Real Interference Alignment

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Abstract—In this paper, we show that the total Degrees-Of-Freedom (DOF) of the $K$-user Gaussian Interference Channel (GIC) can be achieved by incorporating a new alignment technique known as real interference alignment. This technique compared to its ancestor vector interference alignment performs on a single real line and exploits the properties of real numbers to provide optimal signaling. The real interference alignment relies on a new coding scheme in which several data streams having fractional multiplexing gains are sent by transmitters and interfering streams are aligned at receivers. The coding scheme is backed up by a recent result in the field of Diophantine approximation, which states that the convergence part of the Khintchine-Groshev theorem holds for points on non-degenerate manifolds.

Index Terms—Interference channels, interference alignment, number theory, Diophantine approximation.

I. INTRODUCTION

ACHIEVING the optimum throughput of a system requires efficient interference management. Interference alignment is a type of interference management that exploits spatial Degrees-Of-Freedom (DOF) available at transmitters and receivers. In [1], Maddah-Al, Motahari, and Khandani introduced the concept of interference alignment and showed its capability in achieving the full Degrees-Of-Freedom (DOF) for certain classes of two-user $X$ channels.

Interference alignment in $n$-dimensional Euclidean spaces for $n \geq 2$ is studied by several researchers, c.f. [11]–[3]. In this method, at each receiver a subspace is dedicated to interference, then the signaling is designed such that all the interfering signals are squeezed in the interference subspace. Such an approach saves some dimensions for communicating desired signal, while keeping it completely free from the interference. Using this method, Cadambe and Jafar showed that, contrary to the popular belief, a $K$-user Gaussian interference channel with varying channel gains can achieve its total DOF, which is $\frac{K}{2}$. Later, in [4], it is shown that the same result can be achieved using a simple approach based on a particular pairing of the channel matrices. The assumption of varying channel gains, particularly noting that all the gains should be known at the transmitters, is unrealistic, which limits the application of these important theoretical results in practice. This paper aims to remove this shortcoming by proving that the same result can be achieved by relying on a new alignment technique.

Some techniques are proposed for alignment in real domains. In [5] interference alignment is applied in single antenna systems when all receivers are interference free but one. In [6] and [7], the results from the field of Diophantine approximation in Number Theory are used to show that interference can be aligned using properties of rational and irrational numbers and their relations. However, in their examples there is no need for signaling design. The first example of interference alignment in one-dimensional spaces, which requires signaling design, is presented in [8]. Using irrational numbers as transmit directions and applying Khintchine-Groshev theorem, [8] shows the two-user $X$ channel achieves its total DOF. In this paper, we extend the result of [8] and prove the following theorem.

Theorem 1: The total DOF of the $K$-user GIC with real and time invariant channel coefficients is $\frac{K}{2}$ for almost all channel realizations.

Remark 1: In [9], the total DOF of the $K$-user MIMO GIC is addressed. The application of real interference alignment to channels with complex coefficient gain is addressed in [10].

II. REAL INTERFERENCE ALIGNMENT

The ingredients of real interference alignment technique are
1) Decoding based on Diophantine approximation.
2) Alignment based on structural encoding.
3) Achieving asymptotically perfect alignment based on partial interference alignment.

We provide three examples to clarify the impact of the preceding items. We first look at a three-user multiple access channel modeled by $y = x_1 + ax_2 + bx_3 + z$. Let us assume that all three users communicate with the receiver using a single data stream. The data streams are modulated by the constellation $U = A(-Q,Q)Z$ where $A$ is a factor controlling the minimum distance of the received constellation. $(a,b)_Z$ denotes the set of integers between $a$ and $b$.

The received constellation consists of points representable by $A(u_1 + au_2 + bu_3)$ where $u_i$s are integer. Let us choose two distinct points $v_1 = A(u_1 + au_2 + bu_3)$ and $v_2 = A(u_1' + au_2' + bu_3')$ in the received constellation. The distance between these two points is $d = A|u_1 - u_1'| + a|u_2 - u_2'| + b|u_3 - u_3'|$. The following theorem due to Groshev can be used to lower bound the minimum distance of the received constellation.

Theorem 2 (Khintchine-Groshev): The set of $m$-tuple real numbers satisfying
$$|p + v \cdot q| < \frac{1}{Q^{m+e}}$$

(1)
has measure zero for \( p \in \mathbb{Z}, q \in \mathbb{Z}^m \), and \( Q = \max\{|q_1|, \ldots, |q_m|\} \). Place it differently, the theorem states that for almost all \( v \in \mathbb{R}^m \), there is a constant \( \kappa \) which is only related to \( v \) such that

\[
|p + v \cdot q| > \frac{\kappa}{Q^{m+e}}
\]

for all \( p \in \mathbb{Z} \) and \( q \in \mathbb{Z}^m \).

Using the theorem, one can obtain \( d_{\min} \approx \frac{A}{Q^2} \) where \( d_{\min} \) is the minimum distance in the received constellation. It can be shown that if \( d_{\min} \approx 1 \) then the additive gaussian noise can be removed from the received signal using an appropriate coding scheme \([13]\). This condition can be enforced by \( A \approx Q^2 \). In a noise-free environment, the receiver can decode the three messages if there is a one-to-one map from the received signal to the transmit constellation. This condition can be satisfied if \( a \) and \( b \) are rationally independent which in fact holds almost surely. Therefore the receiver can decode all three messages almost surely.

To calculate User \( i \)'s rate \( R_i = \log(2Q-1) \) in terms of \( P \), we need to find a relation between \( Q \) and \( P \). Due to the power constraints, we have \( P = A^2Q^2 \). We showed that \( A \approx Q^2 \). Therefore, \( P \approx Q^6 \). Hence, we have

\[
R_i = \lim_{P \to \infty} \frac{R_i}{0.5 \log P} = \frac{1}{3},
\]

where \( R_i \) is the achievable DOF for User \( i \).

In the preceding example, we implicitly assumed that the pair \( (a,b) \) can take any value in \( \mathbb{R}^2 \). Otherwise, we were not able to apply the Khintchine-Groshev theorem. Let us assume that \( a \) and \( b \) have a relation. For instance, \( b \) is a function of \( a \), say \( b = a^2 \). In this case, the pair \( (a,b) \) lies on a one dimensional manifold in \( \mathbb{R}^2 \), see Figure 1. Since the manifold itself has measure zero, Khintchine-Groshev theorem cannot be applied directly. For such cases, however, there is an extension to Khintchine-Groshev theorem, see \([11]\) and \([12]\), which states that the same lower bound on the minimum distance can be applied when coefficients lie on a non-degenerate manifold and, in fact, the measure of points not satisfying the theorem is zero. In can be shown that if all \( v_i \)'s in \( \mathbb{R}^2 \) are monomials with variables from the set \( g = \{g_1, g_2, \ldots, g_n\} \), then \( v \) lines on a non-degenerate manifold. As a special case when set \( v \) has only one member, i.e. \( v = \{g, g^2, g^3, \ldots\} \).

In real interference alignment we say two data streams are aligned at a receiver if they arrive at the receiver with the same received direction (coefficient). To show this in fact reduces the total dimension of the received interference, we consider the two-user \( X \) channel. In the two-user \( X \) channel, each transmitter has independent messages to both receivers, see Figure 2. Hence, each transmitter has two data streams and they need to be transmitted such that they can be separated in their corresponding receivers. In \([8]\), the following signaling is proposed for the channel:

\[
\begin{align*}
x_1 &= h_{22}u_1 + h_{12}v_1, \\
x_2 &= h_{21}u_2 + h_{11}v_2,
\end{align*}
\]

where \( u_1, u_2 \) and \( v_1, v_2 \) are data streams intended for the first and second receivers, respectively. All data streams are transmitted using the constellation \( U = A(-Q,Q) \mathbb{Z} \), where \( Q \) is an integer and \( A \) is the factor controlling the minimum distance of the received constellation.

Using the above signaling, the received signal can be written as:

\[
\begin{align*}
y_1 &= (h_{11}h_{22})u_1 + (h_{21}h_{12})u_2 + (h_{11}h_{12})(v_1 + v_2) + z_1, \\
y_2 &= (h_{21}h_{12})v_1 + (h_{11}h_{22})v_2 + (h_{21}h_{22})(u_1 + u_2) + z_2.
\end{align*}
\]

The received signals are linear combinations of three terms in which two of them are the intended data streams and one is the sum of interfering signals, see Figure 2. Let us focus on the first receiver. \( y_1 \) resembles the received signal of a multiple access channel with three users. However, there is an important difference between them. In the two-user \( X \) channel the term corresponding to the interfering signals, i.e. \( u_3 = v_1 + v_2 \), is a sum of two data streams. However, we claim that this difference does not change considerably the minimum distance of the received constellation, i.e. \( d_{\min} \). Recall that Khintchine-Groshev theorem is used to bound \( d_{\min} \). The bound is a function of the maximum value that the integers can take. The maximum value of \( u_3 \) is \( 2AQ \), which is different from a single data stream by a factor of two. Since this change only affects the constant term of Khintchine-Groshev theorem, we have \( d_{\min} \approx \frac{A}{Q} \) and the receiver can decode all data streams if each of them have a multiplexing gain of \( \frac{1}{3} \). Therefore, the multiplexing gain of \( \frac{1}{3} \) out of two dimensions
available at both receivers are used for data, which in turn gives us the total DOF of the channel.

In the two user X channel, we have observed that interfering signals from two different sources can be easily aligned at a single receiver. Moreover, two interfering streams are received with the same direction occupying only $\frac{1}{3}$ of the available dimensions of the receivers. This is in fact the best efficiency that one can hope for in reducing the number of waste dimensions. This perfect alignment is not possible in general. In the following example, we show that partial alignment can be used instead to provide the same performance asymptotically.

Let us consider a communication scenario in which three transmitters try to align their signals at two different receivers. The channel is depicted in Figure 3. In order to shed light on the alignment part of the signaling, the intended receivers are removed from the picture.

Alignment can be done at the first receiver by sending a single data stream with direction 1 from each of the transmitters; whereas alignment at the second receiver requires $ac$, $ab$ as chosen transmit directions for first, second, and third transmitters, respectively. In general, it is not possible to simultaneously align three single data streams at two different receivers. Therefore perfect alignment is not feasible by transmitting single data streams from each transmitter.

The solution to this problem is partial alignment, which is first introduced in [3]. In this technique, instead of sending just one data stream, several data streams are transmitted from each transmitter. The idea is to choose the transmit directions based on channel coefficients in such a way that the number of received directions is minimum. For the sake of simplicity, we choose the same directions at all transmitters. Let $T$ denote the set of transmit directions. A direction $T \in \mathcal{T}$ is chosen as a transmit direction if it can be represented as

$$T = a^{s_1}b^{s_2}c^{s_3},$$

where $0 \leq s_i \leq n - 1$ for all $i \in \{1, 2, 3\}$. In this way, the total number of transmit directions is $L_1 = n^3$.

To compute the efficiency of the alignment, one needs to find the set of received directions in the first and second receivers, which are denoted by $T_1$ and $T_2$, respectively. Since all transmit directions arrive at the first receiver intact, $T_1 = \mathcal{T}$. To compute the set of received directions at the second receiver, we look at the received directions due to the first, second, and third transmitters separately. Since all of them are multiplied by $a_i$, the received directions due to the first transmitter are of the form $a^{s_1+1}b^{s_2}c^{s_3}$, where $0 \leq s_i \leq n - 1$ for all $i \in \{1, 2, 3\}$. Similarly, $a^{s_1}b^{s_2+1}c^{s_3}$ and $a^{s_1}b^{s_2}c^{s_3+1}$ are the types of received directions due to the second and third transmitter, respectively. Taking the union of all these directions, one can compute $T_2$. However, we can easily see that the set of directions formed by $a^{s_1}b^{s_2+1}c^{s_3}$, where $0 \leq s_i \leq n$ for all $i \in \{1, 2, 3\}$ includes $T_2$ and can be used as an upper bound on the number of received directions. This set has $(n+1)^3$, which is an upper bound for $L_2$. Hence, we conclude that

$$\eta = \frac{L_1}{L_2} > \left(\frac{n}{n+1}\right)^3.$$
have reliable communication at their maximum rates. The channel’s input-output relation can be stated as follows,
\begin{align*}
y_1 &= h_{11}x_1 + h_{12}x_2 + \ldots + h_{1K}x_K + z_1, \\
y_2 &= h_{21}x_1 + h_{22}x_2 + \ldots + h_{2K}x_K + z_2, \\
&\vdots \\
y_K &= h_{K1}x_1 + h_{K2}x_2 + \ldots + h_{KK}x_K + z_K,
\end{align*}
(5)
where $x_i$ and $y_i$ are input and output symbols of User $i$ for $i \in \{1, 2, \ldots, K\}$, respectively. $z_i$ is Additive White Gaussian Noise (AWGN) with unit power for $i \in \{1, 2, \ldots, K\}$. Transmitters are subject to the power constraint $P_i$, $h_{ii}$ represents the channel gain between Transmitter $i$ and Receiver $j$. It is assumed that all channel gains are real and time invariant. The set of all channel gains is denoted by $\mathbf{h}$, i.e., $\mathbf{h} = \{h_{11}, \ldots, h_{1K}, h_{21}, \ldots, h_{2K}, \ldots, h_{KK}\}$. Since the noise variances are normalized, the Signal to Noise Ratio (SNR) is equivalent to the input power $P$. Hence, we use them interchangeably throughout this paper.

In this paper, we are primarily interested in characterizing the total DOF of the $K$-user GIC. Let $C$ denote the capacity region of this channel. The DOF region associated with the channel is in fact the shape of $C$ in high SNR regimes scaled by $\log$ SNR. Let us denote the DOF region by $\mathcal{R}$. All extreme points of $\mathcal{R}$ can be identified by solving the following optimization problem:
\begin{align*}
r_{\lambda} &= \lim_{\text{SNR} \to \infty} \max_{\mathbf{R} \in C} \frac{\lambda \mathbf{R}}{0.5 \log \text{SNR}}.
\end{align*}
(6)

The total DOF refers to the case where $\lambda = \{1, 1, \ldots, 1\}$, i.e., the sum-rate is concerned. Throughout this paper, $r_{\text{sum}}$ denotes the total DOF of the system.

An upper bound on the DOF of this channel is obtained in [3]. The upper bound states that the total DOF of the channel is less than $\frac{K}{2}$, which means each user can at most enjoy one half of its maximum DOF.

### B. Three-user Gaussian Interference Channel: DOF $= \frac{3}{2}$ is Achievable

In this section, we consider the three-user GIC and explain in detail that, by an appropriate selection of transmit directions, the DOF of $\frac{3}{2}$ is achievable for almost all cases. We will explain in more detail that by an appropriate selection of transmit directions this DOF can be achieved.

In [8], we defined the standard model of the three-user GIC. The definition is as follows:

**Definition 1:** The three user interference channel is called standard if it can be represented as
\begin{align*}
y_1 &= G_1x_1 + x_2 + x_3 + z_1, \\
y_2 &= G_2x_2 + x_1 + x_3 + z_2, \\
y_3 &= G_3x_3 + x_1 + G_0x_2 + z_3,
\end{align*}
(7)
where $x_i$ for User $i$ is subject to the power constraint $P_i$, $z_i$ at Receiver $i$ is AWGN with unit variance.

In [8], it is also proved that every three-user GIC has an equivalent standard channel as far as the DOF is concerned.

As mentioned in the previous section, transmit directions are monomials with variables from channel coefficients. For the three user case, we only use $G_0$ as the generator of transmit directions. Therefore, transmit directions are selected from the set $\mathcal{G}(G_0)$, which is a subset of $\mathcal{G}(G_0, G_1, G_2, G_3)$. Clearly, $\mathcal{G}(G_0) = \{G_0, G_0^2, G_0^3, \ldots\}$.

We only consider the case where $G_0$ is transcendental. In fact, the measure of being algebraic is zero. If $G_0$ is transcendental then all members of $\mathcal{G}(G_0)$ are linearly independent over the field of rational numbers. Hence, we are not limited to any subset of $\mathcal{G}(G_0)$, as far as the independence of transmit directions is concerned. We will show that $\frac{3}{2}$ is an achievable DOF for any $n \in \mathbb{N}$. To this end, we propose a design that is not symmetrical.

Transmitter 1 uses the set of directions $\mathcal{T}_1 = \{G_0, G_0^2, \ldots, G_0^n\}$ to transmit $L_1 = n+1$ to its corresponding receiver. Clearly $\mathcal{T}_1$ satisfies C1. The transmit signal from User 1 can be written as
\begin{align*}
x_1 &= A \sum_{j=0}^{n} G_0^j u_{1j},
\end{align*}
Transmitters 2 and 3 transmit in $L_2 = L_3 = n$ directions using $\mathcal{T}_2 = \mathcal{T}_3 = \{G_0, G_0^2, \ldots, G_0^{n-1}\}$. Clearly both $\mathcal{T}_2$ and $\mathcal{T}_3$ satisfy C1. The transmit signals can be expressed as
\begin{align*}
x_2 &= A \sum_{j=0}^{n-1} G_0^j u_{2j},
\end{align*}
and
\begin{align*}
x_3 &= A \sum_{j=0}^{n-1} G_0^j u_{3j}.
\end{align*}
The received signal at Receiver 1 can be expressed as:
\begin{align*}
y_1 &= A \left( \sum_{j=0}^{n} G_1G_0^j u_{1j} + \sum_{j=0}^{n-1} G_0^j u_{1j} \right) + z_1, \tag{8}
\end{align*}
where $u_{1j}^{(1)} = u_{2j} + u_{3j}$. In fact, transmit signals from Users 2 and 3 are aligned at Receiver 1. This is due to the fact that out of $2n$ possible received directions, only $n$ directions are effective, i.e., $L_1' = n$. One can also confirm that C2 and C3 hold at Receiver 1.

The received signal at Receiver 2 can be expressed as:
\begin{align*}
y_2 &= A \left( \sum_{j=0}^{n-1} G_2G_0^j u_{2j} + \sum_{j=0}^{n} G_0^j u_{2j} \right) + z_2, \tag{9}
\end{align*}
where $u_{2j}^{(2)} = u_{1j} + u_{3j}$ for all $j \in \{0, 1, \ldots, n-1\}$ and $u_{2n} = u_{1n}$. At Receiver 2, transmitted signals from Users 1 and 3 are aligned and the number of effective received directions is $L_2' = n+1$. Moreover, it can be easily seen that C2 and C3 hold at Receiver 2.

The received signal at Receiver 3 can be expressed as:
\begin{align*}
y_3 &= A \left( \sum_{j=0}^{n-1} G_3G_0^j u_{3j} + \sum_{j=0}^{n} G_0^j u_{3j} \right) + z_3, \tag{10}
\end{align*}
where $u_{3j}^{(3)} = u_{1j} + u_{2j}$ for all $j \in \{1, 2, \ldots, n\}$ and $u_{30} = u_{10}$. At Receiver 3, transmitted signals from Users 1 and 2
are aligned and the number of effective received directions is \( L'_{2} = n + 1 \). Clearly, C2 and C3 hold for Receiver 3.

Since C1, C2, and C3 hold at all users, we only need to obtain the number of maximum received directions at all receivers. To this end, we observe that

\[
m = \max\{L_1 + L'_1, L_2 + L'_2, L_3 + L'_3\} = 2n + 1
\]

. Therefore, an application of Theorem \[3\] reveals that the following DOF is achievable.

\[
r_{\text{sum}} = \frac{L_1 + L_2 + L_3}{m} = \frac{3n + 1}{2n + 1}.
\]

(11)

Since \( n \) is an arbitrary integer, one can conclude that \( \frac{3}{2} \) is achievable for the three-user GIC almost surely.

**C. K-user Gaussian Interference Channel: DOF = \( \frac{K}{2} \) is Achievable**

To prove the main result of the paper, we start with selecting the transmit directions for User \( i \). A direction \( T \in \mathcal{G}(h) \) is chosen as the transmit direction for User \( i \) if it can be represented as

\[
T = \prod_{j=1}^{K} h_{sj}^{i},
\]

(12)

where \( s_{j} \)'s are integers satisfying

\[
\begin{align*}
    s_{jj} &= 0 &\forall j \in \{1, 2, \ldots, K\} \\
    0 &\leq s_{j} \leq n - 1 & \forall j \in \{1, 2, \ldots, K\} \ & j \neq i \\
    0 &\leq s_{j} \leq n & \text{Otherwise.}
\end{align*}
\]

The set of all transmit directions is denoted by \( \mathcal{T} \). It is easy to show that the cardinality of this set is

\[
L_i = n^{K-1}(n + 1)^{(K-1)^2}.
\]

(13)

Clearly, \( \mathcal{T} \) satisfies C1 for all \( i \in \{1, 2, \ldots, K\} \).

To compute \( L'_i \) (the number of independent received directions due to interference), we investigate the effect of Transmitter \( k \) on Receiver \( i \). Let us first define \( \mathcal{T}_k \) as the set of directions represented by \[12\] and satisfying

\[
\begin{align*}
    s_{jj} &= 0 &\forall j \in \{1, 2, \ldots, K\} \\
    0 &\leq s_{j} \leq n - 1 & \forall j \in \{1, 2, \ldots, K\} \ & j \neq i \\
    0 &\leq s_{j} \leq n & \text{Otherwise.}
\end{align*}
\]

(14)

We claim that \( \mathcal{T}_{ik} \), the set of received directions at Receiver \( i \) due to Transmitter \( k \), is a subset of \( \mathcal{T}_k \). In fact, all transmit directions of Transmitter \( k \) arrive at Receiver \( i \) multiplied by \( h_{ik} \). Based on the selection of transmit directions, however, the maximum power of \( h_{ik} \) in all members of \( \mathcal{T}_{ik} \) is \( n - 1 \). Therefore, none of the received directions violates the condition of \[14\] and this proves the claim.

Since \( \mathcal{T}_k \) is not related to User \( k \), one can conclude that \( \mathcal{T}_{ik} \subseteq \mathcal{T}_k \) for all \( k \in \{1, 2, \ldots, K\} \) and \( k \neq i \). Hence, we deduce that all interfering users are aligned in the directions of \( \mathcal{T}_k \). Now, \( L'_i \) can be obtained by counting the members of \( \mathcal{T}_k \). It is easy to show that

\[
L'_i = (n + 1)^{K(K-1)}.
\]

(15)

The received directions at Receiver \( i \) are members of \( h_{ii}\mathcal{T}_k \) and \( \mathcal{T}_i \). Since \( h_{ii} \) does not appear in members of \( \mathcal{T}_i \), the members of \( h_{ii}\mathcal{T}_k \) and \( \mathcal{T}_i \) are distinct. Therefore, C2 holds at Receiver \( i \). Since all the received directions are irrationals, C3 does not hold at Receiver \( i \).

Since C1 and C2 hold for all users, we can apply Theorem \[3\] to obtain the DOF of the channel. We have

\[
r_{\text{sum}} = \frac{L_1 + L_2 + \ldots + L_K}{m} = \frac{K n^{K-1}(n + 1)^{(K-1)^2}}{m + 1}.
\]

(16)

where \( m = \max_{i} L_i + L'_i \)

\[
= n^{K-1}(n + 1)^{(K-1)^2} + (n + 1)^{K(K-1)}.
\]

(17)

Combining the two equations, we obtain

\[
r_{\text{sum}} = \frac{K}{1 + \left(\frac{m}{n}\right)^{K-1} + \frac{1}{n^{K-1}(n + 1)(K-1)}}.
\]

(18)

Since \( n \) can be arbitrary large, we conclude that \( \frac{K}{2} \) is achievable for the \( K \)-user GIC.

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