NON-AUTONOMOUS STOCHASTIC EVOLUTION EQUATIONS OF PARABOLIC TYPE WITH NONLOCAL INITIAL CONDITIONS

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Abstract. In this paper, we study the non-autonomous stochastic evolution equations of parabolic type with nonlocal initial conditions in Hilbert spaces, where the operators in linear part (possibly unbounded) depend on time \( t \) and generate an evolution family. New existence result of mild solutions is established under more weaker conditions by introducing a new Green’s function. The discussions are based on Schauder’s fixed-point theorem as well as the theory of evolution family. At last, an example is also given to illustrate the feasibility of our theoretical results. The result obtained in this paper is a supplement to the existing literature and essentially extends some existing results in this area.

1. Introduction. In this paper, we investigate the existence of mild solutions for the following non-autonomous stochastic evolution equations (NSEE) of parabolic type with nonlocal initial conditions

\[
\begin{align*}
    u'(t) - A(t)u(t) & = f(t, u(t)) + g(t, u(t)) \frac{d\mathcal{W}(t)}{dt}, \quad t \in (0, a], \\
    u(0) & = \sum_{k=1}^{m} c_k u(t_k)
\end{align*}
\]

in the real separable Hilbert space \( \mathbb{H} \), where \( A(t) \) is a family of (possibly unbounded) linear operators depending on time and having the domains \( D(A(t)) \) for every \( t \in [0, a] \), \( a > 0 \) is a constant, the state \( u(\cdot) \) takes values in the real separable Hilbert space \( \mathbb{H} \) with inner product \( (\cdot, \cdot) \) and norm \( \| \cdot \| \). Let \( \mathbb{K} \) be another separable Hilbert space with inner product \( (\cdot, \cdot)_\mathbb{K} \) and norm \( \| \cdot \|_\mathbb{K} \). Assume that \( \{\mathcal{W}(t) : t \geq 0\} \) is a cylindrical \( \mathbb{K} \)-valued Wiener process with a finite trace nuclear covariance operator \( Q \geq 0 \) defined on a filtered complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). We are also employing the same notation \( \| \cdot \| \) for the norm of \( \mathcal{L}(\mathbb{K}, \mathbb{H}) \), which denotes the space of all bounded linear operators from \( \mathbb{K} \) into \( \mathbb{H} \). We denote by \( \mathcal{L}(\mathbb{H}) = \mathcal{L}(\mathbb{H}, \mathbb{H}) \). \( f : [0, a] \times \mathbb{H} \to \mathbb{H} \) and \( g : [0, a] \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H}) \) are continuous nonlinear functions, \( 0 < t_1 < t_2 < \cdots < t_m < a, m \in \mathbb{N}, c_k \) are real numbers, \( c_k \neq 0, k = 1, 2, \ldots, m \).

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The theory of nonlocal evolution equations has become an important area of investigation in recent years because they can application to various problems arising in physics, biology, aerospace and medicine. Evolution equations with nonlocal initial conditions are used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion, see McKibben [38]. It is demonstrated that the nonlocal initial condition can be applied in physics with better effect than the classical initial condition \( u(0) = u_0 \). For example, Deng [23] used the nonlocal condition (2) to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, condition (2) allows the additional measurements at \( t_k, k = 1, 2, \cdots, m \), which is more precise than the measurement just at \( t = 0 \). Therefore, the nonlocal condition can be more useful than the standard initial condition \( u(0) = u_0 \) to describe some physical phenomena. For this reason, differential equations with nonlocal initial conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see [5, 6, 7, 13, 25, 26, 33, 45] and the references therein.

In recent years, the stochastic differential and integro-differential equations have attracted great interest because of their practical applications in many areas such as physics, chemistry, economics, social sciences, finance and other areas of science and engineering. For more details about stochastic differential equations we refer to the books by Sobczyk [42], Da Prato and Zabczyk [22], Grecksch and Tudor [31], Mao [37] and Liu [35]. One of the branches of stochastic differential equations is the theory of stochastic evolution equations. Since semilinear stochastic evolution equations are abstract formulations for many problems arising in the domain of engineering technology, biology and economic system etc., stochastic evolution equations have attracted increasing attention in recent years and the existence, uniqueness and asymptotic behavior of mild solutions to stochastic evolution equations have been considered by many authors, see [4, 9, 21, 24, 36, 40, 44] and the references therein for more comments and citations.

In particular, stochastic partial differential equations with nonlocal initial conditions have also been investigated extensively in recent years and some interesting results have been obtained. In 2012, Cui, Yan and Wu [20] studied the existence results of mild solutions for a class of stochastic integro-differential evolution equations with nonlocal initial conditions in Hilbert spaces assuming that the nonlocal item is only continuous but without imposing some compactness and convexity. By using Sadovskii’s fixed point theorem, stochastic analysis theory and fractional calculus under the assumption that the corresponding linear system is approximately controllable, Farahi and Guendouzi [27] investigated the approximate controllability for a class of neutral stochastic fractional differential equations involving nonlocal initial conditions in 2014. Later, Chen and Li [10] obtained the existence of \( \alpha \)-mild solutions for a class of fractional stochastic integro-differential evolution equations with nonlocal initial conditions in a real separable Hilbert spaces by using a new strategy which relies on the compactness of the operator semigroup, Schauder fixed point theorem and approximating techniques. In 2016, Sakthivel et al. [41] investigated the approximate controllability of fractional stochastic differential inclusions with nonlocal conditions with the help of the fixed point theorem for multi-valued operators and fractional calculus. In the same year, by using the concept of \( \alpha \)-order fractional solution operator and \( \alpha \)-resolvent family combined with fractional calculations, Schauder fixed point theorem and stochastic analysis theory, Chen, Zhang...
and Li [11] obtained the existence of mild solutions for a class of fractional stochastic evolution equations with nonlocal initial conditions under the situation that the nonlinear term satisfies some appropriate growth conditions and the \( \alpha \)-order fractional solution operator is compact. In 2017, by establishing a sufficient condition for judging the relative compactness of a class of abstract continuous family of functions on infinite intervals, Chen, Abdelmonem and Li [12] obtained the global existence, uniqueness and asymptotic stability of mild solutions for a class of semilinear evolution equations with nonlocal initial conditions on infinite interval by using stochastic analysis theory, analytic semigroup theory, relevant fixed point theory and the well known Gronwall-Bellman type inequality. Very recently, Zhang, Chen, Abdelmonem and Li [48, 49] investigated the existence of mild solutions for two classes of stochastic evolution equations with nonlocal initial conditions in a real separable Hilbert spaces under the situation that the corresponding operator of semigroups are noncompact.

We point out that among the previous researches, most of researchers focus on the case that the differential operators in the main parts are independent of time \( t \), which means that the problems under considerations are autonomous. However, when treating some parabolic evolution equations, it is usually assumed that the partial differential operators depend on time \( t \) on account of this class of operators appears frequently in the applications, for the details please see Acquistapace [1], Acquistapace and Terreni [2], Amann [3], Tanabe [43], Liang, Liu and Xiao [34], Wang, Ezzinbi and Zhu [46], Wang and Zhu [47], Chen, Zhang and Li [13]-[19], Zhu, Liu and Wu [50] and Fu [29, 30]. Therefore, it is interesting and significant to investigate stochastic non-autonomous evolution equations with nonlocal initial conditions, i.e., the differential operators in the main parts of the considered problems are dependent of time \( t \). However, to the best of the authors’ knowledge, all the existing articles used various methods to study autonomous stochastic evolution equations, i.e., the differential operators in the main parts of the considered problems are independent of time \( t \), but for the case that the corresponding differential operators in the main parts are dependent of time \( t \), we have not seen the relevant papers to study non-autonomous stochastic evolution equations with nonlocal initial conditions. Therefore, inspired by the above mentioned aspects, we are devoted to studying the existence of mild solution for the non-autonomous stochastic evolution equations with nonlocal initial conditions (1)-(2) in this paper. Furthermore, in most of references that investigate evolution equations with nonlocal initial conditions, the expression of mild solutions are very complicated, and this brings a lot of inconvenience to the calculation. In this paper, we will extend and unify the existing results about evolution equations with discrete nonlocal initial conditions by introducing a new Green’s function, which is very important in dealing with such kinds of problems.

2. Preliminaries. We begin with this section by giving some notations. Let \( \mathbb{H} \) and \( \mathbb{K} \) be two real separable Hilbert spaces, we denote by \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_\mathbb{K} \) their inner products, and by \( \| \cdot \| \) and \( \| \cdot \|_\mathbb{K} \) their vector norms, respectively. We use \( \theta \) to present the zero element in \( \mathbb{H} \). We denote by \( \mathcal{L}(\mathbb{H}) \) the Banach space of all linear and bounded operators on \( \mathbb{H} \) endowed with the topology defined by operator norm. Let \( L^1([0, a], \mathbb{H}) \) be the Banach space of all \( \mathbb{H} \)-value Bochner square integrable functions defined on \([0, a]\) with the norm \( \| u \|_{L^1} = \int_0^a \| u(t) \| dt \).

Throughout the paper, we assume that \( \{ A(t) : 0 \leq t \leq a \} \) is a family of closed and densely defined operator on Hilbert space \( \mathbb{H} \), which satisfies the known conditions of Acquistapace and Terreni:
Lemma 2.1. The family of the linear operator \( \{ U(t,s) : 0 \leq s \leq t \leq a \} \) satisfies the following properties:

(i) \( U(t,r)U(r,s) = U(t,s), \) \( U(t,t) = I \) for \( 0 \leq s \leq r \leq t \leq a; \)

(ii) The map \( (t,s) \mapsto U(t,s)x \) is continuous for all \( x \in \mathbb{H} \) and \( 0 \leq s \leq t \leq a; \)

(iii) \( U(t,s) \in C^1((s,\infty), \mathcal{L}(\mathbb{H})), \) \( \frac{\partial U(t,s)}{\partial t} = A(t)U(t,s) \) for \( t > s, \) and \( \| A^k(t)U(t,s) \| \leq M(t-s)^{-k} \) for \( 0 < t-s \leq 1 \) and \( k = 0, 1; \)

(iv) \( \frac{\partial U(t,s)x}{\partial s} = -U(t,s)A(s)x \) for \( t > s \) and \( x \in D(A(s)) \) with \( A(s)x \in D(A(s)). \)

From the property (iii) we know that

\[
\| U(t,s) \|_{\mathcal{L}(\mathbb{H})} \leq M \quad \text{for} \quad 0 \leq s \leq t \leq a.
\]

In (5) and property (iii), \( M > 0 \) is a constant.

Definition 2.2. An evolution family \( \{ U(t,s) : 0 \leq s \leq t \leq a \} \) is said to be compact if for all \( 0 \leq s < t \leq a, \) \( U(t,s) \) is continuous and maps bounded subsets of \( \mathbb{H} \) into precompact subsets of \( E. \)

Lemma 2.3. \((28)\) For each \( t \in [0,a] \) and some \( \lambda \in \rho(A(t)), \) if the resolvent \( R(\lambda, A(t)) \) is a compact operator, then \( U(t,s) \) is a compact operator whenever \( 0 \leq s < t \leq a. \)

Lemma 2.4. \((39)\) Let \( \{ U(t,s) : 0 \leq s \leq t \leq a \} \) be a compact evolution family on \( \mathbb{H}. \) Then for each \( s \in [0,a], \) the function \( t \mapsto U(t,s) \) is continuous by operator norm for \( t \in (s,a]. \)

In this paper, we assume that \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \) is a complete filtered probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets of \( \mathcal{F}. \) Let \( \{ e_k, k \in \mathbb{N} \} \) be a complete orthonormal basis of \( \mathbb{K}. \) Suppose that \( \{ \mathbb{W}(t) : t \geq 0 \} \) is a cylindrical \( \mathbb{K} \)-valued Wiener process defined on the probability space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \) with a finite trace nuclear covariance operator \( Q \geq 0, \) denote \( \text{Tr}(Q) = \sum_{k=1}^{\infty} \lambda_k = \lambda < \infty, \)

\[ M > 0 \]
which satisfies that $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$. Let $\{W_k(t), k \in \mathbb{N}\}$ be a sequence of one-dimensional standard Wiener processes mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ such that
\[
W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} W_k(t) e_k. \tag{6}
\]
We further assume that $\mathcal{F}_t = \sigma\{W(s), 0 \leq s \leq t\}$ is the $\sigma$-algebra generated by $W$ and $\mathcal{F}_0 = \mathcal{F}$. For $\varphi, \psi \in L(\mathbb{K}, \mathbb{H})$, we define $(\varphi, \psi) = \text{Tr}(\varphi Q \psi^*)$, where $\psi^*$ is the adjoint of the operator $\psi$. Clearly, for any bounded operator $\psi \in L(\mathbb{K}, \mathbb{H})$,
\[
\|\psi\|^2_Q = \text{Tr}(\psi Q \psi^*) = \sum_{k=1}^{\infty} \|\psi e_k\|^2. \tag{7}
\]
If $\|\psi\|^2_Q < \infty$, then $\psi$ is called a $Q$-Hilbert-Schmidt operator.

The collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, denoted $L^2(\Omega, \mathbb{H})$, which is a Banach space equipped with the norm $\|u(\cdot)\|_{L^2} = (\mathbb{E}\|u(\cdot,\mathbb{W})\|^2)^{\frac{1}{2}}$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}u = \int_{\Omega} u(\mathbb{W}) d\mathbb{P}$. An important subspace of $L^2(\Omega, \mathbb{H})$ is given by
\[
L^2_0(\Omega, \mathbb{H}) = \{u \in L^2(\Omega, \mathbb{H}) \mid u \text{ is } \mathcal{F}_0\text{-measurable}\}. \tag{8}
\]
We denote by $C([0, a], L^2(\Omega, \mathbb{H}))$ the space of all continuous $\mathcal{F}_t$-adapted measurable processes from $[0, a]$ to $L^2(\Omega, \mathbb{H})$ satisfying $\sup_{t \in [0, a]} \mathbb{E}\|u(t)\|^2 < \infty$. Then it is easy to see that $C([0, a], L^2(\Omega, \mathbb{H}))$ is a Banach space endowed with the supnorm
\[
\|u\|_C = \left(\sup_{t \in [0, a]} \mathbb{E}\|u(t)\|^2\right)^{\frac{1}{2}}. \tag{9}
\]
For any constant $r > 0$, let
\[
B_r = \{u \in C([0, a], L^2(\Omega, \mathbb{H})) : \|u\|^2_C \leq r\}. \tag{10}
\]
Clearly, $B_r$ is a bounded closed convex set in $C([0, a], L^2(\Omega, \mathbb{H}))$.

By [21, Proposition 2.8], we have the following result which will be used throughout this paper.

Lemma 2.5. If $g : [0, a] \times \mathbb{H} \rightarrow L(\mathbb{K}, \mathbb{H})$ is continuous and $u \in C([0, a], L^2(\Omega, \mathbb{H}))$, then
\[
\mathbb{E}\left\|\int_0^a g(t, u(t)) dW(t)\right\|^2 \leq \text{Tr}(Q) \int_0^a \mathbb{E}\|g(t, u(t))\|^2 dt. \tag{11}
\]

Throughout this paper, we assume that

(H1) $\sum_{k=1}^{m} |c_k| < \frac{1}{M}$. 

By the assumption (H1) and (5), we have
\[
\left\|\sum_{k=1}^{m} c_k U(t_k, 0)\right\| \leq M \sum_{k=1}^{m} |c_k| < 1. \tag{12}
\]
By (12) and operator spectrum theorem, we know that
\[
B := \left(I - \sum_{k=1}^{m} c_k U(t_k, 0)\right)^{-1} \tag{13}
\]
exists, bounded and $D(\mathbb{B}) = \mathbb{H}$. Furthermore, by Neumann expression, $\mathbb{B}$ can be expressed by
\[
\mathbb{B} = \sum_{n=0}^{\infty} \left( \sum_{k=1}^{m} c_k U(t_k, 0) \right)^n.
\]
Therefore
\[
\|\mathbb{B}\| \leq \sum_{n=0}^{\infty} \left\| \sum_{k=1}^{m} c_k U(t_k, 0) \right\|^n = \frac{1}{1 - \left\| \sum_{k=1}^{m} c_k U(t_k, 0) \right\|} \leq \frac{1}{1 - M \sum_{k=1}^{m} |c_k|}.
\]
For convenience, we introduce the Green’s function $G(t, s)$ as follows
\[
G(t, s) = \sum_{k=1}^{m} \chi_{t_k}(s) U(t, 0) \mathbb{B} U(t_k, s) + \chi_{t}(s) U(t, s), \quad t, s \in [0, a],
\]
where
\[
\chi_{t_k}(s) = \begin{cases} c_k, & s \in [0, t_k), \\
0, & s \in [t_k, a], \end{cases} \quad \chi_{t}(s) = \begin{cases} 1, & s \in [0, t), \\
0, & s \in [t, a]. \end{cases}
\]
Therefore, by the above discuss and the proof of [6, Lemma 2.2] combined with the aid of the Green’s function $G(t, s)$ defined by (16), we can give the definition of mild solutions for NSEE (1)-(2) as follows.

**Definition 2.6.** An $\mathcal{F}_r$-adapted stochastic process $u : [0, a] \rightarrow \mathbb{H}$ is called a mild solution of NSEE (1.1)-(1.2) if $u(t) \in \mathbb{H}$ has càdlàg paths on $t \in [0, a]$ almost surely and for each $t \in [0, a]$, $u(t)$ $\mathbb{P}$-almost surely satisfies the integral equation
\[
u(t) = \int_{0}^{a} G(t, s) f(s, u(s)) ds + \int_{0}^{a} G(t, s) g(s, u(s)) d\mathbb{W}(s).
\]

**3. Main results.** In this section, we will state and prove the existence result of mild solutions for NSEE (1)-(2). For this purpose, we impose the following restrictions on nonlinear functions $f$ and $g$.

(H2) For some $r > 0$, there exist positive constants $\rho_1, \rho_2$ and functions $\varphi_r, \psi_r \in L([0, a], \mathbb{R}_+)$ such that for all $u \in \mathbb{H}$ satisfying $\mathbb{E}\|u(t)\|^2 \leq r$ and a.e. $t \in [0, a],$
\[
\mathbb{E}\|f(t, u)\|^2 \leq \varphi_r(t) \quad \text{and} \quad \lim_{r \rightarrow +\infty} \inf_{t \in [0, a]} \frac{\|\varphi_r\|_{L([0, a], \mathbb{R}_+)}}{r} := \rho_1 < +\infty,
\]
\[
\mathbb{E}\|g(t, u)\|^2 \leq \psi_r(t) \quad \text{and} \quad \lim_{r \rightarrow +\infty} \inf_{t \in [0, a]} \frac{\|\psi_r\|_{L([0, a], \mathbb{R}_+)}}{r} := \rho_2 < +\infty.
\]
For the convenience, denote
\[
\Lambda = \left[ \frac{M \sum_{k=1}^{m} |c_k|}{1 - M \sum_{k=1}^{m} |c_k|} \right]^2.
\]

**Theorem 3.1.** Assume that the evolution family $\{U(t, s) : 0 \leq s \leq t \leq a\}$ generated by $\{A(t) : 0 \leq t \leq a\}$ is compact. If the assumptions (H1) and (H2) are satisfied, then NSEE (1)-(2) has at least one mild solution on $[0, a]$ provided that
\[
4M^2(\Lambda + 1)(\rho_1 + \text{Tr}(Q)\rho_2) < 1.
\]
Proof. Consider the operator $F : C([0, a], L^2(\Omega, \mathbb{H})) \to C([0, a], L^2(\Omega, \mathbb{H}))$ defined by
\[
(Fu)(t) = \int_0^a G(t, s)f(s, u(s))ds + \int_0^a G(t, s)g(s, u(s))d\mathcal{W}(s), \quad t \in [0, a],
\] (21)
where $G(t, s)$ is the Green’s function defined by (16). By direct calculation, we know that the operator $F$ is well defined in $C([0, a], L^2(\Omega, \mathbb{H}))$. From Definition 2.6, it is easy to see that the mild solution of NSEE (1)-(2) on $[0, a]$ is equivalent to the fixed point of the operator $F$ defined by (21). In what follows, we will prove that the operator $F$ has at least one fixed point by applying the famous Schauder Fixed Point Theorem.

Firstly, we prove that there exists a positive constant $R$ such that the operator $F$ defined by (21) maps the set $B_R$ to itself. If this is not true, then there would exist $t_r \in [0, a]$ and $u_r \in B_r$ such that $\mathbb{E} ||(Fu_r)(t_r)||^2 > r$ for each $r > 0$. By Lemma 2.5, (5), (15), (19), (21) and the assumptions (H1) and (H2), we get that
\[
r < \mathbb{E} ||(Fu_r)(t_r)||^2 \leq 2\mathbb{E} \left( \int_0^a G(t_r, s)f(s, u_r(s))ds \right)^2 \\
+ 2\mathbb{E} \left( \int_0^a G(t_r, s)g(s, u_r(s))d\mathcal{W}(s) \right)^2 \\
\leq 4M^2\|B\|^2 \int_0^a \left( \sum_{k=1}^m \chi_{t_k}(s) \right)^2 \cdot \|U(t_k, s)\|^2 \cdot \mathbb{E} \|f(s, u_r(s))\|^2ds \\
+ 4\int_0^a \sum_{k=1}^m |\chi_{t_k}(s)|^2 \cdot \|U(t_k, s)\|^2 \cdot \mathbb{E} \|f(s, u_r(s))\|^2ds \\
+ 4\text{Tr}(Q)M^2\|B\|^2 \int_0^a \sum_{k=1}^m \chi_{t_k}(s) \cdot \|U(t_k, s)\|^2 \cdot \mathbb{E} \|g(s, u_r(s))\|^2ds \\
+ 4\text{Tr}(Q) \int_0^a \sum_{k=1}^m \chi_{t_k}(s) \cdot \|U(t_r, s)\|^2 \cdot \mathbb{E} \|g(s, u_r(s))\|^2ds \\
\leq 4M^2(\Lambda + 1)\|\varphi_r\|_{L([0, a], \mathbb{R}^+)} + 4M^2\text{Tr}(Q)(\Lambda + 1)\|\psi_r\|_{L([0, a], \mathbb{R}^+)}.\]
(22)
Dividing both side of (22) by $r$ and taking the lower limit as $r \to +\infty$, combined with the assumption (20) we get that
\[
1 \leq 4M^2(\Lambda + 1)(\rho_1 + \text{Tr}(Q)\rho_2) < 1,
\]
which is a contradiction. Therefore, we have proved that $F : B_R \to B_R$.

Secondly, we prove that the operator $F : B_R \to B_R$ is continuous. To this end, let the sequence $\{u_n\}_{n=1}^{\infty} \subset B_R$ such that $\lim_{n \to +\infty} u_n = u$ in $B_R$. By the continuity of the nonlinear functions $f$ and $g$, we have
\[
\lim_{n \to +\infty} f(s, u_n(s)) = f(s, u(s)), \quad \text{a.e. } s \in J
\] (23)
and
\[
\lim_{n \to +\infty} g(s, u_n(s)) = g(s, u(s)), \quad \text{a.e. } s \in J.
\] (24)
From the assumption (H2), we get that for a.e. $s \in J$,
\[
\mathbb{E} \|f(s, u_n(s)) - f(s, u(s))\|^2 \leq 2\mathbb{E} \|f(s, u_n(s))\|^2 + 2\mathbb{E} \|f(s, u(s))\|^2 \leq 4\varphi_R(s)
\] (25)
and
\[
\mathbb{E} \|g(s, u_n(s)) - g(s, u(s))\|^2 \leq 2\mathbb{E} \|g(s, u_n(s))\|^2 + 2\mathbb{E} \|g(s, u(s))\|^2 \leq 4\psi_R(s).
\] (26)
Using the fact that the functions $s \to 4\varphi_R(s)$ and $s \to 4\psi_R(s)$ are Lebesgue integrable for a.e. $s \in [0, t]$ and every $t \in [0, a]$, combined with Lemma 2.5, (5), (15), (19), (21), (23)-(26) and the Lebesgue dominated convergence theorem, we know that

$$E \|(F u_n)(t) - (F u)(t)\|^2 \leq 2E \left\| \int_0^a G(t, s)[f(s, u_n(s)) - f(s, u(s))]ds \right\|^2 + 2E \left\| \int_0^a G(t, s)[g(s, u_n(s)) - g(s, u(s))]dW(s) \right\|^2 \leq 4M^2 \Lambda \int_0^t E\|f(s, u_n(s)) - f(s, u(s))\|^2 ds + 4M^2 \int_0^t E\|g(s, u_n(s)) - g(s, u(s))\|^2 ds + 4M^2 \text{Tr}(Q) \int_0^t E\|g(s, u_n(s)) - g(s, u(s))\|^2 ds \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which means that

$$\|(F u_n) - (F u)\|_C = \left( \sup_{t \in [0, a]} E\|(F u_n)(t) - (F u)(t)\|^2 \right)^\frac{1}{2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, $F : B_R \to B_R$ is a continuous operator.

Thirdly, we demonstrate that the operator $F : B_R \to B_R$ is compact. To prove this, we first show that $\{(F u)(t) : u \in B_R\}$ is relatively compact in $\mathbb{H}$ for every $t \in [0, a]$. It is easy to see from (16)-(18), (21) and Lemma 2.1 that for every $u \in B_R$,

$$(F u)(0) = \int_0^a G(0, s)f(s, u(s))ds + \int_0^a G(0, s)g(s, u(s))dW(s)
= \int_0^a \sum_{k=1}^m \chi_{t_k} U(0, 0)B U(t_k, s)f(s, u(s))ds
+ \int_0^a \sum_{k=1}^m \chi_{t_k} U(0, 0)B U(t_k, s)g(s, u(s))dW(s)
= \sum_{k=1}^m c_k B \int_0^{t_k} U(t_k, s)f(s, u(s))ds + \sum_{k=1}^m c_k B \int_0^{t_k} U(t_k, s)g(s, u(s))dW(s).$$

(27)

For any $0 < \epsilon < t_k$ ($k = 1, 2, \cdots, m$) and $u \in B_R$, we define the operator $F^\epsilon_0$ by

$$(F^\epsilon_0 u)(0) = U(t_k, t_k - \frac{\epsilon}{2})U(t_k - \frac{\epsilon}{2}, t_k - \epsilon) \sum_{k=1}^m c_k B \int_0^{t_k - \epsilon} U(t_k - \epsilon, s) \cdot f(s, u(s))ds + U(t_k, t_k - \frac{\epsilon}{2})U(t_k - \frac{\epsilon}{2}, t_k - \epsilon) \sum_{k=1}^m c_k B \int_0^{t_k - \epsilon} U(t_k - \epsilon, s)g(s, u(s))dW(s).$$

(28)
Since $U(t_k, t_k - \frac{\epsilon}{2}) \in L(H)$ and $U(t_k - \frac{\epsilon}{2}, t_k - \epsilon)$ is compact in $H$, the set $\{(F^*_t u)(0) : u \in B_R\}$ is relatively compact in $H$ for every $\epsilon \in (0, t_k)$ ($k = 1, \cdots, m$). Moreover, for every $u \in B_R$, by Lemma 2.5, (5), (15)-(17), (19), (21), (27), (29) and the assumptions (H1) and (H2), we get that
$$\mathbb{E}\|\mathbb{E}^*_t u(0) - (F u)(0)\|^2$$
\begin{align*}
\leq & \, 4\mathbb{E}\left\|U(t_k, t_k - \frac{\epsilon}{2})U(t_k - \frac{\epsilon}{2}, t_k - \epsilon)\sum_{k=1}^m c_k \mathbb{B} \int_0^{t_k - \epsilon} U(t_k - \epsilon, s)f(s, u(s))ds \right. \\
\quad - & \sum_{k=1}^m c_k \mathbb{B} \int_0^{t_k - \epsilon} U(t_k, s)f(s, u(s))ds \right\|^2 + 4\mathbb{E}\left\|\sum_{k=1}^m c_k \mathbb{B} \int_0^{t_k - \epsilon} U(t_k, s)g(s, u(s))d\mathcal{W}(s) \right. \\
\quad - & \sum_{k=1}^m c_k \mathbb{B} \int_0^{t_k - \epsilon} U(t_k, s)g(s, u(s))d\mathcal{W}(s) \right\|^2 \\
\quad + & \, 4\mathbb{E}\left\|\sum_{k=1}^m c_k \mathbb{B} \int_0^{t_k - \epsilon} U(t_k, s)g(s, u(s))d\mathcal{W}(s) \right\|^2 \\
\leq & \, 4\Lambda \int_{t_k - \epsilon}^{t_k} \varphi_R(s)ds + 4\Lambda\text{Tr}(Q) \int_{t_k - \epsilon}^{t_k} \psi_R(s)ds \\
\to & \, 0 \quad \text{as} \quad \epsilon \to 0.
\end{align*}
(29)

Hence, we have proved that there are relatively compact set $\{(F^*_t u)(0) : u \in B_R\}$ arbitrarily close to the set $\{(F u)(0) : u \in B_R\}$, this means that the set $\{(F u)(0) : u \in B_R\}$ is relatively compact in $H$. Let $0 < t \leq a$ be given, $0 < \epsilon < t$ and $u \in B_R$, we define the operator $F^*_t u$ by
\begin{align*}
(F^*_t u)(t) & = \sum_{k=1}^m c_k U(t, 0) \mathbb{B} \int_0^{t_k} U(t_k, s)f(s, u(s))ds \\
& \quad + U(t, t - \frac{\epsilon}{2})U(t - \frac{\epsilon}{2}, t - \epsilon) \int_0^{t - \epsilon} U(t - \epsilon, s)f(s, u(s))ds \\
& \quad + \sum_{k=1}^m c_k U(t, 0) \mathbb{B} \int_0^{t_k} U(t_k, s)g(s, u(s))d\mathcal{W}(s) \\
& \quad + U(t, t - \frac{\epsilon}{2})U(t - \frac{\epsilon}{2}, t - \epsilon) \int_0^{t - \epsilon} U(t - \epsilon, s)g(s, u(s))d\mathcal{W}(s).
\end{align*}
Since $U(t, t - \frac{\epsilon}{2}) \in L(H)$, $U(t - \frac{\epsilon}{2}, t - \epsilon)$ and $U(t, 0)$ are compact in $H$, the set $\{(F^*_t u)(t) : u \in B_R\}$ is relatively compact in $H$ for every $\epsilon \in (0, t)$. By applying a similar method which used in (29), we can prove that there are relatively compact set $\{(F^*_t u)(t) : u \in B_R\}$ arbitrarily close to the set $\{(F u)(t) : u \in B_R\}$ in $H$ for $0 < t \leq a$. Therefore, the set $\{(F u)(t) : u \in B_R\}$ is also relatively compact in $H$ for $0 < t \leq a$. Hence, combined this with the fact that the set $\{(F u)(0) : u \in B_R\}$ is relatively compact in $H$, we get that the set $\{(F u)(t) : u \in B_R\}$ is relatively compact in $H$ for $0 \leq t \leq a$. 
At least, we show that \( \{Fu : u \in B_R\} \) is a family of equicontinuous functions in \( C([0,a], L^2(\Omega, \mathbb{H})) \). For any \( u \in B_R \) and \( 0 \leq t' < t'' \leq a \), by Lemma 2.5, (5), (15)-(17), (21) and the assumption (H2), we have

\[
\mathbb{E}[(Fu)(t'') - (Fu)(t')]^2 = \mathbb{E} \left| \int_0^a [G(t'', s) - G(t', s)]f(s, u(s))ds \right|^2 \\
+ \int_0^a [G(t'', s) - G(t', s)]g(s, u(s))d\mathcal{W}(s)
\]

\[
\leq 6\mathbb{E} \left| U(t'', 0) - U(t', 0) \right| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)BU(t_k, s)f(s, u(s))ds)^2 \\
+ 6\mathbb{E} \left| \int_0^a \chi_{t'}(s)[U(t'', s) - U(t', s)]f(s, u(s))ds \right|^2 \\
+ 6\mathbb{E} \left| \int_0^a [\chi_{t''}(s) - \chi_{t'}(s)]U(t'', s)f(s, u(s))ds \right|^2 \\
+ 6\mathbb{E} \left| U(t'', 0) - U(t', 0) \right| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)BU(t_k, s)g(s, u(s))d\mathcal{W}(s)\right|^2 \\
+ 6\mathbb{E} \left| \int_0^a \chi_{t'}(s)[U(t'', s) - U(t', s)]g(s, u(s))d\mathcal{W}(s) \right|^2 \\
+ 6\mathbb{E} \left| \int_0^a [\chi_{t''}(s) - \chi_{t'}(s)]U(t'', s)g(s, u(s))d\mathcal{W}(s) \right|^2
\]

\[
\leq 6\mathbb{E} \left| U(t'', 0) - U(t', 0) \right| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)BU(t_k, s)f(s, u(s))ds)^2 \\
+ 6\int_0^{t''} \left| U(t'', s) - U(t', s) \right|^2 \varphi_R(s)ds + 6M^2 \int_0^{t''} \varphi_R(s)ds \\
+ 6\mathbb{E} \left| U(t'', 0) - U(t', 0) \right| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)BU(t_k, s)g(s, u(s))d\mathcal{W}(s)\right|^2 \\
+ 6\mathbb{E} \left| \int_0^{t''} [U(t'', s) - U(t', s)]^2 \psi_R(s)ds + 6M^2 \text{Tr}(Q) \int_0^{t''} \psi_R(s)ds \right| \\
:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\]

where

\[
I_1 = 6\mathbb{E} \left| U(t'', 0) - U(t', 0) \right| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)BU(t_k, s)f(s, u(s))ds)^2,
\]

\[
I_2 = 6\int_0^{t''} \left| U(t'', s) - U(t', s) \right|^2 \varphi_R(s)ds,
\]

\[
I_3 = 6M^2 \int_0^{t''} \varphi_R(s)ds,
\]
In order to prove that $E(\mathcal{F}_u(t'')) - (\mathcal{F}_u(t'))^2 \to 0$ as $t'' - t' \to 0$, we only need to check $I_i \to 0$ independently of $u \in B_R$ when $t'' - t' \to 0$ for $i = 1, 2, \ldots, 6$.

For $I_1$ and $I_4$, by Lemma 2.5, (5), (15)-(17), (20) and the assumption $\text{(H2)}$, we get that

\[ I_1 = 6E \left\| \left[ U(t'', 0) - U(t', 0) \right] \right\|^2, \]

\[ I_5 = 6\text{Tr}(Q) \int_0^{t'} \left\| U(t'', s) - U(t', s) \right\|^2 \psi_R(s) ds, \]

\[ I_6 = 6M^2\text{Tr}(Q) \int_0^{t''} \psi_R(s) ds. \]

Therefore, by Lemma 2.1, (30) and (31) one can easily get that $I_1 \to 0$ and $I_4 \to 0$ as $t'' - t' \to 0$.

For $t' = 0$, $0 < t'' \leq a$, it is easy to see that $I_2 = I_5 = 0$. For $0 < t' < a$ and arbitrary $0 < \delta < t'$, by Lemma 2.4, Lemma 2.5, the assumption $\text{(H2)}$, (5) and the arbitrariness of $\delta$, we get that

\[ I_2 \leq 6 \int_0^{t'-\delta} \left\| U(t'', s) - U(t', s) \right\|^2 \varphi_R(s) ds + 6 \int_{t'-\delta}^{t'} \left\| U(t'', s) - U(t', s) \right\|^2 \varphi_R(s) ds \]

\[ \leq \sup_{s \in [0, t'-\delta]} \left\| U(t'', s) - U(t', s) \right\|^2 \epsilon^{(3)}_{(H)} \cdot 6 \int_0^{t'-\delta} \varphi_R(s) ds + 12M^2 \int_{t'-\delta}^{t'} \varphi_R(s) ds \]

\[ \to 0 \quad \text{as} \quad t'' - t' \to 0 \quad \text{and} \quad \delta \to 0, \]

and

\[ I_5 \leq 6\text{Tr}(Q) \int_0^{t'-\delta} \left\| U(t'', s) - U(t', s) \right\|^2 \psi_R(s) ds \]

\[ + 6\text{Tr}(Q) \int_{t'-\delta}^{t'} \left\| U(t'', s) - U(t', s) \right\|^2 \psi_R(s) ds \]

\[ \leq \sup_{s \in [0, t'-\delta]} \left\| U(t'', s) - U(t', s) \right\|^2 \epsilon^{(3)}_{(H)} \cdot 6\text{Tr}(Q) \int_0^{t'-\delta} \psi_R(s) ds \]

\[ + 12M^2\text{Tr}(Q) \int_{t'-\delta}^{t'} \psi_R(s) ds \]

\[ \to 0 \quad \text{as} \quad t'' - t' \to 0 \quad \text{and} \quad \delta \to 0. \]
For $I_3$ and $I_6$, by the assumption (H2), we get that

$$I_3 = 6M^2 \int_{t'}^{t''} \varphi_R(s)ds \to 0 \quad \text{as} \quad t'' - t' \to 0$$

and

$$I_6 = 6M^2 \text{Tr}(Q) \int_{t'}^{t''} \psi_R(s)ds \to 0 \quad \text{as} \quad t'' - t' \to 0.$$ 

As a result, we have proved that $\mathbb{E}\| (\mathbb{F}u)(t'') - (\mathbb{F}u)(t')\|^2 \to 0$ independently of $u \in B_R$ as $t'' - t' \to 0$, which means that the operator $\mathbb{F} : B_R \to B_R$ is equicontinuous. Hence, combined with the Arzela-Ascoli theorem one gets that $\mathbb{F} : \Omega_R \to \Omega_R$ is a compact operator. Therefore, by Schauder Fixed Point Theorem we know that $\mathbb{F}$ has at least one fixed point $u \in B_R$, which is in turn a mild solution of NSEE (1)-(2) on $[0, a]$. This completes the proof of Theorem 3.1.

**Remark 1.** As the reader can see, Theorem 3.1 in this paper extends the main results of autonomous stochastic evolution equations with nonlocal initial conditions in [10], [11], [12], [20], [27], [41], [48] and [49] to non-autonomous stochastic evolution equations with nonlocal initial conditions. This distinguishes the present paper from earlier works on stochastic evolution equations with nonlocal initial conditions.

**Remark 2.** Note that Theorem 3.1 extends the studying of the papers [15], [16], [17], [18], [34], [46] and [47], which investigate the solvability of non-autonomous parabolic evolution equations with nonlocal initial conditions, to the case of stochastic non-autonomous evolution equations with with nonlocal initial conditions.

4. **An example.** In order to illustrate the applicability of our main results, we consider the following non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions

$$\begin{aligned}
\frac{\partial}{\partial t} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) + a(t)u(x,t) &= \frac{\cos(x,t,u(x,t))}{1+t^2+|x|} + \frac{\sqrt{2}\sin(\pi t)d\mathbb{W}(t)}{1+|u(x,t)|dt}, \quad x \in [0, \pi], \\
u(0,t) &= u(\pi,t) = 0, \quad t \in [0,1], \\
u(x,0) &= \sum_{k=1}^{m} c_k u(x,t_k), \quad x \in [0, \pi],
\end{aligned}$$

(32)

where $\mathbb{W}(t)$ denotes a one-dimensional standard cylindrical Wiener process defined on a stochastic space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $c_k$ are real numbers, $c_k \neq 0$, $k = 1, 2, \cdots, m$, $a : [0,1] \to \mathbb{R}$ is a continuously differentiable function and satisfies

$$a_{\min} := \min_{t \in [0,1]} a(t) > -1. \quad (33)$$

Let $\mathbb{H} = L^2([0, \pi], \mathbb{R})$ with the norm $\| \cdot \|_2$ and inner product $\langle \cdot, \cdot \rangle$. Consider the operator $B$ on $\mathbb{H}$ defined by

$$Bu = \frac{\partial^2}{\partial x^2} u$$

(34)

with domain

$$D(B) = \{ u \in L^2([0, \pi], \mathbb{R}), u'' \in L^2([0, \pi], \mathbb{R}) \text{ and } u(0) = u(\pi) = 0 \}.$$ 

It is well known from Pazy [39] that $B$ has a discrete spectrum, and its eigenvalues are $-n^2$, $n \in \mathbb{N}^+$ with the corresponding normalized eigenvectors $v_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx).$
Define the operator $A(t)$ on $L^2([0,\pi],\mathbb{R})$ by
\[ A(t)u = Bu - a(t)u \] \hspace{1cm} (35)
with domain $D(A(t)) = D(B)$, $t \in [0,1]$. \hspace{1cm} (36)
It follows from [39, Lemma 6.1 in Chapter 7] that there are constants $\theta \in (\frac{\pi}{2},\pi)$ and $M_1 \geq 0$ such that $A(t)$ satisfy the condition (AT\(_1\)). Furthermore, by again [39, Lemma 6.1 in Chapter 7] together with continuously differentiable of coefficient $a(t)$ one know that there exist constants $M_2 > 0$ and $\vartheta, \beta \in (0,1]$ such that for all $\lambda \in \Sigma_\theta$ and $0 \leq s \leq t \leq 1$, the condition (AT\(_2\)) is satisfied. Therefore, the family $\{A(t) : 0 \leq t \leq 1\}$ generates an strongly continuous evolution family $\{U(t,s) : 0 \leq s \leq t \leq 1\}$ defined by
\[ U(t,s)u = \sum_{n=1}^{\infty} e^{-\left(\int_s^t a(\tau)d\tau + n^2(t-s)\right)} \langle u, v_n \rangle v_n, \quad 0 \leq s \leq t \leq 1, \ u \in \mathbb{H}. \] \hspace{1cm} (37)
A direct calculation gives
\[ \|U(t,s)\|_{L(\mathbb{H})} \leq e^{-(1+a_{\min})}, \quad 0 \leq s \leq t \leq 1. \] \hspace{1cm} (38)
(33) and (38) means that \[ M := \sup_{0 \leq s \leq t \leq 1} \|U(t,s)\|_{L(\mathbb{H})} = 1. \] \hspace{1cm} (39)
Note also from [32] that, for each $t, s \in [0,1]$ with $t > s$, the evolution family $U(t,s)$ is a nuclear operator, which implies the compactness of $U(t,s)$ for $t > s$.

Let $u(t) = u(\cdot, t)$, $f(t,u(t)) = \frac{\cos(-t,u(t))}{1+t^2}$, $g(t,u(t)) = \frac{\sqrt{2}\sin(\pi t)}{1+|u(t)|}$, then the non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions (32) can be rewritten into the abstract form of NSEE (1)-(2) in $L^2([0,\pi],\mathbb{R})$.

**Theorem 4.1.** If $\sum_{k=1}^{m} |c_k| < 1$, then the non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions (32) has at least one mild solution.

**Proof.** By the assumption $\sum_{k=1}^{m} |c_k| < 1$ and (39) it is easy to see that the assumption (H1) hold. From the definition of nonlinear functions $f$ and $g$, we can easily to verify that the assumption (H2) hold with
\[ \varphi_r(t) = \pi t^{-\frac{1}{2}}, \quad \psi_r(t) = 2 \sin^2(\pi t), \quad \rho_1 = \rho_2 = 0. \] \hspace{1cm} (40)
From (40) one can easily to verify that the condition (20) is satisfied. Therefore, all the assumptions of Theorem 3.1 are satisfied. Hence, the non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions (32) has at least one mild solution due to Theorem 3.1. This completes the proof of Theorem 4.1.

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REFERENCES

[1] P. Acquistapace, Evolution operators and strong solution of abstract parabolic equations, *Differential Integral Equations*, 1 (1988), 433–457.

[2] P. Acquistapace and B. Terreni, A unified approach to abstract linear parabolic equations, *Rend. Semin. Mat. Univ. Padova*, 78 (1987), 47–107.

[3] H. Amann, Parabolic evolution equations and nonlinear boundary conditions, *J. Differential Equations*, 72 (1988), 201–269.

[4] J. Bao, Z. Hou and C. Yuan, Stability in distribution of mild solutions to stochastic partial differential equations, *Proc. Amer. Math. Soc.*, 138 (2010), 2169–2180.

[5] L. Byszewski, Application of properties of the right hand sides of evolution equations to an investigation of nonlocal evolution problems, *Nonlinear Anal.*, 33 (1998), 413–426.

[6] P. Chen and Y. Li, Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions, *Z. Angew. Math. Phys.*, 65 (2014), 711–728.

[7] P. Chen and Y. Li, Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions, *Collect. Math.*, 66 (2015), 63–76.

[8] P. Chen, X. Zhang and Y. Li, Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions, *Results Math.*, 63 (2013), 731–744.

[9] P. Chen and Y. Li, Existence of solutions for fractional evolution equations with nonlocal conditions, *Fract. Calcu. Appl. Anal.*, 14 (2011), 1507–1526.

[10] P. Chen, A. Abdelmonem and Y. Li, Global existence and asymptotic stability of mild solutions for fractional evolution equations, *Adv. Differential Integral Equations*, 8 (2017), 1259–1284.

[11] P. Chen, X. Zhang, Y. Li, Nonlocal problem for fractional stochastic evolution equations with solution operators, *Fract. Calcu. Appl. Anal.*, 19 (2016), 1507–1526.

[12] P. Chen, X. Zhang and Y. Li, Nonlocal problem for fractional stochastic evolution equations with nonlocal conditions, *J. Integral Equations Appl.*, 29 (2017), 325–348.

[13] P. Chen, X. Zhang, Y. Li, Study on fractional non-autonomous evolution equations with delay, *Comput. Math. Appl.*, 73 (2017), 794–803.

[14] P. Chen, X. Zhang and Y. Li, A blowup alternative result for fractional nonautonomous evolution equation of Volterra type, *Commun. Pure Appl. Anal.*, 17 (2018), 1975–1992.

[15] P. Chen, X. Zhang and Y. Li, Approximate controllability of non-autonomous evolution system with nonlocal conditions, *J. Dyn. Control. Syst.*, 26 (2020), 1–16.

[16] P. Chen, X. Zhang and Y. Li, Fractional non-autonomous evolution equation with nonlocal conditions, *J. Pseudo-Differ. Oper. Appl.*, 10 (2019), 955–973.

[17] P. Chen, X. Zhang and Y. Li, Cauchy problem for fractional non-autonomous evolution equations, *Banach J. Math. Anal.*, 14 (2020), 559–584.

[18] P. Chen, X. Zhang and Y. Li, Existence and approximate controllability of fractional evolution equations with nonlocal conditions via resolvent operators, *Fract. Calcu. Appl. Anal.*, 23 (2020), 268–291.

[19] P. Chen, Y. Li and X. Zhang, Cauchy problem for stochastic non-autonomous evolution equations governed by noncompact evolution families, *Discrete Contin. Dyn. Syst. Ser. B.*, 25 (2020), 1709–1727.

[20] J. Cui, L. Yan and X. Wu, Nonlocal Cauchy problem for some stochastic integro-differential equations in Hilbert spaces, *J. Korean Stat. Soci.*, 41 (2012), 279–290.

[21] R. F. Curtain and P. L. Falb, Stochastic differential equations in Hilbert space, *Proc. Amer. Math. Soc.*, 13 (1962), 141–148.

[22] R. F. Curtain and P. L. Falb, *Stochastic Differential Equations in Hilbert Space*, Academic Press, New York, 1970.

[23] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.*, 179 (1993), 630–637.

[24] M. M. EI-Borai, O. L. Mostafa and H. M. Ahmed, Asymptotic stability of some stochastic evolution equations, *Appl. Math. Comput.*, 144 (2003), 273–286.

[25] K. Ezzinbi, X. Fu and K. Hilal, Existence and regularity in the α-norm for some neutral partial differential equations with nonlocal conditions, *Nonlinear Anal.*, 67 (2007), 1613–1622.

[26] Z. Fan and G. Li, Existence results for semilinear differential equations with nonlocal and impulsive conditions, *J. Funct. Anal.*, 258 (2010), 1709–1727.

[27] S. Farahi and T. Guendouzi, Approximate controllability of fractional neutral stochastic evolution equations with nonlocal conditions, *Results. Math.*, 65 (2014), 501–521.
[28] W. E. Fitzgibbon, *Semilinear functional equations in Banach space*, J. Differential Equations, 29 (1978), 1–14.

[29] X. Fu, Existence of solutions for non-autonomous functional evolution equations with nonlocal conditions, *Electron. J. Differential Equations*, 2012 (2012), No. 110, 15 pp.

[30] X. Fu, Approximate controllability of semilinear non-autonomous evolution systems with state-dependent delay, *Evol. Equ. Control Theory*, 6 (2017), 517–534.

[31] W. Grecksch and C. Tudor, *Stochastic Evolution Equations: A Hilbert Space Approach*, Academic Verlag, Berlin, 1995.

[32] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, New York, 1981.

[33] J. Liang, J. Liu and T.-J. Xiao, Nonlocal Cauchy problems governed by compact operator families, *Nonlinear Anal.*, 57 (2004), 183–189.

[34] J. Liang, J. H. Liu and T.-J. Xiao, Nonlocal Cauchy problems for nonautonomous evolution equations, *Commun. Pure Appl. Anal.*, 5 (2006), 529–535.

[35] K. Liu, *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*, Chapman and Hall/CRC, Boca Raton, FL, 2006.

[36] J. Luo, Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays, *J. Math. Anal. Appl.*, 342 (2008), 753–760.

[37] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Ltd., Chichester, 1997.

[38] M. McKibben, *Discovering Evolution Equations with Applications*, Vol. I Deterministic Models, Chapman and Hall/CRC Appl. Math. Nonlinear Sci. Ser., 2011.

[39] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[40] Y. Ren, Q. Zhou and L. Chen, Existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with poisson jumps and infinite delay, *J. Optim. Theory Appl.*, 149 (2011), 315–331.

[41] R. Sakthivel, Y. Ren, A. Debbouche and N. I. Mahmudov, Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions, *Appl. Anal.*, 95 (2016), 2361–2382.

[42] K. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic Publishers, Dordrecht, 1991.

[43] H. Tanabe, *Functional Analytic Methods for Partial Differential Equations*, Marcel Dekker, New York, USA, 1997.

[44] T. Taniguchi, K. Liu and A. Truman, Existence, uniqueness and asymptotic behavior of mild solutions to stochastic functional differential equations in Hilbert spaces, *J. Differential Equations*, 181 (2002), 72–91.

[45] I. I. Vrabie, Delay evolution equations with mixed nonlocal plus local initial conditions, *Commun. Contemp. Math.*, 17 (2015), 1350035.

[46] R.-N. Wang, K. Ezzinbi and P.-X. Zhu, Non-autonomous impulsive Cauchy problems of parabolic type involving nonlocal initial conditions, *J. Integral Equations Appl.*, 26 (2014), 275–299.

[47] R. N. Wang and P. X. Zhu, Non-autonomous evolution inclusions with nonlocal history conditions: Global integral solutions, *Nonlinear Anal.*, 85 (2013), 180–191.

[48] X. Zhang, P. Chen, A. Abdelmonem and Y. Li, Fractional stochastic evolution equations with nonlocal initial conditions and noncompact semigroups, *Stochastics*, 90 (2018), 1005–1022.

[49] X. Zhang, P. Chen, A. Abdelmonem and Y. Li, Mild solution of stochastic partial differential equation with nonlocal conditions and noncompact semigroups, *Math. Slovaca*, 69 (2019), 111–124.

[50] B. Zhu, L. Liu and Y. Wu, Local and global existence of mild solutions for a class of nonlinear fractional reaction-diffusion equations with delay, *Appl. Math. Lett.*, 61 (2016), 73–79.

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