DETERMINANTAL REPRESENTATIONS OF HYPERBOLIC
PLANE CURVES: AN ELEMENTARY APPROACH

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Abstract. If a real symmetric matrix of linear forms is positive definite at some
point, then its determinant defines a hyperbolic hypersurface. In 2007, Helton
and Vinnikov proved a converse in three variables, namely that every hyperbolic
curve in the projective plane has a definite real symmetric determinantal repre-
sentation. The goal of this paper is to give a more concrete proof of a slightly
weaker statement. Here we show that every hyperbolic plane curve has a definite
determinantal representation with Hermitian matrices. We do this by relating
the definiteness of a matrix to the real topology of its minors and extending a
construction of Dixon from 1902. Like the Helton-Vinnikov theorem, this implies
that every hyperbolic region in the plane is defined by a linear matrix inequality.

1. Introduction

Let $f$ be a real homogeneous polynomial of degree $d$ in $n+1$ variables $x_0, \ldots, x_n$. A Hermitian determinantal representation of $f$ is an expression

$$ f = \det(x_0 M_0 + \cdots + x_n M_n), $$

where $M_0, \ldots, M_n$ are Hermitian $d \times d$ matrices. The representation is definite if there is a point $e \in \mathbb{R}^{n+1}$ for which the matrix $e_0 M_0 + \cdots + e_n M_n$ is positive definite.

The existence of a definite Hermitian determinantal representation imposes an
immediate condition on the complex hypersurface $\mathcal{V}_C(f)$ defined by $f$. Because the eigenvalues of a Hermitian matrix are real, every real line passing through $e$ meets this hypersurface in only real points. A polynomial with this property is called hyperbolic (with respect to $e$). For $n = 2$, we regard $\mathcal{V}_C(f)$ as a projective plane
curve. Hyperbolicity is reflected in the topology of the real points $\mathcal{V}_R(f)$. When the
curve $\mathcal{V}_C(f)$ is smooth, $f$ is hyperbolic if and only if $\mathcal{V}_R(f)$ consists of $\left\lfloor \frac{d}{2} \right\rfloor$ nested
eovals, and a pseudo-line if $d$ is odd.

The Helton-Vinnikov theorem [10] (previously known as the Lax conjecture [11])
says that for $n = 2$, every hyperbolic polynomial possesses a definite determinantal

Figure 1. A quartic hyperbolic hypersurface in $\mathbb{R}^3$ and $\mathbb{P}^2(\mathbb{R})$. 

The Helton-Vinnikov theorem [10] (previously known as the Lax conjecture [11])
says that for $n = 2$, every hyperbolic polynomial possesses a definite determinantal
representation (1.1) with real symmetric matrices. The proof is quite involved and relies on earlier results of Vinnikov [21] and Ball and Vinnikov [1] on Riemann theta functions and the real structure of the Jacobian of the curve $V_C(f)$. The latter had previously been studied by Gross and Harris [8].

Determinantal hypersurfaces are a classical topic of complex algebraic geometry (see Beauville [2] and Dolgachev [4] for a modern presentation). In 1902, Dixon [3] proved that every smooth projective plane curve admits a symmetric determinantal representation. Hermitian and real symmetric representations of real curves were studied in generality only later by Dubrovin [5] and Vinnikov [20, 21].

Recently, questions in convex optimization (semidefinite and hyperbolic programming) and operator theory have been the motivation for more refined questions, especially concerning the definiteness of determinantal representations. If the polynomial $f$ has a definite determinantal representation $x_0M_0 + x_1M_1 + x_2M_2$ with real symmetric matrices, then the real surface defined by $f$ bounds the spectrahedron

$$\{ a \in \mathbb{R}^3 : a_0M_0 + a_1M_1 + a_2M_2 \text{ is positive semidefinite} \}.$$ (1.2)

This convex set is the cone over the region enclosed by the inner oval of the hyperbolic projective curve $V_{\mathbb{C}}(f)$. This realizes the convex region as the feasible set of a semidefinite program, and we say that the region is represented by a linear matrix inequality. In this context, the Helton-Vinnikov theorem [10] says that the convex region of any hyperbolic plane curve can be represented by a linear matrix inequality. The search for a suitable higher-dimensional analogue of this theorem is still an intriguing open problem; see Vinnikov [19] for an excellent review of both the history and recent progress on this problem and Netzer and Thom [12] for further discussion.

In this paper, we give an elementary proof of the fact that every hyperbolic plane curve has a definite representation (1.1) by generalizing a classical construction due to Dixon [3]. The details of this construction and the core of the paper are in Section 4, especially Theorem 4.6. Dixon’s approach is to relate symmetric determinantal representations to families of contact curves. However, explicitly proving the existence of such curves is very difficult. Dixon refers to the theory of theta functions (specifically the existence of a non-vanishing even theta characteristic). On the other hand, we can easily find families of curves that correspond to Hermitian determinantal representations. To construct a definite representation, we need only to start from curves that interlace the given curve. We can, for example, use directional derivatives, which have been used in the study of hyperbolicity cones already in the work of Gårding [7] and later by Renegar [16], Sanyal [17], and others.

Though we construct definite Hermitian (rather than symmetric) matrices, the connections to convex optimization are not lost. As discussed in Section 5, the existence of definite Hermitian representations still implies that the inner oval of any hyperbolic curve is a spectrahedron. Section 2 contains basic facts about hyperbolicity and interlacing polynomials. The connection between the interlacing property and the definiteness of a Hermitian matrix of linear forms is explored in Section 3. We provide a topological characterization of the definiteness of such a matrix by the interlacing of its determinant and comaximal minors (see Theorem 3.3).

Our overall goal is to give a new and self-contained proof of a known result that has attracted the interest of mathematicians from many different areas. In this, we have tried to keep the proofs as algebraic and concrete as possible and keep the use of topology and abstract algebraic geometry to a minimum.
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2. Hyperbolic polynomials and interlacers

Here we introduce the notions of hyperbolicity and interlacing and build up some useful facts about these properties.

Definition 2.1. A homogeneous polynomial \( f \in \mathbb{R}[x] \) in variables \( x = (x_0, \ldots, x_n) \) is called hyperbolic with respect to a point \( e \in \mathbb{R}^{n+1} \) if \( f(e) \neq 0 \) and for every \( a \in \mathbb{R}^{n+1} \), all roots of the univariate polynomial \( f(te + a) \in \mathbb{R}[t] \) are real. Note that since \( f \) is homogeneous, this is equivalent to \( f(e + ta) \) having only real roots.

Definition 2.2. Suppose \( f \) and \( g \) are univariate polynomials of degrees \( d \) and \( d-1 \) (respectively) with only real zeros. Denote the roots of \( f \) by \( \alpha_1 \leq \cdots \leq \alpha_d \) and the roots of \( g \) by \( \beta_1 \leq \cdots \leq \beta_{d-1} \). We say that \( g \) interlaces \( f \) if \( \alpha_i \leq \beta_i \leq \alpha_{i+1} \) for all \( i = 1, \ldots, d-1 \). For multivariate polynomials, if \( f \in \mathbb{R}[x] \) is hyperbolic with respect to \( e \in \mathbb{R}^{n+1} \) and \( g \) is homogeneous of degree \( \deg(f) - 1 \), we say that \( g \) interlaces \( f \) with respect to \( e \) if \( g(te + a) \) interlaces \( f(te + a) \) in \( \mathbb{R}[t] \) for every \( a \in \mathbb{R}^{n+1} \). Note that this implies that \( g \), too, is hyperbolic with respect to \( e \). (See Figure 2).

The most natural examples of interlacing polynomials come from derivatives. If \( f(t) \) is a real univariate polynomial with only real roots, then all the roots of its derivative \( f'(t) \) are real and interlace the roots of \( f \). This easily extends to a multivariate polynomial \( f(x) \) that is hyperbolic with respect to a point \( e \). Since the roots of \( \frac{\partial}{\partial t} f(te + a) \) interlace those of \( f(te + a) \) for all \( a \in \mathbb{R}^{n+1} \), we see that

\[
D_\epsilon f = \sum_{i=0}^{n} c_i \frac{\partial f}{\partial x_i}
\]

interlaces \( f \). This was first noted by Gårding [7] and used extensively in [16].

![Figure 2. A cubic interlacing a quartic in \( \mathbb{R}^3 \) and \( \mathbb{P}^2(\mathbb{R}) \).](image)

Note that if \( f \) and \( g \) are coprime and \( g \) interlaces \( f \) with respect to \( e \), then the roots of \( f(te + a) \) are distinct from the roots of \( g(te + a) \) for points \( a \) in an open dense subset of \( \mathbb{R}^{n+1} \). In particular, this is true of \( g = D_\epsilon f \) when \( f \) is square-free.
We now come to two useful results on interlacing polynomials. The first characterizes the intersection points of $\mathcal{V}_R(f)$ and its directional derivative $\mathcal{V}_R(D_uf)$. 

**Lemma 2.3.** Suppose that $f \in \mathbb{R}[x]_d$ is irreducible and hyperbolic with respect to $e$. Fix $g, h \in \mathbb{R}[x]_{d-1}$ where $g$ interlaces $f$ with respect to $e$. Then $h$ interlaces $f$ with respect to $e$ if and only if $g \cdot h$ is nonnegative on $\mathcal{V}_R(f)$ or nonpositive on $\mathcal{V}_R(f)$. 

**Proof.** To prove this statement, it suffices to restrict to the line $x = te + a$ for generic $a \in \mathbb{R}^{n+1}$. In particular, we may assume that the roots of $f(te + a)$ are distinct from each other and from the roots of $g(te + a) \cdot h(te + a)$.

Suppose that $g \cdot h$ is nonnegative on $\mathcal{V}_R(f)$. By the genericity assumption, the product $g(te + a)h(te + a)$ is positive on all of the roots of $f(te + a)$. Between consecutive roots of $f(te + a)$, the polynomial $g(te + a)$ has a single root and thus changes sign. For the product $g \cdot h$ to be positive on these roots, $h(te + a)$ must also change sign and have a root between each pair of consecutive roots of $f(te + a)$. Hence $h$ interlaces $f$ with respect to $e$.

Conversely, suppose that $g$ and $h$ both interlace $f$. Between any consecutive roots of $f(te + a)$, both $g(te + a)$ and $h(te + a)$ each have exactly one root, and their product has exactly two. It follows that $g(te + a)h(te + a)$ has the same sign on all the roots of $f(te + a)$. Taking $t \to \infty$ shows this sign to be the sign of $(e)h(e)$, independent of the choice of $a$. Hence $g \cdot h$ has the same sign on every point of $\mathcal{V}_R(f)$. ∎

**Lemma 2.4.** Let $f \in \mathbb{R}[x]$ be hyperbolic with respect to $e$. Every real intersection point of $\mathcal{V}_C(f)$ and $\mathcal{V}_C(D_uf)$ is a singular point of $\mathcal{V}_C(f)$.

**Proof.** Suppose for the sake of contradiction that a point $p$ lies in $\mathcal{V}_R(f) \cap \mathcal{V}_R(D_uf)$ and that $p$ is nonsingular in $\mathcal{V}_C(f)$. Then the vector $\nabla f(p)$ is nonzero and orthogonal to $e$. Now consider the affine plane $H = p + \text{span}\{e, \nabla f(p)\}$. The restriction of $\mathcal{V}_C(f)$ to $H$ is a plane curve that is still nonsingular at $p$.

For any $u \in H$ let $\alpha_1(u) \leq \ldots \leq \alpha_d(u)$ be the roots of $f(te + u)$ and for each $j = 1, \ldots, d$, let $q_j(u)$ denote the point $\alpha_j(u)e + u$ in $\mathcal{V}_R(f) \cap H$. Because $p$ lies in the intersection $\mathcal{V}_R(f) \cap \mathcal{V}_R(D_uf)$, the polynomial $f(te + p)$ has a double root at $t = 0$. So for some $k \in \{1, \ldots, d-1\}$, the points $q_k(p)$, $q_{k+1}(p)$, and $p$ are all equal.

For all but finitely many points $u$ in the line $p + \text{span}\{\nabla f(p)\}$, the polynomial $f(te + u)$ has distinct roots. Thus we can take $U$ to be a real open neighborhood of $p$ in this line such that for all $u \in U \setminus \{p\}$ the roots of $f(te + u)$ are distinct. Then the

![Figure 3. A singular hyperbolic curve and close up from Lemma 2.4.](image-url)
maps \( u \mapsto q_k(u) \) and \( u \mapsto q_{k+1}(u) \) give homeomorphisms between \( U \) and different subsets of a neighborhood of \( p \) in \( \mathcal{V}_\mathbb{R}(f) \cap H \).

Since \( p \) is a nonsingular point of \( \mathcal{V}_\mathbb{C}(f) \cap H \), it has an open neighborhood in \( \mathcal{V}_\mathbb{R}(f) \cap H \) that is homeomorphic to a line segment, by the implicit function theorem. However, by the above argument, removing the point \( p \) from this open neighborhood results in at least four connected components: two in \( q_k(U \setminus p) \) and two in \( q_{k+1}(U \setminus p) \).

This contradiction shows that \( p \) must be a singular point of \( \mathcal{V}_\mathbb{C}(f) \). So every real intersection point of the varieties of \( f \) and \( D_e f \) lies in the singular locus of \( \mathcal{V}_\mathbb{C}(f) \). □

3. INTERLACING AND DEFINITENESS

All eigenvalues of a Hermitian matrix are real. On the space of Hermitian matrices, we consider \( \det(X + iY) \) as a polynomial in \( \mathbb{R}[X_{ij}, Y_{ij} : 1 \leq i,j \leq n] \), where \( X = (X_{ij}) \) and \( Y = (Y_{ij}) \) are symmetric and skew-symmetric matrices of variables, respectively. This polynomial is hyperbolic with respect to the identity matrix. In fact, it is hyperbolic with respect to any positive definite matrix. Hence, for any positive semidefinite matrix \( E \neq 0 \), the polynomial

\[
D_E(\det(X + iY)) = \text{tr}(E \cdot (X + iY)^{\text{adj}})
\]

interlaces \( \det(X + iY) \). This holds true when we restrict to linear subspaces. For Hermitian \( d \times d \) matrices \( M_0, \ldots, M_n \) and variables \( x = (x_0, \ldots, x_n) \), denote

\[
M(x) = \sum_{j=0}^{n} x_j M_j.
\]

If \( M(e) \) is positive definite for some \( e \in \mathbb{R}^{n+1} \), then the polynomial \( \det(M(x)) \) is hyperbolic with respect to the point \( e \). If \( E \) as above has rank one, say \( E = \lambda \lambda^T \) where \( \lambda \in \mathbb{C}^d \), and we restrict to the subspace of Hermitian matrices spanned by \( M_0, \ldots, M_n \), then the polynomial (3.1) has the form \( \lambda^T M^{\text{adj}} \lambda \). In Section 4 we use these polynomials to reconstruct the matrix \( M \).

**Definition 3.1.** Let \( M \) be a \( d \times d \) Hermitian matrix of linear forms. With it we associate a family of polynomials,

\[
C(M) = \{ \lambda^T M^{\text{adj}} \lambda \mid \lambda \in \mathbb{C}^d \setminus \{0\} \},
\]

and call \( C(M) \) the system of hypersurfaces associated with \( M \).

Here is a useful identity on these hypersurfaces that goes back to the work of Hesse in 1855 [9]. For completeness, we include the short proof.

**Proposition 3.2.** Let \( M \) be a Hermitian matrix of linear forms. For any \( \lambda, \mu \in \mathbb{C}^d \),

\[
(\lambda^T M^{\text{adj}} \lambda)(\mu^T M^{\text{adj}} \mu) - (\lambda^T M^{\text{adj}} \mu)(\mu^T M^{\text{adj}} \lambda)
\]

is contained in the ideal \( (\det(M)) \). In particular, the polynomial \( (\lambda^T M^{\text{adj}} \lambda)(\mu^T M^{\text{adj}} \mu) \) is nonnegative on the real variety of \( \det(M) \).

**Proof.** Consider a general square matrix of variables \( X = (X_{ij}) \). At a generic point in \( \mathcal{V}_\mathbb{C}(\det(X)) \), the matrix \( X \) has corank one. The identity \( X \cdot X^{\text{adj}} = \det(X)I \) implies that \( X^{\text{adj}} \) has rank one at such a point. In particular, the \( 2 \times 2 \) matrix \((\lambda \mu^T X^{\text{adj}} (X \lambda^T \mu))\) has rank at most one on \( \mathcal{V}_\mathbb{C}(\det(X)) \). Since the polynomial \( \det(X) \) is irreducible, the determinant of this \( 2 \times 2 \) matrix lies in the ideal \( (\det(X)) \). Restricting to \( X = M \) gives the desired identity.
For the claim of nonnegativity, note that \((\mu^T M^{\text{adj}} \lambda) = (\lambda^T M^{\text{adj}} \mu)\). So the polynomial \((\lambda^T M^{\text{adj}} \lambda)(\mu^T M^{\text{adj}} \mu)\) is equal to a polynomial times its conjugate modulo the ideal \((\det(M))\). This shows it to be nonnegative on \(\mathcal{V}_{\mathbb{R}}(\det(M))\). \(\square\)

This simple identity allows us to determine whether or not a determinantal representation \(M\) is definite by examining the real topology of the polynomials in \(\mathcal{C}(M)\).

**Theorem 3.3.** Let \(f \in \mathbb{R}[x]_d\) be irreducible and hyperbolic with respect to \(e\), with a Hermitian determinantal representation \(f = \det(M)\). The following are equivalent:

1. Some polynomial in \(\mathcal{C}(M)\) interlaces \(f\) with respect to \(e\).
2. Every polynomial in \(\mathcal{C}(M)\) interlaces \(f\) with respect to \(e\).
3. The matrix \(M(e)\) is (positive or negative) definite.

**Proof.** (2)\(\Rightarrow\)(1): Clear.

(1)\(\Rightarrow\)(2): Suppose that \(g = \lambda^T M^{\text{adj}} \lambda\) interlaces \(f\). Let \(h\) denote another element of \(\mathcal{C}(M)\), say \(h = \mu^T M^{\text{adj}} \mu\) where \(\mu \in \mathbb{C}^d\). From Proposition 3.2, we see that the product \(g \cdot h\) is nonnegative on \(\mathcal{V}_{\mathbb{R}}(f)\). Then, by Lemma 2.3, \(h\) interlaces \(f\).

(3)\(\Rightarrow\)(2): By switching \(M\) with \(-M\) and \(f\) with \(-f\) if necessary, we can take \(M(e)\) to be positive definite and write it as \(M(e) = \sum_j \mu_j^T \mu_j\) where \(\mu_j \in \mathbb{C}^d\). Then the derivative \(D_e(\det(M))\) equals

\[
D_e(\det(M)) = \text{tr}(M(e) \cdot M^{\text{adj}}) = \text{tr}\left(\sum_j \mu_j^T \mu_j \cdot M^{\text{adj}}\right) = \sum_j \mu_j^T M^{\text{adj}} \mu_j.
\]

Then for any \(\lambda \in \mathbb{C}^d\), the polynomial

\[
(\lambda^T M^{\text{adj}} \lambda) \cdot D_e(\det(M)) = \sum_j (\lambda^T M^{\text{adj}} \lambda)(\mu_j^T M^{\text{adj}} \mu_j)
\]

is nonnegative on \(\mathcal{V}_{\mathbb{R}}(\det(M))\), using Proposition 3.2. Because \(D_e(\det(M))\) interlaces \(\det(M)\) with respect to \(e\), we can then use Lemma 2.3 to see that \(\lambda^T M^{\text{adj}} \lambda\) also interlaces \(\det(M)\) with respect to \(e\).

(2)\(\Rightarrow\)(3): First, let us show that any two elements \(g, h\) of \(\mathcal{C}(M)\) have the same sign at the point \(e\). Since \(f\) is irreducible, the polynomial \(g \cdot h\) cannot vanish on \(\mathcal{V}_{\mathbb{R}}(f)\). By Proposition 3.2, the product \(g \cdot h\) is nonnegative on \(\mathcal{V}_{\mathbb{R}}(f)\) and thus strictly positive on a dense subset of \(\mathcal{V}_{\mathbb{R}}(f)\). Furthermore, because both \(g\) and \(h\) interlace \(f\), they cannot have any zeroes in the component of \(e\) in \(\mathbb{R}^{n+1}\backslash \mathcal{V}_{\mathbb{R}}(f)\). So the product \(g \cdot h\) must be positive on this component of \(e\) in \(\mathbb{R}^{n+1}\backslash \mathcal{V}_{\mathbb{R}}(f)\) and thus at \(e\) itself.

Now consider the Hermitian matrix \(M^{\text{adj}}(e)\). We have shown that the sign of \(\lambda^T M^{\text{adj}}(e) \lambda\) is the same for every \(\lambda \in \mathbb{C}^d\). This shows that the matrix \(M^{\text{adj}}(e)\) is definite, hence so is \(M(e) = f(e)(M^{\text{adj}}(e))^{-1}\). \(\square\)

The diagonal \((d - 1) \times (d - 1)\) minors of \(M\) are elements of \(\mathcal{C}(M)\). So a corollary of Theorem 3.3 is that a linear subspace of Hermitian matrices contains a definite matrix if and only if its diagonal co-maximal minors interlace its determinant. For an alternative proof of this fact, see [19, Theorem 5.3].

We conclude this section with a useful lemma about limits of determinantal representations: The map taking a matrix with linear entries to the determinant is closed when restricted to definite representations, which it need not be in general. This was also shown by Speyer [18, Lemma 8].
Lemma 3.4. Let $e \in \mathbb{R}^{n+1}$. The set of homogeneous polynomials $f \in \mathbb{R}[x]_d$ with $f(e) = 1$ that possess a Hermitian determinantal representation $f = \det(M)$ where $M(e)$ is positive definite is closed in $\mathbb{R}[x]_d$.

Proof. First we observe that if $f(e) = 1$ and $f = \det(M)$ where $M(e) \succ 0$, then $f$ has such a representation $M'$. Now let each sequence $M_j \rightarrow M$ be a determinantal representation of $f$ with $M_j'(e) = I_d$. For each $j$, let $e_j$ denote the $j$th unit vector. Since $f_k(te - e_j)$ is the characteristic polynomial of $M_j^{(k)}$, the eigenvalues of each $M_j^{(k)}$ converge to the zeros of $f(te - e_j)$. It follows that each sequence $(M_j^{(k)})_k$ is bounded. We may therefore assume that the sequence $(M_j^{(k)})$ is convergent (after successively passing to a convergent subsequence of $(M_j^{(k)})_k$) for each $j = 0, \ldots, n$ and conclude that $f = \det(\lim_{k \rightarrow \infty} M^{(k)})$. \hfill \Box

4. Dixon's Construction for Hyperbolic Curves

Here we describe a modification of the classical construction of Dixon [3], which relates determinantal representations of plane curves to contact curves. Dixon considered only determinants of symmetric matrices. As described below, we can use a similar method to construct Hermitian determinantal representations. The exact relation of these determinantal representations to families of “contact” curves is somewhat subtle and has been worked out by Vinnikov in [20]. Here we give an account using only intersection theory of plane curves (all of which can be found for example in [6]), and refer to [20] for more detailed information. Because we now deal only with plane curves, we fix $n = 2$ and replace $(x_0, x_1, x_2)$ by $(x, y, z)$.

As we saw in Proposition 3.2, for any square matrix $M$ of linear forms, the matrix $M^{\text{adj}}$ has rank at most one along $\mathcal{V}_C(\det(M))$. In particular, its $2 \times 2$ minors lie in the ideal generated by $\det(M)$. The main idea of Dixon is to reconstruct $M$ by producing a suitable $A = M^{\text{adj}}$, namely, a $d \times d$ matrix of forms of degree $d - 1$ whose $2 \times 2$ minors lie in the ideal $(\det(M))$. We modify his construction to produce a Hermitian determinantal representation. Theorem 3.3 shows that if the top left entry of $A$ interlaces $f$, then this determinantal representation will be definite.

Here is a summary of the construction. The input is a smooth real form $f$ that is hyperbolic with respect to a point $e = (e_0, e_1, e_2) \in \mathbb{R}^3$, and the output is a definite Hermitian determinantal representation $M$ of $f$.

- Let $a_{11}$ be the form $D_e f = e_0 \frac{\partial f}{\partial x} + e_1 \frac{\partial f}{\partial y} + e_2 \frac{\partial f}{\partial z}$ of degree $(d - 1)$.
- Split the $d(d-1)$ points $\mathcal{V}_C(f) \cap \mathcal{V}_C(a_{11})$ into two disjoint, conjugate sets of points $S \cup \overline{S}$.
- Extend $a_{11}$ to a basis $\{a_{11}, \ldots, a_{1d}\}$ of the forms in $\mathbb{C}[x, y, z]$ of degree $d - 1$ that vanish on the set of points $S$.
- For $1 < j \leq k$, let $a_{jk}$ be a polynomial for which $a_{11}a_{jk} - a_{1j}a_{1k}$ lies in the ideal $(f)$, with $a_{jk}$ real if $j = k$. For $j < k$, define $a_{kj} = \overline{a_{jk}}$ and define $A = (a_{jk})_{j,k}$ to be the resulting $d \times d$ matrix of forms of degree $d - 1$.
- Define $M$ to be the matrix of linear forms obtained by dividing each entry of $A^{\text{adj}}$ by $f^{d-2}$.
We will show that these steps can be carried through and that the resulting matrix $M$ is a definite determinantal representation of $f$. We see that the output depends on some choices, the most important of which is the splitting of the points $V_C(f, a_{11})$. The resulting determinantal representation depends on the divisor of $V_C(f)$ consisting of the points $S$. To discuss this precisely, we use the language of divisors on curves.

For a form $f \in \mathbb{C}[x, y, z]$, let $\text{Div}(f)$ denote the free abelian group over the complex points $V_C(f)$. Thus an element of $\text{Div}(f)$ is an expression $D = \sum_{i=1}^{k} n_i P_i$ with $P_1, \ldots, P_k \in V_C(f)$ and $n_i \in \mathbb{Z}$, called a divisor on $V_C(f)$. The degree of the divisor $D$ is defined by $\deg(D) = \sum_{i=1}^{k} n_i$ and its conjugate divisor is $\overline{D} = \sum_{i=1}^{k} n_i \overline{P_i}$. If $g \in \mathbb{C}[x, y, z]$ is also homogeneous and shares no factors with $f$, then the intersection divisor of $f$ and $g$ is defined by $f \cdot g = \sum_{P \in V_C(f,g)} I_P(f, g) \cdot P$, where $I_P(f, g)$ is the intersection multiplicity of $f$ and $g$ at the point $P$. For two forms $g, h$ in $\mathbb{C}[x, y, z]$ that are coprime to $f$, we have that $f.(gh) = f.g + f.h$ and $f.g = f.(ghf)$. If $f$ has degree $d$ and $g$ has degree $e$, Bézout’s theorem says that $\deg(f.g) = de$. Given a divisor $D = \sum_{i=1}^{k} n_i P_i$, we write $D \geq 0$ if $n_i \geq 0$ for $i = 1, \ldots, k$. For two divisors $D, E \in \text{Div}(f)$, write $E \geq D$ if $E - D \geq 0$. We need the following classical result:

**Theorem 4.1** (Max Noether). Let $V_C(f)$ be a smooth projective plane curve over $\mathbb{C}$ and let $g, h \in \mathbb{C}[x, y, z]$ be homogeneous. Assume that $g$ and $h$ have no irreducible components in common with $f$. If $f.h \geq f.g$, then there exist homogeneous polynomials $a, b \in \mathbb{C}[x, y, z]$ such that $h = af + bg$. If $f, g, h$ are all real, then $a, b$ can also be chosen to be real.

**Proof.** See, for example, [6, §5.5] for the proof. For the reality of $a$ and $b$, note that if $f, g, h$ are all real and $h = af + bg$, then $h = \frac{1}{2}(a + \overline{a})f + \frac{1}{2}(b + \overline{b})g$. \hfill $\square$

The intersection divisors of interest to us come from curves that have special intersection with the set of real points $V_\mathbb{R}(f)$.

**Definition 4.2.** Let $f, g \in \mathbb{R}[x, y, z]$. Then $V_\mathbb{C}(g)$ is a curve of real contact of $V_C(f)$ if there exists a divisor $D \in \text{Div}(f)$ such that $f.g = D + \overline{D}$. In this case, the divisor $D$ is called a real-contact divisor of $V_C(f)$.

In other words, a real plane curve is a curve of real contact of $V_C(f)$ if and only if all real intersection points with $V_C(f)$ have even multiplicity. The simplest example of such a curve $V_C(g)$ is one for which $V_\mathbb{R}(f) \cap V_\mathbb{R}(g) = \emptyset$. For us, the most important examples come from real curves that interlace $f$.

**Proposition 4.3.** Suppose $f \in \mathbb{R}[x, y, z]$ is hyperbolic with respect to $e$ and that the curve $V_C(f)$ has no real singular points. Then any form that interlaces $f$ with respect to $e$ is a curve of real contact of $V_C(f)$.

**Proof.** If $g \in \mathbb{R}[x, y, z]$ interlaces $f$ with respect to $e$, then by Lemma 2.3, the product $g \cdot D_e f$ has constant sign on $V_\mathbb{R}(f)$. If $V_\mathbb{R}(f) \cap V_\mathbb{R}(g)$ is empty, then $g$ is automatically a curve of real contact to $V_C(f)$. On the other hand, suppose there is a point $P$ in $V_\mathbb{R}(f) \cap V_\mathbb{R}(g)$. By Lemma 2.4, $D_e f(P)$ is nonzero. It follows that the restriction of $g$ to $V_C(f)$ has locally constant sign around $P$ and has therefore even vanishing order in $P$. That vanishing order is exactly the intersection multiplicity of $f$ and $g$ in $P$, meaning that every real intersection point of $V_C(f)$ and $V_C(g)$ has even multiplicity. \hfill $\square$
One can also obtain real-contact divisors directly from a Hermitian determinantal representation, as shown in the following proposition. Our eventual goal is to reconstruct the determinantal representation from such a divisor.

**Proposition 4.4.** Let \( \mathcal{V}_C(f) \) be a smooth real projective plane curve of degree \( d \) and let \( f = \det(M) \) be a Hermitian linear determinantal representation. Let \( (a_{11}, \ldots, a_{1d}) \) be the first row of \( M^{\text{adj}} \). For each intersection point \( P \in \mathcal{V}_C(a_{11}, f) \), let \( n_P = \min \{ I_P(f, a_{1j}) \mid j = 1, \ldots, d \} \). Then

\[
D_M = \sum_{P \in \mathcal{V}_C(a_{11}, f)} n_P P
\]

is a real-contact divisor of degree \( d(d - 1)/2 \) on \( \mathcal{V}_C(f) \) with \( f.a_{11} = D_M + \overline{D_M} \).

**Proof.** Using Proposition 3.2, we have that \( a_{jj}a_{kk} - a_{jk}\overline{a}_{jk} \) lies in the ideal \( (f) \) for all \( j, k \), which shows the two intersection divisors \( f.(a_{jj}a_{kk}) \) and \( f.(a_{jk}\overline{a}_{jk}) \) to be equal. Let \( P \in \mathcal{V}_C(a_{11}, f) \). Since \( \mathcal{V}_C(f) \) is smooth, \( M^{\text{adj}}(P) \) is not the zero matrix and there exists \( j \) for which \( f.a_{jj}(P) \neq 0 \). This implies that

\[
I_P(f, a_{11}) = I_P(f, a_{11}a_{jj}) = I_P(f, a_{1j}\overline{a}_{1j}),
\]

which shows that the multiplicity \( I_P(f, a_{11}) \) is even and \( \mathcal{V}_C(a_{11}) \) is a curve of real contact of \( \mathcal{V}_C(f) \). Furthermore, by definition we have that \( I_P(f, a_{1j}\overline{a}_{1j}) \geq n_P + n_P \), which shows that \( f.a_{11} \geq D_M + \overline{D_M} \). On the other hand,

\[
I_P(f, a_{1k}a_{1k}) = I_P(f, a_{11}a_{kk}) \geq I_P(f, a_{11})
\]

holds for any \( k \in \{1, \ldots, d\} \), and thus \( f.a_{11} \leq D_M + \overline{D_M} \). This shows that \( a_{11} \) is a curve of real contact and that \( D_M \) is a real-contact divisor. \( \square \)

If the matrix \( M \) is real, then \( D_M \) equals \( \overline{D_M} \) and \( f.a_{11} \) equals \( 2D_M \), which puts a strong restriction on possible choices of \( a_{11} \). This is the original setting of Dixon’s algorithm. The following is a modification of his construction, which reconstructs the Hermitian determinantal representation \( M \) from a real-contact divisor \( D_M \).

**Construction 4.5** (of \( A = M^{\text{adj}} \)). Let \( D \) be a real-contact divisor of degree \( \binom{d}{2} \) on \( \mathcal{V}_C(f) \). We construct a Hermitian matrix \( A_D \) with entries in \( \mathbb{C}[x, y, z]_{d-1} \) as follows:

Let \( a_{11} \in \mathbb{R}[x, y, z]_{d-1} \) be such that \( f.a_{11} = D + \overline{D} \). Consider the complex vector space \( V \) of polynomials \( g \in \mathbb{C}[x, y, z]_{d-1} \) for which \( f.g \geq D \). The dimension of \( V \) is at least \( \binom{d+1}{2} - \binom{d}{2} = d \) (the dimension of \( \mathbb{C}[x, y, z]_{d-1} \) minus the maximal number \( \binom{d}{2} = \deg(D) \) of linearly independent conditions imposed on \( g \) by \( f.g \geq D \)).

Extend \( a_{11} \) to a linearly independent family \( a_{11}, \ldots, a_{1d} \) in \( V \). For \( 2 \leq j \leq k \leq d \), we have \( f.(\overline{a}_{jj}a_{1k}) \geq D + \overline{D} = f.a_{11} \). Thus we can apply Theorem 4.1 and obtain homogeneous polynomials \( p, q \in \mathbb{C}[x, y, z] \) such that \( \overline{a}_{jj}a_{1k} = pf + qa_{11} \). Put \( a_{jk} = q \). If \( j = k \), then \( a_{11} \) and \( \overline{a}_{1j}a_{1j} \) are both real and we can take \( a_{1j} \) real as well. Finally, put \( a_{kj} = \overline{a}_{jk} \) for \( j < k \) and let \( A_D = (a_{jk})_{j,k} \).

We let \( A_D \) denote any matrix resulting from the above construction. This will be the adjugate matrix of a determinantal representation of \( f \). When \( f \) is hyperbolic and \( a_{11} \) interfaces \( f \), then the representation will be definite.
Theorem 4.6. Let \( \mathcal{V}_C(f) \) be a smooth real projective plane curve of degree \( d \). Suppose \( D \) is a real-contact divisor of degree \( \binom{d}{2} \) of \( \mathcal{V}_C(f) \).

(a) Every entry of the adjugate matrix of \( A_D \) is divisible by \( f^{d-2} \) and the matrix \( M_D = (1/f^{d-2})A_D^{\text{adj}} \) has linear entries. Furthermore there exists \( \gamma \in \mathbb{R} \) such that
\[
\gamma f = \det(M_D).
\]
(b) If \( f \) is hyperbolic and \( a_{11} = (A_D)_{11} \) interlaces \( f \) with respect to \( e \), then \( \gamma \neq 0 \) and the matrix \( M_D(e) \) is (positive or negative) definite.

The following lemma will be essential for the proof of this theorem.

Lemma 4.7. Let \( A \) be a \( d \times d \)-matrix with entries in \( \mathbb{C}[x, y, z] \) with \( d \geq 2 \). Let \( f \in \mathbb{C}[x, y, z] \) be irreducible. If \( f \) divides all \( 2 \times 2 \)-minors of \( A \), then for every \( 1 \leq k \leq d \), the polynomial \( f^{k-1} \) divides every \( k \times k \)-minor of \( A \).

Proof of Lemma. By hypothesis, the claim holds for \( k = 2 \). So assume \( k > 2 \) and suppose that \( f^{k-2} \) divides all \( (k-1) \times (k-1) \)-minors of \( A \). Let \( B \) be a submatrix of size \( k \times k \) of \( A \). From \( B^{\text{adj}} = \det(B) \cdot I_k \) we conclude \( \det(B^{\text{adj}}) = \det(B)^{k-1} \).

Suppose \( \det(B) = f^mg \) where \( f \) does not divide \( g \). Then \( \det(B)^{k-1} = f^{m(k-1)}g^{k-1} \). By assumption \( f^{k-2} \) divides all entries of \( B^{\text{adj}} \), hence \( f^{k(k-2)} \) divides its determinant \( \det(B)^{k-1} \). Since \( f \) is irreducible, \( f \) does not divide \( g^{k-1} \), so \( f^{k(k-2)} \) must divide \( f^{m(k-1)} \). Then \( k(k-2) \leq m(k-1) \) which implies that \( k-1 \leq m \), as claimed.

Proof of Theorem 4.6(a). By construction, the \( 2 \times 2 \) minors of \( A_D \) having the form \( a_{11}a_{jk} - a_{1k}a_{j1} \) are divisible by \( f \). This means that for every point \( p \) in \( \mathcal{V}_C(f) \) with \( a_{11}(p) \neq 0 \), all rows of the matrix \( A_D(p) \) are multiples of the first, and thus \( A_D(p) \) has rank one. Because \( a_{11} \) is not divisible by \( f \), it follows that all the \( 2 \times 2 \) minors of \( A_D \) are divisible by \( f \). Since \( f \) is irreducible in \( \mathbb{C}[x, y, z] \), all \( (d-1) \times (d-1) \)-minors of \( A_D \) are divisible by \( f^{d-2} \), by Lemma 4.7. The entries of \( A_D^{\text{adj}} \) have degree \( (d-1)^2 \) and \( f \) has degree \( d \), so that \( M_D = (1/f^{d-2}) \cdot A_D^{\text{adj}} \) has entries of degree \( (d-1)^2 - d(d-2) = 1 \). Furthermore, by Lemma 4.7, \( \det(A_D) \) is divisible by \( f^{d-1} \). So \( \det(A_D) = cf^{d-1} \) for some \( c \in \mathbb{R}[x, y, z] \) and we obtain
\[
\det(M_D) = \det(f^{2-d}A_D^{\text{adj}}) = f^{d(2-d)}\det(A_D^{\text{adj}}) = f^{d(2-d)}f^{d-1} = c^{d-1}f.
\]
Since \( \det(M_D) \) has degree \( d \), we see that \( c \) is a constant and we take \( \gamma = c^{d-1} \).

This gives us a potential determinantal representation of \( f \). To finish the job, we need to ensure that the constant \( \gamma \) is nonzero. Following Dixon [3], we do this by analyzing the system of curves associated with \( A_D \).

Lemma 4.8. Let \( D \) be a real-contact divisor of degree \( d(d-1)/2 \) of \( \mathcal{V}_C(f) \) and write \( C_D = \{ \lambda^T A_D \overline{\lambda} \mid \lambda \in \mathbb{C}^d \setminus \{0\} \} \).

Every element of \( C_D \) is a curve of real contact of \( \mathcal{V}_C(f) \) of degree \( d-1 \).

Proof. By assumption, the polynomial \( a_{11} = (e_1^T A_D e_1) \) is a curve of real contact to \( f \) with real-contact divisor \( D \). By the preceding proof, the matrix \( A_D \) has rank one on \( \mathcal{V}_C(f) \). From this we see that for every \( \lambda \in \mathbb{C}^d \), we have
\[
a_{11} : (\lambda^T A_D \overline{\lambda}) - (\lambda^T A_D e_1)(\overline{\lambda}^T A_D e_1) \in (f).
\]
Hence \( \lambda^T A_D \overline{\lambda} \) is a curve of real contact with real-contact divisor \( f.(\lambda^T A_D e_1) - D \).
In fact, if \( M \) is a Hermitian determinantal representation of \( f \) and \( D \) is the divisor \( D_M \) defined in Proposition 4.4, then the systems of curves \( C(M) \) and \( C_D \) of Definition 3.1 and Lemma 4.8 are the same. We can tell whether or not \( D \) could come from a determinantal representation by examining the polynomials in \( C_D \).

**Proposition 4.9.** Suppose there exists a real line \( V_C(\ell) \) which meets \( V_C(f) \) in \( d \) distinct real points. If \( D \) is a real-contact divisor of degree \( d(d-1)/2 \) of \( V_C(f) \) and \( \det(M_D) \equiv 0 \), then there exists \( g \in C_D \) with \( \ell^2|g \).

**Proof.** Suppose that \( \det(M_D) \) is identically zero. From the proof of Theorem 4.6(a), we see that \( \det(A_D) \) is zero as well. First we show that there is some polynomial \( g \in C_D \) divisible by \( \ell \). Let \( f, \ell = P_1 + \cdots + P_d \). By construction of \( A_D \), we have \( \text{rank}(A_D(P_j)) \leq 1 \) for \( j = 1, \ldots, d \), so the left kernel of each \( A_D(P_j) \) has dimension at least \( d-1 \). Thus for each \( j \), there is a nonzero vector \( \lambda_j \) in the intersection of the left kernels of the \( d-1 \) matrices \( \{A_D(P_k) : k \neq j \} \). Since \( A_D \) is Hermitian, we see that \( \overline{\lambda_j} \) is then in the right kernel of the matrix \( A_D(P_k) \) for all \( k \neq j \).

Let \( \Lambda \) be the matrix \( (\lambda_1 \ldots \lambda_d) \). Since \( \det(A_D) \equiv 0 \), we know that the determinant of the matrix \( \Lambda^T A_D \overline{\Lambda} \) is identically zero. Moreover, its off diagonal entries \( \lambda_j^T A_D \overline{\lambda_k} \) for \( j \neq k \) vanish at each of the points \( P_1, \ldots, P_d \). Because these entries have degree \( d-1 \), they must vanish on the entire line \( V_C(\ell) \). So modulo the ideal \( (\ell) \) the matrix \( \Lambda^T A_D \overline{\Lambda} \) is diagonal. Because this matrix has determinant zero, we see that \( \ell \) must divide one of the diagonal entries, \( \lambda_j^T A_D \overline{\lambda_j} \), which is an element of \( C_D \).

Now we claim that this element \( g = \lambda_j^T A_D \overline{\lambda_j} \) must be divisible by \( \ell^2 \). We know that \( g = \ell h \) for some \( h \in \mathbb{R}[x, y, z] \). Since \( V_C(g) \) is a curve of real contact of \( V_C(f) \) and the intersection multiplicity of \( \ell \) and \( f \) in each \( P_j \) is equal to 1, we must have \( h(P_j) = 0 \) for \( j = 1, \ldots, d \), so that \( \ell \) divides \( h \) and \( \ell^2 \) divides \( g \). \( \square \)

Using this characterization, we see that if the input divisor \( D \) to Construction 4.5 comes from a curve interlacing \( f \), then the resulting matrix \( M_D \) is indeed a determinantal representation of \( f \).

**Proof of Theorem 4.6(b).** Suppose \( a_{11} = e_1^T A_D e_1 \) interlaces \( f \) with respect to \( e \). Equation (4.1) shows that for every \( \lambda \in \mathbb{C}^d \), the product \( a_{11} \cdot (\lambda^T A_D \overline{\lambda}) \) is nonnegative on \( V_R(f) \). Then, by Lemma 2.3, we see that \( (\lambda^T A_D \overline{\lambda}) \) interlaces \( f \). Since \( V_C(f) \) is smooth, \( f \) is square-free and every polynomial interlacing it must also be square-free. Hence every polynomial in \( C_D \) is square-free. Since \( f \) is hyperbolic, it satisfies the hypothesis of Proposition 4.9, and thus \( \det(M_D) \) cannot be zero.

Now we prove that \( M_D(e) \) is definite. To do this, we show that \( A_D \) is the adjugate matrix of \( M_D \). By construction, \( M_D = f^{2-d} \cdot A_D^{\text{adj}} \). Taking adjugates, we see that

\[
M_{\text{adj}}^D = \frac{1}{f^{(d-2)(d-1)}} \cdot (A_{\text{adj}}^D)^{\text{adj}} = \frac{1}{f^{(d-2)(d-1)}} \cdot \det(A_D)^{d-2} \cdot A_D = c^{d-2} A_D,
\]

where \( \det(A_D) = c f^{d-1} \) as in the proof of Theorem 4.6(a). Thus \( a_{11} \) is a constant multiple of \( e_1^T M_{\text{adj}}^D e_1 \) and belongs to \( C(M_D) \). Since \( a_{11} \) interlaces \( f \) with respect to \( e \), Theorem 3.3 implies that the matrix \( M_D(e) \) is definite. \( \square \)

**Corollary 4.10.** Every hyperbolic plane curve possesses a definite Hermitian determinantal representation.

**Proof.** Suppose \( f \in \mathbb{R}[x, y, z]_d \) is hyperbolic with respect to \( e \) and \( V_C(f) \) is smooth. Then the polynomial \( D_e f \) of (2.1) interlaces \( f \). By Proposition 4.3, \( D_e f \) is a curve of
real contact to $V_C(f)$. Thus, by Theorem 4.6, using any real-contact divisor coming from $D_e f$ as the input for Construction 4.5 will result in a definite determinantal representation of $f$.

Now all that remains is to address singular hyperbolic curves. Let $f \in \mathbb{R}[x, y, z]_d$ be hyperbolic with respect to $e$ with $f(e) = 1$. By Nuij [13], there exists a sequence of polynomials $(f_k) \subset \mathbb{R}[x, y, z]_d$ converging to $f$ such that for all $k$, $f_k$ is hyperbolic with respect to $e$, $f_k(e) = 1$, and $V_C(f_k)$ is smooth. Now each $f_k$ has a Hermitian determinantal representation, hence so does $f$ by Lemma 3.4. □

Remark 4.11. One can analyze the relation between real-contact divisors and Hermitian determinantal representations more precisely than we have done here: If $f = \det(M)$ is a Hermitian determinantal representation with corresponding real-contact divisor $D$, then the matrix $M_D$ is Hermite-equivalent to $M$, which means that there exists $U \in \text{GL}_d(\mathbb{C})$ such that $M = U^T M_D U$. Furthermore, if two Hermitian determinantal representations are Hermite-equivalent, the associated real-contact divisors are linearly equivalent. Conversely, if $D$ and $D'$ are two linearly equivalent real-contact divisors, then $M_D$ is Hermite-equivalent to either $M_{D'}$ or $-M_{D'}$. For a more detailed discussion, see [20, Thm. 8].

Finally, let us see Dixon’s construction in action.

Example 4.12. Here we apply Construction 4.5 to the quartic

\begin{equation}
(4.2) \quad f(x, y, z) = x^4 - 4x^2 y^2 + y^4 - 4x^2 z^2 - 2y^2 z^2 + z^4,
\end{equation}

which is hyperbolic with respect to the point $e = [1 : 0 : 0]$. This curve has two nodes, $[0 : 1 : 1]$ and $[0 : -1 : 1]$, but Dixon’s construction will still work. Figure 3 and Figure 4 show the real curve in the planes \{z = 1\} and \{x = 1\}, respectively.

![Figure 4. The hyperbolic quartic (4.2) and interlacing cubics from $C_D$.](image)

First we define $a_{11}$ to be the directional derivative $\frac{1}{4} D_e f = x^3 - 2xy^2 - 2xz^2$. The intersection divisor of $f$ and $a_{11}$ is the sum of the eight points $[2 : \pm \sqrt{3} : \pm i]$, $[2 : \pm i : \pm \sqrt{3}]$ and the two nodes, $[0 : \pm 1 : 1]$, each with multiplicity 2. By making some arbitrary choices, we can divide these points into two conjugate sets and write the divisor $f.a_{11}$ as $D + \overline{D}$ where

$$D = [0 : 1 : 1] + [0 : -1 : 1] + [2 : \sqrt{3} : i] + [2 : -\sqrt{3} : i] + [2 : i : \sqrt{3}] + [2 : i : -\sqrt{3}].$$
The vector space of cubics in \( \mathbb{C}[x, y, z] \) vanishing on these six points is four dimensional and we extend \( a_{11} \) to a basis \( \{a_{11}, a_{12}, a_{13}, a_{14}\} \) for this space, where
\[
\begin{align*}
a_{12} &= ix^3 + 4ixy^2 - 4x^2z - 4y^2z + 4z^3, \\
a_{13} &= -3ix^3 + 4x^2y + 4ixy^2 - 4y^3 + 4yz^2, \\
a_{14} &= -x^3 - 2ix^2y - 2ix^2z + 4xyz.
\end{align*}
\]

Then, to find \( a_{22} \) for example, we write \( a_{12} \cdot \overline{a_{12}} \) as an element of the ideal \( (f, a_{11}) \),
\[
a_{12} \cdot \overline{a_{12}} = (13x^3 - 14xy^2 - 22xz^2) \cdot a_{11} + (16z^2 - 12x^2) \cdot f,
\]
and set \( a_{22} = 13x^3 - 14xy^2 - 22xz^2 \). Similarly for other \( 2 \leq j \leq k \leq 4 \), we find \( a_{jk} \) by writing \( a_{jk} \cdot \overline{a_{12}} \) as an element of \( (f, a_{11}) \). The output of Construction 4.5 is then the Hermitian matrix of cubics
\[
A_D = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{12} & a_{22} & a_{23} & a_{24} \\
a_{13} & a_{23} & a_{33} & a_{34} \\
a_{14} & a_{24} & a_{34} & a_{44}
\end{pmatrix}
\]
By taking the adjugate of \( A_D \) and dividing by \( f^2 \), we find the desired Hermitian determinantal representation,
\[
M_D = \frac{1}{f^2} \cdot A_D^{\text{adj}} = 2^5 \begin{pmatrix}
14x & 2z & 2ix - 2y & 2i(y - z) \\
2z & x & 0 & -ix + 2y \\
-2ix - 2y & 0 & x & ix + 2y \\
-2i(y - z) & ix + 2y & -ix - 2z & 4x
\end{pmatrix}.
\]

The determinant of \( M_D \) is \( 2^{24} \cdot f \). As promised by Theorems 3.3 and 4.6, the cubics in \( C_D = C(M_D) \) interlace \( f \) (see Figure 4) and the matrix \( M_D \) is positive definite at the point \( (x, y, z) = (1, 0, 0) \).

In general, the challenge of carrying through Construction 4.5 in exact arithmetic is the computation of the intersection points. By contrast, computing a symmetric determinantal representation from a given contact curve is much simpler, but it may be very difficult to find a suitable curve to start from. For further algorithmic results, especially in the case of quartics, see [14] and [15]. Numerical computations seem more promising and we plan to pursue this in a future project.

5. Hyperbolicity cones and spectrahedra

For a polynomial \( f \) that is hyperbolic with respect to \( e \in \mathbb{R}^{n+1} \), the connected component of \( e \) in the complement of the hypersurface \( V_\mathbb{R}(f) \) plays a special role. This is a hyperbolicity cone, denoted \( C(f, e) \) and can also be defined as
\[
C(f, e) = \{a \in \mathbb{R}^{n+1} : f(te - a) \neq 0 \text{ when } t \leq 0\}.
\]

As shown in Gårding [7], \( C(f, e) \) is a convex cone and \( f \) is hyperbolic with respect to any point contained in it.

A hyperbolic program is the problem of optimizing a linear function over an affine slice of a hyperbolicity cone. Hyperbolic programming is a generalization of semi-definite programming, the problem of optimizing a linear function over an affine slice of the cone of positive semidefinite symmetric matrices. Such convex bodies are called spectrahedra. Because the determinant is a hyperbolic polynomial on the
space of real symmetric matrices, we see that every spectrahedral cone is indeed a hyperbolicity cone. A major open question is whether or not the converse holds.

**Generalized Lax Conjecture.** Every hyperbolicity cone is a spectrahedron.

Showing that a hyperbolicity cone $C(f,e)$ is spectrahedral amounts to finding a definite real symmetric determinantal representation for $f$ (or for an appropriate multiple of $f$). For a detailed discussion, see [21, Conjecture 3.3]. The work of Helton-Vinnikov [10] settled this for three dimensional hyperbolicity cones by showing that every hyperbolic polynomial in three variables has a definite symmetric determinantal representation. We conclude by noting that one can obtain the same result from the existence of definite Hermitian determinantal representations.

**Corollary 5.1.** Every three-dimensional hyperbolicity cone is a spectrahedron.

**Proof.** Let $f \in \mathbb{R}[x,y,z]$ be hyperbolic with respect to $e$. By Corollary 4.10, $f$ admits a definite Hermitian determinantal representation $f = \det(M)$. We can write $M = A + iB$, where $A$ is real symmetric and $B$ is real skew-symmetric, and define $N$ to be the real symmetric matrix

$$N = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

By the change of coordinates,

$$U^TNU = \begin{bmatrix} A - iB & 0 \\ 0 & A + iB \end{bmatrix} \quad \text{ where } \quad U = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot I & \frac{i}{\sqrt{2}} \cdot I \\ \frac{i}{\sqrt{2}} \cdot I & \frac{1}{\sqrt{2}} \cdot I \end{bmatrix},$$

we see that $\det(N) = \det(M) \det(M) = f^2$. The hyperbolicity cone of $f$ is the same as that of $f^2$ (with respect to $e$), which is the spectrahedron described by $N$. $\square$

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