We construct a new analytic solution of Einstein-Born-Infeld-dilaton theory in the presence of Liouville-type potentials for the dilaton field. These solutions describe dilaton black holes with nontrivial topology and nonlinear electrodynamics. Black hole horizons and cosmological horizons in these spacetimes, can be a two-dimensional positive, zero or negative constant curvature surface. The asymptotic behavior of these solutions are neither flat nor (A)dS. We calculate the conserved and thermodynamic quantities of these solutions and verify that these quantities satisfy the first law of black hole thermodynamics.

I. INTRODUCTION

Although the nonlinear electrodynamics was first introduced several decades ago by Born and Infeld for the purpose of solving various problems of divergence appearing in the Maxwell theory [1], in recent years, the study of the nonlinear electrodynamics got a new impetus. Strong motivation comes from developments in string/M-theory, which is a promising approach to quantum gravity. It has been shown that the Born-Infeld theory naturally arises in the low energy limit of the open string theory [2,3]. The Born-Infeld action including a dilaton and an axion field, appears in the coupling of an open superstring and an Abelian gauge field theory [2]. This action, describing a Born-Infeld-dilaton-axion system coupled to Einstein gravity, can be considered as a nonlinear extension in the Abelian field of Einstein-Maxwell-dilaton-axion gravity. Although one can consistently truncate such models, the presence of the dilaton field cannot be ignored if one considers coupling of the gravity to other gauge fields, and therefore one remains with Einstein-Born-Infeld gravity in the presence of a dilaton field. Many attempts have been done to construct solutions of Einstein-Born-Infeld-dilaton (EBId) gravity [4,5,6,7,8,9,10,11,12]. The appearance of the dilaton field changes the asymptotic behavior of the solutions to be neither asymptotically flat nor (A)dS. The motivation for studying non asymptotically flat nor (A)dS solutions of Einstein gravity comes from the fact that, these kind of solutions can shed some light on the possible extensions...
of AdS/CFT correspondence. Indeed, it has been speculated that the linear dilaton spacetimes, which arise as near-horizon limits of dilatonic black holes, might exhibit holography [13]. Black hole spacetimes which are neither asymptotically flat nor (A)dS have been explored widely in the literature [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. In the absence of dilaton field, exact solutions of Einstein-Born-Infeld theory with/without cosmological constant have been constructed in [26, 27, 28, 29, 30, 31, 32, 33]. In the scalar-tensor theories of gravity, black hole solutions coupled to a Born-Infeld nonlinear electrodynamics have also been studied recently in [34].

On the other hand, it is a general belief that in four dimensions the topology of the event horizon of an asymptotically flat stationary black hole is uniquely determined to be the two-sphere $S^2$ [35, 36]. Hawking’s theorem requires the integrated Ricci scalar curvature with respect to the induced metric on the event horizon to be positive [35]. This condition applied to two-dimensional manifolds determines uniquely the topology. The “topological censorship theorem” of Friedmann, Schleich and Witt is another indication of the impossibility of non-spherical horizons [37, 38]. However, when the asymptotic flatness of spacetime is violated, there is no fundamental reason to forbid the existence of static or stationary black holes with nontrivial topologies. It has been shown that for asymptotically AdS spacetime, in the four-dimensional Einstein-Maxwell theory, there exist black hole solutions whose event horizons may have zero or negative constant curvature and their topologies are no longer the two-sphere $S^2$. The properties of these black holes are quite different from those of black holes with usual spherical topology horizon, due to the different topological structures of the event horizons. Besides, the black hole thermodynamics is drastically affected by the topology of the event horizon. It was argued that the Hawking-Page phase transition [39] for the Schwarzschild-AdS black hole does not occur for locally AdS black holes whose horizons have vanishing or negative constant curvature, and they are thermally stable [40]. The studies on the topological black holes have been carried out extensively in many aspects [41, 42, 43, 44, 45, 46, 47, 48, 49, 50]. In this Letter we will construct a new analytic solutions in four dimensional EBId theory. These solutions which describe dilaton black holes with nonlinear electrodynamics, and nontrivial topology, have unusual asymptotics. They are neither asymptotically flat nor (A)dS. We compute the conserved quantities of these solutions and find out that they satisfy the first law of black hole thermodynamics.
II. BASIC EQUATIONS AND SOLUTIONS

We examine the action in which gravity is coupled to dilaton and Born-Infeld fields

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( \mathcal{R} - 2g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi) + L(F, \Phi) \right), \]  

(1)

where \( \mathcal{R} \) is the Ricci scalar curvature, \( \Phi \) is the dilaton field and \( V(\Phi) \) is a potential for \( \Phi \). The Born-Infeld \( L(F, \Phi) \) part of the action is given by

\[ L(F, \Phi) = 4\beta^2 e^{2\alpha\Phi} \left( 1 - \sqrt{1 + \frac{e^{-4\alpha\Phi}F^2}{2\beta^2}} \right). \]  

(2)

Here, \( \alpha \) is a constant determining the strength of coupling of the scalar and electromagnetic field and \( F^2 = F_{\mu\nu}F^{\mu\nu} \), where \( F_{\mu\nu} \) is the electromagnetic field tensor. \( \beta \) is called the Born-Infeld parameter with dimension of mass. It is worth noting that we have adopted, following [4], the open string version of the Born-Infeld action coupled to a dilaton field since we would like to examine the pure electric case. Clearly, this version of the Born-Infeld-dilaton action does not enjoy electric-magnetic duality [8]. This form for the Born-Infeld-dilaton term have been investigated previously by a number of authors (see e.g. [4, 5, 6, 7]). In the limit \( \beta \to \infty \), \( L(F, \Phi) \) reduces to the standard Maxwell field coupled to a dilaton field

\[ L(F, \Phi) = -e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu}. \]  

(3)

On the other hand, \( L(F, \Phi) \to 0 \) as \( \beta \to 0 \). It is convenient to set

\[ L(F, \Phi) = 4\beta^2 e^{2\alpha\Phi} \mathcal{L}(Y), \]  

(4)

where

\[ \mathcal{L}(Y) = 1 - \sqrt{1 + Y}, \]  

(5)

\[ Y = \frac{e^{-4\alpha\Phi}F^2}{2\beta^2}. \]  

(6)

By varying the action (1) with respect to the gravitational field \( g_{\mu\nu} \), the dilaton field \( \Phi \) and the electromagnetic field \( A_\mu \) we obtain the equations of motion

\[ \mathcal{R}_{\mu\nu} = 2\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}g_{\mu\nu}V(\Phi) - 4e^{-2\alpha\Phi}\partial_\gamma\mathcal{L}(Y)F_{\mu\gamma}F^{\gamma}_\nu \]  

\[ + 2\beta^2 e^{2\alpha\Phi} \left[ 2Y\partial_\gamma\mathcal{L}(Y) - \mathcal{L}(Y) \right] g_{\mu\nu}, \]  

(7)

\[ \nabla^2 \Phi = \frac{1}{4} \frac{\partial V}{\partial \Phi} + 2\alpha\beta^2 e^{2\alpha\Phi} \left[ 2Y\partial_\gamma\mathcal{L}(Y) - \mathcal{L}(Y) \right], \]  

(8)
\[ \nabla_\mu \left( e^{-2\alpha \Phi} \partial_\mu \mathcal{L}(Y) F^{\mu\nu} \right) = 0. \] (9)

Note that in the case of the linear electrodynamics with \( \mathcal{L}(Y) = \frac{-1}{2} Y \), the system of equations (7)-(9) reduce to the well-known equations of Einstein-Maxwell-dilaton (EMd) gravity [16].

We would like to find topological solutions of the above field equations. The most general such metric can be written in the form

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) d\Omega_k^2, \] (10)

where \( f(r) \) and \( R(r) \) are functions of \( r \) which should be determined, and \( d\Omega_k^2 \) is the line element of a two-dimensional hypersurface \( \Sigma \) with constant curvature,

\[ d\Omega_k^2 = \begin{cases} 
  d\theta^2 + \sin^2 \theta d\phi^2, & \text{for } k = 1, \\
  d\theta^2 + \theta^2 d\phi^2, & \text{for } k = 0, \\
  d\theta^2 + \sinh^2 \theta d\phi^2, & \text{for } k = -1.
\] (11)

For \( k = 1 \), the topology of the event horizon is the two-sphere \( S^2 \), and the spacetime has the topology \( R^2 \times S^2 \). For \( k = 0 \), the topology of the event horizon is that of a torus and the spacetime has the topology \( R^2 \times T^2 \). For \( k = -1 \), the surface \( \Sigma \) is a 2-dimensional hypersurface \( H^2 \) with constant negative curvature. In this case the topology of spacetime is \( R^2 \times H^2 \). First of all, the electromagnetic fields equation (9) can be integrated immediately, where all the components of \( F_{\mu\nu} \) are zero except \( F_{tr} \):

\[ F_{tr} = \frac{\beta q e^{2\alpha \Phi}}{\sqrt{\beta^2 (rR)^4 + q^2}}, \] (12)

where \( q \) is an integration constant related to the electric charge of the black hole. Defining the electric charge via \( Q = \frac{1}{4\pi} \int e^{-2\alpha \Phi} \ast F d\Omega \), we get

\[ Q = \frac{q \omega}{4\pi}, \] (13)

where \( \omega \) represents the area of the constant hypersurface \( \Sigma \). It is worthwhile to note that the electric field is finite at \( r = 0 \). This is expected in Born-Infeld theories. Meanwhile it is interesting to consider three limits of Eq. (12). First, for large \( \beta \) (where the BI action reduces to Maxwell case) we have \( F_{tr} = q e^{2\alpha \Phi}/(rR)^2 \) as presented in [16]. On the other hand, if \( \beta \to 0 \) we get \( F_{tr} = 0 \). Finally, in the absence of the dilaton field \( (\alpha = 0) \), it reduces to the case of Einstein-Born-Infeld theory [31]

\[ F_{tr} = \frac{\beta q}{\sqrt{\beta^2 r^4 + q^2}}. \] (14)
Our aim here is to construct exact topological solutions of the EBI\textsubscript{d} theory with an arbitrary dilaton coupling constant $\alpha$. The case in which we find topological solutions of physically interest is to take the dilaton potential of the form

$$V(\Phi) = 2\Lambda_0 e^{2\zeta_0 \Phi} + 2\Lambda e^{2\zeta \Phi}, \quad (15)$$

where $\Lambda_0$, $\Lambda$, $\zeta_0$ and $\zeta$ are constants. This kind of potential was previously investigated by a number of authors both in the context of Friedman-Robertson-Walker (FRW) scalar field cosmologies [51] and EM\textsubscript{d} black holes (see e.g. [10, 11, 16, 25]). In order to solve the system of equations (7) and (8) for three unknown functions $f(r)$, $R(r)$ and $\Phi(r)$, we make the ansatz [21]

$$R(r) = e^{\alpha \Phi}. \quad (16)$$

A motivation for taking this ansatz is that in the absence of a dilaton field ($\alpha = 0$) it reduces to $R(r) = 1$, as one expected (see Eq. 10). Another motivation comes from the form of the electromagnetic field equation provided one write it explicitly in terms of partial derivative [21]. Inserting (16), the electromagnetic field (12) and the metric (10) into the field equations (7) and (8), one can show that these equations have the following solutions

$$f(r) = -k \frac{\alpha^2 + 1}{\alpha^2 - 1} b^{-2\gamma} r^{2\gamma} - \frac{m}{r^{1-2\gamma}} + \frac{(\Lambda - 2\beta^2)(\alpha^2 + 1)^2 b^{2\gamma}}{\alpha^2 - 3} r^{2-2\gamma}$$

$$-2\beta^2 (\alpha^2 + 1) b^{2\gamma} r^{2\gamma - 1} \int r^{2(1-2\gamma)} \sqrt{1 + \eta} dr, \quad (17)$$

$$\Phi(r) = \frac{\alpha}{\alpha^2 + 1} \ln \left( \frac{b}{r} \right), \quad (18)$$

where $b$ is an arbitrary constant, $\gamma = \alpha^2/(1 + \alpha^2)$, and

$$\eta \equiv \frac{q^2}{\beta^2 b^{4\gamma} r^{4(1-\gamma)}}. \quad (19)$$

In the above expression, $m$ appears as an integration constant and is related to the ADM (Arnowitt-Deser-Misner) mass of the black hole. According to the definition of mass due to Abbott and Deser [52], the mass of the solution (17) is

$$M = \frac{b^{2\gamma} m \omega}{8\pi(\alpha^2 + 1)}. \quad (20)$$

The above solutions will fully satisfy the system of equations (7) and (8) provided we have $\zeta_0 = 1/\alpha$, $\zeta = \alpha$ and $\Lambda_0 = kb^{-2} \alpha^2/(\alpha^2 - 1)$. Notice that $\Lambda$ remains as a free parameter which plays the role of
the cosmological constant. For later convenience, we redefine it as \( \Lambda = -3/l^2 \), where \( l \) is a constant with dimension of length. One may refer to \( \Lambda \) as the cosmological constant, since in the absence of the dilaton field (\( \alpha = 0 \)) the action reduces to the action of EBI gravity with cosmological constant \[31, 32\]. The integral can be done in terms of hypergeometric function and can be written in a compact form. The result is

\[
\begin{align*}
    f(r) &= -k \frac{\alpha^2 + 1}{\alpha^2 - 1} b^{-2\gamma} r^{2\gamma} - \frac{m}{r^{1-2\gamma}} + \frac{\Lambda (\alpha^2 + 1)^2 b^{2\gamma}}{\alpha^2 - 3} r^{-2\gamma - 2} \\
    &\quad - \frac{2\beta^2 (\alpha^2 + 1)^2 b^{2\gamma}}{\alpha^2 - 3} r^{-2\gamma} \times \left( 1 - \frac{1}{2} F_1 \left( \frac{-1}{2}, \frac{1}{4}, \frac{1}{4}, -\eta \right) \right).
\end{align*}
\]

(21)

It is worthwhile to compare our solution to those presented in [8]. The authors of [8] investigated asymptotically flat static and spherically symmetric solutions in Born-Infeld-dilaton theory in the absence of the dilaton potential. They adopted the \( SL(2, R) \) symmetry version of the Born-Infeld-dilaton action. Imposing the asymptotically flat condition, they found a class of black hole solutions in terms of series expansion and explored their properties. Our solutions are different from their solutions for several reasons. First, the coupling of the dilaton to the Born-Infeld term in the action, for which there is no symmetry between electric and magnetic solutions, in contrast to the action of [8] which has \( SL(2, R) \) S-dual symmetry. Second, the presence of the Liouville-type potentials (the negative effective cosmological constant) in our case, which play a crucial role in the existence of these topological dilaton black holes. And third the asymptotic behaviour of the solution which changes to be neither flat nor (A)dS due to the presence of the dilaton potential. One may note that as \( \beta \rightarrow \infty \) these solutions reduce to the topological black hole solutions in EMD gravity given in Ref. [46]. In the absence of a dilaton field (\( \alpha = \gamma = 0 \)), the above solutions reduce to

\[
\begin{align*}
    f(r) &= k - \frac{m}{r} + \frac{r^2}{l^2} + \frac{2\beta^2}{3} r^2 \times \left( 1 - \frac{1}{2} F_1 \left( \frac{-1}{2}, \frac{3}{4}, \frac{1}{4}, -\eta \right) \right),
\end{align*}
\]

(22)

which describes an asymptotically AdS topological Born-Infeld black hole with positive, zero or negative constant curvature hypersurface [32]. Using the fact that \( 2F_1(a, b, c, z) \) has a convergent series expansion for \( |z| < 1 \), we can find the behavior of (21) for large \( r \). This is given by

\[
\begin{align*}
    f(r) &= -k \frac{\alpha^2 + 1}{\alpha^2 - 1} b^{-2\gamma} r^{2\gamma} - \frac{m}{r^{1-2\gamma}} + \frac{\Lambda (\alpha^2 + 1)^2 b^{2\gamma}}{l^2 (\alpha^2 - 3)} r^{-2\gamma - 2} \\
    &\quad + \frac{(\alpha^2 + 1)b^{-2\gamma} q^2}{r^{2\gamma - 2}} \times \frac{(\alpha^2 + 1)^2 b^{-6\gamma} q^4}{4\beta^2 (\alpha^2 + 5) \gamma^6 (1-\gamma)}.
\end{align*}
\]

(23)

Note that for \( \alpha = \gamma = 0 \), the above expression reduces to

\[
\begin{align*}
    f(r) &= k - \frac{m}{r} + \frac{r^2}{l^2} + \frac{q^2}{r^2} - \frac{1}{20\beta^2} \frac{q^4}{r^6},
\end{align*}
\]

(24)
FIG. 1: The function $f(r)$ versus $r$ for $\alpha = 0.5$, $m = 2$, $\beta = 1$ and $q = 1$. $k = -1$ (bold line), $k = 0$ (continuous line) and $k = 1$ (dashed line).

FIG. 2: The function $f(r)$ versus $r$ for $m = 2$, $\beta = 1$, $q = 1$ and $k = 0$. $\alpha = 0$ (bold line), $\alpha = 0.38$ (continuous line) and $\alpha = 0.6$ (dashed line).

which has the form of topological charged black hole in AdS spacetime in the limit $\beta \to \infty$ \cite{43,44}. The last term in the right hand side of the above expression is the leading Born-Infeld correction to the topological black hole in the large $\beta$ limit.

Next we study the physical properties of these solutions. For this purpose, we first look for the curvature singularities. In the presence of dilaton field, the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $r = 0$, it is finite for $r \neq 0$ and goes to zero as $r \to \infty$. Thus, there is an essential singularity located at $r = 0$. The spacetime is neither asymptotically flat nor (A)dS. It is notable to mention that in the $k = \pm 1$ cases this solution does not exist for the string case where $\alpha = 1$. As one can see from Eq. (21), the solution is ill-defined for $\alpha = \sqrt{3}$. The cases with $\alpha < \sqrt{3}$ and $\alpha > \sqrt{3}$ should be considered separately. In the first case where $\alpha < \sqrt{3}$, there exist a cosmological horizon for $\Lambda > 0$, while there is no cosmological horizons if $\Lambda < 0$. In the latter case ($\alpha < \sqrt{3}$...
FIG. 3: The function $m(r_h)$ versus $r_h$ for $\beta = 1$, $q = 1$ and $k = 0$. $\alpha = 0$ (bold line), $\alpha = 0.5$ (continuous line) and $\alpha = 0.7$ (dashed line).

FIG. 4: The function $m(r_h)$ versus $r_h$ for $\beta = 1$, $\alpha = 0.5$ and $k = 0$. $q = 0$ (bold line), $q = 1$ (continuous line) and $q = 1.5$ (dashed line).

FIG. 5: The function $m(r_h)$ versus $r_h$ for $\beta = 1$, $\alpha = 0.5$ and $k = -1$. $q = 0$ (bold line), $q = 1$ (continuous line) and $q = 1.5$ (dashed line).
and $\Lambda < 0$) the spacetimes associated with the solution (21) exhibit a variety of possible casual structures depending on the values of the metric parameters $\alpha$, $m$, $q$ and $k$ (see figs. 1, 2). For simplicity in these figures, we kept fixed the other parameters $l = b = 1$. Figure 1 shows that for fixed value of other parameters, the number of horizons increase with decreasing the constant curvature $k$, while one can see from figure 2 that with increasing $\alpha$, the number of horizons decrease.

In summary, these figures show that our solutions can represent topological black hole, with two horizons, an extreme topological black hole or a naked singularity depending on the values of the metric parameters. In the second case where $\alpha > \sqrt{3}$, the spacetime has a cosmological horizon. Although, in principle, the casual structure of the spacetime can be obtained by finding the roots of $f(r) = 0$, but because of the nature of the dilaton and nonlinear electrodynamic fields in (21), it is not possible to find analytically the location of the horizons. To have further understanding on the nature of the horizons, we plot in figures 3, 4, 5 the mass parameter $m$ as a function of the horizon radius $r_h$ for different value of dilaton coupling constant $\alpha$, charge parameter $q$ and curvature constant $k$. Again, we have fixed $l = b = 1$, for simplicity. It is easy to show that the mass parameter of the black hole can be expressed in terms of the horizon radius $r_h$ as

$$m(r_h) = -\frac{k(\alpha^2 + 1)b^{2\gamma}}{\alpha^2 - 1}r_h + \frac{\Lambda(\alpha^2 + 1)^2b^{2\gamma}}{(\alpha^2 - 3)}r_h^{3-4\gamma}$$

$$-\frac{2\beta^2(\alpha^2 + 1)^2b^{2\gamma}}{(\alpha^2 - 3)}r_h^{3-4\gamma} \times \left(1 - 2F_1\left(-\frac{1}{2}, \frac{\alpha^2 - 3}{4}, \frac{\alpha^2 + 1}{4}, -\eta_h\right)\right).$$

(25)

where $\eta_h = \eta(r = r_h)$. These figures show that for a given value of $\alpha$, the number of horizons depend on the choice of the value of the mass parameter $m$. We see that, up to a certain value of the mass parameter $m$, there are two horizons, and as we decrease the $m$ further, the two horizons meet. In this case we get extremal black hole (see the next section). Figure 3 also shows that, as an example for $k = 0$, with increasing $\alpha$, the $m_{\text{ext}}$ also increases. It is worth noting that in the limit $r_h \to 0$ we have a nonzero value for the mass parameter $m$. This is in contrast to the Schwarzschild black holes in which mass parameter goes to zero as $r_h \to 0$. As we have shown in figure 4, this is due to the effect of the charge parameter $q$ and the nature of the Born-Infeld field, and in the case $q=0$, the mass parameter $m$ goes to zero as $r_h \to 0$. Besides in the case $k = -1$ and $q = 0$, our solution has strange properties. In this case, one can see from figure 5 that the mass of the solution can be negative, however one still has a topological black hole solution with negative curvature horizons. It was argued that this kind of black hole with negative mass can also be formed as a result of gravitational collapse [53].
III. THERMODYNAMICS OF TOPOLOGICAL BLACK HOLES

In this section we would like to study the thermodynamical properties of the topological dilaton black holes we have just found. The Hawking temperature of the topological black hole on the outer horizon \( r_+ \) can be calculated using the relation

\[
T_+ = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi},
\]

where \( \kappa \) is the surface gravity. Then, one can easily show that

\[
T_+ = -\frac{(\alpha^2 + 1)b^{2\gamma}r_{+}^{1-2\gamma}}{4\pi} \left( \frac{kb^{-2\gamma}}{(\alpha^2 - 1)+2\alpha^2(1-\sqrt{1+\eta_+})} + \Lambda - 2\beta^2(1-\sqrt{1+\eta_+}) \right)
\]

\[
= -\frac{k b^{-2\gamma} r_{+}^{2\gamma - 1} - (\alpha^2 - 3)m}{2\pi(\alpha^2 + 1)r_{+}^{2\gamma - 2}} - \frac{q^2 b^{-2\gamma} r_{+}^{2\gamma - 3}}{\pi} \times 2 F_1 \left( \frac{1}{2}, \frac{\alpha^2 + 1}{4}, \frac{\alpha^2 + 5}{4}, -\eta_+ \right),
\]

where \( \eta_+ = \eta(r = r_+) \). The temperature of the black hole is zero in the case of extremal black hole. It is easy to show that

\[
m_{\text{ext}} = -\frac{2k(\alpha^2 + 1)b^{-2\gamma}}{\alpha^2 - 3} r_{+} - \frac{4q^2(\alpha^2 + 1)b^{-2\gamma}}{\alpha^2 - 3} \times 2 F_1 \left( \frac{1}{2}, \frac{\alpha^2 + 1}{4}, \frac{\alpha^2 + 5}{4}, -\eta_+ \right).
\]

Indeed, the metric of Eqs. (10) and (21) can describe a topological dilaton black hole with inner and outer event horizons located at \( r_- \) and \( r_+ \), provided \( m > m_{\text{ext}} \), an extreme topological black hole in the case of \( m = m_{\text{ext}} \), and a naked singularity if \( m < m_{\text{ext}} \). It is worth noting that in the absence of a nontrivial dilaton field \( \alpha = \gamma = 0 \), expressions (25)-(28) reduce to that of an asymptotically AdS topological black hole in Born-Infeld theory [32].

The entropy of the topological black hole still obeys the so called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area [54]. This near universal law applies to almost all kinds of black holes, including dilaton black holes, in Einstein gravity [55]. It is a matter of calculation to show that the entropy of the topological black hole is

\[
S = \frac{b^{2\gamma} r_{+}^{2(1-\gamma)}}{4}.
\]

The electric potential \( U \), measured at infinity with respect to the horizon, is defined by

\[
U = A_{\mu} \chi^\mu \big|_{r=\infty} - A_{\mu} \chi^\mu \big|_{r=r_+},
\]

where \( \chi = \partial_t \) is the null generator of the horizon. One can easily show that the gauge potential \( A_t \) corresponding to the electromagnetic field [12] can be written as

\[
A_t = \frac{q}{r} \times 2 F_1 \left( \frac{1}{2}, \frac{\alpha^2 + 1}{4}, \frac{\alpha^2 + 5}{4}, -\eta \right).
\]
Therefore the electric potential may be obtained as

\[ U = \frac{q}{r_+} \times _2F_1 \left( \frac{1}{2}, \frac{\alpha^2 + 1}{4}, \frac{\alpha^2 + 5}{4}, -\eta_+ \right). \] (32)

In figures 6 and 7 we have shown the behavior of electric potential \( U \) as a function of horizon radius. As one can see from these figures, \( U \) is finite even for \( r_+ = 0 \).

Then, we consider the first law of thermodynamics for the topological black hole. In order to do this, we obtain the mass \( M \) as a function of extensive quantities \( S \) and \( Q \). Using the expression for the charge, the mass and the entropy given in Eqs. (13), (20) and (29) and the fact that \( f(r_+) = 0 \), one can obtain a Smarr-type formula as

\[
M(S, Q) = -\frac{kb^{-\alpha^2}(4S)^{(\alpha^2+1)/2}}{8\pi(\alpha^2 - 1)} - \frac{3(\alpha^2 + 1)b^{\alpha^2}}{8\pi l^2(\alpha^2 - 3)}(4S)^{(3-\alpha^2)/2} \\
- \frac{\beta^2(\alpha^2 + 1)b^{\alpha^2}}{4\pi(\alpha^2 - 3)}(4S)^{(3-\alpha^2)/2} \times \left( 1 - _2F_1 \left( -\frac{1}{2}, \frac{\alpha^2 - 3}{4}, \frac{\alpha^2 + 1}{4}, \frac{-\pi^2 Q^2}{\beta^2 S^2} \right) \right).
\] (33)

We can regard the parameters \( S \), and \( Q \) as a complete set of extensive parameters for the mass \( M(S, Q) \) and define the intensive parameters conjugate to \( S \) and \( Q \). These quantities are the temperature and the electric potential

\[
T = \left( \frac{\partial M}{\partial S} \right)_Q, \quad U = \left( \frac{\partial M}{\partial Q} \right)_S.
\] (34)

Numerical calculations show that the intensive quantities calculated by Eq. (34) coincide with Eqs. (27) and (32). Thus, these thermodynamic quantities satisfy the first law of black hole thermodynamics

\[
dM = T dS + U dQ.
\] (35)

IV. CONCLUSIONS

Black holes in AdS spacetimes are quite different from their counterparts in asymptotically flat spacetime. In AdS spacetimes, there are a kind of black holes with nontrivial topology. The event horizons of these black holes can be a positive, zero or negative constant curvature surface. These black holes are generally called topological black holes. The construction and analysis of these exotic black holes in AdS space is a subject of much recent interest. This is primarily due to their
relevance for the AdS/CFT correspondence. In this letter, we further generalized these exotic black hole solutions by including a dilaton and nonlinear electromagnetic fields in the action. In contrast to the topological black holes in the Einstein-Maxwell theory, which are asymptotically AdS, the topological dilaton black holes we found here, are neither asymptotically flat nor \((A)dS\). Indeed, the Liouville-type potentials (the negative effective cosmological constant) plays a crucial role in the existence of these black hole solutions, as the negative cosmological constant does in the Einstein-Maxwell theory. These solutions do not exist for the string case where \(\alpha = 1\) provided \(k = \pm 1\). Besides they are ill-defined for \(\alpha \neq \sqrt{3}\). In the absence of a dilaton field \((\alpha = \gamma = 0)\), our solutions reduce to the four-dimensional topological black hole solutions of Born-Infeld theory [32], while in the limit \(\beta \to \infty\) they reduce to the topological black holes in Einstein-Maxwell-dilaton gravity [46] (see also [45]). We showed that our solutions can describe topological Born-Infeld black
hole with inner and outer event horizons, an extreme topological black hole or a naked singularity provided the parameters of the solutions are chosen suitably. We also computed the charge, mass, temperature, entropy and electric potential of the topological dilaton black holes and verified that these quantities satisfy the first law of black hole thermodynamics.

Although, in this Letter we constructed the topological Born-Infeld-dilaton black holes in the presence of Liouville-type potentials for the dilaton field, and discussed their thermodynamical properties, many issues however still remain to be investigated. We know that Reissner-Nordstrom AdS black holes undergo Hawking-Page phase transition. This transition gets modified as we include a dilaton and nonlinear electrodynamics fields corrections into account. Indeed, the dilaton field, as well as the Born-Infeld field, can create an unstable phase for the solutions [11]. A detail study on this issue for the case of topological dilaton black holes in the presence of a nonlinear electrodynamics will be addressed elsewhere [56]. It would be also of great interest to generalize these solutions to higher dimensional Einstein-Born-Infeld-dilaton gravity [56].

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