MARGINALLY TRAPPED SURFACES IN \( S^4 \), GAUSS IMAGES AND CONFORMAL INVARIANTS

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Abstract. In this paper we explore the geometry of marginally trapped surfaces in De Sitter 4-space \( S^4_1 \) and its relation to Moebius surface theory in the conformal 3-sphere. Given an oriented marginally trapped surface \( S \subset S^4_1 \) we consider its "spherical Gauss image" i.e. a surface \( S' \) obtained varying a well defined "positive" normal null direction on \( S \). The new surface \( S' \) lies in the conformal 3-sphere \( S^3 \) viewed as the manifold of null directions of Minkowski space \( \mathbb{R}^4_1 \), and so \( S' \) has well defined local conformal or Moebius invariants which encode information of the underlying marginally trapped surface \( S \). We derive an equation relating the conformal invariants of the spherical Gauss image \( S' \) with an intrinsically defined geometric complex quadratic differential \( \delta \) on the underlying marginally trapped surface \( S \). This equation is used to obtain a characterization of marginally trapped surfaces whose spherical Gauss image is Willmore or constrained Willmore [7]. As an application we obtain integrable one-parameter deformations of two classes of marginally trapped surfaces and construct their associated families.

1. INTRODUCTION

The notion of trapped surfaces in a 4-dimensional space-time was introduced by R. Penrose and plays an essential role in the study of singularities of the Einstein equations. Marginally trapped surfaces are spacelike surfaces with lighlike mean curvature vector and they arise as boundaries of black holes [10]. From a pure differential geometric point of view marginally trapped surfaces are interesting objects having no counterpart in riemannian ambients. The marginally trapped equation \( \langle \overrightarrow{H}, \overrightarrow{H} \rangle = 0 \) is the natural generalization of the stationary surface equation \( \overrightarrow{H} = 0 \), hence marginally trapped surfaces arise as natural generalizations of stationary spacelike surfaces.

The purpose of these notes is to explore the geometry of marginally trapped surfaces of De Sitter space-time or pseudo pshere \( S^4_1 \) using Moebius surface theory in the conformal 3-sphere \( S^3 \) [7]. More specifically, given a marginally trapped (conformal) immersion of a Riemann surface \( f : \Sigma \rightarrow S^4_1 \) with a fixed orientation on its lorentzian normal bundle \( \nu(\Sigma) \), each point \( p \in \Sigma \) has a well defined normal "positive null direction" (in a precise sense specified later) \( \ell^+(p) \subset T^+_p \Sigma \). This determines unambiguously a smooth map \( G : p \mapsto \ell^+(p) \) from \( \Sigma \) to the conformal sphere \( S^3 \), viewed as the manifold of null directions of the Minkowski space \( \mathbb{R}^5_1 \) equipped with its natural conformal structure. We call \( G \) the spherical Gauss image of the marginally trapped surface \( f \). Away the zeros of a certain Hopf-type differential of \( f \) we show that \( G : \Sigma \rightarrow S^3 \) is a conformal umbilic-free surface and as such it has well defined local conformal or Moebius invariants which encode the geometry of the marginally trapped surface \( f : \Sigma \rightarrow S^4_1 \). Our construction of the spherical

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Gauss image has been inspired in the paper by Aledo, Galvez and Mira \[1\] in which the authors consider a \textit{hyperbolic Gauss} map to obtain an \textit{extended Weierstrass-Bryant} representation formula for marginally trapped surfaces immersed in the Minkowski spacetime \(\mathbb{R}^4_1\). We mention that it is also possible to construct Gauss images for marginally trapped surfaces in the lorentzian hyperbolic 4-space or anti De Sitter space \(AdS_4\). In this case the manifold of null directions is a lorentzian conformal 3-manifold (other than the conformal 3-sphere) and more general conformal invariants are needed \[17\].

Diverse aspects of the geometry of marginally trapped surfaces have been investigated recently, see for instance \[9\] and the survey \[10\]. The notion of marginally trappedness has also been extended to higher dimension and co-dimensions with very interesting results \[2\], and \[3\]. The paper is organized as follows. In Section 2 we introduce basic notation and derive the structure equations of spacelike surfaces in \(\mathbb{S}^2_4\) with light like or null mean curvature vector field. Section 3 contains a short survey on Moebius geometry of surfaces in the conformal sphere \(\mathbb{S}^3\), in which we state without proofs the main facts on conformal or Moebius invariants of conformally immersed Riemann surfaces. The main references are \[7\] and \[21\].

In Section 4 we obtain the fundamental equation \(43\) relating the conformal invariants of a spherical Gauss image with geometric data (in the form of a complex quadratic differential), of the corresponding marginally trapped surface. As a consequence of this equation we obtain Theorem \(1.2\) which gives a characterization of marginally trapped surfaces whose spherical Gauss images are Willmore or constrained Willmore surfaces in \(\mathbb{S}^3\). Recall that constrained Willmore surfaces are extremes of the Willmore energy with respect to variations preserving the conformal structure \[7\]. Another consequence of equation \(43\) is Theorem \(1.3\) on the congruence of marginally trapped surfaces, i.e. which conditions determine a marginally trapped surface in \(\mathbb{S}^1_4\) up to ambient isometries.

In Section 5 we consider integrable one parameter deformations of marginally trapped and construct corresponding associated families for such surfaces. The deformation parameter enters as a spectral parameter in the structure equations and induces a symmetry of the compatibility equations \[7\]. We investigate here integrable deformations of two classes of marginally trapped surfaces in \(\mathbb{S}^4_1\), namely, those with flat normal bundle and isothermic spherical Gauss image and, surfaces whose spherical Gauss images are constrained Willmore. In the first case the deformation is essentially induced by the Calapso-Bianchi or isothermic T-tranformation \[7\] acting on the spherical Gauss images. In the second case the deformation comes from the associated family of an auxiliary harmonic map with values in a (pseudo riemannian) complex quadric. More precisely, we show that a marginally trapped surface \(f: \Sigma \to \mathbb{S}^4_1\) has constrained Willmore spherical Gauss image if an only if certain map \(\phi\) with values in a complex quadric is harmonic. We use the associated family of \(\phi\) to obtain a symmetry of the compatibility equations of the constrained Willmore Gauss image of \(f\). The symmetry we obtain differs from that considered in \[7\] to describe the associated family of arbitrary constrained Willmore surfaces in \(\mathbb{S}^3\). We conclude with some remarks concerning marginally trapped tori with holomorphic \(\delta\)-differential.

2. Preliminaries

Let \(\mathbb{R}^5_1\) denote the 5-dimensional Minkowski space, that is the real vector space \(\mathbb{R}^5\) with canonical coordinates \((x_0, x_1, x_2, x_3, x_4)\) equipped with the Lorentz inner product

\[
\langle x, y \rangle = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4.
\]

We also consider the complex bilinear extension of the Lorentz metric to \(\mathbb{C}^5\) given by \(\langle z, w \rangle = z_0 w_0 + z_1 w_1 + z_2 w_2 + z_3 w_3 - z_4 w_4\). For our purposes we use also the corresponding (pseudo) hermitian inner product on \(\mathbb{C}^5\) defined by \(\bar{h}(z, w) := \langle z, \bar{w} \rangle\), and we denote by
$\mathbb{C}^5$ the complex 5-space $\mathbb{C}^5$ endowed with the inner product $h$.

De Sitter 4-space or pseudosphere of index one is by definition the quadric

$$S^4_1 = \{ x \in \mathbb{R}^5_1 : \langle x, x \rangle = 1 \}$$

which equipped with the metric induced from the ambient $\mathbb{R}^5_1$ becomes a lorentzian 4-manifold with constant sectional curvature one on which the Lie group $SO(4,1)$ acts transitively. Choosing $e_0 \in S^4_1$ as the base point, then $S^4_1$ is isometric to the pseudoriemannian symmetric space $SO(4,1)/SO(3,1)$. A notion of global time orientedness can be given to $\mathbb{R}^5_1$ and hence to $S^4_1$. A timelike vector $X \in \mathbb{R}^5_1$ is said to be future pointing or positively oriented if $\langle X, e_4 \rangle < 0$. Note that if $\langle X, X \rangle = -1$, then it is future pointing if and only if (its translated) $X$ lies in the (upper) real 4-hyperbolic space

$$\mathbb{H}^4 = \{ x \in \mathbb{R}^5_1 : \langle x, x \rangle = -1, x_4 > 0 \}.$$

Let $\Sigma$ be a connected orientable surface and $f: \Sigma \to S^4_1$ a spacelike immersion i.e. the induced metric $g = f^*\langle ., . \rangle$ is Riemannian and it determines a conformal structure on $\Sigma$. Then $f$ is conformal which means that preserves this conformal structure i.e.

$$\langle f_z, f_z \rangle = 0,$$

and $f_z = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$, are the complex partial operators. Equivalently,

$$(2) \quad \langle f_x, f_y \rangle = 0, \quad \|f_x\|^2 = \|f_y\|^2 > 0.$$  

Conversely, if $f: \Sigma \to S^4_1$ is a conformal immersion from a Riemann surface, then $\langle f_x, f_y \rangle = 0$, and $\|f_x\|^2 = \|f_y\|^2 \neq 0$, for every local complex coordinate $z = x + iy$ on $\Sigma$, where $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$, and $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$.

$$\begin{align*}
2\langle f_{zz}, f_z \rangle &= \partial_z \langle f_z, f_z \rangle = 0 \\
2\langle f_{zz}, f_{\bar{z}} \rangle &= \partial_{\bar{z}} \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0,
\end{align*}$$

hence $f_{zz}$ has no tangential component and so $-e^{2u}f + e^{2u}\bar{H} = f_{zz}$. Since $f : \Sigma \to \mathbb{S}^4_1$ is spacelike, the induced metric on the normal bundle $\nu(f)$ of $f$ is lorentzian, with signature $(+, -)$. Let $\{N_1, N_2\}$ be a local lorentzian orthonormal frame of $\nu(f)$ satisfying

$$\langle N_2, N_2 \rangle = -1, \quad \langle N_1, N_2 \rangle = 0,$$

where $N_2$ is future pointing, that is $N_2 \in \mathbb{H}^4$. Thus if $\{N_1, N_2\}$ has the same orientation as $\nu(f)$ we call it to be positively oriented along the corresponding immersed spacelike surface. If $\{N_1, N_2\}$ has the opposite orientation, then $\{-N_1, N_2\}$ is positively oriented.

Denote by $\mathcal{II}$ the second fundamental form of $f$ and $\bar{H} := \frac{1}{2} \text{trace} \mathcal{II}$ the the mean curvature vector of $f$. Expanding in terms of a positively oriented lorentzian frame $\{N_1, N_2\}$ it follows that,

$$(3) \quad \mathcal{II} = -\langle df, dN_1 \rangle N_1 + \langle df, dN_2 \rangle N_2.$$

Define functions $\xi_1 := \langle f_{zz}, N_1 \rangle$, $\xi_2 := -\langle f_{zz}, N_2 \rangle$. Then the $(2,0)$-part of $\mathcal{II}$ is given by

$$(4) \quad \mathcal{II}(\partial_z, \partial_z) = \xi_1 N_1 + \xi_2 N_2$$

Also defining $h_1 := \langle \bar{H}, N_1 \rangle$, $h_2 := -\langle \bar{H}, N_2 \rangle$, then $\bar{H} = h_1 N_1 + h_2 N_2$, and so

$$(5) \quad \langle \bar{H}, \bar{H} \rangle = h_1^2 - h_2^2.$$
A spacelike surface \( f : \Sigma \to S_1^4 \) is called **marginally trapped** if its mean curvature vector is null or lightlike i.e. \( \langle \vec{H}, \vec{H} \rangle = 0 \). Stationary surfaces are those with vanishing mean curvature: \( \vec{H} = 0 \), thus every stationary surface is also marginally trapped.

If \( \vec{H} \neq 0 \), then after a change of orientation in the normal bundle if necessary, (i.e. after a change of sign \( N_1 \mapsto -N_1 \)) the marginally trapped condition \( \langle \vec{H}, \vec{H} \rangle = h_1^2 - h_2^2 = 0 \), reads \( h_1 = h_2 \), with \( h_1 + h_2 \neq 0 \). In this way the mean curvature vector is given by

\[
\vec{H} = h(N_1 + N_2), \quad \text{with } h = h_1 = h_2.
\]

We call \( h \) the **the mean curvature function** of \( f \) in the lorentzian positively oriented frame \( \{N_1, N_2\} \). From equation \( f_{zz} = -e^{2u}f + e^{2u}\vec{H} \), a formula for the mean curvature function \( h \) in terms of \( f_{zz} \) is obtained which will be of use later namely,

\[
h = e^{-2u}f(N_1, N_2) = -e^{-2u}f(N_1, N_2).
\]

In particular it follows that \( f \) is marginally trapped iif \( \langle f_{zz}, (N_1 + N_2) \rangle = 0 \). Introducing the function \( \sigma := -\langle \partial_z N_1, N_2 \rangle \), the structure equations of the marginally trapped immersion \( f \) are given by,

\[
\begin{align*}
f_{zz} &= 2u_z f_z + \xi_1 N_1 + \xi_2 N_2, \\
f_{z\bar{z}} &= -e^{2u} f + e^{2u} \vec{H}, \\
\partial_z N_1 &= -hf_z - e^{-2u} \xi_1 f_z + \sigma N_2, \\
\partial_z N_2 &= hf_z + e^{-2u} \xi_2 f_z + \sigma N_1.
\end{align*}
\]

The compatibility conditions of this system are the fundamental equations of Gauss, Codazzi and Ricci:

\[
\begin{align*}
\text{Gauss:} & \quad 2u_{zz} = -e^{2u} + e^{-2u}(|\xi_1|^2 - |\xi_2|^2), \\
\text{Codazzi:} & \quad e^{-2u}(\partial_z \xi_1 + \xi_2) = (\partial_z h + \sigma h), \\
& \quad e^{-2u}(\partial_z \xi_2 + \xi_1) = (\partial_z h + \sigma h), \\
\text{Ricci:} & \quad Im(\sigma_z) = e^{-2u} Im(\xi_1 \xi_2).
\end{align*}
\]

The Gaussian curvature of the induced metric \( g \) is given by \( K = -\Delta_u u = -2e^{-2u}u_{zz} \), where \( \Delta_u = 2e^{-2u}\partial_z \partial_{\bar{z}} \), is the Laplace operator of the Riemannian manifold \( (\Sigma, g) \). From Gauss equation (9) we obtain the expression of the Gaussian curvature of the induced metric on \( \Sigma \),

\[
K = 1 - e^{-4u}(|\xi_1|^2 - |\xi_2|^2),
\]

On the other hand if \( \nabla^\perp \) denotes the covariant derivative on the normal bundle \( \nu(f) \), then \( \omega := \langle \nabla^\perp N_2, N_1 \rangle \) is the corresponding connection one form. Fixed an orientation on the normal bundle \( \nu(f) \) the normal curvature is defined by \( d\omega = K^\perp dA_g \), where \( dA_g \) is the area form of the induced metric \( g \). Thus \( \omega = 2\text{Re}(\sigma dz) \), and so \( d\omega = -4\text{Im}(\sigma \bar{z}) dx \wedge dy \). From Ricci equation above it follows that \( \text{Im}(\sigma_z) = e^{-2u} \text{Im}(\xi_1 \xi_2) \). Thus since \( dA_g = 2e^{2u} dx \wedge dy \), the normal curvature function is given by

\[
K^\perp = -e^{-2u} \text{Im}(\sigma_z) = -e^{-2u} \text{Im}(\xi_1 \xi_2).
\]

The \( \nabla^\perp \)-derivative of \( \vec{H} \) can be computed directly from the structure equations of \( f \), and is given by

\[
\nabla^\perp_{\partial_z} \vec{H} = e^{-2u}(h_z + \sigma h)(N_1 + N_2).
\]

Thus \( f \) has parallel mean curvature vector if and only if \( h_z + \sigma h = 0 \).
3. CONFORMAL SURFACE THEORY IN $\mathbb{S}^3$

Here we briefly review the fundamental facts of Moebius surface theory in the conformal sphere $\mathbb{S}^3$. For proofs and further details we refer the reader to [7] and [21]. Consider the null or light cone in $\mathbb{R}^5_1$ defined by

\begin{equation}
\mathcal{L} = \{0 \neq x \in \mathbb{R}^5_1 : \langle x, x \rangle = 0\}.
\end{equation}

The future light cone $\mathcal{L}_+ \subset \mathcal{L}$ consists of future pointing points $x \in \mathcal{L}$ i.e. such that $\langle x, e_4 \rangle < 0$. For every $x \in \mathbb{S}^3 \subset \mathbb{R}^4_1$, the point $(x, 1) \in \mathbb{R}^5_1$ lies in the future light cone $\mathcal{L}_+$. We are using here the fact that any vector $x$ in $\mathbb{R}^5_1$ may be uniquely written as an ordered pair $(x', t)$ with $x' \in \mathbb{R}^4_1$ and $t \in \mathbb{R}$, thus giving rise to an isomorphism $\mathbb{R}^4_1 \oplus \mathbb{R} \rightarrow \mathbb{R}^5_1$. In particular points on $\mathcal{L}$ are of the form $(x, \pm \|x\|^2)$, with $x \in \mathbb{R}^4_1$. The map $\mathbb{S}^3 \ni x \mapsto [(x, 1)]$ identifies the unit round sphere $\mathbb{S}^3 \subset \mathbb{R}^4_1$ with the projectivization of the light cone, $P(\mathcal{L}) \subset \mathbb{R}P^5$. Denote by $O_+(4, 1)$ the group of orthogonal transforations of $\mathbb{R}^5_1$ preserving the time orientation. Then each $F \in O_+(4, 1)$ preserves the light cone $\mathcal{L}$ and maps null lines to null lines. The group $O_+(4, 1)$ acts on $\mathbb{S}^3 \equiv P(\mathcal{L})$ by $g.[x] = [gx]$, and this action is transitive. Here $O_+(4, 1)$ is referred to as the group of Moebius transformations of the conformal sphere $\mathbb{S}^3$. Note that the subgroup of $O(4, 1)$ preserving $P(\mathcal{L}_+)$, is precisely $O_+(4, 1)$. The connected component of $O(4, 1)$ containing the identity $I$ is denoted by $SO_+(4, 1)$. Hence every $T \in SO_+(4, 1)$, satisfies $det(T) = 1$, and preserves the time orientation.

A smooth map $\psi : \Sigma \rightarrow \mathbb{S}^3 \equiv P(\mathcal{L})$ can be viewed as a null line subbundle $\Lambda$ of the trivial bundle $\Sigma \times \mathbb{R}^5_1$ via $\psi(x) = \Lambda_x, x \in \Sigma$. A (local) lift of $\psi$ is a smooth map $X : U \rightarrow \mathcal{L}$ from an open subset $U \subset \Sigma$, such that the null line spanned by $X(x)$ is $\Lambda_x$ for every $x \in U$. The map $\psi$ is a conformal immersion if every local lift $X$ of $\psi$ is conformal i.e. $\langle X_z, X_z \rangle = 0, \langle X_z, X_\bar{z} \rangle > 0$, for every coordinate $z$.

Let $X$ be a conformal lift $X$ of $\psi$, and take a local complex coordinate $z$. Defining $V := \text{span}\{X, dX, X_{\bar{z}}\}$, it can be easily checked that $V$ is in fact independent on the election of any particular conformal lift of $\psi$ and local coordinate $z$. In this way we view $V$ as a vector sub-bundle of the trivial bundle $\mathbb{R}^5_1 \times \Sigma$ on which the ambient metric of $\mathbb{R}^5_1$ induces a vector bundle metric of signature $(3, 1)$. Each fiber $V_x$ determines a Moebius invariant 2-sphere $\mathbb{S}^2(x) \equiv P(V_x \cap \mathcal{L}) \subset P(\mathcal{L}) \equiv \mathbb{S}^3$. These spheres altogether comprise the so-called mean curvature sphere or central sphere congruence of the surface $\psi$ [7]. Respect to a fixed a local coordinate $z$ there is a distinguished local lift $Y$ of $\psi$ taking values in the future light cone $\mathcal{L}_+$ such that

$$\langle Y_\bar{z}, Y_z \rangle = \frac{1}{2},$$

or equivalently $|dY|^2 = |dz|^2$. It is called the canonical lift of the surface $\psi$. The complementary orthogonal line sub-bundle $V^\perp$ is determined by $\Sigma \times \mathbb{R}^5_1 = V \oplus V^\perp$ and the connection $D$ on $V^\perp$ is just orthogonal projection of the usual derivative in $\mathbb{R}^5_1$:

$$D_X v = [d_X v]^\perp, \quad v \in \Gamma(V^\perp), \quad X \in T\Sigma.$$ 

Let $N \in \Gamma(V)$ be the unique section satisfying

$$\langle N, N \rangle = \langle N, Y_\bar{z} \rangle = \langle N, Y_z \rangle = 0, \quad \langle Y, N \rangle = -1.$$
Thus $V = \text{span}\{Y, R(Y), Im(Y), N\}$ and it is shown in [7] that the Moebius invariant frame $\{Y, Y_z, Y_{zz}, N\} \subset \Gamma(V \otimes \mathbb{C})$ satisfies orthogonally relations given by

$$
\begin{align*}
\langle Y, Y \rangle &= \langle N, N \rangle = 0, \quad \langle N, Y \rangle = -1, \\
\langle Y, dY \rangle &= \langle N, dY \rangle = \langle dN, N \rangle = 0, \\
\langle Y_z, Y_z \rangle &= \langle Y_z, Y_{zz} \rangle = 0, \quad \langle Y_z, Y_{zz} \rangle = \frac{1}{2}.
\end{align*}
$$

(14)

A direct consequence of the above equations is that $Y_{zz}$ is orthogonal to $Y, Y_z$ and $Y_{zz}$ and so there is a unique election of a local complex function $s$ on $\Sigma$ for which $Y_{zz} + \frac{s}{2}Y$ is a section of the normal bundle $V^\perp \otimes \mathbb{C}$ namely, $\frac{s}{2} = \langle Y_{zz}, N \rangle$. Thus we arrive at the fundamental equation of Moebius invariant surface geometry:

$$
Y_{zz} + \frac{s}{2}Y = \kappa,
$$

(15)

defining uniquely the complex valued function $s$ and the section $\kappa$ of $V^\perp \otimes \mathbb{C}$, respect to the local coordinate $z$ fixed at the beginning. The function $s$ is interpreted as the schwartzian derivative of the conformal immersion $\psi$ with respect to $z$, and $\kappa$ is identified with the normal valued Hopf differential of $\psi$, expressed in the coordinate $z$. By construction $s$ and $\kappa$ are Moebius invariants of the immersion $\psi$ with respect to a given coordinate $z$.

A useful interpretation of $\kappa$ in terms of euclidean invariants of the immersion $\psi$ is the following. There is a unique conformal immersion $\hat{\psi} : \Sigma \to \mathbb{S}^3 \subset \mathbb{R}^4$ satisfying $[(\hat{\psi}(x), 1)] = \psi(x), \forall x \in \Sigma$. Thus $\phi = (\hat{\psi}(x), 1)$ is a lift of $\psi$, which is called the euclidean lift of $\psi$ [7]. Let $\nu(\hat{\psi})$ denote the normal bundle of the immersed surface $\psi$. Then there is a bundle isomorphism $\nu(\hat{\psi}) \cong V^\perp$ given by

$$
v \mapsto \langle v, \bar{H}(\hat{\psi}, 1) + (v, 0)\rangle,
$$

(16)

where $\bar{H}$ is the mean curvature vector of $\hat{\psi}$. Under this isomorphism $\kappa \in \Gamma(V^\perp \otimes \mathbb{C})$ corresponds to a complex section $\hat{\kappa} \in \nu(\hat{\psi}) \otimes \mathbb{C}$ satisfying $\kappa = \langle \hat{\kappa}, \bar{H}(\hat{\psi}, 1) + (\hat{\kappa}, 0)\rangle$. Using (15) it is shown in [7] that

$$
\hat{\kappa} \frac{dz^2}{dz} = \frac{H^{(2,0)}}{|d\phi|},
$$

where $H^{(2,0)}$ is the $(2, 0)$-part of the normal bundle valued (euclidean) second fundamental form of $\hat{\psi}$. In this way $\kappa$, up to the isomorphism (16), is the trace free part of the second fundamental form, i.e., the normal bundle valued Hopf differential of $\hat{\psi}$, scaled by the square root of the $\hat{\psi}$-induced metric.

The following structural equations of a conformal immersion $\psi : \Sigma \to \mathbb{S}^3$ are consequence of the above orthogonality conditions and were derived in [7]:

$$
\begin{align*}
(i) & \quad Y_{zz} = -\frac{s}{2}Y + \kappa, \\
(ii) & \quad Y_{zz} = -\langle \kappa, \bar{\kappa} \rangle Y + \frac{1}{2}N, \\
(iii) & \quad N_z = -2\langle \kappa, \bar{\kappa} \rangle Y_z - sY_{zz} + 2D_z \kappa.
\end{align*}
$$

(17)

The compatibility equations among these are given by,

$$
\begin{align*}
\text{Conformal Gauss:} & \quad \frac{s}{s} = 3\langle \bar{\kappa}_z, \kappa \rangle + \langle \bar{\kappa}, \kappa_z \rangle, \\
\text{Conformal Codazzi:} & \quad \bar{\text{Im}}(\kappa_{zz} + \frac{s}{2}\kappa) = 0.
\end{align*}
$$

(18)

When the local coordinate changes from $z$ to $w$ the new invariants $s'$ and $\kappa'$ change according to

$$
\begin{align*}
\kappa' &= \kappa \left(\frac{\partial w}{\partial z}\right)^\frac{3}{2} \left(\frac{\partial w}{\partial z}\right)^{-\frac{1}{2}}, \\
s' &= s \left(\frac{\partial w}{\partial z}\right)^2 + S_w(z),
\end{align*}
$$

(19)
where the usual Schwartzian derivative of a meromorphic function \( g : \Sigma \to \mathbb{C} \) is given by \( S_z(g) = (\psi''(\psi')) - \frac{1}{2}(\psi''(\psi'))^2 \).

The importance of the conformal Gauss and Codazzi’s equations is reflected in the following fundamental theorem of conformal surface theory, (see Theorem 3.1 in [7]):

**Theorem 3.1.** Let \( \Sigma \) be a Riemann surface and \( \psi_j : \Sigma \to \mathbb{S}^3 \) be conformal immersed surfaces inducing the same Hopf differentials and the same Schwartzians. Then there is a Moebius transformation \( T : \mathbb{S}^3 \to \mathbb{S}^3 \) with \( T\psi_1 = \psi_2 \). Conversely, let \( \kappa \) and \( s \) be given data on \( \Sigma \) transforming according to (19), which also satisfy the conformal Gauss and Codazzi equations (18). Then there exists a conformal immersion \( x : \Sigma \to \mathbb{S}^3 \) with Hopf differential \( \kappa \) and Schwartzian \( s \).

**Remark 3.1.** It is proved in [4] that \( \kappa \frac{ds^2}{|dz|^2} \) is a globally defined quadratic differential with values in \( L \otimes \mathbb{C} \), where \( L \) is the real line bundle \( (K \otimes \bar{K})^{1/2} \) of densities of conformal weight 1 over \( \Sigma \) [3]. Then for any local coordinate system \((U, z)\), \( \kappa \) is can be viewed just as a local complex function on \( U \subset \Sigma \) which transforms according to (19).

**Remark 3.2.** If a conformal immersion \( \psi : \Sigma \to \mathbb{S}^3 \) has \( \kappa \equiv 0 \), then the image of \( \psi \) is contained in a fixed 2-sphere \( \mathbb{S}^2 \subset \mathbb{S}^3 \), as easily follows from (17). Considering \( \psi \) as a conformal map \( \psi : \Sigma \to \mathbb{S}^2 \equiv \mathbb{C}P^1 \), it is shown in [7] that the usual Schwartzian derivative of \( \psi \) coincides with \( s \), i.e. \( s = (\psi''/\psi') - \frac{1}{2}(\psi''/\psi')^2 \). According to Theorem 3.1 \( s \) determines \( \psi \) up to Moebius transformations of \( \mathbb{C}P^1 \).

**Remark 3.3.** The map \( \gamma : \Sigma \ni p \mapsto V(p) \) with values in the Grassmannian \( G_{3,1}(\mathbb{R}_5^5) \) is called the conformal Gauss map of the immersion \( \psi \) [13], [21]. Since \( G_{3,1}(\mathbb{R}_5^5) \) identifies with the manifold of all real spacelike lines through the origin in \( \mathbb{R}_5^5 \), we see that the projection map \( \mathbb{S}^4 \to G_{3,1}(\mathbb{R}_5^5) \) sending a point \( p \in \mathbb{S}^4 \) to the line \( \mathbb{R}p \) is a lorentzian double cover. Hence the spacelike line \( \gamma = V^\perp \) parallel translated to the origin intersects \( \mathbb{S}^4 \) in two antipodal points \(-\mathcal{G} \) and \(+\mathcal{G} \) which determine the fiber \( \{+\mathcal{G}, -\mathcal{G}\} \) of the covering map. This gives two conformal immersions \( \pm \mathcal{G} : \Sigma \to \mathbb{S}^4 \) which are spacelike and marginally trapped [18], [19]. Both immersions \( \pm \mathcal{G} \) determine the same sphere congruence in \( \mathbb{S}^3 \) with opposite orientations.

In [21] it is shown that \( \gamma \) induces a positive definite conformal metric \( g_\gamma \) on \( \Sigma \) given by \( g_\gamma = \frac{1}{4}(d\gamma, d\gamma) = |\kappa|^2|dz|^2 \). The Willmore energy of the conformal immersion \( \psi \) is defined as the total area of \((\Sigma, g_\gamma)\) and is given by

\[
W(\psi) = \frac{i}{2} \int_{\Sigma} |\kappa|^2 dz \wedge \bar{dz}.
\]

which coincides (up to a constant multiple) with the Willmore energy of the immersion \( \psi \), see [7]. A conformal immersion \( \psi : \Sigma \to \mathbb{S}^3 \) is called a Willmore surface if it extremizes the Willmore energy functional (20). It is known [7] that \( \psi \) is Willmore iif \( \kappa \) and \( s \) satisfy the following stronger version of the conformal Codazzi’s equation:

\[
\kappa_{\bar{z}z} + \bar{s} \kappa = 0.
\]

On the other hand a conformal immersed surface \( \psi : \Sigma \to \mathbb{S}^3 \) is called constrained Willmore if it extremizes the Willmore energy functional with respect to variations through conformal immersions. It is shown in [3] that \( \psi \) is constrained Willmore if and only if

\[
\kappa_{\bar{z}z} + \bar{s} \kappa = Re(\eta \kappa),
\]

where \( \eta \) is the Hopf differential of \( \psi \).
for some holomorphic quadratic differential $\eta dz^2$ on $\Sigma$, where $\kappa$ and $s$ are the Hopf differential and the Schwartzian derivative of $\psi$.

4. THE SPHERICAL GAUSS IMAGE OF A MARGINALLY TRAPPED SURFACE

Let $f : \Sigma \to S^4$ be a conformal immersion which is marginally trapped: $\langle \overrightarrow{H}, \overrightarrow{H} \rangle = 0$. We fix an orientation on the normal bundle $\nu(f)$ and let $\{N_1, N_2\}$ be a positively oriented Lorentzian orthonormal frame of normal vector fields along $f$. Such frame determines a null line span$\{N_1 + N_2\}$, which does not depend on the particular positively oriented orthonormal Lorentzian frame $\{N_1, N_2\}$ chosen. In fact, if $\{N_1', N_2'\}$ is another positively oriented orthonormal Lorentzian frame then both frames are related by a gauge,

$$N_1' = \cosh(s)N_1 + \sinh(s)N_2,$$

$$N_2' = \sinh(s)N_1 + \cosh(s)N_2,$$

hence $N_1' + N_2' = e^s(N_1 + N_2)$, thus $N_1' + N_2'$ and $N_1 + N_2$ generate the same null line.

We define the spherical Gauss image (or spherical Gauss map) $G : \Sigma \to S^3 \equiv P(L)$ of the marginally trapped surface $f$ as the null line sub bundle of the trivial bundle $\Sigma \times \mathbb{R}_1^4$ given by

$$G(x) = \text{the null line generated by } N_1(x) + N_2(x), \quad x \in \Sigma.$$ 

Our previous observation shows that $G(x)$ is independent on any choice of a (local) positively oriented orthonormal Lorentzian frame $\{N_1, N_2\}$ of the normal bundle $\nu(f)$. If we denote by $\widehat{G}$ the unique smooth map from $\Sigma$ to the round Euclidean sphere $S^3 \subset \mathbb{R}^4$ satisfying $\langle [\widehat{G}(x), 1]\rangle = G(x)$ for every $x \in \Sigma$, then $\phi = (\widehat{G}, 1) : \Sigma \to L$ is a section of $G$ which is called the euclidean lift of $G$ [7].

Remark 4.1. A similar map was introduced under the name hyperbolic Gauss map in [1] to obtain a Weierstrass-Bryant representation for marginally trapped surfaces immersed in $\mathbb{R}^4_1$ with flat normal bundle.

Let $\{N_1, N_2\}$ be any positively oriented orthonormal frame of normal fields. Then $F := N_1 + N_2$ is a $\mathcal{L}_+-$valued lift of $G$. Using the structure equations (8) we compute

$$F_z = N_{1,z} + N_{2,z} = (h_2 - h_1)f_z + e^{-2u}(\xi_2 - \xi_1)f_\bar{z} + \sigma F.$$ 

Since $\langle f_z, f_\bar{z} \rangle = \langle f_\bar{z}, f_z \rangle = 0$, $\langle f_z, f_z \rangle = e^{2u}$, then $\langle F, F \rangle = 0$, and $\langle F, f_\bar{z} \rangle = \langle F, f_z \rangle = 0$ and we obtain

$$\langle F_z, F_\bar{z} \rangle = \langle N_{1,z} + N_{2,z}, N_{1,z} + N_{2,z} \rangle = (h_2 - h_1)(\xi_2 - \xi_1)$$

Thus if $f$ is marginally trapped then $\langle F_z, F_\bar{z} \rangle = 0$. On the other hand there is a (local) smooth non-zero function $\lambda$ such that $F = \lambda(\widehat{G}, 1)$. Thus

$$F_z = \lambda_z(\widehat{G}, 1) + \lambda(\widehat{G}_\bar{z}, 0),$$

and from (23) we obtain $0 = \langle F_z, F_\bar{z} \rangle = \lambda^2\langle \widehat{G}_\bar{z}, \widehat{G}_\bar{z} \rangle$, and so $\langle \widehat{G}_\bar{z}, \widehat{G}_\bar{z} \rangle = 0$, where $\langle \cdot, \cdot \rangle$ on the right hand term denotes the round metric on $S^3 \subset \mathbb{R}^3$.

Using again (23) we obtain

$$\lambda^2\langle \widehat{G}_\bar{z}, \widehat{G}_\bar{z} \rangle = e^{-2u}|\xi_1 - \xi_2|^2 = \langle F_z, F_\bar{z} \rangle.$$ 

Thus away from the zeros of $\xi_1 - \xi_2$ it follows that $\langle F_z, F_\bar{z} \rangle > 0$ and also $\langle \widehat{G}_\bar{z}, \widehat{G}_\bar{z} \rangle > 0$. In concordance with [1] we call $q := (\xi_1 - \xi_2)dz^2$ the Hopf differential of the surface $f : \Sigma \to S^4_1$.

We have shown that $F = N_1 + N_2$ is a conformal lift of $G$ for every positive oriented
orthonormal frame \( \{N_1, N_2\} \) of \( \nu(f) \). Note that every lift \( X \) of \( G \) is conformal since it can be written as \( X = \lambda F \), for some nonzero real valued function \( \lambda \). It follows from (25) that \( G, \bar{G} \) are conformal immersions away the zeros of \( q \).

If a spacelike surface \( f : \Sigma \to S_1^4 \) has flat normal bundle then from Ricci’s equation \( \text{Im}(\sigma z) = 0 \). Hence \( \sigma z - \overline{\sigma z} = \bar{\sigma} z = 0 \), and so the real one form \( \eta := \sigma dz + \overline{\sigma d\bar{z}} \) is closed. Thus locally there is a smooth real function \( \beta \) such that \( d\beta = \eta \). Consider the new positively oriented orthonormal lorentzian frame \( \{N_1', N_2'\} \) given by

\[
N_1' = \cosh(\beta)N_1 + \sinh(\beta)N_2, \quad N_2' = \sinh(\beta)N_1 + \cosh(\beta)N_2.
\]

Then the new frame \( \{N_1', N_2'\} \) has structure function \( \sigma' = 0 \). Thus \( \{N_1', N_2'\} \) is a \( \nabla^\perp \)-parallel frame which is unique up to (constant) hyperbolic rotations in the lorentzian normal bundle \( \nu(f) \). Hence if a spacelike surface \( f \) has flat normal bundle, we shall keep denoting by \( \{N_1, N_2\} \) this new positively oriented \( \nabla^\perp \)-parallel orthonormal frame. Thus if \( f \) is marginally trapped with flat normal bundle Codazzi’s equations reduce to

\[
\xi_{1,\bar{z}} = \xi_{2,\bar{z}} = e^{2u}h_z, \quad h = h_1 = h_2,
\]

which imply \( (\xi_2 - \xi_1)_\bar{z} = e^{2u}(h_2 - h_1)_z = 0 \). We have thus proved,

**Lemma 4.1.** Let \( f : \Sigma \to S_1^4 \) be a marginally trapped surface with flat normal bundle. Then the Hopf differential \( q = (\xi_2 - \xi_1)dz^2 \) is holomorphic. Hence if \( q \) does not vanish identically its zeros are isolated.

**Remark 4.2.** i) If \( f : \Sigma \to S_1^4 \) is stationary, then \( q \) is holomorphic as consequence of Codazzi’s equations (9).

ii) If \( f : \Sigma \to S_1^4 \) is marginally trapped immersion satisfying \( \nabla^\perp \hat{H} = 0 \), then from Codazzi’s equations (9) \( q \) is holomorphic if and only if \( f \) has flat normal bundle.

The structure equations of a marginally trapped surface \( f : \Sigma \to S_1^4 \) with flat normal bundle are given by

\[
\begin{align*}
 f_{zz} &= 2u_z f_z + \xi_1 N_1 + \xi_2 N_2, \\
 f_{\bar{z}z} &= -e^{2u} f + e^{2u} h(N_1 + N_2), \\
 \partial_z N_1 &= -h f_z - e^{-2u} \xi_1 f_\bar{z}, \\
 \partial_z N_2 &= h f_z + e^{-2u} \xi_2 f_\bar{z},
\end{align*}
\]

where \( \{N_1, N_2\} \) is a \( \nabla^\perp \)-parallel positively oriented orthonormal frame of the normal bundle of \( f \). From these equations we obtain

\[
\hat{H}_z = h_z (N_1 + N_2) + he^{-2u}(\xi_2 - \xi_1)f_\bar{z} = e^{-2u}\xi_{1,\bar{z}}(N_1 + N_2) + he^{-2u}(\xi_2 - \xi_1)f_\bar{z}.
\]

Since \( h_z = e^{-2u}\xi_{1,\bar{z}} = e^{-2u}\xi_{2,\bar{z}} \), then

\[
\nabla_{\partial_i} \hat{H} = h_z (N_1 + N_2) = e^{-2u}\xi_{i,z}(N_1 + N_2), \quad i = 1, 2,
\]

It follows that if \( f : \Sigma \to S_1^4 \) is marginally trapped with flat normal bundle then \( \nabla^\perp \hat{H} = 0 \), if and only if the mean curvature function \( h \) is constant if \( \xi_1 \) and \( \xi_2 \) are both holomorphic.

**Remark 4.3.** If \( q \equiv 0 \), then by (25) the spherical Gauss image is be constant. Hence \( N_1 + N_2 \) must be a fixed null line for every oriented lorentzian frame \( \{N_1, N_2\} \). Since \( \langle f, N_1 + N_2 \rangle = 0 \), then the surface \( f \) has constant curvature \( K = 1 \) by equation (10) and lies in a degenerated hypersurface \( M_0 \subset S_1^4 \), which is the intersection of the degenerated (fixed) degenerate 4-plane \( [N_1 + N_2] \perp \subset \mathbb{R}_1^5 \) with \( S_1^4 \). For instance this is just the case of
any marginally trapped surface \( f : \mathbb{S}^2 \to \mathbb{S}^4 \) with flat normal bundle. In fact since \( q \) is holomorphic on \( \mathbb{S}^2 \), it must vanish.

4.1. Conformal invariants. Let \( f : \Sigma \to \mathbb{S}^4 \) be a marginally trapped surface whose Hopf differential \( q = (\xi_1 - \xi_2)dz^2 \) is never zero, so that its spherical Gauss image \( G : \Sigma \to \mathbb{S}^3 \) is a conformally immersed surface. Consider the central sphere congruence of \( G \) i.e. the subbundle \( V \subset \Sigma \times \mathbb{R}^4_5 \) and its orthogonal line bundle \( V^\perp \).

On the other hand for each \( x \in \Sigma \) the intersection of the Minkowski vector subspace \( f^\perp(x) \subset \mathbb{R}^4_5 \) with the null cone \( L \) determines the 2-sphere \( S(x) = P(f(x)^\perp \cap L) \subset \mathbb{S}^3 \), i.e. the 2-sphere congruence determined by \( f \). We note that the antipodal surface \( -f \) determines the same sphere congruence. We say that \( S(x) \) is oriented if it is associated to \( f \), and opposite oriented if it is associated to \(-f\). Recall that \( \mathbb{S}^4 \) identifies with the manifold of oriented 2-spheres in \( \mathbb{S}^3 \). We claim that \( f^\perp = V \), i.e. both sphere congruences coincide. To prove this claim recall that for every positively oriented orthonormal lorentzian frame \( \{N_1, N_2\} \) of the normal bundle \( \nu(f) \), \( X := N_1 + N_2 \) is a conformal lift of \( G \) taking values in the future light cone \( L_+ \). Hence \( V = \text{span}\{X, \text{Re}(X), \text{Im}(X), X_{zz}\} \). In particular \( \langle X, f \rangle = 0 \) since \( N_1, N_2 \) are normal to \( f \). On the other hand from (23),

\[
\tag{29}
X_z = e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X.
\]

Hence \( \langle f, X_z \rangle = \langle f, X_{\bar{z}} \rangle = 0 \), or \( \langle f, dX \rangle = 0 \). Taking \( \partial_z \) on (29) and using again (8) yields

\[
X_{zz} = e^{-2u}(\xi_2 - \xi_1)(\xi_1 N_1 + \xi_2 N_2) + \sigma e^{-2u}(\xi_2 - \xi_1)f_z + (\sigma z + |\sigma|^2)X,
\]

from which \( \langle f, X_{zz} \rangle = 0 \) follows and so \( V \subseteq f^\perp \). Thus \( V = f^\perp \) since \( V \) has rank four.

We have proved the following

**Proposition 4.1.** Let \( f : \Sigma \to \mathbb{S}^4 \) be a conformal marginally trapped immersion for which the Hopf differential \( q \) of \( f \) is never zero, so that the spherical Gauss map \( G \) of \( f \) is a conformal immersion.

Then the central sphere congruence of \( G \) coincides with the spherical congruence determined by \( \pm f \).

Now let \( Y \) be the canonical lift of \( G \) respect to a local coordinate \( z \). Then there is a non-zero function \( \tau \) such that \( X = \tau Y \). Using (29), we compute

\[
\tag{30}
\tau_z Y + \tau Y_z = X_z = e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X.
\]

Hence \( \langle X_z, X_{\bar{z}} \rangle = \frac{\tau^2}{\tau} = \tau^2 \langle Y_z, Y_{\bar{z}} \rangle = e^{-2u}|\xi_2 - \xi_1|^2 \), so that

\[
\tag{31}
\tau = \sqrt{2e^{-u}|\xi_2 - \xi_1|}.
\]

In this way we obtain the following expression for the canonical lift of \( G \) in terms of \( X \):

\[
Y = \frac{e^u}{\sqrt{2|\xi_2 - \xi_1|}}(N_1 + N_2).
\]

A routine computation using the structure equations of \( f \) shows that \( Y \) is in fact independent on any particular choice of a positively oriented lorentzian frame \( \{N_1, N_2\} \).

On the other hand

\[
\tau_{zz} Y + 2\tau_z Y_z + \tau Y_{zz} = X_{zz} =
\]

\[
(e^{-2u}(\xi_2 - \xi_1))_z f_{\bar{z}} + e^{-2u}(\xi_2 - \xi_1)(-e^{2u}f + e^{2u}hX) +
\]

\[
\sigma_z X + \sigma(e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X).
\]
Adding and substracting $\tau \hat{\xi} Y$ we obtain

$$(\tau_{zz} - \tau \hat{\xi}) Y + 2\tau_{z} Y_{z} + \tau(Y_{zz} + \hat{\xi} Y) =$$

(32)

$$(e^{-2u}(\xi_{2} - \xi_{1}))_{z}f_{z} + e^{-2u}(\xi_{2} - \xi_{1})(-e^{2u}f + e^{2u}hX) +$$

$$\sigma_{z} X + \sigma\{e^{-2u}(\xi_{2} - \xi_{1})f_{z} + \sigma X\}.$$

Comparing the $V^\perp$ components in this identity we obtain the equality

(33)

$$(\xi_{1} - \xi_{2}) f = \tau(Y_{zz} + \frac{s}{2} Y) = \tau \kappa,$$

From this equation it follows that $q = 0$, if and only if the spherical Gauss image $G$ is totally umbilic: $\kappa = 0$.

Inserting the function $\tau$ obtained in (31) we obtain the following formula for the normal valued Hopf differential of $G$:

(34)

$$\kappa = \frac{(\xi_{1} - \xi_{2})e^{u}}{\sqrt{2} |\xi_{1} - \xi_{2}|} f.$$

Using the polar form $(\xi_{1} - \xi_{2}) = |\xi_{1} - \xi_{2}|e^{i\theta}$, we obtain $\kappa = \frac{e^{u+i\theta}}{\sqrt{2}} f$, and so by remark 3.1 we may (and will) identify

(35)

$$\kappa \equiv \frac{e^{u+i\theta}}{\sqrt{2}}.$$

According to [7] any section $v \in \Gamma(V \otimes \mathbb{C})$ expands in the frame $Y, Y_{z}, Y_{\bar{z}}, N$ according to the following formula:

(36)

$$v = -\langle v, N \rangle Y - \langle v, Y \rangle N + 2\langle v, Y_{z} \rangle Y_{z} + 2\langle v, Y_{\bar{z}} \rangle Y_{\bar{z}}.$$

We use this to expand the particular section $f_{z} \in \Gamma(V \otimes \mathbb{C})$.

Since $i Y = N_{1} + N_{2} = X$, it follows $\langle f_{z}, Y \rangle = 0$. Also from $0 = \langle f, Y_{z} \rangle = \langle f_{z}, Y_{z} \rangle + \langle f, Y_{zz} \rangle$, equation (17) - (i), and $\langle f, Y \rangle = 0$, we compute

$$\langle f_{z}, Y_{z} \rangle = -\langle f, Y_{zz} \rangle = -\langle f, -\frac{s}{2} Y + \kappa \rangle = -\langle f, \kappa \rangle = -\frac{e^{u+i\theta}}{\sqrt{2}}.$$

On the other hand since $0 = \langle f, Y_{z} \rangle = \langle f_{z}, Y_{z} \rangle + \langle f, Y_{zz} \rangle$, then

$$\langle f_{z}, Y_{z} \rangle = -\langle f, Y_{zz} \rangle = |\kappa|^{2}\langle Y, f \rangle - \frac{1}{2}\langle N, f \rangle = 0.$$

Also $\langle f, N \rangle = 0$, implies $\langle f_{z}, N \rangle + \langle f, N_{z} \rangle = 0$. Hence $\langle f_{z}, N \rangle = -\langle f, N_{z} \rangle = -2\langle f, D_{z} \kappa \rangle$. Since $D_{z} \kappa = (u + i\theta)_{z} \kappa$, then

$$\langle f_{z}, N \rangle = -\sqrt{2}(u + i\theta)_{z} e^{u+i\theta}.$$

Using these equations and (35) with $v = f_{z}$, we obtain

(37)

$$f_{z} = \sqrt{2} e^{u+i\theta} \{(u + i\theta)_{z} Y - Y_{z} \}.$$ 

From this we compute,

(38)

$$f_{zz} = \sqrt{2} e^{u+i\theta} \{((u + i\theta)_{z})^{2} + (u + i\theta)_{zz} + \frac{s}{2} \} Y - \sqrt{2} e^{u+i\theta} \kappa.$$
On the other hand using the structure equations of the immersion \( f \) and \( X = N_1 + N_2 = \tau Y \), we obtain
\[
(39) \quad f_{\bar{z}z} = -e^{2u}f + e^{2u}hX = -e^{2u}f + e^{2u}h\tau Y.
\]
Note that \( \sqrt{2}e^{u+i\theta}\kappa = e^{2u}f \), so that equating (38) and (39) gives
\[
e^{2u}h\tau = \sqrt{2}e^{u+i\theta}\{(u+i\theta)_{\bar{z}z}^2 + (u+i\theta)_{\bar{z}z} + \frac{s}{2}\}.
\]
Inserting the function \( \tau \) given by (31) in this expression we obtain the following formula:
\[
(40) \quad h|\xi_2 - \xi_1|e^{-i\theta} = ((u+i\theta)_{\bar{z}z}^2 + (u+i\theta)_{\bar{z}z} + \frac{s}{2}).
\]
or conjugating both sides,
\[
(41) \quad h(\xi_1 - \xi_2) = ((u-i\theta)_{\bar{z}z}^2 + (u-i\theta)_{\bar{z}z} + \frac{s}{2}).
\]
Now recall the connection \( D \) on the normal bundle \( V^\perp \). Any section \( v \in \Gamma(V^\perp) \) can be written as \( v = b\beta \) for some smooth function \( b \). Thus \( d_X(b\beta) = d_X(b\beta) + bd_X\beta \). Condition \( df \perp f \) implies \( D_Xf = 0 \), hence
\[
(42) \quad D_X(v) = (d_Xb)f.
\]
Thus we may identify \( D_X(v) \equiv d_Xb \). According to remark (31) \( \kappa \equiv \frac{e^{u+i\theta}}{\sqrt{2}} \), hence
\[
D_{\bar{z}} D_{\bar{z}} \kappa \equiv \kappa_{\bar{z}\bar{z}} = \left((u+i\theta)^2_{\bar{z}} + (u+i\theta)_{\bar{z}z}\right) \kappa.
\]
Then equation (40) becomes
\[
(43) \quad \kappa_{\bar{z}\bar{z}} + \frac{s}{2} \kappa = h(\xi_1 - \xi_2) \kappa,
\]
where \( h(\xi_1 - \xi_2) \kappa = \frac{e^{u}}{\sqrt{2}}h|\xi_2 - \xi_1| \), and so \( h(\xi_1 - \xi_2) \kappa \) is real valued. Thus (43) is a fundamental equation relating the quadratic differential \( h(\xi_1 - \xi_2)dz^2 \) of a marginally trapped surface \( f : \Sigma \to S^4_1 \) and the conformal invariants \( \kappa, s \) of its spherical Gauss image. Since the quadratic differential \( \delta := h(\xi_1 - \xi_2)dz^2 \) plays a key role in (43), we name it the delta differential of the marginally trapped surface \( f \).

**Remark 4.4.**
1. If the Hopf differential of \( f \) vanish \( q \equiv 0 \), then \( \kappa \equiv 0 \) by (33) and equation (43) still holds.
2. Equation (43) implies the conformal Codazzi equation \( \text{Im}(\kappa_{\bar{z}\bar{z}} + \frac{s}{2} \kappa) = 0 \). The conformal Gauss equation (18) may be recovered from (41) by a tedious calculation using Gauss, Codazzi and Ricci’s equations (9).

**Theorem 4.2.** Let \( f : \Sigma \to S^4_1 \) be a conformal marginally trapped immersion with mean curvature \( h \) and whose Hopf differential \( q = (\xi_1 - \xi_2)dz^2 \) is never zero.
Then \( G \) is a constrained Willmore surface if and only if \( \delta \) is holomorphic. In particular \( G \) is a Willmore surface if and only if \( \delta \equiv 0 \).

**Proof:** We observe first that \( \delta := h(\xi_1 - \xi_2)dz^2 \) is holomorphic if and only if the mean curvature vector of \( f \) is parallel: \( \nabla^\perp H = 0 \). In fact, since \( f \) is marginally trapped Codazzi’s equations reduce to
\[
e^{-2u}(\xi_{1\bar{z}} + \bar{\sigma}\xi_2) = e^{-2u}(\xi_{2\bar{z}} + \bar{\sigma}\xi_1) = h_{\bar{z}} + \sigma h,
\]
which implies \( (\xi_1 - \xi_2)_{\bar{z}} = \bar{\sigma}(\xi_1 - \xi_2) \). Using this we compute
\[
[h(\xi_1 - \xi_2)]_{\bar{z}} = (h_{\bar{z}} + \sigma h)(\xi_1 - \xi_2).
\]
Since by assumption $\xi_1 - \xi_2$ never vanish, then $\delta = h(\xi_1 - \xi_2)dz^2$ is holomorphic if and only if $h_\xi + \bar{\sigma}h = 0$, which from (12) is equivalent to $\nabla h^2 H = 0$. Since we observed before that $h(\xi_1 - \xi_2)\kappa$ is real valued, then equation (13) can be written $\kappa_{\bar{\xi}z} + \frac{1}{2}\kappa = Re(h(\xi_1 - \xi_2)\kappa)$, and so $G$ is constrained Willmore.

On the other hand $\delta \equiv 0$ if and only if $h = 0$, if and only if $f$ is stationary by (5). Thus $\delta \equiv 0$ if and only if (13) becomes $\kappa_{\bar{\xi}z} + \frac{1}{2}\kappa = 0$, which says that $G$ is Willmore. 

\[\square\]

Marginally trapped surfaces in $S^4_1$ with parallel mean curvature vector field were classified in [9], see also [11].

As an application of Theorem 3.1 and formula (13) we solve here the following problem: find conditions which determine a given marginally trapped surface up to ambient isometries of $S^4_1$.

**Theorem 4.3.** Let $f, f' : \Sigma \to S^4_1$ be marginally trapped surfaces with never vanishing Hopf differentials $q, q'$, and assume that $f, f'$ induce the same conformal metric. If, i) $f, f'$ are both non-stationary and they have the same delta differentials $\delta = \delta'$, or ii) $f, f'$ are both stationary with $q = q'$, then there is an isometry $\Phi$ of $S^4_1$ such that $\Phi \circ f = f'$.

**Proof:** Assume first that $f, f'$ are both non-stationary with $\delta = \delta'$ i.e. $h(\xi_1 - \xi_2)dz^2 = h'(\xi'_1 - \xi'_2)dz^2$, hence $h(\xi_1 - \xi_2) = h'(\xi'_1 - \xi'_2)$. Since $h, h'$ are real and non-zero, we may assume they are both positive (if say $h < 0$, we can replace $f$ by its antipodal $-f$ which has mean curvature function $-h > 0$). Since the Hopf differentials $q, q'$ are never zero, we use the polar form $\xi_1 - \xi_2 = |\xi_1 - \xi_2| e^{i\theta}$ and $\xi'_1 - \xi'_2 = |\xi'_1 - \xi'_2| e^{i\theta'}$. Hence the equality $\delta = \delta'$ implies

\[h|\xi_1 - \xi_2| e^{i\theta} = h'|\xi'_1 - \xi'_2| e^{i\theta'}.

It follows that $\theta - \theta' = 2k\pi$ with integer $k$, and since by hypothesis we have $u = u'$, then (35) implies $\kappa = \kappa'$. On the other hand from $\delta = \delta'$ and (13) it follows that $s = s'$. Hence $G, G'$ have the same conformal invariants $\kappa$ and $s$. By Theorem 3.1 there is a Moebius transformation $T$ of $S^3 \equiv P(L)$ such that $TG = G'$. Recall that the group of Moebius transformations of $S^3$ is $O_+(4, 1)$ acting on $S^3$ by $T(x) = [Tx], \forall x \in L$. If $Y$ is the canonical lift of $G$ respect to to a holomorphic coordinate $z$, then $Y' = TY$ is the canonical lift of $G'$ respect to $z$. Since $V = span\{Y, Re(Y_\bar{z}), Im(Y_\bar{z}), Y_\bar{z}\}$, it follows that $TV = V'$ and so $TV^\perp = V'^\perp$. This last equality implies $Tf = \pm f'$. Defining $\Phi = T$, if $Tf = +f$ and $\Phi = -T$, if $Tf = -f$, then $\Phi$ is an isometry of $S^4_1$ satisfying $\Phi \circ f = f'$. If now $f, f'$ are both stationary with $q = q'$, then $|\xi_1 - \xi_2| e^{i\theta} = |\xi'_1 - \xi'_2| e^{i\theta'}$, and so $\theta - \theta'$ is an integer multiple of $2\pi$. Thus since $u = u'$ by hypothesis, (35) implies $\kappa = \kappa'$. Since $f, f'$ are both stationary, then $\delta = \delta' = 0$. Thus from (13), we conclude that $s = s'$, and so $G, G'$ have the same conformal invariants. Following an analogous argument as before we conclude the proof.

\[\square\]

A congruence result for stationary spacelike surfaces in $S^4_1, n \geq 4$ was proved in [16] using a different technique.

5. Associated Families of Marginally Trapped Surfaces

Our goal here is to describe integrable one-parameter deformations of two classes of marginally trapped surfaces in $S^4_1$ namely,

\[\text{1The sign ambiguity here reflects the fact that the sphere congruences determined by } f \text{ and } f' \text{ (modulo Moebius transformations) are equal up to orientation.}\]
From now on we will work with adapted frames. A spacelike surface \( f : \Sigma \to S^1 \) is non-isotropic if the quartic Hopf differential \( Q = (f_{zz}, f_{z})dz^2 \) is never zero on \( \Sigma \) [14]. Since \((f_{zz}, f_{z}) = (\xi_1 - \xi_2)(\xi_1 + \xi_2)\), then if \( f \) is non-isotropic its Hopf differential \( q = (\xi_1 - \xi_2)dz^2 \) is never zero and so the spherical Gauss image of \( f \) is a conformally immersed umbilic free surface in \( S^3 \). For interesting results on isotropic marginally trapped surfaces in \( S^1 \) see [11].

Recall that the structure equations (3) of a marginally trapped conformal immersion \( f : \Sigma \to S^1 \) are given by:

\[
\begin{align*}
f_{zz} &= 2u_z f_z + \xi_1 N_1 + \xi_2 N_2, \\
f_{\bar{z}z} &= -e^{2u} f + e^{2u} H, \\
\partial_z N_1 &= -hf_z - e^{-2u} \xi_1 f_z + \sigma N_2, \\
\partial_z N_2 &= hf_z + e^{-2u} \xi_2 f_z + \sigma N_1,
\end{align*}
\]

where \( \{N_1, N_2\} \) is a positively oriented normal frame along \( f \). The compatibility among these equations are the fundamental equations of Gauss, Codazzi and Ricci of the immersion (4).

\[
\begin{align*}
2u_{\bar{z}z} &= -e^{2u} + e^{-2u} (|\xi_1|^2 - |\xi_2|^2), \\
e^{-2u}(\partial_\bar{z}\xi_1 + \xi_2\bar{\sigma}) &= e^{-2u}(\partial_\bar{z}\xi_2 + \xi_1\bar{\sigma}) = \partial_\bar{z} h + \sigma h, \\
Im(\sigma z) &= e^{-2u} Im(\xi_1 \bar{\xi}_2).
\end{align*}
\]

From now on we will work with adapted frames: a frame \( F = (f, F_1, F_2, N_1, N_2) \in SO_+(4,1) \) (in column notation) will be called adapted to the immersion \( f \), if \( df(T_x \Sigma) = \operatorname{span}\{F_1(x), F_2(x)\} \), for every \( x \in \Sigma \), and the orthonormal lorentzian normal frame \( \{N_1, N_2\} \) is positively oriented along \( f \).

Since \( f \) is conformal we can rotate within the tangent plane \( \operatorname{span}\{F_1, F_2\} \), if necessary, so that \( f_z = \frac{e^{iu}}{\sqrt{2}}(F_1 - iF_2) \). The structure equations for \( f \) in terms of the adapted frame \( F \) can be written as \( F_z = FA \), where

\[
A = \begin{pmatrix}
0 & -e^{iu} & 0 & 0 & 0 \\
e^{iu} & 0 & -a_1 & a_2 \\
e^{-iu} & 0 & -ib_1 & ib_2 \\
0 & a_1 & ib_1 & 0 & \sigma \\
0 & a_2 & ib_2 & \sigma & 0
\end{pmatrix},
\]

with coefficients

\[
\begin{align*}
a_1 &= \frac{e^{uh} - e^{-u}\xi_1}{\sqrt{2}}, \\
b_1 &= \frac{-e^{uh} - e^{-u}\xi_1}{\sqrt{2}}, \\
a_2 &= \frac{e^{uh} + e^{-u}\xi_2}{\sqrt{2}}, \\
b_2 &= \frac{-e^{uh} + e^{-u}\xi_2}{\sqrt{2}}.
\end{align*}
\]

Defining \( B := \overline{A} \), then the compatibility among (3) is given by the matrix differential equation \( Az - Bz = [A, B] \) which encodes Gauss, Codazzi and Ricci’s equations (9). In terms of the \( \mathfrak{so}(4,1) \)-valued one form \( \alpha := Adz + Bd\bar{z} \) the consistence of (3) is expressed by the Maurer-Cartan equation \( d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \), which is the integrability equation for the existence of an adapted frame \( F \) solving the equation \( F^{-1}dF = \alpha \).
5.1. Surfaces with flat normal bundle and isothermic spherical Gauss image. A conformally immersed surface \( \psi : \Sigma \to S^3 \) is isothermic if \( \kappa = \Re \). In this case the conformal Gauss and Codazzi’s equations (18) away of umbilic points reduce to
\[
\begin{align*}
\kappa & = 4(\kappa^2)z, \\
\text{Im}(\kappa^2 + \frac{3}{8} \kappa) & = 0.
\end{align*}
\]
Let \( f : \Sigma \to S^4_t \) be marginally trapped conformal immersion which is non-stationary. We assume \( f \) has isothermic spherical Gauss image \( G \) and flat normal bundle and positive mean curvature function \( h > 0 \). Let \( \kappa \) and \( s \) be the conformal invariants of \( G \). Then \( \kappa \) is real valued and by (34) the Hopf differential \( q = (\xi_1 - \xi_2)dz^2 \) is real valued too i.e. \( (\xi_1 - \xi_2) = (\xi_1 - \xi_2) \). Since \( f \) has flat normal bundle the Hopf differential \( q \) is holomorphic by Lemma 4.1, thus \( \xi_1 - \xi_2 \) must be a non-zero real constant, i.e. \( q = cdz^2 \), with \( c \in \mathbb{R}^\times \), and we may assume \( c > 0 \). The structure equations of \( f \) are given by (8) with \( \xi_1 = \xi + c, \xi_2 = \xi, \sigma = 0 \), in which the positively oriented orthonormal lorentzian frame \( \{N_1, N_2\} \) is \( \nabla^\perp \)-parallel. The compatibility equations (9) reduce to
\[
\begin{align*}
2u_{zz} & = -e^{2u} + e^{-2u}(2c\xi + c^2), \\
2\xi_z & = e^{2u}h_z, \\
0 & = \text{Im}(\langle \xi + c \rangle \xi).
\end{align*}
\]
In terms of an adapted frame \( F = (f, F_1, F_2, N_1, N_2) \) the structure equations of \( f \) read \( F_z = FA \), where the coefficients of the matrix \( A \) in (43) are in this case given by
\[
\begin{align*}
a_1 & = \frac{e^{-u(\xi + c) + u\xi}}{\sqrt{2}}, & b_1 & = \frac{e^{-u(\xi + c) - u\xi}}{\sqrt{2}}, & \sigma & = 0.
\end{align*}
\]
We consider the one-parameter family of matrices given by
\[
A_t = \begin{pmatrix}
0 & -e^u \sqrt{2} & i e^u \sqrt{2} & 0 & 0 \\
0 & 0 & iu_z & -a_1^t & a_2^t \\
-ie^u \sqrt{2} & -iu_z & 0 & -ib_1^t & ib_2^t \\
0 & a_1^t & ib_1^t & 0 & 0 \\
0 & a_2^t & ib_2^t & 0 & 0
\end{pmatrix} \in \mathfrak{so}(4,1)^\mathbb{C}, \quad t \in \mathbb{R},
\]
in which
\[
h^t := h + \frac{t}{2c}, \quad c \in \mathbb{R}^\times, \quad t \in \mathbb{R},
\]
with \( q = cdz^2 \), and
\[
\begin{align*}
a_1^t & = \frac{e^{-u(\xi + c) + u\xi h^t}}{\sqrt{2}}, & b_1^t & = \frac{e^{-u(\xi + c) - u\xi h^t}}{\sqrt{2}}, \\
a_2^t & = \frac{e^{-u\xi + u\xi h^t}}{\sqrt{2}}, & b_2^t & = \frac{-e^{-u\xi - u\xi h^t}}{\sqrt{2}}.
\end{align*}
\]
Note that for \( t = 0 \) we recover \( A \), i.e. \( A_0 = A \).

**Lemma 5.1.** Let \( B_t := \overline{A_t} \) and define a one parameter family of \( \mathfrak{so}(4,1) \)-valued one-forms by
\[
\alpha_t := A_t dz + B_t dz, \quad t \in \mathbb{R}.
\]
Then \( \alpha_t \) satisfies the Maurer-Cartan equation
\[
d\alpha_t + \frac{1}{2} [\alpha_t \wedge \alpha_t] = 0, \quad \forall t \in \mathbb{R},
\]
if and only if \( u, \xi, h \) satisfy the Gauss, Codazzi and Ricci’s equations (18). Also at \( t = 0 \), \( \alpha_0 = \alpha \).
Moreover, since $f(54)$

Since $N$.

which is unique up to left translation by a constant element in $\integrate{\Maurer-Cartan\text{ equation}}{(52)}$ on $\tilde{\Sigma}$ for each and $(54)$ we obtain $F(53)$ $(t$ metric for any $\frame$ for the conformal immersion $f$).

Hence, trapped. On the other hand $A H$.

Thus the mean curvature vector of $f$.

extract $\{\forall\}$ (orthonormal frame with respect to the normal connection $\nabla$ $q$).

Proof. Let $\tilde{\Sigma}$ be the universal covering space of $\Sigma$ (hence $\tilde{\Sigma} = C$, or $\tilde{\Sigma} = \{\text{Im}(z) > 0\}$). We can integrate the Maurer-Cartan equation $(52)$ on $\tilde{\Sigma}$ for each $t$, obtaining $F^t : \tilde{\Sigma} \to SO_+(4, 1)$, which is unique up to left translation by a constant element in $SO_+(4, 1)$. Thus $F^t$ satisfies

$(F^t)^{-1} dF^t = \alpha_t, \ F^0 = F,$

since $\alpha_0 = \alpha$. It is possible to choose the constants of integration so that $t \mapsto F^t(p)$ is $C^\infty$ for every $p \in \tilde{\Sigma} [5, 15]$. Denote by $F^t := (f^t, F^t_1, F^t_2, N^t_1, N^t_2)$ in column notation. Since $N^t_2 = N_2$ is future pointing, then by continuity $N^t_2$ is future pointing for every $t$. Moreover, since $\{N^0_1, N^0_2\} = \{N_1, N_2\}$ is positively oriented, then a simple argument using continuity shows that $\{N^t_1, N^t_2\}$ is positively oriented for every $t \in \mathbb{R}$.

Define $f^t := F^t e_0$, the first column of $F^t$, then

$$f^t_z = F^t_z e_0 = F^t A_t (e_1 - i e_2) = \frac{e^u}{\sqrt{2}} F^t (e_1 - i e_2),$$

hence,

$$\langle f^t_z, f^t_z \rangle = \frac{e^{2u}}{2} (F^t (e_1 - i e_2), F^t (e_1 + i e_2)) = e^{2u},$$

$$\langle f^t_z, f^t_z \rangle = \frac{e^{2u}}{2} (F^t (e_1 - i e_2), F^t (e_1 - i e_2)) = 0,$$

which shows that $f^t$ is a conformal spacelike immersion which induces the same (conformal) metric for any $t$. Since $f^{t=0} = f$, $f^t$ is a one parameter deformation of $f$. Also from $(53)$ and $(54)$ we obtain

$$f^{t}_{zz} = u_z f^t_z + \frac{e^u}{\sqrt{2}} F^t B (e_1 - i e_2), \quad f^{t}_{zz} = u_z f^t_z + \frac{e^u}{\sqrt{2}} F^t A (e_1 - i e_2),$$

which, from the structure of the matrices $A_t, B_t$, become,

$$f^{t}_{zz} = -e^{2u} f^t + e^{2u} h^t (N^t_1 + N^t_2), \quad f^{t}_{zz} = 2 u_z f^t_z + (\xi + c) N^t_1 + \xi N^t_2.$$

Hence the mean curvature vector of $f^t$ is given by $\overrightarrow{H} = h^t (N^t_1 + N^t_2)$ and so $f^t$ is marginally trapped. On the other hand $F^t$ is adapted to $f^t$ since $F^t_z = F^t A_t$. From this equation we extract

$$\partial_z N^t_1 = -a^t_1 F^t_1 - i b^t_1 F^t_2, \quad \partial_z N^t_2 = a^t_2 F^t_1 + i b^t_2 F^t_2,$$

which shows that $f^t$ has flat normal bundle for every $t$ and that $\{N^t_1, N^t_2\}$ is a parallel orthonormal frame with respect to the normal connection $\nabla^t$ of $\nu(f^t)$.

Lemma 5.2. Let $f: \Sigma \to S^1_1$ be a conformal marginally trapped immersion with flat normal bundle and non-vanishing Hopf differential $q$. Let $F = (f, F_1, F_2, N_1, N_2)$ be an adapted frame of $f$, such that the positively oriented lorentzian frame $\{N_1, N_2\}$ is $\nabla^t$-parall$\ell$ and let $\alpha = F^{-1} dF$ be the induced Maurer-Cartan form. If $\alpha_t$ is given by $(51)$, then $d\alpha_t + \frac{1}{2} [\alpha_t \wedge \alpha_t] = 0$, for every $t \in \mathbb{R}$. Moreover, since $\alpha_t$ is $\so(4, 1)$-valued for $t$, and if $F^t$ integrates $\alpha_t$ on the universal covering $\tilde{\Sigma}$, then $F^t = (F^t_0, F^t_1, F^t_2, N^t_1, N^t_2)$ is an adapted frame for the conformal immersion $f^t = F^t e_0 : \tilde{\Sigma} \to S^1_1$ with $\{N^t_1, N^t_2\}$ positively oriented.
and $\nabla^1_t$-parallel. In particular $f^t$ is marginally trapped and has flat normal bundle for any $t \in \mathbb{R}$. Since $\alpha_0 = \alpha$, the original immersion $f$ is recovered at $t = 0$.

The deformation $f^t$ described above is induced essentially by the Calapso-Bianchi T-transform \[7\] of isothermic surface theory. In fact note first that the fundamental equation \[13\] of the spherical Gauss image $G$ of $f$ reads

$$\kappa_{\zeta\zeta} + \frac{\bar{s}}{2}\kappa = c\chi\kappa, \quad 0 < c \in \mathbb{R}_+, \quad \delta = ch\chi^2.$$

The isothermic T-transform deforming the schwartzian is given by,

$$s_t = s + t, \quad t \in \mathbb{R}.$$  \tag{57}$$

Since $(s_t)_\zeta = s_\zeta$, it is immediate that $\kappa, s_t$ satisfy the conformal Gauss and Codazzi’s equations \[13\], and that under a coordinate change they transform according to \[19\]. Thus by Theorem \[34\] the conformal invariants $\kappa, s_t$ determine a unique (up to Moebius transformations of the sphere) conformal immersion $G_t : \Sigma \to \mathbb{S}^3$. Since for $t = 0$ we recover $s$ in \[57\], $G_t$ is the T-transform of the isothermic spherical Gauss image $G$ and gives the associated family of $G$. We claim that $G_t$ is the spherical Gauss image of $f^t$. To prove the claim note that from \[55\] it follows that $q = cd\chi^2$ is the Hopf differential of $f^t$. This together with the fact that $f^t$ induce the same conformal metric for all $t$, imply that $\theta$ in formula \[35\] must be an integer multiple of $2\pi$, and so $\kappa = \frac{e^\theta}{\sqrt{2}}$ is the (common) normal Hopf differential of the spherical Gauss image of all $f^t$. Inserting \[57\] into \[56\] yields,

$$\kappa_{\zeta\zeta} + \frac{\bar{s}}{2}\kappa = c(h + \frac{t}{2c}) = ch^t\kappa, \quad \delta_t = ch^td\chi^2,$$

in which $\delta_t = ch^td\chi^2$ is just the delta differential of $f^t$. Thus we may view the above equation as an evolution of \[56\] and so $s_t$ is just the schwartzian of the spherical Gauss map of $f^t$. Thus $G_t$ has the same conformal invariants $\kappa, s_t$ as the spherical Gauss image of $f^t$. Therefore $G_t$ coincides up to a Moebius transformation of $\mathbb{S}^3$ with the spherical Gaussian image of $f^t$. We have thus proved the claim.

5.2. **Surfaces with constrained Willmore spherical Gauss image.** By Theorem \[14.2\] the spherical Gauss image of marginally trapped surfaces with holomorphic delta differential is a Willmore or constrained Willmore surface in the conformal 3-sphere. We show here that being $\delta$ holomorphic is equivalent to the harmonicity of certain auxiliar map $\phi$ with values in a pseudo riemannian symmetric space (a complex quadric) and as such, has a well understood spectral deformation $\phi^\lambda$ \[3\]. We show that the symmetries of the harmonic map equation of $\phi$ determine a spectral deformation of marginally trapped surfaces with holomorphic $\delta$. This in turn gives rise to a spectral deformation of the conformal invariants of the spherical Gauss map $G$ which gives rise to the associated family $G_\lambda$.

Consider the open submanifold $\mathbb{CP}^4_1 \subset \mathbb{CP}^4$ consisting of all spacelike complex lines through the origin of $\mathbb{C}^5_1$ and define the map

$$\phi : \Sigma \to \mathbb{CP}^4_1, \quad p \mapsto \mathbb{C}f_\zeta(p),$$

where $\mathbb{C}f_\zeta(p) \subset \mathbb{C}^5_1$ is the spacelike isotropic complex line generated by $f_\zeta(p)$. Thus $\phi$ is well defined since it is independent on any particular choice of local coordinate $z$. Moreover $\phi$ factors through the manifold of isotropic spacelike complex lines in $\mathbb{CP}^4_1$ and coincides with the complex quadric

$$Q = \{[z] \in \mathbb{CP}^4_1 : z_0^2 + z_1^2 + z_2^2 + z_3^2 - z_4^2 = 0\}.$$  \tag{59}$$
Hence $Q$ is a complex submanifold, hence totally geodesic in $\mathbb{C}P^4$ on which $SO_+(4,1)$ acts transitively. Taking $\omega := [e_1 - ie_2] \in Q$ as a base point, then $Q$ is diffeomorphic to the pseudo-riemannian symmetric space $SO_+(4,1)/SO(2) \times SO(2,1)$.

Denote by $L \to \mathbb{C}P^4$ the tautological line bundle whose fiber over a point $l \in \mathbb{C}P^4$ is the line $l$ itself and consider the complex line subbundle $\ell := \phi^*(L) \subset \Sigma \times \mathbb{C}P^4$. Denote by $\ell^\perp$ the complementary orthogonal line subbundle so that $\Sigma \times \mathbb{C}P^4 = \ell \oplus \ell^\perp$. Any section $\mu$ of the trivial bundle $\Sigma \times \mathbb{C}P^4$, decomposes uniquely as $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \Gamma(\ell)$ and $\mu_2 \in \Gamma(\ell^\perp)$. The projection maps are defined by $\pi_\ell \mu = \mu_1$ and $\pi_{\ell^\perp} \mu = \mu_2$. Since $Q$ is totally geodesic in $\mathbb{C}P^4$, the map $\phi$ is harmonic as a map into $Q$ if and only if it is harmonic as a map into $\mathbb{C}P^4$. Consider on $\ell$ and $\ell^\perp$ the Koszul-Malgrange complex structure [12].

Recall that a section $s \in \Gamma(\ell)$ (resp. $s \in \Gamma(\ell^\perp)$) is holomorphic if and only if $\pi_\ell(s) = 0$, (resp. $\pi_{\ell^\perp}(s) = 0$). It is known that $\phi : \Sigma \to \mathbb{C}P^4$ is harmonic if and only if the map $$d\phi(\partial_z) : \ell \to \ell^\perp, \quad \mu \mapsto \pi_{\ell^\perp}(\partial_z \mu)$$ is holomorphic, i.e. it sends holomorphic sections of $\ell$, to holomorphic sections of $\ell^\perp$ [4, 12]. Recall the structure equations (8) of the surface $f : \Sigma \to \mathbb{S}^4$ and the corresponding Gauss Codazzi and Ricci’s equations (9). The second structure equation $f_{zz} = -e^{2u}f + e^{2u}H$ implies $\pi_\ell(\partial_z f_z) = 0$, which says that $f_z$ is a holomorphic section of $\ell$. In particular every holomorphic section of $\ell$ is of the form $\zeta f_z$, with $\zeta$ a complex holomorphic function. Thus $\phi$ is harmonic if and only if $\mu := \pi_\ell(\xi f_z)$ is a holomorphic section of $\ell^\perp$. From (8) it follows that $\mu = \xi_1 N_1 + \xi_2 N_2 = H(\partial_z, \partial_z) \in \Gamma(\ell^\perp)$. Using (27) and Codazzi’s equations (9) and (12), we compute

$$\pi_{\ell^\perp}(\partial_z \mu) = \pi_{\ell^\perp}(\partial_z \xi_1 N_1 + \partial_z \xi_2 N_2 + \xi_1 \partial_z N_1 + \xi_2 \partial_z N_2) =$$

$$\pi_{\ell^\perp}(e^{-2u}(h_z + \sigma h) - \xi_2 \sigma N_1 + (e^{-2u}(h_z + \sigma h) - \xi_1 \sigma) N_2) +$$

$$\pi_{\ell^\perp}(\xi_1(-h f_z - e^{-2u}(\xi_1 f_z + \sigma N_2) + \xi_2(h f_z + e^{-2u}(\xi_2 f_z + \sigma N_1)) =$$

$$e^{-2u}(h_z + \sigma h)(N_1 + N_2) = e^{-2u}\nabla_{\partial_z} H.$$

Hence the section $\mu$ is holomorphic if and only if $f$ has parallel mean curvature vector field and this in turn is equivalent to the harmonicity of $\phi$. From Theorem [12] and its proof it follows that $\phi$ is harmonic if and only if the delta differential of $f$ is holomorphic.

Consider the involution $\tau$ of $SO_+(4,1)$ given by $\tau(F) = EFE$, where $E = \text{diag}(1, -1, -1, 1, 1) \in SO_+(4,1)$. The subgroup by $H \subset SO_+(4,1)$ of fixed points of $\tau$ is isomorphic to $SO(2) \times SO(2,1)$, and so the symmetric space $SO_+(4,1)/H$ is diffeomorphic to the complex hyperquadric (59). The $(\pm 1)$-eigenspaces of $\tau$ are given respectively by

$$\mathfrak{m} := \left\{ \begin{pmatrix} 0 & a & b & 0 & 0 \\ -a & 0 & 0 & c & d \\ 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a,b,c,d,e,k \in \mathbb{R} \right\}, \quad \mathfrak{h} := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : s,t,m,n \in \mathbb{R} \right\},$$

which satisfy

$$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

Let $F$ be an adapted frame of $f$ denoted in column notation by $F = (f, F_1, F_2, N_1, N_2)$, then $F$ also frames $\phi$. Consider the $\mathfrak{so}(4,1)$-valued one form $\alpha = F^{-1}dF = Adz + Bd\bar{z}$, where $\alpha_\ell = A(\partial_z, \partial_z) + B(\partial_{\bar{z}}, \partial_{\bar{z}}), \alpha_{\ell^\perp} = A(\partial_{\bar{z}}, \partial_{\bar{z}}) - B(\partial_z, \partial_z)$, and $A$ and $B$ are fixed real valued functions. In particular $A$ is harmonic if and only if

$$A(\partial_z, \partial_z) = e^{-\varphi} \nabla_{\partial_z} \bar{H}.$$
where $A$ is given by \((15)\). Decompose $A = A_m + A_h$, where

$$\begin{pmatrix}
0 & -\frac{e^m}{\sqrt{2}} & i \frac{e^m}{\sqrt{2}} & 0 & 0 \\
-\frac{e^m}{\sqrt{2}} & 0 & 0 & -a_1 & a_2 \\
\frac{e^m}{\sqrt{2}} & 0 & 0 & ib_1 & ib_2 \\
0 & a_1 & ib_1 & 0 & \sigma \\
0 & a_2 & ib_2 & \sigma & 0
\end{pmatrix}, \quad B_m = \overline{A_m},$$

\((61)\)

with coefficients given by \((49)\), and

$$A_h = \text{diag}(0, \begin{pmatrix} 0 & iu_1 \\ -iu_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}), \quad B_h = \overline{A_h}.$$  

\((62)\)

By inspection of the structure of these matrices we observe that $A_m, B_m$ are $m$-valued while $A_h$ and $B_h$ are $h$-valued. Also since $d\tau(x([A_m, B_m]) = [A_m, B_m])$, then $[A_m, B_m]$ is $h$-valued.

Define an $S^1$-loop of $g$-valued one-forms $\alpha_\lambda$ by

$$\alpha_\lambda = \lambda^{-1}\alpha_m + \alpha_h + \lambda\alpha''_m = \lambda^{-1}A_m dz + (A_h dz + B_h d\bar{z}) + \lambda B_m d\bar{z}.$$  

Hence $\alpha_1 = \alpha$, and so $\alpha$ is a one-parameter deformation family of $\alpha$. Note also that $\alpha_\lambda = A^\lambda dz + B^\lambda d\bar{z}$, where $A^\lambda = \lambda^{-1}A_m + A_h$ and $B^\lambda = \overline{A^\lambda}$. According to the splitting $\mathfrak{so}(4,1) = \mathfrak{h} \oplus \mathfrak{m}$ the Maurer-Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$, decomposes into its $\mathfrak{m}$ and $\mathfrak{h}$ parts,

$$\begin{cases}
da_m' + [\alpha_h \land \alpha_m'] + da_m'' + [\alpha_h \land \alpha''_m] = 0, \\
da_h + \frac{1}{2}[\alpha_h \land \alpha_h] + [\alpha_m' \land \alpha''_m] = 0.
\end{cases}$$  

\((64)\)

On the other hand from the general theory \([5]\) we know that the harmonic map equation of $\phi$ in terms of $\alpha$ is expressed by

$$\bar{\partial}\alpha_m' + [\alpha_h \land \alpha_m'] = 0.$$  

Combining \((64)\) with the harmonic map equation for $\phi$ above gives

$$\begin{cases}
(a) \quad da_m'' + [\alpha_h \land \alpha''_m] = 0, \\
(b) \quad da_h + \frac{1}{2}[\alpha_h \land \alpha_h] + [\alpha_m' \land \alpha''_m] = 0.
\end{cases}$$  

\((65)\)

As is well known a spectral parameter can be inserted in these equations. Let $\lambda \in \mathbb{C}$, with $|\lambda| = 1$ and define

$$\lambda \cdot \alpha := \alpha_\lambda = \lambda^{-1}\alpha_p' + \alpha_t + \lambda\alpha''_p.$$  

\((66)\)

Since $\overline{A_p} = B_p$ and $A_p = B_t$, we have $\overline{\alpha''_p} = \alpha''_p$ and $\overline{\alpha_t} = \alpha_t$. Thus $\alpha_\lambda$ is $\mathfrak{so}(4,1)$-valued for every $\lambda \in S^1$. According to \([5]\), $\lambda \cdot \alpha := \alpha_\lambda$ defines an action of $S^1$ on $g$-valued 1-forms which leaves invariant the solution set of equations \((a)\) and \((b)\) above. Comparing coefficients of $\lambda$ it follows that \((a)\) and \((b)\) above hold for $\alpha$ if and only if $\alpha_\lambda$ solves the Maurer-Cartan equation for every $\lambda \in S^1$:

$$da_\lambda + \frac{1}{2}[\alpha_\lambda \land \alpha_\lambda] = 0, \forall \lambda \in S^1.$$  

\((67)\)

Hence the one form $\alpha_\lambda$ determines a flat connection $d + \alpha_\lambda$ on the trivial $SO_+(4,1)$ bundle over $\Sigma$. For each $\lambda$ we can integrate $F^{-1} dF = \alpha_\lambda$ on the universal covering surface $\tilde{\Sigma}$ to get a solution $F^\lambda : \tilde{\Sigma} \to SO_+(4,1)$ (hence a local solution on $\Sigma$). It is possible to choose the constants of integration so that $F^\lambda$ depends smoothly on $\lambda$. In column notation,

$$F^\lambda = (F_0^\lambda, F_1^\lambda, F_2^\lambda, N_1^\lambda, N_2^\lambda).$$  

\((67)\)
An elementary argument shows that \( \{N^\lambda_1, N^\lambda_2\} \) is positively oriented \( \forall \lambda \in S^1 \). Moreover \( F^{\lambda=1} = F \), since \( \alpha_1 = \alpha \). Defining \( \phi_\lambda := F^\lambda \circ \phi \) gives the associated family of \( \phi \), which is a one parameter deformation of \( \phi \) since \( \phi_{(\lambda=1)} = \phi \). By (67) each member \( \phi_\lambda : \Sigma \rightarrow \mathbb{C}P^1_1 \) is harmonic.

Now consider \( f^\lambda := F_0^\lambda = F^\lambda e_0 \), the first column of \( F^\lambda \). Then \( f^\lambda \) is a one parameter deformation of \( f \) since \( F^{\lambda=1} = F \), so that \( F_0^{\lambda=1} = F_0 \). Thus at \( \lambda = 1 \) we recover \( f \), i.e. \( f^1 = f \). We call \( f^\lambda, \lambda \in S^1 \) the associated family of the marginally trapped surface \( f \). Now

\[
f^\lambda_z := F^\lambda z e_0 = F^\lambda A^\lambda e_0 = F^\lambda (\lambda^{-1} A_m + A_b) e_0 = \lambda^{-1} \frac{e^u}{\sqrt{2}} F^\lambda (e_1 - ie_2),
\]

thus, \( F^\lambda \) is adapted to \( f^\lambda \). From the above expression we compute

\[
\langle f^\lambda_z, f^\lambda_z \rangle = \langle \lambda^{-1} \frac{e^u}{\sqrt{2}} (e_1 - ie_2), \lambda^{-1} \frac{e^u}{\sqrt{2}} (e_1 - ie_2) \rangle = 0.
\]

\[
\langle f^\lambda_z, f^\lambda \rangle = \langle F^\lambda (\lambda^{-1} A_m + A_b) e_0, F^\lambda (\lambda B_m + B_b) e_0 \rangle = \langle \lambda^{-1} \frac{e^u}{\sqrt{2}} (e_1 - ie_2), \lambda \frac{e^u}{\sqrt{2}} (e_1 + ie_2) \rangle = e^{2u}.
\]

Hence \( f^\lambda \) is a conformal spacelike immersion which induces the same conformal metric for any \( \lambda \in S^1 \) and so all \( f^\lambda \) have the same (local) conformal factor \( u \). We claim that \( f^\lambda \) is marginally trapped for any \( \lambda \in S^1 \). Denote by \( \tilde{H}_\lambda \) the mean curvature vector of \( f^\lambda \). Since \( f^\lambda \) is conformal and spacelike, it follows that

\[
f^\lambda_{zz} = -e^{2u} f^\lambda + e^{2u} \tilde{H}_\lambda,
\]

hence using (76) we obtain

\[
\tilde{H}_\lambda = e^{-2u} \langle f^\lambda_{zz}, N^\lambda_1 \rangle N^\lambda_1 - e^{-2u} \langle f^\lambda_{zz}, N^\lambda_2 \rangle N^\lambda_2.
\]

On the other hand the structure equations of \( f^\lambda \) are expressed by the system \( F^\lambda_z = F^\lambda A^\lambda = F^\lambda (\lambda^{-1} A_m + A_b) \), which is equivalent to

\[
\begin{align*}
f^\lambda_z &= \frac{1}{\lambda} \frac{e^u}{\sqrt{2}} F^\lambda_1 - i \frac{1}{\lambda} \frac{e^u}{\sqrt{2}} F^\lambda_2, \\
\partial_z F^\lambda_1 &= -\frac{1}{\lambda} \frac{e^u}{\sqrt{2}} f^\lambda - i u_2 F^\lambda_1 + \frac{1}{\lambda} a_1 N^\lambda_1 + \frac{1}{\lambda} a_2 N^\lambda_2, \\
\partial_z F^\lambda_2 &= i \frac{1}{\lambda} \frac{e^u}{\sqrt{2}} f^\lambda + i u_2 F^\lambda_1 + i \frac{1}{\lambda} b_1 N^\lambda_1 + i \frac{1}{\lambda} b_2 N^\lambda_2, \\
\partial_z N^\lambda_1 &= -\frac{1}{\lambda} a_1 F^\lambda_1 - \frac{1}{\lambda} b_1 F^\lambda_2 + \sigma N^\lambda_1, \\
\partial_z N^\lambda_2 &= \frac{1}{\lambda} a_2 F^\lambda_1 + i \frac{1}{\lambda} b_2 F^\lambda_2 + \sigma N^\lambda_2,
\end{align*}
\]

from which we compute

\[
\begin{align*}
\langle f^\lambda_{zz}, N^\lambda_1 \rangle &= -\langle f^\lambda_z, \partial_z N^\lambda_1 \rangle = (a_1 - b_1) \frac{e^u}{\sqrt{2}} = e^{2u} a, \\
\langle f^\lambda_{zz}, N^\lambda_2 \rangle &= -\langle f^\lambda_z, \partial_z N^\lambda_2 \rangle = (b_2 - a_2) \frac{e^u}{\sqrt{2}} = e^{2u} b.
\end{align*}
\]

Thus using (69) we obtain \( \tilde{H}_\lambda = h (N^\lambda_1 + N^\lambda_2) \), which shows that \( f^\lambda \) is marginally trapped for every \( \lambda \in S^1 \), with \( h = h \).

On the other hand since \( \xi_1^\lambda = \langle f^\lambda_{zz}, N^\lambda_1 \rangle \) and \( \xi_2^\lambda = -\langle f^\lambda_{zz}, N^\lambda_2 \rangle \), using (70) we obtain

\[
\begin{align*}
\xi_1^\lambda &= \langle f^\lambda_{zz}, N^\lambda_1 \rangle = -\langle f^\lambda_z, \partial_z N^\lambda_1 \rangle = \lambda^{-2} \frac{e^u}{\sqrt{2}} (a_1 + b_1) = \lambda^{-2} \xi_1, \\
\xi_2^\lambda &= -\langle f^\lambda_{zz}, N^\lambda_2 \rangle = \langle f^\lambda_z, \partial_z N^\lambda_2 \rangle = \lambda^{-2} \frac{e^u}{\sqrt{2}} (a_2 + b_2) = \lambda^{-2} \xi_2,
\end{align*}
\]

and so the \((2, 0)\) part of the second fundamental form \( II^\lambda \) of \( f^\lambda \) satisfies

\[
II^\lambda (\partial_z, \partial_z) = \lambda^{-2} \xi_1 N^\lambda_1 + \lambda^{-2} \xi_2 N^\lambda_2.
\]
From (71) and the above expression we obtain the Hopf differential of $f^\lambda$,
\begin{equation}
q_\lambda = \lambda^{-2}(\xi_1 - \xi_2)dz^2 = \lambda^{-2}q.
\end{equation}
In polar form, $q_\lambda = |\xi_1 - \xi_2|e^{i\theta(\lambda)}dz^2 = \lambda^{-2}|\xi_1 - \xi_2|e^{i\theta}dz^2$. Thus $e^{i\theta(\lambda)} = \lambda^{-2}e^{i\theta}$, and so
\begin{equation}
\theta(\lambda) = \theta - 2\varphi,
\end{equation}
where $\lambda = e^{i\varphi}$. Since $\lambda$ does not depend on $z$, neither does $\varphi$ and so $\theta(\lambda)_z = \theta_z$, and $\theta(\lambda)_{zz} = \theta_{zz}$. Taking this into account and applying (41) to the conformal invariants $\kappa_\lambda, s_\lambda$ of the spherical Gauss map $G_\lambda$ and the delta differential $\delta_\lambda$ of $f^\lambda$, we obtain,
\begin{equation}
h\lambda^{-2}(\xi_1 - \xi_2) = ((u - i\theta)_z)^2 + (u - i\theta)_{zz} + \frac{s_\lambda}{2}.
\end{equation}
Combining the above equation with (41) we obtain the Schwarzian derivative of $G_\lambda$:
\begin{equation}
s_\lambda = s + 2(\lambda^{-2} - 1)h(\xi_1 - \xi_2).
\end{equation}

On the other hand from (35) $\kappa_\lambda$ identifies with $\frac{e^{u+i\theta(\lambda)}}{\sqrt{2}}$, so that from (73) we obtain
\begin{equation}
\kappa_\lambda = \frac{e^{u+i(\theta-2\varphi)}}{\sqrt{2}} = \lambda^{-2}\kappa.
\end{equation}
A straightforward computation using (75) and (76) shows that $\kappa_\lambda, s_\lambda$ obey the fundamental equation (33) namely,
\begin{equation}
(k_\lambda)_zz + \frac{s_\lambda}{2}k_\lambda = h\lambda^{-2}(\xi_1 - \xi_2)k_\lambda, \quad \forall \lambda \in S^1,
\end{equation}
where we observe that $h\lambda^{-2}(\xi_1 - \xi_2)k_\lambda = h(\xi_1 - \xi_2)k$, and so it is real valued for every $\lambda \in S^1$. In particular $\kappa_\lambda, s_\lambda$ obey the conformal Codazzi equation:
\begin{equation}
Im \left( (k_\lambda)_zz + \frac{s_\lambda}{2}k_\lambda \right) = 0, \quad \forall \lambda \in S^1.
\end{equation}
Since $\delta$ is holomorphic, it easily follows that $\kappa_\lambda, s_\lambda$ obey the conformal Gauss equation:
\begin{equation}
\frac{(s_\lambda)_z}{2} = 3(k_\lambda)_z k_\lambda + k_\lambda (k_\lambda)_z.
\end{equation}
Moreover, $\delta_\lambda = \lambda^{-2}\delta = \lambda^{-2}h(\xi_1 - \xi_2)dz^2$ is holomorphic $\forall \lambda \in S^1$, since $\lambda$ does not depend on $z$ and $\delta$ is holomorphic. By Theorem 4.2 it follows that $G_\lambda$ is Willmore ($h = 0$) or constrained Willmore ($h \neq 0$).

**Proposition 5.1.** Let $f : \Sigma \to S^1$ be a marginally trapped surface with holomorphic delta differential. Then each member $f^\lambda$ of the associated family of $f$ constructed above is a marginally trapped surface, which induces the same conformal metric as $f$. Moreover, the delta differential of $f^\lambda$ satisfies $\delta_\lambda = \lambda^{-2}\delta$, hence it is holomorphic for every $\lambda \in S^1$.

If $\kappa, s$ are the conformal invariants of the spherical Gauss map $G$ of $f$, then the system consisting of the fundamental equation (33) and the conformal Gauss and Codazzi equations (38) has the following symmetry for unitary $\lambda \in S^1$:
\begin{equation}
\kappa_\lambda = \lambda^{-2}\kappa, \quad s_\lambda = s + 2(\lambda^{-2} - 1)h(\xi_1 - \xi_2), \quad \delta_\lambda = \lambda^{-2}\delta,
\end{equation}
which describe the associated family $G_\lambda$ of the spherical Gauss image of $f$.

**Remark 5.1.** The symmetry (77) obtained above differs from that given in [7] to describe the associated family of arbitrary constrained Willmore surfaces in $S^3$. In fact, in [7] the authors were able to find (without proof) the following symmetry for the conformal Gauss and Codazzi equations of a constrained Willmore surface $\psi : \Sigma \to S^3$:
\begin{equation}
\kappa_\lambda = \lambda\kappa, \quad s_\lambda = s + (\lambda^2 - 1)\eta, \quad \eta_\lambda = \lambda^2\eta,
\end{equation}
in which $\eta dz^2$ is an holomorphic quadratic differential satisfying $\kappa_{zz} + \frac{3}{2}\kappa = Re(\eta\kappa)$. ```
We close this section with some remarks concerning marginally trapped tori in $S^4$. Let $f : \Sigma \to S^4$ be a marginally trapped conformally immersed surface. By Theorem 5.2 and its proof $G$ is constrained Willmore if and only if $f$ has parallel mean curvature vector: $\nabla^\perp H = 0$. In this case the Codazzi and Ricci's equations imply that $Q$ is holomorphic. If now $\Sigma$ is a two torus $T^2$, then since $f$ is non-isotropic there exists (passing to the universal covering space $\mathbb{C}$) a global coordinate $z$ on $T^2$ such that $Q = dz^4$, i.e. $\langle f_{zz}, f_{zz} \rangle = 1$. We may choose a positively oriented orthonormal lorentzian frame $\{N_1, N_2\}$ normal to the surface along $f$ such that

$$f_{zz} = 2uzf_z + \cosh(C)N_1 + \sinh(C)N_2,$$

where $C = \rho + i\Theta$ is some complex function. Then the new positively oriented lorentzian frame $\{N'_1, N'_2\}$ given by

$$N'_1 = \cosh(\rho)N_1 + \sinh(\rho)N_2,$$
$$N'_2 = \sinh(\rho)N_1 + \cosh(\rho)N_2,$$

has structure function $\sigma' = 0$ and so $\{N'_1, N'_2\}$ is $\nabla^\perp$-parallel along $f$. Then

$$f_{zz} = 2uzf_z + \cos(\Theta)N'_1 + i\sin(\Theta)N'_2,$$

and Ricci's equation now becomes $0 = \cos(\Theta)\sin(\Theta)$, of which $\Theta = 0$ is a solution. For simplicity we drop the primes and keep denoting by $\{N_1, N_2\}$ this new $\nabla^\perp$-parallel frame. The structure equations of $f$ become

$$f_{zz} = 2uzf_z + N_1,$$
$$f_{zz} = -e^{2u}f + e^{2u}h(N_1 + N_2),$$
$$\partial_z N_1 = -hf_z - e^{-2u}f_z,$$
$$\partial_z N_2 = hf_z,$$

with compatibility equation $2u_{zz} = -e^{2u} + e^{-2u}$, with $h$ a constant and $u : \mathbb{C} \to \mathbb{R}$ a doubly periodic solution with respect to some lattice $\Gamma \subset \mathbb{C}$. Since $(N_1 + N_2)_z = -e^{-2u}f_z$, the spherical Gauss map $G$ is not constant and so $f$ can not lie in a singular hypersurface of $S^4$. From (79) the Hopf differential of $f$ is given by $q = dz^2$, hence $\theta$ is an integer multiple of $2\pi$ in (35) and so $\kappa = \bar{\kappa}$ which says that $G : T^2 \to S^3$ is isothermic. The fundamental equation (43) of $f$ reduces in this case to

$$\kappa_{zz} + \frac{s}{2}\kappa = h\kappa, \quad \delta = hdz^2,$$

and so by Theorem 4.2 $G$ is also Willmore or constrained Willmore since $h$ is constant.

**Theorem 5.3.** Let $f : T^2 \to S^4$ be a non-isotropic marginally trapped immersed surface. If $\delta$ is holomorphic, then the spherical Gauss image $G : T^2 \to S^3$ is an isothermic Willmore or isothermic constrained Willmore surface.

By a result in [20] (see also [7]) all constrained Willmore isothermic tori in $S^3$ can be immersed as a constant mean curvature surface in some riemannian space form. We conclude with the following

**Theorem 5.4.** Let $f : T^2 \to S^4$ be a non-isotropic non-stationary marginally trapped surface with holomorphic $\delta$ differential. Then its spherical Gauss image $G : T^2 \to S^3$ is an isothermic constrained Willmore immersed torus and hence it can be immersed as a constant mean curvature torus in some riemannian space form $S(r)$. 
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