ON THE NUMBER OF $k$TH POWERS INSIDE ARITHMETIC PROGRESSIONS

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Abstract. We find upper bounds that are sharp for the number of $k$th powers inside arbitrary arithmetic progressions whose step has $O(1)$ many divisors.

For fixed $k \geq 2$, how many $k$th powers of integers $t^k$ can lie inside an arithmetic progression of length $N$? By considering the progression $\{1, 2, \ldots, N\}$ we see that this number can be as large as $\sim N^{1/k}$. In this note we search for upper bounds, and show that they match the above lower bound in the special case when the step of the progression has $O(1)$ many divisors.

More precisely, let $Q_k(N; q, a)$ denote the number of $k$th powers in the arithmetic progression $a + q, a + 2q, \ldots, a + Nq$. Write

$$Q_k(N) = \sup_{a, q \in \mathbb{N}_{q \neq 0}} Q_k(N; q, a).$$

Rudin [5] has conjectured that $Q_2(N) \sim N^{1/2}$. We may similarly conjecture that for each $k \geq 2$

$$Q_k(N) \lesssim N^{1/k},$$

where $\lesssim$ denotes logarithmic losses of the form $(\log N)^{O(1)}$. The logarithmic loss here is added for extra safety, it is not clear whether it is really needed.

The best known upper bound for $Q_2(N)$ is due to Bombieri and Zannier [2]

$$Q_2(N) \lesssim N^{3/4}.$$ 

This builds on earlier work [1] of Bombieri, Granville and Pintz that proved the result with exponent $\frac{2}{3}$ in place of $\frac{3}{4}$. These rely on deep results in number theory regarding rational points on curves.

The papers [4] and [3] contain a nice discussion on the problem. In particular, they mention a refined version of Rudin’s conjecture, according to which the progression $\{24n + 1 : 0 \leq n \leq N - 1\}$ contains the largest number of squares.

We verify the conjectured inequality (1) in the case when the step $q$ of the progression has $O(1)$ divisors. In fact we observe that the argument extends to arbitrary polynomials. It would be of interest to lower the exponent of $d(q)$ as much as possible.

**Theorem 0.1.** Let $d(q)$ be the number of divisors of $q$. Then for each polynomial $P_k$ of degree $k \geq 1$ with integer coefficients and each $a, q, N \in \mathbb{Z}$ we have

$$|\{t \in \mathbb{Z} : P_k(t) \in \{a + q, a + 2q, \ldots, a + Nq\}\}| \lesssim d(q)^{k-1}N^{1/k}.$$

The implicit constant depends only on $k$.

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Proof We use induction on \( k \). The case \( k = 1 \) is trivial. Assume the statement holds for \( k \), and let \( P_{k+1} \) be a polynomial of degree \( k + 1 \). Fix \( t_0 \neq t \) such that
\[
P_{k+1}(t), P_{k+1}(t_0) \in \{a + q, a + 2q, \ldots, a + Nq\}.
\]
Write
\[
P_{k+1}(t) - P_{k+1}(t_0) = (t - t_0)P_k(t),
\]
for some polynomial \( P_k \) of degree \( k \). We must have
\[
\begin{aligned}
(t - t_0) &= n_1q_1 \\
P_k(t) &= q_2n_2
\end{aligned} \tag{2}
\]
with \( q_1q_2 = q \). Moreover \( n_1n_2 \leq N \). We must have that either
\[
n_1 \leq N^{\frac{k}{k+1}}
\]
or
\[
n_2 \leq N^{\frac{k}{k+1}}.
\]
Let us fix the pair \((q_1, q_2)\). In the first case, there are \( \leq N^{\frac{k}{k+1}} \) possible values of \( t \) (considering only the first equation in (2)), while in the second case there are \( O(d(q_2)^{k-1}N^{\frac{1}{k+1}}) \) values of \( t \), due to the induction hypothesis (considering only the second equation in (2)). Since there are \( d(q) \) many ways to choose the pair \((q_1, q_2)\) we conclude that the total contribution is
\[
\lesssim d(q)(N^{\frac{1}{k+1}} + d(q)^{k-1}N^{\frac{1}{k+1}}) \lesssim d(q)^kN^{\frac{1}{k+1}}.
\]

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