THE LIST CHROMATIC INDEX OF SIMPLE GRAPHS WHOSE ODD CYCLES INTERSECT IN AT MOST ONE EDGE

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Abstract. We study the class of simple graphs $G^*$ for which every pair of distinct odd cycles intersect in at most one edge. We give a structural characterization of the graphs in $G^*$ and prove that every $G \in G^*$ satisfies the list-edge-coloring conjecture. When $\Delta(G) \geq 4$, we in fact prove a stronger result about kernel-perfect orientations in $L(G)$ which implies that $G$ is $(m\Delta(G) : m)$-edge-choosable and $\Delta(G)$-edge-paintable for every $m \geq 1$.

1. Introduction

In this paper all graphs are assumed to be simple unless otherwise indicated.

A fundamental characterization of bipartite graphs, proved by König [7], is that a graph is bipartite if and only if it contains no odd cycle. Hsu, Ikura, and Nemhauser [5], and independently Maffray [8], generalized this result, giving the following structural characterization of the class $G_1$ of graphs containing no odd cycles of length longer than 3. Here, a block of a graph is a maximal 2-connected subgraph, and the join $G \lor H$ of two graphs $G$ and $H$ is the graph obtained from their disjoint union by adding all edges between vertices of $G$ and vertices of $H$.

**Theorem 1.1** (Hsu–Ikura–Nemhauser [5], Maffray [8]). A graph $G$ lies in $G_1$ if and only each of its blocks $B$ satisfies one of the following conditions:

- $B$ is bipartite, or
- $B \cong K_4$, or
- $B \cong K_2 \lor K_r$ for some $r \geq 1$.

In this paper we study graphs where some longer odd cycles are allowed. Let $G^*$ be the class of graphs $G$ in which odd cycles intersect in at most one edge, i.e., for any distinct odd cycles $C_1, C_2$ in $G$, we have $|E(C_1) \cap E(C_2)| \leq 1$. Since any two distinct triangles in a graph intersect in at most one edge, we immediately have that $G_1 \subseteq G^*$. Building on Theorem 1.1, our first result is the following structural characterization of the graphs in $G^*$. Here, for positive integers $p_1, \ldots, p_k$, the $\Theta$-graph $\Theta_{p_1,\ldots,p_k}$ is the graph obtained from a pair of vertices $\{x_1, x_2\}$ joined by $k$ internally disjoint paths, with the $i$th path containing $p_i$ edges. (In particular, if some $p_i = 1$ then the corresponding path is just an edge joining $x_1$ and $x_2$.) Figure 1 shows $\Theta_{1,2,4}$.

**Theorem 1.2.** A graph $G$ lies in $G^*$ if and only if each of its blocks $B$ satisfies one of the following conditions:

- $B$ is bipartite, or
- $B \cong K_4$, or
- $B \cong K_2 \lor K_r$ for some $r \geq 1$.

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we say that $G_k$ the smallest nonnegative $G_4$, we actually prove something stronger than the fact that $G$ graph in $f$ is $G$-edge-choosable: we prove a result about orienting $L$-edge-coloring whenever $\ell(e)$ is an independent set of vertices $S$. It is worth noting however, that $G$ graph also works for multigraphs, whereas our proof of Theorem 1.3 does not.

Figure 1. The graph $\Theta_{1,2,4}$.

- $B \cong \Theta_{1,p_1,\ldots,p_r}$, where $p_1, \ldots, p_r$ are even.

Since $K_r \vee K_r \cong \Theta_{1,2,\ldots,2}$, with the 2 repeated $r$ times, we see that the blocks permitted in Theorem 1.2 generalize the blocks permitted in Theorem 1.1, as required by the inclusion $G_1 \subseteq G^*$. We prove this characterization in Section 2.

We use our structural characterization of the graphs in $G^*$ to prove a result about the list chromatic index of these graphs. A list assignment $L$ to the edges of a graph $G$ is a function $\ell$ that assigns each edge $e \in E(G)$ a list of colors $\ell(e)$. (Typically one uses the letter $L$ to denote a list assignment; we use $\ell$ to avoid conflict with the notation $L(G)$ for line graphs.) An $\ell$-edge-coloring of $G$ is a function $\phi$ defined on $E(G)$ such that $\phi(e) \in \ell(e)$ for all $e$ and such that $\phi(e_1) \neq \phi(e_2)$ whenever $e_1, e_2$ are adjacent. A $k$-edge-coloring of $G$ is an $\ell$-edge-coloring for the list assignment with $\ell(e) = \{1, \ldots, k\}$ for all $e \in E(G)$; the chromatic index of $G$, written $\chi'(G)$, is the smallest nonnegative $k$ such that $G$ admits a $k$-edge-coloring. If $f : E(G) \to \mathbb{N}$ and $G$ has a proper $\ell$-edge-coloring whenever $|\ell(e)| \geq f(e)$ for all $e$, we say that $G$ is $f$-edge-choosable. In particular, if $G$ is $f$-edge-choosable when $f(e) = k$ for all $e$, we say that $G$ is $k$-edge-choosable. The list chromatic index of $G$, written $\chi'_l(G)$, is the smallest nonnegative $k$ such that $G$ is $k$-edge-choosable.

It is clear that $\chi'_l(G) \geq \chi'(G)$ for every graph $G$. The list-edge-coloring conjecture (attributed to many sources, some as early as 1975; see [6]) asserts that equality always holds. In a breakthrough result in 1995, Galvin [4] proved the conjecture for bipartite graphs. Petersen and Woodall [9, 10] later extended this to the class $G_1$. Here we prove, in Section 3, that the list-edge-coloring conjecture holds for $G^*$.

Theorem 1.3. If $G \in G^*$, then $\chi'_l(G) = \chi'(G)$.

As $G_1 \subseteq G^*$, Theorem 1.3 extends the work of the above-mentioned authors, in particular by allowing odd cycles of any length. It is worth noting however, that Petersen and Woodall’s proof for $G_1$ (as well as Galvin’s proof for bipartite graphs) also works for multigraphs, whereas our proof of Theorem 1.3 does not.

In looking at Theorem 1.3 recall that Vizing’s Theorem [12] says that every graph $G$ has chromatic index $\Delta(G)$ or $\Delta(G) + 1$. We shall see that every connected graph in $G^*$, aside from odd cycles, satisfies $\chi'_l(G) = \chi'(G) = \Delta(G)$. When $\Delta(G) \geq 4$, we actually prove something stronger than the fact that $G \in G^*$ is $\Delta(G)$-edge-choosable: we prove a result about orienting $L(G)$ which implies results for two generalized notions of choosability. Before describing these, we first discuss the connection between coloring and kernels in digraphs.

An orientation of a graph $G$ is any digraph obtained by replacing each edge $uv \in E(G)$ with the arc $(u,v)$, the arc $(v,u)$, or both of these arcs. A kernel in a digraph $D$ is an independent set of vertices $S$ such that every vertex in $D - S$
has an out-neighbor in $S$. A digraph $D$ is said to be kernel-perfect if every induced subdigraph of $D$, including $D$ itself, has a kernel. Coloring and kernels are linked by the following lemma of Bondy, Boppana and Siegel (see [4]); here we state the lemma for line graphs (edge-coloring) only.

**Lemma 1.4** (Bondy–Boppana–Siegel). If $D$ is a kernel-perfect orientation of a line graph $L(G)$ and $f(e) = 1 + d_D^+(e)$ for all $e \in E(G)$, then $G$ is $f$-edge-choosable.

**Remark.** A word of caution is in order regarding our use of the word “orientation”. In an orientation of a line graph $L(G)$, we explicitly allow the possibility that both of the arcs $(e, f)$ and $(f, e)$ are present, even when $e$ and $f$ are not parallel edges. This possibility is also allowed by Maffray [5]. However, some papers in the literature implicitly forbid such 2-cycles, such as the paper of Borodin, Kostochka, and Woodall [1] generalizing results of [8]. We mention the distinction here in the hope of avoiding future confusion about which notion of “orientation” is used in this paper.

Let $H$ be a graph, and let $f : V(H) \to \mathbb{N}$. We say an orientation of $H$ is $f$-kernel-perfect if it is kernel-perfect and $f(v) \geq 1 + d_D^+(v)$ for all $v \in V(H)$; if $f(v) = k$ for all $v$ then we say $k$-kernel-perfect. We say a graph $G$ is $f$-edge-orientable if $L(G)$ admits an $f$-kernel-perfect orientation. Using this terminology, the above lemma says that $f$-edge-orientability implies $f$-edge-choosability. In Section 3 we prove the following result:

**Theorem 1.5.** If $G \in G^*$ has $t \geq \max\{4, \Delta(G)\}$, then $G$ is $t$-edge-orientable.

Proving $f$-edge-orientability of a graph also implies two properties stronger than $f$-edge-choosability: $(mf : m)$-edge-choosability and $f$-edge-paintability.

An $m$-tuple coloring of a graph $G$ is a function that assigns to each vertex $v \in V(G)$ a set $\phi(v)$ of $m$ colors such that adjacent vertices receive disjoint sets. For $f : V(G) \to \mathbb{N}$, we say that $G$ is $(f : m)$-choosable if $G$ has an $m$-tuple coloring with $\phi(v) \subseteq \ell(v)$ whenever $\ell(v)$ is a list assignment with $|\ell(v)| \geq f(v)$ for all $v$. In particular, $(f : 1)$-choosability is just ordinary $f$-choosability. This extension of list coloring was first introduced by Erdős, Rubin, and Taylor in [6] alongside the usual notion of $f$-choosability.

A longstanding conjecture from [3] is that every $k$-choosable graph is $(km : m)$-choosable for all $m$; in particular, this conjecture together with the list-edge-coloring conjecture would imply that every graph $G$ is $(m\chi'(G) : m)$-edge-choosable for all $m \geq 1$. It was observed by Galvin [4] that the lemma of Bondy, Boppana, and Siegel applies to $m$-tuple coloring as well: in our language, if $G$ is $f$-edge-orientable, then $G$ is $(mf : m)$-edge-choosable for all $m \geq 1$. Thus, our Theorem 1.5 implies that if $G \in G^*$ with $\Delta(G) \geq 4$, then $G$ is $(m\Delta(G) : m)$-edge-choosable.

Paintability, also known as online list coloring, is another extension of list coloring. Paintability was introduced by Schauz [11] and independently by Zhu [13] and is defined by the following 2-player game on a graph $G$.

The online list coloring game is played by two players, Lister and Painter, over several turns. On each turn, Lister marks some nonempty set $M$ of vertices, which models making the same color available at each vertex of $M$. Then, Painter selects an independent set $S \subseteq M$ and deletes all vertices of $S$ from the graph, which models assigning the given color to the vertices of $S$ in a proper coloring of $G$. Lister then marks some subset of the remaining vertices, and the game continues in
this manner until all vertices have been deleted. Given a function \( f : V(G) \to \mathbb{N} \), the graph \( G \) is said to be \( f \)-paintable if Painter has a strategy ensuring that each vertex \( v \) is marked at most \( f(v) \) times before being deleted. When \( f(v) = k \) for all \( v \in V(G) \), we also say \( k \)-paintable.

Given a fixed list assignment \( \ell \), one strategy available to Lister is to select, on the \( i \)th round, all the vertices \( v \) with \( i \in \ell(v) \); if \( G \) lacks an \( \ell \)-coloring, then this strategy wins for Lister. Thus, every \( f \)-paintable graph is necessarily \( f \)-choosable.

Given \( f : E(G) \to \mathbb{N} \), we say that \( G \) is \( f \)-edge-paintable if its line graph \( L(G) \) is \( f \)-paintable. Schauz \cite{01} proved that Lemma \ref{lemma:1} also works to prove \( f \)-edge-paintability, so that every \( k \)-edge-orientable graph is \( k \)-edge-paintable. Hence, Theorem \ref{thm:1} implies that if \( G \in G^* \) with \( \Delta(G) \geq 4 \), then \( G \) is \( \Delta(G) \)-edge-paintable. (In fact, one can also define \((mf : m)\)-paintability in a manner analogous to \((mf : m)\)-choosability, where each vertex must be “deleted” \( m \) times before it is actually removed. Lemma \ref{lemma:1} then also shows that every \( k \)-edge-orientable graph is \((mf : m)\)-edge-paintable.)

In Section \ref{sec:2} we construct a subcubic graph in \( G_1 \) that is not 3-edge-orientable, which shows that the lower bound on \( t \) in Theorem \ref{thm:1} is sharp.

2. Characterizing graphs in \( G^* \)

It is easy to see that a graph \( G \) lies in \( G^* \) if and only if all its blocks lie in \( G^* \). Thus, in order to prove Theorem \ref{thm:2} it suffices to prove the claim for 2-connected graphs.

**Lemma 2.1.** If \( G \) is 2-connected, then \( G \in G^* \) if and only if one of the following conditions holds:

- \( G \) is bipartite, or
- \( G \cong K_4 \), or
- \( G \cong \Theta_{p_1, \ldots, p_r} \), where \( p_1, \ldots, p_r \) are even.

It is easy to check that all graphs satisfying one of these conditions lie in \( G^* \).

In the rest of this section, we show that if \( G \in G^* \) is 2-connected and \( G \) is neither bipartite nor isomorphic to \( K_4 \), then \( G \) is a \( \Theta \)-graph of the desired form.

Fix some graph \( G \in G^* \) and let \( C \) be a longest odd cycle in \( G \); we also use \( C \) to refer to the vertex set of this cycle. If \( |C| = 3 \), then \( G \in G_1 \), so Theorem \ref{thm:1} immediately implies the claim. Thus, we may assume that \( |C| \geq 5 \). Note that \( G \in G^* \) implies that \( C \) has no chords, that is, the induced subgraph \( G[C] \) is a cycle.

Let \( \overline{C} = V(G) - C \). If \( \overline{C} = \emptyset \), then \( G \cong \Theta_{1, |C|-1} \) and the claim holds, so assume that \( \overline{C} \neq \emptyset \). For \( v \in \overline{C} \) and \( w \in C \), say that \( v \) touches \( w \) if there is a \( v, w \)-path that intersects \( C \) only at \( w \). Let \( T(v) = \{ w \in C : v \text{ touches } w \} \). Since \( G \) is 2-connected, we have \( |T(v)| \geq 2 \) for all \( v \in \overline{C} \).

For \( x, y \in C \), an **external x, y-path** is an \( x, y \)-path \( P \) that meets \( C \) only at its endpoints.

**Lemma 2.2.** Let \( v \in \overline{C} \) and let \( H \) be the component of \( G[\overline{C}] \) containing \( v \). If \( x, y \in T(v) \), then there is an external \( x, y \)-path \( P \) such that the internal vertices of \( P \) all lie in \( H \).

**Proof.** Let \( P_x \) be a \( v, x \)-path and let \( P_y \) be a \( v, y \)-path such that \( P_x \) and \( P_y \) each meet \( C \) only at their endpoints, as guaranteed by the definition of \( T(v) \). Now
Figure 2. Two cycles obtained when $T(v)$ is not a clique, for $v \in \overline{C}$. Wavy lines represent paths; thick lines denote cycle edges.

$P_x \cup P_y$ is a connected subgraph of $G$ containing both $x$ and $y$, and the only $C$-vertices contained in $P_x \cup P_y$ are $x$ and $y$, with all other vertices in $H$. Hence there is an $x, y$-path in $P_x \cup P_y$, and such a path is necessarily an external $x, y$-path with all internal vertices in $H$. □

Lemma 2.3. For all $v \in \overline{C}$, the set $T(v)$ is a clique.

Proof. Suppose to the contrary that $T(v)$ contains nonadjacent vertices $w, z \in C$. Let $Q$ be an external $w, z$-path, as guaranteed by Lemma 2.2. Taking $P_1$ and $P_2$ to be the two internally-disjoint $w, z$-paths in $C$, we see that either $Q \cup P_1$ or $Q \cup P_2$ is an odd cycle, as illustrated in Figure 2. Furthermore, both $Q \cup P_1$ and $Q \cup P_2$ intersect $C$ in at least two edges, since $w$ and $z$ are nonadjacent. Thus, if some $T(v)$ is not a clique then $G \notin G^*$.

Lemma 2.4. If $xy \in E(C)$ and $P$ is an external $x, y$-path, then $P$ has an even number of edges.

Proof. If not, then $(C - xy) \cup P$ is an odd cycle that intersects $C$ in at least 2 edges. □

Lemma 2.5. For all $v, w \in \overline{C}$, we have $T(v) = T(w)$.

Proof. If $v, w$ are in the same component of $G[\overline{C}]$, this is clear. Otherwise, let $v$ and $w$ be vertices in different components of $G[\overline{C}]$ such that $T(v) \neq T(w)$. Write $T(v) = \{x, y\}$ and $T(w) = \{s, t\}$. Let $P$ be an external $x, y$-path and let $Q$ be an external $s, t$-path as guaranteed by Lemma 2.2. Every vertex of $V(P) \cap \overline{C}$ is in the same component of $G[\overline{C}]$ as $v$, while every vertex of $V(Q) \cap \overline{C}$ is in the same component of $G[\overline{C}]$ as $w$. It follows that $P$ and $Q$ are internally disjoint.

By Lemma 2.3 we have $\{xy, st\} \subseteq E(C)$, so by Lemma 2.4 the paths $P$ and $Q$ each have an even number of edges. Thus $(C - \{xy, st\}) \cup (P \cup Q)$ is an odd cycle that shares at least $|C| - 2$ edges with $C$. As $|C| \geq 5$, this contradicts $G \notin G^*$. □

Since Lemma 2.3 states that $T$ is constant on $\overline{C}$, in the rest of the proof we let $\{x_1, x_2\}$ be the constant value of $T$. By Lemma 2.3 we have $x_1x_2 \in E(C)$. Our goal now is to prove that every component of $G[\overline{C}]$ is a path with one endpoint adjacent to $x_1$, the other endpoint adjacent to $x_2$, and all internal vertices nonadjacent to $C$. As such a path must have an even number of edges, this clearly implies that $G$ is a $\Theta$-graph having the desired form. Let $H$ be any component of $G[\overline{C}]$. For $v \in V(G)$, let $N_H(v) = N(v) \cap V(H)$.
Lemma 2.6. If \( v \in N_H(x_1) \), then \( H \) has exactly one path \( P \) such that \( P \) is a \( v,w \)-path with \( w \in N_H(x_2) \). By symmetry, the same statement holds when \( x_1,x_2 \) are interchanged.

Proof. Since \( x_2 \in T(v) \), it is clear that at least one such path exists, so we must show that it is unique. Suppose to the contrary that \( P_1 \) and \( P_2 \) are distinct paths in \( H \) such that \( P_i \) is a \( v,w_i \)-path with \( w_i \in N(x_2) \). (Possibly \( w_1 = w_2 \).) Let \( Q_i \) be the path \( x_1 P_i x_2 \), so that \( Q_1 \) and \( Q_2 \) are distinct external \( x_1,x_2 \)-paths. By Lemma 2.4, the paths \( Q_1 \) and \( Q_2 \) each have an even number of edges, so that \( Q_1 + x_1 x_2 \) and \( Q_2 + x_1 x_2 \) are distinct odd cycles whose intersection contains \( \{ x_1 x_2, x_1 v \} \). This contradicts \( G \in \mathcal{G}^* \).

Corollary 2.7. \( |N_H(x_1)| = |N_H(x_2)| = 1 \).

Proof. Since every vertex of \( H \) touches \( x_1 \) and \( x_2 \), each neighborhood is nonempty. If \( |N_H(x_1)| > 1 \), say, then taking \( v \in N_H(x_2) \) and distinct \( w_1, w_2 \in N_H(x_1) \), the connectedness of \( H \) implies that there is a \( v,w_i \)-path for each \( i \). Since these paths have different endpoints, they are distinct, contradicting Lemma 2.6.

Lemma 2.8. Let \( v_1, v_2 \) be the unique neighbors of \( x_1, x_2 \) in \( H \), respectively. The component \( H \) is a \( v_1,v_2 \)-path.

Proof. Let \( P \) be the unique \( v_1,v_2 \)-path in \( H \), as guaranteed by Lemma 2.6. The uniqueness of \( P \) immediately implies that \( P \) is chordless, i.e., \( H[P] = P \). We prove that no vertex of \( P \) has a neighbor outside \( P \), which implies \( H = P \) since \( H \) is connected.

If \( v_1 \) has an \( H \)-neighbor \( w \) outside of \( P \), then \( v_1 \) lies on every \( v,v_2 \)-path, again by the uniqueness of \( P \). Hence \( G - v_1 \) has no \( w,x_2 \)-path, which contradicts the 2-connectedness of \( G \). The same logic applies to \( v_2 \). Similarly, if \( z \) is an internal vertex of \( P \) with some neighbor \( w \notin P \), then \( z \) lies on every \( w,v_1 \)-path, which implies that \( G - z \) is disconnected, again contradicting the 2-connectedness of \( G \).

Lemma 2.8 completes the proof of Theorem 1.2.

3. PROOF OF THEOREMS 1.3 AND 1.5

Petersen and Woodall [9, 10] proved that a graph \( G \in \mathcal{G}_1 \) is \( \Delta(G) \)-edge-choosable by appealing to the characterization of Theorem 1.1 and defining a locally-stronger notion of edge-choosability that allowed them to prove their result block by block. We extend their approach, defining a locally-stronger notion of edge-orientability, and show that if this stronger definition holds for each block of \( G \), then \( G \) itself is \( k \)-edge-orientable.

A graph \( G \) is strongly \( k \)-edge-orientable if for every \( v \in V(G) \), the graph \( G \) is \( f_{k,v} \)-edge-orientable, where \( f_{k,v} \) is defined by

\[
f_{k,v}(e) = \begin{cases} 
d(v), & \text{if } e \text{ is incident to } v, \\
k, & \text{otherwise.}
\end{cases}
\]

for all \( e \in E(G) \). We call this a strengthening of \( k \)-edge-orientability (defined in the introduction) because we are always interested in \( k \geq \Delta(G) \). (Recall that \( \chi'_k(G) \geq \chi'(G) \geq \Delta(G) \) for every \( G \), with \( k \)-edge-orientability implying \( k \)-edge-choosability.)
When discussing an orientation $D$ of some line graph $L(G)$, we will write $e \to f$ both to stand for the arc $(e, f)$ and as shorthand for the statement that $(e, f) \in E(D)$. We also write $e \leftrightarrow f$ as shorthand for “$e \to f$ and $f \to e$”.

**Lemma 3.1.** Let $H$ be a graph, and let $(X, Y)$ be a partition of $V(H)$. Let $D_{X}$ and $D_{Y}$ be kernel-perfect orientations of $H[X]$ and $H[Y]$, respectively. If $D$ is the orientation of $H$ obtained from $D_{X} \cup D_{Y}$ by orienting every edge of $[X, Y]$ from its endpoint in $Y$ to its endpoint in $X$, then $D$ is a kernel-perfect orientation of $H$.

**Proof.** To see that $D$ is kernel-perfect, let $Z$ be any subset of $V(H)$; we must show that $D[Z]$ has a kernel. Let $Z_X = Z \cap X$. Since $D_X$ is kernel-perfect, $D[Z_X]$ has some kernel $S_X$. (Possibly $Z_X = \emptyset$, in which case we can take $S_X = \emptyset$.) Define $$Z_Y = \{v \in Z \cap Y : N(v) \cap S_X = \emptyset\}.$$ As $D_Y$ is kernel-perfect, $Z_Y$ has some kernel $S_Y$. (As before, if $Z_Y = \emptyset$, then we take $S_Y = \emptyset$.)

Let $S = S_X \cup S_Y$. We claim that $S$ is a kernel for $Z$. Clearly $S \subseteq Z$, and clearly $S$ is independent, since $S_X$ and $S_Y$ are independent, and $N(v) \cap S_X = \emptyset$ for all $v \in S_Y$. It remains to show that every $w \in Z - S$ has an out-neighbor in $S$. If $w \in X$, then we have $w \in Z_X$, so $w$ has an out-neighbor in $S_X$, since $S_X$ is a kernel for $D[Z_X]$. Otherwise, $w \in Y$. If $N(w) \cap S_X \neq \emptyset$, then since all edges in $[X, Y]$ are oriented from their endpoint in $Y$ to their endpoint in $X$, we see that $w$ has an out-neighbor in $S_X$. On the other hand, if $N(w) \cap S_X = \emptyset$, then $w \in Z_Y$, so $w$ has an out-neighbor in $S_Y$, since $S_Y$ is a kernel for $D[Z_Y]$. This completes the proof. \[\square\]

**Lemma 3.2.** If $k \geq \Delta(G)$ and every block of $G$ is strongly $k$-edge-orientable then $G$ is strongly $k$-edge-orientable.

**Proof.** It suffices to show that if $V(G) = A \cup B$ where $A \cap B = \{z\}$, $G[A]$ and $G[B]$ are strongly $k$-edge-orientable, and there are no edges from $A - z$ to $B - z$, then $G$ is strongly $k$-edge-orientable.

Let any $v \in V(G)$ be given; we must show that $G$ is $f_{k, v}$-edge-orientable. Without loss of generality, suppose that $v \in A$. (It is possible that $v = z$, in which case $v \in B$ as well.) Let $D_A$ and $D_B$ be kernel-perfect orientations of $L(G[A])$ and $L(G[B])$, respectively, such that

$$d_{D_A}^+(e) + 1 \leq k$$
for all $e \in E(A)$,

$$d_{D_A}^+(e) + 1 \leq d_{G[A]}(v) \leq k$$
for all $e \in E(A)$ with $v \in e$,

$$d_{D_B}^+(e) + 1 \leq k$$
for all $e \in E(B)$, and

$$d_{D_B}^+(e) + 1 \leq d_{G[B]}(z) \leq k$$
for all $e \in E(B)$ with $z \in e$.

Such orientations exist because $G[A]$ and $G[B]$ are strongly $k$-edge-orientable, and since $d_{G[A]}(v), d_{G[B]}(z) \leq \Delta(G) \leq k$.

The orientations $D_A$ and $D_B$ give a direction for every edge of $L(G)$ except for the edges between $E(A)$ and $E(B)$, all of which correspond to a pair of $G$-edges incident to $z$. Let $D$ be the orientation that agrees with $D_A$ on $E(A)$, agrees with $D_B$ on $E(B)$, and orients every edge between $E(A)$ and $E(B)$ from the edge in $E(B)$ to the edge in $E(A)$. By Lemma 3.1 (applied to the graph $H = L(G))$, the orientation $D$ is kernel-perfect. We claim that $d_{D}^+(e) + 1 \leq k$ for all $e \in E(G)$ and that $d_{D}^+(e) + 1 \leq d(v)$ for all $e$ incident to $v$. \[\square\]
Thus, 

Theorem 1.2. Two of these block types, bipartite graphs and as desired.

Hence the outdegree of \( e \) incident to \( L \) the incident edges have a linear order in our orientation. Hence every clique in \( e \) the color of \( i \) under \( \phi \) hence we can avail ourselves of the following characterization due to Maffray.

\[ d^+_D(e) = d^+_D(e) + d_A(z) \leq d_B(z) - 1 + d_A(z) \leq \Delta(G) - 1 \leq k - 1. \]

Thus, \( d^+_D(e) \leq k - 1 \) for all \( e \in E(G) \).

Now suppose that \( e \) is incident to \( v \). If \( e \in E(A) \), then we immediately have \( d^+_B(e) = d^+_B(e) \leq d(v) - 1 \). If \( e \in E(B) \), then we must have \( v = z \), and so

\[ d^+_B(e) = d^+_B(e) + d_A(v) \leq d_B(v) - 1 + d_A(v) = d_G(v) - 1, \]

as desired. \( \square \)

We now proceed to show orientability for each of the block types present in Theorem 1.2. Two of these block types, bipartite graphs and \( K_4 \), lie in \( G_1 \), and hence we can avail ourselves of the following characterization due to Maffray.

Theorem 3.3 (Maffray [8]). If \( G \in G_1 \) then an orientation of \( L(G) \) is kernel-perfect if and only if every clique in \( L(G) \) has a sink.

For bipartite graphs, the standard proof of Galvin [4] is all that is needed to prove the orientability lemma we need. We briefly include the details below.

Lemma 3.4. Every bipartite graph \( G \) is strongly \( \Delta(G) \)-edge-orientable.

Proof. Let \( G \) be a bipartite multigraph, let \( k = \Delta(G) \), and take any \( v \in V(G) \). We show that \( G \) is \( f_{k,v} \)-edge-orientable, i.e., that \( L(G) \) admits an \( f_{k,v} \)-kernel-perfect orientation.

Since \( G \) is bipartite, it is \( k \)-edge-colorable. Fix such a coloring \( \phi \) where the edges incident to \( v \) have colors \( 1, \ldots, d(v) \). Fix a bipartition \( (X, Y) \) of \( G \) where \( v \in X \). Orient the edges of \( L(G) \) according to the edge-coloring of \( G \): orient from lower color to higher color if the common endpoint lies in \( X \), and from higher color to lower color if the common endpoint lies in \( Y \).

Since \( G \) is triangle-free, every clique in \( L(G) \) corresponds to a vertex in \( G \), and the incident edges have a linear order in our orientation. Hence every clique in \( L(G) \) has a sink, and since \( G \in G_1 \), Theorem 3.3 implies that the orientation is kernel-perfect.

For any edge \( e \in E(G) \), each out-neighbor of \( e \) in \( L(G) \) receives a different color under \( \phi \): higher colors for incidences in \( X \), and lower colors for incidences in \( Y \). Hence the outdegree of \( e \) in our orientation is at most \( k - 1 \).

Now consider an edge \( e \in E(G) \) that is incident to \( v \). We must show that the outdegree of \( e \) in our orientation of \( L(G) \) is at most \( d(v) \). By our choice of coloring, the color of \( e \) is \( i \) for some \( i \in \{1, \ldots, d(v)\} \). As \( v \in X \), the edge \( e \) has at most \( d(v) - i \) neighbors of higher color with their common endpoint in \( X \), and at most \( i - 1 \) neighbors of lower color with their common endpoint in \( Y \), yielding a total of at most \( d(v) - 1 \) out-neighbors, as desired. \( \square \)

Lemma 3.5. \( K_4 \) is strongly 4-edge-orientable.

Proof. Consider the orientation of \( L(K_4) \) shown in Figure 3. There are no induced directed triangles, and hence by Theorem 3.3 the orientation is kernel-perfect. The orientation has maximum outdegree 3, and satisfies \( d^+(e) \leq d(v) - 1 = 2 \) for \( e \) incident to \( v \). \( \square \)
Lemma 3.6. Let \( q_1 \leq \cdots \leq q_r \) be positive even integers, and let \( G = \Theta_{q_1, \ldots, q_r} \). If \( p \geq \max\{4, \Delta(G)\} \), then \( G \) is strongly \( p \)-edge-orientable.

We prove Lemma 3.6 by splitting into three cases: two cases in which \( G \cong K_2 \lor K_k \) for some \( k \), as allowed in Theorem 1.1, and one case in which \( G \) is a larger theta-graph. We state each case as its own lemma, since we will need to refer to some of these results individually later.

Lemma 3.7. Let \( G \) be the graph \( K_2 \lor K_k \), and let \( v \) be a vertex of maximum degree. If \( p \geq \max\{3, \Delta(G)\} \), then \( G \) is \( f_{p,v} \)-edge-orientable.

Proof. If \( \Delta(G) = 2 \), then \( G = K_3 \) and a transitive orientation of \( L(G) \) suffices (since \( p \geq 3 \)). Hence we may assume that \( \Delta(G) \geq 3 \), so that there are only two vertices of maximum degree.

Let \( u_1, u_2 \) be the vertices of maximum degree, and let \( G' \) be the graph obtained from \( G \) by deleting the edge \( u_1u_2 \). Observe that \( \Delta(G') = \Delta(G) - 1 \). We will find an orientation of \( L(G') \) and then extend this orientation to get the desired orientation of \( L(G) \).

As \( G' \) is bipartite, Lemma 3.4 implies that \( L(G') \) admits a kernel-perfect orientation \( D' \) such that:

- \( d_{D'}(e) \leq \Delta(G') - 1 = \Delta(G) - 2 \) for all \( e \in E(G') \), and
- \( d_{D'}(v) \leq d_{G'}(v) - 1 \) for all \( e \in E(G') \) with \( v \in e \).

Now we extend \( D' \) to an orientation \( D \) of \( L(G) \). The only edges of \( L(G) \) that remain to orient are the edges joining \( u_1u_2 \) with the edges of \( G' \); orient all of these towards their endpoint \( u_1u_2 \). The resulting orientation is kernel-perfect, since if an induced subdigraph of \( L(G) \) contains \( K_1, u_1u_2 \), the single vertex \( u_1u_2 \) is a kernel; this also follows from Lemma 3.1. Note that every edge aside from \( u_1u_2 \) (which has outdegree 0) has gained outdegree exactly 1. Since \( d_{G'}(v) = d_{G}(v) - 1 \), the degree bound is also satisfied. \( \square \)

Lemma 3.8. Let \( G \) be the graph \( K_2 \lor K_k \), and let \( v \) be a vertex of degree 2. If \( p \geq \max\{4, \Delta(G)\} \), then \( G \) is \( f_{p,v} \)-edge-orientable.

Proof. As before, we may assume that \( k > 1 \) so that \( G \not\cong K_3 \). Let \( u_1, u_2 \) be the vertices in the copy of \( K_2 \), and let \( G' \) be the graph obtained from \( G \) by deleting
Also as before, we may take $D'$ to be a kernel-perfect orientation of $G'$ such that:

- $d_{D'}^+(e) \leq \Delta(G') - 1 = \Delta(G) - 2$ for all $e \in E(G')$, and
- $d_{D'}^-(e) \leq d_G^-(v) - 1$ for all $e \in E(G')$ with $v \in e$.

Now we extend $D'$ to an orientation $D$ of $L(G)$. The only edges of $L(G)$ that remain to orient are the edges joining $u_1u_2$ with the edges of $G'$. Call these the 

undetermined edges.

Since $d_{D'}^+(u_iv) \leq 1$ for $i \in \{1, 2\}$, and since either $u_1v \to u_2v$ or $u_2v \to u_1v$, we see that there are at most one edge $e^* \in E(G')$ such that $v \notin e^*$ and $u_iv \to e^*$ for some $i$.

Now we orient the undetermined edges. If $f \neq u_1u_2$ is incident to $u_i$ for some $i \in \{1, 2\}$, then:

- If $v \in f$, we orient $u_1u_2 \to f$.
- If $f = e^*$, we orient $u_1u_2 \leftrightarrow f$.
- Otherwise, we orient $f \to u_1u_2$.

Observe that $d_{D'}^+(u_1u_2) \leq 3 \leq p - 1$. Next we consider the outdegree of each vertex in $D'$ as compared to its outdegree in $D$. The edges incident to $v$ have not gained any more outdegree, so $d_{D'}^+(v) \leq d(v) - 1$ for all edges $e$ incident to $v$. The other edges have gained outdegree at most one, so $d_{D'}^+(e) \leq (\Delta(G') - 1) + 1 = \Delta(G) - 1$ for such edges.

We claim that $D$ is kernel-perfect. Since $G \in G_1$, by Theorem 3.3 it again suffices to prove that every clique has a sink. Since $D'$ is kernel-perfect, we need only consider a clique of the form $K \cup \{u_1u_2\}$ where $K$ is a clique in $D'$; suppose $e$ is a sink of $K$ in $D'$. If $u_1u_2 \to e$, then $e$ remains a sink. So suppose that $u_1u_2 \neq e$. We claim that in this case $u_1u_2$ is the required sink. If not, there exists $f$ in $K$ with $f \neq u_1u_2$. This implies that $v \in f$. On the other hand, since $u_1u_2 \neq e$, we have $v \notin e$ and $e \neq e^*$. In particular, since $e \neq e^*$, we have $f \neq e$. This contradicts the assumption that $e$ is a sink in $K$. \qed

**Lemma 3.9.** Let $G$ be the theta-graph $\Theta_{q_1, \ldots, q_r}$. If $\max_i q_i > 2$ and $p \geq \min\{3, \Delta(G)\}$, then $G$ is strongly $p$-edge-orientable.

**Proof.** We prove the theorem by induction on $r$, with trivial base case when $r = 1$ and $G$ is an odd cycle (in this case $\Delta(L(G)) = 2$). We may assume that $q_1 \leq \cdots \leq q_r$, with $q_r > 2$. Let any $v \in V(G)$ be given; we produce a $f_{p,v}$-kernel-perfect orientation of $L(G)$.

First suppose that there is some edge $e^*$ with $d_{L(G)}(e^*) = 2$ and $v \notin e^*$. This implies that both endpoints of $e$ have degree $2$ in $G$. Let $G' = G - e^*$. The graph $G'$ may no longer be $2$-connected; its blocks consist of a $\Theta$-graph $\Theta_{q_1', \ldots, q_r'}$ and possibly some blocks isomorphic to $K_2$. We will show that $G'$ is strongly $p$-edge-orientable. By Lemma 3.2 it suffices to show that the smaller $\Theta$-graph is strongly $p$-edge-orientable, since the $K_2$-blocks clearly are.

**Case 1:** $r = 2$. In this case, the smaller $\Theta$-graph is an odd cycle, and is therefore strongly $p$-edge-orientable since $p \geq 3$.

**Case 2:** $r \geq 3$ and $\max_i q'_i > 2$. In this case, the induction hypothesis immediately implies that the smaller $\Theta$-graph is strongly $p$-edge-orientable.

**Case 3:** $r \geq 3$ and all $q'_i = 2$. In this case, the smaller $\Theta$-graph is $K_2 \vee K_{r-1}$, and in particular has maximum degree $r$. Since $p \geq \Delta(G) = r + 1 \geq 4$, Lemmas 3.7 and 3.8 imply that the smaller $\Theta$-graph is strongly $p$-edge-orientable.
In all three cases, we have argued that $G'$ is strongly $p$-edge-orientable. Let $D'$ be a $f_{p,v}$-kernel-perfect-orientation of $G'$. We extend $D'$ to a $f_{p,v}$-kernel-perfect-orientation of $G$ by orienting both $L(G)$-edges $e^*f$ as $e^* \rightarrow f$. This orientation gives $e^*$ outdegree 2, and it is kernel-perfect since $D'$ was kernel-perfect (by Lemma 3.1). Thus, $G$ is $f_{p,v}$-edge-orientable.

Now suppose no such edge $e^*$ exists. Together with our earlier assumptions, this immediately implies that $q_1 = \ldots = q_{r-1} = 2$, that $q_r = 4$, and that $v$ is the middle vertex of the defining path with 4 edges, as shown in Figure 4.

Since $r \geq 2$, we have $\Delta(G) \geq 3$, so there are exactly two vertices of maximum degree. Let $e$ be the edge joining the two vertices of maximum degree, and let $G' = G - e$. The graph $G'$ is bipartite, so by Lemma 3.4 we can find a $f_{\Delta(G)-1,v}$-edge-orientable orientation $D'$ of $L(G')$. Extend $D'$ to an orientation $D$ of $G$ by making $e$ a sink; by Lemma 3.1 the orientation $D$ is kernel-perfect. Extending $D'$ to $D$ in this manner does not increase the outdegree of any edge incident to $v$, and every edge not incident to $v$ gains at most one new out-neighbor. Since the maximum degree also increased by one, $G$ is $f_{p,v}$-edge-orientable.

We can now prove Theorems 1.5 and 1.3.

Proof of Theorem 1.5. By Theorem 1.2, every block of such a graph is either bipartite, isomorphic to $K_4$, or isomorphic to $\Theta_{\ell_1,\ldots,\ell_r}$ where $\ell_1,\ldots,\ell_r$ are even. We have proved that all such graphs are strongly $t$-edge-orientable, so by Lemma 3.2 $G$ is strongly $t$-edge-orientable.

Proof of Theorem 1.3. If $\Delta(G) \geq 4$ the result follows immediately from Theorem 1.5, so it suffices to prove the result for subcubic graphs. If $G = K_4$ we refer to the reader to the proof by Cariolaro and Lih [2] to see that $G$ is 3-edge-choosable. If $\Delta(G) \leq 2$, then either $G$ has a component which is an odd cycle, or else $G$ is bipartite; either way, $G$ is $\chi'(G)$-edge-choosable. Hence we may assume that $\Delta(G) = 3$.

For the remaining cases, we use induction on the number of blocks in $G$. Our base case (when $G$ is 2-connected) will be handled as a special case of the induction step.

If $G$ is 2-connected (base case), then let $H = G$ and let $v$ be an arbitrary vertex of $G$. If $G$ is not 2-connected (induction step), then let $H$ be a leaf block of $G$ and let $v$ be the cut vertex in $H$.

First suppose that $H$ is $f_{3,v}$-edge-orientable. If $G$ is 2-connected, then this implies that $G$ is 3-edge-choosable, as desired. If $G$ is not 2-connected, then let $\ell$ be the...
any edge list assignment on $G$ with $|\ell(e)| \geq 3$ for all $e$. Let $G' = G - (V(H) - v)$. By the induction hypothesis, $G'$ is $\ell$-choosable. Taking any $\ell$-coloring of $G'$, we see that at most $3 - d_H(v)$ colors appear on the edges of $G'$ incident to $v$. Deleting these colors from the $H$-edges incident to $v$ gives a list assignment $\ell^*$ with $|\ell^*(e)| \geq f_{3,v}(e)$ for all $e \in E(H)$. Since $H$ is $f_{3,v}$-edge-orientable, we can properly color its edges using the colors from $\ell^*$, allowing us to extend the $\ell$-coloring of $G'$ to all of $G$.

It remains to consider the case where $H$ is not $f_{3,v}$-edge-orientable. By Lemma 3.4, Lemma 3.7, and Lemma 3.9, this implies that $H = K_2 \vee K_2$ and $v$ is a vertex of degree 2. We will show that in this case, $H$ is $f_{3,v}$-choosable, even though it is not $f_{3,v}$-orientable. Figure 5(a) shows the list sizes given by $f_{3,v}$. Let $\ell$ be any edge list assignment on $H$ with $|\ell(e)| \geq f_{3,v}(e)$ for all $e$.

Let $\{e_1, f_1\}$ and $\{e_2, f_2\}$ be disjoint matchings in $H$, with each $e_i$ incident to $v$, and let $h$ be the remaining edge of $H$, as shown in Figure 5(b).

First suppose that $\ell(e_i) \cap \ell(f_i) \neq \emptyset$ for some $i \in \{1, 2\}$. Assigning the common color to both $e_i$ and $f_i$ and deleting this color from all other lists, the remaining uncolored edges form a path on 3 vertices with edge list sizes 2, 2, 1. We complete the $\ell$-coloring by coloring this path greedily starting with the list of size 1.

Thus, we may assume that $\ell(e_i) \cap \ell(f_i) = \emptyset$ for $i \in \{1, 2\}$. In this case, assign the edge $h$ a color from $\ell(h)$ not appearing in $\ell(e_i)$ (possible since $|\ell(h)| > |\ell(e_1)|$), and remove this color from all lists in which it appears. At worst, this removes a
4. Sharpness

**Lemma 4.1.** Let \( H = K_2 \lor K_2 \), and let \( v \) be a vertex of degree 2 in \( H \). There is no \( f_3,v \)-kernel-perfect-orientation of \( L(H) \).

**Proof.** Suppose to the contrary that \( D \) is such an orientation. We start by observing that there are 8 edges in \( L(H) \) and that \( \sum_{e \in E(H)} f_3,v(e) = 8 \), that is, the bounds on the outdegree in \( D \) are exactly large enough to orient each edge of \( L(H) \). In particular, this implies that no edge in \( L(H) \) can be bidirected in \( D \), since this would cause the total outdegree over all the \( H \)-edges to exceed 8.

Let \( Q_1 \) be the clique in \( L(H) \) consisting of the two edges incident to \( v \), and let \( vw \) be a source in \( Q_1 \). In an \( f_3,v \)-kernel-perfect-orientation of \( L(H) \), we must have \( d^+(vw) = 1 \), so every \( L(H) \)-edge \((vw)f\) for \( f \notin Q_1 \) must be oriented as \( f \rightarrow vw \). Let \( z \) be the vertex of degree 3 distinct from \( w \), as shown in Figure 7.

Let \( Q_2 \) be the clique of \( L(H) \) consisting of the edges incident to \( z \), and let \( e \) be a source in \( Q_2 \). Since \( d^+(vz) \leq 1 \) (since we have a \( f_3,v \)-kernel-perfect-orientation), we have \( e \neq vz \). Furthermore, since \( d^+(vw) \leq 1 \) and \( vw \rightarrow vz \), we must have \( wz \rightarrow vz \), since otherwise we create a directed triangle in the line graph. Since \( vw \) and \( vz \) are both out-neighbors of \( wz \), there are no other out-neighbors of \( wz \). Hence, if \( u \) is the final vertex in the graph, we know that \( uw, uz \rightarrow wz \). In particular, this means that \( e = uz \), and hence that \( uz \rightarrow vz \). However, now both \( uz \) and \( uw \) have two out-neighbors outside \( \{uw, uz\} \). Since also we must have either \( uz \rightarrow uw \) or \( uw \rightarrow uz \), we get a contradiction. \( \square \)

**Theorem 4.2.** If \( G \) is the graph shown in Figure 8, then \( G \) is not 3-edge-orientable.
Figure 8. A subcubic non-3-edge-orientable graph in $G_1$.

Proof. Suppose to the contrary that $D$ is an orientation of $L(G)$ that is kernel-perfect and has outdegrees at most 2. Let $Q$ be the clique in $L(G)$ consisting of the edges incident to the labeled vertex $u$, let $uv$ be a source in $Q$, and let $H$ be the copy of $K_2 \vee K_2$ containing the vertex $v$. Since $d^+(uv) \leq 2$, both edges $e \in E(H)$ incident to $v$ must have the arc $e(uv)$ oriented as $e \to uv$. Hence, these edges have outdegree at most 1 in the subdigraph $D[E(H)]$. Thus, $D[E(H)]$ is an $f_{3,v}$-kernel-perfect orientation of $L(H)$. By Lemma 4.1, no such orientation of $L(H)$ exists, yielding a contradiction. 

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