On the Degree Sequence and its Critical Phenomenon of an Evolving Random Graph Process

Xian-Yuan Wu\(^1\), Zhao Dong\(^2\), Ke Liu\(^2\) and Kai-Yuan Cai\(^3\)

\(^1\)School of Mathematical Sciences, Capital Normal University, Beijing, 100037, China. Email: wuxy@mail.cnu.edu.cn
\(^2\)Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100190, China. Email: dzhao@amss.ac.cn; kliu@amss.ac.cn
\(^3\)Department of Automatic Control, Beijing University of Aeronautics and Astronautics, Beijing, 100083, China. Email: kycai@buaa.edu.cn

Abstract: In this paper we focus on the problem of the degree sequence for the following random graph process. At any time-step \(t\), one of the following three substeps is executed: with probability \(\alpha_1\), a new vertex \(x_t\) and \(m\) edges incident with \(x_t\) are added; or, with probability \(\alpha - \alpha_1\), \(m\) edges are added; or finally, with probability \(1 - \alpha\), \(m\) random edges are deleted. Note that in any case edges are added in the manner of preferential attachment. We prove that there exists a critical point \(\alpha_c\) satisfying: 1) if \(\alpha_1 < \alpha_c\), then the model has power law degree sequence; 2) if \(\alpha_1 > \alpha_c\), then the model has exponential degree sequence; and 3) if \(\alpha_1 = \alpha_c\), then the model has a degree sequence lying between the above two cases.

1 Introduction and statement of the results

Recently there has been much interest in studying large-scale real-world networks and attempting to model their properties. For a general introduction to this topic, readers can refer to Albert and...
Although the study of real-world networks as graphs can be traced back to long time ago such as the classical model proposed by Erdős and Rényi [14] and Gilbert [16], recent influential activity perhaps started with the work of Watts and Strogatz about the ‘small-world phenomenon’ published in 1998 [27]. Another influential work may be due to the scale-free model proposed by Bollobás and Albert in 1999 [5]. Since then various forms of scale-free phenomenon have been widely revealed. In particular, power law degree distributions have been extensively investigated. Many new models have been introduced to circumvent the shortcomings of the classical models introduced by Erdős and Rényi [14] and Gilbert [16]. One class of these new models was aimed to explain the underlying causes for the emergence of power law degree distributions. This can be observed in ‘LCD model’ [10] and its generalization due to Buckley and Osthus [8], ‘copying’ models of Kumar et al. [19], the very general models defined by Copper and Frieze [12] and the other model with random deletions defined by Copper, Frieze and Vera [13] etc.

For the real-world network of World Wide Web/Internet, experimental studies by Albert, Barabási and Jeong [2], Broder et al. [7] and Faloutsos, Faloutsos and Faloutsos [15] demonstrated that the proportion of vertices of a given degree follows an approximate inverse power law, i.e., the proportion of vertices of degree $k$ is approximately $Ck^{-\alpha}$ for some constants $C$ and $\alpha$. However other forms of the degree distributions can also be observed in real-world networks (see [4] and [25]). For example, Guassian distributions can be observed in the acquaintance network of Mormons [6]; exponential distribution can be observed in the powergrid of southern California [27]. On the other hand, the degree distribution of the network of world airports [4] interpolates between Gaussian and exponential distributions, whereas the degree distribution of the citation network in high energy physics [20] interpolates between exponential and power law distributions. For more forms of degree distributions, readers can refer to [24].

Different models often lead to different forms of degree distributions. An interesting problem arises naturally: does it exist some dynamically evolving random graph process which brings forth various degree distributions by continuous changing of its parameters only? This phenomenon has been numerically investigated in reference [28]: For a general model of collaboration networks in [28], Zhou et al. indicate that, while a relevant parameter $\alpha$ increases from 0 to 1.5, four kinds of degree distributions appear as exponential, arsy-varsy, semi-power law and power law in turn. Note that the above classification is rather rough as no unambiguous borderline between two neighboring patterns is determined. However, to the best of our knowledge, it seems that the problem and its answer have not been formulated in a mathematically rigorous manner. In this paper we focus on a model with edge deletions and provide precise analysis, while a parameter varies, the model exhibits various degree
distributions.

Now, we begin to introduce our model and then state our main results. Consider the following process which generates a sequence of graphs \( G_t = (V_t, E_t) \), \( t \geq 1 \). Write \( v_t = |V_t| \) and \( e_t = |E_t| \).

**Time-Step 1.** Let \( G_1 \) consist of an isolated vertex \( x_1 \).

**Time-Step \( t \geq 2 \).**

1. With probability \( \alpha_1 > 0 \) we add a vertex \( x_t \) to \( G_{t-1} \). We then add \( m \) random edges incident with \( x_t \). In the case of \( e_{t-1} > 0 \), the \( m \) random neighbours \( w_1, w_2, \ldots, w_m \) are chosen independently.

For \( 1 \leq i \leq m \) and \( w \in V_{t-1} \),

\[
\mathbb{P}(w_i = w) = \frac{d_w(t-1)}{2e_{t-1}},
\]

where \( d_w(t-1) \) denotes the degree of vertex \( w \) at the beginning of substep \( t \). Thus neighbours are chosen by preferential attachment. In case of \( e_{t-1} = 0 \), then we add a new vertex \( x_t \) and join it to a randomly chosen vertex in \( V_{t-1} \).

2. With probability \( \alpha - \alpha_1 \geq 0 \) we add \( m \) random edges to existing vertices. If \( e_{t-1} > 0 \), then both endpoints are chosen independently with the same probabilities as in (1.1). Otherwise, we do nothing.

3. With probability \( 1 - \alpha \geq 0 \) we delete \( \min\{m, e_{t-1}\} \) randomly chosen edges from \( E_{t-1} \).

**Remark 1.1** The difference between our model and the model introduced in [13] is that, in our setting, vertex deletions, loop and multi-edge erasures are forbidden, which makes \( \{e_t : t \geq 1\} \) Markovian and makes it possible for us to give exact estimation to \( e_t \).

In order to make the problem meaningful, the following inequalities are natural and necessary:

\[
\frac{1}{2} < \alpha \leq 1; \quad 0 < \alpha_1 \leq \alpha.
\]

(1.2)

For given \( \alpha \) and \( \alpha_1 \) satisfying (1.2), define

\[
\alpha_c := 4\alpha - 2, \quad \eta := \alpha_e m/2,
\]

(1.3)

and choose \( \epsilon = \epsilon(\alpha, \alpha_1) \in (0, \eta) \) such that

\[
\rho_\epsilon := \max \left\{ \frac{m(\alpha_c - \alpha_1)}{2(\eta - \epsilon)}, \frac{1}{2} \right\} < 1.
\]

(1.4)

Note that in case of \( \alpha_1 \geq \alpha_c \), \( \rho_\epsilon = \frac{1}{2} \). Let

\[
\beta = \frac{\alpha_c}{\alpha_c - \alpha_1}; \quad \gamma = 1 - \frac{\alpha_1 - \alpha_c}{2(1 - \alpha)}; \quad \theta = \frac{2\alpha_c - \alpha_1}{2\alpha_c}; \quad \mu = \frac{\alpha_c}{2(1 - \alpha)}.
\]

(1.5)
Obviously, $\beta$ is well defined when $\alpha_1 \neq \alpha_c$ and $0 < \gamma < 1$ when $\alpha_1 > \alpha_c$. To get our main results, besides (1.2), the following condition is necessary

$$\alpha_1 < 2\alpha_c. \quad (1.6)$$

Now, let $D_k(t)$ be the number of vertices with degree $k \geq 0$ in $G_t$ and let $\overline{D}_k(t)$ be the expectation of $D_k(t)$. The main results of this paper follow as

**Theorem 1.1** Assume that (1.2) and (1.6) hold. Then $\alpha_c$ defined in (1.3) is a critical point for the degree sequence of the model satisfying:

1) if $\alpha_1 < \alpha_c$, then there exists a constant $C_1 = C_1(m, \alpha, \alpha_1)$ such that, for any $\nu \in (0, 1 - \rho_c)$,

$$\left| \frac{\overline{D}_k(t)}{t} - C_1 k^{-1-\beta} \right| = O(t^{\alpha_1 + \nu - 1}) + O(k^{-2-\beta}); \quad (1.7)$$

2) if $\alpha_1 > \alpha_c$, then there exists a constant $C_2 = C_2(m, \alpha, \alpha_1)$ such that

$$\left| \frac{\overline{D}_k(t)}{t} - C_2 \gamma^k k^{-1+\beta} \right| = O(t^{-\theta}) + O(\gamma^k k^{-2+\beta}); \quad (1.8)$$

3) if $\alpha_1 = \alpha_c$, then there exists a constant $C_c = C_c(m, \alpha, \alpha_1)$ such that, for any $\nu \in (0, \frac{1}{2})$,

$$\left| \frac{\overline{D}_k(t)}{t} - C_c u_c(k) \right| = O(t^{-\frac{1}{2} + \nu}) \quad (1.9)$$

uniformly in $k$.

Where $u_c(k) = \int_0^1 t^{k-1} e^{-\frac{t}{\mu}} dt$ and $\beta, \gamma, \theta$ and $\mu$ are given in (1.3).

**Remark 1.2** The integral $u_c(k) = \int_0^1 t^{k-1} e^{-\frac{t}{\mu}} dt$ can be rewritten as

$$u_c(k) = \left[ \sum_{i=0}^{k-2} \sum_{l=0}^{k-2-i} \binom{k-1}{i} \binom{k-i-l-1}{k-i} (-1)^{k-i-l-1} e^{-\mu} \right] + \left[ \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{\mu^{k-i-1}}{(k-i)!} \right] \int_1^{+\infty} t^{-2} e^{-\mu t} dt.$$

With help of computer calculation, $u_c(k)$ satisfies

$$\lim_{k \to \infty} \ln u_c(k)/(-k) = \lim_{k \to \infty} (-\ln k)/\ln u_c(k) = 0.$$

Based on Theorem 1.1, we can obtain following two corollaries, which provide a complete distinction with respect to the parameters between the degree sequences for the present model.
Corollary 1.2 If the parameters satisfy that

1) $\alpha > 2/3$; or

2) $\alpha \leq 2/3$ and $\alpha_1 < \alpha_c$,

then the present random graph process has the power law degree sequence (1.7).

Corollary 1.3 Assume $\alpha \leq 2/3$.

1) If $\alpha_c < \alpha_1 < 2\alpha_c$, then the present random graph process has the exponential degree sequence (1.8).

2) If $\alpha_1 = \alpha_c$, then the present random graph process has the critical degree sequence (1.9).

Remark 1.3 When $\alpha > 2/3$, for any $\alpha_1$, the inequality $\alpha_1 \leq \alpha < \alpha_c = 4\alpha - 2$ holds always, therefore, the part 1) of Corollary 1.2 follows from the part 1) of Theorem 1.1. The part 2) of Corollary 1.2 and Corollary 1.3 are straightforward from Theorem 1.1.

Remark 1.4 A special case of the part 1) in Corollary 1.2 is $\alpha = \alpha_1 = 1$. In this case, the model has a power law degree sequence as $Ck^{-3}$, which coincides with the result of [11]. Furthermore, for any $\alpha \in (1/2, 1]$ and $\alpha_1 = 2\alpha - 1$, the model has the degree sequence $Ck^{-3}$.

Remark 1.5 The results are unclear for the following case: $\alpha \leq 2/3$, $2\alpha_c \leq \alpha_1 \leq \alpha$. Clearly, this case can only appear when $\alpha \leq 4/7$. It is natural to conjecture that the model possesses an exponential degree sequence in this case.

The methodology of the proof for the main results follows the standard procedure which can be found in [12] and [13]. The rest of the paper is organized as follows. In Section 2, we bound the degree of vertex in $G_t$. In Section 3, we establish the recurrence for $D_k(t)$ and then derive the approximation of $D_k(t)$ by a recurrence with respect to $k$. Finally, in section 4, we solve the recurrence in $k$ using Laplace’s method [18] and finish the proof of Theorem 1.1.

2 Bounding the Degree

For times $s$ and $t$ with $1 \leq s \leq t$, let $d_{xs}(t)$ be the degree of vertex $x_s$ in $G_t$. If $x_s$ is not added in Time-Step $s$, i.e., at Time-Step $s$, one of the other two substeps is executed, put $d_{xs}(t) = 0$. In this section, we will concentrate on the upper bound of $d_{xs}(t)$.

For the present model, the estimation for $v_t$ is derived in [13] as

$$|v_t - \alpha_1 t| \leq ct^{1/2} \log t, \quad \text{qs},$$
for any constant $c > 0$. We say an event happens quite surely (qs) if the probability of the complimentary set of the event is $O(t^{-K})$ for any $K > 0$.

For the estimation of $e_t$, it can be derived by the same argument as in [13] that

$$|e_t - \eta t| \leq ct^{1/2} \log t, \quad \text{qs,}$$

(2.1)

for any constant $c > 0$.

By a standard argument on large deviation (see e.g. [21] and [23]), one further has: for any $\epsilon > 0$, there exists $c_1, c_2 > 0$ such that

$$P(e_t \leq (\eta - \epsilon)t) \leq c_1 \exp\{-c_2t\},$$

(2.2)

for all $t \geq 1$.

The following is our bounding for $d_{x_s}(t)$, note that our result is based on the exact estimation (2.2) for $e_t$. In our opinion, to bound the degree of vertex effectively, beforehand good estimations for $e_t$ are necessary.

**Lemma 2.1** For any $\alpha \in (1/2, 1]$ and $\alpha_1 \in (0, \alpha]$,

$$d_{x_s}(t) \leq (t/s)^{\alpha_1} (\log t)^3 \quad \text{qs,}$$

(2.3)

where $\rho_\epsilon$ is given in (1.4).

**Proof:** Fix $s \leq t$, suppose that $x_s$ is added in Time-Step $s$. Let $X_{\tau} = d_{x_s}(\tau)$ for $\tau = s, s + 1, \ldots, t$ and let

$$\lambda = \frac{(s/t)^{\rho_\epsilon}}{NM_\epsilon(\log t + 1)},$$

(2.4)

where $N$ be large enough and will be determined later, and $M_\epsilon = \frac{12m_2}{\eta - \epsilon}$. Let $Y$ be the $\{1, 2, 3\}$-valued random variable with $P(Y = 1) = \alpha_1$, $P(Y = 2) = \alpha - \alpha_1$ and $P(Y = 3) = 1 - \alpha$. Then conditional on $X_{\tau} = x$ and $e_{\tau} \geq m$, we have

$$X_{\tau+1} = x + I_{\{Y=1\}} B(m, \frac{x}{2e_{\tau}}) + I_{\{Y=2\}} B(2m, \frac{x}{2e_{\tau}}) - I_{\{Y=3\}} S(m, \frac{x}{e_{\tau}}),$$

(2.5)

where $B(m, p)$ is the Binomial random variable with parameter $(m, p)$ and $S(m, \frac{x}{e_{\tau}})$ is the super geometric random variable with parameter $(e_{\tau}, x, m)$.

Noticing that $\lambda$ is small enough for large $N$, using the basic inequality

$$e^{-y} \leq 1 - y + 2y^2$$

for small $y > 0$.

6
and the fact that $S(m, \frac{x}{e_\tau}) \leq m$, implies
\[ \mathbb{E} \left( \mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_\tau = x, e_\tau) \mid e_\tau \geq m \right) \]
\[ \leq e^{\lambda x} \left\{ \alpha_1 \left[ 1 + \frac{x}{2e_\tau} (e^\lambda - 1) \right]^m + (\alpha - \alpha_1) \left[ 1 + \frac{x}{2e_\tau} (e^\lambda - 1) \right]^{2m} \right. \]
\[ \left. + (1 - \alpha) \left[ 1 - \lambda \mathbb{E} \left( S(m, \frac{x}{e_\tau}) \right) + 2\lambda^2 m \mathbb{E} \left( S(m, \frac{x}{e_\tau}) \right) \right] \right\}. \quad (2.6) \]

Using the inequalities
\[ e^y \leq 1 + y + 2y^2 \text{ for small } y > 0 \]
and
\[ (1 + y)^m \leq 1 + my + \frac{m^2}{2} y^2 \text{ for small } y > 0 \]
to the right hand side of (2.6) in turn, we get
\[ \mathbb{E} \left( \mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_\tau = x, e_\tau) \mid e_\tau \geq m \right) \leq e^{\lambda x} \left\{ 1 + \frac{m \lambda x}{2e_\tau} (\alpha_c - \alpha_1) + 12m\lambda \right\} \]
\[ \leq e^{\lambda x} \left\{ 1 + \frac{m \lambda x}{2e_\tau} \max\left( (\alpha_c - \alpha_1), \frac{\eta - \epsilon}{m} \right) + 12m\lambda \right\} \]
\[ \leq e^{\lambda x} \left\{ 1 + \frac{m \lambda x}{2e_\tau} \max\left( (\alpha_c - \alpha_1), \frac{\eta - \epsilon}{m} \right) (1 + M_\tau \lambda) \right\} \]
\[ \leq \exp \left\{ \lambda x \left[ 1 + \max\left( (\alpha_c - \alpha_1), \frac{\eta - \epsilon}{m} \right) \frac{m}{2e_\tau} (1 + M_\tau \lambda) \right] \right\}. \quad (2.7) \]

Now, we express $\mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_\tau = x)$ as
\[ \mathbb{E} \left( e^{\lambda X_{\tau+1}} \mid X_\tau = x \right) = \mathbb{E} \left( \mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_\tau = x, e_\tau) \right) \]
\[ = \mathbb{E} \left( \mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_\tau = x, e_\tau) \mid e_\tau < m \right) \mathbb{P}(e_\tau < m) \]
\[ + \mathbb{E} \left( \mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_\tau = x, e_\tau) \mid e_\tau \geq m \right) \mathbb{P}(e_\tau \geq m) \]
\[ =: I + II. \quad (2.9) \]

On one hand, conditional on $X_\tau = x$ and $e_\tau < m$, $X_{\tau+1} \leq x + m \leq e_\tau + m \leq 2m$ holds always, so
\[ I \leq e^{2m \mathbb{P}(e_\tau < m)}. \quad (2.10) \]
On the other hand, \( I \) can be expressed as

\[
I = \mathbb{E} \left( \mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_{\tau} = x, e_{\tau}) \mid e_{\tau} \geq m, e_{\tau} \geq (\eta - \epsilon)\tau \right) \\
\times \mathbb{P} \left( e_{\tau} \geq m, e_{\tau} \geq (\eta - \epsilon)\tau \right) \\
+ \mathbb{E} \left( \mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_{\tau} = x, e_{\tau}) \mid e_{\tau} \geq m, e_{\tau} < (\eta - \epsilon)\tau \right) \\
\times \mathbb{P} \left( e_{\tau} \geq m, e_{\tau} < (\eta - \epsilon)\tau \right),
\]

by (2.8) and the fact that \( x \leq e_{\tau} \),

\[
I \leq \exp \left\{ \lambda x \left[ 1 + \frac{\max\{(\alpha_{\epsilon} - \alpha_{1}), \frac{\eta - \epsilon}{m}\} m}{2(\eta - \epsilon)\tau} (1 + M_{\epsilon} \lambda) \right] \right\} \\
\times \exp \left\{ \max\{(\alpha_{\epsilon} - \alpha_{1}), \frac{\eta - \epsilon}{m}\} m \lambda \right\} \mathbb{P}(e_{\tau} < (\eta - \epsilon)\tau)
\]

for some constant \( C' = C'(\alpha, \alpha_{1}, \epsilon, m) > 0 \). By (2.2) and (2.4), choosing \( N \) large enough, then there exists constants \( c_{3}, c_{4} > 0 \) such that

\[
I \leq \exp \left\{ \lambda x \left[ 1 + \frac{\rho_{\epsilon}}{\tau} (1 + M_{\epsilon} \lambda) \right] \right\} + c_{3} \exp\{-c_{4}\tau\}. \quad (2.11)
\]

Combining (2.9)-(2.12), using (2.2) again for (2.10), then there exists constants \( c_{5}, c_{6} > 0 \) such that

\[
\mathbb{E}(e^{\lambda X_{\tau+1}} \mid X_{\tau} = x) \leq \exp \left\{ \lambda x \left[ 1 + \frac{\rho_{\epsilon}}{\tau} (1 + M_{\epsilon} \lambda) \right] \right\} + c_{5} \exp\{-c_{6}\tau\}. \quad (2.13)
\]

Thus

\[
\mathbb{E}(e^{\lambda X_{\tau+1}}) \leq \mathbb{E} \left( \exp \left\{ X_{\tau} \lambda \left[ 1 + \frac{\rho_{\epsilon}(1 + M_{\epsilon} \lambda)}{\tau} \right] \right\} \right) + c_{5} \exp\{-c_{6}\tau\}. \quad (2.14)
\]

Now, put \( \lambda_{t} = \lambda \) and \( \lambda_{\tau-1} = \lambda_{\tau} \left( 1 + \frac{\rho_{\epsilon}(1 + M_{\epsilon} \lambda)}{\tau} \right) \). Obviously, if \( \lambda_{t} \) is small enough, then (2.14) holds for \( \lambda_{t+1}, \tau = s, s+1, \ldots, t-1 \). This will imply that

\[
\mathbb{E}(e^{\lambda X_{t}}) = \mathbb{E}(e^{\lambda_{\tau} X_{\tau}}) \leq \mathbb{E}(e^{\lambda_{\tau} X_{\tau}}) + c_{5} \sum_{\tau=s}^{t} \exp\{-c_{6}\tau\} \leq e^{m\lambda_{s} + C''} \quad (2.15)
\]

for some constant \( C'' > 0 \).

Let \( \Lambda = \frac{10}{N M_{\epsilon}(\log t + 1)} \), note that \( \Lambda \) can be taken small enough uniformly in \( t \) by taking \( N \) large enough. Now provided \( \lambda_{\tau} \leq \Lambda \), we can write

\[
\lambda_{\tau-1} \leq \lambda_{\tau} \left( 1 + \frac{\rho_{\epsilon}(1 + M_{\epsilon} \lambda)}{\tau} \right)
\]
and then
\[
\lambda_s \leq \lambda \prod_{\tau=s}^{t} \left(1 + \frac{\rho_s(1 + M_s \Lambda)}{\tau}\right) \leq 10 \lambda (t/s)^{\rho_s}
\]
which is \(\leq \Lambda\) by the definition of \(\lambda\).

Put \(u = (t/s)^{\rho_s} (\log t)^3\), by (2.15) we get
\[
\mathbb{P}(X_t \geq u) \leq (e^{m \lambda s} + C') e^{-\lambda u} = O(t^{-K})
\]
for any constant \(K > 0\) and the Lemma follows.

\[\square\]

**Remark 2.1** For any \(n\) large enough, \(\rho_s\) can be retaken as \(\max\{\frac{m(\alpha_c - \alpha_1)}{2(\eta - \epsilon)}, \frac{1}{n}\}\), in fact, this can be done by enlarging \(\alpha_c - \alpha_1\) to \(\max\{\frac{2(\eta - \epsilon)}{n m}\}\) instead in (2.7). Thus, in the case of \(\alpha_1 \geq \alpha_c\), \(\rho_s\) can be taken as \(1/n\). Certainly, if this is done as above, constants \(M_s\) and \(N\) should be retaken correspondingly.

### 3  The recurrence for \(\overline{D}_k(t)\)

In this Section, we follow the basic procedures in [13] to establish the recurrence for \(\overline{D}_k(t)\). Put \(D_{-1}(t) = 0\) for all \(t \geq 1\). For \(k \geq 0\), we have
\[
\overline{D}_k(t+1) = \overline{D}_k(t) + (2\alpha - \alpha_1) m \mathbb{E} \left( -\frac{kD_k(t)}{2e_t} + \frac{(k-1)D_{k-1}(t)}{2e_t} + O\left(\frac{\Delta_t}{e_t}\right) \bigg| e_t > 0 \right) \mathbb{P}(e_t > 0)
\]
\[
+ (1 - \alpha) m \mathbb{E} \left( \frac{(k+1)D_{k+1}(t)}{e_t} - \frac{kD_k(t)}{e_t} + O\left(\frac{\Delta_t}{e_t}\right) \bigg| e_t \geq m \right) \mathbb{P}(e_t \geq m)
\]
\[
+ \alpha_1 I_{k=m} \mathbb{P}(e_t > 0) + O(\mathbb{P}(e_t = 0)) + O(\mathbb{P}(e_t < m)).
\]

Here \(\Delta_t\) denotes the maximum degree in \(G_t\) and the term \(O\left(\frac{\Delta_t}{e_t}\right)\) accounts for the probability that we create larger than one degree changes for some vertices at Time-Step \(t+1\). By (2.2) and Lemma 2.1 we have
\[
\frac{\Delta_t}{e_t} \leq O(t^{\rho_s-1}(\log t)^3), \quad qs.
\]

9
The term \( E\left( \frac{kD_k(t)}{e_t} \middle| e_t > 0 \right) \) can be expressed as

\[
E\left( \frac{kD_k(t)}{e_t} \middle| e_t > 0 \right) = E\left( \frac{kD_k(t)}{e_t} \middle| |e_t - \eta t| \leq t^{1/2} \log t \right) P\left( |e_t - \eta t| \leq t^{1/2} \log t \middle| e_t > 0 \right) + E\left( \frac{kD_k(t)}{e_t} \middle| |e_t - \eta t| > t^{1/2} \log t, e_t > 0 \right) P\left( |e_t - \eta t| > t^{1/2} \log t \middle| e_t > 0 \right)
\]

where we used the fact that \( kD_k(t) \leq 2e_t \) to hand the second term. For \( k \geq 1 \), we have \( \overline{D}_k(t) = E(D_k(t) \mid e_t > 0)P(e_t > 0) \), so

\[
E(kD_k(t) \mid |e_t - \eta t| \leq t^{1/2} \log t)P(|e_t - \eta t| \leq t^{1/2} \log t \mid e_t > 0) = k\overline{D}_k(t) - E(kD_k(t) \mid |e_t - \eta t| > t^{1/2} \log t, e_t > 0) \times P(|e_t - \eta t| > t^{1/2} \log t \mid e_t > 0) = k\overline{D}_k(t) + O(t \cdot P(|e_t - \eta t| > t^{1/2} \log t \mid e_t > 0)). \tag{3.3}
\]

Thus, using (2.1), we have for \( k \geq 0 \)

\[
E\left( \frac{kD_k(t)}{e_t} \middle| e_t > 0 \right) = \frac{k\overline{D}_k(t)}{\eta t} + O(t^{-1/2} \log t). \tag{3.5}
\]

Similarly,

\[
E\left( \frac{kD_k(t)}{e_t} \middle| e_t \geq m \right) = \frac{k\overline{D}_k(t)}{\eta t} + O(t^{-1/2} \log t). \tag{3.6}
\]

Substituting (3.2), (3.5) and (3.6) into (3.1), using (2.2) again to the other terms, we derive the following approximate recurrence for \( \overline{D}_k(t) \): \( \overline{D}_{-1}(t) = 0 \) for all \( t > 0 \) and for \( k \geq 0 \)

\[
\overline{D}_k(t + 1) = \overline{D}_k(t) + (A_2(k + 1) + B_2) \overline{D}_{k+1}(t) + (A_1k + B_1 + 1) \overline{D}_k(t) + (A_0(k - 1) + B_0) \overline{D}_{k-1}(t) + \alpha_1 I_{k=m} + O(t^{\rho_1-1}(\log t)^3),
\]

where

\[
A_2 = \frac{1 - \alpha}{2\alpha - 1}; \quad A_1 = -\frac{2 - \alpha}{2(2\alpha - 1)}; \quad A_0 = \frac{2\alpha - \alpha_1}{2(2\alpha - 1)}; \quad B_2 = B_0 = 0 \text{ and } B_1 = -1.
\]
Note that the hidden constant, write as $L$, in term $O(t^{\rho - 1}(\log t)^3)$ of (3.7) is uniform in $k$, which follows from the fact that $e_t = O(t)$ and $kD_k(t) \leq 2e_t = O(t)$ uniformly in $k$.

If we heuristically put $\bar{d}_k = \frac{D_k(t)}{t}$ and assume it is a constant, we get

$$
\bar{d}_k = (A_2(k + 1) + B_2)\bar{d}_{k+1} + (A_1k + B_1 + 1)\bar{d}_k
+ (A_0(k - 1) + B_0)\bar{d}_{k-1} + \alpha_1I_{k=m} + O(t^{\rho - 1}(\log t)^3).
$$

This leads to the consideration of the recurrence in $k$: $d_{-1} = 0$ and for $k \geq -1$,

$$(A_2(k + 2) + B_2)d_{k+2} + (A_1(k + 1) + B_1)d_{k+1} + (A_0k + B_0)d_k = -\alpha_1I_{k=m-1}.
$$

The following Lemma shows that, on certain conditions, (3.8) is a good approximation to (3.7).

Note that our Lemma is a generalization of Lemma 5.1 in [13].

Lemma 3.1 Let $d_k$ be a solution for (3.8) such that $|d_k| \leq \frac{C}{k}$ for $k > 0$ and a constant $C$. We have

1) if $\alpha_1 \leq \alpha_c$, then, for any $\nu \in (0, 1 - \rho_e)$, there exists a constant $M_1 > 0$ such that

$$
|D_k(t) - td_k| \leq M_1t^{\rho_e + \nu},
$$

for all $t \geq 1$ and $k \geq -1$;

2) if $\alpha_c < \alpha_1 < 2\alpha_c$, then there exists a constant $M_2 > 0$ such that

$$
|D_k(t) - td_k| \leq M_2t^{1-\theta},
$$

for all $t \geq 1$ and $k \geq -1$, where $\theta$ is given in (1.5).

Proof. Let $\Theta_k(t) = \overline{D}_k(t) - td_k$ and $k_0 = k_0(t) = \lfloor t^{\rho_e}(\log t)^3 \rfloor$. Lemma 2.1 implies

$$
0 \leq \overline{D}_k(t) \leq t^{-10} \text{ for } k \geq k_0(t).
$$

Proof of part 1): Equation (3.11) and $d_k \leq C/k$ imply that (3.9) holds for $k \geq k_0$ uniformly, i.e., there exists a constant $N_1 > 0$, independent to $k$ and $t$, such that

$$
|\overline{D}_k(t) - td_k| = |\Theta_k(t)| \leq N_1 t^{\rho_e}
$$

for all $k \geq k_0(t)$ and $t \geq 1$.

Recall that the hidden constant in $O(t^{\rho_e - 1}(\log t)^3)$ of (3.7) is denoted by $L$. For any $\nu \in (0, 1 - \rho_e)$, let $R \geq L$ satisfying

$$
Lt^{\rho_e - 1}(\log t)^3 \leq Rt^{\rho_e + \nu - 1}
$$

11
for all $t \geq 1$. Let $N_2 = \frac{R}{\rho - \nu} + 1$, take $\sigma > 0$ such that
\[1 - \frac{R}{N_2} - (1 + \sigma)(1 - \rho - \nu) \geq 0,\tag{3.12}\]
and take $\delta \in (0, 1)$ such that
\[\delta^{1 + \sigma} < e^{-1} < \delta.\tag{3.13}\]
Let $t_1 > 0$ be an integer such that
\[k_0(t) \leq -1 + \frac{1}{2} t = \frac{2(2\alpha - 1)}{2 - \alpha},\tag{3.14}\]
and
\[\delta^{1 + \sigma} \leq \left(1 - \frac{1}{t + 1}\right)^{t + 1}, \quad \left(1 - \frac{1 - R/l}{t + 1}\right)^{\frac{t + 1}{R/l}} \leq \delta\tag{3.15}\]
for all $t \geq t_1$ and $l \geq N_2$.

Now, for the above $t_1$, let $N_3 \geq N_1$ satisfying
\[|\Theta_k(t)| \leq N_3 t^{\rho + \nu} \quad \text{for all } 1 \leq t \leq t_1 \text{ and } k \geq -1.\tag{3.16}\]

Take
\[M_1 = \max\{N_2, N_3\}.\tag{3.17}\]
We will prove that (3.9) holds for the above $M_1$ by induction. Our inductive hypothesis is
\[\mathcal{H}_t^1 : |\Theta_k(t)| \leq M_1 t^{\rho + \nu} \quad \text{for all } k \geq -1.\]

Note that (3.16) and (3.17) imply that $\mathcal{H}_t^1$ holds for $1 \leq t \leq t_1$.

It follows from (3.7) and (3.8) that
\[\Theta_k(t+1) = \Theta_k(t) + A_2(k + 1)\frac{\Theta_{k+1}(t)}{t} + (A_1 k + B_1 + 1)\frac{\Theta_k(t)}{t}
+ A_0 (k - 1)\frac{\Theta_{k-1}(t)}{t} + O(t^{\rho - 1}(\log t)^3).\tag{3.18}\]

For $t \geq t_1$, by (3.14), we have $t + A_1 k + B_1 + 1 \geq 0$ and then (3.18) implies
\[|\Theta_k(t+1)| \leq A_2(k + 1)\frac{|\Theta_{k+1}(t)|}{t} + (t + A_1 k + B_1 + 1)\frac{|\Theta_k(t)|}{t}
+ A_0 (k - 1)\frac{|\Theta_{k-1}(t)|}{t} + R t^{\rho + \nu - 1}
\leq (t + A_2(k + 1) + A_1 k + B_1 + 1 + A_0 (k - 1))M_1 t^{\rho + \nu - 1} + R t^{\rho + \nu - 1}
= (t + A_2 + B_1 + 1 - A_0)M_1 t^{\rho + \nu - 1} + R t^{\rho + \nu - 1}.\]
Let $\varepsilon_0 = A_2 + B_1 + 1 - A_0 = (\alpha_1 - \alpha_c)/\alpha_c$, noticing that $\alpha_1 \leq \alpha_c$, we have $\varepsilon_0 \leq 0$. Then, combining (3.12), (3.15) and (3.17), we have

\[
\frac{(t + \varepsilon_0) M_1 t^{\rho_c + \nu - 1} + R t^{\rho_c + \nu - 1}}{M_1 (t + 1)^{\rho_c + \nu}} \leq \frac{M_1 t^{\rho_c + \nu} + R t^{\rho_c + \nu - 1}}{M_1 (t + 1)^{\rho_c + \nu}}
\]

\[
= \left\{ \left( 1 - \frac{1 - R/M_1}{t + 1} \right) \right\}^{1 - \rho_c/\nu} \left\{ \left( 1 - \frac{1}{t + 1} \right)^{\nu - 1} \right\}^{1 - \rho_c/\nu}
\]

\[
\leq \delta^{1 + \sigma} \left( \frac{1 - \rho_c - \nu}{t + 1} \right)^{(1 + \sigma)(1 - \rho_c - \nu)} \leq \frac{\delta^{1 - e^{-1}}}{t + 1} (1 - \rho_c + \nu) / (t + 1)
\]

\[
\leq 1.
\]

The induction hypothesis $\mathcal{H}_{t+1}$ has been verified and the proof of part 1) is completed.

Proof of part 2): In this case, we have $\alpha_c < \alpha_1 < 2\alpha_c$ and then, for some $\nu \in (0, 1/2)$, $\varepsilon_0 \leq \rho_c + \nu < 1 - \theta$ (note that in this case $\rho_c = 1/2$). Same as what we have done for part 1), for certain $\sigma > 0$ and $\delta \in (e^{-1}, 1)$, we have

\[
\frac{(t + \varepsilon_0) M_2 t^{-\theta} + R t^{-\theta}}{M_2 (t + 1)^{1-\theta}} \leq \delta^{1 - e^{-1}/(1 + \sigma)(1 - \rho_c - \nu)} / (t + 1)
\]

for sufficient large $t$ and $M_2$. This is enough for a inductive proof of (3.10).

Remark 3.1 [Remark 5.2 in [13]] Lemma 3.1 implies that if there is a solution for (3.8) such that $d_k \leq C/k$, then $\lim_{t \to \infty} D_k(t)/t$ exists and equals to $d_k$. In particular, it is shown that: if there exists a solution for (3.8) such that $d_k \leq C/k$, then the solution is unique.

4 Solving (3.8) and the proof of Theorem 1.1

In order to solve (3.8), let us consider the following homogeneous equation

\[
(A_2(k + 2) + B_2) f_{k+2} + (A_1(k + 1) + B_1) f_{k+1} + (A_0 k + B_0) f_k = 0, \quad k \geq 1
\]

which is solved by Laplace’s method as explained in [13].

For $k \geq 1$, we construct function $f_k$ has the following form

\[
f_k = \int_a^b t^{k-1} v(t) dt,
\]

where constants $a$ and $b$, and function $v(t)$ are to be determined later.

Integrating by parts

\[
k f_k = [t^k v(t)]_a^b - \int_a^b t^k v'(t) dt.
\]
Let
\[ \phi_1(t) = A_2 t^2 + A_1 t + A_0, \quad \phi_0(t) = B_2 t^2 + B_1 t + B_0. \]
Substituting (4.2) and (4.3) into (4.1), we obtain
\[
[t^k \phi_1(t) v(t)]^b_a - \int_a^b t^k \phi_1(t) v'(t) dt + \int_a^b t^{k-1} \phi_0(t) v(t) dt = 0. \tag{4.4}
\]
Equation (4.1) will be satisfied if we have
\[
\frac{v'(t)}{v(t)} = \frac{\phi_0(t)}{t \phi_1(t)}, \tag{4.5}
\]
and
\[
[t^k v(t) \phi_1(t)]^b_a = 0. \tag{4.6}
\]
Let \( a = 0 \) and \( b \) equal to a root of \( v(t) \phi_1(t) = 0 \), the parameters \( a \) and \( b \) can be determined satisfying (4.6).

Obviously, \( \phi_0(t) \) and \( \phi_1(t) \) can be rewritten as
\[
\phi_0(t) = -t; \quad \phi_1(t) = At^2 - (A + B)t + B = A(t-1)(t-B/A), \tag{4.7}
\]
where
\[
A = \frac{1 - \alpha}{2\alpha - 1}, \quad B = \frac{2\alpha_1 - \alpha_1}{\alpha_c} = A + \frac{\alpha_c - \alpha_1}{\alpha_c}. \tag{4.8}
\]

Now, we solve the equation (4.1) in the following cases: 1), \( \alpha_1 < \alpha_c \); 2), \( \alpha_1 > \alpha_c \) and 3), \( \alpha_1 = \alpha_c \) respectively.

For case \( \alpha_1 < \alpha_c \), we have \( B > A \), then the differential equation (4.5) is homogeneous and can be integrated to derive
\[
v(t) = (t-1)^{\beta}(t-B/A)^{-\beta}, \tag{4.9}
\]
where \( \beta = 1/(B - A) \) is given by (4.9).

Since in this case \( \beta > 1 \), so by (4.7), the equation
\[
v(t) \phi_1(t) = A(t-1)^{1+\beta}(t-B/A)^{1-\beta} = 0 \tag{4.10}
\]
has a unique root 1. Thus, the parameter \( b = 1 \) satisfies (4.6).

Substituting the parameter \( b \) and the function \( v(t) \) into (4.2) and removing a constant multiplicative factor, we obtain a solution \( u_1(k) \) to (4.1) for \( k \geq 1 \):
\[
u_1(k) = \int_0^1 t^{k-1} \left( \frac{1-t}{1-\zeta t} \right)^\beta dt, \tag{4.11}
\]
where \( \zeta = A/B \).

The order of the function \( u_1(k) \) with respect to \( k \) is given by the following Lemma.
Lemma 4.1 [Lemma 6.1 in [13]] Let $k \geq 1$. Then

$$u_1(k) = (1 + O(k^{-1}))D_1 k^{-(1+\beta)}$$

(4.12)

for $D_1 = D_1(\alpha, \alpha_1)$ a fixed constant.

In Case of $\alpha_1 > \alpha_c$, we have $B < A$, and equation (4.5) has the same solution as (4.9). In addition, under the conditions (1.2) and (1.6), one further has $\beta < -1$, and then the equation (4.10) has a unique root $\gamma := B/A$ as given in (1.5). So we can take $b = \gamma$ to satisfy (4.6). Thus

$$u_2(k) = \int_0^\gamma t^{k-1} \left( \frac{\gamma - t}{1 - t} \right)^{-\beta} dt = \gamma^{k-\beta} \int_0^1 t^{k-1} \left( \frac{1 - t}{1 - \gamma t} \right)^{-\beta} dt$$

is a solution to (4.1) for $k \geq 1$.

By Lemma 4.1 we have

$$u_2(k) = (1 + O(k^{-1}))D_2 \gamma^{k-\beta}$$

(4.13)

for some fixed constant $D_2 = D_2(\alpha, \alpha_1)$.

Finally, we consider the case of $\alpha_1 = \alpha_c$. In this case $A = B$ and the equation (4.5) can be integrated to derive

$$v(t) = e^{-\mu/(1-t)}$$

with $\mu = 1/A$ given in (1.5). With same argument as in cases 1) and 2), take $b = 1$ and define

$$u_c(k) = \int_0^1 t^{k-1} e^{-\mu/(1-t)} dt,$$

then $u_c$ is a solution to (4.1) for $k \geq 1$.

Crudely,

$$u_c(k) \leq \int_0^1 t^{k-1} dt = 1/k.$$  

(4.14)

The precious representation of $u_c(k)$ can be found in Remark 1.2.

Note that in all the three cases, $u_1$, $u_2$ and $u_c$ do not satisfy equation (4.1) when $k = 0$. In fact, as calculated in [13], for $i = 1, 2$ or $c$, we always have

$$2A_2u_i(2) + (A_1 + B_1)u_i(1) = [\phi_1(t)v(t)]_0^B = -\phi_1(0)v(0) \neq 0.$$  

(4.15)

Now, we are going to solve (3.8). By Remark 3.1 we only need to construct a solution for (3.8) which satisfies the requirements of Lemma 3.1. Actually, we will construct such a solution based on the solution of (4.1) given above.

Denote by $g$ the solution for (4.1), i.e., $g = u_1, u_2$ or $u_c$ in the three cases respectively.
For $m > 1$, define $w_k = 0$ for $k \geq m$, $w_{m-1} = -\alpha_1/[(m-1)A_0]$ and for $j = m-2, m-3, \ldots, 1$, let $w_j$ be such that

$$A_2(j+2)w_{j+2} + (A_1(j+1) + B_1)w_{j+1} + A_0jw_j = 0.$$ 

Then $w_k$ satisfies (3.8) for $k \geq 1$. Therefore, any linear combination of $g$ and $w$ is a solution of (3.8) for $k \geq 1$.

Now, let

$$D = -\frac{2A_2 w_2 + (A_1 + B_1)w_1}{2A_2 g(2) + (A_1 + B_1)g(1)}, \quad d = -\frac{A_2(Dg(1) + w_1)}{B_1}.$$ 

Note that $D$ and $d$ depend on $g = u_1, u_2$ and $n_c$ respectively. By (4.15), $D$ is well-defined.

Define

$$d_k = \begin{cases} 
0, & \text{if } k = -1 \\
Dg(k) + w_k, & \text{otherwise} 
\end{cases}$$

It is straightforward to check that $d_k$ given above is the solution of (3.8), by (4.12), (4.13) and (4.14), we know that $d_k$ satisfies the requirements of Lemma 3.1.

For $m = 1$, we can take

$$D = -\frac{\alpha_1}{2A_2 g(2) + (A_1 + B_1)g(1)}, \quad d = -\frac{DA_2 g(1)}{B_1}.$$ 

and directly define

$$d_k = \begin{cases} 
0, & \text{if } k = -1 \\
d, & \text{if } k = 0 \\
Dg(k), & \text{otherwise} 
\end{cases}$$

Similarly, in this case $d_k$ is also a solution to (3.8) which satisfies the condition of Lemma 3.1.

**Proof of Theorem 1.1** By the construction of the solution $d_k$ and Lemma 3.1 the theorem follows immediately by taking

$$C_i = \begin{cases} 
\frac{(2A_2 w_2 + (A_1 + B_1)w_1)D_i}{2A_2 u_i(2) + (A_1 + B_1)u_i(1)}, & \text{for } m > 1 \\
\frac{\alpha_1 D_i}{2A_2 u_i(2) + (A_1 + B_1)u_i(1)}, & \text{for } i = 1, 2, c, \\
\frac{\alpha_1 D_i}{2A_2 u_i(2) + (A_1 + B_1)u_i(1)}, & \text{for } m = 1 
\end{cases}$$

where $D_1$ and $D_2$ are given in (4.12) and (4.13), $D_c = 1$. □

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