EFFICIENCY MEASURES IN FUZZY DATA ENVELOPMENT ANALYSIS WITH COMMON WEIGHTS

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ABSTRACT. This work considers providing a common base for measuring the relative efficiency for all the decision-making units (DMUs) with multiple fuzzy inputs and outputs under the fuzzy data envelopment analysis (DEA) framework. It is shown that the fuzzy DEA model with common weights can be reduced into an auxiliary bi-objective fuzzy optimization problem by considering the most and the least favorable conditions simultaneously. An algorithm with the implementation issue for finding the compromise solution of the fuzzy DEA program is developed. A numerical example is included for illustration and comparison purpose. Our results show that the proposed approach is able to provide decision makers the flexibility in measuring the relative efficiency for DMUs with fuzzy inputs and outputs, which not only differentiates efficient units on a common base but also detects some abnormal efficiency scores calculated from other existing methods.

1. Introduction. Data envelopment analysis (DEA) has been widely applied for measuring the relative efficiencies of a set of decision making units (DMUs) that use multiple inputs to produce multiple outputs. Crisp input and output data are fundamentally indispensable in conventional DEA [5]. However, in a number of applications, the observed values of the input and output data are sometimes imprecise or vague due to the result of measurement inaccuracies, unquantifiable and incomplete. To deal with the imprecise and ambiguous data in DEA, various fuzzy DEA methods have been reported.

Fuzzy DEA was originally proposed by Sengupta [19], in which the fuzzy goal-oriented and constraint-based technique was considered. Kao and Liu [9] developed...
a procedure to measure the efficiencies of DMUs with fuzzy observations. It adopted the fuzzy measurement concept and the extension principle to transform the fuzzy DEA model into a traditional DEA model with parameters of the level $\alpha$. Subsequently, Saati et al. \[18\] and Lertworasirikul et al. \[13, 14\] proposed respectively the fuzzy CCR model with asymmetric triangular fuzzy numbers, and the fuzzy BCC model that uses probability to conduct analysis. They used $\alpha$-cut to transform the fuzzy DEA model into a linear structure model.

As similar to conventional DEA methods, the fuzzy DEA program essentially selects the weights attached to the inputs and outputs for each individual DMU and classifies all DMUs into two groups, namely, efficient and inefficient \[22\] \[4\]. As a considerable number of DMUs are usually categorized as efficient, a procedure for ranking the efficient units is sometimes necessary. Using different sets of weights to classify the DMUs as efficient or inefficient is acceptable to the practitioners. However, if different sets of weights are used for ranking, most practitioners may not agree because every DMU believes that other DMUs will take this advantage to defeat it \[8\]. To provide a common base for measuring the DMUs, both the efficient and inefficient ones, the idea of generating common weights in DEA was first introduced by Roll et al. \[16\]. In this paper we consider providing a common base for ranking the DMUs having multiple fuzzy inputs and outputs. Suppose there are $m$ fuzzy inputs, $s$ fuzzy outputs and $n$ DMUs being evaluated. Denote $\tilde{x}_{i,j}$ as the $i$-th fuzzy input and $\tilde{y}_{r,j}$ as the $r$-th fuzzy output of the $j$-th DMU. The fuzzy DEA model with common weights can be formulated as the following multiple objective programming problem:

$$
\text{max } \tilde{E}(u, v) = \left( \sum_{r=1}^{s} \tilde{y}_{r1} u_r / \sum_{i=1}^{m} \tilde{x}_{i1} v_i, \sum_{r=1}^{s} \tilde{y}_{r2} u_r / \sum_{i=1}^{m} \tilde{x}_{i2} v_i, \ldots \right),
$$

$$
\sum_{r=1}^{s} \tilde{y}_{rn} u_r / \sum_{i=1}^{m} \tilde{x}_{in} v_i
$$

s.t. \[ \sum_{r=1}^{s} \tilde{y}_{rj} u_r / \sum_{i=1}^{m} \tilde{x}_{ij} v_i \leq 1, j = 1, 2, \ldots, n \]

$$
u_r, v_i \geq \varepsilon > 0, r = 1, 2, \ldots, i = 1, 2, \ldots, m,$$

where $u = (u_1, u_2, \ldots, u_s)^T$ and $v = (v_1, v_2, \ldots, v_m)^T$ are the weight vectors, $\tilde{E}(u, v)$ is the efficiency vector calculated from the common weights $(u, v)$, and $\varepsilon$ is a small non-Archimedean quantity \[1\].

It is well known that many decision making problems have multiple objectives which cannot be optimized simultaneously due to the inherent incommensurability and conflict among these objectives. Thus, making a trade off between these objectives becomes a major subject of finding the “best compromise” solution. A variety of methodologies for solving the multi-objective decision making (MODM) problems have been proposed \[10\] \[11\] \[15\] \[17\] \[21\] \[24\] \[25\]. Among them, the goal programming and global criterion methods are the popular approaches. These methods consider only one criterion based on the shortest distance from the given goal or the positive ideal solution. However, in practice, such single criterion may not be enough for a decision maker \[6\]. Instead, the technique for order of preference by similarity to ideal solution (TOPSIS) was first developed by Hwang and Yoon \[7\] to solve a multiple attribute decision making (MADM) problem. It provides the principle of compromise saying that the chosen alternative should have “the shortest distance from the positive ideal solution” and “the farthest distance from the negative ideal
solution.” In this paper the basic principle of compromise of TOPSIS is applied for solving the fuzzy DEA model with common weights \(^{(1)}\). An algorithm with the implementation issue for finding the compromise solution of the fuzzy DEA program \(^{(1)}\) is developed.

The rest of the paper is organized as follows. In Section 2, the compromise solution approach for solving the fuzzy DEA model with common weights \(^{(1)}\) is presented. A solution algorithm and the implementation issue on the algorithm is provided in Section 3. A numerical example is included for illustration and comparison purpose in Section 4. The paper is concluded in Section 5.

2. Solving the fuzzy DEA model with common weights. To solve the fuzzy DEA model \(^{(1)}\), we adopt the principle of compromise, i.e., the chosen solution should have “the shortest distance from the positive ideal solution” and “the farthest distance from the negative ideal solution.”

To define the positive ideal solution and negative ideal solution of the problem \(^{(1)}\), for each \(j, j = 1, 2, \ldots, n\), we consider

\[
E^*_j = \max_s \hat{E}_j(u, v)
\]

s.t. \[
\begin{align*}
\sum_{r=1}^s \hat{y}_{rj} u_r / \sum_{i=1}^m \hat{x}_{ij} v_i & \leq 1, j = 1, 2, \ldots, n \\
u_r, v_i & \geq \varepsilon > 0, r = 1, 2, \ldots, i = 1, 2, \ldots, m,
\end{align*}
\]  

(2)

and

\[
E^-_j = \min_s \hat{E}_j(u, v)
\]

s.t. \[
\begin{align*}
\sum_{r=1}^s \hat{y}_{rj} u_r / \sum_{i=1}^m \hat{x}_{ij} v_i & \leq 1, j = 1, 2, \ldots, n \\
u_r, v_i & \geq \varepsilon > 0, r = 1, 2, \ldots, i = 1, 2, \ldots, m,
\end{align*}
\]  

(3)

where \(\hat{E}_j(u, v) = \sum_{r=1}^s \hat{y}_{rj} u_r / \sum_{i=1}^m \hat{x}_{ij} v_i\). The problem (2) can be regarded as the CCR model with fuzzy inputs and outputs. Let \(E^* = (E^*_1, E^*_2, \ldots, E^*_n)^T \in \mathbb{R}^n\) be the solution vector of (2) which consists of individual best feasible solutions for all objectives. \(E^*\) is then called the positive ideal solution (PIS). Similarly, let \(E^- = (E^-_1, E^-_2, \ldots, E^-_n)^T \in \mathbb{R}^n\) be the solution vector of (3) which consists of individual worst feasible solutions for all objectives. \(E^-\) is then called the negative ideal solution (NIS).

To measure the distances from PIS and NIS to all objectives, the Minkowski’s \(L_p\)-metric is employed, i.e., the distance between \(\hat{E}_j(u, v)\) and \(E^*_j\) (or \(E^-_j\)), \(j = 1, 2, \ldots, n\), is defined by the \(L_p\)-norm with \(p \geq 1\). Moreover, because of the incommensurability among objectives, the component distance from PIS or NIS for each objective is normalized. The following distance functions are then considered:

\[
d_{p}^{PIS}(u, v) = \left\{ \sum_{j=1}^n w_j^p \left[ E^*_j - \hat{E}_j(u, v) / E^*_j - E^-_j \right]^p \right\}^{1/p},
\]

(4)

and

\[
d_{p}^{NIS}(u, v) = \left\{ \sum_{j=1}^n w_j^p \left[ \hat{E}_j(u, v) - E^-_j / E^*_j - E^-_j \right]^p \right\}^{1/p},
\]

(5)
where \( d_{PIS}^p \) and \( d_{NIS}^p \) are the distances from the PIS and NIS to all objectives, respectively, \( w_j \in [0, 1], j = 1, 2, \ldots, n \), is the relative importance of objective function \( j \), and \( p = 1, 2, \ldots, \infty \) is the parameter of norm functions. It should be emphasized that \( w_j \) indicates the degree of importance of the \( j \)th objective. On the other hand, the parameter \( p \) plays the role of the “balancing factor” between the distance \( d_p \) and the objects. One property of the distance parameter \( p \) is that when \( p \) increases, the distance \( d_p \) decreases. i.e., \( d_1 \geq d_2 \geq \cdots \geq d_\infty \) \[20, 23\]. The idea for \( p = 1 \) is similar to the robust approach in statistics, which provides better credibility than others in the measuring concept and emphasizes the sum of individual distances in the utility concept \[6\] \[22\]. The idea for \( p = \infty \) is related to the popular method of least-square and seems to be more acceptable. And, the case of \( p = \infty \) provides more harmonious achieved rates for each objective as discussed in \[6\]. Among all \( p \) values, we will present the cases of \( p = 1, 2 \), and \( \infty \) which are operationally and practically important, and even a well-known standard in the fields of multi-criteria decision making and control theory. Further behavioral interpretations of \( d_p \) on the compromise approach can be found in \[20\] and \[23\].

To approximately minimize \( d_{PIS}^p \), and approximately maximize \( d_{NIS}^p \), the fuzzy DEA model \[1\] can be converted to the following fuzzy bi-objective nonlinear programming problem:

\[
\min_{u,v} \quad \mu_1(u,v) = \begin{cases} 
1, & \text{if } d_{PIS}^p(u,v) < (d_{PIS}^p)^* \\
\frac{(d_{PIS}^p)^* - d_{PIS}^p(u,v)}{(d_{PIS}^p)^* - (d_{PIS}^p)^*}, & \text{if } (d_{PIS}^p)^* \leq d_{PIS}^p(u,v) \leq (d_{PIS}^p)^* \\
0, & \text{if } d_{PIS}^p(u,v) > (d_{PIS}^p)^* 
\end{cases}
\]

(7)

\[
\mu_2(u,v) = \begin{cases} 
1, & \text{if } d_{NIS}^p(u,v) > (d_{NIS}^p)^* \\
\frac{(d_{NIS}^p)^* - d_{NIS}^p(u,v)}{(d_{NIS}^p)^* - (d_{NIS}^p)^*}, & \text{if } (d_{NIS}^p)^* \leq d_{NIS}^p(u,v) \leq (d_{NIS}^p)^* \\
0, & \text{if } d_{NIS}^p(u,v) < (d_{NIS}^p)^* 
\end{cases}
\]

(8)

where

\[
(d_{PIS}^p)^* = \min_{u,v} \quad d_{PIS}^p(u,v)
\]

\[
\text{s.t.} \quad \sum_{r=1}^{\infty} \frac{\tilde{y}_{rj}u_r}{\sum_{i=1}^{n} \tilde{x}_{ij}v_i} \leq 1, j = 1, 2, \cdots, n
\]

(9)

and the solution is \((u,v)^{PIS}\),
\[(d_p^{NIS})^* = \max d_p^{NIS}(u, v)\]
\[
\text{s.t. } \sum_{r=1}^s \tilde{y}_{rj} u_r / \sum_{i=1}^m \tilde{x}_{ij} v_i \leq 1, j = 1, 2, \ldots, n
\]
\[
u_r, v_i \geq \varepsilon > 0, r = 1, 2, \ldots, s, i = 1, 2, \ldots, m.
\]
and the solution is \((u, v)^{NIS}\), \((d_p^{PIS})' = d_p^{PIS}((u, v)^{NIS})\), \((d_p^{NIS})' = d_p^{NIS}((u, v)^{PIS})\).

Let a fuzzy decision \(\tilde{D}\) of the problem (6) be defined as the fuzzy set resulting from the intersection of fuzzy objectives with a corresponding membership function

\[
\mu_{\tilde{D}}(u, v) = \min\{\mu_1(u, v), \mu_2(u, v)\}. \quad \text{(13)}
\]

According to reference \([12, 26]\), a solution, say \((u, v)^*\), of the problem (6) with a degree of satisfaction, \(\mu_{\tilde{D}}((u, v)^*)\), can be taken as the solution with the highest membership in the fuzzy decision set \(\tilde{D}\) and obtained by solving the following problem:

\[
\max \min \{\mu_1(u, v), \mu_2(u, v)\}
\]
\[
\text{s.t. } \sum_{r=1}^s \tilde{y}_{rj} u_r / \sum_{i=1}^m \tilde{x}_{ij} v_i \leq 1, j = 1, 2, \ldots, n
\]
\[
u_r, v_i \geq \varepsilon > 0, r = 1, 2, \ldots, s, i = 1, 2, \ldots, m.
\]

Let \(\alpha = \min[\mu_1(u, v), \mu_2(u, v)]\). The problem (14) is equivalent to the following programming problem:

\[
\max \alpha
\]
\[
\text{s.t. } \mu_1(u, v) \geq \alpha
\]
\[
\mu_2(u, v) \geq \alpha
\]
\[
\sum_{r=1}^s \tilde{y}_{rj} u_r / \sum_{i=1}^m \tilde{x}_{ij} v_i \leq 1, j = 1, 2, \ldots, n
\]
\[
u_r, v_i \geq \varepsilon > 0, r = 1, 2, \ldots, s, i = 1, 2, \ldots, m,
\]
where \(\alpha\) is the degree of satisfactory level for both criteria of “as close to PIS as possible” and “as far away from NIS as possible.”

For the special case of \(p = \infty\), we consider the following problem instead of (6):

\[
\min d_p^{PIS}
\]
\[
\max d_p^{NIS}
\]
\[
\text{s.t. } w_j \left[ \frac{\tilde{E}_j^+ - \tilde{E}_j(u, v)}{E_j^+ - E_j^-} \right] \leq d_p^{PIS}, \forall j = 1, 2, \ldots, n,
\]
\[
w_j \left[ \frac{\tilde{E}_j^-(u, v) - E_j^-}{E_j^+ - E_j^-} \right] \geq d_p^{NIS}, \forall j = 1, 2, \ldots, n,
\]
\[
\sum_{r=1}^s \tilde{y}_{rj} u_r / \sum_{i=1}^m \tilde{x}_{ij} v_i \leq 1, j = 1, 2, \ldots, n,
\]
\[
u_r, v_i \geq \varepsilon > 0, r = 1, 2, \ldots, s, i = 1, 2, \ldots, m.
\]
where \( d_{PIS}^\infty \) and \( d_{NIS}^\infty \) are not real distance, but the largest and smallest components of the \( n \)-dimensional distance functions, respectively. Moreover, \((d_{PIS}^\infty)^*\) and \((d_{NIS}^\infty)^*\) can be obtained by solving the following problems:

\[
(d_{PIS}^\infty)^* = \min d_{PIS}^\infty \quad \text{s.t.} \quad w_j \left[ \frac{E^*_j - \tilde{E}_j(u, v)}{E^*_j - E^-_j} \right] \leq d_{PIS}^\infty, \forall \ j = 1, 2, \cdots, n, \\
\sum_{r=1}^s \tilde{y}_{rj} u_r / \sum_{i=1}^m \tilde{x}_{ij} v_i \leq 1, j = 1, 2, \cdots, n \\
u_r, v_i \geq \varepsilon > 0, r = 1, 2, \cdots, s, i = 1, 2, \cdots, m,
\]

and

\[
(d_{NIS}^\infty)^* = \max d_{NIS}^\infty \quad \text{s.t.} \quad w_j \left[ \frac{\tilde{E}_j(u, v) - E^-_j}{\tilde{E}_j - E^-_j} \right] \geq d_{NIS}^\infty, \forall \ j = 1, 2, \cdots, n, \\
\sum_{r=1}^s \tilde{y}_{rj} u_r / \sum_{i=1}^m \tilde{x}_{ij} v_i \leq 1, j = 1, 2, \cdots, n \\
u_r, v_i \geq \varepsilon > 0, r = 1, 2, \cdots, s, i = 1, 2, \cdots, m.
\]

Since \( d_{PIS}^p = 1 - d_{NIS}^p \) for \( p = 1 \), “\( \min d_{PIS}^1 \)” and “\( \max d_{NIS}^1 \)” are subjected to the same constraints and have the same solution whether the weights of the objectives are the same or not. Thus, the compromise solution of the problem (1) can be obtained by either minimizing \( d_{PIS}^1 \) or maximizing problem \( d_{NIS}^1 \).

3. An algorithm and the implementation issue on the algorithm. Based on the discussion in Section 2, an algorithm for solving the fuzzy DEA program (1) can be described as follows.

Algorithm:

**Step 0.1:** Determine the distance parameter \( p \). If the decision maker emphasizes the sum of individual distances, he should choose \( p = 1 \). On the other hand, if harmony between objectives is considered as one of the most important criteria, \( p = \infty \) should be chosen. Beyond both extreme cases, \( p = 2 \) will be chosen. \( p = 2 \) is similar to the popular least-square approach, and provides an approximation for the case \( 2 < p < \infty \) [6].

**Step 0.2:** Provide a pre-assigned satisfactory level \( \bar{\alpha} \) and the relative importance \( w_j \) of the \( n \) objective functions. (There are various methods including the eigenvector, weighted least square, entropy and LINMAP methods for assessing \( w_j \) [7].)

**Step 1:** Determine the positive ideal solution \((E^*)\) and the negative ideal solution \((E^-)\) by solving (2) and (3).

**Step 2:** If \( p = \infty \), go to Step 3; otherwise, solve the problem (6).

**Step 2.1:** Obtain \((d_{PIS}^p)^*\), \((d_{NIS}^p)^*\), \((d_{PIS}^p)'\) and \((d_{NIS}^p)'\) by solving (9)-(12). Go to Step 2.2.

**Step 2.2:** Obtain membership functions \( \mu_1(u, v) \) and \( \mu_2(u, v) \) by calculating (7) and (8). Go to Step 2.3.

**Step 2.3:** Solve the problem (15). Go to Step 4.

**Step 3:** Solve the problem (16) for \( p = \infty \).
Step 3.1: Obtain \((d_{\infty}^{PIS})^*, (d_{\infty}^{NIS})^*, (d_{\infty}^{PIS})', (d_{\infty}^{NIS})'\) by solving (17), (18), (11) and (12). Go to Step 3.2.

Step 3.2: Obtain membership functions \(\mu_1(u,v)\) and \(\mu_2(u,v)\) by calculating (7) and (8). Go to Step 3.3.

Step 3.3: Solve the following problem:

\[
\begin{align*}
\text{max} & \quad \alpha \\
\text{s.t.} & \quad \mu_1(u,v) \geq \alpha \\
& \quad \mu_2(u,v) \geq \alpha \\
& \quad w_j \left[ \frac{E_j^* - E_j(u,v)}{E_j^* - E_j^-} \right] \leq d_{\infty}^{PIS}, \forall j = 1,2,\cdots,n, \\
& \quad w_j \left[ \frac{E_j^- - E_j(u,v)}{E_j^-} \right] \geq d_{\infty}^{NIS}, \forall j = 1,2,\cdots,n, \\
& \quad \sum_{r=1}^s \tilde{y}_{r,j} u_r / \sum_{i=1}^m \tilde{x}_{i,j} v_i \leq 1, j = 1,2,\cdots,n \\
& \quad u_r, v_i \geq \varepsilon > 0, r = 1,2,\cdots,i = 1,2,\cdots,m.
\end{align*}
\]

(19)

Then go to Step 4.

Step 4: If the value of \(\alpha\) obtained by solving the problem (19) or (19) is greater than or equal to the pre-assigned satisfactory level \(\bar{\alpha}\), stop. Otherwise, decision makers may like to change \(p, w_j\) and/or membership functions. Then, we will go back to Step 0.1, Step 0.2, Step 2.2, or Step 3.2. The solution procedure is then repeated.

3.1. Implementation issue on the algorithm. In Step 1, Step 2 and Step 3 of the proposed algorithm, we face the challenge of solving problems (2), (3), (9)-(12), (15) and (17)-(19). Consider the problem in (2). Since the input and output variables are not known precisely, the decision maker may define the risk-free and impossible bounds for each fuzzy input and output variable. Risk-free bounds are interpreted as the conservative values that are most realistically found, whereas the impossible bounds are associated with the values that are the least realistic. For each fuzzy input and output variable, the change from its risk-free to impossible bounds is represented by its membership function. It is assumed that membership functions are monotonically linear; and are equal to zero, if the input or output bounds are impossible, and are equal to one if they are risk free. Suppose that \(x_{i,j}^L\) and \(x_{i,j}^U\) represent, respectively, the impossible and risk-free bounds of the \(i\)-th fuzzy input of the \(j\)-th DMU. A possible linear membership function associated with the \(i\)-th fuzzy input for the \(j\)-th DMU is given by

\[
\mu_{\tilde{x}_{i,j}}(x) = \frac{x_{i,j}^L - x}{x_{i,j}^U - x_{i,j}^L}, i = 1,2,\cdots,m, j = 1,2,\cdots,n.
\]

Suppose also that \(y_{r,j}^L\) and \(y_{r,j}^U\) represent, respectively, the impossible and risk-free bounds of the \(r\)-th fuzzy output of the \(j\)-th DMU. A possible linear membership function associated with the \(r\)-th fuzzy output for the \(j\)-th DMU is given by

\[
\mu_{\tilde{y}_{r,j}}(y) = \frac{y_{r,j}^L - y}{y_{r,j}^U - y_{r,j}^L}, r = 1,2,\cdots,s, j = 1,2,\cdots,n.
\]

Given the above definitions, we have

\[
\tilde{x}_{i,j} = x_{i,j}^L + \mu_{\tilde{x}_{i,j}}(x_{i,j}^U - x_{i,j}^L), i = 1,2,\cdots,m, j = 1,2,\cdots,n \text{ with } 0 \leq \mu_{\tilde{x}_{i,j}} \leq 1.
\]
\[ y_{r,j} = y_{r,j}^L + \mu_{y_{r,j}}(y_{r,j}^U - y_{r,j}^L), \quad r = 1, 2, \cdots, s, j = 1, 2, \cdots, n \text{ with } 0 \leq \mu_{y_{r,j}} \leq 1. \]

With these transformations, the variables \( \hat{x}_{i,j} \) and \( \tilde{y}_{r,j} \) in (2) are replaced by the new variables \( \hat{x}_{i,j} \) and \( \mu_{y_{r,j}} \), which locate the levels of inputs and outputs within the impossible and risk-free bounds, respectively. The problem (2) can be modified as

\[
\begin{align*}
\max & \quad \sum_{r=1}^{s} u_r(y_{r,j}^L + \mu_{y_{r,j}}(y_{r,j}^U - y_{r,j}^L))/\sum_{i=1}^{m} v_i(x_{i,j}^L + \mu_{x_{i,j}}(x_{i,j}^U - x_{i,j}^L)) \\
\text{s.t.} & \quad \sum_{r=1}^{s} u_r(y_{r,j}^L + \mu_{y_{r,j}}(y_{r,j}^U - y_{r,j}^L))/\sum_{i=1}^{m} v_i(x_{i,j}^L + \mu_{x_{i,j}}(x_{i,j}^U - x_{i,j}^L)) \leq 1, \quad j = 1, 2, \cdots, n, \quad \forall r, i, s, j. \tag{20}
\end{align*}
\]

Let \( q_{i,j} = v_i \mu_{x_{i,j}} \) and \( p_{r,j} = u_r \mu_{y_{r,j}} \), the weighted sum of inputs for unit \( j \) in (20) takes the form

\[
\sum_{i=1}^{m} v_i \hat{x}_{i,j} = \sum_{i=1}^{m} v_i(x_{i,j}^L + \mu_{x_{i,j}}(x_{i,j}^U - x_{i,j}^L)) = \sum_{i=1}^{m} v_i x_{i,j}^L + q_{i,j}(x_{i,j}^U - x_{i,j}^L),
\]

where the new variable \( q_{i,j} \) meets the condition \( 0 \leq q_{i,j} \leq v_i \) for \( i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \). Similarly, the weighted sum of outputs for unit \( j \) takes the form

\[
\sum_{r=1}^{s} u_r \tilde{y}_{r,j} = \sum_{r=1}^{s} u_r(y_{r,j}^L + \mu_{y_{r,j}}(y_{r,j}^U - y_{r,j}^L)) = \sum_{r=1}^{s} u_r y_{r,j}^L + p_{r,j}(y_{r,j}^U - y_{r,j}^L),
\]

with \( 0 \leq p_{r,j} \leq u_r \) for \( r = 1, 2, \cdots, s, j = 1, 2, \cdots, n \).

With the above substitutions, the problem (20) can be transformed into the following linear program:

\[
\begin{align*}
\max & \quad \sum_{r=1}^{s} u_r y_{r,j}^L + p_{r,j}(y_{r,j}^U - y_{r,j}^L) \\
\text{s.t.} & \quad \sum_{i=1}^{s} v_i x_{i,j}^L + q_{i,j}(x_{i,j}^U - x_{i,j}^L) = 1, \\
& \quad \sum_{r=1}^{s} u_r y_{r,j}^L + p_{r,j}(y_{r,j}^U - y_{r,j}^L) - \sum_{i=1}^{s} v_i x_{i,j}^L + q_{i,j}(x_{i,j}^U - x_{i,j}^L) \leq 0, \quad j = 1, 2, \cdots, n, \\
& \quad u_r - v_i \leq 0, \quad i = 1, 2, \cdots, m, j = 1, 2, \cdots, n, \\
& \quad q_{i,j} - v_i \leq 0, \quad i = 1, 2, \cdots, m, j = 1, 2, \cdots, n, \\
& \quad u_r \geq \varepsilon > 0, \quad r = 1, 2, \cdots, s, i = 1, 2, \cdots, m, \\
& \quad p_{r,j} \geq 0, \quad q_{r,j} \geq 0, \quad r = 1, 2, \cdots, s, j = 1, 2, \cdots, m, j = 1, 2, \cdots, n. \tag{21}
\end{align*}
\]

A similar argument holds for solving problems (3), (9)-(12), (15) and (17)-(19).

4. A numerical example. As an illustration and for comparison purposes, an example from [2] is utilized in this section. In that example, five units are considered with a mixture of imprecise and exact data. There are two inputs including one exact and one fuzzy and two outputs including one exact and one ordinal. The entries of the ordinal output are read as follows: 5 - highest rank, \( \cdots, 1 \) - lowest rank. To deal with the input and output data in the example, which is a mixture of fuzzy and ordinal together with exact data, there are some modifications of the implementation on the proposed algorithm based on the discussions in [3]. Table 1 contains the original data.
In Step 1, we determine the positive ideal solution \( (E^+_j) \) and the negative ideal solution \( (E^-_j) \), \( j = 1, 2, \cdots, 5 \), as shown in Table 2. Then, we compute

\[
d^{PIS}_p = (w^1 \cdot \frac{1}{1 - \frac{2000u_1 + p_{21}}{1000u_1 + 0.1q_21 + 0.6v_2}} - 2.001 \times 10^{-7})^p + w^2 \cdot \frac{0.875 - \frac{1000u_1 + p_{22}}{1500u_1 + 0.8v_2 + 0.1q_{22}}}{1 - 9.01 \times 10^{-8}}^p + \\
\frac{w^3 \cdot \frac{1}{1 - \frac{1.201 \times 10^{-7}}{200u_1 + p_{24}}}}{1 - \frac{1}{1500u_1 + 0.1q_{22} + 0.8v_2}}^p + w^4 \cdot \frac{1 - \frac{0.875 - 9.01 \times 10^{-8}}{2000u_1 + p_{22}}}{1 - 9.01 \times 10^{-8}}^p + \frac{w^5 \cdot (0.7 - \frac{900u_1 + p_{25}}{200u_1 + v_2})}{0.7 - 6.01 \times 10^{-8}}^p \right)^{1/p},
\]

and

\[
d^{NIS}_p = \frac{1}{d^{PIS}_p} = (w^1 \cdot \frac{1}{1 - \frac{2000u_1 + p_{21}}{1000u_1 + 0.1q_21 + 0.6v_2}} - 2.001 \times 10^{-7})^p + w^2 \cdot \frac{0.875 - \frac{1000u_1 + p_{22}}{1500u_1 + 0.8v_2 + 0.1q_{22}}}{1 - 9.01 \times 10^{-8}}^p + \\
\frac{w^3 \cdot \frac{1}{1 - \frac{1.201 \times 10^{-7}}{200u_1 + p_{24}}}}{1 - \frac{1}{1500u_1 + 0.1q_{22} + 0.8v_2}}^p + w^4 \cdot \frac{1 - \frac{0.875 - 9.01 \times 10^{-8}}{2000u_1 + p_{22}}}{1 - 9.01 \times 10^{-8}}^p + \frac{w^5 \cdot (0.7 - \frac{900u_1 + p_{25}}{200u_1 + v_2})}{0.7 - 6.01 \times 10^{-8}}^p \right)^{1/p}.\]

Table 2. The positive ideal solution \( (E^+_j) \) and the negative ideal solution \( (E^-_j) \)

| DMU | \( E^+_j \) | \( E^-_j \) |
|-----|---------|---------|
| 1   | 1       | 2.001 \times 10^{-7} |
| 2   | 0.875   | 9.01 \times 10^{-8} |
| 3   | 1       | 1.201 \times 10^{-7} |
| 4   | 1       | 9.01 \times 10^{-8} |
| 5   | 0.7     | 6.01 \times 10^{-8} |

For \( p = 2 \), we obtain \( (d^{PIS}_2)^* = 0.2800, (d^{NIS}_2)^* = 0.2382, (d^{PIS}_2)' = 0.2801, \) and \( (d^{NIS}_2)' = 0.2381. \)
Then we consider solving the problem:

$$\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad 0.2801 - d_{PIS}^2 \geq \alpha \\
& \quad \frac{0.0001}{d_{NIS}^2 - 0.2381} \geq \alpha \\
& \quad 2000u_1 + p_{21} - (100v_1 + 0.6v_2 + 0.1q_{21}) \leq 0, \\
& \quad 1000u_1 + p_{22} - (150v_1 + 0.8v_2 + 0.1q_{22}) \leq 0, \\
& \quad 1200u_1 + p_{23} - (150v_1 + v_2) \leq 0, \\
& \quad 900u_1 + p_{24} - (600v_1 + 0.7v_2 + 0.1q_{24}) \leq 0, \\
& \quad 600u_1 + p_{25} - (200v_1 + v_2) \leq 0, \\
& \quad q_{21} - v_2 \leq 0, \\
& \quad q_{22} - v_2 \leq 0, \\
& \quad q_{24} - v_2 \leq 0, \\
& \quad p_{25} - p_{22} > 0, \\
& \quad p_{23} - p_{21} > 0, \\
& \quad p_{21} - p_{25} > 0, \\
& \quad p_{22} - p_{24} > 0, \\
& \quad u_1, u_2, v_1, v_2, q_{21}, q_{22}, q_{24}, p_{21}, p_{22}, p_{23}, p_{24}, p_{25} \geq 0,
\end{align*}$$

(24)

where $d_{PIS}^2$ and $d_{NIS}^2$ are defined in (22) and (23), respectively, and $\alpha$ is the degree of satisfactory level for both criteria of “as close to PIS as possible” and “as far away from NIS as possible.”

Moreover, for the case $p = \infty$, the following problem is considered:

$$\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad 0.1761 - d_{PIS}^\infty \geq \alpha, \\
& \quad \frac{0.0001}{d_{NIS}^\infty - 0.0312} \geq \alpha, \\
& \quad \frac{1 - \frac{1 - \frac{1}{1000u_1 + v_2}}{2000u_1 + p_{22}}}{1 - \frac{1}{1000u_1 + p_{22}}} \leq d_{PIS}^\infty, \\
& \quad \frac{0.875 - \frac{0.875 - 9.01 \times 10^{-8}}{1200u_1 + p_{22}}}{1 - \frac{1}{1200u_1 + p_{22}}} \leq d_{PIS}^\infty, \\
& \quad \frac{1 - \frac{1 - \frac{1}{150u_1 + v_2}}{500u_1 + q_{24}}}{1 - \frac{1}{150u_1 + v_2}} \leq d_{PIS}^\infty, \\
& \quad \frac{1 - \frac{1 - \frac{1}{200u_1 + v_2}}{500u_1 + q_{24}}}{1 - \frac{1}{200u_1 + v_2}} \leq d_{PIS}^\infty, \\
& \quad \frac{1 - \frac{1}{9.01 \times 10^{-8}}}{1 - \frac{1}{600u_1 + p_{22}}} \leq d_{PIS}^\infty, \\
& \quad \frac{1 - \frac{1}{0.7 \times 600u_1 + p_{22}}}{1 - \frac{1}{2000u_1 + v_2}} \leq d_{PIS}^\infty, \\
& \quad \frac{1 - \frac{1}{2.001 \times 10^{-7}}}{1 - \frac{1}{1000u_1 + p_{22}}} \leq d_{NIS}^\infty, \\
& \quad \frac{1 - \frac{1}{0.857 - 9.01 \times 10^{-8}}}{150u_1 + q_{24} + v_2} \leq d_{NIS}^\infty, \\
& \quad \frac{1 - \frac{1}{1.201 \times 10^{-7}}}{900u_1 + p_{22}} \leq d_{NIS}^\infty, \\
& \quad \frac{1 - \frac{1}{9.01 \times 10^{-8}}}{200u_1 + v_2} \leq d_{NIS}^\infty.
\end{align*}$$

(25)
This example was solved by the IDEA approach in [2] and the Model (6) in [3]. The results for the case of \( p = 1, 2, \infty \), of the proposed approach and other existing methods including fuzzy CCR method are shown in Table 3.

### Table 3. Efficiency scores and the associated rankings (in parentheses) calculated from different methods

| DMU | Fuzy CCR | IDEA approach | Model (6) in [3] | \( p = 1 \) | \( p = 2 \) | \( p = \infty \) |
|-----|----------|----------------|------------------|-------------|-------------|-------------|
| 1   | 1(1)     | 1(1)           | 1(1)             | 1(1)        | 1(1)        | 1(1)        |
| 2   | 0.875(4) | 0.875(4)       | 0.875(4)         | 0.875(4)    | 0.7825(4)   | 0.7563(4)   |
| 3   | 1(1)     | 1(1)           | 1(1)             | 1(1)        | 0.9233(2)   | 0.9062(2)   |
| 4   | 1(1)     | 1(1)           | 1(1)             | 1(1)        | 0.9083(3)   | 0.8943(3)   |
| 5   | 0.7(5)   | 0.7(5)         | 0.7(5)           | 0.7(5)      | 0.6239(5)   | 0.6022(5)   |

The second column of Table 3 shows the efficiency scores of the 5 units calculated from the CCR model with a mixture of imprecise and exact data as shown in Table 1. These scores are the highest values that the units can attain. They are regarded as the positive ideal solution \( (E_j^*) \), \( j = 1, 2, \cdots, 5 \), as shown in Table 2. The numbers in parentheses are the rankings of the corresponding DMUs. There are three efficient units which cannot be differentiated. The third and fourth columns in Table 3 show the efficiency scores of the 5 units obtained from [2] (the IDEA approach) and [3] (model (6)), respectively. They are consistent with results of the fuzzy CCR method. The last three columns of Table 3 show efficiency scores obtained by the proposed approach for \( p = 1, 2 \), and \( \infty \). The rankings of DMUs obtained by using the proposed approach are consistent with those of the existing methods, indicating that the results are reasonable. In addition to this, the three efficient units measured from other methods can be differentiated by the proposed approach for \( p = 2 \) and \( \infty \). Compared to other existing methods, the proposed approach is more informative and practical from the management viewpoint. It not only differentiates the efficient units on a common base but also detects some abnormal efficiency scores calculated from other existing methods. Moreover, since all DMUs can be measured in terms of a common base, the need for a common ranking becomes immaterial.

5. **Conclusions.** This work studies the DEA with common weights in a fuzzy environment. Applying the basic principle of compromise of TOPSIS, the multiple
objective fuzzy DEA program can be reduced into an auxiliary bi-objective fuzzy decision-making problem. To resolve the conflict existing between two distance objective functions, membership functions are developed to model the relationship between distance measures and satisfactory levels. An algorithm with the implementation issue for finding the compromise solution of the fuzzy DEA program is provided. An example from [2] is utilized for illustration and comparison purpose. Our results show that the proposed approach is more informative and practical for management. It provides decision makers the flexibility in measuring the relative efficiency for DMUs with fuzzy inputs and outputs, which not only differentiates efficient units on a common base but also detects some abnormal efficiency scores calculated from other existing methods.

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