RELATIONS BETWEEN THE KÄHLER CONE AND THE BALANCED CONE OF A KÄHLER MANIFOLD

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Abstract. In this paper, we consider a natural map from the Kähler cone of a compact Kähler manifold to its balanced cone. We study its injectivity and surjectivity. We also give an analytic characterization theorem on a nef class being Kähler.

1. Introduction

On a complex \( n \)-dimensional manifold, a balanced metric is a Hermitian metric such that its associated fundamental form \( \omega \) satisfies \( d(\omega^{n-1}) = 0 \). Throughout this paper such an \( \omega \) is called a balanced metric directly. It is easy to see that the existence of a balanced metric \( \omega \) is equivalent to the existence of a \( d \)-closed strictly positive \((n-1,n-1)\)-form \( \Omega \) with the relation \( \Omega = \omega^{n-1} \) (see [21]). Hence, for convenience, each such \( \Omega \) will also be called a balanced metric.

Assume that \( X \) is a compact complex manifold. The (real) \((p,p)\)-th Bott-Chern cohomology group of \( X \) is defined as

\[
H^{p,p}_{BC}(X, \mathbb{R}) = \{ \text{real } d \text{-closed } (p,p)\text{-forms} \}/i\partial\bar{\partial}\{ \text{real } (p-1,p-1)\text{-forms} \},
\]

which can also be defined by currents. Its elements will be denoted by \([\cdot]_{bc}\). It is easy to see that the cohomology classes of all real \((n-1,n-1)\)-forms which are balanced metrics form an open convex cone in \( H^{n-1,n-1}_{BC}(X, \mathbb{R}) \). We denote it by

\[
\mathcal{B} = \{ [\Omega]_{bc} \in H^{n-1,n-1}_{BC}(X, \mathbb{R}) \mid \Omega \text{ is a balanced metric} \}.
\]

It is called the balanced cone of \( X \). Note that the zero cohomology class may be in \( \mathcal{B} \). For example, Fu-Li-Yau [12] constructed a balanced metric \( \omega \) on the connected sum \( Y \) of \( k \geq 2 \) copies of \( S^3 \times S^3 \). Since \( H_{BC}^{2,2}(Y, \mathbb{R}) = 0 \), \( [\omega^2]_{bc} = 0 \in \mathcal{B} \). Clearly, if the zero class belongs to \( \mathcal{B} \), then \( \mathcal{B} = H^{n-1,n-1}_{BC}(X, \mathbb{R}) \). However, if \( X \) is a compact Kähler manifold, then the zero class is never in \( \mathcal{B} \).

Now we assume that \( X \) is a compact Kähler manifold. In this case, by the \( \partial\bar{\partial} \)-lemma, it is well known that \( H^{p,p}_{BC}(X, \mathbb{R}) \) is the same as the cohomology group \( H^{p,p}_{dR}(X, \mathbb{R}) \), the set of de Rham classes represented by a real \( d \)-closed \((p,p)\)-form, see [28]. The Kähler cone \( \mathcal{K} \) of \( X \) is defined to be

\[
\mathcal{K} = \{ [\omega] \in H^{1,1}_{dR}(X, \mathbb{R}) \mid \omega \text{ is a Kähler metric} \},
\]

\[
H_{BC}^{p,p}(X, \mathbb{R}) = \{ \text{real } d \text{-closed } (p,p)\text{-forms} \}/i\partial\bar{\partial}\{ \text{real } (p-1,p-1)\text{-forms} \},
\]
which is an open convex cone in $H^{1,1}_{dR}(X, \mathbb{R})$. It was studied thoroughly by Demailly and Paun in [11]. Since on a Kähler surface, the balanced cone and the Kähler cone coincide by their definitions, we will always assume $n \geq 3$ in the rest of the paper.

The balanced cone $\mathcal{B}$ of a compact Kähler manifold is related to its movable cone $\mathcal{M}$ (cf. [7] for its definition). The first named author learned this notion from Professor Demailly, who mentioned Toma’s paper [25]. Toma observed that every movable curve on a projective manifold can be represented by a balanced metric under the assumption $\mathcal{E}^\vee = \overline{\mathcal{M}}$. This assumption is Conjecture 2.3 in [7]. In fact, the result in [25] holds for all movable classes on any compact Kähler manifold. And along the lines of [25], one can obtain the equivalence of $\mathcal{B}$ and $\mathcal{M}$ under the assumption $\mathcal{E}^\vee = \overline{\mathcal{M}}$ (see the appendix).

In this note, motivated by papers [13, 14], we consider the map $b : \mathcal{K} \to \mathcal{B}$

which maps $[\omega]$ to $[\omega^{n-1}]$. It is clearly well-defined and can be extended to the map $\overline{b} : \overline{\mathcal{K}} \to \overline{\mathcal{B}}$, where $\overline{\mathcal{K}}$ and $\overline{\mathcal{B}}$ are the closures of the corresponding cones. We want to study the properties of $b$ and $\overline{b}$. We will first prove that $b$ embeds $\mathcal{K}$ into $\mathcal{B}$.

**Proposition 1.1.** Let $X$ be a compact Kähler manifold. Then the map $b$ is injective.

The proof of the above proposition contains two key ingredients. The first one is Yau’s celebrated theorem on complex Monge-Ampère equations over compact Kähler manifolds, and the second one is the Arithmetic Mean-Geometric Mean (AM-GM) inequality. Replacing Yau’s theorem by Boucksom-Eyssidieux-Guedj-Zeriahi’s theorem [6] on complex Monge-Ampère equations in big cohomology classes, we can generalize the above proposition on the map $b$ to the map $\overline{b}$. Here we recall that in the Kähler case a cohomology class $[\alpha] \in H^{1,1}_{dR}(X, \mathbb{R})$ is nef if $[\alpha] \in \overline{\mathcal{K}}$, and $[\alpha]$ is big if $[\alpha]$ contains a Kähler current. For compact Kähler manifolds, Demailly and Paun [11] proved that a nef class $[\alpha]$ is big if and only if $\int_X \alpha^n > 0$. In order to generalize the above proposition, we also need a convexity inequality obtained by Gromov [17] and Demailly [9], and use some techniques on currents.

**Theorem 1.2.** Let $X$ be an $n$-dimensional compact Kähler manifold. Then the map $\overline{b}$ is injective when $\overline{b}$ is restricted to the subcone generated by all the nef and big classes.

We remark that the condition “big” is necessary, otherwise, the complex torus $T^n$ gives a counterexample. But it is not clear whether the condition “nef” is necessary.
In general $b$ is not surjective. In fact, we will show that $\overline{B}(\partial K) \cap B$ need not be empty. Let

$$K_{NS} = K \cap NS_R,$$

where $NS_R$ is the real Neron-Severi group of $X$, i.e.,

$$NS_R = (H^{1,1}_{BC}(X, R) \cap H^2(X, \mathbb{Z})) \otimes_{\mathbb{Z}} R.$$

Then, if $X$ is a projective Calabi-Yau manifold (i.e. a projective manifold with $c_1 = 0$), we can characterize when a nef class $[\alpha] \in \partial K_{NS}$ can be mapped into $B$ by $b$. In fact, inspired by the method in [25] and [24], we can give some sufficient conditions when a $d$-closed nonnegative $(n-1, n-1)$-form is a balanced class. Applying these criteria to Proposition 4.1 in [26], we obtain

**Theorem 1.3.** Let $X$ be a projective Calabi-Yau manifold. If $[\alpha] \in \partial K$, then $b([\alpha]) \in B$ implies that $[\alpha]$ is a big class. On the other hand, if $[\alpha] \in \partial K_{NS}$ is a big class, then $b([\alpha]) \in B$ if and only if the exceptional set $Exc(F[\alpha])$ of the contraction map $F[\alpha]$ induced by $[\alpha]$ is of codimension greater than one, i.e., $F[\alpha]$ is a flipping contraction.

For general $[\alpha] \in NS(X)_R$, the map $F[\alpha]$ is described in [26]. In Section 3, we will give some details on it. After proof of the above theorem, then some examples will be given to show that the balanced cone can be bigger than the image of the Kähler cone under the map $b$. We believe that it will be very interesting if one can describe $\overline{B}(\overline{K}) \cap B$ clearly for a compact Kähler manifold.

In the last part we assume that $X$ is an $n$-dimensional Kähler manifold with holomorphically trivial canonical bundle. (Hence, $X$ is a Calabi-Yau manifold.) We will give an analytic method to distinguish the Kähler classes and the nef but not Kähler classes which are mapped into the balanced cone. We fix a Calabi-Yau metric $\omega_0$ satisfying $\int_X \omega_0^n = 1$ and a non-vanishing holomorphic $n$-form $\zeta$ such that $\| \zeta \|_{\omega_0} = 1$. For any Kähler class $[\omega]$, Yau’s theorem states that there exists a unique Calabi-Yau metric $\omega_{CY} \in [\omega]$ such that $\| \zeta \|_{\omega_{CY}}$, the (pointwise) norm of $\zeta$ with respect to $\omega_{CY}$, is a constant. Under the above assumption, this constant can be computed as follows:

$$\| \zeta \|^2_{\omega_{CY}} = \frac{\| \zeta \|^2_{\omega_{0}}} {\omega_0^n} = \frac{\omega_0^n} {\int_X \omega_{CY}^n} = \frac{1} {\int_X \omega^n}.$$ 

We can also ask whether there exists a balanced metric $\Omega_{CY}$ in a given balanced class $[\Omega] \in B$ such that

$$(1.1) \quad \| \zeta \|^2_{\Omega_{CY}} = c$$

is a constant. This is the motivation of papers [13, 14]. There may be infinitely many solutions to equation (1.1) in a given balanced class. For example, Wang, Wu and the first named author [13] proved that if $X$ is a complex $n$-torus, then for a given Kähler metric $\omega$ and for any constant $c \geq (\int_{T^n} \omega^n)^{-1}$, equation (1.1) has solutions in $[\omega^{-1}]$. They also proved
that for any Calabi-Yau manifold $X$ and a given Kähler metric $\omega$ on $X$, if equation (1.1) has a solution in $[\omega^{n-1}]$ for $c \leq (\int_X \omega^n)^{-1}$, then $c = (\int_X \omega^n)^{-1}$ and this solution must be the Calabi-Yau metric. Here we can prove that if $\alpha$ is a nef but not Kähler class and $[\alpha^{n-1}] \in \mathcal{B}$, then there exists no solution in $[\alpha^{n-1}]$ of the equation (1.1) for $c \leq (\int_X \alpha^n)^{-1}$.

**Theorem 1.4.** Let $X$ be an $n$-dimensional Calabi-Yau manifold with a Calabi-Yau metric $\omega_0$ and a non-vanishing holomorphic $n$-form $\zeta$ such that $\int_X \omega_0^n = 1$ and $\| \zeta \|_{\omega_0} = 1$. Let $[\alpha] \in \mathcal{K}$ such that $\mathcal{B}(\{\alpha\}) = [\alpha^{n-1}] \in \mathcal{B}$.

1. If $[\alpha] \in \partial \mathcal{K}$, then equation (1.1) for $c \leq (\int_X \alpha^n)^{-1}$ has no solution in the balanced class $[\alpha^{n-1}]$.
2. If $[\alpha] \in \mathcal{K}$, then there exists a unique solution $\Omega_{CY} \in [\alpha^{n-1}]$ of the equation (1.1) for $c \leq (\int_X \alpha^n)^{-1}$. Actually in this case, $c = (\int_X \alpha^n)^{-1}$ and $\Omega_{CY} = \omega_{CY}^{n-1}$ for the unique Calabi-Yau metric $\omega_{CY}$ in the Kähler class $[\alpha]$.

It is conjectured that for any $c > (\int_X \alpha^n)^{-1}$, the form-type Calabi-Yau equation (1.1) has solutions in the balanced class $[\alpha^{n-1}]$ in the above theorem.

The paper is organized as follows. In Section 2 we prove Proposition 1.1 and Theorem 1.2, which will also be generalized to the Fujiki class $\mathcal{C}$. In Section 3 we prove Theorem 1.3 and give two examples. In Section 4 we will use Theorem 1.2 to prove Theorem 1.4. Finally, for reader’s convenience, we show in the appendix the equivalence of the balanced cone and the movable cone of a Kähler manifold under the assumption $\mathcal{E}^\vee = \mathcal{M}$ following the arguments of Toma.

**Acknowledgments.** We would like to thank Prof. J.-P. Demailly, Zhizhang Wang, D. Wu and Prof. S.-T. Yau for useful discussions and V. Tosatti for useful suggestions and comments. We are also indebted to the referees for helpful comments and suggestions. Fu is supported in part by NSFC grants 11025103 and 11121101.

## 2. Injectivity

In this section, as a warm-up we first prove Proposition 1.1, which states that the map $\mathbf{b}$ is injective. We remark that this is just a special case of Theorem 1.2. By presenting its proof here, we want to emphasize how to apply the solutions of the complex Monge-Ampère equations and the AM-GM inequality to obtain the result.

**Proof.** We need to prove that if $\omega_1$ and $\omega_2$ are two Kähler metrics on $X$ satisfying

\begin{equation}
(2.1) \quad \omega_1^{n-1} = \omega_2^{n-1} + i\partial\overline{\partial}\varphi
\end{equation}

\[\text{Recently, this conjecture has been solved by V. Tosatti and B. Weinkove in [27].}\]
for some real \((n - 2, n - 2)\)-form \(\varphi\), then there exists a smooth function \(f\) on \(X\) such that
\[
\omega_1 = \omega_2 + i\partial\bar{\partial}f.
\]

Let us first recall Yau’s theorem on the complex Monge-Ampère equations on a compact Kähler manifold.

**Lemma 2.1.** \(\textit{([29])}\) Let \(X\) be a compact \(n\)-dimensional Kähler manifold with a Kähler metric \(\omega\). Then for any smooth volume form \(\eta > 0\) satisfying
\[
\int_X \eta = \int_X \omega^n,
\]
there exists a unique Kähler metric \(\tilde{\omega} = \omega + i\partial\bar{\partial}u\) in the Kähler class \([\omega]\) such that \(\tilde{\omega}^n = \eta\).

We use Yau’s theorem as follows. Let \(c\) be the following constant:
\[
\int_X \omega_2^n = c \int_X \omega_1^n.
\]
Without loss of generality, we assume that \(c \geq 1\). Since the class \([\omega_2]\) is Kähler, by Yau’s theorem we can find a representative \(\tilde{\omega}_2 = \omega_2 + i\partial\bar{\partial}u\) of \([\omega_2]\) such that
\[
(2.2) \quad \tilde{\omega}_2^n = c\omega_1^n.
\]
However, the equalities
\[
[\tilde{\omega}_2^{n-1}] = [\omega_2^{n-1}] = [\omega_1^{n-1}]
\]
imply that there exists a real \((n - 2, n - 2)\)-form \(\phi\) such that
\[
(2.3) \quad \tilde{\omega}_2^{n-1} = \omega_1^{n-1} + i\partial\bar{\partial}\phi.
\]
We will use the following notations (see \([13]\)). If \(\Theta\) is a real \((n-1, n-1)\)-form, then \((\Theta_{ij})\) is the matrix whose entries are the coefficients of \(\Theta\), and \((\Theta^{ij})\) is its inverse matrix. We will also denote \(\det \Theta = \det(\Theta_{ij})\). Hence, combining (2.2) with (2.3), we find
\[
\left(\frac{1}{c}\right)^{n-1} = \frac{(\det \omega_1)^{n-1}}{(\det \tilde{\omega}_2)^{n-1}} = \frac{\det \omega_1^{n-1}}{\det \omega_2^{n-1}} = \frac{\det \omega_1^{n-1}}{\det (\omega_1^{n-1} + i\partial\bar{\partial}\phi)}.
\]

Now we follow the proof of Lemma 10 in \([13]\). We apply the AM-GM inequality to obtain
\[
cn \omega_1^n \leq \omega_1^n + i\partial\bar{\partial}\phi \land \omega_1.
\]
Integrating over $X$, since $\omega_1$ is Kähler, we get
\[ c \cdot \frac{n-1}{n} \int_X \omega_1^n \leq \int_X \omega_1^n. \]
This shows that $c = 1$ and a pointwise equality in (2.1) holds. Therefore, (2.3) implies $\tilde{\omega}_2 = \omega_1$ and $\omega_1 = \omega_2 + i\partial \bar{\partial} u$. \hfill \Box

**Remark 2.2.** When $n = 3$, equation (2.1) implies
\[ (\omega_1 - \omega_2) \wedge (\omega_1 + \omega_2) = i\partial \bar{\partial} \varphi. \]
Since $\omega_1 + \omega_2$ is a Kähler metric, by the hard Lefschetz theorem, $\omega_1 + \omega_2$ defines an isomorphism from $H^{1,1}_{dR}(X, \mathbb{R})$ to $H^{2,2}_{dR}(X, \mathbb{R})$. Hence $\omega_1 - \omega_2$ is trivial in $H^{1,1}_{dR}(X, \mathbb{R})$. For $n > 3$, we can rewrite (2.1) as
\[ (\omega_1 - \omega_2) \wedge \left( \sum_{k=0}^{n-2} \omega_1^{n-k-2} \wedge \omega_2^k \right) = i\partial \bar{\partial} \varphi. \]
Here $\sum_{k=0}^{n-2} \omega_1^{n-k-2} \wedge \omega_2^k$ is a $d$-closed strictly positive definite $(n-2, n-2)$-form. In general, such a form cannot be represented by $\omega_0^{n-2}$ for some Hermitian metric $\omega_0$. Otherwise $\omega_0$ is also Kähler (cf. [15]) and then the hard Lefschetz theorem also implies that $\omega_1 - \omega_2$ is trivial. Anyway, we don’t know whether there exists an algebraic proof of Proposition 1.1.

We can generalize the above proposition from the Kähler classes to the nef and big classes. Instead of constructing two equal Kähler metrics, we will construct two equal currents. Hence, we need the following important theorem in [6].

**Lemma 2.3.** ([6]) Let $X$ be a compact $n$-dimensional Kähler manifold and let $\eta$ be a smooth volume form on $X$. Let $[\alpha]$ be a nef and big class on $X$. Then there exists a unique $\alpha$-psh function $u$ with $\sup_X u = 0$ such that
\[ \langle (\alpha + i\partial \bar{\partial} u)^n \rangle = c\eta \quad \text{with} \quad c = \frac{\int_X \alpha^n}{\int_X \eta} > 0. \]
Here $\langle \cdot \rangle$ denotes the non-pluripolar product of positive currents. Moreover, $u$ has minimal singularities and is smooth on $\operatorname{Amp}(\alpha)$, which is a Zariski open set of $X$ depending only on the cohomology class of $\alpha$.

Recall that $u$ is called an $\alpha$-psh function if $\alpha + i\partial \bar{\partial} u$ is a positive current. Let us briefly discuss how the result is obtained. In fact, by Yau’s theorem, the above degenerate complex Monge-Ampère equation can be solved by approximation. Fix a Kähler metric $\omega$ on $X$. If we write $c_t = \int_X (\alpha + t\omega)^n / \int_X \eta$ with $0 < t < 1$, then there exists a unique smooth function $u_t$ with $\sup_X u_t = 0$ such that
\[ (\alpha + t\omega + i\partial \bar{\partial} u_t)^n = c_t \eta. \]
First, by basic properties of plurisubharmonic functions, the family of solutions \( u_t \) is compact in \( L^1(X) \)-topology and then there exists a sequence \( u_{tk} \) such that
\[
\alpha + t_k \omega + i \partial \bar{\partial} u_{tk} \to \alpha + i \partial \bar{\partial} u \quad \text{as currents on } X.
\]
Moreover, by the theory developed in [6] and Yau’s basic estimates in [29], \( u_t \) is compact in \( C^\infty_{\text{loc}}(\text{Amp}(\alpha)) \). Therefore there exists a subsequence of \( u_{tk} \), which is still denoted as \( u_{tk} \), (we will not stress this point in the following,) such that
\[
\alpha + t_k \omega + i \partial \bar{\partial} u_{tk} \to \alpha + i \partial \bar{\partial} u \quad \text{in } C^\infty_{\text{loc}}(\text{Amp}(\alpha)).
\]
Hence \( u \) is smooth on \( \text{Amp}(\alpha) \). Since \( \eta \) is the smooth volume form, \( \alpha + i \partial \bar{\partial} u \) is a Kähler metric on \( \text{Amp}(\alpha) \).

Now we are ready to prove Theorem 1.2. We rephrase it as

**Theorem 2.4.** Let \( X \) be a compact \( n \)-dimensional Kähler manifold. If \([\alpha]\) and \([\beta]\) are two nef and big classes and \([\alpha^{n-1}] = [\beta^{n-1}]\), then \([\alpha] = [\beta]\).

**Proof.** Since \( \alpha \) and \( \beta \) are nef and \([\alpha^{n-1}] = [\beta^{n-1}]\), we have
\[
(2.5) \quad \int_X \beta^n = \int_X \beta \wedge \alpha^{n-1}.
\]
Then by the convexity inequality in [9] or [17], we have
\[
\int_X \beta^n \geq \left( \int_X \beta^n \right)^{\frac{1}{n}} \left( \int_X \alpha^n \right)^{\frac{n-1}{n}},
\]
which implies \( \int_X \beta^n \geq \int_X \alpha^n \). Similarly we also have \( \int_X \alpha^n \geq \int_X \beta^n \). Thus we get
\[
(2.6) \quad \int_X \alpha^n = \int_X \beta^n.
\]

We fix a Kähler metric \( \omega \) and a volume form \( \eta \) on \( X \). We denote for \( 0 < t < 1 \)
\[
c = \frac{\int_X \alpha^n}{\int_X \eta}, \quad c_{\alpha,t} = \frac{\int_X (\alpha + t \omega)^n}{\int_X \eta}, \quad \text{and} \quad c_{\beta,t} = \frac{\int_X (\beta + t \omega)^n}{\int_X \eta}.
\]
Then Lemma 2.1 implies that there exist two families of smooth functions \( u_t \) and \( v_t \) such that, if we denote \( \alpha_t = \alpha + t \omega + i \partial \bar{\partial} u_t \) and \( \beta_t = \beta + t \omega + i \partial \bar{\partial} v_t \), then
\[
\alpha_t^n = c_{\alpha,t} \eta \quad \text{and} \quad \beta_t^n = c_{\beta,t} \eta.
\]
Hence
\[
(2.7) \quad \frac{\alpha_t^n}{\beta_t^n} = c_t
\]
with
\[
(2.8) \quad c_t = \frac{c_{\alpha,t}}{c_{\beta,t}} = \frac{\int_X (\alpha + t \omega)^n}{\int_X (\beta + t \omega)^n}.
\]
Then identity (2.6) implies
\begin{equation}
\lim_{t \to 0} c_t = 1.
\end{equation}

By the assumption \([\alpha^{n-1}] = [\beta^{n-1}]\), there exists a \((n-2, n-2)\)-form \(\phi\) such that \(\alpha^{n-1} = \beta^{n-1} + i \partial \bar{\partial} \phi\). We rewrite it as
\begin{equation}
\alpha_t^{n-1} = \beta_t^{n-1} + \Theta_t
\end{equation}
where
\[
\Theta_t = i \partial \bar{\partial} \phi + \sum_{k=0}^{n-2} C_{n-1}^k (\alpha^k \wedge (t\omega + i \partial \bar{\partial} u_t)^{n-1-k} - \beta^k \wedge (t\omega + i \partial \bar{\partial} u_t)^{n-1-k}).
\]

Then applying the AM-GM inequality to (2.7), we have
\begin{equation}
\frac{c_t^{n-1}}{\det \beta_t^{n-1}} = \left( \frac{\det \beta_t^{n-1} + \Theta_t}{\det \beta_t^{n-1}} \right)^{\frac{1}{n}} \leq 1 + \frac{1}{n} \sum_{i,j} (\beta_t^{n-1})^i j (\Theta_t)^i j.
\end{equation}

We multiply the volume form \(\beta_t^n\) to both sides of the above inequality and get
\begin{equation}
\frac{c_t^{n-1}}{\beta_t^n} \leq \beta_t^n + \beta_t \wedge \Theta_t.
\end{equation}

Next, we consider the limit of \(\beta_t \wedge \Theta_t\) as \(t\) goes to zero. By (2.10), we have
\begin{equation}
\beta_t \wedge \Theta_t = \beta_t \wedge \alpha_t^{n-1} - \beta_t^n.
\end{equation}

It is easy to see the positive measures \(\beta_t \wedge \alpha_t^{n-1}\) and \(\beta_t^n\) have uniformly bounded masses:
\[
||\beta_t \wedge \alpha_t^{n-1}||_{\text{mass}} = \int_X (\beta + t\omega) \wedge (\alpha + t\omega)^{n-1} < \int_X (\beta + \omega) \wedge (\alpha + \omega)^{n-1},
\]
and
\[
||\beta_t^n||_{\text{mass}} = \int_X (\beta + t\omega)^n < \int_X (\beta + \omega)^n.
\]

Hence we can pick a decreasing subsequence \(t_k \to 0\) such that \(\beta_{t_k} \wedge \alpha_{t_k}^{n-1}\) and \(\beta_{t_k}^n\) converge weakly to \(\mu_1\) and \(\mu_2\) respectively. Therefore if we denote \(\mu_0 = \mu_1 - \mu_2\), then as currents,
\[
\beta_{t_k} \wedge \Theta_{t_k} \to \mu_0 \quad \text{when} \quad t_k \to 0.
\]

Moreover, it is not hard to see from (2.9) and (2.12) that \(\mu_0\) is a positive measure on \(X\). Meanwhile, by (2.13) and (2.5),
\[
\int_X \mu_0 = \lim_{t_k \to 0} \int_X (\beta_{t_k} \wedge \alpha_{t_k}^{n-1} - \beta_{t_k}^n) = \int_X (\beta \wedge \alpha^{n-1} - \beta^n) = 0.
\]
Hence \( \mu_0 \) is a zero measure. In particular, since \( \Xi := \text{Amp}(\alpha) \cap \text{Amp}(\beta) \) is a Borel set, we have

\[
\beta_{t_k} \wedge \Theta_{t_k} \to 0 \quad \text{as currents on } \Xi.
\]

On the other hand, by Lemma 2.3, there exists a unique \( \alpha \)-psh function \( u_0 \) with \( \sup_X u_0 = 0 \) and a unique \( \beta \)-psh function \( v_0 \) with \( \sup_X v_0 = 0 \) such that \( u_0 \) (resp. \( v_0 \)) is smooth on \( \text{Amp}(\alpha) \) (resp. \( \text{Amp}(\beta) \)) and

\[
\langle (\alpha + i\partial \bar{\partial} u_0)^n \rangle = c_\eta, \quad \langle (\beta + i\partial \bar{\partial} v_0)^n \rangle = c_\eta.
\]

Here by (2.6), we have

\[
c = \frac{\int_X \alpha^n}{\int_X \eta} = \frac{\int_X \beta^n}{\int_X \eta}.
\]

If we denote \( \alpha_0 = \alpha + i\partial \bar{\partial} u_0 \) and \( \beta_0 = \beta + i\partial \bar{\partial} v_0 \), then as discussed before, there exist subsequences \( \alpha_{t_k} \) of \( \alpha_t \) and \( \beta_{t_k} \) of \( \beta_t \) such that

\[
\alpha_{t_k} \to \alpha_0 \quad \text{in } C^\infty_\text{loc}(\text{Amp}(\alpha))
\]

and

\[
\beta_{t_k} \to \beta_0 \quad \text{in } C^\infty_\text{loc}(\text{Amp}(\beta)).
\]

Thus,

\[
\Theta_{t_k} \to \Theta_0 \quad \text{and} \quad \beta_{t_k} \wedge \Theta_{t_k} \to \beta_0 \wedge \Theta_0 \quad \text{in } C^\infty_\text{loc}(\Xi)
\]

for some smooth form \( \Theta_0 \) which is only defined on \( \Xi \). Combining (2.14) with (2.15) and using uniqueness of the limit, we obtain

\[
\beta_0 \wedge \Theta_0 = 0 \quad \text{on } \Xi.
\]

The above equality and (2.9) imply that on \( \Xi \), if we take the limits of both side of (2.11) as \( t \to 0 \),

\[
1 = \left( \frac{\det \alpha_0^{n-1}}{\det \beta_0^{n-1}} \right)^{1/n} \left( \frac{\det (\beta_0^{n-1} + \Theta_0)}{\det \beta_0^{n-1}} \right)^{1/n} \leq 1 + \frac{1}{n} \sum_{i,j} (\beta_0^{n-1})_{ij} (\Theta_0)_{ij} = 1,
\]

which forces \( \Theta_0 = 0 \) on \( \Xi \). Hence \( \alpha_0^{n-1} = \beta_0^{n-1} \) on \( \Xi \). Since \( \alpha_0 \) and \( \beta_0 \) are Kähler metrics on \( \Xi \), we have \( \alpha_0 = \beta_0 \) on \( \Xi \).

We claim \( \alpha_0 = \beta_0 \) on \( X \). First, we need the following two lemmas.

**Lemma 2.5.** (10) Let \( T \) be a \( d \)-closed \((p,p)\)-current and \( \text{supp} T \) contained in an analytic subset \( A \). If \( \dim A < n - p \), then \( T = 0 \); if \( T \) is of order zero and \( A \) is of dimension \( n - p \) with \((n - p)\)-dimensional irreducible components \( A_1, \cdots, A_k \), then \( T = \sum c_j[A_j] \) with \( c_j \in \mathbb{C} \).

**Lemma 2.6.** (5) Let \([\alpha]\) be a nef and big class, and let \( T_{\text{min}} \) be a positive current in \([\alpha]\) with minimal singularities. Then the Lelong number \( \nu(T_{\text{min}}, x) = 0 \) for any point \( x \in X \).
Now if we let \( T = \alpha_0 - \beta_0 \), then \( T \) is a real \( d \)-closed \((1,1)\)-current and \( \text{supp} T \subset X - \Xi \). If \( X - \Xi \) is of codimension more than one, then the first part of Lemma 2.6 implies \( T = 0 \). Hence \( \alpha_0 = \beta_0 \) on \( X \). If \( X - \Xi \) has irreducible components \( D_1, \ldots, D_k \) of pure codimension one, then the second part of Lemma 2.6 implies \( \alpha_0 - \beta_0 = \sum c_j[D_j] \). We should also consider the following most complicated case: \( X - \Xi \) has irreducible components of codimension one, whose union is denoted by \( D \), and also has of codimension greater than one, whose union is denoted by \( F \). In this case, we use the same argument of the proof of the second part of Lemma 2.6 (cf. page 143 of [10]).

The regular part \( D_{\text{reg}} \) of \( D \) is a complex submanifold of \( X - (D_{\text{sing}} \cup F) \), where \( D_{\text{sing}} \) is the singular part of \( D \), and its connected components are \( D_i \cap D_{\text{reg}} \). Then we apply the second theorem of support (see page 142 of [10]) to get \( \alpha_0 - \beta_0 = \sum c_j[D_j] \) on \( X - (D_{\text{sing}} \cup F) \). Now \( \alpha_0 - \beta_0 - \sum c_j[D_j] \) is a \( d \)-closed current of order 0 and its support is contained in \( D_{\text{sing}} \cup F \) of codimension greater than one. So the current \( \alpha_0 - \beta_0 - \sum c_j[D_j] \) must vanish by the first part of Lemma 2.6. Hence, for the last two cases, we should prove \( c_j = 0 \) for any \( j \).

Since \( \alpha_0 \) and \( \beta_0 \) are real, all \( c_j \)'s can be chosen to be real. If there exists at least one \( c_j > 0 \), we can write this equality as

\[
\alpha_0 - \sum c_j[D_j'] = \beta_0 + \sum c_j[D_j'']
\]

with \( c_j' \leq 0 \) and \( c_j'' > 0 \). Fix one such \( j'' \), which we denote as \( j''_0 \). We take a generic point \( x \in D_{j''_0} \), for example, we can take such a point \( x \) with \( \nu([D_{j''_0}], x) = 1 \) and \( x \notin X - D_{j''_0} \). Then taking the Lelong number at the point \( x \) on both sides of (2.16), we find

\[
\nu(\alpha_0, x) - \sum c_j' \nu([D_j'], x) = \nu(\beta_0, x) + \sum c_j'' \nu([D_j''], x).
\]

Since \( \alpha_0 \) and \( \beta_0 \) are positive currents with minimal singularities in nef and big classes, Lemma 2.6 tells us that \( \nu(\alpha_0, x) = 0 \) and \( \nu(\beta_0, x) = 0 \). The property of \( x \) also implies \( \nu([D_j'], x) = 0 \) and \( \nu([D_j''], x) = 0 \) for all \( j' \) and all \( j'' \neq j''_0 \). All these force \( c_j'' = 0 \), which contradicts our assumption that \( c_j'' > 0 \). Thus we have

\[
\alpha_0 - \sum c_j[D_j'] = \beta_0.
\]

By the same argument, we can also prove \( c_j' = 0 \). Hence \( \alpha_0 = \beta_0 \) on \( X \). Therefore, we have \([\alpha] = [\beta]\) on \( X \).

The above result is also valid if \( X \) is merely in the Fujiki class \( \mathcal{C} \). For a general compact complex manifold, a cohomology class \([\alpha]_{bc} \in H^{1,1}_{BC}(X, \mathbb{R})\) is called nef if for any \( \varepsilon > 0 \), there exists a smooth function \( \psi_\varepsilon \) such that \( \alpha + i\partial\bar{\partial}\psi_\varepsilon > -\varepsilon\omega \).

**Corollary 2.7.** Let \( X \) be a compact complex \( n \)-dimensional manifold in the Fujiki class \( \mathcal{C} \). If \([\alpha]\) and \([\beta]\) are two nef and big classes, and \([\alpha^{n-1}] = [\beta^{n-1}]\), then \([\alpha] = [\beta]\).
Proof. Since $X$ is in the Fujiki class $C$, there exists a proper modification $\mu : \bar{X} \to X$ with $\bar{X}$ a compact Kähler manifold. By assumptions on $\alpha$ and $\beta$, $[\mu^*\alpha]$ and $[\mu^*\beta]$ are also nef and big classes on $\bar{X}$, satisfying

$$[(\mu^*\alpha)^{n-1}] = [(\mu^*\beta)^{n-1}].$$

Then by the theorem above, we have

$$[\mu^*\alpha] = [\mu^*\beta].$$

As $\mu$ is a proper modification, this implies that $[\alpha] = [\beta]$ on $X$. □

Note that on a Moishezon manifold, M. Paun (22) proved that for a holomorphic line bundle $L$, $c_1(L)$ is nef if and only if $L \cdot C \geq 0$ for every irreducible curve $C$. Thus our result yields the following

**Corollary 2.8.** Let $X$ be a compact $n$-dimensional Moishezon manifold. Let $L$ be a big line bundle over $X$ and $L \cdot C \geq 0$ for every irreducible curve $C$ on $X$. Then $c_1(L)^{n-1}$ determines $c_1(L)$.

### 3. The Image of the Boundary of the Kähler Cone

Sometimes it is convenient to consider the Aeppli cohomology groups $V^{p,q}(X, \mathbb{C})$. Since we are interested in the real case, we give the following

**Definition 3.1.** If we denote by $A^{p,q}(X)$ the space of the smooth $\mathbb{C}$-valued $(p,q)$-forms and by $A^{p,p}_\mathbb{R}(X)$ the space of the smooth $\mathbb{R}$-valued $(p,p)$-forms, then

$$V^{p,p}(X, \mathbb{R}) = \frac{\{\phi \in A^{p,p}_\mathbb{R}(X)|\partial\bar{\partial}\phi = 0\}}{\partial A^{p-1,p}(X) + \partial A^{p-1,p}(X)} \cap A^{p,p}_\mathbb{R}(X).$$

We denote the space of $(p,q)$-currents by $D^{p,q}(X)$. Then it is well known that we can also replace $A^{p,q}$ by $D^{p,q}$ in the above definition. We denote an element of the above cohomology groups by $[\cdot]_\alpha$.

We need the following lemma due to Bigolin.

**Lemma 3.2.** (21) Let $X$ be a compact complex $n$-dimensional manifold. The dual space of the $(p,p)$-th Aeppli group is just the $(n-p,n-p)$-th Bott-Chern group, i.e.,

$$V^{p,p}(X, \mathbb{R})' = H^{n-p,n-p}_{BC}(X, \mathbb{R}).$$

In particular, $V^{p,p}(X, \mathbb{R})$ is a finite dimensional vector space. Furthermore, if $X$ satisfies the $\partial\bar{\partial}$-lemma, then $\dim V^{p,p}(X, \mathbb{R}) = h^{p,p}$, where $h^{p,p}$ is the Hodge number. The following lemma is inspired by the method in [25] and [24]. In fact, it is an easy consequence of the Hahn-Banach theorem.

**Lemma 3.3.** Let $X$ be a compact complex $n$-dimensional manifold. Suppose that $\Omega_0$ is a real $d$-closed $(n-1,n-1)$-form satisfying that, for any positive $\partial\bar{\partial}$-closed $(1,1)$-current $T$, $\int_X \Omega_0 \wedge T \geq 0$ and $\int_X \Omega_0 \wedge T = 0$ if and only if $T = 0$. Then $[\Omega_0]$ is a balanced class.
Proof. Fix a Hermitian metric $\omega$ on $X$. We define the following two subsets of $D^{1,1}_R(X)$:

$$D_1 = \{ T \in D^{1,1}_R(X) | \partial \bar{\partial} T = 0, \int_X \Omega_0 \wedge T = 0 \},$$

$$D_2 = \{ T \in D^{1,1}_R(X) | T \geq 0, \int_X \omega^{n-1} \wedge T = 1 \}.$$

Then $D_1$ is a closed subspace and $D_2$ is a compact convex subset under the weak topology of currents. Since $\Omega_0$ is $d$-closed,

$$\{ \partial \bar{S} + \bar{\partial} S | S \in D^{1,0}(X, \mathbb{C}) \} \subset D_1.$$

It is clear that $D_1 \cap D_2$ is empty. By the Hahn-Banach theorem, there exists a smooth real $(n-1,n-1)$-form $\Omega$ such that

$$\Omega|_{D_1} = 0 \quad \text{and} \quad \Omega|_{D_2} > 0 .$$

The identity in (3.2) and (3.1) imply $d\Omega = 0$, and the inequality in (3.2) implies that $\Omega$ is strictly positive. Hence $\Omega$ is a balanced metric.

On the other hand, Lemma 3.2 says that $[\Omega_0]$ and $[\Omega]$ are linear functionals on $V^{1,1}(X, \mathbb{R})$. We have a natural projective map

$$\pi : \{ T \in D^{1,1}_R(X) | \partial \bar{\partial} T = 0 \} \to V^{1,1}(X, \mathbb{R})$$

with $\pi(T) = [T]_a$. Then the definition of $D_1$ implies $\pi(D_1) = \ker[\Omega_0]$, and $\Omega|_{D_1} = 0$ implies $\pi(D_1) \subseteq \ker[\Omega]$. Thus we have

$$\ker[\Omega_0] \subseteq \ker[\Omega] \subseteq V^{1,1}(X, \mathbb{R}).$$

If $\ker[\Omega]$ is the whole Aeppli group, then $[\Omega] = 0$. Since $X$ is compact, there exists an $\varepsilon > 0$ small enough such that $\Omega + \varepsilon \Omega_0 > 0$, i.e., $[\Omega] + \varepsilon[\Omega_0] = \varepsilon[\Omega_0]$ is balanced. If $\ker[\Omega]$ is a proper subspace, since $V^{1,1}(X, \mathbb{R})$ is a finite dimensional vector space, we must have $\ker[\Omega_0] = \ker[\Omega]$. Hence there exists some constant $c$ such that $[\Omega_0] = c[\Omega]$. In this case, if there exists some non-trivial positive $\partial \bar{\partial}$-closed $(1,1)$-current $T$, the constant $c$ must be positive, and this implies that $[\Omega_0]$ is balanced. Otherwise, if there is no non-trivial positive $\partial \bar{\partial}$-closed $(1,1)$-current, then the zero class satisfies our assumption in the lemma and we can repeat our procedure above. We can use the zero class to define the space $D_1$. Hence the zero class is a balanced class. This means that every class in $H^{n-1,n-1}_{BC}(X, \mathbb{R})$ is balanced. Thus we finish the proof of Lemma 3.3. \qed

Remark 3.4. Let $X$ be a compact balanced manifold. If we denote $E_{dd^c} \subseteq V^{1,1}(X, \mathbb{R})$ the convex cone generated by $dd^c$-closed positive $(1,1)$-currents, then the above lemma implies $E_{dd^c} = \overline{B}$.

The above lemma has as corollary the following two interesting propositions. Let $\Omega_0$ be a semi-positive $(n-1,n-1)$-form on $X$ which is strictly positive on $X - V$ for a subvariety $V$ of $X$. If $\text{codim } V > 1$, we first recall Theorem 1.1 in [1].
Lemma 3.5. \cite{1} Let $X$ be a complex $n$-dimensional manifold. Assume $T$ is a $\partial\bar{\partial}$-closed positive $(p, p)$-current on $X$ such that the Hausdorff $2(n - p)$-measure of $\text{supp} T$ vanishes. Then $T = 0$.

Proposition 3.6. Let $X$ be a compact complex $n$-dimensional manifold. If $\Omega_0$ is a $d$-closed semi-positive $(n - 1, n - 1)$-form on $X$ and is strictly positive outside a subvariety $V$ with codim $V > 1$, then $[\Omega_0]$ is a balanced class.

Proof. Fix a $\partial\bar{\partial}$-closed positive $(1, 1)$-current $T$. Then $\Omega_0 \geq 0$ implies that $\int_X \Omega_0 \wedge T \geq 0$. And $\Omega_0 > 0$ on $X - V$ implies that $\int_X \Omega_0 \wedge T = 0$ if and only if $\text{supp} T \subset V$. Hence according to the above lemma, since $T$ is $\partial\bar{\partial}$-closed and codim $V > 1$, we have $T = 0$. Thus $\Omega_0$ satisfies the conditions of Lemma 3.3 and therefore is in the balanced class. \hfill $\square$

If codim $V = 1$ and $\Omega_0$ is a balanced metric, we have $\int_V \Omega_0 > 0$. We want to prove that this is also a sufficient condition when $\Omega_0$ is semi-positive on $X$ and is strictly positive on $X - V$. We need Theorem 1.5 in \cite{1}.

Lemma 3.7. \cite{1} Let $X$ be a complex $n$-dimensional manifold and $E$ a compact analytic subset. Let $E_1, \cdots, E_k$ be the irreducible $p$-dimensional components of $E$. If $T$ is a positive $\partial\bar{\partial}$-closed $(n - p, n - p)$-current such that $\text{supp} T \subset E$, then there exist constants $c_j \geq 0$ such that $T - \sum_{j=1}^k c_j [E_j]$ is a positive $\partial\bar{\partial}$-closed $(n - p, n - p)$-current on $X$, supported on the union of the irreducible components of $E$ of dimension greater than $p$.

Then we have

Proposition 3.8. Let $X$ be a compact complex $n$-dimensional manifold. If $\Omega_0$ is a $d$-closed semipositive $(n - 1, n - 1)$-form on $X$ such that it is strictly positive outside a codimension one subvariety $V$ with irreducible components $E_1, \cdots, E_k$ and $[\Omega_0] \cdot [E_j] > 0$ for $j = 1, \cdots, k$, then $[\Omega_0]$ is a balanced class.

Proof. Since $\Omega_0$ is a semi-positive form on $X$, for any $\partial\bar{\partial}$-closed positive $(1, 1)$-current $T$ on $X$, $\int_X \Omega_0 \wedge T \geq 0$, and $\int_X \Omega_0 \wedge T = 0$ implies $\text{supp} T \subset V$. We need to prove $T = 0$. By the above lemma, there exist constants $c_j \geq 0$ such that

$$T = \sum_{j=1}^k c_j [E_j].$$

Hence $[\Omega_0] \cdot T = 0$ implies that if $[\Omega_0] \cdot [E_j] > 0$, the constants $c_j$ must be zero. This implies $T = 0$. Thus by Lemma 3.3 $[\Omega_0]$ is a balanced class. \hfill $\square$

Before we apply Proposition 3.6 to a nef and big class on a projective Calabi-Yau manifold, we need the following lemma given by Tosatti.

Lemma 3.9. \cite{26} Let $X$ be a projective Calabi-Yau $n$-dimensional manifold and let $[\alpha] \in \partial K_{NS}$ be a big class. Then there exists a smooth form $\alpha_0 \in [\alpha]$ which is nonnegative and strictly positive outside a proper subvariety of $X$. 
For reader’s convenience, we present some details on how to prove the above lemma in \[26\]. First assume that \([\alpha] = c_1(L)\) for some holomorphic line bundle \(L\), which means that \([\alpha]\) lies in the space \(NS(X)_{\mathbb{Z}}\). Hence, \(L\) is nef and big. Now the base point free theorem implies that \(L\) is semiample, so there exists some positive integer \(k\) such that \(kL\) is globally generated. This gives a holomorphic map

\[ F_{[\alpha]} : X \to \mathbb{P}(H^0(X, \mathcal{O}(kL))^*) \]

such that \(F_{[\alpha]}^* \mathcal{O}(1) = kL\). If \([\alpha] \in NS(X)_{\mathbb{Q}}\), then \(l[\alpha] \in NS(X)_{\mathbb{Z}}\) for some positive integer \(l\), and we can also define a holomorphic map \(F_{[\alpha]}\) similarly. Finally if \([\alpha] \in NS(X)_{\mathbb{R}}\), then by Theorem 5.7 in \[18\] or Theorem 1.9 in \[19\], we know that the subcone of nef and big classes is locally rational polyhedral. Hence, \([\alpha]\) lies on a face of this cone which is cut out by linear equations with rational coefficients. It follows that rational points on this face are dense, and it is then possible to write \([\alpha]\) as a linear combination of classes in \(NS(X)_{\mathbb{Q}}\) which are nef and big, with nonnegative coefficients. Notice that all of these classes give the same contraction map, because they lie on the same face. We also denote this map by \(F_{[\alpha]}\). Recall that the exceptional set \(Exc(F_{[\alpha]})\) is defined to be the complement of points where \(F_{[\alpha]}\) is a local isomorphism. It is now clear that we can represent \(\alpha\) by a smooth nonnegative form which is the pull back of Fubini-Study metric (up to scale). And it is strictly positive outside the exceptional set \(Exc(F_{[\alpha]}).\)

In birational geometry (cf. \[20\]), \(F_{[\alpha]}\) is called a divisorial contraction if \(Exc(F_{[\alpha]})\) is of codimension 1 and a flipping contraction if the exceptional set \(Exc(F_{[\alpha]})\) is of codimension greater than 1. We remark that if \(F_{[\alpha]}\) is a divisorial contraction, then the image of \(Exc(F_{[\alpha]})\) under \(F_{[\alpha]}\) is of dimension less than \(n - 1\). In our situation, \(X\) is smooth, thus under divisorial contractions, its image is \(Q\)-factorial and has only weak log-terminal singularities (cf. Proposition 5-1-6 of \[20\]). Thus, its image is \(Q\)-factorial and normal. Then the image of \(Exc(F_{[\alpha]})\) under \(F_{[\alpha]}\) has codimension at least 2 (cf. page 28 of \[8\]). In this case, \([\alpha]^{n-1}\) cannot be a balanced class. Indeed, if \(E_j\) is any codimension 1 component of \(Exc(F_{[\alpha]})\), then we must have \([\alpha]^{n-1} \cdot [E_j] = 0\). Write \(Exc(F_{[\alpha]}) = F_j \cup E_j\) where all irreducible components of \(F\) have codimension at least 2. For a fixed \(j\) and for any \(p \in E_j \setminus (F \cup \cup_{i \neq j} E_i)\), let \(S = F_{[\alpha]}^{-1}(F_{[\alpha]}(p))\) be the fiber over \(F_{[\alpha]}(p)\). Since the image of \(F_{[\alpha]}\) is a normal variety, Zariski’s Main Theorem shows that all irreducible components of \(S\) are positive-dimensional, so there is at least one such component \(S' \subset E_j\) which contains \(p\). Then \(\alpha\) is a smooth semi-positive form in the class \([\alpha]\) and \(\alpha|_{S'} \equiv 0\) since \(S'\) is contained in a fiber of \(F_{[\alpha]}\) and \(\alpha\) is the pull back of Fubini-Study metric. But this means that \((\alpha|_{E_j})^{n-1}(p) = 0\), since \(\alpha|_{E_j}\) has zero eigenvalues in all directions tangent to \(S\). Hence, this is true for all \(p\) in a Zariski open subset of \(E_j\). We conclude that \([\alpha]^{n-1} \cdot [E_j] = \int_{E_j} (\alpha|_{E_j})^{n-1} = 0\).

Now we can prove Theorem \[1.3\].
Proof. By Lemma 3.9, there exists a semipositive $(1,1)$-form $\alpha_0 \in [\alpha]$ such that $\alpha_0$ is strictly positive outside a subvariety $V$. If $V$ is of codimension greater than one, Proposition 3.6 implies that $[\alpha^{n-1}] = [\alpha_0^{n-1}]$ is a balanced metric. If $V$ is of codimension one with irreducible components $E_1, \cdots, E_k$, then $[\alpha^{n-1}] : [E_j] = 0$ for all $1 \leq j \leq k$, thus $[\alpha^{n-1}] \notin \mathcal{B}$. On the other hand, the converse is obvious.

Next, let’s prove $[\alpha^{n-1}] \in \mathcal{B}$ implies that $[\alpha]$ is a big class. Otherwise, we would have $\int_X \alpha^n = 0$. Since $[\alpha]$ is nef, there exists a positive current $T \in \mathcal{A}$. Hence $\int_X \alpha^{n-1} \wedge T = \int_X \alpha^n = 0$. Then $[\alpha^{n-1}] \in \mathcal{B}$ implies $T = 0$. Thus $[\alpha] = [T] = 0$. This is a contradiction. □

We are going to give some examples which show that the holomorphic maps $F_\alpha$ contract high codimensional subvarieties to points, so we can apply Theorem 1.3. The first one is known as a conifold in the physics literature [16] (see also [23]). We learned this from [26]. Let $X_0$ be a nodal quintic in $\mathbb{P}^4$ which has 16 nodal points. Then a smooth Calabi-Yau manifold $X$ is given by a small resolution $f : X \to X_0$, that is a birational morphism which is an isomorphism outside the preimages of the nodes, which are 16 rational curves. Thus we get a contracting map from $X$ to $\mathbb{P}^4$. It is easy to see that the pullback of the Fubini-Study metric is our desired form.

There are also other examples from algebraic geometry (cf. [8], page 24-26). Let $r$ and $s$ be positive integers, let $E$ be the vector bundle on $\mathbb{P}^s$ associated to the locally free sheaf $O_{\mathbb{P}^s} \oplus O_{\mathbb{P}^s}(1)^{r+1}$, and let $Y_{r,s}$ be the smooth $(r+s+1)$-dimensional variety $\mathbb{P}(E^*)$. The projection $\pi : Y_{r,s} \to \mathbb{P}^s$ has a section $P_{r,s}$ corresponding to the trivial quotient of $E$. The linear system $|O_{Y_{r,s}}(1)|$ is base point free. Hence it induces a holomorphic map:

$$C_{r,s} : Y_{r,s} \to \mathbb{P}^{(r+1)(s+1)}.$$  

Moreover, $C_{r,s}$ contracts $P_{r,s}$ to a point and is an immersion on its complement. And its image is the cone over the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$.

Thus, the pull-back of the Fubini-Study metric of $\mathbb{P}^{(r+1)(s+1)}$ is a smooth $(1,1)$-form $\alpha = C_{r,s}^* \omega_{FS}$. Clearly $\alpha$ is pointwise nonnegative on the whole $Y_{r,s}$ and is strictly positive outside $P_{r,s}$ with codimension $r+1$. Thus $[\alpha^{r+s}]$ is a balanced class on $\mathbb{P}(E^*)$. Furthermore, $\int_{P_{r,s}} \alpha^s = 0$ implies $\alpha \in \partial \mathcal{K}(Y_{r,s})$.

In fact, there are a lot of such examples in the Minimal Model Program, encountered when dealing with contraction maps of flipping type ([20]).

The following comment has been formulated by V. Tosatti. In order to produce more examples of birational contraction morphisms as in Lemma 3.9, one can take $X$ more generally to be any smooth projective variety with $-K_X$ nef. This class includes not only Calabi-Yau but also Fano manifolds. Under this assumption, if $L$ is any line bundle on $X$ which is nef and big,
then Kawamata’s base-point-free theorem again gives us that \( L \) is semi-ample and so there is a birational contraction \( F_L \) exactly as in Lemma 3.9. It also works for \( \mathbb{R} \)-linear combinations of line bundles (i.e., big classes on the boundary of \( K_{NS} \)), because again the big points on the boundary of \( K_{NS} \) are locally rational polyhedral (if \( X \) is Fano, then the whole boundary of \( K_{NS} \) is rational polyhedral). Thus if \( X \) has nef anticanonical bundle, we can still apply Theorem 1.3.

4. Characterization theorem on a nef class being Kähler

Using a similar method as in Section 2, we can characterize when a nef class \([\alpha]\) is Kähler under the assumption that \([\alpha^{n-1}]\) is a balanced class.

**Theorem 4.1.** Let \( X \) be a compact \( n \)-dimensional Kähler manifold and \( \eta \) a smooth volume form of \( X \). Assume that \([\alpha]\) is a nef class such that \([\alpha^{n-1}]\) is a balanced class (so \([\alpha]\) is big). If there exists a balanced metric \( \tilde{\omega} \) in \([\alpha^{n-1}]\) (i.e., \( \tilde{\omega}^{n-1} \in [\alpha^{n-1}] \)) such that \( c_{\tilde{\omega}} \geq c_\alpha \) with \( c_{\tilde{\omega}} = \min_X \frac{\tilde{\omega}^n}{\eta} \) and \( c_\alpha = \frac{\int_X \alpha^n}{\int_X \eta} \), then \([\alpha]\) is a Kähler class.

**Proof.** Since \( \tilde{\omega}^{n-1} \in [\alpha^{n-1}] \), there exists a smooth \((n-2, n-2)\)-form \( \phi \) such that

\[
\tilde{\omega}^{n-1} = \alpha^{n-1} + i\partial\bar{\partial}\phi > 0.
\]

Fix a Kähler metric \( \omega \) on \( X \). Then for \( 0 < t \ll 1 \),

\[
(\alpha + t\omega)^{n-1} + i\partial\bar{\partial}\phi = \tilde{\omega}^{n-1} + O(t) > 0.
\]

Thus there exists a balanced metric \( \tilde{\omega}_t \) such that

\[
\tilde{\omega}_t^{n-1} = (\alpha + t\omega)^{n-1} + i\partial\bar{\partial}\phi
\]

and \( \tilde{\omega}_0 = \tilde{\omega} \). Clearly, as \( t \to 0 \), \( \tilde{\omega}_t \to \tilde{\omega} \) in \( C^\infty(\Lambda^{1,1}(X)) \). Then if we let \( F_{\tilde{\omega}_t} := \frac{\tilde{\omega}_t^n}{\eta} \), we have

\[
F_{\tilde{\omega}_t} \to F_{\tilde{\omega}}
\]

in \( C^\infty(X) \) as \( t \to 0 \).

On the other hand, since \([\alpha + t\omega]\) is a Kähler class, by Lemma 2.1 there exists a family of smooth functions \( u_t \) such that \( \alpha + t\omega + i\partial\bar{\partial}u_t \) is Kähler and

\[
(\alpha + t\omega + i\partial\bar{\partial}u_t)^n = c_t\eta
\]

with \( c_t = \frac{\int_X (\alpha + t\omega)^n}{\int_X \eta} \). Moreover, by Lemma 2.3 there also exists an \( \alpha \)-psh function \( u_0 \) such that

\[
\langle (\alpha + i\partial\bar{\partial}u_0)^n \rangle = c_\alpha\eta.
\]

Such \( u_t \) and \( u_0 \) satisfy the following relations

\[
\alpha + t\omega + i\partial\bar{\partial}u_t \to \alpha + i\partial\bar{\partial}u_0 \quad \text{as currents on } X
\]

and

\[
\alpha + t\omega + i\partial\bar{\partial}u_t \to \alpha + i\partial\bar{\partial}u_0 \quad \text{in } C^\infty_{loc}(\text{Amp}(\alpha)).
\]
We denote \( \alpha_t = \alpha + t \omega + i \partial \bar{\partial} u_t \) and \( \alpha_0 = \alpha + i \partial \bar{\partial} u_0 \). Then from (4.1), we have

(4.3) \( \tilde{\omega}_t^{n-1} = \alpha_t^{n-1} + i \partial \bar{\partial} \phi_t \)

for some smooth \((n-2, n-2)\)-form \( \phi_t \) on \( X \).

By the above notations, we have

\[
\tilde{\omega}_t^n \quad \frac{c_t}{\alpha_t^n} = \tilde{\omega}_t^n.
\]

We apply the AM-GM inequality to obtain

\[
\left( \frac{F_{\tilde{\omega}_t}}{c_t} \right)^{\frac{n-1}{n}} = \left( \frac{\det(\alpha_t^{n-1} + i \partial \bar{\partial} \phi_t)}{\det \alpha_t^{n-1}} \right)^{\frac{1}{n}} \leq 1 + \frac{1}{n} \sum_{k,l} (\alpha_t^{n-1})^{kl} (i \partial \bar{\partial} \phi_t)_{kl}.
\]

Equivalently, we have

(4.4) \( \left( \frac{F_{\tilde{\omega}_t}}{c_t} \right)^{\frac{n-1}{n}} \alpha_t^n \leq \alpha_t^n + \alpha_t \wedge i \partial \bar{\partial} \phi_t. \)

We deal with the second term in the above equality, namely

\( \alpha_t \wedge i \partial \bar{\partial} \phi_t = \alpha_t \wedge \tilde{\omega}_t^{n-1} - \alpha_t^n. \)

As discussed in the proof of Theorem 2.4, there exists a convergent subsequence \( \alpha_{t_k} \wedge \tilde{\omega}_{t_k}^{n-1} \) of measures \( \alpha_t \wedge \tilde{\omega}_t^{n-1} \) and a convergent sequence \( \alpha_{t_k}^n \) of measures \( \alpha_t^n \). If we denote their limits by \( \mu_1 \) and \( \mu_2 \), and denote \( \mu_0 = \mu_1 - \mu_2 \), then we have

\( \alpha_{t_k} \wedge i \partial \bar{\partial} \phi_{t_k} \to \mu_0 \) as currents.

Letting \( t = t_k \) in (4.4), integrating with respect to any positive smooth function, and letting \( t_k \) go to zero, we find that the condition \( c_{\tilde{\omega}} \geq c_0 \) implies that \( \mu_0 \) is a positive measure.

Meanwhile, since

\[
\int_X \mu_0 = \lim_{t \to 0} \int_X \alpha_t \wedge \tilde{\omega}_t^{n-1} - \alpha_t^n = \int_X \alpha \wedge (\tilde{\omega}^{n-1} - \alpha^{n-1}),
\]

and as \( \alpha \) is nef and \( \tilde{\omega}^{n-1} \in [\alpha^{n-1}] \), we have \( \int_X \mu_0 = 0 \). Thus \( \mu_0 = 0 \) and \( F_{\tilde{\omega}} = c_0 \) pointwise.

On \( \text{Amp}(\alpha) \), we define a smooth \((1, 1)\)-form

\[
\Psi_0 = \lim_{t \to 0} i \partial \bar{\partial} \phi_t.
\]

Then from (4.3), (4.2) and (4.1), we have

\[
\Psi_0 = \lim_{t \to 0} (\tilde{\omega}_t^{n-1} - \alpha_t^{n-1}) = \tilde{\omega}^{n-1} - \alpha_0^{n-1}.
\]
Hence by uniqueness of the limit, we have on $\text{Amp}(\alpha)$
\[ \alpha_0 \wedge \Psi_0 = 0. \]
Since $F_{\tilde{\omega}} = c_\alpha$, this implies that on $\text{Amp}(\alpha)$,
\[ 1 = \left( \frac{\det \tilde{\omega}^{n-1}}{\det \alpha_0^{n-1}} \right)^{\frac{1}{n}} \leq 1 + \frac{1}{n} \sum_{k,l} (\alpha_0^{n-1})^{k\bar{l}}(\Psi_0)_{k\bar{l}} = 1. \]
Thus $\Psi_0 = 0$. Therefore $\tilde{\omega}^{n-1} = \alpha_0^{n-1}$ or $\tilde{\omega} = \alpha_0$ on $\text{Amp}(\alpha)$.

Since $\tilde{\omega}$ is smooth on $X$ and $d\tilde{\omega} = d\alpha_0 = 0$ on $\text{Amp}(\alpha)$, by continuity, $d\tilde{\omega} = 0$ on $X$, i.e., $\tilde{\omega}$ is a Kähler metric on $X$. However, since $[\tilde{\omega}^{n-1}] = [\alpha^{n-1}]$, by Theorem 1.2, $[\tilde{\omega}] = [\alpha]$. Thus $[\alpha]$ is a Kähler class. \(\square\)

Now we are in a position to conclude the proof of Theorem 1.4.

Proof. We assume that there exists a solution $\Omega_{\text{CY}} \in [\alpha^{n-1}]$ to equation (1.1) for $c \leq (\int_X \alpha^n)^{-1}$. We write $\Omega_{\text{CY}} = \tilde{\omega}^{n-1}$ and then compute
\[ \frac{\tilde{\omega}^n}{\omega_0^n} = \frac{\| \zeta \|^2_{\tilde{\omega}}}{\| \zeta \|^2_{\omega_0}} = \frac{1}{\| \zeta \|^2_{\Omega_{\text{CY}}}} = \frac{1}{c} \geq \frac{\int_X \alpha^n}{\int_X \omega_0^n}. \]
Hence we can use the above theorem. Thus $[\alpha]$ is a Kähler class. Now the proof follows from Theorem 1.3 in [13]. \(\square\)

5. Appendix

In this appendix, we show that the conjectured cone duality $E^\vee = \overline{M}$ in [7] implies that the movable cone $M$ coincides with the balanced cone $B$. Let us first recall the definitions of the pseudoeffective cone and the movable cone of a Kähler manifold.

Definition 5.1. Let $X$ be an $n$-dimensional compact Kähler manifold.

1. The pseudoeffective cone $E \subset H^{1,1}_{BC}(X, \mathbb{R})$ is defined to be the convex cone generated by all positive $d$-closed $(1,1)$-currents.

2. The movable cone $M \subset H^{n-1, n-1}_{BC}(X, \mathbb{R})$ is defined to be the convex cone generated by all positive $d$-closed $(n-1, n-1)$-currents of the form $\mu_*(\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1})$, where $\mu$ ranges among all Kähler modifications from some $\tilde{X}$ to $X$ and $\tilde{\omega}_i$'s are Kähler metrics on $\tilde{X}$.

In [25], Toma observed that every movable curve on a projective manifold can be represented by a balanced metric under the assumption $E^\vee = \overline{M}$. We observe that Toma’s result holds for all movable classes on a compact Kähler manifold. Its proof is along the lines of [25] and the arguments go through mutatis mutandis.

Theorem 5.2. Let $X$ be an $n$-dimensional compact Kähler manifold. Then $E^\vee = \overline{M}$ implies $M = B$. \(\square\)
there exists some current $\tilde{T}$, should show that $dd_j$ of $[T]$ class taking positive values on $\tilde{E}$ is restricted on $BC(X, \mathbb{R})$ is actually an isomorphism (see [2]). Hence when $j$ is restricted on $E$ (which is also denoted by $j$), $j : E \to E_{dd^c}$ is injective. We should show that $j$ is also surjective. For any $[T]_{dd^c} \in E_{dd^c}$ with $T$ positive, there exists some current $S$ such that $d(T + \partial S + \bar{\partial} S) = 0$. We claim that the class $[T + \partial S + \bar{\partial} S]$ is pseudoeffective, i.e., $[T + \partial S + \bar{\partial} S] \in \mathcal{E}$. We need a result in [3], which states that for any modification $\mu : \tilde{X} \to X$ and any positive $dd^c$-closed $(1,1)$-current $T$ on $X$, there exists an unique positive $dd^c$-closed $(1,1)$-current $\tilde{T}$ on $\tilde{X}$ such that $\mu_* \tilde{T} = T$ and $\tilde{T} \in \mu^*[T]_{dd^c}$. Now, take a smooth $(1,1)$-form $\alpha \in [T + \partial \tilde{S} + \bar{\partial} \tilde{S}]$ (which will also be a representative of $[T]_{dd^c}$), $\tilde{T} \in \mu^*[T]_{dd^c}$ implies that there exists some current $\tilde{S}$ such that $\tilde{T} = \mu^* \alpha + \partial \tilde{S} + \bar{\partial} \tilde{S}$. Thus, for any modification $\mu : \tilde{X} \to X$ with $\tilde{X}$ being Kähler, we have

$$\int_X \alpha \wedge \mu_* (\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}) = \int_{\tilde{X}} \mu^* \alpha \wedge \tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}$$

$$= \int_{\tilde{X}} (\mu^* \alpha + \partial \tilde{S} + \bar{\partial} \tilde{S}) \wedge \tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}$$

$$= \int_{\tilde{X}} \tilde{T} \wedge \tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1} \geq 0.$$

By the arbitrariness of $\mu$ and $\tilde{\omega}_i$’s, $\mathcal{E}^\vee = \overline{\mathcal{M}}$ indicates that $[T + \partial \tilde{S} + \bar{\partial} \tilde{S}] \in \mathcal{E}$. This confirms the surjectivity of $j : E \to E_{dd^c}$, and hence $j$ is an isomorphism.

Now, it is easy to see that $\mathcal{M} = \mathcal{B}$. On one hand, since any balanced metric takes positive values on $\mathcal{E} \setminus \{0\}$, $\mathcal{B}$ is obviously contained in the interior of $\mathcal{E}^\vee$, thus $\mathcal{B} \subseteq \mathcal{M}$. On the other hand, $j(\mathcal{E}) = \mathcal{E}_{dd^c}$ yields any movable class taking positive values on $\mathcal{E}_{dd^c} \setminus \{0\}$, hence $\mathcal{E}^\vee_{dd^c} = \overline{\mathcal{B}}$ implies $\mathcal{M} \subseteq \mathcal{B}$. Thus, we obtain $\mathcal{B} = \mathcal{M}$. \hfill $\square$

**Remark 5.3.** In [7], the authors have observed that their conjectured cone duality is true for hyper-Kähler manifolds or Kähler manifolds which are the limits of projective manifolds with maximal Picard number under holomorphic deformations. So in such cases, $\mathcal{B} = \mathcal{M}$ holds.

Inspired by the above theorem, we naturally propose the following problem concerning the balanced cone of a general compact balanced manifold.

**Conjecture 5.4.** Let $X$ be a compact balanced manifold. Then $\mathcal{E}^\vee = \overline{\mathcal{B}}$ holds.

**References**

[1] L. Alessandrini and G. Bassanelli, *Positive $\partial \bar{\partial}$-closed currents and non-Kähler geometry*, J. Geom. Anal. 2 (1992), 291-316.

[2] L. Alessandrini and G. Bassanelli, *Metric properties of manifolds bimeromorphic to compact Kähler spaces*, J. Diff. Geom. 37 (1993), 95-121.
[3] L. Alessandrini and G. Bassanelli, Modifications of compact balanced manifolds, C. R. Math. Acad. Sci. Paris 320 (1995), 1517-1522.
[4] B. Bigolin, Gruppi di Aeppli, Ann. Scuola Norm. Sup. Pisa 23 (1969), 259-287.
[5] S. Bouckoms, Divisorial Zariski decompositions on compact complex manifolds, Ann. Sci. École. Norm. Sup. 37 (2004), 45-76.
[6] S. Bouckoms, P. Eyssidieux, V. Guedj and A. Zeriahi, Monge-Ampère equations in big cohomology classes, Acta Math. 205 (2010), 199-262.
[7] S. Bouckoms, J.-P. Demailly, M. Paun, T. Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, J. Algebraic Geom. 22 (2013) 201-248.
[8] O. Debarre, Higher-Dimensional Algebraic Geometry, Springer-Verlag, 2001.
[9] J.-P. Demailly, A numerical criterion for very ample line bundles, J. Diff. Geom. 37 (1993), 323-374.
[10] J.-P. Demailly, Complex Analytic and Differential Geometry, [http://www-fourier.ujf-grenoble.fr/~demailly/books.html]
[11] J.-P. Demailly and M. Paun, Numerical characterization of the Kähler cone of a compact Kähler manifold, Ann. Math. 159 (2004), 1247-1274.
[12] J. Fu, J. Li and S.-T. Yau, Balanced metrics on non-Kähler Calabi-Yau threefolds, J. Diff. Geom. 90 (2012), 81-130.
[13] J. Fu, Z. Wang and D. Wu, Form-type Calabi-Yau equations, Math. Res. Lett. 17 (2010), 887-903.
[14] J. Fu, Z. Wang and D. Wu, Form-type Calabi-Yau equations on Kähler manifolds of nonnegative orthogonal bisectional curvature, [arXiv:1010.2022v2].
[15] A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Annali Mat. Pura Appl. 123 (1982), 35-58.
[16] B. R. Green, D. R. Morrison and A. Strominger, Black hole condensation and the unification of string vacua, Nuclear Phys. B 451 (1995), 109-120.
[17] M. Gromov, Convex sets and Kähler manifolds, Advances in Differential Geometry and Topology, World Sci. Publishing, Teaneck, NJ (1990), 1-38.
[18] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math. 127 (1988), no. 1, 93-163.
[19] Y. Kawamata, On the cone of divisors of Calabi-Yau fiber spaces, Internat. J. Math. 8 (1997), no. 5, 665-687.
[20] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the Minimal Model Problem, Advanced Studies in Pure Mathematics 10 (1987), 283-360.
[21] M. L. Michelsohn, On the existence of special metrics in complex geometry, Acta. Math. 149 (1982), 261-295.
[22] M. Paun, Sur l’effectivité numérique des images inverses de fibrés en droites, Math. Ann. 310 (1998), 411-421.
[23] M. Rossi, Geometric transitions, J. Geom. Phys. 56 (2006), 1940-1983.
[24] D. Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds, Inventiones Math. 36 (1976), 225-255.
[25] M. Toma, A note on the cone of mobile curves, C. R. Math. Acad. Sci. Paris 348 (2010), 71-73.
[26] V. Tosatti, Limits of Calabi-Yau metrics when the Kähler class degenerates, J. Eur. Math. Soc. 11 (2009), 755-776.
[27] V. Tosatti and B. Weinkove, The Monge-Ampère equation for (n-1)-plurisubharmonic functions on a compact Kähler manifold, [arXiv:1305.7511].
[28] C. Voisin, Hodge Theory and Complex Algebraic Geometry, I, Cambridge Stud. Adv. Math. 76, Cambridge Univ. Press, 2003.
[29] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure. Appl. Math. 31 (1978), 339-411.
