M(atrix) theory : a pedagogical introduction *

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Abstract

I attempt to give a pedagogical introduction to the matrix model of M-theory as developed by Banks, Fischler, Shenker and Susskind (BFSS). In the first lecture, I introduce and review the relevant aspects of D-branes with the emergence of the matrix model action. The second lecture deals with the appearance of eleven-dimensional supergravity and M-theory in strongly coupled type IIA superstring theory. The third lecture combines the material of the two previous ones to arrive at the BFSS conjecture and explains the evidence presented by these authors. The emphasis is not on most recent developments but on a hopefully pedagogical presentation.

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1. INTRODUCTION

Among the multitude of dramatic developments in string duality during the last three years maybe the most striking one has been the return of eleven dimensional supergravity. The strong-coupling limit of the low-energy sector of type IIA superstring is eleven dimensional supergravity. Since eleven dimensional supergravity by itself does not seem to be a consistent quantum theory, while the full superstring theory does, the question immediately arose what is the consistent quantum theory that is the strong-coupling limit of the full type IIA superstring - not only of its low-energy sector. This theory was named M-theory, with many possible interpretations for the letter “M”. Thus the two things we know about M-theory are that it is the strong coupling limit of type IIA superstring and that its low-energy limit is eleven dimensional supergravity. Though one was lacking an intrinsic definition of M-theory in terms of its underlying degrees of freedom, its mere existence led to many powerful predictions or simplifications of superstring dualities.

A major step forward was taken by Banks, Fischler, Shenker and Susskind when they conjectured that the microscopic degrees of freedom of M-theory when described in a certain Lorentz frame are D0-branes. The Lorentz frame in question is the infinite momentum frame (IMF) which allows to interpret the nine space dimensions in which the D0-branes live as the nine transverse dimensions of an eleven dimensional space-time. The dynamics of N such D0-branes was known to be described by a $N \times N$ matrix quantum mechanics. The BFSS conjecture then is that M-theory in the IMF is equivalent to a matrix quantum mechanics of $U(N)$ matrices in the $N \to \infty$ limit, with a particular Hamiltonian that follows from reducing 9+1 dimensional $U(N)$ super Yang-Mills theory to 0+1 dimensions. This conjecture which seems quite bold in the first place passes several tests. First, BFSS have shown that it contains the Fock space of an arbitrary number of supergravitons, i.e. massless supergravity multiplets of 256 states, and that it describes the two graviton scattering correctly. Second, BFSS argue that the matrix model Hamiltonian, always in the $N \to \infty$ limit, reduces to the Hamiltonian of the eleven dimensional supermembrane in the light cone gauge, and hence describes the supermembranes that must be present in M-theory. Since then, many papers have appeared that further confirmed this conjecture and elaborated on many other issues in what has now become known as M(atrix) theory. I will not review these more recent developments in these lectures. These lectures are organised as follows: In the first lecture, I introduce and review D-branes with emphasis on those aspects that will be important

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to the M(atrix) theory. Since D0-branes play a particularly important role they will be given somewhat more attention. This first lecture is essentially a selection from Polchinski’s excellent TASI lectures [2]. In particular, it is shown why a collection of $N$ D$p$-branes is described by a U($N$) super Yang-Mills theory on the $p + 1$ dimensional brane world volume as obtained by dimensionally reducing ten dimensional super Yang-Mills theory. For D0-branes this is just quantum mechanics of nine bosonic U($N$) matrices $X^i$ and their 16 real fermionic partners. The second lecture then is based on Witten’s famous paper [3] where it is shown how eleven dimensional supergravity appears in the strong-coupling limit of low-energy type IIA superstring theory. Here the Kaluza-Klein modes of the eleven dimensional supergravity are identified with the D0-branes of the type IIA superstring. Then it is practically clear that the Kaluza-Klein modes of the eleven dimensional supergravity should be described in terms of the supersymmetric matrix quantum mechanics just mentioned. The third lecture then explains the conjecture of BFSS that this matrix quantum mechanics actually should describe the full eleven dimensional M-theory in the infinite momentum frame.

2. FIRST LECTURE : D-BRANES

I will begin by reviewing some basic aspects of D-branes with emphasis one those that will be important to the matrix model. Most of what will be said in this lecture can be found in Polchinski’s excellent TASI lectures [2].

2.1. T-duality for closed strings

For the closed string the equations of motion $\partial_z \partial_{\bar{z}} X^\mu = 0$ lead to the expansion

$$X^\mu = x^\mu - i \sqrt{\frac{\alpha'}{2}} (\alpha_0^\mu + \bar{\alpha}_0^\mu) \tau$$

$$+ \sqrt{\frac{\alpha'}{2}} (\alpha_0^\mu - \bar{\alpha}_0^\mu) \sigma$$

$$+ i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \left( \frac{\alpha^\mu_m}{m} z^m + \frac{\bar{\alpha}^\mu_m}{m} \bar{z}^m \right)$$

(1)

where $z = e^{\tau - i\sigma}$ and $\bar{z} = e^{\tau + i\sigma}$. We see that $p^\mu = \frac{1}{\sqrt{2\alpha'}} (\alpha_0^\mu + \bar{\alpha}_0^\mu)$. For any non-compact dimension, invariance of $X^\mu$ under $\sigma \rightarrow \sigma + 2\pi$ requires $\alpha_0^\mu = \bar{\alpha}_0^\mu$. However, if we compactify one dimension, say $\mu = 25$ on a circle of radius $R$, $X^{25} + 2\pi R \approx X^{25}$, and invariance under $\sigma \rightarrow \sigma + 2\pi$ only requires $\sqrt{\frac{\alpha'}{2}} (\alpha_0^\mu - \bar{\alpha}_0^\mu) = mR$ for some integer $m$. Thus winding states appear. On the other hand, $p^{25}$ must now be quantized as $n/R$. Then

$$\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}} \left( \frac{n}{R} + m \frac{R}{\alpha'} \right)$$

$$\bar{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}} \left( \frac{n}{R} - m \frac{R}{\alpha'} \right) .$$

(2)

The mass of a given string state is given by the sum $(\alpha_0^{25})^2 + (\bar{\alpha}_0^{25})^2$ plus contributions of the oscillator modes, and hence is invariant under flipping the sign of $\alpha_0^{25}$ which amounts to $n \leftrightarrow m$ and $R \leftrightarrow \frac{R'}{R}$. This is called T-duality: upon exchanging winding and momentum modes, the theory compactified on a circle of radius $R$ and the theory compactified on a circle of radius $\tilde{R} = \alpha'/R$ are equivalent. It is easy to see that this is also an invariance of the interacting theory. For this it is enough to see that the transformation

$$X^{25} \rightarrow X^{25}(z) + X^{25}(\bar{z})$$

$$X^{25} \rightarrow X^{25}(z) - X^{25}(\bar{z})$$

(3)

which changes the signs of all $\hat{\alpha}_m^{25}$ leaves the stress-energy tensor, all operator product expansions and hence all correlation functions invariant: T-duality is a symmetry of perturbative closed string theory. It is a space-time parity operation on the right-moving degrees of freedom only.

2.2. T-duality for open strings

For an open string, to obtain the equations of motion $\partial_x \partial_{\bar{x}} X^\mu = 0$ upon varying the action, one has to impose either of two types of boundary conditions. The usual choice are the Neumann (N) conditions $\partial_x X^\mu = 0$. But one could just as well impose Dirichlet (D) conditions $X^\mu = \text{const}$ at the boundaries. Let’s first consider Neumann conditions and later recover the Dirichlet conditions via T-duality. With the usual N conditions
the string field expansion is
\[ X^{\mu} = x^{\mu} - i\alpha' p^{\mu} \ln z \tau \]
\[ + i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha' \tau^m}{m} (z^{-m} + \tau^{-m}). \] (4)

If one compactifies again $X^{25}$ on a circle of radius $R$, one again has $p^{25} = \frac{1}{R}$, but no winding modes can appear. As the radius $R$ is taken to zero, only the $n = 0$ mode survives and the space-time behaviour of the open string is as if it lived in one dimension less, although the string still vibrates in all 26 dimensions (or rather in all transverse 24 ones). It is similar to what would happen if the endpoints of the open string were stuck to a hyperplane with $D - 1 = 25$ space-time dimensions.

To understand this better, one can introduce the T-dual field $\hat{X}^{25}$ by a transformation similar to eq. (3):
\[ \hat{X}^{25} = \hat{x}^{25} - i\alpha' p^{25} \ln \frac{z}{\tau} \]
\[ + i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha' \tau^m}{m} (z^{-m} - \tau^{-m}). \] (5)

Instead of the boundary condition $\partial_n X^{25} \equiv \partial_n \hat{X}^{25} = 0$ at $\sigma = 0, \pi$ we now have for the T-dual field $\partial_n \hat{X}^{25} = \partial_n \hat{X}^{25} = 0$ where $\partial_n$ and $\partial_{\tau}$ are the normal and tangential derivatives on the boundary of the string world-sheet which we take to be the infinite strip. The boundary conditions for $\hat{X}^{25}$ mean that it is constant along the boundary, hence these are Dirichlet boundary conditions. The difference between the values taken by $\hat{X}^{25}$ at the two boundaries is
\[ \hat{X}^{25}(\pi) - \hat{X}^{25}(0) = 2\pi \alpha' p^{25} = 2\pi \alpha' \frac{n}{R} \]
\[ = 2\pi n R \] (6)

where we used the appropriate T-dual radius $\hat{R} = \alpha'/R$. But in the compactified dual theory $\hat{X}^{25}$ and $\hat{X}^{25} + 2\pi n R$ are to be identified, meaning that both ends of the string lie on one and the same $24 + 1$ dimensional hyperplane. The existence of these T-dual open strings is a logical consequence of the T-duality of the closed string sector contained in open strings. Thus these T-dual open strings must be included, i.e. open strings with Dirichlet boundary conditions on hyperplanes should be included. But now nothing prevents us from taking $\hat{R} \to \infty$ showing that these D boundary conditions should be allowed whether or not $\hat{X}^{25}$ is compactified or not. The $24 + 1$ dimensional hyperplane on which these strings with D boundary conditions end is called a D-brane, or more precisely a D 24-brane. T-dualizing more than one dimension, say $k$ space dimensions, leads to D $p$-branes, with $p = 25 - k$.

A D 25-brane means a $25 + 1$ dimensional hyperplane: this just gives back ordinary open strings.

### 2.3. U(N) gauge symmetry

Open strings can carry Chan-Paton factors at their ends leading to a, say U(N) gauge theory. The open string states then have an additional label $|ij\rangle$ with $i, j = 1, \ldots, N$. Including a background gauge field corresponding to a Wilson line $A_{25} = \text{diag}(\theta_1, \theta_2, \ldots, \theta_N)/(2\pi R)$ generically breaks the gauge symmetry U(N) → U(1)$^N$. Now this is pure gauge, $A_\mu = -iU^{-1} \partial_{\mu} U$ with
\[ U = \text{diag} \left( e^{iX^{25}_1 \theta_1 / 2\pi R}, \ldots, e^{iX^{25}_N \theta_N / 2\pi R} \right) \] (7)

and can be gauged away by this gauge transformation $U$. However, $U$ is not periodic under $X^{25} \to X^{25} + 2\pi R$. Hence under $X^{25} \to X^{25} + 2\pi R$. 

![Figure 1. A Dp-brane is a hypersurface with p space-like and one time-like dimensions on which open strings with Dirichlet boundary conditions end.](image)
\[ X^{25} + 2\pi R \] all fields transforming in the fundamental representation of the gauge group will pick up phases \( e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N} \). Now a state of momentum \( p \) picks up a phase \( e^{i\theta_1} \alpha_p \) under \( x \to x + a \), hence we conclude that now the open string momenta \( p^{25} \) may have fractional parts \( \sim \frac{\theta}{2\pi R} \). More precisely, for an open string whose endpoints are in the state \( |ij\rangle \) the phase is \( e^{i(\theta_j - \theta_i)} \), and possible momenta are \( p^{25} = (2\pi n + \theta_j - \theta_i)/(2\pi R) \). This corresponds to “fractional winding numbers” in the dual picture and, for the positions of the D-branes on which the T-dual open string can end, leads to

\[
\dot{X}^{25}(\pi) - \dot{X}^{25}(0) = (2\pi n + \theta_j - \theta_i)\dot{R}. \tag{8}
\]

This means that now we do not just have a single D-brane but precisely \( N \) D-branes at positions \( \theta_j \dot{R} \), see Fig. 2. Note that \( \theta_j \dot{R} = 2\pi \alpha' \frac{\theta_j}{2\pi R} = 2\pi \alpha' (A_{25})_i \) so that the possible positions are \( (2\pi \alpha') \) times the eigenvalues of \( A_{25} \).

![Figure 2. A Wilson line breaking U(N) to U(1)^N yields N D-branes at positions \( \theta_j \dot{R} \).](image)

In this situation we have \( N \) separate D-branes and the gauge group is \( U(1)^N \). If we now let all \( \theta_i \) coincide the gauge group will no longer be broken and the full \( U(N) \) is restored. At the same time, all \( N \) D-branes are at coinciding positions, all on top of each other. More generally we may only take \( k \) of the \( \theta_i \) to be equal, giving \( U(k) \) for \( k \) coinciding D-branes. The lesson is that separated D-branes correspond to a gauge group which only has \( U(1) \) factors, while the gauge symmetry is enhanced to \( U(k) \) if \( k \) branes coincide. This can also be seen by the following argument. The 25 dimensional mass is

\[
M^2 = (p^{25})^2 + \frac{1}{\alpha'} (N - 1) = \left( \frac{(2\pi n + \theta_j - \theta_i)\dot{R}}{2\pi \alpha'} \right)^2 + \frac{1}{\alpha'} (N - 1) \tag{9}
\]

Then for the vector states \( N = 1 \) with \( n = 0 \), the mass is given by \( \frac{(\theta_j - \theta_i)\dot{R}}{2\pi \alpha'} \). This is just the product of the minimal length of a string stretching between the D-brane hyperplanes at \( x^{25} = \theta_i \dot{R} \) and at \( x^{25} = \theta_j \dot{R} \) times the string tension \( 1/(2\pi \alpha') \). As \( \theta_i \to \theta_j \) new massless vector states appear, corresponding to heavy vector bosons becoming massless and thus increasing the gauge group to \( U(2) \) times the extra \( U(1) \) factors.

### 2.4. Fluctuating D-branes

Generically \( (\theta_i \neq \theta_j) \) the massless vector states only come from strings with both ends on the same D-brane. These strings have vertex operators \( V^{(\mu)} = \partial_\mu X^\mu \), \( \mu = 0, \ldots, 24 \) and \( V^{(25)} = \partial_5 X^{25} = \partial_5 X^{25} \). The first 25 \( V^{(\mu)} \) yield gauge fields in the D-brane while \( V^{(25)} \) due to the appearance of the normal derivative describes transverse fluctuations of the brane. How can this be since we started with a rigid hyperplane? The mechanism is familiar in string theory where one starts e.g. in Minkowski space and then discovers that for the closed string the massless modes describe fluctuations of the space-time geometry. Here \( \partial_5 X^{25} \) similarly describes the transverse fluctuations of the brane. As a result the D-brane becomes dynamical.

### 2.5. D-brane actions

What is the effective action induced on the Dp-brane world-volume that effectively describes low-energy processes of D-branes? One proceeds in exactly the same way as for determining the closed string effective action, except that now boundary terms must be taken into account. We start with a standard \( \sigma \)-model action in the bulk
of the open string world sheet and then add the appropriate boundary couplings \( \delta_i \) are the displacement components of \( A_{\mu} \):

\[
S_{\text{boundary}} \sim \int ds \sum_{m=0}^{25} A_m(x^0, \ldots, x^p) \partial_i X^m + \int ds \sum_{i=p+1}^{25} \delta_i(x^0, \ldots, x^p) \partial_i \hat{X}^i.
\]

(10)

Note that the fields \( A_m, \delta_i \) only depend on the zero-modes \( x^0, \ldots, x^p \) that are in the brane. Requiring the sum of bulk and boundary \( \sigma \)-model to be conformally invariant leads to the \( \beta \)-function equations for \( \phi, G_{\mu\nu}, B_{\mu\nu} \) as well as for \( A_{\mu} \) and \( \delta_i \). Then as usual, these equations can be obtained by varying a certain action functional that is interpreted as the effective action we are looking for. It is given by

\[
S_{\text{Dpbrane}}^{\text{effective}} = -T_p \int d^{p+1} \xi e^{-\phi} \times \left[ \det (g_{mn} + b_{mn} + 2\pi\alpha' F_{mn}) \right]^{1/2}
\]

(11)

where \( g_{mn} = (\partial X^\mu/\partial \xi^m)(\partial X^\nu/\partial \xi^n)G_{\mu\nu}, \) etc are the pull-backs of the space-time fields \( G_{\mu\nu}, B_{\mu\nu} \) to the brane, and \( T_p \) is the Dp-brane tension given below. For trivial metric and antisymmetric ten-

erback as well as constant dilaton field below. For trivial metric and antisymmetric ten-

to the brane, and

\[
S_{\text{Dpbrane}}^{\text{effective}} = -\frac{T_p}{g_s} \int d^{p+1} \xi \frac{1}{4} (2\pi\alpha')^2 F_{mn}^2.
\]

(12)

The action \( S_{\text{Dpbrane}}^{\text{effective}} \) is valid for a single Dp-brane. If there are \( N \) such branes, extra terms appear as we will see next.

2.6. D-brane coordinates as non-commuting matrices

That D-brane coordinates should be represented by matrices might seem strange at first sight. But, following Witten \[5\], I will now show that this follows very naturally from the properties of D-branes I already explained. We have seen that \( N \) D-branes correspond to a gauge group \( U(1)^N \subset U(N) \) with the massless vector states being \( a^\mu_{-1} \otimes |ii\rangle \). As these D-branes coincide, one recovers the full gauge group \( U(N) \). As we have seen, for separated D-branes the breaking of \( U(N) \) is due to the mass terms associated with the off-diagonal \( (A_{\mu})_{ij} \) components which are given by the product of the string tension and the distance between the branes \( i \) and \( j \). For \( N \) coinciding branes, the low-energy effective action must be the non-abelian \( U(N) \) Yang-Mills theory reduced to the brane world-volume. But the world-volume is the same for all \( N \) branes, coinciding or not, so the non-abelian \( U(N) \) YM theory must remain the correct effective action even for separated D-branes.

The point now is that while \( A_{\mu} \) with \( m = 0, 1, \ldots, p \) are actually the gauge fields that live on the brane, the remaining components \( A_i \) with \( i = p + 1, \ldots, 25 \) describe the transverse fluctuations, i.e. the positions of the branes. Before, for a single D-brane, we called them \( \delta_i \) but now it seems more appropriate to call them \( X_i \). More precisely, after eq. (8) we have seen that the correct normalisation includes a factor \( 2\pi\alpha' \):

\[
A_i = \frac{1}{2\pi\alpha'} X_i.
\]

(13)

Since the \( X_i \) are just the components of the 26 dimensional gauge field normal to the brane, it is clear that they, too, must be \( U(N) \) matrices. We will see shortly that for widely separated D branes the eigenvalues of \( X_i \) are just the coordinates of the \( N \) D-branes while the off-diagonal elements take into account the interactions that arise upon integrating out the open strings connecting two different D-branes.

The effective action is just ten dimensional super Yang-Mills theory reduced to \( p + 1 \) dimensions:

\[
S_{\text{YM}}^{(p+1)} \sim -\frac{T_p (2\pi\alpha')^2}{4g_s} \times \int d^{p+1} \xi \text{ tr } (F_{mn}^2 + 2F_{mj}^2 + F_{ij}^2).
\]

(14)

where the overall normalisation is consistent with \( S_{\text{Dpbrane}}^{\text{effective}} \) and \( S_{\text{Dpbrane}}^{\text{effective}} \). I use a metric of signature \( (- + \ldots +) \) and e.g. \( F_{mj}^2 \) is meant to be
\( \sum_{p=1}^{25} \left( -F_{X^p}^2 + \sum_{n=1}^{p} F_{X^p}^2 \right) \), etc. On the D-brane there is no dependence on the zero-modes \( x^i, i = p+1, \ldots, 25 \) because the D boundary condition has removed the zero-modes in the directions normal to the brane. This means that all derivatives in the \( i \) directions disappear. Hence

\[
F_{mn} = \frac{\partial A_m}{\partial x^n} - \frac{\partial A_m}{\partial x^n} + i[A_m, A_n]
\]

\[
(2\pi\alpha') F_{mij} = \frac{\partial X_i}{\partial x^m} + i[A_m, X_j] \equiv D_m X_j
\]

\[
(2\pi\alpha')^2 F_{ij} = i[X_i, X_j].
\]  \(15\)

Upon inserting this into the action \(14\) we get

\[
S_{YM}^{(p+1)} \sim \frac{T_p (2\pi\alpha')^2}{4g_s} \int d^{p+1}x \varepsilon \text{tr} F_{mn}^2
\]

\[
+ \frac{T_p}{g_s} \int d^{p+1}x \varepsilon \text{tr} \left( \frac{1}{2} (D_m X_i)^2 + \frac{1}{4} (X_i, X_j)^2 \right). \]  \(16\)

The first term is just the \( p + 1 \) dimensional YM action on the brane and is the low-energy limit of the obvious non-abelian generalisation of the Born-Infeld effective action we discussed above for a single brane. The second and third term are the effective action governing the D-brane dynamics. In a superstring theory there will be additional fermionic terms. In any case, the scalars \( X_i \) have a potential\(^2\)\(^2\) energy \( \sim -\text{tr} \left( [X_i, X_j]^2 \right) \). In the supersymmetric case, a vacuum with unbroken supersymmetry must have vanishing potential and thus \( [X_i, X_j] = 0 \ \forall \ i, j \). Then all matrices \( X^i \) are simultaneously diagonalisable: \( X^i = \text{diag}(a^i_1, \ldots, a^i_{N_1}) \). We interpret these eigenvalues \( a^i_k \) (\( i = p + 1, \ldots, 25 \)) as giving the coordinates of the \( k \)th brane. Expanding around these vacua gives masses for the off-diagonal modes \( \sim \frac{1}{2\pi\alpha'} |a^{i(k)} - a^{i(l)}| \) as we expect from the string mass formula that yielded masses proportional to the distance between brane \( k \) and brane \( l \). The collection of the \( a^i_k \) parametrizes the moduli space of \( U(N) \) susy vacua rather than really giving the positions of the branes in the quantum theory. When the branes are nearby, many mass terms are small and the massive modes cannot be neglected and one must study the full \( U(N) \) YM theory. This is the origin of the non-commuting matrix character of the “positions” \( X^i \) of the D-branes.

### 2.7. D-branes in superstrings

In type IIA or type IIB superstring theories it is similarly natural to introduce D-branes on which open type I superstrings can end. These D-branes couple naturally to the \( p + 1 \) form RR gauge field. Indeed, D-branes are invariant under half the supersymmetries and hence are BPS states. They must carry conserved abelian charges which are the RR charges in question. The IIA theory has RR gauge fields \( A_\mu, A_{\mu\nu}, A_{\mu\nu\rho\sigma} \) so they couple to D0-branes, D2-branes, D4-branes, etc. The IIB theory has RR gauge fields \( A, A_{\mu\nu}, A_{\mu\nu\rho\sigma} \) so they couple to D(-1)-branes, D1-branes, D3-branes, etc. The D(-1)-branes are D instantons, having also a D boundary condition in the time direction. D0-branes are D particles, while the D1-brane is also called a D string and the D2-brane a D membrane.

Everything we have seen before remains valid with the obvious supersymmetric modifications. In particular, the effective YM action is now replaced by a super YM action which also includes the 16 real component spinors \( \psi \), also in the adjoint representation of \( U(N) \). There is now also a term describing explicitly the couplings of the D-branes to the RR gauge fields. One can compute the various coefficients in front of the effective D-brane actions: the Dp-brane tension \( T_p \) and charge \( \mu_p \). They are given by the appropriate disc diagrams.

I will need the explicit expressions for the tensions. Following the notation of \(13\), \( T_p \) denotes the tension without the factor of \( 1/g_s \) while it is included in \( \tau_p \):

\[
T_p = \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha'}, \quad \tau_p = \frac{T_p}{g_s}.
\]  \(17\)

In particular one has

\[
T_0 = \frac{1}{\sqrt{\alpha'}} \equiv \frac{1}{l_s}.
\]  \(18\)
2.8. Effective D0-brane action

We have seen that the effective Dp-brane action is ten dimensional super YM theory dimensionally reduced to p + 1 dimensions. For p = 0 this gives

\begin{equation}
S^{D0} = \frac{1}{2g_s} \int d^4x \sqrt{g} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi} \Gamma^\mu D_\mu \Psi \right)
\end{equation}

where here and in the following I write

\begin{equation}
c = \frac{1}{2\pi\alpha'}
\end{equation}

for short. The factor of 1/g_s in the effective action comes from computing a disc diagram. A corresponding factor for the fermions has been reabsorbed in the normalisation of \( \Psi \). The general representation of the Clifford algebra is 32 dimensional, but \( \Psi \) is a Majorana-Weyl spinor \( \Psi = \begin{pmatrix} \theta \\ 0 \end{pmatrix} \) with \( \theta \) being a real 16 component spinor. We take the \( \Gamma^\mu \) to be

\begin{equation}
\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix}
\end{equation}

where \( \gamma^i, i = 1, \ldots 9 \) are real symmetric 16 \( \times \) 16 gamma matrices of SO(9). We have again \( F_{ij} = ic^2[X^i, X^j], \) \( F_{0j} = cD_0X^j = c\partial_0X^j + ic[A_0, X^j] \) as well as \( D_i\theta = ic[X_j, \theta] \) and \( D_0\theta = \partial_0\theta + i[A_0, \theta] \) so that

\begin{equation}
S^{D0} = \frac{1}{2g_s} \int d^4x \sqrt{g} \left( \frac{1}{4g_s} (D_0X^j)^2 - i\theta^T D_0\theta \\ c^2((X^i, X^j))^2 + c\theta^T \gamma^j [X_j, \theta] \right).
\end{equation}

This is a supersymmetric quantum mechanics for \( X^i \) and \( \theta \) in the adjoint of U(N), i.e. \( N \times N \) hermitian matrices, and each component of the \( \theta \) matrix is a real 16 component spinor.

3. SECOND LECTURE : THE APPEARANCE OF THE ELEVENTH DIMENSION - M-THEORY

The low-energy effective theory of type IIA superstring is ten dimensional type IIA supergravity. Type IIA supergravity can also be obtained by dimensional reduction of supergravity in eleven dimensions. Since long, this had prompted the question of whether eleven dimensional supergravity is some low-energy effective theory of some consistent quantum theory in eleven dimensions. For some time it had been hoped that this might be a theory with supermembranes as its fundamental objects.

In this lecture, following Witten [3], I will argue that there is indeed such an eleven dimensional theory, called M-theory which could be viewed as the strong-coupling limit of the ten dimensional type IIA superstring.

3.1. Supergravity in eleven and IIA supergravity in ten dimensions

The eleven dimensional supergravity multiplet contains the following massless fields; a metric \( G_{MN} \) or equivalently an elevenbein \( e^M_\alpha \), a three form potential \( A_3 \) with components \( A_{MNP} \) and a Majorana gravitino \( \Psi_M \). To count the physical degrees of freedom of massless fields in dimension \( d \), the simple rule is to do the counting as if one were in \( d - 2 \) dimensions and all components were physical. Hence a symmetric traceless tensor like the metric has \( \frac{1}{2}(d - 2)(d - 1) - 1 \) physical degrees of freedom (dofs). For \( d = 4 \) this gives the familiar two dofs, while it gives 35 dofs in ten dimensions and 44 dofs for the eleven dimensional \( G_{MN} \). The antisymmetric three-index tensor has \( \frac{1}{3}(d - 2)(d - 3)(d - 4) \) dofs, which gives 84 dofs for \( d = 11 \). This makes a total of 128 bosonic dofs for eleven dimensional supergravity. For the fermionic partners the counting is similar: the eleven dimensional Clifford algebra has 32 dimensional spinors. Imposing a Majorana condition gives 32 component real spinors. Due to the Dirac equation only half of them are physical, so a Majorana spinor has 16 real dofs. The gravitino also has a vector index, which contributes a facor of \( d - 2 = 9 \). However one has to project out the spin \( \frac{1}{2} \) components, which leaves us with \( 16 \times 9 - 16 = 128 \) (real) fermionic dofs. The bosonic part of the eleven dimensional supergravity action is schematically (in units where \( \alpha' = 1 \))

\begin{equation}
S^{(11)}_{bos} = \frac{1}{2} \int d^{11}x \sqrt{G} \left( R + |dA_3|^2 \right) + \int A_3 \wedge dA_3 \wedge dA_3
\end{equation}
with the fermionic terms determined by supersymmetry. Here and in the rest of this subsection, I do not care about the precise numerical factors in front of each term in the action.

Next, one reduces this theory to ten dimensions (indices $\mu, \nu, \rho$), i.e. one takes $x^{11}$ on a circle and supposes that nothing depends on $x^{11}$. The eleven dimensional Majorana gravitino $\Psi_M \equiv \begin{pmatrix} \psi_M^1 \\ \psi_M^2 \end{pmatrix}$ gives rise in ten dimensions to a pair of Majorana-Weyl gravitinos of opposite chirality $\psi^a_\mu$ and a pair of Majorana-Weyl spinors $\phi^a_\mu \equiv \psi^a_\mu$, $a = 1, 2$. The eleven dimensional three-form gives rise in ten dimensions to a three form $A_{\mu\nu}$ (56 dofs) and a two-form $B_{\mu\nu} \equiv A_{\mu\nu 11}$ (28 dofs), while the eleven dimensional metric gives in ten dimensions a metric $G_{\mu\nu}$ (35 dofs), a scalar $e^{2\gamma}$ and a vector potential (one form) $A_\mu \equiv -e^{-2\gamma}G_{\mu 11}$ (8 dofs), again a total of 128 bosonic dofs. An important point concerns the interpretation of $e^{2\gamma}$. We take $x^{11}$ to vary from 0 to $2\pi$. However this does not fix the size of the compact dimension because the eleven dimensional line element is

$$ds^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + e^{2\gamma}(dx^{11} - A_\mu dx^\mu)^2$$

and one sees that the eleventh dimension is a circle of radius $e^{-\gamma}$. Equation (24) also shows that

$$\det G_{(11)} = e^{2\gamma} \det G_{(10)}$$

and thus

$$\int d^{11}x \sqrt{G_{(11)}} \cdots = 2\pi \int d^{10}x e^{\gamma} \sqrt{G_{(10)}} \cdots$$

For the bosonic action (23) one gets (remember that I do not care about numerical factors)

$$\int d^{10}x \sqrt{G_{(10)}} \left[ e^{\gamma}(R + |\nabla\gamma|^2 + |dA_3|^2) + e^{3\gamma} |dA|^2 + e^{-\gamma} |dB|^2 \right] + \int B \wedge dA_3 \wedge dA_3 \ .$$

Let me say a word about where the powers of $e^\gamma$ come from. We have already seen that the square-root of the determinant of the metric gives a factor of $e^{\gamma}$. The eleven dimensional curvature gives rise to terms $G^{1111}\partial_\mu G_{\nu\rho\sigma} \sim e^{-2\gamma} e^{2\gamma} \partial_\mu A_\nu \partial_\rho A_\sigma$ plus $(\partial^2 \gamma)$-terms. This gives $e^{2\gamma} \partial A\partial A$, and together with the $e^\gamma$ from the determinant a factor of $e^{2\gamma}$ in front of $|dA|^2$. Similarly, to get $|dB|^2$ from $|dA_3|^2$ one needs to consider $G^{1111}\partial A_{\mu\nu 11} \partial A_{\rho\sigma 11} \sim e^{-2\gamma} |dB|^2$. Together with the $e^\gamma$ from the determinant this gives a factor of $e^{-\gamma}$.

The action (26) contains all terms one wants to obtain for IIA supergravity in ten dimensions. The factors of $e^{-\gamma}$, however, are not what one expects. The usual form of the action for IIA supergravity is (again not worrying about numerical factors)

$$\int d^{10}x \sqrt{g} \left[ e^{-2\phi}(R + |\nabla\phi|^2 + |dB|^2) + |dA_3|^2 + |dA|^2 \right] + \int B \wedge dA_3 \wedge dA_3 \ .$$

To bring the action (26) in the form (27) all one needs to do is to perform a Weyl rescaling of the metric:

$$G_{\mu\nu} = e^{-\gamma} g_{\mu\nu} \ .$$

It then follows that $\sqrt{G_{(10)}} = e^{-\gamma} \sqrt{g}$, while $R[G_{(10)}] = e^{-\gamma} R[g]$ plus terms $\sim |\nabla \phi|^2$. For a $p$ form $A_p$ the kinetic term $|dA_p|^2$ contains $p + 1$ inverse metrics $G^{\mu\nu}$ and hence gives $e^{p \gamma}$ times $|dA_p|^2$ with indices now contracted using $g^{\mu\nu}$. Hence we see that the action (26) takes the desired form if we identify $e^{-3\gamma} = e^{-2\phi}$, i.e.

$$e^\gamma = e^{2\phi/3} \ .$$

3.2. String coupling, radius and KK modes

Recall that $e^{\gamma}$ was the radius of the eleventh dimension, or putting back $\alpha'$ one has $R_{11} = \sqrt{\alpha'} e^{\gamma}$. On the other hand, $\phi$ being the dilaton, $e^{\phi}$ is the coupling constant. In string theory it is the string coupling constant $g_s$, so that one arrives at the relation

$$R_{11} = \sqrt{\alpha'} g_s^{2/3} \ .$$

One has to be a bit careful. This radius is the radius of the eleventh dimension when measured with the eleven dimensional metric $G$. 

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3 To be precise, as one can see from the next equation, $G_{MN}$ is not $G_{MN}$ for $M = \mu$ and $N = \nu$, but it also contains an additional term $-e^{2\gamma} A_\mu A_\nu$. 

If we measure distances instead with the Weyl rescaled (string) metric \( g \) one has \( ds^2 = e^\phi ds^2_G = (e^{\phi/3})^2 ds^2_G \) and thus

\[
R_1^{(g)} = e^{\phi/3} R_1^{(G)} = g_s^{1/3} R_1^{(G)} = \sqrt{\alpha'} g_s .
\]

Introducing the string length \( l_s = \sqrt{\alpha'} \) and string mass \( m_s = 1/\sqrt{\alpha'} = 1/l_s \), this can also be written as

\[
R_1^{(g)} = g_s l_s .
\]

When the eleven dimensional supergravity is compactified on a circle of radius \( R_{11} \), the resulting ten dimensional theory not only has the massless modes described so far that form a supergravity multiplet with 256 states (128 bosonic and 128 fermionic) but there also are all the massive Kaluza-Klein (KK) modes. Since the compactification radius is \( R_{11} \) these KK modes have momenta \( n/R_{11} \). When the momenta are measured with the metric \( g \) they are of course \( n/R_1^{(g)} \). Since the eleven dimensional states are massless, this KK momentum is the only contribution to the ten dimensional mass which thus simply is

\[
M = \frac{n}{R_1^{(g)}} = \frac{n}{\sqrt{\alpha'} g_s} = \frac{n}{l_s g_s} .
\]

Each of these massive KK states is a supermultiplet of 256 states. The important point is that for fixed \( n \) their masses are proportional to the inverse of the string coupling constant, so that in the limit of large string coupling they become very light.

3.3. The strong-coupling limit of the IIA superstring

Let us now consider the low-energy sector of type IIA superstring theory. Its effective action is given by eq. (27). The field \( \phi \) is the dilaton and \( e^\phi \) or rather its expectation value is the string coupling constant \( g_s \). In particular we see that the action is such that all kinetic terms for the NS-NS fields, i.e. the metric, dilaton and two-form potential, have a factor of \( g_s^{-2} = e^{-2\phi} \) in front. The kinetic terms for the RR fields, i.e. the one-form and three-form potentials \( A \) and \( A_3 \), have no \( \phi \) dependence. This “normalisation” of the RR fields is fixed by requiring that their gauge transformation laws be \( \phi \)-independent. Concentrate now on the one-form potential \( A \). There are no states in the perturbative string spectrum that carry a charge for \( A \). These charges are carried by the D0-branes, as we saw that a Dp-brane naturally couples to a \( p + 1 \) form potential via \( \int_{Dp\text{-brane}} A_{p+1} \). As noted in the previous section, the corresponding charge \( \mu_p \) can be computed by a string diagram and is proportional to \( T_p \). We also saw that Dp-branes are BPS states. BPS states break half of the supersymmetries and they saturate the bound on the masses, i.e. \( M = |\mathcal{Z}| \) where \( Z \) is the central charge of the \( \mathcal{N} = 2 \) supersymmetry algebra. This algebra in ten dimension schematically reads \( \{Q, Q\} \sim \{Q', Q'\} \sim P \) and \( \{Q, Q'\} \sim Z \). If this algebra is derived from eleven dimension, the central charge \( Z \) just is the eleventh component of the momentum \( P \). Now the central charge must be made up from the abelian charges, here the RR charge of the D0-brane. Indeed, it is not too difficult to work out the susy algebra for Dp-branes. For a single D0-brane one finds \( Z = P + \tau_0 \) where \( \tau_p \) denotes the Dp-brane tensions, cf eqs. (27,28). For the D0-brane this gives \( Z = \tau_0 = \sqrt{\alpha'} g_s \) and hence

\[
M = \frac{1}{\sqrt{\alpha'} g_s} = \frac{1}{l_s g_s} .
\]

So the type IIA superstring contains states, the D0-branes of masses \( M = 1/(l_s g_s) \) that come in short supermultiplets of \( 2^8 = 256 \) states. It has been shown \( \text{[27]} \) that a system of \( n \) D0-branes has a bound state at threshold, i.e. this bound state has a total mass of exactly \( n \). Again this is a full supermultiplet of 256 states. In the strong coupling limit \( g_s \to \infty \) all these states become very light, and we get infinitely many light states. This is exactly the Kaluza-Klein spectrum of eleven dimensional supergravity we discussed above. This is quite surprising! One knew al-

\footnote{One might ask why not consider the three-form instead. This is simply because \( A_3 \) couples to D2-branes and their masses being equal to their tension times their area, they do not have the same “universal” mass as the D0-branes and thus seem not suited for the identification with the KK states one has in mind.}
ready that simple dimensional reduction of eleven dimensional supergravity gives the massless IIA supergravity in ten dimensions. What Witten has shown, and what I explained here, is that all KK states of the full eleven dimensional supergravity on \( R^{10} \times S^1 \) are contained in the IIA superstring, with each “state” being actually a full supermultiplet of 256 states. Moreover, in the strong-coupling limit \( g_s \to \infty \) these are all low-energy states of the IIA superstring. Also, for \( g_s \to \infty \) one has \( R_{11} \to \infty \) and one gets uncompactified eleven dimensional supergravity: Eleven dimensional supergravity is the low-energy limit of IIA superstring theory at strong coupling with \( R^{(g)}_{11} \sim g_s \). An eleventh dimension has been constructed out of the ten dimensional theory in an entirely non-perturbative way. Clearly, it will be quite non-trivial to see how this eleven dimensional theory, in the uncompactified limit, can manage to have eleven dimensional Lorentz invariance.

Of course, eleven dimensional supergravity is not expected to yield a consistent quantum theory. It should only be the low-energy limit of some consistent theory, baptised M-theory. The latter should then also describe the strong-coupling limit of IIA superstrings, not only at low energies. More precisely, M-theory with its eleventh dimension compactified on a circle of radius \( R_{11} \) should be identical to IIA superstring theory with string coupling \( g_s = R^{(g)}_{11}/\sqrt{\alpha'} \) where \( R^{(g)}_{11} = g_s^{1/2} R_{11} \) is the eleven dimensional radius when measured with the string metric \( g \). This can be taken as the definition of M-theory. What else do we know about it? Since it should describe IIA superstrings which have D0, D2, D4, D6 and D8 branes as well as the solitonic 5-brane and the fundamental string (F1 brane), M-theory should also contain extended objects. Since the higher dimensional Dp-branes are in a certain way dual to the lower dimensional ones, one mainly has to worry about the latter. If M-theory contains pointlike degrees of freedom, as well as membranes (i.e. 2-branes) and 5-branes, then things work out. Indeed, the extended \( p \)-branes of M-theory may or may not be wrapped around the compact \( S^1 \), hence yielding \( p - 1 \) or \( p \) branes in the ten dimensional superstring theory. This gives the branes of IIA superstrings with \( p = 0, 1, 2, 4, 5 \) as it should. The D6 and D8 branes are more subtle.

4. THIRD LECTURE : A MATRIX-MODEL FOR M-THEORY IN THE IMF

This section is based mainly on the BFSS paper \([4]\). So far we have seen that a) in the strong-coupling limit of IIA superstrings an eleventh dimension appears and that the KK states of eleven dimensional supergravity correspond to bound states at threshold of \( n \) D0-branes, and b) a collection of \( n \) D0-brane is described by ten dimensional \( U(n) \) super Yang-Mills theory reduced to \( 0 + 1 \) dimensions, i.e. by \( n \times n \) hermitian matrix quantum mechanics.

There still seems to be a mismatch between the ten dimensions of the matrix model and the eleven dimensions of the supergravity. Here comes the third idea: c) the main idea of BFSS is to interpret the 9 space dimensions (the \( X^i, i = 1,\ldots,9 \) of the D0-brane matrix model as the transverse dimensions of an eleven dimensional theory in the light-cone frame, or more precisely in the infinite momentum frame (IMF). We are familiar with the light-cone quantization of the ordinary string where the light-cone Hamiltonian is \( H = \frac{1}{p} H_\perp \). There, \( H_\perp \) is a Hamiltonian for the \( d-2 \) transverse degrees of freedom. The string theory is nevertheless Lorentz invariant in all \( d \) dimensions (provided \( d = d_{\text{crit}} \)). If one manages to interpret the matrix quantum mechanical Hamiltonian for the nine \( X^i \) as a transverse Hamiltonian, then the full system lives in 11 dimensions and should exhibit eleven dimensional Lorentz invariance. This is the way how the mismatch of dimensions can be resolved. It is thus useful to first recall some facts about the infinite momentum frame.

4.1. The infinite momentum frame (IMF)

The infinite momentum frame (IMF) was introduced in quantum field theory by Weinberg long ago \([3]\) as a mean to simplify perturbation theory. Perturbation theory in the IMF is characterised
by the vacuum being trivial. In particular, Feynman diagrams with vertices where particles are created out of the vacuum are vanishing in the IMF. In this sense, the IMF perturbation theory of QFT looks much like the “old-fashioned” non-covariant perturbation theory, but with the energy denominators replaced by covariant denominators.

For a collection of particles, the IMF is defined to be a reference frame in which the total momentum $P$ is very large. All individual momenta can be written as

$$p_a = \eta_a P + p^a_\perp$$

with $p^a_\perp \cdot P = 0$, $\sum p^2_a = 0$ and $\sum \eta_a = 1$. This means that the observer is moving with high velocity in the $-P$ direction. If the system is boosted sufficiently, all $\eta_a$ are positive. For massive particles it is obvious that a sufficiently large boost in the $P$ direction will eventually make all components of the momenta in the longitudinal $P$ direction positive. For a massless particle the same will be true except if it moves exactly in the $-P$ direction with all transverse $q_a$ vanishing. But this latter case is somewhat degenerate and can simply be avoided by not boosting the system in exactly the opposite of the direction of the momentum of any massless particle.$^6$

Hence we can assume that with a sufficiently large boost all $\eta_a$ are strictly positive. Once $P$ is large enough, further boosting only increases the total momentum $P$ but does not change the $\eta_a$ anymore, and of course, the $q_a$ aren’t changed either.

The energy of any particle is

$$E_a = \sqrt{p^2_a + m^2_a} = \eta_a P + \frac{(p^a_\perp)^2 + m^2_a}{2\eta_a P} + O(P^{-2}) .$$

Apart from the constant $\eta_a P + \frac{m^2_a}{2\eta_a P}$ this has the non-relativistic structure $\frac{1}{2\mu_a} (\Sigma p^2_i)^2$ of a $d-2$ dimensional system with the role of the non-relativistic masses $\mu_a$ played by $\eta_a P$.

Let us now turn to quantum field theory. Then internal lines in Feynman diagrams can carry arbitrary large momenta and for part of the integration range one does not have $\eta_a > 0$. Weinberg has shown, starting from “old-fashioned” perturbation theory with energy denominators that whenever an internal $\eta_i$ is negative the corresponding diagram is suppressed by extra factors of $1/P$. These suppressed diagrams correspond exactly to diagrams with vertices where several particles are created from the vacuum. It is in this sense that in the IMF the vacuum has no non-trivial structure. Hence we conclude that in field theory also the internal lines have momenta with $\eta_a > 0$ only.

It might be useful to compare the IMF to a standard light cone frame. In the latter one again singles out one spatial direction called longitudinal with momentum $p^a_L = \eta_a P$ and defines $p^2_\perp = E^a \pm p^a_L = E^a \pm \eta_a P$. Then the mass shell condition reads $p^2 p^a_\perp - (p^a_\perp)^2 = m^2_a$ or

$$E_a - \eta_a P = \frac{(p^a_\perp)^2 + m^2_a}{p^a_\perp}$$

This is exact whether $p^a_\perp$ is large or not. However, if $P$ and hence $p^a_L = \eta_a P$ is large, one has $E^a \simeq \eta_a P$ and $p^a_\perp \simeq 2\eta_a P$ so that one recovers eq. (36).

4.2. M-theory in the IMF

Now we will consider M-theory in the IMF. We will separate the components of the eleven dimensional momenta as follows: $p_0$, $p_i$, $i = 1, \ldots, 9$ and $p_{11}$. The $p_i$ will collectively be called $p_\perp$. We boost in the 11 direction to the IMF until all momenta in this direction are much larger than any relevant scale in the problem. In particular all $p^a_{11}$ are strictly positive. We also compactify $x^{11}$ on a circle of radius $R$ (we no longer write the subscript “11” because throughout these notes no other dimension will be compactified). To be precise, when I write $R$, I mean $R^{(g)}_{11}$. All momenta $p^a_{11}$ are now quantized as $n_a/R$ with $n_a > 0$. Since there are no eleven dimensional masses $m_a$ the
The energy dispersion relation just is
\[ E - p_{11}^{\text{tot}} = \sum_a \frac{(p_a^1)^2}{2p_{11}}. \]  (38)

Again we see the non-relativistic structure. It exhibits full Galilean invariance in the transverse dimensions. Actually, M-theory in the IMF must have super-Galilean invariance because there are also the 32 real supersymmetry generators. In the IMF, they split into two groups of 16, each transforming as a spinor of \( \text{SO}(9) \): the \( Q_\alpha \) and \( q_A, \alpha, A = 1, \ldots, 16 \) satisfy the algebra
\[ \{ Q_\alpha, Q_\beta \} = \delta_{\alpha\beta} H, \quad \{ q_A, q_B \} = \delta_{AB} P_{11} \]
\[ \{ Q_\alpha, q_A \} = \gamma^*_A P_1. \]  (39)

Recall that the compactification radius \( R \) was related to the string coupling as \( R = l_s g_s \), and that the RR photon of the IIA superstring is the KK photon from compactifying \( x^1 \) on \( S^1 \) of radius \( \tilde{R} \) with the RR charge corresponding to \( p_{11} \). Recall also that no perturbative string states carry RR charge and hence have vanishing \( p_{11} \). RR photon charge is only carried by the D0 branes. A single D0-brane carries one unit of RR charge and thus has \( p_{11} = \frac{1}{R} \). It fills out a whole supermultiplet of 256 states. Since in eleven dimensions it is massless (graviton multiplet), in ten dimensions it is BPS saturated, as we indeed saw. There also are KK states with \( p_{11} = \frac{N}{R} \), \( N \) being an arbitrary integer. For \( N \neq 1 \) this does not correspond to an elementary D0-brane. \( N > 1 \) are bound states of \( N \) D0-branes, while \( N < 0 \) corresponds to anti-D0-branes or bound states thereof. As we take the total \( p_{11} \) to infinity to reach the IMF limit, only positive \( p_{11} \) should appear, i.e. \( N > 0 \). This means that M-theory in the IMF should only contain D0-branes and their bound states. What has happened to the anti-D0-branes and the perturbative string states (\( N = 0 \))? The answer is that these states get boosted to infinite energy and have somehow implicitly been integrated out. This means that the D0-brane dynamics in the IMF should know in some subtle way that before going to the IMF, there was more to M-theory and type IIA superstrings then just D0-branes. This is much as in field theory where the IMF vacuum is trivial, but still, in the end, the amplitudes and cross sections know about vacuum polarisation and all the subtle effects of quantum field theory. Moreover, M-theory should also contain membranes (i.e. 2-branes) and 5-branes. Where are they? We will see below that membranes can effectively be described within the D0-brane quantum mechanics. The 5-brane on the other hand seems to be more subtle.

I can now state the BFSS conjecture: M-theory in the IMF is a theory in which the only dynamical degrees of freedom are D0-branes each of which carries a minimal quantum of \( p_{11} = 1/R \). This system is descried by the effective action for \( N \) D0 branes which is a particular \( N \times N \) matrix quantum mechanics, to be taken in the \( N \to \infty \) limit.

4.3. The matrix model Hamiltonian

The effective action for a system of \( N \) D0-branes was already given in eq. (22). It is this action which is the starting point for the matrix model description of M-theory in the IMF. For convenience, I repeat it again:
\[ S = T_0 \int dt \text{tr} \left( \frac{1}{2g_s} (D_0 X^i)^2 - i \theta^T D_0 \theta \right) + \frac{e^2}{4g_s} |[X^i, X^j]|^2 + e \theta^T \gamma_j [X_j, \theta]. \]  (40)

(Recall that \( e = 1/(2\pi \alpha') \) and \( T_0 = 1/\sqrt{\alpha'} = 1/l_s \).) The indices \( i = 1, \ldots, 9 \) run over the nine transverse directions, and the \( \theta \) are sixteen-component real spinors. The \( X^i \) and \( \theta \) are all in the adjoint representation of the gauge group \( U(N) \), i.e. they are hermitian \( N \times N \) matrices. The covariant derivative \( D_0 \) contains the gauge field \( A_0 \), but let us make the gauge choice \( A_0 = 0 \) to simplify things. Note that the first term in the action just reads \( \int dt \frac{M}{2} (dX^i/dt)^2 \) where \( M = T_0/g_s \) is the D0-brane mass. We also rescale the fields as \( X^i = g_s^{1/3} Y^i \). This corresponds to the Weyl rescaling of the metric, meaning that we now measure lengths with the eleven dimensional supergravity metric \( G \) rather than the ten dimensional string metric \( g \). One can also rescale the time accordingly: \( t = g_s^{-1/3} l_p \tau \) extracting also a factor of eleven dimensional Planck length \( l_p \).
to make $\hat{\tau}$ dimensionless. One has $t_p = g_s^{1/3} t_s$ so that
\[
t = g_s^{2/3} t_s \frac{l_p}{g_s^{1/3}} = g_s^{1/3} \frac{R}{g_s^{1/3}} \hat{\tau}
\]
This is the choice made in ref. [1]. But a dimensionless $\hat{\tau}$ gives a dimensionless Hamiltonian. For the discussion of the spectrum of the Hamiltonian, to be interpreted as energies, it is preferable to work with a Hamiltonian that has the dimension of an energy, as usual (i.e. inverse time or inverse length). So I define instead
\[
t = g_s^{2/3} t_s \frac{T_0 R}{g_s^{1/3}} \tau
\]
Denoting then $\partial / \partial \tau$ simply by a dot, the action becomes
\[
S = T_0^2 \int d\tau \left( \frac{1}{2R T_0^2} \dot{Y}^i \dot{Y}^i - i \frac{1}{T_0} \theta \dot{\theta} - c^2 \frac{R}{4} \left( [Y^i, Y^j] \right)^2 + c R \theta^T \gamma^i [Y_j, \theta] \right)
\]
Defining the conjugate momenta of $Y^i$ and $\theta$ as usual, $\Pi_i = \dot{Y}^i / R$ and $\pi = -i T_0 \theta^T$, one obtains the Hamiltonian
\[
H = R \text{tr} \left( \frac{1}{2} \Pi_i^2 - c^2 \frac{T_0^2}{4} \left( [Y^i, Y^j] \right)^2 - c T_0^2 \theta^T \gamma^i [Y_j, \theta] \right)
\]
\[
\equiv R \tilde{H}
\]
Recall that $-\frac{1}{4} R c^2 T_0^2 \text{tr} \left( [Y^i, Y^j] \right)^2$ is a non-negative potential. In the $R \to \infty$ limit of uncompactified M-theory, finite energy states of $H$ correspond to states whose $\tilde{H}$ energy vanishes. To be more precise, we are looking for states with
\[
\tilde{H} | \Psi \rangle = \frac{\epsilon}{N} | \Psi \rangle \Rightarrow H | \Psi \rangle = \frac{R}{N \epsilon} | \Psi \rangle
\]
with finite $\epsilon$. But recall that for a system of $N$ D0-branes the total $p_{11}$ momentum is
\[
p_{11} = \frac{N}{R}
\]
so that the energy $E$ takes the form
\[
E = \frac{\epsilon}{p_{11}}
\]
Provided we can identify $\epsilon$ with $\frac{1}{2} p_{11}^2$ this gives us the desired dispersion relation of \textit{eleven} dimensions in the IMF. I will return to this point later when discussing the spectrum in more detail.

### 4.4. Coordinate interpretation

In section 2, I already touched upon the interpretation of the $N$ eigenvalues of the $X^i$ as the position vectors of the $N$ D0-branes. Let me now elaborate this point a bit more.

The potential $V(Y) = -\frac{1}{4} R c^2 T_0^2 \left( [Y^i, Y^j] \right)^2$ in the Hamiltonian is the familiar Higgs potential, analogous to $([\phi, \phi^+])^2$ in $\mathcal{N} = 2$ super YM in four dimensions. In field theory, the supersymmetric vacua have $[\phi, \phi^+] = 0$. Here we have quantum mechanics, not quantum field theory and the expectation values of the scalars $Y^i$ do not give superselection sectors (i.e. distinct vacua) but are collective coordinates with corresponding quantum wave functions: they are not frozen at $V(Y) = 0$. Still, $V(Y)$ has flat directions (minima) $[Y^i, Y^j] = 0$ along which the $Y^i$ can be simultaneously diagonalized: $Y^i \equiv \text{diag}(y^i_1, y^i_2, \ldots y^i_N)$ where $y^i_a$ gives the $i$th coordinate of the $a$th D0-brane. More generally, if the branes are far apart from each other, loosely speaking, the $Y^i$ are large and non-commutativity would cost much energy. (This will be seen more precisely below.) In this case commuting $Y^i$ are a good approximation and the D0-brane positions are rather well defined. As they get closer, non-commuting configurations become more important (strings stretching between different branes) and the individual positions can no longer be well defined. Space is intrinsically non-commutative with ordinary commutative space only emerging at long distances. Yet one has the full super-Galilean invariance in the transverse directions. Translations e.g. are given by $Y^i \to Y^i + \vec{v} \cdot \mathbf{1}$ where $\mathbf{1}$ is the unit matrix. This does not affect the kinetic terms nor the commutator terms in the Hamiltonian or action and hence is an invariance of the theory. Similarly, a Galilean boost $Y^i \to Y^i + v^i \mathbf{1}$ only affects the center of mass momentum to be defined below, but not the relative momenta, nor the interaction terms.

Consider configurations where the $N \times N$ matrices $Y^i$ take block-diagonal forms with $n$ blocks of size $N_1, N_2, \ldots N_n$ and $\sum_n N_n = N$. This corresponds to $n$ widely separated clusters of D0-branes. One can define the distance between clus-
\[ r_{ab} = \left| \frac{1}{N_a} \text{tr} Y_a - \frac{1}{N_b} \text{tr} Y_b \right| \tag{48} \]

As an example take \( N = 2 \) and \( N_1 = 1, N_2 = 1 \) and \( Y^i = \left( \alpha_i, \beta_i \right) \). Then \( r_{1,2} = \left( \sum_{i=1}^{9} (\alpha_i - \delta_i)^2 \right)^{1/2} \) is large if at least for one value of \( i \), say \( i_0 \), \( |\alpha_{i_0} - \delta_{i_0}| \) is large. One has \( |\alpha_{i_0} - \delta_{i_0}| \geq \frac{1}{2} r_{1,2} \). Then for any \( j \) one has
\[ - \frac{1}{2} \text{tr} \left( [Y^{i_0}, Y^j] \right)^2 = 4 \text{Im}(\beta_i \beta_j^*)^2 + |\beta_j (\alpha_{i_0} - \delta_{i_0}) - \beta_{i_0} (\alpha_j - \delta_j)|^2 \tag{49} \]

Except for non-generic configurations\(^7\) this will be of order
\[ |\beta_j|^2 (\alpha_{i_0} - \delta_{i_0})^2 \geq \frac{1}{2} |\beta_j|^2 r_{1,2}^2 \tag{50} \]

This generalizes to the general case of larger blocks: for generic configurations, \( \text{tr} \left( [Y^i, Y^j] \right)^2 \) is at least of the order of the modulus squared of the off-block-diagonal elements times the minimum of the \( r_{ab}^2 \) times some numerical constant.

The bottom line is that for well separated clusters of D0 branes - defined by large \( r_{ab} \) - generic off-block diagonal elements must be small or else they give rise to large potential energies \( \sim r_{ab}^2 \).

One might think that this harmonic oscillator type potential would lead to a finite ground state energy, but this is not true due to the supersymmetry of the system. I will now explain why such clusters of D0-branes correspond to a collection of “supergravitons”

4.5. The spectrum of \( H \) and the supergravitons

Begin by considering the simplest case, namely \( N = 1 \), a single D0-brane. Then \( p_{11} = N/R = 1/R \) and
\[ H_{(N=1)} = \frac{R}{2} \Pi_1^2 = \frac{R}{2} p_{11}^2 = \frac{p_{11}^2}{2p_{11}} \tag{51} \]

This is the eleven dimensional relativistically invariant relation between energy and momentum of a massless particle when written in the IMF.

Comparing with eq. (38) we see that the Hamiltonian has been shifted by the constant \( p_{11} \). Moreover, there are also the 16 \( \theta \)'s that generate a \( 2^{16}/2 = 256 \) dimensional supermultiplet. Hence for \( N = 1 \) the spectrum of the Hamiltonian is that of an eleven dimensional massless supermultiplet of 256 states containing up to spin two: this is exactly the supergravity multiplet, and BFSS call it the supergraviton.

For \( N > 1 \) one separates the center-of-mass coordinates and momenta from the relative ones:
\[ Y^i = Y^i_{\text{rel}} + Y^i_{\text{cm}} 1, \quad Y^i_{\text{cm}} = \frac{1}{N} \text{tr} Y^i \]
\[ \Pi_i = \Pi^\text{rel} + \frac{1}{N} P^\text{cm} i, \quad P^\text{cm} = \text{tr} \Pi_i \tag{52} \]
with \( \text{tr} Y^i_{\text{rel}} = \text{tr} \Pi^\text{rel} = 0 \). Plugging this into the Hamiltonian (44) one gets
\[ H = H_{\text{cm}} + H_{\text{rel}} \tag{53} \]
where
\[ H_{\text{cm}} = \frac{R}{2N} \left( P^\text{cm} \right)^2 = \frac{1}{2p_{11}} (P^\text{cm})^2 \tag{54} \]
with the correct factor \( \frac{R}{N} = \frac{1}{p_{11}} \) exactly as it should. The relative part of the Hamiltonian \( H_{\text{rel}}(Y^i_{\text{rel}}, \Pi^\text{rel}) \) exactly looks like the original Hamiltonian \( H(Y^i, \Psi) \) given in (44), except that now all matrices are traceless, i.e. in SU\( (N) \). It has been shown\(^3\)\(^8\) that \( H_{\text{rel}} \) has zero-energy (threshold) bound states. For these bound states the total energy is just given by the center-of-mass energy
\[ E = E_{\text{cm}} = \frac{R}{2N} (P^\text{cm})^2 = \frac{1}{2p_{11}} (p^\text{cm})^2 \tag{55} \]
Again this is a full supergravity multiplet of 256 states, i.e. a supergraviton. Thus for any \( N \) the spectrum contains single supergraviton states. They correspond to the bound state at threshold.

To see that the matrix Hamiltonian can also describe arbitrary many such supergravitons, consider the block decomposition of \( Y^i \) and \( \Pi_i \) discussed in the previous subsection. If these matrices are exactly block-diagonal, i.e. if the off-block-diagonal elements are strictly vanishing\(^9\)
the total Hamiltonian splits into a sum of \( n \) uncoupled Hamiltonians \( H_a \), one for each block of size \( N_a \). Small, but non-vanishing off-block-diagonal elements would correspond to interaction terms between these Hamiltonians. The Hamiltonians \( H_a \) of course have exactly the same form as the initial Hamiltonian \( H \). Again for each block matrix \( Y_a \) one can separate the center-of-mass and relative coordinates and momenta, and arrive at the conclusion that each of the block Hamiltonians \( H_a \) contains a supergraviton in its spectrum. Hence the matrix model can describe several supergravitons, too. Since we want to be able to describe an arbitrary number of them, we must let \( N \) go to infinity. Then we conclude that the matrix model contains the full Fock space of supergravitons. Since this Fock space is embedded into the larger D0-brane quantum mechanics, the theory should be free of UV divergences.

The next question is whether the matrix model also has something to say about the interactions of these supergravitons. These interaction come about via the off-block-diagonal matrix elements. As such they look very non-local in the first place. We will nevertheless see that the matrix model, at low energies, correctly reproduces the (local) supergraviton interaction of supergravity.

### 4.6. Low-energy supergraviton scattering

A convenient way to study the scattering of two supergravitons with low \textit{transverse} velocities is to compute the effective action by expanding the matrix model action around a corresponding classical configuration. I will present the one-loop computation following \cite{9} where further details can be found. For a scattering process with relative transverse velocity \( v \) and impact parameter \( b \) one can e.g. expand the \( X^i \) as follows:

\[
X^8 = \frac{1}{2} v t \sigma_3 + \sqrt{g_s} \delta X^8, \\
X^9 = \frac{1}{2} b \sigma_3 + \sqrt{g_s} \delta X^9, \\
X^i = \sqrt{g_s} \delta X^i, \quad i \neq 8, 9
\]  

(56)

Hamiltonian acting on a state in some Hilbert space, so the appropriate statement should be: “If on a given state \( |\Psi\rangle \) the off-block-diagonal elements of the \( Y^i \) and \( \Pi_i \) are vanishing then \( H|\Psi\rangle = \sum_a H_a |\Psi\rangle \) where each \( H_a \) is a trace over (operator-valued) \( N_a \times N_a \) matrices.

where \( \sigma_3 \) is the Pauli matrix and the \( \delta X \) are the quantum fluctuations around the given classical configuration. It is easy to see that the classical configuration \( \delta X = 0 \) indeed corresponds to the desired scattering process in a reference frame where the total \textit{transverse} center-of-mass momentum and position vanish \( (\text{tr} \sigma_3 = 0) \). Indeed, the \( 2 \times 2 \) matrices are block-diagonal corresponding to two “clusters” of D0-branes with \( N_1 = N_2 = 1 \). According to the definition \cite{15} for the distance between the two supergravitons one indeed finds \( r = r_{1,2} = \sqrt{(vt)^2 + b^2} \). This is appropriate for two particles that do not interact in a first approximation and have impact parameter \( b \). The interaction will manifest itself only through a phase shift.

Expanding the action then yields a collection of massless and massive modes depending on \( b \) and \( t \). As shown in \cite{15} the bosonic fields, including the gauge field, yield 16 modes with masses \( m^2 = r_{1,2}^2 \equiv r^2 \), two modes with \( m^2 = r^2 + 2v \) and two others with \( m^2 = r^2 - 2v \), as well as ten massless modes. There are also eight real fermions with masses \( m^2 = r^2 + v \) and eight real fermions with masses \( m^2 = r^2 - v \), as well as the ghost fields: two complex bosons with \( m^2 = r^2 \) and one massless complex boson. Collecting all the determinants from integrating the massive fields we get

\[
D_{\text{tot}} = D_0^{-3} D_2^{-1} D_{-2}^{-1} D_2^{1} D_{-2}^{1} \quad (57)
\]

where, now with a euclidean time \( \tau \) (and a euclidean velocity, still denoted by \( v \)),

\[
D_\alpha = \det(-\partial_\tau^2 + r^2 + \alpha) = \det(-\partial_\tau^2 + b^2 + v^2 \tau^2 + \alpha). \\

(58)
\]

Note that the sum of the exponents on the right hand side of \cite{15} vanishes, thanks to supersymmetry. Hence \( \log D_{\text{tot}} \) is not affected by an additive (and possibly divergent) ambiguity \( \log D_\alpha \rightarrow \log D_\alpha + d \), provided the constant \( d \) does not depend on \( \alpha \). Thus the (euclidean) effective one-loop action is

\[
S_{\text{eff}} = S_0 - \log D_{\text{tot}} \\

(59)
\]

so that one identifies the one-loop effective potential \( V_{\text{eff}} \) as

\[
- \log D_{\text{tot}} = \int d\tau V_{\text{eff}}(r(\tau))
\]
\[
\begin{align*}
\mathcal{V}_\text{eff}(\sqrt{b^2 + v^2 r^2}) \pm \int \text{d}x \mathcal{V}_\text{eff}(\sqrt{b^2 + v^2 r^2}).
\end{align*}
\]

As usual, to obtain the one-loop effective potential, all one needs to do is to compute the determinants \( D_\alpha \).

To this end, consider the quantum mechanical Hamiltonian of a harmonic oscillator of unit mass
\[
H_\omega = \frac{1}{2} (P^2 + \omega^2 Q^2), \quad [Q, P] = i.
\]

It is well known that the matrix elements of the euclidean evolution operator for a time interval \( 2s \) are
\[
\langle q' | e^{-(2s)H_\omega} | q \rangle = U(\omega, 2s, q', q)
\]
\[
= \left( \frac{\omega}{2 \pi \sinh 2s\omega} \right)^{1/2} \times
\]
\[
\times \exp \left( -\frac{\omega}{2} \left( q^2 + q'^2 \right) \cosh 2s\omega - 2q q' \right) \sinh 2s\omega.
\]

On the other hand one has
\[
\log \det(2H_\omega + \lambda) = \text{tr} \log(2H_\omega + \lambda)
\]
\[
= -\text{tr} \int_0^\infty \frac{ds}{s} e^{-2sH_\omega - s\lambda}
\]
\[
= -\int_0^\infty \frac{ds}{s} e^{-s\lambda} \int_{-\infty}^\infty \text{d}q U(\omega, 2s, q, q).
\]

All these integrals are divergent as \( s \to 0 \). What is finite and makes sense are the derivatives with respect to \( \lambda \). This is what I mean by the “\( \infty \)” sign. Said differently, when expanding in powers of \( \lambda \), one has an equality for all terms with non-zero powers of \( \lambda \), while the \( \lambda \)-independent terms differ by a divergent constant. As remarked above, these divergent constants cancel when computing \( \log D_\text{tot} \), and thus do not affect the validity of the present computation. Inserting then the explicit expression \([52]\) for \( U(\omega, 2s, q, q) \) into \([53]\) and performing the gaussian integration over \( q \), one gets
\[
- \log \det(2H_\omega + \lambda) \simeq \int_0^\infty \frac{ds}{s} e^{-s\lambda} \times
\]
\[
\times \begin{array}{c}
\text{d}x \mathcal{V}_\text{eff}(\sqrt{b^2 + v^2 r^2}) \pm \int \text{d}x \mathcal{V}_\text{eff}(\sqrt{b^2 + v^2 r^2}).
\end{array}
\]

Now observe that, if we replace \( q \) by \( \tau \) so that
\[
P^2 + \omega^2 Q^2 \rightarrow -\partial_\tau^2 + \omega^2 \tau^2,
\]
the determinants \([58]\) we are interested in are
\[
D_\alpha = \det(2H_\omega + b^2 + \alpha)
\]
i.e. \( \lambda = b^2 + \alpha \), so that
\[
- \log D_\text{tot} = \int_0^\infty \frac{ds}{s} e^{-s\lambda} \times
\]
\[
\times ( -6 - 2 \cosh 2sv + 8 \cosh sv ) (66)
\]
As promised, there is no divergence as \( s \to 0 \). For large impact parameter \( b \), only small \( s \) contribute significantly to the integral and
\[
- \log D_\text{tot} \simeq - \frac{\nu^3}{b^6} + \mathcal{O} \left( \frac{\nu^5}{b^{10}} \right).
\]

Equation \((60)\) then gives
\[
\int \text{d}r \mathcal{V}_\text{eff}(\sqrt{b^2 + v^2 r^2}) = - \frac{\nu^3}{b^6} + \mathcal{O} \left( \frac{\nu^5}{b^{10}} \right)
\]
which yields the effective long-range potential
\[
V_\text{eff}(r) = - \frac{15 \nu^4}{16 r^7} + \mathcal{O} \left( \frac{\nu^6}{r^{11}} \right).
\]

What is this supposed to mean for the scattering of eleven dimensional (super)gravitons? Scattering in the IMF should be described by a non-relativistically looking time-independent potential at vanishing \( p_{11} \)-transfer. Remarkably, the potential \([68]\) exactly coincides with the corresponding result from eleven dimensional supergravity. In particular, the factor \( 1/r^7 \) comes from the eleven dimensional propagator of massless fields. It is the time-independent space propagator at vanishing longitudinal momentum \( (p_{11}) \), i.e. integrated over \( x^{11} \): in \( d \) space-time dimensions such a propagator is
\[
\int \frac{d^{d-1} p}{p^2} \delta(p_\perp) e^{i p \cdot x} / p^2 \sim 1/x^{d-4},
\]
which for \( d = 11 \) indeed gives \( 1/r^7 \). The velocity dependence is also correct and, maybe even more remarkably, also the numerical factor. So the one-loop matrix model computation already gives the full and correct answer. Higher loop corrections to the \( v^4 \) term would ruin this agreement. Luckily, at two loops there are none \([9]\), and probably the one-loop result is the full answer. The present computation was done for \( N_1 = N_2 = 1 \) but it is easy to reinstate the dependence on arbitrary cluster sizes \( N_1, N_2 \).

It is remarkable that the simple matrix model knows quite a lot about propagating massless gravitons in eleven dimensions. This is a non-trivial check for the matrix model description.
of M-theory in the IMF. But M-theory also has membrane configurations. How can they appear in the matrix model?

4.7. Membranes in the matrix model

In order to see how the matrix model could describe (super)membranes, let me first discuss the description of the latter. Just as classical superstrings can only exist in certain space-time dimensions, the classical supermembrane also cannot exist for all $d$. But there does exist one in eleven dimensions. It is described by bosonic coordinates $y^\mu(p,q,\tau)$ and their superpartners. Here $p \equiv \sigma^{1}$, $q \equiv \sigma^{2}$ and $\tau$ are the three world-volume coordinates, and the $y^\mu$ describe how the membrane is embedded into the eleven dimensional target space. In a Hamiltonian formalism, no explicit $\tau$ dependence appears, $y^\mu(p,q,\tau) \rightarrow y^\mu(p,q)$, and all $\tau$-derivatives are replaced by the corresponding momenta $\pi_\mu(p,q)$. In the light-cone frame only the transverse components with $i = 1, \ldots 9$ are dynamical and the membrane Hamiltonian is

$$H_m = \frac{1}{2p_{11}} \int \frac{dpdq}{(2\pi)^2} \pi_i^2(p,q) + \frac{(2\pi T_2^m)^2}{2p_{11}} \int \frac{dpdq}{2} \left\{ \{y_i(p,q), y_j(p,q)\}\right\}^2 + \text{fermion terms}$$

(69)

where I used the suggestive (standard) notation

$$\{A, B\} = \partial_q A \partial_p B - \partial_p A \partial_q B$$

(70)

for any two functions $A, B$ of $q$ and $p$ and where $T_2^m$ is the membrane tension as I will show soon. The analogy with the matrix model Hamiltonian is striking. Basically all one needs to do is to trade the two discrete matrix indices for the continuous variables $q, p$ in the limit where the size $N$ of the matrices goes to infinity. The membrane world-volume is taken to factorize as $\Sigma \times \mathbb{R}^2$ with some Riemann surface $\Sigma$. I will only deal with membranes of toric topology, i.e. with $\Sigma$ being a torus. Now let me show that the second term in the membrane Hamiltonian is correctly normalised. For a configuration with vanishing transverse momentum, $\pi_i = 0$, the mass $M$ of the membrane is $M^2 = 2p_{11}H$. If furthermore there are no fermionic excitations, one has from eq. (43)

$$M^2 = \frac{(2\pi T_2^m)^2}{2} \int dpdq \left\{ \{y_i(p,q), y_j(p,q)\}\right\}^2$$

(71)

But the area $A$ of the membrane is given by

$$A^2 = \frac{(2\pi)^2}{2} \int dpdq \sum_{i<j} \left\{ \{y_i(p,q), y_j(p,q)\}\right\}^2$$

$$= \frac{1}{2} \frac{(2\pi)^2}{2} \int dpdq \left\{ \{y_i(p,q), y_j(p,q)\}\right\}^2$$

(72)

as one can see by considering the special case $y^6(p,q) = \frac{2\pi}{L_8}$, $y^6(p,q) = \frac{2\pi}{L_9}$, and all $\tau$-derivatives are replaced by the corresponding momenta $\pi_\mu(p,q)$. In the light-cone frame only the transverse components.

Let me now show how the matrix model can yield the above membrane Hamiltonian and what its prediction for the membrane tension is. For toric membranes, the $y^i(p,q)$ are doubly periodic functions and their expansions yield the Fourier modes $y^i_{mn}$. These form nine $\infty \times \infty$ matrices just as would do the nine $Y^i$ of the matrix model in the $N \rightarrow \infty$ limit. However, one still needs to perform a change of basis, so that the commutator $\{Y^i, Y^j\}$ directly goes over into the bracket $\{y^i, y^j\}$. This is achieved by the following trick.

On the space of $N \times N$ matrices introduce two matrices $U$ and $V$ such that

$$U^N = V^N = 1 \quad \text{and} \quad UV = e^{2\pi i/N} VU$$

(73)

A particular realisation is given by $U_{ij} = 1$ and $V_{ij} = e^{2\pi i(j-1)/N}$ with all other matrix elements vanishing. A more abstract realisation is

$$U = e^{i p} , \quad V = e^{i q} , \quad [p,q] = \frac{2\pi}{N} i$$

(74)

in terms of two operators/matrices that behave like position and momentum on a discrete and compactified space: $U^N = V^N = 1$ implies that $p$ and $q$ have eigenvalues $0, \frac{2\pi}{N}, 2 \frac{2\pi}{N}, \ldots (N-1) \frac{2\pi}{N}$. It follows that

$$\text{tr} U^n V^m = N \delta_{n,0 \text{mod} N} \delta_{m,0 \text{mod} N}.$$
This allows us to “Fourier” decompose any $N \times N$ matrix $Z$ as follows

$$Z = \sum_{n,m=-N/2+1}^{N/2} z_{nm} U^n V^m,$$

$$z_{nm} = \frac{1}{N} \text{tr} U^{-n} Z V^{-m}. \quad (76)$$

If the matrices $U$ and $V$ are written as $e^{ip}$ and $e^{iq}$ then one simply has

$$Z = \sum_{n,m=-N/2+1}^{N/2} z_{nm} e^{ip} e^{imq} \quad (77)$$

Now consider what happens in the $N \to \infty$ limit. As $N \to \infty$, $p$ and $q$ commute and their eigenvalues fill $[0, 2\pi] \times [0, 2\pi]$ with $2\pi$ and $0$ identified, i.e. $(p, q)$ take values on a two-torus. Equation (77) then really is nothing but the standard Fourier decomposition of a double-periodic function on a circle. Let’s call this function

$$z(p, q) = \sum_{n,m=-\infty}^{\infty} z_{nm} e^{inp} e^{imq} \quad (78)$$

with, of course,

$$z_{nm} = \int_0^{2\pi} \int_0^{2\pi} \frac{dp dq}{2\pi^2} z(p, q) e^{-inp} e^{-imq}. \quad (79)$$

Since $\text{tr} Z = N z_{00}$ it follows that in the $N \to \infty$ limit one has

$$\text{tr} Z \to N \int_0^{2\pi} \int_0^{2\pi} \frac{dp dq}{2\pi^2} z(p, q). \quad (80)$$

Next, I will show that the commutator of two matrices goes over to the bracket (74) of the two corresponding functions. First note that $[U^n, V^m] = 2i \sin \frac{2\pi m}{N} e^{inp} e^{imq}$. It follows that

$$\frac{N}{2\pi} [U^n V^k, U^m V^l] = (nl - km) e^{i(n+m)p+i(k+l)q} + O(1/N) \quad (81)$$

But as $N \to \infty$ this precisely goes to the bracket $\{u^n v^k, u^m v^l\}$ where $u(p, q) = e^{ip}$ and $v(p, q) = e^{iq}$ are the classical functions associated to $U$ and $V$. By the bilinearity of the commutator and the bracket it then follows for any two $N \times N$ matrices $Z, W$ and their associated classical functions $z(p, q), w(p, q)$ one has

$$\frac{N}{2\pi} [Z, W] \to \{z(p, q), w(p, q)\}. \quad (82)$$

Now given this correspondence and the form of the matrix model and supermembrane Hamiltonians it is clear that the latter will turn out to be the large $N$ limit of the former. However, one has to carefully define the conjugate momentum: the classical function $\pi_i(p, q)$ corresponding to the matrix $H_i$ is not the canonical conjugate momentum of $y'(p, q)$ corresponding to $y^i$, but they differ by a factor of $N$ as one can see by first working out the large $N$ limit of the Lagrangian (83) (I will only write the bosonic terms):

$$L_{\text{bos}} \to \frac{N}{2R} \int \frac{dp dq}{2\pi} (y'^i(p, q))^2 - \frac{R}{4N} c^2 T_0^2 \int dp dq (\{y^i, y^j\})^2 \quad (83)$$

Recall that $N/R = p_{11}$ so that the momentum conjugate to $y'(p, q)$ is $\pi_i(p, q) = p_{1\overline{i}} y'^i(p, q)$ (and not $\frac{1}{R} \dot{y}^i$). It follows that the $N \to \infty$ limit of the matrix model precisely gives the Hamiltonian (83) of the supermembrane with a membrane tension given by

$$(2\pi T_0^m)^2 = c^2 T_0^2 = \frac{1}{(2\pi \alpha')^2} T_0^2. \quad (84)$$

But according to eq. (17), $T_0/(2\pi \alpha') = 2\pi T_2$ so that the matrix model yields a membrane with tension $T_2 = T_2$ equal to the D2-brane tension, i.e. the correct membrane tension of M-theory.

We have seen how a given matrix configuration in the large $N$ limit yields some membrane configuration, although generally a highly irregular one. Conversely to obtain a given membrane configuration of toroidal topology, one starts with its embedding functions $y'(p, q)$ in $\mathbb{R}^9$, such that they differ by a factor of $N$ as one can see by first working out the large $N$ limit of the Lagrangian (83) (I will only write the bosonic terms):

$$L_{\text{bos}} \to \frac{N}{2R} \int \frac{dp dq}{2\pi} (y'^i(p, q))^2 - \frac{R}{4N} c^2 T_0^2 \int dp dq (\{y^i, y^j\})^2 \quad (83)$$

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We have seen how a given matrix configuration in the large $N$ limit yields some membrane configuration, although generally a highly irregular one. Conversely to obtain a given membrane configuration of toroidal topology, one starts with its embedding functions $y'(p, q)$ in $\mathbb{R}^9$, such that they differ by a factor of $N$ as one can see by first working out the large $N$ limit of the Lagrangian (83) (I will only write the bosonic terms):

$$L_{\text{bos}} \to \frac{N}{2R} \int \frac{dp dq}{2\pi} (y'^i(p, q))^2 - \frac{R}{4N} c^2 T_0^2 \int dp dq (\{y^i, y^j\})^2 \quad (83)$$

Recall that $N/R = p_{11}$ so that the momentum conjugate to $y'(p, q)$ is $\pi_i(p, q) = p_{1\overline{i}} y'^i(p, q)$ (and not $\frac{1}{R} \dot{y}^i$). It follows that the $N \to \infty$ limit of the matrix model precisely gives the Hamiltonian (83) of the supermembrane with a membrane tension given by

$$(2\pi T_0^m)^2 = c^2 T_0^2 = \frac{1}{(2\pi \alpha')^2} T_0^2. \quad (84)$$

But according to eq. (17), $T_0/(2\pi \alpha') = 2\pi T_2$ so that the matrix model yields a membrane with tension $T_2 = T_2$ equal to the D2-brane tension, i.e. the correct membrane tension of M-theory.
the light cone, computes its Fourier coefficients $y_{i}^{m n}$ and defines for every finite $N$ the matrix $Y_{i}^{(N)} = \sum_{m,n=-N/2+1}^{N/2} y_{i}^{m n} U^{m} V^{n}$. In particular, if the membrane is smooth, coefficients $y_{i}^{m n}$ with large $m$ or $n$ will be small, and the information lost about the membrane by including only $|m|, |n| \leq N/2$ will be small. To describe membranes of non-toroidal topology in the matrix model is more subtle. An exception is the plane membrane e.g. obtained from the example following eq. (72) in the limit $L_{8}, L_{9} \to \infty$.

5. CONCLUSIONS AND NO FURTHER DEVELOPMENTS

I will not talk about further developments, not because there are none but because there are too many. I will not even attempt to give references. So let me just briefly conclude what we have seen.

In the first lecture, I briefly reviewed D-branes explaining why $N$ D$p$-branes should be described by ten dimensional $U(N)$ super Yang-Mills theory reduced to $p+1$ dimensions. In the second lecture, I introduced M-theory as the eleven dimensional theory that, when formulated on $R^{10} \times S^{1}$ with $S^{1}$ of radius $R$, is equivalent to the IIA superstring on $R^{10}$ and with string coupling constant $g_{s} = R \sqrt{\alpha^{'}}$ In the third lecture, I developed the ideas of BFSS and described their matrix model for M-theory in the infinite momentum frame, as well as several checks of this conjecture: 1) the matrix model contains the full Fock space of an arbitrary number of supergravitons (supergravity multiplets of 256 states); 2) remarkably, it gives the correct result for low-energy supergraviton scattering (including terms up to $v^{4}$) up to and including a matrix-model two-loop computation; 3) the matrix model contains (super) membranes, and in the large $N$ limit the matrix model dynamics goes over to the dynamics of the corresponding (super)membranes. The tension of these matrix model membranes agrees with the tension of the M-theory membranes.

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