Sum rules for \( CP \) asymmetries of charmed baryon decays

in the \( SU(3)_F \) limit

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Abstract

Motivated by the recent LHCb observation of \( CP \) violation in charm, we study \( CP \) violation in the charmed baryon decays. A simple method to search for the \( CP \) violation relations in the flavor \( SU(3) \) limit, which is associated with a complete interchange of \( d \) and \( s \) quarks, is proposed. With this method, hundreds of \( CP \) violation sum rules in the doubly and singly charmed baryon decays can be found. As examples, the \( CP \) violation sum rules in two-body charmed baryon decays are presented. Some of the \( CP \) violation sum rules could help the experiment to find better observables. As byproducts, the branching fraction of \( \Xi_c^+ \to pK^-\pi^+ \) is predicted to be \( (1.7 \pm 0.5)\% \) in the \( U \)-spin limit and the fragmentation-fraction ratio is determined as \( f_{\Xi_c}/f_{\Lambda_c} = 0.065 \pm 0.020 \) using the LHCb data.

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I. INTRODUCTION

Very recently, the LHCb Collaboration observed the $CP$ violation in the charm sector [1], with the value of

$$\Delta A_{CP} \equiv A_{CP}(D^0 \to K^+K^-) - A_{CP}(D^0 \to \pi^+\pi^-) = (-1.54 \pm 0.29) \times 10^{-3}. \tag{1}$$

It is a milestone of particle physics, since $CP$ violation has been well established in the kaon and $B$ systems for many years [2], while the last piece of the puzzle, $CP$ violation in the charm sector, has not been observed until now. To find $CP$ violation in charm, many theoretical and experimental efforts were devoted in the past decade. On the other hand, with the discovery of doubly charmed baryon [3–5] and the progress of singly charmed baryon measurements [6–22], plenty of theoretical interests focus on charmed baryon decays [23–81]. However, only a few literatures studied the $CP$ asymmetries in charmed baryon decays [62, 73, 82]. The difference between $CP$ asymmetries of $\Lambda_c^+ \to pK^+K^-\) and $\Lambda_c^+ \to p\pi^+\pi^-\ modes has been measured by LHCb collaboration [21] and no signal of $CP$ violation is found:

$$\Delta A_{CP}^{baryon} \equiv A_{CP}(\Lambda_c^+ \to pK^+K^-) - A_{CP}(\Lambda_c^+ \to p\pi^+\pi^-) = (0.30 \pm 0.91 \pm 0.61)\% \tag{2}$$

In charmed and bottomed meson decays, some relations for $CP$ asymmetries in (or beyond) the flavor $SU(3)$ limit are found [83–94]. For example, the direct $CP$ asymmetries in $D^0 \to K^+K^-$ and $D^0 \to \pi^+\pi^-$ decays have following relation in the $U$-spin limit [86, 87]:

$$A_{CP}^{dir}(D^0 \to K^+K^-) + A_{CP}^{dir}(D^0 \to \pi^+\pi^-) = 0. \tag{3}$$

The two $CP$ asymmetries in $\Delta A_{CP}$ have opposite sign and hence constructive in $\Delta A_{CP}$. But the two $CP$ asymmetries in $\Delta A_{CP}^{baryon}$, as pointed out in [73], do not have such relation like the ones in $\Delta A_{CP}$. Prospects of measuring the $CP$ asymmetries of charmed baryon decays on LHCb [92], as well as Belle II [96], are bright. It is significative to study the relations for $CP$ asymmetries in the charmed baryon decays and then help to find some promising observables in experiments.

In Ref. [73], three $CP$ violation sum rules associated with a complete interchange of $d$ and $s$ quarks are derived. In this work, we illustrate that the complete interchange of $d$ and $s$ quarks is a universal law to search for the $CP$ violation sum rules of two charmed hadron decay channels in the flavor $SU(3)$ limit. With the universal law, hundreds of $CP$ violation sum rules can be found in the doubly and singly charmed baryon decays. The $CP$ violation sum rules could be tested in the future measurements or provide a guide to find better observables for experiments. Besides,
the branching fraction $\mathcal{B}(\Xi_c^+ \to pK^-\pi^+)$ and fragmentation-fraction ratio $f_{\Xi_c}/f_{\Lambda_b}$ are estimated in the $U$-spin limit.

The rest of this paper is organized as follows. In Sec. II the effective Hamiltonian of charm decay is decomposed into the $SU(3)$ irreducible representations. In Sec. III we derive the $CP$ violation sum rules for charmed meson and baryon decays and sum up a general law for $CP$ violation sum rules in charm. In Sec. IV we list some results of the $CP$ violation sum rules in charmed baryon decays. Sec. V is a brief summary. And the explicit $SU(3)$ decomposition of the operators in charm decays is presented in Appendix A.

## II. EFFECTIVE HAMILTONIAN OF CHARME DECAY

The effective Hamiltonian in charm quark weak decay in the SM can be written as

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left[ \sum_{q=d,s} V_{q1}^* V_{q2} \left( \sum_{i=1}^2 C_i(\mu) O_i(\mu) \right) - V_{cb}^* V_{ub} \left( \sum_{i=3}^6 C_i(\mu) O_i(\mu) + C_{8g}(\mu) O_{8g}(\mu) \right) \right],$$

where $G_F$ is the Fermi coupling constant, $C_i$ is the Wilson coefficients of operator $O_i$. The tree operators are

$$O_1 = \langle \bar{u}_\alpha q_2 \beta \rangle V_A (\bar{q}_1 \beta c_\alpha), \quad O_2 = \langle \bar{u}_\alpha q_2 \beta \rangle V_A (\bar{q}_1 \beta c_\alpha),$$

in which $\alpha, \beta$ are color indices, $q_{1,2}$ are $d$ and $s$ quarks. The QCD penguin operators are

$$O_3 = \sum_{q'=u,d,s} (\bar{u}_\alpha c_\alpha) V_A (\bar{q}'_2 q'_3) V_A, \quad O_4 = \sum_{q'=u,d,s} (\bar{u}_\alpha c_\beta) V_A (\bar{q}'_3 q'_2) V_A,$$

$$O_5 = \sum_{q'=u,d,s} (\bar{u}_\alpha c_\alpha) V_A (\bar{q}'_2 q'_3) V_A, \quad O_6 = \sum_{q'=u,d,s} (\bar{u}_\alpha c_\beta) V_A (\bar{q}'_3 q'_2) V_A,$$

and the chromomagnetic penguin operator is

$$O_{8g} = \frac{g}{8\pi^2} m_c \bar{u} \sigma_{\mu\nu} (1 + \gamma_5) T^a G^{a\mu\nu} c.$$  

The magnetic-penguin contributions can be included into the Wilson coefficients for the penguin operators following the substitutions

$$C_{3,5}(\mu) \to C_{3,5}(\mu) + \frac{\alpha_s(\mu) 2m_c^2}{8\pi N_c} C_{8g}(\mu), \quad C_{4,6}(\mu) \to C_{4,6}(\mu) - \frac{\alpha_s(\mu) 2m_c^2}{8\pi \langle l^2 \rangle} C_{8g}(\mu),$$

with the effective Wilson coefficient $C_{8g}^{\text{eff}} = C_{8g} + C_5$ and $\langle l^2 \rangle$ being the averaged invariant mass squared of the virtual gluon emitted from the magnetic penguin operator.
The charm quark decays are categorized into three types, Cabibbo-favored(CF), singly Cabibbo-suppressed(SCS), and doubly Cabibbo-suppressed(DCS) decays, with the flavor structures of
\[
c \to s \bar{d}u, \quad c \to \bar{d}d/\bar{s}s, \quad c \to d\bar{s}u,\tag{9}
\]
respectively. In the SU(3) picture, the operators in charm decays embed into the four-quark Hamiltonian,
\[
\mathcal{H}_{\text{eff}} = \sum_{i,j,k=1}^{3} H^k_{ij} O^i_j = \sum_{i,j,k=1}^{3} H^k_{ij}(q^i_k)(q^j c).	ag{10}
\]
Eq. (10) implies that the tensor components of \(H_{ij}^k\) can be obtained from the map \((\bar{u}q_1)(\bar{q}_2c) \to V_{c\bar{q}_2} V_{uq_1}\) in current-current operators and \((\bar{q}q)(\bar{u}c) \to -V_{c\bar{q}_2} V_{u\bar{c}}\) in penguin operators and the others are zero. The non-zero components of the tensor \(H_{ij}^k\) corresponding to tree operators in Eq. (10) are
\[
H^2_{13} = V_{cs}^* V_{ud}, \quad H^2_{12} = V_{cd}^* V_{ud}, \quad H^3_{13} = V_{cs}^* V_{us}, \quad H^3_{12} = V_{cd}^* V_{us},
\]
and the non-zero components of the tensor \(H_{ij}^k\) corresponding to penguin operators in Eq. (10) are
\[
H^1_{11} = -V_{cb}^* V_{ub}, \quad H^2_{21} = -V_{cb}^* V_{ub}, \quad H^3_{31} = -V_{cb}^* V_{ub}.	ag{12}
\]
The operator \(O^i_j\) is a representation of SU(3) group, which is decomposed as four irreducible representations: \(\overline{3} \otimes 3 = \overline{3} \oplus \overline{3} \oplus 6 \oplus 15\). The explicit decomposition is
\[
O^i_j = \delta^i_k \left( \frac{3}{8} O(\overline{3})^i - \frac{1}{8} O(3)^i \right) + \delta^i_k \left( \frac{3}{8} O(\overline{3})^i - \frac{1}{8} O(3)^i \right) + \epsilon^{ijl} O(6)_{lk} + O(15)^{ij}_k.	ag{13}
\]
All components of the irreducible representations are listed in Appendix A. The non-zero components of \(H^k_{ij}\) corresponding to tree operators in the SU(3) decomposition are
\[
H(6)^{22} = -\frac{1}{2} V_{cs}^* V_{ud}, \quad H(6)^{23} = \frac{1}{4} (V_{cd}^* V_{ud} - V_{cs}^* V_{us}), \quad H(6)^{33} = \frac{1}{2} V_{cd}^* V_{us},
\]
\[
H(15)^{11} = -\frac{1}{4} (V_{cd}^* V_{ud} + V_{cs}^* V_{us}) = \frac{1}{4} V_{cb}^* V_{ub}, \quad H(15)^{22} = \frac{1}{2} V_{cs}^* V_{ud}, \quad H(15)^{23} = \frac{1}{2} V_{cd}^* V_{us}, \quad H(15)^{32} = \frac{1}{2} V_{cd}^* V_{us},
\]
\[
H(\overline{3})^1 = V_{cd}^* V_{ud} - \frac{1}{8} V_{cs}^* V_{us}, \quad H(\overline{3})^3 = \frac{3}{8} V_{cs}^* V_{us} - \frac{1}{8} V_{cd}^* V_{ad},
\]
\[
H(\overline{3})^3 = V_{cd}^* V_{ud} + V_{cs}^* V_{us} = -V_{cb}^* V_{ub}.	ag{14}
\]
The non-zero components of \(H_{ij}^k\) corresponding to penguin operators in the SU(3) decomposition are
\[
H^P(\overline{3})^1 = -V_{cb}^* V_{ub}, \quad H^P(\overline{3})^1 = -3 V_{cb}^* V_{ub}.	ag{15}
\]
Here we use superscript \( P \) to differentiate penguin contributions from tree contributions. Eqs. (14) and (15) were derived in [86] for the first time. But the non-zero components \( H^{(1)}_{11} \) and \( H^{(3)}_1 \) in tree operator contributions are missing in [86].

Recent studies for charmed baryon decays in the \( SU(3) \) irreducible representation amplitude (IRA) approach [29, 40, 42, 54, 60, 61, 64, 68, 70, 71, 74, 81] do not analyze \( CP \) asymmetries because they ignore the two 3-dimensional irreducible representations and make the approximation of \( V^\ast_{cs}V_{us} \approx -V^\ast_{cd}V_{ud} \) in the 15- and 6-dimensional irreducible representations, leading to the vanishing of the contributions proportional to \( \lambda_b = V^\ast_{cb}V_{ub} \). If the contributions proportional to \( \lambda_b \) are included, the \( SU(3) \) irreducible representation amplitude approach then can be used to investigate \( CP \) asymmetries in the charmed baryon decays.

III. \( CP \) VIOLATION SUM RULES IN CHARMED MESON/BARYON DECAYS

In this section, we discuss the method to search for the relations for \( CP \) asymmetries of charm decays in the flavor \( SU(3) \) limit. We first analyze the \( CP \) violation sum rules in charmed meson and baryon decays respectively, and then sum up a general law for the \( CP \) violation sum rules in charm decays.

A. \( CP \) violation sum rules in charmed meson decays

Two \( CP \) asymmetry sum rules for \( D \to PP \) decays in the flavor \( SU(3) \) limit have been given in [86, 87]:

\[
A^\text{dir}_{CP}(D^0 \to K^+K^-) + A^\text{dir}_{CP}(D^0 \to \pi^+\pi^-) = 0, \quad (16)
\]
\[
A^\text{dir}_{CP}(D^+ \to K^+\bar{K}^0) + A^\text{dir}_{CP}(D^+_s \to \pi^+K^0) = 0. \quad (17)
\]

To see why the two sum rules correct, we express the decay amplitudes of \( D^0 \to K^+K^- \), \( D^0 \to \pi^+\pi^- \), \( D^+ \to K^+\bar{K}^0 \) and \( D^+_s \to \pi^+K^0 \) modes in the \( SU(3) \) irreducible representation amplitude (IRA) approach. The charmed-meson anti-triplet is

\[
D_i = (D^0, D^+, D^+_s). \quad (18)
\]

The pseudoscalar meson nonet is

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 \\
\frac{1}{\sqrt{2}}\pi^+ \\
-\frac{1}{\sqrt{2}}\pi^- + \frac{1}{\sqrt{6}}\eta_8 \\
K^- \\
K^0 \\
\frac{1}{\sqrt{3}}\eta_8
\end{pmatrix} + 
\begin{pmatrix}
\eta_1 & 0 & 0 \\
0 & \eta_1 & 0 \\
0 & 0 & \eta_1
\end{pmatrix}. \quad (19)
\]
To obtain the $SU(3)$ irreducible representation amplitude of $D \rightarrow PP$ decay, one takes various representations in Eqs. (14) and (15) and contracts all indices in $D^i$ and light meson $P^i_j$ with various combinations:

$$A_{D\rightarrow PP}^{\text{tree}} = a_{15} D^i H_{(15)}^k P^i_j P^i_k + b_{15} D^i H_{(15)}^k P^i_j P^i_k + c_{15} D^i H_{(15)}^k P^i_j (P^i_j)^k$$

$$+ a_6 D^i H_{(6)}^k P^i_j P^i_k + b_6 D^i H_{(6)}^k P^i_j P^i_k + c_6 D^i H_{(6)}^k P^i_j P^i_k$$

$$+ a_3 D^i H_{(3)}^k P^i_j P^i_k + b_3 D^i H_{(3)}^k P^i_j P^i_k + c_3 D^i H_{(3)}^k P^i_j P^i_k$$

$$+ d_3 D^i H_{(3)}^k P^i_j P^i_k.$$  (20)

$$A_{D\rightarrow PP}^{\text{penguin}} = P a_3 D^i H^P_{(3)} P^i_j P^i_k + P b_3 D^i H^P_{(3)} P^i_j P^i_k + P c_3 D^i H^P_{(3)} P^i_j P^i_k$$

$$+ P d_3 D^i H^P_{(3)} P^i_j P^i_k + P a_3 D^i H^P_{(3)} P^i_j P^i_k + P b_3 D^i H^P_{(3)} P^i_j P^i_k$$

$$+ P c_3 D^i H^P_{(3)} P^i_j P^i_k + P d_3 D^i H^P_{(3)} P^i_j P^i_k.$$  (21)

Notice that only the first components of $\mathcal{F}$ and $\mathcal{F}$ irreducible representations are non-zero. Some amplitudes, for example, $a_3$, $Pa_3$ and $Pa_3'$ are always appear simultaneously since they correspond to the same contraction. Noting that $H^P_{(3)} 1 = 3 H^P_{(3)} 1 = H^P_{(3)} 1 = -3 V_{cb} V_{ub}$, if we define

$$Pa = a_3 + Pa_3 + 3 Pa_3', \quad Pb = b_3 + Pb_3 + 3 Pb_3', \quad ...,$$  (22)

the amplitude of $D \rightarrow PP$ decay will be reduced to be

$$A_{D\rightarrow PP}^{\text{tree+penguin}} = a_{15} D^i H_{(15)}^k P^i_j P^i_k + b_{15} D^i H_{(15)}^k P^i_j P^i_k + c_{15} D^i H_{(15)}^k (P^i_j)^k$$

$$+ a_6 D^i H_{(6)}^k P^i_j P^i_k + b_6 D^i H_{(6)}^k P^i_j P^i_k + c_6 D^i H_{(6)}^k P^i_j P^i_k$$

$$+ Pa D^i H_{(3)}^k P^i_j P^i_k + Pb D^i H_{(3)}^k P^i_j P^i_k + Pc D^i H_{(3)}^k P^i_j P^i_k$$

$$+ Pd D^i H_{(3)}^k P^i_j P^i_k.$$  (23)
With Eq. (23), the decay amplitudes of $D^0 \rightarrow K^+K^-$, $D^0 \rightarrow \pi^+\pi^-$, $D^+ \rightarrow K^+\overline{K}^0$ and $D_s^+ \rightarrow \pi^+K^0$ modes read as

$$
\mathcal{A}(D^0 \rightarrow K^+K^-) = -\lambda_d(\frac{1}{8}a_{15} + \frac{3}{8}c_{15} + \frac{1}{4}a_{6} - \frac{1}{4}c_{6}) + \lambda_s(\frac{3}{8}a_{15} + \frac{3}{8}c_{15} + \frac{1}{4}a_{6} - \frac{1}{4}c_{6}) - \lambda_b(2Pa + Pd - a_{15}/4),
$$

(24)

$$
\mathcal{A}(D^0 \rightarrow \pi^+\pi^-) = \lambda_d(\frac{3}{8}a_{15} + \frac{3}{8}c_{15} + \frac{1}{4}a_{6} - \frac{1}{4}c_{6}) - \lambda_s(\frac{1}{8}a_{15} + \frac{1}{8}c_{15} + \frac{1}{4}a_{6} - \frac{1}{4}c_{6}) - \lambda_b(2Pa + Pd - a_{15}/4),
$$

(25)

$$
\mathcal{A}(D^+ \rightarrow K^+\overline{K}^0) = \lambda_d(\frac{3}{8}a_{15} - \frac{1}{8}c_{15} - \frac{1}{4}a_{6} + \frac{1}{4}c_{6}) - \lambda_s(\frac{1}{8}a_{15} - \frac{3}{8}c_{15} - \frac{1}{4}a_{6} + \frac{1}{4}c_{6}) - \lambda_bPd,
$$

(26)

$$
\mathcal{A}(D_s^+ \rightarrow \pi^+K^0) = -\lambda_d(\frac{1}{8}a_{15} - \frac{3}{8}c_{15} - \frac{1}{4}a_{6} + \frac{1}{4}c_{6}) + \lambda_s(\frac{3}{8}a_{15} - \frac{1}{8}c_{15} - \frac{1}{4}a_{6} + \frac{1}{4}c_{6}) - \lambda_bPd,
$$

(27)

in which $\lambda_d = V_{cd}^*V_{ud}$, $\lambda_s = V_{cs}^*V_{us}$, $\lambda_b = V_{cb}^*V_{ub}$. Eqs. (24) ~ (27) are consistent with (26) except for the last terms in Eq. (24) and Eq. (25) because of the non-vanishing $H(15)_{11}$ component in Eq. (14). From above formulas, the $CP$ violation sum rules listed in Eqs. (16) and (17) are derived if the approximation of

$$
\lambda_b\lambda_d = -\lambda_b(\lambda_s + \lambda_b) = -\lambda_b(\lambda_s + \lambda_s^2) \simeq -\lambda_b\lambda_s
$$

(28)

is used. Besides, the decay amplitude of $D^0 \rightarrow K^0\overline{K}^0$ is expressed as

$$
\mathcal{A}(D^0 \rightarrow K^0\overline{K}^0) = -\lambda_b(2Pa + \frac{1}{4}a_{15}).
$$

(29)

The direct $CP$ asymmetry in $D^0 \rightarrow K^0\overline{K}^0$ decay is zero in the flavor $SU(3)$ limit:

$$
A_{CP}^{dir}(D^0 \rightarrow K^0\overline{K}^0) = 0.
$$

(30)

For the $CP$ violation relations (16) and (17), the decay amplitudes of two channels are connected by the interchange of $\lambda_d \leftrightarrow \lambda_s$, and their initial and final are connected by the interchange of $d \leftrightarrow s$:

$$
D^+ \leftrightarrow D_s^+, \quad D^0 \leftrightarrow D_s^0, \quad K^+ \leftrightarrow \pi^+, \quad K^- \leftrightarrow \pi^-, \quad K^0 \leftrightarrow \overline{K}^0.
$$

(31)

For $D^0 \rightarrow K^0\overline{K}^0$ decay, its corresponding mode in the interchange of $d \leftrightarrow s$ is itself. So all $CP$ violation relations in Eqs. (16), (17) and (30) are associated with $U$-spin transformation.

On the other hand, Eqs. (16), (17) and (30) include all the SCS modes without $\pi^0, \eta^{(')}$ in the final states in $D \rightarrow PP$ decays. Mesons $\pi^0$ and $\eta^{(')}$ do not have definite $U$-spin quantum numbers.
Under the interchange of $d \leftrightarrow s$, there are no mesons corresponding to $\pi^0$ and $\eta^{(')}$. For example, $\pi^0$ has the quark constituent of $(\bar{d}d - \bar{u}u)/\sqrt{2}$. Under the interchange of $d \leftrightarrow s$, $(\bar{d}d - \bar{u}u)/\sqrt{2}$ turns into $(\bar{s}s - \bar{u}u)/\sqrt{2}$. No meson has the quark constituent of $(\bar{s}s - \bar{u}u)/\sqrt{2}$. So those decay channels involving $\pi^0$, $\eta^{(')}$ do not have their corresponding modes in the interchange of $d \leftrightarrow s$, and then have no simple $CP$ violation sum rules with two channels.

In fact, not only the $D \to PP$ decays, there are also some $CP$ violation sum rules in the $D \to PV$ decays.

$$A_{CP}^{dir}(D^0 \to \pi^- \rho^+) + A_{CP}^{dir}(D^0 \to K^- K^{*+}) = 0,$$  \hspace{1cm} (32)

$$A_{CP}^{dir}(D^0 \to \pi^+ \rho^-) + A_{CP}^{dir}(D^0 \to K^+ K^{-*}) = 0,$$  \hspace{1cm} (33)

$$A_{CP}^{dir}(D^+ \to \overline{K}^0 K^{*+}) + A_{CP}^{dir}(D^+_s \to \overline{K}^0 \rho^+) = 0,$$  \hspace{1cm} (34)

$$A_{CP}^{dir}(D^+ \to K^+ \overline{K}^{*0}) + A_{CP}^{dir}(D^+_s \to \pi^+ K^{*0}) = 0,$$  \hspace{1cm} (35)

$$A_{CP}^{dir}(D^0 \to K^0 \overline{K}^{*0}) + A_{CP}^{dir}(D^0 \to \overline{K}^0 K^{*0}) = 0.$$  \hspace{1cm} (36)

The detailed derivation of these sum rules is similar to $D \to PP$ and can be found in Ref. [86]. Again, all the $CP$ violation sum rules in $D \to PV$ decays are associated with a complete interchange of $d$ and $s$ quarks, and all the singly Cabibbo-suppressed $D \to PV$ modes with all final states having definite $U$-spin quantum numbers are included in Eqs. (32)~(36).

B. \textit{CP violation sum rules in charmed baryon decays}

In this subsection, we take charmed baryon decays into one pseudoscalar meson and one decuplet baryon as examples to show the complete interchange of $d \leftrightarrow s$ is still valid for the $CP$ violation sum rules in charmed baryon decays. The charmed anti-triplet baryon is expressed as

$$B_{c3} = \begin{pmatrix}
0 & \Lambda_c^+ & \Xi_c^+\\
-\Lambda_c^+ & 0 & \Xi_c^0 \\
-\Xi_c^+ & -\Xi_c^0 & 0
\end{pmatrix}.$$  \hspace{1cm} (37)
Table I: $SU(3)$ irreducible representation amplitudes in $B_{\Xi^0}\to B_{10}M$ decays, in which only those modes that all initial and final states have definite $U$-spin quantum numbers are listed.

| Channel | Amplitude |
|---------|------------|
| $\Lambda^+_c\to \Delta^0\pi^+$ | $\frac{1}{8\sqrt{2}}\lambda_d(6e_1-6e_2+5e_3-2e_4)-\frac{1}{8\sqrt{2}}\lambda_s(2e_1-2e_2-e_3-2e_4)+\frac{1}{\sqrt{6}}\lambda_b Pe$ |
| $\Lambda^+_c\to \Sigma^{++}K^0$ | $\frac{1}{8\sqrt{6}}\lambda_d(6e_1-2e_2+3e_3+2e_4)+\frac{1}{8\sqrt{6}}\lambda_s(2e_1+6e_2-e_3-2e_4)+\frac{1}{\sqrt{6}}\lambda_b Pe$ |
| $\Lambda^+_c\to \Sigma^{*0}K^+$ | $\frac{1}{8\sqrt{6}}\lambda_d(6e_1+2e_2+5e_3-2e_4)-\frac{1}{8\sqrt{6}}\lambda_s(2e_1+6e_2-e_3-2e_4)+\frac{1}{\sqrt{6}}\lambda_b Pe$ |
| $\Lambda^+_c\to \Delta^{++}\pi^-$ | $\frac{1}{8}\lambda_d(2e_1-2e_2+3e_3+2e_4)+\frac{1}{8}\lambda_s(2e_1-2e_2-e_3-2e_4)-\lambda_b Pe$ |
| $\Xi^+_c\to \Xi^{*0}K^+$ | $\frac{1}{8\sqrt{6}}\lambda_d(2e_1-2e_2-e_3-2e_4)+\frac{1}{8\sqrt{6}}\lambda_s(6e_1-6e_2+5e_3-2e_4)+\frac{1}{\sqrt{6}}\lambda_b Pe$ |
| $\Xi^+_c\to \Sigma^{*0}K^0$ | $\frac{1}{8\sqrt{6}}\lambda_d(2e_1+6e_2-e_3-2e_4)+\frac{1}{8\sqrt{6}}\lambda_s(2e_1+6e_2+3e_3+2e_4)+\frac{1}{\sqrt{6}}\lambda_b Pe$ |
| $\Xi^+_c\to \Sigma^{*+}K^-$ | $\frac{1}{8}\lambda_d(2e_1-2e_2-e_3-2e_4)+\frac{1}{8}\lambda_s(2e_1-2e_2+3e_3+2e_4)-\lambda_b Pe$ |
| $\Xi^0_c\to \Sigma^{*+}K^0$ | $\frac{1}{8\sqrt{6}}\lambda_d(2e_1-2e_2+3e_3+2e_4)+\frac{1}{8\sqrt{6}}\lambda_s(6e_1-6e_2+3e_3+2e_4)+\frac{1}{\sqrt{6}}\lambda_b Pe$ |
| $\Xi^0_c\to \Delta^{++}K^0$ | $\frac{1}{8}\lambda_d(2e_1-2e_2+3e_3+2e_4)+\frac{1}{8}\lambda_s(6e_1+2e_2+3e_3+2e_4)+\frac{1}{\sqrt{6}}\lambda_b Pe$ |
| $\Xi^0_c\to \Delta^{*0}K^0$ | $\frac{1}{8\sqrt{6}}\lambda_d(6e_1-6e_2+e_3+2e_4)+\frac{1}{8\sqrt{6}}\lambda_s(2e_1-2e_2+3e_3+2e_4)-\frac{1}{\sqrt{6}}\lambda_b Pe$ |

The light baryon decuplet is given as

$$
\Delta^{++} = B_{10}^{111}, \quad \Delta^0 = B_{10}^{222}, \quad \Omega^- = B_{10}^{333},
$$

$$
\Delta^{+} = \frac{1}{\sqrt{3}}(B_{10}^{112}+B_{10}^{121}+B_{10}^{211}), \quad \Delta^0 = \frac{1}{\sqrt{3}}(B_{10}^{122}+B_{10}^{212}+B_{10}^{222}),
$$

$$
\Sigma^{*+} = \frac{1}{\sqrt{3}}(B_{10}^{113}+B_{10}^{131}+B_{10}^{311}), \quad \Sigma^{*0} = \frac{1}{\sqrt{3}}(B_{10}^{223}+B_{10}^{322}+B_{10}^{332}),
$$

$$
\Xi^{*0} = \frac{1}{\sqrt{6}}(B_{10}^{133}+B_{10}^{313}+B_{10}^{331}), \quad \Xi^{*+} = \frac{1}{\sqrt{6}}(B_{10}^{233}+B_{10}^{323}+B_{10}^{332}),
$$

$$
\Sigma^{*0} = \frac{1}{\sqrt{6}}(B_{10}^{123}+B_{10}^{132}+B_{10}^{213}+B_{10}^{312}+B_{10}^{321}+B_{10}^{331}). \tag{38}
$$

The $SU(3)$ irreducible representation amplitude of $B_{\Xi^0}\to B_{10}M$ decay can be written as

$$
A_{B_{\Xi^0}\to B_{10}M}^{\text{Rec}} = e_1(B_{\Xi^0})_{ij} H(\Xi^0)_{kl} M_{ik}^j B_{10}^{kl} + e_2(B_{\Xi^0})_{ij} H(\Xi^0)_{kl} M_{ik}^j B_{10}^{kl} + e_3(B_{\Xi^0})_{ij} H(\Xi^0)_{kl} M_{ik}^j B_{10}^{kl}
$$

$$
+ e_4(B_{\Xi^0})_{ij} H(6)_{kl} M_{ik}^j B_{10}^{kl} + e_5(B_{\Xi^0})_{ij} H(\Xi^0)_{kl} M_{ik}^j B_{10}^{kl}
$$

$$
+ e_6(B_{\Xi^0})_{ij} H(\Xi^0)_{kl} M_{ik}^j B_{10}^{kl}, \tag{39}
$$

$$
A_{B_{\Xi^0}\to B_{10}M}^{\text{Penguin}} = P e_6(B_{\Xi^0})_{ij} H^P(\Xi^0)_{kl} M_{ik}^j B_{10}^{kl} + P e_7(B_{\Xi^0})_{ij} H^P(\Xi^0)_{kl} M_{ik}^j B_{10}^{kl}. \tag{40}
$$
Similar to $D \rightarrow PP$ decay, if we define
\[ P e = e_6 + P e_6 + 3 P e_7, \] (41)
the amplitude of $B_{\pi} \rightarrow B_{10}M$ decay will be reduced to be
\[
A_{B_{\pi} \rightarrow B_{10}M}^{\text{tree+penguin}} = e_1(B_{\pi})_{ij} H(\Pi)^i_{kl} M_{kl}^{B_{10}} B_{10}^{klm} + e_2(B_{\pi})_{ij} H(\Pi)^k_{km} M_{km}^{B_{10}} B_{10}^{klm} + e_3(B_{\pi})_{ij} H(\Pi)^j_{kl} M_{kl}^{B_{10}} B_{10}^{klm} \\
+ e_4(B_{\pi})_{ij} H(6)^j_{kl} M_{kl}^{B_{10}} B_{10}^{klm} + e_5(B_{\pi})_{ij} H(\Omega)^j_{kl} M_{kl}^{B_{10}} B_{10}^{klm} \\
+ P e(B_{\pi})_{ij} H(3)^i_{kl} M_{kl}^{B_{10}} B_{10}^{klm}, \] (42)
The first four terms are the same with the formula given in [101]. The fifth term is the decay amplitude associated with singlet $\eta_1$, and the sixth term is the amplitude proportional to $\lambda_b$. With Eq. (12), the $SU(3)$ irreducible representation amplitudes of $B_{\pi} \rightarrow B_{10}M$ decays are obtained. The results are listed in Table. I.

From Table. I seven $CP$ violation sum rules in the $SU(3)_F$ limit for the charmed baryon decays into one pseudoscalar meson and one decuplet baryon are found:
\[
A_{CP}^{\text{dir}}(\Lambda^+_{c} \rightarrow \Delta^0 \pi^+) + A_{CP}^{\text{dir}}(\Xi^+_c \rightarrow \Xi^0 K^+) = 0, \] (43)
\[
A_{CP}^{\text{dir}}(\Lambda^+_{c} \rightarrow \Sigma^+ K^0) + A_{CP}^{\text{dir}}(\Xi^+_c \rightarrow \Delta^+ K^0) = 0, \] (44)
\[
A_{CP}^{\text{dir}}(\Lambda^+_{c} \rightarrow \Sigma^0 K^+) + A_{CP}^{\text{dir}}(\Xi^+_c \rightarrow \Sigma^0 K^+) = 0, \] (45)
\[
A_{CP}^{\text{dir}}(\Lambda^+_{c} \rightarrow \Delta^{++} \pi^-) + A_{CP}^{\text{dir}}(\Xi^+_c \rightarrow \Delta^{++} K^-) = 0, \] (46)
\[
A_{CP}^{\text{dir}}(\Lambda^+_{c} \rightarrow \Sigma^- \pi^+) + A_{CP}^{\text{dir}}(\Xi^+_c \rightarrow \Xi^- K^+) = 0, \] (47)
\[
A_{CP}^{\text{dir}}(\Xi^0_c \rightarrow \Delta^0 K^0) + A_{CP}^{\text{dir}}(\Xi^0_c \rightarrow \Xi^0 K^0) = 0, \] (48)
\[
A_{CP}^{\text{dir}}(\Xi^0_c \rightarrow \Sigma^+ \pi^-) + A_{CP}^{\text{dir}}(\Xi^0_c \rightarrow \Delta^+ K^-) = 0. \] (49)

Similar to the charmed meson decays, all the $CP$ violation sum rules are associated with a complete interchange of $d$ and $s$ quarks in the initial and final states. For charmed anti-triplet baryons,
\[
\Lambda^+_c \leftrightarrow \Xi^+_c, \quad \Xi^0_c \leftrightarrow \Xi^0_c. \] (50)
And for light decuplet baryons,
\[
\Delta^0 \leftrightarrow \Xi^0, \quad \Sigma^{++} \leftrightarrow \Delta^+, \quad \Sigma^{0} \leftrightarrow \Sigma^{0}, \quad \Delta^{++} \leftrightarrow \Delta^{++}, \quad \Xi^- \leftrightarrow \Sigma^-, \quad \Delta^- \leftrightarrow \Omega^- . \] (51)
And also, Eqs. (43) ~ (49) include all the SCS modes with all associated particles having definite $U$-spin quantum numbers in $B_{\pi} \rightarrow B_{10}M$ decays.
For other types of charm baryon decay, for example $B_{c3} \rightarrow B_8 M$ decay, multi-body decay and doubly charmed baryon decay, the treatments of their $SU(3)$ irreducible representation amplitudes are similar to $B_{c3} \rightarrow B_{10} M$. Related discussions can be found in Refs. [29, 40, 42, 101]. But notice that the contributions proportional to $\lambda_b$ are neglected in these literatures. To get a complete expression of decay amplitude and then analyze the $CP$ asymmetries, the neglected terms must be found back, just like we have done in this work. One can check that the $CP$ violation sum rules associated with the complete interchange of $d$ and $s$ quarks works in various types of decay.

C. A universal law for $CP$ violation sum rules in charm sector

From above discussions, one can find whether for charmed meson or baryon decays, the $CP$ violation sum rules in the $SU(3)_F$ limit are always associated with a complete interchange of $d$ and $s$ quarks. In this subsection, we illustrate that it is a universal law in charm sector.

Firstly, the complete interchange of $d \leftrightarrow s$ quarks in initial and final states leads to the interchange of $d \leftrightarrow s$ in operators $O_{ij}^k$. It can be understood in following argument. In the IRA approach, each decay amplitude connects to one invariant tensor (in which all covariant indices are contracted with contravariant indices, see Eq. (23) for example), no matter charm meson or baryon decays and two- or multi-body decays. If a complete interchange of $d \leftrightarrow s$ quarks are performed in the tensors corresponding to initial and final states, the complete interchange of $d \leftrightarrow s$ must be performed in tensor $H_{ij}^k$ in order to keep all covariant and contravariant indices contracted. From Eq. (10), one can find $H_{ij}^k$ corresponds to $O_{ij}^k$ one by one. So the $d$ and $s$ quark constituents in operators $O_{ij}^k$ must be interchanged. In physics, if the quark constituents of all initial and final particles in one decay channel are replaced by $d \rightarrow s$ and $s \rightarrow d$, the quark constituents in the effective weak vertexes should be replaced by $d \rightarrow s$ and $s \rightarrow d$ also. The operators $O_{ij}^k$ are abstracted from the effective weak vertexes, so the quark constituents of operators $O_{ij}^k$ transform as a complete interchange of $d \leftrightarrow s$ quarks.

Secondly, the interchange of $d \leftrightarrow s$ in operators $O_{ij}^k$ leads to the decay amplitudes proportional to $\lambda_d/\lambda_s$ are connected by the interchange of $\lambda_d \leftrightarrow \lambda_s$ and the decay amplitudes proportional to $\lambda_b$ are the same in the flavor $SU(3)$ symmetry. The contributions proportional to $\lambda_d/\lambda_s$ in SCS decays are induced by following operators in the $SU(3)$ irreducible representation:

$$O(6)_{23}, \quad O(\overline{15})_{12}^{12}, \quad O(\overline{15})_{13}^{13}.$$ (52)
Under the interchange of \(d \leftrightarrow s\), these operators are transformed as

\[
O(6)_{23} \leftrightarrow -O(6)_{23}, \quad O(\overline{15})_{12}^{12} \leftrightarrow O(\overline{15})_{13}^{13}.
\]  

(53)

These properties can be read from the explicit \(SU(3)\) decomposition of \(O^{ij}_k\), see Appendix A. The corresponding CKM matrix elements then transform as

\[
H(6)_{23}^{23} \leftrightarrow -O(6)_{23}, \quad H(\overline{15})_{12}^{2} \leftrightarrow H(\overline{15})_{13}^{3}.
\]  

(54)

According to Eq. (14), Eq. (54) equals to

\[
\frac{1}{4} (\lambda_d - \lambda_s) \leftrightarrow \frac{1}{4} (\lambda_d - \lambda_s), \quad \frac{3}{8} \lambda_d - \frac{1}{8} \lambda_s \leftrightarrow \frac{3}{8} \lambda_s - \frac{1}{8} \lambda_d.
\]  

(55)

One can find Eq. (54) is equivalent to the interchange of \(\lambda_d \leftrightarrow \lambda_s\). The contributions proportional to \(\lambda_b\) in the SCS decays are induced by following operators:

\[
O(\overline{15})_{11}^{1}, \quad O(\overline{3})^{1}, \quad O(\overline{3})^{1}.
\]  

(56)

Form Appendix A, it is found these operators are invariable under the interchange of \(d \leftrightarrow s\), so as the corresponding CKM matrix elements.

Thirdly, if two decay channels have the relations that their decay amplitudes proportional to \(\lambda_d/\lambda_s\) are connected by the interchange of \(\lambda_d \leftrightarrow \lambda_s\) and the decay amplitudes proportional to \(\lambda_b\) are the same, the sum of their direct \(CP\) asymmetries is zero in the \(SU(3)_F\) limit under the approximation in Eq. (28). For one decay mode with amplitude of

\[
\mathcal{A}(i \rightarrow f) = \lambda_d A + \lambda_s B + \lambda_b C
\]

its \(CP\) asymmetry in the order of \(O(\lambda_b)\) is derived as

\[
A_{CP}^{\text{dir}}(i \rightarrow f) \simeq \frac{|\lambda_d A + \lambda_s B + \lambda_b C|^2 - |\lambda_s A + \lambda_b B + \lambda_d C|^2}{|\lambda_d A + \lambda_s B + \lambda_b C|^2 + |\lambda_s A + \lambda_b B + \lambda_d C|^2}
\]

\[
= 2 \frac{|\lambda_b| |AB| \sin(\delta_A - \delta_B) - |AC| \sin(\delta_A - \delta_C) + |BC| \sin(\delta_B - \delta_C)}{|A|^2 + |B|^2 - 2|AB| \cos(\delta_A - \delta_B)}.
\]

(58)

For one decay mode with amplitude of

\[
\mathcal{A}(i' \rightarrow f') = \lambda_d B + \lambda_s A + \lambda_b C,
\]

which is connected to Eq. (57) by \(\lambda_d \leftrightarrow \lambda_s\), its \(CP\) asymmetry in the order of \(O(\lambda_b)\) is derived as

\[
A_{CP}^{\text{dir}}(i' \rightarrow f') \simeq -2 \frac{|\lambda_d| |AB| \sin(\delta_A - \delta_B) - |AC| \sin(\delta_A - \delta_C) + |BC| \sin(\delta_B - \delta_C)}{|A|^2 + |B|^2 - 2|AB| \cos(\delta_A - \delta_B)}.
\]

(60)
It is apparent that

$$A_{\text{dir}}^{\text{CP}}(i \rightarrow f) + A_{\text{dir}}^{\text{CP}}(i' \rightarrow f') \simeq 0.$$  \hspace{1cm} (61)

Based on above analysis, a useful method to search for the $CP$ violation sum rules with two charmed hadron decay channels is proposed:

- For one type of charmed hadron decay, write down all the SCS decay modes in which the associated hadrons have definite $U$-spin quantum numbers;

- For each decay mode, find the corresponding decay mode in the complete interchange of $d \leftrightarrow s$;

- If there are two decay modes connected by the interchange of $d \leftrightarrow s$, the sum of their direct $CP$ asymmetries is zero in the $SU(3)_F$ limit;

- If the corresponding decay mode is itself, the direct $CP$ asymmetry in this mode is zero in the $SU(3)_F$ limit.

**IV. RESULTS AND DISCUSSION**

With the method proposed in Sec. III one can find many sum rules for $CP$ asymmetries in charm meson/baryon decays. There are hundreds of sum rules for $CP$ asymmetries in the singly and doubly charmed baryon decays. We’re not going to list all the $CP$ violations sum rules, but only present some of them as examples.

Under the complete interchange of $d \leftrightarrow s$, the light octet baryons are interchanged as

$$p \leftrightarrow \Sigma^{\pm}, \quad n \leftrightarrow \Xi^0, \quad \Sigma^- \leftrightarrow \Xi^-.$$  \hspace{1cm} (62)

The sum rules for $CP$ asymmetries in charmed baryon decays into one pseudoscalar meson and one octet baryon are

$$A_{\text{dir}}^{\text{CP}}(A_c^+ \rightarrow \Sigma^+ K^0) + A_{\text{dir}}^{\text{CP}}(\Xi^+_c \rightarrow p\bar{K}^0) = 0,$$  \hspace{1cm} (63)

$$A_{\text{dir}}^{\text{CP}}(A_c^+ \rightarrow n\pi^+) + A_{\text{dir}}^{\text{CP}}(\Xi^+_c \rightarrow \Xi^0 K^+) = 0,$$  \hspace{1cm} (64)

$$A_{\text{dir}}^{\text{CP}}(\Xi^0_c \rightarrow \Sigma^- \pi^+) + A_{\text{dir}}^{\text{CP}}(\Xi^0_c \rightarrow \Xi^- K^+) = 0,$$  \hspace{1cm} (65)

$$A_{\text{dir}}^{\text{CP}}(\Xi^0_c \rightarrow n\bar{K}^0) + A_{\text{dir}}^{\text{CP}}(\Xi^0_c \rightarrow \Xi^0 K^0) = 0,$$  \hspace{1cm} (66)

$$A_{\text{dir}}^{\text{CP}}(\Xi^0_c \rightarrow \Sigma^+ \pi^-) + A_{\text{dir}}^{\text{CP}}(\Xi^0_c \rightarrow pK^-) = 0.$$  \hspace{1cm} (67)
Under the complete interchange of $d \leftrightarrow s$, the doubly charmed baryons are interchanged as

$$
\Xi_{cc}^{++} \leftrightarrow \Xi_{cc}^{++}, \quad \Xi_{cc}^{+} \leftrightarrow \Omega_{cc}^{+}.
$$

(68)

The sum rules for $CP$ asymmetries in doubly charmed baryon decays into one pseudoscalar meson and one charmed triplet baryon are

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow \Lambda_{c}^{+} \pi^{+}) + A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow \Xi_{c}^{+} K^{+}) = 0,
$$

(69)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow \Xi_{c}^{+} K^{0}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Lambda_{c}^{+} K^{0}) = 0,
$$

(70)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{0} \rightarrow \Xi_{c}^{0} K^{+}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{0} \rightarrow \Xi_{c}^{0} \pi^{+}) = 0.
$$

(71)

Under the complete interchange of $d \leftrightarrow s$, the charmed sextet baryons are interchanged as

$$
\Sigma_{c}^{+} \leftrightarrow \Xi_{c}^{+}, \quad \Sigma_{c}^{++} \leftrightarrow \Sigma_{c}^{++}, \quad \Xi_{c}^{0} \leftrightarrow \Omega_{c}, \quad \Sigma_{c}^{0} \leftrightarrow \Omega_{c}^{0}.
$$

(72)

The sum rules for $CP$ asymmetries in doubly charmed baryon decays into one pseudoscalar meson and one charmed sextet baryon are

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow \Sigma_{c}^{+} \pi^{+}) + A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow \Xi_{cc}^{+} K^{+}) = 0,
$$

(73)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow \Sigma_{cc}^{+} \pi^{+}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Sigma_{cc}^{++} K^{-}) = 0,
$$

(74)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{0} \rightarrow \Sigma_{c}^{0} \pi^{+}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{0} \rightarrow \Omega_{c}^{0} K^{+}) = 0,
$$

(75)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow \Xi_{c}^{+} K^{0}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Sigma_{c}^{0} K^{0}) = 0,
$$

(76)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{0} \rightarrow \Xi_{c}^{0} K^{+}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{0} \rightarrow \Xi_{c}^{0} \pi^{+}) = 0.
$$

(77)

With those interchange rules mentioned above, the $CP$ violation sum rules in doubly charmed baryon decays into one charmed meson and one octet baryon are

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow \Sigma^{+} D_{s}^{+}) + A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow p D^{+}) = 0,
$$

(78)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow p D^{0}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Sigma^{0} D^{0}) = 0,
$$

(79)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow n D^{+}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Xi_{c}^{0} D_{s}^{+}) = 0.
$$

(80)

The $CP$ violation sum rules in doubly charmed baryon decays into one charmed meson and one decuplet baryon are

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow \Delta^{+} D^{+}) + A^{\text{dir}}_{CP}(\Xi_{cc}^{++} \rightarrow \Sigma^{++} D_{s}^{+}) = 0,
$$

(81)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow \Delta^{+} D^{0}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Sigma^{+} D^{0}) = 0,
$$

(82)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow \Delta^{0} D^{+}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Xi^{0} D_{s}^{+}) = 0,
$$

(83)

$$
A^{\text{dir}}_{CP}(\Xi_{cc}^{+} \rightarrow \Sigma^{0} D_{s}^{+}) + A^{\text{dir}}_{CP}(\Omega_{cc}^{+} \rightarrow \Sigma^{0} D^{+}) = 0.
$$

(84)

14
For three-body decays, we only list the CP violation sum rules in charmed baryon decays into one octet baryon and two pseudoscalar mesons as examples:

\[
\begin{align*}
A_{CP}^{dir}(\Lambda_c^+ \rightarrow pK^-K^+) + A_{CP}^{dir}(\Xi_c^+ \rightarrow \Sigma^+\pi^-\pi^+) &= 0, \\
A_{CP}^{dir}(\Lambda_c^+ \rightarrow p\pi^-\pi^+) + A_{CP}^{dir}(\Xi_c^+ \rightarrow \Sigma^+K^-K^+) &= 0, \\
A_{CP}^{dir}(\Lambda_c^+ \rightarrow \Sigma^+\pi^-K^+) + A_{CP}^{dir}(\Xi_c^+ \rightarrow pK^-\pi^+) &= 0, \\
A_{CP}^{dir}(\Lambda_c^+ \rightarrow \Sigma^-\pi^+K^+) + A_{CP}^{dir}(\Xi_c^+ \rightarrow \Xi^-K^+\pi^+) &= 0, \\
A_{CP}^{dir}(\Lambda_c^+ \rightarrow nK^+\overline{K}^0) + A_{CP}^{dir}(\Xi_c^+ \rightarrow \Xi^0\pi^+K^0) &= 0, \\
A_{CP}^{dir}(\Xi_c^0 \rightarrow \Sigma^+K^-K^0) + A_{CP}^{dir}(\Xi_c^0 \rightarrow p\pi^-\overline{K}^0) &= 0, \\
A_{CP}^{dir}(\Xi_c^0 \rightarrow \Sigma^-K^+\overline{K}^0) + A_{CP}^{dir}(\Xi_c^0 \rightarrow \Xi^-\pi^+K^0) &= 0, \\
A_{CP}^{dir}(\Xi_c^0 \rightarrow \Xi^0\pi^-K^+) + A_{CP}^{dir}(\Xi_c^0 \rightarrow nK^-\pi^+) &= 0.
\end{align*}
\]

The first three sum rules are the same with [73]. In all above sum rules, the pseudoscalar mesons can be replaced by vector mesons by following correspondence:

\[
\pi^+ \rightarrow \rho^+, \quad \pi^- \rightarrow \rho^-, \quad K^+ \rightarrow K^{*+}, \quad K^- \rightarrow K^{*-}, \quad K^0 \rightarrow K^{*0}, \quad \overline{K}^0 \rightarrow \overline{K}^{*0}.
\]

The CP violation sum rules are derived in the U-spin limit. Considering the U-spin breaking, the CP violation sum rules are no longer valid, as pointed out in [73]. Since the U-spin breaking is sizeable in charm sector, the CP violation sum rules might not be reliable. But they indicate that the CP asymmetries in some decay modes have opposite sign and then can be used to find some promising observables in experiments. In charmed meson decays, Eq. (4) makes the two CP asymmetries in observable \( \Delta A_{CP} \equiv A_{CP}(D^0 \rightarrow K^+K^-) - A_{CP}(D^0 \rightarrow \pi^+\pi^-) \) constructive. Similarly, one can use the CP violation sum rules in charmed baryon decays to construct some observables in which two CP asymmetries are constructive. Some observables are selected for experimental discretion:

\[
\begin{align*}
\Delta A_{CP}^{baryon,1} &= A_{CP}(\Lambda_c^+ \rightarrow \Sigma^+K^{*0}) - A_{CP}(\Xi_c^+ \rightarrow p\overline{K}^{*0}), \\
\Delta A_{CP}^{baryon,2} &= A_{CP}(\Xi_c^0 \rightarrow \Sigma^+\pi^-) - A_{CP}(\Xi_c^0 \rightarrow pK^-), \\
\Delta A_{CP}^{baryon,3} &= A_{CP}(\Lambda_c^+ \rightarrow \Delta^{*+}\pi^-) - A_{CP}(\Xi_c^+ \rightarrow \Delta^{*+}K^-), \\
\Delta A_{CP}^{baryon,4} &= A_{CP}(\Lambda_c^+ \rightarrow \Sigma^+\pi^-K^+) - A_{CP}(\Xi_c^+ \rightarrow pK^-\pi^+).
\end{align*}
\]

If the contributions proportional to \( \lambda_b \) are neglected, the decay amplitudes of the two channels connected by the interchange of \( d \leftrightarrow s \) are the same (except for a minus sign) in the \( SU(3)_F \) limit (see Eq. [57] and Eq. [59]). One can use this relation to predict the branching fractions. As an
example, we estimate the branching fraction of $\Xi^+_c \rightarrow pK^-\pi^+$. The integration over the phase space of the three-body decay $B_c \rightarrow BM_1M_2$ relies on the equation of [2]

$$\Gamma(B_c \rightarrow BM_1M_2) = \int_{m_{12}}^{m_{23}} \frac{|A(B_c \rightarrow BM_1M_2)|^2}{32m^3_B} dm_{12} dm_{23},$$

(98)

where $m_{12} = p_{M_1} + p_{M_2}$ and $m_{23} = p_{M_2} + p_g$. With the experimental data given in [2],

$$Br(\Lambda^+_c \rightarrow \Sigma^+\pi^-K^+) = (2.1 \pm 0.6) \times 10^{-3},$$

(99)

and the relation

$$|A(\Xi^+_c \rightarrow pK^-\pi^+)| \simeq |A(\Lambda^+_c \rightarrow \Sigma^+\pi^-K^+)|,$$

(100)

the branching fraction of $\Xi^+_c \rightarrow pK^-\pi^+$ decay is predicted to be

$$Br(\Xi^+_c \rightarrow pK^-\pi^+) = (1.7 \pm 0.5)\%.$$

(101)

One can find the branching fraction $Br(\Xi^+_c \rightarrow pK^-\pi^+)$ is larger than $Br(\Lambda^+_c \rightarrow \Sigma^+\pi^-K^+)$ because of the larger phase space and the longer lifetime of $\Xi^+_c$. But it is still smaller than the predictions given in [60, 70]. In above estimation, only the $U$-spin symmetry for decay amplitude, but for the phase space, is used. It is plausible since the global fit in Ref. [60, 61, 64, 68, 70, 74, 81] give the reasonable estimations for branching fractions of charmed baryon decays. The uncertainty in Eq. (101) is dominated by the branching fraction of $\Lambda^+_c \rightarrow \Sigma^+\pi^-K^+$ decay and does not include the $U$-spin breaking effects. It is not available to estimate the $U$-spin breaking effects at the current stage since the understanding of the dynamics of charmed baryon decays is still a challenge. Some discussions about the uncertainty induced by $U$-spin breaking can be found in [66].

With the method introduced in [66], and the LHCb data [102]

$$\frac{f_{\Xi_b}}{f_{\Lambda_b}} \frac{Br(\Xi^+_b \rightarrow \Xi^+_c\pi^-)}{Br(\Lambda^+_b \rightarrow \Lambda^+_c\pi^-)} \frac{Br(\Xi^+_c \rightarrow pK^-\pi^+)}{Br(\Lambda^+_c \rightarrow pK^-\pi^+)} = (1.88 \pm 0.04 \pm 0.03) \times 10^{-2},$$

(102)

the fragmentation-fraction ratio $f_{\Xi_b}/f_{\Lambda_b}$ is determined to be

$$f_{\Xi_b}/f_{\Lambda_b} = 0.065 \pm 0.020.$$

(103)

Recent measurement confirmed this result [103]. Our result is consistent with the one obtained via $\Lambda^+_b \rightarrow J/\psi\Lambda^0$ [104], $f_{\Xi_b}/f_{\Lambda_b} = 0.11 \pm 0.03$, the one via $\Xi^-_b \rightarrow J/\psi\Xi^-$ [105], $f_{\Xi_b}/f_{\Lambda_b} = 0.108 \pm 0.034$, and the one via the diquark model for $\Xi^-_b \rightarrow \Lambda^0_b\pi^-$ [106] using the LHCb data [107], $f_{\Xi_b}/f_{\Lambda_b} = 0.08 \pm 0.03$. The detailed comparison for different methods of estimating $f_{\Xi_b}/f_{\Lambda_b}$ can be found in [66].
V. SUMMARY

In summary, we find that if two singly Cabibbo-suppressed decay modes of charmed hadrons are connected by a complete interchange of $d$ and $s$ quarks, the sum of their direct $CP$ asymmetries is zero in the flavor $SU(3)$ limit. According to this conclusion, many $CP$ violation sum rules can be found in the doubly and singly charmed baryon decays. Some of them could help to find better observables in experiments. As byproducts, the branching fraction $Br(\Xi^+_c \to pK^-\pi^+)$ is predicted to be $(1.7 \pm 0.5)\%$ in the $U$-spin limit, and the fragmentation-fraction ratio is determined as $f_{\Xi_b}/f_{\Lambda_b} = 0.065 \pm 0.020$.

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Appendix A: $SU(3)$ decomposition of operator $O^{ij}_k$

All components of the $SU(3)$ decomposition in Eq. (4) are listed following.

$\bar{3}$ presentation:

\[
O(\bar{3})^1 = (\bar{uu})(\bar{uc}) + (\bar{ud})(\bar{dc}) + (\bar{us})(\bar{sc}), \quad O(\bar{3})^2 = (\bar{du})(\bar{uc}) + (\bar{dd})(\bar{dc}) + (\bar{ds})(\bar{sc}), \quad O(\bar{3})^3 = (\bar{su})(\bar{uc}) + (\bar{sd})(\bar{dc}) + (\bar{ss})(\bar{sc}).
\]

$\bar{3}'$ presentation:

\[
O(\bar{3}')^1 = (\bar{uu})(\bar{uc}) + (\bar{dd})(\bar{dc}) + (\bar{ss})(\bar{sc}), \quad O(\bar{3}')^2 = (\bar{du})(\bar{dc}) + (\bar{dd})(\bar{dc}) + (\bar{ds})(\bar{sc}), \quad O(\bar{3}')^3 = (\bar{sd})(\bar{dc}) + (\bar{dd})(\bar{sc}) + (\bar{ss})(\bar{sc}).
\]

$6$ presentation:

\[
O(6)_{11} = \frac{1}{2}[(\bar{du})(\bar{sc}) - (\bar{su})(\bar{dc})], \quad O(6)_{22} = \frac{1}{2}[(\bar{sd})(\bar{uc}) - (\bar{ud})(\bar{sc})],
\]

\[
O(6)_{33} = \frac{1}{2}[(\bar{us})(\bar{dc}) - (\bar{ds})(\bar{uc})],
\]

\[
O(6)_{12} = \frac{1}{4}[(\bar{su})(\bar{uc}) - (\bar{uu})(\bar{sc}) + (\bar{dd})(\bar{sc}) - (\bar{sd})(\bar{dc})],
\]

\[
O(6)_{23} = \frac{1}{4}[(\bar{ud})(\bar{dc}) - (\bar{dd})(\bar{uc}) + (\bar{ss})(\bar{uc}) - (\bar{us})(\bar{sc})],
\]

\[
O(6)_{31} = \frac{1}{4}[(\bar{ds})(\bar{sc}) - (\bar{ss})(\bar{dc}) + (\bar{uu})(\bar{dc}) - (\bar{du})(\bar{uc})].
\]
\[ \mathbf{T_5} \text{ presentation:} \]

\[
O(\mathbf{T_5})_{11}^{1} = \frac{1}{2}(\bar{u}u)(\bar{d}c) - \frac{1}{4}((\bar{u}s)(\bar{sc}) + (\bar{u}d)(\bar{dc}) + (\bar{d}d)(\bar{uc}) + (\bar{s}s)(\bar{uc})),
\]

\[
O(\mathbf{T_5})_{22}^{2} = \frac{1}{2}(\bar{d}d)(\bar{dc}) - \frac{1}{4}((\bar{d}u)(\bar{uc}) + (\bar{d}s)(\bar{sc}) + (\bar{u}u)(\bar{dc}) + (\bar{s}s)(\bar{dc})),
\]

\[
O(\mathbf{T_5})_{33}^{3} = \frac{1}{2}(\bar{s}s)(\bar{sc}) - \frac{1}{4}((\bar{s}u)(\bar{uc}) + (\bar{s}d)(\bar{dc}) + (\bar{u}u)(\bar{sc}) + (\bar{d}d)(\bar{sc})),
\]

\[
O(\mathbf{T_5})_{11}^{1} = \frac{1}{2}((\bar{d}d)(\bar{uc}) + (\bar{u}d)(\bar{sc})),
\]

\[
O(\mathbf{T_5})_{22}^{2} = \frac{1}{2}((\bar{d}d)(\bar{uc}) + (\bar{d}d)(\bar{uc})),
\]

\[
O(\mathbf{T_5})_{33}^{3} = \frac{1}{2}((\bar{s}s)(\bar{sc}) + (\bar{u}u)(\bar{sc})) - \frac{1}{8}((\bar{s}s)(\bar{sc}) + (\bar{s}s)(\bar{uc})),
\]

\[
O(\mathbf{T_5})_{12}^{12} = \frac{3}{8}((\bar{u}u)(\bar{dc}) + (\bar{d}u)(\bar{uc})) - \frac{1}{4}((\bar{dd})(\bar{dc}) - \frac{1}{8}((\bar{d}s)(\bar{sc}) + (\bar{s}s)(\bar{dc})),
\]

\[
O(\mathbf{T_5})_{13}^{13} = \frac{3}{8}((\bar{u}u)(\bar{dc}) + (\bar{d}u)(\bar{uc})) - \frac{1}{4}((\bar{s}s)(\bar{sc}) - \frac{1}{8}((\bar{d}s)(\bar{dc}) + (\bar{d}d)(\bar{sc})),
\]

\[
O(\mathbf{T_5})_{23}^{23} = \frac{3}{8}((\bar{d}d)(\bar{uc}) + (\bar{s}d)(\bar{dc})) - \frac{1}{4}((\bar{s}s)(\bar{uc}) - \frac{1}{8}((\bar{s}u)(\bar{uc}) + (\bar{u}u)(\bar{sc})),
\]

\[
O(\mathbf{T_5})_{33}^{23} = \frac{3}{8}((\bar{d}d)(\bar{uc}) + (\bar{s}d)(\bar{dc})) - \frac{1}{4}((\bar{s}d)(\bar{dc}) - \frac{1}{8}((\bar{d}d)(\bar{uc}) + (\bar{u}u)(\bar{dc})).
\]

(A4)

Above results are consistent with Ref. [86]. Operators \(O(6)_{ij}\) are symmetric in the interchange of their two indices and can be written as \(O(6)_{ij}^{T} = \epsilon^{ij}O(6)_{ik}\). There are 18 operators in irreducible representation \(\mathbf{T_5}\), but only 15 of them are independent because the following equations:

\[
O(\mathbf{T_5})_{11}^{1} = -[O(\mathbf{T_5})_{22}^{2} + O(\mathbf{T_5})_{33}^{3}], \quad O(\mathbf{T_5})_{22}^{2} = -[O(\mathbf{T_5})_{11}^{1} + O(\mathbf{T_5})_{33}^{3}],
\]

\[
O(\mathbf{T_5})_{33}^{3} = -[O(\mathbf{T_5})_{11}^{1} + O(\mathbf{T_5})_{22}^{2}].
\]

(A5)

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