RIGHT COIDEAL SUBALGEBRAS IN $U_q(\mathfrak{sl}_{n+1})$

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Abstract. We offer a complete classification of right coideal subalgebras which contain all group-like elements for the multiparameter version of the quantum group $U_q(\mathfrak{sl}_{n+1})$ provided that the main parameter $q$ is not a root of 1. As a consequence, we determine that for each subgroup $\Sigma$ of the group $G$ of all group-like elements the quantum Borel subalgebra $U_q^+(\mathfrak{sl}_{n+1})$ contains $(n+1)!$ different homogeneous right coideal subalgebras $U$ such that $U \cap G = \Sigma$. If $q$ has a finite multiplicative order $t > 2$, the classification remains valid for homogeneous right coideal subalgebras of the multiparameter version of the Lusztig quantum group $u_q(\mathfrak{g}(n+1))$. In the paper we consider the quantifications of Kac-Moody algebras as character Hopf algebras [V.K. Kharchenko, A combinatorial approach to the quantifications of Lie algebras, Pacific J. Math., 203(1)(2002), 191-233].

1. INTRODUCTION

One of the reasons one-sided coideal subalgebras became more and more important is that Hopf algebras do not have “enough” Hopf subalgebras. The one-sided coideal condition instead plays prominent roles in constructions and developing theory. The very one-sided comodule subalgebras, but not the Hopf subalgebras, turn out to be the Galois objects in the Galois theory for Hopf algebra actions (A. Milinski [37, 38], see also a detailed survey by T.Yanai [57]). In particular, the Galois correspondence theorem for the actions on free algebra set up a one to one correspondence between right coideal subalgebras and intermediate free subalgebras (see, V.O. Ferreira, L.S.I. Murakami, and A. Paques [9]). A recent survey by G. Letzter [39] provides a panorama of the use of one-sided coideal subalgebras in constructing quantum symmetric pairs, in forming quantum Harish-Chandra modules, and in producing quantum symmetric spaces (T. Koornwinder [29], A. Joseph and G. Letzter [16], M. Noumi and T. Sugitani [44], M. Noumi [43], M. Dijkhuizen [7], M.S. Kebé [18, 19], G. Letzter [39, 51, 52], S. Sinelshchikov and L. Vaksman [52], M. Dijkhuizen and M. Noumi [5]).

In the present paper we offer a complete classification of right coideal subalgebras which contain the coradical for the multiparameter version of the quantum group $U_q(\mathfrak{sl}_{n+1})$, see [46, 6, 4], provided that the main parameter $q$ is not a root of 1. If $q$ has a finite multiplicative order $t > 2$, this classification remains valid for homogeneous right coideal subalgebras of the multiparameter version, see [35, 55], of the Lusztig quantum group $u_q(\mathfrak{g}(n+1))$. We are reminded that any Hopf algebra generated by group-like and skew- primitive elements is pointed, while in a pointed Hopf algebra the group-like elements span the coradical, see [41, Definition 5.1.5].
In the second section we introduce main concepts and provide the general results on the structure of the character Hopf algebras that are of use for classification. In Lemma 2.10 we note that if the given character Hopf algebra \( H = A \# k[G] \) is a bosonisation of a quantum symmetric algebra \( A \), then each invariant differential subspace \( U \) of \( A \) defines a right coideal \( U \# k[G] \). This statement allows one to use noncommutative differential calculus, [36] p.6, [39], [24], due to P. Schauenburg’s characterization of quantum Borel subalgebras [19]. The key point of the section is the construction of a PBW-basis over the coradical for a right coideal subalgebra by means of \[27\] [28]. This basis, in particular, provides some invariants for right coideal subalgebras (Definition 2.4).

In the third section we define the multiparameter quantification of a Kac-Moody algebra as a character Hopf algebra. This approach [23] combines and generalizes all known quantifications. We do not put unnecessary restrictions on the characteristic and on the quantification parameters. This allows one, for example, to define a new class of finite Frobenius algebras as the Lusztig quantum groups over a finite field. All their right coideal subalgebras are also Frobenius [23] (finite Frobenius algebras, in turn, have a significant role in the coding theory [13]). In Proposition 3.4 we provide a short proof of the so-called “triangular decomposition” in a quite general form.

In the fourth section (Proposition 4.2) we show that each homogeneous right coideal subalgebra in the quantum Borel algebra \( U_q^+(\mathfrak{sl}_{n+1}) \) has PBW-generators over \( k[G] \) of the following form

\[
(1.1) \quad \Psi^S(k, m) = \left[ \ldots [u[1 + s_r, m], u[1 + s_{r-1}, s_r]], \ldots , u[1 + s_1, s_2]], u[k, s_1]] \right.
\]

where the brackets are defined by the structure of a character Hopf algebra, \([u, v] = uv - \chi^a(g_a)uv \mid u[i, j] = \ldots [x_i, x_{i+1}], \ldots , x_j]\); \( k \leq s_1 < s_2 < \ldots < s_r < m \), \( S \cap \{k, m - 1\} = \{s_1, s_2, \ldots , s_r\} \), while \( x_i, 1 \leq i \leq n \) are the main skew-primitive generators of \( U_q^+(\mathfrak{sl}_{n+1}) \). This certainly implies that the set of all right coideal subalgebras which contain the coradical is finite (Corollary 4.8). To precisely describe the right coideal subalgebras we associate a right coideal subalgebra \( U_{\theta} \) to each sequence of integer numbers \( \theta = (\theta_1, \theta_2, \ldots , \theta_n) \), \( 0 \leq \theta_i \leq n - i + 1, 1 \leq i \leq n \) in the following way.

We define subsets \( R_k, T_k, 1 \leq k \leq n \) of the interval \([k, n]\) thus: if \( \theta_n = 0 \), we put \( R_n = T_n = \emptyset \) and if \( \theta_n = 1 \), we put \( R_n = T_n = \{n\} \). Suppose that \( R_i, T_i, k < i \leq n \) are already defined. If \( \theta_k = 0 \), then we set \( R_k = T_k = \emptyset \). If \( \theta_k \neq 0 \), then by definition \( R_k \) contains \( \bar{\theta}_k = k + \theta_k - 1 \) and all \( m \) that satisfy the following three properties

\[ \begin{align*}
  a) & \ k \leq m < \bar{\theta}_k; \\
  b) & \ \bar{\theta}_k \notin T_{m+1}; \\
  c) & \ \forall r(k \leq r < m) \ \ m \in T_{r+1} \iff \bar{\theta}_k \in T_{r+1}.
\end{align*} \]

Respectively, \( T_k \overset{df}{=} R_k \cup \bigcup_{s \in R_k \setminus \{n\}} T_{s+1} \). The algebra \( U_\theta \) by definition is generated over the coradical by all \( \Psi^T(k, m) \), \( 1 \leq k \leq m \leq n \) with \( m \in R_k \).

Theorem 5.2 shows that if \( q \) is not a root of 1 then all right coideal subalgebras over the coradical have the form \( U_\theta \). In particular, the exact number of right coideal subalgebras, which include the coradical, in the quantum Borel algebra \( U_q^+(\mathfrak{sl}_{n+1}) \) equals \((n+1)!\). If \( q \) has a finite multiplicative order \( t > 2 \), then this is the case for
homogeneous right coideal subalgebras of $U_q^\pm (\mathfrak{sl}_{n+1})$. (If $q$ is not a root of 1 then the all right coideal subalgebras that contain $G$ are homogeneous, Corollary 3.2).

In Section 6 we consider right coideal subalgebras in the quantum Borel algebra that do not contain the coradical. Note that for every submonoid $\Omega \subseteq G$ the set of all linear combinations $k[\Omega]$ is a right coideal subalgebra. We show that if the intersection $\Omega$ of a homogeneous right coideal subalgebra $U$ with $G$ is a subgroup, then $U = U^1_\theta k[\Omega]$. Here $U^1_\theta$ is a subalgebra generated by $g_a^{-1}a$ when $a$ runs through the described above generators of $U_\theta$.

In Section 7 we characterize $\text{ad}$-invariant right coideal subalgebras that have trivial intersection with the coradical in terms of Kébé’s construction [18, 19].

We see that the construction of $U_\theta$ is completely constructive, although it is not straightforward. Hence by means of computer calculations one may find all necessary invariants of the coideal subalgebras and relations between them. In the eighth section we provide a tableaux of the coideal subalgebras and their main characteristics for $n = 3$ that was found by means of computer calculations.

In Sections 9-11 we consider the whole of $U_q(\mathfrak{sl}_{n+1})$. The triangular decomposition,

$$U_q^\pm (\mathfrak{sl}_{n+1}) = U^-_q (\mathfrak{sl}_{n+1}) \otimes_k [k[F] \otimes_k [k[G] U^+_q (\mathfrak{sl}_{n+1}),$$

provides a hope that any (homogeneous) right coideal subalgebra that contains the coradical has the triangular decomposition as well, and for any two right coideal subalgebras $U_\theta \subseteq U^+_q (\mathfrak{sl}_{n+1}), U_\theta' \subseteq U^-_q (\mathfrak{sl}_{n+1})$ the tensor product

$$U = U_\theta' \otimes_k [k[H] \otimes_k [k[G] U_\theta$$

is a right coideal subalgebra. In this hypothesis just one statement fails, the tensor product indeed is a right coideal but not always a subalgebra.

To describe conditions when (1.3) is a subalgebra we display the element $\Psi^S(k, m)$ schematically as a sequence of black and white points labeled by the numbers $k - 1, k, k + 1, \ldots, m - 1, m$, where the first point is always white, and the last one is always black, while an intermediate point labeled by $i$ is black if and only if $i \in S$:

Consider two elements $\Psi^T_k(k, \tilde{\theta}_k)$ and $\Psi^{T'_i}(i, \tilde{\theta}'_i)$, where $T_k, T'_i$ are defined as above by $\theta$ and $\theta'$, respectively. Let us display these elements graphically

$$1.4$$

In Theorem 11.1 we prove that (1.3) is a subalgebra if and only if for each pair $(k, i)$, $1 \leq k, i \leq n$ one of the following two options is fulfilled:

a) Representation (1.4) has no fragments of the form

$$1.$$  

b) Representation (1.4) has the form

where no one of the intermediate columns has points of the same color.
The obtained criterion allows use of the computer in order to find the total number $C_n$ of right coideal subalgebras which contain the coradical:

$$C_2 = 26; \quad C_3 = 252; \quad C_4 = 3,368; \quad C_5 = 58,810; \quad C_6 = 1,290,930; \quad C_7 = 34,604,844.$$ 

**Remark.** If a Hopf algebra $H$ has a Hopf algebra pairing $\langle \cdot, \cdot \rangle : M \times H \to k$ with a Hopf algebra $M$, then $M$ acts on $H$ via $m \mapsto h = \sum h^{(1)}(m, h^{(2)})$. Certainly, in this case each right coideal is $M$-invariant. Conversely, if the pairing is left faithful (that is, $\langle M, h \rangle = 0$ implies $h = 0$) then each $M$-invariant subspace is a right coideal. For $H = U_q(sl_{n+1})$ (or for $H = U_q(sl_{n+1})$ if $q^i = 1$) there exists a Hopf algebra pairing with $M = GL_q(n)$, see [21, 3, 55]. Hence, alternatively, our main result provides a classification of $GL_q(n)$-invariant subalgebras that contain the coradical.

The computer part of this work has been done by the second author, while the proofs are due to the first one.

2. Preliminaries

**PBW-generators.** Let $S$ be an algebra over a field $k$ and $K$ its subalgebra with a fixed basis $\{g_j \mid j \in J\}$. A linearly ordered subset $W \subseteq S$ is said to be a set of $PBW$-generators of $S$ over $K$ if there exists a function $h : W \to \mathbb{Z}^+ \cup \infty$, called the height function, such that the set of all products

$$g_j w_1^{n_1} w_2^{n_2} \cdots w_k^{n_k},$$

where $j \in J$, $w_1 < w_2 < \ldots < w_k \in W$, $n_i < h(w_i), 1 \leq i \leq k$ is a basis of $S$. The value $h(w)$ is referred to as the height of $w$ in $W$. If $K = k$ is the ground field, then we shall call $W$ simply as a set of $PBW$-generators of $S$.

**Character Hopf algebras.** Recall that a Hopf algebra $H$ is referred to as a character Hopf algebra if the group $H$ of all group like elements is commutative and $H$ is generated over $k[G]$ by skew primitive semi-invariants $a_i, i \in I$:

$$\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \quad g_i^{-1}a_i g = \chi^i(g)a_i, \quad g, g_i \in G,$$

where $\chi^i, i \in I$ are characters of the group $G$. By means of the Dedekind Lemma it is easy to see that every character Hopf algebra is graded by the monoid $G^*$ of characters generated by $\chi^i$:

$$H = \bigoplus_{\chi \in G^*} H^\chi, \quad H^\chi = \{a \in H \mid g^{-1}a g = \chi(g)a, \quad g \in G\}.$$

Let us associate a “quantum” variable $x_i$ to $a_i$. For each word $u$ in $X = \{x_i \mid i \in I\}$ we denote by $g_u$ or $gr(u)$ an element of $G$ that appears from $u$ by replacing each $x_i$ with $g_i$. In the same way we denote by $\chi^u$ a character that appears from $u$ by replacing each $x_i$ with $\chi^i$. We define a bilinear skew commutator on homogeneous linear combinations of words by the formula

$$[u, v] = uv - \chi^u(g_v)v u,$$

where sometimes for short we use the notation $\chi^u(g_v) = p_{uv} = p(u, v)$. These brackets satisfy the following Jacobi identities, see [21 (8)]:

$$[u, [v, w]] = [u, [v, w]] + p_{uv}^{-1}[[u, w], v] + ([v, w] - p_{uv}^{-1})[u, w] \cdot v.$$

In particular the following conditional identities are valid

$$[u, [v, w]] = [u, [v, w]],$$

provided that $[u, v] = 0$. 

The brackets are related to the product by the following ad-identities

\[(u \cdot v, w) = p_{uw} [u, w] \cdot v + u \cdot [v, w], \]

\[(u, v \cdot w) = [u, v] \cdot w + p_{uv} v \cdot [u, w]. \]

In particular, if \([u, w] = 0\), we have

\[[u \cdot v, w] = u \cdot [v, w]. \]

The antisymmetry identity takes the form

\[[u, v] = -p_{uv} [v, u], \quad \text{provided that} \quad p_{uv} p_{vu} = 1. \]

The group \(G\) acts on the free algebra \(k\langle X \rangle\) by \(g^{-1} u g = \chi^u(g)u\), where \(u\) is an arbitrary monomial in \(X\). The skew group algebra \(G\langle X \rangle\) has the natural Hopf algebra structure

\[\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g, \quad g \in G. \]

We fix a Hopf algebra homomorphism

\[\xi : G\langle X \rangle \to H, \quad \xi(x_i) = a_i, \quad \xi(g) = g, \quad i \in I, \quad g \in G. \]

**Algebra \(A_n\).** Suppose that the quantification parameters \(p_{ij} \equiv p(x_i, x_j) = \chi^i(g_j)\) satisfy \(p_{ij} p_{ji} = 1\) provided \(|i - j| > 1\). In this case all elements \([x_i, x_j] = x_i x_j - p_{ij} x_j x_i, |i - j| > 1\) are skew primitive. Therefore the ideal of \(G\langle X \rangle\) generated by these elements is a Hopf ideal. We denote by \(A_n\) the quotient character Hopf algebra

\[A_n = G\langle X \rangle / I, \quad I = \langle [x_i, x_j] = 0, \quad j - i > 1 \rangle. \]

**Definition 2.1.** The elements \(u, v\) are said to be separated if there exists an index \(j, 1 \leq j \leq n\), such that either \(u \in k\langle x_i \mid i < j \rangle\), \(v \in k\langle x_i \mid i > j \rangle\) or vice versa \(u \in k\langle x_i \mid i > j \rangle, \quad v \in k\langle x_i \mid i < j \rangle\).

In the algebra \(A_n\) every two separated homogeneous elements \(u, v\) (skew)commute, \([u, v] = 0\), due to (2.9), (2.10).

Let us consider a word \(u(k, m) = x_k x_{k+1} \ldots x_m\). There are a lot of options to rearrange the brackets in this word. For example \([x_k, [x_{k+1}, \ldots, x_{m-1}, x_m] \ldots]\), or \([[\ldots [x_k, x_{k+1}], \ldots, x_{m-1}], x_m] \ldots\).

**Lemma 2.2.** The value in \(A_n\) of the bracketed continuous word \(u(k, m)\) is independent of the alignment of brackets. In particular for the element

\[u[k, m] \equiv [x_k, [x_{k+1}, \ldots, x_{m-1}, x_m] \ldots] = [x_k, u[k + 1, m]] \]

we have the following equalities in \(A_n\) :

\[u[k, m] = [u[k, s], u[s + 1, m]], \quad k \leq s < m. \]

**Proof.** We use induction on \(m - k\). If \([a] = [[v], [w]])\), \(vw = u(k, m)\) then by the inductive supposition it suffices to consider the case when \([v], [w])\) are bracketed like \(u[k, m]\) in (2.14); that is, \([v] = [x_k, [v_1]], [w] = [x_m, [w_1]]\). If \(v_1\) is empty then we come to \(u[k, m]\). If \(v_1\) is not empty, then \(x_k\) and \(w\) are separated, hence \([x_k, [w]) = 0\) in \(A_n\). According to (2.7) this implies \([x_k, [v_1]], [w] = [x_k, [[v_1], [w]])\), and the inductive supposition applied to \([v_1], [w]]) again brings the value to that of \(u[k, m]\). \(\Box\)
**PBW-basis of a character Hopf algebra.** A constitution of a word $u$ in $G \cup X$ is a family of non-negative integers $\{m_x, x \in X\}$ such that $u$ has $m_x$ occurrences of $x$. Certainly almost all $m_x$ in the constitution are zero. We fix an arbitrary complete order, $<$, on the set $X$.

Let $\Gamma^+$ be the free additive (commutative) monoid generated by $X$. The monoid $\Gamma^+$ is a completely ordered monoid with respect to the following order:

$$m_1x_i + m_2x_{i_2} + \ldots + m_kx_{i_k} > m'_1x_i + m'_2x_{i_2} + \ldots + m'_kx_{i_k}$$

if the first from the left nonzero number in $(m_1 - m'_1, m_2 - m'_2, \ldots, m_k - m'_k)$ is positive, where $x_i > x_{i_2} > \ldots > x_{i_k}$ in $X$. We associate a formal degree $D(u) = \sum_{x \in X} m_x x \in \Gamma^+$ to a word $u$ in $G \cup X$, where $\{m_x | x \in X\}$ is the constitution of $u$ (in [10] §2.1 the formal sum $D(u)$ is called the weight of $u$). Respectively, if $f = \sum \alpha_i u_i \in G(X)$, $0 \neq \alpha_i \in k$ then

$$D(f) = \max \{D(u_i)\}.$$ 

On the set of all words in $X$ we fix the lexicographical order with the priority from the left to the right, where a proper beginning of a word is considered to be greater than the word itself.

A non-empty word $u$ is called a standard word (or Lyndon word, or Lyndon-Shirshov word) if $vw > vw$ for each decomposition $u = vw$ with non-empty $v, w$. A nonassociative word is a word where brackets $[\, \, ]$ somehow arranged to show how multiplication applies. If $[u]$ denotes a nonassociative word then by $u$ we denote an associative word obtained from $[u]$ by removing the brackets (of course, $[u]$ is not uniquely defined by $u$ in general, however Lemma 2.2 says that the value of $[u]$ in $A_n$ is uniquely defined provided that $u = u(k, m)$). The set of standard nonassociative words is the biggest set $SL$ that contains all variables $x_i$ and satisfies the following properties.

1) If $[u] = [[v][w]] \in SL$ then $[v], [w] \in SL$, and $v > w$ are standard.

2) If $[u] = [[[v_1][v_2]][w]] \in SL$ then $v_2 \leq w$.

Every standard word has only one alignment of brackets such that the appeared nonassociative word is standard (Shirshov theorem [50]). In order to find this alignment one may use the following procedure: The factors $v, w$ of the nonassociative decomposition $[u] = [[v][w]]$ are the standard words such that $u = vw$ and $v$ has the minimal length ([51], see also [31]).

**Definition 2.3.** A super-letter is a polynomial that equals a nonassociative standard word where the brackets mean $\{\, \, \}$. A super-word is a word in super-letters. A $G$-super-word is a super-word multiplied from the left by a group-like element.

By Shirshov’s theorem every standard word $u$ defines only one super-letter, in what follows we shall denote it by $[u]$. The order on the super-letters is defined in the natural way: $[u] > [v] \iff u > v$.

**Definition 2.4.** A super-letter $[u]$ is called hard in $H$ provided that its value in $H$ is not a linear combination of values of super-words of the same degree (2.17) in smaller than $[u]$ super-letters, and $G$-super-words of smaller degrees.

**Definition 2.5.** We say that a height of a hard in $H$ super-letter $[u]$ equals $h = h([u])$ if $h$ is the smallest number such that: first, $p_{uu}$ is a primitive $t$-th root of 1 and either $h = t$ or $h = tl^r$, where $l = \text{char}(k)$; and then the value in $H$ of $[u]^h$ is a linear combination of super-words of the same degree (2.17) in less than $[u]$
super-letters, and $G$-super-words of smaller degrees. If there exists no such number then the height equals infinity.

Certainly, if the algebra $H$ is homogeneous in each $a_i$ then one may omit the underlined parts of the definitions.

**Theorem 2.6.** ([21 Theorem 2]). *The values of all hard in $H$ super-letters with the above defined height function form a set of PBW-generators for $H$ over $k[G]$.***

**PBW-basis of a coideal subalgebra.** According to [21, 22, Theorem 1.1] every right coideal subalgebra $U$ that contains all group-like elements has a PBW-basis over $k[G]$ which can be extended up to a PBW-basis of $H$.

The PBW-generators $T$ for $U$ can be obtained from the PBW-basis of $H$ given in Theorem 2.6 in the following way.

Suppose that for a given hard super-letter $[u]$ there exists an element $c \in U$ with the leading term $[u]^s$ in the PBW-decomposition given in Theorem 2.6:

$$c = [u]^s + \sum \alpha_i W_i + \ldots \in U,$$

where $W_i$ are the basis super-words starting with less than $[u]$ super-letters, $D(W_i) = sD(u)$, and by the dots we denote a linear combination of $G$-super-words of $D$-degree less than $sD(u)$. We fix one of the elements with the minimal $s$, and denote it by $c_u$. Thus, for every hard in $H$ super-letter $[u]$ we have at most one element $c_u$. We define the height function by means of the following lemma.

**Lemma 2.7.** ([21, 22, Lemma 4.3]). *In the representation (2.18) of the chosen element $c_u$ either $s = 1$, or $p(u, u)$ is a primitive $t$-th root of 1 and $s = t$ or (in the case of positive characteristic) $s = t(\text{char } k)^r$.***

If the height of $[u]$ in $H$ is infinite, then the height of $c_u$ in $U$ is defined to be infinite as well. If the height of $[u]$ in $H$ equals $t$, then, due to the above lemma, $s = 1$ (in the PBW-decomposition (2.18) the exponent $s$ must be less than the height of $[u]$). In this case the height of $c_u$ in $U$ is supposed to be $t$ as well. If the characteristic $l$ is positive, and the height of $[u]$ in $H$ equals $tl^r$, then we define the height of $c_u$ in $U$ to be equal to $tl^r/s$ (thus, in characteristic zero the height of $c_u$ in $U$ always equals the height of $[u]$ in $H$).

**Proposition 2.8.** *The set of all chosen $c_u$ with the above defined height function forms a set of PBW-generators for $U$ over $k[G]$.***

**Proof.** See, [21, 22, Proposition 4.4].

We note that there is an essential freedom in construction of the PBW-generators for a right coideal subalgebra. In particular the PBW-basis is not uniquely defined in the above process. Nevertheless the set of leading terms of the PBW-generators indeed is uniquely defined.

**Definition 2.9.** A hard super-letter $[u]$ is called $U$-effective if there exists $c \in U$ of the form (2.18). The degree $sD(u) \in \Gamma^+$ of $c$ with minimal $s$ is said to be an $U$-root. An $U$-root $\gamma \in \Gamma^+$ is called a simple $U$-root if it is not a sum of two or more other $U$-roots.

Thus, the set of $U$-effective super-letters, the set of $U$-roots, and the set of simple $U$-roots are invariants of any right coideal subalgebra $U$. 

Remark. There is already a fundamental for Lie theory notion of roots associated to semisimple Lie algebras. Certainly, the set of PBW-generators for the universal enveloping algebra \( U(g) \) coincides with a basis of the Lie algebra \( g \). If we apply our definition to \( U(g) \) then \( U(g) \)-roots are the formal degrees of basis elements related to a fixed set of generators \( x_i, i \in I \). At the same time the formal degrees of basis elements for the Borel subalgebra are in one-to-one correspondence with positive roots: to each root \( \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k} \) corresponds a basis element \([x_{i_1}, x_{i_2}, \ldots, x_{i_k}]\), see [15] Chapter IV, §3, Statement XVII. Therefore our definition of a root is a natural generalization of the classical notion. Probably the analogy would be more clear if in our definition of the formal degree we will replace the symbols \( x_i \) with the characters \( \chi^i \) and identify the generators \( g_i \) of the group \( G \) with (exponents of the) basis elements \( h_i \) of the Cartan subalgebra since the classical roots are elements of the dual space \((\sum_i k h_i)\)^*. Lemma 3.1 below shows that this replacement is admissible. We believe that by this very reason in [10] the formal degree is referred to as weight, the notion already well defined in the Lie theory.

Differential calculi. The free algebra \( k\langle X \rangle \) has a coordinate differential calculus
\[
\partial_i(x_j) = \delta_i^j, \quad \partial_i(uv) = \partial_i(u) \cdot v + \chi^u(g_i)u \cdot \partial_i(v).
\]
The partial derivatives connect the calculus with the coproduct on \( G\langle X \rangle \) via
\[
\Delta(u) \equiv u \otimes 1 + \sum_i g_i \partial_i(u) \otimes x_i \quad (\text{mod } G\langle X \rangle \otimes k\langle X \rangle^{(2)}),
\]
where \( k\langle X \rangle^{(2)} \) is the set (an ideal) of noncommutative polynomials without free and linear terms. Symetrically the equation
\[
\Delta(u) \equiv g_u \otimes u + \sum_i g_u g_i^{-1} x_i \otimes \partial_i^u(u) \quad (\text{mod } G\langle X \rangle^{(2)} \otimes k\langle X \rangle)
\]
defines a dual differential calculus on \( k\langle X \rangle \) where the partial derivatives satisfy
\[
\partial_i^u(x_j) = \delta_i^j, \quad \partial_i^u(uv) = \chi^u(g_i)\partial_i^u(u) \cdot v + u \cdot \partial_i^u(v).
\]
Here \( G\langle X \rangle^{(2)}k \) is an ideal of \( G\langle X \rangle \) of elements without free and linear terms. If the kernel of \( \xi \) defined in (2.13) is contained in \( G\langle X \rangle^{(2)} \) then formul\( (2.20) \) and \( (2.21) \) imply that each skew-primitive element \( u \) from \( A^{(2)} = \xi(k\langle X \rangle^{(2)}) \) is a constant with respect to both calculi, \( \partial_i(u) = \partial_i^u(u) = 0 \), \( 1 \leq i \leq n \). More details one can find in [29, 24, 25].

Shuffle representation. Let \( \text{Ker } \xi \subseteq G\langle X \rangle^{(2)} \). In this case there exists a Hopf algebra projection \( \pi : H \to k[G] \), \( a_i \to 0 \), \( g_i \to g_i \). Hence by the Radford theorem we have a decomposition in a biproduct, \( H = A \# k[G] \), by means of the isomorphism \( u \to \psi(u^{(1)}) \# \pi(u^{(2)}) \) with \( \psi(u) = \sum_{\rho} u^{(1)} \pi(S(u^{(2)})) \), see details in [2] §1.5, §1.7.

If \( \text{Ker } \xi \) is the biggest Hopf ideal in \( G\langle X \rangle^{(2)} \), or, equivalently, if \( H \) is a Hopf algebra of type one in the sense of Nichols [42], or, equivalently, if \( A \) is a quantum symmetric algebra (a Nichols algebra [2] §1.3, Section 2), then \( A \) has a shuffle representation as follows.
The algebra \( A \) has a structure of a braided Hopf algebra, \[54\], with a braiding \( \tau(u \otimes v) = p(v, u)^{-1}v \otimes u \). The braided coproduct \( \Delta^b \) is connected with the coproduct on \( H \) in the following way, see \[25\] p. 93, (3.18),
\[
\Delta^b(u) = \sum_{\{u\}} u^{(1)} \GR(u^{(2)})^{-1} \otimes u^{(2)}.
\]

At the same time the tensor space \( T(V), V = \sum_{i} kx_i \) also has a structure of a braided Hopf algebra. This is the quantum shuffle algebra \( Sh(V) \) with the coproduct
\[
\Delta^b(u) = \sum_{i=0}^{m} (z_1 \ldots z_i) \otimes (z_{i+1} \ldots z_m),
\]
where \( z_i \in X, u = (z_1 z_2 \ldots z_{m-1} z_m) \) is the tensor \( z_1 \otimes z_2 \otimes \ldots \otimes z_{m-1} \otimes z_m \) considered as an element of \( Sh(V) \). The map \( a_i \rightarrow (x_i) \) defines an embedding of the braided Hopf algebra \( A \) into the the braided Hopf algebra \( Sh(V) \). More details can be find in \[12\, 56\, 19\, 48\, 11\, 11\, 54\, 12\, 25\, 24\].

**Differential subalgebras.** If \( U \) is a right coideal subalgebra of \( H \) and \( k[G] \subseteq U \), then \( \vartheta(U) \subseteq U \), hence \( U = U_A \# k[G] \) with \( U_A = \vartheta(U) = U \cap A \). Formulae \[2.20\] implies that \( U_A \) is a differential subalgebra of \( A \), that certainly satisfies \( gUAg^{-1} \subseteq U_A \), \( g \in G \). The converse statement is valid if \( \text{Ker}\xi \) is the biggest Hopf ideal.

**Lemma 2.10.** Suppose that \( \text{Ker}\xi \) is the biggest Hopf ideal in \( G(X)^{(2)} \). If \( U \) is a differential subspace of \( A = k\langle a_i \rangle = \vartheta(H) \), and \( gUAg^{-1} \subseteq U, g \in G \), then \( U \# k[G] \) is a right coideal of \( H \).

**Proof.** The braided coproduct \[2.23\] also defines a differential calculus
\[
\Delta^b(u) \equiv u \otimes 1 + \sum_i \frac{\partial^b u}{\partial x_i} \otimes x_i \quad (\text{mod} \ A \otimes A^{(2)}).
\]
In \[24\] Theorem 4.8] this calculus is denoted by \( d^* \). Formulae \[2.20\], \[2.23\], and \[2.25\] imply \( \frac{\partial^b u}{\partial x_i} = g_i \partial_i (u) g_i^{-1} \).

Since \( A \) has a representation as a subalgebra of the quantum shuffle algebra \( Sh(V) \), by \[24\] Theorem 5.1] applied to the calculus \( d^* \) the restriction \( \Omega = \xi |_{k(X)} \) has the following differential form
\[
\Omega(u) = \sum_{i_1, i_2, \ldots, i_n} \frac{(\partial^b u)_u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} (x_{i_1} x_{i_{n-1}} \cdots x_{i_1}), \quad u \in V^\otimes n,
\]
where as above \( (x_{i_1} x_{i_{n-1}} \cdots x_{i_1}) \) is the tensor \( x_{i_1} \otimes x_{i_{n-1}} \otimes \cdots \otimes x_{i_1} \) considered as an element of \( Sh(V) \). By means of \[2.24\] we have
\[
\Delta^b(\Omega(u)) = \sum_{i_1, i_2, \ldots, i_n} \frac{(\partial^b u)_u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} \sum_{k=1}^{n+1} (x_{i_n} \cdots x_{i_k}) \otimes (x_{i_k-1} \cdots x_{i_1})
\]
\[
= \sum_{k=1}^{n+1} \sum_{i_1, i_2, \ldots, i_k-1} \left( \sum_{i_k, i_{k+1}, \ldots, i_n} \frac{(\partial^b u)_u}{\partial x_{i_k} \partial x_{i_{k+1}} \cdots \partial x_{i_n}} (x_{i_n} \cdots x_{i_k-1}) \otimes (x_{i_k-1} \cdots x_{i_1}) \right)
\]
\[
= \sum_{k=1}^{n+1} \sum_{i_1, i_2, \ldots, i_k-1} \Omega \left( \frac{(\partial^b u)_u}{\partial x_{i_1} \partial x_{i_{k-1}}} \right) \otimes (x_{i_k-1} \cdots x_{i_1}).
\]
Since $\Omega$ is a $d^*$-differential map, we have got
\begin{equation}
\Delta^b(w) = \sum_{k=1}^{n+1} \sum_{i_1, i_2, \ldots, i_{k-1}} \frac{(p^b)^{k-1}w}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}} \otimes (x_{i_{k-1}} \cdots x_{i_1}), \quad w = \Omega(u).
\end{equation}
This formula implies that each differential subspace $W \subseteq A$ with respect to $d^*$ is a right coideal with respect to $\Delta^b$. Indeed, $\Delta^b(W) \subseteq (A \otimes A) \cap (W \otimes \text{Sh}_T) = W \otimes A$. Since $\partial^b u/\partial x_i = g_i \partial_i (u) g_i^{-1}$, the space $U$ given in the lemma is a right coideal with respect to the coproduct $\Delta^b$. Now (2.23) shows that $Uk[G]$ is a right coideal of $H$. The lemma is proved.

3. Multiparameter quantification of Kac-Moody algebras as character Hopf algebras

Quantification of Borel subalgebras. Let $C = ||a_{ij}||$ be a symmetrizable by $D = \text{diag}(d_1, \ldots, d_n)$ generalized Cartan matrix, $d_i a_{ij} = d_j a_{ji}$. Denote by $\mathfrak{g}$ a Kac-Moody algebra defined by $C$, see [17]. Suppose that the quantification parameters $p_{ij} = p(x_i, x_j) = \chi^i(g_j)$ are related by
\begin{equation}
 p_{ii} = q^{d_i}, \quad p_{ij} p_{ji} = q^{d_{aij}}, \quad 1 \leq i, j \leq n.
\end{equation}
In this case the multiparameter quantization $U^+_q(\mathfrak{g})$ of the Borel subalgebra $\mathfrak{g}^+$ is a character Hopf algebra defined by Serre relations with the skew brackets in place of the Lie operation:
\begin{equation}
\ldots [x_i, x_j], x_j], \ldots, x_j] = 0, \quad 1 \leq i \neq j \leq n.
\end{equation}

By [20] Theorem 6.1] the left hand sides of these relations are skew-primitive elements in $G(X)$. Therefore the ideal generated by these elements is a Hopf ideal, while $U^+_q(\mathfrak{g})$ indeed has a natural character Hopf algebra structure.

**Lemma 3.1.** If $C$ is a Cartan matrix of finite type (in particular the symmetric matrix $D_C$ is positively defined), and $q$ is not a root of 1 then the grading of $U^+_q(\mathfrak{g})$ by characters $\chi^i(x)$ coincides with the grading by $\Gamma^+$.

**Proof.** Since every homogeneous in $\Gamma^+$ element is homogeneous with respect to $\mathfrak{g}$, it suffices to show that the characters $\chi^i = \chi^i(x)$, $1 \leq i \leq n$ generate a free Abelian group. Suppose in contrary that
\begin{equation}
\chi_1 = (\chi^1)^{k_1} \cdots (\chi^n)^{k_n} = (\chi^1)^{m_1} \cdots (\chi^n)^{m_n}
\end{equation}
where $k_i, m_i \geq 0$, $k_i m_i = 0$, $1 \leq i \leq n$, and one of the $k_i$'s is nonzero. Let $g = g_1^{k_1} \cdots g_n^{k_n}$, $h = g_1^{m_1} \cdots g_n^{m_n}$. By means of (3.1) we have
\begin{equation}
\chi_1(g) = \prod_{1 \leq i, j \leq n} p_{ij}^{k_{ij}} = \prod_{i<j} (p_{ij} p_{ji})^{k_{ij}}, \quad \prod_i p_{ii}^{k_i} = q^N,
\end{equation}
where due to (3.1) for $\tilde{k} = (k_1, \ldots, k_n)$ we have
\begin{equation}
N = \sum_{i<j} d_i a_{ij} k_i k_j + \sum_i d_i k_i^2
= \frac{1}{2} (\sum_{i<j} d_i a_{ij} k_i k_j + \sum_{i<j} d_j a_{ji} k_j k_i + \sum_i a_{ii} d_i k_i^2) = (\tilde{k} DC, \tilde{k}) > 0.
\end{equation}
In the same way $\chi_2(h) = q^M$, $M \geq 0$. Relations (3.1) imply
\[
\chi_2(g)\chi_1(h) = \prod_{1 \leq i,j \leq n} p_{ij}^{m_{ij}k_j} \cdot \prod_{1 \leq i,j \leq n} p_{ij}^{-m_{ij}k_i} = \prod_{1 \leq i,j \leq n} (p_{ij}p_{ji})^{m_{ij}k_j} = q^L,
\]
with $L = \sum_{i,j} d_ia_{ij}m_{ij}k_j \leq 0$ since in the Catran matrix $a_{ij} \leq 0$ for $i \neq j$, while $k_i,m_i = 0$.

We have $\chi_1(g) = \chi_2(g)$, and $\chi_1(h) = \chi_2(h)$. Therefore $q^{M+N} = \chi_1(g)\chi_2(h) = \chi_2(g)\chi_1(h) = q^L$. A contradiction. □

**Remark.** Of course, if the characters $\chi_i$, $1 \leq i \leq n$ generate a free Abelian group then $g_i$, $1 \leq i \leq n$ generate a free Abelian group as well. In particular relations (3.1) imply that $G$ is a free Abelian group with the free generators $g_i$, $1 \leq i \leq n$ provided that $q$ is not a root of 1 and $C$ is of finite type.

**Corollary 3.2.** If $q$ is not a root of 1 and $C$ is of finite type, then every subalgebra $U$ of $U_q(\mathfrak{g})$ containing $G$ is homogeneous with respect to each of the variables $x_i$.

**Proof.** By the above lemma it suffices to note that $U$ is homogeneous with respect to the grading by characters (2.3). If $c = \sum_i c_i \in U$ with $c_i \in H^{\chi_i}$ and different $\chi_i \in G^*$, then
\[(3.4) \quad g^{-1}cg = \sum_i \chi_i(g)c_i \in U, \quad g \in G.
\]
According to the Dedekind Lemma there exist elements $h_i \in G$, such that the matrix $M = \|\chi_i(h_j)\|$ is invertible. Hence we may solve the system of equations (3.4) considering $c_i$ as variables. In particular $c_i \in U$. □

If the multiplicative order $t$ of $q$ is finite, then we define $u_q^+(\mathfrak{g})$ as $G(X)/\Lambda$, where $\Lambda$ is the biggest Hopf ideal in $G(X)^{(2)}$. This is a $\Gamma^+$-homogeneous ideal, see [20, Lemma 2.2]. Certainly $\Lambda$ contains all skew-primitive elements of $G(X)^{(2)}$ (each of them generates a Hopf ideal). Hence by [20, Theorem 6.1] relations (3.2) are still valid in $u_q^+(\mathfrak{g})$.

**Quantification of Kac-Moody algebras.** Consider a new set of variables $X^- = \{x_1, x_2, \ldots, x_n\}$. Suppose that an Abelian group $F$ generated by the elements $f_1, f_2, \ldots, f_n$ acts on the linear space spanned by $X^-$ so that $(x_i)^{f_j} = p_{ji}^{-1}x_i$, where $p_{ij}$ are the same parameters, see (3.1), that define $U_q^+(\mathfrak{g})$. Relations (3.1) are invariant under the substitutions $p_{ij} \leftarrow p_{ji}^{-1}$, $q \leftarrow q^{-1}$. This allows us to define the character Hopf algebra $U_q^-(\mathfrak{g})$ as $U_{q^{-1}}^+(\mathfrak{g})$ with the characters $\chi_i^-$, $1 \leq i \leq n$ such that $\chi_i^-(f_j) = p_{ji}^{-1}$.

We may extend the characters $\chi^i$ on $G \times F$ in the following way
\[(3.5) \quad \chi^i(f_j) \overset{df}{=} p_{ji} \equiv \chi^i(g_i).
\]
Indeed, if $\prod_k f_k^{m_k} = 1$ in $F$, then application to $x_i^-$ implies $\prod_k p^{-m_k}_{ki} = 1$, hence $\chi^i(\prod_k f_k^{m_k}) = \prod p_{ki}^{m_k}$ equals 1 as well. In the same way we may extend the characters $\chi_i^-$ on $G \times F$ so that
\[(3.6) \quad \chi_i^- = (\chi^i)^{-1} \quad \text{as characters of } G \times F.
\]
In what follows we denote by $H$ a quotient group $(G \times F)/N$, where $N$ is an arbitrary subgroup with $\chi^i(N) = 1$, $1 \leq i \leq n$. For example, if the quantification parameters satisfy additional symmetry conditions $p_{ij} = p_{ji}$, $1 \leq i,j \leq n$, as this is...
a case for the original Drinfeld-Jimbo and Lusztig quantifications, then $\chi^i(g_k^{-1}f_k) = p_{ik}^{-1}p_{ki} = 1$, and we may take $N$ to be the subgroup generated by $g_k^{-1}f_k$, $1 \leq k \leq n$. In this particular case the groups $H$, $G$, $F$ may be identified.

In the general case without loss of generality we may suppose that $G, F \subseteq H$. Certainly $\chi^i, 1 \leq i \leq n$ are characters of $H$ and $H$ still acts on the space spanned by $X \cup X^-$ by means of these characters and their inverses. Consider the skew group algebra $H(X \cup X^-)$ as a character Hopf algebra:

\begin{align}
\Delta(x_i) &= x_i \otimes 1 + g_i \otimes x_i, \\
\Delta(x_i^-) &= x_i^- \otimes 1 + f_i \otimes x_i^-,
\end{align}

\begin{align}
g^{-1}x_ig &= \chi^i(g) \cdot x_i, \\
g^{-1}x_i^-g &= (\chi^i)^{-1}(g) \cdot x_i^-,
\end{align}

$g \in H$.

We define the algebra $U_q(g)$ as a quotient of $H(X \cup X^-)$ by the following relations:

\begin{align}
[\ldots[[x_i, x_j], x_j], \ldots, x_j] &= 0, \quad 1 \leq i \neq j \leq n; \\
1-a_{ij} \text{ times}
\end{align}

\begin{align}
[\ldots[[x_i^-, x_j^-], x_j^-], \ldots, x_j^-] &= 0, \quad 1 \leq i \neq j \leq n; \\
1-a_{ij} \text{ times}
\end{align}

\begin{align}
[x_i, x_j^-] &= \delta_i^j(1 - g_i f_i), \quad 1 \leq i, j \leq n
\end{align}

where the brackets are defined on $H(X \cup X^-)$ by the structure of character Hopf algebra, see (2.3). Since due to (3.1) and (20) Theorem 6.1] all polynomials in the above relations are skew primitive in $H(X \cup X^-)$, they define a Hopf ideal of $H(X \cup X^-)$; that is, the natural homomorphism

\begin{align}
H(X \cup X^-) \rightarrow U_q(g)
\end{align}

defines a Hopf algebra structure on $U_q(g)$.

If $q$ has a finite multiplicative order then $u_q(g)$ is defined by relations (3.11) and $u = 0, u \in \Lambda, u^- = 0, u^- \in \Lambda^-$, where $\Lambda, \Lambda^-$ are the biggest Hopf ideals respectively in $G(X)^{(2)}$ and $F(X^-(2))$.

Both algebras $U_q(g)$, and $u_q(g)$ have a grading by the additive group $\Gamma$ generated by $\Gamma^+$, see p.6, provided that we put $D(x_i^-) = -D(x_i) = -x_i, D(H) = 0$ since in this way relations (3.11) become homogeneous.

**Corollary 3.3.** If $q$ is not a root of 1 and the Cartan matrix $C = ||a_{ij}||$ is of finite type then every subalgebra $U$ of $U_q(g)$ containing $H$ is $\Gamma$-homogeneous.

**Proof.** By Lemma 3.1 and definition (3.6) grading by $\Gamma$ coincides with the grading (2.9) by the group of characters (freely) generated by $\chi^i, 1 \leq i \leq n$. Hence every subspace invariant under the conjugations by $H$ is $\Gamma$-homogeneous. □

The defined quantification reduces to known ones under a suitable choice of $x_i, x_i^-$ depending up the particular definition of $U_q(g)$. For example for classical case of one parameter quantification we have $G = F = H$, and in the notations of (36) we may identify

\begin{align}
x_i &= E_i, \\
g_i &= K_i, \\
x_i^- &= F_i K_i (v^{-d_i} - v^{d_i})^{-1}, \\
p_{ij} &= v^{-d_i \alpha_{ij}},
\end{align}

while in the notations of (35) (40) we may take

\begin{align}
x_i &= E_i, \\
g_i &= \tilde{K}_i, \\
x_i^- &= F_i \tilde{K}_i (v_i^{-1} - v_i)^{-1}, \\
p_{ij} &= v^{-(\mu, \nu)}.
\end{align}
For two-parameter quantizations, say in the notations of [3], we may put
\[ x_i \leftarrow e_i, \ g_i \leftarrow \omega_i, \ x_i^+ \leftarrow f_i(\omega_i')^{-1}(r_i - s_i)^{-1}, \ f_i \leftarrow (\omega_i')^{-1}, \]
and find values of parameters \( p_{ij} \) by means of [3] (B2), (C2), (D2)). For the multiparameter case of Reshetikhin or DeConcini-Kac-Procesi in the notations of [6], we may take
\[ x_i \leftarrow E_iL_{\beta_i}, \ g_i \leftarrow L_{\beta_i-\alpha_i+\gamma_i}, \ x_i^- \leftarrow F_iL_{\alpha_i+\beta_i}^{-1}(q_i - q_i^-)^{-1}, \ f_i \leftarrow L_{\gamma_i+\alpha_i,+\beta_i}. \]

**Triangular decomposition.** One may prove that the subalgebra of \( U_q(\mathfrak{g}) \) generated by \( G \) and values of \( x_i, 1 \leq i \leq n \) is isomorphic to \( U^+_q(\mathfrak{g}) \) while the subalgebra generated by \( F \) and values of \( x_i^- \), \( 1 \leq i \leq n \) is isomorphic to \( U^-_q(\mathfrak{g}) \). Moreover, one has the following so-called “triangular decomposition” for both algebras:
\[
U_q(\mathfrak{g}) = U^-_q(\mathfrak{g}) \otimes_{k[F]} k[H] \otimes_{k[G]} U^+_q(\mathfrak{g}),
\]
\[
u_q(\mathfrak{g}) = U^-_q(\mathfrak{g}) \otimes_{k[F]} k[H] \otimes_{k[G]} U^+_q(\mathfrak{g}).
\]
Actually this is not so evident (see [35, 36] for standard one parameter version, [6] for the multiparameter version with Cartan matrix of finite type, [3] for two-parameter version with particular Cartan matrices only). We shall provide here a relatively short proof in the general setting that uses a lemma on tensor decomposition for character Hopf algebras, [23, Lemma 6.2], and (in case \( q^2 = 1 \)) the Hayeneman–Radford theorem.

**Proposition 3.4.** Let \( J \subseteq G(X)^{(2)}, \ J^- \subseteq F(X^-)^{(2)} \) be constitution homogeneous Hopf ideals of \( G(X) \) and \( F(X^-) \) respectively. Denote by \( \mathfrak{A} \) the algebra generated over \( H \) by \( X \cup X^- \) and defined by the relations (3.11) and \( u_s = 0, \ s \in S, \ u^-_s = 0, \ t \in T, \) where \( \{u_s, \ s \in S\} \) (respectively \( \{u^-_s, \ t \in T\} \)) is a set of homogeneous generators of the ideal \( J \) (respectively \( J^- \)). We have
\[
\mathfrak{A} = (F(X^-)/J^-) \otimes_{k[F]} k[H] \otimes_{k[G]} (G(X)/J).
\]

**Proof.** We note first that the algebra \( \mathfrak{A}_1 \) generated over \( H \) by \( X \) and defined by the relations \( u_s = 0, \ s \in S \) has the form \( k[H] \otimes_{k[G]} (G(X)/J) \), while the algebra \( \mathfrak{A}_2 \) generated over \( H \) by \( X^- \) and defined by the relations \( u^-_t = 0, \ t \in T \) has the form \( (F(X^-)/J^-) \otimes_{k[F]} k[H] \). Hence it suffices to show that \( \mathfrak{A} = \mathfrak{A}_2 \otimes_{k[H]} \mathfrak{A}_1 \).

Denote by \( D_i, D^-_i, 1 \leq i \leq n \) the linear maps
\[
D_i : k(X^-) \to H(X^-), \quad D^-_i : k(X) \to H(X)
\]
that satisfy the initial conditions
\[
D_i(x_i^-) = D^-_i(x_i) = \delta_i^j(1 - g_i f_i),
\]
and the skew differential Leibniz rules
\[
D_i(v \cdot w^-) = D_i(v) \cdot w^- + p(x_i, v^-)v \cdot D_i(w^-), \quad v^-, w^- \in k(X^-);
\]
\[
D^-_i(u \cdot v) = p(v, x_i^-)D^-_i(u) \cdot v + u \cdot D^-_i(v), \quad u, v \in k(X).
\]

Lemma 6.2 [23] (under the substitutions \( k \leftarrow n, \ n = 2n, \ x_{n+i} \leftarrow x_i^- \), \( G \leftarrow H, \ H \leftarrow \mathfrak{A} \)) gives the required decomposition provided that there exist homogeneous
defining relations \( \{ \varphi_s = 0, s \in S \} \), and \( \{ \psi_t = 0, t \in T \} \) for \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) respectively, such that

\[
(3.19) \quad D_i(\psi_t) \in H \cdot J, \quad D_i^-(\varphi_s) \in H \cdot J, \quad 1 \leq i \leq n, \quad s \in S, \quad t \in T.
\]

Consider the linear maps

\[
(3.20) \quad \tilde{D}^+_i : u \to \partial^+_i(u) - p_{ii}^{-1}p(u, x_i)\partial_i(u)g_if_i, \quad u \in k\langle X \rangle,
\]

where the partial derivatives are defined in (2.20) and (2.21). We have \( \tilde{D}^-_i(x_j) = \delta^j_i(1 - g_if_i) \), while relations (2.19) and (2.22) imply the differential Leibniz rule \( \tilde{D}^-_i(u \cdot v) = p(x_i, v)\tilde{D}^-_i(u) \cdot v + u \cdot \tilde{D}^-_i(v) \). Since according to (3.5) we have \( p(x_i, v) = p(v, x_i^-) \), the Leibniz rule and initial values for \( D^-_i \) coincide with that for \( \tilde{D}^-_i \). Hence \( D^-_i = \tilde{D}^-_i \). In perfect analogy we have

\[
(3.21) \quad D_i(u^-) = \partial^+_i(u^-)p(x_i, u^-)p_{ii}^{-1} - g_if_i\partial^-_{-i}(u^-), \quad u^- \in k\langle X^- \rangle,
\]

where \( \partial^-_{-i}, \partial^+_i \) are left and right partial derivatives on \( k\langle X^- \rangle \) with respect to \( x_i^- \).

Now if \( u_s, s \in S \) and \( u^-_t, t \in T \) are skew primitive elements (as this is the case for \( \mathfrak{A} = U_q(g) \)) then they are constants for \( \partial_i, \partial^+_i \), and \( \partial^-_{-i}, \partial^+_i \), respectively. Hence (3.20), (3.21) imply \( D^-_i(u_s) = D_i(u^-_t) = 0 \) and [23, Lemma 6.2] applies.

In the general case by the Heyneman-Radford theorem (see [41, Corollary 5.4.7] or very “skew primitive” version [22, Corollary 5.3]) the Hopf ideal \( J \) has a nonzero skew primitive element provided that \( J \neq 0 \). Denote by \( J_1 \) an ideal generated by all skew primitive elements of \( J \). Clearly \( J_1 \) is a Hopf ideal. Since all homogeneous components of a skew primitive element are skew primitive, the Hopf ideal \( J_1 \) is homogeneous. Moreover, we have \( D^-_i(u_s) = 0 \), \( s \in S_1 \) where \( u_s \) run through the set of all homogeneous skew primitive elements of \( J \). Now consider the Hopf ideal \( J/J_1 \) of the quotient Hopf algebra \( G(X)/J_1 \). If \( J \neq J_1 \) then this ideal also has nonzero skew primitive elements. Denote by \( J_2/J_1 \) the ideal generated by all skew primitive elements of \( J/J_1 \), where \( J_2 \) is its preimage with respect to the natural homomorphism \( G(X) \to G(X)/J_1 \). Again we have \( D^-_i(\tilde{u}_s) = 0 \), \( s \in S_2 \) in \( G(X)/J_1 \), where \( \tilde{u}_s \) run through the set of all homogeneous skew primitive elements of \( J/J_1 \). In particular the ideal \( J_2 \) has a set of generators \( u_s, s \in S_1 \cup S_2 \) such that \( D^-_i(u_s) \in J_1 \). Continuing the process we shall find a set of generators \( u_s, s \in S_1 \cup S_2 \cup S_3 \cup \ldots \) for \( J \) such that \( D^-_i(u_s) \in J, \) all \( s \).

In perfect analogy we find a set of generators \( u^-_t \), for \( J^- \) such that \( D_i(u^-_t) \in J^- \), all \( t \). Hence [23, Lemma 6.2] applies.

**Remark.** In the proof we do not use relations (3.1) on the quantification parameters, while relations (3.5), (3.6) on the characters are essential. Also we are reminded that originally the maps \( D_i, D^-_i \) were defined so that \( D_i(u^-) = [x_i, u^-] \), \( D^-_i(u) = [u, x_i^-] \) in the algebra \( H(X \cup X^-)[[x_i, x_i^-]] = \delta^j_i(1 - g_if_i) \). Hence equalities \( D^-_i = \tilde{D}^-_i \) and (3.21) imply a differential representation:

\[
(3.22) \quad [x_i, u^-] = \partial^+_i(u^-)p(x_i, u^-)p_{ii}^{-1} - g_if_i\partial^-_{-i}(u^-), \quad u^- \in k\langle X^- \rangle,
\]

\[
(3.23) \quad [u, x_i^-] = \partial^+_i(u) - p_{ii}^{-1}p(u, x_i)\partial_i(u)g_if_i, \quad u \in k\langle X \rangle.
\]
4. PBW-generators for coideal subalgebras in $U_q^+(\mathfrak{sl}_{n+1})$, $u_q^+(\mathfrak{sl}_{n+1})$

Suppose that the quantification parameters $p_{ij} = p(x_i, x_j) = \chi^j(g_j)$ are connected by the following relations

\begin{equation}
(4.1) \quad p_{ii} = q; \quad p_{i+1j+1i} = q^{-1}; \quad p_{ij}p_{ji} = 1, \quad |i - j| > 1,
\end{equation}

where $q \neq \pm 1$. By definition $U_q^+(\mathfrak{sl}_{n+1})$ as a character Hopf algebra is set up by the relations

\begin{equation}
(4.2) \quad [x_i, x_{i+1}], x_{i+1} = [x_i, x_{i+1}] = [x_i, x_j] = 0, \quad |i - j| > 1.
\end{equation}

The structure of this algebra is defined by the following theorem (see, [23] Theorem A$_n$), or in other terms [1] Lemmas pp. 176, 184, [5]). Recall that $u(k, m) = x_k x_{k+1} \cdots x_m, k \leq m$. The standard word $u(k, m)$ defines a super-letter $u[k, m] = [x_k x_{k+1} \cdots x_m]$, while by definition $u[k, k] = x_k$. Of course the value of $u[k, m]$ in $U_q^+(\mathfrak{sl}_{n+1})$ is independent of the alignment of brackets (Lemma 2.2). By $U_q^+(\mathfrak{sl}_{n+1})$ we denote the ideal $\xi(G(X)^{(2)})$ generated by the values of $x_i x_j$, $1 \leq i, j \leq n$.

**Theorem 4.1.** 1. The values of the super-letters $u[k, m], 1 \leq k \leq m \leq n$ in $U_q^+(\mathfrak{sl}_{n+1})$ form the set of PBW-generators of $U_q^+(\mathfrak{sl}_{n+1})$ over $k[G]$. All heights are infinite.

2. If $q$ is not a root of unity, then the ideal $U_q^+(\mathfrak{sl}_{n+1})^{(2)}$ has no nonzero skew-primitive elements.

3. If $u$ is a standard word then either $u = u(k, m)$ or $[u] = 0$ in $U_q^+(\mathfrak{sl}_{n+1})$.

According to the Heyneman-Radford theorem (see [14], or [41] Corollary 5.4.7) every non-zero bi-ideal of a character Hopf algebra always has nonzero skew-primitive elements. By this reason the second statement implies that $\text{Ker} \xi : G(X) \to U_q^+(\mathfrak{sl}_{n+1})$, is the biggest Hopf ideal in $G(X)^{(2)}$. In particular one can apply Lemma 2.10 to $U_q^+(\mathfrak{sl}_{n+1})$ provided $q$ is not a root of 1.

If $q$ is a root of 1 ($q \neq \pm 1$), then by definition $u_q^+(\mathfrak{sl}_{n+1})$ is a quotient $U_q^+(\mathfrak{sl}_{n+1})/\Lambda$, where $\Lambda$ is the biggest Hopf subideal of $U_q^+(\mathfrak{sl}_{n+1})^{(2)}$. Hence we may apply Lemma 2.10 to $u_q^+(\mathfrak{sl}_{n+1})$ as well.

If $U$ is a right coideal subalgebra of $U_q^+(\mathfrak{sl}_{n+1})$ that contains $k[G]$, then by Proposition 2.8 and Theorem 4.1 it has PBW-generators of the form (2.13):

\begin{equation}
(4.3) \quad c_u = u^s + \sum \alpha_i W_i + \cdots \in U, \quad u = u[k, m].
\end{equation}

By means of relations (4.4) we have $p_{nm} = p(x_k x_{k+1} \cdots x_m, x_k x_{k+1} \cdots x_m) = q$. Thus, if $q$ is not a root of 1, Lemma 2.7 shows that in (4.3) the exponent $s$ equals 1, while all heights of the $c_u$'s in $U$ are infinite.

If $q$ has a finite multiplicative order $t > 2$, then $u[k, m]^t = 0$ in $U_q^+(\mathfrak{sl}_{n+1})$, see for example [26] Theorem 3.2; that is, by [26] Lemma 3.3 the values of $u[k, m]$ are still the PBW-generators of $U_q^+(\mathfrak{sl}_{n+1})$, but all of them have the finite height $t$.

By Lemma 2.7 in (4.3) we have $s \in \{1, t, tl^t\}$. Since $u[k, m]^t = u[k, m]^{tl^t} = 0$, the exponent $s$ in (4.3) equals 1, while all heights of the $c_u$'s in $U$ equal $t$.

Hence in both cases the PBW-generators of $U$ have the following form

\begin{equation}
(4.4) \quad c_u = u[k, m] + \sum \alpha_i W_i + \sum \beta_j V_j \in U.
\end{equation}
where $W_i$ are the basis super-words starting with less than $u[k,m]$ super-letters, $D(W_j) = D(u[k,m]) = x_k + x_{k+1} + \ldots + x_m$, and $V_j$ are $G$-super-words of $D$-degree less than $x_k + x_{k+1} + \ldots + x_m$.

Now, in order to reduce the freedom in construction of the PBW-generators, we are going to show that \((4.3)\) with homogeneous $c_u$ implies that $U$ has an element of the same form that belongs to a special finite set of elements \((4.6)\).

**Proposition 4.2.** If a right coideal subalgebra $\mathbf{U} \supseteq k[G]$ of $U_q^+(\mathfrak{sl}_{n+1})$ or $u_q^+(\mathfrak{sl}_{n+1})$ contains a homogeneous element $c$ with the leading term $u[k,m]$, $k \leq m$, then for a suitable subset $S$ of the interval $[k,m-1]$ the value of the below defined element $\Psi^S(k,m)$ belongs to $U$.

**Definition 4.3.** Let $S$ be a set of integers from the interval $[1,u]$. We define a piecewise continuous word related to $S$ as follows

\[
\Psi^S(k,m) \overset{\text{df}}{=} u(1+s_r,m)u(1+s_{r-1},s_r) \cdots u(1+s_1,s_2)u(k,1),
\]

where $S \cap [k,m-1] = \{s_1,s_2,\ldots,s_r\}$, $k \leq s_1 < s_2 < \ldots < s_{r-1} < s_r < m$.

If the pair $(k,m)$ is fixed, we denote $S_0 = (S \cap [k,m-1]) \cup \{k-1\}$, while $S^* = (S \cap [k,m-1]) \cup \{m\}$; respectively we extend $s_0 = k-1 \in S_0$, and $s_{r+1} = m \in S^*$.

By $\overline{S}$ we denote a complement of $S$ with respect to $[k,m-1]$.

We define a bracketing of a piecewise continuous word by

\[
\Psi^S(k,m) \overset{\text{df}}{=}
[\ldots [u[1+s_r,m],u[1+s_{r-1},s_r]],\ldots ,u[1+s_1,s_2],u[k,1]].
\]

The leading term of $\Psi^S(k,m)$ in the PBW-decomposition given in Theorem \((4.1)\) is proportional to $u[k,m]$. In particular $\Psi^S(k,m)$ has the form \((4.3)\) up to a scalar factor.

The elements $y_i = u[1+s_{i-1}+1], s_{r-2} < s_{r-1} < m$, $1 \leq i \leq r+1$, satisfy $p(y_i,y_j)p(y_j,y_i) = 1$, $[y_i,y_j] = 0$ provided that $|i-j| > 1$. Thus by Lemma \((2.2)\) applied to \((4.3)\) the value in $U_q^+(\mathfrak{sl}_{n+1})$ or $u_q^+(\mathfrak{sl}_{n+1})$ of the bracketing is independent of the alignment of the big brackets. In particular we have the following decomposition

\[
\Psi^S(k,m) = [\Psi^S(1+s_i,m),\Psi^S(k,s_i)].
\]

Of course the word $u(k,m)$ is a piecewise continuous word with empty $S$, or more generally, with $S \cap [k,m-1] = \emptyset$.

Let us choose an arbitrary $s_{r+1}$, $s_r < s_{r+1} < m$, and denote

\[
u = u[1+s_r,s_{r+1}], \quad v = u[1+s_{r+1},m].
\]

Then by \((2.1)\) and decomposition \((4.7)\) we have

\[
\Psi^S(k,m) = [\nu,\Psi^S(k,s_r)],
\]

while

\[
\Psi^{S,u(s_{r+1})}(k,m) = [\nu,u],\Psi^S(k,s_r)].
\]

Since $[\nu,\nu] = -p_{vu}[\nu,\nu] + (1-p_{uv}p_{vu})v \cdot u$ and $p_{uv}p_{vu} = q^{-1}$, using formula \((2.11)\), we get the following recurrence relation for the complete bracketing:

\[
\Psi^{S,u(s_{r+1})}(k,m) = (1 - q^{-1})v \cdot \Psi^S(k,s_{r+1}) - p_{vu} \Psi^S(k,m),
\]

where as above $\nu = u[1+s_r,s_{r+1}], \quad v = u[1+s_{r+1},m]$. 

More generally, if $s_{i-1} < t < s_i$, and we denote $u = u[1+s_{i-1}, t]$, $v = u[1+t, s_i]$, then (4.8) reads

$$\Psi^{S_{ij}(t)}(k, s_i) = (1 - q^{-1})v \cdot \Psi^S(k, t) - p_{vu} \Psi^S(k, s_i).$$

Let us commute this equality with $\Psi^S(1 + s_i, m)$ from the left. Then decomposition (4.7) and (2.10) imply a general formula

$$\Psi^{S_{ij}(t)}(k, m) = (1 - q^{-1})[\Psi^S(1 + s_i, m), v] \cdot \Psi^S(k, t) - p_{vu} \Psi^S(k, m)$$

(4.9)

$$= (1 - q^{-1})\Psi^S(1 + t, m) \cdot \Psi^S(k, t) - p_{vu} \Psi^S(k, m).$$

Denote by $\varphi$ a map from $[1, n]$ to $[1, n]$ given by $\varphi(i) = n - i + 1$. One may replace the main skew primitive generators $x_i, 1 \leq i \leq n$ with $y_i, 1 \leq i \leq n$, where by definition $y_i = x_{\varphi(i)}$. Our basic concepts (Definition 4.3) are not invariant with respect to this replacement. For example,

$$\Psi^\varphi(1, n) = [x_1 x_2 \cdots x_n] = [y_n y_{n-1} \cdots y_1] = \Psi^{[1,n-1]}_y(1, n).$$

This fact signifies that the application of already proved formulae to new generators ought to provide additional information. To get this information we need the following decoding lemma. In what follows by $\sim$ we denote the projective equality: $a \sim b$ if and only if $a = \alpha b$, where $0 \neq \alpha \in k$.

**Lemma 4.4.** We have

$$\Psi^S(k, m) \sim \Psi_{\varphi(S)}^{-1}(\varphi(m), \varphi(k)),$$

where by $\varphi(S) - 1$ we denote $\{\varphi(s) - 1 | s \in S\}$, while the complement is related to the pair $(\varphi(m), \varphi(k))$, see Definition 4.3.

**Proof.** We use induction on $m - k$. If $m = k$, the equality reduces to $x_k = y_{\varphi(k)}$. To make the inductive step we consider two cases.

a). $m - 1 \in S$. In this case $\varphi(m - 1) \in \varphi(S)$. Since $\varphi(m - 1) = \varphi(m) + 1$, we get $\varphi(m) \in \varphi(S) - 1$. In particular we have an equality of sets:

(4.11)

$$[\varphi(m), \varphi(k) - 1] \setminus \{\varphi(S) - 1\} = [\varphi(m) + 1, \varphi(k) - 1] \setminus \{\varphi(S) - 1\}.$$

Using the inductive supposition we have

$$\Psi^S(k, m) \sim [x_m, \Psi^S(k, m - 1)] = [y_{\varphi(m)}, \Psi_{\varphi(S)}^{-1}(\varphi(m) + 1, \varphi(k))].$$

If set (4.11) is empty, the above element equals the element $u_y[\varphi(m), \varphi(k)]$ that, in the context, coincides with $\Psi_{\varphi(S)}^{-1}(\varphi(m), \varphi(k))$. Otherwise denote by $t_1$ the minimal element in (4.11). Using definition (4.6) we may continue

$$= [y_{\varphi(m)}, \Psi_{\varphi(S)}^{-1}(1 + t_1, \varphi(k)), u_y[\varphi(m) + 1, t_1]].$$

Denote $u = y_{\varphi(m)}, v = \Psi_{\varphi(S)}^{-1}(1 + t_1, \varphi(k))$, and $w = u_y[\varphi(m) + 1, t_1]$. Then $[u, v] = 0$ since $u, v$ are separated. By the same reason $p_{uv} p_{vu} = 1$, see (4.11). Hence the conditional Jacobi identity (2.8) implies

$$[u, [v, w]] \sim [v, [u, w]] = \Psi_{\varphi(S)}^{-1}(\varphi(m), \varphi(k)).$$

b). $m - 1 \notin S$. In this case $s_r < m - 1$, and we have

$$\Psi^S(k, m) = [u[1 + s_r, m], \Psi^S(k, s_r)] = [[x_{1+s_r}, u[2 + s_r, m]], \Psi^S(k, s_r)].$$

Hence the projective equality (4.8) reduces to

$$[u[1 + s_r, m], \Psi^S(k, s_r)] = (1 - q^{-1})\Psi^S(k, s_r) \cdot \Psi^S(k, s_r) - p_{vu} \Psi^S(k, s_r).$$

Finally

$$\Psi^S(k, m) = (1 - q^{-1})\Psi^S(k, s_r) \cdot \Psi^S(k, s_r) - p_{vu} \Psi^S(k, s_r).$$
Let us denote \( u = x_{1+s_r}, \ v = u[2+s_r,m], \ w = \Psi^S(k,s_r) \). Then \([v,w] = 0,\ p_{uw}p_{vw} = 1\) since \(v\) and \(w\) are separated. Thus the conditional Jacobi identity (4.4) implies \([u,v],w) \sim [u,w],v)\. Since \([u,w] = \Psi^S(k,1+s_r)\), we may apply the inductive supposition

\[
(4.12) \quad [u,w],v) = \left[\Psi^S_{y}(s_{r+1},(\varphi(s_r) - 1, \varphi(k)), [y_{\varphi(s_r)}-2y_{\varphi(s_r)} - 3\cdots y_{\varphi(m)}]\right).
\]

We know that \(S\) has no one of the points \(1+s_r, 2+s_r, \ldots, m-1\), hence the set \(\varphi(S) - 1\) has no one of the points \(\varphi(s_r) - 2, \varphi(s_r) - 3, \ldots, \varphi(m) + 1, \varphi(m)\), and we have the equality of sets

\[
\{ \varphi(m), \varphi(m) + 1, \ldots, \varphi(s_r) - 2 \} \cup \{ \varphi(s_r) - 1, \varphi(k) - 1 \} \setminus \{ \varphi(S) - 1 \}
\]

Therefore the right hand side of (4.12) takes up the form of the right hand side of (4.11). The decoding lemma is proved.

\[\square\]

**Lemma 4.5.** If \(t \notin S, k \leq t < m\), then

\[
(4.13) \quad \Psi^S(k,m) \sim [\Psi^S(k,t), \Psi^S(1+t,m)]
\]

**Proof.** One should replace variables by means of (4.11), then apply (4.7), and replace back the variables by (4.11). \[\square\]

In particular we have the following formula, convenient for induction.

\[
(4.14) \quad \Psi^S(k,m) \sim \begin{cases} [x_m, \Psi^S(k,m-1)], & \text{if } m-1 \in S; \\ [\Psi^S(k,m-1), x_m], & \text{if } m-1 \notin S. \end{cases}
\]

Symmetrically relations (4.7) and (4.13) imply

\[
(4.15) \quad \Psi^S(k,m) \sim \begin{cases} [x_k, \Psi^S(k+1,m)], & \text{if } k \notin S; \\ [\Psi^S(k+1,m), x_k], & \text{if } k \in S. \end{cases}
\]

Our immediate task is to find a differential subspace generated by \(\Psi^S(k,m)\). To attain these ends we display the element \(\Psi^S(k,m)\) graphically as a sequence of black and white points labeled by the numbers \(k-1, k, k+1, \ldots, m-1, m\), where the first point is always white, and the last one is always black, while an intermediate point labeled by \(i\) is black if and only if \(i \in S\):

\[
(4.16) \quad \begin{array}{cccccccc}
& & & & & k-1 & k & k+1 & k+2 & \cdots & m-2 & m-1 & m \\
& & & & i^{-1} & & & & & & & \\
& & & & \bullet & & & & & & & \bullet
\end{array}
\]

For example the primitive generator \(x_i\) is displayed by two dots

\[
\begin{array}{cccccccc}
& & & & & k-1 & k & k+1 & k+2 & \cdots & m-2 & m-1 & m \\
& & & & i^{-1} & & & & & & & \\
& & & & \bullet & & & & & & & \bullet
\end{array}
\]

The element \([x_kx_{k+1} \cdots x_{m-1}x_m]\) is pictured as the whitest sequence

\[
\begin{array}{cccccccc}
& & & & & k-1 & k & k+1 & k+2 & \cdots & m-2 & m-1 & m \\
& & & & i^{-1} & & & & & & & \\
& & & & \bullet & & & & & & & \bullet
\end{array}
\]

The most black sequence

\[
\begin{array}{cccccccc}
& & & & & k-1 & k & k+1 & k+2 & \cdots & m-2 & m-1 & m \\
& & & & i^{-1} & & & & & & & \\
& & & & \bullet & & & & & & & \bullet
\end{array}
\]

corresponds to the “dual” element \([x_mx_{m-1} \cdots x_{k+1}x_k]\).
In what follows we denote by \( W^S(k, m) \) the set of all elements that are displayed by subsequences of the sequence (4.19) related to \( \Psi^S(k, m) \):

\[
\{ \Psi^S(a, b), \; |k \leq a \leq b \leq m; \; b \in S \; \text{or} \; b = m; \; a - 1 \notin S \; \text{or} \; a = k \}.
\]

The following theorem shows that the differential subspace generated by an element displayed by (4.19) is spanned by all elements corresponding to subsequences of (4.19) and their separated products.

**Definition 4.6.** If \( w \) is a word in \( X \), we define a differential operator \( D_w \) by the recurrence formula \( D_{x_k u} = \partial_k \circ D_u, D_0 = \text{id.} \)

**Theorem 4.7.** The differential subspace generated by \( \Psi^S(k, m) \) in \( U_q^+(\mathfrak{sl}_{n+1}) \) or \( U_q^+(\mathfrak{sl}_{n+1}) \) is spanned by the values of \( \Psi^S(k, m) \) and by the values of all products of pairwise separated (hence \( q \)-commuting, see Definition 2.1) elements from \( \Psi^S(k, m) \).

**Proof.** We shall need the following properties of the partial derivations. If \( u \) is independent of \( x_j \) (or, more generally, if \( \partial_j(u) = 0 \)) then

\[
\partial_j([u, v]) = p(u, x_j) [u, \partial_j(v)], \quad \partial_j([u, x_j]) = 0.
\]

If \( \partial_j(v) = 0 \), then

\[
\partial_j([u, v]) = \partial_j(u) \cdot v - p_{uv}p(v, x_j) v \cdot \partial_j(u);
\]

if additionally \( p(x_j, v)p(v, x_j) = 1 \) (and still \( \partial_j(v) = 0 \)), then

\[
\partial_j([u, v]) = [\partial_j(u), v].
\]

These properties are a straightforward consequence of the definition (2.4) and skew differential Leibniz rule (2.19). Indeed,

\[
\partial_j([u, v]) = \partial_j(u) - p_{uv}\partial_j(vu)
\]

\[
= \partial_j(u) \cdot v + p(u, x_j) u \cdot \partial_j(v) - p_{uv}\partial_j(v) \cdot u - p_{uv}p(v, x_j) v \cdot \partial_j(u),
\]

which easily implies (4.18) - (4.20) since \( p_{uv} = p(u, x_j)p(u, \partial_j(v)) \) provided that \( v \) is dependent on \( x_j \), and \( p_{uv} = p(\partial_j(u), v)p(x_j, v) \) provided that \( u \) is dependent on \( x_j \).

We shall prove by induction on \( m - k \) the following formula

\[
\partial_j(u[k, m]) = \begin{cases} 0, & j \neq k; \\ (1 - q^{-1})u[k + 1, m], & j = k < m; \\ 1, & j = k = m. \end{cases}
\]

If \( k = m \), the formula follows from definition (2.19). Let \( k < m \). According to recurrence definition (2.15) we have \( u[k, m] = [x_k, u[k + 1, m]]. \)

If \( j = k \), then \( \partial_j(u[k + 1, m]) = 0 \) since \( u[k + 1, m] \) is independent of \( x_k \). Hence (4.19) with \( u \leftarrow x_k, v \leftarrow u[k + 1, m] \) implies

\[
\partial_j(u[k, m]) = v - p(x_k, v)p(v, x_k) \cdot v.
\]

By means of (4.1) we have \( p(x_k, v)p(v, x_k) = p_k + 1 \eta p_k + 2 \cdots p_k + 2 k \cdots p_{mk} = p_{k+1}k + 1 k \cdots p_{mk} = q^{-1}. \)

If \( j \neq k \), relation (4.18) with \( u \leftarrow x_k, v \leftarrow u[k + 1, m] \) implies

\[
\partial_j(u[k, m]) = p(u, x_k)[x_k, \partial_j(u[k + 1, m])].
\]

The inductive supposition yields \( \partial_j(u[k + 1, m]) = \alpha u[k + 2, m] \), \( \alpha \in k \). Since \( x_k \) and \( u[k + 2, m] \) are separated, they \( q \)-commute (this is true even if \( u[k + 2, m] \) is
empty: \([x_k,1] = x_k \cdot 1 - 
abla^k (id) 1 \cdot x_k = 0\). Thus \([x_k, \partial_i(u[k+1,m])] = 0\), which completes the proof of (4.21).

Since \(\Psi^S(k,m)\) is a linear combination of products of \(u[1+s_{i-1}, s_i]\), \(1 \leq i \leq r+1\), formula (4.21) implies

\[
\partial_j(\Psi^S(k,m)) = 0, \text{ if } j - 1 \notin S_0 \equiv (S \cap [k, m-1]) \cup \{k-1\}.
\]

Let us consider remaining partial derivatives \(\partial_j\) when \(j - 1 \in S_0\), that is \(j = k\) or \(j = 1 + s_i, s_i \in S \cap [k, m-1]\).

Suppose \(j = k\). Then definition (4.6) implies \(\Psi^S(k,m) = [u,v]\), where \(u = \Psi^S(1+s_1,m), v = u[k,s_1]\). Since \(u\) is independent of \(x_k\), relation (4.15) yields \(\partial_k([u,v]) = p(u,x_k)[u,\partial_k(v)]\). Hence by (4.21) we get

\[
\partial_k(\Psi^S(k,m)) = \begin{cases} 
\lambda \Psi^S(k+1,m), & \text{if } s_1 \neq k < m; \\
0, & \text{if } s_1 = k < m,
\end{cases}
\]

provided that \(S \cap [k, m-1] \neq \emptyset\), where \(\lambda = (1 - q^{-1})p(u(1+s_1,m),x_k)\).

Let \(j = 1 + s_r\). Recurrence formula (4.8) reads

\[
\Psi^S(k,m) = (1 - q^{-1})v \cdot \Psi^S(k,s_r) - p_{uv} \Psi^S(1, s_r)(k,m),
\]

where \(u = u[1+s_{r-1}, s_r], v = u[1+s_r, m]\). By (4.22) we have \(\partial_j(\Psi^S(1, s_r)(k,m)) = 0\).

Since \(\Psi^S(k,s_r)\) is independent of \(x_j\), by means of (4.21) and skew differential Leibniz rule (2.10) we have

\[
\partial_{1+s_r}(\Psi^S(k,m)) = \begin{cases} 
(1 - q^{-1})^2 u[2 + s_r, m] \cdot \Psi^S(k,s_r), & \text{if } 1 + s_r \neq m; \\
(1 - q^{-1}) \Psi^S(k,s_r), & \text{if } 1 + s_r = m.
\end{cases}
\]

Let \(j = 1 + s_i, 1 \leq i < r\). By (4.7) we have \(\Psi^S(k,m) = [u,v]\), where \(u = \Psi^S(1+s_i,m), v = \Psi^S(k,s_i)\). By (4.23) the elements \(\partial_j(u)\) and \(v\) are separated, hence \(v \cdot \partial_j(u) = p(v, \partial_j(u)) \partial_j(u) \cdot v\). Since \(v\) is independent of \(x_j\), one may apply (4.19). We have \(\partial_j([u,v]) = \partial_j(u) \cdot v(1 - p_{uv}p(v,x_j)p(v,\partial_j(u)))\). Due to (4.11) we get \(p_{uv}p(v,x_j)p(v,\partial_j(u)) = p_{uv}p_{uv} = q^{-1}\). Thus formula (4.23) implies

\[
\partial_{1+s_i}(\Psi^S(k,m)) = \begin{cases} 
\mu \Psi^S(2+s_i,m) \cdot \Psi^S(k,s_i), & \text{if } s_{i+1} > 1 + s_i; \\
0, & \text{if } s_{i+1} = 1 + s_i,
\end{cases}
\]

where \(\mu = (1 - q^{-1})^2 p(u(1+s_{i+1},m),x_{1+s_i})\).

Formulæ (4.22–4.25) show that products of pairwise separated elements from \(W^S(k,m)\) span a differential subspace, that contains all first derivatives of \(\Psi^S(k,m)\). Hence by induction it contains the derivatives of higher order as well.

To see that any product of pairwise separated elements from \(W^S(k,m)\) is proportional to some derivative of \(\Psi^S(k,m)\) we shall prove the following relation.

\[
\Psi^S(k,m) \cdot D_w = \alpha \in \mathbb{K}, \alpha \neq 0, \text{ if } w = u^S(k,m).
\]

If \(S \cap [k, m-1] = \emptyset\) then \(w = x_k x_{k+1} \ldots x_m\) and relation follows from (4.21). Let \(S \cap [k, m-1] \neq \emptyset\). By definition (4.3) we have \(w = v'w', \text{ where } v = x_{1+s_1}, x_{2+s_2}, \ldots, x_m, w' = u^S(k,s_r)\). Hence

\[
\Psi^S(k,m) \cdot D_w = \partial_{1+s_r}(\Psi^S(k,m)) \cdot D_v' D_w',
\]
where \( v' = x_{2+s_1} \ldots x_{m} \). By (4.21) the element \( \partial_{1+s_{r}}(\Psi^S(k,m)) \) is proportional to \( u[2+s_{r}] \cdot \Psi^S(k,s_{r}) \). Since \( \Psi^S(k,s_{r}) \) is independent of \( x_j \), \( 2+s_{r} \leq j \leq m \), skew differential Leibniz rule (2.19) implies

\[
(u[2+s_{r}] \cdot \Psi^S(k,s_{r})) \cdot D_{v'} = (u[2+s_{r}] \cdot D_{v'}) \Psi^S(k,s_{r})
\]

By means of the multiple application of (4.21) we see that \( \Psi^S(k,m) \cdot D_{w'} \) is proportional to \( \Psi^S(k,s_{r}) \cdot D_{v'} \). By induction on \( m-k \) we get (4.20).

Now consider a product of separated elements from \( W^S(k,m) \),

\[
(4.27) \quad \Psi^S(a_1,b_1) \cdot \Psi^S(a_2,b_2) \ldots \Psi^S(a_i,b_i), \quad k \leq b_i < a_{i+1} - 1 < m, 1 \leq i \leq l.
\]

We shall prove by induction on \( l \) that there exists a word \( w \) such that \( \Psi^S(k,m) \cdot D_{w} \) is proportional to (4.27).

Assume \( l = 1 \), \( \Psi^S(a,b) \in W^S(k,m) \). Let us prove first that \( \Psi^S(a,b) \) has the required representation. If \( a = k \) there is nothing to prove. Let \( a > k \). In this case by definition \( a-1 \notin S \), say \( s_i < a-1 < s_{i+1} \) for some \( i \), \( 0 \leq i \leq r \), where formally \( s_0 = k-1, s_{r+1} = m \). We have \( s_{i+1} > 1 + s_i \). Hence by (4.23) or by (4.24) provided \( i = 0 \), or by (4.24) provided \( i = r \), the element \( \partial_{1+s_{i}}(\Psi^S(k,m)) \) is proportional to \( \Psi^S(2+s_{i},m) \cdot \Psi^S(k,s_{i}) \), where formally \( \Psi^S(k,s_{i}) = 1 \). Since \( \Psi^S(2+s_{i},m) \) is independent of \( x_j \), \( k \leq j \leq s_{i} \), formula (4.20) shows that \( \Psi^S(k,m) \cdot D_{u}, u = x_{1+s_{i}} \cdot \Psi^S(k,s_{i}) \) is proportional to \( \Psi^S(k+s_{i},m) \). If \( 2+s_{i} = a \), the required representation of \( \Psi^S(a,m) \) is found. If \( 2+s_{i} < a \) then by means of a multiple use of (4.23) we see that \( \Psi^S(2+s_{i},m) \cdot D_{v}, v = x_{2+s_{i}}x_{3+s_{i}} \ldots x_{a-1} \) is proportional to \( \Psi^S(a,m) \).

Similarly we may find a word \( w \) such that \( \Psi^S(a,m) \cdot D_{w} \) is proportional to \( \Psi^S(a,b) \). Indeed, if \( b = m \), we put \( w = \emptyset \). If \( b < m \), then by definition \( b \in S \). Denote by \( s_{i} \) an element from \( S \) such that \( [b,s_{i}] \subseteq S \), and either \( 1+s_{i} \notin S \) or \( 1+s_{i} = m \).

If \( 1+s_{i} \notin m \) then Eq. (4.24) implies that \( \partial_{1+s_{i}}(\Psi^S(a,m)) \) is proportional to \( \Psi^S(2+s_{i},m) \cdot \Psi^S(a,s_{i}) \). Since \( \Psi^S(2+s_{i},m) \) is independent of \( x_{j} \), \( a \leq j \leq s_{i} \), formula (4.20) implies that \( \Psi^S(a,m) \cdot D_{y}, y = x_{1+s_{i}} \cdot \Psi^S(2+s_{i}-a,m) \) is proportional to \( \Psi^S(a,s_{i}) \).

If \( 1+s_{i} = m \) then by (4.24) the element \( \partial_{1+s_{i}}(\Psi^S(a,m)) \) itself is proportional to \( \Psi^S(a,s_{i}) \).

Finally a multiple use of (4.23) shows that \( \Psi^S(a,s_{i}) \cdot D_{z}, z = x_{a}x_{a-1}x_{a-2} \ldots x_{b+1} \) is proportional to \( \Psi^S(a,b) \). This completes the case \( l = 1 \).

Consider (4.27) with \( l > 1 \). By definition \( 1+b_{1} \neq a_{2} \), and \( b_{1} \in S, a_{2} - 1 \notin S \) since obviously \( b_{1} \notin m, a_{2} \notin k \). Denote by \( s_{i} \) a number such that \( [b_{1},s_{i}] \subseteq S, 1+s_{i} \notin S \). Of course \( 1+s_{i} \neq a_{2} - 1 \). Relation (4.23) implies that \( \partial_{1+s_{i}}(\Psi^S(k,m)) \) is proportional to \( \Psi^S(2+s_{i},m) \cdot \Psi^S(k,s_{i}) \). We have \( k \leq a_{1} \leq b_{1} \leq s_{i} \), hence by the considered above case there exists a word \( v \) in \( Y = \{x_{j} | k \leq j \leq a_{1} \text{ or } b_{1} < j \leq s_{i} \} \) such that \( \Psi^S(k,s_{i}) \cdot D_{v} \) is proportional to \( \Psi^S(a_{1},b_{1}) \). Since \( \Psi^S(2+s_{i},m) \) is independent of \( Y \), skew differential Leibniz rule (2.19) shows that \( (\Psi^S(2+s_{i},m) \cdot \Psi^S(k,s_{i})) \cdot D_{v} \) is proportional to \( \Psi^S(2+s_{i},m) \cdot \Psi^S(a_{1},b_{1}) \). Further, by the inductive supposition there exists a word \( u \) in \( Z = \{x_{j} | 2+s_{i} \leq j \leq m \} \) such that \( \Psi^S(2+s_{i},m) \cdot D_{u} \) is proportional to \( \Psi^S(a_{2},b_{2}) \ldots \Psi^S(a_{l},b_{l}) \). Now we see that \( \Psi^S(k,m) \cdot D_{u} \) is proportional to (4.27) because all factors in (4.27) (skew)commute.

**Proof of Proposition 4.2** Since \( c \) is homogeneous, in representation (4.1) there is no the second sum, while by definition the \( W_{i}'s \) are monotonous products of the basis elements \( u[a,b] \). In each \( W_{i} \) exactly one factor depends on \( x_{k} \). This factor has
the form \( u[x_k, b], \, k \leq b < m \), and, as the maximal super-letter of the super-word \( W_t \), it is located at the end of \( W_t \). Hence representation (4.14) of \( c \) takes the form

\[
(4.28) \quad c = u[k, m] + \sum_{i=k}^{m-1} A_i u[k, i],
\]

where \( A_i, \, k \leq i < m \) is a linear combination of monotonous super-words of degree \( x_{i+1} + x_{i+2} + \ldots + x_m \). Each monotonous super-word of that degree has the form

\[
(4.29) \quad u[1 + l_p, m] \cdot u[1 + l_{p-1}, l_p] \cdot \ldots \cdot u[1 + i, l_i] \equiv u[L](1 + i, m),
\]

where \( L = \{l_i < l_2 < \ldots < l_p\} \subseteq [1 + i, m - 1] \). To derivate products of that type we shall prove the following general formula. Assume \( T = \{t_1 < t_2 < \ldots < t_s\} \) is another subset of \( [1 + i, m - 1] \), and let \( w = u^T(1 + i, m) \), see definition (4.5). In this case we have

\[
(4.30) \quad u[L](1 + i, m) \cdot D_w = \begin{cases} \alpha, & \text{if } T \subseteq L; \\ 0, & \text{otherwise,} \end{cases}
\]

where \( \alpha \in \mathfrak{k}, \, \alpha \neq 0 \). Indeed, if \( T = \emptyset \) then \( w = x_{1+i} x_{2+i} \ldots x_m \), while the required equality follows from (1.21) and (2.19). If \( T \neq \emptyset \) then definition (4.5) implies \( w = v \cdot w' \), where \( v = x_{1+t_s} x_{2+t_s} \ldots x_m \), \( w' = u^T(1 + i, t_s) \). If \( t_s \notin L \) then (4.21) implies \( s_{1+t_s}(u[L](1 + i, m)) = 0 \), hence \( u[L](1 + i, m) \cdot D_w = 0 \), which is required. If \( t_s \in L \) say \( t_s = l_j \), then again formulae (4.21) and (2.19) imply \( u[L](1 + i, m) \cdot D_w \sim u[L](1 + i, l_j) \). Therefore \( u[L](1 + i, m) \cdot D_w \sim u[L](1 + i, t_s) \cdot D_w \), which proves (4.30) by evident induction on \( m \).

Suppose that \( A_{s_1}, A_{s_2}, \ldots, A_{s_r} \) are all nonzero terms in (4.28). Denote \( S_0 = \emptyset, \, S_t = \{s_1, s_2, \ldots s_t\}, \, S_r = S \). The decomposition (4.28) shows that

\[
(4.31) \quad \Psi_{S_t}(k, m) = \sum_{i=s_{t+1}}^{m-1} A_i \Psi_{S_t}(k, i) \in U,
\]

where \( t = 0 \). We shall prove in two steps that inclusion (4.31) with a given \( t \), \( 0 \leq t \leq r \) implies the same type of inclusion with \( t = t + 1 \) and proportional \( A_i \). This will imply that (4.31) with \( t = r \), which says \( \Psi_{S_t}(k, m) \in U \), is valid as well.

1. The element \( A_{s_{t+1}} \) is a linear combination of super-words (4.29) with \( i \leftarrow s_{t+1} \). Denote by \( T \) the maximal with respect to inclusion subset of the interval \( [1 + s_{t+1}, m - 1] \), such that in the PBW-decomposition of \( A_{s_{t+1}} \) the super-word \( u^T(1 + s_{t+1}, m) \) appears with a nonzero coefficient. Let us apply \( D_w \), with \( w = u^T(1 + s_{t+1}, m) \) to (4.31). Since \( w \) is independent of \( x_k, x_{1+s_j}, 1 \leq j \leq t \), formula (4.22) with skew differential Leibniz rule (2.19) show that

\[
(4.32) \quad B \cdot D_w = \sum_{i=s_{t+1}}^{m-1} (A_i \cdot D_w) \Psi_{S_t}(k, i),
\]

where \( B \) is the left hand side of (4.31). The constitution of \( w \) contains the constitutions of \( A_i, \, i > s_{t+1} \). Hence all terms in the sum (4.32), except one that corresponds to \( i = s_{t+1} \), are zero. Moreover (4.30) implies that, due to the choice of \( T \), the element \( u^T(1 + s_{t+1}, m) \cdot D_w \) is a nonzero scalar, while \( u[L](1 + s_{t+1}, m) \cdot D_w = 0 \) for any other super-word \( u[L](1 + s_{t+1}, m) \) that appears in the decomposition of \( A_{s_{t+1}} \) with a nonzero coefficient. Hence \( A_{s_{t+1}} \cdot D_w \) is a nonzero scalar \( \mu \). Finally, we have

\[
(4.33) \quad \Psi_{S_t}(k, s_{t+1}) = \mu^{-1} B \cdot D_w \in U,
\]
2. Let us derivate \((4.31)\) by \(x_{1+s_t}\). By formulae \((4.24)\) and \((2.19)\) we have
\[
\mu u[(2 + s_t, m)] \cdot \Psi^{S_{t-1}}(k, s_t)
\]
(4.34)
\[
+ \sum_{i=s_{t+1}}^{m-1} \mu_i A_i u[(2 + s_t, i)] \cdot \Psi^{S_{t-1}}(k, s_t) \in \mathbf{U},
\]
where \(\mu = (1-q^{-1})^2\), \(\mu_i = (1-q^{-1})^2 p(A_i, x_{1+s_t})\) with the only possible exception \(\mu_{s_{t+1}} = (1-q^{-1}) p(A_i, x_{1+s_t})\) provided \(s_{t+1} = 1 + s_t\). Here we may apply \((4.24)\) since the number \(r\) related to \(S_t\) equals \(t\).

Denote by \(z\) the piecewise continuous word \(u^{S_{t-1}}(k, s_t)\). Let us apply \(\cdot D_z\) to \((4.34)\). Formula \((4.26)\) shows that \(\Psi^{S_{t-1}}(k, s_t) \cdot D_z\) is a nonzero scalar. Hence we get
\[
\mu u[(2 + s_t, m)] + \sum_{i=s_{t+1}}^{m-1} \mu_i A_i u[(2 + s_t, i)] \in \mathbf{U}.
\]
Let us apply \(\cdot D_w\) with \(w = u(2 + s_t, s_{t+1})\) to this sum. Formulae \((2.19)\) and \((4.23)\) imply
\[
(1-q^{-1}) u[1 + s_{t+1}, m] + \beta_1 A_{s_{t+1}} + (1-q^{-1}) \sum_{i>s_{t+1}} \beta_i A_i u[1 + s_{t+1}, i] \in \mathbf{U}.
\]
(4.35)
where \(\beta_1 = p(A_{s_{t+1}}, x_{1+s_t}, w) = p_v u, \beta_i = p(A_i, x_{1+s_t}, w) = p_v u\) with
\[
u = u(1 + s_t, s_{t+1}), \quad \nu = u(1 + s_{t+1}, m), \quad v_i = u(1 + i, m).
\]
Let us multiply the element \((4.35)\) from the right by \(\Psi^{S_i}(k, s_{t+1}) \in \mathbf{U}\), see \((4.33)\), and subtract the result from \((4.31)\) multiplied by \(\beta_1\). We get
\[
\beta_1 \Psi^{S_i}(k, m) + (q^{-1}) u[1 + s_{t+1}, m] \cdot \Psi^{S_i}(k, s_{t+1})
\]
(4.36)
\[
+ \sum_{i=s_{t+2}}^{m-1} A_i \{ \beta_1 \Psi^{S_i}(k, i) + (q^{-1} - 1) \beta_i u[1 + s_{t+1}, i] \cdot \Psi^{S_i}(k, s_{t+1}) \} \in \mathbf{U}.
\]
By the recurrence formula \((4.3)\) the first line of the above formula equals \(-\Psi^{S_{t+1}}(k, m)\), while the expression in the braces equals \(-\beta_1 \Psi^{S_{t+1}}(k, i)\). Thus we get the required relation
\[
\Psi^{S_{t+1}}(k, m) + \sum_{i=s_{t+2}}^{m-1} \beta_i A_i \Psi^{S_{t+1}}(k, i) \in \mathbf{U}.
\]
(4.37)
Proposition \((4.2)\) is proved.

**Corollary 4.8.** If \(q\) is not a root of 1, then \(U_q^+ (\mathfrak{sl}_{m+1})\) has just a finite number of right coideal subalgebras that include the coradical. If the multiplicative order of \(q\) equals \(t > 2\), then \(U_q^+ (\mathfrak{sl}_{n+1})\) has just a finite number of homogeneous right coideal subalgebras that include the coradical.

**Proof.** This follows from Lemma \((3.1)\) and Propositions \((2.5)\), \((4.2)\). Indeed, one has \(n(n-1)/2\) options for possible value of an \(\mathbf{U}\)-root (Definition \((2.4)\)). There exists \(2^n(n-1)/2\) variants for sets of \(\mathbf{U}\)-roots. For any given root \(\gamma = x_{k+1} + x_{k+2} + \ldots + x_m\) there exists not more than \(2^{m-k} < 2^n\) options for \(S\) to define a PBW-generator \(\Psi^S(k, m)\). Hence the total number of possible sets of PBW-generators is less than \(n(2^n) \cdot 2^n(n-1)/2\). \(\square\)
5. Root sequence

Our next goal is to show that the exact number of (homogeneous) right coideal subalgebras in \( U_q^+(\mathfrak{sl}(n+1)) \) (in \( u_q^+(\mathfrak{sl}(n+1)) \)) that contain \( k[G] \) equals \((n+1)!\). In what follows for short we shall denote by \([k : m]\) the element \( x_k + x_{k+1} + \ldots + x_m \in \Gamma^+\) considered as an \( U_q^+(\mathfrak{sl}(n+1))\)-root.

**Definition 5.1.** Let \( \gamma_k \) be a simple \( U \)-root of the form \([k : m]\) with the maximal \( m \).

Denote by \( \theta_k \) the number \( m - k + 1 \), the length (weight) of \( \gamma_k \). If there are no simple \( U \)-roots of the form \([k : m]\), we put \( \theta_k = 0 \). The sequence \( r(U) = (\theta_1, \theta_2, \ldots, \theta_n) \) satisfies \( 0 \leq \theta_k \leq n - k + 1 \) and it is uniquely defined by \( U \). We shall call \( r(U) \) a root sequence of \( U \), or just an \( r \)-sequence of \( U \). By \( \theta_k \) we denote \( k + \theta_k - 1 \), the maximal value of \( m \) for the simple \( U \)-roots of the form \([k : m]\) with fixed \( k \).

**Theorem 5.2.** For each sequence \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \), such that \( 0 \leq \theta_k \leq n - k + 1 \), \( 1 \leq k \leq n \) there exists one and only one (homogeneous) right coideal subalgebra \( U \supseteq G \) of \( U_q^+(\mathfrak{sl}(n+1)) \) (respectively, of \( u_q^+(\mathfrak{sl}(n+1)) \)) with \( r(U) = \theta \). In what follows we shall denote this subalgebra by \( U_\theta \).

The proof will result from the following lemmas.

**Lemma 5.3.** If \([k : m]\) is an \( U \)-root, then for each \( r, k \leq r < m \) either \([k : r]\) or \([r + 1 : m]\) is an \( U \)-root.

**Proof.** By Proposition 4.2 we have \( \Psi^S(k, m) \in U \) for a suitable \( S \). If \( r \in S \), then Theorem 4.7 and definition 4.17 imply \( \Psi^S(k, r) \in U \), hence \([k : r]\) is an \( U \)-root. If \( r \notin S \), then again Theorem 4.7 and 4.17 with \( a = r + 1 \) imply \( \Psi^S(r + 1, m) \in U \), hence \([r + 1 : m]\) is an \( U \)-root.

**Lemma 5.4.** If \([k : m]\) is a simple \( U \)-root, then there exists only one subset \( S \) of the interval \([k, m - 1]\), such that \( \Psi^S(k, m) \in U \). Moreover the set \( S \) is uniquely defined by the set of all \( U \)-roots.

**Proof.** Let \( \Psi^S(k, m) \in U \). By the definition of a simple root for each \( r, k \leq r < m \) either \([k : r]\) or \([r + 1 : m]\) is not an \( U \)-root. Hence Lemma 5.3 and Theorem 4.7 provide a criterion for \( r \) to belong to \( S \) if and only if \([k : r]\) is an \( U \)-root.

**Lemma 5.5.** A (homogeneous) right coideal subalgebra \( U \) is uniquely defined by the set of all its simple roots.

**Proof.** Since obviously two subalgebras with the same PBW-basis coincide, it suffices to find a PBW-basis of \( U \) that depends only on a set of simple \( U \)-roots.

We note first that the set of all \( U \)-roots is uniquely defined by the set of simple \( U \)-roots. Indeed, if \([k : m]\) is an \( U \)-root, then there exists a sequence \( k = k_0 < k_1 < \ldots < k_i = m + 1 \) such that \( [k_i : k_{i+1} - 1] \), \( 0 \leq i \leq l \) are simple \( U \)-roots. Conversely, if there exists a sequence \( k = k_0 < k_1 < \ldots < k_i = m + 1 \) such that \( [k_i : k_{i+1} - 1] \), \( 0 \leq i < l \) are simple \( U \)-roots then \( f_l = \Psi^S(k_l, k_{l+1} - 1) \in U \), \( 0 \leq i < l \) for suitable subsets \( S \) of the intervals \([k_i, k_{i+1} - 2]\). By decomposition (4.17) we have

\[
(5.1) \quad \Psi^S(k, m) = [[\ldots [f_{l-1}, f_{l-2}, f_{l-3}], \ldots], f_1], f_0] \in U,
\]

where \( S = \bigcup_{i=1}^{l-1} S_i \cup \{k_i - 1\} \) \( 0 < i < l \). Thus, by definition, \([k : m]\) is an \( U \)-root. Of course the decomposition of \([k : m]\) in a sum of simple \( U \)-roots is not unique in general, however we may fix that decomposition for each non-simple \( U \)-root from the very beginning.
Now if $[k : m]$ is a simple $U$-root, Lemma 5.3 shows that the element $\Psi^R(k, m) \in U$ is uniquely defined by the set of simple $U$-roots. We include this element in the PBW-basis of $U$. If $[k : m]$ is a non-simple $U$-root with the fixed decomposition in sum of simple $U$-roots, then we include in the PBW-basis the above defined element $[k : m]$. □

**Lemma 5.6.** If for (homogeneous) right coideal subalgebras $U$, $U'$ we have $r(U) = r(U')$, then $U = U'$.

**Proof.** By Lemma 5.6 it suffices to show that the $r$-sequence uniquely defines the set of all simple roots. We use the downward induction on $k$, the onset of a simple $U$-root. Suppose $k = n$. Then the only possible root with the onset $n$ is $\gamma = [n : n] = x_n$, in which case $\gamma$ is a simple $U$-root if and only if $\theta_n = 1$.

Let $k < n$. By definition there do not exist simple $U$-roots of the form $[k : m]$, $m > \theta_k$, while $[k, \theta_k]$ is a simple $U$-root.

If $m < \theta_k$, then $[m + 1 : \theta_k]$ is an $U$-root if and only if it is a sum of simple $U$-roots starting with a number greater than $k$. Hence by induction the $r$-sequence defines all roots of the form $[m + 1 : \theta_k], k < m < \theta_k$.

By Lemma 5.3 the weight $[k : m]$ is an $U$-root if and only if $[m + 1 : \theta_k]$ is not an $U$-root (recall that $[k : \theta_k]$ is simple). Hence the $r$-sequence also defines the set of all $U$-roots of the form $[k : m], m < \theta_k$. An $U$-root $[k : m], m < \theta_k$ is simple if and only if there does not exist $r, k \leq r < m$ such that both $[k : r]$ and $[r + 1 : m]$ are $U$-roots. □

Our next goal is a construction of a coideal subalgebra with a given root sequence 

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n),$$

such that $0 \leq \theta_k \leq n - k + 1, 1 \leq k \leq n$.

We shall need the following technical definition.

**Definition 5.7.** By downward induction on $k$ we define subsets $R_k, T_k, 1 \leq k \leq n$ of the interval $[k, n]$ associated to a given sequence $(5.2)$ as follows. If $\theta_n = 0$, we put $R_n = T_n = \emptyset$. If $\theta_n = 1$, we put $R_n = T_n = \{n\}$. Suppose that $R_i, T_i, k < i \leq n$ are already defined. If $\theta_k = 0$, then we set $R_k = T_k = \emptyset$. If $\theta_k \neq 0$, then by definition $R_k$ contains $\theta_k = k + \theta_k - 1$ and all $m$ satisfying the following three properties

1. $a) k \leq m < \theta_k$;
2. $b) \bar{\theta}_k \notin T_{m+1}$;
3. $c) \forall r(k \leq r < m) m \in T_{r+1} \iff \bar{\theta}_k \in T_{r+1}$.

Respectively,

$$T_k = R_k \cup \bigcup_{s \in R_k \setminus \{n\}} T_{s+1}.\tag{5.4}$$

**Example 5.8.** Assume $n = 3, \theta = (3, 1, 0)$.

Since $\theta_3 = 0$, by definition $R_3 = T_3 = \emptyset$.

Let $k = 2$. We have $\theta_2 = 1 \neq 0$, hence $\bar{\theta}_2 = 2 + \theta_2 - 1 = 2 \in R_2$. Certainly there are no points $m$ that satisfy $k = 2 \leq m < \bar{\theta}_2 = 2$, that is $R_2 = \{2\}$: Eq. (5.4) yields

$$T_2 = \{2\} \cup \bigcup_{s \in \{2\} \setminus \{3\}} T_{s+1} = \{2\}.$$
To find $R_1$ we take $k = 1$ and consider $\tilde{\theta}_1 = 1 + 3 - 1 = 3$. Obviously $\theta_1 = 3 \neq 0$; that is, $\tilde{\theta}_1 = 3 \in R_1$. There exist two points $m$ that satisfy $k = 1 \leq m < \tilde{\theta}_1 = 3$, they are $m = 1$, $m = 2$. Both of them satisfy condition b) since $\tilde{\theta}_1 = 3 \notin T_{2+1} = \emptyset$, $\tilde{\theta}_1 = 3 \notin T_{1+1} = \{2\}$.

Let us check condition c) for $m = 1$. The interval $1 = k \leq r < m = 1$ is empty. Therefore the equivalence c) is true (elements from the empty set satisfy all conditions, even $r \neq r$). Thus $1 \in R_1$.

It remains to check condition c) for $m = 2$. The interval $k = 1 \leq r < m = 2$ has only one point $r = 1$. For this point we have $T_{r+1} = T_2$ and $m = 2 \in T_2 = \{2\}$. At the same time $\tilde{\theta}_1 = 3 \notin T_{r+1} = T_2 = \{2\}$; that is, condition c) for $m = 2$ fails. Thus $R_1 = \{1, 3\}$.

Finally by (5.7) we find

$$T_1 = \{1, 3\} \cup \bigcup_{s \in R_1 \setminus \{3\}} T_{s+1} = \{1, 3\} \cup T_2 = \{1, 2, 3\}.$$ 

Thus for $\theta = (3, 1, 0)$ we have $R_3 = T_3 = \emptyset$, $R_2 = T_2 = \{2\}$, $R_1 = \{1, 3\}$, $T_1 = \{1, 2, 3\}$.

**Example 5.9.** Assume $n = 3$, $\theta = (2, 1, 1)$.

Here $\theta_3 \neq 0$, hence by definition $R_3 = T_3 = \{3\}$.

Let $k = 2$. Since $\theta_2 = 1 \neq 0$, we have $\tilde{\theta}_2 = 2 + 1 - 1 = 2 \in R_2$. There are no points that satisfy $k = 2 \leq m < \tilde{\theta}_2 = 2$; that is, $R_2 = \{2\}$. By Eq. (5.7) we have

$$T_2 = \{2\} \cup \bigcup_{s \in \{2\}\setminus\{3\}} T_{s+1} = \{2\} \cup \{3\} = \{2, 3\}.$$ 

To find $R_1$ we take $k = 1$ and consider $\tilde{\theta}_1 = 1 + 2 - 1 = 2$. Since $\theta_1 = 3 \neq 0$, we have $\tilde{\theta}_1 = 2 \in R_1$. There exist only one point $m$ that satisfies $k = 1 \leq m < \tilde{\theta}_1 = 2$, this is $m = 1$. For $m = 1$ we have $\tilde{\theta}_1 = 1 \in T_{1+1} = \{2, 3\}$, hence condition b) fails. Thus $R_1 = \{2\}$. Finally by (5.7) we find

$$T_1 = \{2\} \cup \bigcup_{s \in \{2\}\setminus\{3\}} T_{s+1} = \{2\} \cup T_3 = \{2, 3\}.$$ 

Therefore for $\theta = (2, 1, 1)$ we have $R_3 = T_3 = \{3\}$, $R_2 = \{2\}$, $T_2 = \{2, 3\}$, $R_1 = \{2\}$, $T_1 = \{2, 3\}$.

**Lemma 5.10.** For each sequence $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, such that $0 \leq \theta_k \leq n - k + 1$, $1 \leq k \leq n$ there exists a homogeneous right coideal subalgebra $U$ with $r(U) = \theta$.

**Proof.** Consider a set $W$ of all elements $\Psi^8(k, m)$, $1 \leq k \leq m \leq n$ with

$$(5.5) \quad m \in R_k, \ S = T_k.$$ 

Denote by $U$ a subalgebra generated by values of $W$ in $U^+_q(\mathfrak{sl}_{n+1})$ or in $u^+_q(\mathfrak{sl}_{n+1})$.

We shall check that $U \# U \# k[G]$ is a right coideal subalgebra with $r(U) = \theta$. To attain these ends we shall prove some properties of the sets $R_k, T_k$ by downward induction on $k$.

**Claim 1.** $m \in T_k$ if and only if there exists a sequence $k = k_0 < k_1 < \ldots < k_r = m + 1$, such that $k_{i+1} - k_i \in R_k$, $0 \leq i < r$.

Assume first $m \in T_k$. If $m \in R_k$, we put $k_1 = m + 1, r = 1$. If $m \notin R_k$, then by definition there exists $s \in R_k$, such that $m \in T_{s+1}$. We put $k_1 = s + 1$. By the
The inductive supposition applied to \( k \) remains to apply definition (5.4) with the same time \( k_2 \).

**Claim 2.** If \( s \in T_k \), \( m \in T_{s+1} \), then \( m \in T_k \).

By means of Claim 1 applied to \( s \) we find a sequence \( k_0 = k < k_1 < k_2 < \ldots < k_r = m + 1 \) such that \( k_{i+1} - 1 \in R_{k_i}, 1 \leq r \). Again by Claim 1 we have \( s \in T_{k_r} \).

The inductive supposition applied to \( k_1 \) shows that \( m \in T_{k_1} \). Since \( k_1 - 1 \in R_{k_1} \), it remains to apply definition (5.4) with \( s \leftarrow k_1 - 1 \).

**Claim 3.** If \( m \in T_k \), then for all \( s, k \leq s < m \) either \( s \in T_k \), or \( m \in T_{s+1} \).

By Claim 1 there exists a sequence \( k = k_0 < k_1 < \ldots < k_r = m + 1, k_{i+1} - 1 \in R_{k_i}, 0 \leq i < r \). The same claim implies \( m \in T_{k_1} \), provided that \( r \geq 1 \).

Since \( k \leq s < m \), there exists \( i, 1 \leq i \leq r \), such that \( k_i \leq s < k_{i+1} \). If \( i \geq 1 \), then the inductive supposition applied to \( k_i \) implies that either \( s \in T_k \) or \( m \in T_{s+1} \). If \( m \in T_{s+1} \), we have got the required condition. If \( s \in T_k \), then definition (5.4) with \( s \leftarrow k_1 - 1 \) yields \( s \in T_k \), which required.

Thus it remains to check the case \( i = 0 \); that is, \( k \leq s < k_1, k_1 - 1 \in R_k \). Claim 2 with \( s \leftarrow k_1 - 1, k \leftarrow s + 1 \) says that conditions \( k_1 - 1 \in T_{s+1} \) and \( m \in T_k \) imply \( m \in T_{s+1} \). Hence it is sufficient to show that either \( s \in T_k \) or \( k_1 - 1 \in T_{s+1} \). If \( s = k_1 - 1 \), then of course \( s = k_1 - 1 \in R_k \subseteq T_k \). This allows us to replace \( m \) with \( k_1 - 1 \) and suppose further that \( m \in R_k \), \( i = 0 \). In this case condition (5.3) with \( r \leftarrow s \) is \("m \in T_{s+1} \iff \theta_k \in T_{s+1}"\). Therefore we have to consider only one case \( m = \theta_k \).

Let us suppose that for some \( s, k \leq s \leq \theta_k \) we have \( s \notin T_k \) and \( \theta_k \notin T_{s+1} \). By induction on \( s \), in addition to the downward induction on \( k \), we shall show that these conditions imply \( s \in R_k \), which certainly contradicts to \( s \notin T_k \), see definition (5.4).

Definition (5.3) with \( m = k \) shows that \( k \in R_k \) if and only if \( \theta_k \notin T_{k+1} \). Since in our case \( \theta_k \notin T_{s+1} \), we have \( s \in R_k \), provided that \( s = k \).

Let \( s > k \). Conditions (5.3a) and (5.3b) are valid for \( m \leftarrow s \). Suppose that (5.3c) fails. In this case we may find a number \( t, k \leq t < s \), such that \( \neg(s \in T_{t+1} \iff \theta_k \in T_{t+1}) \).

If \( s \in T_{t+1} \) but \( \theta_k \notin T_{t+1} \), then by the inductive supposition (induction on \( s \)) either \( t \in R_k \) or \( \theta_k \in T_{t+1} \); that is, \( t \in R_k \). Definition (5.4) implies \( s \in T_k \) — a contradiction.

If \( \theta_k \in T_{t+1} \) but \( s \notin T_{t+1} \), then the inductive supposition of the downward induction on \( k \) with \( k \leftarrow t + 1 \) shows that either \( s \in T_{t+1} \) or \( \theta_k \in T_{s+1} \); that is, \( \theta_k \in T_{s+1} \) — again a contradiction.

Thus \( s \) satisfies all conditions (5.3a) — (5.3d), hence \( s \in R_k \).

**Claim 4.** If \( k \leq m < \theta_k \), then \( m \in T_k \) if and only if \( \theta_k \notin T_{m+1} \).

According to Claim 3 one of the conditions \( m \in T_k \) or \( \theta_k \in T_{m+1} \) always holds. If both of them are valid then due to Claim 1 we find a sequence \( k = k_0 < k_1 < \ldots < k_r = m + 1 \) such that \( k_{i+1} - 1 \in R_{k_i}, 0 \leq i < r \). Due to (5.3d), we have \( m \notin R_k \), hence \( r > 1 \). Again by the first claim we have \( m \in T_{k_1} \). Since \( k_1 - 1 \) belongs to \( R_k \), it satisfies condition (5.3b), \( \theta_k \notin T_{k_1} \). However Claim 2 shows that the conditions \( m \in T_{k_1}, \theta_k \notin T_{m+1} \) imply \( \theta_k \notin T_{k_1} \). A contradiction, that proves the claim.
Claim 5. The subalgebra $U'$ generated by $\Psi^S(k, m)$, $1 \leq k \leq m \leq n$, $m \in T_k$, $S = T_k$ is a differential subalgebra.

It suffices to show that all partial derivatives of $\Psi^S(k, m)$ belong to $U$. By Theorem 4.7 we have to check that $\Psi^{T_k}(a, b) \in U$ provided that $b \in T_k$, $a - 1 \notin T_k$, $k \leq a \leq b \leq m$. By definition $\Psi^{T_k}(a, b) \in U$ since due to the third claim $b \in T_a$. If

$$T_k \cap [a, b - 1] = T_a \cap [a, b - 1],$$

then we have nothing to prove. In general, however, just the inclusion $T_k \cap [a, b - 1] \subseteq T_a \cap [a, b - 1]$ holds: if $t \in T_k$, $a \leq t$, then Claim 3 with $s \leftarrow a - 1$ says $t \in T_a$ (since $a - 1 \notin T_k$).

We shall prove $\Psi^{T_k}(a, b) \in U$ by induction on $b - a$. If $b = a$, then (5.6) certainly holds.

Let us choose the minimal $s \in T_a$, $s \notin T_k$. Then $T_k \cap [a, s - 1] = T_a \cap [a, s - 1]$. Hence $\Psi^{T_k}(a, s) = \Psi^{T_k}(a, s) \in U$. By the inductive supposition applied to the interval $[s + 1, b]$ we get $\Psi^{T_k}(s + 1, b) \in U$. By decomposition (4.7) we have

$$\Psi^{T_k \cup (s)}(a, b) = [\Psi^{T_k}(s + 1, b), \Psi^{T_k}(a, s)] \in U.$$

At the same time (4.9) implies

$$\Psi^{T_k \cup (s)}(a, b) - (1 - q^{-1})\Psi^{T_k}(s + 1, b) \cdot \Psi^{T_k}(a, s) = -p_{va}\Psi^{T_k}(a, b).$$

Therefore $\Psi^{T_k}(a, b) \in U$, which is required.

Claim 6. $U = U \# k[G]$ is a right coideal subalgebra.

Since $U$ is homogeneous in each variable, we have $g^{-1}Ug \subseteq U$, $g \in G$. It remains to apply Lemma 2.10.

Claim 7. The set of all $U$-roots is $\{[k : m] \mid m \in T_k\}$. In particular $\{\Psi^{T_k}(k, m) \mid m \in T_k\}$ is a set of PBW-generators of $U$ over $k[G]$.

If $\gamma = [a : b]$ is an $U$-root, then, by definition, in $U$ there exists a homogeneous element (4.4) of degree $\gamma$. Since by definition $U$ is generated by $\{\Psi^{T_k}(k, m) \mid m \in T_k\}$, the degree $\gamma$ is a sum of degrees of the generators: $\gamma = [k_1 : k_2 - 1] + [k_2 : k_3 - 1] + \ldots + [k_{r-1} : k_r - 1], k_{i+1} - 1 \in T_k$, $1 \leq i < r$. The multiple use of Claim 2 yields $k_r - 1 \in T_k$, which is required since $a = k_1$, $b = k_r - 1$.

Claim 8. The set of all simple $U$-roots is $\{[k : m] \mid m \in R_k\}$. In particular $r(U) = \theta$, and $U$ is generated as an algebra by $\Psi^{T_k}(k, m)$, $m \in R_k$, $1 \leq k \leq n$, $k[G]$.

If $\gamma = [k : m]$ is a simple $U$-root, then due to the above claim $m \in T_k$. Hence, according to Claim 1, we may find a sequence $k = k_0 < k_1 < \ldots < k_r = m + 1$, such that $k_{i+1} - 1 \in R_{k_i}$, $0 \leq i < r$. In this case $\gamma = [k : k_1 - 1] + [k_1 : k_2 - 1] + \ldots + [k_r : m]$ is a sum of $U$-roots. Since $\gamma$ is simple this is possible only if $r = 1$, and $m = k_1 - 1 \in R_k$.

Conversely. Let $m \in R_k$. Then by Claim 7 and definition (5.3) $[k : m]$ is an $U$-root. If it is not simple, then it is a sum of two roots $[k : m] = [k : s] + [s + 1 : m]$, $s \in T_k$, $m \in T_{s+1}$. Then Claim 4 implies $\theta_k \notin T_{s+1}$, hence condition (5.3) fails for $r \leftarrow s$.

In Lemma 5.5 we have seen that $U$ is uniquely defined by the simple $U$-roots. In particular formula (5.6) provides a representation of PBW-generators in terms of $\Psi^{T_k}(k, m)$, $m \in R_k$. Lemma 5.10 and Theorem 5.2 are completely proved. □
6. Homogeneous right coideal subalgebras

In this section we consider right coideal subalgebras in $U_q^+(\mathfrak{sl}_{n+1})$, (respectively in $u_q^+(\mathfrak{sl}_{n+1})$ if $q^t = 1, t > 2$) that do not contain the coradical. First of all we note that for every submonoid $\Omega \subseteq G$ the set of all linear combinations $k[\Omega]$ is a right coideal subalgebra. Conversely if $U_0 \subseteq k[G]$ is a right coideal subalgebra then $U_a = k[\Omega]$ for $\Omega = U_0 \cap G$ since $a = \sum_i \alpha_i g_i \in U_0$ implies $\Delta(a) = \sum_i \alpha_i g_i \otimes g_i \in U_0 \otimes k[G]$; that is, $\alpha_i g_i \in U_0$.

**Definition 6.1.** For a sequence $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, such that $0 \leq \theta_k \leq n - k + 1$, $1 \leq k \leq n$ we denote by $U_{\theta}^1$ a subalgebra with 1 generated by $g^{-1}\Psi^B(k, m)$, where $g = g_k g_{k+1} \ldots g_m$, and $\Psi^B(k, m)$ runs through the set of PBW-generators of $U_\theta$, see Theorem 5.2 and Claim 7, Section 5.

**Lemma 6.2.** The subalgebra $U_{\theta}^1$ is a homogeneous right coideal, and $U_{\theta}^1 \cap G = \{1\}$.

**Proof.** The subalgebra $U_{\theta}^1$ is homogeneous since it is generated by homogeneous elements. Its zero homogeneous component equals $k$ since among the generators just one, the unity, has zero degree.

Denote by $A_\theta$ a $k$-subalgebra generated by the PBW-generators $\Psi^B(k, m)$ of $U_\theta$. The algebra $U_{\theta}^1$ is spanned by all elements of the form $g^{-1} a$, $a \in A_\theta$. Since $U_{\theta}^1$ is a right coideal, for any homogeneous $a \in A_\theta$ we have $\Delta(a) = g(a) a' \otimes a''$ where $a' \in A_\theta$, $g_a = g(a) g(a')$. Therefore $\Delta(g^{-1} a) = g(a')^{-1} a' \otimes g^{-1} a''$ with $g(a')^{-1} a' \in U_{\theta}^1$.

**Theorem 6.3.** If $U$ is a homogeneous right coideal subalgebra of $U_q^+(\mathfrak{sl}_{n+1})$ (resp. of $u_q^+(\mathfrak{sl}_{n+1})$) such that $\Omega \subseteq U \cap G$ is a group, then $U = U_{\theta}^1 k[\Omega]$ for a suitable $\theta$.

**Proof.** Let $u = \sum_i h_i a_i \in U$ be a homogeneous element of degree $\gamma \in \Gamma^+$ with different $h_i \in G$, and $a_i \in A$, where by $A$ we denote the $k$-subalgebra generated by $x_i$, $1 \leq i \leq n$. Denote by $\pi_\gamma$ the natural projection on the homogeneous component of degree $\gamma$. Respectively $\pi_\gamma g, g \in G$ is a projection on the subspace $k g$. We have $\Delta(u) \cdot (\pi_\gamma \otimes \pi_{\gamma'}) = h_i a_i \cdot (\pi_\gamma \otimes \pi_{\gamma'})$. Thus $h_i a_i \in U$.

By Theorem 5.2 we have $[k[G]]U = U_\theta$ for a suitable $\theta$. If $u = h a \in U$, $h \in G$, $a \in A$, then $\Delta(h) \cdot (\pi_{ha} \otimes \pi_h) = h g a \otimes h a$. Therefore $h g a \in U \cap G = \Omega$. That is, $u = \omega g^{-1} a$, $\omega \in \Omega$. Since $\Omega$ is a subgroup we get $g^{-1} a \in U$. It remains to note that all elements $g^{-1} a$, such that $h a \in U$ span the algebra $U_{\theta}^1$.

If $U \cap G$ is not a group then $U$ may have a more complicated structure.

**Example 6.4.** Let $\Omega$ be a submonoid of $G$. Denote by $\Omega$ an arbitrary family of sets \{\Omega, $\gamma \in \Gamma^+$\} that satisfies the following conditions

\[ \Omega_0 = \Omega, \quad \Omega_{\gamma} \cdot \Omega_{\gamma'} \subseteq \Omega_{\gamma + \gamma'} \subseteq \Omega_{\gamma} \cap \Omega_{\gamma'}. \]

In this case the linear space $U_{\theta}^1 \Omega$ spanned by the elements $\omega a, \omega \in \Omega$, $a \in U_{\theta}^1$, deg $(a) = \gamma$ is a right coideal subalgebra such that $U_{\theta}^1 \cap G = \Omega$. The $\gamma$-homogeneous component of this algebra equals $\Omega_{\gamma}(U_{\theta}^1)_{\gamma}$. Hence different $\Omega$ define different homogeneous right coideal subalgebras.

Finally we point out a simplest one-parameter family of inhomogeneous right coideal subalgebras that have trivial intersection with the coradical.
Example 6.5. Let $a = g_i^{-1}(x_1 + \alpha)$, $\alpha \in k$. We have
\[
\Delta(a) = g_i^{-1}x_1 \otimes g_i^{-1} + 1 \otimes g_i^{-1}x_1 + \alpha g_i^{-1} \otimes g_i^{-1} = a \otimes g_i^{-1} + 1 \otimes g_i^{-1}x_1.
\]
Therefore the two-dimensional space spanned by $a$ and 1 is a right coideal. Thus the algebra $k[a]$ with 1 generated by $a$ is a right coideal subalgebra, in which case $k[a] \cap G = \{1\}$.

7. Kébé construction and $ad_r$-invariant subalgebras

In this section we characterize $ad_r$-invariant right coideal subalgebras that have trivial intersection with the coradical in terms of Kébé’s construction \[18\] \[19\]. Recall that the right adjoint action of a Hopf algebra $H$ on itself is defined by the formula
\[
(ad_r a)b = \sum \sigma(a^{(1)})ba^{(2)},
\]
where $\sigma$ is the antipode. The map $a \rightarrow ad_r a$ is a homomorphism of algebras $ad_r : H \rightarrow \text{End}H$. In particular a subspace is invariant under the action of all operators $ad_r H$ if and only if it is invariant under the actions of $ad_r h_i$ for some set of generators $\{h_i\}$. For $H = U^+_q(\mathfrak{sl}_{n+1})$ or for $H = u^+_q(\mathfrak{sl}_{n+1})$ we have
\[
(ad_r g)b = g^{-1}bg, \quad g \in G; \quad (ad_r x_i)b = g_i^{-1}(bx_i - x_ib).
\]
The latter equality would be more familiar if we take $b = g_i^{-1}a$ with $a \in A \equiv k(x_1, \ldots, x_n)$:
\[
(ad_r x_i)(g_i^{-1}a) = g_i^{-1}(g_i^{-1}ax_i - x_ig_i^{-1}a) = -g_i^{-1}g_i^{-1}[x_i, a].
\]
In particular the subalgebra $H^1$ generated by $g_a^{-1}a, a \in A$ (in our terms this is $U^+_1$ for $\theta = (1, 1, \ldots, 1)$) is $ad_r$-invariant.

The following construction of $ad_r$-invariant right coideal subalgebras appeared in \[18\] \[19\], see also \[33\] Section 6. Let $\pi$ be a subset of $[1, k]$. Denote by $K(\pi)$ a subalgebra generated by elements of the form
\[
ad_r(x_{i_1}x_{i_2} \ldots x_{i_r})g_j^{-1}x_i, \quad j \in \pi, \quad i_r \notin \pi, \quad 1 \leq r \leq k.
\]
The algebra $K(\pi)$ is $ad_r$-invariant right coideal (see, \[33\] Lemma 1.2] up to a left-right symmetry). This is homogeneous, and $K(\pi) \cap G = \{1\}$ since due to (7.1) the inclusion $K(\pi) \subseteq H^1$ is valid. Thus by Theorem 6.3 we have $K(\pi) = U^+_\theta$ for a suitable $\theta$.

Theorem 7.1. The following conditions on $U = U^+_\theta$ are equivalent
i. $U$ is $ad_r$-invariant.
ii. The sets $T_k$, see Definition 5.7 have the form $T_k = [j(k), n]$, where
\[
j(k) \equiv \min \{j \mid k \leq j, \quad j \in T_j\}.
\]
iii. $U = K(\pi)$ for a suitable $\pi \subseteq [1, n]$.

Proof. iii $\Rightarrow$ i. We have mentioned above.

i $\Rightarrow$ ii. By Claim 7 we have $m \in T_k$ if and only if $\Psi^{T_k}(k, m) \in U$. In particular $j \in T_j$ if and only if $x_j \in U$, or, equivalently, $g_j^{-1}x_j \in U$. If $j \in T_j$ and $k \leq j$, then by (7.1) we have
\[
ad_r(x_{j-1}x_{j-2} \ldots x_k)g_j^{-1}x_j = g_{u[k,j]}^{-1}u[k,j] \in U.
\]
Hence, by definition $[k : j]$ is an $U_\theta$-root; that is, according to Claim 7, we get $j \in T_k$. Moreover, if $i > j$ then by (4.14) we have
\[
\text{ad}_r(x_{j+1}x_{j+2} \ldots x_i) g^{-1}_{u[k,j]} u[k,j] = g^{-1}[x_i, [x_{i-1}, \ldots [x_{j+1}, u[k,j]] \ldots]]
\]
\[
\sim g^{-1} \Psi^{(j+1,j+2, \ldots, i)}(k,i),
\]
where $g = g_k g_{k+1} \ldots g_{i}$. In particular $[k : i]$ is an $U_\theta$-root; that is, again according to Claim 7, we get $i \in T_k$. This proves $[j(k), n] \subseteq T_k$.

If $m$ is the smallest element from $T_k$ then $u[k, m] = \Psi^T_k(k, m) \in U_\theta$, hence by multiple use of (4.14) we get $x_m \in U_\theta$; that is, $m \in T_m$, and $m = j(k)$.

If $i \Rightarrow iii$. Let $\pi = \{j | j \in T_j\}$. For all $k, m, j$ such that $m \in T_k$, $j = j(k)$ we have
\[
\text{ad}_r(x_{j-1}x_{j-2} \ldots x_k x_{j+1}x_{j+2} \ldots x_m) \ g_{j-1}^{-1}x_j
\]
\[
= g^{-1}[x_m, [x_{m-1}, \ldots [x_{j+1}, u[k,j]] \ldots]] = g^{-1} \Psi^T_k(k, m),
\]
where $g = g_m g_{m-1} \ldots g_k$. Since $K(\pi)$ is $\text{ad}_r$-invariant, we get $g^{-1} \Psi^T_k(k, m) \in K(\pi)$. Now Definition 6.1 implies $U \subseteq K(\pi)$.

Since $g_{j-1}^{-1}x_j \in U$, to check $K(\pi) \subseteq U$ it remains to show that $\text{ad}_r(x_i) U \subseteq U$ for $i \notin \pi$. By (2.10) and (4.14) it suffices to prove that $[x_i, \Psi^T_k(k, m)] \in U_\theta$ for $i \notin \pi$, $m \in T_k$.

Let $i = k-1$. Since $k-1 = i \notin \pi$, we have $k-1 \notin T_{k-1}$, and hence $j(k-1) = j(k)$. Eq. (4.13) implies $[x_{k-1}, \Psi^T_k(k, m)] \sim \Psi^T_{k-1}(k-1, m) \in U_\theta$, where $m \in T_{k-1}$ follows from $T_{k-1} = [j(k), n] = T_k$.

If $i < k-1$, then $[x_i, \Psi^T_k(k, m)] = 0$ since $x_i$ and $\Psi^T_k(k, m)$ are separated.

Let $i = m + 1$. Since $m \in T_k = [j(k), n]$, we have $m \geq j(k)$. Therefore $m+1 > j(k)$, and $m+1 \in T_k$. Now formula (4.14) yields
\[
[x_{m+1}, \Psi^T_k(k, m)] \sim \Psi^T_k(k, m+1) \in U_\theta.
\]

If $i > m+1$, then $[x_i, \Psi^T_k(k, m)] = 0$ since $x_i$ and $\Psi^T_k(k, m)$ are separated.

We shall show by induction on $m-k$ that in all remaining cases $[x_i, \Psi^T_k(k, m)] = 0$. More precisely, we prove $[x_i, \Psi^S(k, m)] = 0$ provided that $S$ has the form $[j,n]$, and $k \leq i \leq m$, $i \neq \min \{j, m\}$.

If $m-k = 1$, then for $j \geq m$ we have just one option $i = k$. The required relation takes the form $[x_k, [x_k, x_{k+1}]] = 0$ which is one of the defining relations (4.2). For $j = k$ we also have just one option $i = m = k + 1$. The required relation is $[x_m, [x_m, x_{m-1}]] = 0$. This relation is valid in $U_+^+(\mathfrak{sl}_{n+1})$ since (4.1) imply $[x_m, [x_m, x_{m-1}]] \sim [[x_{m-1}, x_m], x_m]$, see, for example, (2.9) Corollary 4.10.

If $m-k > 2$ then either $i < m-1$ or $i > k+1$. In the former case we have $[x_i, x_m] = 0$, and by the inductive supposition $[x_i, \Psi^S(k, m-1)] = 0$. Hence representation (4.14) implies the required equality. In the latter case we have $[x_i, x_k] = 0$, and by the inductive supposition $[x_i, \Psi^S(k+1, m)] = 0$. In this case representation (4.15) implies the required equality.

Finally, suppose that $m-k = 2$. To simplify the notations we put $k = 1$, $m = 3$.

If $j \geq 3$ then $\Psi^S(1, 3) = [[x_1, x_2], x_3]$, and we have two options $i = 1, i = 2$. If $i = 1$, we have to show $[x_1, [[x_1, x_2], x_3]] = 0$. This relation is valid since $x_1$ (skew)commutes both with $[x_1, x_2]$ and $x_3$ (but not vice versa: $[x_1, x_2]$ does not (skew)commute with $x_3$ since $[[x_1, x_2], x] \neq 0$). Let $i = 2$. We may apply (2.12) since $p_{21}p_{22}p_{23}$ and $p_{12}p_{22}p_{32} = 1$. Thus by (2.12) and (2.7) we have
\[
[x_2, [[x_1, x_2], x_3]] \sim [[[x_1, x_2], x_3], x_2] = [[x_1, [x_2, x_3]], x_2].
\]
The word $x_1x_2x_3x_2$ is standard, and the standard alignment of brackets is precisely $[[x_1, [x_2, x_3]], x_2]$. Hence by the third statement of Theorem 4.1 this is zero in $U^+_q(sl_{n+1})$.

If $j = 2$, then $\Psi^S(1, 3) = [x_3, [x_1, x_2]]$, and we have two options $i = 1, i = 3$. If $i = 1$ then $[x_1, [x_3, [x_1, x_2]]] = 0$ since $x_1$ (skew)commutes both with $[x_1, x_2]$ and $x_3$. Let $i = 3$. By (2.8) we have $[x_3, [x_1, x_2]] = [x_1, [x_3, x_2]]$. Since $x_3$ (skew)commutes both with $x_1$ and $[x_3, x_2]$, we get $[x_3, [x_1, [x_3, x_2]]] = 0$.

If $j = 1$ then $\Psi^S(1, 3) = [[x_3, x_2], x_1]$, and we have two options $i = 2, i = 3$. If $i = 3$ then $[x_3, [x_3, x_2], x_1] = 0$ since $x_3$ (skew)commutes both with $[x_3, x_2]$ and $x_1$. Let $i = 2$. We may use (2.12) since $p_{23}p_{22}p_{21} \cdot p_{32}p_{22}p_{21} = 1$. Thus by (2.12) and (2.7) we have

$$[x_3, [x_3, x_2], x_1] = [[x_3, x_2], x_1] = [x_3, [x_2, x_1]], x_2].$$

This element in new variables $y_1 = x_3, y_2 = x_2, y_3 = x_3$ takes up the form $[[y_1, y_2, y_3], y_2]$. By the third statement of Theorem 4.1 this is zero in $U^+_q(sl_{n+1})$.

8. Examples

In this section by means of Theorem 5.2 we provide some examples of right coideal subalgebras in $U^+_q(sl_n)$ or $U^+_q(sl_n)$ with their main characteristics: PBW-generators, the root sequence $r(U)$, the sets $T_i, R_i$, right coideal subalgebra generators, and maximal Hopf subalgebras. We start with a characterization of $2^n$ “trivial” examples — Hopf subalgebras.

**Proposition 8.1.** A right coideal subalgebra $U = U_\theta$ is a Hopf subalgebra if and only if for every $k$, $1 \leq k \leq n$ either $\theta_k = 0$ or $\theta_k = 1$. An algebra $U_\theta$, with $\theta_i \leq 1$ is generated over $k[G]$ by all $x_k$ with $\theta_k = 1$.

**Proof.** If $\theta_k \leq 1$, $1 \leq k \leq n$, then Definition 5.3 shows that $R_k = \{k\}$ provided that $\theta_k = 1$ and $R_k = 0$ otherwise. Hence by Claim 8 the algebra $U$ is generated over $k[G]$ by all $x_k$ with $\theta_k = 1$. In particular $U$ is a Hopf subalgebra of $U^+_q(sl_n)$.

Conversely, let $U$ be a Hopf subalgebra. According to Claim 8 the algebra $U$ is generated over $k[G]$ by the elements $a$ of the form $\Psi^T_k(k, m)$ with $[k : m]$ being the simple $U$-roots. We have $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$ with $a^{(1)}, a^{(2)} \in U$. Since $[k : m] = D(a) = D(a^{(1)}) + D(a^{(2)})$ and $[k : m]$ is simple, we have either $D(a^{(1)}) = 0$, or $D(a^{(2)}) = 0$. Thus $a$ is a skew primitive element; that is, $a = x_k$ is the only option for $a$ (see the second statement of Theorem 4.1 for $q^t \neq 1$, and comments after that theorem for $q^t = 1$). In particular all simple roots are of length 1, while Definition 5.1 implies $\theta_k \leq 1$.

Now we consider three special cases.

**Example 8.2.** Consider the root sequence with the maximal possible components, $r(U) = (n, n-1, n-2, \ldots, 2, 1)$. In this case by definition $T_n = R_n = \{n\}$. For $k < n$ we have $\theta_k = k + \theta_{k-1} - 1 = n \in R_k$. Moreover by downward induction on $k$ one may prove that $T_k = R_k = \{n\}$. Indeed, the inductive supposition implies that condition (5.3) fails for all $m, k \leq m < n$. This yields $R_k = \{n\}$ Inductive definition (5.1) implies $T_k = \{n\}$ as well. Claim 7 provides a PBW-basis:

$$\{\Psi^{(n)}(k, n) \mid 1 \leq k \leq n\} = \{[x_kx_{k-1} \ldots x_{n-1}x_n] \mid 1 \leq k \leq n\}.$$
Due to formula (4.21) the subalgebra $U$ as a right coideal subalgebra is generated over $k[G]$ by a single long skew commutator $[x_1x_2 \ldots x_n]$ that has the whitest diagram (4.10). The maximal Hopf subalgebra of $U$ is $k[G]\langle x_n \rangle$.

**Example 8.3.** Let $r(U) = (n, 0, 0, \ldots, 0)$. In this case according to definitions (5.3) and (5.4) we have $T_i = R_i = 0$ if $1 < i \leq n$, and $T_1 = R_1 = \{1, 2, \ldots, n-1, n\}$. By means of Claim 7 we obtain a PBW-basis:

$$\{Ψ^{T_1}(1, m) \mid 1 \leq m \leq n\} = \{[x_m x_{m-1} \ldots x_2 x_1] \mid 1 \leq m \leq n\}.$$

As a right coideal subalgebra $U$ is generated by a single long skew commutator $[x_n x_{n-1} \ldots x_1]$ that has the most black diagram (4.10). The maximal Hopf subalgebra of $U$ is $k[G]\langle x_1 \rangle$.

**Example 8.4.** More generally consider a right coideal subalgebra $U^S(k, m)$ generated by $Ψ^S(k, m)$. We claim that the root sequence for this algebra is defined as follows

$$(8.1) \quad θ_i = \begin{cases} m - k + 1, & \text{if } i = k; \\
 m - i + 1, & \text{if } i - 1 \notin S, \ k < i \leq m; \\
 0, & \text{otherwise};
\end{cases}$$

that is, $θ_i$ takes the maximal value if $i - 1$ is a white point on the diagram (4.16), and $θ_i = 0$ if either $i - 1$ is a black point or $i - 1$ is not displayed on the diagram at all.

Let $U_θ$ be a right coideal subalgebra defined by the sequence (8.1) in Lemma 5.10. By downward induction of Lemma 5.10 it is easy to see that

$$(8.2) \quad R_i = T_i = \begin{cases} (S \cap [i, m]) \cup \{m\}, & \text{if } i = k \text{ or } i - 1 \notin S; \\
 \emptyset, & \text{otherwise};
\end{cases}$$

that is, $R_i = T_i$ equals $S^*$ related to the interval $[i, m]$ if $i - 1$ is a white point on the diagram (4.10), and $R_i = T_i$ is empty otherwise.

By Claim 7 the set $W^S(k, m)$ defined in (4.17) is a set of PBW-generators for $U_θ$ over $k[G]$. In particular $U_θ$ contains $Ψ^S(k, m)$, and according to Theorem 4.7 it is generated over $k[G]$ by $Ψ^S(k, m)$ as a right coideal subalgebra.

To describe the maximal Hopf subalgebra of $U^S(k, m)$ we need the following definition.

**Definition 8.5.** A black point $s, s \in S^*$, is said to be a $(k, m)$-entrance into $S$ if $s - 1$ is a white point; that is, either $s = k$ or $s - 1 \notin S$.

**Lemma 8.6.** The maximal Hopf subalgebra of $U^S(k, m)$ is generated by all $x_i$, where $i$ is a $(k, m)$-entrance into $S$.

**Proof.** The element $x_i$ belongs to $U_θ$ if and only if $i \in T_i$. Hence formula (8.2) shows that $x_i \in U_θ$ if and only if $i$ is a $(k, m)$-entrance into $S$. \hfill ∎

If $n = 2$ then by Theorem 5.2 we have totally $3! = 6$ right coideal subalgebras. Among them $2^2 = 4$ are “trivial” cases (Hopf subalgebras) and two more right coideal subalgebras are given in Example 8.2 and Example 8.3.

If $n = 3$ then we have $4! - 2^3 = 16$ proper (not “trivial”) right coideal subalgebras. In the tableaux below we provide main characteristics of these 16 right coideal subalgebras. We mark off by * the $\text{ad}_T$-invariant subalgebras $U_T^1 = K(π)$.
| $r(U)$ | $R$ | $T$ | PBW-generators | r. c. s. generators |
|-------|-----|-----|----------------|-------------------|
| *     | $R_3 = \{3\}$ | $T_3 = \{3\}$ | $x_3, [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
| $(3, 2, 1)$ | $R_2 = \{3\}$ | $T_2 = \{3\}$ | $[x_2x_3], \ldots$ | $[x_1x_2x_3]$ |
|       | $R_1 = \{3\}$ | $T_1 = \{3\}$ | $[x_1x_2x_3]$ | $\circ \circ \bullet \bullet$ |
| *     | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_2, [x_3x_2], \ldots$ | $[x_3x_1x_2]$ |
| $(3, 2, 0)$ | $R_2 = \{2, 3\}$ | $T_2 = \{2, 3\}$ | $[x_1x_2], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \{2, 3\}$ | $T_1 = \{2, 3\}$ | $[x_1x_2x_3]$ | $\circ \circ \bullet \bullet$ |
| $(3, 1, 1)$ | $R_2 = \{3\}$ | $T_2 = \{3\}$ | $x_3, [x_3x_2], \ldots$ | $[x_1x_2x_3], x_2$ |
|       | $R_3 = \{3\}$ | $T_3 = \{3\}$ | $[x_1x_2x_3]$ | $\circ \circ \bullet \bullet$ |
| $(3, 1, 0)$ | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_2, [x_2x_3], \ldots$ | $[x_3x_2x_1], x_2$ |
|       | $R_2 = \{2\}$ | $T_2 = \{2\}$ | $[x_1x_2], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \{1, 3\}$ | $T_1 = \{1, 2, 3\}$ | $x_1, [x_2x_3], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
| $(3, 0, 1)$ | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_3, [x_2x_3], \ldots$ | $[x_3x_2x_1]$ |
|       | $R_2 = \emptyset$ | $T_2 = \emptyset$ | $[x_1x_2], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \emptyset$ | $T_1 = \emptyset$ | $[x_3x_2x_1]$ | $\circ \circ \bullet \bullet$ |
| *     | $R_3 = \{3\}$ | $T_3 = \{3\}$ | $x_3, [x_2x_3], \ldots$ | $[x_3x_2x_1], [x_2x_3]$ |
| $(2, 2, 1)$ | $R_2 = \{3\}$ | $T_2 = \{3\}$ | $[x_2x_3], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \{1, 2, 3\}$ | $T_1 = \{1, 2, 3\}$ | $x_1, [x_2x_3], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
| $(2, 2, 0)$ | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_3, [x_2x_3], \ldots$ | $[x_3x_2x_1], x_2$ |
|       | $R_2 = \{2\}$ | $T_2 = \{2\}$ | $[x_1x_2], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \{2\}$ | $T_1 = \{2\}$ | $[x_1x_2], [x_3x_2], \ldots$ | $\circ \circ \bullet \bullet$ |
| $(2, 1, 1)$ | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_3, [x_3x_2], \ldots$ | $[x_3x_2x_1], x_2$ |
|       | $R_2 = \emptyset$ | $T_2 = \emptyset$ | $[x_1x_2], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \emptyset$ | $T_1 = \emptyset$ | $[x_3x_2x_1], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
| $(2, 0, 0)$ | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_3, [x_3x_2], \ldots$ | $[x_3x_2x_1], x_3$ |
|       | $R_2 = \emptyset$ | $T_2 = \emptyset$ | $[x_1x_2], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \emptyset$ | $T_1 = \emptyset$ | $[x_3x_2x_1], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
| $(2, 0, 1)$ | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_3, [x_3x_2], \ldots$ | $[x_3x_2x_1], x_3$ |
|       | $R_2 = \emptyset$ | $T_2 = \emptyset$ | $[x_1x_2], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \emptyset$ | $T_1 = \emptyset$ | $[x_3x_2x_1], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
| $(2, 0, 0)$ | $R_3 = \emptyset$ | $T_3 = \emptyset$ | $x_3, [x_3x_2], \ldots$ | $[x_3x_2x_1], x_3$ |
|       | $R_2 = \emptyset$ | $T_2 = \emptyset$ | $[x_1x_2], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |
|       | $R_1 = \emptyset$ | $T_1 = \emptyset$ | $[x_3x_2x_1], [x_2x_3], \ldots$ | $\circ \circ \bullet \bullet$ |

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| \( r(U) \) | R | T | PBW-generators | r. c. s. generators |
|---|---|---|---|---|
| * | \( R_3 = \emptyset \) | \( T_3 = \emptyset \) | \( x_2, [x_3 x_2] \) | \( [x_3 x_2], x_1 \) |
| (1, 2, 0) | \( R_3 = \{2, 3\} \) | \( T_2 = \{2, 3\} \) | \( x_1, [x_2 x_1], [x_3 x_2 x_1] \) | \( \bullet \circ \bullet, \circ \bullet \) |
| | \( R_3 = \{1\} \) | \( T_1 = \{1, 2, 3\} \) | \( x_2, [x_3 x_2] \) | \( [x_3 x_2] \) |
| (0, 2, 1) | \( R_3 = \{3\} \) | \( T_2 = \{3\} \) | \( x_3, [x_2 x_3] \) | \( [x_2 x_3] \) |
| | \( R_3 = \emptyset \) | \( T_1 = \emptyset \) | \( x_2, [x_3 x_2] \) | \( [x_3 x_2] \) |
| (0, 2, 0) | \( R_3 = \emptyset \) | \( T_2 = \{2, 3\} \) | \( x_2, [x_3 x_2] \) | \( [x_3 x_2] \) |
| | \( R_3 = \emptyset \) | \( T_1 = \emptyset \) | \( x_2, [x_3 x_2] \) | \( \bullet \circ \bullet \) |

9. Triangular decomposition in \( U_q(\mathfrak{sl}_{n+1}) \)

At this point it is naturally to conjecture that any \((\Gamma\text{-homogeneous})\) right coideal subalgebra of \( U_q(\mathfrak{sl}_{n+1}) \) (of \( u_q(\mathfrak{sl}_{n+1}) \)) that contains \( k[H] \) has the triangular decomposition and for any two right coideal subalgebras \( U^- \subseteq U^-_q(\mathfrak{sl}_{n+1}), U^+ \subseteq U^+_q(\mathfrak{sl}_{n+1}) \) (respectively, \( U^- \subseteq U^-_q(\mathfrak{sl}_{n+1}), U^+ \subseteq U^+_q(\mathfrak{sl}_{n+1}) \) ), the tensor product

\[
U = U^- \otimes_{k[F]} k[H] \otimes_{k[G]} U^+
\]

is a right coideal subalgebra. In this hypothesis just one statement fails, the tensor product indeed is a right coideal but not always a subalgebra.

**Lemma 9.1.** If \( q \) is not a root of 1 then every right coideal subalgebra \( U \) of \( U_q(\mathfrak{sl}_{n+1}) \), \( U \supseteq k[H] \) has a decomposition \((9.2)\), where \( U^+ \supseteq k[G] \) and \( U^- \supseteq k[F] \) are right coideal subalgebras of \( U^+_q(\mathfrak{sl}_{n+1}) \) and \( U^-_q(\mathfrak{sl}_{n+1}) \) respectively. If \( q \) has finite multiplicative order \( t > 2 \), then this is the case for \( \Gamma\text{-homogeneous} \) right coideal subalgebras of \( u_q(\mathfrak{sl}_{n+1}) \).

**Proof.** Due to the triangular decompositions \((9.1)\), \((9.2)\) the values of super-letters \([x_k x_{k+1} \ldots x_m], [x_k^- x_{k+1}^- \ldots x_m^-] \) form a set of PBW-generators over \( k[H] \) for \( U_q(\mathfrak{sl}_{n+1}) \).

Let us fix the following order on the skew-primitive generators

\[
x_1 > x_2 > \ldots > x_n > x_1^- > x_2^- > \ldots > x_n^-.
\]

By Lemma \((9.1)\) and Proposition \((9.2)\) (see the arguments above Eq. \((9.3)\)) the subalgebra \( U \) has PBW-generators of the form

\[
c = [u] + \sum \alpha_i W_i + \sum_j \beta_j V_j \in U,
\]

where \( W_i \) are the basis super-words starting with less than \( [u] \) super-letters, \( D(W_i) = D(u) \), and \( V_j \) are \( G \)-super-words of \( D \)-degree less than \( D(u) \), while the leading term \( [u] \) equals either \([x_k x_{k+1} \ldots x_m]\) or \([x_k^- x_{k+1}^- \ldots x_m^-]\). Certainly the leading terms here are defined by the degree function into the additive monoid \( \Gamma^+ \oplus \Gamma^- \) generated by \( X \cup X^- \) (but not into the group \( \Gamma! \)). In particular all \( W_i \) in \((9.3)\) have the same constitution in \( X \cup X^- \) as the leading term \( [u] \) does. Thus all \( W_i \)'s and the leading term \( [u] \) belong to the same component of the triangular decomposition. Hence it remains to show that there are no terms \( V_j \).

If \( q \) is not a root of 1 then by Corollary \((9.3)\) the algebra \( U \) is \( \Gamma\text{-homogeneous}. \) Hence (in both cases) the PBW-generators may be chosen to be \( \Gamma\text{-homogeneous} \) as well. In this case all terms \( V_j \) have the same \( \Gamma\text{-degree} \) and smaller \( \Gamma^+ \oplus \Gamma^-\text{-degree} \). However this is impossible.
Indeed, if the leading term is \([x_k^- x_{k+1}^- \ldots x_m^-]\) then the \(\Gamma^+ \oplus \Gamma^-\)-degree of \(V_j\) should be less than \(x_k^- + x_{k+1}^- + \ldots + x_m^-\). Hence due to definitions \((9.2)\) and \((2.16)\) we have \(V_j \in U_q^-(\mathfrak{sl}_{n+1})\), (respectively, \(V_j \in u_q^- (\mathfrak{sl}_{n+1})\)), and the \(\Gamma\)-degree of \(V_j\) coincides with the \(\Gamma^+ \oplus \Gamma^-\)-degree. A contradiction.

Suppose that the leading term is \([x_k^+ x_{k+1}^+ \ldots x_m^+]\). Let \(d = \sum s_i x_i + \sum r_i x_i^-\) be the \(\Gamma^+ \oplus \Gamma^-\)-degree of \(V_j\). Since
\[
\begin{align*}
\sum s_i &< x_k^+ + x_{k+1}^+ + \ldots + x_m^+, \\
\sum r_i &< x_k^- + x_{k+1}^- + \ldots + x_m^-,
\end{align*}
\]
we have \(s_i = 0\) provided that \(i < k\). Since the \(\Gamma\)-degree of \(V_j\) equals \(x_k^+ + x_{k+1}^+ + \ldots + x_m^+\), we have \(s_i - r_i = 1\) provided that \(k \leq i \leq m\), and \(r_i = s_i\) otherwise. Inequality \((9.4)\) implies \(s_k \leq 1\). Together with \(s_k - r_k = 1\) this yields \(s_k = 1\), \(r_k = 0\). Hence, again inequality \((9.4)\) implies \(s_{k+1} \leq 1\), and again together with \(s_{k+1} - r_{k+1} = 1\) this yields \(s_{k+1} = 1\), \(r_{k+1} = 0\). In this way we get \(s_i = 1\), \(r_i = 0\), \(k \leq i \leq m\), that contradicts \((9.4)\).

Now we see that each PBW-generator \((9.3)\) belongs to either \(U_q^+ (\mathfrak{sl}_{n+1})\) or \(U_q^- (\mathfrak{sl}_{n+1})\) (respectively, to either \(u_q^+ (\mathfrak{sl}_{n+1})\) or \(u_q^- (\mathfrak{sl}_{n+1})\)). Therefore \(U\) has the decomposition \((9.1)\).

Remark. Certainly, if \(U\) does not contain \(k[H]\) then decomposition \((9.1)\) is not valid. A \(k\)-subalgebra generated by \(g_1 f_1\) has no triangular decomposition of any type provided that \(f_1 \notin G\), \(g_1 \notin F\). In the single parameter case (when \(G = F = H\)) the subalgebra with 1 generated by \(g_1^{-1}(x_1 + x_1)\) provides a similar example. Moreover in the single parameter case many of the \((\text{left})\) coideal subalgebras studied by M. Noumi and G. Letzter do not admit a triangular decomposition.

Corollary 9.2. If \(q\) is not a root of 1, then \(U_q (\mathfrak{sl}_{n+1})\) has just a finite number of right coideal subalgebras containing the coradical. If \(q\) has finite multiplicative order \(t > 2\), then \(u_q (\mathfrak{sl}_{n+1})\) has just a finite number of \(\Gamma\)-homogeneous right coideal subalgebras containing the coradical.

Proof. This follows from the above lemma and Corollary 1.8 applied to \(U_q^\pm (\mathfrak{sl}_{n+1})\), \(u_q^\pm (\mathfrak{sl}_{n+1})\).

Our next goal is to understand when tensor product \((9.1)\) is a subalgebra and then to find a way to calculate the total number of \((\Gamma\)-homogeneous\) right coideal subalgebras.

Lemma 9.3. The tensor product \((9.1)\) is a right coideal subalgebra if and only if
\[
[U^+, U^-] \subseteq U^- \otimes_{k[H]} k[H] \otimes_{k[G]} U^+.
\]

Proof. Of course if \(U\) is a subalgebra then \((9.5)\) holds. Conversely, it is clear that \(U\) is a right coideal. Relation \((9.5)\) implies \(u^+ \cdot v^- = [u^+, v^-] + p(u^+, v^-) u^- \cdot u^+ \in U\), where \(u^+ \in U^+, v^- \in U^-\). Hence \((u^- \cdot u^+)(v^- \cdot v^+) = u^- (u^+ \cdot v^-) v^+ \in U\), with arbitrary \(v^- \in U^+, u^- \in U^-\).

Since \(U = U^- \cdot H \cdot U^+\), it remains to check that \(U^- \cdot H = H \cdot U^-\), and \(U^+ \cdot H = H \cdot U^+\). Since \(U^+\) contains \(G\), it is homogeneous with respect to the grading \((2.3)\). If \(u \in (U^+)\chi, f \in F\), then \(uf = \chi(f)fu\). Hence \(U^+ \cdot F = F \cdot U^+\). Similarly \(U^- \cdot G = G \cdot U^-\).
10. Consistency Condition

In this section we are going to find sufficient condition for consistency relation \((9.5)\) to be valid. In what follows we denote by \(\Psi^s(i, j)\) a polynomial that appears from \(\Psi^S(i, j)\) given in \((4.6)\) under the substitutions \(x_t \leftarrow x_{t-1}, 1 \leq t \leq n\) with skew commutators defined by \((2.4)\) in \(U_q(\mathfrak{sl}_n)\). By \(prW_k^s(m)\) (respectively, \(prW_s(m)\)) we denote a subspace spanned by proper derivatives of \(\Psi^s(k, m)\) (respectively, of \(\Psi^S(i, j)\)), see Theorem \(4.7\). Consider two elements \(\Psi^s(k, m)\) and \(\Psi^T(i, j)\). Let us display them graphically as defined in \((4.16)\):

\[
\begin{array}{ccccccc}
 k & \cdots & i & i+1 & \cdots & m & j \\
 \circ & \circ & \bullet & \circ & \cdots & \bullet & \circ \\
\end{array}
\]

We shall prove that

\[
[\Psi^s(k, m), \Psi^T(i, j)] \in prW^T(i, j) \cdot prW(k, m)
\]

if one of the following two options fulfills:

a) Representation \((10.1)\) has no fragments of the form

\[
\begin{array}{cccc}
 \circ & \cdots & \bullet \\
 \circ & \cdots & \bullet \\
\end{array}
\]

b) Representation \((10.1)\) has the form

\[
\begin{array}{ccccccc}
 k & \cdots & \circ & \circ & \cdots & m \\
 \circ & \cdots & \bullet & \circ & \cdots & \bullet \\
\end{array}
\]

(in particular \(i = k, j = m\)), where no one intermediate column has points of the same color.

Suppose that diagram \((10.1)\) satisfies condition a). In this case all black-black columns are located before all white-white columns. Let us choose the closest black-black and white-white pair of columns. Then \((10.1)\) takes up the form

\[
\begin{array}{cccccccc}
 \circ & \bullet & \cdots & \circ & \bullet & \cdots & \circ & \bullet \\
 \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\end{array}
\]

Here in the “mainly black” zone there are no white-white columns; in the “equality” zone we have just black-white, and white-black columns; while in the “mainly white” zone there are no black-black columns. Of course the “mainly black” zone may be empty. In this case we may omit the “equality” zone as well, since all the diagram has no black-black columns at all. In the same way the “mainly white” zone may be empty too.

Recall that in Definition \(4.3\) for a fixed pair \((k, m)\) we define \(S_0 = (S \cap [k, m - 1]) \cup \{k - 1\}\), while \(S^* = (S \cap [k, m - 1]) \cup \{m\}\); respectively \(s_0 = k - 1 \in S_0\), and \(s_{r+1} = m \in S^*\).

All black-black columns are labeled by numbers from \(S^* \cap T^*\), where the bullets correspond to the pairs \((k, m)\) and \((i, j)\), respectively. Similarly all white-white columns are labeled by numbers from \((S_0 \cap (T_0)\), where \(S, T\) are the complements of \(S, T\) with respect to \([k, m - 1], [i, j - 1]\), respectively. Thus condition a) is equivalent to the inequality

\[
\sup\{S^* \cap T^*\} < \inf\{(S_0 \cap (T_0)\}
\]
We are reminded that the supremum and infimum of the empty set equal $-\infty$ and $\infty$, respectively.

Condition b), in turn, means that $i = k, j = m, T = \mathbb{S}$.

We go ahead with a number of useful notes. If $u$ is a word in $X$, then by $u^-$ we denote a word in $X^-$ that appears from $u$ under the substitution $x_i \leftarrow x_i^-$. We have $p(v, w^-) = \chi^v(f_w) = p(w, v)$, while $p(w^-, v) = (\chi^v)^{-1}(g_v) = p(v, w)^{-1}$. Thus $p(v, w^-)p(w^-, v) = 1$. Therefore the Jacobi and antisymmetry identities take up their original “colored” form (see, (2.5)):

$$
[u, v, w^-] = [u, [v, w^-]] + p_{vw}[[u, w^-], v];
$$

$$
[u, w^-] = -p_{uw}[w^-, u].
$$

In the same way

$$
[u^-, v, w] = [u^-, [v, w]] + p_{vw}^{-1}[[u^-, w], v^-].
$$

Using antisymmetry and (10.9) we have also

$$
[u, [v^-, w^-]] = [[u, v^-], w^-] + p_{uv}[v^-, [u, w^-]].
$$

In these relations $u, v, w$ are words in $X$. To simplify further calculations we may extend the brackets to the set of all $H$-words: We put $\chi^{hu} = \chi^u, gh_u = h_g u, h \in H$, and define the skew-brackets by the same formula (2.4). In this case we have

$$
[u, hv] = \chi^v(h) h[u, v], \quad h \in H;
$$

$$
[hu, v] = h[u, v] + p_{uv}(1 - \chi^v(h)) h \cdot v, \quad h \in H.
$$

To calculate the coefficients it is convenient to have in mind the following consequences of (4.4):

$$
\chi^k(g_{k-1} f_{k-1}) = \chi^{k-1}(g_k f_k) = q^{-1}, \quad \chi^k(g_i f_i) = 1, \quad \text{if } |i - k| > 1.
$$

Of course all basic formulae (2.3), (2.7), (2.10) and their consequences remain valid. However we must stress that once we apply relations (3.11), or other “inhomogeneous in $H$” relations (for example the third option of (10.17), see below), we have to fix the curvature of the brackets as soon as the inhomogeneous substitution applies to the right factor in the brackets:

$$
[u, [x_i, x_i^-]] = u(1 - g_i f_i) - \chi^u(g_i f_i)(1 - g_i f_i)u = (1 - \chi^u(g_i f_i))u,
$$

but not $[u, [x_i, x_i^-]] = [u, 1 - g_i f_i] = [u, 1] - [u, g_i f_i] = 0$. At the same time

$$
[[x_i, x_i^-], u] = (1 - g_i f_i)u - u(1 - g_i f_i) = (\chi^u(g_i f_i) - 1)g_i f_i \cdot u,
$$

and $[[x_i, x_i^-], u] = [1 - g_i f_i, u] = [1, u] - [g_i f_i, u]$ is valid since the inhomogeneous substitution has been applied to the left factor in the brackets. In what follows we shall denote for short $h_i = g_i f_i$, and $h_{k_i} = h_k h_{k+1} \cdots h_{i-1}$, where $k < i$.

Now we consider relation (10.2) when $i = j$.

**Proposition 10.1.** If $k < m$, then $[\Psi^S(k, m), x_i^-] \in \text{pr}W^S(k, m)$ if and only if $i$ is not a $(k, m)$-entrance into $S$ (see Definition 8.3).
Proof. To prove the proposition we shall need the following four formulae (10.17)–(10.20). In that formulae the notation $a \sim b$ means $a = \alpha b$, $0 < \alpha \in \mathbb{k}$.

(10.17) $$[u[k, m], x_k^i] \sim \begin{cases} 0, & \text{if } k < i < m; \\ h_k \cdot u[k + 1, m], & \text{if } i = k < m; \\ u[k, m - 1], & \text{if } k < i = m. \end{cases}$$

For $i = k$ we have $[u[k, m], x_k^i] = [[x_k, u[k + 1, m]], x_k^i]$. If we denote $u = x_k$, $v = u[k + 1, m]$, $w = x_k^i$, then formulae (10.7) and (10.16) imply $[[u, v], w] = \mu_{k,m}[u, w], v] = \mu_{k,m}(q^{-1} - 1) h_k \cdot v$ since $p(v, x_k)p(x_k, v) = q^{-1}$ due to (4.1).

For $k < i = m$ we put $u = u[k, m - 1]$, $v = x_m$, $w = x_m^i$. Then $[u[k, m], x_k^i] = [[u, v], w]$, while (10.7) and (10.15) imply

$$[[u, v], w] = [u, [v, w]] = (1 - q^{-1}) u$$

since according to (4.1) we have $p(u, x_m)p(x_m, u) = q^{-1}$.

Finally, for $k < i < m$ we put $u = u[k, i - 1]$, $v = u[i, m]$, $w = x_i^m$. Then

$$[u[k, m], x_k^i] = [[u, v], w],$$

while (10.7) and considered above case $i = k$ with (10.11) imply

$$[[u, v], w] = [u, [v, w]] \sim [u, h_i u[i + 1, m]] = h_i \chi^u(h_i) \cdot [u, u[i + 1, m]] = 0,$n which proves (10.17).

(10.18) $$[\Psi^S(k, m), x_k^i] \sim \begin{cases} \Psi^S(k + 1, m), & \text{if } s_1 = k; \\ h_k \cdot \Psi^S(k + 1, m), & \text{if } s_1 > k. \end{cases}$$

Let us put $u = \Psi^S(1 + s_1, m)$, $v = u[k, s_1]$, $w = x_k^i$. The definition (4.6) shows that $\Psi^S(k, m) = [u, v]$, while (10.7) implies $[[u, v], w] = [u, [v, w]]$.

If $s_1 = k$, then by (10.15) we have $[u, [v, w]] = u - \chi^u(h_k)u = (1 - q^{-1})u$.

If $s_1 > k$, then (10.17) yields $[v, w] \sim h_k \cdot u[k + 1, s_1]$. Therefore $[u, [v, w]] \sim h_k[u, u[k + 1, s_1]]$, see (10.11). It remains to note that $[u, u[k + 1, s_1]] = \Psi^S(k + 1, m)$ due to (4.6). Thus formula (10.18) is proved.

(10.19) $$[\Psi^S(k, m), x_m^i] \sim \begin{cases} h_m \cdot \Psi^S(k, m - 1), & \text{if } k \leq s_r = m - 1; \\ \Psi^S(k, m - 1), & \text{if } k \leq s_r < m - 1. \end{cases}$$

Let us put $u = u[1 + s_r, m]$, $v = \Psi^S(k, s_r)$, $w = x_m$. Decomposition (4.7) with $i = r$ shows that $\Psi^S(k, m) = [u, v]$, while (10.7) implies $[[u, v], w] \sim [u, [v, w]]$.

If $s_r = m - 1$, then by (10.16) we get $[u, [v, w]] \sim h_m \cdot v$ since $p(v, x_m)p(x_m, v) = q^{-1} \neq 1$.

If $s_r \neq m - 1$, then by (10.17) we have $[u, w] \sim u[1 + s_r, m - 1]$. Therefore by decomposition (4.7), we get $[u, w] \sim \Psi^S(k, m - 1)$, that proves (10.19).

(10.20) $$[\Psi^S(k, m), x_i^i] \sim \begin{cases} h_i \cdot \Psi^S(i + 1, m) \cdot \Psi^S(k, i - 1), & i - 1 \in S, i \notin S; \\ \Psi^S(i + 1, m) \cdot \Psi^S(k, i - 1), & i - 1 \notin S, i \in S; \\ 0, & \text{otherwise}, \end{cases}$$

where of course $k < i < m$.

Suppose that $i - 1 \in S$. Let us put $u = \Psi^S(i, m)$, $v = \Psi^S(k, i - 1)$, $w = x_i^i$. Decomposition (4.7) shows that $\Psi^S(k, m) = [u, v]$, while (10.7) implies $[[u, v], w] \sim [u, [v, w]]$.

If $i \not\in S$, then we apply the second option of (10.18) and (10.15):

$$[[u, w], v] = [h_i \cdot \Psi^S(i + 1, m), v] \sim h_i \Psi^S(i + 1, m) \cdot v$$
This proves the first option of (10.20).

If \( i \in S \) (and still \( i - 1 \in S \)) then we may use the first option of (10.18):

\[
[u, w^-] \sim [\Psi^S(i + 1, m), v] = 0.
\]

Suppose that \( i - 1 \notin S \). If also \( i \notin S \), then (10.20) follows from the first case of (10.17) and definition (4.6).

If \( i \in S \), we put \( u = \Psi^S(i + 1, m) \), \( v = \Psi^S(k, i) \), \( w = x_i^- \). Decomposition (4.7) shows that \( \Psi^S(k, m) = [u, v] \), while (10.7) implies \([u, v, w^-] = [u, v, w^-]\). By (10.19) we have \([v, w^-] \sim \Psi^S(k, i - 1) \). We may use (10.19) taking into account the curvature of the skew commutator:

\[
[u, v, w^-] \sim u \cdot \Psi^S(k, i - 1) - \chi^u(g_vf_w) \Psi^S(k, i - 1) \cdot u.
\]

Since \( \Psi^S(k, i - 1) \) and \( u \) are separated elements, we have

\[
u \cdot \Psi^S(k, i - 1) = \alpha \Psi^S(k, i - 1) \cdot u
\]

with the coefficient \( \alpha = \chi^u(g_vf_w) = \chi^u(h_i) = p(u, x_i)p(x_i, u) = q^{-1} \neq 1 \), which completes the proof of (10.20).

If \( i \) is a \((k, m)\)-entrance into \( S \), then according to definition (4.17) we have \( \Psi^S(k, i) \in \text{prW}^S(k, m) \), while \( \Psi^S(k, i - 1) \notin \text{prW}^S(k, m) \). Hence (10.18) imply \([\Psi^S(k, m), x_i^-] \notin \text{prW}^S(k, m) \).

If we compare formulae (10.18–10.20) with (4.23–4.25), we see that (Ψ, w) = \( [Ψ^S(k, m), x_i^-] = 0 \) prW^S(k, m) only in the following three cases: a) if \( i = k \) and \( k \in S \); b) if \( i = m \) and \( i - 1 \notin S \); c) if \( i - 1 \notin S \) and \( i \in S \), \( k < i < m \). In all these cases \( i \) by definition is a \((k, m)\)-entrance into \( S \).

Now by means of the following lemmas we are going to show that (10.2) fulfills provided that diagram (10.1) takes up the form (10.5).

**Lemma 10.2.** If \( S^* \cap T^* = S_o \cap T_o = \emptyset \) then

\[
(10.21)
\]

\[
[Ψ^S(k, m), Ψ^T(i, j)] = 0.
\]

Here \( S^* \), \( S_o \) and \( T^* \), \( T_o \) correspond to the pair \((k, m)\) and \((i, j)\) respectively, see Definition (4.3).

**Proof.** Since relations (1.1) are invariant under the substitutions \( p_{ij} \leftarrow p_{ji}^{-1}, q \leftarrow q^{-1} \), formulae (10.17, 10.20) remain valid under the substitutions \( x_i \leftarrow x_i^-, x_i^- \leftarrow x_i \).

We show firstly that

\[
(10.22)
\]

\[
[u, k, m], u[i, j^-] = 0, \quad \text{if} \ k \neq i, \ j \neq m.
\]

Indeed, if \( k < i < j \leq m \), or \( i < k \leq m < j \), this follows from the first case of (10.17) and its dual.

If \( k < i = m < j \), we have \([u, k, m], u[m, j^-] = [u, k, m], [x_m, u[m + 1, j^-]] = [u, k, m - 1, u[m + 1, j^-]] = 0\). If \( k < i < m < j \), the induction on \( m - i \) provides \( [u[k, m], u[i, j^-]] = [u[k, m], [x_i^-, u[i + 1, j^-]]] = 0 \).

The remaining case, \( i < k < j < m \), due to (10.8), is dual to one considered above, \( k < i < m < j \). Thus, (10.22) is proved.

If \( S^* \cap T^* = S_o \cap T_o = \emptyset \), then (10.22) implies

\[
[u[1 + s_a, s_{a+1}], u[1 + t_b, t_{b+1}]^-] = 0,
\]
We start with a particular case:

\[\Psi(\alpha, \mu) = 0\]

Relation (10.21) with the only exception, \(\alpha_{\mu+1} = 0\). Here by definition \(h_i = g_i f_i\), \(h_{\nu a} = h_{\nu} h_{\nu+1} \cdots h_{\nu-1}\), and \(\Psi^S(k, -1) = \Psi^T(i, -1) = 1\) (in particular always either the first or the last factor in (10.23) is trivial).

**Proof.** We start with a particular case:

\[\Psi^S(k, m), \Psi^T(i, j)\]

(10.23)

where \(\mu = \min\{m, j\}\), \(\nu = \max\{k, i\}\), while \(\alpha_a = 0\) provided that \(a - 1 \in S \cup T\) with the only exception, \(\alpha_{\mu+1} = 0\). Here by definition \(h_i = g_i f_i\), \(h_{\nu a} = h_{\nu} h_{\nu+1} \cdots h_{\nu-1}\), and \(\Psi^S(k, -1) = \Psi^T(i, -1) = 1\) (in particular always either the first or the last factor in (10.23) is trivial).

Our next step is to prove (10.23) for \(S \neq 0\), \(T = 0\) and \(i = k\):

\[\Psi^S(k, m), \Psi^T(k, -1)\]

(10.25)

where \(\mu = \min\{m, j\}\), while \(\alpha_a = 0\) if \(a - 1 \in S \cup k, m - 1\), and formally \(\Psi^S(m + 1, k, j) = \Psi^S(m + 1, j)\).

Suppose firstly that \(j > m\). In this case \(\mu = m\). We use induction on \(m - k\). If \(m = k\), the formula follows from (10.17). If \(m > k\), we put \(u = \Psi^S(1 + s_1, m)\), \(v = u(k, s_1)\), \(w = u(k, j)\). According to (4.6) we have \(u, v, w = 0\). Hence \(u, v, w = 0\) due to (10.24) with \(T = 0\).

Relation (10.24) with \(T = 0\) implies \(u, u(a, j) = 0\) unless \(a = 1 + s_1\). Using ad-identities (2.10), (10.11) we may continue

\[= \alpha \bar{h}_{k+1} [u, u(1 + s_1, j) -] + \sum_{a=k+1}^{s_1} \alpha_a \bar{h}_{ka} u(a, j) - \Psi^S(a, m),\]
where \( \alpha \neq 0 \). Since \( 1 + s_1 > k \), we may apply the inductive supposition to the first summand.

Let \( j < m \), say \( s_l < j < s_{l+1} \), \( 0 \leq l \leq r + 1 \). Suppose that \( l = r \); that is, \( s_r < j < m \). Denote \( u = u[1 + s_r, m] \), \( v = \Psi^S(k, s_r) \), \( w^− = u[k, j]^- \). By definition \([u, v] = \Psi^S(k, m)\), while \([u, w^−] = 0 \) due to (10.22). By Jacobi identity (10.7) and the considered above case \( j > m \) with \( m \leftarrow s_l \) we get

\[
[\Psi^S(k, m), w^-] = [u, [v, w^-]] = [u, \sum_{a=k+1}^{1+s_l} \alpha_a \bar{h}_{ka} u[a, j]^- \cdot \Psi^S(a, s_l)],
\]

where \( \alpha_a = 0 \) if \( a - 1 \in S \cap [k, s_l - 1] \). Relation (10.22) imply \([u, u[a, j]^-] = 0 \) unless \( a = 1 + s_l \). Hence by ad-identities (2.10), (10.11) we may continue

\[
\alpha \bar{h}_{k+1} \{u, u[1 + s_l, j]\} + \sum_{a=k+1}^{s_l} \alpha_a \bar{h}_{ka} u[a, j]^- \cdot \Psi^S(a, m),
\]

where \( \alpha \neq 0 \), \( \alpha_a = 0 \) if \( a - 1 \in S \cap [k, s_l - 1] \). It remains to apply (10.23) to the first summand.

If, finally, \( l \leq r \), we put \( u = \Psi^S(1 + s_{l+1}, m), v = \Psi^S(k, s_{l+1}), w^− = u[k, j]^- \). Then still \([u, v] = \Psi^S(k, m)\), see decomposition (4.7), and \([u, w^−] = 0 \). By the considered above case with \( m \leftarrow s_{l+1} \) we have

\[
[v, w^-] = \sum_{a=k+1}^{j+1} \alpha_a \bar{h}_{ka} u[a, j]^- \cdot \Psi^S(a, s_{l+1}),
\]

where \( \alpha_a = 0 \) if \( a - 1 \in S \cap [k, s_{l+1} - 1] \). In this case \([u, u[a, j]^-] = 0 \) since \( j < s_{l+1} \). Thus by ad-identities (2.10), (10.11), and decomposition (4.7) we get

\[
[u, [v, w^−]] = \sum_{a=k+1}^{j+1} \alpha_a \bar{h}_{ka} u[a, j]^- \cdot \Psi^S(a, m),
\]

where \( \alpha_a = 0 \) if \( a - 1 \in S \cap [k, s_{l+1} - 1] \). Of course

\[
(S \cap [k, m - 1]) \cap [k, j] = (S \cap [k, s_{l+1} - 1]) \cap [k, j],
\]

hence in the above sum \( \alpha_a = 0 \) if \( a - 1 \in S \cap [k, m - 1] \). Since Jacobi identity (10.7) implies \([\Psi^S(k, m), w^-] = [[u, v], w^-]\), formula (10.25) is proved.

Now we are ready to consider the general case. Since \( S^* \cap T^* = \emptyset \), the intersection \( S_o \cap T_o \) equals either \( \{k - 1\} \) or \( \{i - 1\} \). More precisely, this intersection has the only point \( \{\nu - 1\} \).

Suppose firstly that \( i = k \). In this case we shall prove (10.23) by induction on \( j - k \). If \( j = k \), one may apply the second option of (10.18). Let \( j > i = k \). If \( T = \emptyset \), the formula is already proved, see (10.25). Suppose that \( T \neq \emptyset \). Let us denote \( u = \Psi^S(k, m), v^- = \Psi^T(1 + t_1, j), w^- = u[k, t_1] \). Then according to definition (1.6) the left hand side of (10.23) equals \([u, [v^-, w^-]]\). By (10.21) we have \([u, v^-] = 0 \). Hence Jacobi identity (10.10) implies \([u, [v^-, w^-]] \sim [v^-, [u, w^-]]\). Using (10.25) we get

\[
[u, w^-] = \sum_{a=k+1}^{\mu+1} \alpha_a \bar{h}_{ka} u[a, j]^- \cdot \Psi^S(a, m),
\]

where \( \mu = \min\{m, t_1\} \), while \( \alpha_a = 0 \) if \( a - 1 \in S \cap [k, m - 1] \), and formally \( \Psi^S(m + 1, m) = [j + 1, j]^- = 1 \). Of course, \( m \neq t_1 \) since \( S^* \cap T^* = \emptyset \).
If \( m > t_1 \), then we have
\[
[v^-, [u, w^-]] = \alpha_{1+t_1} [v^-, \bar{h}_{k+1} + t_1] \Psi^S(1+t_1, m) + \sum_{a=k+1}^{t_2} \alpha_a [v^-, \bar{h}_{ka} u[a, t_1]^- \cdot \Psi^S(a, m)].
\]

By (10.8) and (10.21) we have \([v^-, \Psi^S(a, m)] = 0 \) if \( k < a \leq t_1 \). Hence by (10.11) and (2.10) we get \([v^-, \bar{h}_{ka} u[a, t_1]^- \cdot \Psi^S(a, m)] \sim \bar{h}_{ka} \Psi^T_T(a, j) \cdot \Psi^S(a, m). \)

It remains to apply (10.11) and then the inductive supposition to the first summand. If \( m < t_1 \), then
\[
[v^-, [u, w^-]] = \sum_{a=k+1}^{m+1} \alpha_a [v^-, \bar{h}_{ka} u[a, t_1]^- \cdot \Psi^S(a, m)].
\]

Since \( m < t_1 \), we have \([v^-, \Psi^S(a, m)] = 0 \). Hence, again by (10.11) and (2.10), we get
\[
[v^-, \bar{h}_{ka} u[a, t_1]^- \cdot \Psi^S(a, m)] \sim \bar{h}_{ka} \Psi^T_T(a, j) \cdot \Psi^S(a, m),
\]
which completes the case \( i = k \).

Suppose that \( i > k \). In this case \( k-1 \in T \cap [i, j-1] \), say \( k = 1 + t_i \). Let us put \( u = \Psi^S(k, m), v^- = \Psi^T_T(k, j), w^- = \Psi^T_T(i, k-1) \). Decomposition (4.7) implies \( \Psi^T_T(i, j) = [v^-, w^-] \). Since \([u, w^-] = 0\), we have \([u, [v^-, w^-]] = [[u, v^-], w^-] \). To find \([u, v^-] \) we may use already considered case:
\[
[u, v^-] = \sum_{a=k+1}^{\mu+1} \alpha_a \bar{h}_{ka} \Psi^T_T(a, j) \cdot \Psi^S(a, m)
\]
with \( \alpha_a = 0 \) if \( a-1 \in S \cup T, a \neq \mu+1 \). Certainly \([\Psi^S(a, m), w^-] = [\Psi^T_T(a, j), w^-] = 0 \) since \( a > k \). By means of (10.14) we have
\[
\chi^w_-(\bar{h}_{ka}) = \chi^w_-(h_k h_{k+1} \cdots h_{a-1}) = \chi^{k-1}_-(h_k) = q \neq 1.
\]

Now formula (10.12) shows that \([u, v^-], w^-] \sim w^- \cdot [u, v^-] \), which is required.

In perfect analogy, if \( k < i \) then \( i-1 \in S \cap [k, m-1], i = 1 + s_i \). Let us put \( u = \Psi^S(i, m), v = \Psi^S(i, m-1), w^- = \Psi^S_S(i, j) \). Decomposition (4.7) implies \( \Psi^S(k, m) = [u, v] \). Since \([v, w^-] = 0\), by Jacobi identity (10.7) we have \([u, v[w^-]] = [[u, v], w^-] \sim [[u, w^-], v] \).

To find \([u, w^-] \) we use already considered case:
\[
[u, w^-] = \sum_{a=i+1}^{\mu+1} \alpha_a \bar{h}_{ia} \Psi^T_T(a, j) \cdot \Psi^S(a, m)
\]
with \( \alpha_a = 0 \) if \( a-1 \in S \cup T, a \neq \mu+1 \). Of course, \([\Psi^S(a, m), v] = [\Psi^T_T(a, j), v] = 0 \) since \( a > i \). By means of (10.14) we have
\[
\chi^v_-(\bar{h}_{ia}) = \chi^v_-(h_i h_{i+1} \cdots h_{a-1}) = \chi^{i-1}_-(h_i) = q^{-1} \neq 1.
\]

Now formula (10.13) shows that \([u, w^-], v] \sim [u, w^-] \cdot v \), which is required.

We have mentioned above that our main concepts (Definition 1.3) are not invariant with respect to the replacement of \( x_i, x^-_i, 1 \leq i \leq n \) by \( y_i, y^-_i, 1 \leq i \leq n \), where by definition \( y_i = x_{\varphi(i)}, y^-_i = x^-_{\varphi(i)}, \varphi(i) = n - i + 1 \). Hence the application of already proved lemmas to generators \( y_i, y^-_i \) provides an additional information. In this way we are going to prove the following two statements.
Lemma 10.4. If $\mathbf{(S)}_o \cap \mathbf{(T)}_o = (\mathbf{S})^* \cap (\mathbf{T})^* = \emptyset$ then

\[
(10.28) \quad \left[ \Psi^S(k, m), \Psi^T(i, j) \right] = 0.
\]

Lemma 10.5. If $\mathbf{(S)}_o \cap \mathbf{(T)}_o = \emptyset$, while $(\mathbf{S})^* \cap (\mathbf{T})^* \neq \emptyset$ then

\[
(10.29) \quad \Psi^T(\mu + 1, j) \left( \sum_{b=\nu-1}^{\mu-1} \alpha_b \bar{h}_{b+1} \mu + 1 \Psi^T(i, b) \cdot \Psi^S(k, b) \right) \Psi^S(\mu + 1, m)
\]

where $\mu = \min \{m, j\}$, $\nu = \max \{k, i\}$, while $\alpha_b = 0$ provided that $b \notin S \cap T$ with the only exception, $\alpha_{\nu-1} \neq 0$.

Proof. To derive these statements from Lemma 10.2 and Lemma 10.3 we use the decoding lemma, Lemma 4.4. Let us apply (4.10) to the left hand side of (10.29). To derive these statements from Lemma 10.2 and Lemma 10.3 we use the decoding lemma, Lemma 4.4. Let us apply (4.10) to the left hand side of (10.29). In this case we have

\[
(\varphi(S) - 1)_o = \varphi((S)^* + 1), \quad (\varphi(S) - 1)^* = \varphi((S)_o + 1),
\]

where in the left hand sides the operators (Definition 4.3) correspond to $(\varphi(m), \varphi(k))$, while in the right hand sides to $(k, m)$. In particular $(\mathbf{S})^* \cap (\mathbf{T})^* = \emptyset$ is equivalent to $(\varphi(S) - 1)_o \cap (\varphi(T) - 1)_o = \emptyset$, while $(\mathbf{S})_o \cap (\mathbf{T})_o = \emptyset$ is equivalent to $(\varphi(S) - 1)^* \cap (\varphi(T) - 1)^* = \emptyset$. Hence we may use relations (10.21), (10.23). Relation (10.21) proves Lemma 10.4. In the case of Lemma 10.5 we again apply decoding formula (4.10) in order to get (10.29). \(\square\)

Proposition 10.6. Condition a), $p \mathbf{T}$ implies (10.2).

Proof. We have seen that condition a) is equivalent to inequality (10.6).

If $S^* \cap T^* = \emptyset$ then we may use Lemma 10.2 and Lemma 10.3. Let us show that all factors in (10.23) in terms with nonzero $\alpha_a$ belong to either $\text{pr} W^T(i, j)$ or $\text{pr} W^S(k, m)$.

If $a = \mu + 1$, say $a - 1 = m < j$, then $a - 1 = m$ is a white point on the diagram of $\Psi^T(i, j)$. Hence by Theorem 4.7 we have $\Psi^T(m + 1, j) \in \text{pr} W^T(i, j)$. In the same way $a - 1 = j < m$ implies $\Psi^S(j + 1, m) \in \text{pr} W^S(k, m)$ (we note that $j \neq m$ since $S^* \cap T^* = \emptyset$ yet).

If $a - 1 \notin S \cup T$, $a \leq \min \{m, j\}$ then $a - 1$ is a white-white point, hence again Theorem 4.7 implies $\Psi^T(a, j) \in \text{pr} W^T(i, j)$ and $\Psi^S(a, m) \in \text{pr} W^S(k, m)$.

For the first and the last factors (if $k \neq i$, otherwise they do not exist) we have $S_o \cap T_o = \{\nu - 1\}$ (since $S^* \cap T^* = \emptyset$). Hence, if $i < k = \nu$ then we have $\nu - 1 \in T \cap [i, j - 1]$, and by Theorem 4.7 the first factor belongs to $\text{pr} W^T(i, j)$. Similarly, if $k < i = \nu$ then the last factor belongs to $\text{pr} W^S(k, m)$ since $\nu - 1 \in S \cap [k, m - 1]$.

In perfect analogy if $(\mathbf{S})_o \cap (\mathbf{T})_o = \emptyset$, Lemma 10.4 Lemma 10.5, and Theorem 4.7 imply (10.2).

Suppose that both $S^* \cap T^* \neq \emptyset$, and $(\mathbf{S})_o \cap (\mathbf{T})_o \neq \emptyset$. Then the diagram (10.1) takes up the form (10.5) with nonempty “mainly black” and “mainly white” zones.

Let us put $u = \Psi^S(t + 1, m)$, $v = \Psi^S(k, t)$, $w^- = \Psi^T(t + 1, j)$, $z^- = \Psi^T(i, t)$, where $t$ is the label of the last “black-black” column (that is, $t = \max \{S^* \cap T^*\}$).
We have \([u, z^-] = 0, [v, w^-] = 0\). Using decomposition \((4.7)\) and Jacobi identities \((10.7)\) and \((10.10)\) we may write
\[
[\Psi^S(k, m), \Psi_T^T(t + 1, j)] = [[u, v], [w^-, z^-]]
\]
\[
= [u, [v, [w^-, z^-]]] + p_{w^-, v} [[u, [w^-, z^-]], v]
\]
\[
= \alpha[u, [w^-, [v, z^-]]] + \beta[[[u, w^-], z^-], v].
\]
To find \([v, z^-]\) and \([u, w^-]\) we may use \((10.29)\) and \((10.23)\) respectively:
\[
(10.30)
[v, z^-] = \sum_{b=\nu-1}^{t-1} \alpha_b \tilde{h}_{b+1, \mu+1} \Psi_T^T(i, b) \cdot \Psi^S(k, b);
\]
\[
[u, w^-] = \sum_{a=\nu+2}^{\mu+1} \alpha_a \tilde{h}_{t+1, a} \Psi_T^T(a, j) \cdot \Psi^S(a, m).
\]
By means of \((10.11)\) and ad-identity \((2.10)\) we have \([w^-, [v, z^-]] = 0\) since \(w\) is separated both from \(\Psi_T^T(i, b)\), and \(\Psi^S(k, b)\).
At the same time \((10.12)\) and ad-identity \((2.9)\) imply
\[
[[u, w^-], z^-] = \sum_{a=\nu+2}^{\mu+1} \alpha_a \beta_a \tilde{h}_{t+1, a} z^- \cdot \Psi_T^T(a, j) \cdot \Psi^S(a, m)
\]
since due to \((10.14)\) we have \(\chi^- \tilde{h}_{t+1, a} = q \neq 1\). Again by \((10.12)\) and \((2.9)\), taking into account that \(v\) is separated both from \(\Psi^S(a, m)\), and \(\Psi_T^T(a, j)\), we find
\[
[\tilde{h}_{t+1, a} z^- \cdot \Psi_T^T(a, j) \cdot \Psi^S(a, m), v] = \tilde{h}_{t+1, a} z^- \cdot \Psi_T^T(a, j) \cdot v \cdot \Psi^S(a, m)
\]
\[
+ \varepsilon \tilde{h}_{t+1, a} z^- \cdot \Psi_T^T(a, j) \cdot [v, z^-] \cdot \Psi^S(a, m).
\]
By Theorem \((4.7)\) we have \(v \in \text{pr } W^S(k, m)\), and \(z^- \in \text{pr } W_T^T(i, j)\), while \([v, z^-]\) already appears in \((10.30)\). Since all factors in \((10.30)\) with nonzero \(\alpha_k\) belong to either \(\text{pr } W^S(k, m)\), or \(\text{pr } W_T^T(i, j)\), we get \([[u, w^-], z^-], v] \in \text{pr } W_T^T(i, j) \cdot \text{pr } W^S(k, m)\). The proposition is completely proved.

\textbf{Proposition 10.7.} We have
\[
(10.31)
[\Psi^S(k, m), \Psi^S(k, m)] \sim 1 - \tilde{h}_{k, m+1}.
\]
In particular condition \(b)\) of \((10.2)\) implies \((10.23)\).

\textbf{Proof.} We fix \(k\) and use induction on \(m\). If \(m = k\) the statement is clear. Let us put \(u = \Psi^S(k, m - 1), v = x_m, w^- = x_m^-\), \(z^- = \Psi^S(k, m - 1)\) (in fact \(v = w\)). In this case \([v, z^-] = 0, [u, w^-] = 0\).
Suppose firstly that \(m - 1 \notin S\). In this case \(m - 1 \in \overline{S}\), hence \((4.14)\) implies
\[
\Psi^S(k, m) \sim [\Psi^S(k, m - 1), x_m] = [u, v],
\]
\[
\Psi^S(k, m) = [x_m, \Psi^S(k, m - 1)] = [w^-, z^-].
\]
By means of Jacobi identity \((10.10)\) we have
\[
[[u, v], [w^-, z^-]] = [u, [v, [w^-, z^-]]] + p_{w^-, v}[[u, [w^-, z^-]], v]
\]
\[
= [u, [[v, w^-], z^-]] + p_{w^-, v}p_{w^-, u}[[w^-, [u, z^-]], v].
\]
Relations (4.1) imply $p_{zw,vw,u} = p_{uw} \cdot p_{zw} = q \cdot q^{-1} = 1$. Using (10.10) and then (10.11) we get
\[
[u, [[v, w^{-1}], z^{-1}]] = (\chi^+(h_m) - 1)\chi^+(h_m)h_m[u, z^{-1}] = (1 - q^{-1})\varepsilon(1 - \tilde{h}_{km}),
\]
where the inductive supposition yields $[u, z^{-1}] = \varepsilon(1 - \tilde{h}_{km})$, $\varepsilon \neq 0$. Using again the inductive supposition and taking into account the comment to (10.15), we have
\[
[w^{-1}, [u, z^{-1}]] = \varepsilon(1 - \chi^+(h_m))x_m^{-1}.
\]
Thus
\[
[w^{-1}, [u, v]], v] = \varepsilon(1 - \chi^+(h_m))[x_m^{-1}, x_m] = \varepsilon(1 - q)(-q^{-1})(1 - h_m).
\]
Now (10.32) implies
\[
\left[\Psi^S(k, m), \Psi^\Sigma(k, m)\right] \sim [[u, v], [w^{-1}, z^{-1}]] = \varepsilon(1 - q^{-1})(1 - \tilde{h}_{km+1}).
\]
If $m - 1 \in S$, then, of course, $m - 1 \notin S$, hence (4.14) implies
\[
\Psi^S(k, m) = [x_m, \Psi^S(k, m - 1)] = [v, u],
\]
\[
\Psi^\Sigma(k, m) \sim [\Psi^\Sigma(k, m - 1), x_m] = [z^{-1}, w^{-1}].
\]
By means of Jacobi identity (10.10) we have
\[
[[v, u], [z^{-1}, w^{-1}]] = [v, [u, [z^{-1}, w^{-1}]]) + p_{zw,u}[v, [z^{-1}, w^{-1}], u]
\]
\[
(10.33)
\]
Relations (4.1) imply $p_{zw,u} = p_{uw} = q^{-1} \cdot q = 1$. Using the inductive supposition and (10.10), we have
\[
[v, [[u, z^{-1}], w^{-1}]] = \varepsilon(\chi^+(h_{km}) - 1)[x_m, \tilde{h}_{km}x_m] = \varepsilon(q - 1)q^{-1}\tilde{h}_{km}(1 - h_m).
\]
Using (10.15), (10.8), and then the inductive supposition we get
\[
[z^{-1}, [v, w^{-1}], u] = (1 - \chi^+(h_m))[z^{-1}, u] = \varepsilon(1 - q)(-p_{zw})^{-1}(1 - \tilde{h}_{km}).
\]
Now (10.33) implies $[[v, u], [z^{-1}, w^{-1}]] = \varepsilon(1 - q^{-1})(1 - \tilde{h}_{km+1})$. The proposition is proved.

\section{11. Right coideal subalgebras in $U_q(\mathfrak{sl}_{n+1})$, $U_q(\mathfrak{sl}_{n+1})$}

**Theorem 11.1.** Let $U^+_q$, $U^-_q$ be right coideal subalgebras of positive and negative quantum Borel subalgebras defined over the related cordial by r-sequences $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, and $\theta' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$, respectively. The tensor product
\[
U = U^+_q \otimes_{\mathbb{K}[F]} \mathbb{K}[H] \otimes_{\mathbb{K}[C]} U^-_q
\]
is a right coideal subalgebra if and only if for each pair $(k, i)$, $1 \leq k, i \leq n$ one of the following two conditions is fulfilled. In that conditions the sets $T_k$, $T'_k$ are defined according to Definition 7.7 by $\theta$, $\theta'$, respectively, while $\tilde{\theta}_k = k + \theta_k - 1$, $\tilde{\theta}'_k = k + \theta'_k - 1$.

1. \[\sup \left\{ a \mid k \leq a \leq \tilde{\theta}_k, i \leq a \leq \tilde{\theta}'_i, a \in T_k, a \in T'_i \right\} < \inf \left\{ b \mid k - 1 \leq b < \tilde{\theta}_k, i - 1 \leq b < \tilde{\theta}'_i, b \notin T_k, b \notin T'_i \right\};\]
2. \[i = k, \tilde{\theta}'_k = \theta_k, \text{ and} \]
\[
\left\{ \begin{array}{ll}
a \mid k \leq a < \tilde{\theta}_k, a \in T_k, a \in T'_k = \emptyset, \\
a \mid k \leq a < \tilde{\theta}_k, a \notin T_k, a \notin T'_k \end{array} \right\} = \emptyset.
\]
If $q$ is not a root of $1$, then every right coideal subalgebra $U \supseteq k[H]$ of $U_q(\mathfrak{sl}_{n+1})$ has form $(\mathbb{H}, J)$ with $\theta$, $\theta'$ satisfying the above property. If $q$ has finite multiplicative order $t > 2$, then this is the case for $\Gamma$-homogeneous right coideal subalgebras of $u_q(\mathfrak{sl}_{n+1})$.

**Proof.** By Lemma 9.3 we have to show that $[U_q^+, U_q^-] \subseteq U$. The first condition in the theorem means that $\Psi^{T_k}(k, \tilde{\theta}_k)$ and $\Psi^{T_l}(k, \tilde{\theta}'_l)$ satisfy condition a) in (10.4) while the second one is equivalent to condition b) in (3.7) for these elements. By definition $[k : \theta_k]$ is a simple $U_q^+$-root, such that any other simple $U_q^+$-root of the form $[k : m]$ satisfies $m < \theta_k$. Similarly each simple $U_q^-$-root of the form $[i : j]$ satisfies $j \leq \tilde{\theta}'_l$. Condition a) certainly remain valid for subdiagrams, while proper subdiagrams of (10.2) satisfy condition a). Therefore, due to Proposition 10.6 and Proposition 10.7 for each pair of a simple $U_q^+$-root, $[k : m]$, and a simple $U_q^-$-root, $[i : j]$, we have

\[(11.4) \quad \{\Psi^{T_k}(k, m), \Psi^{T_l}(i, j)\} \subseteq U.\]

By Claim 8 the algebras $U_q^+$, and $U_q^-$ are generated by $\Psi^{T_k}(k, m)$, and $\Psi^{T_l}(i, j)$, respectively, where $[k : m]$ and $[i : j]$ run through the sets of simple roots. To show that $[U_q^+, U_q^-] \subseteq U$, it remains to apply ad-identities (2.10) and evident induction on degree (we remark that in (10.2) the degree of factors diminishes).

Conversely, suppose that $[U_q^+, U_q^-] \subseteq U$. Let us choose any pair $(k, i)$, and denote

\[t = \sup \left\{a | k \leq a \leq \tilde{\theta}_k, i \leq a \leq \tilde{\theta}'_l, a \in T_k, a \in T_l^\prime\right\},\]

\[l = \inf \left\{b | k - 1 \leq b < \tilde{\theta}_k, i - 1 \leq b < \tilde{\theta}'_l, b \notin T_k, b \notin T_l^\prime\right\}.\]

If one of these sets is empty then condition (11.2) is valid. Suppose that $t < l$. The point $t$ is a white-white point on the diagram (10.1) of the elements $\Psi^{T_k}(k, \tilde{\theta}_k)$, $\Psi^{T_l^\prime}(i, \tilde{\theta}'_l)$, while $l$ is a black-black one. In particular, due to Theorem 4.7 we have $z \overset{df}{=} \Psi^{T_k}(1 + t, l) \in U_q^+$, $z' \overset{df}{=} \Psi^{T_l^\prime}(1 + t, l) \in U_q^-$. Using both decompositions (1.7) and (1.93) we get

\[(11.5) \quad \Psi^{T_k}(k, \tilde{\theta}_k) \sim [\Psi^{T_k}(1 + l, \tilde{\theta}_k), [\Psi^{T_k}(k, t), z]].\]

\[(11.6) \quad \Psi^{T_l^\prime}(i, \tilde{\theta}'_l) \sim [\Psi^{T_l^\prime}(1 + l, \tilde{\theta}'_l), [\Psi^{T_l^\prime}(i, t), z']].\]

Since between $t$ and $l$ there are no columns marked by one color, we may apply Proposition 10.7 $z, z' = \varepsilon(1 - h)$, where $h = g_z f_z \in H$. This relation and (11.5), (11.6) allow us to consider $z$ and $z' = \varepsilon^{-1}z'$ as a pair of new variables and turn to a new set of variables

\[\{x_1, \ldots, x_t, z, x_{t+1}, \ldots, x_n, x_l^-, \ldots, x_l^- + 1, \ldots, x_n^-\}.\]

In particular we may apply formula (10.20):

\[\Psi^{T_k}(l + 1, \tilde{\theta}_k) \cdot \Psi^{T_k}(k, t) \sim [\Psi^{T_k}(k, \tilde{\theta}_k), z] \subseteq U_q^+ \cap U = U_q^+ .\]

This implies that both $[l + 1 : \tilde{\theta}_k]$ and $[k : t]$ are $U_q^+$-roots. Hence the simple $U_q^+$-root $[k : \tilde{\theta}_k]$ is a sum of three $U_q^+$-roots, $[k : t] + [1 + t : l] + [1 + l : \tilde{\theta}_k]$. This is a contradiction, unless $t = k - 1$, $\tilde{\theta}_k = l$. In perfect analogy we have $t = i - 1$, $\tilde{\theta}'_l = l$; that is, condition (11.3) is valid.

The last statement follows from Lemma 9.1. \[\square\]
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