A Twisted $\mathcal{C}^*$ - algebra formulation of Quantum Cosmology with application to the Bianchi I model

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A twisted $\mathcal{C}^*$- algebra of the extended (noncommutative) Heisenberg-Weyl group has been constructed which takes into account the Uncertainty Principle for coordinates in the Planck length regime. This general construction is then used to generate an appropriate Hilbert space and observables for the noncommutative theory which, when applied to the Bianchi I Cosmology, leads to a new set of equations that describe the quantum evolution of the universe. We find that this formulation matches theories based on a reticular Heisenberg-Weyl algebra in the bouncing and expanding regions of a collapsing Bianchi universe. There is, however, an additional effect introduced by the dynamics generated by the noncommutativity. This is an oscillation in the spectrum of the volume operator of the universe, within the bouncing region of the commutative theories. We show that this effect is generic and produced by the noncommutative momentum exchange between the degrees of freedom in the cosmology. We give asymptotic and numerical solutions which show the above mentioned effects of the noncommutativity.

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I. INTRODUCTION

Reductionism is an essential concept in Physics which has been validated by experiments involving energies ranging from orders of eV’s in molecular and atomic physics to a few TeV in the strong interaction regime. This paradigm has led to such successes of quantum unification as the Standard Model, involving Electromagnetic, Weak and Strong Interactions. However the oldest interaction known to man: Gravity, and its most beautiful geometrical formulation: General Relativity, have to this day avoided quantization and even more so, unification with the other three fundamental forces of Nature. Thus Quantization of Relativity at distances of the order of the Planck length and energies of the order of $10^{16}$ TeV, still remains to be one of the most compelling problems in the field, mainly due to the lack of experimental data that could help shed some more light on which path should one pursue.

Because Quantum Cosmology can be seen as a minisuperspace of Quantum Gravity where most of the degrees of freedom have been frozen and, although there is no a priori reason to assume that the conclusions derived from the former can be readily translated to the later, it is expected that some approaches to Quantum Cosmology can provide a convenient initial framework to investigate quantum processes involving distances of the order of Planck lengths where manifestations of noncommutativity should occur.

The main purpose of this paper is to provide what we consider might be one such self-consistent formulation for Quantum Cosmology that could lead to further insights and directives towards Quantum Gravity at scales where the implications of the Uncertainty Principle of Quantum Mechanics and the Principle of Equivalence of Gravitation become commensurate.

Indeed, regardless of which will be eventually the final and complete Theory for Quantum Gravity, it seems that the present attempts for its formulation have as a common denominator some concept of noncommutativity (see e.g. [1], [2], [3], [4], [5], [6]). Thus, in addition to the fact that Physics is a discipline based on experiment and that a theory needs to be validated or dismissed only on this basis before its ultimate acceptance, it is sensible to expect that the concept of noncommutativity should be a self-consistent part of it. One formulation that appeals to many physicists in the field is String Theory [7]. Several research groups in Relativity on the other hand believe that a more geometrical approach such as Loop Quantum Gravity (LQG) constitutes an equally viable candidate (see e.g. [8]) and, on the other extreme of the theory spectrum, is the Noncommutative Geometry developed by A. Connes and others (see e.g. [9], [10], [11], [12]).

As pointed out in the Review by Douglas and Nekrasov [13], some of the strong arguments in favor of noncommutativity and of further support for Noncommutative Geometry originated from these varied approaches has led to a flurry of activities and trends where mathematical clarity and conceptual self-consistency “appear less central to physical considerations”. Examples of such a case are the earlier quantum cosmology formulations based on a Bopp map deformation of the Wheeler-De Witt equation, resulting from inserting a Moyal $\star$-product between the classical Hamiltonian and the elements of the Hilbert vector space of wave functions. This, from the viewpoint of Deformation Quantization where the Moyal $\star$-product arises as a deformation of the algebra product of the Weyl symbols of quantum operator observables, has no conceptual support. Moreover, as we have shown in [14] (and references therein) a more logical noncommutative replacement for the Schrödinger equation is the $\star$-value equation involving the deformed Moyal $\star$-product of the Weyl symbol of the quantum Hamiltonian operator and the Wigner function. It may be meaningful to notice here also that in a previous work [15] of the type mentioned above, the region close to the singularity has not been explored and the wave functions have branch points which imply an undetermined behavior near the singularity, which could very well be attributed to the authors use of this unsubstantiated Moyal product in the Wheeler-de Witt equation.

Alternatively, the $C^\ast$-algebra $\mathfrak{A}$, on which our approach is based, is in particular a good example of the strategy of Noncommutative Geometry, and a motivational argument for basing our approach on this formalism hinges, on a nut shell, on the theoretical observations that since physically meaningful quantities should be independent of the choice of a gauge, the concepts of gauge potentials or connections had to be incorporated into the formulation of Action
Densities for describing our perception of Nature. This then has led naturally to the formalism of fiber bundles to describe the basic forces of nature and the mathematical physics for dealing with Gauge Theory and Variational Principles in Field Theory. Now, a bundle \( P(M, F, \tau) \) consists of a topological space \( P \), a base \( M \), a typical fiber \( F \) and a continuous surjection \( \tau : P \rightarrow M \), where in semi-classical physics \( M \) is the space-time continuum with a Hausdorff topology. Moreover, it can be shown that a vector bundle over \( M \) can be described purely in terms of concepts pertinent to the commutative \( C^* \)-algebra \( C(M) \) (see e.g. \([16]\)). Furthermore, by the Gel’fand-Naimark Theorem \([17]\): “To every commutative \( C^* \)-algebra with unit there corresponds a Hausdorff space, which implies a complete duality between the category of locally compact Hausdorff spaces and the category of commutative \( C^* \)-algebras \( C(M) \) and \(*\)-homomorphisms. However, at distances of the order of the Planck length, where the Principle of Uncertainty and the Principle of Equivalence become equally important and noncommutativity dominates the dynamics of the system, one needs to generalize the notion of a Hilbert bundle in such a way that the commutative \( C^* \)-algebra \( C(M) \) is replaced by an arbitrary \( C^* \)-algebra \( \mathfrak{A} \), and the dual notion of a Hausdorff topological space \( M \) be replaced by the space of all unitary classes of irreducible representations of \( \mathfrak{A} \) \((\mathfrak{I}, \mathfrak{II}, \mathfrak{III}, \mathfrak{IV})\).

On the basis of the previous remarks and in order to implement this ideas so as to provide the possibility of calculation for observable quantities in physical models, the material in this paper has been structured as follows: In Section II we introduce a projective unitary realization of the generators of the twisted discrete translation group \( C^* \)-algebra of bounded operators with unit, \(*\)-homomorphic to the Heisenberg-Weyl group of deformed quantization. Thus the noncommutative lattices, generated from the primitive spectrum of \( \mathfrak{A} \), are the structure spaces of the \( T_0 \) Jacobson topology and the noncommutative analogue of the Hausdorff topology of the space \( M \) of the Gel’fand-Naimark theorem. In Section III we go on to use the homomorphism obtained in the previous section and the Gel’fand-Naimark-Segal construction to derive the kinematic Hilbert space on which the bounded operators in \( \mathfrak{A} \) act. In addition, the functions resulting from the Pontryagin duality on this Hilbert vector space yield a complete set of functions which satisfy the same orthogonality and summation completeness relations as the algebra of almost periodic functions \([22]\). Section IV begins by considering the ADM reduced classical action of the anisotropic Bianchi I model cosmology coupled to a massless scalar to assume the part of an inner time. We then quantize the system following Dirac’s procedure after expressing the observables of the system in terms of the \( C^* \)-algebra of Hermitized bounded operators previously introduced. Using then the Hamiltonian constraints of the system and applying well documented techniques such as the ones summarized and cited in the text, we derive the physical states of the system from the kinematical states constructed in Sec.III. In Section V the so far inherently discrete system of equations is converted to the continuum by making use of the Feynman Path Integral construction for quantization. It should be noted, however, that the symbol of noncommutativity appears in various terms of the action and acquires different levels of relevance for the different possible stages of evolution of the system, as shown in the later sections. This analysis is in fact carried out extensively in Sections VI and VII, after deriving the equations of motion by applying the method of stationary phase to the action derived in Sec.V. In Section VII, in particular, we consider several scenarios for the system evolution which evidence clearly that noncommutativity, in the form that we have introduced here, not only prevents the singularities that occur in the Classical and Wheeler-DeWitt quantization approach to the Bianchi Cosmology, but it also provides the driving force which, under appropriate boundary conditions, allows the system to leave from a stage of oscillatory evolution within Planck length scales, to stages of regions where noncommutativity becomes negligible and the universe growth is monotonical. In Sec. VIII we summarize what we consider are the main results of this work and possible future lines of research that would extend it.

II. TWISTED DISCRETE TRANSLATION GROUP \( C^* \)-ALGEBRA AND DEFORMATION QUANTIZATION

Let us now consider \([23, 24, 25]\) the twisted (unital, discrete) \( C^* \)-dynamical system \( \Sigma = (\mathcal{A}, G, \alpha, \sigma) \) where the algebra \( \mathcal{A} \) can be related by means of a \( *\)-homomorphism to the \( C^* \)-algebra \( \mathfrak{A} \subset \mathcal{B}(\mathcal{H}) \) of bounded operators with unit, acting on a Hilbert space \( \mathcal{H} \). For this purpose and as a starting point of our analysis we observe that, since the
base topological $M$ space in Classical Bianchi I Cosmology is an $\mathbb{R}^3$, for which translations are isometries, whereas physical space at the Noncommutative Geometry level is described as a sort of a subjacent discrete noncommutative cellular structure (posets), we let $\mathcal{A}$ be the algebra of the noncommutative extended Heisenberg-Weyl group $G$, $G$ be the discrete topological group of translations in $\mathbb{R}^3$, $(\alpha, \sigma)$ the twisted action of $G$ on $\mathcal{A}$, with $\alpha$ denoting the map $\alpha : G \to \text{Aut}(\mathcal{A})$ and $\sigma : G \times G \to T(\mathcal{A})$ is a normalized 2-cocycle on $G$ with values in the multiplicative group $T$ of all complex numbers of unit modules, such that

$$\sigma(x_1, x_2)\sigma(x_1 + x_2, x_3) = \sigma(x_2, x_3)\sigma(x_1, x_2 + x_3), \quad x_1, x_2, x_3 \in G$$

$$\sigma(x, 0) = \sigma(0, x) = 1.$$  \hspace{1cm} (II.1)

In the above we have identified the discrete Abelian group of translations $G$ with the vector space $\mathbf{T}_3$, associated with $\mathbb{R}^3$ as an affine space with a discrete topology and with coset decomposition

$$\mathbf{T}_3 = \sum_{j_1,j_2,j_3=-\infty}^{\infty} (\mu_i j_i) \hat{e}_i, \quad j_i \in \mathbb{Z},$$  \hspace{1cm} (II.2)

where the $\hat{e}_i$ are the basic translations in $\mathbb{R}^3$, the vectors $x_{(i)} = \sum_{j=1}^{3}(\mu_i j_i) \hat{e}_i \in \mathbf{T}_3$ are elements of $\mathbb{R}^3$ as a group and the set $\Gamma : \{\mu_i j_i\}$ form a 3-dimensional cell. We then have

**Definition II.1.** A left $\sigma(x_1, x_2)$-projective unitary representation $\hat{U}$ of $G$ on a (non-zero) Hilbert space $H$ is a map from the group $G$ into the group $U(H)$ of unitaries on $H$ such that

$$U(x_1)U(x_2) = \sigma(x_1, x_2)U(x_1 + x_2).$$  \hspace{1cm} (II.3)

Taking in particular

$$U(H) \ni \sigma_\theta(x_1, x_2) := \sigma(x_1, x_2) = e^{-i\pi x_1^T R x_2} = e^{-i\theta \cdot (x_1 \times x_2)},$$  \hspace{1cm} (II.4)

where $R$ is the anti-symmetric matrix

$$R = \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix},$$  \hspace{1cm} (II.5)

where the $\theta_i$ have been assumed to be Poincaré invariant, as shown in [26], when considering a deformation of the universal enveloping Hopf algebra $U(P)$ of the Poincaré algebra $P$ by means of a Drinfeld twist [27].

**Definition II.2.** A left projective regular unitary realization of the algebra $U(P)$ and $U(P)$ on $l^2(G)$ can be defined as

$$\langle x | \hat{U}_i | \xi \rangle := e^{-2i\pi \epsilon_i x_i} \langle x - \frac{1}{2} \epsilon_i \hat{e}_i \times \xi \rangle = e^{-2i\pi \epsilon_i x_i} \xi(x - \frac{1}{2} \epsilon_i \hat{e}_i \times \theta) = e^{-\xi(x) \theta}; \quad (\xi(x) \in H).$$  \hspace{1cm} (II.6)

Identifying $x$ with the corresponding function on $\mathbf{T}_3$ which is one at $x$ and zero otherwise, i.e. if we let this function be $\delta_x \in l^2(\mathbf{T}_3)$ (the delta function at $x$) then it readily follows that

$$\hat{U}_i \delta_x := e^{-2i\pi \epsilon_i x_i} \delta_x (\frac{\theta}{2} \epsilon_i \hat{e}_i \times \theta + x),$$  \hspace{1cm} (II.7)

and

$$\hat{U}_i | x \rangle = e^{-2i\pi \epsilon_i x_i} | x + \frac{1}{2} \epsilon_i \hat{e}_i \times \theta \rangle.$$  \hspace{1cm} (II.8)

Thus the unitary $\hat{U}_i$ translates the vector $x$ in a direction perpendicular to $\hat{e}_i$ by the amount $\frac{1}{2} \epsilon_i \theta$. It is now fairly straightforward to show, by successive applications of (II.6), that

$$\hat{U}_i \hat{U}_j = e^{-i\pi \epsilon_i \epsilon_j \theta \cdot (\hat{e}_i \times \hat{e}_j)} \hat{U}_{i+j},$$  \hspace{1cm} (II.9)
and interchanging indices and substituting back the result into (II.9) we arrive at
\[ \hat{U}_i \hat{U}_j = e^{-2i\pi \varepsilon_i \varepsilon_j \theta \cdot (\hat{e}_i \times \hat{e}_j)} \hat{U}_j \hat{U}_i. \] (II.10)

Since the parameter of noncommutativity actually has units of length square the quantities \( \varepsilon_i \) must have units of \( \text{length}^{-1} \) and \( \varepsilon_i \hat{e}_i \times \hat{\theta} \) are thus basic vectors in the directions perpendicular to the \( \hat{e}_i \) which determine the fundamental lengths of the lattice.

Extending now the above algebra with the generators \( \hat{V}_i := \hat{V} (\mu_i \hat{e}_i) \) such that
\[ \hat{V}_i |x\rangle = |x + \mu_i \hat{e}_i\rangle, \] (II.11)
so we find that \( \hat{V}_i \) also acts on the kets \( |x\rangle \in \mathcal{H} \) as a translation operator on the vector \( x \) in the direction of \( \hat{e}_i \) by an amount \( \mu_i \). It also follows from (II.11) that
\[ \hat{V}_i \hat{V}_i = \hat{V}_i \hat{V}_i, \] (II.12)
and commuting with \( \hat{U}_i \) as given in (II.8), we arrive at
\[ \hat{U}_i \hat{V}_i = e^{-2i\pi \varepsilon_i \mu_i (\hat{e}_i \cdot \hat{e}_i)} \hat{V}_i \hat{U}_i = e^{-2i\pi \varepsilon_i \mu_i \delta_{ij}} \hat{V}_i \hat{U}_i. \] (II.13)

This is indeed a \( \ast \)-homomorphism between the \( \mathcal{C}^* \)-algebra \( \mathfrak{A} \subset \mathcal{B}(\mathcal{H}) \) of operators generated by the unitaries \( \hat{U}_i \)'s and \( \hat{V}_i \)'s and the extended noncommutative Heisenberg-Weyl algebra \( \mathcal{A} \) of the \( \mathcal{C}^* \)-dynamical system discussed before. Note also that the quantities \( \mu_i \) and \( \varepsilon_i \) introduced in the above relations strictly appear so far as independent parameters of the action of the discrete subgroups of the twisted (extended noncommutative) Heisenberg-Weyl group. This would however imply two different simultaneous noncommutative lattices generated by the unitaries \( \hat{U}_i \)'s and \( \hat{V}_i \)'s. Clearly in order to avoid this the \( \mu_i \) and \( \varepsilon_i \hat{e}_i \cdot (\varepsilon_i \hat{e}_i \times \hat{\theta}) \) must be related. We shall show later on that this relation appears naturally when constructing the Hilbert space on which these operators act.

We also find it important to point out here that, although the expressions (II.9) and (II.10) for the subalgebra of the \( \hat{U}_i \) appear to be the same as that used to describe the quantum torus (cf. e.g. [28]), the realization (II.6) (or (II.8)) introduced here has quite different implications. Indeed, as mentioned in the paper cited above, in the quantum torus formulation the \( \hat{U}_i \) act as Laplacian operators that translate on momentum space, and thus are appropriate to describe noncommutativity in momentum space [29]. On the other hand the realization of the \( \hat{U}_i \) and \( \hat{V}_i \) unitaries in (II.8) and (II.11) is geared to generate a Hilbert space by sequential translations, effected by the noncommutation matrix factor, on a cyclic vector. Thus in this case the noncommutativity is associated with the dynamical configuration variables of our formulation. The strong repercussions for our developments of this choice of realization is evidenced in the analysis presented in the last sections of this work.

### III. GNS-Construction of the Kinematic Hilbert Space

Let us now use this homomorphism to derive explicit forms for the elements of the Hilbert space \( \mathcal{H} \) on which the operators in \( \mathfrak{A} \) act by applying the Gel’fand-Naimark-Segal (GNS) construction [31, 11]. To this end first note that for any state functional \( \phi \) we have that \( \forall \ a \in \mathcal{A} \ \exists \ \phi \) such that \( \phi (a^\ast \ast a) = 1 \). Moreover, since any element \( a \) in the subjacent algebra \( \mathcal{A} \) is unitary, we have that this equality is always true here which, in turn, implies that the left ideal \( \mathcal{I} = \{ a \in \mathcal{A} | \phi (a^\ast \ast a) = 0 \} \) in \( \mathcal{A} \) is empty, so that the quotient space \( \mathcal{N}_\phi = \mathcal{A}/\mathcal{I}_\phi \equiv \mathcal{A} \Rightarrow \phi \) is faithful. Thus, by the GNS construction, we have a pre-Hilbert space with a non-degenerate product defined by
\[ \mathcal{A} \times \mathcal{A} \to \mathbb{C}, \quad (a, b) \mapsto \phi (a^\ast \ast b), \] (III.14)
and where \( \mathcal{H}_\phi \) is the completion of \( \mathcal{A} \) in this norm. Note that the \( \ast \)-homomorphism \( \pi_\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi) \), defines a representation \( (\mathcal{A}, \mathcal{H}_\phi) \) of the \( \mathcal{C}^* \)-algebra \( \mathcal{A} \) by associating to an element \( a \in \mathcal{A} \) an operator \( \pi_\phi (a) \in \mathfrak{A} \subset \mathcal{B}(\mathcal{H}) \) by
\[ \pi_\phi (a) b = a \ast b, \] (III.15)
which is a well defined bounded linear operator in $\mathcal{H}_\phi$. Indeed, from the above definition it follows that

$$\pi_\phi(a_1)\pi_\phi(a_2)(b) = a_1 \ast a_2 \ast b = \pi_\phi(a_1 \ast a_2)b,$$

which shows that $\text{(III.15)}$ is in fact a representation. Note also that in this construction the $C^*$-algebra is itself a Hilbert $\mathcal{A}$-module.

Now, in order to generate the elements of the Hilbert space we start with a distinguished vector $\xi_\phi$ which is cyclic for $\pi_\phi$, i.e. such that $\{\pi(a)\xi_\phi | a \in \mathcal{A}\}$ is dense in $\mathcal{H}_\phi$. Since $\mathcal{A}$ is unital we can chose $\xi_\phi := (x = 0|\xi_\phi) = \xi_\phi(0,0,0) = I$, which is clearly cyclic provided the parameters $\varepsilon_i$ and $\mu_i$, generated by the operators $\pi_\phi(a) = \hat{U}_i$, $\hat{V}_i \in \mathcal{B}(\mathcal{H}_\phi)$, according to $\text{(II.8)}$ and $\text{(II.11)}$ and which translate in directions perpendicular to each other, are appropriately related in order that the set of elements generated by the action of the $\pi_\phi(a)$ on $\xi_\phi$ is indeed dense in $\mathcal{H}_\phi$. It is not difficult to show that such a consistency can be achieved by setting

$$\begin{align*}
\mu_1 &= \frac{n_1}{2} \varepsilon_2 \theta_3 \\
\mu_2 &= \frac{n_2}{2} \varepsilon_1 \theta_3 \\
\mu_3 &= \frac{n_3}{2} \varepsilon_1 \theta_2,
\end{align*}$$

(III.17)

where, as we shall show later on in Section VII, the magnitudes $n_i \in \mathbb{N}^+$ and $\varepsilon_i$ are scale factors of the $\mu_i$’s and $\varepsilon_i$’s determined by the relative relevance of the noncommutative tensor symbol in the different stages of evolution of the dynamical system that we shall consider later on. In fact, we can consider the $\mu_i$’s and $\varepsilon_i$’s as introduced in the formalism to effectively represent a family of continuous projections $\pi_i^{m,n}$ acting on a family of topological spaces $Y^n$ such that

$$\pi_i^{m,n}: Y^m \rightarrow Y^n, \quad n \leq m.$$  

(III.18)

Hence the manifold $M$ with Hausdorff topology ($Y^\infty$) can be recovered as the limiting procedure of the inverse of such a sequence of projectors $\text{[31]}$. Moreover, in the limit $\varepsilon_i \rightarrow 0$ it readily follows that $\text{(III.8)}$ becomes multiplicative and the $\mu_i$ decouple from $\text{(III.17)}$ and $\text{(III.19)}$, so our twisted Heisenberg-Weyl algebra reduces to that in $\text{[32]}$ and the commutative lattices generated by the primitive spectrum of this algebra are now structure spaces of a $T_1$ topology where, as we shall show later on in Sec.VI, the elementary length of the cell induced by the $\mu_i$’s is of $O(\lambda_p)$. Taking the further limit $\mu_i \rightarrow 0$ will then result in the classical Heisenberg-Weyl algebra and a Hausdorff or $T_2$-space.

Note also that in some sense the relations $\text{(III.17)}$ are an equivalent of the improved dynamics introduced in $\text{[33]}$, which in our case appear directly from the consistency required by the translations generated by the noncommutativity. From $\text{(III.17)}$, $\text{(II.8)}$, and $\text{(II.11)}$ we also get

$$\begin{align*}
\varepsilon_2 \theta_3 &= \varepsilon_3 \theta_2 \\
\varepsilon_1 \theta_3 &= \varepsilon_3 \theta_1 \\
\varepsilon_1 \theta_2 &= \varepsilon_2 \theta_1.
\end{align*}$$

(III.19)

Consequently, it follows from the above relations that the subset $\{\pi(\hat{V}_i)\xi_\phi\}$ will be by itself dense in $\mathcal{H}_\phi$ and, by virtue of $\text{(III.15)}$ and $\text{(III.14)}$ (and the GNS Theorem), we have that given a vector-state functional $\phi$ on $\{V_i\} \subset \mathcal{A}$ there is a representation with a distinguished cyclic vector $\xi_\phi \in \mathcal{H}_\phi$ with the property

$$\langle \xi_\phi, \pi_\phi(V_i)\xi_\phi \rangle = \langle I, V_i \rangle = \phi(V_i).$$

(III.20)

Recall now that $\text{(III.11)}$ implies that

$$\langle x_1 = 0|\hat{V}_i|\xi_\phi \rangle = \xi_\phi(0 + \mu_i \hat{e}_i) = \xi_\phi(\mu_i \hat{e}_i),$$

(III.21)
so, if via the algebra *-homomorphism we associate to the element $V_i \in \mathcal{A}$ the operator

$$\pi_\phi(V_i) = \hat{V}(\mu_i \hat{e}_i),$$

then combining (III.20) with (III.21) allows us to identify $\phi(V_i)$ with the character of the discrete translation group, so that

$$\xi^k_\phi(x_n) = e^{2\pi i \sum_{i=1}^3 \mu_i (k_{ij(n)} + j_i)}, \quad f_{(n)i} \in \mathbb{Z}$$

(III.22)

where $k \in \mathbb{R}^3$, and $\mu$ are quantities whose magnitudes determine the size of the fundamental noncommutative lattice cell. Observe also that, since $\mathbb{Z}$ is empty, the representation $(H_\phi, \xi_\phi)$ is irreducible.

The functions $\xi^k_\phi(x)$ in (III.22) are a one-dimensional irreducible regular representation of the operator group $\hat{D}^k(x)$ of the discrete Abelian group of translations. That is

$$\hat{D}^k(x_n) = e^{2\pi i \sum_{i=1}^3 \mu_i (k_{ij(n)} + j_i)}, \quad \text{}(III.23)$$

and satisfies the relations of orthogonality and Poisson summation completeness

$$\int_{-1/2\mu_i}^{1/2\mu_i} \mu_i dk_i \hat{D}^{k_i}(j_{(1)i}) \hat{D}^{k_i}(j_{(2)i}) = \delta_{j_{(1)i}, j_{(2)i}}, \quad l = 1, 2, 3$$

(III.24)

respectively, after noting that the left hand side of the second equation above is a periodic generalized function with period one $\mathbb{Z}$. Observing that since the representations (III.23) of the translation group are invariant under the reciprocal group, the range of fundamental domain of the components of the vector parameter $k$ is $-1/2\mu_i \leq k_i \leq 1/2\mu_i$.

Also, making use of the completeness of the ket space $\{|k]\}$ we can write

$$\hat{D}^{k_i}(j_{(n)i}) = e^{2\pi i j_{(n)i} \mu_i k_i} = \langle \mu_i j_{(n)i} | k_i \rangle = \langle x_{(n)i} | k_i \rangle,$$

(III.25)

with

$$\prod_{i=1}^3 \int_{-1/2\mu_i}^{1/2\mu_i} \mu_i dk_i \langle x_{(n)i} | k_i \rangle \langle k_i | x_{(n')i} \rangle = \langle x_{(n)i} | x_{(n')i} \rangle = \delta_{x_{(n)i}, x_{(n')i}}.$$  

(III.26)

Furthermore, by the Pontryagin duality theorem, the dual of a discrete Abelian group is a compact Abelian group, so by Fourier analysis we can write (for a fixed index $i$)

$$\hat{f}(k_i) = \sum_{j_{(i)i}=-\infty}^{\infty} f(j_{(i)i}) e^{2\pi i j_{(i)i} (2\pi k_i)}, \quad -1/2\mu_i \leq k_i \leq 1/2\mu_i, \quad i = 1, 2, 3,$$

(III.27)

and

$$f(j_{(i)i}) = \int_{-1/2\mu_i}^{1/2\mu_i} dk_i \hat{f}(k_i) e^{-k_i (2\pi \mu_i j_{(i)i})}.$$  

(III.28)

Denote by $\Gamma = \{e^{k_i (2\pi \mu_i j_{(i)i})}\}$ the compact Abelian group of continuous characters dual to the twisted discrete translation group $G$, and let $\hat{G}$ denote the Abelian compact group of all characters, continuous or not, of $G$. Then $\Gamma$ is a continuous isomorphism of $G$ onto a dense subgroup $\beta(G)$ of $\hat{G}$. Thus, since the generators $e^{(2\pi k_i)}$ of the basis of mono-parametric subgroups in (III.27) are isomorphic to the circle group $T$ we have that the $\hat{f}(k_i)$ in (III.27) can be regarded as elements of the dense subgroup of the Bohr compactification of the twisted discrete translation group onto the quantum 3-torus $=\bar{G}$.

In particular, setting $x_{(i)i} := \mu_i j_{(i)i}$, we see that the function $e^{2\pi x_{(i)i} k_i}$ is continuous and periodic in $k_i$, thus the polynomial function $\sum_{i=1}^N f(x_{(i)i}) e^{-2\pi x_{(i)i} k_i}$ is an almost periodic function in the sense of Bohr (cf. [36, 37]).
Furthermore if the latter function converges uniformly to the series \( \sum_{k=1}^{\infty} f(x(t)) e^{2i\pi x(t)k_i} \) when \( N \to \infty \), then the limit function is also almost periodic. Next note that if we now introduce the reciprocal group of the discrete group of translations on the reciprocal lattice

\[
L^R := \{ b^R = b_i/\mu_i, \quad b_i \in \mathbb{Z} \},
\]

it follows immediately from (III.27) that

\[
\hat{f}(k_i) = \hat{f}(k_i + b_i/\mu_i),
\]

which confirms the statement below equation (III.24) regarding the fundamental domain of \( k_i \). In summary, we have seen that the space-space noncommutativity of the Heisenberg algebra can be expressed by a realization of the associated Heisenberg-Weyl group by a \( C^* \)-algebra \( \mathfrak{A} \subset \mathcal{B}(\mathcal{H}) \) of bounded unitary operators with unit, acting on a non-separable Hilbert space where an orthonormal basis is the set of almost periodic functions:

\[
\{ \xi^k(x(t)) = \hat{D}^k(x(t)) = e^{2i\pi x(t)k} \},
\]

given by the characters in (III.22).

### IV. QUANTUM COSMOLOGY FOR THE ANISOTROPIC BIANCHI I MODEL

As it is well known the classical action function, after ADM reduction to canonical form, for a Bianchi I cosmology describing a gravitational field, with space-time metric

\[
g_{\mu\nu} = \begin{pmatrix}
-N^2(t) & 0 & 0 & 0 \\
0 & a^2_1(t) & 0 & 0 \\
0 & 0 & a^2_2(t) & 0 \\
0 & 0 & 0 & a^2_3(t)
\end{pmatrix},
\]

minimally coupled to a massless scalar field \( \varphi(t) \) independent of the spatial coordinates, is given by

\[
S_{grav} + S_\varphi = \left( \frac{e}{G} \right) \int \left( \sum_{ij} \dot{g}_{ij} - \frac{N(t)}{\sqrt{g}} \right) \left[ \frac{1}{2} (\pi^{ij}_k)^2 + \pi^{ij} \pi^{ij} \right] d^4x
\]

\[
+ \hbar \int d^4x \left( p_\varphi \dot{\varphi} - \frac{1}{2} \frac{N(t)}{\sqrt{g}} p^2_\varphi \right),
\]

where (cf. Chapter 21 of [38]) the tensor densities \( \pi^{ij} \) are the canonical momenta conjugate to the metric components \( g_{ij} = a^2_i(t) \) (the square of the Universe radii), \( N(t) \) is the lapse function and \( p_\varphi \) is the canonical momentum conjugate to \( \varphi \), with \( p_\varphi \) being in units of length and \( \varphi \) in units of inverse of length. Moreover, writing the kinematic term in (IV.33) as \( \pi^{ij} \dot{g}_{ij} = 2\pi^{ii}a_i \dot{a}_i \) and making the definition \( 2\pi^{ii}a_i := \pi^i \) we can re-express the gravitational action in (IV.33) in the form

\[
S_{grav} = \frac{1}{2} \left( \frac{e}{G} \right) \int \left( \sum_{i=1}^{3} \dot{\pi}^i a_i - \frac{N(t)}{2\sqrt{3}g} \right) \left[ \frac{1}{2} \left( \sum_{i=1}^{3} \pi^2_i a_i \right) + \sum_{i=1}^{3} (\pi^2_i a^2_i \pi^i) \right] d^4x,
\]

or, observing next from equation (21.91) in [38] that \( \pi^{ij} \) is unitless and therefore that \( \pi^i \) has units of length, we can define a new quantity \( p^i := \frac{e}{G\hbar} \pi^i \), which has units of inverse of length, so (IV.34) can be written as

\[
S_{grav} = \frac{1}{2} \hbar \int \left( \sum_{i=1}^{3} \dot{p}^i a_i - \frac{G\hbar}{2\sqrt{3}e} \right) \left[ \frac{1}{2} \left( \sum_{i=1}^{3} b^2_i a_i \right) + \sum_{i=1}^{3} (b^2_i a^2_i b^i) \right] d^4x.
\]
In addition, the scalar field action can be re-expressed as:

$$S_\phi = \hbar \int d^4x \left( p_\phi \dot{\phi} - \frac{1}{2} \frac{N}{\sqrt{3}g} \left( \frac{Gh}{c^3} \right) \left( \frac{c^3}{Gh} \right) p_\phi^2 \right),$$  \hspace{1cm} (IV.36)

and defining

$$p_\phi := \left( \frac{c^3}{Gh} \right)^{\frac{1}{2}} p_\phi, \quad \text{and} \quad \dot{\phi} := \left( \frac{Gh}{c^3} \right)^{\frac{1}{2}} \dot{\phi},$$  \hspace{1cm} (IV.37)

where both $p_\phi$ and $\dot{\phi}$ are unitless, we arrive at

$$S_\phi = \hbar \int d^4x \left( p_\phi \dot{\phi} - \frac{1}{2} \frac{N}{\sqrt{3}g} \left( \frac{Gh}{c^3} \right) p_\phi^2 \right).$$  \hspace{1cm} (IV.38)

Consequently the total classical Hamiltonian constraint is [39, 40]:

$$C_{\text{grav}} + C_\phi = \frac{N(t)}{2} \left( \frac{Gh}{c^3} \right) \left[ -\frac{1}{2} \left( \sum_{i=1}^{3} p^i a_i \right)^2 + \sum_{i} (p^i a_i^2 p^i) \right] + \frac{1}{2} p_\phi^2 = 0.$$ \hspace{1cm} (IV.39)

If we choose the lapse function to be $N(t)(4(\sqrt{3}g))^{-\frac{1}{4}} = \left( \frac{c^3}{Gh} \right)$ and assume for simplicity the following ordering for the quantum Hamiltonian constraint operator, we therefore have:

$$\hat{C} = \hat{C}_{\text{grav}} + \hat{C}_\phi = \frac{1}{2} \left( -\sum_{i\neq j} p^i p^j a_i a_j + \sum_{i} p^i a_i^2 p^i \right) + \frac{1}{2} p_\phi^2 = 0.$$  \hspace{1cm} (IV.40)

Now, since the action of the $\hat{p}^i$ and $\hat{a}_i$ operators on our Hilbert space basis of kets is to be derived from the unitary operator representations discussed in the previous section and whose action on the Hilbert space is displayed in equations (II.8) and (II.11). For this purpose it is important to notice that the Hilbert space is constructed from the noncommutative group of operators $\mathfrak{A}$. Moreover, due to the noncommutativity, the elements of this group are not exponentials of self adjoint operators. To construct the observables $\hat{a}_i$ we thus take

$$\hat{a}_i := -\frac{\hat{U}_i - \hat{U}_i^\dagger}{2i\varepsilon_i},$$  \hspace{1cm} (IV.41)

so that

$$\hat{a}_i |x(n)\rangle = \frac{1}{2i\varepsilon_i} \left( e^{-2i\pi\varepsilon_i x_i} |x(n)\rangle + \frac{1}{2} \varepsilon_i \hat{e}_i \times \theta \right) - e^{2i\pi\varepsilon_i x_i} |x(n)\rangle + \frac{1}{2} \varepsilon_i \hat{e}_i \times \theta,$$  \hspace{1cm} (IV.42)

and

$$\hat{p}^i := \frac{V_i(\mu_i) - V_i^\dagger(\mu_i)}{2i\mu_i}.$$  \hspace{1cm} (IV.43)

so that

$$\hat{p}^i |x\rangle = \frac{1}{2i\mu_i} (|x + \mu_i \hat{e}_i\rangle - |x - \mu_i \hat{e}_i\rangle).$$  \hspace{1cm} (IV.44)

That (IV.41) reproduces the uncertainty principle for mean-square-deviations of the distributions $\langle \Psi | \hat{a}_i | \Psi \rangle$ and the noncommutative algebra of the $\hat{a}_i$ for the discrete case, can be seen by substituting (IV.41) in the commutator $[\hat{a}_i, \hat{a}_i]$ and making use of (II.8) and (II.9). We then find that

$$\langle j' | [\hat{a}_i, \hat{a}_i] | j \rangle = \left( \frac{2}{\varepsilon_i \hat{e}_i} \right) \sin(\pi \varepsilon_i \epsilon_i \cdot (\hat{e}_i \times \hat{e}_i)) \prod_{m=1}^{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\bar{k}_m e^{2\pi i \bar{k}_m (j' - j)} \cos \left( 2\pi \varepsilon_i \mu_i [j_i + (\frac{1}{2\mu_i}) k \cdot (\hat{e}_i \times \theta)] \right) \times \cos \left( 2\pi \varepsilon_i \mu_i [j_i + (\frac{1}{2\mu_i}) k \cdot (\hat{e}_i \times \theta)] \right) \text{ where } \bar{k}_m := \mu_m k_m.$$  \hspace{1cm} (IV.45)
from where it can be inferred that the quantity
\[
\left( \frac{2}{\varepsilon_i \varepsilon_l} \right) \sin(\pi \varepsilon_i \varepsilon_l \cdot \hat{e}_i \times \hat{e}_l) \cos \left( 2\pi \varepsilon_i \varepsilon_l \mu_i \frac{\hat{e}_l + (1/2\mu_i) \mathbf{k} \cdot \hat{e}_l}{\hat{e}_l \times \hat{e}_l} \right) \times \cos \left( 2\pi \varepsilon_i \varepsilon_l \mu_i \frac{\hat{e}_l + (1/2\mu_l) \mathbf{k} \cdot \hat{e}_l}{\hat{e}_l \times \hat{e}_l} \right)
\]
(IV.46)
is the symbol of the action of the operator commutator on the spectral representation of the product \( \langle j'|j \rangle \). In the limit \( \varepsilon_i \varepsilon_l \theta \cdot (\hat{e}_i \times \hat{e}_l) << 1 \) (since by (III.17) and (III.19) also implies \( \varepsilon_i \varepsilon_l << 1 \)), the above symbol of \([\hat{a}_i, \hat{a}_l]\) is
\[2\pi \theta \cdot (\hat{e}_i \times \hat{e}_l)\].

The expressions (IV.42), (IV.44), are to be substituted into (IV.40) in order to derive the action of the constraint operator on the Hilbert vectors \( |x_{(n)}\rangle \).

To make a detailed connection with other formulations we use the Feynman phase space path integral procedures considered in [32]. The general idea of the group averaging procedure (see e.g. [41]) is that the physical state \( |\Psi_{phys}\rangle \in \mathcal{H}_{phys} \), which is a solution of the constraint equation, is derived by averaging the action of the unitary monoparametric Abelian group \( \exp(i\alpha \hat{C}) \), \( \alpha \in \mathbb{R} \), on a state \( |\Psi_{kin}\rangle \) in an auxiliary kinematic Hilbert space \( \mathcal{H}_{kin} \) dense in \( \mathcal{H}_{phys} \). Thus
\[
|\Psi_{phys}\rangle = \int_{-\infty}^{\infty} d\alpha \exp(i\alpha \hat{C}) |\Psi_{kin}\rangle.
\]
(IV.47)

Heuristically (IV.47) can be justified as a refined algebraic quantization by observing that the integrand can be viewed as a Fourier Dirac delta representation:
\[
\int_{-\infty}^{\infty} d\alpha \exp(i\alpha \hat{C}) \sim \delta(\hat{C}),
\]
(IV.48)
and that by acting on (IV.47) with \( U(\beta) = \exp(i\beta \hat{C}) \) we have
\[
U(\beta)|\Psi_{phys}\rangle = \exp(i\beta \hat{C}) \delta(\hat{C}) |\Psi_{kin}\rangle = \delta(\hat{C}) |\Psi_{kin}\rangle = \int_{-\infty}^{\infty} d\alpha \exp(i\alpha \hat{C}) |\Psi_{kin}\rangle = |\Psi_{phys}\rangle.
\]
(IV.49)

Therefore the unitaries \( U(\beta) \) \( \forall \beta \) act trivially on the physical states defined as in (IV.47), consistent with Dirac’s requirement that physical states be annihilated by the constraints. However, the physical state defined by (IV.47) is not normalizable. Hence, in order to eliminate one of the deltas in the inner product, this is defined according to
\[
(\Phi_{phys}|\Psi_{phys}\rangle := \int_{-\infty}^{\infty} d\alpha \langle \Phi_{kin}|\exp(i\alpha \hat{C})|\Psi_{kin}\rangle.
\]
(IV.50)

Clearly this definition of the inner product has the advantage that it remains the same for any two other physical states of the form \( |\Psi'_{phys}\rangle = \exp(iu\hat{C})|\Phi_{phys}\rangle \).

Now, an orthonormal basis of kinematic quantum states are \( |x, \phi\rangle := |x\rangle|\phi\rangle \), where \( |x\rangle := |\mu_1 j_1, \mu_2 j_2, \mu_3 j_3\rangle \) and \( |\phi\rangle \) are the eigenvectors of the scalar field, such that
\[
\langle x', \phi'|x, \phi\rangle = \delta(x', x) \delta(\phi', \phi).
\]
(IV.51)

We can therefore write (IV.47) in this basis as
\[
(\mathbf{x}, \phi|\Psi_{phys}\rangle = \sum_{x'} \int d\phi' A(\mathbf{x}, \phi; x', \phi') |\Psi_{kin}(x', \phi')\rangle,
\]
(IV.52)
where the Kernel \( A(\mathbf{x}, \phi; x', \phi') \) is given by
\[
A(\mathbf{x}, \phi; x', \phi') = \int d\alpha \langle x, \phi|e^{i\alpha \hat{C}}|x', \phi'\rangle.
\]
(IV.53)
V. THE PATH INTEGRAL APPROACH

We shall follow here the path integral approach, based on [42] and developed for a timeless framework in [52], which consists essentially in replacing the transition function in Feynman’s formalism by the Kernel $A(\mathbf{x}_f, \phi_f; \mathbf{x}_I, \phi_I)$, where the subscripts $f$ and $I$ denote the final and initial states of the system, and regarding the constraint operator $\exp(\imath \alpha \hat{C})$ in (IV.53) in a purely mathematical sense as a Hamiltonian with evolution time equal to one. That is, $e^{\imath \alpha \hat{C}} = e^{\imath t \hat{H}}$ where $\hat{H} = \alpha \hat{C}$ and $t = 1$. Emulating now the standard Feynman construction, we decompose the fictitious evolution into $N$ infinitesimal evolutions of length $\lambda = \frac{1}{N+1}$. Thus we get

\[
\langle \mathbf{x}_f, \phi_f | e^{\imath \alpha \hat{C}} | \mathbf{x}_I, \phi_I \rangle = \sum_{\mathbf{x}_N, \ldots, \mathbf{x}_1} \int d\phi_N \ldots d\phi_1 \times \langle \mathbf{x}_{N+1}, \phi_{N+1} | e^{\imath \lambda \alpha \hat{C}} | \mathbf{x}_N, \phi_N \rangle \ldots \langle \mathbf{x}_1, \phi_1 | e^{\imath \lambda \alpha \hat{C}} | \mathbf{x}_0, \phi_0 \rangle,
\]

(V.54)

where $\langle \mathbf{x}_f, \phi_f \rangle \equiv \langle \mathbf{x}_{N+1}, \phi_{N+1} \rangle$ and $| \mathbf{x}_I, \phi_I \rangle \equiv | \mathbf{x}_0, \phi_0 \rangle$. If we now consider in detail the particular $n$-th term in (V.54) we can readily derive expressions for the remaining other terms. Thus, with $\hat{C}$ as given by (V.40) we get

\[
\langle \mathbf{x}_{n+1}, \phi_{n+1} | e^{\imath \lambda \alpha \hat{C}_{\text{grav}}} | \mathbf{x}_n, \phi_n \rangle = \langle \phi_{n+1} | e^{-\imath \lambda \alpha \hat{p}_n^2} | \phi_n \rangle \langle \mathbf{x}_{n+1} | e^{\imath \lambda \alpha \hat{C}_{\text{grav}}} | \mathbf{x}_n \rangle
\]

\[
= \left( \frac{1}{2\pi} \int dp_n e^{\imath \lambda \alpha \hat{p}_n^2} e^{\imath p_n (\phi_{n+1} - \phi_n)} \right) \langle \mathbf{x}_{n+1} | e^{\imath \lambda \alpha \hat{C}_{\text{grav}}} | \mathbf{x}_n \rangle.
\]

(V.55)

To evaluate the gravitational constraint factor above note that, to order one in $\lambda = \frac{1}{N+1}$ and for $N \gg 1$ we have

\[
\langle \mathbf{x}_{n+1} | e^{\imath \lambda \alpha \hat{C}_{\text{grav}}} | \mathbf{x}_n \rangle \approx \delta_{\mathbf{x}_{n+1}, \mathbf{x}_n} + i \lambda \alpha \langle \mathbf{x}_{n+1} | \hat{C}_{\text{grav}} | \mathbf{x}_n \rangle + O(\lambda^2).
\]

(V.56)

Making use of (V.42), (V.44), as well as of (13.8) – (13.11) we see that there are 16 terms conforming the transition function $\langle \mathbf{x}_{n+1} | \hat{C}_{\text{grav}} | \mathbf{x}_n \rangle$. These terms involve products of the unitaries and/or their conjugates. Let us consider in detail the term of the form

\[
\langle \mathbf{x}_{(n+1)} | \hat{V}_1 \hat{V}_2 \hat{U}_1 \hat{U}_2 | \mathbf{x}_{(n)} \rangle = e^{-\imath \pi \epsilon_i \epsilon_j (\hat{e}_i \times \hat{e}_j) \cdot \theta} e^{-\imath \pi (\epsilon_i e_j (x_{(n)}))} \langle \mathbf{x}_{(n+1)} - \mu_i \hat{e}_i - \mu_j \hat{e}_j | \mathbf{x}_{(n)} + \frac{1}{2} (\epsilon_i \hat{e}_i + \epsilon_j \hat{e}_j) \times \theta \rangle.
\]

(V.57)

Now, as pointed out in Sec.2 we have associated the action of the translation group on itself as leading to an affine space with a discrete topology and with a coset decomposition $T_3 = \sum_{j_1, j_2, j_3 = -\infty}^{\infty} (\mu_i j_i) \hat{e}_i$, where $j_i \in \mathbb{Z}$ and the $\hat{e}_i$ are the basic translations in $\mathbb{R}^3$. The vectors $\mathbf{x}_{(i)} = \sum_{i=1}^{3} (\mu_i j_i) \hat{e}_i \in T_3$ are elements of $\mathbb{R}^3$ as a group and the set $\Gamma : \{ \mu_i j_i \}$ form a 3-dimensional cell. This in turn led us (cf. eqn. (III.20)) to introduce a Kronecker inner product for the space of these vectors. Moreover, when using the GNS construction to derive the kinematic Hilbert space we were also led to require that the translations induced by the Unitary operators $\hat{U}_1$ and $\hat{V}_2$ should be related in order that the “riculations” induced by any of them should coincide. We suggested there that such a coincidence could be achieved by establishing the relations (11.17) and (11.19). This can now be verified directly by noting first that the arguments in the “bra” vectors in (V.57) are clearly integer multiples of the $\mu_i$ and so are the arguments of the “ket” vectors provided the following relations are satisfied:

\[
\frac{\hat{e}_i \cdot [(\epsilon_i \hat{e}_i + \epsilon_j \hat{e}_j) \times \theta]}{2 \mu_i} \in \mathbb{Z}.
\]

(V.58)

These requirements are indeed identically satisfied by the relations (11.17) and (11.19) for all the entries in the transition function in (V.56).

Consequently

\[
\langle \mathbf{x}_{(n+1)} | \hat{V}_1 \hat{V}_2 \hat{U}_1 \hat{U}_2 | \mathbf{x}_{(n)} \rangle = e^{-\imath \pi \epsilon_i \epsilon_j (\hat{e}_i \times \hat{e}_j) \cdot \theta} e^{-2\pi (\epsilon_i e_j (x_{(n)}))} \left( \prod_{l=1}^{3} \int \frac{dp_l}{2\pi} \mu_l dk_{(n)_l} \right) \times e^{-2\pi i \mu_l k_{(n)_l} (j_{(n+1)_l} - j_{(n)_l})} e^{2\pi i k_{(n)_l} (\mu_l \delta_{i_l + \mu_l \delta_{i_l} + \frac{1}{2} (\epsilon_i \hat{e}_i + \epsilon_j \hat{e}_j) \times \theta)},
\]

(V.59)
and making use of (V.41), (V.43) and (V.59) we find that

\[
\sum_{i \neq j} \langle x_{(n+1)} | \hat{P}^i \hat{P}^j \hat{a}_i \hat{a}_j | x_{(n)} \rangle = \frac{1}{2} \sum_{i < j} \cos[\pi \varepsilon_i \varepsilon_j (\hat{e}_i \times \hat{e}_j) \cdot \theta] \int \mu_1 dk_{(n)1} \mu_2 dk_{(n)2} \mu_3 dk_{(n)3} \times e^{-2\pi \sum_{l=1}^{3} \mu_l k_{(n)}(j_{(n)}l - j_{(n)}l)} \sin \left[ 2\pi \varepsilon_i \left( x_{(n)j} + \frac{3}{2} \sum_{l} k_{(n)l} \theta_{lj} \right) \right] (V.60)
\]

\[
\sin \left[ 2\pi \varepsilon_j \left( x_{(n)j} + \frac{1}{2} \sum_{l} k_{(n)l} \theta_{lj} \right) \right] \sin(2\pi k_{(n)j} \mu_i) \sin(2\pi k_{(n)j} \mu_j).
\]

We can now use (V.60) as a master equation to derive the two terms of the gravitational constraint in (V.60). The resulting expression is

\[
\langle x_{(n+1)} | \hat{C}_{\text{grav}} | x_{(n)} \rangle = \prod_{l=1}^{3} \int \mu_l dk_{(n)l} e^{-2\pi ik_{(n)l}(x_{(n+1)l} - x_{(n)l})} \times \left\{ \frac{1}{4} \sum_{i=1}^{3} \frac{1}{\varepsilon_i^2 \mu_i^2} \sin^2 \left[ 2\pi \varepsilon_i \left( x_{(n)l} + \frac{3}{2} \sum_{l} k_{(n)l} \theta_{li} \right) \right] \sin^2(2\pi k_{(n)l} \mu_i) \right. \]

\[
- \frac{1}{2} \sum_{i < j} \cos[2\pi \varepsilon_i \varepsilon_j \theta \cdot (\hat{e}_i \times \hat{e}_j)] \frac{1}{\varepsilon_i} \sin \left[ 2\pi \varepsilon_i \left( x_{(n)l} + \frac{3}{2} \sum_{l} k_{(n)l} \theta_{lj} \right) \right] \times \frac{1}{\varepsilon_j} \sin \left[ 2\pi \varepsilon_j \left( x_{(n)j} + \frac{1}{2} \sum_{l} k_{(n)l} \theta_{lj} \right) \right] \left\{ \frac{1}{\mu_i} \sin(2\pi k_{(n)l} \mu_i) \frac{1}{\mu_j} \sin(2\pi k_{(n)j} \mu_j) \right\} (V.61)
\]

Inserting now (V.61) into (V.56) and exponentiating, we have

\[
\langle x_{(n+1)} | e^{i\lambda \hat{C}_{\text{grav}}} | x_{(n)} \rangle = \prod_{l=1}^{3} \int_{-1/2\mu_l}^{1/2\mu_l} \mu_l dk_{(n)l} e^{-2\pi ik_{(n+1)l}(x_{(n+1)l} - x_{(n)l})} e^{i\lambda \alpha C_g(k_{(n+1)}, x_{(n+1)}; x_{(n)}, x_{(n)})} (V.62)
\]

where $C_g(k_{(n+1)}, x_{(n+1)}, x_{(n)})$ is the infinitesimal spectral contribution of the gravitational part of the constraint, given by the terms inside the braces in (V.61).

Hence, substituting each of the corresponding infinitesimal amplitude terms in (V.62) into the gravitational part of (V.59) yields

\[
\langle x_f | e^{i\alpha \hat{C}_g} | x_f \rangle = \prod_{l=1}^{3} \sum_{j_{N+1}, j_{N+1} = -\infty}^{\infty} \prod_{n=0}^{N} \int_{-1/2\mu_l}^{1/2\mu_l} \mu_l dk_{(n+1)l} e^{-2\pi ik_{(n+1)l}(j_{(n+1)l} - j_{(n)l})} e^{i\lambda \alpha C_g(k_{(n+1)}, x_{(n+1)}; x_{(n)}, x_{(n)})} (V.63)
\]

Now, in order to arrive at an expression involving a proper continuous path integral, we follow the procedure described in [42] and consider first the amplitude (V.63) for the case of no constraint. We then have

\[
\langle x_f | x_f \rangle_0 := \prod_{l=1}^{3} \left[ \sum_{j_{N+1}, j_{N+1} = -\infty}^{\infty} \prod_{n=0}^{N} \int_{-1/2\mu_l}^{1/2\mu_l} dk_{(n+1)l} \right] e^{-2\pi i \sum_{n=0}^{N} k_{(n+1)l}(j_{(n+1)l} - j_{(n)l})} (V.64)
\]

where we have absorbed the $\mu_l$'s in the integrations by redefining $\tilde{k}_{(n+1)l} := \mu_l k_{(n+1)l}$.

Note that the summation in the exponential in (V.64) can be reordered as follows:

\[
\sum_{n=0}^{N} \sum_{l=1}^{3} k_{(n+1)l}(j_{(n+1)l} - j_{(n)l}) = \sum_{l=1}^{3} \left[ \tilde{k}_{(N+1)l}(\bar{j}_{(f)l} - \bar{k}_{(1)l}) - \sum_{n=1}^{N} j_{(n)l}(\bar{k}_{(n+1)l} - \bar{k}_{(n)l}) \right] (V.65)
\]
Substituting this expression back into (V.64) and using the Poisson formula, we arrive at

\[
\langle x_f | x_l \rangle_0 := \prod_{l=1}^{3} \prod_{n=0}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}(n+1)_l \left[ e^{-2\pi i \delta(\bar{k}(n+1)_l - \bar{k}(n)_l + m(n)_l)} \prod_{n=1}^{N} \sum_{m(n)_l = -\infty}^{\infty} \delta(\bar{k}(n+1)_l - \bar{k}(n)_l + m(n)_l) \right],
\]

\[m(n)_l \in \mathbb{Z}. \quad (V.66)\]

Using now the Fourier integral representation of the Dirac delta function we alternatively can write

\[
\langle x_f | x_l \rangle_0 = \prod_{l=1}^{3} \prod_{n=0}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}(n+1)_l e^{-2\pi i \delta(\bar{k}(n+1)_l - \bar{k}(n)_l)}
\times \prod_{n=1}^{N} \left[ \int_{-\infty}^{\infty} d\bar{q}(n)_l \right] \sum_{m(n)_l = -\infty}^{\infty} \left( e^{-2\pi i \sum_{n=1}^{N} \bar{q}(n)_l(\bar{k}(n+1)_l - \bar{k}(n)_l + m(n)_l)} \right), \quad (V.67)
\]

where the unitless \(\bar{q}(n)_l \in \mathbb{R}\). Noting that the integers \(-\infty \leq m(n)_l \leq \infty\) in the sum in the above exponential can be absorbed into the variables \(\bar{k}(n)_l\) for \(1 \leq n \leq N\) so their range of integration is extended to \((-\infty, \infty)\), we therefore can write

\[
\langle x_f | x_l \rangle_0 = \prod_{l=1}^{3} \prod_{n=0}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}(n+1)_l e^{-2\pi i \delta(\bar{k}(n+1)_l - \bar{k}(n)_l)}
\times \prod_{n=1}^{N} \left[ \int_{-\infty}^{\infty} d\bar{q}(n)_l \right] \sum_{m(n)_l = -\infty}^{\infty} \left( e^{-2\pi i \sum_{n=1}^{N} \bar{q}(n)_l(\bar{k}(n+1)_l - \bar{k}(n)_l + m(n)_l)} \right), \quad (V.68)
\]

Rearranging once more the summation in the exponential above, we obtain

\[
\langle x_f | x_l \rangle_0 = \prod_{l=1}^{3} \prod_{n=0}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}(n+1)_l \prod_{n=1}^{N} \left[ \int_{-\infty}^{\infty} d\bar{k}(n)_l \right] \int_{-\infty}^{\infty} d\bar{q}(n)_l e^{-2\pi i \sum_{n=1}^{N} \bar{q}(n)_l(\bar{k}(n+1)_l - \bar{k}(n)_l)} \quad (V.69)
\]

after denoting the end-points as \(\bar{q}(N+1)_l := \bar{j}(f)_l\) and \(\bar{q}(0)_l := \bar{j}(f)_l\).

Comparing now the amplitude (V.69) with (V.63), we note that the sum over the discrete variables \(j(n)_l \in \mathbb{Z}\) in (V.63) is replaced by the continuous \(\bar{q}(n)_l \in \mathbb{R}\) in (V.69). Therefore we can introduce in the summation of the exponential in (V.69) the symbol (the term inside the braces of (V.61)) of the constraint operator \(C_g\) acting on the spectral representation of the infinitesimals \(\langle x_{n+1}_l | x_0 \rangle_0\), after replacing the \(j(n)_l\) discrete variables by the \(\bar{q}(n)_l\) continuous ones. Thus

\[
\langle x_f | e^{i\alpha g} | x_l \rangle = \prod_{l=1}^{3} \prod_{n=0}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}(n+1)_l \prod_{n=1}^{N} \left[ \int_{-\infty}^{\infty} d\bar{k}(n)_l \right] \int_{-\infty}^{\infty} d\bar{q}(n)_l e^{-i \sum_{n=1}^{N} [2\pi \bar{q}(n)_l(\bar{k}(n+1)_l - \bar{k}(n)_l)] - \alpha g(\bar{q}(n+1)_l - \bar{q}(n)_l)}
\times e^{i \alpha g(\bar{k}(n+1)_l - \bar{k}(n)_l)}
\]

\[\times e^{-i \sum_{n=0}^{N} \frac{1}{2}(\bar{k}(n+1)_l - \bar{k}(n)_l)^2} \prod_{n=1}^{N} \int d\phi(n)_l \prod_{n=0}^{N} \int dp \phi(n)_l e^{-i S_N}, \quad (V.70)\]

Making next use of the above expression in the evaluation of (V.54) and (V.55) yields

\[
\langle x_f, \phi_f | e^{i\alpha g} | x_l, \phi_l \rangle = \prod_{l=1}^{3} \prod_{n=0}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}(n+1)_l \prod_{n=1}^{N} \left[ \int_{-\infty}^{\infty} d\bar{k}(n)_l \right] \int_{-\infty}^{\infty} d\bar{q}(n)_l \int_{-\infty}^{\infty} d\bar{p}(n)_l \int_{-\infty}^{\infty} d\phi(n)_l \prod_{n=0}^{N} \int d\phi(n)_l \prod_{n=0}^{N} \int dp \phi(n)_l e^{-i S_N}, \quad (V.71)
\]

with

\[
S_N = -\lambda \sum_{n=0}^{N} \left[ \phi(n+1)_l - \phi(n)_l \right]^{-2\pi} \sum_{l=1}^{3} \bar{k}(n)_l \left( \bar{q}(n+1)_l - \bar{q}(n)_l \right) + \alpha \left( \frac{1}{2} \bar{p}^2 \phi(n)_l + C_g(\bar{k}(n+1)_l, \bar{q}(n)_l, \mu, \varepsilon) \right). \quad (V.72)
\]
The last step in the path integral procedure consists in letting \( \lambda = \Delta \tau \) so that (V.72) reads
\[
S_N = \sum_{n=0}^{N} \Delta \tau \left[ -p_{\phi(n)} \left( \frac{\phi(n+1) - \phi(n)}{\Delta \tau} \right) + 2\pi \sum_{l=1}^{3} \vec{k}(n)_l \left( \frac{\vec{q}(n+1)_l - \vec{q}(n)_l}{\Delta \tau} \right) - \alpha \left( \frac{1}{2}p_{\phi(n)}^2 + C_g(k(n+1)_l, \vec{q}(n)_l, \mu, \varepsilon) \right) \right].
\] (V.73)

Further taking the limit \( N \to \infty \)
\[
S := \lim_{N \to \infty} S_N = \int_{\tau=0}^{\tau=1} d\tau \left[ -p_{\phi} \dot{\phi} + 2\pi \vec{k}(\phi) \cdot \vec{q}(\phi) - \alpha \left( \frac{1}{2}p_{\phi}^2 + C_g(\vec{k}(\phi), \vec{q}(\phi), \mu, \varepsilon) \right) \right]
\] (V.74)
and varying \( p_{\phi} \) results in the equation of motion \( \dot{\phi} = -\alpha p_{\phi} \). Write now
\[
d\tau = d\phi \left( \frac{d\tau}{d\phi} \right) = \frac{d\phi}{\phi},
\] (V.75)
so that
\[
S = \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left[ 2\pi \vec{k}(\phi) \cdot \dot{\vec{q}}(\phi) - \frac{1}{2} \left( \frac{\alpha}{\phi} \right) \left( \frac{1}{2}p_{\phi}^2 + C_g(\vec{k}(\phi), \vec{q}(\phi), \mu, \varepsilon) \right) \right]
\] (V.76)
where from here on “dot” means differentiation with respect to the internal time \( \phi \). With this reparametrization the term in the square brackets in the second equality above is the Hamiltonian of the system, so (V.76) can be written as
\[
S = \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left[ 2\pi \vec{k}(\phi) \cdot \dot{\vec{q}}(\phi) - H \right],
\] (V.77)
where
\[
H = \frac{p_{\phi}^2}{2} - \left( \frac{1}{p_{\phi}} \right) C_g(\vec{k}(\phi), \vec{q}(\phi), \mu, \varepsilon) = E,
\] (V.78)
and the energy \( E \) is a constant of motion. By combining the above different contributions to the action the explicit form of this Hamiltonian is given by
\[
H = \left( \frac{1}{p_{\phi}} \right) \left[ \frac{p_{\phi}^2}{2} + \frac{1}{4} \sum_{i=1}^{3} \frac{1}{\varepsilon_i \mu_i} \sin^2 \left( \frac{1}{2} \varepsilon_i \mu_i \frac{3}{2} \sum_{l=1}^{3} \theta_{ij} \vec{k}_l \right) \sin^2(2\pi \vec{k}_i) \right.
\]
\[
- \frac{1}{2} \left\{ \sum_{i,j=1 \atop i<j}^{3} \cos \left[ 2\pi \varepsilon_i \varepsilon_j \theta \cdot (\vec{\varepsilon}_i \times \vec{\varepsilon}_j) \right] \frac{1}{\varepsilon_i \mu_i} \sin \left[ 2\pi \varepsilon_i \mu_i \left( \vec{q}_i(\phi) - \frac{1}{2} \sum_{l=1}^{3} \theta_{ij} \vec{k}_l \right) \right] \sin(2\pi \vec{k}_i) \right. 
\]
\[
\left. \times \frac{1}{\varepsilon_j \mu_j} \sin \left[ 2\pi \varepsilon_j \mu_j \left( \vec{q}_j(\phi) - \frac{1}{2} \sum_{l=1}^{3} \theta_{ij} \vec{k}_l \right) \right] \sin(2\pi \vec{k}_j) \right\}.
\] (V.79)

In order to get a further physical insight on the terms in (V.79), consider the expectation value of the operator \( \hat{a}_i \) as defined in (IV.41):
\[
\langle \Psi | \hat{a}_i | \Psi \rangle = -\frac{1}{2i \varepsilon_i} \langle \Psi | U_i - U^*_i | \Psi \rangle = -\frac{1}{2i \varepsilon_i} \sum_{j_1,j_2,j_3} \langle \Psi | U_i - U^*_i | x \rangle \langle x | \Psi \rangle
\]
\[
= -\frac{1}{2i \varepsilon_i} \sum_{j_1,j_2,j_3} e^{-2\pi i \varepsilon_i, x} \Psi^* \left( x + \frac{1}{2} \varepsilon_i \vec{\varepsilon}_i \times \theta \right) - e^{2\pi i \varepsilon_i, x} \Psi^* \left( x - \frac{1}{2} \varepsilon_i \vec{\varepsilon}_i \times \theta \right) \Psi(x)
\] (V.80)
and approximated by its argument, it is natural to identify the dimension less quantities $\bar{\alpha}$ since noncommutativity is dominant at distances of the order of a Planck length where the sine function can be well approximated by its argument.

Comparing (V.82) with (V.84) we see that we can identify the function $a_i$ as the symbol of $\hat{a}_i$ (cf. III.28).

Recalling now (III.28) that

$$\Psi(x) = \prod_{l=1}^{3} \int \frac{dk_l}{2\pi} \Phi(k_l) e^{-2\pi i k_l \mu_l j_l},$$

(V.81)

and substituting into (V.80), we get

$$\langle \Psi | \hat{a}_i | \Psi \rangle = \frac{1}{\xi_i} \sum_{j_1, j_2, j_3} \int d^3 k' \int d^3 k \Phi^*(k')\Phi(k) e^{-2\pi i \sum_{j_1, j_2, j_3} \mu_{j_1} j_{2} (k_1 - k'_1)}$$

$$\times \sin \left[ 2\pi \xi_i \mu_i \left( j_i + \frac{k \cdot (\xi_i \times \theta)}{2\mu_i} \right) \right].$$

(V.82)

Consider now the scalar

$$\langle \Psi | \Psi \rangle = \sum_{j_1, j_2, j_3} \langle \Psi | x \rangle \langle x | \Psi \rangle, \quad x = \sum_{l=1}^{3} \mu_j \hat{e}_i,$$

(V.83)

which, making again use of (V.81) and the Poisson sum formula results in the spectral decomposition

$$\langle \Psi | \Psi \rangle = \int d^3 k' \int d^3 k \Phi^*(k')\Phi(k) \int_{-\infty}^{\infty} d^3 q \ e^{-2\pi i \sum_{j_1, j_2, j_3} \mu_{j_1} j_{2} (k_1 - k'_1)}.$$  

(V.84)

Comparing (V.82) with (V.84) we see that we can identify the function

$$(a_i)_{\text{symbol}} := \frac{1}{\xi_i} \sin \left[ 2\pi \xi_i \mu_i \left( \hat{q}_i + \frac{k \cdot (\xi_i \times \theta)}{2\mu_i} \right) \right]$$

(V.85)

as the symbol of $\hat{a}_i$ acting on the spectral representation of $\langle \Psi | \Psi \rangle$, with $j_i = x_i / \mu_i$ going to the continuum limit $j_i \rightarrow \hat{q}_i$. Hence we can infer from (V.79) that this same function is the symbol of $\hat{a}_i(\phi)$. In particular, note that since noncommutativity is dominant at distances of the order of a Planck length where the sine function can be well approximated by its argument, it is natural to identify the dimensionless quantities $\bar{\alpha}$ and $\bar{\beta}$ which satisfy the twisted Poisson bracket algebra $\{\bar{Q}_i, \bar{Q}_j\} = (2\pi)^{-1} \frac{\mu_j}{\mu_i} \delta_{ij}$ and $\{\bar{Q}_i, \bar{k}_j\} = \frac{1}{2\pi} \delta_{ij}$, in the effective Hamiltonian of the path integral formulation. Moreover, recalling that $\bar{Q}_i = \mu_i \bar{Q}_i$ and $\bar{k}_j = \mu_j \bar{k}_j$ we have that the above expressions when appropriately dimensioned as dynamical coordinates of the trajectories and their respective canonical conjugate momenta, become

$$\{Q_i, Q_j\} = (2\pi)^{-1} \theta_{ij} \quad \text{and} \quad \{Q_i, k_j\} = \frac{1}{2\pi} \delta_{ij},$$

(V.87)

which coincide with their Poisson brackets given by a Moyal $\star$-product algebra.

Making next use of these variables and defining

$$\chi_i := \frac{1}{\xi_i \mu_i} \sin(2\pi \xi_i \mu_i \bar{Q}_i) \sin(2\pi \bar{k}_i),$$

(V.88)

and

$$\alpha := \cos[2\pi \xi_1 \xi_2 \theta \cdot (\bar{e}_1 \times \bar{e}_2)]$$

$$\beta := \cos[2\pi \xi_1 \xi_3 \theta \cdot (\bar{e}_1 \times \bar{e}_3)]$$

$$\gamma := \cos[2\pi \xi_2 \xi_3 \theta \cdot (\bar{e}_2 \times \bar{e}_3)],$$

(V.89)
we can rewrite (V.79) as

\[ H = \left( \frac{1}{p_\phi} \right) \left[ \frac{1}{2} p_\phi^2 + \frac{1}{4} \left\{ \chi_1 (\chi_1 - \alpha \chi_2 - \beta \chi_3) + \chi_2 (\chi_2 - \alpha \chi_1 - \gamma \chi_3) + \chi_3 (\chi_3 - \beta \chi_1 - \gamma \chi_2) \right\} \right] = E. \] (V.90)

Furthermore, if we now implement the Hamiltonian constraint strongly, that is to say \( \left( \frac{1}{2} p_\phi^2 + C_g(\bar{k}(\phi), \bar{q}(\phi), \mu, \varepsilon) \right) = 0 \), we have from (V.78) that \( E = p_\phi \). Hence

\[ \frac{p_\phi^2}{2} - C_g(\bar{k}(\phi), \bar{q}(\phi), \mu, \varepsilon) = E p_\phi = p_\phi^2 \] (V.91)

and

\[ \left[ \frac{1}{2} p_\phi^2 + \frac{1}{4} \left\{ \chi_1 (\chi_1 - \alpha \chi_2 - \beta \chi_3) + \chi_2 (\chi_2 - \alpha \chi_1 - \gamma \chi_3) + \chi_3 (\chi_3 - \beta \chi_1 - \gamma \chi_2) \right\} \right] = 0. \] (V.92)

VI. ASYMPTOTICS FOR THE NONCOMMUTATIVE DYNAMICS

The dynamics of our system is given in the stationary phase approximation by the solution of the equations:

\[ \dot{k}_i = -\frac{1}{2p_\phi} \cos(2\pi \varepsilon_i \mu_i \bar{Q}_i) \sin(2\pi \bar{k}_i) R_i, \quad i = 1, 2, 3 \] (VI.93)

where

\[ R_1 := (\chi_1 - \alpha \chi_2 - \beta \chi_3), \quad R_2 := (\chi_2 - \alpha \chi_1 - \gamma \chi_3), \quad R_3 := (\chi_3 - \beta \chi_1 - \gamma \chi_2). \] (VI.94)

\[ \dot{Q}_i = \left( \frac{1}{p_\phi} \right) \left( \frac{1}{2\varepsilon_i \mu_i} \sin(2\pi \varepsilon_i \mu_i \bar{Q}_i) \cos(2\pi \bar{k}_i) R_i - \sum_{j \neq i} \frac{\theta_{ij}}{\mu_i \mu_j} \bar{k}_j \right). \] (VI.95)

Now, to be able to assert the dynamical behavior of the observables \( \bar{Q}_i \) and \( \bar{k}_i \), let us first make use of (V.88) to derive explicitly the time derivative of \( \bar{k}_i \). We get

\[ \dot{\bar{k}}_i = \left( \frac{1}{2\pi} \right) \frac{d}{d\phi} \left( \frac{\varepsilon_i \mu_i \chi_i}{\sin(2\pi \varepsilon_i \mu_i \bar{Q}_i)} \right) \left[ 1 - \left( \frac{\varepsilon_i \mu_i \chi_i}{\sin(2\pi \varepsilon_i \mu_i \bar{Q}_i)} \right)^2 \right]^{-1/2}, \quad i = 1, 2, 3. \] (VI.96)

Substituting (VI.93) into the left hand side of (VI.96) results in

\[ \left( \frac{\pi}{p_\phi} \right) \cos(2\pi \varepsilon_i \mu_i \bar{Q}_i) R_i = \frac{d}{d\phi} \cosh^{-1} \left( \frac{\sin(2\pi \varepsilon_i \mu_i \bar{Q}_i)}{\varepsilon_i \mu_i \chi_i} \right), \quad i = 1, 2, 3 \] (VI.97)

and by integrating yields

\[ \sin \left( 2\pi \varepsilon_i \mu_i \bar{Q}_i \right) = \varepsilon_i \mu_i \chi_i \cosh \left[ \frac{\pi}{p_\phi} \int_{\phi(\tau)}^{\phi(i)} d\phi \cos \left( 2\pi \varepsilon_i \mu_i \bar{Q}_i \right) R_i + B_i \right], \quad i = 1, 2, 3 \] (VI.98)

where \( \phi(I) \) is the inner-time at the boundary conditions, the constant of integration \( B_i \) is the evaluation

\[ B_i = \cosh^{-1} \left( \frac{\sin \left( 2\pi \varepsilon_i \mu_i \bar{Q}_i \right)}{\varepsilon_i \mu_i \chi_i} \right) \bigg|_{\phi(I)}, \] (VI.99)

and the sign of the left hand side of (VI.98) has to be taken consistent with the sign of the \( \chi_i \) on the right hand side. As we show in the paragraph following equation (VI.107) the \( \chi_i \) can be taken consistently to be positive for all times,
thus it follows from (VI.98) that the symbol of $\hat{a}_i$ acting on the spectral representation of $\langle \Psi | \Psi \rangle$ has to satisfy the inequality

$$\left| \frac{\sin (2\pi \varepsilon \mu_i \bar{Q}_i)}{\varepsilon_i} \right| \geq \mu_i \chi_i, \quad (VI.100)$$

as it is also evident from (V.88).

Next, in order to derive the time evolution of the $\hat{k}_i$’s we make use of (VI.93) to write

$$\frac{\dot{k}_i}{\sin (2\pi k_i)} = - \left( \frac{1}{2p_\phi} \right) \cos (2\pi \varepsilon \mu_i \bar{Q}_i) R_i \quad (VI.101)$$

which integrates (for $i=1,2,3$) to

$$\tan (\pi \bar{k}_i (\phi (\tau))) = \tan (\pi \bar{k}_i (\phi (B))) \left[ \exp \left[ - \frac{\pi}{p_\phi} \int_{\phi (I)}^{\phi (\tau)} d\phi \cos (2\pi \varepsilon \mu_i \bar{Q}_i) R_i \right] \right]. \quad (VI.102)$$

To complete this stage of our analysis we need to consider the dynamical evolution of the $\chi_i$’s into which the Hamiltonian constraint is decomposed. Note, by the way, that these quantities turn out to be constants of the motion in the limit of zero noncommutative symbol. Let us then multiply both sides of (VI.95) by $\cot (2\pi \varepsilon \mu_i \bar{Q}_i)$. We get

$$2\pi \varepsilon \mu_i \cot (2\pi \varepsilon \mu_i \bar{Q}_i) \dot{\bar{Q}}_i = \frac{\pi}{p_\phi} \cos (2\pi \varepsilon \mu_i \bar{Q}_i) \cos (2\pi \bar{k}_i) R_i -$$

$$- \left( \frac{2\pi}{p_\phi} \right) \varepsilon_i \cot (2\pi \varepsilon \mu_i \bar{Q}_i) \sum_{j \neq i} \frac{\theta_{ij}}{\mu_j} \bar{k}_j, \quad (VI.103)$$

which can be re-expressed as

$$\frac{d}{d\phi} \ln (\sin (2\pi \varepsilon \mu_i \bar{Q}_i)) = -2\pi \cot (2\pi \bar{k}_i) \dot{k}_i - (2\pi) \sum_{j \neq i} \frac{\theta_{ij}}{\mu_j} \varepsilon_i \cot (2\pi \varepsilon \mu_i \bar{Q}_i) \dot{k}_j, \quad (VI.104)$$

or, passing the first term on the right above as a differential to the left and making use of (V.88) and (VI.93), as

$$\frac{d}{d\phi} \ln (\varepsilon_i \mu_i \chi_i) = \pi \sum_{j \neq i} \varepsilon_i \varepsilon_j \theta_{ij} \chi_j R_j \cot (2\pi \varepsilon \mu_i \bar{Q}_i) \cot (2\pi \varepsilon \mu_j \bar{Q}_j). \quad (VI.105)$$

Multiplying both sides of (VI.105) by $\chi_i R_i$ for $i = 1, 2, 3$ we can eliminate the terms on the right by adding the resulting three equations. Thus we get

$$R_1 \dot{\chi}_1 + R_2 \dot{\chi}_2 + R_3 \dot{\chi}_3 = 0. \quad (VI.106)$$

As a check of consistency note that this result equally follows from differentiating (V.92) with respect to the inner time, since it is easy to show that

$$\frac{d}{d\phi} \left( p_\phi^2 = -\frac{1}{2} (\chi_1 R_1 + \chi_2 R_2 + \chi_3 R_3) \right) \Rightarrow R_1 \dot{\chi}_1 + R_2 \dot{\chi}_2 + R_3 \dot{\chi}_3 = 0. \quad (VI.107)$$

The above makes only sense provided the signs of the $\chi_i$’s in (V.92) and therefore inside the parenthesis in (VI.107) are such that the equation makes sense. To establish this we note that since $p_\phi$ is a constant of the motion and evidently can not be chosen as zero, we are then required that $\frac{1}{2} (\chi_1 R_1 + \chi_2 R_2 + \chi_3 R_3)$ be negative definite at any time $\phi$. It is easy to verify that this implies that none of the $\chi_i$’s can be zero at any time. Indeed, assume that $\chi_1 = 0$, then $p_\phi^2 = -\frac{1}{2} \left[ (\chi_2 - \gamma \chi_3)^2 + \chi_3^2 (1 - \gamma^2) \right]$, which is clearly impossible unless $\chi_2$ and $\chi_3$ are imaginary which is evidently not
Moreover, so far the quantities $\varepsilon_i$ in (V.88) are noncommutative and can not be used as simultaneous observables and also because in the limit of commutativity we have that

$$\chi_i(\phi(B)) = \chi_i(\phi(l)) \exp \left[ \pi \sum_{j \neq i} \varepsilon_i \varepsilon_j \theta_{ij} \int_{\phi(l)}^{\phi(B)} \chi_j R_j \cot(2\pi \varepsilon_i \mu_i \hat{Q}_i) \cot(2\pi \varepsilon_j \mu_j \hat{Q}_j) \, d\phi \right],$$  

(VI.108)

which are therefore always positive and can never reach zero according to our previous considerations.

Next, based on the developments in Sec.V leading to equation (V.83) for the symbols of the operators $\hat{a}_i$, we can define the volume of the Bianchi I Universe as the product of these symbols, i.e., as:

$$V_{\text{symb}} = \prod_{i=1}^{3} (a_i)_{\text{symb}} = \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \left[ \sin(2\pi \varepsilon_1 \mu_1 \hat{Q}_1) \sin(2\pi \varepsilon_2 \mu_2 \hat{Q}_2) \sin(2\pi \varepsilon_3 \mu_3 \hat{Q}_3) \right].$$  

(VI.109)

That this definition is reasonable follows from the fact that the $\hat{a}_i$ are noncommutative and can not be used as simultaneous observables and also because in the limit of commutativity we have that

$$\lim_{\varepsilon \to 0} (V_{\text{symb}}) = \prod_{i=1}^{3} (2\pi \mu_i \hat{Q}_i).$$  

(VI.110)

Moreover, so far the quantities $\varepsilon_i$, $\mu_i$ were introduced in the $C^*$-algebra discussed in Section II in order to account primarily for the proper dimensions in equations (II.6)-(II.11) describing its realization, we can go one step further in our analysis by interpreting $\varepsilon_i$ and $\mu_i$ as scale parameters describing the different stages of evolution of the dynamical system. We shall now express them as scale factors by writing

$$\varepsilon_i = \frac{\bar{\varepsilon}_i}{L_i},$$  

(VI.111)

where $\bar{\varepsilon}_i$ is a constant and $L_i$ is in units of length and magnitude depending on the corresponding scale at which the evolving universe is considered. Correspondingly, since at a scale where noncommutativity is expected to be dominant the $\varepsilon_i$ and the $\mu_i$ are related by equations (III.17) and (III.19), we will have that

$$n_j \varepsilon_i \mu_i = n_i \varepsilon_j \mu_j, \quad i \neq j$$  

(VI.112)

and

$$\mu_1 = \frac{n_1}{2} \frac{\bar{\varepsilon}_2}{L_2} \lambda_3^2 \theta_3, \quad \mu_2 = \frac{n_2}{2} \frac{\bar{\varepsilon}_1}{L_1} \lambda_3^2 \theta_3, \quad \mu_2 = \frac{n_3}{2} \frac{\bar{\varepsilon}_1}{L_1} \lambda_3^2 \theta_2,$$

(VI.113)

(and consistent with our previous notation bared quantities are dimensionless throughout). Thus, in particular, we find that

$$\varepsilon_1 \mu_1 = \frac{n_1}{2} \frac{\bar{\varepsilon}_1 \bar{\varepsilon}_2}{L_1 L_2} \lambda_3^2 \theta_3.$$  

(VI.114)

Noting now that at the Planck length scale the area in the plane perpendicular to the vector $\hat{e}_3$ is related to the symbol of the commutator $[\hat{a}_1, \hat{a}_2]$ we see that when substituting (VI.114) into (IV.46) that

$$(s_3)_0 \approx 2\pi \theta \cdot (\hat{e}_1 \times \hat{e}_2),$$  

(VI.115)
and similarly for the two other planes we have
\[(s_2)_0 \approx 2\pi \theta \cdot (\hat{e}_3 \times \hat{e}_1), \quad (s_1)_0 \approx 2\pi \theta \cdot (\hat{e}_2 \times \hat{e}_3),\] (VI.116)
so that the magnitude of the minimal area of the Bianchi I universe is determined by the noncommutativity and is proportional to the square of the Planck length in magnitude value, similar to expressions obtained by other approaches in different contexts.

One more indicator on the actual values to be assigned to the scale factors \(L_i\) in (VI.111) can be derived from the conceptually expected noncommutativity of the algebras describing physical processes occurring at distances of the order of the Planck length. In mathematical terms this would be equivalent to express the range of validity of the noncommutativity in our equations by introducing a smooth cutoff function in the \(\varepsilon_i\) of (VI.111) with compact support when the universe conforms a region of radial dimensions of the order of Planck lengths. To this end we make use of Theorem 1.4.1 in [43], which shows that a test function \(\psi_i \in C_0^\infty(X)\) of compact support, in an open set in \(\mathbb{R}^3\), can be found with \(0 \leq \psi_i \leq 1\) so that \(\psi_i = 1\) in a neighborhood of a compact subset \(K\) of \(X\). The regularization \(\psi_i\) of \(\varepsilon_i\) is thus obtained by the convolution
\[
\psi_i := \chi_{K_{2\rho}} \ast \varphi_\rho \in C_0^\infty(K_{3\rho}),
\] (VI.117)
where \(\chi_{K_{2\rho}}\) is the characteristic function of
\[
K_{2\rho} := \{y, |x - y| \leq 2\rho, \text{for some } x \in K\},
\] (VI.118)
and \(\varphi_\rho\) is the mollifier
\[
\varphi_\rho(y) = \frac{1}{\rho^3} \exp \left[ -\frac{1}{(1 - \frac{|y|^2}{\rho^2})} \right].
\] (VI.119)
It therefore follows from (VI.117) and (VI.118) that for radii of the order of \(10\lambda_P\) noncommutativity will be supported in a ball of radius \(30\lambda_P\), so we can identify \(\bar{\varepsilon}_i\) with \(\varepsilon_i\), which is equal to one inside the ball and zero outside, and use \(L_i \approx 30\lambda_P\) for the effective regularization cutoff of the noncommutativity terms in our evolution equations; i.e.
\[
\bar{\varepsilon}_i = \psi_i = \int_{B_{L_i}} dy \delta(y - y_0) = \begin{cases} 1 & \text{for } y_0 < \frac{L_i}{2\lambda_P} = 30 \\ 0 & \text{for } y_0 \geq 30 \end{cases}
\] (VI.120)

Thus for \(\bar{Q}_i\) such that \((a_i)_{\text{symb}} < 30\) the argument in the left hand side of (VI.98) becomes, after making use of (VI.114) and (VI.120),
\[
2\pi \varepsilon_i \mu_i \bar{Q}_i \approx \frac{\pi n_i \hat{e}_i \hat{e}_j \hat{e}_k \bar{Q}_i}{900} = \frac{n_i \hat{e}_i \bar{Q}_i}{900} \quad (\text{where } i, j, k \text{ are cyclically ordered}),
\]
while for \(\bar{Q}_i\) such that \((a_i)_{\text{symb}} \geq 30\), since \(\bar{\varepsilon}_i = 0\), we then have
\[
\lim_{\varepsilon_i \to 0} \left( \frac{\sin(2\pi \varepsilon_i \mu_i \bar{Q}_i)}{\varepsilon_i \mu_i} \right) = 2\pi \bar{Q}_i.
\] (VI.121)
Consequently above this cutoff scale we need to replace (VI.98), (VI.102) and (VI.88) by
\[
\bar{Q}_i(\phi(\tau)) = \frac{X_i(\phi(L_i))}{2\pi} \cosh \left[ \frac{\pi}{\rho_\phi} R_i \left( \phi(\tau) - \phi(L_i) \right) + B_i(L_i) \right], \quad i = 1, 2, 3
\] (VI.122)
where here \(B_i(L_i)\) is the evaluation
\[
B_i(L_i) = \cosh^{-1} \left( \frac{\sin(2\pi \varepsilon_i \mu_i \bar{Q}_i)}{\varepsilon_i \mu_i \bar{Q}_i} \right) |_{\phi(L_i)},
\] (VI.123)
\[ \tan(\pi \tilde{k}_i(\phi(\tau))) = \tan(\pi \tilde{k}_i(\phi(L_i))) \left( \exp \left[ -\frac{\pi}{\mu_i} R_i \left( \phi(\tau) - \phi(L_i) \right) \right] \right), \quad (VI.124) \]

\[ \chi_i(\phi(\tau)) = 2\pi \tilde{Q}_i(\phi(\tau)) \sin \left( 2\pi \tilde{k}_i(\phi(\tau)) \right), \quad (VI.125) \]

in our evolution calculations, with \( R_i \) and \( \chi_i \) becoming constants of motion due to the effective absence of noncommutativity beyond this cutoff.

Now observe that \( [VI.110] \) already states the role of the quantities \( 2\pi \mu_i \tilde{Q}_i \) as the physical configuration variables in the limit \( \varepsilon \to 0 \), which in turn imply that in volume and areas in the commutative regime are measured in multiples of an elementary volume \( (2\pi)^3 \mu_1 \mu_2 \mu_3 \) and elementary areas \( (2\pi)^2 \mu_i \mu_j \) respectively. Because this can only be the reminiscence of the minimal areas \( [VI.115] \) and \( [VI.116] \) from the noncommutative regime then

\[ (2\pi)^2 \mu_1 \mu_2 = 2\pi \theta_3, \quad (2\pi)^2 \mu_2 \mu_3 = 2\pi \theta_1, \quad (2\pi)^2 \mu_1 \mu_3 = 2\pi \theta_2, \quad (VI.126) \]

or equivalently

\[ \frac{\theta_3}{\mu_1 \mu_2} = \frac{\theta_1}{\mu_2 \mu_3} = \frac{\theta_2}{\mu_1 \mu_3} = 2\pi. \quad (VI.127) \]

By making use of \( [VI.127] \) along with \( [III.17] \) and \( [III.19] \) it is straightforward to show that \( n_1 = n_2 = n_3 \) and equation \( [VI.112] \) reduces to

\[ \varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 = \varepsilon_3 \mu_3. \quad (VI.128) \]

In order to implement these notions so that the system can be faithfully evolved with the noncommutative equations inside the noncommutative region and with the commutative ones beyond the cutoff, we will require compatible solutions for both scenarios. This compatibility can be achieved through the selection of appropriate boundary values occurring at the cutoff region, which may be obtained by analyzing the behavior of \( \tilde{\chi}_i \).

Because one of the main differences between the noncommutative system and the commutative one is the constancy of all the \( \chi_i \)'s or equivalently \( \tilde{\chi}_i = 0 \) in the commutative case, this also establishes a criteria to determine when and how the noncommutative system can follow the commutative evolution beyond the cutoff. By using eq. \( [VI.105] \) it is immediate that

\[ \tilde{\chi}_i = \pi \sum_{j \neq i} \frac{\theta_{ij}}{\mu_i \mu_j} R_j \cos(2\pi \varepsilon_i \mu_i \tilde{Q}_i) \cos(2\pi \varepsilon_j \mu_j \tilde{Q}_j) \sin(2\pi \tilde{k}_i) \sin(2\pi \tilde{k}_j). \quad (VI.129) \]

From the previous expression we can obtain the values \( \tilde{Q}_i, \tilde{k}_i \) for which \( \tilde{\chi}_i = 0 \), which are clearly given by

\[ \tilde{Q}_i = (-1)^r \frac{2r+1}{4 \varepsilon_i \mu_i}, \quad \tilde{k}_i = \frac{s}{2}, \quad r, s \in \mathbb{Z}, \quad i = 1, 2, 3 \quad (VI.130) \]

where the factor \((-1)^r\) guarantees the positivity of the symbol associated to \( \tilde{\alpha}_i \).

However, because it is precisely when valued at \( [VI.130] \) that \( \tilde{k}_i = 0 \) and the symbols of \( \tilde{\alpha}_i \) reach their maximum and their rate of change becomes zero, there is ambiguity in continuing the evolution of the system beyond such values with expressions \( [VI.122] \) and \( [VI.124] \). To circumvent this difficulty we have to look for more adequate boundary values where the system can be said to be expanding or contracting, but where we still have \( \tilde{\chi}_i \approx 0 \) at any chosen order.

By looking at intervals centered in \( [VI.130] \) we may define the set of boundary conditions

\[ \tilde{Q}_i(0) = (-1)^r \frac{2r+1}{4 \varepsilon_i \mu_i} + \frac{\zeta_i}{2\pi}, \quad \tilde{k}_i(0) = \frac{s}{2} + \frac{\delta_i}{2\pi}, \quad 0 < |\zeta_i| \leq \frac{\pi}{2 \varepsilon_i \mu_i}, \quad 0 < |\delta_i| \leq \frac{\pi}{2}, \quad (VI.131) \]
where expanding solutions correspond to $\zeta_i < 0$ and contracting ones to $\zeta_i > 0$. After substituting this in (VI.129) we get

$$\dot{\chi}_i(0) = \pi \sum_{j \neq i} \frac{\theta_{ij}}{\mu_i \mu_j} R_j \sin(\varepsilon_i \mu_i \zeta_i) \sin(\varepsilon_j \mu_j \zeta_j) \sin(\delta_i) \sin(\delta_j).$$  (VI.132)

Noting from (V.88) that $|\chi_i| \leq \frac{1}{\varepsilon_i \mu_i}$, and consequently $|R_i| \leq \frac{3}{\varepsilon_i \mu_i}$, and using $|\sin(\alpha)| \leq |\alpha|$, we can establish an upper bound for the absolute value of $\dot{\chi}_i(0)$ and using (VI.127) yields

$$|\dot{\chi}_i(0)| = \left| 2\pi^2 \sum_{j \neq i} R_j \sin(\varepsilon_i \mu_i \zeta_i) \sin(\varepsilon_j \mu_j \zeta_j) \sin(\delta_i) \sin(\delta_j) \right| \leq 6\pi^2 \varepsilon_i \mu_i \sum_{j \neq i} |\zeta_i| |\zeta_j| |\delta_i| |\delta_j|,$$  (VI.133)

For an upper bound $M \in \mathbb{R}^+$ such that

$$6\pi^2 \varepsilon_i \mu_i \sum_{j \neq i} |\zeta_i| |\zeta_j| |\delta_i| |\delta_j| \leq M,$$  (VI.134)

the inequalities can be solved to obtain

$$|\zeta_i| |\delta_i| \leq \sqrt{\frac{M}{12\pi^2 \varepsilon_i \mu_i}},$$  (VI.135)

which can be further relaxed if all the $\chi_i$'s are chosen to have the same sign and so $|R_i| \leq \frac{2}{\varepsilon_i \mu_i}$, in which case

$$|\zeta_i| |\delta_i| \leq \sqrt{\frac{M}{8\pi^2 \varepsilon_i \mu_i}},$$  (VI.136)

Finally we need to enforce the cutoff condition in the interval of validity of $\zeta_i$. This is done directly from demanding

$$\frac{1}{\varepsilon_i} \sin(2\pi \varepsilon_i \mu_i \bar{Q}_i(0)) \geq L_i,$$  (VI.137)

or equivalently

$$\frac{1}{\varepsilon_i} \cos(\varepsilon_i \mu_i |\zeta_i|) \geq L_i,$$  (VI.138)

which for our case where $\varepsilon_i \mu_i |\zeta_i| \leq \frac{\pi}{2}$ also implies

$$|\zeta_i| \leq \frac{1}{\varepsilon_i \mu_i} \arccos(\varepsilon_i L_i).$$  (VI.139)

Together, the inequalities (VI.133) and (VI.139) provide the refinement for the admissible intervals of values for $\zeta_i$ and $\delta_i$ expressed now as

$$0 < |\zeta_i| \leq \frac{1}{\varepsilon_i \mu_i} \arccos(\varepsilon_i L_i), \quad 0 < |\delta_i| \leq \sqrt{\frac{M}{8\pi^2 \varepsilon_i \mu_i |\zeta_i|}}.$$  (VI.140)

This criteria provides with the full description of the system below and above the cutoff where from expression (VI.121) the matching boundary conditions at the cutoff region must satisfy

$$(a_i)_{sym}(0) = \frac{1}{\varepsilon_i} \sin(2\pi \mu_i \varepsilon_i \bar{Q}_i(0)) = 2\pi \mu_i \bar{Q}_i(0),$$

$$\chi_i(0) = \frac{1}{\varepsilon_i \mu_i} \sin(2\pi \mu_i \varepsilon_i \bar{Q}_i(0)) \sin(2\pi \bar{k}_i(0)) = 2\pi \bar{Q}_i(0) \sin(2\pi \bar{k}_i(0)),$$  (VI.141)
which implements the change of physical variables when going from below the cutoff to the region above.

In this sense any trajectory governed by the noncommutative algebra evolution of expressions (VI.93) and (VI.95), with boundary values (VI.131) and (VI.140) at the cutoff region, obeys a compatible commutative evolution (to order $M$) outside the Planckian region determined by (VI.122-VI.125).

The results just obtained can be further explained as follows. The system has a 6-dimensional phase-space, of which a suitable parametrization of a projection is the 2-dimensional plot $(\mathcal{V}_{\text{symb}}, \dot{\mathcal{V}}_{\text{symb}})$ shown in Fig. 1 (this phase-space diagram applies to the case discussed in section 8 with reference to Fig. 6). This figure shows a monotone orbit followed by an oscillatory behavior emerging into a new expanding orbit. Even though the quantities $\varepsilon_i, \mu_j$ are linked by the fundamental physics $\theta_{ij}$, strictly from a differential equations point of view we can consider $\theta_{ij} = 0$ with $\varepsilon_i, \mu_j \neq 0$. Then when $\theta_{ij} = 0$, the $R_i$ are constant and the equations (which follow from multiplying (V.88) by $R_i$)

$$R_i \chi_i = \left( \frac{R_i}{\varepsilon_i \mu_i} \right) \sin(2\pi \varepsilon_i \mu_i \bar{Q}_i) \sin(2\pi \bar{k}_i) = \text{const.} \quad (VI.142)$$

provide a family of invariants of the system. Thus in this formulation the universe will oscillate in a quasi-periodic way. Now, when $\theta_{ij} \neq 0$ the tori are subjected to the corresponding Hamiltonian perturbation.

![Phase-space plot of the volume with visible transition from an open collapsing orbit (lower branch) to periodic orbits connecting various invariant tori ending with an open expanding orbit (upper branch).](image)

Consequently the unperturbed orbits have now periods which depend on the amplitude (this can be seen simply by quadrature using (VI.142) for each degree of freedom. Moreover, as the orbits approach the origin in the $\bar{Q}_i$ variables the period becomes longer, since this is a hyperbolic point. Then the classical KAM results ([44]) guarantee the existence of nearby invariant tori for a large (in measure) set of unperturbed tori. In the actual behavior of the solutions we have that, generically, the basic periodic solution of the $i^{th}$ degree of freedom picks up two more periods
due to the interaction with the two other phases. When the invariant tori come close to the separatrix the basic orbit has a long period. These corrections will cause the oscillations. Furthermore, since the basic solutions have long periods, the resulting orbits become very sensitive (as the numerics in the following Section shows) to the parameters and initial conditions. When considering the implications of this behavior in the evolution of the volume, we would expect a relatively fast contracting orbit away from the saddle point merging with a long period resulting thus in a periodic oscillation caused by the noncommutativity and merging again (due to the integrability of the commutative problem) with the expanding solution.

It is important to recall that this behavior is not special but generic and is expected for any noncommutative model with an integrable structure in the commutative limit. We therefore can conclude from the above that generically the noncommutative scenario and its induced evolution of the invariants (VI.142), produces multiple solutions and effective noncommutative lattice structures as a consequence of the cosmology dynamics.

VII. NUMERICAL SOLUTIONS

In order to provide consistent values for the parameters in the equations and for appropriate initial conditions in the interesting parameter regimes described qualitatively in the previous section, let us now recall equations (III.17) and (III.19) which may be written as

\[ \mu_i = n_i \varepsilon_j \theta_k \]

with the indices \( i, j, k \) ordered cyclically. Expressing the above equation in units of Planck lengths we have

\[ \bar{\mu}_i \lambda_P = n_i \varepsilon_j \theta_k \bar{L}_j \lambda_P, \]

(VII.143)

where, as defined previously, bared symbols denote their magnitude and \( \bar{L}_j \) is the magnitude of the scale factor of the \( \varepsilon_j \). Let us next consider the behavior of the two terms in the right of equation (VI.95). In the Planck region the scale magnitude of \( \bar{L}_j \) is of the order of a Planck length so also setting the scale magnitude \( n_i \) of \( \mu_i \) equal to a few Planck lengths we have that \( \mu_i = \varepsilon_j \theta_k \approx 1 \lambda_P = O(\lambda_p) \). Consequently \( \mu_i \varepsilon_i \) is of the order of one in this case. Applying a similar reasoning to the expression

\[ \mu_1 \mu_2 \approx \frac{4 \lambda_P^2}{\varepsilon_1 \varepsilon_2 \theta_2 \theta_3} = O(\lambda_p^2), \]

(VII.144)

which makes it consistent with (VI.127) and, since for calculation simplicity we are taking the tensor of noncommutativity to be of the same magnitude for all three planes, the second term on the right of equation (VI.95) turns out to be commensurate with the first.

To illustrate the possible scenarios and how markedly they depart in the noncommutative case from classical (and non-classical) solutions, consider then the strongly noncommutative solutions of (VI.109) which occur when the noncommutative force term described above is commensurate with the first term in (VI.95) at all times. As mentioned, this corresponds to values of \( \varepsilon_i \) such that \( \varepsilon_i \mu_i \) is of order one. Fig.2 and Fig.3 constitute examples of this regime, with evident similar properties, obtained for numerical values of \( \varepsilon_i = 0.8(\lambda_p)^{-1} \) and \( \varepsilon_i = 0.4(\lambda_p)^{-1} \) respectively. As neither of the solutions can reach the scales that would make noncommutative effects negligible the solutions are confined to Planckian scale volumes.

Although similar, the system in Fig.3 is seen to evolve more diversely than in Fig.2 with global minima and maxima now differing by orders of magnitude. The irregular oscillatory behavior is in both cases the product of the noncommutative force term acting as a drive, modulating the frequencies of the solutions of the independent symbols of the radii of the universe, as can be better observed in Fig.4 where the three independent symbols \( (\alpha_i)_{\text{symb}} \) associated to the volume in Fig.3 have been plotted. This shows explicitly that it is the noncommutativity the agent which eventually drives the universe to scales past the Planckian scale through the smooth cutoff.
FIG. 2. For $\varepsilon_i = 0.8(\lambda_P)^{-1}$, solutions for the Volume (with initial conditions for the radii symbols of order $\lambda_p$) display oscillatory behavior. Maxima and minima are always within the same order of magnitude and the system is confined to Planckian volume scales.

FIG. 3. Solution for $\varepsilon = 0.4(\lambda_P)^{-1}$. For smaller $\varepsilon_i$, the system has access to bigger volumes and constructive interference among the independent symbols of the radii allows the formation of maxima of orders of magnitude greater than the minima. For values of $\varepsilon_i < 1/L_i$ these maxima eventually reach the cutoff region where the solutions are governed by the commutative regime and Eqs. (7.140)-(7.143).

By analyzing the $\chi_i$ variables, which in the commutative case are constants of motion and therefore can be interpreted as action variables, it is observed from Fig. 2 that their behavior in the Planckian regime is not adiabatic and noncommutativity is not simply a perturbation. In fact, the abrupt changes of these variables are associated to minima of the volume where noncommutative effects are stronger, whereas approximately adiabatic regions correspond to maxima of the volume and such regions become more and more dominant at larger scales. It is then that the evolution of the system can continue along commutative states, which is the basis for our selection of boundary values.
FIG. 4. The independent symbols $(a_1)_{\text{symb}}, (a_2)_{\text{symb}}, (a_3)_{\text{symb}}$, associated to the volume in Fig. 3, display complex evolutions due to the noncommutative force term that mixes interactions in the three independent directions at the cutoff, as confirmed by the following cases.

FIG. 5. Plot of $\chi_1, \chi_2, \chi_3$ associated to the volume in Fig. 3 where the approximately adiabatic regions around $\phi \approx \pm 3.3$ correspond to the global maxima seen for the volume.

Thus, let us now consider the evolution when approaching the cutoff from below, i.e. near $\bar{L}_i = 30$ then, by virtue of (VI.121), the first term on the right of (VI.95) becomes $\pi \bar{Q}_i \cos(2\pi \bar{k}_i) R_i$ with $R_i$ given by (VI.94) with $\alpha = \beta = \gamma = 0$ and the $\chi_i$ becoming constants of motion. On the other hand, after observing that (VII.144) is independent of scales, and therefore the coefficients of $\bar{k}_j$ are again of order one and the second term becomes negligible relative to the first one so the evolution beyond this stage is given by equations (VI.122)-(VI.125); In this case $\bar{Q}_i \approx \bar{q}_i$.

Moreover, observe that $\sum_{j \neq i} \frac{\theta_{ij}}{\mu_i \mu_j} \bar{k}_j$ acts as a force with unitless "mass" $\frac{\theta_{ij}}{\mu_i \mu_j}$ and unitless acceleration $\bar{k}_j$ driving the
canonical variables $\hat{Q}_i$ in a direction perpendicular to their $i^{th}$-components. This is made even more transparent when noting that by setting the tensor of noncommutativity equal to zero in (VI.95) the $R_i$ become constants of motion and the remaining first term becomes strictly oscillatory.

To exemplify this kind of solutions consider first the type of bounce depicted in Fig.7. Here we have a scenario where a collapsing trajectory (dashed) enters the noncommutative regime from the left, leading to a noncommutative evolution (solid) below the cutoff, where a number of noncommutative oscillations can be observed, until the effects of the noncommutative force term bring the system to an expansion phase such that it can reach the cutoff region and finally continue along a continuous expansion. Fig.7 provides more insight on the underlying interactions among the independent symbols $(a_i)_{sym}$ that, due to the constructive and destructive interferences, lead to the behavior of the volume shown inside the noncommutative region.

To finalize the discussion regarding this case compare the corresponding evolution of all the $\chi_i$'s in Fig.8 with that of Fig.5 which confirms the fact that at larger scales the adiabatic regions become more dominant and, in particular, it is at both extremes of Fig.8 that the system continues evolving for $\phi \not< 0$ along those constant values of $\chi_i$.

In terms of the stationary phase approximation the solutions so far obtained are for the center of a (gaussian) quantum state moving along classical paths. Thus, in most cases the complete picture of the collapse followed by an expansion is set to occur given decoherence is absent. Our two final examples deal with this possibility. The first case of Fig.9 shows a collapsing solution obtained for boundary conditions with $\zeta_i > 0$ near the cutoff. Because in the commutative regime (dashed) nothing prevents the system from collapsing all the way down to Planckian scales the system will eventually enter the noncommutative regime with boundary values at the cutoff (dot) compatible with a noncommutative evolution (solid) that, just as the previous solutions, avoids singularities and also displays the irregular oscillatory behavior which is the strong indicator of noncommutative effects taking place. As the center of the quantum state remains oscillating within Planck length scales it can be said the state has dissipated due to decoherence.

Time reversing the previous scenario would lead to a situation where the quantum state evolves from decoherence to an expansion. Fig.10 corresponds to the numerical solution for this case characterized by $\zeta_i < 0$ near the cutoff.
FIG. 7. Independent symbols \((a_1)_{\text{symb}}, (a_2)_{\text{symb}}, (a_3)_{\text{symb}}\) for \(\varepsilon = 0.031(\lambda_P)^{-1}\). The constructive (resp. destructive) interference inside the noncommutative regime region leading to the evolution of the volume above (resp. below) the cutoff in (Fig.6) is evidenced.

FIG. 8. Plot of \(\chi_1, \chi_2, \chi_3\) associated to the volume in Fig.6 where simultaneous regions of constant \(\chi_i\) at the left and right of the figure lead to the asymptotic evolution of the volume beyond the cutoff.

Once again the noncommutativity driven oscillations of irregular amplitudes are noted before the system reaches the commutative regime by means of the noncommutative force term discussed previously. Above the cutoff the volume evolves according to (VI.122-VI.125) with boundary values at the cutoff (dot).
FIG. 9. Collapsing solution for $\varepsilon_i = 0.031(\lambda P)^{-1}$. The commutative regime solution (dashed) enters the noncommutative region through the cutoff (dotted) and continues below it along a noncommutative evolution with compatible boundary values (dot). The quantum state undergoes dissipation and cannot bounce back.

FIG. 10. Expanding solution for $\varepsilon_i = 0.031(\lambda P)^{-1}$. For a fixed cutoff value $L_i = 3\lambda P$ the noncommutative regime solution (solid) expands from decoherence reaching the cutoff region (dotted) following a commutative evolution algebra (dashed) compatible with the boundary values (dot).

**VIII. CONCLUSIONS**

In this article we approach Quantum Cosmology from the point of view of a minisuperspace of a theory of Quantum Gravity. We employ in particular the noncommutative $C^*$-algebra $\mathfrak{A}$ outlined in Sections II and III which provides a well founded mathematical structure for introducing the concept of noncommutativity, from the point of view of an
In Sections IV-VII the quantum collapse of a Bianchi I Universe was studied in the context of noncommutative geometry. The noncommutativity of the space variables (the axes of the Bianchi Universe) was taken into account in a consistent way by representing them in terms of the twisted discrete translation group algebra of Sec.II. This representation is then used to construct the transition amplitude by using the Feynman integral formalism, which was shown to be dominated by an effective action that provides a new set of equations that resulted to have a new dynamical behavior that took into account the effect of the noncommutativity. It was shown asymptotically and numerically in a generic case that the noncommutativity induces an oscillatory motion of the volume due to the nontrivial evolution of the action variables which are constant for reticular space commutative theories. We thus have that the dynamical effects of noncommutative produce an oscillatory behavior of the volume in the region of the quantum bounce of reticular space commutative theories. It will be interesting to study if these oscillations in a full quantum field theory with spatial degrees of freedom can be indeed interpreted as a topological change. The differences mentioned above between our formalism and LQC lead to some additional physical implications which result from our GNS construction of the kinematic Hilbert space. The basic point being that the reticulation induced on the arguments of the Hilbert space contain at each point a tower of states, generated by the consistency conditions required between the twisted translations produced the unitaries \( \hat{U} \)'s and the translations due to the \( \hat{V} \)'s. This implies that our reticulation induced by noncommutativity is not the same as that in Ref.\[33\] and allows us to have, within the cosmology, a mechanism which could prevent that all the fluctuations in our Bianchi I universe could grow, thus avoiding to have a bounce at low matter densities. This fundamental characteristic is obtained only in the improved version of the polymeric cosmology of LQC, while in our case it occurs naturally because of the way noncommutativity was implemented. Moreover, in spite of the persistent difficulties inherent to this field of research to obtain experimental information, we could hope that phenomena lying in the interface of general relativity and quantum physics, such as those involving quantum entanglement and quantum coherence and which may be accessible to the experiment in the near-term future, could provide further theoretical insights to a full quantum theory of gravitation. This is suggested by the study of noncommutativity in a simpler problem \[45\] where it was shown that depending on the width of the wave packet of a coherent state one could go from the commutative regime for wide packets to the noncommutative regime for narrow packets. To perform this evolution one needs to find a consistent analogue of the Schrödinger equation in the noncommutative regime, and solve this equation asymptotically as well as numerically in order to understand this transition. This is currently under study.
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