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The role of the coarsest resolution subband in the wavelet-based reconstruction of signals from gradients

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Abstract. An efficient wavelet-based algorithm to reconstruct non-square/ non-cubic signals from gradient data is proposed. This algorithm is motivated by applications such as image or video processing in the gradient domain. In some earlier approaches, the non-square/ non-cubic gradients were extended to enable a square/ cubic Haar wavelet decomposition and the coarsest resolution subband was derived from the mean value of the signal. In this paper, a non-square/non-cubic wavelet decomposition is obtained directly without extending the gradient data. The challenge comes from finding the coarsest resolution subband of the wavelet decomposition and an algorithm to compute this is proposed. The performance of the algorithm is evaluated in terms of accuracy and computation time, and is shown to outperform the considered earlier approaches in a number of cases. Further, a closer look on the role of the coarsest resolution subband coefficients reveals a trade-off between errors in reconstruction and visual quality which has interesting implications in image and video processing applications.

1. Introduction

In many interesting applications from adaptive optics [1], image processing [2], [3], [4] or video processing [5], [6] signal reconstruction from gradient data plays an important role. In some of these applications, such as adaptive optics, the data available is in the form of gradient values instead of signal values. In others, dealing with the human visual system, inherently highly sensitive to contrast, the gradient domain offers an attractive solution domain. Working in the gradient domain, however, requires reconstructing a signal from gradient data. Signal reconstruction from gradient data has been reformulated as a Poisson equation in [2], and several methods have been developed for signal reconstruction. A class of Fourier based techniques [7] with typical complexity $O(N \log N)$, where $N$ is the number of unknowns, have been presented. An algebraic approach based on graph theory is proposed in [8]. Another class consists of iterative solvers [9], which include methods such as Jacobi and Gauss-Seidel, but these are typically slow to converge on large scale problems. A very popular class of fast solvers is the multigrid approach [10], which solves the Poisson problem on diagonally oriented grids, and uses an iterative Poisson solver to smooth the error at each scale and perform the interpolation to obtain solution estimates on the finest grid. Multigrid techniques are $O(N)$.

A wavelet based reconstruction technique was proposed in [11], and used in the context of adaptive optics. This approach also has linear complexity, and was developed and adapted for image [12] and video [5] processing applications. This method is based on obtaining the Haar wavelet decomposition directly from the gradient data. The signal can then be obtained from this wavelet decomposition. This
method deals with non-square (in 2-D) or non-cube (in 3-D) signal by expanding the non-square signal to a square (or cube) of appropriate size. Although this expansion yields satisfactory results, it may require large amounts of additional memory, particularly when editing long video sequences. In this paper, this wavelet-based reconstruction approach is expanded to deal with non-square and non-cube cases without expanding the given data. The approach is based on obtaining the coarsest resolution subband (low-low) of the wavelet decomposition directly from the gradient data. This new approach gives also the opportunity to study closer the effect of errors in the low-low part of the wavelet decomposition. The effects of such errors will be illustrated with examples and the insight gained provides important recommendations for more efficient ways of reconstructing non-square/ non-cubic signals from gradient data, with applications in image or video processing.

The remaining of this paper is organized as follows: Section 2 briefly reviews the wavelet based reconstruction technique. In Section 3, a method is proposed to deal with non-square/non-cube signals. In Section 4, the performance of the technique is evaluated and discussed with respect to accuracy, visual quality and computational resources required. Conclusions are drawn in Section 5.

2. Background information

The Haar wavelet analysis filters are: \( H_L (z) = \left(1 + z^{-1}\right) / \sqrt{2} \) and \( H_H (z) = \left(1 - z^{-1}\right) / \sqrt{2} \). The reconstruction technique in [11], [12], [5] has an analysis and a synthesis step. In the analysis step, the Haar wavelet decomposition of the signal is obtained from the gradient data and the average value of the signal. This step is based on the relation between the gradient and the Haar wavelet analysis filters:

\[
\Phi_d = \sqrt{2} \Phi \otimes H_H (z_d)
\]

where \( \Phi_y, \Phi_x \) are gradient components, and \( \Phi \otimes H_H (z_d) \) denotes 1-D convolution of 2-D signal \( \Phi \) with high pass analysis filter \( H_H (z) \) along direction \( d \in \{y, x\} \). This observation remains valid in higher dimensions. In the synthesis step, the signal is recovered from the wavelet decomposition by wavelet synthesis, with the possibility of including an iterative Poisson solver (Jacobi [9]) at each resolution. Including the iterative Poisson solver is important in gradient based image or video editing applications, where gradient data is typically altered and the zero-curl condition is not satisfied.

In the analysis step of [12] the gradient of non-square signals is first extended to square and then the Haar wavelet decomposition is obtained from the extended data. This has non-negligible implications on memory consumption and computation time. While extending to square is still an option in 2-D, the data size quickly becomes a problem for 3-D signals with significant differences in size, such as video sequences. The reconstruction algorithm proposed in Sec. 3 does not entail signal extension, making this approach more suitable for signals with non-negligible size differences.

3. Obtaining the coarsest resolution subband coefficients from signal derivatives

The first step of the reconstruction is to obtain the Haar wavelet decomposition of the signal directly from the derivatives. For a non-square/ non-cubic signal with size \( 2^M \times 2^N \) (or \( 2^M \times 2^N \times 2^P \)) the equations for finding the “low-high”, “high-low” and “high-high” subbands at all resolutions of the Haar wavelet decomposition from the given gradient components are the same as those for a square (or cubic) signal in[12], [5] provided that the maximum number of levels in the decomposition is \( M \) (if \( M<N \)). What is different in the non-square (non-cubic) case is the way in which the “low-low” (or low-low-low) subband at the coarsest resolution of the Haar wavelet decomposition is found. These subband coefficients are referred to as the coarsest resolution subband. For square (or cubic) signals, the coarsest resolution subband is a scalar proportional to the mean value of the signal. For non-square (or non-cubic) signals, the coarsest resolution subband is a multi-dimensional array. Finding this array of numbers from the signal derivatives and the mean value of the signal is presented next.

3.1. The two dimensional case
Let \( \Phi \) be a 2-D signal, with size \( 2^M \times 2^N \), where \( 1 < M \leq N \) are integers. Let \( y \) and \( x \) be two orthogonal directions. The coarsest resolution low-low subband of the Haar wavelet decomposition of \( \Phi \), denoted \( \Phi^M_{L_yL_x} \), is a signal with size \( 1 \times 2^{N-M} \), obtained by successive filtering with \( H_L(z) \) and subsampling.

It can be shown that it is proportional to the partial sums of all elements in consecutive regions of \( \Phi \), as illustrated in Fig. 1. The objective of this work is to find \( \Phi^M_{L_yL_x} \) from the signal derivatives and its mean value.

Let the given first order directional derivatives of the unknown signal be denoted by \( \Phi_y \) and \( \Phi_x \) (forward difference approximation). Let \( m \) be the mean value of the unknown signal \( \Phi \). Finding \( \Phi^M_{L_yL_x} \) amounts to finding the \( N-M \) partial sums of signal regions illustrated in Fig. 1. This leads to the following algorithm:

**Step 1.** Compute:

\[
c_x = A_x \Phi^T_x B_x
\]

where

\[
A_x = \begin{bmatrix}
1 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1
\end{bmatrix}
\]

\( \in \mathbb{R}^{(2^{N-1}) \times (2^{N-1})} \), and

\[
B_x = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^{2 \times 1}.
\]

**Step 2.** Compute:

\[
u = \frac{1}{2^M} A^{-1} v
\]

where

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
-1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1
\end{bmatrix}
\]

\( \in \mathbb{R}^{2^{N-1} \times 2^{N-1}} \), and

\[
v = \begin{bmatrix} m \cdot 2^{M+N} \\ c_x \end{bmatrix} \in \mathbb{R}^{2^{N-1} \times 1}.
\]

**Step 3.** Compute:

\[
\Phi^M_{L_yL_x} = u^T B
\]

where

\[
B = \begin{bmatrix} B_1 & B_2 & \cdots & B_k \end{bmatrix}^T, \quad k = 2^{N-M} \quad \text{and} \quad B_p = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{R}^{2^{M-1} \times 2^{N-M}}, \quad \text{in}
\]

which the \( p^{th} \) column is an all-ones vector with \( M-1 \) elements.
3.2. The three dimensional case

Let \( \Phi \) be a 3-D signal, with size \( 2^M \times 2^N \times 2^P \) where \( 1 < M \leq N \leq P \) integers. Signal value at a point \((y, x, t)\) is denoted \( \Phi(y, x, t) \) with \( y \) and \( t \) orthogonal directions. The coarsest resolution subband of the Haar wavelet decomposition of \( \Phi \) is denoted \( \Phi_{L_y L_x L_t}^M \) and is a signal with size \( 1 \times 2^{N-M} \times 2^{P-M} \). This signal can be found similarly to the 2-D case, by solving a system of linear equations. This system of equations is obtained from the given discrete partial derivatives and the mean value of the signal. More precisely, the partial derivative \( \Phi_y \) can be used to derive \( 2^{P-M}\left(2^{N-1}-1\right) \) equations, and the partial derivative \( \Phi_t \) can be used to derive \( 2^{N-M}\left(2^{P-1}-1\right) \) equations. The mean value of the signal yields one more equation. In this way, a system of linear equations is formed:

\[
A\lambda = B
\]

where:

\[
A \in \left[ 2^{P-M}\left(2^{N-1}-1\right) + 2^{N-M}\left(2^{P-1}-1\right) + 1 \right]x2^{N+P-2M}
\]

and

\[
B \in \left[ 2^{P-M}\left(2^{N-1}-1\right) + 2^{N-M}\left(2^{P-1}-1\right) + 1 \right]x1
\]

are known quantities obtained from partial derivatives and the mean value of the signal, and \( \lambda \) is the unknown, which, once found is reshaped as \( \Phi_{L_y L_x L_t}^M \). Due to space limits details are not included.

4. Performance evaluation

This section analyzes the performance of different reconstruction algorithms and the role of the coarsest resolution subband coefficients on the quality of signals reconstructed from gradient data. The comparison is between: Algorithm 1: the method in Section 3.1; Algorithm 2: the method in [12], [5] where non-square/ non-cubic gradient data is extended to nearest square/ cube; Algorithm 3: the signal is reconstructed using a coarsest resolution subband with all entries equal to a value proportional to the signal mean. Algorithm 3 is particularly interesting in applications such as image or video editing, as it avoids both the use of additional memory required by data extension to square/ cube and the computations for the exact coarsest resolution subband. Results show that although this approach leads to lower accuracy reconstructed signals, they are still visually acceptable, if an iterative Poisson solver is included in the synthesis step. The benefits of Algorithm 3 become significant in the case of 3-D data such as video sequences, as will be seen shortly.

The reconstruction quality will be determined by analyzing the signal to noise ratio (SNR), measured in dB, between the original signal and the estimate obtained from the derivatives and the mean value of the signal, as an estimate of solution accuracy; and the time required for a MATLAB implementation of the methods, as an estimate of computational speed.

4.1. The two dimensional case

Reconstruction accuracy was first considered for Algorithms 1 and 3, and the results are shown in Fig. 2. The reconstruction accuracy of Algorithm 2 is similar to that of Algorithm 1, and the results were omitted in the interest of not cluttering the figure. In Algorithm 3, the values of \( \Phi_{L_y L_x}^M \) are left 0 before wavelet synthesis, and the average value of the reconstruction is corrected at the end, to match the given average. This corresponds to the partial sums \( s_1, \ldots, s_j \) in Fig. 1 all having the same value.

As can be seen from Fig. 2, obtaining the values of \( \Phi_{L_y L_x}^M \) exactly from the input data results in a very high SNR in the reconstructed signal, and, as expected, leaving the coarsest resolution “low-low”
subband coefficients zero before synthesis and adjusting the signal average value at the end significantly lowers the SNR of the reconstructed signal.

Run times of MATLAB implementations of the three reconstruction algorithms were compared. As Fig. 3 reveals, the performance is similar for square signals, as signal extension is not performed in this case. The advantage of the new approach (i.e., Algorithms 1 and 3) becomes clear when dealing with non-square signals, as a solution is produced faster than when Algorithm 2 is used.

Let us now take a closer look at the reconstruction error in Algorithm 3. In Fig. 4a, a $64 \times 256$ image was obtained from image “threads”[13]. The image reconstructed from its gradient and mean value using Algorithm 1 is shown in Fig. 4b, and is an accurate representation of the image in Fig. 4a. The image reconstructed using Algorithm 3 is shown in Fig. 4c and has visual artifacts due to not obtaining exact values of the coarsest resolution “low-low” subband coefficients. The image reconstructed using Algorithm 3, with an iterative Poisson solver included at each resolution in the synthesis step is shown in Fig. 4d. Including the Poisson solver has the effect of smoothing out the vertical lines noticeable in Fig. 4c. The objective quality of each image was also evaluated by comparison with the original image, using SNR and SSIM[14]. The SNR values agree with the noticeable increase in visual quality (from Fig. 4c to 4d). It is interesting to note that SSIM ranks the result in Fig. 4d of lower quality than that in Fig. 4c, although visual examination indicates otherwise.

![Fig. 2.](image)

**Fig. 2.** Performance evaluation in terms of solution accuracy.

![Fig. 3.](image)

**Fig. 3.** Performance evaluation in terms of speed.

![Fig. 4.](image)

**Fig. 4.** Visualizing the role of the coarsest resolution low subband coefficients and of the Poisson solver. Top to bottom: (a) original image; (b) image reconstructed with exact calculation of $\Phi_{L_y L_x}^M$; (c) image reconstructed without exact calculation of $\Phi_{L_y L_x}^M$ without Poisson solver; (d) image reconstructed without exact calculation of $\Phi_{L_y L_x}^M$, with Poisson solver in the wavelet synthesis.
These experiments reveal the main trade-off of the new approach, and a recommendation can be made. If signal accuracy is of primary importance for the application at hand, the coarsest resolution subband coefficients should be computed exactly before wavelet synthesis. If memory requirements and speed are more important, the coarsest resolution subband coefficients can be approximated, and the result is still visually acceptable if the Poisson solver is included in the wavelet synthesis step.

4.2. The three dimensional case
Reconstructions obtained by computing the coarsest resolution subband using the method in Sec. 3.2. (denoted “With LLL” in Fig. 5) are compared with reconstructions obtained using the coarsest resolution subband with all entries equal to a value proportional to the signal mean (denoted “Without LLL” in Fig. 5). The Poisson solver is included at each resolution. In terms of solution accuracy (SNR) the results are similar to the ones of the 2-D case. In terms of speed, the reconstructions obtained “Without LLL” are compared to the earlier technique in [5], with signal extension to nearest cube. The results are shown in Fig. 6. The results of Fig. 5, 6 indicate that for 3-D signals with large differences between the three dimensions, the new approach produces a solution in significantly less time than the approach in [5]. This is expected since a significant amount of memory is needed to store the extended version of the non-cubic signals, and this requires additional processing time.

5. Conclusions
An efficient way of reconstructing non-square/ non-cubic signals from gradient data was presented. The approach is based on finding the Haar wavelet decomposition of the signal first, and then reconstructing the signal by wavelet synthesis. The focus is on obtaining the coarsest resolution subband of the wavelet decomposition, and on studying the role of these coefficients on the quality of the reconstructions. The new algorithm was compared with a previous reconstruction approach, which extends gradient data to the nearest square or cube. The experiments show that obtaining an exact approximation of the coarsest resolution coefficients generally increases solution accuracy at the expense of computation time. However, studies reveal that in applications such as image or video editing, finding these coefficients exactly has no significant consequence on the visual quality of the reconstructed signal. This finding has great potential in higher dimensional signal processing applications such as light fields, and this is the next step in our research.

![Fig. 5. Performance evaluation in terms of solution accuracy. 3-D case. Poisson solver used in the synthesis.](image1)

![Fig. 6. Performance evaluation in terms of speed. 3-D case. Poisson solver used in the synthesis.](image2)

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