Post-Inflationary Reheating

A. B. Henriques

Departamento de Fisica/CENTRA, Instituto Superior Tecnico, 1096 Lisbon, Portugal

R. G. Moorhouse

Department of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, U.K.

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Abstract

We study the two field model for reheating based on the scalar inflaton field, \( \varphi \), and its interaction with another scalar field, \( \chi \), through a Lagrangian term \( \frac{1}{2} g^2 \chi^2 \varphi^2 \), much investigated for parametric resonance or preheating. Attention is particularly on the quantum excitations of the inflaton field and the metric with a smooth transition from quantum to classical stochastic states, and with reheating followed right through from a specific inflation model to a state including a relativistic (radiation) fluid. The excitations of the metric (but not the inflaton or \( \chi \)) are treated perturbatively and the validity of this approach is assessed. For this model our work points to the possibility of \( \zeta \) (the curvature associated parameter for observed cosmic microwave background radiation anisotropies) changing significantly during reheating with parametric resonance.

I. INTRODUCTION

Cosmic inflation theory combined with the theory of quantum fluctuations of the inflaton field has provided a theoretical basis for the observed cosmic microwave background fluctuations (CMBRF) [1]. In its simplest form with just one scalar field \( \varphi \), the inflaton, the wave lengths of the relevant perturbations become much greater than the horizon distance during inflation and remain thereafter outside the horizon, uninfluenced by cosmic details, until they re-enter the horizon early in the matter era as density and matter perturbations. Among other things this neatly bypasses the complications of reheating when the inflaton field is assumed to thermalize into a radiation fluid. A parameter which is often used to calculate the CMBRF, for the relevant small wave numbers \( k \), is the metric curvature component \( R_k \) which for these small \( k \) is approximately given by \( \zeta \approx -R \) (with \( \zeta \approx -R \)); in the simple theory, having adiabatic dynamics of the physical quantities, this is constant throughout the rest of inflation, reheating and the radiation era.

It has long been recognised [4,5] that the constancy of \( \zeta \) can alter in the presence of another scalar field, \( \chi \). The question as to whether or not this occurs has to be addressed for the details of each particular theory and there has been considerable interest [6–8] in the
effect of parametric resonance [3, 15] on this question, and on the CMBRF generally. Quite
a number of variations on this theme have been studied [12, 14, 16, 6] and some may well be
significant for future directions. But basic scenarios are provided by the most investigated
model where the interaction of the two fields is given by $\frac{1}{2}g^2\chi^2\varphi^2$; parametric resonance
may or may not occur depending on the value of the constant $g^2$ and other parameters of
the theory. It is an object of this paper to investigate the above questions for the $g^2\chi^2\varphi^2$
theories where $\chi$ has no initial classical component.

A problem arising is treatment of the decoherence of the original quantum perturbations
and the consequent stochastic classical variables [17]. In most treatments the CMBRF are
evaluated with the matter era density and metric fluctuations (arising originally from the
inflationary era) as stochastic variations.

In considering non-adiabaticity - whether or not accompanying parametric resonance -
during reheating it is appropriate to consider wave numbers of up to the order of the Hubble
parameter. All waves of $\chi$ and and $\varphi_1$ arise originally as quantum perturbations, where
$\varphi = \varphi_0 + \varphi_1$ with $\varphi_0$ being the classical inflaton field. Then, treating all waves similarly
they eventually smoothly decohere, in the way shown by Polarski and Starobinsky [17],
giving waves expressed by classical stochastic variables $e(k)$ different for each value of the
wave number $k$; the stage of decoherence depends on the number of the field quanta of a
given $k$ as a function of time. We shall retain the stochasticity throughout reheating and
shall develop equations by taking averages over stochastic variables. Typically this will give
rise to terms such as $\langle \chi^2 \rangle$ where $\langle ... \rangle$ denotes the average; this is the equivalent of taking
vacuum expectation values in the quantum case, giving rise to a loop integral and being
sometimes in this context called the Hartree approximation.

We have used stochastic variables, with the ensemble averaging just mentioned; they have
a smooth and well-defined descent from the quantum operators [17]. A simple alternative,
which has been often used, is to insert at some time classical (non-stochastic) fields for $\chi$ and
$\varphi_1$. Besides its intuitive nature that procedure has also the disadvantage of not being very
well defined; so for the present paper we have adopted this alternative excursion into a more
well defined treatment which has however its own approximation in the above averaging.

In our treatment there is no need for a perturbative expansion in $\chi$ and $\varphi_1$. In their
occurrence in the matter tensor $T^\mu_\nu$ of the Einstein equations, or in the scalar equations, the
non-linear terms pose no difficulties since we have the averaging procedure and we get a set
of simultaneous differential equations in the mode functions, as described in section III. For
example $\langle \varphi_1^2 \rangle$ quickly becomes comparable with $\varphi_0^2$ but this does not invalidate the equations.
On the other hand we find it important to take account of the metric perturbation, $\psi$, and
this indeed we have to treat perturbatively and maintaining the perturbative validity imposes

1Such fluctuations have also been derived, by-passing decoherence, by keeping the Heisenberg
representation quantum operators throughout time, and evaluating the final correlation functions
in terms of the mode functions by taking (Heisenberg) vacuum expectation values [18–20].

2The method of Khlebnikov and Tkachev [14] which is also based on stochastic fields is discussed
below in section III.
a strong limitation on the parameters of the theory.

To achieve a state where the $\chi$ and $\varphi_1$ field densities compare with the faded classical inflaton field density is not to complete reheating. The usual view is that reheating is completed when the universe is dominated by a hydrodynamical fluid composed of relativistic particles. We go to such an era through the simplest mechanism of friction or dissipative terms with decay constants $\Gamma_\chi$, $\Gamma_\varphi$ whose contribution to the $\chi$, $\varphi$ equations of motion respectively are proportional to $\Gamma_\chi \chi'$, $\Gamma_\varphi \varphi'$ which operate effectively with rapid oscillations of the fields. Since oscillations, for example those associated with parametric resonance, are prominent features of the reheating process we expect such mechanisms or any dissipative decays, to have a significant influence throughout reheating. Such dissipative processes are entropic.

These features provide a model smooth physical connection of the reheat period to a later era with a radiation fluid. To embed the reheat period precisely in a model cosmic history we need to specify the inflationary era to provide initial conditions for the reheating. We choose power-law inflation (partly for the convenience of an analytic solution) and this provides initial conditions both for the classical inflaton field $\varphi_0$ and its quantum perturbations $\varphi_1$ with the associated metric perturbation. During the inflationary period the $\chi$ field equation of motion has an important term $g_2^2 \chi \varphi_0^2$. For any inflationary era in which $\varphi_0$ is large, $O(m_{\text{Planck}})$, and non-oscillatory this term ensures extreme suppression of any initial $\chi$ field. Thus the $\chi$ field, or equivalently the number of $\chi$ particles, entering the reheat period is nearly zero. A smooth junction with the reheat equations of motion then completes the model embedding of the reheating in the assumed cosmic history.

There now follows Section II in which we treat the relevant relations between quantum states and classical stochastic states. Sections III and IV give the reheat equations, their initial conditions arising from the inflationary model and some details of their solution. In Section V we discuss the results and we conclude with a general summary in Section VI.

II. STOCHASTIC FUNCTIONS IN REHEATING

Observational evidence [21] on the cosmic microwave background fluctuations tends to support the theory that these are due to stochastic perturbations of the cosmic density and metric arising from quantum perturbations of the early classical inflaton field, $\varphi_0(\tau)$ [1].

These can be written as

$$\varphi_1(x, \tau) = \int \frac{d^3k}{(2\pi)^\frac{3}{2}} [c(k) \varphi_k(\tau) \exp(ik.x) + h.c.]$$

(1)

$\varphi_1(x, \tau)$ is an operator in the Heisenberg picture, the time dependence being wholly contained in the mode functions $\varphi_k(\tau)$, and $c(k)$ is a time-independent quantum annihilation operator such that $[c(k), c(k')^\dagger] = \delta^3(k - k')$. $\varphi_k(\tau)$ is a mode such that its $\tau$ dependence, for $\tau \to \infty$, is given by

$$\varphi_k(\tau) \propto a^{-1} \exp(-ik\tau)$$

(2)

We have adopted a conformal time metric whose unperturbed form, having scale factor $a \equiv a(\tau)$, is
The most studied form of interaction has been with another scalar field $\chi$ with the interaction potential given by

$$V_{\text{int}}(\varphi, \chi) = \frac{1}{2} g^2 \varphi^2 \chi^2$$

where $\varphi = \varphi_0 + \varphi_1$ and $\chi$ is in origin a quantum field. We adopt this as our representative of the basic scenario. Like most other authors in this context we do not consider the case where $\chi$ has an aboriginal classical scalar field part; this would change the model to be more sophisticated like some variety of two-field inflation. Thus we write $\chi$ similarly to $\varphi_1$ as

$$\chi(x, \tau) = \int \frac{d^3 k}{(2\pi)^3} [b(k)\chi_k(\tau) \exp(ikx) + h.c.]$$

where $b(k)$ is a quantum annihilation operator, $[b(k), b(k')^\dagger] = \delta^3(k - k')$, and $\chi_k(\tau)$ is a mode function such that its $\tau, k$ dependence, for $\tau \to \infty$, is given by

$$\chi_k(\tau) \propto a^{-1} \exp -ik\tau/\sqrt{2k}$$

Thus we regard $\chi$ as originating like $\varphi_1$. In the absence of a deeper theory, and in view of the seeming success in CMBR calculations of the theory of $\varphi_1$, this seems a most reasonable ansatz. The $\varphi_k(\tau)$ develop through the inflationary era and so do the $\chi_k(\tau)$ mode functions.

Now it is quite legitimate, for linear coupled equations of motion in the various quantum fields, to carry this development right through the reheat and radiation eras to the origin of the CMBR while evaluating the Heisenberg operators and finally taking vacuum expectation values of correlation functions [22,19,20]. In dealing with non-linear coupled equations it can be appropriate to make the transition to the more usual classical stochastic picture of the fields. Let us discuss this using as an example the scalar field equation for $\chi$ which is

$$\chi(x, \tau)'' + 2(a'/a)\chi(x, \tau)' - \nabla^2 \chi(x, \tau) + a^2 [M^2 + g^2(\varphi_0(\tau) + \varphi_1(x, \tau))^2] \chi(x, \tau) = 0$$

This is, from Eqs.(3,4), a non-linear equation in the quantum operators where the non-linearity arises from the $g^2$ coupling of the operators in $\varphi_1$ with those in $\chi$. We eliminate the great quantum field theory difficulties thus arising by taking the Hartree approximation - discussed further below - on the $c, c^\dagger$ in $\varphi_1^2$. That is we replace $(\varphi_0(\tau) + \varphi_1(x, \tau))^2$ by its vacuum expectation value $\langle (\varphi_0(\tau) + \varphi_1(x, \tau))^2 \rangle_0 = \varphi_0(\tau)^2 + \langle \varphi_1(x, \tau)^2 \rangle_0$

$$\langle \varphi_1(x, \tau)^2 \rangle_0 = (2\pi)^{-3} \int d^3 k \varphi_k \varphi_k^*$$

In field theoretic terms this is a loop integral and as stated in Eq.(18) below the same integral form is maintained when we go from quantum operators to stochastic variables. Now take the Fourier transform of the resulting approximation to Eq.(7), leading to an equation linear in the creation and annihilation quantum operators $b_k$ and $b_k^\dagger$. The coefficients of $b_k$ and $b_k^\dagger$ must each vanish giving two equations which are complex conjugates. We could have reversed the order of the derivation and taken the vev after the selection of the coefficients. The resulting $\chi$ mode equation is
\[ \chi'' + 2(a'/a)\chi' + (k^2 + a^2\bar{M}^2)\chi = 0 \] (9)

\[ \bar{M}^2(\tau) = M^2 + g^2 \phi_0(\tau)^2 + g^2 \langle \phi_1(x, \tau) \rangle_0 \] (10)

It is important to note that \( \chi \) is complex, as is the corresponding \( \phi_1 \) in its equation, and the imaginary and the real parts are not constant multiples of each other. We have spelt out this derivation in order to point up the similarities and the contrast to the semi-classical, stochastic, case which now follows.

Polarski and Starobinsky [17] have studied the transition from the quantum to the semi-classical case. For the Heisenberg picture which we have they find that when the mode functions grow large enough so that Planck’s constant, \( h \), can be neglected then the creation and annihilation operators may be replaced with appropriate Gaussian variables to form an equivalent stochastic field. In our formalism Eqs.(1,5) are replaced by

\[ \varphi_1(x, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} [e(k)\varphi_k(\tau) \exp(ik \cdot x) + e^*(k)\varphi_k^*(\tau) \exp(-ik \cdot x)] \] (11)

\[ \chi(x, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} [d(k)\chi_k(\tau) \exp(ik \cdot x) + d^*(k)\chi_k^*(\tau) \exp(-ik \cdot x)] \] (12)

\( e(k) \) and \( d(k) \) are time-independent separately \( \delta \)-correlated Gaussian variables such that, where \( \langle \ldots \rangle \) denotes the average,

\[ \langle e(k)e^*(k') \rangle = \langle d(k)d^*(k') \rangle = \frac{1}{2}\delta^3(k-k'); \] (13)

\[ \langle e(k)e(k') \rangle = \langle d(k)d(k') \rangle = 0; \] (14)

\[ \langle e(k)d(k') \rangle = \langle e(k)d^*(k') \rangle = 0, \] (15)

that is the \( e \) variables and the \( d \) variables are uncorrelated.

If now we insert Eqs.(11,12) into Eq.(9), take the Fourier transform, and average over the Gaussian variables \( e \) we obtain, analogously to Eq.(9),

\[ \hat{\chi}'' + 2(a'/a)\hat{\chi}' + (k^2 + a^2\bar{M}^2)\hat{\chi} = 0 \] (16)

\[ \hat{\chi}_k \equiv d(k)\chi_k(\tau) + d^*(k)\chi_k^*(\tau) \] (17)

where \( \bar{M}^2 \) is given by Eq.(10) since

\[ \langle \phi_1^2 \rangle = (2\pi)^{-3} \int d^3k\varphi_k\varphi_k^* = \langle \phi_1^2 \rangle_0 \] (18)

The coefficients of the uncorrelated variables \((d-d^*)\) and \((d+d^*)\) must each vanish separately and we can thus obtain
\[ \chi''_k + 2(a'/a)\chi'_k + (k^2 + a^2M^2)\chi_k = 0 \]  

(19)

where \( \chi_k \) is complex and because of Eq. (18) this is the same as Eq. (3). Thus we have a smooth transition from quantum to semi-classical or stochastic.

Polarski and Starobinsky have also shown that, in the semi-classical regime, the mode functions can be made real by a time independent phase transformation and they formulate the theory with somewhat different Gaussian variables \( e \) from those of Eqs. (14,15). We can transform our classical stochastic formulation to that of Polarski and Starobinsky:

We can write Eq. (11) as

\[ \varphi_1(x, \tau) = \int \frac{d^3k}{(2\pi)^2} \left[ e(k) \varphi_k(\tau) + e^*(-k) \varphi^{*,-k}(\tau) \right] \exp(ik\cdot x) \]  

(20)

and we can time-independently transform our variables so that \( \varphi_k \) becomes real. Then using the fact that \( \varphi_k = \varphi_{-k} \) we get

\[ \varphi_1(x, \tau) = \int \frac{d^3k}{(2\pi)^2} \tilde{e}(k) \varphi_k(\tau) \exp(ik\cdot x) \]  

(21)

where \( \tilde{e}(k) \equiv e(k) + e^*(-k) \). Thus

\[ \tilde{e}(k) = \tilde{e}^*(-k) \]  

(22)

\[ \langle \tilde{e}(k)e^*(k') \rangle = \delta^3(k - k') \]  

(23)

where the last equation is derived using Eq. (13). Eqs. (21-23) are those of Polarski and Starobinsky (except that we have the factor \( a^{-1}(\tau) \) incorporated in our mode functions).

To summarise: using the Hartree approximation the scalar quantum and classical stochastic mode equations are the same, with complex mode functions, so there is a seamless join. In the classical stochastic regime a conversion to real mode functions can be made yielding completely real equations, but these equations are invalid in the quantum regime. For this reason we prefer to use complex mode functions in reheats studies as not all modes may be classical stochastic through all the relevant period.

We note that if \( \xi(k), \eta(k) \) are semi-classical mode functions, which must be of dimension \( (\text{mass})^{-\frac{1}{2}} \), then we cannot write equations such as \( \xi(k)^\prime \prime + \ldots + g^2 \int \xi(k - k')\eta(k')d^3k'\) which if valid would contain different and interesting rescattering effects. Apart from being dimensionally inconsistent we would have had to take averages over the stochastic variables in their derivation and this, as previously above, would have eliminated all such terms. Of course such equations are perfectly possible if \( \xi, \eta \) are fully classical mode functions of dimension \( (\text{mass})^{-2} \) and indeed have often been written down in the resonant reheating context. But to obtain such such equations, starting from a quantum origin of the modes as we do, requires specific transformations and we are not aware of any suggestions for these for \( k \neq 0 \). Naturally if \( \xi(k) = d(k)\chi_k, \eta(k) = c(k)\varphi_k \) then such equations are valid, but the physical solution involves solving a statistical ensemble of equations, got by sampling \( d, e \) and finally averaging - a seemingly very lengthy procedure. The method of Khlebnikov and Tkachev [14], mentioned also below, seems to provide at least a
partial and more achievable way of implementing the stochasticity in the semi-classical case without resort to the Hartree approximation. However having a system of 7 simultaneous equations with complex mode functions we shall implement the quantum or semi-classical equations which we have illustrated above.

As a final comment we note that it is dimensionally obviously inconsistent to write such coupled classical equations as the above and loop integrals such as \( \text{[18]} \) with the same mode functions.

### III. THE REHEATING EQUATIONS

We now formulate the equations of motion for the reheat period. The Lagrangian for the scalar fields is

\[
L = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \dddot{\varphi}_\alpha \dddot{\varphi}^{\alpha} + \frac{1}{2} \dddot{\chi}_\alpha \dddot{\chi}^{\alpha} - V(\varphi) - V(\chi) - V_{\text{int}}(\varphi, \chi) \right]
\]  

(24)

where \( \varphi = \varphi_0 + \varphi_1 \) and \( V_{\text{int}} \) is given by Eq.(4).

In the perturbed metric, \( g_{\alpha,\beta} \), we use the longitudinal gauge for the perturbation

\[
ds^2 = a(\tau)^2 (1 + 2\phi) d\tau^2 - a(\tau)^2 (1 - 2\psi) \delta_{ij} dx^i dx^j
\]

(25)

Unusually, the scale factor is calculated using the full reheating dynamics as below, Eq.(33). Unlike other treatments, excepting Bassett et al. [6], we include the metric perturbations as dynamical variables in the coupled equations of motion. In this physical system, including the hydrodynamical variables introduced below, there are no space-space non-diagonal components in the energy-momentum tensor. Hence \( \phi = \psi \) [23] and we denote this metric variable by \( \psi \).

We also wish to take account of actual reheating in which the \( \varphi \) and the produced \( \chi \) particles, which may be massive [24], are replaced by a thermal gas of relativistic particles, giving the radiation era. We make no attempt to imagine details of this process but adopt the simple friction mechanism which puts all these details into a black box. The use of this mechanism in general circumstances has been criticized, it being claimed that it is only appropriate when the decaying field is undergoing fast oscillations [13]. In the calculations that we shall describe it is in these circumstances that the main production of a relativistic gas occurs. We describe this gas by the usual hydrodynamical variables \( \rho \), the density, and \( p \) the pressure and in the equations of motion which follow we take that equation of state, \( p = \rho/3 \), which is appropriate for relativistic particles.

We divide the hydrodynamical density into two components

\[
\rho(x, \tau) = \rho_0(\tau) + \rho_1(x, \tau)
\]

(26)

where \( \rho_0(\tau) \) is the homogeneous background density and \( \rho_1(x, \tau) \) is the inhomogeneous part of it; the notation \( \delta \rho \equiv \rho_1/\rho_0 \) is the usual usage. As we shall see in the equations of motion \( \rho_1 \) mainly arises from the inhomogeneous part \( \varphi_1 \) of the inflaton field in accord with the usual theory of the cosmic microwave background fluctuations.
We shall now formulate the equations of motion, being those Einstein equations and scalar wave equations arising from the Lagrangian, Eq.(24), and metric, Eq.(25), with the addition of the hydrodynamical terms. The energy-momentum tensor is

\[ T^\mu_\nu = \varphi^\mu \varphi_\nu + \chi^\mu \chi_\nu - \left[ \frac{1}{2} \varphi^\alpha \varphi_\alpha + \frac{1}{2} \chi^\alpha \chi_\alpha + V(\varphi) + V(\chi) + V_{\text{int}} - p \right] \delta^\mu_\nu + (\rho + p)u^\mu u_\nu \]

(27)

where \( u^\mu \) is the fluid 4-velocity.

First we write 3 spatially homogeneous, or background, equations taking the average over the Gaussian variables as in Eqs(13-18). We use the notation \( ' \) for \( d/d\tau \) and we take the individual field potentials in reheat to be:

\[ V(\varphi) = \frac{1}{2} m^2 \varphi^2; V(\chi) = \frac{1}{2} M^2 \chi^2. \]

(28)

After Eq.(34) below it is noted that \( \varphi_1 \) and \( \psi \) have the same stochastic variables. Thus defining the density and pressure homogeneous parts of the energy-momentum tensor as \( \rho_T(\tau) \equiv -\langle T^0_0 \rangle \) and \( p_T(\tau) \delta_j^i \equiv \langle T^i_j \rangle \) we find, after ensemble averaging, that

\[ \rho_T(\tau) = \frac{1}{2a^2} \left[ \eta + a^2 \bar{m}^2 (\varphi_0^2 + \langle \varphi_1^2 \rangle) + a^2 M^2 (\chi^2) + \langle \varphi_{1,i}^2 \rangle - 4 \varphi_0 \langle \varphi_1 \psi \rangle \right] + \rho_0, \]

(29)

\[ p_T(\tau) = \frac{1}{2a^2} \left[ \eta - a^2 \bar{m}^2 (\varphi_0^2 + \langle \varphi_1^2 \rangle) - a^2 M^2 (\chi^2) - \langle \varphi_{1,i}^2 \rangle / 3 - \langle \chi_{,i}^2 \rangle / 3 - 4 \varphi_0 \langle \varphi_1 \psi \rangle \right] + \rho_0 / 3, \]

(30)

\[ \eta = \varphi_0^2 + \langle \varphi_1^2 \rangle + \langle \chi^2 \rangle \]

(31)

\[ \bar{m}^2 \equiv m^2 + g^2 \langle \chi^2 \rangle. \]

(32)

and we have the Friedmann equation

\[ \left( \frac{a'}{a} \right)^2 = \frac{8\pi G}{3} a^2 \rho_T(\tau). \]

(33)

As in Eq.(18) the averages are independent of \( x \) ensuring the spatially homogeneity of the R.H.S. of Eq.(33). We also see from Eq.(18) that Eq.(33) is equally valid in the quantum regime since the formalism ensures that the vev of the quantum operators equals the average over the stochastic variables.

The other two spatially homogeneous equations are for \( \varphi_0 \) and \( \rho_0 \). Here we introduce the arbitrary decay constants \( \Gamma_\varphi \) and \( \Gamma_\chi \) which serve to create the hydrodynamical radiation gas specified by \( \rho = \rho_0 + \rho_1, p = \rho / 3 \). In Appendix A we show how these are introduced in a way consistent with the Bianchi identities and how the equations for \( \rho_0, \rho_1, \varphi_0, \varphi_1, \) and \( \chi \) are subsequently derived.
\[ \varphi'' + 2(a'/a)\varphi' + a^2\overline{m}^2\varphi - 4\langle \psi'\varphi' \rangle - 4(\psi\nabla^2\varphi) + 2a^2\overline{m}^2\langle \psi\varphi \rangle = -a\Gamma_\varphi(\varphi' + 2\langle \psi\varphi' \rangle) \] (34)

We note the combined reaction of the inhomogeneous fields \( \varphi_1 \) and \( \psi \) on the homogeneous inflaton field through the ensemble averaging, similarly to that in Eqs. (29) and (30).

\[ \rho'_0 + 4(a'/a)\rho_0 = a^{-1}\Gamma_\varphi(\varphi^2_0 + \langle \varphi'^2 \rangle) + a^{-1}\Gamma_\chi(\chi^2) - 2a^{-2}\rho_0\langle \psi\nabla^2\varphi_1 \rangle + 4\langle \psi'\rho \rangle \] (35)

There remain the 4 spatially non-homogeneous equations which we write in the \( k \)-component form. For \( \chi \) and \( \varphi_1 \) these components are specified as the complex mode functions \( \chi_k \) and \( \varphi_k \) of Eqs. (11). Their wave number dependence, given by the succeeding equations, is only on \( k \equiv |k| \).

\[ \chi''_k + 2(a'/a)\chi'_k + (k^2 + a^2\overline{M}^2)\chi_k = -a\Gamma_\chi\chi'_k \] (36)

where \( \overline{M}^2 \) is a function of \( \tau \) given by Eqs. (11).\n\n\[ \varphi''_k + 2(a'/a)\varphi'_k + (k^2 + a^2\overline{m}^2)\varphi_k - 4\varphi'_0\psi'_k + 2a^2\overline{m}^2\varphi_0\psi_k = -a\Gamma_\varphi(\varphi'_k + 2\varphi'_0\psi_k) \] (37)

Though not indicated these equations actually hold for each separate value of \( k \); thus consistency of Eq. (37) indicates that \( \psi_k \), the mode function of the metric perturbation should be associated with the same Gaussian operators (or, in the quantum regime, with the same quantum operators) as \( \varphi_k \) in Eq. (11). Thus

\[ \psi(x, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}}[e(k)\psi_k(\tau) \exp(ik.x) + e^*(k)\psi^*_k(\tau) \exp(-ik.x)] \] (38)

where since \( \psi(x, \tau) \) is dimensionless \( \psi_k \) has dimension \((mass)^{-3/2}\). This associates the metric perturbation with the inhomogeneous part of the inflaton field without assigning priority to either. But the stochastic variables of \( \chi_k \) are independent. Thus no terms in \( \psi_k \) appear in Eq. (55); they are forbidden through ensemble averaging.

The mode equations for \( \psi \) and \( \rho_1 \) are

\[ \psi''_k + (3\psi'_k + (a'/a)\psi_k)(a'/a) = \frac{8\pi G}{3}a^2(\rho_T + 3p_T)\psi_k + 4\pi Ga^2\delta p_k \] (39)

where

\[ \delta p_k = \frac{1}{a^2}[-\xi\psi_k + \varphi'_0\varphi_k' - 2\varphi_k'\psi'_1 - a^2\overline{m}^2\varphi_0\varphi_k] + \rho_k/3 \] (40)

with

\[ \xi \equiv \varphi'^2_0 + \langle \varphi'^2_1 \rangle + \langle \chi'^2 \rangle + \langle \varphi_1,i\varphi_1,i \rangle/3 + \langle \chi,i\chi,i \rangle/3. \] (41)

\[ \rho'_k + 4(a'/a)\rho_k - 4\rho_0\psi'_k = \frac{k^2}{a^2}[(4\pi G)^{-1}(\psi'_k + (a'/a)\psi_k) - (\varphi'_0 + 2\langle \psi\varphi'_1 \rangle)\varphi_k] + 2\Gamma_\varphi\varphi'_0\varphi_k \] (42)
Eqs. (39)-(42) can be deduced by extending the space-time and space-space gauge invariant
Einstein equations given by Mukhanov et al. [23]; the term in square brackets in Eq. (42)
arises from the validity of the space-time equation and vanishes if \( \rho_0 = 0 \).

From the structure of the above equations the Fourier transform of \( \rho_1(x, t) \) has the same
Gaussian variables, \( e(k) \), as have \( \varphi_1(x, t) \) and \( \psi(x, t) \), \( \rho_k \) having dimension \((\text{mass})^{2/2}\); the
quantum operators and consequent Gaussian variables of the \( \chi \) field are independent of the
others. This point also has consequences when we use vacuum expectation values or averages
over the stochastic variables as noted above in connection with Eq. (36).

A number of points arise on our equations of motion in the context of the large body
of work on parametric resonance in reheating since the subject was introduced by Traschen
and Brandenberger [9]:

i. Scale factor. We find the scale factor \( a(\tau) \) by Eq. (33) this being the appropriate
calculation which includes all the relevant dynamical variables, and thus the influence of all
the scalar fields. Generally an approximate ansatz for \( a \) has been used in other paper, an
exception being the work of Boyanovsky et al. [13]. Our equation makes a difference to the
form of \( a \) but this change does not seem to have a critical influence.

ii. The loop integrals (Hartree approximation). For quadratic field forms, in the classical
stochastic regime, we have used the Gaussian average which is a smooth continuation from
the quantum field theory vacuum expectation value. This has been called the ‘Hartree
approximation’ and has been much used in parametric resonance work. (We give some details
of our evaluation in Appendix B.) We note that, for example, the term \( g^2 \varphi_0^2 (\chi^2) \) corresponds
to a first order single loop calculation in quantum field theory as discussed by Kofman et al.
[12]. They have compared the magnitude of this with that of an approximate evaluation of
the single loop second order in \( g^2 \) term, and have concluded that this could be of comparable
magnitude for some resonance modes they studied. In common with Kofman et al. and
other authors we have not included this difficult, though maybe significant, correction. To
include suchlike corrections we would have to add the results of the appropriate field theory
calculations to Eq. (24), thus manufacturing an effective Lagrangian.

As discussed in the last paragraph of section 1 our formalism does not include rescattering corrections of the form \( \int \xi(k-k')\eta(k'-k'')\eta(k'')d^3k'd^3k'' \) as our regime is stochastic. Other authors [12, 6] have written down, though perhaps not fully implemented, equations with such terms which seem to require the fields to be already fully classical, in the sense noted in section 1 together with a question on their provenance. We have found that our emphasis on the Hartree approximation is closely related to the viewpoint of Boyanovsky et al. [13].

iii. The metric perturbation. The coupled equations include the metric perturbation, \( \psi \). As
noted above its quantum or stochastic variables are the same as those of \( \varphi_1 \) whose equation
of motion (37) has an important coupling to \( \psi \) arising from the metric interaction with
\( \varphi_0 \). The equation of motion (39) of \( \psi \) with Eqs. (40, 41) show its coupling to all the other
fields. In the derivation of the coupled equations \( \psi \) is treated as a small perturbation to the
metric, this being in contrast to \( \varphi \) and \( \chi \) which are not treated perturbatively. So the
validity of this has to be maintained throughout reheating and we find this gives significant
restrictions on the parameters of the theory. Much work involving solving equations of motion in this context has ignored the metric perturbation but Bassett et al. [3] have discussed
and emphasized its importance.

**iv. Stochasticity.** Khlebnikov and Tkachev [14], as mentioned also in section II, have an entirely different approach to the equations of motion, which they solve on an x-space lattice. Their method takes account of the stochastic nature of the variables, as exhibited for example in Eq. (11), by taking appropriate initial conditions. For each discretized \( k \)-value they take a random value (chosen from an appropriate distribution) of for example the variable \( e(k)\varphi_k(\tau) \) of Eq. (11). The initial value of \( \varphi(x, \tau) \) is then got by using the discretized Eq. (11) to sum over all those random values. Thus though only one value is used for each \( k \) each \( x \) contains a large random sample. After solution in \( x \)-space the variables can be re-Fourierized to get the \( k \) components. Because of the non-linearity of the \( x \)-space equations this mixes the initial \( k \) components and is not equivalent to solving the \( k \) equations directly. We note that the fields must already have passed through the quantum stage into the classical stochastic phase; the authors do not include the metric perturbation.

**IV. SOLVING THE EQUATIONS**

**A. The Microwave Background Fluctuations**

The present inflationary theory of the origins of the cosmic microwave background fluctuations has so far been successful. In its simplest form a perturbation, \( \delta \varphi_k(t) \) of quantum origin, in the inflaton field develops in the inflationary era in association with the curvature perturbation, \( R_k(t) \). At a certain time, \( t_* \), in the inflationary era before reheat begins, the modes of relevance are such that \( k/a \) becomes equal to, and then rapidly less than, \( \dot{a}/a \equiv H \). Since then \( \lambda_{\text{physical}} = 2\pi a/k \gg H^{-1} \) the latter being, in this context, the Hubble radius and this is sometimes expressed as the perturbations having passed outside the horizon. At around the beginning of the period of matter domination \( H^{-1} \) again becomes greater than \( \lambda_{\text{physical}} \). \( R_k(t_*) = -[H\delta \varphi_k/\dot{\varphi}]_{t=t_*} \) and, with certain assumptions in the simple model such as adiabaticity, \( R_k \) remains the same during that whole period when \( \lambda_{\text{physical}} \gg H^{-1} \) and equivalent [3] to the parameter \( \zeta \): [23,42,25]:

\[
\zeta_k = \frac{2}{3}(H^{-1}\Phi_k + \Phi_k)/(1 + w) + \Phi_k = -R_k \tag{43}
\]

where \( \Phi_k \) is the gauge invariant metric perturbation (equal to the longitudinal gauge metric perturbation \( \psi_k \) of Eq. (25)) and \( w = p/\rho \) being the total pressure to density ratio. This scenario relates the CMBRF rather directly to the inflationary period bypassing reheating complications.

Now the model which we are studying has an extra scalar field, with an interaction \( g^2 \chi^2 \varphi^2 \) known to be capable of giving rise to parametric resonance (preheating [42,12]) in reheating. We consider how far and under what conditions, the constancy of \( \zeta \) for \( \lambda_{\text{physical}} \gg H^{-1} \) remains valid. For this purpose we investigate the conclusion of reheating by particle decay, as mediated by the \( \Gamma_\varphi \) and \( \Gamma_{\chi} \) parameters and for comparison we also look at cases where these are zero.

In what follows we shall largely be engaged with modes which have wavelengths which do not become larger than \( H^{-1} \) before reheat. So we should have an estimate of the magnitude
of all relevant modes at the beginning of reheat and this will be dealt with in the following subsection.

B. Initial Conditions at Reheat

1. $\chi$-field

Whatever we may postulate in detail for the single field potentials $V(\varphi), V(\chi)$ through the inflationary and reheat periods, it is a hypothesis of our model that the $g^2 \varphi^2 \chi^2$ is the significant interaction term in the Lagrangian. This has some immediate consequences for the magnitude of $\chi$ at the (fuzzy) interface between inflation and reheat. A simplified version of the mode equation, Eq.(39) with Eq.(11), in the inflationary era is

$$\left(a \chi_k\right)'' + \left[k^2 + a^2 M^2 + g^2 a^2 \varphi_0^2 - a''/a\right](a \chi_k) = 0 \quad (44)$$

and $\varphi_0^2$ is greater than, or of the order of, $m_{Pl}^2$ through most of the inflationary era. Thus the term in square brackets is large and positive resulting in a quasi-periodic type solution for $a \chi_k$, which has a primordial value given by Eq.(6). The many efold increase of $a$ during the inflationary era indicates an exceedingly small value for $\chi_k$ at the beginning of reheat \[26,27\], which may be as little as $10^{-50}$ of its initial value. This is an important qualitative feature. Firstly for such small mode functions in the beginning of reheat the transition to classical stochastic functions cannot yet be made and the quantum complex formalism should be retained. We have taken the initial value of $\chi$ to be $10^{-n} m_{Planck}^{-1/2}$ where $n$ is of the order of 30.

2. Background metric and the classical inflaton field

For the reasons just outlined we shall assume that it is valid to neglect any influence of the $\chi$-field during inflation on the classical background field $\varphi_0$ and on the quantum or stochastic part $\varphi_1$. So we shall work with the standard single field inflation formalism to formulate conditions at the beginning of reheat for both $\varphi_0$ and $\varphi_1$. It will prove convenient, as we shall see in 4 below, to have a potential, $V(\varphi)$, of different form in the inflation era from that in the reheat era. In a perfect physical model these should be joined by a smooth transition form but as we show we can substitute a sudden (phase change) transition for the purpose of finding the initial conditions of reheat. First we deal with the classical background field which we denote just by $\varphi$ in the rest of this section.

At the boundary between inflation and reheat we can start the reheat era with the same values of $\varphi$ and $\varphi'$ as those at the end of inflation and we can likewise impose continuity of the potential $V$ (for an example see 4 below); of course we also take $a$ continuous. It follows that $a'$ is continuous since

$$\alpha^2 \equiv (a'/a)^2 = \frac{8\pi G}{3} \left[\frac{1}{2} \varphi'^2 + a^2 V\right] \quad (45)$$

We can show that $\alpha'$ is continuous as follows.
2\alpha'\alpha'' = \frac{8\pi G}{3} [\varphi'\varphi'' + a^2 \varphi'(dV/d\varphi) + 2\alpha a^2 V] \quad (46)

Since

\varphi'' + 2\alpha\varphi' + a^2 dV/d\varphi = 0 \quad (47)

it follows that

\gamma \equiv 1 - \frac{\alpha'}{\alpha^2} = 4\pi G(\varphi'/\alpha)^2 \quad (48)

where \gamma, thus defined, is a quantity we shall need in the next section.

Thus from the imposed conditions at the beginning of reheat it follows that \alpha', or equivalently \alpha'', is continuous at the boundary. Thus so is the background curvature, in addition to the metric.

3. \varphi_1 and the metric perturbation

Having a satisfactory background continuation we can now deal with the continuation of the quantum perturbations of the inflationary era. Deruelle and Mukhanov [28] have used the Lichnerowitz curvature conditions [30] to give the necessary continuity conditions on the metric perturbation. These can be summarised as follows [20,29]: \psi, the metric perturbation and \Gamma should be continuous across the boundary where

\Gamma \equiv (\psi'/\alpha + \psi - \nabla^2 \psi / 3\alpha^2) / \gamma. \quad (49)

Since \gamma has just been found to be continuous the last condition can be reduced to the continuity of \psi' + \alpha\psi, which in turn using the time-space Einstein equation [23]

\psi' + \alpha\psi = 4\pi G\varphi_0'\varphi_1 \quad (50)

can be reduced to the continuity of \varphi_1.

Thus from the Lichnerowitz conditions one finds that the values of the mode functions for the metric perturbation \psi and for the non-homogeneous part, \varphi_1, of the inflaton field at the beginning of reheating after the phase change are the same as those at the end of the inflation era; \psi', given by Eq.(50), is also continuous. \varphi_1 does not need to be continuous and in fact is not. Its value can be found in terms of the continuous fields by using the time-time Einstein equation at the beginning of reheating (which differs from that at the end of inflation). At this very beginning of reheat, as at the end of inflation, we validly use linear perturbation theory in \varphi_1 and we can check that the perturbations are adiabatic. This means that we expect \zeta to be constant in the first part of the reheat period, which is what we find as discussed below in section \text{V.C.}

The values of the above quantities with \varphi_0, \varphi_0' and \alpha \equiv (a'/a), plus the input of the very small \chi, \chi', provide all that is needed for the initiation of the reheat equations.
4. Input from power-law inflation

Rather than making some plausible input data from a generality of inflationary calculations we have chosen to use a specific inflationary model. This is power-law inflation with an exponential potential: \( V = U \exp(-\lambda \varphi) \) where \( U \) is a constant. In this there are 2 parameters free to be chosen. These are the specific power of the inflationary scale factor \( a \), and the magnitude of the classical scalar field \( \varphi_0 \) at the junction of inflation and reheat, which we define as where we begin to use the reheat equations of section III.

An advantage of this power-law inflation is that the solutions are analytically expressible \[29\]. For the scale factor \( a \) and the inflaton field \( \varphi_0 \)

\[
a \propto (\tau_i - \tau)^p, \varphi_0' = -\alpha \lambda / \kappa, \tag{51}
\]

where \( \tau_i > \tau \) (end inflation) is a constant and \( p = 2\kappa/(\lambda^2 - 2\kappa) \). With \( \lambda^2 / \kappa \ll 2 \), \( p \) is near to \(-1\) and there is power-law inflation. The imposed continuity of \( V \) at the boundary gives \( U \exp(-\lambda \varphi_0) = \frac{1}{2} m^2 \varphi_0^2 \); thus, given \( m, \lambda \), we can arbitrarily specify \( \varphi_0 \) at the boundary and this condition then merely specifies \( U \). This triviality in making an arbitrary choice of \( \varphi_0 \) at the beginning of reheat is useful.

Also, in the inflation era, the perturbation \( \varphi_1(\mathbf{x}, \tau) \) and the perturbation \( \psi(\mathbf{x}, \tau) \) to the metric can be expressed in terms of a quantum field function

\[
\mu(\mathbf{x}, \tau) = a^{-1} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{2k} [a(\mathbf{k}) \mu_k(\tau) \exp(i\mathbf{k}.\mathbf{x}) + h.c.] \tag{52}
\]

whose \( k \)-modes satisfy

\[
\mu_k'' + (k^2 - a''/a) \mu_k = 0 \tag{53}
\]

This function emerges in the synchronous gauge, a derivation due to Grischuk \[19,20\]; one can convert the resulting formulae to our longitudinal gauge quantities by well known techniques \[23,20\]. These yield

\[
\psi_k = a^{-1} \sqrt{4\pi G/2k} \alpha \sqrt{\gamma} [(\mu_k' - \alpha \mu_k)/k^2] \tag{54}
\]

\[
\varphi_k = a^{-1} (2k)^{-1/2} [\mu + \alpha \gamma (\mu' - \alpha \mu)/k^2] \tag{55}
\]

where \( \gamma \equiv 1 - \alpha' / \alpha^2 = (p + 1)/p \). Since \( a'' / a = \frac{p+1}{p} (\tau_i - \tau)^{-2} \), Eq.\( (53) \) can be expressed as a Bessels equation in terms of

\[
y \equiv k(\tau_i - \tau) = \frac{k}{aH} |p| \tag{56}
\]

and the solutions which satisfy the asymptotic conditions for a scalar field as in Eq.\( \ref{eq:6} \) are

\[
\mu_k(y) = \sqrt{\frac{\pi y}{2}} (J_n - iY_n) \exp[-i(\frac{1}{2} n\pi + \frac{1}{4} \pi)] \tag{57}
\]
where \( n = \frac{1}{2} - p \).

For \( k/a \ll H \) (many times over fulfilled by the \( k \) relevant to the CMBRF observations) \( \mu_k(y) \propto y^p \propto k^p \) and if this \( k \)-dependence passes unaltered through the reheat era into the radiation era it yields a power-law spectrum which scales as \( k^{n-1} \) where \( n \), the spectral index, is given by \( n = 2p + 3 \).

Besides \( \varphi_0 \) and the power \( p \) there is one more parameter that determines the input from inflation and that is the \( m \) of the inflaton field potential in reheat, \( V(\varphi) = \frac{1}{2}m^2\varphi^2 \). This is because it is necessary to know the value of \( y \), defined by Eq.(56), in order to determine the size of the perturbations. From the conditions of continuity of section IV.B.2 it follows that at the junction of inflation and reheat

\[
H = \sqrt{4\pi G m \varphi_0 \sqrt{p/(2p-1)}} \tag{58}
\]

and one can use this in Eq.(56) to give

\[
y = \frac{k}{aH} |p| \tag{59}
\]

To summarize: \( m\varphi_0 \) and \( p \) determine the input of \( \psi \) and \( \varphi_1 \) to the reheat era and the reheating equations of Section III determine subsequent developments.

V. RESULTS

Our interest is in the qualitative features of reheat in a model compatible with known observations rather than in making precise comparisons with data. So we are interested in values of the parameters that roughly supply the usual requirements of inflation and the magnitude of the CMBR fluctuations.

Unless otherwise stated the results we quote are for the values \( m = 10^{-7}, \varphi_0 = 0.3, p = -1.1, M/m = 0.02 \). Throughout we quote dimensionful results and parameters in units such that \( \hbar = c = G = 1 \).

The critical parameters are the coupling constant \( g^2 \) and the frictional decay constants \( \Gamma_\chi \) and \( \Gamma_\varphi \) of the \( \chi \) and \( \varphi \). The quartic coupling constant \( g^2 \) governs the number of \( \chi \) produced and in particular the parametric resonance. We have considered values in the range given by \( 20,000 > g/m > 1000 \). For smaller values there is no appreciable particle production. The range of consideration for the frictional decay constants was \( 0.1 \geq \Gamma_\chi/m \geq 0 \) and \( 0.005 \geq \Gamma_\varphi/m \geq 0 \). Larger values of \( \Gamma_\chi \) tend to suppress parametric resonance while larger values of \( \Gamma_\varphi \) may be unrealistic in the context of the development of the classical inflaton field. We have by no means done a complete scan over this parameter range but in each particular case we have looked at (i) the existence or not of parametric resonance (ii) the development of the metric inhomogeneity \( \psi \) and also \( \zeta \) (iii) the post reheating state; and other significant features. To illustrate significant features the figures show a case with parametric resonance in the production of \( \chi \) particles and \( \Gamma_\chi/m = 0.001, \Gamma_\varphi/m = 0.0004 \) so that no \( \chi \) particles are finally left. If both \( \Gamma \)'s are zero then reheating (as opposed to preheating) does not happen. But it is interesting to inspect this case (also shown in the figures) to look at what the possible variation of \( \zeta \) might be if the decay of the \( \chi \) and \( \varphi \) particles is significantly slower.
A. The Metric Perturbation, $\psi$

As noted in section III the metric is given by $ds^2 = a(\tau)^2(1 + 2\psi(x, \tau))d\tau^2 - a(\tau)^2(1 - 2\psi(x, \tau))\delta_{ij}dx^i dx^j$. From the end of reheat $\psi$ develops through the radiation and matter eras so that the magnitude of the components $\psi(k, \tau)$ for very small $k$ is a determinant of the microwave background perturbation.

1. The magnitude of $\psi$

The validity of the perturbation formalism depends on $\psi(x, \tau)$ being small and for each case we should investigate this, noting that it is not a quite straightforward concept.

We shall discuss $\psi$ as having developed into a classical stochastic field, Eq.(38). The indeterminancy represented by the variables $e$ forces us to consider the average over the product of these variables so that we evaluate $\langle \psi(x, \tau)^2 \rangle$ the result being

$$\langle \psi(x, \tau)^2 \rangle = (2\pi)^{-3} \int d^3k \psi_k(\tau)\psi_k(\tau)^*, \hspace{1cm} (60)$$

the same for every value of $x$. We require that $\sqrt{\langle \psi(x, \tau)^2 \rangle}$ be small compared with unity, since the relevant metric coefficient, Eq.29, is $a(\tau)^2(1 + 2\psi)$. Our viewpoint is that in any particular case satisfaction of this requirement forms sufficient justification for the perturbative approach, because the only way we can mount a comparison of the revised metric coefficient with $a(\tau)^2$ is when we consider the basic equations to be those in configuration space, and then indeed the revision is just a perturbation through all space-time. It is true that if, having written the equations of motion in configuration space, we then project into $k$-space to solve them, as we do, then certain of the $\psi(k)$ will be very large. But at this stage we are operating translation into $k$-space as a device to aid solution. Also then the comparators from $a(\tau)$ will be much larger, these translating either into a $k$-space $\delta$-function or, in the non-ideal case, into a function very strongly peaked round the origin in $k$-space.

In the cases illustrated we see from Figure 3 that the requirement is very well satisfied throughout and from Figure 4 that the requirement is also satisfied though significantly better in the first part of reheating.

B. Production of $\chi$ and Inflaton Particles

It is a generic feature of this model, and probably of most other similar models, that the inflaton particles or inhomogeneous field, $\varphi_1$, that enter the reheating period from inflation attain an energy density comparable or greater than that of the classical inflaton field, $\varphi_0$, during the preheating period and thereafter. In their equation of motion, (37), we note the bilinear couplings to $\varphi_0$ and the metric perturbation, $\psi$. Thus in preheating $\varphi_1$ cannot be treated as a perturbation and we cannot neglect factors like $\langle \varphi_1^2 \rangle$.

The occurrence of parametric resonance for $\chi$ is as expected sensitive to the value of $g$, and it is also, naturally, sensitive to the value of $\Gamma_\chi$ which tends to inhibit it. In the case which we illustrate we see from Figure 1 that parametric resonance occurs markedly
but rather briefly and that afterwards the $\chi$ particles almost vanish under the influence of the friction decay into a radiation fluid. As previously noted $\chi$ and $\psi$ have independent stochastic variables.

C. Magnitude of $\zeta$ During Reheating

It has been found useful to define the parameter $\zeta$ of Eq.43 where $\zeta_k \approx -R_k$, the curvature perturbation $[3,23]$. For adiabatic perturbations with $k^2 \ll a^2H^2$, $\zeta_k$ is constant. For example, in eras where there is only one scalar field or only one hydrodynamic fluid, then the perturbations are adiabatic. This holds before the reheating period because the scalar field $\chi$ is negligible, as we have seen above. However in the reheating period we have two scalar fields $\varphi_1, \chi$ and also the radiation fluid (which arises from dissipative processes) giving rise to an entropy perturbation, $[4]$ and thus the theorem does not apply. Importantly, in addition, $\varphi_1$ can no longer be treated as a perturbation and we do not do so as has been emphasized in section V B.

In the equations of motion the wave number appears as $k^2$ and we evaluate $\zeta$ for a value of $k$ such that $k^2 < 10^{-4}a^2H^2$ throughout the reheating period. We see from Figs.3 and 5 that $\zeta$ has in the radiation era modestly decreased from its value at the end of the inflationary era. When we switch off the dissipative decay into a fluid, so that there is preheating without reheating, $\zeta$ now increases markedly as shown in Figs. 4 and 6. We also see from Figs. 5 and 6 that $\zeta$ is initially constant as it should be since $\varphi_1$ is then small enough to be treated to first order only.

VI. SUMMARY AND CONCLUSIONS

We have studied a single field ($\varphi$) model of inflation, with another scalar field ($\chi$) which is naturally quiescent during inflation and becomes active post inflation with the possibility of parametric resonance and preheating as the classical part, $\varphi_0$, of the inflaton field decreases and oscillates. We have included reheating through friction-type decay mechanisms of the scalar fields into a gas of unspecified relativistic particles. We have adopted a method having stochastic variables naturally succeeding the quantum operators of the early inflation era and have developed equations for the post-inflation period by taking averages over the stochastic bilinear forms in the scalar fields. This enables a non-perturbative treatment of the stochastic scalar fields $\chi$ and $\varphi_1$ (where $\varphi = \varphi_0 + \varphi_1$) though the metric perturbation field, $\psi$, has to be treated perturbatively.

$\zeta$ being a linear function of the curvature perturbation $\psi$ arises with the same quantum operators as the non-homogeneous inflaton field $\varphi_1$ as was noted for $\psi$ in section III. In the successor classical stochastic variables the mode functions of $\varphi_1, \psi$ and $\zeta$ are complex but as we have noted previously can be converted to real by a time independent phase transformation $[17]$. It is this real quantity which we quote here for $\zeta$. 

17
We find that the non-homogeneous scalar field (which might also be described as bearing inflaton particles), $\varphi_1$, remains strong with an energy density which competes (according to the values of the decay parameters) with that of the gas of relativistic particles, during a large part of the reheating period.

There are cases of the parameters in which the metric perturbation $\psi$ markedly resonates, at much the same time as a parametric resonance in $\chi$, and exceeds the perturbative limit. We have to reject such cases from consideration though we cannot say whether the failure be mathematical or physical. There are, however, many valid cases.

The curvature parameter, $\zeta$, for wavelengths relevant to the cosmic microwave background fluctuations occurs very conveniently as a constant in non-entropic single field inflation models. In this present more complicated model it changes from its value in the inflationary period by up to a factor of the order of 10 in the examples presented in this paper.

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FIG. 1. Logarithmic plot of energy densities of $\chi$ (A) and $\varphi_0$ (B) versus the number of e-folds of expansion in the reheat era. The parameters are $g/m = 10^4, \Gamma_\chi/m = 0.01, \Gamma_\varphi/m = 0.004$, with other constants as given in the text, Section V. In all figures densities are in units $m^2m^2_{Planck}$, where $m = 10^{-7}m_{Planck}$.

FIG. 2. $\log \rho = A$ and $\log(\text{energy density}(\varphi_1)) = B$ versus the number of e-folds of expansion in reheating. $\rho$ is the density of the radiation gas and $\varphi_1$ is the inhomogeneous (or particle) part of the inflaton field. The parameters are as for Figure 1.
FIG. 3. $\log(|\psi^2|) = A$ and $\log(\zeta) = B$ where $\psi$ is the metric perturbation and $\zeta$ is defined in the text. The parameters are as for Figure 1.

FIG. 4. As for Figure 3, but with $\Gamma_\chi = 0, \Gamma_\phi = 0$
FIG. 5. log(\(\zeta\)) of Figure 3 shown in more detail for the first 8 efolds of reheating.

FIG. 6. log(\(\zeta\)) of Figure 4 shown in more detail for the first 7 efolds of reheating. Comparison of this with Figure 5 illustrates differences in \(\zeta\) in the cases with and without frictional decay of the scalar fields.

APPENDIX A: BIANCHI IDENTITY AND THE FRICTIONAL DECAY TERMS.

Working to first order in the metric perturbation \(\psi\) a Bianchi identity gives

\[
D_\nu T^\nu_0 \equiv E_\rho + \varphi' S_\varphi + \chi' S_\chi = 0 \tag{A1}
\]
\[ E_\rho \equiv \rho' + (\rho + p)(3\alpha - 3\psi' + (\partial_t - 2\psi, i)u_0 \partial u^i) \]  
(A2)

\[ S_\varphi = (1 - 2\psi)(\varphi'' + 3\alpha \varphi' - 3\psi' \varphi') - \psi' \varphi' + a^2 \tilde{m}^2 \varphi - (1 + 2\psi)\varphi, i, i \]  
(A3)

\[ S_\chi = (1 - 2\psi)(\chi'' + 3\alpha \chi' - 3\psi' \chi') - \psi' \chi' + a^2 \tilde{M}^2 \chi - (1 + 2\psi)\chi, i, i \]  
(A4)

where, from the space-time Einstein equation, the fluid velocity, \( u^i \), is given by

\[ a^{-2}(\rho + p)u_0 u^i = -\varphi' \varphi, i + (4\pi G)^{-1}(\psi' + \alpha \psi). \]  
(A5)

Equating the expressions in (A2, A3, A4) to zero separately gives the fluid equation and the scalar equations. To add friction we use Eq.(A1) to write

\[ [E_\rho - \Gamma_\varphi(\varphi')^2 - \Gamma_\chi(\chi')^2] + \varphi'[S_\varphi + \Gamma_\varphi \varphi'] + \chi'[S_\chi + \Gamma_\chi \chi'] = 0. \]  
(A6)

This equation, which is equivalent to the Bianchi identity, is solved by equating each term in square brackets to zero. This modifies the fluid and scalar equations to give a transfer of fluid and scalar densities which however is modulated by the complexity of our complete set of equations. In our model \( \rho = \rho_0 + \rho_1, \varphi = \varphi_0 = \varphi_1 \) where \( \rho_0, \varphi_0 \) are homogeneous and \( \rho_1, \varphi_1 \) are non-homogeneous and along with \( \psi \) linear in the same stochastic variables - this being forced through consistency of the various equations of the theory; \( \chi \) is linear in different stochastic variables. Each of the first two equations can be divided into a homogeneous part and a part linear in the stochastic variables with the occurrence of quadratic or cubic stochastic variable terms being dealt with by appropriate ensemble averaging. Thus each of the first two equations is divided into two halves which must separately be zero and after some cross substitution we get, together with the equation from the third square bracket the equations for \( \rho_0, \rho_1, \varphi_0, \varphi_1 \), and \( \chi \) which appear in the text.

**APPENDIX B: EVALUATION OF THE LOOP INTEGRALS.**

Integrals such as that in Eq.(20) occur throughout the equations of motion and are evaluated numerically and we have to adopt a finite range of wave number, \( k \). If the integrals diverge as \( k \to \infty \) then the upper limit of the \( k \) integration forms a cut-off which is the crudest way of dealing with such ultra-violet divergences. Two points of view may be taken on this. Firstly this may be considered equivalent to a renormalization procedure in mass and other quantities. Secondly the Lagrangian used may be considered as an effective Lagrangian which has absorbed extra degrees of freedom coming from supersymmetry which eliminate divergences at higher momentum, the cut-off representing this effect. We have taken a cut-off to correspond to a wavelength of \( H^{-1} \) so that \( k_{\text{cutoff}} \approx 2\pi a H \).

Also an infra-red cutoff is needed when the spectral index \( n \) is less than 1 irrespective of the particular inflation model (of the normal type); for the power law inflation model \( n \) is always less than 1. Inflationary theories which are in agreement with COBE observations have a value of \( n \) near, either greater or less than, 1 and thus there is a steep rise of (for a principal example) \( \varphi_k \) as \( k \to 0 \) as mentioned above in section IV B 4. First we
should make the general remark, independent of any particular value of $n$, that for $k = 0$ the inflation era perturbation is homogeneous and may be regarded as incorporable in the classical homogeneous field $\varphi_0$. The problem arises that for $k$ near to 0 the perturbation is nearly homogeneous and also a singularity of the loop integrand can arise at the end of inflation and in reheat. When the actual integral diverges the problem is acute and we deal with it by an infra-red cut-off taking the lower limit of the integration as $k = k_0$ so that Eq.[18]
becomes $\langle \varphi_1^2 \rangle = (2\pi)^{-3} \int_{k_0} d^3 k \varphi_k \varphi^*_k$. The perturbations for $k < k_0$ we regard as hidden by renormalization of the field $\varphi_0$. (In the equations of motion $\langle \varphi_1^2 \rangle$ always occurs in association with $\varphi_2^2$.) $k_0$ should be small enough so that a $\varphi_k$ which gives rise to observable effects is explicitly included in the loop integral. Thus $k_0 < k_m \equiv a_m H_m$ where the suffix $m$ denotes the value at the time when the matter era begins, as observations of the CMBR fluctuations are possible at $k$-values nearly of this order; $k_m$ is tiny compared with the corresponding value of $aH$ at reheat.

Now consider any value of $k$ such that $k \ll aH$ throughout the reheating era; then $\varphi_k \propto k^{n/2-2}$ at the end of inflation and since for such small values of $k$ (wavelength much greater than the Hubble radius) the $k^2$ term in the reheating equations of motion can be neglected means that the initial proportionality is maintained throughout reheating and thus

$$\varphi_k = \tilde{\varphi}(\tau) k^{n/2-2}. $$

We thus evaluate the loop integral as

$$\langle \varphi_1^2 \rangle = (2\pi)^{-1} \int_{k_0}^{k_1} \varphi(\tau) \varphi(\tau)^* k^{n-2} dk + (2\pi)^{-3} \int_{k_1} d^3 k \varphi_k \varphi_k^* \quad \text{(B1)}$$

where $k_1 \ll aH$ throughout reheating and the last term can be evaluated approximately using discrete values of the integrand since the delicate feature is included in the first term on the right hand side.

We shall now treat these small $k$ contributions in more detail, distinguishing the two cases $n < 1$ and $n > 1$.

**Case 1:** $n < 1$. Let $n = 1 - \epsilon$ where $\epsilon > 0$. The analytic integration in Eq.[31] is

$$\int_{k_0}^{k_1} k^{-1-\epsilon} dk = (k_0^{-\epsilon} - k_1^{-\epsilon})/\epsilon. \quad \text{(B2)}$$

Because $k_0 = O(k_m)$ and $\epsilon = O(.1)$ the first term on the right of Eq.[31] contributes a relatively large positive quantity to the loop integral. For the power-law inflation model $n < 1$ so this is the case that applies and is used in this paper.

**Case 2:** $n > 1$. Let $n = 1 + \epsilon$ where $\epsilon > 0$.

$$\int_{k_0}^{k_1} k^{-1+\epsilon} dk = (k_1^{\epsilon} - k_0^{\epsilon})/\epsilon \approx k_1^{\epsilon}/\epsilon. \quad \text{(B3)}$$

So the corresponding contribution is much smaller than that in the preceding case which applies in this paper.
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