Computing Lyapunov functions using deep neural networks

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Abstract: We propose a deep neural network architecture and a training algorithm for computing approximate Lyapunov functions of systems of nonlinear ordinary differential equations. Under the assumption that the system admits a compositional Lyapunov function, we prove that the number of neurons needed for an approximation of a Lyapunov function with fixed accuracy grows only polynomially in the state dimension, i.e., the proposed approach is able to overcome the curse of dimensionality. We show that nonlinear systems satisfying a small-gain condition admit compositional Lyapunov functions. Numerical examples in up to ten space dimensions illustrate the performance of the training scheme.

Keywords: deep neural network, Lyapunov function, stability, small-gain condition, curse of dimensionality, training algorithm

1 Introduction

Lyapunov functions are one of the key tools for the stability analysis of nonlinear systems. They do not only serve as a certificate for asymptotic stability of an equilibrium but also allow to give estimates about its domain of attraction or to quantify its robustness with respect to perturbations, for instance, in the sense of input-to-state stability. However, explicit analytic expressions for Lyapunov functions are often not available. Hence, the numerical computation of Lyapunov functions has attracted significant attention during the last decades. Known approaches use series expansions [29], finite element approaches [4], representations by radial basis functions [11] or piecewise affine functions, see [16], or sum-of-squares techniques, see [3] and the references therein. For a comprehensive overview we refer to the survey by [12]. Often, a characterization of the Lyapunov function via a suitable partial differential equation (PDE) such as Zubov’s equation [39] is used as the basis for these numerical computations.

The usual approaches have in common that the number of degrees of freedom needed for storing the Lyapunov function (or an approximation thereof with a fixed approximation error) grows very rapidly — typically exponentially — with the dimension of the state
space. This is the well known curse of dimensionality, which leads to the fact that the mentioned approaches are confined to low dimensional systems.

In general, the same is true if a deep neural network is used as an approximation architecture. While it is known that such a network can approximate every $C^1$-function arbitrarily well, see [6, 19], the number of neurons needed for this purpose typically grows exponentially with the state dimension, as well, see [29, Theorem 2.1] or Theorem 4.1, below. However, this situation changes if additional structural assumptions are imposed, which is the approach we follow in this paper. Recently, there has been a large activity in exploiting suitable structural properties for solving high-dimensional PDEs using neural networks [7, 10, 13, 18, 20, 22, 21, 33, 37] and since Lyapunov functions can also be represented by PDEs, these results provided the first source of inspiration for this paper.

As we will show in this paper, in the Lyapunov function context a suitable property for making the neural network approach efficient is the existence of what we call a compositional Lyapunov function, cf. Definition 3.1, below. The importance of compositionality for overcoming the curse of dimensionality is explained in [32], and this reference provides the second source of inspiration for this paper. Using similar arguments as in [32], we show that a suitably designed deep neural network can compute approximations of compositional Lyapunov function with a given required accuracy using a number of neurons that grows only polynomially with the dimension of the system. In other words, we show that the curse of dimensionality can be avoided.

The important question then is how restrictive the assumption of the existence of a compositional Lyapunov function is. It turns out that a classical systems theoretic tool for stability analysis of large scale systems, namely small-gain analysis — here in its nonlinear form based on input-to-state stability, see, e.g., [8, 9, 23, 24, 35] — provides conditions on the dynamics under which a compositional Lyapunov function exists. This insight together with the design of a corresponding deep neural network architecture with two hidden layers constitutes the theoretical contribution of this paper. This is complemented by an algorithmic contribution in form of a training algorithm for neural networks that is based on a suitable partial differential inequality, and by numerical tests that illustrate the efficiency of the proposed “DeepLyapunov” method.

There have been earlier attempts to use neural networks for approximating Lyapunov functions. The paper [36] proposes a learning algorithm based on increments instead of derivatives, which relies on successive updates of the network parameters rather than a standard learning algorithm. This paper does not provide numerical examples illustrating the performance of the approach. In [31] only a local Lyapunov function is computed, by using local derivative information in the learning algorithm. In the paper [30] the coefficients of a polynomial Lyapunov functions are computed, rather than representing the Lyapunov function directly by a neural network as in our paper. There are also various papers dealing with the more general problem of computing control Lyapunov functions (clfs). [28] considers this problem by assuming exact representability of the Lyapunov function by a neural network with one hidden layer. The paper [34] considers clfs of a particular quadratic form in discrete time. It implements the decrease condition via classification rather than differential inequalities. The paper [25] considers clfs for models from robotics and optimizes the parameters of a quadratic Lyapunov function candidate. Finally, among the many papers considering neural network based solutions of Hamilton-
Jacobi-Bellmann equations some also yield Lyapunov functions. For instance, this is done in [2], in which neural networks with one hidden layer are considered. As just described, all these references differ in several aspects from the approach proposed in this paper. Yet, the main difference of all these references to our paper is that none of them carries out a complexity analysis or provides a network structure that is provably able to overcome the curse of dimensionality and performs well for higher dimensional nonlinear systems in numerical experiments. This is the distinctive contribution of this paper.

The remainder of the paper is organized as follows. In Section 2 we formulate the problem. In Section 3 we explain the concept of compositional Lyapunov functions and its relation to small-gain theory. Section 4 gives a brief introduction into neural networks, mainly in order to clarify the notation used in Section 5. In this section we propose a neural network architecture and prove that it allows to store approximations to compositional Lyapunov functions avoiding the curse of dimensionality. In Section 6 we propose a training algorithm that allows to actually compute Lyapunov functions using the proposed neural networks. Numerical results illustrating the performance of our approach are given in Section 7. In Section 8 we discuss various aspects and extensions of our approach before we conclude our paper in Section 9. The results from Section 5 are contained in preliminary form in the conference paper [14]. However, [14] did not address training algorithms nor did it present numerical results. Moreover, the proofs in Section 5 are given in more detailed form in the present paper.

2 Problem Formulation

We consider nonlinear ordinary differential equations of the form

$$\dot{x}(t) = f(x(t))$$ (2.1)

with a Lipschitz continuous $f : \mathbb{R}^n \to \mathbb{R}^n$. We assume that $x = 0$ is an asymptotically stable equilibrium and that $K_n \subset \mathbb{R}^n$ is a compact set in its domain of attraction.

It is well known (see, e.g., [17]) that asymptotic stability holds if and only if there exists a $C^1$-function $V : O \to \mathbb{R}$, defined on an open set $O$ containing $K_n$ and satisfying $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, $V(x) \to \infty$ as $\|x\| \to \infty$, and

$$DV(x)f(x) \leq -h(x)$$ (2.2)

for a function $h : \mathbb{R}^n \to \mathbb{R}$ with $h(x) > 0$ for all $x \in K_n$ with $x \neq 0$. It is our goal in this paper to design a neural network that is able to compute an approximation of such a Lyapunov function on $K_n \subset \mathbb{R}^n$ in an efficient manner. Efficient here is meant in the sense that the computational effort as well as the storage effort grow moderately with the space dimension. While this will not be possible in general, we will show that it works for Lyapunov functions satisfying a particular structure, which we call compositional Lyapunov functions. This structure is motivated by recent results on approximation properties of neural networks, but it turns out that it is also well known in systems theory, as it corresponds to a particular kind of a small-gain condition. Throughout this paper, we make the standing assumption

$$\text{there exists } C > 0 \text{ with } K_n \subset [-C, C]^n \text{ for all } n \in \mathbb{N}$$ (2.3)
in order to avoid that the sets $K_n$ grow unboundedly with the dimension $n$.

3 Compositional Lyapunov functions and small-gain theory

The particular compositional structure we consider is motivated by [32], where the approximation of general functions via neural networks is considered. In order to define this structure, the system (2.1) is divided into $s$ subsystems $\Sigma_i$ of dimensions $d_i$, $i = 1, \ldots, s$. To this end, the state vector $x = (x_1, \ldots, x_n)^T$ and the vector field $f$ are split up as

$$x = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_s \end{pmatrix} \quad \text{and} \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_s(x) \end{pmatrix},$$

with $z_i = (x_{d_{i-1}+1}, \ldots, x_{d_i}) \in \mathbb{R}^{d_i}$ and $f_i : \mathbb{R}^n \to \mathbb{R}^{d_i}$ denoting the state and dynamics of each $\Sigma_i$, $i = 1, \ldots, s$, with state dimension $d_i \in \mathbb{N}$ and $\hat{d}_i = \sum_{j=1}^i d_j$. With

$$z_{-i} := \begin{pmatrix} z_1 \\ \vdots \\ z_{i-1} \\ z_{i+1} \\ \vdots \\ z_s \end{pmatrix},$$

and by rearranging the arguments of the $f_i$, the dynamics of each $\Sigma_i$ can then be written as

$$\dot{z}_i(t) = f_i(z_i(t), z_{-i}(t)), \quad i = 1, \ldots, s.$$ Using this decomposition, we can define the following Lyapunov function structure.

**Definition 3.1:** A Lyapunov function $V$ for (2.1) is called **compositional**, if there exist $C^1$-functions $\hat{V}_i : \mathbb{R}^{d_i} \to \mathbb{R}$ such that $V$ is of the form

$$V(x) = \sum_{i=1}^s \hat{V}_i(z_i). \quad (3.1)$$

In the remainder of this section we discuss conditions on $f$ under which a Lyapunov function of the form (3.1) exists.

One situation in which a system (2.1) admits a compositional Lyapunov function is when the $f_i$ do not depend on $z_{-i}$, i.e., if $f_i(z_i, z_{-i}) = f_i(z_i)$. This means that the subsystems are completely decoupled. In this case, consider Lyapunov functions $\hat{V}_i$ of $\dot{x}_i = f_i(x_i)$ on compact sets $\hat{K}_i \subset \mathbb{R}^{d_i}$, and $V$ from (3.1). Then, clearly $V(x) \geq 0$ and $V(x) = 0$ if and only if $x = 0$. Moreover,

$$DV(x)f(x) = \sum_{i=1}^s DV_i(z_i)f_i(z_i) < 0$$
for all $x \in K_n = \prod_{i=1}^s \tilde{K}_i$ with $x \neq 0$.

Assuming that $f$ decomposes into $s$ completely decoupled subsystems is relatively restrictive. Fortunately, a similar construction can also be made if the $f$ are coupled, provided the coupling is such that it does not affect the stability of the overall subsystem. The systems theoretic tool for this approach is nonlinear small-gain theory, which relies on the input-to-state stability (ISS) property introduced in [38]. It goes back to [23, 24] and in the form for large-scale systems we require here it was developed in the thesis [35] and in a series of papers around 2010, see, e.g., [8, 9] and the references therein. ISS small-gain conditions can be based on trajectories or Lyapunov functions and exist in different variants. Here, we use the variant that is most convenient for obtaining approximation results because it yields a smooth Lyapunov function. We briefly discuss one other variant in Section 8(vi).

For formulating the small gain condition, we assume that for the subsystems $\Sigma_i$ yields a smooth Lyapunov function. We briefly discuss one other variant in Section 8(vi).

Theorem 3.3: Assume that the small-gain conditions from Definition 3.2 hold. Then $V$ from (3.1) is a Lyapunov function for the $C^1$-functions $\hat{V}_i : \mathbb{R}^{d_i} \to \mathbb{R}$ given by

$$\hat{V}_i(z_i) := \int_0^{V_i(z_i)} \lambda_i(\tau) d\tau$$

where $\lambda_i(\tau) = \eta_i(\alpha_i(\tau))$. 

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1As usual, we define $\mathcal{K}_\infty$ to be the space of continuous functions $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha$ is strictly increasing to $\infty$.

2A continuous function $\rho : [0, \infty) \to [0, \infty)$ is called positive definite if $\rho(0) = 0$ and $\rho(r) > 0$ for all $r > 0$.  

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In [8], the property from Definition 3.2 is called a weak small-gain condition. This is because if the system (2.1) has an additional input (that is taken into account in the assumptions on the $V_i$), then the construction of $V$ yields an integral ISS Lyapunov function as opposed to an ISS Lyapunov function. Under a stronger version of the small-gain condition, the same construction yields an ISS Lyapunov function. We briefly discuss corresponding extensions of our approach in Section 8(iv).

We note that for various reasons small-gain conditions are tricky to check and to apply: the gains $\gamma_{ij}$ may be difficult to estimate, the functions $\eta_i$ may be hard to find and, above all, appropriate Lyapunov functions $V_i$ for the subsystems may be nontrivial to construct. However, none of these ingredients need to be known for our approach. Moreover, not even the number and the dimension of the subsystems needs to be known and we will also be able to define the $z_i$ in a more general way than we did in this section. All that needs to be assumed in what follows is that a compositional Lyapunov function $V$ of the form (3.1) exists. In summary, the small-gain theory just presented only serves to show that it is realistic to assume the existence of such a $V$, but it is not needed for constructing it.

4 Deep neural networks

A deep neural network is a computational architecture that has several inputs, which are processed through $\ell \geq 1$ hidden layers of neurons. The values in the neurons of the layer with the largest $\ell$ are used in order to compute the output of the network. In this paper, we will only consider feedforward networks, in which the input is processed consecutively through the layers 1, 2, ..., $\ell$. For our purpose of representing Lyapunov functions, we will use networks with the input vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and a scalar output $W(x; \theta) \in \mathbb{R}$. Here, the vector $\theta \in \mathbb{R}^P$ represents the free parameters in the network that need to be tuned (or "learned") in order to obtain the desired output. In our case, the output shall approximate a Lyapunov function, i.e., we want to find $\theta^*$ such that $W(x; \theta^*) \approx V(x)$ for a Lyapunov function $V$ and all $x \in K_n$, where $K_n$ is the compact set on which $V$ shall be computed. Figure 4.1 shows generic neural networks with one and two hidden layers.

Here, the lowest layer is the input layer, followed by one or two hidden layers numbered with $\ell$, and the output layer. The number $\ell_{\text{max}}$ determines the number of hidden layers, here $\ell_{\text{max}} = 1$ or 2. Each hidden layer consists of $N_\ell$ neurons and the overall number of neurons in the hidden layers is denoted by $N = \sum_{\ell=1}^{\ell_{\text{max}}} N_\ell$. The neurons are indexed using the number of their layer $\ell$ and their position in the layer $k$. Every neuron has a scalar value $y_\ell^k \in \mathbb{R}$ and for each layer these values are collected in the vector $y^\ell = (y_\ell^1, \ldots, y_\ell^{N_\ell})^T \in \mathbb{R}^{N_\ell}$. The values of the neurons at the lowest level are given by the inputs, i.e., $y^0 = x \in \mathbb{R}^n$. The values of the neurons in the hidden layers are determined by the formula

$$y_\ell^k = \sigma(\omega_\ell^k \cdot y^{\ell-1} + b_\ell^k),$$

for $k = 1, \ldots, N_\ell$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is a so-called activation function and $\omega_\ell^k \in \mathbb{R}^{N_{\ell+1}}$, $\omega_\ell^k \in \mathbb{R}$ are the parameters of the layer. In our implementation, below, we will use the softplus activation function $\sigma^\ell(r) = \ln(e^r + 1)$ and the linear activation function $\sigma^\ell(r) = r$. With $x \cdot y$ we denote the Euclidean scalar product between two vectors $x, y \in \mathbb{R}^n$. In
The output layer, the values from the topmost hidden layer \( \ell = \ell_{\text{max}} \) are affine linearly combined to deliver the output, i.e.,

\[
W(x; \theta) = \sum_{k=1}^{N_{\ell_{\text{max}}}} a_k y_{k}^{\ell_{\text{max}}} + c = \sum_{k=1}^{N_{\ell_{\text{max}}}} a_k \sigma^{\ell_{\text{max}}} (w_k^{\ell_{\text{max}}} \cdot y_k^{\ell_{\text{max}}-1} + b_k^{\ell_{\text{max}}}) + c. \tag{4.1}
\]

The vector \( \theta \) collects all parameters \( a_k, c, w_k^{\ell}, b_k^{\ell} \) of the network.

In case of one hidden layer, in which \( \ell_{\text{max}} = 1 \) and thus \( y_{1}^{\ell_{\text{max}}-1} = y^0 = x \), we obtain the closed-form expression

\[
W(x; \theta) = \sum_{k=1}^{N_1} a_k \sigma^1 (w_k^1 \cdot x + b_k^1) + c.
\]

The universal approximation theorem states that a neural network with one hidden layer can approximate all smooth functions arbitrarily well. In its qualitative version, going back to [6, 19], it states that the set of functions that can be approximated by neural networks...
with one hidden layer is dense in the set of continuous functions. In Theorem 4.1 below, we state a quantitative version, given as Theorem 1 in [32], which is a reformulation of Theorem 2.1 in [29].

For its formulation consider the compact sets $K_n \subset \mathbb{R}^n$ satisfying (2.3) on which we want to perform our computation. For a continuous function $g : K_n \to \mathbb{R}$ we define the infinity-norm over $K_n$ as

$$\|g\|_{\infty, K_n} := \max_{x \in K_n} |g(x)|.$$ We then define the set of functions

$$W^m_n := \left\{ g \in C^m(K_n, \mathbb{R}) \mid \sum_{1 \leq |\alpha| \leq m} \|D_\alpha g\|_{\infty, K_n} \leq 1 \right\}$$

where $C^m(K_n, \mathbb{R})$ denoted the functions from $K_n$ to $\mathbb{R}$ that are $m$-times continuously differentiable, $\alpha$ are multiindices of length $|\alpha|$ with entries $\alpha_i \in \{1, \ldots, n\}$, $i = 1, \ldots, |\alpha|$ and $D_\alpha g = \partial g^{[\alpha]} / \partial \alpha_1 \ldots \partial \alpha_{|\alpha|}$ denotes the $m$-th directional derivative with respect to $\alpha$.

**Theorem 4.1:** Let $\sigma : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable and not a polynomial. Then, for any $\varepsilon > 0$, a neural network with one hidden layer provides an approximation

$$\inf_{\theta \in \mathbb{R}^P} \|W(x; \theta) - g(x)\|_{\infty, K_n} \leq \varepsilon$$

for all $g \in W^m_n$ with a number of $N$ of neurons satisfying

$$N = \mathcal{O}\left(\varepsilon^{-\frac{n}{m}}\right)$$

and this is the best possible.

**Proof:** See [32, Theorem 1] or [29, Theorem 2.1] for this result with $K_n = [-1, 1]^n$. The extension to $K_n \subset [-C, C]^n$ is straightforward. \(\square\)

Theorem 4.1 implies that one can readily use a network with one hidden layer for approximating Lyapunov functions. However, in general the number $N$ of neurons needed for a fixed approximation accuracy $\varepsilon > 0$ grows exponentially in $n$, and so does the number of parameters in $\theta$. This means that the storage requirement as well as the effort to determine $\theta$ easily exceeds all reasonable bounds already for moderate dimensions $n$. Hence, this approach also suffers from the curse of dimensionality. In the next section, we will therefore exploit the particular structure of compositional Lyapunov functions in order to obtain neural networks with (asymptotically) much lower $N$.

## 5 Neural network structure and complexity analysis

### 5.1 The case of known subsystems

For our first result, for fixed $d_{\text{max}} \in \mathbb{N}$ we consider the family of functions

$$F^d_{\text{max}} := \left\{ f : \mathbb{R}^n \to \mathbb{R}^n \mid n \in \mathbb{N}, f \text{ is Lipschitz and } (2.1) \text{ admits a compositional Lyapunov function } (3.1) \text{ with } \max_{i=1,\ldots,s} d_i \leq d_{\text{max}} \right\}.$$
We assume that for each \( f \in F_{d_{\max}}^1 \) we know the dimensions \( d_i \) and states \( z_i \) of the subsystems \( \Sigma_i, i = 1, \ldots, s \), of the corresponding decomposition. For this situation, we use a network with one hidden layer of the form depicted in Figure 5.1.

\[
\begin{array}{c}
\hat{y}_1^1 & \hat{y}_1^M \\
\hat{y}_s^1 & \hat{y}_s^M \\
\ell = 1 & \\
\end{array}
\]

Figure 5.1: Neural network for Lyapunov functions, \( f \in F_{d_{\max}}^1 \)

In this network, the single hidden layer for \( \ell = 1 \) consists of \( s \) sublayers \( L_1, \ldots, L_s \). The input of each of the neurons in \( L_i \) is the state vector \( z_i = (z_{i,1}, \ldots, z_{i,d_i})^T \) of the subsystem \( \Sigma_i \), which forms a part of the state vector \( x \). We assume that every sublayer \( L_i \) has \( M \) neurons, whose parameters and values are denoted by, respectively, \( \hat{w}_i^k, \hat{a}_i^k, \hat{b}_i^k \) and \( \hat{y}_i^k \), \( k = 1, \ldots, d_i \). Since \( s \leq n \), the layer contains \( N^1 = sM = nM \) neurons, which is also the total number \( N \) of neurons in the hidden layers. The values \( \hat{y}_i^k \) are then given by

\[
\hat{y}_i^k = \sigma^1(\hat{w}_i^k \cdot z_i + \hat{b}_i^k)
\]

and the overall output of the network is

\[
W(x; \theta) = \sum_{i=1}^{s} \sum_{k=1}^{d_i} \hat{a}_i^k \sigma^1(\hat{w}_i^k \cdot z_i + \hat{b}_i^k) + c.
\]

**Proposition 5.1:** Given compact sets \( K_n \subset \mathbb{R}^n \) satisfying (2.3), for each \( f \in F_{d_{\max}}^1 \) there exist a Lyapunov function \( V_f \) such that the following holds. For each \( \varepsilon > 0 \) the network depicted in and described after Figure 5.1 with \( \sigma^1 : \mathbb{R} \to \mathbb{R} \) infinitely differentiable and not polynomial, provides an approximation \( \inf_{\theta \in \mathbb{R}^p} \| W(x; \theta) - V_f(x) \|_{\infty, K_n} \leq \varepsilon \) for all \( f \in F_{d_{\max}}^1 \) with a number of \( N \) of neurons satisfying

\[
N = \mathcal{O} \left( n^{d_{\max}+1} \varepsilon^{-d_{\max}} \right).
\]

**Proof:** Consider the \( C^1 \)-functions \( \hat{V}_i \) from (3.1). We choose \( \mu > 0 \) maximal such that \( \mu \hat{V}_i \) lies in \( W_{d_i}^1 \) and set \( V_f = \mu V \) with \( V = \sum_{i=1}^{s} \hat{V}_i \) from (3.1). Then, by Theorem 4.1 there exist values \( \hat{a}_i^k, \hat{b}_i^k, \hat{w}_i^k, \hat{c}_i, k = 1, \ldots, d_i \), such that the output

\[
o_{L_i}(z_i) := \sum_{k=1}^{d_i} \hat{a}_i^k \sigma^1(\hat{w}_i^k \cdot z_i + \hat{b}_i^k) + \hat{c}_i
\]
of each sublayer $L_i$ satisfies
\[ \left\| a_{L_i} - \mu \widehat{V}_i \right\|_{\infty,K_n} \leq \varepsilon/n \]
for a number of neurons
\[ M = \mathcal{O} \left( (\varepsilon/n)^{-d_{\max}} \right) = \mathcal{O} \left( n^{d_{\max} \varepsilon^{-d_{\max}}} \right). \]
Since this is true for all sublayers $L_1, \ldots, L_s$, by merging the weights $\hat{a}_k$ and $\hat{c}_i$ into the $a_k$ and $c$ in (4.1) we obtain $W(x; \theta) = \sum_{i=1}^{s} a_{L_i}(z_i)$ and thus
\[ \left\| W(x; \theta) - V_f(x) \right\|_{\infty,K_n} \leq \sum_{i=1}^{s} \left\| a_{L_i} - \mu \widehat{V}_i \right\|_{\infty,K_n} \leq s \varepsilon/n \leq \varepsilon \]
with the overall number of neurons $N \leq nM = \mathcal{O} \left( n^{d_{\max}+1} \varepsilon^{-d_{\max}} \right)$. \( \square \)

### 5.2 The case of unknown subsystems

The approach in the previous subsection requires the knowledge of the subsystems $\Sigma_i$ in order to design the appropriate neural network. This is a rather unrealistic assumption that requires a lot of preliminary analysis in order to set up an appropriate network. Fortunately, there is a remedy for this, which moreover applies to a larger family of systems than $F_{1_{d_{\max}}}$ considered above. To this end, we consider the family of maps
\[ F_{2_{d_{\max}}} := \left\{ f : \mathbb{R}^n \to \mathbb{R}^n \mid n \in \mathbb{N}, \text{there is an invertible } T \in \mathbb{R}^{n \times n} \text{ with } \|T\|_{\infty} \leq c, \right\}. \]
Here $\|T\|_{\infty}$ denotes the Matrix norm induced by the vector norm $\|x\|_{\infty} = \max_{i=1,\ldots,n} |x_i|$.

In contrast to Section 5.1, now we do not assume that we know the dimensions $\hat{d}_i$ and states $\hat{z}_i$ of the subsystems $\hat{\Sigma}_i$, and not even their number $\hat{s}$. We also do not need to know the coordinate transformation $T$. The neural network that we propose for $f \in F_{2_{d_{\max}}}$ is depicted in Figure 5.2.

Here, we use different activation functions $\sigma^\ell$ in the different levels. While $\sigma^2$ in layer $\ell = 2$ is chosen like $\sigma^1$ in Proposition 5.1 in Level $\ell = 1$ we use the identity, i.e., the linear activation $\sigma^1(x) = x$. Layer $\ell = 2$ consists of $n$ sublayers $L_1, \ldots, L_n$, each of which has exactly $d_{\max}$ inputs and $M$ neurons. The coefficients and neuron values of each $L_i$ are again denoted with $\hat{\omega}_k^i$, $\hat{a}_k^i$, $\hat{b}_k^i$ and $\hat{y}_k^i$, respectively, for $k = 1, \ldots, d_{\max}$. The $d_{\max}$-dimensional input of each neuron in $L_i$ is given by
\[ (\hat{y}_{(i-1)d_{\max}+1}, \ldots, \hat{y}_{id_{\max}})^T =: \hat{y}_i^1. \]

We note that this network is a special case of the lower network in Figure 4.1.

**Theorem 5.2:** Given compact sets $K_n \subset \mathbb{R}^n$ $K_n \subset \mathbb{R}^n$, for each $f \in F_{2_{d_{\max}}}$ there exist a Lyapunov function $V_f$ such that the following holds. For each $\varepsilon > 0$ the network depicted in and described after Figure 5.2 with $\sigma^2 : \mathbb{R} \to \mathbb{R}$ infinitely differentiable and not polynomial in layer $\ell = 2$ and $\sigma^1(x) = x$ in layer $\ell = 1$, provides an approximation $\inf_{\theta \in \mathbb{R}^p} \|W(x; \theta) - V_f(x)\|_{\infty,K_n} \leq \varepsilon$ for all $f \in F_{2_{d_{\max}}}$ with a number of $N$ of neurons satisfying
Figure 5.2: Neural network for Lyapunov functions, $f \in \mathcal{F}_{2}^{d_{\text{max}}}$

$$N = \mathcal{O} \left( nd_{\text{max}} + n^{d_{\text{max}}+1} \varepsilon^{-d_{\text{max}}} \right).$$

**Proof:** Let $d_i$ be the (unknown) dimensions of the subsystems $\tilde{\Sigma}_i$ and $p_i = 1 + \sum_{k=1}^{i-1} d_k$ the first index of the variables $\tilde{z}_i$ of $\tilde{\Sigma}_i$, i.e., $\tilde{z}_i = (\tilde{x}_{p_i}, \ldots, \tilde{x}_{p_i+1}-1)^T$. Using the notation from above and the fact that $\sigma^1(x) = x$, the value of the inputs $\hat{y}_k^1$ of the sublevels is given by

$$\hat{y}_k^1 = w_k^1 \cdot x + b_k^1.$$

Hence, by choosing $b_k^1 = 0$ and $w_k^1$ as the transpose of the $j$-th row of $T^{-1}$, we obtain $\hat{y}_k^1 = \tilde{x}_j$. Hence, by appropriately assigning all the $w_k^1$, we obtain

$$\tilde{y}_i^1 = \begin{pmatrix} \tilde{z}_i \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the number of the zeros equals $d_{\text{max}} - d_i$. This can be done for $i = 1, \ldots, \tilde{s}$, with $\tilde{s}$ being the number of subsystems. The inputs for the remaining sublayers $L_{\tilde{s}+1}, \ldots, L_n$ are set to 0 by setting the corresponding $w_k^1$ and $b_k^1$ to 0. For this choice of the parameters of the lower layer, each sublayer $L_i$ of the layer $\ell = 2$ receives the transformed subsystem states $\tilde{z}_i$ (and a number of zeros) as input, or the input is 0.

Since the additional zero-inputs do not affect the properties of the network, the upper part of the network, consisting of the hidden layer $\ell = 2$ and the output, has exactly the structure of the network used in Proposition 5.1. We can thus apply this proposition on the sets $\tilde{K}_n = TK_n$ to the upper part of the network and obtain that it can realize a function $W(\tilde{z}; \theta)$ that approximates a Lyapunov function $\tilde{V}$ for $\tilde{f}$ in the sense of Proposition 5.1.
Note that since $\|T\|_{\infty} \leq c$ and $K_n$ satisfies (2.3), we have that $\tilde{K}_n = TK_n \subset [-cC, cC]^n$, hence $\tilde{K}_n$ also satisfies (2.3).

As the lower layer realizes the coordinate transformation $\tilde{x} = Tx$, the overall network $W(x; \theta)$ then approximates the function $V(x) := \tilde{V}(Tx)$. By means of the invertibility of $T$ and the chain rule one easily checks that this is a Lyapunov function for $f$. The claim then follows since the number of neurons $N^2$ in the upper layer is equal to that given in Proposition 5.1, while that in the lower layer equals $N^1 = nd_{\text{max}}$. This leads to the overall number of neurons given in the theorem.

6 Training the network

For training the network in order to actually compute a Lyapunov function we need to specify a loss function $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. Training then consists of finding parameters $\theta$ such that

$$
\frac{1}{m} \sum_{i=1}^{m} L \left( W(x^{(i)}; \theta), DW(x^{(i)}; \theta), x^{(i)} \right)
$$

becomes minimal, where $x^{(i)} \in K_n$ are the elements of a test dataset, which we refer to as test points. In our numerical results in the next section we always use $K_n = [-1, 1]^n$ and the test points $x^{(i)}$ are chosen randomly and uniformly distributed from $K_n$.

Note that in contrast to many other problems in deep learning the loss function $L$ also depends on the values of the derivative of $W$ with respect to $x$ in the test points, which we denote by $DW(x^{(i)}, \theta)$. This is needed because in order to determine whether $W$ is a Lyapunov function, its derivative is needed. For minimizing the expression (6.1) a stochastic gradient algorithm, which is standard in deep learning, can be used. Details are specified in the following section.

The main work is now to design the loss function such that minimizing (6.1) w.r.t. $\theta$ yields a Lyapunov function. To this end, a straightforward idea is to express the Lyapunov function property as a partial differential equation (PDE) and follow the approaches in the literature for solving PDEs with neural networks mentioned in the introduction. A simple PDE that is suitable for this purpose is the Zubov-type equation

$$
DW(x; \theta) = -\|x\|^2,
$$

similar PDEs have been used or discussed, e.g., in [4, 11, 26, 39]. This PDE needs to be complemented by suitable boundary conditions, which in the PDE setting (with $x = 0$ being the equilibrium of interest) are of the form

$$
W(0, \theta) = 0 \quad \text{and} \quad W(x; \theta) > 0 \text{ for all } x \in K_n \setminus \{0\}.
$$

However, in this form the boundary conditions are difficult to be implemented numerically: the equality condition $W(0, \theta) = 0$ is difficult because it is only given in a single point, while the strict “$>$” condition is difficult because numerically only “$\geq$” can be enforced directly. To resolve these problems, we replace the boundary conditions above by the stronger conditions

$$
\alpha_1(\|x\|) \leq W(x; \theta) \leq \alpha_2(\|x\|) \text{ for all } x \in K_n,
$$

where $\alpha_1$ and $\alpha_2$ are suitable functions.
with $\alpha_1, \alpha_2 \in \mathcal{K}$. Of course, the functions $\alpha_i$ have to be chosen appropriately in order to allow for the existence of a solution of (6.2) that satisfies (6.3). However, it follows from [39] that if a Lyapunov function on $K_n$ exists, then it is always possible to find such $\alpha_i$. Loosely speaking, $\alpha_1$ must be sufficiently flat while $\alpha_2$ must be sufficiently steep. In case $x = 0$ is exponentially stable, one can choose the $\alpha_i$ as quadratic functions $\alpha_i(r) = c_ir^2$ with $c_1 > 0$ sufficiently small and $c_2 > 0$ sufficiently large.

Given the vector field $f$ from (2.1), the loss function $L$ is now defined as

$$L(w, p, x) := (pf(x) + \|x\|^2)^2 + \nu \left( [w - \alpha_1(\|x\|)]_+^2 + [w - \alpha_2(\|x\|)]_+^2 \right),$$

where $[a]_- := \min\{a, 0\}$, $[a]_+ := \max\{a, 0\}$, and $\nu > 0$ is a weighting parameter (chosen as $\nu = 1$ in all our numerical examples in the next section). One easily checks that for this $L$ the expression (6.1) is always $\geq 0$ and equals 0 if and only if (6.2) and (6.3) are satisfied for all test points $x^{(i)}$. Conversely, if a Lyapunov function exists for which the bounds (6.3) are feasible, and if this Lyapunov function can be represented by neural network under consideration, the minimizing (6.1) w.r.t. $\theta$ will result in the optimal value of (6.1) being 0.

Unfortunately, while this approach works in principle, it is not necessarily compatible with the complexity analysis from the previous section. The reason is that when a Lyapunov function with the particular small gain structure (3.1) exists, it may not be a solution of (6.2), (6.3). As a consequence, while a solution of (6.2), (6.3) may exist, it may not be representable by the neural network structure from Figure 5.1 or Figure 5.2. Hence, with the choice of $L$ from (6.4), it may not be possible to exploit the low computational complexity provided by this particular network structure. The result depicted in Figure 7.2 below, shows that this indeed happens.

Hence, we need to provide more flexibility to our approach, which we can do by enlarging the set of minima of the loss function. To this end, note that (6.2) is actually a much too strong condition. Requiring the partial differential inequality (PDI)

$$DW(x; \theta) \leq -\|x\|^2,$$

instead of (6.2), will also yield a Lyapunov function. While one may argue that the bound $-\|x\|^{2n}$ on the derivative is somewhat arbitrary here, it is easily seen that by appropriate rescaling any Lyapunov function can be modified such that this bound holds. Hence, modifying the right hand side of (6.5) does not provide more flexibility (but, of course, it affects the set of $\alpha_i$ for which (6.5) and (6.3) together are feasible).

Incorporating (6.5) instead of (6.2) in the loss function $L$ leads to the expression

$$L(w, p, x) := (pf(x) + \|x\|^2)^2 + \nu \left( [w - \alpha_1(\|x\|)]_+^2 + [w - \alpha_2(\|x\|)]_+^2 \right).$$

One easily checks that for this $L$ the expression (6.1) is again always $\geq 0$, but now it equals 0 if and only if (6.5) and (6.3) are satisfied for all test points $x^{(i)}$. As Example 7.1 and Figure 7.1 below, show, this indeed allows to use the network structure from the previous section and it also allows for solving higher dimensional problems, see Example 7.2.
7 Numerical examples

We illustrate the proposed method with two examples, a low-dimensional one that shows that the loss function (6.6) is in general preferable over (6.4) and a larger one that shows the ability of the method to work in finding Lyapunov functions in higher dimensions. All computations were performed with Python 3.7.0 and TensorFlow 2.1.0 on a MacBook Pro (2017, 2.3 GHz Intel Core i5) running macOS Mojave (10.14.6). The python code and the trained networks are available from numerik.mathematik.uni-bayreuth.de/~lgruene/DeepLyapunov/.

Our first example considers a two-dimensional example that has a compositional Lyapunov function consisting of two one-dimensional functions. It is given by

\[
\begin{align*}
\dot{x}_1 &= -x_1 - 10x_2^2 \\
\dot{x}_2 &= -2x_2.
\end{align*}
\] (7.1)

Using the Lyapunov-function candidate

\[ V(x) = x_1^2 + x_2^2 + 13x_4^2, \]

one computes

\[ DV(x)f(x) = -2x_1^2 - 20x_1x_2^2 - 4x_2^2 - 104x_4^2. \]

Since

\[ -x_1^2 - 20x_1x_2^2 - 104x_4^2 \leq -x_1^2 - 20x_1x_2^2 - 100x_2^2 = -(x_1 + 10x_2^2)^2 \leq 0, \]

we obtain \( Dv(x)f(x) \leq -x_1^2 - 4x_2^2 \leq -\|x\|^2. \) Hence, \( V \) is a Lyapunov function and it is obviously of the compositional form (3.1) with \( z_1 = x_1 \) and \( z_2 = x_2. \)

It should thus be able to compute a Lyapunov function with the neural network from Figure 5.2. It turns out that using the loss function (6.6) (with \( \alpha_1(r) = 0 \) and \( \alpha_2(r) = 10r^2 \)) this is indeed possible. Here we used the network structure from Figure 5.2 with \( n = 2 \) and \( d_{\text{max}} = 1, \) with the layers \( L_1 \) and \( L_2 \) consisting of 128 neurons, each, and softplus activation functions \( \sigma^2(r) = \ln(e^r + 1), \) resulting in 775 trainable parameters. The training was performed with 200 000 test points, optimizing with batch size 32 using the Adam optimizer implemented in TensorFlow. The optimization was terminated when the accuracy for the final function \( W(\cdot, \theta^*) \) satisfied

\[ \text{err}_1 := \frac{1}{m} \sum_{i=1}^{m} L\left(W(x^{(i)}, \theta^*), DW(x^{(i)}, \theta^*), x^{(i)}\right) < 10^{-6} \]

and

\[ \text{err}_\infty := \max_{i=1,\ldots,m} L\left(W(x^{(i)}, \theta^*), DW(x^{(i)}; \theta^*), x^{(i)}\right) < 10^{-6}, \]

which was reached after 6 epochs in the run documented here. The time needed for the optimization was 48s. Figure 7.1 shows the computed approximate Lyapunov function \( W(\cdot, \theta^*) \) as a solid surface along with its derivative along the vector field \( DW(x; \theta^*)f(x) \) as a wireframe, shown from two different angles. The graphs illustrate that the method was successful.

\[ \text{err}_1 := \frac{1}{m} \sum_{i=1}^{m} L\left(W(x^{(i)}, \theta^*), DW(x^{(i)}, \theta^*), x^{(i)}\right) < 10^{-6} \]

\[ \text{err}_\infty := \max_{i=1,\ldots,m} L\left(W(x^{(i)}, \theta^*), DW(x^{(i)}; \theta^*), x^{(i)}\right) < 10^{-6}, \]

which was reached after 6 epochs in the run documented here. The time needed for the optimization was 48s. Figure 7.1 shows the computed approximate Lyapunov function \( W(\cdot, \theta^*) \) as a solid surface along with its derivative along the vector field \( DW(x; \theta^*)f(x) \) as a wireframe, shown from two different angles. The graphs illustrate that the method was successful.

\[ \text{err}_1 := \frac{1}{m} \sum_{i=1}^{m} L\left(W(x^{(i)}, \theta^*), DW(x^{(i)}, \theta^*), x^{(i)}\right) < 10^{-6} \]

\[ \text{err}_\infty := \max_{i=1,\ldots,m} L\left(W(x^{(i)}, \theta^*), DW(x^{(i)}; \theta^*), x^{(i)}\right) < 10^{-6}, \]
In contrast to this, performing the computation with the same parameters but with loss function (6.4) fails. As Figure 7.2 shows, the derivative $DW(x; \theta^*) f(x)$ (shown as a wireframe) obviously not satisfy the equation $DW(x; \theta^*) f(x) = -\|x\|^2$. This is also visible in the values $err_1 = 1.363842$ and $err_\infty = 3.110839$ that were reached after 20 epochs. While this alone would not be a problem (as long as $DW(x; \theta^*) f(x)$ is still negative definite), the inability to meet this equation has the side effect that the optimization also does not enforce the inequalities (6.3). As a consequence, the minimum of the computed function is not located in the equilibrium at the origin, as the lateral view on the right of Figure 7.2 shows. This is because it is more difficult to represent a Lyapunov function satisfying $DV(x) f(x) = -\|x\|^2$ with the network structure from Figure 5.2. While this example does, of course, not exclude that the loss function (6.4) works for other parameters, it provides evidence that the advantage in computational complexity offered by our approach is more easily exploited using the loss function (6.6).

In our second example we illustrate the capability of our approach to handle higher dimensional systems and to determine the subspaces for the compositional representation of $V$.

---

5In all runs these error values did not change significantly anymore after epoch 15. In some runs the resulting function had a different shape, but in all cases it visibly violated the required inequalities.
To this end we consider a 10-dimensional example of the form

\[
\dot{x} = f(x) := T^{-1}\hat{f}(Tx) \tag{7.2}
\]

with vector field \( \hat{f} : \mathbb{R}^{10} \to \mathbb{R}^{10} \) given by

\[
\hat{f}(x) = \begin{pmatrix}
-x_1 + 0.5x_2 - 0.1x_2^2 \\
-0.5x_1 - x_2 \\
-x_3 + 0.5x_4 - 0.1x_7^2 \\
-0.5x_3 - x_4 \\
-x_5 + 0.5x_6 + 0.1x_7^2 \\
-0.5x_5 - x_6 \\
-x_7 + 0.5x_8 \\
-0.5x_7 - x_8 \\
-x_9 + 0.5x_{10} \\
-0.5x_9 - x_{10} + 0.1x_2^2
\end{pmatrix}
\]

One easily sees that this system consists of five two-dimensional asymptotically stable linear subsystems that are coupled by four nonlinearities with small gains. It is thus to be expected that on \( K_{10} = [-1, 1]^{10} \) the system is asymptotically stable and a Lyapunov function can be computed using the network from Figure 5.2 five two-dimensional layers \( L_1, \ldots, L_5 \).

The coordinate transformation \( T \in \mathbb{R}^{10 \times 10} \) is given by the (randomly generated) matrix

\[
T = \begin{pmatrix}
-\frac{1}{5} & -\frac{3}{10} & \frac{1}{2} & -\frac{4}{5} & \frac{5}{10} & \frac{7}{10} & \frac{7}{10} & -1 & -\frac{4}{5} \\
\frac{5}{10} & \frac{1}{5} & 0 & \frac{9}{10} & \frac{4}{5} & \frac{3}{10} & \frac{3}{10} & \frac{2}{10} & -\frac{3}{10} \\
-\frac{3}{10} & \frac{3}{10} & 2 & -\frac{5}{10} & 0 & -\frac{2}{10} & \frac{3}{10} & 3 & 1 & -\frac{1}{2} \\
-\frac{7}{10} & -\frac{1}{10} & -\frac{3}{5} & -\frac{1}{2} & -\frac{3}{5} & -\frac{2}{5} & 1 & \frac{1}{10} & -1 & -\frac{3}{10} \\
\frac{7}{10} & \frac{1}{5} & -\frac{6}{5} & \frac{1}{5} & -\frac{3}{5} & -\frac{2}{5} & 1 & \frac{1}{10} & -1 & -\frac{3}{10} \\
\frac{1}{10} & -\frac{3}{10} & -\frac{3}{5} & -\frac{2}{5} & \frac{10}{10} & \frac{7}{10} & \frac{7}{10} & 0 & -\frac{4}{5} \\
\frac{3}{10} & \frac{9}{10} & -\frac{1}{5} & 1 & \frac{2}{5} & 1 & 0 & -\frac{1}{10} & -\frac{2}{5} \cdot 0 \\
-1 & 1 & \frac{7}{10} & \frac{3}{5} & -\frac{4}{5} & -\frac{4}{5} & 0 & -\frac{1}{5} & -\frac{1}{5} & \frac{7}{10} \\
-\frac{9}{10} & \frac{4}{5} & \frac{1}{5} & 1 & -\frac{4}{5} \cdot 5 & -\frac{3}{10} & \frac{7}{10} & 1 & -\frac{4}{5} \cdot 5 \\
\frac{3}{5} & -\frac{1}{10} & -\frac{2}{5} & -\frac{1}{2} & -\frac{3}{10} & -\frac{1}{10} & -\frac{7}{10} & 1 & \frac{4}{5} & -\frac{3}{10} \\
0 & -1 & -\frac{1}{10} & 2 & -\frac{3}{10} & -\frac{1}{10} & -\frac{1}{5} & \frac{7}{10} & -\frac{1}{10} & -\frac{4}{5}
\end{pmatrix}
\]

Explicit formulas for the equations in (7.2) are given in Appendix A.

We have computed a Lyapunov function for this system for the loss function (6.6) with \( \alpha_1(r) = 0.2r^2 \) and \( \alpha_2(r) = 10r^2 \). We used the network structure from Figure 5.2 with \( n = 5 \) and \( d_{\text{max}} = 2 \), with the layers \( L_1, \ldots, L_5 \) consisting of 128 neurons, each, leading to 2671 trainable parameters. The training was performed with 400,000 test points, optimizing over 13 epochs. As for the 2d example, we used batch size 32, the Adam optimizer implemented in TensorFlow, and softplus activation functions \( \sigma^2 \). The time needed for the training was 266s and the resulting function satisfies the inequalities

\[ err_1 < 10^{-6}, \quad err_\infty < 10^{-6}. \]
Figures 7.3 and 7.4 show the resulting function $W(\cdot; \theta^*)$ (solid) and its derivative along $f$ (wireframe) on the $(x_2, x_8)$-plane and the $(x_9, x_{10})$-plane, respectively. The remaining components of $x$ were set to 0 in both figures. Figure 7.5 shows the value of $W(\cdot; \theta^*)$ along three trajectories of (7.2) (computed numerically using the ode45-routine from matlab). It shows the strict decrease that is expected from a Lyapunov function.

Figure 7.3: Lyapunov function for Example (7.2) on $(x_2, x_8)$-plane

Figure 7.4: Lyapunov function for Example (7.2) on $(x_9, x_{10})$-plane

Figure 7.5: Value of Lyapunov function along trajectories for initial values $x_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$, $(0, 1, 0, 1, 0, 1, 0, 1, 0, 1)^T$, $(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ (left to right)
8 Discussion

In this section we discuss a few aspects and possible extensions of the results in this paper.

(i) From the expressions for $N$ in Proposition 5.1 and Theorem 5.2 one sees that for a given $\varepsilon > 0$ the storage effort only grows polynomially in the state dimension $n$, where the exponent is determined by the maximal dimension of the subsystems $d_{\text{max}}$. The proposed approach hence avoids the curse of dimensionality, i.e., the exponential growth of the effort. There is, however, an exponential dependence on the maximal dimension $d_{\text{max}}$ of the subsystems $\Sigma_i$ for the compositional Lyapunov functions (3.1). This is to be expected, because the construction relies on the low-dimensionality of the $\Sigma_i$ and if this is no longer given, we cannot expect the method to work efficiently.

(ii) We stress that our theoretical results only guarantee that the computed functions $W(\cdot; \theta^*)$ are approximations to Lyapunov functions rather than true Lyapunov functions. However, the figures of the graphs of $W$ and $DWf$ as well as further numerical tests suggest that the computed functions are indeed Lyapunov functions, except in small neighbourhoods of the equilibrium $0$. However, it is currently unclear how this can be verified rigorously. In low dimensions a grid based method such as the check of [15] inequality (3) proposed in [15] might be feasible, but in higher dimensions new methods for such a verification need to be developed.

(iii) There have been attempts to use small-gain theorems for grid-based constructions of Lyapunov functions, e.g., in [5] [27]. The problem of such a construction, however, is, that it computes the functions $V_i$ from Theorem 3.3 separately for the subsystems and the small-gain condition has to be checked a posteriori (which is a difficult task). The representation via the neural network does not require to check the small-gain condition nor is the precise knowledge of the subsystems necessary.

(iv) The reasoning in the proofs remains valid if we replace $f(x)$ by $f(x, u)$ and asymptotic stability with ISS. Hence, the proposed network is also capable of efficiently storing ISS and iISS Lyapunov functions. Moreover, an extension to control Lyapunov functions appears attractive, as these functions allow to derive stabilizing feedback laws for nonlinear systems. However, the corresponding extension of the proposed training scheme is nontrivial and is thus subject of future research.

(v) In current neural network applications ReLU activation functions $\sigma(r) = \max\{r, 0\}$ are often preferred over $C^\infty$ activation functions, such as the softplus function used in our implementation (which is, in fact, a smooth approximation to the ReLU activation function). The obvious disadvantage of this concept is that the resulting function $W(x; \theta)$ is nonsmooth in $x$, which implies the need to use concepts of nonsmooth analysis for interpreting it as a Lyapunov function. While one may circumvent the need to compute the derivative of $W$ by means of using nonsmooth analysis or by passing to an integral representation of (2.2), the nonsmoothness implies that the gradient $DW$ in the training scheme needs to be replaced by an appropriate substitute. Details are subject to future research and it remains to be explored whether the difficulties caused by the nonsmoothness of $W$ are compensated by the advantages of ReLU activation functions.
(vi) There are other types of Lyapunov function constructions based on small-gain conditions different from Definition 3.2, e.g., a construction of the form
\[ V(x) = \max_{i=1,...,s} \rho_i^{-1}(V_i(z_i)), \]
found in [9, 35]. Since maximization can also be efficiently implemented in neural networks (via max pooling), such “max-compositional” Lyapunov functions also admit an efficient approximation via deep neural networks. However, when using this formulation we have to cope with two sources of nondifferentiability that complicate the analysis. One source is the maximization in the definition of \( V \) and the other source are the functions \( \rho_i^{-1} \in \mathcal{K}_\infty \), which in most references are only ensured to be Lipschitz.

9 Conclusion

We have proposed a class of deep neural networks that allows for approximating Lyapunov functions \( V \) having a compositional structure. Such Lyapunov functions exist, e.g., when the systems satisfies a small-gain condition. The number of neurons needed for an approximation with fixed accuracy depends exponentially on the maximal dimension of the subsystems in the compositional representation of \( V \), but only polynomially on the overall state dimension. Thus, it provably avoids the curse of dimensionality, a feature that to the best of our knowledge is not available for similar approaches in the literature. The network structure does not need any knowledge about the exact dimensions of the subsystems and even allows for a subsystem structure that only becomes visible after a linear coordinate transformation.

We also presented a training scheme for the proposed architecture that is based on representing a suitable partial differential inequality and boundary conditions in the loss function. By means of numerical examples we demonstrated that this approach is beneficial compared to a loss function based on a partial differential equation and that it produces excellent results in ten space dimensions. This dimension is significantly larger than those reported for other numerical approaches for nonlinear systems in the literature, particularly for grid based methods. As discussed in Section 8, the approach allows for manifold extensions that will be subject of future research.
A Explicit formulas for the vector field from (7.2)

\[
\begin{align*}
\dot{x}_1 &= 0.1989532507x_1^2 + (-0.3746986930x_2 - 0.2736437748x_3 - 0.2102338351x_4 + 0.2970740265x_5 + 0.3127686023x_6 - 0.01669120692x_7 + 0.0699780334x_8 + 0.1066092874x_9 - 0.2932731139x_{10} - 0.9897508136)x_1 + 0.2290160726x_2^2 \\
&+ (0.3043283662x_4 - 0.342739422x_5 - 0.2909684040x_6 - 0.03766820776x_7 - 0.0701314358x_8 - 0.0335077278x_9 + 0.2338743069x_{10} + 0.3113173115x_3 + 2.118681547)x_2 + 0.1313269044x_3^2 + (0.1667773216x_4 - 0.1858768765x_5 - 0.1692031675x_6 + 0.0183558115x_7 + 0.00527957798x_8 - 0.06204132445x_9 + 0.2154590805x_{10} + 0.1829377756)x_3 + 0.1199134206x_4^2 + (2.127824903 \\
&- 0.2551625097x_5 - 0.1918810479x_6 - 0.0685334432x_7 - 0.0782040322x_8 + 0.06778326492x_9 + 0.08851262462x_{10} + 0.1568361752x_2^2 \\
&+ (0.5637474356 + 0.278505234x_6 + 0.05602401930x_7 + 0.1146148356x_8 - 0.01948672995x_9 - 0.1556286023x_{10})x_5 + 0.1435362729x_2^2 + (-0.2301950413 + 0.168765377x_7 + 0.1072316619x_8 + 0.05135244079x_9 - 0.1987347825x_{10})x_6 + 0.02620179645x_4^2 + (-0.4449045860 + 0.03970147617x_8 - 0.08136302008x_9 + 0.05945874215x_{10})x_7 + 0.03840002526x_2^2 + (-0.5863033921 - 0.03022933302x_9 \\
&- 0.008721167594x_{10})x_8 + 0.07453193438x_3^2 + (0.4328101506 - 0.1462574559x_{10})x_9 + 0.1297231241x_{10} - 0.52424980712x_{10}
\end{align*}
\]

\[
\begin{align*}
\dot{x}_2 &= -0.03119605894x_1^2 + (0.03153734707x_2 + 0.02865595927x_3 + 0.02168237588x_4 - 0.02614983431x_5 - 0.04282878632x_6 + 0.02164051571x_7 - 0.03368328076x_8 \\
&- 0.04280870200x_9 + 0.04677950036x_{10} - 0.1917995579)x_1 + 0.0634968205x_2^2 + (-0.09745019444x_4 + 0.04945719986x_5 - 0.00725885789x_6 + 0.0258516683x_7 \\
&- 0.02172930153x_8 - 0.05729940663x_9 - 0.00472767777x_{10} - 0.1008048758x_3 - 0.9781859725x_2 + 0.04847610961x_3^2 + (-0.07334570982x_4 + 0.02014055021x_5 \\
&- 0.02193176669x_6 + 0.00400591788x_7 - 0.02336554649x_8 - 0.02794496258x_9 - 0.01466478759x_{10} + 0.2309047484)x_3 - 0.04468166437x_4^2 + (0.3530648809 \\
&+ 0.03806553631x_5 - 0.00958989098x_6 + 0.01985044356x_7 + 0.0033830209x_8 \\
&- 0.04926706550x_9 + 0.00867868591x_{10})x_4 + 0.02396569262x_2^2 \\
&+ (-0.3232326202 - 0.0328791432x_6 - 0.02185224895x_7 - 0.00628992862x_8 + 0.02470076015x_9 + 0.06350783469x_{10})x_5 - 0.03091558700x_6^2 \\
&+ (0.1323309042 + 0.005453947198x_7 - 0.03389417380x_8 - 0.03330007197x_9 + 0.03192852100x_{10})x_6 - 0.01708198358x_7^2 + (-0.2935693423 + 0.02368730120x_8 \\
&+ 0.05290764966x_9 - 0.2653836321x_{10})x_7 - 0.03264090370x_8^2 \\
&+ (0.3896692795 - 0.04363254132x_9 + 0.01548131327x_{10})x_8 - 0.04769659728x_9^2 \\
&+ (0.0406197547 + 0.05115811655x_{10})x_9 - 0.02359950588x_{10} + 0.3082568899x_{10}
\end{align*}
\]

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\[
\begin{align*}
x_3 &= 0.09126920252x_1^2 + (-0.1837466357x_2 - 0.1402352655x_3 - 0.09333759420x_4 + 0.1367085329x_5 + 0.1392944946x_6 - 0.004746979466x_7 + 0.008913711408x_8 + 0.04466682909x_9 - 0.1419498381x_{10} + 0.4429513679x_1 + 0.08604337194x_2 + (0.1167735712x_4 - 0.1719416349x_5 - 0.1755942083x_6 - 0.01586866373x_7 - 0.06849556991x_8 - 0.023366391257x_9 + 0.1123154339x_{10} - 0.09334227143x_3 + 0.8483458581x_2 + 0.04745814984x_3^2 + (0.01663354444x_4 - 0.06366398541x_5 - 0.09727931370x_6 + 0.03667693045x_7 + 0.00704277082x_8 - 0.08784867008x_9 + 0.1362490790x_{10} + 1.339521793)x_3 + 0.05641452990x_2^2 + (0.4347532629 - 0.1506583242x_3 - 0.1344069171x_6 - 0.05856437772x_7 - 0.07929013343x_8 + 0.05241477736x_9 + 0.01542903344x_{10})x_4 + 0.09141768969x_5^2 + (0.1542947132 + 0.1528726246x_6 + 0.0525455914x_7 + 0.09417777322x_8 - 0.04554532504x_9 - 0.0413046855x_{10})x_5 + 0.06354104563x_6^2 + (-0.2794559621 + 0.03150409182x_7 + 0.05344407299x_8 - 0.02077146017x_9 - 0.06800121637x_{10}x_6 + 0.01925982710x_7^2 + (-0.2336276914 + 0.06731834571x_8 - 0.05515809304x_9 + 0.05114591974x_{10})x_7 + 0.01615005377x_8^2 + (-0.0515408280 + 0.08914645939x_9 + 0.04059226652x_{10})x_8 + 0.04122032728x_9^2 + (0.8823743331 - 0.09773115730x_{10})x_9 + 0.07650499903x_{10}^2 - 1.155048593x_{10}
\end{align*}
\]

\[
\begin{align*}
x_4 &= 0.000109942824x_1^2 + (0.03356003711x_2 + 0.02323754113x_3 + 0.00169044864x_4 - 0.02142491959x_5 - 0.00609162685x_6 - 0.02233368253x_7 + 0.04709505532x_8 + 0.02719344765x_9 + 0.00826521948x_{10} - 0.1137458510)x_1 + 0.03197569451x_2^2 + (0.04898642207x_4 + 0.02341326976x_5 + 0.07907578307x_6 - 0.01112494990x_7 + 0.05515845861x_8 + 0.05594327922x_9 - 0.02797966001x_{10} + 0.07626235173x_{10} - 0.1083576192)x_2 + 0.02909993375x_3^2 + (0.08900483976x_4 - 0.01029067144x_5 + 0.05280784502x_6 - 0.02698774480x_7 + 0.00162851924x_8 + 0.07655040292x_9 - 0.04988825377x_{10} + 0.0239831175)x_3 + 0.001495956764x_4^2 + (-1.358618607 + 0.04667987187x_5 + 0.07423590124x_6 + 0.02888526563x_7 + 0.03992297246x_8 - 0.00050631655x_9 + 0.0122268222x_{10})x_4 - 0.0212929389x_5^2 + (0.3472755899 - 0.03514987692x_6 - 0.01526279489x_7 - 0.00575042410x_8 + 0.01838663898x_9 - 0.01413149723x_{10}x_5 + 0.00514768156x_6^2 + (0.04020270346 - 0.02730319070x_7 + 0.00264941851x_8 + 0.05676910300x_9 - 0.01935545563x_{10})x_6 + 0.005122385702x_7^2 + (0.3644515720 - 0.08075939614x_8 - 0.01655880279x_9 - 0.009837097207x_{10})x_7 + 0.02676284738x_8^2 + (-0.4038672971 + 0.1196289080x_9 - 0.05813179940x_{10})x_8 + 0.02015960523x_9^2 + (-0.2065587443 + 0.01057956599x_{10})x_9 - 0.01601044456x_{10}^2 - 0.008795611x_{10}
\end{align*}
\]
\[ x_5 = -0.235604478 x_1^2 + (0.4368128924 x_2 + 0.3204293413 x_3 + 0.2365468824 x_4 - 0.3493291689 x_5 - 0.3732874301 x_6 + 0.02131713778 x_7 - 0.07468835444 x_8 - 0.1323523275 x_9 + 0.3533971562 x_10 - 0.0147421268) x_1 - 0.2881631989 x_2^2 + (-0.392286193 x_1 + 0.4183395990 x_5 + 0.3325997896 x_6 + 0.0613121538 x_7 + 0.0742319177 x_8 - 0.0300202856 x_9 - 0.2610498978 x_10 - 0.390324168 x_3 - 2.513633472 x_2 - 0.1710301966 x_3^2 + (-0.2039744307 x_4 + 0.2119339561 x_5 + 0.1792833848 x_6 - 0.0226378047 x_7 - 0.03533658575 x_8 + 0.065238073139 x_9 - 0.2597492202 x_{10} + 0.0058670390 x_3 - 0.160784011 x_4^2 + (-2.93211879 + 0.3280215592 x_6 + 0.2249495230 x_6 + 0.110650218 x_7 + 0.0936756110 x_8 - 0.1297453395 x_9 - 0.07572266747 x_{10}) x_4 - 0.1963810687 x_5^2 + (-0.1638866308 + 0.3393330884 x_5 - 0.08319592584 x_7 - 0.1563249011 x_8 + 0.04866923811 x_9 + 0.1643626100 x_{10}) x_5 - 0.17836834645 x_6^2 + (-0.01574759993 - 0.2037362569 x_7 - 0.204200774476 x_6 + 0.008727285361 x_7 - 0.44657264228 x_8 - 0.01664032178 x_9 + 0.03237310870 x_{10} + 0.1579262048) x_1 - 0.019717999780 x_2^2 + (-0.009411239820 x_4 + 0.02263073203 x_5 + 0.03793097508 x_6 - 0.01960010700 x_7 + 0.00553018695 x_8 + 0.04075797187 x_9 - 0.0528394634 x_{10} - 0.3584904003 x_3 + 0.03153139574 x_2 - 0.00395642423 x_3^2 + (-0.04938561448 x_4 + 0.04837433752 x_5 + 0.04476340099 x_6 + 0.008011272630 x_7 + 0.05172384447 x_8 - 0.004710467487 x_9 - 0.006542628740 x_{10} - 0.000004262793 x_3 + 0.009286895212 x_4^2 + (0.4183096855 - 0.02476813038 x_5 + 0.005774852732 x_6 - 0.05415973363 x_7 - 0.004755324375 x_8 + 0.08186419541 x_9 - 0.07189362880 x_{10}) x_4 + 0.002734306575 x_2^2 + (-0.2447161610 - 0.01343966645 x_6 + 0.02761137963 x_7 + 0.04112490938 x_8 - 0.05053718746 x_9 + 0.06648193594 x_{10}) x_5 - 0.009523852257 x_6^2 + (-1.072403800 + 0.01350498951 x_1 - 0.01737425917 x_8 - 0.02618165575 x_9 + 0.04625523647 x_{10}) x_6 + 0.05661597163 x_2^2 + (-0.3402218983 + 0.06127583477 x_7 - 0.02336802754 x_9 - 0.02817641673 x_{10}) x_7 - 0.00578224408 x_2^2 + (-0.4745740131 - 0.07904022308 x_9 + 0.05538492517 x_{10}) x_8 + 0.02058694156 x_2^2 + (-0.09495866683 - 0.03683770877 x_{10}) x_9 + 0.00691259033 x_2^2 + 0.2490206693 x_{10} \]
\[ \dot{x}_7 = 0.1739533759x_7^2 + (-0.3350244126x_2 - 0.2054790966x_3 - 0.1959672537x_4 \\
+ 0.3103898852x_5 + 0.3181229624x_6 + 0.02350429586x_7 + 0.07526826939x_8 \\
+ 0.05519418672x_9 - 0.2381640501x_{10} - 0.5378125466)x_1 + 0.2513311465x_2^2 \\
+(0.3109764237x_4 - 0.2814626018x_5 - 0.1873593607x_6 - 0.02699517830x_7 \\
+ 0.028514000816x_8 + 0.02125324314x_9 + 0.2309699675x_{10} + 0.4020442606x_3 \\
+ 2.585332115)x_2 + 0.1334347191x_3^2 + (0.2919387892x_4 - 0.2617660208x_5 \\
-0.1432345559x_6 - 0.08826316063x_7 - 0.00076569878x_8 + 0.1222633182x_9 \\
+ 0.09091655005x_{10} + 0.2561420917)x_3 + 0.07545659716x_4^3 + (2.542831014 \\
-0.127543760x_5 - 0.07122014199x_6 + 0.00514544561x_7 + 0.09527869569x_8 \\
-0.0104095602x_9 + 0.1738949378x_{10})x_4 + 0.08218611817x_8^2 + (-0.7890598197 \\
+ 0.1888004602x_6 - 0.06126798233x_7 + 0.006889206868x_8 + 0.1371789889x_9 \\
-0.2666446160x_{10})x_5 + 0.1345284350x_6^2 + (0.3109352756 - 0.05845032969x_7 \\
+ 0.0791051252x_8 + 0.1740364179x_9 - 0.2632871889x_{10})x_6 - 0.02871100046x_7^2 \\
+(1.366709846 - 0.04144626131x_8 + 0.05913188175x_9 - 0.0522364083x_{10})x_7 \\
-0.02060051357x_8^2 + (-1.384868757 + 0.1103834035x_9 - 0.1166681038x_{10})x_8 \\
-0.00914009550x_9^2 + (-0.162521828 + 0.00459133672x_{10})x_9 + 0.06107966045x_{10}^2 \\
-0.4706103342x_{10} \\
\]

\[ \dot{x}_8 = 0.01026552266x_1^2 + (-0.03187199823x_2 - 0.000793892722x_3 - 0.006817269314x_4 \\
+ 0.05253478837x_5 + 0.04826223157x_6 + 0.03010087018x_7 - 0.01665094710x_8 \\
-0.2311029685x_9 - 0.01389664226x_{10} + 0.01072576625)x_1 + 0.04465813378x_2^2 \\
+(0.05183681743x_4 - 0.2933197924x_5 + 0.00197779221x_6 - 0.009304296549x_7 \\
+ 0.0393734096x_8 + 0.02260391674x_9 + 0.02473116542x_{10} + 0.01440097493x_3 \\
+ 0.0839783228)x_2 + 0.02013705004x_3^2 + (0.06269877891x_4 - 0.5618655542x_5 \\
-0.00871464069x_6 - 0.04890006622x_7 + 0.02458367844x_8 + 0.07608987663x_9 \\
-0.02040374602x_{10} + 0.1489519871)x_3 + 0.00039422616x_4^2 + (0.2834837759 \\
-0.002012048238x_5 + 0.01705106847x_6 - 0.00613322925x_7 + 0.07468263469x_8 \\
+ 0.01269193177x_9 + 0.02836445274x_{10})x_4 - 0.004844618104x_5^2 + (0.1022313303 \\
+ 0.06472913374x_6 - 0.03959570966x_7 - 0.005062325930x_8 + 0.0577729645x_9 \\
-0.06015458404x_{10})x_5 + 0.01868943770x_6^2 + (0.2943094090 - 0.02766542499x_7 \\
+ 0.0145396782x_8 + 0.0566235001x_9 - 0.05120727162x_{10})x_6 - 0.02522525361x_7^2 \\
+(0.6010179633 + 0.1135788886x_8 + 0.05760306257x_9 - 0.0365036359x_{10})x_7 \\
-0.03613992686x_8^2 + (-0.8923199381 + 0.00752835178x_9 - 0.02308979076x_{10})x_8 \\
-0.02764354738x_9^2 + (0.05671007885 + 0.0366577585x_{10})x_9 - 0.006001738490x_{10}^2 \\
-0.0055618391x_{10} \]
\begin{align*}
\dot{x}_9 &= -0.09497689057x_7^3 + (0.2033638856x_2 + 0.1543259758x_3 + 0.09198433443x_4 \\
&\quad -0.1517072956x_5 - 0.1499516593x_6 - 0.005105339724x_7 + 0.01612675299x_8 \\
&\quad -0.03649260188x_9 + 0.1543198753x_{10} - 0.6170313489x_1 - 0.08415143357x_2^2 \\
&\quad + (-0.1156429920x_4 + 0.1956427857x_5 + 0.2118813826x_6 + 0.01925860567x_7 \\
&\quad + 0.0908343953x_8 + 0.03301652747x_9 - 0.1222703039x_{10} - 0.07532946735x_3 \\
&\quad = -0.7131782561x_2 - 0.0440714271x_3^2 + (0.01795231881x_4 + 0.05905475196x_5 \\
&\quad + 0.1149279646x_6 - 0.04908548282x_7 - 0.0290389967x_8 + 0.1196979096x_9 \\
&\quad = -0.1654653973x_{10} - 0.1337325061x_3 - 0.06074929836x_4^2 + (-0.415392439 \\
&\quad + 0.1951969424x_5 + 0.1712988574x_6 + 0.08779734655x_7 + 0.09882053116x_8 \\
&\quad - 0.08073725812x_9 + 0.00246810218x_{10}x_4 - 0.1093617712x_6^2 + (-0.372897882 \\
&\quad - 0.1799889267x_6 - 0.06834408938x_7 - 0.1323482164x_8 + 0.06613930496x_9 \\
&\quad + 0.0286734767x_{10}x_5 - 0.06810240048x_6^2 + (-0.1121429909 - 0.04569519334x_7 \\
&\quad - 0.0638402097x_8 + 0.04452776278x_9 + 0.06137463992x_{10}x_6 - 0.02015889163x_2^2 \\
&\quad + (0.0856712443 - 0.1156422534x_8 + 0.05954172996x_9 + 0.0653269072x_{10}x_7 \\
&\quad - 0.06709718858x_8^2 + (-0.5124228674 + 0.1555251638x_9 - 0.07569005099x_{10}x_8 \\
&\quad - 0.04354340949x_3^2 + (-1.551265696 + 0.1198169906x_{10}x_9 - 0.09198037687x_{10} \\
&\quad + 0.5566902787x_{10}x_9 \\
\dot{x}_{10} &= -0.09232767692x_7^2 + (0.190427483x_2 + 0.1455960453x_3 + 0.1187073538x_4 \\
&\quad - 0.1333828763x_5 - 0.1297547816x_6 + 0.01600859589x_7 - 0.03772521545x_8 \\
&\quad - 0.04296793828x_9 + 0.1300771420x_{10} + 0.0709262233x_1 - 0.05666641978x_2^2 \\
&\quad + (-0.06203987819x_4 + 0.1430835285x_5 + 0.1866932260x_6 - 0.01452406761x_7 \\
&\quad + 0.07908192648x_8 + 0.0809452091x_9 - 0.1368650892x_{10} - 0.05905368372x_3 \\
&\quad - 0.2049764995x_2 - 0.0226405851x_3^2 + (-0.0164016784x_4 + 0.07648862568x_5 \\
&\quad + 0.1280311791x_6 - 0.03710059262x_7 + 0.05225490874x_8 + 0.09863638176x_9 \\
&\quad - 0.1221892736x_{10} + 0.403379605x_3 - 0.02448892244x_4 + (0.2759614204 \\
&\quad + 0.0989208283x_5 + 0.1278596095x_6 - 0.00461560933x_7 + 0.07971345454x_8 \\
&\quad + 0.04569970809x_9 - 0.07175585739x_{10}x_4 - 0.06881919807x_2^2 + (-0.4598420000 \\
&\quad - 0.1297055428x_6 - 0.02257652510x_7 - 0.0445789173x_8 - 0.000696931411x_9 \\
&\quad + 0.07684569429x_{10}x_5 - 0.04758821665x_7^2 + (-0.03588032938 - 0.02367467985x_7 \\
&\quad - 0.0156654170x_8 + 0.02002302507x_9 + 0.07530333027x_{10}x_6 - 0.01126895539x_7^2 \\
&\quad + (-0.0548398499 - 0.01120011014x_8 + 0.02141068881x_9 - 0.01851571759x_{10}x_7 \\
&\quad - 0.01531694108x_2^2 + (-0.1896173202 + 0.02610044084x_9 + 0.00192671788x_{10}x_8 \\
&\quad - 0.00693600439x_3^2 + (-0.4822097733 + 0.04238410906x_{10}x_9 - 0.05279133763x_{10}^2 \\
&\quad - 0.2873669021x_{10} \\
\end{align*}
References

[1] M. Abadi, A. Agarwal, P. Barham, E. Brevdo, Z. Chen, C. Citro, G. S. Corrado, A. Davis, J. Dean, M. Devin, S. Ghemawat, I. Goodfellow, A. Harp, G. Irving, M. Isard, Y. Jia, R. Jozefowicz, L. Kaiser, M. Kudlur, J. Levenberg, D. Mané, R. Monga, S. Moore, D. Murray, C. Olah, M. Schuster, J. Shlens, B. Steiner, I. Sutskever, K. Talwar, P. Tucker, V. Vanhoucke, V. Vasudevan, F. Viégas, O. Vinyals, P. Warden, M. Wattenberg, M. Wicke, Y. Yu, and X. Zheng, *TensorFlow: Large-scale machine learning on heterogeneous systems*, 2015. Software available from tensorflow.org.

[2] M. Abu-Khalaf and F. L. Lewis, *Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach*, Automatica, 41 (2005), pp. 779–791.

[3] J. Anderson and A. Papachristodoulou, *Advances in computational Lyapunov analysis using sum-of-squares programming*, Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), pp. 2361–2381.

[4] F. Camilli, L. Grüne, and F. Wirth, *A regularization of Zubov’s equation for robust domains of attraction*, in Nonlinear Control in the Year 2000, Volume 1, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, eds., Lecture Notes in Control and Information Sciences 258, NCN, Springer-Verlag, London, 2000, pp. 277–290.

[5] F. Camilli, L. Grüne, and F. Wirth, *Domains of attraction of interconnected systems: a Zubov method approach*, in Proceedings of the European Control Conference — ECC2009, Budapest, Hungary, 2009, pp. 91–96.

[6] G. Cybenko, *Approximation by superpositions of a sigmoidal function*, Math. Control Signals Systems, 2 (1989), pp. 303–314.

[7] J. Darbon, G. P. Langlois, and T. Meng, *Overcoming the curse of dimensionality for some Hamilton-Jacobi partial differential equations via neural network architectures*. Preprint, arXiv:1910.09045, 2019.

[8] S. Dashkovskiy, H. Ito, and F. Wirth, *On a small gain theorem for ISS networks in dissipative Lyapunov form*, Eur. J. Control, 17 (2011), pp. 357–365.

[9] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, *Small gain theorems for large scale systems and construction of ISS Lyapunov functions*, SIAM J. Control Optim., 48 (2010), pp. 4089–4118.

[10] W. E, J. Han, and A. Jentzen, *Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations*, Commun. Math. Stat., 5 (2017), pp. 349–380.

[11] P. Giesl, *Construction of global Lyapunov functions using radial basis functions*, vol. 1904 of Lecture Notes in Mathematics, Springer, Berlin, 2007.

[12] P. Giesl and S. Hafstein, *Review on computational methods for Lyapunov functions*, Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), pp. 2291–2331.
[13] P. Grohs, F. Hornung, and A. Jentzen, *A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black-Scholes partial differential equations*. Preprint, arXiv:1809.02362, 2018.

[14] L. Grüne, *Overcoming the curse of dimensionality for approximating Lyapunov functions with deep neural networks under a small-gain condition*, in Proceedings of the 24th International Symposium on Mathematical Theory of Networks and Systems — MTNS 2020, Cambridge, UK, 2020. Accepted.

[15] S. Hafstein, C. M. Kellett, and H. Li, *Continuous and piecewise affine Lyapunov functions using the Yoshizawa construction*, in Proceedings of the 2014 American Control Conference, 2014, pp. 548–553.

[16] S. F. Hafstein, *An algorithm for constructing Lyapunov functions*, vol. 8 of Electronic Journal of Differential Equations. Monograph, Texas State University–San Marcos, Department of Mathematics, San Marcos, TX, 2007. Available electronically at http://ejde.math.txstate.edu/.

[17] W. Hahn, *Stability of Motion*, Springer–Verlag Berlin, Heidelberg, 1967.

[18] J. Han, A. Jentzen, and W. E, *Solving high-dimensional partial differential equations using deep learning*, Proc. Natl. Acad. Sci. USA, 115 (2018), pp. 8505–8510.

[19] K. Hornik, M. Stinchcombe, and H. White, *Multilayer feedforward networks are universal approximators*, Neural Networks, 3 (1989), pp. 551–560.

[20] C. Hüré, H. Pham, and X. Warin, *Some machine learning schemes for high-dimensional nonlinear PDEs*. Preprint, arXiv:1902.01599v1, 2019.

[21] M. Hutzenthaler, A. Jentzen, and T. Kruse, *Overcoming the curse of dimensionality in the numerical approximation of parabolic partial differential equations with gradient-dependent nonlinearities*. Preprint, arXiv:1912.02571v1, 2019.

[22] M. Hutzenthaler, A. Jentzen, T. Kruse, and T. A. Nguyen, *A proof that rectified deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear heat equations*, SN Partial Differ. Equ. Appl., 10 (2020), p. 34.

[23] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang, *A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems*, Automatica, 32 (1996), pp. 1211–1215.

[24] Z. P. Jiang, A. R. Teel, and L. Praly, *Small-gain theorem for ISS systems and applications*, Math. Control Signals Syst., 7 (1994), pp. 95–120.

[25] S. M. Khansari-Zadeh and A. Billard, *Learning control Lyapunov function to ensure stability of dynamical system-based robot reaching motions*, Robotics and Autonomous Systems, 62 (2014), pp. 752–765.

[26] N. E. Kirin, R. A. Nelepin, and V. N. Bajdaev, *Construction of the attraction region by Zubov’s method*, Differ. Equations, 17 (1982), pp. 871–880.
[27] H. Li, *Computation of Lyapunov functions and stability of interconnected systems*. Dissertation, Universität Bayreuth, Fakultät für Mathematik, Physik und Informatik, 2015.

[28] Y. Long and M. M. Bayoumi, *Feedback stabilization: control Lyapunov functions modelled by neural networks*, in Proceedings of the 32nd IEEE Conference on Decision and Control — CDC 1993, San Antonio, Texas, USA, 1993, pp. 2812–2814.

[29] H. N. Mhaskar, *Neural networks for optimal approximation of smooth and analytic functions*, Neural Computations, 8 (1996), pp. 164–177.

[30] N. Noroozi, P. Karimaghaee, F. Safaei, and H. Javadi, *Generation of Lyapunov functions by neural networks*, in Proceedings of the World Congress on Engineering 2008 Vol I, London, UK, 2008.

[31] V. Petridis and S. Petridis, *Construction of neural network based Lyapunov functions*, in Proceedings of the International Joint Conference on Neural Networks, Vancouver, Canada, 2006, pp. 5059–5065.

[32] T. Poggio, H. Mhaskar, L. Rosaco, M. Brando, and Q. Liao, *Why and when can deep – but not shallow – networks avoid the curse of dimensionality: a review*, Int. J Automat. Computing, 14 (2017), pp. 503–519.

[33] C. Reisinger and Y. Zhang, *Rectified deep neural networks overcome the curse of dimensionality for nonsmooth value functions in zero-sum games of nonlinear stiff systems*. Preprint, arXiv:1903.06652, 2019.

[34] S. M. Richards, F. Berkenkamp, and A. Krause, *The Lyapunov neural network: adaptive stability certification for safe learning of dynamical systems*, in Proceedings of the 2nd Conference on Robot Learning — CoRL 2018, Zürich, Switzerland, 2018. Also available via arXiv:1808.00924.

[35] B. S. Rüffer, *Monotone Systems, Graphs, and Stability of Large-Scale Interconnected Systems*. Dissertation, Fachbereich 3, Mathematik und Informatik, Universität Bremen, Germany, 2007.

[36] G. Serpen, *Empirical approximation for Lyapunov functions with artificial neural nets*, in Proceedings of the International Joint Conference on Neural Networks, Montreal, Canada, 2005, pp. 735–740.

[37] J. Sirignano and K. Spiliopoulos, *DGM: a deep learning algorithm for solving partial differential equations*, J. Comput. Phys., 375 (2018), pp. 1339–1364.

[38] E. D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Autom. Control, 34 (1989), pp. 435–443.

[39] V. I. Zubov, *Methods of A.M. Lyapunov and their Application*, P. Noordhoff, Groningen, 1964.