Photon intrinsic frequency and size in stringy photon model

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Exploiting an open string which performs both rotational and pulsating motions, we investigate a photon intrinsic frequency. Explicitly evaluating the zero point fluctuation of the string which is delineated in terms of the quantum mechanical ground state energy in the vibrational mode of the string and the classical energy in the rotational mode, we find that the intrinsic frequency of the photon is given by $\omega_\gamma = 9.00 \times 10^{23} \text{ sec}^{-1}$ and comparable to those of the baryons such as nucleon and delta baryon. Next, we calculate the photon size $\langle r^2 \rangle^{1/2}(\text{photon}) = 0.17 \text{ fm}$ in a phenomenological stringy photon model.

Keywords: stringy photon model; photon intrinsic frequency; photon size

I. INTRODUCTION

It is well known that in the string theory a particle is assumed to be an extended object [1, 2]. In particular, in this theory the photon is given by the open string described in the extra dimensional space. Making use of the string version of the Hawking-Penrose singularity theorem [3] and the corresponding stringy particle properties, we have studied [4, 5] the stringy cosmology in a higher dimensional total spacetime to suggest that we can describe precisely the stringy congruence in terms of the universe expansion after the Big Bang. Note that in the stringy Hawking-Penrose singularity theorem, we have an advantage that in the early universe we have the degrees of freedom of the rotation and shear of stringy congruence.

Next, the rotating string has been also investigated to obtain the propagator of a string in terms of a set of classical rigid rotators [6]. Here the rotators have been assumed to possess some quantum excitations. Note that, in their approach, they have exploited the normal modes associated with the infinite sum over the product of creation and annihilation operators in quantum Hamiltonian.

On the other hand, the hypersphere soliton model [7–9] has been proposed to construct a topological lower bound on soliton energy and a set of equations of motion through the second class canonical quantization formalism [7]. The baryon physical quantities such as baryon masses, charge radii and magnetic moments, have been later evaluated by making use of the canonical quantization in the hypersphere soliton model, to suggest that a realistic hadron physics can be described in terms of this phenomenological soliton [8]. Recently, we have calculated the intrinsic pulsating frequencies of the baryons [9]. To do this, we have used the hypersphere soliton model, where we have constructed the first class Hamiltonian to quantize the hypersphere soliton [7, 9]. In this first class Dirac quantization scheme, at first we have constructed the axial coupling constant in addition to the above physical quantities. These predictions [9] have been shown to be in good agreement with the corresponding experimental data. Next, we have evaluated explicitly the intrinsic frequencies $\omega_N$ and $\omega_\Delta$ of the nucleon and delta baryon, respectively. Moreover, we have found that the intrinsic frequency for more massive particle is greater than that for the less massive one. To be specific, we have constructed the identity $\omega_\Delta = 2\omega_N$.

In this paper, we will propose a new phenomenological stringy photon model (SPM) to investigate the intrinsic frequency of pulsating photon, which is comparable to those of the baryons such as nucleon and delta baryon. To do this, we will make use of the Nambu-Goto string theory [10, 11], without resorting to the Ramanujan evaluation of the Riemann zeta function related with infinite mode sum [1, 12–15]. Next, in the SPM we will introduce an open string action associated with the photon [16]. Since the zero point energy is described in terms of the energies both for the quantum mechanical ground state and for the corresponding classical one [17], we will calculate the quantum mechanical ground state energy in the vibrational mode channel of the string and the classical energy in the rotational mode one. Making use of these two energies we will evaluate the zero point energy of the string to explicitly yield the photon intrinsic frequency. Here we will also find the formula for the the photon intrinsic frequency in terms of the photon mass, and then we will exploit the approximation that the photon mass is non-vanishing but negligible. Next, assuming that the photon size is given by the string radius in the SPM, we will predict the photon size by making use of the intrinsic frequency of pulsating photon.

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In Sec. II, we will briefly introduce basic features of an open string associated with a photon. In Sec. III, we will recapitulate the rotating string formalism. In Sec. IV, in the SPM we will explicitly construct the photon intrinsic frequency and the photon size. Next, we will discuss the chemical potential associated with the Bose-Einstein statistics for the massive photon with a finite size. Sec. V includes conclusions. In Appendix A, the Riemann zeta function is briefly discussed.

II. SKETCH OF NAMBU-GOTO STRING THEORY

In this section, before we construct the SPM, we will digress to pedagogically summarize a mathematical formalism for the Nambu-Goto open string which is related with a photon. In order to define the action on curved manifold, we introduce \((M, g_{ab})\) which is a \(D\) dimensional spacetime manifold \(M\) associated with the metric \(g_{ab}\). Given \(g_{ab}\), we can have a unique covariant derivative \(\nabla_a\) satisfying\(^{18}\)

\[
\begin{align*}
\nabla_ag_{bc} &= 0, \\
\nabla_a\omega^b &= \partial_a\omega^b + \Gamma^b_{ac}\omega^c, \\
(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c &= R_{abc}^\text{d}\omega_d.
\end{align*}
\]

(2.1)

We parameterize an open string by two world sheet coordinates \(\tau\) and \(\sigma\), and then we have the corresponding vector fields \(\xi^a = (\partial/\partial\tau)^a\) and \(\zeta^a = (\partial/\partial\sigma)^a\). The Nambu-Goto string action is now given by \(^{10, 11}\)

\[
S = -\kappa \int \int d\tau d\sigma \eta f(\tau, \sigma),
\]

(2.2)

where the coordinates \(\tau\) and \(\sigma\) have ranges \(\tau_1 \leq \tau \leq \tau_2\) and \(0 \leq \sigma \leq \pi\) respectively and

\[
f(\tau, \sigma) = [\langle \xi \cdot \zeta \rangle^2 - \langle \xi \cdot \zeta \rangle \langle \zeta \cdot \zeta \rangle]^{1/2}.
\]

(2.3)

Here \(\kappa\) is defined by \(\kappa = \frac{1}{2\alpha'}\), with \(\alpha'\) being the slope of the Regge trajectories.

We now perform an infinitesimal variation of the world sheets \(\gamma_{\alpha}(\tau, \sigma)\) traced by the open string during its evolution in order to find the string geodesic equation from least action principle. Here we impose the restriction that the length of the string is \(\tau\) independent. We introduce the deviation vector \(\eta^a = (\partial/\partial\alpha)^a\) which represents the displacement to an infinitesimally nearby world sheet, and we consider \(\Sigma\) which denotes the three dimensional submanifold spanned by the world sheets \(\gamma_{\alpha}(\tau, \sigma)\). We then may choose \(\tau, \sigma\) and \(\alpha\) as coordinates of \(\Sigma\) to yield the commutator relations

\[
\begin{align*}
\mathcal{L}_{\xi}\eta^a &= \xi^b\nabla_b\eta^a - \eta^b\nabla_b\xi^a = 0, \\
\mathcal{L}_{\eta}\xi^a &= \zeta^b\nabla_b\eta^a - \eta^b\nabla_b\zeta^a = 0, \\
\mathcal{L}_{\zeta}\zeta^a &= \xi^b\nabla_b\zeta^a - \zeta^b\nabla_b\xi^a = 0.
\end{align*}
\]

(2.4)

Now we find the first variation as follows

\[
\frac{dS}{d\alpha} = \int \int d\tau d\sigma \eta_b(\xi^a\nabla_a\eta^b + \zeta^a\nabla_a\pi^b) - \int d\sigma \int d\tau p^b\eta_b|_{\tau=\tau_2} - \int d\tau \int d\sigma \pi^b\eta_b|_{\sigma=0},
\]

(2.5)

where the world sheet currents associated with \(\tau\) and \(\sigma\) directions are respectively given by \(^{19}\),

\[
\begin{align*}
p^a &= \kappa \int \langle (\xi \cdot \zeta) \zeta^a - (\zeta \cdot \zeta) \xi^a \rangle, \\
\pi^a &= \kappa \int \langle (\xi \cdot \zeta) \xi^a - (\zeta \cdot \zeta) \zeta^a \rangle.
\end{align*}
\]

(2.6)

Using the endpoint conditions

\[
\eta^a(\tau = \tau_1; \sigma) = \eta^a(\tau = \tau_2; \sigma) = 0,
\]

(2.7)

and

\[
\pi^a(\tau; \sigma = 0) = \pi^a(\tau; \sigma = \pi) = 0,
\]

(2.8)

we have string geodesic equation

\[
\xi^a\nabla_a\eta^b + \zeta^a\nabla_a\pi^b = 0,
\]

(2.9)

and constraint identities \(^{19}\)

\[
\begin{align*}
p \cdot \zeta &= 0, & p \cdot p + \kappa^2 \zeta \cdot \zeta &= 0, \\
\pi \cdot \zeta &= 0, & \pi \cdot \pi + \kappa^2 \zeta \cdot \zeta &= 0.
\end{align*}
\]

(2.10)
III. ROTATING OPEN STRING IN (3+1) DIMENSIONAL SPACETIME

In this section, to explicitly develop the SPM, we will evaluate the rotational energy of photon in the rotating open string theory defined on the $D = 5$ dimensional total manifold, by restricting ourselves to exploiting one dimensional internal space parameterized by the coordinate $\sigma$, without appealing to the Ramanujan evaluation of the Riemann zeta function whose ambiguity is excluded in (A.17) below. We thus consider the open string in the (3+1) dimensional flat spacetime and delineate the string in terms of the coordinates

$$x_\mu = (x_0, x_i) = (\tau, x_i(\tau; \sigma)), \quad (i = 1, 2, 3). \tag{3.1}$$

We then find that $f(\tau, \sigma)$ in (2.3) is given by

$$f(\tau, \sigma) = \left[(\dot{x}_i \dot{x}'_i)^2 + (1 - \dot{x}_i \dot{x}'_i)\dot{x}'_j \dot{x}'_j\right]^{1/2}. \tag{3.2}$$

Here the overdot and prime denote derivatives with respect to $\tau$ and $\sigma$, respectively. In this paper, we use the metric signature $(+, - , - , -)$. Moreover we proceed to obtain the world sheet currents

$$p_0 = \frac{\kappa}{f} x'_i x'_i, \quad p_i = -\frac{\kappa}{f} [\dot{x}_j x'_j x'_i - (x'_j x'_j) \dot{x}_i],$$

$$\pi_0 = -\frac{\kappa}{f} \dot{x}_i x'_i, \quad \pi_i = -\frac{\kappa}{f} [\dot{x}_j x'_j + (1 - \dot{x}_j \dot{x}_j)x'_i]. \tag{3.3}$$

Inserting $p_\mu$ and $\pi_\mu$ in (3.3) into the string geodesic equation in (2.9), one readily obtains

$$\frac{\partial}{\partial \tau} \left[ \frac{(\dot{x}_j x'_j) x'_i - (x'_j x'_j) \dot{x}_i}{f} \right] + \frac{\partial}{\partial \sigma} \left[ \frac{(\dot{x}_j x'_j) \dot{x}_i + (1 - \dot{x}_j \dot{x}_j)x'_i}{f} \right] = 0. \tag{3.4}$$

Exploiting the boundary conditions in (2.8), one also finds at $\sigma = 0$ and $\sigma = \pi$

$$\dot{x}_i x'_i = 0, \quad (1 - \dot{x}_j \dot{x}_j) x'_i = 0. \tag{3.5}$$

Next, in order to describe an open string, which is rotating in $(x_1, x_2)$ plane and residing on the string center of mass, we take an ansatz

$$x_i^{rot} = (r(\sigma) \cos \omega \tau, r(\sigma) \sin \omega \tau, 0). \tag{3.6}$$

Here we propose that $r(\sigma)$ and $\omega$ represent respectively the diameter and angular velocity of the photon with solid spherical shape which is delineated by the open string. Note that $r(\sigma = \pi/2)$ denotes the center of the diameter of string. More specifically, $r(\sigma = \pi/2)$ is located in the center of the solid sphere which describes the photon. The first boundary condition in (3.5) is trivially satisfied and the second one yields

$$r'(\sigma = 0, \pi) = 0. \tag{3.7}$$

We then obtain $r(\sigma)$ which fulfills the above condition in (3.7)

$$r(\sigma) = \frac{1}{\omega} \cos \sigma. \tag{3.8}$$

Note that the photon has a finite size which is filled with mass.

Next, using the photon configuration in (3.6) and (3.8) together with (3.3), we readily find the rotational energy of the photon

$$E^{rot} = \int_0^\pi d\sigma \ p_0^{rot} = \frac{1}{2\alpha' \hbar \omega}. \tag{3.9}$$

where we have included $\hbar$ factor explicitly, and the value of $\alpha'$ is given by

$$\alpha' = 0.95 \text{ GeV}^{-2}. \tag{3.10}$$

Note that the rotational degrees of freedom of the photon in the early universe have been investigated in Refs. [4, 5].
IV. PHOTON INTRINSIC FREQUENCY AND PHOTON SIZE IN SPM

In this section, we will explicitly evaluate the photon intrinsic frequency in the SPM. To do this, we calculate the vibrational energy of photon by introducing the string coordinate configurations
\[ x_i = x_i^{rot} + y_i, \quad i = 1, 2, 3. \] (4.1)

Exploiting the coordinates in (4.1), we expand the string Lagrangian density
\[ \mathcal{L} = \mathcal{L}_0 + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial x_i \partial x_j} |\dot{y}_i \dot{y}_j| + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial x'_i \partial x'_j} |\dot{y}_i' \dot{y}_j'| + \cdots, \] (4.2)

where the subscript 0 denotes that the terms in (4.2) are evaluated by using the coordinates in (3.6). The ellipsis stands for the higher derivative terms. Here the first term is a constant given by \( \mathcal{L}_0 = \mathcal{L}(x_i^{rot}) \). The first derivative terms vanish after exploiting the string geodesic equations (2.9). Next in order to obtain the intrinsic vibration energy of photon, we define coordinates \( z_i \) which co-rotates with the string itself
\[ \begin{align*}
    z_1 &= y_1 \cos \omega \tau + y_2 \sin \omega \tau, \\
    z_2 &= -y_1 \sin \omega \tau + y_2 \cos \omega \tau, \\
    z_3 &= y_3.
\end{align*} \] (4.3)

After some straightforward algebra, one ends up with the Lagrangian density associated with the coordinates \( z_i \)
\[ \mathcal{L}(z_i) = \frac{\kappa}{2 \sin^2 \sigma} \left[ \frac{1}{\omega} (\dot{z}_2 + \omega z_1)^2 + 2 \sin \sigma \cos \sigma (\dot{z}_1 - \omega z_2) \dot{z}_2' \\
- (\dot{z}_2 + \omega z_1) z_1' - \omega z_2 z_2'' \right] + \frac{\kappa}{2 \omega} (\dot{z}_3^2 - \omega^2 z_3^2). \] (4.4)

The equations of motion for the directions \( z_2 \) and \( z_3 \) are given by
\[ \begin{align*}
    \ddot{z}_2 + \omega^2 z_2 + 2 \omega^2 \cot \sigma z_2' - \omega^2 z_2'' &= 0, \\
    \ddot{z}_3 - \omega^2 z_3'' &= 0.
\end{align*} \] (4.5)

In order to construct the zero point energy \( E^{zero} \) associated with the photon mass \( m_p \), we now investigate the quantum mechanical ground state energy of the string. Note that the photon is assumed to be in the ground state of the string energy spectrum. From (4.4) we find the eigenfunctions for the ground states
\[ \begin{align*}
    z_2 &= c_2 \sin (\omega \tau + \phi_2), \\
    z_3 &= c_3 \cos \sigma \sin (\omega \tau + \phi_3).
\end{align*} \] (4.6)

Here \( \phi_2 \) and \( \phi_3 \) are arbitrary phase constants which are irrelevant to the physics arguments of interest.

It seems appropriate to address comments on the photon vibration modes. The transverse mode \( z_2 \) in (4.6) is independent of the string coordinate \( \sigma \), so that the photon can tremble back and forth with a constant amplitude, while the longitudinal mode \( z_3 \) in (4.7) possesses sinusoidal dependence on \( \sigma \). Here note that \( z_3 \) does not move at the center of the string, namely at \( \sigma = \pi/2 \), independent of \( \tau \) and the other parts of the string oscillate with the frequency \( \omega \). As for the transverse mode \( z_1 \), one can readily find that any value for \( z_1 \) satisfies the Euler-Lagrange equation for \( z_1 \) obtained from the Lagrangian density in (4.4). Up to now we have considered a single massive photon with the solid sphere shape, whose diameter is delineated in terms of the length of the open string. The photon thus has a disk-like cross section on which the coordinates \( z_1 \) and \( z_2 \) resides. Note that, similar to the phonon associated with massive particle lattice vibrations, the photon is massive so that we can have three polarization directions: two transverse directions as in the massless photon case, and an additional longitudinal one. Keeping this argument in mind, we find that there exist two transverse modes \( z_1 \) and \( z_2 \) associated with the photon vibrations on \( z_1-z_2 \) cross sectional disk, in addition to one longitudinal mode \( z_3 \). We thus have the transverse mode in \( z_1 \) direction to yield the eigenfunction for the ground state, with an arbitrary phase constant \( \phi_1 \) similar to \( \phi_2 \) and \( \phi_3 \) discussed above,
\[ z_1 = c_1 \sin (\omega \tau + \phi_1). \] (4.8)

Note that, as in the case of massless photon, \( z_1 \) mode oscillates with the same frequency \( \omega \) as \( z_2 \) mode does.

Note also that the above solutions \( z_i \) satisfy their endpoint conditions at \( \sigma = 0 \) and \( \sigma = \pi \)
\[ \dot{z}_i' = 0, \quad i = 1, 2, 3. \] (4.9)
The energy eigenvalues in the ground states in (4.6)–(4.8) are then given by

\[ E^{\text{vib}}_i = \frac{1}{2} \hbar \omega, \quad i = 1, 2, 3. \]  

(4.10)

Exploiting the energies in (4.10), we arrive at the vibrational energy of the open string ground state

\[ E^{\text{vib}} = \sum_{i=1}^{3} E^{\text{vib}}_i = \frac{3}{2} \hbar \omega. \]  

(4.11)

The zero point energy \( E^{\text{zero}} \) of the string is known to be the difference between the energy for the quantum mechanical ground state and the corresponding classical one [17]. Note that, in the SPM the energy for the quantum mechanical ground state is given by \( E^{\text{vib}} \) in (4.11), while the classical energy is given by \( E^{\text{rot}} \) in (3.9). Moreover, the zero point energy state describes the photon mass \( M_{\gamma} \) to produce \( E^{\text{zero}} = M_{\gamma} \) at \( \omega = \omega_{\gamma} \), so that we can find the photon mass of the form [17]

\[ M_{\gamma} = E^{\text{vib}} - E^{\text{rot}}. \]  

(4.12)

Note that we have already removed the translational degree of freedom, by considering the string observer residing on the photon center of mass. Substituting \( E^{\text{rot}} \) in (3.9) and \( E^{\text{vib}} \) in (4.11) in the case of \( \omega = \omega_{\gamma} \) into (4.12), we find

\[ M_{\gamma} = \frac{3}{3} \hbar \omega_{\gamma} - \frac{1}{2} \alpha' \hbar \omega_{\gamma}, \]  

(4.13)

to yield

\[ \omega_{\gamma} = \frac{1}{3 \hbar} \left[ M_{\gamma} + \left( M_{\gamma}^2 + \frac{3}{3} \alpha' \right)^{1/2} \right]. \]  

(4.14)

On the other hand, one has the upper limit of experimental value for the photon mass [20]

\[ M_{\gamma}^{\text{exp}} = 1.00 \times 10^{-27} \text{ GeV}. \]  

(4.15)

Since in (4.14) the photon mass \( M_{\gamma} \) is non-vanishing but negligible, compared with \( 1/\alpha' \) term, we obtain

\[ \omega_{\gamma} \approx \frac{1}{\hbar (3 \alpha')^{1/2}}. \]  

(4.16)

Inserting the experimental data for \( \hbar \) and \( \alpha' \) in (4.15) into (4.16), we are finally left with

\[ \omega_{\gamma} = 9.00 \times 10^{23} \text{ sec}^{-1}. \]  

(4.17)

As shown in Table I, the predicted value of the intrinsic frequency for the bosonic photon with spin one is comparable to those for the fermionic nucleon and delta baryon with spin 1/2 [9]. To be more specific, the intrinsic frequency with spin one particle is greater than that with spin 1/2 particle. Note that \( \omega_{\gamma} \) is interpreted both as the angular velocity of the rotating solid spherical shape photon in (3.6), and as the intrinsic frequency associated with the vibrational modes in (4.17).

Next, making use of (3.8) and (4.17), we predict the photon size given by the string radius in the SPM

\[ \langle r^2 \rangle^{1/2} \text{(photon)} = \frac{c}{2 \omega_{\gamma}} = 0.17 \text{ fm}, \]  

(4.18)

where we have explicitly included the speed of light \( c \). The above value of \( \langle r^2 \rangle^{1/2} \text{(photon)} \) is 21% of the proton magnetic charge radius \( \langle r^2 \rangle^{1/2} \text{(proton)} = 0.80 \text{ fm} \) [8, 9].

Now, we have a couple of comments to address. First, we investigate the thermodynamic properties of the massive photon with finite size which are described in terms of the Bose-Einstein statistics [21], where the total number of the photons is fixed by the constraint \( \sum n_r = N \). Here \( n_r \) and \( N \) are the number of photons in quantum state \( r \) and the total photon number, respectively. In this case, we exploit the Bose-Einstein distribution \( \bar{n}_s = \frac{1}{e^{(\epsilon_s - \mu)/kT} - 1} \), where \( \beta = 1/kT \) with \( k \) and \( T \) are the Boltzmann constant and temperature, respectively, and \( \epsilon_s (s = 1, 2, ...) \) is the energy in state \( s \). Here \( \mu \) is the chemical potential per the massive photon defined by \( \mu = \frac{\partial F}{\partial N} \), with \( F \) being the Helmholtz
TABLE I: The intrinsic frequencies of particles.

| Particle type | Notation | Intrinsic frequency |
|---------------|----------|---------------------|
| Nucleon      | $\omega_N$ | $0.87 \times 10^{23}$ \text{ sec}^{-1} |
| Delta baryon | $\omega_\Delta$ | $1.74 \times 10^{23}$ \text{ sec}^{-1} |
| Photon        | $\omega_\gamma$ | $9.00 \times 10^{23}$ \text{ sec}^{-1} |

free energy. Note that, in order for the Bose-Einstein distribution to be reduced to the Planck distribution for the massless photon statistics, one should have the vanishing chemical potential which means that we need infinitely many photons.

Second, note that the photon mass creation is possible in the Big Bang epoch [4, 5]. Moreover the equation of state $\frac{D}{D-2} \rho + \frac{D}{D-2} P \geq 0$ derived in the string cosmology [3] yields $\rho + 5P \geq 0$ for the massive extended (not point) particles in the $D = 5$ dimensional total manifold associated with the SPM. Note also that, in the standard cosmology for the point massless particles, the equation of state is described by $\rho + P \geq 0$. Now we propose that all the particles including the photons in the universe are massive so that we can have only a single equation of state $\rho + 5P \geq 0$.

V. CONCLUSIONS

In summary, we have proposed a new approach of the SPM to investigate a photon intrinsic frequency by exploiting a string which performs both rotational and pulsating motions. Here we have interpreted the string as a diameter of a solid spherical shape photon. Explicitly we have found that the intrinsic frequency of the photon is comparable to those of the baryons such as nucleon and delta baryon. Next, we have discussed the chemical potential associated with the Bose-Einstein statistics for the massive photon with a finite size. Moreover, in the SPM we have evaluated the photon size given by the string radius which is approximately 21\% of the proton magnetic charge radius. It will be interesting to search for a strong experimental evidence for the photon size which could be associated with the manifest photon phenomenology such as the photoelectric effect, Compton scattering and Raman scattering, for instance. Once this is done, the algorithm for the stringy photon could give some progressive impacts on the realistic precision optics. Assuming that the SPM exploited in this paper could be a precise description for the photon, the newly evaluated photon intrinsic frequency $\omega_\gamma = 9.00 \times 10^{23}$ \text{ sec}^{-1} and photon size $\langle r^2 \rangle^{1/2}(\text{photon}) = 0.17$ \text{ fm} could be fundamental predictions in the extended object phenomenology, similar to $\omega_N$, $\omega_\Delta$ and the charge radii in the hypersphere soliton model [7–9].

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Appendix A: Ramanujan evaluation for Riemann zeta function

The Riemann zeta function has been developed in theoretical physics in addition to pure mathematics. In this Appendix, we will revisit the Ramanujan result for evaluation the Riemann zeta function which is associated with the (super)string theory defined on the $D = 26$ ($D = 10$) dimensional total manifold. Note that, in the light-cone gauge quantization in the bosonic string theory, the anomaly associated with the commutator of the Lorentz generators has been canceled in the $D = 26$ critical dimensions and with the condition

$$a = -\frac{D-2}{2} \zeta(-1) = 1,$$

where the Ramanujan evaluation for the Riemann zeta function $\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ has been used [1, 13]. After evaluating the Riemann zeta function following the Ramanujan scheme [12, 13] which is a pure number theoretical approach, in this Appendix we will proceed to point out the origin of ambiguity involved in the scheme to clarify the uniqueness of the Riemann zeta function.
1. Evaluation of the Riemann zeta function in the Ramanujan scheme

We will now heuristically investigate the Riemann zeta function, by following the Ramanujan scheme [12] to add up positive integers associated with mode sum in the string theory. To do this, we start with the Riemann zeta function defined as [1, 12–15, 22]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots.$$  \hspace{1cm} (A.2)

Now we are interested in the particular value of the Riemann zeta function

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1 + 2 + 3 + \cdots.$$  \hspace{1cm} (A.3)

To calculate the value of $\zeta(-1)$ defined above, by following the Ramanujan scheme [12], we consider the identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$  \hspace{1cm} (A.4)

to yield

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$  \hspace{1cm} (A.5)

In the case of $x = -1$, (A.5) yields

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}.$$  \hspace{1cm} (A.6)

On the other hand, manipulating the Riemann zeta function $\zeta(-1)$ in (A.3) produces [12]

$$\zeta(-1) - 4\zeta(-1) = -3\zeta(-1) = 1 - 2 + 3 - 4 + \cdots.$$  \hspace{1cm} (A.7)

Combining (A.6) with (A.7) yields [12]

$$\zeta(-1) \equiv 1 + 2 + 3 + \cdots = \frac{1}{12},$$  \hspace{1cm} (A.8)

which has been applied to the string theory associated with the critical dimensions $D = 26$.

Next we consider the infinite sum

$$\sum_{r=1,3,5,\ldots}^{\infty} \frac{r}{2} = \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \cdots = \frac{1}{2} \zeta(-1) - \zeta(-1) = \frac{1}{24},$$  \hspace{1cm} (A.9)

which has been also exploited in the $D = 10$ superstring theory. Note that the Ramanujan result of the Riemann zeta function $\zeta(-1)$ is derived through the number theoretical approach. In other words, the derivation of (A.8) is purely algebraic and it has nothing to do with the regularization scheme related with the Casimir force [14] which will be discussed below. Moreover, the Ramanujan result in (A.8) has been derived without resorting to the analytic continuation procedure [1]. Note that, if we follow the Ramanujan result for evaluation of the Riemann zeta function, we have an ambiguity that $\zeta(-1) = -\frac{1}{12}$ in addition to the well known result $\zeta(-1) = \infty$.

2. Uniqueness of the Riemann zeta function

Next, we will scrutinize the Riemann zeta function, to clarify the ambiguity associated with the Ramanujan result in (A.8). To do this, we consider the infinity $\infty$ which satisfies the basic algebra

$$\infty \pm \alpha = \infty,$$  \hspace{1cm} (A.10)

$$\alpha \times \infty = \infty,$$  \hspace{1cm} (A.11)

$$\infty - \alpha \times \infty \text{ is indeterminate},$$  \hspace{1cm} (A.12)
where $\alpha$ is a finite positive real number. In the case of $\alpha = 4$ in (A.12), we arrive at

$$\infty - 4 \times \infty \text{ is indeterminate.}$$  (A.13)

Now (A.13) can be readily checked if we consider the explicit relations obtained from (A.11), for instance

$$4 \times \infty = 3 \times \infty = 2 \times \infty = \infty.$$  (A.14)

Note that the left hand side of (A.7) is the same as $\infty - 4 \times \infty$, and thus we conclude that, due to (A.13), we cannot uniquely obtain the Ramanujan result in (A.7) to yield

$$\zeta(-1) - 4\zeta(-1) \neq -3\zeta(-1).$$  (A.15)

In other words, we end up with the statement

$$\zeta(-1) - 4\zeta(-1) \text{ is indeterminate.}$$  (A.16)

We thus cannot proceed to evaluate $\zeta(-1) - 4\zeta(-1)$ at this stage, and the ensuing calculations leading to (A.8) and (A.9) are incorrect to yield

$$\zeta(-1) = 1 + 2 + 3 + \cdots - \frac{1}{12},$$

$$\sum_{r=1,3,5,\ldots}^{\infty} \frac{r}{2} = \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \cdots \neq \frac{1}{24}.$$  (A.17)

Since the Ramanujan approach to the Riemann zeta function is now excluded, we then have the uniqueness of the Riemann zeta function

$$\zeta(-1) \equiv 1 + 2 + 3 + \cdots = \infty,$$  (A.18)

without any ambiguity. In other words, we have only one desired unique value $\infty$ for the infinite series $1 + 2 + 3 + \cdots$.

Parenthetically, we make comments on the finite sum

$$\sum_{n=1}^{N} \frac{1}{n-1} = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}.$$  (A.19)

In this case, we arrive at

$$\sum_{n=1}^{N} \frac{1}{n-1} - 4 \sum_{n=1}^{N} \frac{1}{n-1} = -\frac{3N(N+1)}{2}.$$  (A.20)

The incorrectness in $\zeta(-1) - 4\zeta(-1) = -3\zeta(-1)$ involved in (A.7) seems to originate from neglecting the difference between the infinite sum and finite sum. For the correct statement, see (A.15).

Next, we have comments on the calculation of the Riemann zeta function through the regularization scheme [14]. Even though the Ramanujan derivation of (A.8) is purely algebraic, we assume that the regularization scheme could be applicable to evaluation of the Riemann zeta function, and then let us see what happens later. To do this, we start with the regularization procedure of the Riemann zeta function $\zeta(-1)$ by replacing the divergent sum over integers by the expression [14]$	extsuperscript{1}$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots \rightarrow \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} ne^{-\epsilon n} = \lim_{\epsilon \to 0} \left(-\frac{\partial}{\partial \epsilon}\right) \sum_{n=1}^{\infty} e^{-\epsilon n}$$

\textsuperscript{1} In Ref. [14], there exists a typo in the expression in third line of (A.21).
We have used the simple mathematical relation \( \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} = \infty \), and in the last line we have exploited the identity \( (\epsilon^2 - 1)^{-1} = \infty - \frac{1}{12} \), even in the case that we exploit the regularization procedure.

Note that in Ref. \[14\] the term \( \frac{1}{\epsilon^2} \) in \( (A.21) \) has been renormalized away. However there is no reason to discard this divergent term, in mathematical and physical viewpoints, since we need to just add up positive integers associated with mode sums as in the case of the Ramanujan scheme. If we were in the shoes of Tong \[14\] who exploits the renormalization-away method, we could also manipulate the result in \( (A.21) \) as follows

\[
\lim_{\epsilon \to 0} \sum_{n=1}^{\infty} ne^{-\epsilon n} = \lim_{\epsilon \to 0} \left( \frac{1}{\epsilon^2} - \frac{1}{12} - p + O(\epsilon) + p \right) = \lim_{\epsilon \to 0} \left( \frac{1}{\epsilon^2} + p + O(\epsilon) \right),
\]

where we have introduced the definition of \( \tilde{\epsilon} = \frac{1}{\epsilon^2} = \frac{1}{\epsilon^2} - \frac{1}{12} - p \) for a given finite number \( p \). In this case, after renormalizing away the term \( \frac{1}{\epsilon^2} \), we could find the Riemann zeta function \( \zeta(-1) = p \) for any given number \( p \), to yield again the ambiguity in fixing the value of the Riemann zeta function, similar to the ambiguity in Ramanujan scheme associated with \( (A.16) \). Note that the Ramanujan scheme has been employed \[12\] to calculate the Riemann zeta function \( \zeta(-1) \) without resorting to the renormalization-away method.

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