Lamb shift of energy levels in quantum rings

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Abstract
We study the vacuum radiative corrections to energy levels of a confined electron in quantum rings. The calculations are provided for the Lamb shift of energy levels in a low-momentum region of virtual photons and for both one-dimensional and two-dimensional quantum rings. We show that contrary to the well known case of a hydrogen atom the value of the Lamb shift increases with the magnetic momentum quantum number $m$. We also investigate the dependence of the Lamb shift on magnetic flux piercing the ring and demonstrate a presence of magnetic-flux-dependent oscillations. For a one-dimensional ring the value of the shift strongly depends on the radius of the ring. It is small for semiconductor rings but can attain measurable quantities in natural organic ring-shape molecules, such as benzene, cycloalkanes and porphyrins.

Keywords: quantum ring, Lamb shift, quantum electrodynamics

((Some figures may appear in colour only in the online journal)

1. Introduction
Quantum electrodynamics (QED) is the most accurate theory known so far to investigate the fundamental processes of the matter-light interactions \([1, 2]\). QED predicts the possibility of observing effects related to vacuum fluctuations, i.e. the creation and absorption of virtual quanta of an electromagnetic field, referred to as virtual photons. The most notable example is the Lamb radiative energy shift that was first observed in hydrogen atoms \([3]\). Among the other QED effects one can note the appearance of Casimir forces between parallel conducting plates and the phenomenon of vacuum polarization leading to photon–photon scattering in a vacuum due to the creation of a virtual electron–positron pair.

Since the original calculations of the value of the Lamb shift by Bethe and later by Feynman and others, the theory of radiative corrections has been worked out up to a very high precision. The Lamb radiative shift since then has been experimentally investigated for various atomic systems including muonic atoms that provide an ultra-high precision test of the QED \([4, 5]\). The high accuracy of QED predictions for the precise spectroscopy of simple atomic systems allowed the accurate measurements of fundamental physical constants including the Rydberg constant $R_\infty$ from the hydrogen spectrum $\alpha$ from the helium fine structure and the electron mass $m_e$ from the $g$ factor of hydrogen-like ions. The vacuum QED effects in the extremely strong atomic fields has also been measured in experiments with highly charged few-electron ions \([6, 7]\). Additionally, cavity QED allows to study the vacuum radiative shift in interaction of atoms with a single mode electromagnetic field \([8–11]\) and investigate the vacuum Rabi splitting in a system consisting of a single quantum dot placed into an optical semiconductor cavity \([12, 13]\). The modification of the Lamb shift in the presence of external strong laser fields was also investigated \([14–19]\).

Up to now, the study of radiation shifts was mostly restricted to systems consisting of a single atom. However, advances in the engineering of micro- and nanoelectromechanical systems, optical microcavities, superconducting circuits and artificial atoms made actual the problem of the investigation of quantum vacuum effects in these nanodevices. In this context, the Lamb shift has been observed in a superconducting electronic circuit in the form of a superconducting Cooper pair box in a transmission-line resonator \([20]\). It has been also shown that a superconducting qubit strongly coupled to a non-linear resonator can act as a probe of quantum
fluctuations of the intra-resonator field. Theoretical and experimental results have been presented in [21–23].

In addition to these important results it would be interesting to study the vacuum radiative corrections in other types of nano-scale systems. The natural candidates here are Aharonov–Bohm quantum rings, where the spectrum of discrete states can be easily tuned by application of the external magnetic field due to the Aharonov–Bohm effect [24, 25]. The investigation of QED effects in ring-based structures has already started [26]. In particular, it was theoretically demonstrated that in chiral optical resonators the ground state of the electron system in the ring can be associated to non-zero angular momentum [27–29]. Additionally, an interaction of electron with circularly polarized photons has shown to modify charge and spin flow in a quantum ring [30–32]. In this paper we make further contributions to this domain and present the calculation of the vacuum radiative Lamb shifts for 1D and 2D quantum rings placed in the external magnetic field (figure 1) calculating the self-energy of the confined electron and including mass-renormalization procedure. As predicted by QED such bound-state self-energy part is the dominant radiative correction in hydrogen-like systems and gives 98% of the ground-state Lamb shift in atomic hydrogen [33].

We demonstrate that the Lamb shift is minimal for the state with minimal value of \( m + f \), where \( m \) is a value of the electrons angular momentum and \( f \) is a magnetic flux piercing the ring. This is qualitatively different to the case of a hydrogen atom, where the Lamb shift is maximal for \( s \)-states. Besides, we demonstrate that the value of the Lamb shift of momentum levels reveals periodical dependence on \( f \) specific for an Aharonov–Bohm system.

For the 2D quantum ring the energy spectrum consists of the discrete energy levels due to radial motion with the radial quantum numbers \( n = 1, 2, \ldots \), and rotation motion with the quantum numbers \( m = 0, \pm 1, \ldots \). In this case we present calculations in so-called Bethe logarithmic approximation in analogy to consideration of real atoms. This allows us to obtain the general approximate result for the Lamb shift without specification of the confining potential.

2. The interaction Hamiltonian

Let us consider the system shown schematically in figure 1. The Hamiltonian of an electron in a quantum ring interacting with radiation field reads as

\[
H = H_0 + H_R + H_{\text{int}},
\]

(1)

where \( H_0 \) and \( H_R \) are the free Hamiltonians of the quantum ring and the radiation field, and

\[
H_{\text{int}} = \int \vec{A}(\vec{r}) j(\vec{r}) d^3r
\]

describes the interaction of a confined electron with the radiation field, where \( \vec{j} \) is the electron current operator and \( \vec{A} \) is the vector potential. We use the Furry representation for the vector state of the system with the Hamiltonian (1)

\[
|\psi(t)\rangle = U(t)e^{-iH_0t}|\phi(t)\rangle,
\]

(2)

where

\[
U(t) = e^{-iH_Rt}.
\]

(3)

In this representation the dynamic equation for the state \( |\phi(t)\rangle \) reads

\[
i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = H_i(t)|\phi(t)\rangle,
\]

(4)

where

\[
H_i(t) = U^{-1}(t)e^{iH_{\text{int}}t}H_{\text{int}}e^{-iH_{\text{int}}t}U(t)
\]

\[
= \int \vec{A}(r, t) \vec{j}(r, t) dV,
\]

(5)

and the current operator in the Furry representation reads

\[
\vec{j}(r, t) = U^{-1}(t)\vec{j}(r)U(t).
\]

(6)

The system displays cylindrical symmetry and thus we use the circular polarizations, \( \hat{\epsilon}_x = \hat{\epsilon}_x + i\hat{\epsilon}_y, \hat{\epsilon}_e = \hat{\epsilon}_x - i\hat{\epsilon}_y \) and the field operator can be written in the following form:

\[
\vec{A}(r, t) = \frac{\hbar c}{(2\pi)^3} \sum_{\lambda = +, -} \int \frac{d^3k}{\sqrt{2\omega_k}} [\hat{e}_\lambda(k)A_\lambda(k)],
\]

(7)

where

\[
A_\lambda(k) = a_\lambda(k)e^{-i(\omega_k t - \vec{k}\vec{r})} + a_\lambda^+(k)e^{i(\omega_k t - \vec{k}\vec{r})}
\]

\[
A_{-\lambda}(k) = a_{-\lambda}(k)e^{-i(\omega_k t + \vec{k}\vec{r})} + a_{-\lambda}^+(k)e^{i(\omega_k t + \vec{k}\vec{r})}.
\]

(8)

Here: \( a_\lambda^+(k) \) corresponds to the photon creation operator where the sub-script describes the state of photon polarization, while \( a_\lambda(k) \) denotes the photon annihilation operator.

The current operator \( \vec{j} = e\vec{v} \), where \( \vec{v} \) is the velocity operator of confined electron, in the cylindrical coordinates
can be written in two-component form as

$$\vec{j} = e \left( \hat{\rho} \nu_{\rho} + \hat{\phi} \nu_{\phi} \right)$$  (9)

where $\hat{\rho}$, $\hat{\phi}$ are unity vectors, and the interaction Hamiltonian reads:

$$H_{int} = e \sqrt{\frac{\hbar}{2m_e R}} \int \frac{d^3k}{(2\pi)^3} \left[ (A_+(k)e^{i\omega t} + A_-(k)e^{-i\omega t}) \nu_{\rho} + i(A_+(k)e^{i\omega t} - A_-(k)e^{-i\omega t}) \nu_{\phi} \right] dV.$$  (10)

### 3. Matrix elements of the radiative transitions

In this section we derive the matrix elements of transitions between states of confined electron for 1D and 2D models produced by the term of electron–radiation field interaction.

#### 3.1. The case of the 1D quantum ring

The Hamiltonian of an electron confined in a 1D infinitely narrow quantum ring depends only on the polar angle $\varphi$. We consider a general case of the quantum ring pierced by the magnetic flux $\Phi$. The vector-potential is chosen as $\vec{A} = \frac{\varphi}{2R} \vec{r} \times \vec{B}$, where $R$ is the radius of a ring, and electron momentum operator is $\vec{p} = -i\hbar \frac{\partial}{\partial \varphi}$. The corresponding Hamiltonian is given by the expression

$$H_0(\varphi) = \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) - \frac{\hbar^2}{2m_e R^2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + V(\varphi),$$  (11)

where $m_e$ is the electron mass (or the effective mass), $f = \frac{\varphi}{\Phi_0}$ is the number of flux quanta piercing the ring, $\Phi_0 = \hbar/e$. The $2\pi$-periodic eigenfunctions and the energy eigenvalues of the system are

$$\psi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}, \quad \epsilon_m = \epsilon_0 (m + f)^2.$$  (12)

where $\epsilon_0 = \hbar^2/2m_e R^2$.

At first, we calculate the matrix elements of the radiative transitions $(n|\vec{e}_\varphi \vec{v}|m)$ between states $|nm\rangle$ and $|ln\rangle$ of the rotation motion with angular momentum quantum numbers $m = 0, \pm 1, \pm 2, \ldots$. For this goal we consider the Heisenberg equation for the radial vector $\vec{r}(t) = U^{-1}(t)\vec{r}'U(t)$, where $\vec{r}' = \rho \vec{p}$, in the standard form

$$\vec{v} = \frac{d}{dt} \vec{r} = \frac{i}{\hbar} \left[ H_0(\varphi), \vec{r} \right].$$  (13)

Using equation (11) and the expressions for the basis vectors

$$\frac{\partial \hat{\rho}}{\partial \varphi} = \hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \varphi} = -\hat{\rho},$$  (14)

we calculate the velocity operator in the following two-component form

$$\vec{v} = \frac{i\hbar}{2m_e R^2} \left[ \hat{\rho} - 2\hat{\phi} \left( \frac{\partial}{\partial \varphi} + if \right) \right].$$  (15)

Then, the formulas $(\vec{e}_\varphi \vec{p}) = e^{i\omega t}$, $(\vec{e}_\varphi \hat{\phi}) = \pm i e^{i\omega t}$ are used, and integration on the azimuthal angle can be performed using relation

$$\frac{1}{2\pi} \int e^{i(n-m)\varphi} d\varphi = \delta_{n,m},$$  (16)

where $\delta_{n,m}$ is Kronecker delta function. Finally, we obtain the radiative transitions in the following form

$$\langle n | \vec{e}_\varphi \vec{v} | m \rangle = \frac{i\hbar}{2m_e R} [1 \pm 2(m + f)] \delta_{n,m \pm 1}.$$  (17)

#### 3.2. The case of 2D quantum ring

For the 2D quantum ring the Hamiltonian also involves the radial dependence and reads as

$$H_0 = \frac{\hbar^2}{2m_e} \left( \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) - \frac{\hbar^2}{2m_e R^2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + V(\rho),$$  (18)

where $V(\rho)$ is the confining potential. The energy spectrum of $H_0$ consists of the discrete energy levels $E_N = E_{n,m}$ due to radial motion with the radial quantum numbers $n = 1, 2, \ldots$, and rotation motion with the quantum numbers $m = 0, \pm 1, \pm 2, \ldots$. The joint states are denoted as $|N\rangle = |r_{n,m}\rangle |m\rangle$.

The velocity operator $\vec{v}(t) = U^{-1}(t)\vec{v}U(t)$ in this case is calculated in the following form

$$\vec{v}(t) = \frac{i}{\hbar} [H_0, \vec{r}] = -i\hbar \left( \frac{\partial}{\partial \varphi} + \hat{\phi} \frac{\partial}{\partial \rho} \right) \frac{1}{m_e}$$  (19)

that is different from the analogous result for 1D model. Then, by using this formula the matrix elements of the radiative transitions can be calculated as

$$\langle n' | \vec{e}_\varphi \vec{v} | m \rangle = \frac{i}{\hbar} \left[ E_{n',m} - E_{n,m} \right] \delta_{n',m,\pm 1},$$  (20)

where

$$R_{n',m',n,m} = \int_0^\infty R_{n',m'} R_{n,m} \rho^2 d\rho$$  (21)

and the normalization condition for the radial wave functions reads as

$$\int_0^\infty R_{n',m'}^2 R_{n,m} d\rho = \delta_{n',m',n,m}.$$  (22)

### 4. Radiative shifts of the energy levels

In this section we derive the general expression for the radiative shift of a confined electron using the expressions for
the transition matrix elements obtained in the previous section. The time evolution of the system vector state due to interaction with the radiation field in the Furry picture is given by equation (4). The formal solution of this equation can be written as

\[ \Phi(t_2) = U_F(t_2, t_1)\Phi(t_1), \]

\[ U_F(t_2, t_1) = T \exp \left( -i \int_{t_1}^{t_2} dt H_1(t) \right), \]

through the time-evolution matrix \( U_F(t_2, t_1) \), where \( T \) denotes the time-ordering symbol. If the 2D ring is considered in the Furry picture, the confinement potential \( V(\rho) \) is included into the free Hamiltonian.

The radiative shift of the \( E_{n,m} \) energy level for \( |N\rangle = |\Psi_{n,m}(t)\rangle = |R_{n,m}\rangle \) state of an electron confined in the ring is expressed through time-evolution matrix by using the Tomonaga–Schwinger equation

\[ e^{-i\Delta E_{n,m}(t-t_0)} = \langle 0 |N| U_F(t_1, t_0) |N\rangle |0 \rangle \]

\[ \times \langle 0 |N| H_1(t_2) H_1(t_1) |N\rangle |0 \rangle. \]

Here \( |N\rangle \) and \( |0\rangle \) are the vacuum states of an electron and a photon, respectively.

In the second-order of the perturbation theory, the following expression for radiative shifts of \( E_{n,m} \) energy levels may be easily obtained:

\[ (t - t_0) \Delta E_{n,m} = -i \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \]

\[ \times \langle 0 |N| H_1(\tau_2) H_1(\tau_1) |N\rangle |0 \rangle. \]

In this expression we neglected the effects of the vacuum polarization for confined electron. Thus, equation (26) represents only self-energy part of the radiative shift. In figure 2 we show, for completeness, corresponding Feynman diagrams for the self-energy and the electron mass renormalization. Note that for the case of stationary confined potential integration over time in the formula (26) is easily performed leading to the factor \( (t - t_0) \) and hence to the definite result for \( \Delta E_{n,m} \).

Typically, the calculation of the Lamb shift for atomic systems is made by splitting the basic formula for radiative shift in two parts that correspond to the electron–photon interaction in two spectral ranges of virtual photon [34]. For the low-momentum region, up to some momentum of virtual photon \( k \) of the order of \( k_{\text{max}} \leq (aZ) m_e \), the exact wave functions of confined electron including all orders of interaction with virtual \( V \) are used, however in a non-relativistic approach. In high-momentum range \( k > k_{\text{max}} \), the electron wave function can be used in the first approximation of perturbation theory on atomic potential, but in relativistic approach. In general, by adding two parts, the full Lamb shift does not depend on the cut-off parameter \( k_{\text{max}} \).

The analogous approach is used in the case of the quantum ring. In this area we restrict ourselves to the calculation of the non-relativistic part of the Lamb shift. In this way, the standard calculations including the electron mass renormalization procedure lead to the following result \( (\hbar = c = 1) \):

\[ \Delta E_{n,m} = \frac{\alpha}{4\pi^2} \int_0^{k_{\text{max}}} \frac{d^3k}{\omega_\lambda N',\lambda = +, -} \left( E_{N'} - E_N \right) \]

\[ \times \frac{\langle N'| \mathbf{e}_\lambda \mathbf{e'} \rangle |N\rangle \langle \mathbf{e}_\lambda \mathbf{e'} |N'\rangle}{\omega_\lambda - (E_{N'} - E_N - \omega_\lambda)}, \]

where \( \alpha \) is the fine structure constant and \( k_{\text{max}} \) is the cut-off parameter. The matrix elements of transitions are calculated in section 3.

It is well known that the mass renormalization is usually realized by adding the term \( \frac{\delta m}{\pm} |N| \mathbf{v} |N\rangle \) described by figure 2(b) for the \( jA \) version of the electron–radiation field interaction. In this case the low-frequency part of the shift contains only logarithmic divergence that is regularized by the cut-off parameter \( k_{\text{max}} \). The analogous procedure cannot be realized for the dipole version \( Ed \) of the interaction because in this case the formula (27) also contains additional divergences. Note that equation (27) is valid in the long-wavelength approximation \( kR \ll 1 \) and hence the integration takes place until the energy of cutoff \( k_{\text{max}} \). It is reasonable to take \( k_{\text{max}} < \frac{1}{R} \approx \sqrt{m_e c} \), where \( c \) is the characteristic energy of the confined electron. On the other hand, \( k_{\text{max}} \) cannot be arbitrarily small. It is easy to realize assuming that the propagator of the confined electron differs from the propagator of free electron (see figure 2). This consideration leads to the following inequalities \( kp \approx k_{\text{max}} m_e \gg p^2 - m_e^2 \approx m_e v \), where we introduced a four-vector of the electron momentum \( p_x = (m_e + v, 0) \). Thus, we obtain \( e \ll k_{\text{max}} \ll \sqrt{m_e c}. \)

Below we apply the general expression (27) to 1D and 2D models of quantum rings corresponding to the Hamiltonians (11) and (18).

### 4.1. Radiative shift for 1D quantum ring

In this subsection we calculate radiative shift for a one-dimensional quantum ring with the vanishing width \( d \) for which we can consider only azimuthal energy level structure. By using the matrix element (17) for the radiative shift \( \Delta E_{n,f} (f) \) of the energy levels of a quantum ring pierced by the magnetic flux, \( \epsilon_m = \epsilon_0 (m + f)^2, (\epsilon_0 = \hbar^2/2m_e R^2) \), we obtain
from formula (27)
\[ \Delta E_m(f) = \frac{\alpha e_0^2}{2\pi m_e} \left[ (1 - 2(m + f))^3 \ln \frac{k_{\text{max}}}{\epsilon_0 (1 - 2(m + f))} + (1 + 2(m + f))^3 \ln \frac{k_{\text{max}}}{\epsilon_0 (1 + 2(m + f))} \right]. \] (28)

where \( k_{\text{max}} \) is the energy of cut-off, \( \epsilon_0 \ll k_{\text{max}} \ll \sqrt{m_e \epsilon_0} \).

The radiative shift (28) for the lowest energy state corresponding to minimal value \( m + f = 0 \) is periodic in flux with the period \( \Phi_0 \) and reads
\[ \Delta E_m(f = -m) = \frac{\alpha e_0^2}{2m_e} \ln \left( \frac{k_{\text{max}}}{\epsilon_0} \right). \] (29)

We also note the symmetry of the Lamb shifts relative to the sign of azimuthal quantum numbers, \( \Delta E_m(f) = \Delta E_{-m}(-f) \).

We can rewrite equation (28) in the following form:
\[ \Delta E_m(f) = \left[ 1 + 12(m + f)^2 \right] \delta_m(f = -m) \]
\[ - \frac{\alpha e_0^2}{2m_e} \left\{ \frac{A^2}{2} \ln |A_+| + \frac{A^2}{2} \ln |A_-| \right\}, \] (30)

where \( A = 2(m + f) \pm 1 \). Thus, the minimal shift (29) determines the other Lamb shifts and increases with increasing of the parameter \( m + f \).

The behaviour of Aharonov–Bohm oscillations in the Lamb shift is demonstrated in figure 3, where for convenience we introduced a characteristic energy \( \chi = \alpha e_0^2 / m_e \). We note that an exact value of a radiative shift strongly depends on the system parameters which define energy \( \epsilon_0 \). For \( f = 0 \) the result (29) describes the radiative shifts of energetic levels \( \Delta_m(0) \) of an electron in the quantum ring without the magnetic field. In this case the minimal shift is realized for the \( m = 0 \) level. We observe that the radiative shift dependence on the magnetic flux has maxima and minima. In particular, the minima appear at the points \( f = -m \). We find that the Lamb shift is approximately equal to characteristic energy \( \chi \) for small magnetic flux, but grows substantially for higher magnetic fields. Moreover, we find that the effect of vacuum corrections is larger for quantum rings of smaller radius.

First, let us consider a semiconductor quantum ring of radius \( R = 20 \) nm, with corresponding cut-off parameter being \( k_{\text{max}} \approx 10 \) meV. For the given parameters the characteristic energy is equal to \( \chi = 0.13 \) meV, and \( \epsilon_0 = 95 \) meV. Note, that for typical parameters of the semiconductor rings the values of the shifts are extremely small and lie in the range of nanoelectronvolts. However, one can expect that the shift will be dramatically increased in organic ring-shaped molecules, such as benzene, cycloalkanes and porphyrins, for which the value of the radius can become of the order of nanometers. Our estimation gives that for a benzene ring of radius \( R = 0.36 \) nm [35], the characteristic energy can be estimated as \( \chi = 1.3 \) meV, leading to the Lamb shift which is an order of magnitude smaller than in the benzene case.

4.2. Radiative shift for the ring in a cavity

For completeness, we now briefly analyze the case of the quantum 1D ring pierced by the magnetic flux and interacting with single mode in a cavity [36]. We calculate the shift of \( m \)-energetic level due to emission and reabsorption of virtual photons of one-mode cavity by using the \( j^{\dagger} \bar{A} \) version of electron–radiation field interaction. We consider the linearly polarized mode of cavity in the Fock state with \( N \) photons at the frequency \( \omega \) with the vector potential given by
\[ \bar{A}(t) = A_0 \hat{e}_x (ae^{-i\omega t} + a^\dagger e^{i\omega t}). \] (31)

By using the matrix element (17) the radiative shift in the second order of the perturbation theory on interaction of the ring with cavity photons is calculated as
\[ \Delta E_m = \frac{\alpha A_0^2 \epsilon_0}{4m_e} \left\{ (N + 1) \left( \frac{A^2}{\omega + \epsilon_0 A_+} + \frac{A^2}{\omega - \epsilon_0 A_-} \right) - N \left( \frac{A^2}{\omega - \epsilon_0 A_+} + \frac{A^2}{\omega + \epsilon_0 A_-} \right) \right\}, \] (32)

where \( A = 2(m + f) \pm 1 \).

Different from the Lamb shift (27), equation (32) has not included both integration on virtual photon and renormalization procedure. Therefore the result (27) does not contain a small logarithmic factor appearing for the Lamb shift. Besides this the radiative shift for the ring in the cavity can be controlled by the amplitude of field of cavity mode and increases with increasing of the amplitude \( A_0 \).
The vacuum part for $N = 0$ yields
\[ \Delta E'_0 = \frac{\alpha^2 \tilde{\varepsilon}_0}{4m_e} \left( \frac{\Lambda^2}{\omega + \tilde{\varepsilon}_0 \Lambda} + \frac{\Lambda^2}{\omega - \tilde{\varepsilon}_0 \Lambda} \right), \] (33)
with the minimal value
\[ \Delta E'_0 = \frac{\alpha^2 \tilde{\varepsilon}_0}{2m_e} \frac{1}{\omega + \tilde{\varepsilon}_0} \] (34)
that is realized for $m + f = 0$.

4.3. Radiative shift for 2D system

In this section, we consider the Lamb shift of energy levels of an electron confined in a 2D quantum ring. Using equations (20) and (27) one can obtain for the radiative shift of $E_{n,m}$ the following formula:
\[ \Delta E_{n,m} = \frac{\alpha}{\pi} \sum_{n'} \left[ (E_{n',m+1} - E_{n,m})^2 \right] \times \left[ \frac{k_{max}}{|E_{n,m} - E_{n',m+1}|} + \left( E_{n',m-1} - E_{n,m} \right)^2 \right] \times \left[ \frac{k_{max}}{|E_{n,m} - E_{n',m-1}|} \right]. \] (35)

This expression contains the summation over all virtual transitions between electronic states due to emission and reabsorption of virtual photons. To calculate the value of the shift, one needs to find the eigenfunctions and energy levels of the Hamiltonian $H_0$ (18) for some specific confining potential $V(\rho)$. The eigenfunctions of $H_0$ are factorized: $\psi_{n,m}(\rho, \phi) = R_{n,m}(\rho) \Phi_0(\phi)$, and the radial wave function $R_{n,m}(\rho)$ is the solution of the following equation:
\[ -\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m_e^2}{\rho^2} R_{n,m}(\rho) + V(\rho) R_{n,m}(\rho) = \epsilon(n, m) R_{n,m}(\rho). \] (36)

Here $n, m$ are the principal and azimuthal quantum numbers, correspondingly, and the energy levels $E_{n,m}$ can be represented as a sum of azimuthal and radial parts, $E_{n,m} = \epsilon_m + \epsilon(n, m)$.

The Lamb shift physically appears due to the emission and reabsorption of the virtual photons. The corresponding correction to the energy in the case of 2D ring can be divided into two parts:
\[ \Delta E_{n,m} = \Delta E_{n,m}^{(d)} + \Delta E_{n,m}^{(nd)}. \] (37)

The ‘diagonal’ part $\Delta E_{n,m}^{(d)}$ contains the terms in which the absorption of the virtual photon does not change the principal number $n$, while azimuthal number changes by $\pm 1$. Only these terms were accounted for in the case of the 1D ring. However, the rings of the final width include, ‘non-diagonal’ contribution to the Lamb shift given by $\Delta E_{n,m}^{(nd)}$. In this term the absorption of the virtual photon changes the value of $n$, and thus the term involves the summation over intermediate states $\sum_{n'=\pm 1} \ldots$.

Below we consider a narrow ring, without magnetic flux, in which width $d$ is much less than the radius of the ring, $d \ll R$. In this case the characteristic confinement energy of the radial motion $\epsilon(n, m)$ is much greater than the spacing between azimuthal energy levels $\epsilon_m$. Besides, in the case of a narrow ring one can assume that the radial wave functions $R_{n,m}$ depend only on the principal quantum number and the energy spectrum can be represented as a sum of azimuthal and radial contributions $\epsilon(n, m) = \epsilon(n) + \epsilon(m)$. Indeed, the azimuthal motion enters into the equation for radial wavefunction in the form of the additional centrifugal potential $\sim m^2/r^2$ which for the case of a narrow ring of the radius $R \gg d$ can be safely approximated by a constant value of $\sim m^2/R^2$. In this case the energy distances for diagonal transitions $E_{n,m} - E_{n,\pm 1,m} = \epsilon(n, m) - \epsilon(n, m \pm 1)$ are reduced to $\epsilon_{m} - \epsilon_{m \pm 1} = \epsilon_0 \Lambda_{m}$, and the diagonal matrix elements of transitions (21) can be rewritten as
\[ R_{n,m',m} \approx \int_0^\infty V_0(\rho)^2 \rho^2 \mathrm{d}\rho = (\langle \rho \mid R_0 \mid \rho \rangle). \] (38)

On the whole, we get the following expression for the ‘diagonal’ part of the shift:
\[ \Delta E_{n,m}^{(d)} = \frac{\alpha}{\pi} \epsilon_0 \left[ A_+^2 \ln \left( \frac{k_{max}}{\epsilon_0 \Lambda_{+}} \right) + A_-^2 \ln \left( \frac{k_{max}}{\epsilon_0 \Lambda_{-}} \right) \right] \] (39)

In the case of the narrow ring limit only the states with $n = 0$ are occupied. To estimate the corresponding mean radius $\bar{R}_0$ we consider the model potential for a narrow ring in the form of displaced parabola,
\[ V(\rho) = \frac{V_0}{2} (\rho - R)^2. \] (40)

For $d \ll R$, we can use the harmonic approximation of equation (36) with potential (40) for the lowest eigenstate $R_0(\rho)$ and the energy $\epsilon(0)$,
\[ \frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial \rho^2} + \frac{V_0}{2} (\rho - R^2) R_0(\rho) = \epsilon(0) R_0(\rho), \] (41)
for which
\[ \epsilon_0 = \frac{\hbar^2}{2m_e} \frac{h}{2\sqrt{V_0}}, \] (42)
\[ R_0(\rho) = \left( \frac{2}{R d \sqrt{\pi}} \right)^{1/2} e^{-\rho^2/(2d^2)}. \] (43)

This allows to get a simple result $\bar{R}_0 = R$. The value of the diagonal contribution to the Lamb shift thus coincides with earlier obtained result for purely 1D ring.

It should be mentioned that consideration of the simplest form of the potential (40) and the approximate solution (42) of the radial equation (41) prove to be correct for estimation of the effective radius of the ring. However, such approximation in principle is not fully correct. Note also that in
general the 1D ring should be considered as the limit of a
2D ring. However, this procedure can be only per-
formed in detail for each of the given confined potential,
but not in the general form. In particular, for certain sys-
tems the influence of finite ring dimensions was noted [37–40].

In the end of this section we provide the other approach
that allows us to estimate the Lamb shift using equation (35)
in analogy to the standard method of effective logarithm used
in investigation of radiative shifts of atomic spectra [34].

Now let us consider the non-diagonal contribution to
the Lamb shift (see equations (35) and (37) and also
equation (44)) which involves the virtual radiative transitions
between radial wave functions with different principal quan-
tum numbers. We express the logarithms in equation (35) as
the logarithm depending on the characteristic mean energy
in analogy to the standard method of effective logarithm used
Lamb shift (see equations (35) and (37) and also
in investigation of radiative shifts of atomic spectra [34].

The analogous formula is expected to be useful for
comparative analysis of vacuum radiative shifts of various
ring energy levels. It can also be used for estimation of
Lamb shifts for several model potentials. In
particular, its application to the case of parabolic potential
(40) immediately leads to the approximate value of the Lamb
shift of lowest energy level

\[ \Delta E_{0,m} \approx \frac{2a}{\pi m_e^2} V_0 \ln \left( \frac{k_{\text{max}}}{\varepsilon} \right). \]  

We estimate this quantity for the system GaAs quantum ring
embedded in the AlGaAs substrate with parabolic potential
and
\[ V_0 = 1.68 \text{ eV/nm}^2. \]  
The estimation gives the minimal value of
\[ \Delta E_0 = 5.4 \text{ neV}, \]  
which largely overcomes the same quantity
for the 1D case. Next, assuming the porphyrin molecule placed
on the metallic (e.g. aluminium) substrate [41, 42] with
parabolic confinement potential of \[ V_0 = 215 \text{ eV/nm}^2 \] the minimal
radiative shift can be estimated as \[ \Delta E_0 \approx 0.5 \text{ \mu eV}. \]

5. Conclusion

We have studied radiative shifts of energy levels of an elec-
tron confined in a quantum ring. We have analyzed the non-
relativistic part of the Lamb shift corresponding to the low-
momentum spectral ranges of a virtual photon for both 1D
and 2D models. It has been demonstrated that in the absence
of the external magnetic field the minimal Lamb shift corre-
sponds to the state of the minimal energy with \[ m = 0, \] which is
qualitatively different from the case of the hydrogen atom
where Lamb shifts are maximal for s-orbital states. Con-
sidering an Aharonov–Bohm quantum ring pierced by a
magnetic flux we demonstrate flux-dependent oscillations
in the vacuum self-energetic shift of the ground state.
The low-frequency part of the self-energy part calculated in this paper is the dominant radiative correction to energetic levels of the confined electron. Nevertheless, the satisfactory consideration of the Lamb shift must also involve the high-frequency contribution, which will be the subject of future work.

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