\( \alpha_s \) From Hadronic \( \tau \) Decay Data

K. Maltman \(^a\)\(^\ast\), T. Yavin\(^b\)

\(^a\)Dept. of Mathematics and Statistics, York Univ., 4700 Keele St., Toronto, ON CANADA M3J 1P3
\(^b\)Dept. of Physics and Astronomy, York Univ., 4700 Keele St., Toronto, ON CANADA M3J 1P3

We discuss the extraction of \( \alpha_s \) using isovector hadronic \( \tau \) decay data and sum rules constructed specifically to suppress contributions associated with poorly known higher dimension condensates. We show, first, that problems with the treatment of such contributions affect earlier related analyses and, second, that these problems can be brought under good theoretical control through the use of an alternate analysis strategy. Our results, run up to the \( n_f = 5 \) regime, correspond to \( \alpha_s(M_\tau^2) = 0.1187 \pm 0.0016 \), in excellent agreement with the recently updated global fit to electroweak data at the \( Z \) scale and other high-scale direct determinations.

1. Introduction and Background

The strong coupling, \( \alpha_s \), at some conventionally chosen reference scale, is one of the fundamental parameters of the Standard Model (SM). Its value, in the \( n_f = 3 \) regime, can be extracted using hadronic \( \tau \) decay data as a consequence of the finite energy sum rule (FESR) relation

\[
\int_0^{s_0} w(s) \rho(s) \, ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s) \Pi(s) \, ds \tag{1}
\]

which is valid for any analytic weight, \( w(s) \), and any correlator \( \Pi(s) \) without kinematic singularities. In Eq. (1), \( \rho(s) \) is the spectral function of \( \Pi(s) \). The basic idea is to use experimental spectral data on the LHS and, for sufficiently large \( s_0 \), the OPE representation of \( \Pi \) (which involves \( \alpha_s \) on the RHS). The region of applicability of the OPE is extended to lower \( s_0 \) when \( w(s) \) satisfies the condition \( w(s = s_0) = 0 \), which suppresses contributions on the RHS from the region of the contour near the timelike real axis [12].

Experimental input for the LHS of Eq. (1) is available because, in the SM, the kinematics of \( \tau \) decay allows the inclusive rate for decays mediated by the flavor \( ij = ud, us, \) vector (V) or axial vector (A) hadronic currents to be written as kinematically weighted integrals over the spectral functions \( \rho_{V/A;ij}^{(j)}(s) \) of the spin \( J = 0, 1 \) components of the relevant current-current two-point functions [3]. Explicitly, with \( y_\tau = s/m_\tau^2 \) and

\[
R_{V/A;ij} \equiv \frac{\Gamma(\tau \rightarrow \nu_\tau \text{ hadrons} | V/A;ij(\gamma) \rangle)}{\Gamma(\tau \rightarrow \nu_\tau e^- \nu_\nu(\gamma) \rangle)},
\]

we have

\[
R_{V/A;ij} = 12\pi^2 |V_{ij}|^2 S_{EW} \int_0^1 dy_\tau (1 - y_\tau)^2 \left[ (1 + 2y_\tau) \rho_{V/A;ij}^{(0+1)}(s) - 2y_\tau \rho_{V/A;ij}^{(0)}(s) \right] \tag{2}
\]

with \( V_{ij} \) the flavor \( ij \) CKM matrix element, \( S_{EW} \) a short-distance electroweak (EW) correction, and

\[
\rho_{V/A;ij}^{(0+1)}(s) = \rho_{V/A;ij}^{(1)}(s) + \rho_{V/A;ij}^{(0)}(s).
\]

For \( ij = ud \), apart from the \( \pi \) contribution to \( \rho_{A;ud}^{(0)} \), \( \rho_{V;ud}^{(0)}(s) \) and \( \rho_{A;ud}^{(0)}(s) \) are numerically negligible, being proportional to \( O(|m_u \mp m_d|^2) \). The sum of flavor \( ud \) V and A spectral functions, \( \rho_{V+A;ud}^{(0+1)}(s) \), can thus be extracted from the differential decay distribution \( dR_{V+A;ud}/ds \), for all \( s < m_\tau^2 \simeq 3.16 \text{ GeV}^2 \). Further separation into V and A components is unambiguous for \( n\pi \) states, but requires additional input for \( K\pi n\pi \) \((n > 0)\) states, making errors on the experimental distribution smallest for the V+A sum.

Given the spectral functions \( \rho_{V+A;ud}^{(0+1)} \), FESRs for the related correlators, \( \Pi_{V+A;ud}^{(0+1)} \), with \( T = V, A, V + A \), are straightforwardly constructible. For the scales \( s_0 \gtrsim 2 \text{ GeV}^2 \) considered here, the OPE representations of these correlators are strongly dominated by the dimension \( D = 0 \)
contribution, which is entirely determined by \( \alpha_s \), converges well, and is known to \( O(\alpha_s^3) \) \cite{14}. The resulting FESRs are thus well adapted to the determination of \( \alpha_s \). To optimize the precision of this determination, however, care must be taken in evaluating the small, residual higher \( D \) contributions, a \( \sim 1\% \) determination of \( \alpha_s(M_Z^2) \), for example, requiring control of higher \( D \) contributions at the level of \( \sim 0.5\% \) of the \( D = 0 \) term \cite{5}.

For \( ij = ud \) and \( s_0 \gtrsim 2 \) GeV^2, \( D = 2 \) contributions are numerically negligible, being either \( O(m_{u,d}^2) \) or \( O(\alpha_s^2 m_{u,d}^2) \) \cite{6}. \( D = 4 \) contributions are, to very good accuracy, determined by the RG invariant condensates \( \langle m_{\ell\ell} \rangle \), \( \langle m_{\pi i} \rangle \) and \( \langle \alpha G^2 \rangle \), for which phenomenologically input exists (see Ref. \cite{7} for the explicit forms of these contributions, and Ref. \cite{5} for details of the condensate values employed). \( D \geq 6 \) contributions are more problematic since the relevant condensates are poorly known or phenomenologically undetermined. We deal with these contributions by defining effective condensate combinations, \( C_b, C_8, \cdots \), such that \( \langle \Pi(Q^2) \rangle^{OPE}_{D>4} = \sum_{D=6,8,\cdots} C_D/Q^D \) (up to logarithmic corrections) and fitting these quantities to data. This process is greatly facilitated by working with polynomial weights \( w(s) = \sum_{m=0} b_m y^m \) defined in terms of the dimensionless variable \( y = s/s_0 \). For such weights, the integrated \( D \geq 6 \) OPE contributions have the form

\[
-\frac{1}{2\pi i} \int_{|s| = s_0} ds \, w(y) \left[ \langle \Pi(Q^2) \rangle^{OPE}_{D>4} \right]_y = \sum_{k=2} (-1)^k b_k \frac{C_{2k+2}}{s_0^k}
\]

allowing contributions of different \( D \) to be distinguished by their differing \( s_0 \) dependences.

2. Problems With Existing Analyses

Existing analyses are based on the approach pioneered by ALEPH and OPAL \cite{30}. In this approach, OPE contributions with \( D > 8 \) are assumed safely negligible for all weights employed, and the quantities \( \alpha_s(m_Z^2) \), \( \langle \alpha G^2 \rangle \), \( C_6 \) and \( C_8 \) are fitted using the \( s_0 = m_Z^2 \) values of the spectral integrals corresponding to the \((km) = (00), (10), (11), (12) \) and \( (13) \) “\( (km) \) spectral weights”, \( w^{(km)}(y) = (1 - y)^k y^m w^{(00)}(y) \), where \( w^{(00)}(y) = (1 - y)^2(1 + 2y) \) is the kinematic weight occurring on the RHS of Eq. 2. ALEPH \cite{10,11} performed this fit independently for each of the \( V, A \) and \( V+A \) channels, while OPAL \cite{9} performed independent fits for the \( V+A \) and combined \( V \) and \( A \) channels. A potential problem with the assumption that all \( D > 8 \) contributions can be safely neglected is the fact that \( w^{(km)} \) has degree \( 3 + k + m \) which, from Eq. \cite{3}, implies that contributions with \( D \) up to 16 are, in principle, present in at least one of the FESRs considered in the ALEPH and OPAL analyses. Since only the single \( s_0 \) value \( s_0 = m_Z^2 \) is employed, there is no way to prevent the fit from adjusting to the presence of any neglected, but non-negligible, \( D > 8 \) contributions by shifting the lower \( D \) parameters determined in the fit in such a way as to compensate, as best as possible, for the missing terms. Such a problem with the fit can only be exposed by studying the same, or related, FESRs over a range of \( s_0 \), where the different scaling with \( s_0 \) of terms of different \( D \) will become operative.

A simple way to test for the presence (or absence) of such problems in the ALEPH and OPAL fits is to consider the \( s_0 \)-dependent fit-qualities,

\[
F_T^w(s_0) = \frac{I^w_{SPEC}(s_0) - I^w_{OPE}(s_0)}{\delta I^w_{SPEC}(s_0)}
\]

where \( T = V, A \) or \( V + A \), \( I^w_{OPE}(s_0) \) and \( I^w_{SPEC}(s_0) \) are the OPE and spectral integrals appearing, respectively, on the RHS and LHS of the corresponding \( w(s) \)-weighted FESR, and \( \delta I^w_{SPEC}(s_0) \) is the error on \( I^w_{SPEC}(s_0) \), determined using the experimental covariance matrix for \( dR_{T,ud}/ds \). Because of strong correlations between values corresponding to the same \( w(s) \) but different \( s_0 \), a fitted version of the OPE representation should be considered reliable only if \( |F_T^w(s_0)| \) remains \( \lesssim 1 \) for a range of \( s_0 < m_Z^2 \).

In Ref. \cite{3} this condition was shown to be far from satisfied for any of the spectral weights employed in the ALEPH and OPAL analyses. An illustration of the problem is provided in Figure 4 which shows the \( F_T^w(s_0) \) corresponding to the 2005 ALEPH final data and OPE fit for four weights, \( w^{(00)}(y) \), \( w_2(y) = (1 - y)^2 \), \( w_3(y) = 1 - \frac{3}{2}y + \frac{3}{4}y^2 \), and \( w(y) = y(1 - y)^2 \), all having
degree $\leq 3$ (and hence OPE contributions only up to $D = 8$). If the fitted values for the $D \leq 8$ OPE parameters obtained by ALEPH are reliable, one should find an $s_0$ window below $m_\tau^2$ having $|F_{V}^\tau(s_0)| \leq 1$ for all four weights. The results, given by the light lines in the figure, show that no such window exists. In fact, for the weights $w_2$, $w_3$ and $y(1-y)^2$ not employed in the original fit, the fit quality is poor even at $s_0 = m_\tau^2$.

The problem seen in the Figure could be due either to contamination of the $D \leq 8$ OPE fit parameters by neglected, but non-negligible, $D > 8$ contributions, or to OPE breakdown. One may test the latter possibility by performing alternate fits in which potential $D > 8$ contributions, where present, are explicitly taken into account. The results of such fits, discussed in the next section, yield alternate OPE representations in excellent agreement with the corresponding spectral integrals over a range of $s_0$, both for the weights employed in the fits and for related weights with OPE representations determined by the same set of OPE parameters. The resulting alternate fit qualities, $F_{V}^\tau(s_0)$, for the four degree $\leq 3$ weights already discussed above, are shown by the dark lines in Figure 1. The results clearly show no evidence for OPE breakdown.

3. Alternate FESR Analyses

In what follows, we employ the updated charmonium sum rule determination of $\langle aG^2 \rangle$ [12]. The gluon condensate term dominates the $D = 4$ OPE contribution. Details on the input for the small corrections proportional to $\langle m_\ell\ell \ell \rangle$ and $\langle m_\ell s \bar{s} \rangle$ may be found in Ref. [5].

The $D = 0$ contributions are evaluated using the expression for the $D = 0$ Adler function series from Ref. [4]. An $O(\alpha_s^2)$ contribution, employing the estimated value for the corresponding coefficient from Ref. [4], is included for our central fit. We consider both the contour improved (CIPT) and fixed order (FOPT) determinations of the integrated $D = 0$ sum. The reference scale $\nu_f = 3$ coupling needed in the evaluation of the CIPT and FOPT sums (taken to be $\alpha_s(m_\tau^2)$) is a parameter to be determined in the fit.

Since, for V and A correlators, OPE breakdown is expected (and observed) to set in for $s_0$ below $\sim 2$ GeV$^2$ [13,14], a limited window of $s_0$ values is available for use in fitting $\alpha_s$ and the unknown $C_{D>4}$. It is thus convenient to work with FESRs based on the weights

$$w_N(y) = 1 - \frac{N}{N-1} y + \frac{1}{N-1} y^N,$$  \hspace{1cm} (5)

which, like $w^{(00)}(y)$, have a double zero at $s = s_0$ ($y = 1$). From Eq. (3) we see that the $w_N$ FESR involves only a single integrated unknown $D > 4$ OPE contribution, $\langle (-1)^N \alpha^2 C_{2N+2} \rangle (s_0)$. As $N$ is increased, the scaling of this contribution with $s_0$ becomes more and more rapid, aiding in the fitting of $C_{2N+2}$. The decrease in the coefficient factor, $1/(N-1)$, also means that the corresponding FESR is more strongly $D = 0$ dominated, a desirable situation for the determination of $\alpha_s$. The latter effect is dominant for sufficiently large $N$. In addition, as $N$ is increased, $w_N(y) \to (1-y)$, whose single zero at $s = s_0$ provides less strong suppression of contributions from the region of the timelike point on the OPE contour. We thus restrict our attention to the $w_2, \cdots, w_6$ FESRs.

Since $D = 2$ contributions are negligible and $D = 4$ contributions are fixed by phenomenological input, the only OPE parameters to be fit using the $w_N$ FESR are $\alpha_s(m_\tau^2)$ and $C_{2N+2}$. The $C_{2N+2}$ will of course depend on the channel (V, A or V+A) being considered. The values for $\alpha_s(m_\tau^2)$ obtained using the different $w_N$ and/or different channels should, however, be consistent, and this consistency represents an important cross-check on the reliability of the analysis framework.

We have analyzed the $w_2, \cdots, w_6$ FESRs using the final 2005 ALEPH isovector data and covariances, in each of the V, A and V+A channels. A similar analysis has been performed for the V+A channel using the OPAL data and covariances. See Ref. [5] for further details, and a discussion of the reasons for the analysis choices.

We report here only on the results for $\alpha_s(m_\tau^2)$. A full discussion of the errors, and results for the $C_{2N+2}$, may be found in Ref. [5]. Results obtained using the CIPT prescription are presented in Table 1. For each entry, the first error is experimental (computed using the experimental covariance matrix, and including the 0.32% normaliza-
Figure 1. Comparison of the $F_{V}^{w}(s_{0})$ corresponding to (i) our fits and (ii) the 2005 ALEPH fit, and various weights having degree $\leq 3$. The light (heavy) dotted line corresponds to the ALEPH fit (our fit) for $w^{(00)}$, the light (heavy) dashed line to the ALEPH fit (our fit) for $w_{2}$, the light (heavy) dot-dashed line to the ALEPH fit (our fit) for $w_{3}$, and the light (heavy) double-dot-dashed line to the ALEPH fit (our fit) for $w(y) = y(1 - y)^{2}$. 
From the table, one sees that theoretical errors are a factor of $\sim 2$ larger than the corresponding experimental errors. The dominant contribution to the theoretical error is that associated with the truncation uncertainty, while the second is theoretical. The corresponding FOPT-based results may be found in Ref. [5].

From the table we see excellent consistency between the ALEPH-based V, A and V+A results, as well as between the ALEPH and OPAL V+A results. There is also extremely good consistency within each channel between results obtained using the different $w_N$. This consistency is realized only after fitting the small, but non-negligible, $D > 4$ contributions on the OPE sides of the various FESRs [5]. The FOPT results corresponding to different $w_N$ but the same channel display significantly less good consistency [5]. From the table, one sees that theoretical errors are a factor of $\sim 2$ larger than the corresponding experimental errors. The dominant contribution to the theoretical error is that associated with the truncation uncertainty, while the second is theoretical. The corresponding FOPT-based results may be found in Ref. [5].

Table 1

| Data set | Channel | Weight | $\alpha_s (m^2)$ |
|----------|---------|--------|-----------------|
| ALEPH    | V       | $w_2$  | 0.321(7)(12)    |
|          |         | $w_3$  | 0.321(7)(12)    |
|          |         | $w_4$  | 0.321(7)(12)    |
|          |         | $w_5$  | 0.321(7)(12)    |
|          |         | $w_6$  | 0.321(7)(12)    |
| A        | $w_2$  | 0.319(6)(12)    |
|          | $w_3$  | 0.319(6)(12)    |
|          | $w_4$  | 0.319(6)(12)    |
|          | $w_5$  | 0.319(6)(12)    |
|          | $w_6$  | 0.319(6)(12)    |
| V+A      | $w_2$  | 0.320(5)(12)    |
|          | $w_3$  | 0.320(5)(12)    |
|          | $w_4$  | 0.320(5)(12)    |
|          | $w_5$  | 0.320(5)(12)    |
|          | $w_6$  | 0.320(5)(12)    |
| OPAL     | V+A    | $w_2$  | 0.322(7)(12)    |
|          | $w_3$  | 0.322(7)(12)    |
|          | $w_4$  | 0.322(7)(12)    |
|          | $w_5$  | 0.322(7)(12)    |
|          | $w_6$  | 0.322(8)(12)    |

With theory errors dominant, and the $D = 0$ truncation uncertainty dominating the theory error, further improvement will be possible only if a better understanding of the truncation uncertainty can be obtained. In a recent paper, Beneke and Jamin [20] investigated this issue using a model which incorporates the general struc-
tude associated with the first two IR renormalon and leading UV renormalon singularities of the resummed $D = 0$ series. This particular version of what could be a more general model was used to argue in favor of the FOPT over the CIPT prescription. The FOPT evaluation, as well as the resummed model, together with assumed values for $C_6$ and $C_8$, were also used to determine $\alpha_s$. While it was shown in Ref. [5] that the assumed $C_6$ and $C_8$ values are not consistent with the $\alpha_s$ obtained using FOPT, the underlying approach remains extremely interesting. In fact, one can see that the minimal version of the model employed in Ref. [20] predicts CIPT approximations which deviate from the model version of the related resummed series by amounts that are sizeable, and a factor of $\sim 2$ larger for $w^{(00)}$ than for $w_2$ and $w_3$ [21]. The deviations are significantly smaller for the FOPT versions. Were the model to provide a good representation of the true resummed series, one would thus expect no set of $\alpha_s$, $C_6$ and $C_8$ to provide a good quality simultaneous CIPT fit for the $w_2$, $w_3$ and $w^{(00)}$ spectral integrals, while a reasonable quality simultaneous fit would be expected to exist using the FOPT prescription. In fact, just the opposite occurs: the values of $\alpha_s$, $C_6$ and $C_8$ obtained from a combined CIPT fit to the $w_2$, $w_3$ FESRs provide also an excellent representation of the $w^{(00)}$ spectral integrals, while those obtained from the corresponding FOPT fit provide a very poor one. This suggests an extended version of the Beneke-Jamin model, perhaps taking into account more IR and/or UV renormalon singularities, is likely to be needed. The fact that it is possible to reach such a conclusion, however, immediately makes clear that the sets of variously weighted $s_0$-dependent spectral integrals provide significant constraints for use in constructing such generalizations. It is thus likely that such modelling can be improved in future, and used to reduce the truncation uncertainty component in the determination of $\alpha_s$.

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