1 Introduction

Let $R$ be a convex quadrilateral in the $xy$ plane. An ellipse which passes through the vertices of $R$ is called a circumscribed ellipse or ellipse of circumscription. In the book ([D]), Dorrie presents Steiner’s nice characterization of the ellipse of circumscription which has minimal eccentricity, which he calls the most nearly circular ellipse. A pair of conjugate diameters are two diameters of an ellipse such that each bisects all chords drawn parallel to the other. Let $\theta_1$ and $\theta_2$ be the angles which a pair of conjugate diameters make with the positive $x$ axis. Then $\tan \theta_1$ and $\tan \theta_2$ are called a pair of conjugate directions. First, Steiner proves that there is only one pair of conjugate directions, $M_1$ and $M_2$, that belong to all ellipses of circumscription. Then he proves in essence that if there is an ellipse, $E$, whose equal conjugate diameters possess the directional constants $M_1$ and $M_2$, then $E$ must be an ellipse of circumscription which has minimal eccentricity. There are several gaps and missing pieces in Steiner’s result. Steiner does not show that there exists an ellipse of circumscription, $E$, whose equal conjugate diameters possess the directional constants $M_1$ and $M_2$, or that such an ellipse is unique. He also does not prove in general the uniqueness of an ellipse of circumscription which has minimal eccentricity. It is possible that there could exist a circumscribed ellipse of minimal eccentricity that might not have equal conjugate diameters which possess the directional constants $M_1$ and $M_2$.

In Propositions 1 and 2 we fill in these gaps in Steiner’s proof. We prove(Proposition 1), without using the directional constants $M_1$ and $M_2$, that there is a unique ellipse of minimal eccentricity which passes through the vertices of $R$. Then we show(Proposition 2) that there exists an ellipse which passes through the vertices of $R$ and whose equal conjugate diameters possess the directional constants $M_1$ and $M_2$. In addition, our methods enable us to prove(Theorem 2) that there is a unique ellipse of minimal area which passes through the vertices of $R$.

In [H] the author proved numerous results about ellipses inscribed in convex quadrilaterals. We filled in similar gaps in a problem often referred to in the literature as Newton’s problem, which was to determine the locus of centers of ellipses inscribed in $R$. In particular, in [H] the author proved that there exists a unique ellipse of minimal eccentricity, $E_1$, inscribed in $R$. This leads to the last section of this paper, where we discuss a special class of convex quadrilaterals which we call bielliptic and which generalize the bicentric quadrilaterals. A convex quadrilateral, $R$, is called bicentric if there exists a circle inscribed in $R$ and a circle circumscribed about $R$. Let $E_0$ denote the unique ellipse of minimal eccentricity circumscribed about $R$. $R$ is called bielliptic if $E_1$ and $E_0$ have the same eccentricity. We prove(Theorem 4), that there exists a bielliptic convex quadrilateral which is not bicentric. We also prove(Theorem 5), that there exists a bielliptic trapezoid which is not bicentric.

Finally we prove the perhaps not so obvious result(Theorem 3), that if $E_1$ and $E_2$ are each ellipses, with $E_1$ inscribed in $R$ and $E_2$ circumscribed about $R$, then $E_1$ and $E_2$ cannot have the same center.
2 Minimal Eccentricity

We state the following lemma without proof. The details are well-known and can be found in numerous places.

Lemma 1: Consider the ellipse, $E_0$, with equation $Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0$, $A, B > 0$, $AB - C^2 > 0$, $AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF \neq 0$; Let $a$ and $b$ denote the lengths of the semi-major and semi-minor axes, respectively, of $E_0$. Let $\phi$ denote the acute rotation angle of the axes of $E_0$ going counterclockwise from the positive $x$ axis and let $(x_0, y_0)$ denote the center of $E_0$. Then

$$a^2 = \frac{AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF}{2(AB - C^2) \left( (A + B) - \sqrt{(B - A)^2 + 4C^2} \right)}; \quad (a)$$

$$b^2 = \frac{AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF}{2(AB - C^2) \left( (A + B) + \sqrt{(B - A)^2 + 4C^2} \right)}; \quad (b)$$

$$\phi = \frac{1}{2} \cot^{-1} \left( \frac{A - B}{2C} \right), C \neq 0 \text{ and } \phi = 0 \text{ if } C = 0, \quad (\text{phi})$$

and

$$x_0 = -\frac{1}{2} \frac{BD - CE}{AB - C^2}, \quad y_0 = \frac{1}{2} \frac{CD - AE}{AB - C^2} \quad (\text{center})$$

Throughout this section, we let $R$ be a given convex quadrilateral in the $xy$ plane, and we assume throughout that $R$ is not a trapezoid. We use the notation and terminology used by Steiner (see [D]), and we also assume that $R$ has the shape given in [D]. Other shapes for a convex quadrilateral are possible, of course, but we do not consider those cases in the proofs below, the details being similar. Let $OPRQ$ denote the vertices of $R$, in counterclockwise order. By using an isometry of the plane, we can assume that $O = (0, 0)$ and that $P, R, \text{ and } Q$ are in the first quadrant. Let $H = \overrightarrow{QR} \cap \overrightarrow{OP}, K = \overrightarrow{PR} \cap \overrightarrow{OQ}, q = |\overrightarrow{OQ}|, h = |\overrightarrow{OH}|$, and $k = |\overrightarrow{OK}|$. Use the oblique coordinate system with $\overrightarrow{OP}$ as the $x$ axis and $\overrightarrow{OQ}$ as the $y$ axis, with those sides given by $y = 0$ and $x = 0$. The sides $\overrightarrow{PR}$ and $\overrightarrow{QR}$ are given by $z = 0$ and $w = 0$, respectively, where $z = kx + py - kp$ and $w = qx + hy -hq$. It follows that

$$0 < p < h, \quad 0 < q < k. \quad (1)$$

Any ellipse passing through the vertices of $R$ has equation $\lambda xz + \mu yw = 0$, where $\lambda$ and $\mu$ are nonzero real numbers. Letting $v = \frac{\lambda}{\mu}$, the equation becomes $vxz + yw = 0$, or

$$kvx^2 + hy^2 + (vp + q)xy - v{k}px - h{q}y = 0. \quad (\text{ell})$$
Using $A = kv, B = h, C = \frac{1}{2}(vp+q), D = -vp, E = -hq,$ and $F = 0,$ (ell) is the equation of a nontrivial ellipse (that is, not just a single point) if and only if $AB - C^2 = k(v^2(p+q)^2 + q^2(q^2(h-p))^2 > 0$ and $AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF = k(v^2(p+q)^2 + q^2(q^2(h-p))^2 > 0.$ Note that $khv > 0$ implies that $v > 0,$ so that $k(v^2(p+q)^2 + q^2(q^2(h-p))^2 > 0.$ We write the first condition, $AB - C^2 > 0,$ as

$$g(v) > 0, g(v) = -p^2v^2 + (4kh - 2pq)v - q^2.$$ Note that $g(v) = 0 \iff v = \frac{1}{p^2} \left(2kh - pq \pm 2 \sqrt{k(h - pq)}\right).$ Hence $g(v) > 0 \iff v \in I,$ where

$$I = \left(\frac{1}{p^2} \left(2kh - pq - 2 \sqrt{k(h - pq)}\right), \frac{1}{p^2} \left(2kh - pq + 2 \sqrt{k(h - pq)}\right)\right).$$

Note that $(2kh - pq)^2 - 4(k(h - pq)) = q^2p^2 > 0 \Rightarrow 2kh - pq > 2 \sqrt{k(h - pq)}$ since $kh - pq > 0.$ Hence $I \subset (0, \infty).$ Our first main result is the following.

**Proposition 1:** There is a unique ellipse, $E_O,$ of minimal eccentricity which passes through the vertices of $R.$

**Proof:** By Lemma 1,

$$a^2 = \frac{2kh(v^2(p+q)^2 + q^2(q^2(h-p))^2)}{(4kh - (vp + q)^2) \left(kv - (vp + q)^2 + q^2(h - p)^2\right)} \quad \text{(aq)}$$

and

$$b^2 = \frac{2kh(v^2(p+q)^2 + q^2(q^2(h-p))^2)}{(4kh - (vp + q)^2) \left(kv + h\sqrt{(kv - h)^2 + (vp + q)^2}\right)} \quad \text{(bq)}$$

which implies that $\frac{b^2}{a^2} = \frac{kv + h - \sqrt{(kv - h)^2 + (vp + q)^2}}{kv + h + \sqrt{(kv - h)^2 + (vp + q)^2}}.$ Some simplification yields

$$\frac{b^2}{a^2} = f(v) = \frac{g(v)}{(kv + h + \sqrt{(kv - h)^2 + (vp + q)^2})^2}. \quad \text{(fv)}$$

We shall now minimize the eccentricity by maximizing $\frac{b^2}{a^2}.$ Now $f'(v) = -2\frac{(2kh - pq)(vk - q)(2kh - pq)(vk - h) + pqh^2 - q^2k}{\sqrt{(kv - h)^2 + (vp + q)^2} (kv + h + \sqrt{(kv - h)^2 + (vp + q)^2})^2}$ and $f''(v)$...
\[ (2hk - pq)(vk - h) + p^2hv - q^2k = 0 \iff v = v_0, \]
where
\[ v_0 = \frac{q^2k + 2kh^2 - hpq}{2k^2h - kpq + hp^2}. \quad \text{(v0)} \]

Some simplification yields
\[ (kv_0 - h)^2 + (v_0 p + q)^2 = \frac{(ph + qk)^2W}{(2k^2h - kpq + hp^2)^2}, \]
which implies that
\[ g(v_0) = \frac{4kh(kv - pq)W}{(2k^2h - kpq + hp^2)^2}, \quad \text{(gv0)} \]
where
\[ W = 4k^2h^2 + (hp - qk)^2. \quad \text{(W)} \]

Thus \( g(v_0) > 0 \) by (gv0), which implies that \( v_0 \in I \). Note that \( kv + h + \sqrt{(kv - h)^2 + (vp + q)^2} > 0 \) for all \( v > 0 \), and \( g(v) > 0, v \in I \). Thus \( f \) is differentiable on \( I \) and \( f \) has a unique real critical point in \( I \). Since \( g \) vanishes at the endpoints of \( I \), \( f \) also vanishes at the endpoints of \( I \) since the denominator in (fv) is positive at the endpoints of \( I \). Since \( f(v) > 0 \) on \( I \), \( f(v_0) \) must give the unique maximum of \( f \) on \( I \). \( \blacksquare \)

In [D], Steiner shows that the unique pair of conjugate directions that belong to all ellipses which pass through the vertices of \( R \) is given by

\[ M_1 = -\frac{k}{p} + \frac{r}{hp}, M_2 = -\frac{k}{p} - \frac{r}{hp}, \quad \text{where} \quad r = \sqrt{hk\sqrt{hk - pq}}. \quad \text{(M1M2)} \]

**Proposition 2:** There exists an ellipse which passes through the vertices of \( R \) and whose equal conjugate diameters possess the directional constants \( M_1 \) and \( M_2 \).

**Proof:** Let \( E_0 \) denote the unique ellipse from Proposition 1 of minimal eccentricity which passes through the vertices of \( R \), and let \( L \) and \( L' \) denote a pair of equal conjugate diameters of \( E_0 \) with directional constants \( M \) and \( M' \). Let \( \phi \) denote the acute angle of counterclockwise rotation of the axes of \( E_0 \) and let \( a \) and \( b \) denote the lengths of the semi-major and semi-minor axes, respectively, of \( E_0 \). It is known(see, for example [S]) that \( L \) and \( L' \) make equal acute angles, on opposite sides, with the major axis of \( E_0 \). Let \( \theta \) denote the acute angle going counterclockwise from the major axis of \( E_0 \) to one of the equal conjugate diameters, which implies that \( \tan \theta = \frac{b}{a} \). By Lemma 1, with
\[ A = kv, B = h, C = \frac{1}{2}(vp + q), D = -vkp, E = -hq, \]
and \( F = 0, \cot \phi = \frac{kv - h}{vp + q} \). As one would expect from the results in [D], if there is some ellipse whose equal conjugate diameters possess the directional constants \( M_1 \) and \( M_2 \), then that ellipse minimizes the eccentricity among all ellipses of circumscription. By the proof of Proposition 1, the point \( v_0 \) given
Proposition 2, we let \( v = v_0 \). Then \( \cot \phi = \frac{kq - hp}{2kh} \Rightarrow \phi = 1 \frac{1}{2} \cot^{-1} \left( \frac{kq - hp}{2kh} \right) \Rightarrow \cot(2\phi) = \frac{kq - hp}{2kh} \Rightarrow \cot \phi = \frac{1}{2} \cot^{-1} \left( \frac{kq - hp}{2kh} \right) \Rightarrow \cot \phi = \frac{1}{2} \cot^{-1} \left( \frac{kq - hp \pm \sqrt{4k^2h^2 + (kq - hp)^2}}{2kh} \right) = \frac{kq - hp \pm \sqrt{W}}{2kh} \). We first need to determine whether to choose the positive or the negative root. If \( kq - hp > 0 \), then \( \cot(2\phi) = \frac{kq - hp}{2kh} > 0 \Rightarrow 0 \leq 2\phi \leq \frac{\pi}{2} \Rightarrow 0 \leq \phi \leq \frac{\pi}{4} \Rightarrow 1 < \cot \phi < \infty \).

Let \( x = 2kh, y = kq - hp, 0 < x < \infty, 0 < y < \infty \). If \( \cot \phi = \frac{kq - hp - \sqrt{W}}{2kh} \), then \( \cot \phi = \frac{y - \sqrt{x^2 + y^2}}{x} = \frac{y}{x} - \sqrt{1 + \left( \frac{y}{x} \right)^2} = u - \sqrt{1 + u^2} \), where \( u = \frac{y}{x} \), \( 0 < u < \infty \). Let \( z(u) = u - \sqrt{1 + u^2} \). Then \( z'(u) = \frac{\sqrt{1 + u^2} - u}{\sqrt{1 + u^2}} > 0, z(0) = -1, and \lim_{u \to \infty} z(u) = 0 \). Thus \(-1 < z(u) < 0 \Rightarrow -1 < \cot \phi < 0 \), which contradicts \( 1 < \cot \phi < \infty \). If \( kq - hp < 0 \), then \( \cot(2\phi) = \frac{kq - hp}{2kh} < 0 \Rightarrow \frac{\pi}{2} < 2\phi \leq \pi \Rightarrow \frac{\pi}{4} \leq \phi < \frac{\pi}{2} \Rightarrow 0 < \cot \phi < 1 \). Again, if \( \cot \phi = \frac{kq - hp - \sqrt{W}}{2kh} \), then \( \cot \phi = z(u), -\infty < u < 0 \). Since \( z(0) = -1 \) and \( \lim_{u \to -\infty} z(u) = -\infty \), \(-\infty < z(u) < -1 \Rightarrow \cot \phi < -1 \), which contradicts \( 0 < \cot \phi < 1 \). That proves

\[
\cot \phi = \frac{kq - hp + \sqrt{W}}{2kh}.
\]

To finish the proof of Proposition 2, note that \( M_1 = \frac{-kh + \sqrt{kh\sqrt{kh - pq}}}{hp} = \frac{\sqrt{kh\sqrt{kh - pq}}}{hp} < 0 \) and \( M_2 = \frac{-k}{p} - \sqrt{kh\sqrt{kh - pq}} < 0 \). Thus the only way that \( L \) and \( L' \) can form angles of \( \theta \) and \( -\theta \), respectively, with the major axis of \( E_O \) is if the major axis of \( E_O \) has a negative slope. In that case the minor axis of \( E_O \) is rotated by \( \phi \) counterclockwise from the positive \( x \) axis. It follows that the two directional constants, \( M \) and \( M' \), are given by \( \tan \left( \phi + \theta - \frac{\pi}{2} \right) \) and \( \tan \left( \phi - \theta - \frac{\pi}{2} \right) \). We shall prove that \( \tan \left( \phi + \theta - \frac{\pi}{2} \right) = M_1 \). We find it convenient to introduce the following variables:

\[
s = hp + kq, t = hp - kq.
\]

Note that \( 2k^2h - kpq + hp^2 = k(kh - pq) + k^2h + hp^2 > 0 \) by (1). Hence

\[
(kv_0 + h) + \sqrt{(kv_0 - h)^2 + (v_0p + q)^2} = kv_0 + h + \frac{(ph + qk) \sqrt{W}}{2k^2h - kpq + hp^2},
\]

which implies that

\[
\frac{(kv_0 + h) (2k^2h - kpq + hp^2) + (ph + qk) \sqrt{W}}{2k^2h - kpq + hp^2} = \frac{W + (ph + qk) \sqrt{W}}{2k^2h - kpq + hp^2}.
\]
\[
\sqrt{W} \frac{\sqrt{W} + (ph + qk)}{2kh - kpq + hp^2} = \frac{4kh (kh - pq) W}{(2k^2h - kpq + hp^2)^2 W (\sqrt{W} + (ph + qk))^2} = 4r (\sqrt{W} + s)^2 \cdot \frac{2r}{(\sqrt{W} + s)^2}. \]

By (fv) again,

\[
\frac{b}{a} = \frac{2r}{\sqrt{W} + s}. \tag{2}
\]

By (cotphi) and (2),

\[
\tan \left( \phi + \theta - \frac{\pi}{2} \right) = \frac{\tan \theta \tan \phi - 1}{\tan \theta + \tan \phi} = \frac{b + \frac{2kh}{a}}{b + \frac{kq - hp + \sqrt{W}}{a}} = \frac{2kh}{kq - hp + \sqrt{W}}.
\]

\[
\frac{2r}{\sqrt{W} + s} \left( \frac{2kh - t}{\sqrt{W} - t} \right) = \frac{2r}{\sqrt{W} + s} \left( \frac{4kh - \sqrt{W} + s}{\sqrt{W} - t} \right) - \frac{2r}{\sqrt{W} - t} + 2kh \left( \frac{\sqrt{W} + s}{\sqrt{W} - t} \right). \]

Hence \( \tan \left( \phi + \theta - \frac{\pi}{2} \right) - M_1 = \frac{1}{2} \left( \frac{4kh - \sqrt{W} + s}{\sqrt{W} - t} \right) = 0 \Rightarrow \frac{4khr}{hp - h(pst + 2(r-hk)t - 2s(r-hk)kh + (-hp(s-t) - 2r(r-hk) - 2(r-hk)kh)\sqrt{W}} = 0. \]

Similarly, \( \tan \left( \phi + \theta - \frac{\pi}{2} \right) = M_2. \]

By Propositions 1 and 2 and the main result in [D], we have

**Theorem 1:** There exists a unique ellipse, \( E_0 \), which passes through the vertices of \( R \) and whose equal conjugate diameters possess the directional constants \( M_1 \) and \( M_2 \). Furthermore, \( E_0 \) is the unique ellipse of minimal eccentricity among all ellipses which pass through the vertices of \( R \).

### 3 Minimal Area

We now prove a result similar to Proposition 1, but which instead minimizes the area among all ellipses which pass through the vertices of \( R \). This was not discussed by Steiner in [D] and there does not appear to be a nice characterization of the minimal area ellipse. Again we shall prove the case when \( R \) is not a trapezoid. Since convex quadrilaterals and ratios of areas of ellipses are preserved under one–one affine transformations, we may assume throughout this...
section that the vertices of \( R \) are \((0,0), (1,0), (0,1), \) and \((s,t)\) for some positive real numbers \( s \) and \( t \). Furthermore, since \( R \) is convex and not a trapezoid, it follows easily that
\[
s + t > 1 \quad \text{and} \quad s \neq 1 \neq t. \tag{3}
\]

**Lemma 2:** Suppose that the vertices of \( R \) are \((0,0), (1,0), (0,1), \) and \((s,t)\) for some positive real numbers \( s \) and \( t \) satisfying (3). Let
\[
m_{s,t} = \frac{t}{s(s-1)^2} \left( s + t - 1 + st - 2\sqrt{st(s+t-1)} \right)
\]
and
\[
M_{s,t} = \frac{t}{s(s-1)^2} \left( s + t - 1 + st + 2\sqrt{st(s+t-1)} \right).
\]
An ellipse, \( E_0 \), passes through the vertices of \( R \) if and only if \( E_0 \) has the form
\[
stux^2 + sty^2 - (s(1-u) + t(t-1))xy - stux - sty = 0, \quad u \in I_{s,t} = (m_{s,t}, M_{s,t}). \tag{5}
\]

If \( a \) and \( b \) denote the lengths of the semi-major and semi-minor axes, respectively, of \( E_0 \), then
\[
a^2 = \frac{2u(su+tu)s^2t^2(s+t-1)}{4st(s+t-1)u-(s-1)u-(t-1)^2)} \left( st(u+1) - \sqrt{t^2(s+t-1)^2 - 2st(s+t-1)u + s^2(t^2 + (s-1)^2)u^2} \right)
\]
and
\[
b^2 = \frac{2u(su+tu)s^2t^2(s+t-1)}{4st(s+t-1)u-(s-1)u-(t-1)^2)} \left( st(u+1) + \sqrt{t^2(s+t-1)^2 - 2st(s+t-1)u + s^2(t^2 + (s-1)^2)u^2} \right). \tag{6}
\]

Finally, the center of \( E_0 \), \((x_0, y_0)\), is given by
\[
x_0 = \frac{st \left( (2st + s^2 - s)u + (t^2 - t) \right)}{2st \left( st + s + t - 1 \right) u - s^2 \left( s - 1 \right)^2 u^2 - t^2 \left( t - 1 \right)^2} \tag{8}
\]
and
\[
y_0 = \frac{st \left( (2st + t^2 - t)u + (s^2 - s)u^2 \right)}{2st \left( st + s + t - 1 \right) u - s^2 \left( s - 1 \right)^2 u^2 - t^2 \left( t - 1 \right)^2} \tag{9}
\]

**Proof:** Substituting the vertices of \( R \) into the general equation of a conic,
\[
Ax^2 + By^2 + 2Cx x + D x + E y + F = 0,
\]
yields the equations \( F = 0, A + D = 0, B + E = 0, \) and \( As^2 + Bt^2 + 2Cst - As - Bt = 0, \) which implies that \( As(s-1) + Bt(t-1) + 2Cst = 0 \) or \( C = -\frac{As(s-1) + Bt(t-1)}{2st}. \)
Multiplying thru by \( st \) and dividing thru by \( B \) yields the equation in (5), with \( u = \frac{A}{B}. \)

Conversely, any curve satisfying (5) must pass through the vertices of \( R \). The curve defined by \( Ax^2 + By^2 + 2Cx x + D x + E y + F = 0 \) is an ellipse if and only if \( AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF \neq 0 \) and \( AB - C^2 > 0. \) The first condition becomes \( s^2t^2u(s + t - 1)(su + t) \neq 0, \) which clearly holds since \( s + t > 1. \) The second condition becomes
\[
4(\text{stu}) (st -(s(1-u) + t(t-1))^2) = -s^2(s-1)^2u^2 + 2st(st + s + t - 1)u - t^2(t-1)^2 > 0,
\]
which we write as \( \alpha(u) < 0, \) where
\[
\alpha(u) = s^2(s - 1)^2 u^2 - 2st(st + s + t - 1) u + t^2(t - 1)^2.
\]

Now it is easy to show that \( \alpha(u) < 0 \iff m_{s,t} < u < M_{s,t} \). That proves (5). If \( E_0 \) satisfies (5), then (6) and (7) follow immediately from Lemma 1–(asq) and (bsq), and (8) and (9) follow immediately from Lemma 1–center, with \( A = stu, B = st, C = -\frac{1}{2} (s(s - 1)u + t(t - 1)), D = -stu, E = -st \), and \( F = 0 \).

**Theorem 2:** There exists a unique ellipse, \( E_a \), of minimal area which passes through the vertices of \( R \).

**Proof:** By Lemma 2–(6) and (7),
\[
a^2y^2 = \frac{4u^2(su+t)^2s^2t^2(st(s+t-1))^2}{(4stu(st(s+t-1)-4stu(st(t-1)^2))^2(t-1)^2)} \times \\
\frac{(st(u+1)+\sqrt{t^2(s^2+(t-1)^2)}-2st(s+t-1)u+s^2(t^2+(t-1)^2)u^2)^2}{(st(u+1)-\sqrt{t^2(s^2+(t-1)^2)}-2st(s+t-1)u+s^2(t^2+(t-1)^2)u^2)^2} = \\
\frac{4u^2(su+t)^2s^2t^2(st(st(s+t-1))+(su+t)^2-4stu(st(s+t-1)))}{4u^2(su+t)^2s^2t^2(st(s+t-1))^2} = \beta(u),
\]

where
\[
\beta(u) = -\frac{4u^2(su+t)^2s^2t^2(st(s+t-1))^2}{(\alpha(u))^3}.
\]

Note that \( \beta \) is differentiable on \( I_{s,t} \) since \( \alpha(u) < 0 \) there. Also, \( m_{s,t} > 0 \iff s + t - 1 + st > 2\sqrt{st(s+t-1)} \iff (s + t - 1 + st)^2 > 4st(s + t - 1)(s + t - 1), which holds since \( s, t \neq 1 \). Thus \( m_{s,t} > 0 \) and \( M_{s,t} > 0 \), which implies that \( I_{s,t} \in (0, \infty) \). Now \( \lim_{u \to m_{s,t}^{-}} \alpha(u) = \lim_{u \to M_{s,t}^{+}} \alpha(u) = 0 \) thru negative numbers and the numerator of \( \beta(u) \), for given \( s \) and \( t \), satisfies \( 4u^2(su+t)^2s^2t^2(st(s+t-1))^2 > 0 \) for \( u \in I_{s,t} \). Thus \( \lim_{u \to m_{s,t}^{+}} \beta(u) = \lim_{u \to M_{s,t}^{-}} \beta(u) = \infty \), which implies that \( \beta \) must attain its global minimum on \( I_{s,t} \).

\[
\gamma(u) = s^3(s - 1)^2 u^3 + 3s^2t(2s^2 - 3s + st + 1 + t) u^2 - st^2(2s^2 + st - 3t + s + 1) u - t^3(t - 1)^2.
\]

\( \gamma(0) = -t^3(t - 1)^2 < 0 \) and \( \lim_{u \to \infty} \gamma(u) = \infty \), which implies that \( \gamma \) has at least one real root in \( (0, \infty) \). We now show that \( \gamma \) is convex on \( I_{s,t} \) by looking at \( \gamma''(u) = 2s^2d(s, t) \), where \( d(s, t) = 2s^2 + st - 3s + t + 1 \). Now \( \frac{dd}{ds} = 4s + t - 3 \) and \( \frac{dd}{dt} = s + 1 \), which implies that \( d \) has no critical points in \( S = \{(s, t): s + t \geq 1, s \geq 0, t \geq 0\} \). The boundary of \( S \) consists of \( S_1 \cup S_2 \cup S_3 \), where \( S_1 = \{(s, t): s + t = 1, 0 \leq s \leq 1\}, S_2 = \{(s, t): s = 0, t \geq 1\}, S_3 = \{(s, t): s \geq 0, t = 1\} \).
and $S_3 = \{(s,t) : t = 0, s \geq 1\}$. We now check $d$ on $\partial(S)$. On $S_1$ we have $d(1-t, t) = t(1+t) \geq 0$, on $S_2$ we have $d(0, t) = 1+t \geq 0$, and on $S_3$ we have $d(s,0) = (2s-1)(s-1) \geq 0$. Thus $d(s,t) \geq 0$ on $S$, which implies that $\gamma$ is convex on $I_{s,t}$. That in turn implies that $\beta$ has a unique global minimum on $I_{s,t}$, which yields a unique ellipse of minimal area which passes through the vertices of $R$.

**Remark:** In [RP] & [RP2], the authors investigate the problem of constructing and characterizing an ellipse of minimal area containing a finite set of points. The results and methods in this paper are different than those in [RP] & [RP2], but it is worth pointing out some of the small intersection. In particular, for a convex quadrilateral, $R$, the authors in [RP] & [RP2] construct an algorithm for finding the minimal area ellipse containing $R$ and they also prove a uniqueness result. For the case when this ellipse passes thru all four vertices of $R$, this ellipse is then the minimal area ellipse discussed in this paper. However, there are convex quadrilaterals, $R$, for which the minimal area ellipse containing $R$ does not pass thru all four vertices of $R$. In that case, the the minimal area ellipse discussed in this paper is not the same.

### 4 Inscribed versus Circumscribed

In this section and the next, we allow $R$ to be a **trapezoid**, so we shall need a version of Lemma 2 for trapezoids. The proof of Lemma 3 follows immediately from Lemma 1 or from Lemma 2 by allowing $s$ to approach 1. We omit the details here.

**Lemma 3:** Suppose that $R$ is a **trapezoid** with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,t), t \neq 1$.

An ellipse, $E_0$, passes through the vertices of $R$ if and only if $E_0$ has the form

$$tx^2 + ty^2 - t(t-1)xy - tux - ty = 0, u \in I_t = \left(\frac{1}{4} (t-1)^2, \infty\right)$$

(10)

If $a$ and $b$ denote the lengths of the semi–major and semi–minor axes, respectively, of $E_0$, then

$$a^2 = \frac{-2u(u+t)}{\left(-4u + (t-1)^2\right) \left(u + 1 - \sqrt{2 + t^2 - 2t - 2u + u^2}\right)}$$

(11)

and

$$b^2 = \frac{-2u(u+t)}{(t-1)^2 - 4u \left(u + 1 + \sqrt{(t-1)^2 + (u-1)^2}\right)}$$

(12)

Finally, the center of $E_0, (x_0, y_0)$, is given by

$$x_0 = \frac{2u + t - 1}{4u - (t-1)^2}, y_0 = \frac{(1+t)u}{4u - (t-1)^2}$$

(13)
Theorem 3: Let $R$ be a convex quadrilateral in the $xy$ plane which is not a parallelogram. Suppose that $E_1$ and $E_2$ are each ellipses, with $E_1$ inscribed in $R$ and $E_2$ circumscribed about $R$. Then $E_1$ and $E_2$ cannot have the same center.

Proof: Assume first that $R$ is not a trapezoid. Since the center of an ellipse is affine invariant, we may assume that the vertices of $R$ are $(0,0), (1,0), (0,1),$ and $(s,t)$ as above, where $s$ and $t$ satisfy (3). By [H, Theorem 2.3], if $M_1$ and $M_2$ are the midpoints of the diagonals of $R$, then each point on the open line segment, $Z$, connecting $M_1$ and $M_2$ is the center of some ellipse inscribed in $R$. Thus the locus of centers of $E_1$ is precisely $Z$. For $R$ above, the equation of $Z$ is $y = L(x) = \frac{1}{2} s - t + 2x(t - 1) \frac{1}{s - 1}$, where $x$ lies in the open interval connecting $\frac{1}{2}$ and $\frac{1}{2}s$. If $E_1$ and $E_2$ have the same center, then the center of $E_2,(x_0, y_0)$, must lie on $Z$. Hence $L(x_0) = y_0$, which implies that $L(x_0) - y_0 = (t + s) ((s - s^2) u + t^2 - t) (s^2 - s) u + t^2 - t) (s - s^2) u + t^2 - t) = 0$. Thus $(s - s^2) u + t^2 - t = 0$ or $(s^2 - s) u + t^2 - t = 0$, which implies that $u = \frac{t^2 - t}{s^2 - s}$. If $u = \frac{t^2 - t}{s^2 - s}$, then some simplification yields, by (8) in Lemma 2, $x_0 = \frac{1}{2}s$. Similarly, if $u = -\frac{t^2 - t}{s^2 - s}$, then $x_0 = \frac{1}{2}$. But $\frac{1}{2}s$ and $\frac{1}{2}$ do not lie on $Z$, and thus $E_1$ and $E_2$ cannot have the same center. Now suppose that $R$ is a trapezoid, but not a parallelogram. Then we may assume, again by affine invariance, that the vertices of $R$ are $(0,0), (1,0), (0,1),$ and $(1,t), t \neq 1$. The equation of $Z$ is now $x = \frac{1}{2}$, where $y$ lies in the open interval connecting $\frac{1}{2}$ and $\frac{1}{2}$. If $E_1$ and $E_2$ have the same center, then $x_0 = \frac{1}{2}$. By (13) of Lemma 3, $\frac{2u + t - 1}{4u} = \frac{1}{2} \Rightarrow 4u + 2t - 2 = 4u - (t - 1)^2 \Rightarrow t = \pm 1$, which contradicts the assumption that $t > 0, t \neq 1$. Hence again $E_1$ and $E_2$ cannot have the same center.

It is easy to find examples where the center of an ellipse circumscribed about $R$ may lie inside, on, or outside $R$. We make the following conjectures.

Conjecture 1: The center of the ellipse of minimal eccentricity circumscribed about $R$ lies inside $R$.

Conjecture 2: The center of the ellipse of minimal area circumscribed about $R$ lies inside $R$.

5 Bielliptic Quadrilaterals

The following definition is well-known.

Definition 1: Let $R$ be a convex quadrilateral in the $xy$ plane.

(A) $R$ is called cyclic if there is a circle which passes through the vertices of $R$.  

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(B) \( R \) is called tangential if a circle can be inscribed in \( R \).
(C) \( R \) is called bicentric if \( R \) is both cyclic and tangential.

We generalize the notion of bicentric quadrilaterals as follows. In [H, Theorem 4.4] the author proved that there is a unique ellipse, \( E_I \), of minimal eccentricity inscribed in a convex quadrilateral, \( R \). Using Proposition 1 from this paper, we let \( E_O \) be the unique ellipse of minimal eccentricity circumscribed about \( R \).

**Definition 2:** A convex quadrilateral is called bielliptic if \( E_I \) and \( E_O \) have the same eccentricity.

If \( R \) is bielliptic, we say that \( R \) is of class \( \tau \), \( 0 \leq \tau < 1 \), if \( E_I \) and \( E_O \) each have eccentricity \( \tau \).

It is natural to ask the following:

**Question:** Does there exist a bielliptic quadrilateral of class \( \tau \) for some \( \tau, \tau > 0 \)?

We answer this in the affirmative with the following results.

**Theorem 4:** There exists a convex quadrilateral, \( R \), which is bielliptic of class \( \tau \) for some \( \tau > 0 \). That is, there exists a bielliptic convex quadrilateral which is not bicentric.

**Proof:** Consider the convex quadrilateral, \( R \), with vertices \((0,0), (1,0), (0,1), \) and \((s,t)\). We shall show that for some \( s \) and \( t \) satisfying (3), \( R \) is bielliptic of class \( \tau \) for some \( \tau > 0 \). It is easy to show that \( R \) is cyclic if and only if
\[
\left( s - \frac{1}{2} \right)^2 + \left( t - \frac{1}{2} \right)^2 = \frac{1}{2}.
\]
In general, a convex quadrilateral is tangential if and only if opposite sides add up to the same sum. It follows that \( R \) is tangential if and only if \( s = t \). Thus \( s = \frac{1}{2}, t = \frac{1}{2} (1 + \sqrt{2}) \approx 1.207 \) gives a cyclic quadrilateral, and \( s = 2, t = 2 \) gives a tangential quadrilateral. Consider the family of quadrilaterals \( R_r \) given by
\[
s = -\frac{3}{2}r + 2, t = r \left( \frac{1}{2} + \frac{1}{2} \sqrt{2} \right) + 2 - 2r, 0 \leq r \leq 1. \tag{14}
\]
It is easy to see that \( r = 0 \) gives a tangential quadrilateral which is not cyclic, and \( r = 1 \) gives a cyclic quadrilateral which is not tangential. Since the eccentricity of the inscribed and circumscribed ellipses of minimal eccentricity, \( E_I(r) \) and \( E_O(r) \), each vary continuously with \( r \), \( R_r \) must be bielliptic for some \( r, 0 < r < 1 \). More precisely, let \( \epsilon_I(r) \) and \( \epsilon_O(r) \) denote the eccentricities of \( E_I \) and \( E_O \), respectively. Then \( \epsilon_I(0) = 0 \) and \( \epsilon_O(0) > 0 \) since \( E_I(0) \) is a circle and \( E_O(0) \) is not a circle. Similarly, \( \epsilon_I(1) > 0 \) and \( \epsilon_O(1) = 0 \) since \( E_I(1) \) is not a circle and \( E_O(1) \) is a circle. Since \( \epsilon_I(r) \) and \( \epsilon_O(r) \) are each continuous functions of \( r \), by the Intermediate Value Theorem, \( \epsilon_I(r_0) = \epsilon_O(r_0) \) for some \( 0 < r_0 < 1 \). Now if \( s \) and \( t \) satisfy (14), then
\[
(2s - 1)^2 + (2t - 1)^2 - 2 = -\frac{2}{41} \left( -10 + 3 \sqrt{2} \right) (-1 + r) (41r - 40 - 12 \sqrt{2}) = 0
\]
\[\iff r = 1 \text{ or } r = \frac{40}{41} + \frac{12}{3} \sqrt{2} \approx 1.39 > 1. \] So for \( 0 < r < 1 \), \( R_r \) cannot be cyclic. Also, \( s = t \iff -\frac{3}{2}r + 2 = -\frac{3}{2}r + \frac{1}{2} \sqrt{2} + 2 \iff r = 0. \) So for
0 < r < 1, $R_r$ cannot be tangential. It follows that $\epsilon_I(r_0) = \epsilon_O(r_0) = \tau > 0$, which means that $R_{r_0}$ is bielliptic of class $\tau$.

**Theorem 5:** There exists a bielliptic trapezoid of class $\tau$ for some $\tau > 0$.

**Proof:** Consider the trapezoid, $R$, with vertices $(0, 0), (1, 0), (0, 1)$, and $(1, t), t \neq 1$. We shall show that for some $t \neq 1$, $R$ is bielliptic of class $\tau > 0$.

By Lemma 3–(11) and (12), \[
\frac{b^2}{a^2} = \frac{(t - 1)^2 - 4u}{(t - 1)^2 - 4u} \cdot \frac{u + 1 - \sqrt{(t - 1)^2 + (u - 1)^2}}{u + 1 + \sqrt{(t - 1)^2 + (u - 1)^2}}
\]

Hence the square of the eccentricity of an ellipse circumscribed in $R$ is given by \[\epsilon^2(u) = 1 - \frac{b^2}{a^2} = \frac{2\sqrt{(t - 1)^2 + (u - 1)^2}}{u + 1 + \sqrt{(t - 1)^2 + (u - 1)^2}} = \frac{2\sqrt{(t - 1)^2 + (u - 1)^2}}{u + 1 + \sqrt{(t - 1)^2 + (u - 1)^2}} = 0 \iff u = \frac{1}{2} (t^2 - 2t + 3).\]
We shall show that this value of $u$ gives the minimal eccentricity. First, \[
\epsilon\left(\frac{1}{2} (t^2 - 2t + 3)\right) = \frac{2\sqrt{(t - 1)^2 + (u - 1)^2}}{t^2 - 2t + 5 + \sqrt{(t - 1)^2 + (u - 1)^2}} = \frac{2 |t - 1| \sqrt{t^2 - 2t + 5}}{t^2 - 2t + 5 + |t - 1| \sqrt{t^2 - 2t + 5}} = \frac{2 |t - 1|}{\sqrt{(t - 1)^2 + 4 + |t - 1|}} \leq \frac{2 |t - 1|}{|t - 1| + |t - 1|} = 1.
\]
Also, \[
\lim_{u \to (t^2 - 2t + 3)^+} \epsilon(u) = 1 \quad \text{and} \quad \lim_{u \to \infty} \epsilon(u) = 1.
\]
Thus the square of the minimal eccentricity of an ellipse circumscribed in $R$ is given by \[
\epsilon^2(t) = \frac{2 |t - 1|}{\sqrt{(t - 1)^2 + 4 + |t - 1|}} \tag{15}
\]

In [H] the author derived formulas for the eccentricity of the unique ellipse of minimal eccentricity inscribed in a convex quadrilateral, $R$. Those formulas apply when $R$ is not a trapezoid. The methods used in [H] can easily be adapted to the case when $R$ is a trapezoid. The ellipse of minimal eccentricity inscribed in a trapezoid is also unique, and one can derive the following formulas.

Let $I_t$ denote the open interval with $\frac{1}{2}$ and $\frac{1}{2} t$ as endpoints. Let \[
E(k) = \frac{(2k - 1)(2k - t)}{16 (t - 1)^2 k^4 + (8 + 8t^2 + 48t) k^2 - 32t(1 + t) k + 17t^2 - 2t + 1},
\]
\[
\epsilon(k) = \frac{2}{1 + \sqrt{1 - 16t (1 - t)^2 E(k)}} \tag{16}
\]
and \[
c(k) = 16k^3 - 12(t + 1)k^2 + 4(2t - 1)k + t + 1.
\]

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Then \( c(k) \) has a unique root, \( k_0 \), in \( I_k \), and \( \epsilon(k_0) \) equals the square of the minimal eccentricity of an ellipse inscribed in \( R \). By (15) and (16), we want to show that there is a value of \( t \neq 1 \) and \( k \in I_t \) such that \( c(k) = 0 \) and

\[
\frac{2|t|}{\sqrt{(t-1)^2+4+|t-1|}} = \frac{1}{1 + \sqrt{1-16t(1-t)^2} E(k)}.
\]

This is equivalent, after some algebraic simplification, to \( 4t(t-1)^4 E(k) + 1 = 0 \). Some more algebraic simplification yields the equation

\[
16(t-1)^2 k^4 + (16t^5 - 64t^4 + 96t^3 - 56t^2 + 64t + 8) k^2 - 8t(1 + t) (t^2 - 4t + 5) (t^2 + 1) k + 4t^6 - 16t^5 + 24t^4 - 16t^3 + 21t^2 - 2t + 1 = 0
\]

Thus we want a solution to the simultaneous equations (17) and \( c(k) = 0 \), with \( t \neq 1 \) and \( k \in I_t \). Maple gives the following solutions: \( t = 1, k = \frac{1}{2}, k = \frac{1}{2} \), \( t = \frac{1}{2} i + \frac{1}{2} i \), and \( t = \frac{2\rho_2^3 - 3\rho_2^2 + 1 - 2\rho_2}{3\rho_2^3 - 4\rho_2 - 1}, k = \frac{1}{2}\rho_2 \) where \( \rho_2 \) is a root of

\[
p(x) = 32x^{11} - 287x^{10} + 1006x^9 - 1487x^8 + 160x^7 + 1762x^6 - 884x^5 - 822x^4 + 80x^3 + 333x^2 + 150x + 21
\]

\( t = 1 \) or \( t = \frac{1}{2} i \) do not satisfy \( t \) real, \( t \neq 1 \). Since \( p(1) = 64 > 0 \) and \( p(1.5) = -23.07715 < 0 \), \( p \) must have a root, \( x_0 \), between 1 and 2. Numerically \( x_0 \approx 1.232267 \). It appears that the real roots of \( p \) are approximately \(-0.8295535, 1.232267, 1.778672 \), though we don’t need that here. Now \( \rho_2 = 1.232267 \Rightarrow t = \frac{2\rho_2^3 - 3\rho_2^2 + 1 - 2\rho_2}{3\rho_2^3 - 4\rho_2 - 1} \approx 1.658119 \). Then \( k = \frac{1}{2}\rho_2 = 0.6161335 \in I_t \). The corresponding common value of the eccentricity is \( \approx 0.69013 \).

Remark: It is interesting to note here that the bielliptic quadrilateral in Theorem 4 is not a trapezoid. The family of quadrilaterals \( R_t \) given in the proof of Theorem 4 yields a trapezoid if and only if \( s = 1 \) or \( t = 1 \). Now \( s = 1 \iff -\frac{3}{2}r^2 + 2 = 1 \iff r = \frac{2}{3} \) and \( t = 1 \iff r \left( \frac{1}{2} + \frac{1}{\sqrt{2}} \right) + 2 - 2r = 1 \iff r = \frac{2}{3} - \frac{\sqrt{2}}{3} > 1 \). Thus \( R_t \) is a trapezoid \( \iff r = \frac{2}{3} \). Now \( r = \frac{2}{3} \Rightarrow t = 1 + \frac{1}{3}\sqrt{2} \). By (15) in the proof of Theorem 5, the square of the minimal eccentricity of an ellipse circumscribed about \( R_{2/3} \) is

\[
\frac{2|t-1|}{\sqrt{(t-1)^2+4+|t-1|}} = \frac{2}{\sqrt{19} + 1} \approx 0.373.
\]

Also, \( I_t = \left( \frac{1}{2}, \frac{1}{2} t \right) \approx (0.5, 0.736) \). and \( c(k) = 16k^3 + (-24 - 4\sqrt{2}) k^2 + \left( \frac{8\sqrt{2}}{3} + 4 \right) k + 2 + \frac{1}{3}\sqrt{2} = 0 \) has the root \( k \approx 0.5918015 \in I_t \). That yields \( E(k) \approx -1.430 \). By (16) in the proof
of Theorem 5, the square of the minimal eccentricity of an ellipse inscribed in $R_{2/3}$ is $e^2(k) \approx 0.511$. Thus the bielliptic convex quadrilateral from Theorem 4 is not a trapezoid.

Theorems 4 and 5 show the existence of a bielliptic quadrilateral of class $\tau$ for some $0 < \tau < 1$. We cannot yet answer the following:

**Question:** Does there exist a bielliptic quadrilateral of class $\tau$ for each $\tau$, $0 < \tau < 1$?

If $R$ is a bielliptic quadrilateral, is there a nice relationship between the ellipse of minimal eccentricity inscribed in $R$ and the ellipse of minimal eccentricity passing thru the vertices of $R$? This would generalize the known relationship between the inscribed and circumscribed circles of bicentric quadrilaterals.

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