CP Violation in $\tau^\pm \to \pi^\pm K_S^0 \nu$ and $D^\pm \to \pi^\pm K_S^0$

The Importance of $K_S - K_L$ Interference

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The $B$-factories have measured CP asymmetries in the $\tau \to \pi K_S^0 \nu$ and $D \to K_S^0 \pi$ modes. The $K_S^0$ state is identified by its decay to two pions at a time that is close to the $K_S$ lifetime. Within the Standard Model and many of its extensions, the asymmetries in these modes come from CP violation in $K^0 - \bar{K}^0$ mixing. We emphasize that the interference between the amplitudes of intermediate $K_S$ and $K_L$ is as important as the pure $K_S$ amplitude. Consequently, the measured asymmetries depend on the times over which the relevant decay rates are integrated and on features of the experiment.

**Introduction.** The BaBar collaboration has recently announced a measurement of the CP asymmetry in the $\tau \to \pi K_S^0 \nu$ decay $^1$:

$$A_T \equiv \frac{\Gamma(\tau^+ \to \pi^+ K_S^0 \nu) - \Gamma(\tau^- \to \pi^- K_S^0 \nu)}{\Gamma(\tau^+ \to \pi^+ K_S^0 \nu) + \Gamma(\tau^- \to \pi^- K_S^0 \nu)} = (-4.5 \pm 2.4 \pm 1.1) \times 10^{-3},$$

(1)

where the numerical value is an average over the four measurements.

Assuming that direct CP violation in the $\tau$ or $D$ decay plays a negligible role, as is the case in the Standard Model and many of its extensions, then the asymmetries $^1$ and $^2$ arise from CP violation in $K^0 - \bar{K}^0$ mixing $^9$–$^11$. It is important then to realize two facts:

1. The $\tau^+ (\tau^-)$ decay produces initially a $K^0 (\bar{K}^0)$ state, while the $D^+ (D^-)$ decay produces initially a $\bar{K}^0$ ($K^0$) state. (The color and doubly Cabibbo suppressed $D^+ \to K^0 \pi^+$ decay amplitude can be safely neglected.)

2. The intermediate $K_S$-state is not directly observed in the experiments. It is *defined* via a final $\pi^+ \pi^-$ state with $m_{\pi\pi} \approx m_K$ and a time difference between the $\tau$ or $D$ decay and the $K$ decay $t \approx \tau_S$, where $\tau_S$ is the $K_S$ lifetime.

Thus, in the absence of direct CP violation, the asymmetries depend on the integrated decay times, and we have

$$A_T(t_1, t_2) = -A_D(t_1, t_2) = A_e(t_1, t_2),$$

(3)

$$A_e(t_1, t_2) = \frac{\int_{t_1}^{t_2} dt \Gamma(K^0(t) \to \pi\pi) - \Gamma(\bar{K}^0(t) \to \pi\pi)}{\int_{t_1}^{t_2} dt \Gamma(K^0(t) \to \pi\pi) + \Gamma(\bar{K}^0(t) \to \pi\pi)},$$

(4)

where $K^0(t) (\bar{K}^0(t))$ is a time-evolved initially-pure $K^0 (\bar{K}^0)$. The fact that $A_T(t_1, t_2)$ and $A_D(t_1, t_2)$ are predicted to have opposite signs, while the experimental measurements $^1$ and $^2$ carry the same sign is intriguing. The naive expectation that $A_T = -A_D$ is excluded at $3.3\sigma$.

In this work, we derive an explicit expression for the $A_e(t_1, t_2)$ asymmetry and its dependence on the experimentally known mixing parameters $\epsilon$ and $\Delta m$. In doing so, we correct for sign mistakes made in previous literature. We argue that the theoretical prediction depends on $t_1$, $t_2$ and on details of the experiment. Until these subtleties are taken into consideration, it is difficult to assess the significance of $A_T \neq -A_D$.

**The experimental parameters.** The two neutral $K$-meson mass eigenstates, $|K_S\rangle$ of mass $m_S$ and width $\Gamma_S$ and $|K_L\rangle$ of mass $m_L$ and width $\Gamma_L$, are linear combinations of the interaction eigenstates $|K^0\rangle$ (with quark content $\bar{s}d$) and $|\bar{K}^0\rangle$ (with quark content $\bar{d}s$):

$$|K_{S,L}\rangle = p|K^0\rangle \pm q|\bar{K}^0\rangle.$$  

(4)

The average and the difference in mass and width are given by

$$m \equiv \frac{m_S + m_L}{2}, \quad \Gamma \equiv \frac{\Gamma_S + \Gamma_L}{2},$$

$$\Delta m \equiv m_L - m_S, \quad \Delta \Gamma \equiv \Gamma_L - \Gamma_S,$$

$$x \equiv \frac{\Delta m}{\Gamma}, \quad y \equiv \frac{\Delta \Gamma}{2\Gamma}.$$  

(5)

The decay amplitudes into a final state $\pi\pi$ are defined as

$$A_{S,L} \equiv \langle \pi\pi| \mathcal{H} |K_{S,L}\rangle.$$  

(6)
The relevant CP violating parameters are defined as
\[
\frac{|p|^2 - |q|^2}{|p|^2 + |q|^2} \approx 2R\epsilon, \quad \frac{A_L}{A_S} \approx \epsilon, 
\]
where in the first approximation we neglected a correction of relative order $|\epsilon|^2$, and in the second a correction of relative order $\epsilon'/\epsilon$. We obtain:
\[
\Gamma_{\pi\pi}(t) \equiv \Gamma(K^0(t) \rightarrow \pi\pi)
\]
\[
= \frac{|A_S|^2}{4|p|^2} \left( e^{-\Gamma_{st}} + |\epsilon|^2 e^{-\Gamma_L t} + 2Re(e^{i\Delta mt - \Gamma_t \epsilon^*}) \right)
\]
\[
\Gamma_{\pi\pi}(t) \equiv \Gamma(K^0(t) \rightarrow \pi\pi)
\]
\[
= \frac{|A_S|^2}{4|q|^2} \left( e^{-\Gamma_{st}} + |\epsilon|^2 e^{-\Gamma_L t} - 2Re(e^{i\Delta mt - \Gamma_t \epsilon^*}) \right).
\]
For the difference and the sum of these rates, we obtain
\[
D_{\pi\pi}(t) = \frac{\Gamma_{\pi\pi}(t) - \bar{\Gamma}_{\pi\pi}(t)}{N} \quad (9)
\]
\[
= -2Re\epsilon \left( e^{-\Gamma_{st}} + |\epsilon|^2 e^{-\Gamma_L t} \right)
\]
\[
+ 2e^{-\Gamma_t} \left( Re\epsilon \cos(\Delta mt) + Im\epsilon \sin(\Delta mt) \right),
\]
\[
S_{\pi\pi}(t) = \frac{\Gamma_{\pi\pi}(t) + \bar{\Gamma}_{\pi\pi}(t)}{N} \quad (10)
\]
\[
= e^{-\Gamma_{st}} + |\epsilon|^2 e^{-\Gamma_L t} - 4Re\epsilon e^{-\Gamma_t}
\]
\[
\times (Re\epsilon \cos(\Delta mt) + Im\epsilon \sin(\Delta mt)),
\]
where
\[
N = \frac{|A_S|^2}{4} \left( |p|^2 + |q|^2 \right). 
\]
For the sum $S_{\pi\pi}(t)$ of Eq. (10), the interference (and the pure $K_L$) terms are suppressed by $O(\epsilon^2)$ compared to the pure $K_S$ term. For the difference $D_{\pi\pi}(t)$ of Eq. (9), however, this is not the case. The ratio between the second (interference) and first (non-interference) terms in $D_{\pi\pi}(t)$,
\[
R(t) \equiv \frac{-e^{-\Gamma_t} \left( \cos(\Delta mt) + \frac{Im\epsilon}{Re\epsilon} \sin(\Delta mt) \right)}{e^{-\Gamma_{st}} + |\epsilon|^2 e^{-\Gamma_L t}}, 
\]
(12)
is plotted in Fig. 1 as a function of time. In the figure we can observe the following features:

1. Even at very early times, the interference term is not negligible compared to the pure $K_S$ term. For example, at $t = 0$, $R = -1$.
2. In the approximations [good to $O(5\%)$] that $Re\epsilon \approx Im\epsilon$ and $x \approx -y$, the ratio changes sign when $\tan[\frac{\pi}{4}(t/\tau_S)] = -1$, namely $t/\tau_S = 3\pi/2 + 2n\pi$ ($n = 0, 1, 2, \ldots$).
3. For times early enough that the pure $K_L$ term can be neglected ($t \ll 12\tau_S$), $R$ reaches a minimum at $t/\tau_S \sim \pi$, $R \sim -e^{-\pi/2}$, and a maximum at $t/\tau_S \sim 3\pi$, $R \sim +e^{3\pi/2}$.

Since the CP asymmetry depends on the time at which the kaon decays, the final measurement is sensitive to the experimental cuts. To incorporate these cuts, we need to take into account not only the efficiency as a function of the kaon decay time, but also the kaon energy in the lab frame to account for time dilation. We parametrize all of these experiment-dependent effects by a function $F(t)$ such that $t$ is the time in the kaon rest frame and $0 \leq F(t) \leq 1$. We emphasize that this function must be determined as part of the experimental analysis. The experimentally measured asymmetry is thus given by the convolution of the bare asymmetry with $F$:
\[
A_e = \frac{\int_0^\infty F(t)D_{\pi\pi}(t) \, dt}{\int_0^\infty F(t)S_{\pi\pi}(t) \, dt}. 
\]
(13)

While we do not have the function $F(t)$, it is reasonable to approximate it by a double step function,
\[
F(t) = \begin{cases} 
1 & t_1 < t < t_2 \\
0 & \text{otherwise}.
\end{cases} 
\]
(14)
In this case the experimentally measured asymmetry, $A_e$ defined in Eq. (13), coincides with the theoretical one, $A_e(t_1,t_2)$ defined in Eq. (3). When $t_2 \ll \tau_L$, we can
safely neglect terms of $O(e^2)$:

\[
A_e(t_1, t_2) = -2Re(e) \\
\times \left[ 1 - \int_{t_1}^{t_2} dt e^{-\Gamma t} \left( \cos(\Delta m t) + \frac{\Im m(e)}{Re(e)} \sin(\Delta m t) \right) \right].
\]

Neglecting direct CP violation, we can use the model independent relation [12]

\[
\frac{\Im m(e)}{Re(e)} = -\frac{x}{y},
\]

to obtain

\[
A_e(t_1, t_2) = -2Re(e) \left[ 1 - \frac{2(1 - x^2/y) e^{-\Gamma t_1} \cos(\Delta m t_1) - e^{-\Gamma t_2} \cos(\Delta m t_2)}{1 + x^2} - \frac{2(x + x/y) e^{-\Gamma t_1} \sin(\Delta m t_1) - e^{-\Gamma t_2} \sin(\Delta m t_2)}{e^{-\Gamma t_1} e^{-\Gamma t_2}} \right].
\]

A particularly simple result arises when $t_1 \ll \tau_S$ and $\tau_S \ll t_2 \ll \tau_L$, so that we can take $e^{-\Gamma t_1} = 1$, $e^{-\Gamma t_2} = 0$, and $\cos(\Delta m t_1) = 1$. In addition we use $y \simeq -1$, and obtain

\[
A_e(t_1 \ll \tau_S, \tau_S \ll t_2 \ll \tau_L) = +2Re(e) \approx +3.3 \times 10^{-3},
\]

where in the last step we used the experimental value. In Figs. 2 and 3 we investigate the dependence of $A_e(t_1, t_2)$ on the choice of $t_1$ and $t_2$. In Fig. 2 we plot $A_e(t_1, t_2)/(2Re(e))$ as a function of $t_2$ for $t_1 = \tau_S/10$. In Fig. 3 we plot $A_e(t_1, t_2)/(2Re(e))$ as a function of $t_1$ for $t_2 = 10\tau_S$. We emphasize the following points:

1. For $t_2$ large enough that the $e^{-\Gamma t_2}$ term is negligible, and for $t_1/\tau_S \ll 1$, we have

\[
A_e(t_1, t_2) \approx +2Re(e)(1 + t_1/\tau_S).
\]

This linear rise with $t_1$, which can be clearly seen in Fig. 2 is a result of “losing” a fraction $t_1/\tau_S$ of the time independent pure $K_S$ term in the asymmetry.

2. For $t_1$ fixed and small, $A_e$ reaches a maximum at around $t_2 = (3\pi/2)\tau_S$, and then, for higher $t_2$, converges to its asymptotic value of Eq. (19). These features can be clearly seen in Fig. 3 The maximum is enhanced by a factor of about $(1 + \sqrt{2}\exp(-3\pi/4)) \approx 1.13$ compared to the asymptotic value.

Let us comment on previous relevant literature. The idea to measure the CP asymmetry of Eq. (1) was first made in Ref. [9]. In this beautiful work, the importance of the interference term in restoring the CPT constraint is explained. Indeed, the BaBar paper [1] compare their measurement to the prediction given in Eq. (7) of Ref. [9]. We note, however, that both Eq. (6) and Eq. (7) of Ref. [9] have a sign mistake: Both the “pure $K_L$” term and the “pure $K_S$” terms give $|q|^2 - |p|^2 \simeq -2Re(e)$. Yet, when the interference term is taken into account, it approximately reverses the sign of the pure $K_S$ result. Correcting the sign of Eq. (7) in [9] and taking into account the interference term combine to approximately reproduce the numerical prediction quoted in this equation. Further analysis of this asymmetry is given in Ref. [10]. Here the fact that the interference term practically reverses the sign of the “pure $K_S$” asymmetry is nicely pointed out, yet several sign mistakes lead to a wrong sign in the final prediction, see their Eq. (14). The idea to measure the CP asymmetry of Eq. (3) was first made in Ref. [11]. The interference term is not discussed in this work.

**Conclusions.** CP asymmetries of $O(10^{-3})$ in the $\pi^+ \to \pi^\pm K_{S\bar{S}}\nu$ and $D^+ \to \pi^\pm K_S$ decays are predicted within the Standard Model as a result of CP violation in $K^0 - \bar{K}^0$ mixing. A violation of the SM predictions would imply direct CP violation in $\tau$ and/or $D$ decays.

The kaon is identified via final two pions with invariant mass $m_{\pi\pi} \sim m_K$ and decay time $t \sim \tau_S$. In the total decay rate, the contribution of intermediate $K_S$ is strongly dominant. In the CP asymmetry, however, the $K_L - K_S$ interference term is as important as the pure $K_S$ term.
As a consequence of this situation, the asymmetry depends sensitively on the decay time interval over which it is measured, and on details of the experiment. The exact SM prediction can be obtained only if the relevant experimental features are taken into consideration. Generically, we expect the measured asymmetry to be opposite in sign and larger in magnitude than the asymmetry that would arise from the pure $K_S$ contribution.

While we focused here on only the two specific examples of $D$ and $\tau$ decays, the analysis above applies to any measurement of a CP asymmetry that involves $K_S$ in the final state. In particular, similar effects should eventually be taken into account in the determination of the angle $\gamma$ of the unitarity triangle based on $B \to DK$ decays, if the kaon is identified as a $K_S$ or if the $D$ decays into a final state with a $K_S$ such as with the Dalitz decay $D \to \pi^+ \pi^- K_S$ [14]. Another case where the effect should eventually be included is in the determination of $D - \bar{D}$ mixing using $D$ decays into $K_S$. In this case our formalism cannot be directly applied, because at $t = 0$ the kaon state is not a pure $K^0$ (or $\bar{K}^0$), and adjustment to such cases is needed.

The measured asymmetry in $D$ decay seems very consistent with the SM prediction, while the measured asymmetry in $\tau$ decay seems different from the SM prediction by at least $3\sigma$. In view of the potential implications for new, CP violating physics, we urge the experimenters to take into account the subtleties that we point out, and provide not only the measured value of the asymmetry, but also the theoretical prediction which depends on specific experimental features.

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