HERMITE-HADAMARD TYPE INEQUALITIES VIA PREINVEXITY AND PREQUASIINVEXITY

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Abstract. In this paper, we obtain some Hermite-Hadamard type inequalities for functions whose third derivatives in absolute value are preinvex and prequasiinvex.

1. INTRODUCTION

Several researchers have been studied on convexity and a lot of papers have been written on this topic which give new generalizations, extensions and applications. A huge amount of these studies on refinements of celebrated Hermite-Hadamard Inequality for convex functions. Invex functions introduced by Hanson as a generalization of convex functions in [2]. Some properties of preinvex functions have been discussed in the papers [3]-[9]. It is well-known that there are many applications of invexity in nonlinear optimization, variational inequalities and in the other branches of pure applied sciences. 

Now it is time to give the following definitions and results which will be used in this paper (see [5], [6] and [7]):

Let $K$ be a nonempty closed set in $\mathbb{R}^n$. We denote by $\langle ., . \rangle$ and $\| . \|$ the inner product and norm respectively. Let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions.

Definition 1. (See [6]) Let $u \in K$. Then the set $K$ is said to be invex at $u$ with respect to $\eta (.,.)$, if

$$u + t\eta(v, u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$

$K$ is said to be invex set with respect to $\eta$, if $K$ is invex at each $u \in K$. The invex set $K$ is also called a $\eta$-connected set.

Remark 1. (See [5]) We would like to mention that the Definition 1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point $u$ which is contained in $K$. We don’t require that the point $v$ should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that $v$ should be an end point of the path for every pair of points, $u, v \in K$, then $\eta(v, u) = v - u$ and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(v, u) = v - u$, but the converse is not necessarily true.

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Definition 2. (See [6]) The function $f$ on the invex set $K$ is said to be preinvex with respect to $\eta$, if
\[
f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].
\]
The function $f$ is said to be preconcave if and only if $-f$ is preinvex. Note that every convex function is a preinvex function, but the converse is not true. For example, the function $f(u) = -|u|$ is not a convex function, but it is a preinvex function with respect to $\eta$, where
\[
\eta(v, u) = \begin{cases} v - u, & \text{if } v \leq 0, u \leq 0 \\ u - v, & \text{otherwise} \end{cases}.
\]

Definition 3. (See [4]) The function $f$ on the invex set $K$ is said to be prequasiinvex with respect to $\eta$, if
\[
f(u + t\eta(v, u)) \leq \max \{f(u), f(v)\}, \quad \forall u, v \in K, \quad t \in [0, 1].
\]

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality:
\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]
where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$.

Lemma 1. (See [1]) Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a three times differentiable function on $I^\circ$ with $a, b \in I$, $a < b$. If $f''' \in L[a, b]$, then
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx - \frac{b - a}{12} [f'(b) - f'(a)] = \frac{(b - a)^3}{12} \int_0^1 t (1 - t) (2t - 1) f'''(ta + (1 - t)b) dt.
\]

The main purpose of this paper is to prove some new inequalities of Hermite-Hadamard type for preinvex and prequasiinvex functions by using a new version of Lemma 1.

2. Inequalities for preinvex functions

To prove our main results we need the following equality which is a generalization of Lemma 1 to invex sets:

Lemma 2. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(b, a) \neq 0$. Suppose that $f : A \rightarrow \mathbb{R}$ is a three times differentiable function. If $f'''$ is integrable on the $\eta$–path $P_{bc}$, $c = b + \eta(a, b)$, then the following equality holds:
\[
\int_b^{b+\eta(a, b)} f(x)dx - \eta(a, b) \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{(\eta(a, b))^2}{12} [f'(b) - f'(b + \eta(a, b))]
\]
\[
= \frac{(\eta(a, b))^4}{12} \int_0^1 t (1 - t) (2t - 1) f'''(b + t\eta(a, b))dt.
\]

Proof. The proof of Lemma 2 is left to the reader. □
Theorem 1. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: A \times A \rightarrow \mathbb{R}$. Suppose that $f: A \rightarrow \mathbb{R}$ is a three times differentiable function. If $|f''|^q$ is preinvex on $A$, then the following inequality holds:

$$
\left| \int_{b}^{b+\eta(a,b)} f(x) dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))] \right|
\leq \frac{(\eta(a,b))^4}{192} \left[ \frac{|f'''(a)|^q + |f'''(b)|^q}{2} \right]^{\frac{1}{q}}
$$

for $q \geq 1$ and every $a, b \in A$ with $\eta(b,a) \neq 0$.

Proof. Since $a, b \in A$ and $A$ is an invex set with respect to $\eta$, it is obvious that $b + t\eta(a,b) \in A$ for $t \in [0,1]$. By using Lemma 2, the power-mean inequality and preinvexity of $|f''|^q$, we can write

$$
\left| \int_{b}^{b+\eta(a,b)} f(x) dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))] \right|
= \frac{(\eta(a,b))^4}{12} \int_{0}^{1} t (1-t) |2t-1| |f'''(b + t\eta(a,b))| dt
\leq \frac{(\eta(a,b))^4}{12} \left( \int_{0}^{1} t (1-t) |2t-1| |f'''(b + t\eta(a,b))|^q dt \right)^{\frac{1}{q}}
\leq \frac{(\eta(a,b))^4}{12} \left( \int_{0}^{1} t (1-t) |2t-1| |t| |f'''(a)|^q + (1-t) |f'''(b)|^q dt \right)^{\frac{1}{q}}
= \frac{(\eta(a,b))^4}{192} \left[ \frac{|f'''(a)|^q + |f'''(b)|^q}{2} \right]^{\frac{1}{q}}
$$

where we use the fact that

$$
\int_{0}^{1} t (1-t) |2t-1| dt = \frac{1}{16}
$$

and

$$
\int_{0}^{1} t^2 (1-t) |2t-1| dt = \int_{0}^{1} t (1-t)^2 |2t-1| dt = \frac{1}{32}
$$

The proof is completed. □

Corollary 1. In Theorem 1 if we choose $q = 1$, we obtain

$$
\left| \int_{b}^{b+\eta(a,b)} f(x) dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))] \right|
\leq \frac{(\eta(a,b))^4}{384} [|f'''(a)| + |f'''(b)|].
$$

Corollary 2. In Theorem 1 if we take $f'(b) = f'(b + \eta(a,b))$, then we have

$$
\left| \int_{b}^{b+\eta(a,b)} f(x) dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} \right|
\leq \frac{(\eta(a,b))^4}{384} [|f'''(a)|^q + |f'''(b)|^q]^{\frac{1}{q}}.
$$
Theorem 2. Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f''|^q$ is preinvex on $A$, then the following inequality holds:

$$\int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))]| \leq \frac{(\eta(b,a))^4}{24 \times 6^q} \left( \frac{1}{p+1} (p+3) \right)^{\frac{q}{p}} \left[ |f'''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}$$

for every $a, b \in A$ with $\eta(b,a) \neq 0$ where $q > 1$, $p^{-1} + q^{-1} = 1$.

Proof. By using Lemma 2 Hölder inequality and preinvexity of $|f''|^q$, we can write

$$\int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))]| \leq \frac{(\eta(b,a))^4}{12} \left( \int_0^1 t (1-t) |2t-1|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t (1-t) |f''(b + t\eta(a,b))|^q dt \right)^{\frac{1}{q}} \leq \frac{(\eta(b,a))^4}{12} \left( \int_0^1 t (1-t) |2t-1|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t (1-t) [t |f'''(a)|^q + (1-t) |f''(b)|^q] dt \right)^{\frac{1}{q}}.$$

Computing the above integrals, we deduce

$$\int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))]| \leq \frac{(\eta(b,a))^4}{24 \times 6^q} \left( \frac{1}{p+1} (p+3) \right)^{\frac{q}{p}} \left[ |f'''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}.$$
where we used the fact that
\[ \int_0^1 (t-t^2)^p \, dt = \frac{2^{-1-2p} \sqrt{\pi} \Gamma(1+p)}{\Gamma \left( \frac{3}{2} + p \right)} \]
and
\[ \int_0^1 t |2t-1|^q \, dt = \int_0^1 (1-t) |2t-1|^q \, dt = \frac{1}{2(q+1)}. \]

The proof is completed. □

3. Inequalities for prequasiinvex functions

In this section, we obtain Hermite-Hadamard type inequalities for prequasiinvex functions via Lemma

**Theorem 4.** Let \( A \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \to \mathbb{R} \). Suppose that \( f : A \to \mathbb{R} \) is a three times differentiable function. If \( |f'''(\xi)|^q \) is prequasiinvex on \( A \), then the following inequality holds:

\[
\left| \int_b^{b+\eta(a,b)} f(x) \, dx - \eta(a,b) \frac{f(b)+f(b+\eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b+\eta(a,b))] \right| \\
\leq \frac{(\eta(a,b))^4}{192} \left[ \max \{|f'''(\xi)|^q, |f'''(b)|^q\} \right]^{\frac{1}{q}}
\]

for \( q \geq 1 \) and every \( a, b \in A \) with \( \eta(b,a) \neq 0 \).

**Proof.** By using Lemma the power-mean inequality and prequasiinvexity of \( |f'''(\xi)|^q \), we can write

\[
\left| \int_b^{b+\eta(a,b)} f(x) \, dx - \eta(a,b) \frac{f(b)+f(b+\eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b+\eta(a,b))] \right| \\
\leq \frac{(\eta(a,b))^4}{12} \left( \int_0^1 t (1-t) |2t-1| \, dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t (1-t) |2t-1| |f'''(b+t\eta(a,b))|^q \, dt \right)^{\frac{1}{q}} \\
\leq \frac{(\eta(a,b))^4}{12} \left( \int_0^1 t (1-t) |2t-1| \, dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t (1-t) |2t-1| \left[ \max \{|f'''(\xi)|^q, |f'''(b)|^q\} \right] \, dt \right)^{\frac{1}{q}} \\
= \frac{(\eta(a,b))^4}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left[ \max \{|f'''(\xi)|^q, |f'''(b)|^q\} \right]^{\frac{1}{q}}.
\]

The proof is completed. □

**Corollary 3.** In Theorem if we choose \( q = 1 \), we obtain

\[
\left| \int_b^{b+\eta(a,b)} f(x) \, dx - \eta(a,b) \frac{f(b)+f(b+\eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b+\eta(a,b))] \right| \\
\leq \frac{(\eta(a,b))^4}{192} \left[ \max \{|f'''(\xi)|, |f'''(b)|\} \right].
\]
Suppose that
\[ f(a, b) \]
for every
\[ b, a \]
By using Lemma 2, Hölder inequality and preinvexity of
\[ f \]
In Theorem 4, if we take
\[ \text{Corollary 4.} \]
Computing the above integrals, we obtain the desired result.
\[ \Box \]

Theorem 6. Under the assumptions of Theorem 5, we have
\[ \int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} \leq \frac{(\eta(a,b))^4}{192} \left[ \max \left\{ |f'''(a)|^q, |f'''(b)|^q \right\} \right]^\frac{1}{q}. \]

Theorem 5. Let \( A \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \rightarrow \mathbb{R} \). Suppose that \( f : A \rightarrow \mathbb{R} \) is a differentiable function. If \( |f'''|^{\frac{q}{q}} \) is preinvex on \( A \), then the following inequality holds:
\[ \int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} \leq \frac{(\eta(b,a))^4}{24 \times 3^\frac{q}{q}} \left( \frac{1}{(p+1)(p+3)} \right)^\frac{1}{q} \left[ \max \left\{ |f'''(a)|^q, |f'''(b)|^q \right\} \right]^\frac{1}{q} \]
for every \( a, b \in A \) with \( \eta(b,a) \neq 0 \) where \( q > 1 \), \( p^{-1} + q^{-1} = 1 \).

Proof. By using Lemma 2 Hölder inequality and preinvexity of \( |f'''(a)|^q \), we can write
\[ \int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))] \]
\[ \leq \frac{(\eta(b,a))^4}{12} \left( \int_0^1 t (1-t) |2t-1|^p dt \right)^\frac{1}{q} \left( \int_0^1 t (1-t) |f'''(b + t\eta(a,b))|^q dt \right)^\frac{1}{q} \]
\[ \leq \frac{(\eta(b,a))^4}{12} \left( \int_0^1 t (1-t) |2t-1|^p dt \right)^\frac{1}{q} \left( \int_0^1 t (1-t) \left[ \max \left\{ |f'''(a)|^q, |f'''(b)|^q \right\} \right] dt \right)^\frac{1}{q}. \]
Computing the above integrals, we obtain the desired result. \( \Box \)

Theorem 6. Under the assumptions of Theorem 5 we have
\[ \int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))] \]
\[ \leq \frac{(\eta(b,a))^4}{48} \left( \Gamma \left( \frac{1}{p} \right) \Gamma \left( \frac{1}{q} \right) \right)^\frac{1}{q} \left( \frac{1}{q+1} \right)^\frac{1}{q} \left[ \max \left\{ |f'''(a)|^q, |f'''(b)|^q \right\} \right]^\frac{1}{q}. \]

Proof. From Lemma 2 using Hölder inequality and preinvexity of \( |f'''|^{\frac{q}{q}} \), we have
\[ \int_b^{b+\eta(a,b)} f(x)dx - \eta(a,b) \frac{f(b) + f(b + \eta(a,b))}{2} - \frac{(\eta(a,b))^2}{12} [f'(b) - f'(b + \eta(a,b))] \]
\[ \leq \frac{(\eta(b,a))^4}{12} \left( \int_0^1 (t-t^2)^p dt \right)^\frac{1}{q} \left( \int_0^1 |2t-1|^q |f'''(b + t\eta(a,b))|^q dt \right)^\frac{1}{q} \]
\[ \leq \frac{(\eta(b,a))^4}{12} \left( \int_0^1 (t-t^2)^p dt \right)^\frac{1}{q} \left( \int_0^1 |2t-1|^q \left[ \max \left\{ |f'''(a)|^q, |f'''(b)|^q \right\} \right] dt \right)^\frac{1}{q}. \]
Computing the above integrals, we obtain the desired result. \( \Box \)
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