Dynamical Properties of one dimensional Mott Insulators

Davide Controzzi\(^{(a)}\), Fabian H.L. Essler\(^{(b)}\) and Alexei M. Tsvelik\(^{(a)}\)

\(^{(a)}\) Department of Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK
\(^{(b)}\) Department of Physics, Warwick University, Coventry CV4 7AL, UK

Abstract

At low energies the charge sector of one dimensional Mott insulators can be described in terms of a quantum Sine-Gordon model (SGM). Using exact results derived from integrability it is possible to determine dynamical properties like the frequency dependent optical conductivity. We compare the exact results to perturbation theory and renormalisation group calculations. We also discuss the application of our results to experiments on quasi-1D organic conductors.

Lecture given by FHLE at the NATO ASI/EC summer school “New Theoretical Approaches to Strongly Correlated Systems”, Sir Isaac Newton Institute for Mathematical Sciences, Cambridge April 2000

1 1D Mott insulators

The Mott metal-insulator transition is a paradigm for the importance of electron-electron interactions in condensed matter systems. It occurs in a variety of actual materials and has attracted much attention over the last fifty years \(1\). The underlying mechanism that drives the transition is by now well understood, but details on e.g. transport properties remain largely unknown in \(D = 2\) and \(D = 3\) due to the lack of nonperturbative methods for treating strongly correlated electron systems. The situation is more fortunate in two cases: \(D = \infty\), where much progress has been made in recent years \(3\) and \(D = 1\), where nonperturbative methods permit essentially a full solution of the problem. The 1D case is the one we will be concerned with here. A full characterization of the Mott insulating phase requires the knowledge of dynamical correlation functions. The frequency dependent optical conductivity \(\sigma(\omega)\) is one of the most important examples from an experimental point of view. The behaviour of \(\sigma(\omega)\) in the metallic regime is easily understood in terms of the Tomonaga-Luttinger theory \(3, 4\). The situation in the Mott insulating phase is much more complicated due to the spectral gap that is dynamically generated by the electron-electron interactions. Here \(\sigma(\omega)\) has until now only been studied by perturbative methods \(5, 6\), which break down in the most interesting regime of frequencies close to the optical gap. In these proceedings we use methods of integrable quantum field theory to determine \(\sigma(\omega)\) in 1D Mott insulators for all frequencies much smaller than the bandwidth, which is the large scale in the field theory approach to the problem. Some of the results presented here have already appeared in \(7\).

The paradigm of a 1D Mott insulator is the Hubbard model

\[
H = -t \sum_{l,\sigma} \left( c_{l,\sigma}^\dagger c_{l+1,\sigma} + \text{h.c.} \right) + U \sum_l n_{l,\uparrow} n_{l,\downarrow} .
\]  

(1)

Here \(c_{l,\sigma}\) are fermionic annihilation operators of spin \(\sigma = \uparrow, \downarrow\) at site \(l\) of a one-dimensional chain and \(n_{l,\sigma} = c_{l,\sigma}^\dagger c_{l,\sigma}\). The 1D Hubbard model is solvable by the Bethe Ansatz \(8\) and the exact solution establishes the presence of a Mott transition at half-filling (one electron per site) and \(U = 0\). In other words, for a half-filled band the model is insulating for any nonzero value of the on-site repulsion \(U\), whereas it is metallic for a less than half-filled band. Many properties have been determined exactly \(8\), but the asymptotic behaviour of dynamical correlation functions are at present known only in the metallic \(10\) and the gas phase \(11\). In order to make progress in the insulating phase and to clearly expose the mechanism underlying the transition it is very useful to consider the scaling limit of the half-filled Hubbard model. This field theory limit corresponds to weak coupling and can be obtained directly...
from the exact spectrum and scattering matrix of the lattice model [12, 13]. It is defined by
\[
t \to \infty , \quad U/t \to 0 , \quad M = \frac{8t}{\pi} \sqrt{\frac{U}{4t}} \exp (-2\pi t/U) \text{ fixed.}
\] (2)
and has been studied by many authors [14]. On an operator level the field theory can be constructed
along the lines of e.g. chapter 15 of [3]. One starts by splitting the electron operators into fast and slow
components
\[
c_{l,\sigma} \to \sqrt{a_0} [\exp (ik_F x) R_\sigma (x) + \exp (-ik_F x) L_\sigma (x)] .
\] (3)
Here \( k_F \) is the Fermi momentum (it is \( \pi/2 \) for the half-filled band), \( R_\sigma \) and \( L_\sigma \) are right and left moving
electron fields and \( x = la_0 \), where \( a_0 \) is the lattice spacing. Inserting this prescription into the Hamiltonian
(\( \mathcal{H} \)) one obtains
\[
\mathcal{H} = \sum_\sigma v_F \int dx [L_\sigma i \partial x L_\sigma^\dagger - R_\sigma i \partial x R_\sigma^\dagger] + g \int dx \left[ \mathbf{J} \cdot \mathbf{J} - \mathbf{J} \cdot \mathbf{J} \right]
+ \frac{g}{6} \int dx \left[ : \mathbf{I} \cdot \mathbf{I} : + : \mathbf{I} \cdot \mathbf{I} : - : \mathbf{J} \cdot \mathbf{J} : - : \mathbf{J} \cdot \mathbf{J} : \right] ,
\] (4)
where \( v_F = 2ta_0 \) is the Fermi velocity and \( g = 2Ua_0 \). Here \( \mathbf{J} \) and \( \mathbf{I} \) are the chiral components of SU(2)
spin and pseudospin currents
\[
I^3 = \frac{1}{2} \sum_\sigma : L_\sigma^3 L_\sigma : , \quad I^+ = L_\uparrow^3 L_\uparrow^\dagger ,
\]
\[
\bar{I}^3 = \frac{1}{2} \sum_\sigma : R_\sigma^3 R_\sigma : , \quad \bar{I}^+ = R_\uparrow^3 R_\uparrow^\dagger ,
\]
\[
J^3 = \frac{1}{2} \left( L_\uparrow^3 L_\downarrow^\dagger - L_\downarrow^3 L_\uparrow^\dagger \right) , \quad J^+ = L_\uparrow^3 L_\uparrow^\dagger ,
\]
\[
\bar{J}^3 = \frac{1}{2} \left( R_\uparrow^3 R_\downarrow^\dagger - R_\downarrow^3 R_\uparrow^\dagger \right) , \quad \bar{J}^+ = R_\uparrow^3 R_\uparrow^\dagger .
\] (5)
Note that the Hamiltonian (\( \mathcal{H} \)) displays the required SO(4) symmetry [15] of the half-filled Hubbard
model. By employing the Sugawara construction, the Hamiltonian (\( \mathcal{H} \)) can now be split into two parts,
corresponding to the spin and charge sectors respectively [3]
\[
\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s ,
\]
\[
\mathcal{H}_c = \frac{2\pi v_c}{3} \int dx \left[ : \mathbf{I} \cdot \mathbf{I} : + : \mathbf{I} \cdot \mathbf{I} : \right] + g \int dx \mathbf{I} \cdot \mathbf{I} ,
\]
\[
\mathcal{H}_s = \frac{2\pi v_s}{3} \int dx \left[ : \mathbf{J} \cdot \mathbf{J} : + : \mathbf{J} \cdot \mathbf{J} : \right] - g \int dx \mathbf{J} \cdot \mathbf{J} .
\] (6)
Here \( v_s = v_F - Ua_0/2\pi \) and \( v_c = v_F + Ua_0/2\pi \). Apart from the (marginally) irrelevant current-current
interaction in the spin sector and the difference in spin and charge velocities, the Hamiltonian (\( \mathcal{H} \)) is
identical to the one of the SU(2) Thirring model [16]. The latter is integrable [17, 18] and using its
exact solution it is possible to determine dynamical correlation functions via the formfactor bootstrap
approach. For example the optical conductivity was determined in this way and compared to numerical
dynamical density matrix renormalisation group computations on the Hubbard model in [19]. Here we
consider the optical conductivity for a more general model. We note that the optical conductivity is
rather special in that for all cases considered here the electric current operator couples only to the charge
sector, which greatly simplifies all calculations. Let us now consider an extended Hubbard model
\[
H_{\text{ext}} = -t \sum_{l,\sigma} (c_{l+1,\sigma} c_{l,\sigma} + \text{h.c.}) + U \sum_l n_{l,\uparrow} n_{l,\downarrow} + V \sum_l n_l n_{l+1} ,
\] (7)
where $n_l = n_{l,^+} + n_{l,^\downarrow}$. By repeating the above analysis we find that the scaling limit takes the form

$$H_{\text{ext}} = \mathcal{H}_c' + \mathcal{H}_s',$$

$$\mathcal{H}_c' = \frac{2\pi v_c'}{3} \int dx \left[ : \mathbf{I} \cdot \mathbf{I} : + : \mathbf{\bar{I}} \cdot \mathbf{\bar{I}} : \right] + \int dx \left[ g_\perp (I^+ \bar{I}^- + I^- \bar{I}^+) + g_\parallel I^2 \bar{I}^2 \right],$$

$$\mathcal{H}_s' = \frac{2\pi v_s}{3} \int dx \left[ : \mathbf{J} \cdot \mathbf{J} : + : \mathbf{\bar{J}} \cdot \mathbf{\bar{J}} : \right] - 2g_\perp \int dx \mathbf{J} \cdot \mathbf{\bar{J}}.$$  \hspace{1cm} (8)

where

$$g_\perp = (U - 2V) a_0, \quad g_\parallel = (2U + 12V) a_0, \quad v_c' = v_F + \frac{(U + 4V) a_0}{2\pi}. \hspace{1cm} (9)$$

Clearly spin-charge separation still holds, so that the two parts of the Hamiltonian (8) can be bosonized separately. Here we are interested in the charge sector only but note in passing that the spin sector is gapless if $U > 2V$ and gapped otherwise. Applying the standard bosonization rules to $\mathcal{H}_c'$ in (8) one arrives at the Hamiltonian of the SGM \cite{3}.

The electric current operator of the lattice models (1), (7) is given by

$$j = \frac{-i\epsilon t}{\hbar} \sum_{j,\sigma} \left[ c_{j+1,\sigma} c_{j,\sigma} + 1 - c_{j,\sigma} c_{j+1,\sigma} \right], \hspace{1cm} (10)$$

and does not commute with the above Hamiltonians. Using (3) this becomes

$$j = \frac{4\epsilon t}{\hbar} \int dx \left[ I^3(x) - \bar{I}^3(x) \right] \hspace{1cm} (11)$$

in the field theory limit. From now on we drop the factor $\epsilon t/\hbar$ which simply fixes the units in which we measure the current.

## 2 The Sine-Gordon model

The low energy physics of the charge sector of a general, pure, one-dimensional Mott insulator is described by the SGM as we have seen in the previous section for some specific examples. The action is given by

$$S_{\text{SG}} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \phi)^2 - 2\mu \cos(\beta \phi) \right\}. \hspace{1cm} (12)$$

Here we have chosen the normalization of the Bose field following \cite{20, 21} by specifying the short-distance behaviour of the two-point function as

$$\langle e^{i\alpha \phi(x)} e^{-i\alpha \phi(y)} \rangle \rightarrow |x - y|^{-4\alpha^2} \quad \text{as} \quad |x - y| \rightarrow 0. \hspace{1cm} (13)$$

The SGM posesses a conserved (topological) charge

$$Q = \int_{-\infty}^{\infty} j^0 dx = -\frac{\beta}{2\pi} \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial x} dx, \hspace{1cm} (14)$$

where

$$j^\mu = -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi \hspace{1cm} (15)$$

is the Noether current \cite{1}. The electric current (11) is proportional to the Noether current $j^1$

$$j = \sqrt{A} \partial_0 \phi, \hspace{1cm} (16)$$

\footnote{It is normalized such that solitons and antisolitons have charges $\pm 1$.}
where $A$ is some nonuniversal constant\(^2\). The SGM is integrable and has been studied in great detail over the last 25 years\([22, 23, 24]\). Let us review some results obtained from the exact solution\([23]\) that we will need in the following. First of all, some of the results obtained in the repulsive regime $\beta^2 > 1/2$ appear to depend on the regularization scheme\([24, 18]\) employed to deal with the UV divergences. Here we follow the method of\([18]\), which is very natural from a field-theory point of view.

The spectrum of the SGM depends on the value of the coupling constant $\beta^2$ or, alternatively, on

$$\xi = \frac{\beta^2}{1 - \beta^2}. \tag{17}$$

For $\beta^2 < 1$ the cosine term is relevant in the renormalization group (RG) sense and dynamically generates a spectral gap $M$ in the excitation spectrum. In the repulsive regime $1 < \xi < \infty$ the spectrum contains only charged particles of charge $Q = \pm 1$, which are called solitons and antisolitons. In this regime the spectral gap is related to the “optical gap” $\Delta$ i.e. the gap seen in the optical absorption spectra by $\Delta = 2M$. At the so-called “Luther-Emery”\([23]\) (LE) point $\xi = 1$ the SGM is equivalent to a free massive Dirac fermion. In this limit the solitons become non-interacting particles and as we will see the Mott insulator turns into a conventional band insulator. In the limit $\xi \rightarrow \infty$ the sine-Gordon model acquires an SU(2) symmetry and describes the charge sector of the Hubbard model at half-filling in the limit of weak interactions as discussed above. In the attractive regime $0 < \xi < 1$ excitonic soliton-antisoliton bound states are formed and the spectrum becomes more complicated. Here we constrain ourselves to the repulsive regime and refer to\([26]\) for results on dynamical correlation functions in extended Hubbard models that correspond to the attractive regime in the SGM.

As usual in a theory with relativistic dispersion $e(p) = \sqrt{p^2 + M^2}$ it is useful to parametrize the spectrum in terms of a rapidity variable $\theta$ defined by

$$p = M \sinh \theta, \; e = M \cosh \theta. \tag{18}$$

Let us distinguish solitons and antisolitons by an index $\varepsilon = \pm$. The exact 2-particle soliton-antisoliton scattering matrix is then given by\([27]\)

$$S_{\varepsilon_1 \varepsilon_2, \varepsilon_3 \varepsilon_4}^\pm(\theta) = S_{\varepsilon_3 \varepsilon_4, \varepsilon_1 \varepsilon_2}^\pm(\theta) = S_0(\theta), \tag{19}$$

The two-particle S-matrix\([19]\) completely specifies all scattering processes in the SGM as multi-particle scattering is purely elastic and factorizes into two-particle processes. A convenient formalism for the description of a dilute gas of particles with factorizable scattering is obtained in terms of the Zamolodchikov-Faddeev (ZF) algebra. The ZF algebra can be considered to be the logical extension of the algebra of creation and annihilation operators for free fermion or bosons to the case of interacting particles with factorizable scattering. The ZF algebra is usually introduced formally based on the knowledge of the exact spectrum and scattering matrix, which for the SGM was obtained in\([28, 29]\). For the SGM in the repulsive regime the ZF operators (and their hermitian conjugates) thus satisfy the following algebra

$$Z^{\varepsilon_1}(\theta_1) Z^{\varepsilon_2}(\theta_2) = S^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1' \varepsilon_2'}(\theta_1 - \theta_2) Z^{\varepsilon_1'}(\theta_2) Z^{\varepsilon_2'}(\theta_1),$$

\(^2\)We assume that $A$ is nonuniversal because the electric current is not a conserved quantity for the lattice model.
The resolution of the identity is given by

\[ Z_{\varepsilon_1}(\theta_1)Z_{\varepsilon_2}(\theta_2) = Z_{\varepsilon_1}^\dagger(\theta_2)Z_{\varepsilon_2}^\dagger(\theta_1)S_{\varepsilon_1,\varepsilon_2}(\theta_1 - \theta_2), \]

\[ Z^{\varepsilon_1}(\theta_1)Z^{\varepsilon_2}(\theta_2) = Z_{\varepsilon_2}^\dagger(\theta_2)S_{\varepsilon_1,\varepsilon_2}(\theta_2 - \theta_1)Z^{\varepsilon_1}(\theta_1) + (2\pi)\delta_{\varepsilon_2}^\varepsilon_1\delta(\theta_1 - \theta_2), \quad \varepsilon_1, \varepsilon_2 = \pm \frac{1}{2}. \tag{20} \]

Here the two-particle scattering matrices \( S_{\varepsilon_1,\varepsilon_2}(\theta) \) are defined in Eq.(19) and \( \varepsilon_j = \pm \). The factor \( 2\pi \) in the last equation stems from the normalization of the single particle asymptotic states (cf Eq.(24)).

Using the ZF generators a Fock space of states can be constructed as follows. The vacuum is defined by

\[ Z_\varepsilon(\theta)|0\rangle = 0. \tag{21} \]

Multiparticle states are then obtained by acting with strings of creation operators \( Z_\varepsilon(\theta) \) on the vacuum

\[ |\theta_n \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_1} = Z_{\varepsilon_n}^\dagger(\theta_n) \ldots Z_{\varepsilon_1}^\dagger(\theta_1)|0\rangle. \tag{22} \]

We note that (19) together with (22) implies that states with different orderings of two rapidities and indices \( \varepsilon_i \) are related in the following way

\[ |\theta_n \ldots \theta_k\theta_{k+1} \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_k \varepsilon_{k+1} \ldots \varepsilon_1} = S_{\varepsilon_k,\varepsilon_{k+1}}^\varepsilon_{k+1}(\theta_k - \theta_{k+1})|\theta_n \ldots \theta_k\theta_{k+1} \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_k \varepsilon_{k+1} \varepsilon_{k+2} \varepsilon_{k+3} \ldots \varepsilon_1}. \tag{23} \]

The resolution of the identity is given by

\[ \mathbb{1} = \sum_{n=0}^{\infty} \sum_{\varepsilon_i} \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n n!} |\theta_n \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_1} \langle 0| \langle 0| \theta_n \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_1} |0\rangle_{\varepsilon_n \ldots \varepsilon_1}. \tag{24} \]

### 3 Spectral representation of the optical conductivity

An efficient method for the computation of correlation functions in integrable, massive quantum field theories is given by the form factor approach. This approach is based on the spectral representation, that expresses correlation functions in terms of an infinite series over multi-particles states. The two-point correlation function of some operator \( \mathcal{O} \) can be written as

\[ \langle \mathcal{O}(x,t)\mathcal{O}(0,0) \rangle = \sum_{n=0}^{\infty} \sum_{\varepsilon_i} \int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n n!} \exp \left( i \sum_{j=1}^{n} p_j x - e_j t \right) |0\rangle |\theta_n \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_1} |0\rangle |\theta_n \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_1}|^2, \tag{25} \]

where \( p_j \) and \( e_j \) have the form (18)

\[ p_j = M \sinh \theta_j, \quad e_j = M \cosh \theta_j, \tag{26} \]

and

\[ f^{\mathcal{O}}(\theta_1 \ldots \theta_n)_{\varepsilon_1 \ldots \varepsilon_n} \equiv \langle 0|\mathcal{O}(0,0)|\theta_n \ldots \theta_1\rangle_{\varepsilon_n \ldots \varepsilon_1} |0\rangle \]

are the form factors (FF). They can be calculated using a set of “axioms” specifying their analytical properties (see [31], [32] and H. Saleur’s contribution to this volume). From a physical point of view we will be mainly interested in Fourier transforms of retarded two point functions. Their form factor expansions have the form

\[ \chi^{\mathcal{O}}(\omega,q) = i \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dt \ e^{i(\omega+ict)x} \langle [\mathcal{O}(x,t),\mathcal{O}^\dagger(0,0)] \rangle \]

\[ = -2\pi \sum_{n=0}^{\infty} \sum_{\varepsilon_i} \int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n n!} |f^{\mathcal{O}}(\theta_1 \ldots \theta_n)_{\varepsilon_1 \ldots \varepsilon_n}|^2 \]

\[ \times \left\{ \frac{\delta(q-M \sum_j \sinh \theta_j)}{\omega - M \sum_j \cosh \theta_j + i \epsilon} - \frac{\delta(q+M \sum_j \sinh \theta_j)}{\omega + M \sum_j \cosh \theta_j + i \epsilon} \right\}. \tag{28} \]
We will be interested in the optical conductivity which is related to the imaginary part of the retarded current-current correlation function, \( \chi^j(\omega, q) \), by
\[
\sigma(\omega > 0) = \frac{\text{Im} \left\{ \chi^j(\omega, q = 0) \right\}}{\omega}.
\]
(29)

Here the electric current operator \( j \) was defined in (14). Using (28) we obtain the following representation for the optical conductivity
\[
\sigma(\omega) = \frac{2\pi^2}{\omega} \sum_n \sum_{\varepsilon_i} \int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n n!} \left| f^j(\theta_1 \ldots \theta_n)_{\varepsilon_1 \ldots \varepsilon_n} \right|^2 \delta(M \sum_k \sinh \theta_k) \delta(\omega - M \sum_k \cosh \theta_k)
\]
\[
= \sigma_2(\omega) + \sigma_4(\omega) + \ldots
\]
(30)

Here \( f^j(\theta_1 \ldots \theta_n)_{\varepsilon_1 \ldots \varepsilon_n} = \langle 0 | j(0, 0) | \theta_n \ldots \theta_1 \rangle_{\varepsilon_n \ldots \varepsilon_1} \) are the form factors of the electric current operator (10). \( \sigma_2(\omega) \) and \( \sigma_4(\omega) \) represent the contributions from 2 and 4-particle processes and the dots indicate processes involving higher numbers of (anti)solitons. We note that as a consequence of charge conjugation symmetry only intermediate states with an even number of particles contribute to this correlation function (16). From (30) it is easy to see that only 2-particle processes contribute up to energies \( \omega = 4M \), only 2 and 4-particle processes up to \( \omega = 6M \) and so on. It has been previously observed for several models that the FF series converges much more rapidly than expected on the basis of such considerations (33, 34, 35, 36). This may be understood in terms of phase space arguments (33, 37).

The \( n \)-particle form factors (27) for the current operator \( j^\mu \) in the SGM have been determined in (30) and can be used to calculate the first few terms in the expansion (30). The two particle FF is given by
\[
f^j(\theta_1, \theta_2)_{++} = -f^j(\theta_1, \theta_2)_{--} = \frac{4\pi^2 M \sqrt{\frac{A}{\beta}}}{\xi} \cosh \frac{\Delta_1 + \Delta_2}{2} \frac{\cosh \frac{\Delta_1 - \Delta_2}{2\xi}}{\cosh \frac{\varphi - \Delta_1 + \Delta_2}{2\xi}} \zeta(\theta_1 - \theta_2),
\]
(31)

where
\[
\zeta(\theta) = c \sinh \theta/2 \exp \left( \int_0^\infty dk \frac{\sin^2(\frac{\theta}{2}(k/\pi + i)) \sinh(\frac{\theta}{2}k)}{k \sinh(\frac{k}{2}) \sinh(k \cosh(\frac{\theta}{2}))} \right)
\]
(32)
\[
c = \left( \frac{4}{\xi} \right)^{1/4} \exp \left( \frac{1}{4} \int_0^\infty \frac{\sinh(\frac{\theta}{2}) \sinh(\frac{\theta}{2}k)}{k \sinh(\frac{k}{2}) \cosh^2(\frac{\theta}{2})} \right); \quad d = \frac{1}{2\pi\xi c}.
\]
(33)

The function \( \zeta(\theta) \) is analytic in the physical strip \( 0 \leq \text{Im} \theta \leq 2\pi \) and satisfies the following identities
\[
\zeta(\theta) S_0(\theta) = \zeta(-\theta), \quad \zeta(\theta - 2\pi i) = \zeta(-\theta).
\]
(34)

The four particle form-factor is far more complicated and can be represented as (30)
\[
f^j(\theta_1, \ldots, \theta_4)_{---+} = \frac{4\pi^3}{\beta} \xi M \sqrt{Ad} \prod_{k < \ell} \zeta(\theta_k - \theta_1)
\]
\[
\times \prod_{m,n=1,2} (\sinh[(\theta_{2+m} - \theta_n - i\pi)/\xi])^{-1}
\]
\[
\times 2 \sinh[(\theta_4 + \theta_3 - \theta_1 - \theta_2 - 2\pi i)/2\xi]
\]
\[
\times \exp(\frac{1}{\xi} \sum_k \varphi_k) \int_{-\infty}^{\infty} \prod_k \varphi(\alpha - \theta_k) \cosh(\alpha - \frac{1}{2} \sum_k \theta_k)
\]
\[
\times \Delta(\epsilon^{2\alpha/\xi} |e^{2\alpha/\xi}, e^{2\alpha/\xi}, e^{2\alpha/\xi}|),
\]
(35)
Here ζ(θ), c, d have been previously defined, and
\[
\varphi(\theta) = \xi^\frac{1}{2} \exp \left( - \int_0^\infty dk \frac{2}{k} \frac{\sin^2 \left( \frac{\theta}{2} \right) \sinh \left( \frac{\xi}{2} k \right) + \sinh^2 \left( \frac{\xi}{2} \right) \sin \left( \frac{\xi}{2} k \right)}{\sinh \left( \frac{\xi}{2} \right) \sinh (k)} \right)
\]
\[
= \xi^\frac{1}{2} \exp \left( - \int_0^\infty dk \frac{2}{k} \frac{\sin^2 \left( \frac{\theta}{2} \right) \sinh \left( \frac{1}{2} \xi k \right)}{\sinh \left( \frac{\xi}{2} k \right) \sinh (k)} \right)
\]
\[
\frac{\prod_{n=0}^{\infty} \frac{\Gamma \left( \frac{1}{2} + \frac{\xi}{2} n + i \frac{\theta}{2 \pi} \right) \Gamma \left( \frac{1}{2} + \frac{\xi}{2} (n+1) \right)}{\Gamma \left( \frac{1}{2} + \frac{\xi}{2} n \right) \Gamma \left( \frac{1}{2} + \frac{\xi}{2} \frac{\theta}{2 \pi} \right) \Gamma \left( \frac{1}{2} + \frac{\xi}{2} \frac{\theta}{2 \pi} \right)}}, \tag{36}
\]
\[
\Delta(e^{2\alpha/|\xi|} |e^{2\theta_1/\xi}|, e^{2\theta_2/\xi}, e^{2\theta_3/\xi}, e^{2\theta_4/\xi}) = \frac{1}{8} \left[ e^{4\alpha/\xi} (1 + e^{2\pi i/\xi}) \right] - e^{2\alpha/|\xi|} e^{i\pi/\xi} \sum_{j=1}^{4} e^{2\theta_1} (e^{2\theta_1+i\pi/\xi} + e^{2\theta_2+i\pi/\xi} + e^{2\theta_3+i\pi/\xi}). \tag{37}
\]

The function \( \varphi(\theta) \) is even and, as can be easily seen from the last expression, has poles and simple zeros in the physical strip, \( 0 \leq \text{Im} \theta \leq 2\pi \), at the points \( \theta = i\pi/2 + i\pi \xi k, k \geq 0 \) and \( \theta = i3\pi/2 + i\pi \xi k, k \geq 1 \). As \( \theta \to \pm \infty \) it behaves like \( \varphi(\theta) \sim 2 \exp(\pm \frac{\pi}{4}(1 + 1/\xi \theta)) \).

We note that the above integral expression is valid only in the regime \( \xi > 2 \). In general it is necessary to regularize the \( \alpha \)-integral in \( (35) \), for \( \xi > 2 \) no such regularization is required.

The form-factors with different orderings of \( \varepsilon_1, \ldots, \varepsilon_4 \) can be obtained from \( (35) \) using the symmetry property \( (34) \). Using \( (34) \) we obtain
\[
f^j(\theta_1, \theta_2, \theta_3, \theta_4)_{--,+,-} = f^j(\theta_1, \theta_2, \theta_3, \theta_4)_{-+,-,-} + S_{++}^- (\theta_2 - \theta_3) \tag{38}
\]
\[
f^j(\theta_1, \theta_2, \theta_3, \theta_4)_{++,--} = f^j(\theta_2, \theta_3, \theta_1, \theta_4)_{-+,-,-} + S_{++}^- (\theta_1 - \theta_3) S_0 (\theta_1 - \theta_2) \tag{39}
\]
\[
f^j(\theta_1, \theta_2, \theta_3, \theta_4)_{--,-,+,} = f^j(\theta_1, \theta_2, \theta_3, \theta_4)_{--,-,}-S_{++}^- (\theta_1 - \theta_3) S^+_+ (\theta_1 - \theta_2) \tag{40}
\]

The remaining orderings appearing in \( (35) \) can be obtained using the transformation properties of the current form factors under charge conjugation \( (34) \)
\[
f_{\mu}(\theta_1, \ldots, \theta_{2n})_{\varepsilon_1, \ldots, \varepsilon_{2n}} = -f_{\mu}(\theta_1, \ldots, \theta_{2n})_{-\varepsilon_1, \ldots, -\varepsilon_{2n}}. \tag{41}
\]

We can now use the above expressions for the formfactors in the spectral representation for the optical conductivity \( (30) \). The two particle contribution is easily obtained by evaluating the \( \delta \)-functions in \( (30) \)
\[
\sigma_2(\omega) = \frac{2\Theta(\omega - 2M) f(\theta)^2}{\omega^2 \sqrt{\omega^2 - 4M^2}}. \tag{42}
\]
Here \( \Theta(x) \) is the Heaviside function,
\[
f(\theta) = f^j(\theta/2, -\theta/2)_{++} = -f^j(\theta/2, -\theta/2)_{-+} \tag{43}
\]
\[
and \quad \theta = 2 \arccosh (\tilde{\omega}), \quad \tilde{\omega} = \omega / 2M. \tag{44}
\]
As mentioned before, (41) is the full, exact expression for the optical conductivity for frequencies smaller than $4M$.

After some calculations the four particle contribution can be cast in the form

$$
\sigma_4(\omega) = \frac{\Theta(\omega - 4M)}{\omega 192 \pi^2 M^2} \sum_{\epsilon_i} \sum_{\sigma=\pm} \int_{-a}^{a} d\theta \int_{-b(\theta)}^{b(\theta)} d\gamma \\
\times |f^j(g - \frac{\sigma \alpha}{2}, g + \frac{\sigma \alpha}{2}, g + \theta + \gamma, g - \theta + \gamma, \epsilon_1, ..., \epsilon_4)|^2 \\
\times \left\{ \left( \sqrt{\cosh^2 \theta \sinh^2 \gamma + \tilde{\omega}^2 - \cosh \theta \cosh \gamma} \right)^2 - 1 \right\}^{-\frac{1}{2}} \\
\times \left[ \cosh^2 \theta \sinh^2 \gamma + \tilde{\omega}^2 \right]^{-\frac{1}{2}},
$$

(44)

where

$$a = \text{arccosh}(\tilde{\omega} - 1), \quad b(\theta) = \text{arccosh} \left[ \frac{\tilde{\omega}^2 - 1 - \cosh^2 \theta}{2 \cosh \theta} \right],$$

$$g = \ln \left[ \frac{\cosh(\alpha/2) + \exp(-\gamma) \cosh \theta}{\omega} \right],$$

$$\alpha = 2 \text{arccosh} \left[ \sqrt{\cosh^2 \theta \sinh^2 \gamma + \tilde{\omega}^2 - \cosh \theta \cosh \gamma} \right].$$

The remaining integrals in (44) as well as the function $\zeta(\theta)$ have to be evaluated numerically. The latter is easily done to very high precision. The multiple integrals in the expression for the four-particle contribution are much more difficult to evaluate numerically. We estimate the precision of our results to be of the order of $10^{-4}$.

![Figure 1: Two particle (solid line) versus one hundred times the four particle contribution (dashed line) in the form factor expansion for $\beta^2 = 0.9$.](image)

We have evaluated the two and four particle contributions to the optical conductivity for several values of $\beta$. The two and one hundred times the four-particle contributions for $\beta^2 = 0.9$ are presented.
in Fig.1. Most importantly, the square root singularity, being a characteristic feature of band insulators, is suppressed by the momentum dependence of the soliton-antisoliton form factor and reappears only at the LE point $\beta^2 = 1/2$ (the behaviour of the optical conductivity in the vicinity of the LE point is shown in Fig. 2 and discussed in the next section).

The rounding off of the singularity is due to the $\theta$ dependence of the term (41). A similar behavior was previously noted for the Hubbard model at half-filling [19] which corresponds to the special SU(2)-symmetric point $\beta^2 = 1$. We find that for any $\beta^2 \neq 1/2$ there is a square root "shoulder" $\sigma(\omega) \propto \sqrt{\omega - \Delta}$ for $\omega/\Delta - 1 \ll 1$.

The four particle contribution to $\sigma(\omega)$ is seen to be insignificant at low energies and becomes larger than the two particle contribution only at $\omega \approx 180 M$ for $\beta^2 = 0.9$. This suggests that the optical conductivity is well described by the combination of 2 and 4-particle contributions up to several hundreds times the mass gap. Computation of higher order terms in (30) becomes cumbersome and probably of no physical interest, since the previous analysis suggests that they become important outside the region of applicability of the field theory approach to physical systems [2]. Nevertheless it is interesting from a theoretical point of view to determine their importance at very high frequencies, which we will do using a different approach in section 5.

4 Vicinity of the Luther-Emery point

As we have indicated in the previous section, the LE point is quite special. Let us now discuss the behaviour of the optical conductivity in the vicinity of the LE point. This will exemplify some differences between Mott insulators and conventional band insulators.

Let us recall that the SGM is equivalent to the Massive Thirring Model (MTM) [22]

$$S_{MTM} = \int d^2x \left[ i\bar{\psi}\gamma^\mu\partial_\mu \psi - \frac{g}{2} (\bar{\psi}\gamma^\mu\psi)^2 - m\bar{\psi}\psi \right]$$  \hspace{1cm} (45)

with the following identifications

$$j^\mu = \bar{\psi}\gamma^\mu\psi = -\frac{\beta}{2\pi} \epsilon^{\mu\nu}\partial_\nu \phi$$
$$\frac{g}{\pi} = \frac{1}{2} \left( \frac{1 - \xi}{\xi} \right).$$ \hspace{1cm} (46)

Here $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\bar{\psi} = \psi^\dagger \gamma^0$ are the usual two-component Fermi fields and the gamma matrices are chosen as

$$\gamma^0 = \sigma^1, \quad \gamma^1 = i\sigma^2,$$ \hspace{1cm} (47)

where $\sigma^i$ are Pauli matrices. The fermions in the MTM correspond to solitons and antisolitons in the SGM. From (46) it follows that for $\xi = 1$ ($\beta^2 = 1/2$) the fermions become non interacting and the action (45) describes a two-band system of free, spinless fermions. In the absence of doping, in the ground state the upper band is empty while the lower one is completely filled. Thus the LE point describes a conventional band insulator. In fact, at $\xi = 1$ Eq. (41) yields a square root singularity above the threshold, in agreement with this interpretation.

However, as soon as we deviate from the LE point this singularity immediately disappears and the optical conductivity vanishes at the optical gap. Close to the LE point (41) implies the following analytical

Note that in order to determine behaviour just above the threshold we need to consider only the two-particle contribution to the optical conductivity.

The field theory approximation is appropriate as long as the frequencies considered are much less than the band width, which is the UV scale in the problem. In practical applications one is unlikely to encounter a situation where the band width is more than 1000 times the spectral gap.
expression valid for $\tilde{\omega} - 1 \ll 1$

$$\sigma(\omega) \propto \frac{\sqrt{\omega^2 - 1}}{[\tilde{\omega}^2 - 1] + \xi^2 \sin^2 \gamma}, \quad \gamma = \pi \left(\frac{1}{2\beta^2} - 1\right).$$

(48)

The threshold behaviour of $\sigma(\omega)$ for several values of $\beta$ is shown in Fig. 2.

![Figure 2: Threshold behaviour of the optical conductivity close to the Luther-Emergy point for four different values of $\beta$; $\beta = 0.72$ (solid), $\beta = 0.73$ (dotted), $\beta = 0.74$ (dashed) and $\beta = 0.75$ (long dashed).](image)

We see that the square root singularity above $\omega = \Delta$ for $\beta^2 = 1/2$ is replaced by a maximum occurring at $\omega_{\text{max}} = \Delta + O(\gamma^2)$. As we take $\beta^2 \to 1/2$ from above, $\omega_{\text{max}}$ approaches $\Delta$ and $\sigma(\omega_{\text{max}})$ diverges.

## 5 Large energy behavior

In order to obtain a rough estimate of the importance of the contributions involving six or more particles in the FF sum and to obtain a complete picture of the optical conductivity, we now calculate $\sigma(\omega)$ at large energies. A convenient method to do this is Conformal Perturbation Theory (CPT) \cite{38, 39}. Viewing the SGM as a Gaussian model

$$S_{\text{Gauss}} = \int d^2 x \frac{1}{16\pi} (\partial_\nu \phi)^2$$

perturbed by the relevant operator $2 \cos(\beta \phi)$, we can formally obtain correlation functions of the perturbed theory by

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \ldots \mathcal{O}(x_n) \rangle = \frac{\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \ldots \mathcal{O}(x_n) e^{-2\mu \int d^2 x \cos(\beta \phi)} \rangle_{\text{CFT}}}{\langle e^{-2\mu \int d^2 x \cos(\beta \phi)} \rangle_{\text{CFT}}}$$

(50)

We use a normalization in which

$$\langle e^{i\beta \phi(x_1)} \ldots e^{i\beta \phi(x_n)} e^{-i\beta \phi(y_1)} \ldots e^{-i\beta \phi(y_n)} \rangle_{\text{CFT}} = \prod_{i<j}^{n} \frac{|x_i - x_j|^{4\beta^2} |y_i - y_j|^{4\beta^2}}{\prod_{j=1}^{n} |x_i - y_j|^{4\beta^2}}.$$  

(51)

The two-point function of electric currents can for example be obtained by considering

$$\langle j(x) j(x') \rangle = A \lim_{\alpha \to 0} \frac{1}{\alpha^2} \partial_\tau \partial_{\tau'} \langle e^{i\alpha \phi(x, \tau)} e^{-i\alpha \phi(x', \tau')} \rangle.$$  

(52)
Using (50) we have

$$\langle j(x) j(x') \rangle \sim A \lim_{\alpha \to 0} \frac{1}{\alpha^2} \partial_x \partial_{x'} \left( e^{i \alpha \phi(x)} e^{-i \alpha \phi(x')} \right)$$

$$= A \sum_{n} \frac{1}{n!} \left( \frac{\mu^2}{2} \right)^n \int d^2 \omega_1 \cdots d^2 \omega_n \sum_{l_k = \pm 1} \frac{1}{l_k!} \partial_x \partial_{x'} \left( e^{il_1 \beta \phi(\omega_1)} \cdots e^{il_n \beta \phi(\omega_n)} e^{i \alpha \phi(x)} e^{-i \alpha \phi(x')} \right)_{\text{CFT}} \cdot$$

(Eq.53)

Eq. (53) provides a perturbative expansion in integer powers of the scale $\mu$. Using the exact solution of the SGM it is possible to relate $\mu$ to the physical soliton mass $M$ [20].

$$\mu = \frac{\Gamma(\beta^2)}{\pi \Gamma(1 - \beta^2)} \left[ \frac{M \sqrt{\pi \Gamma(1/2 + \xi/2)}}{2 \Gamma(\xi/2)} \right]^{2 - 2 \beta^2}.$$

In our case the CPT expansion is free of ultraviolet divergences (which would cause further complications [20, 11, 21]) as long as $\xi < \infty$ but is known to suffer from infrared divergences. For example, the correlation function of bosonic exponents in the first line of (53) develops infrared divergences at order $O(\mu^{2n})$ for $\beta^2 \leq 1 - 1/2n$ [44]. Here we only consider the term of second order in $\mu$, which is free of divergences as long as $\beta^2 > 1/2$. For more general calculations one could follow the approach suggested in [20] (see also [13]).

We can now calculate the current-current correlation function to leading order in CPT and use the result to determine the optical conductivity at large energies. The $\omega_j$-integrals in (53) are carried out using the methods of [44]. We find

$$\sigma(\omega) = A 2^{9-4\beta^2} \left( \frac{\pi^2 \beta}{\Gamma(2\beta^2)} \right)^2 \mu^2 \omega^{(4\beta^2-5)}$$

$$= \frac{8 \pi^3 \beta^2 A}{\omega \Gamma^2(1 - \beta^2) \Gamma^2(1/2 + \beta^2)} \left[ \frac{\Gamma(\xi)}{2 \sqrt{\pi \Gamma(\xi/2)} \Gamma(\xi/2)} \frac{\omega}{M} \right]^{4\beta^2-4}.$$

We emphasize that the ratio of the coefficients of the high- and low-energy asymptotics (53), (11) is fixed [37, 21]. In other words, the amplitude of the power law in (55) is tied to the overall factor in (11) and the form factor expansion must approach the perturbative result in the large-$\omega$ limit. A comparison between the form factor results and (53) is shown in Fig [3]. We see that the asymptotic regime is not yet reached at energies as high as $\omega \sim 1000M$. In practical terms this implies that perturbation theory (PT) cannot be used to make contact with experiment. We note that the contributions due to intermediate states with 6,8... particles are all positive and will make the agreement of the form factor sum with PT in the region $\omega \approx 1000M$ only worse.

A good way to overcome these deficiencies of bare PT is to carry out a renormalization-group (RG) improvement as performed in [3]. Following [20] we describe the SGM as the $\hat{J} \hat{J}$ perturbation of the $SU(2)_1$ Wess-Zumino-Witten model

$$S_g = S_{SU(2)_1} + \frac{g_{\parallel}}{2 \pi} \int d^2 x \, \hat{J}_0 \hat{J}_0 + \frac{g_{\perp}}{4 \pi} \int d^2 x \, \left( \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ \right).$$

Here $\hat{J}_0$ and $\hat{J}_\alpha$ are the left and right currents of $SU(2)_1$, normalized by the following operator product expansions

$$\hat{J}_0(z) \hat{J}_0(0) = \frac{1}{2z^2} + O(1)$$

$$\hat{J}_0(z) \hat{J}_\pm(0) = \pm \frac{1}{z} \hat{J}_\pm(0) + O(1)$$

$$\hat{J}_+(z) \hat{J}_-(z) = \frac{1}{z^2} + \frac{2}{z} \hat{J}_0(0) + O(1).$$
The SU(2)-symmetric perturbation $g_\perp = g_\parallel$ corresponds to $\xi \rightarrow \infty$. The RG equations for the Sine-Gordon model can be cast in the form \cite{20}

$$\frac{dg_\perp}{dt} = \frac{g_\parallel g_\perp}{1 + \frac{g_\perp}{g_\parallel}}, \quad \frac{dg_\parallel}{dt} = \frac{g_\parallel^2}{1 + \frac{g_\perp}{g_\parallel}}, \quad (58)$$

where $t$ is the RG scale. The solution of (58) is \cite{20}

$$g_\perp = 4 \frac{1 - \beta^2}{\beta^2} \frac{\sqrt{q}}{1 - q}, \quad g_\parallel = 2 \frac{1 - \beta^2}{\beta^2} \frac{1 + q}{1 - q}, \quad (59)$$

where

$$q \left( \frac{(1-q)\beta^2}{4(1-\beta^2)} \right)^{2\beta^2 - 2} = e^{(4-4\beta^2)(t-t_0)}. \quad (60)$$

The perturbative result \cite{20} is expressed in terms of $g_\parallel, g_\perp$ noting that

$$g_\parallel^2 - g_\perp^2 = 4 \frac{\xi^2}{\beta^2}, \quad (61)$$

and by considering $q \rightarrow 0$ in (59), (60). Finally one may fix the scale $t_0$ by simply choosing it the same as in \cite{20} $t - t_0 = \ln \left( \frac{\sqrt{\pi^2/4} \Gamma(1+\xi)}{2\sqrt{\xi}} \right)^4$ and obtain modulo terms of higher order in the couplings

$$\sigma(\omega) = \frac{\pi^3 \beta^6 g_\perp^2}{2\omega \Gamma^2(2-\beta^2)\Gamma^2(\frac{1}{2} + \beta^2)} \left[ \frac{\Gamma(\frac{1}{2})e^{3/4} \sqrt{\xi}}{2^{7/2} \Gamma(\frac{1+\xi}{2})} \right]^{4\beta^2 - 4} \quad (62)$$

\footnote{In order to choose $t_0$ is a meaningful way one needs to calculate the first subleading term in the CPT expansion, which is outside the scope of these proceedings.}
Figure 4: Comparison between the 2 and 2+4-particle contribution to the optical conductivity for $\beta^2 = 0.9$ and the RG improved PT.

The RG improved result (62) for $\sigma(\omega)$ is compared to the form factor result (or more precisely the sum of the two and four-particle contributions) in Fig. 4. We see that on the level of accuracy of a log-log plot the agreement is rather good down to energies of the order of $5M$. Combining the FF results with the RG improved perturbation theory we then obtain a good description of the optical conductivity over the whole frequency range. We believe that the good agreement of RG with the exact result even at rather small frequencies is probably a particular feature of the correlation function considered here. For other correlators like for example the spectral function there is no reason to believe that RG will work as well as it does for the optical conductivity.

6 Applications

Let us now turn to applications of our results for the optical conductivity. There are several materials which are believed to be one-dimensional Mott insulators in one of their phases. These are quasi-1D antiferromagnets like KCuF$_3$ [45], Carbon nanotubes [46], possibly the striped phase in La$_{3.67}$Sr$_{0.33}$NiO$_4$ [17] and organic conductors [48, 49]. Here we will concentrate on the latter due to the availability of extensive optical data. Of particular interest for our purposes are the (TMTTF)$_2$X and (TMTSF)$_2$X families, where X is an inorganic monoanion. These materials exhibit a rich phase diagram as a function of temperature and pressure [19] and at sufficiently high temperatures are believed to be 1D Mott insulators. The (TMTTF)$_2$X family are presumably good examples of 1D Mott insulators, but as the optical gap in these materials is of the order of the bandwidth a field theory description is inappropriate.

For the (TMTSF)$_2$X Bechgaard salts the ratio of optical gap to bandwidth is small and a field theory description is possible. The Bechgaard salts are highly anisotropic materials and can be modelled as weakly coupled, quarter-filled chains. It was suggested in [19] that at energies or temperatures sufficiently far above the 1D-3D crossover scale $E_{cr}$, the interchain coupling becomes ineffective and a description in terms of a purely 1D model with charge sector [12] should be possible. This is a very nontrivial assertion as the microscopic lattice Hamiltonian appropriate for these systems is not given by a simple extended Hubbard model like (7), but includes an explicit dimerization [50]. The low-energy effective field theory obtained by bosonization is therefore not given by the simple form (8), but contains other perturbing operators as well. At present there is also some uncertainty regarding the value of $E_{cr}$ because
interactions can renormalize its bare value, set by the interchain coupling, downwards.

![Graph showing optical conductivity](image)

**Figure 5:** Comparison between the optical conductivity calculated in the SGM for $\beta^2 = 0.9$ (solid lines) and measured optical conductivity for (TMTSF)$_2$PF$_6$ from Ref.[48] (diamonds). The inset shows the same comparison on a logarithmic scale.

There is a lot of ambiguity in fitting our results to the data. The value of the optical gap $2M$ is not known and, as discussed above, we cannot calculate the overall normalization of $\sigma(\omega)$. We therefore use these as parameters in order to obtain a good fit at large $\omega$ (where the theory is expected to work best as 3D effects are unimportant) to the data [48] for any given value of $\beta$. We obtain reasonable agreement with the data for $\beta^2 \approx 0.9$, which corresponds to a Luttinger liquid parameter of $K_\rho = \beta^2/4 \approx 0.23$. This value is consistent with previous estimates (see the discussion in [48]).

As is clear from Fig.5, the model (12) seems to apply well at high energies, but becomes inadequate at energies of the order of about 10 times the Mott gap ($\approx 1600$/cm in (TMTSF)$_2$PF$_6$). Spectral weight is transferred to lower energies and physics beyond that of a pure 1D Mott insulator emerges.

There are at least two mechanisms that should be taken into account in this range of energies. Firstly, as mentioned before, a small dimerization occurs in the 1D chains and will almost certainly affect the structure of $\sigma(\omega)$ around its maximum. Secondly, the interchain hopping is no longer negligible [52] and ought to be taken into account.

**Acknowledgments:**

We are grateful to A. Schwartz for generously providing us with the experimental data. We have benefitted greatly from discussions with F. Gebhard, T. Giamarchi and E. Jeckelmann. We are especially grateful to S. Lukyanov for suggesting the RG analysis to us and for important comments and discussions.

We thank the Isaac Newton Institute for Mathematical Sciences, where this work was completed, for hospitality. F.H.L.E. is supported by the EPSRC under grants AF/100201 and GR/N19359.
References

[1] N. F. Mott, *Metal-Insulator Transitions*, 2nd ed., Taylor and Francis, London (1990); F. Gebhard, *The Mott Metal-Insulator Transition*, Spinger, Berlin (1997).

[2] W. Metzner and D. Vollhardt, Phys. Rev. Lett. **62**, 324 (1989),
D. Vollhardt, Int. Jour. Mod. Phys. **B3**, 2189 (1989),
A. Georges, G. Kotliar, W. Krauth and M.J. Rozenberg, Rev. Mod. Phys. **68**, 13 (1996).

[3] *Bosonization in Strongly Correlated Systems* A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, Cambridge University Press (1999).

[4] F.D.M. Haldane, J. Phys. **C14**, 2585 (1981),
J.V. Emery in *Highly conducting one-dimensional solids*, eds J.T. Devreese, R.E. Evrard and V.E. van Doren, Plenum Press, New York 1979,
J. Voit, Rep. Progr. Phys. **58**, 977 (1995),
J. von Delft and H. Schöller, Annalen Phys. **7**, 225 (1998),
H.J. Schulz, G. Cuniberti, P. Pieri, [cond-mat/9807366](http://arxiv.org/abs/cond-mat/9807366).

[5] T. Giamarchi Phys. Rev. B**44**, 2905 (1991).

[6] T. Giamarchi Phys. Rev. B**46**, 342 (1992); *Physica* B **230-232**, 975 (1997).

[7] D. Controzzi, F.H.L. Essler and A.M. Tsvelik, preprint [cond-mat/0005349](http://arxiv.org/abs/cond-mat/0005349), to appear in PRL.

[8] E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).

[9] *Exactly Solvable Models of Strongly Correlated Electrons*, ed. by V. E. Korepin and F. H. L. Essler (World Scientific, Singapore, 1995).

[10] H. Frahm and V.E. Korepin, Phys. Rev. B**42**, 10553 (1990), *ibid* B**43**, 5653 (1991).

[11] F. Göhmann and V.E. Korepin, Phys.Lett. **A260**, 516 (1999).

[12] F. H. L. Essler and V. E. Korepin, Phys. Rev. Lett. **72**, 908 (1994); Nucl. Phys. B**426**, 505 (1994).

[13] E.Melzer, Nucl. Phys. **443**, 553 (1995).

[14] J.V. Emery, A. Luther and I. Peschel, Phys. Rev. B**13**, 1272 (1976),
V.M. Filev, Theor. Math. Phys. **33**, 119 (1977),
I. Affleck, talk given at the Nato ASI on *Physics, Geometry and Topology*, Banff, August 1989,
F. Woynarovich and P. Forgacs, Nucl.Phys. B **498**, 565 (1997).

[15] O. J. Heilmann and E. H. Lieb, Ann. N.Y. Acad. Sci. **172**, 583 (1971),
C. N. Yang, Phys. Rev. Lett. **63**, 2144 (1989).

[16] P.K. Mitter and P.H. Weisz, Phys. Rev. D**8**, 2781 (1975),
D. Gross and A. Neveu, Phys. Rev. D**10**, 3235 (1974),
R. Dashen and Y. Frishman, Phys. Rev. D**11**, 2781 (1975).

[17] A.A. Belavin, Phys. Lett. B**87B**, 117 (1979),
N. Andrei and J.H. Lowenstein, Phys. Rev. Lett. **43**, 1698 (1979),
N. Andrei and J.H. Lowenstein, Phys. Lett. B**91B**, 401 (1980).

[18] G.I. Japaridze, A.A. Nersesyan and P.B. Wiegmann, Nucl. Phys. B**230**, 511 (1984).

[19] E. Jeckelmann, F. Gebhard and F.H.L. Essler, Phys. Rev. Lett. **85**, 3910 (2000).

[20] ALB. Zamolodchikov, Int. J. Mod. Phys. A **10**, 1125 (1995).
[21] S. Lukyanov, Comm. Math. Phys. 167, 183 (1995),
S. Lukyanov, Mod. Phys. Lett. A12, 2911 (1997).

[22] S. Coleman, Phys. Rev. D11, 2088 (1975),
S. Mandelstam, Phys. Rev. D11, 3026 (1975),
R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D11, 3424 (1975),
L.D. Faddeev and V.E. Korepin, Phys. Rep. 42C, 1 (1978).

[23] A. Luther, Phys. Rev. B14, 2153 (1976),
A.B. Zamolodchikov and Al.B. Zamolodchikov, Annals of Physics 120, 253 (1979),
H. Bergknoff and H. Thacker, Phys. Rev. D19, 3666 (1979),
V.E. Korepin, Theor. Math. Phys. 41, 169 (1979).

[24] V.E. Korepin, Comm. Math. Phys. 76, 165 (1980),
A. G. Izergin and V. E. Korepin, Lett. Math. Phys. 5, 199 (1981),
N. M. Bogoliubov, Theor. Math. Phys. 51, 540 (1982),
V. O. Tarasov, L. A. Takhtajan, and L. D. Faddeev, Theor. Math. Phys. 57, 1059 (1984),
B.M. McCoy and T.T. Wu, Phys. Lett. 87B, 50 (1979),
C. Destri and T. Segalini, Nucl. Phys. B455, 759 (1995).

[25] A. Luther and V.J. Emery, Phys. Rev. Lett. 33, 589 (1974).

[26] F.H.L. Essler, F. Gebhard and E. Jeckelmann, in preparation.

[27] A.B. Zamolodchikov, Prisma ZhETF 25, 499 (1977),
A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann.Phys. 120, 253 (1979).

[28] A.B. Zamolodchikov, JETP Lett. 25, 468 (1977).

[29] M. Karowski, H.-J. Thun, T.T. Truong, P.H. Weisz, Phys. Lett. B67, 321 (1977).

[30] F. A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory, World Scientific,
Singapore (1992).

[31] M. Karowski and P. Weisz, Nucl. Phys. B139, 455 (1978),
B. Berg, M. Karowski, P. Weisz, Phys. Rev. D19, 2477 (1979),
H. Babujian, A. Fring, M. Karowski and A. Zapletal, Nucl. Phys. B538, 535 (1999).

[32] A. Fring, G. Mussardo and P. Simonetti, Nucl.Phys. B393, 413 (1993).

[33] J. Cardy and G. Mussardo, Nucl. Phys. B410, 451 (1993).

[34] F. Lesage, H. Saleur and S. Skorik, Nucl. Phys. B474, 602 (1996).

[35] J. Balog and M. Niedermaier, Nucl. Phys. B500, 421 (1997).

[36] G. Delfino and G. Mussardo, Nucl.Phys.B455, 724 (1995).

[37] G. Mussardo, preprint hep-th/940512.

[38] A.B. Zamolodchikov, Integrable Field Theory from Conformal Field Theory, Advanced Studies in
Pure Mathematics 19, 641 (1989).

[39] Al. B. Zamolodchikov, Nucl. Phys. B348, 619 (1991).

[40] R. M. Konik and A. LeClair, Nucl. Phys. B479, 619 (1996).

[41] G. Delfino, P. Simonetti and J.L. Cardy, Phys. Lett. B387, 327 (1993).
[42] D. Fioravanti, G. Mussardo and P. Simon, cond-mat/0008210.

[43] V. A. Fateev, D. Fradkin, S. Lukyanov, A. B. Zamolodchikov and Al. B. Zamolodchikov, Nucl. Phys. B540, 587 (1999).

[44] Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B240, 312 (1984), ibid B251, 691 (1985), V. Dotsenko, M. Picco and P. Pujol, Nucl. Phys. B455, 701 (1995), R. Guida and N. Magnoli, Int. J. Mod. Phys. A13, 1145 (1998).

[45] D. A. Tennant, R. Cowley, S. E. Nagler and A. M. Tsvelik, Phys. Rev. B52, 13368 (1995), D. C. Dender, P. R. Hammar, D. H. Reich, C. Broholm and G. Aepli, Phys. Rev. Lett. 79, 1750 (1997), C. Kim, Z.-X. Shen, N. Motoyama, H. Eisaki, S. Uchida, T. Tohyama and S. Maekawa, Phys. Rev. B56, 15589 (1997).

[46] D. Cobden, private communication, A. A. Odintsov and H. Yoshioka, preprint cond-mat/9911427.

[47] T. Katsufuji, T. Tanabe, T. Ishikawa, Y. Fukuda, T. Arima and Y. Tokura, Phys. Rev. B54, 14230 (1996).

[48] A. Schwartz, M. Dressel, G. Gruner, V. Vescoli, L. Degiorgi, T. Giamarchi, Phys. Rev. B58, 1261 (1998), V. Vescoli, L. Degiorgi, W. Henderson, G. Gruner, K. P. Starkey, L. K. Montgomery, Science 281, 1181 (1998).

[49] C. Bourbonnais and D. Jerome, in "Advances in Synthetic Metals, Twenty years of Progress in Science and Technology", eds P. Bernier, S. Lefrant, and G. Bidan (Elsevier, New York, 1999), pp. 206-301 and references therein.

[50] K. Penc and F. Mila, Phys. Rev. B50, 11429 (1994).

[51] D. Boies, C. Bourbonnais and A.-M.S. Tremblay, Phys. Rev. Lett. 74, 968 (1995).

[52] A. Georges, T. Giamarchi and N. Sandler, preprint cond-mat/0001063.