HALL ALGEBRA APPROACH TO DRINFELD’S PRESENTATION OF QUANTUM LOOP ALGEBRAS

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Abstract. The quantum loop algebra $U_v(Lg)$ was defined as a generalization of the Drinfeld’s new realization of quantum affine algebra to the loop algebra of any Kac-Moody algebra $g$. Schiffmann [29] has proved (and conjectured) that the Hall algebra of the category of coherent sheaves over weighted projective lines provides a realization of $U_v(Lg)$ for those $g$ associated to a star-shaped Dynkin diagram. In this paper we explicitly find out the elements in the Hall algebra $H(Coh(X))$ satisfying part of Drinfeld’s relations, as addition to Schiffmann’s work. Further we verify all Drinfeld’s relations in the double Hall algebra $DH(Coh(X))$. As a corollary, we deduce that the double composition algebra is isomorphic to the whole quantum loop algebra when $g$ is of finite or affine type.

1. Introduction

1.1. Let $g$ be a Kac-Moody algebra, $U(g)$ be its universal enveloping algebra. The quantum group $U_v(g)$ is defined by a collection of generators and relations (see [3,2]), which is a certain deformation of the Chevalley generators and Serre relations of $U(g)$. When $g$ is of affine type, it is well-known that $g$ can be constructed as (a central extension of) the loop algebra $Lg_0$ of some simple Lie algebra $g_0$. In this case Drinfeld gave another set of generators and relations of $U_v(g)$ known as Drinfeld’s new realization of quantum affine algebras. This new presentation can be treated as a certain deformation of the loop algebra construction. The isomorphism between those two presentations is not obvious and was proved by Beck [2] (also see [13]). He found some "vertex" subalgebras isomorphic to $U_v(\hat{sl}_2)$ embedded into $U_v(g)$. And he used Lusztig’s symmetries associated to some special elements in the extended affine Weyl group, which are crucial in defining those subalgebras.

1.2. The Ringel-Hall algebra approach to quantum groups developed since 1990’s, which shows a deep relationship between Lie theory and finite dimensional hereditary algebras. Precisely speaking, let $Q$ be the quiver whose underlying graph is the Dynkin diagram of the Kac-Moody algebra $g$. Consider the category of finite dimensional representations of $Q$ over a finite field $k = \mathbb{F}_q$, denoted by $\text{mod}(kQ)$. Due to Ringel and Green ([22], [10]), the composition subalgebra of the Hall algebra $H(\text{mod}(kQ))$ is isomorphic to the positive part of the quantum group $U_v^+(g)$ where $v$ specializes to $\sqrt{q}$. This result was generalized to the whole quantum group by using the technique of Drinfeld double for Hopf algebras [30] (see [2,5]).

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Thus it is quite natural to consider the following problem:

**Problem.** How do we understand the Drinfeld’s presentation of quantum affine algebras in the setting of Hall algebras?

Of course we are done if we could explicitly explain the method used by Beck in the language of Hall algebra. For affine type $A$, Hubery has given the answer for the positive part $\mathcal{U}^+_q(\widehat{sl}_n)$ using nilpotent representations of cyclic quivers [11]. But it seems not easy to generalize his method to other affine types. Also we should mention that McGerty [20] has given the Drinfeld generators for the positive part $\mathcal{U}^+_q(\widehat{sl}_2)$ using representations of the Kronecker quiver.

1.3. On the other hand, in his remarkable paper [14] Kapranov observed that there are many similarities between the Hall algebra of the category of coherent sheaves over a smooth projective curve $X$ and Drinfeld’s new realization of quantum affine algebras. And when $X$ is the projective line, he provided an isomorphism between a subalgebra of the Hall algebra and another (compared with the usual one defined as in 5.2) positive part of $\mathcal{U}^+_q(\widehat{sl}_2)$ (also see [1]).

This result was generalized by Schiffmann [29] using the Hall algebra of the category of coherent sheaves over weighted projective lines (first studied in [9]) as following:

Let $\mathcal{L}_g$ be the loop algebra of any Kac-Moody algebra $g$. The quantized enveloping algebra of $\mathcal{L}_g$ generalizes Drinfeld’s presentation of quantum affine algebras (see 3.2). When the Dynkin diagram $\Gamma$ of $g$ is a star-shaped graph, he defined a ”positive part” of $U_v(\mathcal{L}_g)$, denoted by $U_v(\hat{n})$ (Note that there is no standard definition of positive part of $U_v(\mathcal{L}_g)$ since in general $\mathcal{L}_g$ is not a Kac-Moody algebra). And he established an epimorphism from the half quantum loop algebra $U_v(\hat{n})$ to a subalgebra $C(CohX)$ of the Hall algebra $H(CohX)$, where $X$ is a weighted projective line associated to $\Gamma$. Further he proved when $X$ is of parabolic or elliptic type (the corresponding $g$ is of finite or affine type), the epimorphism is in fact an isomorphism (see theorem 5.3). This means that the Drinfeld’s presentation can be thought as in a more general frame, not only for quantum affine algebras but for quantum loop algebras. The problem presented in 1.2 should also be considered in this more general setting.

However, Schiffmann did not explicitly write down all Drinfeld’s generators in terms of Hall algebra elements and check all the relations. Alternatively he use the Chevalley generators for some subalgebras isomorphic to $U_v(\hat{sl}_n)$ embedded into the Hall algebra. And the ”composition subalgebra” $C(Coh(X))$ is actually not generated directly by part of Drinfeld generators (We explain this in 5.0). Moreover, all the results mentioned above are in the half quantum loop algebra. So the Drinfeld’s presentation for the whole quantum loop algebra is not completely understood yet.

1.4. In this paper we first make it explicit that a collection of elements in the Hall algebra $H(CohX)$ satisfy part of Drinfeld’s relations of quantum loop algebra. Then we extend this result to the whole quantum loop algebra. More precisely, we use the construction of double Hall algebra $DH(CohX)$, which is the reduced Drinfeld double of the Hall algebra $H(CohX)$ and show all Drinfeld’s relations are satisfied for a certain collection of elements in $DH(CohX)$ (Theorem 5.4). As a corollary of our main theorem and Schiffmann’s theorem, we know that the double
composition algebra $\text{DC}(\text{Coh}(\mathcal{X}))$ is isomorphic to the whole quantum loop algebra $U_v(\mathcal{L}_q)$, when $\mathcal{X}$ is of parabolic or elliptic type (Corollary 5.5).

Let us briefly explain our method. First we consider the generators and relations in $\text{H}(\text{Coh}(\mathcal{X}))$. Note that the Dynkin diagram of the Kac-Moody algebra $\mathfrak{g}$ we consider is a star-shaped graph $\Gamma$ (see 5.5). Thus each branch corresponds to a subalgebra isomorphic to $U_v(\mathfrak{sl}_n)$. Therefore, we can use the results in [11] to find elements satisfying Drinfeld relations in each subalgebra. For the central vertex $\ast$, we use the generators defined in [29]. Now the remaining work is to check all the relations (some are done in [29], see 7.1). One can see that in our calculation, the extensions between the line bundles $\mathcal{O}(k\vec{c})$ and the elements in each tube are crucial.

To extend the results to the whole quantum loop algebra, we consider the double Hall algebra $\text{DH}(\text{Coh}(\mathcal{X}))$. Of course here the key part is to verify the relations where both positive and negative part are involved. For those relations we have to use the comultiplication, which is much more complicated than the no weight case $\text{Coh}(\mathbb{P}^1)$ studied in [11]. However, fortunately we find that most of the terms appearing in the comultiplication do not effect our calculation (see the proof of lemma 8.2 and 8.7), which makes the problem can be solved.

Note that in a recent work [3] of Burban and Schiffmann, the case of $U_v(\mathfrak{sl}_2)$ has been studied already in the frame of double Hall algebra. Our results coincide with theirs in this case.

1.5. Now we know that there are two Hall algebra realizations for the quantum affine algebras. One is from the representations of tame quivers, the other is from coherent sheaves over weighted projective lines of parabolic type. From [10] we know that these two hereditary categories are derived equivalent. Further, it was proved that the double Hall algebras associated to two derived equivalent hereditary categories are isomorphic (see [4]). In [3] they found that this isomorphism is compatible with Beck’s isomorphism combining with the two Hall algebra realizations (means that the diagram in 9.2 is commutative). Thus the derived equivalence can be thought as a certain ”categorification” of Beck’s isomorphism.

However, one can see in 9.2 that at least for type $D$ and $E$, this two isomorphisms are not compatible.

1.6. The paper is organized as follows: In section 2 we recall the basic notions of Hall algebra and double Hall algebra, for details one can see [23] and [30]. In section 3 we recall the definition of quantum loop algebra, see [29] or the Appendix in the lecture notes [25]. We give a brief review of the theory of coherent sheaves over weighted projective lines in section 4, the main reference for this section is [9]. The main result of the paper (Theorem 5.4) is stated in section 5. The proof of the main theorem consists of the next three sections. More precisely, in section 6 we prove the relations satisfied by those elements arising from one tube, following [11]. Section 7 is devoted to the proof of relations in the positive Hall algebra. All the remaining relations, especially those concerning elements coming from both positive and negative part, are proved in section 8. In the last section 9 we give some remarks for the quantum affine algebras. We explain the Beck’s isomorphism is not compatible with the isomorphism given by derived equivalence for type $D$ and $E$. We also provide an observation that the PBW-bases of two different positive part
of quantum affine algebras are related by the derived equivalence of two hereditary categories.

2. Hall algebras and their Drinfeld doubles

2.1. Hereditary category. Let $k = \mathbb{F}_q$ be a finite field with $q$ elements, and $\mathcal{A}$ be an abelian category. We assume that $\mathcal{A}$ is $k$-linear, Hom-finite and Ext-finite. That is, for all objects $X$, $Y$ and $Z$ in $\mathcal{A}$, the sets $\text{Hom}(X,Y)$ and $\text{Ext}^i(X,Y)$ are finite dimensional $k$-vector spaces and the composition of morphisms $\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \rightarrow \text{Hom}(X,Z)$ is $k$-bilinear. Assume further that $\mathcal{A}$ is hereditary, i.e. $\text{Ext}^i(-,-)$ vanishes for all $i \geq 2$.

Let $\mathcal{P}$ be the set of isomorphism classes of objects in $\mathcal{A}$ and $K_0(\mathcal{A})$ be the Grothendieck group of $\mathcal{A}$. For any $\alpha \in \mathcal{P}$ we choose a representative $V_\alpha \in \alpha$. And for each object $M$ in $\mathcal{A}$, denote by $[M]$ its image in $K_0(\mathcal{A})$. Then the assignment

$$(\alpha, \beta) \mapsto \dim_k \text{Hom}_{\mathcal{A}}(V_\alpha, V_\beta) - \dim_k \text{Ext}^1_{\mathcal{A}}(V_\alpha, V_\beta)$$

gives a well-defined bilinear form $K_0(\mathcal{A}) \times K_0(\mathcal{A}) \rightarrow \mathbb{Z}$, called the Euler form. The symmetric Euler form is defined by $(\alpha, \beta) = (\alpha, \beta) + (\beta, \alpha)$.

2.2. The Hall algebra. For any $\alpha \in \mathcal{P}$, denote by $a_\alpha$ the cardinality of the automorphism group of $V_\alpha$. Set $v = \sqrt{q}$.

For any $\alpha$, $\beta$ and $\gamma$ in $\mathcal{P}$, the Hall number $g_{\alpha\beta}^\gamma$ is defined to be the number of subobjects $X$ of $V_\gamma$ satisfying $X \in \beta$ and $V_\gamma/X \in \alpha$.

The Hall algebra associated to the category $\mathcal{A}$, denoted by $H(\mathcal{A})$, is defined to be the $\mathbb{C}$-algebra with basis $u_\alpha |\alpha \in \mathcal{P}$ and the multiplication:

$$u_\alpha u_\beta = v^{(\alpha, \beta)} g_{\alpha\beta}^\gamma u_\gamma.$$ 

It is easy to see that $H(\mathcal{A})$ is an associative algebra with $1 = u_0$.

2.3. The extended Hall algebra. By adding the Grothendieck group $K_0(\mathcal{A})$ as a torus to the Hall algebra $H(\mathcal{A})$, we obtain the extended Hall algebra $H(\mathcal{A})$. Precisely speaking, $H(\mathcal{A})$ is the $\mathbb{C}$-algebra with basis $\{K_\mu u_\alpha | \mu \in K_0(\mathcal{A}), \alpha \in \mathcal{P}\}$ and the multiplication is extended by

$$K_\mu K_\nu = K_{\mu+\nu}, \quad K_\mu u_\alpha = v^{(\mu, \alpha)} u_\alpha K_\mu.$$ 

When $\mathcal{A}$ is a length category (i.e. each object $M$ has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_\mu = M$ such that $M_{i+1}/M_i$ is a simple object for any $i$), the extended Hall algebra $H(\mathcal{A})$ has a Hopf algebra structure. The comultiplication $\Delta$, counit $\epsilon$ and antipode $S$ are given by:

$$\Delta(K_\mu) = K_\mu \otimes K_\mu,$$
$$\Delta(u_\alpha) = \sum_{\alpha, \beta \in \mathcal{P}} v^{(\alpha, \beta)} a_\alpha a_\beta g_{\alpha\beta}^\gamma u_\alpha K_\beta \otimes u_\beta,$$
$$\epsilon(u_\alpha) = \delta_{\alpha,0}, \quad \epsilon(K_\mu) = 1,$$
$$S(K_\mu) = K_{-\mu},$$
$$S(u_\alpha) = \delta_{\gamma,0} + \sum_{m \geq 1} (-1)^m \sum_{\beta \in \mathcal{P}, \alpha_1 \cdot \cdot \cdot \alpha_m \in \mathcal{P}} v^{2 \sum_{i<j} (\alpha_i, \alpha_j)} a_{\alpha_1} \cdots a_{\alpha_m} g_{\alpha_1}^\gamma \cdots g_{\alpha_m}^\beta K_{-\gamma} u_\beta,$$
where \( \mathcal{P}_1 = \mathcal{P} \setminus \{0\} \) and \( g_{\alpha_1 \cdots \alpha_m}^- \) is the number of all filtrations

\[
0 = M_m \subset M_{m-1} \subset \cdots \subset M_0 = V_{\gamma}
\]
such that \( M_i \setminus M_i \simeq V_{\alpha_i} \) for any \( i \).

**Remark.** (1) The Hall algebra can be defined for any finitary abelian category. However, the heredity is needed to define the comultiplication. For details one can see Schiffrmann’s lecture notes [25].

(2) The comultiplication (resp. antipode) was first defined by Green [10] (resp. Xiao [30]) in the case that \( \mathcal{A} \) is the category of finite dimensional modules over a finite dimensional hereditary algebra.

(3) When \( \mathcal{A} \) is not a length category, or more general, do not satisfy the finite subobjects condition (see [25]), \( \mathcal{H} \) is only a topological bialgebra, see [24].

2.4. The Drinfeld double of the Hall algebra. Now we write \( \mathcal{H}^+(\mathcal{A}) \) for the extended Hall algebra \( \mathcal{H}(\mathcal{A}) \) defined above. And we also write the basis elements \( u_\alpha^+ \) instead of \( u_\alpha \).

The “negative” extended Hall algebra, denoted by \( \mathcal{H}_-^+(\mathcal{A}) \), is defined to be the \( \mathbb{C} \)-algebra with basis \( \{K_\mu u_{\alpha}^- : \mu \in K_0(\mathcal{A}), \alpha \in \mathcal{P} \} \) and multiplication:

\[
u(\alpha, \beta)\gamma \sum_{\gamma \in \mathcal{P}} g_{\alpha, \beta}^\gamma u_{\gamma}^- , \quad K_\mu K_\nu = K_{\mu+\nu},
\]

\[
K_\mu u_{\alpha}^- = u_{-\langle \mu, \alpha \rangle}^- K_\mu; \quad K_\mu u_{\alpha}^+ = u_{-\langle \mu, \alpha \rangle}^+ K_\mu ;
\]

Similarly, if \( \mathcal{A} \) is a length category, \( \mathcal{H}_-^-(\mathcal{A}) \) has a Hopf algebra structure:

\[
\Delta(K_\mu) = K_\mu \otimes K_\mu ,
\]

\[
\Delta(u_{\gamma}^-) = \sum_{\alpha, \beta \in \mathcal{P}} v(\alpha, \beta)\frac{a_{\alpha, \beta}^\gamma}{a_{\gamma}} g_{\beta, \alpha}^- u_{\alpha}^- \otimes u_{\beta}^- K_{-\alpha} ,
\]

\[
\varepsilon(u_{\gamma}^-) = \delta_{\alpha, 0} , \quad \varepsilon(K_\mu) = 1, \quad S(K_\alpha) = K_{-\alpha} ,
\]

\[
S(u_{\gamma}^-) = \delta_{\gamma, 0} + \sum_{m \geq 1} \sum_{a_{\alpha} \in \mathcal{P}} v^2 \sum_{i<j} a_{\alpha_1} \cdots a_{\alpha_m} g_{\alpha_1 \cdots \alpha_m}^- u_{\alpha^-}^- K_{\gamma}.
\]

Actually in this case \( \mathcal{H}_-^-(\mathcal{A}) \) is the dual Hopf algebra of \( \mathcal{H}_-^+(\mathcal{A}) \) with opposite comultiplication.

Following Ringel [23], we define a bilinear form \( \varphi : \mathcal{H}_-^+(\mathcal{A}) \times \mathcal{H}_-^-(\mathcal{A}) \rightarrow \mathbb{C} \) by

\[
\varphi(K_\mu u_{\alpha}^+, K_\nu u_{\beta}^-) = v_{-\langle \mu, \nu \rangle}^-(\alpha, \nu) + (\mu, \beta) \frac{1}{a_{\alpha}} \delta_{\alpha, \beta}.
\]

for any \( \mu, \nu \in K_0(\mathcal{A}) \) and \( \alpha, \beta \in \mathcal{P} \).

The bilinear form defined above is a skew-Hopf pairing on \( \mathcal{H}_-^+(\mathcal{A}) \times \mathcal{H}_-^-(\mathcal{A}) \). It induces a Hopf algebra structure on \( \mathcal{H}_-^+(\mathcal{A}) \otimes \mathcal{H}_-^-(\mathcal{A}) \) as follows: Let \( \text{DH}(\mathcal{A}) \) is the free product of algebras \( \mathcal{H}_-^+(\mathcal{A}) \) and \( \mathcal{H}_-^-(\mathcal{A}) \) subject to the following relations

\[
\sum_{i,j} a_{i}^{(1)}b_{j}^{(2)}\varphi(b_{j}^{(1)}, a_{i}^{(2)}) = \sum_{i,j} b_{j}^{(1)}a_{i}^{(2)}\varphi(b_{j}^{(2)}, a_{i}^{(1)}).
\]
where $\Delta(u_i) = \sum_j a_i^{(1)} \otimes a_i^{(2)}$, $\Delta(u_j) = \sum_j b_j^{(1)} \otimes b_j^{(2)}$.

The double Hall algebra of the category $\mathcal{A}$ is defined to be the quotient of the algebra $\mathcal{D}H(\mathcal{A})$ factoring out the ideal generated by $\{ K_\alpha \otimes 1 - 1 \otimes K_{-\alpha} : \alpha \in K_0(\mathcal{A}) \}$. This algebra is called the reduced Drinfeld double (of the pairing $(\mathcal{H}^+(\mathcal{A}), \mathcal{H}^-(\mathcal{A}), \varphi)$, denoted by $\mathcal{D}H(\mathcal{A})$). For the details, see [30].

The double Hall algebra has a triangular decomposition of the form

$$\mathcal{D}H(\mathcal{A}) = \mathcal{H}^+(\mathcal{A}) \otimes \mathcal{T} \otimes \mathcal{H}^-(\mathcal{A}),$$

where $\mathcal{T}$ is the subalgebra of $\mathcal{D}H(\mathcal{A})$ generated by $\{ K_\mu : \mu \in K_0(\mathcal{A}) \}$, and $\mathcal{H}^+(\mathcal{A})$ (resp. $\mathcal{H}^-(\mathcal{A})$) is the subalgebra of $\mathcal{H}^+(\mathcal{A})$ (resp. $\mathcal{H}^-(\mathcal{A})$) generated by $\{ u_\alpha^+ : \alpha \in \mathcal{P} \}$ (resp. $\{ u_\alpha^- : \alpha \in \mathcal{P} \}$). Also it is easy to see that

$$\mathcal{H}^-(\mathcal{A}) \simeq \mathcal{T} \otimes \mathcal{H}^+(\mathcal{A}), \mathcal{H}^+(\mathcal{A}) \simeq \mathcal{H}^+(\mathcal{A}) \otimes \mathcal{T}.$$  

2.5. **The Hall algebra of $\text{mod}kQ$.** Let $Q$ be a quiver, $I$ be the set of vertices of $Q$. Denote by $\text{mod}kQ$ the category of finite dimensional nilpotent representations of $Q$ over the field $k$. $\text{mod}kQ$ is a hereditary category. Thus we can form the Hall algebra $\mathcal{H}(\text{mod}kQ)$ and the extended Hall algebra $\mathcal{H}(\text{mod}kQ)$. The composition subalgebra $\mathcal{C}(\text{mod}kQ)$ is defined to be the subalgebra of $\mathcal{H}(\text{mod}kQ)$ generated by $u_{S_i}$ ($i \in I$), where $S_i$ is the simple module in $\text{mod}kQ$ corresponding to the vertex $i$. The composition subalgebra provide a realization of the positive part of the quantum group (see the next section):

**Theorem 2.1** ([22], [10]). Let $\mathfrak{g}$ be the Kac-Moody algebra whose corresponding Dynkin diagram is the underlying graph of $Q$. Then we have

$$\mathcal{C}(\text{mod}kQ) \simeq \mathcal{U}_v^+(\mathfrak{g})$$

where $u_{S_i}$ is just sent to the Chevalley generator $E_i$ for each $i$.

Furthermore, $\mathcal{C}(\text{mod}kQ) \otimes \mathcal{T}$ is a Hopf subalgebra of $\mathcal{H}(\text{mod}kQ)$. So we can construct the reduced Drinfeld double $\mathcal{D}C(\text{mod}kQ)$, which is a subalgebra of $\mathcal{D}H(\text{mod}kQ)$. By this construction the Ringel-Green theorem is extended to the whole quantum group.

**Theorem 2.2** ([30]). The double composition algebra is isomorphic to the quantum group

$$\mathcal{D}C(\text{mod}kQ) \simeq \mathcal{U}_v(\mathfrak{g}),$$

where $u_{S_i}^+ \mapsto E_i$, $u_{S_i}^- \mapsto -v^{-1} F_i$, $K_i \mapsto K_i$.

3. **Drinfeld’s presentation of quantum loop algebras**

Throughout this section $v$ will denote an indeterminate.

3.1. **Kac-Moody algebra.** Let $\mathcal{I}$ be a finite set, $C = (c_{ij})_{i,j \in \mathcal{I}}$ be a generalized Cartan matrix. Let $\mathfrak{g} = \mathfrak{g}(C)$ be the Kac-Moody algebra associated to $C$, which is the complex Lie algebra generated by $\{ e_i, f_i, h_i : i \in \mathcal{I} \}$ with relations

$$[h_i, h_j] = 0, \forall i, j \in \mathcal{I};$$

$$[h_i, e_j] = c_{ij} e_j, [h_i, f_j] = -c_{ij} f_j, \forall i, j \in \mathcal{I};$$

$$[e_i, f_j] = \delta_{ij} h_i, \forall i, j \in \mathcal{I};$$

$$(\text{ad} e_i)^{1-c_{ij}} e_i = 0, (\text{ad} f_i)^{1-c_{ij}} f_j = 0, \forall i \in \mathcal{I} \text{ and } j \in \mathcal{I} \text{ with } i \neq j.$$
The root system of \( \mathfrak{g} \) is denoted by \( \Delta \). The simple roots are denoted by \( \alpha_i, i \in I \). \( Q = \oplus_{i \in I} \mathbb{Z} \alpha_i \) is the root lattice. The \( \mathbb{Z} \)-module \( Q \) is equipped with the Cartan bilinear form determined by \( (\alpha_i, \alpha_j) = a_{ij} \).

3.2. Quantum groups. The Drinfeld-Jimbo quantum enveloping algebra \( U_q(\mathfrak{g}) \) of a Kac-Moody Lie algebra \( \mathfrak{g} \) is the \( \mathbb{C}(v) \)-algebra generated by \( \{ E_i, F_i : i \in I \} \) and \( \{ K_\mu : \mu \in \mathbb{Z} I \} \) with the following relations:

\[
K_0 = 1, \quad K_\mu K_\nu = K_{\mu + \nu}, \quad \forall \mu, \nu \in \mathbb{Z} I;
\]

\[
K_\nu E_i = v^{(\mu, i)} E_i K_\nu, \quad K_\mu F_i = v^{-(\mu, i)} F_i K_\mu, \quad \forall i \in I, \quad \mu \in \mathbb{Z} I;
\]

\[
E_i F_j - F_j E_i = \delta_{ij} \left( \frac{K_i - K_i^{-1}}{v - v^{-1}} \right), \quad \forall i, j \in I;
\]

\[
\sum_{p=0}^{1-c_{ij}} (-1)^p \left( \begin{array}{c}
1 - c_{ij} \\
p
\end{array} \right) E_i^p E_j F_i^{1-c_{ij}-p} = 0, \quad \forall i \neq j \in I;
\]

\[
\sum_{p=0}^{1-c_{ij}} (-1)^p \left( \begin{array}{c}
1 - c_{ij} \\
p
\end{array} \right) F_i^p F_j E_i^{1-c_{ij}-p} = 0, \quad \forall i \neq j \in I.
\]

The quantized enveloping algebra admits a natural triangular decomposition

\[
U_v(\mathfrak{g}) = U^-_v(\mathfrak{g}) \otimes U^0_v(\mathfrak{g}) \otimes U^+_v(\mathfrak{g}),
\]

where \( U^+_v(\mathfrak{g}) \) (resp. \( U^-_v(\mathfrak{g}), U^0_v(\mathfrak{g}) \)) is the subalgebra generated by \( E_i \) (resp. \( F_i, K_\mu \)).

3.3. The Loop algebra of \( \mathfrak{g} \). The loop algebra of \( \mathfrak{g} \), denoted by \( \mathcal{Lg} \), is defined to be the complex Lie algebra generated by \( \{ h_{i,k}, e_{i,k}, f_{i,k}, c : i \in I, k \in \mathbb{Z} \} \) subject to the following relations:

\[
[h_{i,k}, h_{j,l}] = k \delta_{k,-l} a_{ij} c,
\]

\[
[e_{i,k}, f_{j,l}] = \delta_{i,j} h_{i,k+l} + k \delta_{k,-l} c,
\]

\[
h_{i,k}, e_{j,l} = a_{ij} e_{j,l+k}, \quad h_{i,k}, f_{j,l} = -a_{ij} f_{j,l+k},
\]

\[
h_{i,k}, e_{k+l, j} = [e_{i,k}, e_{j,l+1}], \quad [f_{i,k+l}, f_{j,l}] = [f_{i,k}, f_{j,l+1}],
\]

\[
[e_{i,k}, e_{k+l}, \ldots, e_{i,k_n}, e_{j,l}, \ldots] = 0, \quad \text{for } n = 1 - a_{ij},
\]

\[
[f_{i,k}, f_{i,k}, \ldots, f_{i,k}, e_{j,l}] = 0, \quad \text{for } n = 1 - a_{ij}.
\]

It is clear that there is an embedding of Lie algebras \( \mathfrak{g} \hookrightarrow \mathcal{Lg} \) which sends \( e_i, f_i, h_i \) to \( e_{i,0}, f_{i,0}, h_{i,0} \) respectively.

Set \( \hat{Q} = Q \oplus \mathbb{Z} \delta \). We can extend the Cartan form to \( \hat{Q} \) by setting \( (\delta, \alpha) = 0 \) for all \( \alpha \in \hat{Q} \). \( \mathcal{Lg} \) is \( \hat{Q} \)-graded by setting \( \deg(e_{i,k}) = \alpha_i + k \delta \), \( \deg(f_{i,k}) = -\alpha_i + k \delta \), \( \deg(h_{i,k}) = k \delta \). Denote by \( \hat{\Delta} \) the root system of \( \mathcal{Lg} \). It is known that \( \hat{\Delta} = \mathbb{Z}^* \delta \cup \{ \Delta + \mathbb{Z} \delta \} \) (See [21]).

For each root \( \alpha \in \hat{Q} \), we call it real if \( (\alpha, \alpha) = 2 \), and imaginary if \( (\alpha, \alpha) \leq 0 \).

Remark. (1) If \( \mathfrak{g} \) is a complex simple Lie algebra, \( \mathcal{Lg} \) is isomorphic to \( \hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} c \), which is an affine Kac-Moody algebra. The isomorphism is given by \( e_{i,k} \mapsto e_{i,t_k}, f_{i,k} \mapsto f_{i,t_k}, h_{i,k} \mapsto h_{i,t_k} \) and \( c \mapsto c \) (See [8]). In fact, this is the motivation of the definition of \( \mathcal{Lg} \), see [25].

(2) In general, \( \mathcal{Lg} \) and \( \hat{\mathfrak{g}} \) are not Kac-Moody algebras. And the above assignment is only a surjective homomorphism \( \mathcal{Lg} \to \hat{\mathfrak{g}} \), whose kernel is described in [21].
3.4. Quantum loop algebras. The quantum loop algebra (with zero central charge) \( U_v(\mathfrak{g}) \) is the \( \mathbb{C}[v] \)-algebra generated by \( x_i^{\pm} = h_i, x_i^{\pm} K_i^\pm \) for \( i \in \mathbb{Z}, k \in \mathbb{Z}, \text{ and } l \in \mathbb{Z}^* \) subject to the following relations:

1. \( [K_i, K_j] = [K_i, h_j, l] = 0 \)
2. \( [h_i, h_j] = 0 \)
3. \( K_i x_j^{\pm} K_i^{-1} = v^{\pm a_{ij}} x_j^{\pm} \)
4. \( [h_i, x_j^{\pm}] = \frac{1}{l} [a_{ij}] x_j^{\pm} \)
5. \( x_{i,k+1} x_j^{\pm} - v^{\pm a_{ij}} x_j^{\pm} x_{i,k+1} = v^{\pm a_{ij}} x_{i,k} x_j^{\pm} - x_{j,l+1} x_{i,k} \)
6. \( [x_{i,k}, x_{j,l}] = \delta_{ij} (\psi_{i,k+l} - \varphi_{i,k+l}) \)

(7) \( \psi_{i,k} = \sum_{t=0}^n (-1)^t \left( \begin{array}{c} n \\ t \end{array} \right) x_{i,k_1}^{\pm} \cdots x_{i,k_t}^{\pm} x_{i,k_{t+1}}^{\pm} \cdots x_{i,k_n}^{\pm} = 0 \)

Remark. One can see that the definition of the quantum loop algebra is obviously motivated by Drinfeld’s new realization of quantum affine algebras, which asserts that when \( \mathfrak{g} \) is a complex simple Lie algebra, \( U_v(\mathfrak{g}) \) is isomorphic to the Drinfeld-Jimbo quantized enveloping algebra \( U_v(\mathfrak{g}) \) (See [7, 2]).

4. THE CATEGORY OF COHERENT SHEAVES ON WEIGHTED PROJECTIVE LINES

Now we introduce the category of coherent sheaves on weighted projective lines as studied in [3]. In this section \( k \) denote an arbitrary field.

4.1. Weighted projective lines. Let \( \mathbf{p} = (p_1, p_2, \cdots, p_n) \in \mathbb{N}^n \). Consider the \( \mathbb{Z} \)-module

\( L(p) = \mathbb{Z}\bar{x}_1 \oplus \mathbb{Z}\bar{x}_2 \oplus \cdots \oplus \mathbb{Z}\bar{x}_n / J \)

where \( J \) is the submodule generated by \( \{ p_1\bar{x}_1 - p_s\bar{x}_s | s = 2, \cdots, n \} \). Set \( \bar{c} = \bar{x}_1 = \cdots = \bar{x}_n \in L(p) \).

The polynomial ring \( k[X_1, \cdots, X_n] \) has a structure of \( L(p) \)-graded algebra by setting \( \deg X_i = \bar{x}_i \). We rewrite it by \( S(p) \).

Let \( \Delta = \{ \lambda_1, \cdots, \lambda_n \} \) be a collection of distinct closed points (of degree 1) on the projective line \( \mathbb{P}^1(k) \). Let \( \mathcal{I}(p, \Delta) \) be the \( L(p) \)-graded ideal of \( \mathbb{I}(p) \) generated by \( \{ X_1 - \lambda_s X_p | s = 3, \cdots, n \} \). Set \( S(p, \Delta) = S(p) / \mathcal{I}(p, \Delta) \), which is also an \( L(p) \)-graded algebra. Denote the image of \( X_i \) in \( S(p, \Delta) \) by \( X_i \), for any \( i \).

Let \( X(p, \Delta) = \text{Spec} \mathbb{R}(p, \Delta) \) be the set of all prime homogeneous ideals in \( S(p, \Delta) \). This is the so-called weighted projective line. \( (p, \Delta) \) is called the weight type of \( X(p, \Delta) \).

The number \( p_i \) is the weight of the point \( \lambda_i \).
In the following we will fix $p$ and $\Delta$, then we write $S = S(p, \Delta)$ and $X = X_{p, \Delta}$ for short.

4.2. Coherent sheaves on $X_{p, \Delta}$. For any homogeneous element $f \in S$, let $V_f = \{ p \in X | f \notin p \}$ and $D_f = X \setminus V_f$.

The structure sheaf $\mathcal{O}_X$ is defined to be the sheaf of $L(p)$-graded algebras on $X$ associated with the presheaf $D_f \mapsto S_f$, where $S_f = \{ g/\pi | g \in S, \pi \in \mathbb{N} \}$. We denote by $\mathcal{O}_X$-$\text{Mod}$ the category of sheaves of $L(p)$-graded $\mathcal{O}_X$-modules on $X$.

For any $\bar{x} \in L(p)$ and any $L(p)$-graded $\mathcal{O}_X$-module $\mathcal{M}$, we denote by $\mathcal{M}(\bar{x})$ the shift of $\mathcal{M}$ by $\bar{x}$ (i.e., $\mathcal{M}(\bar{x})|_{\bar{y}} = \mathcal{M}(|\bar{x} + \bar{y})$). A sheaf $\mathcal{M}$ of $L(p)$-graded $\mathcal{O}_X$-modules is called coherent if there exists an open covering $\{ U_i \}$ of $X$ and for each $i$ an exact sequence

$$\bigoplus_{s=1}^{N} \mathcal{O}_X(I_s)|_{U_i} \to \bigoplus_{t=1}^{M} \mathcal{O}_X(K_t)|_{U_i} \to \mathcal{M}|_{U_i} \to 0.$$  

The category of coherent sheaves on $X$, denoted by Coh$(X)$, is a full subcategory of $\mathcal{O}_X$-$\text{Mod}$. In [9] it is proved that Coh$(X)$ is a $k$-linear hereditary, Hom- and Ext-finite category.

4.3. The structure of the category Coh$(X)$. In this subsection we give a description of the structure of the category Coh$(X)$.

Let $\mathcal{F}$ be the full subcategory of Coh$(X)$ consisting of all locally free sheaves, and $\mathcal{T}$ be the full subcategory consisting of all torsion sheaves. These two subcategories are both extension-closed. Moreover, $\mathcal{T}$ is again hereditary and a length category, while the objects in $\mathcal{F}$ have no simple subobjects. The following lemma is proved in [9]:

**Lemma 4.1.** (1) For any sheaf $\mathcal{M} \in \text{Coh}(X)$, it can be decomposed as $\mathcal{M} = \mathcal{M}_t \oplus \mathcal{M}_f$ where $\mathcal{M}_t \in \mathcal{F}$ and $\mathcal{M}_f \in \mathcal{T}$.

(2) $\text{Hom}(\mathcal{M}_t, \mathcal{M}_f) = \text{Ext}^1(\mathcal{M}_t, \mathcal{M}_f) = 0$, for any $\mathcal{M}_t \in \mathcal{F}$ and $\mathcal{M}_f \in \mathcal{T}$.

To describe $\mathcal{F}$ more precisely, we need a classification of the closed points on $X$. Recall that $X = \text{Spec } \text{gr} S(p, \Delta)$. According to [9], each $\lambda_i$ corresponds to the prime ideal generated by $x_i$ and is called an Exceptional point. All other homogeneous primes are given by a prime homogeneous polynomial $F(x_1^{P_1}, x_2^{P_2}) = k[T_1, T_2]$, which are called ordinary points.

Let $p = \text{lcm}(p_1, \cdots, p_n)$. The degree of a closed point is defined as $\text{deg}(\lambda) = p/p_i$ for any $i$ and $\text{deg}(x) = pd$ for any ordinary point $x$ corresponding to a prime homogeneous polynomial of degree $d$.

Let $C_r$ be the cyclic quiver with $r$ vertices. More precisely, for $r = 1$, $C_1$ is just the quiver with only one vertex and one loop arrow. For $r \geq 2$, the vertices of $C_r$ are indexed by $\mathbb{Z}/r\mathbb{Z}$ and the arrows are from $i$ to $i-1$ for each $i$. Denote by $\text{rep}_0(C_r)_k$ the category of nilpotent representations of $C_r$ over the field $k$. The following lemma, due to [9], describes the structure of the subcategory $\mathcal{T}$.

**Lemma 4.2.** (1) The category $\mathcal{T}$ decomposes as a coproduct $\mathcal{T} = \coprod_{x \in X} \mathcal{T}_x$, where $\mathcal{T}_x$ is the subcategory of torsion sheaves with support at the closed point $x \in X$.

(2) For any ordinary closed point $x$ of degree $d$, let $k_x$ denote the residue field at $x$. Then $\mathcal{T}_x$ is equivalent to the category $\text{rep}_0(C_r)_{k_x}$.

(3) For any exceptional closed point $\lambda_i$ ($1 \leq i \leq n$), the category $\mathcal{T}_{\lambda_i}$ is equivalent to the category $\text{rep}_0(C_{p_i})_{k}$.
4.4. Indecomposable objects in $\mathcal{T}$. We first give a description of the simple objects.

For any ordinary point $x$, let $\pi_x$ denote the prime homogeneous polynomial corresponding to $x$. Then multiplication by $\pi_x$ gives an exact sequence

$$0 \to \mathcal{O}_x \to \mathcal{O}_x (d\vec{c}) \to S_x \to 0.$$ 

$S_x$ is the unique (up to isomorphism) simple sheaf in the category $\mathcal{T}_x$. Moreover, for any $\vec{k} \in L(p)$ we have $S_x(\vec{k}) = S_x$.

For any exceptional point $\lambda_i$, multiplication by $x_i$ leads to exact sequences

$$0 \to \mathcal{O}_x ((j - 1)x_i) \to \mathcal{O}_x (jx_i) \to S_j^i \to 0,$$

for each $j$, $1 \leq j \leq p_i$. And $\{S_j^i | 1 \leq j \leq p_i\}$ is the complete set of non-isomorphic simple sheaves in the category $\mathcal{T}_{\lambda_i}$, for any $i$ ($1 \leq i \leq n$). Moreover, for any $\vec{k} = \sum k_i x_i$ we have $S_j^i(\vec{k}) = S_j^i$. Now we turn to the indecomposable objects. Recall the following well-known results on representation theory of cyclic quivers:

(1). In the category $\text{rep}_0(C_1)$, the set of isomorphic classes of indecomposables is $\{S(a) | a \in \mathbb{N}\}$. $S = S(1)$ is the only simple representation, and $S(a)$ is the unique indecomposable representation of length $a$.

In particular, for any partition $\mu = (\mu_1 \geq \cdots \geq \mu_r)$, set $S(\mu) = \bigoplus_i S(\mu_i)$. Then any nilpotent representation of $C_1$ is isomorphic to $S(\mu)$ for some $\mu$.

(2). In the category $\text{rep}_0(C_r)$, the set of isomorphic classes of indecomposables is $\{S_j(a) | 1 \leq j \leq r, a \in \mathbb{N}\}$. There are $r$ non-isomorphic simple representations $S_j = S_j(1)$ ($1 \leq j \leq r$). And $S_j(a)$ is the unique indecomposable representation with top $S_j$ and length $a$.

The above results, combined with lemma 4.2, tell us all the indecomposable objects in $\mathcal{T}$. We denote by $S_x(a)$ the unique (up to isomorphism) indecomposable object of length $a$ in $\mathcal{T}_x$ for any ordinary point $x$. And for any exceptional point $\lambda_i$, the indecomposable objects in $\mathcal{T}_{\lambda_i}$ are denoted by $S_j^i(a)$ ($1 \leq j \leq p_i, a \in \mathbb{N}$).

4.5. The Grothendieck group and the Euler form. The following proposition (see [9]) gives an explicit description of the Grothendieck group of $\text{Coh}(\mathcal{X})$.

**Proposition 4.3.**

$$K_0(\text{Coh}(\mathcal{X})) \cong (\mathbb{Z}[\mathcal{O}_X] \oplus \mathbb{Z}[\mathcal{O}_X (\vec{c})] \oplus \bigoplus_{i,j} \mathbb{Z}[S_j^i]) / I$$

where $I$ is the subgroup generated by $\{\sum_j [S_j^i] + [\mathcal{O}_X] - [\mathcal{O}_X (\vec{c})] | i = 1, \cdots, n\}$.

In the following we simply write $\mathcal{O}$ for $\mathcal{O}_X$. Set

$$\delta = [\mathcal{O}(\vec{c})] - [\mathcal{O}] = \sum_{j=0}^{p_i-1} [S_j^i], \text{ for } i = 1, \cdots, n.$$ 

To calculate the Euler form on $K_0(\text{Coh}(\mathcal{X}))$, we have the following lemma (see [29]):
Lemma 4.4. The Euler form $\langle -, - \rangle$ is given by
\[ \langle [O], [O] \rangle = 1, \quad \langle [O], \delta \rangle = 1, \quad \langle \delta, [O] \rangle = -1, \]
\[ \langle \delta, \delta \rangle = 0, \quad \langle \delta, [S^i_j] \rangle = 0, \quad \langle [S^i_j], \delta \rangle = 0, \]
\[ \langle [O], [S^i_j] \rangle = \begin{cases} 1 & \text{if } j = p_i \\ 0 & \text{if } j \neq p_i \end{cases} \]
\[ \langle [S^i_j], [O] \rangle = \begin{cases} -1 & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \end{cases} \]
\[ \langle [S^i_j], [S^i'_j] \rangle = \begin{cases} 1 & \text{if } i = i', j = j' \\ -1 & \text{if } i = i', j \equiv j' + 1 \pmod{p_i} \\ 0 & \text{otherwise} \end{cases} \]

5. Main result

5.1. In this section we fix a finite field $k = \mathbb{F}_q$ and set $v = \sqrt{q}$. And we also fix a weight type (see 4.1)
\[ \mathbf{p} = (p_1, \ldots, p_n), \quad \Lambda = \{\lambda_1, \ldots, \lambda_n\} \]
Consider the weighted projective line $X = X_{\mathbf{p}, \Lambda}$ and the category of coherent sheaves $\text{Coh}(X)$. We keep the notions in the last section.
Since $\text{Coh}(X)$ is a $k$-linear hereditary, Hom- and Ext-finite abelian category, we can associate to it the Hall algebra $H(\text{Coh}(X))$ and the double Hall algebra $DH(\text{Coh}(X))$, as in section 2.2 and 2.4.

Note that the subcategories $\mathcal{T}_x$, for any closed point $x$ on $X$, are hereditary length categories. Thus we can define the Hall algebra associated to $\mathcal{T}_x$, denoted by $H(\mathcal{T}_x)$. It is obvious that each $H(\mathcal{T}_x)$ is a subalgebra of $H(\text{Coh}(X))$.

5.2. In this and the next subsection we define some “special” elements $T_r$ for any $r \in \mathbb{N}$ in $H(\mathcal{T}_x)$, following [29].

For an ordinary closed point $x$ of degree $d$, we know that $\mathcal{T}_x$ is equivalent to the category of nilpotent representations of the quiver $C_1$ over $k_x$ (see lemma 4.2). Now $k = \mathbb{F}_q$ hence $k_x = \mathbb{F}_{q^d}$. Thus we have the isomorphism
\[ \Theta_x : H(C_1)_{\mathbb{F}_{q^d}} \rightarrow H(\mathcal{T}_x), \]
where $H(C_1)_{\mathbb{F}_{q^d}}$ is the Hall algebra associated to the the category $\text{rep}_0(C_1)_{\mathbb{F}_{q^d}}$.

Let $\Lambda = \mathbb{C}[z_1, z_2, \ldots]^{S\infty}$ be Macdonald’s ring of symmetric functions (see [19]). The following result is well-known, due to P. Hall.

**Theorem 5.1.** The map $\varphi_1 : \Lambda \rightarrow H(C_1)_{\mathbb{F}_q}$ sending the elementary symmetric function $e_r$ to $v^{r(r-1)}S_x(1^r)$ is an algebra isomorphism.

Let $p_r$ denote the $r$-th power-sum symmetric function and set $h_r = \frac{r}{r} \varphi_1(p_r)$.
Then
\[ h_r = \left[ \frac{r}{r} \right] \sum_{|\mu|=r} n(l(\mu) - 1)S_x(\mu), \]
where $n(l) = \prod_{i=1}^l (1 - v^{2i})$ (see [19]).
For each $r \in \mathbb{N}$, set
\[ h_{r,x} = \begin{cases} 
0 & \text{if } r \nmid d \\
\Theta_x(h_{r/d}) & \text{if } r \mid d 
\end{cases} \]

5.3. For any exceptional point $\lambda_i$, $\mathcal{F}_{\lambda_i}$ is equivalent to the category $\text{rep}_0(C_{p_i})$. And we have the isomorphism
\[ \Theta_{\lambda_i} : \mathbf{H}(C_{p_i}) \to \mathbf{H}(\mathcal{F}_{\lambda_i}). \]

On the other hand, for any positive integer $n$, there is a natural fully faithful functor $\iota_n : \text{rep}_0(C_n) \to \text{rep}_0(C_{n+1})$ whose image is the full subcategory with objects $X$ such that $\text{Hom}(X, S_n) = \text{Hom}(S_{n+1}, X) = 0$.

Then clearly, there is an embedding of algebras $\mathbf{H}(C_n) \to \mathbf{H}(C_{n+1})$ sending
\[ u_S \mapsto \begin{cases} 
u & \text{for } 1 \leq i < n \\
u u_{S_{n+1}} u_S - u_S u_{S_{n+1}} u_S & \text{for } i = n \end{cases} \]

Hence for any $m$, the composition of functors $\iota_1 \circ \iota_2 \cdots \circ \iota_{m-1}$ is an embedding $\text{rep}_0(C_1) \hookrightarrow \text{rep}_0(C_m)$, which induces an embedding of algebras
\[ \Psi : \mathbf{H}(C_1) \to \mathbf{H}(C_m). \]

Set
\[ h_{r,\lambda_i} = \Theta_{\lambda_i} \circ \Psi(h_r). \]

Finally we define
\[ T_r = \sum_{x \in X} h_{r,x} \in \mathbf{H}(\text{Coh}(X)). \]
where the sum is taken over all closed points $x$ on $X$.

5.4. We need more notations in the Hall algebra of $\text{rep}_0(C_m)$, see [11]. For any $1 \leq l \leq m$, let $\mathcal{M}_{l,\alpha}$ be the set of all isomorphism classes of modules $M$ in $\text{rep}_0(C_m)$ such that $\dim M = \alpha$ and $\text{soc}(M) \subseteq S_1 \oplus \cdots \oplus S_l$.

Let $\delta_m$ be the sum of all dimension vectors of simple modules. For any $r \in \mathbb{N}$, set
\[ c_{l,r} = (-1)^{r} v^{-2lr} \sum_{M \in \mathcal{M}_{l,\delta_m}} (-1)^{\dim(M)} a_M u_M \in \mathbf{H}(C_m) \]

Then define $p_{l,r} \in \mathcal{H}(C_m)$ via the following generating function
\[ \sum_{r \geq 1} (1 - v^{-2lr}) p_{l,r} T^{r-1} = \frac{d}{dT} \log C_l(T) \]
where $C_l(T) = 1 + \sum_{r \geq 1} c_{l,r} T^r$.

Set
\[ \pi_{l,r} := \frac{[l,r]}{r} p_{l,r} \]

Remark. It is proved in [12] that there is an algebra isomorphism $\mathbf{H}(C_m) \cong U^+_v(\mathfrak{sl}_m) \otimes \mathcal{Z}$ (where $v$ specializes to $\sqrt{q}$), where $\mathcal{Z} = \mathbb{C}[p_{m,1}, p_{m,2}, \ldots]$ is a central subalgebra of $\mathbf{H}(C_m)$ and the element $p_{m,r}$ is homogeneous of degree $r\delta_m$. 
Now for any exceptional point $\lambda_i$ on $X$ we define
\[ h_{[i,j],k} = \Theta_{\lambda_i}(\pi_{j+1,k} - (v^k + v^{-k})\pi_{j,k} + \pi_{j-1,k}) \in H(\mathcal{G}_{\lambda_i}) \subset H(\text{Coh}(X)). \]

for any $1 \leq j \leq p_i - 1$ and $k \in \mathbb{N}$.

In the following we also write $\pi_{i,j,k} = \Theta_{\lambda_i}(\pi_{j,k})$, thus
\[ h_{[i,j],k} = \pi_{i,j+1,k} - (v^k + v^{-k})\pi_{i,j,k} + \pi_{i,j-1,k}. \]

5.5. Associate to the weight type $(\mathbf{p}, \lambda)$, we have a star-shaped graph $\Gamma$:

![Graph](image)

As marked in the graph, the central vertex is denoted by $\ast$. There are $n$ branches and in each branch there are $p_i - 1$ ($1 \leq i \leq n$) vertices respectively. For the $j$th vertex in the $i$th branch we denote it by $[i,j]$.

Consider $\mathfrak{g} = \mathfrak{g}(\Gamma)$ the Kac-Moody algebra associated to the graph $\Gamma$. As in 3.3 and 3.4, we have the loop algebra $L\mathfrak{g}$ and its quantized enveloping algebra $U^\hbar(L\mathfrak{g})$.

The root systems of $\mathfrak{g}$ and $L\mathfrak{g}$ are denoted by $\Delta$ and $\hat{\Delta}$ respectively. In view of the graph $\Gamma$, the simple roots in $\Delta$ are denoted by $\alpha_\ast$ and $\alpha_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq p_i - 1$. We also know that $\hat{\Delta} = \mathbb{Z}\delta \cup \{\Delta + \mathbb{Z}\delta\}$.

From lemma 4.3 there is a natural identification of $\mathbb{Z}$-modules $K_0(\text{Coh}(X)) \cong \hat{Q}$ given by
\[
[S_j^+] \mapsto \alpha_{ij}, \text{ for } j = 1, \ldots, p_i - 1 \\
[S_0^+] \mapsto \delta - \sum_{j=1}^{p_i-1} \alpha_{ij} \\
\Theta(k\vec{c}) \mapsto \alpha_\ast + k\delta.
\]

Now by lemma 4.4 we have the following

**Lemma 5.2.** The symmetric Euler form on $K_0(\text{Coh}(X))$ coincides with the Cartan form on $\hat{Q}$.

5.6. In [29] (also see [25]) Schiffmann has proved that the Hall algebra $H(\text{Coh}(X))$ provide a realization of the quantized enveloping algebra of a certain nilpotent subalgebra of $L\mathfrak{g}$, denoted by $U^\hbar(\hat{\mathfrak{n}})$.

Let us recall the definition of $U^\hbar(\hat{\mathfrak{n}})$. First, for each $i$, we denote by $U_i$ the subalgebra of $U^\hbar(L\mathfrak{g})$ generated by $x^\pm_{[i,j],k}$, $h_{[i,j],l}$ and $K^\pm_{[i,j]}$ ($1 \leq j \leq p_i - 1$, $k \in \mathbb{Z}$, $l \in \mathbb{Z} \setminus \{0\}$). It is obvious that this subalgebra is isomorphic to $U^\hbar(\mathfrak{sl}_{p_i})$. Denote
the Chevalley generators of $U_v(\mathfrak{sl}_n)$ by $E^+_j$ and $F^+_j$ ($1 \leq j \leq p_i$). Then by Beck’s isomorphism $x^{+}_{[i,j],0}$ corresponds to $E^+_j$ for $1 \leq j \leq p_i - 1$. And the one in $U_x$ corresponding to $E^+_{p_i}$ is not in the set of Drinfeld generators, which we denote by $\varepsilon_i$. Now let $U^+_i$ be the subalgebra generated by $x^{+}_{[k,i],0}$ and $\varepsilon_i$. Thus $U^+_i$ is isomorphic to the standard positive part of $U_v(\mathfrak{sl}_n)$. Finally, $U_v(\mathfrak{h})$ is defined to be the subalgebra of $U_v(\mathcal{L}\mathfrak{g})$ generated by $x^{+}_{[k,i],0}$ and $\varepsilon_i$ for $1 \leq i \leq n$, $1 \leq j \leq p_i$ and $r \in \mathbb{N}$. The following theorem was proved by Schiffmann:

**Theorem 5.3** ([29]). The assignment $x^{+}_{[i,j],0} \mapsto u_{S_i}$ for $1 \leq j \leq p_i - 1$, $\varepsilon_i \mapsto u_{S_{p_i}}$, $x_{+,k} \mapsto u_{\mathcal{E}(k\mathfrak{g})}$, $h_{+,r} \mapsto T_r$, extends to an epimorphism of algebras

$$\Phi : U_v(\mathfrak{h}) \rightarrow \mathbf{C}(\text{Coh}(\mathfrak{X}))$$

Moreover, if $\mathfrak{g}$ is of finite or affine type, $\Phi$ is an isomorphism.

One can see that in the above theorem, the correspondence between generators of $\mathbf{C}(\text{Coh}(\mathfrak{X}))$ and Drinfeld generators of $U_v(\mathcal{L}\mathfrak{g})$ is not completely explicit. Furthermore, we define $\mathbf{D} \mathbf{C}(\text{Coh}(\mathfrak{X}))$ to be the subalgebra of $\mathbf{D} \mathbf{H}(\text{Coh}(\mathfrak{X}))$ generated by $\mathbf{C}(\text{Coh}(\mathfrak{X}))$, $\mathbf{C}^{−}(\text{Coh}(\mathfrak{X}))$ (the corresponding subalgebra in $\mathbf{H}^{−}(\text{Coh}(\mathfrak{X}))$) and the torus $\mathbf{T}$. And we have the following triangular decomposition:

$$\mathbf{D} \mathbf{C}(\text{Coh}(\mathfrak{X})) \simeq \mathbf{C}(\text{Coh}(\mathfrak{X})) \otimes \mathbf{T} \otimes \mathbf{C}^{−}(\text{Coh}(\mathfrak{X}))$$

which can be seen from the relations proved in section 8.

Not surprisingly, we will prove that this construction provide a realization of the whole quantum loop algebra.

### 5.7. We keep the notations in the previous subsections. Note that in 5.3 and 5.4 we have defined the elements $T_k$, $\pi_{j,k}$ in the Hall algebra $\mathbf{H}(\text{Coh}(\mathfrak{X}))$ for any $1 \leq i \leq n$, $1 \leq j \leq p_i - 1$ and $k \geq 1$.

Similarly we can define the elements $T^{-}_k$, $\pi^{-}_{j,k}$ in the negative Hall algebra $\mathbf{H}^{−}(\text{Coh}(\mathfrak{X}))$. Note that to define $\pi^{-}_{j,k}$, we first need to consider the negative hall algebra of cyclic quivers $\mathbf{H}^{−}(C_{p_i})$. Similar to $\Theta_{\lambda_i}$, we have the isomorphism

$$\Theta_{\lambda_i} : \mathbf{H}^{−}(C_{p_i}) \simeq \mathbf{H}^{−}(\mathcal{R}_{\lambda_i})$$

Actually $\pi^{-}_{j,k}$ is in $\mathbf{H}^{−}(C_{p_i})$. By abuse of language, we still write $\pi^{-}_{j,k}$ as for $\Theta_{\lambda_i}(\pi^{-}_{j,k})$.

Moreover, we define

$$\eta^+_{i,j} = v^{1−i} \Theta_{\lambda_i} \left( \sum_{M \in \mathcal{M}_{i+1,k−e}^{+}} (1−v^2)^{\dim \text{End}(M)−1} u^+_M K_{[i,j]} \right),$$

$$\eta^-_{i,j} = −v^{−i} \Theta_{\lambda_i} \left( \sum_{M \in \mathcal{M}_{i+1,k−e}^{−}} (1−v^2)^{\dim \text{End}(M)−1} u^-_M K_{[i,j]} \right).$$

**Theorem 5.4.** For any star-shaped graph $\Gamma$, let $\mathfrak{g}$ be the Kac-Moody algebra and $\mathfrak{X}$ be the weighted projective line associate to $\Gamma$ respectively. Then the following elements in the double Hall algebra $\mathbf{D} \mathbf{H}(\text{Coh}(\mathfrak{X}))$ satisfy the defining relations of $U_v(\mathcal{L}\mathfrak{g})$ (see 3.4). 

Corollary 5.5. When \( g \) is of finite or affine type, the above assignment extends to an isomorphism
\[
U_v(Lg) \simeq \text{DC}(\text{Coh}(\mathbb{X})).
\]

6. Relations in the subalgebras isomorphic to \( U_v(\mathfrak{sl}_p) \)

6.1. Relations in each tube \( \mathcal{T}_{\lambda} \). Recall the star-shaped graph \( \Gamma \) in [5.5]. We can see that for any fixed \( i \in \{1, 2, \ldots, n\} \), the full subgraph consisting of vertices \( \{[i, j] | 1 \leq j \leq p_i - 1\} \) is a Dynkin diagram of type \( A_{p_i - 1} \). Thus the relations to be satisfied by the elements \( x_{[i, j], t}^+, h_{[i, j], r}^+ \) for all \( 1 \leq j \leq p_i - 1, k, l, r \in \mathbb{Z} \), are actually the defining relations of \( U_v(\mathfrak{sl}_p) \). We will prove them in this section.

Note that by definition the elements \( x_{[i, j], k}^+, x_{[i, j], t}^-, h_{[i, j], r}^- \) for \( 1 \leq j \leq p_i - 1, k, l, r \in \mathbb{N} \) are all in the subalgebra \( \mathbf{H}(\mathcal{T}_{\lambda}) \), which is isomorphic to \( \mathbf{H}(C_{p_i}) \). Thus we can use the method developed by Hubery in [11], where he explicitly write down the elements in the Hall algebra of nilpotent representations of cyclic quiver \( C_m \) satisfying Drinfeld relations of \( U_v^+(\mathfrak{sl}_m) \). Then by the isomorphism \( \Theta_{\lambda} \), we can transfer the result to \( \mathbf{H}(\mathcal{T}_{\lambda}) \subset \mathbf{H}(\text{Coh}(\mathbb{X})) \). Namely we have:

Proposition 6.1 ([11]). For any fixed \( i \in \{1, 2, \cdots, n\} \), the elements \( x_{[i, j], k}^+, x_{[i, j], t}^-, h_{[i, j], r}^- \) for \( 1 \leq j \leq p_i - 1, k, l, r \in \mathbb{N} \) satisfy the Drinfeld relations of \( U_v^+(\mathfrak{sl}_p) \).

This result can be easily extended to the whole \( U_v(\mathfrak{sl}_p) \):

Corollary 6.2. For any fixed \( i \in \{1, 2, \cdots, n\} \), the elements \( x_{[i, j], k}^+, h_{[i, j], r}^- \) for all \( 1 \leq j \leq p_i - 1, k \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\} \) satisfy the Drinfeld relations of \( U_v(\mathfrak{sl}_p) \).

The proof will be given in 6.3.
6. Beck’s isomorphism for $U_c(\hat{\mathfrak{sl}}_m)$. In this subsection we briefly recall Beck’s isomorphism in [2].

Let $W$ be the Weyl group of the corresponding finite type Lie algebra $\mathfrak{sl}_m$. $W$ has the simple reflections $s_1, \ldots, s_{m-1}$ as generators. Let $P$ denote the weight lattice and $Q$ denote the root lattice. The fundamental weights are denoted by $\omega_1, \ldots, \omega_{m-1}$.

The extended affine Weyl group is defined to be the semi-direct product $\hat{W} = P \rtimes W$, where the product is given by

$$(x, \omega)(x', \omega') = (x + \omega(x'), \omega\omega').$$

And the affine Weyl group corresponding to $\hat{\mathfrak{sl}}_m$ is the subgroup $\hat{W} = Q \rtimes W$. We have the decomposition $\hat{W} = W \times \mathbb{Z}/m\mathbb{Z}$, where the cyclic group $\mathbb{Z}/m\mathbb{Z}$ has a generator $\tau := (\omega_1, s_1s_2 \cdots s_{m-1})$.

Set $s_m := (\omega_1 + \omega_{m-1}, s_1s_2 \cdots s_{m-1} \cdots s_2s_1)$. Then $\{s_1, s_2, \ldots, s_m\}$ is a set of generators of the affine Weyl group $\hat{W}$. We can extend the length function on $\hat{W}$ to $W$ by setting $l(\tau) = 0$.

We note that the fundamental weights can be considered as elements in $\hat{W}$ and have the following reduced expressions in terms of the generators $s_i$ and $\tau$:

$$\omega_i = \tau^i(s_{m-i} \cdots s_{m-1}) \cdots (s_2 \cdots s_{i+1})(s_1 \cdots s_i).$$

The Braid group associated to $\hat{W}$ is the group on generators $T_\omega(\omega \in \hat{W})$ with the relations $T_\omega T_{\omega'} = T_{\omega + \omega'}$ if $l(\omega) + l(\omega') = l(\omega\omega')$. Following Lusztig, it acts on the Drinfeld-Jimbo presentation of $U_c(\hat{\mathfrak{sl}}_m)$ (see [3.2]) via

$$T_i(E_i) = -F_i K_i, T_i(F_i) = -K_i^{-1} E_i, T_i(K) = K s_i(\alpha),$$

$$T_i(E_j) = \sum_{r+s=-c_{ij}} (-1)^r v^{-r} E_i^s E_j^r, \text{ for } i \neq j,$$

$$T_i(F_j) = \sum_{r+s=-c_{ij}} (-1)^r v^r F_i^s F_j^r, \text{ for } i \neq j,$$

$$T_i(K) = K_{i+1}, T_r(E_i) = E_i+1, T_r(F_i) = F_i+1,$$

where the divided powers $E_i^r := E_i^r/\lfloor r! \rfloor$, $F_i^r := F_i^r/\lfloor r! \rfloor$ and $(c_{ij})$ is the Cartan matrix corresponding to $\hat{\mathfrak{sl}}_m$.

For $1 \leq i \leq m-1$, $k \in \mathbb{Z}$ let

$$x_{i,j}^- = (-1)^{ij} v^{mj} T_{\omega_i}(F_i), \quad x_{i,j}^+ = (-1)^{ij} v^{mj} T_{\omega_i}(E_i).$$

For $1 \leq i \leq m-1$, $k > 0$, define $h_i,k$ via the following generating functions

$$K_i \exp((v - v^{-1}) \sum_{k>0} h_{i,k} u^k) = \sum_{l \geq 0} \psi_{i,l} u^l$$

where $\psi_{i,0} = K_i$.

Similarly, for $1 \leq i \leq n-1$, $k < 0$, define $h_{i,k}$ via

$$K_i^{-1} \exp((v - v^{-1}) \sum_{k<0} h_{i,k} u^{-k}) = \sum_{l \geq 0} \psi_{i,l} u^{-l}$$

where $\varphi_{i,k} = (v - v^{-1}) [F_i, T_{\omega_i}(E_i)]$ and set $\varphi_{i,0} = K_i^{-1}$.

The following is now well-known, see [2] for the proof.

**Theorem 6.3.** $U_c(\hat{\mathfrak{sl}}_m)$ is generated by the elements $x_{i,j}^\pm$, $h_i,k$, $K_i^{\pm 1}$, where $1 \leq i \leq m-1$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}\setminus\{0\}$. The defining relations are Drinfeld relations for
6.3. The proof of Prop 6.1 and Cor 6.2 Proposition 6.1 is a result of Hubery [11]. And corollary 6.2 follows easily by a similar method.

Now we recall the arguments in [11], which we refer to for all the details. Let $C_m$ be the cyclic quiver of $m$ vertices and consider the Hall algebra $H(C_m)$. The composition subalgebra of $H(C_m)$, denoted by $C(C_m)$, is the subalgebra generated by $u_{S_i}$ for all $1 \leq i \leq m$. We know that the composition subalgebra $C(C_m)$ is isomorphic to $U_{\mathfrak{sl}}^+(\mathfrak{sl}_m)$ (see theorem 2.1) and the reduced Drinfeld double $\mathcal{DC}(C_m)$ is isomorphic to $U_v(\mathfrak{sl}_m)$ (see theorem 2.2). Moreover the isomorphism is given by

$$
u_{S_i} \mapsto E_i, \quad -\nu u_{S_i} \mapsto F_i.$$

We also know that $U_v(\mathfrak{sl}_m)$ is isomorphic to $U_v(\mathcal{L}\mathfrak{sl}_m)$. Let

$$\Upsilon : C(C_m) \simeq U_v(\mathcal{L}\mathfrak{sl}_m)$$

be the composition of the two isomorphisms mentioned above.

Now if we can find the inverse image of the elements $x_{i,j}^+, b_{i,k}$ under $\Upsilon$, they should certainly satisfy the Drinfeld relations.

Note that by theorem 6.3 we have

$$x_{i,1}^- = (-1)^i v^m T_{\omega_i}(F_1), \quad x_{i,-1}^+ = (-1)^{-i} v^{-m} T_{\omega_i}(E_i)$$

Recall that $\omega_i = \tau^i(s_{n-i} \cdots s_{n-1}) \cdots (s_2 \cdots s_{i+1})(s_1 \cdots s_i)$. By induction, we can get the following result:

**Lemma 6.4.** For $1 \leq i \leq m-1$, we have

$$T_{\omega_i}(F_i) = -K_i[E_m, E_{m-1}, \ldots, E_{i+1}, E_1, E_2, \ldots, E_{i-1}]_{v^{-i}}$$

$$T_{\omega_i}(E_i) = -[F_i-1, \ldots, F_2, F_1, F_{i+1}, \ldots, F_{m-1}, F_m]_{v^i} K_i^{-1}$$

where $[a, b, c]_{v^{-1}} = [(a, b)_{v^{-1}}, c]_{v^{-1}}$ and $[a, b, c]_{v} = [(a, b)_{v}, c]_{v}$.

We identify $E_i$ with $u_{S_i}^+$ and identify $F_i$ with $-\nu u_{S_i}^-$. Some further calculations yield

$$T_{\omega_i}(F_i) = -v^{-m+i+1} K_i u_{S_{i+1}}^+[u_{S_{i+1}}^+(m-i), u_{S_1}^+, u_{S_2}^+, \ldots, u_{S_{i-1}}^+]_{v^{-1}}$$

$$T_{\omega_i}(F_i) K_i^{-1} = (-1)^i v^{1-m-i} \sum_{M \in \mathcal{M}_{i+1, \delta-e_j}} (1 - v^2)^{\dim \End(M)-1} u_M^+ K_i.$$
The inverse image of the elements \( h_{i,k} \) in the Hall algebra are found by induction using the fact that \( \pi_{n,r} \) (see \( \text{5.3} \)) is central and primitive in Hall algebra. We just give the result:

\[
h_{j,k} = \pi_{j+1,k} - (v^k + v^{-k})\pi_{j,k} + \pi_{j-1,k}, \quad \text{for } k > 0.
\]

And again by a similar method we have

\[
h_{j,-k} = -\pi_{j+1,k} - (v^k + v^{-k})\pi_{j,k} + \pi_{j-1,k}, \quad \text{for } k > 0.
\]

Finally, for the elements \( x^+_{i,j} \) (\( j \neq 0, -1 \)) and \( x^-_{i,j} \) (\( j \neq 0, 1 \)) we can just use the relation \( \text{3.3} \) (4).

7. Relations in \( \mathbf{H}(\text{Coh}(\mathcal{X})) \)

In this section we focus on the elements \( x^+_{i,j,k}, x^-_{i,j,k+1}, h_{i,j,l}, x^+_{i}, \) and \( h_{i,t} \) where \( 1 \leq i \leq n, 1 \leq j \leq p_l - 1, k \geq 0, \) \( l, t \in \mathbb{N} \) and \( r \in \mathbb{Z} \). Those are the elements lie in the positive Hall algebra \( \mathbf{H}(\text{Coh}(\mathcal{X})) \).

7.1. The known relations. For reader's convenience, we list the relations which are already proved. The following relations are proved in \( \text{29} \):

(a). For \( r, r_1 \in \mathbb{N} \) and \( k \in \mathbb{Z} \),

\[
[T_{s,r}, u_{\mathcal{E}(k\mathcal{P})}] = \frac{[2r]}{r} u_{\mathcal{E}((k+r)\mathcal{P})},
\]

\[
[T_{s,r}, x^+_{i[1],r_1}] = -\frac{[r]}{r} x^+_{i[1],r_1+r}.
\]

(b). For \( t_1, t_2 \in \mathbb{Z} \),

\[
u_{\mathcal{E}((t+1)\mathcal{P})} u_{\mathcal{E}(t\mathcal{P})} - v^2 u_{\mathcal{E}(t\mathcal{P})} u_{\mathcal{E}((t+1)\mathcal{P})} = v^2 u_{\mathcal{E}(t\mathcal{P})} u_{\mathcal{E}((t+1)\mathcal{P})} - u_{\mathcal{E}((t+1)\mathcal{P})} u_{\mathcal{E}(t\mathcal{P})}.
\]

(c). For \( r, r_1, r_2 \in \mathbb{N} \) and \( t, t_1, t_2 \in \mathbb{Z} \),

\[
\text{Sym}_{r_1,r_2} \{ x^+_{i[1],r_1}, x^+_{i[1],r_2}, u_{\mathcal{E}(t\mathcal{P})} \}
- [2] u_{\mathcal{E}(t\mathcal{P})} u_{\mathcal{E}(t\mathcal{P})} x^+_{i[1],r_1} + u_{\mathcal{E}(t\mathcal{P})} x^+_{i[1],r_1} x^+_{i[1],r_2} = 0.
\]

\[
\text{Sym}_{t_1,t_2} \{ u_{\mathcal{E}(t_1\mathcal{P})} u_{\mathcal{E}(t_2\mathcal{P})} x^+_{i[1],r} \}
- [2] u_{\mathcal{E}(t_1\mathcal{P})} u_{\mathcal{E}(t_2\mathcal{P})} x^+_{i[1],r} + u_{\mathcal{E}(t_1\mathcal{P})} u_{\mathcal{E}(t_2\mathcal{P})} x^+_{i[1],r} = 0.
\]

(d). For \( r, r_1, r_2 \in \mathbb{N} \) and \( t \in \mathbb{Z} \),

\[
u_{\mathcal{E}((t+1)\mathcal{P})} x^+_{i[1],r} - v^{-1} x^+_{i[1],r} u_{\mathcal{E}((t+1)\mathcal{P})} = v^{-1} u_{\mathcal{E}(t\mathcal{P})} x^+_{i[1],r} + x^+_{i[1],r} u_{\mathcal{E}(t\mathcal{P})}.
\]

\[
x^+_{i[1],r_1}, x^+_{i[1],r_2} - v^2 x^+_{i[1],r_2} x^+_{i[1],r_1} = v^2 x^+_{i[1],r_2} x^+_{i[1],r_1}.
\]

(e). Any two elements arising from different tubes commute because there are no non-trivial extensions between any two torsion sheaves belonging to different tubes. Namely, for \( i \neq j \), we have

\[
h_{i[1],l}, h_{i[j],t} = 0, \quad h_{i[1],l}, x^+_{i[j],t} = 0.
\]

Moreover, note that in the last section we have known the elements in \( \mathbf{H}(T_n) \), for each \( i \), satisfy the require relations.
7.2. The remaining relations. Now we prove the remaining relations.

Lemma 7.1. (1). \([h_{i,j}, h_{i,l}, m] = 0\), for any \(k, m \in \mathbb{N}\).

(2). \([h_{i,t}, h_{s,k}] = 0\), for any \(l \in \mathbb{N}\) and \(k \in \mathbb{Z}\).

Proof. We have \([\pi^i_{j,k}, \pi^l_{i,m}] = 0\) using the algebra embedding

\[\text{H}_e(C_1) \hookrightarrow \text{H}_e(C_2) \cdots \hookrightarrow \text{H}_e(C_p)\]

and the fact that \(\pi^i_{j,k}\) is in the center of \(\text{H}_e(C_j)\) (see the remark in [5,4]). The first equation follows.

For the second one, just see that \([h_{i,t}, h_{s,k}] = [h_{i,t}, \pi^l_{i,k}] = 0\). \(\square\)

Lemma 7.2. \([h_{s,l}, x^+_{i,j}, k] = 0\), for any \(j \geq 2\), \(l \in \mathbb{N}\) and \(k \geq 0\).

Proof. First we prove the case \(k = 0\), we know that \(x^+_{i,j,0} = u_{S_i}\). For any sheaf appearing in \(h_{s,l} = T_l\), we only need to consider the direct summand belonging to \(T_{h_l}\), say \(\mathcal{F}\). We have \(|\mathcal{F}| = 0\) and \(\text{Top}(\mathcal{F}) = S^0_0 \oplus S^0_0 \cdots \oplus S^0_0\). \(\text{Soc}(\mathcal{F}) = S^0_0 \oplus S^0_0 \cdots \oplus S^0_0\). It is clear that \([u_{\mathcal{F}}, u_{S_i}] = 0\). Thus we get \([h_{s,l}, x^+_{i,j,0}] = 0\).

For the general case, one just need to apply \(\text{ad}(h_{i,j}, k)\) to the above formula. \(\square\)

Lemma 7.3. \([h_{s,l}, x^-_{i,j}, k] = \frac{[l]}{l} x^-_{i,j,1+k}, \) for any \(l \in \mathbb{N}\) and \(k \geq 1\).

Proof. Again we first consider the simplest case, namely the case \(k = 1\).

We know \(x^-_{i,j,1} = u_{S^0_0(p_i-1)K_{[i,1]}}\). It is easy to see that \([\pi^l_{2,k}, x^-_{i,j,1}] = 0\), hence

\[h_{s,l}, x^-_{i,j,1} = h_{i,\lambda^l_{1}, x^-_{i,j,1}, 1} = \frac{1}{u^i_{\mathcal{F}}} [h_{i,j,l}, x^-_{i,j,1}] = \frac{[l]}{l} x^-_{i,j,1+k}\]

In general we have

\[h_{s,l}, x^-_{i,j,1} = \frac{-(k-1)}{[2(k-1)]} [h_{i,j,l}, h_{i,1}, x^-_{i,j,1}]\]

\[= \frac{-(k-1)}{[2(k-1)]} [h_{i,j,l}, h_{i,1}, x^-_{i,j,1}, 1] + h_{i,j,l}, x^-_{i,j,1} - \frac{1}{v^i_{\mathcal{F}}} [h_{i,j,l}, h_{i,1}, x^-_{i,j,1}, 1] = \frac{[l]}{l} x^-_{i,j,1+k}\]

\(\square\)

Lemma 7.4. \([h_{s,l}, x^-_{i,j,1}] = 0\), for any \(j \geq 2\), \(l \in \mathbb{N}\) and \(k \geq 1\).

Proof. The case of \(k = 1\) is similar to the proof of Lemma 7.2. And for the general case just apply \(\text{ad}(h_{i,j}, k)\). \(\square\)

Lemma 7.5. \([x^+_{s,k}, x^-_{i,j,1}] = 0\), for any \(k \in \mathbb{Z}\).

Proof. We first consider the case \(j = 1\). Note that \(x^-_{i,j,1} = u_{S^0_0(p_i-1)K_{[i,1]}}\), so we have

\[x^+_{s,k}, u_{S^0_0(p_i-1)K_{[i,1]}} = x^+_{s,k} u_{S^0_0(p_i-1)K_{[i,1]}} - u_{S^0_0(p_i-1)K_{[i,1]}} x^+_{s,k} = (x^+_{s,k} u_{S^0_0(p_i-1)} - v^{-1} u_{S^0_0(p_i-1)} x^+_{s,k}) K_{[i,1]}.\]

We know that

\[\text{Hom}(\mathcal{O}(k\mathcal{C}), S^0_0(p_i-1)) = k,\]

\[\text{Ext}^1(\mathcal{O}(k\mathcal{C}), S^0_0(p_i-1)) = \text{Hom}(S^0_0(p_i-1), \mathcal{O}(k\mathcal{C})) = 0\]
Lemma 7.6. \[ h_{[i,j],l}, x^+_{s,k} = \frac{-[l]}{l} x^+_{s,k+l}, \text{ for any } l \in \mathbb{N} \text{ and } k \in \mathbb{Z} \]

Proof. For fixed \( i \), we already know the following relation holds (see Corollary 6.2):

\[ [x^+_{[i,j],k}, x^-_{[i,j],l}] = \psi_{[i,j],k+l-1} - \delta_{[i,j],l} x^+_{k+l}, \]

Now we set

\[ \xi^i_l = [x^+_{[i,j],k}, x^-_{[i,j],l}] K^{-1}_{[i,j]} = \frac{1}{v - v^{-1}} \psi_{[i,j],r} K^{-1}_{[i,j]} \]

By the definition of \( \psi_{[i,j],r} \), we have

\[ rh_{[i,j],r} = r \xi^i_r - \sum_{s=1}^{r-1} (v - v^{-1}) s h_{[i,j],s} \xi^i_{r-s}. \] (1)

The definition of \( \xi^i_r \) yields

\[ \xi^i_r = x^+_{[i,j],r-1} u_{S_0^0(p_i-1)} - v^2 u_{S_0^0(p_i-1)} x^+_{[i,j],r-1} \]

Thus we have

\[ \xi^i_r u_{\varphi(k\varepsilon)} = x^+_{[i,j],r-1} u_{S_0^0(p_i-1)} u_{\varphi(k\varepsilon)} - v^2 u_{S_0^0(p_i-1)} x^+_{[i,j],r-1} u_{\varphi(k\varepsilon)} = x^+_{[i,j],r-1} u_{\varphi(k\varepsilon)} u_{S_0^0(p_i-1)} - v^2 u_{S_0^0(p_i-1)} x^+_{[i,j],r-1} u_{\varphi(k\varepsilon)} \]

We claim that the following identity holds:

\[ \xi^i_r u_{\varphi(k\varepsilon)} = u_{\varphi(k\varepsilon)} \xi^i_r + (v^{-1} - v) u_{\varphi((k+1)\varepsilon)} \xi^i_{r-1} + v^{-1} (v^{-1} - v) u_{\varphi((k+2)\varepsilon)} \xi^i_{r-2} + \cdots + v^{-(r-2)} (v^{-1} - v) u_{\varphi((k+r-1)\varepsilon)} \xi^i_1 - v^{-(r-1)} u_{\varphi((k+r)\varepsilon)} \] (2)

Now we prove the claim by induction on \( r \).

When \( r = 1 \), we have \( \xi^i_1 = u_{S_1^0(p_i-1)} u_{S_0^0} - v^2 u_{S_0^0(p_i-1)} u_{S_0^1} \).

Also we get

\[ u_{S_1^0} u_{\varphi(k\varepsilon)} = u_{\varphi(k\varepsilon) u_{S_1^0} + u_{\varphi(k\varepsilon+x_i)}} \]

\[ u_{\varphi(k\varepsilon) u_{S_1^0}} = u_{\varphi(k\varepsilon) u_{S_0^0}} \]

\[ u_{\varphi(k\varepsilon+x_i) u_{S_0^0}} = u_{\varphi(k\varepsilon+x_i) u_{S_0^0} + u_{\varphi((k+1)\varepsilon)}} \]

Thus we have

\[ \xi^i_r u_{\varphi(k\varepsilon)} = u_{\varphi(k\varepsilon)} \xi^i_r + (v^{-1} - v) u_{\varphi((k+1)\varepsilon)} \xi^i_{r-1} + v^{-1} (v^{-1} - v) u_{\varphi((k+2)\varepsilon)} \xi^i_{r-2} + \cdots + v^{-(r-2)} (v^{-1} - v) u_{\varphi((k+r-1)\varepsilon)} \xi^i_1 - v^{-(r-1)} u_{\varphi((k+r)\varepsilon)} \]
Thus

\[ \xi^l_1 u_\theta(k\xi) = v u_{S^1} u_\theta(k\xi) u_{S^0(p_i-1)} - v^2 u_{S^0(p_i-1)} u_{S^1} u_\theta(k\xi) \]

\[ = u_\theta(k\xi) u_{S^1} u_{S^0(p_i-1)} + u_\theta(k\xi + x_i) u_{S^0(p_i-1)} \]

\[ - v u_{S^0(p_i-1)} u_\theta(k\xi) u_{S^1} - v^2 u_\theta(k\xi + x_i) \]

\[ = u_\theta(k\xi) u_{S^1} u_{S^0(p_i-1)} + u_\theta(k\xi + x_i) u_{S^0(p_i-1)} - v^2 u_\theta(k\xi + x_i) \]

\[ = \xi^l_1 u_\theta(k\xi) = \xi^l_1 u_\theta(k\xi + x_i) \]

Thus we assume that for \((\ast 2)\) holds for \(r = m - 1\). We will prove the case \(r = m\).

By [29 4.13], we have

\[ \xi^l_1 u_\theta(k\xi) = (u_\theta(k\xi) \xi^l_1 + \xi^l_2 u_\theta((k+1)\xi) - v u_\theta((k+1)\xi) \xi^l_{r-2}) u_{S^0} \]

\[ - v u_{S^0} (u_\theta(k\xi) \xi^l_1 + \xi^l_2 u_\theta((k+1)\xi) - v u_\theta((k+1)\xi) \xi^l_{r-2}) \]

\[ = u_\theta(k\xi) \xi^l_1 - v u_\theta((k+1)\xi) \xi^l_{r-1} + v^{-1} \xi^l_{r-1} u_\theta((k+1)\xi) \]

This complete the proof of \((\ast 2)\).

The lemma is now a consequence of \((\ast 2)\) and \((\ast 1)\). \(\square\)

**Lemma 7.7.** (1) \([x^+_i,j,l, x^+_s,k] = 0\), for \(j \geq 2\), \(l \geq 0\) and \(k \in \mathbb{Z}\).

(2) \([h_{(i,j)},l, x^+_s,k] = 0\), for \(j \geq 2\), \(l \in \mathbb{N}\) and \(k \in \mathbb{Z}\).

**Proof.** First we see that (2) is a consequence of (1), since \([h_{(i,j)},l, x^+_s,k] = 0\) if and only if \([x^+_i,j,k, x^+_i,j,1] = 0\) and we know \([x^+_i,j,1, x^+_s,k] = 0\) by lemma 7.5. Note that when \(l = 0\), \([x^+_i,j,0, x^+_s,k] = 0\) is clearly 0, cause there is no non-trivial extension between \(\theta(k\xi)\) and \(S^1\) for \(j \geq 2\).

Now we argue by induction on \(j\). When \(j = 2\), using lemma 7.6 we have

\[ [x^+_i,j,2,l, x^+_s,k] = -\frac{l}{[l]} [h_{(i,j)},l, u_{S^2}, x^+_s,k] \]

\[ = -\frac{l}{[l]} [h_{(i,j)},l, x^+_s,k, u_{S^2}] = 0. \]

Assume for \(j < m\) the relation holds, we also have \([h_{(i,j)},l, x^+_s,k] = 0\) for \(j < m\).

Hence

\[ [x^+_i,j,m,l, x^+_s,k] = -\frac{l}{[l]} [h_{(i,j,m-1,l,j, x^+_s,k, x^+_s,k}] \]

\[ = -\frac{l}{[l]} [h_{(i,j,m-1,l,j, x^+_s,k, x^+_s,k, x^+_s,k, x^+_s,k, x^+_s,k}] = 0. \]

\(\square\)

**Lemma 7.8.** \([x^-_i,j,l, x^-_i,j] = 0\), for any \(l \geq 1\) and \(k \in \mathbb{Z}\).

**Proof.** For the case \(l = 1\) this is just lemma 7.5. For the general case we apply \(\text{ad}(h_{(i,j),l})\) and use lemma 7.6 and lemma 7.7 (2). \(\square\)
8. Relations in $\mathbf{DH}(\mathbf{Coh}(\mathbb{X}))$

We will prove all the Drinfeld relations in this section. In the last section we have proved the relations in $\mathbf{H}(\text{Coh}(\mathbb{X}))$, thus dually the relations in $\mathbf{H}^-(\text{Coh}(\mathbb{X}))$ are also satisfied. So we focus on the relations where the elements in both the positive and negative part are involved.

8.1. Since we are now in the double Hall algebra, the comultiplication is important for our calculations. By the definition we know that

$$\Delta(h_{s,-l}) = h_{s,-l} \otimes 1 + K_r \delta \otimes h_{s,-l} + \text{rest terms},$$

$$\Delta(u \theta(k \delta)) = u \theta(k \delta) \otimes 1 + \sum_{r=0}^{\infty} \theta \pi K_{\alpha+(k-r)\delta} \otimes u \theta((k-r)\delta) + \text{rest terms},$$

where $\{\theta \pi\}_{r \geq 1}$ are defined by the following generating series

$$\sum_{k \geq 0} \theta \pi u^k = \exp((v - v^{-1}) \sum_{k=1}^{\infty} h_{s,k} u^k).$$

8.2. In this subsection we prove the relation $[h_{s,r}, h_{t,m}] = 0$ for the case $rm < 0$. We assume $m > 0$ and $r < 0$.

We need the following lemma:

**Lemma 8.1.** For fixed $i$, we have

$$[\pi^+_{r,k_1}, \pi^-_{l,k_2}] = 0,$$

**Proof.** For simplicity, we will omit $i$ and write $r$ for $p_i$. Also we write $h_{r,k}$ for $h_{i,r,k}$.

For $m \geq 1$ we have the relation

$$[\pi^+_{r,k_1}, h_{t-m,-k_2}] = 0.$$  

This implies

$$[\pi^+_{r,k_1}, \pi^-_{r-m+1,k_2} - (v^{k_2} + v^{-k_2})\pi^i_{r-m,k_2} + \pi^i_{r-m-1,k_2}] = 0.$$  

Note that $[\pi^+_{r,k_1}, \pi^-_{r,k_2}] = 0$. Hence we get a system of homogeneous linear equations $AX = 0$, where

$$A = \begin{pmatrix}
    a & 1 & 0 & 0 & \cdots & 0 \\
    1 & a & 1 & 0 & \cdots & 0 \\
    0 & 1 & a & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 0 & 1 & a \\
    0 & 0 & \cdots & 0 & 1 & a \\
\end{pmatrix}$$

$$X = (b_{r,r-1}, b_{r,r-2}, \cdots, b_{r,2}, b_{r,1})^t$$

in which

$$a = -(v^{k_2} + v^{-k_2}), \quad b_{r,l} = [\pi^+_{r,k_1}, \pi^-_{l,k_2}].$$

It is clear that $A$ is a nonsingular matrix, thus $[\pi^+_{r,k_1}, \pi^-_{l,k_2}] = 0$ holds for any $1 \leq l \leq r - 1$.

Similarly, the $(r-1) \times (r-1)$ relations $[h_{t_1,k_1}, h_{t_2,-k_2}] = 0$ ($1 \leq l_1 \leq r - 1, 1 \leq l_2 \leq r - 1$) induce a system of inhomogeneous linear equations with $(r-1) \times (r-1)$ variables $[\pi^+_{l_1,k_1}, \pi^-_{l_2,k_2}]$ and the coefficient matrix is also nonsingular. Therefore, for any $1 \leq l_1 \leq r - 1, 1 \leq l_2 \leq r - 1$, we have $[\pi^+_{l_1,k_1}, \pi^-_{l_2,k_2}] = 0$. \qed
Now we have
\[ [h_{s,r}, h_{s,m}] = -\sum_{i=1}^{n} [\pi_{1,-t}, \pi_{1,m}] = 0, \]
\[ [h_{s,r}, h_{[i,j],m}] = -[\pi_{1,-t}, \pi_{j+1,m}] - (v^m + v^{-m})\pi_{j,m} + \pi_{j-1,m} = 0. \]

8.3. In this subsection we deal with the following relation for the rest cases
\[ [h_{s,r}, x_{t,m}^+] = \pm \frac{1}{r} [a_{st}, x_{t,r+m}^+]. \]

**Lemma 8.2.** \[ [h_{s,t}, x_{s,k}^+] = \frac{1}{t} [2l] x_{s,t+k}^+. \] For any \( l < 0 \) and \( k \in \mathbb{Z} \).

**Proof.** Recall that
\[ \Delta(u_{\theta(k\bar{c})}) = u_{\theta(k\bar{c})} \otimes 1 + \sum_{r=0}^{\infty} \theta_{s,r} K_{\alpha_s+(k-r)d} \otimes u_{\theta((k-r)c)} + \text{ other terms}, \]
We first prove the ”rest terms” in \( \Delta(u_{\theta(k\bar{c})}) \) do not effect our calculation. Assume that \( u_A K_{\mu} \otimes u_{\theta(x)} ( \bar{x} \neq s\bar{c} \text{ for any } s \in \mathbb{Z} ) \) occurs in the rest terms, by definition there exists an exact sequence
\[ 0 \rightarrow \mathcal{O}(\bar{x}) \rightarrow \mathcal{O}(k\bar{c}) \rightarrow A \rightarrow 0. \]
Since \( \bar{x} \neq s\bar{c} \), we have
\[ \Ext^1(S_{m,1}, \mathcal{O}(\bar{x})) = \Hom(\mathcal{O}(\bar{x}), S_{m,1}) = 0, \text{ for some } m. \]
Then for any \( u_B \) occurring in the terms of \( h_{i,t}, A \) is not isomorphic to a subobject of \( B \). Otherwise there is an injective map from \( S_{m,1} \) to \( A \). We would have the following commutative diagram where the right square is a pull-back diagram:

\[ \begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}(\bar{x}) \\
\downarrow f & & \downarrow g \\
0 & \longrightarrow & \mathcal{O}(k\bar{c}) \\
\downarrow & & \downarrow \\
& & \mathcal{O}(\bar{x}) \\
& & \mathcal{O}(k\bar{c}) \\
\end{array} \]
\[ \begin{array}{ccc}
& & \mathcal{O}(\bar{x}) \\
& & \mathcal{O}(k\bar{c}) \\
& & \mathcal{O}(\bar{x}) \\
& & \mathcal{O}(k\bar{c}) \\
\end{array} \]
\[ \begin{array}{ccc}
& & \mathcal{O}(\bar{x}) \\
& & \mathcal{O}(k\bar{c}) \\
& & \mathcal{O}(\bar{x}) \\
& & \mathcal{O}(k\bar{c}) \\
\end{array} \]
There is no non-zero morphism from \( S_{m,1} \) to \( \mathcal{O}(k\bar{c}) \), therefore \( gf = 0 \) but \( ip_2 \) is clear not zero, which is a contradiction.

Now using the relations of the Drinfeld double and note that \( (\theta_{s,r}, h_{s,r}) = \frac{2v_r}{t} \),
we obtain the lemma.

**Lemma 8.3.** (1) \[ [h_{s,t}, x_{i,1}^+] = \frac{1}{t} [-l] x_{i,1}^{+}, \text{ for any } l < 0 \text{ and } k \geq 1. \]
(2) \[ [h_{s,t}, x_{i,1}^-] = -\frac{1}{t} [-l] x_{i,1}^{-}, \text{ for any } l < 0 \text{ and } k \geq 1. \]

**Proof.** (1). We have shown \([\pi_{2,-l}, x_{i,1}^+] = 0 \). Dually we have
\[ [\pi_{2,-l}, x_{i,1}^-] = 0. \]

Apply \( \text{ad}(h_{[i,1],k+1}) \) to the above formula, we get
\[ [\pi_{2,-l}, x_{i,1}^+] = 0. \]
Thus
\[ [h_{s,t}, x_{i,1}^+] = [\pi_{2,-l}, x_{i,1}^+] = \frac{-1}{v^l + v^{-l}} [h_{[i,1],l}, x_{i,1}^+] = \frac{1}{t} [-l] x_{i,1}^{+}. \]
(2) We have shown $[\pi_{2,l}^{+}, x_{[1],0}^{+}] = 0$. Dually we have

$$[\pi_{2,l}^{-}, x_{[1],0}^{-}] = 0.$$

Apply $\text{ad}(h_{[1],k})$ to the above formula, we get

$$[\pi_{2,l}^{-}, x_{[1],k}] = 0.$$

Hence we have

$$[h_{*,l}, x_{[1],k}] = [\pi_{2,l}^{-}, x_{[1],k}] = \frac{-1}{v^{l} + v^{-1}} [h_{[1],l}, x_{[1],k}] = -\frac{1}{r}[-l] x_{[1],k+1}.$$

\[\square\]

**Lemma 8.4.** (1). $[h_{*,l}, x_{[1],m}] = 0$, for any $2 \leq m \leq p_i - 1$, $l < 0$ and $k \geq 0$.

(2). $[h_{*,l}, x_{[1],m}] = 0$, for any $2 \leq m \leq p_i - 1$, $l < 0$ and $k \geq 1$.

**Proof.** For the first equation, apply $\text{ad}(h_{[1],k+1})$ to $[h_{*,l}, x_{[1],m}, -1] = 0$. And for the second one, just apply $\text{ad}(h_{[1],k})$ to $[h_{*,l}, x_{[1],m}] = 0$. \[\square\]

**Lemma 8.5.** $[h_{[1],l}, x_{[1],m}] = \frac{1}{l}[-l] x_{[1],k+1}$, for $l < 0$ and $k \in \mathbb{Z}$.

**Proof.** Set $\xi_{l}^{k} = [x_{[1],l}, x_{[1],m}, 2]K_{[1]}$. Note that we have known the relation

$$[x_{[1],l}, x_{[1],2}] = \frac{\phi_{[1],l}}{v^{l} - v^{-1}}$$

By the definition of $\phi$, we have for any $r < 0$,

$$-rh_{[1],r} = -r\xi_{l}^{k} + \sum_{s=1}^{-r} (v - v^{-1})sh_{[1],r} \xi_{r+s}^{k} \quad (\ast)$$

We then claim that the following identity holds:

$$\xi_{l}^{k} x_{[1],k} = x_{[1],k} \xi_{l}^{k} + (v - v^{-1}) x_{[1],k-1} \xi_{l+1}^{k} + v(v - v^{-1}) x_{[1],k-2} \xi_{l+2}^{k} + \cdots + v^{(-r-2)}(v - v^{-1}) x_{[1],k+l} \xi_{r-1}^{k} - v^{(-r-1)} x_{[1],k+l} \xi_{r}^{k} (d1)$$

We prove the claim by induction. When $r = -1$, we have

$$\xi_{-1}^{k} x_{[1],k} = [x_{[1],0}, x_{[1],-1}]K_{[1]} x_{[1],k} = v^{-1} [x_{[1],0}, x_{[1],-1}] x_{[1],0} K_{[1]} = [x_{[1],0}, x_{[1],-1}] K_{[1]} x_{[1],0} + v \xi_{0}^{k} x_{[1],k-1} = x_{[1],k} \xi_{-1}^{k} + x_{[1],k-1}^{+}$$

Now assume (d1) holds for $r = -l + 1$. We will prove the case $r = -l$. By [29] 4.13 and

$$[x_{[1],2l}, x_{[1],k}] = 0$$

...
which will be proved in the next subsection independent of this lemma, we have

\[ \xi^l s,k = [x^+_i, -l, x^+_i, 2l] K[i, 1] x^+_s,k \]

\[ = v^{-1} [x^+_i, -l, x^+_i, 2l] [x^+_s,k] \]

\[ = v^{-1} (x^+_i, -l x^+_i, 2l x^+_s,k - x^+_i, 2l x^+_s,k - l x^+_s,k) K[i, 1] \]

\[ = v^{-1} (x^+_i, -l x^+_s,k x^+_i, 2l - x^+_s,k - l x^+_s,k) K[i, 1] \]

\[ = v^{-1} ((v x^+_i, -l + 1 x^+_s,k - 1 + v x^+_s,k x^+_i, -l - x^+_s,k - l x^+_s,k - l + 1) x^+_i, 2l) \]

\[ x^+_s,k (v x^+_i, -l + 1 x^+_s,k - 1 + v x^+_s,k x^+_i, -l - x^+_s,k - l x^+_s,k - l + 1)) K[i, 1] \]

\[ = x^+_s,k \xi^l - v^{-1} x^+_s,k - 1 \xi^l + v \xi^l x^+_s,k - 1 \]

The lemma is now a consequence of (d1) and (*).

**Lemma 8.6.** \([h[i,m], l, x^+_s,k] = 0, \text{ for } 2 \leq m \leq p_i - 1, l < 0 \text{ and } k \in \mathbb{Z}.\]

**Proof.** When \(m = 2\), we have proved

\[ [x^+_i, 2l, 1, x^+_s,k] = 0, [x^+_i, 2l, 0, x^+_s,k] = 0. \]

Apply \(\text{ad}(h[i,1], l-1)\) to the latter one, we get

\[ [x^+_i, 2l, 1, x^+_s,k] = 0. \]

Then we have \([\varphi[i,2], l, x^+_s,k] = 0\), which is equivalent to \([h[i,2], l, x^+_s,k] = 0\).

Assume for any \(s < p\) the relation \([x^+_i, 2, s, x^+_s,k] = 0\) holds. We shall prove for the case \(s = p\).

Note that the following still holds:

\[ [x^+_i, p, 1, x^+_s,k] = 0, [x^+_i, p, 0, x^+_s,k] = 0 \]

Now apply \(\text{ad}(h[i,p-1], l-1)\) to the latter one, we get

\[ [x^+_i, p, 1, x^+_s,k] = 0. \]

Then we have \([\varphi[i,p], l, x^+_s,k] = 0\), which is equivalent to \([h[i,m], l, x^+_s,k] = 0\).

This completes the proof. \(\square\)

8.4. Now we consider the relation

\[ [x^+_s,k, x^-] = \delta \frac{\psi x^+_s, k + l - \varphi x^+_s, k + l}{v - v^{-1}}. \]

**Lemma 8.7.** For any \(k, l\) such that \(k + l \geq 0\),

\[ [x^+_s,k, x^-] = \frac{\psi x^+_s, k + l - \varphi x^+_s, k + l}{v - v^{-1}}. \]

**Proof.** Recall the comultiplication:

\[ \Delta(u\varphi(k\varphi)) = u\varphi(k\varphi) \otimes 1 + \sum_{r=0}^{\infty} \psi_r \alpha_{(k-r)\varphi} \otimes u\varphi((k-r)\varphi) + \text{rest terms}. \]

\[ \Delta(u\varphi(l\varphi)) = u\varphi(l\varphi) \otimes 1 + \sum_{r=0}^{\infty} \psi_r \alpha_{(l-r)\varphi} \otimes u\varphi((l-r)\varphi) + \text{rest terms}. \]

For any \(A_1 = uB_1 K_\alpha \otimes uC_1\), appearing in the rest terms of \(\Delta(u\varphi(k\varphi))\) and \(A_2 = uB_2 K_\beta \otimes uC_2\) appearing in the rest terms of \(\Delta(u\varphi(-l\varphi))\), \(B_1\) is a nonzero sheaf of finite length and \(C_2\) is a nonzero line bundle. So they are not isomorphic to each
other. And similarly, \( C_1 \) is a nonzero line bundle and \( B_2 \) is a nonzero sheaf of finite length. They are not isomorphic to each other.

Thus we can see that we do not need to consider the rest terms in the calculation.

Now using the relations of the Drinfeld double, we obtain the lemma. \( \square \)

**Lemma 8.8.** \([x^+_{i,k}, x^-_{i,j}, l] = 0\), for any \( k \in \mathbb{Z}, \ l \leq 0 \).

**Proof.** For \( j = 1 \), we have
\[
[x^+_{i,k}, x^-_{i,1}, 1] = 0.
\]
Applying \( \text{ad}(h_{s,t-l}) \), we get the result.
For \( j = 2 \), we have
\[
[x^+_{i,k}, x^-_{i,2}, 1] = 0.
\]
We get the result by applying \( \text{ad}(h_{s,t-l}) \).
Now assume that for \( 2 \leq j < p \), the lemma holds. For \( j = p \), we apply \( \text{ad}(h_{[i,p-1], l-1}) \) to \([x^+_{i,k}, x^-_{i,p}, 1] = 0\), we get the required identity. \( \square \)

8.5. Let us consider the next one:
\[
x^\pm_{i,k+l+1,x_{j,l}} - v^{\pm a_{i,j}} x^\pm_{i,k+l+1,x_{j,l}} = v^{\pm a_{i,j}} x^\pm_{i,k} x^\pm_{i,k+l+1,x_{j,l}} - x^\pm_{i,k+l+1,x_{j,l}}.
\]

**Lemma 8.9.** For any \( l < 0 \) and \( k \geq 0 \),
\[
x^+_{i,k+l+1,x_{j,l}} - v^{-1} x^+_{i,k+l+1,x_{j,l}} = v^{-1} x^+_{i,k} x^+_{i,k+l+1,x_{j,l}} - x^+_{i,k+l+1,x_{j,l}}.
\]

**Proof.** We have known that
\[
x^{+}_{i,k+l+1,x_{j,l}}, 0 - v^{-1} x^{+}_{i,k+l+1,x_{j,l}}, 0 x^{+}_{i,k+1} = v^{-1} x^{+}_{i,k} x^{+}_{i,k+l+1,x_{j,l}} - x^{+}_{i,k+l+1,x_{j,l}}.
\]
Applying \( \text{ad}(h_{[i,1], l-1}) \) we get the required result. \( \square \)

**Lemma 8.10.** \([x^+_i, x^-_j, l] = 0\) for \( 2 \leq j \leq p_i - 1, \ l < 0 \) and \( k \in \mathbb{Z} \).

**Proof.** Just apply \( \text{ad}(h_{[i,1], l-1}) \) to \([x^+_i, x^-_j, 0] = 0\). \( \square \)

8.6. Finally we deal with the following one:
\[
\text{Sym}_{k_1, \ldots, k_n} \sum_{t=0}^{n} (-1)^t \binom{n}{t} \ x_{i,k_1}^{\pm} \cdots x_{i,k_t}^{\pm} x_{i,k_{t+1}}^{\pm} \cdots x_{i,k_n}^{\pm} = 0
\]
where \( i \neq j \) and \( n = 1 - a_{ij} \).

**Lemma 8.11.** For any \( k_1 \leq 0 \),
\[
\text{Sym}_{k_1, k_2} \{ x^+_{i[1], k_1} x^+_{i[1], k_2}, x^+_{i[1], 0}, x^+_{i, t}, x^+_{i[1], k_1} x^+_{i[1], k_2} + x^+_{i, t}, x^+_{i[1], k_1}, x^+_{i[1], k_2} \} = 0
\]

**Proof.** For \( k_1 = 0, \ k_2 = 0 \) and \( t \in \mathbb{Z} \), we know the following holds:
\[
\text{Sym}_{0, 0} \{ x^+_{i[1], 0}, x^+_{i[1], 0}, x^+_{i, t}, x^+_{i[1], 0}, x^+_{i, t}, x^+_{i[1], 0}, x^+_{i[1], 0}, x^+_{i[1], 0} \} = 0.
\]
Applying \( \text{ad}(h_{s,k}) \), we have
\[
0 = - \frac{[k_1]}{k_1} \text{Sym}_{k_1, 0} \{ x^+_{i[1], k_1}, x^+_{i[1], 0}, x^+_{i, t}, x^+_{i[1], 0}, x^+_{i, t}, x^+_{i[1], 0} \}
\]
\[
+ \frac{[k_1]}{k_1} \text{Sym}_{0, 0} \{ x^+_{i[1], 0}, x^+_{i[1], 0}, x^+_{i, t}, x^+_{i[1], 0}, x^+_{i[1], 0}, x^+_{i[1], 0} \}.
\]
Hence

$$\text{Sym}_{k_1,0}\{x_{[1],k_1}^+,x_{[1],0}^+\} = [2]x_{[1],k_1}^+ x_{[1],0}^+ [2]x_{[1],k_1}^+ x_{[1],0}^+ + x_{[1],0}^+ x_{[1],k_1}^+ [2]x_{[1],k_1}^+ x_{[1],0}^+ = 0.$$ 

Then we apply $\text{ad}(h_{s_l})$ to the above equation, this yields

$$0 = -\frac{[k_2]}{k_2} \text{Sym}_{k_1+k_2,0}\{x_{[1],k_1+k_2}^+,x_{[1],0}^+ \} + \frac{[2k_2]}{k_2} \text{Sym}_{k_1,0}\{x_{[1],k_1}^+,x_{[1],0}^+ \} + [k_2] \text{Sym}_{k_1,k_1}\{x_{[1],k_1}^+,x_{[1],0}^+ \}$$

This completes the proof.

Lemma 8.12. For any $r < 0$,

$$\text{Sym}_{t_1,t_2}\{x_{s,t_1}^+,x_{s,t_2}^+,x_{[1],0}^+ \} = 0$$

Proof. For any $t_1$, $t_2$, the following holds:

$$\text{Sym}_{t_1,t_2}\{x_{s,t_1}^+,x_{s,t_2}^+,x_{[1],0}^+ \} = 0$$

Now apply $\text{ad}(h_{s_l})$, we get

$$0 = -\frac{[l]}{l} \text{Sym}_{1+t_2}\{x_{s,t_1}^+,x_{s,t_2}^+,x_{[1],0}^+ \} + \frac{[2l]}{l} \text{Sym}_{1,l}\{x_{s,t_1}^+,x_{s,t_2}^+,x_{[1],0}^+ \}$$

We deduce the relations hold for all $l < 0$.

9. Remarks on derived equivalence and PBW-basis

In this section we restrict to the case that $\mathfrak{g}$ is of finite type, i.e. $\mathcal{L}_\mathfrak{g}$ is an affine Kac-Moody algebra.

9.1. Derived equivalence and double Hall algebra. In this case, the associated star-shaped Dynkin diagram $\Gamma$ is of type A-D-E. Denote by $\hat{\Gamma}$ the extended Dynkin diagram corresponding to $\Gamma$. We know from that the category $\text{Coh}(X)$ is derived equivalent to $\text{mod}\Lambda$ where $\Lambda$ is the path algebra of $\hat{\Gamma}$ (hence $\Lambda$ is a tame hereditary algebra). More precisely, let $\mu$ be the slope function of coherent sheaves and $\chi$ be the Euler characteristic of the weighted projective line $X$ (For the missing definitions one can see [9]). We have the following theorem:

Theorem 9.1 ([9]). The direct sum $T$ of a representative system of indecomposable bundles $E$ with slope $0 \leq \mu(E) < \chi$ is a tilting object of $\text{Coh}(X)$ whose endomorphism ring is isomorphic to $\Lambda$. Thus we have

$$\mathcal{O}^h(\text{Coh}(X)) \simeq \mathcal{O}^h(\text{mod}\Lambda).$$
Recently Cramer has proved the following theorem which asserts that the double Hall algebra is invariant under derived equivalences.

**Theorem 9.2** ([4]). Let \( \mathcal{A} \) and \( \mathcal{B} \) be two \( k \)-linear finitary hereditary categories. Assume one of them is artinian and there is an equivalence of triangulated categories \( D^b(\mathcal{A}) \to \tilde{\mathcal{D}} D^b(\mathcal{B}) \). Let \( R(\mathcal{A}) \to \tilde{\mathcal{R}} R(\mathcal{B}) \) be the induced equivalence of the root categories. Then there is an algebra isomorphism \( F : \text{DH}(\mathcal{A}) \to \text{DH}(\mathcal{B}) \) uniquely determined by the following property. For any object \( X \in \text{OB}(\mathcal{A}) \) such that \( F(X) \simeq \tilde{X}[-n_{\mathcal{F}}(X)] \) with \( \tilde{X} \in \text{OB}(\mathcal{B}) \) and \( n_{\mathcal{F}}(X) \in \mathbb{Z} \) we have:

\[
F(u_{\tilde{X}}^\pm) = v^{-n_{\mathcal{F}}(X)}<\tilde{X},\tilde{X}>_{\tilde{X}}K_{\tilde{X}}^\pm
\]

where \( n_{\mathcal{F}}(X) = + \) if \( n_{\mathcal{F}}(X) \) is even and \( - \) if \( n_{\mathcal{F}}(X) \) is odd. For \( \alpha \in K_0(\mathcal{A}) \), we have \( F(K_\alpha) = K_{\mathcal{F}(\alpha)} \).

9.2. **Compatibility of some isomorphisms.** Now let us consider the following diagram

\[
\begin{array}{ccc}
U_v(\hat{g}) & \xrightarrow{\rho_1} & \text{DC}(\text{mod}\Lambda) \\
\psi \downarrow & & \downarrow \iota_1 \\
U_v(Lg) & \xrightarrow{\rho_2} & \text{DC}(\text{Coh}(X))
\end{array}
\]

\[
\begin{array}{ccc}
& & \text{DH}(\text{mod}\Lambda) \\
\downarrow F & & \downarrow F \\
& & \text{DH}(\text{Coh}(X))
\end{array}
\]

where \( \psi \) is the Drinfeld-Beck isomorphism, \( \rho_1 \) is the isomorphism in theorem 2.2, \( \rho_2 \) is the isomorphism given by corollary 5.5, \( \iota_1 \), \( \iota_2 \) are natural embeddings and \( F \) is the isomorphism in the above theorem.

When \( g = \mathfrak{sl}_2 \), we know that \( \Lambda \) is the path algebra of the Kronecker algebra and \( X \) is just the projective line \( \mathbb{P}^1 \). In this case it is proved in [3] that the above diagram is commutative. This is equivalent to say that the restriction of the isomorphism \( F \) to \( \text{DC}(\text{mod}\Lambda) \) gives the isomorphism

\[
\rho_2 \psi \rho_1^{-1} : \text{DC}(\text{mod}\Lambda) \simeq \text{DC}(\text{Coh}(X)).
\]

However, for the other cases, the above diagram may not commutative. The reason is as follows:

Denote by \( E_m \) and \( F_m \) the Chevalley generators of \( U_v(\hat{g}) \). Here the index \( m \in \{ *, [i,j] | 1 \leq i \leq n, 1 \leq j \leq p_i - 1 \} \cup \{ 0 \} \), where the first set is the vertices of \( \Gamma \) and \( 0 \) corresponds to the extending vertex of \( \hat{\Gamma} \). By definition of \( \rho_1 \), the image of each \( E_m \) in \( \text{DH}(\text{mod}\Lambda) \) is a simple \( \Lambda \)-module.

On the other hand, the Drinfeld-Beck isomorphism \( \psi \) sends \( F_m \) to \( x_{m,0}^+ \) for all \( m \) except the extending vertex. Now for \( m \neq * \), by our theorem 5.4, the image of \( x_{m,0}^+ \) in \( \text{DH}(\text{Coh}(X)) \) under the isomorphism \( \rho_2 \) is a simple sheaf which lies on the bottom of some non-homogeneous tube.

Thus if the diagram is commutative, we should have a derived equivalence \( \mathcal{D}^b(\text{mod}\Lambda) \simeq \mathcal{D}^b(\text{Coh}(X)) \) sending all (except two, corresponding to the vertex *) and the extending vertex) the simple \( \Lambda \)-modules to sheaves on the bottom of non-homogeneous tubes. One can easily see that this is impossible for type \( D \) and \( E \).
9.3. Two PBW-type bases. Now we have two realizations of quantum affine algebras $U_v(\hat{g})$ from the Hall algebras of two different hereditary categories which are derived equivalent. Note that the derived equivalence $D^b(\text{Coh}(\mathcal{X})) \simeq D^b(\text{mod}\Lambda)$ is quite “visible” if one looks at the Auslander-Reiten-quivers. Recall that the AR-quiver of $\text{Coh}(\mathcal{X})$ consisting of two components, the locally free part $\mathcal{F}$ and the torsion part $\mathcal{F}_\omega$. While there are three components in the AR-quiver of $\text{mod}\Lambda$, namely the preprojective component $\mathcal{P}$, the preinjective one $\mathcal{I}$ and the regular one $\mathcal{R}$. Roughly speaking, the derived equivalence is given by splitting the component $\mathcal{F}$ into two pieces. One corresponds to the $\mathcal{P}$, the other is sent to $\mathcal{I}[-1]$, and $\mathcal{F}_\omega$ is just identified with $\mathcal{R}$.

The Hall algebra approach has many advantages. For example, the Hall algebra has a natural basis indexed by the isomorphic classes of objects in the category. Moreover, the structure of the category can give us more information. In particular, one can construct PBW-type basis reflexing the structure of the category. This has been done in [17] using the Hall algebra of $\text{mod}\Lambda$.

Let $P$ be any preprojective module and $I$ be any preinjective module. The notations in the following proposition directly come from [17]. We will not give the explicit definitions. But

**Proposition 9.3 ([17]).** The following set
\[ \{ u_P E_{\pi_1 \in \mathcal{P}} E_{\pi_2 \in \mathcal{P}} \cdots E_{\pi_n \in \mathcal{P}} T_{\omega} u_I \} \]
is a basis of the composition algebra $C(\text{mod}\Lambda)$, i.e. a basis of the (standard) positive part $U_v^+ (\hat{g})$.

We explain the notations in the above proposition. $P$ runs over all preprojective modules and $I$ runs over all preinjective modules. Hence $u_P$, $u_I$ are basis elements arising from preprojective and preinjective components respectively. For each $i$, the element $E_{\pi_i \in \mathcal{P}}$ comes from the non-homogeneous tube $\mathcal{T}_\lambda$. These basis elements were first constructed in [5], we omit the explicit definition here. $\omega = (\omega_1 \geq \omega_2 \cdots \omega_t)$ runs over all partitions of positive integers. And $T_{\omega} = T_{\omega_1} T_{\omega_2} \cdots T_{\omega_t}$. Here $T_{\omega}$ is just the element defined in [5,3] Note that the regular component $\mathcal{R}$ is equivalent to the torsion part $\mathcal{F}_\omega$.

Similarly we can use the Hall algebra of $\text{Coh}(\mathcal{X})$ to give the PBW-type basis of another half of $U_v(\hat{g})$.

**Proposition 9.4.** The following set
\[ \{ u_V E_{\pi_1 \in \mathcal{V}} E_{\pi_2 \in \mathcal{V}} \cdots E_{\pi_n \in \mathcal{V}} T_{\omega_\delta} \} \]
is a basis of the composition algebra $C(\text{Coh}(\mathcal{X}))$, i.e. a basis of $U_v(\hat{n})$.

In this proposition $V$ runs over all vector bundles in $\mathcal{F}$. Other notations are the same as proposition 9.3.

Comparing proposition 9.3 and 9.4 we can see that the PBW-type bases of two different half quantum affine algebra are related by the derived equivalence of the two category $\text{mod}\Lambda$ and $\text{Coh}(\mathcal{X})$.

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