusion of Proposition 3.2 — has dimension at most $17 + 8$.

Case 3.2(4). Here, the associated graded system is generated by $\ell_1^0$ and $c \in \mathbb{k}[\ell_1, \ell_2, \ell_3]$. Taking the limit at zero with respect to a $\mathbb{G}_m$-action with weights of $\ell_2, \ell_3$ equal and smaller than the weight of $\ell_1$, reduces our system to the case $c \in \mathbb{k}[\ell_2, \ell_3]$. Then by classification of cubics in two variables, we have $c = \ell_2^2 + \ell_3^2$ or $c = \ell_2^2 \ell_3$, up to coordinate change. We verify that in both cases the tangent to the fiber has dimension 18, hence same holds for every fiber by semicontinuity, thus — using the result of Proposition 3.2 — the whole locus has dimension at most $18 + 10$.

By Proposition 3.2 there remain three possible types of such homogeneous inverse systems: $\ell_1^0 + \ell_2^0, q$ or $\ell_1^0, \ell_2^0, q$ or $\ell_1^0, \ell_2^0, \ell_3^0$ and in each case the dimension of the homogeneous locus is at most 14. For each inverse system, since we know that $q \notin \mathbb{k}[\ell_1, \ell_2]$, we may choose $\ell_2$ such that the monomial $\ell_2^2$ appears in $q$. Taking a limit of a $\mathbb{G}_m$-action with weights of $\ell_1, \ell_2, \ell_4$ equal and greater than the weight of $\ell_5$, reduces our system to the case $Q = \ell_2^2$. For such a $q$, a direct tangent space check shows that for $I$ annihilating such an inverse system we have dim $\text{Hom}(I, R/I)_{>0}$ equal to 14, 14, 18 respectively. Since $14 + 14 < 30$, we concentrate on the last case. Summing up, from now on we only consider inverse systems whose associated graded has the form $\ell_1^0, \ell_2^0, q$. By Proposition 3.2 the homogeneous such systems form a 13-dimensional family. Again, since every such system degenerates to $\ell_1^0, \ell_2^0$, it is enough to prove the dimension estimate for that system.

We subdivide this case into two. First, we fix a basis $y_1$ with $y_1 = \ell_1$ and $y_2 = \ell_2$. We call a quadric $q$ special if $q(0, 0, y_3, y_4)$ has rank one. Special quadrics form a divisor, hence the graded inverse systems with $q$ special are at most $13 - 1 = 12$ dimensional and the corresponding nongraded systems are at most $12 + 18 = 30$ dimensional by the tangent estimate above. Thus below we may and do consider only non-special quadrics. For such a quadric $q$ take a $\mathbb{G}_m$-action with weights of $y_1, y_2$ equal, weights of $y_3$ and $y_4$ also equal and smaller that the weights of $y_1$. The obtained $\mathbb{G}_m$-limit of the inverse system $y_1^0, y_2^0, q$ is $y_1^0, y_2^0, y_3^0, y_4$. The positive part of the tangent space at $y_1^0, y_2^0, y_3, y_4$ is 17. Hence using the base and tangent-to-fiber estimate we obtain $17 + 13 = 30$.

**Remark 5.17.** In the above proof, one can check that the dimension of the stratum of $y_1^0, y_2^0, y_3^0, y_4$ is exactly 30, so the punctual Hilbert scheme is reducible.

**Proof of Theorem 1.2.** Recall that the expected dimension of $\text{Hilb}_k(A^n, 0)$ is $(k - 1)(n - 1)$ and a locus $Z$ in this Hilbert scheme is negligible if dim $Z \leq (k - 1)(n - 1)$. Suppose that $H$ is such that the stratum of $H$ is not negligible. By Invariance of Codimension we reduce to $H(1) = n$ [BIM19, Proposition A.4]. By irreducibility of $\text{Hilb}_k(A^2, 0)$, we have $H(1) \geq 3$. By Theorem 5.7 we have $H(2) \geq 3$.

Suppose now $k \leq 8$. By Theorem 1.3, we have $H(1) \geq 4$. As $k \leq 8$ this implies that $H = (1, 4, 3)$ and we conclude it is negligible by direct computation of the dimension of this locus, which is $\text{dim Gr}(3, 10) = 21$. So, for $k \leq 8$ and any $n$ the dimension of the Hilbert scheme is the expected one. The stratum of $(1, n, 3)$ for $n \geq 5$ has dimension higher than excepted, see Example 5.19 below. This concludes the part $\tau \geq 3$.

We assume $\tau \leq 2$ and $k$ arbitrary. We first discard several classes of Hilbert functions. If $H(3) = 0$ then $H(2) \leq 2$ since $\tau \leq 2$, a contradiction with the above. If $H(3) = 1$ then Corollary 5.5 shows that $H$ is negligible. If $H = (1, n, 3, 2)$ for some $n$, then Proposition 5.12 concludes.
Suppose now $k \leq 11$. Using Theorem 1.3 we get $H(1) \geq 4$. In conjunction with the above, we have $H(1) \geq 4$, $H(2) \geq 3$ and $H(3) \geq 2$. This immediately concludes the case $k = 9$. For $k = 10$ this leaves, a priori, the single possible case $H = (1, 4, 3, 2)$ but also that case was considered above.

Suppose $k = 11$. Due to the constraints above, we have to consider only the cases $H = (1, 5, 3, 2)$, $H = (1, 4, 4, 2)$, $H = (1, 4, 3, 3)$, $H = (1, 4, 3, 2, 1)$. The case $(1, 5, 3, 2)$ was considered above. The case $(1, 4, 4, 2)$ was considered in Proposition 5.14. The case $(1, 4, 3, 3)$ is impossible to obtain with $r \leq 2$. The case $(1, 4, 3, 2, 1)$ is considered in Proposition 5.16. This concludes the case $k = 11$ and the whole proof of the bound. The violations of the bound for higher $k$ follow from Examples 5.20-5.21. 

\begin{remark}

It is natural to speculate about the cases $\tau = 1, k = 12, 13$ and $\tau = 2, k = 12$. We believe that in these cases the scheme $\text{Hilb}_k^s(A^n, 0)$ has the expected dimension, however the number of cases to consider would increase the length of the paper too greatly and ruin the relative cleanliness of the current proof.

\end{remark}

\begin{proof}[Proof of Theorem 1.1]

The punctual Hilbert schemes $\text{Hilb}_k(P^n, p)$ and $\text{Hilb}_k(A^n, 0)$ have the same dimension [Fog68, Proposition 2.2]. Since $\dim \text{Hilb}_k^s(A^n) = kn$, and the map $A^n \times \text{Hilb}_k^s(A^n, 0) \to \text{Hilb}_k^s(A^n)$ is injective and not dominant, we have $\dim \text{Hilb}_k^s(A^n, 0) \leq (k-1)n-1$. Moreover, if $k \leq 8$, or if $r \leq 2$ and $k \leq 11$, then it follows from Theorem 1.2 that $\dim \text{Hilb}_k^s(A^n, 0) = (k-1)(n-1)$. Therefore the estimates for the theorem follow from Proposition 4.9 in the general case and from Corollary 4.10 in the special cases $k \leq 8$ or $r \leq 2, k \leq 11$.

\end{proof}

\section{Counterexamples}

\begin{example}[\(r \geq 3\)]

The locus of algebras with Hilbert function $H = (1, n, 3)$ in $A^n$ is parametrized by $\text{Gr}(3, (\binom{n+1}{2}))$ whence it has dimension $3 \cdot (\binom{n+1}{2} - 3)$, whereas the expected dimension is $(n-1) \cdot (n+3)$. The difference is

$$\binom{n-1}{2} - 6$$

which is positive for $n \geq 5$; so for $k = n+4 \geq 9$. To get examples for $n > k - 4$ use Invariance of Codimension [BJJM19, Proposition A.4].

\end{example}

\begin{example}[\(r = 1\)]

Consider the Hilbert functions $(1, n, n, 1)$ and $(1, n, n, 1, 1)$. By Proposition 5.1 the dimensions of the corresponding loci minus the expected dimension $(n-1)(k-1)$ is in both cases

$$\binom{n+2}{3} - 1 + \binom{n+2}{2} - n \cdot (2n-1) = \frac{1}{6} n^3 - n^2 + 5 \frac{1}{6} n = \frac{1}{6} n(n-1)(n-5)$$

which is strictly positive for every $n \geq 6$ hence for $k \geq 14$.

\end{example}

\begin{example}[\(r = 2\)]

Take an inverse system for two general polynomials in 5 variables: one of degree 3 and one of degree 2. Then the Hilbert function is $(1, 5, 6, 1)$ and the optimal bound is $4 \cdot 12 = 48$. But the space of homogeneous such choices is $\binom{5}{3} = 1 + (10 - 1) = 43$ dimensional and so the whole space is 52-dimensional, violating the bound. (This example could be generalized to $(1, n, n + 1, 1, 1)$.)

\end{example}

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