A note on perturbation analysis for T-product based tensor singular values

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Abstract

In this note, we present perturbation analysis for the T-product based tensor singular values defined by Lu et al. First, the Cauchy’s interlacing-type theorem for tensor singular values is given. Then, the inequalities about the difference between the singular values of two matrices proposed by Mirsky are extended to tensor cases. Finally, we introduce some useful inequalities for the singular values of tensor products and sums.

Key words: tensor singular values; interlacing theorem; Mirsky inequalities; perturbations

1 Introduction

In 2008, Kilmer, Martin and Perrone [4] proposed the tensor operation T-product and extended the matrix SVD to third-order tensors. Based on this tensor-tensor multiplication, many scholars have contributed to the theoretical results and algebraic analysis [5, 9]. Specific applications of the T-product and the T-SVD can be seen in [12, 13, 15, 16, 17].

Under the T-SVD framework, two different definitions of tensor singular values have been proposed successively by Lu et al. [8] and Qi and Yu [11]. The former definition can be understood as the average of the singular values of several matrices, while the latter definition which is called the T-singular value can be interpreted as a measure of the length of some singular value vectors. In [11], the T-singular values are used to define the tail energy of a third-order tensor, and applied to the error estimation of a tensor sketching algorithm for low-rank tensor approximation. In [6], Ling et al. presented four necessary conditions and a set of sufficient and necessary conditions for s-diagonal tensors via the T-singular values. Contrastly, the main contribution of tensor singular values defined in [8] is to help define the tensor nuclear norm. The definition and properties of the nuclear norm are consistent with the matrix cases. Furthermore, the tensor nuclear norm can be used to solve

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tensor robust principal component analysis problem and applied to image recovery and background modeling problems. We hope to continue along the notion of [3] to discover more properties and effects of the tensor singular values.

Braman [1] introduced the eigendecomposition of \( n \times n \times n \) tensors firstly. A recent work by Liu and Jin [7] studied the T-eigenvalues of third-order tensors in detail. They proved some T-eigenvalue inequalities for Hermitian tensors, including extensions of Weyl’s theorem and Cauchy’s interlacing theorem from the matrix case to the tensor case. Moreover, they described the stability of the T-eigenvalues and studied the Lyapunov equation for tensors. It is therefore natural to consider the extension of the classical perturbation theory for matrix singular value problems. We show herein that the interlacing property and the Mirsky-type inequalities can be recast in the T-product formalism. Also, some singular value inequalities for the tensor products and sums can be generalized.

This paper is organized as follows. In section 2, we review basic definitions and notations. In section 3, we consider the singular values of subtensors. Generalized Mirsky inequalities comparing the differences of two tensors with the singular values of their difference are proposed in section 4. Section 5 shows other useful inequalities for the singular values of the tensor products and sums. A conclusion is given in section 6.

2 Preliminaries

2.1 Notations

The term tensor refers to a multidimensional array of numbers and it is necessary to break a tensor up into various slices and tubal elements, and to have an indexing on those. In our paper, tensors are denoted by calligraphic letters and matrices are denoted by capital letters. We denote \( A^{(i)} \equiv A(i, :, :) \), \( \overrightarrow{A}_i \equiv A(:, i, :) \) and \( A_i \equiv A(:, :, i) \) for the \( i \)-th horizontal, lateral and frontal slice of a third-order tensor \( A \in \mathbb{C}^{m \times n \times p} \) respectively. We also use \( a_{ijk} \) to represent its \((i, j, k)\)-th element, and \( A_{i,j} \equiv A(i, j, :) \) for its \((i, j)\)-th tubal scalar.

For a third-order tensor \( A \), as in [5], it is defined that

\[
\text{unfold}(A) = \begin{bmatrix} A_1 \\ \vdots \\ A_p \end{bmatrix} \in \mathbb{C}^{mp \times n}, \quad \text{fold}(\text{unfold}(A)) = A.
\]

The discrete Fourier transform (DFT) can transform block circulant matrices into block diagonal matrices. Mathematically, this means that if \( F_n = \omega^{(j-1)(k-1)} \in \mathbb{C}^{n \times n} \) with \( \omega = e^{-2\pi i/n} \) denotes the DFT matrix, then we can obtain

\[
(F_p \otimes I_m) \text{bcirc}(A) (F_p^{-1} \otimes I_n) = \text{diag} \left( \hat{A}_1, \hat{A}_2, \ldots, \hat{A}_p \right),
\]

where \( \otimes \) denotes the Kronecker product and \( \text{bcirc}(A) \) is block-circulant matrix of
the form

\[
\text{bcirc}(A) = \begin{bmatrix}
A_1 & A_p & A_{p-1} & \ldots & A_2 \\
A_2 & A_1 & A_p & \ldots & A_3 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
A_p & A_{p-1} & \ldots & A_2 & A_1
\end{bmatrix}.
\]

Using Matlab notations, \( \hat{A} \equiv \text{fft}(A, [ ] , 3) \in \mathbb{C}^{m \times n \times p} \) with \( \hat{A}(; ; i) = \hat{A}_i \) is the tensor obtained by applying the fast Fourier transform (FFT) along each tubal scalar of \( A \).

### 2.2 Definitions and propositions

In this subsection, we give the basic definitions and propositions from [4, 5, 8].

**Definition 2.1.** (T-product) Suppose \( A \in \mathbb{C}^{m \times n \times p} \) and \( B \in \mathbb{C}^{n \times t \times p} \), then the T-product \( A \ast B \) is the tensor in \( \mathbb{C}^{m \times t \times p} \)

\[
A \ast B = \text{fold}(\text{bcirc}(A) \cdot \text{unfold}(B)).
\]

**Lemma 2.1.** Suppose tensors \( A, B \) and \( C \) are well-defined, then

\[
C = A \pm B \iff \hat{C}_i = \hat{A}_i \pm \hat{B}_i, \quad C = A \ast B \iff \hat{C}_i = \hat{A}_i \hat{B}_i.
\]

**Definition 2.2.** (Tensor Transpose) Suppose \( A \in \mathbb{R}^{m \times n \times p} \), then \( A^T \) is the \( n \times m \times p \) tensor obtained by transposing each of the frontal slices and then reversing the order of transposed slices 2 through \( p \).

**Definition 2.3.** (Identity Tensor) The \( n \times n \times \ell \) identity tensor \( I_{nn\ell} \) is the tensor whose first frontal slice is the \( n \times n \) identity matrix, and whose other frontal slices are all zeros.

**Definition 2.4.** (Tensor Inverse) The tensor \( A \in \mathbb{R}^{n \times n \times \ell} \) has an inverse \( B \) provided that

\[
A \ast B = B \ast A = I.
\]

**Definition 2.5.** (Orthogonal Tensor) The tensor \( Q \in \mathbb{R}^{n \times n \times \ell} \) is orthogonal if

\[
Q^T \ast Q = Q \ast Q^T = I.
\]

The tensor \( Q \in \mathbb{R}^{p \times q \times \ell} \) is partially orthogonal if

\[
Q^T \ast Q = I_{qq\ell}.
\]

**Definition 2.6.** (Tensor Frobenius Norm) Suppose \( A = (a_{ijk}) \in \mathbb{R}^{m \times n \times p} \), then

\[
\| A \|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ijk}^2}.
\]

**Theorem 2.1.** If \( Q \) is an orthogonal tensor, then

\[
\| Q \ast A \|_F = \| A \|_F.
\]
Theorem 2.2. Suppose $A \in \mathbb{R}^{m \times n \times p}$ and $\hat{A} = \text{fft}(A, [], 3)$, then
\[
\|A\|_F = \frac{1}{\sqrt{p}} \|\hat{A}\|_F.
\] (2.1)

Theorem 2.3. (T-SVD) Let $A \in \mathbb{R}^{m \times n \times p}$. Then it can be factorized as
\[
A = U \ast S \ast V^T
\]
where $U \in \mathbb{R}^{m \times m \times p}$, $V \in \mathbb{R}^{n \times n \times p}$ are orthogonal, and $S \in \mathbb{R}^{m \times n \times p}$ is an f-diagonal tensor, i.e., each frontal slice is diagonal.

The entries on the diagonal of the first frontal slice $S(:,:,1)$ of $S$ have the decreasing order property, i.e.,
\[
S(1,1,1) \geq S(2,2,1) \geq \cdots \geq S(n',n',1) \geq 0.
\]
where $n' = \min(m, n)$. The above property holds since the inverse FFT gives
\[
S(i,i,1) = \frac{1}{p} \sum_{j=1}^{p} \hat{S}(i,i,j).
\] (2.2)

The entries on the diagonal of $\hat{S}(::,j)$ are the singular values of $\hat{A}(::,j)$.

Definition 2.7. (Tensor singular values [8]) The entries on the diagonal of $S(:,:,1)$ are the singular values of $A$.

3 The singular values of subtensors

Liu and Jin [7, Theorem 4.5] have extended Cauchy’s interlacing theorem from the matrix case to the tensor case. This theorem gives the inequality about the T-eigenvalues of one tensor and its subtensors. Similarly, in this section, we study the relationship between singular values of one tensor and its subtensors. In other words, our theorem is an extension of the matrix singular value interlacing theorem.

Theorem 3.1. Suppose the tensor $A \in \mathbb{R}^{m \times n \times r}$ with singular values
\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n'} \geq 0, \ n' = \min\{m, n\}.
\]
Suppose $B \in \mathbb{R}^{p \times q \times r}$ is a subtensor of $A$ with singular values
\[
\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{p'} \geq 0, \ p' = \min\{p, q\}.
\]
Then,
\[
\alpha_j \geq \beta_j, \ j = 1, 2, \ldots, p', \quad (3.1)
\]
\[
\beta_j \geq \alpha_{j+(m-p)+(n-q)}, \ j = 1, 2, \ldots, \min\{(p+q-m), (p+q-n)\}. \quad (3.2)
\]
Proof. Suppose every frontal slice $\hat{A}_i \in \mathbb{C}^{m \times n}$ of $\hat{A} = \text{fft}(\mathcal{A}, [\ ]_3)$ and $\hat{B}_i \in \mathbb{C}^{p \times q}$ of $\hat{B} = \text{fft}(\mathcal{B}, [\ ]_3)$ have the singular values numbered in decreasing order

$$\sigma_1^i \geq \cdots \geq \sigma_{n'}^i \geq 0, \quad \tau_1^i \geq \cdots \geq \tau_{p'}^i \geq 0, \quad i = 1, 2, \ldots, r.$$ 

From (2.2), the singular values of the tensor $\mathcal{A}$ and subtensor $\mathcal{B}$ are

$$\alpha_j = \frac{1}{r} \sum_{i=1}^{r} \sigma_j^i, \quad j = 1, 2, \ldots, n', \quad \beta_j = \frac{1}{r} \sum_{i=1}^{r} \tau_j^i, \quad j = 1, 2, \ldots, p'.$$

Since the singular value interlacing theorem [13, Theorem 1] tells us that for any $i = 1, 2, \ldots, r$,

$$\sigma_j^i \geq \tau_j^i, \quad j = 1, 2, \ldots, p', \quad (3.3)$$

$$\tau_j^i \geq \sigma_{j+(m-p)+(n-q)}^i, \quad j = 1, 2, \ldots, \min\{(p+q-m), (p+q-n)\}, \quad (3.4)$$

the results (3.1) and (3.2) follows from (3.3) and (3.4) respectively. □

Remark 3.1. This theorem also suggests that if a lateral slice is added to a tensor $\mathcal{A} \in \mathbb{R}^{m \times n \times r}$ with $m > n$, then the largest singular value increases and the smallest one decreases. That is, if $\mathcal{B} = [\mathcal{A}, \mathcal{M}] \in \mathbb{R}^{m \times (n+1) \times r}$, then

$$\sigma_{\text{max}}(\mathcal{B}) \geq \sigma_{\text{max}}(\mathcal{A}), \quad \sigma_{\text{min}}(\mathcal{B}) \leq \sigma_{\text{min}}(\mathcal{A}).$$

When $r = 1$, this conclusion is consistent with standard matrix algebra operations and terminology [2, Corollary 2.4.5].

4 Mirsky-type singular value inequalities for tensors

For any unitarily invariant norm of matrices, it is proved by Mirsky [10] that

$$\| \text{diag}(\sigma_1 - \tilde{\sigma}_1, \ldots, \sigma_n - \tilde{\sigma}_n) \| \leq \| B - \tilde{B} \|, \quad (4.1)$$

where $\sigma_i$'s and $\tilde{\sigma}_i$'s are the corresponding singular values of $B$ and $\tilde{B}$. To the best of our knowledge, there is no straightforward definition of unitarily invariant norm for tensors. Instead, we turn to the Frobenius norm and the spectral norm (also known as the largest singular value [8]) based tensor Mirsky-type inequalities.

Theorem 4.1. Suppose $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{m \times n \times r}$ with $n' = \min\{m, n\}$ and

$$\alpha_1 \geq \cdots \geq \alpha_{n'} \geq 0, \quad \beta_1 \geq \cdots \geq \beta_{n'} \geq 0$$

are the singular values of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then,

$$\left[ \sum_{i=1}^{n'} (\alpha_i - \beta_i)^2 \right]^\frac{1}{2} \leq \| B - \mathcal{A} \|_F, \quad (4.2)$$

$$| \alpha_j - \beta_j | \leq \sigma_{\text{max}}(\mathcal{B} - \mathcal{A}), \quad j = 1, 2, \ldots, n'. \quad (4.3)$$
First, we take the T-SVD of \( \mathcal{A} \) and \( \mathcal{B} \) to get the f-diagonal tensors \( \mathcal{S}_\mathcal{A} \) and \( \mathcal{S}_\mathcal{B} \) with frontal slices

\[
\left( \hat{\mathcal{S}}_\mathcal{A} \right)_i = \text{diag} \left( \sigma_i^1, \ldots, \sigma_{n'}^1 \right), \quad \left( \hat{\mathcal{S}}_\mathcal{B} \right)_i = \text{diag} \left( \tau_i^1, \ldots, \tau_{n'}^1 \right),
\]

where the diagonal elements are arranged in descending order. The Frobenius norm inequality of Mirsky type for matrices reads as

\[
\sum_{j=1}^{n'} \left( \sigma_j^i - \tau_j^i \right)^2 \leq \| \hat{B}_i - \hat{A}_i \|_F^2.
\]

From (2.2), \( \alpha_j = \frac{1}{r} \sum_{i=1}^{r} \sigma_j^i \) and \( \beta_j = \frac{1}{r} \sum_{i=1}^{r} \tau_j^i \) are the \( j \)th singular values of \( \mathcal{A} \) and \( \mathcal{B} \) respectively. Then, it is not hard to show

\[
\sum_{j=1}^{n'} (\alpha_j - \beta_j)^2 = \sum_{j=1}^{n'} \frac{1}{r^2} \left( \sum_{i=1}^{r} (\sigma_j^i - \tau_j^i) \right)^2 \leq \frac{1}{r} \sum_{j=1}^{n'} \sum_{i=1}^{r} (\sigma_j^i - \tau_j^i)^2.
\]

(4.4)

Adjusting the order of summation and regarding \( \mathcal{B} - \mathcal{A} \) as one new tensor, we can get

\[
\frac{1}{r} \sum_{i=1}^{r} \sum_{j=1}^{n'} (\sigma_j^i - \tau_j^i)^2 \leq \frac{1}{r} \sum_{i=1}^{r} \| \hat{B}_i - \hat{A}_i \|_F^2 = \frac{1}{r} \| \hat{B} - \hat{A} \|_F^2 = \| \mathcal{B} - \mathcal{A} \|_F^2,
\]

(4.5)

in which last equality follows from (2.1). Finally, (1.2) is obtained by combining (4.4) and (4.5).

For (1.3), it is direct to show that

\[
| \alpha_j - \beta_j | = \left| \frac{1}{r} \sum_{i=1}^{r} (\sigma_j^i - \tau_j^i) \right| \leq \frac{1}{r} \sum_{i=1}^{r} | \sigma_j^i - \tau_j^i | \leq \frac{1}{r} \sum_{i=1}^{r} \sigma_{\text{max}} \left( \hat{B}_i - \hat{A}_i \right),
\]

where the second inequality is also from (1.1) and the right hand side of the equation is exactly equal to \( \sigma_{\text{max}} (\mathcal{B} - \mathcal{A}) \). That is, \( | \alpha_j - \beta_j | \leq \sigma_{\text{max}} (\mathcal{B} - \mathcal{A}), \ j = 1, 2, \ldots, n' \). \( \square \)

5 Tensor singular values of products and sums

The Weyl’s theorem was extended from matrices to the T-product based tensors by Liu and Jin in [7, Theorem 4.2]. Now, we are interested in developing useful inequalities for the singular values of the products and sums of tensors.

**Theorem 5.1.** Let \( \mathcal{A}, \mathcal{B} \in \mathbb{R}^{m \times n \times r} \) and \( p = \min\{m, n\} \). The following inequalities hold for the decreasingly ordered singular values of \( \mathcal{A}, \mathcal{B}, \mathcal{A} + \mathcal{B} \) and \( \mathcal{A} * \mathcal{B}^T \):

\[
\sigma_{i+j-1} (\mathcal{A} + \mathcal{B}) \leq \sigma_i (\mathcal{A}) + \sigma_j (\mathcal{B}), \quad (5.1)
\]

\[
\sigma_{i+j-1} (\mathcal{A} * \mathcal{B}^T) \leq r \sigma_i (\mathcal{A}) \sigma_j (\mathcal{B}), \quad (5.2)
\]

\[
| \sigma_i (\mathcal{A} + \mathcal{B}) - \sigma_i (\mathcal{A}) | \leq \sigma_1 (\mathcal{B}). \quad (5.3)
\]

\[
\sigma_i (\mathcal{A} * \mathcal{B}^T) \leq r \sigma_i (\mathcal{A}) \sigma_1 (\mathcal{B}), \quad (5.4)
\]

For (5.1) and (5.2), \( 1 \leq i, j \leq p \) and \( i + j - 1 \leq p \). For (5.3) and (5.4), \( i = 1, \ldots, p \).
Proof. As we can see from [3, Theorem 3.3.16], if \( r = 1 \) in our Theorem, all the conclusions are right. Thus, our proof is based on the theorem for matrix cases. Combining the FFT, T-SVD and Lemma 2.1, we can derive that

\[
\sigma_{i+j-1}(A + B) = \frac{1}{r} \sum_{k=1}^{r} \sigma_{i+j-1}(\hat{A}_k + \hat{B}_k)
\]

\[
\leq \frac{1}{r} \sum_{k=1}^{r} \sigma_i(\hat{A}_k) + \sigma_j(\hat{B}_k) = \sigma_i(A) + \sigma_j(B),
\]

\[
\sigma_{i+j-1}(A \ast B^T) = \frac{1}{r} \sum_{k=1}^{r} \sigma_{i+j-1}(\hat{A}_k \hat{B}_k) \leq \frac{1}{r} \sum_{k=1}^{r} \sigma_i(\hat{A}_k) \sigma_j(\hat{B}_k).
\]

The proof of (5.2) is complete by exploiting below the Cauchy-Schwarz inequality for the right-hand side of (5.5),

\[
\sum_{k=1}^{r} \sigma_i(\hat{A}_k) \sigma_j(\hat{B}_k) \leq \sqrt{\left( \sum_{k=1}^{r} \sigma_i(\hat{A}_k) \right)^2 \left( \sum_{k=1}^{r} \sigma_j(\hat{B}_k) \right)^2}
\]

\[
\leq \left( \sum_{k=1}^{r} \sigma_i(\hat{A}_k) \right) \left( \sum_{k=1}^{r} \sigma_j(\hat{B}_k) \right)
\]

\[
= \sum_{k=1}^{r} \sigma_i(\hat{A}_k) \sum_{k=1}^{r} \sigma_j(\hat{B}_k)
\]

\[
= r^2 \sigma_i(A) \sigma_j(B).
\]

The proofs of (5.3) and (5.4) are similar. \( \Box \)

Remark 5.1. If we let \( i = j = 1 \) in (5.1) and let \( i = p \) in (5.3) respectively, it is clear that

\[
\sigma_{\text{max}}(A + B) \leq \sigma_{\text{max}}(A) + \sigma_{\text{max}}(B), \quad \sigma_{\text{min}}(A + B) \geq \sigma_{\text{min}}(A) - \sigma_{\text{max}}(B).
\]

Theorem 5.2. Given \( A \in \mathbb{R}^{m \times n \times r} \) and two partially orthogonal tensors \( U \in \mathbb{R}^{m \times k \times r} \) and \( V \in \mathbb{R}^{n \times k \times r} \), where \( k \leq \min\{m, n\} \). Then,

\[
\sigma_i(U^T \ast A \ast V) \leq \sigma_i(A), \quad i = 1, \ldots, k.
\]

Proof. If \( r = 1 \), this theorem is just the matrix case in [3, Theorem 3.3.1]. It is easy to verify this inequality by the same technique in proving (5.1). \( \Box \)

For a multiplicative perturbation to a tensor, we have the result below, which reduces to the matrix case for \( r = 1 \).

Theorem 5.3. Let \( B' = B + \delta B = U \ast B \ast V \in \mathbb{R}^{m \times n \times r} \). Then,

\[
\sigma_i(B') \leq r^2 \sigma_i(B) \sigma_{\text{max}}(U) \sigma_{\text{max}}(V).
\]

Proof. Since \( B^T \) has the same singular values with \( B \), then using (5.4), we get

\[
\sigma_i(B') = \sigma_i(U \ast B \ast V) = \sigma_i(V^T \ast B^T \ast U^T) \leq r \sigma_i(V^T \ast B^T) \sigma_1(U)
\]

\[
= r \sigma_i(B \ast V) \sigma_1(U) \leq r^2 \sigma_i(B) \sigma_1(V) \sigma_1(U)
\]

\[
= r^2 \sigma_i(B) \sigma_1(V) \sigma_1(U). \Box
\]
6 Concluding remarks

In this paper, we present some perturbation results for the T-product based tensor singular values, including the relationship between the singular values of the subtensor and the original tensor, the sums and products of tensors. The classical results given by Mirsky are also extended to tensors. Extensions of some other relative perturbation results for matrix singular values need to be further studied.

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