Universal contact and collective excitations of a strongly interacting Fermi gas

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We study the relationship between Tan’s contact parameter and the macroscopic dynamic properties of an ultracold trapped gas, such as the frequencies of the collective oscillations and the propagation of sound in one-dimensional (1D) configurations. We find that the value of the contact, extracted from the most recent low-temperature measurements of the equation of state near unitarity, reproduces with accuracy the experimental values of the collective frequencies of the radial breathing mode at the lowest temperatures. The available experiment results for the 1D sound velocities near unitarity are also investigated.

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I. INTRODUCTION

Significant experimental and theoretical work has been devoted in recent years to understand the universal properties of interacting Fermi gases along the BEC-BCS crossover (for a review, see, for example, [1]). More recently, Tan has introduced a new concept for investigating universality based on the so-called contact parameter, which relates the short-range features of these systems to their thermodynamic properties [2, 3]. The universality of the Tan’s relations has been proven in a series of experiments based on the molecular fraction [4], the momentum distribution, the RF spectroscopic rate at high frequencies, the adiabatic sweep and virial theorems [3], the spin structure factor [6] and the equation of state [7]. The temperature dependence of the contact parameter has been the object of recent theoretical [8] and experimental [9] work.

In this paper we discuss the relationship between Tan’s contact parameter and the frequencies of the collective oscillations of a harmonically trapped Fermi gas near unitarity. We also investigate the relationship with the sound velocity in highly elongated configurations. The study of the collective oscillations along the BEC-BCS crossover in terms of the contact parameter has been already addressed in [10] where upper bounds to the collective frequencies were calculated using a sum rule approach. However, the sum rule method developed in this work significantly overestimates the hydrodynamic frequencies of trapped Fermi gases, being consequently ineffective for a useful quantitative comparison with experimental data in the superfluid hydrodynamic regime of low temperatures.

The present approach is based on a perturbative solution of the hydrodynamic equations of superfluids near unitarity [11]. This allows for an exact analytic relationship between Tan’s contact parameter and the deviations of the frequencies of the collective oscillations as well as of the one-dimensional (1D) sound velocity from their values at unitarity. The high precision achievable in the frequency measurements is in particular expected to provide an alternative accurate determination of the contact parameter and to further confirm the universality of this physical quantity.

II. CONTACT, EQUATION OF STATE AND HYDRODYNAMIC EQUATIONS

We start from the following definition of the contact parameter, based on the so-called adiabatic sweep theorem [2]:

\[
\left[ \frac{dE}{d(1/a)} \right]_N = -\frac{\hbar^2 I}{4\pi M},
\]

where \(E\) is the total energy of the system, \(I\) is the contact parameter, \(M\) is the atomic mass, \(a\) is the s-wave scattering length. By using the local density approximation (LDA), the total energy can be calculated as \(E = \int d^3r (\epsilon + nV_{\text{ext}})\), where \(V_{\text{ext}}\) is the trapping potential and \(\epsilon\) is the energy density of a uniform gas. The equilibrium profile, in the LDA, is determined by the equation

\[
\mu(n) + V_{\text{ext}} = \bar{\mu},
\]

where

\[
\mu(n) = \frac{\partial \epsilon(n, a)}{\partial n}
\]

is the chemical potential of uniform matter, providing the equation of state of the gas, while \(\bar{\mu}\) is the chemical potential of the trapped system, fixed by the normalization condition. The derivative of the total energy with respect to \(1/a\) in Eq. (1) can then be conveniently written as

\[
\left[ \frac{dE}{d(1/a)} \right]_N = \int d^3r \left[ \frac{\partial \epsilon(n, a)}{\partial (1/a)} \right]_n,
\]

exploiting the link between the contact parameter and the equation of state.

On the other hand the chemical potential of uniform matter \(\mu(n)\) is a crucial ingredient of the hydrodynamic equations of superfluids. At zero temperature, these
equations actually read \[12\]:

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla (\mathbf{v} n) &= 0, \\
\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{M} \nabla \left[ \frac{1}{2} M \mathbf{v}^2 + \mu(n) + V_{\text{ext}} \right] &= 0,
\end{align*}
\]

(5)

so that an insightful question is to understand the link between the contact parameter and the behavior of the collective modes emerging from the solutions of Eq. (5). In the following we will discuss the problem near unitarity, where we can expand the chemical potential in the form \[2, 11\]

\[
\mu(n) = \frac{\hbar^2}{2M} \xi \left(3\pi^2 n\right)^{2/3} - \frac{2\hbar^2}{5M} \zeta \left(3\pi^2 n\right)^{1/3}
\]

(6)

with \(\xi\) and \(\zeta\) being two universal dimensionless parameters accounting for the effects of the interactions. The first term in (6) exhibits the same density dependence as the ideal Fermi gas with the renormalization factor \(\xi\). The second term (first-order correction in \(1/a\)) is directly related to Tan’s parameter calculated at unitarity. Indeed, using Eqs. (11), one easily finds that the contact, for a harmonically trapped system, is given by

\[
\frac{I}{N k_F} = \frac{512}{175} \xi,
\]

(7)

where we have defined the Fermi wave vector \(k_F = [3\pi^2 n(0)]^{1/3}\) depending on the density in the center of the trap.

The small-amplitude oscillations of the gas can be studied by solving the linearized hydrodynamic equations (5),

\[-\omega^2 \delta n = \frac{1}{M} \nabla \cdot \left( n \nabla \left[ \frac{\partial \mu(n)}{\partial n} \delta n \right] \right),
\]

(8)

where \(\delta n\) is the amplitude of the density oscillations around the equilibrium value \(n\). At unitarity \(\mu(n) \propto n^{2/3}\), and the solutions of Eq. (8), in the presence of harmonic trapping with axial symmetry, exhibit the analytic form

\[
\omega^2(\lambda) = \frac{\omega_z^2}{3} \left[ (4\lambda^2 + 5) \pm \sqrt{16\lambda^4 - 32\lambda^2 + 25} \right],
\]

(9)

with \(\lambda = \omega_z/\omega_\perp\), holding for the lowest \(m = 0\) modes, and include the coupling between the monopole and quadrupole oscillations caused by the non spherical shape of the potential (here \(m\) is the \(z\) component of angular momentum carried by the excitation). Notice that, remarkably, result (9) does not depend on the parameter \(\xi\) characterizing the equation of state at unitarity. This relation for the collective frequencies was actually first obtained in contexts different from the unitary Fermi gas, such as the ideal Bose gas above \(T_c\) \[13, 14\] and the ideal Fermi gas \[15\] in the hydrodynamic regime. In both cases the equation of state \(\mu(n, s)\) actually exhibits the same \(n^{2/3}\) dependence for fixed entropy per particle \(s\) and the hydrodynamic equations yield the same dispersion law \[9\] for the scaling solutions of coupled quadrupole monopole type in the presence of harmonic trapping. The prediction \[11\] for the collective frequencies was checked experimentally at unitarity providing a direct confirmation of the universality exhibited by the unitary Fermi gas \[16, 17\].

The small-amplitude oscillations of the gas can be obtained starting from the expansion of \(\mu(n)\) around the zero-order ground-state density profile \(n_0(r)\) and using the equilibrium condition \[2\] in LDA. This gives

\[
\frac{\delta \omega}{\omega} = -\frac{\int d^3 r \left( \nabla^2 f_0 \right) \left[ n_1 - n_0 \left( \partial n_1/\partial n_0 \right) \right] f_0}{2\omega^2 M \int d^3 r f_0^* \delta n_0}
\]

(11)

for the collective oscillations caused by the perturbation \(\mu_1(n)\) in the chemical potential. In Eq. (11) \(f_0 = (\partial \mu_0/\partial n_0)\delta n_0\) is the zero-order eigenfunction of (5), and \(n_1\) is given by (10). According to Eq. (11), one always has \(\delta \omega = 0\) for the surface modes satisfying the condition \(\nabla^2 f_0 = 0\), as expected, due to their independence of the equation of state. For compression modes one instead expects a correction due to the changes in the equation of state. In general, one can show that the first term in (10) proportional to \(\delta \mu\) gives no contribution to the frequency shift and will be consequently neglected in the following \[20\].

Result (11) is valid for both Bose and Fermi systems. In the case of weakly interacting Bose-Einstein condensed gas, it allows for the calculation of the frequency shifts caused by the Lee-Huang-Yang corrections in the equation of state. In this case, the corresponding density correction, neglecting the term proportional to \(\delta \mu\), can be

**III. COLLECTIVE FREQUENCIES AND SOUND VELOCITY SHIFTS NEAR UNITARITY**

In the following we calculate the deviations of the collective frequencies from the unitary value \[2\] holding for small values of the dimensionless parameter \(1/k_F a\). To this purpose we solve the hydrodynamic equations using a perturbative procedure, which is generally applicable to any equation of state having the form \(\mu(n) = \mu_0(n) + \mu_1(n)\), where \(\mu_1(n) \ll \mu_0(n)\) represents the first-order correction. The density profile \(n(r) = n_0(r) + n_1(r)\), including the first-order correction, can be obtained starting from the expansion of \(\mu(n)\) around the zero-order ground-state density profile \(n_0(r)\) and using the equilibrium condition \[2\] in LDA. This gives

\[
n_1 = \left[ \delta \mu - \mu_1(n_0) \right] \left\{ \frac{\partial \mu_0(n_0)}{\partial n_0} \right\},
\]

(10)

where \(\delta \mu\) is the first-order correction to the chemical potential \(\mu\). Solving the linearized hydrodynamic equation (8) perturbatively, we find the following expression for the frequency shift:

\[
\frac{\delta \omega}{\omega} = -\frac{\int d^3 r \left( \nabla^2 f_0 \right) \left[ n_1 - n_0 \left( \partial n_1/\partial n_0 \right) \right] f_0}{2\omega^2 M \int d^3 r f_0^* \delta n_0}
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written as \( n_1 = -32(n_0 a)^{3/2}/3\sqrt{\pi} \), and using Eq. (11), one finds the frequency shift of the compression mode as derived in [21]. For the Fermi gas near unitarity, we instead employ the expansion of the equation of state around the density profile \( n_0(r) \) calculated at unitarity. Ignoring also in this case the irrelevant term proportional to \( \delta \bar{\mu} \), we find

\[
n(r) = n_0(r) + n_1(r) = n_0(r) - \frac{3\beta}{2\alpha} n_0^{2/3}(r),
\]

where \( \alpha = \hbar^2 \xi(3\pi^2)^{2/3}/2M, \beta = -2\hbar^2 \xi(3\pi^2)^{1/3}/(5Ma) \), and \( n_0 = [\bar{\mu}_0 - V_{\text{ext}}]/\alpha^{3/2} \), with \( \bar{\mu}_0 \) being the chemical potential evaluated for a trapped system at unitarity. After some straightforward algebra, one finds the following expression for the frequency shift of the collective oscillations near unitarity:

\[
\frac{\delta \omega}{\omega} = \frac{\beta}{6\omega^2 M} \int d^3 r \left( \frac{\nabla^2 f_0}{f_0} \right)^{2/3} f_0 \int d^3 r \left( \frac{\nabla^2 f_0 n_0^{2/3}}{f_0} \right).
\]

One can check that this result is equivalent to the result of Eq. (24) and (25) in [11] where a similar expansion was carried out near unitarity.

The eigenfunctions for the \( m = 0 \) modes [9] have the form \( f_0 \sim a + b r^2 + c z^2 \), where

\[
\frac{a}{b} = -\frac{\bar{\mu}_0 [4\lambda^2 + 3(\omega/\omega_\perp)^2 - 10]}{6M\omega_z^2}, \quad \frac{c}{b} = \frac{3(\omega/\omega_\perp)^2 - 10}{2},
\]

After some length but straightforward algebra, one finally obtains the following result for the frequency shift:

\[
\frac{\delta \omega}{\omega} = \left[ \frac{128\xi}{525\pi \xi} \eta_{\pm}(\lambda) \right] (k_F a)^{-1} = \left[ \frac{I/\sqrt{Nk_F^3}}{12\pi \xi^{1/2}} \eta_{\pm}(\lambda) \right] (k_F^0 a)^{-1},
\]

where

\[
\eta_{\pm}(\lambda) = \frac{1}{2} \pm \frac{3}{2\sqrt{16\lambda^2 - 32\lambda^2 + 25}},
\]

with the index \( \pm \) referring to the higher (+) and lower (−) frequencies of Eq. (9). In the second equality of (15), we have used relation (7) for the contact parameter calculated for an harmonically trapped atomic cloud and we have expressed the Fermi momentum \( k_F \) in terms of the ideal Fermi gas wave vector \( k_F^0 = (24N)^{1/6}a_0^{-1} \), with \( a_0 \) being the geometrical average of the harmonic oscillator lengths. For the same total number of atoms, the density \( n(0) \) in the center of the trap for the unitary gas is \( \xi = 3/4 \) times larger than for the ideal gas, yielding \( k_F = k_F^0 \xi^{-1/4} \).

Eq. (15) represents the main result of the present paper. It relates Tan’s contact, a central quantity for the universality relations holding in interacting systems, with the low-energy macroscopic dynamics of the system, namely, the frequencies of the collective oscillations. These equations can be used either to predict theoretically the frequency shifts, once the dimensionless parameters \( \xi \) and \( \zeta \) or Tan’s contact are known, or to determine experimental constraints on the value of \( I \). In Fig. 1 we plot \( \eta_{\pm} \) as a function of the deformation parameter \( \lambda \).

For a spherical trap (\( \lambda = 1 \)) one obtains \( \eta_+ = 1 \) for the monopole mode and \( \eta_- = 0 \), confirming, as already anticipated, that there is no frequency shift for the surface quadrupole mode. When \( \lambda \neq 1 \), the two modes are coupled. In the limit of both spherical traps and highly elongated traps (\( \lambda \ll 1 \)) it reproduces the results of [11].

In Fig. 2 we show the prediction of Eq. (15) in the trapping conditions of the experiment of [17], using the values \( \xi = 0.41, \zeta = 0.93 \) extracted from the direct measure-
ment of the equation of state \([7]\) carried out at the lowest temperatures and yielding the value \(I/Nk_F^0 \approx 3.4\) for the contact parameter. These values differ by a few percent from the most recent theoretical predictions based on Monte Carlo simulations at \(T = 0\) (see, for example, \([22]\) and references therein). The predicted slope \(\delta \omega/\omega \sim 0.11/(k_Fa)\) (black line) of the collective frequencies of the radial breathing mode at unitarity turns out to be in very good agreement with experiments \((\delta \omega/\omega \sim 0.12/(k_Fa)\) \([23]\)).

In order to appreciate the sensitivity of the slope to the choice of the values of \(\xi\) and \(\zeta\), in Fig. \(3\) we also show the predictions [green (light gray) line] for the frequency shifts using the values \(\xi = 0.41\) and \(\zeta = 0.74\), yielding the smaller value \(I/Nk_F^0 \approx 2.7\) for the contact. This value is closer to the measurement of the contact carried out in \([5]\) and \([9]\) at slightly higher values of temperature. The resulting slope \(\delta \omega/\omega \sim 0.09/(k_Fa)\) provides a worse description of the experimental data for the collective frequencies.

The collective oscillations discussed above represent the discretized values of the usual sound waves described by hydrodynamics. It is actually useful to calculate also the changes of the sound velocity of a uniform sample near unitarity in terms of the dimensionless parameters \(\xi\) and \(\zeta\) or, equivalently, Tan’s contact parameter. For bulk Fermi gases, one finds \(\delta c/c = -\zeta/(\delta \xi k_F a)\).

In the case of cylindrical geometry with radial harmonic trapping the sound velocities can be also calculated starting from the 1D hydrodynamic expression \([24]\)

\[
c_1 = \left\{ \frac{1}{M} \int d^2 r_{\perp} n / \int d^2 r_{\perp} \left[ \frac{\partial \mu(n)}{\partial n} \right]^{-1} \right\}^{1/2},
\]

where the the density profile along the radial direction should be evaluated in LDA. The above results hold for sound waves characterized by wavelengths significantly larger than the radial size of the gas. Carrying out a perturbative development around unitarity, similar to the one employed above for the calculation of the excitation frequencies, we obtain the result

\[
\frac{\delta c_1}{c_1} = -\frac{3\zeta}{20\xi} (k_F a)^{-1},
\]

where \(c_1 = (\xi/5)^{1/2}(h k_F/M)\) is the 1D sound velocity depending on the density in the center of the trap via \(k_F\). Notice that \(c_1\) differs from the sound velocity calculated at the central value of the density by the factor \(\sqrt{3/5}\).

In Fig. \(3\) we show the experimental values of \(c_1\) in the unitary regime measured in \([25]\) together with the slope evaluated from Eq. \(13\) using for \(\xi\) and \(\zeta\) the \(T \approx 0\) values obtained from the experiment \([7]\). A quadratic fit is applied to the experiment data (dashed red curve) \([26]\). The figure shows that the value of the slope at unitarity [black curve, \(\delta c_1/c_1 \sim -0.27/(k_F a)\)] overestimates the experimental linear changes in a visible way, suggesting that these experimental data were carried out at relatively higher temperatures, such that they cannot be accurately reproduced by employing the \(T = 0\) values of the contact parameter. Another source of disagreement might be due to the fact that the conditions of applicability of the 1D expression \([17]\) for the sound velocity are not fully satisfied in the experiment of \([25]\).

IV. CONCLUSION

In conclusion our analysis reveals consistency between the experimental results for the contact, obtained through the measurement of the equation of state carried out at \(T \approx 0\) in \([7]\), and the behavior of the collective frequencies carried out in \([17]\) at the lowest temperatures. It would be interesting to extend our analysis of the frequency shifts to finite temperature. The analysis could be simplified by the fact that the scaling modes of monopole and quadrupole types have a universal behavior at unitarity and their frequencies, calculated in the hydrodynamic regime, are independent of temperature. Furthermore they are not coupled to second sound. The proper calculation of the resulting slope at finite temperature and the explicit connection with Tan’s contact parameter at finite temperature will be the object of a future work.

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