A NEW INVARIANT FOR FINITE DIMENSIONAL LEIBNIZ/LIE ALGEBRAS

A. L. AGORE AND G. MILITARU

Abstract. For an $n$-dimensional Leibniz/Lie algebra $\mathfrak{h}$ over a field $k$ we introduce a new invariant $A(\mathfrak{h})$, called the universal algebra of $\mathfrak{h}$, as a quotient of the polynomial algebra $k[X_{ij} | i, j = 1, \cdots, n]$ through an ideal generated by $n^3$ polynomials. Furthermore, $A(\mathfrak{h})$ admits a unique bialgebra structure which makes it an initial object among all bialgebras coacting on $\mathfrak{h}$ through a Leibniz/Lie algebra homomorphism. The bialgebra $A(\mathfrak{h})$ is the key object in approaching three classical and open problems in Lie algebra theory. First, we prove that the automorphisms group $\text{Aut}_{\text{Leib}}(\mathfrak{h})$ of $\mathfrak{h}$ is isomorphic to the group $U(G(\mathfrak{h}^*))$ of all invertible group-like elements of the finite dual $\mathfrak{h}^*$. Secondly, for an abelian group $G$, we show that there exists a bijection between the set of all $G$-gradings on $\mathfrak{h}$ and the set of all bialgebra homomorphisms $A(\mathfrak{h}) \rightarrow k[G]$. Finally, for a finite group $G$, we prove that the set of all actions as automorphisms of $G$ on $\mathfrak{h}$ is parameterized by the set of all bialgebra homomorphisms $A(\mathfrak{h}) \rightarrow k[G]^*$. $A(\mathfrak{h})$ is also used to prove that there exists a universal commutative Hopf algebra associated to any finite dimensional Leibniz algebra $\mathfrak{h}$.

Introduction

Let $A$ be a unital associative algebra over a field $k$. M.E. Sweedler’s result [23, Theorem 7.0.4] which states that the functor $\text{Hom}(\cdot, A) : \text{CoAlg}_k \rightarrow \text{Alg}_k^\text{op}$ from the category of coalgebras over $k$ to the opposite of the category of $k$-algebras has a right adjoint denoted by $M(\cdot, A)$ proved itself remarkable through its applications. Furthermore, $M(A, A)$ turns out to be a bialgebra and the final object in the category of all bialgebras that act on $A$ through a module algebra structure. The dual version was considered by Tambara [20] and in a special (graded) case by Manin [17]. To be more precise, [20, Theorem 1.1] proves that if $A$ is a finite dimensional algebra, then the tensor functor $A \otimes - : \text{Alg}_k \rightarrow \text{Alg}_k$ has a left adjoint denoted by $a(A, -)$. In the same spirit, $a(A, A)$ is proved to be a bialgebra as well and the initial object in the category of all bialgebras that coact on $A$ through a comodule algebra structure. Both objects are very important: as explain in [17], the Hopf envelope of $a(A, A)$ plays the role of a symmetry group in non-commutative geometry. For further details we refer to [1, 2]. A more general construction, which contains all the above as special cases, was recently considered in [3] in the context of $\Omega$-algebras.

2010 Mathematics Subject Classification. 17A32, 17A36, 17A60, 17B05, 17B40, 17B70.

Key words and phrases. Leibniz/Lie algebras, universal constructions, automorphisms group.
The starting point of this paper was an attempt to prove the counterpart of Tambara’s result at the level of Leibniz algebras. Introduced by Bloh [7] and rediscovered by Loday [14], Leibniz algebras are non-commutative generalizations of Lie algebras. This new concept generated a lot of interest mainly due to its interaction with (co)homology theory, vertex operator algebras, the Godbillon-Vey invariants for foliations or differential geometry. Another important concept for our approach is that of a current Lie algebra. Being first introduced in physics [12], current Lie algebras, are Lie algebras of the form \( g \otimes A \), where \( g \) is a Lie algebra, \( A \) is a commutative algebra and the bracket is given by \([x \otimes a, y \otimes b] := [x, y] \otimes ab\), for all \( x, y \in g \) and \( a, b \in A \). They are interesting objects that arise in various branches of mathematics and physics such as the theory of affine Kac-Moody algebras or the structure of modular semisimple Lie algebras (see [25, 26]). Current Leibniz algebras are immediate generalizations: i.e. they are Leibniz algebras of the form \( h \otimes A \) whose bracket is defined as in the case of Lie algebras, where this time \( h \) is a Leibniz algebra and \( A \) a commutative algebra. By fixing a Leibniz algebra \( h \), we obtain a functor \( h \otimes - : \text{ComAlg}_k \to \text{Lbz}_k \) from the category of commutative algebras to the category of Leibniz algebras called the current Leibniz algebra functor. Theorem 2.1 proves that the functor \( h \otimes - : \text{ComAlg}_k \to \text{Lbz}_k \) has a left adjoint, denoted by \( A(h, -) \), if and only if \( h \) is finite dimensional. For an \( n \)-dimensional Leibniz algebra \( h \) and an arbitrary Leibniz algebra \( g \) with \(|I| = \dim_k(g)\), \( A(h, g) \) is a quotient of the usual polynomial algebra \( k[X_{si} | s = 1, \ldots, n, i \in I] \). The commutative algebra \( A(h, g) \) provides an important tool for studying Leibniz/Lie algebras as it captures all essential information on the two Leibniz/Lie algebras. Note for instance that the characters of this algebra parameterize the set of all Leibniz algebra homomorphisms between \( g \) and \( h \) (Corollary 2.3). Theorem 2.1 has obviously a Lie algebra counterpart. In this case, if \( g \) is a Lie algebra and \( m \) a positive integer then the characters of the commutative algebra \( A(g(m, k), g) \) parameterize the space of all \( m \)-dimensional representations of \( g \) (Corollary 2.4). The commutative algebra \( A(h) := A(h, h) \) is called the universal algebra of \( h \); it is a quotient of the polynomial algebra \( M(n) := k[X_{ij} | i, j = 1, \ldots, n] \) through an ideal generated by \( n^3 \) polynomials called the universal polynomials of \( h \). Proposition 2.10 proves that \( A(h) \) has a canonical bialgebra structure such that the projection \( \pi : M(n) \to A(h) \) is a bialgebra map. The first main application of the universal (bi)algebra \( A(h) \) of \( h \) is given in Theorem 2.14 which provides an explicit description of a group isomorphism between the group of automorphisms of \( h \) and the group of all invertible group-like elements of the finite dual \( A(h)^o \):

\[
\text{Aut}_{\text{Lbz}}(h) \cong U(G(A(h)^o)) .
\]

We mention that achieving a complete description of the automorphisms group \( \text{Aut}_{\text{Lbz}}(h) \) of a given Lie algebra \( h \) is a classical [8, 13] and notoriously difficult problem intimately related to the structure of Lie algebras (for more details see the recent papers [4, 5, 11] and their references). The unit of the adjunction depicted in Theorem 2.1, denoted by \( \eta_h : h \to h \otimes A(h) \), endows \( h \) with a right \( A(h) \)-comodule structure and the pair \((A(h), \eta_h)\) is the initial object in the category of all commutative bialgebras that coact on the Leibniz algebra \( h \) (Theorem 2.11). This result allows for two important consequences: Corollary 2.12 proves that for an abelian group \( G \) there exists an explicitly described bijection between the set of all \( G \)-gradings on \( h \) and the set of all bialgebra
homomorphisms \( \mathcal{A}(h) \to k[G] \). Secondly, if \( G \) is a finite group, Corollary 2.13 shows that there exists a bijection between the set of all actions as automorphisms of \( G \) on \( h \) (i.e. morphisms of groups \( G \to Aut_{Lbz}(h) \)) and the set of all bialgebra homomorphisms \( \mathcal{A}(h) \to k[G] \). Related to the last two results we mention that there exists a vast literature concerning the classification of all \( G \)-gradings on a given Lie algebra (see [6, 10, 18] and their references). On the other hand, the study of actions as automorphisms of a group \( G \) on a Lie algebra \( h \) goes back to Hilbert’s invariant theory whose foundation was set at the level of Lie algebras in the classical papers [8, 9, 24]; for further details see [4] and the references therein. Using once again Theorem 2.11 and the existence of a free commutative Hopf algebra on any commutative bialgebra [19, Theorem 65, (2)], we prove in Theorem 2.16 that there exists a universal coacting Hopf algebra on any finite dimensional Leibniz algebra. We point out that, to the best of our knowledge, this is the only universal Hopf algebra associated to a Leibniz algebra appearing in the literature.

1. Preliminaries

All vector spaces, (bi)linear maps, Leibniz, Lie or associative algebras, bialgebras and so on are over an arbitrary field \( k \) and \( \otimes = \otimes_k \). A Leibniz algebra is a vector space \( h \), together with a bilinear map \([-,-]: h \times h \to h\) satisfying the Leibniz identity for any \( x, y, z \in h \):

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y]
\]

Any Lie algebra is a Leibniz algebra, and a Leibniz algebra \( h \) satisfying \( [x, x] = 0 \), for all \( x \in h \) is a Lie algebra. We shall denote by \( Aut_{Lbz}(h) \) (resp. \( Aut_{Lie}(h) \)) the automorphisms group of a Leibniz (resp. Lie) algebra \( h \). Any vector space \( V \) is a Leibniz algebra with trivial bracket \([x, y] := 0\), for all \( x, y \in V \) – such a Leibniz algebra is called abelian and will be denoted by \( V_0 \). For two subspaces \( A \) and \( B \) of a Leibniz algebra \( h \) we denote by \([A, B]\) the vector space generated by all brackets \([a, b]\), for any \( a \in A \) and \( b \in B \). In particular, \( h' := [h, h] \) is called the derived subalgebra of \( h \).

We shall denote by \( Lbz_k \), \( Lie_k \) and \( ComAlg_k \) the categories of Leibniz, Lie and respectively commutative associative algebras. Furthermore, the category of commutative bialgebras (resp. Hopf algebras) is denoted by \( ComBiAlg_k \) (resp. \( ComHopf_k \)). If \( h \) is a Leibniz algebra and \( A \) a commutative algebra then \( h \otimes A \) is a Leibniz algebra with bracket defined for any \( x, y \in h \) and \( a, b \in A \) by:

\[
[x \otimes a, y \otimes b] := [x, y] \otimes ab\]

called the current Leibniz algebra. Indeed, as \( A \) is a commutative and associative algebra, we have:

\[
[[x \otimes a, y \otimes b], z \otimes c] - [[x \otimes a, z \otimes c], y \otimes b]
\]

\[
= [[x, y], z] \otimes abc - [[x, z], y] \otimes acb
\]

\[
= ([[[x, y], z] - [[x, z], y]) \otimes abc
\]

\[
= [x, [y, z]] \otimes abc = [x \otimes a, [y \otimes b, z \otimes c]]
\]

for all \( x, y, z \in h \) and \( a, b, c \in A \), i.e. the Leibniz identity (1) holds for \( h \otimes A \). For a fixed Leibniz algebra \( h \), assigning \( A \mapsto h \otimes A \) defines a functor \( h \otimes - : ComAlg_k \to Lbz_k \)
from the category of commutative $k$-algebras to the category of Leibniz algebras called the current Leibniz algebra functor. If $f : A \rightarrow B$ is an algebra map then $\text{Id}_h \otimes f : h \otimes A \rightarrow h \otimes B$ is a morphism of Leibniz algebras.

For basic concepts on category theory we refer the reader to [16] and for unexplained notions pertaining to Hopf algebras to [21, 23].

2. Universal constructions and applications

Our first result is the Leibniz algebra counterpart of [20, Theorem 1.1].

**Theorem 2.1.** Let $h$ be a Leibniz algebra. Then the current Leibniz algebra functor $h \otimes - : \text{ComAlg}_k \rightarrow \text{Lbz}_k$ has a left adjoint if and only if $h$ is finite dimensional. Moreover, if $h \neq 0$ the functor $h \otimes -$ does not admit a right adjoint.

**Proof.** Assume first that $h$ is a finite dimensional Leibniz algebra and $\dim_k(h) = n$. Fix $\{e_1, \cdots, e_n\}$ a basis in $h$ and let $\{\tau_{i,j}^s | i, j, s = 1, \cdots, n\}$ be the structure constants of $h$, i.e. for any $i, j = 1, \cdots, n$ we have:

$$[e_i, e_j]_h = \sum_{s=1}^n \tau_{i,j}^s e_s. \quad (3)$$

In what follows we shall explicitly construct a left adjoint of the current Leibniz algebra functor $h \otimes -$, denoted by $A(h, -) : \text{Lbz}_k \rightarrow \text{ComAlg}_k$. Let $g$ be a Leibniz algebra and let $\{f_i | i \in I\}$ be a basis of $g$. For any $i, j \in I$, let $B_{i,j} \subseteq I$ be a finite subset of $I$ such that for any $i, j \in I$ we have:

$$[f_i, f_j]_g = \sum_{u \in B_{i,j}} \beta_{i,j}^u f_u. \quad (4)$$

Let $k[X_{si} | s = 1, \cdots, n, i \in I]$ be the usual polynomial algebra and define

$$A(h, g) := k[X_{si} | s = 1, \cdots, n, i \in I]/J \quad (5)$$

where $J$ is the ideal generated by all polynomials of the form

$$P_{(a,i,j)}^{(h,g)} := \sum_{u \in B_{i,j}} \beta_{i,j}^u X_{au} - \sum_{s,t=1}^n \tau_{s,t}^a X_{si}X_{tj}, \quad \text{for all } a = 1, \cdots, n \text{ and } i, j \in I. \quad (6)$$

We denote by $x_{si} := \overline{X_{si}}$ the class of $X_{si}$ in the algebra $A(h, g)$; thus the following relations hold in the commutative algebra $A(h, g)$:

$$\sum_{u \in B_{i,j}} \beta_{i,j}^u x_{au} = \sum_{s,t=1}^n \tau_{s,t}^a x_{si}x_{tj}, \quad \text{for all } a = 1, \cdots, n, \text{ and } i, j \in I. \quad (7)$$

Now we consider the map:

$$\eta_g : g \rightarrow h \otimes A(h, g), \quad \eta_g(f_i) := \sum_{s=1}^n e_s \otimes x_{si}, \quad \text{for all } i \in I. \quad (8)$$
We shall prove first that \( \eta_0 \) is a Leibniz algebra homomorphism. Indeed, for any \( i, j \in I \) we have:

\[
[\eta_0(f_i), \eta_0(f_j)]_{\mathfrak{h} \otimes A(\mathfrak{h}, g)} = \left[ \sum_{s=1}^{n} e_s \otimes x_{si}, \sum_{t=1}^{n} e_t \otimes x_{tj} \right]_{\mathfrak{h} \otimes A(\mathfrak{h}, g)} = \sum_{s,t=1}^{n} [e_s, e_t]_{\mathfrak{h}} \otimes x_{si}x_{tj}
\]

\[
= \sum_{a=1}^{n} e_a \otimes \left( \sum_{s,t=1}^{n} \tau_{s,t}^{a} x_{si}x_{tj} \right) = \sum_{a=1}^{n} e_a \otimes \left( \sum_{u \in B_{i,j}} \beta_{i,j}^{u} x_{au} \right) = \sum_{u \in B_{i,j}} \beta_{i,j}^{u} \eta_0(f_u) = \eta_0([f_i, f_j]_{\mathfrak{g}})
\]

Now we prove that for any Leibniz algebra \( \mathfrak{g} \) and any commutative algebra \( A \) the map defined below is bijective:

\[
\gamma_{\mathfrak{g}, A} : \text{Hom}_{\text{Alg}}(\mathfrak{A}(\mathfrak{h}, \mathfrak{g}), A) \to \text{Hom}_{Lbz}(\mathfrak{h} \otimes A, \mathfrak{h} \otimes A), \quad \gamma_{\mathfrak{g}, A}(\theta) := (\text{Id}_{\mathfrak{h}} \otimes \theta) \circ \eta_0 \quad (9)
\]

To this end, let \( f : \mathfrak{g} \to \mathfrak{h} \otimes A \) be a Leibniz algebra homomorphism. We have to prove that there exists a unique algebra homomorphism \( \theta : \mathfrak{A}(\mathfrak{h}, \mathfrak{g}) \to A \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\eta_0} & \mathfrak{h} \otimes \mathfrak{A}(\mathfrak{h}, \mathfrak{g}) \\
f \downarrow & & \downarrow \text{Id}_{\mathfrak{h}} \otimes \theta \\
\mathfrak{h} \otimes A & \xrightarrow{\eta_0} & A \\
\end{array}
\]

i.e. \( f = (\text{Id}_{\mathfrak{h}} \otimes \theta) \circ \eta_0 \).

Let \( \{d_{si} \mid s = 1, \ldots, n, i \in I\} \) be a family of elements of \( A \) such that for any \( i \in I \) we have:

\[
f(f_i) = \sum_{s=1}^{n} e_s \otimes d_{si} \quad (11)
\]

A straightforward computation shows that for all \( i, j \in I \) we have:

\[
f([f_i, f_j]_{\mathfrak{g}}) = \sum_{a=1}^{n} e_a \otimes \left( \sum_{u \in B_{i,j}} \beta_{i,j}^{u} d_{au} \right) \text{ and } [f(f_i), f(f_j)]_{\mathfrak{h} \otimes A} = \sum_{a=1}^{n} e_a \otimes \left( \sum_{s,t=1}^{n} \tau_{s,t}^{a} d_{si}d_{tj} \right)
\]

Since \( f : \mathfrak{g} \to \mathfrak{h} \otimes A \) is a Leibniz algebra homomorphism, it follows that the family of elements \( \{d_{si} \mid s = 1, \ldots, n, i \in I\} \) need to fulfill the following relations in \( A \):

\[
\sum_{u \in B_{i,j}} \beta_{i,j}^{u} d_{au} = \sum_{s,t=1}^{n} \tau_{s,t}^{a} d_{si}d_{tj}, \quad \text{for all } i, j \in I \text{ and } a = 1, \ldots, n. \quad (12)
\]

The universal property of the polynomial algebra yields a unique algebra homomorphism \( v : k[X_{si} \mid s = 1, \ldots, n, i \in I] \to A \) such that \( v(X_{si}) = d_{si} \), for all \( s = 1, \ldots, n \) and \( i \in I \). It can be easily seen that \( \text{Ker}(v) \supseteq J \), where \( J \) is the ideal generated by all polynomials listed in (6). Indeed, for any \( i, j \in I \) and \( a = 1, \ldots, n \) we have:

\[
v(P_{(a,i,j)}^{(b,j)}) = v \left( \sum_{u \in B_{i,j}} \beta_{i,j}^{u} X_{au} - \sum_{s,t=1}^{n} \tau_{s,t}^{a} X_{si}X_{tj} \right) = \sum_{u \in B_{i,j}} \beta_{i,j}^{u} d_{au} - \sum_{s,t=1}^{n} \tau_{s,t}^{a} d_{si}d_{tj} = 0. \quad (12)
\]
Thus, there exists a unique algebra homomorphism $\theta : \mathcal{A}(\mathfrak{h}, \mathfrak{g}) \to A$ such that $\theta(x_{si}) = d_{si}$, for all $s = 1, \ldots, n$ and $i \in I$. Furthermore, for any $i \in I$ we have:

$$\left(\text{Id}_\mathfrak{h} \otimes \theta\right) \circ \eta_\mathfrak{g}(f_i) = \left(\text{Id}_\mathfrak{h} \otimes \theta\right) \left(\sum_{s=1}^{n} e_s \otimes x_{si}\right) = \sum_{s=1}^{n} e_s \otimes d_{si} \quad (11)$$

Therefore, we have $(\text{Id}_\mathfrak{h} \otimes \theta) \circ \eta_\mathfrak{g} = f$ as desired. Next we show that $\theta$ is the unique morphism with this property. Let $\tilde{\theta} : \mathcal{A}(\mathfrak{h}, \mathfrak{g}) \to A$ be another algebra homomorphism such that $(\text{Id}_\mathfrak{h} \otimes \tilde{\theta}) \circ \eta_\mathfrak{g}(f_i) = f(f_i)$, for all $i \in I$. Then, $\sum_{s=1}^{n} e_s \otimes \tilde{\theta}(x_{si}) = \sum_{s=1}^{n} e_s \otimes d_{si}$, and hence $\tilde{\theta}(x_{si}) = d_{si} = \theta(x_{si})$, for all $s = 1, \ldots, n$ and $i \in I$. Since $\{x_{si} | s = 1, \ldots, n, i \in I\}$ is a system of generators for the algebra $\mathcal{A}(\mathfrak{h}, \mathfrak{g})$ we obtain $\tilde{\theta} = \theta$. All in all, we have proved that the map $\gamma_{\mathfrak{g}, A}$ given by (9) is bijective.

Next we show that assigning to each Leibniz algebra $\mathfrak{g}$ the commutative algebra $\mathcal{A}(\mathfrak{h}, \mathfrak{g})$ defines a functor $\mathcal{A}(\mathfrak{h}, -) : \text{Lbz}_k \to \text{ComAlg}_k$. First, let $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Leibniz algebra homomorphism. Using the bijectivity of the map defined by (9) for the Leibniz algebra homomorphism $f := \eta_{\mathfrak{g}_2} \circ \alpha$, yields a unique algebra homomorphism $\tilde{\theta} : \mathcal{A}(\mathfrak{h}, \mathfrak{g}_1) \to \mathcal{A}(\mathfrak{h}, \mathfrak{g}_2)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{g}_1 & \xrightarrow{\eta_{\mathfrak{g}_1}} & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}, \mathfrak{g}_1) \\
\alpha & & \downarrow \text{Id}_\mathfrak{h} \otimes \theta \\
\mathfrak{g}_2 & \xrightarrow{\eta_{\mathfrak{g}_2}} & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}, \mathfrak{g}_2)
\end{array}
\]

\[\text{i.e.} \quad (\text{Id}_\mathfrak{h} \otimes \theta) \circ \eta_{\mathfrak{g}_1} = \eta_{\mathfrak{g}_2} \circ \alpha \quad (13)\]

We denote this unique morphism $\theta$ by $\mathcal{A}(\mathfrak{h}, \alpha)$ and the functor $\mathcal{A}(\mathfrak{h}, -)$ is now fully defined. Furthermore, the commutativity of the diagram (13) shows the naturality of $\gamma_{\mathfrak{g}, A}$ in $\mathfrak{g}$. It can now be easily checked that $\mathcal{A}(\mathfrak{h}, -)$ is indeed a functor and that $\gamma_{\mathfrak{g}, A}$ is also natural in $A$. To conclude, the functor $\mathcal{A}(\mathfrak{h}, -)$ is a left adjoint of the current Leibniz algebra functor $\mathfrak{h} \otimes -$.

Conversely, assume that the functor $\mathfrak{h} \otimes - : \text{ComAlg}_k \to \text{Lbz}_k$ has a left adjoint. In particular, $\mathfrak{h} \otimes -$ preserves arbitrary products. Now recall that in both categories $\text{ComAlg}_k$ and $\text{Lbz}_k$ products are constructed as simply the products of the underlying vector spaces. Imposing the condition that $\mathfrak{h} \otimes -$ preserves the product of a countable number of copies of the base field $k$ will easily lead to the finite dimensionality of $\mathfrak{h}$.

Assume, now that the functor $\mathfrak{h} \otimes - : \text{ComAlg}_k \to \text{Lbz}_k$ has a right adjoint. This implies that $\mathfrak{h} \otimes -$ preserves coproducts. Now, since in the category $\text{ComAlg}_k$ of commutative algebras the coproduct of two commutative algebras is given by their tensor product, it follows that for any commutative algebras $A$ and $B$ there exists an isomorphism of Leibniz algebras $\mathfrak{h} \otimes (A \otimes B) \cong (\mathfrak{h} \otimes A) \sqcup (\mathfrak{h} \otimes B)$, where we denote by $\sqcup$ the coproduct of two current Leibniz algebras. In particular, for $A = B := k$, we obtain that $\mathfrak{h} \sqcup \mathfrak{h} \cong \mathfrak{h}$, that is $\mathfrak{h} = 0$. The proof is now complete. \hfill \Box

**Remark 2.2.** Theorem 2.1 remains valid in the special case of Lie algebras: if $\mathfrak{h}$ is a finite dimensional Lie algebra, the current Lie algebra functor $\mathfrak{h} \otimes - : \text{ComAlg}_k \to \text{Lie}_k$ has a left adjoint $\mathcal{A}(\mathfrak{h}, -)$ which is constructed as in the proof of Theorem 2.1. We point out, however, that the polynomials defined in (6) take a rather simplified form. The skew
symmetry fulfilled by the bracket of a Lie algebra imposes the following restrictions on
the structure constants:
\[ \tau^s_{i,j} = 0 \quad \text{and} \quad \tau^s_{i,j} = -\tau^s_{j,i} \quad \text{for all} \quad i, j, s = 1, \ldots, n. \]

The commutative algebra \( \mathcal{A}(h, g) \) constructed in the proof of Theorem 2.1 provides an
important tool for studying Lie/Leibniz algebras as it captures most of the essential in-
formation on the two Lie/Leibniz algebras. Indeed, note for instance that the characters
of this algebra (i.e. the algebra homomorphisms \( \mathcal{A}(h, g) \to k \)) parameterize the set of
all Leibniz algebra homomorphisms between the two algebras. This follows as an easy
consequence of the bijection described in (9) by taking \( A := k \):

**Corollary 2.3.** Let \( g \) and \( h \) be two Leibniz algebras such that \( h \) is finite dimensional.
Then the following map is bijective:

\[ \gamma : \text{Hom}_{\text{Alg}}(\mathcal{A}(h, g), k) \to \text{Hom}_{\text{Lbz}}(g, h), \quad \gamma(\theta) := (\text{Id}_h \otimes \theta) \circ \eta_g. \] (14)

In particular, by applying Corollary 2.3 for \( h := gl(m, k) \) and an arbitrary Lie algebra \( g \)
we obtain:

**Corollary 2.4.** Let \( g \) be a Lie algebra and \( m \) a positive integer. Then there exists a
bijective correspondence between the space of all \( m \)-dimensional representations of \( g \) and
the space of all algebra homomorphisms \( \mathcal{A}(g(m, k), g) \to k \).

**Examples 2.5.**
1. If \( h \) and \( g \) are abelian Leibniz algebras then \( \mathcal{A}(h, g) \cong k[X_{si} | s = 1, \ldots, n, i \in I], \) where \( n = \dim_k(h) \) and \( |I| = \dim_k(g) \).
2. Let \( h \) be an \( n \)-dimensional Leibniz algebra with structure constants \( \{\tau^s_{i,j} | i, j, s = 1, \ldots, n\} \). Then \( \mathcal{A}(h, k) \cong k[X_1, \ldots, X_n]/J, \) where \( J \) is the ideal generated by the polynomials \( \sum_{a,t=1}^n \tau^a_{s,t} X_s X_t, \) for all \( a = 1, \ldots, n. \)
3. Let \( g \) be a Leibniz algebra. Then \( \mathcal{A}(k, g) \cong S(g/g') \), the symmetric algebra of \( g/g' \),
where \( g' \) is the derived subalgebra of \( g \). In particular, if \( g \) is perfect (that is \( g' = 0 \)), then
\( \mathcal{A}(k, g) \cong k. \)

Indeed, the functor \( \mathcal{A}(k, -) \) is a left adjoint for the tensor functor \( k \otimes - : \text{ComAlg}_k \to \text{Lbz}_k; \) since the tensor product is also taken over \( k \) this functor is isomorphic to the
functor \( (-) : \text{ComAlg}_k \to \text{Lbz}_k, \) which sends any commutative algebra \( A \) to the abelian
Leibniz algebra \( A_0 := A. \) We shall prove that the functor \( g \to S(g/g') \) is a left adjoint
of \( (-) \). The uniqueness of adjoint functors [16] will then lead to the desired algebra
isomorphism \( \mathcal{A}(k, g) \cong S(g/g') \).

Let \( g \) be a Leibniz algebra and define \( \overline{\tau_0} : g \to S(g/g') \) as the composition \( \overline{\tau_0} := i \circ \pi, \)
where \( \pi : g \to g/g' \) is the usual projection and \( i : g/g' \to S(g/g') \) is the canonical
inclusion of the vector space \( g/g' \) in its symmetric algebra. We shall prove now that the
following map is bijective for any commutative algebra \( A \) and any Leibniz algebra \( g \):

\[ \overline{\tau_0,A} : \text{Hom}_{\text{Alg}}(S(g/g'), A) \to \text{Hom}_{\text{Lbz}}(g, A_0), \quad \overline{\tau_0,A}(\theta) := \theta \circ \overline{\tau_0} \] (15)

This shows that the functor \( g \to S(g/g') \) is a left adjoint of \( (-) \). Indeed, let \( f : g \to A_0 \)
be a Leibniz algebra homomorphism, i.e. \( f \) is a \( k \)-linear map such that \( f([x, y]) = 0, \) for
all \( x, y \in g. \) That is \( \text{Ker}(f) \) contains \( g', \) the derived algebra of \( g. \) Thus, there exists a
unique \( k \)-linear map \( \overline{f} : \mathfrak{g}/\mathfrak{g}' \to A \) such that \( \overline{f} \circ \pi = f \). Now, using the universal property of the symmetric algebra we obtain that there exists a unique algebra homomorphism \( \theta : S(\mathfrak{g}/\mathfrak{g}') \to A \) such that \( \theta \circ i = \overline{f} \), and hence \( \overline{\gamma}_{\mathfrak{g}, A} (\theta) = f \). Therefore, the map \( \overline{\gamma}_{\mathfrak{g}, A} \) is bijective and the proof is now finished.

**Definition 2.6.** Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be Leibniz algebras with \( \mathfrak{h} \) finite dimensional. Then the commutative algebra \( \mathcal{A}(\mathfrak{h}, \mathfrak{g}) \) is called the universal algebra of \( \mathfrak{h} \) and \( \mathfrak{g} \). When \( \mathfrak{h} = \mathfrak{g} \) we denote the universal algebra of \( \mathfrak{h} \) simply by \( \mathcal{A}(\mathfrak{h}) \).

If \( \{ \tau_{i,j}^s \mid i, j, s = 1, \ldots, n \} \) are the structure constants of \( \mathfrak{h} \), where \( n \) is the dimension of \( \mathfrak{h} \), then the polynomials defined for any \( a, i, j = 1, \ldots, n \) by:

\[
P_{(a,i,j)}^{(b)} := \sum_{u=1}^{n} \tau^{u}_{i,j} X_{au} - \sum_{s,t=1}^{n} \tau^{a}_{s,t} X_{si}X_{tj} \in k[X_{ij} \mid i, j = 1, \ldots, n]
\]  

are called the universal polynomials of \( \mathfrak{h} \). It follows from the proof of Theorem 2.1 that \( \mathcal{A}(\mathfrak{h}) = k[X_{ij} \mid i, j = 1, \ldots, n]/J \), where \( J \) is the ideal generated by the universal polynomials \( P_{(a,i,j)}^{(b)} \), for all \( a, i, j = 1, \ldots, n \). Moreover, if \( \{ e_1, \ldots, e_n \} \) is a basis in \( \mathfrak{h} \) then the canonical map

\[
\eta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}), \quad \eta_{\mathfrak{h}}(e_i) := \sum_{s=1}^{n} e_s \otimes x_{si}
\]  

for all \( i = 1, \ldots, n \) is a Leibniz algebra homomorphism. The commutative algebra \( \mathcal{A}(\mathfrak{h}) \) and the family of polynomials \( P_{(a,i,j)}^{(b)} \) are purely algebraic objects that capture the entire information of the Leibniz algebra \( \mathfrak{h} \). Moreover, the universal algebra \( \mathcal{A}(\mathfrak{h}) \) satisfies the following universal property:

**Corollary 2.7.** Let \( \mathfrak{h} \) be a finite dimensional Leibniz algebra. Then for any commutative algebra \( A \) and any Leibniz algebra homomorphism \( f : \mathfrak{h} \to \mathfrak{h} \otimes A \), there exists a unique algebra homomorphism \( \theta : \mathcal{A}(\mathfrak{h}) \to A \) such that \( f = (\text{Id}_{\mathfrak{h}} \otimes \theta) \circ \eta_{\mathfrak{h}} \), i.e. the following diagram is commutative:

\[
\begin{align*}
\mathfrak{h} & \xrightarrow{\eta_{\mathfrak{h}}} \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \\
\downarrow{f} & \quad & \downarrow{(\text{Id}_{\mathfrak{h}} \otimes \theta)} \\
\mathfrak{h} \otimes A & \xrightarrow{} \mathfrak{h} \otimes A
\end{align*}
\]  

**Proof.** Follows straightforward from the bijection given in (9) for \( \mathfrak{g} := \mathfrak{h} \). \( \square \)

**Remark 2.8.** If \( \mathfrak{h} \) is a Lie algebra of dimension \( n \), then the structure constants are subject to the following relations \( \tau_{i,i}^s = 0 \) and \( \tau_{i,j}^s = -\tau_{j,i}^s \), for all \( i, j, s = 1, \ldots, n \). Consequently, we can easily see that the universal polynomials of \( \mathfrak{h} \) fulfill the following conditions:

\[
P_{(a,i,i)}^{(b)} = 0 \quad \text{and} \quad P_{(a,i,j)}^{(b)} = -P_{(a,j,i)}^{(b)}
\]  

for all \( a, i, j = 1, \ldots, n, i \neq j \). Thus, in the case of Lie algebras the universal algebra \( \mathcal{A}(\mathfrak{h}) \) takes a simplified form. We provide further examples in the sequel.
Examples 2.9. 1. Let \( \mathfrak{h} := \mathfrak{aff}(2,k) \) be the affine 2-dimensional Lie algebra with basis \( \{e_1, e_2\} \) and bracket given by \([e_1, e_2] = e_1\). Then, we have:

\[
\mathcal{A}(\mathfrak{aff}(2,k)) \cong k[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{21}, X_{11} - X_{12}X_{22} + X_{12}X_{21}) \\
\cong k[X, Y, Z]/(X - YZ)
\]

Indeed, the non-zero structure constants of \( \mathfrak{h} \) are \( \tau_{1,2}^1 = 1 = -\tau_{2,1}^1 \). Using (19) from the previous remark the only non-zero universal polynomials of the Lie algebra \( \mathfrak{aff}(2,k) \) are \( P_{(1,1,2)} = X_{11} - X_{12}X_{22} + X_{12}X_{21} \), \( P_{(2,1,2)} = X_{21} \), \(-P_{(1,1,2)} \) and \(-P_{(2,1,2)} \). The conclusion now follows.

2. Let \( \mathfrak{h} := \mathfrak{sl}(2,k) \) be the Lie algebra with basis \( \{e_1, e_2, e_3\} \) and bracket \([e_1, e_2] = e_3, [e_3, e_2] = -2e_2, [e_3, e_1] = 2e_1\). A routinely computation proves that \( \mathcal{A}(\mathfrak{sl}(2,k)) \cong k[X_{ij} \mid i, j = 1, 2, 3]/J \), where \( J \) is the ideal generated by the following nine universal polynomials of \( \mathfrak{sl}(2,k) \):

\[
\begin{align*}
X_{13} - 2X_{12}X_{31} + 2X_{11}X_{32}, & \quad 2X_{11} - 2X_{11}X_{33} + 2X_{13}X_{31}, & \quad 2X_{12} - 2X_{13}X_{32} + 2X_{12}X_{33} \\
X_{23} - 2X_{21}X_{32} + 2X_{22}X_{31}, & \quad 2X_{21} - 2X_{23}X_{31} + 2X_{21}X_{33}, & \quad 2X_{22} - 2X_{22}X_{33} + 2X_{23}X_{32} \\
X_{33} - X_{11}X_{22} + X_{12}X_{21}, & \quad 2X_{31} - X_{21}X_{13} + X_{11}X_{23}, & \quad 2X_{32} - X_{12}X_{23} + X_{13}X_{22}.
\end{align*}
\]

We recall that the polynomial algebra \( M(n) = k[X_{ij} \mid i, j = 1, \cdots, n] \) is a bialgebra with comultiplication and counit given by \( \Delta(X_{ij}) = \sum_{s=1}^n X_{is} \otimes X_{sj} \) and \( \varepsilon(X_{ij}) = \delta_{i,j} \), for any \( i, j = 1, \cdots, n \). We will prove now that the universal algebra \( \mathcal{A}(\mathfrak{h}) \) is also a bialgebra.

**Proposition 2.10.** Let \( \mathfrak{h} \) be a Leibniz algebra of dimension \( n \). Then there exists a unique bialgebra structure on \( \mathcal{A}(\mathfrak{h}) \) such that the Leibniz algebra homomorphism \( \eta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \) becomes a right \( \mathcal{A}(\mathfrak{h}) \)-comodule structure on \( \mathfrak{h} \). More precisely, the comultiplication and the counit on \( \mathcal{A}(\mathfrak{h}) \) are given for any \( i, j = 1, \cdots, n \) by

\[
\Delta(x_{ij}) = \sum_{s=1}^n x_{is} \otimes x_{sj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{i,j}
\]

Furthermore, the usual projection \( \pi : M(n) \to \mathcal{A}(\mathfrak{h}) \) becomes a bialgebra homomorphism.

**Proof.** Consider the Leibniz algebra homomorphism \( f : \mathfrak{h} \to \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h}) \) defined by \( f := (\eta_{\mathfrak{h}} \otimes \text{Id}_{\mathcal{A}(\mathfrak{h})}) \circ \eta_{\mathfrak{h}} \). It follows from Corollary 2.7 that there exists a unique algebra homomorphism \( \Delta : \mathcal{A}(\mathfrak{h}) \to \mathcal{A}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h}) \) such that \((\text{Id}_{\mathfrak{h}} \otimes \Delta) \circ \eta_{\mathfrak{h}} = f\); that is, the following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{\eta_{\mathfrak{h}}} & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \\
\downarrow{\eta_{\mathfrak{h}}} & & \downarrow{\text{Id}_{\mathfrak{h}} \otimes \Delta} \\
\mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) & \xrightarrow{\eta_{\mathfrak{h}} \otimes \text{Id}_{\mathcal{A}(\mathfrak{h})}} & \mathfrak{h} \otimes \mathcal{A}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h})
\end{array}
\]
Now, if we evaluate the diagram \((21)\) at each \(e_i\), for \(i = 1, \cdots, n\) we obtain, taking into account \((17)\), the following:

\[
\sum_{t=1}^{n} e_t \otimes \Delta(x_{ti}) = (\eta_h \otimes \text{Id})(\sum_{s=1}^{n} e_s \otimes x_{si}) = \sum_{s=1}^{n} (\sum_{t=1}^{n} e_t \otimes x_{ts}) \otimes x_{si}
\]

and hence \(\Delta(x_{ti}) = \sum_{s=1}^{n} x_{ts} \otimes x_{si}\), for all \(t, i = 1, \cdots, n\). Obviously, \(\Delta\) given by this formula on generators is coassociative. In a similar fashion, applying once again Corollary 2.7, we obtain that there exists a unique algebra homomorphism \(\varepsilon: A(h) \to k\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{\eta_h} & h \otimes A(h) \\
\text{can} \downarrow & & \downarrow \text{Id}_h \otimes \varepsilon \\
\mathfrak{h} \otimes k & \xrightarrow{\varepsilon} & \mathfrak{h} \otimes k
\end{array}
\]

where \(\text{can}: \mathfrak{h} \to \mathfrak{h} \otimes k\) is the canonical isomorphism, \(\text{can}(x) = x \otimes 1\), for all \(x \in \mathfrak{h}\). If we evaluate this diagram at each \(e_t\), for \(t = 1, \cdots, n\), we obtain \(\varepsilon(x_{ij}) = \delta_{ij}\), for all \(i, j = 1, \cdots, n\). It can be easily checked that \(\varepsilon\) is a counit for \(\Delta\), thus \(A(h)\) is a bialgebra. Furthermore, the commutativity of the above two diagrams imply that the canonical map \(\eta_h: \mathfrak{h} \to \mathfrak{h} \otimes A(h)\) defines a right \(A(h)\)-comodule structure on \(\mathfrak{h}\). \(\square\)

We call the pair \((A(h), \eta_h)\), with the coalgebra structure defined in Proposition 2.10, the universal coacting bialgebra of the Leibniz algebra \(\mathfrak{h}\). It fullfils the following universal property which extends Corollary 2.7:

**Theorem 2.11.** Let \(\mathfrak{h}\) be a Leibniz algebra of dimension \(n\). Then, for any commutative bialgebra \(B\) and any Leibniz algebra homomorphism \(f: \mathfrak{h} \to \mathfrak{h} \otimes B\) which makes \(\mathfrak{h}\) into a right \(B\)-comodule there exists a unique bialgebra homomorphism \(\theta: A(h) \to B\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{\eta_h} & h \otimes A(h) \\
\text{f} \downarrow & & \downarrow \text{Id}_h \otimes \theta \\
\mathfrak{h} \otimes B & \xrightarrow{\theta} & B
\end{array}
\]

**Proof.** As \(A(h)\) is the universal algebra of \(\mathfrak{h}\), there exists a unique algebra homomorphism \(\theta: A(h) \to B\) such that diagram \((23)\) commutes. The proof will be finished once we show that \(\theta\) is a coalgebra homomorphism as well. This follows by using again the universal property of \(A(h)\). Indeed, we obtain a unique algebra homomorphism \(\psi: A(h) \to B \otimes B\)
such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{\eta_h} & \mathfrak{h} \otimes A(\mathfrak{h}) \\
\downarrow^{(\text{Id}_B \otimes \Delta_B \circ \theta) \circ \eta_h} & & \downarrow^{\text{Id}_B \otimes \psi} \\
\mathfrak{h} \otimes B \otimes B & & 
\end{array}
\] (24)

The proof will be finished once we show that \((\theta \otimes \theta) \circ \Delta\) makes diagram (24) commutative. Indeed, as \(f : \mathfrak{h} \to \mathfrak{h} \otimes B\) is a right \(B\)-comodule structure, we have:

\[
\begin{align*}
(\text{Id}_h \otimes (\theta \otimes \theta) \circ \Delta) \circ \eta_h &= (\text{Id}_h \otimes \theta \otimes \theta) \circ (\text{Id}_h \otimes \Delta) \circ \eta_h \\
&= (\text{Id}_h \otimes \theta) \circ (\text{Id}_h \otimes \Delta) \circ \eta_h \\
&= (\text{Id}_h \otimes \theta) \circ (\text{Id}_h \otimes \theta) \circ \eta_h \\
&= (f \otimes \text{Id}_B) \circ (\text{Id}_h \otimes \theta) \circ \eta_h \\
&= (\text{Id}_h \otimes \Delta_B) \circ f \\
&= (\text{Id}_h \otimes \Delta_B \circ \theta) \circ \eta_h
\end{align*}
\] (23)

as desired. Similarly, one can show that \(\varepsilon_B \circ \theta = \varepsilon\) and the proof is now finished. \(\square\)

Next we discuss three consequences of our previous results which highlights the power of the universal coacting bialgebra of a Leibniz algebra. The first application of \(A(\mathfrak{h})\) is related to the classification of all \(G\)-gradings on a given Leibniz/Lie algebras. Let \(G\) be an abelian group and \(\mathfrak{h}\) a Leibniz algebra. We recall that a \(G\)-grading on \(\mathfrak{h}\) is a vector space decomposition \(\mathfrak{h} = \bigoplus_{\sigma \in G} \mathfrak{h}_\sigma\) such that \([\mathfrak{h}_\sigma, \mathfrak{h}_\tau] \subseteq \mathfrak{h}_{\sigma \tau}\) for all \(\sigma, \tau \in G\). For more detail on the problem of classifying \(G\)-gradings on Lie algebras see [10] and the references therein. In what follows \(k[G]\) denotes the usual group algebra of a group \(G\).

**Corollary 2.12.** Let \(G\) be an abelian group and \(\mathfrak{h}\) a finite dimensional Leibniz algebra. Then there exists a bijection between the set of all \(G\)-gradings on \(\mathfrak{h}\) and the set of all bialgebra homomorphisms \(A(\mathfrak{h}) \to k[G]\).

The bijection is given such that the \(G\)-grading on \(\mathfrak{h}\) associated to a bialgebra map \(\theta : A(\mathfrak{h}) \to k[G]\) is given for any \(\sigma \in G\) by:

\[
\mathfrak{h}_\sigma := \{ x \in \mathfrak{h} | (\text{Id}_h \otimes \theta) \circ \eta_h(x) = x \otimes \sigma \}
\] (25)

**Proof.** By applying Theorem 2.11 for the commutative bialgebra \(B := k[G]\) yields a bijection between the set of all bialgebra homomorphisms \(A(\mathfrak{h}) \to k[G]\) and the set of all Leibniz algebra homomorphisms \(f : \mathfrak{h} \to \mathfrak{h} \otimes k[G]\) which makes \(\mathfrak{h}\) into a right
$k[G]$-comodule. The proof is finished if we show that the latter set is in bijective correspondence with the set of all $G$-gradings on $h$.

Indeed, it is a well known fact in Hopf algebra theory [21, Exercise 3.2.21] that there exists a bijection between the set of all right $k[G]$-comodule structures $f: h \rightarrow h \otimes k[G]$ on the vector space $h$ and the set of all vector space decompositions $h = \oplus_{\sigma \in G} h_{\sigma}$. The bijection is given such that $x_{\sigma} \in h_{\sigma}$ if and only if $f(x_{\sigma}) = x_{\sigma} \otimes \sigma$, for all $\sigma \in G$. The only thing left to prove is that under this bijection a right coaction $f: h \rightarrow h \otimes k[G]$ is a Leibniz algebra homomorphism if and only if $[h_{\sigma}, h_{\tau}] \subseteq h_{\sigma \tau}$, for all $\sigma, \tau \in G$. To prove this, let $\sigma, \tau \in G$ and $x_{\sigma}, x_{\tau} \in h_{\sigma}, h_{\tau}$, respectively. Then, $[f(x_{\sigma}), f(x_{\tau})] = [x_{\sigma} \otimes \sigma, x_{\tau} \otimes \tau] = [x_{\sigma}, x_{\tau}] \otimes \sigma \tau$. Thus, we obtain that $f([x_{\sigma}, x_{\tau}]) = [f(x_{\sigma}), f(x_{\tau})]$ if and only if $[x_{\sigma}, x_{\tau}] \in h_{\sigma \tau}$. Hence, $f: h \rightarrow h \otimes k[G]$ is a Leibniz algebra homomorphism if and only if $[h_{\sigma}, h_{\tau}] \subseteq h_{\sigma \tau}$, for all $\sigma, \tau \in G$ and the proof is now finished.

Let $G$ be a group and $h$ a Leibniz algebra. We recall that an action as automorphisms of $G$ on $h$ is a morphism of groups $\varphi: G \rightarrow Aut_{Lbz}(h)$.

Corollary 2.13. Let $G$ be a finite group and $h$ a finite dimensional Leibniz algebra with basis $\{e_1, \cdots, e_n\}$. Then there exists a bijection between the set of all actions as automorphisms of $G$ on $h$ and the set of all bialgebra homomorphisms $\mathcal{A}(h) \rightarrow k[G]^\ast$.

The bijection is given such that the group homomorphism $\varphi_{\theta}: G \rightarrow Aut_{Lbz}(h)$ associated to a bialgebra homomorphism $\theta: \mathcal{A}(h) \rightarrow k[G]^\ast$ is given by:

$$\varphi_{\theta}(g)(e_i) = \sum_{s=1}^{n} \langle \theta(x_{si}), g > e_s \rangle \tag{26}$$

for all $g \in G$ and $i = 1, \cdots, n$.

Proof. By applying Theorem 2.11 for the commutative bialgebra $B := k[G]^\ast$ gives a bijection between the set of all bialgebra homomorphisms $\mathcal{A}(h) \rightarrow k[G]^\ast$ and the set of all Leibniz algebra homomorphisms $f: h \rightarrow h \otimes k[G]^\ast$ which make $h$ into a right $k[G]^\ast$-comodule. The proof is finished if we show that the latter set is in bijective correspondence with the set of all group homomorphisms $G \rightarrow Aut_{Lbz}(h)$. This follows by a standard argument in Hopf algebra theory, similar to the one used in [22, Lemma 1]. We indicate very briefly how the argument goes, leaving the details to the reader. Indeed, the category of right $k[G]^\ast$-comodules is isomorphic to the category of left $k[G]$-modules. The left action $\bullet: k[G] \otimes h \rightarrow h$ of the group algebra $k[G]$ on $h$ associated to a right coaction $f: h \rightarrow h \otimes k[G]^\ast$ is given by $g \bullet x := < x_{<1>}, g > x_{<0>}$, where we used the $\sum$-notation for comodules, $f(x) = x_{<0>} \otimes x_{<1>} \in h \otimes k[G]^\ast$ (summation understood). We associate to the action $\bullet$ the map $\varphi_{\bullet}: G \rightarrow Aut_k(h)$, $\varphi_{\bullet}(g)(x) := g \bullet x$, for all $g \in G$ and $x \in h$. Now, it can be easily checked that $f: h \rightarrow h \otimes k[G]^\ast$ being a Leibniz algebra homomorphism is equivalent to $\varphi_{\bullet}(g)$ being an automorphism of the Leibniz algebra $h$, for all $g \in G$ and the proof is finished. □

Recall that for any bialgebra $H$ the set of group-like elements, denoted by $G(H) := \{g \in H \mid \Delta(g) = g \otimes g, \text{ and } \varepsilon(g) = 1\}$, is a monoid with respect to the multiplication of $H$. 
We denote by \( H^o \), the finite dual bialgebra of \( H \), i.e.:

\[
H^o := \{ f \in H^* | f(I) = 0, \text{ for some ideal } I < H \text{ with } \dim_k(H/I) < \infty \}
\]

It is well known (see for instance [21, pag. 62]) that \( G(H^o) = \text{Hom}_{\text{Alg}}(H, k) \), the set of all algebra homomorphisms \( H \to k \). Now, we shall give the third application of the universal algebra of a Leibniz algebra.

**Theorem 2.14.** Let \( \mathfrak{h} \) be a finite dimensional Leibniz algebra with basis \( \{e_1, \cdots, e_n\} \) and let \( U(G(\mathcal{A}(\mathfrak{h})^o)) \) be the group of all invertible group-like elements of the finite dual \( \mathcal{A}(\mathfrak{h})^o \). Then the map defined for any \( \theta \in U(G(\mathcal{A}(\mathfrak{h})^o)) \) and \( i = 1, \cdots, n \) by:

\[
\gamma : U(G(\mathcal{A}(\mathfrak{h})^o)) \to \text{Aut}_{Lbz}(\mathfrak{h}), \quad \gamma(\theta)(e_i) := \sum_{s=1}^{n} \theta(x_{si})e_s
\]

(27)

is an isomorphism of groups.

**Proof.** By applying Corollary 2.3 for \( \mathfrak{g} := \mathfrak{h} \) it follows that the map

\[
\gamma : \text{Hom}_{\text{Alg}}(\mathcal{A}(\mathfrak{h}), k) \to \text{End}_{Lbz}(\mathfrak{h}), \quad \gamma(\theta) = (\text{Id}_\mathfrak{h} \otimes \theta) \circ \eta_\mathfrak{h}
\]

is bijective. Based on formula (17), it can be easily seen that \( \gamma \) takes the form given by (27). As we mentioned above we have \( \text{Hom}_{\text{Alg}}(\mathcal{A}(\mathfrak{h}), k) = G(\mathcal{A}(\mathfrak{h})^o) \). Therefore, since \( \gamma \) is the restriction of \( \gamma \) to the invertible elements of the two monoids, the proof will be finished once we show that \( \gamma \) is an isomorphism of monoids. We mention that the monoid structure on \( \text{End}_{Lbz}(\mathfrak{h}) \) is given by the usual composition of endomorphisms of the Leibniz algebra \( \mathfrak{h} \), while \( G(\mathcal{A}(\mathfrak{h})^o) \) is a monoid with respect to the convolution product, that is:

\[
(\theta_1 \ast \theta_2)(x_{sj}) = \sum_{t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj})
\]

(28)

for all \( \theta_1, \theta_2 \in G(\mathcal{A}(\mathfrak{h})^o) \) and \( j, s = 1, \cdots, n \). Now, for any \( \theta_1, \theta_2 \in G(\mathcal{A}(\mathfrak{h})^o) \) and \( j = 1, \cdots, n \) we have:

\[
(\gamma(\theta_1) \circ \gamma(\theta_2))(e_j) = \gamma(\theta_1)(\sum_{t=1}^{n} \theta_2(x_{tj})e_t) = \sum_{s,t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj})e_s = \sum_{s=1}^{n} \left( \sum_{t=1}^{n} \theta_1(x_{st})\theta_2(x_{tj}) \right) e_s = \sum_{s=1}^{n} (\theta_1 \ast \theta_2)(x_{sj})e_s = \gamma(\theta_1 \ast \theta_2)(e_j)
\]

thus, \( \gamma(\theta_1 \ast \theta_2) = \gamma(\theta_1) \circ \gamma(\theta_2) \), and therefore \( \gamma \) respects the multiplication. We are left to show that \( \gamma \) also preserves the unit. Note that the unit 1 of the monoid \( G(\mathcal{A}(\mathfrak{h})^o) \) is the counit \( \varepsilon_{\mathcal{A}(\mathfrak{h})} \) of the bialgebra \( \mathcal{A}(\mathfrak{h}) \), and we obtain:

\[
\gamma(1)(e_i) = \gamma(\varepsilon_{\mathcal{A}(\mathfrak{h})})(e_i) = \sum_{s=1}^{n} \varepsilon_{\mathcal{A}(\mathfrak{h})}(x_{si})e_s = \sum_{s=1}^{n} \delta_{si} e_s = e_i = \text{Id}_\mathfrak{h}(e_i)
\]

Thus we have proved that \( \gamma \) is an isomorphism of monoids and the proof is finished. \( \square \)
In what follows we construct for any finite dimensional Leibniz algebra \( h \) a universal commutative Hopf algebra \( H(h) \) together with a Leibniz algebra homomorphism \( \lambda_h : h \to h \otimes H(h) \) which makes \( h \) into a right \( H(h) \)-comodule. This is achieved by using the free commutative Hopf algebra generated by a commutative bialgebra introduced in [19, Chapter IV]. Recall that assigning to a commutative bialgebra the free commutative Hopf algebra defines a functor \( L : \text{ComBiAlg}_k \to \text{ComHopf}_k \) which is a left adjoint to the forgetful functor \( \text{ComHopf}_k \to \text{ComBiAlg}_k \) ([19, Theorem 65, (2)]). Throughout, we denote by \( \mu : 1_{\text{ComBiAlg}_k} \to UL \) the unit of the adjunction \( L \dashv U \).

**Definition 2.15.** Let \( h \) be a finite dimensional Leibniz algebra. The pair \( (H(h) := L(A(h)), \lambda_h := (\text{Id}_h \otimes \mu_{A(h)}) \circ \eta_h) \) is called the **universal coacting Hopf algebra of** \( h \).

The pair \( (H(h), \lambda_h) \) fulfills the following universal property which shows that it is the initial object in the category of all commutative Hopf algebras that coact on \( h \).

**Theorem 2.16.** Let \( h \) be a finite dimensional Leibniz algebra. Then, for any commutative Hopf algebra \( H \) and any Leibniz algebra homomorphism \( f : h \to h \otimes H \) which makes \( h \) into a right \( H \)-comodule there exists a unique Hopf algebra homomorphism \( g : H(h) \to H \) for which the following diagram is commutative:

\[
\begin{array}{ccc}
  h & \xrightarrow{\lambda_h} & h \otimes H(h) \\
  f & & \downarrow \text{Id}_h \otimes g \\
  & & h \otimes H
\end{array}
\]  

(29)

**Proof.** Let \( H \) be a commutative Hopf algebra together with a Leibniz algebra homomorphism \( f : h \to h \otimes H \) which makes \( h \) into right a \( H \)-comodule. Using Theorem 2.11 we obtain a unique bialgebra homomorphism \( \theta : A(h) \to H \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
  h & \xrightarrow{\eta_h} & h \otimes A(h) \\
  f & & \downarrow \text{Id}_h \otimes \theta \\
  & & h \otimes H
\end{array}
\]  

i.e. \((\text{Id}_h \otimes \theta) \circ \eta_h = f\)  

(30)

Now the adjunction \( L \dashv U \) yields a unique Hopf algebra homomorphism \( g : L(A(h)) \to H \) such that the following diagram commutes:

\[
\begin{array}{ccc}
  A(h) & \xrightarrow{\mu_{A(h)}} & L(A(h)) \\
  \theta & & \downarrow g \\
  & & H
\end{array}
\]  

i.e. \( g \circ \mu_{A(h)} = \theta \).  

(31)

We are now ready to show that \( g : H(h) = L(A(h)) \to H \) is the unique Hopf algebra homomorphism which makes diagram (29) commutative. Indeed, putting all the above
together yields:
\[
(\text{Id}_h \otimes g) \circ (\text{Id}_h \otimes \mu_A(h)) \circ \eta_h = (\text{Id}_h \otimes g \circ \mu_A(h)) \circ \eta_h
\]
\[
\Rightarrow (\text{Id}_h \otimes \theta) \circ \eta_h = f.
\]

Since \( g \) is obviously the unique Hopf algebra homomorphism which makes the above diagram commutative, the proof is finished. \( \square \)

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Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

E-mail address: ana.agore@vub.be, ana.agore@gmail.com

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

E-mail address: gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com