MARKOV PROCESSES OF CUBIC STOCHASTIC MATRICES:
QUADRATIC STOCHASTIC PROCESSES

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Abstract. We consider Markov processes of cubic stochastic (in a fixed sense) matrices which are also called quadratic stochastic process (QSPs). A QSP is a particular case of a continuous-time dynamical system whose states are stochastic cubic matrices satisfying an analogue of the Kolmogorov-Chapman equation (KCE). Since there are several kinds of multiplications between cubic matrices we have to fix first a multiplication and then consider the KCE with respect to the fixed multiplication. Moreover, the notion of stochastic cubic matrix also varies depending on the real models of application. The existence of a stochastic (at each time) solution to the KCE provides the existence of a QSP. In this paper, our aim is to construct QSPs for two specially chosen notions of stochastic cubic matrices and two multiplications of such matrices (known as Maksimov’s multiplications). We construct a wide class of QSPs and give some time-dependent behavior of such processes. We give an example with applications to the Biology, constructing a QSP which describes the time behavior (dynamics) of a population with the possibility of twin births.

1. Introduction

A Markov process is a random process indexed by time, in which the future is independent of the past, given the present. Thus, Markov processes are the natural stochastic analogs of the deterministic processes described by differential and difference equations. They form one of the most important classes of random processes. If the time space is $T = [0, \infty)$ and the state space is discrete, then Markov processes are known as continuous-time Markov chains.

The Kolmogorov-Chapman equation (KCE) gives the fundamental relationship between the probability transitions (kernels). Namely, it is known that (see e.g. [15]) if each element of a family of matrices satisfying the KCE is stochastic, then it generates a Markov process.

There are many random processes which can not be described by Markov processes of square stochastic matrices (see for example [3, 4, 6, 9]).

To have non-Markov process one can consider a solution of the KCE which is not stochastic for some time as in [2, 11, 13], where a chain of evolution algebras (CEA) is introduced and investigated. Later, this notion of CEA was generalized in [7], where a concept of flow of arbitrary finite-dimensional algebras (i.e. their matrices of structural constants are cubic matrices) is introduced.

By Maksimov [9] some associative multiplication rules of cubic matrices as well as cubic analogues of stochastic or doubly stochastic square matrices are introduced, for which he suggests several possible probability interpretations. Moreover, the concept of a Markov interaction process (MIP) is defined. It is shown that there exists a one-to-one correspondence between the transition matrices defining a MIP and the stochastic cubic matrices of a certain kind.

In this paper we study Markov process of cubic matrices, which is a two-parametric family of cubic stochastic matrices (we fix a notion of stochastic matrix and fix a multiplication rule of cubic matrices) satisfying the KCE.

The paper is organized as follows. In Section 2 we give the main definitions related to Markov processes, cubic matrices, several kinds of multiplications of cubic matrices and Markov processes of cubic matrices which are also called quadratic stochastic processes (QSPs). In Section 3 we

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describe all QSPs of type (3|0), these are solutions of the KCE in the class of 3-stochastic cubic matrices, with respect to the Maksimov’s 0-multiplication (see Section 2 for definitions). In Section 4 we construct some QSPs of type (12|a0), these are solutions of the KCE in the class of (1, 2)-stochastic cubic matrices, with respect to the Maksimov’s a0-multiplication. In Section 5 we give an application of a QSP of type (12|a0) to a population with a possibility of twins birth. For several QSPs we study time-dependent behavior of the processes.

2. Preliminaries

2.1. Markov process of square matrices. Let us recall first the notion of Markov process for square stochastic matrices. This will be useful to compare with Markov processes of cubic matrices.

A square matrix \( U = (U_{ij})_{i,j=1}^m \) is called right stochastic if

\[
U_{ij} \geq 0, \quad \forall i, j = 1, \ldots, m; \quad \sum_{j=1}^m U_{ij} = 1, \quad \forall i = 1, \ldots, m.
\]

Similarly one can define a left stochastic matrix being a non-negative real square matrix, with each column summing to 1 and a doubly stochastic matrix being a square matrix of non-negative real numbers with each row and column summing to 1.

A family of stochastic matrices \( \{U_s^{[s,t]}: s, t \geq 0\} \) is called a Markov process if it satisfies the Kolmogorov-Chapman equation:

\[
U^{[s,t]} = U^{[s,\tau]} U^{[\tau,t]}, \quad \text{for all } 0 \leq s < \tau < t. \tag{2.1}
\]

Let \( I = \{1, 2, \ldots, m\} \). A distribution (or state) of the set \( I \) is a probability measure \( x = (x_1, \ldots, x_m) \), where \( x_i \) is a probability of \( i \in I \). The set of all such vectors is called a simplex and denoted by

\[
S^{m-1} = \left\{ x \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}.
\]

Let \( x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)}) \in S^{m-1} \) be an initial distribution on \( I \). Denote by \( x^{(t)} = (x_1^{(t)}, \ldots, x_m^{(t)}) \in S^{m-1} \) the distribution of the system at the moment \( t \). For arbitrary moments of time \( s \) and \( t \) with \( s < t \) the matrix \( U^{[s,t]} = (U_{ij}^{[s,t]}) \) gives the transition probabilities from the distribution \( x^{(s)} \) to the distribution \( x^{(t)} \). Moreover \( x^{(t)} \) depends linearly from \( x^{(s)} \):

\[
x_k^{(t)} = \sum_{i=1}^m U_{ik}^{[s,t]} x_i^{(s)}, \quad k = 1, \ldots, m.
\]

A Markov chain is a type of Markov process that has either discrete state space or discrete time, but the precise definition of a Markov chain varies (see e.g. [1, 14, 15] for the theory of Markov process).

2.2. Cubic matrices. We consider a cubic matrix \( Q = (q_{ijk})_{i,j,k=1}^m \) as a \( m^3 \)-dimensional vector, i.e. an element of \( \mathbb{R}^{m^3} \), which can be uniquely written as

\[
Q = \sum_{i,j,k=1}^m q_{ijk} E_{ijk},
\]

where \( E_{ijk} \) denotes the cubic unit (basis) matrix, i.e. \( E_{ijk} \) is a \( m^3 \)-cubic matrix whose \( (i, j, k) \)th entry is equal to 1 and all the other entries are equal to 0.

Denoting \( Q_i = (q_{ijk})_{j,k=1}^m \) we can write the cubic matrix \( Q \) in the following form

\[
Q = (Q_1|Q_2| \ldots |Q_m).
\]
Denote by $\mathcal{C}$ the set of all cubic matrices over a field $F$. Then $\mathcal{C}$ is an $m^3$-dimensional vector space over $F$, i.e. for any matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathcal{C}, \lambda \in F$, we have

$$A + B := (a_{ijk} + b_{ijk}) \in \mathcal{C}, \quad \lambda A := (\lambda a_{ijk}) \in \mathcal{C}.$$  

In general, one can fix an $m^3 \times m^3 \times m^3$-cubic matrix $\mu = (C_{ijk,lmr}^{uvw})$ as a matrix of structural constants and give a multiplication of basis cubic matrices as

$$E_{ijk} * \mu E_{lmr} = \sum_{uvw} C_{ijk,lmr}^{uvw} E_{uvw}.$$  

(2.2)

Then the extension of this multiplication by bilinearity to arbitrary cubic matrices gives a general multiplication on the set $\mathcal{C}$ and it becomes an algebra of cubic matrices (ACM), denoted by $\mathcal{C}_\mu$ (see [8] for some basic properties of ACM). Under known conditions (see [5]) on structural constants one can make this general ACM as a commutative or/and associative algebra, etc.

2.3. Maksimov’s multiplications. Introduce some simple versions of multiplications (2.2).

Denote $I = \{1, 2, \ldots, m\}$.

Following [9] define the following multiplications for basis matrices $E_{ijk}$:

$$E_{ijk} *_0 E_{lmr} = \delta_{kl} \delta_{jn} E_{ijr}.$$  

(2.3)

Then for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathcal{C}$ the matrix $A *_0 B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{k=1}^{m} a_{ijk} b_{kjr}.$$  

(2.4)

Consider also

$$E_{ijk} *_a E_{lmr} = \delta_{kl} (E_{ia(j,n)r})_r,$$  

(2.5)

where $a: I \times I \to I$, $(j,n) \mapsto a(j,n), (j,n) \in I$, is an arbitrary associative binary operation and $\delta_{kl}$ is the Kronecker symbol. Note that (2.3) is not a particular case of (2.5).

Denote by $\mathcal{O}_m$ the set of all associative binary operations on $I$.

The general formula for the multiplication is the extension of (2.5) by bilinearity, i.e. for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathcal{C}$ the matrix $A *_a B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{l,n; a(l,n)=j} \sum_{k} a_{ilk} b_{knr}.$$  

Denote by $\mathcal{C}_a \equiv \mathcal{C}_a^m = (\mathcal{C}, *_a), a \in \mathcal{O}_m$, the ACM given by the multiplication $*_a$.

2.4. Markov process as a quadratic stochastic process. Following [7] we define a quadratic stochastic process.

Define several kinds of cubic stochastic matrices (see [9][10]): a cubic matrix $P = (p_{ijk})_{i,j,k=1}^{m}$ is called

(1, 2)-stochastic if

$$p_{ijk} \geq 0, \quad \sum_{i,j=1}^{m} p_{ijk} = 1, \quad \text{for all } k.$$  

(1, 3)-stochastic if

$$p_{ijk} \geq 0, \quad \sum_{i,k=1}^{m} p_{ijk} = 1, \quad \text{for all } j.$$  

(2, 3)-stochastic if

$$p_{ijk} \geq 0, \quad \sum_{j,k=1}^{m} p_{ijk} = 1, \quad \text{for all } i.$$
3-stochastic if
\[ p_{ijk} \geq 0, \quad \sum_{k=1}^{m} p_{ijk} = 1, \quad \text{for all } i, j. \]

The last one can be also given with respect to first and second index.

Maksimov [9] also defined a twice stochastic matrix: a (2,3)-stochastic cubic matrix is called twice stochastic if
\[ \sum_{i=1}^{m} p_{ijk} = \frac{1}{m}, \quad \text{for all } j, k. \]

Denote by \( \mathcal{S} \) the set of all possible kinds of stochasticity and denote by \( \mathbb{M} \) the set of all possible multiplication rules of cubic matrices.

Let parameters \( s \geq 0, t \geq 0 \), are considered as time.
Denote by \( \mathcal{M}^{[s,t]} = \left( P_{ijk}^{[s,t]} \right)_{i,j,k=1}^{m} \) a cubic matrix with two parameters.

**Definition 1** ( [7]). A family \( \{ \mathcal{M}^{[s,t]} : s, t \in \mathbb{R}_+ \} \) is called a Markov process of cubic matrices (or a quadratic stochastic process (QSP)) of type \( (s|\mu) \) if for each time \( s \) and \( t \) the cubic matrix \( \mathcal{M}^{[s,t]} \) is stochastic in sense \( \sigma \in \mathcal{S} \) and satisfies the Kolmogorov-Chapman equation (for cubic matrices):
\[ \mathcal{M}^{[s,t]} = \mathcal{M}^{[\sigma,t]} * \mathcal{M}^{[\tau,t]}, \quad \text{for all } 0 \leq s < \tau < t. \]

with respect to the multiplication \( \mu \in \mathbb{M} \).

We note that this definition of QSP gives an alternative of [10] Definition 3.1.1 and a natural generalization of the Markov process of Subsection 2.1.

In [7] using the QSPs some flows of finite-dimensional algebras are determined and investigated.

2.5. Motivations and interpretations. QSPs arise naturally in the study of biological and physical systems with interactions. Indeed, assume a particle of type \( i \) and a particle of type \( j \) have interaction at time \( s \), as an interaction process, then with probability \( P_{ijk}^{[s,t]} \) a particle of type \( k \) appears at time \( t \). The Kolmogorov-Chapman equation (2.6) gives the time-dependent evolution law of the interacting process (dynamical system).

Let \( x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)}) \in S^{m-1} \) be an initial distribution on \( I \).

Denote by \( x^{(t)} = (x_1^{(t)}, \ldots, x_m^{(t)}) \in S^{m-1} \) the distribution of the system at the moment \( t \). For arbitrary moments of time \( s \) and \( t \), with \( s < t \), the matrix \( \mathcal{M}^{[s,t]} \) gives the transition probabilities from the distribution \( x^{(s)} \) to the distribution \( x^{(t)} \).

Since we should have \( x^{(t)} \in S^{m-1} \), one can consider the following models:

- Consider \( P_{ijk}^{[s,t]} \) as the conditional probability \( P_{ijk}^{[s,t]}(k|i,j) \) that \( i \)th and \( j \)th particles (physics) or species (biology) interbred successfully at time \( s \), then they produce an individual \( k \) at time \( t \).

  Assume the “parents” \( ij \) are independent for any moment of time \( s \) and the matrix \( \left( P_{ijk}^{[s,t]} \right) \) is 3-stochastic, then the probability distribution \( x^{(t)} \) can be found by the total probability as
\[ x^{(t)}_k = \sum_{i,j=1}^{m} P_{ijk}^{[s,t]} x^{(s)}_i x^{(s)}_j, \quad k = 1, \ldots, m, \quad 0 \leq s < t. \]

For 1–stochastic and 2–stochastic it can be defined similarly, by replacing the corresponding indices.

- Consider now a physical (biological, chemical) system where there are \( m \) types of “particles” or molecules, the set of types is denoted by \( I = \{1, \ldots, m\} \), and each particle may split to two new ones having types from \( I \). Consider \( P_{ijk}^{[s,t]} \) as the conditional probability \( P_{ijk}^{[s,t]}(i,j|k) \) that a particle of type \( k \) starts splitting at time \( s \) and finishes splitting at
time $t$ and the result is two particles with $i$th and $j$th types. For a biological model see Section 5

Assume $\left(P_{ikj}\right)^{[s,t]}_{ijk}$ is (1,2)-stochastic then $x^{(t)}$ can be defined by

$$x^{(t)}_k = \frac{1}{2} \sum_{i,j=1}^{m} \left( P_{eij}^{[s,t]} + P_{ikj}^{[s,t]} \right) x^{(s)}_j, \quad k = 1, \ldots, m, \quad 0 \leq s < t. \quad (2.7)$$

For (1,3)-stochastic and (2,3)-stochastic cases one can define similarly by replacing the indices.

Thus finding $P_{ikj}^{[s,t]}$ from the equation (2.4) it is easy to see that if two cubic matrices, say $A$ and $B$, are 3-stochastic then their multiplication $A \cdot B$ is 3-stochastic too.

Section 5.

To construct a QSP of type $\sigma$ one has to solve (2.6). In this paper our aim is to study QSPs, for the following two cases

- $\sigma = 3$-stochastic and $\mu$ is the Maksimov’s multiplication $\mu = 0$ given by (2.3). Call this QSP of type $(3|0)$. Under these conditions the equation (2.6) has the following form

$$P_{ijk}^{[s,t]} = \sum_{k=1}^{m} P_{ijr}^{[s,r]} P_{kjr}^{[r,t]}, \quad \forall i, j, r \in I, \quad (2.8)$$

where

$$P_{ijk}^{[s,t]} \geq 0, \quad \forall i, j, k \in I; \quad \sum_{k=1}^{m} P_{ijk}^{[s,t]} = 1, \quad \text{for all } i, j \in I, \quad 0 \leq s < t. \quad (2.9)$$

Thus a QSP of type $(3|0)$ is a solution to the system (2.8) and (2.9).

- $\sigma = (1,2)$-stochastic and $\mu$ is the Maksimov’s multiplication with the operation $a = a_0$ such that $a_0(i, j) = i$ for any $i, j \in I$. Call this QSP of type $(12|a_0)$. Under these conditions the equation (2.6) has the following form

$$P_{ijr}^{[s,t]} = \sum_{k,n=1}^{m} P_{ijk}^{[s,r]} P_{knr}^{[r,t]}, \quad \forall i, j, r \in I, \quad (2.10)$$

where

$$P_{ijr}^{[s,t]} \geq 0, \quad \forall i, j, r \in I; \quad \sum_{i,j=1}^{m} P_{ijr}^{[s,t]} = 1, \quad \forall r \in I, \quad 0 \leq s < t. \quad (2.11)$$

Hence a QSP of type $(12|a_0)$ is a solution to the system (2.10) and (2.11).

3. QSPs of Type $(3|0)$

For the multiplication (2.4) it is easy to see that if two cubic matrices, say $A$ and $B$, are 3-stochastic then their multiplication $A \cdot B$ is 3-stochastic too.

Let $\mathcal{M}_{[s,t]} = (P_{ikj}^{[s,t]}),_{i,k=1}^{m}$ be the $j$th layer of the matrix $\mathcal{M}_{[s,t]}$. The following proposition characterizes all QSPs of type $(3|0)$.

Proposition 1 (\cite{7}). Any solution of the equation (2.6) for the multiplication (2.3) is a direct sum of solutions of the following $m$ independent equations:

$\mathcal{M}_{j}^{[s,t]} = \mathcal{M}_{j}^{[s,r]} \mathcal{M}_{j}^{[r,t]}, \quad \text{for all } 0 \leq s < t, \quad j = 1, \ldots, m.$
The following lemma is obvious

**Lemma 1.** The matrix $M_{j}^{[s,t]}$ is $3$-stochastic if and only if the square matrix $M_{j}^{[s,t]}$ is right stochastic for any $j = 1,\ldots,m$.

As corollary of Proposition 1 and Lemma 1 we have the following.

**Theorem 1.** Any QSP of type $(3\mid 0)$ is a direct sum of $m$ right stochastic square matrices $M_{j}^{[s,t]}$ satisfying equation (2.1). Consequently, any QSP of type $(3\mid 0)$ consists in independent collection of usual Markov processes (see Subsection 2.1).

The independence mentioned in Theorem 1 allows us to say that the QSPs of type $(3\mid 0)$ are not interesting, because the basic theory of Markov process of square matrices is well developed.

Here we give examples of QSPs of type $(3\mid 0)$. This example also will be used to construct QSPs of type $(12\mid a_{0})$.

**Example 1.** In [12] to construct chains of some algebras, for $m = 2$, a wide class of solutions of (2.1) is presented, many of them are non-stochastic matrices, in general. Here we list the following families of (left, right, doubly) stochastic square matrices (see [12]), which satisfy the equation (2.1), i.e. they generate independently interesting Markov processes:

- $Q_{1}^{[s,t]} = \begin{pmatrix} g(s) & g(s) \\ 1 - g(s) & 1 - g(s) \end{pmatrix}$, where $g(s) \in [0,1]$ is an arbitrary function;
- $Q_{2}^{[s,t]} = \frac{1}{2} \begin{pmatrix} 1 + \frac{\Psi(t)}{\Psi(s)} & 1 - \frac{\Psi(t)}{\Psi(s)} \\ 1 - \frac{\Psi(t)}{\Psi(s)} & 1 + \frac{\Psi(t)}{\Psi(s)} \end{pmatrix}$,

where $\Psi(t) > 0$ is an arbitrary decreasing function of $t \geq 0$;

- $Q_{3}^{[s,t]} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } s \leq t < b, \\ \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } t \geq b, \end{cases}$, where $b > 0$;

- $Q_{4}^{[s,t]} = \begin{pmatrix} 1 & 0 \\ 1 - \frac{\psi(t)}{\psi(s)} & \psi(t) \frac{\psi(t)}{\psi(s)} \end{pmatrix}$,

where $\psi(t) > 0$ is a decreasing function of $t \geq 0$;

- $Q_{5}^{[s,t]} = \begin{pmatrix} f(t) & 1 - f(t) \\ f(t) & 1 - f(t) \end{pmatrix}$, where $f(t) \in [0,1]$ is an arbitrary function;

- $Q_{6}^{[s,t]}(\lambda, \mu) = \begin{pmatrix} 1 - \frac{\lambda}{2(\lambda - \mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) & \frac{\lambda - 2\mu}{2(\lambda - \mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) \\ \frac{\lambda - 2\mu}{2(\lambda - \mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) & 1 - \frac{\lambda}{2(\lambda - \mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) \end{pmatrix}$,
where \( \lambda, \mu \) are real parameters such that \( 0 < 2\mu < \lambda \) and \( \theta(t) > 0 \) is an arbitrary decreasing function:

\[
Q_7^{[s,t]} = \begin{cases} 
(1 \ 0) , & \text{if } s \leq t < a, \\
(0 \ 1) , & \text{if } t \geq a, \\
g(t) \ 1 - g(t) & \text{if } t \geq a, \\
g(t) \ 1 - g(t) & \text{if } t \geq a,
\end{cases}
\]

Using the right stochastic matrices we can construct the following QSPs of type (3|0):

\[
\mathcal{M}^{[s,t]} = \left( \mathcal{M}_1^{[s,t]} \mid \mathcal{M}_2^{[s,t]} \right), \quad \text{with any } \mathcal{M}_1^{[s,t]}, \mathcal{M}_2^{[s,t]} \in \left\{ Q_2^{[s,t]}, Q_3^{[s,t]}, \ldots, Q_7^{[s,t]} \right\}.
\]

We note that the matrices \( Q_i^{[s,t]}, i = 1, \ldots, 7 \), generate interesting usual Markov processes: some of them independent on time, some depend only on \( t \), but many of them non-homogenously depend on both \( s, t \). Depending on the statistical models of real-world processes one can choose parameter functions (i.e. \( g, \Psi, \psi, f, \theta \)) and be able then to control the evolution (with respect to time) of such Markov processes. Then the evolution of the QSP of type (3|0) will be given by the evolution of two independent Markov processes.

4. QSPs of Type (12|a0)

Let \( \mathcal{M}^{[s,t]} = \left( P_{ijk}^{[s,t]} \right) \) be a cubic matrix, define the square matrix \( \overline{\mathcal{M}}^{[s,t]} = (\overline{c}_{ik}^{[s,t]}) \) with

\[
\overline{c}_{ik}^{[s,t]} = \sum_{j=1}^{m} P_{ijk}^{[s,t]}, \quad i, k = 1, \ldots, m.
\]

**Proposition 2**. Any solution of equation (2.10) for the multiplication of type \( a_0 \) (equivalently equation (2.10)) can be given by a solution of the system (4.1) with a matrix \( \overline{\mathcal{M}}^{[s,t]} = (\overline{c}_{ik}^{[s,t]}) \) which satisfies (2.11).

From this proposition it follows that the family of matrices \( \overline{\mathcal{M}}^{[s,t]} = (\overline{c}_{ik}^{[s,t]}) \) is a Markov process if and only if the matrices are left stochastic.

The following lemma gives a connection between left stochastic and (1,2)-stochastic matrices.

**Lemma 2.** The matrix \( \mathcal{M}^{[s,t]} = \left( P_{ijk}^{[s,t]} \right) \), with \( P_{ijk}^{[s,t]} \geq 0 \), is (1,2)-stochastic if and only if the corresponding matrix \( \overline{\mathcal{M}}^{[s,t]} \) is left stochastic.

**Proof.** It is consequence of the equality (4.11). \( \square \)

4.1. Two-dimensional cases. Now we construct QSPs of type (12|a0) corresponding to the left stochastic matrices mentioned in Example 1. Write a cubic matrix \( \mathcal{M}^{[s,t]} \) for \( m = 2 \) in the following convenient form:

\[
\mathcal{M}^{[s,t]} = \begin{pmatrix} 
P_{111}^{[s,t]} & P_{112}^{[s,t]} & P_{121}^{[s,t]} & P_{122}^{[s,t]} \\
P_{121}^{[s,t]} & P_{122}^{[s,t]} & P_{211}^{[s,t]} & P_{212}^{[s,t]} \\
P_{211}^{[s,t]} & P_{212}^{[s,t]} & P_{221}^{[s,t]} & P_{222}^{[s,t]} \\
P_{221}^{[s,t]} & P_{222}^{[s,t]} & P_{211}^{[s,t]} & P_{212}^{[s,t]} \end{pmatrix}.
\]

Case \( Q_1^{[s,t]} \): Let \( \overline{\mathcal{M}}^{[s,t]} = Q_1^{[s,t]} \). Then from (2.10) by (4.11) we get

\[
P_{ij1}^{[s,t]} = g(s)P_{ij1}^{[s,t]} + (1 - g(s))P_{ij2}^{[s,t]}, \quad i, j = 1, 2,
\]

\[
P_{ij2}^{[s,t]} = g(s)P_{ij1}^{[s,t]} + (1 - g(s))P_{ij2}^{[s,t]}, \quad i, j = 1, 2.
\]
Consequently \( P_{ij1}^{[s,t]} = P_{ij2}^{[s,t]} \). Therefore, by the last system we have \( P_{ij1}^{[s,t]} = P_{ij1}^{[s,\tau]} \). Hence \( P_{ij1}^{[s,t]} \) should not depend on \( t \), i.e. there exists a function \( u_{ij}(s) \) such that
\[
P_{ij1}^{[s,t]} = P_{ij1}^{[s,\tau]} = u_{ij}(s). \tag{4.3}
\]

By (4.1) and (4.3) we shall have
\[
P_{111}^{[s,t]} + P_{121}^{[s,t]} = P_{112}^{[s,t]} + P_{122}^{[s,t]} = u_{11}(s) + u_{12}(s) = g(s),
\]
\[
P_{211}^{[s,t]} + P_{221}^{[s,t]} = P_{212}^{[s,t]} + P_{222}^{[s,t]} = u_{21}(s) + u_{22}(s) = 1 - g(s).
\]

Consequently the matrix \((\ref{4.2})\) has the following form:
\[
\mathcal{M}_{(1)}^{[s,t]} = \left( \begin{array}{cc} u_{11}(s) & u_{11}(s) \\ g(s) - u_{11}(s) & g(s) - u_{11}(s) \end{array} \right) \left( \begin{array}{cc} u_{21}(s) & u_{21}(s) \\ 1 - g(s) - u_{21}(s) & 1 - g(s) - u_{21}(s) \end{array} \right), \tag{4.4}
\]
where \( u_{11} \) and \( u_{21} \) are arbitrary functions of \( s \geq 0 \).

Thus we proved the following.

**Proposition 3.** Let \( g(s) \in [0,1] \) be an arbitrary function. The family of matrices \((\ref{4.4})\), \( \mathcal{M}_{(1)}^{[s,t]} \), is a QSP of type (12)(a0) if and only if the functions \( u_{11}(s) \) and \( u_{21}(s) \) are such that
\[
0 \leq u_{11}(s) \leq g(s), \quad 0 \leq u_{21}(s) \leq 1 - g(s).
\]

For this QSP \( \mathcal{M}_{(1)}^{[s,t]} \), using (2.7), let us give the time behavior of the distribution \( x^{(t)} = (x_1^{(t)}, x_2^{(t)}) \in S^1 \). Fix \( s \geq 0 \) and by taking a vector \( x^{(s)} = (x_1^{(s)}, x_2^{(s)}) \in S^1 \), then by formula (2.7) independently on the vector \( x^{(s)} \), for any \( t > s \), we get
\[
x_1^{(t)} = A(s) = \frac{1}{2} (g(s) + u_{11}(s) + u_{21}(s)),
\]
\[
x_2^{(t)} = 1 - A(s) = 1 - \frac{1}{2} (g(s) + u_{11}(s) + u_{21}(s)).
\]

Thus the time behavior of \( x^{(t)} \) is clear: start process at time \( s \) with an arbitrary initial distribution vector \( x^{(s)} \) then as soon as the time \( t \) turns on the distribution of the system goes to the distribution \( (A(s), 1 - A(s)) \) and this distribution remains stable during all time \( t > s \).

**Case** \( Q_2^{[s,t]} \): Let \( \mathcal{M}_{(2)}^{[s,t]} = Q_{ij}^{[s,t]} \). Then from (2.10) by (4.1) we get
\[
P_{ij1}^{[s,t]} = \frac{1}{2} P_{ij1}^{[s,\tau]} \left( 1 + \frac{\Psi(t)}{\Psi(\tau)} \right) + \frac{1}{2} P_{ij2}^{[s,\tau]} \left( 1 - \frac{\Psi(t)}{\Psi(\tau)} \right), \quad \text{for } i,j = 1,2,
\]
\[
P_{ij2}^{[s,t]} = \frac{1}{2} P_{ij1}^{[s,\tau]} \left( 1 - \frac{\Psi(t)}{\Psi(\tau)} \right) + \frac{1}{2} P_{ij2}^{[s,\tau]} \left( 1 + \frac{\Psi(t)}{\Psi(\tau)} \right), \quad \text{for } i,j = 1,2.
\]

Denoting \( \alpha_{ij}(s,t) = P_{ij1}^{[s,t]} - P_{ij2}^{[s,t]} \) and \( \beta_{ij}(s,t) = P_{ij1}^{[s,t]} + P_{ij2}^{[s,t]} \), from the last system of equations we get
\[
\frac{\alpha_{ij}(s,t)}{\Psi(t)} = \frac{\alpha_{ij}(s,\tau)}{\Psi(\tau)}, \quad \beta_{ij}(s,t) = \beta_{ij}(s,\tau).
\]

It follows from the last equalities that \( \frac{\alpha_{ij}(s,t)}{\Psi(t)} \) and \( \beta_{ij}(s,t) \) do not depend on \( t \), i.e. there are functions \( \gamma_{ij}(s) \) and \( \zeta_{ij}(s) \) such that
\[
\alpha_{ij}(s,t) = \gamma_{ij}(s) \Psi(t), \quad \beta_{ij}(s,t) = \zeta_{ij}(s).
\]

Consequently,
\[
P_{ij1}^{[s,t]} = \frac{1}{2} (\gamma_{ij}(s) \Psi(t) + \zeta_{ij}(s)), \quad P_{ij2}^{[s,t]} = \frac{1}{2} (\zeta_{ij}(s) - \gamma_{ij}(s) \Psi(t)). \tag{4.5}
\]
By these equalities from (4.1) (for $\mathcal{M}^{[s,t]}(a_{2}) = Q_{2}^{[s,t]}$) we get

$$
\begin{align*}
(\gamma_{11}(s) + \gamma_{12}(s) - \frac{1}{\Psi(s)}) & \Psi(t) + \zeta_{11}(s) + \zeta_{12}(s) = 1, \\
\zeta_{11}(s) + \zeta_{12}(s) - (\gamma_{11}(s) + \gamma_{12}(s) - \frac{1}{\Psi(s)}) & \Psi(t) = 1, \\
(\gamma_{21}(s) + \gamma_{22}(s) + \frac{1}{\Psi(s)}) & \Psi(t) + \zeta_{21}(s) + \zeta_{22}(s) = 1, \\
\zeta_{21}(s) + \zeta_{22}(s) - (\gamma_{21}(s) + \gamma_{22}(s) + \frac{1}{\Psi(s)}) & \Psi(t) = 1.
\end{align*}
$$

From this system we obtain

$$
\begin{align*}
\zeta_{11}(s) + \zeta_{12}(s) &= \zeta_{21}(s) + \zeta_{22}(s) = 1, \\
\gamma_{11}(s) + \gamma_{12}(s) &= -(\gamma_{21}(s) + \gamma_{22}(s)) = \frac{1}{\Psi(s)}.
\end{align*}
$$

Using these equalities and (4.5) the matrix (4.2) can be written in the following form:

$$
\mathcal{M}^{[s,t]}_{(2)} = \frac{1}{2}
\begin{pmatrix}
\zeta_{11}(s) + \gamma_{11}(s)\Psi(t) & \zeta_{11}(s) - \gamma_{11}(s)\Psi(t) \\
1 - \zeta_{11}(s) + \left(\frac{1}{\Psi(s)} - \gamma_{11}(s)\right)\Psi(t) & 1 - \zeta_{11}(s) - \left(\frac{1}{\Psi(s)} - \gamma_{11}(s)\right)\Psi(t) \\
\zeta_{21}(s) + \gamma_{21}(s)\Psi(t) & \zeta_{21}(s) - \gamma_{21}(s)\Psi(t) \\
1 - \zeta_{21}(s) - \left(\frac{1}{\Psi(s)} + \gamma_{21}(s)\right)\Psi(t) & 1 - \zeta_{21}(s) + \left(\gamma_{21}(s) + \frac{1}{\Psi(s)}\right)\Psi(t)
\end{pmatrix},
$$

(4.6)

where $\Psi(t) > 0$ is a decreasing function, $\gamma_{11}$, $\gamma_{21}$, $\zeta_{11}$ and $\zeta_{21}$ are arbitrary functions of $s \geq 0$.

**Proposition 4.** The family of matrices $\mathcal{M}^{[s,t]}_{(2)}$ (with $\Psi(t) > 0$ a decreasing function), is a QSP of type (12)$a_{0}$ if and only if for the functions $\gamma_{11}$, $\gamma_{21}$, $\zeta_{11}$ and $\zeta_{21}$ the following conditions hold:

$$
\begin{align*}
\frac{1}{2} \left(\frac{1}{\Psi(s)} - \frac{1}{\Psi(t)}\right) & \leq \gamma_{11}(s) \leq \frac{1}{2} \left(\frac{1}{\Psi(s)} + \frac{1}{\Psi(t)}\right), \quad \forall s, t, \ 0 \leq s < t; \\
-\frac{1}{2} \left(\frac{1}{\Psi(s)} + \frac{1}{\Psi(t)}\right) & \leq \gamma_{21}(s) \leq \frac{1}{2} \left(\frac{1}{\Psi(s)} - \frac{1}{\Psi(t)}\right), \quad \forall s, t, \ 0 \leq s < t;
\end{align*}
$$

$$
\begin{align*}
0 & \leq \zeta_{11}(s) \leq 1, \quad \forall s, t, \quad 0 \leq \zeta_{21}(s) \leq 1, \quad \forall s, t.
\end{align*}
$$

**Proof.** It is easy to see that the matrix $\mathcal{M}^{[s,t]}_{(2)}$ satisfies $\sum_{i,j} P^{[s,t]}_{ij k} = 1$, for $k = 1, 2$. Therefore we shall show that $P^{[s,t]}_{ij k} \geq 0$. The system of inequalities $P^{[s,t]}_{1 ij} \geq 0$, $i, j = 1, 2$, is equivalent to

$$
\begin{align*}
0 & \leq \zeta_{11}(s) + \gamma_{11}(s)\Psi(t) \leq 1 + \frac{\Psi(t)}{\Psi(s)}, \\
0 & \leq \zeta_{11}(s) - \gamma_{11}(s)\Psi(t) \leq 1 - \frac{\Psi(t)}{\Psi(s)}.
\end{align*}
$$

Solving this system of inequalities with respect to $\zeta_{11}(s)$ and $\gamma_{11}(s)$ we get the conditions mentioned in the proposition. The conditions for $\zeta_{21}(s)$ and $\gamma_{21}(s)$ can be obtained similarly from the system of inequalities $P^{[s,t]}_{2 ij} \geq 0$, $i, j = 1, 2$. \qed
Remark 1. If $\Psi(t) > 0$ is a bounded function, say $\Psi(0) \leq \Psi(t) \leq \Psi(\infty)$, then the condition of Proposition 3 can be given uniformly with respect to $t$, i.e. one gets

$$\frac{1}{2} \left( \frac{1}{\Psi(s)} - \frac{1}{\Psi(\infty)} \right) \leq \gamma_1(s) \leq \frac{1}{2} \left( \frac{1}{\Psi(s)} + \frac{1}{\Psi(\infty)} \right), \quad \text{for all } s \geq 0;$$

$$-\frac{1}{2} \left( \frac{1}{\Psi(s)} + \frac{1}{\Psi(\infty)} \right) \leq \gamma_2(s) \leq \frac{1}{2} \left( \frac{1}{\Psi(s)} - \frac{1}{\Psi(0)} \right), \quad \text{for all } s \geq 0.$$

Now let us give, for the QSP $\mathcal{M}^{[s,t]}$, the time behavior of the distribution $x^{(t)} = (x_1^{(t)}, x_2^{(t)}) \in S^1$. Fix $s \geq 0$ and by taking an initial distribution $x^{(s)} = (x_1^{(s)}, x_2^{(s)}) \in S^1$, then by formula (2.7) for any $t > s$, we get

$$x_1^{(t)} = \frac{1}{4} \left( 1 + \zeta_1(s) + \zeta_2(s) - \left\{ \gamma_1(s) + \gamma_2(s) + \frac{1}{\Psi(s)} \right\} \Psi(t) \right) x_1^{(s)} + \frac{1}{4} \left( 1 + \zeta_1(s) + \zeta_2(s) - \left\{ \gamma_1(s) + \gamma_2(s) + \frac{1}{\Psi(s)} \right\} \Psi(t) \right) x_2^{(s)},$$

$$x_2^{(t)} = \frac{1}{4} \left( 3 - \zeta_1(s) - \zeta_2(s) - \left\{ \gamma_1(s) + \gamma_2(s) + \frac{1}{\Psi(s)} \right\} \Psi(t) \right) x_1^{(s)} + \frac{1}{4} \left( 3 - \zeta_1(s) - \zeta_2(s) + \left\{ \gamma_1(s) + \gamma_2(s) + \frac{1}{\Psi(s)} \right\} \Psi(t) \right) x_2^{(s)}.$$

Thus the time behavior of $x^{(t)}$ depends on the function $\Psi(t)$:

- If $\Psi(t)$ has a limit, say $\Psi(\infty)$, then depending on the initial vector $x^{(s)}$ we have the following limit distribution:

$$\lim_{t \to \infty} x^{(t)} = (x_1^{(\infty)}(s), 1 - x_1^{(\infty)}(s)),$$

where

$$x_1^{(\infty)}(s) = \frac{1}{4} \left( 1 + \zeta_1(s) + \zeta_2(s) - \left\{ \gamma_1(s) + \gamma_2(s) + \frac{1}{\Psi(s)} \right\} \Psi(\infty) \right) x_1^{(s)} + \frac{1}{4} \left( 1 + \zeta_1(s) + \zeta_2(s) - \left\{ \gamma_1(s) + \gamma_2(s) + \frac{1}{\Psi(s)} \right\} \Psi(\infty) \right) x_2^{(s)}.$$

- If $\Psi(t)$ is a periodic function, then for any $t > s$ the behavior of $x^{(t)}$ will be periodic.

Thus if one starts the process at time $s$ with an arbitrary initial distribution vector $x^{(s)}$ then the distribution of the system goes to a limit distribution if $\Psi$ has a limit, otherwise, the set of limit points of $x^{(s)}$ is equivalent to the set of limit points of $\Psi(t)$. Concluding, we say that choosing the parameter functions $\Psi$, $\zeta_1$, $\zeta_2$, $\gamma_1$, and $\gamma_2$, one can control the behavior of the distribution $x^{(t)}$ during all time $t > s$.

Case $\mathcal{Q}_3^{[s,t]}$: Let $\mathcal{M}^{[s,t]} = \mathcal{Q}_3^{[s,t]}$. Then from (2.10) by (4.1) for $t < b$ we get

$$P_{ij_1}^{[s,t]} = P_{ij_1}^{[s,r]}, \quad P_{ij_2}^{[s,t]} = P_{ij_2}^{[s,r]}, \quad i, j = 1, 2.$$

Consequently, there are $\eta_{ij}(s)$ and $\xi_{ij}(s)$ such that

$$P_{ij_1}^{[s,t]} = \eta_{ij}(s), \quad P_{ij_2}^{[s,t]} = \xi_{ij}(s), \quad i, j = 1, 2.$$

Moreover, by (4.1), for any $s \geq 0$, we shall have

$$\eta_{11}(s) + \eta_{12}(s) = 1, \quad \eta_{21}(s) + \eta_{22}(s) = 0,$$

$$\xi_{11}(s) + \xi_{12}(s) = 0, \quad \xi_{21}(s) + \xi_{22}(s) = 1.$$
In case \( t \geq b \) the solution of the equation can be reduced to the case \( Q_1^{[s,t]} \) with \( g(s) \equiv \frac{1}{2} \). Therefore, we get

\[
M_{[s,t]}^{[3]} = \begin{cases}
\begin{pmatrix}
\eta_{11}(s) & \xi_{11}(s) \\
1 - \eta_{11}(s) & -\xi_{11}(s)
\end{pmatrix}
& \text{if } s \leq t < b, \\
\begin{pmatrix}
\kappa_{11}(s) & \kappa_{11}(s) \\
\frac{1}{2} - \kappa_{11}(s) & \frac{1}{2} - \kappa_{11}(s)
\end{pmatrix}
& \text{if } t \geq b,
\end{cases}
\]

where \( b > 0 \).

The following proposition is obvious.

**Proposition 5.** The family of matrices \( M_{[s,t]}^{[3]} \) is a QSP of type \((12|a_0)\) if and only if

\[
\eta_{11}(x), \xi_{21}(s) \in [0,1], \quad \eta_{21}(s) = \xi_{21}(s) \equiv 0, \quad \kappa_{11}(x), \kappa_{21}(s) \in [0,1/2].
\]

4.2. \textit{m-dimensional case.} For arbitrary \( m \) the following theorem gives an example of time non-homogenous QSP of type \((12|a_0)\):

**Theorem 2.** Let \( \{A^t\} = (a^t_{ij}), t \geq 0 \) be a family of invertible \( m \times m \) square matrices (for all \( t \)), and let \( (A^t)^{-1} = (b^t_{ij}) \) denote the inverse of \( A^t \). Assume that

(i) The square matrix \( M_{[s,t]}^{[m]} = A^s (A^t)^{-1} \) is left stochastic for any \( s < t \).

(ii) Take arbitrary functions \( \beta^{(s)}_{ijk}, i,j,k = 1, \ldots, m \), such that

\[
\sum_{j=1}^{m} \beta^{(s)}_{ijk} = a^{s}_{ik}, \quad \text{for any } i,k \text{ and } s,
\]

\[
\sum_{k=1}^{m} \beta^{(s)}_{ijk} b^{[t]}_{kr} \geq 0, \quad \text{for any } i,j,r \text{ and } s < t.
\]

Then the cubic matrix

\[
M_{[s,t]} = \left( \sum_{k=1}^{m} \beta^{(s)}_{ijk} b^{[t]}_{kr} \right)_{i,j,r=1}^{m}
\]

(4.7)

generates a QSP of type \((12|a_0)\).

**Proof.** By \cite{1} Theorem 1 it is known that the matrix \( (4.7) \) satisfies the equation \( (2.6) \) for the multiplication \( a_0 \). By the conditions (i), (ii) and Lemma 2 we conclude that the matrix \( (4.7) \) is \((1,2)\)-stochastic. Thus this matrix satisfies conditions \((2.10)\) and \((2.11)\), i.e. is a QSP of type \((12|a_0)\). \(\square\)

Since the inverse of a stochastic matrix may not be stochastic, one wants to have an example of a family of matrices \( A^t \) satisfying conditions of Theorem 2. The following proposition gives such an example.

**Proposition 6.** Let \( m = 2 \) and suppose that the matrix \( A^t, t \geq 0 \), has the form

\[
A^t = \begin{pmatrix}
a(t) & 1 - b(t) \\
1 - a(t) & b(t)
\end{pmatrix},
\]

where \( a(t), b(t) \in (0,1) \) are arbitrary increasing (resp. decreasing) functions such that \( a(t) + b(t) > 1 \) (resp. \( a(t) + b(t) < 1 \), \( \forall t \in (0,1) \). Then the matrix \( A^t \) satisfies condition (i) of Theorem 2.

**Proof.** From condition \( \det(A^t) = a(t) + b(t) - 1 \neq 0 \) it follows that \( A^t \) is invertible for any \( t \in (0,1) \). We shall prove that it satisfies the condition (i) of Theorem 2. We have

\[
(A^t)^{-1} = \frac{1}{a(t) + b(t) - 1} \begin{pmatrix}
b(t) & -1 + b(t) \\
-1 + a(t) & a(t)
\end{pmatrix}.
\]
Using this equality we get
\[
A^{[s]}(A^{[t]})^{-1} = (A^{[s,t]}_{ij})_{i,j=1,2}
\]
\[
= \frac{1}{a(t) + b(t) - 1} \begin{pmatrix}
(a(s)b(t) - (1 - b(s))(1 - a(t))) & a(t)(1 - b(s)) - a(s)(1 - b(t)) \\
(b(t)(1 - a(s)) - b(s)(1 - a(t))) & a(t)b(s) - (1 - a(s))(1 - b(t))
\end{pmatrix}.
\]

It is easy to see that \(A^{[s,t]}_{ij} + A^{[s,t]}_{kj} = 1, j = 1, 2\). Therefore it remains to check that \(A^{[s,t]}_{ij} \geq 0\).

We assume \(a(t) + b(t) > 1, \forall t \in (0, 1)\) (the case \(a(t) + b(t) < 1\) can be considered similarly).

Then, since \(0 < a(t), b(t), 1 - a(t), 1 - b(t) < 1\) for all \(t \geq 0\), we have
- the inequality \(A^{[s,t]}_{11} \geq 0\) is equivalent to \(a(s)b(t) - (1 - b(s))(1 - a(t)) \geq 0\) which is true since by our assumption we have \(a(s) > 1 - b(s)\) and \(b(t) > 1 - a(t)\).
- the inequality \(A^{[s,t]}_{21} \geq 0\) is equivalent to \(\frac{b(t)}{1-a(t)} \geq \frac{b(s)}{1-a(s)}\), for all \(s < t\). The last inequality follows from our condition that \(a(t) > a(s)\) and \(b(t) > b(s)\), for all \(t > s\) (increasing functions).
- the inequality \(A^{[s,t]}_{12} \geq 0\) is equivalent to \(\frac{a(t)}{1-b(t)} \geq \frac{a(s)}{1-b(s)}\), for all \(s < t\). The last inequality follows again from the condition that \(a\) and \(b\) are increasing functions.
- the inequality \(A^{[s,t]}_{22} \geq 0\) is equivalent to \(a(t)b(s) - (1 - a(s))(1 - b(t)) \geq 0\) which is true since by our assumption we have \(a(t) > 1 - b(t)\) and \(b(s) > 1 - a(s)\).

This completes the proof. \(\square\)

Denote
\[
\alpha(s) = \rho^{(s)}_{11}, \quad \beta(s) = \rho^{(s)}_{112}, \quad \gamma(s) = \rho^{(s)}_{211}, \quad \delta(s) = \rho^{(s)}_{212}.
\]

Using Theorem 2 and Proposition 6 we construct the following cubic matrix
\[
\mathcal{A} = \left( P^{[s,t]}_{ijk} \right) = \frac{1}{a(t) + b(t) - 1}.
\]
\[
\begin{pmatrix}
\alpha(s)b(t) + \beta(s)(a(t) - 1) & \alpha(s)(b(t) - 1) + \beta(s)a(t) \\
(a(s) - \alpha(s))b(t) + (1 - b(s) - \beta(s))(a(t) - 1) & (a(s) - \alpha(s))(b(t) - 1) + (1 - b(s) - \beta(s))a(t) \\
\gamma(s)b(t) + \delta(s)(a(t) - 1) & \gamma(s)(b(t) - 1) + \delta(s)a(t) \\
(1 - a(s) - \gamma(s))b(t) + (b(s) - \delta(s))(a(t) - 1) & (1 - a(s) - \gamma(s))(b(t) - 1) + (b(s) - \delta(s))a(t)
\end{pmatrix}.
\]

The following proposition illustrates Theorem 2.

**Proposition 7.** Let \(a(t), b(t) \in (0, 1)\) be functions such that \(a(t) + b(t) - 1 > 0\) for any \(t \geq 0\).

The family of matrices \(\mathcal{A}^{[s,t]}\) is a QSP of type \((12|a_0)\) if and only if the functions \(\alpha(s), \beta(s), \gamma(s)\) and \(\delta(s)\) satisfy the following
\[
0 \leq \alpha(s) \leq a(s), \quad 0 \leq \beta(s) \leq 1 - b(s), \quad \forall s \geq 0; \quad \text{(4.8)}
\]
\[
0 \leq \gamma(s) \leq 1 - a(s), \quad 0 \leq \delta(s) \leq b(s), \quad \forall s \geq 0. \quad \text{(4.9)}
\]

**Proof.** It is easy to see that the matrix \(\mathcal{A}^{[s,t]}\) satisfies \(\sum_{i,j} P^{[s,t]}_{ijk} = 1, \) for \(k = 1, 2\). Therefore we shall show that \(P^{[s,t]}_{ijk} \geq 0\). The system of inequalities \(P^{[s,t]}_{ij} \geq 0, i,j = 1, 2,\) is equivalent to the following system of inequalities with respect to \(\alpha(s)\) and \(\beta(s)\):
\[
0 \leq b(t)\alpha(s) - (1 - a(t))\beta(s) \leq a(s)b(t) - (1 - a(t))(1 - b(s)),
\]
\[
0 \leq -(1 - b(t))\alpha(s) + a(t)\beta(s) \leq -a(s)(1 - b(t)) + a(t)(1 - b(s)).
\]

By dividing the first inequalities by \(1 - a(t) > 0\) and by dividing the second inequalities by \(a(t) > 0\) and by summing the resulting inequalities, we get
\[
0 \leq \frac{a(t) + b(t) - 1}{(1 - a(t))\alpha(s)} \leq a(s)\frac{a(t)}{a(t) - 1} + b(t) - 1.
\]
Since \( \frac{a(t) + b(t) - 1}{(1 - a(t))a(t)} > 0 \), we get the first inequalities of (4.8). The second inequalities of (4.8) can be obtained similarly.

Now the system of inequalities \( P_{2ij}^{[s,t]} \geq 0 \), \( i, j = 1, 2 \), is equivalent to the following system of inequalities with respect to \( \gamma(s) \) and \( \delta(s) \):

\[
0 \leq b(t)\gamma(s) - (1 - a(t))\delta(s) \leq (1 - a(s))b(t) - (1 - a(t))b(s),
\]

\[
0 \leq -(1 - b(t))\gamma(s) + a(t)\delta(s) \leq -(1 - a(s))(1 - b(t)) + a(t)b(s).
\]

By dividing the first (resp. second) inequalities by \( 1 \) (resp. second) inequalities, we get

\[
0 \leq a(t) + b(t) - 1 \quad (1 - a(s))\gamma(s) \leq (1 - a(t))a(t) \gamma(s) - (1 - a(s))b(t),
\]

\[
0 \leq -a(t) + b(t) \quad (1 - a(s))\delta(s) \leq (1 - a(t))a(t) \delta(s) - (1 - a(s))b(s).
\]

Again since \( \frac{a(t) + b(t) - 1}{(1 - a(t))a(t)} > 0 \), we get the first inequalities of (4.9). The second inequalities of (4.9) can be obtained similarly.

\[
\begin{align*}
I & = \{0, 1, 2\} \\
J & = \{0, 1, 2\} \\
K & = \{0, 1, 2\}
\end{align*}
\]

\[
M_{\alpha}^{[s,t]} = \begin{pmatrix}
1 & a_{1}^{[s,t]} & 0 & 0 & a_{1}^{[s,t]} & 0 & 0 & u_{1}^{[s,t]} & 0 \\
0 & b_{1}^{[s,t]} & 0 & 0 & b_{1}^{[s,t]} & 0 & 0 & v_{1}^{[s,t]} & 0 \\
0 & c_{1}^{[s,t]} & 0 & 0 & c_{1}^{[s,t]} & 0 & 0 & w_{1}^{[s,t]} & 0
\end{pmatrix},
\]

(5.1)

It is easy to check that the functions \( P_{ijk}^{[s,t]} \) for \( k = 0 \) and \( k = 2 \), satisfy equation (2.10) and condition (2.11), where one should use sums for \( i, j = 0, 1, 2 \). The equation (2.10) for \( P_{ij1}^{[s,t]} \)
taking into account the matrix \([5.1]\), can be written as the following system of nine equations:

\[
a^{[s,t]} = a^{[r,t]} + b^{[r,t]} + c^{[r,t]} + a^{[s,r]} (\alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]}) + u^{[r,t]} + v^{[r,t]} + w^{[r,t]}, (5.2)
\]

\[
b^{[s,t]} = a^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right), \quad c^{[s,t]} = c^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right); (5.3)
\]

\[
a^{[s,t]} = a^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right), \quad \beta^{[s,t]} = \beta^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right), \quad \gamma^{[s,t]} = \gamma^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right); \quad (5.4)
\]

\[
u^{[s,t]} = u^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right), \quad \beta^{[s,t]} = \beta^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right), \quad \gamma^{[s,t]} = \gamma^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right). \quad (5.5)
\]

Now we shall solve this system of two-variable-functional equations. By condition (2.11) we should only consider non-negative solutions which for any \(0 \leq s < t\) satisfy

\[
a^{[s,t]} + b^{[s,t]} + c^{[s,t]} + a^{[s,t]} + \beta^{[s,t]} + \gamma^{[s,t]} + u^{[s,t]} + v^{[s,t]} + w^{[s,t]} = 1. \quad (5.6)
\]

Denoting

\[
f(s,t) = \alpha^{[s,t]} + \beta^{[s,t]} + \gamma^{[s,t]}
\]

from system \([5.4]\) we get

\[
f(s,t) = f(s,\tau)f(\tau,t).
\]

This equation is known as Cantor’s second equation which has a very rich family of solutions:

(a) \(f(s,t) \equiv 0;\)

(b) \(f(s,t) = \frac{\Phi(t)}{\Phi(s)},\) where \(\Phi\) is an arbitrary function with \(\Phi(s) \neq 0;\)

(c) \(f(s,t) = \begin{cases} 
1, & \text{if } s \leq t < a, \\
0, & \text{if } t \geq a.
\end{cases}\)

**Case of the solution (a):** In this case we get from system \([5.2]–[5.5]\) and \([5.6]\) that

\[
a^{[s,t]} = 1, \quad b^{[s,t]} = c^{[s,t]} = a^{[s,t]} = \beta^{[s,t]} = \gamma^{[s,t]} = u^{[s,t]} = v^{[s,t]} = w^{[s,t]} \equiv 0.
\]

Thus we constructed a QSP of type \((12|a_0).\) To give a biological interpretation, let us compute distributions \(x^{(t)} = (x^{(t)}_0, x^{(t)}_1, x^{(t)}_2).\) By using formula \((2.7)\) we get

\[
x_0^{(t)} = \frac{1}{2} \sum_{i,j=0}^{2} \left( P^{[s,t]}_{0ij} + P^{[s,t]}_{0ji} \right) x_j^{(s)} = x_0^{(s)} + x_1^{(s)} + x_2^{(s)} = 1,
\]

\[
x_1^{(t)} = x_2^{(t)} = 0, \quad 0 \leq s < t.
\]

Since \(x_0^{(s)} = 1\) is the probability to have 0 type, the biological interpretation of the process is clear: independently on initial distribution \(x^{(s)},\) the population will die as soon as the time \(t > s\) turns on.

**Case of the solution (b):** In this case from system \([5.4]\) we get

\[
b^{[s,t]} = b^{[s,r]} \left( \alpha^{[r,t]} + \beta^{[r,t]} + \gamma^{[r,t]} \right) = b^{[s,r]} f(\tau,t) = b^{[s,r]} \frac{\Phi(t)}{\Phi(\tau)},
\]

consequently,

\[
\frac{b^{[s,t]}}{\Phi(t)} = \frac{b^{[s,r]}}{\Phi(\tau)}.
\]
Thus we obtained a solution of the system (5.2)–(5.5), for which the condition (5.6) has the form
\[ \kappa = \frac{b^t}{\Phi(t)} \Rightarrow b^t = b(s)\Phi(t). \]

Similarly one can prove that there are functions \( c(s), \alpha(s), \beta(s), \ldots, w(s) \) such that
\[
\begin{align*}
  c^{s,t} &= c(s)\Phi(t), & \alpha^{s,t} &= \alpha(s)\Phi(t), & \beta^{s,t} &= \beta(s)\Phi(t), & \gamma^{s,t} &= \gamma(s)\Phi(t), \\
  u^{s,t} &= u(s)\Phi(t), & v^{s,t} &= v(s)\Phi(t), & w^{s,t} &= w(s)\Phi(t).
\end{align*}
\]

By definition of \( f(s, t) \) we shall have
\[ f(s, t) = \alpha^{s,t} + \beta^{s,t} + \gamma^{s,t} = \Phi(t)(\alpha(s) + \beta(s) + \gamma(s)) = \frac{\Phi(t)}{\Phi(s)}, \]
i.e.
\[ \alpha(s) + \beta(s) + \gamma(s) = \frac{1}{\Phi(s)}. \]

By using equalities (5.7) and (5.8) from (5.2) we get
\[ a^{s,t} = a^{\tau,t} + a^{\tau,t}\frac{\Phi(t)}{\Phi(\tau)} + \Phi(t)\left(b(\tau) + c(\tau) + u(\tau) + v(\tau) + w(\tau)\right). \]

Denoting \( g(s, t) = \frac{a^{s,t}}{\Phi(t)} \) from the last equation we get
\[ g(s, t) = g(s, \tau) + g(\tau, t) + b(\tau) + c(\tau) + u(\tau) + v(\tau) + w(\tau). \]

This equation has the following solution\(^1\)
\[ g(s, t) = \kappa(t) - \kappa(s) - b(s) - c(s) - u(s) - v(s) - w(s), \]
where \( \kappa(t) \) is an arbitrary function. Consequently, we get
\[ a^{s,t} = \Phi(t)\left(\kappa(t) - \kappa(s) - b(s) - c(s) - u(s) - v(s) - w(s)\right). \]

Thus we obtained a solution of the system (5.2)–(5.5), for which the condition (5.6) has the form
\[ \Phi(t)\left(\kappa(t) - \kappa(s) + \frac{1}{\Phi(s)}\right) = 1, \]
i.e.
\[ \kappa(s) - \frac{1}{\Phi(s)} = \kappa(t) - \frac{1}{\Phi(t)}. \]

This equality says that the function \( \kappa(t) - \frac{1}{\Phi(t)} \) should not depend on \( t \), i.e. there is a constant \( K \) such that
\[ \kappa(t) = K + \frac{1}{\Phi(t)}. \]

Thus
\[ a^{s,t} = \Phi(t)\left(\frac{1}{\Phi(t)} - \frac{1}{\Phi(s)} - b(s) - c(s) - u(s) - v(s) - w(s)\right). \]

Now we are ready to write an explicit formula for the corresponding cubic matrix:

\(^1\)The equation \( g(s, t) = g(s, \tau) + g(\tau, t) \) is known as Cantor's first equation. It is easy to check that this equation has very rich class of solutions, i.e. \( g(s, t) = \kappa(t) - \kappa(s) \) is a solution for an arbitrary function \( \kappa \). For Cantor's first and second equations, see [http://eqworld.ipmnet.ru/en/solutions/eqindex/eqindex-fe.htm](http://eqworld.ipmnet.ru/en/solutions/eqindex/eqindex-fe.htm)
Proposition 8. The family of matrices \( \mathcal{M}^{[s, t]} = \Phi(t) \)

\[
\begin{pmatrix}
\frac{1}{\Phi(t)} - \frac{1}{\Phi(s)} - b(s) - c(s) - u(s) - v(s) - w(s) & 1 \\
b(s) & 0 \\
c(s) & 0 \\
0 & \alpha(s) & 0 & 0 \\
0 & \beta(s) & 0 & 0 \\
0 & \frac{1}{\Phi(s)} - \alpha(s) - \beta(s) & 0 & 0 \\
\end{pmatrix}
\]

Thus we have proved the following.

**Proposition 8.** The family of matrices \( \mathcal{M}^{[s, t]}, (5.9) \), is a QSP of type (12|a0) if and only if \( b(t), c(t), \alpha(t), \beta(t), u(t), v(t), w(t) \in [0,1], \Phi(t) > 0 \), are arbitrary functions such that

\[
\frac{1}{\Phi(t)} - \frac{1}{\Phi(s)} \geq b(s) + c(s) + u(s) + v(s) + w(s),
\]

\[
\alpha(t) + \beta(t) \leq \frac{1}{\Phi(t)}, \quad \text{for all } 0 \leq s < t.
\]

Time behavior of this QSP depends on fixed functions. Let us give some interesting interpretations:

- Assume the following limit exists

\[
\lim_{t \to \infty} \Phi(t) = \Phi(\infty) > 0.
\]

In this case, if, for example, \( \beta(s) > 0 \) then the population has a positive probability, \( P_{111}^{[s, \infty]} = \Phi(\infty)\beta(s) > 0 \) to have twins (i.e. female-female twins). In this case, the condition \( \gamma(s) + v(s) > 0 \) gives positive probability \( \Phi(\infty)(\gamma(s) + v(s)) \) of having female-male twins, and similarly if \( w(s) > 0 \) then \( \Phi(\infty)w(s) > 0 \) is the positive probability of male-male twins.

- If at the initial time \( s \) some probability is positive, for example, \( \Psi(t)\beta(s) > 0 \), then during all fixed time \( t \), with \( t > s \), this probability remains positive. For this example, it means that if initially, the population had positivity, to have a female-female twin, then it will have this possibility always, although this might not be true in the limiting case.

- If \( \Phi(\infty) = 0 \) then the population asymptotically dies.

- It is known that the human twin birth rate is about 2 percent of standard one-child birth. This can be used to choose our parameter functions. For example, one can take \( \beta(t) = 0.02\alpha(t) \), etc.

**Case of the solution (c):** In this case for \( t < a \) we have

\[
a^{[\tau, t]} + \beta^{[\tau, t]} + \gamma^{[\tau, t]} = 1, \quad 0 \leq \tau < t < a.
\]

Then from the system \( (5.2) \) we get that there are functions \( b_0(s), c_0(s), \ldots, w_0(s) \) such that

\[
b^{[s, t]}_0 = b_0(s), \quad c^{[s, t]}_0 = c_0(s), \quad a^{[s, t]}_0 = a_0(s), \quad \beta^{[s, t]}_0 = \beta_0(s), \quad \gamma^{[s, t]}_0 = \gamma_0(s),
\]

\[
u^{[s, t]}_0 = u_0(s), \quad v^{[s, t]}_0 = v_0(s), \quad w^{[s, t]}_0 = w_0(s), \quad 0 \leq \tau < t < a.
\]

Then by equation \( (5.2) \) we get that there is \( \kappa_0(t) \) such that

\[
a^{[s, t]}_0 = \kappa_0(t) - \kappa_0(s) - b_0(s) - c_0(s) - u_0(s) - v_0(s) - w_0(s).
\]

Then by condition \( (5.6) \) we get that \( \kappa_0(t) = \kappa_0(s) \). To make the corresponding matrix a \( (1,2) \)-stochastic we need \( b_0(s), c_0(s), u_0(s), v_0(s), w_0(s) \in [0,1] \) and by previous results we get

\[
a^{[s, t]}_0 = -b_0(s) - c_0(s) - u_0(s) - v_0(s) - w_0(s),
\]

which is non-negative if and only if

\[
b_0(s) = c_0(s) = u_0(s) = v_0(s) = w_0(s) \equiv 0.
\]
Thus, for \( t < a \), the cubic matrix has the following form:

\[
\mathcal{M}_1^{[s,t]} = \begin{pmatrix}
1 & 0 & 1 & 0 & \alpha_0(s) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_0(s) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 - \alpha_0(s) - \beta_0(s) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

The case \( t \geq a \) is simpler, because in this case \( f(s,t) = 0 \) and the solution of the system is

\[
a^{[s,t]} = 1, \quad b^{[s,t]} = \cdots = w^{[s,t]} = 0.
\]

Thus, for \( t \geq a \), the cubic matrix has the following form:

\[
\mathcal{M}_0^{[s,t]} = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Define now

\[
\mathcal{M}^{[s,t]} = \begin{cases}
\mathcal{M}_1^{[s,t]}, & \text{if } s \leq t < a, \\
\mathcal{M}_0^{[s,t]}, & \text{if } t \geq a,
\end{cases}
\]

where \( a > 0 \).

Thus we have proved the following.

**Proposition 9.** The family of matrices \( \mathcal{M}^{[s,t]} \), (5.10), is a QSP of type \((12|a_0)\) if and only if \( \alpha_0(t), \beta_0(t) \in [0, 1] \), are arbitrary functions such that

\[
\alpha_0(t) + \beta_0(t) \leq 1, \quad \text{for all } 0 \leq s < t < a.
\]

The time behavior of this QSP depends on fixed functions. But it is simpler than the previous case. Let us give some interesting interpretations:

- Start the process at time \( s \), then for any \( t < a \), the probabilities \( \{P_{101}^{[s,t]}, P_{111}^{[s,t]}, P_{121}^{[s,t]}\} = \{\alpha_0(s), \beta_0(s), 1 - \alpha_0(s) - \beta_0(s)\} \) are independent on time \( t < a \). While all the other probabilities independent on both \( s \) and \( t \). Thus the process is stable for any \( t < a \), i.e. until \( t = a \).
- Start the process at time \( s \) as soon as \( t \geq a \) then the population immediately dies. This phenomenon reminds a cataclysm (catastrophe): “everything is going good, good, \ldots, died”.

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