Intermediately subcritical branching process in random environment: the initial stage of the evolution

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Abstract

We consider branching process evolving in i.i.d. random environment. It is assumed that the process is intermediately subcritical case. We investigate the initial stage of the evolution of the process given its survival up to the distant moment of time.

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1 Introduction and main result

Consider a branching processes in random environment (BPRE). For the first time BPRE have been introduced in [3, 13] as a model for the development of a population. Particles in this process reproduce independently of each other according to some random reproduction laws which can vary from one generation to the other. The complete and detailed construction of the model of branching process in random environment is given in the monograph [18]. We give here a short description of it. Denote by Δ the space of all probability measures on \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \). Equipped with a metric, it is a Polish space. Let \( Q \) be a random variable taking values in Δ. An infinite sequence \( \Pi = (Q_1, Q_2, \ldots) \) of i.i.d. copies of \( Q \) is called a random environment and \( Q_n \) is the (random) offspring distribution of a particles in generation \( n-1 \). Let \( Z_n \) be the number of particles in generation \( n \).

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A sequence of non-negative integer-valued random variables $Z_0, Z_1, \ldots$ is called a branching process in the random environment $\Pi$, if $Z_0$ is independent of $\Pi$ and, given $\Pi$ the process $Z = (Z_0, Z_1, \ldots)$ is a Markov process with the law

$$
\mathcal{L}(Z_n \mid Z_{n-1} = z, \Pi = (q_1, q_2, \ldots)) = q_n^z
$$

for every $n \in \mathbb{N} = \{1, 2, \ldots\}$, $z \in \mathbb{N}_0$ and $q_1, q_2, \ldots \in \Delta$, where $q_n^z$ is the $z$-fold convolution of the measure $q$.

We denote by $P$ the corresponding probability measure on the underlying probability space. For convenience we assume that $Z_0 = 1$ and $P(Q(0) = 1) < 1$ (here and in what follows we use the notation $Q(y), q(y)$ for $Q\{\{y\}\}, q\{\{y\}\}$).

For a probability measure $q \in \Delta$ we define its mean

$$
m(q) := \sum_{y=0}^{\infty} y q(y).
$$

Introduce the so-called associated random walk $S = (S_n)_{n \geq 0}$. This random walk has increments given by

$$
X_n := \log m(Q_n), \quad n \in \mathbb{N}_0.
$$

These increments are i.i.d. probabilistic copies of $X := \log m(Q)$. Assume that $S_0 = 0$.

A BPRE is called critical if $S$ is an oscillating random walk. A BPRE is called subcritical if its oscillating random walk drifts to $-\infty$ $P$-a.s. as $n \to \infty$. A BPRE is weakly subcritical if $\mathbb{E}[X e^X] > 0$, intermediately subcritical if $\mathbb{E}[X e^X] = 0$, and strongly subcritical if $\mathbb{E}[X e^{-X}] < 0$.

Clearly

$$
\mathbb{E}[Z_n \mid \Pi] = \prod_{k=1}^{n} m(Q_k) = \exp(S_n) \quad P\text{-a.s.,}
$$

and as $n \to \infty$

$$
P(Z_n > 0 \mid \Pi) \leq \min_{0 \leq k \leq n} \mathbb{E}[Z_k \mid \Pi] \leq \exp \left( \min_{0 \leq k \leq n} S_k \right) \to 0 \quad P\text{-a.s.}
$$

for critical and subcritical BPRE. We will assume that the following restriction is valid.

**Assumption A1.** The process $Z$ is intermediately subcritical, i.e.

$$
\mathbb{E}[X] < 0, \quad \mathbb{E}[X e^X] = 0.
$$
One of the essential instruments for the study of the properties of PBRE is a change of measure. We also use this approach and introduce along with the measure $\mathbb{P}$, one more measure $\mathbb{P}$, setting for every $n \in \mathbb{N}$ and every bounded measurable function $\varphi : \Delta^n \times \mathbb{N}_0^{n+1} \to \mathbb{R}$

$$E[\varphi(Q_1, \ldots, Q_n, Z_0, \ldots, Z_n)] = \gamma^{-n}E[\varphi(Q_1, \ldots, Q_n, Z_0, \ldots, Z_n)e^{S_n-S_0}],$$

where

$$\gamma = E[e^X].$$

(We include $S_0$ in this expression, since later on we will also consider cases where $S_0 \neq 0$.) Note that under this change of measure the relation $E[Xe^X] = 0$ gives

$$E[X] = 0.$$  \hspace{1cm} (3)

Thus, the random walk $S$ is recurrent under $\mathbb{P}$. Note, that (2) yields

$$E[Z_n] = \gamma^n.$$

We also need the following assumption.

**Assumption A2.** The distribution of $X$ is absolutely continuous and belongs with respect to $\mathbb{P}$ to the domain of attraction of a stable law with index $\alpha \in (1,2]$.

It follows from (3) that there is an increasing sequence of positive numbers

$$a_n = n^{1/\alpha}\ell_n,$$

where $\ell_1, \ell_2, \ldots$ is a sequence slowly varying at infinity such that as $n \to \infty$ the sequence $\frac{1}{a_n}S_n$ weakly converges, i.e.

$$\mathbb{P}\left(\frac{1}{a_n}S_n \in dx\right) \to s(x) dx,$$

where $s(x)$ is the density of the limiting stable law. Note that in the case of finite variance $\sigma^2 = E[X^2] < \infty$ we have $\ell_n = \sigma$.

We also need the following moment restriction on the distribution of the random variable

$$\zeta(a) := \frac{1}{m(Q)^2} \sum_{y=a}^{\infty}y^2Q(y), \quad a \in \mathbb{N}.$$
Assumption A3. There exist $\varepsilon > 0$ and $a \in \mathbb{N}$, such that $E[(\log^+ \zeta(a))^{a+\varepsilon}] < \infty$, where $\log^+ x = \log(x \vee 1)$.

Let
$$\tau_n = \min\{k \leq n \mid S_k \leq S_0, S_1, \ldots, S_n\}$$
be the moment, when $S$ takes its minimum for the first time on the interval $[0, n]$.

Let $r_n \in \mathbb{N}_0$, $n = 1, 2, \ldots$, and $r_n \to \infty$, $r_n = o(n)$, $n \to \infty$. For brevity we will use the notation
$$r = r_n, \tau = \tau_r.$$

The main result of this paper is the following statement in which the symbol $\xrightarrow{d}$ denotes weak convergence.

**Theorem 1** If Assumptions A1 – A3 are valid and $r = r_n = o(n)$ as $n \to \infty$, then

1) there is a random variable $\kappa_1$ with values in $\mathbb{N}$ such that as $n \to \infty$
$$\left( Z_{r_n} \mid Z_n > 0 \right) \xrightarrow{d} \kappa_1;$$

2) there is a positive random variable $\kappa_2$ such that as $n \to \infty$
$$\left( \frac{Z_r}{e^{S_r-S_{\tau r}}} \mid Z_n > 0 \right) \xrightarrow{d} \kappa_2.$$

Observe, that Theorem 1 characterizes the behaviour of an intermediatly subcritical branching process $Z$ in random environment at the initial stage of the evolution of the process given its survival up to time $n \to \infty$. This result complements the results of [2], which investigates the behaviour of a subcritical branching process $Z$ in random environment at moments $r_n = t_i n$, $t_i \in [0, 1], i = 1, \ldots, k$, under the condition of the survival of the process up to $n \to \infty$ (see for this matter the remark after Lemma 1). Note also that the distribution of the number of particles at the initial period of the evolution for critical and weakly subcritical BPRE given their survival up to a distant moment were investigated in [16] and [17].

## 2 Auxiliary results

In this section we assume that the associated random walk $S$ can start from arbitrary value $S_0 = x$, and use the symbols $P_x(\cdot)$ and $E_x[\cdot]$ to denote the corresponding probabilities and expectations. Thus $P = P_0$. 

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For $n \geq 1$ set

$$L_n := \min(S_1, \ldots, S_n), \quad M_n := \max(S_1, \ldots, S_n).$$

Introduce the renewal functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$, defined by

$$u(x) = 1 + \sum_{k=1}^{\infty} P(-S_k \leq x, M_k < 0), \quad x \geq 0; \quad u(x) = 0, \quad x < 0,$$

$$v(x) = 1 + \sum_{k=1}^{\infty} P(-S_k > x, L_k > 0), \quad x < 0; \quad v(x) = 0, \quad x > 0,$$

$$v(0) = E[v(X); X < 0].$$

It is known [2], that $E[u(x + X); X + x \geq 0] = u(x), \quad x \geq 0,$

$$E[u(x + X); X + x < 0] = v(x), \quad x \leq 0,$$

that allows to specify two new measures $P^+$ and $P^-$. The construction procedure of these measures is standard and explained in detail, for example, in [1], [5]. We give a sketch of this construction.

Consider the filtration $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$, where $\mathcal{F}_n = \sigma(Q_1, \ldots, Q_n, Z_0, \ldots, Z_n)$. Thus, $S$ is adapted to $\mathcal{F}$, and $X_{n+1}$ (as well as the measure $Q_{n+1}$) is independent of $\mathcal{F}_n$ for all $n \geq 0$. For every bounded, $\mathcal{F}_n$-measurable random variable $R_n$ set

$$E^+_x[R_n] := \frac{1}{u(x)} E_x[R_n u(S_n); L_n \geq 0], \quad x \geq 0, \quad x \geq 0,$$

$$E^+_x[R_n] := \frac{1}{v(x)} E_x[R_n v(S_n); M_n < 0], \quad x \leq 0.$$

In other words, $P^+_x$ and $P^-_x$ correspond to conditioning the random walk $S$, not to enter $(-\infty, 0)$ and $[0, \infty)$, respectively.

Note that by duality (see [19], ch. XII, §2)

$$P(\tau_n = n) = P(M_n < 0).$$

We need the following results.

**Lemma 1** ([2]) Under Assumption A2 there are real numbers

$$b_n = n^{1-\alpha-1} \ell_n', \quad n \geq 1,$$
where \((\ell_n', n \geq 1)\) is a sequence slowly varying at infinity, such that for every \(x \geq 0\) as \(n \to \infty\)
\[
P(M_n < x) \sim v(-x)b_n^{-1}.
\]
Moreover, there is a constant \(c > 0\) such that for all \(x \geq 0\)
\[
P(M_n < x) = P_x(M_n < 0) \leq cv(-x)b_n^{-1}.
\]

Lemma 2 (\cite{2}, Lemma 2.3) Assume A2 and let \(U_1, U_2, \ldots\) be a sequence of uniformly bounded random variables, adapted to the filtration \(\mathcal{F}\). If \(U_n \to U_\infty\) \(\mathbb{P}^+\)-a.s. for some limiting random variable \(U_\infty\), then as \(n \to \infty\)
\[
\mathbb{E}[U_n \mid L_n \geq 0] \to \mathbb{E}^+ [U_\infty].
\]
Similarly, if \(U_n \to U_\infty\) \(\mathbb{P}^-\)-a.s., then as \(n \to \infty\)
\[
\mathbb{E}[U_n \mid M_n < 0] \to \mathbb{E}^- [U_\infty].
\]

Let \(D[0, 1]\) be the space of c.d.l.g functions on \([0,1]\). If we equippe space \(D[0, 1]\) with the Skorohod metric we get the Skorohod space \(D[0,1]\). For convinien we assume that \(S_{nt} = S_{\lfloor nt\rfloor}\). It follows from Assumption A2 (see, for example, \cite{6}), that there exists a Levy-process \(L = (L_t)_{0 \leq t \leq 1}\) such that the process \(S_n := (\frac{1}{a_n} S_{nt})_{0 \leq t \leq 1}\) converges in distribution to \(L\) in the Skorohod space \(D[0,1]\). Let \(L^- = (L^-_t)_{0 \leq t \leq 1}\) be the corresponding non-positive Levy meander. This is the process \((L^-_t)_{0 \leq t \leq 1}\), conditioned on the event \(\sup_{t \leq 1} L_t \leq 0\) (see \cite{8} and \cite{4}). Let \(\Rightarrow\) denote the weak convergence in the Skorohod space \(D[0,1]\).

We need the following result.

Lemma 3 (\cite{2}, Lemma 2.4) Under Assumptions A1 and A2 for every \(x \geq 0\) as \(n \to \infty\)
\[
\left(\frac{1}{a_n} S_n \mid M_n < x\right) \Rightarrow L^-
\]
in the Skorohod space \(D[0,1]\).

Recall that the random walk \(S\) satisfies, under the measure \(\mathbb{P}\), Assumption A2. Let \(S^{(1)}\) and \(S^{(2)}\) be two independent probability copies of the random walk \(S\). For \(S^{(2)}\) we denote by \(\tau^{(2)}_n\) and \(M^{(2)}_n\) analogies of the random variables \(\tau_n\) and \(M_n\).

Lemma 4 Under Assumptions A1 and A2
\[
P \left( \min_{1 \leq t \leq \tau_n} \left( S^{(1)}_t - S^{(1)}_{\tau_n} \right) \leq S^{(2)}_n \mid \tau^{(2)}_n = n \right) \to 0, \ n \to \infty.
\]
Proof. Using duality and taking into account that \( r = r_n = o(n) \), we have

\[
P \left( \min_{1 \leq i \leq r} \left( \frac{S_i^{(1)} - S_r^{(1)}}{a_n} \right) \leq \frac{S_n^{(2)}}{a_n}; \tau_n^{(2)} = n \right) / P (\tau_n^{(2)} = n)
\]

\[
= P \left( \min_{1 \leq i \leq r} \left( \frac{S_i^{(1)} - S_r^{(1)}}{a_n} \right) \leq \frac{S_n^{(2)}}{a_n}; M_n^{(2)} < 0 \right) / P (\tau_n^{(2)} = n)
\]

\[
= P \left( \min_{1 \leq i \leq r} \left( \frac{S_i^{(1)} - S_r^{(1)}}{a_n} \right) \leq \frac{S_n^{(2)}}{a_n} \mid M_n^{(2)} < 0 \right) \frac{P \left( M_n^{(2)} < 0 \right)}{P (\tau_n^{(2)} = n)}
\]

\[
= P \left( \min_{1 \leq i \leq r} \left( \frac{S_i^{(1)} - S_r^{(1)}}{a_n} \right) \leq \frac{S_n^{(2)}}{a_n} \mid M_n^{(2)} < 0 \right) \to 0, n \to \infty,
\]

since \( \min_{1 \leq i \leq r} \left( \frac{S_i^{(1)} - S_r^{(1)}}{a_n} \right) / a_n \to 0 \) as \( n \to \infty, r = o(n) \), and given \( M_n^{(2)} < 0 \) the distribution of \( \frac{S_n^{(2)}}{a_n} \) converges weakly to an absolutely continuous distribution on \([0, \infty)\).

The lemma is proved.

Let \( Q_j = Q_1 \) for \( j \leq 0 \).

Lemma 5 If Assumptions A1 and A2 are valid, then for any \( m \geq 0, k \geq 1 \) given the event \( \tau_{n-m} = n - m \) for \( n \to \infty \) distribution of

\[
\left( \left( Q_{\tau+1}, \ldots, Q_{\tau+k} \right), \left( Q_{\tau}, \ldots, Q_{\tau-k+1} \right), \left( \frac{S_{\tau}}{a_r}, \frac{S_{n-m}}{a_n} \right) \right)
\]

converges weakly to a probability measure \( \mu_k^+ \otimes \mu_k^- \otimes \mu \), where \( \mu_k^+ \) and \( \mu_k^- \) are the distributions of \( (Q_1, \ldots, Q_k) \) under the probability measures \( P^+ \) and \( P^- \), respectively, and \( \mu \) is probability measure on \( \mathbb{R}^2 \).

Proof. For \( l \geq 0 \) set

\[
Q^+ (l) := (Q_{l+1}, \ldots, Q_{l+k}) , \quad Q^- (l) := (Q_l, \ldots, Q_{l-k+1}) .
\]

Let \( \phi_1, \phi_2 : \Delta^k \to \mathbb{R} \) be bounded functions and \( \phi_3, \phi_4 : \mathbb{R} \to \mathbb{R} \) be bounded continuous functions. We have

\[
E \left[ \phi_1(Q^- (\tau_r)) \phi_2(Q^+ (\tau_r)) \phi_3(S_{\tau_r}/a_r) \phi_4(S_{\tau_{n-m}} - S_\tau/a_n); \tau_{n-m} = n - m \right]
\]

\[
= \sum_{l=0}^{r} E \left[ \phi_1(Q^- (l)) \phi_2(Q^+ (l)) \phi_3(S_{\tau_r}/a_r) \phi_4(S_{\tau_{n-m}} - S_\tau/a_n); \tau_r = l, \tau_{n-m} = n - m \right] .
\]

\[(5)\]
Introduce the random walk

\[ S'_k = S_{k+l} - S_l, \quad k \geq 0. \]

We will use prime to denote the random variables related to this random walk. It is easy to see that

\[ \mathbb{E}[\phi_1(Q^-(l))\phi_2(Q^+(l))\phi_3(S_{\alpha l})\phi_4(\frac{S_{n-m}-S_l}{a_n}); \tau_r = l, \tau_{n-m} = n-m] \]

\[ = \mathbb{E}[\phi_1(Q^-(l))\phi_3(S_{\alpha l}); \tau_l = l] \mathbb{E}[\phi_2(Q^+(0))\phi_4(\frac{S_{n-m-1}}{a_n}); L'_{r-l} \geq 0, \tau'_{n-m-l} = n-m-l] \]

\[ = \mathbb{E}[\phi_1(Q^+(0))\phi_3(S_{\alpha l}); M_l < 0] \mathbb{E}[\phi_2(Q^+(0))\phi_4(\frac{S_{n-m-1}}{a_n}); L'_{r-l} \geq 0, \tau'_{n-m-l} = n-m-l] . \quad (6) \]

Note that for \( l > k \)

\[ \frac{\mathbb{E}[\phi_1(Q^+(0))\phi_3(S_{\alpha l}); M_l < 0]}{\mathbb{P}(M_l < 0)} \]

\[ = \mathbb{E}[\phi_1(Q^+(0))\phi_3(S_{\alpha l}) | M_{l-k} < 0] \frac{\mathbb{P}(M_{l-k} < 0)}{\mathbb{P}(M_l < 0)}; M_k < 0] . \]

Therefore by Lemmas 1, 3 and the dominated convergence theorem, if \( l = L_n \sim t \tau_n \) for some \( 0 < t < 1 \), then \( a_{L_n}/a_{\tau_n} \sim t^{\frac{1}{2}} \) and

\[ \frac{\mathbb{E}[\phi_1(Q^+(0))\phi_3(S_{\alpha l}); M_{L_n} < 0]}{\mathbb{P}(M_{L_n} < 0)} \]

\[ \rightarrow \mathbb{E}[\phi_1(Q^+(0))v(S_k); M_k < 0] \mathbb{E}[\phi_3(t^{\frac{1}{2}}L_1^{-})] \]

\[ = \mathbb{E}^{-}[\phi_1(Q^+(0))] \mathbb{E}[\phi_3(t^{\frac{1}{2}}L_1^{-})] . \quad (7) \]

Now consider the expectation

\[ \mathbb{E}[\phi_2(Q^+(0))\phi_4(\frac{S_{n-m-1}}{a_n}); L'_{r-l} \geq 0, \tau'_{n-m-l} = n-m-l] . \]

Set \( R = r-l, N = n-m-l \). Introduce the random walk \( S''_i := S'_{R+i} - S'_R, \quad i \geq 0 \). Let

\[ A_0 := \{ S'_1 \geq 0, S'_2 \geq 0, \ldots, S'_R \geq 0 \} , \]

\[ A_1 := \{ S''_1 \geq S''_{N-R}, S''_2 \geq S''_{N-R}, \ldots, S''_{N-R-1} \geq S''_{N-R-1} \} , \]

\[ B := \{ S''_{N-R} < S'_R \} , \quad \overline{B} = \{ S''_{N-R} \geq S'_R \} . \]

Clearly, the events \( A_0 \) and \( A_1 \) are independent and

\[ \{ L'_{r-l} \geq 0, \tau'_{n-m-l} = n-m-l \} = A_0 \cap A_1 \cap B. \]
Therefore,
\[
\begin{align*}
\mathbb{E}[\phi_2(Q^+(0))\phi_4\left(\frac{S_{n-R}}{a_n}\right); L'_{r-l} & \geq 0, \tau'_{n-m-l} = n - m - l] \\
& = \mathbb{E}[\phi_2(Q^+(0)); A_0] \mathbb{E}[\phi_4\left(\frac{S_{n-R}}{a_n}\right); A_1] - \mathbb{E}[\phi_2(Q^+(0))\phi_4\left(\frac{S_{n-R}}{a_n}\right); A_0 \cap A_1 \cap \overline{B}]. 
\end{align*}
\]

(8)

Applying Lemma 4 we see that, as \( n \to \infty \)
\[
\mathbb{E}[\phi_2(Q^+(0))\phi_4\left(\frac{S_{n-R}}{a_n}\right); A_0 \cap A_1 \cap \overline{B}]
\leq c \mathbb{P} (A_1 \cap \overline{B}) = o (\mathbb{P} (\tau_N = N)) = o (\mathbb{P} (\tau_n = n)),
\]
where \( c > 0 \). In much the same way as in proving of (7) it follows from Lemma 2.3 in [11] that
\[
\frac{\mathbb{E}[\phi_2(Q^+(0)); A_0]}{\mathbb{P}(L_R \geq 0)} \to \mathbb{E}^+[\phi_2(Q^+(0))], \ n \to \infty,
\]
(10)
\[
\frac{\mathbb{E}[\phi_4\left(\frac{S_{n-R}}{a_n}\right); A_1]}{\mathbb{P}(\tau_{N-R} = N - R)} \sim \frac{\mathbb{E}[\phi_4\left(\frac{S_n}{a_n}\right); \tau_n = n]}{\mathbb{P}(\tau_n = n)} \to \mathbb{E}[\phi_4(L^-_1)], \ n \to \infty.
\]
(11)

Relations (8) - (11) yield
\[
\frac{\mathbb{E}[\phi_2(Q^+(0))\phi_4\left(\frac{S_{n-m-l}}{a_n}\right); L'_{r-l} \geq 0, \tau'_{n-m-l} = n - m - l]}{\mathbb{P}(L'_R \geq 0)\mathbb{P}(\tau_{n-m} = n - m)} \\
\to \mathbb{E}^+[\phi_2(Q^+(0))\phi_4(L^-_1)], \ n \to \infty.
\]

(12)

It follows from Assumption A2 (see [6]), that \( \tau_n/n \) converges in distribution to a Beta-distribution with a density, which we denote by \( g(t) \). Since \( \mathbb{P}(M_{l_n} < 0)\mathbb{P}(L_{r-l} \geq 0) = \mathbb{P}(\tau_r = l_n) \), it follows from (10) - (11) and (12) that
\[
\mathbb{E}[\phi_1(Q^-(\tau_r))\phi_2(Q^+(\tau_r))\phi_3\left(\frac{S_{n-R}}{a_n}\right)\phi_4\left(\frac{S_{n-m-S_{\tau_r}}}{a_n}\right)|\tau_{n-m} = n - m] \\
\to \mathbb{E}^-[\phi_1(Q_1, ..., Q_k)]\mathbb{E}^+[\phi_2(Q_1, ..., Q_k)] \\
\times \mathbb{E}[\phi_4(L^-_1)] \int_0^1 \mathbb{E}[\phi_3(t \bar{x} L^-_1)]g(t) \, dt.
\]

The lemma is proved.

Remark. Note that despite on the seeming similarity of the formulations of Lemma 5 and Lemma 2.6 in [2] the statements proved in it are essentially different, since the measure \( \mu \) appearing as the limiting in Lemma 5 is essentially different from the corresponding limiting measure \( \mu \), arising in Lemma 2.6 in [2].
Set
\[ \eta_k := \sum_{y=0}^{\infty} y(y-1)Q_k(y) / \left( \sum_{y=0}^{\infty} yQ_k(y) \right)^2, \quad k \geq 1. \]

Lemma 6 ([1], [2]) Assume Assumptions A1 – A3. Then for all \( x \geq 0 \)
\[ \sum_{k=0}^{\infty} \eta_{k+1} e^{-S_k} < \infty \quad \mathbb{P}^+_x - a.s. \]
and for all \( x \leq 0 \)
\[ \sum_{k=1}^{\infty} \eta_k e^{S_k} < \infty \quad \mathbb{P}^-_x - a.s. \]

3 Trees with stem

We need the following construction considered in [2] which is in spirit to the models from [9], [10]. For \( n = 0, 1, \ldots, \infty \) let \( T_n \) be the set of all ordered rooted trees of height exactly \( n \), whose edges are directe from the root. The precise definition of ordered rooted trees was given by Neven [12]. Let \( T_{\geq n} = T_n \cup T_{n+1} \cup \cdots \cup T_\infty \) be the set of ordered rooted trees of at least height \( n \), whose edges are directed from the root. Denote by \([t]_n \in T_n \) the tree obtained from the tree \( t \in T_{\geq n} \) by eliminating from the tree \( t \) all edges and nodes of a height exceeding \( n \). We will say that the tree \([t]_n \) is obtained by pruning a tree \( t \) to the tree \([t]_n \) on the level \( n \).

For \( n = 0, 1, \ldots, \infty \) a tree with a stem of height \( n \), or shortly an o-tree of height \( n \), is a pair
\[ t = (t, k_0k_1 \ldots k_n), \]
where \( t \in T_{\geq n}, k_0, \ldots, k_n \) are nodes in \( t \) such that \( k_0 \) is the root, and nodes \( k_{i-1} \) and \( k_i, i = 1, \ldots, n \), are connected by the edge directed from \( k_{i-1} \) to \( k_i \), i.e. the node \( k_i \) belongs to the generation \( i \). We call \( k_0 \ldots k_n \) the stem within o-tree \( t \) (which, evidently, is determined by \( k_n \)). Let \( T'_n \) be the set of all o-trees with the treests of height \( n \).

Pruning an o-tree \( t = (t, k_0k_1 \ldots k_n) \) of the height \( n \) at level \( m \leq n \), we obtain the o-tree
\[ [t]_m = ([t]_m, k_0 \ldots k_m) \]
of the height \( m \).
Every tree $t \in \mathcal{T}_{\geq n}$ includes a unique o-tree

$$\langle t \rangle_n = ([t]_n, k_0(t) \ldots k_n(t))$$

of height $n$, where $k_0(t) \ldots k_n(t)$ is the leftmost stem, which can be fitted into $[t]_n$.

In the sequel it will be convinient to assume that there is a particle in every node of the tree $t$. The particle located at the node $a$ is the child of the particle in the node $b \in t$, if the nodes $a$ and $b$ are connected by the edge directed from $a$ to $b$.

Let $\pi = (q_1, q_2, \ldots)$ be a fixed environment. Define the discrete distribution $\tilde{q}_i$ by its weights

$$\tilde{q}_i(y) = \frac{1}{m(q_i)} q_i(y), \quad y = 0, 1, \ldots.$$ 

Then an oLLP-tree (Lyons-Pemantle-Peres o-tree) corresponding to the distribution $(q_1, q_2, \ldots)$ is the random o-tree $\tilde{T} = (\tilde{T}, \tilde{K}_0 \tilde{K}_1 \ldots)$ with values in $\mathcal{T}_\infty$, satisfying the following properties:

Given $\Pi = (q_1, q_2, \ldots)$

- The offspring numbers of all particles are independent random variables.
- The offspring number of $\tilde{K}_{i-1}$ has distribution $\tilde{q}_i$ and the offspring number of any other particle in generation $i - 1$ has distribution $q_i$.
- The node $\tilde{K}_i$ is uniformly distributed among all children of $\tilde{K}_{i-1}$.

In other words the particles being in the nodes of infinite stem reproduce according to the distribution $(\tilde{q}_1, \tilde{q}_2, \ldots)$, and all other particles reproduce according to the distribution $(q_1, q_2, \ldots)$.

Recall some useful properties of oLLP-trees. Let $\tilde{Z}_n$ be the population size of the oLLP-tree in generation $n$.

We have the following statement.

**Lemma 7** ([2]) If Assumptions A1 to A3 are valid, then as $n \to \infty$

$$e^{-S_n \tilde{Z}_n} \to W^+ \quad \mathbb{P}^+-a.s.,$$

where the random variable $W^+$ is such that $W^+ > 0 \mathbb{P}^+-a.s.$
We use the representation

\[ \tilde{Z}_n = 1 + \sum_{i=0}^{n-1} \tilde{Z}_i^n, \]

where \( \tilde{Z}_i^n \) is the number of particles in generation \( n \) other than \( \tilde{K}_n \), which are descent of the particle from \( \tilde{K}_i \), but not of the particle from \( \tilde{K}_{i+1} \). Note that \( E[\tilde{Z}^i_{n+1} \mid \Pi] = \sum_y y \tilde{Q}^i_{n+1}(y) - 1 = e^{X_{i+1} \eta_{i+1}} \) and a.s.

\[ E[\tilde{Z}_n \mid \Pi] = e^{S_n - S_{i+1}} E[\tilde{Z}^i_{n+1} \mid \Pi] = \eta_{i+1} e^{S_n - S_i}. \] (13)

Clearly, that given the environment, the sequence \( e^{-S_n} \sum_{i=k}^{n-1} \tilde{Z}^i \) is for \( n > k \) a non-negative submartingale. From here and Doob’s inequality it follows that for every \( \varepsilon \in (0, 1) \)

\[ P\left( \max_{k \leq m \leq n} e^{-S_m} \sum_{i=k}^{m-1} \tilde{Z}^i \geq \varepsilon \mid \Pi \right) \leq \frac{1}{\varepsilon} \sum_{i=k}^{n-1} e^{-S_n} E[\tilde{Z}^i_n \mid \Pi] \leq \frac{1}{\varepsilon} \sum_{i \geq k} \eta_{i+1} e^{-S_i}. \] (14)

and

\[ P^+ \left( \sup_{m > k} e^{-S_m} \sum_{i=k}^{m-1} \tilde{Z}^i_m \geq \varepsilon \right) \leq \frac{1}{\varepsilon} E^+ \left[ 1 \wedge \sum_{i \geq k} \eta_{i+1} e^{-S_i} \right]. \] (15)

It follows from (5) that for sufficiently large \( k \)

\[ P^+ \left( \sup_{m > k} e^{-S_m} \sum_{i=k}^{m-1} \tilde{Z}^i_m \geq \varepsilon \right) \leq \varepsilon. \]

To approximate the conditional distribution of the considered branching process \( Z = (Z_0, Z_1, \ldots) \) given \( Z_n > 0 \) we use its approximation by oLLP-trees. Let \( T \) be the tree corresponding to the considered branching process \( Z \) in random environment \( \Pi \).

We need the following statement.

**Theorem 2** ([2]) Let \( 0 \leq d_n < n \) be a sequence of natural numbers with \( d_n \rightarrow \infty \) as \( n \rightarrow \infty \), and \( Y_n \) be uniformly bounded random variables of the form \( Y_n = \varphi(Q_1, \ldots, Q_{n-d_n}) \), and let \( B_n \subset T_{n-d_n}', n \geq 1 \). If Assumptions A1 – A3 are valid and for some \( \ell \geq 0 \)

\[ \mathbb{E}[Y_n; \tilde{T}_{n-d_n} \in B_n \mid \tau_{n-m} = n - m] \rightarrow \ell \]
for all $m \geq 0$, then

$$E[Y_n; (T)_n; n \in B_n \mid Z_n > 0] \rightarrow \ell.$$ 

Here the set $B_n$ may be random, depending only on the environment $\Pi$.

4 Proof of Theorem 1.4

Let $T$ be an oLLP-tree. Recall that for $i < j$ $Z_i^j$ is the number of the particles in generation $j$ other than the particle in $K_j$, which descent from a particle located in $K_i$ but not from a particle located in $K_{i+1}$. For convenience we put $Z_i^j = 0$ for $i \geq j$.

Recall also that $r = r_n \rightarrow \infty$, $n \rightarrow \infty$, $r_n = o(n)$, $\tau_r = \tau_{n_r}$.

Lemma 8 For every $\varepsilon > 0$ there is a natural number $a$ such that for any natural numbers $m$ and $\varsigma \in [\tau_r, r]$ for sufficiently large $n$ (depending on $\varepsilon, a$ and $m$)

$$P\left( \sum_{i:|i-\tau_r| \geq a} \frac{Z_i^\varsigma}{e^{S_{i}-S_{\tau_r}}} \geq \varepsilon \mid \tau_{n-m} = n - m \right) \leq \varepsilon,$$

here $\varsigma$ may be random, depending only on the random environment $\Pi$.

Proof. For $0 \leq j < n$ set

$$L_{j,n} = \min(S_{k+1}, ..., S_n) - S_n, L_{n,n} = 0.$$ 

from Markov inequality and (13) we have for $0 < \varepsilon \leq 1$ and $m \leq n - r_n$

$$\varepsilon P\left( \sum_{|i-\tau_r| \geq a} \frac{Z_i^\varsigma}{e^{S_{i}-S_{\tau_r}}} \geq \varepsilon; \tau_{n-m} = n - m \right) \leq E\left[ 1 \wedge \sum_{i:|i-\tau_r| \geq a} \eta_{i+1}e^{S_{\tau_r}-S_i}; \tau_{n-m} = n - m \right]$$

$$\leq \sum_{j \leq r} E\left[ 1 \wedge \sum_{i:|i-j| \geq a} \eta_{i+1}e^{S_j-S_i}; \tau_j = j, L_{j,r} \geq 0 \right] \times P(\tau_{n-r-m} = n - r - m).$$
Further,
\[
\sum_{j \leq r} E \left[ 1 \land \sum_{i \leq \xi, |i-j| \geq a} \eta_i e^{S_j - S_i}; \tau_j = j, L_{j,r} \geq 0 \right] = \\
= \sum_{j \leq r} E \left[ 1 \land \sum_{i=0}^{j-a} \eta_i e^{S_j - S_i}; \tau_j = j \right] P(L_{r-j} \geq 0) \\
+ \sum_{j \leq r} P(\tau_j = j) E \left[ 1 \land \sum_{i=j+a}^{\zeta} \eta_i e^{S_j - S_i}; L_{j,r} \geq 0 \right].
\]

Using duality, we obtain
\[
\sum_{j \leq r} E \left[ 1 \land \sum_{i \leq \xi, |i-j| \geq a} \eta_i e^{S_j - S_i}; \tau_j = j, L_{j,r} \geq 0 \right] = \\
\leq \sum_{a \leq j \leq r} E \left[ 1 \land \sum_{i=a}^{j-a} \eta_i e^{S_i}; M_j < 0 \right] P(L_{r-j} \geq 0) \\
+ \sum_{a \leq k \leq r} P(\tau_r - k = r - k) E \left[ 1 \land \sum_{i=a}^{k} \eta_i e^{S_i}; L_k \geq 0 \right].
\]

By Lemmas 2 and 6 we may choose \(a\) so large that for a given \(\delta > 0\) and all \(j, k > a\)
\[
E \left[ 1 \land \sum_{i=a}^{j} \eta_i e^{S_i}; M_j < 0 \right] \leq \delta P(M_j < 0), \\
E \left[ 1 \land \sum_{i=a}^{k} \eta_i e^{S_i}; L_k \geq 0 \right] \leq \delta P(L_k \geq 0).
\]

Duality yields
\[
\sum_{j \leq r} E \left[ 1 \land \sum_{i \leq \xi, |i-j| \geq a} \eta_i e^{S_j - S_i}; \tau_j = j, L_{j,r} \geq 0 \right] \\
\leq \delta \sum_{a \leq j \leq r} P(\tau_j = j) P(L_{r-j} \geq 0) \\
+ \delta \sum_{a \leq k \leq r} P(\tau_r - k = r - k) P(L_k \geq 0) \leq 2\delta.
\]
and
\[
P \left( \sum_{|i-\tau_r| \geq a} \frac{\tilde{Z}_i}{e^{S_\tau_r-S_\zeta}} \right) \geq \varepsilon; \tau_{n-m} = n - m \right)
\leq \frac{2\delta}{\varepsilon} P(\tau_{n-r-m} = n - r - m). \]

Since $P(\tau_n = n)$ is regularly varying and $r = o(n), n \to \infty$, the right-hand side is bounded by $\varepsilon P(\tau_n = n)$, if $\delta$ is chosen small enough. The lemma is proved.

We now come back to the proof of the first part of Theorem 1. Let, as before, $\tilde{Z}_j$ be the number of nodes in generation $j$ of the oLPP-tree $\tilde{T}$, i.e.,
\[
\tilde{Z}_j = 1 + \sum_{k=0}^{j-1} \tilde{Z}_j^k.
\]
Fix an $\varepsilon > 0$. In view of the preceding lemma with $\zeta = \tau_r$ there is a natural number $a$ such that given $\tau_{n-m} = n - m$ the probability is at least $1 - \varepsilon$ that the event
\[
\tilde{Z}_{\tau_r} = 1 + \sum_{|k-\tau_r| \leq a} \tilde{Z}_{\tau_r}^k = 1 + \sum_{k=\tau_r-a}^{\tau_r} \tilde{Z}_{\tau_r}^k
\]
holds. Note that given the environment $\Pi$ the distribution of
\[
1 + \sum_{k=\tau_r-a}^{\tau_r} \tilde{Z}_{\tau_r}^k
\]
only depends on $(Q_{\tau_r-a}, \ldots, Q_{\tau_r})$. By Lemma 5 given $\tau_{n-m} = n - m$ the vector $(Q_{\tau_r-a}, \ldots, Q_{\tau_r})$ converges in distribution to a limit.

These observations hold for every $\varepsilon > 0$. Therefore we may summarize our arguments as follows: For all $m \geq 1$
\[
(\tilde{Z}_{\tau_r} | \tau_{n-m} = n - m) \overset{d}{\to} \zeta_1,
\]
where the random variable $\zeta_1$ has the properties as claimed in Theorem 1. Now Theorem 2 gives the claim of the first part of Theorem 1.

Now we pass to the proof of the second part of Theorem 1. Let for fixed $a$
\[
\tilde{Z}_{a,k} = \sum_{i:|i-\tau_r| \leq a} \tilde{Z}_i
\]
and
\[
\alpha_{a,r} = e^{S_{\tau_r}-S_r} \tilde{Z}_{a,r} \quad \text{and} \quad \beta_{a,r} = e^{S_{\tau_r}-S_{\tau_r+a}} \tilde{Z}_{a,\tau_r+a}.
\]
We need the following statement.
Lemma 9. Let $m \geq 1$ and $\varepsilon \in (0, 1)$. Then, if $a$ is sufficiently large
\[
\limsup_{n \to \infty} P(|\alpha_{a,r} - \beta_{a,r}| > \varepsilon \mid \tau_{n-m} = n - m) \leq \varepsilon. \tag{16}
\]

Proof. In virtue of Markov inequality and (13), (14) and (15)
\[
P(\beta_{a,r} > d \mid \tau_{n-m} = n - m)
\leq P(e^{s_{r_r}}S_{r_r}^+ E[\hat{Z}_{a,r} \mid \Pi] > \sqrt{d} \mid \tau_{n-m} = n - m) + \frac{1}{\sqrt{d}}
\leq P\left(\sum_{i \geq 1} \eta_i e^{S_{r_r}^+} > \sqrt{d} \mid \tau_{n-m} = n - m\right) + \frac{1}{\sqrt{d}}.
\]

It follows from Lemma 5 that the sum under the sign of probability converges in distribution as $n \to \infty$ and
\[
\limsup_{n \to \infty} P(\beta_{a,n} > d \mid \tau_{n-m} = n - m)
\leq P^\left(\sum_{i \geq 1} \eta_i e^{S_{r_r}^+} \geq \sqrt{d} \right) + P^\left(\sum_{i \geq 0} \eta_i e^{S_{r_r}^+} \geq \sqrt{d} \right) + \frac{1}{\sqrt{d}}.
\]

By Lemma 6 there exists $d < \infty$ such that for all $a > 0$
\[
\limsup_{n \to \infty} P(\beta_{a,r} > d \mid \tau_{n-m} = n - m) < \varepsilon / 2.
\]

Further, since $r = o(n)$ and $P(\tau_n = n)$ is regularly varying at infinity we have for sufficiently large $n$
\[
P(\tau_r \in [r - a, r] \mid \tau_{n-m} = n - m) = \frac{P(\tau_r \in [r - a, r], \tau_{n-m} = n - m)}{P(\tau_{n-m} = n - m)}
\leq \sum_{k=r-a}^r P(\tau_r = k) \frac{P(\tau_{n-m-r} = n - m - r)}{P(\tau_{n-m} = n - m)}
\leq 2 \sum_{k=r-a}^r P(\tau_r = k) = 2P(\tau_r \in [r - a, r]). \tag{17}
\]

Since the distribution of the ratio $\tau_n/n$ converges as $n \to \infty$ to a Beta distribution (17) implies
\[
P(\tau_r + a \geq r \mid \tau_{n-m} = n - m) \to 0, \ n \to \infty.
\]

Therefore
\[
P(|\beta_{a,r} - \alpha_{a,r}| > \varepsilon \mid \tau_{n-m} = n - m)
\leq \frac{\varepsilon}{2} + P(|\alpha_{a,r} - \beta_{a,r}| > \varepsilon, \beta_{a,r} \leq d, \tau_r + a \leq r \mid \tau_{n-m} = n - m) \tag{18}
\]

16
for sufficiently large \( n \). Note, that given \( \Pi, \hat{Z}_{a,\tau_r+a} \) and \( \tau_r + a \leq r \), the process \( \hat{Z}_{a,k} \), \( k \geq \tau_r + a \) is a branching process in varying environment. Therefore

\[
\mathbb{E}[\alpha_{a,r} \mid \Pi, \hat{Z}_{a,\tau_r+a}] = \beta_{a,r} \text{ a.s. and }
\]

\[
\frac{\text{Var}(Z_n \mid Z_0 = z, \Pi)}{\mathbb{E}[Z_n \mid Z_0 = 1, \Pi]^2} = z \left( e^{-S_n} + \sum_{i=0}^{n-1} \eta_{i+1} e^{-S_i} - 1 \right) .
\]

Inserting this estimate into (18), we have

\[
P(\left| \beta_{a,r} - \alpha_{a,r} \right| > \varepsilon; \tau_{n-m} = n - m)
\leq \frac{\varepsilon}{2} P(\tau_{n-m} = n - m) + \frac{d}{\varepsilon^2} \mathbb{E} \left[ 1 \wedge \left( e^{-(S_r-S_{r'})} \right) \right.
+ \sum_{i=\tau_r+a}^r \eta_{i+1} e^{-(S_i-S_{r'})} ; \tau_r + a \leq r, \tau_{n-m} = n - m \right] . \tag{19}
\]

Thus to finish the proof of Lemma it is sufficient to show that the term

\[
\mathbb{E} \left[ 1 \wedge \left( e^{-(S_r-S_{r'})} + \sum_{i=\tau_r+a}^r \eta_{i+1} e^{-(S_i-S_{r'})} \right) ; \tau_r + a \leq r, \tau_{n-m} = n - m \right]
= \sum_{j \leq r-a} \mathbb{E} \left[ 1 \wedge \left( e^{-(S_r-S_j)} + \sum_{i=j+a}^r \eta_{i+1} e^{-(S_i-S_j)} \right) ; \tau_r = j, \tau_{n-m} = n - m \right]
\]

may be made arbitrary small.

Since for \( j \leq r = o(n) \)

\[
\sup_{0 \leq j \leq r} \frac{P(\tau_{n-m-j} = n - m - j)}{P(\tau_{n-m} = n - m)} \rightarrow 1
\]
as \( n \rightarrow \infty \), we have for all sufficiently large \( n \) and \( r = o(n) \)

\[
\mathbb{E} \left[ 1 \wedge \left( e^{-(S_r-S_j)} + \sum_{i=j+a}^r \eta_{i+1} e^{-(S_i-S_j)} \right) ; \tau_r = j, \tau_{n-m} = n - m \right]
\leq \mathbb{E} \left[ 1 \wedge \left( e^{-(S_r-S_j)} + \sum_{i=j+a}^r \eta_{i+1} e^{-(S_i-S_j)} \right) ; \tau_r = j \right] P(\tau_{n-m-j} = n - m - j)
\leq 2 \mathbb{E} \left[ 1 \wedge \left( e^{-(S_r-S_j)} + \sum_{i=j+a}^r \eta_{i+1} e^{-(S_i-S_j)} \right) ; \tau_r = j \right] P(\tau_{n-m} = n - m) . \tag{20}
\]
Further,
\[ \mathbb{E} \left[ 1 \land \left( e^{-(S_r-S_j)} + \sum_{i=j+1}^{r} \eta_{i+1}e^{-(S_i-S_j)} \right); \tau_r = j \right] \]
\[ = \mathbb{P}(\tau_j = j) \mathbb{E} \left[ 1 \land \left( e^{-(S_r-S_j)} + \sum_{i=j+1}^{r} \eta_{i+1}e^{-(S_i-S_j)} \right); L_{j,r} \geq 0 \right] \]
\[ \leq \mathbb{P}(\tau_j = j) \mathbb{P}(L_{r-j} \geq 0) \]
\[ \times \left( \mathbb{E} \left[ e^{-S_{r-j}}|L_{r-j} \geq 0 \right] + \mathbb{E} \left[ 1 \land \left( \sum_{i=j+1}^{r} \eta_{i+1}e^{-S_{i-j}} \right)|L_{r-j} \geq 0 \right] \right). \]

Note, that by Lemma 2 with \( U_n = e^{-S_n} \) we have for sufficiently large \( a \)
\[ \sup_{0 \leq j \leq r-a} \mathbb{E} \left[ e^{-S_{r-j}}|L_{r-j} \geq 0 \right] \leq \sup_{a \leq k \leq \infty} \mathbb{E} \left[ e^{-S_k}|L_k \geq 0 \right] \leq \varepsilon^3/(8d). \] (22)

Finally,
\[ \mathbb{E} \left[ 1 \land \left( \sum_{i=j+1}^{r} \eta_{i+1}e^{-S_{i-j}} \right)|L_{r-j} \geq 0 \right] = \mathbb{E} \left[ 1 \land \left( \sum_{k=a}^{r-j} \eta_{k+1}e^{-S_k} \right)|L_{r-j} \geq 0 \right] \]
and
\[ \lim_{r-j \to \infty} 1 \land \left( \sum_{k=a}^{r-j} \eta_{k+1}e^{-S_k} \right) = 1 \land \left( \sum_{k=a}^{\infty} \eta_{k+1}e^{-S_k} \right) \]
\( \mathbb{P}^+ - a.s. \) Hence, applying Lemma 2 we obtain
\[ \lim_{r-j \to \infty} \mathbb{E} \left[ 1 \land \left( \sum_{k=a}^{r-j} \eta_{k+1}e^{-S_k} \right)|L_{r-j} \geq 0 \right] = \mathbb{E}^+ \left[ 1 \land \left( \sum_{k=a}^{\infty} \eta_{k+1}e^{-S_k} \right) \right]. \]

Since
\[ \sum_{k=a}^{\infty} \eta_{k+1}e^{-S_k} < \infty \mathbb{P}^+ - a.s., \]
it follows that
\[ \lim_{a \to \infty} 1 \land \left( \sum_{k=a}^{\infty} \eta_{k+1}e^{-S_k} \right) = 0 \mathbb{P}^+ - a.s. \]

Hence, by the dominated convergence theorem we get
\[ \lim_{a \to \infty} \mathbb{E}^+ \left[ 1 \land \left( \sum_{k=a}^{\infty} \eta_{k+1}e^{-S_k} \right) \right] = \mathbb{E}^+ \left[ \lim_{a \to \infty} \left( 1 \land \left( \sum_{k=a}^{\infty} \eta_{k+1}e^{-S_k} \right) \right) \right] = 0. \]
Thus,
\[
\lim_{a \to \infty} \lim_{r-j \to \infty} \mathbb{E} \left[ 1 \wedge \left( \sum_{k=a}^{r-j} \eta_{k+1} e^{-S_k} \right) \mid L_{r-j} \geq 0 \right] = 0.
\]

Therefore, given \( r-j \geq 2a \) we have for sufficiently large \( a \)
\[
\mathbb{E} \left[ 1 \wedge \left( \sum_{k=a}^{r-j} \eta_{k+1} e^{-S_k} \right) \mid L_{r-j} \geq 0 \right] \leq 2 \mathbb{E}^+ \left[ 1 \wedge \left( \sum_{k=a}^{\infty} \eta_{k+1} e^{-S_k} \right) \right] \leq \varepsilon^3 / (8d) .
\]

(23)

Thus, it remains to consider the case
\[
r - 2a \leq j \leq r - a.
\]

Since the random variable \( \tau_n/n \) converges in distribution as \( n \to \infty \) to Beta distribution we have for sufficiently large \( r \)
\[
\sum_{r-2a \leq j \leq r-a} \mathbb{E} \left[ 1 \wedge \left( e^{-(S_r - S_j)} + \sum_{i=j+a}^{r} \eta_{i+1} e^{-(S_i - S_j)} \right) ; \tau_r = j \right] \\
\leq \sum_{r-2a \leq j \leq r-a} \mathbb{P} (\tau_r = j) = \mathbb{P} (r - 2a \leq \tau_r \leq r - a) \leq \varepsilon^3 / (4d) .
\]

This gives the needed result
\[
\sum_{j \leq r-a} \mathbb{E} \left[ 1 \wedge \left( e^{-(S_r - S_i)} + \sum_{i=j+a}^{r} \eta_{i+1} e^{-(S_i - S_j)} \right) ; \tau_r = j \right] \\
\leq \varepsilon^3 \mathbb{P} (\tau_r \leq r - 2a) / (4d) + \varepsilon^3 / (4d) \leq \varepsilon^3 / 2 .
\]

(24)

Relations (19) – (24) yield (16).

We are now ready to finish the proof of the second part of Theorem 1.

From \( \tilde{Z}_r = 1 + \tilde{Z}_{a,r} + \sum_{i:|i-\tau_r|>a} \tilde{Z}_r^i \) we have
\[
\mathbb{P} \left( e^{S_r-S_r} \tilde{Z}_r - \beta_{a,r} \geq 3 \varepsilon \mid \tau_{n-m} = n - m \right) \\
\leq \mathbb{P} \left( e^{S_r-S_r} \geq \varepsilon \mid \tau_{n-m} = n - m \right) \\
+ \mathbb{P} \left( |\alpha_{a,r} - \beta_{a,r}| \geq \varepsilon \mid \tau_{n-m} = n - m \right) \\
+ \mathbb{P} \left( e^{S_r-S_r} \sum_{i:|i-\tau_r|>a} \tilde{Z}_r^i \geq \varepsilon \mid \tau_{n-m} = n - m \right) .
\]
Fix an $\varepsilon > 0$. Then
\[
P(e^{S_{\tau_r} - S_r} \geq \varepsilon \mid \tau_{n-m} = n - m) \geq \frac{1}{P(\tau_{n-m} = n - m)} P\left(\frac{S_{\tau_r} - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}, \tau_{n-m} = n - m\right).
\]

(25)

Clearly,
\[
P\left(\frac{S_{\tau_r} - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}, \tau_{n-m} = n - m\right) = \sum_{k=0}^{r} P\left(\frac{S_k - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}; \tau_r = k, \tau_{n-m} = n - m\right).
\]

(26)

Let $S'_i = S_{r+i} - S_r, n \geq 0$, and
\[
\tau'_n = \min \left\{ l \in [0, n]: S'_l = \min_{0 \leq i \leq n} S'_i \right\}.
\]

For a fixed $k$ we write the following chain of relations
\[
P\left(\frac{S_k - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}; \tau_r = k, \tau_{n-m} = n - m\right) \leq P\left(\frac{S_k - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}; \tau_r = k, \tau'_{n-m-r} = n - m - r\right)
\]
\[
= P\left(\frac{S_k - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}; \tau_r = k\right) P\left(\tau'_{n-m-r} = n - m - r\right)
\]
\[
= P\left(\frac{S_k - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}; \tau_r = k\right) P\left(\tau_{n-m-r} = n - m - r\right).
\]

(27)

Since the probability $P(\tau_r = n) = P(M_n < 0)$ is regularly varying as $n \to \infty$ and $r = o(n)$, we have
\[
\lim_{n \to \infty} \frac{P(\tau_{n-m-r} = n - m - r)}{P(\tau_r = n)} = 1.
\]

It follows now from (25)–(27) that for large $n$
\[
P(e^{S_{\tau_r} - S_r} \geq \varepsilon \mid \tau_{n-m} = n - m) \leq 2 \sum_{k=0}^{r} P\left(\frac{S_k - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}; \tau_r = k\right)
\]
\[
= 2P\left(\frac{S_{\tau_r} - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}\right).
\]

Since
\[
\lim_{n \to \infty} P\left(\frac{S_{\tau_r} - S_r}{a_r} \geq \log \frac{\varepsilon}{a_r}\right) = P\left(\inf_{t \in [0,1]} L_t - L_1 \geq 0\right) = 0,
\]

20
where $L_t$ is a Levy process generated by the stable distribution with index $\alpha$, we have

$$
\lim_{n \to \infty} P(e^{S_{\tau_r} - S_r} \geq \varepsilon \mid \tau_{n-m} = n - m) = 0.
$$

Combining this estimate with Lemmas 8 and 9 we conclude that for all $\varepsilon > 0$ there is a natural number $a$ such that for sufficiently large $n$

$$
P\left(\left|e^{S_{\tau_r} - S_r} \tilde{Z}_r - \beta_{a,r}\right| \geq \varepsilon/2 \mid \tau_{n-m} = n - m\right) \leq \varepsilon/2. \tag{28}
$$

Moreover, from Lemma 5 we see that $\beta_{a,r}$, conditioned on $\tau_{n-m} = n - m$, converges in distribution for every $a$. This implies that $e^{S_{\tau_r} - S_r} \tilde{Z}_r$, conditioned on $\tau_{n-m} = n - m$ converges in distribution. According to Lemma 7 there is a $\delta > 0$ such that

$$
P^+\left(e^{-S_a} \sum_{1 \leq i \leq a} \tilde{Z}_i < \delta\right) < \varepsilon/2 \tag{29}
$$

if $a$ is sufficiently large.

For sufficiently small $\varepsilon > 0$ let

$$
\delta' = \delta - \varepsilon/2 > 0.
$$

In view of (28) and (29)

$$
\lim_{n \to \infty} P(\beta_{a,r} < \delta' \mid \tau_{n-m} = n - m)
\leq \lim_{n \to \infty} P(e^{S_{\tau_r} - S_r} \tilde{Z}_r < \delta' + |e^{S_{\tau_r} - S_r} \tilde{Z}_r - \beta_{a,r}| \mid \tau_{n-m} = n - m)
\leq \lim_{n \to \infty} P(e^{S_{\tau_r} - S_r} \tilde{Z}_r < \delta \mid \tau_{n-m} = n - m) + \varepsilon/2 \leq \varepsilon.
$$

Thus

$$
P(\beta_{a,r} < \delta' \mid \tau_{n-m} = n - m) \leq \varepsilon
$$

if $n$ is sufficiently large. Therefore the limiting distribution of $e^{S_{\tau_r} - S_r} \tilde{Z}_r$ conditioned on $\tau_{n-m} = n - m$ has no atom in zero. Applying Theorem 2 with $d_n = n - r_n$, we obtain the needed result. The theorem is proved.

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