Collapse Models

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Abstract. This is a review of formalisms and models (nonrelativistic and relativistic) which modify Schrödinger's equation so that it describes wavefunction collapse as a dynamical physical process.

1 Representing The World

One of the major goals of theoretical physics is to find a useful mathematical object to represent the physical state of our world. In classical physics the object is a point in phase space. So far it has been possible to represent the world at an "instant," i.e., on a spacelike hypersurface\(^1\). One thing “useful” means is that a dynamical equation can be found for the object, enabling predictions about the state of the world on a future spacelike hypersurface.

Is there such an object in quantum physics? In pursuing the research discussed here I have made some bets as to the nature of an eventually satisfactory physical theory. One of them is that there is such an object, a statevector in a suitable Hilbert space plus something more. I shall argue that more must be added because standard quantum theory (SQT) is a theory of choices without a chooser: more is a chooser. Along the way I shall make a few opinionated remarks about other points of view. However, mostly I shall try to give a coherent account of rationale (section 1), formalisms (section 2), nonrelativistic (including gravitational, section 3) and relativistic (section 4) models for a particular way of altering Schrödinger’s equation so the statevector behaves in a manner consistent with this bet.

There is a point of view that the statevector does not represent the world but is only a mathematical tool enabling people to calculate the statistical outcomes of experiments\(^1\). There is a point of view that only an ensemble of experiments is represented by the statevector\(^2, 3, 4\). I disagree with both because I find it hard to believe such a successful theory as quantum theory either has no relevance to the unobserved world or to the individual world. There is a point

\(^1\) One can imagine other possibilities, e.g., a representation might only be possible on the spacetime volume between two hypersurfaces; there might be another time parameter whose specification is also required; etc.
of view that it is the density matrix which is the proper object to represent the world. There is a point of view that only ensembles of worlds can be represented. I shall comment on these later on in this section.

I have of necessity been loose about what I mean by the part of “the world” whose state the statevector represents. It should contain at least what we observe exists and what we thereby infer exists. I will take as a minimal characterization that this means all particles and structures whose existence physicists at present generally agree upon. These do not include the “more” mentioned above, which is posited to be an additional structure in the world. However, it may be possible to include the “more” in the statevector as well (see section 2.3h).

Essential is the view that there is a specifiable “real world out there”[5] whose behavior is completely consistent with our observations. Implied is that we are fair observers (what we see is there) and sound reasoners about the world so our conclusions can be trusted. Not implied is that we are in any way essential to the description of the world. Rather it is the theory of the world that is essential to the description of us and our observations (what has been called empirical reality[6]). I bet against the ultimate value of any theory whose logical consistency requires human beings[7].

All statevectors having the same “direction” in Hilbert space characterize the same state of the world. This generalizes the “ray” (overall phase) independence of the SQT description to norm independence. In SQT the norm (squared) must be 1 because it represents the sum of probabilities, but that is not required when the statevector represents actuality. If two statevectors differ in direction by the slightest bit they represent two different states of the world. However, any one of a “small” ball of directions will suffice to describe the same state of empirical reality[8, 9], i.e., an object may be empirically here when a small amount of its wavefunction is elsewhere.

It is very possible that the statevector should represent more particles and structures than physicists now agree exist. However, I don’t consider that the statevector should represent more variant replicas of our world, as in the many-worlds interpretation of the statevector[10, 11] because it is likely that the foundations of a coherent many-worlds picture requires human beings[12, 13, 14]. But, even with such an interpretation, one of the statevectors in the many-worlds superposition describes our world, and it is this statevector I want to talk about.

The assumption that the statevector gives a complete specification of the familiar particles and structures in the world excludes from discussion the deBroglie-Bohm model[15, 16, 17, 18, 19]. This model is akin to the models I discuss in that both contain an extra structure, a chooser, but the deBroglie-Bohm chooser is the positions of particles. I am betting that the statevector alone satisfactorily describes the particles in the world. Both the many-worlds and deBroglie-Bohm approaches to the central problem addressed here are interesting and have successes and difficulties, but I won’t discuss them.

This central problem is arguably the most important unsolved problem in the foundations of quantum mechanics. It is brought about by certain features
of the world as evidenced by our observations and inferences. These are that events occur in the world, that in certain circumstances they occur in a fundamentally unpredictable (random) way, and that macroscopic objects (choose your own definition) are almost always highly spatially localized—even objects whose location is an unpredictable event.

This latter behavior means that certain statevectors cannot represent possible states of the world, namely statevectors whose evolution describes a macroscopic object continually in a superposition of here and there with comparably sized coefficients. The problem arises because, using the otherwise estimable SQT, one may start with a statevector representing a state of the world, in a situation such as a measurement in which uncontrollable events occur, and it evolves via Schrödinger’s equation into a statevector which does not represent a state of the world, but rather the sum of such states.

The founding fathers of quantum theory had a cure for this. It is to modify the evolution of the statevector so that it instantly “collapses” (or “jumps” or “reduces”) in such circumstances to a viable statevector representing the state of the world. The problem with this modification is that it is terribly ill-defined. It is supposed to be invoked whenever a measurement has been completed, but no one has been able to define what a measurement is or when completion is. Ad hoc means “for this case only,” and the prescription given is very ad hoc, with every situation requiring its own assessment. Perhaps due to the persuasive powers of John Bell[7], more physicists nowadays think this collapse postulate is unsatisfactory.

There have been two approaches which can be said to follow in the footsteps of the founding fathers.

One approach is to try to make the instantaneous collapse postulate well defined without the need for ad hoc information. This is my view of the “consistent histories” program[20, 21, 22]. It has been described as a promissory note[23] which has serious problems in being fulfilled[24]. The problem is that Nature selects a unique set of consistent collapse possibilities but the theory does not.

There is a test I think should be applied to all theories with fundamental pretensions[25]. If, confronted with an initial statevector of a seriously complicated and realistic part of the world (e.g., the local galactic group), are there well-defined procedures for constructing the mathematical quantities which correspond to the possible real events which take place in the world? The point of this test is to see if a theory can do more than just handle some simplified models into which ad hoc features creep. So far the consistent histories approach does not pass this test.

This is as good a place as any to mention that the somewhat related “environmental decoherence” scheme[26, 27] does not pass this test either[28]: so far the choice of “environment” is ad hoc. This scheme is a prime example of trying to use the density matrix to represent the world, which has the following problems. If the density matrix is, in the Victorian characterization of d’Espagnat[29] “Pure and Proper”, i.e., $= |\psi, t > < \psi, t |$, then it is equivalent
to a statevector and must undergo collapse in order to represent the world. If it is Impure (mixed) and Proper then it may either represent an ensemble of worlds or our ignorance about our world, but not the world itself. But if it is e.g., a density matrix traced over the “environment,” it begins as a Pure and Improper representation of our world and ends up interpreted as an Impure and Improper representation of an ensemble of worlds or of our ignorance about the world. The interpretation of a consistent theory should not undergo dynamical evolution. This is why I do not believe a density matrix is the appropriate object to represent the world.

2 Describing Collapse

A second approach following the founding fathers, and espoused here, is to agree with them that the evolution of the statevector should be modified but to do it so the collapse is not instantaneous but, rather, follows a well-defined dynamics of a modified Schrödinger equation. This is a bet that there is a real physical process which causes events to occur which is not yet in physics and it is worthwhile to try to make a phenomenological model of it. Such a model is very strongly constrained because it must agree with empirical reality, which includes all of tested physics, the random choices made by nature and the localization of macroscopic objects.

In order to describe the random choices made by nature the Schrödinger equation must have a chooser in it. The first such model[30] used some hidden variables. Subsequently, I tried to use the phases of the superposition[31] as chooser (which is appealing because nothing “more” must be added beyond what is in the statevector). Then, betting that the chooser is one of many things in nature which fluctuate randomly, I settled on modelling it by external random noise[32], a choice which has been adopted in subsequent work[33, 34, 35, 36, 37]. In this section I shall discuss variations on the theme of collapse formalism.

2.1 Gambler’s Ruin Game

With external random noise as the chooser, it turns out that the mechanism for obtaining agreement with the predictions of quantum theory is very simple[38] which suggests one is on the right track.

The mechanism is the gambler’s ruin game[39]. Suppose one gambler has $36 and another has $64, and they toss a coin to determine who gives a dollar to the other. Their dollar amounts fluctuate. Eventually one gambler wins all the money. It is readily shown that the probability of winning all the money is .36 for the first (.64 for the second).

Precisely analogous is the modelled evolution of the initial statevector

\[ |\psi, 0 > = .6|a_1 > + .8|a_2 > . \] (1)
Under the influence of the random noise, the amplitudes multiplying $|a_1>$ and $|a_2>$ fluctuate. Eventually the statevector ends up as $|a_1>$ with probability .36 (ends up as $|a_2>$ with probability .64). This final result is just what the SQT collapse postulate accomplishes.

2.2 Simple Model

Here is a simplest example I can give of such a collapse dynamics.

Two equations are required for a collapse model. The first is the dynamical equation which replaces Schrödinger’s equation. Remarkably, it is possible to write this as a linear equation for the statevector\[40, 41\]. This simplicity is very useful e.g., it makes possible the relativistic collapse models discussed in section 4. The dynamical equation depends upon a classical white noise function $w(t)$. However, for this simple example, the solution actually depends only upon $\int_0^t w(t')dt' \equiv B(t)$ which is a classical Brownian motion function. Here is the solution $|\psi, t> = |\psi, 0>$ when the initial state is (1):

$$|\psi, t> = e^{-\frac{\tau}{4}[B(t)-2\lambda a_1]^2}|a_1>, \quad |\psi, t> = e^{-\frac{\tau}{4}[B(t)-2\lambda a_2]^2}|a_2>.$$  \hspace{1cm} (2)

As will soon be seen, the operator $A$ determines the choices while $B(t)$ is the chooser. $A$’s eigenstates, $|a_1>$, $|a_2>$ (eigenvalues $a_1$, $a_2$) are the states to which collapse occurs. $A$’s eigenvalue differences, together with $\lambda$ (a parameter of the theory) determine the collapse rate.

The evolution equation (2) tells us what the initial statevector evolves into under a particular $B(t)$. The evolution is not unitary, so statevectors evolving under different $B(t)$’s have different norms. This plays a role in the second required equation, the probability rule. It gives the probability density for $B(t)$ to be the actual noise that occurs in nature:

$$P_t\{B\} \equiv \psi, t|\psi, t> = e^{-\frac{\tau}{4}[B(t)-2\lambda a_1]^2} + .64 e^{-\frac{\tau}{4}[B(t)-2\lambda a_2]^2}.$$  \hspace{1cm} (3)

Eq. (4) says that the statevectors with the largest norm are the most likely to occur. The total probability, according to (5), is

$$\frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} dB P_t\{B\} = 1.$$  \hspace{1cm} (5)

To see how these equations work, from Eq. (5) we note that the most probable $B(t)$’s occur if $B(t) = 2\lambda a_1$ or $B(t) = 2\lambda a_2$ plus or minus a few standard deviations $(\lambda t)^{\frac{1}{2}}$. For small $t$ these regions overlap, but for large $t$ they don’t. For example, set $B(t) = B_0(t) + 2\lambda a_1$, where the range of $B_0(t)$ is a small integer $\times (\lambda t)^{\frac{1}{2}}$. Then, for large $t$, Eqs. (3), (5) become...
\[ |\psi', t > B > \approx .6|a_1 > e^{-\frac{1}{2\pi} B_0(t)^2} + .8|a_2 > e^{-\lambda|a_1 - a_2|^2} \]

\[ \mathcal{P}_t \{ B \} \approx .36 e^{-\frac{1}{2\pi} B_0(t)^2} + .64 e^{-2\lambda|a_1 - a_2|^2} . \]

Thus the probability associated with \( B_0(t) \) in this range approaches .36 and the statevector approaches \( |a_1 > \). Of course, a similar argument holds for \( B(t) = B_0(t) + 2\lambda t a_2 \), resulting in collapse to \( |a_2 > \) with probability .64. For other ranges of \( B(t) \) the associated probability approaches 0 for large \( t \). This, then, is the way \( B(t) \) acts as chooser: it chooses either \( 2\lambda t a_1 \) or \( 2\lambda t a_2 \) to fluctuate around, and this determines the collapse outcome.

Collapse models such as this provide an explanation for the random results which occur in nature and are unexplained by SQT: the result was this rather than that because the noise fluctuated this way rather than that way. What is left unexplained is at the next level: why did the noise fluctuate this way rather than that way? A future theory may address this question by identifying the noise with something physical (e.g., gravitational fluctuations: see section 3.1) and having dynamics for it.

It is useful to discuss various features of collapse dynamics in the context of this simple model before considering more sophisticated models.

2.2a Density Matrix The density matrix describes the ensemble of statevectors which arise from all possible noises. For our simple example it is

\[ \rho(t) = \frac{1}{\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} dB \mathcal{P}_t \{ B \} dB |\psi, t > B > B < \psi, t |_{B < \psi, t, t > B} \]

\[ = \frac{1}{\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} dB |\psi, t > B > B < \psi, t | \]

\[ = .36|a_1 > < a_1 | + .64|a_2 > < a_2 | \]

\[ + .48 |a_1 > < a_2 | + |a_2 > < a_1 | e^{-\frac{1}{2\pi}(a_1 - a_2)^2} \]

which follows from (4), (3). One clearly sees in the exponential decay of the off-diagonal density matrix elements that the collapse rate increases with increasing difference of \( A \)'s eigenvalues.

It should be emphasized that, while collapse dynamics give rise to such density matrix behavior, it does not follow that such density matrix behavior implies collapse dynamics[42]. Thus, the unitarily evolving statevector

\[ |\psi', t > B_0 = e^{-i B_0(t) A} |\psi, 0 > \]

\[ = .6|a_1 > e^{-i B_0(t) a_1} + .8|a_2 > e^{-i B_0(t) a_2} \]

where \( B_0(t) \) has probability density
\[ P_t \{ B_0 \} = e^{-\frac{B_0^2(t)}{2\lambda t}} \]

has no collapse behavior, just increasingly random phases, yet at every instant its density matrix is equal to (7):

\[ \rho'(t) = \frac{1}{\sqrt{2\pi \lambda t}} \int_{-\infty}^{\infty} dB_0 e^{-\frac{B_0^2(t)}{2\lambda t}} |\psi', t > B_0 < \psi', t| = \rho(t). \]

It is often said that if the same density matrix arises from such different ensembles of statevectors as

Ensemble A:  \(|a_1 >\) (probability .36) plus \(|a_2 >\) (probability .64)
Ensemble B:  .6|a_1 > e^{i\theta_1} + .8|a_2 > e^{i\theta_2} (random \theta_1, \theta_2)

then the ensembles describe the same physics because they predict (statistically) identical experimental outcomes. The crucial point is that, for this argument to hold, it is necessary for the theory to say that experiments have outcomes.

Suppose the state (1) represents the initial result of a measurement interaction where \(|a_1 >, |a_2 >\) describe macroscopically different apparatus states, and suppose A, B represent possible final apparatus ensembles. Each member of ensemble A represents a specific measurement outcome and each member of ensemble B does not. This conclusion is the same if a second apparatus is introduced to measure the state of the first apparatus and the two evolutions take place as did the first (collapse for A, noncollapse for B). Thus one cannot make the argument cited above that ensembles A and B describe the same physics because the argument depends upon both ensembles having a collapse evolution, which B does not have.

In a recent stimulating series of papers, Mermin implies that an important lesson of quantum theory is that there can be no physical difference if there is no experimental difference[43]. This is in keeping with the lesson learned by the previous generation of physicists from the Ether issue. Well, each succeeding generation may well unlearn the lesson of the previous generation, e.g., when it’s worth going to war. The position taken here is the direct opposite, that an important lesson of quantum theory is that just because we cannot measure a difference does not mean that there is no difference. After all, why should we be able to measure all that the world contains? Thus, consider the ensembles A and B but now take both ensembles as subject to the collapse dynamics (2), (4). Although the density matrix for both ensembles remains .36|a_1 > < a_1| + .64|a_2 > < a_2| during the evolution, the dynamics recognizes that the states that make up A and B are different and it treats them differently, leaving the states in A alone and collapsing the states in B.
2.2b Fourier Form It is useful and interesting to write the statevector evolution (2) as a Fourier Transform:

\[
|\psi, t > = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\eta e^{-\eta^2/2} e^{i\eta \int_0^t \frac{1}{\lambda t} [B(t) - 2\lambda t A]} |\psi, 0 > .
\]  

(8)

This says that the collapse evolution can be viewed as a gaussian-weighted sum of unitary evolutions. This is useful because unitary evolutions are well understood—for example, one may use Feynman diagram techniques with them (see section 4.2a). It also may be viewed more profoundly, as a possible natural generalization of the usual unitary evolution of SQT. In SQT the statevector not only represents the world but it carries the extra burden of representing a probability, so its norm must be one and its evolution is limited to being unitary. When the statevector represents reality and is freed from representing a probability it is also freed to have a more general evolution than unitary.

The density matrix can likewise be written as a Fourier transform using (6) and (8):

\[
\rho(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\eta e^{-\eta^2} e^{-i\eta \sqrt{2\lambda t} A} \rho(0) e^{i\eta \sqrt{2\lambda t} A}.
\]  

(9)

Again, as discussed in the previous section, although this is a collapse evolution, the Fourier form of the density matrix displays it as a Gaussian-weighted sum of non-collapse (unitary) evolutions.

2.2c Norm The statevector evolving according to (2) gets diminished in norm (e.g., from 1 to \( \approx .6 \) or \( \approx .8 \) in the example). Further similar evolutions decrease the norm further. However, as we have mentioned, the norm has no importance in a theory with collapse (e.g., amplitude ratios, operator eigenstates are norm-independent). Because the evolution (2) is linear in \( |\psi > \), the right side may be multiplied by a function \( F[t, B(t)] \) and the probability density (4) scaled by \( F^{-2} \) without physical effect. For example, if \( F = \exp[B^2(t)/4\lambda t] \), the evolution equation becomes

\[
|\psi, t >' = e^{B(t) A - \lambda t A^2} |\psi, 0 >
\]

so the final statevectors grow in norm (e.g., \( \sim \exp \lambda t a_r^2 \)). Incidentally, this may be written as

\[
|\psi, t >' = e^{A \int_0^t dt (\frac{dB(t)}{\lambda t} - \lambda A)} |\psi, 0 >
\]

showing that (2) has a time-translation invariant form.

If, because of prior experience with SQT, one is more comfortable with normalized eigenstates, one may feel free to normalize the statevector at any time, or even set \( F = B < \psi, t | \psi, t > \) so that the statevector is normalized at every
instant of time. However, then manifest scale invariance is lost and the evolution equation is nonlinear in $|\psi\rangle$.

For evaluating collapse dynamics with a computer[44] it is useful to have the statevector always of norm 1 since computers don’t handle well numbers which get too small or large. Also computers most easily generate random noise functions which average around 0, and also most easily calculate dynamical evolution in small steps. To do this, one may consider evolution between $t$ and $t + dt$ and write the noise $dB(t)$ in this time interval as

$$dB(t) = dB_0(t) + 2\lambda dt \bar{A}(t)$$
$$\bar{A}(t) = \frac{B < \psi, t|A|\psi, t >_B}{B < \psi, t|\psi, t >_B}$$

thereby causing the probability density to be $\exp\left(-\frac{dB_0^2(t)}{2\lambda dt}\right)$, belonging to an increment of Brownian motion. The statevector may be multiplied by an $F$ chosen to normalize it. The result[45, 33] is the nonlinear equation

$$d|\phi, t >_{B_0} = \left\{ dB_0(A - \bar{A}) - \lambda(A - \bar{A})^2 dt \right\} |\phi, t >_B . \quad (10)$$

In a recent paper[46] the authors wrote that a linear evolution such as (2) is “merely a mathematical relation” and went on to say that “to be truly useful” states should be normalized. Computational usefulness should not be confused with physical importance. The evolution (2) with initial state (1) models a physical quantity $dB/dt$ whose random fluctuations (both positive and negative) about one of the eigenvalues of $A$ drive the statevector toward the associated eigenvector. In the evolution (10) with initial state (1), the noise fluctuates about zero with positive excursions enhancing the eigenstate with the larger eigenvalue and negative excursions enhancing the eigenstate with the smaller eigenvalue. Indeed, although (10) is computationally equivalent to (2) and (4), one might argue that (10) is “merely a mathematical relation” since a symmetrical effect of positive and negative fluctuations about a mean value (such as is embodied in (2)) is the usual type of behavior for a random physical quantity. As another example of the questionable physical nature of such descriptions as (10), it has been shown[47] that such an equation can never lead to a relativistic collapse dynamics, unlike the situation with the linear equation (section 4).

### 2.3 Less Simple Models

The model upon which more sophisticated collapse dynamics is based has the statevector evolution

$$|\psi, t >_w = T e^{-\frac{1}{\hbar^2} \int_0^t dt[w(t)-2\lambda A(t)]^2} |\psi, 0 >$$

($T$ is the time-ordering operator). This evolution is in what may be called the “collapse interaction picture,” where $A(t) \equiv \exp iHtA\exp -iHt$ is a Heisenberg
operator and the statevector only evolves because of the collapse dynamics. The probability density for \( w \) is

\[
P_t\{w\} \equiv <\psi, t|\psi, t >_w.
\]

(12)

This is more complicated than the previous model because the state vector (2) is a function of \( B(t) \) but the statevector (11) is a functional of \( w(t) \). The probability that this noise lies between \( w(t) \) and \( w(t) + dw \) is

\[
\prod_{t'=0}^t \left( \frac{2\pi \lambda/dt}{2\pi} \right)^{-1/2} dw(t') P_t\{w\}.
\]

One imagines the time interval \( t \) divided into infinitesimal segments \( dt \), with each \( w(t') \) as an independent variable.

However, the previous model is a special case of this, when \( H = 0 \) so \( A \) is time independent. Then, writing \( B(t) = \int_0^t dt w(t) \), (11) becomes

\[
|\psi, t >_w = e^{-\frac{1}{4\lambda} \int_0^t dt w^2(t) - \frac{B^2(t)}{4\lambda}} e^{-\frac{1}{4\lambda} [B(t) - 2\lambda A]^2} |\psi, 0 >.
\]

This is just the evolution (2) multiplied by a time-dependent norm factor. Apart from this factor, the evolution only depends upon \( B(t) \) and not upon \( B \) at earlier times so one may integrate the probability density over \( B \) at earlier times (using \( w(t') = B(t') - B(t' - dt) \)), obtaining the probability (4) of \( B(t) \) alone.

One may think of (11) as describing collapse like (2) over each brief interval \( dt \). The equation attempts collapse to \( A(t) \)'s instantaneous eigenstates, with a rate proportional to the squared difference of its eigenvalues. Since \( A(t) \) is changing with time, its eigenvalues provide a “moving target” for \( w(t) \) to fluctuate around. Collapse to one state occurs if the collapse rate characterized by \( \lambda \) is faster than the transition rate between these states characterized by \( H \). Most generally, there is a competition between collapse and Hamiltonian dynamics. But, often states do not appreciably change during collapse, so the behavior is well approximated by setting \( H = 0 \).

2.3a Density Matrix The density matrix which follows from (11), (12) is

\[
\rho(t) = \int_{-\infty}^{\infty} \prod_{0}^{t} \frac{dw(t)}{\sqrt{2\pi \lambda/dt}} |\psi, t >_w <\psi, t | \int_0^t dt e^{-\frac{1}{4\lambda} [B(t) - 2\lambda A]^2} \rho(0).
\]

(14)

\( A \otimes B \rho \equiv A \rho B \) and \( T \) implies time ordering for operators to the left of \( \rho(0) \) and time-reversed ordering for operators to the right).

2.3b Fourier Form The Fourier transform of (11) is found by regarding each \( w(t) \) as an independent variable with conjugate variable \( \eta(t) \), resulting in

\[
|\psi, t >_w = \int_{-\infty}^{\infty} D\eta e^{-\lambda \int_0^t dt \eta^2(t)} T e^{i \int_0^t dt \eta(t)[w(t) - 2\lambda A(t)]} |\psi, 0 >.
\]

(15)
expressed as a superposition of unitary transformations.

The density matrix is easily expressed in Fourier form using (13), (15):

\[
\rho(t) = \int_0^t D\eta e^{-2\lambda \int_0^t dt \eta^2(t)} T e^{-i2\lambda \int_0^t dt \eta(t) A(t)} \rho(0) T_R e^{i2\lambda \int_0^t dt \eta(t) A(t)} \]

\[
= \int_0^t D\eta e^{-2\lambda \int_0^t dt \eta^2(t)} T e^{-i2\lambda \int_0^t dt \eta(t) [A(t) \otimes 1 - 1 \otimes A(t)]} \rho(0)
\]

(\(T_R\) is the time-reversed ordering operator). Of course, if the \(\eta\)'s are integrated over, the result is (14). But, the form (16) displays the density matrix as a superposition of noncollapse unitary evolutions, each describing the interaction of the operator \(A(t)\) with Gaussian-weighted noise \(\eta(t)\).

### 2.3c Many \(A\)'s

To model the collapse of a system where states differ in a number of collapse-relevant features, each feature characterized by an operator \(A_n\), the evolution (11) may be generalized to

\[
|\psi, t >_{w'} = T e^{-\frac{\lambda}{2} \sum_n \int_0^t dt [w_n(t) - 2\lambda A_n(t)]^2} |\psi, 0 > . \tag{17}
\]

The \(A_n\)'s are usually taken to commute at equal times. It is to their joint eigenstates that collapse tends, with a rate proportional to the sum of the squares of their eigenvalue differences.

As an example of how this works, suppose the \(A_n(t)\)'s are time independent and the initial state is \(|\psi, 0 > = \sum_i c_i |i >\) where \(|i > = |a_i^{(1)} > |a_i^{(2)} > ... |A_n|a_i^{(n)} > >= a_i^{(n)}|a_i^{(n)} > . Then the analog of (3) follows from (17):

\[
|\psi, t >_B = \sum_i c_i |i > e^{\frac{\lambda}{2} \sum_n |B_n(t) - 2\lambda a_i^{(n)}|^2} .
\]

Collapse takes place to \(|i >\) with probability \(|c_i|^2\) when \(B_n(t) \approx 2\lambda a_i^{(n)}\). Then the magnitude of the \(j\)th eigenbasis vector decreases \(\sim \exp -\lambda \sum_n |a_i^{(n)} - a_j^{(n)}|^2\), showing that each different feature in \(|i >, |j > increases the collapse rate.

An equivalent form of (17) may be obtained in terms of new operators \(A_n'(t)\) and noise functions \(w_n'(t)\) related to the old ones by a nonsingular matrix transformation \(A_n(t) = \sum_j K_{nj}(t) A_n'(t)\), \(w_n(t) = \sum_j K_{nj}(t) w_n'(t)\):

\[
|\psi, t >_{w'} = T e^{-\frac{\lambda}{2} \sum_j \int_0^t dt [w_j'(t) - 2\lambda A_j'(t)]^2 G_{ij}[w_j'(t) - 2\lambda A_j'(t)]^2} |\psi, 0 > \tag{18}
\]

where the matrix \(G = K^T K\) is symmetric and positive.

It is possible that one may choose \(K\) to be singular. For example, if \(A_1\)'s eigenvalues are all equal then \(A_1\) and \(w_1\) play no role in collapse. Then one may choose \(K_{1j} = 0\), so \(G_{1j} = 0\) and \(A_1\) and \(w_1\) are excised from (18). (Of course, there is one less \(w'\) to integrate over.)
2.3d NonMarkovian Evolution  Another generalization[48] of the evolution (11) is

$$|\psi, t > w = T e^{-\frac{\lambda}{2} \int_0^t dt_1 dt_2 |w(t_1) - 2\lambda A(t_1)|^2 G(t_1 - t_2) |w(t_2) - 2\lambda A(t_2)|^2} |\psi, 0 > . \quad (19)$$

This is a nonMarkovian evolution because it does not satisfy the usual Markov property that two evolutions, one from $t_0$ to $t_1$ followed by one from $t_1$ to $t$, are equivalent to a single evolution from $t_0$ to $t$. In Eq. (19),

$$G(t_1 - t_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t_1 - t_2)} \tilde{G}(\omega) \quad (20)$$

where $G(t - t')$ is real, symmetric under exchange of $t$ and $t'$ and positive, i.e., $\int_{-\infty}^{\infty} dt dt' f(t) G(t - t') f(t') \sim |f(\omega)|^2 \tilde{G}(\omega) > 0$ for arbitrary $f(t)$, so $\tilde{G}(\omega)$ must be real, symmetric and positive. When $\tilde{G}(\omega) = 1$ so $G(t_1 - t_2) = \delta(t_1 - t_2)$, (19) reduces to the Markovian (11). A reason for using the more general form (19) is to adjust the spectrum of $w$, which may be used to lower the amount of energy a system gets from $w$ during collapse.

To see how (19) gives rise to collapse, consider again the initial statevector (1), let $A$ be independent of $t$ and, for definiteness, choose $G(t - t') = (\alpha/2) \exp -\alpha|t - t'|$ ($\tilde{G}(\omega) = \alpha^2/(\omega^2 + \alpha^2)$). Then (19) yields

$$|\psi, t > w \sim \sum_{n=1}^{2} c_n(0) a_n > e^{-\frac{\{B'(t) - 2\lambda a_n |t - \alpha^{-1}(1 - e^{-\alpha t})\|^2}{4\lambda |t - \alpha^{-1}(1 - e^{-\alpha t})|}}$$

where $B'(t) \equiv \int_0^t dt_1 w(t_1) [1 - \frac{1}{2}(e^{-\alpha t_1} + e^{-\alpha(t-t_1)})]$.

For $t >> \alpha^{-1}$ this becomes (3), whose description of statevector collapse was discussed in section 2.2.

In the Markovian case, the probability rule (12) applied at any time $T > t$ gives the same probability for the noise $w(t)$. In the nonMarkovian case this is not so. In the present example

$$\mathcal{P}_T \{w\} = \sum_{n=1}^{2} |c_n(0)|^2 e^{-\frac{\lambda}{2} \int_0^T dt_1 dt_2 |w(t_1) - 2\lambda a_n|^2 e^{-\alpha |t_1 - t_2|} |w(t_2) - 2\lambda a_n|}$$

For fixed $t$, as $T$ increases past $t$, the probability of $w(t)$ only “settles down” (becomes essentially independent of $T$) when $T - t >> \alpha^{-1}$.

What does this mean? We interpret the probability as a measure of rational belief based upon present information[49, 50]. The future is not known at the present; present probability is conditional upon present time. Although the most we can ever know about $\{w\}$ is given by $\mathcal{P}_\infty$, at time $t$ we use $\mathcal{P}_t$ given by (12) because it represents all that can be known at time $t$. In the above example, if $\alpha^{-1}$ is small compared to the collapse time, the difference is not significant.
2.3e Fourier Form of the NonMarkovian Evolution  The Fourier transformation representations of the statevector (19) and its associated density matrix are

\[ |\psi, T >_w = \int_{-\infty}^{\infty} D\eta e^{-\lambda} \int_0^T dt \delta(t) G^{-1}_{0,T}(t,t') \eta(t') \]

and

\[ \rho(T) = \int_{-\infty}^{\infty} D\eta e^{-2\lambda} \int_0^T dt \delta(t) \int G^{-1}_{0,T}(t,t') \eta(t') \]

\[ \mathcal{T} e^{i \int_0^T dt \delta(t) [\omega(t) - 2\lambda \Lambda(t)]} |\psi, 0 > \]

(21)

(22)

\[ G^{-1}_{0,T}(t_1, t_2) \] is the inverse of \( G \) over the interval \((0, T)\):

\[ \int_0^T dt_1 G(t, t_1) G^{-1}_{0,T}(t_1, t') = \delta(t - t') \]

(23)

and depends upon the differences of \( T, 0, t_1, t_2 \). For our example,

\[ G^{-1}_{0,T}(t, t') = \left( 1 - \frac{1}{\alpha^2} \frac{\partial^2}{\partial t^2} \right) \delta(t - t') - \frac{1}{\alpha^2} \left\{ \delta(T - t) \delta'(T - t') + \delta(t) \delta'(t') \right\} \]

\[ - \frac{1}{\alpha} \left\{ \delta(T - t) \delta(T - t') + \delta(t) \delta(t') \right\} . \]

The inverse of \( G \) over an infinite interval, \( \equiv G^{-1}(t - t') \), is much easier to find than its inverse over a finite interval. Indeed, from (20) and (23) with the integral limits \((-\infty, \infty)\), it follows that \( G^{-1}(t - t') \) is the Fourier transform of \( 1/G(\omega) \).

Because \( G^{-1} \) is so simple, it is useful to note that Eqs. (21), (22) may be written with \( G^{-1}_{0,T} \) replaced by \( G^{-1} \) and the double integral limits replaced by \((-\infty, \infty)\). This can be seen from the identity

\[ e^{-\frac{i}{2} \int_{-\infty}^{\infty} dt \delta'(t) G(t-t') f(t')} = \int D\eta e^{-\frac{i}{2} \int_{-\infty}^{\infty} dt \delta(t) G^{-1}(t-t') \eta(t')} e^{i \int_{-\infty}^{\infty} \eta(t) f(t)} \]

With \( f(t) \) nonvanishing only in the interval \( 0 \leq t \leq T \) this becomes

\[ e^{-\frac{i}{2} \int_{0}^{T} dt \delta'(t) G(t-t') f(t')} = \int D\eta e^{-\frac{i}{2} \int_{-\infty}^{\infty} dt \delta(t) G^{-1}(t-t') \eta(t')} e^{i \int_{0}^{T} \eta(t) f(t)} \]

which must be equivalent (manifest when the \( \eta \)'s are integrated over for time values outside the interval \((0, T)\)) to
In the additional \( \eta \) to be an arbitrary function its asymmetrical time integrals are required to make the trace of the associated density matrix \( = 1 \). This density matrix is readily shown to be 

\[
\rho(T) = T e^{-\frac{i}{2} \int_0^T dt e^{i\lambda \int_0^T dt' \psi(t)G(t,t')\psi(t')}} \cdot e^{-\lambda \int_0^T dt \eta(t)G^{-1}(t,t')\eta(t')}
\]

where \( \lambda A_0,T(t_1) \equiv A(t_1) \) for \( 0 \leq t_1 \leq t \) and \( A_0,T(t_1) \equiv 0 \) elsewhere.

In the additional \( \eta \)-independent term, \( A \) interacts with the noise \( w \) through an nonHermitian operators. Nonetheless \( A \) interacts with the noise \( w \) through an nonHermitian operators. Nonetheless

\[
|\psi, T >_w = \int D\eta e^{-\frac{i}{2} \int_0^T dt e^{i\lambda \int_0^T dt' \psi(t)G(t,t')\psi(t')}} \cdot T e^{i \int_0^T dt e^{i\lambda \int_0^T dt' \psi(t)G(t,t')\psi(t')}} \cdot e^{i \int_0^T \eta(t)F(t)} \cdot e^{i \int_0^T dt F(t,t')} w(t') |\psi, 0 >
\]

This is a special case of an even more general nonMarkovian density matrix first discussed by Strunz[52], whose statevector evolution is based upon complex noise and allows nonHermitian operators.

2.3f All-Time Dependent Evolution A variant of the nonMarkovian evolution (19) makes the statevector evolving from time 0 to time \( t \) nonetheless dependent upon \( w \) for all time:

\[
|\psi, t >_w = T e^{-\frac{i}{2} \int_{-\infty}^\infty \int_0^T dt_1 dt_2 w(t_1)G(t_1-t_2)w(t_2) \rho(0)}
\]

where \( \rho(0) \) is the initial state. If the evolution is Markovian then the statevector (24) differs from the statevector (11) only by an inessential numerical factor 

\[
e^{-\frac{i}{2} \int_{-\infty}^0 dt w^2(t_1) + \int_0^\infty dt w^2(t_1)}
\]

i.e., although the expression for the statevector depends upon past and future \( w \)'s, these have no effect. The interpretive issue lies with the probability rule (12). The probability for \( w(t) \) given by \( P_T(w) \) changes quite abruptly as \( T \) increases.
from less than \( t \) to greater than \( t \). However, the interpretation of \( \mathcal{P}_T(w) \) is no different than already stated for a non-Markovian evolution, namely it is a conditional probability, conditioned upon present information. At unaccessed times \( \mathcal{P}_T(w) \) just gives a “neutral probability estimate” of \( w(t) \), i.e., that of white noise with zero mean.

If the evolution is non-Markovian, there is a new wrinkle. The statevector (24) and its associated probability at time \( T \) depend upon \( w(t) \) in the future of \( T \) and the past of \( 0 \) in a non-neutral way, at least for a brief interval (\( \approx \alpha^{-1} \) in our example). There’s nothing contradictory about this, it’s just the way things are according to the model: at the statevector level the present depends at least a little bit on the future.

However, the evolution (24) gives the density matrix

\[
\rho(T) = \int_{-\infty}^{\infty} D\eta e^{-2\lambda \int_{-\infty}^{\infty} dt_1 dt_2 \eta(t_1) G^{-1}(t_1-t_2) \eta(t_2)} \mathcal{T} e^{-\frac{12\lambda}{\pi} \int_0^T dt \eta(t) [A(t) \otimes 1 \otimes A(t)]} \rho(0)
\]

which is equal to the density matrix (22) (see explanation in previous section) which results from the evolution (19). Therefore, at the ensemble level, there is no effect of the future upon the present.

### 2.3g Time-Smeared Evolution

Another non-Markovian evolution form equivalent to (24) is suggested by the observation that, since \( G \) is positive, we may construct \( G^\frac{1}{2} \):

\[
G^\frac{1}{2}(t-t') \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} |\tilde{G}(\omega)|^\frac{1}{2}
\]

so \( \int_{-\infty}^{\infty} dt_1 G^\frac{1}{2}(t-t_1)G^\frac{1}{2}(t_1-t') = G(t-t') \). For our example, \(|\tilde{G}(\omega)|^\frac{1}{2} = \alpha/(\omega^2 + \alpha^2)\) and \( G^\frac{1}{2}(t-t') = (\alpha/\pi)K_0[\alpha(t-t')] \).

If \( w'(t) \equiv \int dt G^\frac{1}{2}(t-t_1)w(t_1) \), \( A_s(t) \equiv \int dt G^\frac{1}{2}(t-t_1)A_0,T(t_1) \) which are time-smeared variables are introduced into (24) we obtain

\[
|\psi, t > w = \mathcal{T} e^{-\frac{1}{2\lambda} \int_0^T dt [w(t) - 2\lambda A_s(t)]^2} |\psi, 0 >.
\]

(26)

Although this looks Markovian, of course it is not. Because \( \mathcal{T} \) time-orders \( A(t) \), not \( A_s(t) \), two sequential evolutions are not equivalent to one evolution since, for \( t \geq t' \), \([\mathcal{T} A_s(t)][\mathcal{T} A_s(t')] \neq [\mathcal{T} A_s(t)A_s(t')] \). We remark that (26) may be useful in a relativistic model where spacelike smearing of an operator should be accompanied by timelike smearing.
2.3h Quantizing the Noise  Although it may appear that the following is not a collapse model, I shall argue that it is.

Consider an operator \( W(t) \) to represent quantized noise\[^41, 53, 54\]:

\[
W(t) \equiv \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} d\omega \left[ e^{-i\omega t} a(\omega) + e^{i\omega t} a^\dagger(\omega) \right]
\]  

(27)

where \( a(\omega), a^\dagger(\omega) \) are annihilation and creation operators \( [a(\omega), a^\dagger(\omega')] = \delta(\omega - \omega') \), etc.) of a mode of frequency \( \omega \) which can take on positive or negative values.

This makes \( W(t) \) unusual for a quantum field, which usually only has positive frequencies. As a result, \( [W(t), W(t')] = 0 \) which also is unusual for a quantum field. But it is a nice property for a quantum field which has classical aspirations since a complete eigenbasis for \( W(t) \) is labelled by eigenvalues at each time:

\[
W(t) |w > = w(t) |w >, \quad < w |w' > = \prod_{t=-\infty}^{\infty} \delta[w(t) - w'(t)].
\]

If \( |0 > \) is the state with no noise mode excited \( (a(\omega)) |0 > = 0 \), one readily finds

\[
<w|0 > = e^{-\frac{1}{4\lambda} \int_{-\infty}^{\infty} dt w^2(t)}.
\]  

(28)

\[dW(t)/dt \] commutes with \( W(t) \): one must look elsewhere for \( W \)'s conjugate operator. It is

\[
\Pi(t) \equiv \frac{i}{8\pi\lambda} \int_{-\infty}^{\infty} d\omega \left[ -e^{-i\omega t} a(\omega) + e^{i\omega t} a^\dagger(\omega) \right]
\]  

(29)

where \( [\Pi(t), \Pi(t')] = 0 \) and \( [W(t), \Pi(t')] = i\delta(t - t') \).

Now, consider the unitary statevector evolution

\[
|\Psi, T > = Te^{-i2\lambda \int_{0}^{T} dt A(t) \Pi(t)} |0 > |\Psi, 0 >
\]

\[
= \int Dw|w > Te^{-i2\lambda \int_{0}^{T} dt A(t) \Pi(t)} < w|0 > |\Psi, 0 >
\]

\[
= \int Dw|w > Te^{-\frac{1}{4\lambda} \int_{-\infty}^{\infty} dt[w(t)-2\lambda A_0(t)]^2} |\psi, 0 >
\]  

(30)

where the last step follows from (28). Eq. (30) says \( |\Psi, T > = \int Dw|w > |\psi, T >_w \), where \( |\psi, t >_w \) is identical to (24) (Markovian version). The standard probability measure of the state \( |w > |\psi, T >_w \) arising from (30) is equivalent to the probability rule (12).

In section 1 I emphasized that such a superposition as (30) cannot represent a possible state of the world. Then, what is the meaning of (30)?
In his famous “cat paradox” paper [55], Schrödinger wrote “For each measurement one is required to ascribe to the $\psi$–function (= the prediction catalog) a characteristic, quite sudden change... The abrupt change by measurement... is the most interesting part of the entire theory.” (30) does not represent a possible state of the world but, in Schrodinger’s terminology, it is the “prediction catalog” of such states. Each $|\psi, t \rangle_w$ in the superposition represents a possible state of the world: one of these is realized, the rest are not. Because of the dynamics presented here, an unambiguous reality assignment to one of the terms in the superposition (30) is allowed which is not possible with SQT. Instead of measurement-induced (whatever that means) abrupt collapses (occurring at ambiguous times), **collapses occur every $dt$ sec.** Each $|\psi, t \rangle_w$ “branches” into a family of possible $|\psi, t + dt \rangle_w$’s with the “label state” $|w \rangle$ distinguishing among them. The states $|\psi, t \rangle_w$ in (30) describe the choices and the attached noise state $|w \rangle$ is the chooser. One may think of an Omar Khayyamish “moving finger” going from $t$ to $t + dt$ and choosing the actual $|w \rangle$’s new $w(t + dt)$ along the way. It is this feature which is either lacking or which is present in poorly defined form in interpretations of SQT without collapse dynamics.

Eq. (30) has a number of features which facilitate this interpretation.

1. For those enamored of environmental decoherence, the noise is effectively a universal environment (which does not have to be defined on an ad hoc basis for each physical situation). This is so desirable it suggests looking for a universal environment as the physical source for the noise (gravity is an attractive possibility—see section 3.1d).
2. The states $|\psi, t \rangle_w |w \rangle$ do not interfere with one another. This allows the interpretation that one of them corresponds to reality and the rest to unrealized choices (one would think that an unrealized choice should not influence reality).
3. The attached state $|w \rangle$ acts as a complete label of the past history of the state $|\psi, t \rangle_w$, making identification of its evolution unambiguous.
4. Each state $|\psi, t \rangle_w$ is physically sensible in that macroscopic objects are (almost always) localized (see section 3.1a).
5. The states $|w \rangle |\psi, t \rangle_w$ are always orthogonal because the $|w \rangle$’s are always orthogonal. The states $|\psi, t \rangle_w$ representing reality are not always orthogonal, e.g., if the initial state $|\psi, 0 \rangle = \alpha|a_1 \rangle + \beta|a_2 \rangle$ evolves continuously under different $w$’s to $|a_1 \rangle$ or $|a_2 \rangle$, these states obviously cannot be orthogonal during the evolution. (Indeed, it is my opinion that what has made a sensible SQT interpretation impossible is that the states which represent reality are not always orthogonal.)

It is possible, in a future more sophisticated theory, that some of these attractive features may be altered, with attendant complications. But, it is interesting that a collapse model based upon classical noise and a completely quantum model can be essentially equivalent. Although I shall not pursue this any further here, a similar completely quantum construction can be made for the nonrelativistic and relativistic collapse models in the next sections.
3 Nonrelativistic Collapse Models

The first Galilean-invariant collapse model was constructed by Ghirardi, Rimini and Weber (GRW)\[56, 57\]. In their “Spontaneous Localization” (SL) model they introduced two important concepts which may be regarded as connected to the two parameters which characterize their model.

The first parameter, $\lambda^{-1} \approx 10^{16} \text{ sec} \approx 3 \times 10^8 \text{ yrs}$, accompanies the concept that collapse is caused by a physical process which acts on all particles. $\lambda$ is the collapse rate for a single particle, chosen small enough so that an individual particle in a superposition is negligibly affected, but large enough so that there is a dramatic effect on a macroscopic collection of $n$ particles in a superposed state, namely collapse in $(\lambda n)^{-1} \text{ sec}$.

The second parameter, the mesoscopic distance $a \approx 10^{-5} \text{ cm}$, accompanies the concept that the collapse process narrows wavefunctions so that particles are spatially localized, but not localized too much. A particle’s widely spaced wavefunction is narrowed to dimensions $\approx a$ by collapse. Since a narrowed wavefunction has increased energy by the uncertainty principle, if $a$ were too small then particles would get too much energy from the process (see section 3.1c). If $a$ were too large, then macroscopic objects would not be well localized by the process.

I shall not discuss the SL model any further as it is not in one of the forms considered here. It has a problem (i.e., the process destroys the symmetry of the wavefunction under particle exchange) and has been superseded by the CSL model discussed in the next section. (See however references \[58, 59\] for reviews of this seminal model.)

3.1 CSL

In order to construct a Galilean-invariant model with features of SL in the form of a type we are considering, take the evolution equation (17) with many $A_n$’s and choose $n$ to be a spatial index so that $A_n(t) \to A(x, t)$:

$$A(x, t) \equiv \frac{1}{(\pi a^2)^{3/4}} \int dz N(z, t) e^{-\frac{1}{2a^2} (x-z)^2}.$$  \hspace{1cm} (31)

Essentially, $A(x, t)$ is proportional to the number of particles/vol in a spherical volume of radius $a$ centered upon $x$—say, for now, that the particles are nucleons. In (31), $N(z) \equiv \xi^\dagger(z)\xi(z)$ is the nucleon number density operator, $(\xi(z), \xi^\dagger(z)$ are the nucleon annihilation and creation operators at $z$) and $N(z, t) = \exp iHtN(z) \exp -iHt$ is the associated Heisenberg operator.

The statevector evolution

$$|\psi, T >_w = T e^{\frac{-1}{\hbar} \int_{T_0}^T dt dx [w(x, t) - 2 \lambda A(x, t)]^2} |\psi, T_0 > \hspace{1cm} (32)$$
and the probability rule (12) constitute what I called the “Continuous Spontaneous Localization” (CSL) model\([60, 45]\) (in appreciation of GRW’s SL model).

In CSL the classical field \(w(x, t)\) chooses to fluctuate around the nucleon number density \((× \text{ a constant, smeared over a volume } \sim a^3)\) of one of the states in a superposition, thereby causing collapse to that state.

### 3.1a Collapse Rate in CSL

It remains to explore consequences of the model. In doing so it is useful to work with the density matrix

\[
\rho(T) = T e^{-\frac{\lambda T}{2} \int_0^T dt dx |A(x, t)\otimes 1 - 1 \otimes A(x, t)|^2} \rho(0). \tag{33}
\]

Suppose that \(|\psi, 0\rangle = \sum_k c_k(0)|n_k\rangle\) where the normalized basis vectors \(|n_k\rangle\) correspond to different nucleon number density distributions: \(N(x)|n_k\rangle = n_k(x)|n_k\rangle\). Setting \(H = 0\), the off-diagonal elements of (33) decay as

\[
<n_j|\rho(T)|n_k> = c_j(0)c_k^*(0)e^{-\frac{\lambda T}{2} \int dx |a_j(x) - a_k(x)|^2} \tag{34}
\]

where \(a_k(x)\) is given by (31) with \(N(z, t)\) replaced by \(n_k(z)\).

Consider \(n\) particles in a small clump of size \(< a\). Suppose the initial state is a superposition of such clumps with widely separated centers \(x_k\), where \(|x_j - x_k| >> a\). We may make the approximation \(n_k(z) \approx n\delta(z - x_k)\). Then (31), (34) give

\[
<n_j|\rho(T)|n_k> \approx c_j(0)c_k^*(0)e^{-\lambda T n^2}
\]

showing that the rate of collapse to one clump is proportional to \(n^2\).

Now consider an extended homogeneous object of uniform number density \(n(z) = \rho\), where the states \(|n_j\rangle\), \(|n_k\rangle\) describe the object in two different places. Let their overlapping volume be \(V_0\) and their nonoverlapping volume be \(V_1\). Then, by (31), \(a_j(x) \approx (4\pi)^{\frac{3}{2}} a^\frac{3}{2} \rho\) inside the volume of each. Outside the volume of each, \(a_j(x) \approx 0\). The integral over the overlapping volume \(V_0\) gives no contribution in (34), and we obtain

\[
<n_j|\rho(T)|n_k> \approx c_j(0)c_k^*(0)e^{-(4\pi)^{\frac{3}{2}} \lambda T n^2(V_1)\rho}. \tag{35}
\]

Thus the collapse rate for such an extended object is proportional to the number \(V_1\rho\) of nonoverlapping particles (and to the number of particles in a volume \(\sim a^3\)). Even if the object is small, a cube with sides \(10^{-4}\) cm in length, with modest nucleon density \(\rho = 10^{25}\) nucleons/cc it has a collapse rate of \(\approx 10^{-8}\) sec. This is the reason for the assertion that macroscopic objects are localized except for very brief intervals.

But, what is meant by localized? A collapsed statevector, corresponding to an object considered to be localized here, generally has a “tail.” This is a “small”
piece of the wavefunction which is not here and which is generally decreasing exponentially with time at the collapse rate (as evidenced by the above off-diagonal density matrix behavior). For a discussion of criteria for the “smallness” of the tails see [8, 9].

In CSL the collapse rate increases as superposed states differ more by having more particles in different places. In this way the interaction of a system with its environment may increase the collapse rate[61, 62]. However, the prime initiator of the collapse is the system itself when it is in a superposition of spatially different states.

Some examples of superpositions have been given where the authors have felt that the superposed states are macroscopic enough so that collapse ought to occur (e.g., a superposition here + there of many photons[63]; a complex molecule in a superposition of excited plus unexcited states[64]) and criticized CSL for not predicting collapse in these cases. But it has been argued that there is no conflict with experience in these cases because collapse indeed occurs when the full experimental situation (e.g., observation of the photons by a human detector[65]; detection of the state of the complex molecule[66]) are taken into account: with CSL, opinion can be replaced by calculation.

3.1b Nonlocality and Locality in CSL Collapse of an individual statevector is a nonlocal process: something you do here affects something there. For example, if a particle is in a superposition of wavepackets here and there and you turn on a position measuring device located near here, then the wavepacket there is affected. Either the particle is detected so the wavepacket there disappears or the particle is not detected and the wavepacket there is all that survives. In either case the particle was neither here nor there before the measurement and is either here or there after the measurement, so the reality status of the particle there has been changed by an action here.

The model clearly displays accord with Bell’s theorem: to produce agreement with the predictions of quantum theory there is nonlocal influence. The nonlocality arises in two ways. One is from the direct product structure of states which requires the apparatus here to be multiplied by the particle state there so that collapse engendered by the former affects the latter. The other way is through the probability rule which requires high probability $w(x, t)$’s everywhere. Thus a high probability $w(x, t)$ at the site of the apparatus is (nonlocally) correlated with a high probability $w(x, t)$ at the site of the particle.

However, this nonlocality on the statevector level cannot be used to send signals from here to there because one has no control over the field $w(x, t)$. This is evident by considering the density matrix in Fourier form:

$$
\rho(T) = \int D\eta e^{-2\lambda \int_0^T dt d\eta \sqrt{\eta}(x, t)} U_{\eta}(T) \rho(0) U_{\eta}^\dagger(T)
$$

where $U_{\eta}(T) = \mathcal{T} \exp{-i2\lambda \int_0^T dt d\eta \sqrt{\eta}(x, t) A(x, t)}$. As is well known, such a unitary transformation does not allow nonlocal influences, nor does a superposition
3.1c Experimental Tests of CSL Since CSL has different dynamics than SQT it provides testably different predictions. A straightforward test might involve interference of a sufficiently large object [67, 68, 69]. For example, if an n-particle bound state wavefunction passes through a two-slit screen, the two emerging packets will fluctuate in amplitude (play the gambler’s ruin game) according to CSL, so they will not have the same amplitude at a distance from the screen. The resulting interference pattern will therefore be different (diminished in contrast) from that predicted by SQT where the packets have the same amplitude. Such an experiment is difficult to do.

A more readily performed experiment is to measure the increased energy of particles due to the collapse process [58, 70]. The expectation value of the energy, $\bar{H}(T) \equiv \text{Trace} H \rho(T)$, can be found using (33). The potential energy part of $H$ commutes with $A(x, t)$ and, when the exponential in (33) is expanded in a power series in $\lambda$ the kinetic energy part of $H$ gives $\int dx [A(x, t), [A(x, t), H]] = \text{const} \times \int dx N(x)$ so all terms of higher order than the first vanish. The result for $n$ particles of mass $m_p$ is

$$\bar{H}(T) = \bar{H}(0) + \frac{3}{4} \lambda T n \frac{\hbar^2}{2m_p a^2}.$$ 

This says $10^{24}$ nucleons gain an average of $\approx 0.3$ eV/sec which corresponds to a temperature increase of $\approx 0.001^\circ$K. Although the average energy increase is quite small, infrequently a particle suddenly gains a large amount of energy and that can be looked for.

We now note that different particles may have different collapse rates. To accommodate this we replace the particle number operator $N(z, t)$ in (31) with $\sum_\alpha g_\alpha N_\alpha(z, t)$, where $N_\alpha$ is the particle number operator for particles of type $\alpha$: a particle’s collapse rate is $\lambda g_\alpha^2$, with $g_e$ for electrons, $g_p$ for protons, $g_n$ for neutrons.

Using (33), the probability/sec, $\Gamma$, of excitation of a bound state $|\psi\rangle$ to an excited state $|\phi\rangle$ (irrespective of the center of mass behavior) is found to be [71]

$$\Gamma = \frac{\lambda}{2a^2} < \phi \sum_j g_{\alpha(j)}(x_j - Q)|\psi >|^2$$

to lowest order in $(\text{size of bound state}/a)^2$. Here the sum is over all particles in the bound state, $x_j$ is the position coordinate of the $j$th particle and $Q = \sum_j m_j x_j / \sum_j m_j$ is the center of mass operator. This rate vanishes identically if $g_{\alpha(j)} \sim m_j$.

A number of experiments currently being done for other purposes (e.g., dark matter searches, neutrino detection, proton decay, etc.) look for the sudden appearance of energy in a volume of matter. To date, for purposes of testing CSL,
the most sensitive experiment has been one which looks for X-rays appearing in a slab of Germanium[72]. If a 1s electron in a Ge atom is ionized by the collapse process, the result is an X-ray pulse of magnitude 11.1 keV (the 1s electron’s ionization energy, emitted as photons from the Ge ion decaying to its ground state) plus the kinetic energy of the ejected electron (converted to photons by the electron’s collision with Ge atoms). The excitation rate of such an electron is found from the equation above to have its largest value

\[ \approx 5000\left(\frac{g_e}{g_p}\right) - \left(\frac{m_e}{m_p}\right)^2 \]

counts/keV-kg-day at \( \approx 11.1 \) keV, assuming that \( a \) has the GRW value and that \( \lambda \) has the GRW value for nucleons (so \( g_p = g_n = 1 \))[73].

The best experimental limit[74] is that the rate at \( \approx 11.1 \) keV is less than \( \approx 1.2 \) counts/keV-kg-day, resulting in \( g_e/g_p \leq 13m_e/m_p \). Thus, according to CSL and experiment, electrons collapse much less rapidly than nucleons. This is suggestive (but only suggestive, given the assumptions about the parameter values) of mass-proportional coupling, 

\[ g_\alpha \sim \frac{m_\alpha}{m_p} \]

This means that \( N(z,t) \) in (31) is to be replaced by \( M(z,t)/m_p \), where \( M \) is the mass density operator. Mass-proportional coupling suggests that collapse is connected to gravity.

### 3.1d Collapse and Gravity

It is hoped that collapse dynamics will some day be seen to arise in a natural way from another area of physics. It has frequently been suggested that this area is gravity[75, 76, 77, 78, 79, 80, 81, 35, 36, 37]. Failing gravity at present to move toward collapse, collapse may move toward gravity as follows.

Let us make a change of variables of type illustrated in (18). We change from \( A(x,t) \) and \( w(x,t) \) to \( M(x,t) \) and \( w'(x,t) \):

\[
A(x,t) = \frac{1}{m_p(\pi a^2)^{3/4}} \int dM(z,t)e^{-\frac{1}{a^2}(x-z)^2} \]

\[
w(x,t) = \frac{2\lambda}{m_p(\pi a^2)^{3/4}} \int dw'(z,t)e^{-\frac{1}{a^2}(x-z)^2} .
\]

Now, \( \lambda a/c \approx 10^{-32} \) is a dimensionless number not far from \( Gm_p^2/\hbar c \approx 10^{-38} \), as was pointed out by Diosi[77]. By substituting for the new variables in Eq. (32) and replacing \( \lambda \) by \( Gm_p^2/\hbar a \) we obtain as evolution equation

\[
|\psi,T > \rightarrow \psi(T) = T e^{-\frac{i}{\hbar} \int_0^T \int dtdz' [w'(z,t) - M(z,t)] \frac{i}{4a^2} e^{-\frac{(z-z')^2}{4a^2}} [w'(z',t) - M(z',t)]} |\psi(T_0) >
\]

(35)

2 Unlike (18), this transformation is singular because \( G \) has some vanishing eigenvalues: its eigenvalue spectrum is \( \exp -a^2k^2/2 \) which vanishes at \( |k| = \infty \). This can be handled technically in various ways and, as discussed following (18), causes no essential difficulty.
We may think of (35) as the nonrelativistic limit of a General Relativistic collapse formulation, where \( M \) is replaced by the trace of the energy-momentum-stress tensor \((\gamma c^2)\), and \( w' \) is regarded as the curvature scalar \((x c^2/8\pi G)\). That is, the collapse is due to fluctuations in the curvature scalar about its “classical value,” an eigenvalue of the operator representing the traced stress tensor.

Ghirardi, Grassi and Rimini\[78\] was the first to suggest that fluctuations in the metric tensor could be related to collapse. Squires and I\[71\] gave a crude model for such fluctuations. In it, Planck mass \((m_{pl})\) point particles randomly appear and disappear from the vacuum in the neighborhood of e.g., a proton, remaining on average for a Planck time with an average density of one “Planckon” per proton Compton wavelength which means that particles get too much energy due to collapse.

It was pointed out by Ghirardi, Grassi and Rimini\[78\] that equations were obtained for the two CSL parameters: the comparable CSL expressions with certain numerical coefficients, so that two features of CSL might arise from an understanding of curvature fluctuations. Although this model should not be taken very seriously, it does suggest that Gravitational quantities other than curvature may be considered to give rise to collapse. Diosi\[77\] was the first to propose a gravitational collapse model of the form discussed here, with fluctuations in the scalar potential about its “classical value,” an eigenvalue of the operator representing the traced stress tensor. His evolution equation may be written as

\[
|\psi; T > = T e^{\int_{t_0}^{T} dt dx dx' [w(x,t) - \phi(x,t)]} \frac{\phi(x,t)}{\frac{1}{2} \sqrt{m_{pl} G m_p}} \approx 1.4 \times 10^{-5} \text{cm}.
\]

Although this model should not be taken very seriously, it does suggest that features of CSL might arise from an understanding of curvature fluctuations.

Gravitational quantities other than curvature may be considered to give rise to collapse. Karolyhazy\[75\] was the first to suggest that fluctuations in the metric tensor (\(-m_p \nabla \phi\) exerted by the Planckons), match the comparable CSL expressions with certain numerical coefficients, so that two equations were obtained for the two CSL parameters:

\[
\lambda = \frac{1}{2(3\pi)^{3/2}} \frac{G m_p^2}{\hbar a}, \quad a = \left( \frac{3}{\pi^2} \right)^{1/4} \frac{\hbar}{4 m_p c} \sqrt{\frac{m_{pl}}{m_p}} \approx 1.4 \times 10^{-5} \text{cm}.
\]

With a suitable change of variable, \( w(x,t) = G \int d\mathbf{z}' d\mathbf{x}' [w'(x',t) - \phi(x',t)]\) (36) can look much like (35). Using \( \nabla_x \cdot \nabla x' = -\nabla^2_x \) in the integrand of (36) and integration by parts one obtains

\[
|\psi; T > = T e^{\int_{t_0}^{T} dt dx dx' [w'(x',t) - \phi(x',t)]} \frac{G m_p}{|x - x'|^{\frac{5}{2}}} [w'(x',t) - \phi(x',t)] |\psi; T_0 >
\]

which has a similar interpretation to (35).
Penrose’s suggestion\cite{76} that collapse arises from the gravitational energy associated with the difference of gravitational fields of superposed states may be implemented by

$$
|\psi, T >_w = T e^{-\frac{1}{\hbar} \int_{T_0}^{T} dt dx dx' [w(x, t) - \nabla \phi(x, t)]^2} |\psi, T_0 > \tag{37}
$$

where fluctuations in the gravitational field $w \equiv g$ about its “classical value” causes collapse.

The density matrices associated with the evolutions (35), (36) and (37) are respectively

$$
\rho(T) = T e^{-\frac{1}{\hbar} \int_{T_0}^{T} dt dx dx' \Delta M(z, t) \frac{\sqrt{s}}{a^2} \Delta M(z', t)} \rho(0) \tag{38}
$$

where $\Delta M(z, t) \equiv [M(z, t) \otimes 1 - 1 \otimes M(z, t)]$, and

$$
\rho(T) = T e^{-\frac{1}{\hbar} \int_{T_0}^{T} dt dx dx' \Delta \phi(z, t) \frac{\sqrt{s}}{a^2} \Delta \phi(z', t)} \rho(0) \tag{39}
$$

where $\Delta \phi(z, t) \equiv [\phi(z, t) \otimes 1 - 1 \otimes \phi(z, t)]$, and

$$
\rho(T) = T e^{-\frac{1}{\hbar} \int_{T_0}^{T} dt dx dx' [\Delta \nabla \phi(z, t)]^2} \rho(0) \tag{40}
$$

where $\Delta \nabla \phi(z, t) \equiv [\nabla \phi(z, t) \otimes 1 - 1 \otimes \nabla \phi(z, t)]$. Eqs. (39) and (40) are the same, as may be seen by integrating by parts in (39). For purposes of comparison with (38) they also may be written as

$$
\rho(T) = T e^{-\frac{1}{\hbar} \int_{T_0}^{T} dt dx dx' \Delta M(z, t) \frac{\sqrt{s}}{a^2} \Delta M(z', t)} \rho(0). \tag{41}
$$

Consider the collapse rate ($\times \hbar$) for a superposition of two states of different spread out $> a$ mass distributions. According to (38), this is essentially $\sum G(\Delta M)^2/a$, where the sum is over cells of volume $a^3$ into which space is divided, and $\Delta M$ is the difference of the state’s masses in each cell, i.e., the sum of the gravitational mass-difference self-energy of each cell. According to (41), the collapse rate ($\times \hbar$) is essentially the gravitational energy associated with the mass-difference (since the integral over $s \approx 1$). The collapse rate for (38) is, so to speak, a local gravitational energy and the collapse rate for (41) is a global gravitational energy.

Anandan\cite{81} has suggested that a natural generalization of Penrose’s suggestion to General Relativity is that collapse arises because of differences in the connection $\Gamma^\rho_{\mu\nu}$. Unfortunately, $[\Delta \Gamma]^2$ is not positive definite so this suggestion cannot be implemented with the formalism presented here.
There have been other proposals regarding gravitational collapse. Collapse based upon energy differences (not energy density differences) was first proposed by Bedford and Wang[82]. It has received an elegant formulation by Hughston[36] which, however, is equivalent to (11) with \( A(t) = H \) and \( \lambda^{-1} = \hbar \mu_{\text{pl}} c^2 \). The problem with such a proposal is that, in collapse as in real estate, location is everything and energy localization does not lead to spatial localization. The proposal of Percival[35] is based upon energy localization for small distances: it’s extension to large distances has not been completed. The proposal of Fivel[37] is based upon collapse acting to decrease a measure of entanglement and has also not been completed. One cannot say at present whether these two models localize satisfactorily.

4 Relativistic Collapse Models

One reason for trying to construct a special relativistically invariant collapse model is that, like Mount Everest, the symmetry is there. But there are other reasons as well. It seems that collapse and relativity are intimately related. In nonrelativistic quantum theory, one cannot communicate to there by initiating collapse here, for any experimental setup whatsoever. This is remarkable: why should a nonrelativistic theory be prevented from long-distance communication in this way when one can communicate long-distance another way, by just sending a particle from here to there with sufficient velocity? It looks like this noncommunicability is a holdover from a relativistic theory where long-distance (superluminal) communication could make it difficult to have a consistent theory.

It might be possible to have collapse in a preferred reference frame, i.e., the comoving frame of the universe, and yet have the experimental results (but not the theoretical structure) agree with relativity. Aharonov and Albert[83] have shown that an instantaneous collapse postulate in a preferred frame will not predict experimental results which disagree with special relativity. This assumes that collapse produces well defined and sensible states, which so far has not been achieved by any instantaneous collapse postulate but has been achieved by collapse models. However, collapse models also bring along with them other effects, and these may not be relativistically invariant in a preferred-frame collapse model. For example, the spontaneous excitation rate of atoms in CSL can be regarded as a clock whose ticking is a measure of the collapse rate. Then a moving atom’s rate will be time-dilated and one may discern the special frame as the one in which the rate is fastest.

Of course there are many phenomena which are preferred frame dependent, e.g., the 2.7°K blackbody radiation. There is nothing intrinsically wrong with having collapse produce preferred frame phenomena. Still, it would be rather peculiar (although the universe is peculiar) for collapse, which allows quantum

\underline{3} This was mentioned to me some time ago by Renata Grassi. It suggests that the collapse rate of any moving object is time dilated—see section 4.1d.
theory to be compatible with relativity, to carry along with it phenomena which are not relativistically invariant. So, it is not unreasonable to guess that collapse may be completely relativistically invariant and to construct relativistic collapse models.

4.1 RCSL1

The first relativistic collapse model[79, 84] has a Markovian evolution:

\[ |\psi, \sigma > = \mathcal{T} e^{-\frac{1}{2\gamma} \int_{\sigma_0}^{\sigma} dx \left[ w(x) - 2\gamma \phi(x) \right]^2} |\psi, \sigma_0 > . \]  

(42)

(\sigma_0, \sigma, represent spacelike hypersurfaces, \( x = (x, t) \), \( dx = dx dt \) and \( \phi \) is a Heisenberg scalar relativistic quantum field). Here collapse occurs because of differences of scalar field amplitude (not because of differences of particle number density as in (32)). The reason for the scalar field is to obtain the parameter \( a \) in a natural relativistic way. \( a \) is the Compton wavelength corresponding to \( \phi \)’s mass \( \mu = \frac{\hbar}{ac} \approx 1 \text{eV} \). The Hamiltonian under which \( \phi \) evolves includes the interaction \( g\phi \bar{\psi}\psi \): of a fermion field with \( \phi \) so that \( \phi \) “dresses” a fermion, i.e., surrounds it with an average Yukawa field of extension \( a \).

Collapse described by Eq. (42) works as follows. A Fermion in a superposition of here plus there is entangled with its scalar field: \( |\psi, \sigma_0 > = |\text{Fermion here}> |\text{scalar here}> + |\text{Fermion there}> |\text{scalar there}> \). \( w \) “chooses” to fluctuate around one of these scalar fields causing collapse to its state and, in so doing, it collapses the attached Fermion state as well. The collapse rate of a massive Fermion in a superposition of locations \( x_1, x_2 \) is[79]:

\[ \frac{\gamma g^2}{2} \int dx_1 \left[ \frac{e^{-|x-x_1|/a}}{4\pi|x-x_1|} - \frac{e^{-|x'-x_1|/a}}{4\pi|x'-x_1|} \right]^2 = \frac{\gamma g^2 a}{16\pi} \left[ 1 - e^{-|x-x'|/a} \right]. \]

In the comparable CSL expression, \( \gamma g^2 a/16\pi \rightarrow \lambda \) and \( \exp -|x-x'|/a \rightarrow \exp -(x-x')^2/2a \).

This is the good news: collapse works well.

4.1a Vacuum Excitation The bad news is that collapse works too well. For simplicity, remove the Fermions from the model. The \( \phi \)-vacuum state may be written as a superposition of eigenstates of \( \phi(x) \). The evolution (42) acting on the vacuum proceeds to collapse it toward one of these eigenstates. This amounts to excitation of the vacuum: \( \phi \)-particles are created. Each momentum \( k \) mode of the vacuum may be regarded as a harmonic oscillator in its ground state. It gets excited so that its average energy increases as \( \bar{H} = \frac{1}{2} \mu \lambda t \), a modest rate of \( \approx 1 \text{eV}/300 \text{ million years} \). However there are an infinite number of modes so the total average energy increases as
\[ \hat{H} = \frac{\lambda \mu t}{2} \sum_{\text{modes}} 1 = \frac{\lambda \mu t}{2} \frac{V}{(2\pi)^3} \int dk \]

i.e., at an infinite energy/sec-vol rate. When Fermions are restored to the model they also are produced at an infinite energy/sec-vol rate.

In a relativistic theory, when any vacuum excitation occurs it must be infinite excitation. (A particle of a particular four-momentum produced from the vacuum in one frame has a different momentum in another frame so, since all frames are equivalent, all frames must have particles of all momenta come out of the vacuum.) Therefore, to make a sensible relativistic theory all vacuum excitation must be eliminated.

4.1b Pictures of Relativistic Collapse Before looking at the cause of vacuum excitation and exploring how to eliminate it, I want to briefly mention a few features of the RCSL1 collapse process. The elucidation of these and other features is mainly due to Ghirardi and coworkers[84, 47, 85, 86] (see his contribution in this volume). The features mentioned here turn out to embody a proposal of Aharonov and Albert[87, 88] regarding instantaneous collapse in a (noninstantaneous) collapse model.

Essentially, in each reference frame the collapse process works as in nonrelativistic CSL (except that the collapse time for a moving object is time-dilated—see section 4.2d). This means that the picture of collapse dynamics is frame dependent. Suppose a particle is in a superposition of wavepackets here and there and that apparatuses here and there are set up to detect the particle at the same time in a particular reference frame. In this frame at that time both apparatuses simultaneously trigger collapse: say, the packet here grows inside the apparatus here while the packet there fades away inside the apparatus there. However, in any other reference frame, only one of the apparatuses triggers the collapse and the packet not in it grows or fades before it reaches the other apparatus. Thus, unlike CSL, in RCSL what triggers collapse and when or where the collapse takes place are frame-dependent. The model does not give objective (observer-independent) answers to what, when and where but, of course, these are not measurable properties.

4.2 Removing Vacuum Excitation in Lowest Order: the Tachyon

To see the reason for the infinite vacuum excitation, and how to cure it, we begin by generalizing the evolution (42), using a combination of the generalizations (18) and (19):\(^4\)

\(^4\) For a more detailed presentation of the material in this and the remaining sections, see reference [89]. In what follows we set \(\hbar = c = 1\) unless mentioned otherwise.
\[
|\psi, \sigma \rangle = T e^{-\frac{1}{4\gamma} \int_{\sigma_0}^\sigma dx dx' [w(x)-2\gamma \phi(x)]G(x-x')[w(x')-2\gamma \phi(x')]} |\psi, \sigma_0 \rangle \tag{43}
\]

where \( G(x - x') = \frac{1}{(2\pi)^4} \int dk e^{ik(x-x')} \tilde{G}(k^2) \). \( \tag{44} \)

The non-Markovian evolution (43) reduces to the Markovian evolution (42) when \( \tilde{G}(k^2) = 1 \) \((G(x-x') = \delta(x-x'))\).

The density matrix associated with (43), in Fourier form, is

\[
\rho(\sigma) = T \int D\eta e^{-2\gamma \int_\sigma_{-\infty} dx dx' \eta(x)G^{-1}(x-x')\eta(x')}
\int_{\sigma_0}^\sigma dx\eta(x)\phi(x) e^{i2\gamma \int_{\sigma_0}^\sigma dx\eta(x)\phi(x)} \rho(\sigma_0) \tag{45}
\]

We see from (45) that the density matrix is the same as for a non-collapse unitary evolution where the scalar field \( \phi(x) \) interacts with a classical noise \( \eta(x) \) with correlation function \( \eta(x)\eta(x') = (4\gamma)^{-1}G(x-x') \) and spectrum \( (4\gamma)^{-1}G(k^2) \).

As is well known, noise will cause a transition between two states if it supplies a four-momentum equal to their four-momenta difference. To create a particle out of the vacuum, the noise has to supply the four-momentum of the particle. For the Markovian evolution, \( \tilde{G}(k^2) = 1 \) so the noise \( \eta \) supplies every four-momentum. To prevent \( \eta \) from creating \( \phi \)-particles out of the vacuum one must set \( \tilde{G}(\mu^2) = 0 \).

### 4.2a Feynman Diagrams

It is possible to use Feynman diagrams to visualize calculations because the density matrix (45) is expressed as the sum of unitary evolutions. Suppose at first that there are no Fermions. The perturbation expansion of (45) provides two kinds of lines. One is \( \phi \), representing a \( \phi \)-particle of four-momentum \( k' \). The other is \( \eta \) representing the propagator \( \tilde{G}(k^2) \): when the integrals over \( \eta \) are performed, only even powers of \( \eta \) give a contribution, and its correlation function \( G(x-x') \) acts like a propagator connecting two \( \phi \)-lines.

The unusual feature is that the Feynman diagrams represent a density matrix. For example, the diagram showing the lowest order vacuum excitation of a single \( \phi \)-particle is \( \downarrow \). One must think of a vertical line down the middle of the diagram, separating it into the parts representing the terms to the left and right of the initial density matrix \( \rho(T_0) \). To the left (right), the unitary transformation is time-ordered (time-reverse-ordered), so one must think of the four-momentum coming in from the future to the past (\( \downarrow \)) to the right and going from the past to the future (\( \uparrow \)) to the left. The only thing that can cross from right to left are \( G \)-propagator lines.

In the diagram \( \downarrow \), four-momentum is conserved because there is zero four-momentum in the initial vacuum state (below the lines) and zero four-momentum
going out: $k^\nu$ goes in at the right, then it passes through the G-propagator and
goes out at the left. The density matrix associated with this diagram is
\[
\rho(T) = \gamma \tilde{G}(\mu^2) T \int \frac{dk}{\omega(k)} a^\dagger(k)|0><0|a(k)
\]
($a(k)$ is the annihilation operator for the $\phi$-particle). The resulting energy in-
crease is
\[
\text{Trace} \int dk \omega(k) a^\dagger(k)a(k)\rho(T) = \gamma \tilde{G}(\mu^2) TV \frac{1}{(2\pi)^3} \int dk.
\]
Thus, as mentioned earlier, if $\tilde{G}(\mu^2) = 0$ the vacuum excitation vanishes. How-
ever, this is only if there are no Fermions, and of course we must have Fermions
since the whole point of collapse models is to collapse their states.

4.2b Timelike Four-Momenta Cause Vacuum Excitation When there are
Fermions, then Fermion pairs can be excited from the vacuum since Fermions
are coupled to the $\phi$-particles. Pair production is represented by the diagram $\bigotimes$
with $\triangledown \triangledown$ attached to the ends of the two $\phi$-lines. Since all four-momentum passes
through the G-propagator, the diagram amplitude is proportional to $\tilde{G}[(p_1+p_2)^2]$
where $p_1$, $p_2$ are the Fermion pair four-momenta. To make this vanish we must
have $\tilde{G}(k^2)$ vanish for $k^2 \geq (2m)^2$.

But, that’s not the end of it. Pairs of $\phi$ particles are produced via the diagram
$\bigotimes$ with $\bigotimes \bigotimes$ on the ends of the $\phi$-lines ($\bigotimes$ is a closed Fermion loop). To avoid this
we must have $\tilde{G}(k^2) = 0$ for $k^2 \geq (2\mu)^2 = (2eV)^2$. That leaves very little of the
timelike spectrum on which $\tilde{G}(k^2)$ doesn’t vanish. Indeed, it must vanish there
too. When the Fermion is charged, photon production from the vacuum (via the
previous diagram where a single photon emerges from the closed Fermion loop)
occurs unless $\tilde{G}(k^2) = 0$ for $k^2 \geq 0$.

4.2c Enter the Tachyon By removing timelike four-momenta from the noise
spectrum $\tilde{G}(k^2)$, vacuum production of particles with timelike four-momenta
is avoided. But, with only spacelike four-momenta remaining in the spectrum,
the immediate question is whether the nonMarkovian evolution (45) describes
collapse and gives CSL in the nonrelativistic limit. The answer is yes. Timelike
four-momenta excite the vacuum and spacelike four-momenta cause collapse.

Consider the $c \to \infty$ limit of $G$:
\[
G(x - x', t - t') = \frac{1}{(2\pi)^4} \int d\omega dk e^{i\omega(t-t')-ik(x-x')} \tilde{G}(\frac{\omega^2}{c^2} - k^2)
\]
\[
\rightarrow \delta(t-t') \frac{1}{(2\pi)^3} \int dk e^{-ik(x-x')} \tilde{G}(-k^2).
\]
(46)
In this limit, $G$ is Markovian. If we choose $\tilde{G}(k^2)$ to be nonvanishing at only a single spectrum four-momentum $-\mu^2$:

$$\tilde{G}(k^2) = \delta(k^2_0 - k^2 + \mu^2) \tag{47}$$

then it follows from (46) that

$$G(x - x') \to \delta(t - t') \frac{1}{2\pi^2} \frac{\sin[|x - x'|/a]}{|x - x'|}. \tag{48}$$

The nonrelativistic limit (48) is a fine replacement for the CSL Gaussian[90]. (For the record, the exact expression for $G(x)$ is $-(8\pi^2ax)^{-1}N_1(x/a)$ for spacelike $x \equiv (x^2 - x_0^2)^{1/2}$ and $G(x) = -(4\pi^3ax)^{-1}K_1(x/a)$ for timelike $x \equiv (x_0^2 - x^2)^{1/2}$, whose spacelike oscillations with period $a$ and timelike decay with time constant $a/c$ gives rise to (48)).

(47) is the spectrum of a tachyon of mass $\mu = 1/a$. With this choice, the scalar field of mass $\mu$ is redundant: it isn’t needed any longer to get the parameter $a$ into the model. One may remove $\phi$ from the evolution equation (43), replacing it with $N(x) \equiv \psi(x)\overline{\psi}(x)$: (then $\gamma$ is dimensionless) or the trace of the stress tensor $M(x) \equiv m \psi(x)\overline{\psi}(x)$: (then $\gamma$ has the dimensions of $G$, i.e., mass$^{-2}$).

The vacuum pair production diagram $\triangledown\triangledown$ vanishes because the spacelike four-momentum of the tachyon propagator (formerly called the G-propagator) cannot equal the timelike four-momentum of the fermion pair.

If asked to propose a natural nonlocal relativistic structure, it is likely you would mention the tachyon. It is interesting that the need here for the tachyonic structure did not arise from any abstract desire for it but rather from the needs to eliminate vacuum excitation in lowest order and to obtain the nonrelativistic CSL limit. Tachyons and collapse are two distinct theoretical structures requiring careful treatment lest superluminal communication and the difficulties of causal loops creep in: it is interesting that they are conjoined here.

**4.2d Collapse Rate and Energy Production Rate** We now have a model with evolution equation

$$|\psi, T >_w = T e^{-\frac{\gamma}{2}\int_0^T dx dx' [w(x) - 2\gamma N(x)]G(x-x')}|\psi, T_0 > \tag{49}$$

(we have used $N$ rather than $M$ to facilitate comparison with CSL’s Eq. (32)) which is finite in lowest order, and we can calculate things. The associated density matrix in Fourier form is

$$\rho(T) = T \int D\eta e^{-2\gamma \int_{-\infty}^\infty dx dx' \eta(x)G^{-1}(x-x')\eta(x')}$$
\[ e^{-2\gamma \int_{T_0}^T dx \eta(x)[N(x) \otimes 1 \otimes 1 \otimes N(x) - 1 \otimes 1 \otimes N(x)]} \rho(T_0) \]  
\[ = \rho(T_0) - \frac{\gamma}{2!} \int_{T_0}^T dx dx' G(x-x') T[N(x), [N(x'), \rho(T_0)]] + \ldots \]  

(50)

CSL’s comparable expressions are of the same form, where one replaces \( G(x-x') \) by \( \delta(t-t') \exp - (x-x')^2 / 4a^2 \), and \( N \) by the nonrelativistic particle number density operator.

The Feynman diagrams corresponding to (51), describing collapse of a single Fermion in lowest order, are \[ \square \] and \[ \triangle \] which give the rate of depletion of the initial state (coming from (51)’s \( N(x)N(x') \rho(T_0) \) and \( \rho(T_0)N(x)N(x') \) respectively) and \[ \blacksquare \] which gives the transition rate to the new state (coming from \( N(x)\rho(T_0)N(x') \)).

In the following calculations I have replaced \( N(x) = \bar{\psi}(x)\psi(x) \) with \( N(x) = 2m\psi^2(x) \) where \( \psi \) is a Boson field to simplify the calculation by avoiding Dirac algebra. The relativistic energy increase of the particle is found to be

\[ \frac{d}{dt} \bar{H}(t) = \frac{1}{2\pi^2} \frac{\gamma \mu^3}{m} \sqrt{1 + (\frac{\mu}{2m})^2} \]

which is independent of the particle’s four-momentum (\( d\bar{H}(t)/dt \) is an invariant). For \( \mu/m << 1 \) and \( \gamma = \lambda \mu^{-1} \) this is the same (apart from numerical factors) as the nonrelativistic rate \( 3\lambda/8ma^2 \) of section 3.1c.

The relativistic collapse rate of a particle in an initial state \( \alpha|L\rangle + \beta|R\rangle \), where \( |L\rangle \) and \( |R\rangle \) represent two widely separated identical wavepackets with momentum-space wavefunction \( \Psi(p) \), is characterized by the decay of the off-diagonal density matrix element:

\[ \frac{d}{dt} <L | \rho(T) | R> = -\alpha \beta^* \frac{1}{\pi} \gamma \mu \sqrt{1 + (\frac{\mu}{2m})^2} \int dp m |E| \Psi(p)|^2 \]

In the nonrelativistic limit one can set \( E = m \) in the integrand and the result is the same as the CSL result \(-\alpha \beta^* \lambda\) (see the equation following (34) in section 3.1a). More generally, if the particle has a fairly well defined momentum we may make the approximation \( |\Psi(p)|^2 \approx \delta(p - p_0) \), with the result that the collapse rate, \( \sim \sqrt{1 - v_0^2} \), is time dilated.

### 4.3 Finite RCSL Model for Free Particles

Beyond lowest order there are diagrams with internal Fermion lines and, like the Hydra, vacuum excitation rears its head again. Pairs are produced from the vacuum via the second order diagram \( \Psi \).

Moreover, a free Fermion can gain infinite energy. From the first order diagram \( \square \) (utilized in the last section), where the incoming and outgoing lines are on-shell, easy kinematics shows that an incident Fermion at rest gains a small
energy $\mu^2/2m$ ($\approx 10^{-6}\text{eV}$ for an electron) and momentum $\mu \sqrt{1 + (\mu/2m)^2}$ by emitting a (negative energy) tachyon. However, a Fermion can gain arbitrary energy via the second order diagram $\square$.

There are two reasons for this. One is that the sum of four-momenta of two tachyons is capable of adding up to any four momentum, e.g., the sum or difference of two Fermion four-momenta. The other is that a Feynman Fermion propagator $\sim (p^2 - m^2 + i\epsilon)^{-1}$ is capable of carrying away from a vertex any four momentum, e.g., the sum or difference of a Fermion and tachyon four-momentum.

At present I know of three approaches to try solving this problem, only one of which I have spent some time at. In one (largely unexplored) approach, the idea is to limit the four-momentum a particle propagator can transfer by introducing a relativistically invariant cutoff, effectively “smearing” $\int \cdots$ $: \overline{\psi}(x_1)\psi(x_2):$. A second (unexplored) approach has been suggested by Diosi (private communication). He notes that the density matrix for electrodynamics may be traced over the electromagnetic field, resulting in a more general nonMarkovian density matrix than we have been using here for RCSL[91]. However, it is a form for which Strunz[52] has given a collapse dynamics (and for which a simpler collapse dynamics than Strunz’s is given at the end of section 2.3e). This statevector and density matrix evolution describes just the particle behavior in electrodynamics (as in Wheeler and Feynman’s classical action-at-a-distance electrodynamics) and is relativistically invariant. The collapse dynamics replaces the decoherence due to radiation by particles. There is no energy creation problem—indeed, the extra $F$ term effectively absorbs energy, accounting for the radiative loss of electrodynamics. This is not a good candidate for a fundamental collapse model in that e.g., a massive body in its ground state in a superposition of here plus there will stay that way. But, it suggests that it may be possible to construct a satisfactory collapse model with the extra freedom of a more general nonMarkovian density matrix.

### 4.3a Removing Time Ordering

The approach I shall discuss is unconventional, but it may be simply stated. The evolution equation is (49) in Fourier form, except with the time ordering removed:

$$|\psi, T >_w = \int D\eta e^{-2\gamma \int_{-\infty}^{\infty} dx dx' \eta(x) G^{-1}(x-x') \eta(x')} \cdot e^{i \int_{T_0}^{T} dx \eta(x) [w(x) - 2\gamma N(x)]} |\psi, T_0 > .$$

Likewise, the associated density matrix is (50) with $T$ removed.

What does this achieve? The Feynman diagrams with or without time-ordering are the same. However, the off-shell propagator $<0 | T \psi(x) \overline{\psi}(x') | 0>$ is replaced by the on-shell propagator $<0 | \psi(x) \overline{\psi}(x') | 0>$. That is, even when particle lines are internal, each vertex $\square$ describes a process where a real Fermion emits/absorbs a real tachyon, thereby gaining a limited amount of energy (it
is readily shown that $\Delta E/E \leq \mu/m$ for $\mu << m$). Moreover, there is no pair production, real or virtual, at any vertex: $\nabla$ vanishes since a spacelike four-momentum cannot equal a timelike four-momentum:

$$\int^T_{-T} dx' G(x - x') \psi_+^2 (x') \to 0 \quad \text{as } T \to \infty$$  \hfill (53)

($\psi_+(x)$ is the positive frequency—the annihilation—part of $\psi(x)$). The result is a completely finite S-matrix: for large $T$, because of (53),

$$\rho(T/2) = \sum_{n=0}^{\infty} \frac{(-4\gamma)^n}{(2n)!} \int_{-T/2}^{T/2} dx_1 ... dx_{2n} \sum_C G(x_{i_1} - x_{i_1}) ... G(x_{i_{2n-1}} - x_{i_{2n}})$$

\[ \cdot \left[ \tilde{\psi}_-(x_1) \psi_+(x_1), ... \left[ \tilde{\psi}_-(x_{2n}) \psi_+(x_{2n}), \rho(-T/2) \right] ... \right] \]  \hfill (54)

($C$ means the sum is over all combinations of pairs of indices; for simplicity I have assumed that no antiparticles are initially present and so have omitted $\psi_- \tilde{\psi}_+$ terms). We see from (54) that there is no need for renormalization. One may also set $N(x) = \tilde{\psi}_-(x) \psi_+(x)$ in the evolution equation (52).

Removing the time-ordering operation still gives an evolution which describes collapse. Indeed, in the usual test of collapse one sets $H = 0$ and then, since operators are time-independent, time-ordering is irrelevant.

It is worth remarking that removal of time ordering in a usual field theory results in the trivial S-matrix $S = 1$ since energy-momentum conservation prevents a real process at each vertex such as a charged particle emitting a photon. It is also worth remarking that reinstating time-ordering in e.g., (54) would result in a nonrelativistic expression since $T \psi_+(x) \tilde{\psi}_-(x')$ is not Lorentz invariant ($\psi_+(x)$, $\tilde{\psi}_-(x')$ do not commute at spacelike separation). A non-trivial finite relativistic S-matrix without time-ordering is possible because of the tachyon.

This model, however, is only for free particles. If the Fermion interacts with other particles, the free-particle field $\psi$ should be replaced by the Heisenberg field $\psi_H$. The removal of time-ordering only applies to the $\psi_H$’s involved in the collapse interaction, not to those involved in e.g., the electromagnetic interaction, or else the usual electrodynamic physics will be destroyed. But this allows the Feynman off-shell propagator to connect two vertices, one where a tachyon is emitted, the other where a photon is emitted, with the result that the photon can get infinite energy.

### 4.3b Time-Ordering and No Time-Ordering

Removing time-ordering introduces nonlocality. How bad is it?

To see what no time-ordering does, we begin by comparing the two perturbation series:

$$\mathcal{T} \{e^{-i \int_0^T dtV(t)} \} = \sum_n (-i)^n e^{tH_0 T} \int_0^T dt_n ... \int_0^{t_3} dt_2 \int_0^{t_2} dt_1$$
\[ e^{-iH_0(T-t_n)}V(0) \cdots e^{-iH_0(t_3-t_2)}V(0)e^{-iH_0(t_2-t_1)}V(0)e^{-iH_0t_1} \]

\[ e^{-i \int_0^T dt V(t)} = \sum_n \left(-\frac{i}{n!}\right) e^{iH_0T} \int_0^T dt_n e^{-iH_0(T-t_n)}V(0)e^{-iH_0t_n} \cdots \]

\[ e^{iH_0T} \int_0^T dt_2 e^{-iH_0(T-t_2)}V(0)e^{-iH_0t_2}e^{iH_0T} \int_0^T dt_1 e^{-iH_0(T-t_1)}V(0)e^{-iH_0t_1} \]

With time-ordering, particles evolve freely for time \( t_1 \), exchange momenta, evolve freely for time \( t_2 - t_1 \), exchange momenta, etc., until time \( T \). Without time-ordering, particles evolve freely for time \( t_1 \), exchange momenta, evolve freely for time \( T - t_1 \) and then evolve backwards in time for time \( T \), and then do it all over again.

The hallmark of no time-ordering is backwards in time evolution. Repeated forwards and backwards in time evolutions can take a particle outside its light cone. However, as shall now be explained, in the present model this occurs with very small probability.

4.3c Particle Evolution Consider the wavefunction of a particle which starts out centered at \( x = 0 \) and with momentum centered at zero as well. Consider how this wavefunction spreads in lowest order. This turns out to be the same as the spread of a classical ensemble of particles. In this ensemble, each particle starts out at \( x = 0 \) with velocity \( v = 0 \). At random times in the interval \( (0, T) \), particles suddenly emit a negative energy tachyon in a random direction, thereby gaining velocity \( \approx (\mu/m)c \) (assuming \( \mu << m \)). Thereafter each particle continues moving freely forward in time for the rest of the time interval and, following this, it moves freely backward in time for \( T \) sec.

A particle which received its impulse at \( t = 0 \) travels a distance \((\mu/m)cT\) by time \( T \) but, in its subsequent time-reversed evolution, returns to \( x = 0 \). At the other extreme, a particle which receives its impulse at time \( T \) ends up, after its time-reversed motion, at a distance \((\mu/m)cT\) from the origin in the direction opposite to its velocity. Thus the particles end up distributed in a sphere of radius \((\mu/m)cT\), with velocities of magnitude \((\mu/m)c\) directed radially inward. These distributions describe the wavefunction. (If the wavefunction is that of a particle initially moving with some particular nonzero velocity, the description is the appropriately Lorentz invariant one, e.g., the spread of the wavefunction is Lorentz contracted in the direction of motion.)

If the particle emits/absorbs n tachyons, it undergoes n successive evolutions as described above. The sum of ladder diagrams \([- + \square + \ldots\] essentially describes the particle as undergoing a random walk with spread \( \approx (\mu/m)cT \) (this spreads faster than the usual random walk \( \sim \sqrt{T} \)). The superluminal part of the wavefunction is negligibly small— a tail.

4.3d Approximate Locality Consider two separate clumps of particles, L and R, at \( t = 0 \), and the light cones emanating from them. Assume the light
cones do not overlap at time $t = T$. Suppose the statevector evolution is unitary, $|\psi, T > = U(T)|\psi, 0 >$. One may demonstrate locality, that anything going on in R during the interval $(0, T)$ doesn’t affect what goes on in L, by showing that $\text{Trace}_R\{U(T)\rho(0)U^\dagger(T)\} = U^\prime_L(T)\text{Trace}_R\{\rho(0)\}U^\dagger_L(T)$, where $U^\prime_L$’s operators only act upon the particles in L.

In a standard quantum field theory, where the operators in $U$ are time-ordered, one may write $U(T)|\psi, 0 > = U_R^\prime(T)U^\prime_L(T)|\psi, 0 >$, where the operators in $U_R^\prime(T)$ ($U_L^\prime(T)$) act only within R’s (L’s) lightcone, and the above result follows. One can do this because the operators within R’s lightcone commute with the operators within L’s lightcone and because the operators in $U$ which are not in these lightcones and which may not commute with R’s or L’s operators do not contribute to the evolution of $|\psi, T >$ (they act on the no-particle state in their region of spacetime and their contribution gets negligibly small for $T >> \hbar/2mc^2$, the virtual pair production time).

In the model presented here, the density matrix in Fourier form is a Gaussian-weighted superposition of unitary transformations $U_n$, where the operators in $U$ are not time-ordered. Because of the attendant forward and backward in time evolution, the operators in $U$ which act on e.g., R’s particles, occupy more than R’s lightcone. Neglecting the tails, R’s particles at time $T$ are within R’s lightcone, but the operators that got them there occupy a cylinder whose cross-section at any time is the cross-section of the lightcone at time $T$. While all the operators within R’s cylinder do not commute with all the operators within L’s cylinder, nonetheless the operators within R’s cylinder do not act on L-particles, and vice versa. Therefore it is still true in this case that $U(T)|\psi, 0 > = U^\prime_R(T)U^\prime_L(T)|\psi, 0 >$, where the operators in $U_R^\prime(T)$ ($U_L^\prime(T)$) act only within R’s (L’s) cylinder, and the above-mentioned demonstration of locality (or rather approximate locality, since we have neglected the tails) goes through.

This shows that it is possible to make a finite relativistic collapse model. As more such models are constructed, common features may be separable from model-specific ones. This can lead to a better understanding of what is necessary in such models and, perhaps, a model with particularly satisfying features may stand out.

4.4 Concluding Remarks

There is a big difference between a conditional statement and an absolute statement: “if you win the lottery then you will get ten million dollars” can’t compare with “you have won the lottery and you get ten million dollars.”

The statements of SQT are conditional. Faced with the statevector $c_1|a_1 > + c_2|a_2 >$, SQT says “if this is the description of a completed measurement then the physical state is $|a_1 >$ or $|a_2 >$. But actually, what the if is conditioned upon, what the words “a completed measurement” mean, lies outside the theory’s ken. SQT is not a complete description of nature because it fails to predict a physical phenomenon, namely that an event does—or does not—occur.
Phenomenological models are introduced into physics to describe phenomena that present theory fails to adequately treat. Collapse models are phenomenological models. Their statements are absolute. Faced with the statevector $c_1|a_1> + c_2|a_2>$, the collapse model says that represents the physical state. If you wait a bit, it may happen that the statevector is unchanged, and that’s that. Or, it may occur that the statevector rapidly evolves to $|a_1>$ or $|a_2>$, and again that’s that. By “that’s that” is meant, in all cases, that the statevector represents the physical state: the model tells you whether or not an event occurred.

A phenomenological model becomes convincing when it produces experimentally verified predictions beyond those engineered into its construction and achieves respectability when it is wedded to the rest of physics. So far, neither of these has happened for collapse models, but they are fairly young. However, the predictions are there (section 3.1c) and perhaps, in the hints of gravity (sections 3.1c,d) or tachyons (sections 4.2—in string theory models, tachyonic modes of strings appear—and are gotten rid of) there may be the seeds of a connection to another physics domain.

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