Estimates for Schur Multipliers and Double Operator Integrals—A Wavelet Approach

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Abstract. We discuss the work of Birman and Solomyak on the singular numbers of integral operators from the point of view of modern approximation theory, in particular, with the use of wavelet techniques. We are able to provide a simple proof of norm estimates for integral operators with kernel in $B_{p,p}^{1/p-1/2}$, which recovers, extends, and sheds new light on a theorem of Birman and Solomyak. We also use these techniques to provide a simple proof of Schur multiplier bounds for double operator integrals with bounded symbol in $B_{2p/(2-2p),p}^{1/(2-2p)}$, which extends Birman and Solomyak’s result to symbols without compact domain.

Key words: double operator integrals, wavelets, Besov spaces.

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1. Introduction

Through their study of differentiability of functions of Hermitian operators, Daleckii and Krein ([9], [10]) were led to initiate the theory of double operator integrals, a powerful tool built from spectral theory, which has now found itself deeply embedded in pseudodifferential operator theory, harmonic analysis, and mathematical physics. Although not the first works studying these methods, the series of papers [2], [3], and [4] by Birman and Solomyak undoubtedly form foundations for the further study of double operator integrals (from here on referred to as DOIs). These papers were revolutionary at the time and are still fundamental, but they suffer from technically challenging proofs, which are difficult to further build upon. We wish to rectify this, combining advances in nonlinear approximation and wavelet analysis in order to simplify proofs and strengthen the beautiful results which lie at the center of Birman and Solomyak’s theory.

Here we consider a question which forms the basis for the further study of DOIs: What norm bounds does a DOI admit for a given symbol? Following the footsteps of Birman and Solomyak, we are able to provide a simple proof technique, building upon the machinery afforded to us through wavelet analysis and nonlinear approximation theory, as developed in the 1980s and 1990s.

Unfortunately, so far as the authors are aware, the use of these techniques to study Schur multipliers has been very limited. However, there are some clear examples, such as the early work of Peng [22], who used wavelet bases to find Schatten–von Neumann class norm estimates for certain integral operators. Only recently, these techniques were adapted in order to prove new Lipschitz estimates in Schatten–von Neumann ideals [19].

We mark three contributions to the theory of DOIs. The first contribution is extending Schatten–von Neumann class and Schur multiplier estimates to DOIs having symbol with noncompact support. Secondly, we loosen the smoothness restrictions on the symbol, and finally, we provide clear and succinct proofs of these results, building a new methodology.

It is only through powerful wavelet analysis and approximation techniques that we are able to both simplify and extend the results of Birman and Solomyak, and we hope that in turn, this new approach will allow working mathematicians and physicists, not already familiar with the technicalities of the theory, to develop new applications for DOIs.
1.1. Overview of the results. Recall that if $T \in K(H)$ is a compact linear operator on a Hilbert space $H$, then the singular value sequence $\mu(T)$ is defined by

$$\mu(T) = (\mu(n,T))_{n=0}^\infty, \quad \mu(n,T) = \inf\{\|T - R\|_\infty : \text{rank}(R) \leq n\},$$

where $\|\cdot\|_\infty$ is the operator norm. For $0 < p < \infty$, an operator $T$ is said to belong to the Schatten–von Neumann $\mathcal{L}_p(H)$-class if $\|T\|_p = (\sum_{n=0}^\infty \mu(n,T)^p)^{1/p} < \infty$. For $1 \leq p < \infty$, $\mathcal{L}_p$ is a Banach ideal of the algebra of all bounded linear endomorphisms of $H$ and is a quasi-Banach ideal for $0 < p < 1$. Similarly, $\mathcal{L}_{p,q}(H)$ is defined as the space of operators $T$ with singular value sequence belonging to the Lorentz sequence space $\ell_{p,q}$ (see, for example, [18] for the definition).

For a square-integrable function $k \in L_2(\mathbb{R}^2)$, we denote by $\text{Op}(k)$ the corresponding integral operator on $L_2(\mathbb{R})$. That is,

$$\text{Op}(k)(f)(y) = \int_\mathbb{R} k(x,y) f(x) \, dx, \quad f \in L_2(\mathbb{R}), \ y \in \mathbb{R}.$$ 

Here, and throughout, we understand a function $k$ in some vector-valued Besov class $B^s_{p,q}(\mathbb{R}; L_\tau(\mathbb{R}))$ as being a function over $\mathbb{R}^2$. Consider a function $k: \mathbb{R}^2 \to \mathbb{R}$. Clearly, for any $x \in \mathbb{R}$, $k(x, \cdot)$ is a function over $\mathbb{R}$, and in this sense we may identify any function over $\mathbb{R}^2$ with a function-valued function. Conversely, let $k \in B^s_{p,q}(\mathbb{R}; L_\tau(\mathbb{R}))$. Then, for each $s \in \mathbb{R}$, there exists a function $k_s: \mathbb{R} \to \mathbb{R} \in L^\tau(\mathbb{R})$ such that $k(s) = g_s$. Thus, for each $t \in \mathbb{R}$, we may treat $k$ as a function over $\mathbb{R}^2$ by identifying $k$ with $k(s, t) = k_s(t)$. We will make use of this identification throughout.

One of the primary goals of Birman and Solomyak’s 1977 survey [5] is to give sufficient conditions on $k$ under which $\text{Op}(k) \in \mathcal{L}_p(L_2(\mathbb{R}))$. This is achieved by highly technical results of piecewise polynomial approximation. The essential idea is to find conditions on $k$ under which there exists a sequence of functions of the form $(t,s) \mapsto \sum_{j=1}^n \xi_j(t)\eta_j(s)$ which approximate $k$ sufficiently quickly. Numerous results of this nature have been obtained. Our aim here is to shed new light on these ideas through modern techniques of nonlinear approximation theory and wavelet analysis. Notation and definitions concerning Besov spaces will be given in the next section.

The following is a special case of [5; Proposition 2.1].

**Theorem 1.1.** Let $I$ be an open bounded interval in $\mathbb{R}$, and let $0 < p \leq 2$. Let $k \in L_2(\mathbb{R}^2)$ belong to the vector-valued Besov class $B^2_{q,q}(I, L_2(I))$, where

$$1/p = \alpha + 1/2, \quad q \geq \max\{2, 1/\alpha\}.$$ 

Then $\text{Op}(k) \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}))$.

**Remark 1.2.** The original statement of Birman and Solomyak applied to integral operators on $L_2(I^d)$ for $d \geq 1$. Here we restrict ourselves to the one-dimensional case for simplicity. Moreover, in [5] the space $B^\alpha_{p,p}(I, L_2(I))$ was defined as a Sobolev–Slobodetskii space, which is smaller than the Besov space when $\alpha$ is an integer.

Throughout the paper, let us write that $a \lesssim b$ if there exists a constant $C$ such that $a \leq Cb$, and $a \approx b$ if there exists a constant $C$ such that $C^{-1}b \leq a \leqCb$. This constant may change from line to line. If, for some given variables $p$ and $q$, the constant is dependent upon those variables, we may write $\lesssim_{p,q}$ or $\approx_{p,q}$.

In this paper we give a new perspective on and strengthen Theorem 1.1 by proving that if

$$k \in L_2(\mathbb{R}^2) \cap B^{1/p-1/2}_{p,p}(\mathbb{R}, L_2(\mathbb{R})), \quad$$

then $\text{Op}(k)$ belongs to $\mathcal{L}_p$. The following is our first key result; it is proved in Section 4.
Theorem 1.3. Let $0 < p \leq 2$, and let $k \in L^2_2(\mathbb{R}^2)$ belong to the vector-valued homogeneous Besov class $B_{p,p}^{1/p'-1/2}(\mathbb{R}, L_2(\mathbb{R}))$. Then $\text{Op}(k)$ belongs to $\mathcal{L}_p$, and

$$
\| \text{Op}(k) \|_p \lesssim \| k \|_2 + \| k \|_{B_{p,p}^{1/p'-1/2}(\mathbb{R}, L_2(\mathbb{R}))}.
$$

Another important contribution of Birman and Solomyak concerns estimates for Schur multipliers. For $0 < p \leq 2$, the $\mathcal{M}_p$-norm of a bounded measurable function $k$ on $I^2$, where $I \subseteq \mathbb{R}$, is given by

$$
\| k \|_{\mathcal{M}_p(I^2)} = \sup_{\| \phi \|_{L_p(I^{2\cdot 1}))} \| \text{Op}(\phi k) \|_{L_p(I^2)}.
$$

We review material concerning Schur multipliers in Section 5 below.

For $0 < p < 2$, we denote $p' = 2p/(2 - p)$.

Theorem 1.4 ([5; Theorem 9.2]; see also [2; Theorems 3,9]). For any index $0 < p \leq 1$, let $\alpha > 1/p'$. Over any bounded open interval $I$, for any function $k$ on $I \times I$,

$$
\| k \|_{\mathcal{M}_p(I^2)} \lesssim I \| k \|_{B_{p,p}^{\alpha}(I,L_\infty(I))}.
$$

Remark 1.5. (i) The dependence of the constant on the choice of the interval $I$ prevents the proof technique from being adapted to functions without compact support, as limiting arguments cannot be applied. Our proof technique thus provides a substantial improvement by removing this limitation, so that we can consider functions without compact support.

(ii) As before, we state here only the 1-dimensional variant of Theorem 9.2 in [5]. Note that in Birman and Solomyak’s paper $B_{2,\infty}^\alpha(I)$ is called a Nikolskii–Besov space and denoted by $H_2^\alpha(I)$; it should not be confused with the Bessel potential space $B_{2,2}^\alpha(I)$, which is typically also denoted by $H_2^\alpha(I)$.

(iii) Our approach can be further extended to study Besov spaces over $\mathbb{R}^d$, for any dimension $d$; however, here we only cover the 1-dimensional variant for simplicity of exposition and to avoid unnecessary technicalities.

The following is our second key result; it is proved below in Section 5.

Theorem 1.6. For any index $p \in (0,2)$, if $k \in B_{p,p}^{1/p'}(\mathbb{R}, L_\infty(\mathbb{R})) \cap L_\infty(\mathbb{R}^2)$, then $k$ is an $\mathcal{L}_p$-Schur multiplier with the quasi-norm estimate

$$
\| k \|_{\mathcal{M}_p(\mathbb{R}^2)} \lesssim_p \| k \|_{B_{p,p}^{1/p'}(\mathbb{R}, L_\infty(\mathbb{R}))} + \| k \|_\infty.
$$

2. Wavelets and Vector-Valued Besov Spaces

Here we approach the construction of vector-valued Besov spaces through Meyer’s wavelet characterization. Recall that an orthonormal wavelet is a function $\varphi \in L_2(\mathbb{R})$ such that the family

$$
\{ \varphi_{j,k}(x) = 2^{k/2} \varphi(2^k x - j) : j, k \in \mathbb{Z} \}
$$

of translations and dilations of $\varphi$ forms an orthonormal basis of $L_2(\mathbb{R})$. We refer to such a family of functions as a wavelet system.

A celebrated theorem of Daubechies [11] states that, for every $N \geq 0$, there exists an $N$-times continuously differentiable compactly supported wavelet. If $f$ is a locally integrable function and $\varphi$ is a compactly supported continuous wavelet, then the wavelet coefficient $\langle \psi_{j,k}, f \rangle$ is well defined.

Recall that the homogeneous Besov space $B_{p,q}^{s}(\mathbb{R})$ for $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ can be described as the class of tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$(2^{ns} \| \Delta_n f \|_{L_p(\mathbb{R}))}_{n \in \mathbb{Z}} \in \ell_q(\mathbb{Z}).$$
Here \( \{\Delta_n\}_{n \in \mathbb{Z}} \) is a Littlewood–Paley decomposition of \( \mathbb{R} \); see, e.g., [15; Sec. 2.2.1] or [25; Sec. 2.4] for details. The \( \ell_q \)-quasi-norm of the above sequence is \( |f|_{\dot{B}_{p,q}^s} \), the homogeneous Besov semi-quasi-norm of \( f \). Observe that as polynomial functions have Fourier transform supported at \( \{0\} \), it follows by definition that all polynomials belong to \( \dot{B}_{p,q}^s(\mathbb{R}) \) with vanishing semi-quasi-norm.

Homogeneous Besov spaces admit simple characterizations in terms of wavelet coefficients. The first such characterization is due to Meyer [20], who proved that if \( \psi \) is a Schwartz class wavelet, then a distribution \( f \in \mathcal{S}'(\mathbb{R}) \) belongs to \( \dot{B}_{p,q}^s(\mathbb{R}) \) if and only if

\[
(2^{j(s+1/2-1/p)} \| (f, \psi_{j,k}) \|_{\ell_p(\mathbb{Z})})_{j \in \mathbb{Z}} \in \ell_q(\mathbb{Z});
\]

see also [25; Sec. 2.4]. A Schwartz class wavelet \( \psi \) necessarily has vanishing moments of all orders (that is, \( \int_\mathbb{R} t^l \psi(t) \, dt = 0 \) for all \( l \geq 0 \)) [21; Chap. 3, Sec. 7], and hence this characterization is at least consistent with the fact that all polynomials belong to \( \dot{B}_{p,q}^s(\mathbb{R}) \).

In our case, we will require that all wavelets be compactly supported. A compactly supported wavelet \( \varphi \) can have at most finite degree of regularity, because otherwise all moments of \( \varphi \) would vanish, and this would imply that \( \varphi = 0 \) [16; Theorem 3.8].

The homogeneous Besov seminorm can be estimated in terms of wavelet coefficients with finite degree of smoothness, as in Meyer’s book [21; Chap. 6, Sec. 10]. This issue is somewhat more subtle than in the case of Schwartz class wavelets. The first reason is that if \( \varphi \) is a \( C^N \) wavelet, then the wavelet coefficient \( (f, \varphi_{j,k}) \) is defined only for tempered distributions of sufficiently high regularity, rather than for all of them. The second issue is that wavelets with finite degree of smoothness have only a finite number of vanishing moments.

**Theorem 2.1.** Let \( s \in \mathbb{R} \) and \( 0 < p, q \leq \infty \) satisfy \( s > \max\{1/p - 1, 0\} \). Let \( \varphi \) be a compactly supported \( C^N \) wavelet, where \( N > |s| \).

A distribution \( f \in \mathcal{S}'(\mathbb{R}) \) belongs to the homogeneous Besov space \( \dot{B}_{p,q}^s(\mathbb{R}) \) if and only if there exists a sequence of polynomials \( (P_{j,k})_{j,k \in \mathbb{Z}} \) and constants \( (c_{j,k})_{j,k \in \mathbb{Z}} \) such that

\[
f = \sum_{j,k \in \mathbb{Z}} c_{j,k} \varphi_{j,k} + P_{j,k},
\]

where the series converges with respect to the topology of \( \mathcal{S}'(\mathbb{R}) \) and

\[
(2^n(s+1/2-1/p) \| (c_{j,k})_{k \in \mathbb{Z}} \|_{\ell_p(\mathbb{Z})})_{n \in \mathbb{Z}} \in \ell_q(\mathbb{Z}).
\]

The infimum of the \( \ell_q(\mathbb{Z}) \) quasi-norms of the above sequence over all representations of \( f \) is equivalent to the corresponding Besov semi-quasi-norm of \( f \).

This theorem is stated in essentially the same form as in [21; pp. 201], with modifications for the case \( p < 1 \) as in [8; Theorem 3.7.7]. The case \( s \leq \max\{1/p - 1, 0\} \) is discussed in [8; Remark 3.7.5], but it is not relevant to our present applications.

If \( E \) is a Banach space, then the space \( \mathcal{S}'(\mathbb{R}, E) \) of \( E \)-valued tempered distributions is defined as the space of all continuous linear maps \( T: \mathcal{S}(\mathbb{R}) \rightarrow E \). The space \( \mathcal{S}'(\mathbb{R}, E) \) is equipped with the topology of pointwise norm-convergence; see [17; Definition 2.4.24]. We shall take the wavelet characterization of Besov spaces as motivation for the following definition.

**Definition 2.2.** Let \( E \) be a Banach space, and let \( f \in \mathcal{S}'(\mathbb{R}, E) \). Let \( s \in \mathbb{R} \) and \( 0 < p, q \leq \infty \) satisfy \( s > \max\{1/p - 1, 0\} \). Let \( \varphi \) be a compactly supported \( C^N \)-wavelet, where \( N > |s| \). We say that \( f \) belongs to the homogeneous Besov space \( \dot{B}_{p,q}^s(\mathbb{R}, E) \) if \( f \) can be represented as

\[
f = \sum_{j,k \in \mathbb{Z}} c_{j,k} \varphi_{j,k} + P_{j,k},
\]
where the series converges with respect to the topology of $S'(\mathbb{R}, E)$, the $P_{j,k}$ are polynomials with coefficients in $E$, and the $c_{j,k} \in E$ are such that

$$\left\| \left( 2^{n(s+1/2-1/p)} \right) \left( \| c_{j,k} \|_E \right)_{k \in \mathbb{Z}} \right\| \ell_p(\mathbb{Z}) < \infty.$$ 

The Besov semi-quasi-norm $|f|_{B^s_p(E)}$ of $f$ is defined to be the infimum of the above quantity over all representations of $f$. For our present purposes, we will only need $E = L_2(\mathbb{R})$ and $E = L_\infty(\mathbb{R})$.

### 3. n-Term Approximation of $L_2(\mathbb{R})$-Valued Functions

The ideas at the heart of Birman and Solomyak’s original proofs revolve around developments in nonlinear approximation theory for piecewise polynomials.

Let $(X, \| \cdot \|_X)$ be a quasi-normed linear space. A sequence $(X_n)_{n=0}^\infty$ of subsets of $X$ will be called an approximation scheme for $X$ if

(i) $X_0 = \{ 0 \}$;
(ii) for all $n \geq 0$, $X_n \subseteq X_{n+1}$;
(iii) for all $n \geq 0$ and any $a \in \mathbb{C}$, $aX_n = X_n$;
(iv) there exists an integer $k > 0$ such that, for any $n \in I$, $X_n + X_n \subseteq X_{kn}$;
(v) the set $\bigcup_{n=0}^\infty X_n$ is dense in $X$.

If $X_n + X_n = X_n$ for every $n \geq 0$, we say that $(X_n)_{n=0}^\infty$ is a linear approximation scheme, and otherwise we call it nonlinear.

**Definition 3.1.** Given a quasi-normed space $(X, \| \cdot \|_X)$ and an approximation scheme $(X_n)_{n=0}^\infty$ of (possibly nonlinear) subsets of $X$, the $n$th approximation number for an element $f \in X$ is defined by

$$E_n(f)_X = \inf_{g \in X_n} \| f - g \|_X.$$ 

We denote by $E(f)_X = (E_n(f)_X)_{n=0}^\infty$ the sequence of approximation numbers.

For $\alpha > 0$ and $q \in (0, \infty]$, we define the quasi-norm

$$\| f \|_{A^\alpha_q} = \| E(f)_X \|_{\ell_{\alpha^{-1},q}}.$$ 

The approximation space

$$A^\alpha_q = A^\alpha_q(X, (X_n)_{n \in \mathbb{N}})$$ 

is the set of all $f \in X$ such that $\| f \|_{A^\alpha_q} < \infty$.

Let us now consider a nonlinear approximation scheme $(T_n \otimes E)_{n \geq 0}$ for some Banach space $E$, where, for each $n \in \mathbb{N}$,

$$T_n = \left\{ f = \sum_{l=1}^n c_l \varphi_{j_l,k_l} : c_l \in \mathbb{C}, j_l, k_l \in \mathbb{Z} \right\}$$

(3.1)

is the space of all functions given by a linear combination of any $n$ functions in the wavelet system.

Before we state our approximation theorem, we recall the discrete version of the Hardy inequality.

**Lemma 3.2** (see, for example, [13; Chapter 2, Lemma 3.4]). For any two positive monotonically decreasing sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ of nonnegative real numbers, if there exist constants $\mu, r, C > 0$ such that

$$a_n \leq Cn^{-r} \left( \sum_{k=n}^\infty k^{r\mu-1} b_k^\mu \right)^{1/\mu}$$

for all $n \geq 1$, then it follows for all $0 < q \leq \infty$ and $s > r$ that

$$\| (a_n)_{n \in \mathbb{N}} \|_{\ell_{s-1,q}} \lesssim_{\mu, r, C} \| (b_n)_{n \in \mathbb{N}} \|_{\ell_{s-1,q}}.$$
The following theorem characterizes the approximation spaces for the space $L_2(\mathbb{R}, L_2(\mathbb{R}))$ with respect to the approximation scheme $(T_n \otimes L_2(\mathbb{R}))_{n=0}^\infty$ (see (3.1)). The proof is identical to that in the corresponding scalar-valued case \cite[Sec. 7.6]{12}; we give it here for convenience.

**Theorem 3.3.** For any index $p \in (0,2)$,

$$
\dot{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R})) \cap L_2(\mathbb{R}^2) = \mathcal{A}_p^{1/p-1/2}(L_2(\mathbb{R}, L_2(\mathbb{R})), (T_n \otimes L_2(\mathbb{R}))_{n \in \mathbb{N}}).
$$

In particular, the space $\dot{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R})) \cap L_2(\mathbb{R}^2)$ is the space of all $L_2(\mathbb{R})$-valued functions $f$ such that

$$
\{ \| \langle \varphi_{j,k}, f \rangle \|_2 \}_{j,k \in \mathbb{Z}} \in \ell_p(\mathbb{Z}^2) \quad \text{and} \quad \| f \|_{L_2(\mathbb{R}^2)} < \infty.
$$

**Proof.** It follows from Definition 2.2 that $f \in \dot{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R}))$ if and only if

$$
\sum_{j,k \in \mathbb{Z}} \| \langle \varphi_{j,k}, f \rangle \|_2^2 < \infty.
$$

Let $(b_n)_{n \in \mathbb{N}}$ denote the decreasing rearrangement of the sequence

$$
\{ \| \langle \varphi_{j,k}, f \rangle \|_2 \}_{j,k \in \mathbb{Z}}.
$$

Since the spaces $\varphi_{j,k} \otimes L_2(\mathbb{R})$ are orthogonal in $L_2(\mathbb{R}, L_2(\mathbb{R}))$, it follows by the definition of the approximation numbers that

$$
E_n(f) = \inf \{ \| f - g \|_{L_2(\mathbb{R}, L_2(\mathbb{R}))} : f \in T_n \otimes L_2(\mathbb{R}) \} = \left( \sum_{k=n+1}^\infty b_k^2 \right)^{1/2}.
$$

By the discrete Hardy inequality, it follows for any $r < 2$ and $0 < q \leq \infty$ that

$$
\left\| \left\{ \frac{E_n(f)}{(n+1)^{1/2}} \right\}_{n \in \mathbb{N}} \right\|_{\ell_{r,q}} \lesssim \|(b_n)_{n \in \mathbb{N}}\|_{\ell_{r,q}}.
$$

In particular,

$$
\| f \|_{\mathcal{A}_q^{1/q-1/2}(\mathbb{R}, L_2(\mathbb{R})), (T_n \otimes L_2(\mathbb{R}))_{n \in \mathbb{N}}} \lesssim \|(b_n)_{n \in \mathbb{N}}\|_{\ell_{r,q}}.
$$

In the case when $p = q = r$, it follows from the definition of the sequence $(b_n)_{n \in \mathbb{N}}$ and Definition 2.2 that

$$
\| f \|_{\mathcal{A}_p^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R})), (T_n \otimes L_2(\mathbb{R}))_{n \in \mathbb{N}}} \lesssim |b_0| + \left( \sum_{j,k} \| \langle \varphi_{j,k}, f \rangle \|_2^2 \right)^{1/p}
$$

$$
\approx |b_0| + |f|_{\dot{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R}))}.
$$

Given that the sequence $(b_n)_{n \in \mathbb{N}}$ is monotonically decreasing, we have $b_{2n}^2 \leq b_k^2$ for all $0 \leq k < 2n$. In turn,

$$
nb_{2n}^2 \leq \sum_{k=n}^{2n-1} b_k^2 \leq \sum_{k=n}^{\infty} b_k^2 = E_{n-1}(f)^2,
$$

where the approximation numbers are defined by the wavelet nonlinear approximation scheme. Thus, $n^{1/2}b_{2n} \leq E_{n-1}(f)$, and so $(2n)^{1/2}b_{2n} \lesssim E_{n-1}(f)$.

The reverse inclusion then follows, because $b_0 = \| f \|_{L_2(\mathbb{R}^2)}$ and

$$
|f|_{\dot{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R}))} \approx \|(b_n)_{n \in \mathbb{N}}\|_{\ell_p} \leq \| f \|_{\mathcal{A}_p^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R})), (T_n \otimes L_2(\mathbb{R}))_{n \in \mathbb{N}}}.
$$

This completes the proof. \qed
4. Approximation of Singular Values for Integral Operators

Both Birman and Solomyak’s initial research into DOIs ([2], [3]) and their later investigations into integral operators [5] capitalize upon the insight that approximation theory techniques allow one to effectively find norm estimates for integral operators.

Observe that the nested family
\[ \mathcal{R}_n = \{ T \in \mathcal{K}(\mathcal{H}) : \text{rank}(T) \leq n \}, \quad n \geq 0, \]
of subsets is an approximation scheme for \( \mathcal{K}(\mathcal{H}) \), and the singular value sequence \( \mu(T) \) is precisely the sequence \( E(T)_{\mathcal{K}(\mathcal{H})} \) of approximation numbers. Correspondingly, the Schatten–von Neumann ideals can equivalently be described as approximation spaces:
\[ \mathcal{L}_{p,q}(\mathcal{H}) = \mathcal{A}_q^{1/p}(\mathcal{K}(\mathcal{H}), (\mathcal{R}_n)_{n=0}^{\infty}). \]

The primary goal of Birman and Solomyak’s survey [5] is to estimate the singular values of integral operators \( \text{Op}(k) \) in terms of \( k \). The main difficulty is that there is no sufficiently general estimate for the operator norm of an integral operator in terms of its kernel.

However, it is well known that the Hilbert–Schmidt norm is given by
\[ \| \text{Op}(k) \|_2 = \| k \|_{L_2(\mathbb{R}^2)}. \]
The \( L_2 \)-approximation numbers are then given by
\[ e_n(T) = \inf \{ \| T - R \|_2 : \text{rank}(R) \leq n \} \]
for each \( n \geq 0 \). Of course, these are the approximation numbers for \( \mathcal{L}_2 \) with respect to the approximation scheme \( (\mathcal{R}_n)_{n=0}^{\infty} \). That is, \( e_n(T) = E_n(T)_{\mathcal{L}_2(\mathcal{H})} \).

The following simple lemma is at the heart of Birman and Solomyak’s results and allows us to replace the operator norm with the Hilbert–Schmidt norm.

**Lemma 4.1** ([5; Lemma 1.3]). For any compact operator \( T \) and every \( n \geq 1 \),
\[ \mu(2n, T) \leq n^{-1/2} e(n, T). \]

**Proof.** For any operator \( R \) with \( \text{rank}(R) \leq n \),
\[ \mu(2n, T) \leq \mu(n, T - R) + \mu(n, R) = \mu(n, T - R) \leq n^{-1/2} \| T - R \|_2. \]
The result follows by taking the infimum over all such \( R \). \( \square \)

It is remarkable that this simple lemma is sufficiently sharp for estimates of \( L_{p,q} \)-norms.

**Proposition 4.2.** For any compact operator \( T \) and any indices \( p \) and \( q \) such that \( 0 < p < 2 \) and \( 0 < q \leq \infty \),
\[ \| T \|_{L_{p,q}} \approx \left\| \left( \frac{e(n, T)}{(n + 1)^{1/2}} \right)_{n \in \mathbb{N}} \right\|_{\ell_{p,q}} = \| (e(n, T))_{n \in \mathbb{N}} \|_{\ell_{(1/p-1/2)^{-1},q}}. \]

**Proof.** To start, note that
\[ e(n, T) = \left( \sum_{k=n+1}^{\infty} \mu(k, T)^2 \right)^{1/2} \]
for any \( n \geq 0 \).
It then follows by the application of the discrete Hardy inequality (Lemma 3.2) to \( \mu = 2 \) and \( r = 1/2 \) that
\[
\left\| \left( \frac{e(n, T)}{(n + 1)^{1/2}} \right)_{n \geq 0} \right\|_{\ell_{p,q}} \lesssim \| (\mu(n, T))_{n \geq 0} \|_{\ell_{p,q}}
\]
for all \( p \in (0, 2) \) and all \( 0 < q \leq \infty \).

Applying Lemma 4.1, we obtain the quasi-norm equivalence
\[
\left\| \left( \frac{e(n, T)}{(n + 1)^{1/2}} \right)_{n \geq 0} \right\|_{\ell_{p,q}} \approx \| (\mu(n, T))_{n \geq 0} \|_{\ell_{p,q}}
\]
for all \( p \in (0, 2) \) and all \( 0 < q \leq \infty \), which is equivalent to the stated result.

The following result implies many of the estimates of Birman and Solomyak [5; Theorem 2.4] and Pietsch ([23], [24], [14]).

Let us note how short the proof of the following result is in comparison with the other approaches just mentioned.

**Theorem 4.3.** For any index \( p \in (0, 2) \), the map
\[
\text{Op}: \hat{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R})) \cap L_2(\mathbb{R}^2) \to \mathcal{L}_p(L_2(\mathbb{R}))
\]
is a continuous embedding.

**Proof.** Let \( \Sigma = (\Sigma_n)_{n \in \mathbb{N}} \) be the nested family of subsets of \( L_2(\mathbb{R}^2) \) defined for each \( n \geq 0 \) by
\[
\Sigma_n = \left\{ k(t, s) = \sum_{j=1}^{n} x_j(t)y_j(s) : x_j, y_j \in L_2(\mathbb{R}) \right\}.
\]
That is to say that \( \Sigma_n \) consists of linear combinations of at most \( n \) elementary tensors of functions in \( L_2(\mathbb{R}) \). Under \( \text{Op} \), \( \Sigma_n \) is sent precisely to the space \( \mathcal{R}_n \) of linear operators of rank at most \( n \).

It is then immediate from Proposition 4.2 that, for any fixed index \( 0 < q \leq \infty \), we can restrict the isometry
\[
\text{Op}: L_2(\mathbb{R}^2) \to \mathcal{L}_2(L_2(\mathbb{R}))
\]
to the isomorphism
\[
\text{Op}: \mathcal{A}_q^{1/p-1/2}(L_2(\mathbb{R}^2), \Sigma) \cong \mathcal{L}_{p,q}(L_2(\mathbb{R})).
\]
Observe that, for every \( n \geq 0 \), we have
\[
T_n \otimes L_2(\mathbb{R}) \subset \Sigma_n.
\]
It is therefore immediate that we have a continuous inclusion
\[
\mathcal{A}_q^{1/p-1/2}(L_2(\mathbb{R}^2), (T_n \otimes L_2(\mathbb{R}))_{n \in \mathbb{N}}) \hookrightarrow \mathcal{A}_q^{1/p-1/2}(L_2(\mathbb{R}^2), \Sigma),
\]
and we know from Theorem 3.3 that
\[
\hat{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R})) \cap L_2(\mathbb{R}^2) = \mathcal{A}_q^{1/p-1/2}(L_2(\mathbb{R}^2), (T_n \otimes L_2(\mathbb{R}))_{n=0}^\infty).
\]
Thus,
\[
\text{Op}: \hat{B}_{p,p}^{1/p-1/2}(\mathbb{R}, L_2(\mathbb{R})) \cap L_2(\mathbb{R}^2) \to \mathcal{L}_p(L_2(\mathbb{R})).
\]

This concludes the proof of Theorem 1.3, which is simply a restatement of the above result.

**Remark 4.4.** To conclude this section, let us note that Proposition 4.3 gives a sufficient condition for an integral operator to be trace class. Namely, if \( k \in \hat{B}_{1,1}^{1/2}(\mathbb{R}, L_2(\mathbb{R})) \), then \( \text{Op}(k) \in \mathcal{L}_1 \). Furthermore, the operator trace of \( \text{Op}(k) \) is given by the formula
\[
\text{Tr}(\text{Op}(k)) = \int_{\mathbb{R}} k(x, x) \, dx,
\]
provided that one understands the restriction of \( k \) to the diagonal \( \{(x, x) : x \in \mathbb{R}\} \) in the correct sense. For further details, see [7].
5. Double Operator Integrals and Schur Multipliers

Our final task is to show that if a DOI has symbol in some vector-valued Besov class, then it is an $L_p$-Schur multiplier. This difficult task was accomplished in Birman and Solomyak’s original papers [2]–[5] through a careful argument, which shows, in general terms, that by replacing multiplication by an $L_2$-operator with a weighted measure, norm-estimates for operators with nonsmooth symbol can be replaced by those for some corresponding smooth symbol. Here we are able to avoid all of these measure-theoretic difficulties by applying the wavelet characterization of wavelet vector-valued Besov classes. In this way, not only is the result achieved more easily, but also we remove the need for the symbol to be compactly supported. For background on Schur multipliers, see the survey [1] of Aleksandrov and Peller. Recall that, for $0 < p \leq 2$, a bounded Borel measurable function $k$ on $\mathbb{R}^2$ is called an $L_p$-Schur multiplier, $k \in \mathcal{M}_p$, if there exists a constant $C$ such that

$$\| \text{Op}(k\phi) \|_p \leq C \| \text{Op}(\phi) \|_p$$

for all $\phi \in L_2(\mathbb{R}^2)$ such that $\text{Op}(\phi) \in L_p(L_2(\mathbb{R}))$. The $\mathcal{M}_p$-quasi-norm of $k$ is defined as

$$\| k \|_{\mathcal{M}_p} = \sup_{\| \text{Op}(\phi) \|_p \leq 1} \| \text{Op}(k\phi) \|_p.$$  \hspace{1cm} (5.1)

It follows directly from the properties of $\mathcal{L}_p$ that, for $0 < p < 1$, $\mathcal{M}_p$ is a quasi-Banach space and obeys the $p$-triangle inequality

$$\left\| \sum_{j=0}^{\infty} k_j \right\|_{\mathcal{M}_p}^p \leq \sum_{j=0}^{\infty} \| k_j \|_{\mathcal{M}_p}^p.$$  \hspace{1cm} (5.2)

For $1 \leq p \leq 2$, $\mathcal{M}_p$ is a Banach space, and $\mathcal{M}_2 = L_\infty(\mathbb{R}^2)$.

An important feature of the $\mathcal{M}_p$-quasi-norm when $0 < p < 1$ is that it suffices to compute the supremum in (5.1) over rank-one operators. The following fundamental fact is the relation [5; Theorems 8.1 and 8.2]

$$\| k \|_{\mathcal{M}_p} = \sup_{\xi, \eta} \| \text{Op}(k(\xi \otimes \eta)) \|_{\mathcal{L}_p},$$  \hspace{1cm} (5.2)

where the supremum is taken over all $\xi$ and $\eta$ in the unit ball of $L_2(\mathbb{R})$. This is an immediate consequence of the Schmidt decomposition of a compact operator and the $p$-triangle inequality; see [19; Lemma 2.2.1] for a proof of the corresponding matrix case.

The notion of an $\mathcal{L}_p$-Schur multiplier is related to the idea of a double operator integral in the following sense. Let $M_x$ denote the unbounded self-adjoint operator on $L_2(\mathbb{R})$ of multiplication by the coordinate variable $M_x \xi(t) := t\xi(t)$. Then

$$T^{M_x,M_x}_k(\text{Op}(\phi)) = \text{Op}(k\phi), \quad \phi \in L_2(\mathbb{R}^2).$$

It follows that, for all $0 < p \leq 2$, we have

$$\| k \|_{\mathcal{M}_p} = \| T^{M_x,M_x}_k \|_{\mathcal{L}_p \to \mathcal{L}_p}.$$  \hspace{1cm} (5.2)

Similarly, if $A$ and $B$ are unbounded self-adjoint operators on a Hilbert space $\mathcal{H}$ of the same spectral type as $M_x$, then

$$\| T^{A,B}_k \|_{\mathcal{L}_p \to \mathcal{L}_p} = \| k \|_{\mathcal{M}_p}.$$  \hspace{1cm} (5.2)

For more details on the relationship between Schur multipliers and double operator integrals, see [6; Sec. 1.4].

Recall that, for any $p \in (0,2)$, we let $p^* = 2p/(2-p)$, in order that $1/p^* = 1/p - 1/2$. Our first result is a continuous extension of Theorem 3.7 in [1], which has essentially the same proof.
Theorem 5.1. Fix any function $k \in L_\infty(\mathbb{R}^2)$ and any index $0 < p \leq 1$. For any partition of $\mathbb{R}$ into intervals $I_j = [t_j, t_{j+1})$, $j \in \mathbb{Z}$, let

$$k_j(t,s) = \chi_{I_j}(t)k(t,s), \quad t,s \in \mathbb{R}, \quad j \in \mathbb{Z}.$$  

The following estimate then holds:

$$\|k\|_{\mathfrak{M}_p} \leq \|(\|k_j\|_{\mathfrak{M}_p})_{j \in \mathbb{Z}}\|_{\ell_p^p}.$$  

Proof. Here we use (5.2) to compute the $\mathfrak{M}_p$-multiplier quasi-norm. Let $\xi, \eta \in L_2(\mathbb{R})$. It follows from the decomposition $k = \sum_{j \in \mathbb{Z}} k_j$ and the triangle inequality that

$$\| \text{Op}(k(\xi \otimes \eta)) \|_{\mathcal{L}_p}^p \leq \sum_{j \in \mathbb{Z}} \| \text{Op}(k_j(\xi \otimes \eta)) \|_{\mathcal{L}_p}^p.$$  

It is clear that, for any $j \in \mathbb{Z}$, we have $k_j(\xi \otimes \eta) = k_j(\chi_{I_j}\xi \otimes \eta)$, so that

$$\| \text{Op}(k(\xi \otimes \eta)) \|_{\mathcal{L}_p}^p \leq \sum_{j \in \mathbb{Z}} \|k_j\|_{\mathfrak{M}_p^p} \|\chi_{I_j}\xi\|_2 \|\eta\|_2^p.$$  

Applying Hölder’s inequality to the right-hand side, we see that

$$\| \text{Op}(k(\xi \otimes \eta)) \|_{\mathcal{L}_p}^p \leq \sum_{j \in \mathbb{Z}} \|k_j\|_{\mathfrak{M}_p}\|\chi_{I_j}\xi\|_2 \|\eta\|_2.$$  

However, the intervals $I_j$ are disjoint, so that $(\chi_{I_j}\xi)_{j \in \mathbb{Z}}$ is a sequence of functions orthogonal in $L_2(\mathbb{R})$, which in turn gives us

$$\|\|(\chi_{I_j}\xi)\|_2\|_{j \in \mathbb{Z}}\|_2 = \|\xi\|_2.$$  

The estimate

$$\| \text{Op}(k(\xi \otimes \eta)) \|_{\mathcal{L}_p}^p \leq \sum_{j \in \mathbb{Z}} \|k_j\|_{\mathfrak{M}_p} \|\xi\|_2 \|\eta\|_2$$  

then gives the result once we take the supremum over all $\xi$ and $\eta$ in the unit ball of $L_2(\mathbb{R})$. □

Corollary 5.2. For any sequence $(k_j)_{j \in \mathbb{Z}} \subseteq L_\infty(\mathbb{R})$, let $\psi$ be a compactly supported bounded function on $\mathbb{R}$, and let

$$k(t,s) = \sum_{j \in \mathbb{Z}} \psi(t-j)k_j(s).$$  

Then the following estimate holds:

$$\|k\|_{\mathfrak{M}_p} \leq \psi \|(\|k_j\|_{\infty})_{j \in \mathbb{Z}}\|_{\ell_p^p}.$$  

Proof. If the functions $(t \mapsto \psi(t-j))_{j \in \mathbb{Z}}$ are disjointly supported, the result follows immediately from Theorem 5.1. If not, we may choose some sufficiently large $N > 0$, such that $(t \mapsto \psi(t-Nj))_{j \in \mathbb{Z}}$ is a disjointly supported sequence of functions. Then let $k = \sum_{l=0}^{N-1} k^{(l)}$, where

$$k^{(l)}(t,s) = \sum_{j \in \mathbb{Z}} \psi(t-l-Nj)k_j(s)$$  

for all $s,t \in \mathbb{R}$. The result then follows from the individual bounds on each $k^{(l)}$. □

Let us now return to wavelet decompositions. For each $j,l \in \mathbb{Z}$ and any function $k$ on $\mathbb{R}^2$, we define

$$k_{j,l}(s) = \int_{\mathbb{R}} \varphi_{j,k}(t)k(t,s) \, dt$$  

for all $s \in \mathbb{R}$. The next corollary then follows immediately by rescaling, as $\varphi$ is a compactly supported wavelet, where $\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k)$. □
Corollary 5.3. For any bounded function \( k \in \dot{B}^{1/p'}_{p,p}(\mathbb{R}, L^\infty(\mathbb{R})) \) and any \( j \in \mathbb{Z} \), let

\[
k_j(t, s) = \sum_{l \in \mathbb{Z}} \varphi_{j,l}(t)k_{j,l}(s)
\]

for all \( s, t \in \mathbb{R} \). Then

\[
\|k_j\|_{\dot{B}^0_{p,p}} \lesssim 2^{j/2}\|\|k_j\|\|_{\dot{B}^0_{p,p}}\|_{\ell^p_{j,l}}
\]

for all \( j \in \mathbb{Z} \).

Before we proceed, let us note that, for a function \( k \in \dot{B}^{s,t}_{p,q}(\mathbb{R}, L^\infty(\mathbb{R})) \), it need not be the case that the decomposition \( k = \sum_{j \in \mathbb{Z}} k_j \) holds (the failure of such a decomposition in the scalar-valued case was detailed in [21; Chapter 3, Proposition 4]). Despite this, it follows from an obvious modification of Lemma 4.1.4 in [19] that if \( k \in L^\infty(\mathbb{R}^2) \), then there exists a function \( c \in L^\infty(\mathbb{R}) \) such that

\[
k(t, s) = c(t) + \sum_{j \in \mathbb{Z}} (k_j(t, s) - k_j(t, 0))
\]

for all \( t, s \in \mathbb{R}^2 \) and

\[
\|c\|_{\infty} \lesssim \|k\|_{\infty} + |k|_{\dot{B}^s_{p,q}(\mathbb{R}, L^\infty(\mathbb{R}))}.\]

Finally, we can prove our Theorem 1.6 stated in the introduction and strengthening Theorem 9.2 of [5]. Note that the condition that the function \( k \) is bounded cannot be removed, as the space of \( \mathcal{L}_2 \)-Schur multipliers is precisely the set of bounded functions.

**Proof of Theorem 1.6.** Given that \( k \) is bounded, we have

\[
k(t, s) = c(t) + \sum_{j \in \mathbb{Z}} (k_j(t, s) - k_j(t, 0))
\]

for all \( s, t \in \mathbb{R} \), where \( c \) is bounded and satisfies

\[
\|c\|_{\infty} \lesssim \|k\|_{\infty} + |k|_{\dot{B}^s_{p,q}(\mathbb{R}, L^\infty(\mathbb{R}))}.\]

By Corollary 5.3 and the \( p \)-triangle inequality, we have

\[
\|k\|_{\dot{B}^0_{p,p}} \lesssim \|c\|_{\infty} + \sum_{j \in \mathbb{Z}} (\|k_j\|_{\dot{B}^0_{p,p}} + \|k_j\|_{\infty})
\]

\[
\lesssim \|c\|_{\infty} + \sum_{j \in \mathbb{Z}} 2^{jp/2}\left(\sum_{l \in \mathbb{Z}} \|k_{j,l}\|_{\dot{B}^0_{p,p}}\|_{\ell^p_{j,l}}\right)^{p/p'}
\]

\[
\lesssim \|k\|_{\infty} + |k|_{\dot{B}^{1/p'}_{p,p}(\mathbb{R}, L^\infty(\mathbb{R}))} + \sum_{j \in \mathbb{Z}} 2^{jp/2}\left(\sum_{l \in \mathbb{Z}} \|k_{j,l}\|_{\dot{B}^0_{p,p}}\|_{\ell^p_{j,l}}\right)^{p/p'}
\]

where the last inequality follows by the definition of the \( L^\infty \)-valued Besov space.

**Remark 5.4.** Let \( D_x \) denote the unbounded self-adjoint operator on \( L^2(\mathbb{R}) \) given by \( D_x \xi(t) = -i\xi'(t) \). It was observed by Birman and Solomyak that if \( k \) is a Schur multiplier of \( \mathcal{L}_1 \), then the transformer \( T_k^{M_* D_x} \) acts boundedly in the operator norm and \( T_k^{M_* D_x}(1) \) coincides with a pseudodifferential operator with symbol function \( k \); see [6; Sec. 6]. Theorem 1.6 therefore gives a new proof of the result that a pseudodifferential operator with symbol function bounded and belonging to \( \dot{B}^{1/2}_{2,1}(\mathbb{R}, L^\infty(\mathbb{R})) \) defines a bounded linear operator on \( L^2(\mathbb{R}) \) (see [6; Eq. (6.3)] for a weaker example of such a result).

This result should also be compared with Sugimoto’s similar estimates for pseudo-differential operators [26].

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References

[1] A. B. Aleksandrov and V. V. Peller, “Hankel and Toeplitz–Schur multipliers”, Math. Ann., 324:2 (2002), 277–327.
[2] M. Ś. Birman and M. Z. Solomyak, “Double Stieltjes operator integrals”, Problems of Mathematical Physics, no. 1: Spectral Theory and Wave Processes, Izdat. Leningrad. Univ., Leningrad, 1966, pp. 33–67. (in Russian)
[3] M. Ś. Birman and M. Z. Solomyak, “Double Stieltjes operator integrals. II”, Problems of Mathematical Physics, no. 2: Spectral Theory, Diffraction Problems, Izdat. Leningrad. Univ., Leningrad, 1967, pp. 26–66. (in Russian)
[4] M. Ś. Birman and M. Z. Solomyak, “Double Stieltjes operator integrals. III,”, Problems of Mathematical Physics, no. 6, Izdat. Leningrad. Univ., Leningrad, 1973, pp. 27–53. (in Russian)
[5] M. Ś. Birman and M. Z. Solomyak, “Estimates for the singular numbers of integral operators”, Uspekhi Mat. Nauk, 32:1(193) (1977), 17–84; English transl.: Russian Math. Surveys, 32:1 (1977), 15–89.
[6] M. Ś. Birman and M. Z. Solomyak, “Double operator integrals in a Hilbert space”, Integral Equations Operator Theory, 47:2 (2003), 131–168.
[7] C. Brislawn, “Kernels of trace class operators”, Proc. Amer. Math. Soc., 104:4 (1988), 1181–1190.
[8] A. Cohen, “Numerical analysis of wavelet methods”, Studies in Mathematics and its Applications, no. 32, North-Holland, Amsterdam, 2003.
[9] Yu. L. Dalecki˘ı and S. G. Kre˘ın, “Formulas of differentiation according to a parameter of functions of Hermitian operators”, Doklady Akad. Nauk SSSR (N.S.), 76:1 (1951), 13–16.
[10] Yu. L. Dalecki˘ı and S. G. Kre˘ın, “Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations”, Trudy Sem. Funktsional. Anal., vol. 1, Voroneˇz. Gos. Univ., Voroneˇz, 1956, pp. 81–105.
[11] I. Daubechies, “Orthonormal bases of compactly supported wavelets”, Comm. Pure Appl. Math., 41:7 (1988), 909–996.
[12] R. A. DeVore, “Nonlinear approximation”, Acta Numer., 7 (1998), 51–150.
[13] R. A. DeVore and G. G. Lorentz, Constructive Approximation, Grundlehren der Mathematischen Wissenschaften, no. 303, Springer-Verlag, Berlin, 1993.
[14] D. Elstner and A. Pietsch, “Eigenvalues of integral operators. III”, Math. Nachr., 132 (1987), 191–205.
[15] L. Grafakos, Modern Fourier Analysis, Graduate Texts in Mathematics, no. 250, Springer-Verlag, New York, 2014.
[16] E. Hernández and G. Weiss, “A First Course on Wavelets”, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.
[17] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, “Analysis in Banach spaces. Vol. I: Martingales and Littlewood–Paley theory”, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, no. 63, Springer, Cham, 2016.
[18] J. Lindenstrauss and L. Tzafriri, “Classical Banach spaces. II”, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 97, Springer-Verlag, Berlin–New York, 1979.
[19] E. McDonald and F. A. Sukochev, Lipschitz estimates in quasi-Banach Schatten ideals, arXiv: 2009.08069.
[20] Y. Meyer, “Principe d’incertitude, bases hilbertiennes et algèbres d’opérateurs”, Astérisque, 145–146 (1987), 209–223; Séminaire Bourbaki, 1985/86, exp. n°662.
[21] Y. Meyer, Wavelets and Operators, Cambridge Studies in Advanced Mathematics, no. 37, Cambridge University Press, Cambridge, 1992.
[22] L. Z. Peng, “Wavelets and para-commutators”, Ark. Mat., 31:1 (1993), 83–99.
[23] A. Pietsch, “Eigenvalues of integral operators. I”, Math. Ann., 247:2 (1980), 169–178.
[24] A. Pietsch, “Eigenvalues of integral operators. II”, *Math. Ann.*, **262**:3 (1983), 343–376.
[25] Y. Sawano, “Theory of Besov Spaces”, Developments in Mathematics, no. 56, Springer, Singapore, 2018.
[26] M. Sugimoto, “$L^p$-boundedness of pseudodifferential operators satisfying Besov estimates. I”, *J. Math. Soc. Japan*, **40**:1 (1988), 105–122.

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