Essential dimension of Hermitian spaces

N. Semenov, K. Zainoulline*

Abstract

Given an hermitian space we compute its essential dimension, Chow motive and prove its incompressibility in certain dimensions.

The notion of an essential dimension \( \dim_{\text{es}} \) is an important birational invariant of an algebraic variety \( X \) which was introduced and studied by N. Karpenko, A. Merkurjev, Z. Reichstein, J.-P. Serre and others. Roughly speaking, it is defined to be the minimal possible dimension of a rational retract of \( X \). In particular, if it coincides with the usual dimension, then \( X \) is called incompressible.

In general, this invariant is very hard to compute. As a consequence, there are only very few known examples of incompressible varieties: certain Severi-Brauer varieties and projective quadrics. In the present paper we provide new examples of incompressible varieties: Hermitian quadrics of dimensions \( 2^r - 1 \). We also give an explicit formula for the essential dimension of a Hermitian form in the sense of O. Izhboldin, hence, providing a Hermitian version of the result of Karpenko-Merkurjev [KM03]. At the end we discuss the relations with Higher forms of Rost motives of Vishik [Vi00].

We follow the notation of [Kr07]. Let \( F \) be a base field of characteristic not 2 and let \( L/F \) be a quadratic field extension. Let \( (W, h) \) be a non-degenerate \( L/F \)-Hermitian space of rank \( n \) and let \( q \) be the quadratic form associated to the hermitian form \( h \) via \( q(v) = h(v, v), v \in W \).

The main objects of our study are the following two smooth projective varieties over \( F \):

- the variety \( V(q) \) of \( q \)-isotropic \( F \)-lines in \( W \), i.e. a projective quadric;

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the variety $V(h)$ of $h$-isotropic $L$-lines in $W$ called a Hermitian quadric.

Observe that $V(q)$ has dimension $(2n - 2)$ and $V(h)$ is a $(2n - 3)$-dimensional projective homogeneous variety under the action of the unitary group associated with $h$. It is also a twisted form of the incidence variety that is the variety of flags consisting of a dimension one and codimension one linear subspaces in an $n$-dimensional vector space.

The forms $q$ and $h$ are closely related by the following celebrated result of Milnor-Husemoller (see [Le79]):

A quadratic form $q$ on an $F$-vector space $W$ is the underlying form of a Hermitian form over a quadratic field extension $L = F(\sqrt{a})$ iff $\dim W = 2n$, $q_L$ is hyperbolic, and $\det q = (-a)^n \mod F^2$.

1. Incompressibility A smooth projective $F$-variety $X$ is called incompressible if any rational map $X \dashrightarrow X$ is dominant. The basic examples of such varieties are anisotropic quadrics of dimensions $2^r - 1$ and Severi-Brauer varieties of division algebras of prime degrees.

Theorem (A). Assume that the variety $V(h)$ is anisotropic and $\dim V(h) = 2^r - 1$ for some $r > 0$. Then $V(h)$ is incompressible.

Proof. The key idea is that a Hermitian quadric which is purely a geometric object can be viewed as a twisted form of a Milnor hypersurface $M$ – a topological object, namely, a generator of the Lazard ring of algebraic cobordism of M. Levine and F. Morel [LM].

More precisely, by [LM 2.5.3] the variety $M$ is the zero divisor of the line bundle $\mathcal{O}(1) \otimes \mathcal{O}(1)$ on $\mathbb{P}^{n-1}_F \times \mathbb{P}^{n-1}_F$, i.e. it is given by the equation

$$\sum_{i=0}^{n-1} x_i y_i = 0, \quad (1)$$

where $[x_0 : \ldots : x_{n-1}]$ and $[y_0 : \ldots : y_{n-1}]$ are the projective coordinates of the first and the second factor respectively.

From another hand side, the Hermitian quadric $V(h)$ is a twisted form of the incidence variety $X = \{W_1 \subset W_{n-1}\}$, where $\dim W_i = i$. Taking $[x_0 : \ldots : x_{n-1}] = W_1$ and $[y_0 : \ldots : y_{n-1}]$ to be the normal vector to $W_{n-1}$ we obtain that $X$ is given by the same equation (1), therefore, $X \simeq M$. 

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By [Mc02, Prop.7.2] we obtain the following explicit formula for the Rost characteristic number $\eta_2$ of $M$

$$\eta_2(M) := \frac{c_{\dim M}(-T_M)}{2} = \frac{1}{2} \left( \binom{2(n-1)}{n-1} \right) \mod 2.$$ 

It has the following property:

$$\eta_2(M) \equiv 1 \mod 2 \iff \dim M = 2^r - 1 \text{ for some } r > 0. \quad (2)$$

Since $\eta_2$ doesn’t depend on the base change, $\eta_2(M) = \eta_2(V(h))$.

We apply now the standard arguments involving the Rost degree formula (see [Mc03, §7]). Let $f : V(h) \to V(h)$ be a rational map. By the degree formula:

$$\eta_2(V(h)) \equiv \deg f \cdot \eta_2(V(h)) \mod n_{V(h)}, \quad (3)$$

where $n_{V(h)}$ is the greatest common divisor of degrees of all closed points of $V(h)$. Since $V(h)$ becomes isotropic over $L$, $n_{V(h)} = 2$.

Assume now that $\dim(V(h)) = 2^r - 1$ for some $r > 0$. Then, by (2) $\eta_2(V(h)) \equiv 1$ and by (3) $\deg f \neq 0$ which means that $f$ is dominant. This finishes the proof of the theorem. \qed

2. Essential dimension

Following O. Izhboldin we define the essential dimension of a Hermitian space $(W, h)$ as

$$\dim_{es}(h) := \dim V(h) - i(q) + 2,$$

where $i(q)$ stands for the first Witt index of the form $q$ (cf. [KM03]). The following theorem provides a Hermitian version of the main result of [KM03]

**Theorem** (B). Let $Y$ be a complete $F$-variety with all closed points of even degree. Suppose that $Y$ has a closed point of odd degree over $F(V(h))$. Then $\dim_{es}(h) \leq \dim Y$. Moreover, if $\dim_{es}(h) = \dim Y$, then $V(h)$ is isotropic over $F(Y)$.

**Proof.** In [Kr07] D. Krashen constructed a $\mathbb{P}^1$-bundle

$$Bl_{S}(V(q)) \to V(h), \quad (4)$$

where $Bl_{S}(V(q))$ is the blow-up of the quadric $V(q)$ along the linear subspace $S = \mathbb{P}^{n-1}_L$. In particular, the function field of $V(q)$ is a purely transcendental extension of the function field of $V(h)$ of degree 1, and, therefore, our theorem follows from [KM03, Theorem 3.1]. \qed
Using Theorem (B) we can give an alternative proof of Theorem (A):

Another proof of (A). Let $Y$ be the closure of the image of a rational map $V(h) \rightarrow V(h)$. Then by Theorem (B) the incompressibility of $V(h)$ follows from the equality $\dim_{es}(h) = \dim V(h)$. The latter can be deduced from the Hoffmann’s conjecture (proven in [Ka03]) if $V(h)$ is anisotropic and $\dim V(h) = 2^r - 1$. Indeed, if $\dim V(h) = 2^r - 1$, then $\dim q = 2^r + 2$. Therefore, $i(q) = 1$ or 2. But by the result of Milnor-Husemoller $i(q)$ must be even. Hence, $\dim_{es}(h) = \dim V(h)$.

3. Chow motives

We follow the notation of [CM06, §6]. As a direct consequence of the fibration (4) and the Krull-Schmidt theorem proven in [CM06] we obtain the following expressions for the Chow motives of $V(q)$ and $V(h)$:

**Theorem (C).** There exists a motive $N_h$ such that

\[ M(V(q)) \simeq \begin{cases} N_h \oplus N_h \{1\}, & \text{if } n \text{ is even;} \\ N_h \oplus M(\text{Spec } L)\{n-1\} \oplus N_h \{1\}, & \text{if } n \text{ is odd;} \end{cases} \tag{5} \]

and

\[ M(V(h)) \simeq \begin{cases} N_h \oplus \bigoplus_{i=0}^{(n-4)/2} M(\mathbb{P}_L^{n-1})\{2i+1\}, & \text{if } n \text{ is even;} \\ N_h \oplus \bigoplus_{i=0}^{(n-3)/2} M(\mathbb{P}_L^{n-2})\{2i+1\}, & \text{if } n \text{ is odd.} \end{cases} \tag{6} \]

Observe that by the projective bundle theorem $M(\mathbb{P}_L^{n}) \simeq \bigoplus_{i=0}^{n} M(\text{Spec } L)\{i\}$.

**Proof.** Using the $\mathbb{P}^1$-fibration (4) D. Krashen provided the following formula relating the Chow motives of $V(q)$ and $V(h)$:

\[ M(V(q)) \oplus \bigoplus_{i=1}^{n-2} M(\mathbb{P}_L^{n-1})\{i\} \simeq M(V(h)) \oplus M(V(h))\{1\}. \tag{7} \]

Observe that the motives of all varieties participating in the formula split over $L$ into direct sums of twisted Tate motives $\mathbb{Z}_L$. For each such decomposition $M_L \simeq \bigoplus_{i \geq 0} \mathbb{Z}_L\{i\}^{\oplus a_i}$ we define the respective Poincaré polynomial by $P_{M_L}(t) := \sum_{i \geq 0} a_i t^i$. Using the standard combinatorial description of the cellular structure on $V(q)_L$, $V(h)_L$ and $\mathbb{P}_L^{n-1}$ (see [CM06]) we obtain the following explicit formulae:

\[ P_{V(q)_L}(t) = \frac{(1-t^n)(1+t^{n-1})}{1-t}, \quad P_{V(h)_L}(t) = \frac{(1-t^n)(1+t^{n-1})}{(1-t)^2} \quad \text{and} \quad P_{\text{Spec } L}(t) = 2. \tag{8} \]
Consider the subcategory of the category of Chow motives with \( \mathbb{Z}/2 \)-coefficients generated by \( M(V(h); \mathbb{Z}/2) \). Since \( V(h) \) is a projective homogeneous variety, the Krull-Schmidt theorem and the cancellation theorem hold in this subcategory by [CM06, Cor.35]. In particular, two decompositions of the formula (7) have to consist from the same indecomposable summands.

Analyzing their Poincaré polynomials over \( L \) using (8) we obtain the formulae (5) and (6) for motives with \( \mathbb{Z}/2 \)-coefficients. Finally, applying [PSZ, Thm.2.16] for \( m = 2 \) we obtain the desired formulae integrally.

4. Higher forms of Rost motives

In [Vi00, Thm.5.1] A. Vishik proved that given a quadratic form \( q \) over \( F \) divisible by an \( m \)-fold Pfister form \( \varphi \), that is \( q = q' \otimes \varphi \) for some form \( q' \), there exists a direct summand \( N \) of the motive \( M(Q_q) \) of the projective quadric \( Q_q \) associated with \( q \) such that

\[
M(Q_q) \simeq \begin{cases} 
N \otimes M(\mathbb{P}_F^{m-1}), & \text{if } \dim q' \text{ is even;} \\
(N \otimes M(\mathbb{P}_F^{m-1})) \oplus M(Q_\varphi)\{\frac{\dim q}{2} - 2^{m-1}\}, & \text{if } \dim q' \text{ is odd.}
\end{cases}
\]

In view of the Milnor-Husemoller theorem mentioned in the beginning, formula (5) implies a shortened proof of Vishik’s result for \( m = 1 \).

References

[CM06] Chernousov V., Merkurjev A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. Transf. Groups 11 (2006), no. 3, 371–386.

[Ka03] Karpenko N. On the first Witt index of quadratic forms. Invent. Math. 153 (2003), no. 2, 455–462.

[KM03] Karpenko N., Merkurjev A. Essential dimension of quadrics. Invent. Math. 153 (2003), no. 2, 361–372.

[Kr07] Krashen D. Motives of unitary and orthogonal homogeneous varieties. J. of Algebra 318 (2007), 135–139.

[Le79] Lewis D. A note on Hermitian and quadratic forms. Bull. London Math. Soc. 11 (1979), no. 3, 265–267.
[LM] Levine M., Morel F. Algebraic cobordism. Springer-Verlag Berlin Heidelberg, 2007.

[Me02] Merkurjev A. Algebraic oriented cohomology theories. In Algebraic number theory and algebraic geometry, Contemp. Math., 300, Amer. Math. Soc., Providence, RI, 2002, 171–193.

[Me03] Merkurjev A. Steenrod operations and degree formulas. J. Reine Angew. Math. 565 (2003), 13–26.

[PSZ] Petrov V., Semenov N., Zainoulline K. J-invariant of linear algebraic groups. Ann. Sci. Ec. Norm. Sup. (4), 42pp. to appear.

[Vi00] Vishik A. Motives of quadrics with applications to the theory of quadratic forms. In Geometric methods in the algebraic theory of quadratic forms. Lens, France, June 2000.

N. Semenov, Mathematisches Institut der LMU München

K. Zainoulline, Mathematisches Institut der LMU München, Theresienstr. 39, 80333 München; e-mail: kirill@mathematik.uni-muenchen.de