New solutions to the confluent Heun equation and quasiexactly solvability

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Abstract: We construct new solutions in series of confluent hypergeometric functions for the confluent Heun equation (CHE). Some of these solutions are applied to the one-dimensional stationary Schrödinger equation with hyperbolic and trigonometric quasiexactly solvable potentials.

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I. INTRODUCTION

By means of an integral transformation, recently [1] we have found a new solution for the spheroidal wave equation and pointed out that such solution can generate a new group of solutions for the confluent Heun equation (CHE) – this equation is more general than the spheroidal one. Now we establish explicitly that group of solutions and discuss, as illustration, possible applications to the one-dimensional stationary Schrödinger equation with hyperbolic and trigonometric quasiexactly solvable potentials.

For the CHE, or generalized spheroidal wave equation [2], we use the form [3]

\[ z(z - z_0) \frac{d^2U}{dz^2} + (B_1 + B_2z) \frac{dU}{dz} + \left[B_3 - 2\omega \eta (z - z_0) + \omega^2 z(z - z_0)\right] U = 0, \]  

where \( z_0, B_i, \eta \) and \( \omega \) are constants. In applications, we consider a particular case known as Whittaker-Hill equation (WHE) or Hill’s equation with three terms. It is written in the form [4, 5]

\[ \frac{d^2W}{du^2} + \zeta^2 \left[ \vartheta - \frac{1}{8} \epsilon^2 - (p + 1)\xi \cos(2\zeta u) + \frac{1}{8} \epsilon^2 \cos(4\zeta u) \right] W = 0, \quad (\zeta = 1, i) \]  

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where \( \vartheta, \xi \) and \( p \) are parameters; if \( u \) is a real variable, this represents the WHE when \( \varsigma = 1 \) and the modified WHE when \( \varsigma = i \). The substitutions

\[
W(u) = U(z), \quad z = \cos^2(\varsigma u), \quad [\varsigma = 1, i]
\]

(3a)

transform the WHE (2) into the CHE (1) with

\[
z_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = 1, \quad B_3 = \frac{(p+1)(\xi - \vartheta)}{4}, \quad i\omega = \frac{\xi}{2}, \quad i\eta = \frac{p+1}{2}.
\]

(3b)

The (ordinary) spheroidal wave equation \([15]\), given in Appendix B, is another case of CHE which has only three parameters as the WHE.

We consider solutions for the CHE given by series whose coefficients satisfy three-term recurrence relations. Regarding the range of values assumed by the summation index \( n \), we distinguish three types of series: two-sided infinite series if \( n \) runs from minus to plus infinity, one-sided infinite series if \( n \geq 0 \), and finite series if \( n \) has a lower and an upper limit. Two-sided infinite series are necessary to assure the convergence of solutions for equations in which there is no free parameter. However, when we truncate the series on the left-hand side by requiring that \( n \geq 0 \), the two-sided series give one-sided infinite series which are suitable for equations with a free parameter; in turn, one-sided infinite series become finite series for special values of the equation parameters (Appendix A).

In section II we study sets of one-sided expansions in series of confluent hypergeometric functions. Each set is constituted by three solutions, one in series of irregular confluent hypergeometric functions, and two in series of regular functions: in general the former converges near the infinity, whereas the others converge for finite values of \( z \). In section III we show how such solutions can be transformed into two-sided series; these must satisfy the ratio test for \( n \to \infty \) and for \( n \to -\infty \), a fact which restricts the regions of convergence of some solutions (in comparison with the regions of one-sided series). In both cases, the solutions are valid under some constraints on the parameters of the CHE. Consequences of these constraints appear in sections IV and V where we apply the three types of series to the stationary Schrödinger equation with quasiexactly solvable (QES) potentials. Finally, in section VI we present some conclusions and mention issues which deserve further investigations.

We recall that for QES quantum-mechanical problems one part of energy spectrum and the respective eigenfunctions can be computed explicitly \([7, 8]\). If a QES problem obeys an equation of the Heun family \([9]\), that part of the spectrum may be derived from finite-series solutions if these are known. Indeed, a problem is QES if it admits solutions given by finite series whose coefficients necessarily satisfy three-term or higher order recurrence relations, and the problem is exactly solvable if admits solutions given by hypergeometric functions \([10]\). This definition opens the possibility of finding the remaining part of the spectrum from one- or two-sided infinite-series solutions for the Heun equations.

We write the one-dimensional Schrödinger equation for a particle with mass \( M \) and energy \( E \) as

\[
\frac{d^2\psi}{du^2} + [E - V(u)]\psi = 0, \quad u = ax, \quad E = \frac{2M}{\hbar^2a^2}E, \quad V(u) = \frac{2M}{\hbar^2a^2}V(x),
\]

(4)

where \( a \) is a constant with inverse-length dimension, \( \hbar \) is the Plank constant divided by \( 2\pi \), \( x \) is the spatial coordinate and \( V(x) \) is the potential. For \( V(u) \), in section IV we choose the Ushveridze potential \([8]\)

\[
V(u) = 4\beta^2\sinh^4 u + 4\beta[\beta - 2(\gamma + \delta) - 2\ell]\sinh^2 u + 4\delta - \frac{3}{4}\left[\delta - \frac{3}{4}\right] \frac{1}{\sinh u} - 4\left[\gamma - \frac{5}{4}\right] \left[\gamma - \frac{5}{4}\right] \frac{1}{\cosh u},
\]

(5)

where \( \beta, \gamma \) and \( \delta \) are real constants with \( \beta > 0 \) and \( \delta \geq 1/4 \). Ushveridze stated that this potential is QES if \( \ell = 0, 1, 2, \ldots \). However, we have found \([11]\) that, for

\[
(\gamma, \delta) = \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{3}{4}\right),
\]

(6)

the potential \([3]\) is QES also when \( \ell \) is a positive half-integer. The above cases will be called Razavy-type potential because they have the same form as a hyperbolic potential found by Razavy \([12]\) corresponding to a Schrödinger equation which reduces to the modified WHE \([2]\).

For these Razavy-type potentials \([3]\) there are even and odd finite-series solutions which are bounded for any value of the independent variable. These afford a finite number of energy levels characteristic of QES problems. Recently we have also found even infinite-series solutions \([11]\), valid for any admissible value of \( z \). We can get odd infinite-series solutions only if we use two solutions covering different intervals of \( z \). We will see that the new expansions do not modify such results. However, by excluding the cases \([3]\), in addition to finite-series solutions we have already found
infinite-series solutions bounded and convergent for \(1/4 \leq \delta < 1/2\) and \(1/2 < \delta \leq 3/4\), without restrictions on the parameter \(\gamma\)\footnote{11}. Now we will see that the new solutions lead to a result less general in the sense that infinite-series solutions for \(1/2 < \delta \leq 3/4\) impose restrictions on \(\gamma\).

In the section V, we briefly consider the Ushveridze trigonometric potential \footnote{8}. Alternative forms for these solutions follow from the relations

\[
\Psi(a, c; u) = e^{u} \Phi(a, c, -a, c; -u) - u^{1-c} \Phi(a, c, 1+a - c, 2-c; u),
\]

which admits the solutions

\[
\begin{align*}
\varphi^1(u) &= \Phi(a, c; u), & \varphi^2(u) &= u^{1-c} \Phi(1+a - c, 2-c; u), \\
\varphi^3(u) &= \Psi(a, c; u), & \varphi^4(u) &= e^{u} u^{1-c} \Psi(1-a, 2-c; -u),
\end{align*}
\]

We take \(\varphi^1, \varphi^2\) and \(\varphi^3\) as bases for the series expansions for the solutions of the CHE: in this manner we form a set of three expansions, one in series of irregular functions and two in series of confluent hypergeometric functions and Coulomb wave functions may be used to deal with this problem, we conclude that we can simply use the Baber-Hassé expansions in power series \footnote{9, 13}.

\section{One-sided Series Solutions}

First we present the confluent hypergeometric equation, remind the transformations of the CHE and establish some notations. Thence we write the expansions in series of confluent hypergeometric functions and notice that, in each set, one expansion has properties similar to the Baber-Hassé series. After this, we show how the first set is constructed and regard the convergence of the solutions.

\subsection{Confluent hypergeometric functions and symmetries of the CHE}

First, the regular and irregular confluent hypergeometric functions used as bases for the solutions of the CHE are denoted by \(\Phi(a, c; u)\) and \(\Psi(a, c; u)\), respectively. They satisfy the confluent hypergeometric equation \footnote{13}

\[
u \frac{d^2 \varphi(u)}{du^2} + (c - u) \frac{d \varphi(u)}{du} - a \varphi(u) = 0
\]

which admits the solutions

\[
\begin{align*}
\varphi^1(u) &= \Phi(a, c; u), & \varphi^2(u) &= u^{1-c} \Phi(1+a - c, 2-c; u), \\
\varphi^3(u) &= \Psi(a, c; u), & \varphi^4(u) &= e^{u} u^{1-c} \Psi(1-a, 2-c; -u),
\end{align*}
\]

Alternative forms for these solutions follow from the relations

\[
\begin{align*}
\Phi(a, c; u) &= e^{u} \Phi(c-a, c; -u), & \Psi(a, c; u) &= u^{1-c} \Psi(1+a - c, 2-c; u).
\end{align*}
\]

Second, if \(U(z) = U(B_1, B_2, B_3; z_0, \omega, \eta; z)\) denotes one solution of the CHE, the following four transformations \(T_i\) leave invariant the form of the CHE \footnote{9, 15, 10}:

\[
\begin{align*}
T_1 U(z) &= z^{1+B_1} z_0^{1-B_1} U(C_1, C_2, C_3; z_0, \omega, \eta; z), \\
T_2 U(z) &= (z - z_0)^{1-B_2-B_3} U(B_1, D_2, D_3; z_0, \omega, \eta; z), \\
T_3 U(z) &= U(B_1, B_2, B_3; z_0, -\omega, -\eta; z), \\
T_4 U(z) &= U(-B_1 - B_2 z_0, B_2, B_3 + 2\eta \omega z_0; z_0, -\omega, \eta; z_0 - z),
\end{align*}
\]

where

\[
\begin{align*}
C_1 &= -B_1 - 2z_0, & C_2 &= 2 + B_2 + \frac{2B_1}{z_0}, & C_3 &= B_3 + \left[1 + \frac{B_1}{z_0}\right] \left[B_2 + \frac{B_3}{z_0}\right], \\
D_2 &= 2 - B_2 - \frac{2B_1}{z_0}, & D_3 &= B_3 + \frac{B_1}{z_0} \left[B_2 + B_2 - 1\right].
\end{align*}
\]
Using the transformations (12), from an initial set of three solutions it is possible to obtain a group constituted by 16 sets. For one-sided series solutions, these sets of one-sided series solutions are denoted by

\[
\tilde{U}_i(z) = \left[ \tilde{U}_i^{\infty}(z), \tilde{U}_i(1), \tilde{U}_i(z) \right],
\]

where the solutions \( \tilde{U}_i^{\infty} \) are given by series of irregular confluent hypergeometric functions, whereas the solutions \( \tilde{U}_i \) and \( \tilde{U}_1 \) are given by series of regular confluent hypergeometric functions. The ring over the symbols (\( \tilde{U} \), for example) means that we are dealing with one-sided series solutions (\( n \geq 0 \)). The infinity symbol appearing in \( \tilde{U}_i^{\infty}(z) \) indicates that the series associated with such solutions are possibly convergent at \( z = \infty \). We write only the 8 sets, namely,

\[
\begin{align*}
\tilde{U}_1(z), & \quad \tilde{U}_2(z) = T_1 \tilde{U}_1(z), & \quad \tilde{U}_3(z) = T_2 \tilde{U}_2(z), & \quad \tilde{U}_4(z) = T_1 \tilde{U}_3(z); \\
\tilde{U}_5(z) & = T_4 \tilde{U}_1(z), & \quad \tilde{U}_6(z) = T_4 \tilde{U}_2(z), & \quad \tilde{U}_7(z) = T_4 \tilde{U}_3(z), & \quad \tilde{U}_8(z) = T_4 \tilde{U}_4(z),
\end{align*}
\]

(15)

since others are obtained by the transformation \( T_3 \) which changes \( (\eta, \omega) \) by \( (-\eta, -\omega) \) in the above sets.

In order to write the three solutions of a fixed set in terms of the same series coefficients \( \tilde{b}_n \) (where \( n \) represents the summation index), instead of \( \Phi(a, b; y) \) we use the function \( \tilde{\Phi}(a, b; y) \) defined by

\[
\tilde{\Phi}(a, c; u) = \frac{\Gamma(c - a)}{\Gamma(c)} \Phi(a, c; u) = \frac{\Gamma(c - a)}{\Gamma(c)} \left[ 1 + \frac{a u}{c} + \frac{a(a + 1)}{c(c + 1)} u^2 + \cdots \right].
\]

(16)

Thence, the series coefficients satisfy three-term recurrence relations having the form

\[
\alpha^i_n \tilde{b}_1^n + \beta^i_n \tilde{b}_0^n + \gamma^i_n \tilde{b}_{n-1}^i = 0 \quad (n \geq 1),
\]

(17)

where \( \alpha^i_n, \beta^i_n \) and \( \gamma^i_n \) depend on the parameters of the differential equation. If some of these parameters is arbitrary it may be determined from the characteristic equation given by the infinite continued fraction \[3\]

\[
\tilde{\beta}_0 = \tilde{\alpha}_0^{1,1} \tilde{\alpha}_0^{1,2} \tilde{\alpha}_0^{1,3} \cdots,
\]

(18)

which is equivalent to the vanishing of the determinant of the tridiagonal matrix \[4\] associated with the previous relations.

B. The sets of one-sided solutions and relations with power series

In each of the following sets, the solutions \( \tilde{U}_i^{\infty}(z) \) and \( \tilde{U}_i(z) \) are valid only if the parameters of the CHE satisfy certain restrictions which avoid that the hypergeometric functions reduce to polynomials of fixed degree \( l \). In effect, we have the relations \[14\]

\[
\tilde{\Phi}(l, \alpha + 1; u) = (-1)^l \tilde{\Psi}(l, \alpha + 1; u) = l! \left[ l^\alpha(u) \right], \quad l = 0, 1, 2, \cdots
\]

(19)

where the \( L_l^{(\alpha)}(u) \) denote generalized Laguerre polynomials of degree \( l \). Then, if \( l \) is fixed (i. e., if \( l \) does not depend on the summation index \( n \) ) \( \tilde{\Phi} \) and \( \tilde{\Psi} \) are polynomials of fixed degree and, so, cannot be used as bases for series expansions. In addition, we have to take into account that the functions \( \tilde{\Phi}(a, c; u) \) in general are well defined only if \( c \) is not zero or negative integer. Further, there are cases for which the definition \[10\] is unsuitable because the argument of \( \Gamma(c - a) \) is zero or a negative integer: then, the solutions must be rewritten as in the case considered in Appendix B.

First set: \( 2 + i\eta - \frac{B}{z} \neq 0, -1, \cdots \) in \( \tilde{U}_i^{\infty}(z) \) and \( \tilde{U}_2(z) \).

\[
\begin{align*}
\tilde{U}_1^{\infty}(z) &= e^{i\omega z} \left[ z^{1 - B} \sum_{n=0}^{\infty} \frac{\tilde{b}_n}{\tilde{b}_n} \right] \\
\tilde{U}_1(z) &= e^{i\omega z} \left[ z^{1 - B} \sum_{n=0}^{\infty} \frac{\tilde{b}_n}{(2i\omega z)^n} \right] \tilde{\Phi}(n + 1 + i\eta - \frac{B}{z_0} - \frac{B}{z_0}; -2i\omega z),
\end{align*}
\]

(20a)
where series coefficients $\hat{\beta}_n^1$ satisfy the recurrence relations (17) with

$$\hat{\beta}_n^1 = n \left[ n + 1 - B_2 - \frac{2B_1}{z_0} + 2i\omega z_0 \right] + \left[ i\omega z_0 - 1 - \frac{B_1}{z_0} \right] \left[ 2 - B_2 - \frac{B_1}{z_0} \right] + 2 - B_2 + B_3,$$

$$\hat{\alpha}_n^1 = -2i\omega z_0 (n + 1), \quad \hat{\gamma}_n^1 = - \left[ n + i\eta - \frac{B_1}{z_0} - \frac{B_2}{z_0} \right] \left[ n + 1 - B_2 - \frac{B_1}{z_0} \right]. \quad (20b)$$

This set is obtained in section II.B; in [1] it has been written without further details.

**Second set:** $1 + i\eta - \frac{B_2}{z_0} - \frac{B_1}{z_0} \neq 0, -1, \cdots$ in $\hat{U}_2^\infty(z)$ and $\hat{U}_2(z)$.

$$\begin{bmatrix} \hat{U}_2^\infty(z) \\ \hat{U}_2(z) \end{bmatrix} = e^{i\omega z} \left[ z - z_0 \right]^{1-B_2 - \frac{B_1}{z_0}} \sum_{n=0}^{\infty} \hat{b}_n^2 \left[ 2i\omega z \right]^n \begin{bmatrix} \Psi \left( 1 + i\eta - \frac{B_2}{z_0} - \frac{B_1}{z_0} - n; -2i\omega \right) \\ \hat{\Phi} \left( 1 + i\eta - \frac{B_2}{z_0} - \frac{B_1}{z_0} - n; -2i\omega \right) \end{bmatrix},$$

$$\hat{U}_2(z) = e^{i\omega z} \left[ z - z_0 \right]^{1-B_2 - \frac{B_1}{z_0}} \sum_{n=0}^{\infty} \hat{b}_n^2 \left[ 2i\omega z \right]^n \begin{bmatrix} \Psi \left[ n + 2 + i\eta - \frac{B_2}{z_0} - n + \frac{B_1}{z_0} + 2; -2i\omega \right] \\ \hat{\Phi} \left[ n + 2 + i\eta - \frac{B_2}{z_0} - n + \frac{B_1}{z_0} + 2; -2i\omega \right] \end{bmatrix}. \quad (21a)$$

**Third set:** $i\eta + \frac{B_2}{z_0} \neq 0, -1, \cdots$ in $\hat{U}_3^\infty(z)$ and $\hat{U}_3(z)$.

$$\begin{bmatrix} \hat{U}_3^\infty(z) \\ \hat{U}_3(z) \end{bmatrix} = e^{i\omega z} \sum_{n=0}^{\infty} \hat{b}_n^3 \begin{bmatrix} \Psi \left( i\eta + \frac{B_2}{z_0} - \frac{B_1}{z_0} - n; -2i\omega \right) \\ \hat{\Phi} \left( i\eta + \frac{B_2}{z_0} - \frac{B_1}{z_0} - n; -2i\omega \right) \end{bmatrix},$$

$$\hat{U}_3(z) = e^{i\omega z} \sum_{n=0}^{\infty} \hat{b}_n^3 \left[ 2i\omega z \right]^n \begin{bmatrix} \Psi \left[ n + 1 + i\eta + \frac{B_2}{z_0} + \frac{B_1}{z_0} - n + \frac{B_1}{z_0} + 2; -2i\omega \right] \\ \hat{\Phi} \left[ n + 1 + i\eta + \frac{B_2}{z_0} + \frac{B_1}{z_0} - n + \frac{B_1}{z_0} + 2; -2i\omega \right] \end{bmatrix}. \quad (22a)$$

**Fourth set:** $i\eta + 1 + \frac{B_2}{z_0} + \frac{B_1}{z_0} \neq 0, -1, \cdots$ in $\hat{U}_4^\infty(z)$ and $\hat{U}_4(z)$.

$$\begin{bmatrix} \hat{U}_4^\infty(z) \\ \hat{U}_4(z) \end{bmatrix} = e^{i\omega z} \sum_{n=0}^{\infty} \hat{b}_n^4 \begin{bmatrix} \Psi \left( i\eta + 1 + \frac{B_2}{z_0} + \frac{B_1}{z_0} + 2 + \frac{B_1}{z_0} - n; -2i\omega \right) \\ \hat{\Phi} \left( i\eta + 1 + \frac{B_2}{z_0} + \frac{B_1}{z_0} + 2 + \frac{B_1}{z_0} - n; -2i\omega \right) \end{bmatrix},$$

$$\hat{U}_4(z) = e^{i\omega z} \sum_{n=0}^{\infty} \hat{b}_n^4 \left[ 2i\omega z \right]^n \begin{bmatrix} \Psi \left[ n + i\eta + \frac{B_2}{z_0} - n - \frac{B_1}{z_0}; -2i\omega \right] \\ \hat{\Phi} \left[ n + i\eta + \frac{B_2}{z_0} - n - \frac{B_1}{z_0}; -2i\omega \right] \end{bmatrix}. \quad (23a)$$

**Fifth set:** $2 + i\eta - \frac{B_2}{z_0} \neq 0, -1, \cdots$ in $\hat{U}_5^\infty(z)$ and $\hat{U}_5(z)$.

$$\begin{bmatrix} \hat{U}_5^\infty(z) \\ \hat{U}_5(z) \end{bmatrix} = e^{i\omega z} f(z) \sum_{n=0}^{\infty} \hat{b}_n^5 \begin{bmatrix} \Psi \left( 2 + i\eta - \frac{B_2}{z_0} - 2 - B_2 - \frac{B_1}{z_0} - n; 2i\omega(z_0 - \z) \right) \\ \hat{\Phi} \left( 2 + i\eta - \frac{B_2}{z_0} - 2 - B_2 - \frac{B_1}{z_0} - n; 2i\omega(z_0 - \z) \right) \end{bmatrix},$$

$$\hat{U}_5 = e^{i\omega z} \sum_{n=0}^{\infty} \hat{b}_n^5 \left( 2i\omega (z - z_0) \right)^n \begin{bmatrix} \Psi \left[ n + 1 + i\eta + \frac{B_2}{z_0} - n + \frac{B_1}{z_0} + B_2 + \frac{B_1}{z_0} 2i\omega (z_0 - \z) \right] \\ \hat{\Phi} \left[ n + 1 + i\eta + \frac{B_2}{z_0} - n + \frac{B_1}{z_0} + B_2 + \frac{B_1}{z_0} 2i\omega (z_0 - \z) \right] \end{bmatrix}. \quad (24a)$$
where \( f(z) = z^{1+\frac{B_i}{2\pi}}[z - z_0]^{-1-B_{2,\frac{B_i}{2\pi}}} \) and

\[
\tilde{\beta}^5_n = n \left[ n + 1 + B_2 + \frac{B_2}{2} - 2i\omega z_0 \right] + \frac{B_1}{2\pi} \left[ B_2 + \frac{B_1}{2\pi} + 1 - i\omega z_0 \right] + 2\eta_0 z_0 - 2i\omega z_0 + B_2 + B_3, \quad \tilde{\alpha}^5_n = 2i\omega z_0(n+1), \quad \tilde{\gamma}^5_n = -\left[ n + i\eta + \frac{B_3}{2} + \frac{B_1}{2\pi} \right] \left[ n + 1 + \frac{B_i}{2\pi} \right]. \tag{24b}
\]

Sixth set: \( \eta + 1 + \frac{B_2}{2\pi} + \frac{B_i}{2\pi} \neq 0, -1, \cdots \) in \( \dot{U}_6^\infty(z) \) and \( \dot{U}_6(z) \).

\[
\begin{align*}
\dot{U}_6^\infty &= e^{i\omega z} z^{1+\frac{B_i}{2\pi}} \sum_{n=0}^\infty \tilde{\beta}^5_n \begin{bmatrix}
\Psi \left( 1 + \eta + \frac{B_1}{2\pi} + \frac{B_2}{2\pi}, B_2 + \frac{B_1}{2\pi} - n; 2i\omega (z_0 - z) \right) \\
\Phi \left( 1 + \eta + \frac{B_1}{2\pi} + \frac{B_2}{2\pi}, B_2 + \frac{B_1}{2\pi} - n; 2i\omega (z_0 - z) \right)
\end{bmatrix},
\end{align*}
\tag{25a}
\]

\[
\dot{U}_6 = e^{i\omega z} f(z) \sum_{n=0}^\infty \tilde{\beta}^5_n [2i\omega (z - z_0)]^n \tilde{\Phi} \left[ n + 2 + \eta - \frac{B_2}{2}, n + 2 + B_3 - \frac{B_1}{2\pi}, 2i\omega (z_0 - z) \right]
\]

where \( f(z) \) is given as in \( \dot{U}_4^\infty \), and

\[
\tilde{\beta}^6_n = n \left[ n + 3 - 2i\omega z_0 - B_2 \right] - i\omega B_1 + 2\eta_0 z_0 - 2i\omega z_0 + B_3 - B_2 + 2, \quad \tilde{\alpha}^6_n = 2i\omega z_0(n+1), \quad \tilde{\gamma}^6_n = -\left[ n + i\eta + 1 - \frac{B_3}{2} \right] \left[ n + 1 + \frac{B_i}{2\pi} \right]. \tag{25b}
\]

Seventh set: \( \eta + \frac{B_3}{2\pi} \neq 0, -1, \cdots \) in \( \dot{U}_7^\infty(z) \) and \( \dot{U}_7(z) \).

\[
\begin{align*}
\dot{U}_7^\infty &= e^{i\omega z} z^{1+\frac{B_i}{2\pi}} \sum_{n=0}^\infty \tilde{\beta}^7_n \begin{bmatrix}
\Psi \left( \eta + \frac{B_1}{2\pi}, B_2 + \frac{B_1}{2\pi} - n; 2i\omega (z_0 - z) \right) \\
\Phi \left( \eta + \frac{B_1}{2\pi}, B_2 + \frac{B_1}{2\pi} - n; 2i\omega (z_0 - z) \right)
\end{bmatrix},
\end{align*}
\tag{26a}
\]

\[
\dot{U}_7 = g(z) \sum_{n=0}^\infty \tilde{\beta}^7_n [2i\omega (z - z_0)]^n \tilde{\Phi} \left[ n + 1 + \eta - \frac{B_3}{2\pi} - \frac{B_1}{2\pi}, n + 2 + B_3 - \frac{B_1}{2\pi}, 2i\omega (z_0 - z) \right]
\]

where \( g(z) = e^{i\omega z}[z - z_0]^{-1-B_{2,\frac{B_i}{2\pi}}} \) and

\[
\tilde{\beta}^7_n = n \left[ n + 1 - 2i\omega z_0 - B_2 - \frac{B_1}{2\pi} \right] + i\omega B_1 + 2\eta_0 z_0 + B_3 + \frac{B_1}{2\pi} \left[ \frac{B_1}{2\pi} + B_2 - 1 \right], \quad \tilde{\alpha}^7_n = 2i\omega z_0(n+1), \quad \tilde{\gamma}^7_n = -\left[ n + i\eta - \frac{B_3}{2\pi} - \frac{B_1}{2\pi} \right] \left[ n - 1 + \frac{B_i}{2\pi} \right]. \tag{26b}
\]

Eighth set: \( \eta + 1 - \frac{B_3}{2\pi} - \frac{B_1}{2\pi} \neq 0, -1, \cdots \) in \( \dot{U}_8^\infty(z) \) and \( \dot{U}_8(z) \).

\[
\begin{align*}
\dot{U}_8^\infty &= g(z) \sum_{n=0}^\infty \tilde{\beta}^8_n \begin{bmatrix}
\Psi \left( \eta + 1 - \frac{B_3}{2\pi} - \frac{B_1}{2\pi}, B_3 - \frac{B_1}{2\pi} - n; 2i\omega (z_0 - z) \right) \\
\Phi \left( \eta + 1 - \frac{B_3}{2\pi} - \frac{B_1}{2\pi}, B_3 - \frac{B_1}{2\pi} - n; 2i\omega (z_0 - z) \right)
\end{bmatrix},
\end{align*}
\tag{27a}
\]

\[
\dot{U}_8 = e^{i\omega z} \sum_{n=0}^\infty \tilde{\beta}^8_n [2i\omega (z - z_0)]^n \tilde{\Phi} \left[ n + \eta + \frac{B_3}{2\pi}, n + B_3 + \frac{B_1}{2\pi}, 2i\omega (z_0 - z) \right]
\]

where \( g(z) \) is given in \( \dot{U}_7 \) and

\[
\begin{align*}
\tilde{\beta}^8_n &= n \left[ n - 1 + B_2 - 2i\omega z_0 \right] + i\omega B_1 + 2\eta_0 z_0 + B_3, \quad \tilde{\alpha}^8_n = 2i\omega z_0(n+1), \quad \tilde{\gamma}^8_n = -\left[ n - 1 + i\eta + \frac{B_3}{2} \right] \left[ n - 1 + \frac{B_i}{2\pi} \right]. \tag{27b}
\end{align*}
\]

In each set, the solution \( \dot{U}_i \) is the only one affording finite series which can be expressible as a sum of polynomials. In effect, if the parameter \( a \) of \( \Phi(a, c; y) \) in \( \dot{U}_i \) is \( a = n - N \) (where \( N = 0, 1, 2, \cdots \)), then \( \dot{U}_i \) is given by finite series since \( \tilde{\gamma}^i_n \propto (n - N - 1) \), a fact which implies series terminating at \( n = N \) (see Appendix A). In this case, the functions \( \Phi(n-N, c; y) \) reduce to the Laguerre polynomials \( L_n \) with \( l = N - n \) (0 \( \leq n \leq N \)). This fact allows us to obtain some relations among the solutions \( \dot{U}_i(z) \) and the Baber-Hassé solutions in power series for the CHE. 

We denote the latter by \( U^\text{baber}_1(z) \) and take

\[
U^\text{baber}_1(z) = e^{i\omega z} \sum_{n=0}^\infty \dot{\alpha}^1_n(z - z_0)^n \quad \text{with} \quad z_0 \left[ n + B_2 + \frac{B_1}{2\pi} \right] [n+1] \dot{\alpha}^1_{n+1} + \frac{\beta^4}{2} \dot{\alpha}^1_n + 2i\omega \left[ n + i\eta + \frac{B_3}{2} - 1 \right] \dot{\alpha}^1_{n-1} = 0, \quad \dot{\alpha}^1_{-1} = 0. \tag{28b}
\]
where \( \beta_n^4 \) is given in (23a). Eight solutions are generated as in (15), that is,

\[
\begin{align*}
U_1^{\text{baber}}, & \quad U_2^{\text{baber}} = T_1 U_1^{\text{baber}}, & \quad U_3^{\text{baber}} = T_2 U_2^{\text{baber}}, & \quad U_4^{\text{baber}} = T_1 U_3^{\text{baber}}, \\
U_5^{\text{baber}} = T_4 U_1^{\text{baber}}, & \quad U_6^{\text{baber}} = T_4 U_2^{\text{baber}}, & \quad U_7^{\text{baber}} = T_4 U_3^{\text{baber}}, & \quad U_8^{\text{baber}} = T_4 U_4^{\text{baber}},
\end{align*}
\]

while additional solutions are obtained by applying the transformation \( T_3 \) on the above ones. The solution \( U_5^{\text{baber}} \), to be used later, reads

\[
U_5^{\text{baber}}(z) = e^{i\omega z} \sum_{n=0}^{\infty} (-1)^n a_n^1 z^n \quad \text{with}
\]

\[
z_0 \left[ n - \frac{B_1}{z_0} \right] [n + 1] a_{n+1}^5 + \beta_n^8 a_n^5 - 2i\omega \left[ n + i\eta + \frac{B_2}{2} - 1 \right] a_{n-1}^1 = 0, \quad a_{-1}^5 = 0,
\]

where \( \beta_n^8 \) is given in (27b). Notice that the coefficient \( \beta_n \) are the same of the solutions \( \tilde{U}_i \). In fact, the same \( \beta_n \) appears in the following pairs of solutions:

\[
\begin{align*}
U_1 & \leftrightarrow U_4^{\text{baber}}, & \quad U_2 & \leftrightarrow U_3^{\text{baber}}, & \quad U_3 & \leftrightarrow U_2^{\text{baber}}, & \quad U_4 & \leftrightarrow U_1^{\text{baber}}, \\
U_5 & \leftrightarrow U_8^{\text{baber}}, & \quad U_6 & \leftrightarrow U_7^{\text{baber}}, & \quad U_7 & \leftrightarrow U_6^{\text{baber}}, & \quad U_8 & \leftrightarrow U_5^{\text{baber}}.
\end{align*}
\]

Notice that the solutions \( \tilde{U}_i \) in general are valid only if the parameter \( c \) of \( \tilde{\Phi}(a,c;y) \) is not zero or negative integer, a restriction not required by \( U_i^{\text{baber}} \). Nevertheless, if both solutions of a given pair are valid, we verify that

- Both solutions in each pair (28) are given only by finite series or by infinite series.
- In each pair, the product \( \alpha_n^1 \gamma_{n+1} \) is the same for both solutions which, by this reason, present the same characteristic equation having the form (15).
- As we will see, the infinite-series solutions \( \tilde{U}_i \) converge for any finite \( z \), just as the corresponding solutions \( U_i^{\text{baber}} \).

For the Whittaker-Hill equation \( (z_0 = 1, B_1 = -1/2, B_2 = 1) \), all the above pairs are valid, as we can check. In addition, according to Appendix B, for the spheroidal equation we can take \( z_0 = 1, B_2 = -2B_1 \) and \( \eta = 0 \); hence, the solutions \( U_2, U_4, U_6 \) and \( U_8 \) reduce to Baber-Hassé expansions up to a redefinition of the series coefficients.

### C. Construction of the first set of solutions

It is necessary to study only the first set of solutions; the others follow from the transformations (12). Thus, to get the solutions (20a), first we accomplish the substitutions

\[
U(z) = e^{i\omega z} z^{1+\frac{d}{dz}(z-z_0)^{-1-B_2-\frac{d}{dz}}} Y(y), \quad y = -2i\omega z
\]

which, when inserted into (11), lead to

\[
\begin{align*}
(y + 2i\omega z_0) \left( \frac{d^2 Y}{dy^2} - \frac{d Y}{dy} \right) + \left( -2i\omega \bar{C}_1 + \bar{C}_2 y \right) \frac{d Y}{dy} + \left[ \bar{C}_3 + i\omega \bar{C}_1 + 2\eta\omega z_0 - \frac{1}{2} (\bar{C}_2 + 2i\eta) y \right] Y &= 0, \quad (32)
\end{align*}
\]

where \( \bar{C}_1 = -B_1 - 2z_0, \bar{C}_2 = 4 - B_2 \) and \( \bar{C}_3 = B_3 + 2 - B_2 \). In the second place, to obtain \( \tilde{U}_1 \) and \( \tilde{U}_1 \), we expand \( Y(y) \) as

\[
Y(y) = \sum_{n=0}^{\infty} b_n^1 \mathcal{F}_n(y), \quad \mathcal{F}_n(y) = \Psi(\alpha, \beta - n; y) \quad \text{or} \quad \mathcal{F}_n(y) = \tilde{\Phi}(\alpha, \beta - n; y)
\]

and show that

\[
\alpha = 2 + i\eta - (B_2/2), \quad \beta = 2 + (B_1/z_0),
\]

\[
(33a) \quad \text{or} \quad (33b)
\]
while the coefficients $b^1_n$ satisfy the relations (17) with (20b). To this end, we use (8)
\[ y \frac{d^2 F_n}{dy^2} - y \frac{d F_n}{dy} = (\alpha - \beta) F_n + \alpha F_n, \quad \text{[see Eq. (8)]} \]
\[ \frac{d F_n}{dy} = F_{n-1}, \]
\[ y F_{n-1} = (n + 1 - \beta) F_{n+1} + (y + \beta - n - 1) F_n. \] (34)

Inserting $Y(y) = \sum b^1_n F_n(y)$ into (32) and using (34), we find
\[ \sum_{n=0}^{\infty} b^1_n \alpha_{n-1} F_{n-1}(y) + \sum_{n=0}^{\infty} b^1_n \beta_{n-1} F_n(y) + \sum_{n=0}^{\infty} b^1_n \gamma_{n+1} F_{n+1}(y) + \sum_{n=0}^{\infty} b^1_n \left[ \alpha - i \eta - \frac{1}{2} \bar{C}_2 \right] y F_n(y) = 0, \] (35)
or, equivalently,
\[ \bar{a}^1_{-1} b^1_0 F_{-1}(y) + \left[ \bar{a}^1_{0} b^1_1 + \bar{\beta}^1_{1} b^1_0 \right] F_0 + \sum_{n=1}^{\infty} \left[ \bar{a}^1_{n} b^1_{n+1} + \bar{\beta}^1_{n} b^1_1 + \bar{\gamma}^1_{n+1} b^1_{n-1} \right] F_n(y) + \sum_{n=0}^{\infty} b^1_n \left[ \alpha - i \eta - \frac{1}{2} \bar{C}_2 \right] y F_n(y) = 0, \] (36)
where
\[ \bar{\beta}^1_n = 2i \omega z_0 [n + \alpha - \beta] + [n + 1 - \beta] \left[ n - \beta + \bar{C}_2 \right] + 2 \eta \omega z_0 - i \omega \bar{C}_1 + \bar{C}_4, \]
\[ \bar{a}^1_{n} = -2i \omega z_0 \left( n + 1 - \beta - \frac{i \eta}{z_0} \right), \quad \bar{\gamma}^1_{n} = - \left[ n + \alpha - \beta \right] \left[ n - 1 - \beta + \bar{C}_2 \right]. \]

Now we determine the constants $\alpha$ and $\beta$ by equating to zero the first and the last terms of Eq. (36). More precisely, we take
\[ \bar{a}^1_{-1} = 2i \omega z_0 \left( \beta + \frac{1}{z_0} \bar{C}_1 \right) = 0, \quad \alpha - i \eta = \frac{1}{2} \bar{C}_2 = 0, \]
what demands that $\beta = 2 + (B_1 / z_0)$ and $\alpha = 2 + i \eta - (B_2 / 2)$, as stated in (33b). Thence, Eq. (36) is satisfied if the $b^1_n$ obey the relations (17) with the coefficients (20b).

On the other hand, by supposing that Eq. (11) is valid, we can obtain $U_1$ as a linear combination of $\tilde{U}_1$ and $\tilde{U}^\infty$, although the validity of $U_1$ does not depend on Eq. (11). Actually, up to a multiplicative constant, we have
\[ U_1(z) = e^{i \omega z} \bar{z}^1 + \frac{B^1_1}{z_0} \bar{z}^2 \left[ z - z_0 \right] \bar{z}^2 \sum_{n=0}^{\infty} \left( -1 \right)^n b^1_n \left( -2i \omega \bar{z} \right)^{n-1} \times \]
\[ \bar{\Phi} \left( n + 1 + i \eta - \frac{B_1}{z_0} \bar{z}^1 - \frac{B_2}{z_0}, n - \frac{B_2}{z_0}, -2i \omega \bar{z} \right). \] (37a)

Thus, in Eq. (31)
\[ Y(y) = \sum_{n=0}^{\infty} \left( -1 \right)^n b^1_n \phi \left( a_n, c_n; y \right), \quad a_n = n + 1 + i \eta - \frac{B_1}{z_0} - \frac{B_2}{z_0}, \quad c_n = n - \frac{B_1}{z_0}. \] (37b)

If, by definition we take
\[ F_n(y) := \left( -1 \right)^n y^{c_n-1} \bar{\Phi} (a_n, c_n; y) \quad \Rightarrow \quad Y(y) = \sum_{n=0}^{\infty} b^1_n F_n(y), \] (37b)
then we find the relations
\[ y \frac{d^2 F_n(y)}{dy^2} - y \frac{d F_n(y)}{dy} = (c_n - 2) \frac{d F_n(y)}{dy} + (a_n + 1 - c_n) F_n(y), \]
\[ \frac{d F_n(y)}{dy} = F_{n-1}(y) \quad \text{and} \quad y F_{n-1}(y) = a_n F_{n+1}(y) - (c_n - 1 - y) F_n(y). \] (38)

The first equation expresses the fact that $F_n(y)$ is basically the solution $\varphi^2(u)$ given in Eq. (9) for equation (8) with $u = y, \quad a = a_n + 1 - c_n$ and $c = 2 - c_n$. The second and third relations result, for example, from Eqs. (13.3.21) and (13.3.14) of Ref. (4) combined with definition (37b). However, by using the parameters $\alpha$ and $\beta$ given in (33b), we find that Eqs. (34) and (36) are formally identical. Therefore, $Y(y) = \sum b^1_n F_n(y)$ is also solution of Eq. (32).
D. Convergence of the one-sided solutions

Before studying the convergence, notice that the function $Ψ(a,c;y)$ is satisfactory in the neighborhood of infinity but it may become infinity at $y = 0$; in particular, it presents logarithmic terms at $y = 0$ if $c = 0, 1, 2$ [6]. Thus, we suppose that the solutions $U_i^∞$ are not valid when $y = 0$. Furthermore, we will see that the the ratio test for convergence is inapplicable to solutions (20a) when $z → ∞$. For this reason, we consider the behaviour of each solution when $z → ∞$. In this manner, we will find that: (i) the solutions $U_1$ and $U_0$ converge only for finite values of $z$, (ii) the solution $U_i^∞$ converges for $z ≠ 0$; it converges also at $z = ∞$ only if $\Re(iη - B_2/2) < -1$. As to the other solutions, see the end of this subsection.

The convergence of an one-sided infinite series $\sum_{n=0}^{∞} f_n(z)$ is obtained by using the limit

$$L_1(z) = \lim_{n \rightarrow ∞} \left| \frac{f_{n+1}(z)}{f_n(z)} \right|.$$  \hspace{3cm} (39)

By the D’Alembert ratio test, the series converges in the region where $L_1(z) < 1$ and diverges where $L_1(z) > 1$; if $L_1 = 1$, the test is inconclusive. By the Raabe test to get the convergence of $˚\Phi$ results from the fact that $\lim_{n \rightarrow ∞} \frac{˚\Phi_{n+1}(z)}{˚\Phi_n(z)} = \frac{1}{n}$, if

$$L_1(z) = 1 + \frac{A}{n}$$ \hspace{3cm} (40)

(where $A$ is constant), the series converges if $A < -1$ and diverges if $A > -1$; if $A = -1$, the test is inconclusive. First we use the Raabe test to get the convergence of $U_1^∞(z)$ and $˚\Phi(z)$; second, we use the D’Alembert test to study the convergence of $U_1(z)$.

For $n → ∞$ the recurrence relations for $b_n^1$ lead to

$$-2iωz_0 \left[ 1 + \frac{1}{n} \right] \left( \frac{b_{n+1}^1}{b_n^1} \right) + \left[ n + 1 + 2iωz_0 - B_2 - \frac{2B_1}{z_0} \right] \left[ n + 1 + iη - \frac{2B_1}{z_0} - \frac{3B_2}{2} \right] \frac{b_{n+1}^1}{b_n^1} = 0,$$ \hspace{3cm} (41)

whose solutions, when $n → ∞$, satisfy

$$\frac{b_{n+1}^1}{b_n^1} \sim 1 + \frac{1}{n} \left( iη - \frac{B_2}{2} \right) \Rightarrow \frac{b_{n+1}^1}{b_n^1} \sim 1 - \frac{1}{n} \left( iη - \frac{B_2}{2} \right) \text{ or}$$ \hspace{3cm} (42a)

$$\frac{b_{n+1}^1}{b_n^1} \sim \frac{n}{2iωz_0} \left[ 1 - \frac{1}{n} \left( B_2 + \frac{2B_1}{z_0} \right) \right] \Rightarrow \frac{b_{n+1}^1}{b_n^1} \sim \frac{2iωz_0}{n} \left[ 1 + \frac{1}{n} \left( B_2 + \frac{2B_1}{z_0} \right) \right].$$ \hspace{3cm} (42b)

On the other hand, from the last equation given in (34) we find $(y = -2iωz)$

$$\left[ n + 1 + iη - \frac{B_1}{z_0} - \frac{B_2}{2} \right] \frac{˚\Phi_{n+1}(y)}{˚\Phi_n(y)} - \left[ n - 1 + 2iωz - \frac{B_1}{z_0} + 2iωz \frac{˚\Phi_{n-1}(y)}{˚\Phi_n(y)} \right] = 0.$$ \hspace{3cm} (43)

If $z$ is bounded (that is, if $2iωz/n → 0$), then when $n → ±∞$ [13] is satisfied by two expressions for $˚\Phi_{n+1}/˚\Phi_n$ or $˚\Phi_n/˚\Phi_{n-1}$, namely,

$$\frac{˚\Phi_{n+1}}{˚\Phi_n} \sim 1 + \frac{1}{n} \left( B_2 - 2 - iη \right) \Leftrightarrow \frac{˚\Phi_{n-1}}{˚\Phi_n} \sim 1 - \frac{1}{n} \left( B_2 - 2 - iη \right),$$ \hspace{3cm} (44a)

$$\frac{˚\Phi_{n+1}}{˚\Phi_n} \sim \frac{2iωz}{n} \left( 1 + \frac{B_1}{z_0} \right) \Leftrightarrow \frac{˚\Phi_{n-1}}{˚\Phi_n} \sim \frac{n}{2iωz_0} \left[ 1 - \frac{1}{n} \left( B_2 + \frac{B_1}{z_0} \right) \right].$$ \hspace{3cm} (44b)

When $n → ∞$, we have to choose the first possibility for both cases, that is,

$$\lim_{n → ∞} \frac{˚\Phi_{n+1}}{˚\Phi_n} = 1 + \frac{1}{n} \left( B_2 - 2 - iη \right) \text{ for } ˚\Phi_n = Ψ \text{ and } ˚\Phi_n = ˚Φ.$$ \hspace{3cm} (44c)

The ratio for $˚Φ$ results from the fact that $\lim_{n → ∞} = Φ(a,c;y) = 1$ if $a$ and $y$ remain fixed and bounded [14]. For $˚\Phi_n = Ψ$ that choice is consistent with the fact that, if $|c| → ∞$ while $a$ and $y$ remain fixed and bounded, then [14]

$$\Psi(a,c;y) \sim e^{-a} \left[ (-1)^{-a} + \frac{a^c}{Γ(c)} \left( \frac{c}{y} \right)^{c+a-\frac{1}{2}} y^{a-\frac{1}{2}} e^{y+a-\frac{1}{2}} \right] \left[ 1 + O \left( \frac{1}{n^2} \right) \right],$$ \hspace{3cm} (45)

$[c → ∞; \ a ≠ 0, -1, -2, \ldots; \ |\arg(±c)| < π].$

Thus, using (42a) and (44c), we find

$$\lim_{n → ∞} \frac{b_{n+1}^1}{b_n^1} \frac{˚\Phi_{n+1}}{˚\Phi_n} = 1 - \frac{2}{n} + O \left( \frac{1}{n^2} \right) \text{ for } U_1^∞ \text{ and } U_1.$$ \hspace{3cm} (46)
On the other hand, for \( \hat{U}_1 \) we use the possibility corresponding to (44), that is,

\[
\lim_{n \to +\infty} \frac{F_{n+1}(y)}{F_n(y)} = \frac{2i\omega z}{n} \left( 1 + \frac{B_1}{n z_0} \right),
\]

since this is the only one compatible with

\[
\lim_{n \to \infty} F_n(y) = \frac{\Gamma(\frac{B_2}{2}) e^{-2i\omega z (2i\omega z)^{n-1}\frac{B_1}{z_0}}}{\Gamma[\frac{B_2}{2}]} \lim_{n \to \infty} \Phi \left( \frac{B_2}{2} - 1 - i\eta, n - \frac{B_1}{z_0}, -2i\omega z \right)
\]

\[
= \frac{\Gamma(\frac{(B_2/2)-1-in}{2}) e^{-2i\omega z (2i\omega z)^{n-1}\frac{B_1}{z_0}}}{\Gamma[(B_2/2)]} \cdot \frac{(\frac{B_2}{2} - 1 - i\eta, n - \frac{B_1}{z_0}, -2i\omega z)}{\Gamma(\frac{B_2}{2})}
\]

where we have used (40). In this manner,

\[
\lim_{n \to \infty} \frac{\hat{b}_{n+1} F_{n+1}}{\hat{b}_n F_n} = \frac{2i\omega z}{n} + O\left( \frac{1}{n^2} \right) \quad \text{for} \quad \hat{U}_1.
\]

As the ratios (46) and (47) are not valid at \( z = \infty \), we compute the asymptotic behaviour of the solutions. So, from \( \lim_{y \to \infty} \Psi(a, c; y) = y^{-a} \), we find

\[
\lim_{n \to \infty} \hat{U}_1^\infty(z) = e^{i\omega z} z^{-i\eta - \frac{B_2}{2}} \sum_{n=0}^{\infty} \frac{\hat{b}_n}{n} = 1 + \frac{1}{n} \text{Re} \left( i\eta - \frac{B_2}{2} \right),
\]

(48)

Thus, by the Raabe test, the series in \( \hat{U}_1^\infty \) converge at \( z = \infty \) if \( \text{Re}(i\eta - B_2/2) < -1 \). To examine the asymptotic behaviour of \( \hat{U}_1 \) and \( \hat{U}_1 \) we use (4)

\[
\lim_{y \to \infty} \Phi(a, c; y) \propto \frac{\Gamma(c-a)}{\Gamma(a)} e^{iy^{a-c}} e^{\pm i\pi y^{a}} \quad \text{[a \neq 0, -1, ... ; c - a \neq 0, -1, ...]}
\]

where the upper sign holds for \(-\pi/2 < \arg y < 3\pi/2\) and the lower sign, for \(-3\pi/2 < \arg y \leq -\pi/2\). Thence, we see that it is not possible to prove that \( \hat{U}_1 \) and/or \( \hat{U}_1 \) converge at \( z = \infty \). We get, for example,

\[
\lim_{z \to \infty} \hat{U}_1(z) \propto e^{i\omega z} z^{-i\eta - \frac{B_2}{2}} \sum_{n=0}^{\infty} \frac{\hat{b}_n}{n} = \frac{1}{n} \text{Re} \left( i\eta - \frac{B_2}{2} \right),
\]

(49)

where the convergence of the second series becomes indefinite when \( n \to \infty \) since

\[
\lim_{n \to \infty} \frac{\Gamma(n+1) \hat{b}_{n+1} (-2i\omega z)^{n+1}}{\Gamma(n) \hat{b}_n (-2i\omega z)^n} \propto \frac{2i\omega z}{n}, \quad (z \to \infty).
\]

By using the transformations of the CHE as in (105) we can extend the study of convergence for the other solutions. We find that:

- The one-sided infinite-series solutions \( \hat{U}_1 \) and \( \hat{U}_1 \) converge for finite values of \( z \).
- The one-sided infinite-series solutions \( \hat{U}_1^\infty \) in terms of irregular functions converge for finite values of \( z \), excepting possibly the points \( z = 0 \) (if \( i = 1, 2, 3, 4 \)) and \( z = z_0 \) (if \( i = 5, 6, 7, 8 \)); the \( \hat{U}_1^\infty \) converge also at \( z = \infty \) if

\[
\text{Re} \left[ i\eta - \frac{B_2}{2} + 1 \right] < 0: \quad \hat{U}_1^\infty, \hat{U}_5^\infty; \quad \text{Re} \left[ i\eta - \frac{B_2}{z_0} - \frac{B_3}{2} \right] < 0: \quad \hat{U}_3^\infty, \hat{U}_7^\infty; \quad \text{Re} \left[ i\eta + \frac{B_3}{z_0} + \frac{B_2}{2} \right] < 0: \quad \hat{U}_4^\infty, \hat{U}_6^\infty.
\]

(50)

The above conditions also assure the convergence at \( z = \infty \) of the two-sided series solutions \( \hat{U}_1^\infty \) discussed in the following.
III. TWO-SIDED SERIES SOLUTIONS

In the present section, each set of the one-sided series solutions \((\tilde{U}_i)\) is transformed into two-sided series \((U_i)\) by replacing the summation index \(n\) by \(n + \nu_i\), where the parameter \(\nu_i\) must be chosen appropriately. We write down only the first set of solutions and establish the modifications in the domains of convergence which result when we apply the ratio test for \(n \to -\infty\).

We use the convention
\[
\tilde{U}_i(z) = [U_i^\infty(z), U_i(z), U_i(\bar{z})]\tag{51}
\]
and denote the respective series coefficients by \(b^i_n\); these satisfy recurrence relations having the form
\[
\alpha^i_n b^i_{n+1} + \beta^i_n b^i_n + \gamma^i_n b^i_{n-1} = 0, \quad [-\infty < n < \infty]\tag{52}
\]
where \(\alpha_n^i, \beta_n^i, \gamma_n^i\) depend on the parameters of the differential equation as well as on the parameter \(\nu_i\). This parameter \(\nu_i\) must be such that \(\alpha_n^i \neq 0\) and \(\gamma_n^i \neq 0\), on the contrary the series truncates on the left \((\alpha_n^i = 0)\) or on the right-hand side \((\gamma_n^i = 0)\) according to Appendix B. From relations \((52)\) it results a transcendental (characteristic) equation given as a sum of two infinite continued fractions \(3\), namely,
\[
\beta_0^i = \frac{\alpha_{-1}^i + \gamma_{-1}^i}{\beta_{-1}^i} - \frac{\alpha_{-2}^i \gamma_{-2}^i}{\beta_{-2}^i} - \cdots + \frac{\alpha_{-1}^i \gamma_{-1}^i}{\beta_{1}^i - \beta_{2}^i - \cdots} \cdots , \tag{53}\]
If the differential equation has no free constant, the above equation may be used to determine \(\nu_i\); only for such values the solutions are valid. However, if the equation presents an arbitrary constant, we can ascribe convenient values for \(\nu_i\); then, Eq. \((53)\) permits to find values for that constant: in section IV, this fact is used to get bounded infinite-series eigenfunctions for the QES potentials \(3\).

A. The initial set of two-sided solutions

According to the the previous prescription, the first set of solutions is
\[
U^\infty_1 = e^{i\omega z} z^1 + \frac{b^1_0}{\alpha_0^1} (z - z_0)^{-1} B_2 - \frac{B_1}{\alpha_0^1} \sum_{n = -\infty}^{\infty} b^1_n \Phi \left(2 + i\eta - \frac{B_2}{2} \right) 2 + \frac{B_1}{\alpha_0^1} - n - \nu_1 + 2i\omega z , \tag{54}
\]
\[
U_1 = e^{i\omega z} z^1 + \frac{b^1_0}{\alpha_0^1} (z - z_0)^{-1} B_2 - \frac{B_1}{\alpha_0^1} \sum_{n = -\infty}^{\infty} b^1_n \Phi \left(2 + i\eta - \frac{B_2}{2} \right) 2 + \frac{B_1}{\alpha_0^1} - n - \nu_1 + 2i\omega z , \tag{54}
\]
\[
U_1 = e^{i\omega z} z^1 + \frac{b^1_0}{\alpha_0^1} (z - z_0)^{-1} B_2 - \frac{B_1}{\alpha_0^1} \sum_{n = -\infty}^{\infty} b^1_n (2i\omega z)^{n+\nu_1}
\]
\[
\times \Phi \left(n + \nu_1 + 1 + i\eta - \frac{B_2}{2} - \frac{B_1}{\alpha_0^1} - n - \nu_1 - 2i\omega z \right) ,
\]
with
\[
\alpha^1_n = -2i\omega z_0(n + \nu_1 + 1), \quad \gamma^1_n = - \left[n + \nu_1 + i\eta - \frac{B_1}{2} \right] \left[n + \nu_1 + 1 - B_2 - \frac{B_1}{\alpha_0^1} \right],
\]
\[
\beta^1_n = (n + \nu_1) \left[n + \nu_1 + 1 - B_2 - \frac{2B_1}{\alpha_0^1} + 2i\omega z_0 \right] + \left[i\omega z_0 - 1 - \frac{B_1}{\alpha_0^1} \right] \left[2 - B_2 - \frac{B_1}{\alpha_0^1} \right] + 2 - B_2 + B_3 , \tag{55}
\]
in the recurrence relations \((52)\). As in the one-sided series, \(2 + i\eta - (B_2/2) \neq 0, -1, \cdots\) for \(U^\infty_1(z)\) and \(U_1(z)\). In fact, the restrictions imposed on the parameters of the hypergeometric functions of the the pairs \((U^\infty_i, U_i)\) are valid also for the two-sided solutions \((U^\infty_i, U_i)\).

Solutions \((54)\) can be checked by the same steps of section II.B, with slight modifications. For example, instead of Eq. \((33)\) we find
\[
\sum_{n = -\infty}^{\infty} b^1_n \alpha^1_{n+1} \Phi_{-1}(y) + \sum_{n = -\infty}^{\infty} b^1_n \beta^1_n \Phi_n(y) + \sum_{n = -\infty}^{\infty} b^1_n \gamma^1_{n+1} \Phi_{n+1}(y) = 0
\]
where now
\[
\Phi_n(y) = \Psi \left(2 + i\eta - \frac{B_2}{2} + \frac{B_1}{\alpha_0^1} - n - \nu_1; y \right) \text{ or } \Phi_n(y) = \Phi \left(2 + i\eta - \frac{B_2}{2} + \frac{B_1}{\alpha_0^1} - n - \nu_1; y \right) \]
For two-sided series \((-\infty < n < \infty)\), the above equation can be written as

\[ \sum_{n=-\infty}^{\infty} \left[ a_n b_{n+1}^1 + \beta_n b_n^1 + \gamma_n b_{n-1}^1 \right] \mathcal{F}_n(y) = 0, \]

which is satisfied by the recurrence relations \((52)\) with \(i = 1\).

**B. Convergence of the two-sided series**

For the series \(\sum_{n=-\infty}^{\infty} f_n(z)\), in addition to the ratio \((59)\), we need to compute the ratio

\[ L_2(z) = \left| \lim_{n \to -\infty} f_{n-1}(z)/f_n(z) \right|. \]  \((56)\)

By the Raabe test, if

\[ L_2(z) = 1 + (B/|n|) \]  \((57)\)

(where \(B\) is a constant) the series converges when \(B < -1\) and diverges when \(B > -1\); for \(B = -1\) the test is inconclusive. Hence, the convergence of two-sided series is given by the intersection of the regions obtained when \(n \to \infty\) with the region obtained when \(n \to -\infty\).

When \(n \to \infty\) the ratio test does not alter the convergence, that is, the regions stated at the end of section II.D hold for two-sided expansions as well. On the other hand, when \(n \to -\infty\), the \(U_i(z)\) converge for any finite value of \(z\) as \(U_i(z)\) do, but the domains of convergence of \(U_i^\infty(z)\) and \(U_i(z)\) are restricted. Then, if the parameters of the confluent hypergeometric functions satisfy the same conditions as in the case of one-sided solutions, the actual regions of convergence are given by:

- The two-sided solutions \(U_i\) converge for any finite value of \(z\); the \(U_i\) converge only for the finite values of \(z\) which satisfy the restrictions given in \((55)\) and \((61)\).
- The two-sided solutions \(U_i^\infty\) converge at \(z = \infty\) if the conditions \((57)\) are fulfilled; these \(U_i^\infty\) converge only for finite values of \(z\) which satisfy the restrictions specified in \((56)\) and \((59)\).

\[ |z| \geq |z_0| \text{ if } \begin{cases} \text{Re} \left[ B_2 + \frac{B_1}{z_0} \right] > 1 \text{ for } (U_1^\infty, U_1), \text{ and } (U_2^\infty, U_2), \\ \text{Re} \left[ B_2 + \frac{B_1}{z_0} \right] < 1 \text{ for } (U_3^\infty, U_3), \text{ and } (U_4^\infty, U_4), \end{cases} \]  \((58)\)

\[ |z - z_0| \geq |z_0| \text{ if } \begin{cases} \text{Re} \left[ \frac{B_3}{z_0} \right] < -1 \text{ for } (U_5^\infty, U_5), \text{ and } (U_6^\infty, U_6), \\ \text{Re} \left[ \frac{B_3}{z_0} \right] > -1 \text{ for } (U_7^\infty, U_7), \text{ and } (U_8^\infty, U_8), \end{cases} \]  \((59)\)

where the constraints on \(B_2 + (B_1/z_0)\) or \(B_1/z_0\) are necessary only to assure the convergence at \(|z| = |z_0|\) or \(|z - z_0| = |z_0|\), respectively, that is: without these constraints, the solutions indicated in \((55)\) converges for \(|z| > |z_0|\), while the ones indicated in \((56)\) converges for \(|z - z_0| > |z_0|\).

Notice that, if for a given problem a solution \(U_i^\infty\) converges at \(z = \infty\), then this \(U_i^\infty\) and \(U_i\) cover the entire complex plane \(z\). This fact does not occur with the two-sided expansions in series of Coulomb wave functions \((11)\).

Let us find the restrictions \((55)\) and \((59)\). It is sufficient to consider explicitly only the firts set; the results for the other sets follow by the transformations \((12)\). In the first place, for \(n \to \pm \infty\) instead of \((11)\) we have

\[ 2i\omega z_0 \left[ 1 + \frac{1}{n} \right] b_n^{1+1} - \left[ n + \nu_1 + 1 + 2i\omega z_0 - B_2 - \frac{2B_1}{z_0} \right] + \left[ n + \nu_1 + 1 + i\eta - \frac{2B_1}{z_0} - \frac{3B_2}{2} \right] b_n^{1+1} = 0, \]

whose minimal solution behaves as

\[ \frac{b_n^{1+}}{b_n^{1+}} \sim 2i\omega z_0 \left[ 1 + \frac{1}{n} \left( 1 + B_2 + \frac{2B_1}{z_0} - \nu_1 \right) \right], \quad \frac{b_n^{1+}}{b_n^{1+}} \sim \frac{n}{2i\omega z_0} \left[ 1 - \frac{1}{n} \left( B_2 + \frac{2B_1}{z_0} - \nu_1 \right) \right]. \]  \((60)\)

On the other hand, relation \((13)\) is replaced by \((y = -2i\omega z)\)

\[ \left[ n + \nu_1 + 1 + i\eta - \frac{B_1}{z_0} - \frac{B_2}{2} \right] \mathcal{F}_{n+1}(y) - \left[ n + \nu_1 + 1 + 2i\omega z - \frac{B_1}{z_0} \right] \mathcal{F}_n(y) = 0. \]  \((61)\)
where, now,
\[ \mathcal{F}_n(y) = \Psi \left( 2 + i\eta - \frac{B_2}{2}, 2 + \frac{B_1}{z_0} - n - \nu_1; y \right) \]
or
\[ \mathcal{F}_n(y) = \tilde{\Psi} \left( 2 + i\eta - \frac{B_2}{2}, 2 + \frac{B_1}{z_0} - n - \nu_1; y \right). \]

If \( z \) is bounded (that is, if \( 2i\omega z/n \to 0 \)), then Eq. \( (\ref{58}) \) when \( n \to \pm \infty \) gives
\[ \mathcal{F}_{n+1}/\mathcal{F}_n \sim 1 + \frac{1}{n} \left( \frac{B_1}{z_0} - \nu_1 \right) \Rightarrow \mathcal{F}_{n+1}/\mathcal{F}_n \sim 1 - \frac{1}{n} \left( \frac{B_1}{z_0} - \nu_1 \right), \]
\[ \frac{\mathcal{F}_{n+1}}{\mathcal{F}_n} \sim \frac{2i\omega z}{n} \left[ 1 + \frac{1}{n} \left( \frac{B_1}{z_0} - \nu_1 \right) \right] \Rightarrow \frac{\mathcal{F}_{n+1}}{\mathcal{F}_n} \sim \frac{n}{2i\omega z} \left[ 1 - \frac{1}{n} \left( 1 - \nu_1 + \frac{B_1}{z_0} \right) \right]. \]

For \( n \to -\infty \) we have to select
\[ \frac{\mathcal{F}_{n+1}}{\mathcal{F}_n} \sim \frac{n}{2i\omega z} \left[ 1 + \frac{1}{n} \left( \nu_1 - 1 - \frac{B_1}{z_0} \right) \right] \text{ if } \mathcal{F}_n = \Psi, \]
\[ \frac{\mathcal{F}_{n+1}}{\mathcal{F}_n} \sim 1 - \frac{1}{n} \left( \frac{B_1}{z_0} - \nu_1 \right) \text{ if } \mathcal{F}_n = \tilde{\Psi} \hspace{1cm} (62) \]

Thence, using also the ratio \( (\ref{59}) \), we find
\[ \lim_{n \to -\infty} \frac{b_{n-1}^1 \mathcal{F}_{n-1}}{b_n^1 \mathcal{F}_n} = \frac{z_0}{z} \left[ 1 + \frac{1}{n} \left( \frac{B_1}{z_0} - \nu_1 \right) \right] + O \left( \frac{1}{n^2} \right), \quad [\mathcal{F}_n = \Psi \mapsto U^\infty_1] \]
\[ \lim_{n \to -\infty} \frac{b_{n-1}^1 \mathcal{F}_{n-1}}{b_n^1 \mathcal{F}_n} = \frac{2i\omega z_0}{n} + O \left( \frac{1}{n^2} \right), \quad [\mathcal{F}_n = \tilde{\Psi} \mapsto U_1]. \]

The first relation leads to the restriction \( (\ref{58}) \) for \( U^\infty_1 \); the second relation does not restrict the convergence of \( U_1 \). To show that the convergence of \( U_1 \) is restricted as in \( (\ref{58}) \), we define \( F_n(y) \) as
\[ F_n(y) = (2i\omega z)^{n+\nu_1} \tilde{\Psi} \left( n + \nu_1 + 1 + i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, n + \nu_1 - \frac{B_1}{z_0} - 2i\omega z \right). \]

This \( F_n \) satisfies \( (\ref{61}) \) which, for \( n \to -\infty \), gives
\[ \frac{F_{n+1}}{F_n} \sim \frac{n}{2i\omega z} \left[ 1 + \frac{1}{n} \left( \nu_1 - 1 - \frac{B_1}{z_0} \right) \right]. \]

Together with the ratio \( (\ref{60}) \) the previous limit produces the condition given in \( (\ref{58}) \). On the other side, by using \( \lim_{y \to \infty} \tilde{\Psi}(a, c; y) = y^{-a} \), we find
\[ \lim_{z \to \infty} U^\infty_1(z) = e^{i\omega z} \left( z^{-i\eta - \frac{B_1}{z_0}} \sum_{n = -\infty}^{\infty} b_n^1 \mathcal{F}_n \right) \lim_{n \to -\infty} \frac{b_{n-1}^1 \mathcal{F}_{n-1}}{b_n^1 \mathcal{F}_n} \left| \frac{2i\omega z_0}{n} \right| \to 0. \]

Therefore, \( U^\infty_1 \) converges at \( z = \infty \) if the first condition given in \( (\ref{59}) \) is satisfied.

**IV. SCHröDINGER EQUATION FOR HYPERBOLIC POTENTIALS**

For the hyperbolic potential \( (\ref{5}) \), the Schrödinger equation \( (\ref{1}) \) becomes
\[ \frac{d^2 \psi}{du^2} + \left\{ \mathcal{E} - 4\beta^2\sinh^4 u - 4\beta \left[ \beta - 2(\gamma + \delta + \ell) \right] \sinh^2 u \right. \]
\[ -4 \left[ \delta - \frac{1}{4} \right] \left[ \delta - \frac{3}{4} \right] \frac{1}{\sinh^4 u} + 4 \left[ \gamma - \frac{1}{4} \right] \left[ \gamma - \frac{3}{4} \right] \frac{1}{\cosh^4 u} \right\} \psi = 0. \]

The substitutions
\[ z = \cosh^2 u, \quad \psi(u) = \psi[u(z)] = z^{\gamma - \frac{1}{4}}(z - 1)^{\delta - \frac{1}{4}}U(z), \quad [z \geq 1] \]
transform the preceding equation into a confluent Heun equation with
\[ z_0 = 1, \quad B_1 = -2\gamma, \quad B_2 = 2\gamma + 2\delta, \quad B_3 = \frac{\ell}{2} + \left( \gamma + \delta - \frac{1}{4} \right)^2, \]
\[ i\omega = \pm \beta, \quad i\eta = \pm(\ell + \delta + \gamma), \]
where the upper or the lower sign for \( (\eta, \omega) \) must be taken throughout.
A. The Ushveridze potential

Now we exclude the four cases given in Eq. (63) and suppose that \( \ell \) is a non-negative integer. Due to the factor \( \exp(i\omega z) \), in order to obtain bounded solutions when \( z \to \infty \) we take

\[
i \omega = -\beta, \quad i\eta = -\ell - \gamma - \delta, \quad [\ell = 0, 1, 2, \cdots].
\]

in Eq. (63).

When \( \delta \geq 1/4 \), the above family of potentials is quasiexactly solvable because it admits bounded wavefunctions given by finite series which allow to determine only a finite number of energy levels for any real value of the parameter \( \gamma \). In addition, for \( 1/4 \leq \delta < 1/2 \) and \( 1/2 < \delta \leq 3/4 \) we have found two-sided infinite series of Coulomb wave functions which are convergent and bounded for any \( z \geq 1 \) this gives the possibility of determining the remaining part of the energy spectra as solutions of a characteristic equation. For \( \delta = 1/2 \), the Raabe test is inconclusive. We will show that these results are more general than the ones obtained from the new two-sided solutions for the CHE.

We will see that it is advantageous to use the Baber-Hassé expansions to get finite-series solutions. Then, by inserting the solution \( \psi_1^{\text{baber}} \) given in (28a) into Eq. (63), we find

\[
\psi_1^{\text{baber}}[u(z)] = e^{-\beta z}z^{-\frac{\delta}{2}}(z-1)^{\frac{\delta}{2}-\frac{1}{4}} \sum_{n=0}^{\ell} a_n^1(z-1)^n, \quad \delta \geq \frac{1}{4}
\]

where the coefficients satisfy the relations \( a_{-1}^1 = 0 \)

\[
A_n^1 a_{n+1}^1 + B_n^1 a_n^1 + G_n^1 a_{n-1}^1 = 0, \quad 0 \leq n \leq \ell \quad \text{with}
\]

\[
A_n^1 = (n+1)(n+2\delta), \quad B_n^1 = n(n+2\gamma + 2\delta - 1 - 2\beta) + \frac{\delta}{4} + \left(\gamma + \delta - \frac{1}{2}\right)^2 - 2\beta \delta, \\
G_n^1 = -2\beta(n-\ell-1).
\]

According to Appendix A, the series in (65a) ends at \( n = \ell \) because \( G_\ell^1 = 0 \) when \( n = \ell + 1 \). Since \( \beta > 0 \), the previous eigenfunctions are bounded for all \( z \geq 1 \) (including \( z = 1 \)) provided that \( \delta \geq 1/4 \). In fact, the condition \( A_n^1 G_n^1 > 0 \) \( (0 \leq n \leq \ell - 1) \) assures that \( \psi_1^{\text{baber}} \) represents \( \ell + 1 \) solutions, each one with a different real energy determined from the vanishment of the determinant of the tridiagonal matrix \( [A, b] \) corresponding to (65b).

From (60) we see that there is a finite-series solution \( \psi_4 \) associated with the the expansion \( \hat{U}_4 \) in series of confluent hypergeometric functions:

\[
\hat{\psi}_4[u(z)] = e^{-\beta z}z^{\gamma - \frac{1}{4}}(z-1)^{\delta - \frac{1}{2}} \sum_{n=0}^{\ell} b_n^4 (-2\beta z)^n \Phi(n-\ell, n+2\gamma; 2\beta z),
\]

where the \( b_n^4 \) satisfy \( (17) \) with

\[
\dot{a}_n^4 = 2\beta(n+1), \quad \dot{b}_n^4 = B_n^1, \quad \dot{\gamma}_n^4 = -[n-1+2\delta][n-\ell-1],
\]

Contrary to (65b), these wavefunctions are valid only if \( 2\gamma \neq 0, -1, -2, \cdots \). In this sense, the Baber-Hassé solutions are more general when we are concerned with finite-series solutions, but they do not afford infinite series convergent at \( z = \infty \).

To get infinite series we use the two-sided solutions. We impose the conditions \( (50) \) and \( (53) \), and require that \( a \) is not zero or a negative integer in \( \Psi(a, c; y) \). Thus, for \( U_2^\infty \) Eq. (63) becomes

\[
\psi_2[u(z)] = e^{-\beta z}z^{\gamma - \frac{1}{4}}(z-1)^{\delta - \frac{1}{2}} \sum_{n=-\infty}^{\infty} b_n^2 \Psi(1-\ell-2\delta, 2\gamma - n-2\nu_2; 2\beta z), \quad [1/2 < \delta \leq 3/4]
\]

where the \( b_n^2 \) satisfy (52) with

\[
\alpha_n^2 = 2\beta(n+\nu_2+1), \quad \gamma_n^2 = -[n+\nu_2+1-2\gamma-2\delta][n+\nu_2+1-\ell-2\delta], \\
\beta_n^2 = [n+\nu_2][n+\nu_2+3-2\gamma-2\delta-2\beta] + \frac{\delta}{4} + \left(\gamma + \delta - \frac{1}{2}\right)^2 - 2\beta [1-\delta] + 2 - 2\gamma - 2\delta.
\]

According to (55), the series converges at \( z = 1 \) if \( \delta > 1/2 \); in addition, the solution is bounded at \( z = 1 \) if \( \delta \leq 3/4 \): hence the series converges also at \( z = \infty \) because in such cases \( \ell + 2\delta > 0 \).
as required by \[51\]. Furthermore, to have two-sided infinite series, the parameter \(\nu_2\) cannot allows that \(\alpha_n^4\) and \(\gamma_n^4\) vanish, that is,

\[
\nu_2, \quad \nu_2 - 2\gamma - 2\delta, \quad \nu_2 - 2\delta \quad \text{are not integers.} \quad (66c)
\]

On the other side, using \(U_4^{\infty}\) we find

\[
\psi_4[u(z)] = e^{-\beta z} z^{3/2 - \gamma} (z - 1)^{-\delta/4} \sum_{n = -\infty}^{\infty} b_n^4 \Psi \left(1 - \ell - 2\gamma, 2\gamma - n - \nu_4; 2\beta z\right), \quad (67a)
\]

\[
\left[1/4 \leq \delta < 1/2, \ell + 2\gamma > 0, \ell + 2\gamma \neq 1, 2, 3, \ldots \right]
\]

where the \(b_n^4\) satisfy \[52\] with

\[
\alpha_n^4 = 2\beta(n + \nu_4 + 1), \quad \gamma_n^4 = - [n + \nu_4 - 1 + 2\delta] [n + \nu_4 - 1 - \ell],
\]

\[
\beta_n^4 = [n + \nu_4] [n + \nu_4 - 1 + 2\gamma + 2\delta - 2\beta] + \frac{\delta}{4} + \left[\gamma + \delta - \frac{1}{2}\right]^2 - 2\beta\delta. \quad (67b)
\]

The condition \(\delta < 1/2\) assures that the series converges at \(z = 1\), while \(1/4 \leq \delta\) assure that the solution is bounded at \(z = 1\); the condition \(\ell + 2\gamma > 0\) assures convergence at \(z = \infty\), while \(\ell + 2\gamma \neq 1, 2, 3, \ldots\) guarantees that \(\Psi(1 - \ell - 2\gamma, c; 2\beta z)\) is not a polynomial of fixed degree. To have two-sided infinite series, \(\nu_4\) cannot allows that \(\alpha_n^4\) and \(\gamma_n^4\) vanish, a condition that is fulfilled if \(\nu_4\) and \(\nu_4 + 2\delta\) are not integers. \quad (67c)

The restrictions on the parameter \(\gamma\) are sufficient to see that the solutions \[67a\] are less general than the expansions in series of Coulomb wave functions.

B. Whittaker-Hill equation for a hyperbolic Razavy potential

Let us consider only the case \(\gamma = \delta = 1/4\) in \[51\]. We get

\[
\mathcal{V}(u) = 4\beta^2 \sinh^4 u + 4\beta [\beta - 1 - 2\ell] \sinh^2 u, \quad \ell = 0, 1/2, 1, 3/2, \ldots
\]

where \(u \in (-\infty, \infty)\). The substitutions

\[
z = \cosh^2 u, \quad \psi(u) = \psi[u(z)] = U(z), \quad [z \geq 1] \quad (68a)
\]

transform the Schrödinger equation for the preceding potential into the CHE \[11\] with

\[
z_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = 1, \quad B_3 = \frac{\ell}{4}, \quad i\omega = \pm\beta, \quad i\eta = \pm(\ell + \frac{1}{2}), \quad (68b)
\]

that is, into the Whittaker-Hill equation \[2\]. To obtain bounded solutions for \(z = \cosh^2 u \geq 1\) we will choose

\[
i\omega = -\beta, \quad i\eta = -\ell - (1/2).
\]

By using expansions in series of Coulomb wave functions we have seen that, for \(\ell\) integer or half-integer, this problem admits: (i) even and odd finite-series solutions (with respect to \(u \mapsto -u\)) which are convergent and bounded for all \(z \geq 1\), (ii) even infinite-series solutions which are convergent and bounded for all \(z \geq 1\). To find odd infinite-series solutions it is necessary to use two solutions valid for different intervals of \(z\). Below we find similar results by using the solutions for the CHE given in the present study.

In fact, there are several possibilities to form finite-series solutions for this case. One can use, for example, the one-sided series solutions \(U_i^1\) with \(i = 1, \ldots, 4\) as explained at the end of section II.B: notice that we cannot use the \(U_i\) if \(i = 5, \ldots, 8\), because these do not satisfy the conditions \(\alpha_i^4 \gamma_i^{4+1} > 0\) which assure real energies \[4\]. The final result is: \(U_1^1\) and \(U_3^1\) yield, respectively, odd and even finite-series solutions when \(\ell\) is half-integer; if \(\ell\) is a non-negative integer we can use \(U_2^1\) (odd) and \(U_4^1\) (even).

To find even infinite-series solutions bounded for any \(z = \cosh^2 u \geq 1\) we use the two-sided expansions \(U_3^{\infty}\) and \(U_4^{\infty}\) for \(\ell\) half-integer and integer, respectively: these satisfy the conditions \[50\] and \[55\] at the same time and, consequently, converge at \(z = \infty\) and \(z = 1\). Thus

\[
U_3^{\infty}[u(z)] = e^{-\beta \cosh^2 u} \sum_{n = -\infty}^{\infty} b_n^4 \Psi (-\ell, 1/2 - n - \nu_3; 2\beta \cosh^2 u), \quad [\ell = 1/2, 3/2, 5/2, \ldots] \quad (69a)
\]
with
\[
\alpha_n^3 = 2\beta(n + \nu_3 + 1), \quad \beta_n^3 = (n + \nu_3)[n + \nu_3 + 1 - 2\beta] + \frac{1}{2}\left(\frac{3}{2} - \beta\right) + \frac{\beta}{2},
\]
\[
\gamma_n^3 = -\left[n + \nu_3 - \ell - \frac{1}{2}\right]\left[n + \nu_3 - \frac{1}{2}\right],
\]
(69b)
in the recurrence relations (52). The summation runs from \(-\infty\) to \(+\infty\) if we choose \(\nu_3\) in such a way that \(\alpha_n\) and \(\gamma_n\) do not vanish, for example, \(\nu_3 \in (0, 1/2)\). On the other side, if \(\ell\) is integer, we find the infinite series
\[
U_4^\infty[z] = \cosh u e^{-\beta \cosh^2 u} \sum_{n=-\infty}^{\infty} b_n^4 \Psi \left(\frac{1}{2} - \ell, \frac{3}{2} - n - \nu_4; 2\beta \cosh^2 u\right), \quad \left[\ell = 0, 1, 2, \cdots\right]
\]
(70a)
with
\[
\alpha_n^4 = 2\beta(n + \nu_4 + 1), \quad \beta_n^4 = (n + \nu_4)[n + \nu_4 - 2\beta] - \frac{1}{2}\beta + \frac{\nu_4}{4},
\]
\[
\gamma_n^4 = -\left[n + \nu_4 - \ell - 1\right]\left[n + \nu_4 - \frac{1}{2}\right]
\]
(70b)
in the recurrence relations (52). The parameter \(\nu_4\) can be chosen as above, \(\nu_4 \in (0, 1/2)\).

Odd infinite-series solutions are obtained if we use two-sided solutions which are valid for different ranges of \(z\) but have the same characteristic equation, namely: the sets \(U_1\) and \(U_2\) for \(\ell\) half-integer and integer, respectively. Thus, as an example we suppose that \(\ell\) is half-integer. Then, we have the odd set
\[
U_1^\infty[z(u)] = \sinh(2u) e^{-\beta \cosh^2 u} \sum_{n=-\infty}^{\infty} b_n^1 \Phi \left(1 - \ell, \frac{3}{2} - n - \nu_1; 2\beta \cosh^2 u\right),
\]
\[
U_1[z(u)] = \sinh(2u) e^{-\beta \cosh^2 u} \sum_{n=-\infty}^{\infty} b_n^1 \Phi \left(1 - \ell, \frac{3}{2} - n - \nu_1; 2\beta \cosh^2 u\right),
\]
(71a)
\[
U_1[z(u)] = \sinh u e^{-\beta \cosh^2 u} \sum_{n=-\infty}^{\infty} b_n^1 [-2\beta \cosh^2 u]^{n+\nu_4} \times
\]
\[
\Phi \left(n + \nu_1 - \ell + \frac{1}{2}, n + \nu_1 + \frac{1}{2}; 2\beta \cosh^2 u\right),
\]
with
\[
\alpha_n^1 = 2\beta(n + \nu_1 + 1), \quad \beta_n^1 = (n + \nu_1)[n + \nu_1 + 1 - 2\beta] - \frac{3}{2}\beta + \frac{\nu_4+1}{4},
\]
\[
\gamma_n^1 = -\left[n + \nu_1 - \ell - \frac{1}{2}\right]\left[n + \nu_1 + \frac{1}{2}\right]
\]
(71b)
in the recurrence relations (52). The summation runs from \(-\infty\) to \(+\infty\) if we choose \(\nu_1\) in such a way that \(\alpha_n\) and \(\gamma_n\) do not vanish, for example, \(\nu_1 \in (0, 1/2)\). The solution \(U_1^\infty\) converges at \(z = \cosh^2 u = \infty\) because it satisfies the condition \([\ell > 0]\) (\(\ell > 0\)); it does not converge at \(z = 1\) because the condition given \([32]\) is not fulfilled. On the other hand, the solution \(U_1\) is valid for any finite value of \(z\). Therefore, these two solutions cover all of the admissible values for \(z\); furthermore, by means of relation \([11]\), \(U_1\) can be written as a linear combination of \(U_1^\infty\) and \(U_1\) in the region where \(z \neq 1\) and \(z \neq \infty\).

V. SCHRODINGER EQUATION FOR TRIGONOMETRIC POTENTIALS

For the trigonometric potential \([7]\), the Schrödinger equation \([1]\) becomes
\[
\frac{d^2\psi}{du^2} + \left\{\mathcal{E} + 4\beta^2 \sin^4 u - 4\beta[\beta + 2(\gamma + \delta + \ell)] \sin^2 uight.
\]
\[
- 4\left[\delta - \frac{1}{4}\right]\left[\delta - \frac{3}{4}\right] \frac{1}{\sin^2 u} - 4\left[\gamma - \frac{1}{4}\right]\left[\gamma - \frac{3}{4}\right] \frac{1}{\cos^2 u}\right\} \psi = 0.
\]
The substitutions
\[
z = \cos^2 u, \quad \psi(u) = \psi[u(z)] = z^{\delta-\frac{1}{4}}(z - 1)^{\gamma-\frac{1}{4}} \psi(U(z)), \quad [0 \leq z \leq 1]
\]
(72a)
transform the Schrödinger equation into a confluent Heun equation with
\[
z_0 = 1, \quad B_1 = -2\delta, \quad B_2 = 2\gamma + 2\delta, \quad B_3 = -\frac{\delta}{4} + (\gamma + \delta - \frac{1}{2})^2,
\]
\[
\imath \omega = \pm \beta, \quad \imath \eta = \mp(\ell + \delta + \gamma),
\]
(72b)
where the upper or the lower sign for \((i\eta, i\omega)\) must be taken throughout. Usveridze supposed that \(\ell = 0, 1, 2, \cdots\) but, if \(\gamma\) and \(\delta\) are given by \([3]\), the potential is quasisolvable even when \(\ell\) is a positive half-integer. These trigonometric Razavy potentials are ruled by Whittaker-Hill equations and will not be considered in the following. We will see that it is convenient to express the solutions as expansions in power series.

A. Solutions in power series (Baber-Hassé)

Finite-series solutions are obtained by taking \(i\omega = \beta\) and \(i\eta = -\ell - \gamma - \delta\) in \(U^\text{baber}_1\) and \(U^\text{baber}_5\). However, only \(U^\text{baber}_1\) satisfies the condition \(\alpha_n\gamma_{n+1} > 0\) for real energies; the corresponding solution, denoted by \(\psi_5(u)\), is

\[
\psi_5^\text{baber}(u) = e^{\beta \cos^2 u} \left[ \cos^2 u \right]^{\delta - 2} \left[ 1 - \cos^2 u \right]^{\gamma - 2} \left( \sum_{n=0}^{\ell} (-)^n a_n^5 \left[ \cos^2 u \right]^n, \ [\delta \geq 1, \ \gamma \geq 1/2] \right) \tag{73a}
\]

where recurrence relations for the coefficients are given by \((a^5_{-1} = 0)\)

\[
(n + 2\delta) (n + 1) a_{n+1}^5 + B_n^5 a_n^5 - 2\beta (n - \ell - 1) a_{n-1}^5 = 0, \quad \text{where}
\]

\[
B_n^5 = n (n + \ell + 2\delta - 1 - 2\beta) - \frac{\ell}{4} + \left( \gamma + \delta - \frac{1}{2} \right)^2 + 2\beta (\ell + \gamma).
\tag{73b}
\]

The conditions \(\delta \geq 1/4\) and \(\gamma \geq 1/4\) assure that the solutions are bounded at \(\cos u = 0\) and \(\cos u = \pm 1\), respectively.

Infinite-series solutions results from \(i\omega = -\beta\) and \(i\eta = \ell + \gamma + \delta\) in \(U^\text{baber}_1\) and \(U^\text{baber}_5\). Denoting these solutions by \(\bar{\psi}_1\) and \(\bar{\psi}_5\), respectively, we have

\[
\bar{\psi}_1^\text{baber}(u) = e^{-\beta \cos^2 u} \left[ \cos^2 u \right]^{\delta - 2} \left[ 1 - \cos^2 u \right]^{\gamma - 2} \left( \sum_{n=0}^{\ell} (-1)^n a_n^1 \left[ 1 - \cos^2 u \right]^n, \right) \tag{74a}
\]

where the coefficients are given by \((a^1_{-1} = 0)\)

\[
(n + 2\gamma) (n + 1) a_{n+1}^1 + B_n^1 a_n^1 - 2\beta (n + \ell + 2\gamma + 2\delta - 1) a_{n-1}^1 = 0 \quad \text{with}
\]

\[
B_n^1 = n (n + \ell + 2\gamma + 2\delta - 1 - 2\beta) - \frac{1}{4} + \left( \gamma + \delta - \frac{1}{2} \right)^2 + 2\beta(\ell + \gamma + 2\delta).
\tag{74b}
\]

and

\[
\bar{\psi}_5^\text{baber}(u) = e^{-\beta \cos^2 u} \left[ \cos^2 u \right]^{\delta - 2} \left[ 1 - \cos^2 u \right]^{\gamma - 2} \left( \sum_{n=0}^{\ell} a_n^5 \left[ - \cos^2 u \right]^n \right) \tag{75a}
\]

satisfying the recurrence relations \((a^5_{-1} = 0)\)

\[
(n + 2\delta) (n + 1) a_{n+1}^5 + B_n^5 a_n^5 + 2\beta (n + \ell + 2\gamma + 2\delta - 1) a_{n-1}^5 = 0 \quad \text{with}
\]

\[
B_n^5 = n (n + 2\gamma + 2\delta - 1 + 2\beta) - \frac{\ell}{4} + \left( \gamma + \delta - \frac{1}{2} \right)^2 + 2\beta(\ell + \gamma + 2\delta).
\tag{75b}
\]

For \(\delta \geq 1/4\) and \(\gamma \geq 1/4\), the solutions \(\psi_1\) and \(\psi_5\) are both convergent and bounded for whole interval \(0 \leq \cos^2 u \leq 1\). In this manner, we have found only one expression for finite-series solutions, but two expressions for infinite-series solutions, \(\psi_1\) and \(\psi_5\). However, the latter are formally identical, as we will see.

In effect, by using the relation

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{x^k}{k!}
\]

and taking \(z = \cos^2 u\) in \(\psi_1\), we find

\[
\sum_{n=0}^{\ell} (-)^n a_n^1 \left[ 1 - z \right]^n = \sum_{n=0}^{\ell} (-)^n n! a_n^1 \sum_{k=0}^{n} (-z)^k \frac{n!}{(n-k)!} \frac{x^k}{k!} = \sum_{n=0}^{\ell} (-)^n n! a_n^1 \sum_{k=0}^{n} (-z)^k \frac{(m+k)!}{m!} \frac{x^m}{m!}
\]

\[
= \sum_{n=0}^{\ell} (-z)^n \left[ \frac{1}{n!} \sum_{m=0}^{\infty} \frac{(-m+n+1)!}{m!} \frac{(m+n)!}{a_{m+n}} \right].
\]
Hence, both \( \tilde{\psi}_1 \) and \( \tilde{\psi}_5 \) are given by the same series of \((-z)^n\) provided that

\[
\tilde{a}_n^5 = \frac{(-)^n}{n!} \sum_{m=0}^{\infty} \frac{(-)^m (m+n)!}{m!} \tilde{a}_{n+m}^5.
\]  

(76)

Similarly, by starting from \( \tilde{\psi}_5 \), we get

\[
\sum_{n=0}^{\infty} \tilde{a}_n^5 \cdot (-z)^n = \sum_{n=0}^{\infty} (-)^n [1-z]^n \left[ \frac{1}{n!} \sum_{m=0}^{\infty} \frac{(-)^{m+n} (m+n)!}{m!} \tilde{a}_{n+m}^5 \right]
\]

and, as a consequence, both \( \tilde{\psi}_1 \) and \( \tilde{\psi}_5 \) are given by the same series of \((1-z)^n\) if

\[
\tilde{a}_n^1 = \frac{(-)^n}{n!} \sum_{l=0}^{\infty} \frac{(-)^l (l+n)!}{l!} \tilde{a}_{n+l}^5.
\]  

(77)

B. Solutions in series of confluent hypergeometric functions

Using the correspondences (30), we replace the preceding Baber-Hassé expansions by series \( \hat{U}_i \) of regular confluent hypergeometric functions as follows:

\[
U_5^{\text{baber}} \mapsto \hat{U}_8 \text{ with } (i\omega, i\eta) = (\beta, -\ell - \gamma - \delta) : \text{ finite series } \psi_8,
\]

\[
(U_5^{\text{baber}}, U_5^{\text{baber}}) \mapsto \left( \hat{U}_4, \hat{U}_5 \right) \text{ with } (i\omega, i\eta) = (-\beta, \ell + \gamma + \delta) : \text{ infinite series } (\tilde{\psi}_4, \tilde{\psi}_5).
\]

Thus, instead of (76), we get the finite-series solutions

\[
\psi_8(u) = e^{\beta \cos^2 u} \left[ \cos^2 u \right]^{\gamma - \frac{1}{2}} \left[ \sin^2 u \right]^{\gamma - \frac{1}{2}} \sum_{n=0}^{\infty} \hat{b}_n^8 \left[ -2\beta \sin^2 u \right]^n \hat{\Phi} \left( n - \ell + n + 2; 2\beta \cos^2 u \right),
\]

(78a)

where, in the recurrence relations (17) for \( \hat{b}_n^8 \),

\[
\hat{a}_n^8 = 2\beta (n + 1), \quad \hat{b}_n^8 = B_n^5 \text{ of Eq. (73b),} \quad \hat{c}_n^8 = - (n - \ell - 1) (n - 1 + 2\delta).
\]

(78b)

For infinite series, we obtain

\[
\tilde{\psi}_4(u) = e^{-\beta \cos^2 u} \left[ \cos^2 u \right]^{\gamma - \frac{1}{2}} \left[ \sin^2 u \right]^{\gamma - \frac{1}{2}} \times \sum_{n=0}^{\infty} \tilde{b}_n^4 \left[ -2\beta \cos^2 u \right]^n \tilde{\Phi} \left( n + \ell + 2\gamma + 2\delta, n + 2\delta; 2\beta \cos^2 u \right),
\]

(79a)

where, in the recurrence relations (14) for \( \tilde{b}_n^4 \),

\[
\tilde{a}_n^4 = 2\beta (n + 1), \quad \tilde{b}_n^4 = B_n^1 \text{ of Eq. (74b),} \quad \tilde{c}_n^4 = - (n + \ell - 1 + 2\gamma + 2\delta) (n - 1 + 2\gamma),
\]

(79b)

and

\[
\tilde{\psi}_8(u) = e^{-\beta \cos^2 u} \left[ \cos^2 u \right]^{\gamma - \frac{1}{2}} \left[ \sin^2 u \right]^{\gamma - \frac{1}{2}} \times \sum_{n=0}^{\infty} \tilde{b}_n^8 \left[ 2\beta \sin^2 u \right]^n \tilde{\Phi} \left( n + \ell + 2\gamma + 2\delta, n + 2\gamma; -2\beta \sin^2 u \right),
\]

(80a)

where, in the recurrence relations (14) for \( \tilde{b}_n^8 \),

\[
\tilde{a}_n^8 = -2\beta (n + 1), \quad \tilde{b}_n^8 = B_n^6 \text{ of Eq. (75b),} \quad \tilde{c}_n^8 = - \left[ n + \ell - 1 + 2\gamma + 2\delta \right] [n - 1 + 2\delta].
\]

(80b)

The solutions \( \tilde{\psi}_4 \) and \( \tilde{\psi}_8 \) are both convergent and bounded for whole interval \( 0 \leq \cos^2 u \leq 1 \). In this manner, we have one expression for finite-series solutions, but two expressions for infinite-series solutions, \( \psi_4 \) and \( \psi_8 \). Up to now we have not been able to show that \( \tilde{\psi}_4 \) and \( \tilde{\psi}_8 \) are linearly dependent. In any case, the power series solutions (74a) and (75a) do not present this ambiguity.
VI. CONCLUSION

We have found new solutions for the CHE and applied them to the Schrödinger equation with quasixactly solvable potentials. To get suitable infinite-series solutions for the hyperbolic potential we have used two-sided series. For the hyperbolic Razavy-type potentials we have found only even infinite-series solutions convergent and bounded for any value of the independent variable; to get odd solutions it is necessary to use two solutions with different domains of convergence. Excluding the Razavy-type potentials, we have seen that it is advantageous to use the expansions in series of Coulomb wave functions to get infinite-series solutions appropriate for $\frac{2}{4} < \delta < \frac{3}{4}$.

For the trigonometric potential, we can use one-sided series to obtain bounded infinite-series solutions. In fact, the Schrödinger equation may be solved using the Baber-Hassé expansions in power series provided that we take the minimal solutions of the recurrence relations for the series coefficients (then, the series converge for any finite value of $z$). We can also use expansions in series of regular hypergeometric functions but, in this case, it is necessary additional study to decide on the duplicity of solutions referred to in section V.B. This issue occurs also for the trigonometric Razavy-type potentials which are given by expression with the restrictions: solutions for this problem need further considerations.

Another question consists in knowing if the one-sided expansions $\tilde{U}_i^\infty$ in series of irregular confluent hypergeometric functions can play the role of the Hylleraas and Jaffé solutions which so far have been used to represent bound states of hydrogen molecule-like ions. While it is difficult to determine the conditions for the convergence of the Hylleraas and Jaffé solutions, the conditions for the convergence of $\tilde{U}_i^\infty$ follow from the Raabe test.

On the other side, in section III we have seen that the expansions $U_i^\infty$ and $U_i$ of a given set of two-sided solutions may cover the entire complex plane $z$, contrary to the solutions in series of Coulomb wave functions. This suggests that such solutions could be useful for some astrophysical problems. In effect, in order to solve the radial Teukolsky equations, Ochik proposed to match an expansion in series of hypergeometric functions and another in series of Coulomb functions, both having the same series coefficients but converging in different regions (see also [22, 23]). The use of $U_i^\infty$ and $U_i$ would give, by means of (11), a solution $U_i$ which is a linear combination of $U_\infty$ and $U_i$.

At last we mention that the new solutions for the CHE can afford solutions for an equation called reduced CHE. We write the latter in the form

$$z(z - z_0)\frac{d^2U}{dz^2} + (B_1 + B_2z)\frac{dU}{dz} + [B_3 + q(z - z_0)]U = 0, \quad [q \neq 0]$$

(81)

where $z_0$, $B_i$, and $q$ ($q \neq 0$) are constants. In Eqs. (11) and (81), if $z_0 \neq 0$ then $z = 0$ and $z = z_0$ are regular singular points with exponents $(0, 1 + B_1/z_0)$ and $(0, 1 - B_2 - B_1/z_0)$, respectively. However, at the irregular singular point $z = \infty$ the expected behavior of the solutions are [25]

$$U(z) \sim e^{\pm i\omega z} z^{1/2 - i\eta - (B_2/2)}$$

for Eq. (11),

$$U(z) \sim e^{\pm 2i\sqrt{q} z^{(1/4) - (B_2/2)}}$$

for Eq. (81).

Eq. (81) is obtained by applying the so-called Whittaker-Ince limit [9, 25]

$$\omega \to 0, \quad \eta \to \infty \quad \text{such that} \quad 2\eta\omega = -q, \quad \text{[Whittaker-Ince limit]}$$

(82)

to equation (11). By accomplishing the above limit in the same manner we have done with regard to the expansions in Coulomb wave functions [11], we will find that the solutions in series of confluent hypergeometric functions give new expansions in series of Bessel functions for the reduced CHE. In particular, we will get new solutions in series of Bessel functions also for the Mathieu equation by the reason that this is a special case of the reduced CHE [11].
Appendix A: Recurrence relations in matrix form

The three-term recurrence relations having the form [14] can be written as

\[
\begin{align*}
\begin{bmatrix}
\hat{\gamma}_0 & \hat{\beta} & 0 \\
\hat{\gamma}_1 & \hat{\beta}_1 & \hat{\alpha} \\
0 & \hat{\beta}_2 & \hat{\alpha}_2 \\
\ddots & \ddots & \ddots \\
\hat{\gamma}_N & \hat{\beta}_N & \hat{\alpha}_N \\
\end{bmatrix}
\begin{bmatrix}
\hat{\alpha}_0 \\
\hat{\alpha}_1 \\
\hat{\alpha}_2 \\
\ddots \\
\hat{\alpha}_N \\
\end{bmatrix} =
\begin{bmatrix}
\hat{b}_0 \\
\hat{b}_1 \\
\hat{b}_2 \\
\ddots \\
\hat{b}_N \\
\end{bmatrix}
\end{align*}
\]

(A.1)

where we have suppressed the upper indices and split the matrix into blocks.

The previous relations hold only if the series begin at \( n = 0 \). In fact, in some cases the series truncate on the left-hand side and, then, the series begin at \( n > 0 \) (see page 171 of [3], Ex. 2); in other cases cases the series truncate on the right-hand side leading to finite series (p. 146 of [3]). Specifically, for one-sided series solutions,

- If \( \hat{\alpha}_{n=N} = 0 \) for some \( N \geq 0 \), then the series begins at \( n = N + 1 \), that is, \( \hat{b}_{N+1} = \cdots = \hat{b}_N = 0 \). In this case, only the right lower block of the matrix is relevant. If we set \( n = m + N \) and relabel the series coefficients, we reobtain series beginning at \( m = 0 \) with recurrence relations like [17].
- If \( \hat{\gamma}_{n=N+1} = 0 \) for some \( N \geq 0 \), then the series terminates at \( n = N \) (\( 0 \leq n \leq N \)), that is, \( \hat{b}_N = \hat{b}_{N+1} = \cdots = 0 \). In this case, only the left upper block of the matrix is relevant.

On the other hand, for two-sided series solutions we rewrite the relations [52] as

\[
\begin{align*}
\begin{bmatrix}
\hat{\gamma}_n & \hat{\beta}_n & \hat{\alpha}_n \\
\hat{\gamma}_{n+1} & \hat{\beta}_{n+1} & \hat{\alpha}_{n+1} \\
\hat{\gamma}_{n+2} & \hat{\beta}_{n+2} & \hat{\alpha}_{n+2} \\
\ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
\hat{b}_{n-1} \\
\hat{b}_n \\
\hat{b}_{n+1} \\
\end{bmatrix} = 0 \quad \text{[} -\infty < n < \infty \text{]} \quad (A.2)
\end{align*}
\]

where \( \mathbf{0} \) denotes the null column vector. In this case, \( \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n \) depend on a parameter \( \nu \). By determining \( \nu \) such that \( \hat{\alpha}_{n-1} = 0 \), we obtain one-sided series expansions \( n \geq 0 \) out of two-sided series expansions: for the expansions in series of hypergeometric functions considered in section III, we find \( \nu = 0 \).

Appendix B: Remarks on the spheroidal equation

The (ordinary) spheroidal wave equation [3],

\[
\frac{d}{dx} \left[ (1 - x^2) \frac{dX(x)}{dx} \right] + \left[ \gamma^2 (1 - x^2) + \lambda - \frac{\mu^2}{1 - x^2} \right] X(x) = 0, \quad (B.1)
\]

is a particular case of the CHE. In effect, the substitutions

\[
x = 1 - 2z, \quad X(x) = z^{\mu/2} (z - 1)^{\nu/2} U(z), \quad (B.2)
\]

transform equation \((B.1)\) into

\[
z(z - 1) \frac{d^2 U}{dz^2} + [- (\mu + 1) + 2 (\mu + 1) z] \frac{dU}{dz} + [\mu (\mu + 1) - \lambda + 4 \gamma^2 z (z - 1)] U = 0,
\]

which is a CHE \((1)\) with

\[
z_0 = 1, \quad B_2 = -2B_1 = 2(\mu + 1), \quad B_3 = \mu (\mu + 1) - \lambda, \quad \eta = 0, \quad \omega^2 = 4 \gamma^2. \quad (B.3)
\]
Therefore, the spheroidal equation is a CHE with \( z_0 = 1, \eta = 0 \) and \( B_2 = -2B_1 \), that is,
\[
z(z-1)\frac{d^2 U}{dz^2} + (B_1 - 2B_1 z) \frac{dU}{dz} + [B_3 + \omega^2 z(z-1)] U = 0 \quad \text{(spheroidal equation)} \quad (B.4)
\]

For some solutions of the spheroidal equation, the definition \([10]\) for \( \Phi(a,c; \gamma) \) becomes inappropriate because \( c-a \) in \( \Gamma(c-a) \) is zero or a negative integer. To avoid this problem, first we rewrite the solutions in terms of \( \Phi(a,c; \gamma) \) by putting
\[
\dot{b}_n^i \Phi(a,c; y) = \frac{b_n^i}{\Gamma(c)} \Phi(a,c; y) \quad \text{for} \quad \dot{U}_1(z), \dot{U}_3(z), \dot{U}_5(z), \dot{U}_7(z);
\]
\[
\dot{b}_n^i \Phi(a,c; y) = \bar{b}_n^i \Phi(a,c; y) \quad \text{for} \quad \dot{U}_2(z), \dot{U}_4(z), \dot{U}_6(z), \dot{U}_8(z),
\]
where the equations for \( \dot{b}_n^i \) and \( \bar{b}_n^i \) are obtained from the ones for \( b_n^i \) by taking
\[
\dot{b}_n^i = \Gamma(c-a) \bar{b}_n^i, \quad \bar{b}_n^i = \Gamma(c-a) \dot{b}_n^i / \Gamma(c). \]

Hence, the forms of the recurrence relations, similar to \([17]\), are \( \dot{b}_{-1}^i = \bar{b}_{-1}^i = 0 \)
\[
\hat{\alpha}_n^i \dot{b}_{n+1}^i + \hat{\beta}_n^i \dot{b}_n^i + \hat{\gamma}_n^i \bar{b}_{n-1}^i = 0, 
\]
\[
\hat{\alpha}_n^i \bar{b}_{n+1}^i + \hat{\beta}_n^i \dot{b}_n^i + \hat{\gamma}_n^i \bar{b}_{n-1}^i = 0. \]

We find that the solutions \( \dot{U}_1(z) \) of the second case given in \([B.5]\) reduce to power-series expansions since \( \Phi(a,c; y) = \Phi(a,c; y) \exp(y) \), while the solutions \( \dot{U}_1(z) \) may be expressed as expansions in series of incomplete gamma functions, \( \Gamma(\beta, y) \) and \( \gamma(\beta, y) \), by means of \([6]\)
\[
\Gamma(\beta, y) = e^{-y} y^\beta \Psi(1, 1 + \beta, y); \quad \gamma(\beta, y) = e^{-y} y^\beta \Phi(1, 1 + \beta, y) / \beta.
\]
The function \( \gamma(\beta, y) \) is valid only if \( \beta \) is not zero or negative integer: in this case
\[
\Gamma(\beta, y) + \gamma(\beta, y) = \Gamma(\beta).
\]

As examples, below we write two sets of solutions. The first one is
\[
\begin{bmatrix} \dot{U}_1(z) \\ \dot{U}_1(z) \end{bmatrix} = e^{i\omega z} [z - 1]^{1 + B_1} \sum_{n=0}^{\infty} \frac{b_n^i}{\Gamma(2 + B_1, 2 + B_1 - n; -2i\omega z)} \Phi(2 + B_1, 2 + B_1 - n; -2i\omega z), \quad (B.8)
\]
\[
\dot{U}_1(z) = e^{i\omega z} [z - 1]^{1 + B_1} \sum_{n=0}^{\infty} \frac{\dot{b}_n^i}{\Gamma(2 + B_1, 2 + B_1 - n; -2i\omega z)} \Phi(n + 1, n - B_1; -2i\omega z),
\]
where the coefficients \( \dot{b}_n^i \) satisfy the recurrence relations \([17]\) with
\[
\hat{\alpha}_n^i = -2i\omega (n + 1), \quad \hat{\beta}_n^i = n [n + 1 + 2i\omega] + i\omega [2 + B_1] - B_1 [1 + B_1] + B_3,
\]
\[
\hat{\gamma}_n^i = -n [n + 1 + B_1], \quad (B.9)
\]
whereas the \( \bar{b}_n^i \) satisfy \([16]\) with
\[
\hat{\alpha}_n^i = 2i\omega (n + 1)^2, \quad \hat{\beta}_n^i = \hat{\beta}_n^i, \quad \hat{\gamma}_n^i = n + 1 + B_1. \quad (B.10)
\]
The second set takes the form
\[
\begin{bmatrix} \dot{U}_2(z) \\ \dot{U}_2(z) \end{bmatrix} = e^{i\omega z} [z - 1]^{1 + B_1} \sum_{n=0}^{\infty} \frac{\dot{b}_n^i}{\Gamma(1, -B_1 - n; -2i\omega z)} \Phi(1, -B_1 - n; -2i\omega z), \quad (B.11)
\]
\[
\dot{U}_2(z) = e^{-i\omega z} [z - 1]^{1 + B_1} \sum_{n=0}^{\infty} \frac{\bar{b}_n^i}{\Gamma(1, -B_1 - n; -2i\omega z)} [2i\omega]^n,
\]
where the \( \bar{b}_n^i \) satisfy the recurrence relations \([17]\) with
\[
\hat{\alpha}_n^i = -2i\omega (n + 1), \quad \hat{\beta}_n^i = n [n + 3 + 2B_1 + 2i\omega] + i\omega [2 + B_1] + 2 + 2B_1 + B_3,
\]
\[
\hat{\gamma}_n^i = -[n + 1 + B_1]^2, \quad (B.12)
\]
and \( b_n^2 \) satisfy \( B.7 \) with
\[
\tilde{\alpha}_n^2 = -2i\omega (n + 1)[n + 2 + B_1], \quad \tilde{\beta}_n^2 = \tilde{\beta}_n^2, \quad \tilde{\gamma}_n^2 = -[n + 1 + B_1].
\] (B.13)

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