ZEROS OF PARTIAL SUMS OF THE DEDEKIND ZETA FUNCTION OF A CYCLOTOMIC FIELD

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Abstract. In this article, we study the zeros of the partial sums of the Dedekind zeta function of a cyclotomic field $K$ defined by the truncated Dirichlet series

$$\zeta_{K,X}(s) = \sum_{\|a\| \leq X} \frac{1}{\|a\|^s},$$

where the sum is to be taken over nonzero integral ideals $a$ of $K$ and $\|a\|$ denotes the absolute norm of $a$. Specifically, we establish the zero-free regions for $\zeta_{K,X}(s)$ and estimate the number of zeros of $\zeta_{K,X}(s)$ up to height $T$.

1. Introduction and statement of results

A first generalization of the Riemann zeta-function $\zeta(s)$ is provided by the Dirichlet $L$-functions. Subsequently, Dedekind studied the zeta function $\zeta_K(s)$ of an arbitrary algebraic number field $K$, defined for $\text{Re}(s) > 1$ by

$$\zeta_K(s) = \sum_a \frac{1}{\|a\|^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where the first sum is to be taken over all nonzero integral ideals $a$ of $K$ and where $\|a\|$ denotes the absolute norm of $a$. In the second sum, $a(n)$ is used to denote the number of integral ideals $a$ with norm $\|a\| = n$.

As in the particular case $K = \mathbb{Q}$, where $\zeta(s) = \zeta_{\mathbb{Q}}(s)$, the function $\zeta_K(s)$ is analytic everywhere except solely for a simple pole at $s = 1$. (See Davenport [4] and Neukrich [12].) The residue of this pole is given by the analytic class number formula

$$\text{Res}_{s=1} \zeta_K(s) = \frac{2^r \pi^{n_0 - r} R_K h_K}{w_K \sqrt{|d_K|}},$$

where $r = r_1 + r_2$ (with $r_1$ being the number of real embeddings and $r_2$ being the number of complex conjugate pairs of complex embeddings of $K$), $n_0 = [K : \mathbb{Q}]$ denotes the degree of $K/\mathbb{Q}$, $R_K$ denotes the regulator, $h_K$ denotes the class number, $w_K$ denotes the number of roots of unity in $K$, and $d_K$ denotes the discriminant of $K$. (See Neukrich [12] page 467.)

For $\zeta(s)$, Hardy and Littlewood [2] provided the approximate functional equation

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + \pi^{s-1/2} \Gamma\left(\frac{(1-s)/2}{\Gamma(s/2)}\right) \sum_{n \leq Y} \frac{1}{n^{1-s}} + O(X^{-\sigma}) + O(Y^{\sigma-1}|t|^{-\sigma+1/2}),$$

where $s = \sigma + it$, $0 \leq \sigma \leq 1$, $X > H > 0$, $Y > H > 0$, and $2\pi XY = |t|$, with the constant implied by the big-$O$ term depending on $H$ only. Such approximate functional equations motivate the

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study of properties of the partial sums $F_X(s)$ of $\zeta(s)$ defined by

$$F_X(s) := \sum_{n \leq X} \frac{1}{n^s}.$$ 

Gonek and one of the authors [5] studied the distribution of zeros of the partial sums $F_X(s)$. The authors denote the number of typical zeros $\rho_X = \beta_X + i\gamma_X$ of the partial sums $F_X(s)$ with ordinates $0 \leq \gamma_X \leq T$ by $N_X(T)$. In the case that $T$ is the ordinate of a zero, they define $N_X(T)$ as $\lim_{\epsilon \to 0^+} N_X(T + \epsilon)$. In [5], the authors are concerned with results on $N_X(T)$ as both $X$ and $T$ tends to infinity.

Theorem 1 in [5] collects together a number of known results on the zeros of $F_X(s)$ (see Borwein, Fee, Ferguson and Waal [1], Montgomery [10], and Montgomery and Vaughan [11]), which can be summarized as follows:

The zeros of $F_X(s)$ lie in the strip $\alpha < \sigma < \beta$, where $\alpha$ and $\beta$ are the unique solutions of the equations $1 + 2^{-\sigma} + \cdots + (X - 1)^{-\sigma} = X^{-\sigma}$ and $2^{-\sigma} + 3^{-\sigma} + \cdots + X^{-\sigma} = 1$, respectively. In particular, $\alpha > -X$ and $\beta < 1.72865$. Furthermore, there exists a number $X_0$ such that if $X \geq X_0$, then $F_X(s)$ has no zeros in the half-plane

$$\sigma \geq 1 + \left(\frac{4}{\pi} - 1\right) \frac{\log \log X}{\log X}.$$ 

On the other hand, for any constant $C$ satisfying the inequalities $0 < C < 4/\pi - 1$ there exists a number $X_0$ depending on $C$ only such that if $X \geq X_0$, then $F_X(s)$ has zeros in the half-plane

$$\sigma > 1 + \frac{C \log \log X}{\log X}.$$ 

Theorem 2 in [5] (see also Langer [9]) can be summarized as follows:

If $X$ and $T$ are both greater than or equal to 2, then one has

$$\left| N_X(T) - \frac{T}{2\pi} \log |X| \right| < \frac{X}{2}.$$ 

Here and henceforth, $[X]$ denotes the greatest integer less than or equal to $X$. Chandrasekharan and Narasimhan [2] gave an approximate functional equation for the Dedekind zeta function

$$(1) \quad \zeta_K(s) = \sum_{n \leq X} \frac{a(n)}{n^s} + B 2^{s-1} A(1-s) \sum_{n \leq Y} \frac{a(n)}{n^{1-s}} + O(X^{1-\sigma-1/n_0} \log X),$$

where $A(s) = \Gamma(s/2)\Gamma'(s), B = 2\pi^s n_0/\sqrt{|d_K|}$, $X > H > 0$, $Y > H > 0$, $XY = |d_K|(|t|/2\pi)^{n_0}$, and $C_1 < X/Y < C_2$ for some constants $C_1$ and $C_2$. In the present article, we investigate the distribution of zeros of the partial sums of the function $\zeta_K(s)$ defined by

$$\zeta_{K,X}(s) := \sum_{\|a\| \leq X} \frac{1}{\|a\|^s} = \sum_{n \leq X} \frac{a(n)}{n^s},$$

which appears in the approximate functional equation (1). Our purpose is to determine whether the partial sums $\zeta_{K,X}(s)$ exhibit similar properties. To this end, we denote the number of non-real zeros $\rho_{K,X} = \beta_{K,X} + i\gamma_{K,X}$ of the partial sums $\zeta_{K,X}(s)$ with ordinates $0 \leq \gamma_{K,X} \leq T$ by $N_{K,X}(T)$. If $T$ is the ordinate of a zero, then $N_{K,X}(T)$ is to be defined by $\lim_{\epsilon \to 0^+} N_{K,X}(T + \epsilon)$.

Our first result about the zeros of $\zeta_{K,X}(s)$ is summarized as follows.
Proposition 1. Let $K$ be an arbitrary algebraic number field of degree $n_0 = [K: Q]$ over the field $Q$ of rational numbers, let $X$ be a real number greater than or equal to 2, and denote by $s$ the complex variable $\sigma + it$. Then there exist two real numbers $\alpha$ and $\beta$, with $\alpha$ depending on $n_0$ and $X$ only and with $\beta$ depending on $n_0$ only, such that the zeros of the partial sums $\zeta_{K,X}(s)$ all lie within the rectilinear strip of the complex plane given by the inequalities $\alpha < \sigma < \beta$.

Our second theorem provides an approximate formula for $N_{K,X}(T)$, the number of zeros of the partial sums $\zeta_{K,X}(s)$ in the rectangle determined by the inequalities $\alpha < \sigma < \beta$ and $0 < t < T$, where $\alpha$ and $\beta$ are provided in Proposition 1. Let $K$ be any algebraic number field of degree $n_0 = [K: Q]$ over the field $Q$ of rational numbers. In a similar fashion to the case of Riemann zeta function (see [5] and [9]), it can be shown that

(2) \[ |N_{K,X}(T) - \frac{T}{2\pi} \log[X] | \leq \frac{X}{2}, \]

where $X$ and $T$ both go to infinity together. However, if $K = Q(\zeta_q)$ is a cyclotomic field, where $q \geq 2$, we can significantly improve the error term in (2).

Theorem 1. Let $q \geq 2$, let $\zeta_q$ be a primitive root of unity of order $q$, let $K = Q(\zeta_q)$, and let $T, X \geq 3$. Let, further, $N$ be the largest integer less than or equal to $X$ such that $a(N) \neq 0$. We have

(3) \[ N_{K,X}(T) = \frac{T}{2\pi} \log N + O_q \left( X \left( \frac{\log \log X}{\log X} \right)^{1-\phi(q)} \right), \]

where $\phi$ is Euler’s totient function.

2. Preliminary Results

To prove Theorem 1, we will make use of two auxiliary lemmas.

Lemma 1. Fix a positive integer $q \geq 2$. We have

\[ \# \{ n \leq y : \mu(n) \neq 0 \text{ and } p \mid n \text{ imply } p \equiv 1 \pmod{q} \} = O_q \left( y \left( \frac{\log \log y}{\log y} \right)^{1-\phi(q)} \right), \]

where $\mu$ denotes the M"obius function.

Proof. Fix a positive integer $q \geq 2$ and define

\[ B(q,y) := \{ n \leq y : \mu(n) \neq 0 \text{ and } p \mid n \text{ imply } p \equiv 1 \pmod{q} \}. \]

We apply Brun’s pure sieve to estimate the size of the set $B(q,y)$. (See Murty and Cojocaru [3] page 86.) Let $A$ be the set of all positive integers $n \leq y$. Let $P$ be the set of all primes $p$ congruent to 1 modulo $q$. Let $A_p$ be the set of elements of $A$ which are divisible by $p$. Let, further, $A_1 := A$ and $A_d := \bigcap_{p \mid d} A_p$, where $d$ is a square-free positive integer composed of a list of prime factors from $P$. For any positive real number $z$, we define

\[ S(A, P, z) := A \setminus \bigcup_{p \mid P(z)} A_p, \]

where

\[ P(z) := \prod_{p \in P} p. \]

We consider the multiplicative function $\omega$ defined for all primes $p$ by $\omega(p) := 1$. We have

\[ \#A_d = \# \{ n \leq y : n \equiv 0 \pmod{d} \} = \frac{\omega(d)}{d} y + R_d, \]
where
\[ |R_d| \leq \omega(d). \]

From Mertens’s estimates, we have
\[ \sum_{p \in P, p < z} \frac{\omega(p)}{p} = \frac{\phi(q) - 1}{\phi(q)} \log \log z + O(1). \]

For the sake of brevity, we let
\[ W(z) := \prod_{p \mid P(z)} \left( 1 - \frac{\omega(p)}{p} \right). \]

By Brun’s pure sieve, we have
\[ \#S(A, P, z) = y W(z) \left( 1 + O \left( \frac{z}{\log z} \right) \right) + O \left( \frac{1}{\phi(q)} \right), \]
where \( A = \eta \log \eta \) and, for some \( \alpha < 1 \), \( \eta = \frac{\alpha \log y}{\log z \log \log z} \).

Since \( \omega(p) = 1 \), Mertens’s estimates yield
\[ W(z) = O \left( \frac{1}{(\log z)^{1-1/\phi(q)}} \right). \]

We now choose \( \log z = c \log y / \log \log y \). Then for a suitable positive and sufficiently small constant \( c \) and from (4) and (5), we have
\[ \#S(A, P, z) = O \left( y \left( \frac{\log \log y}{\log y} \right)^{1-1/\phi(q)} \right). \]

Since \( B(q, y) \subseteq S(A, P, z) \), we have \( \#B(q, z) \leq \#S(A, P, z) \). Employing this last inequality together with (6), we complete the proof of Lemma 1. \( \square \)

**Lemma 2.** Let \( q \geq 2 \) and let \( K = Q(\zeta_q) \). Let, further,
\[ \zeta_K(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \]

We have
\[ \# \{ n \leq x : a(n) \neq 0 \} = O_q \left( x \left( \frac{\log \log x}{\log x} \right)^{1-1/\phi(q)} \right). \]

**Proof.** Let \( K = Q(\zeta_q) \), where \( \zeta_q \) is a primitive root of unity of order \( q \). We have
\[ \zeta_K = \prod_{P \mid q} \left( 1 - \frac{1}{P^s} \right)^{-1} F_q(s), \]
where
\[ F_q(s) = \prod_{\chi \equiv \mod{q}} L(s, \chi). \]

(See [12 page 468].) For \( \sigma > 1 \), we have
\[ F_q(s) = \prod_{\chi \equiv \mod{q}} \prod_{p \text{ prime}} \left( 1 - \frac{\chi(p)}{p^s} \right). \]
Hence, for $\sigma > 1$, we have

$$\log F_q(s) = - \sum_{\chi \equiv \pmod{q}} \sum_{p \text{ prime}} \log \left(1 - \frac{\chi(p)}{p^s}\right)$$

$$= \sum_{\chi \equiv \pmod{q}} \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{\chi(p)}{mp^{ms}}$$

$$= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \sum_{\chi \equiv \pmod{q}} \chi(p^m),$$

where

$$\sum_{\chi \equiv \pmod{q}} \chi(p^m) = \begin{cases} \phi(q), & \text{if } p^m \equiv 1 \pmod{q}; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\log F_q(s) = \sum_{p \text{ prime}, m \geq 1 \atop p^m \equiv 1 \pmod{q}} \frac{\phi(q)}{mp^{ms}}.$$ 

Hence, we have

$$F_q(s) = \exp \left( \sum_{p \text{ prime}, m \geq 1 \atop p^m \equiv 1 \pmod{q}} \frac{\phi(q)}{mp^{ms}} \right).$$

Now, for $\sigma > 1$,

$$F_q(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \ldots \right).$$

Thus, we have

$$\log F_q(s) = \sum_{p \text{ prime}} \log \left(1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \ldots \right)$$

$$= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} (-1)^m \frac{(c(p))^{m}}{m} \left( \frac{c(p^2)}{p^{2s}} + \ldots \right)^m,$$

and hence

$$c(p) = \begin{cases} \phi(q), & \text{if } p \equiv 1 \pmod{q}; \\ 0, & \text{if } p \not\equiv 1 \pmod{q}. \end{cases}$$

For all $n$ such that $c(n) \neq 0$, we have $n = AB$, where $A$ is coprime to $B$, $A$ is squareful, and $B$ is square-free, that is, $\mu(B) \neq 0$. Furthermore, all the prime factors of $B$ are congruent to 1 modulo $q$. Letting

$$H(x) := \prod_{p \leq x, p \text{ prime} \atop p \equiv 1 \pmod{q}} p,$$
we have
\[
\#\{n \leq x: c(n) \neq 0\} \leq \#\{(A, B): A \text{ squareful, } \mu(B) \neq 0, AB \leq x, B \mid H(x)\}
\]
\[
= \sum_{A \leq x} \sum_{\substack{B \leq x/A \mid H(x)\ A \text{ squareful}}} 1
\]
\[
= \sum_{A \leq x} B\left(q, \frac{x}{A}\right)
\]
\[
= \sum_{A \leq \sqrt{x} \log x} B\left(q, \frac{x}{A}\right) + \sum_{\sqrt{x} \log x \leq A \leq x} B\left(q, \frac{x}{A}\right).
\]

We examine the sums on the far right-hand side separately.

Using Lemma \[\Pi\], we see that
\[
\sum_{A \leq \sqrt{x} \log x} B\left(q, \frac{x}{A}\right) = O\left(\sum_{A \leq \sqrt{x} \log x} x\left(\log \log \frac{x}{\log x}\right)^{1-1/\phi(q)}\right)
\]
\[
= O\left(x\left(\frac{\log \log x}{\log x}\right)^{1-1/\phi(q)} \sum_{A \leq \sqrt{x} \log x} \frac{1}{A}\right)
\]
\[
= O\left(x\left(\frac{\log \log x}{\log x}\right)^{1-1/\phi(q)} \sum_{a \geq 1, b \geq 1} \frac{1}{a^2 b^3}\right)
\]
\[
= O\left(x\left(\frac{\log \log x}{\log x}\right)^{1-1/\phi(q)}\right).
\]

Furthermore, we have
\[
\sum_{\sqrt{x} \log x \leq A \leq x} B\left(q, \frac{x}{A}\right) \leq \sum_{\sqrt{x} \log x \leq A \leq x} \frac{x}{A}
\]
\[
\leq \sum_{\sqrt{x} \log x \leq A \leq x} \frac{x}{\sqrt{x} \log x}
\]
\[
\leq \frac{x}{\log x} \#\{A \leq x: A \text{ squareful}\}
\]
\[
= O\left(\frac{x}{\log x}\right).
\]

Suppose that \(\mathcal{P}_1, \ldots, \mathcal{P}_r\) are the prime ideals in the ring of integers of \(K\) lying over the prime factors of \(q\) and consider the Dirichlet series
\[
\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \prod_{\mathcal{P} \mid q} \left(1 - \frac{1}{\|\mathcal{P}\|^s}\right)^{-1}.
\]

For all \(z\), we have
\[
\#\{n \leq z: b(n) \neq 0\} \leq \#\{n \leq z \text{ with all prime factors of } n \text{ in the sets } \mathcal{P}_1, \ldots, \mathcal{P}_r\}.
\]
It is well-known that the right-hand side of (7) is $O_q((\log z)^r)$. Thus, we have
\[
\#\{n \leq z: b(n) \neq 0\} = O_q((\log z)^r).
\]

For brevity’s sake, we let
\[
A = \{n: a(n) \neq 0\}, \quad B = \{m: b(m) \neq 0\}, \quad C = \{k: c(k) \neq 0\},
\]
and denote
\[
A_\omega = A \cap [1, \omega], \quad B_\omega = B \cap [1, \omega], \quad C_\omega = C \cap [1, \omega].
\]
Here, we note that
\[
\#B_\omega = O_r((\log \omega)^r)
\]
and
\[
\#C_\omega = O_q\left(\omega \left(\frac{\log \log \omega}{\log \omega}\right)^{1-1/\phi(q)}\right).
\]
Furthermore, we have
\[
\zeta_K(s) = \sum_{n \in A} a(n) n^s = \sum_{m \in B} b(m) m^s \sum_{k \in C} c(k) k^s.
\]

On noting that $A \subseteq BC$, where $BC = \{bc: b \in B, c \in C\}$, we have $A_x \subseteq (BC)_x$. It follows that
\[
\#A_x \leq \#(BC)_x,
\]
where
\[
(BC)_x = \sum_{b \leq L} \sum_{c \leq x/b} 1 = \sum_{b \leq L} \sum_{c \leq x/b} 1 + \sum_{L < b \leq x} \sum_{c \leq x/b} 1,
\]
with $1 \leq L \leq x$ (to be chosen later). By (8), we have
\[
\sum_{b \leq L} \sum_{c \leq x/b} 1 \leq \sum_{b \leq L} \sum_{c \leq x/b} \#C_{x/b} = O\left(\sum_{b \leq L} \frac{x}{b} \left(\frac{\log \log (x/b)}{\log (x/b)}\right)^{1-1/\phi(q)}\right).
\]

Since $b \leq L$, we have
\[
\left(\frac{\log x}{b}\right)^{1-1/\phi(q)} > \left(\frac{\log x}{L}\right)^{1-1/\phi(q)}.
\]
Hence, we have
\[
\sum_{b \leq L} \sum_{c \leq x/b} 1 = O\left(x \left(\frac{\log \log x}{\log x/L}\right)^{1-1/\phi(q)} \sum_{b \leq L} \frac{1}{b}\right) = O\left(x \left(\frac{\log \log x}{\log (x/L)}\right)^{1-1/\phi(q)}\right),
\]
since
\[
\sum_{b \leq L} \frac{1}{b} < \infty.
\]

Next, we have
\[
\sum_{L < b \leq x} \sum_{c \leq x/b} 1 = \sum_{L < b \leq x} \sum_{c \leq x/b} \#C_{x/b} \leq \sum_{L < b \leq x} \frac{x}{b} \leq \frac{x}{L} \#B_x = O\left(\frac{x(\log x)^r}{L}\right).
\]
In view of (9), we substitute (11) and (12) into (10) to obtain
\[
\#A_x = O \left( \frac{x(\log x)^r}{L} \right) + O \left( x \left( \frac{\log \log x}{\log(x/L)} \right)^{1-1/\phi(q)} \right).
\]
Then choosing \( L = (\log x)^{r+1} \), we obtain
\[
\#A_x = O \left( x \left( \frac{\log \log x}{\log x} \right)^{1-1/\phi(q)} \right).
\]
This finishes the proof of Lemma 2. \( \square \)

3. Proof of Proposition 1

We show separately that \(|\zeta_{K,X}(s)| > 0\) in the right half-plane \( \sigma \geq \beta \) and in the left-half plane \( \sigma \leq \alpha \). More specifically, we want to find a \( \beta \) so that
\[
1 - \sum_{2 \leq n \leq X} \frac{a(n)}{n^\sigma} > 0,
\]
for \( \sigma \geq \beta \). Toward this end, we employ the upper bound \( a(n) \leq d(n)^{n_0-1} \), where \( d(n) \) denotes the number of divisors of \( n \) (see Chandrasekharan and Narasimhan [2], Lemma 9) and satisfies the upper bound \( d(n) \leq C_{\epsilon_0} n^{\epsilon_0} \) for all positive \( \epsilon_0 \) (see Hardy and Wright [6], Chapter XVIII, Theorem 317). Hence, we have \( a(n) \leq C_{\epsilon_0,n_0} n^{\epsilon_0 n_0} \).

It is enough to show that
\[
C_{\epsilon_0,n_0} \sum_{n=2}^{\infty} \frac{1}{n^{\sigma-\epsilon_0 n_0}} < 1.
\]
If we let \( \epsilon_0 < 1/n_0 \), then for \( \sigma \geq \beta \) we have
\[
\sum_{n=2}^{\infty} \frac{1}{n^{\sigma-\epsilon_0 n_0}} \leq \sum_{n=2}^{\infty} \frac{1}{n^{\beta-\epsilon_0 n_0}} \leq \frac{1}{2^\beta D_{\epsilon_0,n_0}},
\]
where
\[
D_{\epsilon_0,n_0} = \sum_{n=2}^{\infty} \frac{4}{n^{2-\epsilon_0 n_0}}.
\]
In order to obtain (13), it is enough to have
\[
\beta > \frac{\log C_{\epsilon_0,n_0} D_{\epsilon_0,n_0}}{\log 2}.
\]
We have
\[
\sum_{n=2}^{\infty} \frac{d(n)^{n_0}}{n^{\beta}} \leq C_{\epsilon_0,n_0} \sum_{n=2}^{\infty} \frac{1}{n^{\beta-\epsilon_0 n_0}} = \frac{1}{2^\beta} C_{\epsilon_0,n_0} D_{\epsilon_0,n_0}.
\]
Then for \( \sigma \geq \beta \), we have
\[
\left| \sum_{2 \leq n \leq X} \frac{a(n)}{n^s} \right| \leq \sum_{2 \leq n \leq X} \frac{d(n)^{n_0}}{n^{\beta}} < 1,
\]
and hence
\[
|\zeta_{K,X}(s)| \geq 1 - \left| \sum_{2 \leq n \leq X} \frac{a(n)}{n^s} \right| > 0.
\]
Therefore, \( \zeta_{K,X}(s) \neq 0 \) on the right-half plane \( \sigma \geq \beta \).
Next, let $N$ be the largest positive integer less than or equal to $X$ for which the coefficient $a(N)$ is nonzero. Since

$$|ζ_K,X(s)| ≥ \frac{a(N)}{N^σ} - \left| \sum_{1 ≤ n ≤ N-1} \frac{a(n)}{n^σ} \right|,$$

it is enough to find an $α$ such that

$$\frac{1}{N^σ} > \sum_{1 ≤ n ≤ N-1} \frac{a(n)}{n^σ},$$

for $σ ≤ α$.

To this end, let us fix $δ_0 > 0$. Then there exist constants $C_{δ_0} > 0$ and $n_{δ_0} ∈ \mathbb{Z}^+$ such that for all $1 ≤ n < n_{δ_0}$, we have

$$d(n) ≤ C_{δ_0}n(δ_0+log 2)/log log n,$$

and that for all $n ≥ n_{δ_0}$, we have

$$d(n) ≤ n^{(δ_0+log 2)/log log n}.$$

(See Wigert [15].)

It suffices to have

$$\frac{1}{N^σ} > C_{δ_0}^{n_{δ_0}} \sum_{1 ≤ n ≤ n_{δ_0} - 1} n^{(δ_0+log 2)n_{δ_0}/log log n} \sigma n^{σ} + \sum_{n_{δ_0} ≤ n ≤ N-1} n^{(δ_0+log 2)n_{δ_0}/log log n} \sigma n^{σ} = 1 + C_{δ_0}^{n_{δ_0}} S_I(n_{δ_0}, δ_0, n_{δ_0}, σ) + S_{II}(n_{δ_0}, δ_0, σ),$$

for $σ ≤ α$, where

$$S_I(n_{δ_0}, δ_0, n_{δ_0}, σ) = \sum_{2 ≤ n ≤ n_{δ_0} - 1} n^{(δ_0+log 2)n_{δ_0}/log log n} \sigma n^{σ}$$

and

$$S_{II}(n_{δ_0}, δ_0, σ) = \sum_{n_{δ_0} ≤ n ≤ N-1} n^{(δ_0+log 2)n_{δ_0}/log log n} \sigma n^{σ}.$$

This would follow from the inequality

$$\frac{1}{N^σ} > 1 + C_{δ_0}^{n_{δ_0}} S_I(n_{δ_0}, δ_0, n_{δ_0}, α) + S_{II}(n_{δ_0}, δ_0, α),$$

since, for any $σ ≤ α$,

$$\frac{1}{N^σ} > \frac{1}{N^{σ-α}} \left[ 1 + C_{δ_0}^{n_{δ_0}} S_I(n_{δ_0}, δ_0, n_{δ_0}, α) + S_{II}(n_{δ_0}, δ_0, α) \right] \frac{1}{N^{σ-α} n^{σ}} + \sum_{n_{δ_0} ≤ n ≤ N-1} n^{(δ_0+log 2)n_{δ_0}/log log n} \sigma n^{σ-α} n^{α}$$

$$= 1 + C_{δ_0}^{n_{δ_0}} \sum_{2 ≤ n ≤ n_{δ_0} - 1} n^{(δ_0+log 2)n_{δ_0}/log log n} \sigma n^{σ-α} n^{α} + \sum_{n_{δ_0} ≤ n ≤ N-1} n^{(δ_0+log 2)n_{δ_0}/log log n} \sigma n^{σ-α} n^{α}$$

$$= 1 + C_{δ_0}^{n_{δ_0}} S_I(n_{δ_0}, δ_0, n_{δ_0}, σ) + S_{II}(n_{δ_0}, δ_0, σ).$$

Thus, it is enough to find $α$ such that

(15) $$\frac{1}{N^α} > 2 + 2C_{δ_0}^{n_{δ_0}} S_I(n_{δ_0}, δ_0, n_{δ_0}, α)$$

and such that

(16) $$\frac{1}{N^α} > 2S_{II}(n_{δ_0}, δ_0, α).$$
It is enough to have
\begin{equation}
\frac{1}{N^{\alpha}} > 2 + 2C_{\delta_0}^{n_0} \frac{1}{n_{\delta_0}} \sum_{2 \leq n \leq n_{\delta_0} - 1} n^{(\delta_0 + \log 2)n_0/\log \log n},
\end{equation}
since the right-hand side of (17) is greater than the right-hand side of (15).

The inequality in (17) holds for any fixed \( \alpha < 0 \) and for all \( N \) large enough in terms of \( n_0, \delta_0, C_{\delta_0}, \) and \( \alpha \). Therefore, we may take any fixed \( \alpha < 0 \) as a function of \( N, n_0, \) and \( \delta_0 \) for which (16) holds true. For \( n_{\delta_0} \geq 16 \), we see that
\begin{equation}
\sum_{n_{\delta_0} \leq n \leq N - 1} n^{(\delta_0 + \log 2)n_0/\log \log n} \leq \sum_{n_{\delta_0} \leq n \leq N - 1} \frac{N^{(\delta_0 + \log 2)n_0/\log \log N}}{n^{\alpha}} < N^{(\delta_0 + \log 2)n_0/\log \log N} \sum_{n_{\delta_0} \leq n \leq N - 1} \frac{1}{n^{\alpha}}.
\end{equation}

It remains to examine the sum on the far-right hand side of (18).

For \( \alpha < 0 \), we have
\begin{equation}
\sum_{n_{\delta_0} \leq n \leq N - 1} \frac{1}{n^{\alpha}} \leq (N - 1)^{-\alpha} + \int_{n_{\delta_0}}^{N - 1} \frac{dy}{y^{\alpha}} < (N - 1)^{-\alpha} \left( \frac{N - \alpha}{1 - \alpha} \right).
\end{equation}

It follows from (18) that (16) is consequence of
\begin{equation}
N^{-\alpha} > 2N^{(\delta_0 + \log 2)n_0/\log \log N} (N - 1)^{-\alpha} \left( \frac{N - \alpha}{1 - \alpha} \right).
\end{equation}

One sees that an admissible choice of \( \alpha \) is given by
\[ \alpha = -3(\delta_0 + \log 2)n_0 \frac{N \log N}{\log \log N}. \]
Then \( \zeta_{K,X}(s) \neq 0 \) in the left-half plane \( \sigma \leq \alpha \). This completes the proof of Proposition 1.

4. Proof of Theorem 1

Assuming for simplicity’s sake that \( T \) does not coincide with the ordinate of any zero, we have
\[ N_{K,X}(T) = \frac{1}{2\pi i} \int_R \frac{\zeta'_{K,X}(s)}{\zeta_{K,X}(s)} ds, \]
where \( R \) is the rectangle with vertices at \( \alpha, \beta, \beta + iT, \) and \( \alpha + iT \). Thus, we have
\begin{equation}
2\pi N_{K,X}(T) = \int_R \text{Im} \left( \frac{\zeta'_{K,X}(s)}{\zeta_{K,X}(s)} \right) ds = \Delta_R \text{arg} \zeta_{K,X}(s),
\end{equation}
where \( \Delta_R \) denotes the change in \( \text{arg} \zeta_{K,X}(s) \) as \( s \) traverses \( R \) in the positive sense.

Since \( \zeta_{K,X}(s) \) is real and nonzero on \( [\alpha, \beta] \), we have
\begin{equation}
\Delta_{[\alpha,\beta]} \text{arg} \zeta_{K,X}(\sigma) = 0.
\end{equation}

As \( s \) describes the right edge of \( R \), we observe from (14) that
\[ |\zeta_{K,X}(s) - 1| < 1. \]
It follows that \( \text{Re} \zeta_{K,X}(\beta + it) > 0 \) for \( 0 \leq t \leq T \). Hence, we have
\begin{equation}
\Delta_{[0,T]} \text{arg} \zeta_{K,X}(\beta + it) = O(1).
\end{equation}
Furthermore, along the top edge of $R$, to estimate the change in $\arg \zeta_{K,X}(s)$ we decompose $\zeta_{K,X}(s)$ into its real part and its imaginary part. We have

$$
\zeta_{K,X}(s) = \sum_{n \leq [X]} a(n) \exp\{-\sigma t \log n\} = \sum_{n \leq [X]} a(n) \frac{\cos(\sigma t \log n) - i \sin(\sigma t \log n)}{n^\sigma}.
$$

so that

$$
\text{Im}(\zeta_{K,X}(\sigma + iT)) = - \sum_{n \leq [X]} a(n) \frac{\sin(T \log n)}{n^\sigma}.
$$

By a generalization of Descartes’s Rule of Signs (see Pólya and Szegö [13], Part V, Chapter 1, No. 77), the number of real zeros of $\text{Im}(\zeta_{K,X}(\sigma + iT))$ in the interval $\alpha \leq \sigma \leq \beta$ is less than or equal to the number of nonzero coefficients $a(n) \sin(T \log n)$. By Lemma 2 the number of nonzero coefficients $a(n)$ is $O(X(\log \log X/(\log X)^{1-1/\phi(q)})$ at most.

Since the change in argument of $\zeta_{K,X}(\sigma + iT)$ between two consecutive zeros of $\text{Im}(\zeta_{K,X}(\sigma + iT))$ is at most $\pi$, it follows that

$$
\triangle_{[\alpha, \beta]} \arg \zeta_{K,X}(\sigma + iT) = O \left(X \left(\frac{\log \log X}{\log X}\right)^{1-1/\phi(q)}\right).
$$

As in the proof of Proposition 1, we let $N$ be the largest integer less than or equal to $X$ so that $a(N) \neq 0$. Along the left edge of $R$, we have

$$
\zeta_{K,X}(\alpha + it) = \left[1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right] a(N)N^{-\alpha - it}.
$$

Therefore, we have

$$
\triangle_{[0, T]} \arg \zeta_{K,X}(\alpha + it) = \triangle_{[0, T]} \arg \left[1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right]
+ \triangle_{[0, T]} \arg a(N)N^{-\alpha - it}.
$$

In the proof of Proposition 1 we noticed that

$$
\frac{a(N)}{N^\alpha} > \sum_{1 \leq n \leq N-1} \frac{a(n)}{n^\alpha}.
$$

Thus, for any $t$, we have

$$
\left|1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right| < 1,
$$

and hence

$$
\triangle_{[0, T]} \arg \left[1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right] = O(1).
$$

Finally, we have

$$
\triangle_{[0, T]} \arg a(N)N^{-\alpha - it} = \triangle_{[0, T]} \arg a(N)N^{-\alpha} \exp\{-it \log N\}
= \triangle_{[0, T]} \arg \exp\{-it \log N\}
= -T \log N.
$$

Then substituting (24) and (25) into (23), we obtain

$$
\triangle_{[0, T]} \arg \zeta_{K,X}(\alpha + it) = -T \log N + O(1).
$$

By a generalization of Descartes’s Rule of Signs (see Pólya and Szegö [13], Part V, Chapter 1, No. 77), the number of real zeros of $\text{Im}(\zeta_{K,X}(\sigma + iT))$ in the interval $\alpha \leq \sigma \leq \beta$ is less than or equal to the number of nonzero coefficients $a(n) \sin(T \log n)$. By Lemma 2 the number of nonzero coefficients $a(n)$ is $O(X(\log \log X/(\log X)^{1-1/\phi(q)})$ at most.

Since the change in argument of $\zeta_{K,X}(\sigma + iT)$ between two consecutive zeros of $\text{Im}(\zeta_{K,X}(\sigma + iT))$ is at most $\pi$, it follows that

$$
\triangle_{[\alpha, \beta]} \arg \zeta_{K,X}(\sigma + iT) = O \left(X \left(\frac{\log \log X}{\log X}\right)^{1-1/\phi(q)}\right).
$$

As in the proof of Proposition 1, we let $N$ be the largest integer less than or equal to $X$ so that $a(N) \neq 0$. Along the left edge of $R$, we have

$$
\zeta_{K,X}(\alpha + it) = \left[1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right] a(N)N^{-\alpha - it}.
$$

Therefore, we have

$$
\triangle_{[0, T]} \arg \zeta_{K,X}(\alpha + it) = \triangle_{[0, T]} \arg \left[1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right]
+ \triangle_{[0, T]} \arg a(N)N^{-\alpha - it}.
$$

In the proof of Proposition 1 we noticed that

$$
\frac{a(N)}{N^\alpha} > \sum_{1 \leq n \leq N-1} \frac{a(n)}{n^\alpha}.
$$

Thus, for any $t$, we have

$$
\left|1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right| < 1,
$$

and hence

$$
\triangle_{[0, T]} \arg \left[1 + \frac{1 + a(2)2^{-\alpha - it} + \ldots + a(N - 1)(N - 1)^{-\alpha - it}}{a(N)N^{-\alpha - it}}\right] = O(1).
$$

Finally, we have

$$
\triangle_{[0, T]} \arg a(N)N^{-\alpha - it} = \triangle_{[0, T]} \arg a(N)N^{-\alpha} \exp\{-it \log N\}
= \triangle_{[0, T]} \arg \exp\{-it \log N\}
= -T \log N.
$$

Then substituting (24) and (25) into (23), we obtain

$$
\triangle_{[0, T]} \arg \zeta_{K,X}(\alpha + it) = -T \log N + O(1).
$$
Since
\[ \Delta_R \arg \zeta_{K,X}(s) = \Delta_{[\alpha, \beta]} \arg \zeta_{K,X}(\sigma) + \Delta_{[0,T]} \arg \zeta_{K,X}(\beta + it) - \Delta_{[\alpha, \beta]} \arg \zeta_{K,X}(\sigma + iT) - \Delta_{[0,T]} \arg \zeta_{K,X}(\alpha + it), \]
we may now substitute (20), (21), (22), (26) into (19) to obtain Theorem 1.

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