On the asymptotically discretely self-similar solutions of the Navier-Stokes and the Euler equations

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Abstract

We study scenarios of self-similar type blow-up for the incompressible Navier-Stokes and the Euler equations. The previous notions of the discretely (backward) self-similar solution and the asymptotically self-similar solution are generalized to the locally asymptotically discretely self-similar solution. We prove that there exists no such locally asymptotically discretely self-similar blow-up for the 3D Navier-Stokes equations if the blow-up profile is a time periodic function belonging to $C^1(\mathbb{R}; L^3(\mathbb{R}^3) \cap C^2(\mathbb{R}^3))$. For the 3D Euler equations we show that the scenario of asymptotically discretely self-similar blow-up is excluded if the blow-up profile satisfies suitable integrability conditions.

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1 The main theorems

1.1 Navier-Stokes equations

We consider the Cauchy problem of the 3D Navier-stokes equations.

\begin{align*}
\partial_t v + v \cdot \nabla v &= -\nabla p + \Delta v, \\
\text{div } v &= 0, \\
v(x, 0) &= v_0(x),
\end{align*}

where $v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t))$ is the velocity, $p = p(x, t)$ is the pressure, and $v_0(x)$ is the initial data satisfying $\text{div} v_0 = 0$. We study the possibility of finite

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time blow-up of smooth solution of (NS). By translation of time we may assume that the solution is smooth for \( t < 0 \), and the blow-up happens at \( t = 0 \). We say a solution \( v(x, t) \) to (NS) is a (backward) self-similar blowing up solution at \( t = 0 \) if there exists \((V, P)\) such that

\[
  v(x, t) = \frac{1}{\sqrt{-t}} V\left(\frac{x}{\sqrt{-t}}\right), \quad p(x, t) = -\frac{1}{t} P\left(\frac{x}{\sqrt{-t}}\right), \quad (1.1)
\]

For such solution we have \( \lambda v(\lambda x, \lambda^2 t) = v(x, t) \) for all \( \lambda \in \mathbb{R} \) and for all \((x, t) \in \mathbb{R}^3 \times (-\infty, 0)\). The nonexistence of nontrivial solution given by (1.6) was shown by Nečas-Rážička-Šverák[14], and Tsai[17](see also [13]). It can also be deduced from the result of [8]. These results are generalized by introduction of the more general notion of the asymptotically self-similar blow-up solutions, and their exclusion in [2, 4, 1], which was motivated by earlier study of asymptotically self-similar solutions for the nonlinear heat equation by Giga and Kohn[9, 10](see also [12]). A different notion of the asymptotically self-similar blow-up solutions, and their exclusion in [8] can also be deduced from the result of [15], as is shown in the next section.

**Theorem 1.1.** Let \( \varepsilon > 0 \). If \( v \in C(-\varepsilon, 0; L^3(\mathbb{R}^3)) \) is a solution to (NS), which blows up at \( t = 0 \), then \( t = 0 \) is not a time for discretely self-similar blow up.

Next we consider more general possibility of locally asymptotically discretely self-similar blow-up. Let \( \varepsilon > 0, q \in [1, \infty) \), and \( v \in C(-\varepsilon, 0; L^q_{\text{loc}}(\mathbb{R}^3)) \) be a solution of (NS), which blows up at \( t = 0 \). We say that \((x, t) = (0, 0)\) is a space-time point of locally asymptotically discretely self-similar blow-up in the sense of \( L^3 \) if there exist \( R > 0 \) and a solenoidal vector field \( \mathbf{V}(y, s) \in C(\mathbb{R}; L^q(\mathbb{R}^3)) \) with \( \mathbf{V}(y, s) = \mathbf{V}(y, s + S_0) \) for some \( S_0 \neq 0 \) and for all \((y, s) \in \mathbb{R}^3 \times \mathbb{R}\) such that

\[
  \lim_{t \to 0} \sup_{t < \tau < 0} \left\| v(\cdot, \tau) - \frac{1}{\sqrt{-\tau}} \mathbf{V}\left(\frac{\cdot}{\sqrt{-\tau}}, - \log(-\tau)\right)\right\|_{L^q(B(0, R\sqrt{-t}))} = 0. \quad (1.4)
\]
Note that this is much more general notion than the discrete self-similarity, which corresponds to the equality inside of the norm in (1.4) for all \( R > 0 \). The following is a generalization of Theorem 1.2 of [1] and Theorem 1.1 above.

**Theorem 1.2.** Let \( \varepsilon > 0 \), and \( v \in C(-\varepsilon, 0; L^3_{\text{loc}}(\mathbb{R}^3)) \) be a solution to (NS), which blows up at \( t = 0 \), then it is not a time for locally asymptotically discretely self-similar blow up in the sense of \( L^q(\mathbb{R}^3) \) if \( q \in [2, \infty] \), and the blow-up profile satisfies
\[
V \in C(\mathbb{R}; L^3(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)).
\]

### 1.2 The Euler equations

Here we study the possibility of existence of self-similar type blow-up for a solution to the 3D Euler equations.

\[
(E) \begin{cases}
\partial_t v + v \cdot \nabla v = -\nabla p, \\
\text{div } v = 0, \\
v(x, 0) = v_0(x)
\end{cases}
\]

We say a solution \( v \) is a self-similar blowing up solution at \( t = 0 \) if there exists \((V, P)\) such that
\[
v(x, t) = \frac{1}{(-t)^{\frac{1}{1+\alpha}}} V \left( \frac{x}{(-t)^{\frac{1}{1+\alpha}}} \right), \quad p(x, t) = \frac{1}{(-t)^{\frac{2\alpha}{1+\alpha}}} P \left( \frac{x}{(-t)^{\frac{1}{1+\alpha}}} \right).
\]

For such \((v, p)\) we have the following scaling invariance \( v(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t) \) and \( p(x, t) = \lambda^{2\alpha} v(\lambda x, \lambda^{\alpha+1} t) \) for all \( \lambda > 0, \alpha \in \mathbb{R} \), and for all \((x, t) \in \mathbb{R}^3 \times (-\infty, 0) \). The nonexistence of nontrivial self-similar blowing up solution under suitable assumption on the blow-up profile \( V \) was obtained in [3, 5]. A discretely self-similar solution \( v \) is a solenoidal vector field, for which there exist \( \lambda > 0, \alpha > -1 \) such that \( v(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t) \) \( \forall (x, t) \in \mathbb{R}^3 \times (-\infty, 0) \). If we represent \((v, p)\) by
\[
v(x, t) = \frac{1}{(-t)^{\frac{1}{1+\alpha}}} V(y, s), \quad p(x, t) = \frac{1}{(-t)^{\frac{2\alpha}{1+\alpha}}} P(y, s),
\]
where \( y = x/(-t)^{\frac{1}{1+\alpha}} \) and \( s = -\log(-t) \). then the discrete self-similarity of the solution \((v, p)\) is equivalent to that \((V, P)\) is a solution to
\[
\begin{cases}
V_s + \frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (y \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \\
\text{div } V = 0,
\end{cases}
\]
which satisfies \( V(y, s) = V(y, s + S_0) \) for all \((y, s) \in \mathbb{R}^{3+1}\) with \( S_0 = -(\alpha + 1) \log \lambda \). The following result on the nonexistence of nontrivial discretely self-similar blow-up for the 3D Euler equations is proved in [6].

**Theorem 1.3.** Let \( V(y, s) \in C^1(\mathbb{R}^{3+1}) \) be a time periodic solution of (1.7) with period \( S_0 > 0 \).
(i) Let $V$ satisfies either one of the following conditions:

(a) $V(y, s) \in C^2_y C^1_t (\mathbb{R}^{3+1})$, and

$$
\Omega := \text{curl} V \in \bigcap_{r > 0} \cap_{0 < q < r} L^q(\mathbb{R}^{3+1} \times [0, S_0]).
$$

(b) $\lim_{|y| \to \infty} \sup_{0 < s < S_0} \{ |y| |V(y, s)| + |\nabla V(y, s)| \} = 0$, and $\Omega \in L^q(\mathbb{R}^3 \times [0, S_0])$.

Then, $\Omega = 0$. If we further assume $\lim_{|y| \to \infty} V(y, s) = 0 \forall s \in [0, S_0)$, then $V = 0$.

(ii) Let $-1 < \alpha \leq 3/p$ or $3/2 < \alpha < \infty$, and $V \in L^p(\mathbb{R}^3 \times [0, S_0])$ for some $3 \leq p \leq \infty$. We also assume that the pressure is given by $-\Delta_y P(\cdot, s) = \sum_{i,j} \partial_i \partial_j (V_i V_j(\cdot, s))$. Then, $V = 0$.

In the next section we will prove following theorem, which is an extension of the part (i)(b) of the above theorem.

**Theorem 1.4.** Let $V(y, s) \in C^1(\mathbb{R}^{3+1})$ be a time periodic solution of (1.7) with period $S_0 > 0$ satisfying

$$
\lim_{|y| \to \infty} \sup_{0 < s < S_0} \{ |y| |V(y, s)| + |\nabla V(y, s)| \} = 0,
$$

and

$$
\int_0^{S_0} \int_{\mathbb{R}^3} |\Omega(y, s)|^q (1 + |y|)^{\eta} dyds < \infty,
$$

where $q \in (0, \frac{3+\eta}{1+\alpha})$ with $\alpha > -1, \eta > -3$. Then, $\Omega = 0$. Therefore if we further assume $\lim_{|y| \to \infty} V(y, s) = 0 \forall s \in [0, S_0)$, then $V = 0$.

Note that the case of $\eta = 0$ of the above theorem corresponds to the part (i)(b) of Theorem 1.3. We now introduce more general notion of the asymptotically discretely self-similar blow-up for the 3D incompressible Euler equations. Let $v \in C^1(\mathbb{R}^{3+1})$ be a solution to (E), which blows up at $t = 0$. We say that $t = 0$ is a time for *asymptotically discretely self-similar blow-up* if there exist $\alpha > -1$ and a solenoidal vector field $\nabla \in C^1(\mathbb{R}^{3+1})$, which is a time periodic solution to (1.7), and satisfies the following convergence conditions:

$$
\lim_{t \to 0} (-t) \left\| \nabla y \left. \frac{1}{t} \nabla \nabla \left( \frac{1}{(t)^{\frac{\eta}{\alpha + 1}}}, -\log(-t) \right) \right\|_{L^\infty} = 0,
$$

and there exists $\epsilon_0 > 0$ such that

$$
\sup_{-\epsilon_0 < t < 0} (-t)^{\frac{\eta - \alpha}{\alpha + 1}} \left\| v(\cdot, t) - \frac{1}{(t)^{\frac{\eta}{\alpha + 1}}} \nabla \left( \frac{1}{(t)^{\frac{\eta}{\alpha + 1}}}, -\log(-t) \right) \right\|_{L^1} < \infty.
$$

The following is a generalization of the corresponding results in [1] [2].

**Theorem 1.5.** Let $T > 0$ and $v \in C([-T, 0); H^m(\mathbb{R}))$, $m > 5/2$, be a classical solution to (E), which blows up at $t = 0$. Then, $t = 0$ is not a time for asymptotically discretely self-similar blow-up if the blow-up profile $\nabla \in C^1(\mathbb{R}^{3+1})$ satisfies one of the conditions of (i)-(ii) of Theorem 1.3, or the condition of Theorem 1.4.
2 Proof of the main theorems

Proof of Theorem 1.1 We recall the main result in [15], saying that $t = 0$ is the blow-up time for the solution $v \in C(-\varepsilon, 0; L^3(\mathbb{R}^3))$ only if
\[
\lim_{t \to 0} \|v(\cdot, t)\|_{L^3} = \infty. \tag{2.1}
\]
By the discrete self-similarity there exists $0 < \lambda \neq 1$ such that
\[
\lambda^k v(\lambda^k x, \lambda^{2k} t) = v(x, t) \quad \forall k \in \mathbb{Z}.
\]
Hence,
\[
\|v(\cdot, \lambda^{2k} t)\|_{L^3} = \|v(\cdot, t)\|_{L^3} \quad \forall t \in (-\infty, 0). \tag{2.2}
\]
Passing $k \to \infty$ if $\lambda \in (0, 1)$, while passing $k \to -\infty$ if $\lambda \in (1, \infty)$, one has for $t_k = \lambda^k t$,
\[
\|v(\cdot, t)\|_{L^3} = \lim_{t_k \to 0} \|v(\cdot, t_k)\|_{L^3} = \infty \quad \forall t \in (-\infty, 0),
\]
which is a contradiction to the fact that $v \in C(-\infty, 0; L^3(\mathbb{R}^3))$. \(\Box\)

For the proof of Theorem 1.2 we recall the following result, which is Theorem 1.1 of [11].

Lemma 2.1. Let $q \in [3/2, \infty]$. Suppose $v$ is a suitable weak solution of (NS) in a cylinder, say $Q = B(0, r_1) \times (-r_1^2, 0)$ for some $r_1 > 0$. Then, there exists a constant $\eta = \eta(q) > 0$ such that if
\[
\limsup_{r \downarrow 0} \left\{ r^{\frac{q-3}{q}} \sup_{-r^2 < t < 0} \|v(\cdot, t)\|_{L^q(B(0, r))} \right\} \leq \eta, \tag{2.3}
\]
then $v$ is Hölder continuous both in space and time variables near $(0, 0) \in \mathbb{R}^{3+1}$.

Proof of Theorem 1.2 We use the self-similar variables $y = x/\sqrt{-t}$, $s = -\log(-t)$, and transform $(v, p) \to (V, P)$ as previously,
\[
v(x, t) = \frac{1}{\sqrt{-t}} V \left( \frac{x}{\sqrt{-t}}, -\log(-t) \right), \quad p(x, t) = -\frac{1}{t} P \left( \frac{x}{\sqrt{-t}}, -\log(-t) \right). \tag{2.4}
\]
Then, substituting $(v, p)$ into (NS), we find that $(V, P)$ satisfies (1.3). We observe that the condition (1.4) for some $R \in (0, \infty)$ is equivalent to
\[
\lim_{t \to 0} (-t)^{\frac{q-3}{2q}} \sup_{t < \tau < 0} \left\|v(\cdot, \tau) - \frac{1}{\sqrt{-\tau}} V \left( \frac{\cdot}{\sqrt{-\tau}}, -\log(-\tau) \right) \right\|_{L^q(B(0, R\sqrt{-\tau}))} = 0 \tag{2.5}
\]
for all $R \in (0, \infty)$(see e.g. the argument in the proof of Theorem 1.5, pp. 446, [2]), which implies, in terms of $V$, that
\[
\lim_{s \to \infty} \|V(\cdot, s) - V(\cdot, s)\|_{L^q(B(0, R))} = 0 \tag{2.6}
\]
Therefore there exists $\bar{\phi}$ and (2.8). Integrating by parts in (2.7) and (2.8) for all $\psi \in C_c^1(\mathbb{R}^3)$, and then we integrate by parts to obtain:

$$
\begin{align*}
- \int_0^{S_0} \int_{\mathbb{R}^3} \xi_s(s)\phi(y) \cdot V(y, s + n) + \frac{1}{2} \int_0^{S_0} \int_{\mathbb{R}^3} \xi(s) V(y, s + n) \cdot \phi(y) \, dy \, ds

\quad + \frac{1}{2} \int_0^{S_0} \int_{\mathbb{R}^3} \xi(s) V(y, s + n) \cdot (y \cdot \nabla)\phi(y) \, dy \, ds

\quad - \int_0^{S_0} \int_{\mathbb{R}^3} \xi(s) [V(y, s + n) \cdot (V(y, s + n) \cdot \nabla)\phi(y)] \, dy \ \, ds

= \int_0^{S_0} \int_{\mathbb{R}^3} \xi(s) V(y, s + n) \cdot \Delta\phi(y) \, dy \, ds
\end{align*}
$$

(2.7)

Similarly we multiply the second equations of (1.3) by $\xi(s - n)\psi(y)$, and integrate it over $\mathbb{R}^3 \times [n, n + S_0]$, we have

$$
\int_0^{S_0} \int_{\mathbb{R}^3} V(s + n) \cdot \nabla\psi(y)\xi(s) \, dy = 0.
$$

(2.8)

Passing to the limit $n \to \infty$ in (2.7)-(2.8), and observing that $V(\cdot, s + n) \to \bar{V}(\cdot, s)$ in $L^q_{\text{loc}}(\mathbb{R}^3) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3)$ for $q \geq 2$, we find that $\bar{V} \in C^1_c(\mathbb{R}^3)$ satisfies (2.7) and (2.8). Integrating by parts in (2.7) and (2.8) for $\bar{V}$ respectively, we obtain

$$
\int_{\mathbb{R}^3} \int_0^{S_0} \left[ \nabla_s + \frac{1}{2} \nabla + \frac{1}{2} (y \cdot \nabla) \nabla + (\nabla \cdot \nabla) \nabla - \Delta \nabla \right] \cdot \phi(y) \xi(s) \, dy \, dt = 0
$$

(2.9)

for all vector test function $\phi \in C^1_c(\mathbb{R}^3)$ with $\text{div} \ \phi = 0$, and $\xi \in C^1_{0}(0, S_0)$, and also for all $\psi \in C^2(\mathbb{R}^3)$ we have

$$
\int_0^{S_0} \int_{\mathbb{R}^3} [\text{div} \ \bar{V}] \psi(y)\xi(s) \, dy \, ds = 0.
$$

(2.10)

Therefore there exists $\bar{P} \in C^1(\mathbb{R}^3 \times [0, S_0])$ such that

$$
\nabla \bar{s} + \frac{1}{2} \nabla + \frac{1}{2} (y \cdot \nabla) \nabla + (\nabla \cdot \nabla) \nabla - \Delta \nabla = -\nabla \bar{P}, \quad \text{div} \ \bar{V} = 0.
$$

Since $\bar{V}$ is a $C^1(\mathbb{R}^3; L^3(\mathbb{R}^3) \cap C^2(\mathbb{R}^3))$ solution of (1.3) satisfying the periodic condition, $\bar{V}(y, s) = \bar{V}(y, s + S_0)$, we find that $\bar{V} = 0$ by Theorem 1.1. Therefore, the assumption (1.4) is reduced to

$$
\lim_{t \to 0} \left\{ (-t)^{\frac{q-3}{q}} \sup_{t \leq \tau < 0} \| v(\cdot, \tau) \|_{L^q(\mathbb{R}^3 \setminus B(\sqrt{-t}))} \right\} = 0
$$

(2.11)

for all $R \in (0, \infty)$. We set $R = 1$ and $\sqrt{-t} = r$ in (2.11), then we obtain

$$
\lim_{r \downarrow 0} \left\{ r^{\frac{q-3}{q}} \sup_{-r^2 \leq \tau < 0} \| v(\cdot, \tau) \|_{L^q(B(0, r))} \right\} = 0.
$$

(2.12)
Applying Lemma 2.1, we find that the space-time point \((0, 0)\) is not a blow-up space-time point. □

In order to prove Theorem 1.3 we recall the following blow-up criterion for the Euler equations, which corresponds to Lemma 2.1 for the Navier-Stokes equations.

**Lemma 2.2.** Let \(m > 5/2, \varepsilon > 0\) and \(v \in C((-\varepsilon, 0); H^m(\mathbb{R}^3))\) be a classical solution to \((E)\). Suppose the following inequality holds

\[
\limsup_{t \uparrow 0} (-t) \| \nabla v(t) \|_{L^\infty} < 1,
\]

then there exists no blow-up at \(t = 0\).

For the proof see the proof of Theorem 1.1 in \(\Pi\) (see also \([12]\)).

**Proof of Theorem 1.3** We transform the solution \((v, p) \rightarrow (V, P)\) as in \((1.6)\). Then, \((V, P)\) satisfies \((1.7)\). The convergence conditions \((1.9)-(1.10)\) can be written in the self-similar form as

\[
\lim_{s \to \infty} \| \nabla V(\cdot, s) - \nabla V(\cdot, s) \|_{L^\infty} = 0,
\]

and

\[
\sup_{-\log(c_0) < s < \infty} \| V(\cdot, s) - \overline{V}(\cdot, s) \|_{L^1} < \infty
\]

respectively. From \((2.14)\) and \((2.15)\), using the interpolation for the \(L^p\) spaces, we have that

\[
\lim_{s \to \infty} \| V(\cdot, s) - \overline{V}(\cdot, s) \|_{L^2(B_R)} = 0 \quad \forall R > 0,
\]

and repeating the argument of the proof of Theorem 1.2 word by word, we find that \(\overline{V}\) is \(C^1(\mathbb{R}^3 \times (0, T_0))\) solution of \((1.7)\), satisfying the time periodicity and one of the conditions \((a)-(c)\) of Theorem 1.3. Therefore \(\overline{V} = 0\), and the condition \((1.9)\) reduces to

\[
\lim_{t \uparrow 0} (-t) \| \nabla v(\cdot, t) \|_{L^\infty} = 0.
\]

Thanks Lemma 2.2 we can conclude that \(t = 0\) is not a blow-up time. □

**Proof of Theorem 1.4** We consider the vorticity equation of \((1.7)\),

\[
\begin{cases}
\Omega_s + \Omega + \frac{1}{\alpha + 1} (y \cdot \nabla)\Omega + (V \cdot \nabla)\Omega - (\Omega \cdot \nabla)V = 0, \\
\text{div } V = 0, \quad \text{curl } V = \Omega.
\end{cases}
\]

We introduce a cut-off function \(\sigma \in C^\infty_0(\mathbb{R}^N)\) such that

\[
\sigma(|x|) = \begin{cases}
1 & \text{if } |x| < 1, \\
0 & \text{if } |x| > 2,
\end{cases}
\]

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and $0 \leq \sigma(x) \leq 1$ for $1 < |x| < 2$. For each $R > 0$, we define $\sigma_R(x) := \sigma \left( \frac{|x|}{R} \right)$. Given $\rho > 0$, we also define a function $\psi = \psi_\rho(y)$ as follows.

$$\psi_\rho(y) = \begin{cases} 1, & \text{if } |y| > \rho + \pi \\ \sin(|y| - \rho), & \text{if } \rho < |y| \leq \rho + \pi \\ 0, & \text{if } |y| \leq \rho. \end{cases}$$

We take $L^2(\mathbb{R}^3 \times [0, S_0])$ inner product the first equations of (2.17) by $\Omega|\Omega|^{q-2}|y|^q \psi_\rho \sigma_R$, and integrate by part to obtain

$$0 = \left( 1 - \frac{3 + \eta}{q(\alpha + 1)} \right) \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^q \psi_\rho \sigma_R dy ds - \frac{1}{\alpha + 1} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^q (y \cdot \nabla) \psi_\rho \sigma_R dy ds$$

$$- \frac{1}{\alpha + 1} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^q \psi_\rho (y \cdot \nabla) \sigma_R dy ds - \frac{\eta}{q} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^q (y \cdot \nabla) \psi_\rho \sigma_R dy ds$$

$$- \frac{1}{q} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^q (V \cdot \nabla) \psi_\rho \sigma_R dy ds - \frac{1}{q} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^q \psi_\rho (V \cdot \nabla) \sigma_R dy ds$$

$$- \int_0^{S_0} \int_{\mathbb{R}^3} (\Omega \cdot \nabla)V \cdot \Omega |\Omega|^{q-2}|y|^q \psi_\rho \sigma_R dy ds$$

$$:= I_1 + \cdots + I_7.$$ (2.19)

We first observe that for all $\eta \in \mathbb{R}$

$$\int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^q \psi_\rho dy ds \leq \int_0^{S_0} \int_{|y| \geq \rho} |\Omega|^q |y|^q dy ds$$

$$\leq \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q (1 + |y|)^q dy ds < \infty.$$ (2.20)

Therefore we can apply the dominated convergence theorem in the followings.

$$|I_3| \leq \frac{||\nabla \sigma||_{L^\infty}}{R(\alpha + 1)} \int_0^{S_0} \int_{\{R \leq |y| \leq 2R\}} |\Omega|^q |y|^q \psi_\rho |y| dy ds$$

$$\leq \frac{2||\nabla \sigma||_{L^\infty}}{(\alpha + 1)} \int_0^{S_0} \int_{\{R \leq |y| \leq 2R\}} |\Omega|^q |y|^q \psi_\rho dy ds$$

$$\to 0.$$ (2.21)

as $R \to \infty$, and

$$|I_6| \leq \frac{||\nabla \sigma||_{L^\infty}}{qR} \int_0^{S_0} \int_{\{R \leq |y| \leq 2R\}} |\Omega|^q |y|^q \psi_\rho |V| dy ds$$

$$\leq \frac{||\nabla \sigma||_{L^\infty}}{qR} \sup_{R \leq |y| \leq 2R} \frac{||V(y)||}{|y|} \int_0^{S_0} \int_{\{R \leq |y| \leq 2R\}} |\Omega|^q |y|^q \psi_\rho |y| dy ds$$

$$\leq \frac{2||\nabla \sigma||_{L^\infty}}{q} \sup_{R \leq |y| \leq 2R} \frac{||V(y)||}{|y|} \int_0^{S_0} \int_{\{R \leq |y| \leq 2R\}} |\Omega|^q |y|^q \psi_\rho dy ds$$

$$\to 0.$$ (2.22)
as $R \to \infty$. Therefore, passing $R \to \infty$, and applying the dominated convergence theorem to the other terms of (2.19), we find that

$$0 = \left(1 - \frac{3 + \eta}{q(\alpha + 1)}\right) \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} \psi_\rho dyds - \frac{1}{\alpha + 1} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} (y \cdot \nabla) \psi_\rho dyds$$

$$- \frac{\eta}{q} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q V \cdot \frac{y}{|y|} |y|^{\eta-1} \psi_\rho dyds - \frac{1}{q} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} (V \cdot \nabla) \psi_\rho dyds$$

$$- \int_0^{S_0} \int_{\mathbb{R}^3} \Omega \cdot \nabla V \cdot \Omega |y|^{\eta-2} |y|^{\eta} \psi_\rho dyds$$

$$:= J_1 + \cdots + J_5. \quad (2.23)$$

Under our hypothesis we have $J_1 \leq 0$. Since $\psi_\rho(y)$ is radially non-decreasing, we also have $J_2 \leq 0$, and

$$J_2 = - \frac{1}{\alpha + 1} \int_0^{S_0} \int_{\{\rho \leq |y| \leq \rho + \pi\}} |\Omega|^q |y|^{\eta+1} \psi_\rho' dyds. \quad (2.24)$$

We have

$$|J_3| \leq \frac{|\eta|}{q} \sup_{|y| \geq \rho, s \in [0, S_0]} \frac{|V(y, s)|}{|y|} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} \psi_\rho dyds \leq \frac{1}{4} |J_1| \quad (2.25)$$

for sufficiently large $\rho$. We compute

$$|J_4| \leq \frac{1}{q} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} |V| \psi_\rho' dyds$$

$$\leq \frac{1}{q} \sup_{|y| \geq \rho, s \in [0, S_0]} \frac{|V(y, s)|}{|y|} \int_0^{S_0} \int_{\{\rho \leq |y| \leq \rho + \pi\}} |\Omega|^q |y|^{\eta+1} \psi_\rho' dyds$$

$$\leq \frac{1}{2} |J_2| \quad (2.26)$$

for sufficiently large $\rho$. For $J_5$ we obtain

$$|J_5| \leq \int_0^{S_0} \int_{\mathbb{R}^3} \nabla V ||\Omega|^q |y|^{\eta} \psi_\rho dyds$$

$$\leq \sup_{|y| \geq \rho, s \in [0, S_0]} |\nabla V(y, s)| \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} \psi_\rho dyds$$

$$\leq \frac{1}{4} |J_1| \quad (2.27)$$

for sufficiently large $\rho$. Taking into account the estimates (2.25)-(2.27) in (2.23), we find that there exists $\rho_0 > 0$ such that

$$0 \geq \frac{1}{2} \left(\frac{3 + \eta}{q(\alpha + 1)} - 1\right) \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} \psi_\rho dyds$$

$$+ \frac{1}{2(\alpha + 1)} \int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q |y|^{\eta} (y \cdot \nabla) \psi_\rho dyds \quad (2.28)$$

for all $\rho \geq \rho_0$. Hence, $\Omega(y, s) = 0$ for all $(y, s) \in \{y \in \mathbb{R}^3 | |y| > \rho_0\} \times [0, S_0]$. Applying Theorem 1.3 (i)(a), we conclude $\Omega = 0$. \(\square\)
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