On Stein fillings of the 3-torus $T^3$

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Abstract

Topological properties of Stein fillings of contact 3-manifolds diffeomorphic to the 3-torus $T^3$ are determined. We show that for a Stein filling $S$ of $T^3$ the first Betti number $b_1(S)$ is two, while $Q_S = \langle 0 \rangle$. In the proof we also show that if $S$ is Stein and $\partial S$ is diffeomorphic to the Seifert fibered 3-manifold $-\Sigma(2, 3, 11)$ then $b_1(S) = 0$ and $Q_S = H$. Similar results are obtained for $S^1 \times S^2$. Finally, we describe Stein fillings of the Poincaré homology sphere $\pm \Sigma(2, 3, 5)$; in studying these fillings we apply recent gauge theoretic results, and prove our theorems by determining certain Seiberg-Witten invariants.

1 Introduction

Suppose that $M$ is a closed, oriented 3-manifold and $\xi$ is a 2-plane field on $M$. This $\xi$ is called a contact structure if it is completely nonintegrable, i.e., for the 1-form (locally) defining $\xi$ as ker $\alpha$, the expression $\alpha \wedge d\alpha$ is nowhere 0. (For more about contact manifolds see [1].) A 3-manifold $M$ in a Kähler surface $X$ inherits a natural contact structure provided that $M$ is convex, that is, there exists a vector field $v$ on $X$ transverse to $M$ such that $\mathcal{L}_v \omega = \omega$ for the Kähler form $\omega$ ($\mathcal{L}$ stands for the Lie derivative). In this case the complex lines in $TM$ form a 2-plane field $\xi$ satisfying the definition of a contact structure. For example, if $X$ admits a proper biholomorphic embedding into $\mathbb{C}^n$ for some $n$ (that is, $X$ is a Stein surface) then for the distance function $f = \| \cdot - p \|^2: X \to [0, \infty)$ (for $p \in \mathbb{C}^n$ generic) the submanifolds $f^{-1}(t)$ will be convex, hence contact away from the critical points. In fact, this property characterizes Stein surfaces:

Theorem 1.1 ([13]). The (noncompact) complex surface $X$ is Stein if and only if there is a proper Morse function $f: X \to [0, \infty)$ such that away from
the critical points the submanifolds $f^{-1}(t) \subset X$, with the plane fields induced by the complex tangent lines of $X$ in $TM$, are contact 3-manifolds.

Suppose that $t \in \mathbb{R}$ is a regular value of the above Morse function $f : X \to [0, \infty)$. The manifold (with boundary) $S = f^{-1}[0, t] \subset X$ is called a Stein domain, and it can be regarded as the compact version of Stein surfaces. (For more about Stein surfaces and Stein domains see \cite{13, 14}.)

**Definition 1.2.** The contact manifold $(M, \xi)$ is **Stein fillable** if there is a Stein domain $S$ such that $(M, \xi)$ is contactomorphic to $\partial S$ (with the induced contact structure on it). In this case $S$ is called a **Stein filling** of $(M, \xi)$.

**Remark 1.3.** We always assume that $M$ is oriented and $\xi$ respects this orientation through the requirement that $\alpha \wedge d\alpha > 0$ for any 1-form $\alpha$ defining $\xi$. Since $S$ has a natural orientation (as a complex surface), it induces an orientation on $\partial S$. We require that the above contactomorphism is orientation preserving.

It is expected that the knowledge of all contact structures on $M$ will tell us something about its geometry. To achieve this goal it seems reasonable to study all Stein fillings of a given 3-manifold. On the other hand, Stein domains can be regarded as analogues of minimal complex surfaces of general type in the category of manifolds with boundary. Therefore the study of Stein domains is interesting from the 4-dimensional point of view as well. The *geography problem* for surfaces of general type asks the possible values of $b_1$, $c_2^1$ and $c_2$ of such manifolds. Extending this problem we get:

**Problem 1.4 (The geography problem for Stein domains).** Fix a contact 3-manifold $(M, \xi)$ and describe characteristic numbers of Stein fillings of it.

In this paper we will address the problem of describing Stein domains with the 3-torus $T^3$, $S^1 \times S^2$, $-\Sigma(2, 3, 11)$ and $\pm \Sigma(2, 3, 5)$ as contact boundary. (For a possible definition of these Seifert fibered manifolds see Figure 1.) The problem of Stein fillability (and more generally, symplectic fillability) of contact 3-manifolds has been extensively studied recently, see for example [4, 6, 18, 19, 20, 21]. We only mention a prototype result here:

**Theorem 1.5 (3).** If $W$ is a Stein domain with $\partial W = S^3$ the 3-dimensional sphere then $W$ is diffeomorphic to the 4-dimensional disk $D^4$. □
In the following we will prove a similar (but substantially weaker) statement for the 3-torus $T^3$, $S^1 \times S^2$ and for the Seifert fibered 3-manifolds $-\Sigma(2,3,11)$ and $\pm \Sigma(2,3,5)$. Our main result determines homological properties of Stein fillings of the 3-torus $T^3$. The intersection form of a 4-manifold $X$ will be denoted by $Q_X$.

**Theorem 1.6.** If $S$ is a Stein filling of $T^3$ then $b_1(S) = 2$ and $Q_S = \langle 0 \rangle$; in particular, $\chi(S) = 0$ and $\sigma(S) = 0$. Moreover, an appropriate Stein structure on $D^2 \times T^2$ provides a Stein filling of $T^3$ with the above properties.

**Remark 1.7.** In our proof we show that, in fact, $\pi_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ also holds. Using this isomorphism and Freedman’s Classification Theorem one can prove that the Stein filling $S$ is homeomorphic to $D^2 \times T^2$. We hope to return to this question later, see Remark 4.3 and [29].

A similar argument will show

**Theorem 1.8.** If $S$ is a Stein filling of $S^1 \times S^2$ then $\pi_1(S) \cong \mathbb{Z}$ and $b_2(S) = 0$.

**Remark 1.9.** With a little more care one can actually determine Stein fillings of $\# nS^1 \times S^2$ up to diffeomorphism, and get that if $S$ is Stein and $\partial S \cong \# nS^1 \times S^2$ then $S$ is diffeomorphic to the $n$-fold boundary connected sum of $S^1 \times D^3$. For details see [29].

The proof of Theorem 1.6 rests on the following result.

**Theorem 1.10.** If $S$ is a Stein filling of $-\Sigma(2,3,11)$ for some contact structure on it, then $b_1(S) = 0$ and $Q_S = H$. Moreover, there is a Stein domain $S$ with $b_1(S) = 0$, $Q_S = H$ and $\partial S = -\Sigma(2,3,11)$.

(Here, as usual, $H$ denotes the hyperbolic plane $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Below $E_8$ stands for the symmetric bilinear form defined by the negative definite Cartan matrix of the exceptional Lie algebra $E_8$. For the fixed orientation of the 3-manifolds $\Sigma(2,3,5)$ and $\Sigma(2,3,11)$ see text following Theorem 2.3.) Similar arguments as applied in the above theorem show the following — these statements were already known [3, 18, 25].

**Theorem 1.11.** If $S$ is a Stein filling of $\Sigma(2,3,5)$ then $b_1(S) = 0$ and $Q_S = E_8$. The 3-manifold $-\Sigma(2,3,5)$ admits no Stein filling.
Remark 1.12. Notice the difference between symplectic and Stein fillings. Blowing up is a symplectic operation which ruins the Stein structure. Even if we disregard blow-ups, no finiteness can be expected for symplectic fillings for $-\Sigma(2, 3, 11)$ or $T^3$ since by fiber summing the obvious fillings with elliptic surfaces infinitely many symplectic fillings with distinct $b_2^+$-invariant can be constructed.

In computing the first Betti numbers of various Stein fillings we will verify the following more general statement:

**Proposition 1.13.** If $S$ is a Stein filling of a contact 3-manifold $(M, \xi)$ then the homomorphism $i_*: \pi_1(M) \to \pi_1(S)$ induced by the inclusion $i: M \to S$ is a surjection. Consequently $i_*: H_1(M; \mathbb{Z}) \to H_1(S; \mathbb{Z})$ is onto, hence $b_1(S) \leq b_1(M)$.

In proving Theorem 1.10 we will use recent results in gauge theory. The relevant theorems and constructions will be summarized in Section 4. Section 3 deals with fillings of $-\Sigma(2, 3, 11)$ while Section 4 contains the proof of our main result Theorem 1.6. In the final section we prove Theorem 1.11.

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## 2 Gauge theoretic backgrounds

We will frequently invoke the following celebrated result of Donaldson:

**Theorem 2.1 ([5]).** If $X$ is a smooth, closed 4-manifold with negative definite intersection form, then $Q_X$ is standard, that is, isomorphic to $\oplus_{i=1}^{b_2^+(X)} \langle -1 \rangle$. If $X$ is a smooth, simply connected spin 4-manifold with $b_2^+(X) = 1$ then $Q_X$ is isomorphic to $H$.

**Remark 2.2.** Using the monopole equations rather the instantons Donaldson originally used in his proof, Furuta [11] extended Theorem 2.1 by showing that if a smooth spin 4-manifold $X$ has $Q_X = 2kE_8 \oplus lH$ then $l \geq 2|k| + 1$.

At one point we will appeal to the following famous result of Rohlin:
Theorem 2.3 ([28]). If $X$ is a smooth spin 4-manifold then the signature $\sigma(X)$ of $X$ is divisible by 16.

The 3-manifolds $\Sigma(2, 3, 5)$ and $\Sigma(2, 3, 11)$ are defined as oriented boundaries of the complex manifolds $M_c(2, 3, 5) = \{ (x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = \varepsilon, \ |x|^2 + |y|^2 + |z|^2 \leq 1 \}$ and $M_c(2, 3, 11) = \{ (x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^{11} = \varepsilon, \ |x|^2 + |y|^2 + |z|^2 \leq 1 \}$ (with $|\varepsilon| \ll 1$), i.e., as the boundaries of the corresponding (compactified) Milnor fibers. Alternatively, $-\Sigma(2, 3, 5)$ and $-\Sigma(2, 3, 11)$ are the oriented boundaries of the the nuclei $N_1 \subset E(1)$ and $N_2 \subset E(2)$, where $N_1$ and $N_2$ are given by the Kirby diagrams of Figure 1.

In our subsequent arguments we will apply product formulae for Seiberg-Witten invariants when we cut 4-manifolds along $\Sigma(2, 3, 5)$ or $\Sigma(2, 3, 11)$. For the sake of completeness we sketch the definition of Seiberg-Witten invariants and analyze the 3-dimensional equations for $\Sigma(2, 3, 5)$ and $\Sigma(2, 3, 11)$ more carefully.

Suppose that $X$ is a smooth, closed, oriented 4-manifold with no 2-torsion in $H_1(X; \mathbb{Z})$. A characteristic element $K$ in $H^2(X; \mathbb{Z})$ uniquely determines a spin$^c$ structure on $X$, and once a connection $A$ on the line bundle $L$ with $c_1(L) = K$ is fixed, a metric on $X$ gives rise to a twisted Dirac operator $\slashed{D}_A$. The space of connections on the complex line bundle $L$ will be denoted by $\mathcal{A}_L$, the curvature of $A \in \mathcal{A}_L$ is $F_A$, and $F_A^+$ stands for its self-dual part. Let
W^+ denote the positive spinors corresponding to the spin^c structure defined by \( K \). Then the Seiberg-Witten equations for a pair \((A, \psi) \in A_L \times \Gamma(W^+)\) read as
\[
(\ast) \quad \overline{\partial}_A \psi = 0 \quad \text{and} \quad F^+_A = iq(\psi).
\]
(Here \( q: \Gamma(W^+) \to \Gamma(\Lambda^+) \) is a certain quadratic map.) If \( K^2 - (3\sigma(X) + \chi(X)) = 0 \), the solution space \( \mathcal{M}_K \) of the equations (mod symmetries of the equations) is a compact 0-dimensional manifold. An orientation of \( H_2(W^+; \mathbb{R}) \otimes H^1(X; \mathbb{Z}) \) fixes an orientation on this solution space, and provided \( b_2^+(X) > 1 \), the algebraic sum of the points of \( \mathcal{M}_K \) turns out to be a smooth invariant of \( X \), denoted by \( SW_X(K) \in \mathbb{Z} \). Similar, but somewhat more complicated procedure provides \( SW_X(K) \in \mathbb{Z} \) for \( K \) with \( K^2 - (3\sigma(X) + 2\chi(X)) > 0 \). (If \( K^2 - (3\sigma(X) + 2\chi(X)) < 0 \) or \( K \) is not characteristic, then \( SW_X(K) = 0 \) by definition.) The class \( K \) is called a basic class of \( X \) if \( SW_X(K) \neq 0 \). It can be shown that \( SW_X(K) = (-1)^{\frac{\sigma(X) + \chi(X)}{4}} SW_X(-K) \), therefore \( K \) and \( -K \) are basic classes at the same time. We say that \( X \) is of simple type if for a basic class \( K \) of \( X \) the equation \( K^2 = 3\sigma(X) + 2\chi(X) \) holds. Following ideas of Fintushel and Stern \[9\] we can associate a formal series to any 4-manifold \( X \) with \( b_2^+(X) > 1 \): If \( \{\pm K_1, ..., \pm K_n\} \) are the nonzero basic classes of \( X \), then take
\[
\mathcal{SW}(X) = SW_X(0) + \sum_{i=1}^{n} SW_X(K_i) \exp(K_i) + SW_X(-K_i) \exp(-K_i).
\]
It has been proved \[32\] that if \( X \) is a minimal surface of general type then it has two basic classes \( \pm c_1(X) \), and \( SW_X(\pm c_1(X)) = \pm 1 \). Therefore in that case \( \mathcal{SW}(X) = \exp(c_1(X)) \pm \exp(-c_1(X)) \). (For other complex surfaces we only know that \( \pm c_1(X) \) are basic classes.) It is also known that the K3-surface \( Y \) has a single basic class which is \( c_1(Y) = 0 \), hence \( \mathcal{SW}(Y) = 1 \). For a more thorough study of Seiberg-Witten theory, see \[14, 22, 32\].

In a similar vein the 3-dimensional analogue of Seiberg-Witten equations can be defined. For a 3-manifold \( M \) the spin^c structures are parametrized by \( H^2(M; \mathbb{Z}) \) and if \( W \to M \) denotes the spinor bundle then the Seiberg-Witten equations for \((A, \psi) \in A_{\det W} \times \Gamma(W)\) are
\[
(\ast\ast) \quad \overline{\partial}_A \psi = 0 \quad \text{and} \quad * F_A = iq(\psi).
\]
(As usual, \( F_A \) denotes the curvature 2-form of the connection \( A \in A_{\det W} \), and \( * \) stands for the Hodge \( * \)-operator given by a metric on \( M \). Now \( q \) maps
from $\Gamma(W)$ to $\Gamma(\Lambda^1M)$.) These equations have been solved for $\Sigma(2,3,11)$ in [24]. Notice that since $\Sigma(2,3,11)$ is an integral homology sphere, it admits a unique spin$^c$ structure. After substituting the Levi-Civita connection with a suitable connection in the definition of $\tilde{\theta}_A$, in [24] it was shown that \((**)\) admits 3 solutions (up to gauge equivalence): one of them is the trivial solution $\theta$, which is the trivial connection with vanishing spinor field; the other two will be denoted by $\alpha$ and $\overline{\alpha}$.

**Remarks 2.4.**

- Such a perturbation of the Seiberg-Witten equations over a three-manifold can be naturally extended to give a perturbation of the Seiberg-Witten moduli space over 4-manifolds containing long necks. It is proved [24] that this perturbation over a smooth closed 4-manifold with $b_2^+ > 1$ gives a compact moduli space which is smoothly cobordant to the unperturbed Seiberg-Witten moduli space. This implies that such a perturbation can be used to compute the Seiberg-Witten invariants.

- Because of the presence of a positive scalar curvature metric, one can easily show that the Seiberg-Witten equations on $\Sigma(2,3,5)$ admit a unique solution $\theta$, which is the trivial connection with vanishing spinor field.

By finding relations between the $L^2$ moduli spaces of Seiberg-Witten solutions over a 4-manifold $X$ with boundary diffeomorphic to $\Sigma(2,3,11)$, in [30] relative invariants, relative basic classes and the (formal) series $\text{SW}(X)$ has been defined for a smooth 4-manifold $X$ with boundary diffeomorphic to $\pm \Sigma(2,3,11)$. The relation between absolute and relative invariants is given by

**Theorem 2.5 ([30]).** If the closed 4-manifold $Z$ decomposes as $Z = X \cup_{\Sigma(2,3,11)} Y$ with $b_2^+(X), b_2^-(Y) > 0$ then $\text{SW}(Z) = \text{SW}(X) \cdot \text{SW}(Y)$, that is, the product of the relative invariants equals the absolute invariant of the closed 4-manifold $Z$. $\square$

There are three more important ingredients of the proofs we will describe in the following sections. The first theorem (due to Lisca and Matić) provides a Kähler embedding of a Stein domain into a minimal surface of general type, more precisely
Theorem 2.6 ([20]). For a Stein domain $S$ there exists a minimal surface $X$ of general type and a Kähler embedding $f: S \to X$. Moreover, we can assume that $X - f(S)$ is not spin and $b^+_{2}(X - f(S)) > 1$. □

The next theorem is a special case of a result of Ozsváth and Szabó which describes restrictions on the embedding of certain circle bundles over surfaces. Suppose that $M_{e,1}$ is a circle bundle over the torus with Euler number $e$.

Theorem 2.7 ([27]). If the minimal surface $X$ of general type decomposes as $X = X_1 \cup_{M_{e,1}} X_2$ along the 3-manifold $M_{e,1}$ with $|e| \geq 1$ then either $b^+_{2}(X_1) = 0$ or $b^+_{2}(X_2) = 0$. □

Combining Theorems 2.6 and 2.7 we get (see also [3]):

Corollary 2.8. If $S$ is a Stein domain with $\partial S = M_{e,1}$ and $|e| \geq 1$ then $b^+_{2}(S) = 0$. □

Finally we invoke a result of Morgan and Szabó which characterizes homotopy K3-surfaces:

Theorem 2.9 ([23]). Suppose that $X$ is a simply connected spin 4-manifold of simple type with $Q_X = 2kE_8 \oplus lH$ and $SW_X(0) = \pm 1$. Then $Q_X = 2E_8 \oplus 3H$. □

(Notice that since $X$ is of simple type and 0 is a basic class, it follows that $l = 4k - 1$. The theorem of Morgan and Szabó proves that $SW_X(0)$ is even once $k > 1$.)

3 Fillings of $-\Sigma(2, 3, 11)$

We begin our study of Betti numbers of Stein fillings by proving an estimate on $b_1$.

Proof of Proposition [4]. It is well-known that a Stein domain $S$ can be built up using 0-, 1- and 2-handles only; for such manifolds the surjectivity of $\pi_1(\partial S) \to \pi_1(S)$ is obvious. This surjection now trivially implies that $H_1(\partial S; \mathbb{Z}) \to H_1(S; \mathbb{Z})$ is also a surjection, hence $b_1(S) \leq b_1(\partial S)$. □

Since $\pm \Sigma(2, 3, 5)$ and $-\Sigma(2, 3, 11)$ are integral homology spheres, the above theorem shows that Stein fillings of these Seifert fibered 3-manifolds have trivial first homology, hence vanishing first Betti number.

Next we consider intersection forms of fillings of $-\Sigma(2, 3, 11)$.
Proposition 3.1. If $X$ is a smooth 4-manifold with $\partial X = -\Sigma(2, 3, 11)$ then $b_2^+(X) > 0$.

Proof. It is a standard fact that the K3-surface $Y$ contains three disjoint copies of the nucleus $N_2$ (recall that $\partial N_2 = -\Sigma(2, 3, 11)$); and the intersection form of the manifold $Y - 3 \text{ int } N_2$ is negative definite and nonstandard. So if $X$ is negative definite with $\partial X = -\Sigma(2, 3, 11)$ then the 4-manifold $(Y - 3N_2) \cup 3X$ we get by replacing the nuclei in $Y$ by $X$ is a smooth manifold with nonstandard negative definite intersection form. The existence of such a manifold, however, contradicts Donaldson’s Theorem 2.1, showing that $X$ is not negative definite.

Proof of Theorem 1.10. Let $S$ be a Stein filling of $-\Sigma(2, 3, 11)$ and consider the Kähler embedding $S \to X$ where $X$ is a minimal surface of general type — the existence of such an embedding is guaranteed by Theorem 2.6. Recall that we can assume that $X - S$ is nonspin with $b_2^+(X - S) > 1$. The product formula of Theorem 2.3 shows that $\text{SW}(S) = \pm 1$: We use the fact that $X$, as a minimal surface of general type has only two basic classes $\pm c_1(X)$; therefore $\text{SW}(X)$ is nondivisible, but since $X - S$ is nonspin, $0$ is not characteristic and so $\text{SW}(X - S) \neq 1$. Notice that this computation shows that $c_1(S) = 0$ is the unique basic class for $S$, in particular, $S$ is spin. (The product formula of Theorem 2.3 applies since $b_2^+(S) > 0$ by Proposition 3.4 and $b_2^+(X - S) > 0$ by Theorem 2.6.) Now consider $Z = (Y - N_2) \cup S$ — as before, $Y$ stands for the K3-surface. The product formula shows that $\text{SW}(Z) = \pm 1$, and easy handle calculus verifies that $Z$ is simply connected: $Y - N_2$ is simply connected and we can build $Z$ on the top of it by adding only 2-, 3- and 4-handles, since $S$ is Stein. In conclusion, for the simply connected spin manifold $Z$ we have that $\text{SW}_Z(0) = \pm 1$; applying Theorem 2.3 of Morgan and Szabó this fact implies that $Q_Z = 2E_8 \oplus 3H$ and since $Q_{Y - N_2} = 2E_8 \oplus 2H$, we get that $Q_S \cong Q_{N_2} = H$. 

Remarks 3.2. 1. Figure 2 demonstrates that, in fact, $N_2$ carries a Stein structure. (For handle calculus of Stein domains, see [13, 14].) This provides a Stein filling of $-\Sigma(2, 3, 11)$ as stated.

2. It is known [10] that $-\Sigma(2, 3, 11)$ carries a unique Stein fillable (in fact, a unique tight) contact structure.
3. It seems natural to conjecture that the Stein filling $S$ of $-\Sigma(2,3,11)$ is diffeomorphic to $N_2$, although the techniques applied in the above proof seem to be weak to verify such a conjecture.

4 Stein fillings of $T^3$

Using a famous result of Eliashberg [7] together with the classification result due to Giroux and Kanda (given below), now we can prove our result regarding Stein fillings of the 3-torus $T^3$. Our proof of Theorem 1.6 will heavily rely on Theorem 1.10.

**Theorem 4.1** ([7, 12, 17]). The contact structures $\xi_n = \ker(\cos(2\pi nt)dx - \sin(2\pi nt)dy)$ on $T^3$ (in coordinates $(x, y, t)$ on $T^3$) are all noncontactomorphic and comprise a complete list of tight contact structures on the 3-torus $T^3$. If $(T^3, \xi_n)$ admits a Stein filling then $n = 1$. [□]

The contact structure $\xi_1$ can be given as the boundary of the Stein domain given by Figure 3(i). (For the relation of Kirby diagrams and Stein structures and for the definition of the Thurston-Bennequin invariant $tb$ of a Legendrian knot, see [13, 14].) Before turning to the proof of Theorem 1.6 we need a
Lemma 4.2. (a) Gluing a 2-handle along the fine curve a (or b) of Figure 3(ii) with framing $tb(a) - 1 = -1$ results a handlebody with boundary $M_{1,1}$ of Theorem 2.7. 

(b) Gluing 2-handles along the three fine curves a,b,c of Figure 3(ii) (with framings $-1, -1$ and $-2$, resp.) results a handlebody with boundary $-\Sigma(2, 3, 11)$. 

Proof of Theorem 1.6. Fix a contactomorphism between $\partial S$ and the boundary of the Stein domain of Figure 3(i). For determining $\chi(S)$ consider $W = S \cup$ three 2-handles attached along $a, b, c$ with framing $tb - 1$. The result is a Stein filling of $-\Sigma(2, 3, 11)$, and according to Theorem 1.10 it has Euler characteristic 3. (Since we glue the 2-handles along Legendrian curves with framing $tb-1$, the existence of a Stein structure on $W$ follows from by now standard arguments discussed in [13, 14].) Removing the three 2-handles from $W$ we arrive to the conclusion $\chi(S) = 0$. This fact implies that
Figure 4: Proof of Lemma 4.2(b) with Kirby diagrams
$b_1(S) \geq 1$, since $b_1(S) = 0$ and $\chi(S) = 0$ would imply $b_2(S) = -1$. Therefore $S$ admits unramified covers of any degree. Notice that since $b_1(S) \leq 3$ by Theorem 1.13 (and so $b_2(S) \leq 2$), we get that $|\sigma(S)| \leq 2$. Consider a 3-fold unramified cover of $\partial S$ — the result is a Stein filling $\overline{S}$ of the 3-fold cover of $\partial S$, which is $T^3$ again. Therefore all the above said — in particular $|\sigma(\overline{S})| \leq 2$ — holds for $\overline{S}$. Since $\sigma(\overline{S}) = 3\sigma(S)$, we conclude that $\sigma(S) = 0$.

Finally we show that $b_1(S) = 2$ for a Stein filling $S$ of $T^3$. According to Theorem 1.13 the map $H_1(T^3; \mathbb{Z}) \rightarrow H_1(S; \mathbb{Z})$ is onto — we claim that the (image of the) circles $a$ and $b \subset T^3$ of Figure 3(ii) remain essential in $H_1(S; \mathbb{Q})$ while $c$ becomes 0. Since $\xi_n$ can be given as a pull-back of $\xi_1$ under unramified cover along the $t$ coordinate, if $i_*(c) \neq 0$ in $\pi_1(S) = H_1(S; \mathbb{Z})$ then a corresponding $n$-fold cover of $S$ provides a Stein filling of $T^3$ equipped with $\xi_n$. This contradicts Theorem 4.1 once $n > 1$, hence $c = 0$ in $H_1(S; \mathbb{Z})$. If $a = 0$ in $H_1(S; \mathbb{Q})$ then attaching a 2-handle along $a$ with $tb(a) - 1$ we get a Stein domain $\tilde{S}$ with $\partial \tilde{S} = M_{1,1}$ of Lemma 2.7 and $b_1(\tilde{S}) > 0$, since the surface in $S$ with boundary $a$ together with the core of the handle and the dual torus of $a$ in $T^3$ give a hyperbolic pair in $\tilde{S}$. This fact contradicts Corollary 2.8, therefore $a \neq 0$ in $H_1(S; \mathbb{Q})$. The role of $b$ is analogous, hence $b \neq 0$ in $H_1(S; \mathbb{Q})$. It follows now that $S$ admits a CW decomposition with two 1-cells, and so the number of 2-cells is two in this decomposition (since $\chi(S) = 0$). Since $\pi_1(S)$ is Abelian (being a factor of $\pi_1(T^3) \cong \mathbb{Z}^3$), we get that the attaching circle of this 2-cell is homologically trivial, therefore $H_1(S; \mathbb{Z}) \cong \pi_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}$. In particular, $b_1(S) = 2$. This implies $b_2(S) = 1$, and so the intersection form $Q_S$ can be easily identified with $Q_S = \langle 0 \rangle$. The proof is now complete.

**Remark 4.3.** Notice that the above proof, in fact, showed that $\pi_1(S) \cong H_1(S; \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$; moreover we proved that a Stein filling of $T^3$ embeds into a Stein filling of $-\Sigma(2,3,11)$. Since for this latter 3-manifold all Stein fillings are spin, we conclude that a Stein filling of $T^3$ is spin. Since $\sigma(S) = 0$, it follows that the induced spin structure on $\partial S$ is diffeomorphic to the the one $\partial(Y - \nu F)$ inherits from $Y - \nu F$; here $Y$ is the K3-surface and $F$ is a regular fiber in an elliptic fibration on $Y$. Therefore $Z = S \cup_{T^3} (Y - \nu F)$ is a (simply connected) spin 4-manifold, and using the gauge theoretic results discussed in [31] one can verify that $SW_Z(0)$ is $\pm 1$. Consequently Theorem 2.9 implies that $Z$ is homeomorphic to $Y$. Motivated by this homeomorphism one can conjecture that the Stein filling $S$ is diffeomorphic to $D^2 \times T^2$ — the Stein filling shown by Figure 3(i).
fact, the extension of Freedman’s Classification Theorem for 4-manifolds with boundary and nontrivial fundamental group shows that a Stein filling $S$ of $T^3$ is homeomorphic to $D^2 \times T^2$, see [29].

We close this section with the proof of Theorem 1.8.

Proof of Theorem 1.8. Since $S^1 \times S^2$ admits a unique tight contact structure, adding a 1-handle and a 0-framed 2-handle (as shown by Figure 3(i)) to $\partial S$ results a Stein filling $W$ of $T^3$. Since $W = S \cup 1$-handle $\cup 2$-handle, we obviously have $\chi(S) = \chi(W) = 0$. This again shows that $b_1(S) \geq 1$, which implies $\pi_1(S) \cong \mathbb{Z}$, since — according to Proposition 1.13 — the fundamental group $\pi_1(S)$ is the factor of $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$. Now $b_2(S) = 0$ trivially follows.

Remark 4.4. Using Theorem 1.5 of Eliashberg one can also show that a Stein filling of $S^1 \times S^2$ is diffeomorphic to $S^1 \times D^3$, see [29]. Similar argument determines the Stein filling of $\# nS^1 \times S^2$ for all $n \geq 1$ up to diffeomorphism.

5 Appendix: Stein fillings of $\pm \Sigma(2, 3, 5)$

Finally we turn to the proof of Theorem 1.11. The theorem was already proved by various authors (see Remark 5.3); we include it here because the proof given below is very similar in spirit to the proof of Theorem 1.10. We begin our proof with the following “folk” theorem. (For a complete proof see [18, 25].)

Lemma 5.1. If $S$ is a Stein filling of a 3-manifold which admits positive scalar curvature then $S$ is negative definite.

Proof (sketch). Consider the Kähler embedding $S \to X$ where $X$ is a minimal surface of general type with $b_2^+(X - S) > 1$. Since a 3-manifold with positive scalar curvature metric cannot divide a 4-manifold with nonzero Seiberg-Witten invariants into two pieces both with $b_2^+ > 0$, the lemma follows.

Notice that $\pm \Sigma(2, 3, 5)$ (as the quotient of $S^3$) admits a metric with positive scalar curvature.

Proposition 5.2. If $S$ is a Stein filling of $\Sigma(2, 3, 5)$ (for some contact structure) then $S$ is negative definite and spin.
Proof. The fact that \( b^+_2(S) = 0 \) follows from Lemma 5.1. Recall that the Seiberg-Witten equations admit a unique (up to gauge equivalence) solution on \( \Sigma(2, 3, 5) \). Now consider an embedding \( S \to X \) where \( X \) is a minimal surface of general type. Grafting solutions for the spin\(^c\) structures \( \pm c_1(S) \) and \( c_1(X - S) \) together we get 4 basic classes unless \( c_1(S) = 0 \) or \( c_1(X - S) = 0 \). Since \( X \) has exactly two basic classes and \( c_1(X - S) \) is nonspin (therefore \( c_1(X - S) \neq 0 \)) we get that \( c_1(S) = 0 \), consequently \( S \) is spin. \( \square \)

Proof of Theorem 1.11. It can be easily verified that the negative definite \( E_8 \)-plumbing \( E \) (see Figure 5) embeds in the K3-surface \( Y \). For a Stein filling \( S \) consider \( Z = S \cup (Y - E) \). Since \( S \) is spin (and \( \partial S = \Sigma(2, 3, 5) \) admits a unique spin structure) it is spin and \( \pi_1(S \cup (Y - E)) = 1 \). Since \( S \) is negative definite, we have that \( Q_Z = (k+1)E_8 \oplus 3H \). Furuta’s Theorem (see Remark 2.2) shows that \( k \leq 1 \); since Rohlin’s famous Theorem 2.3 excludes \( k = 0 \), we conclude that \( Q_S = E_8 \), proving Theorem 1.11 for \( \Sigma(2, 3, 5) \). The part of Theorem 1.11 about \( -\Sigma(2, 3, 5) \) follows from the fact that \( -\Sigma(2, 3, 5) \) does not bound negative definite 4-manifold at all: If \( \partial X = -\Sigma(2, 3, 5) \) and \( b^+_2(X) = 0 \) then \( X \cup -\Sigma(2,3,5) E \) violates Donaldson’s Theorem 2.1. \( \square \)

Remarks 5.3. 1. A similar proof of the above theorem was already found by Ohta and Ono \[25\].

2. In the above proof one can also argue that a negative definite spin 4-manifold with boundary diffeomorphic to \( \Sigma(2, 3, 5) \) has intersection form isomorphic to \( E_8 \) along the lines developed by Frøyshov, see \[10\].

3. According to \[16\], the 3-manifold \( \Sigma(2, 3, 5) \) admits a unique fillable (in fact, a unique tight) contact structure. Notice that in our proof we made no assumption on the contact structure on \( \Sigma(2, 3, 5) \).
4. The $E_8$-plumbing $E$ supports a Stein structure providing a filling for $\Sigma(2, 3, 5)$. It seems natural to expect that any Stein filling of $\Sigma(2, 3, 5)$ is diffeomorphic to $E$.

5. Using Theorem 1.13 and analyzing possible quotients of $\pi_1(\Sigma(2, 3, 5))$ we get that for a Stein filling $S$ of $\Sigma(2, 3, 5)$ the group $\pi_1(S)$ is trivial. Therefore Freedman’s Theorem implies that $S$ is, in fact, homeomorphic to the $E_8$-plumbing $E$. For the details of the above argument see [29].

6. For $-\Sigma(2, 3, 5)$ Lisca proved that it does not admit a symplectic semi-filling, while Etnyre and Honda showed [8] that it supports no tight contact structure at all.

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