New proof of Weyl’s theorem

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Abstract

Let \( lu = -u'' + q(x)u \), where \( q(x) \) is a real-valued \( L^2_{\text{loc}}(0, \infty) \) function. H. Weyl has proved in 1910 that for any \( z, \text{Im}z \neq 0 \), the equation \( (l - z)w = 0, \ x > 0 \), has a solution \( w \in L^2(0, \infty) \).

We prove this classical result using a new argument.

1 Introduction

Let \( lu = -u'' + q(x)u \), where \( q(x) \in L^2_{\text{loc}} \) is a real-valued function. Fix an arbitrary complex number \( z, \text{Im}z > 0 \), and consider the equation

\[
lw - zw = 0, \quad x > 0
\]  

(1.1)

H. Weyl proved \( [3] \) that equation (1.1) has a solution \( w \in L^2(0, \infty) \), which is called a Weyl’s solution. He gave the limit point-limit circle classification of the operator \( l \): if equation (1.1) has only one solution \( w \in L^2(0, \infty) \), then it is a limit point case, otherwise it is a limit circle case.

Weyl’s theory is presented in several books, e.g. in \( [4], [3] \). This theory is based on some limiting procedure \( b \to \infty \) for the solutions to (1.1) on a finite interval \( (0, b) \). In \( [3] \) a nice different proof is given for continuous \( q(x) \).

The aim of our paper is to give a new method for a proof of Weyl’s result.

Theorem 1.1. Equation (1.1) has a solution \( w \in L^2(0, \infty) \).

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Let us outline the new approach and the steps of the proof.

Since \( q(x) \) is a real-valued function, symmetric operator \( l_0 \) defined on a linear dense subset \( C_0^\infty(0, \infty) \) of \( H = L^2(0, \infty) \) by the expression \( lu = -u'' + q(x)u \) has a selfadjoint extension, which we denote by \( l \). Therefore the resolvent \((l - z)^{-1}\) is a bounded linear operator on the Hilbert space

\[
H = L^2(0, \infty), \quad \|(l - z)^{-1}\| \leq |\text{Im}z|^{-1}.
\]

This operator is an integral operator with the kernel \( G(x, y; z) \), which is a distribution satisfying the equation

\[
(l - z)G(x, y; z) = \delta(x - y), \quad G(x, y; z) = G(y, x; z).
\]

We will prove that

\[
\int_0^\infty |G(x, y; z)|^2 dy \leq c(x; z) \quad \forall x \in (0, \infty), \quad \text{Im}z > 0,
\]

where \( c(x; z) = \text{const} > 0 \).

The kernel \( G(x, y; z) \), which is the Green function of the operator \( l \), can be represented as

\[
G(x, y; z) = \varphi(y; z)w(x; z), \quad x > y,
\]

where \( w \) and \( \varphi \) are linearly independent solution to (1.1), so that \( w(x; z) \neq 0 \). From (1.3) it follows that

\[
w(x; z) \in L^2(0, \infty).
\]

A detailed proof is given in section 2.

One may try to prove the existence of a Weyl’s solution as follows: take an \( h \in L^1_{\text{loc}}(0, \infty) \), \( h = 0 \) for \( x > R \), \( h \neq 0 \), and let \( W := W(x, z) := (l - z)^{-1}h, \, \text{Im}z > 0 \). Then \( W \) solves (1.1) for \( x > R \) and \( W \in L^2(0, \infty) \) since \( l \) is a selfadjoint operator in \( H \). However, one has to prove then that \( W \) does not vanish identically for \( x > R \), and this will be the case not for an arbitrary \( h \) with the above properties. In our paper the role of \( h \) is played by the delta-function, and since \( \varphi(y; z) \) and \( w \) in (1.4) are linearly independent solutions of (1.1), one concludes that \( w \) does not vanish identically.

2 Proofs

Lemma 2.1. If \( q(x) \in L^1_{\text{loc}}(0, \infty) \) and \( q(x) \) is real-valued, then symmetric operator

\[
l_0u := -u'' + q(x)u, \quad D(l_0) = \{ u : u \in C_0^\infty(0, \infty), \quad l_0u \in H := L^2(0, \infty) \}
\]

is defined on a linear dense in \( H \) subset, and admits a selfadjoint extension \( l \).
Proof. This result is known: the density of the domain of definition of the symmetric operator $l_0$ mentioned in Lemma 1 and the existence of a selfadjoint extension are proved in [2]. The defect indices of $l_0$ are (1,1) or (2,2), so that by von Neumann extension theory $l_0$ has selfadjoint extensions (see [2]). Actually we assume in the Appendix that $q \in L^2_{\text{loc}}(0, \infty)$, in which case the conclusion of Lemma 2.1 is obvious: $C^\infty_0(0, \infty)$ is the linear dense subset in $H$ on which $l_0$ is defined. ◻

Let $l$ be a selfadjoint extension of $l_0$, $(l - z)^{-1}$ be its resolvent, $Imz > 0$, and $G(x, y; z)$ be the resolvent’s kernel (in the sense of distribution theory) of $(l - z)^{-1}$, $G(x, y; z) = G(y, x; z)$.

**Lemma 2.2.** For any fixed $x \in [0, \infty)$ one has

$$\left( \int_0^\infty |G(x, y; z)|^2 \, dy \right)^{\frac{1}{2}} \leq c, \quad c = c(x; z) = \text{const} > 0. \quad (2.1)$$

Proof. Let $h \in C^\infty_0(0, \infty)$ and $u := (l - z)^{-1} h$, so

$$u(x; z) = \int_0^\infty G(x, y; z) h(y) \, dy, \quad (l - z)u = h. \quad (2.2)$$

Let us prove that:

$$|u(x; z)| \leq c(x; z) \|h\|, \quad (2.3)$$

where $x \in [0, \infty)$ is an arbitrary fixed point, $c(x) = \text{const} > 0$, $\|h\| := \|h\|_{L^2(0, \infty)}$.

If (2.3) is proved, then

$$|(G(x, y; z), h)| \leq c(x; z) \|h\|. \quad (2.4)$$

From (2.4) the desired conclusion (2.1) follows immediately by the Riesz theorem about linear functionals in $H$.

To complete the proof, one has to prove estimate (2.3).

This estimate follows from the inequality:

$$\|u\|_{C(D_1)} \leq c \left( \|u'' + q(x)u - zu\|_{L^2(D_2)} + \|u\|_{L^2(D_2)} \right) \leq c \left( 1 + \frac{1}{|Imz|} \right) \|h\|, \quad (2.5)$$

where $c = c(D_1, D_2) = \text{const} > 0$, $D_1 \subset D_2$, $D_2 \subset [0, \infty)$, $D_1$ is a strictly inner open subinterval of $D_2$.

Indeed, since $l$ is selfadjoint, (2.2) implies:

$$\|u\| \leq \frac{\|h\|}{|Imz|}. \quad (2.6)$$
Moreover
\[-u'' + qu - zu = h, \quad (2.7)\]
so, using (2.6), one gets:
\[
\|u\|_{L^2(D_2)} + \|u'' + qu - zu\|_{L^2(D_2)} \leq \frac{\|h\|}{|Im z|} + \|h\| \leq \left(1 + \frac{1}{|Im z|}\right) \|h\|, \quad (2.8)
\]
From (2.5), (2.6) and (2.8) one gets (2.3).

Let us finish the proof by proving (2.5).

In fact, inequality (2.5) is a particular case of the well-known elliptic estimates (see e.g. [1, pp. 239-241]), but an elementary proof of (2.5) is given below in the Appendix.

Lemma 2 is proved.

\[\square\]

Proof of Theorem 1.1

Equation (1.2) implies that
\[G(x, y; z) = \varphi(x; z)w(y; z), \quad y \geq x,\]
where \(w(y; z)\) solves (1.1), and the function \(\varphi(x; z)\) is also a solution to (1.1). Inequality (2.1) implies \(w \in L^2(0, \infty)\) if \(Im z > 0\).

Theorem 1.1 is proved. \[\square\]

To make this paper self-contained we give an elementary proof of inequality (2.5) in the Appendix. This proof allows one to avoid reference to the elliptic inequalities [4], the proof of which in [4] is long and complicated (in [4] the multidimensional elliptic equations of general form are studied, which is the reason for the complicated argument in [4]).

Appendix: An elementary proof of inequality (2.5).

Since \(u(x) \in C^1_{loc}(0, \infty)\) it is sufficient to prove (2.5) assuming that \(D_1 = (a, b)\) and \(b - a\) is arbitrarily small. Let \(\eta(x) \in C^\infty_0(a, b)\) be a cut-off function, \(0 \leq \eta \leq 1, \eta(x) = 1\) in \((a + \delta, b - \delta), 0 < \delta < \frac{b-a}{4}\), \(\eta(x) = 0\) in a neighborhoods of points \(a\) and \(b\).

Let \(v = \eta u\). Then (2.2) implies:
\[lv = \eta h - 2\eta' u' - \eta'' u, \quad v(a) = v'(a) = 0.\]
Thus
\[v'' = qv - zu - \eta h + \eta'' u + 2\eta' u', \quad (A.1)\]
and
\[
|v(x)| = \left|\int_a^x (x - s)v''(s)ds\right| \leq c_1 \int_a^b [qv] + |z||v| ds + c_2, \\
\int_a^b |h| ds + c_2 \int_a^b |u| ds + c_2 \int_a^b |u'| ds.
\]
(A.2)
Here
\[ c_1 = b - a, \quad c_2 = \max_{a \leq x \leq b} [\eta(x)] + 2|\eta'|. \]

If \( b - a \) is sufficiently small, then
\[ c_1 \int_a^b (|q| + |z|) \, dx \max_{a \leq x \leq b} |v(x)| < \gamma \max_{a \leq x \leq b} |v(x)|, \quad 0 < \gamma < 1. \]

Therefore (A.1) implies
\[ \max_{a \leq x \leq b} |v(x)| \leq c_3 \left[ \|h\|_{L^2(a,b)} + \|u\|_{L^2(a,b)} = \|u'\|_{L^2(a,b)} \right], \quad (A.3) \]
where \( c_3 = c_3(a, b; z) \). From (A.3) and (2.6) it follows that inequality (2.5) holds, provided that:
\[ \|u'\|_{L^2(a,b)} \leq \varepsilon \|h\| + \delta \|u\|_{L^\infty}. \quad (A.4) \]

The last estimate is proved as follows. Multiply (2.2) by \( \bar{\eta}u \) (the bar stands for complex conjugate and \( \eta \) is a cut-off function, \( \eta \in C_0^\infty(a, b) \)) and integrate over \((a, b)\) to get
\[
\int_a^b |u'|^2 \eta \, dx = \int_a^b u' \bar{\eta} u' \, dx + \int_a^b \eta h \bar{u} \, dx + z \int_a^b \eta |u|^2 \, dx - \int_a^b q |u|^2 \eta \, dx := I_1 + I_2 + I_3 + I_4.
\]

One has, using the inequality \( |uv| \leq \varepsilon |u|^2 + \frac{|v|^2}{4\varepsilon}, \varepsilon > 0 \),
\[
|I_1| \leq c \left( \varepsilon \|u'\|^2 + \frac{1}{4\varepsilon} \|u\|^2 \right), \quad c = \max |\eta'|,
\]
\[
|I_2| + |I_3| \leq c \left( \|h\| \|u\|_{L^\infty} \|u\| \right) \leq c_1 \|h\|^2,
\]
where (2.6) was used,
\[
|I_4| \leq \|qu\| \|u\| \leq \|q\|_{L^2} \|u\|_{L^\infty} \|u\|.
\]

Thus, if \( a < a_1 < b_1 < b \), where \( \eta = 1 \) on \([a_1, b_1]\), one gets
\[
\int_{a_1}^{b_1} |u'|^2 \, dx \leq C \left( \|h\|^2 + \|u\|_{L^\infty} \|h\| \right) \leq \delta \|u\|_{L^\infty}^2 + C \|h\|^2, \quad (A.5)
\]
where \( C = C(\varepsilon, z, a, b, \delta) = \text{const} > 0, \ 0 < \delta \) can be chosen arbitrarily small. Inequality (A.5) implies (A.4).

Inequality (2.5) is proved. \( \square \)
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