A Mathematical Theory of Co-Design

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Abstract—This paper describes a mathematical theory of “co-design”, in which the objects of investigation are “design problems”, defined as tuples of “functionality space”, “implementation space”, and “resources space”, together with a feasibility relation. “Monotone” design problems are those for which functionality and resources are partially ordered and related by an order-preserving map. This condition is essentially the core of engineering: increasing the functionality does not decrease the resources needed. Specific examples are given for the domain of Robotics. Everything generalizes to everything else in engineering.

Monotone Co-Design Problems (MCDPs) are defined as the composition of monotone design problems by three operations, equivalent to the concepts of series, parallel, and feedback. Monotonicity is preserved by these operations. Furthermore, the invariance group for these properties are all order isomorphisms; thus this is a completely intrinsic theory.

As a class of optimization problems, MCDPs are multi-objective, nonconvex, nondifferentiable, noncontinuous, and not even defined on continuous spaces. Yet there exists a complete solution: if there exists a procedure to solve the primitive design problems, then there exists a systematic procedure to solve the larger MCDP. The solution of an MCDP can be cast as the problem of finding the least fixed point of a certain map on the set of resources antichains, a concept similar to Pareto fronts. The solution can be obtained by the iterative application of a map. In particular, no differential operators are needed. The iteration always converges: if it converges to finite values, that is guaranteed to be the set of minimal resources. If it converges to infinity (in a sense to be specified), then the sequence is a certificate of infeasibility. In general, no differential operators are needed. The iteration always converges: if it converges to finite values, that is guaranteed to be the set of minimal resources. If it converges to infinity (in a sense to be specified), then the sequence is a certificate of infeasibility. In general, these iterations are in a space that is not finitely representable; fortunately, very natural finite approximations can be found, which give upper and lower bounds for the solution.

These results make us much more optimistic about the problem of designing “complex” systems.

I. INTRODUCTION

THE title of this paper is a humble and respectful nod to Shannon’s paper [1]. Shannon considered what was a very “messy” problem, the problem of “communication”. He found a great formalization of the problem, which allowed to focus on the fundamental principles and challenges, abstracting away from the details. In the 21st century, the great engineering challenge is dealing with the design of “complex” systems. A complex system is complex because its components cannot be decoupled; otherwise, it would be just a (simple) product of simple systems. The design of a complex system is complicated because of the “co-design constraints”. This paper is an attempt towards defining, formalizing and systematically solving the problem of “co-design”.

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found, there are no claims of optimality, for example expressed as minimality of resources usage (power, money, time).

**Generalization to everything else:** The same pattern applies to everything else. Complex systems are created out of components. Each component provides functionality and requires resources. Co-design constraints are constraints between functionality and resources of different components. This point of view is compositional and applies to multiple scales, from the level of a battery to the level of a city (Fig. 2).

### B. Contribution: A Principled Theory of Co-Design

This paper describes a theory to deal with arbitrarily complicated co-design problems in a principled way.

A “design problem” is defined as a tuple of “functionality space”, “implementation space”, and a “resources space”, plus the two maps that relate implementations to functionality and implementation to resources. A design problem defines a set of optimization problems, equivalent to “find the minimal resources needed to implement a given functionality”. A “co-design problem” is the composition of design problems using three operations, equivalent to “series”, “parallel”, and “feedback”, which allows expressing recursive co-design constraints.

“Monotone” design problems are those where both functionality and resources are partially ordered, and the relation between function implemented and resources needed is monotone (order-preserving). The first main result in this paper (Theorem 3) is that the interconnection of any number of monotone design problems is still monotone. The class of Monotone Co-Design Problems (MCDPs) is closed to interconnection. The second main result (Theorem 4) is that if we have a procedure to solve the primitive design problems, then there exists a systematic procedure to solve any MCDP defined from them.

### C. Overview

- Sec. II recalls the necessary background about partial orders and fixed points.
- Sec. III defines the abstract notion of a “design problem”.
- Sec. IV defines the idea of a “co-design problem” and “co-design constraints”.
- Sec. V defines the notion of monotonicity and states the invariance properties.
- Sec. VI describes exact solution algorithms for MCDPs.
- Sec. VII describes approximated solution algorithms.
- Sec. VIII contains modeling examples for the Robotics domain.

### II. Background

Boyd and Vandenberghe [2] provide a great introduction to the field of convex optimization. In the book, the reader can find definitions for the following concepts: convex functions, convex sets, affine sets, cones, log-convexity, duality, etc., as well as an explanation of the most widely-used algorithms, such as gradient descent, Newton’s algorithm, interior-point methods, etc.

We are not going to use any of that. In particular, we are not going to use any of these symbols:

$$\partial, \nabla, \nabla^2.$$

What we are going to use are basic facts about order theory. Davey and Priestley [3] and Roman [4] are possible reference texts. These notions are objectively easy—certainly easier than $$\nabla^2$$, but they are not usually taught in engineering. I promise that it is worth it. And I also promise that this is going to be fun, though that depends on whether the reader shares our notion of “fun”.

#### A. Posets, antichains and upper sets

In the following, let $$\langle \mathcal{P}, \preceq \rangle$$ be a poset. “$$\preceq$$” will be written as “$$\leq$$” if the context is clear. Let $$\mathcal{P}(\mathcal{P})$$ be the power set of $$\mathcal{P}$$. We shall be working with the set of antichains of $$\mathcal{P}$$, denoted here as $$\mathcal{A}\mathcal{P}$$, and the set of upper sets, denoted $$\mathcal{U}\mathcal{P}$$.

**Definition 1** (Antichains). $$S \subseteq \mathcal{P}$$ is an antichain if no elements are comparable: for $$x, y \in S$$, $$x \leq y$$ implies $$x = y$$. Call $$\mathcal{A}\mathcal{P}$$ the set of all antichains in $$\mathcal{P}$$.

**Definition 2** (Upper sets). $$S \subseteq \mathcal{P}$$ is an upper set if $$x \in S$$ and $$x \leq y$$ implies $$y \in S$$.

Call $$\mathcal{U}\mathcal{P}$$ the set of upper sets of $$\mathcal{P}$$.

$$\mathcal{U}\mathcal{P}$$ is a lattice with $$\preceq_{\mathcal{U}\mathcal{P}} \equiv \supseteq$$, $$\bot_{\mathcal{U}\mathcal{P}} \equiv \mathcal{P}, \top_{\mathcal{U}\mathcal{P}} \equiv \emptyset$$, $$\wedge_{\mathcal{U}\mathcal{P}} \equiv \cap, \vee_{\mathcal{U}\mathcal{P}} \equiv \cup$$. Note that we are setting $$\preceq_{\mathcal{U}\mathcal{P}} = \subseteq$$ rather than $$\preceq_{\mathcal{U}\mathcal{P}} = \supseteq$$, which is the usual way of making $$\mathcal{U}\mathcal{P}$$ a lattice. This will make some things easier later, as we will deal with monotone maps, rather than antitone maps.

Two operators that are needed later are the upper closure operator $$\uparrow$$, which maps any set to an upper set, and the Min operator that maps any set to an antichain.

**Definition 3** (Upper closure). $$\uparrow : \mathcal{P}(\mathcal{P}) \to \mathcal{U}\mathcal{P}$$, $$S \mapsto \{ y \in \mathcal{P} : \exists x \in S : x \preceq y \}.$$
Definition 4 (Minimal elements). The minimal elements of $S \subseteq P$ are
\[
\text{Min } S \ni \{ x \in S : (y \in S) \land (y \preceq x) \Rightarrow (x = y) \}.
\]

Note that Min refers to the minimal elements (elements that are not dominated), while “min” is the least element (an element that dominates all others). If $\text{min } S$ exists, then $\text{Min } S = \{ \text{min } S \}$. However, $\text{Min } S \neq \emptyset$ does not imply that $\text{min } S$ exists.

Definition 7 (Directed set). A set $S \subseteq P$ is directed if each pair of elements in $S$ has an upper bound. In other words, for all $a, b \in S$, there exists $c \in S$ such that $a \preceq c$ and $b \preceq c$.

Definition 8. A poset is a directed complete partial order (DCPO) if each of its directed subsets has a supremum (least of upper bounds). It is a complete partial order (CPO) if it also has a bottom element $\bot$.

Example 1. This is a simple construction that illustrates the principle and is needed for the successive examples. The set of real numbers $\mathbb{R}$ is not a CPO, because it lacks a bottom element. The set of nonnegative reals $\mathbb{R}^+ = \{ x \in \mathbb{R} : x \geq 0 \}$ has a bottom element $\bot = 0$, however, it is not a DCPO because some of its directed subsets do not have a supremum. For example, take $\mathbb{R}^+$, which is a subset of $\mathbb{R}^+$. Then $\mathbb{R}^+$ is directed, because for each $a, b \in \mathbb{R}^+$, there exists $c = \max \{a, b\} \in \mathbb{R}^+$ for which $a \preceq c$ and $b \preceq c$. One way to make $(\mathbb{R}^+, \preceq)$ a CPO is by adding to it an artificial top element $\top$. Define $\mathbb{R}^+ \triangleq \mathbb{R}^+ \cup \{ \top \}$, and extend the partial order $\preceq$ so that $\top \preceq \top$ and $a \preceq \top$ for all $a \in \mathbb{R}^+$. Now $\mathbb{R}^+$ is a CPO, because each directed subset has a supremum.

The equivalent of contractivity (Def. 5) is monotonicity (Def. 9), or the stronger property of Scott continuity (Def. 10).

Definition 9. A map $f : P \to Q$ between two posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ is monotone (or “order-preserving”) iff $x \preceq_P y$ implies $f(x) \preceq_Q f(y)$.

Definition 10 (Scott continuity). A map $f$ between two DCPOs $P$ and $Q$ is Scott continuous if for each directed subset $D \subseteq P$, the image $f(D) \subseteq Q$ is directed, and
\[
f(\sup D) = \sup f(D).
\]

Scott continuity implies monotonicity (Def. 7).

In order theory there are a many “fixed-point theorems”. The results were stated multiple times in different fields with slightly different assumptions and conclusions [3]. In this context, what is needed is Kleene’s theorem, which allows the computation of the least fixed points.

Definition 11. A least fixed point of $f : P \to P$ is the minimum (if it exists) of the set of fixed points of $f$:
\[
\text{lfp}(f) \doteq \min \{ x \in P : f(x) = x \}.
\]

Monotonicity of the map $f$ plus completeness of the poset is sufficient to ensure the existence of the least fixed point.

Lemma 2 ([3], CPO Fixpoint Theorem II, 8.22)). If $P$ is a CPO and $f : P \to P$ is monotone, then $f$ has a least fixed point.

With the additional assumption of Scott-continuity, it is possible to obtain a systematic procedure to find the least fixed point.

Theorem 2 (Kleene’s fixed-point theorem [3], CPO fixpoint theorem I, 8.15)). Assume $P$ is a CPO, and $f : P \to P$ is
Scott-continuous. Then $f$ has a least fixed point, which is the supremum of the Kleene ascent sequence

$$
\bot \leq f(\bot) \leq f(f(\bot)) \leq \cdots \leq f^n(\bot) \leq \ldots.
$$

There are many other interesting connections between contraction theory and order theory. For example, Kleene’s theorem might appear to be weaker, but it is possible to prove Banach’s theorem directly from Kleene’s theorem \[9\].

C. Remarks on Scott-continuity

Scott-continuity is not equivalent to topological continuity.

**Lemma 3.** A map from the CPO $\langle \mathbb{R}, \leq \rangle$ to itself is Scott-continuous if and only if it is nondecreasing and left-continuous (topologically).

An example is the ceiling function (Fig. 4).

![Fig. 4. The ceiling function is Scott-continuous.](image)

**Remark 2 (⋆ Measure theory has it backwards).** A related concept used in probability is the class of càdlàg functions. This is French for “continue à droite, limite à gauche”, which means right-continuous, with existence of left limits. For example, cumulative distribution functions (cdfs) are usually defined as càdlàg. The convention here is different; Scott-continuous implies left-continuous. This divergence is unfortunate. It is particularly unfortunate because cdfs are also great examples of monotone discontinuous functions. Can the conflict be resolved? Yes. The convention used in probability is arbitrary. You can switch the order of integration by integrating starting from $+\infty$ rather than $-\infty$, and everything in probability will work. In this alternative probability theory, the cumulative distribution functions are càglàd (right-continuous). In formulas, instead of using

$$
cdf(y) = \int_{-\infty}^{y} p(x) \, dx,
$$

use the following:

$$
cdf(y) = 1 - \int_{+\infty}^{y} p(x) \, dx.
$$

In other words, take your probability book (Fig. 5), first flip it horizontally (Fig. 5b), and then flip it vertically (Fig. 5c). The composition of these two affine transformations is, of course, a 180° rotation of the book.

Fig. 5.

**Remark 3 (⋆ What was Dana Scott thinking?).** Once we have understood all of the above, one question still remains: why did Dana Scott choose the name “continuous” for a notion that, when projected to $\mathbb{R}$, does not imply topologically continuous? This confused me for a while. I have asked friends of his, and the answer was: “He just thinks abstractly”.

III. DESIGN PROBLEMS

The basic objects considered in this paper are design problems. The following is the definition of a simple ("atomic") design problem; the next sections shows how design problems can be composed together.

**Definition 12.** A design problem is a tuple

$$
dp = \langle \mathcal{F}, \mathcal{R}, \mathcal{I}, \text{exec}, \text{eval} \rangle
$$

where (Fig. 6):

- $\mathcal{F}$ is a set, called “functionality space” (or “function”, “functional requirements”);
- $\mathcal{R}$ is a poset, called “resources space” (or “costs”);
- $\mathcal{I}$ is a set, called “implementation space”;
- the map exec: $\mathcal{I} \to \mathcal{F}$, mnemonics for “execution”, maps an implementation to the function it realizes;
- the map eval: $\mathcal{I} \to \mathcal{R}$, mnemonics for “evaluation”, evaluates the resources needed for an implementation.

The set of all design problems is called $\mathcal{DP}$.

![Fig. 6.](image)

A design problem defines a family of optimization problems, parametrized by the desired functionality $f$.

**Problem 1.** Given $f \in \mathcal{F}$, find the implementations in $\mathcal{I}$ that realize the functionality $f$ with minimal resources (or provide a proof that there are none):

$$
\begin{aligned}
\text{using } & \quad i \in \mathcal{I}, \\
\text{Min}_{\mathcal{R}} & \quad \text{eval}(i), \\
\text{s.t.} & \quad \text{exec}(i) = f.
\end{aligned}
$$

(1)
Note the use of “Minₗₘₕ” in (1), which indicates the set of minimal (non-dominated) elements. In all problems in this paper, the goal is to find the optimal trade-off (“Pareto front”).

**Example 2 (Motor design).** Suppose we need to choose a motor for our robot. One way to abstract the problem is as in Fig. [10]. The functional requirement for a motor is that it must provide a certain torque at a certain speed. The resources needed are cost ($), weight (g), and the electrical properties. These include, at a minimum, the input voltage and the input current. The set \( \mathcal{I} \) is the set of all motors available (which could be either the set of all motors that could ever be designed; or just the set of motors available in the catalogue at hand). The function \( \text{exec} : \mathcal{I} \rightarrow \mathcal{F} \) assigns to each motor its function, and the function \( \text{eval} : \mathcal{J} \rightarrow \mathcal{R} \) assigns to each motor its resources requirement.

![functions](image1)

![implementations](image2)

![resources](image3)

**Example 3 (Chassis design).** Suppose we want to choose the chassis for our robot. For the purpose of this example, the “chassis” includes all physical components (wheels, gears, etc.), except motors and battery. The functional requirements (set \( \mathcal{F} \)) for a chassis could be “transport a certain payload \( P \) (g)” and “at a given speed \( v \) (m/s)”. More refined functional requirements would include maneuverability, the cargo volume, etc. The “resources” needed (set \( \mathcal{R} \)) are: 1) the cost ($); 2) for each motor, the required velocity (rad/s) and torque (Nm); 3) the total weight (g) (Fig. [9]).

![functions](image4)

![implementations](image5)

![resources](image6)

**IV. Co-design problems**

A “co-design” problem is the composition of two or more design problems, interconnected in a way that the resources required by one become the functional requirement of another.

It is useful to introduce a graphical notation that will allow to reason about composition. Represent a design problem as a rounded oval box, with incoming arrows describing functions, and outgoing arrows describing resources (Fig. [9]).

![functionality](image7)

![design problem](image8)

![resource](image9)

**Remark 4.** This graphical notation is not to be confused with a signal flow diagram, in which the boxes represent oriented systems and the edges represent signals.

Using this graphical notation, a co-design problem can be specified as the interconnection of diagrams. Before getting to the formal definition of the composition rules, an example is useful to understand the desired semantics of interconnection.

**Example 4 (Chassis plus motor).** One simple example consists in the co-design of chassis (Example 3) plus motor (Example 2). Using the diagrammatic notation, the simplest version of the design problem for a car has speed and torque as the provided functionality (what the motor must provide), and cost, weight, voltage, and current as the required resources (Fig. [10]).

![speed (rad/s)](image10)

![torque (Nm)](image11)

![cost ($)](image12)

![weight (g)](image13)

![voltage (V)](image14)

![current (A)](image15)

For the chassis, the provided functionality is parameterized by the weight of the payload and the velocity. The required resources include the cost. A chassis needs a motor to function. Hence the resources also include the required motor speed and torque.

![payload weight (g)](image16)

![velocity (m/s)](image17)

![cost ($)](image18)

![total weight](image19)

![required motor speed (m/s)](image20)

![required motor torque (Nm)](image21)

Some of the resources needed by the chassis (actuation torque and speed) become functional requirements for the motor. In Fig. [12], this interconnection is indicated with the symbol “\( \subseteq \)” in a rounded box. Intuitively, this means that the motor needs to have at least the given torque and speed. This type of interconnection will be later formally defined as the “series” interconnection (Def. [14]).

![payload](image22)

![velocity](image23)

![cost](image24)

Notice that there are a few “dangling” connections in Fig. [12]. One would like to sum together cost of chassis and cost of motor, to obtain the total cost. This does not introduce a loop. A “co-design” problem is one where there are recursive constraints. In this case, we might set the payload to be transported to be the sum of the motor weight plus some extra payload. Now there is a cycle in the graph (Fig. [13]).
This formalism makes it easy to abstract away the details in which we are not interested. Once a diagram like Fig. 13 is obtained, we can draw a box around it and consider the composed problem (Fig. 14).

The rest of this section shows that these operations on the diagrams can be given a precise semantics.

**Definition 13.** The set of co-design problems CDP is defined recursively (Fig. 15), starting from the set of design problems DP, and applying two binary operations (par and series) and a unary operation loop:

\[ CDP \equiv DP \cup \{\text{series}(dp_1, dp_2) \mid dp_1, dp_2 \in CDP\} \cup \{\text{par}(dp_1, dp_2) \mid dp_1, dp_2 \in CDP\} \cup \{\text{loop}(dp) \mid dp \in CDP\}. \]

The operations par, series, and loop are defined in Def. 14–16.

**Definition 14 (series).** Given two design problems

\[ dp_1 = \langle F_1, R_1, J_1, \text{exec}_1, \text{eval}_1 \rangle, \]
\[ dp_2 = \langle F_2, R_2, J_2, \text{exec}_2, \text{eval}_2 \rangle, \]

then if \( F_2 = R_1 \), define

\[ \text{series}(dp_1, dp_2) = \langle F_1, R, J, \text{exec}, \text{eval} \rangle, \]

where:

\[ F = F_1, \]
\[ R = R_2, \]
\[ J = \{ (i_1, i_2) \in J_1 \times J_2 \mid \text{eval}_1(i_1) \leq \text{eval}_2(i_2) \}, \]
\[ \text{exec} : (i_1, i_2) \mapsto \text{exec}_1(i_1), \]
\[ \text{eval} : (i_1, i_2) \mapsto \text{eval}_2(i_2). \]

**Definition 15 (par).** Given two design problems

\[ dp_1 = \langle F_1, R_1, J_1, \text{exec}_1, \text{eval}_1 \rangle, \]
\[ dp_2 = \langle F_2, R_2, J_2, \text{exec}_2, \text{eval}_2 \rangle, \]

define

\[ \text{par}(dp_1, dp_2) = \langle F, R, J, \text{exec}, \text{eval} \rangle, \]

where:

\[ F = F_1 \times F_2, \]
\[ R = R_1 \times R_2, \]
\[ J = J_1 \times J_2, \]
\[ \text{exec} : \langle i_1, i_2 \rangle \mapsto \langle \text{exec}_1(i_1), \text{exec}_2(i_2) \rangle, \]
\[ \text{eval} : \langle i_1, i_2 \rangle \mapsto \langle \text{eval}_1(i_1), \text{eval}_2(i_2) \rangle. \]

The par composition is clearly the equivalent of a product in category theory, as one can see from the numerous “×” in Fig. 17. The equivalent of a “coproduct” (see, e.g., [10, Section 2.4]) between two design problem, a construction that is not needed for the current paper, would be a design problem with the implementation space \( J = J_1 \sqcup J_2 \), signifying two possible alternative families of designs for the same function.

**Definition 16 (loop).** Suppose \( dp \) is a design problem defined on the factored function space \( F_1 \times R \):

\[ dp = \langle F_1 \times R, R, J, \text{exec}, \text{eval} \rangle. \]

Define the new design problem \( \text{loop}(dp) \) as

\[ \text{loop}(dp) = \langle F_1 \times R, R', J, \text{exec}', \text{eval} \rangle, \]

where \( J' \) limits the implementations to those that respect the additional constraint \( \text{eval}(i) \leq \text{exec}_2(i) \):

\[ J' = \{ i \in J \mid \text{eval}(i) \leq \text{exec}_2(i) \}. \]

This is equivalent to “closing a loop” with the constraint \( f_2 \geq r \) (Fig. 18).

**Remark 5.** Note that the more general form in Fig. 19a can be obtained by writing it as in Fig. 19b.
V. MONOTONE CO-DESIGN PROBLEMS (MCDPs)

This section defines the class of Monotone Co-Design Problems (MCDPs). The monotonicity property is essentially the core of engineering: increasing the functionality does not decrease the resources needed. The first main result of this paper is that an arbitrary composition of a set of monotone design problems is monotone (Theorem 3), because monotonicity is preserved for the set of graph building operations par, series, loop.

A. Definition of monotonicity

Monotonicity of a design problem is defined as a property of the map $h_{dp}$ that associates to each functionality $f$ the minimal elements for the objective $[1]$. The definition is:

\[ h_{dp} : \mathcal{F} \rightarrow 2^R, \quad f \mapsto \text{Min}\{\text{eval}(i) \mid (i \in \mathcal{I}) \land (\text{exec}(i) = f)\}. \]

Using the map $h_{dp}$, we can abstract over the space of implementations and reason only with the space of functionality and resources (Fig. 20). Throughout the paper, the illustrations of the trade-off curves are for intuition only. In particular, it is not assumed that the curve $h_{dp}(f_1)$ is continuous, or that there are only a finite number of solutions.

A particular class of design problems are those that are “monotone”, which means that there is a partial order structure on $\mathcal{F}$ and that increasing the provided functionality increases the required resources.

Definition 18 (Monotone Design Problem). A design problem $dp = \langle \mathcal{F}, \mathcal{R}, \mathcal{I}, \text{exec}, \text{eval} \rangle$ is monotone if $\mathcal{F}$ and $\mathcal{R}$ are CPOs and the map $h_{dp}$ is monotone and Scott-continuous (Def. [10]).

Example 5 (Example 2 continued). In the case of the motor design problem, the map $h_{dp}$ assigns to each pair of (speed, torque) the achievable trade-off of cost, weight, and required voltage/current (Fig. 21). The design problem is monotone if increasing the required speed and torque increases the resources needed.

Remark 6 (Non-monotone co-design problems). It is easy to come up with examples of monotone co-design problems. Are there co-design problems that are not monotone?

Some technologies have non-monotone efficiency curves. An internal combustion engine at constant load has a peak efficiency as a function of velocity. There is a non-monotone relation between vehicle speed (m/s or mph) and fuel economy (km/l or mpg) [11]. Consequently, if one takes the functionality space to be parametrized as “Travel at speed $V$ from point A to B”, and the resource space as “Fuel consumed $F$”, then it might not be true that there is a monotone relation between $V$ and $F$.

B. Invariance of monotonicity

Let atoms : $CDP \rightarrow DP^*$ be the map that returns the set of primitive atoms of a co-design problem. The first main result of this paper is that if all subproblems of a co-design problem are monotone, then the co-design problem is monotone.

Theorem 3. For a co-design problem $cdp \in CDP$, if all $dp \in \text{atoms}(cdp)$ are monotone, then $cdp$ is monotone.

Proof. A co-design problem $cdp$ is defined recursively as the application of the operators par, series, and loop (Def. [13]) starting from a set of atoms $\text{atoms}(cdp)$. By assumption, the atoms are monotone. If all three operators preserve monotonicity, then the co-design problem $cdp$ is monotone. The proofs for the three operators par, series, and loop are given as Prop. [11][13].

Proposition 1. If $dp_1$ and $dp_2$ are monotone, then $\text{par}(dp_1, dp_2)$ is monotone.

Proof. For two sets $A, B$, write their product as $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

With this notation, the map $h_{\text{par}}(dp_1, dp_2)$ can be written as

\[ h_{\text{par}}(dp_1, dp_2) : \mathcal{F}_1 \times \mathcal{F}_2 \rightarrow 2^{(\mathcal{R}_1 \times \mathcal{R}_2)}, \quad (f_1, f_2) \mapsto h_{dp_1}(f_1) \times h_{dp_2}(f_2). \]

It is a routine check that all products on the right are antichains in $2^{(\mathcal{R}_1 \times \mathcal{R}_2)}$, and that the map so defined is continuous iff $h_{dp_1}$ and $h_{dp_2}$ are.

Proposition 2. If $dp_1$ and $dp_2$ are monotone, then $\text{series}(dp_1, dp_2)$ is monotone.

Proof. If $dp_1, dp_2$ are monotone, then the two maps $h_{dp_1}$ and $h_{dp_2}$ are monotone. We have to prove that $h_{\text{series}}(dp_1, dp_2)$ is monotone as well. Call these functions just $h_1, h_2,$ and $h$ for brevity. For a fixed $f_1$, the map $h$ traces an antichain in $\mathcal{R}_2$ (Fig. 22). What we need to prove is that if $f_1$ increases in $\mathcal{F}_1$, then $h(f_1)$ increases according to the partial order on antichains.

This property is plausible, considering that $h$ is the composition of operations that are monotone (Fig. [23]), but it will be
From continuity of \( \sigma \), which is less obvious.

From the definition of series (Def. 14), the semantics of the interconnection is captured by this problem:

\[
\begin{align*}
\text{minimize} & \quad r_1, r_2, \quad \text{s.t.} \quad r_1 \in h_1(f_1), \\
& \quad r_1 \preceq_{R_1} r_2, \\
& \quad h_1(f_1) \preceq_{\mathcal{F}_1} h_2(f_2).
\end{align*}
\]

Note that (3) describes a map from \( f_1 \) to the solution(s) of an optimization problem. The optimization problem specifies (with the “using” clause) the type of the variables involved: \( r_1, f_2, \) and \( r_2 \). In this proof we will manipulate (3) as to obtain an explicit expression for \( h \).

In general, if \( S_1 \subseteq S_2 \subseteq \mathcal{P} \) then \( \uparrow S_2 \preceq_{\mathcal{P}} \uparrow S_1 \). From \( r_1 \in h_1(f_1) \) in (3), it follows

\[
\uparrow h_1(f_1) \preceq_{\mathcal{U}_{R_1}} \uparrow r_1. \quad (4)
\]

From \( r_2 \in h_2(f_2) \) in (3), it follows

\[
\uparrow h_2(f_2) \preceq_{\mathcal{U}_{R_2}} \uparrow r_2. \quad (5)
\]

From \( r_1 \preceq_{R_1} f_2 \) in (3), it follows

\[
\uparrow r_1 \preceq_{\mathcal{U}_{R_1}} \uparrow f_2. \quad (6)
\]

From (4) and (6), it follows

\[
\uparrow h_1(f_1) \preceq_{\mathcal{U}_{R_1}} \uparrow f_2. \quad (7)
\]

Using \( \bar{r} \) as defined in (17) and property (19), it follows

\[
\bar{r}_2(\uparrow r_1) \preceq_{\mathcal{U}_{R_2}} \bar{r}_2(\uparrow f_2). \quad (8)
\]

From (7) and (5), it follows

\[
\bar{r}_2(\uparrow f_2) \preceq_{\mathcal{U}_{R_2}} \uparrow r_2. \quad (9)
\]

From (6) and (6), it follows

\[
\bar{r}_2(\uparrow r_1) \preceq_{\mathcal{U}_{R_2}} \bar{r}_2(\uparrow r_1). \quad (10)
\]

From continuity of \( \bar{r}_2 \) and (4), it follows

\[
\bar{r}_2(\uparrow h_1(f_1)) \preceq_{\mathcal{U}_{R_2}} \bar{r}_2(\uparrow r_1). \quad (11)
\]

From (10) and (11), it follows

\[
\bar{r}_2(\uparrow h_1(f_1)) \preceq_{\mathcal{U}_{R_2}} \uparrow r_2. \quad (12)
\]

From (18) and (12), it follows

\[
\bar{r}_2(\uparrow h_1(f_1)) = \uparrow \bar{r}_2(h_1(f_1)). \quad (13)
\]

From (13) and (12), it follows

\[
\uparrow \bar{r}_2(h_1(f_1)) \preceq_{\mathcal{U}_{R_2}} \uparrow r_2. \quad (14)
\]

The left-hand side is a constant that depends only on \( f_1 \), which is fixed during the optimization; call it \( B_{f_1} \in \mathcal{U}_{R_2} \):

\[
B_{f_1} = \uparrow \bar{r}_2(h_1(f_1)).
\]

The optimization problem (3) can then be rewritten as

\[
\begin{align*}
\text{minimize} & \quad r_2, \quad \text{s.t.} \quad B_{f_1} \preceq_{\mathcal{U}_{R_2}} r_2, \\
& \quad h(f_1) \preceq_{\mathcal{U}_{\mathcal{F}_1}} h_2(f_2).
\end{align*}
\]

From this, it is clear that \( h \) is monotone because it is the composition of monotone functions.

\[\text{Lemma 4. Let } f : \mathcal{P} \to \mathcal{Q}. \text{ Define } \bar{f} : \mathcal{P}(\mathcal{P}) \to \mathcal{P}(\mathcal{Q}), \quad \bar{f} = \bigcup_{x \in \mathcal{P}} f(x). \]

Then, if \( f \) is Scott-continuous:

1) If \( S \in \mathcal{P} \) then \( \bar{f}(S) \in \mathcal{Q} \).
2) \( \bar{f} \) is monotone and Scott-continuous as a map from \( \mathcal{P} \) to \( \mathcal{Q} \).
3) \( \bar{f} \) commutes with \( \uparrow : \)

\[\bar{f}(\uparrow S) = \uparrow \bar{f}(S) \quad (18)\]

4) For any point \( x \in \mathcal{P} \):

\[\uparrow f(x) = \bar{f}(\uparrow x) \quad (19)\]

\[\text{Proof. (Monotonicity) By contradiction. Suppose that } \bar{f} \text{ is not monotone. Then there exist two upper sets } A, B \in \mathcal{P} \text{ such that } A \preceq_{\mathcal{P}} B \text{ for which } \bar{f}(A) \not\preceq_{\mathcal{Q}} \bar{f}(B). \text{ This is equivalent to say that there exists a } z \in \mathcal{Q} \text{ such that } z \in \bar{f}(B) \text{ but } z \not\in \bar{f}(A). \text{ From } z \in \bar{f}(B) \text{ it follows that there exists a } b \in B \text{ such that } f(b) \preceq_{\mathcal{Q}} z. \text{ Because } B \subseteq A, \text{ it follows this } b \text{ is also in } A \text{ and } f(b) \in \bar{f}(A). \text{ Because } \bar{f}(A) \text{ is an upper set, any point } x \text{ such that } f(b) \preceq_{\mathcal{Q}} x \text{ is also in } \bar{f}(A). \text{ In particular, } z \in \bar{f}(A) \text{ (contradiction.)} \]

\[\text{Proposition 3. If } dp \text{ is monotone, so is } \text{loop}(dp). \]

\[\text{Proof. This is a particularly important point. There are two ways to prove these properties: using antichains and using upper sets. This proof uses antichains.}\]
We need to describe the map \( h_{\text{loop(dp)}} : f_1 \rightarrow \mathcal{A} \mathcal{R} \). The semantics of MCDPs make it possible to describe \( h_{\text{loop(dp)}} \) as the solution of an optimization problem that depends on \( f_1 \):

\[
\begin{align*}
\text{using} & \quad r, f_2 \in \mathcal{R}, \\
\text{s.t.} & \quad r \in \mathcal{h}_{\text{dp}}(f_1, f_2), \\
& \quad r \leq_{\mathcal{R}} f_2.
\end{align*}
\]

For a fixed \( f_1 \), the set \( h_{\text{loop(dp)}}(f_1) \) is an antichain by construction (because of the operator \( \mathbb{Min}_{\leq_{\mathcal{R}}} \)).

Denote by \( h_{f_1} \) the map \( h_{\text{dp}} \) with the first element fixed:

\[
h_{f_1} : \mathcal{R} \rightarrow \mathcal{A} \mathcal{R},
\]

\[
f_2 \mapsto \mathcal{h}_{\text{dp}}(f_1, f_2).
\]

Rewrite \( r \in \mathcal{h}_{\text{dp}}(f_1, f_2) \) in (20) as

\[
r \in h_{f_1}(f_2).
\]

Let \( r \) be a feasible solution, but not necessarily optimal.

Then the feasibility constraints can be rewritten as \( \uparrow r = \uparrow h_{f_1}(f_2) \cap \uparrow r \). Because \( f_2 \geq r \), and \( h_{f_1} \) is Scott-continuous, it follows that \( h_{f_1}(f_2) \leq h_{f_1}(r) \), and

\[
\uparrow r = \uparrow h_{f_1}(r) \cap \uparrow r.
\]

This is now a condition that feasible resources must satisfy.

Take the \( \text{Min} \) of both sides in (22). Because \( h_{f_1}(r) \) is an antichain, \( \text{Min} \uparrow h_{f_1}(r) = h_{f_1}(r) \), hence

\[
\{ r \} = h_{f_1}(r) \cap \uparrow r.
\]

Now, consider an antichain of feasible resources, \( A \), and say that \( r \) is the generic element of \( A \):

\[
A = \bigcup_{r} \{ r \}.
\]

We can substitute (23) in (24) to obtain:

\[
A = \bigcup_{r} \{ r \} = \bigcup_{r} h_{f_1}(r) \cap \uparrow r.
\]

This is a recursive constraint for \( A \), of the kind \( \Phi(A) = A \). Rewrite it in this way:

\[
\bigcup_{r \in A} h_{f_1}(r) \cap \uparrow r = A.
\]

Define the map \( \Phi_{f_1} \) as

\[
\Phi_{f_1} : \mathcal{A} \mathcal{R} \rightarrow \mathcal{A} \mathcal{R},
\]

\[
A \mapsto \bigcup_{r \in A} h_{f_1}(r) \cap \uparrow r.
\]

We have concluded that an antichain of feasible resources satisfies \( \Phi_{f_1}(A) = A \), which means that \( A \) is a fixed point of \( \Phi_{f_1} \). The **minimal** set of resources is the least fixed point of \( \Phi_{f_1} \). Therefore, the map \( h_{\text{loop(dp)}} \) can be written as

\[
h_{\text{loop(dp)}} : f_1 \mapsto \text{lfp}(\Phi_{f_1}).
\]

**Lemma 5.** Let \( (\mathcal{P}, \preceq_{\mathcal{P}}), (\mathcal{Q}, \preceq_{\mathcal{Q}}) \) be CPOs and \( f : \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{Q} \) be Scott-continuous. For each \( x \in \mathcal{P} \), define

\[
f_x : \mathcal{Q} \rightarrow \mathcal{Q},
\]  
\[
y \mapsto f(x, y).
\]

Then there exists a continuous map \( f^\dagger : \mathcal{P} \rightarrow \mathcal{Q} \) that associates to each \( x \in \mathcal{P} \) the least fixed point of \( f_x \):

\[
f^\dagger : \mathcal{P} \rightarrow \mathcal{Q},
\]  
\[
x \mapsto \text{lfp}(f_x).
\]

**Proof.** Davey and Priestly [3] leave this as Exercise 8.26. A proof is found in Gierz et al. [12, Exercise II-2.29]. \qed

**Example 6.** Let us finish assembling our robot. We have already defined the design problem for the chassis (Example 3) and the motor (Example 2). A motor needs a motor control board (Fig. 24). The functional requirements are the (peak) output current and the output range.

We also need a battery, or generally a power supply unit. The requirements for the power supply are the output current, the output voltages, and the capacity. The resources are at least the cost and the weight (Fig. 25).

**Fig. 26** shows the interconnection of MCB and PSU.

Sums and products are degenerate monotone design problems. Components’ power consumptions are added together. Multiplication is used to model relations such as current \( \times \) voltage = power and power \( \times \) time = energy.

Connect the dangling arrows in Fig. 26 as in Fig. 27 to obtain a composite design problem with functions (voltage, current, mission duration) and output (weight, cost).

---

**Fig. 27.**

**Fig. 24.**

**Fig. 25.**

**Fig. 26.**

**Fig. 23**
Draw a box around the diagram, and call it “MCB+PSU”; then interconnect it with the “chassis+motor” diagram obtained before (Fig. 25). The dangling arrows on the right are weight and cost. This means that we are looking for all possible trade-offs of weight and cost. All other constraints about voltage, current, torque, etc. have been abstracted away.

This paper did not go into the mathematical details for each of the design problems to state this formally, but: pending a more accurate formalization, all of the design problems considered are monotone. Consequently, in the diagram in Fig. 28 there is a monotonic (order-preserving) map between the set of functions (incoming dangling arrows) and the set of resources (outgoing dangling arrows).

Also note that when all functions are fixed, there is still a trade-off of resources. For the diagram in Fig. 28 this means that there might be several designs that do not dominate each other according to the (weight, cost) trade-off.

All optimization problems so far were carefully formulated as to always consider trade-offs explicitly. The only case when there is a “unique solution”, in the sense of a unique design that is optimal, is when the resource space of the composed problem is a chain, which means that everything has been flattened down to a scalar objective.

This scalar objective is usually measured in $. For example, a conversion from weight to cost exists and it is called “shipping” (Fig. 29). Depending on the destination, the conversion factor is between $0.5/lbs , using USPS, to $10k/lbs for sending your robot to low Earth orbit.

C. MCDPs encode nonconvex constraints

MCDPs describe highly nonconvex sets. I will give an example of a 1-dimensional problem for which the feasible set is disconnected, even with an $h_{dp}$ that is topologically continuous.

Let us consider in detail the case of one design problem, with only one cycle (Fig. 30).

Because there are no incoming arrows, the semantics of the diagram is the description of a set of resources; more specifically, the diagram describes a set of minimal resources $M \subseteq \mathcal{R}$, formally specified as

$$ M = \left\{ \begin{array}{ll}
\text{using } & f, r \in \mathcal{F} = \mathcal{R}, \\
\min_x & r, \\
& r \in h(f), \\
& r \leq f.
\end{array} \right. $$

By construction, because of the “Min” operator, the set of minimal resources $M$ is an antichain:

$$ M \in \mathcal{A}\mathcal{R}. $$

I will describe a dynamical system that computes $M$. Towards this end, it is useful to proceed pedantically.

Consider the feasible set $\Phi \subseteq \mathcal{F} \times \mathcal{R}$, as the set of functions and resources that satisfy the constraints:

$$ \Phi = \{ (f, r) \in \mathcal{F} \times \mathcal{R} : (r \in h(f)) \land (r \leq f) \}. $$

The projection $P$ of $\Phi$ to the set of resources is

$$ P = \pi_2\Phi = \{ f : (f, r) \in \Phi \}. $$

The set $M$ is the set of minimal elements of $P$:

$$ M = \min_x M \preceq \pi_2\Phi. $$

Example 7. Let $\mathbb{R}_+^+$ be the completion of the set of nonnegative real numbers, as defined in Example 1. Note that

$$ \mathcal{A}\mathbb{R}_+^+ \simeq \mathbb{R}_+^+. $$

Therefore, we already know that if $M$ is nonempty, it is a singleton, and thus it is connected.

For $\mathcal{F} = \mathbb{R} = \mathbb{R}_+^+$, because of (29), the map $h_{dp} : \mathcal{F} \to \mathcal{A}\mathbb{R}$ is simply a map $h_{dp} : \mathbb{R}_+^+ \to \mathbb{R}_+^+$. Scott-continuous is equivalent to the map $h_{dp}$ being nondecreasing and left-continuous in the topological sense (Lemma 3). That is, there exist topologically discontinuous functions that are Scott-continuous. However, it is not necessary to consider discontinuous functions to find a disconnected feasible set. From (27) and (28), the set $P$ is simply

$$ P = \{ x \in \mathbb{R}_+^+ : h_{dp}(x) \leq x \}. $$

Consider the continuous, nondecreasing, Scott-continuous function $h_{dp}$ in Fig. 31a. That function oscillates above
and below the diagonal. Consequently, the set \( P \) is disconnected (Fig. 31b).

![Graph](image1)

Fig. 31. A topologically continuous \( h_{dp} \) inducing a disconnected \( P \).

This is thus an example for which:

- \( \mathcal{F}, \mathcal{R} \) are one dimensional,
- \( h_{dp} \) is topologically continuous,
- \( M \) is a singleton,
- \( \Phi \) is disconnected,
- \( P \) is disconnected.

In a successive example (Sec. VII-A), I will show that 2 dimensions are sufficient to also have \( M \) disconnected.

VI. EXACT SOLUTION OF MCDPs

A. Reduction to the base cases

The second main result of this paper is that the solution of a co-design problem defined as the composition of monotone problems can be obtained from the solution of the primitive problems.

**Theorem 4.** Let \( \text{cdp} \in \text{CDP} \) and suppose that all atoms are monotone. Then \( h_{cdp} \) has an explicit expression in terms of the functions \( \{ h_{dp} \mid dp \in \text{atoms}(\text{cdp}) \} \).

**Proof.** By recursion. The statement holds for the base case of \( \text{cdp} = dp \). Explicit expressions for the composite \( h \) were derived in the proofs of Prop. 1–3.

**Parallel composition:** In this case, \( h_{\text{par}(dp_1, dp_2)} \) is simply the product of \( h_{dp_1} \) and \( h_{dp_2} \), as was derived in (3):

\[
h_{\text{par}(dp_1, dp_2)} : f \mapsto h_{dp_1}(f) \times h_{dp_2}(f).
\]

**Series composition:** From (16) we have

\[
h_{\text{series}(dp_1, dp_2)} : f \mapsto \min_{\leq s_2} h_{dp_2}(h_{dp_1}(f)).
\]

Thus also \( h_{\text{series}(dp_1, dp_2)} \) can be evaluated explicitly knowing \( h_{dp_1} \) and \( h_{dp_2} \).

**Feedback composition:** the proof of Prop. 3 already derived as (26):

\[
h_{\text{loop}(dp)} : f \mapsto \text{lfp}(\Phi_f).
\]

for \( \Phi_f \) defined as in (25), thus establishing the need to solve a least-fixed-point problem in the space of antichains.

The procedure outline above is not yet a practical algorithm, for the following two reasons:

1) **Non-finitely representable sets:** The antichains are, in general, not finitely representable. Therefore, even with no loops, in general there is a problem with representation that needs to be addressed.

2) **Nested loops:** Each loop requires \( \infty \) steps. This is not a problem by itself, because with a few further assumptions we can make approximation bounds on the convergence. The problem arises if loops are nested; for example, to evaluate \( h_{\text{loop}(\text{loop}(dp))} \) takes \( \infty \infty \) steps. These points will be addressed later in the paper.

B. Solving a single loop

Consider again the case of a single loop represented in Fig. 30, reproduced here for convenience:

![Graph](image2)

The following result describes explicitly a dynamical system that computes \( M \). Most of the complexity in the proof is implicitly contained in Prop. 3. This is an independent treatment that is more explicit from the point of view of computation.

**Proposition 4.** Suppose \( h_{dp} \) is Scott-continuous. Define \( \varphi \) as follows:

\[
\varphi : \mathcal{P}(\mathcal{R}) \rightarrow \mathcal{A}\mathcal{R}, \quad S \mapsto \min_{\leq \infty} \bigcup_{s \in S} h_{dp}(s) \cap (\uparrow s).
\]

Then: (a) the least fixed point of \( \varphi \) exists and is \( M \):

\[
\text{lfp}(\varphi) = M.
\]

Furthermore, define the sequence \( \{ S_0, S_1, \ldots \} \in \mathcal{A}\mathcal{R}^+ \) as

\[
S_0 = \{ \bot \} \mathcal{R}, \quad S_{k+1} = \varphi(S_k).
\]

Then: (b) the sequence is increasing:

\[
S_{k+1} \succeq S_k.
\]

And: (c) the sequence converges (in the sense of \( \text{sup} \)) to \( M \):

\[
\text{sup}_k S_k = \text{lfp}(\varphi) = M.
\]

**Proof.** \( \varphi \) is Scott-continuous. This implies directly point (b): the sequence is increasing. Together with (a) + Kleene’s theorem, this implies (c). The hard part will be proving (a), done in Lemma 6.

**Lemma 6.** \( \text{lfp}(\varphi) = M \).

**Proof.** Because \( \varphi \) is monotone, the least fixed point \( \text{lfp}(\varphi) \) exists. Call it \( Q \):

\[
Q \triangleq \text{lfp}(\varphi).
\]
We will show that \( Q = M \).

From the definition of least fixed point, we have
\[
Q = \min \{ S \in \mathbb{A} \mathbb{R} : \varphi(S) \preceq_{\mathbb{A} \mathbb{R}} S \}.
\]

Note the use of “\( \min \)”, not “\( \text{Min} \)” as before. By definition, the \textit{least} fixed point is unique.

Note also that \( Q \) is itself an antichain. In general, \( Q \) contains many points:
\[
Q = \{ r_1, r_2, \ldots \} \subseteq \mathbb{R}.
\]

At the end of the proofs, we will show that the \( r \) are all and only minimal resources. To reiterate:

There is a unique least fixed point, \( Q \), which lives in the space of antichains \( \mathbb{A} \mathbb{R} \). The least fixed point \( Q \in \mathbb{A} \mathbb{R} \) describes possibly multiple minimal resources \( r \), which are solutions of the original design problems \( \text{dp} \).

(It can not be simpler than this.)

Now that we have clarified the terms, only computations are left. Lemma 8 shows that \( Q \subseteq M \) and Lemma 9 shows that \( Q \supseteq M \), hence \( Q = M = \text{lp} \varphi \).

\textbf{Lemma 7.} \( Q \subseteq P \). In words: all elements of \( Q \) are feasible resources.

\textit{Proof.} The statement means that for all \( q \in Q \), there exists \( f \preceq q : q \in \text{dp}(f) \).

Take any \( q \in Q \). I will show that \( q \in \text{dp}(q) \).

Because \( Q \) is the least fixed point of \( \varphi \):
\[
Q = \varphi(Q).
\]

Because \( q \in Q \), this implies:
\[
q \in \text{Min} \bigcup_{r \in Q} \text{dp}(r) \cap \uparrow r.
\]

Because \( Q \) is an antichain, for any other \( r \in Q \), \( q \) and \( r \) are not comparable: \( q \parallel r \).

Therefore, \( q \notin \uparrow r \), for \( r \neq q \).

Therefore,
\[
q \in \text{Min} \left( \text{dp}(q) \cap \uparrow q \right)
\]

which implies
\[
q \in \text{Min} \text{dp}(q)
\]

By definition \( \text{dp} \) is the identity on antichains, so
\[
q \in \text{dp}(q).
\]

\( \Box \)

\textbf{Lemma 8.} \( Q \subseteq M \subseteq P \). In words: all elements of \( Q \) are also minimal feasible resources.

\textit{Proof.} So far, by Lemma 7, we know \( Q \subseteq P \). I will show \( Q \subseteq M \subseteq P \) by contradiction. Suppose that \( Q \not\subseteq M \). Then this means that there are some elements of \( Q \) that are not minimal resources. Then there exists \( x \in Q \) such that \( x \notin M \) (Fig. 32).

If \( x \in P \) but \( x \notin M \), but \( M = \text{Min} P \), it means that there exists \( y \in P \) such that \( y \prec x \), in the sense that \( y \preceq x \) and \( y \neq x \). Because \( Q \) is an antichain, and \( x \in Q \), then \( y \notin Q \) (Fig. 33).

Define a new antichain \( \tilde{Q} \) by including \( y \) in \( Q \) (Fig. 34):
\[
\tilde{Q} = \text{Min}(\{ y \} \cup Q).
\]

By construction, \( \tilde{Q} \leq Q \). Because there exists at least \( y \notin Q \), \( \tilde{Q} \neq Q \) and therefore \( \tilde{Q} \prec Q \). Write \( \hat{Q} \) as the union of \( \{ y \} \) and a set \( Q_1 \subseteq Q \):
\[
\hat{Q} = \{ y \} \cup Q_1.
\]

Consider \( \varphi(\hat{Q}) \):
\[
\varphi(\hat{Q}) = \text{Min} \left\{ \left[ \text{dp}(y) \cap \uparrow y \right] \cup \bigcup_{r \in Q_1} \left[ \text{dp}(r) \cap \uparrow r \right] \right\}
= \text{Min} \varphi(\{ y \}) \cup \varphi(Q_1). \tag{30}
\]

Because \( y \in M \), \( y \in P \), and so \( y \in \text{dp}(y) \). Therefore \( \{ y \} \) is a fixed point of \( \varphi \):
\[
\varphi(\{ y \}) = \text{Min} \left[ \text{dp}(y) \cap \uparrow y \right]
= \{ y \}.
\]

The set \( \{ y \} \) is a fixed point of \( \varphi \), but it is not necessarily the least fixed point. The same reasoning is valid for \( Q_1 \); it is also a fixed point. From (30) it follows
\[
\varphi(\hat{Q}) = \text{Min} \varphi(\{ y \}) \cup \varphi(Q_1)
= \text{Min} \{ y \} \cup Q_1
= \text{Min} \hat{Q}
= \hat{Q}.
\]

In summary, we know the following facts:

1) By assumption, \( Q \) is the least fixed point of \( \varphi \).
2) If \( Q \not\subseteq M \), it is possible to construct a \( \tilde{Q} \) such that:
   a) \( \tilde{Q} \) is a fixed point of \( \varphi \); and
   b) \( \tilde{Q} \prec Q \).

Therefore, \( Q \subseteq M \).\( \Box \)
Lemma 9. Let $Q \supseteq M$. In other words, all minimal solutions are contained in the least fixed point of $\varphi$.

Proof. This is another proof by contradiction. I know, I know. It is the second proof by contradiction in the section. I feel like my mathematician friends will judge me for this. Anyway, suppose $Q \not\supseteq M$. Then there exists $\tilde{r}$ such that $\tilde{r} \in M$ and $\tilde{r} \not\in Q$.

There are two cases: $\tilde{r} \in \uparrow Q$ and $\tilde{r} \not\in \uparrow Q$. Yes, dear mathematician friend, we are now doing case analysis in a proof by contradiction. Welcome to engineering! The motto is: “We’ll get things done, somehow”.

First case ($\tilde{r} \in M, \tilde{r} \not\in Q, \tilde{r} \in \uparrow Q$): If $\tilde{r} \in \uparrow Q$, then $\exists q \in Q$ such that $q \leq \tilde{r}$ and $q \neq \tilde{r}$. Because $q \in Q$, $q \in h_{dpa}(q)$. This implies $q \in P$. This is a contradiction with $\tilde{r} \in M = \text{Min} P$, as there exists a $q \prec \tilde{r}, q \in P$.

Second case ($\tilde{r} \in M, \tilde{r} \not\in Q, \tilde{r} \not\in \uparrow Q$): Construct a $\hat{Q}$ as follows:

$$\hat{Q} = \text{Min}(\{\tilde{r}\} \cup Q).$$

Then $\hat{Q} \not\subseteq \mathcal{A}_R Q$ by construction. Moreover, $\hat{Q} \neq Q$, because $\tilde{r} \in Q$ and $\tilde{r} \not\in Q$. Thus $\hat{Q} \prec Q$. Following a similar reasoning as in the proof of Lemma 8, one can show that also $\hat{Q}$ is a fixed point of $\varphi$. This contradicts the assumption that $Q$ is the least fixed point. \qed

C. Complexity of the solution for a simple loop

It is easy to derive worst-case bounds for the complexity of the procedure above, based on the concepts of poset width and height.

Definition 19 (Width and height of a poset). Let $P$ be a poset. Then the width of $P$, $\text{width}(P)$, is the maximum cardinality of an antichain in $P$ and the height of $P$, $\text{height}(P)$, is the maximum cardinality of a chain in $P$.

Everything that is representable in a physical computer has finite cardinality. So, in practice, cardinalities are finite numbers. However, the following applies equally well in theory, for countably finite and countably infinite sets:

$$\text{height}(\text{int32}) \leq 2^{32},$$
$$\text{height}(\text{float32}) \leq 2^{32},$$
$$\text{height}(\mathbb{N}) = \aleph_0,$$
$$\text{height}(\mathbb{R}) = \aleph_1 > \aleph_0.$$

With this nomenclature, it is easy to state worst-case bounds for the procedure described in Prop. 4.

Proposition 5. The sequence constructed in Prop. 4 is a chain of antichains in $\mathcal{R}$. Therefore, there are at most $\text{height}(\mathcal{A}_R)$ steps in the sequence. At each step, the state $S_k$ is at most $\text{width}(\mathcal{R})$ times; call the quantity $c$. Then the computational complexity for the basic Kleene ascent algorithm is

$$\text{width}(\mathcal{R}) \times \text{height}(\mathcal{A}_R) \times c.$$

Example 8. In $\mathbb{N}$, consider an MCDP equivalent to $f(x) \leq x$, with $f : x \mapsto x + 1$. This takes $\text{height}(\mathbb{N}) = \aleph_0$ steps to converge to $\top$.

Making more precise claims requires instantiating the MCDP theory more concretely, for example by assuming that there is also a metric on $\mathcal{R}$.

Remark 7. What is not easy to do is to derive bounds for the solution of an arbitrary MCDP. To solve an MCDP, the first step is to write the graph as a tree with junctions series, par, loop, with only one instance of the loop junction. There are multiple arbitrary choices to be made; there are many tree representations of the same graph. An extensive discussion is beyond the scope of this paper.

D. Numerical example: Optimizing over the natural numbers

This section describes numerical examples. These examples were chosen to highlight the technical details rather than the modeling power of MCDPs.

The optimization problem in this section is over the discrete CPO $\mathbb{N} = \mathbb{N} \cup \{\top\}$. Let $c \in \mathbb{R}$ be any nonnegative real constant. We will consider a family of optimization problems as a function of $c$:

$$M(c) = \begin{cases} \text{Min}_{\leq \pi} \langle x, y \rangle, \\ \text{s.t.} \quad x + y \geq \sqrt{x} + \sqrt{y} + c \end{cases}$$

(31)

This means that $M(c) \subseteq \mathbb{N} \times \mathbb{N}$ is the set of minimal elements for which the constraint holds. That is, the feasible set $F(c)$ is

$$F(c) = \{\langle x, y \rangle \in \mathbb{N} \times \mathbb{N} : x + y \geq \sqrt{x} + \sqrt{y} + c\},$$

and the minimal set is

$$M(c) = \text{Min}_{\leq \pi}^c F(c).$$

The cardinality of $M(c)$ changes as a function of $c$. For example, for $c = 0$, the point $\langle 0, 0 \rangle$ is a solution, and it is definitely the only minimal solution:

$$M(0) = \{\langle 0, 0 \rangle\}.$$

For $c = 1$, there are two minimal solutions:

$$M(1) = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}.$$

This can be verified by direct computation. For $c = 1$, $\langle 0, 0 \rangle$ is not feasible. The two points $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$ are both feasible. Therefore, they are the two minimal solutions. It is not needed to look for the other elements of $\mathbb{N} \times \mathbb{N}$, because they are all dominated by $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$. More formally,

$$\uparrow \{\langle 0, 0 \rangle, \langle 0, 1 \rangle\} = (\mathbb{N} \times \mathbb{N}) / \{\langle 0, 0 \rangle\},$$

and so we only have to check $\langle 0, 0 \rangle$, which is not feasible.

Remark 8 (The trick). After a seminar I gave about this material, an esteemed colleague, who is an expert in optimization, told me “I don’t buy it!”. The answer to the question “What is the trick?” is, of course, “there is no trick”. However, it is worth pointing out a nonobvious subtle point, that very well constitutes the “trick”:

To compute the set of minimal resources, it is not necessary to compute the set of all feasible resources.
In the example, to compute $M(1) = \text{Min } F(1)$, it is not necessary to compute $F(1)$. In general, it might be that $M$ is much simpler than $F$ (in the extreme case, $F$ is not computable, while $\text{Min } F$ is). Also notice the following:

While the feasible set is non-convex, the “objective function” is, essentially, the identity function.

These two remarks should allow the reader to relax, with the confidence that there is no contradiction between this work and everything else in optimization.

Let us now return to the numerical example. It is reasonable to expect that the problem (31) is an MCDP, because it is an example in a paper about MCDPs. This is correct, and it can be proved by exhibiting a diagram of monotone design problems that are equivalent to (31). One particular diagram equivalent to (31) is shown in Fig. 35.

Fig. 35.

In this diagram, there are three types of primitive design problem. The first is square root plus ceiling function:

\[ h_1 : \mathbb{N} \rightarrow \mathbb{N}, \quad a \mapsto \lceil \sqrt{x} \rceil. \]

The second is addition:

\[ h_+ : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}, \quad (a, b) \mapsto a + b. \]

The third is the dual of addition, here denoted $h^d$:

\[ h^d : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}, \quad s \mapsto \{(a, b) : a + b = s\}. \]

Therefore, it is possible to apply the algorithm in Prop. 4 to compute the minimal solutions $M(c)$, for any choice of $c$.

Fig. 35 shows the sequence $S_k$, which converges to a fixed point at $k = 5$.

E. Dealing with nested loops

Let us deal with the problem of nested loops first. This is actually easier than it looks. In short, one can always rewrite two nested loops as only one loop. A graphical proof is entirely satisfying. A diagram like the one in Fig. 37a can be rewritten as Fig. 37b. In other words, it is possible to describe an arbitrary graph of design problems using only one instance of the operator loop.

This construction is analogous to the construction used for the analysis of process networks.

To make this work, some “plumbing” is required. The plumbing consists of the two connectors shown in Fig. 38. These should be considered two degenerate design problems. In particular Fig. 38b inside the loop there is the series of the connector in Fig. 38a and the original dp, and the connector in Fig. 38b.
This could be formalized as a design problem by letting \( \mathcal{F} = \mathbb{R}_{\text{distance}} \) and \( \mathcal{R} = \mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{speed}} \). The relation among the quantities is obviously
\[
d = Tv.
\]
This could be formalized as a design problem by letting \( \mathcal{F} = \mathbb{R}_{\text{distance}} \) and \( \mathcal{R} = \mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{speed}} \). The map \( h \) that describes the problem is
\[
h : \mathbb{R}_{\text{distance}} \to \mathbb{A}(\mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{speed}}),
\]
\[
d \mapsto \{ (T,v) \in \mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{speed}} : d = Tv \}.
\]
It can be easily verified that this map is Scott-continuous.

If we fix \( d = 1 \) km, there is a continuum of solutions. All pairs \((T,v)\) for which \(Tv = 1\) km belong to the antichain \(h(1)\) km.

There are two approaches one can take to deal with the problem of infinite antichains.

1) Discretizing the spaces: First, we could discretize the functionality \( \mathcal{F} \) and the resources \( \mathcal{R} \) by sampling and/or coarsening. Because all spaces are finite we can create a proper “algorithm” and not just an abstract dynamical system that computes the solution. Nevertheless, this solution does not get to the core of the issue.

2) Bounding the design problem: The core of the issue is that the computation budget we have available might be smaller than the sets that need to be represented.

A possible way that works in practice is to approximate the design problem itself, rather than the spaces \( \mathcal{F}, \mathcal{R} \), which are left as possibly infinite.

The basic idea is that an infinite antichain can be bounded from below and from above by two antichains that have a finite number of points.

Let \( n_L \) and \( n_U \) be the “computation budget”: the number of points used to represent the upper and lower approximation. For any possibly infinite antichain \( S \in \mathbb{A}\mathcal{R} \), one can find antichains \( S_{L}^{n_L}, S_{U}^{n_U} \in \mathbb{A}\mathcal{R} \) that provide bounds of the kind
\[
S_{L}^{n_L} \preceq_{\mathbb{A}\mathcal{R}} S \preceq_{\mathbb{A}\mathcal{R}} S_{U}^{n_U}.
\]

This formulation can be formalized as a design problem by letting
\[
\mathcal{F} = \mathbb{R}_{\text{distance}}, \quad \mathcal{R} = \mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{speed}},
\]
\[
h : \mathbb{R}_{\text{distance}} \to \mathbb{A}(\mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{speed}}),
\]
\[
d \mapsto \{ (T,v) \in \mathbb{R}_{\text{time}} \times \mathbb{R}_{\text{speed}} : d = Tv \}.
\]
It can be easily verified that this map is Scott-continuous.

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\[
S_{L}^{n_L} \preceq_{\mathbb{A}\mathcal{R}} S \preceq_{\mathbb{A}\mathcal{R}} S_{U}^{n_U}.
\]

Moreover, the inequalities will become equalities as the number of points \( n_L, n_U \to \infty \):
\[
\sup_{n_L} S_{L}^{n_L} = S = \inf_{n_U} S_{U}^{n_U}.
\]

In practice, the construction of these sequences depends on the “shapes” of the antichains generated by each design problem. A formal treatment of this is beyond the scope of this paper. If we know how to find bounding sequences of the kind \((33)\) and \((34)\), then the construction can be lifted to the maps \( h \), and then to the design problems themselves.

For any map \( h_{dp} : \mathcal{F} \to \mathbb{A}\mathcal{R} \), we can find bounds
\[
h_{dp}^{L} \preceq h_{dp} \preceq h_{dp}^{U}.
\]
This induces bounds on the design problems itself; for each \( dp \), there are \( dp_{L}^{n_L} \) and \( dp_{U}^{n_U} \) such that
\[
\sup_{n_L} dp_{L}^{n_L} \preceq dp \preceq \sup_{n_U} dp_{U}^{n_U},
\]
in the sense of \((35)\), and for which
\[
\sup_{n_L} dp_{L}^{n_L} = dp = \inf_{n_U} dp_{U}^{n_U}.
\]
This suggests the following way to deal with a finite computation budget \( n \). Choose \( n_L, n_U \) such that \( n_L + n_U = n \). Solve \( dp_{L}^{n_L} \) to obtain \( S_{L}^{n_L} \) and solve \( dp_{U}^{n_U} \) to obtain \( S_{U}^{n_U} \). By construction, the non-finitely representable solution \( S^* \) of \( dp \) will be bounded by the two finite approximations:
\[
S_{L}^{n_L} \preceq S^* \preceq S_{U}^{n_U}.
\]
This provides a clear framework for dealing with finite computation budgets. The details are beyond the scope of this paper.

A. Numerical example: Optimizing over the reals

Take \((33)\) and lift it to the real numbers \( \mathbb{R} \) instead of \( \overline{\mathbb{R}} \):
\[
M(c) = \left\{ \begin{array}{ll}
\min \left\langle x, y \right\rangle & x + y \geq \left[ \sqrt{x} \right] + \left[ \sqrt{y} \right] + c, \\
\text{s.t.} & \\
\end{array} \right.
\]

The feasible set is now
\[
F(c) = \{ (x, y) \in \mathbb{R} \times \mathbb{R} : x + y \geq \left[ \sqrt{x} \right] + \left[ \sqrt{y} \right] + c \}.
\]
Note that on the reals the map \( x \mapsto \left[ \sqrt{x} \right] \) is topologically discontinuous but Scott-continuous.

Fix \( c = 4 \). Because \( F(4) \) is a subset of \( \mathbb{R}^2 \), it makes sense to ask: Is the set \( F(4) \) convex? This is tricky to see, so the best course of action is to ask Mathematica to plot the inequality
\[
x + y \geq \left[ \sqrt{x} \right] + \left[ \sqrt{y} \right] + 4.
\]
Mathematica’s plot is shown in Fig. 40a. From this picture, one can conclude that the set \( F(4) \) is not convex. Unfortunately,
the plot in Fig. 40b is not complete. It misses a few points that belong to \( F(4) \). The points, found by direct inspection, are \( \{(0, 7), (4, 4), (7, 0)\} \). Mathematica misses these points because they are isolated points of \( F(4) \).

Because there are 4 disconnected components, we can conclude that the set \( F(4) \) is not only nonconvex in this parameterization, but remains nonconvex in any other continuous reparameterization.

More importantly, the set is also not finite. Whether it is “finitely representable” depends, of course, on the representation that one has available. In fact, the set is finitely representable if the representation includes points and segments. It is, however, easy to show that if one changed slightly \( \sqrt{a} \), the segments would not be straight anymore.

Rather than fixing a representation and asking what problems are finitely representable, it is more interesting to consider the general case, in which the available representation is not sufficient to represent the solution.

In particular, the problem in this particular example lies with the \( +^d \) map defined in Eq. (32): for each “functionality” \( s \), there are infinite combinations of \( (a, b) \) such that \( a + b = s \).

The strategy is as follows: define two sequences of design problems \( +^d_L(n_L) \) and \( +^d_U(n_U) \) that are bounds for \( +^d \):

\[
+^d_L(n_L) \leq +^d \leq +^d_U(n_U).
\]

Furthermore, the two sequences should converge to \( +^d \) as \( n_L, n_U \) increase to infinity.

There are many possibilities to build such sequences. For each antichain, each subset of the antichain is an upper bound. Thus, an upper bound for \( +^d \) with computation budget \( n_U \) can be found by just sampling \( n_U \) points on \( a + b = s \). To find a lower bound for the antichain, one can employ a construction such as the one shown in Fig. 40.

Then one creates a new MCDP with the approximations. For example, using \( +^d_U \) instead of \( +^d \), the MCDP obtained is in Fig. 42.

The resulting fixed point for \( n_U = 10 \) is shown in Fig. 43c. Note how this finite antichain is an upper bound for the infinite antichain \( M(4) \). Analogously, constructing a lower bound approximation \( +^d_L(n_L) \) for \( n_L = 100 \) will induce a fixed point as in Fig. 43b.

VIII. EXAMPLES OF MODELING WITH MCDPS

This last section describes a few examples of modeling the co-design constraints in robotic systems.

Example 10 (Seabed surveying). This example describes a scenario with a partial order of resources (time, energy, cost). Consider the scenario of seabed surveying using a AUV [13], formalized as an area coverage problem. Many other scenarios are equivalent, such as the problem of autonomous lawn mowing, and aerial inspection for search and rescue. Assume that the vehicle moves at velocity \( v \) [m/s] with respect to the seabed, at a fixed depth, such that the field of view is \( r \) [m] (Fig. 44a). The functionality of the system is to sweep an area \( A \) [m²] (Fig. 44b). We will characterize the resources (time, energy, cost) and then ask under what condition the design is feasible based on the customer’s preferences.

(a) Seabed surveying using an AUV
(b) Sweeping strategy

Fig. 44. The AUV seabed surveying scenario. Figures adapted from [13], with permission of the authors.

All the quantities in this problem are related by cyclic co-design constraints (Fig. 45a). Read the diagram from the left.
The area $A$ is a parametrization of the function. Neglecting the turning time, and assuming there are no currents, the constraint between velocity $v$, time $T$, and field of view $r$ is $vTr \geq cA$, for a constant $c$. The actuation power $P_a$ [W] is a function $\psi$ of $v$. We can assert that this function $\psi$ is monotonically nondecreasing, even without any knowledge of hydrodynamics. This gives the constraint $P_a \geq \psi(v)$. If there is a maximum velocity $v_{\text{max}}$, let $\psi(v) = T$ for $v \geq v_{\text{max}}$. Similarly, the sensing power $P_s$ is monotonically nondecreasing in $r$. The energetics constraint is $E \geq (P_a + P_s)T$. We can also consider the cost of the sensor, as a monotonically nondecreasing function of $r$. For clarity, the costs for the other components are neglected. Together, these relations define a constraint of the kind

$$\langle E, T, S \rangle \geq h(A).$$

The other half of the story are the customer’s preferences. Suppose we create a system that sweeps an area $A$ for a certain time, energy, and cost. Fixed ($T, E, S$), what is the minimum area for which the customer accepts the system? In general, there is a nonlinear constraint of the kind $A \geq \varphi(T, E, S)$ for a monotonic function $\varphi$, which describes the customer. Putting the two constraints together gives a fixed-point constraint of the type $A \geq (\varphi \circ h)(A)$, which one can solve to find the minimum functionality $A$ (if it exists) at which the system is accepted by the customer, and the corresponding minimum resources needed $h(A)$.

These two roles can be formalized as monotone co-design constraints. Assume functionality and costs are any poset. Then the engineer is equivalent to a map $h$ that describes the constraint

$$\text{costs} \succeq h(\text{functionality}).$$

The customer is equivalent to a map $\varphi$ that describes the constraint

$$\text{functionality} \succeq \varphi(\text{costs}).$$

If the diagram (Fig. 45b) corresponds to a feasible MCDP, then there exists a solution that is technically feasible and works from the business point of view.

A. Modeling sensing and computation

What is missing to obtain a complete robot design problem is: the sensors, the computing platform, plus all of the software, including perception, planning, control (Fig. 46). Can all of these be described as monotone co-design problems, interconnected through functions and resources that are posets?

For robotic sensors, Lavalle and O’Kane [14, 15] described quite in detail the lattice of sensors, and their monotonicity properties. All that work should be reusable in this framework.

Having a computer on board (the “computation” box in Fig. 46) adds to the payload and power requirements, all concrete quantities measurable in grams and watts. But what is the “functionality” of a computer? One answer is that the function of a computer is to run a program.

Do programs live on posets? It depends on the representation. Some representations of programs developed in the field of embedded systems [7, 16] such as as “computation graphs” have a poset structure and thus can be used naturally in this framework [17]. A computation graph is a graph where nodes are computations (with label “number of operations”) and edges are signals (with label “size”, in bytes). Equivalently, an architecture graph is a graph where nodes are processors (with label “flops”) and edges are network links (with labels “latency” and “bandwidth”).

The scheduling problem studied in embedded systems is monotone between these two posets: to run a larger computation graph (in the sense that it has more nodes, or larger nodes, or thicker edges), with the same performance, specified in terms of latency or throughput, the hardware requirements do not decrease. Therefore, the “choose the processor and networks” design problem is monotone with the interface represented in Fig. 47. The additional functionality are the performance requirements (latency and throughput). The additional resource is power consumption.
IX. Conclusions

This paper introduced a mathematical theory of co-design. The basic intuition is that a complex system can be described in terms of simpler systems that interact by providing functionality and requiring resources. This point of view is completely orthogonal with respect to other compositional meta-models of systems, such as bond graphs. This approach is well suited to the problem of designing a system, rather than simulation or control.

The resulting class of optimization problems is called Monotone Co-Design Problems (MCDP). These were shown to be nonconvex, noncontinuous, and even not finitely representable. Yet, MCDP are solvable for finite posets, and there are well-behaved finite approximations for the MCDPs that are defined on infinite spaces.

This formalization is able to represent the co-design constraints among heterogeneous domains. Examples were given in the domain of robotics; everything extends naturally to all other engineering domains.

A. Future work

1) The dual formulation: Throughout this paper, for convenient exposition there was a preferred "direction": fix the functionality, then minimize resources. However, everything applies equally well in the other direction. The dual formulation is: fix resources and maximize functionality. To obtain the dual formulation, just re-read this paper by inverting all arrows, use "lower sets" instead of "upper sets", and "min" instead of "max".

At a slightly higher level of abstraction, there is a more general formulation that is completely symmetric in terms of functionality and resources. Category theory is the right formalism: functionality and resources are the objects in a category and the design problems are the morphisms of the category.

2) Performance guarantees: This paper did not provide an intrinsic, quantitative bound on the accuracy of the Kleene ascent. There are two main difficulties.

First, no metric was defined; therefore, concepts such as "linear" or "quadratic" convergence do not make sense.

Second, the solution outlined is not intrinsic. An MCDP is a graph of design problems. To find the solutions, one needs to:

1) Rewrite the graph of design problems as a tree, using the three operators series, par, loop.
2) Flatten the nested loops to only one loop (Sec. [VI-E]). There are many arbitrary choices in these two steps. Each set of choices gives a slightly different fixed-point iteration with different computational properties.

3) An interval algebra for design problems: The solution of an MCDP is, in general, an infinite antichain.

One way to deal with this is to make additional assumptions about the blocks of the MCDP as to assure that the antichain is finitely representable. This is a vast research direction which will give efficient algorithms for special cases.

A different approach is to look for "general purpose" approximations such as the one explained in Sec. [VI-E]. This is another vast research direction.

4) More efficient algorithms: The "Kleene ascent" described here is the equivalent of "gradient descent" for convex optimization. What is the equivalent of Newton’s method for MCDP?

5) Parallelization: An interesting aspect is that the fixed point iteration is parallelizable. The information structure allows very flexible parallelization of the work. Consider first the case where there is a master and several slaves. Say that a master node keeps a copy of the current antichain $S_k$. The master distributes each slave computes $\varphi(\{s^k_i\})$ and the master updates $S_{k+1}$. Note that by construction $\{s^k_i\}$ is already a lower bound for $\varphi(\{s^k_i\})$. Therefore, if a slave fails, the master can simply use $\{s^k_i\}$ as a placeholder for $\varphi(\{s^k_i\})$. This and other properties make parallelization of MCDPs another vast research direction.

6) Additional structure: It will be interesting to consider "MCDP plus $x$", for $x$ = convex, contractive, submodular, etc.

Of course, where there is more structure, the algorithms can be more efficient. This is the usual generality-performance trade-off. A related trade-off that will be interesting to understand is the trade-off between

prior knowledge vs performance.

For example, assuming that the problem is also $x$, we would like to know what the performance penalty in not knowing that it is also $x$.

7) The user experience: Last, but not least, there is the "user experience". It is very important to have intuitive tools for practitioners. Inspired by CVX and "disciplined convex programming" [18] [19], I have developed MCDPL, a modeling language to describe MCDPs. This is really the most disciplined language possible, given that it is not possible to describe anything that is not monotone. For example, multiplying by a negative number is a syntax error. An example of a model is in Fig. [15].

The software is available at [http://mcdp.mit.edu](http://mcdp.mit.edu).

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Fig. 47.

To find the solutions, one needs to:

1) Rewrite the graph of design problems as a tree, using the three operators series, par, loop.
2) Flatten the nested loops to only one loop (Sec. [VI-E]).
 endurance extra power extra payload

Fig. 48. Example of a MCDPL model. Interpreter and solver are available at [http://mcdp.mit.edu](http://mcdp.mit.edu).

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