A distribution-function-valued SPDE and its applications

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Abstract. In this paper we further study the stochastic partial differential equation first proposed by Xiong [22]. Under localized conditions on its coefficients, we prove a comparison theorem on its solutions and show that the solution is in fact distribution-function-valued. We also establish pathwise uniqueness of the solution. As applications we obtain the well-posedness of martingale problems for two classes of measure-valued diffusions: interacting super-Brownian motions and interacting Fleming-Viot processes. Properties of the two superprocesses such as the existence of density fields and the survival-extinction behaviors are also studied.

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1 Introduction

It is well known that the density process \( \{X_t(x) : t > 0, x \in \mathbb{R}\} \) of a one-dimensional binary branching super-Brownian motion solves the following non-linear stochastic partial differential equation (SPDE):

\[
\frac{\partial}{\partial t} X_t(x) = \frac{1}{2} \Delta X_t(x) + \sqrt{X_t(x)} \dot{W}_t(x), \quad t > 0, \ x \in \mathbb{R},
\]  

(1.1)

where \( \Delta \) denotes the Laplacian operator and \( \{\dot{W}_t(x) : t \geq 0, x \in \mathbb{R}\} \) is the derivative of a space-time Gaussian white noise. This SPDE was first derived and studied independently by Konno and Shiga [9] and Reimers [20]. The weak uniqueness of solution to (1.1) follows from that of a martingale problem for super-Brownian motion. But the strong (pathwise) uniqueness of nonnegative solution to (1.1) remains open even though it has been studied by many authors. The main difficulty comes from the unbounded drift coefficient and the non-Lipschitz diffusion coefficient. Progresses have been made in considering modified forms of SPDE (1.1). When the random field \( \{W_t(x) : t \geq 0, x \in \mathbb{R}\} \) is colored in space and white in time, the strong uniqueness of nonnegative solution to (1.1) and to more general equations were studied in [17, 21]. When \( \{W_t(x) : t \geq 0, x \in \mathbb{R}\} \) is a space-time Gaussian white noise and solutions are allowed to take both positive and negative values with \( \sqrt{X_t(x)} \) replaced by \( \sigma(t, x, X_t(x)) \) \( \sigma(\cdot, \cdot, u) \) is Hölder continuous in \( u \) of index \( \beta_0 > 3/4 \) in [11], the pathwise uniqueness was proved by Mytnik and Perkins [16] and further investigated by Mytnik and Neuman [15]. Recently, some negative
results were obtained. When $\sqrt{X_t(x)}$ is replaced by $|X_t(x)|^{\beta_1}$ in (1.1), nonuniqueness results were obtained for $0 < \beta_1 < 3/4$ in [11]. It was also shown in Chen [2] that the solution is a super-Brownian motion with immigration and the pathwise nonuniqueness holds when a positive function is added on the right-hand side of (1.1). We refer to Li [10] and Xiong [24] for introductions on superprocesses and the related SPDEs.

A novel approach for studying the strong uniqueness of (1.1) was proposed by Xiong [22], where an SPDE for the distribution-function process of measure-valued super-Brownian motion was formulated and the strong existence and uniqueness for this SPDE were established. Similar stochastic equations of distribution-function processes for superprocesses were also established in Dawson and Li [3]. He et al. [7] showed that the distribution-function process of a one-dimensional super-Lévy process with general branching mechanism is the unique strong solution to another SPDE. The uniqueness of solution to the martingale problem of a superprocess with interactive immigration mechanism was first established in Mytnik and Xiong [18], and then in Xiong and Yang [24] (for a more general process) by studying the corresponding SPDE for its distribution-function process.

Let $X_0(\mathbb{R})$ be the Hilbert space consisting of all measurable functions $f$ on $\mathbb{R}$ satisfying $\int_\mathbb{R} |f(x)| e^{-|x|} dx < \infty$. Let $D(\mathbb{R})$ be the set of bounded right-continuous nondecreasing functions $f$ on $\mathbb{R}$ satisfying $f(-\infty) = 0$. Let $D_c(\mathbb{R})$ be the subset of $D(\mathbb{R})$ consisting of continuous functions.

Given a Polish space $E$ and a $\sigma$-finite Borel measure $\pi$ on $E$, consider the following SPDE:

$$Y_t(y) = Y_0(y) + \frac{1}{2} \int_0^t \Delta Y_s(y) ds + \int_0^t \int_E G(u, Y_s(y)) W(ds, du), \quad Y_0 \in X_0(\mathbb{R}), \quad (1.2)$$

where $\{W(ds, du) : s \geq 0, u \in E\}$ is a Gaussian white noise with intensity $d\pi du$ and $G$ is a Borel function on $E \times \mathbb{R}_+$ ($\mathbb{R}_+ := [0, \infty)$). Then an $X_0(\mathbb{R})$-valued process $\{Y_t(y) : t \geq 0, y \in \mathbb{R}\}$ is a solution to (1.2) if for any $f \in C^2_0(\mathbb{R})$ (to be defined at the end of this section) with $\int_\mathbb{R} |f(x)| dx < \infty$,

$$\langle Y_t, f \rangle = \langle Y_0, f \rangle + \frac{1}{2} \int_0^t \langle Y_s, f'' \rangle ds + \int_0^t \int_E \left[ \int_\mathbb{R} G(u, Y_s(y)) f(y) dy \right] W(ds, du), \quad \text{P-a.s.} \quad (1.3)$$

Xiong [22] proved that (1.2) has a unique strong $X_0(\mathbb{R})$-valued solution if $G$ satisfies the following conditions: there is a constant $C > 0$ so that

$$\int_E |G(u, x)|^2 \pi(du) \leq C(1 + x^2), \quad x \in \mathbb{R} \quad (1.4)$$

and

$$\int_E |G(u, x_1) - G(u, x_2)|^2 \pi(du) \leq C|x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}. \quad (1.5)$$

For $G(u, v) = 1_{\{0 \leq u \leq v\}}$ and $G(u, v) = 1_{\{0 \leq u \leq v \leq 1\}} - v$, the solutions to (1.2) are the distribution-function processes of super-Brownian motion and Fleming-Viot process, respectively.

In this paper we further improve the results in [22]. In particular, we establish a comparison theorem which shall be of independent interest. Using it we prove that (1.2) indeed has a unique strong $D_c(\mathbb{R})$-valued solution when $G$ satisfies certain conditions. We then apply the results to show the well-posedness of martingale problems for an interacting super-Brownian motion and an interacting Fleming-Voit process. We summarize the main results in the following.
Our first main result of this paper, Theorem 2.3, shows that SPDE (1.2) has a unique strong $D_c(\mathbb{R})$-valued solution when $G$ satisfies the following conditions:

$$
\int_E |G(u, 0)| \pi(du) = 0,
$$

for each $k \geq 1$ there is a constant $C_k > 0$ so that

$$
\int_E |G(u, x)|^2 \pi(du) \leq C_k, \quad x \in [0, k]
$$

and

$$
\int_E |G(u, x_1) - G(u, x_2)|^2 \pi(du) \leq C_k |x_1 - x_2|, \quad x_1, x_2 \in [0, k].
$$

In our second main results of this paper, Theorems 3.2 and 4.2 we prove that the martingale problems for both the interacting super-Brownian motion and the interacting Fleming-Viot process are well-posed by first associating the martingale problems with the corresponding SPDEs (1.2) with $E = \mathbb{R}_+$, $G(u, x) = 1_{u \leq x} \sqrt{\sigma(u)}$ and with $E = [0, 1]^2$, $G(a, b, x) = 1_{a \leq x \leq b} \sqrt{\gamma(a, b)}$ for nonnegative functions $\sigma$ and $\gamma$ (see Lemmas 3.1 and 4.1), respectively, and then applying Theorem 2.3. They partially generalize the recent work of [18, 24].

We want to point out that, in our paper, the existence of solutions to the martingale problems follows directly from the relationship with their corresponding SPDEs and the existence of solutions to these SPDEs, which is different from the approach of approximating martingale problem for the classical superprocesses in [18, Proposition 2.4] and the approach of approximating branching interacting particle system in Xiong and Yang [24, Theorem 3.3].

Our last main results, Theorems 3.3–3.5 and 4.3 concern properties such as the existence of density fields and survival-extinction behaviors of the interacting super-Brownian motions and Fleming-Viot processes. In particular, using martingale arguments we discuss the survival-extinction properties of the interacting super-Brownian motion $\{X_t : t \geq 0\}$ determined by the local martingale problem (3.3)–(3.4). We show that the total mass process $\{X_t(1) : t \geq 0\}$ satisfies $X_t(1) = 0$ for all $t$ large enough if $\int_0^\infty \sigma(y) dy$ decreases to 0 slow enough as $x \to 0+$ and $X_t(1) > 0$ for all $t$ if $\int_0^\infty \sigma(y) dy$ decreases to 0 fast enough.

The rest of the paper is organized as follows. In Section 2, the comparison theorem and the strong uniqueness of $D_c(\mathbb{R})$-valued solution to (1.2) are established. The interacting super-Brownian motions and Fleming-Viot processes are studied in Sections 3 and 4, respectively.

Notations: We always assume that all random elements are defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual hypotheses. For a topological space $V$ let $B(V)$ be the space of Borel measurable functions and Borel sets on $V$. Let $B(V)$ be the collection of bounded functions on $V$ furnished with the supremum norm $\|\cdot\|$ and $C(V)$ the space of bounded continuous functions on $V$. For $f, g \in B(\mathbb{R})$ write $\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) dx$ whenever it exists. Let $C^2(\mathbb{R})$ be the space of twice continuously differentiable functions on $\mathbb{R}$. Let $C_0(\mathbb{R})$ be the subset of $C(\mathbb{R})$ consisting of functions vanishing at infinity and let $C^k_0(\mathbb{R})$ be the subset of functions with derivatives up to order $k$ belonging to $C_0(\mathbb{R})$. We use the superscript “$+$” to denote the subsets of non-negative elements of the function spaces. Denote $M(\mathbb{R})$ for the space of finite Borel measures on $\mathbb{R}$ equipped with the topology of weak convergence. Then there is an obvious one-to-one correspondence between $M(\mathbb{R})$ and $D(\mathbb{R})$ by associating a measure with its distribution function. We endow $D(\mathbb{R})$ with the topology induced by this
correspondence from the weak convergence topology on $M(\mathbb{R})$. Then for any $M(\mathbb{R})$-valued stochastic process \(\{X_t : t \geq 0\}\), its distribution-function process \(\{Y_t : t \geq 0\}\) is a $D(\mathbb{R})$-valued stochastic process. For $\mu \in M(\mathbb{R})$ and $f \in B(\mathbb{R})$ write $\mu(f) \equiv \int_\mathbb{R} f(x) \mu(dx)$. Write $f(\infty) := \lim_{x \to \infty} f(x)$ and $f(-\infty) := \lim_{x \to -\infty} f(x)$ for $f \in D(\mathbb{R})$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. Throughout this paper we use $C$ to denote a positive constant whose value might change from line to line. We also write $C_n$ if the constant depends on $n \geq 1$.

## 2 Existence and strong uniqueness of solution to SPDE

In this section we establish the existence and strong uniqueness of $D(\mathbb{R})$-valued solutions to the SPDE \((1.2)\) under conditions \((1.4)\)–\((1.6)\) and conditions \((1.6)\)–\((1.8)\), respectively. Xiong \cite{22} proved that \((1.2)\) has a unique strong $X_0(\mathbb{R})$-valued solution under conditions \((1.4)\)–\((1.5)\). So we only need to show that the $X_0(\mathbb{R})$-valued solution is in fact $D(\mathbb{R})$-valued under additional condition \((1.6)\) and further prove that \((1.2)\) has a unique strong $D_c(\mathbb{R})$-valued solution under conditions \((1.6)\)–\((1.8)\).

For this purpose, we first establish the following comparison theorem for \((1.2)\) under the two sets of conditions, which is of independent interest. The corresponding strong uniqueness then follows.

**Proposition 2.1** (i) Let $\{\bar{Y}_{0,t} : t \geq 0\}$ and $\{\bar{\bar{Y}}_{0,t} : t \geq 0\}$ be any two $X_0(\mathbb{R})$-valued solutions of \((1.2)\) under conditions \((1.4)\)–\((1.5)\). If $\bar{Y}_{0,0}(x) \leq \bar{\bar{Y}}_{0,0}(x)$ $P$-a.s. for $\lambda$-a.e. $x$ and

$$
\sup_{t \in [0,T]} \left[ \int_\mathbb{R} \bar{Y}_{0,t}(x)^2 e^{-|x|} dx + \int_\mathbb{R} \bar{\bar{Y}}_{0,t}(x)^2 e^{-|x|} dx \right] < \infty \quad P\text{-a.s.}
$$

(2.1)

for all $T > 0$, then for any $t > 0$,

$$
P\{\bar{Y}_{0,t}(x) \leq \bar{\bar{Y}}_{0,t}(x) \text{ for } \lambda\text{-a.e. } x\} = 1.
$$

(ii) Let $\{\bar{Y}_{1,t} : t \geq 0\}$ and $\{\bar{\bar{Y}}_{1,t} : t \geq 0\}$ be any two $D(\mathbb{R})$-valued solutions of \((1.2)\) under conditions \((1.7)\)–\((1.8)\). If $\bar{Y}_{1,0}(x) \leq \bar{\bar{Y}}_{1,0}(x)$ $P$-a.s. for all $x \in \mathbb{R}$ and

$$
\sup_{t \in [0,T]} [\bar{Y}_{1,t}(\infty) + \bar{\bar{Y}}_{1,t}(\infty)] < \infty \quad P\text{-a.s.}
$$

(2.2)

for all $T > 0$, then for any $t > 0$,

$$
P\{\bar{Y}_{1,t}(x) \leq \bar{\bar{Y}}_{1,t}(x) \text{ for all } x \in \mathbb{R}\} = 1.
$$

**Proof.** The proof is inspired by that of \cite{24} Lemma 4.10 and \cite{18} Proposition 3.1. It is divided into four steps.

**Step 1.** For $n \geq 1$ and $x, y \in \mathbb{R}$ let

$$
g_n(y) := g_n(x - y) := \sqrt{\frac{n}{2\pi}} \exp \left\{ - \frac{n(x - y)^2}{2} \right\}, \quad \phi_n(x) := \int_{-\infty}^\infty [(x - y) \vee 0]g_n(y)dy.
$$

Then

$$
\phi_n'(x) = \int_{-\infty}^x g_n(y)dy, \quad \phi_n''(x) = g_n(x) \geq 0, \quad \phi_n(x) \to x \vee 0
$$

for
as \( n \to \infty \) and \( |x| \phi''_{n}(x) \leq 1 \) for all \( n \geq 1 \) and \( x \in \mathbb{R} \). Define

\[
J(x) := \int_{\mathbb{R}} e^{-|y|} \rho_{0}(x - y) dy
\]

with the mollifier \( \rho_{0} \) given by

\[
\rho_{0}(x) := \tilde{C} \exp \{ -1/(1 - x^{2}) \} 1_{\{|x|<1\}},
\]

where \( \tilde{C} \) is a constant so that \( \int_{\mathbb{R}} \rho_{0}(x) dx = 1 \). By (2.1) in [12], for each \( n \geq 0 \) there exist constants \( \tilde{c}_{n}, \tilde{C}_{n} > 0 \) so that the \( n \)-th derivative \( J^{(n)} \) of \( J \) satisfies

\[
\tilde{c}_{n} e^{-|x|} \leq |J^{(n)}(x)| \leq \tilde{C}_{n} e^{-|x|}, \quad x \in \mathbb{R},
\]

which implies

\[
|J^{(n)}(x)| \leq \tilde{C}_{n} J(x), \quad x \in \mathbb{R}.
\]

**Step 2.** Let \( t > 0 \) be fixed in the following. For \( k \geq 1 \) define stopping times

\[
\tau_{0,k} := \inf \left\{ s \geq 0 : \langle \bar{Y}_{0,s}^{2}, J \rangle + \langle \bar{Y}_{0,s}^{2}, J \rangle > k \right\}
\]

and

\[
\tau_{1,k} := \inf \left\{ s \geq 0 : \bar{Y}_{1,s}(\infty) + \bar{Y}_{1,s}(\infty) > k \right\}
\]

with the convention \( \inf \emptyset = \infty \). It follows from (2.1) and (2.2) that

\[
\lim_{k \to \infty} \tau_{0,k} = \infty \quad \text{and} \quad \lim_{k \to \infty} \tau_{1,k} = \infty \quad \text{P.a.s.}
\]

For \( x \in \mathbb{R}, m \geq 1 \) and \( i = 0, 1, s \geq 0 \) let

\[
Y_{i,s} := \bar{Y}_{i,s} - \bar{Y}_{i,s}, \quad \bar{G}_{i}(s, u, x) := G(u, \bar{Y}_{i,s}(x)) - G(u, \bar{Y}_{i,s}(x)), \quad v^{m}_{i,s}(x) := \langle Y_{i,s}, g^{m}_{u} \rangle.
\]

By conditioning we may assume that \( \bar{Y}_{i,0} \) and \( \bar{Y}_{i,0} \) are deterministic. It follows from (1.3) and Itô’s formula that for each \( x \in \mathbb{R} \),

\[
\phi_{n}(v^{m}_{i,t \wedge \tau_{i,k}}(x)) = \phi_{n}(v^{m}_{i,0}(x)) + \frac{1}{2} \int_{0}^{t \wedge \tau_{i,k}} \phi'_{n}(v^{m}_{i,s}(x)) \Delta v^{m}_{i,s}(x) ds + \frac{1}{2} \int_{0}^{t \wedge \tau_{i,k}} \phi''_{n}(v^{m}_{i,s}(x)) ds \int_{E} \left[ \int_{\mathbb{R}} \bar{G}_{i}(s, u, y) g^{m}_{u}(y) dy \right]^{2} \pi(du) + \text{mart.}
\]

Therefore,

\[
\mathbb{E}\left\{ \int_{\mathbb{R}} \phi_{n}(v^{m}_{i,t \wedge \tau_{i,k}}(x)) J(x) dx \right\}
\]

\[
= \int_{\mathbb{R}} \phi_{n}(v^{m}_{i,0}(x)) J(x) dx + \frac{1}{2} \mathbb{E}\left\{ \int_{0}^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi'_{n}(v^{m}_{i,s}(x)) \Delta v^{m}_{i,s}(x) J(x) dx \right\}
\]

\[
+ \frac{1}{2} \mathbb{E}\left\{ \int_{0}^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi''_{n}(v^{m}_{i,s}(x)) J(x) dx \int_{E} \left[ \int_{\mathbb{R}} \bar{G}_{i}(s, u, y) g^{m}_{u}(y) dy \right]^{2} \pi(du) \right\}
\]

\[
=: I_{0}(i, m, n) + I_{1}(i, m, n, k, t) + I_{2}(i, m, n, k, t).
\]
Step 3. Observe that
\[
\int_{\mathbb{R}} \phi_n'(v_{i,s}^m(x)) \Delta v_{i,s}^m(x) J(x) \, dx = \int_{\mathbb{R}} (\phi_n(v_{i,s}^m(x)))'' J(x) \, dx - \int_{\mathbb{R}} \phi_n''(v_{i,s}^m(x)) |\nabla v_{i,s}^m(x)|^2 J(x) \, dx.
\]
Since \(\phi_n'' = g_n \geq 0\), the second term on the right-hand side is negative. Thus
\[
2I_1(i, m, n, k, t) \leq E \left\{ \int_{0}^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} (\phi_n(v_{i,s}^m(x)))'' J(x) \, dx \right\} = E \left\{ \int_{0}^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi_n(v_{i,s}^m(x)) J(x) \, dx \right\} \leq C E \left\{ \int_{0}^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi_n(v_{i,s}^m(x)) J(x) \, dx \right\},
\] (2.7)
where integration by parts and (2.4) are used. By the Hölder inequality and conditions (1.5) and (1.8),
\[
2I_2(i, m, n, k, t) \leq E \left\{ \int_{0}^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi_n''(v_{i,s}^m(x)) J(x) \, dx \int_{\mathbb{R}} g_n^x(y) dy \int_{\mathcal{E}} G_i(s, u, y) 2\pi(du) \right\} \leq C_k E \left\{ \int_{0}^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi_n''(v_{i,s}^m(x)) |Y_{i,s}|^2 g_n^x J(x) \, dx \right\}.
\] (2.8)
For \(i = 0\), by the fact \(\|\phi_n'|| \leq 1\), \(\|\phi_n''|| \leq C_n\) and (22) Lemma 2.1, it is elementary to see that for \(0 \leq s \leq \tau_{0,k}\),
\[
\langle v_{0,s}^m, J \rangle \rightarrow \langle Y_{0,s}, J \rangle, \quad \langle \phi_n(v_{0,s}^m) - \phi_n(Y_{0,s}), J \rangle \leq \langle |v_{0,s}^m - Y_{0,s}|, J \rangle \rightarrow 0,
\] (2.9)
and
\[
\left| \int_{\mathbb{R}} \phi_n''(v_{0,s}^m(x)) |Y_{0,s}|^2 g_n^x J(x) \, dx - \int_{\mathbb{R}} \phi_n''(Y_{0,s}(x)) |Y_{0,s}(x)|^2 J(x) \, dx \right| \leq \int_{\mathbb{R}} \left| \phi_n''(v_{0,s}^m(x)) - \phi_n''(Y_{0,s}(x)) \right| |Y_{0,s}(x)|^2 J(x) \, dx \leq \|\phi_n''\| \left( \int_{\mathbb{R}} \left| \phi_n''(v_{0,s}^m(x)) - \phi_n''(Y_{0,s}(x)) \right|^2 J(x) \, dx \right)^{1/2} \rightarrow 0
\] (2.10)
as \(m \rightarrow \infty\). Observe that
\[
\phi_n(x) \leq |x| + \int_{\mathbb{R}} |y| g_n(y) \, dy \leq |x| + 1
\] (2.11)
for all \(n \geq 1\). Then by (2.13) in (22),
\[
\sup_{m \geq 1} \left| \int_{\mathbb{R}} \phi_n(v_{0,s}^m(x)) J(x) \, dx \right| + \int_{\mathbb{R}} \phi_n''(v_{0,s}^m(x)) \langle |Y_{0,s}|, Y_{0,s} \rangle J(x) \, dx < \infty, \quad \text{on} \quad \{s \leq \tau_{0,k}\}.
\] (2.12)
By (24) Lemma 4.5), one can check that (2.9)–(2.10) and (2.12) also hold for \(i = 1\). Combining this with (2.7), (2.8) and dominated convergence we get
\[
\lim_{m \rightarrow \infty} E \left\{ \int_{\mathbb{R}} \phi_n(v_{i,t \wedge \tau_{i,k}}^m(x)) J(x) \, dx \right\} = E \left\{ \int_{\mathbb{R}} \phi_n(Y_{i,t \wedge \tau_{i,k}}(x)) J(x) \, dx \right\},
\]
and
\[ \lim_{m \to \infty} I_0(i, m, n) = (\phi_n(Y_{i,0}), J) = J_0(i, n), \]
\[ \limsup_{m \to \infty} I_1(i, m, n, k, t) \leq C E \left\{ \int_0^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi_n(Y_{i,s}(x)) J(x) dx \right\} =: J_1(i, n, k, t), \]
\[ \limsup_{m \to \infty} I_2(i, m, n, k, t) \leq C_k E \left\{ \int_0^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \phi''_n(Y_{i,s}(x)) |Y_{i,s}(x)| J(x) dx \right\} =: J_2(i, n, k, t) \]
for both \( i = 0, 1. \)

Now taking \( m \to \infty \) in \( (2.7) \) we have
\[ E \left\{ \int_{\mathbb{R}} \phi_n(Y_{i,t \wedge \tau_{i,k}}(x)) J(x) dx \right\} \leq J_0(i, n) + J_1(i, n, k, t) + J_2(i, n, k, t). \tag{2.13} \]

**Step 4.** Since \( 0 \leq |\phi''_n(x)| \leq 1 \) for all \( x \in \mathbb{R}, n \geq 1 \) and \( \lim_{n \to \infty} |\phi''_n(x)| = 0 \) for each \( x \in \mathbb{R} \), by dominated convergence we have
\[ \lim_{n \to \infty} J_2(i, n, k, t) \leq C_k E \left\{ \int_0^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} \lim_{n \to \infty} \phi''_n(Y_{i,s}(x)) |Y_{i,s}(x)| J(x) dx \right\} = 0. \tag{2.14} \]

Using dominated convergence and \( (2.11) \) we know that
\[ \lim_{n \to \infty} J_0(i, n) = \int_{\mathbb{R}} (Y_{i,0}(x))^+ J(x) dx = 0, \]
\[ \lim_{n \to \infty} J_1(i, n, k) = C E \left\{ \int_0^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} (Y_{i,s}(x))^+ J(x) dx \right\}, \]
\[ \lim_{n \to \infty} E \left\{ \int_{\mathbb{R}} \phi_n(Y_{i,t \wedge \tau_{i,k}}(x)) J(x) dx \right\} = E \left\{ \int_{\mathbb{R}} (Y_{i,t \wedge \tau_{i,k}}(x))^+ J(x) dx \right\}. \]

Combining with \( (2.13)-(2.14) \) we get
\[ E \left\{ 1_{\{t \leq \tau_{i,k}\}} \int_{\mathbb{R}} (Y_{i,t}(x))^+ J(x) dx \right\} \leq E \left\{ \int_{\mathbb{R}} (Y_{i,t \wedge \tau_{i,k}}(x))^+ J(x) dx \right\} \leq C E \left\{ \int_0^{t \wedge \tau_{i,k}} ds \int_{\mathbb{R}} (Y_{i,s}(x))^+ J(x) dx \right\} = C E \left\{ \int_0^t \left[ 1_{\{s \leq \tau_{i,k}\}} \int_{\mathbb{R}} (Y_{i,s}(x))^+ J(x) dx \right] ds \right\}. \]

Then by Gronwall’s lemma,
\[ E \left\{ 1_{\{t \leq \tau_{i,k}\}} \int_{\mathbb{R}} (Y_{i,t}(x))^+ J(x) dx \right\} = 0, \]
which implies that
\[ 1_{\{t \leq \tau_{i,k}\}} \int_{\mathbb{R}} (Y_{i,t}(x))^+ J(x) dx = 0, \quad \text{P-a.s.} \]

Letting \( k \to \infty \) we see that \( \int_{\mathbb{R}} (Y_{i,t}(x))^+ J(x) dx = 0 \) P-a.s. by \( (2.5) \). Then the desired result follows. \( \square \)
Lemma 2.2 Suppose that conditions (1.4)–(1.6) hold and \( Y = \{Y_t : t \geq 0\} \) is a solution of (1.2). It then follows from Proposition 2.1(i) that similarly, by dominated convergence and which implies

Without loss of generality we assume that \( Y \) satisfies

\[ \sup_{t \in [0, T]} E\{|Y_t^2, e^{-|t|}|\} < \infty \] for all \( T > 0 \). If \( Y_0 \in D(\mathbb{R}) \), then \( Y \) has a \( C([0, \infty) \times \mathbb{R}) \)-valued modification \( \tilde{Y} = \{\tilde{Y}_t : t \geq 0\} \) with \( \tilde{Y}_0 \in D(\mathbb{R}) \) (i.e. \( P\{(|Y_t - \tilde{Y}_t|, 1) = 0\} = 1 \) for all \( t \geq 0 \)). Moreover, \( P\{\tilde{Y}_t \in D_c(\mathbb{R}) \text{ for all } t > 0\} = 1 \) and \( \sup_{t \in [0, T]} \tilde{Y}_t(\infty) < \infty \), \( P \)-a.s.

Proof. The proof is proceeded in four steps.

1. **Step 1.** By the assertion and proof of [22], Theorem 1.2], the solution \( Y \) of (1.2) has a \( C([0, \infty) \times \mathbb{R}) \)-valued modification \( \tilde{Y} = \{\tilde{Y}_t : t \geq 0\} \). Observe that \( \tilde{Y}_t \equiv 0 \) is also a solution of (1.2). It then follows from Proposition 2.1(i) that \( P\{\tilde{Y}_t(\infty) = 0\} = 1 \) for all \( t \geq 0 \) and \( x \in \mathbb{R} \).

Fixed \( y_1 \geq y_2 \), let \( Y_{t,1}(x) := Y_t(x + y_1) \) and \( Y_{t,2}(x) := Y_t(x + y_2) \). It follows from Proposition 2.1(i) again that \( Y_{1,0}(x) \geq Y_{2,0}(x) \) for all \( x \in \mathbb{R} \) and

\[ P\{Y_t(x + y_1) \geq Y_t(x + y_2) \text{ for } \lambda \text{-a.e. } x\} = 1, \]

which implies

\[ P\{\tilde{Y}_t(x) \geq \tilde{Y}_t(y) \text{ for all } x \geq y \in \mathbb{R}\} = 1. \]

2. **Step 2.** It follows from (2.12) in [22] that, for each \( |f| \leq CJ \), we have for any \( t > 0 \),

\[ \langle Y_t, f \rangle = \langle Y_0, P_t f \rangle + \int_0^t \int_E \left[ \int_{\mathbb{R}} G(u, Y_s(z)) P_{t-s} f(z) dz \right] W(ds, du) \text{ P-a.s.} \]

Without loss of generality we assume that \( Y_0 \) is deterministic in the following. Thus \( E\{\langle Y_t, J_n \rangle\} = \langle Y_0, P_t J_n \rangle \) for \( J_n(x) := J(x - n) \). By monotone convergence and \( (1, J) = 2 \) we have

\[ \lim_{n \to \infty} E\{\langle Y_t, J_n \rangle\} = \lim_{n \to \infty} E\left\{ \int_{\mathbb{R}} \tilde{Y}_t(x + n) J(x) dx \right\} = \langle 1, J \rangle E\{\tilde{Y}_t(\infty)\} = 2E\{\tilde{Y}_t(\infty)\} \]

and

\[ \lim_{n \to \infty} \langle Y_0, P_t J_n \rangle = \lim_{n \to \infty} \int_{\mathbb{R}} Y_0(z + n) dz \int_{\mathbb{R}} p_t(y) J(z - y) dy = 2Y_0(\infty), \]

which implies

\[ E\{\tilde{Y}_t(\infty)\} = Y_0(\infty). \quad (2.15) \]

Similarly, by dominated convergence and \( (1, J_n) = 2 \) we also get

\[ 2E\{\tilde{Y}_t(-\infty)\} = E\left[ \lim_{n \to -\infty} \langle Y_t, J_n \rangle \right] = \lim_{n \to -\infty} E\{\langle Y_t, J_n \rangle\} = \lim_{n \to -\infty} \langle Y_0, P_t J_n \rangle = 0. \quad (2.16) \]

3. **Step 3.** In this step we show that \( \sup_{t \in [0, T]} \tilde{Y}_t(\infty) < \infty \) for each \( T > 0 \). By (1.3) and the Burkholder-Davis-Gundy inequality,

\[ E\left\{ \sup_{t \in [0, T]} \langle Y_t, J_n \rangle \right\} \leq \langle Y_0, J_n \rangle + \frac{1}{2} \int_0^T E\{\langle Y_s, |J'_n| \rangle\} ds + CE\left\{ \int_0^T ds \int_E \langle G(u, Y_s), J_n \rangle^2 \pi(du) \right\}^{\frac{1}{2}}. \quad (2.17) \]
Under conditions (1.5)–(1.6), for each \( y \geq 0 \),
\[
\int_E G(u, y)^2 \pi(du) = \int_E |G(u, y) - G(u, 0)|^2 \pi(du) \leq C|y|.
\]

It then follows from the Hölder inequality and \( \langle 1, J_n \rangle = 2 \)
that
\[
\int_E \langle G(u, Y_s), J_n \rangle^2 \pi(du) \leq 2 \int_E \langle G(u, Y_s) \rangle^2, J_n \rangle \pi(du)
= 2 \int J_n(x) dx \int_E \langle G(u, Y_s(x)) \rangle^2 \pi(du) \leq C \int \langle Y_s(x) \rangle J_n(x) dx.
\]

Putting together (2.17) and (2.18) we have
\[
\mathbb{E}\left\{ \sup_{t \in [0, T]} \langle Y_t, J_n \rangle \right\} \leq \int \mathbb{E}\{Y_0(x + n) J(x)dx + C \int_0^T dt \int \mathbb{E}\{Y_t(x + n)\} J(x) dx \right\}
+ C \int_0^T dt \int \mathbb{E}\{Y_t(x + n)\} J(x) dx \right\}^{\frac{1}{2}}.
\]

(2.18)

It then follows from (2.15) and (2.18) that for each \( T > 0 \),
\[
\sup_{n \geq 1} \mathbb{E}\left\{ \sup_{t \in [0, T]} \langle Y_t, J_n \rangle \right\} \leq 2Y_0(\infty) + CTY_0(\infty) + C[TY_0(\infty)]^{\frac{1}{2}}.
\]

Thus, by monotone convergence
\[
\langle 1, J \rangle \mathbb{E}\left\{ \sup_{t \in [0, T]} \tilde{Y}_t(\infty) \right\} = \mathbb{E}\left\{ \sup_{t \in [0, T], n \geq 1} \int \tilde{Y}_t(x + n) J(x) dx \right\}
= \mathbb{E}\left\{ \lim_{n \to \infty} \sup_{t \in [0, T]} \int \tilde{Y}_t(x + n) J(x) dx \right\} = \lim_{n \to \infty} \mathbb{E}\left\{ \sup_{t \in [0, T]} \langle Y_t, J_n \rangle \right\},
\]

(2.19)

which implies that \( \sup_{t \in [0, T]} \tilde{Y}_t(\infty) < \infty, \mathbb{P}\)-a.s.

**Step 4.** By (2.16), (2.18) and dominated convergence
\[
\langle 1, J \rangle \mathbb{E}\left\{ \sup_{t \in [0, T]} \tilde{Y}_t(-\infty) \right\} \leq \mathbb{E}\left\{ \sup_{t \in [0, T]} \int \tilde{Y}_t(-\infty) J(x) dx \right\}
\leq \mathbb{E}\left\{ \lim_{n \to \infty} \sup_{t \in [0, T]} \int \tilde{Y}_t(x + n) J(x) dx \right\} \leq \lim_{n \to \infty} \mathbb{E}\left\{ \sup_{t \in [0, T]} \langle \tilde{Y}_t, J_n \rangle \right\} = 0,
\]

which implies \( \sup_{t \in [0, T]} \tilde{Y}_t(-\infty) = 0, \mathbb{P}\)-a.s. Therefore, \( \mathbb{P}\{\tilde{Y}_t \in D(\mathbb{R})\} = 1 \) for each \( t > 0 \), which finishes the proof. \( \square \)

Now we are ready to present the main theorem.

**Theorem 2.3** Suppose that conditions (1.6)–(1.8) hold. Then for any \( Y_0 \in D(\mathbb{R}) \), SPDE (1.2) has a pathwise unique continuous \( D_c(\mathbb{R}) \)-valued solution \( \{Y_t : t > 0\} \).

**Proof.** The pathwise uniqueness follows immediately from Proposition 2.1(ii). We prove the existence of a \( D_c(\mathbb{R}) \)-valued solution to (1.2) in the following. We assume that \( Y_0 \) is deterministic.
By [22, Theorem 1.2] and Lemma 2.2, the following SPDE has a unique strong continuous \(D_c(\mathbb{R})\)-valued solution:

\[
Y_t(y) = Y_0(y) + \frac{1}{2} \int_0^t \Delta Y_s(y) ds + \int_0^t \int_E G_m(u, Y_s(y)) W(ds, du), \tag{2.20}
\]

where \(G_m(u, y) := G(u, y \wedge m)\) for \(m \geq 1\). For each \(m \geq 1\), let \(\{Y^m_t : t \geq 0\}\) be a strong continuous \(D_c(\mathbb{R})\)-valued solution to (2.20) with \(Y^m_0 := Y_0\). For each \(m, n \geq 1\), define stopping time \(\tau^m_n := \inf\{t \geq 0 : Y^m_t(\infty) \geq n\}\). It then follows from the uniqueness of strong solution to (2.20) that \(\mathbb{P}\text{-a.s.}\)

\[
Y^m_{t \wedge \tau^m_n \wedge \tau^m_{n+1}} = Y^{m+1}_{t \wedge \tau^m_n \wedge \tau^m_{n+1}}, \quad t \geq 0.
\]

If \(\tau^m_m \leq \tau^m_{m+1}\), then \(Y^{m+1}_m(\infty) = Y^m_m(\infty) = m\), which, by the definition of \(\tau^m_{m+1}\), implies that \(\tau^m_m \geq \tau^m_{m+1}\). Therefore, \(\tau^m_n = \tau^m_{m+1}\) and \(Y^m_n = Y^{m+1}_{m+1}\). This implies that \(\tau^m_m\) is increasing in \(m\). Observe that for \(J_n(x) = J(x - n)\) we have \(\langle Y^m_t, J_n \rangle = \int_{\mathbb{R}} Y^m_t(y + n) J(y)dy\). It then follows from (2.20) that

\[
\int_{\mathbb{R}} Y^m_t(y + n) J(y)dy = \int_{\mathbb{R}} Y_0(y + n) J(y)dy + \frac{1}{2} \int_0^t \int_{\mathbb{R}} Y^m_s(y + n) J''(y)dy ds + \int_0^t \int_E \left[ \int_{\mathbb{R}} G_m(u, Y^m_s(y + n)) J(y)dy \right] W(ds, du).
\]

Letting \(n \to \infty\), we have

\[
Y^m_m(\infty) = Y_0(\infty) + \int_0^t \int_E G_m(u, Y^m_m(\infty)) W(ds, du),
\]

which implies \(\mathbb{E}\{Y^m_{m \wedge M}(\infty)\} = Y_0(\infty)\). Thus

\[
m\mathbb{P}\{\tau^m_m < t\} = \mathbb{E}\{Y^m_{m \wedge M}(\infty)1_{\{\tau^m_m < t\}}\} \leq \mathbb{E}\{Y^m_{m \wedge M}(\infty)\} = Y_0(\infty).
\]

Letting \(t \to \infty\), we have

\[
\mathbb{P}\{\tau^m_m < \infty\} \leq Y_0(\infty)/m \to 0
\]
as \(m \to \infty\). So, we must have \(\tau^m_m \uparrow \infty\). Therefore, we can define the solution to (1.2) by \(Y^m_t = Y^m_{t \wedge \tau^m_m}\) for \(t \leq \tau^m_m\). This completes the proof. \(\Box\)

### 3 Interacting super-Brownian motions

In the early work of Méléard and Roelly [11] a measure-valued branching process with mean field interaction was introduced. Using particle system approximation, it was shown that if \(\tilde{\sigma} \in C(M(\mathbb{R}) \times \mathbb{R})^+\), then the continuous \(M(\mathbb{R})\)-valued solution \(\{X_t : t \geq 0\}\) to the following martingale problem exists: for any \(f \in C^2(\mathbb{R})\),

\[
M^f_t := X_t(f) - X_0(f) - \frac{1}{2} \int_0^t X_s(f'')ds, \quad t \geq 0 \tag{3.1}
\]
is a square-integrable continuous martingale with quadratic variation process
\[ \langle M^f \rangle_t = \int_0^t \langle X_s, \hat{\sigma}(X_s)f^2 \rangle ds. \]  
(3.2)

However, owing to the interaction, the fundamental multiplicative property of the measure-valued branching process disappears and one cannot associate the interacting measure-valued branching processes with a cumulant semigroup, which can be characterized as the unique solution of a non-linear partial differential equation. The uniqueness of solution to the above martingale problem is still unknown. Using snake representation and random time change techniques, Delmas and Dhersin [4] proved the existence of solution to a similar martingale problem for interacting super-Brownian motion. But again, the uniqueness of solution is left open in general.

In this section, we consider an interacting super-Brownian motion \( \{X_t : t \geq 0\} \), which is a continuous \( M(\mathbb{R}) \)-valued process solving the following local martingale problem: for any \( f \in C^2(\mathbb{R}) \),
\[ M_t(f) := X_t(f) - X_0(f) - \frac{1}{2} \int_0^t X_s(f''(x))ds, \quad t \geq 0 \]  
(3.3)
is a continuous local martingale with quadratic variation process
\[ \langle M(f) \rangle_t = \int_0^t ds \int_{\mathbb{R}} \sigma(X_s, x)f(x)^2X_s(dx) := \int_0^t ds \int_{\mathbb{R}} \sigma(X_s((-\infty, x]))f(x)^2X_s(dx), \]  
(3.4)
where the branching rate function \( \sigma \in \mathcal{B}(\mathbb{R}_+)^+ \) is bounded on \([0, n]\) for each \( n \geq 1 \) and
\[ \lambda\{u \in \mathbb{R}_+ : \sigma(u) = 0\} = 0. \]

Note that the local martingale problem (3.3)–(3.4) is not covered by that of (3.1)–(3.2). For constant \( \sigma \), super-Brownian motion with branching rate \( \sigma \) is the unique solution to the above-mentioned martingale problem.

We will establish both the existence and uniqueness of solution to martingale problem (3.3)–(3.4) by associating its solution with an SPDE of type (1.2). Then properties of the interacting super-Brownian motion will be investigated.

### 3.1 Well-posedness of the martingale problem

The next result is on the connection between the local martingale problem and the corresponding distribution-function-valued SPDE. We say that a \( D(\mathbb{R}) \)-valued process \( \{Y_t : t \geq 0\} \) is a distribution-function process of a \( M(\mathbb{R}) \)-valued process \( \{X_t : t \geq 0\} \) if \( Y_t(x) = X_t((-\infty, x]) \) for all \( t \geq 0 \) and \( x \in \mathbb{R} \).

**Lemma 3.1** A continuous \( D(\mathbb{R}) \)-valued process \( \{Y_t : t \geq 0\} \) is the distribution-function process of the interacting super-Brownian motion if and only if there is, on an enlarged probability space, a Gaussian white noise \( \{W(dt, du) : t \geq 0, u \geq 0\} \) with intensity \( dtdu \) so that \( \{Y_t : t \geq 0\} \) solves the following SPDE:
\[ Y_t(x) = Y_0(x) + \frac{1}{2} \int_0^t \Delta Y_s(x)ds + \int_0^t \int_0^\infty 1_{\{u \leq Y_s(x)\}} \sqrt{\sigma(u)}W(ds, du). \]  
(3.5)
Remark Choose $E = \mathbb{R}^+$, $\pi =$Lebesgue measure on $\mathbb{R}^+$ and
\[ G(u, x) = 1_{\{u \leq x\}} \sqrt{\sigma(u)}, \quad u, x \in \mathbb{R}^+. \]
Then $\{Y_t : t \geq 0\}$, the $D(\mathbb{R})$-valued solution to SPDE (1.2) under conditions (1.6)–(1.8), is in fact the distribution-function process of the measure-valued process $\{X_t : t \geq 0\}$ which solves the local martingale problem (3.3)–(3.4). Consequently, the existence of solution to the local martingale problem follows from the existence of solution to (1.2), which is different from the approaches in [4 11], and the uniqueness of solution follows from the strong uniqueness of the SPDE. The following theorem partially generalizes the results of [18 24] in which $\sigma$ is required to be bounded.

**Theorem 3.2** The local martingale problem (3.3)–(3.4) is well-posed.

**Proof.** The result follows from Theorem 2.3 and Lemma 3.1 immediately. The uniqueness of solution to the local martingale problem can alternatively be shown by [18 Theorem 1.2(b)] and a standard localization argument. \hfill \Box

**Proof of Lemma 3.1.** The proof is a modification of those of [22 Theorem 1.3] and [7 Theorem 3.1]. Suppose that $\{Y_t : t \geq 0\}$ is a continuous $D(\mathbb{R})$-valued solution of (3.3). By integration by parts, $X_t(f) = -\langle Y_t, f' \rangle$ for each $f \in C^0_0(\mathbb{R})$. It then follows from (3.5) that
\[
X_t(f) = -\langle Y_0, f' \rangle - \frac{1}{2} \int_0^t \langle Y_s, f'' \rangle ds - \int_0^t \int_0^\infty \left[ \int_\mathbb{R} f'(x) 1_{\{u \leq Y_s(x)\}} dx \right] \sqrt{\sigma(u)} W(ds, du)
\]
\[
= X_0(f) + \frac{1}{2} \int_0^t X_s(f'') ds - \int_0^t \int_0^\infty \left[ \int_\mathbb{R} f'(x) 1_{\{Y_s^{-1}(u) \leq x\}} dx \right] \sqrt{\sigma(u)} W(ds, du)
\]
\[
= X_0(f) + \frac{1}{2} \int_0^t X_s(f'') ds + \tilde{I}_t(f),
\]
where
\[
Y_t^{-1}(u) := \inf\{x \in \mathbb{R} : Y_t(x) \geq u\} \quad \text{ (3.6)}
\]
with the convention $\inf \emptyset = \infty$ and
\[
\tilde{I}_t(f) := \int_0^t \int_0^\infty f(Y_s^{-1}(u)) \sqrt{\sigma(u)} W(ds, du).
\]
One can see that $\{\tilde{I}_t(f) : t \geq 0\}$ is a continuous local martingale with quadratic variation process
\[
\langle \tilde{I}(f) \rangle_t = \int_0^t \int_0^\infty \sigma(u) f(Y_s^{-1}(u))^2 du = \int_0^t ds \int_\mathbb{R} \sigma(X_s(-\infty, x)) f(x)^2 X_s(dx).
\]
By an approximation argument, one can see the above relation remains true for all $f \in C^2(\mathbb{R})$. Therefore, $\{X_t : t \geq 0\}$ is an interacting super-Brownian motion.

Suppose that $\{X_t : t \geq 0\}$ is an interacting super-Brownian motion determined by the local martingale problem (3.3)–(3.4). We will show that the distribution-function process $\{Y_t : t \geq 0\}$ of $\{X_t : t \geq 0\}$ solves (3.5). We assume that $X_0$ is deterministic in the following.

Observe that
\[
X_t(1) = X_0(1) + M_t(1),
\]
where
\[
\tilde{I}_t = \int_0^t \int_0^\infty \frac{1}{2} \sigma(u) f(Y_s^{-1}(u))^2 du + \int_0^t ds \int_\mathbb{R} \sigma(X_s(-\infty, x)) f(x)^2 X_s(dx).
\]
where \{M_t(1) : t \geq 0\} is a continuous local martingale and \{M_{t\wedge \tau_n}(1) : t \geq 0\} is a continuous martingale for each \( n \geq 1 \) with stopping time \( \tau_n \) defined by \( \tau_n := \inf \{ t \geq 0 : X_t(1) \geq n \} \). It then follows that \( \mathbf{E}\{X_{t\wedge \tau_n}(1)\} = X_0(1) \). Since \( t \mapsto X_t(1) \) is continuous, \( \tau_n \to \infty \) as \( n \to \infty \). Thus by Fatou’s lemma,

\[
\mathbf{E}\{X_t(1)\} = \mathbf{E}\left\{ \liminf_{n \to \infty} X_{t\wedge \tau_n}(1) \right\} \leq \liminf_{n \to \infty} \mathbf{E}\{X_{t\wedge \tau_n}(1)\} = X_0(1). \tag{3.7}
\]

We can complete the proof in four steps.

**Step 1.** We say \( \{\tilde{M}_t(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\} \) is a continuous local martingale measure if there are stopping times \( \tilde{\sigma}_n \) with \( \tilde{\sigma}_n \to \infty \) \( \mathbf{P} \)-a.s. so that for each \( n \geq 1 \), \( \{M_{t\wedge \tilde{\sigma}_n}(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\} \) is a continuous martingale measure. That is for each \( B \in \mathcal{B}(\mathbb{R}) \), \( \tilde{M}_{t\wedge \tilde{\sigma}_n}(B) : t \geq 0 \) is a square-integrable continuous martingale and for every disjoint sequence \( \{B_1, B_2, \cdots \} \in \mathcal{B}(\mathbb{R}) \) we have

\[
\tilde{M}_{t\wedge \tilde{\sigma}_n}\left( \bigcup_{k=1}^{\infty} B_k \right) = \sum_{k=1}^{\infty} \tilde{M}_{t\wedge \tilde{\sigma}_n}(B_k)
\]

by the convergence in \( L^2(\Omega, \mathbf{P}) \). For any \( n \geq 1 \) and \( f \in B(\mathbb{R}) \), let \( M^n_t(f) = M_{t\wedge \tilde{\sigma}_n}(f) \) which determines a continuous martingale measure \( \{M^n_t(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\} \) by the same argument as in the proof of [10, Theorem 7.25]. Then by the same argument as in [14, Lemma 5.6] we have \( M^n_t = M^{n+1}_t \) for \( n \geq 1 \). Thus we can define the continuous local martingale measure \( \{M_t(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\} \) by \( M_t = M^n_t \) for \( t \leq \tau_n \).

**Step 2.** Let \( B(\mathbb{R}_+ \times \mathbb{R} \times \Omega) \) be the space of progressively measurable functions on \( \mathbb{R}_+ \times \mathbb{R} \times \Omega \) and

\[
\mathcal{F} = \left\{ F \in B(\mathbb{R}_+ \times \mathbb{R} \times \Omega) : \int_0^t ds \int_{\mathbb{R}} \sigma(X_s(-\infty, x)) F(s, x) X_s(dx) < \infty, \ t > 0, \ \mathbf{P} \text{-a.s.} \right\},
\]

\[
\mathcal{F}_b = \left\{ F \in B(\mathbb{R}_+ \times \mathbb{R} \times \Omega) : \mathbf{E}\left\{ \int_0^t ds \int_{\mathbb{R}} \sigma(X_s(-\infty, x)) F(s, x) X_s(dx) \right\} < \infty, \ \forall t > 0 \right\}.
\]

In this step we show that the stochastic integral of \( F \in \mathcal{F} \) with respect to \( \{M(ds, dx) : s \geq 0, x \in \mathbb{R}\} \) is well defined. By the same argument as in [10, pp.160–166], the stochastic integral of \( F \in \mathcal{F}_b \) with respect to \( \{M(ds, dx) : s \geq 0, x \in \mathbb{R}\} \) is well defined. For \( F \in \mathcal{F} \) define stopping times \( \tau'_n \) by

\[
\tau'_n = \inf \left\{ t \geq 0 : \int_0^t ds \int_{\mathbb{R}} \sigma(X_s(-\infty, x)) F(s, x) X_s(dx) \geq n \right\}.
\]

Then \( \tau'_n \to \infty \) \( \mathbf{P} \)-a.s. and the stochastic integral of \( F \in \mathcal{F} \) with respect to \( \{M(ds, dx) : s \geq 0, x \in \mathbb{R}\} \) is defined by

\[
\int_0^t F(s, x) M(ds, dx) = \int_0^t F(s, x) 1_{\{s \leq \tau'_n\}} M(ds, dx), \quad t \leq \tau'_n.
\]

**Step 3.** For \( g \in B(\mathbb{R}_+) \) define continuous local martingale \( t \mapsto Z_t(g) \) by

\[
Z_t(g) = \int_0^t \int_{\mathbb{R}} g(Y_s(x)) \sigma(Y_s(x))^{-\frac{1}{2}} 1_{\{\sigma(Y_s(x)) \neq 0\}} M(ds, dx), \quad (3.8)
\]

Observe that the quadratic variation process of \( t \mapsto Z_t(g) \) satisfies

\[
\langle Z_t(g) \rangle_t = \int_0^t ds \int_{\mathbb{R}} g(Y_s(x))^2 \sigma(Y_s(x))^{-1} 1_{\{\sigma(Y_s(x)) \neq 0\}} \sigma(Y_s(x)) X_s(dx).
\]
\[
\int_0^t ds \int_\mathbb{R} g(Y_s(x))^2 \sigma(Y_s(x))^{-1} 1_{\{\sigma(Y_s(x)) \neq 0\}} dY_s(x) \\
= \int_0^t ds \int_\mathbb{R} g(u)^2 1_{\{\sigma(u) \neq 0, u \leq Y_s(\infty)\}} du = \int_0^t ds \int_\mathbb{R} g(u)^2 1_{\{u \leq Y_s(\infty)\}} du,
\]
where \(\lambda \{u \in \mathbb{R}_+: \sigma(u) = 0\} = 0\) was used in the last equation. Combining this with (3.11) one sees that \(\{Z_t(g) : t \geq 0\}\) is a martingale for each \(g \in B(\mathbb{R}_+)\). Then the family \(\{Z_t(g) : t \geq 0, g \in B(\mathbb{R}_+)\}\) determines a martingale measure \(\{Z(dt, dx) : t \geq 0, x \geq 0\}\). Then by [5] Theorem III-6, on an extended probability space, there is a Gaussian white noise \(\{W(ds, du) : s \geq 0, u > 0\}\) so that

\[
Z_t(g) = \int_0^t \int_0^\infty g(u) 1_{\{u \leq Y_s(\infty)\}} W(ds, du).
\]

Then we can extend the definition of the stochastic integral of

\[
g \in \mathcal{F}_0 := \left\{ g \in B(\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega) : \int_0^t ds \int_\mathbb{R} g(s, u)^2 1_{\{u \leq Y_s(\infty)\}} du < \infty \text{ P-a.s., } \forall t \geq 0 \right\}
\]

with respect to the martingale measure \(\{Z(dt, dx) : t \geq 0, x \geq 0\}\) and

\[
\int_0^t \int_0^\infty g(s, x) Z(ds, dx) = \int_0^t \int_0^\infty g(s, u) 1_{\{u \leq Y_s(\infty)\}} W(ds, du). \tag{3.9}
\]

It follows from Step 2 and (3.8) that for each \(g \in \mathcal{F}_0\),

\[
\int_0^t \int_0^\infty g(s, u) Z(dt, du) = \int_0^t \int_\mathbb{R} g(s, Y_s(x)) \sigma(Y_s(x))^{-\frac{1}{2}} 1_{\{\sigma(Y_s(x)) \neq 0\}} M(ds, dx).
\]

Combining this with (3.9) we know that

\[
\int_0^t \int_\mathbb{R} g(s, Y_s(x)) \sigma(Y_s(x))^{-\frac{1}{2}} 1_{\{\sigma(Y_s(x)) \neq 0\}} M(ds, dx) \\
= \int_0^t \int_0^\infty g(s, u) 1_{\{u \leq Y_s(\infty)\}} W(ds, du), \quad g \in \mathcal{F}_0. \tag{3.10}
\]

**Step 4.** Observe that for each \(x \in \mathbb{R}\),

\[
\int_0^t \int_\mathbb{R} 1_{\{y \leq x\}} M(ds, dy) \\
= \int_0^t \int_\mathbb{R} 1_{\{Y_s(y) \leq Y_s(x)\}} M(ds, dy) - \int_0^t \int_\mathbb{R} 1_{\{Y_s(y) \leq Y_s(x), y > x\}} M(ds, dy) \\
= \int_0^t \int_\mathbb{R} \left[ 1_{\{Y_s(y) \leq Y_s(x)\}} \sigma(Y_s(y))^{\frac{1}{2}} \sigma(Y_s(y))^{-\frac{1}{2}} 1_{\{\sigma(Y_s(y)) \neq 0\}} \right] M(ds, dy) \\
+ \int_0^t \int_\mathbb{R} 1_{\{Y_s(y) \leq Y_s(x), \sigma(Y_s(y)) = 0\}} M(ds, dy) - \int_0^t \int_\mathbb{R} 1_{\{Y_s(y) \leq Y_s(x), y > x\}} M(ds, dy) \\
=: M_1(t, x) + M_2(t, x) - M_3(t, x). \tag{3.11}
\]

Letting \(g(s, u) = 1_{\{u \leq Y_s(x)\}} \sigma(u)^{\frac{1}{2}}\) in (3.10) we obtain

\[
\tilde{M}_1(t, x) = \int_0^t \int_0^\infty 1_{\{u \leq Y_s(x)\}} \sigma(u)^{\frac{1}{2}} W(ds, du). \tag{3.12}
\]
Observe that
\[ \mathbb{E}\{\tilde{M}_2(t, x)^2\} = \mathbb{E}\left\{ \int_0^t ds \int_{\mathbb{R}} 1_{\{Y_s(y) \leq Y_s(x), \sigma(Y_s(y)) = 0\}} \sigma(Y_s(y)) dy \right\} = 0, \]
which implies that
\[ \tilde{M}_2(t, x) = 0, \quad \mathbb{P}\text{-a.s.} \tag{3.13} \]

Observe that the quadratic variation process \( \langle \tilde{M}_3(x) \rangle_t \) of \( t \to \tilde{M}_3(t, x) \) satisfies
\[
\langle \tilde{M}_3(x) \rangle_t = \int_0^t ds \int_{\mathbb{R}} \sigma(X_s(-\infty, y]))1_{\{Y_s(y) \leq Y_s(x), y > x\}} X_s(dx)
= \int_0^t ds \int_{\mathbb{R}} \sigma(Y_s(y))1_{\{Y_s(y) \leq Y_s(x), y > x\}} dY_s(y) = 0,
\]
which implies \( \tilde{M}_3(t, x) = 0 \) \( \mathbb{P}\)-a.s. Combining (3.11)–(3.13) one has
\[
\int_0^t \int_{\mathbb{R}} 1_{\{y \leq x\}} M(ds, dy) = \int_0^t \int_{\mathbb{R}} 1_{\{u \leq Y_s(x)\}} \sigma(u)^{\frac{1}{2}} W(ds, du), \quad \mathbb{P}\text{-a.s.}
\]
Then by stochastic Fubini’s theorem, for \( f \in C^2_0(\mathbb{R}) \) with \( ||f||, 1 \) < \( 1 \) and \( \tilde{f}(y) := \int_y^\infty f(x)dx \),
\[
M_t(\tilde{f}) = \int_0^t \int_{\mathbb{R}} \tilde{f}(y)M(ds, dy) = \int_0^t \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f(x)1_{\{y \leq x\}} dx \right] M(ds, dy)
= \int_{\mathbb{R}} f(x)dx \int_0^t \int_{\mathbb{R}} 1_{\{y \leq x\}} M(ds, dy)
= \int_{\mathbb{R}} f(x)dx \int_0^t \int_{\mathbb{R}} 1_{\{y \leq Y_s(x)\}} \sigma(u)^{\frac{1}{2}} W(ds, du), \quad \mathbb{P}\text{-a.s.}
\]
By integration by parts, we have \( \mathbb{P}\)-a.s.
\[
M_t(\tilde{f}) = X_t(\tilde{f}) - X_0(\tilde{f}) - \frac{1}{2} \int_0^t X_s((\tilde{f})^\prime) ds = \langle Y_t, f \rangle - \langle Y_0, f \rangle - \frac{1}{2} \int_0^t \langle Y_s, f'' \rangle ds,
\]
which yields that \( \{Y_t : t \geq 0\} \) solves (3.5). \( \square \)

3.2 Properties

In this subsection we discuss the existence of density field and long time survival-extinction behaviors of the interacting super-Brownian motion.

**Theorem 3.3** Suppose that \( \{X_t : t \geq 0\} \) is the interacting super-Brownian motion with distribution-function process \( \{Y_t : t \geq 0\} \) solving SPDE (3.5). Then for any \( f \in C^2(\mathbb{R}) \), \( \mathbb{P}\)-a.s.
\[
X_t(f) = X_0(f) + \frac{1}{2} \int_0^t X_s(f'') ds + \int_0^t \int_0^\infty f(Y_s^{-1}(u)) \sqrt{\sigma(u)} W(ds, du), \quad t \geq 0, \tag{3.14}
\]
and for any \( f \in B(\mathbb{R}) \) and \( t > 0 \),
\[
X_t(f) = X_0(P_tf) + \int_0^t \int_0^\infty P_{t-s} f(Y_s^{-1}(u)) \sqrt{\sigma(u)} W(ds, du) \quad \mathbb{P}\text{-a.s.}, \tag{3.15}
\]

15
where $P_t$ denotes the transition semigroup for Brownian motion and recall that $Y_t^{-1}(u)$ is defined by \eqref{eq:Y_def}. Moreover, for each $t > 0$ the random measure $X_t(dx)$ is absolutely continuous with respect to the Lebesgue measure and the density $X_t(x)$ satisfies

$$X_t(x) = \int_{\mathbb{R}} p_t(x - y) X_0(dy) + \int_0^t \int_0^\infty p_{t-s}(x - Y_s^{-1}(u)) \sqrt{\sigma(u)} W(ds, du) \ P\text{-a.s.}, \quad (3.16)$$

where $p_t$ is the transition density function for Brownian motion.

**Proof.** By the proof of Lemma \ref{lem:3.1} we easily get \eqref{eq:3.14}. We then finish the proof in the following two steps.

**Step 1.** Let $t > 0$ be fixed. For $n \geq 1$ define stopping time $\tau_n$ by

$$\tau_n = \inf\{s \geq 0 : X_s(1) \geq n\}.$$ 

Then $\tau_n \to \infty$ as $n \to \infty$. In this step we want to show that for each $f \in C^2(\mathbb{R})$ and $n \geq 1$,

$$X_{t\wedge \tau_n}(P_{t-(t\wedge \tau_n)}) = X_0(P_t f) + \int_0^t \int_0^\infty P_{t-s} f(Y_s^{-1}(u)) \sqrt{\sigma(u)} 1_{\{s \leq \tau_n\}} W(ds, du) \ P\text{-a.s.} \quad (3.17)$$

Consider a partition $\Delta_m := \{0 = t_0 < t_1 < \cdots < t_m = t\}$ of $[0, t]$. Let $|\Delta_m| := \max_{1 \leq i \leq m} |t_i - t_{i-1}|$. It is obvious that $\frac{dP_t f}{ds} = \frac{1}{2} \Delta P_t f$ for $s \geq 0$. It then follows from \eqref{eq:3.14} that

$$\begin{align*}
X_{t\wedge \tau_n}(P_{t-(t\wedge \tau_n)}) f &= X_0(P_t f) + \sum_{i=1}^m X_{t_i \wedge \tau_n}(P_{t-(t_i \wedge \tau_n)}) f - P_{t-(t_i \wedge \tau_n)} f \\
&\quad + \sum_{i=1}^m \left[ X_{t_i \wedge \tau_n}(P_{t-(t_i \wedge \tau_n)}) f - X_{t_{i-1} \wedge \tau_n}(P_{t-(t_i \wedge \tau_n)}) f \right] \\
&= X_0(P_t f) + \frac{1}{2} \sum_{i=1}^m \int_{t-(t_i \wedge \tau_n)}^{t_i \wedge \tau_n} X_{t_i \wedge \tau_n}(\Delta P_s f) ds + \frac{1}{2} \sum_{i=1}^m \int_{t_i-1}^{t_i} X_s(\Delta P_{t-(t_i \wedge \tau_n)}) 1_{\{s \leq \tau_n\}} ds \\
&\quad + \sum_{i=1}^m \int_{t_i-1}^{t_i} \int_0^\infty P_{t-(t_i \wedge \tau_n)} f(Y_s^{-1}(u)) \sqrt{\sigma(u)} 1_{\{s \leq \tau_n\}} W(ds, du) \\
&= X_0(P_t f) + \frac{1}{2} \int_0^t \sum_{i=1}^m I_i(s) \left[ X_s(\Delta P_{t-(t_i \wedge \tau_n)}) f - X_{t_i \wedge \tau_n}(\Delta P_{t-s}) f \right] 1_{\{s \leq \tau_n\}} ds \\
&\quad + \int_0^t \int_0^\infty \sum_{i=1}^m I_i(s) P_{t-(t_i \wedge \tau_n)} f(Y_s^{-1}(u)) \sqrt{\sigma(u)} 1_{\{s \leq \tau_n\}} W(ds, du) \\
&= X_0(P_t f) + \tilde{I}_1(t, m, n)/2 + \tilde{I}_2(t, m, n), \quad (3.18)
\end{align*}$$

where $I_i(s) := 1_{(t_{i-1}, t_i)}(s)$. Observe that by the dominated convergence and the continuities of $s \mapsto P_s f$ and $s \mapsto X_s(P_t f)$ for $t' > 0$, we have

$$\begin{align*}
\lim_{|\Delta_m| \to 0} \left| \tilde{I}_1(t, m, n) \right| &\leq \int_0^t \lim_{|\Delta_m| \to 0} \sum_{i=1}^m I_i(s) \left[ \left| X_s(\Delta P_{t-(t_i \wedge \tau_n)}) f - X_s(\Delta P_{t-s}) f \right| \\
&\quad + \left| X_s(\Delta P_{t-s}) f - X_{t_i \wedge \tau_n}(\Delta P_{t-s}) f \right| \right] 1_{\{s \leq \tau_n\}} ds \\
\end{align*}$$

and

$$\begin{align*}
\lim_{|\Delta_m| \to 0} \mathbb{E}\left\{ \left[ \tilde{I}_2(t, m, n) - \int_0^t \int_0^\infty P_{t-s} f(Y_s^{-1}(u)) \sqrt{\sigma(u)} 1_{\{s \leq \tau_n\}} W(ds, du) \right]^2 \right\}
\end{align*}$$
\[
\lim_{|\Delta m| \to 0} E\left[ \left( \int_0^t \sum_{i=1}^m I_i(s) \left[ (P_t-(t_{i-1} \wedge \tau_n))f(Y_s^{-1}(u)) - P_{t-s}f(Y_s^{-1}(u)) \right] \sqrt{\sigma(u)}1_{s \leq \tau_n}W(ds,du) \right]^2 \right]
\] = \lim_{\Delta m \to 0} \left( \int_0^t \sum_{i=1}^m I_i(s) \left[ (P_t-(t_{i-1} \wedge \tau_n))f(Y_s^{-1}(u)) - P_{t-s}f(Y_s^{-1}(u)) \right] \sqrt{\sigma(u)}1_{s \leq \tau_n}W(ds,du) \right)
\]

Thus, \((3.17)\) follows by letting \(|\Delta m| \to 0\) in \((3.18)\).

**Step 2.** Equation \((3.17)\) holds for all \(f \in B(\mathbb{R})\) by an approximation argument and then \((3.16)\) follows by letting \(n \to \infty\). We now prove the last assertion. By \((3.17)\) we have that for any \(f \in B(\mathbb{R})\),

\[
E\left\{ \langle X_{t \wedge \tau_n}, P_{t-(t \wedge \tau_n)}f \rangle \right\} = X_0(P_tf).
\]

It follows from Fatou’s lemma that for \(f \in B(\mathbb{R})^+\),

\[
E\{X_t(f)\} = \lim_{n \to \infty} E\{\langle X_{t \wedge \tau_n}, P_{t-(t \wedge \tau_n)}f \rangle \} \leq \liminf_{n \to \infty} E\{\langle X_{t \wedge \tau_n}, P_{t-(t \wedge \tau_n)}f \rangle \} = X_0(P_tf).
\]

Thus,

\[
E\left\{ \int_{\mathbb{R}} f(x)dx \int_0^t 1_{s \leq \tau_n}ds \int_0^\infty p_{t-s}(x-Y_s^{-1}(u))2\sigma(u)du \right\}
\]

\[
\leq E\left\{ \int_{\mathbb{R}} f(x)dx \int_0^t (t-s)^{-\frac{1}{2}}ds \int_0^\infty p_{t-s}(x-Y_s^{-1}(u))\sigma(u)1_{s \leq \tau_n}du \right\}
\]

\[
\leq E\left\{ \int_{\mathbb{R}} f(x)dx \int_0^t (t-s)^{-\frac{1}{2}}ds \int_0^\infty p_{t-s}(x-y)\sigma(Y_s(y))1_{s \leq \tau_n}X_s(dy) \right\}
\]

\[
\leq c_n E\left\{ \int_{\mathbb{R}} f(x)dx \int_0^t (t-s)^{-\frac{1}{2}}ds \int_0^\infty p_{t-s}(x-y)X_s(dy) \right\}
\]

\[
\leq c_n E\left\{ \int_0^t (t-s)^{-\frac{1}{2}}X_s(P_{t-s}f)ds \right\} \leq c_n \mu(P_tf) \int_0^t (t-s)^{-\frac{1}{2}}ds < \infty,
\]

where \(c_n := \sup_{x \in [0,n]} \sigma(x)\). Then by [10] Theorem 7.24, for each \(t_0 > 0\) we have

\[
\langle X_{t_0 \wedge \tau_n}, P_{t_0-(t_0 \wedge \tau_n)}f \rangle = \int_{\mathbb{R}} f(x)K_{t_0}(t_0 \wedge \tau_n, x)dx \quad \text{P-a.s.}
\]

where

\[
K_{t_0}(t, x) := \int_{\mathbb{R}} p_{t_0}(x-y)X_0(dy) + \int_0^t \int_0^\infty p_{t-s}(x-Y_s^{-1}(u))\sqrt{\sigma(u)}W(ds,du).
\]

Letting \(n \to \infty\) in \((3.19)\) we get \(X_t(f) = \int_{\mathbb{R}} f(x)K_t(t, x)dx \quad \text{P-a.s.}\), which finishes the proof.

Write \(\sigma_0(x) := \int_0^x \sigma(y)dy\) for \(x \geq 0\). Notice that \(\sigma_0(x)\) is an increasing continuous function and \(\sigma_0(x) = 0\) if and only if \(x = 0\). In the following of this subsection we always assume that \(\mu \in M(\mathbb{R})\) and \(\{X_t : t \geq 0\}\) is an interacting super-Brownian motion with \(X_0 = \mu\). Taking \(f = 1\) in \((3.14)\) we get

\[
X_t(1) = X_0(1) + \int_0^t \int_0^{X_s(1)} \sqrt{\sigma(u)}W(ds,du).
\]
Thus the total mass process \( \{X_t(1) : t \geq 0\} \) is a nonnegative continuous local martingale with quadratic variation process

\[
\langle X(1) \rangle_t = \int_0^t ds \int_0^t \sigma(y) dy = \int_0^t \sigma_0(X_s(1)) ds.
\]

(3.20)

By the uniqueness of solution to the local martingale problem, if \( \mu \) is a zero measure, then \( \mathbb{P}\{X_t(1) = 0 \text{ for all } t > 0\} = 1 \). So we assume that \( \mu(1) > 0 \) in the rest of the subsection.

For \( a \geq 0 \) let

\[
\hat{\tau}_a := \inf\{t \geq 0 : X_t(1) = a\}
\]

with the convention \( \inf \emptyset = \infty \).

**Theorem 3.4** For any finite measure \( \mu \) and constant \( a > 0 \) satisfying \( \mu(1) > a > 0 \), we have \( \mathbb{P}\{\hat{\tau}_a < \infty\} = 1 \). Further, \( \mathbb{P} \)-a.s. \( X_t(1) \to 0 \) as \( t \to \infty \).

**Proof.** Notice that for any \( \tilde{\lambda} > 0 \), process

\[
\exp\left\{ -\tilde{\lambda} X_t(1) - \frac{\tilde{\lambda}^2}{2} \int_0^t \sigma_0(X_s(1)) ds \right\}
\]
is a bounded continuous martingale. Then by optional sampling,

\[
\mathbb{E}\left\{ \exp\left[ -\tilde{\lambda} X_t(1) - \frac{\tilde{\lambda}^2}{2} \int_0^{t \wedge \hat{\tau}_a} \sigma_0(X_s(1)) ds \right] \right\}.
\]

Letting \( t \to \infty \), by dominated convergence we have

\[
\mathbb{E}\left\{ \exp\left[ -\tilde{\lambda} a - \frac{\tilde{\lambda}^2}{2} \int_0^{\hat{\tau}_a} \sigma_0(X_s(1)) ds \right] \right\}
\]

\[
= \lim_{t \to \infty} \mathbb{E}\left\{ \exp\left[ -\tilde{\lambda} X_t(1) - \frac{\tilde{\lambda}^2}{2} \int_0^{t \wedge \hat{\tau}_a} \sigma_0(X_s(1)) ds \right] \right\} = e^{-\tilde{\lambda} \mu(1)}.
\]

Now letting \( \tilde{\lambda} \to 0 \) we have

\[
\mathbb{P}\left\{ \int_0^{\hat{\tau}_a} \sigma_0(X_s(1)) ds < \infty \right\} = 1.
\]

Observe that by the assumption on function \( \sigma \) we have \( \inf_{y \geq x} \sigma_0(y) > 0 \) for any \( x > 0 \), which implies that

\[
\{\hat{\tau}_a = \infty\} \subset \left\{ \int_0^{\hat{\tau}_a} \sigma_0(X_s(1)) ds = \infty \right\}.
\]

Then we have \( \mathbb{P}\{\hat{\tau}_a < \infty\} = 1 \), which proves the first assertion.

For any \( n \geq 1 \) and small \( \varepsilon > 0 \) let

\[
A_n := \{\hat{\tau}_{\varepsilon^{n+1}} < \infty, \hat{\tau}_\varepsilon \circ \theta_{\hat{\tau}_{\varepsilon^{n+1}}} < \hat{\tau}_{\varepsilon^{n+2}} \circ \theta_{\hat{\tau}_{\varepsilon^{n+1}}} \},
\]

where \( \theta_t, t \geq 0 \), denotes the usual shift operator, that is, \( Y \equiv \theta_t(X) \) is the process such that \( Y_s = X_{t+s} \) for all \( s \geq 0 \).
Since \( \{X_t(1) : t \geq 0\} \) is a continuous local martingale, then by optional stopping we have
\[
\varepsilon^{n+1} = X_{\tilde{\tau}_{\varepsilon^{n+1}}(1)} = E[X_{\tilde{\tau}_{\varepsilon^{n+1}}(1)}|\mathcal{F}_{\tilde{\tau}_{\varepsilon^{n+1}}}]
\geq E[X_{\tilde{\tau}_{\varepsilon^{n+1}}(1)}1_{\{\tilde{\tau}_{\varepsilon^{n+1}} < \tilde{\tau}_{\varepsilon^{n+1}}\}}|\mathcal{F}_{\tilde{\tau}_{\varepsilon^{n+1}}}]
= \varepsilon P[\tilde{\tau}_{\varepsilon^{n+1}} < \tilde{\tau}_{\varepsilon^{n}}|\mathcal{F}_{\tilde{\tau}_{\varepsilon^{n+1}}}],
\]
where \( \tilde{\tau}_{\varepsilon^{n}} := \tilde{\tau}_{\varepsilon^{n+1}} \circ \theta_{\tilde{\tau}_{\varepsilon^{n+1}}} \) and \( \tilde{\tau}_{\varepsilon^{n+1}} := \tilde{\tau}_{\varepsilon^{n+2}} \circ \theta_{\tilde{\tau}_{\varepsilon^{n+1}}} \). Then we know that
\[
P\{A_n\} \leq E\{P[\tilde{\tau}_{\varepsilon^{n}} < \tilde{\tau}_{\varepsilon^{n+1}}|\mathcal{F}_{\tilde{\tau}_{\varepsilon^{n+1}}}]\} \leq \varepsilon^n.
\]
It follows from the Borel-Cantelli lemma that \( P\{A_n \) infinitely often\} = 0. Then by the first assertion, for all \( n \) large enough,
\[
\tilde{\tau}_{\varepsilon^n} < \infty, \quad \tilde{\tau}_{\varepsilon^{n+1}} < \tilde{\tau}_{\varepsilon^n} \circ \theta_{\tilde{\tau}_{\varepsilon^n}}, \quad \text{P-a.s.}
\]
Therefore, P-a.s. for \( n \) large enough \( X_t(1) < \varepsilon \) for all \( t \in [\tilde{\tau}_{\varepsilon^n}, \tilde{\tau}_0) \), where \( \tilde{\tau}_0 = \lim_{n \to \infty} \tilde{\tau}_{\varepsilon^n} \). If \( \tilde{\tau}_0 < \infty \), then \( X_t(1) = 0 \) for all \( t \geq \tilde{\tau}_0 \); otherwise, \( X_t(1) \to 0 \) as \( t \to \infty \) but \( X_t(1) > 0 \) for all \( t \). Putting them together, we have \( X_t(1) \to 0 \).

The extinction behavior of \( \{X_t : t \geq 0\} \) depends on the branching rate when the total mass is close to 0. So, it depends on how fast \( \sigma_0(x) \) converges to 0 when \( x \to 0^+ \).

**Theorem 3.5** (i) If there exists a constant \( \gamma_1 \in [0, 2) \) so that \( \liminf_{x \to 0^+} \sigma_0(x)/x^{\gamma_1} > 0 \), we have \( P\{\tilde{\tau}_0 < \infty\} = 1 \) and \( X_t(1) = 0 \) for all large enough \( t \). (ii) If there exists a constant \( \gamma_2 \in [2, \infty) \) so that \( \limsup_{x \to 0^+} \sigma_0(x)/x^{\gamma_2} < \infty \), we have \( P\{\tilde{\tau}_0 = \infty\} = 1 \) and \( X_t(1) > 0 \) for all \( t > 0 \).

**Corollary 3.6** If there is a constant \( \gamma' \geq 0 \) so that \( \sigma(x) = x^{\gamma'} \) for all \( x \geq 0 \), then \( P\{\tilde{\tau}_0 < \infty\} = 1 \) for \( \gamma' < 1 \) and \( P\{\tilde{\tau}_0 = \infty\} = 1 \) for \( \gamma' \geq 1 \).

**Proof.** Since \( \sigma_0(x) = \int_0^x \sigma(y)dy = x^{\gamma'+1}/(\gamma' + 1) \), the assertion follows from Theorem 3.5 immediately.

We end this section with the proof of Theorem 3.5.

**Proof of Theorem 3.5** (i) Fix an \( \varepsilon > 0 \) small enough so that \( \sigma_0(x) \geq b_1 x^{\gamma_1} \) for all \( 0 < x \leq \sqrt{\varepsilon} \) and some \( b_1 > 0 \), let \( T' := \tilde{\tau}_{\varepsilon^{1+\delta}} \land \tilde{\tau}_{\varepsilon^{1/2}} \) for \( \delta > 0 \). By (3.20) one can see that
\[
\exp\left\{-\tilde{\lambda}X_t(1) - \frac{\tilde{\lambda}^2}{2} \langle X(1) \rangle_t\right\} = \exp\left\{-\tilde{\lambda}X_t(1) - \frac{\tilde{\lambda}^2}{2} \int_0^t \sigma_0(X_s(1))ds\right\}
\]
is a continuous local martingale. Then by optional sampling, for \( \mu(1) = \varepsilon \) we have
\[
e^{-\tilde{\lambda} x} = E\left\{\exp\left[-\tilde{\lambda}X_{T'}(1) - \frac{\tilde{\lambda}^2}{2} \int_0^{T'} \sigma_0(X_s(1))ds\right]\right\}
\leq E\left\{\exp\left[-\tilde{\lambda}^{1+\delta} - \frac{\tilde{\lambda}^2 x^{\gamma_1(1+\delta)}\tilde{\tau}_{\varepsilon^{1+\delta}}}{2}\right]1_{\{\tilde{\tau}_{\varepsilon^{1+\delta}} < \tilde{\tau}_{\varepsilon^{1/2}}\}}\right\} + P\{\tilde{\tau}_{\varepsilon^{1+\delta}} > \tilde{\tau}_{\varepsilon^{1/2}}\}. \quad (3.21)
\]
Let \( \tilde{\tau}_{3,\varepsilon} := \tilde{\tau}_{\varepsilon^{1+\delta}} \land \tilde{\tau}_{\varepsilon^{1/2}} \). Then by optional stopping again,
\[
\varepsilon = E\{X_{\tilde{\tau}_{3,\varepsilon}}(1)\} \geq E\{X_{\tilde{\tau}_{\varepsilon^{1/2}}}(1)1_{\{\tilde{\tau}_{\varepsilon^{1+\delta}} > \tilde{\tau}_{\varepsilon^{1/2}}\}}\} = \sqrt{\varepsilon} P\{\tilde{\tau}_{\varepsilon^{1+\delta}} > \tilde{\tau}_{\varepsilon^{1/2}}\},
\]

\[19\]
which implies
\[ \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} > \hat{\tau}_{\varepsilon^{1/2}}\} \leq \sqrt{\varepsilon}. \] (3.22)

Since \( 0 \leq \gamma_1 < 2 \), we can choose \( 0 < \delta < 1/2 \) so that
\[ \gamma_1(1 + \delta) < 2 - 4\delta. \] (3.23)

Then by (3.21)–(3.23), for \( \hat{\lambda} = \varepsilon^{-1+\delta} \) we have
\[
1 - 2\varepsilon^\delta 
\leq e^{-\hat{\lambda}(\varepsilon^{-1+\delta})} 
\leq \mathbb{E}\left\{ \exp \left[ -2 - b_1\varepsilon^{-2}\hat{\tau}^{1+\delta} \right] \right\} \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} < \hat{\tau}_{\varepsilon^{1/2}}\} 
\leq \mathbb{E}\left\{ \exp \left[ -2 - b_1\varepsilon^{-2}\hat{\tau}^{1+\delta} \right] \right\} + \sqrt{\varepsilon e^{\hat{\lambda}(\varepsilon^{-1+\delta})}} 
\leq \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} \leq \varepsilon^\delta\} + \exp \left[ -2 - b_1\varepsilon^{-2}\varepsilon^\delta \right] \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} > \varepsilon^\delta\} + \sqrt{\varepsilon e^{\hat{\lambda}(\varepsilon^{-1+\delta})}} 
= 1 - \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} > \varepsilon^\delta\} + \exp \left[ -2 - b_1\varepsilon^{-2}\varepsilon^\delta \right] \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} > \varepsilon^\delta\} + \sqrt{\varepsilon e^{\hat{\lambda}(\varepsilon^{-1+\delta})}}. \] (3.24)

Solving inequality (3.24) for \( \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} > \varepsilon^\delta\} \), then for small \( \varepsilon \),
\[ \mathbb{P}\{\hat{\tau}_{\varepsilon^{1+\delta}} > \varepsilon^\delta\} \leq \frac{2\varepsilon^\delta + \sqrt{\varepsilon e^{\hat{\lambda}(\varepsilon^{-1+\delta})}}}{1 - \exp \left[ -2 - b_1\varepsilon^{-2}\varepsilon^\delta \right]} \leq 3\varepsilon^\delta. \]

Put \( x_n := \varepsilon^{(1+\delta)n} \) for \( n \geq 1 \). Then by the Markov property of \( \{X_t(1) : t \geq 0\} \),
\[ \mathbb{P}\{\hat{\tau}_{x_{n+1}} - \hat{\tau}_x > x_n^\delta|\mathcal{F}_{\hat{\tau}_x}\} = \mathbb{P}\{\hat{\tau}_{x_{n+1}} \circ \theta_{\hat{\tau}_x} > x_n^\delta|\mathcal{F}_{\hat{\tau}_x}\} \leq 3x_n^\delta, \]
which implies that
\[ \sum_{n=0}^{\infty} \mathbb{P}\{\hat{\tau}_{x_{n+1}} - \hat{\tau}_x > x_n^\delta\} = \sum_{n=0}^{\infty} \mathbb{E}\left\{ \mathbb{P}\{\hat{\tau}_{x_{n+1}} - \hat{\tau}_x > x_n^\delta|\mathcal{F}_{\hat{\tau}_x}\} \right\} \leq \sum_{n=0}^{\infty} 3x_n^\delta < \infty. \]

Then by the Borel-Cantelli lemma we have
\[ \mathbb{P}\{\{\hat{\tau}_x < \infty\} \cap \{\hat{\tau}_{x_{n+1}} - \hat{\tau}_x > x_n^\delta\} \text{ infinitely often}\} = 0. \]

It follows from Theorem 3.3 that there are at most finitely many \( n \) so that \( \hat{\tau}_x < \infty \) and \( \hat{\tau}_{x_{n+1}} - \hat{\tau}_x > x_n^\delta \), \( \mathbb{P} \)-a.s. Since \( \mathbb{P}\{\hat{\tau}_x < \infty \text{ for all } n\} = 1 \) and \( \sum_{n=0}^{\infty} x_n^\delta < \infty \), then
\[ \hat{\tau}_0 = \lim_{n \to \infty} \hat{\tau}_x = \hat{\tau}_x + \sum_{n=0}^{\infty} [\hat{\tau}_{x_{n+1}} - \hat{\tau}_x] \leq \hat{\tau}_x + \sum_{n=0}^{\infty} x_n^\delta < \infty, \quad \mathbb{P} \)-a.s.

and we have the desired result for \( \mu \) with \( \mu(1) = \varepsilon \). Then the result for general \( \mu \) follows.

(ii) Given small \( \varepsilon, \delta > 0 \) so that \( \sigma_0(x) \leq b_2x^{\gamma_2} \) for all \( 0 < x < \varepsilon^{1-\delta} \) and some \( b_2 > 0 \), for any finite measure \( \mu \) on \( \mathbb{R} \) with \( \mu(1) = \varepsilon \) and \( T'' := \hat{\tau}^{1+\delta} \land \hat{\tau}^{1-\delta} \). By Ito’s formula we have
\[ X_{t\wedge T''}(1)^{-1} = X_0(1)^{-1} - \int_0^{t\wedge T''} X_s(1)^{-2}dX_s(1) + \int_0^{t\wedge T''} \sigma_0(X_s(1))X_s(1)^{-3}ds. \]

Then by integration by parts,
\[ X_{t\wedge T''}(1)^{-1} \exp \left\{ -\int_0^{t\wedge T''} \sigma_0(X_s(1))X_s(1)^{-2}ds \right\} \]
is a martingale. Then by optional sampling

\[ \varepsilon^{-1} = \mathbb{E}\{X_{t\wedge T''}(1)^{-1} \exp \left( - \int_0^{t\wedge T''} \sigma_0(X_r(1)) X_r(1)^{-2}dr \right) \} \]

Letting

\[ \hat{\tau}_{\varepsilon_1+\delta} := \hat{\tau}_{\varepsilon_1+\delta} \]

which implies

\[ \mathbb{E}\{ \exp \left( - b_2 \varepsilon^{(1-\delta)(\gamma_2-2)} \hat{\tau}_{\varepsilon_1+\delta} \right) \} \leq \varepsilon^\delta. \] (3.25)

Applying optional sampling we have

\[ \varepsilon = \mathbb{E}\{X_{T''}(1)\} \geq \mathbb{E}\{X_{\hat{\tau}_{\varepsilon_1+\delta}}(1) \} = \varepsilon^{1-\delta} \mathbb{P}\{\hat{\tau}_{\varepsilon_1+\delta} < \hat{\tau}_{\varepsilon_1+\delta}\}, \]

which implies \( \mathbb{P}\{\hat{\tau}_{\varepsilon_1-\delta} < \hat{\tau}_{\varepsilon_1+\delta}\} \leq \varepsilon^\delta. \) It then follows from the Markov inequality that

\[ \mathbb{P}\{T'' < 1\} \]

where we have used (3.25) for the fourth inequity.

Taking \( x_n = \varepsilon^{(1+\delta)^n} \) and repeating the above argument with \( T'' \) replaced by \( T_n := \hat{\tau}_{x_n+1} \wedge \hat{\tau}_{x_n+1} \) for \( n \geq 1 \), we have

\[ \mathbb{P}\{\hat{\tau}_{x_{n+1}} - \hat{\tau}_{x_n} < 1|\mathcal{F}_{x_n}\} \leq \mathbb{P}\{T_n \circ \theta_{x_n} < 1|\mathcal{F}_{x_n}\} \leq (1+e^{b_2})\varepsilon^\delta = (1+e^{b_2})\varepsilon^{(1+\delta)^n}. \]

It then follows that

\[ \mathbb{P}\{\hat{\tau}_{x_n} < \infty, \hat{\tau}_{x_{n+1}} - \hat{\tau}_{x_n} < 1\} \leq \mathbb{P}\{\hat{\tau}_{x_{n+1}} - \hat{\tau}_{x_n} < 1\} \]

where we have used (3.25) for the fourth inequity.
By the Borel-Cantelli lemma,
\[ P\left\{ \hat{\tau}_{x_n} < \infty \right\} \cap \left\{ \hat{\tau}_{x_{n+1}} - \hat{\tau}_{x_n} < 1 \right\} \text{ infinitely often} = 0. \]

It thus follows from Theorem 3.4 that for \( n \) large enough, \( \hat{\tau}_{x_{n+1}} - \hat{\tau}_{x_n} \geq 1 \). We can thus conclude that \( P\{\hat{\tau}_0 = \infty\} = 1 \). Then the desired result follows for any positive measure \( \mu \). The assertion on behavior of the total mass process follows immediately from the end of the proof of Theorem 3.5. □

4 Interacting Fleming-Viot processes

Let \( D_1(\mathbb{R}) \) be the subset of \( D(\mathbb{R}) \) consisting of continuous functions \( f \) with \( f(\infty) = 1 \) and \( M_1(\mathbb{R}) \) be the space of probability measures on \( \mathbb{R} \) equipped with the topology of weak convergence. Then there is an obvious one-to-one correspondence between \( D_1(\mathbb{R}) \) and \( M_1(\mathbb{R}) \) by associating a probability measure to its distribution function. We endow \( D_1(\mathbb{R}) \) with the topology induced by this correspondence from the weak convergence topology on \( M_1(\mathbb{R}) \).

In this section we study the continuous \( M_1(\mathbb{R}) \)-valued solution to the following martingale problem for interacting Fleming-Viot process \( \{X_t : t \geq 0\} \): for any \( f \in C^2(\mathbb{R}) \),
\[
N_t(f) := X_t(f) - X_0(f) - \frac{1}{2} \int_0^t X_s(f')ds, \quad t > 0
\]

is a continuous martingale with quadratic variation process
\[
\langle N(f) \rangle_t = \int_0^t ds \int_{\mathbb{R}} X_s(dx) \int_{\mathbb{R}} (f(y) - f(x))^2 \gamma_0(X_s, x, y)X_s(dy),
\]
where \( \gamma_0(\mu, x, y) := \gamma(\mu(-\infty, x], \mu(-\infty, y]) \) for \( \gamma \in B([0, 1]^2)^+ \), \( \mu \in M_1(\mathbb{R}) \) and \( x, y \in \mathbb{R} \). If \( \gamma \) is a positive constant function, then the solution \( \{X_t : t \geq 0\} \) is the so-called Fleming-Viot process with constant resampling rate \( \gamma \) and Brownian mutation on type space \( \mathbb{R} \); see [6] for details. In the following two subsections we show that the martingale problem (4.1)–(4.2) is well-posed and investigate some properties of this process.

4.1 Well-posedness of the martingale problem

The following lemma is on the connection between the martingale problem (4.1)–(4.2) and the distribution-function-valued SPDE.

**Lemma 4.1** A continuous \( D_1(\mathbb{R}) \)-valued process \( \{Y_t : t \geq 0\} \) is the distribution-function process of the interacting Fleming-Viot process if and only if there is, on an enlarged probability space, a Gaussian white noise \( \{W(ds, da, db) : s \geq 0, a, b \in [0, 1]\} \) with intensity \( dsdadb \) so that \( \{Y_t : t \geq 0\} \) solves equation
\[
Y_t(y) = Y_0(y) + \frac{1}{2} \int_0^t \Delta Y_s(y)ds + \int_0^t \int_0^1 \int_0^1 1_{\{a \leq Y_s(y) \leq b\}} \sqrt{\gamma(a, b)}W(ds, da, db).
\]

**Theorem 4.2** The martingale problem (4.1)–(4.2) is well posed.
Proof. Let $Y_0 \in D_1(\mathbb{R})$. Take $E = [0, 1]^2$, $\pi =$Lebesgue measure on $[0, 1]^2$ and 

$$G(a, b, x) = 1_{\{a \leq x \leq b\}} \sqrt{\gamma(a, b)}, \quad x, a, b \in [0, 1].$$

Then the conditions in Theorem [2.3] hold. In fact, it is elementary to check that for each $x_1, x_2 \in [0, 1]$ with $x_1 \leq x_2$,

$$\int_0^1 da \int_0^1 G(a, b, x)^2 db \leq \|\gamma\| \int_0^{x_1} da \int_{x_1}^1 db \leq \|\gamma\| x_1$$

and

$$\int_0^1 da \int_0^1 |G(a, b, x_1) - G(a, b, x_2)|^2 db \leq \|\gamma\| \int_0^1 da \int_0^1 |1_{\{a \leq x_1 \leq b\}} - 1_{\{a \leq x_2 \leq b\}}| db$$

$$\leq \|\gamma\| \int_0^{x_1} da \int_x^{x_2} db + \|\gamma\| \int_0^{x_2} da \int_{x_1}^{x_2} db \leq \|\gamma\| [x_2 - x_1].$$

This means that conditions (1.7) and (1.8) hold. It is obvious that $G(a, b, 0) = 0$ for all $a \in (0, 1]$ and $b \in [0, 1]$, which implies that condition (1.6) holds. Therefore, equation (4.3) has a pathwise unique continuous $D(\mathbb{R})$-valued solution $\{Y_t : t \geq 0\}$ with $Y_0 \in D_1(\mathbb{R})$.

Observe that the process $\{\tilde{Y}_t : t \geq 0\}$, defined by $\tilde{Y}_t(x) = 1$ for all $x \in \mathbb{R}$ and $t \geq 0$, is a solution to (1.3) with $\tilde{Y}_0 = 1$. Since $Y_0 \leq 1$, then by Proposition 2.1 ii), $Y_t \leq \tilde{Y}_t = 1$ $\mathbf{P}$-a.s., especially $Y_t(\infty) \leq 1$. Moreover, we known from (2.15) that

$$\mathbf{E}\{Y_t(\infty)\} = Y_0(\infty),$$

thus $Y_0(\infty) = 1$ implies that $Y_t(\infty) = 1$ $\mathbf{P}$-a.s. for each $t > 0$, which means that (1.3) has a unique strong continuous $D_1(\mathbb{R})$-valued solution as $Y_0 \in D_1(\mathbb{R})$. Then the assertion follows from Lemma 4.1 immediately.

**Proof of Lemma 4.2.** The proof is carried out in two steps.

**Step 1.** Suppose that $\{Y_t : t \geq 0\}$ is a solution to (1.3), and $\{X_t : t \geq 0\}$ is the corresponding measure-valued process. By integration by parts, for any $f \in C_0^3(\mathbb{R})$, we have $X_t(f) = -\langle Y_t, f' \rangle$, and so (1.3) yields

$$X_t(f) = -\langle Y_t, f' \rangle - \frac{1}{2} \int_0^t \langle Y_s, f'' \rangle ds$$

$$- \int_0^t \int_0^1 \int_0^1 \left[ \int_{\mathbb{R}} 1_{\{a \leq Y_s(y) \leq b\}} f'(y) dy \right] \sqrt{\gamma(a, b)} W(ds, da, db)$$

$$= X_0(f) + \frac{1}{2} \int_0^t X_s(f'') ds$$

$$+ \int_0^t \int_0^1 \int_0^1 \left[ \int_{\mathbb{R}} 1_{\{a \leq Y_s(y) \leq b\}} f'(y) dy \right] \sqrt{\gamma(a, b)} W(ds, da, db),$$

which implies that

$$\tilde{N}_t(f) := X_t(f) - X_0(f) - \frac{1}{2} \int_0^t X_s(f'') ds$$

$$= \int_0^t \int_0^1 \int_0^1 (f(Y^{-1}(b)) - f(Y^{-1}(a))) \sqrt{\gamma(a, b)} W(ds, da, db)$$

which is a solution to (1.3).
is a square-integrable continuous martingale with
\[ \langle \tilde{N}(f) \rangle_t = \int_0^t ds \int_{\mathbb{R}} X_s(dx) \int_{\mathbb{R}} (f(y) - f(x))^2 \gamma_0(X_s, x, y) X_s(dy), \]
where \( Y_t^{-1}(u) \) is defined by [40]. By an approximation argument, one can see the above relation remains true for any \( f \in C^2(\mathbb{R}) \). Thus, \( \{X_t : t \geq 0\} \) is an interacting Fleming-Viot process.

**Step 2.** Suppose that \( \{X_t : t \geq 0\} \) is an interacting Fleming-Viot process and \( Y_t(x) := X_t(-\infty, x] \) for \( x \in \mathbb{R} \) and \( t \geq 0 \). Let \( \mathcal{S}(\mathbb{R}) \) be the Schwartz space and \( \mathcal{S}'(\mathbb{R}) \) the space of Schwartz distributions. Let \( f \in \mathcal{S}(\mathbb{R}) \) and \( \tilde{f}(y) = \int_y^\infty f(x) dx \). Then
\[
\langle Y_t, f \rangle = X_t(\tilde{f}) = X_0(\tilde{f}) + \frac{1}{2} \int_0^t X_s(\tilde{f}'') ds + M_t(\tilde{f})
\]
where \( \{M_t(\tilde{f}) : t \geq 0\} \) is a martingale with quadratic variation process
\[
\langle M_t(\tilde{f}) \rangle_t = \int_0^t ds \int_{\mathbb{R}} X_s(dx) \int_{\mathbb{R}} (\tilde{f}(y) - \tilde{f}(x))^2 \gamma_0(X_s, x, y) X_s(dy), \quad t \geq 0.
\]
Define the \( \mathcal{S}'(\mathbb{R}) \)-valued process \( \{N_t : t \geq 0\} \) by \( N_t(f) = \tilde{I}_t(\tilde{f}) \) for any \( f \in \mathcal{S}(\mathbb{R}) \). Then \( \{N_t : t \geq 0\} \) is a square-integrable continuous \( \mathcal{S}'(\mathbb{R}) \)-valued martingale with
\[
\langle N(f) \rangle_t = \int_0^t ds \int_{\mathbb{R}} \int_{\mathbb{R}} [\tilde{f}(y) - \tilde{f}(x)]^2 \gamma_0(X_s, x, y) X_s(dy) X_s(dx)
\]
Define the \( \mathcal{S}'(\mathbb{R}) \)-valued process \( \{N_t : t \geq 0\} \) by \( N_t(f) = \tilde{I}_t(\tilde{f}) \) for any \( f \in \mathcal{S}(\mathbb{R}) \). Then \( \{N_t : t \geq 0\} \) is a square-integrable continuous \( \mathcal{S}'(\mathbb{R}) \)-valued martingale with
\[
\langle N(f) \rangle_t = \int_0^t ds \int_{\mathbb{R}} \int_{\mathbb{R}} [\tilde{f}(y) - \tilde{f}(x)]^2 \gamma_0(X_s, x, y) X_s(dy) X_s(dx)
\]
\[
= \int_0^t ds \int_0^1 da \int_0^1 [\int_{\{a \leq Y_s(y) \leq b\}} f(y) dy]^2 \gamma(a, b) db
\]
\[
= \int_0^t ds \int_0^1 da \int_0^1 da [\int_{\{a \leq Y_s(y) \leq b\}} f(y) dy]^2 \gamma(a, b) db.
\]
Let \( f \in L^2(\mathbb{R}) \) be the subset of \( \mathcal{S}(\mathbb{R}) \) consisting of functions \( f \) with \( \langle f^2, 1 \rangle < \infty \) and \( L^2([0,1]^2) \) be defined similarly. Similar to the martingale representation theorem (see [3] Theorem 3.3.6] or [5] Theorem III-6), there exist a \( L^2([0,1]^2) \)-cylindrical Brownian motion \( \{B_t : t \geq 0\} \) (possibly on an extended probability space) so that
\[
\langle N(f) \rangle_t = \int_0^t \langle \Phi(s) f, dB_s \rangle_{L^2([0,1]^2)},
\]
where \( \Phi(s) \) is linear map from \( L^2(\mathbb{R}) \) to \( L^2([0,1]^2) \) so that for \( f \in L^2(\mathbb{R}) \),
\[
(\Phi(s) f)(a, b) = \sqrt{\gamma(a, b)} \int_\mathbb{R} 1_{\{a \leq Y_s(y) \leq b\}} f(y) dy.
\]
Let \( \{h_j\} \) be a complete orthonormal system of the Hilbert space \( L^2([0,1]^2) \) and define random measure \( \{W(ds, da, db) : s \geq 0, a, b \in [0,1]\} \) as
\[
W([0,t] \times A) = \sum_{j=1}^\infty \langle 1_A, h_j \rangle B_t^{h_j}, \quad t \geq 0, A \in \mathcal{B}([0,1]^2).
\]
It is easy to show that \( \{W(ds, da, db) : s \geq 0, a, b \in [0,1]\} \) is a Gaussian white noise with intensity \( dsdada \). Furthermore,
\[
N_t(f) = \int_0^t \int_0^1 \int_0^1 [\int_{\mathbb{R}} 1_{\{a \leq Y_s(y) \leq b\}} f(y) dy] \sqrt{\gamma(a, b)} W(ds, da, db),
\]
which implies the desired result. \( \square \)
4.2 Properties

Theorem 4.3 Suppose that \( \{X_t : t \geq 0\} \) is the interacting Fleming-Viot process with distribution-function process \( \{Y_t : t \geq 0\} \) solving (4.3). Then for any \( f \in C^2(\mathbb{R}) \)
\[
X_t(f) = X_0(f) + \frac{1}{2} \int_0^t X_s(f'') ds \\
+ \int_0^t \int_0^1 \int_0^1 [f(Y_s^{-1}(b)) - f(Y_s^{-1}(a))] \sqrt{\gamma(a, b)} W(ds, da, db),
\]
and for any \( f \in B(\mathbb{R}), \ P\text{-a.s.}, \)
\[
X_t(f) = X_0(P_t f) + \int_0^t \int_0^1 \int_0^1 [P_{t-s} f(Y_s^{-1}(b)) - P_{t-s} f(Y_s^{-1}(a))] \sqrt{\gamma(a, b)} W(ds, da, db),
\]
where \( Y_s^{-1}(u) \) is defined by (3.6). Moreover, for each \( t > 0 \) the random measure \( X_t(dx) \) is absolutely continuous with respect to the Lebesgue measure and the density \( X_t(x) \) satisfies
\[
X_t(x) = \int_{\mathbb{R}} p_t(x-y) X_0(dy) + \int_0^t \int_0^1 \int_0^1 [p_{t-s}(x-Y_s^{-1}(b)) - p_{t-s}(x-Y_s^{-1}(a))] \sqrt{\gamma(a, b)} W(ds, da, db) \ 
\] \( \text{P-a.s.} \) \ (4.6)

Proof. The proof is similar to that of Theorem 3.3. By the Step 1 of the proof of Lemma 4.1, we obtain (4.4). By a similar argument as in the proof of (3.15), we obtain (4.5) immediately. By conditioning we may assume that \( X_0 \) is deterministic and \( X_0 = \mu \in M_1(\mathbb{R}) \). It follows from (4.5) that \( \mathbb{E}\{X_t(f)\} = \mu(P_t f) \) for \( f \in B(\mathbb{R}) \). Thus for \( f \in B(\mathbb{R})^+ \),
\[
\mathbb{E}\left\{ \int_{\mathbb{R}} f(x) dx \int_0^t ds \int_0^1 da \int_0^1 [p_{t-s}(x-Y_s^{-1}(b)) - p_{t-s}(x-Y_s^{-1}(a))]^2 \gamma(a, b) db \right\} \\
\leq 2(2\pi)^{-\frac{1}{2}} \|\gamma\| \mathbb{E}\left\{ \int_{\mathbb{R}} f(x) dx \int_0^t (t-s)^{\frac{1}{2}} ds \int_0^1 p_{t-s}(x-Y_s^{-1}(a)) da \right\} \\
= 2(2\pi)^{-\frac{1}{2}} \|\gamma\| \mathbb{E}\left\{ \int_{\mathbb{R}} f(x) dx \int_0^t (t-s)^{\frac{1}{2}} ds \int_{\mathbb{R}} p_{t-s}(x-y) X_s(dy) \right\} \\
\leq \|\gamma\| \int_0^t (t-s)^{-\frac{1}{2}} \mathbb{E}\{X_s(P_{t-s}f)\} ds = \|\gamma\| \mu(P_t f) \int_0^t (t-s)^{-\frac{1}{2}} ds < \infty.
\]
Now using [10] Theorem 7.24 we obtain
\[
X_t(f) = \int_{\mathbb{R}} f(x) \tilde{I}_t(x) dx,
\]
where \( \tilde{I}_t(x) \) equals to the right-hand side of (4.6). This gives the desired result. \( \square \)

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