Siegel’s problem for $E$-functions

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Périodes, motifs et équations différentielles : entre arithmétique et géométrie

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Definition 1

A power series \( F(z) = \sum_{n=0}^{\infty} a_n z^n / n! \in \overline{\mathbb{Q}}[[z]] \) is an \( E \)-function if

(i) \( F(z) \) is solution of a non-zero linear differential equation with coefficients in \( \overline{\mathbb{Q}}(z) \).

(ii) There exists \( C > 0 \) such that for any \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and any \( n \geq 0 \), \( |\sigma(a_n)| \leq C^{n+1} \).

(iii) There exists a sequence of positive integers \( d_n \), with \( d_n \leq C^{n+1} \), such that \( d_n a_m \) are algebraic integers for all \( m \leq n \).

Siegel's definition was more general: the two bounds (\( \cdots \)) \( \leq C^{n+1} \) are replaced by: for all \( \varepsilon > 0 \), (\( \cdots \)) \( \leq n!^\varepsilon \) for all \( n \geq N(\varepsilon) \).

\( E \)-functions are entire functions. They form a ring stable under \( \frac{d}{dz} \) and \( \int_0^z \). If \( F(z) \) is an \( E \)-function and \( \alpha \in \overline{\mathbb{Q}} \), then \( F(\alpha z) \) is an \( E \)-function.

A power series \( \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]] \) is a \( G \)-function if \( \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \) is an \( E \)-function (in the sense of Definition 1).
Examples

**E-functions:** polynomials in $\overline{\mathbb{Q}}[z]$,

$$
\begin{align*}
\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\
L(z) &:= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \right) \frac{z^n}{n!}, \\
H(z) &:= \sum_{n=0}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{k} \right) \frac{z^n}{n!}, \\
J_0(z) &:= \sum_{n=0}^{\infty} \frac{(iz/2)^{2n}}{n!^2}.
\end{align*}
$$

**G-functions:** algebraic functions over $\overline{\mathbb{Q}}(z)$ regular at 0, \log(1 - z) = - \sum_{n=1}^{\infty} z^n / n and (multiple) polylogarithms

$$
\begin{align*}
\text{Li}_s(z) &:= \sum_{n=1}^{\infty} \frac{z^n}{n^s} \quad (s \in \mathbb{Z}), \\
\sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \quad (s_1, s_2, \ldots, s_k \in \mathbb{Z}), \\
\frac{1}{\pi} \int_{0}^{1} \frac{\sqrt{x(1-x)}}{1-zx} \, dx.
\end{align*}
$$

The intersection of both classes of series is reduced to $\overline{\mathbb{Q}}[z]$. 
Why are $E$- and $G$-functions interesting?

**Theorem 1 (Lindemann-Weierstrass)**

If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independent, then $(e^{\alpha_1 z}, \ldots, e^{\alpha_n z}$ are $\overline{\mathbb{Q}(z)}$-algebraically independent and) $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$ are $\overline{\mathbb{Q}}$-algebraically independent.

Consequences:

- For any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, $\exp(\alpha) \notin \overline{\mathbb{Q}}$.

- For any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, $\log(\alpha) \notin \overline{\mathbb{Q}}$ for any given determination of the logarithm.

Recall that $\exp(z)$ is an $E$-function while $\log(1 - z)$ is a $G$-function: Siegel’s aim was to generalize the above statements.
The Siegel-Shidlovskii Theorem

Theorem 2 (Siegel-Shidlovskii, 1929-1956)

Let \( Y = ^t(F_1, \ldots, F_n) \) be a vector of \( E \)-functions (in Siegel’s sense) and \( A \in M_{n \times n}(\mathbb{Q}(z)) \) such that \( Y' = AY \).

Let \( T \in \overline{\mathbb{Q}}[z] \setminus \{0\} \) a common denominator of the entries of \( A \), of minimal degree.

Then, for all \( \alpha \in \overline{\mathbb{Q}} \) such that \( \alpha T(\alpha) \neq 0 \),

\[
\deg \text{tr}_{\mathbb{Q}(z)\mathbb{Q}(z)}(F_1(z), \ldots, F_n(z)) = \deg \text{tr}_{\mathbb{Q}(z)\mathbb{Q}(z)}(F_1(\alpha), \ldots, F_n(\alpha)).
\]

We obtain (a version of) the Lindemann-Weierstrass Theorem with \( F_j(z) = e^{\alpha_j z} \), \( A = \text{Diag}(\alpha_j) \) and \( \alpha = 1 \).

Siegel, 1929: The \( E \)-functions \( J_0(z) \) et \( J'_0(z) \) are \( \overline{\mathbb{Q}}(z) \)-algebraically independent and

\[
\begin{pmatrix} J'_0(z) \\ J''_0(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{z} \end{pmatrix} \begin{pmatrix} J_0(z) \\ J'_0(z) \end{pmatrix}, \ T(z) = z.
\]

For all \( \alpha \in \overline{\mathbb{Q}} \setminus \{0\} \), the numbers \( J_0(\alpha) \) et \( J'_0(\alpha) \) are \( \overline{\mathbb{Q}} \)-algebraically independent.
After the Siegel-Shidlovskii Theorem

André obtained in 2000 a new proof of the Siegel-Shidlovskii Theorem (in the restricted sense). He used the special properties of the differential equations satisfied by such $E$-functions.

These properties are inherited from those of the diff equations satisfied by $G$-functions, found in the 80’s by André, Bombieri, Chudnovsky, Galochkin, Katz: The non-zero minimal differential equation satisfied by a given $G$-function is fuchsian with rational exponents.

Beukers, 2006: If $Y = t(F_1, \ldots, F_n)$ is a vector of $E$-functions (in the restricted sense) such that $Y' = AY$ and the $F_j$’s are linearly independent over $\overline{\mathbb{Q}}(z)$, then for any $\alpha \in \overline{\mathbb{Q}}^*$ not a singularity of $A$, the numbers $F_1(\alpha), \ldots, F_n(\alpha)$ are linearly independent over $\overline{\mathbb{Q}}$.

Consequence: for any non-polynomial $E$-function $F(z)$, there are only finitely many $\alpha \in \overline{\mathbb{Q}}$ such that $F(\alpha) \in \overline{\mathbb{Q}}$. This is not a consequence of the Siegel-Shidlovskii Theorem. An exotic evaluation: $J_0^{(4)}(\pm \sqrt{3}) = 0$.

In 2014, André extended Beukers’ lifting theorem to the case of $E$-functions in Siegel’s sense.
Chudnovsky’s Theorem

Chudnovsky “completed” Siegel’s program for $G$-functions.

**Theorem 3 (Chudnovsky 1984)**

Let $Y(z) = ^t(F_1(z), \ldots, F_S(z))$ be a vector of $G$-functions solution of

$$Y'(z) = A(z)Y(z), \quad A(z) \in M_S(\overline{\mathbb{Q}}(z)).$$

Assume $F_1(z), \ldots, F_S(z)$ to be $\overline{\mathbb{Q}}(z)$-algebraically independent.

For any $d$, there exists $C_{Y,d} > 0$ such that, for any $\alpha \in \overline{\mathbb{Q}}$ of degree $\leq d$ with

$$0 < |\alpha| < \exp \left( - C_{Y,d} \log (H(\alpha))^{\frac{4S}{4S+1}} \right),$$

there does not exist a polynomial relation of degree $\leq d$ between the values $1, F_1(\alpha), \ldots, F_S(\alpha)$ over $\mathbb{Q}(\alpha)$.

A condition like (1) is unavoidable: there exist transcendental $G$-functions that take algebraic values on a dense set of algebraic points in the disk of convergence (Wolfart).
Hypergeometric $E$-functions

Set $(x)_m := x(x + 1) \cdots (x + m - 1)$.

Siegel: the “hypergeometric” series

\[ pF_q \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z^{q-p+1} \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{n!(b_1)_n \cdots (b_q)_n} z^{n(q-p+1)}, \]

is an $E$-function when $q \geq p \geq 1$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all $j$.

$L(z)$ and $H(z)$ are not of $pF_q(z^{q-p+1})$ type but

\[ \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \frac{(n+k)}{n} \right) z^n \frac{1}{n!} = e^{(3-2\sqrt{2})z} \cdot {}_1F_1 \left[ \begin{array}{c} 1/2 \\ 1 \end{array} ; 4\sqrt{2}z \right], \]

\[ \sum_{n=0}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{k} \right) z^n \frac{1}{n!} = ze^z \cdot {}_2F_2 \left[ \begin{array}{c} 1, 1 \\ 2, 2 \end{array} ; -z \right]. \]
Siegel’s question

Question 1 (Siegel, 1949)

Is it possible to write every $E$-function (in Siegel’s sense) as a polynomial with coefficients in $\mathbb{Q}$ of series $\,_{p}F_{q}[a_1, \ldots, a_p; b_1, \ldots, b_q; \lambda z^{q-p+1}]$, with $q \geq p \geq 1$, $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$?

Such a representation may not be unique. For instance

$$J_0(z) := \,_{1}F_{2}\left[\frac{1}{1, 1}; \left(iz/2\right)^2\right] = e^{-iz} \cdot \,_{1}F_{1}\left[\frac{1/2}{1}; 2iz\right].$$

Gorelov, 2004: the answer is yes if the $E$-function (in Siegel’s sense) is solution of a differential equation of order $\leq 2$ with coefficients in $\overline{\mathbb{Q}}(z)$.

In 2019, Fischler and myself gave a strong reason to believe that the answer was negative in general for $E$-functions of differential order $\geq 4$.

The answer was then shown to be negative by Fresán and Jossen in 2020, who produced an explicit counter-example.

In the rest of the talk, I will explain our 2019 result. From now on, $E$-functions are always understood in the restricted sense.
Rings of special values

\( \mathbb{G} \) the ring of values taken at algebraic points by analytic continuations of \( \mathbb{G} \)-functions. Algebraic numbers, \( \Gamma(a/b)^b \) (\( a, b \in \mathbb{N} \)) and \( \pi \) are units of \( \mathbb{G} \).

\( \mathbb{H} \) the ring generated by \( \overline{\mathbb{Q}} \), \( 1/\pi \) and \( \Gamma^{(n)}(r) \), \( r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0} \), \( n \in \mathbb{N} \). Algebraic numbers and \( \Gamma(r) \) (\( r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0} \)) are units of \( \mathbb{H} \).

\( \mathbb{S} \) the \( \mathbb{G} \)-module generated by \( \Gamma^{(n)}(r) \), \( r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0} \), \( n \in \mathbb{N} \). It is a ring. \( \mathbb{G} \) and \( \mathbb{H} \) are subrings of \( \mathbb{S} \).

**Proposition 1**

(i) \( \mathbb{H} \) is generated by \( \overline{\mathbb{Q}} \), \( 1/\pi \) and

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\text{Li}_s(e^{2i\pi r}) & s \in \mathbb{N}^*, \ r \in \mathbb{Q}, \ (s, e^{2i\pi r}) \neq (1, 1) \\
\log(q) & q \in \mathbb{N}^*
\end{array} \right. \\
&\Gamma(r) \\
&\gamma := -\Gamma'(1) \quad (Euler's \ constant)
\end{align*}
\]

(ii) \( \mathbb{S} \) is the \( \mathbb{G}[\gamma] \)-module generated by \( \Gamma(r) \), \( r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0} \).
Theorem 4 (Fischler-R., 2019)

At least one of the following statements is true:

(i) \( G \subset H \);

(ii) Siegel’s question has a negative answer.

(i) is very unlikely. It contradicts a conjecture on exponential periods that generalizes Grothendieck’s periods conjecture.

If there exist \( s \in \mathbb{N}^* \) and \( \alpha \in \overline{\mathbb{Q}} \) such that \( \text{Li}_s(\alpha) \in G \) is not in \( H \), then the \( E \)-function

\[
\sum_{n=2}^{\infty} \left( \sum_{k=1}^{n-1} \frac{\alpha^k}{k^s} \right) \frac{z^n}{n!}
\]

is a counter-example, of differential order (at most) \( s + 3 \).

I will outline the proof of Theorem 4 when in Siegel’s question we further assume that \( p = q \).

The proof of the general case is based on the case \( p = q \) together with more complicated arguments.
Asymptotic expansions in large sectors

**Definition 2**

Let $\theta \in \mathbb{R}$. We write

$$f(z) \sim \sum_{\rho \in \mathbb{C}} e^{\rho z} \sum_{\alpha \in \mathbb{C}} z^{\alpha} \sum_{i \in \mathbb{N}} \log(z)^i \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n}(\theta)/z^n$$

where the sums on $\rho, \alpha, i$ are finite, and say (in this talk) that the RHS is the asymptotic expansion of $f$ at $\infty$ in a large sector bisected by the direction $\theta$, when there exist $\varepsilon, R, B, C > 0$ and certain functions $f_{\rho}(z)$ holomorphic in the sector

$$U := \{ z \in \mathbb{C}, |z| \geq R, \theta - \pi/2 - \varepsilon \leq \arg(z) \leq \theta + \pi/2 + \varepsilon \},$$

such that $f(z) = \sum_{\rho} e^{\rho z} f_{\rho}(z)$ and

$$\left| f_{\rho}(z) - \sum_{\alpha \in \mathbb{C}} z^{\alpha} \sum_{i \in \mathbb{N}} \log(z)^i \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n}(\theta)/z^n \right| \leq C^N N! |z|^{B-N}, \quad z \in U, \ N \geq 1.$$

If such an expansion of $f(z)$ exists in a large sector, it is unique in this sector.
Theorem 5

(i) (André, 2000) Let $f(z)$ be an $E$-function. There exists a finite set $A$ such that, for any $\theta \in (-\pi, \pi) \setminus A$,

$$f(z) \sim \sum_{\rho \in \overline{Q}} e^{\rho z} \sum_{\alpha \in \mathbb{Q}} z^{\alpha} \sum_{i \in \mathbb{N}} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\rho,\alpha,i,n}(\theta)}{z^n},$$

in a large sector bisected by the direction $\theta$, where (Fischler-R., 2016) the coefficients

$$c_{\rho,\alpha,i,n}(\theta) \in S.$$

(ii) (Fischler-R., 2019) Let $\xi \in G$. There exists an $E$-function $F(z)$ and a finite set $S$ such that for any $\theta \in (-\pi, \pi) \setminus S$, $\xi$ is one of the $c_{\rho,\alpha,i,n}(\theta)$ of the expansion of $F(z)$ in a large sector bisected by $\theta$. 

Asymptotic expansions of $E$-functions
Theorem 6

Let $\theta \in (-\pi, \pi) \setminus \{0\}$, and $f(z)$ be a hypergeometric series $pF_p(z)$ with rational parameters. Then,

$$f(z) \sim \sum_{\rho \in \{0,1\}} e^{\rho z} \sum_{\alpha \in \mathbb{Q}} z^\alpha \sum_{i \in \mathbb{N}} \log(z)^i \sum_{n=0}^{\infty} \frac{c_{\rho,\alpha,i,n}(\theta)}{z^n}$$

in a large sector bisected by the direction $\theta$ where (Fischler-R., 2019) the coefficients

$$c_{\rho,\alpha,i,n}(\theta) \in \mathbb{H}.$$ 

It is a consequence of Barnes and Wright’s classical results, with refinements coming from the theory of Meijer’s $G$-function.
Proof of Theorem 4 in the case $p = q$

Let $\xi \in \mathbf{G}$.

By Theorem 5(ii), there exist an $E$-function $F(z)$ and a finite set $S$ such that for any $\theta \in (-\pi, \pi) \setminus S$, $\xi$ is a coefficient of the expansion of $F(z)$ in a large sector bisected by $\theta$.

Assume that Siegel’s question has a positive answer (in the case $p = q$).

There exist $pF_p$-hypergeometric series $f_1, \ldots, f_n$ with rational parameters, algebraic numbers $\lambda_1, \ldots, \lambda_n$, and a polynomial $P \in \mathbb{Q}[X_1, \ldots, X_n]$, such that

$$F(z) = P(f_1(\lambda_1z), \ldots, f_n(\lambda_nz)).$$

Choose $\theta \in (-\pi, \pi) \setminus S$ such that $\theta + \arg(\lambda_i) \not\in \pi\mathbb{Z}$ for every $i$. By Theorem 6, the expansion of each $f_i(\lambda_iz)$ in a large sector bisected by $\theta$ has coefficients in $\mathbf{H}$. The same holds for $F(z)$ because $\mathbf{H}$ is a $\overline{\mathbb{Q}}$-algebra.

Such an expansion being unique, the coefficient $\xi$ belongs to $\mathbf{H}$.
A Siegel like problem for $G$-functions

The generalized hypergeometric series

\[ p+1 F_p \left[ \begin{array}{c} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array} ; z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n z^n}{n!(b_1)_n \cdots (b_p)_n}, \]

is a $G$-function when $p \geq 0$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all $j$.

**Question 2**

Is it possible to write any $G$-function as a polynomial with coefficients in $\mathbb{Q}$ of series of the form $p+1 F_p[a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; \lambda(z)]$, with $a_j, b_j \in \mathbb{Q}$ and $\lambda(z)$ algebraic over $\mathbb{Q}(z)$, regular at $0$ and such that $\lambda(0) = 0$?

**Theorem 7 (Fischler-R., 2019)**

At least one of the following statements is true:

(i) $G \subset H$

(ii) Question 2 has a negative answer under the further assumption that the algebraic functions $\lambda$ have a common singularity in $\overline{\mathbb{Q}}^* \cup \{\infty\}$ at which they all tend to $\infty$.

If there exist $s \in \mathbb{N}^*$ and $\alpha \in \overline{\mathbb{Q}}$ such that $\text{Li}_s(\alpha) \in G$ is not in $H$, then $\text{Li}_s\left(\frac{\alpha z}{z-\alpha}\right)$ is a counter-example of differential order $s + 1$. 
Why is the inclusion $G \subset H$ unlikely, according to Yves André

“The inclusion $G \subset H$ does not contradict Grothendieck’s period conjecture but it contradicts its extension to exponential motives. In the description of $H$ given in Proposition 1, we find

1/π, a period of the Tate motive,

$\text{Li}_s(e^{2i\pi r})$, periods of a mixed Tate motive over $\mathbb{Z}[1/r]$,

$log(q)$, a period of a 1-motive over $\mathbb{Q}$,

$\Gamma(r)$, whose suitable powers are periods of Abelian varieties with complex multiplication by $\mathbb{Q}(e^{2i\pi r})$,

$\gamma$, a period of an exponential motive, which is a non-classical extension of the Tate motives.

Let $M$ be the Tannakian category of mixed motives over $\overline{\mathbb{Q}}$ generated by all these motives. Consider a non CM elliptic curve over $\overline{\mathbb{Q}}$ and $E$ its motive. The periods of $E$ are in $G$.

If $G \subset H$, the periods of $E$ are in $H$. By the exponential period conjecture, $E$ would be in $M$. This is impossible because the motivic Galois group of $M$ is pro-solvable, while that of $E$ is $GL_2$. ”