Slow-roll k-essence

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We derive slow-roll conditions for thawing k-essence with a separable Lagrangian \( p(X, \phi) = F(X)V(\phi) \). We examine the evolution of the equation of state parameter, \( w \), as a function of the scale factor \( a \), for the case where \( w \) is close to \(-1\). We find two distinct cases, corresponding to \( X \approx 0 \) and \( F_X \approx 0 \), respectively. For the case where \( X \approx 0 \) the evolution of \( \phi \) and hence \( w \) is described by only two parameters, and \( w(a) \) is model-independent and coincides with similar behavior seen in thawing quintessence models. This result also extends to non-separable Lagrangians where \( X \approx 0 \). For the case \( F_X \approx 0 \), an expression is derived for \( w(a) \), but this expression depends on the potential \( V(\phi) \), so there is no model-independent limiting behavior. For the \( X \approx 0 \) case, we derive observational constraints on the two parameters of the model, \( w_0 \) (the present-day value of \( w \)), and the \( \mathcal{K} \), which parametrizes the curvature of the potential. We find that the observations sharply constrain \( w_0 \) to be close to \(-1\), but provide very poor constraints on \( \mathcal{K} \).

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I. INTRODUCTION

Cosmological data from a wide range of sources including type Ia supernovae [1, 2, 3], the cosmic microwave background [4], baryon acoustic oscillations [5, 6], cluster gas fractions [7, 8] and gamma ray bursts [9, 10] seem to indicate that at least 70% of the energy density in the universe is in the form of an exotic, negative-pressure component, called dark energy. (See Ref. [11] for a recent review.)

The dark energy component is usefully parameterized by its equation of state (EoS) parameter \( w \), defined as the ratio of its pressure to its density. Observations constrain \( w \) to be very close to \(-1\). For example, if \( w \) is assumed to be constant, then \(-1.1 \lesssim w \lesssim -0.9 \) [12, 13].

On the other hand, a variety of models have been proposed in which \( w \) is time varying. A common approach is to use a scalar field as the dark energy component. The class of models in which the scalar field is canonical is dubbed quintessence [14, 15, 16, 17] and has been extensively studied. In most of these models the field rolls slowly on a flat potential. It is also possible to have models in which the field is trapped in a false vacuum (see [18] and references therein for examples of such models).

A related, yet somewhat different approach is phantom dark energy, i.e., a component for which \( w < -1 \), as first proposed by Caldwell [19]. Such models have well-known problems [20, 21, 22, 23] (however see [24] for recent attempts to construct a stable model), but nevertheless have been widely studied as potential dark energy candidates.

In both the above approaches, the potential energy of the scalar field is responsible for bringing about the late-time acceleration of the Universe. A third type of model utilizes the kinetic energy of the field through the use of a non-canonical kinetic term in the Lagrangian. Such fields were first used in the context of inflation, in a scenario that is called k-inflation [25]. They have since been studied as dark energy candidates [26, 27, 28] and these models are called “k-essence”.

Given the considerable freedom that exists in choosing the potential function of the scalar field \( V(\phi) \), as well as the kinetic function \( F(X) \) for k-essence, it would be interesting to find any model-independent evolution for \( w(a) \). Some recent work has uncovered such model-independent evolution for quintessence and phantom models when the field evolves in the vicinity of extrema of the potential, and \( w \sim -1 \). Ref. [29] considered the evolution of a scalar field, initially at rest, in a potential satisfying the “slow-roll” conditions:

\[
\frac{1}{V} \frac{dV}{d\phi} \ll 1, \tag{1}
\]

\[
\frac{1}{V} \frac{d^2V}{d\phi^2} \ll 1. \tag{2}
\]

The first condition ensures that \( w \) is close to \(-1\), while the two conditions taken together indicate that \((1/V) (dV/d\phi)\) is nearly constant. In the terminology of Ref. [30], these are “thawing” models.

For all potentials satisfying these conditions, it was shown in [29] that the behavior of \( w \) can be accurately described by a unique expression depending only on the...
present-day values of $\Omega_\phi$ and the initial value of $w$. In this result was extended to phantom models satisfying Eqs. [12], and the $w$ dependence of these phantom models was shown to be described by the same expression as in the quintessence case.

The slow roll conditions, Eqs. [12], while sufficient to ensure $w \simeq -1$ today, are not necessary. In [32], a second possibility was considered, in which equation [2] holds, but equation [1] is relaxed. This corresponds to a quintessence field rolling near a local minimum of its potential. As in the case of slow-roll quintessence, this case can be solved analytically. In this case, there is an extra degree of freedom, the value of $(1/V)(\partial^2 V/\partial \phi^2)$, so that instead of a single solution for the evolution of $w$, one obtains a family of solutions that depend on the present-day values of $\Omega_\phi$ and $w$ and the value of $(1/V)(\partial^2 V/\partial \phi^2)$ at the maximum of the potential. This family of solutions includes the slow-roll solution as a special case in the limit where $(1/V)(\partial^2 V/\partial \phi^2) \to 0$. The corresponding result for phantom fields, where the field rolls near the minimum of its potential was derived in [33] - and an identical expression for $w(a)$ was obtained. In both the above cases, it was shown that the expression for $w(a)$ reduces to the corresponding ones in [29] and [31] as the potential gets flat, i.e., $(1/V)(\partial^2 V/\partial \phi^2) \to 0$.

Ref. [34] examined the opposite case of dark energy models in which a quintessence or a phantom field rolls near the vicinity of a local minimum or maximum, respectively, of its potential. It was shown that as long as Eq. [11] is satisfied, (although Eq. [2] need not be), the evolution of $w$ is described by an expression identical to the one in [32] [33]. In these cases the evolution of $w$ was found to encompass a richer set of behaviors, including oscillating solutions.

In [35], one of us (TC) showed that the expression derived in [32], [33], [34] has a wider applicability than the cases of fields rolling close to extrema in their potentials. The following more general slow-roll conditions on the potential are derived there:

$$\epsilon \equiv \frac{V''}{6H^2V}; \quad \epsilon \ll 1,$$

$$\eta \equiv \frac{V''}{3H^2}; \quad |\eta| \ll 1,$$

and while retaining the assumption that $w \approx -1$, dropped the assumption that the field is close to a local extremum in the potential. Interestingly, once again the expression for the evolution of $w$ under these more general conditions was found to coincide exactly with the one in refs. [32], [33], [34].

In this paper we investigate thawing models in k-essence. We derive slow-roll conditions for thawing k-essence analogous to equations [30] - [41], and show that, when $w \sim -1$, in some cases one does obtain the same model independent evolution of $w$ seen in the above references.

## II. SLOW-ROLL THAWING K-ESSENCE

The Lagrangian density of k-essence $p(\phi, X)$, where $X = -\nabla^\mu \phi \nabla_\mu \phi/2$. The pressure $p_\phi$ of the scalar field $\phi$ is given by $p(\phi, X)$ and the energy density $\rho_\phi$ is given by $\rho_\phi = 2X(\partial p/\partial X) - p$ [21], [20], so that the equation of state parameter, $w$, is

$$w = \frac{p}{2X(\partial p/\partial X) - p}. \quad (5)$$

Working in units of $8\pi G = 1$, the basic equations in a flat universe are

$$\frac{\dot{\phi}}{X} \left( \frac{\partial p}{\partial X} + \frac{2}{3} \frac{\partial^2 p}{\partial X^2} \right) + 3H \frac{\partial p}{\partial X} + \frac{\partial^2 p}{\partial \phi \partial X} - \frac{\partial p}{\partial \phi} = 0, \quad (6)$$

$$H^2 = \left( \frac{\dot{\phi}}{a} \right)^2 = \frac{1}{3} (\rho_B + \rho_\phi), \quad (7)$$

$$\frac{\dot{\phi}}{a} = -\frac{1}{6} (\rho_B + 3p_B + \rho_\phi + 3p_\phi) = -\frac{1}{6} ((1 + 3w_B)\rho_B + (1 + 3w)\rho_\phi), \quad (8)$$

where $\rho_B$ and $p_B$ are the energy density and the pressure of the background matter and/or radiation, respectively.

We now proceed to derive slow-roll conditions for k-essence with the following factorized form of $p(\phi, X)$:

$$p(\phi, X) = V(\phi)F(X). \quad (9)$$

The equation of motion of the scalar field is then written as

$$\ddot{\phi} (FX + 2XF_{XX}) + 3HF_{XX} \dot{\phi} + (2XF_F - F) \frac{V'}{V} = 0, \quad (10)$$

where $FX = dF/dX$ and $V' = dV/d\phi$. We also introduce the sound speed of k-essence, which is the relevant quantity for the growth of density perturbations,

$$c_s^2 = \frac{\partial p/\partial X}{\partial \rho/\partial X} = \frac{FX}{2XF_{XX} + FX}. \quad (11)$$

Using $c_s^2$, The equation of motion Eq. [10] is rewritten as

$$\ddot{\phi} + 3c_s^2H\dot{\phi} + c_s^2 \frac{2XF_F - FV'}{FX} = 0. \quad (12)$$

### A. Slow-Roll Conditions for K-Essence

By slow-roll k-essence, we mean a model of k-essence whose equation of state $w$ is close to $-1$ so that

$$|XF_F| \ll |F|. \quad (13)$$
Thawing models correspond to the equation of state \( w = p_\phi/\rho_\phi \) very close to \(-1\), so that the Hubble friction is not effective and hence \( \dot{\phi} \) is not necessarily small compared with \( 3H\dot{\phi} \) in Eq. (12).

We derive the slow-roll conditions for thawing k-essence during the matter/radiation dominated epoch. Generalizing the corresponding expression for quintessence [32–34, 37], we first introduce the following function:

\[
\beta = \frac{\dot{\phi}}{3c_s^2 H \phi}.
\]  

(14)

As stated above, for thawing models, \( \beta \) is a quantity of \( O(1) \). We assume \( \beta \) is approximately constant in the sense that \( |\beta| \ll H|\beta| \), and the consistency of this assumption will be checked later. In terms of \( \beta \), from Eq. (12) using Eq. (13), \( \dot{\phi} \) is written as

\[
\dot{\phi} = \frac{F V'}{3(1 + \beta) H F_X V},
\]  

(15)

and the slow-roll condition Eq. (13) becomes

\[
\epsilon = \frac{|F| V'^2}{6 H^2 F_X |V|^2} \ll 1,
\]  

(16)

where we have omitted \( 1 + \beta \) since it is an \( O(1) \) quantity, and we have introduced the factor of 1/6 so that \( \epsilon \) coincides with the slow-roll parameter for thawing quintessence [32, 33], \( \epsilon = \frac{1}{6}(V'^2/H^2 V) \). 1 Eq. (10) is a k-essence counterpart of the quintessence slow-roll condition \( V'^2/H^2 V \ll 1 \).

Similar to the case of inflation, the consistency of Eq. (13) and Eq. (12) should give the second slow-roll condition. In fact, from the time derivative of Eq. (15) we obtain

\[
\beta = \left( \frac{V''}{V} - \frac{V'^2}{V^2} \right) \frac{F}{9(1 + \beta)^2 H^2 F_X} + \frac{(1 + w_B)}{2},
\]  

(17)

where we have used \( \dot{H}/H^2 \simeq -3(1 + w_B)/2 \) from Eqs. [7–8] and have assumed \( \dot{\beta} \ll H \beta \). While the left-hand-side of Eq. (17) is an almost time-independent quantity by assumption, the first term in the right-hand-side is a time-dependent quantity in general. Therefore the equality holds if the first term is negligible, which requires in addition to Eq. (10)

\[
\eta = -\frac{F V''}{3H^2 F_X V}, \quad |\eta| \ll 1,
\]  

(18)

so that \( \beta \) becomes

\[
\beta = \frac{1 + w_B}{2},
\]  

(19)

or \( \eta \) itself becomes a constant so that

\[
\eta = -3(1 + \beta) \left( \beta - \frac{1}{2}(1 + w_B) \right).
\]  

(20)

The former condition would correspond to the slow-roll models with \( X \simeq 0 \), while the latter corresponds to the slow-roll models with \( F_X \simeq 0 \). The expression for \( \beta \) given by Eq. (19) is approximately constant, which is consistent with our assumption. Here the factor \(-1/3\) is introduced in Eq. (18) so that \( \eta \) coincides with the slow-roll parameter for thawing quintessence [33, 34], \( \eta = \frac{1}{3}(V''/H^2) \). 2 Eq. (13) is a k-essence counterpart of the quintessence slow-roll condition \( |V''/H^2| \ll 1 \).

Eq. (10) and Eq. (18) constitute the slow-roll conditions for thawing k-essence during the matter/radiation epoch. Note that these expressions have assumed a negligible contribution to the expansion rate from the k-essence itself, and so \( \beta \) is no longer a constant and Eq. (10) or Eq. (20) becomes progressively less accurate as the k-essence begins to dominate at late times. In what follows, we do not make the assumption of matter/radiation domination, so that our results will be accurate up to the present.

### B. Parametrizing the Equation of State

Next we derive general solutions for \( \dot{\phi} \) in the limit where \( |1 + w| \ll 1 \), and we derive \( w \) as a function of \( a \). We note that \( 1 + w = 0 \) implies (a) \( X = 0 \) or (b) \( F_X = 0 \). In the following we consider each case.

**Case (a):** First we consider the case where \( X \simeq 0 \). In this case, \( \epsilon \simeq 1 \), and Eq. (12) simplifies to

\[
\dot{\phi} + 3H\dot{\phi} - \frac{F(0)V'}{F_X(0)V} = 0.
\]  

(21)

The Hubble friction term in Eq. (12) can be eliminated by the following change of variable [32]

\[
u = (\phi - \phi_i)a^{3/2},
\]  

(22)

where \( \phi_i \) is an arbitrary constant, which is introduced for later use, and then Eq. (12) becomes

\[
\ddot{u} + \frac{3}{4}(p_B + p_\phi)u - a^{3/2} \frac{F(0)V'}{F_X(0)V} = 0.
\]  

(23)

We assume a universe consisting of matter and k-essence with \( w \simeq -1 \). Then the pressure is well approximated by a constant: \( p_B + p_\phi \simeq p_\phi \simeq -\rho_\phi \), where \( \rho_\phi \) is the

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1 A canonically normalized scalar field \( \varphi \) corresponds to \( d\varphi = V^{1/2} d\phi \) for \( F = X = 1 \).

2 In terms of a canonically normalized scalar field \( \varphi \), \( 3\eta H^2 = V_{,\varphi\varphi} + \frac{3}{2}V_{,\varphi}^2/V \), and when combined with \( \epsilon \ll 1 \), \( \eta \simeq \frac{1}{3}(V_{,\varphi\varphi}/H^2) \).
nearly constant density contributed by the k-essence in the limit \( w \approx -1 \). Eq. (23) then becomes

\[
\ddot{\phi} - 3 \frac{\rho_{\phi 0}}{a} u + a^{3/2} \frac{F(0)V'}{F_X(0)V} = 0.
\] (24)

Since we consider a slow-roll scalar field \( (X \approx 0) \), the potential may be generally expanded around some value \( \phi_i \), which we identify with the initial value, in the form (up to quadratic order)

\[
V(\phi) = V(\phi_i) + V'(\phi_i)(\phi - \phi_i) + \frac{1}{2} V''(\phi_i)(\phi - \phi_i)^2.
\] (25)

Substituting the expansion given by Eq. (26) into Eq. (24) and taking \( \rho_{\phi 0} = -F(0)\dot{V}(\phi_i) \) gives

\[
\ddot{\phi} + \left( -\frac{F(0)V''(\phi_i)}{F_X(0)V(\phi_i)} + \frac{3}{4} F(0)V(\phi_i) \right) \dot{\phi} = \frac{F(0)V(\phi_i)}{F_X(0)V(\phi_i)} a^{3/2}.
\] (26)

This equation is identical to Eq. (19) in Ref. 35 by the substitution: \( V \rightarrow -V, V'' \rightarrow -V''/F_X, \dot{V} \rightarrow -V'/F_XV \). Therefore, the evolution of \( \phi \) is the same (in functional form) and the equation of state is again given by the same functional form derived in 32, 35:

\[
1 + w(a) = (1 + w_0) a^{3(K-1)} \left( \frac{(K - F(a))(F(a) + 1)^K + (K + F(a))(F(a) - 1)^K}{(K - \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2} + 1)^K + (K + \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2} - 1)^K} \right)^2.
\] (27)

where \( K \) and \( F(a) \) [not to be confused with \( F(X) \)] are defined by

\[
K = \sqrt{1 - \frac{4}{3} \frac{V''(\phi_i)}{F_X(0)V(\phi_i)^2}},
\] (28)

\[
F(a) = \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}}.
\] (29)

In this case, assuming \( X \approx X_m \), Eq. (12) simplifies to

\[
\dot{X} + 3H(X - X_m) - \frac{F(X_m)}{2X_m F_X(X_m)} \frac{d}{dt}(\ln V) = 0.
\] (31)

Therefore, the evolution of \( X \) depends on the shape of \( V(\phi) \), which is no longer Taylor expanded because \( \phi \) can evolve significantly in this case.

However, one can still derive an analytic expression for \( w(X) \). Starting from Eq. (25), one can write \( w(X) \) as

\[
w(X) = \frac{F(X)}{2F_X(X) - F(X)}.
\] (32)

For \( F_X(X) \rightarrow 0 \), this expression can be expanded about the point \( X = X_m \). In this case terms up to second order must be retained, because while \( (X - X_m)^2 \) is small, the second Taylor coefficient can be large leading to a large contribution. The resulting expansion gives:

\[
w(X) = -1 - \left[ \frac{2X_m F_X(X_m)}{F(X_m)} \right] (X - X_m) - \left[ \frac{2F_X(X_m)}{F(X_m)} + \frac{4X_m^2F_X^2(X_m)}{F^2(X_m)} \right] (X - X_m)^2.
\] (33)
Our numerical results indicate that equation (27) is an excellent approximation for the evolution of $w$ for the case where $F_X \to 0$ (see Fig. 4 below). However, it is of limited usefulness, since $w(a)$ in this case depends on $X(a)$, and $X(a)$, in turn, depends on the functional form of $V(\phi)$.

III. COMPARISON TO NUMERICAL RESULTS

We now turn to numerically solving the equations of motion in order to compare Eq. (27) against the exact evolution. We consider a Universe consisting of perfect fluid dark matter and k-essence dynamical dark energy.

We work with models which satisfy the slow roll conditions derived above, i.e., Eq. (10) and Eq. (15). For the cases where $X \approx 0$, the three specific models considered are listed below.

1. Case 1:
   \[ F(X) = \sqrt{1 - mX}, \quad V(\phi) = Ae^{-\phi^2/\sigma^2}, \]
   see Fig. (1). This is the rolling tachyon Lagrangian suggested by the boundary string field theory.

2. Case 2:
   \[ F(X) = \sqrt{1 - mX}, \quad V(\phi) = A\phi^{-\alpha}, \]
   see Fig. (2). This is the model studied in Ref. [40].

3. Case 3:
   \[ p(\phi, X) = mT(\phi) - mT(\phi)(1 - 2X/X(\phi))^{1/2} - V(\phi), \]
   \[ T(\phi) = \phi^4, \quad V(\phi) = \mu^2\phi^2 \]
   see Fig. (3). This is the Dirac-Born-Infeld (DBI) model discussed in Refs. [41, 42]

where $m = \pm 1$ for all three cases. In the first two cases, we choose $\phi(t = t_i) = 1$ and $\phi(t = t_i) = 0$. The constants in the potentials ($4, \sigma$ and $\alpha$) are then adjusted to give $\rho_0 = \rho_\Lambda$ and $w = -1$ at $t = t_i$ and $w = -0.9$ or $w = -1.1$ at $t = t_0$. For the third case, we choose $\phi(t = t_i) = 0$ and adjust $\phi_i$ and $\mu$ to get the above initial and final conditions.

In all these cases we find excellent agreement between the numerical and analytic results, i.e., $\delta w/w \leq 0.01$. The success of our approximation for the DBI case indicates that, as noted earlier, it is not just restricted to separable Lagrangians. Note that for the second case it is found in Ref. [10] that models have a unique equation of state, which corresponds to $K = 1$ in Eq. (27).

As noted earlier, for the set of models where $F_X \approx 0$, we do not expect the evolution of $w(a)$ to converge to a common potential-independent behavior. This is evident from Fig. (4) where we plot the $w(a)$ behavior for two different potentials, $V(\phi) = Ae^{-\phi^2/\sigma^2}$ and $V(\phi) = A\phi^{-\alpha}$, and in both cases we choose $F(X) = X_m + (X - X_m)^2$.

We take initial conditions $\phi_i = 1$ and $X = X_m$ and adjust the constants appropriately. The $w(a)$ behaviors turn out to be very different, indicating that there is no common model-independent behavior for this class of models. On the other hand, our analytic approximation (equation 27) does give excellent agreement, although of course the approximation is itself a function of $V(\phi)$ in this case.

IV. OBSERVATIONAL CONSTRAINTS IN THE ($w_0, K$) PLANE

The results of this paper, combined with previous studies, indicate that Eq. (27) applies both to quintessence models and to a subset of k-essence models with $w \approx -1$. Hence Eq. (27) is a useful and physically well-motivated parametrization for $w(a)$ that can be compared with the observations. So, in this section, we present the obser-
vational constraints on the equation of state parameters \( w_0 \) and on \( K \).

First, we note that the cosmological constant corresponds to a line in the \((w_0, K)\) plane: \( w_0 = -1 \) irrespective of \( K \). This can be understood for a canonical scalar field by noting that \( w_0 = -1 \) corresponds to the case where the scalar field sits at the minimum \((K < 1)\) or the maximum \((K > 1)\) of the potential.

As observational data we consider the recent compilation of 397 Type Ia supernovae (SNIa), called the Constitution set with the light curve fitter SALT, by Hicken et al.\(^2\) and the measurements of baryon acoustic oscillations (BAO) from the SDSS data.\(^3\) Uncertainties in the distance modulus of a supernova include uncertainties in light curve fitting parameters (the maximum magnitude, stretch parameter, color correction parameter) and due to the peculiar velocity (400 kms\(^{-1}\)) as given in Fig. 5.

BAO measurements from the SDSS data provide a constraint on the distance parameter \( A \) defined by

\[
A(z) = (\Omega_m H_0^2)^{1/2} \left( \frac{1}{H(z) z^2} \int_0^z \frac{dz'}{H(z')} \right)^{2/3}
\]

(34)

to be \( A(z = 0.35) = 0.469 \pm 0.017.\(^4\)

The joint constraints from SNIa and BAO are shown in Fig. 6. We marginalize over \( \Omega_m \) to derive the constraint. The allowed range of \( w_0 \) is narrow: \(-1.04 \lesssim w_0 \lesssim -0.86(1\sigma).\(^5\) We find that the cosmological constant \( w_0 = -1 \) is fully consistent with the current data. Note that \( K \), which parametrizes the curvature of \( V(\phi) \), is not well-constrained by current SNIa and BAO data.

V. CONCLUSIONS

Our results indicate that k-essence models with \( w \) near -1 can be divided into two broad categories: models with

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\(^3\) We do not consider BAO distance measurements by Percival et al.\(^4\) because some points of tension were noted between the data sets.

\(^4\) Note that \( A \) changes only slightly in varying the spectral index of the matter power spectrum, \( n \). This value is for \( n = 0.98 \), which changes to \( A = 0.472 \) if \( n = 0.96 \).

\(^5\) We note that SNIa data alone do not constrain \( w_0 \) much: \(-1.2 \lesssim w_0 \lesssim -0.7 \).
\[ X \approx 0 \text{ and those with } F_X \approx 0. \] In the former case, we find a generic evolution for \( w(a) \) which is identical to the previously-derived evolution for quintessence. This strengthens the case that \( \Phi \) is a useful and physically well-motivated parametrization for \( w(a) \) that can be compared with the observations, since it applies both to quintessence models and to a subset of k-essence models with \( w \approx -1 \). Applying this parametrization to SNIa data and BAO, we find that the present-day value of \( w \) is constrained to lie near \(-1\), while the curvature parameter \( K \) is poorly constrained by the observations. Further, we see that the cosmological constant limit of models with \( K \) can be compared with the observations, since it applies physically well-motivated parametrization for \( w(a) \) that is identical to the previously-derived evolution for quintessence. This strengthens the case that equation (27) is a useful and physically well-motivated parametrization for \( w(a) \) that can be compared with the observations, since it applies both to quintessence models and to a subset of k-essence models with \( w \approx -1 \). Applying this parametrization to SNIa data and BAO, we find that the present-day value of \( w \) is constrained to lie near \(-1\), while the curvature parameter \( K \) is poorly constrained by the observations. Further, we see that the cosmological constant limit of these models is consistent with the current data.

On the other hand, k-essence models with \( F_X \approx 0 \) can demonstrate quite different behavior. In this case, the evolution of \( w(a) \) is strongly dependent on the particular potential, and there is no “generic” behavior.

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