Realisation of Hardy’s Thought Experiment

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We present an experimental realisation of Hardy’s thought experiment [Phys. Rev. Lett. 68, 2981 (1992)], using photons. The experiment consists of a pair of Mach-Zehnder interferometers that interact through photon bunching at a beam splitter. A striking contradiction is created between the predictions of quantum mechanics and local hidden variable based theories. The contradiction relies on non-maximally entangled position states of two particles.

Quantum mechanics poses a challenge to the notion that objects carry with them values of observables, such as position, that both determine the outcomes of measurements and that are local, i.e. uninfluenced by events that happen outside the object’s backward light cone. It was first pointed out by Bell [1] that the predicted correlations between outcomes of measurements on two spatially separated systems prepared in an entangled quantum state, were too strong to be reproduced by any theory based on local ‘hidden’ variables (LHVs). He formulated an inequality which places a bound on the correlations predicted by any such theory, opening the possibility of performing experimental tests whose realisation [2] decided in favour of quantum mechanics. By considering three particles in an entangled quantum state Greenberger, Horne and Zeilinger [3] later proposed a scheme in which quantum mechanics and LHV theories predict opposite measurement outcomes, leading to an even stronger contrast between the two. The predictions were verified with polarisation entangled photons [4].

More recently, Hardy formulated a thought experiment [5] that involves only two spatially separated particles, like in the Bell case, but which leads to a strong contradiction between the LHV and quantum mechanical predictions in a similar spirit to the GHZ scheme. A crucial feature in Hardy’s thought experiment is that the two particles are non-maximally entangled. He later generalised his scheme in a way that could be tested with polarisation entangled photons [6, 7].

Here we present an experimental realisation of Hardy’s original thought experiment, using photons. It differs from the polarisation-based scheme in that the variables used are paths taken by photons. This makes the contradiction particularly striking, since position is an external variable that translates intuitively to its classical equivalent.

The basic building block of Hardy’s thought experiment is a Mach-Zehnder interferometer for quantum particles (Fig. 1, Left). The interferometer is tuned so that particles entering in arm a exit in arm c:

\[ |a⟩ \rightarrow \frac{|v⟩ + i|w⟩}{\sqrt{2}} \rightarrow \frac{(|d⟩ + i|c⟩) + i(|c⟩ + i|d⟩)}{2} = i|c⟩. \]

If the amplitude for the particle in one arm, say w, were to be obstructed by a second particle in arm b that collides with it (Fig. 1, Right), only the v amplitude would reach the second beam splitter, and would split into arms c and d with equal amplitude. The detection of a particle in arm d would thus indicate the presence of the obstructing particle without the latter being affected.

Hardy’s original thought experiment (Fig. 2) has two interferometers, one for electrons and one for positrons, arranged in such a way that their w arms intersect. If both the electron and the positron take arms w in their respective interferometers, they will annihilate with certainty to produce gamma radiation: \(|w^+⟩|w^−⟩ \rightarrow |γ⟩\). Therefore the presence of either particle in its w-arm will
implies the presence of the obstructing electron in $u^-$.

A paradox then arises because sometimes (Eq. 1) the particles do emerge simultaneously at $d^+$ and $d^-$ (with probability $p = \frac{1}{4}$). Quantum mechanically, the $|d^+\rangle|d^-\rangle$ term arises in fact from the non-maximally entangled nature $\mathcal{E}$ of the state just before the final beam splitters $|v^+\rangle|v^-\rangle + i|u^+\rangle|v^-\rangle + i|v^+\rangle|u^-\rangle$.

It is instructive to analyse a single run of the experiment from the point of view of different frames of reference. An inertial frame of reference can always be chosen in which one particle, say the positron, reaches a detector before the other reaches the final beam splitter. In that frame upon recording a click at $d^+$ one can make the prediction that the electron is in arm $u^-$ with probability $p = 1$ since the state of the electron is projected onto $|u^-\rangle$. Alternatively one can choose a frame moving in the opposite direction and, upon recording a $d^-$ event, predict with certainty that the positron is in arm $u^+$. Thinking locally, one would then argue that each particle must have travelled in its $w$ arm. However by comparing results in the different frames one then runs into a contradiction $\mathcal{E}$ because had they come from $w^+$ and $w^-$, they would have annihilated. Changing frames in this way allows the paradox to be established for a single setup in which the final measurement is conducted in the $c/d$ basis.

Our scheme to implement the thought experiment, which follows essentially the proposal of Ref. $\mathcal{E}$, uses indistinguishable photons as a substitute for the electron and positron and photon bunching at a beam splitter $\mathcal{E}$ as the annihilating interaction. The central part of our setup is a set of seven beam splitters arranged as in Fig. $\mathcal{E}$. The two interferometers share a central beam splitter where the bunching occurs and can be identified as the sets of four beam splitters on the left (unshaded) and on the right (shaded). The outermost beam splitters balance the losses through the cen-
eral beam splitter. The path-lengths are tuned so that photons entering $e^+$, if not lost through the central or outer beam splitters, emerge exclusively in arm $c^+$. Similarly light entering $e^−$ emerges in $c^-$. In the experiment, pairs of identical photons arrive simultaneously in arms $e^+$ and $e^−$ and enter their respective interferometers: $(e^+|e^−) \rightarrow \frac{1}{\sqrt{2}}((v^+) + i(w^+))(|v^−) + i|w^−))$. As in the electron-positron case, four terms can be identified corresponding to the four combinations of the paths the two photons can take.

The $|w^+|w^−)$ term will bunch at the central beam splitter, $|w^+|w^−) \rightarrow \frac{1}{\sqrt{2}}((2u^+)+ (2u^−))$. This excludes the possibility of detecting a photon leaving each interferometer simultaneously. The absence of such a coincidence click plays an equivalent role to the electron-positron annihilation. The $|v^+|v^−) term evolves into a superposition of states in which neither, one or both photons are lost through the outermost beam splitters. The cases in which one or both are lost do not give rise to a coincidence click and are therefore not counted. The $|v^+|v^−) term then simply picks up a reduction in amplitude and a change in phase from the reflections: $|v^+|v^−) \rightarrow \frac{1}{2}(|v^+)|v^−)$. Finally, the $|v^+|w^−) terms also evolve into a superposition of states in which one photon is lost through the outermost beam splitter or one photon crosses over to the other’s interferometer or the photons end up one in a $u^±$ and the other in a $u^\mp$ arm. Post-selection on coincidence counts gives the evolution: $|w^+|w^−) \rightarrow \frac{1}{2}(|u^+|u^−)$. Combining these terms, we get the desired post-selected state of the thought experiment:

$$\frac{1}{\sqrt{3}}[(v^+|v^−) + i(u^+|v^−) + i(u^−|u^+)].$$

(2)

cf. second line of Eq. 14, dropping the $|γ) term which does not give rise to a coincidence click. The paradox is the same: $[d^± → u^±, p(d^±; d^±) = \frac{1}{2}], p(u^+|u^−) = 0].$

In practice, neither the bunching nor the implications $[d^± → u^±] will be perfect. It is therefore necessary to derive an inequality describing the predictions of LHV theories. An LHV theory simultaneously predicts the results for the complementary $c/d$ and $u/v$ measurements for any given value of the hidden variables. The predicted results on one side must be independent of the measurement performed on the other side. Probabilities such as $p(u^+, c^−; u^−, c^−)$ denote the fraction of all hidden variable values that give the results shown in the brackets for the respective measurements. These probabilities are not directly measurable, however the values of measurable probabilities, such as $p(u^+; u^−)$, are derived from them according to a simple rule:

$$p(u^+; u^−) = p(u^+, d^±; u^−, d^±) + p(u^+, d^±; u^−, c^−) + p(u^+, c^+; u^−, d^−) + p(u^+, c^+; u^−, c^−),$$

(3)

one adds all possible outcomes for the complementary measurements on each side. Since probabilities are positive, this implies:

$$p(u^+; u^−) ≥ p(u^+, d^±; u^−, d^±)$$

(4)

The expression for $p(d^±; d^−) = p(u^+, d^±; u^−, d^±) + p(u^+, d^±; u^−, d^−) + p(v^+, d^±; v^−, d^−) + p(v^+, d^±; v^−, d^−)$ allows us to rewrite Eq. 4 as:

$$p(d^±; d^−) ≤ p(u^+; u^−) + p(u^+, d^±; v^−, d^−) + p(v^+, d^±; u^−, d^−) + p(v^+, d^±; v^−, d^−).$$

(5)

To bound the last three terms on the right hand side, one can use the equalities:

$$p(d^±; v^−) = p(u^+, d^±; v^−, c^−) + p(u^+, d^±; v^−, d^−)$$

$$+ p(v^+, d^±; v^−, c^−) + p(v^+, d^±; v^−, d^−),$$

$$p(v^+, d^−) = p(v^+, c^+; u^−, d^−) + p(v^+, c^+; v^−, d^−)$$

$$+ p(v^+, d^±; v^−, d^−) + p(v^+, d^±; v^−, d^−),$$

(6)

derived as above, to obtain:

$$p(d^±; v^−) + p(v^+, d^±; v^−, d^−) ≥ p(u^+, d^±; v^−, d^−)$$

$$+ p(v^+, d^±; u^−, d^−) + p(v^+, d^±; v^−, d^−).$$

(7)

Using this inequality in Eq. 5 gives the final result:

$$p(d^±; d^−) ≤ p(u^+; u^−) + p(d^±; v^−) + p(v^+, d^−),$$

(8)

which is similar to the Clauser-Horne inequality [12].

We now discuss the experimental requirements to violate this inequality. The quality of the bunching depends on the distinguishability of the photons emerging from the central beam splitter. The parts of their wave-packets which are distinguishable do not bunch and will either both be reflected, both be transmitted or both end up on the same side. The case in which they are both reflected is equivalent to each particle remaining in its own interferometer and therefore leads to a $c^+c^−$-click. The case in which they are both transmitted however, leads to the photons each emerging randomly from the last beam splitters giving an equal amount of $c^+c^−$, $c^+d^−$, $d^+c^−$ and $d^+d^−$ clicks. The implications $[d^± → u^±] remain unaltered since the only way a $d^±$ or a $d^−$ click can arise is from the amplitude in which both photons swap interferometers, which can only have occurred if the photons emerge in arms $u^+$ and $u^−$. Consequently, the quality of the implications depends only on the quality of the interferometers, which in turn depends on their alignment and can be made high. By contrast, the quality of the annihilation poses a stronger restriction than might be expected. A $u^+u^−$ event arising from the distinguishable amplitudes both swapping interferometers only leads to a $d^±d^−$ click with a probability $p = \frac{1}{4}$, contributing four times as many $u^+u^−$ than $d^±d^−$ events. Even assuming perfectly working interferometers ($p(d^±; u^±) = 0)$, this leads to the requirement that the probability of the
photons being distinguishable \( p(\text{disting.}) \) be less than \( \frac{1}{8} = 12.5\% \) for the inequality \( \mathbf{3} \) to be violated.

In our experimental setup (Fig. 3) a Ti:Sapphire mode-locked laser produces light pulses of 120fs duration, centered at a wavelength of 780nm, with a repetition rate of 82MHz. The light is passed through a \( \beta \)-Barium Borate (\( \beta \)-BBO) crystal where it is frequency doubled. The frequency doubled light then arrives at a second, 2mm thick \( \beta \)-BBO crystal where it is down-converted \( [14] \) to produce pairs of near-degenerate photons having orthogonal polarization. To make the photons less distinguishable, two 3nm bandwidth interference filters were placed in arms \( e^+ \) and \( e^- \), together with a half-wave plate at 45° in arm \( e^+ \) to align the polarizations. For a detailed discussion on the effect of filters on bunching in this type of system, see \( [12] \). The light then passes through the setup and is detected at \( c^+, c^-, d^+, d^- \). The outermost beam splitters are mounted on piezo-electrically driven translation stages used to tune the length of the \( v \)-arms of the interferometers.

We measure simultaneous clicks between detectors on the left and on the right \( (c^+c^-, e^+d^-, c^-d^+, d^+d^-) \). Coincidence logic allows the distinction of genuine two-photon events from random dark counts. To measure in the \( c/d \) or \( u/v \) bases we close the shutters in arms \( u \) and \( v \) as appropriate. For example, if the \( w^- \) arm is blocked, then all the photons reaching \( c^- \) and \( d^- \) must have come from arm \( w^- \). Thus to measure, say, the number of \( d^+v^- \): \( N(d^+u^-) \) we measure \( N(d^+c^-) + N(d^+d^-) \) with the shutter in arm \( w^- \) closed.

For a fair measurement of the contradiction \( N(d^+d^-) > N(u^+u^-) + N(d^+v^-) + N(v^+d^-) \) we need to make sure that the right hand side is not underestimated by our measurement technique. To do this we measured the rates \( N(d^+d^-) \), \( N(c^+d^-) \), \( N(d^+c^-) \) and \( N(c^+c^-) \) under all combinations of blocking arms \( u \) and \( v \) and ensured the efficiencies were all within 10% of each other, with the \( d^+d^- \) efficiency less than the others in all configurations. Fig. 3 shows the quality of the interferometers and of the bunching at the time of the experiment. The probability of getting a \( d^+d^- \) click from two photons accidentally emerging in the \( d \) arms is less than 0.55%. The quality of the bunching is above the threshold required to measure a violation \( p(\text{disting.}) = 8\% < 12.5\% \).

Fig. 4 shows a comparison between the measured \( N(d^+d^-) \) and the LHV bound on it. The bound is given by the sum of the measured \( N(d^+v^-) \), \( N(d^-v^+) \) and \( N(u^+u^-) \). We find a violation of the LHV inequality (Eq. 4) by 12 standard deviations. The violation is consistent with the quantum mechanical predictions based on the probability of bunching and the detection efficiencies in our setup.

In conclusion, we have implemented Hardy’s thought experiment consisting of two interacting Mach-Zehnder interferometers, demonstrating the contradiction between quantum mechanics and LHV theories in a striking way. It should be noted that the concept of creating entanglement by influencing a single photon interferometer with another photon also plays a crucial role in optical approaches to quantum logic gates \( [17] \).

We would like to acknowledge Ian Walmsley for useful discussions, support from NSF grants 0304678 and 0404440, and Perkin-Elmer regarding the SPCM-AQR-13-FC detectors. W.I. acknowledges Elsag s.p.a. for support under MIUR’s grant n.67679/L.488/92. J.H. acknowledges support from Lucent Technologies CRFP.
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