The Representation Theory of Co-triangular Semisimple Hopf Algebras

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1 Introduction

In [EG1, Theorem 2.1] we prove that any semisimple triangular Hopf algebra $A$ over an algebraically closed field of characteristic 0 (say the field of complex numbers $\mathbb{C}$) is obtained from a finite group after twisting the ordinary comultiplication of its group algebra in the sense of Drinfeld [D]; that is $A = \mathbb{C}[G]'$ for some finite group $G$ and a twist $J \in \mathbb{C}[G] \otimes \mathbb{C}[G]$. In [EG2] we show how to construct twists for certain solvable non-abelian groups by iterating twists of their abelian subgroups, and thus obtain new non-trivial semisimple triangular Hopf algebras. We also show how any non-abelian finite group which admits a bijective 1-cocycle with coefficients in an abelian group, gives rise to a non-trivial semisimple minimal triangular Hopf algebra. Such non-abelian groups (which are necessarily solvable [ESS]) exist in abundance and were constructed in [ESS] in connection with set-theoretical solutions to the quantum Yang-Baxter equation.

If $A$ is minimal triangular then $A$ and $A^{*\text{op}}$ are isomorphic as Hopf algebras. But any non-trivial semisimple triangular $A$ which is not minimal, gives rise to a new Hopf algebra $A^*$, which is also semisimple by [LR]. These are very interesting semisimple Hopf algebras which arise from finite groups, and they are abundant by the constructions given in [EG2]. Generally, the dual Hopf algebra of a triangular Hopf algebra is called co-triangular in the literature.

In this paper we explicitly describe the representation theory of co-triangular semisimple Hopf algebras $A^* = (\mathbb{C}[G]'')^*$ in terms of representations of some associated groups. As a corollary we prove that Kaplansky’s 6th conjecture [K] holds for $A^*$; that is that the dimension of any irreducible representation of $A^*$ divides the dimension of $A$.

We note that we have used in an essential way the results of the paper [Mo], from which we learned a great deal.
2 Preliminaries

2.1 Projective Representations and Central Extensions

Here we recall some basic facts about projective representations and central extensions. They can be found in textbooks, e.g. [CR, Section 11E].

A projective representation of a group $\Gamma$ is a vector space $V$ together with a homomorphism of groups $\pi_V : \Gamma \to PGL(V)$, where $PGL(V) \cong GL(V)/C$ is the projective linear group.

A linearization of a projective representation $V$ of $\Gamma$ is a central extension $\hat{\Gamma}$ of $\Gamma$ by a central subgroup $\zeta$ together with a linear representation $\tilde{\pi}_V : \hat{\Gamma} \to GL(V)$ which descends to $\pi_V$. If $V$ is a finite-dimensional projective representation of $\Gamma$ then there exists a linearization of $V$ such that $\zeta$ is finite (in fact, one can make $\zeta = \mathbb{Z}/(\dim V)\mathbb{Z}$).

Any projective representation $V$ of $\Gamma$ canonically defines a cohomology class $[V] \in H^2(\Gamma, \mathbb{C}^*)$. The representation $V$ can be lifted to a linear representation of $\Gamma$ if and only if $[V] = 0$.

2.2 The Algebras Associated With a Twist

Let $H$ be a finite group, and let $J \in \mathbb{C}[H] \otimes \mathbb{C}[H]$ be a minimal twist (see [EG1]). That is, the right (and left) components of the R-matrix $R = J_{21}^{-1}J$ span $\mathbb{C}[H]$. Define two coalgebras $(A_1, \Delta_1, \epsilon), (A_2, \Delta_2, \epsilon)$ as follows: $A_1 = A_2 = \mathbb{C}[H]$ as vector spaces, the coproducts are determined by

$$\Delta_1(x) = (x \otimes x)J, \ \Delta_2(x) = J^{-1}(x \otimes x)$$

for all $x \in H$, and $\epsilon$ is the ordinary counit of $\mathbb{C}[H]$. Note that since $J$ is a twist, $\Delta_1$ and $\Delta_2$ are indeed coassociative. Clearly the dual algebras $A_1^*$ and $A_2^*$ are spanned by $\{\delta_h | h \in H\}$, where $\delta_h(h') = \delta_{hh'}$.

**Theorem 2.2.1** Let $A_1^*$ and $A_2^*$ be as above. The following hold:

1. $A_1^*$ and $A_2^*$ are $H$–algebras via

$$\rho_1(h)\delta_y = \delta_{hy}, \ \rho_2(h)\delta_y = \delta_{yh^{-1}}$$

respectively.

2. $A_1^* \cong A_2^{\text{op}}$ as $H$–algebras (where $H$ acts on $A_2^{\text{op}}$ as it does on $A_2^*$).

3. The algebras $A_1^*$ and $A_2^*$ are simple, and are isomorphic as $H$–modules to the regular representation $R_H$ of $H$. 
Proof: The proof of part 1 is straightforward.

The proof of part 3 follows from the results in [Mo]. Namely, it follows from [Mo, Proposition 14] that in the case of a minimal twist the group $St$ defined in [Mo] (which is, by definition, a subgroup of $H$) coincides with $H$. Therefore, by [Mo, Propositions 6,7] the algebras $A_1^*$ and $A_2^*$ are simple. Furthermore, by [Mo, Propositions 11,12], the actions of $H$ on $A_1^*$ and $A_2^*$ are isomorphic to the regular representation of $H$.

Let us prove part 2. Let $S_0, \Delta_0, m_0$ denote the standard antipode, coproduct and multiplication of $C[H]$, and define $Q = m_0(S_0 \otimes I)(J)$. Then it is straightforward to verify that $Q$ is invertible, and $(S_0 \otimes S_0)(J) = (Q \otimes Q)(J^{21}) \Delta_0(Q)^{-1}$ (see e.g. (2.17) in [Ma, Section 2.3]). Hence the map $A_2^* \to A_1^{\text{op}}$, $\delta_x \mapsto \delta_{S_0(x)Q^{-1}}$ determines an $H$-algebra isomorphism.

Corollary 2.2.2 Let $A$ be a semisimple minimal triangular Hopf algebra over $C$ with Drinfeld element $u$. If $u = 1$ then $\dim A$ is a square, and if $u \neq 1$ then $2\dim A$ or $\dim A$ is a square.

Proof: The first statement follows from [EG1, Theorem 2.1] and part 3 of Theorem 2.2.1. To prove the second statement, let $(A, R, u)$ be a semisimple minimal triangular Hopf algebra with $u \neq 1$, and $(A, R', u')$ be obtained from $(A, R, u)$ by changing $R$ so that the new Drinfeld element $u' = 1$ (as in [EG1]). Then $(A, R') = (C[H]^J, J^{21})$ for some finite group $H$. Let $A_{min} = C[H]^J$ be the minimal triangular Hopf subalgebra of $(A, R')$ where $H' \subset H$ is a subgroup, and $J$ is a minimal twist for $H'$. It is clear that $H$ is generated by $H'$ and $u$. Since $u$ is a central group-like element of order 2, we get that the index of $H'$ is at most 2. This implies our statement.

Let $A_1^*, A_2^*$ be the $H$-algebras as in Theorem 2.2.1. Since the algebras $A_1^*, A_2^*$ are simple, the actions of $H$ on $A_1^*, A_2^*$ give rise to projective representations $H \to PGL(|H|^{1/2}, C)$. We will denote these projective representations by $V_1, V_2$ (they can be thought of as the simple modules over $A_1^*, A_2^*$, with the induced projective action of $H$). Note that part 2 of Theorem 2.2.1 implies that $V_1, V_2$ are dual to each other, hence that $[V_1] = -[V_2]$.

3 The Main Result

Let $(A, R)$ be a semisimple triangular Hopf algebra over $C$, and assume that the Drinfeld element $u$ is 1 (this can be always achieved by a simple modification of $R$, without changing the Hopf algebra structure [EG1]). Then by [EG1, Theorem 2.1], there exist finite groups $H \subset G$ and a minimal twist $J \in C[H] \otimes C[H]$ such that $(A, R) \cong (C[G]^J, J^{21})$ as triangular Hopf algebras. So from now on we will assume that $A$ is of this form.

Consider the dual Hopf algebra $A^*$. It has a basis of $\delta$-functions $\delta_g$. The first simple but important fact about the structure of $A^*$ as an algebra is:

Proposition 3.1 Let $Z$ be a double coset of $H$ in $G$, and $A^*_Z = \oplus_{g \in Z} C\delta_g \subset A^*$. Then $A^*_Z$ is a subalgebra of $A^*$, and $A^* = \oplus_Z A^*_Z$ as algebras.
Proof: Straightforward. □

Thus, to study the representation theory of \( A^* \), it is sufficient to describe the representations of \( A^*_Z \) for any \( Z \).

Let \( Z \) be a double coset of \( H \) in \( G \), and let \( g \in Z \). Let \( K_g = H \cap gHg^{-1} \), and define the embeddings \( \theta_1, \theta_2 : K_g \to H \) given by \( \theta_1(a) = g^{-1}ag, \theta_2(a) = a \). Denote by \( W_i \) the pullback of the projective \( H \)-representation \( V_i \) to \( K_g \) by means of \( \theta_i, i = 1, 2 \).

Our main result is the following theorem, which is proved in the next section.

**Theorem 3.2** Let \( W_1, W_2 \) be as above, and let \( (\tilde{K}_g, \tilde{\pi}_w) \) be any linearization of the projective representation \( W = W_1 \otimes W_2 \) of \( K_g \). Let \( \zeta \) be the kernel of the projection \( \tilde{K}_g \to K_g \), and \( \chi : \zeta \to C^* \) be the character by which \( \zeta \) acts in \( W \). Then there exists a 1-1 correspondence between isomorphism classes of irreducible representations of \( A^*_Z \) and isomorphism classes of irreducible representations of \( \tilde{K}_g \) with \( \zeta \) acting by \( \chi \). If a representation \( Y \) of \( A^*_Z \) corresponds to a representation \( X \) of \( \tilde{K}_g \) then \( \dim Y = \frac{|H|}{|K_g|} \dim X \).

As a corollary we get Kaplansky’s 6th conjecture [K] for semisimple co-triangular Hopf algebras.

**Corollary 3.3** The dimension of any irreducible representation of a semisimple co-triangular Hopf algebra divides the dimension of the Hopf algebra.

**Proof:** Since \( \dim X \) divides \( |K_g| \) (see e.g. [CR, Proposition 11.44]), we have that \( \frac{|G|}{|K_g|} \dim X = \frac{|G|}{|H|} \dim X \) and the result follows. □

In some cases the classification of representations of \( A^*_Z \) is even simpler. Namely, let \( \overline{g} \in \text{Aut}(K_g) \) be given by \( a \mapsto g^{-1}ag \). Then we have:

**Corollary 3.4** If the cohomology class \( [W_1] \) is \( \overline{g} \)-invariant then irreducible representations of \( A^*_Z \) correspond in a 1-1 manner to irreducible representations of \( K_g \), and if \( Y \) corresponds to \( X \) then \( \dim Y = \frac{|H|}{|K_g|} \dim X \).

**Proof:** For any \( \alpha \in \text{Aut}(K_g) \) and \( f \in \text{Hom}((K_g)^n, C^*) \), let \( \alpha \circ f \in \text{Hom}((K_g)^n, C^*) \) be given by \( (\alpha \circ f)(h_1, \ldots, h_n) = f(\alpha(h_1), \ldots, \alpha(h_n)) \) (which determines the action of \( \alpha \) on \( H^i(K_g, C^*) \)). Then it follows from the identity \( [V_1] = -[V_2] \), given at the end of Section 2, that \( [W_1] = -\overline{g} \circ [W_2] \). Thus, in our situation \( [W] = 0 \), hence \( W \) comes from a linear representation of \( K_g \). Thus, we can set \( \tilde{K}_g = K_g \) in the theorem, and the result follows. □

**Example 3.5** Let \( p > 2 \) be a prime number, and \( H = (\mathbb{Z}/p\mathbb{Z})^2 \) with the standard symplectic form \( (,): H \times H \to \mathbb{C}^* \) given by \( ((x, y), (x', y')) = e^{2\pi i (xy' - yx')/p} \). Then the element
Proof: It follows from Proposition 4.1 that $A \circ \rho$ is a minimal twist for $C[H]$. Let $g \in GL_2(\mathbb{Z}/p\mathbb{Z})$ be an automorphism of $H$, and $G_0$ be the cyclic group generated by $g$. Let $G$ be the semidirect product of $G_0$ and $H$. It is easy to see that in this case, the double cosets are ordinary cosets $g^kH$, and $K_{g^k} = H$. Moreover, one can show either explicitly or using [Mo, Proposition 9], that $[W_1]$ is a generator of $H^2(H, C^*)$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The element $g^k$ acts on $[W_1]$ by multiplication by $\det(g^k)$. Therefore, by Corollary 3.4, the algebra $A_{g^kH}^*$ has $p^2$ 1-dimensional representations (corresponding to linear representations of $H$) if $\det(g^k) = 1$.

However, if $\det(g^k) \neq 1$, then $[W]$ generates $H^2(H, C^*)$. Thus, $W$ comes from a linear representation of the Heisenberg group $\hat{H}$ (a central extension of $H$ by $\mathbb{Z}/p\mathbb{Z}$) with some central character $\chi$. Thus, $A_{g^kH}^*$ has one p-dimensional irreducible representation, corresponding to the unique irreducible representation of $\hat{H}$ with central character $\chi$ (which is $W$).

4 Proof of Theorem 3.2

Let $Z \subset G$ be a double coset of $H$ in $G$, and let $A_1, A_2$ be as in Subsection 2.2. For any $g \in Z$ define the linear map

$$F_g : A_Z^* \rightarrow A_2^* \otimes A_1^*, \quad \delta_g \mapsto \sum_{h,h' \in H: y = hgh'} \delta_h \otimes \delta_{h'}.$$ 

Proposition 4.1 Let $\rho_1, \rho_2$ be as in Theorem 2.2.1. Then:

1. The map $F_g$ is an injective homomorphism of algebras.

2. $F_{\rho_2}(\varphi) = (\rho_2(a) \otimes \rho_1(a')^{-1})F_g(\varphi)$ for any $a, a' \in H, \varphi \in A_Z^*$.

Proof: 1. It is straightforward to verify that the map $F_g^* : A_2 \otimes A_1 \rightarrow A_Z$ is determined by $h \otimes h' \mapsto hgh'$, and that it is a surjective homomorphism of coalgebras. Hence the result follows.

2. Straightforward. ■

For any $a \in K_g$ define $\rho(a) \in Aut(A_2^* \otimes A_1^*)$ by $\rho(a) = \rho_2(a) \otimes \rho_1(a^g)$, where $a^g = g^{-1}ag$ and $\rho_1, \rho_2$ are as in Theorem 2.2.1. Then $\rho$ is an action of $K_g$ on $A_2^* \otimes A_1^*$.

Proposition 4.2 Let $U_g = (A_2^* \otimes A_1^*)^{\rho(K_g)}$ be the algebra of invariants. Then $Im(F_g) = U_g$, so $A_Z^* \cong U_g$ as algebras.

Proof: It follows from Proposition 4.1 that $Im(F_g) \subseteq U_g$, and $rk(F_g) = dimA_Z^* = \frac{|H|^2}{|K_g|}$. On the other hand, by Theorem 2.2.1, $A_1^*, A_2^*$ are isomorphic to the regular representation $R_H$.
of $H$. Thus, $A_1^*, A_2^*$ are isomorphic to $\frac{|H|}{|K_g|} R_{K_g}$ as representations of $K_g$, via $\rho_1(a), \rho_2(a^g)$. Thus, $A_2^* \otimes A_1^* \cong \frac{|H|^2}{|K_g|^2} (R_{K_g} \otimes R_{K_g}) \cong \frac{|H|^2}{|K_g|^2} R_{K_g}$. So $U_g$ has dimension $|H|^2/|K_g|$, and the result follows. 

Now we are in a position to prove Theorem 3.2. Since $W_i \otimes W_i^* \cong A_i^*$ for $i = 1, 2$, it follows from Theorem 2.2.1 that $W_1 \otimes W_2 \otimes W_1^* \otimes W_2^* \cong \frac{|H|^2}{|K_g|^2} (R_{K_g} \otimes R_{K_g})$, thus, if $\chi_W$ is the character of $W = W_1 \otimes W_2$ as a $\hat{K}_g$ module then $|\chi_W(x)|^2 = 0$, $x \notin \zeta$ and $|\chi_W(x)|^2 = |H|^2$, $x \in \zeta$.

Therefore, $\chi_W(x) = 0$, $x \notin \zeta$ and $\chi_W(x) = |H| \cdot x_{W}$, $x \in \zeta$, where $x_{W}$ is the root of unity by which $x$ acts in $W$. Now, it is clear from the definition of $U_g$ (see Proposition 4.2) that $U_g = \text{End}_{\hat{K}_g}(W)$. Thus if $W = \bigoplus_{M \in \text{Irr}(\hat{K}_g)} W(M) \otimes M$, where $W(M) = \text{Hom}_{K_g}(M, W)$ is the multiplicity space, then $U_g = \bigoplus_{M:W(M) \neq 0} \text{End}_{\mathbb{C}}(W(M))$. So \{W(M)|W(M) \neq 0\} are the irreducible representations of $U_g$. Thus the following implies the theorem:

Lemma.

1. $W(M) \neq 0$ if and only if for all $x \in \zeta$, $x_{|M} = x_{|W}$.

2. If $W(M) \neq 0$ then $\text{dim} W(M) = \frac{|H|}{|K_g|} \text{dim} M$.

Proof of the Lemma. The ”only if” part of 1 is clear. For the ”if” part compute $\text{dim} W(M)$ as the inner product $(\chi_W, \chi_M)$. We have

$$(\chi_W, \chi_M) = \sum_{x \in \zeta} \frac{|H|}{|K_g|} x_{|W} \cdot \text{dim} M \cdot x_{|M}.$$ 

If $x_{|M} = x_{|W}$ then

$$(\chi_W, \chi_M) = \sum_{x \in \zeta} \frac{|H|}{|K_g|} \text{dim} M = \frac{|H|}{|K_g|} \text{dim} M = \frac{|H|}{|K_g|} \text{dim} M.$$ 

This proves part 2 as well, and hence concludes the proof of the theorem. ■
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