The Resolvent Algebra: Ideals and Dimension

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Abstract

Let \((X, \sigma)\) be a symplectic space admitting a complex structure and let \(\mathcal{R}(X, \sigma)\) be the corresponding resolvent algebra, \textit{i.e.} the C*-algebra generated by the resolvents of self-adjoint operators satisfying canonical commutation relations associated with \((X, \sigma)\). In previous work this algebra was shown to provide a convenient framework for the analysis of quantum systems. In the present article its mathematical properties are elaborated with emphasis on its ideal structure. It is shown that \(\mathcal{R}(X, \sigma)\) is always nuclear and, if \(X\) is finite dimensional, also of type I (postliminal). In the latter case \(\text{dim}(X)\) labels the isomorphism classes of the corresponding resolvent algebras. For \(X\) of arbitrary dimension, principal ideals are identified which are the building blocks for all other ideals. The maximal and minimal ideals of the resolvent algebra are also determined.

Keywords: resolvent algebra, symplectic space, canonical commutation relations, C*-algebra, ideal, representation, isomorphism class
1 Introduction

In [4] we have defined and analyzed the resolvent algebra of the canonical commutation relations. Apart from the applications in that paper, this algebra has already demonstrated its usefulness elsewhere. For example, on its basis one can model in a C*-context superderivations which occur in supersymmetry, cf. [3], as well as in BRST-constraint theory, cf. [6]. It occurs also naturally in the representation theory of abelian Lie algebras of derivations acting on a C*-algebra [5]. Here we continue our analysis of the resolvent algebra, with particular emphasis on its ideal structure.

We review the background which motivates the study of the resolvent algebra. Canonical systems of operators have always been a central ingredient in the modeling of quantum systems. These systems of operators may all be presented in the following general form: there is a real linear map \( \phi \) from a given symplectic space \((X, \sigma)\) to a linear space of selfadjoint operators on some common dense invariant core \(D\) in a Hilbert space \(\mathcal{H}\), satisfying the relations

\[
[\phi(f), \phi(g)] = i\sigma(f, g)1, \quad \phi(f)^\ast = \phi(f) \quad \text{on} \quad D.
\]

In the case that \(X\) is finite dimensional, one can reinterpret this relation in terms of the familiar quantum mechanical position and momentum operators, and if \(X\) consists of Schwartz functions on some manifold one may consider \(\phi\) to be a bosonic quantum field. The observables of the system are then constructed from the operators \(\{\phi(f) : f \in X\}\), usually as polynomial expressions. Since one wants to study a variety of representations of such systems, it is convenient to cast the algebraic information of the canonical systems into C*-algebras, given the rich source of mathematical tools available there.

The obvious way to take this step is to form suitable bounded functions of the generically unbounded fields \(\phi(f)\). In the approach introduced by Weyl, this is done by considering the C*-algebra generated by the set of unitaries

\[
\{ \exp(i\phi(f)) : f \in X \}.
\]

Regarded as an abstract algebra, it is the familiar Weyl algebra [11], denoted by \(\mathcal{W}(X, \sigma)\). The Weyl algebra suffers, however, from several well-known flaws with regard to physics. First and foremost, it does not admit the definition of much interesting dynamics (one-parameter automorphism groups), cf. [4],[8]. Second, natural observables such as bounded functions of the Hamiltonian are not in \(\mathcal{W}(X, \sigma)\). Third, the Weyl algebra has a vast number of representations in which representers of its generators \(\phi(f)\) cannot be defined [11],[9],[10].
These nonregular representations describe situations where the field $\phi$ has “infinite strength”. Whilst this is sometimes useful for idealizations, cf. for example the discussion of plane waves in [1] or of quantum constraints in [10], the majority of nonregular representations of the Weyl algebra is of no interest.

This motivates the consideration of alternative $C^*$-algebraic versions of the canonical commutation relations. Instead of taking the $C^*$-algebra generated by exponentials of the underlying generators, as for the Weyl algebra, we propose to consider the $C^*$-algebra generated by their resolvents [4]. These are given on the underlying Hilbert space $\mathcal{H}$ by

$$\{ R(\lambda, f) \doteq (i\lambda \mathbf{1} - \phi(f))^{-1} : \lambda \in \mathbb{R}\setminus\{0\}, f \in X \}.$$  

All algebraic properties of the fields can be expressed in terms of relations amongst these resolvents and this fact allows one to define the unital $C^*$-algebra generated by the resolvents also in representation independent terms. The structure of the resulting resolvent algebra will be studied below.

In contrast to the Weyl algebra, which is simple, the resolvent algebra has ideals. This feature agrees with the observation that a unital $C^*$-algebra which admits the definition of a sufficiently diverse variety of dynamics cannot be simple [4, p. 2767]. In the present article we analyze the ideal structure of the resolvent algebra and show that it depends sensitively on the size of the underlying quantum system. By a study of its primitive ideals we find that the resolvent algebra is of type I (postliminal) if the dimension of $X$ is finite and that it is merely nuclear if $X$ is infinite dimensional. Moreover, the specific nesting of its primitive ideals encodes information about the dimension of the underlying space $X$. As a matter of fact, this dimension, if it is finite, is an algebraic invariant which labels the isomorphism classes of the associated resolvent algebras. We also analyze a distinguished family of principal ideals, generated by resolvents, which are the building blocks of all other ideals. Based on these results we determine the maximal and minimal ideals of the resolvent algebras, the latter being trivial if $X$ is infinite dimensional. So, in summary, each resolvent algebra comprises very specific information about the underlying quantum system.

The article is organized as follows. We recall in the subsequent section some definitions and facts concerning the resolvent algebra which were established in [4]; readers familiar with these results may skip this section. In Sect. 3 we analyze the structure of its irreducible representations and of its primitive ideals. Sect. 4 contains the analysis of its elemental principal ideals and further resultant information about its ideal structure. A brief discussion concludes this article.
2 Resolvent algebra – definitions and facts

For the convenience of the reader, we compile in this section some definitions and facts which were established in [4], where proofs of the claims and further details can be found. Let $X$ be a real vector space and let $\sigma : X \times X \to \mathbb{R}$ be a nondegenerate symplectic form; in order to avoid pathologies, we make the standing assumption that $(X, \sigma)$ admits some complex structure [13]. Just as the Weyl algebra can be abstractly defined by the Weyl relations, the $C^*$-algebra of resolvents is abstractly defined by its generators and relations.

2.1 Definition Given a symplectic space $(X, \sigma)$, $\mathcal{R}_0$ is the universal unital *-algebra generated by the set $\{R(\lambda, f) : \lambda \in \mathbb{R}\{0\}, f \in X\}$ and the relations

\begin{align*}
R(\lambda, f) - R(\mu, f) &= i(\mu - \lambda)R(\lambda, f)R(\mu, f) \quad (2.1) \\
R(\lambda, f)^* &= R(-\lambda, f) \quad (2.2) \\
[R(\lambda, f), R(\mu, g)] &= i\sigma(f, g)R(\lambda, f)R(\mu, g)^2R(\lambda, f) \quad (2.3) \\
\nu R(\nu \lambda, \nu f) &= R(\lambda, f) \quad (2.4) \\
R(\lambda, f)R(\mu, g) &= R(\lambda + \mu, f + g)(R(\lambda, f) + R(\mu, g) + i\sigma(f, g)R(\lambda, f)^2R(\mu, g)) \quad (2.5) \\
R(\lambda, 0) &= -\frac{i}{\lambda}1 \quad (2.6)
\end{align*}

where $\lambda, \mu, \nu \in \mathbb{R}\{0\}$ and $f, g \in X$, and for (2.5) we require $\lambda + \mu \neq 0$. That is, start with the free unital *-algebra generated by $\{R(\lambda, f) : \lambda \in \mathbb{R}\{0\}, f \in X\}$ and factor out by the ideal generated by the relations (2.1) to (2.6) to obtain the *-algebra $\mathcal{R}_0$.

2.2 Remarks (a) Relations (2.1), (2.2) encode the algebraic properties of the resolvent of some selfadjoint operator, (2.3) encodes the canonical commutation relations and relations (2.4) to (2.6) express the linearity of the underlying map $\phi$ on $X$.

(b) The *-algebra $\mathcal{R}_0$ is nontrivial, because it has nontrivial representations. For instance, in a Fock representation $\pi$ of the canonical commutation relations over $(X, \sigma)$ one has selfadjoint operators $\phi_\pi(f), f \in X$, satisfying the commutation relations on a sufficiently big domain so that one can define $\pi(R(\lambda, f)) \doteq (i\lambda 1 - \phi_\pi(f))^{-1}$ to obtain a representation of $\mathcal{R}_0$.

Let $\mathcal{S}$ denote the set of positive, normalized functionals $\omega$ of $\mathcal{R}_0$. By [4, Prop. 3.3] their GNS–representations $(\pi_\omega, \mathcal{H}_\omega)$ are uniformly bounded w.r.t. $\mathcal{S}$, so one can define:
2.3 Definition Let $(X, \sigma)$ be a symplectic space and let $\mathcal{R}_0$ be the corresponding *-algebra. Then
\[ \|A\| = \sup_{\omega \in \Theta} \|\pi_\omega(A)\|_{\mathcal{H}_\omega}, \quad A \in \mathcal{R}_0 \]
defines a $C^*$-seminorm on $\mathcal{R}_0$. The resolvent algebra $\mathcal{R}(X, \sigma)$ is defined as the $C^*$-completion of the quotient algebra $\mathcal{R}_0 / \text{Ker} \| \cdot \|$, where here and in the following the symbol Ker denotes the kernel of the respective map.

2.4 Remark It follows from [4, Thm. 3.6] that the functions $\lambda \mapsto R(\lambda, f)$ on $\mathbb{R}\setminus\{0\}$ can be analytically continued within $\mathcal{R}(X, \sigma)$ to the domain $\mathbb{C}\setminus i\mathbb{R}$. Obvious extensions of relations (2.1) to (2.6) hold also for these continuations.

As mentioned, the resolvent algebra is not simple, in contrast to the Weyl algebra. In order to explore its rich ideal structure we need to study its representations. In this context the notion of a regular representation is of particular interest.

2.5 Definition A representation $(\pi, \mathcal{H})$ of $\mathcal{R}(X, \sigma)$ is regular on a set $S \subseteq X$ if
\[ \text{Ker} \pi(R(\lambda, f)) = \{0\} \quad \text{for all } f \in S \text{ and some (hence all) } \lambda \in \mathbb{R}\setminus\{0\}. \]
A state $\omega$ of $\mathcal{R}(X, \sigma)$ is regular on a set $S \subseteq X$ if its GNS-representation $\pi_\omega$ is regular on $S \subseteq X$. A representation (resp. state) which is regular on all of $X$ is said to be regular.

This definition draws upon the fact that, as a consequence of the resolvent equations (2.1), (2.2) all operators $\pi(R(\lambda, f))$, $\lambda \in \mathbb{R}\setminus\{0\}$, have a common range and a common null space for fixed $f \in X$. Moreover, if $\text{Ker} \pi(R(\lambda, f)) = \{0\}$ for some (hence for all) $\lambda \in \mathbb{R}\setminus\{0\}$, they are resolvents of some generator $\phi_\pi(f)$, cf. [14, Ch. VIII.4, Thm. 1]. It is given by
\[ \phi_\pi(f) = i\lambda 1 - \pi(R(\lambda, f))^{-1} \quad (2.7) \]
on the dense domain $\text{Dom} \phi_\pi(f) = \pi(R(\lambda, f)) \mathcal{H}$, $\lambda \in \mathbb{R}\setminus\{0\}$. Some relevant properties of these (selfadjoint) operators have been established in [4, Thm. 4.2]; in particular, they provide a one-to-one correspondence between the regular representations of $\mathcal{R}(X, \sigma)$ and of $\mathcal{W}(X, \sigma)$. Further properties of the resolvent algebra $\mathcal{R}(X, \sigma)$ which are used in the present analysis are, cf. [4, Thm. 4.9]:

2.6 Proposition Let $(X, \sigma)$ be a symplectic space of arbitrary dimension and let $S \subset X$ be any finite dimensional nondegenerate subspace (including the zero-dimensional space $\{0\}$). Then:
(i) The norms of $\mathcal{R}(X, \sigma)$ and of $\mathcal{R}(S, \sigma)$ coincide on $\ast$-$\text{Alg}\{ R(\lambda, f) : f \in S, \lambda \in \mathbb{R}\setminus\{0\}\}$. Thus one has the containment $\mathcal{R}(S, \sigma) \subset \mathcal{R}(X, \sigma)$.

(ii) $\mathcal{R}(X, \sigma)$ is the inductive limit of the net of all $\mathcal{R}(S, \sigma)$ where $S \subset X$ ranges over all finite dimensional nondegenerate subspaces of $X$ (including $X$ if it is finite dimensional).

(iii) Every regular representation of $\mathcal{R}(X, \sigma)$ is faithful.

This result also has structural consequences. Since the irreducible Fock representation induces a regular representation of $\mathcal{R}(X, \sigma)$, point (iii) implies that $\mathcal{R}(X, \sigma)$ has faithful irreducible representations (i.e. it is a primitive algebra). Thus its center must be trivial and every nonzero closed two–sided ideal $J \subset \mathcal{R}(X, \sigma)$ is essential (i.e. the left, hence also the right annihilator of $J$ is zero since there exist cyclic vectors for $J$ in every faithful irreducible representation). Consequently the intersection $J_1 \cap J_2$ of any two nonzero closed two–sided ideals $J_1, J_2 \subset \mathcal{R}(X, \sigma)$ is also nonzero (i.e. $\mathcal{R}(X, \sigma)$ is prime), cf. Lemma 3.3(i) below.

3 Primitive ideals and the dimension of $X$

In this section we determine the primitive ideals of $\mathcal{R}(X, \sigma)$ when the dimension $\dim(X)$ of $X$ is finite. The results enable us to show that $\dim(X)$ distinguishes the isomorphism classes of $\mathcal{R}(X, \sigma)$. As a further consequence we find that $\mathcal{R}(X, \sigma)$ is of type I (postliminal).

By definition, the primitive ideals are the kernels of irreducible representations, including the trivial ideal $\{0\}$. To determine them we rely on the subsequent lemma, established in [4, Prop. 4.7], where we use following notation: The symplectic complement of any subspace $S \subseteq X$ is denoted by $S^\perp \doteq \{ f \in X : \sigma(f, S) = 0 \}$. The expression $X = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ says that all subspaces $S_i \subset X$ are nondegenerate, $S_i \subseteq S_j^\perp$ if $i \neq j$, and each $f \in X$ has a unique decomposition $f = f_1 + f_2 + \cdots + f_n$ with $f_i \in S_i, i = 1, \ldots, n$. Note that the zero–dimensional subspace $\{0\} \subset X$ is admitted here for notational convenience.

3.1 Lemma Let $X$ be of finite or infinite dimension and let $(\pi, \mathcal{H})$ be a representation of $\mathcal{R}(X, \sigma)$.

(i) The set $X_R \doteq \{ f \in X : \text{Ker} \pi(R(\lambda, f)) = \{0\} \}$ is a linear space which is independent of $\lambda \in \mathbb{R}\setminus\{0\}$. So for its complement $X_S \doteq X \setminus X_R$ one has $X_S + X_R = X_S$. 

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(ii) The set \( X_T \triangleq \{ f \in X : \text{Ker} \, \pi(R(\lambda, f)) = \{0\} \text{ and } \pi(R(\lambda, f)^{-1}) \subseteq X_R \} \subseteq X_R \) is a linear space which is independent of \( \lambda \in \mathbb{R}\setminus\{0\} \). Moreover, \( \pi(R(\mu, g)) = 0 \) for all \( \mu \in \mathbb{R}\setminus\{0\} \) and \( g \in X \) with \( \sigma(g, X_T) \neq 0 \). Hence \( \sigma(X_R, X_T) = 0 \).

(iii) If \( \pi \) is factorial, then \( \pi(R(\lambda, f)) = 0 \) for all \( f \in X_S \), and \( \pi(R(\lambda, f)) \in \mathbb{C} \setminus \{0\} \) for all \( f \in X_T \) and any \( \lambda \in \mathbb{R}\setminus\{0\} \). Moreover \( X_T = X_R \cap X_R^\perp \).

(iv) Let \( X \) be finite dimensional. If \( \pi \) is factorial one can augment \( X_T \) by a complementary space \( X_T^\wedge \subseteq X \) of the same dimension such that the space \( Q = X_T + X_T^\wedge \) is nondegenerate. Moreover, one has the decomposition

\[
X = Q \oplus (Q^\perp \cap X_R) \oplus (Q^\perp \cap X_R^\perp)
\]

into nondegenerate spaces and \( Q^\perp \cap X_R \subseteq (X_R \setminus X_T) \cup \{0\} \), \( Q^\perp \cap X_R^\perp \subseteq X_S \cup \{0\} \).

Clearly \( X_R \) is the part of \( X \) on which \( \pi \) is regular, \( X_T \) is the part on which it is “trivially regular”, \( X_S \) is the part on which it is singular, and these have a particularly nice form when \( \pi \) is factorial. Refining this, we can in fact fully characterize all irreducible representations and their respective kernels. However, we first need to define an object which will be a convenient index set of the unitary equivalence classes of irreducibles.

### 3.2 Definition

Given a symplectic space \((X, \sigma)\), let \( I(X, \sigma) \) denote the set of all pairs \((Y, \chi)\), where \( Y \subseteq X \) is any subspace and \( \chi \) any pure state (character) on the abelian \( C^* \)-algebra \( C^* \{ R(\lambda, f) : \lambda \in \mathbb{R}\setminus\{0\}, f \in Y \cap Y^\perp \} \) which does not vanish on any of its generating resolvents. Note that, as a consequence of the relations (2.1) to (2.6), there is a linear functional \( \varphi : Y \cap Y^\perp \to \mathbb{R} \) such that \( \chi(R(\lambda, f)) = (i \lambda - \varphi(f))^{-1} \) for \( \lambda \in \mathbb{R}\setminus\{0\}, f \in Y \cap Y^\perp \).

Let \((\pi, \mathcal{H})\) be any irreducible representation of \( \mathcal{R}(X, \sigma) \) and let \( X_R, X_T \) be the associated subspaces of \( X \), defined in the preceding lemma. According to part (iii) of this lemma \( X_T = X_R \cap X_R^\perp \) and \( \pi(\mathcal{R}_T) \in \mathbb{C} \setminus \{0\} \), where \( \mathcal{R}_T \triangleq C^* \{ R(\lambda, f) : \lambda \in \mathbb{R}\setminus\{0\}, f \in X_T \} \). Taking into account that \( \pi \) is a homomorphism and \( \mathcal{R}_T \) an abelian \( C^* \)-algebra it follows that there is a unique pure state \( \chi_T \) on \( \mathcal{R}_T \) such that \( \pi \upharpoonright \mathcal{R}_T = \chi_T(\cdot) \mathbf{1}_\mathcal{H} \). Moreover, as an immediate consequence of the definition of \( X_T \) and the resolvent equation (2.7), \( \chi_T \) does not vanish on any of the generating resolvents of \( \mathcal{R}_T \). Thus one can assign to each irreducible representation of \( \mathcal{R}(X, \sigma) \) a space and state with properties described in the preceding definition: \((\pi, \mathcal{H}) \longmapsto (X_R, \chi_T) \in I(X, \sigma) \). Note that all representations in the unitary equivalence class
of \((\pi, \mathcal{H})\) give rise to the same element of \(I(X, \sigma)\) since the associated spaces \(X_R, X_T\) and functionals \(\chi_T\) remain fixed under arbitrary unitary transformations of the representation. Thus, denoting the set of unitary equivalence classes of irreducible representations (the spectrum) of \(\mathcal{R}(X, \sigma)\) by \(\hat{\mathcal{R}}(X, \sigma)\), the preceding assignment determines a map

\[
\iota : \hat{\mathcal{R}}(X, \sigma) \rightarrow I(X, \sigma).
\]

Its properties are studied in the subsequent proposition for the case of finite dimensional symplectic spaces.

**3.3 Proposition** Let \((X, \sigma)\) be any given symplectic space of finite dimension. Then the map \(\iota : \hat{\mathcal{R}}(X, \sigma) \rightarrow I(X, \sigma)\) defined above is a bijection.

*Proof:* In a first step we prove that \(\iota\) is injective. Let \((\pi, \mathcal{H})\) and \((\pi', \mathcal{H}')\) be two irreducible representations of \(\mathcal{R}(X, \sigma)\) determining the same element \((X_R, \chi_T) \in I(X, \sigma)\). We need to show that these representations are equivalent. Since a representation is uniquely specified by its values for \(\{R(\lambda, f) : \lambda \in \mathbb{R}\setminus\{0\}, f \in X\}\), we focus on this set. According to part (iii) of Lemma 3.1 and the resolvent equation (2.1) we have \(\pi(R(\lambda, f_S)) = 0\) for \(f_S \in X_S = X \setminus X_R\); moreover, \(\pi(R(\lambda, f_T)) = \chi_T(R(\lambda, f_T)) 1_\mathcal{H} = (i\lambda - \varphi(f_T))^{-1} 1_\mathcal{H}\) for \(f_T \in X_T = X_R \cap X_R^\perp\), where \(\varphi : X_T \rightarrow \mathbb{R}\) is a linear functional. By part (iv) of Lemma 3.1 there is a nondegenerate subspace \(X_N = Q^\perp \cap X_R \subseteq X_R\) (depending on the chosen extension \(Q\) of \(X_T\)) such that each \(f_R \in X_R\) can uniquely be decomposed into \(f_R = f_T + f_N\) where \(f_T \in X_T\) and \(f_N \in X_N\). Plugging this into relation (2.7) and bearing in mind the linearity of the generators \(\phi_\pi\) we obtain \(\phi_\pi(f_R) = \varphi(f_T) 1_\mathcal{H} + \phi_\pi(f_N)\) which yields

\[
\pi(R(\lambda, f_R)) = \pi(R(\lambda + i\varphi(f_T), f_N)). \tag{3.2}
\]

It is an immediate consequence of these observations that \(\pi(\mathcal{R}(X, \sigma)) = \pi(\mathcal{R}(X_N, \sigma))\), hence \(\pi \upharpoonright \mathcal{R}(X_N, \sigma)\) is still irreducible. The same arguments apply to the representation \((\pi', \mathcal{H}')\), so one can exchange everywhere in the preceding relations \(\pi\) by \(\pi'\) and \(1_\mathcal{H}\) by \(1_{\mathcal{H}'}\). Now since \(X_N\) is finite dimensional and non–degenerate, all regular irreducible representations of \(\mathcal{R}(X_N, \sigma)\) are unitarily equivalent by the von Neumann uniqueness theorem, cf. [1] Cor. 4.4]. Thus there is an isometry \(V\) mapping \(\mathcal{H}\) onto \(\mathcal{H}'\) which intertwines the representations \(\pi \upharpoonright \mathcal{R}(X_N, \sigma)\) and \(\pi' \upharpoonright \mathcal{R}(X_N, \sigma)\). The relations established above then imply that \(V \pi(R(\lambda, f)) = \pi'(R(\lambda, f)) V\) for all \(\lambda \in \mathbb{R}\setminus\{0\}, f \in X\). Hence the representations \((\pi, \mathcal{H})\) and \((\pi', \mathcal{H}')\) of the given algebra \(\mathcal{R}(X, \sigma)\) are equivalent, proving the injectivity of \(\iota\).
For the proof of surjectivity, let \((Y, \chi) \in I(X, \sigma)\) be given and let \(Z \doteq Y \cap Y^\perp\). We have to show that there exists some irreducible representation \((\pi, \mathcal{H})\) of \(\mathcal{R}(X, \sigma)\) such that the associated spaces coincide with the given ones, i.e. \(X_R = Y\), \(X_T = Z\), \(X_S = X \setminus Y\), and \(\pi \upharpoonright \mathcal{R}_T = \chi(\cdot) \mathbf{1}_\mathcal{H}\). In a first step we establish a decomposition of \(X\) similar to that given in relation (3.1): Making use of [1, Lem. A.1(iii)], we pick some subspace \(Z^\sim \subset X\) which has the same dimension as \(Z\) and which is complementary to \(Z\) in the sense that the linear space \(Q \doteq Z + Z^\sim\) is nondegenerate. Then each \(f \in Y\) can uniquely be decomposed into \(f = f_Z + f_N\) where \(f_Z \in Z \subset Y\) and \(f_N \in Z^\sim^\perp \cap Y = Q^\perp \cap Y\). Moreover, the space \(Q^\perp \cap Y\) is also nondegenerate; for if \(\sigma(f_N, g) = \sigma(f_N, g_N) = 0\) for all \(g = g_Z + g_N \in Y\) it follows that \(f_N \in Q^\perp \cap Y \cap Y^\perp = Q^\perp \cap Z = \{0\}\).

Now let \((\pi_0, \mathcal{H})\) be any regular irreducible representation of \(\mathcal{R}(Q^\perp \cap Y, \sigma)\), e.g. the Schrödinger representation, and let \(\varphi : Z \to \mathbb{R}\) be the linear functional fixed as above by the pure state \(\chi\) on \(Z \doteq C^*\{R(\lambda, f) : \lambda \in \mathbb{R} \setminus \{0\}, f \in Z\}\). With this input and the preceding information we can define a representation \((\pi, \mathcal{H})\) of the \(*\)-algebra \(\mathcal{R}_0\) generated by the resolvents \(R(\lambda, f)\) with \(\lambda \in \mathbb{R} \setminus \{0\}\), \(f \in X\), putting in analogy to relation (3.2)

\[
\pi(R(\lambda, f)) \doteq \begin{cases} 
\pi_0(R(\lambda + i\varphi(f_Z), f_N)) & \text{if } f = f_Z + f_N \in Y \\
0 & \text{if } f \in X \setminus Y 
\end{cases}
\tag{3.3}
\]

and extending this definition of \(\pi\) to arbitrary finite sums and products of resolvents by linearity and multiplicativity. This definition is consistent with relations (2.1) to (2.6) and thus determines a representation of \(\mathcal{R}_0\) on \(\mathcal{H}\) which can be extended to \(\mathcal{R}(X, \sigma)\) by continuity. By construction, \(\pi(\mathcal{R}(X, \sigma)) = \pi_0(\mathcal{R}(Q^\perp \cap Y, \sigma))\), hence \((\pi, \mathcal{H})\) is also irreducible.

It remains to show that the element \((X_R, \chi_T) \in I(X, \sigma)\) associated with \((\pi, \mathcal{H})\) coincides with the given \((Y, \chi)\). Since \((\pi_0, \mathcal{H})\) is a regular representation of \(\mathcal{R}(Q^\perp \cap Y, \sigma)\), it follows from the defining relation (3.3) that \(\text{Ker} (\pi(R(\lambda, f))) = \{0\}\) iff \(f \in Y\), hence \(X_R = Y\). Moreover, by Lemma 3.1(iii) \(X_T = X_R \cap X_R^\perp = Y \cap Y^\perp = Z\) and there exists a pure state \(\chi_T\) on the abelian algebra \(\mathcal{R}_T = C^*\{R(\lambda, f) : \lambda \in \mathbb{R} \setminus \{0\}, f \in X_T = Z\} = Z\) such that \(\pi \upharpoonright \mathcal{R}_T = \chi_T(\cdot) \mathbf{1}_\mathcal{H}\). Now according to relation (3.3)

\[
\pi(R(\lambda, f_Z)) = (i\lambda - \varphi(f_Z))^{-1} \mathbf{1}_\mathcal{H} = \chi(R(\lambda, f_Z)) \mathbf{1}_\mathcal{H}, \quad f_Z \in Z = X_T,
\]

and \(\chi\) is, by assumption, a pure state on \(Z = \mathcal{R}_T\). Hence \(\chi = \chi_T\), completing the proof. 

Note that if \((X, \sigma)\) is infinite dimensional, the map \(\iota\) is definitely not injective, as there exist inequivalent regular irreducible representations. There is then no simple characterization
of the spectrum of $R(X, \sigma)$. Focusing on the finite dimensional case, we determine next the primitive ideals of $R(X, \sigma)$, i.e. the kernels of its irreducible representations. Before entering into this analysis let us recall two basic facts about (throughout this paper always closed and two–sided) ideals, cf. [7] Add. 1.9.12 and [2] Prop. II.8.2.4.

3.4 Lemma Let $A$ be a $C^*$-algebra,

(i) Given ideals $J_1$ and $J_2$ of $A$, then $J_1 \cap J_2 = J_1 \cdot J_2 = J_2 \cdot J_1$.

(ii) Let $\{A_i : i \in I\}$ be a set of $C^*$-subalgebras such that $A_0 = \bigcup_{i \in I} A_i$ is a dense *-subalgebra of $A$. If $J$ is any ideal of $A$, then $J \cap A_0 = \bigcup_{i \in I} (A_i \cap J)$ is dense in $J$.

We can now prove:

3.5 Proposition Let $(X, \sigma)$ be a finite dimensional symplectic space and let $\hat{\pi} \in R(X, \sigma)$ with associated pair $\iota(\hat{\pi}) = (X_R, \chi_T) \in I(X, \sigma)$. Then

(i) Ker $\hat{\pi}$ (i.e. the common kernel of all irreducible representations $(\pi, \mathcal{H})$ belonging to the class $\hat{\pi}$) is the (possibly zero) ideal generated in $R(X, \sigma)$ by

$$S(X_R, \chi_T) = \{R(\lambda, f) : f \in X_S\} \cup \{(R(\mu, g) - \chi_T(R(\mu, g)) \cdot 1) : g \in X_T\}.$$ 

This ideal does not depend on the choice of $\lambda, \mu \in \mathbb{R}\setminus\{0\}$.

(ii) The map $\hat{\pi} \mapsto$ Ker $\hat{\pi}$ is a bijection from $R(X, \sigma)$ to the set of primitive ideals of $R(X, \sigma)$.

Proof: (i) Let $J(X_R, \chi_T) \subset R(X, \sigma)$ denote the ideal generated by the set $S(X_R, \chi_T)$. It follows by a routine computation from the resolvent equation (2.1) and the fact that $\chi_T$ is a pure state on the abelian algebra $R_T = C(\{R(\mu, g) : \mu \in \mathbb{R}\setminus\{0\}, g \in X_T\}$ that $J(X_R, \chi_T)$ does not depend on the choice of $\lambda, \mu \in \mathbb{R}\setminus\{0\}$.

Let $(\pi, \mathcal{H})$ be any irreducible representation in the class of the given $\hat{\pi}$. By the very definition of the pair $(X_R, \chi_T)$ the set $S(X_R, \chi_T)$ lies in the kernel of $\pi$ for any $\lambda, \mu \in \mathbb{R}\setminus\{0\}$, hence $J(X_R, \chi_T) \subseteq$ Ker $\pi$. For the proof that one has equality, we choose a space $Q$ as in relation (3.1) and make use of the unique decomposition of $X_R$ into $X_T = X_R \cap X_R^\perp$ and $X_N = Q^\perp \cap X_R$. Let $f_R = f_T + f_N \in X_R$ with $f_T \in X_T$, $f_N \in X_N$ and let $\lambda, \mu \in \mathbb{R}\setminus\{0\}, \lambda \neq \mu$. It follows from the resolvent equation (2.1) that $(1 - i\mu R(\lambda, f_T))^{-1} = (1 + i\mu R(\lambda - \mu, f_T))$ and taking also into account that $\sigma(f_R, f_T) = 0$ equation (2.5) implies

$$R(\lambda, f_R) - R(\mu, f_N) = (1 - i\mu R(\lambda, f_T)) \cdot (R(\lambda, f_R) - R(\mu, f_N) - i\mu R(\lambda, f_R)R(\mu, f_N)).$$
Because of the analyticity properties of the resolvents this equality extends to \( \mu \in \mathbb{C} \setminus i \mathbb{R} \), hence putting \( \mu = -i \chi_T(R(\lambda, f_T))^{-1} \) one obtains \( (R(\lambda, f_R) - R(\mu, f_N)) \in \mathcal{J}(X_R, \chi_T) \). Since also \( R(\lambda, f_S) \in \mathcal{J}(X_R, \chi_T) \) for \( f_s \in X_S \) each element of the dense subalgebra \( \mathcal{R}_0 \subset \mathcal{R}(X, \sigma) \) generated by all polynomials in \( R(\lambda, f) \) for \( \lambda \in \mathbb{R} \setminus \{0\}, f \in X \) can be decomposed into a sum of elements of \( \mathcal{R}(X_N, \sigma) \) and of \( \mathcal{J}(X_R, \chi_T) \). Now the representation \( \pi | \mathcal{R}(X_N, \sigma) \) is, by the very definition of the symplectic subspace \( X_N \), regular and hence faithful according to Proposition 2.6(iii) and \( \pi \upharpoonright \mathcal{J}(X_R, \chi_T) = 0 \). So this decomposition extends by the continuity of \( \pi \) uniquely to all elements of \( \mathcal{R}(X, \sigma) \). Consequently \( \text{Ker} \pi \subseteq \mathcal{J}(X_R, \chi_T) \), completing the proof of part (i).

(ii) By definition, the map \( \hat{\pi} \to \text{Ker} \hat{\pi} \) is a surjection from \( \mathcal{R}(X, \sigma) \) to the set of primitive ideals of \( \mathcal{R}(X, \sigma) \), so we only need to prove injectivity. Let \( \hat{\pi}, \hat{\pi}' \in \mathcal{R}(X, \sigma) \) such that \( \text{Ker} \hat{\pi} = \text{Ker} \hat{\pi}' \) and let \( \iota(\hat{\pi}) = (X_R, \chi_T) \), \( \iota(\hat{\pi}') = (X'_R, \chi'_T) \) be the associated pairs in \( I(X, \sigma) \). We pick representations \( (\pi, \mathcal{H}), (\pi', \mathcal{H}') \) in the classes of of \( \hat{\pi} \) and \( \hat{\pi}' \), respectively. According to Lemma 3.4 \( \pi(R(\lambda, f)) \neq \{0\} \) iff \( f \in X_R \) and, similarly, \( \pi'(R(\lambda, f)) \neq \{0\} \) iff \( f \in X'_R \). Since \( \text{Ker} \pi = \text{Ker} \pi' \) it follows that \( X_R = X'_R \), hence \( X_T = X'_T \). Moreover, in view of the inclusion \( S(X_R, \chi_T) \subseteq \text{Ker} \pi = \text{Ker} \pi' \) we have

\[
(\chi'_T(R(\lambda, g)) - \chi_T(R(\lambda, g))) 1_{\mathcal{H}'} = \pi'(R(\lambda, g) - \chi_T(R(\lambda, g)) 1) = 0, \quad g \in X_T.
\]

i.e. \( \chi_T = \chi'_T \) and consequently \( I(\hat{\pi}) = I(\hat{\pi}') \). According to Proposition 3.5(i) this implies \( \hat{\pi} = \hat{\pi}' \), completing the proof.

Property (ii) in this proposition is a remarkable feature of the resolvent algebra, shared with the abelian \( C^* \)-algebras. It rarely holds for noncommutative algebras. Consider for example the Weyl algebra which, being simple, has only faithful representations. As a matter of fact, property (ii) does not hold either for the resolvent algebra if the underlying symplectic space is infinite dimensional, cf. the remark made after Proposition 3.3.

Next, we consider the partial order of the set of primitive ideals in \( \mathcal{R}(X, \sigma) \), \( \text{Ker} \hat{\pi} \subseteq \text{Ker} \hat{\pi}' \), where strict inclusions will be denoted by \( \text{Ker} \hat{\pi} \subset \text{Ker} \hat{\pi}' \).

3.6 Definition Let \( \mathcal{A} \) be a unital \( C^* \)-algebra and let \( \hat{\mathcal{A}} \) be the corresponding set of equivalence classes of irreducible representations of \( \mathcal{A} \). The maximal length of strictly increasing chains of (possibly zero) primitive ideals in \( \mathcal{A} \) is denoted by \( L(\mathcal{A}) \), i.e.

\[
L(\mathcal{A}) = \sup \{ n \in \mathbb{N} : \text{Ker} \hat{\pi}_1 \subset \cdots \subset \text{Ker} \hat{\pi}_n, \hat{\pi}_1, \ldots, \hat{\pi}_n \in \hat{\mathcal{A}} \}
\]

and \( L(\mathcal{A}) = \infty \) in case the supremum does not exist. The quantity \( L(\mathcal{A}) \) is clearly an isomorphism invariant of \( C^* \)-algebras.
We are now in a position to prove the main result of this section.

3.7 Theorem Let \((X, \sigma)\) be a symplectic space of arbitrary dimension. Then

(i) \(L(\mathcal{R}(X, \sigma)) = \dim(X)/2 + 1\) if \(\dim(X) < \infty\) and \(L(\mathcal{R}(X, \sigma)) = \infty\) otherwise.

(ii) The isomorphism classes of the set of all resolvent algebras associated with finite-dimensional symplectic spaces are completely characterized by \(L(\mathcal{R}(X, \sigma))\).

Proof: (i) First, let \(X\) be finite dimensional, let \((\pi, \mathcal{H}), (\pi', \mathcal{H}')\) be irreducible representations in the classes \(\widehat{\pi}, \widehat{\pi}' \in \mathcal{R}(X, \sigma)\), respectively, and let \(\text{Ker} \widehat{\pi} \subseteq \text{Ker} \widehat{\pi}'\). According to Proposition 3.5(i) the kernels of the two representations coincide with the two-sided ideals generated by the corresponding sets \(S(X_R, \chi_T)\) and \(S(X_R', \chi_T')\), respectively. Now if \(f \in X_S = X \setminus X_R\), i.e. \(\pi(R(\lambda, f)) = 0\) for \(\lambda \in \mathbb{R} \setminus \{0\}\), one also has \(\pi'(R(\lambda, f)) = 0\), i.e. \(f \in X_S' = X \setminus X_R'\), proving \(X_S \subseteq X_S'\) which implies \(X_R \supseteq X_R'\). Similarly, if \(g \in X_T = X_R \cap X_R^\perp\) one has \(\pi(R(\mu, g) - \chi_T(R(\mu, g))) = 0\) for \(\mu \in \mathbb{R} \setminus \{0\}\), hence \(\pi'(R(\mu, g) - \chi_T(R(\mu, g))) = 0\). But \(\chi_T(R(\mu, g)) \neq 0\), so it follows that \(g \in X_T'\), hence \(X_T \subseteq X_T'\), and \(\chi_T' \uparrow X_T = \chi_T\). Moreover, in case of a strict inclusion \(\text{Ker} \widehat{\pi} \subsetneq \text{Ker} \widehat{\pi}'\) one has \(X_R \supsetneq X_R'\); for otherwise \(S(X_R, \chi_T) = S(X_R', \chi_T')\) in conflict with Proposition 3.5.

Now according to Lemma 3.1(iv) one can always extend the space \(X_T = X_R \cap X_R^\perp\) to a nondegenerate subspace \(Q \subset X\) by augmenting it with some complementary space \(X_T^\sim\) of the same dimension. The space \(X_N = Q^\perp \cap X_R = X_T^\sim \cap X_R\) is nondegenerate by construction and has the dimension \(\dim(X_N) = (\dim(X_R) - \dim(X_T))\) which is even and bounded by \(0 \leq \dim(X_N) \leq \dim(X)\). The same statements apply mutatis mutandis to the spaces \(X_R', X_T'\) and \(X_N'\) affiliated with the second representation. In case of a strict inclusion \(\text{Ker} \widehat{\pi} \subsetneq \text{Ker} \widehat{\pi}'\) it follows from the preceding discussion that \(X_R \supsetneq X_R'\). Since \(X_T \subseteq X_T'\) this implies \(\dim(X_N) > \dim(X_N')\), so the length \(l\) of any given strictly increasing chain of primitive ideals in \(\mathcal{R}(X, \sigma)\) complies with the upper bound \(l \leq \dim(X)/2 + 1\). In order to exhibit a chain where one has equality we choose any strictly decreasing sequence of nondegenerate subspaces \(X_0 = X \supseteq X_1 \supseteq \cdots \supseteq X_{\dim(X)/2} = \{0\}\) and consider the family of representations \((\pi_n, \mathcal{H}_n)\) of \(\mathcal{R}(X, \sigma)\) for which \(\pi_n \uparrow \mathcal{R}(X_n, \sigma)\) is regular, acts irreducibly on \(\mathcal{H}_n\) and \(\pi_n(R(\lambda, f)) = 0\), \(f \in X \setminus X_n\) for \(n = 0, \ldots, \dim(X)/2\), cf. definition (3.3). The resulting strictly increasing chain of primitive ideals \(\text{Ker} \pi_n, n = 0, \ldots, \dim(X)/2\), has the desired length, hence \(L(\mathcal{R}(X, \sigma)) = \dim(X)/2 + 1\), as claimed. It also follows from the latter construction that \(L(\mathcal{R}(X, \sigma)) = \infty\) if \(X\) is infinite dimensional, thereby completing the proof of the first part of the statement.
(ii) As we have seen, $L$ defines an isomorphism invariant from which the dimension of the symplectic space underlying a resolvent algebra can be recovered. Conversely, let $(X, \sigma)$, $(X', \sigma')$ be symplectic spaces of equal finite dimension and let $\mathcal{R}(X, \sigma)$, $\mathcal{R}(X', \sigma')$ be the corresponding resolvent algebras. There then exists a symplectic transformation $\Gamma : X \to X'$ mapping the first space onto the second one and satisfying $\sigma'(\Gamma f, \Gamma g) = \sigma(f, g), \ f, g \in X$. This transformation induces a bijection between the generating elements of the resolvent algebras given by $\gamma(R(\lambda, f)) = R'(\lambda, \Gamma f)$, cf. [4, Thm. 5.3(ii)], which is compatible with the defining relations. It therefore extends to an isomorphism $\gamma : \mathcal{R}(X, \sigma) \to \mathcal{R}(X', \sigma')$, as claimed. Thus $L$ provides a complete algebraic invariant for the family of resolvent algebras associated with finite dimensional symplectic spaces.

Another consequence of the preceding results is the following theorem which throws further light on the structure of resolvent algebras. It ought to be mentioned in this context that the resolvent algebras are not separable [4, Thm. 5.3].

**3.8 Theorem** Let $(X, \sigma)$ be a symplectic space of arbitrary dimension. Then

(i) $\mathcal{R}(X, \sigma)$ is a nuclear C*-algebra,

(ii) $\mathcal{R}(X, \sigma)$ is a Type I (postliminal) C*-algebra iff $\dim(X) < \infty$.

*Proof:* As Type I C*-algebras are nuclear, it follows from part (ii) and Proposition [2, 2.6(ii)] that $\mathcal{R}(X, \sigma)$ is an inductive limit of nuclear algebras, hence nuclear, cf. [2, Prop. IV.3.1.9]. Thus we only need to prove part (ii).

By Theorem IV.1.5.7 and IV.1.5.8 (nonseparable case) in [2] we know that $\mathcal{R}(X, \sigma)$ is a Type I C*-algebra iff it is GCR, i.e. its image for each irreducible representation has nonzero intersection with the compacts [2, Def. IV.1.3.1]. Let $\dim(X) < \infty$, then for an irreducible representation $(\pi, \mathcal{H})$ we know from relation (3.2) and the subsequent remarks that $\pi(\mathcal{R}(X, \sigma)) = \pi(\mathcal{R}(X_N, \sigma))$, where $X_N \subseteq X$ is a nondegenerate subspace. Moreover, $\pi \upharpoonright \mathcal{R}(X_N, \sigma)$ is a regular irreducible representation, hence its range contains the compacts according to [4, Thm. 5.4(i)]. So $\mathcal{R}(X, \sigma)$ is GCR hence Type I if $X$ is finite dimensional. Conversely, if $(\pi, \mathcal{H})$ is a faithful irreducible representation of $\mathcal{R}(X, \sigma)$ such that $\pi(\mathcal{R}(X, \sigma))$ contains the compacts, then their respective pre–image constitutes a non–zero ideal in $\mathcal{R}(X, \sigma)$ which is minimal in view of Lemma [4, 3.4(i)]. But as we will show in Theorem [4, 3.5(ii)] there are no such ideals if $\dim(X) = \infty$, hence $\mathcal{R}(X, \sigma)$ is not GCR in this case. 

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4 Principal ideals

In the preceding section we have characterized all the primitive ideals of \( \mathcal{R}(X, \sigma) \) for the case \( \dim(X) < \infty \). If \( \dim(X) = \infty \), an exhaustive characterization of these ideals seems a hopeless task, however. The concept of principal ideals, \( i.e. \) ideals generated by some element of the algebra, is more appropriate then for structural analysis.

Throughout this section, we consider symplectic spaces \((X, \sigma)\) of arbitrary dimension unless otherwise stated. Let \( \mathcal{J} \subset \mathcal{R}(X, \sigma) \) be any ideal. Since \( \mathcal{R}(X, \sigma) \) is the C*-inductive limit of the algebras \( \mathcal{R}(S, \sigma) \) based on all finite dimensional nondegenerate subspaces \( S \subset X \), it follows from Lemma 3.4(ii) that \( \mathcal{J} \) is the C*-inductive limit of the ideals \( \mathcal{J}(S) \doteq \mathcal{J} \cap \mathcal{R}(S, \sigma) \) in \( \mathcal{R}(S, \sigma) \). As we have seen, the latter ideals are built from principal ideals generated by the operators \((R(\lambda, f) - \rho \mathbf{1})\), where \( \rho \) belongs to \( \text{spec}(R(\lambda, f)) \), the spectrum of the operator \( R(\lambda, f) \) which according to relations (2.2) and (2.3) is normal. Hence these principal ideals are building blocks of all ideals. It is therefore warranted to have a closer look at their structure. We begin with a preparatory lemma which slightly generalizes Theorem 4.1(iv) and Proposition 8.1(ii) in [4].

4.1 Lemma Let \( f \in X \setminus \{0\} \), \( \lambda \in \mathbb{R} \setminus \{0\} \), and let \( \rho \in \text{spec}(R(\lambda, f)) \).

(i) There exists a pure state \( \omega \) on \( \mathcal{R}(X, \sigma) \) such that \( (R(\lambda, f) - \rho \mathbf{1}) \in \text{Ker} \ \omega \).

(ii) Given a state \( \omega \) on \( \mathcal{R}(X, \sigma) \) such that \( (R(\lambda, f) - \rho \mathbf{1}) \in \text{Ker} \ \omega \), then \( \pi_{\omega}(R(\lambda, f) - \rho \mathbf{1}) = 0 \), where \( (\pi_{\omega}, \mathcal{H}_{\omega}) \) denotes the GNS representation induced by \( \omega \). Moreover, if \( \rho \neq 0 \) then \( \pi_{\omega}(R(\mu, g)) = 0 \) for all \( \mu \in \mathbb{R} \setminus \{0\} \) and \( g \in X \) satisfying \( \sigma(g, f) \neq 0 \).

Proof: (i) Since \( \rho \) is contained in the spectrum of \( R(\lambda, f) \), there exists a pure state \( \chi \) on the abelian C*-algebra generated by \( R(\mu, f) \), \( \mu \in \mathbb{R} \setminus \{0\} \) and \( \mathbf{1} \) such that \( \chi(R(\lambda, f) - \rho \mathbf{1}) = 0 \). By the Hahn–Banach theorem, \( \chi \) can be extended to a pure state \( \omega \) on \( \mathcal{R}(X, \sigma) \).

(ii) Since \( R(\lambda, f) \) is a normal operator it follows from the resolvent equations (2.1), (2.2) that its spectral values \( \rho \) satisfy \( \overline{\rho} - \rho = 2i\lambda |\rho|^2 \). Hence one obtains for the given \( \omega \)

\[
\omega((R(\lambda, f) - \rho \mathbf{1})^*(R(\lambda, f) - \rho \mathbf{1})) = (2i\lambda)^{-1}\omega(R(\lambda, f)^* - R(\lambda, f)) - |\rho|^2 = 0 .
\]

Let \( \Omega_{\omega} \in \mathcal{H}_{\omega} \) be the GNS–vector derived from \( \omega \). The preceding equality and the fact that \( R(\lambda, f) \) is normal implies \( \pi_{\omega}(R(\lambda, f) - \rho \mathbf{1}) \Omega_{\omega} = 0 = \pi_{\omega}(R(\lambda, f) - \rho \mathbf{1})^* \Omega_{\omega} \). It will be shown in two steps that these equalities imply \( \text{Ker} \pi_{\omega}(R(\lambda, f) - \rho \mathbf{1}) = \mathcal{H}_{\omega} \). First, let \( \rho = 0 \). It then follows from relation (2.3) that \( \text{Ker} \pi_{\omega}(R(\lambda, f)) \) is stable under the action of \( \pi_{\omega}(R(\mu, g)) \) for
any $\mu \in \mathbb{R}\backslash \{0\}$, $g \in X$. Since $\Omega_\omega \in \text{Ker } \pi_\omega(R(\lambda, f))$ is cyclic for $\pi_\omega(R(X, \sigma))$ this implies $\pi_\omega(R(\lambda, f)) = 0$. Second, if $\rho \neq 0$ it is still true that $\text{Ker } \pi_\omega(R(\lambda, f) - \rho \mathbf{1})$ is stable under the action of the operators $\pi_\omega(R(\mu, g))$ whenever $\sigma(f, g) = 0$. So let $g \in X$ be such that $\sigma(f, g) \neq 0$ and let $\mu \in \mathbb{R}\backslash \{0\}$. By relation (2.3)

$$0 = \omega([R(\lambda, f), R(\mu, g)]) = \sigma(f, g) \rho^2 \omega(R(\mu, g)^2),$$

where it has been used that $\omega(R(\lambda, f)A) = \omega(A R(\lambda, f)) = \rho \omega(A)$ for $A \in \mathcal{R}(X, \sigma)$. Hence $\omega(R(\mu, g)^2) = 0$. Since $\mu \mapsto R(\mu, g)$ is differentiable and $i \frac{d}{d\mu} R(\mu, g) = R(\mu, g)^2$ according to relation (2.1) one arrives at $i \frac{d}{d\mu} \omega(R(\mu, g)) = 0$. Thus $\mu \mapsto \omega(R(\mu, g)) = \text{const} = 0$, where the second equality follows from the bound $\|R(\mu, g)\| \leq |\mu|^{-1}$ for large $\mu$. According to the first step this entails $\pi_\omega(R(\mu, g)) = 0$, hence $\text{Ker } \pi_\omega(R(\lambda, f) - \rho \mathbf{1})$ is stable under the action of all operators $\pi_\omega(R(\mu, g))$ with $\mu \in \mathbb{R}\backslash \{0\}$, $g \in X$. Since $\Omega_\omega \in \text{Ker } \pi_\omega(R(\lambda, f) - \rho \mathbf{1})$ it is then clear that $\pi_\omega(R(\lambda, f) - \rho \mathbf{1}) = 0$ for $\rho \neq 0$ as well. The last part of the statement has already been established in the preceding step, completing the proof.

4.2 Proposition Let $\lambda \in \mathbb{R}\backslash \{0\}$, $f \in X \backslash \{0\}$, $\rho \in \text{spec}(R(\lambda, f))$ and let $\mathcal{I}(\lambda, f; \rho)$ be the ideal generated by $(R(\lambda, f) - \rho \mathbf{1})$, viz. $\mathcal{I}(\lambda, f; \rho) = [\mathcal{R}(X, \sigma)(R(\lambda, f) - \rho \mathbf{1})\mathcal{R}(X, \sigma)]$

(i) $\mathcal{I}(\lambda, f; \rho)$ is proper

(ii) $\mathcal{I}(\lambda, f; \rho) = [\mathcal{R}(X, \sigma)(R(\lambda, f) - \rho \mathbf{1})] = [(R(\lambda, f) - \rho \mathbf{1})\mathcal{R}(X, \sigma)]$

(iii) $(R(\mu, f) - \frac{\rho}{1+i(\mu-\lambda)} \rho \mathbf{1}) \in \mathcal{I}(\lambda, f; \rho)$ for all $\mu \in \mathbb{R}\backslash \{0\}$

(iv) If $\rho \neq 0$ then $R(\mu, g) \in \mathcal{I}(\lambda, f; \rho)$ for all $\mu \in \mathbb{R}\backslash \{0\}$ and $g \in X$ such that $\sigma(g, f) \neq 0$.

Proof: (i) According to part (i) of the preceding lemma there exists a state $\omega$ on $\mathcal{R}(X, \sigma)$ such that $(R(\lambda, f) - \rho \mathbf{1}) \in \text{Ker } \omega$. Hence, by part (ii) of the lemma, $\pi_\omega(R(\lambda, f) - \rho \mathbf{1}) = 0$. Thus $\mathcal{I}(\lambda, f; \rho)$, being non–trivial and contained in the kernel of the representation $\pi_\omega$, is a proper ideal.

(ii) Any closed left ideal is the intersection of the left kernels of all states which contain it, cf. [12, Lem. 3.13.5]. Let $\omega$ be any state on $\mathcal{R}(X, \sigma)$ such that $[\mathcal{R}(X, \sigma)(R(\lambda, f) - \rho \mathbf{1})] \subseteq \mathcal{L}_\omega$, where $\mathcal{L}_\omega$ denotes the left kernel of $\omega$. Then $[\mathcal{R}(X, \sigma)(R(\lambda, f) - \rho \mathbf{1})\mathcal{R}(X, \sigma)] \subseteq \mathcal{L}_\omega$ according to Lemma 4.1(iii). Taking intersections with regard to all such states $\omega$, one arrives at $[\mathcal{R}(X, \sigma)(R(\lambda, f) - \rho \mathbf{1})\mathcal{R}(X, \sigma)] \subseteq [\mathcal{R}(X, \sigma)(R(\lambda, f) - \rho \mathbf{1})]$. The opposite inclusion is

\footnote{Here and in the following $[\cdot]$ denotes the closed linear span of its argument}
trivial, hence \([R(X, \sigma)(R(\lambda, f) - \rho 1)] = [R(X, \sigma)(R(\lambda, f) - \rho 1)R(X, \sigma)]\). Replacing in this equality \(R(\lambda, f)\) by \(\overline{R(\lambda, f)} = R(-\lambda, f)\), \(\rho\) by its complex conjugate \(\overline{\rho}\) and recalling that any closed two–sided ideal of a unital C*-algebra is stable under taking adjoints one also obtains \([\overline{(R(\lambda, f) - \rho 1)R(X, \sigma)}] = [R(X, \sigma)(R(\lambda, f) - \rho 1)R(X, \sigma)],\) as claimed.

(iii) Relation \((2.1)\) implies \((1+i(\mu - \lambda)\rho)R(\mu, f) - \rho 1 = (1+i(\lambda - \mu) R(\mu, f))(R(\lambda, f) - \rho 1)\) from which the assertion follows.

(iv) If \(\rho \neq 0\) and \(\omega\) is any state such that \([R(X, \sigma)(R(\lambda, f) - \rho 1)] \subseteq \mathcal{L}_\omega\) it follows from the last part of Lemma 4.1(ii) that \(R(\mu, g) \in \mathcal{L}_\omega\) for \(\mu \in \mathbb{R}\{0\}\) and any \(g \in X\) satisfying \(\sigma(f, g) \neq 0\). Consequently \(R(\mu, g) \in \overline{[R(X, \sigma)(R(\lambda, f) - \rho 1)]}\).

It is apparent from the preceding discussion that the principal ideals under consideration are in one–to–one correspondence with representations in which the underlying generators \(\phi(f)\) of the resolvents have sharp values (including the singular case \(\infty\)). We turn now to the analysis of intersections of these principal ideals which turn out to be principal ideals as well. Before proving this we need to establish the following lemma.

4.3 Lemma Let \(f \in X \setminus \{0\}, \lambda \in \mathbb{R} \setminus \{0\}\) and \(\rho \in \text{spec}(R(\lambda, f))\), and let \(\mathcal{I}(\lambda, f; \rho)\) be the ideal defined in the preceding proposition. For any ideal \(\mathcal{J} \subset \mathcal{R}(X, \sigma)\) one has
\[
\mathcal{I}(\lambda, f; \rho) \cap \mathcal{J} = [(R(\lambda, f) - \rho 1)\mathcal{J}] = [\mathcal{J}(R(\lambda, f) - \rho 1)] = [\mathcal{J}(R(\lambda, f) - \rho 1)\mathcal{J}].
\]

Proof: According to Lemma 3.4(i) \(\mathcal{I}(\lambda, f; \rho) \cap \mathcal{J} = \mathcal{I}(\lambda, f; \rho) \cdot \mathcal{J}\). Hence, making use of part (ii) of the preceding proposition, one obtains
\[
\mathcal{I}(\lambda, f; \rho) \cap \mathcal{J} = [(R(\lambda, f) - \rho 1)\mathcal{R}(X, \sigma)]\mathcal{J} \subseteq [(R(\lambda, f) - \rho 1)\mathcal{R}(X, \sigma)\mathcal{J}] = [(R(\lambda, f) - \rho 1)\mathcal{J}].
\]
Clearly \([R(\lambda, f) - \rho 1]\mathcal{J} \subseteq \mathcal{I}(\lambda, f; \rho) \cap \mathcal{J}\), so the first equality in the statement follows and in a similar manner one obtains the second equality. For the last equality one makes use of the fact that \([\mathcal{J}(R(\lambda, f) - \rho 1)] = \mathcal{I}(\lambda, f; \rho) \cap \mathcal{J}\) is an ideal, hence
\[
[\mathcal{J}(R(\lambda, f) - \rho 1)] \cap \mathcal{J} \subseteq [[\mathcal{J}(R(\lambda, f) - \rho 1)] \mathcal{J}] \subseteq [\mathcal{J}(R(\lambda, f) - \rho 1)],
\]
and the opposite inclusion holds since any ideal \(\mathcal{J}\) has an approximate identity.

4.4 Proposition Let \(f_m \in X \setminus \{0\}, \lambda_m \in \mathbb{R} \setminus \{0\}, \rho_m \in \text{spec}(R(\lambda_m, f_m))\) and let \(\mathcal{I}(\lambda_m, f_m; \rho_m)\) be the corresponding ideals introduced in Proposition 4.3, \(m = 1, \ldots, n\). Then
\[
\bigcap_{m=1}^{n} \mathcal{I}(\lambda_m, f_m; \rho_m) = [R(X, \sigma) \prod_{m=1}^{n} (R(\lambda_m, f_m) - \rho_m 1)] \mathcal{R}(X, \sigma)
\]
\[
= [\prod_{m=1}^{n} (R(\lambda_m, f_m) - \rho_m 1)] \mathcal{R}(X, \sigma)] = [R(X, \sigma) \prod_{m=1}^{n} (R(\lambda_m, f_m) - \rho_m 1)],
\]
where the order of the operators \((R(\lambda_m, f_m) - \rho_m 1)\) in the products is arbitrary.
Proof: The proof proceeds by induction in \( n \). For \( n = 1 \) the statement was established in Proposition 4.2. Putting \( J_n = \bigcap_{m=1}^{n} I(\lambda_m, f_m; \rho_m) \) it follows from the preceding lemma and the induction hypothesis that

\[
J_{n+1} = I(\lambda_{n+1}, f_{n+1}; \rho_{n+1}) \bigcap J_n = \left[ (R(\lambda_{n+1}, f_{n+1}) - \rho_{n+1} \mathbf{1}) \right] J_n
\]

Similarly \( J_{n+1} = [R(X, \sigma) \bigcap \Pi_{m=1}^{n} (R(\lambda_m, f_m) - \rho_m \mathbf{1}) (R(\lambda_{n+1}, f_{n+1}) - \rho_{n+1} \mathbf{1})] \). Moreover, since \( J_{n+1} \) is an ideal, the preceding equality implies

\[
J_{n+1} = [R(X, \sigma) \bigcap \Pi_{m=1}^{n} (R(\lambda_m, f_m) - \rho_m \mathbf{1}) (R(\lambda_{n+1}, f_{n+1}) - \rho_{n+1} \mathbf{1})] R(X, \sigma) \bigcap [R(X, \sigma) \bigcap \Pi_{m=1}^{n} (R(\lambda_m, f_m) - \rho_m \mathbf{1}) (R(\lambda_{n+1}, f_{n+1}) - \rho_{n+1} \mathbf{1})]
\]

Since the intersection of sets is stable under their permutation the order of the factors \( (R(\lambda, f) - \rho, \mathbf{1}) \) in the products is arbitrary, completing the proof of the statement.

As already mentioned at the end of Sec. 2 the intersection of any two nonzero ideals in \( \mathcal{R}(X, \sigma) \) is nonzero. The situation changes, however, if one proceeds to infinite intersections. There one encounters a marked difference between the resolvent algebras of finite and of infinite systems.

4.5 Theorem Let \( J \subset \mathcal{R}(X, \sigma) \) be the intersection of all nonzero ideals of \( \mathcal{R}(X, \sigma) \).

(i) If \( \dim(X) < \infty \) then the ideal \( J \) is isomorphic to the \( C^* \)-algebra \( K \) of compact operators.

(ii) If \( \dim(X) = \infty \) then \( J = \{0\} \). In fact, there exists no nonzero minimal ideal of \( \mathcal{R}(X, \sigma) \)
in this case.

Proof: (i) According to Theorem 3.3(ii) the algebra \( \mathcal{R}(X, \sigma) \) is of type I (postliminal) if \( \dim(X) < \infty \). Picking any faithful irreducible representation \( (\pi, \mathcal{H}) \) of \( \mathcal{R}(X, \sigma) \) one has \( \mathcal{K} \subset \pi(\mathcal{R}(X, \sigma)) \), so the pre-image \( \pi^{-1}(\mathcal{K}) \) is a nonzero ideal of \( \mathcal{R}(X, \sigma) \). Given any other nonzero ideal \( \mathcal{I} \subset \mathcal{R}(X, \sigma) \) one has \( \mathcal{I} \cap \pi^{-1}(\mathcal{K}) \neq \{0\} \) since \( \mathcal{R}(X, \sigma) \) is prime and consequently \( \pi(\mathcal{I}) \cap \mathcal{K} \neq \{0\} \) since \( \pi \) is faithful. So the ideal \( \pi(\mathcal{I}) \subset B(\mathcal{H}) \) contains some nontrivial compact operator and consequently \( \mathcal{K} \subset \pi(\mathcal{I}) \) since \( \pi \) is irreducible. Hence \( \pi^{-1}(\mathcal{K}) \subset \mathcal{I} \) for any nonzero ideal \( \mathcal{I} \), proving the first statement.

(ii) Let \( J \) be a nonzero minimal ideal of \( \mathcal{R}(X, \sigma) \). It follows from Proposition 2.6(ii) and Lemma 3.4(ii) that there exists a finite dimensional non-degenerate subspace \( S_0 \subset X \) such that \( J \cap \mathcal{R}(S_0, \sigma) \neq \{0\} \). Hence \( J \cap \mathcal{R}(S_0, \sigma) \) is a nonzero ideal of \( \mathcal{R}(S_0, \sigma) \) which,
according to the first step, contains a distinguished algebra \( \mathcal{K}(S_0) \) which is isomorphic to the algebra of compact operators. Hence \([\mathcal{R}(X, \sigma)\mathcal{K}(S_0)\mathcal{R}(X, \sigma)] \subseteq \mathcal{J} \) and since \( \mathcal{J} \) is minimal one has equality, \([\mathcal{R}(X, \sigma)\mathcal{K}(S_0)\mathcal{R}(X, \sigma)] = \mathcal{J} \). The same reasoning applies to all larger finite dimensional and non–degenerate subspaces \( S \supset S_0 \). But, as we shall see, one has the strict inclusion \([\mathcal{R}(X, \sigma)\mathcal{K}(S)\mathcal{R}(X, \sigma)] \subsetneq [\mathcal{R}(X, \sigma)\mathcal{K}(S_0)\mathcal{R}(X, \sigma)] \) whenever \( S_0 \subsetneq S \). Hence no nonzero ideal of \( \mathcal{R}(X, \sigma) \) can be minimal if \( X \) is infinite dimensional.

For the proof of the above inclusion we recall that \( \mathcal{K}(S) \) is contained in all nonzero ideals of \( \mathcal{R}(S, \sigma) \), so one clearly has \( \mathcal{K}(S) \subset [\mathcal{R}(S, \sigma)\mathcal{K}(S_0)\mathcal{R}(S, \sigma)] \subset [\mathcal{R}(X, \sigma)\mathcal{K}(S_0)\mathcal{R}(X, \sigma)] \) and consequently \([\mathcal{R}(X, \sigma)\mathcal{K}(S)\mathcal{R}(X, \sigma)] \subset [\mathcal{R}(X, \sigma)\mathcal{K}(S_0)\mathcal{R}(X, \sigma)] \) if \( S_0 \subset S \). In order to see that this inclusion is strict if \( S_0 \subset S \) we pick any pure state \( \omega_S \) on \( \mathcal{R}(S, \sigma) \) which is regular on \( \mathcal{R}(S_0, \sigma) \) and has in its kernel the resolvents \( R(\lambda, f) \) with \( \lambda \in \mathbb{R}\setminus\{0\} \), \( f \in S\setminus S_0 \), cf. Proposition 3.5. We extend \( \omega_S \) to a state \( \omega \) on \( \mathcal{R}(X, \sigma) \) and consider its GNS–representation \((\pi_\omega, \mathcal{H}_\omega)\). According to Lemma 4.1(ii) the resolvents \( R(\lambda, f) \) and therefore also the spaces \([\mathcal{R}(S, \sigma) R(\lambda, f) R(S, \sigma)] \), \( f \in S\setminus S_0 \) are contained in the kernel of \( \pi_\omega \). But each space \([\mathcal{R}(S, \sigma) R(\lambda, f) R(S, \sigma)] \) is a nonzero ideal of \( \mathcal{R}(S, \sigma) \) and consequently \( \mathcal{K}(S) \) lies in the kernel of \( \pi_\omega \) as well. So \( \pi_\omega \upharpoonright [\mathcal{R}(X, \sigma)\mathcal{K}(S)\mathcal{R}(X, \sigma)] = 0 \). On the other hand \( \omega \upharpoonright \mathcal{K}(S_0) = \omega_S \upharpoonright \mathcal{K}(S_0) \) is regular by construction, hence \( \pi_\omega \upharpoonright [\mathcal{R}(X, \sigma)\mathcal{K}(S_0)\mathcal{R}(X, \sigma)] \) is different from 0. So the respective ideals are different, which completes the proof of the statement.

Having clarified the structure of the basic principal ideals, of their intersections and of the minimal ideals of the resolvent algebras \( \mathcal{R}(X, \sigma) \) we will determine now the maximal ideals. We will show that these are generated by certain specific subfamilies of the basic principal ideals.

**4.6 Theorem** (i) Let \( Z \subset X \) be an isotropic subspace, i.e. \( \sigma(f, g) = 0 \) for all \( f, g \in Z \), let \( \mathcal{A}(Z) \subset \mathcal{R}(X, \sigma) \) be the abelian \( C^* \)-algebra generated by \( R(\lambda, f) \) where \( \lambda \in \mathbb{R}\setminus\{0\} \), \( f \in Z \) and let \( \chi \) be a pure state on \( \mathcal{A}(Z) \) which has none of these generating resolvents in its kernel. The closed two–sided ideal \( \mathcal{J} \) generated by \( \{ (R(\lambda, f) - \chi(R(\lambda, f)))1 : \lambda \in \mathbb{R}\setminus\{0\} , f \in Z \} \) and \( \{ R(\mu, g) : \mu \in \mathbb{R}\setminus\{0\} , g \in X\setminus Z \} \) is a proper maximal ideal of \( \mathcal{R}(X, \sigma) \).

(ii) Conversely, let \( \mathcal{J} \subset \mathcal{R}(X, \sigma) \) be a proper maximal ideal of \( \mathcal{R}(X, \sigma) \). There exists an isotropic subspace \( Z \subset X \) and a pure state \( \chi \) on the abelian \( C^* \)-algebra \( \mathcal{A}(Z) \) generated by \( R(\lambda, f) \) with \( \lambda \in \mathbb{R}\setminus\{0\} , f \in Z \), which has none of these generating resolvents in its kernel, such that \( \mathcal{J} \) coincides with the corresponding ideal defined in (i).
Proof: (i) For the proof that \( \mathcal{J} \) is a proper ideal we construct a representation \( \pi_{\mathcal{J}} \) of \( \mathcal{R}(X, \sigma) \) which has \( \mathcal{J} \) in its kernel. This representation acts on the one–dimensional Hilbert space \( \mathbb{C} \) and is fixed by setting \( \pi_{\mathcal{J}}(R(\lambda, f)) = \chi(\lambda(\lambda, f)) \) for \( \lambda \in \mathbb{R} \setminus \{0\}, f \in Z \) and \( \pi_{\mathcal{J}}(R(\mu, g)) = 0 \) for \( \mu \in \mathbb{R} \setminus \{0\}, g \in X \setminus Z \). Since \( \chi : \mathcal{A}(Z) \to \mathbb{C} \) is a homomorphism one easily verifies that \( \pi_{\mathcal{J}} \) extends by linearity and multiplicativity to the *–algebra \( \mathcal{R}_0 \) generated by the resolvents, i.e. its definition is consistent with the defining relations (2.1) to (2.6). By continuity it can therefore be extended to all of \( \mathcal{R}(X, \sigma) \). By construction \( \pi_{\mathcal{J}} \upharpoonright \mathcal{J} = \{0\} \), i.e. \( \mathcal{J} \) is a proper ideal. For the proof that it is maximal we recall that \( \mathcal{R}(X, \sigma) \) is the closure of all polynomials formed out of resolvents and the identity operator. Replacing in any given polynomial the resolvents \( R(\lambda, f) \) by \( (\lambda R(\lambda, f) - \chi(\lambda(\lambda, f))) \mathbf{1} + \chi(\lambda(\lambda, f)) \mathbf{1} \) if \( \lambda \in \mathbb{R} \setminus \{0\}, f \in Z \) and keeping the other resolvents untouched it is apparent that \( \mathcal{J} + \mathbb{C} \mathbf{1} = \mathcal{R}(X, \sigma) \). Hence the codimension of the proper ideal \( \mathcal{J} \) is 1, so it cannot be extended any further to a proper ideal, i.e. it is maximal.

(ii) For the proof of the second statement we make use of the fact that every maximal ideal \( \mathcal{J} \) is primitive, i.e. kernel of some irreducible representation \( (\pi_{\mathcal{J}}, \mathcal{H}_{\mathcal{J}}) \), cf. [2, Subsec. II.6.5.3]. It is an immediate consequence of Lemma 3.1(iii) and the spectral theorem for normal operators that for each resolvent \( R(\lambda, f) \in \mathcal{R}(X, \sigma) \) with \( \lambda \in \mathbb{R} \setminus \{0\}, f \in X \) there is a spectral value \( \rho \in \text{spec}(R(\lambda, f)) \) such that the inverse of \( \pi_{\mathcal{J}}(R(\lambda, f) - \rho \mathbf{1}) \) does not exist as a bounded operator on \( \mathcal{H}_{\mathcal{J}} \). As a matter of fact, for any such \( \rho \) one must have \( (R(\lambda, f) - \rho \mathbf{1}) \in \mathcal{J} \). For otherwise there exist in view of Proposition 1.2(ii) and the fact that \( \mathcal{J} \) is a maximal ideal sequences \( \{J_n \in \mathcal{J}\}_{n \in \mathbb{N}}, \{R_n \in \mathcal{R}(X, \sigma)\}_{n \in \mathbb{N}} \) such that \( J_n + (R(\lambda, f) - \rho \mathbf{1}) R_n \to \mathbf{1} \) in the norm topology. Since the invertible elements in a C*-algebra form an open set it follows that for sufficiently large \( n \) one has \( (J_n + (R(\lambda, f) - \rho \mathbf{1}) R_n)^{-1} \in \mathcal{R}(X, \sigma) \). Thus there are \( J \in \mathcal{J}, R \in \mathcal{R}(X, \sigma) \) such that \( J + (R(\lambda, f) - \rho \mathbf{1}) R = \mathbf{1} \). But this implies \( \pi_{\mathcal{J}}(R(\lambda, f) - \rho \mathbf{1}) \pi_{\mathcal{J}}(R) = \mathbf{1}_{\mathcal{H}_{\mathcal{J}}}, \) i.e. \( \pi_{\mathcal{J}}(R(\lambda, f) - \rho \mathbf{1}) \) has a bounded right inverse. In a similar manner one shows that \( \pi_{\mathcal{J}}(R(\lambda, f) - \rho \mathbf{1}) \) has also a bounded left inverse, which is in conflict with the initial choice of \( \rho \). Hence \( (R(\lambda, f) - \rho \mathbf{1}) \in \mathcal{J} \), the kernel of \( \pi_{\mathcal{J}} \), and consequently \( \pi_{\mathcal{J}}(R(\lambda, f)) \in \mathbb{C} \) for \( \lambda \in \mathbb{R} \setminus \{0\}, f \in X \), i.e. \( (\pi_{\mathcal{J}}, \mathcal{H}_{\mathcal{J}}) \) is a one–dimensional representation of \( \mathcal{R}(X, \sigma) \).

According to Lemma 3.1(ii) the set \( Z \subset X \) for which \( \pi_{\mathcal{J}}(R(\lambda, f)) \in \mathbb{C} \setminus \{0\} \) if \( \lambda \in \mathbb{R} \setminus \{0\}, f \in Z \), is an isotropic subspace. Let \( \mathcal{A}(Z) \subset \mathcal{R}(X, \sigma) \) be the abelian C*-algebra generated by the corresponding resolvents and let \( \chi \doteq \pi_{\mathcal{J}} \upharpoonright \mathcal{A}(Z) \). Since \( \pi_{\mathcal{J}} : \mathcal{A}(Z) \to \mathbb{C} \) is a homomorphism, \( \chi \) is a pure state which does not vanish on any of the generating resolvents. According
to the preceding step the ideal generated by \( \{(R(\lambda, f) - \chi(R(\lambda, f)))1 : \lambda \in \mathbb{R}\backslash\{0\}, f \in Z\} \)
and \( \{R(\mu, g) : \mu \in \mathbb{R}\backslash\{0\}, g \in X\backslash Z\} \) is a proper maximal ideal of \( \mathcal{R}(X, \sigma) \). But, as has been shown, these generating elements are also contained in the given maximal ideal \( \mathcal{J} \). So the two ideals must coincide, completing the proof of the statement.

As has been shown in the preceding proof, each maximal ideal of \( \mathcal{R}(X, \sigma) \) coincides with the kernel of a one–dimensional representation and hence corresponds to a character. Moreover, these characters separate the generating resolvents \( \{R(\lambda, f) : \lambda \in \mathbb{R}\backslash\{0\}, f \in X\} \), so there are many of them. We conclude this analysis of the ideal structure of the resolvent algebra with a brief discussion of its commutator ideal, \( \text{i.e.} \) the ideal which is generated by the commutators of all its elements.

4.7 Proposition
The ideal \( \mathcal{J}_c \) generated by \( \{[R, R'] : R, R' \in \mathcal{R}(X, \sigma)\} \) is proper. It coincides with the ideal generated by \( \{R(\lambda, f)R(\mu, g) : \lambda, \mu \in \mathbb{R}\backslash\{0\}, f, g \in X \text{ with } \sigma(f, g) \neq 0\} \) and is contained in all maximal ideals of \( \mathcal{R}(X, \sigma) \).

Proof: Since the algebra \( \mathcal{R}_0 \) of all polynomials in the basic resolvents is dense in \( \mathcal{R}(X, \sigma) \) it follows that \( \mathcal{J}_c \) coincides with the ideal generated by \( \{[R(\lambda, f), R(\mu, g)] : \lambda, \mu \in \mathbb{R}\backslash\{0\}, f, g \in X\} \). According to relation (2.3) the latter ideal in turn coincides with the ideal generated by \( \{R(\lambda, f)R(\mu, g)^2R(\lambda, f) : \lambda, \mu \in \mathbb{R}\backslash\{0\}, f, g \in X \text{ with } \sigma(f, g) \neq 0\} \). Now by Proposition 4.4 the ideals generated by \( R(\lambda, f)R(\mu, g)^2R(\lambda, f) \), being equal to \( \mathcal{I}(\lambda, f, 0) \cap \mathcal{I}(\mu, g, 0) \), coincide with the ideals generated by \( R(\lambda, f)R(\mu, g) \) for \( \lambda, \mu \in \mathbb{R}\backslash\{0\} \) and \( f, g \in X \). Hence \( \mathcal{J}_c \) is equal to the ideal generated by \( \{R(\lambda, f)R(\mu, g) : \lambda, \mu \in \mathbb{R}\backslash\{0\}, f, g \in X \text{ with } \sigma(f, g) \neq 0\} \).

It remains to show that the ideal \( \mathcal{J}_c \) is contained in all maximal ideals determined in the preceding theorem (which also implies that \( \mathcal{J}_c \) is proper). But this instantly follows from the fact, established above, that all maximal ideals coincide with the kernels of one–dimensional representations of \( \mathcal{R}(X, \sigma) \). These annihilate all commutators and consequently also \( \mathcal{J}_c \).

Denoting by \( R_c(\lambda, f) \) the class of \( R(\lambda, f) \) modulo \( \mathcal{J}_c \) where \( \lambda \in \mathbb{R}\backslash\{0\}, f \in X \) one easily checks that these operators satisfy the defining relations (2.1) to (2.6) with \( \sigma \equiv 0 \) and they generate the abelian \( \mathbb{C}^* \)–algebra \( \mathcal{R}(X, \sigma)/\mathcal{J}_c \). The symplectic form \( \sigma \) only enters in the additional relation \( R_c(\lambda, f)R_c(\mu, g) = 0 \) if \( \sigma(f, g) \neq 0 \). Hence the operators assigned to such incompatible elements of \( X \) have disjoint spectral supports, reflecting the incommensurability of the underlying quantum observables in the abelian quotient algebra.
5 Conclusions

In the present investigation we have clarified the ideal structure of the resolvent algebra $\mathcal{R}(X, \sigma)$. All of its ideals are built in a simple, physically significant manner from principal ideals generated by the basic resolvents. Moreover, the nesting of its primitive ideals encodes precise information about the dimension of the symplectic space $X$, i.e. of the size of the underlying quantum system. There is a sharp algebraic distinction between quantum systems with finitely many particles, where $\dim(X) < \infty$ and the resolvent algebra is postliminal (type I), and the case of quantum field theory, respectively infinitely many particles, where $\dim(X) = \infty$ and the resolvent algebra is no longer postliminal, but it is still nuclear.

Another prominent difference between these two cases consists of the following fact: If $\dim(X) < \infty$ the resolvent algebra $\mathcal{R}(X, \sigma)$ contains a non–trivial minimal ideal $\mathcal{K}$ which is isomorphic to the compacts. This ideal carries the regular representations of $\mathcal{R}(X, \sigma)$ in the sense that these are precisely the unique extensions of the (nondegenerate) representations of $\mathcal{K}$. If $\dim(X) = \infty$ there exists no such ideal in $\mathcal{R}(X, \sigma)$, however. Yet since $\mathcal{R}(X, \sigma)$ is the $\mathbb{C}^*$–inductive limit of its subalgebras $\mathcal{R}(S, \sigma)$ for all finite dimensional nondegenerate subspaces $S \subset X$ there is a certain substitute. Each subalgebra $\mathcal{R}(S, \sigma) \subset \mathcal{R}(X, \sigma)$ contains its own compact ideal $\mathcal{K}(S)$. Let $\mathcal{L} \subset \mathcal{R}(X, \sigma)$ be the $\mathbb{C}^*$–algebra which is generated by all algebras $\mathcal{K}(S), S \subset X$. One can show that $\mathcal{L}$ is a bimodule for $\mathcal{R}(X, \sigma)$ and lies in the kernel of the one–dimensional representation which annihilates all resolvents, hence $\mathcal{L}$ is a proper ideal. It carries the regular representations of $\mathcal{R}(X, \sigma)$ in the sense that these are precisely the unique extensions of the representations of $\mathcal{L}$ whose restrictions to all subalgebras $\mathcal{K}(S) \subset \mathcal{L}$, $S \subset X$ are nondegenerate.

The properties of the ideal $\mathcal{L}$ and of its subalgebras are also a key to the construction of interesting automorphism groups (dynamics) of the resolvent algebra [4]. We hope to return to this physically important issue in a future publication.

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