HOMOLOGY OF FORMAL DEFORMATIONS OF PROPER ÉTALE LIE GROUPOIDS

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Abstract. In this article, the cyclic homology theory of formal deformation quantizations of the convolution algebra associated to a proper étale Lie groupoid is studied. We compute the Hochschild cohomology of the convolution algebra and express it in terms of alternating multi-vector fields on the associated inertia groupoid. We introduce a noncommutative Poisson homology whose computation enables us to determine the Hochschild homology of formal deformations of the convolution algebra. Then it is shown that the cyclic (co)homology of such formal deformations can be described by an appropriate sheaf cohomology theory. This enables us to determine the corresponding cyclic homology groups in terms of orbifold cohomology of the underlying orbifold. Using the thus obtained description of cyclic cohomology of the deformed convolution algebra, we give a complete classification of all traces on this formal deformation, and provide an explicit construction.

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1. Introduction

In symplectic geometry and mathematical physics one often encounters, for example by reduction, Poisson spaces which are singular. One of the easiest examples of such singular spaces is given by a symplectic orbifold; this an orbifold which admits a covering by orbifold charts equipped with an invariant symplectic structure. Therefore, the study of deformation quantization and index theory of such spaces appears naturally.

To address such questions, one first has to decide what “algebra of smooth functions” on an orbifold one wants to consider. Any orbifold has a natural sheaf of functions which locally can be lifted to smooth invariant functions on any orbifold chart. For this algebra on a symplectic orbifold, a deformation quantization was constructed in Pf03, generalizing Fedosov’s method Fe96 on smooth manifolds. This case was further studied by Fedosov–Schulze–Tarkhanov in FeSchTa, making several interesting conjectures on the related index problem.

However from the point of view of noncommutative geometry Co94, an orbifold presents one of the prime examples of a “noncommutative manifold”: its “algebra of smooth functions” is given by the noncommutative convolution algebra on a proper étale Lie groupoid, whose quotient space identifies with the underlying orbifold (cf. Mo). This construction generalizes the crossed product of the algebra of smooth functions on a manifold by a finite group. As shown in Ta04b, a Poisson structure on the orbifold induces a natural noncommutative Poisson structure on this algebra in the sense of Block–Getzler BlGe and Xu Xu, which admits a deformation quantization. It is the properties of this deformed algebra that we study in this paper. Notice that it contains the deformed algebra of Pf01 as the subalgebra of invariants under the groupoid.

The first step in understanding the deformation quantization of a groupoid algebra is to count how many noncommutative Poisson structures it has. We partially answer this question by calculating the Hochschild cohomology of the groupoid algebra of a proper étale groupoid. When the groupoid is a manifold, this is given by the Hochschild–Kostant–Rosenberg Theorem. In the case of an orbifold, partial results have already appeared in the literature, e.g. CaGiWi for the case of a global quotient of an algebraic variety by a finite group. In Section 3, we present a calculation in the case of a proper étale Lie groupoid. By Teleman’s localization technique, we relate this cohomology to the sheaf cohomology of the multivector fields on the corresponding inertia groupoid. This is the first step to classify the Poisson structures on a groupoid algebra. We leave the study of the Gerstenhaber bracket and possible extensions to a “noncommutative formality theorem” for future research.

The second step in understanding the quantization of an algebra is its semiclassical geometry, or in other words its noncommutative Poisson geometry. In this paper, we introduce a noncommutative Poisson homology generalizing Brylinski’s definition on a Poisson manifold Br. This noncommutative Poisson homology appears as the $E^1$-term of the spectral sequence associated to the $\hbar$-adic filtration in the Hochschild homology of the deformed algebra. We calculate this Poisson homology in case of the noncommutative Poisson structure on the convolution algebra of a groupoid associated to an orbifold with a Poisson structure. Our calculation uses the methods developed by Connes, Burghelea, Brylinski, Nistor, and Crainic in calculating the cyclic homology of an étale groupoid. We track the change of
the Poisson differential in the various steps of the calculation of the Hochschild homology of the groupoid algebra, and relate it to the sheaf homology of Brylinski’s complex on the corresponding inertia groupoid.

Next, we compute the Hochschild and cyclic (co)homology of the deformation quantization of the convolution algebra of a symplectic orbifold. Our computation draws upon two ideas: one is the spectral sequence introduced in \cite{BrGe} and \cite{NeTs} associated to the $\hbar$-adic filtration of the Hochschild complex and its relation to the noncommutative Poisson homology of the previous section. Second is the localization to the inertia groupoid, as used in the calculation of the Hochschild and cyclic homology of the “classical” groupoid algebra of Brylinski–Nistor \cite{BrNi} and Crainic \cite{Cr}. The main difficulty here is that, due to the noncommutative nature of the sheaf of “quantized functions”, one has to add “quantum corrections” to this localization map. Interestingly, all our results are given in terms of the orbifold cohomology of Chen–Ruan \cite{ChRu}.

The calculation of the cyclic cohomology gives in particular a complete classification of the traces of the deformed groupoid algebra. In the last section we give an explicit construction of all traces, building on earlier work by Fedosov \cite{Fe00}. Finally, our constructions also clarify a conjecture in \cite{FeSchTa} on a certain “Picard group” acting on the space of traces.

Since our computations use quite some machinery of groupoids and cyclic homology developed by several people over the years, we have included, for the convenience of the reader, a rather detailed section devoted to these subjects. Its main purpose is to give an introduction into the ideas of Connes, Burghelea, Brylinski, Nistor, and Crainic in calculating the cyclic homology of étale groupoids, and to set up the notation.

Our paper is related to the recent paper \cite{DoEt} by Dolgushev–Etingof, where for the quotient of a smooth complex affine symplectic variety $X$ by a finite group the Hochschild cohomology of the convolution algebra and of the algebra of invariant regular functions is computed. Note however, that due to the algebraic nature of their setup the methods used there are quite different to the ones used here.

Let us also mention at this point, that some of the results presented here have been obtained by the fourth author in his PhD-thesis and, independently, by the collaboration of the remaining authors. During a conference in Luminy, where three of us, namely M.J.P., H.P. and X.T. met, we then decided to continue our work together and write a joint paper about our results.

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2. Preliminaries

2.1. Notation. For clarity, we collect here some of the notation used throughout the paper.

- $X$ denotes an orbifold,
- $\tilde{X}$ its inertia orbifold as constructed in [ChRu],
- By $G$ we denote a smooth étale groupoid modeling $X$,
- $\mathcal{A}$ is the $G$-sheaf of smooth functions on $G_0$, if not stated otherwise,
- and $\mathcal{A}^\hbar$ is the $G$-sheaf of a formal deformation of $\mathcal{A}$.
- Concerning products, $f \cdot g$ denotes the pointwise product of two functions $f,g \in C^\infty(G_1)$,
- whereas $f \ast g$ is the convolution product on the groupoid $G$,
- and, finally, $f \star g$ is the star product associated to a formal deformation.

2.2. Orbifolds. The notion of an orbifold was introduced by Satake [Sa]. Roughly speaking, an orbifold is a second countable Hausdorff space $X$ which is locally modeled on the quotient of open subsets of $\mathbb{R}^n$ by a finite group. In this article we will use the language of groupoids to model orbifolds, following the approach by Moerdijk [Mo] (see also [MoMr, Chap. 5] for an introduction). To set up notation and for the convenience of the reader let us recall some basic notions from the theory of groupoids.

A groupoid $G$ is a small category in which every morphism is invertible. More explicitly, denote by $G_0$ the space of objects and by $G_1$ the space of arrows in $G$. The groupoid structure is encoded by the following five maps:

$G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \xrightarrow{i} G_1 \xrightarrow{s} G_0 \xrightarrow{t} G_1$

Here, $s$ and $t$ are the source and target map, $m$ is the multiplication resp. composition $(g,h) \mapsto m(g,h) := gh := g \circ h$, $i$ denotes the inverse which is given by $g \mapsto i(g) := g^{-1}$ and finally $u$ is the inclusion of objects by identity arrows, i.e., $u(x) := \text{id}_x$ for all $x \in G_0$. An arrow $g \in G_1$ with $s(g) = x$ and $t(g) = y$ will often be denoted by $g : x \to y$. Moreover, if no confusion can arise, we write $G$ instead of $G_1$.

A groupoid $G$ is a Lie groupoid (also called a smooth groupoid) if both $G_0$ and $G_1$ are smooth manifolds, all the structure maps are smooth and $s$ and $t$ submersions. It then follows that $u$ is an immersion and that $i$ is a diffeomorphism. A Lie groupoid is called proper when the map $(s,t) : G_1 \to G_0 \times G_0$ is proper. It is called a foliation groupoid, if every isotropy group $G_x$ is discrete. An étale groupoid is a special type of foliation groupoid for which $s$ and $t$ are local diffeomorphisms.

**Definition 2.1.** [Mo, Def. 3.1] An orbifold groupoid is a proper foliation groupoid.

More precisely, one proves that the orbit space $X : = G_0/G_1$ has a canonical orbifold structure. This description of orbifolds by groupoids goes back to [MoPr]. The fundamental idea is that the orbifold structure on $X$ in fact only depends on the Morita equivalence class of the groupoid $G$. For an introduction to the theory of Morita equivalence of groupoids we refer to [Mo]. We merely remark that this allows us to choose a proper étale groupoid $G$ representing the orbifold $X$.

This groupoid has the property that for each $x \in G_0$ there exists a neighbourhood $U_x \subset G_0$ such that the restriction $G|_{U_x}$ is isomorphic to a translation groupoid $\Gamma_x \ltimes U_x$, with $\Gamma_x$ a finite group. This property gives the connection with the usual
definition of orbifolds in terms of local charts. In the following, we will denote by \( \pi : G \to X \) the projection onto the orbit space of \( G \).

### 2.3. Sheaves on orbifolds

The theory of sheaves on orbifolds is discussed in \([129MoPr]\). One can use an orbifold groupoid to model the category of sheaves. In general a \( G \)-sheaf \( S \) on an étale groupoid \( G \) is a sheaf on \( G_0 \) with a right action of \( G \). This means that any arrow \( g : x \to y \) induces a morphism on stalks \( S_y \to S_x \) satisfying the obvious properties. A section \( a \) of a sheaf is said to be invariant, if on the level of germs one has \([a]_y = [a]_x \) for every arrow \( g : x \to y \). The abelian category of \( G \)-sheaves of abelian groups is denoted by \( \text{Sh}(G) \). There is a left exact functor \( \Gamma_{\text{inv}} : \text{Sh}(G) \to \text{Ab} \), where \( \text{Ab} \) is the category of abelian groups, given by associating to a \( G \)-sheaf \( S \) its global invariant sections. Its right derived functors define the groupoid cohomology groups \( H^\bullet(G, S) \).

Likewise, we have the compactly supported cohomology groups: consider the functor \( \Gamma_{\text{inv},c} : \text{Sh}(G) \to \text{Ab} \), defined by

\[
\Gamma_{\text{inv},c}(S) = \{ a \in \Gamma_{\text{inv}}(S) \mid \pi(\text{supp}(a)) \text{ is compact in } X \}.
\]

Since \( G \) is a proper groupoid, this functor is left exact and the compactly supported cohomology groups \( H^\bullet_{\text{c}}(G, S) \) are defined as the right derived functors of \( \Gamma_{\text{inv},c} \). This definition extends in the usual way to define the hypercohomology groups \( H^\bullet(G, S^\bullet) \) and \( \mathbb{H}^\bullet(G, S^\bullet) \) of any cochain complex \( S^\bullet \) of \( G \)-sheaves.

**Remark 2.2.** The cohomology groups \( H^\bullet_{\text{c}}(G, S) \) are isomorphic to the (re-indexed) homology groups of \([1900CrMo]\), see Section 4.9 and 4.10 of that paper. In view of the cohomological indexing, we use the invariant instead of the coinvariant sections; the two are isomorphic as one can show by an averaging argument (properness is essential for this, cf. e.g. \([100Lo]\), p. 280)). The homology theory of \([1900CrMo]\) is defined for any étale groupoid, but not a derived functor in general.

More generally, one can associate to every morphism \( f : G \to H \) of étale groupoids a functor

\[
f_! : \text{Sh}(G) \to \text{Sh}(H).
\]

For its construction, observe first that the functors \( f_* \) and \( f^{-1} \) can be extended to the category of sheaves on étale groupoids in the obvious manner. The sheaf \( f_! S \in \text{Sh}(H) \) then has stalk at \( x \in G_0 \) given by

\[
(f_! S)_x := H^0_c(x/f, \pi_1^{-1} S),
\]

where \( x/f \) denotes the comma groupoid over \( x \), that is the fiber of \( f \) over \( x \):

\[
\begin{array}{ccc}
  x/f & \xrightarrow{x} & H \\
  \downarrow & & \downarrow f \\
  1 & \xrightarrow{x} & G.
\end{array}
\]

Analogously, the right derived functors \( R^k f_! \) have a similar construction using the higher compactly supported cohomology groups of the comma groupoids. The Leray spectral sequence generalizes to the category of sheaves on étale groupoids. In particular, in the case of orbifolds this spectral sequence degenerates for the projection \( \pi : G \to X \) and induces an isomorphism

\[
H^\bullet_{\text{c}}(G, S) \cong H^\bullet_{\text{c}}(X, \pi_! S).
\]

(2.2)
Let us briefly discuss the Bar-complexes used in explicit computations of these cohomology groups. Denote by $G^{(k)}$ the space of $k$-composable arrows:

$$G^{(k)} = \{(g_1, \ldots, g_k) \in G^k \mid s(g_i) = t(g_{i+1}), \ i = 1, \ldots, k - 1\}.$$ 

These spaces are part of a simplicial space $d_i : G^{(k)} \to G^{(k-1)}$, $i = 0, \ldots, k$ defined as usual by

$$d_i(g_1, \ldots, g_k) = \begin{cases} (g_2, \ldots, g_k), & \text{for } i = 0, \\ (g_1, \ldots, g_i \cdot g_{i+1}, g_{i+2}, \ldots, g_k), & \text{for } 1 \leq i \leq k - 1, \\ (g_1, \ldots, g_{k-1}), & \text{for } i = k. \\ \end{cases}$$

Note that $d_0, d_1 : G_1 \to G_0$ are simply the source and target map. The geometric realization of this simplicial space is a model for the classifying space $BG$ of the groupoid $G$.

We have two maps $\epsilon_k, \tau_k : G^{(k)} \to G_0$, which send a string

$$x_0 \xleftarrow{g_1} x_1 \xleftarrow{g_2} \cdots \xleftarrow{g_k} x_k$$

to $x_k$ resp. $x_0$. Let $\mathcal{S}$ be a $G$-sheaf. Define $\mathcal{S}^k := \tau_k^{-1}\mathcal{S}$ and put

$$B_k(G, \mathcal{S}) := \Gamma_c(G^{(k)}, \mathcal{S}^k).$$

These vector spaces can be turned into a simplicial space by observing that there are isomorphisms $d_1^* \mathcal{S}^{k-1} \cong \mathcal{S}^k$ which act on the stalks as identity for $i \neq 0$, but by $g_1$, if $i = 0$. Using this isomorphism, the simplicial maps $d_i$ induce differentials in the obvious way, and its associated homology groups compute the homology $H_\bullet(G, \mathcal{S})$, in case $\mathcal{S}$ is c-soft.

**Definition 2.3.** (Cf. [CrMo00, Sec. 3] and [Cr] Sec. 2.3.) Let $\mathcal{S}_\bullet$ be a (bounded below) chain complex of c-soft $G$-sheaves. The *hyperhomology* of $\mathcal{S}_\bullet$ on $G$ then is defined as the total homology of the double complex $B_\bullet(G, \mathcal{S}_\bullet)$, i.e.,

$$\mathbb{H}_\bullet(G, \mathcal{S}_\bullet) := H_\bullet(\text{Tot}(B_\bullet(G, \mathcal{S}_\bullet))).$$

### 2.4. Orbifold cohomology

The notion of twisted sectors of an orbifold plays an important role in index theory [KA]. Loosely speaking, it is a geometric way of dealing with the “stacky aspects” of an orbifold, that is, the automorphisms of points. As before, we let $X$ be an orbifold, represented by a groupoid $G$. Denote by $B^{(0)} \subset G_1$ the “space of loops”

$$B^{(0)} = \{g \in G \mid s(g) = t(g)\}.$$ 

The groupoid $G$ acts on $B^{(0)}$ by conjugation and one defines

$$\Lambda G := B^{(0)} \rtimes G.$$ 

This groupoid has “loops” in $G$ as objects, and the space of arrows can be identified with

$$\Lambda G_1 = \{(g_1, g_2) \in G^{(2)} \mid g_1 \in B^{(0)}\}.$$ 

One observes that $\Lambda G$ is again an orbifold groupoid which comes equipped with a canonical morphism $\beta : \Lambda G \to G$. The orbifold underlying $\Lambda G$ is denoted $\tilde{X}$, and is called the inertia orbifold [ChRu]. There is a canonical open-closed embedding $G \hookrightarrow \Lambda G$ of groupoids. In fact, this embedding is induced by the partition of $B^{(0)}$ into so-called sectors of $G$:

$$B^{(0)} = \coprod \mathcal{O},$$

(2.3)
where each $\mathcal{O}$ is a $G$-saturated open-closed subset of $B^{(0)}$ and minimal among such sets with respect to set-theoretic inclusion. We denote the set of sectors of $G$ by $\text{Sec}(G)$. Note that the above decomposition of $B^{(0)}$ induces similar decompositions of the inertia groupoid $\Lambda G$ and the orbifold $\tilde{X}$, where each component of $\Lambda G$ can be identified as $\mathcal{O} \rtimes G$. The components besides $G_0 \subset \Lambda G_0$ are called "twisted sectors". The twisted sectors play an important role in orbifold cohomology, which we will now define. In the following, let $\ell(\mathcal{O})$ denote the codimension of a sector $\mathcal{O}$ inside $G$.

**Definition 2.4.** Let $X$ be an orbifold represented by a groupoid $G$. The orbifold cohomology groups and the orbifold cohomology groups with compact support of $X$ are defined as

$$H^\bullet_{\text{orb}}(X, \mathbb{C}) := H^\bullet(\Lambda G, \mathbb{C}) = \bigoplus_{\mathcal{O}} H^\bullet(\mathcal{O} \rtimes G, \mathbb{C}),$$

$$H^\bullet_{\text{orb, c}}(X, \mathbb{C}) := \bigoplus_{\mathcal{O}} H^{-\ell(\mathcal{O})}_{\text{c}}(\mathcal{O} \rtimes G, \mathbb{C}).$$

Notice that the right hand sides are equal to cohomology groups of the inertia orbifold $\tilde{X}$. Our shifting of degrees differs from [CHR1], but likewise we have a Poincaré duality $H^k_{\text{orb}}(X, \mathbb{C}) \times H^{\dim X-k}_{\text{orb, c}}(X, \mathbb{C}) \to \mathbb{C}$. In [CHR1], a remarkable cup product is defined on these orbifold cohomology groups, which however will not be used in this paper.

2.5. **Cyclic homology.** Here we give a quick review of the basic definitions of Hochschild and cyclic (co)homology. For more details, one should consult e.g. [LO].

**Cyclic objects.** Let $r \in \mathbb{N}^* \cup \{\infty\}$. An $r$-cyclic object in a category then is a simplicial object $(X_\bullet, d, s)$ together with automorphisms (cyclic permutations) $t_k : X_k \to X_k$ satisfying the identities

$$d_i t_{k+1} = \begin{cases} t_{k-1} d_{i-1} & \text{for } i \neq 0, \\ d_k & \text{for } i = 0, \end{cases}$$

$$s_i t_k = \begin{cases} t_{k+1} s_{i-1} & \text{for } i \neq 0, \\ t_{k+1}^2 s_k & \text{for } i = 0, \end{cases}$$

$$t_r^{(k+1)} = 1, \quad \text{if } r \neq \infty.$$ 

Just like a simplicial object is a contravariant functor from the simplicial category, an $r$-cyclic object is nothing but a functor from the $r$-cyclic category (cf. [CON3]).

**Mixed complexes.** A mixed complex $(X_\bullet, b, B)$ in an abelian category is a graded object $(X_k)_{k \in \mathbb{N}}$ equipped with maps $b : X_k \to X_{k-1}$ of degree $-1$ and $B : X_k \to X_{k+1}$ of degree $+1$ such that $b^2 = B^2 = bB + Bb = 0$. A mixed complex gives rise
to a first quadrant double complex $B_{\bullet\bullet}(X)$

\[
\begin{array}{cccc}
X_3 & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & X_0 \\
B & & & & B & & \\
X_2 & \rightarrow & X_1 & \rightarrow & X_0 \\
B & & & & B \\
X_1 & \rightarrow & X_0 \\
B & \\
X_0 \\
\end{array}
\]

**Definition 2.5.** The Hochschild homology $HH_{\bullet}(X)$ of a mixed complex $X = (X_{\bullet\bullet}, b, B)$ is the homology of the $(X_{\bullet\bullet}, b)$-complex, which sometimes will also be denoted by $(C_{\bullet\bullet}(X), b)$. The cyclic homology $HC_{\bullet}(X)$ is defined as the homology of the total complex associated to the double complex $B_{\bullet\bullet}$.

Looking at the double complex above, it is clear that there is a short exact sequence of complexes

\[
0 \rightarrow (X_{\bullet\bullet}, b) \xrightarrow{L} (\operatorname{Tot}_{\bullet}B_{\bullet\bullet}(X), b + B) \xrightarrow{S} (\operatorname{Tot}_{\bullet}B_{\bullet\bullet}(X)[-2], b + B) \rightarrow 0.
\]

With the definitions above, the associated long exact sequence in homology reads

\[
\cdots \xrightarrow{B} HH_k(X) \xrightarrow{L} HC_k(X) \xrightarrow{S} HC_{k-2}(X) \xrightarrow{B} HH_{k-1}(X) \xrightarrow{L} \cdots.
\]

This sequence is called the SBI sequence and relates Hochschild and cyclic homology. Stabilizing with respect to the shift operator $S$, the homology of the inverse limit complex

\[
\lim_{\rightarrow} \operatorname{Tot}_{\bullet}B_{\bullet\bullet}(X)[-2k],
\]

is called the periodic cyclic homology $HP_{\bullet}(X)$. Note that periodic cyclic homology is only $\mathbb{Z}_2$-graded. Alternatively, it is the homology of the super complex $\tilde{X} = \Pi_{\bullet\bullet} X_{\bullet\bullet}$ with differential $B - b$.

In case $r \neq \infty$, an $r$-cyclic object $(X_{\bullet\bullet}, d, s, t)$ gives rise to a mixed complex (cf. [PETS A.3.2] and [CR 3.1.2]). Define

\[
\begin{align*}
&b' := \sum_{i=0}^{k-1} (-1)^i d_i, \quad b := \sum_{i=0}^{k} (-1)^i d_i, \quad N = \sum_{i=0}^{(k+1)r-1} (-1)^i k_t^i, \\
&\text{and finally put } B = (1 + (-1)^{k} t_k) s N. \quad \text{One checks that the triple } (X_{\bullet\bullet}, b, B) \text{ is a mixed complex.}
\end{align*}
\]
Example 2.6. Let $A$ be a unital algebra equipped with an automorphism $\alpha$. Define

the simplicial object $A^\bullet_A = (A^\bullet_A, d, s)$ by $A^\cdot_{\alpha,k} := A^{\otimes (k+1)}$ with face maps given by

$$d_i(a_0 \otimes \ldots \otimes a_k) = \begin{cases} a_0 \otimes \ldots \otimes a_{i+1} \otimes \ldots \otimes a_k, & \text{if } 0 \leq i \leq k-1, \\ \alpha(a_k)a_0 \otimes \ldots \otimes a_{k-1}, & \text{if } i = k, \end{cases}$$

and degeneracies given by

$$s_i(a_0 \otimes \ldots \otimes a_{k+1}) = a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_k.$$

The differential on the associated Hochschild complex $(C^\bullet_A(A), b_\alpha) := (A^\bullet_A, b_\alpha)$ then reads as follows:

$$b_\alpha(a_0 \otimes \ldots \otimes a_k) = \sum_{i=0}^k a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k + (-1)^k \alpha(a_k)a_0 \otimes \ldots \otimes a_{k-1}. \quad (2.4)$$

Its homology is denoted by $HH^\bullet(A)$. The map

$$t_k(a_0 \otimes \ldots \otimes a_k) = \alpha(a_k) \otimes a_0 \otimes \ldots \otimes a_{k-1}$$

defines an $r$-cyclic structure, where $r$ denotes the order of $\alpha$. When $r$ is finite, one defines Hochschild, cyclic and periodic cyclic homology as in Definition 2.5 using this cyclic object. When $\alpha = 1$, the chain complex $(C^\bullet_A(A), b) := (A^\bullet_A, b)$ is nothing but the usual Hochschild complex. The double complex $B^\bullet\cdot(A)$ associated to the mixed complex $(A^\bullet_A, b, B)$ is called Connes’ $(b, B)$-complex. In this case one denotes the homologies simply by $HH^\bullet(A), HC^\bullet(A), HP^\bullet(A)$.

2.6. Cyclic homology of orbifold groupoids. Here we briefly review the computations of $\text{BrNi} \ [\text{CR}]$ of the Hochschild and cyclic homology of étale groupoids. To be precise, this is the homology of the convolution algebra of the groupoid that means of $C^\infty_c(G)$ with the multiplication

$$(a_1 \ast a_2)(g) = \sum_{g_1 g_2 = g} a_1(g_1) a_2(g_2), \quad \text{where } a_1, a_2 \in C^\infty_c(G), \ g \in G. \quad (2.5)$$

Let us first describe the general idea behind the computation, which we will need in particular in Sec. 6 when computing the Hochschild and cyclic homology of deformations of the convolution algebra.

The convolution algebra is a special case of the “crossed product” algebra $A \rtimes G$ associated to any c-soft $G$-sheaf $A$ of unital algebras. As a vector space, one has $A \rtimes G := \Gamma_c(G_1, s^*A)$, and the multiplication $\ast$ is determined by

$$[a_1 \ast a_2]_g = \sum_{g_1 g_2 = g} ([a_1]_{g_1} [a_2]_{g_2}), \quad \text{for } a_1, a_2 \in \Gamma_c(G_1, s^*A), \ g \in G. \quad (2.6)$$

where $[a]_g$ denotes the germ of $a$ at $g$. Indeed, for the sheaf of smooth functions, one recovers the convolution algebra $\text{BrNi}$ in this way. The computations of the cyclic homology involve one piece of machinery, introduced in full generality in $\text{CR}$, that we briefly introduce now.

Cyclic groupoids. A cyclic groupoid is an étale groupoid $G$ equipped with a continuous map $\theta : G_0 \to G_1$, $x \mapsto \theta_x$ such that $s(\theta_x) = t(\theta_x) = x$ and $g \theta_x = \theta_y$ for all $g \in G_1$ with $s(g) = x$ and $t(g) = y$. If $\text{ord}(\theta_x) < \infty$ for all $x \in G_0$, we say that $G$ resp. $\theta$ is elliptic. Of course, any étale groupoid has a cyclic structure with $\theta_x = u(x)$. The main example of a nontrivial cyclic groupoid is $\Lambda G$ with
\[ \theta(g) = (g, g) \text{ for a loop } g \in B^{(0)}. \] Notice that \( \theta \) is elliptic in this case, since we deal only with proper étale groupoids, i.e., orbifolds. Next, define a \( \theta \)-cyclic sheaf on a cyclic groupoid \( G \) to be an \( r \)-cyclic object \( \mathcal{X} \) in \( \text{Sh}(G) \) such that for all \( x \in G_0 \), the morphism \( [t_k]_x : (X_k)_x \to (X_k)_x \) is given by the action of \( \theta_x \). As explained in Sec. 2.6, this gives rise to a mixed complex \( (\mathcal{X}, b, B) \) of sheaves on \( G \), if \( \theta \) is elliptic. We then denote the associated Hochschild complex by \( \mathcal{C} B \) and the associated double complex by \( B_{\bullet, \bullet}(\mathcal{X}) \). We now define, using the hyperhomology from Def. 2.3.

**Definition 2.7.** The Hochschild and cyclic homology groups of a \( \theta \)-cyclic sheaf \( \mathcal{X} \) on an elliptic cyclic groupoid \( G \) are defined by

\[
HH_\bullet(G, \theta; \mathcal{X}) := \mathbb{H}_\bullet(G, (\mathcal{C}_\bullet(\mathcal{X}), b)),
HC_\bullet(G, \theta; \mathcal{X}) := \mathbb{H}_\bullet(G, (\text{Tot}_\bullet B_{\bullet, \bullet}(\mathcal{X}), b + B)).
\]

Notice that the boundary operators \( b \) and \( B \) involve the twisting by the cyclic structure \( \theta \). Of course, for an étale groupoid with the trivial cyclic structure this twisting is trivial. As explained in Sec. 2.6, for a complex of \( c \)-soft sheaves, homology is computed from the Bar-complex of \( G \). Therefore, we have the following consequence of the Eilenberg–Zilber Theorem which will be used implicitly several times throughout this paper.

**Proposition 2.8.** (Cf. [GR, 3.2.8].) If \( \mathcal{X} \) is a \( \theta \)-cyclic sheaf on an elliptic cyclic étale Lie groupoid \( G \) such that each \( \mathcal{X}_k \) is \( c \)-soft, then \( HH_\bullet(G, \theta; \mathcal{X}) \), \( HC_\bullet(G, \theta; \mathcal{X}) \) (and \( HH_\bullet(G, \theta; \mathcal{X}) \)) are computed by the diagonal of the bisimplicial vector space \( B_{\bullet, \bullet}(G, \mathcal{X}_\bullet) \), i.e. by the cyclic vector space

\[
\Gamma_\bullet(G, \theta; \mathcal{X}) : \cdots \longrightarrow \Gamma_c(G^{(2)}, \mathcal{X}_2) \longrightarrow \Gamma_c(G^{(1)}, \mathcal{X}_1) \longrightarrow \Gamma_c(G^{(0)}, \mathcal{X}_0).
\]

Let us now recall the basic idea behind the computations of Hochschild and cyclic homology. We have groupoids and maps as follows:

\[
\begin{array}{ccc}
\Lambda G & \xrightarrow{\alpha} & \mathcal{X} \\
\downarrow \quad \quad & \downarrow \beta & \downarrow \\
NG & \xrightarrow{\tau} & G
\end{array}
\]

(2.7)

where \( NG \) is the groupoid obtained from \( \Lambda G \) by dividing out the action of the cyclic structure. It has the same space of objects \( B^{(0)} \), but \( NG_1 = \Lambda G_1 / \mathbb{Z} \), where \( n \cdot g = \theta^n \cdot g, n \in \mathbb{Z} \). Then, consider the functor

\[
\tau := \alpha \circ \beta^{-1} : \text{Sh}(G) \to \text{Sh}(NG).
\]

(2.8)

Any sheaf of algebras \( \mathcal{A} \) gives rise to a cyclic sheaf \( \mathcal{A}^\bullet \) in \( \text{Sh}(G) \) by putting \( \mathcal{A}^\bullet := \Delta_{k+1} \mathcal{A}^{2\otimes(k+1)} \), where \( \Delta_k : G_0 \to (G_0)^k \) is the diagonal embedding.

**Remark 2.9.** Notice that in the case of sheaves of locally convex topological algebras, e.g. the sheaf of smooth functions, one has to make a choice with respect to the topology on the tensor product. In most cases, one chooses the projective tensor product topology [GR, Chap. 1, §I, Def. 1], but in our setting the inductive tensor product topology [GR, Chap. 1, §3, Def. 3] is more natural. To circumvent such subtleties bornological instead of topological tensor products could be used. For a discussion on this point see for instance [ME].
The pull-back sheaf $A^*_w := \beta^{-1}A^2$ to the cyclic groupoid $\Lambda G$ carries a natural $\theta$-cyclic structure; the stalk of $A^*_w$ at $g \in B^{(0)}$ is the $r$-cyclic vector space $(A^*_w)_{\theta_g}$ as in Example 2.4, with the automorphism given by $\theta_g$. Applying the functor $\alpha_!$, one can kill the twisting and compute the twisted cyclic homology of $A^*_w$ as sheaf homology over $NG$. (This follows from [CR] Prop 3.3.12 and the fact that $\Lambda G$ is elliptic for $G$ proper.)

The general idea in the computation of Hochschild and cyclic homology of the crossed product $\mathcal{A} \rtimes G$ is to relate it to sheaf homology of $A^*_w$ over $\Lambda G$. This proceeds in two steps:

**Step I. Reduction to loops.** Consider Burghelea’s space

$$B^{(k)} := \{(g_0, \ldots, g_k) \in G^{(k+1)}, \ t(g_0) = s(g_k)\}.$$ 

There is a map $\sigma_k : B^{(k)} \to (G_0)^{k+1}, \ \sigma_k(g_0, \ldots, g_k) = (s(g_0), \ldots, s(g_k))$. For any $c$-soft sheaf of unital algebras $\mathcal{A}$, we define the vector spaces

$$\Gamma_c \Lambda^2_k \mathcal{A} := \Gamma_c(B^{(k)}, \sigma_k^{-1} \mathcal{A}^\otimes(k+1)).$$ 

(2.9)

By construction, $\Gamma_c \Lambda^2_k \mathcal{A}$ is the space of global sections (with compact support) of the sheaf $\Lambda^2_k \mathcal{A}$ on $B^{(k)}$ which for $(g_0, \ldots, g_k) \in B^{(k)}$ has stalks $(\Lambda^2_k \mathcal{A})_{(g_0, \ldots, g_k)}$ given by germs

$$[a_0 \otimes \cdots \otimes a_k]_{(g_0, \ldots, g_k)},$$ 

(2.10)

where each $a_i$ is an element of $\mathcal{A}(U_i)$ defined over an open neighborhood $U_i$ of $s(g_i)$. As explained in [CR] Sec. 3.4, the vector spaces $\Gamma_c \Lambda^2_k \mathcal{A}$ carry a canonical cyclic structure, combining the structure maps from the underlying cyclic manifold $B^{(k)}$ with the structure maps from the cyclic sheaf $A^2$. The associated Hochschild and cyclic homology is denoted by $HH_* (\Gamma_c \Lambda^2_k \mathcal{A})$ and $HC_* (\Gamma_c \Lambda^2_k \mathcal{A})$.

In case of the convolution algebra, that means in case $\mathcal{A}$ is the sheaf of smooth functions with compact support, we use, as stated above, the completed inductive topological tensor product $\otimes$ in the definition of the cyclic vector space. The completed inductive tensor product has the crucial property that

$$\mathcal{C}^\infty_c(M) \otimes \mathcal{C}^\infty_c(N) \cong \mathcal{C}^\infty_c(M \times N)$$

for two smooth manifolds $M$ and $N$. We therefore have topological linear isomorphisms

$$(\mathcal{A} \rtimes G)_k^2 = \Gamma_c(G, s^{-1} \mathcal{A})^\otimes \cdots \otimes \Gamma_c(G, s^{-1} \mathcal{A}) \cong \Gamma_c(G^{k+1}, s_{k+1}^{-1} \mathcal{A}^\otimes(k+1)),$$

where $s_k = s \times \cdots \times s$. The “reduction to loops” now is the natural projection

$$p : \Gamma_c(G^{k+1}, s_{k+1}^{-1} \mathcal{A}^\otimes(k+1)) \to \Gamma_c(B^{(k)}, \sigma_k^{-1} \mathcal{A}^\otimes(k+1)),$$

(2.11)

given by the restriction $B^{(k)} \subset G^{k+1}$. This defines a map of cyclic vector spaces $(\mathcal{A} \rtimes G)_k^2 \to \Gamma_c \Lambda^2_k \mathcal{A}$. The essential point proved in [BRN] Prop. 3.2 and [CR] Prop. 4.1.1] is that for $\mathcal{A}$ the sheaf of smooth functions this map induces isomorphisms in Hochschild and cyclic homology.

**Step II. Relation to sheaf cohomology.** The Hochschild and cyclic homology of the cyclic vector space $\Gamma_c \Lambda^2_k \mathcal{A}$, defined in [CR], turn out to be related to sheaf homology on the inertia groupoid $\Lambda G$:
Proposition 2.10. (Cf. [CR]) For any c-soft sheaf of unital algebras $A$ on a proper étale groupoid $G$, there are natural isomorphisms

$$HH_\bullet(\Gamma, A^2_\infty A) \cong HH_\bullet(\Lambda G, \theta; A^2_w) \cong HH_\bullet(NG, \tau(\mathcal{A}^2)).$$

(2.12)

Sketch of Proof. There is an isomorphism $B^{(k)} \cong (\Lambda G)_k$ of cyclic manifolds, which induces an isomorphism of $\infty$-cyclic vector spaces

$$\Gamma_c A^2_k A = \Gamma_c (B^{(k)}, \sigma^{-1}_k A^2(k+1)) \rightarrow \Gamma_c (\Lambda G_k, \beta^{-1} A^2_k).$$

Over the stalk at $(g_0, \cdots, g_k) \in B^{(k)}$, the isomorphism is given as follows:

$$(A^2_k A)_{(g_0, \cdots, g_k)} \rightarrow (A^2_{tw,k})_{(g_1 \cdots g_k g_0, g_1, \cdots, g_k)} = (\beta^{-1} A^2_k)_{(g_1 \cdots g_k g_0, g_1, \cdots, g_k)}$$

$$[a_0 \otimes \cdots \otimes a_k]_{(g_0, \cdots, g_k)} \mapsto [(a_0 g_1 \cdots g_k g_0) \otimes \cdots \otimes (a_k g_0)]_{(g_1 \cdots g_k g_0, g_1, \cdots, g_k)}.$$

(2.13)

By Proposition 2.8 the cyclic vector spaces on the right hand side computes the $\theta$-twisted Hochschild and cyclic homology of $A^2_w$ on $\Lambda G$. By applying $\alpha_1$ one obtains the second isomorphism in Hochschild and cyclic homology.

Applied to the sheaf of smooth functions, the twisted Hochschild–Kostant–Rosenberg Theorem [CR Lem. 3.1.5] shows that $A^2_w$ is quasi-isomorphic ($\mathcal{C}^\infty_{\Lambda G}$)$^2$, the usual cyclic sheaf associated to the commutative sheaf of smooth functions on $\Lambda G$. Applying the Hochschild–Kostant–Rosenberg–Comes quasi-isomorphism $\mathcal{C}_\bullet(\mathcal{C}^\infty_{\Lambda G}, b) \cong (\mathcal{O}_{\Lambda G}, 0)$, which turns the $B$-operator into the de Rham differential, one finds the additive isomorphism

$$HC_\bullet(A \times G) \cong H^*_\bullet(\Lambda G, \mathbb{C}) = H^*_\bullet(\tilde{X}, \mathbb{C}).$$

Localization. As explained in [BRN] Prop. 3.3, the partition $\mathcal{O}$ of $B^{(0)}$ into sectors induces decompositions

$$HH_\bullet(A \times G) \cong \bigoplus_\mathcal{O}_\mathcal{O} HH_\bullet(A \times G)_\mathcal{O} := \bigoplus_\mathcal{O} HH_\bullet(\Lambda G_\mathcal{O}, \theta; A^2_w),$$

$$HC_\bullet(A \times G) \cong \bigoplus_\mathcal{O}_\mathcal{O} HC_\bullet(A \times G)_\mathcal{O} := \bigoplus_\mathcal{O} HC_\bullet(\Lambda G_\mathcal{O}, \theta; A^2_w),$$

and similar for periodic cyclic homology. Of course, this can be read off from the final result of the computation in terms of orbifold cohomology, but it also follows by acting with idempotents $e_\mathcal{O} \in (\beta^{-1} A)(\Lambda G)$ having support $\mathcal{O}$ on the complexes $\Gamma_\bullet(\Lambda G, \theta; A^2_w)$. In fact, this works for any fine sheaf of unital algebras.

2.7. Quantization of proper étale groupoids. In [TAOED], the last author considered deformation quantization of a pseudo étale groupoid and proved that one can construct star products for such groupoids. As a special case one obtains that every proper étale Lie groupoid with an invariant Poisson structure has a formal deformation quantization. In this section, we will recall the basic concepts and constructions from [TAOED].

Definition 2.11. (Cf. [BEG] and [XU]) A Poisson structure on an associative complete locally convex topological algebra $\mathcal{A}$ is an element $[\Pi]$ of the (continuous) Hochschild cohomology group $H^2(\mathcal{A}, \mathcal{A})$ such that the cohomology class of the Gerstenhaber bracket $[[\Pi, \Pi]]$ vanishes.
Remark 2.12. If $\Pi \in Z^2(A,A)$ is a Hochschild cocycle representing a Poisson structure on $A$, one has $[\Pi,\Pi] = \delta(\Theta)$ for some Hochschild cochain $\Theta$ on $A$; in the following we will occasionally make use of this fact. By slight abuse of language, we sometimes also call $\Pi$ a Poisson structure on $A$.

Definition 2.13. (Cf. [BlGe, Ta04b].) Let $(A,\Pi)$ be a noncommutative Poisson algebra, and $A[[\hbar]]$ the space of formal power series with coefficients in $A$. A formal deformation quantization of $(A,\Pi)$ (or in other words star product) then is an associative product $\star : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]]$, $(a_1,a_2) \mapsto a_1 \star a_2 = \sum_{k=0}^\infty \hbar^k c_k(a_1,a_2)$ satisfying the following properties:

1. Each one of the maps $c_k : A[[\hbar]] \otimes A[[\hbar]] \to A[[\hbar]]$ is $\mathbb{C}[[\hbar]]$-bilinear.
2. One has $c_0(a_1,a_2) = a_1 \cdot a_2$ for all $a_1,a_2 \in A$.
3. The relation $a_1 \star a_2 - c_0(a_1,a_2) - \frac{i}{2} \hbar \Pi(a_1,a_2) \in \hbar^2 A[[\hbar]]$

holds true for some representative $\Pi \in Z^2(A,A)$ of the Poisson structure and all $a_1,a_2 \in A$.

From now on we consider a proper étale Lie groupoid $G$ and let $A$ denote the sheaf of smooth functions on $G_0$.

Definition 2.14. A Poisson (resp. symplectic) structure on $G$ is a Poisson (resp. symplectic) structure $\Pi$ on the unit space $G_0$ which is invariant under the local diffeomorphisms induced by the source and target maps.

One easily checks that in the symplectic case, this notion is equivalent to the definition of a symplectic orbifold. Note that an invariant Poisson bivector on $G_0$ has a canonical lift to a Poisson bivector on $G_1$. Having this in mind define a Hochschild 2-cochain on $A \rtimes G$ by

$$\tilde{\Pi}(a_1,a_2)(g) = \sum_{g_1,g_2 = g} \Pi(g) ([da_1]_{g_1} \otimes [da_2]_{g_2}), \quad g \in G_1, \ a_1,a_2 \in A \rtimes G, \quad (2.14)$$

where $[da_1]_{g_1}$ and $[da_2]_{g_2}$ have been pulled back to $g$ along the maps $t$ and $s$. In [Ta04b], it was proved that this Hochschild 2-cochain gives rise to a Poisson structure on the convolution algebra indeed. For convenience, we will simply denote the Poisson structure $\tilde{\Pi}$ on $A \rtimes G$ by $\Pi$ as well; this will not cause any confusion. As proved in [BlGe], the center of a noncommutative Poisson algebra carries a natural Poisson structure in the commutative sense. For proper étale Lie groupoids, the center equals $C^\infty(X)$, the smooth functions on the orbifold with the Poisson structure considered in [PeFo3].

In [Ta04b] it has been shown that the above Poisson structure on the groupoid algebra of a proper étale groupoid admits a formal deformation quantization. Such a deformation can be constructed as follows: first construct a deformation quantization of the Poisson manifold $G_0$, invariant under the action of the groupoid. In the symplectic case this can be done by Fedosov’s construction [Fe94] associated to an invariant symplectic connection. This defines a fine, so in particular $c$-soft, sheaf of noncommutative algebras $A^\hbar \in \text{Sh}(G)$. The associated crossed product...
algebra $A^h \rtimes G$, as in [26], quantizes the convolution algebra $A \rtimes G$ with the Poisson structure $\Pi$. We denote the multiplication on $A^h \rtimes G$ which combines the star product $\ast$ on $A^h$ with convolution on $G$ by $\ast_c$. Notice that $\Gamma_{\text{inv},c}(A^h)$ is the deformation quantization of $\mathcal{C}_c^\infty(X)$ with the induced Poisson structure as the center of $\mathcal{C}_c^\infty(G)$; the thus obtained star product algebra coincides with the formal deformation quantization studied in [10].

3. Hochschild cohomology of étale groupoids

Given a proper étale Lie groupoid $G$, we will determine in this section the Hochschild cohomology $H^\bullet(A \rtimes G, A \rtimes G)$, where $A$ is the $G$-sheaf of smooth functions on $G_0$. Recall that the (continuous) Hochschild cohomology of $A \rtimes G$ with values in a locally convex topological $(A \rtimes G)$-bimodule $M$ is defined as the cohomology of the cochain complex $(C^\bullet(A \rtimes G, M), \beta)$, where

$$C^k(A \rtimes G, M) = \text{Hom}_{(A \rtimes G) -(A \rtimes G)}( (A \rtimes G)^{\otimes (k+2)}, M),$$

and $\beta$ is the standard Hochschild coboundary map (see [LG Sec. 1.5]). Hereby, we have denoted by $\text{Hom}_{(A \rtimes G) -(A \rtimes G)}(N, M)$ the vector space of continuous $(A \rtimes G)$-bimodule maps between two locally convex topological $(A \rtimes G)$-bimodules $N$ and $M$.

**Remark 3.1.** Even though $A \rtimes G$ is usually nonunital, the standard complexes derived from the Bar resolution can be used to compute Hochschild (co)homology, since $A \rtimes G$ has local units, hence is $H$-unital (cf. [CRM01 Prop. 2]). The same holds for the deformed convolution algebra $A^h \rtimes G$. Later, we will tacitly make use of this fact when we determine the Hochschild homology of $A^h \rtimes G$.

For the computation of $H^\bullet(A \rtimes G, A \rtimes G)$ we proceed in several steps.

**Step 1.** In the following we provide a more convenient description of the cochain complex $C^\bullet := C^\bullet(A \rtimes G, A \rtimes G)$ and identify it with the complex of global section spaces of some sheaf complex $K^\bullet$ on the orbit space $X := G_0/G_1$. To this end first note that the Fréchet space $C^\infty(G_1)$ inherits from the convolution algebra $A \rtimes G$ the structure of a locally convex $(A \rtimes G)$-bimodule. More generally, observe that for every open subset $U \subset X$ the Fréchet space $C^\infty(U_1)$ of smooth functions on the preimage $U_1 := (\pi \circ s)^{-1}(U) \subset G_1$ becomes a topological $(A \rtimes G)$-bimodule by the formula

$$(a_1 \ast a \ast a_2)(g) = \sum_{h_1 \cdot h_2 = g} a_1(h_1) a(h) a_2(h_2),$$

(3.1)

where $a_1, a_2 \in A \rtimes G$, $a \in C^\infty(U_1)$ and $g \in G_1$. Now, choose for every compact $K \subset G_1$ a smooth function $\varphi_K : G_0 \to [0, 1]$ with compact support such that $\varphi_K(x) = 1$ for all $x \in s(K) \cup t(K)$. Let $\varphi_K \delta_u : G_1 \to [0, 1]$ be the function which coincides with $\varphi_K$ on $u(G_0)$ and vanishes elsewhere. Then $\varphi_K \delta_u$ is an element of the convolution algebra, hence for every cochain $\Phi \in C^k$ one obtains a continuous linear map $\hat{\Phi} : (A \rtimes G)^{\otimes k} \to C^\infty(G_1)$ by putting

$$\hat{\Phi}(a_1 \otimes \cdots \otimes a_k)|_K = \Phi(\varphi_K \delta_u \otimes a_1 \otimes \cdots \otimes a_k \otimes \varphi_K \delta_u)|_K,$$

where $K$ runs through the compact subsets of $G_1$. One checks immediately, that $\hat{\Phi}$ is well-defined and continuous indeed. Moreover, it is easy to prove that the map
identifies \( C^k \) with \( \text{Hom}((A \times G)^{\otimes k}, C^\infty(G_1)) \). Having this identification in mind we now put for every open \( U \subset X \):

\[
K^k(U) := \text{Hom}((A \times G)^{\otimes k}, C^\infty(U_1)), \quad U_1 := (\pi \circ s)^{-1}(U),
\]

where \( C^\infty(U_1) \) carries the \( A \times G \)-bimodule structure given by Eq. (3.2). Since the Hochschild coboundary is functorial with respect to restriction maps, and since the smooth functions on \( G_1 \) form a sheaf, \( K^\bullet \) is a complex of sheaves on \( X \). By the above identification it is clear that \( K^\bullet(X) \) can be naturally identified with \( C^\bullet \).

Step 2. In this part we prove a localization result for Hochschild (co)homology of the convolution algebra. To this end we need some more notation. First let us fix a smooth function \( \eta : \mathbb{R} \to [0, 1] \) which has support in \( (-\infty, \frac{1}{2}] \) and which satisfies \( \eta(r) = 1 \) for \( r \leq \frac{1}{2} \). For \( \varepsilon > 0 \) we denote by \( \eta_\varepsilon \) the rescaled function \( r \mapsto \eta(r/\varepsilon) \).

Next choose a \( G \)-invariant metric \( d \) on \( G_0 \) such that \( d^2 \) is smooth, and set for every \( k \in \mathbb{N}, i = 1, \ldots, k \) and \( \varepsilon > 0 \):

\[
\Psi_{k,i,\varepsilon}(g_0, g_1, \ldots, g_k) = \prod_{j=0}^{i-1} \eta_\varepsilon(d^2(s(g_j), t(g_{j+1}))), \quad \text{where } g_{k+1} := g_0.
\]

Moreover, put \( \Psi_{k,\varepsilon} := \Psi_{k,k+1,\varepsilon} \).

Given a Hochschild chain \( c \) resp. a Hochschild cochain \( F \) (of degree \( k \)) we now define \( \Psi_{k,\varepsilon} c \in C_k := C_k(A \times G, A \times G) \) resp. \( \Psi_{k,\varepsilon} F \in C^k := C^k(A \times G, A \times G) \) as follows:

\[
(\Psi_{k,\varepsilon} c)(g_0, g_1, \ldots, g_k) := \Psi_{k,\varepsilon}(g_0, g_1, \ldots, g_k) \cdot c(g_0, g_1, \ldots, g_k),
\]

\[
(\Psi_{k,\varepsilon} F(a_1 \otimes \cdots \otimes a_k))(g_0) := F(\Psi_{k,\varepsilon}(g_0^{-1}, -1, \ldots, -1) \cdot (a_1 \otimes \cdots \otimes a_k))(g_0),
\]

\[
(g_0, g_1, \ldots, g_k) \in G^{k+1}, \quad a_1, \ldots, a_k \in C^\infty(G).
\]

One immediately checks then that the operations \( \Psi_{\bullet,\varepsilon} \) and \( \Psi_{\varepsilon,\bullet} \) are both chain maps on the Hochschild chain resp. cochain complex.

Let us now construct a homotopy between the identity operator and \( \Psi_{\bullet,\varepsilon} \) resp. \( \Psi_{\varepsilon,\bullet} \). To this end define maps \( \eta_{k,i,\varepsilon} : C_k \to C_{k+1} \) for \( 1 \leq i \leq k+1 \) and maps \( \eta^{k,i,\varepsilon} : C^k \to C^{k-1} \) for \( 1 \leq i \leq k \) as follows:

\[
\eta_{k,i,\varepsilon}(c)(g_0, g_1, \ldots, g_{k+1}) = \begin{cases} 
\Psi_{k+1,i,\varepsilon}(g_0, g_1, \ldots, g_{k+1}) \cdot c(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{k+1}) \cdot \delta_u(g_i), & \text{if } i < k + 1, \\
\Psi_{k+1,k+1,\varepsilon}(g_0, g_1, \ldots, g_{k+1}) \cdot c(g_0, \ldots, g_k) \cdot \delta_u(g_{k+1}), & \text{if } i = k + 1,
\end{cases}
\]

and

\[
\eta^{k,i,\varepsilon}(F)(a_1 \otimes \cdots \otimes a_{k-1})(g_0) = \begin{cases} 
F(\Psi_{k,i,\varepsilon}(g_0^{-1}, -1, \ldots, -1) \cdot (a_1 \otimes \cdots \otimes a_{i-1} \otimes \delta_u \otimes a_{i} \otimes \cdots \otimes a_{k-1}))(g_0), & \text{if } i < k, \\
F(\Psi_{k,k,\varepsilon}(g_0^{-1}, -1, \ldots, -1) \cdot (a_1 \otimes \cdots \otimes a_{k-1} \otimes \delta_u))(g_0), & \text{if } i = k.
\end{cases}
\]

Hereby, \( \delta_u \in C^\infty(G) \) denotes the function

\[
g \mapsto \begin{cases} 
1, & \text{if } g = u(x) \text{ for some } x \in G_0, \\
0, & \text{else.}
\end{cases}
\]

By a somewhat lengthy, but straightforward computation one then proves the following result.
Proposition 3.2. The maps
\[ H_{k,\varepsilon} := \sum_{i=1}^{k+1} (-1)^{i+1} \eta_{k,i,\varepsilon} : C_k \to C_{k+1} \quad \text{and} \]
\[ H^{k,\varepsilon} := \sum_{i=1}^{k} (-1)^{i+1} \eta^{k,i,\varepsilon} : C^k \to C^{k-1} \]
form a homotopy between the identity and the localization morphism \( \Psi_{\bullet,\varepsilon} \) resp. \( \Psi^{\bullet,\varepsilon} \).
More precisely,
\[ (b_{k+1} H_{k,\varepsilon} + H_{k-1,\varepsilon} b_k) c = c - \Psi_{\bullet,\varepsilon} c \quad \text{for all } c \in C_k, \quad (\beta^{k-1} H^k_{\gamma,\varepsilon} + H^{k+1,\varepsilon} \beta^k) F = F - \Psi^{\bullet,\varepsilon} F \quad \text{for all } F \in C^k. \quad (3.3) \]

Sketch of Proof. Let \( d_{k,j} : C_k \to C_{k-1} \) be the face maps of Example 2.6. Then one easily checks the following commutation relations for \( i = 1 \)
\[ (d_{k+1,j} \eta_{k,1,\varepsilon} c) (g_0, \ldots, g_k) = \begin{cases} c(g_0, \cdots, g_k), & \text{if } j = 0, \\ \left( \Psi_{k,1,\varepsilon} c \right) (g_0, \cdots, g_k), & \text{if } j = 1, \\ \left( \eta_{k-1,1,\varepsilon} d_{k,j-1} c \right) (g_0, \cdots, g_k), & \text{if } 1 < j \leq k + 1, \end{cases} \]
for \( i = 2, \ldots, k \)
\[ (d_{k+1,j} \eta_{k,i,\varepsilon} c) (g_0, \cdots, g_k) = \begin{cases} \left( \eta_{k-1,i-1,\varepsilon} d_{k,j} c \right) (g_0, \cdots, g_k), & \text{if } 0 \leq j < i - 1, \\ \left( \Psi_{k,i-1,\varepsilon} c \right) (g_0, \cdots, g_k), & \text{if } j = i - 1, \\ \left( \Psi_{k,i,\varepsilon} c \right) (g_0, \cdots, g_k), & \text{if } j = i, \\ \left( \eta_{k-1,i,\varepsilon} d_{k,j-1} c \right) (g_0, \cdots, g_k), & \text{if } i < j \leq k + 1, \end{cases} \]
and for \( i = k + 1 \)
\[ (d_{k+1,j} \eta_{k,k,\varepsilon} c) (g_0, \ldots, g_k) = \begin{cases} \left( \eta_{k-1,k,\varepsilon} d_{k,j} c \right) (g_0, \cdots, g_k), & \text{if } 0 \leq j \leq k, \\ \left( \Psi_{k,k,\varepsilon} c \right) (g_0, \cdots, g_k), & \text{if } j = k, \\ \left( \Psi_{k,k+1,\varepsilon} c \right) (g_0, \cdots, g_k), & \text{if } j = k + 1. \end{cases} \]
From these commutation relations one immediately derives Eq. (3.3).

Now let us consider the dual case. Let \( \sigma^{k,j} : C^k \to C^{k+1}, j = 0, \cdots, k + 1 \) be the face maps of the cosimplicial vector space \( C^\bullet \), i.e., let
\[ \sigma^{k,j} F(a_1 \otimes \cdots \otimes a_{k+1}) = \begin{cases} a_1 \ast F(a_2 \otimes \cdots \otimes a_{k+1}), & \text{if } j = 0, \\ F(d_{k+1,j}(a_1 \otimes \cdots \otimes a_{k+1})), & \text{if } 1 < j < k + 1, \\ F(a_1 \otimes \cdots \otimes a_k) \ast a_{k+1}, & \text{if } j = k + 1. \end{cases} \]
For \( i = 2, \cdots, k \) and \( j = 0 \) one then computes
\[ (\eta^{k+1,i,\varepsilon} \sigma^{k,j} F)(a_1 \otimes \cdots \otimes a_k) (g_0) = \]
\[ = (\sigma^{k,0} F)(\Psi_{k+1,i,\varepsilon} (g_0^{-1}, - \cdots, -) \cdots (a_1 \otimes \cdots \otimes a_{i-1} \otimes \delta_u \otimes a_i \otimes \cdots \otimes a_k) (g_0) \]
\[ = \sum_{h, h' = g_0} a_1(h) \cdot F(\Psi_{k+1,i,\varepsilon} ((h')^{-1}, - \cdots, -) \cdots (a_2 \otimes \cdots \otimes a_{i-1} \otimes \delta_u \otimes a_i \otimes \cdots \otimes a_k) (h')) \]
\[ = \sum_{h, h' = g_0} a_1(h) \cdot F(\Psi_{k-1,i,\varepsilon} ((h')^{-1}, - \cdots, -) \cdots (a_2 \otimes \cdots \otimes a_{i-1} \otimes \delta_u \otimes a_i \otimes \cdots \otimes a_k) (h')) \]
\[ = (\sigma^{k-1,j} \eta^{k,i-1,\varepsilon} F)(a_1 \otimes \cdots \otimes a_k) (g_0). \]
By computations of this type and the corresponding relations in the homology case one obtains for $i = 1$

$$(\eta^{k+1,1,\varepsilon} \sigma^{k,j} F)(a_1 \otimes \cdots \otimes a_k)(g_0) =$$

$$= \begin{cases} 
F(a_1 \otimes \cdots \otimes a_k)(g_0), & \text{if } j = 0, \\
F(\Psi_{k,1,\varepsilon}(g_0^{-1}, -, \cdots, -) \cdot (a_1 \otimes \cdots \otimes a_k))(g_0), & \text{if } j = 1, \\
(\sigma^{k-1,j-1} \eta_{k,1,\varepsilon} F)(a_1 \otimes \cdots \otimes a_k)(g_0), & \text{if } 1 < j \leq k + 1,
\end{cases}$$

for $i = 2, \cdots, k$

$$(\eta^{k+1,1,\varepsilon} \sigma^{k,j} F)(a_1 \otimes \cdots \otimes a_k)(g_0) =$$

$$= \begin{cases} 
(\sigma^{k-1,j} \eta^{k,j-1,\varepsilon} F)(a_1 \otimes \cdots \otimes a_k)(g_0), & \text{if } 0 \leq j < i - 1, \\
F(\Psi_{k,i-1,\varepsilon}(g_0^{-1}, -, \cdots, -) \cdot (a_1 \otimes \cdots \otimes a_k))(g_0), & \text{if } j = i - 1, \\
F(\Psi_{k,i,\varepsilon}(g_0^{-1}, -, \cdots, -) \cdot (a_1 \otimes \cdots \otimes a_k))(g_0), & \text{if } j = i, \\
(\sigma^{k-1,j-1} \eta^{k,j,\varepsilon} F)(a_1 \otimes \cdots \otimes a_k)(g_0), & \text{if } i < j \leq k + 1,
\end{cases}$$

and for $i = k + 1$

$$(\eta^{k+1,1,\varepsilon} \sigma^{k,j} F)(a_1 \otimes \cdots \otimes a_k)(g_0) =$$

$$= \begin{cases} 
(\sigma^{k-1,j} \eta^{k,k,\varepsilon} F)(a_1 \otimes \cdots \otimes a_k)(g_0), & \text{if } 0 \leq j < k, \\
F(\Psi_{k,k,\varepsilon}(g_0^{-1}, -, \cdots, -) \cdot (a_1 \otimes \cdots \otimes a_k))(g_0), & \text{if } j = k, \\
F(\Psi_{k,k+1,\varepsilon}(g_0^{-1}, -, \cdots, -) \cdot (a_1 \otimes \cdots \otimes a_k))(g_0), & \text{if } j = k + 1.
\end{cases}$$

Using $\beta^k = \sum (-1)^j \sigma^{k,j}$, these commutation relations immediately entail Eq. (3.4). $\square$

Denote by $C_k^\varepsilon$ the subspace of all Hochschild chains with support in the complement of

$$U_{k+1,\varepsilon} := \{(g_0, \cdots, g_k) \in G^{k+1} \mid d^2(s(g_0), t(g_1)) + \cdots + d^2(s(g_k), t(g_0)) < \varepsilon\}$$

and by $C_k^\varepsilon$ the space of all Hochschild cochains having support in the complement of

$$\bar{U}_{k+1,\varepsilon} := \{(g_0, \cdots, g_k) \in G^{k+1} \mid d^2(s(g_0), t(g_1)) + \cdots + d^2(s(g_k), t(g_0)) < \varepsilon\}.$$ 

Moreover, let $C_0^\varepsilon$ resp. $C_k^\varepsilon$ be the union of all $C_k^\varepsilon$ resp. $C_k^\varepsilon$, where $\varepsilon$ runs through all positive real numbers. Then the proposition entails

**Corollary 3.3.** The subcomplexes $C_0^\varepsilon$ and $C_0^\varepsilon$ are acyclic. In particular, the quotient maps

$$C_\bullet \to C_\bullet/C_0^\varepsilon \quad \text{and} \quad C^\bullet \to C^\bullet/C_0^\varepsilon$$

are quasi-isomorphisms.

**Remark 3.4.** Originally, Brylinski–Nistor have shown in [BRN] Prop. 3.2 that $C_\bullet \to C_\bullet/C_0^\varepsilon$ is a quasi-isomorphism and used this result to compute the Hochschild homology $HH_\bullet(\mathcal{A} \rtimes G)$.

**Remark 3.5.** In the case, where $G$ is the Lie groupoid whose objects and arrows are given by the points of a smooth manifold $M$, one recovers the well-known localization scheme for Hochschild homology à la Telemann [TE]. In the following, we will freely make use of this fact.
Step 3. In the third step we restrict our considerations to the case, where \( G \) is a transformation groupoid \( \Gamma \ltimes M \) of a finite group \( \Gamma \) acting on a smooth manifold \( M \). Recall that then \( G_1 = \Gamma \times M, \ G_0 = M \) and that every \( \gamma \in \Gamma \) acts on \( A := C^\infty(M) \) by

\[
\gamma a(p) = a(\gamma^{-1} p), \quad \text{where } a \in A, \ p \in M.
\]

Moreover, every element \( a \) of the convolution algebra \( A \ltimes \Gamma \) has a unique representation of the form

\[
a = \sum_{\gamma \in \Gamma} f_\gamma \delta_\gamma,
\]

where \( f_\gamma \in A \) and where \( f_\gamma \delta_\gamma \) is the function which satisfies \( f_\gamma \delta_\gamma(\gamma, p) = f_\gamma(\gamma p) \) and vanishes elsewhere. One easily computes that then

\[
f_1 \delta_{\gamma_1} * f_2 \delta_{\gamma_2} = f_1(\gamma_1 f_2) \delta_{\gamma_1 \gamma_2} \quad \text{for all } f_1, f_2 \in A \text{ and } \gamma_1, \gamma_2 \in \Gamma.
\]

Concerning the topological tensor product considered, one should observe in the following that the completed inductive tensor product \( A \hat{\otimes} A \) and the completed projective tensor product \( A \hat{\otimes} A \) coincide, since \( A \) is a (nuclear) Fréchet space.

Lemma 3.6. Let \( \Gamma \) act on the space \( \text{Hom}(A \hat{\otimes}^k, A \ltimes \Gamma) \) (of continuous linear maps) as follows:

\[
(\gamma \phi)(f_1 \otimes \cdots \otimes f_k) = \delta_\gamma * \phi(\gamma^{-1} f_1 \otimes \cdots \otimes \gamma^{-1} f_k) * \delta_{\gamma^{-1}}.
\]

Then the relation

\[
f \delta_e * \gamma \phi * f' \delta_e = \gamma((\gamma^{-1} f) \delta_e * \phi * (\gamma^{-1} f') \delta_e)
\]

holds true for all \( \phi \in \text{Hom}(A \hat{\otimes}^k, A \ltimes \Gamma) \) and \( f, f' \in A \).

Proof. The claim is an immediate consequence of Eq. 3.6. \( \square \)

Consider now the vector spaces \( C^n_{\Gamma,m} = \text{Hom}(\mathbb{C}^\Gamma^n, \text{Hom}(A \hat{\otimes}^m, A \ltimes \Gamma)) \), where \( n, m \in \mathbb{N} \). Using the \( \Gamma \)-action on \( \text{Hom}(A \hat{\otimes}^k, A \ltimes \Gamma) \) from above, the simplicial structures coming from group cohomology and Hochschild cohomology then induce
on $C^{\bullet,\bullet}_T$ the structure of a bicosimplicial vector space as follows (where $\Psi \in C^{m,n}_T$):

$$d^i_v : C^{m,n}_T \rightarrow C^{m+1,n}_T, \quad d^i_v \Psi(\gamma_1, \ldots, \gamma_{m+1}) =$$

$$= \begin{cases} 
\gamma_1(\Psi(\gamma_2, \ldots, \gamma_{m+1})), & \text{if } i = 0, \\
\Psi(\gamma_1, \ldots, \gamma_i \gamma_{i+1}, \ldots, \gamma_{m+1}), & \text{if } 1 \leq i \leq m, \\
\Psi(\gamma_1, \ldots, \gamma_m), & \text{if } i = m + 1,
\end{cases} \quad (3.8)$$

$$d^i_h : C^{m,n}_T \rightarrow C^{m,n+1}_T, \quad d^i_h \Psi(\gamma_1, \ldots, \gamma_m) (f_1 \otimes \cdots \otimes f_{n+1}) =$$

$$= \begin{cases} 
f_1 \delta_e * \Psi(\gamma_1, \ldots, \gamma_m)(f_2 \otimes \cdots \otimes f_{n+1}), & \text{if } j = 0, \\
\Psi(\gamma_1, \ldots, \gamma_m)(f_1 \otimes \cdots \otimes f_{j+1} \otimes \cdots \otimes f_{n+1}), & \text{if } 1 \leq j \leq n, \\
\Psi(\gamma_1, \ldots, \gamma_m)(f_1 \otimes \cdots \otimes f_n) * f_{n+1} \delta_e, & \text{if } j = n + 1,
\end{cases} \quad (3.9)$$

$$s^i_v : C^{m,n}_T \rightarrow C^{m-1,n}_T, \quad s^i_v \Psi(\gamma_1, \ldots, \gamma_{m-1}) =$$

$$= \begin{cases} 
\Psi(e, \gamma_1, \ldots, \gamma_{m-1}), & \text{if } i = 0, \\
\Psi(\gamma_1, \ldots, \gamma_i, e, \gamma_{i+1}, \ldots, \gamma_{m-1}), & \text{if } 1 \leq i \leq m - 1,
\end{cases} \quad (3.10)$$

$$s^i_h : C^{m,n}_T \rightarrow C^{m,n-1}_T, \quad s^i_h \Psi(\gamma_1, \ldots, \gamma_m)(f_1 \otimes \cdots \otimes f_{n-1}) =$$

$$= \begin{cases} 
\Psi(\gamma_1, \ldots, \gamma_m)(1 \otimes f_1 \otimes \cdots \otimes f_{n-1}), & \text{if } j = 0, \\
\Psi(\gamma_1, \ldots, \gamma_m)(f_1 \otimes \cdots \otimes f_j \otimes 1 \otimes f_{j+1} \otimes \cdots \otimes f_{n-1}), & \text{if } 1 \leq j \leq n - 1.
\end{cases} \quad (3.11)$$

The $d^i_v$ (resp. $d^i_h$) form the vertical (resp. horizontal) face maps of the bicosimplicial vector space, the $s^i_v$ (resp. $s^i_h$) the vertical (resp. horizontal) degeneracies. Using Lemma 3.6 it is easy to show that every vertical structure map commutes with every horizontal structure map, hence $C^{\bullet,\bullet}_T$ is a bicosimplicial vector space indeed.

For example, let us show that $d^0_v$ and $d^0_h$ commute:

$$(d^0_v d^0_h \Psi)(\gamma_1, \ldots, \gamma_{m+1})(f_1 \otimes \cdots \otimes f_{n+1}) =$$

$$= \delta_{\gamma_1} * (\gamma_1^{-1} f_1 \delta_e) * \Psi(\gamma_2, \ldots, \gamma_{m+1})(\gamma_1^{-1} f_2 \otimes \cdots \otimes \gamma_{m+1}^{-1} f_{n+1}) * \delta_{\gamma_1^{-1}}$$

$$= f_1 \delta_e * \delta_{\gamma_1} * \Psi(\gamma_2, \ldots, \gamma_{m+1})(\gamma_1^{-1} f_2 \otimes \cdots \otimes \gamma_{m+1}^{-1} f_{n+1}) * \delta_{\gamma_1^{-1}}$$

$$= (d^0_v d^0_h \Psi)(\gamma_1, \ldots, \gamma_{m+1})(f_1 \otimes \cdots \otimes f_{n+1}) \quad (3.12)$$

At this point recall that the bicosimplicial space $C^{\bullet,\bullet}_T$ induces the structure of a cosimplicial space on the diagonal $C^\bullet_T := \text{diag}(C^{\bullet,\bullet}_T)$ (see [WE] Sec. 8.5)). Its structure maps are given by $d^i = d^i_h d^i_v$ and $s^i = s^i_h s^i_v$.

**Proposition 3.7.** Define for every $\Phi \in C^k(A \times \Gamma, A \times \Gamma)$ an element $\hat{\Phi} \in C^k_T = \text{Hom}(\mathbb{C}^k, \text{Hom}(A^k, A \times \Gamma))$ as follows:

$$\hat{\Phi} (\gamma_1, \ldots, \gamma_k)(f_1 \otimes \cdots \otimes f_k) =$$

$$= \Phi((\gamma_1 \gamma_2 \ldots \gamma_k f_1) \delta_{\gamma_1} \otimes (\gamma_2 \ldots \gamma_k f_2) \delta_{\gamma_2} \otimes \cdots \otimes (\gamma_k f_k) \delta_{\gamma_k}). \quad (3.13)$$

Then $^\wedge : C^\bullet(A \times \Gamma, A \times \Gamma) \rightarrow C^\bullet_T$ is a cosimplicial map.
Proof. Denote by $b^i$ the face maps of Hochschild cohomology, which act on a cochain $\Phi \in C^k(A \rtimes \Gamma, A \rtimes \Gamma)$ as follows:

$$b^i \Phi (f_1 \delta_{\gamma_1} \cdots \cdot f_{k+1} \delta_{\gamma_{k+1}}) =
\begin{cases}
  f_1 \delta_{\gamma_1} \Phi(f_2 \delta_{\gamma_2} \cdot \cdots \cdot f_{k+1} \delta_{\gamma_{k+1}}) & \text{if } i = 0 \\
  \Phi(f_1 \delta_{\gamma_1} \cdots \cdot f_i \delta_{\gamma_i} f_{i+1} \cdots \cdot f_{k+1} \delta_{\gamma_{k+1}}) & \text{if } 1 \leq i < k,
  \\
  \Phi(f_1 \delta_{\gamma_1} \cdots \cdot f_k \delta_{\gamma_k}) \cdot f_{k+1} \delta_{\gamma_{k+1}} & \text{if } i = n
\end{cases}
$$

Then compute for $1 \leq i < k$:

$$\left( b^i \Phi \right)^\gamma(\gamma_1, \ldots, \gamma_{k+1}) (f_1 \otimes \cdots \otimes f_{k+1}) =
\begin{align*}
  &= b^i \Phi((\gamma_1 \cdots \gamma_{k+1} f_1) \delta_{\gamma_1} \cdots \cdots (\gamma_{k+1} f_{k+1}) \delta_{\gamma_{k+1}}) \\
  &= \Phi((\gamma_1 \cdots \gamma_{k+1} f_1) \delta_{\gamma_1} \cdots \cdots (\gamma_{k+1} \cdots \gamma_{k+1} f_{k+1}) (a_1 \delta e \cdots \cdots a_k \delta e)) \\
  &= d^e_{f_i} d^e_{f_i} \Phi(\gamma_1, \ldots, \gamma_{k+1})(a_1 \delta e \cdots \cdots a_k \delta e).
\end{align*}
$$

By a similar computation one shows that $^\gamma$ preserves all the other face and the degeneracy maps. This proves the claim. 

We now have the tools to show the following result.

**Proposition 3.8.** (Cf. [LAGVIV Prop. 4.1]) Let $\Gamma$ be finite group acting by diffeomorphisms on the manifold $M$, and $A$ the Fréchet algebra of smooth functions on $M$. Then the Hochschild cohomology $H^\bullet(A, A \rtimes \Gamma)$ carries a natural $\Gamma$-action such that

$$H^\bullet(A \rtimes \Gamma, A \rtimes \Gamma) \cong H^\bullet(A, A \rtimes \Gamma)^\Gamma.
$$

On the level of cochains, this isomorphism is induced by the following chain map:

$$C^k(A \rtimes \Gamma, A \rtimes \Gamma) \to C^k(A, A \rtimes \Gamma), \quad F \mapsto (a_1 \delta e \cdots \cdots a_k \delta e) \mapsto F(a_1 \delta e \cdots \cdots a_k \delta e),
$$

where $a_i \delta e$ denotes the smooth function on $\Gamma \times M$ which coincides with $a_i$ on the unit space $u(M)$ and vanishes elsewhere.

**Proof.** By the Eilenberg-Zilber Theorem one has

$$H^\bullet \text{diag}(C^\bullet_{\Gamma^\bullet}) = H^\bullet(\text{Tot} C^\bullet_{\Gamma^\bullet}).$$

Moreover, there is a spectral sequence

$$E_1^{m,n} = H^m_{\Gamma}(C^\bullet_{\Gamma^\bullet}), \quad E_2^{m,n} = H^m_{\Gamma} H^n_{\Gamma}(C^\bullet_{\Gamma^\bullet}) \Rightarrow H^{m+n} \text{diag}(C^\bullet_{\Gamma^\bullet}).$$

Now recall that the group cohomology of a finite group vanishes in degrees $\geq 1$. Using the $\Gamma$-action from Lemma 3.6 we thus obtain the following chain of natural isomorphisms:

$$H^n(A \rtimes \Gamma, A \rtimes \Gamma) \cong H^n \text{diag}(C^\bullet_{\Gamma^\bullet}) \cong H^n_{\Gamma} H^n_{\Gamma}(C^\bullet_{\Gamma^\bullet}) = (H^\bullet(A, A \rtimes \Gamma))^\Gamma.
$$

This proves the first claim; the second is a direct consequence of Prop. 3.7 and the spectral sequence argument leading to Eq. (3.16).

\[\square\]
**Corollary 3.9.** Let $\Gamma, M, A$ as above and $A_c := C_c^\infty(M)$ the algebra of smooth functions with compact support on $M$. Then there exists a commutative diagram of canonical isomorphisms:

$$
\begin{array}{ccc}
HH^k(A \times \Gamma, A \times \Gamma) & \longrightarrow & HH^k(A_c \times \Gamma, A_c \times \Gamma) \\
\downarrow & & \downarrow \\
HH^k(A, A \times \Gamma)^\Gamma & \longrightarrow & HH^k(A_c, A_c \times \Gamma)^\Gamma.
\end{array}
$$

**Proof.** Using the identification $\sim$ of the first step one checks that the following morphisms are chain maps:

$$
C^k(A \times \Gamma, A \times \Gamma) \to \text{Hom}(\langle A_c \times \Gamma \rangle, A \times \Gamma) \cong C^k(A_c \times \Gamma, A_c \times \Gamma)
$$

$$
F \mapsto (A_c \times \Gamma \ni a_1 \otimes \ldots \otimes a_k \mapsto F(a_1 \otimes \ldots \otimes a_k) \in A \times \Gamma),
$$

and

$$
C^k(A, A \times \Gamma) \to \text{Hom}(\langle A_c \langle \rangle \rangle, A \times \Gamma) \cong C^k(A_c, A_c \times \Gamma)
$$

$$
F \mapsto (A \ni f_1 \otimes \ldots \otimes f_k \mapsto F(f_1 \otimes \ldots \otimes f_k) \in A \times \Gamma).
$$

With the help of the localization maps $\Psi^{\ast \circ}$ (associated to a complete $\Gamma$-invariant metric $d$ on $M$) and an appropriate invariant smooth partition of unity on $M$ one can construct quasi-inverses to the chain maps $C^\bullet(A \times \Gamma, A \times \Gamma) \to C^\bullet(A_c \times \Gamma, A_c \times \Gamma)$ and $C^\bullet(A, A \times \Gamma) \to C^\bullet(A_c, A_c \times \Gamma)$. Thus, the two horizontal arrows in the above diagram are isomorphisms. The left vertical arrow is an isomorphism by the preceding proposition, hence the induced right vertical arrow has to be an isomorphism as well. \qed

**Step 4.** According to Prop. 3.8 it suffices to compute the (invariant part of the) cohomology of the cochain complex $C^\bullet(A, A \times \Gamma)$, if $G$ is a translation groupoid $\Gamma \ltimes M$. To this end we specialize the situation further and assume that $M$ is an open $\Gamma$-invariant neighborhood of the origin of some finite dimensional linear $\Gamma$-representation space $V$. We choose a $\Gamma$-invariant scalar product on $V$ and orthonormal linear coordinates $x_1, \ldots, x_n$ of $V$ such that $x_1, \ldots, x_i$ span the fixed point space $V^\Gamma$ and $x_{i+1}, \ldots, x_n$ span $W$, the subspace orthogonal to $V^\Gamma$. We assume further that $M$ has the form $M^\Gamma \times N$ with $M^\Gamma$ an open ball in $V^\Gamma$ and $N$ an open ball in $W$.

For the computation of $H^\bullet(A, A \times \Gamma)$ we will use the (topologically projective) resolution of $A$ given by the Koszul complex $(K_{\bullet}, \partial)$ associated to the regular sequence $(x_1 \otimes \text{id} - \text{id} \otimes x_1, \ldots, x_n \otimes \text{id} - \text{id} \otimes x_n)$ in $A \hat{\otimes} A$. More precisely, the resolution of $A$ by $K_{\bullet}$ has the form

$$
0 \longrightarrow A \hat{\otimes} A \otimes \Lambda^n V^* \xrightarrow{\partial} \cdots \xrightarrow{\partial} A \hat{\otimes} A \otimes \Lambda^1 V^* \xrightarrow{\partial} \cdots \xrightarrow{\partial} A \hat{\otimes} A \xrightarrow{m} A \longrightarrow 0
$$

with differential $\partial : A \hat{\otimes} A \otimes \Lambda^k V^* \to A \hat{\otimes} A \otimes \Lambda^{k-1} V^*$ given by

$$
f_1 \otimes f_2 \otimes dx_{i_1} \wedge \ldots \wedge dx_{i_k} \mapsto \sum_{j=1}^k (-1)^j (x_{i_j} f_1 \otimes f_2 - f_1 \otimes x_{i_j} f_2) \otimes dx_{i_1} \wedge \ldots \wedge dx_{i_{j-1}} \wedge dx_{i_{j+1}} \wedge \ldots \wedge dx_{i_k},
$$

where $\Lambda^n V^*$ is the exterior algebra on $V^*$ with $n$ generators.

The resolution $\hat{\otimes}$ is induced by the $\Gamma$-invariant vector field $\gamma$ on $\partial M$, which is the fixed point of an element $\iota : \Gamma \to \Gamma$ that acts on $M$ by translation on $M^\Gamma$ and fixes $N$.

This completes the proof of the corollary.
Let us provide another description of the Koszul complex \((K_\bullet, \partial)\). Denote by \(E_k\) the pull-back bundle \(pr^*_2(\Lambda^kT^*M)\), where \(\Lambda^kT^*M\) is the exterior product of the cotangent bundle of \(M\), and \(pr_2 : M \times M \to M\) is the projection on the second coordinate. Then the vector field

\[
\xi : M \times M \to V, \quad (p, q) \mapsto \xi(p, q) = \sum_{i=1}^{n} (x_i(p) - x_i(q)) \frac{\partial}{\partial x_i}, \tag{3.19}
\]

comprises a section of \(E^*_1\) which does not vanish outside the diagonal. Moreover, \(K_k\) can be naturally identified with the sectional space \(\Gamma^\infty(E_k)\), and \(\partial\) is the insertion of the vector field \(\xi\).

The cohomology \(H^\bullet(A, A \times \Gamma)\) now is given as the direct sum over the elements \(\gamma \in \Gamma\) of the cohomologies of the cochain complexes \((\text{Hom}(K_\bullet, A), \partial^\gamma)\), where \(A_\gamma\) coincides with \(A\) as a Fréchet space and carries the following \(A\)-bimodule structure:

\[
(f_1 * a * f_2)(p) = f_1(\gamma p) a(p) f_2(p) \quad \text{for all } p \in M, \quad a \in A_\gamma, \quad f_1, f_2 \in A. \tag{3.20}
\]

This entails immediately that for every natural \(k\) there is a canonical isomorphism

\[
\eta_k : \Gamma^\infty(\Lambda^kT^*M) \to \text{Hom}_{A}(K_k, A_\gamma), \quad \tau \mapsto \eta(\tau),
\]

which is uniquely determined by the relation

\[
\eta(\tau)(\omega) = \langle \Delta^*_\omega, \tau \rangle \quad \text{for all } \omega \in \Gamma^\infty(E_k).
\]

Hereby, \(\langle - , - \rangle : \Omega^k(M) \times \Lambda^kT^*M \to C^\infty(M)\) denotes the canonical fiberwise pairing, \(\Delta^*_\gamma : M \to M \times M = M\) is the embedding \(p \mapsto (\gamma p, p)\), and \(\Delta^*_\omega\) is defined by \(\langle \Delta^*_\omega(p), v \rangle = \langle \omega(\Delta^*_\gamma(p)), v \rangle\) for every \(p \in M\) and \(v \in T_pM \cong V\). Clearly, \(\eta_k\) is injective. Let us show that it is surjective as well. Let \(F\) be a continuous \(A\)-bimodule map from \(\Gamma^\infty(E_k)\) to \(A_\gamma\) and define for all multiindices \(1 \leq i_1 < \ldots < i_k \leq n\) coefficients \(\tau_{i_1, \ldots, i_k}\) by \(\tau_{i_1, \ldots, i_k} := F(\text{pr}_2^*(dx_{i_1} \wedge \ldots \wedge dx_{i_k}))\). Then \(\eta\) maps the multivectorfield

\[
\tau := \sum_{i_1 < \ldots < i_k} \tau_{i_1, \ldots, i_k} \frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_k}}
\]

to \(F\), hence \(\eta\) is surjective.

Now let \(\kappa\) be the vector field on \(M\) defined by

\[
\kappa(p) = \xi(\gamma p, p) = \sum_{i=l+1}^{n} (x_i(\gamma p) - x_i(p)) \frac{\partial}{\partial x_i}. \tag{3.21}
\]

Under the isomorphism \(\eta\), the cohomological differential \(\partial^\gamma\) corresponds to the operation \(\kappa \wedge -\). To check this, let \(\omega \in \Gamma^\infty(E_{k+1})\) and compute:

\[
(\partial^\gamma \eta(\tau))(\omega) = \eta(\tau)(i_\xi \omega) = \langle \Delta^*_\omega, \partial^\gamma \tau \rangle = \langle i_\xi^* \Delta^*_\omega, \tau \rangle = \langle \Delta^*_\omega, \kappa \wedge \tau \rangle = \eta(\kappa \wedge \tau)(\omega),
\]

which proves the claim. Hence it remains to determine the cohomology of the cochain complex

\[
(\Gamma^\infty(\Lambda^*T^*M), \kappa \wedge -). \tag{3.22}
\]

But this complex is a dual Koszul complex. To compute its cohomology observe first that the decomposition \(V = V^\gamma \oplus W\) induces a decomposition of the alternating multivector fields on \(M\) as follows:

\[
\Gamma^\infty\Lambda^k(T^*M) \cong \bigoplus_{p=0}^{k} \Lambda^p(V^\gamma) \otimes \Gamma^\infty(\Lambda^{k-p} \text{pr}_N^* T^*N), \tag{3.23}
\]
where \( \text{pr}_N : M \to N \) is the projection onto \( N \) along \( M^\gamma \). Under this isomorphism, the differential \( \kappa \wedge \cdot \) acts only on the second components. Hence one can interpret the cohomology of (3.22) as the total cohomology of the double complex \( D^{p,q} = \Lambda^p(V^\gamma) \otimes \Gamma^\infty(\Lambda^q \text{pr}_N^* TN) \), which has 0-differential in the \( p \)-direction and differential \( \kappa \wedge \cdot \) in \( q \)-direction. Since \( \text{pr}_N^* TN \) is a trivial vector bundle with fiber dimension \( n - l^\gamma \), the sectional space \( \Gamma^\infty(\Lambda^q \text{pr}_N^* TN) \) is isomorphic to \( \mathbb{C}^\infty(M) \otimes \Lambda^q \mathbb{R}^{n-l^\gamma} \). Together with Eq. (3.24) this implies that \( (\Gamma^\infty(\Lambda^q \text{pr}_N^* TN), \kappa \wedge \cdot) \) is the dual Koszul complex of the algebra \( \mathbb{C}^\infty(M) \) associated to the regular sequence

\[
(\gamma^{-1}x_{l^\gamma+1} - x_{l^\gamma+1}, \ldots, \gamma^{-1}x_n - x_n).
\]  

(3.24)

The cohomology of this dual Koszul complex is well-known (cf. [21, Sec. 17.2]). It does not vanish only for \( q = n - l^\gamma \), where it is given as the quotient of \( \mathbb{C}^\infty(M) \) by the (closed) ideal generated by the regular sequence (3.24), i.e., by the algebra \( \mathbb{C}^\infty(M^\gamma) \).

Using the spectral sequence of the double complex \( D^{p,q} \) one then concludes that

\[
H^k(A,A) \cong H^k(\Gamma^\infty(\Lambda^\bullet TM), \kappa \wedge \cdot) \cong \Gamma^\infty(M^\gamma, \Lambda^{k-n+l^\gamma} TM^\gamma). 
\]  

(3.25)

Using a standard localization argument for Hochschild cohomology (see Remark 3.5), one now infers from this equation and Prop. 3.10 the following result.

**Proposition 3.10.** Let \( \Gamma \) be a finite group acting on a smooth manifold \( M \). Then the Hochschild cohomology \( H^k(A \rtimes \Gamma, A \rtimes \Gamma) \) can be naturally identified as follows with spaces of invariant multivectorfields:

\[
H^k(A \rtimes \Gamma, A \rtimes \Gamma) = \bigoplus_{<\gamma> \in \text{Conj}(\Gamma)} \bigoplus_{M^\bullet \in \text{Comp}(M^\gamma)} \Gamma^\infty(M^\bullet, \Lambda^{k-\dim M^\bullet+n} TM^\bullet)^{Z(\gamma)}, 
\]  

(3.26)

where \( \text{Conj}(\Gamma) \) is the set of conjugacy classes of \( \Gamma \), \( \text{Comp}(M^\gamma) \) the set of connected components of \( M^\gamma \), and \( Z(\gamma) \subset \Gamma \) the centralizer of \( \gamma \) in \( \Gamma \).

**Step 5.** From now on we consider again the general case of a proper étale Lie groupoid and use all the previous results to prove the following main theorem.

**Theorem 3.11.** Let \( G \) be a proper étale Lie groupoid. Then the Hochschild cohomology of the convolution algebra \( \mathcal{A} \rtimes G \) with values in \( \mathcal{A} \rtimes G \) is naturally given as follows:

\[
H^k(\mathcal{A} \rtimes G, \mathcal{A} \rtimes G) \cong \bigoplus_{\mathcal{O} \in \text{Sec}(G)} \Gamma^\infty(\Lambda^{k-\ell(\mathcal{O})} T\mathcal{O}), 
\]  

(3.27)

where the sum is taken over the sectors of \( G \).

**Proof.** Consider the complex \( \mathcal{K}^\bullet \) of sheaves on the orbit space \( X \) constructed in Step 1., and define a second sheaf complex \( \mathcal{H}^\bullet \) on \( X \) (with differential the zero map) as follows:

\[
\mathcal{H}^k(U) := \bigoplus_{\mathcal{O} \in \text{Sec}(G)} \Gamma^\infty(\mathcal{O} \cap \Lambda U_0, \Lambda^{k-\ell(\mathcal{O})} T\mathcal{O}), 
\]

where \( U \) runs through the open subsets of \( X \) and \( \Lambda U_0 := (\pi \circ \beta \circ s)^{-1}(U) \). Observe now that both sheaf complexes \( \mathcal{K}^\bullet \) and \( \mathcal{H}^\bullet \) are fine, since the sheaf of smooth functions on the orbifold \( X \) is fine. Moreover, note that the global section space of the cohomology sheaf of \( \mathcal{K}^\bullet \) is the cohomology we want to compute and that \( \mathcal{H}^\bullet(X) \) is the graded vector space we claim the cohomology to coincide with. Hence, if one
can construct a morphism of sheaf complexes \( \Xi^* : K^* \to \mathcal{H}^* \) which locally is a quasi-isomorphism, the claim is proved by [Sp, Chap. 6, Sec. 8, Thm. 9]. Thus it remains to construct \( \Xi \) and prove that, locally, \( \Xi \) is a quasi-isomorphism. Before we come to the details of the construction we need two lemmas.

**Lemma 3.12.** Assume \( U \subset X \) to be open, let \( U_0 := \pi^{-1}(U) \) and \( U_1 := (\pi \circ s)^{-1}(U) \). Denote by \( G_{U_1} \), the restriction of the groupoid \( G \) to \( U_1 \) and let \( C^\infty_{U_0} \) be the \( G_{U_1}\)-sheaf of smooth functions on \( U_0 \). Then the embedding \( C^\infty_{U_0} \otimes G_{U_1} \hookrightarrow \mathcal{A} \rtimes G \) induces a quasi-isomorphism

\[
K^*(U) \to C^*(C^\infty_{U_0} \otimes G_{U_1}, C^\infty(U_1)).
\]

**(3.28)**

**Proof of the Lemma.** Note first that \( C^*(C^\infty_{U_0} \otimes G_{U_1}, C^\infty(U_1)) \) is the global section space of the sheaf complex \( K_U^* \), which for \( V \subset U \) open and \( k \in \mathbb{N} \) has section space

\[
K^k_U(V) := \text{Hom}( (C^\infty_{U_0} \otimes G_{U_1})^\otimes k, C^\infty(V_1)), \quad V_1 := (\pi \circ s)^{-1}(V),
\]

and which has the Hochschild coboundary as its differential. Note also that \( K_U^* \) and the restriction \( K_U^* \) to \( U \) are both complexes of fine sheaves, since the sheaf of smooth functions on \( X \) is fine. If we can now show that the natural morphism of sheaf complexes \( K_U^* \to K_U^* \) is locally a quasi-isomorphism the claim is proved by [Sp, Chap. 6, Sec. 8, Thm. 9].

To verify this it suffices to check that \( K_U^*(V) \to K_U^*(V) \) is a quasi-isomorphism for every relatively compact connected open subset \( V \subset U \). To this end choose a complete metric \( d_X \) on \( X \) and a complete metric \( d \) on \( G_0 \) (for which \( d^2 \) is smooth) such that \( d(x, y) \geq d_X(\pi(x), \pi(y)) \) for all \( x, y \in G_0 \). By the assumptions on \( V \), there exists an \( \varepsilon > 0 \) such that

\[
d(t(g), t(h)) \geq d_X(\pi(t(g), \pi(t(h))) > \varepsilon \quad \text{for all } g \in V_1 \text{ and } h \in G_1 \setminus U_1.
\]

Moreover, since the preconditions of Step. 2 are satisfied, we have the localization functions \( \Psi_{k, \varepsilon} \) at our disposal. With their help define now for every \( N \in \mathbb{N}^* \) chain maps \( \Theta_{k, \varepsilon}^N : K_U^N(V) \to K_U^N(V) \) between the cut-off chain complexes as follows:

\[
\Theta_{k, \varepsilon}^N(F)(a_0 \otimes \ldots \otimes a_k)(g_0) = F(\Psi_{k, \varepsilon/N}(g_0^{-1}, \ldots, -) \cdot (a_0 \otimes \ldots \otimes a_k))(g_0),
\]

where \( k \leq N, F \in K_U^k(V), a_1, \ldots, a_k \in \mathcal{A} \rtimes G \) and \( g_0 \in V_1 \). By [Sp, 210] one concludes that

\[
\Psi_{k, \varepsilon/N}(g_0^{-1}, g_1, \ldots, g_k) \cdot a_1(g_1) \cdot \ldots \cdot a_k(g_k) = 0,
\]

if \( g_0 \in V_1 \) and \( g_1, \ldots, g_k \in G_1 \) with some \( g_i \in G \setminus U_1 \), hence \( \Theta_{k, \varepsilon}^N(F) \) is well-defined indeed for \( k \leq N \). Prop. 3.2 now entails that \( K_U^*(V) \to K_U^*(V) \) is a quasi-isomorphism in degrees \( k < N \). Since \( N \) was arbitrary, it is a quasi-isomorphism in all degrees, and the claim follows.

Since \( G \) is a proper étale Lie groupoid, there exists for every point \( \hat{g} \in G_0 \) an open contractible neighborhood \( M_{\hat{g}} \subset G_0 \), a smooth action of the isotropy \( G_{\hat{g}} \) on \( M_{\hat{g}} \subset G_0 \) and a monomorphism of groupoids

\[
\iota_{\hat{g}} : G_{\hat{g}} \ltimes M_{\hat{g}} \hookrightarrow G
\]

which induces a Morita equivalence of groupoids from \( G_{\hat{g}} \ltimes M_{\hat{g}} \) to the restricted groupoid \( G_{U_{\hat{g}}, 1} \), where \( U_{\hat{g}} := \pi(M_{\hat{g}}) \) and \( U_{\hat{g}, 1} = (\pi \circ s)^{-1}(U_{\hat{g}}) \).
Lemma 3.13. The Morita equivalence $\iota_{\tilde{x}}$ gives rise to a quasi-isomorphism

$$\iota^{*} : K_{U_{\tilde{x}}}^{*}(U_{\tilde{x}}) \rightarrow C^{*}(C^{\infty}_{c}(M_{\tilde{x}}) \times G_{\tilde{x}}, C^{\infty}(M_{\tilde{x}}) \times G_{\tilde{x}})$$  \hspace{1cm} (3.30)

which associates to every $F \in K_{U_{\tilde{x}}}^{k}(U_{\tilde{x}})$ the cochain

$$\iota^{*}(F) : (C^{\infty}_{c}(M_{\tilde{x}}) \times G_{\tilde{x}})^{\otimes k} \rightarrow C^{\infty}(M_{\tilde{x}}) \times G_{\tilde{x}},$$

where we have put for $a \in C^{\infty}_{c}(M_{\tilde{x}}) \times G_{\tilde{x}}$ and $g \in U_{\tilde{x},1}$

$$\iota_{*}(a)(g) = \begin{cases} a(\iota_{\tilde{x},1}(g)), & \text{if } g \in \text{im}\iota_{\tilde{x}}, \\ 0, & \text{else} \end{cases}$$

Proof of the Lemma. For the proof of the claim we use the language of Hilsum-Skandalis maps (i.e biprincipal bundles) and their associated Morita bimodules as explained in MR. As shown in MR Sec. 1], the Morita equivalence $\iota : G_{\tilde{x}} \times M_{\tilde{x}} \hookrightarrow G_{U_{\tilde{x}}}$ induces a principal $G_{\tilde{x}}$-$G_{\vert U_{\tilde{x},1}}$-bibundle $\langle \iota \rangle$ as follows:

$$\langle \iota \rangle = \{(g, p) \in G_{1} \times M_{\tilde{x}} \mid s(g) = p \} \cong \{g \in G_{1} \mid s(g) \in M_{\tilde{x}}\},$$

$$G_{1} \times G_{0} \langle \iota \rangle \rightarrow \langle \iota \rangle, \quad (g', g) \mapsto g'g,$n

$$\langle \iota \rangle \times M_{\tilde{x}} (G_{\tilde{x}} \times M_{\tilde{x}}) \rightarrow \langle \iota \rangle, \quad (g, \gamma, p) \mapsto g \cdot \iota(\gamma, p).$$

Then, by MR Sec. 2], the locally convex topological vector space $C^{\infty}_{c}((\iota))$ carries the structure of a $(C^{\infty}_{c}(U_{x,0}) \times G_{\vert U_{x,1}})$-$(C^{\infty}_{c}(M_{\tilde{x}}) \times G_{\tilde{x}})$-bimodule and forms a Morita equivalence between these two algebras. Now, the chain map $\iota^{*}$ is induced by this Morita equivalence, as one checks by an immediate but somewhat tedious computation (see LG Sec. 1.2]). Hence, $\iota^{*}$ is a quasi-isomorphism. \hfill \Box

Now we come back to the construction of the morphism of sheaf complexes $\Xi$. Let $U \subset X$ be open, and choose for every $x \in U$ an open neighborhood $U_{x} \subset U$, a point $\tilde{x} \in \pi^{-1}(x)$ together with an open neighborhood $M_{\tilde{x}} \subset G_{0}$, an action of the isotropy group $G_{\tilde{x}}$ on $M_{\tilde{x}}$ and, finally, a Morita equivalence of Lie groupoids

$$\iota_{\tilde{x}} : G_{\tilde{x}} \times M_{\tilde{x}} \rightarrow G_{\vert U_{x,1}}.$$ 

Then one has for every one of the $U_{x}$ a sequence of natural chain maps:

$$K^{*}(U) \rightarrow K^{*}(U_{x}) \xrightarrow{\delta_{1}} K_{U_{\tilde{x}}}^{*}(U_{\tilde{x}}) \xrightarrow{\iota_{\tilde{x}}} C^{*}(C^{\infty}_{c}(M_{\tilde{x}} \times G_{\tilde{x}}), C^{\infty}(M_{\tilde{x}}) \times G_{\tilde{x}}) \rightarrow$$

$$C^{*}(C^{\infty}_{c}(M_{\tilde{x}}), C^{\infty}(M_{\tilde{x}}) \times G_{\tilde{x}}) \rightarrow \Gamma^{\infty}(\Lambda^{*}TM_{\tilde{x}}) \rightarrow$$

$$\bigoplus_{\gamma \in G_{\tilde{x}}, \Gamma_{\tilde{x},\alpha} \in \text{Comp}(M_{\tilde{x}})} \Gamma^{\infty}(\Lambda^{\ast-\dim G+\dim M_{\tilde{x}},\alpha}T_{M_{\tilde{x}},\alpha}^{\gamma}M_{\tilde{x}}) \rightarrow$$

$$\bigoplus_{<\gamma> \in \text{Conj} G_{\tilde{x}}, \Gamma_{\tilde{x},\alpha} \in \text{Comp}(M_{\tilde{x}})} \Gamma^{\infty}(\Lambda^{\ast-\dim G+\dim M_{\tilde{x}},\alpha}T_{M_{\tilde{x}},\alpha}^{\gamma}M_{\tilde{x}})$$

where the arrow in the last line is the projection onto the invariant part obtained by averaging over $G_{\tilde{x}}$. By naturality of all constructions involved one checks that
for all \( x, y \in U \) the following diagram commutes:

\[
\begin{array}{ccc}
K^\bullet(U) & \longrightarrow & H^\bullet(U_y) \\
\downarrow & & \downarrow \\
H^\bullet(U_x) & \longrightarrow & H^\bullet(U_x \cap U_y).
\end{array}
\]

Hence, by the sheaf property of \( H^\bullet \) one can glue together these maps to a chain map \( \Xi(U) : K^\bullet(U) \rightarrow H^\bullet(U) \). By construction, the \( \Xi(U) \) commute with restriction maps, hence one obtains a morphism of sheaf complexes \( \Xi : K^\bullet \rightarrow H^\bullet \). By the above lemmas and Steps 3. and 4. it is clear that for every \( x \in U \) the chain map \( \Xi(U_x) : K^\bullet(U_x) \rightarrow H^\bullet(U_x) \) has to be a quasi-isomorphism. This finishes the last part of the proof and thus entails the claim. □

4. Noncommutative Poisson homology

This section is divided into two parts. In the first part, we introduce a Poisson homology for a noncommutative Poisson algebra. In the second part, we calculate this homology for the Poisson algebra constructed from a Poisson structure on a proper étale Lie groupoid (recall Sec. 2.7 for definitions).

4.1. Poisson homology. In [BR], Brylinski defined Poisson homology on a Poisson manifold \((M, \Pi)\) as the homology of the complex \( b_{\Pi} : \Omega^\bullet(P) \rightarrow \Omega^\bullet(P) \), where

\[
b_{\Pi}(f_0 df_1 \wedge df_2 \wedge \cdots \wedge df_k) = \sum_{j=1}^{k} (-1)^{j-1} d(f_0, f_j) df_1 \wedge \cdots \wedge df_j \wedge \cdots \wedge df_k + \\
+ \sum_{i<j} (-1)^{i+j-1} f_0 df_i \wedge df_1 \cdots \wedge df_i \wedge df_j \cdots \wedge df_k.
\]

\[\quad \tag{4.1}\]

In the following, we define a noncommutative analog of this Poisson homology. Like in the manifold case, we start from a Poisson structure \([\Pi] \in HH^2(A, A)\) of the algebra \( A \) and let it act on \( HH^\bullet(A) \), the noncommutative analog of differential forms. Before we introduce the precise definition, we first recall a well-known action of the Hochschild cochains on Hochschild chains.

\[\text{Definition 4.1.} \quad \text{For an element } \varphi \in C^k(A, A), \text{ define } d_\varphi : C_n(A) \rightarrow C_{n-k+1}(A) \text{ by } d_\varphi(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \]

\[\sum_{i=0}^{n-k+1} (-1)^i(k-1)a_0 \otimes \cdots \otimes \varphi(a_i \otimes \cdots \otimes a_{i+k-1}) \otimes \cdots \otimes a_n + \\
+ \sum_{i=2}^{k} (-1)^{(n-k+i)(k-i+1)} \varphi(a_{n-k+i} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}) \otimes \cdots \otimes a_{n-k+i-1}.\]

\[\text{Theorem 4.2.} \quad \text{For } \phi \in C^k(A, A), \psi \in C^l(A, A), \text{ we have } d_\phi \circ d_\psi - (-1)^{k-1(l-1)}d_\psi \circ d_\phi = d_{[\phi, \psi]} .\]
Proof. We prove this property on $a_0 \otimes a_1 \otimes \cdots \otimes a_n$.

(1) \(d_\phi \circ d_\psi (a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \)

\[= \sum_{i=0}^{n-l} (-1)^{(l-1)} d_\phi (a_0 \otimes \cdots \otimes \psi (a_i \otimes a_{i+1} \otimes \cdots \otimes a_{i+l-1}) \otimes \cdots) + \]

\[+ \sum_{i=1}^{l} (-1)^{(n-i+1)(l-1)} d_\psi (a_{n-l+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}) \otimes a_{i-1} \otimes \cdots \otimes a_{n-l+i-1}) \]

\[= \sum_{i=0}^{n-l} (-1)^{(l-1)} \left( \sum_{j=0}^{l} (-1)^{(l-1)} \right) \]

\[\left( a_0 \otimes \cdots \otimes \psi (a_i \otimes \cdots \otimes a_{j+l-1}) \otimes \cdots \otimes a_{i+l-1} \otimes \cdots) \right) + \]

\[+ \sum_{i=1}^{l} (-1)^{(n-i+1)(l-1)} \psi (a_{n-l+i} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}) \otimes a_{i-1} \otimes \cdots \otimes \phi (a_j \otimes \cdots \otimes a_{i+k-1}) \otimes \cdots) + \]

\[+ \sum_{j=i+k+1}^{n} (-1)^{(l+k-1)(k-1)} \psi (a_{n-l+i} \otimes \cdots \otimes a_{n+k-1} \otimes \cdots) \right) \]

\[= \sum_{i=0}^{n-l} (-1)^{(l-1)} \sum_{j=0}^{l} (-1)^{(l-1)} \]

\[\left( a_0 \otimes \cdots \otimes \psi (a_i \otimes \cdots \otimes a_{j+l-1}) \otimes \cdots \otimes a_{i+l-1} \otimes \cdots) \right) + \]

\[+ \sum_{i=1}^{l} (-1)^{(n-i+1)(l-1)} \psi (a_{n-l+i} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}) \otimes a_{i-1} \otimes \cdots \otimes \phi (a_j \otimes \cdots \otimes a_{i+k-1}) \otimes \cdots) + \]

\[+ \sum_{j=i+k+1}^{n} (-1)^{(l+k-1)(k-1)} \psi (a_{n-l+i} \otimes \cdots \otimes a_{n+k-1} \otimes \cdots) \right) \]

(2) \(d_\psi \circ d_\phi (a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \)

\[= \sum_{i=0}^{n-k} (-1)^{(k-1)} \sum_{j=0}^{l} (-1)^{(l-1)} \]

\[\left( a_0 \otimes \cdots \otimes \psi (a_i \otimes \cdots \otimes a_{j+l-1}) \otimes \cdots \otimes a_{n+k-1} \right) \]

\[+ \sum_{j=i+k+1}^{n} (-1)^{(l+k-1)(k-1)} \psi (a_{n-l+i} \otimes \cdots \otimes a_{n+k-1} \otimes \cdots) \right) \]

(3) \(d_\phi \circ d_\psi - (-1)^{(k-1)(l-1)} d_\psi \circ d_\phi. \)

Note that there are two types of terms:

(a) \(\cdots \otimes \phi (\cdots) \otimes \cdots \otimes \psi (\cdots) \otimes \cdots \)

or \(\cdots \otimes \psi (\cdots) \otimes \cdots \otimes \phi (\cdots) \otimes \cdots \).

(b) \(\cdots \otimes \psi (\cdots) \otimes \cdots \)

or \(\cdots \otimes \phi (\cdots) \otimes \cdots \).

For type (a), they appear in both \(d_\phi \circ d_\psi\) and \(d_\psi \circ d_\phi\), and differ by the sign \((-1)^{(k-1)(l-1)}\). Therefore, they get cancelled in the sum. For type (b), terms like

\[\cdots \otimes \phi (\cdots) \otimes \cdots \]

appear in \(d_\phi \circ d_\psi\), while terms like

\[\cdots \otimes \psi (\cdots) \otimes \cdots \]
appear in \( d_\phi \circ d_\psi \). They come out in pairs. It is straightforward to check that their signs match with those of \( d_{[\phi, \psi]} \). Therefore, we conclude that
\[
d_\phi \circ d_\psi - (-1)^{(k-1)(l-1)} d_\psi \circ d_\phi = d_{[\phi, \psi]}.
\]

\[\square\]

**Remark 4.3.** Definition 4.1 is a special case of a general theory of Nest and Tsygan [NeTs99] of operations on Hochschild and cyclic complexes. Moreover, Theorem 4.2 shows that by Definition 4.1 one obtains an \( L_\infty \)-module structure for the differential graded Lie algebra \( (C^\bullet(A, A), \beta, [\ , \ ] ) \) and also a Lie module structure on \( HH_\bullet(A) \) for the super Lie algebra \( (H^\bullet(A, A), [\ , \ ] ) \).

On an associative algebra \( A \), there is a natural 2-cocycle \( m \) associated to the multiplication defined by \( m(a_1 \otimes a_2) = a_1 a_2 \). It is easy to check that \( d_m \) is the Hochschild differential on \( C^\bullet(A) \), so we will simply write \( b \) instead of \( d_m \).

By taking \( \phi = m, \psi = \Pi \), where \( \Pi \in Z_2(A, A) \) is a representative of the Poisson structure \( \pi \), Theorem 4.2 now gives
\[
b \circ d_\Pi + d_\Pi \circ b = d_{[\phi, \psi]} = 2d_\Pi \circ d_\Pi = d_{\beta(\Theta)} = b \circ d_\phi + d_\phi \circ b.
\]
This proves that \( d_\Pi^2 = 0 \) in \( HH_\bullet(A) \). Finally, we obtain for \( \phi = m, \psi = \beta(\eta) \)
\[
b \circ d_\eta - d_\eta \circ b = d_{[\phi, \eta]} = d_{[\phi, \psi]} = d_\psi,
\]
which shows that two representatives of the Poisson structure define the same homology in \( HH_\bullet(A) \). In other words this means that noncommutative Poisson homology defined below depends only on the Poisson structure \( [\Pi] \in H^2(A, A) \) and not on the particular choice of a representative \( \Pi \in Z^2(A, A) \).

**Definition 4.4.** For a noncommutative Poisson structure \( [\Pi] \) on an associative algebra \( A \), its Poisson homology is defined as the homology of the differential complex \( d_\Pi : HH_\bullet(A) \to HH_{\bullet-1}(A) \), where \( \Pi \) is a cocycle representing the Poisson structure.

In the case of a Poisson manifold \( M \), our definition of the Poisson homology on the Hochschild homology of \( C^\infty_c(M) \) is compatible with the one defined on differential forms by Brylinski.

**Proposition 4.5.** Let \( M \) be a Poisson manifold with Poisson bivector \( \Pi \), and \( A_c = C^\infty_c(M) \) the algebra of compactly supported smooth functions together with the noncommutative Poisson structure induced by \( \Pi \). Then the following diagram commutes:
\[
\begin{array}{ccc}
HH_k(A_c) & \xrightarrow{\epsilon_k} & \Omega^k_c(M) \\
\downarrow{d_\Pi} & & \downarrow{2(k-1)!\pi}\n\\
HH_{k-1}(A_c) & \xrightarrow{\pi_{k-1}} & \Omega^{k-1}_c(M)
\end{array}
\]
where $b_{1}$ is Brylinski’s Poisson differential, $\epsilon_{k}$ is the antisymmetrization map defined as follows

$$\epsilon_{k}(f_{0}df_{1} \wedge \cdots \wedge df_{k}) := \sum_{\sigma \in S_{k}} \text{sgn}(\sigma) f_{0} \otimes f_{\sigma^{-1}(1)} \otimes f_{\sigma^{-1}(2)} \otimes \cdots \otimes f_{\sigma^{-1}(k)},$$

and $\pi_{k-1}$ is the projection defined by

$$\pi_{k-1}(f_{0} \otimes f_{1} \otimes \cdots \otimes f_{k-1}) := f_{0}df_{1} \wedge \cdots \wedge df_{k-1}.$$

Since $\epsilon_{k}$ resp. $\pi_{k}$ gives rise to an isomorphism between $HH_{k}(A_{c})$ and $\Omega_{c}^{k}(M)$, these maps also induce an isomorphism between the Poisson homologies by the above diagram.

Proof. See Theorem 3.1.1 in [BR] and also [BRGE]. □

Morita equivalence is an important notion in the study of algebras by methods of noncommutative geometry. In the rest of this section, we will briefly look at Morita invariance of Poisson homology. With respect to Poisson geometry, there exists quite some work on the invariance of Poisson (co)homology under (weak) Morita equivalences between Poisson manifolds (cf. [Xu]). In this paper we will now consider algebraic versions of Morita invariance within noncommutative Poisson homology.

It is well-known that Morita equivalent algebras have isomorphic Hochschild (co)homologies (see [Lo] 1.2.4, 1.2.7 and 1.5.6).

**Proposition 4.6.** A Morita equivalence bimodule between algebras with local units $A$ and $B$ defines an isomorphism between the sets of Poisson structures.

Proof. The Hochschild cohomology $H^{k}(A, A)$ is isomorphic to $Ext_{A}^{k}(A, A)$. In [Kë], Keller shows that for derived equivalent algebras (which is more general than Morita equivalence), the canonical isomorphism defined by tensoring an extension by the equivalent bimodule preserves the corresponding $G$-brackets. This result implies that a Morita equivalence bimodule between $A$ and $B$ defines an isomorphism between $H^{\bullet}(A, A)$ and $H^{\bullet}(B, B)$ as Lie algebras, which induces an isomorphism on the sets of corresponding Maurer-Cartan elements. We know that in $H^{\bullet}(A, A)$ and $H^{\bullet}(B, B)$ Maurer-Cartan elements are Poisson structures. Therefore, we have isomorphic sets of Poisson structures. □

**Proposition 4.7.** Under the isomorphism between the Hochschild cohomologies of Prop. 4.6, the corresponding noncommutative Poisson structures have isomorphic Poisson homologies.

Proof. Consider the following diagram

$$
\begin{array}{ccc}
HH_{\bullet}(A) & \xrightarrow{d_{A}^{\bullet}} & HH_{\bullet-1}(A) \\
\downarrow{\sigma_{\bullet}} & & \downarrow{\sigma_{\bullet-1}} \\
HH_{\bullet}(B) & \xrightarrow{d_{B}^{\bullet}} & HH_{\bullet-1}(B)
\end{array}
$$

where $\sigma_{\bullet}$ is an isomorphism constructed by a Morita equivalence. The claim of the proposition says that the above diagram commutes. The proof of this goes along the same lines as the proof of [LO] (1.2.7). Instead of working out the general case,
we will only look at the special case where $B = M_n(A)$, the $n \times n$ matrix algebra of $A$.

For $A$ and $M_n(A)$, following (1.2.4) of Loday [LO], we define $\sigma_k : HH_k(A) \to HH_k(M_n(A))$ as follows (where $E^n_{i1}$ denotes the matrix with $a$ at the $(1,1)$ position and 0 elsewhere)

$$\sigma_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k) := E^n_{i1} \otimes E^n_{ij} \otimes \cdots \otimes E^n_{i1},$$

and a generalized trace map $\tau : HH_k(M_n(A)) \to HH_k(A)$ by

$$\tau(a_0 \otimes a_1 \otimes \cdots \otimes a_k) := (a_0)_{i0i1} \otimes (a_1)_{i1i2} \otimes \cdots \otimes (a_k)_{i_ki0}.$$

It is shown in (1.2.4) of [LO] that both $\tau$ and $\sigma_k$ induce isomorphisms in Hochschild homology, and that their dual versions give rise to isomorphisms in cohomology. Under the corresponding isomorphisms in Hochschild cohomology, a Poisson structure $\Pi$ on $A$ is transformed to $\tilde{\Pi}$, a Poisson structure on $M_n(A)$, by the following formula

$$\tilde{\Pi}(a, b) = \left( \sum_{l} \Pi(a_{il}, b_{lj}) \right)_{ij}.$$

One can easily show that $\tilde{\Pi}$ is a Poisson structure on $M_n(A)$ indeed.

Since $\sigma$ and $\tau$ are inverse to each other, the claim is proved, if one can show that $d\Pi = \tau \circ d\tilde{\Pi} \circ \sigma$. But this formula is obvious from the definition of $\tilde{\Pi}$.

**Remark 4.8.** To prove Morita invariance for noncommutative Poisson structures in general, one has to construct chain homotopies which entail the above diagram to be commutative. The corresponding constructions are similar to those of (1.2.7) in [LO].

**4.2. Poisson homology of the noncommutative Poisson algebra.** Assume to be given a proper étale Lie groupoid $G$ together with an invariant Poisson bivector. For $A$ the $G$-sheaf of smooth functions we compute in this part the Poisson homology of the noncommutative Poisson algebra.

**Definition 4.9.** Let $(X_\bullet, b, d)$ be a triple consisting of a graded vector space $X_\bullet = \bigoplus_{k \in \mathbb{N}} X_k$ and two homogeneous maps $b : X_\bullet \to X_\bullet_{\pm1}$, $d : X_\bullet \to X_\bullet_{\pm1}$, both either of degree $+1$ or $-1$. Then $(X_\bullet, b, d)$ is called an almost bicomplex, if the following relations hold true for some $h : X_\bullet \to X_\bullet_{-1}$ (resp. some $h : X_\bullet \to X_\bullet_{+1}$):

$$b^2 = 0, \quad db + bd = 0, \quad d^2 = bh + hb.$$

An almost bicomplex $(X_\bullet, b, d)$ gives rise to two complexes $H^b_c(X)$ and $H^d_c(X)$, where $H^b_c(X)$ is defined by $(X_\bullet, b)$, while $H^d_c(X)$ is defined by $(X_\bullet, d)$.

**Example 4.10.** If $A$ is an algebra with a Poisson structure induced by a Hochschild cocycle $\Pi$, and $d\Pi$ is the differential of Poisson homology as defined above, then the triple $(C_\bullet(A), b, d\Pi)$ is an almost bicomplex.

In the following computations, we will frequently use the next result.

**Lemma 4.11.** Assume to be given two almost bicomplexes $(X_\bullet^i, b^i, d^i)$, $i = 1, 2$ and a quasiisomorphism $\Psi$ between $(X_\bullet^1, b^1)$ and $(X_\bullet^2, b^2)$. If $\Psi$ commutes with $d^i$ on the homologies $H^i_c(X_\bullet^i, b^i)$, i.e. if $\Psi d^i = d^i \Psi$, then $\Psi$ induces an isomorphism between $(H^i_c(X_\bullet^1, d^1))$ and $(H^i_c(X_\bullet^2, d^2))$. 

Proof. The proof of this lemma is obvious, since $\Psi$ is an isomorphism between the homologies and commutes with the differentials $d^i$.

Remark 4.12. Of course we can allow for more general morphisms between bicomplexes to induce isomorphisms on Poisson homology. In particular, the quasi-isomorphism $\Psi$ in the lemma is allowed to “commute up to homotopy”, i.e., $d^1 \Psi = \Psi d^2 + b^1 H + H b^2$, for some $H : X^1 \to X^2_\bullet$. The main difficulty of the computation below is to show that the “reduction to loops” morphism \((2.11)\) which computes the Hochschild homology of the convolution algebra is a morphism of this kind: since the morphism does not preserve the Poisson differential, a homotopy as above is required.

With the above preparations, we are now ready to determine the Poisson homology on the convolution algebra of the proper étale groupoid $G$. Our strategy is to track the change of the Poisson differential in the various steps of the computation.

Recall that $HH_\bullet(A \times G)$ is calculated by the Bar complex \((\langle A \times G \rangle^2_k, b)\) the components of which are isomorphic to the vector spaces $\Gamma_c(G^{k+1}; s_k^+ A^{2(k+1)})$.

Under these isomorphisms, the Poisson differential on \((\langle A \times G \rangle^2_k\rangle\) has the following form:

$$d^1_\Pi := \sum_{i=0}^k (-1)^i d^i_{1\Pi},$$

where

$$d^i_{1\Pi}(a_0, \ldots, a_k) := \begin{cases} (a_0 \otimes \cdots \otimes \Pi(a_i, a_{i+1}) \otimes \cdots \otimes a_k), & \text{if } 0 \leq i \leq k - 1, \\ (\Pi(a_k, a_0) \otimes a_1 \otimes \cdots \otimes a_{k-1}), & \text{if } i = k. \end{cases}$$

We now proceed in three major steps.

Step I. Reduction to loops. Recall from the Preliminaries (Sec. 2.6 Step I) the method of reduction to loops for the computation of Hochschild and cyclic homology of an étale groupoid. This method shows that the cyclic vector space \((\langle A \times G \rangle^2_k\rangle\) is quasi-isomorphic to $\Gamma_c \Lambda^k A$ via the natural restriction $p : (\langle A \times G \rangle^2_k\rangle \to \Gamma_c \Lambda^k A$. Now, one observes that $p$ naturally induces a Poisson differential $d^1_\Pi$ on $\Gamma_c \Lambda^k A$ by putting

$$d^1_\Pi(p(a)) := p(d^1_\Pi(a)) \quad \text{for all } a \in (\langle A \times G \rangle^2_k\rangle.$$

Note that $d^1_\Pi$ is well-defined indeed, since $p(a)$ and $p(d^1_\Pi(a))$ are, respectively, the germs of $a$ and $d^1_\Pi(a)$ on $B^{(k)}$. Using Lemma 4.11 we now conclude that the homology of $d^1_\Pi$ on $HH_\bullet(\Gamma_c \Lambda^k A)$ is equal to the homology of $d^1_\Pi$ on $HH_\bullet(A \times G)$.

Step II. Homology of the cyclic groupoid. Recall from the proof of Prop. 2.10 in the Preliminaries that the $\infty$-cyclic vector spaces $\Gamma_c \Lambda^k A$ and $\Gamma_\bullet(A \equiv \theta, \Lambda^k_{tw})$ are isomorphic, with isomorphism over the stalk at $(g_0, \ldots, g_k) \in B^{(0)}$ given by Eq. 2.13. By this isomorphism, the Poisson differential $d^1_\Pi$ gives rise to a noncommutative Poisson differential $d^1_{tw}$ on $(A^{2}_{tw}, b_{tw})$, by which we can define a Poisson homology. The explicit formulas (with $(g_0, g_1, \ldots, g_k) \in \Lambda G_k$)

$$(d^1_{tw})_{i}(a_0 \otimes \cdots \otimes a_k)(g_0, g_1, \ldots, g_k) =$$

$$= \begin{cases} \{ a_0, a_1 \} g_1 \otimes a_2 g_1 \otimes \cdots \otimes a_k g_1(g_{0, g_1, g_2, \ldots, g_k}^{-1}), & \text{for } i = 0, \\ a_0 \otimes \cdots \otimes \{ a_i, a_{i+1} \} \otimes \cdots \otimes a_k(g_{0, 1, \ldots, g_i, g_{i+1}, \ldots, g_k}), & \text{for } 1 \leq i \leq k - 1, \\ \{ a_k, a_0 \} \otimes a_1 \otimes \cdots \otimes a_{k-1}(g_{0, g_1, \ldots, g_{k-1}}), & \text{for } i = k, \end{cases}$$

where $\{ , , \}$ is the Poisson bracket on $G_0$. 

Since the above isomorphism is a local diffeomorphism which maps Poisson structures naturally, we conclude by Lemma [4.14] that \((\Gamma_\bullet(\Lambda G, \theta, \mathcal{A}^\ell_{tw}), b_{tw}, d^w_{tw})\) calculates the Poisson homology of \(A \rtimes G\). We define the Poisson homology \(H^\Pi_\bullet(\Lambda G)\) as the homology of \(d^w_{tw}\) on \(HH^\bullet(\Lambda G, \theta, \mathcal{A}^\ell_{tw}) = H^\bullet_\bullet(\Gamma_\bullet(\Lambda G, \theta, \mathcal{A}^\ell_{tw}), b_{tw})\).

From the above considerations one can now immediately derive the following localization property similarly to the corresponding one for Hochschild homology (cf. [7,11,13,14]):

**Theorem 4.13.** Let \(B^{(0)} = \bigcup_{\mathcal{O} \in \text{Sec}(G)} \mathcal{O}\) be the decomposition of \(B^{(0)}\) into sectors. Then

\[
H^\Pi_\bullet(A \rtimes G) = \bigoplus_{\mathcal{O} \in \text{Sec}(G)} H^\Pi_\bullet(A \rtimes G)_\mathcal{O}.
\]

**Step III. Inertia groupoid.** Since the groupoid is proper, one knows that the Poisson structure on \(G_0\) defines a natural invariant Poisson structure on \(B^{(0)}\), which gives rise to an invariant Poisson bivector \(\Pi\) on \(NG_0\) (see [10,11,13] for a detailed proof in the general case and Lemma 5.4 in the following section for groupoids with a symplectic structure). Now, over an invariant open-closed subset \(\mathcal{O}\) of \(NG_0\), we consider \(C^\infty_\mathcal{O}\), the sheaf of smooth functions, together with the Hochschild differential \(d\) and the Poisson differential \(d_{tw}\) defined by the Poisson structure \(\Pi\). It is straightforward to check that \(((C^\infty_\mathcal{O})^\bullet, b, d_{tw})\) forms an almost bicomplex. We define the Poisson homology \(H^\Pi_\bullet(NG_\mathcal{O})\) to be the homology of \(d_{tw}\) on \(HH^\bullet(NG, (C^\infty_\mathcal{O})^\bullet)\). By Proposition 4.3 and Lemma 4.4 we conclude that \(((C^\infty_\mathcal{O})^\bullet, b, d_{tw})\) is quasi-isomorphic (as an almost bicomplex) to the bicomplex \((\Omega^\bullet_\mathcal{O}, 0, b_{tw})\), hence \(H^\Pi_\bullet(NG_\mathcal{O})\) is given by the homology of \(b_{tw}\) on \(H^\bullet_\bullet(NG, \Omega^\bullet_\mathcal{O})\), which we will denote by \(H^\Pi_\bullet(NG)_\mathcal{O}\).

**Theorem 4.14.** For an invariant open-closed subset \(\mathcal{O} \subset B^{(0)}\), one has

\[
H^\Pi_\bullet(A \rtimes G)_\mathcal{O} = H^\Pi_\bullet(NG)_\mathcal{O} = H^\Pi_\bullet(NG)_\mathcal{O}.
\]

**Proof.** The second equality in the claim has been shown above, so it remains to prove the first one. To this end recall first the twisted Hochschild–Kostant–Rosenberg Theorem [13,14,15] which entails that the natural restriction of the germ of a smooth function to \(\mathcal{O}\) induces a quasi-isomorphism \(\rho : (\mathcal{A}^\ell_{\mathcal{O}}, b_{tw}) \to ((C^\infty_\mathcal{O})^\bullet, b)\). Below, we will show that via \(\rho\) one can pushforward \(d^w_{tw}\) to a Poisson differential \(d'_{tw}\) on \(HH^\bullet(NG, (C^\infty_\mathcal{O})^\bullet)\), which then calculates the Poisson homology of the convolution algebra \(A \rtimes G\). Moreover, we will show that \(d'_{tw}\) is equal to \(d_{tw}\) on \(HH^\bullet(NG, (C^\infty_\mathcal{O})^\bullet)\). This will prove the claim. Note that the problem one has to cover here is the fact that due to the existence of normal directions, \(\rho\) does not induce a Poisson map from \((A(\mathcal{O}), \Pi)\) to \((C^\infty_\mathcal{O}(\mathcal{O}), \Pi_\mathcal{O})\).

Let us now construct \(d'_{tw}\) in detail. For all \(a \in \Gamma_\bullet(NG, (C^\infty_\mathcal{O})^\bullet)\) with \(b(a) = 0\) one can find an \(x \in \Gamma_\bullet(\Lambda G, \mathcal{A}^\ell_{\mathcal{O}})\) with \(\rho(x) = a\) and \(b_{tw}(x)|_\mathcal{O} = 0\). We define \(d'_{tw}\) on \(a\) by

\[
d'_{tw}(a) := \rho(d^w_{tw}(x)).
\]

The following is a list of properties of \(d'_{tw}\):

1. Since \(b_{tw}(x)|_\mathcal{O} = 0\), we have that \(d^w_{tw}(b_{tw}(x))\) vanishes on \(\mathcal{O}\) as well, hence one has \(\rho(d^w_{tw}(b_{tw}(x))) = 0\). This implies the following equality:

\[
b(d'_{tw}(a)) = b(\rho(d^w_{tw}(x))) = b(b_{tw}(d^w_{tw}(x))) = -\rho(d^w_{tw}(b_{tw}(x))) = 0,
\]
Proof of the Lemma. Since we have used that \( \rho \) commutes with \( b \), and in the third one we have used that \( d_{II}^{tw} \) anti-commutes with \( b_{tw} \).

(ii) Since \( \rho \) is an isomorphism on the Hochschild homology, we can choose for any \( [a] \in HH_\bullet(NG, (C^\infty)^2) \) a representative \( a \in \Gamma_\bullet(NG, (C^\infty)^2) \) such that there is \( x \in \Gamma_\bullet(AG, A^2_O) \) satisfying \( \rho(x) = a \) and \( b_{tw}(x) = 0 \). Thus, we obtain

\[
\begin{align*}
b_{tw}(d_{II}^{tw}(x)) &= -d_{II}^{tw} (b_{tw}(x)) = 0.
\end{align*}
\]

Hence, by construction,

\[
d_{II_N} \circ d_{II_N}(a) = \rho(d_{II}^{tw}(d_{II}^{tw}(x))) = \rho(b_{tw}(h(x)) + b_{tw}(x))) = b(h(x)),
\]

where \( h \) is the homotopy associated to \( (d_{II}^{tw})^2 \). This shows that \( d_{II_N} \circ d_{II_N} = 0 \) in \( HH_\bullet(NG, (C^\infty)^2) \).

(iii) To prove that \( d_{II_N} \) is well defined, we need to show that our definition is independent of the choices of \( a \) and \( x \). We will use the following lemmas:

**Lemma 4.15.** Let \( y \in \Gamma_\bullet(AG, A^2_O) \) with \( \rho(y) = 0 \) and \( b_{tw}(y) = 0 \). Then there exists \( z \in \Gamma_\bullet(AG, A^2_O) \) such that \( y = b_{tw}(z) \).

**Proof of the Lemma.** Since \( \rho \) is a quasi-isomorphism and \( \rho(y) = 0 \), \( y \) has to be a boundary in \( \Gamma_\bullet(AG, A^2_O) \). Therefore, there is \( z \in \Gamma_\bullet(AG, A^2_O) \) such that \( y = b_{tw}(z) \). \( \square \)

By the lemma one now concludes that for any \( x, y \in \Gamma_\bullet(AG, A^2_O) \) with \( \rho(x) = \rho(y) = a \) and \( b_{tw}(x) = b_{tw}(y) = 0 \) there exists \( z \) in \( \Gamma_\bullet(AG, A^2_O) \), such that \( x - y = b_{tw}(z) \). Therefore,

\[
\begin{align*}
\rho(d_{II}^{tw}(x)) &= \rho(d_{II}^{tw}(y)) + \rho(d_{II}^{tw}(x - y)) \\
&= \rho(d_{II}^{tw}(y)) + \rho(d_{II}^{tw}(b_{tw}(z))) \\
&= \rho(d_{II}^{tw}(y)) + \rho(d_{II}^{tw}(b_{tw}(z))) \\
&= \rho(d_{II}^{tw}(y)) - \rho(b_{tw}(d_{II}^{tw}(z))) \\
&= \rho(d_{II}^{tw}(y)) - b(\rho(d_{II}(z))).
\end{align*}
\]

Hence, the homology class of \( d_{II_N}(a) \) is independent of the lift of \( a \). To show that it is also independent of the representative \( a \) in the homology class \( [a] \), we prove the following proposition.

**Lemma 4.16.** \( d_{II_N} \) is equal to the Poisson differential \( d_{II_N} \) on \( HH_\bullet(NG, (C^\infty)^2) \).

**Proof of the Lemma.** For any \( a \in \Gamma_\bullet(NG, (C^\infty)^2) \) with \( b(a) = 0 \) we construct a particular lift \( x \) in \( \Gamma_\bullet(AG, (A^2_O)) \), such that \( b_{tw}(x) = 0 \). To achieve this, recall that \( O \) is embedded in \( G^{(k+1)} \) with normal bundle being the set of all nontrivial representations of \( G \) on \( (TG^{(k+1)})_\mathbb{O} \). By the tubular neighborhood theorem, one can find a function \( x \in C^\infty_c(G^{(k+1)}) \) which is equal to the pull back of \( a \) in a tubular neighborhood of \( O \). Hence, \( b(x) = 0 \) in the tubular neighborhood, and \( d_{II_N}(x)_O \) is equal to \( d_{II_N}(a) \). Therefore, we have \( d_{II_N}(a) = d_{II_N}(x)_O = d_{II_N}(a) \). \( \square \)

By Lemma 4.16 and the fact that \( d_{II_N} \) is well-defined on \( HH_\bullet(NG, (C^\infty)^2) \), we obtain that \( d_{II_N} \) acts on \( HH_\bullet(NG, (C^\infty)^2) \) independent of the choice of representatives \( a \). In other words, \( d_{II_N} \) is well-defined on \( HH_\bullet(NG, (C^\infty)^2) \) and equal to \( d_{II_N} \). Altogether, this finishes the proof. \( \square \)
Remark 4.17. Lemma 4.11 now entails that we can use $\left(\Gamma \bullet (NG, (C^\infty_C)^\lambda), b, d_{ab}\right)$ to calculate the Poisson homology of the localized convolution algebra $A \rtimes G$.

5. Hochschild and cyclic homology of the quantized algebra

In this section we present the computation of Hochschild and cyclic homology of a formal deformation quantization of the convolution algebra on a proper étale Lie groupoid $G$ representing a symplectic orbifold $X$. By $\omega$ we denote the symplectic form on $G_0$, and by $A_0$, as before, the $G$-sheaf of smooth functions on $G_0$. The deformation quantization is constructed as explained in Section 2.7. We choose a $G$-invariant star product $\star$ on the sheaf $A_{[[\hbar]]}$ and to the deformed global crossed product algebra $A_{\hbar} \rtimes G$ as in Eq. (2.6).

5.1. Periodic cyclic homology.

The computation of the periodic cyclic homology groups of $A_{\hbar} \rtimes G$ follows at once from the "classical" computations of the periodic cyclic homology of étale groupoids in [BrNi, Cr] by the following rigidity property [Ge, NeTs95, Thm. A2.2]: For any formal deformation quantization $A_{\hbar} := (A[[\hbar]], \star)$ of an algebra $A$, one has an isomorphism

$$HP_\bullet(A_{\hbar}) \cong HP_\bullet(A) \otimes \mathbb{C}[[\hbar]].$$

Therefore, one easily finds

**Proposition 5.1.** The periodic cyclic homology groups of $A_{\hbar} \rtimes G$ are given by:

$$HP_\bullet(A_{\hbar} \rtimes G) = \prod_k H^{2k+\bullet}_{n,c}(X, \mathbb{C}[[\hbar]]).$$

A similar rigidity property of (algebraic) $K$-theory was proved in [Ro]. Recall that the Chern–Connes character maps $K^{alg}_*(A \rtimes G)$ to $HP_\bullet(A \rtimes G)$. The two rigidity isomorphisms are compatible with this character map.

5.2. Computation of Hochschild homology.

The computation of Hochschild and cyclic homology is more involved. The main tools in the computation are:

1) The "quantum to classical" spectral sequence induced by the $\hbar$-adic filtration introduced in [BrGe].

2) The "classical" computation of cyclic homology of étale groupoids of [BrNi, Cr] using the language of sheaves. The computation exactly follows the steps of these computations: we first “localize” to a sheaf cohomology computation on the inertia groupoid, and then use the $\hbar$-filtration to reduce the outcome to orbifold cohomology.

In the following, we will occasionally work over the field $\mathbb{C}((\hbar))$, so let us put $A^{((\hbar))} := A_{\hbar} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]^{-1}]$. Then one has $A^{((\hbar))} \in \mathcal{S}(G)$ as well. For the following, notice that when $G$ has an invariant symplectic form $\omega$, the pull-back $\beta^* \omega$ defines an invariant symplectic structure on $\Lambda G$, which descends to $NG$, cf. also Prop. 5.4 below.

**Theorem 5.2.** Let $G$ be a proper étale Lie groupoid representing a symplectic orbifold $X$ of dimension $2n$. Then the Hochschild homology of the deformed convolution
algebra $A((\hbar)) \rtimes G$ is given by
\[
HH_*((A((\hbar)) \rtimes G)) \cong H_{\text{orb},c}^{2n-*}(X, \mathbb{C}((\hbar)))
\]

Proof. Consider the spectral sequence induced by the $\hbar$-adic filtration on the Hochschild complex of a formal deformation $A^\hbar = (A[[\hbar]], \ast)$ of a noncommutative Poisson algebra $(A, [\Pi])$ in the sense of Definition 2.13. Clearly, in degree zero one finds the classical Hochschild complex of $A$ and a straightforward computation shows that the differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is given by the noncommutative Poisson differential $d_{\Pi} : HH^{p+q}(A) \to HH^{p+q-1}(A)$ of Definition 4.4. In our case, with $A^\hbar = A^\hbar \rtimes G$, $A = C^\infty_{c}(G)$, one finds $E^1_{p,q} = HH^{p+q}(A) \cong \Omega^{p+q}_{\text{inv},c}(NG)$.

The last isomorphism follows from the computations of [CR]. For general étale groupoids this also includes higher cohomology groups $H^k_{\text{c}}(NG, \Omega^l)$, but these all vanish here by the following argument. Using the projection $\pi : G \to X$ onto the orbifold, one identifies $\pi_!\Omega^*_{\text{c}}$ as the sheaf on $X$ of invariant forms, which is fine, see [PF01], and vanishing of cohomology follows from (2.2).

As explained in Sec. 4, the differential $d^1$ is nothing but Brylinski’s Poisson differential (4.1) on the invariant differential forms on the groupoid $NG$, which is well-defined because the induced symplectic form is invariant. Of course, this is the image of the sheaf version of Brylinski’s complex $(\Omega^*_{NG}, d_{\Pi})$ in $\text{Sh}(NG)$, under the functor $\Gamma_{\text{inv},c}$. Using the fact that $NG$ is symplectic, there is a quasi-isomorphism between this sheaf complex and the de Rham complex, as in [BR],

\[
(\Omega^*_{\text{c}}, d_{\Pi}) \cong (\Omega^\text{dim} \circ -_{\text{c}}, d_{dRb}).
\]

Here, we restrict to a connected component of the inertia groupoid because the components may have different dimensions, affecting the degree shift in the isomorphism. By the fact that $\Omega^*_{\text{c}}$ is $\Gamma_{\text{inv},c}$-acyclic, on therefore finds
\[
E^2_{p,q} = H^{2n-p-q}_{\text{orb},c}(X, \mathbb{C}((\hbar))).
\]

As in the case of smooth manifolds, we claim that the spectral sequence degenerates at this stage. When $NG$ has a finite number of components, the argument is the same as that of [NETS95]. If the spectral sequence does not degenerate at this stage, one has
\[
\dim_{\mathbb{C}((\hbar))} HH_{i}(A^\hbar \rtimes G) < b_{\text{orb},c}^{2n-i}(X),
\]
where $b_{\text{orb},c}^{i}(X) = \dim H_{\text{orb},c}^{i}(X)$. But this would imply that
\[
\dim_{\mathbb{C}((\hbar))} HP_{i}(A^\hbar \rtimes G) < \sum_{k} b_{\text{orb},c}^{2k+i}(X),
\]
contradicting Prop. 5.1. The last inequality hereby follows from the spectral sequence from Hochschild homology to cyclic homology obtained by filtering the $(b, B)$-complex by columns. When $\tilde{X}$ however does have infinite many connected components, one uses the decomposition of Hochschild homology induced by the $G$-invariant decomposition (2.3) of $NG$:
\[
HH_*((A^\hbar \rtimes G)) = \bigoplus_{O} HH_*((A^\hbar \rtimes G)_{O}).
\]

This decomposition can be deduced from the similar decomposition in Poisson homology, cf. Theorem 4.13 by the spectral sequence above. The star product is
given in terms of local multidifferential operators on $G_0$, and therefore the decomposition is preserved by the higher differentials. By Proposition \[5.1\], the periodic cyclic homology has a similar decomposition, and it is not too difficult to see that both are compatible. Since $G$ is proper, each $\mathcal{O}$ has a finite number of connected components, and the above dimension argument, applied to each component separately, proves that the spectral sequence degenerates at the second stage. The theorem follows. \hfill \Box

\textbf{Remark 5.3.} We expect a similar result to hold for Poisson structures, provided one can prove the corresponding analogue of Prop. \[5.4\] below. In that case, it follows by formality \cite{Dol2} that the Hochschild homology is given by the Poisson homology of the groupoid $NG$.

5.3. Cyclic homology. Next, we proceed to compute the cyclic homology, analogously to the computations of \cite{BrNi} and \cite{Cr}. We begin with a detailed analysis of some natural sheaves on the space of loops $B^{(0)}$. Consider the morphism $\beta^{-1} : \text{Sh}(G) \to \text{Sh}(\Lambda G)$. Applied to the quantum sheaf of algebras $\mathcal{A}^h$ on $G$, we get a $\theta$-cyclic sheaf $\beta^{-1} \left( (\mathcal{A}^h)^\Theta \right)$ on $\Lambda G$. Again, its stalk at $g \in B^{(0)}$ is given by $\left( \mathcal{A}^h_{\beta g} \right)^\Theta$, cf. \cite{Cr} 3.4.1 and Sec. 2.6.

\textbf{Proposition 5.4.} The inertia groupoid $\Lambda G$ carries a natural symplectic structure $\omega_{(0)} := \beta^* \omega$, which descends to $NG$. If $\mathcal{A}_{(0)}$ denotes the sheaf of smooth functions on $B^{(0)}$, then there is a Poisson morphism of sheaves on $\Lambda G$

$$\phi_0 : \mathcal{A}_{(0)} \to \beta^{-1} \mathcal{A}, \quad (5.1)$$

where $\mathcal{A}_{(0)}$ carries the Poisson structure induced by $\omega_{(0)}$, and $\beta^{-1} \mathcal{A}$ inherits the Poisson structure from $\omega$. Moreover, the sheaf $\mathcal{A}^h = \beta^{-1} \mathcal{A}^h$ is isomorphic to a formal deformation quantization with coefficients of the symplectic structure $\beta^* \omega$.

\textit{Proof.} The first and last claim are essentially local statements and therefore we can restrict to the case of a translation groupoid $G = \Gamma \ltimes M$ by the action of a finite group. In this case one has a decomposition

$$B^{(0)} = \coprod_{\gamma \in \Gamma} M^\gamma.$$

As shown in \cite{Fe} Sec. 5], the pull back of the symplectic form along the embedding $M^\gamma \hookrightarrow M$ for all $\gamma \in \Gamma$ gives $B^{(0)}$ a symplectic structure and $\beta^{-1} \mathcal{A}^h$ is a formal deformation quantization of $(\Lambda G, \omega_{(0)})$ with coefficients in the normal bundle with respect to this embedding.

For the construction of $\phi_0$ choose a $G$-invariant Riemannian metric on $G_0$. Let $g \in B^{(0)}$ and consider a germ $[f]_g \in (\mathcal{A}_{(0)})_g$. Let $U_g \subset B^{(0)}$ be an open neighborhood of $g$ on which $f$ is defined and put $M_g := s(U_g)$. Then $M_g$ is a submanifold of $G_0$ and there is a projection $\pi_g : T_g \to M_g$ from a tubular neighborhood $T_g$ to $M_g$ along geodesics of the chosen $G$-invariant Riemannian metric. We now define $\phi_0([f]_g)$ as the germ $[f \circ s^{-1} \circ \pi_g]_g$. Since the normal bundle to $M_g$ (with respect to the above Riemannian metric) is a symplectic bundle by the symplectic slice theorem, it is clear, that the thus defined sheaf morphism is Poisson. \hfill \Box

\textbf{Remark 5.5.} The morphism $\phi_0$ is a right inverse to the restriction map $f \mapsto f|_{B^{(0)}}$ used by Brylinski–Nistor, which is a quasi-isomorphism on Hochschild homology,
Next, we want to lift this Poisson morphism to a “quantum morphism” $\phi : A^h(0) \to \beta^{-1}A^h$, where $A^h(0)$ is some suitable deformation quantization of $(B^{(0)}, \omega(0))$. Consider now the characteristic class $[\ast] \in \frac{\omega}{h^2} + H^2(G_0, \mathbb{C}[[h]])$ of the star product on $A^h$, and let $c \in \frac{\omega(0)}{h} + H^2(B^{(0)}, \mathbb{C}[[h]])$ be the pullback class $\beta^*[\ast]$. Since $\ast$ is $G$-invariant, there exists a $G$-invariant Fedosov star product $\ast(0)$ on $A(0)$ with characteristic class given by $[\ast(0)] = c$. Denote the resulting deformed sheaf of algebras by $A^h(0)$. By construction of $\ast(0)$ and Thm. 5.4 in [CR], one now concludes that the Poisson morphism $\phi_0 : A(0) \to \beta^{-1}A$ from Prop. 5.3 indeed can be extended to a morphism of sheaves of algebras

$$
\phi = \sum_{k=0}^{\infty} \phi_k h^k : A^h(0) \to \beta^{-1}A^h,
$$

where each $\phi_k$ is a linear morphism of sheaves from $A(0)$ to $\beta^{-1}A$. We now have:

**Proposition 5.6.** The morphism $\phi : A^h(0) \to \beta^{-1}A^h$ induces a quasi-isomorphism $\phi_* : (A^h(0))^2 \to (A^h)^2_{tw}$ of $\theta$-cyclic sheaves on $\Lambda G$, where the left hand side carries the trivial cyclic structure.

**Proof.** Let us start with the Hochschild complex. Clearly, the map $\phi$ induces a map on the (twisted) Hochschild homology. Consider now the spectral sequence induced by the $h$-adic filtration. Since the morphism $\phi$ induces an isomorphism at level $E^1$, see Remark 5.5, it must induce an isomorphism on the level of Hochschild homology of the quantum sheaves by the spectral sequence comparison theorem, cf. [VE] Thm. 5.2.12. As for cyclic homology, since the morphism of mixed complexes induced by $\phi$ is a quasi-isomorphism on Hochschild homology, it must induce an isomorphism on cyclic homology as well (cf. Prop. 2.5.15 in [LC]). □

Next, we consider the “reduction to loops” map

$$
p : (A^h \rtimes G)^{\mathbb{Z}}_k \to \Gamma_c(B^{(k)}, \sigma^{-1}_{k+1}(A^h)^{\mathbb{Z}(k+1)}),
$$

(5.2)

restricting sections over $G^{k+1}$ to $B^{(k)}$. As explained in Section 2.6, the right hand side equals the Bar complex computing the $\theta$-twisted homology of the twisted cyclic sheaf $\beta^{-1}A^h$ on $\Lambda G$. Now we are in a position to prove:

**Proposition 5.7.** Reduction to loops induces a quasi-isomorphism on Hochschild and cyclic homology:

$$
HH_\bullet(A^{((k))} \rtimes G) \cong HH_\bullet(\Lambda G, \theta, (A^{((k))})^2_{tw}),
$$

$$
HC_\bullet(A^{((k))} \rtimes G) \cong HC_\bullet(\Lambda G, \theta, (A^{((k))})^2_{tw}).
$$

**Proof.** Again, we start with Hochschild homology. Using the quasi-isomorphism of Prop. 5.7, the right hand side of the first equation above is isomorphic to the homology of sheafified Hochschild complex associated to a formal deformation quantization of the inertia groupoid $(\Lambda G, \omega(0))$. But this sheaf homology is readily computed, cf. Prop. 5.8 below, to give the orbifold cohomology as in Theorem 5.2. It is
not difficult to show that in fact the isomorphism in Theorem 5.7 is equal to the map induced by \( p \) in (5.2). Therefore we conclude that reduction to loops induces an isomorphism on Hochschild homology. However, given that it is a quasi-isomorphism on Hochschild homology, it induces an isomorphism on cyclic homology as well. □

Having reduced the computation cyclic homology to sheaf cohomology, we can do the computations locally for \( A_{(0)} \) and then take cohomology. The local results are given in the following proposition. For completeness, we also state the analogous results for Hochschild homology.

**Proposition 5.8.** On a proper étale Lie groupoid with symplectic structure of dimension \( 2n \) there are quasi-isomorphisms of complexes of sheaves:

\[
(C_\bullet(A(\hbar)), b) \cong \mathbb{C}((\hbar))[2n],
\]

\[
(Tot_\bullet(B_\bullet(A(\hbar))), b + B) \cong \bigoplus_{k \in \mathbb{N}} \mathbb{C}((\hbar))[2n + 2k].
\]

**Proof.** There are obvious inclusions of the right hand side into the sheaf complexes on the left hand side. To prove that these are quasi-isomorphisms, one needs to prove that they induce isomorphisms of the stalk-wise cohomology, or, equivalently, of the homology sheaves. However, locally the sheaf \( A(\hbar) \) is isomorphic to the Weyl algebra \( W(\hbar) \) of formal Laurent series on \( \mathbb{C}^n \). Its Hochschild and cyclic homology are given as follows (see [NeTs 95, Sec. 3.2]), where the second identity follows from the first by the spectral sequence induced by filtering the \((b,B)\)-complex by columns.

\[
HH_k(W(\hbar)) = \begin{cases} \mathbb{C}((\hbar)), & \text{for } k = 2n, \\ 0, & \text{else}, \end{cases}
\]

\[
HC_k(W(\hbar)) = \begin{cases} \mathbb{C}((\hbar)), & \text{for } k = 2n + 2l, \\ 0, & \text{else}. \end{cases}
\]

This proves that the canonical inclusion is a quasi-isomorphism. □

**Theorem 5.9.** Let \( G \) be a proper étale Lie groupoid and \( A^\hbar \) like above. Then the cyclic homology of \( A^\hbar \rtimes G \) is given by

\[
HC_\bullet(A(\hbar)) \rtimes G = \bigoplus_{k \geq 0} H^*_{orb,c}(X, \mathbb{C}((\hbar))).
\]

**Proof.** By Propositions 5.4, 5.7 and 5.6 the cyclic homology of \( A(\hbar) \rtimes G \) equals the hyperhomology of the total complex of the sheaffied Connes’ \((b,B)\)-complex of a formal deformation of the sheaf of smooth functions on \( \Lambda G_0 \). The preceding Proposition then entails the claim. □

5.4. Cohomology. Having computed the Hochschild and cyclic homology, the dual results may be computed analogously, as in [Cr]. Recall that the Hochschild cohomology of an algebra is computed from the complex \( A_k^\natural := \text{Hom}(A^k, \mathbb{C}) \) dual to the Hochschild complex, with the corresponding differential. Of course, in our case we use topological tensor products and consider only continuous functionals. Again using the \( \hbar \)-adic filtration, one observes that the \( E^1 \)-term in the spectral sequence is given by the complex of de Rham currents \((\Omega_\bullet, b'_H)\) on \( NG \) with the dual Poisson differential. The isomorphism of [Br] dualizes on each component \( O \).
to give \((\Omega_\bullet, b'_\bullet) \cong (\Omega_{\dim c - \bullet}, d'_{\dim b}) \cong (\Omega_\bullet, d_{\dim b})\). As for homology, the spectral sequence collapses at the second stage and one proves the first part of the following

**Theorem 5.10.** The Hochschild and cyclic cohomology of \(A^h \rtimes G\) are given by

\[
HH^\bullet(A^{((h))} \rtimes G) \cong H^\bullet_{orb}(X, \mathbb{C}(\langle h \rangle)),
\]

\[
HC^\bullet(A^{((h))} \rtimes G) \cong \bigoplus_{k \geq 0} H_{orb}^\bullet - 2k(X, \mathbb{C}(\langle h \rangle)).
\]

Furthermore, the pairing between homology and cohomology is given by Poincaré duality for orbifolds.

**Proof.** Observe that the isomorphism in Hochschild cohomology described above is induced by the maps dual to the “reduction to loops” from Eq. \((\ref{eq:reduction-to-loops})\):

\[
\Gamma_c(B^{(k)}, \sigma_{k+1}^{-1}(A^{((h)))} \mathbb{E}^{(k+1)})' \rightarrow (A^{((h)))} \rtimes G)^k,
\]

where \((\cdot)'\) denotes the topological dual. Notice that the sheaf of distributions \(A'\) has the property that

\[
\Gamma(X, A') = \Gamma_c(X, A)'.
\]

Therefore, we can realize the left hand side of Eq. \((\ref{eq:reduction-to-loops})\) as the space of sections over \(B^{(k)}\) of the pullback via \(\sigma_{k+1}\) of the sheaf of distributional Laurent-series, dual to \(A^{((h)))}\). On a groupoid, the sheaf of distributions has a natural \(G^{\text{op}}\)-structure. This property relates, as in \([\text{CR}]\) for the sheaf of smooth functions, the left hand side of \((\ref{eq:reduction-to-loops})\) to the twisted cyclic cohomology of \((A^{((h)))}'\) on \(\Delta G\). Dualizing Prop. \(5.6\), one obtains a quasi-isomorphism from \((A^{((h)))}'_{\text{tw}}\) to \((A^{((h)))}'_{2})\) with the trivial \(\theta\)-cyclic structure. To compute the hypercohomology of the resulting sheaf complex \(\text{Tot}^\bullet B^{**}(A^{((h)))}_{(0)}\) dual to the \(\text{Tot}^\bullet B_{**}(A^{((h)))}_{(0)}\)-complex, we use the following quasi-isomorphism which is dual to the one of Prop. \(5.8\) and which is obtained by computing the cyclic cohomology of the formal Weyl algebra of \(\mathbb{C}^n\) with compact support:

\[
\text{Tot}^\bullet (B^{**}(A^{((h)))}_{(0)}) \cong \bigoplus_{k \in \mathbb{N}} \mathbb{C}(\langle h \rangle)[2k].
\]

Taking the hypercohomology of this complex over \(\Delta G\) gives the result. \(\square\)

### 5.5. Examples

In this section we will give some examples of the computations of the Hochschild and cyclic homology, as well as the noncommutative Poisson homology. All are related to so-called transformation groupoids, i.e., the groupoid \(G = \Gamma \rtimes M \equiv M\) associated to a proper action of a discrete group \(\Gamma\) on a manifold \(M\). In this case the underlying orbifold is simply the quotient \(X = M/\Gamma\). An invariant Poisson structure on \(M\) leads, by equation \((\ref{eq:invariant-poisson})\), to a noncommutative Poisson structure on \(C_c^\infty(G) = A_c \rtimes \Gamma\), where \(A_c = C_c^\infty(M)\), already stated in \([\text{L1}]\).

As a special case of the quantization procedure for Poisson groupoids \([\text{L04b}]\), a \(\Gamma\)-invariant deformation quantization \(A_{\mathbb{C}}^h\) of \(A_c\) defines a quantization of the noncommutative Poisson algebra \(A_c \rtimes \Gamma\) by taking the crossed product \(A_{\mathbb{C}}^h \rtimes \Gamma\). We discuss our computations in the following special cases:

**Example 5.11.** (Free action) Notice that in case of a free action \(\Gamma\) must be a finite group, since the groupoid \(G\) is assumed to be proper. Trivially, we have that \(B^{(0)} = M\) and \(\Delta G = \Gamma \rtimes M\), i.e., \(X = X\) in this case. Using the computation of the Hochschild homology of \(A_c \rtimes \Gamma\) in \([\text{BEN}]\) \(\text{[CR]}\), and the Leray spectral sequence associated to the groupoid morphism \(\pi : \Gamma \rtimes M \rightarrow M/\Gamma\) sending a point of \(M\)
to its image in the quotient space \( M/\Gamma \), cf. [40], one obtains an isomorphism \( \pi : HH_* (A_c \rtimes \Gamma) \to HH_* (M/\Gamma) \). It is easy to check that under \( \pi \), the Poisson differential on \( HH_* (A_c) \) is mapped to the Poisson differential on \( HH_* (M/\Gamma) \), since a section of a sheaf on \( M/\Gamma \) can be identified with an invariant section of the corresponding \( G \)-sheaf on \( \Gamma \times M \). Therefore, the Poisson homology of \( A_c \rtimes \Gamma \) is equal to the Poisson homology of the quotient space \( M/\Gamma \). When the Poisson structure is in fact symplectic, the Poisson homology on \( M/\Gamma \) is dual to the de Rham cohomology of \( M/\Gamma \) by Brylinski’s result in [BR].

**Remark 5.12.** We can also use Proposition [14] to obtain this result. We know that \( A_c \rtimes \Gamma \) is Morita equivalent to \( C_c^\infty (M/\Gamma) \) by the bimodule \( C_c^\infty (M) \). As shown in [Xu], under this Morita equivalence the Poisson structure \( \Pi \) on \( A_c \rtimes \Gamma \) is mapped to the Poisson structure \( \Pi \) on \( M/\Gamma \) coming from the projection \( \pi : M \to M/\Gamma \). By Prop. [67], the Poisson homology of \( (A_c \rtimes \Gamma, \Pi) \) is isomorphic to the Poisson homology of \( (C_c^\infty (M/\Gamma), \Pi) \).

Next, we assume the Poisson structure to be symplectic, so that \( M \) is a manifold of dimension \( 2n \). For a free action, the projection \( \pi : M \to M/\Gamma = X \) gives rise to a quasi-isomorphism \( \pi_* \Omega_M \cong \Omega_X \in \mathcal{Sh}(X) \), where the right hand side is the sheaf of de Rham forms. Then the computations of the Hochschild and cyclic homology of the algebra \( A^{(h)} \rtimes \Gamma \) give:

\[
\begin{align*}
HH_* (A^{(h)} \rtimes \Gamma) &\cong H^{2n-*} (X, \mathbb{C}) \otimes \mathbb{C}(h), \\
HC_* (A^{(h)} \rtimes \Gamma) &\cong \bigoplus_{k \geq 0} H^{2n+2k-*} (X, \mathbb{C}) \otimes \mathbb{C}(h).
\end{align*}
\]

As for the computation of the Poisson homology, this could have been deduced at once by the observation that the quotient \( A^{(h)} \rtimes \Gamma \) gives a quantization of \( C_c^\infty (X) \) which is Morita equivalent to the crossed product \( A^{(h)} \rtimes \Gamma \) by the equivalence bimodule \( A^{(h)} \rtimes \Gamma \). Therefore, the Hochschild and cyclic homology may be computed from the deformation quantization of \( C_c^\infty (X) \), for which the computations in [Nets95] give the results above.

As can be seen from above, for a trivial group our computations reduce to the well-known statements in [Nets95]. We leave the statements about Hochschild and cyclic cohomology to the reader. Finally, notice that the space of traces on \( A^{(h)} \rtimes \Gamma \), i.e., \( HC^0 (A^{(h)} \rtimes \Gamma) \) or the dual of \( HH_0 (A^{(h)} \rtimes \Gamma) \), is one-dimensional: all traces are proportional to each other. (See section 3)

**Remark 5.13.** It is easy to generalize this discussion to groupoids with trivial isotropy groups, for example those obtained from a covering of a manifold. Again by Morita invariance, the computations of Poisson, Hochschild and cyclic homology reduce to the results of [Nets95] on the underlying smooth manifold.

**Example 5.14.** (Proper action, cf. [BacCo] for the undeformed case) One easily checks that for a proper action of a discrete group, the transformation groupoid \( \Gamma \times M \rightrightarrows M \) is proper and étale. Since we no longer assume the action to be free, \( B^{(0)} \) is usually bigger than \( M \), more precisely one has \( B^{(0)} = \{(\gamma, x) \in \Gamma \times M \mid \gamma x = x\} \). Now, \( B^{(0)} \) has the following decomposition into sectors:

\[
B^{(0)} = \coprod_{(\gamma) \in \text{Conj}(\Gamma)} O(\gamma),
\]
where \( \langle \gamma \rangle \) denotes the conjugacy class of \( \gamma \) in \( \Gamma \), and
\[
\mathcal{O}_{\langle \gamma \rangle} := \prod_{\gamma' \in \langle \gamma \rangle} M^{\gamma'}.
\]
According to Theorem 4.13, we therefore get the decomposition:
\[
H_{\Pi}^n (A_c \rtimes \Gamma) = \bigoplus_{\langle \gamma \rangle \in \text{Conj}(\Gamma)} H_{\Pi}^n (A_c \rtimes \Gamma) \mathcal{O}_{\langle \gamma \rangle}.
\]
For \( \gamma \in \Gamma \), we define
\[
Z_\gamma := \{ \gamma' \in \Gamma \mid \gamma' \gamma = \gamma \gamma' \},
\]
and
\[
N_\gamma = Z_\gamma / \langle \gamma \rangle.
\]
We have Morita equivalences
\[
\Lambda_G \mathcal{O}_{\langle \gamma \rangle} \cong Z_\gamma \triangleright M_\gamma
\]
and
\[
N_\gamma \mathcal{O}_{\langle \gamma \rangle} \cong N_\gamma \triangleright M_\gamma,
\]
which induce isomorphisms between the corresponding Poisson homologies by a similar argument as above for a free action. Therefore, we get
\[
H_{\Pi}^n (A_c \rtimes \Gamma) = \bigoplus_{\langle \gamma \rangle \in \text{Conj}(\Gamma)} H_{\Pi}^n (N_\gamma \triangleright M_\gamma),
\]
and the right hand can be computed from the Poisson differential on the invariant differential forms on \( M_\gamma / N_\gamma \). The decomposition of \( B^0 \) above into sectors, together with the Morita equivalences, give a decomposition of the inertia orbifold \( \tilde{X} \) of \( X = M / \Gamma \) as
\[
\tilde{X} = \coprod_{\langle \gamma \rangle \in \text{Conj}(\Gamma)} M_\gamma / N_\gamma.
\]
Therefore, we have for the Hochschild homology
\[
HH_n (A_c ((\hbar)) \rtimes \Gamma) \cong \bigoplus_{\langle \gamma \rangle \in \text{Conj}(\Gamma)} H_c^{\dim(M_\gamma)} (M_\gamma / N_\gamma, \mathbb{C}) \otimes \mathbb{C}((\hbar)).
\]

6. TRACES ON THE DEFORMED GROUPOID ALGEBRA

Traces on an algebra obtained by deformation quantization form an important ingredient in index theory. Since such functionals are nothing but cyclic cocycles of degree zero, Theorem 5.10 gives a complete classification of traces on the deformed groupoid algebra \( A_c \rtimes \Gamma \), that means of maps \( \text{tr} : A_c \rtimes \Gamma \rightarrow \mathbb{C}((\hbar)) \) such that
\[
\text{tr}(a \ast c b) = \text{tr}(b \ast c a), \quad \text{for all } a, b \in A_c \rtimes \Gamma.
\]
(6.1)

In this section, we will be concerned with the actual construction of all traces. Our discussion somewhat parallels with the constructions in \[\text{FESCHLA}\], and also uses in an essential way the paper \[\text{FE00}\], however notice that \[\text{FESCHLA}\] is only concerned with the subalgebra of \( A_\hbar \rtimes \Gamma \) of invariant quantized functions on the underlying orbifold \( X \). The full algebra \( A_\hbar \rtimes \Gamma \) contains more information which we believe to be essential for index theory.

6.1. Traces on finite transformation groupoids. We work in the situation of Sec. 5.5 and use the notation from there. Additionally, we assume (for notational convenience only) that each fixed point manifold \( M_\gamma \subset M \) with \( \gamma \in \Gamma \) has constant dimension. We then consider the crossed product algebra \( (A_c \rtimes \Gamma, \ast_c) \), where the star product \( \ast \) on \( A_c \) has been obtained by a \( \Gamma \)-invariant Fedosov construction.

As explained above, the cyclic cohomology group \( HC^0 (A_c ((\hbar)) \rtimes \Gamma) \) determines the space of traces on \( A_c \rtimes \Gamma \) and is given as follows:
\[
HC^0 (A_c ((\hbar)) \rtimes \Gamma) \cong H^0_{\text{orb}} (M / \Gamma, \mathbb{C}) ((\hbar)) = \bigoplus_{\langle \gamma \rangle \in \text{Conj}(\Gamma)} H^0 (M_\gamma / N_\gamma, \mathbb{C}) ((\hbar)).
\]
(6.2)
Hence, the space of traces has dimension
\[ \dim_{\mathbb{C}((\hbar))} H^0_{\text{orb}}(M/\Gamma, \mathbb{C})((\hbar)) = \# \text{Comp}(M/\Gamma), \]
the number of connected components of the inertia orbifold.

Let us now examine the space of traces on \( A^p_\epsilon \times \Gamma \) in some more detail. To this end we will use in the remainder of this section the following notation. Like in Sec. 3 Eq. 3.3, we expand elements \( a \in A_\epsilon \times \Gamma \) as sums \( \sum_{\gamma \in \Gamma} f_\gamma \delta_\gamma \) with \( f_\gamma \in A_\epsilon \) and extend this decomposition to formal Laurent series \( a = \sum_k a_k \hbar^k \in A^p((\hbar)) \times \Gamma \) with \( a_k \in A_\epsilon \times \Gamma \) as follows:
\[
a = \sum_{\gamma \in \Gamma} f_\gamma \delta_\gamma, \quad \text{where } f_\gamma = \sum_k f_{k,\gamma} \hbar^k \text{ and } a_k = \sum_{\gamma \in \Gamma} f_{k,\gamma} \delta_\gamma. \quad (6.3)
\]

Following \( \text{Fe00} \) Sec. 1 and \( \text{FeSchTa} \) we now consider a family \( (\tau_\gamma)_{\gamma \in \Gamma} \) of linear forms on \( A^p((\hbar)) \) with the following properties:
\[
\tau_\gamma(f \ast f') = \tau_\gamma(f' \ast f) \quad \text{for all } \gamma \in \Gamma \text{ and } f, f' \in A_\epsilon, \quad (6.4)
\]
\[
\tau_{\gamma'}(f) = \tau_{\gamma' \gamma^{-1}}(\gamma f) \quad \text{for all } f \in A_\epsilon \text{ and } \gamma, \gamma' \in \Gamma. \quad (6.5)
\]

The following result can now be verified by a straightforward computation.

**Proposition 6.1.** Under the assumptions stated above let \( (\tau_\gamma)_{\gamma \in \Gamma} \) be a family of linear forms on \( A^p((\hbar)) \) which satisfies the assumptions (6.4) and (6.5). Then the functional
\[
\text{tr} : A^p((\hbar)) \times \Gamma \rightarrow \mathbb{C}((\hbar)), \quad a \mapsto \sum_{\gamma \in \Gamma} \tau_\gamma(f_\gamma) \quad (6.6)
\]
is a trace on \( A^p((\hbar)) \times \Gamma \). Vice versa, given a trace \( \text{tr} : A^p((\hbar)) \times \Gamma \rightarrow \mathbb{C}((\hbar)) \), one obtains a family \( (\tau_\gamma)_{\gamma \in \Gamma} \) satisfying the above conditions by defining
\[
\tau_\gamma(f) := \text{tr}(f \delta_\gamma) \quad \text{for all } f \in A^p((\hbar)). \quad (6.7)
\]

Finally, the trace corresponding to the so defined family of linear forms coincides with the originally given one.

In order to construct all traces on the quantized convolution algebra we thus have to find functionals \( (\tau_\gamma)_{\gamma \in \Gamma} \) satisfying the assumptions made above. In \( \text{Fe00} \) Fedosov has explicitly constructed such functionals. Let us recall Fedosov’s construction. To this end we restrict our assumptions further and assume that \( M \) is an open \( \Gamma \)-invariant convex neighborhood of the origin of some symplectic vector space \( V \). Then it is well-known that over \( M \), the star product \( \ast \) on \( A^k \) is equivalent to the Weyl star product \( \ast_W \) coming from \( V \). Let \( S = 1 + \sum_{k=1}^\infty S_k \hbar^k : C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]] \) be an equivalence from \( \ast \) to the Weyl star product \( \ast_W \). Next choose a \( \Gamma \)-invariant complex structure \( J \) on \( V \), and consider the Hermitian product induced by \( J \) on the symplectic vector space \( V \). Since then, \( \Gamma \) acts unitarily on \( V \), one has for every \( \gamma \in \Gamma \) a decomposition \( V = V^\gamma \oplus V^\perp \), where \( V^\gamma \) is the fixed point subspace of \( \gamma \), and \( V^\perp \) its orthogonal complement. Now, \( \gamma \) leaves \( V^\perp \) invariant and acts on \( V^\perp \) via a matrix \( \gamma_L \). Finally, choose complex unitary coordinates \( z_{\text{inv}} \) of \( V^\gamma \), \( z_\perp \) of \( V^\perp \), and put \( z = (z_{\text{inv}}, z_\perp) \). With these notations, one can define functionals \( \tau^W_\gamma : C^\infty(M)((\hbar)) \rightarrow \mathbb{C}((\hbar)) \) as follows (with integration induced by the real part of
satisfies the conditions (6.4) and (6.5) above. Moreover, for every $f \in A_c$ the functional
\[ \tau^W_\gamma(f) = \frac{1}{(2\pi\hbar)^k} \int_{V^*} \frac{1}{\det(1 - \gamma_1^{-1})} \exp \left( \hbar \frac{\partial}{\partial z_\perp} \left( 1 + \gamma_1^{-1} \frac{\partial}{\partial z_\perp} \right) f \right) \Bigg|_{z_\perp = \tau z_\perp = 0} dz_{\text{inv}} d\tau_{\text{inv}}. \] 
\[ (6.8) \]

According to Eq. (2.17) and Prop. 2.5 of \[ \text{FeSchTa}, \] the thus obtained family $\tau^W_\gamma \in \Gamma$ satisfies properties (6.4) and (6.5) above with respect to the Weyl star product. The following result is essentially a reformulation of \[ \text{FeSchTa} \] Prop. 2.5 and \[ \text{FeSchTa} \] Cor. 7.5.

**Proposition 6.2.** Let $\Gamma, V, M \subset V$ and $(A^h, \ast)$ as before, and choose $S, J$ and $z = (z_{\text{inv}}, z_\perp)$ like above. Then the family $\tau^F_\gamma$ of functionals on $A_c^{(h)}$ defined by
\[ \tau^F_\gamma(f) := \tau^W_\gamma(Sf) \quad \text{for all } f \in A_c \]
satisfies the conditions (6.3) and (6.5) above. Moreover, for every $\gamma$, the restriction of the functional $\tau^F_\gamma$ to the $\Gamma$-invariant elements of $A_c^{(h)}$ does not depend on the choice of $J$, $z$ and $S$.

Now consider a family $\kappa = (\kappa_{(\gamma)})_{(\gamma) \in \text{Conj}(\Gamma)}$ of complex coefficients. Then the functional
\[ \text{tr}_\kappa : A_c^{(h)} \times \Gamma \to \mathbb{C}((h)), \quad a = \sum_{\gamma \in \Gamma} f_\gamma \delta_\gamma \mapsto \sum_{\gamma \in \Gamma} \kappa_{(\gamma)} \tau^F_\gamma(f_\gamma) \]
has to be a trace on the deformed crossed product algebra by the preceding propositions, since the family $(\kappa_{(\gamma)} \tau^F_\gamma)_{\gamma \in \Gamma}$ satisfies conditions (6.4) and (6.5). One even has more.

**Corollary 6.3.** With notations from above, the functionals $\text{tr}_\kappa$ have the following properties.

1. Every trace on $A_c^{(h)} \times \Gamma$ is of the form $\text{tr}_\kappa$ with a uniquely determined family $\kappa \in \mathbb{C}^{\text{Conj}(\Gamma)}$.
2. The traces $\text{tr}_\kappa$ are invariant in the following sense. Let $V'$ be another symplectic vector space, $\Gamma'$ a finite group acting by linear symplectomorphisms on $V'$, and let $\ast'$ be a Fedosov star product on $V'$. Assume further that $F : M \to V'$ is an open embedding with the following properties:
   (a) $F$ is equivariant with respect to an injective homomorphism $\iota : \Gamma \to \Gamma'$,
   (b) $F$ is symplectic,
   (c) the pull-back via $F$ induces a homomorphism of star product algebras
   \[ F^* : (C^\infty(V')[[\hbar]], \ast') \to A^h, \]
   (d) the induced quotient map $\overline{F} : M/\Gamma \to V'/\Gamma'$ is an open embedding.

Then for every $a = \sum_{\gamma \in \Gamma} f_\gamma \delta_\gamma \in A_c^{(h)} \times \Gamma$ the equality
\[ \text{tr}_{\kappa \ast F}(F_a) = \text{tr}_\kappa(a) \]
holds true, where $F_\ast(a) = \sum_{\gamma \in \Gamma} f_\gamma \circ F^{-1} \delta_{\iota(\gamma)}$ and $\iota \ast \kappa$ is the family
\[ (\kappa_{(\gamma')})_{(\gamma') \in \text{Conj}(\Gamma')} \]
defined by
\[ \kappa_{(\gamma')} = \begin{cases} \kappa_{(\gamma)}, & \text{if } \gamma' \sim \iota(\gamma) \text{ for some } \gamma \in \Gamma, \\ 0, & \text{else}. \end{cases} \]
we can choose the sets \( U \) is an open invariant set of some symplectic \( \Gamma \)
then \( \tau_\gamma(f) = \iota_\gamma^F(f \circ F'^{-1}) \)
holds for all \( \Gamma \)-invariant \( f \in \mathcal{A}^{(h)}(U) \) and all \( \gamma \in \Gamma \). By the first claim and the definition of \( \iota_\kappa \), this entails Eq. (6.11). \( \square \)

6.2. Traces in the general case. Let now \( G \) be an arbitrary proper étale Lie groupoid with a symplectic structure and let \( \star \) be an invariant star product on \( A \).

We now want to construct all traces on the crossed product \( \mathcal{A}^{(h)}(\Gamma) \rtimes G \). To this end first fix a dense countable family \((x_i)_{i \in J}\) of points of \( G_0 \) and an open covering \( U = (U_i)_{i \in I} \) of \( G_0 \) such that \( x_i \in U_i \) for all \( i \in I \) and such that one has isomorphisms \( G_{U_i} \cong \Gamma_i \ltimes U_i \), where \( \Gamma_i \) is the isotropy group \( G_{x_i} \).

By appropriate choices we can even achieve, that each \( U_i \) is symplectomorphic to an open ball around the origin of some symplectic \( \Gamma_i \)-representation space \( V_i \), that \( x_i \) corresponds to the origin under this symplectomorphism, and that for every pair \( U_i, U_j \) with \( \pi(U_i) \cap \pi(U_j) \neq \emptyset \) there exists an open connected subset \( W_{ij} \subset G_0 \) and a finite isotropy group \( \Gamma_{ij} := G_{x_{ij}} \) (with \( x_{ij} \in W_{ij} \)) acting symplectically on \( W_{ij} \) such that \( G_{W_{ij}} \cong \Gamma_{ij} \ltimes W_{ij} \) and such that \( \pi(U_i), \pi(U_j) \subset \pi(W_{ij}) \). Moreover, we can assume that \( W_{ij} = W_{ji} \) and that \( W_{ij} \) is an open invariant set of some symplectic \( \Gamma_{ij} \)-representation space \( V_{ij} \). Finally, we can choose the sets \( U_i \) so small such that for all \( i, j \in I \) with \( \pi(U_i) \cap \pi(U_j) \neq \emptyset \) there exist bisections \( s_{ij} : U_i \to W_{ij} \) with \( t \circ s_{ij}(x) \in W_{ij} \) for all \( x \in U_i \). Now put \( F_{ij} := t \circ s_{ij} \).

Using that the underlying groupoid is proper étale and the assumptions on the covering \( (U_i)_{i \in I} \) one then immediately checks the following properties of the maps \( F_{ij} \):

(a) each \( F_{ij} \) is an embedding and equivariant with respect to the monomorphism \( \Gamma_i \to \Gamma_{ij} \) induced by the composition on \( G_1 \),
(b) each \( F_{ij} \) is symplectic,
(c) since the star product on \( G_0 \) is \( G \)-invariant, the pull-back via \( F_{ij} \) induces a homomorphism \( F_{ij}^* : (\mathcal{C}^\infty(W_{ij})[[\hbar]], \star) \to (\mathcal{C}^\infty(U_i)[[\hbar]], \star) \),
(d) the induced quotient map \( \overline{F}_{ij} : U_i/\Gamma_i \to W_{ij}/\Gamma_{ij} \) is the natural inclusion of open subsets of \( X \).

Using Cor. 6.3 (2), these properties will later guarantee that one can glue together local traces on \( \mathcal{A}^{(h)}(\Gamma) \rtimes G \).

Associated to the covering \( U \) is the groupoid \( G_U \) with objects and morphisms given by

\[
\begin{align*}
(G_U)_0 = \coprod_{i \in I} U_i, & \quad (G_U)_1 = \coprod_{i,j \in I} s^{-1}(U_i) \cap t^{-1}(U_j).
\end{align*}
\]  

(6.13)

The obvious morphism \( G_U \to G \) then is a weak equivalence. The sheaf of quantum algebras \( \mathcal{A}^{(h)}(\Gamma) \) restricts over every open subset \( U_i \) to define a sheaf, also denoted by \( \mathcal{A}^{(h)}(\Gamma) \) and therefore defines a crossed product \( \mathcal{A}^{(h)}(\Gamma) \rtimes G_U \). Let us write elements in \( \mathcal{A}^{(h)}(\Gamma) \rtimes G_U \) as \( a = (a_{ij})_{i,j \in I} \), etc. Denote by \( \star_U \) the multiplication on \( \mathcal{A}^{(h)}(\Gamma) \rtimes G_U \) obtained by combining the \( \star \)-product with the convolution product. This product

then reads as follows (using germs in the notation):
\[
[(a \star_U b)_{ij}]_g = \sum_k \sum_{g_1 g_2 = g} [a_{ik}]_{g_1} \ast [b_{kj}]_{g_2}, \quad s(g) \in U_i, \ t(g) \in U_j. \quad (6.14)
\]

We will now construct an injective homomorphism \( \mathcal{A}^{(h)}(G) \rightarrow \mathcal{A}^{(h)}(G)_U \).
Consider a partition of unity \( (\varphi_i)_{i \in I} \) subordinate to \( U \), satisfying \( \sum_i \varphi_i^2 = 1 \). Define the following formal power series on \( G_0 \):
\[
\Phi_i = \left( \sum_k \varphi_k \ast \varphi_k \right)^{-1/2} \ast \varphi_i, \quad \Psi_i = \varphi_i \ast \left( \sum_k \varphi_k \ast \varphi_k \right)^{-1/2}.
\]

Notice that the inverse of the square root exists, since \( \sum_k \varphi_k \ast \varphi_k - 1 \in \mathcal{h}A^0(G_0) \).
By construction, we have \( \text{supp}(\Phi_i) \subseteq U_i, \text{supp}(\Psi_i) \subseteq U_i \), and \( \sum_i \Psi_i \ast \Phi_i = 1 \).
From these properties it is easy to deduce that for the “convolution \( \ast \)-product” in \( \mathcal{A}^{(h)}(G) \) we have
\[
\sum_i (\Psi_i \ast_c \Phi_i) = \delta_u,
\]
where the \( \Psi_i \) and \( \Phi_i \) have been extended by 0 outside \( G_0 \), and where \( \delta_u \) is the “unit in the convolution algebra” from Step 2., Sec. 3.
By inspection of the multiplication \((\Phi_i \ast_c a \ast_c \Psi_j)_{ij}\) it follows that the map
\[
a \mapsto (\Phi_i \ast_c a \ast_c \Psi_j)_{ij}
\]
defines a homomorphism \( \Phi_U : \mathcal{A}^{(h)}(G) \rightarrow \mathcal{A}^{(h)}(G) \times G_U \). With this notation, we have the following final result.

**Theorem 6.4.** Let \( G, U, \mathcal{A}^{(h)} \) be like above and \( \kappa : B^{(0)} \rightarrow \mathbb{C} \) a locally constant \( G \)-invariant function. Then the restriction of \( \kappa \) to \( B^{(0)} \cap s^{-1}(U_i) \) induces for every \( i \in I \) a family \( \kappa_i \in \mathbb{C}^{\text{Conj}(\Gamma_i)} \). Moreover, the formula
\[
\text{tr}_\kappa(a) = \sum_i \text{tr}_{\kappa_i}(\Phi_i \ast_c a \ast_c \Psi_i) \quad (6.15)
\]
defines a trace on \( \mathcal{A}^{(h)}(G) \), and every trace on \( \mathcal{A}^{(h)}(G) \times G \) is equal to such a \( \text{tr}_\kappa \) with unique \( \kappa \).

**Proof.** The \( \kappa_i \) induce traces \( \text{tr}_{\kappa_i} \) on \( \mathcal{A}^{(h)}(G \times \Gamma_i) \), \( A_{c,i} := C^\infty_c(U_i) \) by Cor. 6.3. Since \( \Phi_U \) is a homomorphism of algebras, \( \text{tr}_\kappa \) is proved to be a trace, if the functional
\[
\text{tr}_U : \mathcal{A}^{(h)}(G) \rightarrow \mathbb{C}((h)), \quad (a_{ij}) \mapsto \sum_i \text{tr}_{\kappa_i}(a_{ii}) \quad (6.16)
\]
is a trace. To this end it suffices to show that for all \( (a_{ij}), (b_{ij}) \in \mathcal{A}^{(h)}(G) \times G_U \) one has
\[
\text{tr}_{\kappa_i}(a_{ij} \ast_c b_{ij}) = \text{tr}_{\kappa_j}(b_{ji} \ast_c a_{ij}), \quad (6.17)
\]
if \( \pi(U_i) \cap \pi(U_j) \neq \emptyset \). For the proof of this equality we use the equivariant embeddings \( F_{ij} : U_i \rightarrow W_{ij} \) constructed above and apply Cor. 6.3. More precisely, let \( \text{tr}_{ij} \) be the trace on \( \mathcal{A}^{(h)}_{c,ij}(G) \times \Gamma_{ij}, \ A_{c,ij} := C^\infty_c(W_{ij}) \) induced by the restriction of \( \kappa \) to \( B^{(0)} \cap s^{-1}(W_{ij}) \). Cor. 6.3 (2) entails that the left hand side of Eq. (6.16) coincides with \( \text{tr}_{ij}(F_{ij} \ast_c a_{ij} \ast_c F_{ij} \ast_c b_{ij}) \), and the right hand side with \( \text{tr}_{ij}(F_{ij} \ast_c b_{ij} \ast_c F_{ij} \ast_c a_{ij}) \). By the trace property of \( \text{tr}_{ij} \), Eq. (6.16) follows, and \( \text{tr}_{ij} \) is a trace indeed.

Since the map \( \kappa \rightarrow \text{tr}_\kappa \) clearly is injective, the second claim now follows easily from the fact that \( HC^0(\mathcal{A}^{(h)}(G), \mathbb{C}) \) has dimension (over \( \mathbb{C}((h)) \)) equal to the
number of components of the inertia orbifold, and the fact that the latter number gives also the complex dimension of the space of locally constant invariant functions from $B^{(0)}$ to $\mathbb{C}$.

6.3. On a conjecture of Fedosov, Schulze and Tarkhanov. Unlike in the case of a (connected) symplectic manifold, where the space of traces on a deformed algebra of compactly supported smooth functions is one dimensional, the space of traces on $A^{(\hbar)} \rtimes G$ has dimension $> 1$, since by Theorem 5.10 this dimension is given by the number of connected components of the inertia orbifold $\tilde{X}$. In [FESCHTA], Fedosov, Schulze and Tarkhanov show that a certain abelian group of isomorphism classes of line bundles on a symplectic orbifold acts nontrivially on the space of traces of the deformed convolution algebra and conjecture that “this ambiguity in traces is the only possible one”. In our framework such type of questions can be answered naturally.

We start with a different view on orbifold cohomology. Consider the representation ring sheaf $R_{\mathbb{C}}$ on $X$ whose stalk at $x \in X$ is given by the complexified representation ring $R_{\mathbb{C}}(G_x) = R(G_x) \otimes_{\mathbb{Z}} \mathbb{C}$ of the isotropy group $G_x$, a finite group. As explained in [MO], Sec. 6.4], the Leray spectral sequence associated to the morphism of groupoids $\beta : AG \rightarrow G$ yields an isomorphism

$$H^b_{\text{orb}}(X, \mathbb{C}) \cong H^b(X, R_{\mathbb{C}}),$$

where the right hand side is simply sheaf cohomology on the space $X$. Consider now the abelian group $P_G := H^1(G, S^1)/H^1(X, S^1)$. By Sec. 2.3 $H^1(G, S^1)$ classifies the $G$-line bundles on $G$, whereas $H^1(X, S^1)$ gives the set of isomorphism classes of line bundles on $X$, which, by pull-back along the projection $\pi : G \rightarrow X$, identifies with the set of isomorphism classes of $G$-line bundles with trivial action of the isotropy groups. Thus, $P_G$ is the Picard group as defined in [FESCHTA], however we do not use this terminology here in view of the Picard group in Poisson geometry (cf. [BUWE]) which is a completely different group. Now, there is a natural homomorphism from $P_G$ into the group of units of the ring $H^0(X, R_{\mathbb{C}})$. For its construction observe that every $G$-line bundle gives rise to a representation of the isotropy group $G_x$ for every $x \in X$, hence, by taking the character at every point, there is a canonical map $H^1(G, S^1) \rightarrow H^0(X, R_{\mathbb{C}})$. As its kernel is given by $H^1(X, S^1)$, the existence of the injection $P_G \rightarrow H^0(X, R_{\mathbb{C}})$ follows.

Since traces on an algebra are nothing but cyclic $0$-cocycles, Theorem 5.10 entails that the space of traces on $A^{(\hbar)} \rtimes G$ is isomorphic to $H^0_{\text{orb}}(X, \mathbb{C}) \otimes \mathbb{C}(\hbar)$. By Eq. (6.18), the conjecture of [FESCHTA] can now be reformulated as the statement that the image of $P_G$ in $H^0(X, R_{\mathbb{C}})$ forms a basis. But since irreducible representations of a finite group are necessarily one dimensional only if the underlying group is abelian, the claim holds true in general, if and only if every isotropy group $G_x$ is abelian. Therefore, the conjecture in [FESCHTA] is true for $G$ with abelian isotropy groups, but not otherwise.

Strictly speaking, the paper [FESCHTA] is only concerned with the algebra $\Gamma_{\text{inv}, c}(A^{\hbar})$ of invariant sections of $A^{\hbar}$, or in other words with the deformation quantization of $\mathcal{C}_c^{\infty}(X)$ constructed in [PR], cf. Sec. 2.7]. However, the conclusion above remains true also in this case in view of the following:

**Proposition 6.5.** In case the proper étale Lie groupoid $G$ is reduced, i.e. if each isotropy group $G_x$ acts faithfully on a neighborhood of $x \in G_0$ (cf. [MO] Sec. 1.5]), then the algebras $A^{(\hbar)} \rtimes G$ and $\Gamma_{\text{inv}, c}(A^{(\hbar)})$ are Morita equivalent.
Sketch of Proof. The equivalence bimodule is given by $\mathcal{A}((\hbar))(G_0)$, the quantization of the symplectic manifold $G_0$. To prove that this really defines a Morita equivalence one first observes that it suffices to prove the claim locally. One can check this for example by using the groupoid $G_U$ and the partition of unity associated to a covering of $G_0$. Using a covering by open subsets over which the restricted groupoid is isomorphic to a translation groupoid by a faithful action of a finite group, the claim is proved locally as in [DoEt] by the fact that the deformed algebra $\mathcal{A}((\hbar))(M)$ of a symplectic manifold is simple. The latter holds true since there are no nontrivial Poisson ideals in the ring of smooth functions on a symplectic manifold. $\square$
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