The reliability model of the fault-tolerant computing system with triple-modular redundancy based on the independent nodes

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Abstract. This paper deals with a reliability model of the restorable non-stop computing system with triple-modular redundancy based on independent computing nodes, taking into consideration the finite time for node activation and different node failure rates in the active and passive states. The obtained by authors generalized reliability model and calculation formulas for reliability indices for the system based on identical and independent computing nodes with the given threshold for quantity of active nodes, at which system is considered as operable, are also discussed. Finally, the application of the generalized model to the particular case of the non-stop restorable computing system with triple-modular redundancy based on independent nodes and calculation examples for reliability indices are also provided.

1. Introduction
In the modern world, a rapid development of information technologies and their application in different spheres of human activities is observed. A person has a deal with a large amount of information almost every day. He creates, stores, processes and transmits it by using personal computers and mobile devices. Moreover, the enterprises use specialized data storage and processing systems [1, 2], which are used as the base for information systems to provide enterprise business processes.

A special place in the modern world is occupied by the distributed data processing systems [3, 4], such as high-availability clusters and fault-tolerant computing systems, which provides fault-tolerant data processing. For such systems, it is important to know the reliability indices to assess the risks for the business processes and the degree of reduction of such risks due to the application of fault-tolerant technologies. In this situation, the development of reliability models and analysis of reliability indices for the fault-tolerant computing systems is quite an urgent need of the day.

Currently, a set of fundamental reliability models and calculation methods for reliability indices are discussed in Russian textbooks [5, 6] and several specialized models for the computer systems and networks are available in international literature [7, 8]. However, such models are based on the simplified two-state model of the repairable element and do not take into consideration the specific technical issues of modern data processing systems, based on a set of computing nodes: the finite time for node activation and different failure rates of the node in active and passive states.
Within the research work of the authors, in the field of fault-tolerant data storage, transmission and processing systems \[9, 10\], the reliability model for restorable non-stop computing systems with triple-modular redundancy, based on independent nodes, taking into consideration the finite time for activation of computing nodes and different node failure rates in active and passive states, was developed. A base three-state reliability model of the computing node and calculation formulas for reliability indices were derived by the authors. Later, this model and formulas were generalized for the specific case of the system, based on independent and identical nodes, with the given bottom threshold for the quantity of active nodes, at which the computing system is considered as operable. At last, the generalized model and formulas were applied for the case of a computing system with triple-modular redundancy.

2. The base three-state reliability model of the computing node

At first, let us introduce a reliability model of the computing node based on Markov chain with the next state-space and conditions of transitions between states:

- State P (passive) – the node is up, but passive, and does not provide computing operations because of software initialization. From this state, the node can either pass to state A with rate \( \gamma_N \) (activation rate), or pass to state F with rate \( \lambda_P \) (failure rate in the passive state).
- State A (active) – the node is up and active, and provides the computing operations. From this state, the node can pass to state F with rate \( \lambda_A \) (failure rate in the active state).
- State F (fault) – the node is down. From this state, the node can pass to state P with rate \( \mu_N \) (rate of the repair operations).

Figure 1 shows Markov chain, which graphically represents the state-space and transition conditions according to the discussed reliability model of the computing node:

![Figure 1. The base three-state reliability model of the computing node.](image)

Accordingly, the differential equations of Kolmogorov-Chapman for this Markov chain is as follows:

\[
\begin{align*}
  P_P(0) &= 1; \quad P_A(0) = 0; \quad P_F(0) = 0; \\
  \frac{dP_P(t)}{dt} &= -\lambda_P P_P(t) + \gamma_N P_F(t) + \mu_N P_F(t) \\
  \frac{dP_A(t)}{dt} &= \gamma_N P_F(t) - \lambda_A P_A(t) \\
  \frac{dP_F(t)}{dt} &= \lambda_P P_P(t) + \lambda_A P_A(t) - \mu_N P_F(t).
\end{align*}
\]  

\[ (1) \]

Where \( \lambda_A \) - the failure rate of the computing node in the active state, \( \lambda_P \) - the failure rate of the computing node in the passive state, \( \mu_N \) - the repair rate of the computing node, and \( \gamma_N \) - the activation rate of the computing node (the rate of passing from the passive state to the activate state).

Considering \( t \to \infty \), when Markov process reaches the steady state and derivatives of probability functions tending to zero, we obtain the following formulas for stationary probabilities of the states:
\[
\begin{align*}
P_F(\infty) & = \frac{\mu_N \lambda_A}{\mu_N \gamma_N + \lambda_A (\mu_N + \gamma_N + \lambda_p)}; \\
P_A(\infty) & = \frac{\mu_N \gamma_N}{\mu_N \gamma_N + \lambda_A (\mu_N + \gamma_N + \lambda_p)}; \\
P_F(\infty) & = \frac{\lambda_A (\gamma_N + \lambda_p)}{\mu_N \gamma_N + \lambda_A (\mu_N + \gamma_N + \lambda_p)}.
\end{align*}
\] (2)

Considering state A as the only operable state for the computing node, we also obtain the formula for the stationary availability factor of the computing node:
\[
K = P_A(\infty) = \frac{\mu_N \gamma_N}{\mu_N \gamma_N + \lambda_A (\mu_N + \gamma_N + \lambda_p)}. \tag{3}
\]

Moreover, the formula for the mean time to failure of the computing node can be easily obtained by using the topological method for the reliability models based on the Markov chains. It can be calculated as a ratio of the availability factor to the weighted sum of probabilities of all operable states, each of which is multiplied by the sum of all rates of transitions to all inoperable states. In our case, we have only one operable state, A, with only one transition to state F with rate \( \lambda_A \), thus we obtain the following simple formula for the mean time to failure:
\[
T_F = \frac{P_A(\infty)}{\lambda_A P_A(\infty)} = \frac{1}{\lambda_A}. \tag{4}
\]

At last, we can easily obtain the formula for the mean time to the recovery of the computing node, which takes into consideration both repair and activation rates, by using the fundamental relation between the availability factor, mean time to failure and mean time to recovery:
\[
T_R = \frac{1 - K}{K}T_F = \frac{\mu_N + \gamma_N + \lambda_p}{\mu_N \gamma_N}. \tag{5}
\]

3. A generalized reliability model of the system based on independent computing nodes

Let us now introduce a computing system with \( n \) independent and identical computing nodes. Each of the nodes can independently be in one of the three states and reliability parameters of all nodes are identical – it is our main assumption within this scientific research.

Next, let us introduce a state-space for the computing system with the two-dimensional numbering scheme, in which each states represent some quantity of \( i \) active nodes, \( j \) faulty nodes, \( 0 \leq i \leq n \), \( 0 \leq j \leq n \), \( i + j \leq n \), and \( n - i - j \) passive nodes. The total number of states is equal to \((n + 1)(n + 2)/2\). From state \((i, j)\), the system can pass to one of the neighbor states:

- State \((i - 1, j + 1)\) with rate \( \lambda_A \) (failure of one of active nodes).
- State \((i + 1, j)\) with rate \( (n - i - j)\gamma_N \) (activation of one of passive nodes).
- State \((i, j + 1)\) with rate \( (n - i - j)\lambda_p \) (failure of one of passive nodes).
- State \((i, j - 1)\) with rate \( j\mu_N \) (completion of repair of one of faulty nodes).

Moreover, let us consider the computing system as operable only if at least \( s \) nodes are active (provide computing operations), \( 1 \leq s \leq n \). Accordingly, states \((i \geq s, j)\) of the system are considered as operable, and states \((i < s, j)\) - as inoperable.

Figure 2 shows Markov chain for the reliability model of the computing system with independent and identical computing nodes, and it graphically represents the state-space and transition conditions according to the discussed reliability model of the computing system.
Figure 2. The reliability model of the system based on \( n \) identical and independent computing nodes.

Mention must be made, due to node independency and identity we do not need to construct and solve the Kolmogorov-Chapman system for the large Markov chain. The formula for stationary probability of state \((i, j)\) can be obtained as a product of stationary probabilities of \( i \) nodes in the active state, stationary probabilities of \( j \) nodes in the faulty state, stationary probabilities of \( n - i - j \) nodes in passive states, multiplied by the number of all possible combinations \( n!/i!j!(n-i-j)! \) of active, faulty and passive nodes. Finally, we obtain the following formula for probability of state \((i, j)\):

\[
P_{i,j}(\infty) = \frac{n!P_A^i(\infty)P_F^j(\infty)P_P^{n-i-j}(\infty)}{i!j!(n-i-j)!} = \frac{n!\mu_N^i\lambda_N^i\gamma_N^i(\gamma_N + \lambda_P)^j\mu_N^{n-i-j}\lambda_A^{n-i-j}}{i!j!(n-i-j)!(\mu_N\gamma_N + \lambda_A(\mu_N + \gamma_N + \lambda_P))^n};
\]

\(i = 0\ldots n; \ j = 0\ldots n; \ i + j \leq n.\)  

To obtain the formulas for reliability indices of the system, we also need an additional formula for the sum of probabilities of all states for the given \( i \) active nodes:

\[
\sum_{j=0}^{n-i} P_{i,j}(\infty) = \frac{n!P_A^i(\infty)(P_F(\infty) + P_P(\infty))^{n-i}}{i!(n-i)!} = \frac{C_A^i\mu_N^i\gamma_N^i\lambda_A^{n-i}(\mu_N + \gamma_N + \lambda_P)^{n-i}}{(\mu_N\gamma_N + \lambda_A(\mu_N + \gamma_N + \lambda_P))^n}.
\]

For given bottom threshold \( s \) for the quantity of active nodes, at which system is considered as operable, we obtain the following formula for the stationary availability factor of the system as a sum of the probabilities of all operable states \((i \geq s, 0 \leq j \leq n-i)\):

\[
K = \sum_{i=s}^{n} \sum_{j=0}^{n-i} P_{i,j}(\infty) = \sum_{i=s}^{n} \sum_{j=0}^{n-i} \frac{C_A^i\mu_N^i\gamma_N^i\lambda_A^{n-i}(\mu_N + \gamma_N + \lambda_P)^{n-i}}{(\mu_N\gamma_N + \lambda_A(\mu_N + \gamma_N + \lambda_P))^n}.
\]

Accordingly, the mean time to failure can be calculated as a ratio of the stationary availability factor to the weighted sum of probabilities of the «border» operable states \((i = s, 0 \leq j \leq n-i)\), each of which is multiplied by according rate \(i\lambda_A\) of transition to the inoperable state. It may be noted that
for all «border» states, the transition rate to the inoperable state is the same and is equal to $s\lambda_A$. So, we obtain the following formula for the mean time to failure of the discussed computing system:

$$T_F = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} P_{i,j}(\infty) = \frac{\sum_{i=0}^{n-1} C_n^i \mu_N^{i-s} \gamma_N^{s-i} (\mu_N + \gamma_N + \lambda_P)^{n-i}}{s\lambda_A \sum_{j=0}^{n-1} P_{s,j}(\infty)}.$$

Finally, the formula for the mean time to recovery can be easily obtained by using the fundamental relation between the availability factor, the mean time to failure and the mean time to recovery:

$$T_R = \frac{1 - K}{K} T_F = \frac{\sum_{i=0}^{s-1} C_n^i \mu_N^{i-s} \gamma_N^{s-i} (\mu_N + \gamma_N + \lambda_P)^{n-i}}{s\lambda_A \sum_{j=0}^{n-1} P_{s,j}(\infty)}.$$

4. The reliability model of the restorable non-stop computing system with triple-modular redundancy

Lastly, let us introduce a computing system with three identical and independent nodes and triple-modular redundancy. All three nodes provide the same computing operations for the same data, and the final result is selected by using some majority voting circuit, so the system provides a valid result only if at least two of three nodes are active (provide computing operations).

It should be noted that for our reliability model we consider the computing system as a non-stop system. The system will not be halted in case of failure of two nodes, when computing results become invalid. Moreover, the system will continue functioning even if the third node fails. So, the system is «on-line» in any case, but is considered as operable (able for providing valid computing results) only if at least two of three nodes are active. This is quite an important assumption for the discussed fault-tolerant computing system with three nodes and the majority voting circuit, which is slightly different from the generic triple-modular redundancy system. In addition, we consider that the majority voting circuit is ideal and its failure rate is negligible within the scope of our reliability model.

Accordingly, the reliability model of the computing system with triple-modular redundancy is a particular case of the generalized model described above with the number of nodes equal to $n = 3$ and bottom threshold $s = 2$ for the quantity of active nodes.

Accordingly, the formula for the stationary availability factor of the computing system can be obtained by substituting $n = 3$ and $s = 2$ in formula 8:

$$K = \frac{\mu_N^2 \gamma_N^2 (\mu_N + \gamma_N + 3\lambda_A (\mu_N + \gamma_N + \lambda_P))}{(\mu_N + \gamma_N + \lambda_A (\mu_N + \gamma_N + \lambda_P))^3}.$$

Similarly, we can obtain the formula for the mean time to failure from formula 9:

$$T_F = \frac{\mu_N^2 \gamma_N^2 (\mu_N + \gamma_N + \lambda_P)}{6\lambda_A^2 (\mu_N + \gamma_N + \lambda_P)}.$$

Finally, we can obtain the formula for the mean time to recovery from formula 10:

$$T_R = \frac{3\mu_N \gamma_N + \lambda_P (\mu_N + \gamma_N + \lambda_P)}{6\mu_N^2 \gamma_N^2}.$$

5. The reliability indices calculation example

A restorable non-stop computing system with triple-modular redundancy is given. The failure rate of the computing node in the active state is $\lambda_A = 1/2920$ hour$^{-1}$ (on average, one failure per 4 months), the failure rate in the passive state is $\lambda_P = 1/8760$ hour$^{-1}$ (on average, one failure per year), node activation rate $\gamma_N = 20$ hour$^{-1}$ (on average, one activation per 3 minutes) and node repair rate
μ_N = 1/24 hour^{-1} (on average, one repair per 24 hours). The calculations of the reliability indices for the given computing system with triple-modular redundancy by using the obtained above formulas give the following values:

- Stationary availability factor by formula 11 is equal to: K_F ≈ 0.999801.
- Mean time to failure by formula 12 is equal to: T_F ≈ 60548 hours.
- Mean time to recovery by formula 13 is equal to: T_R ≈ 12 hours.

6. Conclusion

Within the scope of this article, a reliability model of the restorable non-stop computing system with triple-modular redundancy based on independent computing nodes, taking into consideration the finite time for node activation and different node failure rates in the active and passive states, is discussed. The generalized reliability model obtained by authors and calculation formulas for reliability indices for the system based on identical and independent computing nodes with the given threshold for the quantity of active nodes, at which the system is considered as operable, are also overviewed. Finally, calculation examples for reliability indices are also provided.

Scientific results, obtained by the authors, were used for designing fault-tolerant computing systems of Moscow Power Engineering Institute, Nuclear Power Plant “Balakovo” and several other medium and large-scale enterprises.

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