REGULARIZED ASYMPTOTIC DESCENTS FOR A CLASS OF NONCONVEX OPTIMIZATION PROBLEMS

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Abstract. We propose and analyze regularized asymptotic descent (RAD) methods for finding the global minimizer of a class of possibly nonconvex, nonsmooth, or even discontinuous functions. Such functions are extended from strongly convex functions with Lipschitz-continuous gradients. We establish an explicit representation for the solution of regularized minimization, so that the method can find the global minimizer without being trapped in saddle points, local minima, or discontinuities. The main theoretical result shows that the method enjoys the global linear convergence with high probability for such functions. Besides, the method is derivative-free and its per-iteration cost, i.e., the number of function evaluations, is bounded, so it has a complexity bound $O(\log \frac{1}{\epsilon})$ for finding a point such that the optimality gap at this point is less than $\epsilon > 0$. Numerical experiments in up to 500 dimensions demonstrate the benefits of the method.

Key words. nonconvex optimization, nonsmooth, derivative-free, convergence rate, complexity

AMS subject classifications. 65K05, 68Q25, 90C26, 90C56

1. Introduction. In this paper, we propose and analyze regularized asymptotic descent (RAD) methods for solving the optimization problem

$$x_* = \arg \min_{x \in \mathbb{R}^d} f(x),$$

where the objective $f : \mathbb{R}^d \to \mathbb{R}$ comes from a class of possibly nonconvex, nonsmooth, or even discontinuous functions (see Assumption 2.2); such a function has a unique global minimum and possibly multiple local minima. And our goal here is to find this minima without being trapped in saddle points, local minima, or discontinuities.

To the best of our knowledge, our methods are the first algorithms that enjoy the global linear convergence for a certain class of nonconvex and nonsmooth functions with multiple local minima. More specifically, the RAD method requires only $O(\log \frac{1}{\epsilon})$ function evaluations to find a point $x$ such that $\|x - x_*\|^2 < \epsilon$ under Assumption 2.2. Before reviewing the related work, we first introduce our motivation and contributions.

1.1. Motivation. Our motivation is first to consider the regularized iteration

$$x_{k+1} = x_k + r_k,$$

where $r_k$ is defined as

$$r_k = \arg \min_{r \in \mathbb{R}^d} \left( f(x_k + r) + \frac{\lambda}{2} ||r||^2 \right)$$

for a usually small scalar $\lambda > 0$; and then to employ an explicit asymptotic formula to approximate the solution of the possibly nonconvex regularized minimization problem (3). This is the reason that the method is referred to as the regularized asymptotic descent method.

1.2. Contributions. Under such a basic framework described above, the main contributions of this work can be summarized as the following four aspects:

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Asymptotic solution of regularized problems. The regularized iteration (2) can be rewritten as

$$\text{arg min}_{x \in \mathbb{R}^d} \left( f(x) + \frac{\lambda}{2} \|x - x_k\|^2 \right)$$  \text{for fixed } x_k \in \mathbb{R}^d \text{ and } \lambda > 0.$$  

We prove that (Theorem 3.1), if the problem (4) has a unique solution $s_*$, and $f$ is bounded below and continuous at $s_*$, then the solution can be represented as

$$s_* = \lim_{\alpha_k \to \infty} \frac{\int_{\mathbb{R}^d} x \exp \left( - \alpha_k \left( f(x) + \frac{\sigma}{2} \|x - x_k\|_2^2 \right) \right) dx}{\int_{\mathbb{R}^d} \exp \left( - \alpha_k \left( f(x) + \frac{\sigma}{2} \|x - x_k\|_2^2 \right) \right) dx}.$$  

Regularized asymptotic descent. Inspired by the asymptotic formula (5), we propose a derivative-free strategy (Algorithm 1) for solving the problem (1). More specifically, with an initial point $x_1$, a fixed regularization parameter $\lambda$, and two scalar sequences $\{\alpha_k\}$ and $\{n_k\}$, the RAD methods are characterized by the iteration

$$x_{k+1} = \frac{\sum_{i=1}^{n_k} x_i \exp \left( - \alpha_k f(x_i) \right)}{\sum_{i=1}^{n_k} \exp \left( - \alpha_k f(x_i) \right)}, \text{ where } x_i \sim \mathcal{N} \left( x_k, \frac{1}{\alpha_k \lambda} I_d \right).$$  

Here, $I_d \in \mathbb{R}^{d \times d}$ is an identity matrix. Obviously, we replaced the two integrals of the ratio in (5) with two Monte Carlo estimates for a fixed $\alpha_k$. So $x_{k+1}$ is an estimate for

$$s(\alpha_k) := \frac{\int_{\mathbb{R}^d} x \exp \left( - \alpha_k f(x) \right) dP_X(x)}{\int_{\mathbb{R}^d} \exp \left( - \alpha_k f(x) \right) dP_X(x)} = \frac{\int_{\mathbb{R}^d} x \exp \left( - \alpha_k \left( f(x) + \frac{\lambda}{2} \|x - x_k\|_2^2 \right) \right) dx}{\int_{\mathbb{R}^d} \exp \left( - \alpha_k \left( f(x) + \frac{\lambda}{2} \|x - x_k\|_2^2 \right) \right) dx}.$$  

We show that (Corollary 4.4), under Assumption 2.2, $f$ has the global minimizer $x_*$, whether or not the solution of (4) is unique, the gap between $s(\alpha_k)$ and $x_*$ satisfies

$$\|s(\alpha_k) - x_*\|^2 \leq \rho'(\lambda) \|x_k - x_*\|^2,$$  

where $\alpha_k = \frac{l + \lambda}{\lambda^2 \|x_k - x_*\|^2}$ with a fixed $l > 0$.

Here, $\rho'(\lambda)$ is given by (30) and there must be some $\lambda > 0$ such that $\rho'(\lambda) < 1$.

Global linear convergence. Our main result (Theorem 4.5) shows that, under Assumption 2.2, suppose the regularization parameter $\lambda > 0$ is small enough so that $\rho'(\lambda) < \frac{1}{16}$, then, for all $C > 0$ and $k \in \mathbb{N}$, if the stepsize parameter is chosen as

$$\alpha_k = \frac{l + \lambda}{\lambda^2 \|x_k - x_*\|^2}$$  

and the length of the inner cycle is chosen as

$$n_k = \max \left\{ \frac{4C^2 \mathbb{V}[\phi_k(x)]}{\mathbb{E}[\psi_k(x)]}, \frac{16C^2 \mathbb{V}[\phi_k(x)]}{\|x_k - x_*\|^2 \mathbb{E}[\psi_k(x)]} \right\}, \text{ where } x \sim \mathcal{N} \left( x_k, \frac{1}{\alpha_k \lambda} I_d \right),$$  

then with probability at least $1 - \frac{1}{k^4}$, the iteration (6) satisfies

$$\|x_{k+1} - x_*\|^2 \leq \rho \|x_k - x_*\|^2,$$  

where $\rho = 8 \rho'(\lambda) + \frac{1}{2} < 1$, $\phi_k$ and $\psi_k$ are defined by (23). And we consider a realistic algorithm in section 5 and numerical experiments in up to 500 dimensions in section 6.

Complexity bound. We further show that (Theorem 4.6), the per-iteration cost $n_k$, i.e., the number of function evaluations, is bounded, so that the iteration (6) has a complexity bound $\mathcal{O}(\log \frac{1}{\epsilon})$ for finding a point $x$ such that $\|x - x_*\|^2 < \epsilon$. 

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1.3. Related Work. We first discuss the relationships between the new method and two closely related ideas, which are the proximal point methods and the asymptotic solution of minimization problems. And then, we comment several popular methods for finding a first-order critical point or second-order stationary point in a nonconvex setting, which include derivative-based descent methods, perturbed gradient descent methods and derivative-free descent methods.

Proximal point methods. The proximal point method (e.g., [25]), which can be traced back to Martinet [14] in the context of convex minimization and Rockafellar [27] in the general setting of maximal monotone operators, is a conceptually simple approach for minimizing a function \( f \) on \( \mathbb{R}^d \). Given an iterate \( x_k \), the method defines \( x_{k+1} \) to be any minimizer of the proximal iteration

\[
\arg\min_{x \in \mathbb{R}^d} \left( f(x) + \frac{\lambda}{2} \|x - x_k\|^2 \right) \text{ for an appropriate } \lambda > 0,
\]

which can be seen as minimizing \( \Psi(x_k) := \min_x (f(x) + \frac{\lambda}{2} \|x - x_k\|^2) \) by applying the gradient descent with the stepsize \( \frac{1}{\lambda} \) [25]; or in other words, the term proximal refers to the presence of the regularization term with a large \( \lambda \), which encourages the new iterate to be close to \( x_k \) [2, 21]. However, a significant feature of our method is that the regularization parameter \( \lambda \) is kept very small. Therefore, the meaning of the regularization term in (8) is far different from that in our regularized iteration (4). This reflects the difference between local and global perspectives.

Asymptotic solution of minimization problems. In 1967, Pincus proved that \( [23] \), if \( \Omega \subset \mathbb{R}^d \) is a bounded domain, \( f \) is a continuous function on \( \Omega \) and has a unique minimizer \( s_* \) over \( \Omega \), then the minimizer can be represented as

\[
s_* = \lim_{\alpha \to \infty} \frac{\int_{\Omega} x \exp\left( -\alpha f(x) \right) dx}{\int_{\Omega} \exp\left( -\alpha f(x) \right) dx}.
\]

And later, Pincus [24] also suggested a similar Monte Carlo estimate to approximate the minimizer \( s_* \). However, this idea did not receive enough attention because it is not sufficiently efficient [31]. The major reason is that, for building such an estimate, one has to keep sampling uniformly on the entire domain \( \Omega \), but when \( \alpha \) goes large, the main contributors in these samples are only those sufficiently close to the minimizer \( s_* \).

In order to resolve this problem, we consider the regularized minimization (4), and its regularization term leads to a normal sampling distribution so that the corresponding samples will gather in the vicinity of \( x_k \) as \( \alpha_k \) increases.

Derivative-based descent methods. For convenience we call an \( \epsilon \)-approximate first-order critical point \( \epsilon \)-solution temporarily. It is known that the gradient method can find an \( \epsilon \)-solution in \( \mathcal{O}(\epsilon^{-2}) \) iterations for every objective function \( f \) with Lipschitz continuous gradients [18]. If \( f \) additionally has Lipschitz continuous Hessian, then the accelerated gradient method [3] can achieve the complexity \( \mathcal{O}(\epsilon^{-7/4} \log \frac{1}{\epsilon}) \); by using Hessians, the cubic regularization of Newton method [19, 6] can find an \( \epsilon \)-solution in \( \mathcal{O}(\epsilon^{-3/2}) \) iterations. More generally, the \( p \)th-order regularization methods [1] can find an \( \epsilon \)-solution in \( \mathcal{O}(\epsilon^{-(p+1)/p}) \) iterations for every \( f \) with Lipschitz continuous derivatives up to order \( p \geq 1 \), and this complexity cannot be further improved [4]. Furthermore, the first-order methods cannot achieve the complexity \( \mathcal{O}(\epsilon^{-8/5}) \) for arbitrarily smooth functions [5]. Hence, the correlation between the degree of difficulty of a nonconvex problem and the smoothness of the corresponding objective function is not significant. Compared with Assumption 2.2, it is somewhat difficult to identify which nonconvex problems can be more efficiently solved by derivative-based methods.
Perturbed gradient descent methods. In nonconvex settings, convergence to first-order critical points is not yet satisfactory. In 1988, Pemantle [22] realized that by adding zero-mean noise perturbations, a gradient descent method can circumvent strict saddle points with probability one. More recently, it is shown that the perturbed gradient method converges to an $\epsilon$-second-order stationary point with high probability in $O(\epsilon^{-2})$ iterations for twice-differentiable strict saddle functions [11]. Further, even without adding noise perturbations, gradient descent with random initialization [13] can also avoid strict saddle points with probability one. Although not all objectives satisfy the strict saddle property, the perturbed gradient method is still more feasible in practice than more costly trust-region techniques [29, 30]. In contrast, the RAD method will not be trapped at any saddle point for arbitrarily functions because of its inherent asymptotic property.

Derivative-free descent methods. The derivative-free descent methods (e.g., [8, 25]), which are also known as zero-order methods [9, 28] in the literature or bandit optimization in the machine learning literature [10, 28], were among the first schemes suggested in the early days of the development of optimization theory [15]. One of the most typical derivative-free methods is established by the finite-difference method [7, 20, 25], and its descent direction can be seen as an asymptotically unbiased estimate of the smoothed gradient [17, 20]. Therefore, the finite-difference derivative-free descent (FD-DFD) method can be regarded as a smoothed extension of the gradient descent method. In nonconvex settings, the FD-DFD method can also find an $\epsilon$-solution in $O(\epsilon^{-2})$ iterations for every function with Lipschitz continuous gradients [20]. The FD-DFD method can obviously be used to solve nonsmooth problems, and it seems intuitive that a sufficiently large smoothing parameter may help the FD-DFD method to stride saddle points, discontinuities or even some local minima, but further research is needed for related issues.

1.4. Paper Organization. The remainder of the paper is organized as follows. In the next section, we extends strongly convex functions with Lipschitz-continuous gradients to a class of possibly nonconvex, nonsmooth, or discontinuous functions. In section 3, we propose regularized asymptotic descent methods after building an asymptotic formula for the solution of the regularized minimization problem. Then we establish the global linear convergence and logarithmic work complexity in section 4. And finally, we draw some conclusions in section 5.

2. Assumption. Here we introduce a class of possibly nonconvex, nonsmooth, or even discontinuous objective functions extended from strongly convex functions with Lipschitz-continuous gradients. Minimization of such a function can be effectively solved by RAD methods. Let us start with the strongly convex and smooth functions.

2.1. Strong convex objectives with Lipschitz-continuous gradients. We first introduce a basic assumption of smoothness and strong convexity of the objective function. It is essential for convergence analyses of gradient-based methods [2, 18].

**Assumption 2.1 (Strong convex objectives with Lipschitz-continuous gradients).** The objective function $f : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable and there exist $0 < l \leq L < \infty$ such that, for all $x, y \in \mathbb{R}^d$,

\begin{align}
\|\nabla f(x) - \nabla f(y)\|_2 & \leq L\|x - y\|_2 \\
\|\nabla f(x) - \nabla f(y)\|_2 & \leq L\|x - y\|_2 \\
\|\nabla f(x) - \nabla f(y)\|_2 & \leq L\|x - y\|_2 \\
\|\nabla f(x) - \nabla f(y)\|_2 & \leq L\|x - y\|_2 \tag{10}
\end{align}

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\|\nabla f(x) - \nabla f(y)\|_2 & \leq L\|x - y\|_2 \tag{11}
\end{align}

where $L$ is the Lipschitz constant and $l$ is the strong convexity parameter.
First, note that (10) ensures that the gradient of the objective $f$ is bounded and does not change arbitrarily quickly with respect to the parameter vector. Moreover, (10) also implies the inequality (see (4.3) in [2] or Theorem 2.1.5 in [18])

\[(12) \quad f(x) \leq f(y) + \nabla f(y)^T(x-y) + \frac{L}{2}\|x-y\|_2^2 \quad \text{for all } x, y \in \mathbb{R}^d.\]

Clearly, (12) is trivial if $f$ is twice continuously differentiable with $\|\nabla^2 f(x)\|_2 \leq L$.

Second, the strong convexity condition, i.e., (11), is often used to ensure a linear convergence rate for gradient methods [26] as well as a sublinear convergence rate for the stochastic gradient methods [2, 16].

Assumption 2.1 implies that there is $0 < l \leq L < \infty$ such that, for all $x, y \in \mathbb{R}^d$,

\[(13) \quad f(y) + \nabla f(y)^T(x-y) + \frac{l}{2}\|x-y\|_2^2 \leq f(x) \leq f(y) + \nabla f(y)^T(x-y) + \frac{L}{2}\|x-y\|_2^2;\]

that is, $f$ has quadratic upper and lower bounds at every $x \in \mathbb{R}^d$. Further, when $f$ is twice continuously differentiable, this means that $l \leq \|\nabla^2 f(y)\|_2 \leq L$ for all $x \in \mathbb{R}^d$.

### 2.2. Nonconvex objectives with strong convex bounds.

Inspired by (13), we introduce a class of nonconvex objectives, in each of which only has similar upper and lower bounds if $y$ equals the global minimizer $x_*$.

**Assumption 2.2** (Nonconvex objectives with quadratic upper and lower bounds). The objective function $f: \mathbb{R}^d \to \mathbb{R}$ satisfies that there exist $x_* \in \mathbb{R}^d$ and $0 < l \leq L < \infty$ such that for all $x \in \mathbb{R}^d$,

\[(14) \quad f_* + \frac{l}{2}\|x-x_*\|_2^2 \leq f(x) \leq f_* + \frac{L}{2}\|x-x_*\|_2^2;\]

Hence, $f$ has a unique global minimizer $x_*$ with $f_* := f(x_*)$.

**Remark 2.1.** Actually, it is sufficient for the subsequent analyses that (14) holds almost everywhere on $\mathbb{R}^d$, except a sufficiently small neighborhood of $x_*$. And in the literature, e.g. [31], it is sometimes used as a local property, i.e., suppose that (14) holds in a neighborhood of $x_*$.

Obviously, such a objective function $f$ is continuously differentiable at the global minimizer $x_*$ but may be nonsmooth or even discontinuous elsewhere. And these upper and lower bounds are critical to the convergence analysis of the RAD methods.

### 3. Methods.

As mentioned above, the solution of the regularized minimization (3) can be represented as the limit of the ratio of the two integrals. And we establish the RAD iteration by replacing these two integrals with the corresponding Monte Carlo estimates. Let us begin with building this asymptotic solution.

#### 3.1. An asymptotic solution of regularized problems.

We here focus on establishing an asymptotic formula for the solution of the regularized minimization problem (3). First, using the substitution $x = x_k + r$, (3) can be rewritten as

\[(15) \quad \min_{x \in \mathbb{R}^d} \left( f(x) + \frac{\lambda}{2}\|x-x_k\|_2^2 \right) \quad \text{for fixed } x_k \in \mathbb{R}^d \text{ and } \lambda > 0.\]

Suppose that (15) has a unique global minimizer $s_*(x_k, \lambda)$ over $\mathbb{R}^d$; for convenience we abbreviate $s_*(x_k, \lambda) = s_*$. If $f$ is further bounded below by a scalar $f_{\inf}$, then

\[
\exp \left[ -\alpha \left( f(x) + \frac{\lambda}{2}\|x-x_k\|_2^2 \right) \right] \quad \text{and} \quad \|x-s_*\|_2^2 \exp \left[ -\alpha \left( f(x) + \frac{\lambda}{2}\|x-x_k\|_2^2 \right) \right]
\]

are integrable on $\mathbb{R}^d$ for any $\alpha > 0$. And we have the following asymptotic formula:
THEOREM 3.1. Suppose that the regularized minimization \((15)\) has a unique global minimizer \(s_*\). If \(f\) is bounded below and continuous at \(s_*\), then the minimizer can be represented as

\[
s_* = \lim_{\alpha \to \infty} \frac{\int_{\mathbb{R}^d} x \exp \left[ - \alpha \left( f(x) + \frac{1}{2} \| x - x_k \|^2 \right) \right] dx}{\int_{\mathbb{R}^d} \exp \left[ - \alpha \left( f(x) + \frac{1}{2} \| x - x_k \|^2 \right) \right] dx}.
\]

Proof. For convenience we define \(\tau(x) = \exp \left[ - \left( f(x) + \frac{1}{2} \| x - x_k \|^2 \right) \right]\) and \(m^{(\alpha)}(x) = \frac{\tau^{\alpha}(x)}{\int_{\mathbb{R}^d} \tau^{\alpha}(x) dx}\).

Clearly, \(m^{(\alpha)}(x) > 0\) for all \(x \in \mathbb{R}^d\) and \(\int_{\mathbb{R}^d} m^{(\alpha)}(x) dx = 1\), then according to the convexity of \(\| \cdot \|_2^2\) and Jensen’s inequality for convex functions, one obtains

\[
\left\| \int_{\mathbb{R}^d} x \cdot m^{(\alpha)}(x) dx - s_* \right\|_2^2 \leq \int_{\mathbb{R}^d} \| x - s_* \|_2^2 m^{(\alpha)}(x) dx.
\]

We decompose \(\mathbb{R}^d\) into two domains to establish an upper bound for the integral on the right-hand side of the inequality above. For all \(\delta > 0\) we define the open domain

\[\Omega_\delta = \{ x \in \mathbb{R}^d : \tau(x) > \tau(s_*) - \delta \}\]

with its complement \(\Omega'_\delta = \mathbb{R}^d - \Omega_\delta\).

Since \(f\) is continuous at \(s_*\), we observe that, for small \(\epsilon > 0\), there exists \(\delta(\epsilon) > 0\) such that \(\mu(\Omega_{\delta(\epsilon)}) > 0\) and \(\| x - s_* \|_2^2 < \frac{\epsilon}{2}\) for all \(x \in \Omega_{\delta(\epsilon)}\); and we have

\[
\int_{\mathbb{R}^d} \| x - s_* \|_2^2 m^{(\alpha)}(x) dx = \int_{\Omega_{\delta(\epsilon)}} \| x - s_* \|_2^2 m^{(\alpha)}(x) dx + \int_{\Omega'_{\delta(\epsilon)}} \| x - s_* \|_2^2 m^{(\alpha)}(x) dx.
\]

For the first integral on the right-hand side of \((18)\), we clearly have

\[
\int_{\Omega_{\delta(\epsilon)}} \| x - s_* \|_2^2 m^{(\alpha)}(x) dx < \frac{\epsilon}{2} \int_{\Omega_{\delta(\epsilon)}} m^{(\alpha)}(x) dx < \frac{\epsilon}{2} \int_{\mathbb{R}^d} m^{(\alpha)}(x) dx = \frac{\epsilon}{2}.
\]

For the second integral on the right-hand side of \((18)\), we obtain

\[
\int_{\Omega'_{\delta(\epsilon)}} \| x - s_* \|_2^2 m^{(\alpha)}(x) dx = \frac{\int_{\Omega'_{\delta(\epsilon)}} \| x - s_* \|_2^2 \tau^{\alpha}(x) dx}{\int_{\mathbb{R}^d} \tau^{\alpha}(x) dx} \leq \frac{\int_{\Omega'_{\delta(\epsilon)}} \| x - s_* \|_2^2 \tau^{\alpha}(x) dx}{\int_{\Omega_{\delta(\epsilon)}} \tau^{\alpha}(x) dx}.
\]

When \(\mu(\Omega_{\delta(\epsilon)}) > 0\), \(\tau^{\alpha}(x) \exp(\| x - x_k \|_2^2)\) and \(\| x - s_* \|_2^2 \exp(-\| x - x_k \|_2^2)\) are integrable on \(\mathbb{R}^d\), hence, by the mean value theorem for integrals, there is \(\xi \in \Omega'_{\delta(\epsilon)}\) such that

\[
\int_{\Omega'_{\delta(\epsilon)}} \| x - s_* \|_2^2 \tau^{\alpha}(x) dx = \tau^{\alpha}(\xi) \| x - s_* \|_2^2 \int_{\Omega'_{\delta(\epsilon)}} \exp(-\| x - x_k \|_2^2) dx \\
< \tau^{\alpha}(\xi) \| x - s_* \|_2^2 \int_{\mathbb{R}^d} \exp(-\| x - x_k \|_2^2) dx.
\]

and similarly, there is \(\zeta \in \Omega_{\delta(\epsilon)}\) such that

\[
\int_{\Omega_{\delta(\epsilon)}} \tau^{\alpha}(x) dx = \tau^{\alpha}(\zeta) \mu(\Omega_{\delta(\epsilon)}).
\]
Thus, we obtain
\[
\int_{\Omega_{\xi(\epsilon)}} \|x - s_*\|^2 m^{(\alpha)}(x)dx < \left(\frac{\tau(\xi)}{\tau(\xi_1)}\right)^\alpha \exp\left(\frac{\|\xi - x_k\|_2^2}{\mu(\Omega_{\xi(\epsilon)})}\right) I_{s_* x_k},
\]
where \(\tau(\xi) < \tau(\xi_1)\) and \(I_{s_* x_k} = \int_{\mathbb{R}^d} \|x - s_*\|^2 \exp(-\|x - x_k\|_2^2)dx < \infty\). Therefore, there exists a fixed \(\alpha > 0\) such that for every \(\alpha > \alpha_\epsilon\), it holds that
\[
\int_{\Omega_{\xi(\epsilon)}} \|x - s_*\|^2 m^{(\alpha)}(x)dx < \frac{\epsilon}{2}.
\]
Finally, from (17)–(20), we observe that for small \(\epsilon > 0\), there is a fixed \(\alpha > 0\) such that for all \(\alpha > \alpha_\epsilon\), it holds that
\[
\left\|\int_{\mathbb{R}^d} x \cdot m^{(\alpha)}(x)dx - s_*\right\|^2_2 < \epsilon,
\]
and the proof is complete.

### 3.2. Regularized asymptotic descent algorithms

Inspired by the explicit asymptotic formula built in the previous section, our RAD methods are procedures in which each iterate is chosen as a weighted average of normally distributed samples with mean equal to the latest iterate. Specifically, with an initial point \(x_1\), a regularization parameter \(\lambda\), scalar sequences \(\{\alpha_k\}\) and \(\{\eta_k\}\) that are either predetermined or set dynamically, the methods are characterized by the iteration
\[
x_{k+1} = \frac{\sum_{i=1}^{n_k} x_i \exp\left(-\alpha_k f(x_i)\right)}{\sum_{i=1}^{n_k} \exp\left(-\alpha_k f(x_i)\right)}, \quad \text{where} \quad x_i \sim \mathcal{N}\left(x_k, \frac{1}{\alpha_k \lambda} I_d\right).
\]
Here, \(I_d \in \mathbb{R}^{d \times d}\) is an identity matrix so that \(\mathcal{N}(x_k, \alpha_k^{-1} \lambda^{-1} I_d)\) is a spherical normal distribution. To establish this iteration, we replaced the two integrals of the ratio in (16) with two Monte Carlo estimates for a fixed \(\alpha_k\), that is,
\[
\frac{1}{n_k} \sum_{i=1}^{n_k} h(x_i) \exp\left(-\alpha_k f(x_i)\right) \approx \left(\frac{\alpha_k \lambda}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} h(x) \exp\left[-\alpha_k \left(f(x) + \frac{\lambda}{2} \|x - x_k\|_2^2\right)\right]dx,
\]
where \(h(t)\) equals 1 or \(t\), and \(x_i \sim \mathcal{N}(x_k, \alpha_k^{-1} \lambda^{-1} I_d)\). So \(x_{k+1}\) is an estimate for
\[
s(\alpha_k) := \frac{\int_{\mathbb{R}^d} x \exp\left[-\alpha_k \left(f(x) + \frac{\lambda}{2} \|x - x_k\|_2^2\right)\right]dx}{\int_{\mathbb{R}^d} \exp\left[-\alpha_k \left(f(x) + \frac{\lambda}{2} \|x - x_k\|_2^2\right)\right]dx}.
\]
In Corollary 4.4, we will see that \(s(\alpha_k)\) satisfies the following inequality
\[
\|s(\alpha_k) - x_*\|^2_2 \leq \rho'(\lambda)\|x_k - x_*\|^2_2,
\]
provided the stepsize parameter is chosen as
\[
\alpha_k = \frac{l + \lambda}{\lambda^2\|x_k - x_*\|^2_2},
\]
where \(\rho'(\lambda)\) is given by (30) and there is some \(\lambda > 0\) such that \(\rho'(\lambda) < 1\). Compared with the asymptotic formula (16), this contraction inequality (22) provides a more intuitive understanding for the global linear convergence of RAD methods.
Obviously, this iteration (21) is derivative-free and can be applied to nonsmooth or even discontinuous problems. And now, we can define our generalized RAD method as Algorithm 1. The regularization parameter $\lambda$ and the stepsize parameter $\alpha_k$ jointly determine the exploration radius of each outer cycle, and $n_k$ determines the number of explorations of each inner cycle.

**Algorithm 1** Regularized Asymptotic Descent Method

1: Choose an initial iterate $x_1$ and a regularization parameter $\lambda > 0$.
2: \textbf{for} $k = 1, 2, \cdots$ \textbf{do}
3: \hspace{0.5cm} Choose a stepsize parameter $\alpha_k > 0$ and a size parameter $n_k \in \mathbb{N}$.
4: \hspace{0.5cm} Set $w_k = 0$ and $v_k = (0, \ldots, 0)^T \in \mathbb{R}^d$.
5: \hspace{0.5cm} \textbf{for} $i = 1, \ldots, n_k$ \textbf{do}
6: \hspace{1.0cm} Generate a realization $x_i$ of the random vector from $\mathcal{N}(x_k, \alpha_k^{-1}\lambda^{-1}I_d)$.
7: \hspace{1.0cm} Compute a function value $f(x_i)$.
8: \hspace{1.0cm} Update $w_k = w_k + \exp[-\alpha_k f(x_i)]$ and $v_k = v_k + x_i \exp[-\alpha_k f(x_i)]$.
9: \hspace{0.5cm} \textbf{end for}
10: Set the new iterate as $x_{k+1} = \frac{w_k}{w_k}$. 
11: \textbf{end for}

Usually, Algorithm 1 is run with an increasing stepsize parameter sequence $\{\alpha_k\}$ so that the exploration radius gradually decreases, and further, a decreasing radius sequence could effectively reduce the length of the inner cycle $n_k$.

**4. Analyses.** In this section, we provide insights into the behavior of an RAD method by establishing its convergence property and complexity bound. We start by analyzing its convergence property.

As we mentioned above, the RAD method enjoys the global linear convergence. Usually, a typical linear convergence can be described as

$$\|x_{k+1} - x_*\|_2^2 \leq \rho \|x_k - x_*\|_2^2$$

for a certain $0 < \rho < 1$, which represents a contraction relationship between $\|x_{k+1} - x_*\|_2^2$ and $\|x_k - x_*\|_2^2$.

Before we establish a prototype of this relationship by two key lemmas, we pause to introduce two $d$-dimensional integrals that occur many times in subsequent analysis.

**4.1. Preliminary.**

**Lemma 4.1.** For any $\alpha, \beta, \gamma \in \mathbb{R}$ and $u, v \in \mathbb{R}^d$, if $\alpha(\beta + \gamma) > 0$ and

$$\varphi(x) = \exp \left[ -\alpha \left( \frac{\beta}{2}\|x - u\|_2^2 + \frac{\gamma}{2}\|x - v\|_2^2 \right) \right],$$

then the integrals

$$\int_{\mathbb{R}^d} \varphi(x)dx = \exp \left( -\frac{\alpha \beta \gamma \|u - v\|_2^2}{2(\beta + \gamma)} \right) \left( \frac{2\pi}{\alpha(\beta + \gamma)} \right)^{\frac{d}{2}}$$

and

$$\int_{\mathbb{R}^d} \|x - u\|_2^2 \varphi(x)dx = \exp \left( -\frac{\alpha \beta \gamma \|u - v\|_2^2}{2(\beta + \gamma)} \right) \left( \frac{2\pi}{\alpha(\beta + \gamma)} \right)^{\frac{d}{2}} \left( \frac{d}{\alpha(\beta + \gamma)} + \frac{\gamma^2 \|u - v\|_2^2}{(\beta + \gamma)^2} \right).$$

See Appendix A for a proof. These two $d$-dimensional integrals are fundamental to the analyses of RAD algorithms.
4.2. Two key lemmas. Now we introduce a critical medium $I_k$ to establish a relationship between $\|x_{k+1} - x_*\|^2_2$ and $\|x_k - x_*\|^2_2$, which is described by Lemmas 4.2 and 4.3. For convenience we first define
\begin{equation}
(23) \phi_k(x) = \|x - x_*\|^2 \exp(-\alpha_k f(x)) \quad \text{and} \quad \psi_k(x) = \exp(-\alpha_k f(x))
\end{equation}
with $x \sim \mathcal{N}(x_k, \frac{1}{\alpha_k} V I_d)$, where $x_*$ is the global minimizer of $f$ under Assumption 2.2; then the medium $I_k$ can be defined as
\begin{equation}
I_k := \frac{\mathbb{E}[\phi_k]}{\mathbb{E}[\psi_k]} = \frac{\int_X \phi_k(x) dP_X(x)}{\int_X \psi_k(x) dP_X(x)}.
\end{equation}
Or equivalently, (24) can also be rewritten as
\begin{equation}
I_k = \int_{\mathbb{R}^d} \|x - x_*\|^2 \exp \left(-\alpha_k \left(f(x) + \frac{\alpha}{2} \|x - x_*\|^2\right)\right) dx
\end{equation}
\begin{equation}
\int_{\mathbb{R}^d} \exp \left(-\alpha_k \left(f(x) + \frac{\alpha}{2} \|x - x_*\|^2\right)\right) dx.
\end{equation}

The following lemma first establishes an upper bound for $\|x_{k+1} - x_*\|^2_2$.

**Lemma 4.2** (Relationship between $\|x_{k+1} - x_*\|^2_2$ and $I_k^2$). Suppose that the RAD method (Algorithm 1) is run with a natural number sequence $\{n_k\}$ such that, for all $k \in \mathbb{N}$ and $C > 0$, the length of the inner cycle
\begin{equation}
(26) n_k \geq \frac{4C^2 \mathbb{V}[\phi_k]}{(\mathbb{E}[\psi_k])^2}.
\end{equation}

Then with probability at least $1 - \frac{1}{C^2}$, the iterates of RAD satisfy for all $k \in \mathbb{N}$:
\begin{equation}
\|x_{k+1} - x_*\|^2_2 \leq \frac{8I_k^2 + 8C^2 \mathbb{V}[\phi_k]}{(\mathbb{E}[\psi_k])^2} \frac{1}{n_k},
\end{equation}
where $I_k$ is defined by (24), $\phi_k$ and $\psi_k$ are defined by (23).

**Proof.** It follows from the iteration (21) that
\begin{equation}
x_{k+1} - x_* = \frac{\sum_{i=1}^{n_k} (x_i - x_*) \exp(-\alpha_k f(x_i))}{\sum_{i=1}^{n_k} \exp(-\alpha_k f(x_i))},
\end{equation}

then together with the convexity of $\|\cdot\|_2$ and Jensen’s inequality for convex functions, one obtains
\begin{equation}
\|x_{k+1} - x_*\|^2_2 = \frac{\sum_{i=1}^{n_k} (x_i - x_*) \exp(-\alpha_k f(x_i))}{\sum_{i=1}^{n_k} \exp(-\alpha_k f(x_i))} \leq \frac{\sum_{i=1}^{n_k} \|x_i - x_*\|_2 \exp(-\alpha_k f(x_i))}{\sum_{i=1}^{n_k} \exp(-\alpha_k f(x_i))},
\end{equation}
which can be rewritten as
\begin{equation}
(27) \|x_{k+1} - x_*\|^2_2 \leq \frac{\tilde{\phi}_k}{\tilde{\psi}_k}, \quad \text{where} \quad \tilde{\phi}_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \phi_k(x_i), \quad \tilde{\psi}_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \psi_k(x_i).
\end{equation}

Furthermore, notice that
\begin{equation}
\mathbb{E}[	ilde{\phi}_k] = \mathbb{E}[\phi_k], \quad \mathbb{V}[	ilde{\phi}_k] = \mathbb{V}[\phi_k]/n_k \quad \text{and} \quad \mathbb{E}[	ilde{\psi}_k] = \mathbb{E}[\psi_k], \quad \mathbb{V}[\tilde{\psi}_k] = \mathbb{V}[\psi_k]/n_k,
\end{equation}
it follows from the Chebyshev’s inequality that, for all \( C > 0 \),
\[
\mathbb{P}\left( |\bar{\phi}_k - \mathbb{E}[\phi_k]| \geq C \sqrt{\mathbb{V}[\phi_k]/n_k} \right) \leq \frac{1}{C^2} \quad \text{and} \quad \mathbb{P}\left( |\bar{\psi}_k - \mathbb{E}[\psi_k]| \geq C \sqrt{\mathbb{V}[\psi_k]/n_k} \right) \leq \frac{1}{C^2},
\]
that is, with probability at least 1 − \( \frac{1}{C^2} \), it hold that
\[
|\bar{\phi}_k - \mathbb{E}[\phi_k]| \leq C \sqrt{\mathbb{V}[\phi_k]/n_k}, \quad |\bar{\psi}_k - \mathbb{E}[\psi_k]| \leq C \sqrt{\mathbb{V}[\psi_k]/n_k},
\]
and further,
\[
\frac{\bar{\phi}_k}{\psi_k} \leq \frac{\mathbb{E}[\phi_k] + C \sqrt{\mathbb{V}[\phi_k]/n_k}}{\mathbb{E}[\psi_k] - C \sqrt{\mathbb{V}[\psi_k]/n_k}} = \frac{\mathbb{E}[\phi_k] + C \sqrt{\mathbb{V}[\phi_k]/n_k}}{\mathbb{E}[\psi_k] - C \sqrt{\mathbb{V}[\psi_k]/n_k}} \leq \frac{\mathbb{E}[\phi_k]}{\mathbb{E}[\psi_k] - C \sqrt{\mathbb{V}[\psi_k]/n_k}} = \left( I_k + \frac{C \sqrt{\mathbb{V}[\phi_k]}}{\mathbb{E}[\psi_k] - C \sqrt{\mathbb{V}[\psi_k]/n_k}} \right) \sqrt{\frac{n_k}{\mathbb{E}[\psi_k] - C \sqrt{\mathbb{V}[\psi_k]/n_k}}}
\]
since \( \sqrt{\frac{t}{\sqrt{t-s}}} \) is monotonically decreasing with respect to \( t \) when \( t > s \) for any \( s \in \mathbb{R} \), the condition (26) guarantees that
\[
\sqrt{\frac{n_k}{\mathbb{E}[\psi_k] - C \sqrt{\mathbb{V}[\psi_k]/n_k}}} \leq 2,
\]
thus, we obtain
\[
(28) \quad \frac{\bar{\phi}_k}{\psi_k} \leq 2I_k + 2 \frac{C \sqrt{\mathbb{V}[\phi_k]}}{\mathbb{E}[\psi_k]} \frac{1}{\sqrt{n_k}}.
\]
By noting that the Arithmetic Mean Geometric Mean inequality, it follows from (27) and (28) that, with probability at least 1 − \( \frac{1}{C^2} \),
\[
\|x_{k+1} - x_*\|^2 \leq \left( \frac{\bar{\phi}_k}{\psi_k} \right)^2 \leq \left( 2I_k + 2 \frac{C \sqrt{\mathbb{V}[\phi_k]}}{\mathbb{E}[\psi_k]} \frac{1}{\sqrt{n_k}} \right)^2 \leq 8I_k^2 + 8 \frac{C^2 \mathbb{V}[\phi_k]}{[\mathbb{E}[\psi_k]^2 n_k},
\]
and the proof is complete.

The following lemma shows that for a properly selected stepsize parameter \( \alpha_k \), \( I_k^2 \) can be bounded by the product of \( \|x_k - x_*\|^2 \) and a scalar, and this scalar could be less than one for some suitable regularization parameter \( \lambda \).

**Lemma 4.3** (Relationship between \( I_k^2 \) and \( \|x_k - x_*\|^2 \)). Under Assumption 2.2, suppose that the RAD method (Algorithm 1) is run with a sequence \( \{\alpha_k\} \) such that, for all \( k \in \mathbb{N} \), the stepsize parameter
\[
(29) \quad \alpha_k = \frac{l + \lambda}{\lambda^2 \|x_k - x_*\|^2}.
\]
Then the iterates of RAD satisfy the following inequality for all \( k \in \mathbb{N} \):
\[
I_k^2 \leq \rho'(\lambda) \|x_k - x_*\|^2/2,
\]
where \( I_k \) is defined by (25) and
\[
(30) \quad \rho'(\lambda) = \frac{(d + 1)\lambda^2}{(l + \lambda)^2} \left( \frac{L + \lambda}{l + \lambda} \right)^{\frac{d}{2}} \exp \left( \frac{L - l}{2(L + \lambda)} \right).
\]
Remark 4.1. Note that
\[ \rho'(\lambda) < \lambda^2 \frac{d + 1}{l^2} \left( \frac{L}{l} \right)^{\frac{d}{2}} \exp \left( \frac{L - l}{2L} \right), \]
for any \( 0 < \epsilon < 1 \), there is \( \lambda_\epsilon > 0 \) such that for every \( \lambda < \lambda_\epsilon \), it holds that \( \rho'(\lambda) < \epsilon \).

Proof. According to the definition (25) of \( I_k \) and Jensen’s inequality for convex functions, it follows that
\[ I_k^2 \leq \frac{\int_{\mathbb{R}^d} \|x - x_*\|^2 \exp \left( -\alpha_k (f(x) + \frac{\lambda}{2} \|x - x_k\|^2) \right) dx}{\int_{\mathbb{R}^d} \exp \left( -\alpha_k (f(x) + \frac{\lambda}{2} \|x - x_k\|^2) \right) dx}. \]
According to Assumption 2.2, i.e.,
\[ f_* + \frac{L}{2} \|x - x_*\|^2 \leq f(x) \leq f_* + \frac{L}{2} \|x - x_*\|^2, \]
one can first observe that, for the fraction on the right-hand side of (31), we have the upper bound of the numerator
\[ \exp(-\alpha_k f_*) \int_{\mathbb{R}^d} \|x - x_*\|^2 \exp \left( -\alpha_k \left( \frac{L}{2} \|x - x_*\|^2 + \frac{\lambda}{2} \|x - x_k\|^2 \right) \right) dx \]
and the lower bound of the denominator
\[ \exp(-\alpha_k f_*) \int_{\mathbb{R}^d} \exp \left( -\alpha_k \left( \frac{L}{2} \|x - x_*\|^2 + \frac{\lambda}{2} \|x - x_k\|^2 \right) \right) dx. \]
So it follows from (31)–(33) that
\[ I_k^2 \leq \frac{\int_{\mathbb{R}^d} \|x - x_*\|^2 \exp \left( -\alpha_k \left( \frac{L}{2} \|x - x_*\|^2 + \frac{\lambda}{2} \|x - x_k\|^2 \right) \right) dx}{\int_{\mathbb{R}^d} \exp \left( -\alpha_k \left( \frac{L}{2} \|x - x_*\|^2 + \frac{\lambda}{2} \|x - x_k\|^2 \right) \right) dx}. \]
Further, according to Lemma 4.1, the numerator of the fraction on the right-hand side of (34) equals to
\[ \exp \left( -\frac{\alpha_k L \|x_k - x_*\|^2}{2(L + \lambda)} \right) \left( \frac{2\pi}{\alpha_k (L + \lambda)} \right)^{\frac{d}{2}} \left( \frac{d}{\alpha_k (L + \lambda)} + \frac{\lambda^2 \|x_k - x_*\|^2}{(L + \lambda)^2} \right) \]
and the corresponding denominator equals to
\[ \exp \left( -\frac{\alpha_k L \|x_k - x_*\|^2}{2(L + \lambda)} \right) \left( \frac{2\pi}{\alpha_k (L + \lambda)} \right)^{\frac{d}{2}}. \]
Thus, we obtain
\[ I_k^2 \leq \exp \left( \frac{\alpha_k (L - l)^2 \|x_k - x_*\|^2}{2(L + \lambda)(l + \lambda)} \right) \left( \frac{L + \lambda}{l + \lambda} \right)^{\frac{d}{2}} \left( \frac{d}{\alpha_k (L + \lambda)} + \frac{\lambda^2 \|x_k - x_*\|^2}{(l + \lambda)^2} \right). \]
By taking the stepsize parameter policy (29), i.e., \( \alpha_k = \frac{l + \lambda}{\lambda^2 \|x_k - x_*\|^2} \), we finally get
\[ I_k^2 \leq \frac{(d + 1)^2}{(l + \lambda)^2} \left( \frac{L + \lambda}{l + \lambda} \right)^{\frac{d}{2}} \exp \left( \frac{L - l}{2(L + \lambda)} \right) \|x_k - x_*\|^2 = \rho'(\lambda) \|x_k - x_*\|^2, \]
as claimed. \( \square \)
The following corollary establishes a linear convergence behavior of asymptotic formula (16) for a fixed $\alpha_k$ in each iteration.

**Corollary 4.4.** Under Assumption 2.2, there must be some $\lambda > 0$ such that $\rho'(\lambda) < 1$; moreover, if the policy (29) is further employed, then whether or not the minima of (15) is unique, we have the following contraction inequality

$$\|s(\alpha_k) - x_k\|^2 \leq \rho'(\lambda)\|x_k - x^*_k\|^2,$$

where $\rho'(\lambda)$ is defined by (30) and

$$s(\alpha_k) = \frac{\int_{R^d} x \exp \left[ -\alpha_k (f(x) + \frac{\lambda}{2}\|x - x_k\|^2) \right] dx}{\int_{R^d} \exp \left[ -\alpha_k (f(x) + \frac{\lambda}{2}\|x - x_k\|^2) \right] dx}.$$

**Proof.** According to Jensens inequality for convex functions, it follows that

$$\|s(\alpha_k) - x_k\|^2 \leq \frac{\int_{R^d} \|x - x_k\|^2 \exp \left[ -\alpha_k (f(x) + \frac{\lambda}{2}\|x - x_k\|^2) \right] dx}{\int_{R^d} \exp \left[ -\alpha_k (f(x) + \frac{\lambda}{2}\|x - x_k\|^2) \right] dx}.$$

Note that the right-hand side of the inequality above is the same as the right-hand side of (34), so the desired result can be obtained in the same way as the proof of Lemma 4.3. \qed

### 4.3. Convergence

According to Lemmas 4.2 and 4.3, we could establish the main convergence theorem for RAD algorithms.

**Theorem 4.5** (Global linear convergence). Under Assumption 2.2, suppose that the RAD method (Algorithm 1) is run with a regularization parameter $\lambda > 0$ such that

$$\rho'(\lambda) := \frac{(d + 1)\lambda^2}{(l + \lambda)^2} \left( \frac{L + \lambda}{l + \lambda} \right)^{\frac{d}{2}} \exp \left( \frac{L - l}{2(L + \lambda)} \right) < \frac{1}{16},$$

Then, for all $C > 0$ and $k \in \mathbb{N}$, if the stepsize parameter is chosen as $\alpha_k = \frac{L + \lambda}{\lambda \|x_k - x^*_k\|}$ and the length of the inner cycle is chosen as

$$n_k = \max \left\{ \frac{4C^2\mathbb{V}[\psi_k(x)]}{(\mathbb{E}[\psi_k(x)])^2}, \frac{16C^2\mathbb{V}[\phi_k(x)]}{\mathbb{E}[\psi_k(x)]^2} \right\}, \quad x \sim \mathcal{N}(x_k, \frac{1}{\alpha_k} I_d),$$

then with probability at least $1 - \frac{1}{e^C}$, the iterates of RAD method satisfy the global linear convergence property

$$\|x_{k+1} - x^*_k\|^2 \leq \rho\|x_k - x^*_k\|^2,$$

where $\rho = 8\rho'(\lambda) + \frac{1}{2} < 1$, $\phi_k$ and $\psi_k$ are defined by (23).

**Proof.** Note that $n_k \geq \frac{4C^2\mathbb{V}[\psi_k(x)]}{(\mathbb{E}[\psi_k(x)])^2}$ from (37), it follows from Lemmas 4.2 and 4.3 that, with probability at least $1 - \frac{1}{e^C}$,

$$\|x_{k+1} - x^*_k\|^2 \leq 8\rho'(\lambda)\|x_k - x^*_k\|^2 + 8C^2\mathbb{V}[\phi_k] \frac{1}{(\mathbb{E}[\psi_k])^2 n_k},$$

together with $\frac{1}{n_k} < \frac{\|x_k - x^*_k\|^2 (\mathbb{E}[\psi_k])^2}{16C^2\mathbb{V}[\phi_k]}$, one obtains

$$\|x_{k+1} - x^*_k\|^2 \leq 8\rho'(\lambda)\|x_k - x^*_k\|^2 + 8\mathbb{V}[\phi_k] \frac{1}{\mathbb{E}[\psi_k]^2 n_k} \|x_k - x^*_k\|^2 \leq \rho\|x_k - x^*_k\|^2,$$

where $\rho = 8\rho'(\lambda) + \frac{1}{2} < 1$, and the proof is complete. \qed
Since $\rho'(\lambda)$ obviously depends on the dimension $d$, a feasible choice of $\lambda$ may also depend on $d$; however, $\rho'(\lambda)$ is just a very crude bound such that the desired contraction relationship (35) holds strictly, so it does not mean that a feasible choice of $\lambda$ must be exponentially dependent on $d$. Therefore, the correct understanding of the choice of $\lambda$ is that for a given initial point $x_1$, $\lambda$ should be small enough so that the global minimizer $x_*\in\mathbb{R}^d$ can be covered in the sampling range in the first iteration.

Even with a linear convergence, it is too early to say that the RAD algorithm is efficient because the per-iteration cost $n_k$ may increase without bound; and in this case, it may not make sense to describe a method as linearly convergent. To address this problem, we will establish a bound for $n_k$ as well as a complexity bound for the RAD algorithm in the next subsection.

4.4. Complexity. To obtain a complexity bound, we need to establish an upper bound for the per-iteration cost, i.e., the length of the inner cycle $n_k$.

**Theorem 4.6** (Upper bound for $n_k$). Suppose the conditions of Theorem 4.5 hold. The iterates of RAD (Algorithm 1) satisfy that, for all $k \in \mathbb{N}$ and $C > 0$,

$$n_k \leq 4C^2 \left( \frac{(L + \lambda)^2}{\lambda(2L + \lambda)} \right)^{\frac{d}{2}} \exp \left( \frac{Ll + L\lambda - l\lambda}{\lambda(L + \lambda)} \right),$$

which is independent of $k$.

**Remark 4.2.** This upper bound is very crude; but fortunately, there is a very realistic way to choose $n_k$ in practical situations, see the next subsection for details.

**Proof.** According to the policy (37), the length of the inner cycle is chosen as

$$n_k = \max \left\{ \frac{4C^2 \mathbb{V}[\psi_k(x)]}{(\mathbb{E}[\psi_k(x)])^2}, \frac{16C^2 \mathbb{V}[\phi_k(x)]}{(\mathbb{E}[\psi_k(x)])^2} \right\}, \ x \sim \mathcal{N}(x_k, \frac{1}{\alpha_k \lambda}I_d).$$

By further noting that

$$\frac{\mathbb{V}[\psi_k]}{(\mathbb{E}[\psi_k])^2} \leq \frac{\mathbb{E}[\psi_k^2]}{(\mathbb{E}[\psi_k])^2} \quad \text{and} \quad \frac{\mathbb{V}[\phi_k]}{(\mathbb{E}[\psi_k])^2} \leq \frac{\mathbb{E}[\phi_k^2]}{(\mathbb{E}[\psi_k])^2},$$

we need to establish a lower bound for $\mathbb{E}[\psi_k]$ and upper bounds for $\mathbb{E}[\psi_k^2]$ and $\mathbb{E}[\phi_k^2]$.

We first establish a lower bound for $\mathbb{E}[\psi_k]$. Note that

$$\mathbb{E}[\psi_k] = \mathbb{E} \left[ \exp \left( -\alpha_k f(x) \right) dP_X(x) \right] = \left( \frac{\alpha_k \lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left[ -\alpha_k \left( f(x) + \frac{\lambda}{2} \|x - x_k\|_2^2 \right) \right] dx.$$

together with Assumption 2.2, then yields

$$\mathbb{E}[\psi_k] \geq E_k \left( \frac{\alpha_k \lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left[ -\alpha_k \left( \frac{L}{2} \|x - x_*\|_2^2 + \frac{\lambda}{2} \|x - x_k\|_2^2 \right) \right] dx,$$

where $E_k = \exp(-\alpha_k f_*)$; further, according to Lemma 4.1, one obtains

$$\mathbb{E}[\psi_k] \geq E_k \left( \frac{\lambda}{L + \lambda} \right)^{\frac{d}{2}} \exp \left( -\alpha_k L\lambda \|x_k - x_*\|_2^2 \frac{2}{2(L + \lambda)} \right),$$
by taking $\alpha_k = \frac{l+\lambda}{\lambda||x_k-x_*||^2}$, we get a lower bound of $\mathbb{E}[\psi_k]$, i.e.,

$$(39) \quad \mathbb{E}[\psi_k] \geq E_k \left( \frac{\lambda}{L+\lambda} \right)^{\frac{d}{2}} \exp \left( -\frac{L(l+\lambda)}{2\lambda(L+\lambda)} \right).$$

Now we establish an upper bound for $\mathbb{E}[\psi_k^2]$. Note that

$$\mathbb{E}[\psi_k^2] = \int_X \exp \left( -2\alpha_k f(x) \right) dP_X(x)$$

$$= \left( \frac{\alpha_k \lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left[ -\alpha_k \left( 2f(x) + \frac{\lambda}{2} ||x-x_k||^2 \right) \right] dx,$$

together with Assumption 2.2 and Lemma 4.1, then yields

$$\mathbb{E}[\psi_k^2] \leq E_k^2 \left( \frac{\alpha_k \lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left[ -\alpha_k \left( l||x-x_*||^2 + \frac{\lambda}{2} ||x-x_k||^2 \right) \right] dx$$

$$\leq E_k^2 \left( \frac{\lambda}{2l+\lambda} \right)^{\frac{d}{2}} \exp \left( -\frac{\alpha_k l\lambda ||x_k-x_*||^2}{2l+\lambda} \right).$$

By taking $\alpha_k = \frac{l+\lambda}{\lambda||x_k-x_*||^2}$, we have

$$(40) \quad \mathbb{E}[\psi_k^2] \leq E_k^2 \left( \frac{\lambda}{2l+\lambda} \right)^{\frac{d}{2}} \exp \left( -\frac{l(l+\lambda)}{\lambda(2l+\lambda)} \right).$$

Therefore, according to (39) and (40), and together with

$$(41) \quad \exp \left( -\frac{l(l+\lambda)}{\lambda(2l+\lambda)} \right) \left[ \exp \left( -\frac{L(l+\lambda)}{2\lambda(L+\lambda)} \right) \right]^{-2} \leq \exp \left( \frac{Ll + L\lambda - l\lambda}{\lambda(L+\lambda)} \right),$$

we get the following bound

$$(42) \quad \frac{\mathbb{V}[\psi_k]}{(\mathbb{E}[\psi_k])^2} \leq \left( \frac{(L+\lambda)^2}{\lambda(2l+\lambda)} \right)^{\frac{d}{2}} \exp \left( \frac{Ll + L\lambda - l\lambda}{\lambda(L+\lambda)} \right).$$

Similarly, we establish an upper bound for $\mathbb{E}[\phi_k^2]$. Note that

$$\mathbb{E}[\phi_k^2] = \int_X ||x-x_*||^2 \exp \left( -2\alpha_k f(x) \right) dP_X(x)$$

$$= \left( \frac{\alpha_k \lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} ||x-x_*||^2 \exp \left[ -\alpha_k \left( 2f(x) + \frac{\lambda}{2} ||x-x_k||^2 \right) \right] dx,$$

together with Assumption 2.2 and Lemma 4.1, then yields

$$\mathbb{E}[\phi_k^2] \leq E_k^2 \left( \frac{\alpha_k \lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} ||x-x_k||^2 \exp \left[ -\alpha_k \left( l||x-x_*||^2 + \frac{\lambda}{2} ||x-x_k||^2 \right) \right] dx$$

$$\leq E_k^2 \left( \frac{\lambda}{2l+\lambda} \right)^{\frac{d}{2}} \exp \left( -\frac{\alpha_k l\lambda ||x_k-x_*||^2}{2l+\lambda} \right) \left( \frac{d}{\alpha_k (2l+\lambda)} + \frac{\lambda^2 ||x_k-x_*||^2}{(2l+\lambda)^2} \right),$$

where $\mathbb{V}[\psi_k]$ and $\mathbb{V}[\phi_k]$ represent the variance of $\psi_k$ and $\phi_k$, respectively.
By taking \( \alpha_k = \frac{t+\lambda}{\lambda^2 k^2} \), this yields

\[
\mathbb{E}[\phi_k^2] \leq (d + 1) E_k^2 \left( \frac{\lambda}{2l + \lambda} \right)^{\frac{d}{2}} \exp \left( - \frac{l(\lambda + \lambda)}{\lambda(2l + \lambda)} \right) \frac{\lambda^2}{(l + \lambda)^2} \frac{\|x_k - x_*\|^2}{(l + \lambda)^2}.
\]

Therefore, according to (39), (41), and (43), we get

\[
\mathbb{V}[\phi_k] \leq (d + 1) \mathbb{E}[\phi_k] \left( \frac{\lambda}{2l + \lambda} \right)^{\frac{d}{2}} \exp \left( \frac{L l + \lambda - \lambda}{\lambda(2l + \lambda)} \right).
\]

Further, note that (36) implies \( \frac{(d+1)\lambda^2}{(l+\lambda)^2} < \frac{1}{16} \), then we get the following bound

\[
\mathbb{V}[\phi_k] \leq \frac{1}{16} \left( \frac{L l + \lambda - \lambda}{\lambda(2l + \lambda)} \right)^{\frac{d}{2}} \exp \left( \frac{L l + \lambda - \lambda}{\lambda(2l + \lambda)} \right).
\]

Finally, combining (38), (42), and (44), one can obtain the desired bound for \( n_k \).

Now, the following corollary is immediate from Theorems 4.5 and 4.6. It provides a total work complexity bound for the RAD methods.

**Corollary 4.7 (Complexity bound).** Suppose the conditions of Theorem 4.5 hold. Then the number of function evaluations of an RAD (Algorithm 1) required to achieve \( F(x_k) - F_* \leq \epsilon \) is \( O(\log(1/\epsilon)) \).

**5. A realistic algorithm.** Algorithm 1 has a hidden danger that the increase of \( \alpha_k \) might cause instability or overflow inside the algorithm. To address this problem, we define the revised RAD method as Algorithm 2, which allows more convenience in the choice of parameters. In each iteration of this version, the variance of function value sequence is used to enhanced stability, and the regularization parameter \( \lambda \) is further chosen as \( \alpha_k \). The performance of this algorithm is shown in section 6.

**Algorithm 2 Revised RAD Method**

1. Choose an initial iterate \( x_1 \) and preset the parameters \( \alpha_0 > 0, q > 1, n \in \mathbb{N} \).
2. for \( k = 1, 2, \cdots \) do
3. Set the stepsize parameter \( \alpha_k = q^{k-1} \alpha_0 \).
4. Generate \( n \) realizations \( \{x_i\}_{i=1}^n \) of the random vector from \( N(x_k, \alpha_k^{-2} I_d) \).
5. Compute function value sequence \( \{f(x_i)\}_{i=1}^n \) with its mean \( \mu_k \) and variance \( \sigma_k \).
6. Compute \( w_k = \sum_{i=1}^n \exp \left( -\frac{f(x_i) - \mu_k}{\sigma_k} \right) \) and \( v_k = \sum_{i=1}^n x_i \exp \left( -\frac{f(x_i) - \mu_k}{\sigma_k} \right) \).
7. Set the new iterate as \( x_{k+1} = \frac{v_k}{w_k} \).
8. end for

First, \( \alpha_0^{-2} \) determines an initial variance of the exploration, which depends on how wide a user expects to make a detection. And it is worth noting that, if any iterate is found to be outside of this preset range, e.g., \( \|x_k - x_i\| > 3/\alpha_0 \) according to the three-sigma rule of thumb, one may need to restart with a smaller \( \alpha_0 \) at \( x_k \). Second, the parameter \( q \) is the contraction factor of each iteration and \( n \) is the number of function evaluations required for each iteration corresponding to \( q \). Clearly, if the objective is differentiable at \( x_k \), then \( \sigma_k \) is proportional to \( 1/\alpha_k \) when \( \alpha_k \) is large.

**6. Numerical experiments.** Now we illustrate the numerical performance of the algorithm. We consider the revised Rastrigin function in \( \mathbb{R}^d \) defined as

\[
f(x) = \|x\|_2^2 - \frac{1}{2} \sum_{i=1}^d \cos(5\pi x^{(i)}) + \frac{d}{2}, \quad \text{where} \ x^{(i)} \ \text{be the} \ i \ \text{th component of} \ x.
\]
This function satisfies Assumption 2.2. It has a unique global minima located at the origin and very many local minima. Only in the hypercube $[-1, 1]^d$, the number of its local minima reaches $5^d$, e.g., about $3.055 \times 10^{349}$ for $d = 500$. In the following, every initial iterate is selected on a $d$-dimensional sphere of radius $\sqrt{d}$ centered at the origin. Therefore, finding the global minima is extremely difficult for a large $d$.

In experiments below, the random vectors in each iteration are generated by a halton sequence with RR2 scramble type [12]. Figure 1 shows the intuitive convergence behavior of the RAD algorithm for the revised Rastrigin function in 2 dimensions. One can see that the RAD algorithm not only guarantees global linear convergence, but also tends to converge directly towards the global minimizer, despite the existence of numerous local minima. The factor $q$ determines the convergence rate when the remaining parameters $\alpha_0$ and $n$ are properly selected.

Figure 2 shows the performance of the RAD algorithm in various dimensions from 10 to 500. These experiments further demonstrate global linear convergence. And it is worth noting that, the difficulty setting of these experiments increases as the dimension $d$ increases because the distance between the initial iterate and the global minimizer is getting farther as $d$ increases. Moreover, one does not need a very large $n$ to guarantee convergence. These results seem to suggest that $n$ satisfying the convergence condition is of the order $\mathcal{O}(d)$, but further research is needed.
Fig. 2. Performance of RAD method for the revised Rastrigin function in various dimensions (from 10 to 500), every initial iterate is randomly selected on a sphere of radius $\sqrt{d}$ centered at the origin, the parameter $\alpha_0 = \sqrt{d}$, and three different settings for the remaining parameters $q$ and $n$ are run independently for each plot.
7. Conclusions. In this work we have proposed regularized asymptotic descent methods for finding the global minima of a class of possibly nonconvex, nonsmooth, or even discontinuous functions, and these derivative-free methods have a total work complexity bound $O(\log \frac{1}{\epsilon})$ to find a point such that the optimality gap at this point is less than $\epsilon$. Numerical experiments in up to 500 dimensions demonstrate both the global linear convergence with high probability and the logarithmic work complexity of the proposed method.

The algorithm is implemented in Matlab. The source code of the implementation is available at https://github.com/xiaopenglou/rad.

Future research is currently being conducted in the following areas. One of the attempts is to establish an adaptive selection strategy for parameters. The empirical choice of parameters depends on a large number of comparative experiments, this requires a lot of computational cost. A successful achievement will make our methods more suitable for large-scale applications, meanwhile, it also helps to reduce the length of the inner cycle, i.e., $n_k$, as much as possible.

Second, we are considering how to extend the assumption of our methods without significantly increasing the computational cost. It is very valuable to efficiently find the best local minima in a certain range for a general nonconvex problem. And our methods increase the possibility of achieving this purpose.

Third, we also hope to investigate further properties of the proposed asymptotic formula. Our work obviously relies on some interesting properties of this formula. It is the key to transform from the differential viewpoint to the integral viewpoint. And further exploration may lead to other ideas for essential nonconvex and nonsmooth optimization problems.

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Appendix A. Proof.

Lemma 4.1. Let $a^{(i)}$ be the $i$th component of a vector $a \in \mathbb{R}^d$, then for any $1 \leq i \leq d$, when $\alpha (\beta + \gamma) > 0$, one obtains

$$I_1^{(i)} := \int_{\mathbb{R}} \exp \left[ -\alpha \left( \frac{\beta}{2} (x^{(i)} - u^{(i)})^2 + \frac{\gamma}{2} (x^{(i)} - v^{(i)})^2 \right) \right] dx^{(i)}$$

$$= \int_{\mathbb{R}} \exp \left[ -\frac{\alpha (\beta + \gamma)}{2} \left( x^{(i)} - \frac{\beta u^{(i)} + \gamma v^{(i)}}{\beta + \gamma} \right)^2 - \frac{\alpha \beta \gamma (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right] dx^{(i)}$$

$$= \exp \left( -\frac{\alpha \beta \gamma (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right) \int_{\mathbb{R}} \exp \left[ -\frac{\alpha (\beta + \gamma)}{2} \left( x^{(i)} - \frac{\beta u^{(i)} + \gamma v^{(i)}}{\beta + \gamma} \right)^2 \right] dx^{(i)},$$

using the substitution

$$t = x^{(i)} - \frac{\beta u^{(i)} + \gamma v^{(i)}}{\beta + \gamma},$$

this yields

$$I_1^{(i)} = \exp \left( -\frac{\alpha \beta \gamma (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right) \int_{\mathbb{R}} \exp \left[ -\frac{\alpha (\beta + \gamma)}{2} t^2 \right] dt$$

$$= \exp \left( -\frac{\alpha \beta \gamma (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right) \left( \frac{2\pi}{\alpha (\beta + \gamma)} \right)^{\frac{1}{2}};$$
and similarly,
\[
I_2^{(i)} := \int_{\mathbb{R}} (x^{(i)} - u^{(i)})^2 \exp \left[ -\alpha \left( \frac{\beta}{2} (\alpha^{(i)} - u^{(i)})^2 + \frac{\gamma}{2} (x^{(i)} - v^{(i)})^2 \right) \right] \, dx^{(i)}
\]
\[
= \exp \left( -\frac{\alpha \beta \gamma (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right) \int_{\mathbb{R}} \left( t + \frac{\gamma (v^{(i)} - u^{(i)})^2}{\beta + \gamma} \right)^2 \exp \left[ -\frac{\alpha (\beta + \gamma) t^2}{2} \right] \, dt
\]
\[
= \exp \left( -\frac{\alpha \beta \gamma (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right) \left( \frac{2\pi}{\alpha(\beta + \gamma)} \right)^{\frac{d}{2}} \left( \frac{1}{\alpha(\beta + \gamma)} + \frac{\gamma^2 (v^{(i)} - u^{(i)})^2}{(\beta + \gamma)^2} \right).
\]

Thus, it follows that
\[
\int_{\mathbb{R}^d} \varphi(x) \, dx = \prod_{i=1}^d I_1^{(i)} = \left( \frac{2\pi}{\alpha(\beta + \gamma)} \right)^{\frac{d}{2}} \prod_{i=1}^d \exp \left( -\frac{\alpha \beta \gamma (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right)
\]
\[
= \left( \frac{2\pi}{\alpha(\beta + \gamma)} \right)^{\frac{d}{2}} \exp \left( -\frac{\alpha \beta \gamma \sum_{i=1}^d (u^{(i)} - v^{(i)})^2}{2(\beta + \gamma)} \right)
\]
\[
= \left( \frac{2\pi}{\alpha(\beta + \gamma)} \right)^{\frac{d}{2}} \exp \left( -\frac{\alpha \beta \gamma \|u - v\|_2^2}{2(\beta + \gamma)} \right);
\]
and similarly,
\[
\int_{\mathbb{R}^d} \|x - u\|^2_2 \varphi(x) \, dx = \sum_{i=1}^d \int_{\mathbb{R}^d} (x^{(i)} - u^{(i)})^2 \exp \left[ -\alpha \left( \frac{\beta}{2} \|x - u\|_2^2 + \frac{\gamma}{2} \|x - v\|_2^2 \right) \right] \, dx
\]
\[
= \sum_{i=1}^d \left( I_2^{(i)} \prod_{j \neq i} I_1^{(j)} \right)
\]
\[
= \left( \frac{2\pi}{\alpha(\beta + \gamma)} \right)^{\frac{d}{2}} \exp \left( -\frac{\alpha \beta \gamma \|u - v\|_2^2}{2(\beta + \gamma)} \right) \left( \frac{d}{\alpha(\beta + \gamma)} + \frac{\gamma^2 \|u - v\|_2^2}{(\beta + \gamma)^2} \right).
\]

so the proof is complete.

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