Horndessence: ΛCDM Cosmology from Modified Gravity

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Rather than obtaining cosmic acceleration with a scalar field potential (quintessence) or non-canonical kinetic term (k-essence), we can do it purely through a modified gravity braiding of the scalar and metric, i.e. the \(G_3\) Horndeski action term. Such “Horndessence” allows an exact \(\Lambda\)CDM cosmological expansion without any cosmological constant, and by requiring shift symmetry we can derive the exact form of \(G_3\). We find that this route of deriving \(G_3(X)\) leads to a functional form far from the usual simple assumptions such as a power law. Horndessence without any kinetic term or potential has the same number of parameters as \(\Lambda\)CDM and makes an exact prediction for the expansion history (\(\Lambda\)CDM) and modified gravity cosmic growth history; we show the viable gravitational strength for cosmic structure growth and light propagation.

I. INTRODUCTION

Current cosmic acceleration could be due to a cosmological constant, a constant vacuum energy density. Quintessence poses an alternative where there is a dynamical scalar field, rolling in a potential. Cosmic acceleration is also possible without a potential, through changing the kinetic structure of the scalar field, as in purely kinetic k-essence.\(^\text{[1–4]}\). Here we explore using neither the potential nor kinetic structure, but modified gravity to deliver an effective cosmological constant, without any actual vacuum energy, is therefore an interesting idea to pursue, especially if we do so within the framework of a shift symmetric theory, which ameliorates many quantum corrections.

Here we investigate the cosmology where Horndeski gravity acts like an effective cosmological constant, exploring its observational impact on the modified gravitational strengths for cosmic structure growth and light propagation and prediction for growth rate measurable by redshift space distortions in galaxy surveys, the consequences for the form of the Horndeski functions, and theoretical soundness. Section II lays out the basic equations of motion and classes of theories, while Section III derives the solutions and observational impacts. Section IV treats the soundness of the theories and we conclude in Section V.

II. HOW TO GET A COSMOLGICAL CONSTANT

The Horndeski action is

\[
S = \int d^4x \sqrt{-g} \left[ G_4(\phi) R + K(\phi, X) - G_3(\phi, X) \Box \phi + \mathcal{L}_m[g_{\mu\nu}] \right],
\]

where \(\phi\) is the scalar field, \(X = -(1/2)\partial^\mu \phi \partial^\nu \phi\), \(R\) is the Ricci scalar, \(\mathcal{L}_m\) the matter Lagrangian, and \(K, G_3, G_4\) the Horndeski terms. We will work within a Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology. Note that taking \(G_5 = 0\) and \(G_4 = G_4(\phi)\) ensures the speed of gravitational wave propagation is equal to the speed of light.

The equations of motion, writing \(2G_4 = M^2_{\text{pl}} + A(\phi)\), are

\[
3H^2(M^2_{\text{pl}} + A) = \rho_m + 2XK_X - K + 6H\dot{\phi}XG_{3X} - 2XG_{3\phi} - 3H\dot{\phi}A_{\phi}
\]

\[
-2H(M^2_{\text{pl}} + A) = \ddot{\phi}(A_{\phi}) - H\dot{\phi}(A_{\phi} - 6XG_{3X}) + 2X(A_{\phi} - 2G_{3\phi}) + 2XK_X + \rho_m + P_m
\]

\[
0 = \ddot{\phi} \left[ K_X + 2XK_{XX} - 2G_{3\phi} - 2XG_{3\phi,X} + 6H\dot{\phi}(G_{3X} + XG_{3XX}) \right]
\]

\[
+ 3H\dot{\phi}(K_X - 2G_{3\phi} + 2XG_{3\phi,X}) - K_{\phi}
\]

\[
+ 2X \left[ K_{\phi,X} - G_{3\phi,X} + 3G_{3X}(\dot{H} + 3H^2) \right] - 3A_{\phi}(\dot{H} + 2H^2).
\]

\text{(1)} \hspace{1cm} \text{(2)} \hspace{1cm} \text{(3)} \hspace{1cm} \text{(4)}
Thus the Horndeski and matter terms determine the cosmic expansion rate $H$ and scalar field evolution $\dot{\phi}$. Note the mix of dependent variables: we have for example $G_3(\phi, X)$ but $\rho_m(a), H(a)$. We must solve the coupled equations to figure out how they are related, i.e. $a(\phi, X(\phi))$. This can be quite involved except for the simplest forms of the Horndeski functions, e.g. power laws.

Here we will take a different approach, and specify not the Horndeski functions but the Hubble parameter $H$, in particular as exactly that of $\Lambda$CDM:

$$3M_{pl}^2 H^2 = \rho_m + \rho_\Lambda = (3M_{pl}^2 H_0^2 - \rho_\Lambda) a^{-3} + \rho_\Lambda,$$

where $H_0 = H(a = 1)$ and $\rho_\Lambda$ is a constant (effective) energy density. From this we will attempt to derive the Horndeski functional forms and scalar field evolution.

This approach cannot be done in general. Consider the effective dark energy density and pressure,

$$\rho_{de} = 2X K_X - K + 6H \dot{\phi} X G_{3X} - 2XG_{3\phi} - 3H \dot{\phi} A_\phi$$

$$\rho_{de} + P_{de} = \ddot{\phi} (A_\phi - 2X G_{3X}) - H \dot{\phi} (A_\phi - 6X G_{3X}) + 2X (A_{\phi\phi} - 2G_{3\phi}) + 2X K_X.$$

For our effective cosmological constant, $\rho_{de} + P_{de} = 0$, but we have a mix of $\dot{\phi}(t)$ and the Horndeski functions which cannot be separately determined without further assumptions.

Following the idea of quintessence, which uses only the kinetic function in the action Eq. (1), (in the form $K = \dot{\phi}^2 / 2 - V(\phi)$), or purely kinetic k-essence, which uses only the kinetic function $K(X)$, let us explore one Horndeski function at a time. If we try to use only $G_4(\phi)$, then the scalar field equation (4) is merely $A_\phi (\dot{H} + 2H^2) = 0$; this is only valid for FLRW cosmology if $A_\phi = 0$, i.e. $G_4$ is just the usual constant Planck mass squared (over 2). If we take $G_3(\phi)$, i.e. no dependence on $X$, this can be converted to k-essence [5] (though not purely kinetic k-essence in general, rather $K(\phi, X)$). For the (noncanonical) kinetic term alone, the $\dot{H}$ equation (3) gives simply $X K_X = 0$ for an effective cosmological constant; then Eq. (4) implies $K_\phi = 0$ too, so there is no such cosmological constant behavior solution.

Thus we are left with $G_3(X)$. This also has the nice property of being shift symmetric, which we can relax that at times. Such an action – though with a kinetic term – has been used in kinetic gravity braiding dark energy [6], inflation [7], and to address the original cosmological constant problem [8].

Now Eq. (7) becomes

$$0 = -2X G_{3X} (\ddot{\phi} - 3H \dot{\phi}),$$

with solution

$$\ddot{\phi} = 3H \dot{\phi}$$

$$\dot{\phi} = \dot{\phi}_i \left( \frac{a}{a_i} \right)^3.$$

With the solution for $\dot{\phi}(a)$ (and implicitly $\phi(a)$), we can then use Eq. (6) (or equivalently Eq. 4) to determine $G_3(X)$. This demonstrates the inverse path of the usual construction – starting with $\Lambda$CDM and deriving $G_3(X)$, rather than starting with the Horndeski functions and deriving the cosmic expansion evolution $H(a)$.

Interestingly, the scalar field motion is independent of the form of $G_3$: the kinetic energy simply grows with scale factor as $X \sim a^6$ (recall no potential was introduced). This is like a time reversed version of “skating” [9, 10], where a field on a flat potential glides with diminishing kinetic energy $X \sim a^{-6}$; here the modified gravity speeds up the field rather than the field slowing due to Hubble friction ultimately to a stop (cosmological constant).

### III. DERIVING $G_3$ AND COSMOLOGICAL INFLUENCES

Let us proceed to derive $G_3(X)$ without allowing any other Horndeski term, including the kinetic term $K(X)$. While we hold to no $K(X)$, we will also explore the effect of having a potential as part of $K$ (recall that for quintessence, for example, $K = X - V$).

#### A. Zero Potential

In the absence of any potential, Eq. (6) becomes

$$6\sqrt{2}X^{3/2}G_{3X} = \rho_\Lambda.$$  (11)
Using Eq. (5) and recalling that $\rho_\Lambda$ is a constant, the solution is

$$G_3(X) = G_3(X_i) - \frac{\rho_\Lambda}{\rho_{m,0}} \sqrt{\frac{2 M^2_{pl}}{3 X_i a_i^{-3}}} \left[ \sqrt{\frac{\rho_{m,0}(X/X_i)^{-1/2} + \rho_\Lambda - \sqrt{\rho_{m,0} + \rho_\Lambda}}{\rho_{m,0} + \rho_\Lambda}} \right]. \tag{12}$$

Recall that $\rho_{m,0} = 3 M^2_{pl} H_0^2 - \rho_\Lambda$. It is interesting that even in this extremely simple situation of a single Horndeski function of a single variable, the functional form corresponding to the concordance cosmology is not trivial. That is, if one attempted to parametrize the Horndeski functions a priori, one might not guess forms like $K(X)$ is a constant, the solution is $G_3(X)$ for a simple physical system with kinetic gravity braiding [11], for $f(R)$ gravity [12, 13], and for some DHOST theories [14]. Our result agrees with [15], who also analyzed the no potential case.

### B. Linear Potential

We can make a slight elaboration by adding a linear potential. Such a potential $V = \lambda^3 \phi$ is shift symmetric and has been used for cosmic acceleration in inflation and dark energy [16, 17]. We keep $K_X = 0$ but now $K_\phi = -\lambda^3$.

This does not affect Eq. (8) or its solution Eq. (10) for $\dot{\phi}(a)$, but does impact the solution for the functional form $G_3(X)$. Equation (11) now becomes

$$6 \sqrt{2} H X^{3/2} G_{3X} = \rho_\Lambda - \lambda^3 \phi. \tag{13}$$

To solve this we need $\phi(X(a))$. The integral of Eq. (10) yields

$$\phi(a) = \phi(a_i) + \dot{\phi}_i a_i^{-3} \int_{a_i}^a dt \ A^2 \ \frac{3 M^2_{pl}}{\sqrt{\rho_{m,0} A^{-3} + \rho_\Lambda}} \tag{14}$$

$$= \phi(a_i) + \dot{\phi}_i a_i^{-3} \int_{a_i}^a dA \ A^2 \ \frac{3 M^2_{pl}}{\sqrt{\rho_{m,0} A^{-3} + \rho_\Lambda}} \tag{15}$$

$$= \phi(a_i) - \frac{\dot{\phi}_i a_i^{-3}}{2 \rho_\Lambda} \rho_{m,0} \sqrt{\frac{M^2_{pl}}{3 \rho_\Lambda}} \left[ \ln \left( \frac{1 + \sqrt{\Omega_\Lambda}}{1 - \sqrt{\Omega_\Lambda}} \right) \right] \tag{16}$$

Recall that in terms of $X$,

$$H^2(X) = \frac{\rho_{m,0} (X/X_i)^{-1/2} + \rho_\Lambda}{3 M^2_{pl}} \tag{18}$$

$$\Omega_\Lambda(X) = \left[ 1 + \frac{\rho_{m,0} a_i^{-3}}{\rho_\Lambda} \frac{X}{X_i} \right]^{-1/2} \tag{19}$$

Using that

$$G_{3X} = \frac{dG_3}{dH^2} \frac{dH^2}{dX} = \frac{dG_3}{dH^2} X^{-3/2} \rho_{m,0} \sqrt{X a_i^{-3}} \frac{M^2_{pl}}{3 M^2_{pl}}, \tag{20}$$

gives to $G_3(a)$ a contribution from the second term of Eq. (13) as an integral of the form

$$\int dH \phi = \sqrt{\frac{\rho}{3 M^2_{pl}}} \left[ (y + 1) \ln(y + 1) - (y - 1) \ln(y - 1) - (y_i + 1) \ln(y_i + 1) - (y_i - 1) \ln(y_i - 1) - \ln \frac{y^2 - 1}{y_i^2 - 1} \right], \tag{21}$$
where \( y = \sqrt{1/\Omega_\Lambda} = \sqrt{3M_{\text{pl}}^2H^2/\rho_\Lambda} \). The form of \( G_3 \) is then

\[
G_3(X) = G_3(X_i) - \frac{\rho_\Lambda}{\rho_{m,0}} \sqrt{\frac{2M_{\text{pl}}^2}{3X_i a_i^6}} \left[ \sqrt{\rho_{m,0}(X/X_i)^{-1/2} + \rho_\Lambda} - \sqrt{\rho_{m,0} + \rho_\Lambda} \right] \\
+ \frac{\lambda^3 M_{\text{pl}}^2 \sqrt{2}}{\rho_{m,0} \sqrt{X_i a_i^6}} \left\{ \phi_i (H - H_i) + \phi_i a_i^{-3} \rho_{m,0} \frac{M_{\text{pl}}^2}{2\rho_\Lambda} \left[ \ln \frac{1}{\sqrt{1/\Omega_\Lambda,i}} + 1 - \frac{2\sqrt{\Omega_{\Lambda,i}}}{1/\Omega_\Lambda,i - 1} \right] (H - H_i) \\
- \frac{\phi_i a_i^{-3} \rho_{m,0}}{6\rho_\Lambda} \left[ (\sqrt{1/\Omega_\Lambda} + 1) \ln(1/\sqrt{1/\Omega_\Lambda}) + (\sqrt{1/\Omega_\Lambda,i} + 1) \ln(1/\sqrt{1/\Omega_\Lambda,i}) \\
- (\sqrt{1/\Omega_\Lambda} - 1) \ln(1/\sqrt{1/\Omega_\Lambda} - 1) + (\sqrt{1/\Omega_\Lambda,i} - 1) \ln(1/\sqrt{1/\Omega_\Lambda,i} - 1) + \ln \frac{1}{\Omega_\Lambda,i - 1} \right] \right\}. \tag{22}
\]

For the full glory (see Appendix A), one would expand the notation using Eqs. (18) and (19) to show \( G_3(X) \) for this very simple model, with only a linear potential and no \( K(X) \) or \( G_4(\phi) \), or \( G_3(\phi) \) – we emphasize that \( G_3(X) \) in the above expression, with \( \phi(X) \) from Eq. (17) and \( \dot{\phi} = (2X)^{1/2} \) in Eq. (10), gives the \( \Lambda \text{CDM} \) solution to the Friedmann equations. It seems quite unlikely that someone starting from \( G_3(X) \) would have chosen such a functional form; we see that assuming simple, e.g. power law, forms should have no expectation of capturing accurately the detailed physics of a cosmic expansion near \( \Lambda \text{CDM} \).

### C. Cosmological Impact

Apart from the expansion history, modified gravity affects the growth of cosmic structure through the time varying effective gravitational strength – the gravity history. Generally there are two gravitational strengths, seen in two modified Poisson equations: those relating the time-time metric potential to the matter density perturbation and relating the sum of the time-space and space-space metric potentials to the matter density perturbation. These can be abbreviated \( G_{\text{matter}} \) and \( G_{\text{light}} \) respectively.

They are related to the Horndeski functions and their derivatives, but a compact notation in terms of property functions \( \alpha_i \) was given by [18]. For computing the gravitational strengths we need

\[
\alpha_M = \frac{d \ln G_4}{d \ln a} , \tag{23}
\]

\[
\alpha_B = \frac{\phi XG_{3X}}{HG_4} , \tag{24}
\]

for our class of modified gravity, where \( \alpha_T = 0 \) since we took the speed of gravitational waves to be the speed of light. As our \( 2G_4 = M_{\text{pl}}^2 \) is constant, we also have \( \alpha_M = 0 \). Thus our theory is part of the No Run Gravity class [19], and \( G_{\text{matter}} = G_{\text{light}} \) (thus there is no gravitational slip) so we will simply refer to \( G_{\text{eff}} \).

Evaluating our solutions for \( G_3 \), we have

\[
\alpha_B(a) = \Omega_\Lambda(a) \left( 1 - \frac{\lambda^3 \phi}{\rho_\Lambda} \right) , \tag{25}
\]

where \( \phi(a) \) is given by Eq. (16) or (17). Note the beautifully simple form for the no potential case: \( \alpha_B(a) = \Omega_\Lambda(a) \). This would go to zero in the early universe, indicating an unmodified general relativity, and freeze to unity in the de Sitter future.

The gravitational strength for the No Run Gravity class is

\[
G_{\text{eff}} = \frac{\alpha_B + \alpha_B'}{\alpha_B - \alpha_B'/2 + \alpha_B} , \tag{26}
\]

where \( G_{\text{eff}} \) is in units of Newton’s constant and a prime denotes \( d/d\ln a \). For the no potential and linear potential models this yields

\[
G_{\text{eff}} = \frac{\Omega_\Lambda(4 - 3\Omega_\Lambda)(1 - \lambda^3 \phi/\rho_\Lambda) - \lambda^3 \phi_i a_i^{-3}(\Omega_\Lambda/\rho_\Lambda)(a^3/H)}{\sqrt{\Omega_\Lambda(4 - 3\Omega_\Lambda)(1 - \lambda^3 \phi/\rho_\Lambda) - \lambda^3 \phi_i a_i^{-3}(\Omega_\Lambda/\rho_\Lambda)(a^3/H) - (1/2)\Omega_\Lambda^2(1 - \lambda^3 \phi/\rho_\Lambda)^2}} . \tag{27}
\]
At early times the $\Omega^2_\Lambda$ term from $\alpha^2_B$ is negligible compared to the linear $\alpha_B$ term, and $G_{\text{eff}} \rightarrow 1$. Again this indicates general relativity holds in the early universe. At late times, $G_{\text{eff}} \rightarrow 2$ in the no potential case ($\lambda^3 = 0$) and $G_{\text{eff}} \rightarrow 0$ in the linear potential case (not surprising, as the field rolls to infinity and $G_3$, and hence $\alpha_B$, must grow to match it in order to keep $\rho_{\text{de}} = \rho_\Lambda$ for the $\Lambda$CDM background). For the no potential case the gravitational strength takes a simple form,

$$G_{\text{eff}} = \frac{1 - (3/4)\Omega_\Lambda}{1 - (7/8)\Omega_\Lambda}. \quad \text{[No potential]} \quad (28)$$

Figure 1 plots the braiding property function $\alpha_B$ and the gravitational strength $G_{\text{eff}}$ for the no potential and linear potential cases. For the linear potential model there are two parameters, essentially the initial energy density $\lambda^3 \phi_i$ and initial velocity $\dot{\phi}_i$. We use dimensionless parameters $\kappa_i = \lambda^3 \phi_i/\rho_\Lambda$ and $\kappa_0 = \lambda^3 \phi(a = 1)/\rho_\Lambda$ for better physical interpretation, and set $\Omega_{\Lambda,0} = 0.7$.

All cases indeed behave as general relativity in the past. The no potential case has no free parameters, and shows strengthening gravity, $G_{\text{eff}} > 1$, with $G_{\text{eff}}(z = 0.5) = 1.08$. It asymptotes to $G_{\text{eff}} \rightarrow 2$ in the future, where $\alpha_B \rightarrow 1$. The linear potential cases exhibit weakened gravity, with the behavior shown for a variety of initial conditions given by $\kappa_i$ and velocities, or field distance rolled, characterized by $\kappa_0 - \kappa_i$. Increasing $\kappa_i$ causes the deviation from general relativity to occur earlier, while increasing $\kappa_0 - \kappa_i$ increases the deviation nearer the present and determines the future behavior, though they all eventually asymptote to $G_{\text{eff}} \rightarrow 0$ in the future (while $\alpha_B \rightarrow -\infty$). However, during the observable epoch these models are all consistent with growth measurements, e.g. $G_{\text{eff}} \in [0.93, 1]$ for $z > 0.5$.

Figure 2 shows the growth rate $f\sigma_8(z)$ as can be measured from redshift space distortions in galaxy redshift surveys. The ongoing survey with the Dark Energy Spectroscopic Instrument (DESI [20, 21]) can make percent level measurements over a wide range of redshifts. The low redshift region where the modified gravity effects are strongest can gain further precision from peculiar velocity surveys [22–24]. Note that the different shapes of the predicted curves help lift degeneracy with the amplitude of density perturbations, e.g. $\sigma_{8,0}$. 
FIG. 2. The cosmic growth rate $f\sigma_8$ is plotted vs redshift, for general relativity (GR), the Horndessence no potential theory, and two Horndessence linear potential models corresponding to the same color curves as in Fig. 1 and labeled by $(\kappa_i, \kappa_0 - \kappa_i)$.

IV. SOUNDNESS

The property functions $\alpha_i$ are also convenient for checking the soundness of the theory, specifically whether it is ghost free and stable to scalar perturbations. The ghost free condition is given by $\mathcal{g} \equiv \alpha_K + (3/2)\alpha_B^2 \geq 0$.

For our class of theory,

$$\alpha_K = \frac{12\phi X(G_{4X} + XG_{4XX})}{H M_{pl}^2}.$$  (29)

For the no potential and linear potential models this becomes

$$\alpha_K = -\frac{3}{2} \Omega_\Lambda (1 + \Omega_\Lambda) \left(1 - \frac{\lambda^3 \phi}{\rho_\Lambda}\right) - \frac{\lambda^3 \dot{\phi}/H}{3 M_{pl}^2 H^2}.$$  (30)

Combining this with Eq. (25), the ghost free condition is

$$\mathcal{g} = -\frac{3}{2} \Omega_\Lambda + \frac{3}{2} \Omega_\Lambda (1 - \Omega_\Lambda) \frac{\lambda^3 \phi}{\rho_\Lambda} \frac{\lambda^3 \dot{\phi}/H}{3 M_{pl}^2 H^2} + \frac{3}{2} \left(\Omega_\Lambda \frac{\lambda^3 \phi}{\rho_\Lambda}\right)^2 \geq 0.$$  (31)

We immediately see that this is violated for the no potential ($\lambda^3 = 0$) model. For the linear potential model it can be ghost free. In the asymptotic future, $\phi \sim a^3$ and the last term dominates so the condition holds. In the asymptotic past, $\phi \sim \phi_i + a^{9/2}$, and

$$\mathcal{g} \rightarrow -\frac{3}{2} \Omega_\Lambda \left(1 - \frac{\lambda^3 \phi_i}{\rho_\Lambda}\right).$$  (32)

So we would require the field to start from a frozen state with $\phi_i > \rho/\lambda^3$ to have a ghost free theory.
To ensure stability against scalar perturbations, the sound speed squared must be nonnegative,
\[ g c_s^2 = \frac{\alpha_B}{2} (3\Omega_A - 1 - \alpha_B) + a_B' \geq 0. \]  
(33)

The expression evaluates to
\[ g c_s^2 = \frac{5}{2} \Omega_A \left( 1 - \frac{\lambda^3\phi}{\rho_A} \right) - \frac{\Omega_A^2}{2} \left( 1 - \frac{\lambda^3\phi}{\rho_A} \right)^2 - \frac{3\Omega_A^2}{2} \left( 1 - \frac{\lambda^3\phi}{\rho_A} \right) \frac{\Omega_A}{\rho_A} \frac{\lambda^3\phi_i(a/a_i)^3}{H}. \]  
(34)

Recalling that at late times \( \phi \sim a^3 \), we see the second term dominates and \( g c_s^2 \to -(1/2)(\lambda^3\phi/\rho_A)^2 < 0 \), giving a late time instability. At early times \( g c_s^2 \to (5/2)\Omega_A(1 - \lambda^3\phi_i/\rho_A) \) so we would require \( \phi_i < \rho/\lambda^3 \) – the exact opposite of the ghost free condition! Thus neither the no potential nor the linear potential model is sound.

Can we extend this no go situation to an arbitrary potential (giving up the desired property of shift symmetry)? For a general potential \( V(\phi) \), the solution \( G_3(X) \) will change, but we can write the \( \alpha_i \) and the soundness conditions without solving for \( G_3(X) \), just using Eq. (13) with \( \lambda^3\phi \) replaced with \( V \). This gives the minor change
\[ \alpha_B = \Omega_A \left( 1 - \frac{V}{\rho_A} \right), \]  
(35)
\[ \alpha_K = -\frac{3}{2} \Omega_A(1 + \Omega_A) \left( 1 - \frac{V}{\rho_A} \right) - \frac{\dot{V}/H}{3M_{pl}^2 H^2}. \]  
(36)

The no ghost condition is Eq. (31), simply with \( \lambda^3\phi \) replaced with \( V \) (so \( \lambda^3\phi \to \dot{V} \)); the stability condition is Eq. (34) with the same substitution (noting that \( \lambda^3\phi_i(a/a_i)^3 = \lambda^3\phi \to \dot{V} \)).

However, the new element is that we now have freedom in \( \dot{V} \): it no longer has to go as \( \dot{\phi} \sim a^3 \). Studying the equations, we see that at early times
\[ g \to -\frac{3\Omega_A}{2} \left[ 1 - \frac{V}{\rho_A} - \Omega_A \left( \frac{V}{\rho_A} \right)^2 \right] - \frac{\Omega_A \dot{V}}{\rho_A H} \]  
(37)
\[ g c_s^2 \to \frac{5\Omega_A}{2} \left( 1 - \frac{V}{\rho_A} \right) - \frac{\Omega_A^2}{2} \left( 1 - \frac{V}{\rho_A} \right)^2 - \frac{\Omega_A \dot{V}}{\rho_A H}. \]  
(38)

If \( V \lesssim \rho_A \) then we need \( \dot{V} < 0 \). and the magnitude must be \( |\dot{V}| \gtrsim H\rho_A \gtrsim HV \). But with \( V \) changing rapidly it will eventually break the criterion \( V \lesssim \rho_A \). If \( V \gg \rho_A \) then if \( \Omega_A V/\rho_A < 1 \) we need \( \dot{V} < 0 \) and \( |\dot{V}| > HV \), and eventually \( \Omega_A V/\rho_A < 1 \) is overturned. If \( \Omega_A V/\rho_A > 1 \) then we need \( \dot{V} < 0 \) with \( \Omega_A |\dot{V}|/(\rho_A H) > (\Omega_A V/\rho_A)^2 \), or \( |\dot{V}| > HV [V/(M_{pl}^2 H^2)] \).

At late times,
\[ g \to \frac{3}{2} \left[ \left( \frac{V}{\rho_A} \right)^2 - 1 \right] - \frac{\dot{V}}{\rho_A H}, \]  
(39)
\[ g c_s^2 \to -\frac{1}{2} \left[ \left( \frac{V}{\rho_A} \right)^2 - 1 \right] - \frac{\dot{V}}{\rho_A H}. \]  
(40)

If \( V < \rho_A \) then we need \( \dot{V} < 0 \) and \( |\dot{V}| > \rho_A H \), which again will eventually become inconsistent with \( V < \rho_A \). For \( V > \rho_A \), we need \( \dot{V} < 0 \) and \( |\dot{V}| \geq HV \). It’s unclear if this can be realized: as \( V \) is driven smaller by \( \dot{V} < 0 \), it may become inconsistent with \( V > \rho_A \). However, if \( V \to \rho_A \) then both the ghost free and stability conditions may be satisfied. Another possibility is to allow a negative potential, so \( V \) is just driven more negative. (This does not have the usual Big Crunch doomsday since the modified gravity term in Eq. 13 with \( \lambda^3\phi \) replaced by \( V \) compensates, so the cosmic expansion remains as in $\Lambda$CDM.)

V. CONCLUSIONS

Horndessence generalizes scalar field evolution to achieve cosmic acceleration in a manner parallel to how quintessence does with a nonconstant potential or k-essence does with a noncanonical kinetic term. Here it is modified gravity through the Horndeski braiding function \( G_3(X) \) that pushes the scalar field. While usually one would start with a Lagrangian function \( G_3(X) \), somehow motivated, and derive the resulting cosmic expansion, here
we explore the inverse path of requiring a ΛCDM expansion history, as consistent with observations, and investigating the derived necessary function $G_3(X)$.

The results show that even for this simple expansion behavior, the functional form of $G_3(X)$ can be more involved than a simple a priori parametrization such as a power law or polynomial in $X$ — see Eq. (22). In fact, it is not even a rational function. Adding the simplest possible elaboration in terms of a linear (shift symmetric) potential greatly increases the complication. Thus one must take care when starting from parametrization of the Horndeski action (or effective field theory property) functions – it is not clear that any measure or prior on some simple functional parametrization space will properly sample the observably viable cosmology.

Horndessence as treated in this article was limited to a theory that preserves shift symmetry. This made it extremely predictive: the cosmic expansion history is that of ΛCDM by construction, but the gravitational strength for matter (cosmic growth of structure) and light (gravitational lensing) differ from general relativity. We have $G_{\text{matter}} = G_{\text{light}} \neq G_{\text{Newton}}$ so there is no gravitational slip, and Horndessence is a type of No Run Gravity so gravitational wave propagation is not affected. General relativity holds in the early universe. We exhibit the braiding evolution $\alpha_H(a)$ and $G_{\text{eff}}(a)$, as well as the observable cosmic structure growth rate $f\sigma_8(a)$, for the two models of no potential and linear potential. For the no potential model there are the same number of free parameters as in ΛCDM.

The two simple, shift symmetric models, with no potential and linear potential, however cannot satisfy the ghost free and stability conditions at all times. We have outlined a way around this by using a more general (not shift symmetric) potential. One could also increase the complexity of the action by allowing a $K(X)$ term. (Note that there is some nice exploration of this in [15], without a potential and assuming some $X(a)$ or $G_3(X)$. We leave such extensions to future work, but do not expect them to change the key result that cosmic expansion behavior such as ΛCDM does not lead to simple functional parametrizations of action functions like $G_3(X)$.

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Appendix A: $G_3(X)$ for ΛCDM Expansion

To exhibit the full complexity of a Horndeski action function such as $G_3$ corresponding to even a simple cosmological model such as ΛCDM expansion, we expand Eq. (22) fully to show its explicit dependence on $X$:

$$G_3(X) = G_3(X) - \frac{\rho}{\rho_{m,0}} \sqrt{2} \sqrt{\frac{M^2_{\text{pl}}}{3X_i a_i^{-6}}} \left[ \sqrt{\rho_{m,0}(X/X_i)^{-1/2} + \rho_a} - \sqrt{\rho_{m,0} + \rho_a} \right]$$

$$+ \frac{\lambda^3 M^2_{\text{pl}}}{\rho_{m,0} X_i a_i^{-6}} \left[ \sqrt{\rho_{m,0}(X/X_i)^{-1/2} + \rho_a} - \sqrt{\rho_{m,0} + \rho_a} \right]$$

$$\times \left( \phi_i + \phi_i a_i^{-3} \frac{\rho}{\rho_{m,0}} \sqrt{\frac{M^2_{\text{pl}}}{3\rho_a}} \ln \left[ \frac{1 + (\rho_{m,0}/\rho_a) a_i^{-3}}{1 + (\rho_{m,0}/\rho_a) a_i^{-3}} \right] - \frac{2}{(\rho_{m,0}/\rho_a) a_i^{-3}} \right)$$

$$- \frac{\rho_{m,0}}{6\rho_a} \left[ \sqrt{1 + (\rho_{m,0}/\rho_a) a_i^{-3} (X/X_i)^{-1/2} + 1} \ln \left( \sqrt{1 + (\rho_{m,0}/\rho_a) a_i^{-3} (X/X_i)^{-1/2} + 1} \right) - \left( \sqrt{1 + (\rho_{m,0}/\rho_a) a_i^{-3} (X/X_i)^{-1/2} + 1} \right) \ln \left( \sqrt{1 + (\rho_{m,0}/\rho_a) a_i^{-3} (X/X_i)^{-1/2} + 1} \right) \right] \right) \right).$$

(A1)
[1] C. Armendariz-Picon, V. Mukhanov, P. Steinhardt, Dynamical Solution to the Problem of a Small Cosmological Constant and Late-Time Cosmic Acceleration, Phys. Rev. Lett. 85, 4438 (2000) [arXiv:astro-ph/0004134]
[2] T. Chiba, T. Okabe, M. Yamaguchi, Kinetically Driven Quintessence, Phys. Rev. D 62, 023511 (2000) [arXiv:astro-ph/9912463]
[3] R. de Putter, E.V. Linder, Kinetic k-essence and Quintessence, Astropart. Phys. 28, 263 (2007) [arXiv:0705.0400]
[4] G. Gubitosi, E.V. Linder, Purely Kinetic Coupled Gravity, Phys. Lett. B 703, 113 (2011) [arXiv:1106.2815]
[5] S. Appleby, E.V. Linder, The Well-Tempered Cosmological Constant: The Horndeski Variations, JCAP 2012, 036 (2020) [arXiv:2009.01720]
[6] C. Deffayet, O. Pujolas, I. Sawicki, A. Vikman, Imperfect Dark Energy from Kinetic Gravity Braiding, JCAP 1010, 026 (2010) [arXiv:1008.0048]
[7] T. Chiba, T. Okabe, M. Yamaguchi, Kinetic Dark Energy: inflation driven by the Galileon field, Phys. Rev. Lett. 105, 231302 (2010) [arXiv:1008.0603]
[8] S. Appleby, E.V. Linder, The Well-Tempered Cosmological Constant, JCAP 1807, 034 (2018) [arXiv:1805.00470]
[9] M. Sahlén, A.R. Liddle, E.V. Linder, Direct reconstruction of the quintessence potential, Phys. Rev. D 72, 083511 (2005) [astro-ph/0506696]
[10] E.V. Linder, Curved Space or Curved Vacuum?, Astropart. Phys. 24, 391 (2005) [astro-ph/0508333]
[11] O. Pujolas, I. Sawicki, A. Vikman, The Imperfect Fluid behind Kinetic Gravity Braiding, JHEP 1111, 156 (2011) [arXiv:1103.5360]
[12] T. Multamaki, I. Vilja, Cosmological expansion and the uniqueness of gravitational action, Phys. Rev. D 73, 024018 (2006) [arXiv:astro-ph/0506692]
[13] P.K.S. Dunsby, E. Elizalde, R. Goswami, S. Odintsov, D. Saez-Gomez, On the LCDM Universe in f(R) gravity, Phys. Rev. D 82, 023519 (2010) [arXiv:1005.2205]
[14] A.D. Linde, Inflation And Quantum Cosmology, in “Three hundred years of gravitation”, (eds. S.W. Hawking, W. Israel, Cambridge Univ. Press, 1987), p. 604
[15] R. Kallosh, J. Kratichovil, A. Linde, E.V. Linder, M. Shmakova, Observational Bounds on Cosmic Doomsday, JCAP 0310, 015 (2003) [arXiv:astro-ph/0307185]
[16] E. Bellini, I. Sawicki, Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity, JCAP 1407, 050 (2014) [arXiv:astro-ph/1404.3713]
[17] E.V. Linder, No Run Gravity, JCAP 1907, 034 (2019) [arXiv:1903.02010]
[18] E.V. Linder, Complementarity of Peculiar Velocity Surveys and Redshift Space Distortions for Testing Gravity, Phys. Rev. D 101, 023516 (2020) [arXiv:1911.09121]
[19] C. Howlett, L. Staveley-Smith, C. Blake, Cosmological Forecasts for Combined and Next Generation Peculiar Velocity Surveys, MNRAS 464, 2517 (2017) [arXiv:1609.08247]
[20] C. Howlett, A.S.G. Robotham, C.D.P. Lagos, A.G. Kim, Measuring the growth rate of structure with Type IA Supernovae from LSST, ApJ 847, 128 (2017) [arXiv:1708.08236]