ADDITIVITY AND RELATIVE KODAIRA DIMENSIONS

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Dedicated to Shing-Tung Yau on the occasion of his 60th birthday

1. Introduction

The notion of Kodaira dimension has been defined for complex manifolds in [16], for symplectic 4−manifolds in [23] (see also [35], [17]). It is shown in [5] (and [23]) that these two definitions are compatible in dimension 4. Furthermore, we calculate it for some (4-dimensional) Lefschetz fibrations when the base has positive genus. In [50], this notion is extended to 3−dimensional manifolds via geometric structures in the sense of Thurston. All these Kodaira dimensions are “absolute” invariants, taking values in the set

\[ \{-\infty, 0, 1, \cdots, \left\lfloor \frac{n}{2} \right\rfloor\} \]

where \( n \) is the real dimension of the manifold, and \( \left\lfloor x \right\rfloor \) is the largest integer bounded by \( x \).

We will review them and introduce a few more for logical convenience in section 2. We also point out in section 2 that they are invariant under covering, and further show that they are additive for many fiber bundles.

In recent years, the study of relative invariants for a pair of symplectic (projective) manifold with a codimension 2 symplectic submanifold (smooth divisor) becomes increasingly important, especially in Gromov-Witten theory ([20], [36], [6], [21]). The relative invariants are used to calculate the absolute invariants via fiber sum and its reverse, symplectic cut (degeneration). From this point of view, the paper [31] by the first author and Yau can be viewed as a first step towards a possible definition of relative Kodaira dimension for symplectic 4−manifolds.

In this paper, motivated by [31], we introduce in section 3 the notion of relative Kodaira dimension \( \kappa^s(M^4, \omega, F^2) \) for a symplectic 4−manifold \( (M^4, \omega) \) with a possibly disconnected embedded symplectic surface \( F \). They take the same set of values as in (1).

To define it we need to establish several homological properties of embedded symplectic surfaces in 3.1-2. These properties are formulated in terms of the formal Kodaira dimension [5]. It should be mentioned that symplectic spheres do not satisfy most of these properties. We also formulate in 3.3 the notion of relative minimal model, and prove the existence and the somewhat surprising uniqueness.
We then define $\kappa_s(M^4, \omega, F^2)$ in 3.4. One notable feature is that the sphere components of $F^2$ have to be discarded, which resembles the definition of Thurston norm of 3–manifolds. For a symplectic 4–manifold constructed via a positive genus fiber sum, the main result in [31] can then be interpreted as a simple expression of its Kodaira dimension in terms of the relative Kodaira dimensions of the summands (Theorem 3.24).

Another motivation comes from the paper [5] by the second author and J. Dorfmeister concerning the additivity of the Kodaira dimensions for a 4–dimensional Lefschetz fibration with singular fibers. In that paper, the additivity is shown to hold in many cases, while there is only a supadditivity relation in some cases. It was speculated by the second author whether this defect can be remedied if using appropriate relative Kodaira type invariants. For this purpose, we also introduce relative Kodaira dimension for a 2–manifold with a $\mathbb{Q}$–linear combination of points. The well definedness is immediate in this case. We demonstrate that this notion of relative Kodaira dimension can indeed be used to calculate the Kodaira dimension of the total space for several kinds of fibrations over surfaces with singular fibers.

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2. Kodaira Dimensions and Fiber Bundles

The goal of this section is to recall briefly the definitions of various Kodaira dimensions mentioned in the introduction, and establish the additivity for appropriate classes of fiber bundles.
2.1. $\kappa^h$ for complex manifolds and $\kappa^t$ up to dimension 3.

2.1.1. The holomorphic Kodaira dimension $\kappa^h$. Let us first recall the original Kodaira dimension in complex geometry.

**Definition 2.1.** Suppose $(M, J)$ is a complex manifold of real dimension $2m$. The holomorphic Kodaira dimension $\kappa^h(M, J)$ is defined as follows:

$$\kappa^h(M, J) = \begin{cases} 
-\infty & \text{if } P_l(M, J) = 0 \text{ for all } l \geq 1, \\
0 & \text{if } P_l(M, J) \in \{0, 1\}, \text{ but } \neq 0 \text{ for all } l \geq 1, \\
k & \text{if } P_l(M, J) \sim c_l^k; c > 0.
\end{cases}$$

Here $P_l(M, J)$ is the $l$–th plurigenus of the complex manifold $(M, J)$ defined by $P_l(M, J) = h^0(K_{J} \otimes J^l)$, with $K_J$ the canonical bundle of $(M, J)$.

2.1.2. The topological Kodaira dimension $\kappa^t$ for manifolds up to dimension 3. As mentioned there are other situations where a similar notion can be defined. Let $M$ be a closed, smooth, oriented manifold. To begin with, we make the following definition for logical compatibility.

**Definition 2.2.** If $M = \emptyset$, then its Kodaira dimension is defined to be $-\infty$.

The only closed connected 0–dimensional manifold is a point, and the only closed connected 1–dimensional manifold is a circle.

**Definition 2.3.** If $M$ has dimension 0 or 1, then its Kodaira dimension $\kappa^t(M)$ is defined to be 0.

The 2–dimensional Kodaira dimension is defined by the positivity of the Euler class. We write $K = -e$.

**Definition 2.4.** Suppose $M^2$ is a 2–dimensional manifold with Euler class $e(M^2)$. Write $K = -e(M^2)$ and define

$$\kappa^t(M^2) = \begin{cases} 
-\infty & \text{if } K < 0, \\
0 & \text{if } K = 0, \\
1 & \text{if } K > 0.
\end{cases}$$

It is easy to see that for any complex structure $J$ on $M^2$, $K$ is its canonical class, and $\kappa^h(M^2, J) = \kappa^t(M^2)$. $\kappa^t(M^2)$ can be further interpreted from other viewpoints: symplectic structure ($K$ is also the symplectic canonical class), the Yamabe invariant, geometric structures and etc.

Recall that the Yamabe invariant is defined in the following way ([15], [11]):

$$Y(M) = \sup_{[g] \in C} \inf_{g \in [g]} \int_M s_g dV_g,$$

where $g$ is a Riemannian metric on $M$, $s_g$ the scalar curvature of $g$, $[g]$ the conformal class of $g$, and $C$ the set of conformal classes on $M$.

A basic fact is that $Y(M) > 0$ if and only if $M$ admits a metric of positive scalar curvature. Thus $Y(M)$ is non-positive if $M$ does not admit
metrics of positive scalar curvature. Furthermore, in this case, another basic fact is that $Y(M)$ is the supremum of the scalar curvatures of all unit volume constant-scalar-curvature metrics on $M$ (such metrics exist due to the resolution of the Yamabe conjecture). It immediately follows that, in dimension two, the sign of $Y(M^2)$ completely determines $\kappa^t(M^2)$.

We move on to dimension 3. In this dimension the definition of the Kodaira dimension in [50] by the second author is based on geometric structures in the sense of Thurston. Divide the 8 Thurston geometries into 3 categories:

- $-\infty$ : $S^3$ and $S^2 \times \mathbb{R}$;
- $0$ : $\mathbb{E}^3$, Nil and Sol;
- $1$ : $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL_2(\mathbb{R})}$ and $\mathbb{H}^3$.

Given a 3-manifold $M^3$, we decompose it first by a prime decomposition and then further consider a toroidal decomposition for each prime summand, such that at the end each piece has a geometric structure either in group (1), (2) or (3) with finite volume. The following definition was introduced in [50], where the well definedness was also checked.

**Definition 2.5.** For a 3-dimensional manifolds $M^3$, we define its Kodaira dimension as follows:

1. $\kappa^t(M^3) = -\infty$ if for any decomposition, each piece has geometry type in category $-\infty$,
2. $\kappa^t(M^3) = 0$ if for any decomposition, we have at least a piece with geometry type in category 0, but no piece has type in category 1,
3. $\kappa^t(M^3) = 1$ if for any decomposition, we have at least one piece in category 1.

In this dimension, $Y(M^3)$ is also closely related to geometric structure of $M^3$, at least when $M^3$ is irreducible (see the discussions in [1] by Anderson). However, as observed in [50], the number $Y(M^3)$ does not completely determine $\kappa^t(M^3)$. For $\Sigma_g \times S^1$, it has vanishing Yamabe invariant if $g \geq 1$. But $\kappa^t(\Sigma_g \times S^1) = 0$ if $g = 1$, $\kappa^t(\Sigma_g \times S^1) = 1$ if $g \geq 2$. In this case, $\kappa^t$ is still determined by (2) if we distinguish whether the supremum is attainable by a metric. But this refinement of $Y(M^3)$ will still not determine $\kappa^t$ since a Nil 3-manifold like a non-trivial $S^1$-bundle over $T^2$ has Yamabe invariant 0 which is not attainable by any metric.

Notice that here we use $\kappa^t$ to denote the Kodaira dimension for smooth manifolds in dimensions 0, 1, 2, 3. Here $t$ stands for *topological*, because in these dimensions homeomorphic manifolds are actually diffeomorphic.

For a possibly disconnected manifold, we define its Kodaira dimension to be the maximum of that of its components. In summary, we have defined the Kodaira dimension for all the closed, oriented manifolds with dimension less than 4.
2.2. \( \kappa^s \) for symplectic 4–manifolds. In \[23\], the first author systematically investigated the notion of symplectic Kodaira dimension for symplectic 4–manifolds. To define it we need to first recall the notion of minimality.

2.2.1. Minimality in dimension 4.

**Definition 2.6.** Let \( \mathcal{E}_M \) be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection \(-1\). \( M \) is said to be (smoothly) minimal if \( \mathcal{E}_M \) is the empty set.

Equivalently, \( M \) is minimal if it is not the connected sum of another manifold with \( \mathbb{CP}^2 \). We say that \( N \) is a minimal model of \( M \) if \( N \) is minimal and \( M \) is the connected sum of \( N \) and a number of \( \mathbb{CP}^2 \).

We also recall the notion of minimality for \((M, \omega)\). \((M, \omega)\) is said to be (symplectically) minimal if \( \mathcal{E}_\omega \) is the empty set, where

\[ \mathcal{E}_\omega = \{ E \in \mathcal{E}_M | E \text{ is represented by an embedded } \omega–\text{symplectic sphere} \}. \]

A basic fact proved using SW theory (\[14\], \[28\], \[24\]) is: \( \mathcal{E}_\omega \) is empty if and only if \( \mathcal{E}_M \) is empty. In other words, \((M, \omega)\) is symplectically minimal if and only if \( M \) is smoothly minimal.

2.2.2. Definitions.

**Definition 2.7.** For a minimal symplectic 4–manifold \((M^4, \omega)\) with symplectic canonical class \( K_\omega \), the Kodaira dimension of \((M^4, \omega)\) is defined in the following way:

\[
\kappa^s(M^4, \omega) = \begin{cases} 
-\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega \cdot K_\omega < 0, \\
0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega \cdot K_\omega = 0, \\
1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega = 0, \\
2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega > 0.
\end{cases}
\]

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

Here \( K_\omega \) is defined as the first Chern class of the cotangent bundle for any almost complex structure compatible with \( \omega \).

We here offer an interpretation of \( \kappa^s \) which relates it to the 2–dimensional \( \kappa^t \). Define the (symplectic) Kodaira dimension for a number \( k \) (or equivalently, a top dimensional cohomology class of a closed oriented manifold) in the following way:

\[
\kappa^s(r) = \begin{cases} 
-\infty & \text{if } r < 0, \\
0 & \text{if } r = 0, \\
1 & \text{if } r > 0.
\end{cases}
\]

Then for a 2–dimensional manifold \( F^2 \), we have

\[
\kappa^t(F^2) = \kappa^s(-e(F^2)) = \kappa^s(-\chi(F^2)),
\]
where $\chi$ denotes the Euler characteristic. Furthermore, for a 4–dimensional minimal symplectic manifold $(M^4, \omega)$,

\begin{equation}
\kappa^s(M^4, \omega) = \kappa^s(K_2^\omega) + \kappa^s(K_\omega \cdot [\omega]).
\end{equation}

We further make a couple of easy observations based on (4).

**Lemma 2.8.** Let $(M^4, \omega)$ be a minimal symplectic manifold. If $K_2^\omega < 0$, then $\kappa^s(M^4, \omega) = \kappa^s(K_2^\omega) = -\infty$. If $K_2^\omega \geq 0$, then

\begin{equation}
\kappa^s(\kappa^s(M^4, \omega)) = \kappa^s(K_\omega \cdot [\omega]).
\end{equation}

Due to the properties of $\kappa^s$ listed in [23], such as the diffeomorphism invariance of $\kappa^s$, we can yet regard $\kappa^s$ as an invariant of a large class of smooth 4–manifolds in the following way.

**Definition 2.9.** Suppose that $M^4$ is a 4–dimensional closed, oriented manifold admitting symplectic structures (compatible with the orientation). $M^4$ is said to have symplectic Kodaira dimension $\kappa^s = -\infty$ if $M^4$ is rational or ruled. Otherwise, first suppose that $M^4$ is smoothly minimal. Then the Kodaira dimension $\kappa^s$ of $M^4$ is defined as follows:

\[
\kappa^s(M^4) = \kappa^s(M^4, \omega) = \begin{cases} 
0 & \text{if } K_\omega \text{ is torsion,} \\
1 & \text{if } K_\omega \text{ is non-torsion but } K_2^\omega = 0, \\
2 & \text{if } K_2^\omega > 0.
\end{cases}
\]

Here $\omega$ is any symplectic form on $M^4$ compatible with the orientation.

For a general $M^4$, $\kappa^s(M^4)$ is defined to be $\kappa^s(N^4)$, where $N^4$ is a smooth minimal model of $M^4$.

Here a rational 4–manifold is $S^2 \times S^2$, or $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ for some non-negative integer $k$. A ruled 4–manifold is the connected sum of a number of (possibly zero) $\mathbb{CP}^2$ with an $S^2$–bundle over a Riemann surface.

It was verified in [5] that $\kappa^s = \kappa^h$ whenever both are defined. In fact it was shown earlier in [10] that $\kappa^h(M^4, J)$ (even the plurigenera) only depends on the oriented diffeomorphism type of $M^4$.

LeBrun ([19]) calculated $Y(M^4)$ when $M^4$ admits a Kähler structure, from which he concluded that (2) completely determines $\kappa^h$. As $\kappa^s = \kappa^h$ for a Kähler surface, we can rephrase LeBrun’s calculation in the following way: If $M^4$ admits a Kähler structure, then

\begin{equation}
\kappa^s(M^4) = \begin{cases} 
-\infty & \text{if } Y(M^4) > 0, \\
0 & \text{if } Y(M^4) = 0 \text{ and } 0 \text{ is attainable by a metric,} \\
1 & \text{if } Y(M^4) = 0 \text{ and } 0 \text{ is not attainable,} \\
2 & \text{if } Y(M^4) < 0.
\end{cases}
\end{equation}

However, (6) does not determine $\kappa^s(M^4)$ for all symplectic $M^4$: All $T^2$–bundle over $T^2$ have $\kappa^s = 0$ (see [23]) while most of them do not have any
zero scalar curvature metrics. But the question of LeBrun in [18] still makes sense: if \( M^4 \) admits a symplectic structure and \( Y(M^4) < 0 \), is \( \kappa^s(M^4) = 2? \)

A related question is whether we can extend \( \kappa^s \) and \( \kappa^h \) to \( \kappa^d \) for all smooth 4-manifolds (here \( d \) standing for diffeomorphic).

2.2.3. Higher Dimension. In higher dimension, Kodaira dimension is only defined for complex manifolds. And \( \kappa^h \) is known not to be a diffeomorphism invariant. Here is a specific example following [40].

Consider a Fano surface \( (M^4 = \mathbb{CP}^2 \# 5 \mathbb{CP}^2, J_1) \), and a complex surface \( (N^4, J_2) \) of general type homeomorphic to \( M^4 \) as constructed by J. Park et al([39]). Then \( (M^4, J_1) \times (T^2, j) \) and \( (N^4, J_2) \times (T^2, j) \) are complex manifolds, and they are diffeomorphic by the \( s \)-cobordism theorem (as they are \( h \)-cobordant and their Whitehead groups vanish, for details see [40]). However, their complex Kodaira dimensions are different due to the additivity property of \( \kappa^h \) for a product. Similarly, the pair of diffeomorphic 5-manifolds \( M^4 \times S^1 \) and \( N^4 \times S^1 \) tells us that, there is no smoothly invariant definition of Kodaira dimension in dimension 5 if we require the very natural additivity for a product manifold.

Thus we can only expect to have a notion of Kodaira dimension for manifolds with some structures such as complex structures or symplectic structures (for the latter case see the proposal in [29]).

2.3. Additivity for a fiber bundle. We discuss additivity of the Kodaira dimensions \( \kappa^h, \kappa^t, \kappa^s \) for appropriate classes of fiber bundles.

2.3.1. Additivity for \( \kappa^h \). Let us start with the holomorphic Kodaira dimension \( \kappa^h \). A classical theorem says that the additivity holds for a holomorphic fiber bundle (see Theorem 15.1 in [46] for example). Especially, \( \kappa^h \) is covering invariant.

2.3.2. Covering invariance. We start with fiber bundles with 0-dimensional fibers, namely, unramified coverings.

**Proposition 2.10.** The Kodaira dimensions \( \kappa^t, \kappa^s \) are covering invariant.

**Proof.** For 0- and 1-manifolds, it is obvious. For 2-dimensional manifolds it follows from the fact that \( \chi(M) = n\chi(M) \) if \( f: \tilde{M} \to M \) is a degree \( n \) covering map. For 3-dimensional manifolds, it is more or less clear from the definition and was verified in [50].

It remains to check \( \kappa^s \). First of all, if \( f: \tilde{M}^4 \to M^4 \) is a covering map and \( \omega \) is a symplectic form on \( M^4 \), then \( f^*\omega \) is a symplectic form on \( \tilde{M}^4 \), and thus \( \kappa^s(\tilde{M}^4) \) is defined.

One characterization of \( \kappa^s = -\infty \) manifolds is the existence of an embedded symplectic sphere with non-negative self-intersection ([33, 28]). Suppose \( \kappa^s(M^4, \omega) = -\infty \) and \( F \subset (M^4, \omega) \) is an embedded symplectic sphere with \( [F]^2 \geq 0 \). As \( F \) is simply connected, \( f^{-1}(F) \subset (\tilde{M}^4, f^*\omega) \) consists of
\( l = \deg(f) \) symplectic spheres, each still with self-intersection \([F]^2\). Thus
\[ \kappa^s(\tilde{M}^4, f^*\omega) = -\infty = \kappa^s(M^4, \omega). \]

In fact, we can easily enumerate all the coverings in the case \( \kappa^s(M^4, \omega) = \infty \). Assume first that \((M^4, \omega)\) is minimal. Then \( M^4 = \mathbb{CP}^2, S^2 \times S^2 \) or an \( S^2 \)-bundle over \( \Sigma_{h \geq 1} \). If \( M^4 = \mathbb{CP}^2, S^2 \times S^2 \), then so is \( \tilde{M}^4 \). If \( M^4 \) is an \( S^2 \)-bundle over \( \Sigma_{h \geq 1} \), then \( \tilde{M}^4 \) is an \( S^2 \)-bundle over \( \Sigma_{h' \geq 1} \), induced by a covering \( \Sigma_{h'} \to \Sigma_h \). In particular, \((\tilde{M}^4, f^*\omega)\) is still minimal.

For a non-minimal \((M^4, \omega)\) we have the following general observation: When \((M^4, \omega)\) is a blow up of \((N^4, \tau)\) around a symplectic ball \( B^4 \), we observe that \((\tilde{M}^4, f^*\omega)\) is the blow up of \((\tilde{N}^4, g^*\tau)\), where \( \tilde{N}^4 \) is obtained by gluing \( \deg(f) \) copies of \( B^4 \) to \( \tilde{f}^{-1}(N^4 - B^4) \), and \( g : \tilde{N}^4 \to N^4 \) is the obvious covering map.

To prove (7) when \( \kappa^s(M, \omega) \geq 0 \), we need the following fact.

**Lemma 2.11.** \((\tilde{M}^4, f^*\omega)\) is minimal if and only if \((M^4, \omega)\) is minimal.

Let us first assume Lemma 2.11. Using the fact \( K_{f^*\omega} = f^*K_{\omega} \), we have
\[ K_{f^*\omega} \cdot [f^*\omega] = \deg(f) K_{\omega} \cdot [\omega], \quad K_{f^*\omega} \cdot K_{f^*\omega} = \deg(f) K_{\omega} \cdot K_{\omega}. \]
Together with Lemma 2.11, it follows that
\[ (7) \quad \kappa^s(\tilde{M}^4, f^*\omega) = \kappa^s(M^4, \omega) \]
when \((M^4, \omega)\) is minimal.

Now, (7) for a general \((M^4, \omega)\) is a consequence of the observation made before Lemma 2.11.

It only remains to prove Lemma 2.11.

**Proof.** Suppose \((M^4, \omega)\) is not minimal. Then there is a symplectic \(-1\) sphere \( S \) in \((M^4, \omega)\). As \( S \) is simply connected, \( f^{-1}(S) \subset (\tilde{M}^4, f^*\omega) \) consists of \( l = \deg(f) \) symplectic spheres, each still with self-intersection \(-1\).

Suppose \((M^4, \omega)\) is minimal. We want to prove that \((\tilde{M}^4, f^*\omega)\) is also minimal. The case \( \kappa^s(M^4, \omega) = -\infty \) is already settled. When \( \kappa^s(M^4, \omega) \geq 0 \), for a generic \( \omega \)-compatible almost complex structure \( J \), according to Taubes ([11]), \( K_{\omega} \) is represented by a \( J \)-holomorphic submanifold \( C \), possibly disconnected and empty, but without sphere components. Let \( \tilde{J} = f^*J \). Then \( \tilde{C} = f^{-1}(C) \) is a \( \tilde{J} \)-holomorphic submanifold of \((\tilde{M}^4, \tilde{J})\) representing \( K_{f^*\omega} \). Notice that \( \tilde{C} \) still has no sphere components. If \((M^4, f^*\omega)\) is not minimal and \( \tilde{E} \in E_{f^*\omega} \), then there is a \( \tilde{J} \)-holomorphic curve \( V \) in the class of \( \tilde{E} \). The curve \( V \) could be singular and reducible, but every component of \( V \) has to have genus 0. In particular, \( V \) and \( \tilde{C} \) have no common components. By the positivity of intersection of distinct irreducible pseudo-holomorphic curves, \([V] \cdot [\tilde{C}] \geq 0 \). But this contradicts to \( K_{f^*\omega} \cdot \tilde{E} = -1 \).

We note that the notion of Kodaira dimension does not depend on the orientation of the manifold in dimension at most 3. So we could extend it
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Let us mention that the sign of the Yamabe invariant $Y(M)$ is generally not a covering invariant. In dimension 4, LeBrun in [17] constructed a reducible non-symplectic manifold $M^4$ with $Y(M^4) < 0$, whose universal covering is $k\mathbb{CP}^2 \# l\mathbb{CP}^2$, hence having positive Yamabe invariant. This example also shows that the condition that $M$ admits symplectic structures in Lemma 2.11 is necessary.

2.3.3. Bundles in dimensions at most four. The following is essentially contained in [50].

**Proposition 2.12.** $\kappa^t$ is additive for any fiber bundle in dimension at most 3.

The statement is obvious when the base is 0 dimensional. When fibers are 0--dimensional, it is just Proposition 2.10.

It is also obvious that $\kappa^t$ is additive for any circle bundle when the total space is of dimension 1 or 2, even if the bundle is not orientable.

There are two kinds of bundles in dimension 3: circle bundles over surface and surface bundles over circle. In both cases the additivity of $\kappa^t$ is shown in [50]. Circle bundles are special Seifert fiber spaces. See 4.1.2 for related discussions.

In dimension 4 we have the following additivity results.

**Proposition 2.13.** Suppose $M^3 \times S^1$ has a symplectic (complex) structure, then $\kappa^{s(h)}(M^3 \times S^1) = \kappa^t(S^1) + \kappa^t(M^3) = \kappa^t(M^3)$.

**Proposition 2.14.** Suppose $M^4$ is a surface bundle over surface and it has a symplectic (complex) structure, then $\kappa^{s(h)}(M^4) = \kappa^t(\text{base}) + \kappa^t(\text{fiber})$.

Proposition 2.13 is contained in [50]. For symplectic case, it depends on the resolution of the Taubes conjecture by Friedl and Vidussi ([9]). For complex case it depends on [8]. Hopefully, we can generalize it to $M^3$ bundles over $S^1$ or $S^1$ bundles over $M^3$.

Proposition 2.14 is established in [5] when the base surface has positive genus. When the base is $S^2$, the total space is either a ruled manifold which is symplectic and complex and has $\kappa^s = \kappa^h = -\infty$, or a Hopf surface which is complex and has $\kappa^h = -\infty$ (the latter case occurs when the fiber is $T^2$ and homologically trivial).

3. Embedded symplectic surfaces and relative Kod. dim. in dim. 4

In this section $M$ denotes a smooth, oriented, closed and connected 4–manifold, $\omega$ denotes a symplectic form on $M$ compatible with the orientation.
We often identify a degree 2 homology class with its Poincaré dual, and vice versa. We denote by \( \cdot \) the pairing between a degree 2 homology class and a degree 2 cohomology class, the intersection product of two degree 2 homology classes, as well as the cup product of two degree 2 cohomology classes.

3.1. Embedded symplectic surfaces and maximality.

3.1.1. Embedded symplectic surfaces.

**Definition 3.1.** Suppose \( F \subset (M, \omega) \) is a symplectically embedded surface (possibly disconnected). Its genus is defined by

\[
2g(F) - 2 = K_\omega \cdot [F] + [F]^2.
\]

More generally, for a class \( e \in H_2(M) \) we use \( [F]^2 \) to define the \( \omega \)-genus \( g_\omega(e) \) of \( e \).

If \( F \) is connected, \( [F]^2 \) is just the adjunction formula, and thus the (formal) genus \( g(F) \) defined by \( [F]^2 \) is just the usual genus of \( F \). Observe also that if \( F = \sqcup F_i \) with connected components \( F_i \), then

\[
2g(F) - 2 = K_\omega \cdot [F] + [F]^2 = \sum (K_\omega \cdot [F_i] + [F_i]^2) = \sum (2g(F_i) - 2).
\]

In particular, we have

**Lemma 3.2.** Suppose \( F = \sqcup F_i \) with connected components \( F_i \).

(i) If each \( F_i \) has positive genus, then \( g(F) \geq 1 \).

(ii) If \( F \) admits a degree \( d \) map to a connected surface of genus \( h \), then \( g(F) \geq dh - d + 1 \).

Recall that a degree 2 class is called GW stable in [25] if certain GW invariant of this class is nonzero. The next lemma is well-known (cf. [27], [34]).

**Lemma 3.3.** The following classes are GW stable classes.

- The class of an embedded symplectic sphere with non-negative self intersection.
- Any symplectic \(-1\) class \( E \in \mathcal{E}_{M,\omega} \).
- \( K_\omega - E_1 - \cdots - E_p \) with \( E_i \neq E_j \in \mathcal{E}_{M,\omega} \) when \( \kappa^s(M,\omega) \geq 0 \) and \( b^+ > 1 \).
- \( 2K_\omega - E_1 - \cdots - E_p \) with \( E_i \neq E_j \in \mathcal{E}_{M,\omega} \) when \( \kappa^s(M,\omega) \geq 0 \) and \( b^+ = 1 \).
- \( -K_\omega \) when \( (M,\omega) \) is minimal and \( K_\omega^2 \geq 0 \).

The following simple fact was observed in [25].

**Lemma 3.4.** If \( \alpha \in H_2(M;\mathbb{Z}) \) with \( \alpha^2 \geq 0 \) is represented by an embedded symplectic surface, then \( \alpha \) pairs non-negatively with any GW stable class.
Finally, for a possibly disconnected embedded surface $F = \sqcup F_i$ in $M$ with connected components $F_i$, let $F^+$ be the union of $F_i^+$, where

$$F_i^+ = \begin{cases} F_i & \text{if } \kappa^i(F_i) \neq -\infty, \\ \emptyset & \text{if } \kappa^i(F_i) = -\infty. \end{cases}$$

3.1.2. $\kappa^s(K_\omega \cdot [F])$ and $\kappa^s(K_\omega \cdot [\omega])$.

**Lemma 3.5.** Let $(M, \omega)$ be a minimal symplectic manifold with $K^2 \geq 0$. Suppose $S$ is a symplectic surface with $S^2 > 0$. We further suppose that there is a relatively minimal Lefschetz fibration on $\tilde{M} = M^2 k\mathbb{CP}^2$ such that the class of a fiber $\tilde{S}$ satisfies $\pi_*[\tilde{S}] = [S]$, where $\pi_* : H_2(\tilde{M}) \to H_2(M)$ is the natural homomorphism. Then

$$\kappa^s(K_\omega \cdot [S]) = \kappa^s(\kappa^s(M, \omega)).$$

**Proof.** First of all, under the assumption that $(M, \omega)$ is minimal and $K^2 \geq 0$, by Lemma 3.3, $K_\omega$ or $2K_\omega$ is a GW stable class if $\kappa^s(M, \omega) \geq 0$, and $-K_\omega$ is a GW stable class if $\kappa^s(M, \omega) = -\infty$. By Lemma 3.4, $K_\omega \cdot [S] \geq 0$ if $\kappa^s(M, \omega) \geq 0$, and $K_\omega \cdot [S] \leq 0$ if $\kappa^s(M, \omega) = -\infty$.

Thus, if $K_\omega \cdot [S] < 0$, we must have $\kappa^s(M, \omega) = -\infty$. Conversely, if $\kappa^s(M, \omega) = -\infty$, since $[S]^2 > 0$ and $K_\omega \cdot [S] \leq 0$, by the light cone lemma, we have $K_\omega \cdot [S] < 0$.

If $\kappa^s(M, \omega) = 0$, then $K_\omega$ is a torsion class. So $K_\omega \cdot [S] = 0$ in this case.

To prove (10), what remains to show is that if $\kappa^s(M, \omega) \geq 1$, then $K_\omega \cdot [S] > 0$. It is here that we need the assumption that $[S]$ lifts to the fiber class $[\tilde{S}]$ of a Lefschetz fibration on $(\tilde{M} = M^2 k\mathbb{CP}^2, \tilde{\omega})$. Notice that since $[S]^2 > 0$, $[S]$ itself cannot be the fiber class, thus we must have $k > 0$. Notice also that $K_\omega \cdot [S] = \pi^*K_\omega \cdot [\tilde{S}]$.

In this case, $\kappa^s(\tilde{M}, \tilde{\omega}) \geq 1$. Then $\pi^*K_\omega$, or $\pi^*(2K_\omega)$ in the case $b^+ = 1$, is still a GW stable class in the blow up $(\tilde{M}, \tilde{\omega})$. Here $\pi^* : H^2(\tilde{M}) \to H^2(M)$ is the natural inclusion. Choose an almost complex structure $J$ on $\tilde{M}$ making the Lefschetz fibration $J$-holomorphic. What can a $J$-holomorphic representative of $\pi^*K_\omega$ ($2\pi^*K_\omega$) be? If it is in a fiber or a union of several fibers, then its square is at most 0, and its square is 0 only if it is a union of fibers. Thus if $\kappa^s(M, \omega) = 2$, this is impossible. If $\kappa^s(M, \omega) = 1$, it still violates the fact that the intersection number of $\pi^*K_\omega$ with any $-1$ class of $(\tilde{M}, \tilde{\omega})$ is 0. Thus $K_\omega$ must have a multi-section component. This shows that $\pi^*K_\omega \cdot [\tilde{S}] > 0$, and hence $K_\omega \cdot [S] > 0$.

Following from Lemma 2.8, we have

**Corollary 3.6.** Under the assumption of Lemma 3.5

$$\kappa^s(K_\omega \cdot [S]) = \kappa^s(K_\omega \cdot [\omega]).$$

Any member of Lefschetz pencil on $(M, \omega)$ satisfies the assumption of Lemma 3.5. In this case, Gompf (12) showed that there is a symplectic
form $\tau$ on $M$ in the positive ray of $[S]$. It would be interesting to see whether this remains to be true for any $S$ as in Lemma 3.5.

Suppose $E_i$ are the classes of symplectic $-1$ spheres in $(\tilde{M}, \tilde{\omega})$ that are blown down to obtain $(M, \omega)$. Since $[\tilde{S}]^2 = 0$,

\begin{equation}
2g(\tilde{S}) - 2 = K_\omega \cdot [\tilde{S}] = (\pi^*K_\omega + \sum E_i)(\iota_* [S] - \sum ([\tilde{S}] \cdot E_i)E_i),
\end{equation}

where $\iota_* : H_2(M) \to H_2(\tilde{M})$ is the natural inclusion. Thus we can express $\kappa^s(K_\omega \cdot [\omega])$ in terms of $g(\tilde{S})$ and $c_i = [\tilde{S}] \cdot E_i$,

\begin{equation}
\kappa^s(K_\omega \cdot [\omega]) = \kappa^s(2g(\tilde{S}) - 2 - \sum c_i).
\end{equation}

3.1.3. Maximal surfaces.

**Definition 3.7.** Suppose $F \subset (M, \omega)$ is a symplectically embedded surface without sphere components. $F$ is called maximal if $[F] \cdot E \neq 0$ for any $E \in \mathcal{E}_\omega$.

For a general embedded symplectic surface $F$, it is called maximal if $F^+$ is maximal.

Any member of a relatively minimal Lefschetz pencil or a fiber of a relatively minimal Lefschetz fibration is maximal. Notice that if $F^+ = \emptyset$, then $F$ is maximal if and only if $(M, \omega)$ is minimal.

Let $F_i$ be the connected components of an embedded symplectic surface $F$. Because the $F_i$ are disjoint and embedded symplectic surfaces, we can choose an almost complex structure $J$ to make each $F_i$ $J$-holomorphic.

**Claim 3.8.** Suppose the genus of each $F_i$ is positive. Then for any $E \in \mathcal{E}_\omega$, we can further assume that $J$ is chosen such that both $F$ and an embedded representative of $E$ are $J$–holomorphic.

**Proof.** This can be done, for example, by Proposition 4.1 in [37]. We recall the argument here: Without loss of generality, we assume that $F$ is connected.

First, we choose a $J_0$ such that $F$ is $J_0$ holomorphic. We can assume that $J_0$ is generic outside a small neighborhood $U$ of $F$ so that any simple $J_0$ holomorphic curve which are not contained in $U$ are transversal. Suppose $E$ and $[F]$ can not be represented by $J$–holomorphic curves simultaneously. Choose a sequence of $J_n$ converging to $J_0$ such that $E$ is represented by the embedded $J_n$–holomorphic $-1$–curve $E_n$ for all $n$. Then $E_n$ converges to the image of a stable map $\sum m_iB_i$, where $B_i$’s are simple. Here $\sum m_i > 1$.

Now we show that one of $\{B_i\}$ is contained in $U$. If not, they are transversal by our genericness assumption of $J_0$. Hence, for $n$ large enough, $B_i$ deform to $J_n$–holomorphic $B'_i$. Thus $E_n$ and $\sum m_iB'_i$ are both $J_n$–holomorphic curves representing class $E$. If $E_n$ does not appear in $\{B_i\}$, then $-1 = E_n^2 = [E_n] \cdot \sum m_i[B'_i] \geq 0$, a contradiction. So there is an $i$ so that $B_i = E_n$. It never happens because symplectic area only depends on the homology class and $\sum m_i > 1$. 

\[12\]
Now, note that each component of the stable map above is of genus 0 and at least one of them is possibly a multiple cover of $F$, whose genus is positive. This is impossible. □

Thus we can conclude

**Lemma 3.9.** Suppose $F \subset (M, \omega)$ is a symplectically embedded surface without sphere components.

- $F \neq \emptyset$ is maximal in $(M, \omega)$ in the sense of Definition 3.7 if and only if $[F] \cdot E > 0$ for any $E \in \mathcal{E}_\omega$.
- If $[F] \cdot E = 0$ for some $E \in \mathcal{E}_\omega$, then we can blow down a symplectic sphere in the class $E$ which is disjoint from $F$.

Here is another useful consequence of Claim 3.8.

**Lemma 3.10.** Suppose $(N, \sigma)$ is obtained from $(M, \omega)$ by blowing down a finite set of disjoint symplectic $-1$ spheres in the classes $E_i$. Then for any embedded symplectic surface $F \subset (M, \omega)$, possibly disconnected but with each component positive genus, there is an embedded symplectic surface $F' \subset (N, \sigma)$ with each component positive genus such that

$$[F] = \iota_*[F'] - \sum_{i=1}^k c_iE_i.$$

Here $\iota_* : H^2(N) \to H^2(M)$ is the natural inclusion.

**Proof.** To apply Lemma 3.9 we blow down the $-1$ classes successively. We choose an $\omega$–tamed almost complex structure $J$ as in the proof of Claim 3.8 such that $E_1$ is represented by an embedded $J$–holomorphic sphere $S_1$, and $F$ is $J$–holomorphic. By a small isotopy of $F$, we can further assume that $F$ is symplectic and intersects $S_1$ transversally and non-negatively. We can then perform blow down such that $F$ becomes an immersed symplectic surface with only positive nodal points and still without sphere components. Here it is convenient to view blowing down as fiber summing with the pair $\mathbb{C}P^2$ along a line, and from this point of view, the immersed symplectic surface is obtained from Gompf’s pairwise fiber sum construction ([11]). Observe that Claim 3.8 actually generalizes to a positively immersed symplectic surface as it still can be made pseudo-holomorphic. Then we repeat this process to finally obtain an immersed symplectic surface $F_{\text{red}}$ in $(N, \sigma)$ with only positive nodal points and still without sphere components. By Corollary 3.4 in [30], we can perturb it to an embedded symplectic surface $F'$. Notice that, if $c_i = F \cdot E_i$, then,

$$[F] = \iota_*[F_{\text{red}}] - \sum_{i=1}^k c_iE_i = \iota_*[F'] - \sum_{i=1}^k c_iE_i.$$

□
3.2. The adjoint class $K_\omega + [F]$. The following definition was introduced in [31] for a connected surface.

**Definition 3.11.** Let $F$ be an embedded symplectic surface in $(M, \omega)$ with each component positive genus. The adjoint class of $F$ is defined as $K_\omega + [F]$.

- $F$ is called maximal if for any symplectic $-1$ class $E$,
  $$(K_\omega + [F]) \cdot E \geq 0.$$
- $F$ is called special if $(K_\omega + [F])^2 = 0$.
- $F$ is called distinguished if $K_\omega + [F]$ is rationally trivial.

As $K_\omega \cdot E = -1$ for any $E \in E_M$, by Lemma 3.9, the two notions of maximality in Definitions 3.7 and 3.11 coincide when $F$ is an embedded symplectic surface without sphere components.

In this subsection we assume that $F$ is a maximal symplectic surface in $(M, \omega)$ with each component positive genus.

3.2.1. $\kappa^s((K_\omega + [F])^2)$. We now discuss the sign of $(K_\omega + [F])^2$ for a maximal symplectic surface $F$, in other words, we calculate $\kappa^s((K_\omega + [F])^2)$.

**Proposition 3.12.** Suppose $\kappa^s(M, \omega) \geq 0$, $F = \bigsqcup F_i$ is a maximal symplectic surface and each $F_i$ of positive genus. Then we have

$$(15) \quad (K_\omega + [F])^2 \geq 0.$$

**Proof.** Notice that when $F$ is connected the statement is contained in [31]. We point out however when $[F]^2 < 0$ some further arguments, e.g. those in the appendix in [4], are needed to complete the proof there. We here offer an alternative argument for this more general (possibly disconnected) situation.

Let us rewrite

$$(16) \quad (K_\omega + [F])^2 = K_\omega^2 + K_\omega \cdot [F] + (K_\omega \cdot [F] + [F]^2)$$

as a sum of three terms.

First let us suppose that $F$ is connected. By (8) the last term is non-negative as $g(F) \geq 1$. Let us argue that

$$\text{(17)} \quad K_\omega \cdot [F] \geq 0.$$

When $[F]^2 \geq 0$, it is due to lemmas 3.3 and 3.3; when $[F]^2 < 0$, because $g(F) \geq 1$, it is due to the adjunction formula (8).

If we further assume that $(M, \omega)$ is minimal, then $K_\omega^2 \geq 0$ as well. Thus we can conclude that (15) holds when $F$ is connected and $(M, \omega)$ is minimal.

For a disconnected symplectic surface $F = \bigsqcup F_i$, as each connected component $F_i$ has positive genus, we still have by the adjunction formula,

$$(18) \quad (K_\omega \cdot [F] + [F]^2) = \sum (K_\omega \cdot [F_i] + [F_i]^2) \geq 0.$$

Moreover, $K_\omega \cdot [F_i] \geq 0$, so

$$(19) \quad K_\omega \cdot [F] = \sum K_\omega \cdot [F_i] \geq 0.$$
Thus if \((M, \omega)\) is minimal, all three terms in (16) are still non-negative.

In summary we have shown that (15) holds when \((M, \omega)\) is minimal.

Now we assume that \((M, \omega)\) is non-minimal and \(E_\omega = \{E_i\}\). Then \(K_\omega^2\) could be negative. However, as the 3rd term in (16) is always non-negative, it suffices to prove that the sum of the 1st and the 2nd terms

\[
K_\omega^2 + K_\omega \cdot [F]
\]

in (16) is non-negative.

Let \((N, \sigma)\) be the minimal model of \((M, \omega)\) and \(K_\sigma\) be its symplectic canonical class. Then

\[
K_\omega = \pi^* K_\sigma + \sum_{1}^{k} E_i,
\]

where \(\pi^*: H^2(N) \rightarrow H^2(M)\) is the natural inclusion. By Lemma 3.10, there is an embedded symplectic surface \(F' \subset (N, \sigma)\) such that

\[
[F] = \iota_* [F'] - \sum c_i E_i,
\]

As argued above, we have

\[
K_\sigma^2 + K_\sigma \cdot [F'] \geq 0.
\]

The contribution of \(E_i\) to \(K_\sigma^2\) is \(-k\), to \(K \cdot [F]\) is \(\sum_{i} c_i\). Because \(F\) is maximal, \(c_i \geq 1\). Thus the difference of (20) and (22) is non-negative.

\[
K_\sigma^2 + K_\sigma \cdot [F'] \geq 0.
\]

From the arguments above, it is easy to determine when \((K_\omega + [F])^2 = 0\).

**Proposition 3.13.** Suppose \(\kappa^s(M, \omega) \geq 0\), \(F = \sqcup F_i\) is a maximal symplectic surface and each \(F_i\) of positive genus. If \((K_\omega + [F])^2 = 0\), then \(\kappa^s(M, \omega) = 0\) or 1, and each \(F_i\) is a torus.

Moreover, suppose \(\kappa^s(M, \omega) = 0\) or 1, \(F = \sqcup F_i\) is a maximal symplectic surface and each \(F_i\) is a torus. If \((M, \omega)\) is minimal, then \((K_\omega + [F])^2 = 0\) if and only if \([F_i]^2 = 0\); and in general, suppose \((N, \sigma)\) is the minimal model, then \((K_\omega + [F])^2 = 0\) if and only if there is a partition \(\{E_{ij}\}\) of \(E_{M,\omega}\) such that \([F_i] = \iota_* [F'_i] - \sum_j E_{ij}\), where \(F'_i\) are disjoint square 0 symplectic tori in \((N, \sigma)\).

When \(\kappa^s(M, \omega) = -\infty\) we also have (15) except in one case.

**Proposition 3.14.** Suppose \(\kappa^s(M, \omega) = -\infty\), \(F = \sqcup F_i\) is a maximal symplectic surface and each \(F_i\) of positive genus. If \(F\) is not a section of a genus \(g \geq 1\) \(S^2\) bundle, then we have (15).

**Proof.** This is also proved in [31] under the assumption that \(F\) is connected. The argument is a case by case analysis. Our argument here is also a case by case analysis. As in [31], we observe that (15) is equivalent to

\[
K_\omega^2 - [F]^2 \geq 4(1 - g(F)).
\]
Here $g(F)$ is the genus defined in Definition 3.1. We need to point out however that there is a misprint in (8) in [31]: the lefthand side should be $K^2_\omega - |F|^2$.

With Lemma 3.2 understood we can check that the argument for (23) in [31] for a connected $F$ remains valid in each case for a disconnected $F$. □

We offer another argument using SW theory as in [26].

**Lemma 3.15.** Suppose $\kappa^s(M, \omega) = -\infty$, $e$ is a class with positive $\omega$–genus $g_\omega(e)$ (see Definition 3.1), $e \cdot [\omega] > 0$, and $e \cdot E > 0$ for any $E \in E_{M, \omega}$. If $e$ is not the class of a section of a genus $g \geq 1$ $S^2$ bundle, then $K_\omega + e$ is represented by $\bigsqcup G_i$ with each $G_i$ a symplectic surface satisfying

$$[G_i]^2 \geq 0, \quad -K_\omega \cdot [G_i] + [G_i]^2 \geq 0.$$  
In particular, $(K_\omega + e)^2 = \sum [G_i]^2 \geq 0$.

**Proof.** Recall that for a symplectic 4–manifold $(M, \omega)$, there is a canonical bijection between Spin$^c$ structures and $H_2(M; \mathbb{Z})$. Recall also when $b^+(M) = 1$ (which is our case here), for each Spin$^c$ structure (equivalently, a class in $H_2(M; \mathbb{Z})$), there are two SW invariants, one of which is $SW_\omega$. By the celebrated result of [14], if $SW_\omega(\alpha) \neq 0$, then $\alpha \cdot [\omega] > 0$. Moreover, if $\alpha \cdot E \geq 0$ for any $E \in E_{M, \omega}$, $\alpha$ is represented by a possibly disconnected symplectic submanifold $\bigsqcup G_i$ satisfying (24).

We will show that $SW_\omega(K_\omega + e) \neq 0$ under the assumption that $g_\omega(e) > 0$ and $e \cdot [\omega] > 0$. Since we also assume $e \cdot E \geq 1$ for any $E \in E_{M, \omega}$, the conclusion of Lemma 3.15 will then follow.

We first calculate the Seiberg-Witten dimension of the Spin$^c$ structure $K_\omega + e$,

$$\dim_{SW}(K_\omega + e) = -K_\omega \cdot (K_\omega + e) + (K_\omega + e)^2 = e(K_\omega + e) = 2g_\omega(e) - 2.$$  
Since $g_\omega(e)$ is assumed to be positive, $\dim_{SW}(K_\omega + e) \geq 0$. Notice that $K_\omega - (K_\omega + e) = -e$. Thus we have

$$|SW_\omega(K_\omega + e) - SW_\omega(-e)| = \begin{cases} 1 & \text{if } (M, \omega) \text{ rational,} \\ |1 - (e \cdot T)| & \text{if } (M, \omega) \text{ irrationally ruled,} \end{cases}$$

where $T$ is the unique positive fiber class of irrationally ruled manifolds (see [35]). Since $(-e) \cdot [\omega] < 0$ by assumption, we have $SW_\omega(-e) = 0$.

Hence we can conclude that unless $(M, \omega)$ is irrationally ruled and $e \cdot T = 1$, we have $SW_\omega(K_\omega + e) \neq 0$. It remains to show that $e \cdot T = 1$ only if $(M, \omega)$ is an $S^2$ bundle over a positive genus surface, i.e. $(M, \omega)$ is minimal. This follows immediately from the fact that if $(M, \omega)$ is not minimal, there are two classes $E_1, E_2 \in E_{M, \omega}$ with $E_1 + E_2 = T$. □

**Corollary 3.16.** Suppose $\kappa^s(M, \omega) = -\infty$, $F = \bigsqcup F_i$ is a maximal symplectic surface and each $F_i$ of positive genus. Then $(K_\omega + [F])^2 = 0$ if and only if $K_\omega + [F]$ is represented by a disjoint union of symplectic spheres in the same class with square 0, or a disjoint union of symplectic tori whose classes have square 0 and proportional to each other.
Proof. By Lemma 3.15 $K_{\omega} + [F]$ is represented by an embedded symplectic surface $\sqcup G_i$ satisfying (24). Since $\sum [G_i]^2 = 0$, we have $[G_i]^2 = 0$ for each $i$. Apply (24) and the adjunction formula, we find that the genus of each $G_i$ is either all equal to 0 or 1. Moreover, by the light cone lemma, the classes $[G_i]$ must be proportional to each other. \qed

3.2.2. $\kappa^s((K_{\omega} + [F]) \cdot [\omega])$. We now discuss the sign of $(K_{\omega} + [F]) \cdot [\omega]$ for a maximal symplectic surface $F$, in other words, we calculate $\kappa^s((K_{\omega} + [F]) \cdot [\omega])$.

**Proposition 3.17.** If $\kappa^s(M, \omega) \geq 0$, then

\begin{equation}
(K_{\omega} + [F]) \cdot [\omega] \geq 0,
\end{equation}

with equality holds if and only if $(M, \omega)$ is minimal with $\kappa^s = 0$ and $F$ is empty.

**Proof.** When $\kappa^s(M, \omega) \geq 1$, we have $K_{\omega} \cdot [\omega] > 0$ and hence $(K_{\omega} + [F]) \cdot [\omega] > 0$.

When $\kappa^s(M, \omega) = 0$, then $(K_{\omega} + [F]) \cdot [\omega] = 0$ only when $(M, \omega)$ is minimal and $[F] \cdot [\omega] = 0$. As $F$ is symplectic, this is possible only if $F$ is empty. \qed

**Proposition 3.18.** Suppose $\kappa^s(M, \omega) = -\infty$ and $F = \sqcup F_i \subset (M, \omega)$ is a possibly disconnected maximal symplectic surface with $g(F_i) \geq 1$. If $F$ is not a section of a genus $g \geq 1$ $S^2$ bundle, then we have

\begin{equation}
(K_{\omega} + [F]) \cdot [\omega] \geq 0.
\end{equation}

Moreover, equality holds only if $[F] = -K_{\omega}$ and each $F_i$ is a torus.

**Proof.** We first characterize those with $(K_{\omega} + [F]) \cdot [\omega] = 0$. If $F$ is not a section of a genus $g \geq 1$ $S^2$ bundle, then $(K_{\omega} + [F])^2 \geq 0$ by Proposition 3.16. Notice that $b^+(M) = 1$. As $(K_{\omega} + [F])^2 \geq 0$ and $[\omega]^2 > 0$, we can apply the light cone lemma to $(K_{\omega} + [F]) \cdot [\omega] = 0$ to conclude that $K_{\omega} + [F]$ is a torsion class. Since $M$ has no torsion in homology, in fact, $[F] = -K_{\omega}$.

For any component $F_i$ of $-K_{\omega}$, we have $-K_{\omega} \cdot [F_i] = [F_i]^2$. Thus its genus is still 1.

It remains to prove (26).

By Proposition 3.18 and Lemma 3.2 it suffices to show that if $F$ is maximal and

\begin{equation}
(K_{\omega} + [F]) \cdot [\omega] < 0 \quad \text{and} \quad (K_{\omega} + [F])^2 \geq 0,
\end{equation}

then we obtain a contradiction, often in the form $g(F) < 1$, i.e. $2g(F) - 2 < 0$. Our argument is a case by case analysis.

- $S^2 \times S^2$.

In this case $K_{\omega} = -2H_1 - 2H_2$, $[F] = aH_1 + bH_2$ for some integers $a, b$. Here $H_1, H_2$ are classes of $S^2$ factors with positive symplectic area. Then

\begin{align*}
K_{\omega} + [F] &= (a - 2)H_1 + (b - 2)H_2, \\
(K_{\omega} + [F])^2 &= 2(a - 2)(b - 2).
\end{align*}
As $H_1$, $H_2$ have positive symplectic area,

$$\begin{align*}
[\omega] = xH_1 + yH_2, & \quad y > 0, x > 0, \\
(K_\omega + [F]) \cdot [\omega] = x(b - 2) + y(a - 2).
\end{align*}$$

Then (27) becomes that

$$x(b - 2) + y(a - 2) < 0, \quad (a - 2)(b - 2) \geq 0,$$

which implies that $a, b \leq 2$ and at most one of them gets the value 2.

If $[F]^2 \geq 0$, since $H_1$ and $H_2$ are GW stable classes, by Lemma 3.4 we know that $a, b \geq 0$. If $[F]^2 < 0$, then $ab < 0$, and so one of them should be 1. It is straightforward to check that in both cases, we have

$$2g(F) - 2 = (K_\omega + [F]) \cdot [F] = (a - 2)b + (b - 2)a < 0.$$  

**$\mathbb{CP}^2|_k\mathbb{CP}^2$**

Let $E_i$ be the positive generators of $H_2$ of the $\mathbb{CP}^2$ factors. In this case $K_\omega = -3H + \sum_i c_i E_i$ and $[F] = d[H] - \sum_i c_i E_i$ for some $d > 0$ and $c_i \geq 1$. Then

$$\begin{align*}
K_\omega + [F] &= (d - 3)H - (c_i - 1)E_i, \\
(K_\omega + [F])^2 &= (d - 3)^2 - \sum(c_i - 1)^2, \\
[\omega] &= xH - \sum z_i E_i, \quad x > 0, z_i > 0, x^2 > \sum z_i^2, \\
(K_\omega + [F]) \cdot [\omega] &= (d - 3)x - \sum z_i(c_i - 1).
\end{align*}$$

(27) becomes

$$(d - 3)x < \sum z_i(c_i - 1), \quad (d - 3)^2 \geq \sum(1 - c_i)^2.$$  

Hence, when $d \geq 3$ we have the following absurd inequality

$$(d - 3)^2x^2 < (\sum z_i(c_i - 1))^2 \leq \sum z_i^2 \cdot \sum(c_i - 1)^2 = (d - 3)^2x^2.$$  

In fact, what is behind the inequality is the light cone lemma. Finally, if $0 < d < 3$ then

$$2g(F) - 2 = (d - 3)d - \sum(c_i - 1)c_i < 0.$$  

**Non-trivial $S^2$--bundle over $\Sigma_h$ with $h \geq 1$.**

In this case let $U$ be the class of a section with square 1, $T$ be the class of a fiber, both with positive symplectic area. Then $K_\omega = -2U + (2h - 1)T$, and $[F] = aU + bT$ for some integers $a$ and $b$. Now

$$\begin{align*}
K_\omega + [F] &= (a - 2)U + (2h - 1 + b)T, \\
(K_\omega + [F])^2 &= (a - 2)(a - 2 + 4h - 2 + 2b).
\end{align*}$$

As $U, T$ have positive symplectic area,

$$\begin{align*}
[\omega] &= xU + yT, \quad x > 0, x + y > 0, x + 2y > 0, \\
(K_\omega + [F]) \cdot [\omega] &= (x + y)(a - 2) + x(2h - 1 + b).
\end{align*}$$  

Then (27) becomes

\[ 2h - 1 + b < -\frac{(x + y)(a - 2)}{x}, \quad (a - 2)(2h - 1 + b) \geq 0, \]

which implies that \( a \leq 2 \) and \( 2h - 1 + b \leq 0 \), and at most one equality holds.

To proceed we compute that

\[ 2g(F) - 2 = (K_\omega + [F]) \cdot [F] = a(a - 2 + 2h - 1 + b) + b(a - 2). \]

If \([F]^2 \geq 0\), we also have \( a \geq 0 \) by Lemma 3.4 since \( T \) is a stable class.

When \( a = 0 \), then \( b \) has to be positive as \([F] \cdot [\omega] > 0\), and by (29), \( 2g(F) - 2 = -2b < 0 \); if \( a = 2 \) then \( 2h - 1 + b < 0 \) by (28), and by (29), \( 2g(F) - 2 = 2(b + 2h - 1) < 0 \).

If \([F]^2 < 0\), we have \( a(a + 2b) < 0 \). When \( a < 0 \), \( a + 2b > 0 \), then \( g(F) < 0 \) by (29). The case when \( a > 0 \) but \( a \neq 1 \) is already analyzed above.

Finally, we analyze the case \( a = 1 \). If \( F \) is connected, then it is a section. If it is not connected, then there is a component with \( a \neq 1 \). But the genus of such a component (which is automatically maximal as \( M \) is minimal) is not positive as already shown.

- \( S^2 \times \Sigma_h, h \geq 1 \).
  - This case is similar to the previous case.
- \( (S^2 \times \Sigma_h)^{k \mathbb{CP}^2} \)
  - Let \( E_i \) be the positive generators of \( H_2 \) of the \( \mathbb{CP}^2 \) factors. In this case let \( U \) be the class of a section with square 0, \( T \) be the class of a fiber, both with positive symplectic area. Then \( \mathcal{E}_\omega = \{ E_i, T - E_i \} \) and
  \[
  K_\omega = -2U + (2h - 2)T + \sum_{i=1}^{k} E_i.
  \]

Thus \( F \) is maximal if and only if

\[ [F] = aU + bT - \sum_{i=1}^{k} c_i E_i, \quad a > c_i \geq 1. \]

We explicitly compute,

\[
\begin{align*}
K_\omega + [F] &= (a - 2)U + (2h - 2 + b)T + \sum_{i=1}^{k}(1 - c_i)E_i, \quad a > c_i \geq 1, \\
(K_\omega + [F])^2 &= 2(a - 2)(2h - 2 + b) - \sum(c_i - 1)^2, \\
[\omega] &= xU + yT - \sum z_i E_i, \quad x, y, z_i > 0, 2xy - \sum z_i^2 > 0, \\
(K_\omega + F) \cdot [\omega] &= (a - 2)y + (2h - 2 + b)x - \sum z_i(c_i - 1).
\end{align*}
\]

Then (27) becomes

\[ (a - 2)y + (2h - 2 + b)x - \sum z_i(c_i - 1) < 0, \quad \sum(c_i - 1)^2 \leq 2(a - 2)(b + 2h - 2). \]
When \((a - 2)y + (2h - 2 + b)x \geq 0\),
\[
((a - 2)y + (2h - 2 + b)x)^2 < \left( \sum z_i(c_i - 1) \right)^2 
\leq \sum z_i^2 \cdot \sum (c_i - 1)^2 
< 2xy \cdot 2(a - 2)(b + 2h - 2).
\]

This is equivalent to saying that
\[
((a - 2)y + (2h - 2 + b)x)^2 < 0,
\]
which is a contradiction! Again what is hidden behind is the light cone lemma.

Now let us suppose \((a - 2)y + (2h - 2 + b)x < 0\). If \(a < 2\), then by the maximality condition \(a > c_i \geq 1\), \((M, \omega)\) is in fact minimal in which case we have treated above. Now, we assume that \(a \geq 2\) and \(2h - 2 + b < 0\). Then
\[
\sum (c_i - 1)^2 \leq 2(a - 2)(b + 2h - 2) \leq 0.
\]
This forces \(c_i = 1\) and \(a = 2\). In this case \(2g(F) - 2 = 2(2h - 2 + b) < 0\). □

3.3. Existence and Uniqueness of relatively minimal model. Any surface can be made maximal by blowing down.

Lemma 3.19. Suppose \(F \subset (M, \omega)\) is a symplectic surface without sphere components. Denote the set of \(E\) with \(F \cdot E = 0\) by \(\mathcal{E}_\omega^F\). Suppose \(\{E_i\} \subset \mathcal{E}_\omega\) is a maximal subset of pairwise orthogonal elements. Blow down a set of symplectic \(-1\) spheres \(S_i\) in the classes \(\{E_i\}\), which are disjoint from each other and from \(F\), to obtain \((M', \omega')\). If we denote the same surface in \((M', \omega')\) by \(F'\), then \(F'\) is maximal in \((M', \omega')\).

Proof. When \(F\) is connected, this is Theorem 1.1(ii) in [33]. For a disconnected \(F\) it follows from Theorem 3.4 in [33], with \(\Lambda\) there being the subgroup orthogonal to the subgroup generated by \([F_i]\).

Definition 3.20. Suppose \(F \subset (M, \omega)\) is a symplectic surface without sphere components. \((M', \omega', F')\) in Lemma 3.19 is called a relative minimal model of \((M, \omega, F)\).

For a general symplectic surface \(F\), a relative minimal model of \((M, \omega, F)\) is defined to be a relative minimal model of \((M, \omega, F^+)\).

It is well known that in the case of \(\kappa^s = -\infty\), there are more than one minimal models. So the following uniqueness of relative minimal model when \(F^+\) is not empty is surprising.

Theorem 3.21. If \(F^+\) is nonempty, there is a unique relative minimal model.

Proof. Without loss of generality we can assume \(F = F^+\).

Recall that
\[
\mathcal{E}_\omega^F = \{E \in \mathcal{E}_\omega | E \cdot [F] = 0\}.
\]
When $M$ is not rational or ruled, the classes in $\mathcal{E}_\omega$ are pairwisely orthogonal, and represented by disjoint symplectic $-1$ spheres. Of course the same is true for $\mathcal{E}_F'$. Thus there is a unique way to make $F$ maximal.

When $M$ is irrationally ruled, $\mathcal{E}_\omega$ can be described as

$$\{E_1, T - E_1, \ldots, E_l, T - E_l\},$$

where $T$ is the unique $\omega-$positive fiber class. If $\mathcal{E}_F'$ contains both $E_1$ and $T - E_1$, then $[F] \cdot T = 0$. As $T$ is a GW stable class, we have $F_i \cdot T = 0$ as well by Lemma 3.4. It follows that each $[F_i]$ is of the form $a_i T - \sum c_j E_j$, $a_i \geq 0$.

But by the adjunction formula, such a component has genus at most zero. As each component of $F$ is of positive genus, $\mathcal{E}_F'$ contains only pairwisely orthogonal classes. Again there is a unique way to make $F$ maximal in this case.

If $M = \mathbb{CP}^2 \# \mathbb{CP}^2$, there is a unique class in $\mathcal{E}_\omega$ and hence at most one class in $\mathcal{E}_F'$.

The remaining case is $M = \mathbb{CP}^2 \# l \mathbb{CP}^2$ with $l \geq 2$. The proof is based on the properties of the adjoint class of a maximal surface established in 3.2.

Suppose $(M', \omega', F')$ is a relative minimal model of $(M, \omega, F)$. Then by Lemmas 3.16 and 3.18 we can assume that

$$\begin{align*}
(K_{\omega'} + [F'])^2 &\geq 0, \\
(K_{\omega'} + [F']) \cdot [\omega'] &\geq 0,
\end{align*}$$

since $M$ is not an $S^2-$bundle over $\Sigma_h$ with $h \geq 1$. Let $S_i$ be a set of disjoint symplectic $-1$ spheres which are blown down to obtain $(M', \omega')$. Notice that the $S_i$ are also assumed to be disjoint from $F$. Let $U = \{E_i = [S_i]\}$. Then

$$K_\omega = \pi^* K_{\omega'} + \sum E_i,$$

Suppose $G \in \mathcal{E}_F'$ and is distinct from $E_i$. Suppose also that there is some $E_i \in U$ such that $E_i \cdot G > 0$. After choosing a symplectic $-1$ sphere in the class $G$ which is disjoint from $F$ and intersects the $S_i$ transversally and non-negatively, by Lemma 3.10 we see that there is possibly immersed symplectic surface $C$ in $(M', \omega')$ with the following properties:

- $C$ is disjoint from $F' = F$, so

$$[F'] \cdot [C] = 0.$$

- $[C]$ is related to $G$ via

$$G = \iota_* [C] - \sum (E_i \cdot G) E_i.$$

- By (33), we have

$$[C]^2 = [G]^2 + 2 \sum (E_i \cdot G)^2 - \sum (E_i \cdot G)^2$$

$$= -1 + \sum (E_i \cdot G)^2$$

$$\geq 0.$$
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• By (31) and (33), we have

\[ K_{\omega'} \cdot [C] = \pi^* K_{\omega} \cdot \iota_\ast [C] \]
\[ = (K_{\omega} - \sum E_i)(G + \sum (E_i \cdot G)E_i) \]
\[ = K_{\omega} \cdot G + \sum (E_i \cdot G)(K_{\omega} \cdot E_i - E_i^2 - 1) \]
\[ = K_{\omega} \cdot G - \sum (E_i \cdot G) \]
\[< 0 \]

• By (32) and (35) we conclude

\[ (K_{\omega'} + [F']) \cdot [C] < 0. \]

Notice that both \([C], K_{\omega'} + [F']\) have non-negative square by (33) and (30), and both pair positively with \([\omega']\) by (30). Since \(b + (M) = 1\), (36) violates the light cone lemma. This contradiction again shows that \(E_F\) contains only pairwisely orthogonal classes. Therefore there is a unique way to make \(F\) maximal in this case.

We remark that there is an alternative argument when \(b + (M) = 1\) and there is a component, say \(F_1\), with \([F_1]\) \(\geq 0\). In this case we can directly show that the classes in \(E_F\) are pairwise orthogonal. Suppose \(G_1, G_2 \in E_F\) and \(G_1 \cdot G_2 \neq 0\). Then \(G_1 \cdot G_2 > 0\). If \(G_1 \cdot G_2 \geq 2\), then \((G_1 + G_2)^2 > 0\). Since \(b + (M) = 1\), this contradicts to the light cone lemma as \([F_1]^2 \geq 0\) and \([F_1] \cdot (G_1 + G_2) = 0\). If \(G_1 \cdot G_2 = 1\) then \((G_1 + G_2)^2 = 0\), we still get a contradiction unless \([F_1]\) and \(G_1 + G_2\) are proportional to each other. However, this is impossible due to the adjunction formula and \(K_{\omega} \cdot (G_1 + G_2) = -2\).

When \([F'] = -K_{\omega'}\) we can also directly argue that if \(G\) is a \(-1\) class of \((M, \omega)\) distinct from \(E_i\), then \(G\) does not lie in \(E_F\). Notice that

\[ G \cdot E_i \geq 0 \quad \text{and} \quad K_{\omega} \cdot G = -1, \]

and hence by (31)

\[ -[F] \cdot G = \pi^* K_{\omega'} \cdot G = K_{\omega} \cdot G - \sum (E_i \cdot G) \leq -1. \]

3.4. \(\kappa^s(M, \omega, F)\). In this subsection we define the relative Kodaira dimension of a 4–dimensional symplectic manifold relative to a possibly disconnected, embedded symplectic surface.

3.4.1. Definition for a maximal \(F\) without sphere components. We first assume that \(F\) is maximal and has no sphere components.

Definition 3.22. Let \(F \subset (M, \omega)\) be a maximal symplectic surface without sphere components. Then the relative Kodaira dimension of \((M, F, \omega)\) is defined in the following way: if \(F\) is empty, then \((M, \omega)\) is necessarily minimal and \(\kappa^s(M, \omega, F)\) is defined to be \(\kappa^s(M, \omega)\). Otherwise,
Thus, in this case, we also have \( \kappa(M, \omega, F) \) as in the proof of Proposition 3.12 that \( \kappa(M, \omega, F) \) certainly holds when \( \kappa(M, \omega, F) \leq 0 \) and \( (K_\omega + [F])^2 = 0 \).

\[ \kappa(M, \omega, F) = \begin{cases} 
-\infty & \text{if } (K_\omega + [F]) \cdot \omega < 0 \text{ or } (K_\omega + [F])^2 > 0, \\
0 & \text{if } (K_\omega + [F]) \cdot \omega = 0 \text{ and } (K_\omega + [F])^2 = 0, \\
1 & \text{if } (K_\omega + [F]) \cdot \omega > 0 \text{ and } (K_\omega + [F])^2 = 0, \\
2 & \text{if } (K_\omega + [F]) \cdot \omega > 0 \text{ and } (K_\omega + [F])^2 > 0.
\]

Next, we prove that the above definition is well defined.

**Theorem 3.23.** Definition 3.31 is well-defined.

**Proof.** The only thing we need to check is that there is no maximal surface without sphere components \( F \subset (M, \omega) \) with \( (K_\omega + [F]) \cdot [\omega] = 0 \) and \( (K_\omega + [F])^2 > 0 \).

By Proposition 3.17 it remains to discuss the case when \( M \) is rational or ruled. As \( b^+(M) = 1 \) in this case, the statement follows from the light cone lemma and \( [\omega]^2 \geq 0 \).

As mentioned in the introduction, the main result in [31] has the following simple interpretation.

**Theorem 3.24.** Let \((M, \omega)\) be a 4–dimensional relatively minimal fiber sum of \((M_1, \omega_1)\) and \((M_2, \omega_2)\) along connected genus \( g \geq 1 \) symplectic surfaces \( F_i \subset (M_i, \omega_i) \). Then

\[
\kappa(M, \omega) = \max\{\kappa(M_1, \omega_1, F_1), \kappa(M_2, \omega_2, F_2)\}.
\]

**3.4.2.** Comparing the relative and absolute Kodaira dimensions.

**Theorem 3.25.** Assume \( F \) is a maximal symplectic surface without sphere components in \((M, \omega)\), then

\[
\kappa(M, \omega, F) \geq \kappa(M, \omega).
\]

**Proof.** (38) certainly holds when \( \kappa(M, \omega, F) = 2 \).

To deal with the case of \( \kappa^s(M, \omega, F) = 1 \), let us introduce \((N, \sigma), K_\sigma, c_i, F'\) as in the proof of Proposition 3.12. We can assume that \( \kappa^s(M, \omega) \geq 0 \), otherwise the inequality (38) holds automatically. Recall that it is shown in the proof of Proposition 3.12 that \( K_\sigma \cdot [F'] \geq 0 \). As \( (K_\omega + [F])^2 = 0 \), it follows that

\[
K_\sigma^2 = (K_\sigma + [F'])^2 - 2K_\sigma \cdot [F'] - (K_\sigma + [F'])^2
\]

\[
= (K_\omega + [F])^2 + \sum (c_i - 1)^2 - (K_\sigma \cdot [F'] + [F']^2) - K_\sigma \cdot [F']
\]

\[
= \sum (c_i - 1)^2 - (K_\omega \cdot [F] + [F]^2) - \sum (c_i - 1)^2 - K_\sigma \cdot [F']
\]

\[
\leq \sum (1 - c_i) \leq 0.
\]

Thus, in this case, we also have \( \kappa^s(M, \omega) = \kappa^s(N, \sigma) \leq 1 = \kappa^s(M, \omega, F) \).

If \( \kappa^s(M, \omega, F) = 0 \), then \( (K_\omega + [F]) \cdot [\omega] = 0 \). By Proposition 3.17, \( \kappa^s(M, \omega) = -\infty \) when \( F \neq \emptyset \), and \( \kappa^s(M, \omega) = 0 \) when \( F = \emptyset \).
Now let us check the case of \( \kappa^s(M, \omega, F) = -\infty \). If \( (K_\omega + [F]) \cdot \omega < 0 \), we have \( \kappa^s(M, \omega) = -\infty \) by Proposition 3.17. If \( (K_\omega + [F])^2 < 0 \), then \( \kappa^s(M, \omega) = -\infty \) by Propositions 3.12 and 3.18.

\[ \Box \]

3.4.3. Classification when \( \kappa^s(M, \omega, F) = -\infty \).

**Theorem 3.26.** Suppose a nonempty surface \( F \subset (M, \omega) \) is maximal with each component positive genus. Then \( \kappa^s(M, \omega, F) = -\infty \) if and only if \( M \) is a genus \( h \) \( S^2 \) bundle with \( h \geq 1 \), and \( F \) is a section.

**Proof.** By Theorem 3.25 \( M \) satisfies \( \kappa^s(M) = -\infty \). Thus the only if part of the statement is a direct consequence of Propositions 3.16 and 3.18.

Let us verify the if part. Suppose either \( M \) is \( S^2 \times \Sigma_h, [F] = [\Sigma_h] + b[S^2] \), or \( M \) is a nontrivial \( S^2 \) bundle over \( \Sigma_h, [F] = U + bT \).

We check the case of \( S^2 \times \Sigma_h \), the other case is similar. As in 3.2, we compute in this case

\[ \begin{align*}
(K_\omega + [F])^2 &= -(b+2h-2), \\
(K_\omega + [F]) \cdot [\omega] &= -y + (b+2h-2)x.
\end{align*} \]

If \( (K_\omega + [F])^2 < 0 \), then \( \kappa^s(M, \omega, F) = -\infty \). If \( (K_\omega + [F])^2 \geq 0 \), then \( b + 2h - 2 \leq 0 \). Since \( x > 0, y > 0 \), we have \( (K_\omega + [F]) \cdot [\omega] < 0 \), so \( \kappa^s(M, \omega, F) = -\infty \) as well.

\[ \Box \]

**Remark 3.27.** Notice that this classification in Theorems 3.26 is independent of \( \omega \). This may not be so obvious, and actually it follows from Theorem 3.24 as summing with an \( S^2 \) bundle along a section is the so called smoothly trivial sum.

3.4.4. Classification when \( \kappa^s(M, \omega, F) = 0 \). By Theorem 3.25 and Proposition 3.18 we have

**Theorem 3.28.** Suppose a nonempty surface \( F \subset (M, \omega) \) is maximal with each component positive genus. \( \kappa^s(M, \omega, F) = 0 \) if and only if

\[ \kappa^s(M, \omega) = -\infty \quad \text{and} \quad [F] = -K_\omega. \]

3.4.5. Dependence on \( F \).

**Proposition 3.29.** Suppose \( F_1, F_2 \subset (M, \omega) \) are maximal symplectic surfaces without sphere components. If \( [F_1] = [F_2] \), then \( \kappa^s(M, \omega, F_1) = \kappa^s(M, \omega, F_2) \).

**Proof.** By the classification Theorems 3.26 and 3.28 we can assume that \( \kappa^s(M, \omega, F_i) \geq 1 \) for \( i = 1, 2 \). Suppose \( \kappa^s(M, \omega, F_1) = 1 \), then

\[ (K_\omega + [F_2])^2 = (K_\omega + [F_1])^2 = 0, \]

so \( \kappa^s(M, \omega, F_2) \) is at most 1. Thus \( \kappa^s(M, \omega, F_2) \) must be equal to 1 as well. \[ \Box \]
3.4.6. Non-maximal surface. We have defined $\kappa^s(M, \omega, F)$ when $F$ is empty, or maximal and without sphere components. As a direct consequence of Lemma 3.9 and Theorem 3.21, we can extend $\kappa^s(M, \omega, F)$ to any embedded symplectic surface $F$ without sphere components.

**Definition 3.30.** Suppose $F \subset (M, \omega)$ is a symplectic surface without sphere components. If $F = \emptyset$, then the relative Kodaira dimension of $(M, \omega, F)$, $\kappa^s(M, \omega, F)$, is defined to be $\kappa^s(M, \omega)$. Otherwise, let $(M', \omega', F')$ be the unique relative minimal model of $(M, \omega, F)$, and define $\kappa^s(M, \omega, F)$ to be $\kappa^s(M', \omega', F')$.

It is easy to see that all the results for maximal surfaces hold for general surfaces with obvious modifications.

3.4.7. $F$ possibly with sphere components. Recall that $F^+$ is the surface obtained from $F$ by removing the sphere components.

**Definition 3.31.** Let $F \subset (M, \omega)$ be an embedded symplectic surface. Then the relative Kodaira dimension of $(M, \omega, F)$, $\kappa^s(M, \omega, F)$, is defined to be $\kappa^s(M, \omega, F^+)$.

It is not hard to check that it is still well-defined and all the results still hold in this more general setting with obvious modifications.

We notice that the above definition is similar in one aspect to the definition of the Thurston norm of 3-manifolds: the 2-spheres have to be discarded. One explanation is that a 2-sphere has $\kappa^t = -\infty$, so it behaves like the empty set in some sense.

It is also necessary in our case for two reasons, one is the positive genus assumption in several results in section 3, e.g. Lemma 3.9. Another is that there are the following three special situations with $F$ a sphere, which would have relative dimension $-\infty$ if we had defined it “naively”:

1. $K_2^2 = 0$, $K_\omega \cdot [F] = 0$, $[F]^2 = -2$.
2. $K_2^2 = 0$, $K_\omega \cdot [F] = 1$, $[F]^2 = -3$.
3. $K_2^2 = 1$, $K_\omega \cdot [F] = 0$, $[F]^2 = -2$.

An example for (1) is $M = E(2)$ and $F$ a $-2$ sphere, and an example for (2) is $M = E(3)$ and $F$ a $-3$ sphere.

Due to Proposition 3.29 it is possible to extend $\kappa^s(M, \omega, F)$ to the case of $F$ being a symplectic surface with pseudo-holomorphic singularities, or a weighted symplectic surface. We should also mention that the notion of the logarithmic Kodaira dimension of a noncomplete variety introduced by Iitaka (see [14]) should be closely related to our relative Kodaira dimension $\kappa^s(M, \omega, F)$. All these will be studied elsewhere.

4. Relative Kod. dim. in dim. 2 and fibrations over a surface

In this section we introduce Kodaira dimension for a 2-manifold relative to a rational linear combination of points, and discuss how it might be used...
to compute the Kodaira dimension of the total space of certain fibrations with a 2–dimensional base or a 2–dimensional fiber.

In general, our viewpoint for a fibration is: “good” fibers and a “singular” base. More precisely, we first project the singular fibers to the base to obtain a finite set. We then assign a rational weight for each point of the image, subject to the requirement that the weight is positive and only depends on the type (local data) of the singular fiber. For any such assignment, we get an effective $\mathbb{Q}$–divisor on the base, hence relative Kodaira dimension for the base along with absolute Kodaira dimensions for the fiber and the total space. What we are able to show is that often there is a way (and sometimes unique) to assign the weight so that these three quantities together form an additivity relation. We also note that this scheme does not work in all cases. For example, we observe that for a genus two 4–dimensional Lefschetz fibration over $S^2$ with non-minimal total space, we have to further modify this scheme taking into account the total intersection numbers of $-1$ classes with the fiber class, in particular, we also need to relativize the Kodaira dimension of the fiber. It indicates that relative Kodaira dimension might be related to divisor contractions.

4.1. $\kappa^t(F,D)$, Riemann-Hurwitz formula and Seifert fibrations. In dimension 2, codimension 2 submanifolds are just points.

**Definition 4.1.** Let $F$ be a closed oriented real surface. A $\mathbb{Q}$–linear combination of points on $F$ of the form $D = \sum_{i=1}^{k} m_i x_i, x_i \in F, m_i \in \mathbb{Q}$ is called a $\mathbb{Q}$–divisor on $F$. Denote by $c(D) = \sum_{i=1}^{k} m_i$. The set $\{x_i\}$ is called the support of $D$. $D$ is called effective or positive if $m_i \geq 0$, and $D$ is called negative if $m_i \leq 0$.

**Definition 4.2.** Let $F$ be a closed oriented real surface of genus $g$ and $D$ a $\mathbb{Q}$–divisor. Define

$$\kappa^t(F,D) = \begin{cases} 
-\infty & \text{if } 2g - 2 + c(D) < 0, \\
0 & \text{if } 2g - 2 + c(D) = 0, \\
1 & \text{if } 2g - 2 + c(D) < 0.
\end{cases}$$

$D$ is allowed to be the empty set, and in this case, $\kappa^t(F,\emptyset) = \kappa^t(F)$. Clearly, $\kappa^t(F,D) \geq \kappa^t(F)$ if $D$ is effective, and $\kappa^t(F,D) \leq \kappa^t(F)$ if $D$ is negative.

For an integral and effective $D$, there are simple analogues of 4–dimensional results. For instance, if a nonempty $D$ is integral and effective, then $\kappa^t(F,D) = -\infty$ if and only if $F = S^2$ and $D = x$ for some $x \in F$. Notice that if we view $S^2$ as an $S^2$–bundle over a point, then this simple fact exactly corresponds to Theorem 3.26.

We also observe that the relative Kodaira dimension fits well with the connected sum construction (compare with Theorem 3.24).
Proposition 4.3. Suppose $F$ is the connected sum of $F_1$ and $F_2$ along $p_1, ..., p_n \in F_1, q_1, ..., q_n \in F_2$, then $\kappa^t(F) = \max\{\kappa^t(F_1, D_1), \kappa^t(F_2, D_2)\}$ with $D_1 = \sum_{i=1}^n p_i, D_2 = \sum_{i=1}^n q_i$.

If $n = 0$, then $D_1 = D_2 = \emptyset$. By the definition of $\kappa^t$ for a disconnected manifold, $\kappa^t(F) = \kappa^t(F_1 \sqcup F_2) = \max\{\kappa^t(F_1), \kappa^t(F_2)\}$. When $n = c(D_1) = c(D_2)$ is positive, this can also be easily checked.

As mentioned in the introduction, $\kappa^t(F, D)$ is introduced to achieve additivity of Kodaira dimensions for a fibration where $F$ is either the base or a smooth fiber. When $F$ is the base, the support of $D$ is often the image of the singular fibers, and each weight $m_i$ is positive. It might be delicate to determine the exact value of $m_i$ in each specific case. We will illustrate this idea by investigating several types of important fibrations. We begin with ramified coverings in dimension 2.

4.1.1. Ramified coverings and the Riemann-Hurwitz formula. Let $S', S$ be oriented surfaces and $\pi : S' \rightarrow S$ a ramified cover of degree $N$. Suppose the ramification set is $\{p_i\}$ and denote by $e_{p_i}$ the ramification index of $p_i$. Then we have the famous Riemann-Hurwitz formula:

$$\chi(S') = N\chi(S) - \sum(e_{p_i} - 1) = N(\chi(S) - \frac{1}{N} \sum(e_{p_i} - 1)).$$

A ramified cover is often viewed as a fibration with “good” base and some “bad” fibers. However, we would like to think of the base surface $S$ as a “relative surface” $(S, D)$ with

$$D_\pi = \sum_{\{p_i\}} e_{p_i} - \frac{1}{N} p_i.$$ 

With this natural choice of $D_\pi$, the Riemann-Hurwitz formula (39) can be interpreted as

$$\kappa^t(S') = \kappa^t(S, D_\pi) + \kappa^t(\text{fiber}) = \kappa^t(S, D_\pi).$$

4.1.2. Seifert fibrations. A Seifert fibration on a 3–manifold $M^3$ is a fibration $\pi : M^3 \rightarrow B$ to a closed surface $B$ with circle fibers. The singular fibers are all multiple fibers. Suppose the singular fibers have images $p_1, ..., p_n \in B$ and multiplicities $a_1, ..., a_n$. Classically, $B$ is viewed as an orbifold with orbifold points $\{p_i\}$, and with orbifold Euler characteristic

$$\chi^{\text{orb}}(B) = \chi(B) - \sum(1 - \frac{1}{a_i}).$$

Our view is slightly different, viewing the base as a relative surface with the natural choice of divisor, $D_\pi = \sum_{i=1}^n (1 - \frac{1}{a_i}) p_i$, suggested by the definition of $\chi^{\text{orb}}(B)$ above.

Proposition 4.4. With the set up above, we have

$$\kappa^t(M^3) = \kappa^t(B, D_\pi) + \kappa^t(\text{fiber}) = \kappa^t(B, D_\pi).$$
The argument is similar to the special case of $S^1$-bundles in [30]. Notice that $\kappa^i(B, D)_{\pi}$ only depends on the sign of $\chi^{orb}(B)$.

When $\chi^{orb}(B) > 0$, by the classification of Seifert fibre spaces, $M^3$ has $S^3$ geometry if $\pi_1(M^3)$ is finite, and $S^2 \times \mathbb{R}$ geometry if $\pi_1(M^3)$ is infinite. In this case, $\kappa^i(M^3) = \kappa^i(B, D_{\pi}) = -\infty$.

When $\chi^{orb}(B) = 0$, again by the classification, the possible geometries for $M^3$ are Euclidean or Nil. In this case, $\kappa^i(M^3) = \kappa^i(B, D_{\pi}) = 0$.

Finally, when $\chi^{orb}(B) < 0$, $M^3$ has geometry of type $\mathbb{H}^2 \times \mathbb{R}$ or $SL_2(\mathbb{R})$. In this case, $\kappa^i(M^3) = \kappa^i(B, D_{\pi}) = 1$.

4.2. Lefschetz fibrations. Now we investigate several kinds of 4-dimensional Lefschetz fibrations. We will denote a 4-dimensional Lefschetz fibration by $\pi : M^4 \to B$, and a general smooth fiber by $F$. It suffices to restrict to relatively minimal Lefschetz fibrations. By Proposition 2.14, we can also assume that there is at least a singular fiber.

4.2.1. When $\kappa^i(F) = -\infty$. Notice that if $\kappa^i(F) = -\infty$, if there is a singular fiber, then it is not relatively minimal. So we also assume from now on that $\kappa^i(F) \geq 0$.

4.2.2. When $\kappa^i(B) = 1$ and $\kappa^i(F) \geq 0$. In this case it was shown in [3] that if $M^4$ admits a symplectic (complex) structure, then

\[(40) \quad \kappa^{s(h)}(M^4) = \kappa^i(F) + \kappa^i(B).\]

Since for any effective divisor $D$ of the base surface $B$, we have $\kappa^i(B, D) = \kappa^i(B) = 1$, if we assign any positive weight $b_i$, we still have

\[(41) \quad \kappa^{s(h)}(M^4) = \kappa^i(F) + \kappa^i(B, D_{\pi, b_i}).\]

4.2.3. When $\kappa^i(B) = 0$ and $\kappa^i(F) = 0$. In this case, it was calculated in [3] that $\kappa^s(M^4) = 1$. Thus for any positive assignment $b_i$, we have

\[\kappa^s(M^4) = 1 + 0 = \kappa^i(B, D_{\pi, b_i}) + \kappa^i(F).\]

4.2.4. When $\kappa^i(B) = -\infty$ and $\kappa^i(F) = 0$. In this case there is a unique choice of weights. Notice that there is only one type of elliptic Lefschetz singular fibers. Thus the weight $b$ is determined by a fibration $\pi : K3 \to S^2$ with 24 singular fibers: If the additivity holds for this fibration, then $\kappa^i(S^2, D_{\pi, b}) = \kappa^s(K3) - \kappa^i(F) = 0 - 0 = 0$, which means that $-\chi(S^2) - c(D_{\pi, b}) = 2 - 24b = 0$, i.e. $b = \frac{1}{12}$. Then it is easy to check that with this choice of weight, the additivity also holds for all relatively minimal elliptic Lefschetz fibrations over $S^2$, namely, $\pi : E(n) \to S^2$ with $12n$ singular fibers, as $\kappa^s(E(n)) = \kappa^s(n - 2)$.

In the remaining cases we assume the fibration is hyperelliptic.
4.2.5. When \( \kappa^t(B) = 0 \) and \( \kappa^t(F) = 1 \) and the fibration is hyperelliptic. In this case, it was calculated in [5] that \( \kappa^s(M^4) = 2 \). Thus for any positive assignment \( b_i \), we have

\[
\kappa^s(M^4) = 1 + 0 = \kappa^t(B, D_{\pi, b_i}) + \kappa^t(F).
\]

4.2.6. When \( \kappa^t(B) = -\infty, \kappa^t(F) = 1 \) and the fibration is hyperelliptic with minimal total space. In this case, since \( F \) is not a torus, the total space \( M^4 \) admits a compatible symplectic structure \( \omega \). We further observe

**Lemma 4.5.** For any genus \( g \geq 2 \) Lefschetz fibration with minimal total space \( M \) and a compatible symplectic form \( \omega \),

\[
\kappa^s(M, \omega) = \kappa^s(K_\omega^2) + 1 = \kappa^s(K_\omega^2) + \kappa^t(F).
\]

**Proof.** (42) certain holds if \( K_\omega^2 < 0 \). If \( K_\omega^2 = 0 \), then \( \kappa^s(M) = 0 \) or 1. Since \( g \geq 2 \), by the adjunction formula, we actually must have \( \kappa^s(M) = 1 = \kappa^s(0) + 1 \). The remaining case is \( K_\omega^2 > 0 \). If \( \kappa^s(M, \omega) = -\infty \) and \( K_\omega^2 > 0 \), then \( M \) is \( \mathbb{CP}^2 \) or an \( S^2 \)-bundle over \( S^2 \) or \( T^2 \). It is easy to check that there are no square 0 symplectic surfaces with genus at least 2 in such a \( (M, \omega) \). Thus, we must have in this case \( \kappa^s(M) = 2 = 1 + 1 = \kappa^s(K_\omega^2) + 1 \). \( \square \)

It remains to show that we can assign \( b_i \) to each \( x_i \) such that \( b_i \) only depends on the singularity type of \( \pi^{-1}(x_i) \) and

\[
\kappa^s(S^2, D_{\pi, b_i}) = \kappa^s(K_\omega^2).
\]

We take clues from Endo’s signature formula for hyperelliptic fibration over \( S^2 \) ([7]):

\[
\sigma(M) = -\frac{g + 1}{2g + 1} a + \sum_{p=1}^{[\frac{g}{2}]} \left( \frac{4p(g - p)}{2g + 1} - 1 \right)s_p,
\]

where \( a \) is the number of non-separating singular fibers, and \( s_p \) is the number of separating fibers of type \( (p, g - p) \). The formula for \( K_\omega^2 \) is calculated in [5] to be

\[
K_\omega^2 = 3\sigma(M) + 2\chi(M)
\]

\[
= 2(2 - 2g)) + \frac{g - 1}{2g + 1} a + \sum_{p=1}^{[\frac{g}{2}]} \frac{6p(2g - 2p) + 2g(p - 1) + (4g - 1)}{2g + 1} s_p.
\]

Let \( b_{g, n, s} \) be the weight for a non-separating fiber and \( b_{g, p} \) be the weight for a separating fiber of type \( (p, g - p) \). By (45), it is natural to propose that

\[
b_{g, n, s} = \frac{g - 1}{(4g - 1)(2g + 1)} = \frac{1}{4(2g + 1)},
\]

\[
b_{g, p} = \frac{6p(2g - 2p) + 2g(p - 1) + (4g - 1)}{(4g - 4)(2g + 1)},
\]

and it is easy to check that, with this choice of \( b_i \), (43) holds.

In fact, \( b_i \) defined by (46) should be the unique weight such that (43) holds. We have indeed verified the uniqueness for genus 2 fibrations. In this case, in (46), \( b_{2, n, s} = \frac{1}{20} \), and \( b_{2, 1} = \frac{7}{20} \). Our strategy is simple. First consider a self fiber sum of a genus two holomorphic Lefschetz fibration
with no separating fibers and 20 non-separating singular fiber in \([3]\). It is minimal by \([47]\) and has \(K_\omega^2 = 0\). Thus it follows \(b_{2,ns}\) has to be \(\frac{1}{20}\). We next consider a self fiber sum of a genus two Lefschetz fibration with 2 separating fiber and 6 non-separating singular fiber in \([32]\) and \([38]\). It is minimal again by \([47]\) and also has \(K_\omega^2 = 0\). With \(b_{2,ns}\) already determined to be \(\frac{1}{20}\), \(b_{2,1}\) has to be \(\frac{7}{20}\).

4.2.7. When \(\kappa^t = -\infty, \kappa^t(F) = 1\) and the fibration is hyperelliptic. An interesting discovery here is that we also need to use the relative Kodaira dimension of a generic smooth fiber. Here the support of the divisor is the intersection with a maximal set of disjoint \(-1\) spheres, and the coefficients are negative. Let \((M', \omega')\) be a minimal model of \((M, \omega)\) and \(E_i\) the classes of the symplectic \(-1\) spheres in \((M, \omega)\) that are blown down to obtain \((M, \omega)\). Let \(c\) be the number of those \(-1\) spheres. Since \(\kappa^s(M, \omega) = \kappa^s(M', \omega')\), we can compute it using the expression

\[
\kappa^s(K_\omega^2) + \kappa^s(K_{\omega'} \cdot [\omega']).
\]

Now, let us first compute \(\kappa^s(K_{\omega'}^2)\). First, we have

\[
(47) \quad K_{\omega'}^2 = (K_\omega - \sum E_i)^2 = K_\omega^2 + c.
\]

Notice that we can fiber sum \((M, \omega, F)\) with itself to get a minimal manifold \((D M, \tau)\). \((D M, \tau)\) also has a genus \(g\) hyperelliptic Lefschetz fibration structure with twice of the singular fibers. It is minimal by the result of \([47]\). In addition,

\[
(48) \quad K_\tau^2 = 2(K_\omega + [F])^2.
\]

Using the hyperelliptic Lefschetz fibration structure on \((D M, \tau)\), we can also compute \(K_\tau^2\) by \([45]\) and \([46]\).

\[
(49) \quad K_\tau^2 = 2(\sum b_i - 1)(4g - 4)
\]

Thus, combine \([47], [48], [49]\), we have

\[
(50) \quad \kappa^s(-2 + \sum b_i + \frac{c}{4g - 4}) = \kappa^s(K_{\omega'}^2).
\]

Regard \(\kappa^s(-2 + \sum b_i + \frac{c}{4g - 4})\) as the relative Kodaira dimension of the base. When \(c = 0\), this is just what we have previously.

Now we turn to \(\kappa^s(K_{\omega'} \cdot [\omega'])\). Let \(F'\) be the symplectic surface in \((M', \omega')\) obtained by blowing down \(F\) and smoothing. Let \(c' = \sum [F] \cdot [E_i]\), by \([13]\) applied to \(F' \subset (M', \omega')\), when \(K_{\omega'}^2 \geq 0\), we have

\[
\kappa^s(K_{\omega'} \cdot [\omega']) = \kappa^s(K_{\omega'} \cdot [F']) = \kappa^s(2g - 2 - c').
\]

Thus \(\kappa^s(K_{\omega'} \cdot [\omega'])\) can be viewed as the relative Kodaira dimension of the fiber relative to the pencil points but with “negative mass”, at least when \(K_{\omega'}^2 \geq 0\). In particular, we have in this case

\[
\kappa^s(M, \omega) = \kappa^s(-2 + \sum b_i + \frac{c}{4g - 4}) + \kappa^s(2g - 2 - c').
\]
This continues to hold when $K_w^2 < 0$ as both sides are equal to $-\infty$.

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