On Integral Forms of Specht Modules
Labelled by Hook Partitions
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Abstract

We investigate integral forms of simple modules of symmetric groups over fields of characteristic \(0\) labelled by hook partitions. Building on work of Plesken and Craig, for every odd prime \(p\), we give a set of representatives of the isomorphism classes of \(\mathbb{Z}_p\)-forms of the simple \(\mathbb{Q}_p\mathfrak{S}_n\)-module labelled by the partition \((n-k,1^k)\), where \(n \in \mathbb{N}\) and \(0 \leq k \leq n-1\). We also settle the analogous question for \(p=2\), assuming that \(n \not\equiv 0 \pmod{4}\) and \(k \in \{2, n-3\}\). As a consequence this leads to a set of representatives of the isomorphism classes of \(\mathbb{Z}\)-forms of the simple \(\mathbb{Q}\mathfrak{S}_n\)-modules labelled by \((n-2,1^2)\) and \((3,1^{n-3})\), again assuming \(n \not\equiv 0 \pmod{4}\).

Keywords: integral representation, integral form, Jordan–Zassenhaus, symmetric group, Specht module, hook partition

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1 Introduction

Suppose that \(R\) is a principal ideal domain and \(K\) its field of fractions, and let \(G\) be a finite group. As is well known, every finitely generated \(KG\)-module \(V\) admits an \(R\)-form, that is, a finitely generated \(RG\)-module \(M\) that is \(R\)-free of finite rank and satisfies \(V \cong K \otimes_R M\). In general, \(R\)-forms of \(V\) are far from being unique. However, under suitable conditions on \(R\) and \(K\), the Jordan–Zassenhaus Theorem ensures that there are only finitely many \(RG\)-isomorphism classes of \(R\)-forms of \(V\). By [8, Theorem (24.1), Theorem (24.7)] this holds, in particular, if \(K\) is a global field, or if \(R\) is a complete discrete valuation ring and \(K\) is a local field. A finitely generated \(R\)-free \(RG\)-module of finite \(R\)-rank will be called an \(RG\)-lattice throughout.

In light of the Jordan–Zassenhaus Theorem one is immediately led to asking for the precise number of isomorphism classes of \(R\)-forms of \(V\), and possibly concrete representatives of these isomorphism classes. In such generality this is of course a completely hopeless task. So one might impose restrictions on the \(KG\)-modules under consideration, starting with simple \(KG\)-modules. It turns out that even then not too much is known when it comes to determining all isomorphism classes of \(R\)-forms of \(V\) or their number.

A list of some known results in this direction can be found in [8, §34] and [25]. Moreover, if \(\mathfrak{G}_n\) is the symmetric group of degree \(n \in \mathbb{N}\) and \(V\) is the natural simple \(\mathbb{Q}\mathfrak{G}_n\)-module, then the isomorphism classes of \(\mathbb{Z}\)-forms of \(V\) have been determined independently by Craig [6] and Plesken [23, 24]. This was generalized by Feit first to the reflection representation of Weyl groups of indecomposable root systems, see [11], and later to the natural representation of some complex reflection groups, see [12].

In this article we are concerned with the case where \(G\) is the symmetric group \(\mathfrak{G}_n\) of degree \(n \in \mathbb{N}\), and \(R\) is the ring of integers \(\mathbb{Z}\) or its \(p\)-adic completion \(\mathbb{Z}_p\), for some prime number \(p\). The simple \(\mathbb{Q}\mathfrak{G}_n\)-modules have been well studied for more than a century, and are known as Specht modules. Their isomorphism classes are in bijection with the partitions of \(n\), and the Specht \(\mathbb{Q}\mathfrak{G}_n\)-module labelled by a partition \(\lambda\) will be denoted by \(S^\lambda_{\mathfrak{G}_n}\). Specht modules have a number of remarkable properties: for instance, they are absolutely simple, self-dual, and each
\( \mathbb{Q}\mathfrak{S}_n \)-module \( S^1_\mathbb{Q} \) already comes with a distinguished \( \mathbb{Z} \)-form, which we shall denote by \( S^1_\mathbb{Z} \) and whose definition will be recalled in 5.2.

The aim of this article now is to investigate the \( \mathbb{Z} \)-forms of the Specht \( \mathbb{Q}\mathfrak{S}_n \)-modules labelled by hook partitions, that is, partitions of shape \((n-k, 1^k)\), for \( k \in \{0, \ldots, n-1\} \). Since \( S^{(n)}_\mathbb{Q} \) is just the trivial \( \mathbb{Q}\mathfrak{S}_n \)-module and \( S^{(1^n)}_\mathbb{Q} \) is the one-dimensional module affording the sign representation, each of these clearly has only one \( \mathbb{Z} \)-form up to isomorphism. The modules \( S^{(n-1,1)}_\mathbb{Q} \) and \( S^{(2,1^{n-2})}_\mathbb{Q} \) are precisely those dealt with by Plesken and Craig mentioned above; the number of isomorphism classes of \( \mathbb{Z} \)-forms of each of them is the number of positive divisors of \( n \), and both Plesken and Craig give explicit representatives.

Starting from Plesken’s and Craig’s results, we shall thus focus on the case where \( k \geq 2 \). Our overall strategy is as follows: first, in order to determine the isomorphism classes of \( \mathbb{Z} \)-forms of \( S^{(n-k, 1^k)}_\mathbb{Q} \), it suffices to determine the isomorphisms classes of \( \mathbb{Z}_p \)-forms of the \( p \)-adic completion \( S^{(n-k, 1^k)}_\mathbb{Q}_p := \mathbb{Q}_p \otimes_\mathbb{Q} S^{(n-k, 1^k)}_\mathbb{Q} \), for every prime number \( p \); we shall explain this in more detail in Section 3. Given \( p \), every \( \mathbb{Z}_p \)-form of \( S^{(n-k, 1^k)}_\mathbb{Q}_p \) is isomorphic to a full-rank sublattice of any fixed \( \mathbb{Z}_p \)-form of \( S^{(n-k, 1^k)}_\mathbb{Q}_p \); see Remark 2.11.

Thus, for every prime number \( p \), it then suffices to determine the full-rank \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattices of the given \( \mathbb{Z}_p \)-form \( S^{(n-k, 1^k)}_\mathbb{Z}_p := \mathbb{Z}_p \otimes_\mathbb{Z} S^{(n-k, 1^k)}_\mathbb{Z} \) of \( S^{(n-k, 1^k)}_\mathbb{Q} \), up to isomorphism. It turns out that the cases \( p = 2 \) and \( p \geq 3 \) behave completely differently, the main issue being the poorly understood behaviour of the Specht lattices \( s^{(n-k, 1^k)}_\mathbb{Z}_2 \) after 2-modular reduction. While for odd \( p \) we are able to give representatives of the isomorphism classes of \( \mathbb{Z}_p \)-forms of \( S^{(n-k, 1^k)}_\mathbb{Q}_p \) for all \( k \in \{2, \ldots, n-2\} \), for \( p = 2 \) we only get partial information and only in the case where \( k \in \{2, n-3\} \). As the first main result of this paper we obtain the following theorem, which will be proved in Section 8. Here, for a prime number \( p \) and a natural number \( n \in \mathbb{N} \), we denote by \( \nu_p(n) \) the \( p \)-adic valuation of \( n \), that is, \( \nu_p(n) = \max \{ k \in \mathbb{N}_0 : p^k \mid n \} \).

1.1 Theorem. Let \( n \in \mathbb{N} \) be such that \( n \geq 3 \), and let \( k \in \{1, \ldots, n-2\} \). Let \( h_p(k) \) be the number of isomorphism classes of \( \mathbb{Z}_p \)-forms of \( S^{(n-k, 1^k)}_\mathbb{Q}_p \).

- (a) If \( p \) is odd, then \( h_p(k) = \nu_p(n) + 1 \).
- (b) Assume that \( p = 2 \), \( n \geq 5 \), \( n \not\equiv 0 \pmod{4} \) and \( k \in \{2, n-3\} \).
  - (i) If \( n \geq 5 \) is odd, then \( h_2(k) = 3 \).
  - (ii) If \( n \equiv 2 \pmod{4} \), then \( h_2(k) = 4 \).

In fact, part (a) will follow from the results of Plesken and Craig concerning the case \( k = 1 \), and our investigation of the \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattices of \( S^{(n-k, 1^k)}_\mathbb{Z}_p \) in Section 6. In Theorem 6.1 we shall prove that, for \( p \geq 3 \) and \( k \in \{1, \ldots, n-2\} \), there is a bijection between the isomorphism classes of \( \mathbb{Z}_p \)-forms of \( S^{(n-1,1)}_\mathbb{Q}_p \) and those of \( S^{(n-k, 1^k)}_\mathbb{Q}_p \). The key step here is Theorem 4.5 on exterior powers, which should also be of independent interest.

Part (b) of Theorem 1.1 will be a consequence of a very detailed study of the structure of the \( \mathbb{Z}_2 \mathfrak{S}_n \)-lattice \( S^{(n-2, 1^2)}_\mathbb{Q}_2 \) in Section 7, the main results there being Theorem 7.10 and Theorem 7.16. Unfortunately, at present, we are not able to settle the case where \( p = 2 \) and \( n \equiv 0 \pmod{4} \). However, based on computational data we shall state a conjecture at the end of Section 7.

As well, the case \( k \in \{3, \ldots, n-4\} \) and \( p = 2 \) remains open so far, due to a lack of knowledge of the structure of the \( \mathbb{Z}_2 \mathfrak{S}_n \)-lattices \( S^{(n-k, 1^k)}_\mathbb{Q}_2 \) and their 2-modular reductions.
Conjecture 7.18(b) concerns the number of isomorphism classes of $\mathbb{Z}_2$-forms of $S_{Q_2}^{(n-3,1^3)}$ and $S_{Q_2}^{(4,1^{n-4})}$, and is also based on computer calculations.

As an immediate consequence of Theorem 1.1 and Corollary 3.4, we get

1.2 Corollary. Let $n \in \mathbb{N}$ be such that $n \geq 4$ and $n \not\equiv 0 \pmod{4}$. Let $k \in \{2, n-3\}$ and denote by $j(k)$ the number of isomorphism classes of $\mathbb{Z}$-forms of $S_q^{(n-k,1^k)}$, and by $d(n)$ the number of divisors of $n$ in $\mathbb{N}$.

(a) If $n \geq 5$ is odd, then $j(k) = 3d(n)$.
(b) If $n \equiv 2 \pmod{4}$, then $j(k) = 2d(n)$.

In fact, our proof of Theorem 1.1 will not only reveal the number of isomorphism classes of $\mathbb{Z}_p$-forms of the $Q_p$-modules in question, but will provide explicit representatives of their isomorphism classes. Hence we have the following, a more precise statement being the content of Theorem 8.3.

1.3 Theorem. Let $n \in \mathbb{N}$ be such that $n \geq 5$ and $n \not\equiv 0 \pmod{4}$. For $k \in \{2, n-3\}$, we can explicitly construct representatives of the isomorphism classes of $\mathbb{Z}$-forms of $S_q^{(n-k,1^k)}$.

The present paper is organized as follows: in Section 2, we summarize the necessary background on $RG$-lattices and $KG$-modules that will be used throughout. In Section 3 we explain how the $\mathbb{Z}$-forms of an absolutely simple $QG$-module $V$ are related to the $\mathbb{Z}_p$-forms of its $p$-adic completion; specifically Proposition 3.5 will be of great importance for our proof of Theorem 1.3.

Since the Specht module $S_q^{(n-k,1^k)}$, for $k \in \{1, \ldots, n-2\}$, arises as the $k$th exterior power of $S_q^{(n-1,1)}$, in Section 4 we give a brief overview of exterior powers of $KG$-modules and $RG$-lattices. In Section 5 we introduce Specht modules and Specht lattices, and recall the connection of Specht modules labelled by hook partitions with exterior powers just mentioned. Sections 6 and 7 are then devoted to analyzing the $\mathbb{Z}_pSG_n$-lattices $S_{\mathbb{Z}_p}^{(n-k,1^k)}$ in the case where $p \geq 3$ and $p = 2$, respectively. The main results of these two sections will pave the way towards proving Theorem 1.1 and Theorem 1.3 in Section 8.

We conclude this article with an appendix concerning dual Specht $ZSG_n$-lattices. More precisely, in unpublished work [28] Wildon has presented, for every partition $\lambda$ of $n$, a concrete $\mathbb{Z}SG_n$-monomorphism that embeds the $\mathbb{Z}$-linear dual $(S_\lambda^\ell)^*$ into $S_\lambda$. Wildon’s proof refers to the proof of [13, §7.4, Lemma 5], which, however, seems to contain some subtleties. Wildon’s embedding $(S_\lambda^\ell)^* \to S_\lambda$ has been one of the key steps in our proof of Theorem 7.16, and should also be of independent interest. Therefore, we consider it to be worthwhile giving some more details on the construction of the embedding, following the lines of [28] and [18, Theorem 6.7]. We also mention that similar constructions can be found in work of Fayers [10, Section 4].

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2 Preliminaries on modules and lattices

Throughout this paper let $R$ be a principal ideal domain and $K$ its field of fractions. We start by summarizing some well-known facts concerning $RG$-lattices and $KG$-modules that we shall
need throughout. We assume the reader to be familiar with the basic notions on modules over group algebras of finite groups, and refer to [8, 22] for background.

By \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{N}_0 \) we denote the set of integers, the set of positive integers and the set of non-negative integers, respectively. For every prime number \( p \), we further denote by \( \mathbb{Z}_p \) the ring of \( p \)-adic integers and by \( \mathbb{Q}_p \) the field of \( p \)-adic numbers, the \( p \)-adic valuation will be denoted by \( \nu_p \). The localization of \( \mathbb{Z} \) at the prime ideal \( (p) \) will be denoted by \( \mathbb{Z}_{(p)} \).

### 2.1 Notation

Let \( G \) be a finite group.

(a) An \( RG \)-lattice \( L \) is always understood to be a left \( RG \)-module that is finitely generated and free over \( R \); the \( R \)-rank of \( L \) will be denoted by \( \text{rk}_R(L) \).

(b) Suppose that \( F \) is any field. By an \( FG \)-module we shall always mean a finitely generated left \( FG \)-module. If \( V \) is an \( FG \)-module, then we denote by \( \text{Rad}(V) \) the (Jacobson) radical of \( V \) and by \( \text{Hd}(V) := V/\text{Rad}(V) \) the head of \( V \).

For \( i \geq 0 \), we denote the \( i \)th radical of \( V \) by \( \text{Rad}^i(V) \), where \( \text{Rad}^0(V) := V \). Suppose that \( V \) has Loewy length \( l \geq 1 \) with Loewy layers \( \text{Rad}^{l-1}(V)/\text{Rad}^l(V) \cong D_{i_1} \oplus \cdots \oplus D_{i_r} \), for \( i \in \{1, \ldots, l\} \), \( r_1, \ldots, r_l \in \mathbb{N} \) and simple \( FG \)-modules \( D_{i_1}, \ldots, D_{i_r} \). Then we shall write

\[
V \sim \begin{bmatrix} D_{i_1} \oplus \cdots \oplus D_{i_{r_1}} \\ \vdots \\ D_{i_1} \oplus \cdots \oplus D_{i_{r_l}} \end{bmatrix},
\]

and say that \( V \) has Loewy series (1).

### 2.2. Change of coefficient rings

Let \( S \) be a principal ideal domain domain, and let \( \rho : R \to S \) be a unitary ring homomorphism, so that \( S \) becomes an \( (R, R) \)-bimodule via \( \rho \).

(a) With the above notation, we shall identify the \( S \)-algebra \( S \otimes_R RG \) with the group algebra \( SG \) in the usual way.

Suppose that \( L \) is an \( RG \)-lattice with \( R \)-basis \( \{b_1, \ldots, b_k\} \). Then the \( S \)-lattice \( L_S := S \otimes_R L \) becomes naturally an \( SG \)-lattice with \( S \)-basis \( \{1 \otimes b_1, \ldots, 1 \otimes b_k\} \).

If \( L_1 \) and \( L_2 \) are \( RG \)-lattices then there is an isomorphism of \( SG \)-lattices \( S \otimes_R (L_1 \otimes_R L_2) \cong (S \otimes_R L_1) \otimes_S (S \otimes_R L_2) \).

(b) Suppose that \( \rho \) is injective. Then we may view \( L \) as an \( RG \)-submodule of \( S \otimes_R L \), by identifying \( x \in L \) with \( 1 \otimes x \). With these conventions, we then have \( L_S = S \otimes_R L = SL = S \langle b_1, \ldots, b_k \rangle \).

(c) Consider the special case where \( S = K \) and \( \rho \) is the inclusion map. Every \( KG \)-module \( V \) admits an \( RG \)-lattice \( L \) such that \( V \cong KL \) and \( \text{rk}_R(L) = \dim_K(V) \); see, for instance, [7, Theorem (73.6)]. One calls \( L \) an \( R \)-form of \( V \).

(d) Suppose that \( L \) is an \( RG \)-lattice, and denote by \( L^* := \text{Hom}_R(L, R) \) the dual \( RG \)-lattice. If \( \{b_1, \ldots, b_k\} \) is an \( R \)-basis of the \( RG \)-lattice \( L \) then we denote by \( \{b_1^*, \ldots, b_k^*\} \) the \( R \)-basis of \( L^* \) that is dual to \( \{b_1, \ldots, b_k\} \). If \( L \cong L^* \) as \( RG \)-lattices, then \( L \) is called self-dual.

Furthermore, we always obtain an \( SG \)-isomorphism

\[
S \otimes_R L^* \to (S \otimes_R L)^* = \text{Hom}_S(S \otimes_R L, S),
\]

sending \( 1 \otimes b_i^* \) to \( (1 \otimes b_i)^* \), for \( i \in \{1, \ldots, k\} \).
(e) If $S$ is flat as a right $R$-module via $\rho$ and if $L_1$ and $L_2$ are $RG$-lattices, then there is an isomorphism of $S$-modules

$$\Psi : S \otimes_R \text{Hom}_R(L_1, L_2) \cong \text{Hom}_S(S \otimes_R L_1, S \otimes_R L_2)$$

such that $(\Psi(\alpha \otimes \varphi))(\beta \otimes x) = \alpha \beta \otimes \varphi(x)$, for $\alpha, \beta \in S$, $\varphi \in \text{Hom}_R(L_1, L_2)$, and $x \in L_1$; see [22, Theorem 1.11.7].

(f) Let $a$ be an ideal in $R$, let $S := R/a$, and let $\rho : R \to S$ be the canonical projection. Suppose that $L$ is an $RG$-lattice. Then the factor $RG$-module $L/aL$ naturally becomes an $SG$-isomorphism

$$L/aL \cong S \otimes_R L;$$

see [22, Theorem 1.9.17]. As well, note that, by the Third Isomorphism Theorem, every $SG$-sublattice of $L/aL$ is of the form $M/aL$, where $M$ is an $RG$-sublattice of $L$ containing $aL$.

2.3 Lemma. Let $M$ be an $RG$-lattice.

(a) If $N$ is a maximal $RG$-sublattice of $M$, then there exists a maximal ideal $m$ of $R$ such that $mM \subseteq N \subseteq M$.

(b) Let $N$ be an $RG$-sublattice of $M$ with $mM \subseteq N \subseteq M$, for some maximal ideal $m$ of $R$. Then the following are equivalent:

(i) $N$ is a maximal $RG$-sublattice of $M$;

(ii) $N/mM$ is a maximal $(R/m)G$-submodule of $M/mM$;

(iii) $M/N$ is a simple $(R/m)G$-module.

Proof. Part (a) is consequence of Nakayama’s Lemma; a proof can be found in [16, Lemma 8.3]. Part (b) is obvious.

2.4. Order ideal and index. Suppose that $M$ and $N$ are $R$-lattices of the same rank $n \in \mathbb{N}$ with $N \subseteq M$. Then the factor module $M/N$ is a torsion $R$-module and, hence, admits a decomposition

$$M/N \cong \prod_{i=1}^{m} R/(r_i),$$

for suitable $m \in \mathbb{N}$ and $r_1, \ldots, r_m \in R$. The ideal $(r_1 \cdots r_m)$ of $R$ is independent of the chosen decomposition; one sets $(M : N) := (M : N)_R := (r_1 \cdots r_m)$, and calls $(M : N)$ the order ideal of $N$ in $M$. For details concerning order ideals see [8, §4D].

Note that one always has an $R$-endomorphism $\phi : M \to M$ with $\phi(M) = N$. Using this, a connection between the determinant of any such $R$-endomorphism of $M$, the order ideal of $N$ in $M$ and the ordinary index $[M : N]$ from group theory is given by the next proposition. In the proof we shall use that $[M : N] = \prod_{i=1}^{m} |R/(r_i)| = |R/(r_1 \cdots r_m)| = |R/(M : N)|$, provided that every proper factor ring of $R$ is finite. This is due to the fact that $R$ is a unique factorization domain, and can be deduced immediately from the following observation and the Chinese Remainder Theorem: Suppose that every proper factor ring of $R$ is finite. Suppose further that $p$ is a prime element in $R$ and $k \in \mathbb{N}$. Then one has a surjective $R$-module homomorphism $R/(p^k) \to (p^{k-1})/(p^k)$, given by multiplication with $p^{k-1}$, with kernel $(p)/(p^k)$. This gives $R/(p) \cong (p^{k-1})/(p^k)$ as $R$-modules, and implies $|R/(p^k)| = |R/(p)| \cdot \cdots \cdot |(p)/(p^2)| \cdots \cdot |(p^{k-1})/(p^k)| = |R/(p)|^k$.
2.5 Proposition. Let $M$ and $N$ be $RG$-lattices such that $N \subseteq M$ and $\text{rk}_R(M) = \text{rk}_R(N) = n \in \mathbb{N}$. Let further $\phi : M \to M$ be an $R$-endomorphism of $M$ with $\phi(M) = N$. Then one has $(M : N) = (\det(\phi))$. If, moreover, every proper factor ring of $R$ is finite, then

$$|R/(M : N)| = |M : N| = |R/(\det(\phi))|;$$

in particular, if $R = \mathbb{Z}$, then $|M : N| = |\det(\phi)|$. If $R = \mathbb{Z}_p$ for some prime number $p$, then $|M : N| = p^{\nu_p(\det(\phi))}$. 

Proof. By [8, Proposition (4.20a)], we have $(M : N) = (\det(\phi))$. Now suppose that every proper factor ring of $R$ is finite. Let $M/N \cong \prod_{i=1}^m R/(r_i)$ be a decomposition into cyclic torsion $R$-modules as in 2.4. Then $r_i \neq 0$, for all $i \in \{1, \ldots, m\}$. Since all proper quotients of $R$ are finite and since $R$ is a principal ideal domain, we have $|M : N| = \prod_{i=1}^m |R/(r_i)| = |R/(r_1 \cdots r_m)| = |R/(M : N)|$, as noted in 2.4. 

2.6 Lemma. Let $m = (p)$ be a maximal ideal in $R$, and let $M$ and $N$ be $R$-lattices of rank $n \in \mathbb{N}$ such that $mM \subseteq N \subseteq M$. Let further $k$ be the residue field $R/m$, and let $s := \dim_k(M/N)$.

(a) There exists an $R$-basis $\{v_1, \ldots, v_n\}$ of $M$ and an $R$-basis $\{w_1, \ldots, w_n\}$ of $N$ such that $v_i = w_i$, for $i \in \{1, \ldots, n-s\}$, and $w_i = pv_i$, for $i \in \{n-s+1, \ldots, n\}$.

(b) Suppose that $s = 1$, and let $\{v_1, \ldots, v_n\}$ be any $R$-basis of $M$. Then there exists an $R$-basis $\{w_1, \ldots, w_n\}$ of $N$ and $r_1, \ldots, r_n \in R$ such that $w_j = pv_j$, for some $j \in \{1, \ldots, n\}$, and $v_i = v_i + r_jv_j$, for all $i \neq j$.

Proof. In the proof we shall make use of the Hermite and Smith normal form, see [5, Section 2.4] and [1, Sections 5.2, 5.3].

(a) Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be arbitrary $R$-bases of $M$ and $N$ respectively, and let $T = (t_{ij}) \in \text{Mat}_{n \times n}(R)$ be the (uniquely determined) matrix with $\sum_{j=1}^n t_{ij}x_j = y_i$, for $i \in \{1, \ldots, n\}$. That is, $T$ is the matrix of the $R$-endomorphism $\phi : M \to M$, $x_i \mapsto y_i$ with respect to the basis $\{x_1, \ldots, x_n\}$ of $M$. By the theory of the Smith normal form, there exists a diagonal matrix $S = \text{diag}(s_1, \ldots, s_n) \in \text{Mat}_{n \times n}(R)$ and invertible matrices $U_1, U_2 \in \text{GL}_n(R)$ with $U_2 \cdot T \cdot U_1 = S$. Moreover, $s_i | s_{i+1}$, for $i \in \{1, \ldots, n-1\}$ and $M/N \cong \prod_{i=1}^m R/(s_i)$ as $R$-modules. But as $k$-vector spaces and $R$-modules we also have $M/N \cong k^s = (R/(p))^s$.

So, by [1, Corollary 7.6], we may assume that $s_1 = 1$ for $i \in \{1, \ldots, n-s\}$, and $s_i = p$ for $i \in \{n-s+1, \ldots, n\}$. Now if $U_1 = (u_{ij})$ and $U_2^{-1} = (u'_{ij})$, then we set $v_i := \sum_{j=1}^n u'_{ij}x_j$ and $z_i := \sum_{j=1}^n u_{ij}x_j$. Then $\{z_1, \ldots, z_n\}$ and $\{v_1, \ldots, v_n\}$ are $R$-bases of $M$ and $S$ is the matrix of $\phi$ with respect to these bases. Thus $\phi(z_i) = s_i v_i$, for $i \in \{1, \ldots, n\}$. Setting $w_i := s_i v_i$, for $i \in \{1, \ldots, n\}$, the claim of (a) follows.

(b) We proceed as in (a) but use the Hermite normal form instead of the Smith normal form. Let $\{v_1, \ldots, v_n\}$ be the fixed $R$-basis of $M$, let $\{y_1, \ldots, y_n\}$ be an arbitrary $R$-basis of $N$, and let $T \in \text{Mat}_{n \times n}(R)$ be such that $\sum_{j=1}^n t_{ij}v_j = y_i$, for $i \in \{1, \ldots, n\}$.

Now let $P \subseteq R$ be a set of representatives of the equivalence classes modulo $R$-associates. We may choose $1 \in P$ (as a representative of the equivalence class of the units in $R$) as well as $p \in P$. For each $r \in P$, let further $P(r)$ be a set of representatives of the residue classes of $R/(r)$; in particular, we choose $P(1) = \{0\}$. By appealing to the Hermite normal form, there exists a transformation $U \in \text{GL}_n(R)$ such that $H = TU$ is a lower triangular matrix with the following properties: the diagonal entries are non-zero elements of $P$, and if $r_i \in P$ is the entry at position $(i,i)$, then all entries at the positions $(i,1), \ldots, (i,i-1)$ are elements of $P(r_i)$. By our hypothesis, we have $M/N \cong R/(p)$ as $R$-modules and $k$-vector spaces. Hence
\((p) = (M : N) = (\det(T)) = (\det(H))\), by Proposition 2.5. With our choice of \(P\) this forces that there is a unique diagonal entry of \(H\) not equal to 1, say at position \((j, j)\), and this entry is equal to \(p\). If \(H = (h_{ij})\), the claim then follows with \(w_i := \sum_{l=1}^{n} h_{il}v_l\) for \(i \in \{1, \ldots, n\}\), since then \(w_j = pw_j\) and \(w_i = v_i + h_{ji}v_j\) for \(i \neq j\). \(\square\)

The next results will provide the key method for determining \(R\)-forms of simple \(KG\)-modules in subsequent sections. These statements go back to [23] in case \(R = \mathbb{Z}\) and can easily be generalized to the setting of the present paper; see also [15, Lemma 8.4].

2.7 Proposition. Let \(m\) be a maximal ideal in \(R\), and let \(k\) be the residue field \(R/m\). Moreover, let \(M\) be an \(RG\)-lattice, and let \(\pi : M \to M/mM\) be the canonical projection.

(a) Let \(D\) be a simple \(kG\)-module, and suppose that there is a non-zero \(kG\)-homomorphism \(\phi \in \text{Hom}_{kG}(M/mM, D)\). Let \(N := \pi^{-1}(\ker(\phi))\). Then \(N\) is a maximal \(RG\)-sublattice of \(M\) with \(mM \subseteq N \subseteq M\) and \(M/N \cong D\) as \(kG\)-modules; in particular, \(m^{\dim_k(D)} = (M : N)\).

(b) Conversely, let \(N\) be a maximal \(RG\)-sublattice of \(M\) such that \(mM \subseteq N \subseteq M\), and let \(s \in \mathbb{N}\) be such that \((M : N) = m^s\). Then there is a simple \(kG\)-module \(D\) and some \(\phi \in \text{Hom}_{kG}(M/mM, D)\) such that \(\dim_k(D) = s\) and \(N = \pi^{-1}(\ker(\phi))\).

2.8 Corollary. Let \(m\) be a maximal ideal in \(R\), let \(k\) be the residue field \(R/m\), and let \(M\) be an \(RG\)-lattice.

(a) Suppose that \(D\) is an absolutely simple \(kG\)-module occurring as a composition factor of \(\text{Hd}(M/mM)\) with multiplicity one. Then there is a unique maximal \(RG\)-sublattice \(N\) of \(M\) such that \(mM \subseteq N \subseteq M\) and \(M/N \cong D\).

(b) Suppose that \(\text{Hd}(M/mM)\) is multiplicity-free with absolutely simple composition factors \(D_1, \ldots, D_l\) of \(\text{Hd}(M/mM)\). Then \(M\) has exactly \(l\) maximal \(RG\)-sublattices \(M_1, \ldots, M_l\) with \(mM \subseteq M_i \subseteq M\) for \(i \in \{1, \ldots, l\}\). Moreover, after reordering, one has \(M/M_i \cong D_i\) for \(i \in \{1, \ldots, l\}\).

Proof. To simplify the notation, let us write \(\bar{M} = M/mM\). First note that if \(D\) is a simple \(kG\)-module, then \(\text{Hom}_{kG}(\bar{M}, D) \neq \{0\}\) if and only if \(D\) is a composition factor of \(\text{Hd}(M)\).

(a) By Proposition 2.7, it suffices to show that \(\dim_k(\text{Hom}_{kG}(\bar{M}, D)) = 1\). Then every non-zero element of \(\text{Hom}_{kG}(\bar{M}, D)\) will have the same kernel. Since \(D\) is absolutely simple and occurs with composition multiplicity one in \(\text{Hd}(M)\), we have \(\dim_k(\text{Hom}_{kG}(\text{Hd}(\bar{M}), D)) = 1\).

The kernel of every non-zero \(kG\)-homomorphism \(M \to D\) is a maximal submodule of \(M\), thus factors through \(\text{Hd}(\bar{M}) = \bar{M}/\text{Rad}(\bar{M})\). Hence the canonical map \(\text{Hom}_{kG}(\bar{M}, D) \to \text{Hom}_{kG}(\text{Hd}(\bar{M}), D)\) is an isomorphism of \(k\)-vector spaces.

Assertion (b) now follows from (a). \(\square\)

Let \(M\) and \(N\) be \(RG\)-lattices. Via the conventions in 2.2, we may and shall from now on identify the \(K\)-vector spaces \(\text{Hom}_{kG}(KM, KN)\) and \(K \otimes_R \text{Hom}_{RG}(N, M)\). Note that every \(RG\)-homomorphism of \(RG\)-lattices \(\phi : M \to N\) has a unique extension to a homomorphism of \(KG\)-modules \(KM \to KN\), which, by abuse of notation, will also be denoted by \(\phi\).

2.9 Lemma. Let \(M\) and \(N\) be \(RG\)-lattices.

(a) One has
\[
\text{Hom}_{RG}(M, N) = \{\phi|_M : \phi \in \text{Hom}_{kG}(KM, KN), \phi(M) \subseteq N\}
\]
and \( \text{rk}_R(\text{Hom}_{RG}(M,N)) = \dim_K(\text{Hom}_{KG}(KM,KN)) \).

(b) Suppose that \( \text{Hom}_{RG}(M,N) \) is an \( R \)-lattice of rank one, generated by \( \phi \). If \( 0 \neq r \in R \setminus R^\times \), then the \( KG \)-homomorphism \( r^{-1}\phi \in \text{Hom}_{KG}(KM,KN) \) is not the extension of an element in \( \text{Hom}_{RG}(M,N) \).

(c) Suppose that \( \text{Hom}_{RG}(M,N) \) is an \( R \)-lattice of rank one. Suppose further that there is an \( RG \)-isomorphism \( \phi : M \to N \). Then \( \text{Hom}_{RG}(M,N) = R(\phi) \).

\textbf{Proof}: Part (a) is clear.

As for (b), note that if \( \text{Hom}_{RG}(M,N) \) has \( R \)-rank 1, then \( \text{Hom}_{KG}(KM,KN) \) has \( K \)-dimension 1, and is spanned by \( r^{-1}\phi \), for every \( 0 \neq r \in R \). Thus, if \( (r^{-1}\phi)|_M \in \text{Hom}_{RG}(M,N) \), then \( \phi(x) = rs\phi(x) \), for some \( s \in R \) and all \( x \in M \). But this forces \( \phi = rs\phi \), \( rs = 1 \) and \( r \in R^\times \).

To prove (c), let \( \{\psi\} \) be an \( R \)-basis of \( \text{Hom}_{RG}(M,N) \), and let \( r \in R \) be such that \( \phi = r\psi \). Then \( rN \subseteq N = \phi(M) = (r\psi)(M) = r\psi(M) \subseteq rN \), which implies \( rN = N \). Hence \( r \in R^\times \) and \( \{\phi\} \) is also an \( R \)-basis of \( \text{Hom}_{RG}(M,N) \).

\( \square \)

\textbf{2.10 Proposition}. Let \( V_1 \) and \( V_2 \) be simple \( KG \)-modules with \( R \)-forms \( L_1 \) and \( L_2 \), respectively. If \( \phi : L_1 \to L_2 \) is a non-zero \( RG \)-homomorphism, then \( \phi \) is injective.

\textbf{Proof}: Assume that \( x \in L_1 \) is such that \( x \neq 0 \) and \( \phi(x) = 0 \), and view \( \phi \) as a \( KG \)-homomorphism \( KL_1 \to KL_2 \) as before. Since \( KL_1 \cong V_1 \) and \( KL_2 \cong V_2 \), Schur’s Lemma implies that \( \phi \) is either a \( KG \)-isomorphism or the zero map. Since \( 0 \neq x \in KL_1 \) and \( \phi(x) = 0 \), we must have \( \phi = 0 \), both as \( KG \)-homomorphism and as \( RG \)-homomorphism, a contradiction. Therefore, \( \phi \) is injective.

\( \square \)

In the next section we shall explain how to obtain \( \mathbb{Z} \)-forms of absolutely simple \( \mathbb{Q}G \)-modules from \( p \)-local data. There and in the following we shall make use of the following observation:

\textbf{2.11 Remark}. Let \( M \) be an \( RG \)-lattice of rank \( n \in \mathbb{N} \). Assume that also \( N \) is an \( RG \)-lattice and there is a \( KG \)-isomorphism \( \phi : KN \to KM \). Since both \( M \) and \( \phi(N) \) are full-rank \( RG \)-sublattices of \( KM \), there exists \( 0 \neq r \in R \) such that \( \phi(rN) = r\phi(N) \subseteq M \). Moreover, \( \phi(rN) \cong rN \cong N \) as \( RG \)-lattices. This shows that every \( RG \)-lattice \( N \) with \( KN \cong KM \) as \( KG \)-modules is isomorphic to an \( RG \)-sublattice of the fixed \( RG \)-lattice \( M \).

Conversely, if \( N \subseteq M \) is an \( RG \)-sublattice of rank \( n \), then \( KN \) is a \( KG \)-submodule of \( KM \) and both modules have \( K \)-dimension \( n \). Thus \( KM = KN \). Consequently, determining all isomorphism classes of \( RG \)-lattices \( N \) with \( KN \cong KM \) as \( KG \)-modules is equivalent to determining all isomorphism classes of \( RG \)-sublattices of \( M \) of full rank.

We conclude this section with the following consequence of a theorem of Brauer-Nesbitt.

\textbf{2.12 Proposition}. Suppose that \( R \) is a discrete valuation ring with maximal ideal \( m \) and residue field \( k := R/m \). Let \( V \) be any \( KG \)-module.

(a) If \( M \) and \( N \) are \( RG \)-forms of \( V \), then the \( kG \)-modules \( M/mM \) and \( N/mN \) have the same composition factors.

(b) Suppose that \( V \) is absolutely simple. Suppose further that, for an \( R \)-form \( M \) of \( V \), the \( kG \)-module \( M/mM \) is also absolutely simple. Then there is only one isomorphism class of \( R \)-forms of \( V \). If, moreover, \( V \) is self-dual, then so is every \( R \)-form of \( V \).
Let $G$ be a finite group, and suppose that $V$ is a simple $\mathbb{Q}G$-module. As mentioned in 2.2, $V$ always admits a $\mathbb{Z}$-form, which is in general far from being unique, not even up to isomorphism. However, the Theorem of Jordan–Zassenhaus ensures that there are only finitely many $\mathbb{Z}$-forms of $V$ up to $\mathbb{Z}G$-isomorphism. A similar result holds in the local setting: if $p$ is a prime number then every simple $\mathbb{Q}_pG$-module admits only finitely many $\mathbb{Z}_p$-forms up to $\mathbb{Z}_pG$-isomorphism. For both statements we refer to [8, Theorem (24.1), Theorem (24.7)], which holds in much greater generality. We content ourselves with the rational version, since this is the one we shall need in Section 8.

In the following we shall be concerned with the case that $V$ is absolutely simple, that is, $K \otimes_{\mathbb{Q}} V$ is a simple $KG$-module, for every extension field $K$ of $\mathbb{Q}$. We shall explain in more detail how $\mathbb{Z}$-forms of $V$ and $\mathbb{Z}_p$-forms of the $p$-adic completion $V_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} V$ are related. These methods will provide our strategy towards proving Corollary 1.2 and Theorem 1.3 in Section 8. Our main reference here is [8, §30, §31].

3.1. Localization and $p$-completion. (a) Let $M$ be a $\mathbb{Z}$-form of $V$, and let $p$ be a prime number. Then $M_{\mathbb{Z}(p)}$ is a $\mathbb{Z}_p$-form of $V$ as well, and $M_{\mathbb{Z}_p}$ is a $\mathbb{Z}_p$-form of $V_{\mathbb{Q}_p}$.

(b) Suppose further that $N \subseteq M$ is also a $\mathbb{Z}$-form of $V$, and let $\phi : M \to M$ be a $\mathbb{Z}$-endomorphism with $\phi(M) = N$. Since $\mathbb{Z}(p)$ and $\mathbb{Z}_p$ are both flat over $\mathbb{Z}$, we can regard $N_{\mathbb{Z}(p)}$ as a $\mathbb{Z}_pG$-sublattice of $M_{\mathbb{Z}(p)}$, and $N_{\mathbb{Z}_p}$ as a $\mathbb{Z}_pG$-sublattice of $M_{\mathbb{Z}_p}$. The $\mathbb{Z}$-endomorphism $\phi$ of $M$ yields a $\mathbb{Z}_p$-endomorphism of $M_{\mathbb{Z}(p)}$ with image $N_{\mathbb{Z}(p)}$, as well as a $\mathbb{Z}_p$-endomorphism of $M_{\mathbb{Z}_p}$ with image $N_{\mathbb{Z}_p}$. By [8, Corollary (4.18), Proposition (4.20a)], one has $[M : N] = \mathbb{Z} \cdot \det(\phi)$, $(M_{\mathbb{Z}(p)} : N_{\mathbb{Z}(p)}) = \mathbb{Z}_p \cdot \det(\phi)$ and $(M_{\mathbb{Z}_p} : N_{\mathbb{Z}_p}) = \mathbb{Z}_p \cdot \det(\phi)$.

In particular, if $\det(\phi) \in \mathbb{Z}_p^\times$, then one has $M_{\mathbb{Z}_p} = N_{\mathbb{Z}_p}$, by Proposition 2.5.

If $[M : N]$ is a $p$-power, then Proposition 2.5 implies $[M : N] = [M_{\mathbb{Z}(p)} : N_{\mathbb{Z}(p)}] = [M_{\mathbb{Z}_p} : N_{\mathbb{Z}_p}]$.

As a converse of (a), one has

3.2 Proposition. Let $V$ be an absolutely simple $\mathbb{Q}G$-module, let $p$ be a prime number, and let $N$ be a $\mathbb{Z}_p$-form of $V_{\mathbb{Q}_p}$. Then there is a $\mathbb{Z}$-form $M$ of $V$ and a $\mathbb{Z}(p)$-form $L$ of $V$ such that $M_{\mathbb{Z}(p)} \cong L$ and $M_{\mathbb{Z}_p} \cong L_{\mathbb{Z}_p} \cong N$.

Proof. The assertion follows from [8, Proposition (23.13), Corollary (30.10)].

3.3 Proposition. Let $V$ be an absolutely simple $\mathbb{Q}G$-module with $\mathbb{Z}$-forms $M$ and $N$. Let further $\{p_1, \ldots, p_g\}$ be the set of prime numbers dividing $|G|$. Then the following are equivalent:

(i) $M \cong N$ as $\mathbb{Z}G$-lattices;

(ii) for all prime numbers $p$, one has $M_{\mathbb{Z}(p)} \cong N_{\mathbb{Z}(p)}$ as $\mathbb{Z}(p)G$-lattices;
(iii) for all $i \in \{1, \ldots, g\}$, one has $M_{\mathbb{Z}p_i} \cong N_{\mathbb{Z}p_i}$ as $\mathbb{Z}_{p_i}$-lattices;
(iv) for all prime numbers $p$, one has $M_{\mathbb{Z}p} \cong N_{\mathbb{Z}p}$ as $\mathbb{Z}_p$-lattices;
(v) for all $i \in \{1, \ldots, g\}$, one has $M_{\mathbb{Z}p_i} \cong N_{\mathbb{Z}p_i}$ as $\mathbb{Z}_{p_i}$-lattices.

**Proof.** The equivalences (ii)$\iff$(iv) and (iii)$\iff$(v) follow from [8, Proposition (30.17)], while the equivalences (ii)$\iff$(iii) and (iv)$\iff$(v) follow from [8, Proposition (31.2)]. Lastly, By Maranda’s Theorem [7, Theorem (81.5)], also (i) and (ii) are equivalent. \hfill \Box

As a consequence of Proposition 3.3 and the considerations in [8, §31A., page 51], one obtains

**3.4 Corollary.** Let $V$ be an absolutely simple $\mathbb{Q}G$-module. For each prime number, let $h_p(V)$ be the number of isomorphism classes of $\mathbb{Z}_p$-forms of $V_{\mathbb{Q}_p}$, and let $j(V)$ be the number of isomorphism classes of $\mathbb{Z}$-forms of $V$. If \{1, $g\}$ is the set of prime divisors of $|G|$, then one has

$$j(V) = \prod_{i=1}^{g} h_{p_i}(V).$$

The following result shows how Corollary 3.4 can be strengthened to give explicit representatives for the isomorphism classes of $\mathbb{Z}$-forms of $V$. It is implicitly contained in [24, 23] (see also [15]). Since it will be essential in Section 8, for the readers’ convenience we give a proof.

**3.5 Proposition.** Let $V$ be an absolutely simple $\mathbb{Q}G$-module, and let $M$ be a $\mathbb{Z}$-form of $V$. Let \{1, $g\}$ be the set of prime divisors of $|G|$. Suppose that, for each $p \in \{1, \ldots, g\}$, one is given a set $X_p$ of $\mathbb{Z}$-forms of $V$ contained in $M$ such that, for all $N \in X_p$, the index $[M : N]$ is a $p$-power and \{1, $g\}$ is a set of representatives of the isomorphism classes of $\mathbb{Z}_p$-forms of $V_{\mathbb{Q}_p}$. Then

$$\{N_1 \cap \cdots \cap N_g : N_i \in X_{p_i}, 1 \leq i \leq g\}$$

is a set of representatives of the isomorphism classes of $\mathbb{Z}$-forms of $V$.

**Proof.** By Proposition 3.3, the $\mathbb{Z}$-forms $L$ and $N$ of $V$ are isomorphic if and only if the $\mathbb{Z}_p$-forms $L_{\mathbb{Z}p}$ and $N_{\mathbb{Z}p}$ of $V_{\mathbb{Q}_p}$ are isomorphic, for all $p \in \{1, \ldots, g\}$. Moreover, by Corollary 3.4, the number of isomorphism classes of $\mathbb{Z}$-forms of $V$ is equal to $\prod_{i=1}^{g} |X_{p_i}|$. Let us write $X := \{N_1 \cap \cdots \cap N_g : N_i \in X_{p_i}, 1 \leq i \leq g\}$. It suffices to show that $|X| = \prod_{i=1}^{g} |X_{p_i}|$ and that no two elements of $X$ are isomorphic. To this end, let $L = N_1 \cap \cdots \cap N_g \in X$ with $N_i \in X_{p_i}$ for $i \in \{1, \ldots, g\}$. Let $i \in \{1, \ldots, g\}$. By assumption, for $j \neq i$, we have that $(M_{\mathbb{Z}p_i} : (N_j)_{\mathbb{Z}p_i}) = \mathbb{Z}_{p_i}[M : N_j]$, where $[M : N_j]$ is a power of $p_j \in \mathbb{Z}_{p_i}$, hence $(M_{\mathbb{Z}p_i} : (N_j)_{\mathbb{Z}p_i}) = \mathbb{Z}_{p_i}$, and $(N_j)_{\mathbb{Z}p_i} = M_{\mathbb{Z}p_i}$. This yields

$$L_{\mathbb{Z}p_i} = (N_1 \cap \cdots \cap N_g)_{\mathbb{Z}p_i} = (N_i)_{\mathbb{Z}p_i} \cap M_{\mathbb{Z}p_i} = (N_i)_{\mathbb{Z}p_i}.$$

This shows that $X$ has the correct cardinality and the elements of $X$ are pairwise non-isomorphic. \hfill \Box

In general, given a principal ideal domain, its field of fractions $K$ and an absolutely simple $KG$-module $V$, it is quite difficult to determine all isomorphism classes of $R$-forms of $V$. We record here two special cases, where $R$ is local and where the reduction modulo the maximal
ideal of $R$ reveals enough information to determine all isomorphism classes. Both will be applied in the setting of Specht modules in Sections 6 and 7, where $G$ is a symmetric group and $R = \mathbb{Z}_p$, for some prime number $p$.

3.6 Proposition. Suppose that $R$ is local with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Let $V$ be a $kG$-module, and let $M$ be an $R$-form of $V$. Let $n := \text{rk}_R(M) \in \mathbb{N}$.

(a) Suppose that, for every $R\mathfrak{g}$-lattice $N$ with $\mathfrak{m}M \leq N \leq M$, the $kG$-module $N/\mathfrak{m}N$ is uniserial. Then the set of $R\mathfrak{g}$-sublattices of $M$ of rank $n$ is totally ordered. Moreover, every $R\mathfrak{g}$-sublattice of $M$ of rank $n$ is of the form $\mathfrak{m}^iN$, for some $i \in \mathbb{N}_0$ and some $R\mathfrak{g}$-sublattice $N$ of $M$ with $\mathfrak{m}M \leq N \leq M$.

(b) Suppose that the set of $R\mathfrak{g}$-sublattices of $M$ of rank $n$ is totally ordered. Then the set of $R\mathfrak{g}$-lattices $\{ N : \mathfrak{m}M \leq N \leq M \}$ contains a set of representatives of the isomorphism classes of $R$-forms of $V$.

Proof. First recall from Remark 2.11 that, up to isomorphism, the $R$-forms of $V$ are precisely the $R\mathfrak{g}$-sublattices of $M$ of full rank $n$. Lifting any composition series of the $kG$-module $M/\mathfrak{m}M$ to $RG$, we obtain an $r \in \mathbb{N}$ and $R\mathfrak{g}$-lattices $M_1, \ldots, M_r$ such that $\mathfrak{m}M = M_r \leq M_{r-1} \leq \cdots \leq M_2 \leq M_1 = M$ and such that $M_i$ is a maximal $R\mathfrak{g}$-sublattice of $M_{i-1}$, for $i \in \{2, \ldots, r\}$.

(a) Since, in particular, $M/\mathfrak{m}M$ is uniserial, we deduce that $M_1, \ldots, M_r$ are precisely the $R\mathfrak{g}$-sublattices of $M$ containing $\mathfrak{m}M$. We claim that

$$\{ \mathfrak{m}^jM_i : j \in \mathbb{N}_0, 1 \leq i \leq r \}$$

is the set of all $R\mathfrak{g}$-sublattices of $M$ of full rank $n$, which is clearly totally ordered. To this end, for every $R\mathfrak{g}$-sublattice $L$ of $M$ with $\text{rk}_R(L) = n$, we define $\ell(L)$ to be the minimal integer $s \in \mathbb{N}_0$ such that there exists a chain of $R\mathfrak{g}$-lattices $L = L_s \leq \cdots \leq L_0 = M$, where $L_i$ is a maximal $R\mathfrak{g}$-sublattice of $L_{i-1}$, for all $i \in \{1, \ldots, s\}$. We argue by induction on $\ell(L)$ to show that $L \in \{ \mathfrak{m}^jM_i : j \in \mathbb{N}_0, 1 \leq i \leq r \}$. The case $\ell(L) = 0$ is trivial, and if $\ell(L) = 1$ then $L = M_2$, since $M_2$ is the unique maximal $R\mathfrak{g}$-sublattice of $M$. So suppose now that $\ell(L) \geq 2$. Then there is some $R\mathfrak{g}$-sublattice $L'$ of $M$ of full rank such that $\ell(L') \leq \ell(L) - 1$ and such that $L$ is maximal in $L'$. By induction, $L' = \mathfrak{m}^jM_i$ for some $j \in \mathbb{N}_0$ and some $i \in \{1, \ldots, r\}$. We shall show that $L'$ has a unique maximal sublattice, namely $\mathfrak{m}^jM_{i+1}$ if $i < r$, and $\mathfrak{m}^j+1M_i$ if $i = r$.

Let $p \in R$ be such that $(p) = \mathfrak{m}$. Then we have the $RG$-isomorphism $M_i \to \mathfrak{m}^jM_i = L'$, $m \mapsto p^i/m$. Hence we may suppose that $j = 0$, so that $L' = M_i$. By assumption, $M_i/\mathfrak{m}M_i$ is a uniserial $kG$-module. Therefore, $M_i$ has a unique maximal $R\mathfrak{g}$-sublattice; by construction, the latter equals $M_{i+1}$ if $i < r$ and $\mathfrak{m}M = \mathfrak{m}M_1$ if $i = r$. This completes the proof of (a).

(b) Since $L \cong \mathfrak{m}^jL$, for every $R\mathfrak{g}$-lattice $L$ and every $j \in \mathbb{N}_0$, the hypotheses of (b) imply that $\{ \mathfrak{m}^jM_i : j \in \mathbb{N}_0, 1 \leq i \leq r \}$ is the set of $R\mathfrak{g}$-sublattices of $M$ of rank $n$, and the assertion follows.

3.7 Proposition. Suppose that $R$ is local with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Let $V$ be a $kG$-module, and let $M$ be an $R$-form of $V$. Suppose that $M/\mathfrak{m}M = D_1 \oplus D_2$, for absolutely simple $kG$-modules $D_1$ and $D_2$ with $D_1 \not\cong D_2$. Then the $R\mathfrak{g}$-lattice $M$ has precisely two maximal sublattices $N_1$ and $N_2$. If, moreover, $N_1/\mathfrak{m}N_1$ and $N_2/\mathfrak{m}N_2$ are both uniserial $kG$-modules, then $M, N_1$ and $N_2$ are representatives of the isomorphism classes of $R$-forms of $V$. 
Hence, by Corollary 2.8, the \(RG\) proposition now follows.

Now suppose that \(N_1/mN_1\) and \(N_2/mN_2\) are uniserial \(kG\)-modules. Then they must have Loewy series

\[N_1/mN_1 = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \quad \text{and} \quad N_2/mN_2 = \begin{bmatrix} D_2 \\ D_1 \end{bmatrix}.
\]

By Corollary 2.8, for \(i \in \{1, 2\}\), the \(RG\)-lattice \(N_i\) has a unique maximal sublattice, which must then be \(mM\). As in the proof of Proposition 3.6(a) we deduce that the \(RG\)-sublattices of \(M\) of rank \(rk_R(M)\) are precisely those in \(\{m^jM, m^jN_1, m^jN_2 : j \in \mathbb{N}_0\}\), which by Remark 2.11 contain representatives of the isomorphism classes of \(R\)-forms of \(V\). The \(RG\)-lattices \(N_1, N_2\) and \(M\) are pairwise non-isomorphic, since their reductions modulo \(m\) are pairwise non-isomorphic as \(kG\)-modules. Since \(mL \cong L\), for every \(RG\)-lattice \(L\) and every \(j \in \mathbb{N}\), the assertion of the proposition now follows.

\[\square\]

4 Exterior Powers

In this short section we collect a number of important properties of exterior powers of \(RG\)-lattices, most of which can be found in [26, Section 9.8]. They will be crucial for our study of Specht modules labelled by hook partitions in subsequent sections.

4.1 Exterior powers of lattices and modules. As before, let \(R\) be a principal ideal domain, let \(G\) be a finite group, and let \(M\) be an \(RG\)-lattice of rank \(n \in \mathbb{N}\).

(a) The exterior algebra \(\bigwedge(M)\) of \(M\) is a graded \(R\)-algebra, with multiplication denoted by

\[\bigwedge(M) \times \bigwedge(M) \to \bigwedge(M), \ (x, y) \mapsto x \wedge y.\]

For \(k \geq 0\), the \(k\)-th homogeneous component of \(\bigwedge(M)\) is denoted by \(\bigwedge^k(M)\), and is called the \(k\)-th exterior power of \(M\).

Suppose that \(\{b_1, \ldots, b_n\}\) is an \(R\)-basis of \(M\), and let \(k \in \mathbb{N}\). If \(k \leq n\), then \(\bigwedge^k(M)\) is an \(RG\)-lattice with \(R\)-basis \(\{b_1 \wedge \cdots \wedge b_i : 1 \leq i_1 < \cdots < i_k \leq n\}\); the \(RG\)-action on \(\bigwedge^k(M)\) is given by

\[g \cdot (m_1 \wedge \cdots \wedge m_k) = gm_1 \wedge \cdots \wedge gm_k,\]

for \(g \in G\) and \(m_1, \ldots, m_k \in M\). If \(k > n\) then \(\bigwedge^k(M) = \{0\}\). Thus, for every \(k \in \mathbb{N}_0\), the exterior power \(\bigwedge^k(M)\) is an \(RG\)-lattice of \(R\)-rank \(\binom{n}{k}\), where \(\binom{n}{k} := 0\), for \(k > n\).

We also recall that \(\wedge\) is alternating, that is, for \(k \in \{2, \ldots, n\}\), \(m_1, \ldots, m_k \in M\) and \(1 \leq i < j \leq k\), one has \(m_1 \wedge \cdots \wedge m_k = -(m_1 \wedge \cdots \wedge m_{i-1} \wedge m_j \wedge m_{i+1} \wedge \cdots \wedge m_{j-1} \wedge m_i \wedge m_{j+1} \wedge \cdots \wedge m_k)\).

(b) Let \(M\) and \(N\) be \(RG\)-lattices, and let \(\phi : M \to N\) be an \(R\)-linear map. As well, let \(k \in \mathbb{N}_0\). Then there is a unique \(R\)-linear map \(\bigwedge^k(\phi) : \bigwedge^k(M) \to \bigwedge^k(N)\) such that

\[\bigwedge^k(\phi)(m_1 \wedge \cdots \wedge m_k) = \phi(m_1) \wedge \cdots \wedge \phi(m_k),\]

for all \(m_1, \ldots, m_k \in M\). If \(M\) and \(N\) are also \(RG\)-lattices and \(\phi\) is an \(RG\)-homomorphism, then so is \(\bigwedge^k(\phi)\).

(c) Exterior powers commute with taking the dual, that is, if \(M\) is an \(RG\)-lattice an \(k \in \mathbb{N}_0\), then there is an \(RG\)-isomorphism \(\bigwedge^k(M^*) \cong (\bigwedge^k(M))^*\). For \(k > n\), both modules
are \{0\}. For \(k \leq n\), let \(\{b_1, \ldots, b_n\}\) be an \(R\)-basis of \(M\), and consider the \(R\)-linear map \(\bigwedge^k(M^*) \to (\bigwedge^k(M))^*\) mapping the basis element \(b_{i_1}^* \wedge \cdots \wedge b_{i_k}^*\) of \(\bigwedge^k(M^*)\) to the basis element \((b_{i_1} \wedge \cdots \wedge b_{i_k})^*\) of \((\bigwedge^k(M))^*\), for \(1 \leq i_1 < \cdots < i_k \leq n\). This map is obviously bijective, and it is routine to check that it is also an \(RG\)-homomorphism.

(d) Consider principal ideal domains \(S\) and \(R\) and a unitary ring homomorphism \(\rho : R \to S\). Let further \(M\) be an \(R\)-lattice with \(R\)-basis \(\{b_1, \ldots, n_n\}\). In analogy to 2.2(a), we obtain an isomorphism of \(S\)-lattices \(S \otimes_R \bigwedge^k(M) \to \bigwedge^k(S \otimes_R M)\) mapping the basis element \(1 \otimes (b_{i_1} \wedge \cdots \wedge b_{i_k})\) to the basis element \((1 \otimes b_{i_1}) \wedge \cdots \wedge (1 \otimes b_{i_k})\), for \(1 \leq i_1 < \cdots < i_k \leq n\). If \(M\) is an \(RG\)-lattice, then this yields an isomorphism of \(SG\)-lattices. As an immediate consequence, we mention the following:

4.2 Proposition. Let \(R\) be a principal ideal domain and \(K\) its field of fractions. Let further \(V\) be a \(KG\)-module, and let \(M\) be an \(R\)-form of \(V\). For every \(k \in \mathbb{N}\), the exterior power \(\bigwedge^k(M)\) is then an \(R\)-form of the \(KG\)-module \(\bigwedge^k(V)\).

4.3 Proposition. Let \(M\) and \(N\) be \(R\)-lattices, and let \(k \in \mathbb{N}\). Moreover, let \(\phi : M \to N\) be an \(R\)-linear map.

(a) If \(\phi\) is injective (respectively, surjective), then also the \(R\)-linear map \(\bigwedge^k(\phi) : \bigwedge^k(M) \to \bigwedge^k(N)\) is injective (respectively, surjective).

(b) Suppose that \(N = M\), and let \(rk_R(M) = n \in \mathbb{N}\). Then one has \(\det(\bigwedge^k(\phi)) = \det(\phi)^{(n-1)}\).

(c) Suppose that \(rk_R(M) = rk_R(N) = n \in \mathbb{N}\). Suppose further that \(\{v_1, \ldots, v_n\}\) is an \(R\)-basis of \(M\) and \(\{w_1, \ldots, w_n\}\) is an \(R\)-basis of \(N\) such that the matrix of \(\phi\) with respect to these bases is a diagonal matrix with diagonal entries \(d_1, \ldots, d_n \in R\). Then, with respect to the \(R\)-bases \(\{v_1 \wedge \cdots \wedge v_k : 1 \leq i_1 < \cdots < i_k \leq n\}\) and \(\{w_1 \wedge \cdots \wedge w_k : 1 \leq i_1 < \cdots < i_k \leq n\}\) of \(\bigwedge^k(M)\) and \(\bigwedge^k(N)\), respectively, the \(R\)-linear map \(\bigwedge^k(\phi)\) is represented by a diagonal matrix with diagonal entries \(d_{i_1} \cdots d_{i_k}\), for \(1 \leq i_1 < \ldots, i_k \leq n\).

Proof. Assertion (a) can be found in [4, III, §7, no. 8–9], assertion (b) can be found in [2, §38]. Assertion (c) is clear.

In consequence of Proposition 4.3(a), whenever we have \(RG\)-lattices \(M\) and \(N\) with \(N \subseteq M\), we may view \(\bigwedge^k(N)\) as an \(RG\)-sublattice of \(\bigwedge^k(M)\), for every \(k \in \mathbb{N}_0\). The next result will be important in Section 6.

4.4 Lemma. Suppose that all proper factor rings of \(R\) are finite. Let \(M\) and \(N\) be \(RG\)-lattices of rank \(n \in \mathbb{N}\) such that \(N \subseteq M\). For \(k \in \mathbb{N}\), one then has

\[
\left[\bigwedge^k(M) : \bigwedge^k(N)\right] = [M : N]^{(n-1)}.
\]

Proof. Let \(\phi\) be an \(R\)-endomorphism of \(M\) with \(\phi(M) = N\). Suppose that \(\{v_1, \ldots, v_n\}\) is an \(R\)-basis of \(M\). If for \(i \in \{1, \ldots, n\}\), we set \(w_i := \phi(v_i)\). Then \(\{w_1, \ldots, w_n\}\) is an \(R\)-basis of \(N\), since \(\phi(M) = N\). As \(\bigwedge^k(\phi)(v_{i_1} \wedge \cdots \wedge v_{i_k}) = w_{i_1} \wedge \cdots \wedge w_{i_k}\), for \(1 \leq i_1 < \cdots < i_k \leq n\), and since \(\{w_1 \wedge \cdots \wedge w_{i_1} : 1 \leq i_1 < \cdots < i_n \leq n\}\) is an \(R\)-basis of \(\bigwedge^k(N)\), we have that \(\bigwedge^k(\phi)\) is an \(R\)-endomorphism of \(\bigwedge^k(M)\) with \(\bigwedge^k(\phi)(\bigwedge^k(M)) = \bigwedge^k(N)\). The assertion of the lemma is now a consequence of Proposition 2.5 and Proposition 4.3(b).
The next theorem will be crucial in the proof of our main result in Section 6, but should also be of independent interest. We do not expect Theorem 4.5 to be new. However, we did not find an appropriate reference in the literature, and thus provide a proof here.

4.5 Theorem. Let $R$ be a principal ideal domain an $K$ its field of fractions. Moreover, let $V$ be an absolutely simple $KG$-module with $R$-forms $M$ and $N$. Suppose that $k \in \{1, \ldots, \dim_K(V) - 1\}$ is such that also $\bigwedge^k(V)$ is an absolutely simple $KG$-module. Then one has $M \cong N$ if and only if $\bigwedge^k(M) \cong \bigwedge^k(N)$, as $RG$-lattices.

Proof: By assumption, the $K$-vector spaces $\text{End}_{KG}(V)$ and $\text{End}_{KG}(\bigwedge^k(V))$ are of dimension one. Since $K$ is a torsion-free $R$-module, we further have $K$-vector space isomorphisms

$$K \otimes_R \text{Hom}_{RG}(M, N) \cong \text{Hom}_{KG}(K \otimes_R M, K \otimes_R N) \cong \text{End}_{KG}(V)$$

and

$$K \otimes_R \text{Hom}_{RG} \left( \bigwedge^k(M), \bigwedge^k(N) \right) \cong \text{End}_{KG} \left( \bigwedge^k(V) \right),$$

by [22, Theorem 1.11.12] and 4.1(c). Thus $\text{Hom}_{RG}(M, N)$ and $\text{Hom}_{RG}(\bigwedge^k(M), \bigwedge^k(N))$ are $R$-lattices of rank one each. Suppose that $\phi : \bigwedge^k(M) \to \bigwedge^k(N)$ is an $RG$-isomorphism. By Lemma 2.9, $\phi$ spans $\text{Hom}_{RG}(\bigwedge^k(M), \bigwedge^k(N))$ as an $R$-module.

Now let $\{\psi\}$ be an $R$-basis of $\text{Hom}_{RG}(M, N)$. We shall show that $\psi$ is an $RG$-isomorphism. In order to do so, it suffices to verify that $\psi$ is an $R$-isomorphism.

Since $\phi$ spans $\text{Hom}_{RG}(\bigwedge^k(M), \bigwedge^k(N))$, there is some $r \in R$ such that $\bigwedge^k(\psi) = r\phi$. By appealing to the Smith normal form, we may choose $R$-bases of $M$ and $N$ in such a way that the matrix $A \in \text{Mat}_{n \times n}(R)$ representing $\psi$ with respect to these bases is diagonal, say $A = \text{diag}(d_1, \ldots, d_n)$. Since $\psi$ also induces a non-zero $KG$-homomorphism $KM \to KN$ and $KM \cong V \cong KN$ is simple, $\psi$ actually induces a $KG$-isomorphism $KM \to KN$. Thus we must have $d_i \neq 0$, for all $i \in \{1, \ldots, n\}$.

By Proposition 4.3(c), the $R$-linear map $\bigwedge^k(\psi)$ can be represented by a diagonal matrix with diagonal entries $d_1 \cdots d_k$, for $1 \leq i_1 < \cdots < i_k \leq n$. Hence also $\phi$ can be represented by a diagonal matrix, say $D := \text{diag}(u_1, \ldots, u_s) \in \text{Mat}_{s \times s}(R)$, where $s := (\binom{n}{k})$. Since $\phi$ is an $RG$-isomorphism, we have $u_1 \cdots u_s = \det(D) \in R^\times$, thus $u_1, \ldots, u_s \in R^\times$. Since $r\phi = \bigwedge^k(\psi)$, we may suppose that $d_1 \cdots d_k = ru_1$. Moreover, for every $j \in \{k + 1, \ldots, n\}$ and every $i \in \{1, \ldots, k\}$, the element $d_1 \cdots d_{i-1}d_i d_{i+1} \cdots d_k$ is a diagonal entry of the fixed matrix representing $\bigwedge^k(\psi)$, so that $d_1 \cdots d_{i-1}d_i d_{i+1} \cdots d_k = ru_{s_{i,j}}$, for some $s_{i,j} \in \{1, \ldots, s\}$. This in turns implies

$$\frac{d_j}{d_i} = \frac{d_1 \cdots d_{i-1}d_i d_{i+1} \cdots d_k}{d_1 \cdots d_k} = \frac{ru_{s_{i,j}}}{ru_1} \in R^\times,$$

for $j \in \{k + 1, \ldots, n\}$ and $i \in \{1, \ldots, k\}$. This shows that any two entries of $A$ only differ by a unit in $R$. Thus $A/d_1 \in \text{Mat}_{n \times n}(R)$. Therefore, the $KG$-homomorphism $\psi/d_1 \in \text{Hom}_{KG}(KM, KN)$ is the extension of an element in $\text{Hom}_{RG}(M, N)$. By Lemma 2.9, this implies $d_1 \in R^\times$, and we conclude $\det(A) \in R^\times$, so that $\psi$ is indeed an $R$-isomorphism, thus an $RG$-isomorphism, as desired.

Conversely, if $M \cong N$, then clearly $\bigwedge^k(M) \cong \bigwedge^k(N)$, by Proposition 4.3(a). □
5 Specht lattices

Throughout this section, let \( n \in \mathbb{N} \). The symmetric group of degree \( n \) will be denoted by \( S_n \). For background on the representation theory of symmetric groups and the known results stated below, we refer the reader to [18, 27].

By \( \mathcal{P}(n) \) we denote the set of partitions of \( n \). If \( p \) is a prime number and \( \lambda \in \mathcal{P}(n) \) is such that each non-zero part of \( \lambda \) occurs at most \( p - 1 \) times, then \( \lambda \) is called \( p \)-regular.

5.1. Young tableaux and tabloids. (a) Given \( \lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(n) \), let \( t \) be a \( \lambda \)-tableau; recall that \( t \) is obtained by taking the Young diagram

\[ [\lambda] := \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq l, 1 \leq j \leq \lambda_i\} \]

and replacing each node bijectively by a number in \( \{1, \ldots, n\} \).

The conjugate partition \( \lambda' \) is the partition of \( n \) whose Young diagram is obtained by transposing \([\lambda]\). For every \( \lambda \)-tableau \( t \), we thus obtain a \( \lambda' \)-tableau \( t' \) by transposing \( t \).

A \( \lambda \)-tableau \( t \) is called standard if its entries strictly increase along every row from left to right, and down every column from top to bottom.

(b) Let \( \lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(n) \). The symmetric group \( S_n \) acts naturally on the set of \( \lambda \)-tableaux. For every \( \lambda \)-tableau \( t \), one denotes by \( R_t \) and \( C_t \) its row stabilizer and column stabilizer, respectively. That is, \( R_t \) contains the permutations in \( S_n \) fixing the rows of \( t \) setwise, and \( C_t \) contains the permutations fixing the columns of \( t \) setwise. For every \( \pi \in S_n \), one obviously has \( R_{\pi t} = \pi R_t \pi^{-1} \) as well as \( C_{\pi t} = \pi C_t \pi^{-1} \). Furthermore, \( C_t \cap R_t = \{1\} \).

For every \( \lambda \)-tableau \( t \), one defines

\[ \rho_t := \sum_{\sigma \in R_t} \sigma \in ZS_n \quad \text{and} \quad \kappa_t := \sum_{\sigma \in C_t} \text{sgn}(\sigma)\sigma \in ZS_n. \]

(c) The set \( \{\sigma \cdot t : \sigma \in R_t\} \) is called the \( \lambda \)-tabloid corresponding to \( t \), and will be denoted by \( \{t\} \). If \( \{t\} \) contains a standard \( \lambda \)-tableau, then \( \{t\} \) is called a standard \( \lambda \)-tabloid. The set of \( \lambda \)-tabloids will be denoted by \( T(\lambda) \).

5.2. Young permutation modules and Specht modules. Let \( \lambda \in \mathcal{P}(n) \).

(a) The \( S_n \)-action on the set of \( \lambda \)-tableaux induces a transitive \( S_n \)-action on the set \( T(\lambda) \) of \( \lambda \)-tabloids. The resulting permutation \( ZS_n \)-lattice will be denoted by \( M^\lambda_Z \), and is called a Young permutation \( ZS_n \)-lattice. We shall call \( T(\lambda) \) the standard basis of \( M^\lambda_Z \).

For every \( \lambda \)-tableau \( t \), one defines the corresponding \( \lambda \)-polytabloid

\[ e_t := \kappa_t \cdot \{t\} = \sum_{\sigma \in C_t} \text{sgn}(\sigma)\{\sigma t\} \in M^\lambda_Z. \]

If \( t \) is a standard \( \lambda \)-tableau, then \( e_t \) is called a standard \( \lambda \)-polytabloid. The definition of \( e_t \) depends on the tableau \( t \). However, if \( \pi \in S_n \), then one has \( \pi \cdot e_t = e_{\pi t} \). This shows that if \( s \) is also a \( \lambda \)-tableau, then the cyclic \( ZS_n \)-sublattices of \( M^\lambda_Z \) generated by \( e_t \) and \( e_s \), respectively, coincide; one calls \( ZS_n \langle e_t \rangle \subseteq M^\lambda_Z \) the Specht \( ZS_n \)-lattice corresponding to \( \lambda \) and denotes it by \( S^\lambda_Z \). In consequence of [18, Section 8], the standard \( \lambda \)-polytabloids form a \( Z \)-basis of \( S^\lambda_Z \).

(b) Let \( R \) be any principal ideal domain, which is naturally a \((Z, Z)\)-bimodule. The \( R S_n \)-lattices \( R \otimes_Z M^\lambda_Z \) and \( R \otimes_Z S^\lambda_Z \) will be denoted by \( M^\lambda_R \) and \( S^\lambda_R \), respectively. Let \( \iota : S^\lambda_Z \rightarrow M^\lambda_Z \) be the inclusion map. Then the \( R S_n \)-homomorphism \( \text{id}_R \otimes \iota : S^\lambda_R \rightarrow M^\lambda_R \) is always injective,
independently of \( R \). To see this, write \( e_t \), for a standard \( \lambda \)-tableau \( t \), as a \( \mathbb{Z} \)-linear combination of tabloids. Then \( \{ t \} \) occurs with coefficient 1, and every other tableau occurring is strictly smaller than \( \{ t \} \) in the total order on tabloids defined in [18, Definition 3.10]; see [18, Lemma 8.3]. Moreover, each tableau occurring in \( e_t \) has coefficient 1 or \(-1\). For \( \lambda \in \mathcal{P}(n) \), one calls \( S^\lambda_R \) the Specht \( R\mathfrak{S}_n \)-lattice labelled by \( \lambda \).

(c) By \( \text{sgn} \) we denote the \( \mathbb{Z}\mathfrak{S}_n \)-lattice of rank one on which \( \sigma \in \mathfrak{S}_n \) acts by multiplication with \( \text{sgn}(\sigma) \). Then, for every \( \lambda \in \mathcal{P}(n) \), there is an isomorphism of \( \mathbb{Z}\mathfrak{S}_n \)-lattices \( S^\lambda_Z \otimes \text{sgn} \cong (S^\lambda_Z)^* \), which by extension of scalars induces an isomorphism \( S^\lambda_R \otimes \text{sgn}_R \cong (S^\lambda_R)^* \) for any principal ideal domain \( R \); for a proof, see Proposition A.9.

5.3 Remark. In consequence of 5.1(b), we may view the Specht \( R\mathfrak{S}_n \)-lattice \( S^\lambda_R \) as an \( R\mathfrak{S}_n \)-sublattice of \( M^\lambda_R \) and the standard of \( \lambda \)-polytabloids as an \( R \)-basis of \( S^\lambda_R \), for every \( \lambda \in \mathcal{P}(n) \).

As well, \( S^\lambda_R = R\mathfrak{S}_n(e_t) \), for every \( \lambda \)-polytabloid \( t \).

In later sections it will be important to investigate the structure of Specht lattices over various coefficient rings and fields. To this end, we mention the following well-known properties of Specht modules over fields.

5.4 Theorem. Let \( F \) be a field.

(a) If \( \text{char}(F) = 0 \) then the Specht \( F\mathfrak{S}_n \)-modules \( S^\lambda_F \), for \( \lambda \in \mathcal{P}(n) \), yield a set of representatives of the isomorphism classes of absolutely simple \( F\mathfrak{S}_n \)-modules. Moreover, every Specht \( F\mathfrak{S}_n \)-module is self-dual.

(b) If \( \text{char}(F) = p > 0 \) and if \( \lambda \in \mathcal{P}(n) \) is \( p \)-regular, then the Specht module \( S^\lambda_F \) has a unique simple quotient module \( D^\lambda_F \). If \( \lambda \) varies over the set of \( p \)-regular partitions of \( n \), then \( D^\lambda_F \) varies over a set of representatives of the isomorphism classes of absolutely simple \( F\mathfrak{S}_n \)-modules. Moreover, every simple \( F\mathfrak{S}_n \)-module is self-dual.

(c) If \( \text{char}(F) \geq 3 \), then the Specht \( F\mathfrak{S}_n \)-modules \( S^\lambda_F \), for \( \lambda \in \mathcal{P}(n) \), are pairwise non-isomorphic and absolutely indecomposable. Moreover, for \( \lambda \in \mathfrak{S}_n \), one has \( \text{End}_{F\mathfrak{S}_n}(S^\lambda_F) \cong F \).

(d) If \( \text{char}(F) = 2 \) and if \( \lambda \in \mathcal{P}(n) \) is \( 2 \)-regular, then one has \( \text{End}_{F\mathfrak{S}_n}(S^\lambda_F) \cong F \); in particular, \( S^\lambda_F \) is then absolutely indecomposable.

Proof. Assertions (a) and (b) can be found in [18, Theorem 4.12, Theorem 11.5]. Assertions (c) and (d) follow from [18, Corollary 13.17].

It should be emphasized that Specht modules over fields of characteristic 2 are, in general, not indecomposable; the first examples of decomposable Specht modules can be found in work of G. Murphy [21]. This lack of knowledge also causes problems when trying to determine the isomorphism classes of \( \mathbb{Z} \)-forms of Specht \( \mathcal{Q}\mathfrak{S}_n \)-modules. We shall come back to this at the end of Section 7.

Next we shall focus on Specht lattices labelled by hook partitions, that is, partitions of the form \( (n-k, 1^k) \), for \( k \in \{0, \ldots, n-1\} \). As before, for every \( \lambda \in \mathcal{P}(n) \) and every principal ideal domain \( R \), we consider the Young permutation \( R\mathfrak{S}_n \)-lattice \( M^\lambda_R \) with its standard \( R \)-basis \( \mathcal{T}(\lambda) \).

5.5. Hook partitions and exterior powers. Let \( n \geq 2 \).

(a) Consider the partition \( (n-1, 1) \) of \( n \). Each \( (n-1, 1) \)-tableau \( \{ t \} \) is uniquely determined by the entry in the second row of any \( (n-1, 1) \)-tableau in \( \{ t \} \). In this way, one identifies the
set $T((n - 1, 1))$ with the set $\{1, \ldots, n\}$, and the $\mathfrak{S}_n$-action on $T((n - 1, 1))$ corresponds just to the natural $\mathfrak{S}_n$-action on $\{1, \ldots, n\}$.

Similarly, every $(n - 1, 1)$-polytabloid is uniquely determined by the first column of the underlying tableau. If $i, j \in \{1, \ldots, n\}$ are such that $i \neq j$ and if $t$ is any $(n - 1, 1)$-tableau with first column entries $i$ and $j$, then we shall denote the corresponding polytabloid by $e(i, j)$. Note that $e(i, j) = j - i \in M^{(n-1,1)}_R$. Further note that $e(1, 2), \ldots, e(1, n)$ are precisely the standard $(n - 1, 1)$-polytabloids.

One calls $M^{(n-1,1)}_R$ the natural permutation $R\mathfrak{S}_n$-lattice, and $S^{(n-1,1)}_R$ the natural Specht $R\mathfrak{S}_n$-lattice.

(b) Set $M := M^{(n-1,1)}_R$ and $S := S^{(n-1,1)}_R$. Let $k \in \{1, \ldots, n\}$. We have the $R\mathfrak{S}_n$-lattice $\bigwedge^k(M)$ with $R$-basis $\{i_1 \land \cdots \land i_k : 1 \leq i_1 < \cdots < i_k \leq n\}$. By Proposition 4.3, we can also regard $\bigwedge^k(S)$ as an $R\mathfrak{S}_n$-sublattice of $\bigwedge^k(M)$, and $\bigwedge^k(S)$ has $R$-basis $\{e(1, i_1) \land \cdots \land e(1, i_k) : 2 \leq i_1 < \cdots < i_k \leq n\}$. For $2 \leq i_1 < \cdots < i_k \leq n$, we set

$$b(i_1, \ldots, i_k) := e(1, i_1) \land \cdots \land e(1, i_k) = (i_1 - 1) \land \cdots \land (i_k - 1).$$

Now let $k \in \{0, \ldots, n-1\}$, and consider the hook partition $(n - k, 1^k)$ of $n$. As in the case $k = 1$, every $(n - k, 1^k)$-polytabloid is determined by the first-column entries of the underlying tableau. If $i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}$ are pairwise distinct, then we denote the $(n - k, 1^k)$-polytabloid corresponding to the $(n - k, 1^k)$-tableau with first-column entries $i_1, \ldots, i_{k+1}$ by $e(i_1, \ldots, i_{k+1})$. With this convention, the standard $(n - k, 1^k)$-polytabloids are precisely those of the form $e(1, i_1, \ldots, i_k)$, for $2 \leq i_1 < \cdots < i_k \leq n$. Moreover, the following $R$-linear map defines an $R\mathfrak{S}_n$-isomorphism:

$$\bigwedge^k(S^{(n-1,1)}_R) \rightarrow S^{(n-k,1^k)}_R, \quad e(1, i_1) \land \cdots \land e(1, i_k) \mapsto e(1, i_1, \ldots, i_k),$$

where $2 \leq i_1 < \cdots < i_k \leq n$; for a proof see [20, Proposition 2.3], which is stated under the assumption that $R$ is a field, but works for every principal ideal domain. In particular, $S^{(n-k,1^k)}_R$ has R-rank $(\binom{n-1}{k})$, for $k \in \{0, \ldots, n - 1\}$.

It has proved to be very useful to identify the Specht lattice $S^{(n-k,1^k)}_R$ with the exterior power $\bigwedge^k(S^{(n-1,1)}_R)$ via the isomorphism (2), and we shall exploit this repeatedly in the following.

5.6 Remark. It is clear from the definition, that $S^{(n)}_R$ is isomorphic to the trivial $R\mathfrak{S}_n$-lattice, and $S^{(1^n)}_R$ is isomorphic to the sign $R\mathfrak{S}_n$-lattice.

Below we summarize some known results concerning the structure of hook Specht modules over fields of characteristic $p \geq 3$.

5.7 Theorem ([18, Theorem 23.7, Theorem 24.1]). Let $F$ be a field of characteristic $p \geq 3$.

(a) Suppose that $p \nmid n$. Then the Specht modules $S^{(n-k,1^k)}_F$, for $k \in \{0, \ldots, n - 1\}$, are pairwise non-isomorphic absolutely simple $F\mathfrak{S}_n$-modules.

(b) Suppose that $p \mid n$. For $k \in \{1, \ldots, n - 2\}$, the Specht $F\mathfrak{S}_n$-module $S^{(n-k,1^k)}_F$ has a unique simple submodule $D(k)_F$, and the quotient module $\overline{D}(k)_F := S^{(n-k,1^k)}_F/D(k)_F$ is also simple and not isomorphic to $D(k)_F$. Moreover, one has $D(1)_F \cong F$, $\overline{D}(k-1)_F \cong D(k)_F$ for $k \in \{2, \ldots, n - 2\}$, and $\overline{D}(n-2)_F \cong S^{(1^n)}_F := D(n-1)_F$. 17
As an immediate consequence, we also mention the following, which will be needed in Section 6.

5.8 Corollary. Let $F$ be a field of characteristic $p \geq 3$. For $k \in \{1, \ldots, n - 2\}$, the simple $F\mathfrak{S}_n$-module $D(k)$ in Theorem 5.7(b) has dimension $\binom{n-2}{k-1}$.

6 The case $p \geq 3$

In this section, let $p$ be an odd prime. We shall determine a set of representatives of the isomorphism classes of $\mathbb{Z}_p$-forms of the Specht $\mathbb{Q}_p\mathfrak{S}_n$-modules $S^{(n-k,1^k)}_p$, for $k \in \{1, \ldots, n - 2\}$.

For ease of notation, for $k \in \{0, \ldots, n - 1\}$, we set $S(k)_R := S^{(n-k,1^k)}_R$, for every principal ideal domain $R$ under consideration. Moreover, we identify $S(k)_R$ with the exterior power $\wedge^k(S(1)_R)$ via the $R\mathfrak{S}_n$-isomorphism (2) in 5.5.

The aim of this section is to prove Theorem 6.1 below. Together with the results of Craig [6] and Plesken [24, Theorem 5.1] concerning the isomorphism classes of $\mathbb{Z}$-forms of $S(1)_{\mathbb{Q}}$ and $S(n-2)_{\mathbb{Q}}$, this will enable us to prove Theorem 1.1(a) in Section 8.

6.1 Theorem. Let $n \in \mathbb{N}$ with $n \geq 3$, and let $k \in \{1, \ldots, n - 2\}$. Moreover let $p \geq 3$ be a prime number. If $L_1, \ldots, L_s$ are representatives of the $\mathbb{Z}_p\mathfrak{S}_n$-isomorphism classes of $\mathbb{Z}_p$-forms of $S(1)_{\mathbb{Q}_p}$, then $\wedge^k(L_1), \ldots, \wedge^k(L_s)$ are representatives of the $\mathbb{Z}_p\mathfrak{S}_n$-isomorphism classes of $\mathbb{Z}_p$-forms of $S(k)_{\mathbb{Q}_p}$; in particular, the number of isomorphism classes of $\mathbb{Z}_p$-forms of $S(1)_{\mathbb{Q}_p}$ and the number of isomorphism classes of $\mathbb{Z}_p$-forms of $S(k)_{\mathbb{Q}_p}$ coincide.

As the case $p \mid n$ will be dealt with by Proposition 2.12 and Theorem 5.4(a), we shall from now on suppose that $p \nmid n$, for the remainder of this section.

6.2 Remark. Let $k \in \{1, \ldots, n - 2\}$. Recall from Theorem 5.7 that the $\mathbb{F}_p\mathfrak{S}_n$-module $S(k)_{\mathbb{F}_p} \cong S(k)_{\mathbb{Z}_p}/pS(k)_{\mathbb{Z}_p}$ is uniserial with Loewy series

$$S(k)_{\mathbb{F}_p} \simeq \begin{bmatrix} D(k + 1) \\ D(k) \end{bmatrix},$$

where $D(k) \neq D(k + 1)$. Thus, for every $\mathbb{Z}_p$-form $L$ of the $\mathbb{Q}_p\mathfrak{S}_n$-module $S(k)_{\mathbb{Q}_p}$, we know that the $\mathbb{F}_p\mathfrak{S}_n$-module $L/pL$ has composition factors $D(k)$ and $D(k + 1)$ as well; in particular, there are the following three possibilities for the Loewy series of $L/pL$:

$$L/pL \sim \begin{bmatrix} D(k + 1) \\ D(k) \end{bmatrix} \text{ or } L/pL \sim \begin{bmatrix} D(k) \\ D(k + 1) \end{bmatrix} \text{ or } L/pL \cong D(k) \oplus D(k + 1);$$

in particular, by Proposition 2.7, $L$ has either one or two $\mathbb{Z}_p\mathfrak{S}_n$-sublattices $N$ such that $pL \subseteq N \subseteq L$, and these are precisely the maximal $\mathbb{Z}_p\mathfrak{S}_n$-sublattices of $L$.

6.3 Lemma. Let $k \in \{1, \ldots, n - 2\}$, and let $p \mid n$. Let further $L$ be a $\mathbb{Z}_p$-form of $S(k)_{\mathbb{Q}_p}$. Then one has

- (a) $L/pL \sim \begin{bmatrix} D(k + 1) \\ D(k) \end{bmatrix}$ if and only if $L \cong S(k)_{\mathbb{Z}_p}$;
- (b) $L/pL \sim \begin{bmatrix} D(k) \\ D(k + 1) \end{bmatrix}$ if and only if $L \cong S(k)_{\mathbb{Z}_p}^*$.
Thus, by (a), we have
\[ L \]
Then the dual lattice
\[ \mathbb{F}_p \mathcal{S}_n \text{-module} \text{ is self-dual, } S(k)_{Z_p}^* / pS(k)_{Z_p}^* \cong S(k)_{Z_p}^* \text{ is uniserial with Loewy series } \left[ \begin{array}{c} D(k + 1) \\ D(k) \end{array} \right] \text{ and since, by Theorem 5.4, every simple } \mathbb{F}_p \mathcal{G}_n \text{-module is self-dual, } S(k)_{Z_p}^* / pS(k)_{Z_p}^* \cong S(k)_{Z_p}^* \text{ is uniserial with Loewy series } \left[ \begin{array}{c} D(k) \\ D(k + 1) \end{array} \right] .

Conversely, let L be a \( Z_p \)-form of \( S(k)_{Q_p} \) with the same Loewy structure as \( M := S(k)_{Z_p} \). Denote by \( \sim : \mathbb{Z}_p \to \mathbb{F}_p \) the canonical projection. By Lemma 2.9, \( \text{Hom}_{\mathbb{Z}_p \mathcal{G}_n}(L, M) \cong \mathbb{Z}_p \) as \( \mathbb{Z}_p \)-lattice. Thus, by [22, Lemma 4.8.8], we have an injective \( \mathbb{F}_p \)-linear map
\[ \Phi : \mathbb{F}_p \cong \mathbb{F}_p \otimes_{\mathbb{Z}_p} \text{Hom}_{\mathbb{Z}_p \mathcal{G}_n}(L, M) \to \text{Hom}_{\mathbb{F}_p \mathcal{G}_n}(L / pL, M / pM); \tag{3} \]
in particular, there is some non-zero homomorphism \( \phi \in \text{Hom}_{\mathbb{F}_p \mathcal{G}_n}(L / pL, M / pM) \). If \( \phi \) was not an isomorphism, we would have \( \text{Im}(\phi) \not\cong D(k) / \ker(\phi) \cong D(k + 1) \), a contradiction. Hence \( \phi \) is an isomorphism, implying \( \text{End}_{\mathbb{F}_p \mathcal{G}_n}(L / pL, M / pM) \cong \text{End}_{\mathbb{Z}_p \mathcal{G}_n}(S(k)_{F_p}) \) as \( \mathbb{F}_p \)-vector spaces. By Theorem 5.4, we know that \( \text{End}_{\mathbb{Z}_p \mathcal{G}_n}(S(k)_{F_p}) \cong \mathbb{F}_p \), so that the \( \mathbb{F}_p \)-monomorphism in (3) actually has to be an isomorphism. Hence, there is some \( \psi \in \text{Hom}_{\mathbb{Z}_p \mathcal{G}_n}(L, M) \) such that \( \Phi(1 \otimes \psi) = \phi \). Let \( \{b_1, \ldots, b_s\} \) be a \( Z_p \)-basis of \( L \) and let \( \{c_1, \ldots, c_s\} \) be a \( Z_p \)-basis of \( M \). Let further \( A \in \text{Mat}_{s \times s}(\mathbb{Z}_p) \) be the matrix of \( \psi \) with respect to these bases. We identify \( L / pL \) with \( \mathbb{F}_p \otimes_{\mathbb{Z}_p} L \) and \( M / pM \) with \( \mathbb{F}_p \otimes_{\mathbb{Z}_p} M \) as in 2.2(f). Then \( \{1 \otimes b_1, \ldots, 1 \otimes b_s\} \) and \( \{1 \otimes c_1, \ldots, 1 \otimes c_s\} \) are \( \mathbb{F}_p \)-bases of \( \mathbb{F}_p \otimes_{\mathbb{Z}_p} L \) and \( \mathbb{F}_p \otimes_{\mathbb{Z}_p} M \), respectively. The matrix \( A \) of \( \Phi(1 \otimes \psi) = \phi \) with respect of these bases is obtained by reducing the entries of \( A \) modulo \( p \). In particular, \( 0 \neq \det(A) = \det(\tilde{A}) \). Hence \( \det(A) \in \mathbb{Z}_p^{*} \), implying that \( \psi \) is a \( \mathbb{Z}_p \mathcal{G}_n \)-isomorphism.

To prove assertion (b), suppose that \( L \) is a \( Z_p \)-form of \( S(k)_{Q_p} \) such that \( L / pL \sim \left[ \begin{array}{c} D(k) \\ D(k + 1) \end{array} \right] \). Then the dual lattice \( L^* \) is also a \( Z_p \)-form of \( S(k)_{Q_p}^* \), and the \( \mathbb{F}_p \mathcal{G}_n \)-module \( L^* / pL^* \cong \mathbb{F}_p \otimes_{\mathbb{Z}_p} L^* \cong (\mathbb{F}_p \otimes_{\mathbb{Z}_p} L)^* \cong (L / pL)^* \) is uniserial with Loewy series \( \left[ \begin{array}{c} D(k + 1) \\ D(k) \end{array} \right] \). Thus, by (a), we have \( L^* \cong S(k)_{Z_p}^* \), hence \( L \cong S(k)_{Z_p}^* \), as claimed.

\[ \left( \right. \]

6.4 Proposition. Let \( k \in \{1, \ldots, n - 2\} \), and let \( p \mid n \). Let \( N \) be a \( Z_p \)-form of \( S(1)_{Q_p} \). Then \( \wedge^k(N) \) is a \( Z_p \)-form of \( S(k)_{Q_p} \). Moreover
\[ \left( \right. \]
(a) the following are equivalent
\[ \left( \right. \]
(i) \( N / pN \sim \left[ \begin{array}{c} D(2) \\ D(1) \end{array} \right] ; \]
(ii) \( N \cong S(1)_{Z_p} ; \]
(iii) \( \wedge^k(N) \cong S(k)_{Z_p} ; \]
(iv) \( \wedge^k(N) / p \wedge^k(N) \sim \left[ \begin{array}{c} D(k + 1) \\ D(k) \end{array} \right] ; \]
(b) the following are equivalent
\[ \left( \right. \]
(i) \( N / pN \sim \left[ \begin{array}{c} D(1) \\ D(2) \end{array} \right] ; \]
(ii) \( N \cong S(1)_{Z_p} ; \]
(iii) \( \wedge^k(N) \cong S(k)_{Z_p} ; \]
Lemma 6.3. This implies

\[
\Lambda^k(N)/p \Lambda^k(N) \sim \begin{bmatrix} D(k) \\ D(k+1) \end{bmatrix}.
\]

(c) one has

\[
N/pN \cong D(1) \oplus D(2) \iff \bigwedge^k(N)/p \bigwedge^k(N) \cong D(k) \oplus D(k+1).
\]

**Proof.** Let \( N \) be a \( \mathbb{Z}_p \)-form of \( S(1)_{\mathbb{Q}_p} \). By Proposition 4.2, we know that \( L := \Lambda^k(N) \) is a \( \mathbb{Z}_p \)-form of \( \Lambda^k(S(1)_{\mathbb{Q}_p}) \cong S(k)_{\mathbb{Q}_p} \).

To prove (a), suppose that \( N/pN \) has Loewy series \( \begin{bmatrix} D(2) \\ D(1) \end{bmatrix} \). Then \( N \cong S(1)_{\mathbb{Z}_p} \), by Lemma 6.3. This implies \( \Lambda^k(N) \cong S(k)_{\mathbb{Z}_p} \), by 5.5(b), and then \( \Lambda^k(N)/p \Lambda^k(N) \) has Loewy series \( \begin{bmatrix} D(k+1) \\ D(k) \end{bmatrix} \), by Theorem 5.7. Conversely, if \( \Lambda^k(N)/p \Lambda^k(N) \) has Loewy series \( \begin{bmatrix} D(k+1) \\ D(k) \end{bmatrix} \), then \( \Lambda^k(N) \cong S(k)_{\mathbb{Z}_p} \), by Lemma 6.3 again. So \( \Lambda^k(N) \cong \Lambda^k(S(1)_{\mathbb{Z}_p}) \), which forces \( N \cong S(1)_{\mathbb{Z}_p} \), by Theorem 4.5.

Analogously one obtains (b).

Lastly suppose that \( N/pN \) is semisimple, that is, \( N/pN \cong D(1) \oplus D(2) \). Then \( S(1)_{\mathbb{Z}_p} \neq N \cong S(1)_{\mathbb{Z}_p} \), by Lemma 6.3. Hence \( S(k)_{\mathbb{Z}_p} \cong \Lambda^k(S(1)_{\mathbb{Z}_p}) \neq \Lambda^k(N) \neq \Lambda^k(S(1)_{\mathbb{Z}_p}) \cong \Lambda^k(S(1)_{\mathbb{Z}_p}) \cong \Lambda^k(S(1)_{\mathbb{Z}_p}) \cong \Lambda^k(S(1)_{\mathbb{Z}_p}) \cong \Lambda^k(S(1)_{\mathbb{Z}_p}) \). Therefore, Lemma 6.3 and Remark 6.2 show that \( \Lambda^k(N)/p \Lambda^k(N) \cong D(k) \oplus D(k+1) \). Conversely, if \( \Lambda^k(N)/p \Lambda^k(N) \cong D(k) \oplus D(k+1) \) then we get \( S(k)_{\mathbb{Z}_p} \neq \Lambda^k(N) \neq S(k)_{\mathbb{Z}_p} \), thus \( S(1)_{\mathbb{Z}_p} \neq N \neq S(1)_{\mathbb{Z}_p} \), by Theorem 4.5, and then \( N/pN \cong D(1) \oplus D(2) \), by Lemma 6.3.

**6.5 Notation.** Suppose that \( G \) is any finite group and \( M \) is a \( \mathbb{Z}_p \)-sublattice of \( M \). Suppose further that \( N \) is a \( \mathbb{Z}_p \)-sublattice of \( M \) such that \( N \subseteq p^i M \), for some \( i \geq 1 \). Then \( \{p^{-i} x : x \in N\} \subseteq \mathbb{Q}_p M \) is also a \( \mathbb{Z}_p \)-sublattice of \( M \) isomorphic to \( N \). We shall use the notation \( N/p^i := \{p^{-i} x : x \in N\} \). Note that \( N/p^i \cong p^i(N/p^i) = N \) as \( \mathbb{Z}_p \)-sublattices.

In the proof of the next proposition we shall have to assume \( k < n - 3 \). However, this is not really a restriction, since the case \( k = n - 2 \) has been dealt with anyway by Plesken and Craig, as we shall see in the proof of Theorem 6.1.

**6.6 Proposition.** Let \( p \mid n \), and let \( k \in \{1, \ldots, n-3\} \). Let \( M \) be a \( \mathbb{Z}_p \)-form of \( S(1)_{\mathbb{Q}_p} \).

(a) If \( M \) has only one maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattice \( N \) and if \( N \) has index \( p \) in \( M \), then \( \Lambda^k(N) \) is the only maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattice of \( \Lambda^k(M) \).

(b) If \( M \) has only one maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattice \( N \) and if \( N \) has index \( p \) in \( M \), then \( \Lambda^k(N)/p^{k-1} \) is the only maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattice of \( \Lambda^k(M) \).

(c) If \( M \) has two maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattices \( N_1 \) and \( N_2 \) of index \( p \) and \( p^{n-2} \), respectively, then \( \Lambda^k(N_1) \) and \( \Lambda^k(N_2)/p^{k-1} \) are the maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattices of \( \Lambda^k(M) \). Moreover \( \Lambda^k(N_1) \neq \Lambda^k(N_2)/p^{k-1} \).

In particular, every maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattice of \( \Lambda^k(M) \) is isomorphic to \( \Lambda^k(N) \) for some maximal \( \mathbb{Z}_p \mathfrak{S}_n \)-sublattice \( N \) of \( M \).

**Proof.** (a) Suppose that \( M \) has a unique maximal sublattice \( N \). Then, by Proposition 2.7, the \( \mathbb{F}_p \mathfrak{S}_n \)-module \( N/pN \) has a simple head. If \( N \) has index \( p \) in \( M \) then \( \dim_{\mathbb{F}_p}(\Hd(N)) = 1 \).
by Proposition 2.7. Thus Proposition 6.4 implies $M \cong S(1)_{\mathbb{Z}_p}$ and $\wedge^k(M) \cong S(k)_{\mathbb{Z}_p}$. So by Theorem 5.7 and Proposition 2.7 again, $\wedge^k(M)$ also has a unique maximal sublattice, which must have index $p^{\dim_{\mathbb{F}_p}(D(k))} = p^{\binom{n-2}{k-2}}$; in particular, there is a unique sublattice of $\wedge^k(M)$ of index $p^{\binom{n-2}{k-2}}$. Moreover, by Lemma 4.4, $\wedge^k(N)$ is a $\mathbb{Z}_p\mathfrak{S}_n$-sublattice of $\wedge^k(M)$ of index $p^{\binom{n-2}{k-2}}$ as well. This proves (a).

(b) In analogy to (a) we deduce that in this case we must have $M \cong S(1)^*_{\mathbb{Z}_p}$, $\wedge^k(M) \cong S(k)^*_{\mathbb{Z}_p}$, and $\wedge^k(M)$ has a unique maximal sublattice $\tilde{N}$. Moreover $\tilde{N}$ has index $p^{\dim_{\mathbb{F}_p}(D(k+1))} = p^{\binom{n-2}{k-2}}$.

We now want to show that $p^k \wedge^k(M) \subseteq \wedge^k(N) \subseteq p^{k-1} \wedge^k(M)$. As $M/N$ is an $\mathbb{F}_p$-vector space of dimension $n-2$, by Lemma 2.6, we can find $\mathbb{Z}_p$-bases $\{v_1, \ldots, v_{n-1}\}$ and $\{w_1, \ldots, w_{n-1}\}$ of $M$ and $N$, respectively, such that $w_1 = v_1$ and $w_i = pv_i$, for $2 \leq i \leq n-1$. Thus, for $1 \leq i_1 < \cdots < i_k \leq n-1$, we have

$$p^k(v_{i_1} \wedge \cdots \wedge v_{i_k}) = \begin{cases} p^k(v_{i_1} \wedge \cdots \wedge v_{i_k}), & \text{if } i_1 \neq 1 \\ p(v_{i_1} \wedge \cdots \wedge v_{i_k}), & \text{otherwise}, \end{cases}$$

and

$$w_{i_1} \wedge \cdots \wedge w_{i_k} = \begin{cases} p^{k-1}(v_{i_1} \wedge \cdots \wedge v_{i_k}), & \text{if } i_1 = 1 \\ p^k(v_{i_1} \wedge \cdots \wedge v_{i_k}), & \text{otherwise}; \end{cases}$$

in particular, $p^k(v_{i_1} \wedge \cdots \wedge v_{i_k}) \in \wedge^k(N)$ and $w_{i_1} \wedge \cdots \wedge w_{i_k} \in p^{k-1} \wedge^k(M)$. Hence we obtain $p^k \wedge^k(M) \subseteq \wedge^k(N) \subseteq p^{k-1} \wedge^k(M)$, that is,

$$p^k \wedge^k(M) \subseteq \frac{\wedge^k(N)}{p^{k-1}} \subseteq \wedge^k(M).$$

By Lemma 4.4, the index of $\frac{\wedge^k(N)}{p^{k-1}}$ in $\wedge^k(M)$ is

$$\left[\frac{\wedge^k(M)}{\wedge^k(N)}\right]_{\left(p^{k-1}\right)^{\left(\binom{n-2}{k-2}\right)}} = p^{(n-2)\binom{n-2}{k-2} - (k-1)\binom{n-1}{k}}.$$ 

A quick calculation reveals that

$$(n-2)\binom{n-2}{k-1} - (k-1)\binom{n-1}{k} = \binom{n-2}{k},$$

and we conclude that $\tilde{N}$ and $\wedge^k(N)/p^{k-1}$ have the same index in $\wedge^k(M)$. Since $\tilde{N}$ is unique with this index, we have $\tilde{N} = \wedge^k(N)/p^{k-1}$.

(c) Using Proposition 2.7 and Lemma 6.3 we see that $\wedge^k(M)$ has exactly two maximal $\mathbb{Z}_p\mathfrak{S}_n$-sublattices $\tilde{N}_1$ and $\tilde{N}_2$. If $\tilde{N}_1$ and $\tilde{N}_2$ have different index in $\wedge^k(M)$, then we may argue as in the proof of (a) and (b) above to show that $\{\wedge^k(N_1), \wedge^k(N_2)/p^{k-1}\} = \{\tilde{N}_1, \tilde{N}_2\}$. If $\tilde{N}_1$ and $\tilde{N}_2$ have the same index in $\wedge^k(M)$, that is, if $\binom{n-2}{k-1} = \binom{n-2}{k}$, then we only get $\{\wedge^k(N_1), \wedge^k(N_2)/p^{k-1}\} \subseteq \{\tilde{N}_1, \tilde{N}_2\}$. To get equality, it suffices to show that $\wedge^k(N_1) \neq \wedge^k(N_2)/p^{k-1}$.

We first apply Lemma 2.6(a) to get $\mathbb{Z}_p$-bases $\{v_1, \ldots, v_{n-1}\}$ and $\{w_1, \ldots, w_{n-1}\}$ of $M$ and $N_2$, respectively, as in the proof of (b). Since $N_1$ has index $p$ in $M$, we can then apply
Lemma 2.6(b) to find a $\mathbb{Z}_p$-basis $\{u_1, \ldots, u_{n-1}\}$ of $N_1$ such that $u_j = pv_j$ for some $1 \leq j \leq n-1$, and $u_i = u_1 + \lambda_i v_j$ for $i \neq j$ and suitable $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{Z}_p$. Now, since $k < n - 2$, we can choose $1 \leq i_1 < \cdots < i_k \leq n - 1$ with $1, j \notin \{i_1, \ldots, i_k\}$ and consider the element

$$\alpha = u_{i_1} \wedge \cdots \wedge u_{i_k} = (v_{i_1} + \lambda_{i_1} v_j) \wedge \cdots \wedge (v_{i_k} + \lambda_{i_k} v_j) = (v_{i_1} \wedge \cdots \wedge v_{i_k}) + w \in \bigwedge^k(N_1),$$

where $w$ does not involve a basis element of the form $v_1 \wedge \cdots \wedge v_k$. To show that $\alpha \notin \bigwedge^k(N_2)/p^{k-1}$ let us assume the contrary. Then, for $1 \leq j_1 < \cdots < j_k \leq n - 1$, there exists $\lambda_{j_1, \ldots, j_k} \in \mathbb{Z}_p$ such that we can write

$$\alpha = (v_{i_1} \wedge \cdots \wedge v_{i_k}) + w = \sum_{1 < j_1 < \cdots < j_k \leq n} \frac{\lambda_{j_1, \ldots, j_k}}{p^{k-1}} \cdot w_{j_1} \wedge \cdots \wedge w_{j_k}.$$

Comparing the coefficients at $v_{i_1} \wedge \cdots \wedge v_{i_k}$ we see that $1 = \lambda_{j_1, \ldots, j_k} \cdot p$, which is not possible. This completes the proof of (c).

\[\square\]

We are now in the position to prove Theorem 6.1.

**Proof of Theorem 6.1.** Let first $p \nmid n$. Then, for $k \in \{2, \ldots, n - 2\}$, by Proposition 2.12 and Theorem 5.7(a), both $S(1)_{\mathbb{Q}_p}$ and $S(k)_{\mathbb{Q}_p}$ have only one $\mathbb{Z}_p$-form up to isomorphism. Thus the assertion follows in this case.

Now let $p \mid n$ and $k < n - 2$. Let further $L_1, \ldots, L_s$ be representatives of the isomorphism classes of $\mathbb{Z}_p$-forms of $S(1)_{\mathbb{Q}_p}$. The exterior powers $\bigwedge^k(L_1), \ldots, \bigwedge^k(L_s)$ are $\mathbb{Z}_p$-forms of $\bigwedge^k(S(1)_{\mathbb{Q}_p}) \cong S(k)_{\mathbb{Q}_p}$, by Proposition 4.2. By Theorem 4.5 and Theorem 5.4(a), $\bigwedge^k(L_1), \ldots, \bigwedge^k(L_s)$ are pairwise non-isomorphic.

Conversely, let $L$ be any $\mathbb{Z}_p$-form of $S(k)_{\mathbb{Q}_p}$, and let $M$ be any $\mathbb{Z}_p$-form of $S(1)_{\mathbb{Q}_p}$. Then, by Remark 2.11, $L$ is isomorphic to a $\mathbb{Z}_p\mathbb{S}_n$-sublattice of the $\mathbb{Z}_p$-form $\bigwedge^k(M)$ of $S(k)_{\mathbb{Q}_p}$, and we may thus suppose that $L \subseteq \bigwedge^k(M)$. Then there exists an $l \in \mathbb{N}_0$ and $\mathbb{Z}_p\mathbb{S}_n$-sublattices $N_1, \ldots, N_l$ of $\bigwedge^k(M)$ such that

$$L = N_l \subseteq N_{l-1} \subseteq \cdots \subseteq N_0 = \bigwedge^k(M)$$

and such that $N_{i+1}$ is maximal in $N_i$, for $i \in \{0, \ldots, l-1\}$. We need to show that $L \cong \bigwedge^k(L_j)$, for some $j \in \{1, \ldots, s\}$. To do so we argue by induction in $l$. If $l = 0$ then the assertion is trivial. So let $l > 0$. By induction, for every $i \in \{0, \ldots, l-1\}$, there is some $j_i \in \{1, \ldots, s\}$ such that $N_i \cong \bigwedge^k(L_{j_i})$. We may suppose that $N_{l-1} \cong \bigwedge^k(L_{j_{l-1}})$. Thus, by Proposition 6.6, there is some $j_l \in \{1, \ldots, s\}$ such that $L = N_l \cong \bigwedge^k(L_{j_l})$ or $L = N_l \cong \bigwedge^k(L_{j_l})/p^{k-1}$. Since $L_{j_l} \cong \bigwedge^k(L_{j_l})/p^{k-1}$, the assertion of the theorem follows, for $1 \leq k \leq n - 3$.

It remains to deal with the case $k = n - 2$. But, by Theorem 5.4(a) and Proposition A.9 (see also [18, Theorem 6.7]), we have $S(n-2)_{\mathbb{Q}_p} \cong S(1)_{\mathbb{Q}_p} \otimes \text{sgn}_{\mathbb{Q}_p}$. From this we deduce that $L_1 \otimes \text{sgn}_{\mathbb{Q}_p}, \ldots, L_s \otimes \text{sgn}_{\mathbb{Q}_p}$ must be representatives of the isomorphism classes of $\mathbb{Z}_p$-forms of $S(n-2)_{\mathbb{Q}_p}$. On the other hand, by Theorem 4.5 and Proposition 4.2 again, $\bigwedge^k(L_1), \ldots, \bigwedge^k(L_s)$ are pairwise non-isomorphic $\mathbb{Z}_p$-forms of $S(n-2)_{\mathbb{Q}_p}$, hence have to be representatives of the isomorphism classes of such forms as well. This completes the proof of the theorem. \[\square\]
7 The case $p = 2$

In this section we now turn to the case where $p = 2$ and $n \geq 4$. We shall determine a set of representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of the Specht $\mathbb{Q}_2 \mathfrak{S}_n$-module $S_{\mathbb{Q}_2}^{(n-2,1^2)}$, in the case where $n \not\equiv 0 \pmod{4}$, thereby proving Theorem 1.1(b). We retain the notation introduced in 5.5. For ease of notation, we shall use the following convention throughout this section: we set $S_R := S_R^{(n-2,1^2)}$, for every principal ideal domain $R$ under consideration. We shall also regard $S_R$ as an $R\mathfrak{S}_n$-sublattice of $\bigwedge^2(M^{(n-1,1)})$ as in 5.5(b), and work with our standard $R$-basis 

\[ b(i_1, i_2) = (i_1 - 1) \wedge (i_2 - 1) = (i_1 \wedge i_2) + (1 \wedge i_1) - (1 \wedge i_2), \]

for $2 \leq i_1 < i_2 \leq n$.

7.1 Remark. In the proof of the next proposition, we shall use a result of Müller and Orlob on Young modules in [19]. Suppose that $\lambda \in \mathcal{P}(n)$, and let $F$ be any field. In any given indecomposable direct sum decomposition of the $F\mathfrak{S}_n$-module $M_\lambda^F$, there is a unique indecomposable direct summand containing $S_\lambda^F$ as a submodule. This $F\mathfrak{S}_n$-module is unique up to isomorphism and is called the Young $F\mathfrak{S}_n$-module labelled by $\lambda$; see [9, 14]. It is usually denoted by $Y_\lambda^F$.

7.2 Proposition. Let $F$ be a field of characteristic 2, and let $n \geq 4$.

(a) If $n \equiv 1 \pmod{4}$, then $S_F$ is uniserial with Loewy series

\[ \begin{bmatrix} F \\ D_F^{(n-2,1^2)} \\ F \end{bmatrix}. \]

(b) If $n \equiv 3 \pmod{4}$, then $S_F \cong F \oplus D_F^{(n-2,1^2)}$.

(c) If $n \equiv 2 \pmod{4}$, then $S_F$ is uniserial with Loewy series

\[ \begin{bmatrix} F \\ D_F^{(n-2,1^2)} \\ F \\ D_F^{(n-1,1)} \end{bmatrix}. \]

(d) If $n \equiv 0 \pmod{4}$, then $S_F$ is indecomposable with Loewy series

\[ \begin{bmatrix} F \oplus D_F^{(n-2,1^2)} \\ D_F^{(n-1,1)} \end{bmatrix}. \]

Proof. The Specht module $S_F$ is isomorphic to a submodule of the Young module $Y_F^{(n-2,1^2)}$. In [19, Theorem 1.1] the complete submodule lattice of $Y_F^{(n-2,1^2)}$ has been determined. From this one obtains that $Y_F^{(n-2,1^2)}$ has a unique submodule of dimension $\dim_F(S_F) = \binom{n-1}{2}$, and this module has the claimed Loewy series. \hfill \square
We now start to construct representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of $S_{\mathbb{Z}_2}$ along the lines of Proposition 2.7, analyzing the 2-modular reduction of $S_{\mathbb{Z}_2}$. We shall deal with the case where $n$ is odd first, and mention the following useful observation beforehand.

7.3 Lemma. Suppose that $F$ is a field of characteristic 2, and let $k \in \{1, \ldots, n\}$. Then $\bigwedge^k(M_F^{(n-1,1)})$ is a transitive permutation $FG_{\mathcal{S}_n}$-module and is isomorphic to the induced module $\text{Ind}^G_{\mathcal{S}_k \times \mathcal{S}_{n-k}}(F)$. Furthermore,

$$T_k := \{i_1 \wedge \cdots \wedge i_k : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a transitive permutation basis of $\bigwedge^k(M_F^{(n-1,1)})$.

Proof: Since $\text{char}(F) = 2$, it is clear that the $F$-basis $T_k$ of $\bigwedge^k(M_F^{(n-1,1)})$ is a transitive $\mathcal{S}_n$-set. Moreover, the stabilizer of $1 \wedge \cdots \wedge k$ in $\mathcal{S}_n$ is obviously $\mathcal{S}_k \times \mathcal{S}_{n-k}$, whence the claim. \hfill \Box

7.4 Lemma. Let $n \geq 4$, let $F$ be a field of characteristic 2, and consider the following $F$-subspaces of $S_F$:

$$U_1 := \left\{ \sum_{2 \leq i_1 < i_2 \leq n} \beta(i_1, i_2)b(i_1, i_2) : \beta(i_1, i_2) \in F, \sum_{2 \leq i_1 < i_2 \leq n} \beta(i_1, i_2) = 0 \right\},$$
$$U_2 := \left\{ \beta \sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) : \beta \in F \right\}.$$

(a) The $F$-vector space $U_1$ is an $FG_{\mathcal{S}_n}$-submodule of $S_F$ of codimension 1.

(b) If $n$ is odd, then $U_2$ is the unique trivial $FG_{\mathcal{S}_n}$-submodule of $S_F$.

(c) If $n \equiv 1 \pmod{4}$, then $\{0\} \subseteq U_2 \subseteq U_1 \subseteq S_F$ is the unique composition series of $S_F$; in particular, $S_F/U_1 \cong F \cong U_2$ and $U_1/U_2 \cong D_F^{(n-2,2)}$.

(d) If $n \equiv 3 \pmod{4}$, then $S_F = U_1 \oplus U_2$; in particular, $U_1 \cong D_F^{(n-2,2)}$.

Proof: Set $M := \bigwedge^2(M_F^{(n-1,1)})$. By Lemma 7.3, $M$ is a permutation $FG_{\mathcal{S}_n}$-module with transitive permutation basis $T_2$. Hence

$$V_1 := \left\{ \sum_{1 \leq i_1 < i_2 \leq n} \gamma(i_1, i_2)i_1 \wedge i_2 : \sum_{1 \leq i_1 < i_2 \leq n} \gamma(i_1, i_2) = 0 \right\}$$

is the unique $FG_{\mathcal{S}_n}$-submodule of $M$ with trivial factor module. Moreover,

$$V_2 := \left\{ \gamma \sum_{1 \leq i_1 < i_2 \leq n} i_1 \wedge i_2 : \gamma \in F \right\}$$

is the unique trivial submodule of $M$.

(a) It is clear that $U_1$ has codimension 1 in $S$, since it is the kernel of the $F$-epimorphism $S \rightarrow F$, $\sum_{2 \leq i_1 < i_2 \leq n} \beta(i_1, i_2)b(i_1, i_2) \mapsto \sum_{2 \leq i_1 < i_2 \leq n} \beta(i_1, i_2)$. 

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We show that $U_1 = V_1 \cap S$, so that $U_1$ is an $F \mathfrak{S}_n$-submodule of $S$.

So let $x := \sum_{2 \leq i_1 < i_2 \leq n} \beta(i_1, i_2)b(i_1, i_2) \in U_1$. We write $x$ as an $F$-linear combination of the permutation basis $T_2$ of $M$. For $i \in \{2, \ldots, n\}$, we set

$$\beta(1, i) := \sum_{j=i+1}^{n} \beta(i, j) - \sum_{j=2}^{i-1} \beta(j, i) \in F.$$ 

Observe that, for $2 \leq i_1 < i_2 \leq n$, we see $\beta(i_1, i_2)$ as a summand of $\beta(1, i_1)$, and $-\beta(i_1, i_2)$ as a summand of $\beta(1, i_2)$. Hence we obtain $x = \sum_{1 \leq j_1 < j_2 \leq n} \beta(j_1, j_2)j_1 \wedge j_2$, and

$$\sum_{1 \leq j_1 < j_2 \leq n} \beta(j_1, j_2) = \sum_{2 \leq i_1 < i_2 \leq n} \beta(1, i_2) + \sum_{i=2}^{n} \beta(1, i) = 0 + 0 = 0,$$

so that $x \in V_1$.

This proves $U_1 \subseteq V_1 \cap S$. Since, for instance, $b(n-1, n) \notin V_1$, we have $S \nsubseteq V_1$. Thus, since $V_1$ has codimension 1 in $M$, also $V_1 \cap S$ has codimension 1 in $S$. Hence $\dim_F(U_1) = \dim_F(V_1 \cap S)$, and $U_1 = V_1 \cap S$.

(b) Let $n$ be odd. It suffices to show that we have $U_2 = V_2$, which holds, since

$$\sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) = \sum_{2 \leq i_1 < i_2 \leq n} (i_1 \wedge i_2 + 1 \wedge i_1 + 1 \wedge i_2)$$

$$= \sum_{2 \leq i_1 < i_2 \leq n} i_1 \wedge i_2 + (n-2) \sum_{i=2}^{n} 1 \wedge i = \sum_{1 \leq j_1 < j_2 \leq n} j_1 \wedge j_2.$$

(c) If $n \equiv 1 \pmod{4}$, then \binom{n-1}{2} is even, so that $U_2 \subseteq U_1$. We thus get a strictly ascending chain of $F \mathfrak{S}_n$-submodules $\{0\} \subseteq U_2 \subseteq U_1 \subseteq S$, and the assertion follows from Proposition 7.2(a).

(d) Lastly, if $n \equiv 3 \pmod{4}$, then \binom{n-1}{2} is odd, so that $U_2 \nsubseteq U_1$. Hence, $U_1 + U_2 = U_1 \oplus U_2 = S$, comparing dimensions. By (b), we have $U_2 \cong F$, thus $U_1 \cong D_F^{(n-2), 2}$, by Proposition 7.2(b).

7.5 Lemma. Let $n \geq 4$, and let $S := S_{\mathfrak{Z}_2}$. Let further

$$S_1 := \left\{ \sum_{2 \leq i_1 < i_2 \leq n} \alpha(i_1, i_2)b(i_1, i_2) : \alpha(i_1, i_2) \in \mathbb{Z}_2, \sum_{2 \leq i_1 < i_2 \leq n} \alpha(i_1, i_2) \equiv 0 \pmod{2} \right\} \subseteq S.$$ 

Then

(a) $S_1$ is a maximal $\mathbb{Z}_2 \mathfrak{S}_n$-sublattice of $S$; in particular, $2S \subseteq S_1$. Moreover, $S_1/2S \cong U_1$ as $\mathbb{F}_2 \mathfrak{S}_n$-module;

(b) a $\mathbb{Z}_2$-basis of $S_1$ is given by

$$\{ b(i_1, i_2) + b(n-1, n), 2b(n-1, n) : 2 \leq i_1 < i_2 \leq n, (i_1, i_2) \neq (n-1, n) \}.$$ 

(4)

Proof. We have the canonical projection

$$\pi : S \to S/2S \cong S_{\mathbb{F}_2},$$

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which is a $\mathbb{Z}_2\mathcal{G}_n$-epimorphism. By Lemma 7.4, we have the $\mathbb{F}_2\mathcal{G}_n$-submodule $U_1$ of $S_{\mathbb{F}_2}$, and $S_{\mathbb{F}_2}/U_1 \cong \mathbb{F}_2$. Obviously, $S_1$ is the preimage of the $U_1$ under $\pi$. Consequently, by Lemma 2.3, $S_1$ is a maximal $\mathbb{Z}_2\mathcal{G}_n$-submodule of $S$.

It is clear that the elements in (4) are linearly independent over $\mathbb{Z}_2$ and contained in $S_1$. If $x = \sum_{2 \leq i_1 < i_2 \leq n} \beta(i_1, i_2)b(i_1, i_2) \in S_1$, then we have $\sum_{2 \leq i_1 < i_2 \leq n} \beta(i_1, i_2) = 2\alpha$, for some $\alpha \in \mathbb{Z}_2$. Setting $y := \sum_{(j_1, j_2) \neq (n-1, n)} \beta(j_1, j_2)(b(j_1, j_2) + b(n-1, n))$ we deduce that $x - y = 2\alpha b(n-1, n)$, hence $x$ is a $\mathbb{Z}_2$-linear combination of the elements in (4).

7.6 Lemma. Let $n \geq 4$ be odd. Let further

$$S_2 := \left\{ \sum_{2 \leq i_1 < i_2 \leq n} \alpha(i_1, i_2)b(i_1, i_2) : \alpha(i_1, i_2) \equiv \alpha(n-1, n) \pmod{2}, \text{ for all } i_1, i_2 \right\} \subseteq S.$$ Then

(a) $S_2$ is a $\mathbb{Z}_2\mathcal{G}_n$-submodule of $S$ such that $2S \subseteq S_2$. Moreover, $S_2/2S \cong U_2$ as $\mathbb{F}_2\mathcal{G}_n$-module;

(b) one has

$$S_2 = \left\{ \alpha \sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) : \alpha \in \mathbb{Z}_2 \right\} + 2S;$$

(c) a $\mathbb{Z}_2$-basis of $S_2$ is given by

$$\left\{ \sum_{2 \leq j_1 < j_2 \leq n} b(j_1, j_2), 2b(i_1, i_2) : 2 \leq i_1 < i_2 \leq n, (i_1, i_2) \neq (n-1, n) \right\}. \quad (5)$$

**Proof.** Let $n$ be odd. It is easily verified that $S_2 = \{\alpha \sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) : \alpha \in \mathbb{Z}_2\} + 2S$. By Lemma 7.4(b), we have the unique trivial submodule $U_2$ of $S_{\mathbb{F}_2}$. Moreover, $S_2$ is the preimage of $U_2$ under the canonical $\mathbb{Z}_2\mathcal{G}_n$-epimorphism

$$\pi : S \to S/2S \cong S_{\mathbb{F}_2}.$$ So $S_2$ is a $\mathbb{Z}_2\mathcal{G}_n$-submodule of $S$.

It is clear that the elements in (5) are linearly independent over $\mathbb{Z}_2$ and contained in $S_2$. Since $\{2b(i_1, i_2) : 2 \leq i_1 < i_2 \leq n\}$ is a $\mathbb{Z}_2$-basis of $2S$, it suffices to show that $2b(n-1, n)$ is a $\mathbb{Z}_2$-linear combination of the elements in (5). But this is clear, since $2b(n-1, n) = 2\sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) - \sum_{(j_1, j_2) \neq (n-1, n)} 2b(j_1, j_2)$. This completes the proof of the lemma. \[\square\]

7.7 Lemma. Let $n \geq 5$ be odd, and let $S_1$ and $S_2$ be the $\mathbb{Z}_2\mathcal{G}_n$-submodules of $S$ defined in Lemma 7.5 and Lemma 7.6, respectively. Then the $\mathbb{F}_2\mathcal{G}_n$-module $S_1/2S_1$ does not have a trivial factor module, and the $\mathbb{F}_2\mathcal{G}_n$-module $S_2/2S_2$ does not have a trivial submodule.

**Proof.** Assume first that $S_2/2S_2$ has a trivial submodule. Then there is some $x \in S_2 \setminus 2S_2$ such that $x - \sigma \cdot x \in 2S_2$, for all $\sigma \in \mathcal{G}_n$. We show that this forces $x \in 2S$. To this end, we write $x = \sum_{2 \leq i_1 < i_2 \leq n} \alpha(i_1, i_2)b(i_1, i_2)$, for uniquely determined $\alpha(i_1, i_2) \in \mathbb{Z}_2$. By Lemma 7.6, we have $\alpha(i_1, i_2) \equiv \alpha(n-1, n) \pmod{2}$, for all $2 \leq i_1 < i_2 \leq n$. Thus, if $\alpha(n-1, n) \equiv 0$
(mod 2), then \( x \in 2S \). So assume that \( \alpha(i_1, i_2) \in \mathbb{Z}_2^\times \), for all \( 2 \leq i_1 < i_2 \leq n \). Then we may further assume that \( \alpha(n-1, n) = 1 \), and get

\[
x - (n-1, n)x = 2b(n-1, n) + \sum_{i=2}^{n-2} (\alpha(i, n-1) - \alpha(i, n))b(i, n-1) + \sum_{i=2}^{n-2} (\alpha(i, n) - \alpha(i, n-1))b(i, n);
\]

in particular, the coefficient of \( x - (n-1, n)x \) at the basis element \( b(n-3, n-2) \) of \( S \) is 0, the coefficient at \( b(n-1, n) \) is 2. But, since \( x - (n-1, n)x \in 2S_2 \), Lemma 7.6 would then imply \( 2 \equiv 0 \pmod{4} \), a contradiction. Hence \( x \in 2S \), so that \( x + 2S_2 \) also spans a trivial submodule of the \( \mathbb{F}_2 \mathfrak{S}_n \)-module \( 2S/2S_2 \cong S/S_2 \).

We now distinguish two cases: if \( n \equiv 1 \pmod{4} \), then, by Lemma 7.4 and Proposition 7.2, we have \( 2S \subseteq S_2 \subseteq S_1 \subseteq S \) and \( S/S_2 \cong (S/2S)/(S_2/2S) \cong S_{F_2}/U_2 \) is uniserial with simple socle \( D^{(n-2,2)}_{F_2} \); in particular, \( S/S_2 \) does not have a trivial submodule, a contradiction.

If \( n \equiv 3 \pmod{4} \), then, by Lemma 7.4, we have

\[
S/S_2 \cong S_{F_2}/U_2 \cong U_1 \cong D^{(n-2,2)}_{F_2} \not\cong \mathbb{F}_2,
\]

and we obtain again a contradiction.

It remains to show that \( S_1/2S_1 \) does not have a trivial factor module. Assume it does, so that there is an \( \mathbb{F}_2 \mathfrak{S}_n \)-epimorphism

\[
\varphi : S_1/2S_1 \to \mathbb{F}_2.
\]

Thus \( \varphi(\sigma x + 2S_1) = \sigma \varphi(x + 2S_1) = \varphi(x + 2S_1) \), for all \( x \in S_1 \) and all \( \sigma \in \mathfrak{S}_n \). In the following, let

\[
- : S_1 \to S_1/2S_1
\]

be the canonical epimorphism, and recall the \( \mathbb{Z}_2 \)-basis of \( S_1 \) from (4). If \( \varphi(b(i_1, i_2) + b(n-1, n)) = 0 \), for all \( 2 \leq i_1 < i_2 \leq n \) with \( (i_1, i_2) \neq (n-1, n) \), then also

\[
0 = (n-1, n)\varphi(b(n-3, n-2) + b(n-1, n)) = \varphi(b(n-2, n-3) - b(n-1, n)),
\]

and then \( 0 = \varphi(2b(n-1, n)) \). But this would imply \( \varphi = 0 \), a contradiction. Thus, we have some \( 2 \leq i_1 < i_2 \leq n \) with \( (i_1, i_2) \neq (n-1, n) \) such that \( \varphi(b(i_1, i_2) + b(n-1, n)) = 1 \). We distinguish three cases:

**Case 1:** \( \{i_1, i_2\} \cap \{n-1, n\} = \emptyset \). Then we have \( 1 = (i_1, i_2)\varphi(b(i_1, i_2) + b(n-1, n)) = \varphi(-b(i_1, i_2) + b(n-1, n)) \), hence \( 0 = \varphi(2b(n-1, n)) \). Moreover,

\[
1 = (i_2, n-1)\varphi(b(i_1, i_2) + b(n-1, n)) = \varphi(b(i_1, n-1) + b(i_2, n)) \]

\[
= (i_2, n)\varphi(b(i_1, n-1) + b(i_2, n)) = \varphi(b(i_1, n-1) - b(i_2, n))
\]

\[
= \varphi(b(i_1, n-1) + b(n-1, n)) - \varphi(b(i_2, n) + b(n-1, n)).
\]

But then

\[
\varphi(b(i_1, n-1) + b(n-1, n)) = (i_1, i_2)(n-1, n)\varphi(b(i_1, n-1) + b(n-1, n))
\]

\[
= \varphi(b(i_2, n) - b(n-1, n))
\]

\[
= \varphi(b(i_2, n) + b(n-1, n)) - \varphi(2b(n-1, n))
\]

\[
= \varphi(b(i_2, n) + b(n-1, n)),
\]

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which yields the contradiction $1 = 0$.

Case 2: $i_2 = n$, $i_1 < n - 1$. Then
\[
\varphi(b(i_1, n - 1) - b(n - 1, n)) = (n - 1, n)\varphi(b(i_1, n) + b(n - 1, n)) = 1 \\
= (i_1, n)\varphi(b(i_1, n) + b(n - 1, n)) = \varphi(-b(i_1, n) - b(i_1, n - 1)) \\
= \varphi(b(i_1, n) + b(i_1, n - 1)),
\]
and hence
\[
0 = \varphi(b(i_1, n) + b(i_1, n - 1)) - \varphi(b(i_1, n - 1) - b(n - 1, n)) \\
= \varphi(b(i_1, n) + b(n - 1, n)) = 1,
\]
a contradiction.

Case 3: $i_2 = n - 1$. Let $j \in \{2, \ldots, n\} \setminus \{i_1, n - 1, n\}$. This time we get
\[
\varphi(b(j, n - 1) + b(n - 1, n)) = (i_1, j)\varphi(b(i_1, n - 1) + b(n - 1, n)) = 1 \\
= (j, n)\varphi(b(i_1, n - 1) + b(n - 1, n)) = \varphi(b(i_1, n - 1) - b(j, n - 1)),
\]

hence we derive the contradiction
\[
0 = \varphi(b(i_1, n - 1) - b(j, n - 1)) + \varphi(b(j, n - 1) + b(n - 1, n)) \\
= \varphi(b(i_1, n - 1) + b(n - 1, n)) = 1.
\]

This finally proves that $S_1/2S_1$ cannot have a trivial factor module.

\[\square\]

**7.8 Lemma.** Let $n \geq 5$ be such that $n \equiv 1 \pmod 4$, and let $S_1$ and $S_2$ be the $\mathbb{Z}_2\mathfrak{S}_n$-sublattices of $S$ defined in Lemma 7.5 and Lemma 7.6, respectively. Then $2S/2S_1$ is the unique trivial submodule of the $\mathbb{F}_2\mathfrak{S}_n$-module $S_1/2S_1$, and $(S_2/2S_2)/(2S/2S_2)$ is the unique trivial factor module of the $\mathbb{F}_2\mathfrak{S}_n$-module $S_2/2S_2$.

**Proof.** Suppose that $n \equiv 1 \pmod 4$. In consequence of Lemma 7.5 and Lemma 7.6, we have $2S \subseteq S_2 \subseteq S_1 \subseteq S$, since $\binom{n-1}{2}$ is even.

We first show that $2S/2S_1$ is the unique trivial submodule of $S_1/2S_1$. By Lemma 7.4, we have
\[
2S/2S_1 \cong S/S_1 \cong (S/2S)/(S_1/2S) \cong S_{\mathbb{F}_2}/U_1 \cong \mathbb{F}_2,
\]
so that $2S/2S_1$ is indeed a trivial submodule of $S_1/2S_1$. Now let $x \in S_1$ be such that $x + 2S_1$ spans a trivial submodule of $S_1/2S_1$, that is, $x \notin 2S_1$ and $x - \sigma x \in 2S_1$, for all $\sigma \in \mathfrak{S}_n$. We need to show that $x \in 2S$. Assume not. Since $2S_1 \subseteq 2S$, $x + 2S$ then also spans a trivial submodule of $S/2S$. By Proposition 7.2 and Lemma 7.4, the $\mathbb{F}_2\mathfrak{S}_n$-module $S/2S$ is uniserial, and $S_2/2S \cong U_2$ is its unique trivial submodule. Hence $x + 2S \in S_2/2S$ and $x \in S_2$. So, by Lemma 7.6, we have
\[
x = \sum_{2 \leq i_1 < i_2 \leq n} \alpha(i_1, i_2)b(i_1, i_2),
\]
for $\alpha(i_1, i_2) \in \mathbb{Z}_2$ such that $\alpha(i_1, i_2) \equiv \alpha(n - 1, n) \pmod 2$, for all $2 \leq i_1 < i_2 \leq n$. Since we are assuming $x \notin 2S$, we must have $\alpha(i_1, i_2) \in \mathbb{Z}_2^\times$, for all $2 \leq i_1 < i_2 \leq n$. We may thus suppose that $\alpha(n - 1, n) = 1$. Then we get
\[
x - (n - 1, n)x = 2b(n - 1, n) + \sum_{i = 2}^{n-2} (\alpha(i, n - 1) - \alpha(i, n))b(i, n - 1) + \sum_{i = 2}^{n-2} (\alpha(i, n) - \alpha(i, n - 1))b(i, n).
\]
But then the sum of the coefficients at the basis elements of $S$ is $2 \not\equiv 0 \pmod{4}$, so that $x - (n-1, n)x \not\in 2S_1$, a contradiction.

Consequently, we must have $x \in 2S$, as claimed.

Now we show that $(S_2/2S_2)/(2S/2S_2)$ is the unique trivial factor module of $S_2/2S_2$. To this end, let

$$\varphi : S_2/2S_2 \to \mathbb{F}_2$$

be an $\mathbb{F}_2 \mathfrak{S}_n$-epimorphism. We show that $\ker(\varphi)$ contains $2S/2S_2$. Since, by Lemma 7.6, we have

$$(S_2/2S_2)/(2S/2S_2) \cong S_2/2S \cong U_2 \cong \mathbb{F}_2,$$

this forces $\ker(\varphi) = 2S/2S_2$, and we are done.

Let $- : S_2 \to S_2/2S_2$ be the canonical epimorphism. Since $S_2/2S_2$ has composition factors $\mathbb{F}_2$ with multiplicity 2 and $D_{\mathbb{F}_2}^{(n, -2)}$ with multiplicity 1 and since, by Lemma 7.7, $S_2/2S_2$ does not have a trivial submodule, $S_2/2S_2$ has a unique simple $\mathbb{F}_2 \mathfrak{S}_n$-submodule, namely

$$2S_1/2S_2 \cong S_1/S_2 \cong (S_1/2S)/(S_2/2S) \cong U_1/U_2 \cong D_{\mathbb{F}_2}^{(n, -2, 2)}.$$

Thus $2S_1/2S_2 \subseteq \ker(\varphi)$. Recall that $b(i_1, i_2) + b(n-1, n) \in S_1$, hence $2b(i_1, i_2) + 2b(n-1, n) \in 2S_1$ for all $2 \leq i_1 < i_2 \leq n$. This implies

$$0 = \varphi(2b(i_1, i_2) + 2b(n-1, n)) = \varphi(2b(i_1, i_2)) + \varphi(2b(n-1, n)),$$

that is, $\varphi(2b(i_1, i_2)) = \varphi(2b(n-1, n))$, for all $2 \leq i_1 < i_2 \leq n$. Assume that $\varphi(2b(n-1, n)) = 1$. Then we get

$$\varphi\left( \sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) \right) + 1 = \varphi\left( \sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) - 2b(n-1, 1) \right)$$

$$= \varphi\left( \sum_{(i_1, i_2) \neq (n-1, n)} b(i_1, i_2) - b(n-1, 1) \right)$$

$$= (n-1, n) \varphi\left( \sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) \right) = \varphi\left( \sum_{2 \leq i_1 < i_2 \leq n} b(i_1, i_2) \right) \cdot$$

a contradiction. Therefore, $\varphi(2b(i_1, i_2)) = 0$, for all $2 \leq i_1 < i_2 \leq n$, proving $2S/2S_2 \subseteq \ker(\varphi)$. \hfill \Box

**7.9 Proposition.** Let $n \geq 5$ be odd, and let $S_1$ and $S_2$ be the $\mathbb{Z}_2 \mathfrak{S}_n$-sublattices of $S$ defined in Lemma 7.5 and Lemma 7.6, respectively. Then the $\mathbb{F}_2 \mathfrak{S}_n$-modules $S_1/2S_1$ and $S_2/2S_2$ are uniserial. More precisely,

(a) if $n \equiv 1 \pmod{4}$, then $S_1/2S_2$ and $S_2/2S_2$ have the following Loewy series:

$$S_1/2S_1 \sim \begin{bmatrix} D_{\mathbb{F}_2}^{(n, -2, 2)} \\ \mathbb{F}_2 \\ \mathbb{F}_2 \end{bmatrix}, \quad S_2/2S_2 \sim \begin{bmatrix} \mathbb{F}_2 \\ D_{\mathbb{F}_2}^{(n, -2, 2)} \end{bmatrix};$$

(b) if $n \equiv 3 \pmod{4}$, then $S_1/2S_2$ and $S_2/2S_2$ have the following Loewy series:

$$S_1/2S_1 \sim \begin{bmatrix} D_{\mathbb{F}_2}^{(n, -2, 2)} \\ \mathbb{F}_2 \\ \mathbb{F}_2 \end{bmatrix}, \quad S_2/2S_2 \sim \begin{bmatrix} \mathbb{F}_2 \\ \mathbb{F}_2 \end{bmatrix}. $$

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Proof. (a) First suppose that \( n \equiv 1 \pmod{4} \). Since \( S_1 \) and \( S_2 \) are \( \mathbb{Z}_2\mathfrak{S}_n \)-lattices of finite index in \( S \), both of them are \( \mathbb{Z}_2 \)-forms of \( S_{Q_2} \). By Proposition 7.2, the \( \mathbb{F}_2\mathfrak{S}_n \)-modules \( S_1/2S_1 \) and \( S_2/2S_2 \) thus have three composition factors: \( \mathbb{F}_2 \) with multiplicity 2, and \( D_{\mathbb{F}_2}^{(n-2,2)} \) with multiplicity 1. In consequence of Lemma 7.7 and Lemma 7.8, we also know that \( S_1/2S_1 \) has a unique trivial submodule and no trivial factor module, while \( S_2/2S_2 \) has a unique trivial factor module and no trivial submodule. Therefore, \( S_1/2S_1 \) and \( S_2/2S_2 \) both have to be uniserial with the claimed Loewy series.

(b) If \( n \equiv 3 \pmod{4} \), then Proposition 7.2 shows that \( S_1/2S_1 \) and \( S_2/2S_2 \) have composition factors \( \mathbb{F}_2 \) and \( D_{\mathbb{F}_2}^{(n-2,2)} \), both occurring with multiplicity 1. Now Lemma 7.7 immediately implies the assertion.

We are now in the position to establish the first main result of this section:

7.10 Theorem. Let \( n \geq 5 \) be odd, and let \( S_1 \) and \( S_2 \) be the \( \mathbb{Z}_2\mathfrak{S}_n \)-sublattices of \( S := S_{Q_2} \) defined in Lemma 7.5 and Lemma 7.6, respectively. Then \( S, S_1 \) and \( S_2 \) are representatives of the isomorphism classes of \( \mathbb{Z}_2 \)-forms of \( S_{Q_2} \).

Proof. Suppose that \( n \equiv 1 \pmod{4} \) first. Then \( S/2S \) is uniserial with composition series \( \{0\} \subseteq U_2 \subseteq U_1 \subseteq S \), by Lemma 7.4. Thus, by Lemma 7.5 and Lemma 7.6, we deduce that the only \( \mathbb{Z}_2\mathfrak{S}_n \)-sublattices of \( S \) containing \( 2S \cong S \) are \( 2S_2, S_1 \) and \( S \). By Proposition 7.9 and Proposition 7.2, \( S/2S, S_1/2S_1 \) and \( S_2/2S_2 \) are uniserial \( \mathbb{F}_2\mathfrak{S}_n \)-modules. Therefore, the assertion of the theorem follows from Proposition 3.6.

If \( n \equiv 3 \pmod{4} \), then, by Proposition 7.2, we know that \( S/2S \cong S_{\mathbb{F}_2} \cong \mathbb{F}_2 \oplus D_{\mathbb{F}_2}^{(n-2,2)} \) as \( \mathbb{F}_2\mathfrak{S}_n \)-module. Since \( S \) is a \( \mathbb{Z}_2 \)-form of \( S_{Q_2} \), the assertion follows from Proposition 7.9 and Proposition 3.7.

Next we turn to the case where \( n \equiv 2 \pmod{4} \), and start with the following immediate consequence of Proposition 7.2(c).

7.11 Lemma. Let \( n \geq 6 \) be such that \( n \equiv 2 \pmod{4} \) and let \( S := S_{Q_2} \). There are uniquely determined \( \mathbb{Z}_2\mathfrak{S}_n \)-lattices \( T_1, T_2, T_3 \) such that

\[
2S =: T_4 \subseteq T_3 \subseteq T_2 \subseteq T_1 \subseteq T_0 := S
\]

and such that \( T_i \) is maximal in \( T_{i-1} \), for \( i \in \{1, \ldots, 4\} \). Moreover, one has

\[
S/T_1 \cong \mathbb{F}_2, \quad T_1/T_2 \cong D_{\mathbb{F}_2}^{(n-2,2)}, \quad T_2/T_3 \cong \mathbb{F}_2, \quad T_3/2S \cong D_{\mathbb{F}_2}^{(n-1,1)}
\]

as \( \mathbb{F}_2\mathfrak{S}_n \)-modules; in particular, \( T_1 = S_1 \), where \( S_1 \) is the \( \mathbb{Z}_2\mathfrak{S}_n \)-sublattice of \( S \) defined in Lemma 7.5.

7.12 Remark. Note that the \( \mathbb{Z}_2 \)-forms defined in Lemmas 7.5, 7.6 and 7.11 are all of the form \( \hat{L}_{Q_2} \), for some \( \mathbb{Z} \)-form \( \hat{L} \) of \( S_Q \) with \( \hat{L} \subseteq S_Z \), and \( [S_Z : \hat{L}] = [S_{Q_2} : \hat{L}_{Q_2}] \); in particular, the indices \([S_Z : \hat{L}]\) are 2-powers. While this is obvious for \( S_1, S_2 \), for \( T_1, T_2, T_3 \) this can be seen by lifting the unique composition series of \( S_Z/2S_Z = S_{Q_2}/2S_{Q_2} \) along \( \mathbb{Z} \rightarrow \mathbb{F}_2 \) to a chain of \( \mathbb{Z} \)-forms of \( S_Q \) instead of lifting it along \( Z \rightarrow Z_Q \rightarrow S_Q \). In this way, we, in particular, have \( \mathbb{F}_2\mathfrak{S}_n \)-isomorphisms \( \hat{S}_i/2\hat{S}_i \cong S_{Q_2}/2S_{Q_2} \) for \( n \equiv 1 \pmod{2} \) and \( i \in \{1, 2\} \), and \( \hat{T}_j/2\hat{T}_j \cong T_j/2T_j \) for \( n \equiv 2 \pmod{4} \) and \( j \in \{1, 2, 3\} \).
Lemma 7.11 thus provides us with \( \mathbb{Z}_2 \text{-forms } S, T_1, T_2, T_3 \) of the Specht \( \mathbb{Q}_2 \mathcal{S}_n \)-module \( S_{\mathbb{Q}_2} \).

Our aim now is to show that these form a set of representatives of the isomorphism classes of \( \mathbb{Z}_2 \text{-forms of } S_{\mathbb{Q}_2} \).

### 7.13 Remark.
Recall that we are still identifying the Specht lattice \( \mathbb{Z}_2 \). Our aim now is to show that these form a set of representatives of the isomorphism classes of \( \mathbb{Z}_2 \)-forms of \( S_{\mathbb{Q}_2} \).

In the following we shall also identify the dual Specht lattice \( S^* \) with \( (\wedge^2(S_{\mathbb{Z}_2}^{(n-1,1)}))^* \) via the isomorphism in 4.1(c). So if \( \{e(1, i_1, i_2) : 2 \leq i_1 < i_2 \leq n\} \) denotes the \( \mathbb{Z}_2 \)-basis of \( S^* \) dual to the standard polytabloid basis of \( S \) and if \( \{b(i_1, i_2)^* : 2 \leq i_1 < i_2 \leq n\} \) denotes the \( \mathbb{Z}_2 \)-basis of \( (\wedge^2(S_{\mathbb{Z}_2}^{(n-1,1)}))^* \) dual to our standard basis of \( \wedge^2(S_{\mathbb{Z}_2}^{(n-1,1)}) \), then we have \( e(1, i_1, i_2)^* = b(i_1, i_2)^* \), for all \( 2 \leq i_1 < i_2 \leq n \).

### 7.14 Proposition.
Let \( n \geq 6 \) be such that \( n \equiv 2 \pmod{4} \), and let \( S := S_{\mathbb{Z}_2} \). With the notation of Lemma 7.11, the \( \mathbb{Z}_2 \mathcal{S}_n \)-lattice \( T_3 \) is isomorphic to the dual \( \mathbb{Z}_2 \mathcal{S}_n \)-lattice \( S^* \).

**Proof.** Let \( \lambda := (n-2, 1^2) \) and consider the \( \mathbb{Z}_2 \mathcal{S}_n \)-monomorphism \( \varphi := \varphi^\lambda_{\mathbb{Z}_2} : S^* \to S \) in Corollary A.11. In consequence of Example A.12, for \( 2 \leq i < j \leq n \), we get:

\[
\varphi(b(i, j)^*) = \varphi(e(1, i, j)^*) = \sum_{\sigma \in \mathcal{S}(\{1, \ldots, n\} \setminus \{i, j\})} \sigma \cdot e(1, i, j)
\]

\[
= \sum_{\sigma \in \mathcal{S}(\{1, \ldots, n\} \setminus \{i, j\})} \sigma \cdot b(i, j)
\]

\[
= \sum_{\sigma \in \mathcal{S}(\{1, \ldots, n\} \setminus \{i, j\})} \sigma \cdot ((i \wedge j) + (1 \wedge i) - (1 \wedge j))
\]

\[
= (n-3)! \sum_{\substack{k=1 \atop i \neq k \neq j}}^n ((i \wedge j) + (k \wedge i) - (k \wedge j)).
\]

For \( 2 \leq i < j \leq n \), we set

\[
f(i, j) := \sum_{\substack{k=1 \atop i \neq k \neq j}}^n ((i \wedge j) + (k \wedge i) - (k \wedge j)) \in S.
\]

Then the elements \( f(i, j) \in S \), with \( 2 \leq i < j \leq n \), are linearly independent over \( \mathbb{Z}_2 \), and span a \( \mathbb{Z}_2 \mathcal{S}_n \)-sublattice \( T \) of \( S \) isomorphic to \( \varphi(S^*) \cong S^* \). It remains to verify that \( 2S \subseteq T \).

By 5.2(b), \( 2S \) is a cyclic \( \mathbb{Z}_2 \mathcal{S}_n \)-module generated by any of the basis elements \( 2b(i, j) \), where \( 2 \leq i < j \leq n \). Thus, it suffices to show that \( 2b(2, 3) \in T \). In the following, we shall show that

\[
n b(2, 3) = 3f(2, 3) + \sum_{k=4}^n (f(2, k) - f(3, k)) \in T. \tag{6}
\]

Since \( n \equiv 2 \pmod{4} \), we have \( n = 2u \), for some \( u \in \mathbb{Z}_2^* \), so that (6) then implies \( 2b(2, 3) \in T \).
as well. By definition,

\[
f(2,3) = b(2,3) + \sum_{k=4}^{n}((2 \land 3) + (k \land 2) - (k \land 3))
\]

\[
= b(2,3) + \sum_{k=4}^{n}((3 \land k) + (2 \land 3) - (2 \land k))
\]

\[
= b(2,3) + \sum_{k=4}^{n}(b(3,k) + b(2,3) - b(2,k)) = (n - 2)b(2,3) + \sum_{k=4}^{n}(-b(2,k) + b(3,k)).
\]

For \(k \in \{4, \ldots, n\}\), we have \((3,k) \cdot b(2,3) = b(2,k)\), hence also \((3,k) \cdot b(2,3)^* = b(2,k)^*\), by Example A.12, and \((n - 3)!f(2,k) = \varphi(b(2,k)^*) = (n - 3)!f(2,3)\) as well as \(f(2,k) = (2, k) \cdot f(2,3)\). Analogously, for \(k \in \{4, \ldots, n\}\), we have \((2,3) \cdot b(2,k) = b(3,k)\), thus \((2,3) \cdot b(2,k)^* = b(3,k)^*\) and \((2,3) \cdot f(2,k) = f(3,k)\). From this we now deduce the following

\[
f(2,k) = (n - 2)b(2,k) - b(2,3) - \sum_{i=4}^{k-1}b(2,i) - \sum_{i=k+1}^{n}b(2,i) - \sum_{i=3}^{k-1}b(i,k) + \sum_{i=k+1}^{n}b(k,i)
\]

\[
f(3,k) = (n - 2)b(3,k) + b(2,3) - \sum_{i=4}^{k-1}b(3,i) - \sum_{i=k+1}^{n}b(3,i) - \sum_{i=2}^{k-1}b(i,k) + \sum_{i=k+1}^{n}b(k,i).
\]

Using these equations one now verifies (6).

Therefore, we have now shown that \(2S \subseteq T \cong S^*\). It remains to prove that actually \(T = T_3\). By Proposition 7.2, the \(\mathbb{F}_2 \mathfrak{S}_n\)-module \(S_{\mathbb{F}_2}\) is not self-dual. Thus, by 2.2(d) the \(\mathbb{Z}_2 \mathfrak{S}_n\)-module \(S\) cannot be self-dual either. In particular, \(S \neq T \neq 2S\). So, from Lemma 7.11, we deduce that \(T = T_i\), for some \(i \in \{1, 2, 3\}\). Set \(D := D_{\mathbb{F}_2}^{(n-2,2)}\) and \(D' := D_{\mathbb{F}_2}^{(n-1,1)}\), for the remainder of this proof.

By Proposition 7.2, the \(\mathbb{F}_2 \mathfrak{S}_n\)-module \(S/2S \cong S_{\mathbb{F}_2}\) has simple socle \(D'\), so that \((T/2T) \cong (S/2S)^\ast\) has simple head \((D')^\ast \cong D'\). Since \(T/2S \neq \{0\}\) and \(T/2S \cong (T/2T)/(2S/2T)\), also \(T/2S\) has a simple factor module isomorphic to \(D'\). Since \(S/2S\) is uniserial, by Proposition 7.2, and \(T_i/2S\), for \(i \in \{1, 2, 3\}\). More precisely, Lemma 7.11 shows that \(T_i/2S\) is uniserial with head isomorphic to \(D \neq D'\), and \(T_3/2S\) is uniserial with head isomorphic to \(\mathbb{F}_2\). Therefore, we must have \(T = T_3\), and the proof of the proposition is complete.

**7.15 Corollary.** Let \(n \geq 6\) be such that \(n \equiv 2 \pmod{4}\), and let \(S := S_{\mathbb{F}_2}\). Let \(T_1, T_2\) and \(T_3\) be the \(\mathbb{Z}_2 \mathfrak{S}_n\)-sublattices of \(S\) given in Lemma 7.11. Then the \(\mathbb{F}_2 \mathfrak{S}_n\)-modules \(T_1/2T_1\), \(T_2/2T_2\) and \(T_3/2T_3\) are uniserial with the following Loewy series:

\[
T_1/2T_1 \sim \left[ \begin{array}{c} D_{\mathbb{F}_2}^{(n-2,2)} \\ F_2 \\ D_{\mathbb{F}_2}^{(n-1,1)} \end{array} \right], \quad T_2/2T_2 \sim \left[ \begin{array}{c} F_2 \\ D_{\mathbb{F}_2}^{(n-1,1)} \\ F_2 \\ D_{\mathbb{F}_2}^{(n-2,2)} \end{array} \right], \quad T_3/2T_3 \sim \left[ \begin{array}{c} F_2 \\ D_{\mathbb{F}_2}^{(n-1,1)} \\ F_2 \\ D_{\mathbb{F}_2}^{(n-2,2)} \end{array} \right].
\]

**Proof.** Let \(D := D_{\mathbb{F}_2}^{(n-2,2)}\) and \(D' := D_{\mathbb{F}_2}^{(n-1,1)}\). Recall that every simple \(\mathbb{F}_2 \mathfrak{S}_n\)-module \(S/2S \cong S_{\mathbb{F}_2}\) is uniserial with Loewy series

\[
S/2S \sim \left[ \begin{array}{c} F_2 \\ D \\ F_2 \\ D' \end{array} \right].
\]
By Proposition 7.14 and 2.2(d), we further know that \( T_3/2T_3 \cong (S/2S)^* \), so that also \( T_3/2T_3 \) is uniserial with Loewy series as claimed. It remains to treat \( T_1/2T_1 \) and \( T_2/2T_2 \). To this end, we first observe the following:

(i) Since \( (T_3/2T_3)/(2T_1/2T_3) \cong T_3/2T_1 \subseteq T_1/2T_1 \), the \( F_2 S_n \)-module \( T_1/2T_1 \) has a uniserial submodule with Loewy series

\[
\begin{bmatrix}
D' \\
F_2 \\
D
\end{bmatrix},
\]

the corresponding factor module is isomorphic to \( T_1/2T_1 \subseteq S/T_3 \cong (S/2S)/(T_3/2S) \), hence uniserial as well with Loewy series

\[
\begin{bmatrix}
D \\
F_2 \\
D'
\end{bmatrix}.
\]

(ii) We have \( F_2 \cong S/2T_1 \subseteq T_1/2T_1 \), and the factor module \( (T_1/2T_1)/(2S/2T_1) \cong T_1/2S \) is uniserial with Loewy series

\[
\begin{bmatrix}
D \\
F_2 \\
D'
\end{bmatrix}.
\]

Analogously,

(iii) \( T_2/2T_2 \) has a uniserial \( F_2 S_n \)-submodule \( V \) with Loewy series

\[
\begin{bmatrix}
F_2 \\
D
\end{bmatrix},
\]

and the corresponding factor module is uniserial with Loewy series

\[
\begin{bmatrix}
F_2 \\
D'
\end{bmatrix};
\]

(iv) \( T_2/2T_2 \) has the uniserial \( F_2 S_n \)-submodule \( U := T_3/2T_2 \) with Loewy series

\[
\begin{bmatrix}
D' \\
F_2 \\
D
\end{bmatrix},
\]

and \( (T_3/2T_2)/(T_3/2T_2) \cong T_2/T_3 \cong F_2 \).

In consequence of (iv), we either have \( U = \text{Rad}(T_2/2T_2) \) or \( \text{Rad}(U) = \text{Rad}(T_2/2T_2) \). In the second case, we would have \( (T_2/2T_2)/\text{Rad}(T_2/2T_2) \cong D' \oplus F_2 \), forcing \( T_2/2T_2 \) to have Loewy series

\[
\begin{bmatrix}
D' \oplus F_2 \\
F_2 \\
D
\end{bmatrix}.
\]

By (iii) we also get

\[
\text{Rad}(T_2/2T_2)/(\text{Rad}(T_2/2T_2) \cap V) \cong (\text{Rad}(T_2/2T_2) + V)/V \subseteq \text{Rad}((T_2/2T_2)/V) \cong D'.
\]

On the other hand, the factor module \( \text{Rad}(T_2/2T_2)/(\text{Rad}(T_2/2T_2) \cap V) \) either has to be \( \{0\} \), or has a composition factor isomorphic to \( D \) or \( F_2 \). Since \( F_2 \not\cong D' \not\cong D \), we must have
Rad\((T_2/2T_2)/(\mathrm{Rad}(T_2/2T_2) \cap V) = \{0\}\), thus \(\mathrm{Rad}(T_2/2T_2) \subseteq V\). Comparing \(\mathbb{F}_2\)-dimensions this implies \(\mathrm{Rad}(T_2/2T_2) = V\), so that \((T_2/2T_2)/V \cong D' \oplus \mathbb{F}_2\), which is not uniserial, a contradiction to (iii). Therefore, we conclude that \(U = \mathrm{Rad}(T_2/2T_2)\), and the assertion concerning the Loewy series of \(T_2/2T_2\) follows.

Similarly, considering socle series instead of Loewy series one shows that also \(T_1/2T_1\) is uniserial with the claimed Loewy series.  

7.16 Theorem. Let \(n \geq 6\) be such that \(n \equiv 2 \pmod{4}\), and let \(S := S_{2n}\). Let \(T_1, T_2\) and \(T_3\) be the \(Z_2\)-sublattices of \(S\) given in Lemma 7.11. Then \(S, T_1, T_2\) and \(T_3\) are representatives of the isomorphism classes of \(Z_2\)-forms of the Specht \(\mathbb{Q}_2\mathfrak{S}_n\)-module \(S_{2n}\).

Proof. By Lemma 7.11, \(S, T_1, T_2\) and \(T_3\) are the only \(Z_2\)-sublattices of \(S\) properly containing \(2S\). By Proposition 7.2 and Corollary 7.15, the \(\mathbb{F}_2\mathfrak{S}_n\)-modules \(S/2S, T_1/2T_1, T_2/2T_2\) and \(T_3/2T_3\) are uniserial and pairwise non-isomorphic; in particular, the \(Z_2\)-lattices \(S, T_1, T_2\) and \(T_3\) are pairwise non-isomorphic. Now Proposition 3.6 yields the assertion of the theorem.

7.17 Remark. (a) We would like to mention that the definition of the \(Z_2\)-lattices \(S_1\) and \(S_2\) in Lemma 7.5 and Lemma 7.6, respectively, as well as our proofs of Lemma 7.7 and Lemma 7.8 are reminiscent of [23, Satz (I.11)]. There the \(Z\)-forms of a certain absolutely simple \(\mathbb{Q}[\mathfrak{S}_2 \wr \mathfrak{S}_n]\)-module have been determined.

(b) Since \((S(2))_{2n}\) is self-dual, by Theorem 5.4(a), for every \(Z_2\)-form \(L\) of \((S(2))_{2n}\), the dual lattice \(L^*\) has to be a \(Z_2\)-form of \((S(2))_{2n}\) as well.

Suppose first that \(n\) is odd. Then the natural Specht module \((S(1))_{2n}\) is absolutely simple and self-dual, by Theorem 5.4, and the Specht \(\mathbb{F}_2\mathfrak{S}_n\)-module \((S(1))_{2n}\) is also absolutely simple, by [18, Theorem 23.7]. Thus, by Proposition 2.12, \((S(1))_{2n}\) is the only \(Z_2\)-form of \((S(1))_{2n}\) and \((S(1))_{2n} \equiv (S(1))_{2n}^*\). So, by 2.2(d), also \((S(2))_{2n} \equiv \bigwedge^2(S(1))_{2n}\) is self-dual. By Proposition 7.9, the \(\mathbb{F}_2\mathfrak{S}_n\)-modules \(S_1/2S_1\) and \(S_2/2S_2\) are not self-dual, thus the \(Z_2\)-lattices \(S_1\) and \(S_2\) are not self-dual either. Hence we must have \(S_1 \cong S_2^*\), by Theorem 7.10.

Now suppose that \(n \equiv 2 \pmod{4}\). By Proposition 7.14, we know that \((S(2))_{2n}\) is self-dual. Corollary 7.15 shows that \(T_1/2T_1\) and \(T_2/2T_2\) are not self-dual, thus \(T_1\) and \(T_2\) are not self-dual. By Theorem 7.16, this gives \(T_1 \cong T_2^*\).

(c) In consequence of Theorem 7.10 and Theorem 7.16, as far as the determination of the isomorphism classes of \(Z_2\)-forms of \((S(2))_{2n}\) is concerned, the case \(n \equiv 0 \pmod{4}\) remains open so far. The Loewy series of the \(\mathbb{F}_2\mathfrak{S}_n\)-module \((S(2))_{2n}\) in Proposition 7.2(d) already indicates that the lattice of \(Z_2\)-sublattices of \((S(2))_{2n}\) of full rank will have a much more complicated structure in this case.

As well, for \(k \geq 3\), we are currently neither able to determine representatives of the isomorphism classes of \(Z_2\)-forms of \((S(k))_{2n}\) nor their number. However, based on computer calculations with MAGMA [3], we would like to state the following conjecture. Using the algorithms developed in [15], we have checked part (a) of the conjecture for \(n \leq 52\), and part (b) for \(n \leq 23\).

7.18 Conjecture. Let \(n \in \mathbb{N}\), and let \(n \geq 5\). For \(k \in \{1, \ldots, n-2\}\), denote by \(h_2(k)\) the number of isomorphism classes of \(Z_2\)-forms of \((S(k))_{2n}\).

(a) If \(n \equiv 0 \pmod{4}\) and \(k \in \{2, n-3\}\), then \(h_2(k) = 3\nu_2(n) + 1\).

(b) For \(k \in \{3, n-4\}\), the following holds:

(i) If \(n \geq 7\) is odd, then \(h_2(k) = 3\).
(ii) If \( n \equiv 2 \pmod{4} \), then \( h_2(k) = 8 \).
(iii) If \( n \equiv 0 \pmod{4} \), then \( h_2(k) = 9\nu_2(n) + 1 \).

8 Proofs of the main results

This section is now devoted to the proofs of Theorem 1.1 and Corollary 1.2, and to establishing a precise version of Theorem 1.3. To do so, we shall apply Theorems 6.1, 7.10 and 7.16, and invoke the results of Plesken [23, Satz (I.9), Korollar (I.10)], [24, Theorem 5.1] and Craig [6], who determined the isomorphism classes of \( \mathbb{Z} \)-forms of the natural Specht \( \mathbb{Q} \mathbb{S} \)-module \( S(1)_{\mathbb{Q}} = S_{\mathbb{Q}}^{(n-1,1)} \). In fact, both Plesken and Craig work with the Specht module \( S(1)_{\mathbb{Q}} \otimes \text{sgn}_Q \cong S(n-2)_{\mathbb{Q}} = S_{\mathbb{Q}}^{(2,1^{n-2})} \) rather than \( S(1)_{\mathbb{Q}} \). Translated into our terminology, the result reads as follows:

8.1 Theorem. Let \( n \in \mathbb{N} \), and let \( M := S(1)_{\mathbb{Z}} \). For every divisor \( d \in \mathbb{N} \) of \( n \), let

\[
M_d := \mathbb{Z} \left( \sum_{i=2}^{n} b(i) \right) + dM \subseteq M.
\]

Then
(a) \( \{ M_d : d \mid n \} \) is a set of representatives of the isomorphism classes of \( \mathbb{Z} \)-forms of \( S(1)_{\mathbb{Q}} \);
(b) for every \( d \mid n \), one has \( |M : M_d| = d^{n-2} \);
(c) if \( p \leq n \) is a prime number, then \( \{ (M_p)_{\mathbb{Z}_p} : 0 \leq i \leq \nu_p(n) \} \) is a set of representatives of the isomorphism classes of \( \mathbb{Z}_p \)-forms of \( S(1)_{\mathbb{Q}_p} \).

Proof. (a) It is easily checked that each \( M_d \), for \( d \mid n \), is a \( \mathbb{Z} \mathbb{G} \)-sublattice of \( M \) with \( \mathbb{Z} \)-basis \( \{ \sum_{i=2}^{n} b(i), \, db(2), \ldots, \, db(n-1) \} \). Thus each \( M_d \) is a \( \mathbb{Z} \)-form of \( S(1)_{\mathbb{Q}} \), by Remark 2.11. Moreover, if \( d \mid n \) and \( a \in \mathbb{Z} \setminus \{ 1, -1 \} \), then \( M_d \) is not contained in \( aM \), since then \( \sum_{i=2}^{n} b(i) \notin aM \). Thus, by [24, Proposition 2.3], the \( \mathbb{Z} \mathbb{G}_n \)-lattices in \( \{ M_d : d \mid n \} \) are pairwise non-isomorphic. But, by [23, Korollar (I.10)], the number of isomorphism classes of \( \mathbb{Z} \)-forms of \( S(1)_{\mathbb{Q}} \) equals the number of positive divisors of \( n \), whence (a).

Part (b) follows immediately from Proposition 2.5 using the \( \mathbb{Z} \)-basis of \( M_d \) just mentioned.

To prove (c), let \( d \mid n \), and let \( p \) be a prime number such that \( d = pq \), for some \( q \in \mathbb{N} \) with \( p \nmid q \). Then \( dM \subseteq p^2M \), hence also \( M_d \subseteq M_p \) and \( (M_d)_{\mathbb{Z}_p} \subseteq (M_p)_{\mathbb{Z}_p} \subseteq \mathbb{M}_{\mathbb{Z}_p} \). As well, \( (M_{\mathbb{Z}_p} : (M_p)_{\mathbb{Z}_p}) = d^{n-2} \cdot \mathbb{Z}_p = p^i \cdot \mathbb{Z}_p = (M_{\mathbb{Z}_p} : (M_p)_{\mathbb{Z}_p}) \). Thus Proposition 2.5 gives \( (M_d)_{\mathbb{Z}_p} = (M_p)_{\mathbb{Z}_p} \). On the other hand, in consequence of (a) and Proposition 3.3, we obtain that the \( \mathbb{Z} \mathbb{G}_n \)-lattices in \( \{ (M_p)_{\mathbb{Z}_p} : 0 \leq i \leq \nu_p(n) \} \) have to be pairwise non-isomorphic, since \( (M_p)_{\mathbb{Z}_p} \cong \mathbb{M}_{\mathbb{Z}_p} \), for all \( i \in \{ 0, \ldots, \nu_p(n) \} \) and every prime number \( q \neq p \). Now Proposition 3.2 implies the assertion in (c). \( \square \)

8.2 Remark. By Theorem 5.4 and Proposition A.9, we know that \( S(k)_{\mathbb{Q}} \cong S(n-k+1)_{\mathbb{Q}} \otimes \text{sgn}_Q \) as well as \( S(k)_{\mathbb{Q}_p} \cong S(n-k+1)_{\mathbb{Q}_p} \otimes \text{sgn}_{Q_p} \), for every \( k \in \{ 1, \ldots, n-2 \} \) and every prime number \( p \). These isomorphisms entail a bijection between the set of isomorphism classes of \( \mathbb{Z} \)-forms of \( S(k)_{\mathbb{Q}} \) and the set of isomorphism classes of \( \mathbb{Z} \)-forms of \( S(n-k+1)_{\mathbb{Q}} \), as well as a bijection between the set of isomorphism classes of \( \mathbb{Z}_p \)-forms of \( S(k)_{\mathbb{Q}_p} \), and the set of isomorphism classes of \( \mathbb{Z}_p \)-forms of \( S(n-k+1)_{\mathbb{Q}_p} \).

Hence, together with Theorem 8.1 we now get:
Proof of Theorem 1.1. Part (a) follows from Theorem 6.1 and Theorem 8.1(c). Part (b) follows from Theorem 7.10 and Theorem 7.16.

Proof of Corollary 1.2. If \( n \in \mathbb{N} \) has prime factor decomposition \( n = p_1^{a_1} \cdots p_r^{a_r} \) and if \( d(n) \) denotes the number of divisors of \( n \) in \( \mathbb{N} \), then \( d(n) = \prod_{i=1}^{r} (a_i + 1) \). Thus the corollary follows from Theorem 1.1 together with Corollary 3.4.

We conclude by proving Theorem 8.3. Here we shall use the following notation: for every prime number \( p \geq 3 \), we set

\[
X_p := \left\{ \bigwedge_i (M_{p^i}) : 0 \leq i \leq \nu_p(n) \right\}.
\]

Furthermore, we set

\[
X_2 := \begin{cases} \{S(2)_Z, \tilde{S}_1, \tilde{S}_2\} & \text{if } n \equiv 1 \pmod{2} \\ \{S(2)_Z, \tilde{T}_1, \tilde{T}_2, \tilde{T}_3\} & \text{if } n \equiv 2 \pmod{4} \end{cases};
\]

here \( \tilde{S}_1, \tilde{S}_2, \tilde{T}_1, \tilde{T}_2, \tilde{T}_3 \) are the \( \mathbb{Z} \)-forms of \( S(2)_\mathbb{Q} \) in Remark 7.12. With this notation, we have

8.3 Theorem. Let \( n \geq 5 \) be such that \( n \equiv 0 \pmod{4} \), and let \( L \) be a \( \mathbb{Z} \)-form of \( S(2)_\mathbb{Q} \). Let \( \{p_1, \ldots, p_g\} \) be the set of prime divisors of \( n \). Then for each \( j \in \{1, \ldots, g\} \), there exists a unique \( N_{p_j} \in X_{p_j} \) such that

\[
L \cong \bigcap_{j=1}^{g} N_{p_j}.
\]

Proof. This follows from Theorems 6.1, 7.10 and 7.16, together with Proposition 3.5.

By Remark 8.2, Theorem 8.3 also provides representatives of the isomorphism classes of \( \mathbb{Z} \)-forms of \( S(n-3)_\mathbb{Q} \).

A Dual Specht lattices

Let \( n \in \mathbb{N} \), and let \( \lambda \) be a partition of \( n \). The aim of this appendix is to construct an explicit \( \mathbb{Z}\mathfrak{S}_n \)-monomorphism

\[
\varphi^\lambda_Z : (S^\lambda_Z)^* \rightarrow S^\lambda_Z.
\]

Since \( S^\lambda_Z \) is self-dual, it is clear that such a monomorphism has to exist, by Remark 2.11. The particular monomorphism \( \varphi^\lambda_Z \) in Theorem A.10, which has been essential in the proof of Proposition 7.14, can be found in unpublished work of Wildon [28]. It can be obtained by a close inspection of the arguments in the proof of [18, Theorem 6.7]. In the following we aim to work out the details of the construction of \( \varphi^\lambda_Z \) following the lines of [28].

We start by summarizing some properties of bilinear forms of permutation modules that we shall need throughout this appendix.

A.1. Permutation modules and bilinear forms. Let \( G \) be any finite group, let \( R \) be a principal ideal domain and \( K \) its field of fractions. Suppose that \( L \) is a permutation \( RG \)-lattice with permutation basis \( \{\omega_1, \ldots, \omega_k\} \).
(a) Again, denote by \( \{ \omega_1^*, \ldots, \omega_k^* \} \) the dual \( R \)-basis of \( L^* \). Moreover, consider the canonical \( R \)-bilinear form
\[
\langle \cdot \mid \cdot \rangle : L \times L \to R, \quad (\omega_i, \omega_j) \mapsto \delta_{ij},
\]
which is symmetric, non-degenerate and \( G \)-invariant. Whenever \( N \) is an \( RG \)-sublattice of \( L \), we have the \( RG \)-sublattice \( N^\oplus := \{ f \in L^* : f(x) = 0, \text{ for all } x \in N \} \) of \( L^* \) as well as the \( RG \)-sublattice \( N^\perp := \{ x \in L : \langle x \mid y \rangle = 0, \text{ for all } y \in N \} \) of \( L^* \). The \( RG \)-isomorphism
\[
\psi : L \to L^*, \quad x \mapsto (y \mapsto \langle x \mid y \rangle)
\]
maps \( N^\perp \) to \( N^\oplus \).

(b) Now suppose that \( \{ \omega_1, \ldots, \omega_k \} \) is a transitive permutation basis of \( L \). Then \( L \) is clearly a cyclic \( RG \)-lattice, generated by any basis element \( \omega_i \). Since \( \psi(\omega_i) = \omega_i^* \), for \( i \in \{1, \ldots, k\} \), also \( L^* \) is a cyclic \( RG \)-lattice, generated by any \( \omega_i^* \). Note that if \( i, j \in \{1, \ldots, k\} \) and \( g \in G \) are such that \( g\omega_i = \omega_j \), then also \( g\omega_i^* = \omega_j^* \).

(c) Lastly, note that the \( R \)-bilinear form (7) naturally extends to a symmetric, non-degenerate \( G \)-invariant \( K \)-bilinear form on \( KL \), which we denote by \( \langle \cdot \mid \cdot \rangle_K \). Thus, if \( N \) is an \( RG \)-sublattice of \( L \), then we have the \( RG \)-sublattice \( N^\perp \) of \( L \), on the one hand, and the \( KG \)-submodule \( (KN)^\perp \) of \( KL \), on the other hand. The next lemma explains how these modules are related.

**A.2 Lemma.** Let \( G \) be a finite group, and let \( R \) be a principal ideal domain with field of fractions \( K \). Let further \( L \) be a \( permutation \) \( RG \)-lattice, and let \( N \) be an \( RG \)-sublattice of \( L \). With the notation of A.1, one has \( (KN)^\perp = KN^\perp \); in particular,
\[
\rk_R(L) = \rk_R(N) + \rk_R(N^\perp).
\]

**Proof.** Let \( x \in N^\perp \), and let \( y \in KN \). Then \( \alpha y \in N \), for some \( \alpha \in R \) with \( \alpha \neq 0 \). For \( \beta \in K \) we get
\[
\langle \beta x \mid y \rangle_K = \beta \alpha^{-1} \langle x \mid \alpha y \rangle_K = \beta \alpha^{-1} \langle x \mid \alpha y \rangle = 0,
\]
which shows that \( \beta x \in (KN)^\perp \).

Conversely, let \( x \in (KN)^\perp \), and let \( y \in N \). Moreover, let \( \alpha \in R \) be such that \( \alpha \neq 0 \) and \( \alpha x \in L \). Then
\[
\langle \alpha x \mid y \rangle = \langle \alpha x \mid y \rangle_K = \alpha \langle x \mid y \rangle_K = 0,
\]
thus \( \alpha x \in N^\perp \) and \( x = \alpha^{-1} \alpha x \in KN^\perp \).

This proves \( (KN)^\perp = KN^\perp \), and we deduce
\[
\rk_R(L) = \dim_K(KL) = \dim_K(KN) + \dim_K((KN)^\perp) = \dim_K(KN) + \dim_K(KN^\perp) = \rk_R(N) + \rk_R(N^\perp).
\]

\[\square\]

For the remainder of this appendix we consider \( M := M_2^\times \) and \( S := S_3^\times \). As in A.1, we denote by \( \langle \cdot \mid \cdot \rangle \) the non-degenerate symmetric \( \mathfrak{S}_n \)-invariant \( \mathbb{Z} \)-bilinear form on \( M \) with respect to the standard basis \( T(\lambda) \). As an immediate consequence of the \( \mathfrak{S}_n \)-invariance of this bilinear form, one has the following well-known fact:

**A.3 Lemma.** Let \( t \) be any \( \lambda \)-tableau, and let \( x, y \in M \). Let \( \kappa_t \) and \( \rho_t \) be as in 5.1(b). Then
\[
\langle \kappa_t x \mid y \rangle = \langle x \mid \kappa_t y \rangle \quad \text{and} \quad \langle \rho_t x \mid y \rangle = \langle x \mid \rho_t y \rangle.
\]
The next result has already been stated in [28, Lemma 2], and is an immediate generalization of the classical Submodule Theorem [17]; we include a short proof for completeness.

**A.4 Theorem (Integral Version of the Submodule Theorem).** Let $U$ be a $\mathbb{Z}\mathfrak{S}_n$-sublattice of $M$. If $U \not\subseteq S^1$, then there is some $m \in \mathbb{N}$ such that $mS \subseteq U$.

**Proof.** Recall that $e_t \in S$ denotes the $\lambda$-polytabloid labelled by $t$. Let $u \in U$. Then, by [27, Corollary 2.4.3], there is some $q \in \mathbb{Q}$ such that $\kappa_t \cdot u = qe_t$. Thus $r\kappa_t u = me_t$, for some $r, m \in \mathbb{Z}$ such that $m \geq 0$ and $r \neq 0$. If $m \neq 0$, then we have $mS = \mathbb{Z}\mathfrak{S}_n(me_1) \subseteq U$.

So suppose now that $\kappa_t u = 0$, for all $u \in U$ and every $\lambda$-tableau $t$, so that

$$0 = \langle \kappa_t u | t \rangle = \langle u | \kappa_t | t \rangle = \langle u | e_t \rangle,$$

by Lemma A.3. This shows that $U \subseteq S^1$ in this case.

Throughout this section, we now fix a $\lambda$-tableau $t$. The corresponding $\lambda'$-tableau obtained by transposing $t$ will again be denoted by $t'$ as before. Moreover, by $\text{sgn}$ we denote the $\mathbb{Z}\mathfrak{S}_n$-lattice of rank one on which $\sigma \in \mathfrak{S}_n$ acts by multiplication with $\text{sgn}(\sigma)$. The respective $\mathbb{Q}\mathfrak{S}_\sigma$-module will be denoted by $\text{sgn}_Q$.

**A.5 Lemma.** There is a $\mathbb{Z}\mathfrak{S}_n$-epimorphism

$$f_t : M \to S^\lambda_Z \otimes \text{sgn}$$

such that $f_t(\{t\}) = e_{t'} \otimes 1$ and $\ker(f_t) = S^1$.

**Proof.** As in the proof of [18, Theorem 6.7], we set

$$f_t(\{\pi t\}) := \text{sgn}(\pi)(e_{\pi t'} \otimes 1),$$

for $\pi \in \mathfrak{S}_n$, and then extend this map $\mathbb{Z}$-linearly. Note that $\text{sgn}(\pi)(e_{\pi t'} \otimes 1) = \pi \cdot (e_{t'} \otimes 1) = \pi p_t(\{t'\} \otimes 1)$, for $\pi \in \mathfrak{S}_n$. We first show that $f_t$ is well defined. To this end, let $\pi_1, \pi_2 \in \mathfrak{S}_n$ be such that $\{\pi_1 t\} = \{\pi_2 t\}$, that is, $\pi_1^{-1} \pi_2 \in R_t = C_{t'}$. Hence

$$\pi_1^{-1} \pi_2 (e_{t'} \otimes 1) = (\pi_1^{-1} \pi_2 e_{t'}) \otimes \text{sgn}(\pi_1^{-1} \pi_2) = (\text{sgn}(\pi_1^{-1} \pi_2) e_{t'}) \otimes \text{sgn}(\pi_1^{-1} \pi_2) = e_{t'} \otimes 1,$$

and

$$f_t(\pi_2 t') = f_t(\{\pi_1 \pi_1^{-1} \pi_2 t\}) = (\pi_1 \pi_1^{-1} \pi_2) \cdot (e_{t'} \otimes 1) = \pi_1 (\pi_1^{-1} \pi_2 (e_{t'} \otimes 1)) = \pi_1 (e_{t'} \otimes 1) = f_t(\{\pi_1 t\}).$$

This implies that $f_t$ is well defined. By definition, $f_t$ is also a $\mathbb{Z}\mathfrak{S}_n$-homomorphism, and we have $f_t(\{t\}) = e_{t'} \otimes 1$. Since $S^\lambda_Z \otimes \text{sgn} = \mathbb{Z}(e_{\pi t'} \otimes 1 : \pi \in \mathfrak{S}_n)$, we deduce that $f_t$ is, in fact, a $\mathbb{Z}\mathfrak{S}_n$-epimorphism. It remains to determine the kernel of $f_t$. Note that $f_t(e_t) = (\rho_{\pi' \kappa_t} t') \otimes 1 \in M^\lambda_Z \otimes \text{sgn}$. Moreover, Lemma A.3 gives

$$\langle \rho_{\pi' \kappa_t} t' | t' \rangle = \langle \kappa_t t' | \rho_{\pi'} t' \rangle = \langle e_{t'} | R_{\pi'} t' \rangle = |R_{\pi'}|.$$

So, writing $f_t(e_t)$ as a $\mathbb{Z}$-linear combination of the standard $\mathbb{Z}$-basis of $M^\lambda_Z \otimes \text{sgn}$, the basis element $\{t'\} \otimes 1$ occurs with non-zero coefficient $|R_{\pi'}|$; in particular, $f_t(e_t) \neq 0$ as well as
\( f_t(me_t) \neq 0 \), for all \( m \in \mathbb{N} \). Therefore, \( mS \not\subset \ker(f_t) \), for all \( m \in \mathbb{N} \), implying \( \ker(f_t) \subset S^\perp \), by Theorem A.4.

To show that we actually have equality, note first that \( \text{rk}_Z(\ker(f_t)) = \text{rk}_Z(M) - \text{rk}_Z(S^\perp_Z \otimes \text{sgn}) = \text{rk}_Z(M) - \text{rk}_Z(S) \). On the other hand, Lemma A.2 also gives \( \text{rk}_Z(S^\perp) = \text{rk}_Z(M) - \text{rk}_Z(S) \). Hence, \( \text{rk}_Z(\ker(f_t)) = \text{rk}_Z(S^\perp) \), so that \( S^\perp / \ker(f_t) \) is a finite abelian group of order \( r \in \mathbb{N} \), say. If \( r > 1 \) then there would be some \( x \in S^\perp \) such that \( f_t(x) \neq 0 \), but \( 0 = f_t(rx) = rf_t(x) \), a contradiction. This shows that, indeed, \( S^\perp = \ker(f_t) \).

**A.6 Remark.** Let \( k := \text{rk}_Z(S) \). In the following it will be useful to replace the standard \( \mathbb{Z} \)-basis of \( S = S^\lambda_Z \) by a slightly modified one. In general, writing a standard \( \lambda \)-polytabloid as a \( \mathbb{Z} \)-linear combination of \( \lambda \)-tabloids, several standard \( \lambda \)-tabloids will occur with non-zero coefficients. However, the proof of [18, Corollary 8.12] shows that \( S \) admits a \( \mathbb{Z} \)-basis \( \{ b_1, \ldots, b_k \} \) such that, for each \( i \in \{ 1, \ldots, k \} \), there is a unique standard \( \lambda \)-tabloid \( \{ t_i \} \) such that \( \{ b_i \mid \{ t_i \} \} = 1 \) and \( \{ b_i \mid \{ s \} \} = 0 \) for every standard \( \lambda \)-tabloid \( \{ s \} \neq \{ t_i \} \). Moreover, \( \{ t_i \} \neq \{ t_j \} \), for \( i \neq j \). Denote the dual basis of \( S^* \) by \( \{ b_1^*, \ldots, b_k^* \} \).

For each \( i \in \{ 1, \ldots, k \} \), consider the element \( \{ t_i \}^* \in M^* \) of the \( \mathbb{Z} \)-basis of \( M^* \) that is dual to the tabloid basis of \( M \). Then \( \{ t_i \}^*(b_j) = \delta_{ij} \), for \( i, j \in \{ 1, \ldots, k \} \), so that \( b_i^* \) is precisely the restriction of \( \{ t_i \}^* \) to \( S \).

**A.7 Lemma.** The map 
\[
g : M \to S^*, \ m \mapsto (x \mapsto \langle x \mid m \rangle)
\]
is a \( \mathbb{Z}\mathfrak{S}_n \)-epimorphism with kernel \( S^\perp \).

**Proof.** The map \( g \) is clearly \( \mathbb{Z} \)-linear with kernel \( S^\perp \), and due to the \( \mathfrak{S}_n \)-invariance of \( \langle \cdot \mid \cdot \rangle \) the map \( g \) is also a \( \mathbb{Z}\mathfrak{S}_n \)-homomorphism. Remark A.6 shows, moreover, that \( g \) is surjective. \( \square \)

**A.8 Lemma.** There is a \( \mathbb{Z}\mathfrak{S}_n \)-epimorphism 
\[
h_t : M^\lambda_Z \otimes \text{sgn} \to S
\]
such that \( h_t(\{ t' \} \otimes 1) = e_t \). Moreover, the restriction of \( h_t \) to \( S^\lambda_Z \otimes \text{sgn} \) is injective.

**Proof.** By Lemma A.5, we have the \( \mathbb{Z}\mathfrak{S}_n \)-epimorphism \( f_t : M^\lambda_Z \to S^\lambda_Z \otimes \text{sgn} \), inducing a \( \mathbb{Z}\mathfrak{S}_n \)-epimorphism \( f_t \otimes \text{id} : M^\lambda_Z \otimes \text{sgn} \to S^\lambda_Z \otimes \text{sgn} \otimes \text{sgn} \). Finally, \( S^\lambda_Z \otimes \text{sgn} \otimes \text{sgn} \cong S^\lambda_Z \), via the \( \mathbb{Z} \)-linear map sending \( e_s \otimes 1 \otimes 1 \) to \( e_s \), for every standard \( \lambda \)-tableau \( s \). In total, this yields a \( \mathbb{Z}\mathfrak{S}_n \)-epimorphism \( h_t : M^\lambda_Z \otimes \text{sgn} \to S \) such that \( h_t(\{ t' \} \otimes 1) = e_t \), as desired.

It remains to show that \( h_t \) restricts to an injective \( \mathbb{Z}\mathfrak{S}_n \)-homomorphism \( S^\lambda_Z \otimes \text{sgn} \to S^\lambda_Z \). Denote this restriction by \( \phi \); for ease of notation, we suppress the index \( t \) for the time being. Then we have
\[
\phi(e_t \otimes 1) = h_t(e_t \otimes 1) = h_t(( \sum_{\sigma \in C_t^\nu} \text{sgn}(\sigma)\{ \sigma t' \} ) \otimes 1) = h_t( \sum_{\sigma \in C_t^\nu} \sigma \cdot (\{ t' \} \otimes 1) )
\]
\[
= \sum_{\sigma \in C_t^\nu} \sigma \cdot h_t(\{ t' \} \otimes 1) = \sum_{\sigma \in C_t^\nu} \sigma e_t
\]
\[
= \sum_{\sigma \in C_t^\nu} \sigma \sum_{\pi \in C_t} \text{sgn}(\pi)\pi \{ t \} = \sum_{\sigma \in \mathcal{B}_t} \sum_{\pi \in C_t} \text{sgn}(\pi)(\sigma \pi t) \in S^\lambda_Z \subset M^\lambda_Z.
\]
The coefficient at the basis element \( \{t\} \) of \( M^\lambda_2 \) in \( h_t(e^t \otimes 1) \) equals

\[
\sum_{\sigma \in R_t \atop \pi \in C_t \atop \sigma \pi \in R_t} \text{sgn}(\pi).
\]

If \( \sigma \in R_t \) and \( \pi \in C_t \) are such that \( \sigma \pi \in R_t \) then \( \pi \in R_t \cap C_t = \{1\} \). Hence the coefficient at \( \{t\} \) in \( h_t(e^t \otimes 1) \) is \( |R_t| \neq 0 \); in particular, \( \phi \neq 0 \).

Now \( \phi \) induces a \( \mathbb{Q}\mathfrak{S}_n \)-homomorphism

\[
\phi_Q : S^\lambda_Q \otimes \text{sgn}_Q \to S^\lambda_Q
\]
such that \( \phi_Q(e^t \otimes 1) \neq 0 \). Consequently, \( \phi_Q \) is an isomorphism, since \( S^\lambda_Q \otimes \text{sgn}_Q \) and \( S^\lambda_Q \) are simple \( \mathbb{Q}\mathfrak{S}_n \)-modules. Since \( S^\lambda_Q \otimes \mathbb{Z} \text{sgn}_Q \) is a \( \mathbb{Z} \)-form of \( S^\lambda_Q \otimes \text{sgn}_Q \) and \( S^\lambda_Q \) is a \( \mathbb{Z} \)-form of \( S^\lambda_Q \).

Proposition 2.10 implies that \( \phi \) has to be an injective \( \mathbb{Z}\mathfrak{S}_n \)-homomorphism, and the proof of the lemma is complete.

As an immediate consequence of Lemma A.5 and Lemma A.7 one has the following, which is [18, Theorem 8.15] in the case where \( R \) is a field and [10, Proposition 4.8] in the case where \( R = \mathbb{Z} \):

A.9 Proposition. For every \( \lambda \in \mathcal{P}(n) \) and every principal ideal domain \( R \), one has an isomorphism

\[
S^\lambda_R \otimes \text{sgn}_R \cong (S^\lambda_R)^*
\]
of \( RG \)-lattices.

Proof. For \( \lambda \in \mathcal{P}(n) \), Lemma A.5 and Lemma A.7 yield a \( \mathbb{Z}\mathfrak{S}_n \)-isomorphism \( S^\lambda_Q \otimes \text{sgn}_Q \cong (S^\lambda_Q)^* \). Hence the assertion of the proposition follows from 2.2(a),(d).

We are now in the position to prove Theorem A.10 below. Recall from Remark A.6 that, as \( s \) varies over the set of standard \( \lambda \)-tableaux, the \( \mathbb{Z} \)-linear maps

\[
\{s\}^*: \begin{cases} S^\lambda_2 \to \mathbb{Z}, \ x \mapsto \langle x | \{s\} \rangle 
\end{cases}
\]

vary over a \( \mathbb{Z} \)-basis of \( S^\lambda_2 \).

A.10 Theorem (Wildon [28]). Let \( \lambda \) be a partition of \( n \). Then there is a \( \mathbb{Z}\mathfrak{S}_n \)-monomorphism

\[
\varphi : (S^\lambda_2)^* \to S^\lambda_2
\]
such that

\[
\varphi\{s\}^* = \sum_{\sigma \in R_s} e_{\sigma s},
\]

for every \( \lambda \)-tableau \( s \).

Proof. Let \( t \) be a fixed \( \lambda \)-tableau, as before. By Lemma A.5 and Lemma A.7, we have \( \mathbb{Z}\mathfrak{S}_n \)-isomorphisms

\[
\tilde{f}_t : M^\lambda_2 / (S^\lambda_2)^{\perp} \to S^\lambda_Q \otimes \text{sgn}_Q \ , \ m + (S^\lambda_2)^{\perp} \mapsto f_t(m)
\]
Now let

\[ \pi \]

and then

\[ Z \]

Furthermore, by Lemma A.8, we have the \( \mathbb{Z} \mathfrak{S}_n \)-monomorphism

\[ (h_t)|_{S^\lambda_Y \otimes \text{sgn}} \rightarrow S^\lambda_Z. \]

We set \( \varphi := (h_t)|_{S^\lambda_Y \otimes \text{sgn}} \circ \bar{f}_t \circ \bar{g}^{-1} \), and show that \( \varphi \) has the desired property (independently of \( t \)). First note that

\[ \varphi(t^*_{|S^\lambda_Z}) = h_t(f_t(t) + (S^\lambda_Z)^0) = h_t(f_t(t)) = h_t(e_t \otimes 1) \]

= \[ \sum_{\sigma \in C_{\pi'}} \sigma e_t = \sum_{\sigma \in R_t} \sigma e_t = \sum_{\sigma \in R_t} e_{\sigma t}. \]

Now let \( u \) be any \( \lambda \)-tableau. Then there is some \( \pi \in \mathfrak{S}_n \) such that \( \pi t = u \), thus also \( \pi \{ t \} = \{ u \} \), and then \( \pi \{ t \}^* = \{ u \}^* \), by A.1(b) Furthermore, \( R_u = R_{\pi t} = \pi R_t \pi^{-1} \). Consequently,

\[ \varphi(u^*_{|S^\lambda_Z}) = \varphi(\pi \{ t \}^*_{|S^\lambda_Z}) = \pi \cdot \varphi(\{ t \}^*_{|S^\lambda_Z}) = \pi \cdot \sum_{\sigma \in R_t} e_{\sigma t} \]

= \[ \sum_{\sigma \in R_t} e_{\sigma t} = \sum_{\sigma \in R_t} e_{\pi \sigma t} = \sum_{\tau \in R_u} e_{\tau u}. \]

Now the assertion of the theorem follows. \( \square \)

A.11 Corollary. Let \( \lambda \in \mathcal{P}(n) \), and let \( R \) be a principal ideal domain that is a flat right \( \mathbb{Z} \)-module. Then there is an \( R \mathfrak{S}_n \)-monomorphism

\[ \varphi^\lambda_R : (S^\lambda_R)^* \rightarrow S^\lambda_Z. \]

Proof. By 2.2(d), we have an \( R \mathfrak{S}_n \)-isomorphism \( (S^\lambda_R)^* = (R \otimes \mathbb{Z} S^\lambda_Z)^* \cong R \otimes \mathbb{Z} (S^\lambda_Z)^* \). Since \( R \) is a flat right \( \mathbb{Z} \)-module, the \( \mathbb{Z} \mathfrak{S}_n \)-monomorphism \( \varphi^\lambda_R \) from Theorem A.10 induces an \( R \mathfrak{S}_n \)-monomorphism \( i_{\lambda R} \otimes \varphi^\lambda_Z : R \otimes \mathbb{Z} (S^\lambda_Z)^* \rightarrow R \otimes \mathbb{Z} S^\lambda_Z = S^\lambda_R. \)

A.12 Example. Suppose that \( \lambda \) is a hook partition of \( n \), that is, \( \lambda = (n - k, 1^k) \), for some \( 0 \leq k \leq n - 1 \).

(a) Let \( t \) be a standard \( \lambda \)-tableau. When writing the standard polytabloid \( e_t \) as a \( \mathbb{Z} \)-linear combination of \( \lambda \)-tableaux, \( \{ t \} \) is the unique standard \( \lambda \)-tabloid occurring with non-zero coefficient; in fact, the coefficient at \( \{ t \} \) is 1. In particular, in this case, we have

\[ \{ t \}_{|S^\lambda_Z}^* = e_t^* \in (S^\lambda_Z)^*, \]

for every standard \( \lambda \)-tableau \( t \).

Recall, moreover, that \( S^\lambda_Z \) is a cyclic \( \mathbb{Z} \mathfrak{S}_n \)-lattice, generated by any (standard) \( \lambda \)-polytabloid \( e_t \). If \( s \) is also a standard \( \lambda \)-tableau, then we have \( s \cdot t = s \), for some \( \sigma \in \mathfrak{S}_n \), hence also \( \sigma \{ t \} = \{ s \} \). This in turn implies \( \sigma \{ t \}^* = \{ s \}^* \), see A.1(b), thus \( e_t^* = \sigma \cdot e_t^* \). This shows that \( (S^\lambda_Z)^* \) is a cyclic \( \mathbb{Z} \mathfrak{S}_n \)-module, generated by \( e_t^* \), for any standard \( \lambda \)-polytabloid \( e_t \). So, by Theorem A.10, the element \( \sum_{\sigma \in R_t} e_{\sigma t} \) then generates a \( \mathbb{Z} \mathfrak{S}_n \)-sublattice of \( S^\lambda_Z \) that is isomorphic to \( (S^\lambda_Z)^* \).

(b) Let \( R \) be a principal ideal domain that is flat as right \( \mathbb{Z} \)-module. For each \( \lambda \)-tableau \( t \), we identify the element \( 1 \otimes e_t \in S^\lambda_R \) simply with \( e_t \) as in 5.2, and then the element \( e_t^* \in (S^\lambda_R)^* \) with \( 1 \otimes e_t^* \in R \otimes (S^\lambda_Z)^* \) via the \( R \mathfrak{S}_n \)-isomorphism in 2.2(d). With this notation, the \( R \mathfrak{S}_n \)-monomorphism \( \varphi^\lambda_R \) in Corollary A.11 again maps \( e_t^* \) to \( \sum_{\sigma \in R_t} e_{\sigma t}. \)
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