About the exotic structure of $Z_{cs}$

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Abstract

Very recently a new hadronic structure around 3.98 GeV was observed in BESIII experiment. From its decay modes, it is reasonable for people to assign it to the category of exotic state, say $Z_{cs}^+$, the stranged-partner of $Z_c(3900)$. This finding indicates for the first time the exotic state with strange quark in charm sector, and hence has a peculiar importance. By virtue of the QCD Sum Rule technique, we analyze the $Z_{cs}^+$ about its possible configuration and physical properties, and find it could be configured as a mixture of two types of structures, $[1_c]_{cu} \otimes [1_c]_{sc}$ and $[1_c]_{cc} \otimes [1_c]_{su}$, or $[3_c]_{cu} \otimes [\overline{3}_c]_{sc}$ and $[3_c]_{cc} \otimes [\overline{3}_c]_{su}$, with $J^P = 1^+$. Physically, it then appears to be the emergence of a compound of four possible currents in each configuration, which tells the single current evaluation of hadron spectroscopy and their decay properties are sometimes not enough. We find in both cases the energy spectra may fit well with the experimental observation, i.e. 3.98 GeV, within the uncertainties, while noted the former is not favored by vector meson exchange model. Various $Z_{cs}^+(3980)$ decay modes are evaluated, which are critical for pinning down its configuration and left for experimental verification. We also predict the mass of $Z_{cs}^0$, the neutral partner of $Z_{cs}^+(3980)$, and analyze its dominant decay probabilities.

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The establishment of quark model (QM) in the 50s of last century is is a milestone in the exploration of micro world [1, 2]. The spectroscopy of conventional meson and baryon in QM are as of yet gradually confirmed in experiment and are going to be complete. Entering the new millennium, with the development of technology the so-called exotic state emerges in experiment, like \( X(3872) \) [3], and new ones tend to appear more frequently. Now we already have a bunch of exotic-state candidates waiting for clarification, similar to the phase of "particle zoo" in last century. To discover more exotic states and explore their properties are one of the most intriguing and important topics in particle physics for nowadays physicists, which may promote our understanding of quantum chromodynamics (QCD) and enrich our knowledge of hadron spectroscopy.

In light hadron sector, because normally the spacings between various states are small, it is hard to discriminate the exotic states from the conventional hadrons, except the former possess some peculiar quantum numbers. In contrast, the exotic states in heavy hadron sector may have relatively distinct signatures. Indeed, in recent years a bunch of so-called plethora charmonium-/bottomonium-like states XYZ are observed in experiment [3–7], which provides a new horizon for our understanding of the emergence of structures in quantum chromodynamics (QCD).

Very recently, by scrutinizing the \( D^- \), \( D_s^- \) and \( K^- \)-meson production in electron-position collision, BESIII Collaboration observed a structure in \( D \) and \( D_s \) invariant mass of about 3.982 GeV with 5.3 \( \sigma \) significance and decay width of some 12.8 MeV [8]. If it is true, the new structure should be a charged hidden charm state, and hence named as \( Z_{cs}^+ (3980) \). From its known decay products the new state most likely possesses a quantum number of 1\(^+\). If the BESIII observation is further confirmed to be a hadronic structure, rather the kinematic effect, it turns out to be a remarkable discovery in the exploration of hadron spectroscopy, the novel strange-hidden-charm state.

About strange-hidden-charm(bottom) states there have been some investigations in the literature [9–17]. Nevertheless, before the experimental evidence appearing, theoretical investigations tend to be with large uncertainties, say diverse yields in different theoretical frameworks. The BESIII observation in some sense rules out the tetraquark octet-octet configuration [16], at least the \( Z_{cs}^+ (3980) \) has a weak coupling to that kind of current. Considering
the previous calculation results deviate more or less from the experimental measurement, we
find the new structure could be a compound of states from four molecular currents, which
may then be evaluated by virtue of the model independent Shifman, Vainshtein and Zakharov
(SVZ) QCD sum rule technique [18].

The SVZ sum rule, viz QCD sum rule, has some peculiar advantages in exploring hadron
properties involving nonpertubative QCD. It is a QCD based theoretical framework which
incorporates nonperturbative effects universally order by order, rather a phenomenological
model, and has already achieved a lot in the study of hadron spectroscopy and decays. To
establish the sum rules, the starting point is to construct the proper interpolating currents
corresponding to the hadron of interest. Using the current, one can then construct the two-
point correlation function, which has two representations: the QCD representation and the
phenomenological representation. Then, roughly speaking, by equating these two representa-
tions, the QCD sum rules will be formally established, from which the hadron mass and
decay width may be deduced.

In this work, we make a thorough analysis on the $Z_{c_s}^+(3980)$ in the framework of QCD sum
rule, including mass spectroscopy and decay properties. Composite currents in molecular and
tetraquark configurations are taken into account. Its neutral partner $Z_{c_s}^0$ is also evaluated for
future confirmation. In order to analyze the mass spectrum of $Z_{c_s}^+$ state, one has to construct
the appropriate current for it. The lowest order possible interpolating currents of $1^+$ charged
charmonium-like strange molecular state take the following four forms:

$$j^{D^*D^+}_\mu = i[c_a\gamma_\mu c_a][\bar{s}_b\gamma_5 c_b], \quad (1)$$
$$j^{D^0D_s^+}_\mu = i[c_a\gamma_5 u_a][\bar{s}_b\gamma_\mu c_b], \quad (2)$$
$$j^{J/\psi K^+}_\mu = i[c_a\gamma_5 c_a][\bar{s}_b\gamma_5 u_b], \quad (3)$$
$$j^{\eta_c K^{**}}_\mu = i[c_a\gamma_5 c_a][\bar{s}_b\gamma_\mu u_b], \quad (4)$$

where $a$ and $b$ are color indices, $\mu$ denotes Lorentz index. Therefore, the interpolating current
for $[1_c] \otimes [1_c]$ state of $Z_{c_s}^+$ can be expressed as mixing of the currents in Eqs. (1)-(4), i.e.,

$$j^M_\mu = A_1 j^{D^*D^+}_\mu + B_1 j^{D^0D_s^+}_\mu + C_1 j^{J/\psi K^+}_\mu + D_1 j^{\eta_c K^{**}}_\mu. \quad (5)$$
On the other hand, the possible tetraquark interpolating currents can be constructed as

\[ j^{A}_\mu = i\epsilon_{abc}\epsilon_{dec}[u^T a C\gamma_5 c b][\bar{s}_d\gamma_\mu \bar{c}, c^T], \]  
\[ j^{B}_\mu = i\epsilon_{abc}\epsilon_{dec}[u^T a C\gamma_\mu c b][\bar{s}_d\gamma_5 \bar{c}, c^T], \]  
\[ j^{C}_\mu = i\epsilon_{abc}\epsilon_{dec}[u^T a C\gamma_5 c b][\bar{s}_d\gamma_\mu \gamma_5 \bar{c}, c^T], \]  
\[ j^{D}_\mu = i\epsilon_{abc}\epsilon_{dec}[u^T a C\gamma_\mu \gamma_5 c b][\bar{s}_d \bar{c}, c^T]. \]  

Here, the superscripts \( A, B, C, \) and \( D \) indicate the currents composed of \( 0^+ \otimes 1^+ \), \( 1^+ \otimes 0^+ \), \( 0^- \otimes 1^- \), and \( 1^- \otimes 0^- \), respectively, and \( C \) represents the charge conjugation matrix. Therefore, the interpolating current for \([3_c] \otimes [\bar{3}_c]\) state of \( Z_{cs}^+ \) can be expressed as mixing of the currents in Eqs. (6)-(9), i.e.,

\[ j^T_\mu = A_2 j^{A}_\mu + B_2 j^{B}_\mu + C_2 j^{C}_\mu + D_2 j^{D}_\mu. \]  

Here, the superscript \( M \) and \( T \) denote the molecular and tetraquark state, respectively. It is noteworthy that in the literature most of the calculations were performed by single-current analysis. However, in fact, different inner configurations may yield different physical results, not to say their interference. Before the advent of experimental measurement, to make a comprehensive analysis on a typical hadron is usually unrealistic, but now for \( Z_{cs}^+ \) we can do so, which is important in order to know its real structure.

With the currents of (5) and (10), the two-point correlation function can be readily established, i.e.,

\[ \Pi_{\mu\nu}(q) = i \int d^4xe^{iq\cdot x}\langle 0 | T\{ j_\mu(x), j^\dagger_\nu(0)\} | 0 \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2), \]  

where \(|0\rangle\) denotes the physical vacuum. In the partonic representation, the dispersion relation may express the correlation function \( \Pi(q^2) \) as

\[ \Pi_i^{OPE}(q^2) = \int_{s_{\min}}^\infty ds \rho_i^{OPE}(s) \frac{s}{s - q^2} + \Pi_i^{sum}(q^2). \]  

Here, \( \rho_i^{OPE}(s) = \text{Im}[\Pi_i^{OPE}(s)]/\pi \) and \( \Pi_i^{sum}(q^2) \) is the sum of those contributions in the correlation function that have no imaginary part but have nontrivial magnitudes after the Borel transformation. \( s_{\min} \) is a kinematic limit, which usually corresponds to the square of
the sum of the current quark masses of the hadron \[ s \min = (2m_c + m_s + m_u)^2 \]. In (12) the subscript \( i \) represents \( M \) and \( T \) for molecular and tetraquark states, respectively. By applying the Borel transformation to (12), we then have

\[
\Pi_i^{OPE}(M_B^2) = \int_{s_{\min}}^{\infty} ds \rho_i^{OPE}(s)e^{-s/M_B^2} + \Pi_i^{\text{sum}}(M_B^2). \tag{13}
\]

In the hadronic representation, after isolating the ground state contribution from the hadronic state, we obtain the correlation function \( \Pi(q^2) \) in dispersion integral over the physical region, i.e.,

\[
\Pi_i(q^2) = \frac{\lambda_i^2}{M_i^2 - q^2} + \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\rho_i(s)}{s - q^2}, \tag{14}
\]

where \( M_i \) denotes the mass the lowest lying hadronic state, either molecule-like or tetraquark state, \( \rho_i(s) \) is the spectral density that contains the contributions from higher excited states and the continuum states above the threshold \( s_0 \). The coupling constant \( \lambda_i \) is defined through

\[
\langle 0|j_{\mu}^i|Z^{+}\rangle = \lambda_i^\epsilon^\mu.
\]

By performing the Borel transform on the hadronic side, Eq. (14), and matching it to Eq. (13), we can then obtain the mass and the coupling constant of the tetraquark state,

\[
M_i(s_0, M_B^2) = \sqrt{\frac{L_{i1}(s_0, M_B^2)}{L_{i0}(s_0, M_B^2)}}, \tag{15}
\]

\[
\lambda_i^2 e^{-M_i^2/M_B^2} = L_{i0}(s_0, M_B^2), \tag{16}
\]

where the moments \( L_1 \) and \( L_0 \) are, respectively, defined as:

\[
L_{i0}(s_0, M_B^2) = \int_{s_{\min}}^{s_0} ds \rho_i^{OPE}(s)e^{-s/M_B^2} + \Pi_i^{\text{sum}}(M_B^2), \tag{17}
\]

\[
L_{i1}(s_0, M_B^2) = \frac{\partial}{\partial \frac{1}{M_B^2}} L_{i0}(s_0, M_B^2). \tag{18}
\]

In the numerical calculation of QCD sum rules, the values of input parameters we take are: \( m_u = 2.3 \) MeV, \( m_d = 6.4 \) MeV, \( m_s(2 \text{ GeV}) = (95 \pm 5) \) MeV, \( m_c(m_c) = \overline{m}_c = (1.275 \pm 0.025) \) GeV, \( \langle \bar{q}q \rangle = -(0.24 \pm 0.01)^3 \) GeV\(^3\), \( \langle \bar{s}s \rangle = (0.8\pm0.1)\langle \bar{q}q \rangle \), \( \langle g_s^2 G^2 \rangle = 0.88 \) GeV\(^4\), \( \langle \bar{g}_s \sigma \cdot Gq \rangle = m_0^2 \langle \bar{q}q \rangle \), \( \langle \bar{s}g_s \sigma \cdot Gs \rangle = m_0^2 \langle \bar{s}s \rangle \), \( \langle g_s^3 G^3 \rangle = 0.045 \) GeV\(^6\), and \( m_0^2 = 0.8 \) GeV\(^2\), in which the \( \overline{\text{MS}} \) running heavy quark masses is adopted.

Moreover, there exist two additional parameters \( M_B^2 \) and \( s_0 \) introduced in establishing the sum rules, which will be fixed in light of the so-called standard procedures abiding by two
FIG. 1: (a) The ratios $R_{i}^{OPE}$ and $R_{i}^{PC}$ as functions of the Borel parameter $M_{B}^{2}$ for different values of $\sqrt{s_{0}}$, where blue lines represent $R_{i}^{OPE}$ and red lines denote $R_{i}^{PC}$. (b) The mass of $Z_{cs}^{+}$ as a function of the Borel parameter $M_{B}^{2}$ for different values of $\sqrt{s_{0}}$.

Various $s_{0}$ satisfying above constraints should be taken into account in the numerical analysis. Among these values, we need to pick up the one which yields an optimal window for Borel parameter $M_{B}^{2}$. That is to say, in the optimal window, the tetraquark mass $M_{Z}$ is somehow independent of the Borel parameter $M_{B}^{2}$. In practice, we may vary $\sqrt{s_{0}}$ by 0.1 GeV in numerical calculation, which sets the upper and lower bounds and hence the uncertainties of $\sqrt{s_{0}}$.

With above preparation we numerically evaluate the mass spectrum of the $Z_{cs}^{+}$ with different $(A_{1}, B_{1}, C_{1}, D_{1})$ and $(A_{2}, B_{2}, C_{2}, D_{2})$ for molecular and tetraquark states, respectively. The ratios $R_{i}^{OPE}$ and $R_{i}^{PC}$ of the molecular state $Z_{cs}^{+}$ are shown as functions of Borel parameter $M_{B}^{2}$ in Fig. 1(a) with $A_{1} = 0.47$, $B_{1} = -0.47$, $C_{1} = 0.53$, and $D_{1} = -0.53$ and with different values of $\sqrt{s_{0}}$, 4.6, 4.7, and 4.8 GeV. The dependency relations between $Z_{cs}^{+}$ mass
and parameter $M_B^2$ are given in Fig. 1(b). The optimal window for Borel parameter was found at $2.5 \leq M_B^2 \leq 3.3 \text{ GeV}^2$, and the mass and coupling constant of $Z_{cs}^+$ are extracted as

\begin{align}
M_{Z_{cs}^+}^M &= (3.98 \pm 0.14) \text{ GeV} , \\
\lambda_{Z_{cs}^+}^M &= (1.81 \pm 0.12) \times 10^{-2} \text{ GeV}^5 .
\end{align}

(21)

(22)

Replace $u$–quark with $d$–quark in Eq.(5), its neutral partner $Z_{cs}^0$ will be obtained. With the same mixing coefficients, the ratios $R_{\text{OPE}}^M$ and $R_{\text{PC}}^M$ of $Z_{cs}^0$ are shown as functions of Borel parameter $M_B^2$ in Fig. 2(a) with different values of $\sqrt{s_0}$, 4.7, 4.8, and 4.9 GeV and the dependency relations between $Z_{cs}^0$ mass and parameter $M_B^2$ are given in Fig. 2(b). The optimal window for Borel parameter was found at $2.6 \leq M_B^2 \leq 3.4 \text{ GeV}^2$, and the mass and coupling constant of $Z_{cs}^0$ are extracted as

\begin{align}
M_{Z_{cs}^0}^M &= (4.03 \pm 0.16) \text{ GeV} , \\
\lambda_{Z_{cs}^0}^M &= (1.95 \pm 0.11) \times 10^{-2} \text{ GeV}^5 .
\end{align}

(23)

(24)

On the other hand, the ratios $R_{T}^{\text{OPE}}$ and $R_{T}^{\text{PC}}$ of the compact tetraquark, the diquark-antidiquark, state $Z_{cs}^+$ are shown as functions of Borel parameter $M_B^2$ in Fig. 3(a) with $A_2 = 0.61$, $B_2 = -0.61$, $C_2 = -0.36$, and $D_2 = 0.36$ and with different values of $\sqrt{s_0}$, 4.4, 4.5, and 4.6 GeV. The dependency relations between $Z_{cs}^+$ mass and parameter $M_B^2$ are given in Fig. 3(b). The optimal window for Borel parameter is found at $2.1 \leq M_B^2 \leq 3.0 \text{ GeV}^2$, and the mass and coupling constant of $Z_{cs}^+$ are extracted as

\begin{align}
M_{Z_{cs}^+}^T &= (3.98 \pm 0.08) \text{ GeV} , \\
\lambda_{Z_{cs}^+}^T &= (1.29 \pm 0.10) \times 10^{-2} \text{ GeV}^5 .
\end{align}

(25)

(26)
Its neutral partner $Z^0_{cs}$ will be obtained by replacing $u-$quark with $d-$quark in Eq. (10). With the same mixing coefficients, the ratios $R^{OPE}$ and $R^{PC}$ of $Z^0_{cs}$ are shown as functions of Borel parameter $M_B^2$ in Fig. 4(a) with different values of $\sqrt{s_0},$ 4.5, 4.6, and 4.7 GeV and the dependencies between $Z^0_{cs}$ mass and parameter $M_B^2$ are given in Fig. 4(b). The optimal window for Borel parameter is found at $2.1 \leq M_B^2 \leq 3.0$ GeV$^2$, and the mass and coupling constant of $Z^0_{cs}$ are therefore obtained as

\begin{align}
M_{Z^0_{cs}}^T &= (4.04 \pm 0.09) \text{ GeV} , \\
\lambda_{Z^0_{cs}}^T &= (1.43 \pm 0.10) \times 10^{-2} \text{ GeV}^5 .
\end{align}

In above results, (22)-(28), errors stem from the uncertainties of the quark masses, the condensates and the threshold parameter $\sqrt{s_0}$.

Note, as shown in Appendices, that the cross terms $A_1 B_1 (A_2 C_2)$ and $C_1 D_1 (B_2 D_2)$ give no contribution to the mass spectrum, while $A_1 C_1, A_1 D_1, B_1 C_1 (A_2 B_2, A_2 D_2, B_2 C_2)$, and
The calculation of the decay vertex starts from the three-point correlation function,

$$\Pi^{i\mu\nu}(q, q_1, q_2) = \int d^4x d^4y \, e^{iq_1 \cdot x + iq_2 \cdot y} \Pi^{i\mu\nu}_\Pi(x, y) \, ,$$

(29)

where \(\Pi^{i\mu\nu}_\Pi(x, y) = \langle 0 | T[\bar{j}^\mu D^* (x) j^\nu_{5s} (y)] 0 \rangle\) and \(q = q_1 + q_2\). The interpolating currents of \(\bar{D}^*\) and \(D^+_s\) take the following forms:

\[
\begin{align*}
\bar{j}^\mu \bar{D}^* &= \bar{c}a \gamma^\mu u_a, \\

j^{D^+_s}_\mu &= i \bar{s}b \gamma^5 c_b. 
\end{align*}
\]

(30)

(31)

On the phenomenological side of the QCD sum rules, we insert the intermediate states into the correlation function (29), and obtain

$$\Pi^{\text{phen}}_{\mu\nu} = -\chi^{Z^+_cs} m_{D^*} f_{D^*} f_{D^+_s} m^2_{D^*_s} g^{Z^+_cs \bar{D}^* D^+_s} \left( \frac{m_c + m_s}{q^2 - M^2_{Z^+_cs}} \right) \left( q^2_1 - m^2_{D^*} \right) \left( q^2_2 - m^2_{D^+_s} \right) \times \left( -g_{\mu\alpha} \frac{q_{1\alpha}}{m^2_{D^*}} \right) \left( -g_{\nu\alpha} \frac{q_{2\alpha}}{M^2_{Z^+_cs}} \right).$$

(32)

Here, \(g^{Z^+_cs \bar{D}^* D^+_s}\) presents \(Z^+_{cs}\) decay form factor; \(f_{D^*}\) and \(f_{D^+_s}\) are meson decay constants defined as:

\[
\begin{align*}
\langle \bar{D}^* D_s | Z^+_{cs} \rangle &= g^{Z^+_cs \bar{D}^* D^+_s} \varepsilon^\alpha_{\mu}(q_1) \varepsilon^\alpha(q_1) \, , \\

\langle 0 | j^\mu_{\bar{D}^*} | \bar{D}^* \rangle &= m_{D^*} f_{D^*} \varepsilon^\alpha_{\mu}(q_1) \, , \\

\langle 0 | j^\mu_{D^+_s} | D_s \rangle &= \frac{f_{D^+_s} m^2_{D^+_s}}{m_c + m_s}. 
\end{align*}
\]

(33)

(34)

(35)

On the OPE side of QCD sum rules, the three-point function can be formulated as

$$\Pi^{\text{OPE}} = \int dv ds \, \rho^{\text{OPE}}(s, v) \left( \frac{s - q^2_1}{v - q^2_2} \right).$$

(36)

Performing Borel transforms \(q^2_1 = q^2_2 \to M^2_B\) on both side of (32) and (36), we obtain the form factor \(g^{Z^+_cs \bar{D}^* D^+_s}\) by equating \(\Pi^{\text{OPE}}(s_0, M^2_B)\) to \(\Pi^{\text{phen}}(s_0, M^2_B)\), where \(s_0\) is the continuum threshold and \(M^2_B\) is the Borel parameter of the \(D\) meson. Details of the decay widths calculation are presented in the Appendix for reference.

In numerical evaluation of the \(Z^+_{cs}\) decays, we adopt the mass and decay constants employed in Refs.\[29–33\], i.e., \(m_{D^*} = 2.01\text{ GeV}\), \(m_{D^+_s} = 1.97\text{ GeV}\), \(m_{D^*} = 2.11\text{ GeV}\),
TABLE I: Form factors and the decay widths of Zcs.

| Decay mode | Form factor g (GeV) | Decay width Γ (MeV) |
|------------|---------------------|---------------------|
| $Z_{cs}^M \rightarrow D^* D_s$ | 3.72 ± 0.78 | 3.23 ± 1.29 |
| $Z_{cs}^M \rightarrow \bar{D} D_s^*$ | -4.18 ± 0.88 | 4.08 ± 1.65 |
| $Z_{cs}^M \rightarrow J/\psi K^+$ | 1.68 ± 0.39 | 4.92 ± 2.23 |
| $Z_{cs}^M \rightarrow \eta_c K^*$ | -2.84 ± 0.69 | 8.42 ± 3.84 |
| $Z_{cs}^T \rightarrow \bar{D}^* D_s$ | 2.09 ± 0.42 | 1.14 ± 0.44 |
| $Z_{cs}^T \rightarrow \bar{D} D_s^*$ | 1.30 ± 0.30 | 0.40 ± 0.18 |
| $Z_{cs}^T \rightarrow J/\psi K^+$ | 0.74 ± 0.22 | 0.98 ± 0.54 |
| $Z_{cs}^T \rightarrow \eta_c K^*$ | 3.83 ± 1.07 | 15.60 ± 8.08 |

$m_D = 1.86 \text{ GeV}, m_{J/\psi} = 3.07 \text{ GeV}, m_{\eta_c} = 2.98 \text{ GeV}, m_{K^*} = 0.89 \text{ GeV}, m_K = 0.49 \text{ GeV},$
$f_{D^*} = 0.24 \text{ GeV}, f_{D_s} = 0.24 \text{ GeV}, f_{D_s^*} = 0.33 \text{ GeV}, f_D = 0.18 \text{ GeV}, f_{D^*} = 0.41 \text{ GeV},$
$f_{D_s} = 0.35 \text{ GeV}, f_{K^*} = 0.22 \text{ GeV},$ and $f_K = 0.16 \text{ GeV}.$

With the above inputs and formula in Appendix we can readily obtain the various form factors and decay widths, as presented in Table I. It is worth mentioning that the molecular Zcs decay widths and form factors are give for the first time in this work. The tetraquark Zcs decays were once evaluated by Dias, et al. [10], however since they adopted only two currents, that is only [6] and [7], the Zcs to $J/\psi K^+$ and $\eta_c K^*$ decay widths are quite different. Furthermore, we find there is a misprint in the contribution of mixed condensate in ref. [10], the gluon in light quark radiation is missed, whatsoever its numerical effects are tiny.

In summary, we perform a complete analysis on the $Z_{cs}^+(3980)$ in the framework of QCD sum rule, which matches well the recent experimental observation in BESIII experiment. In our calculation, the full leading order currents in various configurations are taken into account, their relative weights are estimated by fitting the QCD sum rule results to the experimental measurements. We consider both molecular and tetraquark structures with different configurations, that is set $Z_{cs}^+(3980)$ in $[1_c]_{cu} \otimes [1_c]_{sc}$ and $[1_c]_{cc} \otimes [1_c]_{su},$ or $[3_c]_{cu} \otimes [\bar{3}_c]_{sc}$ and $[3_c]_{cc} \otimes [\bar{3}_c]_{su}$ representation with $J^P = 1^+.$ Within the error of uncertainties, the summed width of four dominant decay channels falls in the experimental measurement, that is $\Gamma_M = (20.65 \pm 9.01) \text{ MeV}$ and $\Gamma_T = (18.12 \pm 9.24) \text{ MeV}$ for molecular and tetraquark...
The mass spectrum of the $Z_{cs}^+$’s neutral partner $Z_{cs}^0$ is also calculated. Note, with the results of this work, future experimental measurements on $Z_{cs}^+$ dominant decay channels may pin down its inner configurations.

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Appendix A: The spectral densities of $Z_{car{s}}$

In order to calculate the spectral density of the operator product expansion (OPE) side, the light quark ($q = u, d$ or $s$) and heavy-quark ($Q = c$ or $b$) full propagators $S^q_{jk}(x)$ and $S^Q_{jk}(p)$ are employed, say

\[
S^q_{jk}(x) = \frac{i\delta_{jk} \not{p} - \delta_{jk}m_q}{2\pi^2 x^4} - \frac{\delta_{jk}m_q}{4\pi^2 x^2} - \frac{i\bar{q} G^a_{\alpha \beta}}{32\pi^2 x^2} \sigma^{\alpha \beta} \not{f} + \frac{\delta_{jk} (\bar{q}q)}{12} + \frac{i\delta_{jk} \not{p} - \delta_{jk}x^2}{48 m_q (\bar{q}q)} - \frac{\delta_{jk}x^2}{192} (g_s \bar{q} \sigma \cdot Gq) + \frac{i\delta_{jk}x^2}{152} m_q (g_s \bar{q} \sigma \cdot Gq) - \frac{t^a_{jk} G_{\alpha \beta}}{192} (g_s \bar{q} \sigma \cdot Gq) + \frac{i\delta_{jk} (\sigma_{\alpha \beta} \not{f} + \not{f} \sigma_{\alpha \beta})m_q (g_s \bar{q} \sigma \cdot Gq)}{168} , \quad (A1)
\]

\[
S^Q_{jk}(p) = \frac{i\delta_{jk} (\not{p} + m_Q)}{p^2 - m_Q^2} - \frac{i\delta_{jk} g_s^2 G^2}{4 (p^2 - m_Q^2)^2} \sigma^{\alpha \beta} (\not{p} + m_Q) + (\not{p} + m_Q) \sigma^{\alpha \beta} + \frac{i\delta_{jk} m_Q (g_s^2 G^2)}{12 (p^2 - m_Q^2)^2} \left[ 1 + \frac{m_Q (\not{p} + m_Q)}{p^2 - m_Q^2} \right] + \frac{i\delta_{jk} (\not{p} + m_Q) [\not{p}(p^2 - 3m_Q^2) + 2m_Q (2p^2 - m_Q^2)]}{48 (p^2 - m_Q^2)^6} \times (\not{p} + m_Q) \} (g_s^3 G^3) . \quad (A2)
\]

Here, the vacuum condensates are explicitly shown. For more explanation on above propagators, readers may refer to Refs. [19].

\[
\rho^{OPE}(s) = \rho^{pert}(s) + \rho^{(\bar{q}q)}(s) + \rho^{(\bar{s}s)}(s) + \rho^{(G^2)}(s) + \rho^{(\bar{q}Gq)}(s) + \rho^{(\bar{s}G\bar{s})}(s) + \rho^{(\bar{q}q\bar{q})}(s) + \rho^{(G^3)}(s) + \rho^{(\bar{q}q\bar{q})G^2}(s) + \rho^{(\bar{s}s)(G^2)}(s) + \rho^{(\bar{s}s)(G^2)}(s) + \rho^{(\bar{q}q\bar{q})(G^2)}(s) + \rho^{(\bar{s}s)(G^2)}(s) + \rho^{(\bar{q}q\bar{q})(G^2)}(s) , \quad (A3)
\]

\[
\Pi^{sum}(q^2) = \Pi^{(\bar{q}q)(\bar{s}s)}(q^2) + \Pi^{(G^3)}(q^2) + \Pi^{(\bar{q}q\bar{q})G^2}(q^2) + \Pi^{(\bar{s}s)(G^2)}(q^2) + \Pi^{(\bar{s}s)(G^2)}(q^2) + \Pi^{(\bar{q}q\bar{q})G^2}(q^2) . \quad (A4)
\]

1. The spectral densities for molecular state

The spectral density $\rho^{OPE}(s)$ is calculated up to dimension eight.
\[
\rho_{\text{pert}}(s) = \frac{1}{3 \times 2^{13} \pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \frac{d\alpha}{\alpha^3} \int_{\beta_{\text{min}}}^{1-\alpha} \frac{d\beta}{\beta^3} F_{\alpha\beta}^2 (\alpha + \beta - 1) \left\{ 18A^2 \left[ F_{\alpha\beta}(1 + \alpha + \beta) - 4m_c(2m_q + \beta m_s + \alpha m_q(1 + \alpha + \beta)) \right] \right. \\
+ 6 \left[ 3(C^2 + D^2) F_{\alpha\beta}(1 + \alpha + \beta) + 4m_c^2 (\alpha + \beta - 1) [3C^2 + D^2 (2 + \alpha + \beta)] \right] \\
+ \Delta \left( F_{\alpha\beta}(1 + \alpha + \beta) + 12m_c [m_c (\alpha + \beta - 1) - (m_q + m_s) (\beta^2 + \beta + \alpha + 2 \alpha)] \right) \\
+ B C \left( F_{\alpha\beta}(1 + \alpha + \beta) + 12m_c [m_c (\alpha + \beta - 1) - (m_q + m_s) (\alpha^2 + \alpha + \beta + 2 \beta)] \right) \\
+ \Delta D \left( F_{\alpha\beta}(1 + \alpha + \beta) + 4m_c [m_c (\alpha + \beta - 1) (\alpha + \beta + 2) - 3 (\alpha + \beta) [2m_q + m_s (1 + \alpha + \beta)] \right] + B D \left( 3F_{\alpha\beta}(1 + \alpha + \beta) \\
+ 4m_c (m_c (\alpha + \beta - 1) (\alpha + \beta + 2) - 3 (\alpha + \beta) [2m_q + m_s (1 + \alpha + \beta)] \right) \right\}, \tag{A5}
\]

\[
\rho_{\text{qs}}(s) = \frac{\langle \bar{q} q \rangle}{2^9 \pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \frac{d\alpha}{\alpha^2} \int_{\beta_{\text{min}}}^{1-\alpha} \frac{d\beta}{\beta^2} \left\{ 6A^2 F_{\alpha\beta}(2m_c F_{\alpha\beta} + \beta m_q F_{\alpha\beta} - 4m_c^2 m_s) \right. \\
+ 6B^2 F_{\alpha\beta}(2m_c (\alpha + \beta) F_{\alpha\beta} + \beta m_q F_{\alpha\beta} - 4m_c^2 m_s) \right. \\
+ 6C^2 F_{\alpha\beta}(m_q F_{\alpha\beta} + 2m_c^2 m_s (\alpha + \beta - 4m_c^2 m_s)) \right. \\
+ \Delta A F_{\alpha\beta}(F_{\alpha\beta}(m_q - 2m_s) + 2m_c F_{\alpha\beta} (\alpha + \beta + 2 \beta) \right. \\
+ A D F_{\alpha\beta}(F_{\alpha\beta} + 2m_c^2) (m_q - 2m_s) + 2m_c F_{\alpha\beta} (\alpha + \beta + 2 \beta) \right. \\
+ B C F_{\alpha\beta}(F_{\alpha\beta} + 2m_c^2) (m_q - 2m_s) + 2m_c F_{\alpha\beta} (\alpha^2 + \alpha + \beta + 2 \beta) \right. \\
+ B D F_{\alpha\beta}(F_{\alpha\beta} + 2m_c^2) (m_q - 2m_s) + 2m_c F_{\alpha\beta} (\alpha^2 + \alpha + \beta + 2 \beta) \right. \\
+ \frac{\mathcal{H}^2}{\alpha (\alpha - 1)} \left. \left( 6 (A^2 + B^2 + C^2 + D^2) m_q - 12C^2 m_s \right) \right\}, \tag{A6}
\]

\[
\rho_{\text{ss}}(s) = \frac{\langle s s \rangle}{2^9 \pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \frac{d\alpha}{\alpha^2} \int_{\beta_{\text{min}}}^{1-\alpha} \frac{d\beta}{\beta^2} \left\{ -24A^2 m_q^2 m_s F_{\alpha\beta} \right. \\
+ 6B^2 F_{\alpha\beta}(m_s F_{\alpha\beta} - 4m_q^2 m_q + 2m_c^2 m_s (\alpha + \beta)) \right. \\
+ 6C^2 F_{\alpha\beta}(m_s F_{\alpha\beta} - 4m_q^2 m_q + 2m_c^2 m_s (\alpha + \beta)) \right. \\
+ \frac{\mathcal{H}^2}{\alpha (\alpha - 1)} \left. \left( 6 (A^2 + B^2 + C^2 + D^2) m_q - 12C^2 m_s \right) \right\}, \tag{A6}
\]

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\[
\frac{\rho^{(G^2)}(s)}{32} = \frac{\langle g_s^2 G^2 \rangle}{32} \times \frac{1}{24 \pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \left\{ \int_{\beta_{\min}}^{1-\alpha} d\beta \frac{\alpha^2 \beta^2}{\alpha^3 \beta^3} \left\{ 36 A^2 \left[ \alpha \beta F_{\alpha \beta} \left[ 2 \alpha \beta m_c m_s (\alpha + \beta - 2) - F_{\alpha \beta} (\alpha^2 - 2 \alpha + 4 \alpha \beta + 3 \beta^2) \right] - m_c (\alpha + \beta - 1)(\alpha + \beta + 1) \left[ 3 \beta m_s \cdots \right] \right] \right. \\
\left. + 36 B^2 \left[ \alpha \beta F_{\alpha \beta} \left[ 2 \beta F_{\alpha \beta} + 6 \alpha \beta m_c m_s \right] - \alpha^2 \beta^2 (\alpha + \beta + 2)(\alpha^2 - \alpha \beta + \beta^2) + 18 \alpha \beta F_{\alpha \beta}^2 (3 \alpha^2 + 3 \beta^2 + 8 \alpha \beta - 3) \right] \\
+ 36 m_c^2 F_{\alpha \beta} \left[ 2 \alpha^5 + 7 \alpha^4 \beta - 4 \alpha^3 + 13 \alpha^2 \beta^2 + 2 \beta^3 - 4 \beta^3 + 2 \beta^5 \right] \\
+ \alpha^2 \beta ((\beta - 1)(13 \beta + 9) + 2) + \alpha \beta (7 \beta^3 - 9 \beta + 4) \right] + D^2 \left[ 18 \alpha \beta F_{\alpha \beta} (F_{\alpha \beta} (8 \alpha \beta \cdots) \\
+ (\alpha - 4) \alpha + \beta^2 - 4 \beta + 3) + 12 \alpha \beta m_c^2 \right] + 36 m_c^2 (\alpha + \beta - 1)(F_{\alpha \beta} (\alpha^4 \\
+ \alpha^3 (\beta + 4) + 3 \alpha^2 (\beta - 1) + \alpha \beta^2 (\beta + 3) + \beta^2 (\beta + 4) - 3) \\
+ m_c^2 (\alpha + \beta - 1)(\alpha + \beta)(\alpha^2 - \alpha \beta + \beta^2) \right] - AC \left[ 6 m_c (\alpha + \beta - 1)(3 m_s F_{\alpha \beta} (2 \alpha^3 \\
+ (\alpha + 1) \beta^3 + \beta^4) - m_c^3 (\alpha^3 + 3 \beta^3)(m_c (\alpha + \beta - 1) - m_s (\beta^2 + \beta + 2 \alpha \beta) \\
+ \alpha^3 (3 \alpha + \beta - 1))(\alpha + \beta)(\alpha^2 + \beta^3)(m_c (\alpha + \beta - 1) - m_s (\beta^2 + \beta + 2 \alpha \beta) \\
+ \alpha^3 (\beta + 4) - 3 \alpha \beta m_c (\alpha^4 + \beta^2 (\beta - 1) + \alpha \beta^2 (\beta + 3) \\
+ \beta^2 (\beta (\beta + 4) - 3)) \right] + 3 \alpha \beta F_{\alpha \beta} \left[ 5 \alpha^2 + \alpha (8 \beta - 4) + \beta (5 \beta - 4) + 3 \right] \\
+ \alpha \beta (2 \alpha - 1) (\beta^2 + 2 (\alpha - 1) \alpha \beta + \alpha ((\alpha - 2) \alpha + 3) + \beta^3 + 3 \beta - 2) \\
+ m_s (\alpha + \beta)(\alpha^2 + 2 \alpha (\beta + 2) + \beta(\beta + 4) - 3) \right] + AD \left[ - 3 \alpha \beta F_{\alpha \beta}^2 (3 \alpha^2 \\
+ \alpha (8 \beta - 4) + \beta (11 \beta + 4) - 3) + 6 m_c^2 (\alpha + \beta - 1)(\alpha + \beta)(\alpha^2 \\
+ \alpha \beta)(\alpha + \beta - 1) - m_s (\alpha (\beta + 2) + \beta^2 + \beta) \right] + 6 m_c F_{\alpha \beta}^2 (m_c (\alpha^5 \\
+ \alpha^4 (6 \beta + 3) + \alpha^3 (\beta + 1)(9 \beta - 7) + \alpha^2 (\beta + 1)(\beta (5 \beta - 9) + 3) }
\]
\[
\rho^{(qGq)}(s) = \frac{\langle q\bar{q} \cdot Gq \rangle}{3 \times 2^9 \pi^4} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \int_{\beta_{\text{min}}}^{1-\alpha} \frac{d\beta}{\beta^2} \left\{ 16 B^2 m_c (2 \beta m_c m_q - F_{\alpha \beta}(2 \alpha + 3 \beta)) \right\} + 6 D^2 m_q^2 + 2 A C 2 F_{\alpha \beta}(\alpha + \beta)(m_c(\alpha + 2 \beta - 1) - \alpha m_s) - \frac{\alpha^2}{2} m_c^2 m_q^2 \\
- 6 D^2 m_q^2 m_c + A D \frac{m_c}{(6 \alpha^2 + 11 \alpha \beta + 2 \beta^2) + 2 \beta m_c(\alpha + \beta)(\beta m_c - 3 m_s)} + B C \frac{m_c}{(6 \alpha^2 + 11 \alpha \beta + 2 \beta^2)} + 2 \beta m_c(\alpha + \beta)(\beta m_c - 3 m_s)) \\
- C \frac{m_c}{(6 \alpha^2 + 11 \alpha \beta + 2 \beta^2)} + 2 \beta m_c(\alpha + \beta)(\beta m_c - 3 m_s)) \\
- 12 D^2 (\alpha - m_c^2)(m_q - 3 m_s) + 6 (A^2 + B^2) \left[ \frac{3 m_c H_\alpha}{1 - \alpha} - 2 m_q H_\alpha + m_c^2 (m_q - 3 m_s) \right] \\
+ A C [2 m_c^2 (m_q - 3 m_s) + H_\alpha (3 m_s - 2 m_q)] - \frac{H_\alpha (2 m_c - m_s)}{\alpha (\alpha - 1)} \\
+ A D [2 m_c^2 (m_q - 3 m_s) + H_\alpha (3 m_s + 2 m_c (2 + \alpha) + 2 (m_s + m_q (\alpha - 1) - 3 m_s))] \\
+ B C - 2 \alpha H_\alpha (m_c + (\alpha - 1) m_q) + (6 (\alpha - 1) \alpha - 3) H_\alpha m_s + 2 (\alpha - 1) a m_c^2 (m_q - 3 m_s) \\
\frac{\alpha (\alpha - 1)}{\alpha (\alpha - 1)} \right\}
\]
\( \rho^{(8\text{G}s)}(s) = \frac{\langle q\bar{q} \cdot G_s \rangle}{3 \times 2^6 \pi^2} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \left\{ \int_{\beta_{\text{min}}}^{1-\alpha} d\beta \left\{ -48 \frac{F_{\alpha\beta}m_c(3\alpha + 2\beta)}{\alpha^2} \right\} \right\} \),

\[ \rho^{(q\bar{q}ss)}(s) = \frac{\langle q\bar{q} \rangle \langle ss \rangle}{3 \times 2^6 \pi^2} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \left\{ \int_{\beta_{\text{min}}}^{1-\alpha} d\beta \left\{ 6A^2 m_c(-2\alpha - m_q + \alpha m_q + 2am_s) \right\} \right\} \),

\[ \rho^{(G^3)}(s) = \frac{\langle q^3 \bar{q}^3 \rangle}{3 \times 2^5 \pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \int_{\beta_{\text{min}}}^{1-\alpha} d\alpha \int_{\beta_{\text{min}}}^{1-\alpha} d\beta \left\{ 6A^2 m_c(-2\alpha - m_q + \alpha m_q + 2am_s) \right\} \),

\]
\[ + \, 6B^2(\alpha + \beta - 1)[F_{\alpha\beta}(\alpha + \beta)(\alpha + \beta + 1) (\alpha^2 - \alpha \beta + \beta^2)
+ 2m_c (m_c(\alpha + \beta + 1) (\alpha^4 + \beta^4) - \beta m_s (\alpha^3 + 6\beta^3))]
+ 6C^2(\alpha + \beta - 1)[F_{\alpha\beta}(\alpha + \beta)(\alpha + \beta + 1) (\alpha^2 - \alpha \beta + \beta^2)
+ 2m_c^2(2\alpha^5 + \alpha^4(3\beta + 2) + \alpha^3 (\beta^2 + 2) + \alpha^2 \beta^3
+ \alpha \beta^3(3\beta + 1) + 2\beta^3(\beta^2 + \beta - 1))]/ + 6D^2(\alpha + \beta - 1)[F_{\alpha\beta}(\alpha + \beta)(\alpha + \beta + 1)
\times (\alpha^2 - \alpha \beta + \beta^2) + 2m_c^2(\alpha^5 + \alpha^4(\beta + 4) + 3\alpha^3(\beta - 1) + \alpha \beta^3(\beta + 3)
+ \beta^3(\beta(4 + 3)))] + AC(\alpha + \beta - 1)[F_{\alpha\beta}(\alpha + \beta)(\alpha + \beta + 1) (\alpha^2 - \alpha \beta + \beta^2)
+ m_c(2m_c(\alpha^3 \beta^2 + \alpha^3(3\alpha + 1)\beta + (\alpha^2 + \alpha - 2) \beta^3 + 2\alpha^3 (\alpha^2 + \alpha - 1))
+ (3\alpha + 2)\beta^4 + 2\beta^5) - m_s(\alpha + \beta + 1) (2\alpha^2 + 3\alpha \beta + 2\beta^2) (3\alpha^2 - 4\alpha \beta + 3\beta^2))]
+ AD(\alpha + \beta - 1)[F_{\alpha\beta}(\alpha + \beta)(\alpha + \beta + 1) (\alpha^2 - \alpha \beta + \beta^2)
+ m_c(2m_c(\alpha^5 + \alpha^4(\beta + 4) + 3\alpha^3(\beta - 1) + \alpha \beta^3(\beta + 3) + 3\beta(\beta + 4) - 3))
- m_s(\alpha^4(\alpha + 12) + \alpha^3 \beta(\beta + 1) + 2\alpha \beta^3(3\beta + 1) + 6\beta^4(\beta + 1))]
+ BC(\alpha + \beta - 1)[F_{\alpha\beta}(\alpha + \beta)(\alpha + \beta + 1) (\alpha^2 - \alpha \beta + \beta^2)
+ 2m_c(\alpha \beta^2 + \alpha^3(3\alpha + 1)\beta + (\alpha^2 + \alpha - 2) \beta^3 + 2\alpha^3 (\alpha^2 + \alpha - 1))
+ (3\alpha + 2)\beta^4 + 2\beta^5) - m_s(6\alpha^4 + 3\alpha \beta + 3\alpha \beta^2 + 6\beta^4)]
+ BD(\alpha + \beta - 1)[F_{\alpha\beta}(\alpha + \beta)(\alpha + \beta + 1) (\alpha^2 - \alpha \beta + \beta^2)
+ m_c(2m_c(\alpha^5 + \alpha^4(\beta + 4) + 3\alpha^3(\beta - 1) + \alpha \beta^3(\beta + 3) + 3\beta(\beta + 4) - 3))
- m_s(6(\alpha + 1)\alpha^4 + 2(3\alpha + 1)\alpha^3 \beta + (\alpha + 12)\beta^4 + (\alpha + 1)(\alpha \beta^3))]\right\}, \tag{A12}
\]

\[
\rho(\bar{q}q)(G^2)(s) = \frac{\langle \bar{q}q \rangle (G^2)}{\sqrt{s} \times 2^{10} \pi} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \left\{ \int_{\beta_{\text{min}}}^{1-\alpha} d\beta \left\{ -18A^2 \frac{\alpha m_c}{\beta^2} + 6B^2 \frac{m_c(3\alpha^2 + 3\alpha \beta - 2\beta^2)}{\beta^2}
- 6D^2 m_s + 3AC \frac{m_c(\alpha + \beta)(\alpha^2 + \beta^2)}{\alpha^2 \beta^2} + AD \frac{m_c(\alpha + \beta)(3\alpha^2 + 3\beta^2 - 2\alpha \beta(1 + \beta))}{\alpha^2 \beta^2}
- m_s + BC \frac{m_c(\alpha + \beta) (3\alpha^2 - \alpha \beta + 3\beta^2)}{\alpha^2 \beta^2} + BD [-m_s
+ m_c(\alpha + \beta) (3\alpha^2 + 3\beta^2 - 2\alpha \beta(1 + \alpha))]} \right\} + (18A^2 - 6B^2)m_c + (18C^2 - 6D^2)m_s
+ 3AC[m_s + \frac{m_c}{\alpha(1 - \alpha)}] + AD [-m_s + \frac{m_c(3 - 4\alpha)}{\alpha(1 - \alpha)}] + BC [3m_s + \frac{-m_c}{\alpha(1 - \alpha)}]
+ BD [-m_s + \frac{m_c(4\alpha - 1)}{\alpha(1 - \alpha)}] \right\}, \tag{A13}
\]
\[
\rho^{(ss)}(G^2) (s) = \frac{\langle ss \rangle (G^2)}{3^2 \times 2^{11} \pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \left\{ \int_{\beta_{\min}}^{1-\alpha} d\beta \left\{ \frac{12A^2 m_c(3\beta^2 + 3\alpha \beta - \alpha^2)}{\alpha^2} + 36B^2 m_c \beta^2 \right\} + 18\alpha^2 m_s + 6\Delta^2 m_s + AC[3m_s + \frac{2m_c(\alpha + \beta)(3\alpha^2 - \alpha \beta + 3\beta^2)}{\alpha^2\beta^2}] + AD[m_s + \frac{2m_c(\alpha + \beta)(3\alpha^2 - \alpha \beta + 3\beta^2)}{\alpha^2\beta^2}] + BC[3m_s + \frac{6m_c(\alpha + \beta)(\alpha^2 + 3\beta^2)}{\alpha^2\beta^2}] + BD |m_s + \frac{2m_c(\alpha + \beta)(\alpha^2 + 3\beta^2)}{\alpha^2\beta^2})| \right\} + A^2(18\alpha m_s - 12m_c) - 18\alpha^2 m_s + 6\Delta^2 m_s + 2AC \frac{m_c}{\alpha(\alpha - 1)} + 2AD[(3\alpha - 1)m_s
+ \frac{me(4\alpha - 3)}{\alpha(\alpha - 1)} - 6BC[m_s + \frac{m_c}{\alpha(1 - \alpha)}] + 2BD \frac{m_c(1 - 4\alpha) + \alpha m_s(-2 + 5\alpha - 3\alpha^2)}{\alpha(1 - \alpha)}\right\},
\]

\[
\rho^{(qq/\bar{s}G)} (s) = \frac{\langle \bar{q}q \rangle (\bar{s}G_s)}{2^7 \pi^2} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \left\{ (\alpha - 1)(2AC + 2BC + AD + BD + BD^2) \right\},
\]

\[
\rho^{(ss/\bar{q}Gq)} (s) = \frac{\langle ss \rangle (\bar{q}Gq)}{2^7 \pi^2} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \left\{ \alpha(1 - 24C^2 + BD^2) \right\},
\]

\[
\Pi^{(qq)} (M_B^2) = \frac{m^2_c \langle \bar{q}q \rangle \langle ss \rangle}{3 \times 2^{6} \pi^2} \int_0^1 d\alpha e^{-\frac{m^2_c}{M_B^{1-\alpha}} \beta} \left\{ - (AC + AD + BC + BD)(m_q + m_s)
+ 6(A^2 + B^2)[(\alpha - 1)m_q - \alpha m_s]\right\},
\]

\[
\Pi^{(G^2)} (M_B^2) = \frac{m^2_c \langle \bar{q}q \rangle \langle G^2 \rangle}{3^2 \times 2^{11} \pi^4} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left\{ e^{-\frac{m^2_c}{M_B^{1-\alpha}} \beta} \right\} \left\{ 18A^2 \beta m_s[\alpha^5 + \alpha^4(\beta - 1)
+ \alpha \beta^3(\beta + 2) + \beta^4(\beta + 1)] + 36B^2(\alpha^4 + \beta^4) + C^2(-6\alpha^6 m_c
- 12\alpha^5 \beta m_c - 6\alpha^5 m_c - 6\alpha^4 \beta^2 m_c - 6\alpha^4 \beta m_c + 12\alpha^4 m_c - 6\alpha^2 \beta^4 m_c - 12\alpha^5 m_c
- 6\alpha^4 m_c - 6\beta m_c - 6\alpha^4 m_c + 12\alpha^4 m_c) + D^2(-18\alpha^5 m_c - 18\alpha^4 m_c
+ 18\alpha^4 m_c - 18\alpha^3 m_c - 18\beta^5 m_c + 18\beta^4 m_c) + AC[3m_s(\alpha + \beta)(\alpha + \beta + 1)(\alpha^4 + \beta^4)
- m_c(\alpha + \beta - 1)(\alpha^4 \beta + (\alpha + 2)\alpha^4 + 3\beta^4)] + AD[3m_s(\alpha^4 + \beta^4)(\alpha(\alpha + \beta) + \beta^2 + \beta)
- m_c(\alpha + \beta - 1)(3\alpha^4 + 3\beta^4)] + BC[6m_s(\alpha + \beta)(\alpha^4 + \beta^4)
- m_c(\alpha + \beta - 1)(\alpha^4 \beta + (\alpha + 2)\alpha^4 + 3\beta^4)] + BD[3m_s(\alpha^4 + \beta^4)(\alpha(\alpha + \beta) + 2\beta)
- m_c(\alpha + \beta - 1)(3\alpha^4 + 3\beta^4)]\right\},
\]

\[
\Pi^{(qq)} (M_B^2) = \frac{m^2_c \langle \bar{q}q \rangle \langle G^2 \rangle}{3^2 \times 2^{11} \pi^4} \int_0^1 d\alpha \left\{ \int_0^{1-\alpha} d\beta \left\{ \frac{12A^2 m_c(3\beta^2 + 3\alpha \beta - \alpha^2)}{\alpha^2\beta^2} - \frac{m^2_c}{M_B^\alpha} \right\} \right\}
\]
\[ \Pi^{(s \bar{s})(G^2)}(M_B^2) = \frac{m_s^2}{3 \times 2^{11} \pi^4} \int_0^1 da \left\{ \int_0^{1-a} \frac{d\beta}{a^3 \beta \bar{M}_B^2} e^{-\frac{m_s^2}{M_B^2 \alpha \beta}} \left\{ 6\beta A^2 M_B^2 (\alpha^3 + \beta^3) \right\} \right. \\
\times (2m_c(\alpha + \beta) + 3b^2) + 6\beta B^2 M_B^2 (\alpha^3 + \beta^3) (2m_c + 3m_s) \\
+ 6D^2 m_s (m_s^2 (\alpha^4 + \alpha^3 \beta + \alpha \beta^3 + \beta^4) - \alpha \beta (4\alpha^3 + 3\alpha^2 \beta + 3\alpha \beta (\beta + 2) + 4\beta^3) M_B^2) \\
+ AC(\alpha + \beta)(m_s^2 m_s(\alpha^3 + \beta^3)) - 2M_B^2 (m_c(\alpha^4 + \alpha^3 \beta + \alpha \beta^3 + \beta^4) \\
+ 2\alpha \beta m_s (\alpha^2 + \alpha \beta + \beta^2)) + AD[m_s^2 m_s(\alpha^3 + \beta^3)] \\
- M_B^2 (2m_c(\alpha + \beta)(\alpha \beta + \alpha + \beta^2)(\alpha^2 - \alpha \beta + \beta^2) \\
+ \alpha \beta m_s (\alpha^3 + \alpha^2 (2\beta + 3) + \beta^2 (\beta + 3))) + BC(\alpha + \beta)(m_s^2 m_s(\alpha^3 + \beta^3)) \\
- 2M_B^2 (m_c(\alpha^3 + \beta^3) + \alpha \beta m_s (2\alpha^2 + \alpha \beta + 2\beta^2)) + BD[m_s^2 m_s(\alpha^3 + \beta^3)] \\
- M_B^2 (2m_c(\alpha + \beta)(\alpha^2 - \alpha \beta + \beta^2)(\alpha + \beta + \beta) \\
+ \alpha \beta m_s (\alpha^3 + (2\alpha + 3)\beta^2 + \beta^3)) \} \\
+ \frac{m_s e^{-\frac{m_s^2}{M_B^2 (\alpha - 1)}}}{\alpha^2 (\alpha - 1) \beta^2} \left\{ A^2 (-6\alpha^3 + 24\alpha^2 - 18\alpha + 6) + B^2 (-6\alpha^3 + 24\alpha^2 - 18\alpha + 6) \\
+ (C^2 + D^2)(18\alpha^2 - 18\alpha + 6) + AC((\alpha - 1)\alpha + 1) + AD((\alpha - 1)\alpha (8\alpha - 9) - 1) \\
+ BC(9(\alpha - 1)\alpha + 1) + BD((\alpha - 1)\alpha (8\alpha + 1) + 1) \right\} , \]
Π^{(qq)⟨sGα⟩}(M_B^2) = \frac{m_c⟨\bar{q}q⟩⟨\bar{s}Gα⟩}{32 \times 2^{8-2}} \int_0^1 \frac{dα}{α^2(α-1)^2M_B^4} e^{-\frac{m_c^2}{M_B^2(1-α)^α}} \left\{ 18A^2m_c[2(α-2)(α-1)αM_B^4 \\
+ αm_cM_B^2(-2m_c - αm_q + m_q) + m_c^2m_q] + 18B^2[2αM_B^2(m_c \\
+ (α-1)m_q)((α-1)αM_B^2 - m_c^2) + m_c^4m_q] + 36C^2m_c[19(α-1)αM_B^2 + 5m_c^2] \\
+ 36D^2m_c[18(α-1)αM_B^2 + 5m_c^2] - AC[(α-1)αM_B^2(m_c^2(12m_c \\
- 3m_q - 4m_s) - 2(α-1)αM_B^2(15m_c - 2m_s)) + m_c^4(3m_q + 2m_s)] \\
- AD[(α-1)αM_B^2(2(α-1)αM_B^2(-12m_c + 3αm_q + 2αm_s) \\
+ m_c^2(12m_c - (α + 1)(3m_q + 2m_s))) + m_c^4(3m_q + 2m_s)] \\
+ BC[2(α-1)αM_B^2(3(α-1)αM_B^2(5m_c - m_q)) + m_c^2(-6m_c + 3m_q + m_s)] \\
- m_c^4(3m_q + 2m_s)] + BD[(α-1)αM_B^2(12m_c + (α-1)(3m_q + 2m_s)) \\
+ m_c^2(-12m_c - (α-2)(3m_q + 2m_s))) - m_c^4(3m_q + 2m_s)] \right\} , \tag{A21}

Π^{(ss)⟨\bar{q}Gα⟩}(M_B^2) = \frac{m_c⟨\bar{s}s⟩⟨\bar{q}Gα⟩}{2^{7-2}} \int_0^1 \frac{dα}{α^2(α-1)^2M_B^4} e^{-\frac{m_c^2}{M_B^2(1-α)^α}} \left\{ -A^2[(α-1)αM_B^2 \\
× (2(α-1)αM_B^2(αm_s - m_c) + m_c^2(2m_c + (α-1)m_q - 2αm_s)) \\
+ m_c^4(-αm_q + m_q + αm_s)] + B^2[αm_c^2M_B^2(-2(α-1)m_c - 2(α-1)^2m_q \\
+ α(α+1)m_s) + (α-1)αM_B^2(2(α+1)m_c + (α-1)(2(α-1)m_q - αm_s)) \\
+ m_c^4(α-1)m_q - αm_s)] + 2C^2m_c[5m_c^2 + 19α(α-1)M_B^2] \\
+ 2D^2m_c[5m_c^2 + 18α(α-1)M_B^2] \right\} , \tag{A22}

where \(M_B\) is the Borel parameter introduced by the Borel transformation, \(q = u \) or \(d\). Here, we also have the following definitions:

\[ F_{αβ} = (α + β)m_c^2 - αβs \}, \mathcal{H}_α = m_c^2 - α(1 - α)s \, , \tag{A23} \]

\[ α_{min} = \left(1 - \sqrt{1 - 4m_c^2/s}\right)/2 \right., \mathcal{C}_α = \left(1 + \sqrt{1 - 4m_c^2/s}\right)/2 \, , \tag{A24} \]

\[ β_{min} = αm_c^2/(α - m_c^2). \tag{A25} \]

2. The spectral densities for tetraquark state

\[ ρ_{pert}(s) = \frac{1}{3 \times 10^{-6}} \int_{α_{min}}^{α_{max}} \frac{dα}{α^3} \int_{β_{min}}^{β_{max}} \frac{dβ}{β^3} F_{αβ}^2(α + β - 1) \right\} \times \frac{3A^2}{3A^2} F_{αβ}(1 + α + β - 4m_cF_{αβ}[2βm_s} \right\} \tag{21} \]
\[ \rho^{(qq)}(s) = \frac{\langle \bar{q} q \rangle}{2 \pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \frac{F_{\alpha\beta}}{\alpha^2 \beta^2} \{ A^2 \left[ -2 \alpha^2 m_c F_{\alpha\beta} - 2 \alpha^2 \beta m_q F_{\alpha\beta} + \alpha \beta m_q F_{\alpha\beta} + 4 \alpha \beta^2 m_q m_{s} \right] + B^2 \left[ -2 \alpha m_c F_{\alpha\beta} + 2 \alpha \beta^2 m_q m_{s} - \alpha m_q F_{\alpha\beta} + 4 \alpha \beta m_q m_{c} \right] + C^2 \left[ 2 \alpha^2 m_c F_{\alpha\beta} - 2 \alpha \beta^2 m_q m_{s} + \alpha m_q F_{\alpha\beta} + 4 \alpha m_q m_{c} \right] + D^2 \left[ 2 \alpha m_c F_{\alpha\beta} - 2 \alpha \beta^2 m_q m_{s} - \alpha m_q F_{\alpha\beta} + 4 \alpha \beta m_q m_{c} \right] + A D \left[ (2 \alpha m_q m_c + 2 \beta F_{\alpha\beta})(\alpha m_q + (\alpha + 1)m_c) \right] + B C \left[ (2 \alpha m_q m_c - 2 \beta F_{\alpha\beta})(-\alpha m_q + (\alpha + 1)m_c) \right] \}, \quad (A27) \]

\[ \rho^{(ss)}(s) = \frac{\langle \bar{s} s \rangle}{2 \pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \frac{F_{\alpha\beta}}{\alpha^2 \beta^2} \{ - A^2 \left[ 2 \beta m_c F_{\alpha\beta} - 2 \alpha^2 \beta m_q m_{s} + \alpha m_q F_{\alpha\beta} \right] - B^2 \left[ 2 \alpha m_c F_{\alpha\beta} + 2 \beta^2 F_{\alpha\beta} + \alpha m_q F_{\alpha\beta} - 4 \alpha m_q m_c \right] - C^2 \left[ -2 \beta m_c F_{\alpha\beta} + 2 \alpha^2 \beta m_q m_{s} + \alpha m_q F_{\alpha\beta} - 4 \alpha m_q m_c \right] - D^2 \left[ -2 \alpha m_c F_{\alpha\beta} - 2 \beta^2 F_{\alpha\beta} + \alpha m_q F_{\alpha\beta} - 4 \alpha m_q m_c \right] + A D \left[ (2 \alpha m_q m_c + 2 \beta F_{\alpha\beta})(-\beta m_q + (\alpha + 1)m_c) \right] + B C \left[ (2 \alpha m_q m_c + 2 \alpha F_{\alpha\beta})(\beta m_q + (\alpha + 1)m_c) \right] \}, \quad (A28) \]

\[ \rho^{(g^2)}(s) = \frac{\langle g_s^2 G_s^2 \rangle}{32 \times 2 \pi^6} \int_{\alpha_{\min}}^{\alpha_{max}} d\alpha \int_{\beta_{\min}}^{\alpha_{\max}} d\beta \frac{F_{\alpha\beta}}{\alpha^2 \beta^3} \left[ 6 A^2 \left[ \alpha \beta F_{\alpha\beta}((\alpha + 1)(3 \alpha + \beta) - 2) \right] - 2 F_{\alpha\beta} m_c (\alpha + 1)(\alpha + 1)(\alpha^2 - \alpha \beta + 2 \beta + 2 \alpha^2 \beta - 3 \alpha (\alpha^2 - \alpha) - 2) \right] + 2 m_c^2 (\alpha + \beta) \]
\(-1)(m_c(\alpha^3 + \beta^3)(\alpha m_q(\alpha + \beta + 1) + 2\beta m_s) - 6\alpha \beta m_q m_s(\alpha^2 + \beta^2))\]
\[+ 6B^2\left[\alpha \beta F_{\alpha \beta}^2(4\alpha \beta + (\alpha - 2)\alpha + 3\beta^2) - 2F_{\alpha \beta} m_c(3\alpha^2 m_q(-2\alpha \beta - 2(\alpha - 1)\alpha + \beta^2) + m_c(\alpha + \beta - 1)(\alpha + \beta)(\alpha + \beta + 1)(\alpha^2 - \alpha \beta + \beta^2) + \beta^2 m_s(\alpha^3 + 2\alpha^2(\beta + 1) - 6\alpha \beta^2 - 3\beta^3 + 3\beta)) + 2m_c(\alpha + \beta - 1)(2\alpha m_q(m_c(\alpha^2 + \beta^3) - 3\beta m_s(\alpha^2 + \beta^2)) + \beta m_c m_s(\alpha + \beta)(\alpha + \beta + 1)(\alpha^2 - \alpha \beta + \beta^2))\right]
\[+ 6C^2\left[\alpha \beta F_{\alpha \beta}^2((\alpha + \beta)(3\alpha + \beta) - 2\beta) - 2F_{\alpha \beta} m_c(\alpha^2 m_q(6\alpha^2 \beta + 3\alpha(\alpha^2 - 1)) + 2(\alpha + 1)\beta^2 - \beta^3) + m_c(\alpha + \beta - 1)(\alpha + \beta)(\alpha + \beta + 1)(\alpha^2 - \alpha \beta + \beta^2) + \beta^2 m_s(-3\alpha^2 + 2\alpha \beta + (2(\alpha - 1)\beta)) - 2m_c^2(\alpha + \beta - 1)(\alpha m_q(m_c(\alpha + \beta)(\alpha + \beta + 1)(\alpha^2 - \alpha \beta + \beta^2) + 6\beta m_s(\alpha^2 + \beta^2)) + 2\beta m_Q m_s(\alpha^3 + \beta^3)\right]
\[+ 6D^2\left[\alpha \beta F_{\alpha \beta}^2(4\alpha \beta + (\alpha - 2)\alpha + 3\beta^2) - F_{\alpha \beta} m_c(3\alpha^2 m_q(2\alpha \beta + 2(\alpha - 1)\alpha - \beta^2) + m_c(\alpha + \beta - 1)(\alpha + \beta)(\alpha + \beta + 1)(\alpha^2 - \alpha \beta + \beta^2) + \beta^2 m_s(-3\alpha^2 + 2\alpha \beta + 6\alpha \beta^2 + 3\beta(\beta^2 - 1))) - m_c^2(\alpha + \beta - 1)(2\alpha m_q(m_c(\alpha^3 + \beta^3) + 3\beta m_s(\alpha^2 + \beta^2)) + \beta m_c m_s(\alpha + \beta)(\alpha + \beta + 1)(\alpha^2 - \alpha \beta + \beta^2))\right]
\[+ \alpha \beta \alpha \beta \left[\alpha \beta \alpha \beta(4m_c^3 - 3m_q) + 3\beta^2(3m_c - 2m_q)(2\alpha m_c + m_c - \alpha m_s) - 3\beta(m_q - 2m_c)((\alpha(3\alpha + 4) - 3)m_c - 2\alpha(\alpha + 1)m_s) + (\alpha - 1)m_c((\alpha(4\alpha + 13) - 5)m_q - 3\alpha(\alpha + 3)m_s))\right] + \alpha \beta \alpha \beta \left[m_c(\alpha + \beta - 1)(\alpha m_q(m_c(\alpha + \beta - 1)(12(\alpha - 1)\alpha^2 - (5\alpha + 12)\beta^2 - 5(\alpha - 1)\alpha \beta + 12\beta^3) + 36((\alpha - 1)\alpha^3 m_s - (\beta - 1)\beta^3 m_q)) + 4m_c(\alpha^3 + \beta^3)(m_c^2(\alpha + \beta - 1)^2 + 3m_c(\alpha + \beta - 1)(\alpha m_s - \beta m_q) - 6\alpha \beta m_q m_s))\right]
\[+ \alpha \beta \alpha \beta \left[m_c(\alpha + \beta - 1)(\alpha m_q(m_c(\alpha + \beta - 1)(12(\alpha - 1)\alpha^2 - (5\alpha + 12)\beta^2 - 5(\alpha - 1)\alpha \beta + 12\beta^3) + 36((\alpha - 1)\alpha^3 m_s - (\beta - 1)\beta^3 m_q)) + 4m_c(\alpha^3 + \beta^3)(m_c^2(\alpha + \beta - 1)^2 - 3m_c(\alpha + \beta - 1)(\alpha m_s - \beta m_q) - 6\alpha \beta m_q m_s))\right]
\[+ \alpha \beta \alpha \beta \left[F_{\alpha \beta}(\beta^3 m_c(4m_c + 3m_q) + 3\beta^2(3m_c + 2m_q)(2\alpha m_c + m_c + \alpha m_s) + 3\beta(m_q + 2m_c)((\alpha(3\alpha + 4) - 3)m_c + 2\alpha(\alpha + 1)m_s) + (\alpha - 1)m_c((\alpha(4\alpha + 13) - 5)m_c + 3\alpha(\alpha + 3)m_s))]\right] \right\} \right), \quad (A31)
\[ \rho^{(Gq)}(s) = \frac{(g_s/g_q \cdot G_q)}{3 \times 2^3 \pi^4} \int^{\alpha_{\text{max}}}_{\alpha_{\text{min}}} \int^{1-\alpha}_{\beta_{\text{min}}} \frac{d\beta}{\alpha^2 \beta^2} \left\{ 12A^2 \alpha^2 m_c((\alpha + 2\beta)F_{\alpha\beta} - \beta m_c m_s) 
- 12D^2 \alpha^2 m_c((\alpha + 2\beta)F_{\alpha\beta} + \beta m_c m_s) + AB^2 F_{\alpha\beta}[(2\alpha + \beta - 1)m_c - \alpha m_s] 
- CD^2 F_{\alpha\beta}[(2\alpha + \beta + 1)m_c + \alpha m_s] + AD\beta[6\alpha F_{\alpha\beta}(m_c - \alpha m_c - \alpha m_s)] 
- m_c F_{\alpha\beta}(1 + 19\alpha) + 4\alpha^2 m_q m_c - \beta^2 m_c F_{\alpha\beta} + BC\beta[6\alpha F_{\alpha\beta}(m_c - \alpha m_c + \alpha m_s)] 
+ m_c F_{\alpha\beta}(1 + 19\alpha) - 4\alpha^2 m_q m_c + \beta^2 m_c F_{\alpha\beta} \right\} + \frac{4}{\alpha - 1} \left\{ A^2[H_\alpha(3m_c + 2m_q(\alpha - 1))] 
+ m_c(\alpha - 1)(-m_c m_q + 3m_c m_s + m_s m_q(\alpha - 1)) + B^2[H_\alpha(3m_c + 2m_q(\alpha - 1))] 
+ m_c(\alpha - 1)(-m_c m_q + 3m_c m_s + m_s m_q(\alpha - 1)) + C^2[H_\alpha(3m_c + 2m_q(\alpha - 1))] 
- m_c(\alpha - 1)(m_c m_q - 3m_c m_s + 2m_s m_q(\alpha - 1)) + D^2[H_\alpha(3m_c + 2m_q(\alpha - 1))] 
- m_c(\alpha - 1)(m_c m_q - 3m_c m_s + m_s m_q(\alpha - 1)) + AD m_s(\alpha - 1)(3H_\alpha + \alpha m_c m_q) 
+ BC m_s(\alpha - 1)(3H_\alpha - \alpha m_c m_q) + 2H_\alpha[AB(-m_c + m_s) + CD(m_c + m_s)] \right\}, \quad (A32) \]

\[ \rho^{(Gs)}(s) = \frac{(g_s/g_\sigma \cdot G_\sigma)}{3 \times 2^3 \pi^4} \int^{\alpha_{\text{max}}}_{\alpha_{\text{min}}} \int^{1-\alpha}_{\beta_{\text{min}}} \frac{d\beta}{\alpha^2 \beta^2} \left\{ 12B^2 \beta^2 m_c((2\alpha + \beta)F_{\alpha\beta} - \alpha m_c m_s) 
- 12D^2 \beta^2 m_c((2\alpha + \beta)F_{\alpha\beta} + \alpha m_c m_s) + AB^2 F_{\alpha\beta}[(\alpha + 2\beta + 1)m_c - \beta m_c] 
- CD^2 F_{\alpha\beta}[(\alpha + 2\beta + 1)m_c + \alpha m_s] + AD\alpha[-6\beta^2 F_{\alpha\beta} m_q - 4\alpha^2 m_c^2 m_s] 
- m_c F_{\alpha\beta}(\alpha + \alpha^2 + 19\alpha \beta + 6\beta^2 - 6\beta) - BC\alpha[6\beta^2 F_{\alpha\beta} m_q + 4\alpha^2 m_c^2 m_s] 
+ m_c F_{\alpha\beta}(\alpha + \alpha^2 + 19\alpha \beta + 6\beta^2 - 6\beta) \right\} + \frac{4}{\alpha} \left\{ A^2[H_\alpha(-3m_c + 2m_q\alpha)] 
+ m_c\alpha(-m_c m_s + 3m_c m_q - m_s m_q\alpha)] + B^2[H_\alpha(-3m_c + 2m_q\alpha)] 
+ m_c\alpha(-m_c m_s + 3m_c m_q - 2m_s m_q\alpha)] + C^2[H_\alpha(3m_c + 2m_q\alpha)] 
+ m_c\alpha(-m_c m_s + 3m_c m_s + m_s m_q\alpha)] + D^2[H_\alpha(3m_c + 2m_q\alpha)] 
+ m_c\alpha(-m_c m_s + 3m_c m_s + 2m_s m_q\alpha)] + AD m_q\alpha(3H_\alpha + (\alpha - 1)m_c m_q) 
+ BC m_q\alpha(3H_\alpha - (\alpha - 1)m_c m_s) + 2H_\alpha[AB(-m_c + m_s) + CD(m_c + m_s)] \right\}, \quad (A33) \]

\[ \rho^{(q\bar{q})_{ss}}(s) = \frac{(g_\bar{q}g_{\bar{s}} \cdot G_{\bar{s}})}{3 \times 2^3 \pi^4} \int^{\alpha_{\text{max}}}_{\alpha_{\text{min}}} da \left\{ A^2[2m_c(2m_c - \alpha m_s) + (\alpha - 1)m_q(4m_c - 3\alpha m_s)] 
+ B^2[4m_c(m_c - \alpha m_s) + (\alpha - 1)m_q(2m_c - 3\alpha m_s)] 
+ C^2[2m_c(2m_c + \alpha m_s) - (\alpha - 1)m_q(4m_c + 3\alpha m_s)] 
+ D^2[4m_c(m_c + \alpha m_s) - (\alpha - 1)m_q(2m_c + 3\alpha m_s)] 
+ AD[4H_\alpha + 2m_c m_s(\alpha - 1) + 2m_c m_q\alpha] \right\].
\[ +\, BC[4H_\alpha - 2m_c m_s(\alpha - 1) - 2m_c m_q \alpha]\right\}, \quad (A34) \]

\[ \rho^{(G^3)}(s) = \frac{(g_s^3 G^3)}{3 \times 2^{12} \pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{1-\alpha} \frac{d\beta}{\beta^3} (\alpha + \beta - 1) \left\{ A^2 \left[ 2F_{\alpha \beta}(\alpha + \beta + 1) + m_c [4am_c(\alpha + \beta + 1) - m_q(6\alpha^2 + 6\alpha + 7\alpha \beta + \beta^2)] + B^2 \left[ 2F_{\alpha \beta}(\alpha + \beta + 1) + m_c [4am_c(\alpha + \beta + 1) - m_q(6\alpha^2 + 6\alpha + 7\alpha \beta + \beta^2)] + C^2 \left[ 2F_{\alpha \beta}(\alpha + \beta + 1) + m_c [4am_c(\alpha + \beta + 1) + 2m_s(6\alpha + \beta) + m_q(6\alpha^2 + 6\alpha + 7\alpha \beta + \beta^2)] \right] + D^2 \left[ 2F_{\alpha \beta}(\alpha + \beta + 1) + m_c [4am_c(\alpha + \beta + 1) + 2m_s(6\alpha + \beta) + m_q(6\alpha^2 + 6\alpha + 7\alpha \beta + \beta^2)] \right] + \frac{(AD + BC)m_c(m_q - m_s)(6\alpha^2 - 6\alpha + 7\alpha \beta + \beta^2) - \beta)}{\partial^2} \right\}, \quad (A35) \]

\[ \rho^{(gq)(G_s)}(s) = \frac{(gq)(G_s)}{3 \times 2^{12} \pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \left\{ \alpha (AB + CD) - 2(AD + BC)(6\alpha^2 - 7\alpha - 1) \right\} \], \quad (A36) \]

\[ \rho^{(gs)(Gq)}(s) = \frac{(gs)(Gq)}{3 \times 2^{12} \pi^6} \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \left\{ (1 - \alpha)(AB + CD) - 2(AD + BC)(6\alpha^2 - 5\alpha) \right\} \], \quad (A37) \]

\[ \Pi^{(G^2)}(M_B^2) = \frac{m_s m_q m_s (g_s^2 G^2)}{3 \times 2^{12} \pi^6} \int_0^1 \frac{d\alpha}{\alpha^3} \int_0^{1-\alpha} \frac{d\beta}{\beta^3} e^{\frac{m_s^2 (\alpha + \beta)}{M_B^2 \alpha (\alpha - 1)}} (A^2 + B^2 + C^2 + D^2) \]

\[ \times \left[ \alpha + A^2(\beta - 1) \right] \], \quad (A38) \]

\[ \Pi^{(gq)}(M_B^2) = \frac{m_s^2 m_q m_s (g_s g_q \cdot G_q)}{3 \times 2^{12} \pi^6} \int_0^1 \left\{ - \frac{e^{\frac{m_s^2 (\alpha - 1)}{M_B^2 \alpha (\alpha - 1)}}}{\alpha} (A^2 + B^2 - C^2 - D^2) \right\}, \quad (A39) \]

\[ \Pi^{(gs)}(M_B^2) = \frac{m_s^2 m_q m_s (g_s \bar{s} G_s \cdot G_s)}{3 \times 2^{12} \pi^6} \int_0^1 \left\{ \frac{e^{\frac{m_s^2 (\alpha - 1)}{M_B^2 \alpha (\alpha - 1)}}}{\alpha - 1} (A^2 + B^2 - C^2 - D^2) \right\}, \quad (A40) \]

\[ \Pi^{(gq)}(M_B^2) = \frac{m_s^2 (gq)(ss)}{3 \times 2^{12} \pi^6} \int_0^1 \left\{ \frac{e^{\frac{m_s^2 (\alpha - 1)}{M_B^2 \alpha (\alpha - 1)}}}{\alpha} \left[ (A^2 + B^2)[2am_c m_s + m_q (\alpha - 1)(-2m_c + 3am_s)] \right] + (C^2 + D^2)[-2am_c m_s + m_q (\alpha - 1)(2m_c + 3am_s)] \right\} \], \quad (A41) \]

\[ \Pi^{(G^3)}(M_B^2) = \frac{m_s^2 (g_s^3 G_s^3)}{3^2 \times 2^{12} \pi^6} \int_0^1 \frac{d\alpha}{\alpha} \int_0^{1-\alpha} \frac{d\beta}{\beta^3} e^{\frac{m_s^2 (\alpha + \beta)}{M_B^2 \alpha (\alpha - 1)}} (A^2 [6(\alpha + \beta)m_c m_s M_B^2 + 3m_q(4m_c^2 m_s - 12\beta m_s M_B^2 + m_c M_B^2 (\alpha + \beta + 1)(\alpha + \beta))] + B^2 [6(\alpha + \beta)]) + 7\alpha \beta + \beta^2) \right\}] + \left[ 4(AD + BC) [\alpha \beta m_q m_s - m_c^2 (\alpha + \beta - 1)^2] \right] + \left[ (AD - BC) m_c (m_q - m_s)(6\alpha^2 - 6\alpha + 7\alpha \beta + \beta^2) \right] + \left[ \frac{(AD - BC)m_c(m_q - m_s)(6\alpha^2 - 6\alpha + 7\alpha \beta + \beta^2) - \beta)}{\partial^2} \right\], \quad (A35) \]
\[
\Pi^{\langle qq \rangle \langle sG s \rangle}(M_B^2) = \frac{m_c \langle q \bar{q} \rangle \langle sG s \rangle}{3^2 \times 2^7 \pi^2} \int_0^1 \frac{d\alpha}{\alpha^2(\alpha - 1)^3 M_B^6} e^{-\frac{m_c^2}{M_B^2(\alpha - 1)\alpha}} \left\{ -4A^2(\alpha - 1)[m_c m_q + 2\alpha M_B^2] + 3(\alpha - 1)M_B^2[2\alpha m_c M_B^2(-m_c^2 + \alpha(\alpha - 1)M_B^2) + m_q(m_c^4 - 2\alpha(\alpha - 1)m_c^2 M_B^2) - 1)(3m_c - 2\alpha m_s) + (\alpha - 1)^2 M_B^2(-3m_c(\alpha - 1) + 2\alpha^2 m_s)] + 4C^2(\alpha - 1)[-m_c m_s + 3(\alpha - 1)M_B^2][2\alpha m_c M_B^2(m_c^2 - \alpha(\alpha - 1)M_B^2) + m_q(m_c^4 - 2\alpha(\alpha - 1)m_c^2 M_B^2 + 2\alpha^2(\alpha - 1)^2 M_B^4)] - 4D^2(\alpha - 1)(m_c m_q + 2\alpha M_B^2)(m_c^4 m_s - m_c^2 M_B^2(\alpha - 1)(3m_c + 2\alpha m_s) + (\alpha - 1)^2 M_B^2(3m_c(\alpha - 1) + 2\alpha^2 m_s)) + A\beta(\alpha - 1)M_B^2[-2m_q m_c^2 + M_B^2(\alpha - 1)(-4m_c + \alpha m_q)] + C\alpha^2(\alpha - 1)M_B^2[2m_q m_c^2 - M_B^2(\alpha - 1)(4m_c + \alpha m_q)] + A\beta(\alpha - 1)M_B^2[4(\alpha - 1)m_c^3 m_q m_s + 2M_B^2 m_c(4(\alpha - 1)^2 m_c m_s - 2\alpha(\alpha - 1)^2 m_q m_c + \alpha(6\alpha - 7) m_q m_c) - \alpha(\alpha - 1)M_B^2((\alpha(24\alpha - 31) + 6)m_q + 8(\alpha - 1)(2(\alpha - 1)m_s - 3m_c)) + B\alpha(\alpha - 1)M_B^2[4(\alpha - 1)m_c^3 m_q m_s - 2M_B^2 m_c(4(\alpha - 1)^2 m_c m_s + 2\alpha(\alpha - 1)^2 m_q m_c + \alpha(\alpha - 1)M_B^2((\alpha(24\alpha - 31) + 6)m_q + 8(\alpha - 1)(2(\alpha - 1)m_s + 3m_c)) \right\},
\]
\[
\Pi^{\langle ss \rangle \langle G G \rangle}(M_B^2) = \frac{m_c \langle s \bar{s} \rangle \langle G G \rangle}{3^2 \times 2^7 \pi^2} \int_0^1 \frac{d\alpha}{\alpha^2(\alpha - 1)^3 M_B^6} e^{-\frac{m_c^2}{M_B^2(\alpha - 1)\alpha}} \left\{ -4A^2(\alpha - 1)[m_c m_q + 2\alpha M_B^2] + 2(\alpha - 1)M_B^2[3\alpha m_c M_B^2(-m_c^2 + \alpha^2 M_B^2) + m_q(m_c^4 - 2\alpha(\alpha - 1)m_c^2 M_B^2)
\]
\]

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$$\begin{align*}
+ 2\alpha^2(\alpha - 1)^2M_B^4 \right] + 4B^2(\alpha - 1)(-m_c m_q + 3\alpha M_B^2) \left[m_c^2 m_s + m_c^2 M_B^2(\alpha - 1)(m_c - m_s) + 2\alpha(\alpha - 1)^2M_B^2(-m_c + m_s) \right] \\
+ 4C^2(\alpha - 1)[-m_c m_s + 2(\alpha - 1)M_B^2] \left[3\alpha m_c M_B^2(m_c^2 - \alpha^2 M_B^2) + m_q(m_c^4 - 2\alpha(\alpha - 1)m_c^2 M_B^2 + 2\alpha^2(\alpha - 1)^2M_B^4) \right] \\
- 4D^2(\alpha - 1)(m_c m_q + 3\alpha M_B^2) \left[m_c^4 m_s - 2m_c^2 M_B^2(\alpha - 1)3m_c + m_s) + 2\alpha(\alpha - 1)^2M_B^2(m_c^2 + m_s) \right] \\
+ AB(\alpha - 1)^3M_B^4][-2m_q m_c^2 + M_B^2 \alpha(4m_c + (\alpha - 1) m_s)] \\
+ CD(\alpha - 1)^3M_B^4[2m_q m_c^2 + M_B^2 \alpha(4m_c + (\alpha - 1) m_s)] \\
+ AD(\alpha - 1)^2M_B^2 \left[4\alpha m_c^3 m_q m_s + 2M_B^2 m_c(4\alpha^2 m_c m_s) - 2\alpha^2(\alpha - 1)m_q m_s + (\alpha - 1)(6\alpha + 1)m_c m_s) - \alpha(\alpha - 1)M_B^4((\alpha(24\alpha - 17) - 1)m_s + 8\alpha(2m_c m_s - 3m_c) \right] \\
+ BC(\alpha - 1)^2M_B^4 \left[4\alpha m_c^3 m_q m_s - 2M_B^2 m_c(4\alpha^2 m_c m_s) + 2\alpha^2(\alpha - 1)m_q m_s + (\alpha - 1)(6\alpha + 1)m_c m_s) + \alpha(\alpha - 1)M_B^4((\alpha(24\alpha - 17) - 1)m_s + 8\alpha(2m_c m_s + 3m_c) \right] \right],
\end{align*}$$

(A44)

**Appendix B: The decay spectral densities of Z+cs**

1. The decay spectral densities of Z+cs for molecular state

To calculate the decay spectral densities of Z+cs in molecular structure, we isolate the g_{\mu\nu} structure of both side of Eqs. (32) and (36). On the OPE side of QCD sum rules for the current (5), the three-point function of Z+cs \( \to \bar{D}DS^+_D \) after Borel transformation may write:

$$\begin{align*}
\Pi^{pert}(s_0, v_0, M_{1B}^2, M_{2B}^2) &= \frac{9\Lambda}{8\pi^2} \int_{m_c^2}^{s_0} ds \int_{m_c^2}^{v_0} dv \int_0^\Lambda d\alpha \int_0^\Lambda d\beta \\
&\times e^{-\frac{s_0}{M_{1B}^2} - \frac{v_0}{M_{2B}^2}} [m_c^2 - (1 - \alpha)s][m_c^2 - (1 - \beta)v],
\end{align*}$$

(B1)

$$\begin{align*}
\Pi^{(\bar{q}q)}(s_0, v_0, M_{1B}^2, M_{2B}^2) &= \frac{3m_c(\bar{q}q)Ae^{-\frac{m_c^2}{M_{1B}^2}}}{\pi^2} \int_{m_c^2}^{s_0} ds \int_0^\Lambda d\alpha e^{-\frac{s}{M_{1B}^2} [m_c^2 - (1 - \alpha)s]},
\end{align*}$$

(B2)
\[\Pi^{(ss)}(s_0, v_0, M_{1B}^2, M_{2B}^2) = \frac{3m_c(\bar{s}s)A e^{-m^2_{C\bar{B}}}}{2\pi^2} \int_{m_c^2}^{v_0} dv \int_0^\Lambda d\beta e^{-\frac{v}{M_{1B}^2}} e^{-\frac{m^2_c}{M_{2B}^2}} (1 - (1 - \beta)v), \quad (B3)\]

\[\Pi^{(G^2)}(s_0, v_0, M_{1B}^2, M_{2B}^2) = -\frac{m_c^2(\bar{G}G)A}{64\pi^4} \left\{ \int_{m_c^2}^{s_0} ds \int_0^\Lambda d\alpha \int_0^1 d\beta e^{-\frac{m^2_c}{M_{1B}^2}} e^{-\frac{m^2_c}{(1-\beta)M_{2B}^2}} \times \frac{(m^2_c - (1 - \alpha)s)}{(1 - \beta)^2 M_{2B}^2} + \int_{m_c^2}^{v_0} dv \int_0^1 d\alpha \int_0^\Lambda d\beta e^{-\frac{v}{M_{1B}^2}} e^{-\frac{m^2_c}{(1-\alpha)M_{1B}^2}} \times \frac{(m^2_c - (1 - \beta)v)}{(1 - \alpha)^2 M_{1B}^2} \right\}, \quad (B4)\]

\[\Pi^{(\bar{q}Gq)}(s_0, v_0, M_{1B}^2, M_{2B}^2) = -\frac{3m_c^3(\bar{q}Gq)A e^{-m^2_{C\bar{B}}}}{4\pi^2 M_{1B}^2} \int_{m_c^2}^{s_0} ds \int_0^\Lambda dae^{-\frac{m^2_c}{M_{1B}^2}} [m^2_c - (1 - \alpha)s] \quad (B5)\]

\[\Pi^{(sGs)}(s_0, v_0, M_{1B}^2, M_{2B}^2) = -\frac{3m_c^3(\bar{q}Gq)A e^{-m^2_{C\bar{B}}}}{8\pi^2 M_{1B}^2} \int_{m_c^2}^{v_0} dv \int_0^\Lambda d\beta e^{-\frac{v}{M_{1B}^2}} e^{-\frac{m^2_c}{M_{1B}^2}} [m^2_c - (1 - \beta)d] \quad (B6)\]

\[\Pi^{(\bar{q}q)(s\bar{s})}(s_0, v_0, M_{1B}^2, M_{2B}^2) = 4m_c^2A(\bar{q}q)(\bar{s}s)e^{-\frac{m^2_c}{M_{1B}^2}} \frac{m^2_c}{M_{1B}^2} \quad (B7)\]

where \(s_0\) and \(v_0\) are the continuum threshold of \(D_s\) and \(\bar{D}^*\), and \(M_{1B}\) and \(M_{2B}\) are their Borel parameters, respectively, and \(\Lambda = (1 - m^2_c/s)\). Since \(m_{D_s} \approx m_{\bar{D}^*}\), we can set \(s_0 = v_0\) and \(M_{1B}^2 = M_{2B}^2 \equiv M_B^2\). After Borel transformation on the Eq. (32), the phenomenological side of three-point is then obtained as

\[\Pi^{phen}(s_0, M_B^2) = \frac{3\lambda_{Z_{cs}} m_{D^*} f_{D^*} f_{D_s} m^2_{D_s} g_{Z_{cs} D^* D_s}}{4(m_c + m_s)(m^2_c/Z_{cs}^2/4 - m^2_{D^*})} \times (e^{-m^2_{D^*}/M_B^2} - e^{-m^2_{Z_{cs}}/(4M_B^2)})e^{-m^2_{D_s}/M_B^2}. \quad (B8)\]

In our calculation, \(\sqrt{s_0} = 2.1\) GeV and 1.5 GeV \(\leq M_B^2 \leq 2.5\) GeV is the proper Borel window for decay process.

The OPE side of the three-point function of the decay process \(Z_{cs}^+ \rightarrow \bar{D} D_s^*\) is just the transformation of three-point function of the decay process \(Z_{cs}^+ \rightarrow \bar{D}^* D_s^+\) with \(\langle \bar{q}q \rangle \leftrightarrow \langle s\bar{s} \rangle\), \(\langle \bar{q}Gq \rangle \leftrightarrow \langle \bar{s}Gs \rangle\), and \(A \rightarrow B\), and the phenomenological side of the three-point function of \(Z_{cs}^+ \rightarrow \bar{D} D_s^*\) is just the transformation of Eq. (B8) with \(D_s^+ \rightarrow \bar{D}\) and \(\bar{D}^* \rightarrow D_s^*.\) In numerical analysis, \(\sqrt{s_0} = 2.5\) GeV and 1.7 GeV \(\leq M_B^2 \leq 2.3\) GeV is the proper Borel window for decay process.
The OPE side of the three-point function of the decay process $Z_{cs}^+ \to J/\psi K^+$ reads as

\[
\Pi^{\text{pert}}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{9C}{16\pi^4} \int_{4m_c^2}^{s_0} ds \int_{m_c^2}^{v_0} dv \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \, e^{-\frac{3m_K^2}{M_{1B}^2} - \frac{3m^2_v}{M_{2B}^2} (m_c^2 - \mathcal{H}_\alpha)} v, \quad (B9)
\]

\[
\Pi^{(qq)}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{9m_s\langle \bar{q}q \rangle C}{2\pi^2} \int_{4m_c^2}^{s_0} ds \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \, e^{-\frac{m_K^2}{M_{1B}^2} (\mathcal{H}_\alpha - m_c^2)}, \quad (B10)
\]

\[
\Pi^{(ss)}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{9m_s\langle \bar{s}s \rangle C}{4\pi^2} \int_{4m_c^2}^{s_0} ds \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \, e^{-\frac{m_K^2}{M_{1B}^2} (m_c^2 - \mathcal{H}_\alpha)}, \quad (B11)
\]

\[
\Pi^{(G^2)}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{(G^2)C}{128\pi^3} \left\{ \int_{m_c^2}^{v_0} dv \int_0^1 d\alpha \, e^{-\frac{m_c^2}{\alpha(1-\alpha)M_{1B}^2} - \frac{m^2_v}{M_{2B}^2} \left( 3v(m_c^2 - \alpha(\alpha - 1)M_{1B}^2) \right)} \right. \\
+ \left. \frac{3v m_c^2 (1 - 3\alpha + 3\alpha^2) + 2\alpha M_{1B}^2 (2\alpha^3 - 4\alpha^2 + 3\alpha - 1)}{128\alpha^3(\alpha - 1)^3 M_{1B}^4} \right\}, \quad (B12)
\]

\[
\Pi^{(G^3)}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{(G^3)C}{512\pi^3} \int_{m_c^2}^{v_0} dv \int_0^1 d\alpha \, v \frac{m_c^2}{\alpha(1-\alpha)M_{1B}^2} - \frac{m^2_v}{M_{2B}^2} \left[ 6m_c^2 (2\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 1) + \alpha M_{1B}^2 (4\alpha^5 - 12\alpha^4 + 59\alpha^3 - 98\alpha^2 + 62\alpha - 15) \right], \quad (B13)
\]

where $s_0$ and $v_0$ are the continuum threshold of $J/\psi$ and $K^+$, and $M_{1B}^2$ and $M_{2B}^2$ are their Borel parameters, respectively. After employing Borel transformation to Eq. (32) with $D^* \to J/\psi$ and $D_s \to K^+$, the phenomenological side of three-point function of $Z_{cs}^+ \to J/\psi K^+$ will be obtained.

\[
\Pi^{\text{phen}}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{3\lambda_{Z_{cs}^+} m_{J/\psi} f_{J/\psi} f_{K^+} m_{K}^2 + g_{Z_{cs}^+} J/\psi K^+}{m_s (m_{Z_{cs}^+} - m_{J/\psi}^2)} \times \left( e^{-m_{Z_{cs}^+}^2/M_{1B}^2} - e^{-m_{J/\psi}^2/M_{2B}^2} \right) e^{-m_{K}^2/M_{1B}^2}. \quad (B14)
\]

In numerical analysis, $\sqrt{s_0} = 3.2$ GeV, $\sqrt{v_0} = 0.6$ GeV, 2.0 GeV$^2 \leq M_{1B}^2 \leq 3.0$ GeV$^2$, and 2.0 GeV$^2 \leq M_{2B}^2 \leq 3.0$ GeV$^2$ is the proper Borel window for decay process. Here we set $v_0 = (m_K + m_c)^2$.

The OPE side of the three-point function of the decay process $Z_{cs}^+ \to \eta_c K^{*+}$ reads:

\[
\Pi^{\text{pert}}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{3D}{16\pi^4} \int_{4m_c^2}^{s_0} ds \int_{m_c^2}^{v_0} dv \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \, e^{-\frac{\eta_c}{M_{1B}^2} - \frac{v}{M_{2B}^2} (2m_c^2 - 3\mathcal{H}_\alpha)} v, \quad (B15)
\]

\[
\Pi^{(qq)}(s_0, v, M_{1B}^2, M_{2B}^2) = \frac{3m_s\langle \bar{q}q \rangle D}{2\pi^2} \int_{4m_c^2}^{s_0} ds \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \, e^{-\frac{\eta_c}{M_{1B}^2} (3\mathcal{H}_\alpha - 2m_c^2)}, \quad (B16)
\]
\[ \Pi^{(s)}(s_0, v_0, M^2_{1B}, M^2_{2B}) = \frac{3m_c^2 \langle \bar{s}s \rangle D}{4\pi^2} \int_{m_c^2}^{s_0} ds \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \ e^{-\frac{m_c^2}{M^2_{1B}}(2m_c^2 - 3\mathcal{H}_\alpha)}, \]  
\[ \Pi^{(G^2)}(s_0, v_0, M^2_{1B}, M^2_{2B}) = \frac{\langle G^2 \rangle D}{128\pi^4} \left\{ \int_{m_c^2}^{s_0} dv \int_0^1 d\alpha \ e^{-\frac{m_c^2}{\alpha(1-\alpha)M^2_{1B}}} - \frac{6}{M^2_{2B}} \left( \frac{3v(m_c^2 - \alpha(\alpha - 1)M^2_{1B})}{128\alpha(\alpha - 1)M^2_{1B}} \right) \right. 
+ \left. \int_{4m_c^2}^{s_0} ds \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \ e^{-\frac{m_c^2}{M^2_{1B}}(2m_c^2 - 3\mathcal{H}_\alpha)} \right\}, \]  
\[ \Pi^{(G^3)}(s_0, v_0, M^2_{1B}, M^2_{2B}) = \frac{\langle G^3 \rangle D}{512\pi^4} \int_{m_c^2}^{s_0} dv \int_0^1 d\alpha \ \frac{v m_c^2}{\alpha^4(\alpha - 1)^4M^2_{1B}} \left[ \frac{4m_c^2(2\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 1)}{128\alpha^4} + \frac{41\alpha^3 - 62\alpha^2 + 38\alpha - 9}{128\alpha} \right], \]  
where \( s_0 \) and \( v_0 \) are the continuum threshold of \( \eta_c \) and \( K^{*+} \), and \( M^2_{1B} \) and \( M^2_{2B} \) are their Borel parameters, respectively. Taking the transformation of Eq. (B14) with \( J/\psi \to K^{*+} \) and \( K^+ \to \eta_c \), the phenomenological side of decay process \( Z_{c^+}^+ \to \eta_c K^{*+} \) will be obtained. In numerical analysis, \( \sqrt{s_0} = 3.1 \text{ GeV} \), \( \sqrt{v_0} = 1.0 \text{ GeV} \), 2.0 GeV\(^2 \leq M^2_{1B} \leq 3.0 \text{ GeV}^2 \), and 2.0 GeV\(^2 \leq M^2_{2B} \leq 3.0 \text{ GeV}^2 \) is the proper Borel window for decay process \( Z_{c^+}^+ \to \eta_c K^{*+} \).

2. The decay spectral densities of \( Z_{c^+}^+ \) for tetraquark state

The Ref. [10] discussed that only the color-connected diagrams which give the nontrivial color-structure should be considered for tetraquark states, which suggests that only the condensates of \( \langle \bar{q}Gq \rangle \) and \( \langle \bar{s}Gs \rangle \) of OPE side need to be considered. For the color-connected diagrams, we need to isolate the \( q^i_1q^i_2 \) structure of both sides of Eqs. (32) and (36).

On the OPE side of QCD sum rules for the current Eq. (10), the three-point function of \( Z_{c^+}^+ \to D^* D_s^+ \) after Borel transformation is

\[ \Pi^{(\bar{q}Gq)+\langle \bar{s}Gs \rangle}(M_B^2) = \frac{m_c^2 \langle \bar{q}Gq \rangle + \langle \bar{s}Gs \rangle}{64\pi^2} (A - B + C - D) \int_0^1 d\alpha \ e^{\frac{(2-\alpha) m_c^2}{\alpha(\alpha - 1)M_B^2}} \frac{1 + \alpha}{1 - \alpha} + \frac{m_c^2 \langle \bar{s}Gs \rangle}{32\pi^2} (A - B) \int_0^1 d\alpha \ e^{\frac{(2-\alpha) m_c^2}{\alpha(\alpha - 1)M_B^2}} \frac{1 - 3\alpha}{1 - \alpha}. \]
After Borel transformation on the Eq.(32), the phenomenological side of three-point reads:

\[
\Pi^{\text{phen}}(M_B^2) = \frac{3\lambda Z_{cs}^+ m_D f_D f_{D_s} m_{D_s}^2 g_{Z_{cs}^+} D^* D_s}{4 m_{Z_{cs}^+}^2 (m_c + m_s)(m_{D_s}^2/4 - m_{D_s}^2)} \times (e^{-m_{D_s}^2/M_B^2} - e^{-m_{Z_{cs}^+}^2/(4M_B^2)}) e^{-m_{D_s}^2/M_B^2}.
\]  \(\text{(B21)}\)

In our calculation \(2 \text{ GeV}^2 \leq M_B^2 \leq 3 \text{ GeV}^2\) is the proper Borel window for decay process.

The OPE side of the three-point function of the decay process \(Z_{cs}^+ \to \bar{D}D_s^*\) is just the transformation of three-point function of the decay process \(Z_{cs}^+ \to \bar{D}^* D_s^+\) with \(\langle \bar{q}Gq \rangle \leftrightarrow \langle \bar{s}Gs \rangle\), and the phenomenological side of the three-point function of \(Z_{cs}^+ \to \bar{D}D_s^*\) is just the transformation of Eq.(B21) with \(D_s^+ \to \bar{D}\) and \(\bar{D}^* \to D_s^*\). In numerical analysis \(2 \text{ GeV}^2 \leq M_B^2 \leq 3 \text{ GeV}^2\) is the proper Borel window for decay process.

The OPE side of the three-point function of the decay process \(Z_{cs}^+ \to J/\psi K^+\) is

\[
\Pi^{\langle \bar{q}Gq \rangle + \langle \bar{s}Gs \rangle}(M_B^2) = \frac{m_c (\langle \bar{q}Gq \rangle + \langle \bar{s}Gs \rangle)}{32\pi^2} (A - B + C - D) \int_0^1 d\alpha \frac{e^{a(\alpha - 1)M_B^2}}{\alpha}.
\]  \(\text{(B22)}\)

After employing Borel transformation to Eq.(B22) with \(\bar{D}^* \to J/\psi\) and \(D_s \to K^+\), the phenomenological side of three-point function of \(Z_{cs}^+ \to J/\psi K^+\) will be obtained.

\[
\Pi^{\text{phen}}(M_B^2) = \frac{3\lambda Z_{cs}^+ m_{J/\psi} m_K f_{J/\psi} f_K m_{K^+}^2 g_{Z_{cs}^+} J/\psi K^+}{m_{Z_{cs}^+}^2 m_{Z_{cs}^+} (m_{Z_{cs}^+}^2 - m_{J/\psi}^2)} \times (e^{-m_{Z_{cs}^+}^2/M_B^2} - e^{-m_{J/\psi}^2/M_B^2}).
\]  \(\text{(B23)}\)

While the \(m_{K^+}\) is very small, the contribution of the \(e^{-m_{K^+}^2/M_B^2}\) is very close to 1 which is neglecting in the Eq.(B23). In our calculation \(2 \text{ GeV}^2 \leq M_B^2 \leq 3 \text{ GeV}^2\) is the proper Borel window for decay process.

The OPE side of the three-point function of the decay process \(Z_{cs}^+ \to \eta_c K^{++}\) writes as

\[
\Pi^{\langle \bar{q}Gq \rangle + \langle \bar{s}Gs \rangle}(M_B^2) = \frac{m_c (\langle \bar{q}Gq \rangle + \langle \bar{s}Gs \rangle)}{32\pi^2} (A - B + C + D) \int_0^1 d\alpha \frac{e^{a(\alpha - 1)M_B^2}}{1 - \alpha}.
\]  \(\text{(B24)}\)

Taking the transformation of Eq.(B23) with \(J/\psi \to K^{++}\) and \(K^+ \to \eta_c\), the phenomenological side of decay process \(Z_{cs}^+ \to \eta_c K^{++}\) will be obtained. In our calculation \(2 \text{ GeV}^2 \leq M_B^2 \leq 3 \text{ GeV}^2\) is the proper Borel window for decay process.