A concentration of measure result for non-catalytic decoupling via approximate unitary $t$-designs

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Abstract

Decoupling theorems are an important tool in quantum information theory where they are used as building blocks in a host of information transmission protocols. A decoupling theorem takes a bipartite quantum state shared between a system and a reference, applies some local operation on the system, and then, if suitable conditions are met, proves that the resulting state is close to a product state between the output system and the untouched reference. The theorem is said to be non-catalytic if it does not require an additional input of a quantum state, in tensor with the given input state, in order to perform the decoupling. Dupuis [Dup10] proved an important non-catalytic decoupling theorem where the operation on the system was a Haar random unitary followed by a fixed superoperator, unifying many decoupling results proved earlier. He also showed a concentration result for his decoupling theorem viz. with probability exponentially close to one a Haar random unitary gives rise to a state close to a product state.

In this paper we give a new concentration result for non-catalytic decoupling by showing that, for suitably large $t$, a unitary chosen uniformly at random from an approximate $t$-design gives rise to a state close to a product state with probability exponentially close to one. A unitary $t$-design is a finite set of unitaries with the property that the first $t$-moments of the matrix entries have the same expectation under the uniform distribution on the finite set as under the Haar measure over the full unitary group. Our concentration, though exponential, is less than that of Dupuis. However for many important applications it uses less random bits than Dupuis. In particular, we prove that approximate $|A_1|$-designs decouple a quantum system in the Fully Quantum Slepian Wolf (FQSW) theorem wherein the fixed superoperator traces out the subsystem $A_2$ from a system $A_1 \otimes A_2$. This immediately leads to a saving in the number of random bits to $O(|A_1| \log(|A_1||A_2|))$ from $\Omega(|A_1|^2|A_2|^2 \log(|A_1||A_2|))$ required by Haar random unitaries. Moreover, if $|A_1| = \text{polylog}(|A_2|)$, efficient constructions of approximate $|A_1|$-designs exist. Furthermore, this result implies that approximate unitary $|A_1|$-designs achieve relative thermalisation in quantum thermodynamics with exponentially high probability. Previous works using unitary designs [SDTR13, NHMW17] did not obtain exponentially high concentration.

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I. INTRODUCTION

A peculiar characteristic of quantum information theory is that many information transmission protocols, be it compression of quantum messages or sending quantum information through unassisted quantum channels, can be constructed by first removing correlations of a particular system from some other systems around it. This behooves us to prove general theorems that take a bipartite quantum state shared between a system \( A \) (e.g. the “particular system” above) and a reference \( R \) (e.g. the “some other systems” above), apply some local operation on \( A \), and then, if suitable conditions are met, prove that the resulting state is close to a product state between the output system \( B \) and the untouched reference \( R \). This process of removing quantum correlations i.e. obtaining a state close to a product state, is referred to as decoupling. Decoupling theorems play a vital role in proving achievability bounds for several quantum information theory protocols as well as thermalisation results in quantum thermodynamics. In particular, the so-called Fully Quantum Slepian Wolf (FQSW) protocol [ADHW09], which has been hailed as the mother protocol of quantum information theory, is constructed via a decoupling argument. In the FQSW problem, the system \( A \) is thought of as a bipartite system \( A = A_1 \otimes A_2 \) and the fixed superoperator is nothing but tracing out \( A_2 \). The FQSW protocol is used as a building block for many other important protocols in quantum information theory in the asymptotic iid setting e.g. noisy teleportation, noisy super dense coding, distributed compression, entanglement unassisted and assisted quantum channel coding, one way entanglement distillation, reverse Shannon theorem etc. Asymptotic iid setting means that the given messages / channels are of the tensor power form \((\cdot)^\otimes n\) for large \( n \). However the basic decoupling and FQSW results are actually one-shot results where the given message / channel is to be used only once. The one shot FQSW result can be immediately used to obtain one-shot relative thermalisation results in quantum thermodynamics [dHRW16], where a system \( A = A_1 \otimes A_2 \) (with \( A_1 \) being the subsystem of physical interest e.g. the subsystem with a certain energy bound) initially starts out in a correlated state together with its environment \( R \) but very soon evolves into something close to a completely mixed state on \( A_1 \) (called a relative thermal state) tensored with the reduced state on the environment \( R \).

In this paper, we build on the following important decoupling theorem proved by Dupuis in his doctoral thesis [Dup10].
Fact 1. Consider a quantum state $\rho^{AR}$ shared between a system $A$ and a reference $R$. Let $\mathcal{T}^{A\rightarrow B}$ be a completely positive trace preserving superoperator (aka CPTP map aka quantum operation) with input system $A$ and output system $B$. Define

$$\sigma^{BR} := (\mathcal{T}^{A\rightarrow B} \otimes \mathbb{I}^R)((U^A \otimes I^R)\rho^{AR}(U^A \dagger \otimes I^R)).$$

Then,

$$\mathbb{E}_{U^A}||\sigma^{BR} - \omega B \otimes \rho^R||_1 \leq 2^{-\frac{1}{2}H_2(A|R)_{\rho} - \frac{1}{2}H_2(A'|B)_{\omega}},$$

where the expectation is taken over the Haar measure on unitary operators on $A$, $\mathbb{I}^R$ is the identity superoperator on $R$, $\mathbb{I}^R$ is the identity operator on $R$, $\omega^{AB} := (\mathcal{T}^{A\rightarrow B} \otimes \mathbb{I}^{A'})(|\Phi\rangle\langle \Phi|^{AA'})$, $|\Phi\rangle^{AA'} := |A|^{-1/2} \sum_a |a\rangle^A \otimes |a\rangle^{A'}$ is the standard EPR state on system $AA'$ where $A'$ has the same dimension as $A$, and $H_2(\cdot|\cdot)$ is the conditional Rényi 2-entropy defined in Definition 1 below. We remark that $H_2(A|R)_{\rho} = -2\log\|\tilde{\rho}^{AR}\|_2$ and $H_2(A'|B)_{\omega} = -2\log\|\tilde{\omega}^{AB}\|_2$, where $\tilde{\rho}^{AR}$ and $\tilde{\omega}^{AB}$ are certain positive semidefinite matrices defined in Definition 1.

Informally speaking, the above theorem states that if some entropic conditions are met then, in expectation, the state $\sigma^{BR}$ obtained by first applying a Haar random unitary $U^A$ on the initial state $\rho^{AR}$ followed by a CPTP map $\mathcal{T}^{A\rightarrow B}$ is close to the decoupled state $\omega B \otimes \rho^R$. Here, $\omega^B = \mathcal{T}^{A\rightarrow B}(I^A|\rangle\langle A|)$ is the state obtained by applying $\mathcal{T}$ to the completely mixed state on $A$. Intuitively, a Haar random unitary $U^A$ ‘randomises’ the state on $A$ to give the completely mixed state which is then sent to $\omega^B$ by $\mathcal{T}^{A\rightarrow B}$. So it is reasonable to believe that the local state on $B$ should be $\omega^B$. Notice that the local state on $R$ after applying $U^A$ and $\mathcal{T}^{A\rightarrow B}$ is always $\rho^R$. The punch of the decoupling theorem is that the global state is close to the desired tensor product state.

The distance of the actual global state from the desired tensor product state is upper bounded by two quantities. The first quantity $H_2(A|R)_{\rho}$ is usually negative, which signifies that $A$ and $R$ are entangled in the initial state $\rho$. To decouple $A$ from $R$ we start by applying a Haar random unitary $U$ to the system $A$. A single unitary cannot decouple $A$ from $R$, and that is why the decoupling theorem above also has the CPTP map $\mathcal{T}$. Now in an intuitive sense, the EPR state $\Phi^{AA'}$ is the ‘most entangled state’. So if a Haar random unitary $U$ on the system $A$ of $\Phi^{AA'}$ followed by the CPTP map $\mathcal{T}$ can decouple the output system $B$
from $R$, then it must be able to decouple $B$ from $R$ when the input is any entangled state $\rho^{AR}$, provided that the ‘amount of entanglement’ between $A$ and $R$ in $\rho$ is less than the ‘amount of entanglement’ between $A$ and $A'$ in $\Phi$. This explains the quantity $H_2(A'|B)_{\omega}$ in the expression above. To counteract a negative $H_2(A|R)_{\rho}$, the quantity $H_2(A'|B)_{\omega}$ had better be positive which signifies that $A'$ is mostly decoupled from $B$ in the state $\omega$.

Dupuis showed in his doctoral thesis how the decoupling theorem above can be used to recover in a unified fashion several previously known results, as well as obtain some totally new results in quantum information theory. Szehr et al. [SDTR13] extended the decoupling theorem by showing that the expectation can be taken over approximate unitary 2-designs (defined formally in Definition 10 below) instead of over Haar random unitaries. The advantage of unitary 2-designs is that efficient constructions for them exist unlike the case with Haar random unitaries. Szehr et al. also bounded the expectation in terms of smooth entropic quantities which have better mathematical properties compared to the non-smooth ones. In particular, in the asymptotic iid limit, the smooth entropic quantities are suitably bounded by $n$ times the corresponding Shannon entropies which is not the case with the non-smooth quantities. Their result is stated below.

**Fact 2.** Under the setting of Fact 1 above,

$$
\mathbb{E}_{U^A}[\|\sigma^{BR} - \omega^B \otimes \rho^R\|_1] \leq 2^{-\frac{1}{2}H_{\min}(A|R)_{\rho}} - \frac{1}{2}H_{\min}(A'|B)_{\omega} + 12\epsilon,
$$

where the expectation is taken over the Haar measure on unitary operators on $A$. The same result holds if the expectation is taken over the uniform choice of a unitary from an exact 2-design. The bound gets multiplied by a dimension dependent multiplicative factor if the expectation is taken over the uniform choice of a unitary from a $\delta$- approximate 2-design. The smooth conditional min-entropy terms appearing in the bound are defined in [SDTR13].

In a different vein Anshu and Jain [AJ18] showed, extending earlier work by Ambainis and Smith [AS04], that it is possible to add a small ancilla $C$ in tensor product with $A$, apply an efficient unitary to $A \otimes C$ and then trace out $C$ so that $A$ is now decoupled from $R$ even before applying the CPTP map $T$. The difference between Ambainis and Smith’s or Anshu and Jain’s works, and Dupuis’, Szehr et al.’s or our works is that we want a single unitary on the system $A$ to achieve decoupling and not the average of a number of unitaries on $A$ or, more generally, a unitary on a larger system $A \otimes C$. A single unitary cannot
decouple $A$ from $R$. That is why the decoupling theorem above also has the CPTP map $T$. The single unitary followed by CPTP map form of the decoupling theorem is required for applications where there is no entanglement assistance e.g. sending quantum information over an unassisted quantum channel.

After obtaining the decoupling result in expectation above, it is natural to ask whether such a theorem also holds with high probability over the choice of the random unitary $U^A$. Dupuis [Dup10] answered this question in the affirmative for the Haar measure.

**Fact 3.** Under the setting of Fact 1 above, we have

$$\mathbb{P}_{U^A}[\|\sigma^{BR} - \omega^B \otimes \rho^R\|_1 > 2^{-\frac{1}{2}H_2(A|R)\rho - \frac{1}{2}H_2(A'|B)\omega + \delta}] \leq 2 \exp\left(-\frac{|A|\delta^2}{16\|\rho^A\|_\infty}\right),$$

where $\|\rho^A\|_\infty$ is the so-called Schatten $\ell_\infty$-norm aka operator norm of $\rho^A$ and is equal to the largest eigenvalue of $\rho^A$, and the probability is taken over the Haar measure on $U^A$.

The concentration of measure result for the decoupling theorem above immediately implies an exponential concentration result for the FQSW problem, which further implies that relative thermalisation occurs for a system in contact with a heat bath for all but an exponentially small fraction of unitary evolutions of the system as long as the system is assumed to evolve according to a Haar random unitary. However this is not a very satisfactory explanation from a physical and computational point of view as Haar random unitaries are provably impossible to implement by quantum circuits with size polylogarithmic in the dimension of the system. Also, Haar random unitaries on a system $A$ require $\Omega(|A|^2 \log |A|)$ number of random bits for a precise description. This leads us to wonder if relative thermalisation can be achieved with high probability by simpler unitary evolutions of the system $A$. Nakata et al. [NHMW17] gave an affirmative answer by showing that decoupling can indeed be achieved by choosing unitaries diagonal in the Pauli $X$ and $Z$ bases, and these unitaries can be implemented by quantum circuits of size polylog($|A|$). However the fraction of unitaries which achieve decoupling is not strongly concentrated near one.

**A. Our results**

1. In this paper, we prove for the first time that approximate unitary $t$-designs for suitable values of $t$ achieve decoupling with probability exponentially close to one. An exact $t$-
design of \( n \times n \) unitaries can be described using \( O(t \log n) \) random bits [Kup06] as opposed to \( \Omega(n^2 \log n) \) random bits required to describe a Haar random unitary to reasonable precision. Thus for many applications our result implies a substantial saving in the number of random bits compared to Dupuis’ result. However, the concentration guaranteed by our result is less than that guaranteed by Dupuis even though it is exponential. Our concentration bound for decoupling via unitary designs is expressed in terms of smooth entropic quantities.

**Theorem 1.** Consider a quantum state \( \rho^{AR} \) shared between a system \( A \) and a reference \( R \). Let \( T^{A\rightarrow B} \) be a completely positive trace preserving superoperator with input system \( A \) and output system \( B \). Let \( U \) be a unitary on the system \( A \). Define the function

\[
    f(U) := \|(T^{A\rightarrow B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^A)\otimes I^R) - \omega^B \otimes \rho^R\|_1,
\]

where \( I^R \) is the identity superoperator on \( R \) and \( I^R \) is the identity operator on \( R \). Let \( A' \) be a new system having the same dimension as \( A \). Define \( \omega^{\overline{A}B} := (T^{A\rightarrow B} \otimes I^A')(|\Phi\rangle\langle\Phi|^{A'A'}) \), where \( |\Phi\rangle^{A'A'} := |A\rangle^{-1/2} \sum_a |a\rangle^A \otimes |a\rangle^{A'} \) is the standard EPR state on system \( AA' \). Let \( 0 < \epsilon, \delta < 1/3 \). Let \( \kappa > 0 \). Then,

\[
    \mathbb{P}_{U^A}[f(U) > 2^{-\frac{1}{2}H_2(A|R)_\rho - \frac{1}{2}(H_2)^{\epsilon,\delta}(A'|B)_{\omega} + 1} + 14 \sqrt{\epsilon} + 2\kappa] \leq \exp(-800^{-1}a).
\]

where \( U^A \) is chosen uniformly at random from a \(|A|, s, \lambda, 4m\)-qTPE defined in Definition 11 below,

\[
    a := |A|\kappa^22^{H_2(A|R)_\rho - (1+\delta)(H'_{\max})^\epsilon(B)_{\omega}} m := \lfloor 300^{-1}2^{-4}a \rfloor, 0 \leq \lambda \leq 2^{-4}|A|^{-8}|B|^{-4}\mu^2\kappa^2a.
\]

The entropic quantities \( H_2(A|R)_\rho, (H_2)^{\epsilon,\delta}(A'|B)_{\omega} \) and \((H'_{\max})^\epsilon(B)_{\omega}\) are defined in Definitions 1, 3 and 2 below. The quantity \( \mu \) is defined as \( \mu := \mathbb{E}_{\text{Haar}}[g(U)] \) for a related function \( g(U) \) defined in Equation 2 below. It satisfies \( \mu^2 \leq \mathbb{E}_{\text{Haar}}[(g(U))^2] \leq 6\mu \) if

\[
    |A| > 2400 \cdot 2^{(1+\delta)(H'_{\max})^{\epsilon}(B)_{\omega} - H_2(A|R)_\rho}(2 \log |A|) \cdot ((1+\delta)(H'_{\max})^{\epsilon}(B)_{\omega} - H_2(A|R)_\rho) \cdot (- \log \mathbb{E}_{\text{Haar}}[(g(U))^2]).
\]
Moreover,
\[
E_{\text{Haar}}[(g(U))^2] = \alpha \| (\tilde{\omega})^R \|_2^2 + \beta \| (\tilde{\rho})^{AR} \|_2^2 - \| (\tilde{\omega})^B \|_2 \| (\tilde{\rho})^R \|_2^2 < \| (\tilde{\omega})^{AB} \|_2 \| (\tilde{\rho})^{AR} \|_2 = 2^{-\frac{1}{2}(\tilde{H}^R_2)_{\epsilon,\delta}(A|B)_{\omega}} \cdot 2^{-\frac{1}{2}(\tilde{H}^B_2)_{\epsilon}(A|R)_{\rho}},
\]
where \(\alpha, \beta\) are defined in Proposition 1 below. The positive semidefinite matrices \((\tilde{\omega})^{AB}, (\tilde{\rho})^{AR}\) are defined in Equations 4, 3 below.

In the asymptotic iid setting, we can infer the following corollary of our main result.

The statement of the corollary is in terms of the standard Shannon entropies.

**Corollary 1.** Consider the setting of Theorem 1 above. Consider the density matrix \(\omega^{AB}\). Let \(|w_j\)^{AB}, \(j \in [[A]|B]\) be the eigenvectors of \(\omega^{AB}\) with eigenvalues \(q_j\). For \(j \in [[A]|B]\), define \(\theta_j^B := \text{Tr}_A[|w_j\rangle^{AB}\langle w_j|]\). Let \(p_j, j \in [[A]|B]\) be the probability distribution on \([[B]]\) obtained by measuring \(\theta_j\) in the eigenbasis of \(\omega^B\). Let \(0 < \epsilon, \delta < 1/3\) and Define \(q_{\text{min}} := \min_{j \in [[A]|B]} q_j, p_{\text{min}} := \min_{j \in [[A]|B]} \min_{i \in [[B]]} p_j(i) > 0\). Let \(n := 2^5 q_{\text{min}}^{-1} p_{\text{min}}^{-1} \delta^{-2} \log(|A||B|/\epsilon)\). Consider the \(n\)-fold tensor power \(\omega^{(A)^nB^n} := (\omega^{AB})^\otimes n\). Let \(\epsilon' := 7(n + |A||B|)^{|A||B|^{1/4}}\). Define \(\sigma^{B^nR^n} := (\sigma^{BR})^\otimes n, \rho^{A^nR^n} := (\rho^{AR})^\otimes n\). Then,

\[
\mathbb{P}_{U^A^n}[(\| \sigma^{B^nR^n} - \omega^{B^n} \otimes \rho^{R^n} \|_1 < 2^{-\frac{1}{2}(H(A|R)_{\rho} - \delta(3H(AR)_{\rho} + 7H(R)_{\rho})) - \frac{1}{2}(H(A'|B)_{\omega} - \delta(3H(A'B)_{\omega} + 7H(B)_{\omega})) + 14 \sqrt{\epsilon} + 2\kappa}) + \exp(-800^{1-a})],
\]

where \(U^A^n\) is chosen uniformly at random from a \((|A|^n, s, \lambda, 4m)\)-qTPE,

\[
a = |A|^n \kappa^{2n(H(A|R)_{\rho} - \delta(3H(AR)_{\rho} + 7H(R)_{\rho})) - nH(B)_{\omega}(1+\delta)},
\]

and \(\lambda, m\) are defined in Theorem 1 above.

The proof of our main result and the analysis of its iid limit requires us to define two novel one-shot entropic quantities that we call smooth modified conditional Rényi 2-entropy \((H'_2)^{\epsilon,\delta}(\cdot|\cdot)\) and smooth modified max-entropy \((H'_{\max})^{\epsilon}(\cdot)\). Their definitions and techniques used in our proofs should be of independent interest.
2. Our concentration result for decoupling immediately implies that approximate unitary $|A_1|$-designs decouple a quantum system in the Fully Quantum Slepian Wolf (FQSW) theorem with probability $1 - \exp(-\Theta(|A_1|))$, where the system $A$ is expressed as a tensor product $A_1 \otimes A_2$ and the superoperator simply traces out $A_2$.

**Theorem 2** (FQSW concentration under design). Consider the setting of Theorem 1. Consider the FQSW decoupling function

$$f(U) = f_{\text{FQSW}}(U^{A_1A_2}) := \|\text{Tr}_{A_2}[(U^{A_1A_2} \otimes I^R) \circ \rho^{A_1A_2R}] - \pi^{A_1} \otimes \rho^R\|_1.$$ 

Suppose we are promised that $\|\tilde{\rho}^R\|_2^2 < 0.9|A_1||A_2|\|\tilde{\rho}^R\|_2^2$ and $|A_1| \geq 2$. The following concentration inequality holds when $U$ is chosen from a $(d_A, s, \lambda, 4m)$-qTPE:

$$\mathbb{P}_{U \sim \text{TPE}}[f_{\text{FQSW}}(U) > \sqrt{|A_1|/|A_2|} \cdot 2^{-\frac{1}{2}H_2^s(\mathcal{A}|R)_\rho + 1} + 14\sqrt{\epsilon} + 2\kappa] \leq \exp(-800a^{-1}),$$

where $a$, $m$, $\lambda$ are defined in Theorem 1. Moreover, if $|A_1| \leq \text{polylog}(|A_2|)$, then efficient constructions for such qTPEs exist.

The above result immediately leads to a saving in the number of random bits to $O(|A_1| \log(|A_1||A_2|))$ from $\Omega(|A_1|^2|A_2|^2 \log(|A_1||A_2|))$ required by Haar random unitaries. If $|A_1| = \text{polylog}|A_2|$, then efficient algorithms exist for implementing approximate unitary $|A_1|$-designs [BHH12, Sen18b]. Thus, for small values of $|A_1|$ our result shows that relative thermalisation can indeed be achieved by efficiently implementable unitaries with probability exponentially close to one, the first result of this kind. More generally, our result implies that approximate unitary $|A_1|$-designs achieve relative thermalisation with exponentially high probability. Earlier it was unknown whether exponentially high probability was achievable by anything other than Haar random unitaries.

**B. Proof technique**

We now give a high level description of the proof of our main result. For a positive semidefinite matrix $\sigma$, we use $\sigma^{-1}$ to denote the operator which is the orthogonal direct sum
of the inverse of $\sigma$ on its support and the zero operator on the orthogonal complement of the support. For a unitary $U$ on the system $A$, we define the value taken by the decoupling function at $U$ as follows:

$$f(U) := \| (\mathcal{T}_{A \rightarrow B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^{A\dagger} \otimes I^R)) - \omega^B \otimes \rho^R \|_1.$$  

We wish to prove a tail bound for $f(U)$ where $U$ is chosen uniformly from a unitary design. For this, it is easier to first prove a tail bound for a related function $g(U)$:

$$g(U) := \| ((\tilde{T})^{A \rightarrow B} \otimes I^R)((U^A \otimes I^R)(\tilde{\rho}^{AR}(U^{A\dagger} \otimes I^R)) - (\tilde{\omega}')^B \otimes (\tilde{\rho}')^R \|_2,$$

where $(\tilde{T})^{A \rightarrow B}$, $(\tilde{\rho}')^{AR}$, $(\tilde{\omega}')^{A'B}$ will be defined later in Section III. We will have, for all probability distributions on $U^A$,

$$\mathbb{P}_{U^A}[f(U) > 2^{-\frac{1}{2}H_2(A|R)\rho - \frac{1}{2}(H_2')(\omega)_{a+1} + 14\sqrt{\epsilon+2\theta}}] \leq \mathbb{P}_{U^A}[g(U) > \| (\tilde{\rho}')^{AR}\|_2 \cdot \| (\tilde{\omega}')^{A'B}\|_2 + \theta].$$

We then bound $\mathbb{P}_{U^A}[g(U) > \| (\tilde{\rho}')^{AR}\|_2 \cdot \| (\tilde{\omega}')^{A'B}\|_2 + \theta]$ where $U^A$ is chosen according to the Haar measure. For this we need to upper bound the Lipschitz constant of $g(U)$, which we do in Lemma 2. Then Levy’s lemma (Fact 19) gives an exponential concentration result for $g(U)$ under the Haar measure. Using techniques from [Low09], [Sen18a], we obtain upper bounds on the centralised moments of $(g(U))^2$ under the Haar measure. Observe now that $(g(U))^2$ is a balanced degree two polynomial (for the precise meaning see Definition 9) in the matrix entries of $U$. We then use Low’s [Low09] derandomisation technique in order to obtain an exponential concentration result for $(g(U))^2$ when the unitary $U^A$ is chosen uniformly from $t$-designs with the value of $t$ stated above. This then leads to a similar exponential concentration result for $f(U)$ when $U^A$ is chosen uniformly from a $t$-design, completing the proof of Theorem 1.

C. Organisation of the paper

Section II describes some notations, definitions and basic facts required for the paper. Section III proves the main result on one-shot decoupling with exponentially high concentration using unitary $t$-designs. The bounds obtained are described using smooth versions
of variants of one-shot Rényi 2-entropies and max entropies. Section IV considers the main decoupling result in the iid limit and obtains bounds in terms of the more familiar Shannon entropic quantities. Section V shows how to apply the main result in order to obtain an exponential concentration for FQSW theorem for unitary designs. It also discusses implications of FQSW concentration to relative thermalisation in quantum thermodynamics. Section VI concludes the paper and discusses directions for further research.

II. PRELIMINARIES

A. Notation

All vector spaces considered the paper are finite dimensional inner product spaces, aka finite dimensional Hilbert spaces, over the complex field. We use $|V|$ to denote the dimension of a Hilbert space $V$. Letters $c_1, c_2, c'_1, c'_2, \ldots$ denote positive universal constants. Logarithms are all taken to base two. We tacitly assume that the ceiling is taken of any formula that provides dimension or value of $t$ in unitary $t$-design. The symbols $\mathbb{E}$, $\mathbb{P}$ denote expectation and probability respectively. The notation ”:=” is used to denote the definitions of the underlying mathematical quantities.

The notation $\mathcal{L}(A_1, A_2)$ denotes the Hilbert space of all linear operators from Hilbert space $A_1$ to Hilbert space $A_2$ with the inner product being the Hilbert-Schmidt inner product $\langle M, N \rangle := \text{Tr}[M^\dagger N]$. For the special case when $A_1 = A_2$ we use the phrase operator on $A_1$ and the symbol $\mathcal{L}(A_1)$. Further, when $A_1 = A_2 = \mathbb{C}^m$, $\mathcal{M}_m$ denotes vector space of all $m \times m$ matrices. The symbol $I_A$ denotes the identity operator on vector space $A$. The matrix $\pi^A$ denotes the so-called completely mixed state on system $A$, i.e., $\pi^A = \frac{I_A}{|A|}$. We use the notation $U \circ A$ as a short hand to denote the conjugation of the operator $U$ on the operator $A$, that is, $U \circ A := UAU^\dagger$.

The symbol $\rho$ usually denotes a quantum state aka density matrix which is nothing but a Hermitian positive semidefinite matrix with unit trace, and $\mathcal{D}(\mathbb{C}^d)$ denotes the set of all $d \times d$ density matrices. The symbol $\text{Pos}(\mathbb{C}^d)$ denotes the set of all $d \times d$ positive semidefinite matrices, and the symbol $\mathcal{U}(d)$ denotes the set of all $d \times d$ unitary matrices with complex entries. The symbol $|v\rangle$ denotes a vector $v$ of unit $\ell_2$-norm, and $\langle v|$ denotes the corresponding linear functional. A rank one density matrix is called a pure quantum state. Often, in what
is a loose notation, a pure quantum state $|v\rangle\langle v|$ is denoted by just the vector $|v\rangle$. For two Hermitian matrices $A$, $B$ of the same dimension, we use $A \geq B$ as a shorthand for the statement that $A - B$ is positive semidefinite.

Let $M \in \mathcal{L}(A)$. The symbol $\text{Tr} M$ denotes the trace of operator $M$. Trace is a linear map from $\mathcal{L}(A)$ to $\mathbb{C}$. Let $A$, $B$ be two vector spaces. The partial trace $\text{Tr}_B[\cdot]$ obtained by tracing out $B$ is defined to be the unique linear map from $\mathcal{L}(A \otimes B)$ to $\mathcal{L}(A)$ satisfying $\text{Tr}_B[M \otimes N] = (\text{Tr} N) M$ for all operators $M \in \mathcal{L}(A)$, $N \in \mathcal{L}(B)$.

A linear map $\mathcal{T} : \mathcal{M}_m \to \mathcal{M}_d$ is called a superoperator. A superoperator $\mathcal{T}$ is said to be positive if it maps positive semidefinite matrices to positive semidefinite matrices, and completely positive if $\mathcal{T} \otimes I$ is a positive superoperator for all identity superoperators $I$. Completely positive and trace preserving (abbreviated as CPTP) superoperators are called quantum operations or quantum channels. In this paper we only consider completely positive and trace non-increasing superoperators. Note that both trace and partial trace defined in the previous paragraph are quantum channels.

The adjoint of a superoperator is defined with respect to the Hilbert-Schmidt inner product on matrices. In other words, if $\mathcal{T} : \mathcal{M}_m \to \mathcal{M}_d$ is a superoperator, then its adjoint $\mathcal{T}^\dagger : \mathcal{M}_d \to \mathcal{M}_m$ is a superoperator uniquely defined by the property that $\langle \mathcal{T}^\dagger(A), B \rangle = \langle A, \mathcal{T}(B) \rangle$ for all $A \in \mathcal{M}_d$, $B \in \mathcal{M}_m$.

We will be using the Stinespring representation of a superoperator, which we state as the following fact:

**Fact 4.** Any superoperator $\mathcal{T}^{A \to B}$ can be represented as:

$$\mathcal{T}^{A \to B}(M^A) = \text{Tr}_Z V^{AC \to BZ}_{\mathcal{T}} (M^A \otimes (|0\rangle\langle 0|)^C) W^{AC \to BZ}_{\mathcal{T}} \dagger$$

where $V_{\mathcal{T}}$, $W_{\mathcal{T}}$ are operators that map vectors from $A \otimes C$ to vectors in $B \otimes Z$. Systems $C$ and $Z$ are considered as the input and output ancillary systems respectively, such that $|A||C| = |B||Z|$. Without loss of generality, $|C| \leq |B|$ and $|Z| \leq |A|$. Furthermore, in the following special cases $V_{\mathcal{T}}$, $W_{\mathcal{T}}$ have additional properties.

1. $\mathcal{T}$ is completely positive iff $V_{\mathcal{T}} = W_{\mathcal{T}}$.

2. $\mathcal{T}$ is trace preserving iff $V_{\mathcal{T}}^{-1} = W_{\mathcal{T}}^\dagger$. Thus, $\mathcal{T}$ is completely positive and trace preserving iff $V_{\mathcal{T}} = W_{\mathcal{T}}$ and are unitary operators.
3. $T$ is completely positive and trace non-decreasing iff $V_T = W_T$ and $\|V_T\|_\infty \leq 1$.

For $p \geq 1$, the Schatten $p$-norm for any operator $M \in \mathcal{L}(A_1, A_2)$ is defined as $\|M\|_p \triangleq \left[ \text{Tr} \left( (M^* M)^{p/2} \right) \right]^{1/p}$. In other words, $\|M\|_p$ is nothing but the $\ell_p$-norm of the tuple of singular values of $M$. The Schatten $\infty$-norm is defined by taking the limit $p \to \infty$. The Schatten 2-norm, aka the Hilbert Schmidt norm, is nothing but the $\ell^2$-norm of the tuple obtained by stretching out the entries of the matrix into a vector. The Schatten $\infty$-norm is nothing but the operator norm $\|M\|_\infty = \max_{\|v\|_2 = 1} \|Mv\|_2$. We have the norm properties $|\text{Tr} M| \leq \|M\|_1$, $\|M\|_1 \leq \sqrt{\text{Tr} M^* M}_2$, $\|M \otimes N\|_p = \|M\|_p \cdot \|N\|_p$, $\|M\|_p \leq \|M\|_q$ if $p \geq q$ and $\|MN\|_p \leq \min\{\|M\|_p\|N\|_\infty, \|M\|_\infty\|N\|_p\}$.

**B. Matrix manipulation**

**Fact 5.** [Wat04, Theorem 4.3] For Hilbert spaces $\mathcal{H}_X$, $\mathcal{H}_Y$ suppose that vectors $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}_X \otimes \mathcal{H}_Y$ satisfy:
$$\text{Tr}_Y(\langle \psi | \psi \rangle) = \text{Tr}_Y(\langle \phi | \phi \rangle)$$
then there exists a unitary operator $U$ on $\mathcal{H}_Y$ such that $|\psi\rangle = (I_X \otimes U^Y)|\phi\rangle$.

Fix an orthonormal basis $\{|a\rangle^A\}_a$ of $A$ and $\{|z\rangle^Z\}_z$ of $Z$. Consider the tensor basis $\{|a\rangle^A \otimes |z\rangle^Z\}_{a,z}$ of the Hilbert space $A \otimes Z$. The isometric linear map $\text{vec}^{A,Z} : \mathcal{L}(Z, A) \to A \otimes Z$ is defined as the unique linear map satisfying $\text{vec}^{A,Z}(|a\rangle^A \langle z|) := |a\rangle^A \otimes |z\rangle^Z$ [Wat04]. The inverse linear map is denoted by $(\text{vec}^{A,Z})^{-1}$. It is also an isometry. We will be using the following property of the vec$^{-1}$ map which we state as a fact here. A simpler version of this fact was used in [ASW10].

**Fact 6.** For any two vectors $|x\rangle^{AZ}$, $|y\rangle^{AZ}$ on a bipartite Hilbert space $A \otimes Z$,
$$(\text{Tr}_Z(|x\rangle^{AZ} \langle y|))^{A \times A} = (\text{vec}^{-1}(|x\rangle))^{A \times Z}((\text{vec}^{-1}(|y\rangle)))^{A \times Z}^\dagger$$
where $\text{vec}^{-1} : A \otimes Z \to A \times Z := \mathcal{L}(Z, A)$.

**Proof.** Fix orthonormal bases $\{|a\rangle^A\}_a, \{|z\rangle^Z\}_z$ for $A, Z$. We can write
$$|x\rangle^{AZ} = \sum_{az} x_{az} |a\rangle^A |z\rangle^Z, \quad |y\rangle^{AZ} = \sum_{az} y_{az} |a\rangle^A |z\rangle^Z.$$
This gives
\[ \text{vec}^{-1}(|x\rangle) = \sum_{az} x_{az} |a\rangle^A \langle z|^Z, \quad \text{vec}^{-1}(|y\rangle) = \sum_{az} y_{az} |a\rangle^A \langle z|^Z, \]
\[ \Rightarrow (\text{vec}^{-1}(|x\rangle))(\text{vec}^{-1}(|y\rangle))^\dagger = \sum_{aa'} \sum_z x_{az} y_{az}^* |a\rangle^A \langle a'|. \]

On the other hand
\[ \text{Tr}_Z(|x\rangle^A_Z \langle y|) = \text{Tr}_Z \left( \sum_{aa'} \sum_{zz'} x_{az} y_{az}^* |a\rangle^A \langle a'| \otimes |z|^Z \langle z' | \right) = \sum_{aa'} x_{az} y_{az}^* |a\rangle^A \langle a'|. \]

This completes the proof.

We now state the so called polar decomposition of any linear operator.

**Fact 7. Polar Decomposition** [Wil17, Theorem A.0.1] Any operator \( M \) can be expressed as \( M = VQ \), known as the left polar decomposition, where \( V \) is a unitary matrix and \( Q \) is a positive semidefinite matrix. Also, \( M \) can be expressed as \( M = PU \), where \( P \) is a positive semidefinite matrix and \( U \) is a unitary matrix. This is known as the right polar decomposition.

Next, we state four useful facts from Dupuis’ thesis [Dup10].

**Fact 8 ([Dup10, Lemma I.1]).** Let \( \rho, \rho' \) and \( \sigma \) be positive semidefinite operators on \( \mathcal{H} \) such that \( \text{Tr}[\rho'] \leq \text{Tr}[\sigma] \) and \( \rho' \geq \rho \). Then, \( \|\rho' - \sigma\|_1 \leq 2\|\rho - \sigma\|_1. \)

**Fact 9 ([Dup10, Lemma I.2]).** Let \( \rho^{AB} \) be a positive semidefinite operator, and let \( 0 \leq P^B \leq I^B \). Then, \( \text{Tr}_B[(P^B \otimes I^A)\rho^{AB}(P^B \otimes I^A)] \leq \rho^A. \)

**Proof.** We give a more direct and elementary proof of this fact than what was given in [Dup10]. The proof is a simple application of the definition of the partial trace and the fact that \( P^B \leq I^B \Rightarrow (P^B)^2 \leq I^B \). By spectral theorem for positive semidefinite matrices, we express \( P^B \) in its eigen basis as \( P^B = \sum_{i=1}^{\|B\|} p_i |b_i\rangle^B \langle b_i|. \) Since \( P^B \leq I^B \), therefore \( p_i \leq 1, \forall i. \)

Now we express \( \rho^{AB} \) in block diagonal form with \( \{b_j\}_{j=1}^{\|B\|} \) as the orthonormal basis for \( \mathcal{H}_B \):
\[ \rho^{AB} = \sum_{j,j' = 1}^{\|B\|} A_{j,j'}^A \otimes |b_j\rangle^B \langle b_{j'}| \]
\[ \Rightarrow \rho^A = \sum_j^{\|B\|} A_{j,j}^A. \]
The block matrices $A_{j,j}$ are positive semidefinite. Now evaluating $\text{Tr}_B[P^B \cdot \rho^{AB}]$:

$$\text{Tr}_B[P^B \cdot \rho^{AB}] = \sum_{j,j',k,l=1}^{\vert B \vert} p_k p_l A_{j,j'}^{A} \langle b_k | b_{j'} \rangle \langle b_{j'} | b_l \rangle \langle b_l | b_k \rangle = \sum_{j=1}^{\vert B \vert} p_j a_{j,j} \leq \sum_{j=1}^{\vert B \vert} A_{j,j}^A = \rho^A$$

where (a) holds since $p_j \leq 1$ and $A_{j,j}$ are positive semidefinite matrices for all $j$. This completes the proof. 

**Fact 10** ([Dup10, Lemma I.3]). Let $|\psi^{AB}\rangle \in A \otimes B$, $\rho^A \in \text{Pos}(A)$ such that $\rho^A \leq \psi^A$. Then, there exists an operator $P^B$ on $B$ such that $0^B \leq P^B \leq I^B$ and $\text{Tr}_B[(P^B \otimes I^A)|\psi^{AB}\rangle \langle \psi^{AB}|(P^B \otimes I^A)] = \rho^A$.

**Proof.** We give a more simpler and direct proof of this fact than given in [Dup10]. Since $\psi^A \geq \rho^A$, therefore, there exists a positive semidefinite matrix $\sigma^A$ such that $\psi^A = \rho^A + \sigma^A$. Let the pure state $|\rho^{AB}\rangle$ be a purification of $\rho^A$ and $|\sigma^{AB}\rangle$ be a purification of $\sigma^A$. Now let $Q = \mathbb{C}^2$ be the system representing a qubit. We define the pure state $|\theta^{ABQ}\rangle$ as:

$$|\theta^{ABQ}\rangle \triangleq |\rho^{AB} \otimes |0\rangle_Q + |\sigma^{AB} \otimes |1\rangle_Q$$

It follows that $|\theta^{ABQ}\rangle$ is a purification of the state $\psi^A$ and so is the state $|\psi^{AB} \otimes |0\rangle_Q$. Thus by Fact 5 there exists a unitary matrix $U^{BQ}$ on the composite system $BQ$ satisfying:

$$|\theta^{ABQ}\rangle = (I^A \otimes U^{BQ})(|\psi^{AB}\rangle|0\rangle_Q)$$

Now we define a POVM measurement that first appends the ancilla $Q$ initialized to state $|0\rangle_Q$ to the state $|\psi^{AB}\rangle$, followed by applying the unitary $I^A \otimes U^{BQ}$ on the state $|\psi^{AB} \otimes |0\rangle_Q$ and finally measuring the ancilla system $Q$ of the resultant state in computational basis $\{|0\rangle Q, |1\rangle Q\}$. We get the desired result given that we see the outcome 0 in the ancilla register. This can formally be seen by defining an operator $M^B$ as:

$$(I^A \otimes M^B)|\psi^{AB}\rangle \triangleq (I^{AB} \otimes \langle 0 \rangle_Q)(I^A \otimes U^{BQ})|\psi^{AB}\rangle |0\rangle_Q^Q$$
From the above expression it can be seen that $M^B = (I^B \otimes \langle 0|Q)U^B Q(I^B \otimes |0)_Q)$ and $\|M^B\|_\infty \leq 1$. We thus have:

$$\text{Tr}_B[(I^A \otimes M^B)|\psi\rangle^A B \langle \psi|(I^A \otimes M^B)] = \text{Tr}_B[(I^{AB} \otimes \langle 0|Q)(I^A \otimes U^{BQ})(|\psi\rangle^A B \langle 0|Q)]$$

$$= \text{Tr}_B[(I^{AB} \otimes \langle 0|Q)|\theta\rangle^A B \langle \theta|(I^{AB} \otimes |0)_Q)]$$

$$= \text{Tr}_B[|\rho\rangle^A B \langle \rho|]$$

$$= \rho^A$$

Now, to come up with $P^B$ as mentioned in the statement of the fact we express $M^B = U^B_M^B P^B$, using the polar decomposition from Fact 7, with $P_B \geq 0$. Since $\|M\|_\infty \leq 1$, therefore $P^B \leq I^B$. Thus we get,

$$\rho^A = \text{Tr}_B[(I^A \otimes M^B)|\psi\rangle^A B \langle \psi|(I^A \otimes M^B)]$$

$$= \text{Tr}_B[(I^A \otimes U^B_M)(I^A \otimes P^B)|\psi\rangle^A B \langle \psi|(I^A \otimes U^B_M^B)]$$

$$\overset{a}{=} \sum_i(I^A \otimes |i\rangle^B_M^B)(I^A \otimes P^B)|\psi\rangle^A B \langle \psi|(I^A \otimes P^B)(I^A \otimes U^B_M^B)(I^A \otimes |i\rangle^B)$$

$$\overset{b}{=} \sum_i(I^A \otimes (U^B_M^B|i\rangle \langle i|)^B)(I^A \otimes P^B)|\psi\rangle^A B \langle \psi|(I^A \otimes P^B)(I^A \otimes (U^B_M^B|i\rangle \langle i|)^B)$$

$$= \text{Tr}_B[(I^A \otimes P^B)|\psi\rangle^A B \langle \psi|(I^A \otimes P^B)]$$

where (a) follows by the basis dependent definition of $\text{Tr}_B$ by fixing $\{|i\rangle\}_{i=1}^{|i|}$ as an orthonormal basis for system $B$; (b) holds since $U^B_M^B$ is a unitary matrix that maps orthonormal basis $|i\rangle \mapsto ket_u_i$ with $\{|u_i\rangle\}_{i=1}^{|B|}$ also forming the orthonormal basis for $B$.

We thus have a $0 \leq P^B \leq I^B$ satisfying the fact. This completes the proof.

Fact 11 ([Dup10, Lemma 3.5]). Let $\rho^{AB} \in \text{Pos}(A \otimes B)$ and let $\rho^B := \text{Tr}_A \rho^{AB}$. Then,

$$|A|^{-1} \leq \frac{\|\rho^{AB}\|_2^2}{\|\rho^B\|_2^2} \leq |A|.$$

In order to upper bound Schatten 1-norm of an operator, sometimes it is more convenient to upper bound Schatten 2-norm of a slightly modified operator. The following fact, which
is nothing but an application of the Cauchy-Schwarz inequality, allows us to do so.

**Fact 12.** Let $M \in \mathcal{L}(\mathcal{H})$ and $\sigma \in \mathcal{D}(\mathcal{H})$. Then $\|M\|_1 \leq \|\sigma^{-1/4}M\sigma^{-1/4}\|_2$.

We will also need Winter’s gentle measurement lemma [Win99].

**Fact 13.** Let $P$ be a positive operator such that $P \leq I$. For any density matrix $\rho$, satisfying $\text{Tr}[P\rho P] \geq 1 - \epsilon$, it holds that $\|\rho - P\rho P\|_1 \leq 2\sqrt{\epsilon}$.

We now state an important geometric fact about how a pair of subspaces of a Hilbert space interact. This fact, first discovered by Jordan a hundred and fifty years ago but which has since been independently rediscovered many times, defines canonical angles between a pair of subspaces. These angles are sometimes called as chordal angles.

**Fact 14.** Let $A, B$ be subspaces of a Hilbert space $\mathcal{H}$. Then there is a decomposition of $\mathcal{H}$ as an orthogonal direct sum of the following types of subspaces:

1. One dimensional spaces orthogonal to both $A$ and $B$;
2. One dimensional spaces contained in both $A$ and $B$;
3. One dimensional spaces contained in $A$ and orthogonal to $B$;
4. One dimensional spaces contained in $B$ and orthogonal to $A$;
5. Two dimensional spaces intersecting $A$, $B$ each in one dimensional spaces.

Moreover, the one dimensional spaces in (2) and (3) above together with the one dimensional intersections of the spaces in (5) with $A$ form an orthonormal basis of $A$. A similar statement holds for $B$.

We end this section by stating two properties of the so-called swap trick that will be useful later on.

**Fact 15** ([Dup10, Lemma 3.3]). For two operators $M^A, N^A \in \mathcal{L}(A)$, we have $\text{Tr}[(MN)^A] = \text{Tr}[(M^A_1 \otimes N^A_2)F^{A_1A_2}]$, where $A_1, A_2$ are two Hilbert spaces of the same dimension as $A$ and $F^{A_1A_2}$ swaps the tensor multiplicand systems $A_1$ and $A_2$.

**Fact 16.** For an operator $M^{AR} \in \mathcal{L}(A \otimes R)$, we have

$$\|\text{Tr}_{R_1R_2}[(I^{A_1}A_2 \otimes F^{R_1R_2})(M^{A_1R_1} \otimes (M^\dagger)^{A_2R_2})]\|_1 \leq |A|\|M^{AR}\|_2.$$
Proof. Fix an orthonormal basis \( \{|r\}\rangle_r \) for the system \( R \). Let \( M^{AR} = \sum_{rr'} M_{rr'}^A \otimes |r\rangle^R \langle r'| \), where \( M_{rr'}^A \) is an operator in \( A \) for every \( r, r' \). Then,

\[
\begin{align*}
\text{Tr}_{R_1R_2}[\left(I^{A_1A_2} \otimes F^{R_1R_2}\right)(M^{A_1R_1} \otimes (M^\dagger)^{A_2R_2})] &= \sum_{rr''r'''} (M_{rr''}^A \otimes (M_{r'r''r'''}^\dagger)^{A_2}) \text{Tr}_{R_1R_2}[F^{R_1R_2}(|r\rangle^{R_1} \langle r' \rangle \otimes |r''\rangle^{R_2} \langle r'''\rangle)] \\
&= \sum_{rr''r'''} (M_{rr''}^A \otimes (M_{r'r''r'''}^\dagger)^{A_2}) \text{Tr}_{R_1R_2}[F^{R_1R_2}(|r\rangle^{R_1} \langle r''\rangle^{R_2} \langle r' \rangle \langle r'''\rangle)] = \sum_{rr'} M_{rr'}^A \otimes (M_{rr'}^\dagger)^{A_2}.
\end{align*}
\]

So,

\[
\|\sum_{rr'} M_{rr'}^A \otimes (M_{rr'}^\dagger)^{A_2}\|_1 \leq \sum_{rr'} \|M_{rr'}^A \otimes (M_{rr'}^\dagger)^{A_2}\|_1 = \sum_{rr'} \|M_{rr'}^A\|_1 \leq \sum_{rr'} |A||M_{rr'}^A|_2^2 = |A||M^{AR}|_2^2.
\]

This completes the proof. \( \square \)

C. Entropic quantities

The Shannon entropy of a random variable \( X \) with probability distribution \( (p_x)_x \) is defined as \( H(X)_p := -\sum_x p_x \log p_x \). For a quantum system \( B \) in a state \( \omega^B \), the Shannon entropy is defined analogously as \( H(B)_\omega := -\text{Tr}[\omega \log \omega] \). For a bipartite quantum system \( AB \) in a state \( \omega^{AB} \), the conditional Shannon entropy is defined as \( H(A|B)_\omega := H(AB)_\omega - H(B)_\omega \).

We recall the definition of the smooth conditional Rényi 2-entropy from [Dup10].

**Definition 1.** Let \( 0 \leq \epsilon < 1 \). The \( \epsilon \)-smooth conditional Rényi 2-entropy for a bipartite positive semidefinite operator \( \rho^{AR} \) on systems \( A \) and \( R \) is defined as:

\[
H_2^\epsilon(A|R)_\rho := -2 \log \min_{\sigma^{AR} \in \mathcal{D}(AR): \|\rho^{AR} - \omega^{AR}\|_1 \leq \epsilon} \{\|(\omega^R \otimes \mathbb{I}^A)^{-1/4} \sigma^{AR} (\omega^R \otimes \mathbb{I}^A)^{-1/4}\|_2\}.
\]

When \( \epsilon = 0 \), we simply refer to the above quantity as conditional Rényi 2-entropy and denote it by \( H_2(A|R)_\rho \) and define \( \omega^{AR} := (\omega^R \otimes \mathbb{I}^A)^{-1/4} \rho^{AR} (\omega^R \otimes \mathbb{I}^A)^{-1/4} \).
We also recall the definition of the $\epsilon$-smooth max-entropy defined in [TCR09].

\[
H^\epsilon_{\text{max}}(B)_\omega := 2 \log \min_{\sigma^B \in \text{Pos}(B) : \|\rho^B - \sigma^B\|_1 \leq \epsilon} \{ \text{Tr} \sqrt{\omega} \}
\]

We now define a new quantity that we call the smooth modified max-entropy.

**Definition 2.** The $\epsilon$-smooth modified max-entropy of system $B$ under a quantum state $\omega^B$ is defined as:

\[
(H'_{\text{max}})^\epsilon(B)_\omega := \log \|((\omega''^B)^{-1})^\|_\infty^1,
\]

where $(\omega''^B)$ is the positive semidefinite matrix obtained by zeroing out those smallest eigenvalues of $\omega^B$ that sum to less than or equal to $\epsilon$. The $\epsilon$-smooth modified max-entropy of a probability distribution can be defined similarly.

It is easy to see that $H^\epsilon_{\text{max}}(B)_\omega \leq (H'_{\text{max}})^\epsilon(B)_\omega \leq \log(|B|/\epsilon)$ for any state $\omega^B$.

We next define a novel entropic quantity called smooth modified conditional Rényi 2-entropy.

**Definition 3.** Let $0 \leq \epsilon, \delta < 1$. The $(\epsilon, \delta)$-smooth modified conditional Rényi 2-entropy for a bipartite positive semidefinite operator $\omega^{AB}$ on systems $A$ and $B$ is defined as:

\[
(H_2')^{\epsilon,\delta}(A|B)_\omega := -2 \log \max_{\eta^{AB}, 0 \leq \eta^{AB} \leq \omega^{AB}, \|\omega^{AB} - \eta^{AB}\|_1 \leq \epsilon} \| (I^A \otimes (\omega''_{\epsilon,\delta}^B)^{1/4} \eta^{AB} (I^A \otimes (\omega''_{\epsilon,\delta}^B)^{1/4})_2^2^2 \geq 1 - \epsilon
\]

where $(\omega''_{\epsilon,\delta}^B)$ is the positive semidefinite operator obtained by zeroing out those eigenvalues of $\omega^B$ that are smaller than $2^{-(1+\delta)(H_{\text{max}}')^\epsilon(B)_\omega}$.

Observe that $(\omega''_{\epsilon,\delta}^B) \geq (\omega''^B)$, where $(\omega''^B)$ is defined in Definition 2 above. It is easy to see, via Fact 13, that for any state $\omega^{AB}$, $H_2'(A|B)_\omega \geq (H_2')^{6\epsilon,\delta}(A|B)_\omega$ for any $\delta > 0$.

**D. Types and typicality**

The smooth entropic quantities defined in the previous section are suitably bounded by the standard Shannon entropic quantities in the iid limit, as will be shown in Section IV. In
order to lay the groundwork for the proofs in Section IV, we recall the definitions of types, typical sequences and subspaces.

**Definition 4.** Let $X$ be a finite set. Fix a probability distribution $p$ on $X$. The Shannon entropy of $X$ is defined as $H(X) := -\sum_{x \in X} p(x) \log p(x)$. Let $n$ be a positive integer. Let $X^n$ denote the random variable corresponding to $n$ independent copies of $X$. The notation $x^n$ shall represent a sequence of length $n$ over the alphabet $X$. Let $N(a|x^n)$ denote the number of occurrences of the symbol $a \in X$ in the sequence $x^n$. The vector $(N(a|x^n))_{a \in X}$ is called the type of $x^n$. The set of all possible types is nothing but the set of all possible $|X|$-tuples of non-negative integers summing up to $n$.

**Definition 5.** Let $0 < \delta < 1$. The set of strongly $\delta$-typical types of length $n$ over the alphabet $X$ pertaining to the distribution $p$ is defined as $\{ (m_a)_{a \in X} : \forall a \in X, m_a \in np(a)(1 \pm \delta) \}$. A sequence $x^n$ is said to be strongly $\delta$-typical if its type is strongly $\delta$-typical. The set of strongly $\delta$-typical sequences is denoted by $T^{X^n}_{p,\delta}$.

Let $p^n$ denote the $n$-fold tensor power of probability distribution $p$. The strongly typical sequences satisfy the following property which is called as Asymptotic Equipartition Property (AEP) in classical Shannon theory.

**Fact 17** ([EGK11, Sen12]). The number of types is $\binom{n+|X|-1}{|X|-1}$. The set of all possible sequences $X^n$ is partitioned into a disjoint union, over all possible types, of sequences having a given type. Let $0 < \epsilon, \delta < 1/2$. Define $p_{\min} := 2^{-H_{\max}(1/2)(X)p}$. Let $n \geq 4p_{\min}^{-1} \delta^{-2} \log(|X|/\epsilon)$. Then,

$$\sum_{x^n \in T^{X^n}_{p,\delta}} p^n(x^n) \geq 1 - \epsilon,$$

$$\forall x^n \in T^{X^n}_{p,\delta} : 2^{-nH(X)(1+\delta)} \leq p^n(x^n) \leq 2^{-nH(X)(1-\delta)},$$

$$2^{nH(X)(1-\delta)(1-\epsilon)} \leq |T^{X^n}_{p,\delta}| \leq 2^{nH(X)(1+\delta)}.$$

In the quantum setting, we extend the notion of types and typical sequences with respect to a particular distribution to the notion of type subspaces and typical subspaces with respect to the $n$-fold tensor product of a quantum state.

**Definition 6.** Let $\rho$ be a density matrix over a Hilbert space $B$. Consider a canonical eigenbasis $B = \{|\chi_1\rangle, \ldots, |\chi_{|B|}\rangle\}$ of $\rho$. Consider the diagonalisation $\rho = \sum_{\chi \in B} q(\chi) |\chi\rangle \langle \chi|$. 


where the set of eigenvalues \( \{ q(\chi) \}_\chi \) can be treated as a probability distribution over \( \mathcal{B} \). The quantum analogue of Shannon entropy, sometimes called von Neumann entropy, is defined to be the entropy of the probability distribution on \( \mathcal{B} \) viz. \( H(B)_\rho := -\text{Tr}[\rho \log \rho] = H(B)_q \).

Given a type \( (m(\chi))_\chi \in \mathcal{B} \), which is nothing but a \(|\mathcal{B}|\)-tuple of non-negative integers summing to \( n \), we define the corresponding type subspace to be the span of all \( n \)-fold tensor products of vectors from \( \mathcal{B} \) having the given type.

**Definition 7.** Let \( 0 < \epsilon, \delta < 1/2 \). The strongly \( \delta \)-typical subspace of \( B^\otimes n \) corresponding to the \( n \)-fold tensor power operator \( \rho^\otimes n, T^{B^n}_{\rho,\delta} \), is defined as the orthogonal direct sum of all type subspaces with strongly \( \delta \)-typical types with respect to the probability distribution \( q \) on \( \mathcal{B} \).

Let \( \Pi^{B^n}_{\rho,\delta} \) denote the orthogonal projection onto \( T^{B^n}_{\rho,\delta} \). The typical projector satisfies the following so called quantum AEP analogous to that of Fact 17:

**Fact 18 ([Sen12, Fact 2]).** The number of types is \( \left( \binom{n+|\mathcal{B}|-1}{|\mathcal{B}|-1} \right) \). The Hilbert space \( B^\otimes n \) can be decomposed into an orthogonal direct sum, over all possible types, of type subspaces. Let \( 0 < \epsilon, \delta < 1/2 \). Let \( \rho \) be a quantum state. Define \( q_{ \min } \sim q_{ \min } := 2^{-H^{\prime}_{\max}(B)/\epsilon} \). Suppose that \( n \geq 4q_{ \min }^{-1} \delta^{-2} \log(|\mathcal{B}|/\epsilon) \). Then,

\[
\text{Tr} [\rho^\otimes n \Pi^{B^n}_{\rho,\delta}] \geq 1 - \epsilon, \\
2^{-nH(X)(1+\delta)} \Pi^{B^n}_{\rho,\delta} \leq \Pi^{B^n}_{\rho,\delta} = \Pi^{B^n}_{\rho,\delta} \rho^\otimes n \Pi^{B^n}_{\rho,\delta} = 2^{-nH(X)(1-\delta)} \Pi^{B^n}_{\rho,\delta}, \\
2^{nH(X)(1-\delta)}(1 - \epsilon) \leq \text{Tr} \Pi^{B^n}_{\rho,\delta} \leq 2^{nH(X)(1+\delta)}.
\]

**E. Concentration of measure**

We state the main tool for concentration of measure of Lipschitz functions defined on the sphere or on the unitary group in high dimensions.

**Definition 8.** A complex valued function \( f \) defined on a subset of \( \mathbb{C}^n \) is said to be \( L \)-Lipschitz, with Lipschitz constant \( L \), if \( \forall x, y \in \mathbb{C}^n \) it satisfies the following inequality:

\[
|f(x) - f(y)| \leq L \|x - y\|_2.
\]

**Fact 19.** (Levy’s Lemma [AGZ09]) Let \( f \) be an \( L \)-Lipschitz function on \( \mathcal{U}(n) \) where the metric on \( \mathcal{U}(n) \) is induced by the embedding of \( \mathcal{U}(n) \) into \( \mathbb{C}^{n^2} \). In other words, the metric on
\( U(n) \) is taken to be the Schatten 2-norm. Consider the Haar probability measure on \( U(n) \). Let the mean of \( f \) be \( \mu \). Then:

\[
P(\left| f(U) - \mu \right| \geq \lambda) \leq 2 \exp\left(-\frac{n\lambda^2}{4L^2}\right).
\]

The following fact can be used to compute upper bounds on the centralised moments of Lipschitz functions. The proof follows Bellare and Rompel’s seminal work on concentration for sums of \( t \)-wise independent random variables [BR94, Lemma A.1] and its quantum adaptation by Low [Low09, Lemma 3.3]. However, inspired by a technique from [Sen18a], we extend the earlier results in an important and essential way by computing upper bounds on the centralised moments of squares of Lipschitz functions also, which will be required in Section III.B.

**Fact 20.** Let \( X \) be a non-negative random variable. Suppose there is a positive number \( \mu \) satisfying, for any \( \kappa > 0 \), the tail bound \( \mathbb{P}(\left| X - \mu \right| > \kappa) \leq C \exp(-a\kappa^2) \) for some positive constants \( C, a \). Let \( m \) be a positive integer satisfying \( 64m < 9a \). Then

\[
\mathbb{E}[(X - \mu)^{2m}] \leq C(m/a)^m, \quad \mathbb{E}[(X^2 - \mu^2)^{2m}] \leq C(2m + 1)(9m\mu^2/a)^m.
\]

**Proof.** Let \( \Omega \) with a probability measure \( d\omega \) be the sample space serving as the domain of the measurable function \( X \). Then,

\[
\mathbb{E}[(X - \mu)^{2m}] = \int_{\Omega} (X(\omega) - \mu)^{2m} d\omega = \int_{\Omega} \int_0^{\mu(x) - \mu(x)} dx d\omega = \int_0^{\infty} \int_{\mu(x) - \mu(x)}^{\infty} d\omega dx
\]

\[
= \int_0^{\infty} \mathbb{P}[(X - \mu)^{2m} \geq x] dx = \int_0^{\infty} \mathbb{P}[\left| X - \mu \right| \geq x^{1/2m}] dx \leq C \int_0^{\infty} \exp(-ax^{1/m}) dx
\]

\[
= Cm a^{-m} \int_0^{\infty} e^{-y^{m-1}} dy = Cm a^{-m} \Gamma(m + 1) \leq C(m/a)^m.
\]

Let \( A := \{\omega : 0 \leq X(\omega) < 2\mu\} \) and \( \bar{A} \) denote its complement in \( \Omega \). We have,

\[
\mathbb{E}[(X^2 - \mu^2)^{2m}] = \int_{\Omega} (X(\omega)^2 - \mu^2)^{2m} d\omega = \int_A (X(\omega)^2 - \mu^2)^{2m} d\omega + \int_{\bar{A}} (X(\omega)^2 - \mu^2)^{2m} d\omega,
\]
\[
\int_{A} (X(\omega)^2 - \mu^2)^{2m} d\omega \\
= \int_{A} (X(\omega) - \mu)^{2m} (X(\omega) + \mu)^{2m} d\omega \\
\leq (3\mu)^{2m} \int_{\Omega} (X(\omega) - \mu)^{2m} d\omega \\
\leq C(3\mu)^{2m} (m/a)^m = C\left(\frac{9\mu^2 m}{a}\right)^m.
\]

\[
\int_{A} (X(\omega)^2 - \mu^2)^{2m} d\omega \\
= 2m\mu^{4m} \int_{A} \int_{0 \leq x \leq ((X(\omega))^{2}/\mu^2 - 1)} x^{2m-1} dx d\omega \\
\leq 2m\mu^{4m} \int_{3}^{\infty} \int_{X(\omega) \geq \mu(1 + x^{1/2}/2)} x^{2m-1} d\omega dx \\
\leq 2Cm\mu^{2m} \int_{3}^{\infty} \exp(-a\mu x/4) x^{2m-1} dx \\
\leq 2Cm\mu^{2m} (4/a)^{2m} \Gamma(2m) \leq 2Cm\left(\frac{64m^2 \mu^2}{a^2}\right)^m \\
\leq 2Cm\mu^{2m} (4/a)^{2m} \Gamma(2m) \leq 2Cm\left(\frac{9\mu^2 m}{a}\right)^m.
\]

Thus,

\[
\mathbb{E}[(X^2 - \mu^2)^{2m}] \leq C(2m + 1)\left(\frac{9\mu^2 m}{a}\right)^m,
\]

completing the proof of the fact. \(\square\)

### F. Unitary t-designs

**Definition 9.** A monomial in elements of a matrix \(U\) is of degree \((r, s)\) if it contains \(r\) conjugated elements and \(s\) unconjugated elements of \(U\). We call it balanced if \(r = s\) and will simply say a balanced monomial has degree \(t\) if it is of degree \((t, t)\). A polynomial is of degree \(t\) if it is a sum of balanced monomials of degree at most \(t\).

**Definition 10.** A probability distribution \(\nu\) on a finite set of \(d \times d\) unitary matrices is said to be a \(\epsilon\)-approximate unitary \(t\)-design if for all balanced monomials \(M\) of degree at most \(t\), the following holds \([\text{Low09}]\):

\[
|\mathbb{E}_\nu(M(U)) - \mathbb{E}_{\text{Haar}}(M(U))| \leq \frac{\epsilon}{d^t}
\]

If \(\epsilon = 0\), we say that \(\nu\) is an exact unitary \(t\)-design, or just unitary \(t\)-design.
For technical ease, we use quantum tensor product expanders (qTPEs) in place of unitary designs in our actual proofs. The formal definition of a qTPE follows.

**Definition 11.** A quantum \(t\)-tensor product expander (\(t\)-qTPE) in \(H\), \(|H| = d\), of degree \(s\) can be defined as a quantum operation \(G : L(H^\otimes t) \to L(H^\otimes t)\) that can be expressed as
\[
G(M) = \frac{1}{s} \sum_{i=1}^{s} (U_i)\otimes M(U_i^{-1})\otimes, \quad \text{for any matrix } M \in L(H^\otimes t), \quad \text{where } \{U_i\}_{i=1}^{s} \text{ are } d \times d \text{ unitary matrices.}
\]
The qTPE is said to have second singular value \(\lambda\) if \(\|G - I\|_{\infty} \leq \lambda\), where \(I\) is the ‘ideal’ quantum operation defined by its action on a matrix \(M\) by \(I(M) := \int_{U \in U(D)} U^\otimes t M(U^\dagger)^\otimes d\text{Haar}(U)\). In other words, if \(M \in L(H^\otimes t)\), then \(\|G(M) - I(M)\|_2 \leq \lambda\|M\|_2\). We use the notation \((d, s, \lambda, t)\)-qTPE to denote such a quantum tensor product expander.

A \((d, s, \lambda, t)\)-qTPE can be sequentially iterated \(O\left(\frac{t\log d + \log \epsilon^{-1}}{\log \lambda^{-1}}\right)\) times to obtain an \(\epsilon\)-approximate unitary \(t\)-design [Low09, Lemma 2.7]. For \(t = \text{polylog}(d)\), efficient construction of \(t\)-qTPEs are known [BHH12, Sen18b].

**III. PROOF OF THEOREM 1**

The proof of our main result viz. Theorem 1 is broken into three subsections. In the first subsection, we show that, for any probability distribution on \(U^A\), instead of proving a tail bound for the given random variable \(f(U)\), it suffices to prove a tail bound for a related random variable \(g(U)\), where \(f(U), g(U)\) were informally defined just below the statement of Theorem 1 above. In the second subsection, we first obtain an upper bound on the Lipschitz constant of \(g(U)\) which by Levy’s lemma leads to a tail bound for \(g(U)\) where \(U\) is chosen from the Haar measure. We then obtain upper bounds on the centralised moments of \((g(U))^2\) under the Haar measure. Now \((g(U))^2\) is a balanced degree two polynomial in the matrix entries of \(U\). In the final subsection, we apply Low’s method to finally obtain a tail bound for \((g(U))^2\) for a uniformly random \(U\) chosen from a unitary design. This finishes the proof of Theorem 1.
A. From $f(U)$ to $g(U)$

Recall that for a unitary $U$ on the system $A$, we define the value taken by the decoupling function at $U$ as follows:

$$f(U) := \| (\mathcal{T}^{A \to B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^{A\dagger} \otimes I^R)) - \omega^B \otimes \rho^R \|_1.$$  

Let $\eta^{AB} \leq \omega^{AB}$ be the positive semidefinite operator achieving the optimum in Definition 3. Let $V_{T}^{AC \to BZ}$ be a unitary Stinespring dilation of the CPTP map $\mathcal{T}^{A \to B}$ provided by Fact 4. Thus

$$(\mathcal{T}^{A \to B} \otimes I^A')(M^{AA'}) = \text{Tr}_Z[(V_{T}^{AC \to BZ} \otimes I^A')(M^{AR} \otimes (|0\rangle^C \langle 0|))(V_{T}^{AC \to BZ} \otimes I^A')\dagger]$$

for any $M^{AA'} \in \mathcal{L}(A \otimes A')$. Recall that $\omega^{AB} := (\mathcal{T}^{A \to B} \otimes I^A')(\Phi^{AA'})$, where $\Phi^{AA'}$ is the standard EPR pure state on $AA'$. By Fact 10, there exists a POVM element on $Z$, $0^Z \leq P^Z \leq I^Z$, such that

$$(\hat{\mathcal{T}}^{A \to B} \otimes I^A')(\Phi^{AA'}) := \text{Tr}_Z[(P^Z \otimes I^{BA'})(V_{T}^{AC \to BZ} \otimes I^A')(M^{AR} \otimes (|0\rangle^C \langle 0|))(V_{T}^{AC \to BZ} \otimes I^A')\dagger(P^Z \otimes I^{BA'})] = \eta^{BA'}.$$ 

The superoperator $\hat{\mathcal{T}}^{A \to B}$ is completely positive and trace non-increasing. Define the function

$$\hat{f}(U) := \| (\hat{\mathcal{T}}^{A \to B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^{A\dagger} \otimes I^R)) - \omega^B \otimes \rho^R \|_1.$$  

By Fact 9,

$$(\hat{\mathcal{T}}^{A \to B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^{A\dagger} \otimes I^R)) \leq (\mathcal{T}^{A \to B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^{A\dagger} \otimes I^R)).$$

Hence by Fact 8, $f(U) \leq 2\hat{f}(U)$.

Let $\Pi_{\omega_{\epsilon,\delta}}^B$ be the projector onto the support of $(\omega_{\epsilon,\delta})^B$. Define the completely positive
Define the function

\[(\mathcal{T}')^{A\to B} := \Pi_{\omega_{\epsilon,\delta}}^{B} \circ \hat{\mathcal{T}}^{A\to B}.\]

From Definition 3 and Fact 13, we have

\[\|(\hat{\mathcal{T}}^{A\to B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^A_\dagger \otimes I^R)) - ((\mathcal{T}')^{A\to B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^A_\dagger \otimes I^R))\|_1 \leq 2\sqrt{\epsilon}.\]

Define the function

\[f'(U) := \|(\mathcal{T}')^{A\to B} \otimes I^R)((U^A \otimes I^R)\rho^{AR}(U^A_\dagger \otimes I^R)) - \omega^B \otimes \rho^R\|_1.\]

By triangle inequality, \(|f'(U) - \hat{f}(U)| \leq 2\sqrt{\epsilon}\) which further implies that \(f(U) \leq 2f'(U) + 4\sqrt{\epsilon}\).

Define the states \((\rho')^{AR}, \xi^R\) to be the ones achieving the optimum in Definition 1 of \(H_2^\epsilon(A|R)_\rho\) i.e.

\[\|(I^A \otimes \xi^R)^{-1/4}(\rho')^{AR}(I^A \otimes \xi^R)^{-1/4}\|_2 = 2^{-\frac{1}{2}H_2^\epsilon(A|R)_\rho}.\]

Define the function

\[f''(U) := \|(\mathcal{T}')^{A\to B} \otimes I^R)((U^A \otimes I^R)(\rho')^{AR}(U^A_\dagger \otimes I^R)) - \omega^B \otimes (\rho')^R\|_1.\]

By triangle inequality, \(|f''(U) - f'(U)| \leq 2\epsilon\) which implies that \(f(U) \leq 2f''(U) + 8\sqrt{\epsilon}\).

Now define the positive semidefinite matrix \((\omega')^{A'B} := ((\mathcal{T}')^{A\to B} \otimes I^{A'})((\Phi^{AA'})\). Observe that \((\omega')^{A'B} = \Pi_{\omega_{\epsilon,\delta}}^{B} \circ \eta^{A'B}\). From Definition 3 and Fact 13, we have \(\|(\omega')^{A'B} - \eta^{A'B}\|_1 \leq 2\sqrt{\epsilon}\) which further implies that \(\|(\omega')^{A'B} - \omega^{A'B}\|_1 \leq 3\sqrt{\epsilon}\). Define the function

\[f'''(U) := \|(\mathcal{T}')^{A\to B} \otimes I^R)((U^A \otimes I^R)(\rho')^{AR}(U^A_\dagger \otimes I^R)) - (\omega')^B \otimes (\rho')^R\|_1.\]

By triangle inequality, \(|f'''(U) - f''(U)| \leq 3\sqrt{\epsilon}\) which implies that \(f(U) \leq 2f'''(U) + 14\sqrt{\epsilon}\).

Observe now that the range space of \((\mathcal{T}')^{A\to B}\) is contained in the support of \(\omega''_{\epsilon,\delta}'\). By Fact 12, we can upper bound \(f'''(U)\) by the function \(g(U)\) defined by

\[g(U) := \|(\mathcal{T}')^{A\to B} \otimes I^R)((U^A \otimes I^R)(\rho')^{AR}(U^A_\dagger \otimes I^R)) - (\omega')^B \otimes (\rho')^R\|_2, \quad (2)\]
where \((\tilde{T}^t)^{A\rightarrow B} := (\omega_{t\delta}^m)^{B} \circ (T^t)^{A\rightarrow B}\),

\[
(\tilde{\rho}')^{AR} := (I^{A} \otimes \xi^{R})^{-1/4}(\rho')^{AR}(I^{A} \otimes \xi^{R})^{-1/4},
\]

and

\[
(\tilde{\omega}')^{A'B} := ((\tilde{T}^t)^{A\rightarrow B} \otimes I^{A'})(\Phi^{AA'}) = (I^{A'} \otimes (\omega_{t\delta}^m)^{B})^{-1/4}(\omega')^{A'B}(I^{A'} \otimes (\omega_{t\delta}^m)^{B})^{-1/4}.
\]

Thus, \(f(U) \leq 2g(U) + 14\sqrt{\epsilon}\). Recall from Definitions 1 and 3 respectively, that

\[
2^{-\frac{1}{2}H_2^2(A|R)_{\rho}} = \|(\tilde{\rho}')^{AR}\|_2; \text{ and}
\]

\[
2^{-\frac{1}{2}H_2^2(A'|B)_{\omega}} = \|(\tilde{\omega}')^{A'B}\|_2.
\]

We have thus shown the following lemma.

**Lemma 1.** Let \(\mu, \kappa > 0\). For all probability distributions on \(U^A\),

\[
\mathbb{P}_{U^A}[f(U) > 2\mu + 14\sqrt{\epsilon} + 2\kappa] \leq \mathbb{P}_{U^A}[g(U) > \mu + \kappa].
\]

In particular this holds for

\[
\mu = 2^{-\frac{1}{2}H_2^2(A|R)_{\rho}} - \frac{1}{2}H_2^2(A'|B)_{\omega} = \|(\tilde{\rho}')^{AR}\|_2 \cdot \|(\tilde{\omega}')^{A'B}\|_2.
\]

**B. Bounding centralised moments of \((g(U))^2\) under Haar measure**

We now upper bound the tail of \(g(U)\) when \(U^A\) is chosen from the Haar measure. For this we need to upper bound the Lipschitz constant of \(g(U)\) as follows.

**Lemma 2.** The Lipschitz constant \(L_g\) of function \(g(U)\) satisfies

\[
L_g \leq 2^{\frac{14}{2\omega}(H_{\max}')(B)_{\omega} - \frac{1}{2}H_2^2(A|R)_{\rho} + 1}.
\]
Proof. Write \((\tilde{\rho}')^{AR}\) in any canonical tensor basis for \(A \otimes R\):

\[
(\tilde{\rho}')^{AR} = \sum_{ij} \sum_{kl} \tilde{\rho}_{ijkl}^{'} |i\rangle \langle j| \otimes |k\rangle \langle l| = \sum_{kl} (\tilde{M}'_{kl})^{A} \otimes |k\rangle \langle l| = \sum_{kl} s_{x}^{kl} |a_{x}^{kl}\rangle \langle b_{x}^{kl}| \otimes |k\rangle \langle l|,
\]

where \(\tilde{M}'_{kl} := \sum_{ij} \tilde{\rho}_{ijkl}^{'} |i\rangle \langle j|\), and \(\tilde{M}'_{kl} = \sum_{x} s_{x}^{kl} |a_{x}^{kl}\rangle \langle b_{x}^{kl}|\) is the singular value decomposition of \(\tilde{M}'_{kl}\).

Let \(W^{AC \rightarrow BZ}_{T}\) be a Stinespring dilation of the completely positive trace non-increasing map \(\tilde{T}^{A \rightarrow B}\) provided by Fact 4. Thus \((\tilde{T}')^{A \rightarrow B}(M^{A}) = \text{Tr}_{Z}[W_{\tilde{T}}, \circ (M^{A} \otimes 0^{C})]\), where \(0^{C} := |0\rangle^{C} \langle 0|\). Note that

\[
W^{AC \rightarrow BZ}_{\tilde{T}} = ((\omega_{\epsilon,\delta}^{m})^{B} \otimes I^{Z})^{-1/4}(\Pi_{\omega_{\epsilon,\delta}^{m}}^{B} \otimes I^{Z})(I^{B} \otimes P^{Z})V^{AC \rightarrow BZ}_{\tilde{T}},
\]

where \(V^{AC \rightarrow BZ}_{\tilde{T}}\) is the unitary Stinespring dilation of \(\tilde{T}^{A \rightarrow B}\) provided by Fact 4. We have

\[
\|W^{AC \rightarrow BZ}_{\tilde{T}}\|_{\infty} \leq \|((\omega_{\epsilon,\delta}^{m})^{B} \otimes I^{Z})^{-1/4}\|_{\infty} = \|(\omega_{\epsilon,\delta}^{m})^{B})^{-1}\|_{\infty}^{1/4} \leq 2^{1+\beta}(H_{\max})^{\epsilon}(B)_{\omega},
\]

where the last inequality follows from the definition of \((\omega_{\epsilon,\delta}^{m})^{B}\) and

\[
\|W^{AC \rightarrow BZ}_{\tilde{T}}\|_{2} \leq \sqrt{|B| |Z|} \|W^{AC \rightarrow BZ}_{\tilde{T}}\|_{\infty} \leq \sqrt{|B| |A|} \cdot 2^{1+\beta}(H_{\max})^{\epsilon}(B)_{\omega},
\]

where we used that \(|Z| \leq |A|\) guaranteed by Fact 4.

Let \(U^{A}, V^{A}\) be two unitaries on \(A\). Then,

\[
|g(U) - g(V)| \leq \|((\tilde{T}')^{A \rightarrow B} \otimes I^{R})((U^{A} \otimes I^{R}) \circ (\tilde{\rho}')^{AR}) - ((\tilde{T}')^{A \rightarrow B} \otimes I^{R})((V^{A} \otimes I^{R}) \circ (\tilde{\rho}')^{AR})\|_{2}
\]

\[
= \|((\tilde{T}')^{A \rightarrow B} \otimes I^{R})(((U^{A} \otimes I^{R}) \circ (\tilde{\rho}')^{AR}) - ((V^{A} \otimes I^{R}) \circ (\tilde{\rho}')^{AR})\|_{2}
\]

\[
\leq \|((\tilde{T}')^{A \rightarrow B} \otimes I^{R})((U^{A} \otimes I^{R})(\tilde{\rho}')^{AR}(U^{A} \otimes I^{R})^{\dagger} - (U^{A} \otimes I^{R})(\tilde{\rho}')^{AR}(V^{A} \otimes I^{R})^{\dagger})\|_{2}
\]

\[
+ \|((\tilde{T}')^{A \rightarrow B} \otimes I^{R})((U^{A} \otimes I^{R})(\tilde{\rho}')^{AR}(V^{A} \otimes I^{R})^{\dagger} - (V^{A} \otimes I^{R})(\tilde{\rho}')^{AR}(V^{A} \otimes I^{R})^{\dagger})\|_{2}.
\]

We now upper bound

\[
\|((\tilde{T}')^{A \rightarrow B} \otimes I^{R})((U^{A} \otimes I^{R})(\tilde{\rho}')^{AR}(U^{A} \otimes I^{R})^{\dagger} - (U^{A} \otimes I^{R})(\tilde{\rho}')^{AR}(V^{A} \otimes I^{R})^{\dagger})\|_{2}
\]
Fix $k$, $l$. For ease of notation drop the superscript $kl$ below. We now upper bound

$$\| \sum_x s_x (\hat{T}^x)^{\lambda \rightarrow B} ((U|a_x^k)^A (b_x^k|U^\dagger)) - (U|a_x^k)^A (b_x^k|V^\dagger)) \|_2 = \sqrt{\sum_k \sum_x s_x (\hat{T}^x)^{\lambda \rightarrow B} ((U|a_x^k)^A (b_x^k|U^\dagger)) - (U|a_x^k)^A (b_x^k|V^\dagger)) ^2}.$$

where

(a) $$P_{x,U}^{B \times Z} := (\text{vec}^{B,Z})^{-1} (W_\lambda^{AC \rightarrow BZ} (U^A \otimes I^C) (|a_x^k\rangle \otimes |0\rangle^C)),$$

$$Q_{x,U}^{B \times Z} := (\text{vec}^{B,Z})^{-1} (W_\lambda^{AC \rightarrow BZ} (U^A \otimes I^C) (|b_x^k\rangle \otimes |0\rangle^C)),$$

and $Q_{x,V}^{B \times Z}$ is defined similarly. The above operators map system $Z$ to system $B$ or are $B \times Z$ matrices for fixed bases of $B$ and $Z$. The equality holds due to Fact 6.

(b) Let $Q$ be a single qubit register and $x$ range over the computational basis of $A$.

$$P_{U}^{B Q \times Z A} := \sum_x s_x (P_{x,U}^{B \times Z} \otimes |0\rangle^Q \langle x|^A),$$

$$Q_{U}^{B Q \times Z A} := \sum_x (Q_{x,U}^{B \times Z} \otimes |0\rangle^Q \langle x|^A),$$

and $Q_{V}^{B Q \times Z A}$ is defined similarly.

(c) By Equation 8, $(\omega_{\epsilon,\delta}^{\mu \nu})^B$,

$$\| P_{U,k,l}^{B Q \times Z A} \|_2^2 = \sum_x (s_x^k)^2 \| P_{x,U,k,l}^{B \times Z} \|_2^2 = \sum_x (s_x^k)^2 \| W_\lambda^{AC \rightarrow BZ} (U^A \otimes I^C) (|a_x^l\rangle \otimes |0\rangle^C) \|_2^2.$$
Lemma 3. Hence

\[ \|W_{\mathcal{T}'}^{AC\rightarrow BZ}\|_2^2 \sum_x (s_x^{kl})^2 \leq 2^{1+\delta}(H_{\max}^{\ast})_\omega \|\tilde{M}_{kl}'\|_2^2, \]

This implies that

\[
\|Q_U^{BQ\times ZA} - Q_V^{BQ\times ZA}\|_2^2 \\
= \sum_x \|Q_{x,x'}^{BQ\times Z} - Q_{x,x'}^{BQ\times Z}\|_2^2 \\
= \sum_x \|W_{\mathcal{T}'}^{AC\rightarrow BZ}(U^A \otimes I^C)(|b_x\rangle \otimes |0\rangle^C) - W_{\mathcal{T}'}^{AC\rightarrow BZ}(V^A \otimes I^C)(|b_x\rangle \otimes |0\rangle^C)\|_2^2 \\
\leq \|W_{\mathcal{T}'}^{AC\rightarrow BZ}\|_2^2 \sum_x \|((U^A \otimes I^C) - (V^A \otimes I^C))(|b_x\rangle \otimes |0\rangle^C)\|_2^2 \\
\leq \|((\omega'_{\epsilon,\delta})^B)^{-1}\|_2^2 \sum_x \|((U^A \otimes I^C) - (V^A \otimes I^C))(|b_x\rangle \otimes |0\rangle^C)\|_2^2 \\
\leq \|((\omega'_{\epsilon,\delta})^B)^{-1}\|_2^2 \sum_x \|(U - V)^A|b_x\rangle\|_2^2 = \|((\omega'_{\epsilon,\delta})^B)^{-1}\|_2^2 \sum_x \langle b_x| (U - V)\dag (U - V)|b_x\rangle \\
= \|((\omega'_{\epsilon,\delta})^B)^{-1}\|_2^2 \text{Tr}[(U - V)^\dag (U - V)] \leq 2^{1+\delta}(H_{\max}^{\ast})_\omega \|U - V\|_2^2.
\]

This completes the proof of the lemma. \(\square\)

Lemma 3. For any unitary \(U \in \mathcal{U}(A)\),

\[ g(U) \leq (2|A|)^{1/2} \cdot 2^{1+\delta}(H_{\max}^{\ast})_\omega \|\rho'^{AR}\|_2 \|U - V\|_2. \]
Proof. Define the Hermitian matrix $\gamma^{AR} := (\tilde{\rho}')^{AR} - \pi^A \otimes (\tilde{\rho}')^R$. We have

$$
\|\gamma^{AR}\|_2^2 = \|(\tilde{\rho}')^{AR}\|_2^2 + \|\pi^A \otimes (\tilde{\rho}')^R\|_2^2 - \langle(\tilde{\rho}')^{AR}, (\pi^A \otimes (\tilde{\rho}')^R)\rangle \leq \|\tilde{\rho}'\|_2^2 + \|\pi^A \otimes (\tilde{\rho}')^R\|_2^2 \leq 2\|\tilde{\rho}'\|_2^2,
$$

where we used the fact that $(\tilde{\rho}')^{AR}$, $(\pi^A \otimes (\tilde{\rho}')^R)$ are positive semidefinite matrices in the first inequality and Fact 11 in the second inequality.

Observe that $g(U) = \|(\mathcal{T})^{A\rightarrow B} \otimes I^R)((U^A \otimes I^R) \circ \gamma^{AR})\|_2$. Arguing similarly as in the proof of Lemma 2, we can conclude that

$$
g(U) \leq |A|^{1/2} \cdot 2^{\frac{1}{4}(H_{\text{max}}^\gamma(B)\omega - (H_{\text{max}}^\gamma(B)\omega - H_2^\gamma(A|R))\rho}.
$$

□

We now apply Levy’s Lemma (Fact 19) to obtain an exponential upper bound on the deviation of $g(U)$ about its expectation $\mu$ when $U$ is chosen from the Haar measure.

**Proposition 1.** Define $\mu := E_{U \sim \text{Haar}}[g(U)]$. Let $\kappa > 0$. Define

$$
a := |A|\kappa^2 2^{H_2^\gamma(A|R)\rho - (1+\delta)(H_{\text{max}}^\gamma(B)\omega - H_2^\gamma(A|R)\rho)}.
$$

Then

$$
\mathbb{P}_{U \sim \text{Haar}}[|g(U) - \mu| > \kappa] \leq 2 \exp(-2^{-4a}).
$$

Note that

$$
\mu^2 \leq E_{\text{Haar}}[(g(U))^2] = \alpha\|(\tilde{\rho}')^R\|_2^2 + \beta\|(\tilde{\rho}')^{AR}\|_2^2 - \|(\tilde{\rho}')^B\|_2^2\|(\tilde{\rho}')^R\|_2^2 < \|(\tilde{\rho}')^{A'B}\|_2^2 \cdot \|(\tilde{\rho}')^{AR}\|_2^2 = 2^{-(H_2^\gamma(A'|B)\omega - (H_2^\gamma(A|R)\rho)},
$$

where $(\tilde{\rho}')^{A'B}$, $(\tilde{\rho}')^{AR}$ are defined in Equations 4, 3 respectively,

$$
\alpha := \|(\tilde{\rho}')^B\|_2^2 \frac{|A|^2 - |A|\eta}{|A|^2 - 1}, \quad \beta := \|(\tilde{\rho}')^{A'B}\|_2^2 \frac{|A|^2 - |A|\eta^{-1}}{|A|^2 - 1}, \quad \eta := \|(\tilde{\rho}')^{A'B}\|_2^2
$$

Proof. The proof of Fact 1 in [Dup10] implies the equality and upper bound for $E_{\text{Haar}}[(g(U))^2]$. 

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given above. Fact 19 applied to the function \( g(U) \) with upper bound \( L_g \) on the Lipschitz constant given by Lemma 2 gives the desired concentration result.

We now evaluate the higher order moments of the functions \( g(U) - \mu \) and \( (g(U))^2 - \mu^2 \).

**Lemma 4.** Define \( \mu := \mathbb{E}_{U \sim \text{Haar}}[g(U)] \). Let \( m \) be a positive integer satisfying \( 64m < 9|A| \cdot 2^{-(1+\delta)(H'_{\max})^*(B)\omega+H_2'(A|R)\rho+4} \). Then the \((2m)\)th moment of the functions \( g(U) - \mu \) and \( (g(U))^2 - \mu^2 \) are upper bounded by

\[
\mathbb{E}_{\text{Haar}}(|g(U) - \mu|^{2m}) \leq 2(m2^{(1+\delta)(H'_{\max})^*(B)\omega-H_2'(A|R)\rho+4}|A|^{-1})^m.
\]

\[
\mathbb{E}_{\text{Haar}}(|(g(U))^2 - \mu^2|^{2m}) \leq 2(2m + 1)(9m\mu^22^{(1+\delta)(H'_{\max})^*(B)\omega-H_2'(A|R)\rho+4}|A|^{-1})^m.
\]

**Proof.** Applying Fact 20 to the non-negative random variable \(|g(U) - \mu|\) with concentration given by Proposition 1 gives the bounds of the lemma.

**Lemma 5.** Define \( \mu := \mathbb{E}_{\text{Haar}}[g(U)] \). Suppose

\[
|A| > 2400\cdot 2^{(1+\delta)(H'_{\max})^*(B)\omega-H_2'(A|R)\rho}(2 \log |A|) \cdot (1+\delta)(H'_{\max})^*(B)\omega-H_2'(A|R)\rho \cdot (-\log \mathbb{E}_{\text{Haar}}[(g(U))^2]).
\]

Then \( \mu^2 \leq \mathbb{E}_{\text{Haar}}[(g(U))^2] \leq 6\mu \).

**Proof.** The first inequality follows by convexity of the square function. We now prove the second inequality. Let \( m \) be a positive integer. By Lemma 4, we get

\[
\mathbb{P}_{\text{Haar}}[(g(U))^2 - \mu^2 > 2\mu] \\
\leq \mathbb{P}_{\text{Haar}}[(g(U))^2 - \mu^2]^{2m} > (2\mu)^{2m} \leq \frac{\mathbb{E}_{\text{Haar}}[(g(U))^2 - \mu^2]^{2m}}{(2\mu)^{2m}} \\
\leq 2(2m + 1)(9m\mu^22^{(1+\delta)(H'_{\max})^*(B)\omega-H_2'(A|R)\rho+4}|A|^{-1})^m \\
\leq 2(2m + 1)(3m2^{(1+\delta)(H'_{\max})^*(B)\omega-H_2'(A|R)\rho+4}|A|^{-1})^m.
\]

Define

\[
a := |A| \cdot 2^{-(1+\delta)(H'_{\max})^*(B)\omega+H_2'(A|R)\rho}, \quad m := \left[300^{-1} \cdot 2^{-4} \cdot a\right].
\]

This gives:

\[
\mathbb{P}_{\text{Haar}}[(g(U))^2 > \mu^2 + 2\mu] \leq 2(2m + 2)10^{-2m} \leq 9^{-2m}.
\]
Using Lemma 3,

\[ \mathbb{E}_{\text{Haar}}[(g(U)^2)] \leq \mu^2 + 2\mu + (2|A|) \cdot 2^{(1+\delta)(H_{\text{max}})^2(B)_{\omega}-H_2(A|R)_{\rho}} \cdot 9^{-2m} \leq 3\mu + \frac{\mathbb{E}_{\text{Haar}}[(g(U)^2)]}{2}. \]

This finishes the proof of the lemma. \( \Box \)

C. Concentration of \((g(U))^2\) under t-design

In this section we finally obtain an exponential concentration for \((g(U))^2\) when \(U\) is chosen uniformly at random from a unitary t-design for suitable \(t\). We first prove the following lemma.

**Lemma 6.** Let \(\mathcal{T}^{A\rightarrow B}\) be a completely positive superoperator with Stinespring dilation \(W_{\mathcal{T}}^{AC\rightarrow BZ} \geq 0\), where \(|A||C| = |B||Z|\), the input ancillary system is \(C\) and the output ancillary system is \(D\). Let \(F^{A_1A_2}\) and \(F^{B_1B_2}\) be the appropriate swap operators. Then

\[ \|((\mathcal{T}^\dagger)_{B_1\rightarrow A_1} \otimes (\mathcal{T}^\dagger)_{B_2\rightarrow A_2})(F^{B_1B_2})\|_2 = \|((\mathcal{T}^{A_1\rightarrow B_1} \otimes \mathcal{T}^{A_2\rightarrow B_2})(F^{A_1A_2})\|_2 \leq \|W_{\mathcal{T}}\|^4. \]

**Proof.** By Stinespring representation of \(\mathcal{T}\) as given in Fact 4, \(\mathcal{T}^{A\rightarrow B}(M^A) = \text{Tr}_Z[W_{\mathcal{T}}(M^A \otimes |0\rangle \langle 0|)W_{\mathcal{T}}]\) for any \(M^A \in \mathcal{L}(A)\). Expressing the swap operator \(F^{A_1A_2}\) in computational basis, we have

\[ F^{A_1A_2} = \sum_{a} (|a\rangle \langle a|)^{A_1A_2}(|\langle a| a\rangle) = \sum_{a'} |a'\rangle^{A_1} \langle a| \otimes |a\rangle^{A_2} \langle a'|. \]

Note that the swap operator is Hermitian.

Observe that

\[ \|((\mathcal{T}^\dagger)_{B_1\rightarrow A_1} \otimes (\mathcal{T}^\dagger)_{B_2\rightarrow A_2})(F^{B_1B_2})\|_2 \]

\[ \overset{a}{=} \text{Tr}[((((\mathcal{T}^\dagger)_{B_1\rightarrow A_1} \otimes (\mathcal{T}^\dagger)_{B_2\rightarrow A_2})(F^{B_1B_2}))^2] \]

\[ \overset{b}{=} \text{Tr}[((((\mathcal{T}^\dagger)_{B_1\rightarrow A_1} \otimes (\mathcal{T}^\dagger)_{B_2\rightarrow A_2})(F^{B_1B_2})) \otimes (((\mathcal{T}^\dagger)_{B_1'\rightarrow A_1'} \otimes (\mathcal{T}^\dagger)_{B_2'\rightarrow A_2'})(F^{B_1'B_2'})))F^{(A_1A_2)(A_1'A_2')}] \]

\[ \overset{c}{=} \text{Tr}[(F^{B_1B_2} \otimes F^{B_1'B_2'})(((\mathcal{T}^{A_1\rightarrow B_1} \otimes \mathcal{T}^{A_2\rightarrow B_2})) \otimes (\mathcal{T}^{A_1'\rightarrow B_1'} \otimes \mathcal{T}^{A_2'\rightarrow B_2'}))(F^{(A_1A_2)(A_1'A_2')})] \]

\[ \overset{d}{=} \text{Tr}[(F^{B_1B_1'}(B_2B_2'))(((\mathcal{T}^{A_1\rightarrow B_1} \otimes \mathcal{T}^{A_1'\rightarrow B_1'})) \otimes (\mathcal{T}^{A_2\rightarrow B_2} \otimes \mathcal{T}^{A_2'\rightarrow B_2'}))(F^{A_1A_1'} \otimes F^{A_2A_2'})] \]

\[ = \text{Tr}[(F^{B_1B_1'}(B_2B_2'))(((\mathcal{T}^{A_1\rightarrow B_1} \otimes \mathcal{T}^{A_1'\rightarrow B_1'}))(F^{A_1A_1'}) \otimes (\mathcal{T}^{A_2\rightarrow B_2} \otimes \mathcal{T}^{A_2'\rightarrow B_2'}))(F^{A_2A_2'})] \]

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\[ \begin{align*}
&= \text{Tr} \left[ (\mathcal{T}^{A_1 \rightarrow B_1} \otimes \mathcal{T}^{A_1' \rightarrow B_1'}) (F^{A_1 A_1'})^2 \right] \\
&= \| (\mathcal{T}^{A_1 \rightarrow B_1} \otimes \mathcal{T}^{A_2 \rightarrow B_2}) (F^{A_1 A_2}) \|^2_2,
\end{align*} \]

where in

(a) we use the fact that \( \mathcal{T}^\dagger \) is completely positive as \( \mathcal{T} \) is completely positive and the fact that the swap operator is Hermitian,

(b) we take \( A_1', A_2' \) to be two new systems of the same dimension as \( A \), \( F^{(A_1 A_2)(A_1' A_2')} \) as the operator swapping \( (A_1 A_2) \) with \( (A_1' A_2') \) and Fact 15,

(c) we use the definition of the adjoint of a superoperator under the Hilbert-Schmidt inner product,

(d) we use a property of the swap operator,

(e) we use Fact 15.

This proves the first of the equalities asserted above.

Finally,

\[ \begin{align*}
\| (\mathcal{T}^{A_1 \rightarrow B_1} \otimes \mathcal{T}^{A_2 \rightarrow B_2}) (F^{A_1 A_2}) \|^2_2 &
\leq \sum_{aa'} \| (P_a P_a^\dagger) B_1 \otimes (P_a P_a^\dagger) B_2 \|^2_2 \\
&= \sum_{aa'} \| (P_a P_a^\dagger) B_1 \|^2_2 \cdot \| (P_a P_a^\dagger) B_2 \|^2_2 \\
&= \left( \sum_{a} \| (W_T (|a\rangle \otimes |0\rangle)^C B_1 \|^2_2 \right) \cdot \left( \sum_{a} \| (W_T (|a\rangle \otimes |0\rangle)^C B_2 \|^2_2 \right) \\
&= \left( \text{Tr} \left[ W_T^\dagger W_T \right] \right)^2 = \| W_T \|^4_2,
\end{align*} \]

where in (a) we define \( P_a^{B \times B} := (\vec{B},Z)^{-1}((W_T (|a\rangle \otimes |0\rangle)^C B Z) \) and use Fact 6. This completes the proof of the present lemma.
Note that \((g(U))^2\) is a balanced degree two polynomial in the matrix entries of \(U\). We now find out how close the moments of \((g(U))^2\) under Haar measure are to their counterparts under \(t\)-design.

**Lemma 7.** Let \(i\) be a positive integer. Consider a \(|A|, s, \lambda, 4i\)-qTPE for some positive integer \(s\) and \(\lambda \geq 0\). Then,

\[
|\mathbb{E}_{U \sim \text{qTPE}}[(g(U))^2] - \mathbb{E}_{U \sim \text{Haar}}[(g(U))^2]| \leq (2|A|^3|B|^2)^i \cdot \lambda \cdot 2^{i(1+\delta)(H_{\text{max}}^r(B)_r - iH_2^r(A|R)_r)}.
\]

**Proof.** Define the Hermitian matrix \(\gamma^{AR} := (\rho')^{AR} - \pi^A \otimes (\rho')^R\). Let \(i\) be a positive integer. Observe that

\[
(g(U))^2 = (||((\tilde{T}^t)^A \otimes B \otimes I^R)((U^A \otimes I^R) \circ (\tilde{\rho}')^{AR} - (\tilde{\omega}'^B) \otimes (\tilde{\rho}')^R)||^2_2)^i.
\]

\[
= (\text{Tr} [(((\tilde{T}^t)^A \otimes B \otimes I^R)((U^A \otimes I^R) \circ \gamma^{AR}))^2])^i.
\]

\[
= (\text{Tr} [(((\tilde{T}^t)^A \otimes B \otimes I^R)((U^A \otimes I^R) \circ \gamma^{AR})) \otimes (((\tilde{T}^t)^A_2 \otimes B_2 \otimes I^R_2)((U^A_2 \otimes I^R_2) \circ \gamma^{A_2R_2}))])
\]

\[
(F^{B_1R_2} \otimes F^{R_1R_2})^i
\]

\[
= \text{Tr} [((I^{A_1A_2} \otimes F^{R_1R_2})(\gamma^{A_1R_1} \otimes \gamma^{A_2R_2}))
\]

\[
= \text{Tr} [\otimes_{j=1}^{i}((I^{A_1(j)A_2(j)} \otimes F^{R_1(j)R_2(j)}(\gamma^{A_1(j)R_1(j)} \otimes \gamma^{A_2(j)R_2(j)}))
\]

\[
= \text{Tr} [\otimes_{j=1}^{i}(((U^t)^{A_1(j)} \otimes (U^t)^{A_2(j)}) \circ (((\tilde{T}^t)^{B_1(j) \rightarrow A_1(j)} \otimes ((\tilde{T}^t)^{B_2(j) \rightarrow A_2(j)}(F^{B_1(j)B_2(j)})))
\]

\[
= \text{Tr} [((\gamma^{AR})^i)^i]
\]

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\[\left((U^\dagger)^{A_1} \otimes (U^\dagger)^{A_2}\right)^{\otimes i} \circ \left(((\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right),\]

where \((\gamma')^{A_1 A_2} := \text{Tr}_{R_1 R_2}(I^{A_1 A_2} \otimes F^{R_1 R_2})(\gamma_{A_1 R_1} \otimes \gamma_{A_2 R_2})\). Hence,

\[
|E_{U \sim \text{TPE}}[(g(U))^{2i}] - E_{U \sim \text{Haar}}[(g(U))^{2i}]|
\]

\[
= |\text{Tr}_{U \sim \text{TPE}}\left[((\gamma')^{A_1 A_2})^{\otimes i}\left(((U^\dagger)^{A_1} \otimes (U^\dagger)^{A_2})^{\otimes i} \circ \left(((\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right)\right] - \text{Tr}_{U \sim \text{Haar}}\left[((\gamma')^{A_1 A_2})^{\otimes i}\left(((U^\dagger)^{A_1} \otimes (U^\dagger)^{A_2})^{\otimes i} \circ \left(((\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right)\right]||_1
\]

\[
\leq \|(\gamma')^{A_1 A_2}\|_1^{\cdot \|E_{U \sim \text{TPE}}\left[((U^\dagger)^{A_1} \otimes (U^\dagger)^{A_2})^{\otimes i} \circ \left(((\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right] - \text{Tr}_{U \sim \text{Haar}}\left[((U^\dagger)^{A_1} \otimes (U^\dagger)^{A_2})^{\otimes i} \circ \left(((\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right]||_\infty
\]

\[
\leq \|(\gamma')^{A_1 A_2}\|_1^{\cdot \|E_{U \sim \text{TPE}}\left[((U^\dagger)^{A_1} \otimes (U^\dagger)^{A_2})^{\otimes i} \circ \left(((\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right] - \text{Tr}_{U \sim \text{Haar}}\left[((U^\dagger)^{A_1} \otimes (U^\dagger)^{A_2})^{\otimes i} \circ \left(((\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right]||_2
\]

\[
\leq \left(\|A\| \cdot |\gamma'|^{2i} \cdot \lambda \cdot \|W_{\gamma'}^{\dagger}\|_2^{\dagger} \cdot \left(\|(\tilde{T}')^\dagger)^{B_1 \rightarrow A_1} \otimes ((\tilde{T}')^\dagger)^{B_2 \rightarrow A_2}(F^{B_1 B_2})^{\otimes i}\right)\right)\|_2
\]

\[
= (2|A|^{3}|B|^2)^i \cdot \lambda \cdot 2^{i(1+\delta)(H_{\text{max}})^{\gamma}(B)}(\|A\| \cdot |B|^2)^i,
\]

where

(a) follows from Fact 16 and Definition 11,

(b) follows from Equation 10,

(c) follows from Lemma 6,

(d) follows from Equation 9.
This completes the proof of the lemma.

Now we upper bound the centralised \((2m)\)th moment of \((g(U))^2\) under the approximate unitary design.

**Lemma 8.** Let \(m\) be a positive integer satisfying \(64m < 9|A|\cdot 2^{-(1+\delta)(H_{\text{max}}')^r(B)_{\omega} + H_2^2(A|R)_{\rho} + 4}\), and \(1 > \mu := \mathbb{E}_{U \sim \text{Haar}}[|g(U)|]\) and \(H_2^2(A|R)_{\rho} < 0\). Suppose \(1 > \mu := \mathbb{E}_{U \sim \text{Haar}}[|g(U)|]\) and \(H_2^2(A|R)_{\rho} < 0\). Consider a \((|A|, s, \lambda, 4m)\)-qTPE for some positive integer \(s\) and

\[
0 \leq \lambda^{1/m} \leq 2^{-4}|A|^{-7}|B|^{-4} \mu^2 2^{-(1+\delta)(H_{\text{max}}')^r(B)_{\omega} + H_2^2(A|R)_{\rho}}.
\]

Then,

\[
\mathbb{E}_{U \sim \text{TPE}}[((g(U))^2 - \mu^2)^{2m}] \leq 2(2m + 2)(9m \mu^2 2^{(1+\delta)(H_{\text{max}}')^r(B)_{\omega} - H_2^2(A|R)_{\rho} + 4|A|^{-1}})^m.
\]

**Proof.** From Lemma 7, we get

\[
| \mathbb{E}_{U \sim \text{TPE}}[((g(U))^2 - \mu^2)^{2m}] - \mathbb{E}_{U \sim \text{Haar}}[((g(U))^2 - \mu^2)^{2m}] |
\]

\[
= \sum_{i=0}^{2m} \binom{2m}{i} (-\mu^2)^{2m-i} \mathbb{E}_{U \sim \text{TPE}}[((g(U))^2)^i] - \mathbb{E}_{U \sim \text{Haar}}[((g(U))^2)^i] |
\]

\[
\leq \sum_{i=0}^{2m} \binom{2m}{i} (\mu^2)^{2m-i} \mathbb{E}_{U \sim \text{TPE}}[((g(U))^2)^i] - \mathbb{E}_{U \sim \text{Haar}}[((g(U))^2)^i] |
\]

\[
\leq \lambda \sum_{i=0}^{2m} \binom{2m}{i} (\mu^2)^{2m-i} (2|A|^3|B|^2)^i \cdot 2^{(1+\delta)(H_{\text{max}}')^r(B)_{\omega} - iH_2^2(A|R)_{\rho}}
\]

\[
= \lambda (\mu^2 + (2|A|^3|B|^2) \cdot 2^{(1+\delta)(H_{\text{max}}')^r(B)_{\omega} - H_2^2(A|R)_{\rho}) 2m
\]

\[
\leq \lambda (4|A|^3|B|^2) \cdot 2^{(1+\delta)(H_{\text{max}}')^r(B)_{\omega} - H_2^2(A|R)_{\rho}) 2m.
\]

Using Lemma 4 and the above inequality, we get

\[
\mathbb{E}_{U \sim \text{TPE}}[((g(U))^2 - \mu^2)^{2m}]
\]

\[
\leq \lambda (4|A|^3|B|^2) 2^{2(1+\delta)(H_{\text{max}}')^r(B)_{\omega} - 2H_2^2(A|R)_{\rho}) m
\]

\[
+ 2(2m + 1)(9m \mu^2 2^{(1+\delta)(H_{\text{max}}')^r(B)_{\omega} - H_2^2(A|R)_{\rho} + 4|A|^{-1}})^m
\]

\[
\leq 2(2m + 2)(9m \mu^2 2^{(1+\delta)(H_{\text{max}}')^r(B)_{\omega} - H_2^2(A|R)_{\rho} + 4|A|^{-1}})^m.
\]
This completes the proof of the lemma.

\textbf{Proof of Theorem 1.} Using Lemma 8, we get
\[
\mathbb{P}_{\text{TPE}}[g(U) - \mu > \kappa] 
\leq \mathbb{P}_{\text{TPE}}[(g(U))^2 > (\mu + \kappa)^2] 
\leq \mathbb{P}_{\text{TPE}}[(g(U))^2 - \mu^2 > 2\mu\kappa] 
\leq \mathbb{P}_{\text{TPE}}[((g(U))^2 - \mu^2)^{2m} > (2\mu\kappa)^{2m}] 
\leq \mathbb{E}_{\text{TPE}}[((g(U))^2 - \mu^2)^{2m}] 
\leq \frac{2(2m + 2)(9m\mu^2(1+\delta)(H'_{\text{max}})'(B)_{\omega} - H_{Z}'(A|R)_{\rho} + 4|A|^{-1})^m}{(2\mu\kappa)^{2m}} 
\leq 2(2m + 2)(3m\kappa^{-2}2^{(1+\delta)(H'_{\text{max}})'(B)_{\omega} - H_{Z}'(A|R)_{\rho} + 4|A|^{-1})^m}.
\]

Define
\[
a := |A|\kappa^2 \cdot 2^{- (1+\delta)(H'_{\text{max}})'(B)_{\omega} + H_{Z}'(A|R)_{\rho}}; \quad m := \lceil 300^{-1} \cdot 2^{-4} \cdot a \rceil.
\]

This gives:
\[
\mathbb{P}_{\text{TPE}}[g(U) > \mu + \kappa] \leq 2(2m + 2)10^{-2m} \leq 9^{-2m}.
\]

The above requires us to use a \((|A|, s, \lambda, 4m)\)-qTPE with
\[
0 \leq \lambda \leq (|A|^{-8}|B|^{-4}\mu^2 \cdot 300m)^m.
\]

Combined with Lemma 1 and Proposition 1, we finally get
\[
\mathbb{P}_{\text{TPE}}[f(U) > 2^{- \frac{1}{2}H_{Z}'(A|R)_{\rho} - \frac{1}{2}(H_{Z}')^{\epsilon, \delta}(A'|B)_{\omega} + 1} + 14\sqrt{\epsilon} + 2\kappa] \leq \exp(-800^{-1}a).
\]

Together with Lemma 5, this finishes the proof of Theorem 1.

\section{The Asymptotic IID Case}

\subsection{Asymptotic smoothing of \((H'_{\text{max}})^\epsilon\) and \((H_{Z}^{'})^{\epsilon, \delta}\)}

In this section, we use the properties of typical sequences and subspaces to find an upper bound on \((H'_{\text{max}})^\epsilon\) and a lower bound on \((H_{Z}^{'})^{\epsilon, \delta}\) in the asymptotic limit of many iid copies of the underlying quantum states. The bounds obtained will be the Shannon entropic quantities that one would expect. We first prove a few essential lemmas.
Lemma 9. Suppose we have a density matrix \( \omega \) on the system \( AB \). Let \( |w_j\rangle^{AB} \), \( j \in [||A||B]| \) be the eigenvectors of \( \omega^{AB} \) with eigenvalues \( q_j \). For \( j \in [||A||B]| \), define \( \theta_j^B := \text{Tr}_A [ |w_j\rangle^{AB} \langle w_j | ] \). Let \( p_j \), \( j \in [||A||B]| \) be the probability distribution on \([|B|] \) obtained by measuring \( \theta_j \) in the eigenbasis of \( \omega^B \). Let \( 0 < \epsilon, \delta < 1/3 \). Define \( q_{\min} := 2^{-(H_{\max})^{y/2}(AB)\omega} \), \( p_{\min} := \min_{j \in [||A||B]|} 2^{-((H_{\max})^{y/2}(|B|)_{pj})} \). Let \( n \geq 2^{5q_{\min}^{-1}p_{\min}^{-1}\delta^{-2}\log(||A||B)|/\epsilon} \). Consider the \( n \)-fold tensor power \( \omega^{nB^n} := (\omega^{AB})^{\otimes n} \). Let \( \tau \) be a strongly \( \delta \)-typical type of an eigenvector sequence of \( \omega^{nB^n} \). Let \( (|v_1\rangle \cdots \otimes |v_n\rangle)^{A^nB^n} \) be an eigenvector sequence of type \( \tau \). Let \( \sigma^{B^n} := \Pi_{\omega,3\delta}^B \omega^{B^n} \Pi_{\omega,3\delta}^B \). Let \( \Pi_{\sigma}^B \) be the orthogonal projection onto the support of \( \sigma^{B^n} \). Then,

\[
\text{Tr} \left( (I^{A^n} \otimes \Pi_{\sigma}^B) (|v_1\rangle \cdots |v_n\rangle)^{A^nB^n} (\langle v_1\rangle \cdots \langle v_n |) \right) \geq 1 - \epsilon.
\]

Proof. Since \( \tau \) is a strongly \( \delta \)-typical type, the number of occurrences \( n_j \) of each \( |w_j\rangle^{AB} \) in the sequence \( |v_1\rangle^{AB}, \ldots, |v_n\rangle^{AB} \) is \( nq_j(1 \pm \delta) \). After a suitable rearranging, we can write

\[
(|v_1\rangle \otimes \cdots \otimes |v_n\rangle)^{A^nB^n} = (|w_1\rangle^{B_1} \otimes \cdots \otimes |w|_{|A||B|}^{B_n})^{A^nB^n}.
\]

To prove the lemma, it suffices to show that

\[
\text{Tr} \left( \Pi_{\sigma}^B (\theta_1^B)^{\otimes n_1} \cdots (\theta_{|A||B|}^B)^{\otimes n_{|A||B|}} \right) \geq 1 - \epsilon.
\]

Let \( \Pi_{\sigma}^B \) be the projector onto the eigenvectors of \( (\omega^B)^{\otimes n_j} \) that are strongly \( \delta \)-typical according to \( p_j \). Then, \( \text{Tr} \left( \Pi_{\sigma}^B (\theta^B)^{\otimes n_j} \right) \geq 1 - \frac{\epsilon}{|A||B|} \). Thus,

\[
\text{Tr} \left( (\Pi_{\sigma}^B)^{\otimes n_1} \cdots (\Pi_{|A||B|}^B)^{\otimes n_{|A||B|}} \right) (\theta_1^B)^{\otimes n_1} \cdots (\theta_{|A||B|}^B)^{\otimes n_{|A||B|}} \right) \geq 1 - \epsilon.
\]

Let \( |x_1\rangle^B, \ldots, |x_{|B|}\rangle^B \) be the eigenbasis of \( \omega^B \) with eigenvalues \( r_1, \ldots, r_{|B|} \). Observe that we have the operator equality \( \sum_{j=1}^{||B||} q_j \theta_j^B = \omega^B \). Now consider the matrices \( \theta_j^B \) in the basis \( |x_1\rangle^B, \ldots, |x_{|B|}\rangle^B \). Thus for any \( i \in [|B|], \sum_{j=1}^{||B||} q_j \theta_j^B(i) = r_i \). Fix an eigenvector in the support of \((\Pi_{\sigma}^B)^{\otimes n_1} \cdots (\Pi_{|A||B|}^B)^{\otimes n_{|A||B|}} \); the eigenvector can be viewed as a sequence of length \( n \). Then the number of occurrences of \( |x_i\rangle \) in the sequence is

\[
\sum_{j=1}^{||B||} nq_j(1 \pm \delta)p_j(i)(1 \pm \delta) = r_i(1 \pm 3\delta).
\]

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This shows that the eigenvector is strongly \((3\delta)\)-typical for the state \(\omega^{B^n}\). In other words,

\[
\Pi_1^{B^n_1} \otimes \cdots \otimes \Pi_{|A|}^{B^n_{|A|}|B|} \leq \Pi_{\omega,3\delta} = \Pi^B_{\sigma}.
\]

This completes the proof of the lemma. \(\square\)

**Lemma 10.** \textit{Consider the setting of Lemma 9. Let }\(V_\tau \leq A^n B^n\text{ denote the type subspace corresponding to type }\tau\text{. Then there is a subspace }\hat{V}_\tau \leq V_\tau, |\hat{V}_\tau| \geq (1 - \sqrt{\epsilon})|V_\tau|\text{ such that for every vector }|v\rangle \in \hat{V}_\tau,\]

\[
\|(I^{A^n} \otimes \Pi^B_{\sigma})|v\rangle\|_2^2 \geq 1 - \sqrt{\epsilon}.
\]

**Proof.** We invoke Fact 14 with \(A := V_\tau\) and \(B := I^{A^n} \otimes \Pi^B_{\sigma}\) in order to prove this lemma. Take the basis for \(A\) provided by Fact 14. Call it \(\{|a\rangle\}_a\). Observe that the vectors \(\Pi_B|a\rangle\) are pairwise orthogonal (some of them may be the zero vector). From Lemma 9, we know that \(\text{Tr}[\Pi_B|a\rangle\langle a|]\geq 1 - \epsilon\). By Markov’s inequality, there is a subset \(S\) of the basis vectors of \(A\), \(|S| \geq (1 - \sqrt{\epsilon})|A|\) such that for all \(a \in S\), \(\|\Pi_B|a\rangle\|_2^2 = \text{Tr}[\Pi_B|a\rangle\langle a|]\geq 1 - \sqrt{\epsilon}\).

Define the subspace \(\hat{A} := \text{span}_{a \in S}|a\rangle\). From the above observation, for any vector \(|v\rangle \in \hat{A}\), \(\|\Pi_B|v\rangle\|_2^2 \geq 1 - \sqrt{\epsilon}\). This subspace \(\hat{A}\) serves as the subspace \(\hat{V}_\tau\) required by the lemma. \(\square\)

**Lemma 11.** \textit{Let }\(0 < \epsilon < 1\text{. Let }|v_1\rangle, \ldots, |v_t\rangle\text{ be orthonormal vectors lying in a Hilbert space }\mathcal{H}\text{. Suppose there is a subspace }B \leq \mathcal{H}\text{ with the property that }\|\Pi_B|v_i\rangle\|_2^2 \geq 1 - \epsilon\text{ for all }i \in [t].\text{ Let }|v\rangle\text{ be a unit vector lying in the span of the vectors }|v_i\rangle\text{. Then, }\|\Pi_B|v\rangle\|_2^2 \geq 1 - 7t\sqrt{\epsilon}\text{.}\}

**Proof.** Let \(|v\rangle = \sum_{i=1}^t \alpha_i |v_i\rangle\) where \(\sum_{i=1}^t |\alpha_i|^2 = 1\). Define the column \(t\)-tuple \(\alpha := (\alpha_1, \ldots, \alpha_t)^T\), and the \(t \times t\)-matrix \(M\) with \(M_{ij} := \langle v_i|\Pi_B|v_j\rangle\). Note that \(M\) is Hermitian. Then,

\[
\|\Pi_B|v\rangle\|_2^2 = \sum_{i,j=1}^t \alpha_i^* \alpha_j \langle v_i|\Pi_B|v_j\rangle = \alpha^\dagger M \alpha.
\]

We have \(M_{ii} \geq 1 - \epsilon\). For \(i \neq j\), we use triangle inequality and Fact 13 to obtain

\[
\|\Pi_B|v_i\rangle - \Pi_B|v_j\rangle\|_2 \geq \||v_i\rangle - |v_j\rangle\|_2 - \|\Pi_B|v_i\rangle - |v_i\rangle\|_2 - \|\Pi_B|v_j\rangle - |v_j\rangle\|_2 \geq \sqrt{2} - 4\sqrt{\epsilon},
\]

which implies that

\[
2 - 8\sqrt{2\epsilon} \leq \|\Pi_B|v_i\rangle - \Pi_B|v_j\rangle\|_2^2 \leq 2 - 2|M_{ij}|,
\]

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which further implies that $|M_{ij}| \leq 4\sqrt{2}$. By Gershgorin’s theorem, the smallest eigenvalue of $M$ is larger than $1 - \epsilon - 4(t-1)\sqrt{2} \geq 1 - 7t\sqrt{\epsilon}$. So $\|\Pi_B|v\|_2^2 = \alpha^t M \alpha \geq 1 - 7t\sqrt{\epsilon}$, completing the proof of the lemma.

**Proposition 2.** Suppose we have a density matrix $\omega$ on the system $B$. Let $0 < \epsilon, \delta < 1/3$ and Define $q_{\min} := 2^{-(H_{\max}^{(B)})^2/2} \omega$. Let $n \geq 4q_{\min}^{-1} \delta^{-2} \log(\|B\|/\epsilon)$. Consider the $n$-fold tensor power $\omega^{B^n} := (\omega^B)^{\otimes n}$. Then, $(H'_{\max})^\epsilon (B^n)_{\omega} \leq (1 + \delta)H(B)_{\omega}$.

**Proof.** Consider the eigenvalues of $\omega^{B^n}$ that are not strongly $\delta$-typical; call them atypical. By Fact 18, the atypical eigenvalues sum to less than or equal to $\epsilon$ and the smallest typical eigenvalue is at least $2^{-n(1+\delta)H(B)_{\omega}}$. Hence the eigenvalues less than $2^{-n(1+\delta)H(B)_{\omega}}$ add up to less than or equal to $\epsilon$. This completes the proof the proposition.

**Proposition 3.** Suppose we have a density matrix $\omega$ on the system $AB$. Let $|w_j\rangle^{AB}$, $j \in [|A||B|]$ be the eigenvectors of $\omega^{AB}$ with eigenvalues $q_j$. For $j \in [|A||B|]$, define $\theta^B_j := \text{Tr}_A [|w_j\rangle^{AB}\langle w_j|]$. Let $p_j, j \in [|A||B|]$ be the probability distribution on $ [|A||B|]$ obtained by measuring $\theta^B_j$ in the eigenbasis of $\omega^B$. Let $0 < \epsilon, \delta < 1/3$ and Define $q_{\min} := \min_{j \in [|A||B|]} q_{\min} |q_j > 0 q_j$, $p_{\min} := \min_{j \in [|A||B|]} \min_{i \in [|B|]} p_{\min} > 0 p_j(i)$. Let $n := 2^5 q_{\min}^{-1} p_{\min}^{-1} \delta^{-2} \log(\|A||B|/\epsilon)$. Consider the $n$-fold tensor power $\omega^{A^n B^n} := (\omega^{AB})^{\otimes n}$. Let $\epsilon' := 7(n + |A||B|) |A||B|^{1/4}$. Then,

$$H_{2}^{-2\epsilon^2} (A^n|B^n)_{\omega} \geq (H'_{2})^{\epsilon',5\delta} (A^n|B^n)_{\omega} \geq nH(A|B)_{\omega} - n\delta (3H(AB)_{\omega} + 7H(B)_{\omega}),$$

**Proof.** We use Lemma 10. For a type $\tau$ of $\omega^{A^n B^n}$, define $p_{\tau} := \text{Tr}_A [\Pi_{V_\tau} \omega^{A^n B^n}]$. By Fact 18,

$$\omega^{A^n B^n} = \bigoplus_{\tau} \Pi_{V_\tau} \omega^{A^n B^n} = \bigoplus_{\tau} p_{\tau} \Pi_{V_\tau} |V_\tau\rangle,$$

where the direct sum is over all types $\tau$. Now define $\eta^{A^n B^n} = \bigoplus_{\tau} p_{\tau} \Pi_{V_\tau} |V_\tau\rangle$, where the sum is only over strongly $\delta$-typical types $\tau$. We have

$$\eta^{A^n B^n} \leq \omega^{A^n B^n}, \quad \|\eta^{A^n B^n} - \omega^{A^n B^n}\|_1 \leq \sqrt{\epsilon}.$$

Let $\sigma^{B^n} := \Pi^{B^n}_{\omega,3\delta} \omega^{B^n}$. Let $\Pi^{B^n}_{\sigma}$ be the orthogonal projection onto the support of $\sigma^{B^n}$. By Fact 18, we have

$$\sigma^{B^n} \leq \omega^{B^n}, \quad \|\omega^{B^n} - \sigma^{B^n}\|_1 \leq \epsilon, \quad \|\sigma^{B^n}\|_{\infty} \leq 2^{nH(B)_{\omega}(1+3\delta)}.$$
From Lemma 9, we already know that for any strongly $\delta$-typical type $\tau$, for any vector $|w_\tau\rangle \in \hat{V}_\tau$, $\|(I^{A_n} \otimes \Pi^B_\sigma)|w_\tau\rangle\|_2^2 \geq 1 - \sqrt{\epsilon}$. We now have to show a similar result for an arbitrary linear combination of vectors $|w_\tau\rangle$ over all strongly $\delta$-typical types $\tau$. For this we invoke Lemma 11 and Fact 18. We thus conclude that for any vector $|v\rangle \in \text{supp}(\eta^{A_n B^n})$,

$$
\|(I^{A_n} \otimes \Pi^B_\sigma)|v\rangle\|_2 \geq 1 - 7\left( \frac{n + |A||B| - 1}{|A||B| - 1} \right)^{1/4} \geq 1 - \epsilon'.
$$

By Fact 18, the smallest non-zero eigenvalue of $(\omega'_{\epsilon,\delta})^{B^n}$ is less than $2^{-nH(B)\omega(1-\delta)}$. Again invoking Fact 18, we conclude that $\text{supp}((\omega''_{\epsilon,5\delta})^{B^n}) \geq \text{supp}(\sigma^{B^n})$. Thus, for any vector $|v\rangle \in \text{supp}(\eta^{A_n B^n})$, $\|(I^{A_n} \otimes \Pi^B_{\omega''_{\epsilon,5\delta}})|v\rangle\|_2 \geq 1 - \epsilon'$. Moreover, by Proposition 2

$$
\log\|\omega''_{\epsilon,5\delta}^{B^n}\|^{-1}_\infty = (1 + 5\delta) \log\|\omega''_{\epsilon,\delta}^{B^n}\|^{-1}_\infty = (1 + 5\delta)(H_{\text{max}}')^{(B^n)} \omega \leq n(1 + 7\delta)H(B)\omega.
$$

Again using Fact 18, we get

$$
(H_2')^{\epsilon',5\delta} (A^n|B^n)\omega \\
\geq -2 \log\|\Pi^{A_n B^n} (I^{A_n} \otimes (\omega''_{\epsilon,5\delta}^{B^n})^{(B^n)} - 1/4) B^n\|_2 \\
\geq -2 \log\|\Pi^{A_n B^n} (I^{A_n} \otimes (\omega''_{\epsilon,5\delta}^{B^n})^{(B^n)} - 1/4) B^n\|_2 \\
= -2 \log\|\Pi^{A_n B^n} (I^{A_n} \otimes (\omega''_{\epsilon,5\delta}^{B^n})^{(B^n)} - 1/4) B^n\|_2 \\
\geq -nH(B)\omega(1 + 7\delta) - 2 \log\|\Pi^{A_n B^n} (I^{A_n} \otimes (\omega''_{\epsilon,5\delta}^{B^n})^{(B^n)} - 1/4) B^n\|_2
$$

This completes the proof of the proposition.

**Remark:** Consider fixed systems $A$, $B$ and a fixed state $\omega^{AB}$. For a fixed $\delta$, divide the smooth modified conditional Rényi 2-entropy, the smooth conditional Rényi 2-entropy and
the smooth modified max-entropy by $n$ and let $\epsilon \to 0$. Then $\epsilon' \to 0$. Finally, let $\delta \to 0$. This shows that in the asymptotic iid limit, the smooth conditional Rényi 2-entropy divided by $n$ is lower bounded by the smooth modified conditional Rényi 2-entropy divided by $n$ which is further lower bounded by the conditional Shannon entropy, and the smooth modified max-entropy divided by $n$ is upper bounded by the Shannon entropy.

B. Proof of Corollary 1: IID extension of Theorem 1

In this section we take our main one-shot concentration result and apply it in the asymptotic iid setting. That is, we take the $n$-fold tensor product copy of the channel $\mathcal{T}$ and the state $\rho^{AR}$, apply Theorem 1 to it, and obtain bounds in terms of the standard Shannon entropies.

Proof of Corollary 1. The proof follows by a direct application of Theorem 1 and Propositions 2 and 3. We get

$$
\mu \leq 2^{-\frac{1}{2}H^2_{\max}(A^p|B^n)_{\omega\otimes n} - \frac{1}{2}H^2_{\max}(A^n|R^n)_{\rho\otimes n}}
$$

$$
a = |A|^{n} \kappa_{2}^{2} H^2_{\max}(A^n|R^n)_{\rho\otimes n} - (1 + \delta) H^2_{\max}(B^n)_{\omega\otimes n}
$$

$$
\geq |A|^{n} \frac{1}{\kappa}^{2} H^{2}(A|R)_{\rho} - \delta(3H(AR)_{\rho} + 7H(R)_{\rho}) - nH(B_{\omega})(1 + \delta)^{2}
$$

Substituting the above expressions in Theorem 1 proves the desired corollary. $\Box$

V. FQSW AND RELATIVE THERMALISATION

In this section, we use the following definition for the decoupling function $f(U)$ in the Fully Quantum Slepian Wolf (FQSW) problem.

$$
f(U) = f_{FQSW}(U^{A_1A_2}) := \|\text{Tr}_{A_2}[U^{A_1} \otimes I^{R} \circ \rho^{A_1A_2R}] - \pi^{A_1} \otimes \rho^{R}\|_{1}.
$$

Now we demonstrate how our main concentration result under approximate unitary $t$-designs can be used directly to obtain a concentration result for the FQSW problem. In order to apply Theorem 1 we identify the following parameters defined therein as:
• Unitary group over which function $f$ is defined is $U(A_1 A_2)$ i.e. the input system $A$ to the superoperator is $A := A_1 \otimes A_2$.

• Output system $B := A_1$ and superoperator $T^{A \to A_1} := \text{Tr}_{A_2}$.

• The state $\omega^{A' A_1} = (T^{A \to A_1} \otimes I')\Phi^{A' A'} = \Phi^{A_1 A'_1} \otimes \pi^{A'_2}$.

• $f(U) = \| (\text{Tr}_{A_2} \otimes \mathbb{I}^R)((U^A \otimes I^R) \circ \rho^{AR}) - \pi^{A_1} \otimes \rho^R \|_1$.

• $(H'_{\max})^\epsilon(B)_\omega = \log |A_1|$ as the the reduced state $\omega^{A_1} = \pi^{A_1}$,

• Take $\delta = 0$,

• $g(U) = |A_1|^{1/2}\| (\text{Tr}_{A_2} \otimes \mathbb{I}^R)((U^A \otimes I^R) \circ (\tilde{\rho}')^{AR}) - \pi^{A_1} \otimes (\tilde{\rho}')^{R} \|_2$.

• Observe that $(\omega^{\epsilon\rho}_0)^{A_1} = \pi^{A_1}$. Define $(\tilde{\omega}')^{A' A_1} := I^{A'_1} \otimes I^{A'_2} \otimes (\omega^{\epsilon\rho}_0)^{A_1})^{-1/4} \circ (\Phi^{A_1 A'_1} \otimes \pi^{A'_2})$. Note that $(H'_{\max})^{\epsilon,0}(A|A_1)_\omega = -2 \log \| (\tilde{\omega}')^{A' A_1} \|_2 - 2 \log (1 - \epsilon)$. We have,

\[
\| (\tilde{\omega}')^{A' A_1} \|_2 = \| |A_1|^{1/2}\Phi^{A_1 A'_1} \otimes \pi^{A'_2} \|_2 = \sqrt{|A_1|/|A_2|},
\]

\[
\| (\tilde{\omega}')^{A_1} \|_2 = \| |A_1|^{1/2}\Phi^{A_1} \|_2 = 1,
\]

\[
\eta = \frac{\| (\tilde{\omega}')^{A' A_1} \|_2^2}{\| (\tilde{\omega}')^{A_1} \|_2} = \frac{|A_1|}{|A_2|},
\]

\[
\alpha = \frac{\| (\tilde{\omega}')^{A_1} \|_2^2 |A|^2 - |A_1|^2}{|A|^2 - 1} = \frac{|A_1|^2 |A_2|^2 - |A_2|^2}{|A|^2 - 1},
\]

\[
\beta = \frac{\| (\tilde{\omega}')^{A' A_1} \|_2^2 |A|^2 - |A_1|^2 |A_2|^2 - |A_1|^2}{|A|^2 - 1} = \frac{|A_1|^2 |A_2|^2 - |A|^2}{|A|^2 - 1}.
\]

\[
\mathbb{E}_{\text{Haar}}[(g(U))^2] = \alpha \| (\tilde{\rho}')^{R} \|_2^2 + \beta \| (\tilde{\rho}')^{AR} \|_2^2 - \| (\tilde{\omega}')^{A_1} \|_2 \| (\tilde{\rho}')^{R} \|_2^2
\]

\[
= - \frac{|A|^2 - 1}{|A|^2 |A_2|^2 - 1} \| (\tilde{\rho}')^{R} \|_2^2 + \frac{|A_1|^2 |A_2|^2 - |A|^2}{|A|^2 |A_2|^2 - 1} \| (\tilde{\rho}')^{AR} \|_2^2
\]

\[
\leq \frac{|A_1|^2}{|A_2|^2} \cdot \| (\tilde{\rho}')^{AR} \|_2^2.
\]

• Suppose we are promised that $\| (\tilde{\rho}')^{R} \|_2^2 < 0.9 |A_1| |A_2| \| (\tilde{\rho}')^{AR} \|_2^2$ and $|A_1| \geq 2$. Then,

\[
\mathbb{E}_{\text{Haar}}[(g(U))^2] \geq \frac{0.1 |A_1|}{|A_2|^2} \cdot \frac{|A_1|^2 |A_2|^2 - |A_2|^2}{|A|^2 |A_2|^2 - 1} \| (\tilde{\rho}')^{AR} \|_2^2 \geq \frac{0.07 |A_1|}{|A_2|^2} \cdot \| (\tilde{\rho}')^{AR} \|_2^2.
\]
• Define $\mu := \mathbb{E}_{\text{Haar}}[g(U)]$. Then, $\mu^2 \leq \mathbb{E}[(g(U))^2] \leq 6\mu$.

• The tail parameter $a$ from Theorem 1 becomes

$$a = |A|\kappa^2|A_1|^{-\frac{1}{2}H_2^{(A|R)}(\rho)} = |A_2|\kappa^22^{H_2^{(A|R)}(\rho)}.$$  

Proof of Theorem 2. Now substituting these parameters in Theorem 1 we get

$$\mathbb{P}_{\text{TPE}}[f(U) > 2\sqrt{|A_1|2^{-\frac{1}{2}H_2^{(A|R)}(\rho)} + 14\sqrt{\epsilon} + 2\kappa}] \leq \exp(-800^{-1}a)$$

for $U$ chosen uniformly at random from a $(|A_1||A_2|, s, \lambda, 4m)$-qTPE where $m = \lceil 300^{-1}2^{-4}a \rceil$ and

$$0 \leq \lambda \leq (|A_1|^{-11}|A_2|^{-9} \cdot 2^{-H_2^{(A|R)}(\rho)} \cdot m)^m \leq (|A_1|^{-12}|A_2|^{-8}\mu^2 \cdot 300m)^m.$$  

Observe that if $|A_1| \leq \text{polylog}(|A_2|)$, then efficient constructions for such qTPEs exist [BHH12, Sen18b]. This completes the proof. \qed

An immediate application of measure concentration of FQSW lies in quantum thermodynamics, in describing a process called relative thermalisation [dHRW16]. One of the most fundamental questions in quantum thermodynamics is how a small system starting out in a particular quantum state spontaneously thermalises when brought in contact with a much larger environment e.g. a bath. More precisely when brought in contact with a bath, the small system decouples from any another system, which we may call as the reference system, it may be initially entangled with. The formal definition of relative thermalisation is as follows:

Definition 12. Let system $S$, environment $E$ and reference $R$ be quantum systems and $\Omega \subseteq S \otimes E$ be a subspace corresponding to a physical constraint such as total energy. The global system is in a state $\rho^{\Omega R}$, supported in the Hilbert space $\Omega \otimes R$. The time evolution is described by a unitary on $S \otimes E$. The state after time evolution is denoted by $\sigma^{\Omega R}$. The system $S$ is said to be $\delta$-thermalised relative to $R$ in state $\sigma^{\Omega R}$ if:

$$\|\sigma^{S R} - \pi^S \otimes \sigma^R\|_1 \leq \delta$$

where $\sigma^{S R} := \text{Tr}_E[\sigma^{\Omega R}]$ and $\pi^S \triangleq \text{Tr}_E[I^{\Omega R}]$ is the so called local microcanonical state.
Thus, relative thermalisation requires that, after the environment $E$ is traced out, the system $S$ should be close to the state $\pi^S$ and should not have strong correlations with the reference $R$. If the time-evolution of $S \otimes E$ is modelled by a Haar random unitary on $\Omega$, then Fact 1 guarantees that relative thermalisation occurs in expectation over the Haar measure. Furthermore, Fact 3 says that $1 - \exp\left(-\frac{\delta^2 |\Omega|^2}{16\|\rho\|_\infty^4}\right)$ fraction of Haar random unitaries achieve relative thermalisation.

Since Haar random unitaries are computationally inefficient, it is natural to wonder whether nature truly evolves via Haar random unitary. Hence, the work of Nakata et al. [NHMW17] investigates what happens if the evolution of system plus environment is modelled by unitary chosen from an efficiently implementable approximate unitary 2-design. Their unitary acts on the subspace $\Omega$ only. They show that relative thermalisation indeed takes place but for a much smaller fraction $1 - \exp\left(-\frac{\delta^2 |\Omega|^2}{218|\Omega|^2\|\rho\|_\infty^4}\right)$ of unitaries. It is reasonable to expect that $\|\rho^\Omega\|_\infty \geq \frac{1}{\sqrt{|\Omega|}}$, in which case the fraction of unitaries achieving relative thermalisation is only guaranteed to be at least $1 - \exp\left(-\frac{\delta^2 |\Omega|^2}{218|\Omega|^2}\right)$, which is almost zero for large $|\Omega|$.

Suppose the local microcanonical state $\pi^S$ is completely mixed on $S$. Then Theorem 2 achieves intermediate performance between the result of Dupius (Fact 3) and the result of Nakata et al. [NHMW17] in the following senses:

1. In our result, the system plus environment evolves according to a unitary chosen uniformly at random from an approximate unitary $t$-design for moderate values of $t$. Our unitary acts on the subspace $\Omega$ only. Our unitaries require less random bits than the Haar random unitaries used by Dupuis, but more random bits than the approximate 2-design used by Nakata et al. Our unitaries are not known to be efficiently implementable unless $|S| \leq \text{polylog}(|R|)$. Note however that the Haar random unitaries used by Dupuis are known to be inefficient to implement.

2. Our Theorem 2 shows that relative thermalisation still takes place for the fraction $1 - \exp(-800^{-1}a)$ of unitaries, where $a = |\Omega|\delta^2|S|^{-1}2H_2(|\Omega|,|R|,\rho)$. Note that $H_2(|\Omega|,|R|,\rho) \geq -\log |\Omega|$ for any state $\rho^\Omega R$. The equality is achieved when $\rho^\Omega R$ is maximally entangled on $R$. Under the reasonable assumption that $H_2(|\Omega|,|R|,\rho) \geq -0.5 \log |\Omega|$, the fraction of unitaries that achieve relative thermalisation is at least $1 - \exp(-800^{-1} \cdot |\Omega|^{1/2}\delta^2|S|^{-1})$, as environment $E$ is generally of a much larger dimension than the
system $S$, it is reasonable to expect that $|S| < |Ω|^{1/4}$. In that case, the fraction of unitaries that achieve relative thermalisation in our result is guaranteed to be at least $1 - \exp(-800^{-1} \cdot |Ω|^{1/4} \delta^2)$, which is nearly one for large $|Ω|$. Our decoupling result is much better than that of Nakata et al. which can only guarantee that $1 - \exp\left(-\frac{\delta^4}{2^{18} |Ω|}\right) \approx 0$ fraction of unitaries achieve relative thermalisation. However, our result is worse than that of Dupuis which guarantees that $1 - \exp\left(-\frac{|Ω|^{3/2} \delta^2}{16}\right)$ fraction of Haar random unitaries achieve relative thermalisation.

VI. CONCLUSION

In this work we obtain a novel concentration result for one-shot non-catalytic decoupling via approximate unitary $t$-designs for moderate values of $t$. Our bounds are stated in terms of one-shot smooth variants of Rényi 2-entropies and max-entropies. We then consider the asymptotic iid limit of our concentration result and show that the bounds reduce to the standard Shannon entropies. Finally, we apply our concentration result to the Fully Quantum Slepian Wolf problem. This leads to a new result on relative thermalisation of quantum systems. In particular for systems that are much smaller than their reference or partner systems, we show that relative thermalisation can be achieved with probability exponentially close to one using efficiently implementable unitaries. This is the first result of this kind.

For larger systems, it is unknown whether suitable efficient approximate $t$-designs exist. Hence the question of whether relative thermalisation can be achieved by efficiently implementable unitaries with exponentially high probability in the general case still remains open.

Several applications of the original decoupling theorem in expectation are known in the literature. Our result can be applied to many of them obtaining, for the first time, corresponding concentration results via approximate unitary $t$-designs. Whether these concentration results have any operational significance is a topic left for future research.
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