A DISPROOF OF HOOLEY’S CONJECTURE

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Abstract. Define $G(x; q)$ to be the variance of primes $p \leq x$ in the arithmetic progressions modulo $q$, weighted by $\log p$. Hooley conjectured that as soon as $q$ tends to infinity and $x \geq q$, we have the upper bound $G(x; q) \ll x \log q$. In this paper we show that the upper bound does not hold in general, and that $G(x; q)$ can be asymptotically as large as $x(\log q + \log \log \log x)^2/4$.

1. Introduction and statement of results

For $x > q \geq 3$, we define the variance

$$G(x; q) := \sum_{a \mod q} \left| \sum_{p \leq x \atop (a,q)=1} \log p - \frac{x}{\phi(q)} \right|^2,$$

as well as the closely related (and perhaps slightly more natural)

$$V_\Lambda(x; q) := \sum_{a \mod q} \left| \sum_{n \leq x \atop (a,q)=1} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq x \atop (n,q)=1} \Lambda(n) \right|^2.$$

Since the pioneering work of Barban, Davenport and Halberstam [Ba, DH], the study of this variance has seen a long line of developments, and continues to be an active research topic. To cite a few of the numerous papers written over the years, we mention the works [M, H1, FG1, FG2, GV, V, Liu, Pe, HS], as well as Hooley’s series of 19 research papers and 2 survey papers (see for instance [H2, H5, H7, H1, HM]). We also have the recent works [MP, KR, BaF, BW, Sm, CLLR], which explore sparse averages over $q$, as well as analogues for number fields and function fields.

Hooley conjectured [HM, p. 217], [H5, equation (2)] that as soon as $q$ tends to infinity with $x$, we have the upper bound

$$G(x; q) \ll x \log q.$$  \hfill (1)

Our main result is a disproof of this conjecture.

Theorem 1.1. There exists an infinite sequence of pairs $(q_j, x_j)$, with $q_j < x_j$ both tending to infinity, for which

$$\lim_{j \to \infty} \frac{G(x_j; q_j)}{x_j \log q_j} = \infty.$$  \hfill (2)

The same holds for $V_\Lambda(x_j; q_j)$ in place of $G(x_j; q_j)$.

Date: August 14, 2020.
One can ask whether the size of $G(x; q)$ and $V_\Lambda(x; q)$ in (2) can be made more explicit in terms of $x$ and $q$. Moreover, in light of the work of Keating and Rudnick [KR, Section 2.2] and of the first author [F, Conjecture 1.1], one might be curious as to the corresponding size of $q$ relative to $x$. Finally, one can ask whether the density of the set of moduli $q_j$ described in Theorem 3.9 can be quantified.

These questions are answered in the following more precise version of Theorem 1.1, in which we believe the range of $q$ in terms of $x$ is essentially best possible. To make this range explicit, we will consider functions $h: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$h(x) \text{ is increasing to infinity, and } h(e^y) \ll_A h(e^y) \text{ for every } A > 1,$$

(3)

the prototypical example of which is $h(x) = \max\{\log \log x, 1\}$.

**Theorem 1.2.** Let $h(x)$ be a function satisfying equation (3), and let $\varepsilon > 0$. If

$$\frac{\varepsilon \log \log x}{\log \log \log x} \leq h(x) \leq \log \log x$$

(4)

when $x \geq e^3$, then for a positive proportion of moduli $q$, there exist associated values $x_q$ such that $q \asymp h(x_q)$ and

$$G(x_q; q) \gg \varepsilon x_q \log q \cdot \frac{\log \log x_q}{q}.$$  

(5)

In particular, when $\delta > 0$ is sufficiently small, equation (1) cannot hold in any range of $q$ that satisfies $q < \delta \log \log x$.

If on the other hand the function

$$h(x) \leq \varepsilon \frac{\log \log x}{\log \log \log x},$$

for $x \geq e^3$, then for a positive proportion of moduli $q$, there exist associated values $x_q$ such that $q \asymp h(x_q)$ and

$$G(x_q; q) \geq \left(\frac{1}{4} - \varepsilon\right)x_q \cdot (\log q + \log \log \log x_q)^2 \gg x_q(\log q)^2.$$  

The same statements hold for $V_\Lambda(x_q; q)$ in place of $G(x_q; q)$.

**Remark 1.3.** Under GRH, the generalized Riemann hypothesis for Dirichlet $L$-functions, omega results similar to Theorem 1.2 hold for all moduli $q$. Indeed, we will show in Theorem 3.9 that for any function $h$ satisfying equation (3) and $h(x) \leq \delta \log \log x$ with $\delta > 0$ small enough, there exists a sequence $\{x_q\}_{q \geq 1}$ such that $\phi(q) \asymp h(x_q)$ and having the property that $G(x_q; q)/(x_q \log q)$ tends to infinity as $q \to \infty$.

Our method also produces an omega result in a wider range, namely for

$$\frac{\log \log x}{\log \log \log x} \leq q \leq (\log x)^{\delta}$$

with $\delta$ small enough. Indeed, $G(x; q)$ can be as large as $(x \log \log x)(\log q)/\phi(q)$ (combine Theorem 3.9 with Proposition 2.3). However, in this range sharper omega results will be obtained in [BrF] for a weighted variant of $V_\Lambda(x; q)$ and all higher even moments.

One might wonder whether omega results analogous to Theorem 1.2 also hold for fixed values of $q$, as $x \to \infty$. For instance, in the case $q = 4$,

$$V_\Lambda(x; 4) = \frac{|\psi(x, \chi_{-4})|^2}{2} = \Omega(x(\log \log \log x)^2)$$

with $

|\psi(x, \chi_{-4})|^2 = \Omega(x(\log \log \log x)^2).$
by Littlewood’s celebrated result [Lit]. This result has been generalized by Davidoff to all real Dirichlet characters (private communication), and as a result, an application of Parseval’s identity results in

\[ V_\Lambda(x; q) = \Omega \left( \frac{x (\log \log \log x)^2}{\phi(q)} \right), \tag{6} \]

which is comparable to [FG1, equation (1.22)]. In the next theorem we generalize the results of Littlewood and Davidoff to all characters modulo \( q \). We also show that the large values of \( |\psi(x, \chi)| \) with \( \chi \) running over the Dirichlet characters modulo \( q \) can be synchronized, and as a result we obtain an omega result sharper than (6).

**Theorem 1.4.** Let \( q \geq 3 \) be fixed. For any fixed nonprincipal character \( \chi \mod q \),

\[ \Re(\theta(x, \chi)) = \Omega - \left( x \frac{1}{2} \log \log \log x \right) \tag{7} \]

Moreover,

\[ G(x; q) = \Omega \left( x (\log \log \log x)^2 \right). \tag{8} \]

The same oscillation results hold with \( \psi(x, \chi) \) and \( V_\Lambda(x; q) \) in place of \( \theta(x, \chi) \) and \( G(x; q) \), respectively.

We remark that the sequence of \( x \)-values implied by these oscillation results depends upon \( \chi \) or \( q \); the implied \( \Omega \)-constants, however, are absolute.

Our final theorem is an omega result on the average of \( V(x; q) \) over \( q \). Hooley conjectured the upper bound (1) based on his result [H5, Theorem 1], which states (assuming GRH) that uniformly for \( q \leq x \), one has the upper bound

\[ \frac{1}{T} \int_{\log 2}^{T} e^{-y} G(e^y; q) \, dy \ll \log q, \]

that is, equation (1) holds on average over \( \log x \). A natural question to ask here would be whether replacing this average with a more classical \( q \)-average would result in the same upper bound, that is, whether

\[ \frac{1}{Q} \sum_{q \leq Q} G(x; q) \ll x \log Q. \tag{9} \]

As it turns out, this assertion is also false, as we show below.

**Theorem 1.5.** Let \( \varepsilon > 0 \) be small enough, and let \( Q : \mathbb{R}_{\geq 0} \to \mathbb{N} \) be a monotonic function with the property (3) and such that \( Q(x) \leq \varepsilon (\log \log x)^{1/2} / (\log \log \log x)^{1/2} \). Then we have the omega result

\[ \frac{1}{Q(x)} \sum_{Q(x/2 \leq q \leq 2Q(x)} G(x; q) = \Omega \left( x (\log \log \log x)^2 \right), \]

and the same statement holds for \( V_\Lambda(x; q) \) in place of \( G(x; q) \).

Let us briefly describe the tools used in the proofs of Theorems 1.2, 1.4, and 1.5. The first step, which is carried out in Section 2, is to show that if GRH is false, then a stronger but ineffective omega result holds. Indeed, if \( L(s, \chi) \) has a non-trivial zero \( \rho_\chi = \Theta_\chi + i\gamma_\chi \) off the critical line, then by Landau’s theorem \( |\theta(x, \chi) - 1_{\chi=\chi_0} x| \) can be as large as \( x^{\Theta_\chi - \varepsilon} \). (Here \( 1_{\chi=\chi_0} \) equals 1 if \( \chi = \chi_0 \) and 0 otherwise.) This works well for fixed moduli \( q \) (as in Theorem 1.4); however, one needs to modify this approach to have a result which is uniform
in the range \( q \leq x^{o(1)} \) (for Theorem 1.2). To achieve this, we apply Parseval’s identity (11) and positivity.

Let us use GRH(\( \chi \)) to denote the generalized Riemann hypothesis for a specific Dirichlet L-function \( L(s, \chi) \). If \( q_\chi \) is the least modulus for which a character \( \chi_\chi \) exists such that GRH(\( \chi_\chi \)) is false, then \( \chi_\chi \) will induce a character modulo every multiple of \( q_\chi \) whose associated Dirichlet L-function also violates GRH. As a result, we will deduce (see Proposition 2.3) that \( G(x; q) \) can be as large as \( x^{2\Theta_\chi} e^{-\varepsilon} / \phi(q) \); since \( \Theta_\chi \) is independent of \( q \) and \( x \), this deduction will violate Hooley’s conjecture (1) in the range \( q \leq x^{o(1)} \) for a positive proportion of moduli \( q \).

In other words, Hooley’s conjecture in any range of the form \( q \approx x^{o(1)} \) is stronger than GRH; indeed, Hooley’s conjecture in a range of the form \( q \approx x^{\delta} \) implies the zero-free strip \( \Re(s) > \frac{1}{2} + \frac{\delta}{2} \) for all Dirichlet L-functions modulo \( q \) (see Proposition 2.2).

A difficulty arises in this approach when one is looking for a result which holds for many values of \( q \). Indeed, if one uses the oscillations of \( \theta(x, \chi_\chi) - 1_{\chi=\chi_0} x \) to create large values of \( G(x; q) \), say on a sequence \( x_j \), then the condition \( q \approx h(x_q) \) will force \( q \) to be in a set which is possibly thin, since Landau’s theorem alone does not give the rate of growth of \( x_j \).

To circumvent this possible issue we applied a refined omega result of Kaczorowski and Pintz [KP], which gives a rate of growth for \( x_j \); however, their main theorem requires the assumption that \( L(\sigma, \chi_\chi) \neq 0 \) for \( 1/2 \leq \sigma < 1 \). Fortunately, for our purposes it is sufficient to apply a weaker result, which as we will show can be proven unconditionally.

In the second step, which is more intricate, we assume that GRH holds. Our general strategy in Section 3 is to apply the explicit formula and to synchronize the summands using homogeneous Diophantine approximation. This goes back to Littlewood’s original approach, modified as in [MV, Theorem 15.11] and [RS, Lemma 2.4]. However, working uniformly in \( q \) poses significant new challenges. In the classical proofs, the quantity \( x^{-\frac{1}{2}} (\psi(x) - x) \) is expressed as a sum over zeros of \( \zeta(s) \), which under the Riemann hypothesis is approximated by

\[
- \sum \frac{x^{i\gamma}}{\rho} \approx - \sum \frac{x^{i\gamma}}{1 - \rho} = -2 \sum_{\gamma > 0} \frac{\sin(\gamma \log x)}{\gamma}.
\]

If one applies the same approximation to \( \psi(x, \chi) \), then one runs into the problem of potential real zeros \( \rho_\chi \approx \frac{1}{2} \), that is, \( \gamma_\chi = 0 \). Even assuming Chowla’s conjecture \( L(1/2, \chi) \neq 0 \), we would still not be able to control the ordinate of the smallest zero uniformly in \( q \). Moreover, the approximation (10) has an error term of size \( x^{\frac{1}{2}} \), which translates to \( x^{\frac{1}{2}} \log q \) in the case of \( \psi(x, \chi) \). This error term would need to be smaller than a main term \( \approx x^{\frac{1}{2}} \log(q \log x) \), which is not the case in the range \( \log \log x \) \( \leq q \leq \log \log x \).

In order to circumvent these issues, we keep the term on the left hand side of equation (10) as it is, and bound the contribution of real zeros by applying a generalization of the 1-level density result of Hughes and Rudnick [HR], as well as a result of Selberg [Se] on multiplicities of zeros of \( L(s, \chi) \) (see Lemma 3.1 for both applications). We compute the average of \( \psi(e^{\gamma}, \chi) \) in suitable short intervals, which will be determined by an application of homogeneous Diophantine approximation, to synchronize the frequencies \( y\gamma/2\pi \mod 1 \) simultaneously for all zeros \( \rho_\chi = \frac{1}{2} + i\gamma_\chi \), with height at most \( T \), of all \( L(s, \chi) \) with \( \chi \mod q \). Interestingly, in certain ranges, rather than synchronizing an unbounded number of frequencies for a single character, we synchronize a large enough but bounded number of frequencies for each character modulo \( q \). This step forces the value of \( \log x \) to be as large as \( \exp(c\phi(q)T \log(qT)) \) for
some constant $c > 0$, which explains the range $q \leq \log \log x / \log \log \log x$ in the second part of Theorem 1.2.

We also localize the large values fairly precisely in Theorem 1.2, that is, we obtain two-sided bounds on $x$ in terms of $q$ and $T$. The key observation here is that we can exploit the almost-periodicity of $\psi(e^y, \chi)$ as in [RS, Section 2.2] by finding many values of $n$ in the Diophantine approximation step, which will force one of these values to be $\geq \exp \left( \frac{1}{2} \phi(q) T \log(qT) \right)$. Once this is done, the last step is to estimate the resulting sums using the Riemann–von Mangoldt formula and an evaluation of the average log-conductor [FiMa, Proposition 3.3], which yields the second part of Theorem 1.2.

In order to obtain the full range in Theorem 1.2 (that is, $q \leq \varepsilon \log \log x$), this approach needs to be further modified. Indeed, the fact that we are synchronizing the frequencies $y\gamma_{\chi}/2\pi \mod 1$ for all characters $\chi \mod q$ forces $x$ to be as large as $\exp(qO(q))$. To reduce this bound, we instead synchronize a subset of characters modulo $q$, resulting in the weaker omega result (5), which is still strong enough to contradict Hooley’s conjecture (1). This approach introduces additional difficulties in the estimation of the average of the log-conductor, which are overcome by applying the recent statistical results [BrF, Lemma 3.1] and [F, Lemma 3.2] (see Lemmas 3.2 and 3.6 below).

In summary, in Section 2 we establish propositions that imply our main theorems when GRH is false; in Section 3 we prove more delicate results that imply our main theorems when GRH is true. In particular, our main technical result is Theorem 3.9, after which we deduce Theorems 1.1, 1.2, 1.4, and 1.5.

2. HOOLEY’S CONJECTURE AND GRH

The goal of this section is to show that Hooley’s conjecture (1) in any range of the form $q \asymp x^{o(1)}$ is stronger than GRH; as a result, in the proof of Theorem 1.1 (that is, when disproving Hooley’s conjecture) we will be able to assume GRH in subsequent sections. We will use the classical notation

$$
\psi(x; q, a) := \sum_{n \leq x \atop n \equiv a \mod q} \Lambda(n); \quad \theta(x; q, a) := \sum_{p \leq x \atop p \equiv a \mod q} \log p,
$$

and for a Dirichlet character $\chi \mod q$,

$$
\psi(x, \chi) := \sum_{n \leq x} \chi(n) \Lambda(n); \quad \theta(x, \chi) := \sum_{p \leq x} \chi(p) \log p.
$$

We record the Parseval identities

$$
V_\Lambda(x; q) = \frac{1}{\phi(q)} \sum_{\chi \mod q \atop \chi \neq \chi_0} |\psi(x, \chi)|^2; \quad G(x; q) = \frac{1}{\phi(q)} \sum_{\chi \mod q} |\theta(x, \chi) - 1_{\chi = \chi_0} x|^2. \quad (11)
$$

Our first step is to see that if $\chi \mod q$ is a character for which $L(s, \chi)$ does not satisfy the Riemann Hypothesis, then $|\psi(x, \chi)|$ and $|\theta(x, \chi) - 1_{\chi = \chi_0} x|$ have large values. This will follow from a result of Kaczorowski and Pintz [KP] which we will adapt in order to obtain an unconditional statement. One should keep in mind that this will give no information about uniformity in $q$. We also mention that for $\chi = \chi_0$, we have the more precise results of Pintz [Pi] and Schlage–Puchta [SP].
Lemma 2.1. Fix \( \varepsilon > 0 \), let \( q \geq 1 \), and let \( \chi \) be a character mod \( q \). Define \( \Theta_{\chi} \geq \frac{1}{2} \) to be the supremum of the real parts of the zeros of \( L(s, \chi) \). Then, for every large enough \( X \) (in terms of \( \chi \) and \( \varepsilon \)), there exists \( x \in [X^{1-\varepsilon}, X] \) such that

\[
\Re(\psi(x, \chi)) - 1_{\chi=\chi_0}x < -x^{\Theta_{\chi} - \varepsilon}.
\]

Proof. Suppose first that \( L(\Theta_{\chi}, \chi) \neq 0 \). Then the claim follows from setting \( f(x) = \Re(\psi(x, \chi)) - 1_{\chi=\chi_0}1_{x \geq 1} \) and applying [KP, Theorem 1]. Indeed,

\[
\int_0^\infty f(x)x^{-s-1}dx = -\frac{1}{2s}\left(\frac{L'(s, \chi)}{L(s, \chi)} + \frac{L'(s, \overline{\chi})}{L(s, \overline{\chi})}\right) - \frac{1_{\chi=\chi_0}}{s-1},
\]

which is regular in the half plane \( \Re(s) > \Theta_{\chi} \) but not in any half plane of the form \( \Re(s) > \Theta_{\chi} - \varepsilon \). Here we used the fact that the residues of \( L'(s, \chi)/L(s, \chi) \) are nonnegative, since \( L(s, \chi) - 1_{\chi=\chi_0}/(s-1) \) is entire (in other words, the poles of \( L'(s, \chi)/L(s, \chi) \) and \( L'(s, \overline{\chi})/L(s, \overline{\chi}) \) cannot cancel each other).

Suppose otherwise that \( L(\Theta_{\chi}, \chi) = 0 \), and thus \( \chi \neq \chi_0 \). The explicit formula [MV, Theorems 12.5 and 12.10] implies that for \( T \geq 1 \),

\[
\psi(x, \chi) = -\sum_{|\rho_{\chi}| \leq T} \frac{x^{\rho_{\chi}}}{\rho_{\chi}} + O\left(\log(qx) + \frac{x(\log(qxT))^2}{T}\right).
\]

We deduce that for any \( 0 < T_1 < T_2 \),

\[
\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} e^{-\Theta_{\chi}t}\psi(e^{t}, \chi) dt = -\sum_{\rho_{\chi} \neq \Theta_{\chi}} \frac{e^{T_2(\rho_{\chi} - \Theta_{\chi})} - e^{T_1(\rho_{\chi} - \Theta_{\chi})}}{\rho_{\chi}(\rho_{\chi} - \Theta_{\chi})(T_2 - T_1)} - \frac{\text{ord}_{s=\Theta_{\chi}}L(s, \chi)}{\Theta_{\chi}} + O\left(e^{-\Theta_{\chi}T_1}\log(qT_2)\right).
\]

Taking \( T_1 = (1 - \varepsilon)\log X \) and \( T_2 = \log X \), and noting that the infinite sum over zeros in the last equation converges absolutely, we deduce that for large enough \( X \) there exists \( x \in [X^{1-\varepsilon}, X] \) for which

\[
x^{-\Theta_{\chi}}\Re(\psi(x, \chi)) < -\frac{\text{ord}_{s=\Theta_{\chi}}L(s, \chi)}{2\Theta_{\chi}}.
\]

The claim follows. \( \square \)

With this omega result in hand, we will deduce that in certain ranges, the upper bound (1) is stronger than GRH. This is made precise in the following proposition. The goal here is to overcome the uniformity problems caused by the fact that \( \Theta_{\chi} \) depends on \( \chi \) in Lemma 2.1. This will be done by noticing that for many moduli \( q \), characters of small conductor occur in the sums in equation (11).

Proposition 2.2. Fix \( 0 < \delta < 1 \), and assume that equation (1) holds in the range \( x^\delta \leq q \leq 2x^\delta \). Then every Dirichlet \( L \)-function is nonvanishing in the half-plane \( \Re(s) > \frac{1}{2} + \frac{\delta}{2} \). If one replaces \( G(x; q) \) with \( V_{\chi}(x; q) \) in equation (1), then the same half-plane is zero-free for all Dirichlet \( L \)-functions corresponding to nonprincipal characters.

Proof. We prove the contrapositive. Suppose that there exists a primitive Dirichlet \( L \)-function \( L(s, \chi_0) \) of conductor \( q_e \geq 1 \) that has a zero with real part \( \beta_e > \frac{1}{2} + \frac{\delta}{2} \). By
Lemma 2.1, for every $0 < \varepsilon < \beta_e$ there exists an increasing sequence $\{x_j\}_{j \geq 1}$ tending to infinity such that
\[
|\theta(x_j, \chi_e) - 1_{x=x_0}x_j| \geq x_j^{\beta_e - \varepsilon}.
\]
We may assume that each $x_j > q_e^{1/\delta}$, so that any interval of length $x_j^{\delta}$ contains a multiple of $q_e$. For each $j \geq 1$, choose an integer $x_j^{\delta} \leq q_j \leq 2x_j^{\delta}$ that is a multiple of $q_e$, and let $\chi_j$ be the character mod $q_j$ induced by $\chi_e$. Note that
\[
|\theta(x_j, \chi_e) - \theta(x_j, \chi_j)| \leq \log q_j \ll \delta \log x_j,
\]
and hence for $j$ large enough in terms of $\chi_e$ and $\varepsilon$,
\[
|\theta(x_j, \chi_j) - 1_{x=x_0}x_j| \geq x_j^{\beta_e - \varepsilon}
\]
as well. Consequently, when $j$ is large enough we have that
\[
G(x_j; q_j) = \frac{1}{\phi(q_j)} \sum_{\chi \mod q_j} |\theta(x_j, \chi) - 1_{x=x_0}x_j|^2
\]
\[
\geq \frac{|\theta(x_j, \chi_j) - 1_{x=x_0}x_j|^2}{q_j} \geq \frac{x_j^{2(\beta_e - \varepsilon)}}{2x_j^{\delta}} = \frac{x_j^{2\beta_e - \delta - 2\varepsilon}}{2},
\]
so that
\[
G(x_j; q_j) \gg \frac{x_j^{2\beta_e - 1 - \delta - 2\varepsilon}}{\log q_j} \gg \frac{x_j^{2\beta_e - 1 - \delta - 2\varepsilon}}{\delta \log x_j}.
\]
By assumption, $2\beta_e - 1 - \delta > 0$, and so the exponent $2\beta_e - 1 - \delta - 2\varepsilon$ is positive as long as $\varepsilon$ is chosen small enough. Therefore
\[
\lim_{j \to \infty} \frac{G(x_j; q_j)}{x_j \log q_j} = \infty,
\]
contradicting equation (1). The proof is identical for $V_{\Lambda}(x; q)$.

We now adapt the arguments in the proof of Proposition 2.2 to prove a proposition that is more suitable for the proof of Theorem 1.1.

**Proposition 2.3.** Assume that GRH is false. Then there exists an absolute constant $\delta > 0$ with the following property. Let $h(x)$ be an increasing function tending to infinity such that $h(x) = o(x^{\delta})$ as $x \to \infty$. For a positive proportion of moduli $q$, there exist associated values $x_q$ such that $h(x_q^{1-\delta}) \leq q \leq h(x_q)$ and
\[
G(x_q; q) \geq x_q^{1+\delta}.
\]
If GRH($\chi$) is false for some nonprincipal character $\chi$, then the same lower bound holds with $V_{\Lambda}(x_q; q)$ in place of $G(x_q; q)$.

**Proof.** Fix a modulus $q_e \geq 1$ for which there exists an associated primitive character $\chi_e$ such that $L(s, \chi_e)$ has a zero with real part $\beta_e > \frac{1}{2}$. Fix a positive number $\varepsilon < \beta_e - \frac{1}{2}$, and choose a positive number $\delta < \beta_e - \frac{1}{2} - \varepsilon \leq \frac{1}{2}$. Now let $q$ be any large enough multiple of $q_e$; the set of such moduli $q$ has positive (though ineffective) density in $\mathbb{N}$. By Lemma 2.1, there exists $x_q \in [h^{-1}(q), h^{-1}(q)^{1+\varepsilon}]$ such that
\[
|\theta(x_q, \chi_e)| > x_q^{\beta_e - \varepsilon}.
\]
Note that this implies that $q \in [h(x_1^{-\delta}), h(x_q)]$. Denote by $\chi_q$ the character mod $q$ induced by $\chi_e$. Then the calculations in equations (12) through (14) apply exactly to this situation; we conclude that

$$G(x_q; q) \gg x_q^{2\beta_e - \delta - 2\varepsilon},$$

and the right-hand side is eventually larger than $x_j^{1+\delta}$ by our choice of $\delta$, establishing the asserted lower bound. The proof for $V_{A}(x; q)$ is identical as long as $\chi_e$ is nonprincipal. □

To end this section, we further adapt Proposition 2.2 with the aim of proving Theorem 1.4. This situation is much easier since there are no uniformity issues (that is, $q$ is fixed).

**Proposition 3.4.** Fix $q \geq 1$, and assume that there exists a character $\chi_e \mod q$ such that GRH($\chi_e$) is false. Then, there exists a sequence $\{x_i\}_{i \geq 1}$, depending on $q$, such that for each $\varepsilon > 0$,

$$G(x_i; q) \geq \frac{1}{\phi(q)} |\theta(x_i; \chi_e)| - 1_{\chi_e = \chi_0} x_i |\phi_e - \varepsilon.\$$

Here, $\Theta_{\chi_e}$ is the supremum of real parts of zeros of $L(s, \chi_e)$ with $\chi \mod q$. Similarly for $V_A(x; q)$ and $\psi(x, \chi)$.

**Proof.** This follows at once from equation (11) and Lemma 2.1. □

3. **Explicit formulas and homogeneous Diophantine approximation**

The goal of this section is to show that GRH implies Theorem 1.2. Our goal will be to synchronize the arguments of the summands in the explicit formula, but only for a subset $\mathcal{F}_q$ of the set $\mathcal{X}_q$ of characters modulo $q$.

Throughout, $\gamma_\chi$ denotes the imaginary part of a nontrivial zero of $L(s, \chi)$, and $q_\chi$ denotes the conductor of $\chi$. We let $\|t\|$ denote the distance from $t$ to the nearest integer, and we use the shorthand $\log_2 t = \log \log t$ and $\log_3 t = \log \log \log t$.

**Lemma 3.1.** Let $q \geq 3$ be an integer. If $\chi \mod q$ is a nonprincipal character, then assuming GRH($\chi$) we have the bound

$$\text{ord}_{s=\frac{1}{2}} L(s, \chi) \ll \frac{\log q}{\log_2 q}.$$

Moreover, if GRH($\chi$) is true for all nonprincipal $\chi \mod q$, then

$$\sum_{\chi \mod q} \text{ord}_{s=\frac{1}{2}} L(s, \chi) \leq \left(\frac{1}{2} + o_{q \to \infty}(1)\right) \phi(q).$$

**Proof.** The first bound is well known (see [Se] or [IK, Proposition 5.21]). As for the second, it follows from [FiMi, Theorem 2.1] (which generalizes [HR] to composite moduli). Indeed, letting $f(x) := (\sin(2\pi x)/2\pi x)^2$, we have

$$\hat{f}(x) = \begin{cases} \frac{1}{2} - \frac{|x|}{4}, & \text{if } |x| \leq 2, \\ 0, & \text{otherwise}. \end{cases}$$

(16)
Since $\zeta(\frac{1}{2}) \neq 0$ and we are assuming that all non-trivial zeros of $L(s, \chi)$ with $\chi \neq \chi_0$ have real part $\frac{1}{2}$, it follows that

$$
\sum_{\chi \mod q} \text{ord}_{s=\frac{1}{2}} L(s, \chi) \leq \sum_{\chi \mod q, \chi \neq \chi_0} f\left(\frac{\log q}{2\pi} \gamma_\chi\right),
$$

which by [FiMi, Theorem 2.1] is $\lesssim \phi(q)/2$. \qed

We will also need to control the conductors of the characters in $\mathcal{F}_q$. We define $\Phi_q = \#\mathcal{F}_q$. We will require $\mathcal{F}_q$ to have the property that

$$
\chi \in \mathcal{F}_q \text{ if and only if } \overline{\chi} \in \mathcal{F}_q,
$$

(17)

**Lemma 3.2.** Let $w(q) > 0$ be any function tending to zero as $q \to \infty$. For each $q \geq 3$, there exists a subset $\mathcal{F}_q \subset \chi_q$ of the set of characters modulo $q$ of cardinality

$$
\Phi_q = \#\mathcal{F}_q \geq \phi(q)(1 - O(w(q)^2)),
$$

(18)

having the property (17), such that $\log q_\chi = \log q + O(w(q)^{-1} \log_2 q)$ for each character $\chi \in \mathcal{F}_q$.

**Proof.** By [BrF, Lemma 3.1], we have the estimate

$$
\frac{1}{\phi(q)} \sum_{\chi \mod q} (\log q_\chi - \log q)^2 \ll (\log_2 q)^2.
$$

(19)

Combined with [FiMa, Proposition 3.3] and Chebyshev’s inequality, this yields

$$
\frac{1}{\phi(q)} \#\{\chi \mod q: |\log q_\chi - \log q| > w(q)^{-1} \log_2 q\} \ll w(q)^2.
$$

Since $q_\overline{\chi} = q_\chi$, the characters not included in the above set have the property (17), and the desired estimate (18) follows. \qed

We are now ready to estimate the logarithmic averages of $\psi(x, \chi)$ and $\theta(x, \chi)$ over a short interval. By carefully choosing this interval to synchronize the frequencies in the explicit formula, we will ultimately create large values of $G(x; q)$ and $V_\Lambda(x; q)$. The Riemann–von Mangoldt formula

$$
N(T, \chi) := \#\{\rho_\chi: |3(\rho_\chi)| \leq T\} = \frac{T}{\pi} \log \left(\frac{q_\chi T}{2\pi e}\right) + O(\log(qT))
$$

(20)

for $T \geq 2$ will be central in our analysis. The next lemma records some estimates that follow easily from this asymptotic formula and partial summation.

**Lemma 3.3.** For any real parameters, $0 < \delta < 1$ and $T \geq \delta^{-1}$,

$$
\sum_{|\gamma_\chi| > T} e^{i\gamma_\chi} \rho_\chi^2 \left(\frac{i \sin(\delta \gamma_\chi)}{\delta} + \frac{\cos(\delta \gamma_\chi)}{2}\right) \ll \frac{1}{\delta} \sum_{|\gamma_\chi| > T} \frac{1}{\gamma_\chi^2} \ll \log q T \delta^{-1};
$$

$$
\max \left\{ \sum_{0 \leq \gamma_\chi \leq T} \frac{\sin^2(\delta \gamma_\chi)}{\delta |\rho_\chi|^4}, \sum_{0 \leq \gamma_\chi \leq T} \frac{\gamma_\chi |\sin(\delta \gamma_\chi) \cos(\delta \gamma_\chi)|}{|\rho_\chi|^4} \right\} \ll \delta \sum_{0 \leq \gamma_\chi \leq T} \frac{1}{\gamma_\chi^2} \ll \delta \log q.
$$
Moreover, property (denotes the primitive character inducing $\chi$ and will contain a large value of those functions.

**Lemma 3.4.** Let $q \geq 1$ be an integer, and let $F_q$ be a set of characters modulo $q$ with the property (17) such that GRH($\chi$) is true for all $\chi \in F_q$. Moreover, let $0 < \delta < 1$, $T \geq \delta^{-1}$, and $M \geq 1$ be real parameters, and define

$$R_\delta(y) := \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} \sum_{\chi \in F_q} (\psi(e^t, \chi) - 1_{\chi=\chi_0} e^t) \, dt;$$

$$S_\delta(y) := \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} \sum_{\chi \in F_q} (\theta(e^t, \chi) - 1_{\chi=\chi_0} e^t) \, dt.$$

(a) For all $y \geq 0$,

$$S_\delta(y) = R_\delta(y) + O(e^{\frac{y}{2}} \Phi_q).$$

(b) If $n \in \mathbb{N}$ satisfies $\|n\gamma T/2\pi\| < M^{-1}$ for each $0 \leq \gamma \leq T$ with $\chi \in F_q$, then $y = (n+1)\delta$ has the property that

$$R_\delta(y) = -e^{\frac{y}{2}} \sum_{\chi \in F_q} \sum_{0 \leq \gamma \leq T} \left(2\gamma \sin(\delta \gamma T) \phi(\gamma) \right) + \left(\frac{1}{4} - \gamma^2 \right) \frac{\sin^2(\delta \gamma T)}{\rho_x^2} + O\left(e^{\frac{y}{2}} \Phi_q \left(\frac{\log(qT)}{\delta T} + \min \left(\frac{\phi(q)}{\Phi_q}, \frac{\log q}{\log_2 q} \right) + \frac{\log q}{\log M} + \delta \log q \right) \right).$$

**Proof.** We begin by noting that part (a) is a direct consequence of the bound

$$|\psi(x, \chi) - \theta(x, \chi)| \leq \sum_{p \leq x} \log p \ll x^{\frac{1}{2}}.$$

As for part (b), we can transfer the question to primitive characters by noting that if $\chi^*$ denotes the primitive character inducing $\chi$, then

$$\sum_{\chi \in F_q} \psi(e^t, \chi) - \sum_{\chi \in F_q} \psi(e^t, \chi^*) \ll \Phi_q \log q.$$

Moreover, $L(s, \chi)$ and $L(s, \chi^*)$ have the same zeros on the critical line. Hence, the explicit formula [MV, Theorems 12.5 and 12.10] gives that for $y \geq 1$ and $S \geq 1$,

$$R_\delta(y) = -\frac{1}{2\delta} \int_{y-\delta}^{y+\delta} \sum_{\chi \in F_q} \sum_{\rho_x} \frac{e^{iy\rho_x}}{\rho_x} dt + O\left(\Phi_q y \log q + \frac{\Phi_q e^y (\log(\rho_x^2S))^2}{S} \right)$$

$$= -\frac{1}{2\delta} \sum_{\chi \in F_q} \sum_{\rho_x} \frac{e^{iy\rho_x}}{\rho_x^2} (e^{\delta \rho_x} - e^{-\delta \rho_x}) + O\left(\Phi_q y \log q + \frac{\Phi_q e^y (\log(\rho_x^2S))^2}{S} \right)$$

$$= -\frac{1}{2\delta} \sum_{\chi \in F_q} \sum_{\rho_x} \frac{e^{iy\rho_x}}{\rho_x^2} (e^{\delta \rho_x} - e^{-\delta \rho_x}) + O(\Phi_q y \log q).$$
after taking $S \to \infty$. Using $e^{\pm \delta p x} = e^{\pm i \delta \gamma x} (1 \pm \frac{\delta}{2} + O(\delta^2))$ and truncating the infinite sum over zeros using Lemma 3.3, we deduce the estimate

$$R_\delta(y) = -e^\frac{2}{7} \sum_{\chi \in \mathcal{F}_q} \sum_{\rho_x} \frac{e^{i \gamma r x}}{\rho_x^2} \left( \frac{i \sin(\delta \gamma x)}{\delta} \left( \frac{e^{i \gamma r x}}{\rho_x^2} - \frac{e^{-i \gamma r x}}{\rho_x^2} \right) + \frac{\cos(\delta \gamma x)}{2} \left( \frac{e^{i \gamma r x}}{\rho_x^2} + \frac{e^{-i \gamma r x}}{\rho_x^2} \right) \right) + O\left( \Phi_q \log q + \Phi_q \delta e^\frac{2}{7} \log q \right)$$

$$= -e^\frac{2}{7} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \leq \gamma_x \leq T} \left( \frac{i \sin(\delta \gamma x)}{\delta} \left( \frac{e^{i \gamma r x}}{\rho_x^2} - \frac{e^{-i \gamma r x}}{\rho_x^2} \right) + \frac{\cos(\delta \gamma x)}{2} \left( \frac{e^{i \gamma r x}}{\rho_x^2} + \frac{e^{-i \gamma r x}}{\rho_x^2} \right) \right) + O\left( \frac{e^\frac{2}{7}}{M} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \leq \gamma_x \leq T} \frac{1}{|\rho_x|} \right)$$

$$= -e^\frac{2}{7} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \leq \gamma_x \leq T} \left\{ \frac{-2 \sin(\delta \gamma x)}{\delta|\rho_x|^4} \left( \frac{1}{4} - \frac{\gamma_x^2}{4} \right) \sin(\delta \gamma x) - \gamma_x \cos(\delta \gamma x) \right\} + \frac{\cos(\delta \gamma x)}{|\rho_x|^4} \left( \left( \frac{1}{4} - \frac{\gamma_x^2}{4} \right) \cos(\delta \gamma x) + \gamma_x \sin(\delta \gamma x) \right) \right\} + O\left( \Phi_q \frac{e^\frac{2}{7}}{M} \log(qT) \log T \right)$$

$$= -e^\frac{2}{7} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \leq \gamma_x \leq T} \left( \frac{2 \gamma_x \sin(\delta \gamma x) \sin(\delta \gamma x) + \cos(\delta \gamma x)}{\delta|\rho_x|^4} \left( \frac{1}{4} - \frac{\gamma_x^2}{4} \right) \cos(\delta \gamma x) \right) + O\left( \Phi_q \frac{e^\frac{2}{7}}{M} \log(qT) \log T + \Phi_q \delta e^\frac{2}{7} \log q \right)$$

by Lemma 3.3. □

Now that we have expressed averages of $\theta(x, \chi)$ and $\psi(x, \chi)$ in suitable short intervals in terms of sums over zeros, our strategy is to estimate the sums in equation (22) using the Riemann–von Mangoldt formula (20).

**Lemma 3.5.** Let $q \geq 1$ be an integer, and let $\mathcal{F}_q$ be a set of characters modulo $q$ with the property (17) such that GRH($\chi$) is true for all $\chi \in \mathcal{F}_q$. For any $0 < \delta \leq e^{-1}$ and $T \geq e\delta^{-1}$,

$$\delta \sum_{\chi \in \mathcal{F}_q} \sum_{0 \leq \gamma_x \leq T} \frac{2 \gamma_x^4 \sin^2(\delta \gamma x)}{|\rho_x|^4 (\delta \gamma x)^2} = \frac{\Phi_q}{2} \log(q\delta^{-1})$$

$$+ O\left( \delta \Phi_q \log(qT) \log T + \delta^\frac{2}{7} \Phi_q \log(q\delta^{-1}) + (E_q + \Phi_q) \log(T\delta) + \frac{\Phi_q \log(qT)}{\delta T} \right), \quad (24)$$

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where

\[ E_q := \sum_{\chi \in \mathcal{F}_q} (\log q - \log q_\chi), \quad (25) \]

**Proof.** We will see that the main contribution in the sum on the left hand side of (24) comes from zeros \( \gamma_\chi \) of intermediate size. We can therefore discard low-lying zeros as follows:

\[
\delta \sum_{\chi \in \mathcal{F}_q, 0 \leq \gamma_\chi \leq T} 2\gamma_\chi^4 \sin^2(\delta \gamma_\chi) = \delta \sum_{\chi \in \mathcal{F}_q, \frac{1}{2} < \gamma_\chi \leq T} \frac{2\gamma_\chi^4 \sin^2(\delta \gamma_\chi)}{|ho_\chi|^4 (\delta \gamma_\chi)^2} + O\left( \delta \sum_{\chi \in \mathcal{F}_q} N(\delta^{-\frac{1}{2}}, \chi) \right)
\]

\[
= \delta \sum_{\chi \in \mathcal{F}_q, \delta^\frac{1}{2} < \gamma_\chi \leq T} \left( 2 + O\left( \frac{1}{\gamma_\chi^2} \right) \right) \frac{\sin^2(\delta \gamma_\chi)}{(\delta \gamma_\chi)^2} + O(\Phi_q \delta^\frac{3}{2} \log(q\delta^{-\frac{1}{2}}))
\]

\[
= 2\delta \sum_{\chi \in \mathcal{F}_q, \delta^\frac{1}{2} < \gamma_\chi \leq T} \frac{\sin^2(\delta \gamma_\chi)}{(\delta \gamma_\chi)^2} + O(\Phi_q \delta^\frac{3}{2} \log(q\delta^{-\frac{1}{2}}) + \Phi_q \delta^\frac{3}{2} \log(q\delta^{-1})),
\]

and the first error term is smaller than the second. Define

\[
N(t, \mathcal{F}_q) := \sum_{\chi \in \mathcal{F}_q} N(t, \chi) = \sum_{\chi \in \mathcal{F}_q} \left( \frac{t}{\pi} \log \left( \frac{q_\chi t}{2\pi e} \right) + O(\log(qt)) \right)
\]

\[
= \sum_{\chi \in \mathcal{F}_q} \frac{t}{\pi} \log(qt) - \sum_{\chi \in \mathcal{F}_q} \frac{t}{\pi} (\log q - \log q_\chi) + O\left( \sum_{\chi \in \mathcal{F}_q} (t + \log(qt)) \right)
\]

\[
= \Phi_q t \frac{t}{\pi} \log(qt) + O\left( E_q t + \Phi_q (t + \log(qt)) \right)
\]

by the asymptotic formula (20). Notice that counting only zeros above the real axis would yield \( \frac{1}{2} N(t, \mathcal{F}_q) \) in place of \( N(t, \mathcal{F}_q) \) thanks to the property (17) and the functional equation for Dirichlet L-functions. We may now compute

\[
2\delta \sum_{\chi \in \mathcal{F}_q, \delta^{-\frac{1}{2}} < \gamma_\chi \leq T} \frac{\sin^2(\delta \gamma_\chi)}{(\delta \gamma_\chi)^2} = 2\delta \int_{\delta^{-\frac{1}{2}}}^{T} \frac{\sin^2(\delta t)}{(\delta t)^2} d\left( \frac{1}{2} N(t, \mathcal{F}_q) \right)
\]

\[
= \delta N(t, \mathcal{F}_q) \left. \frac{\sin^2(\delta t)}{(\delta t)^2} \right|_{\delta^{-\frac{1}{2}}}^{T} - \delta \int_{\delta^{-\frac{1}{2}}}^{T} \left( \frac{2 \sin(\delta t) \cos(\delta t)}{\delta t^2} - \frac{2 \sin^2(\delta t)}{\delta^2 t^3} \right) N(t, \mathcal{F}_q) dt
\]

\[
= -\delta \int_{\delta^{-\frac{1}{2}}}^{T} 2 \left( \frac{\sin(2\delta t)}{2\delta t^2} - \frac{\sin^2(\delta t)}{\delta^2 t^3} \right) N(t, \mathcal{F}_q) dt + O\left( \Phi_q \log(q^{-\frac{1}{2}}) + \frac{\Phi_q \log(qT)}{\delta T} \right)
\]

\[
= -\frac{\Phi_q \delta}{\pi} \int_{\delta^{-\frac{1}{2}}}^{T} 2 \left( \frac{\sin(2\delta t)}{2\delta t^2} - \frac{\sin^2(\delta t)}{\delta^2 t^3} \right) t \log(qt) dt
\]

\[+ O\left( \Phi_q \left( \delta \log(qT) \log T + \log(T\delta) \right) + E_q \log(T\delta) + \delta^2 \Phi_q \log(q\delta^{-1}) + \frac{\Phi_q \log(qT)}{\delta T} \right),
\]
using $\sin u \ll \min\{1, \frac{1}{|u|}\}$. Finally, integrating by parts in the other direction, the main term in this expression equals

$$-\frac{\Phi_q\delta}{\pi}t \log(qt) \frac{\sin^2(\delta t)}{(\delta t)^2} \bigg|_{\delta = \frac{1}{2}} + \frac{\Phi_q\delta}{\pi} \int_{\delta = \frac{1}{2}}^{T} \frac{\sin^2(\delta t)}{(\delta t)^2} (\log(qt) + 1) \, dt$$

$$= \frac{\Phi_q}{\pi} \int_0^\infty \frac{\sin^2 u}{u^2} (\log(q\delta^{-1}u) + 1) \, du + O\left(\delta T \Phi_q \log(q\delta^{-\frac{1}{2}}) + \frac{\Phi_q \log(qT)}{\delta T}\right)$$

$$= \frac{\Phi_q}{2} \log(q\delta^{-1}) + O\left(\delta \Phi_q \log(q\delta^{-1}) + \frac{\Phi_q \log(qT)}{\delta T} + \Phi_q\right)$$

by the evaluation $\int_0^\infty \frac{\sin^2 u}{u^2} \, du = \frac{\pi}{2}$ (which can be derived from equation (16)).

Now that we have evaluated some of the main terms in equation (22), we can deduce a precise estimate of the values attained by $R_\delta(y)$ and $S_\delta(y)$ in Lemma 3.4.

**Lemma 3.6.** Let $q \geq 1$ be an integer, and let $\mathcal{F}_q$ be a set of characters modulo $q$ with the property (17) such that GRH(\chi) is true for all $\chi \in \mathcal{F}_q$. Let $\delta > 0$ be small enough (in absolute terms), $T \geq e^\delta^{-1}$ and $M \geq 1$. If $n \in \mathbb{N}$ satisfies $\|n\gamma_\chi / 2\pi\| < M^{-1}$ for each $0 \leq \gamma_\chi \leq T$ with $\chi \in \mathcal{F}_q$, then $y = (n+1)\delta$ has the property that

$$R_\delta(y) = -\frac{\Phi_q \log(q^2\delta^{-1})}{2} + O\left(\frac{\log(qT)}{\delta T} \log(T\delta) + \frac{y \log q}{e^\delta} + \frac{\log(qT) \log T}{M} + \delta \Phi_q \log(qT) + \frac{\log q}{\log_2 q}\right),$$

where $E_q$ was defined in equation (25).

**Proof.** We first truncate part of the sum in Lemma 3.4, in order to expand $\sin(\delta \gamma_\chi)$ and $\cos(\delta \gamma_\chi)$ into Taylor series. Doing so, we see that the main term in the estimate (22) equals

$$-e^\frac{\gamma_\chi}{2} \sum_{\chi \in \mathcal{F}_q, 0 \leq \gamma_\chi \leq \delta^{-\frac{1}{2}}} \left(2\gamma_\chi \sin(\delta \gamma_\chi) \cos(\delta \gamma_\chi) + \left(\frac{1}{4} - \gamma_\chi^2\right) \frac{\cos^2(\delta \gamma_\chi)}{|\rho_\chi|^4}\right)$$

$$-e^\frac{\gamma_\chi}{2} \sum_{\chi \in \mathcal{F}_q, 0 \leq \gamma_\chi \leq T} \frac{2\gamma_\chi^2 \sin^2(\delta \gamma_\chi)}{|\rho_\chi|^4} (\delta \gamma_\chi)^2 + O\left(e^\frac{\gamma_\chi}{2} \delta \Phi_q \log(q\delta^{-1})\right)$$

$$= -e^\frac{\gamma_\chi}{2} \sum_{\chi \in \mathcal{F}_q, 0 \leq \gamma_\chi \leq T} \frac{2\gamma_\chi^2 \sin^2(\delta \gamma_\chi)}{|\rho_\chi|^4} (\delta \gamma_\chi)^2 - e^\frac{\gamma_\chi}{2} \sum_{\chi \in \mathcal{F}_q, 0 \leq \gamma_\chi \leq \delta^{-\frac{1}{2}}} \frac{1}{4} \gamma_\chi^2 + O\left(e^\frac{\gamma_\chi}{2} \delta \Phi_q \log(q\delta^{-1})\right)$$

$$= -e^\frac{\gamma_\chi}{2} \sum_{\chi \in \mathcal{F}_q, \gamma_\chi \geq 0} \frac{2\gamma_\chi^4 \sin^2(\delta \gamma_\chi)}{|\rho_\chi|^4} (\delta \gamma_\chi)^2 - e^\frac{\gamma_\chi}{2} \sum_{\chi \in \mathcal{F}_q, \gamma_\chi \geq 0} \frac{1}{4} \gamma_\chi^2 + O\left(e^\frac{\gamma_\chi}{2} \delta \Phi_q \log(q\delta^{-1}) + e^\frac{\gamma_\chi}{2} \Phi_q \log(qT)\right).$$

The second double sum can be treated by using the functional equation (with Lemma 3.1 accounting for possible zeros at $s = \frac{1}{2}$), and applying [FiMa, Lemma 3.5] and the Littlewood
bound $L'(1, \chi)/L(1, \chi) \ll \log_2 q$; we obtain
\[
\sum_{\chi \in \mathcal{F}_q} \sum_{\gamma \geq 0} \frac{1}{|\rho_{\chi}|^2} = \frac{1}{2} \sum_{\chi \in \mathcal{F}_q} \sum_{\gamma} \frac{1}{|\rho_{\chi}|^2} + O(\Phi_q \log q) = \frac{1}{2} \sum_{\chi \in \mathcal{F}_q} \log q_{\chi} + O(\Phi_q \log q) = \frac{\Phi_q \log q}{2} + O(\frac{\Phi_q \log q}{\log_2 q} + E_q).
\]

Combining this evaluation with Lemma 3.5, we find that the sum of the first two terms in equation (27) equals
\[
- e^x \frac{\Phi_q \log(q^2 \delta^{-1})}{2} + O\left(e^x \Phi_q \left(\delta \log(q T) \log T + \frac{\log(q T)}{\delta T} + \left(\frac{E_q}{\Phi_q} + 1\right) \log(T \delta) + \frac{\log q}{\log_2 q}\right)\right). \quad \Box
\]

Now that we know exactly how large $R_0(y)$ and $S_0(y)$ can be, it is time to understand more precisely the set of $y$ which are admissible. In particular, it is important for us to localize these values in terms of the modulus $q$. This will be done using a counting argument, inspired by the proof of [RS, Lemma 2.4].

**Lemma 3.7.** Let $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$ be a set of real numbers. For any positive integers $M$ and $N$,
\[
\#\{n \leq N : \|n\lambda\| \leq M^{-1} \text{ for all } \lambda \in \Lambda\} \geq \frac{N}{M^k} - 1.
\]

**Proof.** Consider the integer multiples $n \mathbf{v}$, with $1 \leq n \leq N$, of the vector $\mathbf{v} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k/\mathbb{Z}^k$. If we divide $\mathbb{R}^k/\mathbb{Z}^k$ into $M^k$ cubes of side length $M^{-1}$, then one of these cubes will contain $s \geq N/M^k$ multiples of $\mathbf{v}$. If the integers producing these multiples are $m_1 < m_2 < \ldots < m_s$, then we have
\[
\{n \leq N : \|n\lambda\| \leq M^{-1} \text{ for all } \lambda \in \Lambda\} \supset \{m_2 - m_1, m_3 - m_1, \ldots, m_s - m_1\},
\]
and the cardinality of the right-hand side is at least $N/M^k - 1$. \quad \Box

**Proposition 3.8.** Fix $\varepsilon > 0$ sufficiently small, and let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ be two functions such that $f$ is minorized by a large enough constant, and such that $(\varepsilon^{-1} \log q)/\phi(q) \leq g(q) \leq \log q$. If GRH is true, then for each sufficiently large $q$ there exists $x_q$ satisfying
\[
\log_2 x_q \asymp \varepsilon \phi(q) f(q) g(q) \left(1 + \frac{\log(f(q))}{\log q}\right)
\]
with the property that
\[
\frac{G(x_q, q)}{x_q} \geq \left(\frac{1}{4} - 2\varepsilon\right) \frac{g(q)(\log(q^2 f(q))^2)}{\log q}.
\]

Under the slightly weaker assumption that GRH($\chi$) is true for every nonprincipal character $\chi$, the same statement holds with $G(x_q, q)$ replaced by $V_\chi(x_q, q)$. \quad \Box

**Proof.** We apply Lemma 3.2 with $w(q) = (\log_2 q)^{-1}$. We deduce the existence of a set $\mathcal{G}_q$ of characters modulo $q$ for which $|\mathcal{G}_q| \geq \phi(q)(1 - K(\log_2 q)^{-2})$, where $K > 0$ is an absolute constant, such that $\log q_{\chi} = \log q + O((\log_2 q)^2)$ for each $\chi \in \mathcal{G}_q$. We can assume that all elements of $\mathcal{G}_q$ are complex characters, since there are $\ll 2^{\omega(q)} \ll \sqrt{q}$ real characters.
modulo \(q\). Since \(q_{\chi} = q^\chi\), we can also assume that \(G_q\) has the property (17). We extract a subset \(F_q \subset G_q\) of cardinality

\[
\Phi_q = 2\left\lfloor \frac{\phi(q)g(q)}{2\log q}(1 - K(\log_2 q)^{-2}) \right\rfloor
\]

for which (17) holds, where the right-hand side is at least 2 when \(q\) is sufficiently large. By Lemma 3.2, the error term (25) then satisfies the bound

\[
E_q \ll \Phi_q(\log_2 q)^2.
\]

We now apply Lemma 3.7 to the set \(\Lambda = \{0 \leq \gamma \chi \leq T, \chi \in F_q\}\). For \(T\) large enough, it follows from equation (20) that the set \(S\) of values of \(n \leq N\) for which \(\|n\delta^\gamma \chi /2\pi\| < (2\pi M)^{-1}\) has at least \(NM^{-1 + o(1)}\Phi_qT\log(qT)/\pi - 1\) elements. Taking \(N = M^{\Phi_qT\log(qT)/\pi}\), we obtain that \(S \cap [N^{1/2}, N] \neq \emptyset\). Then, Lemma 3.6 implies that for \(y = \delta(n + 1)\) with \(n \in S\),

\[
S_\delta(y) = -\frac{e^{\frac{1}{2}}\Phi_q \log(q^2\delta^{-1})}{2} + O\left(e^{\frac{1}{2}}\Phi_q \left(\frac{\log(qT)}{\delta T} + (\log_2 q)^2 \log(\delta T) + \frac{y \log q}{e^\frac{1}{2}} + \frac{\log(qT)}{M} + \delta^\frac{1}{2} \log(qT) + \frac{\log q}{\log_2 q}\right)\right).
\]

If \(C = C(\varepsilon) > 0\) is large enough, then picking \(T = C/\delta\), \(M = C \log T\) and \(0 < \delta < C^{-2}\) will result, for \(q\) and \(y\) large enough, in the bound

\[
S_\delta(y) \leq -e^{\frac{1}{2}}\Phi_q \log(q^2\delta^{-1}) \left(\frac{1 - 3\varepsilon}{4}\right)^\frac{1}{2}.
\]

Now \(S_\delta(y)\) is the average of the function \(\sum_{\chi \in F_q} (\theta(e^t, \chi) - 1_{\chi = \chi_0} e^t)\) over the short interval \(e^t \in [e^{y-\delta}, e^{y+\delta}]\), and hence this function itself has such a large negative value in that interval. In other words, there exists a value \(x = e^y(1 + O(\delta))\) such that

\[
\sum_{\chi \in F_q} (\theta(x, \chi) - 1_{\chi = \chi_0} x) \geq -x\frac{1}{2} \Phi_q \log(q^2\delta^{-1}) \left(\frac{1 - 3\varepsilon}{4}\right)^\frac{1}{2},
\]

since \(C\) is large enough in terms of \(\varepsilon\). Using positivity in (11) and applying the Cauchy-Schwarz inequality, we obtain that

\[
G(x; q) \geq \frac{1}{\phi(q)} \sum_{\chi \in F_q} |\theta(x, \chi) - 1_{\chi = \chi_0} x|^2 \geq \frac{1}{\phi(q)\Phi_q} \sum_{\chi \in F_q} (\theta(x, \chi) - 1_{\chi = \chi_0} x)^2 \geq x\frac{\Phi_q}{\phi(q)} (\log(q^2\delta^{-1}))^2 \left(\frac{1 - 2\varepsilon}{4}\right).
\]

Since \(y = \delta(n + 1)\) with \(n \in [N^{1/2}, N]\), it follows that the associated \(x\) satisfies

\[
\log_2 x \approx _\varepsilon \delta^{-1} \log_2 (\delta^{-1})\Phi_q \log(q\delta^{-1}).
\]

The result follows from taking \(\delta = f(q)^{-1} \log_2 f(q)\).

The proof is identical for \(V_\Lambda(x; q)\). \(\square\)

We are now ready to prove our main technical theorem, at which point we will be able to deduce Theorems 1.1, 1.2, 1.4, and 1.5.

**Theorem 3.9.** Assume GRH, and fix \(\varepsilon > 0\) small enough.
(a) If $h(x)$ is an increasing function satisfying
\[ \varepsilon \frac{\log_2 x}{\log_3 x} \leq h(x) \leq (\log x)^2 \tag{32} \]
for all $x \geq e^3$, then for all moduli $q$ there exist associated values $x_q$ satisfying
\[ h(\exp((\log x_q)\varepsilon^{-1})) \leq \phi(q) \leq h(\exp((\log x_q)\varepsilon^{-1})) \]
such that
\[ G(x_q; q) \gg \varepsilon x_q \log q \cdot \frac{\log_2 x_q}{\phi(q)}. \]

(b) If $h(x)$ is a function with the property (3) and satisfying
\[ h(x) \leq \varepsilon \frac{\log_2 x}{\log_3 x} \tag{33} \]
for all $x \geq e^3$, then for all sufficiently large moduli $q$ there exist associated values $x_q$ satisfying $\phi(q) \asymp \varepsilon h(x_q)$ such that
\[ G(x_q; q) \geq \left( \frac{1}{4} - \varepsilon \right)x_q(\log q + \log_3 x_q)^2. \]

These results hold with $V_{\Lambda}(x_q; q)$ in place of $G(x_q; q)$, under the weaker assumption that GRH($\chi$) is true for every nonprincipal character $\chi$.

Proof. Under the condition (32), we apply Proposition 3.8 with $f(q)$ equal to a sufficiently large absolute constant, and with $g(q) = \varepsilon \log_2(h^{-1}(\phi(q)))/(2\phi(q))$. Note that the inequality $h(\exp(q^{-1}\phi(q))) \geq \phi(q)$ holds for $q$ large enough, and thus $g(q) \leq \log q$. Moreover, $h(\exp(q^{-2})) \leq q^2 \leq \phi(q)$, and hence $g(q) \geq (\varepsilon^{-1}\log q)/\phi(q)$. Therefore, the hypotheses of Proposition 3.8 are satisfied. We deduce the existence of a sequence $\{x_q\}_{q \geq q_0}$ such that $\log_2 x_q \asymp \phi(q)g(q)$ and
\[ G(x_q; q) \gg g(q) \log q \asymp \frac{\log_2 x_q \cdot \log q}{\phi(q)}, \]
establishing part (a).

On the other hand, under the conditions (3) and (33), we make the choices $g(q) = \log q$ and
\[ f(q) := \begin{cases} \log_2(h^{-1}(\phi(q))), & \text{if } \log_2(h^{-1}(\phi(q))) \leq q^2, \\ \frac{\phi(q) \log q}{\log_2(h^{-1}(\phi(q)))}, & \text{otherwise}. \end{cases} \]
The condition (33) ensures that $f(q)$ is minorized by a large enough positive constant when $q$ is sufficiently large. Moreover, one can check that
\[ \phi(q)f(q) \log q \cdot \left(1 + \frac{\log(f(q))}{\log q}\right) \asymp \log_2(h^{-1}(\phi(q))). \]
Hence, Proposition 3.8 (applied with $\frac{\xi}{x}$ in place of $2\varepsilon$) yields a real number $x_q$ satisfying $\log_2 x_q \simeq \log_2 (h^{-1}(\phi(q)))$ with the property that if $\log_2 (h^{-1}(\phi(q))) > q^2$,
\[
\frac{G(x_q; q)}{x_q} \geq \left( \frac{1}{4} - \frac{\varepsilon}{2} \right) \left( \log (q^2 f(q)) \right)^2
\geq \left( \frac{1}{4} - \frac{\varepsilon}{2} \right) \left( \log \left( \frac{q \log_2 (h^{-1}(\phi(q)))}{\log_3 (h^{-1}(\phi(q)))} \right) \right)^2 \geq \left( \frac{1}{4} - \varepsilon \right) \left( \log \left( \frac{q \log_2 (h^{-1}(\phi(q)))}{\log_3 (h^{-1}(\phi(q)))} \right) \right)^2,
\]
when $q$ is sufficiently large, and similarly when $\log_2 (h^{-1}(\phi(q))) \leq q^2$. Moreover, the estimate $\log_2 x_q \simeq \log_2 (h^{-1}(\phi(q)))$ combined with the property (3) implies that $\phi(q) \simeq h(x_q)$, establishing part (b).

Proof of Theorems 1.1 and 1.2. We prove Theorem 1.2 which implies Theorem 1.1. If we assume that GRH is false, then the desired result for $G(x; q)$ follows from Proposition 2.3. On the other hand, if we assume that GRH holds, then the desired result follows from applying Theorem 3.9, which holds for all moduli $q$, and then restricting to the positive proportion of moduli $q$ that satisfy $\phi(q) \geq \frac{1}{2} q$, say. (The constant $\frac{1}{2}$ is unimportant here; any constant less than 1 suffices, since we know [Sch2, Theorem 1, §8] (see also [Sch1, Section 5]) that the limiting distribution function $\phi(q)/q$ is strictly increasing on $(0, 1)$.)

The proof is similar for $V_\Lambda(x; q)$, and the Riemann hypothesis for principal characters $\chi_0$ is never needed (see the formulas (11)). □

Proof of Theorem 1.4. If GRH($\chi$) is false, then the desired result for $\theta(x, \chi)$ follows from Proposition 2.4. If GRH($\chi$) is true, then we argue analogously to the proof of Proposition 3.8.

Take $F_q = \{ \chi, \overline{\chi} \}$ in Lemma 3.6, as well as $T = C/\delta$, $M = C \log T$ and $\delta < C^{-2}$ with $C$ large enough. Take moreover $N = M T \log(qT)/\pi$ in Lemma 3.7. Hence, there exists $n \in [N^{\frac{1}{4}}, N]$ such that $y = (n + 1)\delta$ has the property that
\[
S_\delta(y) \leq -\frac{1}{2} e^{\frac{\pi}{4}} \log(\delta^{-1}).
\]

Since $\log_2 x = \log y \simeq q^2 \log(\delta^{-1}) \log_2 (\delta^{-1})$, we have that $e^{\frac{\pi}{4}} \log(\delta^{-1}) \simeq q \log_2 x$ and the lower bound (7) follows. The proof of the lower bound (8) is similar, this time taking $F_q$ to be the set of all characters modulo $q$ and applying equation (30). □

Proof of Theorem 1.5. If $Q(x)$ is bounded, and thus is eventually constant $Q(x) = Q_0$, then the result follows from the bound
\[
\sum_{Q_0 < q \leq 2Q_0} G(x; q) \geq G(x; 2Q_0)
\]
and Theorem 1.4.

We now assume that $Q(x)$ tends to infinity. If GRH is false, we let $\chi_e \mod q_e$ be a primitive character for which $L(s, \chi_e)$ has a non-trivial zero off the critical line. Then, for $x$ large enough the interval $(Q(x), 2Q(x)]$ will contain a multiple $q_j$ of $q_e$. Hence, if $\chi_j \mod q_j$ is the character induced by $\chi_e$, we have that
\[
\sum_{Q(x) < q \leq 2Q(x)} G(x; q) \geq \frac{1}{\phi(q)} \left( |\theta(x, \chi_j) - \delta|_{\chi_j = \chi_0} |^2 \right),
\]
and the rest of the proof proceeds as in the proof of Proposition 2.2.
If GRH is true, then we argue as in the proof of Proposition 3.8. We apply Lemma 3.7 to the set \( \Lambda = \{ 0 \leq \gamma \leq T : \chi \mod q, Q < q \leq 2Q \} \). Taking \( N = M^{2Q^2\log(QT)/\pi} \), we see that the set \( S \) of values of \( n \leq N \) for which \( \| n\delta_q \gamma / 2\pi \| < (2\pi M)^{-1} \) has at least one element exceeding \( N^{\frac{1}{2}} \). Then we set \( T = C/\delta_q, M = C \log T \) and \( \delta_q \leq C^{-2} \) with \( C \) large enough in Lemma 3.6, and obtain that for \( y = \delta_q (n + 1) \) with \( n \in S \),

\[
S_{\delta_q}(y) \leq -\left( \frac{1}{2} - \varepsilon \right) e^{\frac{y}{2}} \phi(q) \log(q^2\delta_q^{-1}).
\]

Hence, as in (30), for each large enough \( Q \) there exists \( x_Q \) such that

\[
\frac{1}{Q} \sum_{Q < q \leq 2Q} G(x_Q; q) \gg \frac{x_Q}{Q} \sum_{Q < q \leq 2Q} (\log(q^2\delta_q^{-1}))^2 \gg x_Q (\log(Q^2\delta_q^{-1}))^2,
\]

and for which

\[
\log x_Q \asymp Q^2\delta_q^{-1} \log(Q\delta_q^{-1}).
\]

The rest of the proof proceeds as in the proof of Theorem 3.9. \( \square \)

Acknowledgements

The work of the first author was supported at the University of Ottawa by an NSERC discovery grant and at the Institut Mathématique de Jussieu by a postdoctoral fellowship from the Fondation Sciences Mathématiques de Paris, and was initiated at the University of Michigan. The second author was supported in part by a Natural Sciences and Engineering Research Council of Canada Discovery Grant. We would like to thank Régis de la Bretèche for helpful discussions and for his comments.

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