RIEMANN-ROCH FOR STACKY MATRIX FACTORIZATIONS

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Abstract. We establish a Hirzebruch-Riemann-Roch type theorem and Grothendieck-Riemann-Roch type theorem for matrix factorizations on quotient Deligne-Mumford stacks. For this we first construct a Hochschild-Kostant-Rosenberg type isomorphism explicit enough to yield a categorical Chern character formula. We next find an expression of the canonical pairing of Shklyarov under the isomorphism.

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1. Introduction

1.1. Main results. Let $k$ be an algebraically closed field of characteristic zero. The main interest of this paper is an LG model, $(\mathcal{X}, w)$, where $\mathcal{X}$ is a smooth separated Deligne-Mumford stack of finite type over $k$ and a regular function $w$ with no other critical values but zero.

By a matrix factorization for $(\mathcal{X}, w)$ we mean a pair $(P, \delta_P)$ of a locally free coherent $G$-graded sheaf $P$ on $\mathcal{X}$ and a curved differential $\delta_P$ whose...
square is \( w \cdot \text{id}_P \). Here \( \mathbb{G} \) can be either the group \( \mathbb{Z} \) or \( \mathbb{Z}/2 \) depending on \( w \). There is the notion of the coderived category of matrix factorizations \( \text{DMF}(\mathcal{X}, w) \) and its dg enhancement defined as the dg quotient of the dg category of matrix factorizations by the subcategory of coacyclic or equivalently locally contractible matrix factorizations. Later we will introduce its dg-enhancement denoted by \( \text{MF}_{dg}(\mathcal{X}, w) \); see [15, 32] and also Definition 4.6.

1.1.1. HKR and a Chern character formula. Associated to the dg category \( \text{MF}_{dg}(\mathcal{X}, w) \) there is the so-called mixed Hochschild chain complex \( \text{MC}(\text{MF}_{dg}(\mathcal{X}, w)) \). It has been expected that \( \text{MC}(\text{MF}_{dg}(\mathcal{X}, w)) \) should be quasi-isomorphic to the \( dw \)-twisted de Rham mixed complex \( (\Omega^*_1, -dw|_{\mathcal{I}X}, d) \) of the inertia stack \( \mathcal{I}X \) of \( \mathcal{X} \). However only particular cases have been proven so far. In this paper we verify that the expectation is indeed true.

We first introduce some notation. Let \( \rho_X : \mathcal{I}X \rightarrow \mathcal{X} \) be the natural forgetful morphism. Write \( P|_{\mathcal{I}X} \) for \( \rho_{\mathcal{I}X}^*P \) and let \( \text{can}_{P|_{\mathcal{I}X}} \) be the canonical automorphism of \( P|_{\mathcal{I}X} \); see § 3. Let \( \text{tr} \) denote the supertrace morphism

\[ \mathbb{R}\text{Hom}(P|_{\mathcal{I}X}, P|_{\mathcal{I}X} \otimes (\Omega^*_1, -dw|_{\mathcal{I}X})) \rightarrow \mathbb{H}^*(\mathcal{I}X, (\Omega^*_1, -dw|_{\mathcal{I}X})) \]

and let \( \hat{\text{at}}(P|_{\mathcal{I}X}) \) denote the Atiyah class of the matrix factorization \( P|_{\mathcal{I}X} \) for \( (\mathcal{I}X, w|_{\mathcal{I}X}) \); see [16, 23, 24] and § 5.2.

Theorem 1.1. Suppose \( \mathcal{X} \) is smooth and has the resolution property. Then there is an isomorphism

\[ \text{MC}(\text{MF}_{dg}(\mathcal{X}, w)) \cong \mathbb{R}\Gamma(\Omega^*_1, -dw|_{\mathcal{I}X}, d) \]

in the derived category of mixed complexes. Under the isomorphism the Hochschild homology valued Chern character \( \text{ch}_{HH}(P) \) is representable by

\[ \text{tr}(\text{can}_{P|_{\mathcal{I}X}} \exp(\hat{\text{at}}(P|_{\mathcal{I}X}))) \]

in \( \mathbb{H}^*(\mathcal{I}X, (\Omega^*_1, -dw|_{\mathcal{I}X})) \) after the appropriate sense of the exponential operation \( \exp \) is taken into account; see § 5.2.

The history of related works are very rich. Here we mention only the case of stacky matrix factorizations. In the local case Theorem 1.1 is proven by Polishchuk and Vaintrob [33]. There are works of Căldăraru, Tu and Segal [8, 37] for HKR type isomorphisms in affine cases with a finite group action. The paper [3] of Ballard, Favero, and Katzarkov shows a HKR type isomorphism for the graded cases on linear spaces. This result has also been obtained by Halpern-Leistner and Pomerleano [18, Remark 3.20] and [17, Corollary 4.6]. We note that there is a difference in the map constructed in the current text and [17] (see (4.3) and Remark 4.11 ). Theorem 1.1 is proven by Kuerak Chung, Taejung Kim, and the second author in [10], when one considers quotient stacks of the form \( [X/G] \) where \( X \) is a smooth variety with a finite group action. In [24] a HKR type isomorphism and Chern character formula including the case for the graded matrix factorizations are obtained by the universal Atiyah class.
1.1.2. **HRR and GRR.** Further assume that the smooth separated DM stack $\mathcal{X}$ is a stack quotient of a smooth variety by an action of an affine algebraic group and the critical locus of $w$ is proper over $k$. When $G = \mathbb{Z}/2$, we shall further assume that the morphism $w : \mathcal{X} \to \mathbb{A}^1_k$ is flat. We call the pair $(\mathcal{X}, w)$ a proper LG model. We define the Euler characteristic $\chi(P, Q)$ of the pair $(P, Q)$ by the alternating sum of the dimensions of higher sheaf cohomology:

$$\chi(P, Q) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathbb{R}^i \text{Hom}(P, Q).$$

For a vector bundle $E$ on $\mathcal{I}X$ let $\text{at}(E) \in \text{Ext}^1(E, E \otimes \Omega^1_{\mathcal{I}X})$ denote the usual Atiyah class of $E$. Let

$$\text{ch}_{tw}(E) := \text{tr}(\text{can}_E \text{exp}(\text{at}(E))) \in \mathbb{H}^*(\mathcal{I}X, (\Omega^*_{\mathcal{I}X}, 0)).$$

For a virtual vector bundle $E$ define $\text{ch}_{tw}(E)$ by linearity. We define the Todd class $\text{td}(T_{\mathcal{I}X})$ of $T_{\mathcal{I}X}$ by the formulation of Todd class in terms of the Chern character $\text{ch}$.

**Theorem 1.2.** Let $P^\vee$ denote the matrix factorization $(P^\vee, \delta_P^\vee)$ for $(\mathcal{X}, -w)$ dual to $(P, \delta_P)$, let $N^\vee_{\mathcal{I}X/\mathcal{X}}$ denote the normal vector bundle of $\mathcal{I}X$ to $\mathcal{X}$ via $\rho_X$, let $\dim_{\mathcal{I}X}$ be the locally constant function for local dimensions of $\mathcal{I}X$, and let $\lambda_{-1}(N^\vee_{\mathcal{I}X/\mathcal{X}})$ be the alternating sum of exterior powers of $N^\vee_{\mathcal{I}X/\mathcal{X}}$. Then

$$\chi(P, Q) = \int_{\mathcal{I}X} (-1)^{\left(\dim_{\mathcal{I}X} + 1\right)} \text{ch}_{HH}(Q) \wedge \text{ch}_{HH}(P^\vee) \wedge \frac{\text{td}(T_{\mathcal{I}X})}{\text{ch}_{tw}(\lambda_{-1}(N^\vee_{\mathcal{I}X/\mathcal{X}}))}.$$

Here RHS is the composition of the following two operations:

$$\wedge : \mathbb{H}^*(\mathcal{I}X, (\Omega^*_{\mathcal{I}X}, -dw|_{\mathcal{I}X})) \otimes \mathbb{H}^*(\mathcal{I}X, (\Omega^*_{\mathcal{I}X}, dw|_{\mathcal{I}X})) \otimes \mathbb{H}^*(\mathcal{I}X, (\Omega^*_{\mathcal{I}X}, 0)) \to \bigoplus \text{H}_c^*(\mathcal{I}X, \Omega^p_{\mathcal{I}X}[p])$$

$$\int_{\mathcal{I}X} : \bigoplus_{p \in \mathbb{Z}} H_c^*(\mathcal{I}X, \Omega^p_{\mathcal{I}X}[p]) \xrightarrow{\text{projection}} H_c^0(\mathcal{I}X, \Omega^p_{\mathcal{I}X}[n]) \xrightarrow{\text{tr}_{\mathcal{I}X}, k}.$$
**Theorem 1.3.** (=Theorem 6.11) The following diagram is commutative:

\[
\begin{array}{ccc}
K_0(\text{MF}_{dg}(\mathcal{X}, w)) & \xrightarrow{f_!} & K_0(\text{MF}_{dg}(\mathcal{Y}, v)) \\
\Delta_{HH} & & \Delta_{HH} \\
\mathbb{H}^0(I\mathcal{X}, (\Omega^\bullet_{IX}, -dw|_{IX})) & \xrightarrow{\int_{ff}(-1)^{\dim ff}\cdot\wedge_{\text{Id}(T_{ff})}} & \mathbb{H}^0(I\mathcal{Y}, (\Omega^\bullet_{IY}, -dv|_{IY})).
\end{array}
\]

When $G = \mathbb{Z}$ (and hence $w = 0$), various versions of Riemann-Roch theorem on DM stacks are proven by Kawasaki [19], Toënn [40]; and Edidin and Graham [11, 12, 13]. When $w$ has one critical point, Theorem 1.2 is proven by Polishchuk and Vaintrob [33].

### 1.2. On the proofs and pertinent works.

#### 1.2.1. HKR

For the computation of Hochschild homology of the category of matrix factorizations, there are at least three known approaches by (1) finding a suitable flat resolution of the diagonal module [3, 26, 33], (2) using the quasi-Morita equivalence [5, 8, 10, 14, 37], and (3) using the universal Atiyah classes [23, 24] which goes back to [7, 29]. In this paper we take the second approach by achieving a correct globalization of Baranovsky’s map [4] closely following the proof of Proposition 2.13 of [18] and [17, Corollary 4.6]. Combining this with a chain-level map from [5, 10] we obtain a boundary-bulk map formula as well as a Chern character formula; see §5.

#### 1.2.2. HRR

For any proper smooth dg category $\mathcal{A}$ there is a categorical HRR theorem by Shklyarov [38]. Let $\mathcal{A}^{op}$ denote the opposite category of $\mathcal{A}$. Let $\langle \cdot, \cdot \rangle_{\text{can}}$ be the canonical pairing of Shklyarov:

\[
\langle \cdot, \cdot \rangle_{\text{can}} : HH_*(\mathcal{A}) \otimes HH_*(\mathcal{A}^{op}) \to k.
\]

Then the categorical HRR theorem is the equality

\[
\chi(P, Q) = \langle \text{Ch}_{HH}(Q), \text{Ch}_{HH}(P^\vee) \rangle_{\text{can}} \forall P, Q \in \mathcal{A}.
\]

There is a characterization property of the canonical pairing in terms of the Chern character of diagonal bimodule; see for example §6.1.2. Let $\mathcal{A}$ be the dg category of matrix factorizations for $(\mathcal{X}, w)$ localized by coacyclic matrix factorizations. When $\mathcal{X}$ is local, using the characterization property Polishchuk and Vaintrob [33] show that the canonical pairing becomes up to sign the residue pairing under their HKR type isomorphism. In the nonstacky local case, there is also a work of Brown and Walker [6] identifying the canonical pairing with the residue pairing under the HKR type isomorphism.

When $\mathcal{X}$ is a smooth variety, using the deformation to the normal cone as well as the characterization property of the canonical pairing, the second author [22] shows that the canonical pairing becomes a trace map under the HKR type isomorphism up to a Todd correction term. When $\mathcal{X}$ is stacky, furthermore using the deformation to the normal cone for local immersions.
[25, 41] and the Chern character formula in Theorem 1.1, we are able to prove Theorem 1.2.

1.2.3. **GRR.** The proper morphism $f$ in Theorem 1.3 induces a dg functor from $\text{MF}_{dg}(\mathcal{X}, w)$ to $\text{MF}_{dg}(\mathcal{Y}, v)$. The induced homomorphism

$$HH_* (\text{MF}_{dg}(\mathcal{X}, w)) \to HH_* (\text{MF}_{dg}(\mathcal{Y}, v))$$

in Hochschild homology has a description § 6.1.1 in terms of the canonical pairing of $\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \otimes 1 + 1 \otimes v)$ and the categorical Chern character of the matrix factorization associated to the graph morphism $\Gamma_f : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$. Under the HKR isomorphisms, the deformation to the normal cone allows us to interpret the description as the pushforward by $f$ of twisted Hodge cohomology up to a Todd correction term.

1.3. **Conventions and notation.** Let the ground field $k$ be an algebraically closed field of characteristic zero. Let $\mu_r$ denote the group of $r$-th roots of unity over the field $k$. Throughout this paper, let $\mathcal{X}$ be a finite type separated DM stack over $k$. We denote by $I \mathcal{X}$ the inertia stack of $\mathcal{X}$. Let $\rho \mathcal{X} : I \mathcal{X} \to \mathcal{X}$ denote the natural representable morphism, which is finite and unramified; [1, § 3]. Let $G$ be a finite group which acts on a scheme $Y$ of finite type over $k$. For a quotient stack $[Y/G]$, let

$$I[Y/G] = [Y/G].$$

We write $\tilde{G}$ for the character group $\text{Hom}(G, \mathbb{G}_m)$ of $G$. The label $\sim$ on an arrow indicates the arrow is a quasi-isomorphism. By a vector bundle we mean a locally free coherent sheaf.

For a local immersion (i.e., an unramified representable morphism) $f : \mathcal{X} \to \mathcal{Y}$ between DM stacks, we denote $C_{\mathcal{X}/\mathcal{Y}}$ be the normal cone to $\mathcal{X}$ in $\mathcal{Y}$; see [25, 41]. If $f$ is a regular local immersion, then we write $N_f$ or $N_{\mathcal{X}/\mathcal{Y}}$ for the vector bundle $C_{\mathcal{X}/\mathcal{Y}}$ on $\mathcal{X}$.

For a vector bundle $E$, we often write $1_E$ for the identity morphism $\text{id}_E$ of $E$. For a dg category $\mathcal{A}$, its homotopy category is denoted by $[\mathcal{A}]$.

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2. Mixed Hochschild complexes and Chern characters

Unless otherwise stated, we follow notation and conventions of [5, 10] for curved dg (in short cdg) categories $\mathcal{A}$, the mixed Hochschild complexes of $\mathcal{A}$, and the category of mixed complexes. We briefly recall notation therein and some foundational facts which we are going to use later.

2.1. Mixed Hochschild complexes. For a curved dg category $\mathcal{A}$, we use the following notation:

- $\text{C}(\mathcal{A})$ (MC(\mathcal{A})) : (mixed) Hochschild complex.
- $\overline{\text{C}}(\mathcal{A})$ (MC(\mathcal{A})) : (mixed) normalized Hochschild complex.
- $\text{C}^{II}(\mathcal{A})$ (MC$^{II}(\mathcal{A}))$ : (mixed) Hochschild complex of the second kind.
- $\overline{\text{C}}^{II}(\mathcal{A})$ (MC$^{II}(\mathcal{A}))$ : (mixed) normalized Hochschild complex of the second kind.

For notational convenience we let $\text{C}^{1}$ denote either $\text{C}$, $\overline{\text{C}}$, $\text{C}^{II}$ or $\overline{\text{C}}^{II}$ and MC$^{1}$ denote either MC, MC, MC$^{II}$, or MC$^{II}$. The normalized negative cyclic complex is denoted by $\text{C}[u]$ where $u$ is a formal variable of degree 2. For a mixed complex $(C,b,B)$ we simply write $\text{C}[u]$ for the complex $(C[u],b + uB)$.

2.2. Foundational facts frequently in use.

2.2.1. Invariance of the natural projections. The projection MC($D$) $\to$ MC($D$) for a dg category $D$ and the projection MC$^{II}(\mathcal{A})$ $\to$ MC$^{II}(\mathcal{A})$ for a cdg category $\mathcal{A}$ are quasi-isomorphisms [5, 34].

2.2.2. (Quasi-)Morita invariance. If a dg functor $D \to D'$ is Morita-equivalent (i.e., the induced functor $D(D) \to D(D')$ of derived categories of $D$ and $D'$ is an equivalence), then the induced morphism MC($D$) $\to$ MC($D'$) of mixed complexes is a quasi-isomorphism [20]. If $\mathcal{A} \to \mathcal{A}'$ is a pseudo-equivalence of cdg categories, then the induced morphism MC$^{II}(\mathcal{A})$ $\to$ MC$^{II}(\mathcal{A}')$ is a quasi-isomorphism [34]. This invariance is dubbed as quasi-Morita invariance.

2.2.3. Localization in cyclic homology. If $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence of exact dg categories, then it induces an exact triangle of mixed complexes

$$MC(\mathcal{A}) \to MC(\mathcal{B}) \to MC(\mathcal{C}) \to MC(\mathcal{A})[1];$$

see [21, § 5.6].

2.2.4. Local description of the inertia stack. Let $G$ be a finite group acting on a $k$-scheme $X$ and let $G/G$ denote the set of conjugacy classes of $G$. Then the following holds $I[X/G] \cong \sqcup_{g \in G/G} [X^g/C_G(g)]$. 
2.2.5. Invariants and coinvariants. Let \( G \) be a group with a linear action upon \( V \) a not-necessarily finite dimensional vector space. Define the invariant space \( V^G := \text{Hom}_G(k,V) \) and the coinvariant space \( V_G := k \otimes_G V \). If \( G \) is a finite group, then the composition of the natural homomorphisms \( V^G \to V \to V_G \) is an isomorphism.

2.3. Categorical Chern characters. For \( P \in \mathcal{A} \) the cycle class represented by the identity morphism \( 1_P \) in the normalized Hochschild complex \( \overline{C}(\mathcal{A}) \) (resp. the normalized negative cyclic complex \( \overline{C}(\mathcal{A})[[u]] \)) of \( \mathcal{A} \) is denoted by \( \text{Ch}_{HH}(P) \) (resp. \( \text{Ch}_{HN} \)).

3. The canonical central automorphism

3.1. The central embedding. The inertia stack has a decomposition:

\[ I\mathcal{X} = \sqcup_{i=1}^{\infty} I_{\mu_i} \mathcal{X}. \]

An object of \( I_{\mu_i} \mathcal{X} \) over a \( k \)-scheme \( T \) is a pair \((\xi, \alpha)\) of an object \( \xi \in \mathcal{X}(T) \) and an injective morphism of group-schemes \( \alpha : \mu_r \times T \to \text{Aut}_T(\xi) \); see [1, § 3]. Note that for all but finitely many \( r \), \( I_{\mu_r} \mathcal{X} \) is empty. An automorphism of the pair \((\xi, \alpha)\) in \( I_{\mu_i} \mathcal{X} \) is by definition an automorphism \( f \in \text{Aut}_T(\xi) \) such that

\[ f \circ \alpha \circ f^{-1} = \alpha. \]

In other words the automorphism group-scheme \( \text{Aut}_T(\xi, \alpha) \) of \((\xi, \alpha)\) over \( T \) is the centralizer of \( \alpha \) in \( \text{Aut}_T(\xi) \). We have a canonical central embedding \( \mathfrak{c} : \mu_r \times T \to \text{Aut}_T(\xi, \alpha) \). This gives a natural morphism

\[ \mu_r \times T \to T \times I_{\mu_r} \mathcal{X} T; (\xi, t) \mapsto (t, \mathfrak{c}(\xi, t)). \]

3.2. The central automorphism. Let \( T \to I_{\mu_r} \mathcal{X} \) be an étale surjection and let \( \text{pr}_i : T \times I_{\mu_r} \mathcal{X} T \to T \) be the \( i \)-th projection. A vector bundle \( E \) on \( I_{\mu_r} \mathcal{X} \) amounts to a vector bundle \( F \) on \( T \) with an isomorphism \( \phi^F \in \text{Isom}_T(\text{pr}_1^* F, \text{pr}_2^* F) \) satisfying the cocycle condition. By pulling back the isomorphism \( \phi^F \) to \( \mu_r \times T \) we obtain a morphism of group-schemes \( \mu_r \times T \to \text{Aut}_T(F) \). Here \( \text{Aut}_T(F) \) denotes the group of automorphisms of \( F \) fixing \( T \). Since \( \mathfrak{c} \) is central, the homomorphism descends to a homomorphism \( \mu_r \to \text{Aut} I_{\mu_r} \mathcal{X}(E) \). Denote by

\[ \text{can}_E \in \text{Aut} I_{\mu_r} \mathcal{X}(E) \]

the image of the chosen \( r \)-th root \( e^{2\pi i/r} \) of unity.

According to the action \( \mu_r \) upon \( E \), the bundle \( E \) is decomposable into eigenbundles

\[ \bigoplus_{\chi \in \hat{\mu}_r} E_{\chi}, \]

where \( \hat{\mu}_r \) is the character group of \( \mu_r \). Then we have

\[ \text{can}_E = \bigoplus_{\chi \in \hat{\mu}_r} \chi(e^{2\pi i/r}) \text{id}_E \in \text{Hom}_{I_{\mu_r} \mathcal{X}}(E, E). \]

In turn this gives an automorphism \( \text{can}_P \) of \( P \) for every object \( P \) of \( \text{MF}_{dg}(I\mathcal{X}, w) \). It is a morphism in the category \( \text{MF}_{dg}(I\mathcal{X}, w) \).
Note that
\[(3.1) \quad \text{can}_{P'} \circ a = a \circ \text{can}_P \quad \forall a \in \text{Hom}_{I\mathcal{X}}(P, P').\]

Hence the assignments \(P \mapsto \text{can}_P\) yield a natural transformation \(\text{can}\) between the identity functor \(id : MF_{dg}(I\mathcal{X}, \mathcal{W}) \to MF_{dg}(I\mathcal{X}, \mathcal{W})\). We will often drop the subscript \(P\) in \(\text{can}_P\) for simplicity.

3.3. The local description. Locally the automorphisms are described as follows. Suppose that \(\mathcal{X} = [X/G]\) where \(G\) is a finite group and \(X\) is a scheme. Let \(g \in G\) with order \(r\) and write \(C_G(g)\) for the centralizer of \(g\) in \(G\). For the component \([X^g/C_G(g)]\) of \(I_{\mu, \mathcal{X}}\) and a \(C_G(g)\)-equivariant sheaf \(E\) on \(X^g\), we have an isomorphism
\[
(g^{-1})^* E \xrightarrow{\varphi^E_g} E
\]
from the equivariant structure of \(E\). Since \(g\) acts trivially on \(X^g\), \((g^{-1})^* E = E\). And hence \(\varphi^E_g\) is an automorphism of \(E\), which is the automorphism \(\text{can}_E\). Since any element of \(C_G(g)\) commutes with \(g\), the homomorphism \(\varphi^E_g\) is \(C_G(g)\)-equivariant. Thus \(\varphi^E_g \in \text{Hom}_{I_{\mu, \mathcal{X}}}(E, E)\). For any \(C_G(g)\)-equivariant sheaf \(E'\) on \(X^g\) and any \(C_G(g)\)-equivariant \(\mathcal{O}_{X^g}\)-module homomorphism \(a : E \to E'\), note that \(\varphi^E_g \circ a = a \circ \varphi^E_g\).

3.4. \(T_{I\mathcal{X}} \cong IT_{\mathcal{X}}\). In this subsection let \(\mathcal{X}\) be smooth over \(k\). We prove that there is a natural isomorphism \(T_{I\mathcal{X}} \cong IT_{\mathcal{X}}\).

3.4.1. Note first that there is a natural short exact sequence of vector bundles
\[
0 \to T_{I\mathcal{X}} \to \rho_X^* T_{\mathcal{X}} \to N_{\rho_X} \to 0.
\]
In fact this sequence splits. The reason is that according to the canonical automorphism of \(\rho_X^* T_{\mathcal{X}}\) there is a decomposition of \(\rho_X^* T_{\mathcal{X}}\) into the fixed part and the moving part, which are naturally isomorphic to \(T_{I\mathcal{X}}\) and \(N_{\rho_X}\), respectively.

3.4.2. Consider a commuting diagram of natural morphisms
\[
\begin{array}{ccc}
IT_{\mathcal{X}} & \xrightarrow{\phi} & T_{I\mathcal{X}} \\
\downarrow & & \downarrow \\
T_{\mathcal{X}} & \xrightarrow{\rho_X^*} & T_{\mathcal{X}} \\
\downarrow & & \downarrow \\
I\mathcal{X} & \xrightarrow{\rho_X} & \mathcal{X},
\end{array}
\]
where the square is a fiber square.

**Lemma 3.1.** The morphism \(\phi\) induces an isomorphism \(IT_{\mathcal{X}} \cong T_{I\mathcal{X}}\).
Proof. First note that it is enough to check the isomorphism over the étale site of the coarse moduli space of $\mathcal{X}$. Since $\mathcal{X}$ is separated, then $\mathcal{X}$ is étale locally a quotient of a nonsingular variety $Y$ by a finite group $G$ action. Hence we may assume that $\mathcal{X} \sim Y/G$. Since $\mathcal{X}$ is separated, then $\mathcal{X}$ is étale locally a quotient of a nonsingular variety $Y$ by a finite group $G$ action. Hence we may assume that $\mathcal{X} \sim Y/G$. Since $\mathcal{X}$ is separated, then $\mathcal{X}$ is étale locally a quotient of a nonsingular variety $Y$ by a finite group $G$ action. Hence we may assume that $\mathcal{X} \sim Y/G$.

we have

$$IT[Y/G] \cong \bigsqcup_{g \in G/G} [(T_Y)^g/C_G(g)]$$

Since $(T_Y)^g \cong T_Y^g$, we conclude the proof. $\square$

Remark 3.2. When $\mathcal{X}$ is the global quotient $[Y/G]$ by a finite group $G$, $N_{\rho_X} \cong [N_{Y^g}/C_G(g)]$.

4. Stacky Hochschild-Kostant-Rosenberg

In this section, we assume $\mathcal{X}$ is a smooth separated DM stack and the critical value of $w$ is over 0. If $G = \mathbb{Z}$, then $w = 0$. If $G = \mathbb{Z}/2$, then we furthermore assume that $w : \mathcal{X} \to \mathbb{A}^1$ is flat.

4.1. Matrix factorizations and their derived categories. Whenever an LG model $(\mathcal{X}, w)$ is given, we consider a sheaf of curved differential graded (CDG for short) algebra $(\mathcal{O}\mathcal{X}, -w)$ over $\mathcal{X}$. It is concentrated in degree 0 with zero differential and a curvature $-w$.

Definition 4.1. A quasi-differential graded module (QDG-module for short) over $(\mathcal{O}\mathcal{X}, -w)$ is a pair $(P, \delta_P)$ of an $\mathcal{O}\mathcal{X}$-module $P$ and an $\mathcal{O}\mathcal{X}$-linear degree 1 endomorphism $\delta_P$. We say a QDG module is

- (quasi-)coherent if $P$ is (quasi-)coherent,
- locally free if $P$ is locally free,
- matrix quasi-factorization if $P$ is locally free of finite rank.

We denote a category of QDG modules over $(\mathcal{O}\mathcal{X}, -w)$ by $q\text{Mod}(\mathcal{X}, w)$. It is a CDG category whose morphisms and differentials are

$$\text{Hom}_{q\text{Mod}}((P, \delta_P), (Q, \delta_Q)) = (\text{Hom}_{\mathcal{O}\mathcal{X}}(P, Q), \delta),$$

$$\delta(f) = \delta_Q \circ f - (-1)^{|f|} f \circ \delta_P.$$

The curvature element $h_{(P, \delta_P)}$ of $(P, \delta_P)$ is defined as $\delta_P^2 + \rho_{-w} \in \text{End}(P)$, where $\rho_{-w}$ is the multiplication map by $-w$.

Definition 4.2. A QDG-module $(P, \delta_P)$ is called a factorization if its curvature is zero. We define (quasi-)coherent or locally free factorizations as in 4.1. In particular, we call it a matrix factorization if $P$ is a locally free of finite rank.
By definition, factorizations form a DG subcategory inside $q\text{Mod}(\mathcal{X}, w)$ denoted by $\text{Mod}(\mathcal{X}, w)$. We denote a full DG subcategory of (quasi-) coherent and matrix factorizations by $\text{QCoh}(\mathcal{X}, w)$, $\text{Coh}(\mathcal{X}, w)$ and $\text{MF}(\mathcal{X}, w)$ respectively.

We recall constructions of the derived category of factorizations following [35]. Let $[\text{QCoh}(\mathcal{X}, w)]$ be the homotopy category of $\text{QCoh}(\mathcal{X}, w)$. Denote by $\text{AbsAcyc}(\mathcal{X}, w)$ the smallest triangulated subcategory containing the totalizations of all short exact sequences in $Z^0\text{QCoh}(\mathcal{X}, w)$. Its object are called absolutely acyclic factorizations. Also, denote by $\text{CoAcyc}(\mathcal{X}, w)$ the smallest triangulated subcategory containing the totalizations of all acyclic factorizations which is closed under infinite direct sum, its object are called a coacyclic factorizations.

**Definition 4.3.** The absolute derived category of $\text{QCoh}(\mathcal{X}, w)$ is the Verdier quotient

$$D^\text{abs}(\text{QCoh}(\mathcal{X}, w)) := [\text{QCoh}(\mathcal{X}, w)]/\text{AbsAcyc}(\mathcal{X}, w).$$

The coderived category of $\text{QCoh}(\mathcal{X}, w)$ is the Verdier quotient

$$D^\text{co}(\text{QCoh}(\mathcal{X}, w)) := [\text{QCoh}(\mathcal{X}, w)]/\text{CoAcyc}(\mathcal{X}, w).$$

We define an absolute/coderived category of $\text{Coh}(\mathcal{X}, w)$ and $\text{MF}(\mathcal{X}, w)$.

**Definition 4.4.** The derived category of of matrix factorizations denoted by $D\text{MF}(\mathcal{X}, w)$ is the smallest full triangulated subcategory of $D^\text{co}(\text{QCoh}(\mathcal{X}, w))$ which contains $D^\text{abs}(\text{MF}(\mathcal{X}, w))$.

**Remark 4.5.** Relations between various categories are well-known. We only recall a few facts we will going to use later. If $\mathcal{X}$ is smooth, then it is known that Verdier localization $D^\text{abs}(\text{QCoh}(\mathcal{X}, w)) \to D^\text{co}(\text{QCoh}(\mathcal{X}, w))$ is an equivalence, and an image of $D^\text{abs}(\text{Coh}(\mathcal{X}, w))$ consists of compact generators inside $D^\text{co}(\text{QCoh}(\mathcal{X}, w))$ (see [35, § 3.6]). If $\mathcal{X}$ has the resolution property, then $D^\text{abs}(\text{MF}(\mathcal{X}, w)) \to D^\text{abs}(\text{Coh}(\mathcal{X}, w))$ is an equivalence. (See [32])

4.2. Čech model. In this subsection, we recall the Čech type DG enhancement of $D\text{MF}(\mathcal{X}, w)$ as in [10].

Fix an affine étale surjective morphism $p : \mathfrak{U} \to \mathcal{X}$ from a $k$-scheme $\mathfrak{U}$. Let $\mathfrak{U}^r$ denote the $r$-th fold product of $\mathfrak{U}$ over $\mathcal{X}$ and let $p_r : \mathfrak{U}^r \to \mathcal{X}$ denote the projection. For a vector bundle $E$ on $\mathcal{X}$, let $\check{C}^r(E) = p_{r*}p^*_r E$ and

$$\check{C}(E) := \left( \bigoplus_{r \geq 1} \check{C}^r(E), d_{\check{\text{Cech}}} \right) = [0 \to p_{1*}p^*_1 E \to p_{2*}p^*_2 E \to \cdots]$$

a Čech complex.

Now let $(E, \delta_E)$ be a matrix quasi-factorization over $(\mathcal{O}_\mathcal{X}, -w)$. Observe that $\check{C}(\mathcal{O}_\mathcal{X})$ can be viewed as a sheaf of $\mathcal{O}_\mathcal{X}$-algebras equipped with Alexander-Whitney product. By projection formula $\check{C}(E) = E \otimes_{\mathcal{O}_\mathcal{X}} \check{C}(\mathcal{O}_\mathcal{X})$
and $\check{C}(E)$ carries a natural $\check{C}(\mathcal{O}_X)$-module structure. We equip $\check{C}(E)$ with a curved differential

$$\delta_{\check{C}(E)} := \delta P \otimes 1 + 1 \otimes d_{\check{Cech}}.$$ 

We regard $(\check{C}(E), \delta_{\check{C}(E)})$ as a QDG-module over $(\check{C}(\mathcal{O}_X), w)$. Notice that the curvature of $(\check{C}(E), \delta_{\check{C}(E)})$ coincides with the curvature of $E$ under the canonical map $\text{End}_{\mathcal{O}_X}(E) \to \text{End}_{\check{C}(\mathcal{O}_X)}(\check{C}(E))$.

**Definition 4.6.** For a fixed affine étale open cover $\mathfrak{U}$, a Čech model CDG category $q\text{MF}_{dg}(\mathcal{X}, w)$ is a CDG category whose objects are matrix quasi-factorizations and its $\mathbb{G}$-graded Hom spaces are defined as usual

$$\text{Hom}_{q\text{MF}_{dg}}(P, Q) := (\text{Hom}_{\mathcal{O}_X}(\check{C}(P), \check{C}(Q)), \delta)$$

$$\delta = \delta_{\check{C}(P)} \circ f - (-1)^{|f|} f \circ \delta_{\check{C}(P)}$$

Similarly, we define a Čech model dg category $\text{MF}_{dg}(\mathcal{X}, w)$ of matrix factorizations for $(\mathcal{X}, w)$ as a full DG subcategory of $q\text{MF}_{dg}(\mathcal{X}, w)$ consisting of matrix factorizations for $(\mathcal{X}, w)$. When it is necessary to specify the covering $\mathfrak{U}$, we write $\text{MF}_{dg}(\mathcal{X}, w; \mathfrak{U})$ for $\text{MF}_{dg}(\mathcal{X}, w)$.

**Remark 4.7.** We view $\text{Hom}_{\text{MF}_{dg}}(P, Q)$ as a $\mathbb{Z} \otimes \mathbb{Z}$-graded bicomplex. This complex is not bounded above in $\mathbb{Z}$-direction because a Čech cover $\mathfrak{U}$ of a stack is genuinely unordered. One can go around the subtleties by taking a suitable truncation. Suppose $(E, \delta_E)$ is a matrix factorization. Define

$$\tau \check{C}(E) := \tau_{\leq \dim \mathcal{X}} \left( \bigoplus_{r \geq 1} \check{C}^r (E), d_{\check{Cech}} \right)$$

where $\tau_{\leq \dim \mathcal{X}}$ denotes the truncation. Note that under the assumptions in this text all stacks have finite cohomological dimension, hence the truncated one does compute the sheaf cohomology of the quasi-coherent sheaf $E$, since the map to the coarse moduli $\mathcal{X} \to \mathcal{X}$ is cohomologically affine. Therefore, the induced map between the spectral sequences associated to Čech filtrations

$$\text{Hom}(\check{C}(P), \check{C}(Q)) \to \text{Hom}(\tau \check{C}(P), \check{C}(Q))$$

are isomorphisms at the first page and hence, are quasi-isomorphisms themselves. Also notice that $\tau \check{C}(E) \to \check{C}(E)$ is a quasi-isomorphism in $\text{DMF}(\mathcal{X}, w)$.

It is not hard to see that $\text{MF}_{dg}(\mathcal{X}, w)$ is a DG enhancement of the derived category of $\text{DMF}(\mathcal{X}, w)$. Note that

1. $P$ is coacyclic if and only if $P|_\mathfrak{U}$ is coacyclic, since the natural morphism $P \to \check{C}(P)$ is termwise exact. See [9, Proposition 2.2.6].
2. TFAE: $P|_\mathfrak{U}$ is coacyclic, $P|_\mathfrak{U}$ is absolutely acyclic, and $P|_\mathfrak{U}$ is contractible by [35, § 3.6];
(3) TFAE: $P|_{\mathcal{U}}$ is coacyclic, $P$ is locally contractible by [9, Proposition 2.2.6] and (1). Here $P$ is called locally contractible if there is an open covering $\mathcal{U}$ in smooth topology of $\mathcal{X}$ such that $P|_{\mathcal{U}}$ is contractible; see [32].

Therefore if $P \in \text{MF}_{dg}(\mathcal{X}, w)$ represents a coacyclic object in $\text{DMF}(\mathcal{X}, w)$, then $P$ is locally contractible and hence, $\text{Hom}_{\text{MF}_{dg}}(P, P) \simeq 0$.

Remark 4.8. Consider another affine covering $\mathcal{U}' \to \mathcal{X}$. Let $\mathcal{U}'' = \mathcal{U}' \times_{\mathcal{X}} \mathcal{U}$. Then there is a natural dg functor $\text{MF}_{dg}(\mathcal{X}, w, \mathcal{U}') \to \text{MF}_{dg}(\mathcal{X}, w, \mathcal{U}'')$, which is a quasi-equivalence. The induced chain map $\text{MC}'(\text{MF}_{dg}(\mathcal{X}, w, \mathcal{U})) \to \text{MC}'(\text{MF}_{dg}(\mathcal{X}, w, \mathcal{U}''))$ between mixed complexes is quasi-isomorphism. For the mixed complex of the first kind it follows from the Morita invariance. For that of the either kind consider the Hochschild complex $C'(\text{MF}_{dg}(\mathcal{X}, w))$ filtered by Čech degree. Since the filtration is bounded (See 4.7), we may apply the Eilenberg-Moore comparison theorem ([42, Theorem 5.5.11]) to check that the induced chain map is a quasi-isomorphism. A similar argument shows that the natural chain map

$$\text{MC}'(q\text{MF}_{dg}(\mathcal{X}, w, \mathcal{U})) \to \text{MC}'(q\text{MF}_{dg}(\mathcal{X}, w, \mathcal{U}''))$$

is also a quasi-isomorphism.

4.3. Twisting. For a morphism from a scheme $U$ to $\mathcal{X}$, write $IU$ for the fiber product $U \times_{\mathcal{X}} I\mathcal{X}$. Following Toën, Halpern-Leistner and Pomerleano [18] we consider the assignments

$$U \mapsto C'(q\text{MF}_{dg}(IU, w|_{IU}, \mathcal{U} \times_{\mathcal{X}} IU))$$

for all étale morphisms $U \to \mathcal{X}$. They form a presheaf of mixed complexes on the small étale site $\mathcal{X}$, which we denote by

$$\text{MC}'(q\text{MF}_{I\mathcal{X}, w}(\mathcal{X}, w))$$

Denote the associated cochain complex by $C'(q\text{MF}_{I\mathcal{X}, w}(\mathcal{X}, w))$. There is a natural morphism of mixed complexes

$$\text{nat} : \text{MC}'(q\text{MF}_{dg}(I\mathcal{X}, w)) \to (\Gamma(\mathcal{X}, \text{MC}'(q\text{MF}_{I\mathcal{X}, w}(\mathcal{X}, w)))),$$

where the Čech model $q\text{MF}_{dg}(I\mathcal{X}, w)$ uses the affine cover $\mathcal{U} \times_{\mathcal{X}} I\mathcal{X} \to I\mathcal{X}$.

Define a $k$-linear map $\text{tw} : C'(q\text{MF}_{dg}(I\mathcal{X}, w)) \to C'(q\text{MF}_{dg}(I\mathcal{X}, w))$ associated to $\text{can}$ by

$$a_0[a_1|\cdots|a_n] \mapsto a_0[a_1|\cdots|\text{can} \circ a_n].$$

Note that by (3.1) $b \circ \text{tw} = \text{tw} \circ b$, i.e., $\text{tw}$ is a chain automorphism of the Hochschild complex $C'(q\text{MF}_{dg}(I\mathcal{X}, w))$.

Consider the composition $\tau'$ of a sequence of chain maps

$$(4.3) \quad C'(q\text{MF}_{dg}(\mathcal{X}, w)) \xrightarrow{\text{pullback}} C'(q\text{MF}_{dg}(I\mathcal{X}, w)) \xrightarrow{\text{tw}} C'(q\text{MF}_{dg}(I\mathcal{X}, w)) \xrightarrow{\text{nat}} \Gamma(\mathcal{X}, C'(q\text{MF}_{I\mathcal{X}, w}(\mathcal{X}, w))).$$
Proposition 4.9. The chain map $\tau^{II}$ is a quasi-isomorphism when $\mathcal{X}$ is of form $[\text{Spec} A/G]$ for some commutative $k$-algebra $A$ with a finite group $G$ action.

A proof of the above proposition will be given § 4.4 and § 4.5. For simplicity we will often write $\tau$ for $\tau'$ when there is no risk of confusion.

4.4. Local case. Let $\mathcal{X} = [\text{Spec} A/G]$ and let $w \in A^G$ a $G$-invariant element of $A$ as in Proposition 4.9. Let $\text{MF}^G_{dg}(A, w)$ denote the dg category of $G$-equivariant factorizations $P$ for $(A, w)$ which are projective as $A$-modules. The Hom space from $P$ to $Q$ is the $G$-invariant part of $\text{Hom}_{A}(P, Q)$ of $G$-graded $A$-module homomorphisms. Likewise we have the cdg category $\text{MF}^G_{dg}(A, w)$ of $G$-equivariant quasi-modules for $(A, w)$ which are projective as $A$-modules. In fact these coincide with the Čech models $\text{MF}^G_{dg}(\mathcal{X}, w)$ and $\text{MF}^G_{dg}(\mathcal{X}, w)$ with respect to the natural choice of an affine cover: $\text{Spec} A \to \mathcal{X}$.

Let $I_g$ be the ideal of $A$ generated by $a - ga$ for all $a \in A$. Denote $A_g := A/I_g$ and $w_g := w|_{A_g} \in A_g$. We regard the pair $(A, w)$ (resp. $(A_g, w_g)$) as a curved dg algebra $A$ (resp. $A_g$) with zero differential and curvature $w$ (resp. $w_g$).

The algebra $A$ has the induced left $G$-action. Note that for $a, b \in A$ and $g, h \in G$,

$$g(a \cdot h(b)) = g(a) \cdot gh(b).$$

The cross product algebra $A \times G := A \otimes k[G]$ has the multiplication defined by $(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh$. We also view $A \times G$ as a right $A$-module with a left $G$-action by

$$(a \otimes g) \cdot b = ag(b) \otimes g \text{ and } h \cdot (a \otimes g) = a \otimes gh^{-1}.$$  

Equivalently, $A \times G$ is a right $A \times G$-module by the multiplication

$$(a \otimes g) \cdot (b \otimes h') = ag(b) \otimes gh'.$$

Note that the curvature of $A \times G$ with zero differential as a right quasi-module over $(A \times G, w \otimes 1)$ is $-w \otimes 1$.

Denote by $\{(A \times G, -w \otimes 1)\}$ the full subcategory of $q\text{MF}_{dg}(A \times G, w \otimes 1)$ consisting of the indicated object $A \times G$ with zero differential and curvature $-w \otimes 1$. The embedding $\{(A \times G, -w)\} \hookrightarrow q\text{MF}_{dg}(A \times G, w \otimes 1)$ is called a quasi-Yoneda embedding. It is a pseudo-equivalence; see [34]. Consider the embedding $q\text{MF}_{dg}(A \times G, w \otimes 1) \hookrightarrow q\text{MF}^G_{dg}(A, w)$, which is also a pseudo-equivalence; see [10]. Hence by the quasi-Morita invariance the induced morphism of mixed complexes

$$\overline{\text{MC}}^{II}(A \times G, -w) \to \overline{\text{MC}}^{II}(q\text{MF}^G_{dg}(A, w))$$

is a quasi-isomorphism.

Consider an embedding $\{(A_g \times G, -w_g \otimes 1)\} \hookrightarrow q\text{MF}_{dg}(A_g, w_g)$. This is a pseudo-equivalence, since $(A_g, -w_g)$ is a direct summand of $(A_g \times G, -w_g \otimes 1)$.
as a right quasi-module over \((A_g, -w_g)\). Hence by the quasi-Morita invariance the induced morphism of mixed complexes
\[
\overline{MC}^I(\text{End}_{A_g}(A_g \times G), -w_g \otimes 1) \to \overline{MC}^I(q\text{MF}_{dg}(A_g, w_g))
\]
is a quasi-isomorphism.

We have a diagram of chain maps
\[(4.4)\]
\[
\begin{array}{ccc}
\overline{C}^I(q\text{MF}_{dg}(A, w)) & \xrightarrow{\tau} & (\oplus G \overline{C}^I(q\text{MF}_{dq}(A, w_q)))_G \\
\downarrow \quad \quad \quad \downarrow \tau|_{A \times G} & & \downarrow \text{natural} \\
\overline{C}^I(A \times G, -w \otimes 1) & \xrightarrow{\psi_A} & (\oplus G \overline{C}^I(\text{End}_{A_g}(A_g \times G), -w_g \otimes 1))_G \\
\end{array}
\]

where:
- two vertical chain maps are quasi-isomorphisms as explained already;
- the middle horizontal map \(\tau|_{A \times G}\) is induced from the composition \(\text{natural} \circ \tau\);
- \(\psi_A\) is the quasi-isomorphic chain map defined by Baranovsky [4, page 799] and Căldăraru and Tu [8, Proof of 6.3];
- \(\text{Tr}\) is the generalized trace map.

**Lemma 4.10.** The triangle in \((4.4)\) is commutative. Hence \(\tau|_{A \times G}\) is a quasi-isomorphism.

**Proof.** This will be clear since \(g\) acts on \(A_g\) trivially. Note that the following diagram commutes,
\[(4.5)\]
\[
\begin{array}{ccc}
|G| & |G| & |G| \\
\downarrow \quad \quad \quad \downarrow \text{Tr} & & \downarrow \text{Tr} \\
\oplus_{g \cdot \prod_{i=0}^{n-1} a_i = 0} g \text{-th component} & & \oplus_{g \cdot \prod_{i=0}^{n-1} a_i = 0} g \text{-th component} \\
\end{array}
\]

where \(\text{Tr}\) denotes the element in \(A_g\) associated to \(a\). Here the right bottom corner is meant to have the \(g\)-th component
\[
\begin{cases} 
|G| \cdot \prod_{i=0}^{n-1} a_i 
= \begin{cases} 
|G| \cdot \prod_{i=0}^{n-1} a_i & \text{if } g = g_0 \cdots g_n \\
0 & \text{otherwise}
\end{cases}
\end{cases}
\]

**Remark 4.11.** Note that for for example when \(A = k\) with a nontrivial \(G\), diagram \((4.5)\) is not commutative without the twisting \(\text{tw}\) insertion in the definition of \(\tau\).

4.5. **Proof of Proposition 4.9.** Due to Remark 4.8 and the compatibility of the map \(\tau\) with Čech differentials, the proof follows from Lemma 4.10.
4.6. **The role of can at the level of category.** As we have already re-
marked, the insertion of central automorphism \( \text{tw} \) was essential. We would 
like to sketch how does it appear naturally in the computation of Hochschild 
invariants in order to clarify its role. The proof of Proposition 4.9 can be 
interpreted as a two-step process.

The first step is purely categorical. Suppose a \( k \)-linear category \( \mathcal{C} \) carries 
a strict \( G \)-action of a finite group \( G \). We still denote corresponding endo-
functors by \( g : \mathcal{C} \to \mathcal{C} \), \( g \in G \). We also denote its category of \( G \)-equivariant 
objects by \( \mathcal{C}_G \). Its object consists of a pair \( (E, \phi_g^E) \) where \( E \) is an object of 
\( \mathcal{C} \) and \( \phi_g^E \) is an isomorphism \( \phi_g^E : E \simeq gE \) satisfying a cocycle condition. 
The morphism is defined as usual.

There is a natural functor

\[
\sim : \mathcal{C} \to \mathcal{C}_G
\]

called *linearization* which is defined on objects as

\[
\tilde{E} = \left( \bigoplus_{h \in G} hE, \phi_g^h \right), \quad \phi_g^h : \bigoplus_{h \in G} hE = \bigoplus_{h \in G} g(hE) \simeq \bigoplus_{h \in G} (gh)E.
\]

It is not hard to see that the linearization is a both left and right adjoint to 
the forgetful functor. Its essential image generates \( \mathcal{C}_G \) and 

\[
\text{Hom}_{\mathcal{C}_G}(\tilde{E}_1, \tilde{E}_2) \simeq \text{Hom}_{\mathcal{C}}(E_1, E_2)^G.
\]

This fact leads to the following simple description of Hochschild homology 
of \( \mathcal{C}_G \). (See [31])

\[
(4.6) \quad HH_*(\mathcal{C}_G) \simeq \left( \bigoplus_{g \in G} HH_*(\mathcal{C}, g) \right)^G \simeq \bigoplus_{g \in \text{Conj}(G)} HH_*(\mathcal{C}_C(g), g).
\]

Here, \( HH_*(\mathcal{C}, g) \) denotes Hochschild homology with coefficient \( g \), where the 
endofunctor \( g \) is considered as a bimodule.

Second observation is geometric. For simplicity, let \( \mathcal{C} = D(X) \) be a dg 
category of coherent sheaves on a smooth affine scheme \( X = \text{Spec}(A) \) acted 
on by a finite group \( G \). One can easily extend the discussion to the case of matrix factorizations. Each components of (4.6) has a simpler description:

\[
(4.7) \quad HH_*(D_{C(g)}(X), g) \xrightarrow{\text{res}} HH_*(D_{C(g)}(X^g), g)
\]

Notice that the action of \( g \) on \( X^g \) is trivial. If \( (E, \{ \phi_{\overline{h}}^E \}_{\overline{h} \in G}) \) is a \( G \)-equivariant 
sheaves on \( X \), then \( \varphi_{\overline{g}}^E = (\phi_{\overline{g}}^E)^{-1} \) restricts to the central automorphism \( \text{can}_{E|X^g} \) of \( E|X^g \). In fact, any \( C(g) \)-equivariant object \( (F, \{ \phi_{\overline{h}}^F \}_{\overline{h} \in C(g)}) \) on \( X^g \) 
carries a distinguished automorphism \( \text{can}_F = (\phi_{\overline{g}}^F)^{-1} \). This assignment is 
viewed as a natural transformation between identity functors, or an element 
of zeroth Hochschild cohomology;

\[
[\text{can}] \in HH^0(D_{C(g)}(X^g)).
\]

The map \( \text{tw} \) on Hochschild chains is a cap product with \([\text{can}].\)
Lastly, observe that $D_{C(g)}(X^g)$ is generated by $O_{X^g}$. Notice that $g$-action on $O_{X^g}$ is trivial so can could be ignored. This implies

\[(4.8) \quad HH_*^g(A_g) \xrightarrow{\text{inc}} HH_*^g(D_{C(g)}(X^g), g).\]

### 4.7. Mixed complex case

In general, the map $\tau$ is not a morphism of mixed complexes. In this subsection we modify $\tau$ to get a morphism of mixed complexes.

For $\chi \in \hat{\mu}_r$ let $q\text{MF}^\chi_{dg}(I_{\mu_r} X, w)$ be the full subcategory of $q\text{MF}_{dg}(I_{\mu_r} X, w)$ consisting $\chi$-eigenobjects of $q\text{MF}_{dg}(I_{\mu_r} X, w)$. The map $\tau$ restricted to the subcomplex $C(q\text{MF}^\chi_{dg}(I_{\mu_r} X, w), \chi)$, denoted by $\tau_{\chi}$, is nothing but the multiplication by $\chi(e^{2\pi i/r})$. Hence

$$\tau_{\chi} : \text{MC}(q\text{MF}^\chi_{dg}(I_{\mu_r} X)) \rightarrow \text{MC}(q\text{MF}^\chi_{dg}(I_{\mu_r} X))$$

is an automorphism of the mixed Hochschild complex.

Consider the composition $\tau_m$ of a sequence of morphisms of mixed complexes

$$\tau_m : \text{MC}^H(q\text{MF}_{dg}(X, w)) \xrightarrow{\text{pullback}} \bigoplus_{\chi, \chi} \text{MC}^H(q\text{MF}^\chi_{dg}(I_{\mu_r} X, w)) \xrightarrow{\tau_{\chi}} \bigoplus_{\chi, \chi} \text{MC}^H(q\text{MF}^\chi_{dg}(I_{\mu_r} X, w)) \xrightarrow{\text{nat}} \bigoplus_{\chi} \Gamma(I_{\mu_r} X, \text{MC}^H(q\text{MF}_{I_{\mu_r} X, w}(-))),$$

where the natural map $\text{nat}$ is defined by setting

$$\text{nat}(\sum_{\chi} a^\chi_0 [\pi_1] \ldots [\pi_n]) = (U \mapsto \sum_{\chi} a^\chi_0 [\pi^\chi_1]|_{I_{\mu_r} X} \ldots [\pi^\chi_n]|_{I_{\mu_r} X}).$$

**Remark 4.12.** While the cochain map

$$C^H(q\text{MF}_{dg}(I_{\mu_r} X, w)) \xrightarrow{\text{Tr}_1} \bigoplus_{\chi} C^H(q\text{MF}^\chi_{dg}(I_{\mu_r} X, w))$$

is an isomorphism, $\overline{\text{C}}^H(q\text{MF}_{dg}(I_{\mu_r} X, w)) \xrightarrow{\text{Tr}_1} \bigoplus_{\chi} \overline{\text{C}}^H(q\text{MF}^\chi_{dg}(I_{\mu_r} X, w))$ is not an isomorphism in general but a quasi-isomorphism from the facts that $C^H \rightarrow \overline{\text{C}}^H$ is a quasi-isomorphism and the above $\text{Tr}$ is an isomorphism.

**Proposition 4.13.** Suppose that $X$ is of form $[\text{Spec} A/G]$ as in Proposition 4.9. The morphism $\tau_m$ in the category of mixed complexes is a quasi-isomorphism.

**Proof.** By the definition, we need to show that $\tau_m$ is a quasi-isomorphism between Hochschild-type chain complexes. Replacing $\tau$ by $\tau_m$ and $\overline{\text{C}}^H$ by $\overline{\text{MC}}^H$ in diagram (4.4) we conclude the proof. \[\square\]

### 4.8. Global case

Let $X$ denote the coarse moduli space of $X$. For an étale map $V \rightarrow X$ let $X_V := V \times_X X$. We take the sheafification $\overline{\text{MC}}^H(q\text{MF}_{dg}(X, w))$ (resp. $\overline{\text{MC}}^H(q\text{MF}_{I_{\mu_r} X, w})$) of the presheaf

$$V \mapsto \overline{\text{MC}}^H(q\text{MF}_{dg}(X_V, w)) \text{ resp. } (V \mapsto \Gamma(I_{\mu_r} X, \overline{\text{MC}}^H(q\text{MF}_{I_{\mu_r} X, w}(-))).$$
both on the étale site of $\mathcal{X}$. We take the sheaf homomorphism $\tau_m$ induced from $\tau_m$

\[ \tau_m : \text{MC}(\mathcal{MF}_{dg}(\mathcal{X}, w)) \to \text{MC}(\mathcal{MF}_{I\mathcal{X}, w}(-)). \]

**Lemma 4.14.** Suppose that $\mathcal{X}$ is smooth over $k$. Then the induced morphism $R\Gamma(\tau_m)$ fits into a diagram of isomorphisms in the derived category of mixed complexes:

\[
\begin{array}{ccc}
\text{MC}(\mathcal{MF}_{dg}(\mathcal{X}, w)) & \cong & R\Gamma(\text{MC}(\mathcal{MF}_{I\mathcal{X}, w}(-))) \\
\cong & \cong & R\Gamma(\text{MC}(\mathcal{MF}_{I\mathcal{X}, w}(U))) \\
\end{array}
\]

**Proof.** The right vertical map is a quasi-isomorphism by the quasi-Morita invariance and the fact that for each étale morphism $U \to \mathcal{X}$ the Yoneda embedding $(\mathcal{O}_{I\mathcal{X}}(U), -w) \to \mathcal{MF}_{I\mathcal{X}, w}(U)$ is a pseudo-equivalence; see [5, Proposition 3.25] and [34]. It remains to show that the left vertical map is a quasi-isomorphism. Let $\pi : \mathcal{X} \to \mathcal{X}$ be the coarse moduli space. By [17, Corollary 4.6], the presheaf $(V \mapsto C(\mathcal{MF}_{dg}(V \times \mathcal{X}, \mathcal{X}, w))$ is a sheaf on the étale site of $\mathcal{X}$. It is thus enough to show that the left vertical map is a quasi-isomorphism when $\mathcal{X} = [X/G]$ for a smooth variety $X$ and a finite group $G$ which follows from [10, Theorem 6.9].

**Theorem 4.15.** Suppose that $\mathcal{X}$ is smooth over $k$. Then the isomorphism

\[
\text{MC}(\mathcal{MF}_{dg}(\mathcal{X}, w)) \cong R\Gamma(\text{MC}(\mathcal{MF}_{I\mathcal{X}, w}(U)))
\]

in the derived category of mixed complexes induces the isomorphism

\[
\text{MC}(\mathcal{MF}_{dg}(\mathcal{X}, w)) \cong R\Gamma(\Omega_{I\mathcal{X}}^1, -dw|_{I\mathcal{X}}, ud).
\]

**Proof.** The proof follows from Lemma 4.14 and the HKR-type isomorphism ([8, 37]) for affine orbifolds.

5. **Chern character formulae**

Let $\mathcal{X}$ be a smooth separated finite-type DM stack over $k$ and let $w : \mathcal{X} \to \mathbb{A}^1_k$ be an algebraic function on $\mathcal{X}$ with only critical value 0.

5.1. **A formula via Čech model and Chern-Weil theory.** We fix an affine étale surjective morphism $p : \mathcal{U} \to \mathcal{X}$ from a $k$-scheme $\mathcal{U}$ as in § 4.2. Since $\mathcal{U}$ is an affine scheme over $k$, every $E|_{\mathcal{U}}$ has a connection

\[ \nabla_{E|_{\mathcal{U}}} : E|_{\mathcal{U}} \to E|_{\mathcal{U}} \otimes \Omega^1_{\mathcal{U}}. \]

Define a connection

\[ \nabla_{E|_{\mathcal{U}}} : E|_{\mathcal{U}} \to E|_{\mathcal{U}} \otimes \Omega^1_{\mathcal{U}}. \]
by letting $\nabla_{E|_w} = p_1^* \nabla_{E|_w}$, where $p_1$ is the first projection $\mathfrak{M}^\ast \to \mathfrak{M}$. This gives rise to a connection

$$\nabla_E : \hat{\mathcal{C}}(E) \to \Omega^1_{\mathcal{X}} \otimes \hat{\mathcal{C}}(E)$$

where $\hat{\mathcal{C}}(E) := (\bigoplus_{r \geq 0} \hat{\mathcal{C}}^r(E), d_{\text{cch}})$ and $\hat{\mathcal{C}}^r(E) = p_r^* p^*_\ast E$. For every $E$, fix such a connection once and for all.

Let $\mathfrak{M}$ denote the affine scheme $\mathfrak{M} \times_{\mathcal{X}} I\mathcal{X}$. Using this affine covering of $I\mathcal{X}$, we have the Čech resolution $\hat{\mathcal{C}}(E|_{I\mathcal{X}})$ and the connection

$$\nabla_{E|_{I\mathcal{X}}} : \hat{\mathcal{C}}(E|_{I\mathcal{X}}) \to \Omega^1_{I\mathcal{X}} \otimes \hat{\mathcal{C}}(E|_{I\mathcal{X}}).$$

In general, for every vector bundle $F$ on $I\mathcal{X}$, we can choose a connection $\nabla_F : \hat{\mathcal{C}}(F) \to \Omega^1_{I\mathcal{X}} \otimes \hat{\mathcal{C}}(F)$.

For each $P \in q\text{MF}_d(I\mathcal{X}, w)$, choose a connection $\nabla_P$ as above once and for all. Let $R = u \nabla^2_x + [\nabla_P, \delta_{\hat{\mathcal{C}}(P)}]$ a kind of the total curvature of $\nabla_P$. By a straightforward generalization of the definition of a chain map $tr_{\mathcal{V}}$ in [10] to the stacky case, we obtain a $k[[u]]$-linear map

$$tr_{\mathcal{V},I\mathcal{X}} : C(q\text{MF}_d(I\mathcal{X}, w))[[u]] \to \Gamma(\hat{\mathcal{C}}(\Omega^\bullet_{I\mathcal{X}}))[[u]]$$

mapping $a_0[a_1] \cdots [a_n]$ for $a_i \in \text{End}_{\hat{\mathcal{C}}(P)}(P)$, $P \in q\text{MF}_d(I\mathcal{X}, w)$ to

$$\sum_{(j_0, \ldots, j_n) : \sum j_i \in \mathbb{Z}_{\geq 0}} (-1)^{j_0 + \cdots + j_n} \cdot \frac{\text{tr}(a_0 R^{j_0} [\nabla_P, a_1] R^{j_1} [\nabla_P, a_2] \cdots [\nabla_P, a_n] R^{j_n})}{(n + j_0 + \cdots + j_n)!}.$$

It is clear how to map an arbitrary element of $C(q\text{MF}_d(I\mathcal{X}, w))[[u]]$. By the same proof in [10, Appendix B] the map $tr_{\mathcal{V},I\mathcal{X}}$ is a chain map. Likewise we have a chain map

$$tr_{\mathcal{V}} : \overline{C}$$

where $p : I\mathcal{X} \to \mathcal{X}$ is the natural morphism.

Consider a diagram of chain maps in negative cyclic type complexes (5.1)

$$\begin{array}{ccc} C(q\text{MF}_d(\mathcal{X}, w))[[u]] & \xrightarrow{\text{tw} \circ \rho_X^\ast} & C(q\text{MF}_d(I\mathcal{X}, w))[[u]] \\
\downarrow & & \downarrow \\
\Gamma(\overline{\mathcal{C}}) & \xrightarrow{\overline{\mathcal{C}}(\mathcal{X}, w))[[u]] & \Gamma(\overline{\mathcal{C}}) \\
\Gamma(tr_{\mathcal{V}}) & \xrightarrow{\Gamma(tr_{\mathcal{V}})} & \Gamma(\overline{\mathcal{C}}) \\
\end{array}$$

Note that the diagram is commutative and all arrows possibly but two top horizontal arrows are quasi-isomorphisms. Hence we have the following corollaries.

**Corollary 5.1.** The chain map

$$tr_{\mathcal{V},I\mathcal{X}} \circ \text{tw} \circ \rho_X^\ast : C(q\text{MF}_d(\mathcal{X}, w))[[u]] \to \Gamma(\hat{\mathcal{C}}(\Omega^\bullet_{I\mathcal{X}}))[[u]]$$

is a quasi-isomorphism.
Corollary 5.2. Under the isomorphism in (4.9), the Chern character \( \text{ch}_{HN}(P) \) of \( P \in \text{MF}_{dg}(X, w) \) is the class represented by Čech cocycle

\[
\text{tr} \left( \text{can}_{P|_{IX}} \circ \exp(-u \nabla_{P|_{IX}}^2 + [\nabla_{P|_{IX}}, \delta_{P|_{IX}} + d_{\text{Čech}}]) \right)
\]

in \( H(\Omega, (\Omega^\bullet_{IX}, -dw|_{IX})) \).

Example 5.3. Consider a DM stack \( X \) of the form \([X/G]\) with \( X \) quasi-projective and \( G \) a linearly reductive group. Then there is a finite collection \( \Omega = \{U_i\}_{i \in I} \) of \( G \)-invariant affine open subset \( U_i \) of \( X \) such that \( \bigcup_i U_i = X \). On the other hand there is a finite subset \( S \) of \( G \) such that

\[
IX = \sqcup_{g \in S}[X^g/C_G(g)].
\]

Note that \( I \Omega = \{U_i^g : i \in I, g \in S\} \). Instead of affine étale covering we may use the affine smooth covering \( \Omega \) for a Čech model of \( q \text{MF}_{dg}(X, w) \) and the chain map \( \text{tr}_\nabla|_{IX} \). Since \( G \) is linearly reductive, each \( P|_{U_i} \) has a \( G \)-equivariant connection, which in turn gives a \( C_G(g) \)-equivariant connection \( \nabla_{i,g} \) on \( P|_{U_i^g} \). This is because of the surjection of the canonical map \( \text{Hom}_{\hat{\Omega}_{X}}^\bullet(E, J(E)) \rightarrow \text{Hom}_{\hat{\Omega}_{X}}^\bullet(E, E) \) induced from the jet sequence

\[
0 \rightarrow \Omega^1_X \otimes O_X E \rightarrow J(E) \rightarrow E \rightarrow 0.
\]

We have

\[
\text{ch}_{HN}(P) = \bigoplus_{g \in S} \text{tr} \left( g \circ \exp(-u \Pi_{i \in I} \nabla^2_{i,g} - [\Pi_{i \in I} \nabla_{i,g}, \delta_{P|_{U_i^g}} + d_{\text{Čech}}]) \right).
\]

When \( X \) itself is affine, then the formula simplifies to

\[
\text{ch}_{HN}(P) = \bigoplus_{g \in S} \text{tr} \left( g \circ \exp(-u \nabla^2_{g} - [\nabla_{g}, \delta_{P|_{U_i^g}}]) \right),
\]

taking into account the fact that \( [\nabla_{g}, d_{\text{Čech}}] = 0 \).

Let \( a \in \bigoplus \mathbb{R}^2 \text{End}(P) \), then it determines a class in \( H^*(IX, (\Omega^\bullet_{IX}, -dw|_{IX})) \) under the HKR isomorphism. Denote the class by \( \tau(a) \). The assignment \( a \mapsto \tau(a) \) is called the boundary-bulk map.

Corollary 5.4. (The boundary-bulk map formula)

\[
\tau(a) = \text{tr} \left( a \circ \text{can}_{P|_{IX}} \circ \exp(-[\nabla_{P|_{IX}}, \delta_{P|_{IX}} + d_{\text{Čech}}]) \right).
\]

5.2. A formula via Atiyah class. Let \( P \in \text{MF}_{dg}(X, w) \) and let

\[
\Omega^-_{X} := [\Omega^1_X \xrightarrow{0} O_X]
\]

be the matrix factorization for \((X, 0)\) located at amplitude \([-1, 0]\). The Atiyah class \( \alpha_{P|_{X}} \) defined in \([16, \text{Appendix B}]\) is a suitable element of

\[
\text{Ext}^1(P, P \otimes \Omega^-_{X}).
\]

When \( w = 0 \), we have the decomposition

\[
\text{Ext}^1(P, P \otimes \Omega^-_{X}) = \text{Ext}^0(P, P) \oplus \text{Ext}^1(P, P \otimes \Omega^1_X)
\]
and let
\[ \hat{\text{at}}(P) = \text{proj} \circ \hat{\text{at}}(P) \in \text{Ext}^1(P, P \otimes \Omega^1_{X^w}), \]
where \( \text{proj} \) is the projection. For example when \( P \) is a vector bundle \( F \) and \( X \) is non-stacky, then \( \hat{\text{at}}(F) \) is the usual Atiyah class \([2]\). In this case \( \hat{\text{at}}(F) = 1_F + \hat{\text{at}}(F) \).

**Definition 5.5.** Taking into account the convention of the exponential \( \exp \) of \( \hat{\text{P}} \) as explained in \([16, 23]\), we define a naive Chern character of \( P \) by
\[ \text{ch}(P) := \text{tr}(\exp(\hat{\text{at}}(P))) \in H^*(X, (\Omega^*_X, -dw)). \]
For simplicity we abuse notation writing \( \exp(\hat{\text{at}}(P)) \) for \( \exp(\hat{\text{at}}(P)) \).

The correct formula for \( \text{ch}_{HH}(P) \) in \([24]\) is
\[ \text{ch}_{HH}(P) = \text{tr} \left( \text{can}P|_{IX} \circ \exp(\hat{\text{at}}(P)_{|IX}) \right) = \text{ch}(P) + \text{twisted part}. \]
We note that this formula agrees with the formula in Corollary 5.2, since Atiyah class \( \hat{\text{at}}(P|_{IX}) \) is representable as
\[ (\text{id}_P, [\nabla P|_{IX}, \delta P|_{IX} + d_{\text{Cech}}]) \in \Gamma(IX, \text{End}(P|_{IX}) \otimes \Omega^{-|dw|}_{IX} \otimes \check{\mathcal{C}}(O_{IX})) \]
in Čech cohomology \( \check{\mathcal{H}}(IX, \Omega^*_X, -dw|_{IX}) \); see \([23, \text{Proposition 1.3}] \) and
\[ \exp(\text{id}_P, [\nabla P|_{IX}, \delta P|_{IX} + d_{\text{Cech}}]) = \exp(-[\nabla P|_{IX}, \delta P|_{IX} + d_{\text{Cech}}]). \]
The boundary bulk map formula can also be written in terms of the Atiyah class:
\[ \tau(a) = \text{tr} \left( \text{can}P|_{IX} \circ \exp(\hat{\text{at}}(P|_{IX})) \right). \]

**Definition 5.6.** For a vector bundle \( E \) on \( IX \) we define
\[ \text{ch}_{tw}(E) := \text{tr}(\text{can}E \exp(\hat{\text{at}}(E)) \)
and Todd class \( \text{td}_{tw}(E) \) of \( E \) by the expression of Todd class in terms of the Chern character \( \text{ch}_{tw}(E) \). For example, \( \text{td}_{tw}(T_{IX}) \) is defined. Since \( T_{IX} \) is fixed under the canonical automorphism, we simply write \( \text{td}(T_{IX}) \) for \( \text{td}_{tw}(T_{IX}) \).

5.3. **Proof of Theorem 1.1.** The first statement of Theorem 1.1 is Theorem 4.15. The second statement follows from (5.2).

5.4. **Compactly supported case.** Let \( Z \) be a closed substack of \( X \) proper over \( k \). Let \( P \) be a matrix factorization for \((X, w)\) which is coacyclic over \( X - Z \). Note that
\[ \hat{\text{at}}(P|_{IX}) \in \text{Ext}^1(P|_{IX}, P|_{IX} \otimes \Omega^{-|dw|}_{IX}) = \text{Ext}^1_{IZ}(P|_{IX}, P|_{IX} \otimes \Omega^{-|dw|}_{IX}). \]
To emphasize that \( \hat{\text{at}}(P|_{IX}) \) can be considered as an \( IZ \)-supported extension class write \( \hat{\text{at}}_Z(P|_{IX}) \) for \( \hat{\text{at}}(P|_{IX}) \). Let \( \text{MF}_{dg}(X, w)|_Z \) be the full subcategory of \( \text{MF}_{dg}(X, w) \) consisting of all matrix factorization for \( (X, w) \) that are coacyclic over \( X - Z \).
Corollary 5.7. There is an isomorphism 
\[ \text{MC}(\text{MF}_{dg}(X, w))_Z \cong \mathbb{R}\Gamma_Z(\Omega^*_X, -dw|_{IX}, d) \]
in the derived category of mixed complexes. Under the isomorphism \( \text{ch}_{HH}(P) \) is equal to 
\[ \text{tr}(\text{can}_{P|_{IX}} \exp(\hat{\text{at}}_Z(P|_{IX}))) \]
in \( \mathbb{H}^*_Z(IX, (\Omega^*_X, -dw|_{IX})) \).

Proof. The first statement immediately follows from Theorem 1.1. The second statement follows from a concrete chain map for the Hochschild type chain complexes; see [10, § 6.2] for some details. \( \square \)

6. Stacky Riemann-Roch

6.1. The categorical HRR. The results in this subsection are taken from [33, 38] with a weaker condition on a dg category \( \mathcal{A} \). Instead assuming that \( \mathcal{A} \) is saturated, we assume that \( \mathcal{A} \) is a locally proper and smooth.

Definition 6.1. Let \( \mathcal{A} \) be a locally proper dg category: i.e., for every \( x, y \in \mathcal{A} \), the dimension \( \sum_{i \in \mathbb{Z}} \dim H^i \text{Hom}_\mathcal{A}(x, y) \) of total cohomology of \( \text{Hom}_\mathcal{A}(x, y) \) is finite. Let

\[ \langle \cdot, \cdot \rangle_{\text{can}} : HH_*(\mathcal{A}) \times HH_*(\mathcal{A}^{\text{op}}) \rightarrow k \]

be the canonical pairing of \( \mathcal{A} \) defined by Shklyarov. It is a \( k \)-linear pairing.

6.1.1. Transformations by bimodules.

Definition 6.2. For a dg category \( \mathcal{C} \), we take the projective model structure on the category \( \text{Mod}(\mathcal{C}) \) of right \( \mathcal{C} \)-modules. The cofibrant objects are exactly the summands of semi-free dg-modules. A right \( \mathcal{C} \)-module \( N \) is called perfect if \( N \) is a cofibrant object which is compact in the derived category \( D(\mathcal{C}) \) of right \( \mathcal{C} \)-modules. Let \( \text{Mod}_{dg}(\mathcal{C}) \) be the dg category of right \( \mathcal{C} \)-modules. Let \( \text{Perf}(\mathcal{C}) \) be the full subcategory of \( \text{Mod}_{dg}(\mathcal{C}) \) consisting of all perfect \( \mathcal{C} \)-modules. We call a dg category \( \mathcal{C} \) is smooth if the diagonal bimodule \( \Delta_{\mathcal{A}} \) is a perfect bimodule.

From now on let \( \mathcal{A} \) and \( \mathcal{B} \) be locally perfect and smooth dg categories unless otherwise stated.

Lemma 6.3. The total dimension of Hochschild homology \( HH_*(\mathcal{A}) \) of \( \mathcal{A} \) is finite. The dg category \( \text{Perf}(\mathcal{A} \otimes \mathcal{B}) \) is locally prefect and smooth.

Proof. The first claim amounts \( \Delta_{\mathcal{A}} \otimes_{\mathcal{A}^{\text{op}} \otimes \mathcal{A}} \Delta_{\mathcal{A}} \) is a perfect dg \( k \)-module, which follows from tensor-hom adjunction and the conditions on \( \mathcal{A} \). The second claim follows from [28, Lemma 2.13, 2.14, 2.15], since \( k \) is a field. \( \square \)

For a right \( \mathcal{A}^{\text{op}} \otimes \mathcal{B} \)-module \( M \) there is a dg functor \( T_M : \text{Perf}(\mathcal{A}) \rightarrow \text{Mod}_{dg}(\mathcal{B}) \) sending \( N \) to \( N \otimes_{\mathcal{A}} M \). If \( M \) is representable, then \( T_M \) factors though \( \text{Perf}(\mathcal{B}) \), since \( \mathcal{A} \) is locally proper. Hence this is the case for every perfect \( \mathcal{A}^{\text{op}} \otimes \mathcal{B} \)-module \( M \).
Let $M \in Perf(A^{op} \otimes B)$ and let $\text{Ch}(M) = \sum_i \gamma_i \otimes \gamma^i$ under the K"unneth isomorphism

$$HH_*(Perf(A^{op} \otimes B)) \cong HH_*(Perf(A^{op})) \otimes HH_*(Perf(B)).$$

Let

$$HH(T_M) : HH_*(Perf(A)) \to HH_*(Perf(B))$$

be the induced homomorphism from $T_M : Perf(A) \to Perf(B)$.

**Proposition 6.4.** If $\langle \cdot, \cdot \rangle_{can}$ denotes the canonical pairing of $Perf(A^{op} \otimes B)$, then for every $\sigma \in HH_*(Perf(A))$ we have

$$HH_*(T_M)(\sigma) = \sum_i \langle \sigma, \gamma_i \rangle_{can} \gamma^i.$$ 

6.1.2. The characterization. There are natural isomorphisms

$$HH_*(Perf(A)) \cong HH_*(A^{op} \otimes A) \cong HH_*(A^{op}) \otimes HH_*(A)$$

by the Morita invariance and the K"unneth isomorphism. Write

$$\text{Ch}_{HH}(\Delta_A) = \sum_i T^i \otimes T_i \in HH_*(A^{op}) \otimes HH_*(A).$$

Then by Proposition 6.4 we obtain this.

**Corollary 6.5.** The canonical pairing $\langle \cdot, \cdot \rangle_{can}$ is characterized by two conditions: (1) it is non-degenerate (2) it satisfies the ‘diagonal decomposition’ property:

$$\sum_i \langle \gamma, T^i \rangle \langle T_i, \gamma' \rangle = \langle \gamma, \gamma' \rangle$$

for every $\gamma \in HH_*(A), \gamma' \in HH_*(A^{op})$.

6.1.3. The Cardy condition. Consider objects $x, y \in A$. Let $a$ and $b$ be closed endomorphisms of $x$ and $y$, respectively. Let

$$L_b \circ R_a : \text{Hom}_A(x, y) \to \text{Hom}_A(x, y), \ (1)\ a[c] \to b \circ c \circ a.$$ 

**Theorem 6.6.** We have

$$\text{tr}(L_b \circ R_a) = \langle [b], [a] \rangle_{can}.$$ 

For the identities $a = 1_x, b = 1_y$, it is specialized to

$$\chi(x, y) = \langle \text{Ch}_{HH}(y), \text{Ch}_{HH}(x) \rangle_{can}.$$ 

6.2. On $\text{MF}_{dg}(X, w)$. From this point in the text all DM stacks considered are assumed to be quotient stacks that satisfy the resolution property.

**Definition 6.7.** An LG model is a pair $(\mathcal{X}, w)$ of a smooth separated DM stack $\mathcal{X}$ over $k$ and a regular function $w$ on $\mathcal{X}$. We further assume that $\mathcal{X}$ is a quotient stack that satisfies the resolution property. We assume that the critical value is over 0. If $G = \mathbb{Z}$, then $w = 0$. If $G = \mathbb{Z}/2$, then we furthermore assume that $w : \mathcal{X} \to \mathbb{A}^1$ is flat. The pair $(\mathcal{X}, w)$ will be called a proper LG model if the critical locus of $w$ is proper over $k$. 
Consider two proper LG models \((\mathcal{X}, w)\) and \((\mathcal{Y}, v)\). We want to show that 
\(\text{MF}_{dg}(\mathcal{X}, w)\) is locally proper, smooth; and the following \((\dagger)\) and \((\star)\) hold:

\((\dagger)\) There is a natural dg functor 
\[
\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, w \boxplus v) \to \text{Perf}(\text{MF}_{dg}(\mathcal{X}, w) \otimes \text{MF}_{dg}(\mathcal{Y}, v))
\]
defined by 
\[
E \mapsto \Psi(E) : x \otimes y \mapsto \text{Hom}_{\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, w \boxplus v)}(x \boxtimes y, E).
\]
Here \(w \boxplus v\) denotes \(w \otimes 1 + 1 \otimes v\).

\((\star)\) The triangulated category \([\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, w \boxplus v)]\) is the smallest full triangulated subcategory containing all exterior products closed under finite coproducts and summands. Here an object of \([\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, w \boxplus v)]\) is called an exterior product if it is isomorphic to \(x \boxtimes y\) for some \(x \in \text{MF}_{dg}(\mathcal{X}, w), y \in \text{MF}_{dg}(\mathcal{Y}, v)\).

**Lemma 6.8.**

1. \((\star)\) implies \((\dagger)\).
2. \((\dagger)\) implies that the smoothness of \(\text{MF}_{dg}(\mathcal{X}, w)\).

**Proof.** (1) is clear. Let \(\Delta : \mathcal{X} \to \mathcal{X}^2\) be the diagonal morphism. Then (2) follows from that \(\Psi(\Delta_\mathcal{X})\) is quasi-isomorphic to the diagonal bimodule. ∎

Since \(\text{MF}_{dg}(\mathcal{X}, w)\) is clearly locally proper, it is enough to show \((\star)\). We check this when \(\mathcal{X}\) is a stack quotient \([\mathcal{X}/G]\) of a smooth variety by an action of an affine algebraic group \(G\).

### 6.2.1. The case when \(G = \mathbb{Z}\).

Then \(w = 0\). Note that \((\star)\) holds true by Theorem 2.29 and Corollary 4.21 of [3].

### 6.2.2. The case when \(G = \mathbb{Z}/2\).

Then \(w\) is a \(G\)-invariant function on \(X\), not identically zero on any component of \(X\). Note that \((\star)\) holds by Theorem 2.29 and Lemma 4.23 of [3].

### 6.3. A geometric realization of the diagonal module.

Consider two proper LG models \((\mathcal{X}, w), (\mathcal{Y}, v)\). Suppose that \(\mathcal{X}, \mathcal{Y}\) are stack quotients of smooth varieties by actions of affine algebraic groups. Let \(f : \mathcal{X} \to \mathcal{Y}\) be a proper morphism with \(f^*v = w\). We call \(f : (\mathcal{X}, w) \to (\mathcal{Y}, v)\) an **proper LG morphism**. Choose an affine étale cover \(\mathfrak{U} \to \mathcal{X}\) and \(\mathfrak{U}' \to \mathcal{Y}\). Denote
\[
\mathcal{A} := \text{MF}_{dg}(\mathcal{X}, w), \quad \mathcal{B} := \text{MF}_{dg}(\mathcal{Y}, v).
\]
They are locally proper and smooth as seen in § 6.2.

Let \(-w \boxplus v := -w \otimes 1 + 1 \otimes v\) and let \(\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)\) be the Čech dg model of the matrix factorizations for \((\mathcal{X} \times \mathcal{Y}, -w \boxplus v)\) with respect to the affine cover \(\mathfrak{U} \times \mathfrak{U}' \to \mathcal{X} \times \mathcal{Y}\). Then by \((\dagger)\) we have a natural dg functor
\[
\Psi : \text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v) \to \text{Perf}(\mathcal{A}^{op} \otimes \mathcal{B}).
\]
Let $D : \mathcal{A}^{\text{op}} \to \text{MF}_{dg}(\mathcal{X}, -w)$ be the duality functor. Then we have a commuting diagram of isomorphisms

\[
\begin{array}{ccc}
HH_*(\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)) & \xrightarrow{HH(\Phi)} & HH_*(\text{Perf}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})) \\
\downarrow \text{hkr} & & \downarrow \text{hkr} \circ HH(D) \otimes \text{id} \otimes \text{k"unneth} \\
\mathbb{H}^{-*}((\mathcal{O}_X^* \boxtimes I_Y), d(w \boxplus -v)) & \xrightarrow{\text{k"unneth}} & \mathbb{H}^{-*}((\mathcal{O}_X^*, dw) \otimes \mathbb{H}^{-*}((\mathcal{O}_Y^*, -dv)),
\end{array}
\]

where hkr and k"unneth are the HKR type isomorphisms in § 4.8 and the K"unneth isomorphisms, respectively.

Consider a matrix factorization $K$ for $(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)$. For example, we have a coherent factorization

\[
\mathcal{O}_{\Gamma, Y} := (\Gamma_Y)_* \mathcal{O}_X \text{ for } (\mathcal{X} \times \mathcal{Y}, -w \boxplus v).
\]

Since $\mathcal{X} \times \mathcal{Y}$ satisfies the resolution property by [3, Theorem 2.29], $\mathcal{O}_{\Gamma, Y}$ is quasi-isomorphic to a matrix factorization.

For all $x \in \mathcal{A}, y \in \mathcal{B}$ there is a natural quasi-isomorphism

\[
\mathbb{R}\text{Hom}(y, q_* (K \otimes p^* x)) \simeq_{\text{qiso}} \text{Hom}_{\text{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)}(x^\vee \boxtimes y, K)
\]

functorial under the morphisms in the categories $\mathcal{B}$ and $\mathcal{A}$. This shows the following, which will be used later.

**Lemma 6.9.** For easy notation, write $T_K$ for $T_{\Psi(K)}$. Then:

1. The transformation $T_K : \text{Perf}(\mathcal{A}) \to \text{Perf}(\mathcal{B})$ is a dg enhancement of the Fourier-Mukai transform $[\mathcal{A}] \to [\mathcal{B}]$ attached to the kernel $K$.
   In particular, $T_{\mathcal{O}_{\Gamma, Y}}$ represents $\mathbb{R}f_* : [\mathcal{A}] \to [\mathcal{B}]$.

2. The bimodule $\Psi(\mathcal{O}_{\Gamma, \text{id}})$ and the diagonal bimodule $\Delta_\mathcal{A}$ are quasi-isomorphic.

The second statement in the above lemma is also in Lemma 5.24 of [3].

**6.4. An explicit realization of the canonical pairing.**

**Theorem 6.10.** Let $(\mathcal{X}, w)$ be a proper LG model. Assume that $\mathcal{X}$ is a smooth quotient DM stack which satisfies the resolution property. Then the canonical pairing coincides with the paring defined by

\[
\int_{\mathcal{I}_X} (-1)^{\left(\dim_{\mathcal{I}_X} + 1\right)} \wedge \chi_{t_1} \cdot \cdots \cdot \chi_{t_{\dim_{\mathcal{I}_X}}} \cdot \text{td}(T_{\mathcal{I}_X}),
\]

where $\dim_{\mathcal{I}_X}$ is the locally constant dimension function of $\mathcal{I}_X$.

**Proof.** We prove the characterization Corollary 6.5 for the pair $(6.2)$. The nondegeneracy follows from Serre duality [30] as argued in [16, § 4.1]. By Lemma 6.9 the ‘diagonal decomposition’ is

\[
\sum_i \int_{\mathcal{I}_X} \gamma^i \cdot \chi_{t_1} \cdot \text{td}(T_{\mathcal{I}_X}) \int_{\mathcal{I}_X} \gamma^i \cdot \chi_{t_1} \cdot \text{td}(T_{\mathcal{I}_X}) = \int_{\mathcal{I}_X} (-1)^{\left(\dim_{\mathcal{I}_X} + 1\right)} \gamma'' \cdot \text{td}(T_{\mathcal{I}_X}),
\]
where
\[
\sum_i t^i \otimes t_i = \text{ch}_{HH}(\Delta_* \mathcal{O}_\mathcal{X}) \in \mathbb{H}^*(\mathcal{I}\mathcal{X}, (\Omega^\bullet_{\mathcal{I}\mathcal{X}}, -dw|_{\mathcal{I}\mathcal{X}})) \otimes \mathbb{H}^*(\mathcal{I}\mathcal{X}, (\Omega^\bullet_{\mathcal{I}\mathcal{X}}, dw|_{\mathcal{I}\mathcal{X}}))
\]
\[
\gamma \in \mathbb{H}^*(\mathcal{I}\mathcal{X}, (\Omega^\bullet_{\mathcal{I}\mathcal{X}}, dw|_{\mathcal{I}\mathcal{X}})), \gamma' \in \mathbb{H}^*(\mathcal{I}\mathcal{X}, (\Omega^\bullet_{\mathcal{I}\mathcal{X}}, -dw|_{\mathcal{I}\mathcal{X}}))
\]
(6.4) \[\tilde{\text{td}}(\mathcal{T}_{\mathcal{I}\mathcal{X}}) := \frac{\text{td}(\mathcal{T}_{\mathcal{I}\mathcal{X}})}{\text{ch}_{tw} (\lambda_1 (N^\mathcal{I}_{\mathcal{I}\mathcal{X}/\mathcal{X}}))} \]

To show (6.3) we use the deformation to the normal cone.

Over \( \mathbb{P}^1 \) there is a deformation stack \( \mathcal{M}^\circ \) to the normal cone \( N_{\mathcal{X}/\mathcal{X}^2} \): the general fiber is \( \mathcal{X}^2 \) and the special fiber, say over \( \infty \), is the vector bundle stack \( N_{\mathcal{X}/\mathcal{X}^2} \cong T_{\mathcal{X}} \). It comes with a natural morphism \( h : \mathcal{M}^\circ \to \mathcal{X}^2 \), a flat morphism \( pr : \mathcal{M}^\circ \to \mathbb{P}^1 \), and a morphism \( \tilde{\Delta} : \mathcal{X} \times \mathbb{P}^1 \to \mathcal{M}^\circ \) such that \( (h, pr) \circ \tilde{\Delta} = \Delta \times \text{id}_{\mathbb{P}^1} \). Consider the fiber square diagram

\[
\begin{array}{ccc}
\mathcal{X} \times 0 & \to & \mathcal{X} \times \mathbb{P}^1 & \leftarrow & \mathcal{X} \times \infty \\
\Delta & & \tilde{\Delta} & & \delta \\
\mathcal{X}^2 & \to & \mathcal{M}^\circ & \leftarrow & T_{\mathcal{X}} \\
\rho_{\mathcal{X}^2} & & \rho_{\mathcal{M}^\circ} & & \rho_{T_{\mathcal{X}}} \\
I\mathcal{X}^2 & \to & I\mathcal{M}^\circ & \leftarrow & IT_{\mathcal{X}}.
\end{array}
\]

Here we use facts that \( I\mathcal{X}^2 \cong I\mathcal{X} \times I\mathcal{X} \) and \( T_{\mathcal{I}\mathcal{X}} \cong IT_{\mathcal{X}} \) by Lemma 3.1.

Let
\[
\pi_{\mathcal{X}} : T_{\mathcal{X}} \to \mathcal{X} \quad \text{and} \quad \pi_{\mathcal{I}\mathcal{X}} : T_{\mathcal{I}\mathcal{X}} \to I\mathcal{X}
\]
be the projections from vector bundles. Then LHS of (6.3) becomes
(6.5) \[\int_{N_{\mathcal{I}\mathcal{X}/\mathcal{X}^2}} \pi^*_\mathcal{I}\mathcal{X} (\gamma'')(\text{ch}_{tw}(\mathcal{L}_\rho^* T_{\rho} \delta_* \mathcal{O}_\mathcal{X})) \cdot \pi^*_\mathcal{I}\mathcal{X} (\tilde{\text{td}}_{\mathcal{I}\mathcal{X}})^2;\]

by the Tor independence of the pair \( (\mathcal{X} \times \mathbb{P}^1, \mathcal{M}^\circ \times_{\mathbb{P}^1} p) \) over \( \mathcal{M}^\circ \) for \( p = 0, \infty \) and the base change II in § 7.1.1; for details see the proof of [22, § 3.3]. Let \( \sigma \) be the diagonal section of the vector bundle \( \pi^*_\mathcal{X} T_{\mathcal{X}} \) on \( T_{\mathcal{X}} \) and let \( \text{Kos}(\sigma) \) denote the Koszul complex associated to the section \( \sigma \). Then (6.5) becomes
\[
\int_{N_{\mathcal{I}\mathcal{X}/\mathcal{X}^2}} \pi^*_\mathcal{I}\mathcal{X} (\gamma'')(\text{ch}_{tw}(\rho_{\mathcal{I}\mathcal{X}}^* \text{Kos}(\sigma))) \cdot \pi^*_\mathcal{I}\mathcal{X} (\tilde{\text{td}}_{\mathcal{I}\mathcal{X}})^2;
\]

which equals, by the functoriality and the projection formula § 7.1.1,
(6.6) \[\int_{I\mathcal{X}} (\gamma'' \cdot \tilde{\text{td}}(T_{\mathcal{I}\mathcal{X}})) \int_{\pi_{\mathcal{I}\mathcal{X}}} (\text{ch}_{tw}(\rho_{\mathcal{I}\mathcal{X}}^* \text{Kos}(\sigma))) \cdot \pi^*_\mathcal{I}\mathcal{X} (\tilde{\text{td}}_{\mathcal{I}\mathcal{X}}).\]

Let \( I\sigma \) be the diagonal section of the vector bundle \( \pi^*_\mathcal{X} T_{\mathcal{I}\mathcal{X}} \) on \( T_{\mathcal{I}\mathcal{X}} \). From the short exact sequence in § 3.4.1, we have a short exact sequence
(6.7) \[0 \to \pi^*_\mathcal{I}\mathcal{X} T_{\mathcal{I}\mathcal{X}} \to \pi^*_\mathcal{I}\mathcal{X} (T_{\mathcal{X}}|_{I\mathcal{X}}) \to \pi^*_\mathcal{I}\mathcal{X} N_{I\mathcal{X}/\mathcal{X}} \to 0;\]
with \( i(I\sigma) = \pi^*_\mathcal{I}\mathcal{X} \sigma \).
and an equality
\[ T_X^{\text{fixed}} = T_{IY}. \]
Then (6.6) becomes, by (6.7) & (6.8),
\[ \int_{IY} (\gamma'' \cdot \widetilde{td}(T_{IY})) \int_{IY} \text{ch}(\text{Kos}(I\sigma))\pi_i^X \cdot \widetilde{td}(T_{IY}) \]
which becomes, by § 7.1.2,
\[ \int_{IY} (-1)^{\dim_{IY} X + 1} \gamma'' \cdot \widetilde{td}(T_{IY}). \]

This completes the proof. \( \square \)

### 6.5. Proof of Theorem 1.2

This follows from Theorems 6.6 and 6.10.

### 6.6. GRR

Consider a proper morphism \( f : X \to Y \) with \( f^*v = w \) as in § 6.3. Let \( K_0(A) \) be the Grothendieck group of the homotopy category of a pretriangulated dg category \( A \). Denote \( f_! : K_0(MF_d(X, w)) \to K_0(MF_d(Y, v)) \) be the homomorphism in the Grothendieck groups induced from \( Rf_* \).

**Theorem 6.11.** (==Theorem 1.3) The diagram

\[
\begin{array}{ccc}
K_0(MF_d(X, w)) & \xrightarrow{f_!} & K_0(MF_d(Y, v)) \\
\text{Ch}_{HH} \downarrow & & \downarrow \text{Ch}_{HH} \\
HH_*(MF_d(X, w)) & \xrightarrow{HH(Rf_*)} & HH_*(MF_d(Y, v)) \\
\downarrow \text{I}_{HKR} & & \downarrow \text{I}_{HKR} \\
\mathbb{H}^*(I_{X}, (\Omega^*_I, -dw|_{IY})) & \xrightarrow{\text{I}_{f!}(-)^{\dim_{IY}} \cdot \widetilde{td}(T_{IY})} & \mathbb{H}^*(I_{Y}, (\Omega^*_I, -dv|_{IY}))
\end{array}
\]

is commutative. Here \( \widetilde{td}(T_{IY}) := \text{td}(T_{IY})/f^* \cdot \text{td}(T_{IY}) \) and \( \text{dim}_f = \dim_{IY} - \dim_{IY} \), where \( \widetilde{td}(T_{IY}) \) is \( \text{td} \) for \( T_I \) in (6.4).

**Proof.** The proof is parallel to that of Theorem 3.6 of [22]. The upper rectangle is clearly commutative. We show the commutativity of the lower rectangle. For \( \gamma \in HH_*(MF_d(X, w)) \) let \( \alpha := \text{I}_{HKR}(\gamma) \), \( \alpha' := \text{I}_{HKR}(HH(Rf_*)(\gamma)) \) and let
\[
\text{ch}(\Psi(O_{f!})) = \sum_i T^i \otimes T_i \in \mathbb{H}^*(I_{X}, (\Omega^*_I, dw|_{IY})) \otimes \mathbb{H}^*(I_{Y}, (\Omega^*_I, -dv|_{IY})).
\]
then by Proposition 6.4 and Theorem 6.10 we have for \( \beta \in \mathbb{H}^*(I_{Y}, (\Omega^*_I, dv|_{IY})) \)
\[ \int_{IY} \alpha' \wedge \beta \wedge \text{td}(T_{IY}) = \sum_i \int_{IY} (-1)^{\dim_{IY} + 1} \alpha \wedge T^i \wedge \text{td}(T_{IY}) \int_{IY} T_i \wedge \beta \wedge \text{td}(T_{IY}). \]
Let $\pi$ denote the projection $If^*T_Y \to IX$ and let $s$ be the diagonal section of $\pi^*If^*T_Y$ on $If^*T_Y$. Then the deformation argument for $\Gamma_f : X \to X \times Y$ as in the proof of Theorem 6.10 shows that

\[
RHS \text{ of } (6.9) = \int_{IX \times IY} (-1)^{\dim X + 1}(\alpha \otimes \beta) \wedge \text{ch}(O^*_f) \wedge (\text{td}(TIX) \otimes \text{td}(TICY))
\]

\[
= \int_{f^*T_Y} (-1)^{\dim f^* + 1}(\alpha \wedge f^* \beta \wedge \text{td}(If^*T_Y) \wedge \text{td}(TIX)) \wedge \text{ch}(\text{Kos}(s))
\]

\[
= \int_{IX} (-1)^{\dim f^* + 1}(\alpha \wedge f^* \beta \wedge \text{td}(TIX) = \int_{IX} (\int_{If} (-1)^{\dim f^* + 1}\alpha \wedge \text{td}(TIX)) \wedge \beta.
\]

This completes the proof. \qed

**Remark 6.12.** We briefly discuss how the GRR for $\Delta$ would compute the canonical pairing, which shows some relationship between GRR and the canonical pairing.

Consider the Riemann-Roch map

\[
\text{ch}^\tau : K_0(X, w) \to H^*(IX, (\Omega^*_X, -dw))
\]

\[
E \mapsto \text{ch}_H(E) \text{td}(TIX).
\]

Suppose that we have a GRR type theorem for the diagonal map $\Delta : X \to X^2$:

$$
\Delta_\ast \text{ch}^\tau(O_X) = \text{ch}^\tau(\Delta_\ast O_X) = \frac{\text{td}(IX^2) \text{ch}_H(\Delta_\ast O_X)}{\text{ch}_H(N_{IX^2/X^2})}.
$$

(6.10)

This yields a formula

\[
\text{ch}_H(\Delta_\ast O_X) = \Delta_\ast \left( \frac{\text{ch}_w(N_{IX/X})}{\text{td}(IX)} \text{ch}_H(O_X) \right) = \Delta_\ast \frac{\text{ch}_w(N_{IX/X})}{\text{td}(IX)},
\]

since $\text{ch}_H(O_X) = 1$. Denote $\text{td} = \text{td}TIX$. Then

\[
\int_{IX} (-1)^{\dim X + 1}\gamma \cdot t^i \cdot \text{td} \int_{IX} (-1)^{\dim X + 1}\gamma' \cdot t^j \cdot \text{td}
\]

\[
= \int_{IX} \gamma \otimes \gamma' \cdot \Delta_\ast \frac{\text{ch}_w(N_{IX/X})}{\text{td}(IX)} \cdot \text{td} \otimes \text{td} = \int_{IX} (-1)^{\dim X + 1}\gamma \cdot \gamma' \cdot \text{td},
\]

which is the characterization property of the canonical pairing. Thus (6.10) implies that the canonical pairing is $\int_{IX} (-1)^{\dim X + 1} \cdot \gamma \cdot \gamma' \cdot \text{td}(TIX)$.

7. **Pushforward in Hodge cohomology**

7.1. **A functor $f^!$ and its base change.** In 6.4, we implicitly use the existence of the right adjoint functor $f^!$ of $Rf_*$ and its base change formula. In order to simplify notations, all functors in this subsection are considered as derived functors unless stated otherwise.
Theorem 7.1. (See [30]) Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a proper morphism between Deligne-Mumford stacks and let \( F \in D^+_c(\mathcal{X}) \) and \( G \in D^+_c(\mathcal{Y}) \). Then there exists a right adjoint functor \( f^! \) of \( f_* \) such that the composition

\[
 f_* \text{Hom}_\mathcal{X}(F, f^! G) \xrightarrow{\text{nat}} \text{Hom}_\mathcal{Y}(f_* F, f_* f^! G) \xrightarrow{\text{tr}_f} \text{Hom}_\mathcal{Y}(f_* F, G)
\]

is an isomorphism. Here, the map \( \text{tr}_f \) is the counit of the adjoint pair \( f_* \dashv f^! \).

Suppose that we have a tor-independent Cartesian diagram of DM stack

\[
(7.1) \quad \begin{array}{ccc}
\mathcal{X}' & \xrightarrow{v} & \mathcal{X} \\
\downarrow g & & \downarrow f \\
\mathcal{Y}' & \xrightarrow{u} & \mathcal{Y}
\end{array}
\]

such that \( f \) is proper. Because it is tor-independent, we have a base change isomorphism

\[
(7.2) \quad \sigma : g_* v^* \xrightarrow{\sim} u^* f_*
\]

We also have a unit and counit map from the adjoint pair \( f_* \dashv f^! \)

\[
(7.3) \quad \epsilon_f : \text{Id} \rightarrow f^! f_* \quad \eta_f (= \text{tr}_f) : f_* f^! \rightarrow \text{Id}.
\]

In this setup, we want to prove the following base change formula for Grothendieck duality which is well-known schemes. We prove a minor extension for Deligne-Mumford stacks under a mild assumption on \( u \).

Proposition 7.2. (See [27]) Further assume that the map \( u \) in 7.1 is representable affine of finite-Tor dimension. Then the canonical base change

\[
(7.4) \quad g_* \text{Hom}_{\mathcal{X}'}(v^* F, v^* f^! G) \xrightarrow{g_* \beta} g_* \text{Hom}_{\mathcal{X}'}(v^* F, g^! u^* G)
\]

is an isomorphism.

Proof. Observe that \( u \), and hence \( v \), is reflexive, which means that if \( v_* \phi \) is an isomorphism, then so is \( \phi \). Therefore, \( \beta \) is an isomorphism if \( v_* \beta : v_* v^* f^! \rightarrow v_* g^! u^* \) is an isomorphism. Applying Yoneda lemma and adjunction formula for \( v^* \dashv v_* \), it is enough to prove that for \( F \in D^+_c(\mathcal{X}), G \in D^+_c(\mathcal{Y}) \), the induced map

\[
\beta : \text{Hom}_{\mathcal{X}'}(v^* F, v^* f^! G) \rightarrow \text{Hom}_{\mathcal{X}'}(v^* F, g^! u^* G)
\]

is an isomorphism. Consider the following commutative diagram of sheaves on \( \mathcal{Y}' \):

\[
(7.4) \quad g_* \text{Hom}_{\mathcal{X}'}(v^* F, v^* f^! G) \xrightarrow{g_* \beta} g_* \text{Hom}_{\mathcal{X}'}(v^* F, g^! u^* G) \xrightarrow{\sim} \text{Hom}_{\mathcal{Y}'}(g_* v^* F, u^* G)
\]
The morphism $\tilde{\beta}$ is an instance of a sheaf version of base change morphism. It is defined as a composition

\begin{equation}
\tilde{\beta}_{(F',G)} : g_*\text{Hom}_{X'}(F', v^*f^!G) \to \text{Hom}_{X'}(g_*F', g_*v^*f^!G)
\end{equation}

\[ \xrightarrow{\sigma} \text{Hom}_{Y'}(g_*F', u^*f_*^!G) \xrightarrow{\text{t.f.}} \text{Hom}_{Y'}(g_*F', u^*G). \]

for each $F' \in D^+_c(Y'), G \in D^+(Y)$. Then $\tilde{\beta} = \tilde{\beta}_{(v^*F,G)}$. One can easily check that it is an isomorphism from the following commutative diagram;

\begin{equation}
g_*\text{Hom}_{X'}(v^*F, v^*f^!G) \xrightarrow{\tilde{\beta}} \text{Hom}_{Y'}(g_*v^*F, u^*G) \xrightarrow{\sim} \text{Hom}_{X'}(v^*F, v^*f^!G) \xrightarrow{\sim} \text{Hom}_{Y'}(u^*f_*F, u^*G) \xrightarrow{\sim} u^*\text{Hom}_{Y'}(f_*F, G).
\end{equation}

Here, the vertical isomorphisms on the top left and bottom right follows from the fact that the natural morphism of sheaves

$u^*\text{Hom}_{Y'}(F, G) \to \text{Hom}_{Y'}(u^*F, u^*G)$

is an isomorphism because $u$ (and hence, $v$) is of finite tor-dimension. \qed

We apply 7.2 to the deformation to the normal cone.

7.1.1. Some basic properties. In this section all stacks are assumed to be smooth separated DM stacks of finite type over $k$. Let $f : X \to Y$ be a morphism. Assume that they are pure dimensional and let $d$ be $\dim X - \dim Y$.

**Definition 7.3.** Once we have the right adjoint functor $f^!$ of $Rf_*$, as in [22] we can define

\[ \int_f : H^q_{\mathbb{Z}_1}(X, \Omega^p_X) \to H^q_{\mathbb{Z}_2}(Y, \Omega^p_{Y^d}) \]

where ($\mathbb{Z}_1, \mathbb{Z}_2$) is either $(c, c)$ or $(cf, \emptyset)$. When $Y$ is $\text{Spec } k$, write $\int_X$ for $\int_f$.

**Remark 7.4.** In the construction of $\int_f$, Nagata’s compactification and the resolution of singularities were used. In our separated DM stack setup both are known by [36] and [39], respectively.

The following can be straightforwardly proven as in [22, §3.6].

1. (Base change I) Consider a Cartesian diagram (7.1). Assume that $f$ is a flat, proper and l.c.i morphism. Then

\[ \int_{X'} v^*(\gamma) = u^*(\int_f \gamma). \]
(2) (Base change II) Consider a Cartesian diagram \((7.1)\). Assume that \(f\) is a flat morphism, \(\mathcal{Y}\) is a connected 1-dimensional smooth scheme, and \(u\) is the embedding of a closed point \(\mathcal{Y}'\) of \(\mathcal{Y}\). Then
\[
\int_{\mathcal{X}'} v^* (\gamma) = u^* \left( \int_f \gamma \right) \in k.
\]

(3) (Functoriality) Let \(X \xrightarrow{f} Y \xrightarrow{g} Z\) be morphisms. Then
\[
\int_g \circ \int_f = \int_{g \circ f}.
\]

(4) (Projection formula) Let \(X \xrightarrow{f} Y\) be a morphisms. Then
\[
\int_f (f^* \sigma \wedge \gamma) = \sigma \wedge \int_f \gamma
\]
for \(\gamma \in H^d_{cf}(X, \Omega^d_X)\) and \(\sigma \in H^q(Y, \Omega^p_Y)\).

7.1.2. Computation. Let \(E\) be a vector bundle on \(X\) of rank \(n\), let \(\pi : E \to X\) be the projection, and let \(s\) be the diagonal section of \(\pi^* E\). Since \(\pi\) is representable, we have
\[
\int_{\pi} \text{ch}(\text{Kos}(s)) \text{td}(\pi^* E) = (-1)^{\binom{n+1}{2}}
\]
by the base change I in § 7.1.1 and the computation of [22, § 3.6.6].

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