Weak Epigraphical Solutions to Hamilton-Jacobi-Bellman Equations on Infinite Horizon

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Abstract

In this paper we show a uniqueness result for weak epigraphical solutions of Hamilton-Jacobi-Bellman (HJB) equations on infinite horizon for a class of lower semicontinuous functions vanishing at infinity. Weak epigraphical solutions of HJB equations, with time-measurable data and fiber-convex, turn out to be viscosity solutions – in the classical sense – whenever they are locally Lipschitz continuous. Here we extend the notion of locally absolutely continuous tubes to set-valued maps with continuous epigraph of locally bounded variations. This new notion fits with the lack of uniform lower bound of the Fenchel transform of the Hamiltonian with respect to the fiber. Controllability assumptions are considered.

Keywords: Weak solutions; Hamilton-Jacobi-Bellman equations; Locally bounded variations set-valued maps; Representation results.

MSC: 70H20 · 49L25 · 49J24.

1. Introduction

Since the 80s the investigation of existence and uniqueness of weak solutions to first-order partial differential equations on finite/infinite horizon was carried out by the meaning of viscosity solutions in the pioneer works of Crandall, Evans, Barles and Lions (3, 9, 10). Such weak solutions – also called viscosity solutions – known in the context of control theory, calculus of variations, mean field games, etc... focus on the use of super-sub/solutions. Advances have been made in the direction of HJB equations with autonomous Hamiltonian, by Souganidis and Ishii (16, 17, 21). The investigation of weak solutions of HJB equations when the Hamiltonian is only time-measurable
has become increasingly fundamental due to its applications in applied sciences as in macroeconomy and engineering, although the classical notion of weak solution is unsatisfactory and challenging to manage. In fact, the value function, which is a viscosity solution of the HJB equation, loses its differentiability property – even in the absence of state constraints – whenever there are several optimal solutions to the same initial datum or when state constraints are imposed. Avoiding the usage of “test” functions, by using geometric arguments, the definition of a weak solution can be equivalently stated in terms of “normals” to the epigraph (and hypograph) of the solution, (cfr. [7, 13, 15] and the references therein), i.e.,

\[
F(t, x, -\mathcal{F}_{\text{graph}} u(t, x, u(t, x))) = \{0\} \quad \text{in } ]0, +\infty[ \times \Omega 
\]

where \(\mathcal{F}_E^-(y)\) stands for negative polar of the Boulingad tangent cone of a set \(E\) at \(y\) and \(\Omega \subset \mathbb{R}^n\) is an open subset (cfr. Section 2 and Definition 3.1 below). We point out that, when the dynamics is time-measurable, this equivalence may not be true.

Nevertheless, the study of the uniqueness of weak solutions can be conducted using non-smooth analysis techniques as in the definition reported in [6]. To deal with Hamiltonians measurable in time, Ishii [7] proposed a new notion of weak solution in the class of continuous functions studying the existence and uniqueness of viscosity solutions of HJB equations for the stationary-evolutionary case in finite horizon and on infinite horizon with free state constraints. In the case of Bellman equations, associated with optimal control problems, it is recognized that the viscosity solutions are represented as the corresponding value function. In the general case, the viscosity solutions of the evolutionary equations of HJB with the fiber-convex Hamiltonian, on infinite horizon with state constraints, are represented as the value function of a suitable optimal control problem. In finite horizon, this point of view has been studied by Ishii [8] for the convex case providing a Hölder continuous representation, and in [9] for Hamiltonians not necessarily convex, but the Lagrangian is simply continuous and the control space is infinite dimensional. On the other hand, in [10] the author constructs a faithful representation, Lipschitz continuous in the state and control. Frankowska et al. [11] studied faithful representations of Hamiltonians measurable in time and their stability, giving clear results on the Lipschitz constants of these representations. In the recent work [4], under weaker hypotheses and assuming the boundedness from above of the Fenchel transform - with respect to the
fiber - of the Hamiltonian on its domain, the author has extended the previous representation result, constructing a faithful epigraphical representation (see the references therein [4] for further discussions).

To deal with the study of weak solutions in terms of their characterization with the value function of a particular optimal control problem, conditions on the Fenchel transform in the fiber of the Hamiltonian are assumed in the previous works. The uniform lower bound assumption on the Fenchel transform of the Hamiltonian with respect to the state and the fiber, allows the use of the known viability theorems for tubes investigated by Frankowska et al. in [14, 15]. Indeed, in this setting (cfr. also [4]), the epigraph of the value function is locally absolutely continuous. However, the assumption of boundedness from below of the Fenchel transform is a not satisfactory condition for many applications, as in mechanics (cfr. [5]). Indeed, in such a context, the epigraph of the value function of Bellman problems on infinite horizon is at least merely continuous, for Lipschitz problem data. Therefore, it is needed to extends the notion of locally absolute continuity property of the epigraph, and so further investigations on viability results have to be considered. For this purpose, we give a viability result for continuous tubes of locally bounded variations (see Section 3 and Definition 3.2 below).

The central result of this paper is a uniqueness theorem, under controllability assumptions, for weak epigraphical solutions vanishing at infinity (see Theorem 3.3 below) of HJB equations (1). By a representation result in [4], each weak epigraphical solution turn out to be an upper and lower weak solution of the value function associated with such representation. Skipping the uniform bound from below of the Fenchel transform, the weak epigraphical solutions are considered merely lower semicontinuous with continuous epigraph of locally bounded variations.

The outline of this paper is as follows. In Section 2 notations and some known results on non-smooth analysis are collected. In Section 3 we give preliminary definitions with the statement of the main result of this paper. The Section 4 is devoted to the proofs.

2. Preliminaries and Notations

\( \mathbb{N} \), |·|, and \( \langle \cdot, \cdot \rangle \) stands for the set of natural numbers, the Euclidean norm, and the scalar product, respectively. Let \( E \subset \mathbb{R}^k \) be a subset and \( x \in \mathbb{R}^k \). The closed ball in \( \mathbb{R}^k \) of radius \( r > 0 \) and centered at \( x \) is denoted by \( B^k(x,r) \) (\( \mathbb{B}^k := B^k(0,1) \) and \( \mathbb{S}^{k-1} := \text{bdr} \ B^k(0,1) \)). cl \( E \), int \( E \), bdr \( E \),
$E^c$, and $co E$ stands, respectively, for the closure, the interior, the boundary, the complement, and the convex hull of $E$. The set $E^- = \{ p \in \mathbb{R}^k \mid \langle p, e \rangle \leq 0 \text{ for all } e \in E \}$ is the negative polar of $E$. $\mu$ denotes the Lebesgue measure.

Consider a closed subset $I \subset \mathbb{R}$, $C \subset \mathbb{R}^k$ non-empty, and $a < b$. We take the following notation:

- $\mathcal{L}^1(I; C) = \{ u : I \to C \text{ Lebesgue integrable} \}$.
- $\mathcal{L}_{loc}^1(I; C) = \{ u : I \to C \mid u \in \mathcal{L}^1(J; C) \forall J \subset I \text{ compact} \}$.
- $\mathcal{L}_{loc} = \{ u \in \mathcal{L}_{loc}^1([0, +\infty[; [0, +\infty[) \mid \lim_{\varepsilon \to 0} \sup_{J \subset [0, +\infty[} \int_J u(s) \, ds = 0 \}$.
- $\mathcal{W}^{1,1}([a, b]; C) = \{ u : [a, b] \to C \text{ absolutely continuous} \}$ endowed with the norm $\| u \|_{\mathcal{W}^{1,1},[a,b]} := |u(a)| + \int_a^b |u'(s)| \, ds$.
- $\mathcal{W}_{loc}^{1,1}([a, +\infty[; C) = \{ u : [a, +\infty[ \to C \mid u \in \mathcal{W}^{1,1}([a, t]; C) \forall t \geq a \}$.

Let $A, B \subset \mathbb{R}^k$ be two non-empty sets and $x, y \in \mathbb{R}^k$. We define the following Euclidean distances: $d(x, y) := |x - y|$, $\text{dist}(x, B) := \inf \{ d(x, b) \mid b \in B \}$, and $\text{dist}(A, B) := \inf \{ d(a, b) \mid a \in A, b \in B \}$. We use the same notation for the distance of a point from a set and between two sets – it does not generate confusion in the context where it is used. We define the so-called excess of $A$ beyond $B$ by

$$\text{exc}(A|B) := \sup \{ \text{dist}(a, B) \mid a \in A \} \in \{ 0, +\infty \} \cup \{ +\infty \}.$$  

We recall the following $\text{exc}(A|B) = \inf \{ \varepsilon > 0 : A \subset B + \varepsilon \mathbb{B} \}$ where $\inf \emptyset = +\infty$, by convention. Moreover, the Pompeiu–Hausdorff distance between $A$ and $B$ is defined by

$$d_{\mathcal{H}}(A, B) := \text{exc}(A|B) \vee \text{exc}(B|A) \in \{ 0, +\infty \} \cup \{ +\infty \}.$$  

Let $X, Y$ be two normed spaces and $D \subset X$ be a non-empty closed set. Continuity properties of a set-valued mapping $S : D \subset X \rightharpoonup Y$ can be defined on the basis of Painlevé-Kuratowski set convergence. For any $\bar{x} \in D$ we define, respectively, the upper an lower limit of $S(x)$ when $x \to \bar{x}$ by

$$\text{Lim sup}_{x \to \bar{x}} S(x) := \{ y \in Y \mid \lim_{x \to \bar{x}^+} d(y, S(x)) = 0 \}$$

$$\text{Lim inf}_{x \to \bar{x}} S(x) := \{ y \in Y \mid \lim_{x \to \bar{x}^+} d(y, S(x)) = 0 \}$$
We say that $S$ is outer semicontinuous (osc) at $\bar{x}$ when $\limsup_{x \to \bar{x}} S(x) \subset S(\bar{x})$ and inner semicontinuous (isc) at $\bar{x}$ when $\liminf_{x \to \bar{x}} S(x) \supset S(\bar{x})$. It is called Painlevé-Kuratowski continuous at $\bar{x} \in D$ when it is both osc and isc at $\bar{x} \in D$. Continuity is taken to refer to Painlevé-Kuratowski continuity, unless otherwise specified. We say that $S$ is continuous if it is continuous at $\bar{x}$ for any $\bar{x} \in D$.

Consider a non-empty subset $E \subset \mathbb{R}^k$ and $x \in \text{cl} E$. The Boulingad tangent cone (or contingent cone) and the Clarke tangent cone to $E$ at $x$ are defined, respectively, by

$$T_E(x) := \limsup_{t \to 0^+} t^{-1} (E - x)$$
$$T^C_E(x) := \liminf_{t \to 0^+, y \to E} t^{-1} (E - y).$$

The limiting normal cone and the regular (or Clarke) normal cone to $E$ at $x$ are defined, respectively, by

$$N_E(x) := \limsup_{y \to E} T_E(y)^{-}$$
$$N^C_E(x) := T^C_E(x)^{-}.$$  

It is known that $T^C_E(x) = N_E(x)^{-} \subset T_E(x)$ and $T^C_E(x)^{-} = \text{cl co} N_E(x)$ whenever $E$ is closed ([19, Chapter 6]).

We denote for any $\eta, r \geq 0$ the sets

$$N_E(x; \eta) := \{ n \in S^{k-1} | n \in \text{cl co} N_E(y), y \in (\text{bdr} E) \cap B^k(x, \eta) \}$$
$$\Sigma_E(x; \eta, r) := \{ p \in \mathbb{R}^k | \forall n \in N_E(x; \eta), \langle p, n \rangle \geq r \}$$
$$\Gamma_E(x; \eta) := \{ p \in \mathbb{R}^k | \exists n \in N_E(x; \eta), \langle p, n \rangle \leq 0 \}.$$

Let $\varphi : \mathbb{R}^k \to \mathbb{R} \cup \{ \pm \infty \}$ be an extended real function. We write dom $\varphi$, epi $\varphi$, hypo $\varphi$, and graph $\varphi$ for the domain, the epigraph, the hypograph, and the graph of $\varphi$, respectively. The Fenchel transform (or conjugate) of $\varphi$, written $\varphi^*$, is the extended real function $\varphi^* : \mathbb{R}^k \to \mathbb{R} \cup \{ \pm \infty \}$ defined by

$$\varphi^*(v) := \sup_{p \in \mathbb{R}^k} \{ \langle v, p \rangle - \varphi(p) \}.$$  

3. The Main Result

Consider a closed non-empty subset $\Omega \subset \mathbb{R}^n$. We focus our analysis on HJB equation (1) where

$$F(t, x, r, p, q) = r + H(t, x, p, q), \quad r \in \mathbb{R}$$  

(2)
and 
\[ H : [0, +\infty[ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \]  

is a given Hamiltonian. For any \((t, x, q) \in [0, +\infty[ \times \mathbb{R}^n \times \mathbb{R}\), we denote by \(H^*(t, x, ., q) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\}\) the Fenchel transform of \(H(t, x, ., q)\).

**Definition 3.1 (Weak Epigraphical Solution).** We say that a lower semi-continuous function \(u : [0, +\infty[ \times \Omega \to \mathbb{R} \cup \{\pm \infty\}\) is an epigraphical weak solution of the HJB equation (1) if it satisfies:

for a.e. \(t \geq 0\) and all \(x \in \text{bdr} \ \Omega\) such that \((t, x) \in \text{dom} \ u\)
\[
F(t, x, -\varphi) \geq 0 \text{ for all } \varphi \in T_{\text{epi} u}(t, x, u(t, x))^{-}
\]

and

for a.e. \(t \geq 0\) and all \(x \in \text{int} \ \Omega\) such that \((t, x) \in \text{dom} \ u\)
\[
F(t, x, -\varphi) = 0 \text{ for all } \varphi \in T_{\text{epi} u}(t, x, u(t, x))^{-}.
\]

**Definition 3.2 (Set-Valued Maps of Locally Bounded Variations).** Consider a closed interval \(I \subset \mathbb{R}\). We say that a set-valued map \(\Phi : I \rightrightarrows \mathbb{R}^k\) is of locally bounded variations (LBV) if it satisfies the following:

- \(\Phi\) takes non-empty closed images;
- for any \([a, b] \subset I\),
\[
\sup_{i=1}^{m-1} \sum_{i+1} e\text{xc}(\Phi(t_{i+1}) \cap \mathcal{H}|\Phi(t_i)) \vee e\text{xc}(\Phi(t_{i}) \cap \mathcal{H}|\Phi(t_{i+1})) < +\infty
\]

where the supremum is taken over all compact subset \(\mathcal{H} \subset \mathbb{R}^k\) and all finite partition \(a = t_1 < t_2 < \ldots < t_{m-1} < t_m = b\).

By \(\mathcal{E}pi_{loc}([0, +\infty[ \times \Omega)\) we denote the family of all lower semicontinuous functions \(u : [0, +\infty[ \times \Omega \to \mathbb{R} \cup \{\pm \infty\}\) such that
\[
(t \mapsto \text{epi} \ u(t, .)) \text{ is continuous and of LBV on } [0, +\infty[.
\]

The following assumptions are considered on \(H\):
H.1 for any \( t \geq 0, x, p \in \mathbb{R}^n \), and \( q > 0 \):

(a) \( H(., x, p, q) \) is Lebesgue measurable,
(b) \( H(t, x, ., q) \) is convex,
(c) \( H(t, x, ..) \) is positively homogeneus.

H.2 there exist \( \sigma_X, \sigma_P, \hat{\sigma} : [0, +\infty[ \to [0, +\infty[ \) measurable, with \( \sigma_X \in \mathcal{L}_{loc} \) and \( \hat{\sigma} \) locally bounded, such that for any \( q > 0 \):

for all \( t \geq 0 \) and \( x_i, p_i, x, p \in \mathbb{R}^n \)

(a) \(|H(t, x_1, p, q) - H(t, x_2, p, q)| \leq \sigma_X(t)(1 + |p|)|x_1 - x_2|

(b) \(|H(t, x, p_1, q) - H(t, x, p_2, q)| \leq \sigma_P(t)(1 + |x|)|p_1 - p_2|

and for a.e. \( t \geq 0 \), all \( y \in \mathbb{R}^n, x \in \text{bdr} \, \Omega \), and \( p \in \text{dom} \, H^*(t, x, ., q) \)

(c) \( H^*(t, y, p, q) \leq \hat{\sigma}(t)(1 + |y|) \).

The next controllability assumptions are also taken into account:

C.1 there exists \( \sigma_{\text{bdr}} \in \mathcal{L}_{loc} \) such that

\[ |p| + |H^*(t, x, p, 1)| \leq \sigma_{\text{bdr}}(t) \]

for a.e. \( t \geq 0 \), for all \( y \in \mathbb{R}^n, x \in \text{bdr} \, \Omega \), and \( p \in \text{dom} \, H^*(t, x, .., 1) \);

C.1 there exist \( \eta > 0, r > 0, M \geq 0 \) such that

for a.e. \( t > 0, \forall y \in \text{bdr} \, \Omega + \eta \mathbb{R}^n, \forall p \in \text{dom} \, H^*(t, y, .., 1) \cap \Gamma_{\Omega}(y; \eta), \exists w \in \text{dom} \, H^*(t, y, .., 1) \cap B^n(p, M) : w, w - p \in \Sigma_{\Omega}(y; \eta, r) \).

Theorem 3.3. Consider the HJB equation (1)-(3) with the general assumptions H.1-2, C.1-2.

Suppose that for every \((t, x) \in [0, +\infty[ \times \Omega \)

(a) \( \lim_{T \to +\infty} \int_t^T H^*(s, \xi(s), \xi'(s), 1) \, ds \) exists for any \( \xi \in \mathcal{W}_{loc}^{1,1}( [t, +\infty[ ; \Omega) \) such that \( \xi(t) = x \),

(b) the infimum \( \alpha(t, x) \) defined by

\[ \inf \left\{ \lim_{T \to +\infty} \int_t^T H^*(s, \xi(s), \xi'(s), 1) \, ds \bigg| \xi \in \mathcal{W}_{loc}^{1,1}( [t, +\infty[ ; \Omega), \xi(t) = x \right\} \]

is finite.
Then there exists only one weak epigraphical solution \( u \in E_{\text{pi} \text{loc}}(\mathbb{R}^{n+1}) \) of the HJB equation with vanishing condition:

\[
F(t, x, -T_{\text{graph}} u(t, x), u(t, x)) - F(t, x, x, u(t, x)) = 0 \quad \text{in} \quad [0, +\infty[ \times \Omega
\]

\[
\lim_{{t \to +\infty}} \sup_{{x \in \Omega}} \abs{u(t, x)} = 0. \tag{4}
\]

**Remark 3.4.**

(a) In the previous result no regularity assumptions are made w.r.t. the \( q \)-dependence on the Hamiltonian \( H \), beyond those which are made in \( \text{H.1-}(a), (b) \). Moreover, condition \( \text{H.1-}(c) \) does not ensures non-negative values for \( H^*(t, x, ., q) \).

(b) We underline that, from assumption \( \text{H.2-}(c) \), for all \( t \in I, x \in \mathbb{R}^n, q > 0, \) and \( \bar{q} \in \text{cl dom} H^*(t, x, ., q) \) there exists \( \varepsilon > 0 \) such that

\[
\sup \{ H^*(t, x, p, q) | p \in \text{dom} H^*(t, x, ., q) \cap B^n(\bar{q}, \varepsilon) \} < +\infty.
\]

4. Proofs

In this section we provide the proofs of the results of this paper. We recall that, in the literature, a map \( E : [0, +\infty[ \to \mathbb{R}^d \) is also called tube.

4.1. Viability for Continuous Tubes of Locally Bounded Variations

The following result is well known (cfr. [11, Chapter 3]).

**Lemma 4.1** (Characterization of Continuity of Set-Valued Maps, [11]). Let \( D \subset \mathbb{R}^n \) be a closed non-empty set. Consider a set-valued map \( S : D \subset \mathbb{R}^n \to \mathbb{R}^m \) and let \( \bar{x} \in D \cap \text{dom}S \).

Then \( S \) is continuous at \( \bar{x} \) if and only if the function \( \text{dist}(x, S(\cdot)) \) is continuous at \( \bar{x} \) in \( D \) for every \( x \in \mathbb{R}^m \).

Lemma below provide a generalization of Lemma 4.8 in [15] to continuous tubes of LBV.

**Lemma 4.2.** Assume that \( E : [0, +\infty[ \to \mathbb{R}^d \) is continuous of locally bounded variations and the set-valued map \( Y : [0, +\infty[ \to \mathbb{R}^d \) satisfies

(a) \( Y(\cdot) \) is measurable for every \( x \in \mathbb{R}^d \) with non-empty compact images;
(b) there exists $\rho \in L_{loc}^{1}([0, +\infty[; [0, +\infty[)$ such that $d_{\mathcal{H}}(Y(t), Y(s)) \leq \int_{s}^{t} \rho(h)dh$ for all $0 \leq s \leq t$.

Let $\Psi$ be defined by

$$t \mapsto \Psi(t) := \text{dist} (E(t), Y(t)).$$

Then $\Psi$ is continuous and of locally bounded variations on $[0, +\infty[$.

Proof. We first show that the function $(t, x) \mapsto \text{dist}(x, E(t))$ is continuous in $[0, +\infty[ \times \mathbb{R}^{d}$. (5)

Indeed, from the continuity of $E$ and Lemma 4.1 we have that the function $t \mapsto \text{dist}(0, E(t))$ is continuous on $[0, +\infty[$. Furthermore, for any $x \in \mathbb{R}^{n}$, it follows that the map $t \mapsto \text{dist}(x, E(t))$ is continuous on $[0, +\infty[$ because, by applying the same argument above, $t \mapsto E(t) - \{x\}$ is continuous as well. Hence, since for any $t, s \in [0, +\infty[$ and $x, y \in \mathbb{R}^{d}$ the triangular inequality yield

$$|\text{dist}(x, E(t)) - \text{dist}(y, E(s))| \leq |\text{dist}(x, E(t)) - \text{dist}(x, E(s))| + |x - y|,$$

it follows (5).

Now, fix $[a, b] \subset [0, +\infty[$. Notice that, by the triangular inequality and our assumptions, for every $a \leq s \leq b$

$$d_{\mathcal{H}}(\{0\}, Y(s)) \leq d_{\mathcal{H}}(\{0\}, Y(b)) + d_{\mathcal{H}}(Y(s), Y(b))$$

$$\leq d_{\mathcal{H}}(\{0\}, Y(b)) + \int_{a}^{b} \rho(h)dh =: r,$$

so we deduce that $Y(s) \subset r\mathbb{B}^{d}$ for all $s \in [a, b]$. Using (5) and the Weierstrass theorem, we put $R := r + \max\{\text{dist}(x, E(t)) : x \in r\mathbb{B}^{d}, t \in [a, b]\}$. Hence, it follows for any $s \in [a, b]$

$$\text{dist}(E(s) \cap R\mathbb{B}^{d}, Y(s)) = \text{dist}(E(s), Y(s)).$$

Since for any $s, t \in [0, +\infty[$ and any compact $\mathcal{K} \subset \mathbb{R}^{d}$

$$E(s) \cap \mathcal{K} \subset E(t) + \text{exc}(E(s) \cap \mathcal{K} | E(t)) \mathbb{B}^{d},$$
keeping $\mathcal{K} = \mathbb{R}^d$, we get
\[
\begin{align*}
\text{dist} (Y(s), E(s)) \\
\leq \text{dist} (Y(s), E(t)) + \text{exc} (E(s) \cap \mathcal{K} | E(t)) \vee \text{exc} (E(t) \cap \mathcal{K} | E(s))
\end{align*}
\]
and
\[
\begin{align*}
\text{dist} (Y(s), E(t)) \\
\leq \text{dist} (Y(s), E(s)) + \text{exc} (E(s) \cap \mathcal{K} | E(t)) \vee \text{exc} (E(t) \cap \mathcal{K} | E(s))
\end{align*}
\]
Thus, for every $s, t \in [0, +\infty[
\[
|\text{dist} (Y(s), E(t)) - \text{dist} (Y(s), E(s))| \\
\leq \text{exc} (E(s) \cap \mathcal{K} | E(t)) \vee \text{exc} (E(t) \cap \mathcal{K} | E(s))
\]
Finally, there exists $M > 0$, depending only on $[a, b]$, such that, for any partition $a = t_1 < t_2 < ... < t_{m-1} < t_m = b$,
\[
\begin{align*}
\sum_{i=1}^{m-1} |\Psi(t_{i+1}) - \Psi(t_i)| \\
\leq \sum_{i=1}^{m-1} |\text{dist} (Y(t_{i+1}), E(t_{i+1})) - \text{dist} (Y(t_{i+1}), E(t_i))| \\
+ \sum_{i=1}^{m-1} d_{\mathcal{K}} (Y(t_{i+1}), Y(t_i)) \\
\leq \sum_{i=1}^{m-1} \text{exc} (E(t_{i+1}) \cap \mathcal{K} | E(t_i)) \vee \text{exc} (E(t_i) \cap \mathcal{K} | E(t_{i+1})) \\
+ \int_a^b \rho(h) dh \\
\leq M.
\end{align*}
\]
Hence, the locally bounded variations property for real valued functions follows.

Next we show that $\Psi$ is uniformly continuous in $[a, b]$. Fix $\varepsilon > 0$. For any $\tau \in [a, b]$ consider $y_\varepsilon(\tau) \in Y(\tau)$ such that
\[
\text{dist}(Y(\tau), E(\tau)) + \frac{\varepsilon}{4} \geq \text{dist}(y_\varepsilon(\tau), E(\tau)).
\]
Then, for any \(s, t \in [a, b]\) and any \(x_s \in Y(s)\)
\[
\Psi(s) - \Psi(t) \\
\leq \text{dist}(Y(s), E(s)) - \text{dist}(y_\varepsilon(t), E(s)) + \frac{\varepsilon}{4} \\
\leq \text{dist}(x(s), E(s)) - \text{dist}(y_\varepsilon(t), E(t)) + \frac{\varepsilon}{4} \\
\leq |x_s - y_\varepsilon(t)| + |\text{dist}(y_\varepsilon(t), E(s)) - \text{dist}(y_\varepsilon(t), E(t))| + \frac{\varepsilon}{4}. 
\]
(7)

We notice that, from assumption (b), there exists \(\delta > 0\), depending only on \([a, b]\), such that for every \(s, t \in [a, b]\) with \(|s - t| \leq \delta\)
\[
d_{\mathscr{F}}(Y(s), Y(t)) \leq \frac{\varepsilon}{4}. 
\]
(8)

Furthermore, applying the triangle inequality and the Lipschitz continuity of the distance function, for any \(s, t \in [a, b]\)
\[
|\text{dist}(y_\varepsilon(t), E(s)) - \text{dist}(y_\varepsilon(t), E(t))| \\
\leq |\text{dist}(y_\varepsilon(s), E(s)) - \text{dist}(y_\varepsilon(t), E(s))| \\
+ |\text{dist}(y_\varepsilon(s), E(s)) - \text{dist}(y_\varepsilon(t), E(t))| \\
\leq |y_\varepsilon(s) - y_\varepsilon(t)| + |\text{dist}(y_\varepsilon(s), E(s)) - \text{dist}(y_\varepsilon(t), E(t))|. 
\]
(9)

We recall that, from [5], the map \((\tau, x) \mapsto \text{dist}(x, E(\tau))\) is uniformly continuous on \([a, b] \times R^{2d}\), with \(R > 0\) depending on \([a, b]\) as above. Hence, replacing \(\delta\) with a sufficiently small one and using (8), for any \(s, t \in [a, b]\) with \(|s - t| \leq \delta\) holds
\[
|\text{dist}(y_\varepsilon(s), E(s)) - \text{dist}(y_\varepsilon(t), E(t))| \leq \frac{\varepsilon}{4}. 
\]
(10)

Moreover, from assumption (a) and applying the Measurable Selection Theorem ([2, Theorem 8.1.3]), we can find a measurable selection \(x(\tau) \in Y(\tau)\) for all \(\tau \in [0, +\infty[\). Thus, keeping \(x_s = x(s)\) in (7), using (8), (9), and (10), we conclude that
\[
\Psi(s) - \Psi(t) \leq \varepsilon \quad \forall s, t \in [a, b] \text{ such that } |s - t| \leq \delta. 
\]

From the symmetry with respect to \(s\) and \(t\) in the previous inequality, the conclusion follows. \(\square\)
In what follows, we denote by
\[ D^+\Psi(t) := \limsup_{h \to 0^+} \frac{\Psi(t + h) - \Psi(t)}{h}, \] the right Dini derivative at \( t \in \mathbb{R} \) of a real valued function \( \Psi(\cdot) \). Before to state the main result of this section, we need the following

**Lemma 4.3.** Let \( \Psi : [\tau, T] \to \mathbb{R} \) be a continuous function and \( \alpha, \beta : [\tau, T] \to \mathbb{R} \) be two locally bounded functions, with \( \alpha(\cdot) \geq 0 \), such that
\[ D^+\Psi(t) \leq \alpha(t)\Psi(t) + \beta(t) \quad \text{for all } t \in [\tau, T]. \]

Then, for every \( t \in [\tau, T] \),
\[ \Psi(t) \leq \Psi(\tau)e^{\alpha(t-\tau)} + \int_\tau^t e^{\alpha(t-r)}\beta\,dr \]
where \( \alpha := \sup_{s \in [\tau,T]} \alpha(s) \) and \( \beta := \sup_{s \in [\tau,T]} |\beta(s)| \).

**Proof.** Let \( \delta > 0 \) and define \( \varphi_\delta(t) := (\Psi(\tau) + \delta)e^{\alpha(t-\tau)} + \int_\tau^t e^{\alpha(t-r)}(\beta + \delta)\,dr \).

Then, \( \varphi_\delta'(t) = \alpha\varphi_\delta(t) + \beta + \delta \) in \([\tau, T] \) and \( \varphi_\delta(t) > \Psi(t) \) for all \( t \in [\tau, T] \) close to \( \tau \). We show that \( \varphi_\delta(\cdot) \geq \Psi(\cdot) \) for any \( \delta > 0 \). By contradiction, assume that there are some \( \bar{t} \in [\tau, T] \) and \( \delta > 0 \) with \( \varphi_\delta(\bar{t}) < \Psi(\bar{t}) \). Setting \( s := \inf \{ t \in [\tau, \bar{t}] \mid \varphi_\delta(t) < \Psi(t) \} \), we obtain that \( \varphi_\delta(s) = \Psi(s) \) and \( \tau < s < \bar{t} \).

Thus, from the definition of \( s \),
\[ \varphi_\delta'(s) = \liminf_{h \to 0^+} \frac{\varphi_\delta(s + h) - \varphi_\delta(s)}{h} \leq \limsup_{h \to 0^+} \frac{\Psi(s + h) - \Psi(s)}{h} \]
\[ \leq \alpha(s)\Psi(s) + \beta(s) \]
\[ \leq \alpha \varphi_\delta(s) + \beta, \]
i.e., \( \alpha \varphi_\delta(s) + \beta + \delta \leq \alpha \varphi_\delta(s) + \beta \). Then a contradiction follows.

Below we give a relaxation of the result in [15], in which the stronger condition of the absolutely continuity of the tube \( E \) is considered. The proof is a mild adaptation, based in turn on [18].

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Proposition 4.4. Assume that $E : [0, +\infty] \rightarrow \mathbb{R}^d$ is continuous and of locally bounded variations in sense of Definition 3.2, let $t_0 \in [0, +\infty[$, $x_0 \in E(t_0)$, and $\Phi : [0, +\infty[ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a set-valued map with non-empty convex closed values such that

(i) $\Phi(.,x)$ is measurable for any $x$;

(ii) $\exists \rho \in L^1_{loc}([0, +\infty[; [0, +\infty[)$

such that for a.e. $t$

(a) $\sup_{v \in \Phi(t,x)} |v| \leq \rho(t)(1 + |x|)$,

(b) $\Phi(t,.)$ is $k_r(t)$-Lipschitz on $r\mathbb{B}^d$.

Then, if for a.e. $t > 0$ and all $y \in E(t)$

$$\text{cl} \text{ co } \mathcal{T}_{\text{graph}} E(t, y) \cap \{1\} \times \Phi(t, y) \neq \emptyset,$$

for any $T > t_0$ and $t_0 \leq t_i < t_{i+1} \leq T$, $i = 1, ..., m$, there exists on $[t_0, T]$ a solution $x(\cdot)$ of

$$x'(t) \in \Phi(t, x(t)) \text{ a.e. } t \quad (11)$$

satisfying $x(t_0) = x_0$ and

$$x(t_i) \in E(t_i) \text{ for all } i = 1, ..., m+1.$$

Proof. First of all, we notice that, by the Gronwall’s Lemma and assumption (ii)-(a), for all $r > 0$ there exists $R > r$ such that if an absolutely continuous function $x : [0, t_1] \rightarrow \mathbb{R}^d$ satisfies $|x'(t)| \leq \rho(t)(1 + |x(t)|)$ a.e. in $[0, t_1]$, $x(0) = x_0$, and $|x_0| \leq r$, then $|x(t)| \leq R$ for all $t \in [0, t_1]$. Moreover, observe that $\Phi$ is integrably bounded on $[0, t_1]$, i.e. for almost all $t \in [0, t_1]$ and all $x \in r\mathbb{B}^d$, $|v| \leq \rho(t)(1 + R) := \rho_R(t)$ for any $v \in \Phi(t, x)$.

Fix $t_0 \in [0, +\infty[ \text{ and } x_0 \in E(t_0)$. Consider the map $\Psi$ of the Lemma 4.2 applied with $Y(\cdot)$ defined by the reachable set

$$Y(s) = R[t_0, x_0](s) := \{x(s) \mid x(\cdot) \text{ solution of } (11), x(t_0) = x_0\}.$$

We first claim that

$$\Psi(t) = 0 \text{ } \forall t \in [t_0, +\infty[,$$

$$13$$
arguing by contradiction. Suppose that \( T > t_0 \) with \( \Psi(T) > 0 \) and consider 
\[ t \in \text{sup} \{ t < T \mid \Psi(t) = 0 \} \]. So, \( \Psi > 0 \) on \([\tau, T]\) and \( \Psi(\tau) = 0 \). We divide the proof of the claim (12) into two steps.

**Step 1 (Estimate on differentiability points):** From Lemma 4.2, the well known Scorza-Dragoni property (cfr. e.g. [12, Chapter 2], [20]), and the Mean Value Theorem for set-valued maps (cfr. [1]), there exists a subset \( C \subset [0, T] \) of full measure such that for all \( t \in C \) and \( x \in \mathbb{R}^d \) the following three properties hold: \( \Psi \) is differentiable at \( t \); for every \( v \in \Phi(t, x) \) there exists on \([t, T]\) a solution to the problem \( y' \in \Phi(t, y) \), \( y(t) = x \), \( y'(t) = v \); for every \( y(\cdot) \) solution of (11) on \([t_0, t]\) satisfying \( y(t) = x \) and for every sequence \( h_i \to 0+ \) we have \( \emptyset \neq \text{Limsup}_{i \to \infty} \left\{ \frac{y(t - h_i) - z}{h_i} \right\} \subset -\Phi(t, x) \). Consider \( t \in C \) and let \( z \in Y(t), y \in E(t) \) satisfy \( \Psi(t) = |z - y| \) and put \( p = \frac{z - y}{|z - y|} \). We first prove that for all \( (u, w) \in \mathcal{F}_{\text{graph}} E(t, y) \)

\[
\Lambda(u, w) = \Phi(t, z) \text{ if } u \geq 0 \text{ and } \Lambda(u, w) \neq \emptyset \text{ if } u < 0 \tag{13}
\]

where we denoted \( \Lambda(u, w) := \{ v \in \Phi(t, z) : \Psi'(t)u \leq \langle p, uv - w \rangle \} \). Indeed, let \( (u, w) \in \mathcal{F}_{\text{graph}} E(t, y) \) and \( h_i \to 0+, u_i \to u, w_i \to w \) satisfying \( y + h_iw_i \in E(t + h_iu_i) \) for all \( i \in \mathbb{N} \). Suppose that there exists a subsequence \( \{u_{i_k}\}_{k \in \mathbb{N}} \) with \( u_{i_k} \geq 0 \) for all \( k \in \mathbb{N} \). Let \( v \in \Phi(t, z) \) and \( x(\cdot) \) a solution of (11) on \([t, T]\) such that \( x(t) = z \) and \( x'(t) = v \). Thus

\[
\Psi(t + h_{i_k}u_{i_k}) - \Psi(t) \leq |x(t + h_{i_k}u_{i_k}) - y - h_{i_k}w_{i_k}| - |z - y|.
\]

Dividing by \( h_{i_k} \) and taking the limit we get \( \Psi'(t)u \leq \langle p, uv - w \rangle \). Otherwise, we have \( u_i < 0 \) for all \( i \) large enough. In this case, consider a solution \( \bar{x}(\cdot) \) of (11) on \([t_0, t]\), \( \{i_k\}_{k \in \mathbb{N}} \), and \( \bar{v} \in \Phi(t, z) \) such that \( \bar{x}(t_0) = x_0, \bar{x}(t) = z, \) and

\[
\lim_{k \to \infty} \frac{\bar{x}(t + h_{i_k}u_{i_k}) - z}{h_{i_k}} = uv. \text{ Hence for all } k \in \mathbb{N} \text{ sufficiently large}
\]

\[
\Psi(t + h_{i_k}u_{i_k}) - \Psi(t) \leq |\bar{x}(t + h_{i_k}u_{i_k}) - y - h_{i_k}w_{i_k}| - |z - y|.
\]

Dividing by \( h_{i_k} \) and taking the limit we get \( \Psi'(t)u \leq \langle p, uv - w \rangle \). Hence, it follows (13).

Now, consider \( e_j \geq 0 \) and \( (u_j, w_j) \in \mathcal{F}_{\text{graph}} E(t, y) \) for \( j = 0, \ldots, d \) such that \( \sum_{j=0}^d e_j = 1 \) and \( u := \sum_{j=0}^d e_ju_j > 0 \). Without loss of generality, we may assume that for some natural number \( 0 \leq N < d \) and all \( j = 1, \ldots, N \) we have \( u_j \geq 0 \) and \( u_j < 0 \) for all \( j = N + 1, \ldots, d \). From (13), for every \( j = N + 1, \ldots, d \) there exists \( \bar{v}_j \in \Phi(t, z) \) such that
\[ \Psi'(t)u_j \leq \langle p, u_jv_j - w_j \rangle . \] Thus, applying again (13) it follows that
\[ \Psi'(t) (N \sum_{j=0}^{N} e_ju_j) \leq \langle p, N \sum_{j=0}^{N} e_ju_j \rangle, \quad \forall v \in \Phi(t, z), \]  
\[ \Psi'(t) (d \sum_{j=N+1}^{d} e_ju_j) \leq \langle p, d \sum_{j=N+1}^{d} e_ju_j \rangle. \]

Notice that, since \( e_{N+1}|u_{N+1}| + \ldots + e_d|u_d| = |e_{N+1}u_{N+1} + \ldots + e_d u_d| < e_0 u_0 + \ldots + e_N u_N \), we have
\[ \theta_j := \frac{e_j|u_j|}{\sum_{j=0}^{N} e_ju_j} \rightarrow \sum_{j=N+1}^{d} \theta_j \in \left[ 0, 1 \right] \]
that, due to convexity of \( \Phi(t, z) \), it implies in turn that \( (1 - \sum_{j=N+1}^{d} \theta_j)v + \sum_{j=N+1}^{d} \theta_j\bar{v}_j \in \Phi(t, z) \) for every \( v \in \Phi(t, z) \). Hence, recalling (13) and (14), we obtain for all \( v \in \Phi(t, z) \)
\[ \Psi'(t)u = \Psi'(t) (d \sum_{j=0}^{d} e_ju_j) \leq \langle p, (\sum_{j=0}^{N} e_ju_j - \sum_{j=N+1}^{d} e_j|u_j|)v \rangle + \sum_{j=N+1}^{d} e_j(|u_j| + u_j)\bar{v}_j - \sum_{j=0}^{d} e_jw_j \]
\[ = \langle p, uv - \sum_{j=0}^{d} e_jw_j \rangle. \]

So, we have that
\[ \forall t \in C, \forall (u, w) \in \text{cl co } \mathcal{T}_{\text{graph}} E(t, y) \text{, with } u > 0 \]
\[ \exists h_i \rightarrow 0+, \exists u_i \rightarrow u, \exists w_i \rightarrow w : \]
\[ y + h_i w_i \in E(t + h_i u_i) \text{ for all } i \in \mathbb{N} \text{ and } \]
\[ \lim_{i \rightarrow +\infty} \frac{\Psi(t+h_i u_i) - \Psi(t)}{h_i} = \Psi'(t)u \leq \langle p, uv - w \rangle, \quad \forall v \in \Phi(t, z) \]
where \( z \in Y(t), y \in E(t), \) satisfy \( \Psi(t) = |z - y|, \) and \( p := \frac{z - y}{|z - y|} \).

**Step 2 (Upper estimate of the Right Dini Derivative):** Fix \( \epsilon > 0 \) and \( t \in [\tau, T] \). Keep a sequence of positive numbers \( h_k \rightarrow 0+ \) such that
\[
\lim_{k \to +\infty} \frac{\Psi(t+h_k)-\Psi(t)}{h_k} = D^+\Psi(t). \]
From Lemma 4.2, for all \( k \in \mathbb{N} \) there exists \( \delta_k > 0 \) such that for any sequences \( \{s_k\}_k, \{\tilde{s}_k\}_k \subseteq [\tau, T] \)
\[
|s_k - \tilde{s}_k| \leq \delta_k \quad \forall k \iff |\Psi(s_k) - \Psi(\tilde{s}_k)| \leq o(h_k) \quad \forall k. \tag{15}
\]
Moreover, applying again Lemma 4.2, we can find a sequence of differentiability points \( \{t_k\}_{k \in \mathbb{N}} \subseteq \mathcal{C} \) for \( \Psi \) such that
\[
\varepsilon h_k + \Psi(t) \geq \Psi(t_k) \text{ and } |t_k - (t + h_k)| \leq \frac{\delta_k}{3} \text{ for all } k \in \mathbb{N}. \tag{16}
\]
For any \( k \in \mathbb{N} \) consider \( z_k \in Y(t_k) \) and \( y_k \in E(t_k) \) such that \( \Psi(t_k) = |z_k - y_k| \) and put \( p_k = \frac{z_k - y_k}{|z_k - y_k|} \). From Step 1, for any \( k \in \mathbb{N} \) and any \( (u_k, w_k) \in \text{cl co } \mathcal{G}_{\text{graph}} E(t_k, y_k) \), with \( u_k > 0 \), there exist \( h_j^{(k)} \to 0^+, u_j^{(k)} \to u_k \), and \( w_j^{(k)} \to w_k \) satisfying
\[
y_k + h_j^{(k)} w_j^{(k)} \in E(t_k + h_j^{(k)} u_j^{(k)}) \quad \forall j \in \mathbb{N}
\]
and
\[
\lim_{j \to +\infty} \frac{\Psi(t_k + h_j^{(k)} u_j^{(k)}) - \Psi(t_k)}{h_j^{(k)}} \leq \langle p_k, u_k v - w_k \rangle \quad \forall v \in \Phi(t_k, z_k).
\]
In particular, it follows that for any \( k \in \mathbb{N} \) we can choose \( j_k \in \mathbb{N} \) satisfying
\[
|h_j^{(k)}| \leq h_k, \quad |h_j^{(k)} u_j^{(k)}| \leq \frac{\delta_k h_k}{3}, \quad \forall j \geq j_k. \tag{17}
\]
Hence, from (15) and (16), we have for any large \( k \in \mathbb{N} \) and any \( j \geq j_k \)
\[
\frac{\Psi(t + h_k) - \Psi(t_k + h_j^{(k)} u_j^{(k)})}{h_k} = o(h_k)
\]
and
\[
\frac{\Psi(t + h_k) - \Psi(t)}{h_k} \leq \frac{\Psi(t + h_k) - \Psi(t + h_j^{(k)} u_j^{(k)})}{h_k} + \frac{\Psi(t + h_j^{(k)} u_j^{(k)}) - \Psi(t)}{h_k} + \varepsilon. \tag{18}
\]
From (17) and Step 1, we get for every large \( k \in \mathbb{N} \)
\[
\limsup_{j \to +\infty} \frac{\Psi(t_k + h_j u_j^{(k)}) - \Psi(t_k)}{h_k} \leq \langle p_k, u_k v - w_k \rangle, \quad \forall v \in \Phi(t_k, z_k).
\]

Using that, inequality in (16), assumption (ii)-(b), and since for all \( k \in \mathbb{N} \) we can find \((1, v_k) \in \text{cl co} \ \mathcal{G}_{\text{graph}} E(t_k, y_k)\) with \( v_k \in \Phi(t_k, y_k) \), passing in (18) to the upper limit as \( j \to \infty \) we get
\[
\Psi(t + h_k) - \Psi(t) \leq o(h_k) + k_r(t_k)\Psi(t_k) + \varepsilon \leq o(h_k)(1 + k_r\varepsilon) + \varepsilon + k_r\Psi(t) \quad \text{for all large } k \in \mathbb{N}
\]
where \( k_r = \sup_{t \in [\tau, T]} |k_r(t)| \). Passing first to the limit in (19) as \( k \to +\infty \), and then using the arbitrariness of \( \varepsilon \), we get
\[
D^+\Psi(t) \leq k_r \Psi(t).
\]

From that and Lemma 4.3, the claim (12) follows immediately.

To conclude the proof, consider \( t_0 \leq t_i < t_{i+1} \leq T, \ i = 1, \ldots, m \). Following the induction argument, assume that for some \( j \geq 0 \) there exists an absolutely continuous trajectory \( y : [t_0, t_i] \to \mathbb{R}^d \) solving (11) such that \( y(t_i) \in E(t_i) \) for all \( i \leq j \). From the above claim applied with \((t_0, x_0)\) replaced by \((t_j, y(t_j))\) we can find an absolutely continuous trajectory \( \bar{y} : [t_j, T] \to \mathbb{R}^d \) solving (11) such that \( \bar{y}(t_j) = y(t_j) \) and \( \bar{y}(t_{j+1}) \in E(t_{j+1}) \). Thus we can extend \( y \) on the time interval \([t_j, t_{j+1}]\) by setting \( y(t) = \bar{y}(t) \) for all \( t \in [t_j, t_{j+1}] \), and the proof is now complete. \( \square \)

**Corollary 4.5** (Viability for Continuous of LBV Tubes). *If all the assumptions of Proposition 4.4 hold, then for any \( t_0 \in [0, +\infty[ \) and \( x_0 \in E(t_0) \) there exists an absolutely continuous viable solution*

\[
 x'(t) \in \Phi(t, x(t)) \quad \text{for a.e. } t > t_0 \]
\[
 x(t_0) = x_0 \]
\[
 x(t) \in E(t) \quad \forall t > t_0.
\]

**Proof.** We take the same notations as in the proof of Proposition 4.4. Let \( t_0 \in [0, +\infty[ \), \( x_0 \in E(t_0) \), \( T > t_0 \), and \( \varepsilon > 0 \). Pick \( \delta > 0 \) such that the map \( \Psi \) in the Lemma 4.2 is uniformly continuous in \([t_0, T]\). By replacing \( \delta \) with
a suitable small one, we can assume that \( \int_{s}^{\bar{s}} \rho(t)\,dt \leq \varepsilon \) for any \( s, \bar{s} \in [t_0, T] \) such that \( |s - \bar{s}| \leq \delta \). Let a finite partition \( t_0 < t_1 < \cdots < t_m = T \) be such that \( t_{i+1} - t_i < \delta \). Then, from the choice of \( \delta \), the Gronwall’s Lemma, and Proposition 4.3, there exist a constant \( c > 0 \) (depending only on \( |x_0| \) and \( T \)) and a trajectory \( x_\varepsilon(\cdot) \) solving (11) on \([t_0, T]\) such that for any \( t \in [t_0, T] \) and \( i \) with \( t_i \leq t \leq t_{i+1} \)

\[
\text{dist}(x_\varepsilon(t), E(t)) \leq \varepsilon + c \int_{t_i}^{t} \rho(s)\,ds \leq (1 + c)\varepsilon
\]

where \( Y_i(s) = R[t_i, x(t_i)](s) \). Now, applying again the Gronwall’s Lemma, consider a sequence \( \{x_\varepsilon_j(\cdot)\}_{j \in \mathbb{N}} \) converging weakly to some absolutely continuous function \( x : [t_0, T] \to \mathbb{R}^d \), where \( \varepsilon_j \to 0^+ \). It follows that \( x \) solves (11), with \( x(t_0) = x_0 \), and \( x(t) \in E(t) \) for all \( t \in [t_0, T] \). Using an iterative argument, we can extend such solution to the whole halfline \([t_0, +\infty[\) in order to get the statement. \( \square \)

4.2. Proof of Theorem 3.3

Next, we recall a Representation result of time-measurable and fiber-convex Hamiltonians recently investigated in ([4, Proposition 4.1]).

**Proposition 4.6** (Representation, [4]). Assume \( \text{H.1-(a),(b)} \) and \( \text{H.2} \). Then, there exists an epigraphical representation

\[
(\mathcal{F}_#, \mathcal{L}_#) : I \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}^n \times \mathbb{R}
\]

with \((\mathcal{F}_#, \mathcal{L}_#)(., x, q, \theta)\) measurable for any \( x \in \mathbb{R}^n \), \( q > 0 \), \( \theta \in \mathbb{R}^{n+1} \) and satisfying:

(i) for any \( t \in I \), \( x, p \in \mathbb{R}^n \), and \( q > 0 \)

\[
H(t, x, p, q) = \sup \{ \langle (p, -1), (\mathcal{F}_#(t, x, q, \theta), \mathcal{L}_#(t, x, q, \theta)) \rangle \mid \theta \in \mathbb{R}^{n+1} \};
\]

(ii) \(|(\mathcal{F}_#, \mathcal{L}_#)(t, x_1, q, \theta_1) - (\mathcal{F}_#, \mathcal{L}_#)(t, x_2, q, \theta_2)| \leq C(t, x_1, x_2, \theta_1, \theta_2, q)\) for any \( t \in I \), \( x_1, x_2 \in \mathbb{R}^n \), and \( \theta_1, \theta_2 \in \mathbb{R}^{n+1} \), where
\( C(t, x_1, x_2, \theta_1, \theta_2, q) := 5(n + 1)(\sigma_X(t)|x_1 - x_2| + |\eta(t, x_1)\theta_1 - \eta(t, x_2)\theta_2|) \)

\( \eta(t, x) := \sigma_P(t)(1 + |x|) + \gamma(t, x) + |H(t, x, 0, q)| \)

\( \gamma(t, x) := 0 \lor \sup \left\{ H^*(t, x, p, q) \mid p \in \text{dom } H^*(t, x, .., q) \right\}; \)

(iii) for any \( t \in I, x \in \mathbb{R}^n, \) and \( q > 0 \)

(a) \( \text{dom } H^*(t, x, .., q) = \mathcal{F}_#(t, x, q, \mathbb{B}^{n+1}) \)

(b) \( \text{graph } H^*(t, x, .., q) \subset (\mathcal{F}_#, \mathcal{L}_#)(t, x, q, \mathbb{B}^{n+1}) \)

(c) \( \text{epi } H^*(t, x, .., q) = (\mathcal{F}_#, \mathcal{L}_#)(t, x, q, \mathbb{R}^{n+1}). \)

Moreover, if in addition \( H.1-(c) \) holds, then we have the following representation

\( H(t, x, p, q) = \sup \left\{ \left( p, -q \right), \left( f(t, x, \theta), \mathcal{L}(t, x, \theta) \right) \right\} \mid \theta \in \mathbb{B}^{n+1} \}

\( = \sup \left\{ \left( p, -q \right), \left( f(t, x, \theta), \mathcal{L}(t, x, \theta) \right) \right\} \mid \theta \in \mathbb{B}^{n+1} \}

where

\( f(t, x, \theta) := \mathcal{F}_#(t, x, 1, \theta) \quad \& \quad \mathcal{L}(t, x, \theta) := \mathcal{L}_#(t, x, 1, \theta) \)

\( \mathcal{L}(t, x, \theta) := H^*(t, x, f(t, x, \theta), 1). \)

Proof. Let \((\mathcal{F}_#, \mathcal{L}_#)\) be the representation given by [4, Theorem 4.1] and satisfying the statements (i)-(iii). Assuming further \( H.1-(c) \), from (i) we have for any \( q > 0 \)

\( H(t, x, p, q) = qH(t, x, \frac{p}{q}, 1) \)

\( = q \sup \left\{ \left( \frac{p}{q}, \mathcal{F}_#(t, x, 1, \theta) \right) - \mathcal{L}_#(t, x, 1, \theta) \mid \theta \in \mathbb{B}^{n+1} \right\}. \)

Thus, from (iii) and the proof of [4, Theorem 4.1], we get

\( H(t, x, p, q) = \sup \left\{ \left( p, \mathcal{F}_#(t, x, 1, \theta) \right) - q\mathcal{L}_#(t, x, 1, \theta) \mid \theta \in \mathbb{B}^{n+1} \right\} \)

\( = \sup \left\{ \left( p, -q \right), \left( f(t, x, \theta), H^*(t, x, f(t, x, \theta), 1) \right) \right\} \mid \theta \in \mathbb{B}^{n+1} \}. \)

\( \square \)
In what follows, we consider the representation associated with the Hamiltonian $H$

$$(f, \mathcal{L}) : [0, +\infty] \times \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}^n \times \mathbb{R}$$

provided by Proposition 4.6. For any $(t, x) \in [0, +\infty] \times \Omega$, we denote by $\mathcal{U}_\Omega(t, x)$ the – possibly empty – set of all pairs $(\xi, \theta) : [t, +\infty] \to \mathbb{R}^n \times \mathbb{R}^{n+1}$ such

$$\begin{align*}
\xi'(s) &= f(s, \xi(s), \theta(s)), \quad \theta(s) \in \mathbb{R}^{n+1} \quad \text{for a.e. } s \in [t, +\infty[ \\
\xi(t) &= x \\
\xi(\cdot) &\subset \Omega.
\end{align*}$$

(20)

The value function $V_{f, \mathcal{L}} : [0, +\infty] \times \Omega \to \mathbb{R} \cup \{\pm \infty\}$ associated with the representation $(f, \mathcal{L})$ is defined by

$$V_{f, \mathcal{L}}(t, x) := \inf \{ \lim_{T \to +\infty} \int_t^T \mathcal{L}(s, \xi(s), \theta(s)) \, ds \mid (\xi, \theta) \in \mathcal{U}_\Omega(t, x) \}$$

where $\inf \emptyset = +\infty$ by convention.

Now, consider two epigraphical weak solutions of the HJB equation, namely $v$ and $w$, satisfying the vanishing condition in (4). It is sufficient to show that $v \leq V_{f, \mathcal{L}} \leq w$. Fix $(t_0, x_0) \in [0, +\infty[ \times \Omega$. By our assumptions, there exists $T > t_0$ such that

$$|w(t, y)| \leq \varepsilon, \quad |v(t, y)| \leq \varepsilon \quad \forall t \geq T, \forall y \in \Omega.$$ 

(21)

We divide the proof of Theorem 3.3 into parts (A), (B), and (C).

(A): We have

$$w(t_0, x_0) \geq V_{f, \mathcal{L}}(t_0, x_0).$$

(22)

If $w(t_0, x_0) = +\infty$, then $w(t_0, x_0) \geq V_{f, \mathcal{L}}(t_0, x_0)$. So, assume that $(t_0, x_0) \in \text{dom } w$. In order to prove (22), we show the following

$$\exists (\xi, \theta) : [t_0, +\infty[ \to \mathbb{R}^n \times \mathbb{R}^{n+1} \text{ solving } (20) :$$

$$w(t_0, x_0) \geq w(t, \xi(t)) + \int_{t_0}^t \mathcal{L}(s, \xi(s), \theta(s)) \, ds \quad \forall t \geq t_0.$$

(23)

If $u \in \mathcal{L}^1_{\text{loc}}([a, +\infty[ ; \mathbb{R})$, we denote by $\int_a^\infty u(s) \, ds := \lim_{b \to +\infty} \int_a^b u(s) \, ds$, provided this limit exists.
We postpone the proof of (23) and we assume temporarily it is valid. Using the vanishing condition and passing to the upper limit in (23) as \( t \to \infty \) yields, for every \((\xi(\cdot), \theta(\cdot)) \in \mathcal{U}_\Omega(t_0, x_0),\)

\[
w(t_0, x_0) \geq \limsup_{t \to +\infty} \int_{t_0}^t \mathcal{L}(s, \xi(s), \theta(s)) \, ds.
\]

In particular, it follows that

\[
w(t_0, x_0) \geq \inf \left\{ \limsup_{t \to +\infty} \int_{t_0}^t H^*(s, \xi(s), \xi'(s), 1) \, ds \mid \xi \in \mathcal{W}^{1,1}_{loc}([t_0, +\infty[; \Omega), \xi(t_0) = x_0 \right\} = \alpha(t_0, x_0).
\]

By our assumptions, \( \alpha(t_0, x_0) > -\infty \). Fix \( \varepsilon > 0 \) and consider a trajectory \( \xi \in \mathcal{W}^{1,1}_{loc}([t_0, +\infty[; \mathbb{R}^n) \) with \( \xi(t_0) = x_0 \) and \( \xi(\cdot) \subset \Omega \) satisfying

\[
\int_{t_0}^{+\infty} H^*(s, \xi(s), \xi'(s), 1) \, ds < \alpha(t_0, x_0) + \varepsilon.
\]

We have that \((\xi'(s), z'(s)) \in \text{graph } H^*(s, \xi(s), ., 1)\) for a.e. \( s \geq t_0 \), where we put \( z(s) := \int_t^s H^*(\tau, \xi(\tau), \xi'(\tau), 1) \, d\tau \) for all \( s \geq t_0 \). Applying now Proposition 4.6-(iii)-(b) and the Measurable Selection Theorem, we have that there exists a measurable function \( \theta : [t_0, +\infty[ \to \mathbb{B}^{n+1} \) such that \((\xi'(s), z'(s)) = (f(s, \xi(s), \theta(s)), \mathcal{L}(s, \xi(s), \theta(s)))\) for a.e. \( s \geq t_0 \). So, for all \( t \geq t_0 \)

\[
\int_{t_0}^t H^*(s, \xi(s), \xi'(s), 1) \, ds = \int_{t_0}^t z'(s) \, ds = \int_{t_0}^t \mathcal{L}(s, \xi(s), \theta(s)) \, ds,
\]

that imply

\[
\int_{t_0}^{+\infty} H^*(s, \xi(s), \xi'(s), 1) \, ds \geq \mathcal{V}_{f, \mathcal{L}}(t_0, x_0).
\]

We get \( \alpha(t_0, x_0) + \varepsilon > \mathcal{V}_{f, \mathcal{L}}(t_0, x_0) \). Since \( \varepsilon \) is arbitrary, we have \( \alpha(t_0, x_0) \geq \mathcal{V}_{f, \mathcal{L}}(t_0, x_0) \). Recalling (24), it follows the inequality in (22).

To conclude the proof of (A), we have only to show (23). Since \( w \) is a weak epigraphical solution of the HJB, applying the representation result
Proposition 4.6 there exists a set \( C \subset [0, +\infty[ \) with \( \mu(C) = 0 \) such that for all \( (t, x) \in \text{dom } w \cap (([0, +\infty[ \setminus C) \times \Omega) \)

\[
F(t, x, \varphi) = (r, p, q)) \\
= -r + \sup \{ (f(t, x, \theta), -p) + q \mathcal{L}(t, x, \theta) : \theta \in \mathbb{B}^{n+1} \} \geq 0 \\
\forall (r, p, q) \in \mathcal{F}(t, x, w(t, x))^-.
\]

Hence, from (25), the representation result Proposition 4.6 and the Separation Theorem, we deduce that

\[
\left( \{1\} \times \hat{\Phi}(t, x) \right) \cap \text{cl co } T_{\text{epi } w}(t, x, w(t, x)) \neq \emptyset
\]

for all \( (t, x) \in \text{dom } w \cap (([0, +\infty[ \setminus C) \times \Omega) \)

Putting, for all \( (t, x, z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \)

\[
\Phi(t, x, z) := \hat{\Phi}(t, x),
\]

from assumptions H.1-2 and applying Corollary 4.5 with \( E(t) = \text{epi } w(t, \cdot), \)

there exists an absolutely continuous trajectory \( X_0(\cdot) = (\xi_0(\cdot), z_0(\cdot)) \) solving

\[
\begin{align*}
X'(t) & \in \Phi(t, X(t)) \quad \text{for a.e. } t \in [t_0, t_0 + 1] \\
\xi(t_0) & = x_0, \quad z(t_0) = w(t_0, x_0) \\
\xi(t) & \in \Omega, \quad z(t) \geq w(t, \xi(t)) \quad \forall t \in [t_0, t_0 + 1].
\end{align*}
\]

We claim that for any positive \( j \in \mathbb{N} \) the trajectory \( X_0 \) admits an extension on the interval \([t_0, t_0 + j]\) to a trajectory \( X_j(\cdot) = (\xi_j(\cdot), z_j(\cdot)) \) satisfying (27) on \([t_0, t_0 + j]\). We proceed by the induction argument on \( j \in \mathbb{N} \). Let \( j \in \mathbb{N} \) and suppose that \( X_j(\cdot) = (\xi_j(\cdot), z_j(\cdot)) \) satisfies the claim. Then, using (26) and applying again Corollary 4.5 on the time interval \([t_0 + j, t_0 + j + 1]\), we can find a trajectory \( X(\cdot) = (\xi(\cdot), z(\cdot)) \) satisfying

\[
\begin{align*}
X'(t) & \in \Phi(t, X(t)) \quad \text{for a.e. } t \in [t_0 + j, t_0 + j + 1] \\
\xi(t_0 + j) & = \xi_j(t_0 + j), \quad z(t_0 + j) = z_j(t_0 + j) \\
\xi(t) & \in \Omega, \quad z(t) \geq w(t, \xi(t)) \quad \forall t \in [t_0 + j, t_0 + j + 1].
\end{align*}
\]
Putting $X_{j+1}(t) = (\xi_j(t), z_j(t))$ if $t \in [t_0, t_0 + j]$ and $X_{j+1}(t) = (\xi(t), z(t))$ if $t \in ]t_0 + j, t_0 + j + 1]$, we deduce that $X_{j+1}()$ satisfies our claim. Now, consider the trajectory $X(t) = (\xi(t), z(t))$ given by $X(t) = X_j(t)$ if $t \in [t_0 + j, t_0 + j + 1]$. By the Measurable Selection Theorem, there exist two measurable functions $\theta(\cdot)$ and $r(\cdot)$, with $\theta(t) \in \mathbb{B}^{n+1}$ and $r(t) \in [0, \sigma(t)(1 + |\xi(t)|) - \mathcal{L}(t, \xi(t), \theta(t))]$ for a.e. $t \geq t_0$, such that

$$z(t) = w(t_0, x_0) - \int_{t_0}^{t} \mathcal{L}(t, \xi(s), \theta(s)) \, ds - \int_{t_0}^{t} r(s) \, ds \geq w(t, \xi(t))$$

for all $t \geq t_0$. Hence, inequality in (23) immediately follows.

(B): We show

$$v(t_0, x_0) \leq \mathcal{V}_{f,E}(t_0, x_0).$$

If $\mathcal{V}_{f,E}(t_0, x_0) = +\infty$, then $\mathcal{V}(t_0, x_0) \geq v(t_0, x_0)$. So, let us assume that $(t_0, x_0) \in \text{dom} \mathcal{V}_{f,E}$. Fix $\varepsilon > 0$. Let $(\bar{\xi}(\cdot), \bar{\theta}(\cdot))$ be an optimal trajectory-control pair at $(t_0, x_0)$ for $\mathcal{V}_{f,E}$ and consider $s_i \to +\infty$ with $\{s_i\}_{i \in \mathbb{N}} \subseteq ]T, +\infty[$. Put $\bar{X}(\cdot) = (\bar{\xi}(\cdot), \bar{z}(\cdot))$ where $\bar{z}(t) = -\int_{t_0}^{t} \mathcal{L}(t, \bar{\xi}(s), \bar{\theta}(s)) \, ds$. From assumptions H.1-2 and C.1-2, applying the Neighboring Feasible Trajectory result Proposition 4.8 in Appendix, we deduce that for any $i \in \mathbb{N}$ there exists a trajectory $X_i(\cdot) = (\xi_i(\cdot), z_i(\cdot))$ solving

$$X_i'(t) \in \Phi(t, X_i(t)) \quad \text{for a.e. } t \in [t_0, s_i]$$
$$X_i(s_i) = (\bar{\xi}(s_i), \bar{z}(s_i))$$
$$\xi_i(t) \in \text{int } \Omega \quad \forall t \in [t_0, s_i]$$

and

$$\lim_{i \to \infty} \sup \{|X_i(s) - \bar{X}(s)| \mid s \in [t_0, s_i]\} = 0.$$ 

Hence, by the Measurable Selection Theorem, for any $i \in \mathbb{N}$ there exists a measurable selection $\theta_i(t) \in \mathbb{B}^{n+1}$ such that $(\xi_i(\cdot), \theta_i(\cdot))$ satisfies

$$\xi_i'(t) = f(t, \xi_i(t), \theta_i(t)) \quad \text{for a.e. } t \in [t_0, s_i]$$
$$\xi_i(s_i) = \bar{\xi}(s_i)$$
$$\xi_i(t) \in \text{int } \Omega \quad \forall t \in [t_0, s_i]$$

and

$$\lim_{i \to \infty} \xi_i(t_0) = \bar{\xi}(t_0),$$

23
\[
\lim_{i \to \infty} \int_{t_0}^{s_i} \mathcal{L}(t, \xi(t), \tilde{\theta}(t)) \, ds = \int_{t_0}^{\infty} \mathcal{L}(t, \xi(t), \tilde{\theta}(t)) \, ds.
\]

(30)

Now, fix \( i \in \mathbb{N} \) and consider \( \{\tau_j\}_j \subset \mathbb{T}, s_i, \) with \( \tau_j \to s_i \). Note that, by the dynamic programming principle, \( \xi_i(\tau_j) \in \text{dom} \mathcal{V}_{f, \mathcal{L}}(\tau_j, \cdot) \) for all \( j \in \mathbb{N} \). We need the following

**Lemma 4.7.** For any \( 0 < \tau_0 < \tau_1 \) and any pair \((\xi, \theta)\) solution of

\[
\begin{align*}
\xi'(s) &= f(s, \xi(s), \theta(s)), \quad \theta(s) \in \mathbb{B}^{n+1} \quad \text{for a.e. } s \in [\tau_0, \tau_1] \\
\xi([\tau_0, \tau_1]) &\subset \text{int } \Omega \\
(\tau_0, \xi(\tau_0)) &\in \text{dom } v,
\end{align*}
\]

we have

\[
(\xi(t), v(\tau_0, \xi(\tau_0)) - \int_{\tau_0}^{t} \mathcal{L}(s, \xi(s), \theta(s)) \, ds) \in \text{epi } v(t, \cdot) \quad \forall t \in [\tau_0, \tau_1].
\]

**Proof.** Since \( v \) is an epigraphical solution and from Proposition 4.6, there exists a set \( C \subset [0, +\infty[ \) with \( \mu(C) = 0 \) such that for all \((t, x) \in \text{dom } v \cap (\mathbb{B}^{n+1} \cap \text{int } \Omega) \)

\[
F(t, x, r, p, q) = -r + \sup \{ f(t, x, \theta) - p \} + q \mathcal{L}(t, x, \theta) : \theta \in \mathbb{B}^{n+1} \} = 0
\]

\[
\forall (r, p, q) \in \mathcal{T}_{\text{epi } v}(t, x, v(t, x))^{-}.
\]

Notice that, by the separation theorem, this is equivalent to

\[
\{ -1 \} \times ( -\tilde{\mathcal{L}}(t, x) ) \subset \text{cl } \text{co } \mathcal{T}_{\text{epi } v}(t, x, y)
\]

for all \( y \geq v(t, x) \) and all \((t, x) \in (\mathbb{B}^{n+1} \cap \text{int } \Omega) \cap \text{dom } v \). Let \( 0 < \tau_0 < \tau_1 \). Thus

\[
(1, f^0(t, x, \theta), \mathcal{L}^0(t, x, \theta)) \in \text{cl } \text{co } \mathcal{T}_{\text{graph } E}(t, x, y)
\]

(32)

for a.e. \( t \in [0, \tau_1 - \tau_0] \), any \((x, y) \in E(t) \cap (\text{int } \Omega \times \mathbb{R}) \), and any \( \theta \in \mathbb{B}^{n+1} \), where \( f^0(t, x, \theta) := -f(\tau_1 - t, x, \theta), \mathcal{L}^0(t, x, u) := \mathcal{L}(\tau_1 - t, x, \theta), \) and \( E(t) := \text{epi } v(\tau_1 - t, \cdot) \). Consider a trajectory-control pair \((\xi(\cdot), \theta(\cdot))\) solving (31), with \( \xi([\tau_0, \tau_1]) \subset \text{int } \Omega \) and \((\tau_1, \xi(\tau_1)) \in \text{dom } v\). Denote by \( z \) the solution of

\[
\begin{align*}
z'(t) &= -\mathcal{L}(t, \xi(t), \theta(t)) \quad \text{for a.e. } t \in [\tau_0, \tau_1] \\
z(\tau_1) &= v(\tau_1, \xi(\tau_1)).
\end{align*}
\]
Put $\theta^\circ(\cdot) = \theta(\tau_1 - \cdot)$, and $\xi^\circ(\cdot) = \xi(\tau_1 - \cdot)$, $z^\circ(\cdot) = z(\tau_1 - \cdot)$ the unique solutions of $\xi^\circ(t) = f^\circ(t, \xi^\circ(t), \theta^\circ(t))$, $z^\circ(t) = \mathcal{L}^\circ(t, \xi^\circ(t), \theta^\circ(t))$ a.e. $t \in [0, \tau_1 - \tau_0]$, respectively, with $\xi^\circ(0) = \xi(\tau_1)$ and $z^\circ(0) = v(\tau_1, \xi(\tau_1))$. Applying Corollary 4.5 with $\Phi$ given by the single-valued map

$$(t, x) \leadsto \{((f^\circ(t, \xi^\circ(t), \theta^\circ(t)), \mathcal{L}^\circ(t, \xi^\circ(t), \theta^\circ(t)))\},$$

it follows the conclusion. \qed

We continue the proof of part (B). Consider the solution $g_j(\cdot)$ of the Cauchy problem

$$g'(t) = -\mathcal{L}(t, \xi_i(t), \theta_i(t)) \quad \text{for a.e. } t \in [t_0, \tau_j]$$

$$g(\tau_j) = v(\tau_j, \xi_i(\tau_j)).$$

From Lemma 4.7 above and since $\mathcal{L} \geq \mathcal{L}$, we have that

$$\int_{t_0}^{\tau_j} \mathcal{L}(s, \xi_i(s), \theta_i(s)) \, ds + v(\tau_j, \xi_i(\tau_j)) \geq v(t_0, \xi_i(t_0)) \quad \forall j \in \mathbb{N}.$$ 

Hence, by (21),

$$\int_{t_0}^{\tau_j} \mathcal{L}(s, \xi_i(s), \theta_i(s)) \, ds + \varepsilon \geq v(t_0, \xi_i(t_0)) \quad \forall j \in \mathbb{N},$$

and taking the limit as $j \to \infty$ we get $\int_{t_0}^{s_i} \mathcal{L}(s, \xi_i(s), \theta_i(s)) \, ds + \varepsilon \geq v(t_0, \xi_i(t_0))$.

Passing now to the lower limit as $i \to \infty$, using (29), (30), and the lower semicontinuity of $v$, we have $\int_{t_0}^{\infty} \mathcal{L}(s, \bar{\xi}(s), \bar{\theta}(s)) \, ds + \varepsilon \geq v(t_0, x_0)$, i.e., $\mathcal{V}_{f, \mathcal{L}}(t_0, x_0) + \varepsilon \geq v(t_0, x_0)$. Since $\varepsilon$ is arbitrary, we conclude

$$\mathcal{V}_{f, \mathcal{L}}(t_0, x_0) \geq v(t_0, x_0).$$

(C): From parts (A) and (B) we have that $v \leq \mathcal{V}_{f, \mathcal{L}} \leq w$ on $]0, +\infty[ \times \Omega$, that in turn imply $v = w$ on $]0, +\infty[ \times \Omega$. Finally, since $t \leadsto \text{epi } w(t, \cdot)$ is continuous, $w$ is lower semicontinuous, and applying Lemma 4.1, we have $\liminf_{s \to 0^+, y \to \Omega x_0} w(s, y) = w(0, x)$ for all $x \in \Omega$. So, for any $x_0 \in \Omega$

$$w(0, x_0) = \liminf_{s \to 0^+, y \to \Omega x_0} w(s, y) = \liminf_{s \to 0^+, y \to \Omega x_0} v(s, y) = v(0, x_0).$$

25
Appendix

We provide here a proof of a neighboring feasible trajectory result (used in the proof of Theorem 3.3) involving uniform linear estimates on intervals \( I = [t_0, t_1], \) with \( 0 \leq t_0 < t_1, \) for state constrained differential inclusion of the form

\[
x'(t) \in Q(t, x(t)) \quad \text{a.e. } t \in I \\
x(t) \in A \quad \forall t \in I
\]

where \( Q : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a given set-valued map and \( A \subset \mathbb{R}^n \) is a closed non-empty set. A function \( x : [t_0, t_1] \rightarrow \mathbb{R}^n \) is said to be an \( Q \)-trajectory if it is absolutely continuous and \( x'(t) \in Q(t, x(t)) \) for a.e. \( t \in [t_0, t_1], \) and a feasible \( Q \)-trajectory if \( x(\cdot) \) is an \( Q \)-trajectory and \( x([t_0, t_1]) \subset A. \)

We consider the following assumptions on \( Q(\cdot, \cdot): \)

**Ax.1** \( Q \) has closed, non-empty values, a sub-linear growth, and \( Q(\cdot, x) \) is Lebesgue measurable for all \( x \in \mathbb{R}^n; \)

**Ax.2** there exists \( \varphi \in \mathcal{L}_{\text{loc}} \) such that \( Q(t, \cdot) \) is \( \varphi(t)-\text{Lipschitz continuous} \) for a.e. \( t \in [0, +\infty[. \)

We consider also the following controllability assumptions:

**Ax.C.1** \( \exists q \in \mathcal{L}_{\text{loc}} \) such that \( Q(t, x) \subset q(t) \mathbb{R}^n \) for all \( x \in \text{bdr } A, \) and a.e. \( t \in [0, +\infty[; \)

**Ax.C.2** there exist \( \eta > 0, r > 0, M \geq 0 \) such that

for a.e. \( t \in [0, +\infty[, \) all \( y \in \text{bdr } A + \eta \mathbb{B}^n \) and \( v \in Q(t, y) \cap (-\Gamma_A(y; \eta)) \)

there exists \( w \in Q(t, y) \cap B^n(v, M) : w, w - p \in (-\Sigma_A(y; \eta, r)). \)

**Proposition 4.8** (Neighboring Feasible Trajectory). Let us assume **Ax.1-2, Ax.C.1-2.** Then for all \( \delta > 0 \) there exists a constant \( \beta > 0 \) such that for any \( [t_0, t_1] \subset [0, +\infty[, \) with \( t_1 - t_0 = \delta, \) any \( Q \)-trajectory \( \hat{x}(\cdot) \) on \( [t_0, t_1] \) with \( \hat{x}(t_0) \in A, \) and any \( \rho > 0 \) with

\[
\rho \geq \text{dist}(\hat{x}(t), A) \quad \forall t \in [t_0, t_1],
\]

there exists a \( Q \)-trajectory \( x(\cdot) \) on \( [t_0, t_1], \) with \( x(t_0) = \hat{x}(t_0), \) satisfying

\[
x(t) \in \text{int } A \quad \forall t \in [t_0, t_1]
\]

\[
|\hat{x}(s) - x(s)| \leq \beta \rho \quad \forall s \in [t_0, t_1].
\]
Proof. The proof is inspired by [6] (see also the references therein for further Neighboring Feasible Trajectory results). We denote by \( \theta_g(.) \) the modulus of continuity of a real function \( g \). Fix \( \delta > 0 \) and let \( k > 0, \Delta > 0 \) be such that \( k > 1/r \), and

\[
(i) \quad 8\Delta M \leq \eta; \quad (ii) \quad 2e^{\theta_{\varphi}(\Delta)}\theta_{\varphi}(\Delta)M < r; \quad (iii) \quad 2e^{\theta_{\varphi}(\Delta)}\theta_{\varphi}(\Delta)Mk < rk - 1. \tag{34}
\]

We notice that, from Ax.2 and Ax.C.1, for any \( \alpha > 0 \) there exists \( q_{\alpha} \in \mathcal{L}_{\text{loc}} \) such that \( Q(t, x) \subset q_{\alpha}(t) \mathbb{B}^n \) for a.e. \( t > 0 \) and all \( x \in \text{bdr} \ A + \alpha \mathbb{B}^n \). So, by the proof of [6, Theorem 1], without loss of generality we can replace \( M \) with a suitable greater constant and suppose that \( \delta \leq \Delta \), \( \int_J q_{\eta} \leq M \) for any \( J \subset [0, +\infty) \) with \( \mu(J) = \delta \). Moreover, by (34)-(i) we have that if \( \hat{x}(t_0) \in A \setminus (\text{bdr} \ A + \eta/4 \mathbb{B}^n) \) then the conclusion follows taking \( x(.) = \hat{x}(.) \).

Then we suppose that \( \hat{x}(t_0) \in (\text{bdr} \ A + \eta/4 \mathbb{B}^n) \cap A \). Let us denote the non-smooth orientated distance \( \text{Dist}_A(.) \) from the boundary of \( A \) by\(^2\)

\[ \text{Dist}_A(.) := \text{dist}(., A) - \text{dist}(., A^c) \]

and define the measurable set

\[ A_+ = \left\{ s \in [t_0, t_1] \cap J \left| \begin{array}{l} \exists x \in (\text{bdr} \ A) + B^n(\hat{x}(s), \eta), \\
\exists n \in \mathcal{N}_A^c(x) \cap \mathbb{S}^{n-1}, \\
\langle n, \hat{x}'(s) \rangle \geq 0 \\
\end{array} \right. \right\}. \]

where \( J = \{ s \in [t_0, t_1] | \hat{x}'(s) \text{ exists} \} \). We show the stated uniform estimate by distinguish the cases \( \mu(A_+) = 0 \) or \( \mu(A_+) > 0 \).

**Case 1:** \( \mu(A_+) = 0 \).

Let \( t \in (t_0, t_1) \). Then applying the Main Value Theorem (cfr. [8]), for some \( z(t) \in [\hat{x}(t_0), \hat{x}(t)] \) and \( \xi(t) \in \partial^C \text{Dist}_A(z(t)) \) (where \( \partial^C \) stands for the generalized subdifferential, cfr. [2, 8]), we have

\[ \text{Dist}_A(\hat{x}(t)) = \text{Dist}_A(\hat{x}(t_0)) + \langle \xi(t), \hat{x}(t) - \hat{x}(t_0) \rangle. \]

We recall any vector of the form \( v = (x - \tilde{x})/\text{dist}(x, E) \), with \( \tilde{x} \) in the projection set of \( x \neq \tilde{x} \) onto \( E \), is a proximal normal to \( E \) at \( \tilde{x} \), and so it

\(^2\)We refer to [8, 22] for the well known properties about such distance.
lies in $\cl co \mathcal{M}_e(x)$. Hence, from [19, Theorem 8.49] and the well known characterization of the subdifferential of the distance function in terms of projection sets (cfr. [19, pp. 340-341]), we have that there exist $\lambda_i > 0$, $y_i$ in the set of projections of $z(t)$ onto $\text{bdr } A_i$, and $\xi_i \in \mathcal{N}^C_A(y_i) \cap \mathbb{S}^{n-1}$, for $i = 1, \ldots, N$, with $1 \leq N \leq n+1$, such that $\sum_{i=1}^N \lambda_i = 1$ and $\xi(t) = \sum_{i=1}^N \lambda_i \xi_i$. Moreover, for any $s \in [t_0, t_1]$

$$|y_i - \dot{x}(s)| \leq |y_i - z(t)| + |z(t) - \dot{x}(s)|$$

$$\leq \text{dist}(z(t), \text{bdr } A_i) + |\dot{x}(t_0) - \dot{x}(s)|$$

$$\leq \text{dist}(\dot{x}(t_0), \text{bdr } A_i) + |z(t) - \dot{x}(t_0)| + |\dot{x}(t_0) - \dot{x}(s)|$$

$$\leq \frac{\eta}{4} + 2\frac{\eta}{8} = \eta.$$

Hence, $\text{Dist}_A(\dot{x}(t)) = \text{Dist}_A(\dot{x}(t_0)) + \sum_{i=1}^N \lambda_i \int_{t_0}^t (\xi_i, \dot{x}(s)) \, ds < 0$ for all $t \in [t_0, t_1]$, and $x(\cdot) = \dot{x}(\cdot)$ satisfies the conclusions.

**Case 2:** $\mu(A_+) > 0$.

Applying the Measurable Selection Theorem (cfr. [2]), let $w : A_+ \to \mathbb{R}^n$ be the Lebesgue measurable selection function satisfying $w(t) \in Q(t, \dot{x}(t))$ for a.e. $t \in A_+$, as in **Ax.C.2.** Define

$$\tau := \begin{cases} t_1 & \text{if } \mu(A_+) \leq k\rho \\ \min \{t \in [t_0, t_1] \mid \mu(A_+ \cap [t_0, t]) = k\rho \} & \text{otherwise,} \end{cases}$$

and keep $y : [t_0, t_1] \to \mathbb{R}^n$ the arc satisfying $y(t_0) = \dot{x}(t_0)$ and

$$y'(t) := \begin{cases} w(t) & \text{if } t \in A_+ \cap [t_0, \tau] \\ \dot{x}(t) & \text{if } t \in [t_0, t_1] \setminus (A_+ \cap [t_0, \tau]) \cap J. \end{cases}$$

(35)

So, we have for all $t \in [t_0, t_1]$

$$\|\dot{x} - y\|_{W^{1,1}[t_0, t]} = |y(t_0) - \dot{x}(t_0)| + \int_{t_0}^t |y'(s) - \dot{x}'(s)| \, ds$$

$$= \int_{t_0}^{\tau \land t} |w(s) - \dot{x}'(s)| \chi_{A_+ \cap [t_0, \tau]}(s) \, ds \leq 2M \mu(A_+ \cap [t_0, \tau \land t]).$$

(36)

Applying Filippov’s Theorem (cfr. [2]), we have that there exists a $Q$-trajectory $x(\cdot)$ on $[t_0, t_1]$ such that $x(t_0) = y(t_0)$ and

$$\|y - x\|_{W^{1,1}[t_0, t]} \leq e^\int_{t_0}^t \varphi(r) \, dr \int_{t_0}^t \text{dist}(y'(s), Q(s, y(s))) \, ds$$

(37)
Thus, by (37), we have
\[ \xi_t \text{ for all } t \in [t_0, t_1]. \]

Using (38) and (36), it follows that
\[ \text{dist}(y'(s), Q(s, y(s))) \leq \text{dist}(y'(s), Q(s, \hat{x}(s))) + \varphi(s) \int_{t_0}^{s} |y'(\xi) - \hat{x}'(\xi)| \, d\xi. \]

So, taking note of (35), we have for a.e. \( s \in [t_0, t_1] \)
\[
dist(y'(s), Q(s, y(s))) \leq \text{dist}(y'(s), Q(s, \hat{x}(s))) + \varphi(s) \int_{t_0}^{s} |y'(\xi) - \hat{x}'(\xi)| \, d\xi. \]

Using (38) and (36), it follows that
\[ \text{dist}(y'(s), Q(s, y(s))) \leq 2\varphi(s)M_{\mu}(A_+ \cap [t_0, \tau \wedge s]) \]
for a.e. \( s \in [t_0, t_1] \). Hence, we obtain the estimates for all \( t \in [t_0, t_1] \)
\[
\int_{t_0}^{t} \text{dist}(y'(s), Q(s, y(s))) \, ds \leq 2\theta_{\varphi}(\Delta)M_{\mu}(A_+ \cap [t_0, \tau \wedge t]).
\]

Thus, by (37), we have
\[ \|y - x\|_{W^{1,1}([t_0, t])} \leq 2e^{\theta_{\varphi}(\Delta)}\theta_{\varphi}(\Delta)M_{\mu}(A_+ \cap [t_0, \tau \wedge t]) \]
for all \( t \in [t_0, t_1] \), and, using (36), we get
\[ \|\hat{x} - x\|_{W^{1,1}([t_0, t])} \leq \beta \rho \text{ with } \beta = 2M(e^{\theta_{\varphi}(\Delta)}\theta_{\varphi}(\Delta) + 1)k. \]

To conclude the proof, we show that
\[ x(t) \in \text{int } A \quad \forall t \in [t_0, t_1]. \]

Consider \( t \in [t_0, \tau] \). Then, applying the main value theorem, we have
\[ \text{Dist}_A(y(t)) = \text{Dist}_A(y(t_0)) + \langle \xi(t), y(t) - y(t_0) \rangle \]
for some \( z(t) \in [y(t_0), y(t)] \) and \( \xi(t) \in \partial C \text{ Dist}_A(z(t)). \)

Now, we have that there exist \( \lambda_i > 0, y_i \) in the set of projections of \( z(t) \) onto \( \text{bdr } A \), and \( \xi_i \in \mathcal{N}_A^C(y_i) \cap S^{n-1} \), for \( i = 1, \ldots, N \),
with \( 1 \leq N \leq n + 1 \), such that \( \sum_{i=1}^{N} \lambda_i = 1 \) and \( \xi(t) = \sum_{i=1}^{N} \lambda_i \xi_i. \)
Since for all \( s \in [t_0, t_1] \) and all \( i = 1, \ldots, N \),
\[ |y_i - z(t)| \leq |y_i - \hat{x}(s)| + |\hat{x}(s) - \hat{x}(t_0)| + |\hat{x}(t) - \hat{x}(t_0)| \]
and
\[
|y_i - z(t)| \leq \text{dist}(y(t), \text{bdr } A) + |\text{dist}(z(t), \text{bdr } A) - \text{dist}(y(t), \text{bdr } A)|
\leq \text{dist}(y(t), \text{bdr } A) + |y(t) - y(t_0)|
\leq \text{dist}(y(t_0), \text{bdr } A) + 2|y(t) - y(t_0)|,
\]
we have that $|y_i - \hat{x}(s)| \leq \frac{n}{4} + 2\frac{n}{8} + 2\frac{n}{8} < \eta$. It follows that

$$\langle \xi(t), y(t) - y(t_0) \rangle = \int_{[t_0, t] \cap A_+} \langle \xi(t), \hat{x}'(s) \rangle \, ds + \int_{[t_0, t] \cap \partial A_+} \langle \xi(t), w(s) \rangle \, ds$$

$$= \sum_{i=1}^{N} \int_{[t_0, t] \cap A_+} \langle \xi_i, \hat{x}'(s) \rangle \, ds + \sum_{i=1}^{N} \int_{[t_0, t] \cap A_+} \lambda_i \langle \xi_i, w(s) \rangle \, ds$$

$$
\leq -r \mu(A_+ \cap [t_0, t]).
$$

(39)

Hence, using (34)-(ii),

$$Dist_A(x(t)) \leq |x(t) - y(t)| + Dist_A(y(t))$$

$$\leq (2e^{\theta_0(\Delta)} \theta_0(\Delta) M - r) \mu(A_+ \cap [t_0, t]) \leq 0.$$  

(40)

The equality in (39) occurs only if $\mu([t_0, t] \setminus A_+) = 0$, and in that case it follows that $\mu(A_+ \cap [t_0, t])$ has positive measure. It follows that the inequality in (40) is strict.

Consider $t \in ]\tau, t_1]$. By the Main Value Theorem (cfr. [8, Theorem 2.3.7]), for some $z(t) \in [\hat{x}(t), y(t)]$ and $\xi(t) \in \hat{x}^C Dist_A(z(t))$, we have $Dist_A(y(t)) = Dist_A(\hat{x}(t)) + \langle \xi(t), y(t) - \hat{x}(t) \rangle$. Furthermore, arguing as in the previous case, consider $\lambda_i > 0, y_i$ in the set of projections of $z(t)$ onto $\partial A$, and $\xi_i \in A^C_A(y_i) \cap S^{n-1}$, for $i = 1, ..., N$, with $1 \leq N \leq n + 1$, such that $\sum_{i=1}^{N} \lambda_i = 1$ and $\xi(t) = \sum_{i=1}^{N} \lambda_i \xi_i$. We notice that for all $i = 1, ..., N$ and $s \in ]\tau, t_1]$, $|y_i - \hat{x}(s)| \leq |y_i - z(t)| + |z(t) - \hat{x}(s)|$ and

$$|z(t) - \hat{x}(s)|$$

$$\leq \frac{1}{2} (|z(t) - y(t)| + |y(t) - y(t_0)| + |\hat{x}(t_0) - \hat{x}(s)| + |\hat{x}(s) - \hat{x}(t)|$$

$$+ |\hat{x}(t) - z(t)|)$$

$$\leq |\hat{x}(t) - y(t)| + \frac{1}{2} (|y(t) - y(t_0)| + |\hat{x}(t_0) - \hat{x}(s)| + |\hat{x}(s) - \hat{x}(t)|).$$

Now, since $z(t) = ay(t) + (1 - a)\hat{x}(t)$, where $a \in [0, 1]$, we have that $|y_i - z(t)| \leq Dist_A(\hat{x}(t_0)) + |\hat{x}(t_0) - z(t)| \leq \frac{n}{4} + |y(t) - y(t_0)| + |\hat{x}(t) - \hat{x}(t_0)|$. 

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Summing up we obtain $|y_i - \hat{x}(s)| \leq \left( \frac{\eta}{4} + 2\frac{\eta}{8} \right) + 2\frac{\eta}{8} + \frac{1}{2} \frac{3\eta}{8} < \eta$. Then

$$\langle \xi(t), y(t) - \hat{x}(t) \rangle = \int_{t_0}^{t} \langle \xi(t), y'(s) - x'(s) \rangle \, ds$$

$$= \int_{A_+ \cap [t_0, \tau]} \langle \xi(t), w(s) - x'(s) \rangle \, ds$$

$$= \sum_{i=1}^{N} \int_{A_+ \cap [t_0, \tau]} \lambda_i \langle \xi_i, w(s) - x'(s) \rangle \, ds$$

$$\leq -r \mu(A_+ \cap [t_0, \tau]).$$

Finally, by (34)-(iii),

$$\text{Dist}_A(x(t)) \leq |x(t) - y(t)| + \text{Dist}_A(y(t))$$

$$\leq 2e^{\theta_\varphi(\Delta)} \theta_\varphi(\Delta) M k \rho + \text{Dist}_A(y(t))$$

$$\leq \text{Dist}_A(\hat{x}(t)) + (2e^{\theta_\varphi(\Delta)} \theta_\varphi(\Delta) M - r) k \rho$$

$$\leq (1 + (2e^{\theta_\varphi(\Delta)} \theta_\varphi(\Delta) M - r)k) \rho < 0.$$

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