The Fourier Space Statistics of Seed-like Cosmological Perturbations

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Abstract

We propose a new test for distinguishing observationally cosmological models based on seed-like primordial perturbations (like cosmic strings or textures), from models based on Gaussian fluctuations. We investigate analytically the Fourier space statistical properties of temperature or density fluctuation patterns generated by seed-like objects and compare these properties with those of Gaussian fluctuations generated during inflation. We show that the proposed statistical test can easily identify temperature fluctuations produced by a superposition of a small number of seeds per horizon scale for any observational angular resolution and any seed geometry. However, due to the Central Limit Theorem, the distinction becomes more difficult as the number of seeds in the fluctuation pattern increases.

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1 Introduction

One of the directly measurable features of the primordial fluctuations that gave rise to galaxies and large scale structure formation is the probability distribution and the corresponding moments of the primordial perturbation field $\delta(x)$ and its Fourier transform $\tilde{\delta}(k)$.

The phases $\phi_k$ of the Fourier modes $\tilde{\delta}(k)$ are usually assumed to be uncorrelated and randomly distributed according to a uniform probability distribution. This assumption, based on the prediction of inflationary models, leads by the Central Limit Theorem to a Gaussian probability distribution for the field $\delta(x)$.

There are two main advantages of such Gaussian models: First, all the statistical information about the field $\delta(x)$ is encoded in a single function: the two point correlation function (or equivalently the power spectrum). Second, when combined with Cold Dark Matter (CDM), Gaussian models are in reasonable agreement with small and intermediate scale observations (White et. al. 1987). However, observations on large scales (larger than 10$h^{-1}\text{Mpc}$) have consistently indicated that Gaussian CDM models lack power on large scales.

One approach to the resolution of this problem is to retain the Gaussian nature of the primordial perturbations while modifying other aspects of the model in an effort to transfer power to large scales. This has led to the construction of the ‘hybrid models’ which attempt through the introduction of additional parameters (like a component of Hot Dark Matter (HDM)) to reconcile Gaussian models with large scale structure observations.

The other approach, is the consideration of non-Gaussian primordial perturbations. A class of non-Gaussian perturbations which is well motivated physically is seed-like perturbations. These primordial perturbations may be naturally provided by topological defects (e.g. cosmic strings (Kibble 1976; Vilenkin 1981; Brandenberger 1992) or textures (Turok 1989)) produced during phase transitions in the early universe. Other interesting mechanisms (e.g. primordial black holes (Carr & Rees 1984)) can also produce seed-like perturbations. Models based on cosmic strings for example, have been shown to have several interesting features that make them worth of further investigation. After appropriately normalizing the single free parameter of the model, cosmic strings can naturally provide concentrations of galaxies on sheets (string wakes) with typical dimensions $40 \times 40 \times 4h^{-1}\text{Mpc}^3$ (Vachas-
pati 1986; Stebbins et. al. 1987; Perivolaropoulos, Brandenberger & Stebbins 1990; Vollick 1992; Hara & Miyoshi 1993), they are consistent with the recent detection of anisotropy by COBE (Bouchet, Bennett & Stebbins 1988; Bennett, Stebbins & Bouchet 1992; Perivolaropoulos 1993a), and they are in reasonable agreement with observations of peculiar velocities regarding measurements of the Cosmic Mach Number (Perivolaropoulos & Vachaspati 1993). However, like the CDM model, the cosmic string model is not free from problems. As pointed out by Albrecht & Stebbins (1992a) the power spectrum of density fluctuations produced by cosmic strings in a universe consisting mostly of CDM appears to have too much power on small scales. This problem was shown to be resolved however, if CDM is substituted by HDM (Albrecht & Stebbins 1992b). In addition, Perivolaropoulos & Vachaspati (1993) have recently pointed out that cosmic strings can not explain the observed magnitude of peculiar velocity flows on scales larger than $50h^{-1}\text{Mpc}$ if normalized from peculiar velocity observations on scales $5 - 20h^{-1}\text{Mpc}$. This problem however, also appears in the CDM model and may be resolved by assuming velocity bias.

Perturbations in seed-based models may be represented as a superposition of localized fluctuations with geometry that depends on the model under consideration. The most sensitive way to distinguish observationally models based on Gaussian perturbations from models based on seeds is by investigating the statistical properties of the perturbations. In fact, interesting statistical tests have been proposed that attempt to provide ways to efficiently make this distinction (Coles 1988; Lucchin, Matarrese & Vittorio 1988; Scherrer, Melott & Shandarin 1991; Gaztanaga & Yokoyama 1992; Luo & Schramm 1992; Perivolaropoulos 1993b).

However, there are two main problems that tend to decrease the sensitivity of these tests. The first comes from the Central Limit Theorem which predicts that as the number of superimposed seeds increases, the resulting perturbations look more like Gaussian. The second comes from the finite resolution of observational experiments. Observations effectively average over patches in the sky and associate with each patch a measurement that may correspond to a temperature, a density or a velocity field. By the Central Limit Theorem such averaging tends to reduce the non-Gaussian signature of seed based models.

The statistical test we discuss in this paper is an attempt to evade the second problem. By studying the statistical properties of Fourier modes we
can isolate the effects of low resolution, to high wavenumber $k$ Fourier modes and focus on the low $k$ modes that remain unaffected by the smoothing on small scales.

In what follows we consider perturbation patterns produced by a random superposition of $N$ identical seed perturbations and derive the probability distribution and moment generating function of the Fourier modes that correspond to the pattern. The pattern of perturbations investigated here is known in the literature as ‘shot noise’ (Campbell 1909; Rice 1944) and appears in several and diverse problems. The statistical properties of shot noise have been studied previously (Rice 1944) mainly in coordinate space and in the large $N$ limit, showing strong Gaussian behavior. In the present analysis we focus instead on the statistical properties in Fourier space. The results presented here are fairly general in that they are valid for any value of $N$ and any shape of the superimposed seeds. For simplicity we focus on the one dimensional case but we show that the analysis can be easily generalized to higher dimensions.

2 The Large $N$ Limit

Consider the random function

$$f(x) = \sum_{n=1}^{N} f_1(x - x_n)$$ (1)

where $f_1(x)$ is a seed function superimposed randomly at positions $x_n$ such that $-l \leq x_n \leq l$. Both $f(x)$ and $f_1(x)$ are defined within the interval $[−l, l]$ and periodic boundary conditions are used during the superposition. The Fourier expansion of $f_1(x)$ is:

$$f_1(x) = \sum_{k=-\infty}^{+\infty} g_1(k)e^{ikx}$$ (2)

with

$$g_1(k) = \frac{1}{2l} \int_{-l}^{+l} dx f_1(x)e^{-ikx}$$ (3)
The fact that $f_1(x)$ is real implies that $g_1^*(k) = g_1(-k)$. Using (1) and (2) the random function $f(x)$ can be expanded as:

$$f(x) = \sum_{k=-\infty}^{+\infty} g_1(k) e^{ik\pi l} Q(k)$$  \hspace{1cm} (4)$$

with

$$Q(k) = \sum_{n=1}^{N} e^{-ik\pi x_n} \equiv q_1(k; x_1...x_N) + iq_2(k; x_1...x_N)$$  \hspace{1cm} (5)$$

Thus all the random properties of $f(x)$ have been transferred to $Q(k)$ which is independent of the shape of the seed function $f_1(x)$ and can be viewed as the final position of a $N$ step random walk in the two dimensional $q_1 - q_2$ plane. Notice that the reality condition $Q^*(k) = Q(-k)$ which is trivially satisfied implies that the end points of random walks with negative $k$ are simply obtained by reflection with respect to the $q_1$ axis of the corresponding positive $k$ end points. We are interested in the joint probability distribution $P(q_1, q_2)$ and the corresponding moment generating function.

The general case of arbitrary $N$ is treated in the next section. Here we study the special case $N \to \infty$ for which there are results available in the literature (Rice 1944). For $N \to \infty$ it is easy to show that the random variables $q_1, q_2$ become independent i.e.

$$P(q_1, q_2) = P(q_1)P(q_2)$$  \hspace{1cm} (6)$$

In the same limit, the Central Limit Theorem implies that both $q_1$ and $q_2$ (being sums of identically distributed random variables) are distributed according to the Gaussian

$$P(q_i) \to \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right) e^{-\frac{(q_i - \mu)^2}{2\sigma^2}}$$  \hspace{1cm} (7)$$

where $\mu = < q_i >$ and $\sigma^2 = < q_i^2 >$ ($i = 1, 2$). Since the probability distribution of $x_n$ is uniform in the interval $[-l, l]$ we can find $\mu$ and $\sigma^2$ as:

$$\mu = < q_1 > = \frac{1}{2l} l \int_{-l}^{l} dx_1...dx_N \left( \sum_{n=1}^{N} \cos \frac{k\pi}{l} x_n \right) = N\delta_{k0}$$  \hspace{1cm} (8)$$

and

$$\sigma^2 = < q_1^2 >= \frac{N}{2}$$  \hspace{1cm} (9)$$
Similar results are also easily shown to hold for \( q_2 \). Thus for \( k \neq 0 \) we find

\[
P(q_1, q_2) \rightarrow \frac{1}{\pi N} e^{-\frac{q^2}{N}}
\]

where \( q^2 \equiv q_1^2 + q_2^2 \). Clearly, the probability distribution is independent of the phase \( \phi_k \equiv \tan^{-1}(\frac{q_1}{q_2}) \) of the Fourier modes. Therefore, for \( N \rightarrow \infty \) the Fourier phases are distributed uniformly while the Fourier mode magnitude \( q(k) \) has a Gaussian distribution. It may be easily seen by visualizing the random walk \( Q(k) \) that the probability distribution of \( \phi_k \) will in fact be uniform for any \( N \). However, the rest of the results of this section are not valid for finite \( N \) since the independence of the variables \( q_1, q_2 \) (expressed through (6)) breaks down in that case. This will be shown rigorously in the following section.

### 3 Arbitrary \( N \)

The Fourier transform of \( P(q_1, q_2) \) may be written for any \( N \) as:

\[
\tilde{P}(p_1, p_2) = \int_{-N}^{+N} dq_1 dq_2 P(q_1, q_2) e^{i \frac{p_1}{N} q_1} e^{i \frac{p_2}{N} q_2}
\]

which implies that

\[
P(q_1, q_2) = \left( \frac{1}{2N} \right)^2 \sum_{p_1, p_2 = -\infty}^{+\infty} \tilde{P}(p_1, p_2) e^{-i \frac{p_1}{N} q_1} e^{-i \frac{p_2}{N} q_2}
\]

where \( p_1, p_2 \) are integer variables, Fourier conjugate to \( q_1, q_2 \). By inspection of (11) it becomes clear that \( \tilde{P}(p_1, p_2) \) is also the moment generating function for the distribution \( P(q_1, q_2) \). In fact, it is easy to see that

\[
< q_1^m q_2^n > = \left( \frac{i\pi}{N} \right)^{-(n+m)} \frac{\partial^{n+m} \tilde{P}(p_1, p_2)}{\partial p_1^m \partial p_2^n} \big|_{p_1 = p_2 = 0}
\]

But the same moments are also generated by the function

\[
R(p_1, p_2) = \left( \frac{1}{2l} \right)^N \int_{-l}^{+l} dx_1 \ldots dx_N e^{i \frac{p_1}{N} q_1(k; x_1 \ldots x_N)} e^{i \frac{p_2}{N} q_2(k; x_1 \ldots x_N)}
\]
since \( \frac{1}{2l} \) is the probability that \( x_n \) will be in the range \([x_n, x_n + dx_n]\). It is easy to check that (13) also holds with \( \bar{P}(p_1, p_2) \) substituted by \( R(p_1, p_2) \). Since the moment generating function that corresponds to a given probability distribution is uniquely defined (Feller 1971) we must have

\[
\bar{P}(p_1, p_2) = R(p_1, p_2)
\]  \hspace{1cm} (15)

By expanding \( q_1(k; x_1...x_N) \) and \( q_2(k; x_1...x_N) \) according to (5) we obtain using (14) and (15)

\[
\bar{P}(p_1, p_2) = \left( \frac{1}{2\pi k} \int_{-k\pi}^{k\pi} d\xi e^{it_1 \cos \xi + t_2 \sin \xi} \right)^N
\]  \hspace{1cm} (16)

where \( t_i \equiv p_i \frac{\pi}{N} \) \((i = 1, 2) \) and \( \xi = \frac{k\pi x_l}{N} \). Let now

\[
t_1 = t \cos \delta \hspace{1cm} t_2 = t \sin \delta
\]  \hspace{1cm} (17)

Using the periodicity of \( \cos \xi \) and the fact that \( k \) is integer (16) becomes

\[
\bar{P}(p_1, p_2) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} d\xi e^{it \cos \xi} \right)^N
\]  \hspace{1cm} (19)

or

\[
\bar{P}(p_1, p_2) = (J_0(t))^N
\]  \hspace{1cm} (20)

where

\[
t = \left( \frac{\pi}{N} \right) \sqrt{p_1^2 + p_2^2} \equiv \left( \frac{\pi}{N} \right) p
\]  \hspace{1cm} (21)

The generating function (20) is one of the central results of this paper. It is valid for any \( N \) and clearly depends only on the magnitude \( p \) of the vector \((p_1, p_2)\). This implies that its Fourier transform \( P(q_1, q_2) \) will similarly be a function of the magnitude \( q \) only and there will be no dependence on the phase of the vector \((q_1, q_2)\). Thus, the Fourier phases of seed induced perturbations obey a uniform distribution for any number \( N \) of superimposed seeds. Obviously, this statement applies to each mode \( k \) individually and does not imply that there will be no correlations among the phases of different modes. Such correlations will clearly exist for seed perturbations but are not the subject of the present study.
Our results can explain the numerical simulations of Suginohara & Suto (1991) where it was found that even in strongly non-Gaussian evolved density fields the phases $\phi_k$ are uniformly distributed. The authors of that paper had concluded that the investigation of the distribution function of the phases $\phi_k$ does not provide a sensitive test of the non-Gaussian behavior in the strongly non-linear regime but no clear explanation was given of this fact. Since the density field in the non-linear regime can be viewed as a superposition of dense lumps (seeds), the above analysis is applicable and predicts exactly the uniform distribution of phases seen in the simulations of Suginohara & Suto (1991).

The probability distribution $P(q_1, q_2) = P(q)$ is obtained by Fourier transforming the generating function (20) as follows:

$$P(q) = \left(\frac{1}{2N}\right)^2 \sum_{p_1, p_2=-\infty}^{+\infty} (J_0(t(p_1, p_2)))^N e^{i\vec{q}\cdot\vec{t}}$$

(22)

where $\vec{t} = \frac{\pi}{N}(p_1, p_2)$ and $\vec{q} = (q_1, q_2)$. For $N > 1$ we may approximate the sum (22) by an integral and reduce it to

$$P(q) = \frac{1}{2\pi} \int_0^{\infty} dp \ p(J_0(p))^N J_0(pq)$$

(23)

It is straightforward to verify that for $N > 1$

$$\int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 P(q_1, q_2) = \int_0^{\infty} dp p(J_0(p))^N \int_0^{\infty} dq J_0(pq) = 1$$

(24)

which is to be expected since $P(q_1, q_2)$ is a probability distribution.

Directly measurable quantities from a given fluctuation pattern are the moments of the fluctuation probability distribution. The moments generated by the function (20) can easily be obtained and compared with the moments of the Gaussian probability distribution. Since $\vec{P}(p_1, p_2)$ depends only on the magnitude $p$, it is easy to show using (13) that for any positive integers $m$ and $N$ we have

$$< q_1^m > = < q_2^m > = \frac{\partial^m (J_0(t/i))^N}{\partial t^m} \bigg|_{t=0}$$

(25)

By expanding $(J_0(t/i))^N$ in powers of $t$ we obtain

$$(J_0(t/i))^N = 1 + \frac{(t\sqrt{N/2})^2}{2} + (t\sqrt{N/2})^4 \left(\frac{1}{8} - \frac{1}{16N}\right) + \ldots$$

(26)
From (25) and (26) it may be shown that the moments of the appropriately normalized variables $r_i \equiv \frac{q_i}{\sqrt{N/2}}$ ($i = 1, 2$) for $k \neq 0$ are

\begin{align*}
<r_i^{2m+1}> &= 0 \
<r_i^2> &= 1 \
<r_i^4> &= 3(1 - \frac{1}{2N})
\end{align*}

The kurtosis (defined as $(<r_i^4> - 3)$) is negative for all finite $N$ and approaches the Gaussian value 0 for large $N$. Also, the skewness $<r_i^3>$ is 0 for all $N$. By expanding the generating function further, the higher moments may also be obtained. The negative sign of the kurtosis is to be contrasted with the corresponding sign of the kurtosis of seed perturbations in coordinate space where several cases of interest have been shown to have positive kurtosis (Scherrer & Bertschinger 1991; Luo & Schramm 1992; Perivolaropoulos 1993b).

By numerically evaluating the integral (23) we plot the probability distribution $P(q)$ and compare it with the Gaussian. This is shown in Figure 1 (dotted line) for $N = 10$. The corresponding Gaussian distribution with the same standard deviation (obtained from (10) with $N = 10$) is also shown for comparison (continuous line).

It is of interest to obtain the generating function for the moments of the normalized variables $r_i$. This is easily shown to be

$$\tilde{P}(t_1, t_2) = (J_0(\frac{t}{i\sqrt{N/2}}))^N = (1 + \frac{t^2}{2N} + ...)^N \rightarrow e^{\frac{t^2}{2}}$$

where the limit, indicated by the arrow, leading to the standard Gaussian generating function $e^{\frac{t^2}{2}}$, is obtained for $N \gg 1$. Thus, the generating function approaches, as expected, the Gaussian for large $N$.

Let us demonstrate the utility of these results in a somewhat realistic case. Consider an one dimensional pattern of fluctuations in Fourier space. In a realistic case, these fluctuations will be a superposition of a Gaussian noise random variable $q_n$ and a signal $q_s$. Let the signal to noise ratio be

$$\gamma \equiv \frac{<q_s^2>}{<q_n^2>}$$
assumed known. The measured variable at each pixel is \( q = q_n + q_s \). We want to test the hypothesis that \( q_s \) is produced by a superposition on \( N \) seeds. The moment generating functions for the variables \( q_n \) and \( q_s \) are:

\[
M_{q_n}(t) = e^{\alpha t^2/2}
\]

and

\[
M_{q_s}(t) = (J_0(t/i))^{N}
\]

Therefore \( \gamma = \frac{\langle q_s^2 \rangle}{\langle q_n^2 \rangle} = \frac{N}{2\alpha} \). Since the variables \( q_n \) and \( q_s \) are independent, the moment generating function for the measured variable \( q \) is:

\[
M_q(t) = (J_0(t/i))^{N} e^{\alpha t^2/2}
\]

It is now straightforward to expand \( M_q(t) \) and thus obtain the kurtosis \( \kappa \) for the random variable \( q \):

\[
\kappa = \frac{\langle q^4 \rangle}{\langle q^2 \rangle^2} = 3(1 - \frac{\gamma^2}{2(1 + \gamma)^2 N})
\]

The kurtosis \( \kappa \) is measurable, and any constraint on it can be translated using (35) to a constraint on \( N \), the number of superimposed seeds on the pattern under consideration. Given that different seed-based models predict widely different number of seeds per Hubble scale (according to simulations, there are 0.04 textures unwinding per Hubble volume per Hubble time while the corresponding number for long strings is about 10), this test can be used to rule in favour of a particular seed-based model or, if \( N \) is found too large to rule out such models. For example, the number of textures predicted to have unwound in 10 \( \times \) 10 degree MBR sky map between the time of recombination and today is less than 8. In fact, if reionisation is realized, as required by the texture model (Turok & Spergel 1990), this number will be much less than 8. This implies that the proposed test may be efficiently used in this case since the predicted value of the kurtosis can be smaller by more than 10% compared to the Gaussian for \( \gamma \approx 1 \).

Expressions similar to (35) may be easily obtained for higher moments of \( q \). Using such expressions the proposed test can be applied even in cases where the signal to noise ratio \( \gamma \) is not known, by using the measured constraints on higher moments of \( q \) to eliminate \( \gamma \).
It is straightforward to generalize the above analysis to higher dimensional cases. In fact we will show that the form of the generating function is the same in any number of dimensions. We will demonstrate the three dimensional case, applicable to large scale structure considerations. The two dimensional case corresponding to the MBR follows trivially from the three dimensional analysis.

Consider a three dimensional rectangular area with coordinates $\vec{x} = (x_1, x_2, x_3)$ such that $-l \leq x_i \leq l$ ($i = 1, 2, 3$). In this case the wavenumber $k$ becomes $\vec{k} = (k_1, k_2, k_3)$ and using the same analysis as in the one dimensional case it can be shown that

$$\bar{P}(p_1, p_2) = R(p_1, p_2) = \left(\frac{1}{2l}\right)^3 \int_{-l}^{+l} dx_1 \int_{-l}^{+l} dx_2 \int_{-l}^{+l} dx_3 e^{i(t_1 \cos(\vec{k}\vec{x}) + t_2 \sin(\vec{k}\vec{x}))} N$$

Define now

$$\xi_1 \equiv \frac{\pi}{l}(k_1 x_1 + k_2 x_2 + k_3 x_3)$$
$$\xi_2 \equiv \frac{\pi}{l} k_2 x_2$$
$$\xi_3 \equiv \frac{\pi}{l} k_3 x_3$$

A change of variables from $(x_1, x_2, x_3)$ to $(\xi_1, \xi_2, \xi_3)$ leads to

$$\bar{P}(p_1, p_2) = \left(\frac{1}{2\pi}\right)^3 \frac{1}{k_1 k_2 k_3} \int_{-k_3\pi}^{+k_3\pi} d\xi_3 \int_{-k_2\pi}^{+k_2\pi} d\xi_2 \int_{-k_1\pi + \xi_2 + \xi_3}^{+k_1\pi + \xi_2 + \xi_3} d\xi_1 e^{i(t_1 \cos \xi_1 + t_2 \sin \xi_1)}$$

which leads to a result identical to the one dimensional result (19) since $k_1, k_2, k_3$ are integers. It is trivial to see that the same is true for the two dimensional case.

4 Discussion-The Power Spectrum

So far we have investigated the statistical properties of the random function $Q(k) = q_1(k) + iq_2(k)$ which is only part of the Fourier modes of the perturbations. In fact we are interested in the full mode function

$$g(k) \equiv g_1(k)Q(k)$$
Since the only random part of $g(k)$ is $Q(k)$, the statistical properties of $g(k)$ are fully specified once we know such properties for $Q(k)$. For example, the power spectrum of perturbations defined as

$$P(k) = \langle |g(k)|^2 \rangle$$  \hspace{1cm} (42)

is easily found using (29) to be

$$P(k) = |g_1(k)|^2 < q^2(k) >= N|g_1(k)|^2$$  \hspace{1cm} (43)

for $k \neq 0$ (obviously $P(0) = N^2|g_1(k)|^2$ since $Q(0) = N$). It is possible to obtain the same result for the power spectrum by simply Fourier transforming the two point correlation function in coordinate space (Rice 1944; Scherrer & Bertschinger 1991).

In conclusion, we have proposed a new statistical test for the identification of signatures of seed-based models in cosmological observations. The main advantages of investigating the statistics of perturbations in Fourier space rather than in coordinate space is that in Fourier space the analysis is valid for any geometry of superimposed seeds and can be directly applied to any particular experiment by simply selecting the Fourier modes for which the resolution and sky coverage of the experiment is most sensitive. No smoothing is needed as would be the case for coordinate space statistics.

The statistical properties of the seed-like perturbations were shown to approach the Gaussian for a large number $N$ of superimposed seeds. Thus, these properties can only distinguish efficiently models where the perturbations are produced by a small number of seeds per horizon scale. For an alternative statistic which can efficiently distinguish particular seed based models for larger $N$ see Moessner, Perivolaropoulos & Brandenberger (1993).

So far we have considered superposition of identical (but of any shape) seeds. However, our results can be easily generalized to variable seed magnitude and extend in space. Such generalizations are shown in the Appendix.

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Appendix A

Generalizations

An interesting generalization of our results can be provided by considering seeds of variable size. For example the size of perturbations induced by topological defects increases with cosmic time due to the growth of the horizon. Such effect can be taken into account by generalizing the Fourier space variable $g_1(k)Q(k)$ to

$$\sum_{j=0}^{M} g_1(2^j k)Q(2^j k)$$

which corresponds to repeating the superposition of $N$ seeds, $M$ times while each time modifying the spatial scale of each seed by a factor of 2 in order to take into account the horizon growth (Vachaspati 1992; Perivolaropoulos 1993; Moessner, Perivolaropoulos & Brandenberger 1993).

In this case the moment generating function $P_{sum}(p_1, p_2)$ for the sum of random variables is given (Feller 1971) by the product of the individual generating functions i.e.

$$\bar{P}_{sum}(p_1, p_2) = \prod_{j=0}^{M} (J_0(|g_1(2^j k)|t))^N$$

where the factor $|g_1(2^j k)|$ appears because we are now interested in the distribution of the variable (44) as opposed to simply the variable $Q(k)$.

Finally, it is also straightforward to generalize our analysis to the case of seeds of variable magnitude. Such generalization would be needed in order to take into account the variable velocities of long cosmic strings. Consider for example the superposition of $N$ seeds with Fourier transforms $\lambda_i g_1(k)$ ($i = 1, \ldots, N$) where the coefficients $\lambda_i$ represent the relative magnitude of seed fluctuations. In this case the Fourier mode $k$ becomes

$$g_1(k) \sum_{n=1}^{N} \lambda_n e^{i \frac{2\pi n}{k}} = g_1(k)Q_\lambda(k)$$

and the generating function for the variable $Q_\lambda(k)$ is

$$\bar{P}_\lambda(p_1, p_2) = \prod_{j=1}^{N} J_0(\frac{\lambda_j t}{k})$$
The above discussion is an attempt to show that our results are fairly general and can be easily adapted to the cases of particular seed-based models. Clearly, further work is needed to adapt the above analysis to any particular model. Work in that direction for the cosmic string case is currently in progress.

**Figure Captions**

**Figure 1:** A comparison of the Gaussian (continuous line) with $P(q)$ for $N = 10$. 
References

Albrecht A. & Stebbins A. 1992a Phys. Rev. Lett. 68, 2121. Albrecht A. & Stebbins A. 1992b Phys. Rev. Lett. 69, 2615. Bennett D. P., Stebbins A. & Bouchet F. R. 1992. Ap. J. Lett. 399, L5.
Bouchet F. R., Bennett D. P. & Stebbins A. 1988. Nature 335, 410.
Brandenberger R. 1992. ‘Topological Defect Models of Structure Formation after the COBE discovery of CMB Anisotropies’, (Invited Talk at Erice Course, Sep. 1992) BROWN-HET-881.
Campbell N. 1909. Proc. Cambridge Phil. Soc. 15 117.
Carr B. J. & Rees M. J. 1984. MNRAS 206 801.
Coles P. 1988. M.N.R.A.S 234, 509.
Feller W. 1971. ‘An Introduction to Probability Theory and its Applications’, New York: Willey.
Gaztanaga E., Yokoyama J. 1992 ‘Probing the Statistics of Primordial Fluctuations and its Evolution’, Fermilab preprint, PUB-92-71-A.
Hara T. & Miyoshi S. 1993, Ap. J. 405, 419. Kibble T. W. B. 1976, J.Phys. A9, 1387.
Lucchin F., Matarrese S. & Vittorio N. 1988. Ap. J. 330, L21.
Luo X., Schramm D. 1992. ‘Kurtosis, Skewness and non-Gaussian Cosmological Density Perturbations’ Fermilab preprint PUB-92-214-A.
Moessner R., Perivolaropoulos L. & Brandenberger R., 1993. Ap. J. in press.
Perivolaropoulos L. 1993a. Phys. Lett. B298, 305.
Perivolaropoulos L. 1993b. Phys. Rev. D48, 1530.
Perivolaropoulos L., Brandenberger R. & Stebbins A. 1990 Phys. Rev. D41, 1764.
Perivolaropoulos L. & Vachaspati T. 1993. submitted to Ap. J. Lett..
Rice S. 1944. Bell System Tech J. 23 282.
Scherrer R. J. & Bertschinger E. 1991. Ap. J. 381 349.
Scherrer R. J., Melott A. & Shandarin S. 1991. Ap. J. 377 29.
Stebbins A., Veeraraghavan S., Brandenberger R., Silk J. & Turok N. 1987 Ap. J. 322 1.
Suginohara T. & Suto Y. 1991. Ap. J. 371 470.
Turok N. 1989. Phys. Rev. Lett. 63 2625.
Turok N. & Spergel D. 1990. Phys. Rev. Lett. 64 2736.
Vachaspati T. 1986 Phys. Rev. Lett 57 1655.
Vachaspati T. 1992. *Phys. Lett* B282, 305.
Vilenkin A. 1981. *Phys. Rev. Lett.* 46 1169.
Vollick D. N. 1992, Ap. J. 397, 14
White S., Davis M., Efstathiou G., Frenk C. 1987. *Nature* 330, 451.
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