Let \( R \) be a commutative Noetherian ring with non-zero identity, \( \mathfrak{a} \) an ideal of \( R \), \( M \) a finitely generated \( R \)-module, and \( a_1, \ldots, a_n \) an \( \mathfrak{a} \)-filter regular \( M \)-sequence. The formula

\[
H^i_{\mathfrak{a}}(M) \cong \begin{cases} 
H^i_{(a_1, \ldots, a_n)}(M) & \text{for all } i < n, \\
H^{i-n}_{\mathfrak{a}}(H^n_{(a_1, \ldots, a_n)}(M)) & \text{for all } i \geq n,
\end{cases}
\]

is known as Nagel-Schenzel formula and is a useful result to express the local cohomology modules in terms of filter regular sequences. In this paper, we provide an elementary proof to this formula.

1. Introduction

Throughout \( R \) will denote a commutative Noetherian ring with non-zero identity, \( \mathfrak{a} \) and \( \mathfrak{b} \) two ideals of \( R \), \( X \) an arbitrary \( R \)-module which is not necessarily finitely generated, and \( M \) a finitely generated \( R \)-module. Recall that the \( i \)-th local cohomology functor \( H^i_{\mathfrak{a}} \) is the \( i \)-th right derived functor of the \( \mathfrak{a} \)-torsion functor \( \Gamma_{\mathfrak{a}} \). For basic results, notations and terminology not given in this paper, the reader is referred to [1], [2], and [3].
The concept of an \(\mathfrak{a}\)-filter regular sequence is a generalization of the concept of a filter regular sequence which has been studied in [3] and [4], and has led to some interesting results. Let \(a_1, \ldots, a_n \in \mathfrak{a}\). Recall that \(a_1, \ldots, a_n\) is an \(\mathfrak{a}\)-filter regular \(M\)-sequence if

\[
\text{Supp}_R \left( \frac{(a_1, \ldots, a_{i-1})M :_M a_i}{(a_1, \ldots, a_{i-1})M} \right) \subseteq \text{Var}(\mathfrak{a})
\]

for all \(1 \leq i \leq n\), where \(\text{Var}(\mathfrak{a})\) denotes the set of prime ideals of \(R\) containing \(\mathfrak{a}\). Let \(a_1, \ldots, a_n\) be an \(\mathfrak{a}\)-filter regular \(M\)-sequence. Then, by [3, Proposition 1.2], we have

\[
\tag{1}
H^i_{a+b}(Q) \cong H^{i+1}_{a+b}(X)
\]

which is known as Nagel-Schenzel formula. This formula was first obtained by Nagel and Schenzel, in [4, Lemma 3.4], in the case where \(R\) is a local ring with maximal ideal \(\mathfrak{m}\) and \(a = \mathfrak{m}\). Both of them used the Grothendieck spectral sequence

\[
E^{p,q}_2 := H^p_a(H^q_b(X)) \Rightarrow H^{p+q}(M)
\]

to prove (1). In this paper, we provide an elementary proof to this formula.

2. An elementary proof of (1)

The following lemmas are needed in our proof of Nagel-Schenzel formula.

**Lemma 2.1.** Let \(t\) be a non-negative integer such that \(H^{t-i}_a(H^i_b(X)) = 0\) for all \(0 \leq i \leq t\). Then \(H^t_{a+b}(X) = 0\).

**Proof.** We prove by using induction on \(t\). The case \(t = 0\) is clear because \(\Gamma_a(\Gamma_b(X)) = \Gamma_{a+b}(X)\). Suppose that \(t > 0\) and that \(t - 1\) is settled. Assume that \(X = X/\Gamma_b(X)\) and \(Q = E_R(X)/X\) where \(E_R(X)\) is an injective hull of \(X\). Since \(\Gamma_b(X) = 0 = \Gamma_{a+b}(X)\), \(\Gamma_b(E_R(X)) = 0 = \Gamma_{a+b}(E_R(X))\). Applying the derived functors of \(\Gamma_b(\cdot)\) and \(\Gamma_{a+b}(\cdot)\) to the short exact sequence

\[
0 \rightarrow X \rightarrow E_R(X) \rightarrow Q \rightarrow 0,
\]

we obtain the isomorphisms

\[
\tag{2}
H^i_b(Q) \cong H^{i+1}_b(X)
\]

and

\[
\tag{3}
H^i_{a+b}(Q) \cong H^{i+1}_{a+b}(X)
\]

for all \(i \geq 0\). From the isomorphisms (2), for all \(0 \leq i \leq t - 1\), we have

\[
H^{(t-1)-i}_a(H^i_b(Q)) \cong H^{t-(i+1)}_a(H^{i+1}_b(X))
\]
which is zero by the assumptions. Thus, from the induction hypothesis on $Q$, we have $H_{a+b}^{t-1}(Q) = 0$. Therefore $H_{a+b}^t(X) = 0$ by the isomorphisms (3). Now, by the short exact sequence

$$0 \rightarrow \Gamma_b(X) \rightarrow X \rightarrow \overline{X} \rightarrow 0,$$

we get the long exact sequence

$$\cdots \rightarrow H_{a+b}^t(\Gamma_b(X)) \rightarrow H_{a+b}^t(X) \rightarrow H_{a+b}^t(\overline{X}) \rightarrow \cdots .$$

Since $H_{a+b}^t(\Gamma_b(X)) = H_a^t(\Gamma_b(X)) = 0$, the above long exact sequence shows that $H_{a+b}^t(X) = 0$. □

**Lemma 2.2.** Let $s$ and $t$ be non-negative integers such that

1. $H_a^{s+t-i}(H_b^i(X)) = 0$ for all $i \neq t$,
2. $H_a^{s+t-i+1}(H_b^i(X)) = 0$ for all $i < t$, and
3. $H_a^{s+t-i-1}(H_b^i(X)) = 0$ for all $i > t$.

Then we have the isomorphism $H_a^s(H_b^t(X)) \cong H_{a+b}^{s+t}(X)$.

**Proof.** Let $\overline{X} = X/\Gamma_b(X)$ and $Q = E_R(\overline{X})/\overline{X}$ where $E_R(\overline{X})$ is an injective hull of $\overline{X}$. We prove by using induction on $t$. In the case that $t = 0$, we have $H_{a+b}^{s-1}(X) = 0 = H_{a+b}^s(X)$ from hypothesis (iii) and (i), and Lemma 2.1. Since $H_{a+b}^s(\Gamma_b(X)) = H_a^s(\Gamma_b(X))$, the assertion follows by the exact sequence

$$H_{a+b}^{s-1}(X) \rightarrow H_{a+b}^s(\Gamma_b(X)) \rightarrow H_{a+b}^s(X) \rightarrow H_{a+b}^s(\overline{X}),$$

obtained from the short exact sequence

$$0 \rightarrow \Gamma_b(X) \rightarrow X \rightarrow \overline{X} \rightarrow 0.$$

Suppose that $t > 0$ and that $t - 1$ is settled. From the isomorphisms (2) and the assumptions, we have

- $H_a^{s+(t-1)-i}(H_b^i(Q)) = H_a^{s+t-(i+1)}(H_b^{i+1}(X)) = 0$ for all $i \neq t - 1$,
- $H_a^{s+(t-1)+1-i}(H_b^i(Q)) = H_a^{s+t+1-(i+1)}(H_b^{i+1}(X)) = 0$ for all $i < t - 1$, and
- $H_a^{s+(t-1)-1-i}(H_b^i(Q)) = H_a^{s+t-1-(i+1)}(H_b^{i+1}(X)) = 0$ for all $i > t - 1$.

Thus we get $H_{a+b}^{s+(t-1)}(Q) \cong H_a^s(H_b^{t-1}(Q))$ by the induction hypothesis on $Q$. Therefore $H_{a+b}^{s+t}(\overline{X}) \cong H_a^s(H_b^t(X))$ from the isomorphisms (2) and (3). On the other hand, by assumptions (i) and (ii), and the exact sequence

$$H_{a+b}^{s+t}(\Gamma_b(X)) \rightarrow H_{a+b}^{s+t}(X) \rightarrow H_{a+b}^{s+t}(\overline{X}) \rightarrow H_{a+b}^{s+t+1}(\Gamma_b(X))$$

obtained from the short exact sequence

$$0 \rightarrow \Gamma_b(X) \rightarrow X \rightarrow \overline{X} \rightarrow 0,$$
we get $H_{a+b}^{s+t}(X) \cong H_{a+b}^{s+t}(X)$. Hence $H_a^s(H_b^t(X)) \cong H_{a+b}^{s+t}(X)$ which completes the proof. $\square$

**Lemma 2.3.** Let $a_1, \ldots, a_n$ be an $a$–filter regular $M$–sequence. Then, for all $0 \leq i \leq n-1$, $\text{Supp}_R(H^j_{(a_1, \ldots, a_n)}(M)) \subseteq \text{Var}(a)$. In particular,

$$H^j_{(a_1, \ldots, a_n)}(M) \cong \begin{cases} H^j_{(a_1, \ldots, a_n)}(M) & \text{if } j = 0, \\ 0 & \text{if } j > 0, \end{cases}$$

for all $0 \leq i \leq n-1$.

**Proof.** Let $0 \leq i \leq n-1$ and $p \in \text{Supp}_R(H^j_{(a_1, \ldots, a_n)}(M))$. Assume contrarily that $p \notin \text{Var}(a)$. Thus $p \in \text{Spec}(R) \setminus \text{Var}(a)$ and so $\frac{a_1}{1}, \ldots, \frac{a_n}{1}$ is a weak $M_p$–sequence. Hence $H^j_{(\frac{a_1}{1}, \ldots, \frac{a_n}{1})}(M_p) = 0$. Therefore we get $(H^j_{(a_1, \ldots, a_n)}(M))_p = 0$. This contradiction shows that $p \in \text{Var}(a)$. $\square$

Now we are ready to give an elementary and simple proof for $(\text{III})$.

**Proof of Nagel-Schenzel formula.** Let $i < n$ (resp. $i \geq n$). Consider Lemma 2.3 and apply Lemma 2.2 with $s = 0$, $t = i$, and $b = (a_1, \ldots, a_n)$ (resp. $s = i-n$, $t = n$, and $b = (a_1, \ldots, a_n)$).

**References**

[1] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge, 1998.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1998.

[3] K. Khashyarmanesh and Sh. Salarian, *Filter regular sequences and the finiteness of local cohomology modules*, Comm. Algebra 26 (1998) 2483–2490.

[4] U. Nagel and P. Schenzel, *Cohomological annihilators and Castelnuovo-Mumford regularity*, Commutative algebra: Syzygies, multiplicities, and birational algebra, Contemp. Math. 159 (1994) 307–328.

[5] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, San Diego, 1979.

[6] P. Schenzel, N. V. Trung, and N. T. Cuong, *Verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr. 85 (1978) 57–73.

[7] J. Stuckrad and W. Vogel, *Buchsbaum Rings and Applications*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.

**Alireza Vahidi**

Department of Mathematics,
Payame Noor University (PNU),
P.O.BOX, 19395-4697,
Tehran, Iran.
vahidi.ar@pnu.ac.ir