THE LAGRANGIAN OF \( q \)-POINCARÉ GRAVITY

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Abstract

The gauging of the \( q \)-Poincaré algebra of ref. \( \text{[1]} \) yields a non-commutative generalization of the Einstein-Cartan lagrangian.

We prove its invariance under local \( q \)-Lorentz rotations and, up to a total derivative, under \( q \)-diffeomorphisms. The variations of the fields are given by their \( q \)-Lie derivative, in analogy with the \( q = 1 \) case. The algebra of \( q \)-Lie derivatives is shown to close with field dependent structure functions.

The equations of motion are found, generalizing the Einstein equations and the zero-torsion condition.
We describe in this Letter a geometric procedure to gauge the quantum Poincaré algebra found in ref. The lagrangian we obtain is a generalization of the Einstein-Cartan lagrangian, and has the same kind of symmetries (now \( q \)-deformed symmetries) as its classical counterpart: it is invariant under local Lorentz rotations and \( q \)-diffeomorphisms.

As one could expect, the differential calculus on the \( q \)-deformed Poincaré group is the correct framework for the program of finding a \( q \)-generalization of Einstein gravity. It was not obvious that this program could be carried to the end: in fact it can be done. We refer to ref. for most of the technicalities regarding the inhomogeneous quantum groups \( ISO_q(N) \) and their differential calculus. Here we concentrate directly on the \( ISO_q(3, 1) \) quantum Lie algebra, and discuss its gauging.

The method we follow is a natural \( q \)-extension of the geometric procedure described in ref. for classical gauge and (super)gravity theories. The starting point is the \( q \)-algebra \( ISO_q(3, 1) \) of ref.:

\[
\begin{align}
[\chi_{ab}, \chi_{cd}] &= C_{bc}\chi_{ad} + C_{ad}\chi_{bc} - C_{bd}\chi_{ac} - C_{ac}\chi_{bd} \\
[\chi_{12}, \chi_a]_{q^{-1}} &= q^{-\frac{1}{2}}C_{2a}\chi_1 - q^{-\frac{1}{2}}C_{1a}\chi_2 \\
[\chi_{13}, \chi_a]_{q^{-1}} &= q^{-\frac{1}{2}}C_{3a}\chi_1 - q^{-\frac{1}{2}}C_{1a}\chi_3 \\
[\chi_{14}, \chi_a] &= C_{4a}\chi_1 - C_{1a}\chi_4 \\
[\chi_{23}, \chi_a] &= C_{3a}\chi_2 - C_{2a}\chi_3 \\
[\chi_{24}, \chi_a] &= q^{\frac{1}{2}}C_{4a}\chi_2 - q^{\frac{1}{2}}C_{2a}\chi_4 \\
[\chi_{34}, \chi_a] &= q^{\frac{1}{2}}C_{4a}\chi_3 - q^{\frac{1}{2}}C_{3a}\chi_4 \\
[\chi_1, \chi_2]_{q^{-1}} &= 0, \quad [\chi_1, \chi_3]_{q^{-1}} = 0 \\
[\chi_1, \chi_4]_{q^{-2}} &= 0, \quad [\chi_2, \chi_3] = 0 \\
[\chi_2, \chi_4]_{q^{-1}} &= 0, \quad [\chi_3, \chi_4]_{q^{-1}} = 0
\end{align}
\]

where \([A, B]_s \equiv AB - sBA\). The subalgebra spanned by the Lorentz generators \( \chi_{ab} (= -\chi_{ba}) \) is classical; the deformation parameter \( q \) appears only in the commutation relations (2) and (3), involving the momenta \( \chi_a \). The metric \( C_{ab} \) is given by

\[
C_{ab} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

with Lorentz signature \((+, +, +, -)\). Only in the classical limit \( q \to 1 \) can one redefine the generators so as to diagonalize (4). The fact that the metric is diagonal

\[\text{1 the so-called “group manifold approach” was initiated in ref.s \[3\].}\]
in the indices 2,3 (and not completely antidiagonal as for the $q$-groups defined in [4]) is due to the existence of a particular real form on $SO_q(4; \mathbb{C})$. This real form, first discussed in ref. [5] for the uniparametric $q$-groups $SO_q(2n; \mathbb{C})$, was extended to the multiparametric case and to $ISO_q(2n; \mathbb{C})$ in [1], and allows to redefine antihermitian (linear combinations of the) generators, bringing the antidiagonal metric of ref. [4] in the hybrid form (4).

The algebra (1)-(3) was obtained in ref. [1] via a consistent projection from the $q$-Lie algebra of a particular multiparametric deformation of $SO(6)$, for which the $R$ matrix takes a very simple form: it is diagonal and satisfies $\hat{R}^2 = 1$, with $\hat{R} \equiv PR (\hat{R}^{ab \ cd} \equiv R^{ba \ cd})$.

The $q$-Lie algebra (1)-(3) has the form
\[ \chi_i \chi_j - \Lambda_{ij}^{kl} \chi_k \chi_l = C_{ij}^k \chi_k \]
where i,j... are adjoint indices running on the 10 values corresponding to the indices (a,ab) of the generators of $ISO_q(3,1)$. The non-vanishing components of the braiding matrix $\Lambda$ and the structure constants $C$, implicitly defined by (1)-(3), are given below (no sum on repeated indices):

\[ \Lambda_{ab \ cd \ ef \ gh} = \delta^a_b \delta^c_d \delta^e_f \delta^h_g \]
\[ \Lambda_{a bc \ de f} = (\alpha_{de})^{-2} \delta^a_f \delta^b_c \delta^d_e \]
\[ \Lambda_{bc \ a \ de f} = (\alpha_{de})^{-2} \delta^b_f \delta^c_d \delta^a_e \]
\[ \Lambda_{ab \ cd} = \beta_{cd} \delta^a_b \delta^c_d \]

\[ C_{ab \ cd} = \frac{1}{4} \left[ C_{ad} \delta^c_b \delta^d_f + C_{bc} \delta^a_f \delta^e_d - C_{ac} \delta^e_b \delta^d_f - C_{bd} \delta^a_c \delta^d_f \right] - (e \leftrightarrow f) \]
\[ C_{ab \ cd} = \frac{1}{2} \alpha_{ab} \left( C_{bc} \delta^d_a - C_{ac} \delta^d_b \right) \]
\[ C_{ab \ cd} = -\frac{1}{2} \alpha_{ab}^{-1} \left( C_{bc} \delta^d_a - C_{ac} \delta^d_b \right) \]

with
\[ \alpha_{12} = \alpha_{13} = q^{-\frac{1}{2}}, \quad \alpha_{24} = \alpha_{34} = q^{\frac{1}{2}}, \quad \alpha_{14} = \alpha_{23} = 1 \]
\[ \beta_{12} = \beta_{13} = \beta_{24} = \beta_{34} = q^{-1}, \quad \beta_{14} = q^{-2}, \quad \beta_{23} = 1, \quad \beta_{ab} = \beta_{ba}^{-1} \]

Note that the $\Lambda$ tensor has unit square, i.e.
\[ \Lambda_{ij}^{kl} \Lambda_{kl}^{mn} = \delta^i_m \delta^j_n \]
so that the algebra in (1)-(3) is a minimal deformation of $ISO(3,1)$. Deformations of Lie algebras whose braiding matrix has unit square were considered some time ago by Gurevich [3].
The Λ and C components in (16)-(19) satisfy the following conditions

\[ C_{ri} n C_{nj} = \Lambda_{kl} L_{ij} C_{nk} C_{sj} \quad (q\text{-Jacobi identities}) \quad (16) \]

\[ \Lambda_{nm} \Lambda_{ik} = \Lambda_{nk} \Lambda_{ij} (Yang-Baxter) \quad (17) \]

\[ C_{im} \Lambda_{mj} + \Lambda_{il} C_{mj} = \Lambda_{pq} \Lambda_{lk} C_{rp} + C_{mk} \Lambda_{is} \quad (18) \]

\[ C_{rk} m \Lambda_{ns} = \Lambda_{ij} C_{mk} \quad (19) \]

These are the “bicovariance conditions”, see refs. [7, 8, 9], necessary for the existence of a bicovariant differential calculus (see also the discussion in Appendix B of [1]). Whenever we have a set of matrices Λ_{ij} and C_{kij} satisfying (16)-(19) we can construct a differential calculus on the quantum group \( \text{Fun}_{q}(M_{ij}) \), generated by the elements (adjoint representation of the q-groups) \( M_{ij} \) satisfying the “ΛMM” relations:

\[ M_{ij} M_{ij} M_{ij} = M_{ij} \quad (20) \]

Consistency of these relations is ensured by the QYB equations (17). One can define in the usual way a coproduct \( \Delta(M_{ij}) = M_{ij} \otimes M_{ij} \) and a counit \( \varepsilon(M_{ij}) = \delta_{ij} \). When \( \Lambda^{2} = 1 \) one can also define a coinverse \( \kappa(M_{ij}) \) with \( \kappa^{2} = 1 \) (This is done by enlarging the algebra \( \text{Fun}_{q}(M_{ij}) \), see Appendix B of [1]).

The generators \( \chi_{i} \) of the q-Lie algebra (5) are functionals on \( \text{Fun}_{q}(M_{ij}) \):

\[ \chi_{j}(M_{ik}) = C_{ij} \quad (21) \]

We recall that products of functionals are defined via the coproduct \( \Delta \), i.e. \( \chi_{i} \chi_{j} = (\chi_{i} \otimes \chi_{j}) \Delta \), whereas functionals act on products as \( \chi_{i}(ab) = \Delta'(\chi_{i})(a \otimes b) \), \( a, b \in \text{Fun}_{q}(M_{ij}) \) (see below the definition of \( \Delta' \)).

Next we introduce new functionals \( f^{i}_{j} \) via their action on the basis \( M_{k,j} \):

\[ f^{i}_{j}(M_{k,j}) = \Lambda_{ij} k \quad (22) \]

The co-structures of \( \chi \) and \( f \) are given by:

\[ \Delta'(\chi_{i}) = \chi_{j} \otimes f^{j}_{i} + I' \otimes \chi_{i} \quad (23) \]

\[ \varepsilon'(\chi_{i}) = 0 \quad (24) \]

\[ \kappa'(\chi_{i}) = -\chi_{j} \kappa'(f^{j}_{i}) \quad (25) \]

\[ \Delta'(f^{i}_{j}) = f^{i}_{k} \otimes f^{k}_{j} \quad (26) \]

\[ \varepsilon'(f^{i}_{j}) = \delta^{i}_{j} \quad (27) \]

\[ \kappa'(f^{i}_{j}) = f^{i}_{j} \circ \kappa \quad (28) \]

The algebra generated by the \( \chi \) and \( f \) is a Hopf algebra (the \( \chi, f \) and \( f, f \) commutations are given in [8, 9, 1]), and defines a bicovariant differential calculus.
on the $q$-group generated by the $M_{ij}$ elements. For example, one can introduce left-invariant one-forms $\omega^i$ as duals to the “tangent vectors” $\chi_i$, an exterior product

$$\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}_{kl} \omega^k \otimes \omega^l,$$

an exterior derivative on $\text{Fun}_q(M_{ij})$ as

$$da = (id \otimes \chi_i) \Delta(a) \omega^i, \quad a \in \text{Fun}_q(M_{ij})$$

and so on. The commutations between one-forms and elements $a \in \text{Fun}_q(M_{ij})$ are given by:

$$\omega^i a = (id \otimes f^i_j) \Delta(a)$$

The exterior derivative can be extended to the (left-invariant) one-forms via the deformed Cartan-Maurer equations [7, 9]

$$d\omega^i + C_{ijk} \omega^j \wedge \omega^k = 0$$

The $C$ structure constants appearing in the Cartan-Maurer equations are related to the $C$ constants of the $q$-Lie algebra as [9]:

$$C_{jk}^i = C_{jk}^i - \Lambda^{rs}_{jk} C_{rs}^i$$

In the particular case $\Lambda^2 = I$ it is not difficult to see that $C = \frac{1}{2}C$.

The procedure we have advocated in ref.s [10] for the “gauging” of quantum groups essentially retraces the steps of the group-geometric method for the gauging of usual Lie groups, described for instance in ref.s [2].

We consider one-forms $\omega^i$ which are not left-invariant any more, so that the Cartan-Maurer equations are replaced by:

$$R^i = d\omega^i + C_{jk} \omega^j \wedge \omega^k$$

where the curvatures $R^i$ are now non-vanishing, and satisfy the $q$-Bianchi identities:

$$dR^i - C_{jk} \omega^j \wedge R^k + C_{jk} \omega^j \wedge R^k = 0$$

due to the Jacobi identities on the structure constants $C_{ijk}^k$ [7]. As in the classical case we can write the $q$-Bianchi identities as $\nabla R^i = 0$ (these define the covariant derivative $\nabla$).

Consider now the definition (34) of the curvature $R^i$, and apply it to the $q$-Poincaré algebra of (1)-(3): the one-forms dual to $\chi_{ab}$, $\chi_a$ are respectively denoted by $\omega^{ab}$, $V^a$ and the corresponding curvatures read (we omit wedge symbols):

$$R^{ab} = d\omega^{ab} + C_{cd} \omega^{ac} \omega^{db}$$

$$R^a = dV^a + \alpha_{af} C_{fb} \omega^{af} V^b$$

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where $V_a \equiv C_{ab}V^b$, $\alpha_{af}$ and $C_{ab}$ are given in (13) and (4), and we used $C_{ij}^k = \frac{1}{2}C_{ij}^k$. We have also rescaled $\omega^{ab}$ by a factor $\frac{1}{2}$ to obtain standard normalizations. $\tilde{R}^{ab}$ is the $q$-Lorentz curvature, coinciding with the classical one (as a function of $\omega^{ab}$), and $R^a$ is the $q$-deformed torsion.

The Bianchi identities, deduced from (35), are:

\[
\begin{align*}
    dR^{ab} - C_{fe}R^{af} \omega^{eb} + C_{fe}\omega^{af} R^{eb} &= 0 \\
    dR^a + \alpha_{af}C_{fb}R^{af} V^b - \alpha_{af}C_{fb}\omega^{af} R^b &= 0
\end{align*}
\]

From the definition (29) of the exterior product we see that for $\Lambda^2 = I$ the one-forms $\omega^i$ $q$-commute as:

\[
\omega^i\omega^j = -\Lambda_{ij}^{kl}\omega^k\omega^l
\]

Inserting the $\Lambda$ tensor corresponding to (6)-(9) we find:

\[
\begin{align*}
    V^a\omega^{12} &= -q^{-1}\omega^{12}V^a \\
    V^a\omega^{13} &= -q^{-1}\omega^{13}V^a \\
    V^a\omega^{14} &= -\omega^{14}V^a \\
    V^a\omega^{23} &= -\omega^{23}V^a \\
    V^a\omega^{24} &= -q\omega^{24}V^a \\
    V^a\omega^{34} &= -q\omega^{34}V^a \\
    V^2V^1 &= -q^{-1}V^1V^2 \\
    V^3V^1 &= -q^{-1}V^1V^3 \\
    V^4V^1 &= -q^{-2}V^1V^4 \\
    V^3V^2 &= -V^2V^3 \\
    V^4V^2 &= -q^{-1}V^2V^4 \\
    V^4V^3 &= -q^{-1}V^3V^4
\end{align*}
\]

and usual anticommutations between the $\omega^{ab}$ (components of the Lorentz spin connection). The exterior product of two identical one-forms vanishes (this is not true in general when $\Lambda^2 \neq I$). As a consequence, the exterior product of five vielbeins is zero.

We are now ready to write the lagrangian for the $q$-gravity theory based on $ISO_q(3,1)$. The lagrangian looks identical to the classical one, i.e.:

\[
\mathcal{L} = R^{ab}V^cV^d\varepsilon_{abcd}
\]

The Lorentz curvature $R^{ab}$, although defined as in the classical case, has non-trivial commutations with the $q$-vielbein:

\[
V^aR^{12} = q^{-1}R^{12}V^a
\]
\[ V^a R^{13} = q^{-1} R^{13} V^a \]
\[ V^a R^{14} = R^{14} V^a \]
\[ V^a R^{23} = R^{23} V^a \]
\[ V^a R^{24} = q R^{24} V^a \]
\[ V^a R^{34} = q R^{34} V^a \]
\[ \text{(44)} \]

deducible from the definition (36). As in ref. [10, 9], we make the assumption that the commutations of \(d\omega^i\) with the one-forms \(\omega^j\) are the same as those of \(C_{jk} \omega^j \omega^k\) with \(\omega^i\), i.e. the same as those valid for \(R^i = 0\). For the definition of \(\varepsilon_{abcd}\) in \((13)\), see below.

We discuss now the notion of \(q\)-diffeomorphisms. It is known that there is a consistent \(q\)-generalization of the Lie derivative (see ref.s [8, 12, 11, 1]) which can be expressed as in the classical case as:

\[ \ell_V = i_V d + d i_V \]
\[ \text{(45)} \]

where \(i_V\) is the \(q\)-contraction operator defined in ref.s [3, 11], with the following properties:

\[ i) \quad i_V(a) = 0, \quad a \in A, \quad V \text{ generic tangent vector} \]
\[ ii) \quad i_{t_i} \omega^j = \delta_i^j I \]
\[ iii) \quad i_{t_i}(\theta \wedge \omega^k) = i_{t_i}(\theta)\omega^j \Lambda^k_{ij} + (-1)^p \theta \, \delta_i^k, \quad \theta \text{ generic p-form} \]
\[ iv) \quad i_V(a\theta + \theta') = ai_V(\theta) + i_V\theta', \quad \theta, \theta' \text{ generic forms} \]
\[ v) \quad i_{t_V} = \lambda i_V, \quad \lambda \in \mathbb{C} \]
\[ vi) \quad i_{\varepsilon_V}(\theta) = i_V(\theta)\varepsilon, \quad \varepsilon \in A \]
\[ \text{(46)} \]

As a consequence, the \(q\)-Lie derivative satisfies:

\[ i) \quad \ell_V a = i_V(da) \equiv V(a) \]
\[ ii) \quad \ell_V d = d\ell_V \]
\[ iii) \quad \ell_V(\lambda \theta + \theta') = \lambda \ell_V(\theta) + \ell_V(\theta') \]
\[ iv) \quad \ell_{\varepsilon_V}(\theta) = (\ell_V\theta)\varepsilon - (-1)^p i_V(\theta)d\varepsilon, \quad \theta \text{ generic p-form} \]
\[ v) \quad \ell_{t_i}(\theta \wedge \omega^k) = (\ell_{t_i}\theta) \wedge \omega^j \Lambda^k_{ij} + \theta \wedge \ell_{t_i}\omega^k \]
\[ \text{(47)} \]

In analogy with the classical case, we define the \(q\)-diffeomorphism variation of the fundamental field \(\omega^i\) as

\[ \delta \omega^k \equiv \ell_{\varepsilon_{t_i}} \omega^k \]
\[ \text{(48)} \]

where according to iv) in (37):

\[ \ell_{\varepsilon_{t_i}} \omega^k = (i_{t_i} d\omega^k + di_{t_i} \omega^k)\varepsilon^i + d\varepsilon^k = (i_{t_i} d\omega^k)\varepsilon^i + d\varepsilon^k \]
\[ \text{(49)} \]

As in the classical case, there is a suggestive way to write this variation:

\[ \ell_{\varepsilon_{t_i}} \omega^k = i_{\varepsilon_{t_i}} R^k + \nabla \varepsilon^k \]
\[ \text{(50)} \]
where
\[
\nabla \varepsilon^k \equiv d\varepsilon^k - C_{rs}^k i_t (\omega^r \wedge \omega^s) \varepsilon^i = d\varepsilon^k - C_{rs}^k \varepsilon^r \omega^s + C_{rs}^k \omega^r \varepsilon^s
\]
(51)

Proof: use the Bianchi identities (35) and iii) in (46).

Notice that if we postulate:
\[
\Lambda^{rk} l_i \omega^l \varepsilon^i = \varepsilon^r \omega^k
\]
\[
\Lambda^{rk} l_i \omega^l \wedge d\varepsilon^i = - d\varepsilon^r \wedge \omega^k
\]
(52)
we find
\[
\delta (\omega^j \wedge \omega^k) = \delta \omega^j \wedge \omega^k + \omega^j \wedge \delta \omega^k
\]
(53)
i.e. a rule that any “sensible” variation law should satisfy. To prove (52) use iv) and v) of (47). The \(q\)-commutations (52) were already proposed in [10] in the context of \(q\)-gauge theories. A consequence of (52) are the following commutations between the variation parameter and \(d\omega^j\):
\[
\Lambda^{rk} l_i d\omega^l \varepsilon^i = \varepsilon^r d\omega^k
\]
(54)
As discussed in ref.s [10], it is consistent to postulate that \(R^i\) has the same commutations with \(\varepsilon^j\) as \(d\omega^j\):
\[
\Lambda^{rk} l_i R^l \varepsilon^i = \varepsilon^r R^k
\]
(55)

We have now all the tools we need to investigate the invariances of the \(q\)-gravity lagrangian (43). The result will be analogous to the classical one: after imposing the horizontality conditions
\[
i_{ab} R^{cd} = i_{ab} R^c = 0
\]
(56)
along the Lorentz directions one finds that, \textit{provided} the \(\varepsilon\) tensor in (43) is appropriately defined, the lagrangian is invariant under \(q\)-diffeomorphisms and local Lorentz rotations. Note that, as in the \(q = 1\) case, the horizontality conditions (56) can be obtained as field equations (see later).

We first consider Lorentz rotations. Under these, the curvature \(R^{ab}\) and the vielbein \(V^c\) transform as:
\[
\delta R^{ab} \equiv \ell_{gh} t_{gh} R^{ab} = C_{ef}^{gh} a^{ab} R^{ef} \varepsilon^{gh} - C_{gh}^{ef} a^{ab} \varepsilon^{gh} R^{ef}
\]
(57)
\[
\delta V^c \equiv \ell_{gh} t_{gh} V^c = - C_{ef}^{gh} e^{ef} V^g + C_{gh}^{ef} e^{ef} V^g \varepsilon^{ef}
\]
(58)
To obtain these variations, use the definition (43), iv) of (47), the Bianchi identity (38) and the horizontality conditions (56).
Now we have the lemma:
\[ \delta \mathcal{L} = \left[ \left( \delta R^{ab} \right) V^c V^d + R^{ab} \left( \delta V^c \right) V^d + R^{ab} V^c \left( \delta V^d \right) \right] \varepsilon_{abcd} \] (59)

Proof: use v) of (17) and the first of (52).

Inserting the variations (57) and (58) inside (59) we find, after ordering the terms as
\[ \varepsilon RV \] with (52) and (55):
\[ \delta \mathcal{L} = 2 \left( C_{ef gh}^{\ ab} \varepsilon_{abcd} - C_{gh cd}^{\ e} \varepsilon_{efsd} - C_{rs \ d}^{\ p} \Lambda^{\ rs q \ gh} c \varepsilon_{efqp} \right) \varepsilon^{gh} \mathcal{R}^{ef} V^c V^d \] (60)

Using the explicit form of the \( \Lambda \) and \( \mathbf{C} \) tensors into (6)-(12) and imposing \( \delta \mathcal{L} = 0 \), we find a set of equations for the \( \varepsilon_{abcd} \) tensor. These can in fact be solved and yield:
\[ \begin{align*}
\varepsilon_{1234} &= 1, & \varepsilon_{1243} &= -q, & \varepsilon_{1324} &= -1, & \varepsilon_{1342} &= q, \\
\varepsilon_{1423} &= q, & \varepsilon_{1432} &= -q, & \varepsilon_{2134} &= -q, & \varepsilon_{2143} &= 1, \\
\varepsilon_{2314} &= q, & \varepsilon_{2341} &= -q, & \varepsilon_{3124} &= q, & \varepsilon_{3142} &= -q, \\
\varepsilon_{3214} &= q, & \varepsilon_{3241} &= -q, & \varepsilon_{4123} &= q, & \varepsilon_{4132} &= q, \\
\varepsilon_{4213} &= -q, & \varepsilon_{4231} &= -q, & \varepsilon_{3412} &= q, & \varepsilon_{3421} &= -q.
\end{align*} \] (61)

Consider next the variation of \( \mathcal{L} \) under \( q \)-diffeomorphisms, i.e.:
\[ \delta \mathcal{L} = \ell_{\varepsilon t_g} \mathcal{L} = \left( \ell_{t_g} \mathcal{L} \right) \varepsilon^g - \left( i_{t_g} \mathcal{L} \right) d\varepsilon^g = \] (62)
\[ = d \left[ i_{t_g} \left( R^{ab} V^c V^d \varepsilon_{abcd} \right) \varepsilon^g \right] + i_{t_g} \left[ d \left( R^{ab} V^c V^d \varepsilon_{abcd} \right) \varepsilon^g \right] \] (63)

Then the variation \( \delta \mathcal{L} \) is a total derivative if
\[ d \left( R^{ab} V^c V^d \varepsilon_{abcd} \right) = 0 \] (64)

After using the expression for \( dR^{ab} \) given by the Bianchi identity (38) and the torsion definition (37) to find \( dV^a \) (note that \( R^{ab} R^c V^d \varepsilon_{abcd} = 0 \) because of horizontality of \( R^{ab}, R^c \) and the vanishing of the product of five vielbeins), eq. (64) yields a set of conditions on \( \varepsilon_{abcd} \). These conditions in fact coincide with those found to ensure local Lorentz invariance of the \( q \)-lagrangian. This is not a miracle: indeed we could have computed the Lorentz variation of \( \mathcal{L} \) in the same way as in (83); the total derivative term would have vanished because \( i_{t_g} \left( R^{ab} V^c V^d \varepsilon_{abcd} \right) = 0 \) (horizontality of \( R^{ab} \)), and we would have found again eq. (64) as a condition for \( \delta \mathcal{L} = 0 \).

Thus the lagrangian (13) with the \( \varepsilon_{abcd} \) tensor as given in (61) is invariant (up to a total derivative) also under \( q \)-diffeomorphisms.

We discuss now the algebra of \( q \)-Lie derivatives. We have the theorem, analogous to the classical one:
\[ \ell_i \ell_j - \Lambda^{kl}_{ij} \ell_k \ell_l = \ell_i \left( C_{ij}^{\ \ e} - R^e_{ij} \right) \varepsilon^n \] (65)
with

\[ R^i \equiv R^i_{jk} \omega^j \land \omega^k \]  
\[ R^i_{jk} \equiv R^i_{jk} - \Lambda_{rs}^{jk} R^i_{rs} \]  

As for the structure constants, we have

\[ R^i_{jk} = \frac{1}{2} R^i_{jk} \] when \( \Lambda^2 = 1 \). The proof of the composition law (65) is computational: one applies its left-hand side to \( \omega^k \), and uses the properties of the \( q \)-Lie derivative. \textit{Hint 1}: rewrite the Lie derivative of \( \omega^k \) as:

\[ \ell_t \omega^k = (C_{rj}^k - R^k_{rj}) \omega^r \]  

\textit{Hint 2}: use the following expression for \( \Lambda_{ij}^{kl} \) (no sums on repeated indices):

\[ \Lambda_{ij}^{kl} = [kl] \delta_i^l \delta_j^k \]  

and the identities

\[ [kl] = \frac{1}{[lk]} \]  
\[ C_{rk}^s[lk][lr] = C_{rk}^s[ls] \]  

due to \( \Lambda^2 = 1 \) and the bicovariance condition (19).

From the \( q \)-algebra (63) it is not difficult to find the following composition law for \( q \)-variations:

\[ \ell_{t_j} \omega^k = (C_{rj}^k - R^k_{rj}) \omega^r \]  

The order of the factors is important in the composite parameter \( (C_{ij}^n - R^n_{ij}) \varepsilon^i_2 \varepsilon^j_1 \). Note also that

\[ (C_{ij}^n - R^n_{ij}) \varepsilon^i_2 \varepsilon^j_1 = \frac{1}{2} (C_{ij}^n - R^n_{ij}) (\varepsilon^i_2 \varepsilon^j_1 - \varepsilon^j_2 \varepsilon^i_1) \]  

if we postulate the commutation rule

\[ \varepsilon^i_1 \varepsilon^j_2 = \Lambda_{ij}^{kl} \varepsilon^k_2 \varepsilon^l_1 \]  

Indeed \( (C_{ij}^n - R^n_{ij}) \Lambda_{ij}^{kl} = -(C_{kl}^n - R^n_{kl}) \) (due to \( \Lambda^2 = 1 \)). Then the composite parameter is explicitly \( (1 \leftrightarrow 2) \) antisymmetric.

Let us derive the equations of motion corresponding to the \( q \)-lagrangian (18). We assume the same variational rule as with the Lie derivative. The \( q \)-Einstein equations are obtained by varying the vielbein in \( \mathcal{L} \):

\[ \delta \mathcal{L} = R^{ab} (\delta V^c V^d + V^c \delta V^d) \varepsilon_{abcd} = 0 \]  

Postulating that \( \delta \omega^i \) has the same commutations as \( \omega^i \), and noticing that

\[ \varepsilon_{abfc} = -\Lambda^{cd} \varepsilon_{ef} \varepsilon_{abcd} \]
(use the explicit entries in (9) and (11), or notice that since $\varepsilon_{abcd}$ multiplies $V^cV^d$ in (13), it must be $\Lambda$-antisymmetric in the indices c,d), one arrives at:

$$R^{ab}V^e\varepsilon_{abef} = 0$$  \[77\]

The $q$-Einstein equations are found as in the classical case: expand (77) along three vielbeins:

$$R^{ab}_{cd}V^cV^dV^e\varepsilon_{abef} = 0,$$  \[78\]
multiply by another vielbein $V^g$ and use:

$$\varepsilon^{cd eg}V^1V^2V^3V^4 \equiv V^cV^dV^eV^g$$  \[79\]

(N.B.: the entries of $\varepsilon^{abcd}$ are different from those of $\varepsilon_{abcd}$) so that finally we have:

$$R^{ab}_{cd}\varepsilon^{cd ef} = 0$$  \[80\]

The contraction of the two alternating tensors yields a $q$-weighted product of Kronecker deltas. We leave to the reader to find the final form of the $q$-Einstein equations. Expanding (77) on $\omega V V V$ yields instead the horizontality condition on $R^{ab}$.

The torsion equation is obtained by varying (43) in the spin connection $\omega^{ab}$. The final result is again an equation that formally looks identical to the classical one:

$$R^cV^d\varepsilon_{abcd} = 0$$  \[81\]

As for $q = 1$, this equation implies that the torsion vanishes as a two-form: $R^c = 0$ (hence also horizontality of $R^c$).

**Note 1:** The $q$-volume 4-form $V^1V^2V^3V^4 = f(q)V^aV^bV^cV^d\varepsilon_{abcd}$ (with $[f(q)]^{-1} = \varepsilon^{abcd}\varepsilon_{abcd} = 2q^{-\frac{q}{2}}(2q^{-\frac{q}{2}} + q^{-1} + 2q^{-\frac{3}{2}} + 2 + 2q^{1/2} + q + 2q^{3/2})$) is invariant under Lorentz rotations and, up to a total derivative, under $q$-diffeomorphisms. The proof is similar to the one used for the lagrangian. This means that the $q$-symmetries allow a cosmological constant term $V^aV^bV^cV^d\varepsilon_{abcd}$.

**Note 2:** It would be worthwhile to give a recipe for extracting numbers out of a $q$-theory of the kind discussed in this Letter. A possible way of doing this would be to find a consistent definition of path-integral on $q$-commuting fields, leading to $\mathbb{C}$-number amplitudes. On this question, see for example [13].

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