Ordinary Least Squares based on Fourier Series Residuals for fitting the Modified Exponential Growth Model

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Abstract. It is well-known that grey system theory was put forward in the statistical literature to overcome the problem of partially unknown parameters, therefore it is given the attention of the researchers in many scientific areas. Therefore, many methods have been presented in which the grey system is combined with the Fourier series and others for more accurate prediction. This paper presents a methodology to modify the method of least squares based on Fourier series residuals that are computed by using grey system theory. The objective of this methodology is to improve the estimates of a modified exponential growth model. The performance of the proposed method by using simulation is compared with two famous methods and the results show our proposed method outperforms more than others and it is highly efficient and reliable. On the practical, the new method was used to predict the price of a barrel of crude oil for the OPEC basket for the period from Jul 2019 to Dec 2019, and It can be seen that OPEC basket prices will decline in the end of 2019.

1. Introduction

The grey system theory is used in a number of varying scientific fields and was presented for the first time in 1982 by Deng [1]. Since then, it has gained a good reputation for its ability to deal with systems where parameters are partially unknown. It outperformed conventional statistical models, as the grey models require only limited data to estimate the unknown behaviour of systems.

The grey system theory has been used in a lot of scientific research and has been presented at several scientific conferences, and it was included in a number of Australian, Chinese, Japanese, Taiwanese and American university curricula in the early 1990s. A private association was founded for the grey system in China in 1996 and began to spread in scientific journals, such as Grey System Magazine, which has been published in Britain since 1989 [2]. Wu and Du [3] explained that using the Fourier series with grey systems produces accurate estimates when the sample has significant fluctuations. Huang and Jane [4] adopted a hybrid model to predict the stock market forecasting and portfolio selection mechanism in Taiwan. It was a combination of a moving average autoregressive model (ARX), Grey systems (GM) and Rough Set (RS) theories. They show that the hybrid method has a better forecasting than the GM(1,1) method but obtains a greater rate of the selected stocks. Huang and Lee [5] displayed a volatile time series of the demand for tourism in Taiwan, so they combined the grey forecasting model (GM) and a modified Fourier series residual model. This study improved the accuracy of short-term forecasting in cases involving sample data with high fluctuation. Mohammad et al. [6] have predicted CO2 emissions in Iran based on Grey Systems and the
Autoregressive Integrated Moving Average. The results show the accuracy of the Grey System forecasting method is higher than the others. The forecast of carbon dioxide emissions will reach up to 925.68 million tons in 2020, which include an increase of 66 percent compared to 2010. Shaghayegh et al. [7] presented the application of a Grey Verhulst model in studying the global ICT development. Moreover, a new Grey Verhulst model and Fourier Residual Grey Verhulst model were suggested as a way to improve the forecasting accuracy. Data of the world’s fixed telephone subscriptions, which is an ICT indicator, from 2001 to 2010, were used as a forecasted example. They increased the precision by employing the Grey-Markov model, which is a mixture of the $GM(1,1)$ and a Markov model.

To reach the aim, this paper was presented as follows. In Section 2, Modified Exponential Growth Curve and Fourier Series Residual Modification Grey Forecasting Model (FGM) are presented. In addition to that, we explained the Modified Least Squares Estimation by using Fourier Series Residuals. In Section 3, simulation techniques was used to perform the required comparisons, and in Section 4 we illustrated the performance of our method via analysis of the simulated dataset. We conclude with a brief conclusion in Section 6.

2. Methods

2.1. Modified Exponential Growth Curve

The growth curve can be defined as any expression of a population size that can be represented as a variable function of time ($t$) so that it describes the path of growth [8]. Growth curves are considered to be amongst the most important mathematical functions that deal with longitudinal data of repeated observations, depending on the effective time that can be formulated as a stochastic process [9]. This type of data differs from most data types, as they must relate the current response to the past time.

This curve is one of the most famous exponential growth curves and has been used in a wide range of applications and represents the amount of growth declining at a constant rate. The curve has an upper limit that cannot be exceeded [10].

The general formula is:

$$f(t) = \alpha + \lambda t$$  \hspace{1cm} (1)

Where $\alpha$ is the maximum size of the population after a period of time ($t \to \infty$), $\lambda$ is the size at time zero, and $\eta$ is the relative Growth Rate.

The advantage of this curve is that it fits the growth of populations in which the size does not exceed the upper limit. However, it is not suitable for those where the relative growth rate is not decreasing ($\eta > 1$) and the amount of growth is not constant.

2.2. Fourier Series Residual Modification Grey Forecasting Model (FGM)

The grey model $GM(1,1)$ is one of the most widely used Grey forecasting models and was developed by Deng (1982). It is employed as a specialised model to predict a time-series and includes a set of differential equations that are adapted for this purpose and can deal with series that include more than four observations [11].

The Grey prediction model is based on three processes: (1) Accumulated generation, (2) Inverse accumulated generation, and (3) Grey modelling [12].

Assuming the data is a series, it can be described as the process of construction $GM(1,1)$ as follows [13]:

$$X^{(0)} = (X_1^{(0)}, X_2^{(0)}, \ldots, X_n^{(0)})$$  \hspace{1cm} (2)

Where the series is non-negative and $n$ is the sample size.

Accumulating Generation Operation (AGO) is used, depending on the previous series’ data, to obtain a new increasingly generating series, by taking the symbol $(X^{(1)})$, and:
\[ X^{(1)} = \left( X_1^{(1)}, X_2^{(1)}, \ldots, X_n^{(1)} \right) \]  
\[ X_k^{(1)} = \sum_{i=1}^{k} X_i^{(0)}, \quad k = 1,2,\ldots,n \]  
\[ X_1^{(1)} = X_1^{(0)} \]  
\[ \hat{X}_k^{(1)} = \hat{X}_{k+1}^{(1)} - \hat{X}_1^{(1)} \]  
\[ e^{(0)} = X_k^{(0)} - \hat{X}_k^{(0)}, \quad k = 2,3,\ldots,n \]  
\[ e_k^{(0)} = \frac{1}{2} \hat{\alpha}_0 + \sum_{i=1}^{n} \left[ \hat{\alpha}_i \cos \left( \frac{2\pi i}{T} k \right) + \hat{\beta}_i \sin \left( \frac{2\pi i}{T} k \right) \right], \quad k = 2,3,\ldots,n \]  
\[ P = \begin{bmatrix} \frac{1}{2} \cos \left( \frac{2\pi}{T} 1 \right) \sin \left( \frac{2\pi}{T} 1 \right) & \cdots & \cos \left( \frac{2\pi}{T} 2 \right) \sin \left( \frac{2\pi}{T} 2 \right) \\ \frac{1}{2} \cos \left( \frac{2\pi}{T} 3 \right) \sin \left( \frac{2\pi}{T} 3 \right) & \cdots & \cos \left( \frac{2\pi}{T} 3 \right) \sin \left( \frac{2\pi}{T} 3 \right) \\ \vdots & \vdots & \vdots \\ \frac{1}{2} \cos \left( \frac{2\pi}{T} n \right) \sin \left( \frac{2\pi}{T} n \right) & \cdots & \cos \left( \frac{2\pi}{T} n \right) \sin \left( \frac{2\pi}{T} n \right) \end{bmatrix} \]  
\[ C = \begin{bmatrix} \hat{\alpha}_0, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \ldots, \hat{\alpha}_n, \hat{\beta}_n \end{bmatrix} \]  
\[ \text{By solving equation (10) using the Ordinary Least Squares method, we get the following equation:} \]  
\[ C = (P^T P)^{-1} P^T \hat{e}^{(0)} \]  
\[ \hat{\alpha}_i, \hat{\beta}_i \]  
\[ \hat{\varepsilon}_k^{(0)} = \left( \hat{\varepsilon}_2^{(0)}, \hat{\varepsilon}_3^{(0)}, \ldots, \hat{\varepsilon}_n^{(0)} \right) \]  
\[ \hat{X} f_k = \hat{X}_k^{(1)} = \hat{X}_1^{(1)} \]  
\[ \hat{X} f_k = \hat{X}_k^{(0)} + \hat{\varepsilon}_k^{(0)}, \quad k = 2,3,\ldots,n \]  
\[ f(t) = a + \lambda t + \varepsilon \]  
\[ \hat{X} = f(t; \hat{\theta}) + \hat{\varepsilon} \]
Where \( \xi = (\xi_1, \xi_2, \ldots, \xi_k) \), \( \theta = (\alpha, \eta, \lambda) \), and \( \xi \) represented the random errors of rank \((n \times 1)\). and \( f(t; \theta) = \alpha + \eta t^\lambda \), \( E(X) = f(t; \theta) \)

It is assumed that \( \varepsilon \sim N(0, \sigma^2 I_n) \).

Rewritten the equation (16) for the observation \((i)\) as follows:

\[
X_i = f(t_i; \theta) + \varepsilon_i
\]

(17)

Where \( f(t_i; \theta) \) is the nonlinear function [16].

Using the Taylor’s series, the nonlinear function \( f(t_i; \theta) \) can be rewritten as follows [17]:

\[
f(t_i; \theta) = f(t_i; \theta^{(0)}) + \sum_{j=1}^{p} \left[ \frac{\partial f(x; \theta)}{\partial \theta_j} \right]_{\theta = \theta^{(0)}} (\theta_j - \theta_j^{(0)})
\]

(18)

Briefly

\[
X_i - f_i^{(0)} = \sum_{j=1}^{p} D_{ij}^{(0)} \varphi_j^{(0)} + \varepsilon_i
\]

(19)

Where \( f_i^{(0)} = f(t_i; \theta^{(0)}) \), \( \varphi_j^{(0)} = \theta_j - \theta_j^{(0)} \) and \( D_{ij}^{(0)} = \left[ \frac{\partial f(x; \theta)}{\partial \theta_j} \right]_{\theta = \theta^{(0)}} \)

At this stage, we employ the least squares method to estimate the parameters \((\varphi_j^{(0)})\).

Let

\[
D^{(0)} = \begin{bmatrix}
D_{11}^{(0)} & D_{12}^{(0)} & \cdots & D_{1p}^{(0)} \\
D_{21}^{(0)} & D_{22}^{(0)} & \cdots & D_{2p}^{(0)} \\
\vdots & \vdots & \ddots & \vdots \\
D_{n1}^{(0)} & D_{n2}^{(0)} & \cdots & D_{np}^{(0)}
\end{bmatrix}
\]

and \( \hat{\varphi}^{(0)} = \begin{bmatrix}
\hat{\varphi}_1^{(0)} \\
\hat{\varphi}_2^{(0)} \\
\vdots \\
\hat{\varphi}_p^{(0)}
\end{bmatrix} \)

Where \( \hat{\varphi}^{(0)} \) is the least squares estimation of the parameters \((\varphi^{(0)})\) of rank \((p \times 1)\).

Assuming \( X' = X - f^{(0)} \) then:

\[
X' = D^{(0)} \varphi^{(0)} + \varepsilon
\]

(20)

We can use the Modified Exponential Growth model with the Fourier series residuals \((FOLS)\) by employing the previous steps of the Grey model modified by Fourier series residuals. The residuals resulting from the estimation exponential growth model by using the least squares method \((OLS)\) can be used instead of the Inverse Accumulating Generation Operation \((IAGO)\), described in equation (5) for obtaining the residual series [5].

3. Simulation Study

The data was generated in five different samples (using sizes 8, 10, 15, 20 and 25) consistent with the modified exponential growth model based on the initial values, noting that the observations represented the income growth data for a special case [18]. As shown in Table 1, eight models were generated based on equation (15) relying on the error that follows the normal distribution with mean zero and standard deviation (0.5, 1.5).
Table 1. Initial values of the exponential growth model parameters

| Model       | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|
| Modified    | α     | 14.1  | 14.1  | 14.1  | 14.5  | 14.5  | 14.5  | 14.5  |
|             | η     | 3.2   | 3.2   | 3.5   | 3.5   | 3.2   | 3.2   | 3.5   |
| Exponential | λ     | 1.2   | 1.2   | 1.2   | 1.2   | 1.2   | 1.2   | 1.2   |
|             | σ     | 0.5   | 1.5   | 0.5   | 1.5   | 0.5   | 1.5   | 1.5   |

We estimated modified exponential growth model parameters by using $FGM$, $OLS$ and $FOLS$ depending on 1000 run size, and various criteria were used to compare these methods.

The Mean Absolute Percentage Error ($MAPE$) were used to compare between the methods [2][7] that shown in Table 2. The results shown that 36 out of 40 give preference to $FOLS$. As well that 31 out of 40 give preference to $OLS$ when comparing $FGM$ with $OLS$. Note that $FGM$ is better in the case of generated models of even sequences (standard deviation 1.5) with a small sample size (i.e. 8 and 10 only).

With regards to Table 3 (including Root Mean Squares Error ($RMSE$)), the results have given preference to $FOLS$ compared with other methods for all $RMSE$. 


Table 2. Mean Absolute Percentage Error (MAPE)

| n  | Model  | FGME | OLS  | FOLS | Best |
|----|--------|------|------|------|------|
| 8  | 1      | 0.1843| 0.1807| 0.1628| FOLS |
|    | 2      | 0.1689| 0.4489| 0.2504| FGME |
|    | 3      | 0.1855| 0.1761| 0.1400| FOLS |
|    | 4      | 0.1822| 0.3661| 0.2581| FGME |
|    | 5      | 0.1805| 0.1569| 0.1452| FOLS |
|    | 6      | 0.2707| 0.3968| 0.1744| FOLS |
|    | 7      | 0.1825| 0.1797| 0.1595| FOLS |
|    | 8      | 0.1898| 0.4190| 0.3102| FGME |
| 10 | 1      | 0.2069| 0.1425| 0.1222| FOLS |
|    | 2      | 0.1842| 0.2493| 0.1694| FOLS |
|    | 3      | 0.1721| 0.1504| 0.1269| FOLS |
|    | 4      | 0.1737| 0.1971| 0.1458| FOLS |
|    | 5      | 0.1553| 0.1410| 0.1271| FOLS |
|    | 6      | 0.2321| 0.2914| 0.2365| FGME |
|    | 7      | 0.2344| 0.1664| 0.1538| FOLS |
|    | 8      | 0.2118| 0.2506| 0.1609| FOLS |
| 15 | 1      | 0.2688| 0.0961| 0.0818| FOLS |
|    | 2      | 0.2692| 0.1713| 0.0684| FOLS |
|    | 3      | 0.2605| 0.0955| 0.0727| FOLS |
|    | 4      | 0.2385| 0.2392| 0.1062| FOLS |
|    | 5      | 0.2577| 0.1083| 0.0717| FOLS |
|    | 6      | 0.2501| 0.1572| 0.0735| FOLS |
|    | 7      | 0.2485| 0.1094| 0.0823| FOLS |
|    | 8      | 0.3203| 0.1681| 0.1250| FOLS |
| 20 | 1      | 1.3944| 0.0894| 0.0588| FOLS |
|    | 2      | 1.4909| 0.1970| 0.1161| FOLS |
|    | 3      | 1.5337| 0.1022| 0.0655| FOLS |
|    | 4      | 1.5272| 0.1385| 0.0659| FOLS |
|    | 5      | 1.3579| 0.0857| 0.0598| FOLS |
|    | 6      | 1.3640| 0.1065| 0.0590| FOLS |
|    | 7      | 1.4702| 0.0825| 0.0777| FOLS |
|    | 8      | 1.5072| 0.1489| 0.0925| FOLS |
| 25 | 1      | 7.2085| 0.0810| 0.0654| FOLS |
|    | 2      | 7.3380| 0.1262| 0.0490| FOLS |
|    | 3      | 7.5586| 0.0622| 0.0402| FOLS |
|    | 4      | 7.8484| 0.1460| 0.0847| FOLS |
|    | 5      | 7.1013| 0.0659| 0.0446| FOLS |
|    | 6      | 7.0131| 0.1161| 0.0565| FOLS |
|    | 7      | 7.4854| 0.0731| 0.0568| FOLS |
|    | 8      | 7.4953| 0.1248| 0.0612| FOLS |
| n   | Model | FGM   | OLS   | FOLS  | Best |
|-----|-------|-------|-------|-------|------|
|     |       |       |       |       |      |
| 8   | 1     | 0.2140| 0.3764| 0.2131| FOLS |
|     | 2     | 0.6545| 1.1465| 0.6512| FOLS |
|     | 3     | 0.2174| 0.3808| 0.2173| FOLS |
|     | 4     | 0.6369| 1.110 | 0.6361| FOLS |
|     | 5     | 0.2214| 0.3810| 0.2206| FOLS |
|     | 6     | 0.6544| 1.1472| 0.6489| FOLS |
|     | 7     | 0.2149| 0.3778| 0.2149| FOLS |
|     | 8     | 0.6462| 1.1268| 0.6449| FOLS |
| 10  | 1     | 0.1917| 0.3980| 0.1900| FOLS |
|     | 2     | 0.5682| 1.1896| 0.5663| FOLS |
|     | 3     | 0.1938| 0.4015| 0.1935| FOLS |
|     | 4     | 0.5945| 1.2078| 0.5931| FOLS |
|     | 5     | 0.1946| 0.4007| 0.1941| FOLS |
|     | 6     | 0.5869| 1.2132| 0.5845| FOLS |
|     | 7     | 0.1994| 0.4029| 0.1973| FOLS |
|     | 8     | 0.5956| 1.2239| 0.5952| FOLS |
| 15  | 1     | 0.1130| 0.4448| 0.1034| FOLS |
|     | 2     | 0.2966| 1.3141| 0.2929| FOLS |
|     | 3     | 0.1123| 0.4490| 0.1029| FOLS |
|     | 4     | 0.3136| 1.3239| 0.3094| FOLS |
|     | 5     | 0.1104| 0.4442| 0.1008| FOLS |
|     | 6     | 0.2923| 1.3115| 0.2903| FOLS |
|     | 7     | 0.1103| 0.4465| 0.1010| FOLS |
|     | 8     | 0.3239| 1.3284| 0.3189| FOLS |
| 20  | 1     | 0.5121| 0.4763| 0.1395| FOLS |
|     | 2     | 0.6499| 1.4014| 0.4187| FOLS |
|     | 3     | 0.5895| 0.4725| 0.1435| FOLS |
|     | 4     | 0.6818| 1.3961| 0.4169| FOLS |
|     | 5     | 0.5074| 0.4714| 0.1400| FOLS |
|     | 6     | 0.6181| 1.4008| 0.4097| FOLS |
|     | 7     | 0.5691| 0.4738| 0.1407| FOLS |
|     | 8     | 0.6634| 1.3966| 0.4183| FOLS |
| 25  | 1     | 2.8797| 0.4810| 0.0790| FOLS |
|     | 2     | 2.8715| 1.4206| 0.2457| FOLS |
|     | 3     | 3.1846| 0.4778| 0.0774| FOLS |
|     | 4     | 3.1670| 1.4291| 0.2362| FOLS |
|     | 5     | 2.8584| 0.4825| 0.0816| FOLS |
|     | 6     | 2.8539| 1.4361| 0.2394| FOLS |
|     | 7     | 3.1663| 0.4783| 0.0774| FOLS |
|     | 8     | 3.1655| 1.4418| 0.2469| FOLS |
4. Real Data

Crude oil prices are the first catalyst in the growth or decline of the economies, and given the importance of oil and its impact on the global economy in general, and the economies of industrialized countries in particular, has been studying the prices of OPEC basket of oil for its importance in the global economy to reach the future forecast for the next six months of crude oil prices in US dollars [19].

Table 4. Average monthly OPEC basket crude oil price from June 2018 to June 2019 (in U.S. dollars per barrel)

| Year | Month | Price |
|------|-------|-------|
| 2018 | Jun   | 73.22 |
|      | Jul   | 73.27 |
|      | Aug   | 72.26 |
|      | Sept  | 77.18 |
|      | Oct   | 79.39 |
|      | Nov   | 65.33 |
|      | Dec   | 56.94 |
| 2019 | Jan   | 58.74 |
|      | Feb   | 63.83 |
|      | Mar   | 66.37 |
|      | Apr   | 70.78 |
|      | May   | 69.97 |
|      | Jun   | 62.92 |

Table 4 represents monthly data to reach the required estimates of oil prices for the OPEC basket for the last six months of 2019. This was based on the fitting of Modified Exponential growth model according to the Ordinary Least Squares method and the Modified Least Squares.

Table 5 shows the forecasted of OPEC basket oil barrel prices in US dollars for the period Jul 2019 - Dec 2019.

Table 5. Forecasted of OPEC basket oil barrel prices in US dollars for the period Jul 2019 - Dec 2019

| Month | FGM | OLS | FOLS |
|-------|-----|-----|------|
| Jul   | 62.70 | 74.41 | 63.12 |
| Aug   | 61.73 | 73.57 | 62.05 |
| Sept  | 66.89 | 72.72 | 67.03 |
| Oct   | 69.13 | 71.87 | 69.18 |
| Nov   | 55.30 | 71.02 | 55.18 |
| Dec   | 46.94 | 70.17 | 46.73 |

With regards to Table 6 (including MAPE and RMSE), the results have given preference to FOLS compared with other methods.

Table 6. Mean Absolute Percentage Error and Root Mean Squares Error of OPEC
The Absolute Percentage Error (APE) was calculated and shown in Figure 1. It is clear that \textit{FOLS} has given a significantly lower result, compared with other methods, as well as it can be seen that \textit{FGM} has shown a higher APE result than the other methods.

|                      | \textit{FGM} | \textit{OLS} | \textit{FOLS} |
|----------------------|--------------|--------------|---------------|
| \textit{MAPE}       | 0.1453       | 16.9555      | 0.1166        |
| \textit{RMSE}       | 0.0608       | 0.6635       | 0.0463        |

\textbf{Figure 1.} Absolute Percentage Error
5. Conclusion

When analyzing the MAPE and RMSE, the Modified Least Squares Residuals by using Fourier Series Residuals reached more efficient estimates compared with the other methods, regardless of the sample size or standard deviation value. Depending on the real data for average monthly OPEC basket crude oil price from June 2018 to June 2019 (in U.S. dollars per barrel), we can be seen FOLS reached more efficient estimates compared with the other methods. In addition, the estimation methods have given volatile APE results, especially when using OLS, and it is clear that FOLS has given a significantly lower result, compared with other methods. As can be seen, FGM estimates have given higher APE than other methods, as well as it can be seen that OPEC basket prices will decline in the end of 2019.

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