FEW LONG LISTS FOR EDGE CHOOSABILITY OF PLANAR CUBIC GRAPHS

LUIS GODDYN AND ANDREA SPENCER

Abstract. It is known that every loopless cubic graph is 4-edge choosable. We prove the following strengthened result.

Let $G$ be a planar cubic graph having $b$ cut-edges. There exists a set $F$ of at most $\frac{5}{2}b$ edges of $G$ with the following property. For any function $L$ which assigns to each edge of $F$ a set of 4 colours and which assigns to each edge in $E(G) - F$ a set of 3 colours, the graph $G$ has a proper edge colouring where the colour of each edge $e$ belongs to $L(e)$.

1. Introduction

We assume here that graphs are finite and loopless. An edge list assignment for a graph $G$ is a function $L$ which maps each edge of $G$ a set of colours. An $L$-edge colouring is a proper edge colouring $c$ of $G$ for which $c(e) \in L(e)$ for each $e \in E(G)$. Let $f : E(G) \to \mathbb{N}$ be an edge weighting with positive integers. We say that $G$ is $f$-edge choosable if $G$ has an $L$-edge colouring for every edge list assignment $L$ satisfying $|L(e)| \geq f(e)$ for each $e \in E(G)$. For $k \in \mathbb{N}$, we say that $G$ is $k$-edge choosable if $G$ is $f$-edge choosable for some $f$ satisfying $\max f \leq k$. If, additionally, $G$ has a most $s$ edges $e$ for which $f(e) = k$, then we say that $G$ is $s$-nearly $(k - 1)$-edge choosable. We consider the problem finding a good upper bound on the quantity

$$s(G, k) := \min\{s \mid G \text{ is } s\text{-nearly } k\text{-edge choosable}\}.$$

The notion of vertex choosability is defined similarly, with reference to vertex colouring. An extension of Brooks’ theorem to vertex choosability asserts that every simple connected graph $H$ with maximum degree $\Delta$ is $\Delta$-vertex choosable unless $H$ is a complete graph or a cycle. Applying this to the line-graph of a cubic graph, it follows that every loopless cubic graph is 4-edge choosable. However, a typical cubic graph $G$ is $s$-nearly 3-edge choosable.

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where $s$ is somewhat smaller than the order of $G$. For example, a cubic graph $G$ satisfies $s(G, 3) = 0$ (that is, $G$ is 3-edge choosable) if $G$ is either bipartite [7], or planar and 2-connected [6]. The latter result strengthens the 4-colour theorem. If $G$ has a cut-edge, then $s(G, 3) > 0$ since $G$ is not 3-edge colourable. One easily constructs cubic graphs $G$ having $b$ cut-edges for which $s(G, 3) \geq 2b$. For example, if each connected component of $G$ has exactly one cut-edge, then $G$ is $f$-edge choosable only if each of the $2b$ leaf-blocks of $G$ contains at least one edge $e$ for which $f(e) \geq 4$. (This is because no leaf block is 3-edge colourable). In this paper we show that planar cubic graphs $G$ satisfy $s(G, 3) \leq \frac{5}{2}b$.

**Theorem 1.1.** Let $G$ be a planar cubic graph having $b$ cut-edges. Then $G$ is $f$-edge choosable for some function $f : E(G) \to \{1, 2, 3, 4\}$ that has average value 3, and $|f^{-1}(4)| \leq \frac{5}{2}b$.

2. The Polynomial Method

Let $e_1, e_2, \ldots, e_m$ be the edges of a graph $G$, and let $x_i$ be an indeterminate associated with the edge $e_i$. The **edge monomial** of $G$ is the polynomial in $\mathbb{R}[x_1, \ldots, x_m]$ defined by

$$
\epsilon(G) = \prod_{1 \leq i < j \leq m} (x_i - x_j)^{c(i,j)}.
$$

Here $c(i,j) \in \{0, 1, 2\}$ is the number of vertices incident to both $e_i$ and $e_j$. Note that $\epsilon(G)$ is a homogeneous polynomial of degree $\epsilon(G)$. Furthermore, $\epsilon(G)$ is well defined (up to negation) regardless of the edge ordering $e_1, \ldots, e_m$. Each term in the standard expansion of $\epsilon(G)$ takes the form $\alpha_w x^w := \alpha_w \prod_{1 \leq i \leq m} x_i^{w(e_i)}$ where the exponent function $w : E(G) \to \{0, 1, \ldots\}$ is a nonnegative integer weighting of the edges of $G$. We shall write $w + 1$ for the function $e \mapsto w(e) + 1$. The Combinatorial Nullstellensatz [1] [2] for edge choosability asserts the following.

**Lemma 2.1.** Let $G$ be a loopless graph, and let $\alpha_w x^w$ be a nonzero term in the expansion of $\epsilon(G)$. Then $G$ is $f$-edge choosable, where $f = w + 1$.

The **polynomial method** for proving that $G$ is $f$-edge choosable typically involves selecting an exponent function $w$ satisfying $w + 1 \leq f$, and showing that $\alpha_w \neq 0$. To evaluate $\alpha_w$, Ellingham and Goddyn [4] provide a combinatorial interpretation of $\alpha_w$ in terms of **star labellings** which we describe below. Let $v$ be a vertex of degree $d$ in $G$. A **star labelling at** $v$ is a bijective function $\pi_v$ from the edges incident with $v$ to the integers $\{0, 1, \ldots, d - 1\}$. A **star labelling** of $G$ is a set $\pi = \{\pi_v : v \in V(G)\}$ where each $\pi_v$ is a star labelling at $v$. The **exponent** of a star labelling $\pi$ is the edge weighting $w = w_\pi$ defined by $w(e_i) = \pi_u(e_i) + \pi_v(e_i)$,
for $e_i = uv \in E(G)$. The sign, $\text{sgn}(\pi_v)$, of a star labelling at $v$ is the sign of the permutation of $\{0, 1, \ldots, d - 1\}$ defined by $j \mapsto \pi_v(e_i)$, where $e_{i_0}, e_{i_1}, \ldots, e_{i_{d-1}}$ are the edges incident with $v$, and $i_0 < i_1 < \cdots < i_{d-1}$. The sign of a star labelling of $G$ is defined by $\text{sgn}(\pi) = \prod_{v \in V(G)} \text{sgn}(\pi_v)$.

**Lemma 2.2 [4]**. For any loopless graph $G$ we have

$$(2) \quad \epsilon(G) = \sum_{\pi} \text{sgn}(\pi) x^{w_{\pi}},$$

where the sum is taken over the set of star labellings of $G$.

The reader should notice that, up to negation, the edge monomial $\epsilon(G)$ does not depend on the particular ordering $e_1, e_2, \ldots, e_m$ of $E(G)$. A novel feature of this paper is our use of several coefficients of $\epsilon(G)$ in the polynomial method. If a set of coefficients $\{\alpha_{w_i} \mid 1 \leq i \leq k\}$ has a nonzero sum, then at least one coefficient $\alpha_{w_i}$ is not zero. This gives a multi-term version of Lemma 2.1.

**Corollary 2.3.** Let $W = \{w_1, \ldots, w_k\}$ be a set of edge weightings for $G$, and let $\Pi(W)$ be the set of star labellings $\pi$ of $G$ such that the exponent of $\pi$ is a member of $W$. If the integer

$$(3) \quad \sum_{\pi \in \Pi(W)} \text{sgn}(\pi)$$

is not zero, then $G$ is $(w_i + 1)$-edge choosable, for some $i \in \{1, 2, \ldots, k\}$.

**Proof.** Let $\Pi(w_i)$ be the set of star labellings of $G$ having exponent $w_i$. If (3) is not zero, then $\sum_{\pi \in \Pi(w_i)} \text{sgn}(\pi) \neq 0$, for some $i \in \{1, 2, \ldots, k\}$. By Lemma 2.2 this last sum equals the coefficient of the term $\alpha_{w_i} x^{w_i}$ in the expansion of $\epsilon(G)$, and the result follows from Lemma 2.1. \qed

To prove our main result, we will construct an appropriate set $W$ of edge weightings of a planar cubic graph $G$, and show that the signed sum (3) is positive.

### 3. Weightings and Star Labellings of Threads

A flag of a graph $G$ is a pair $(v, e) \in V(G) \times E(G)$ whose members are incident in $G$. We may write $ve$ instead of $(v, e)$. It is convenient to regard a star labelling of $G$ to be a nonnegative integer function $\pi : F(G) \rightarrow \{0, 1, \ldots, \Delta(G) - 1\}$, where $F(G)$ is the set of flags of $G$. For $m \geq 0$, a thread of order $m$ is the graph $T_m$ obtained from a path $v_0e_0v_1e_1 \ldots e_mv_{m+1}$ by adding new vertices $w_k$ and new edges $f_k = v_kw_k$ ($1 \leq k \leq m$). See
Figure 1. In particular the trivial thread $T_0$ is the path $v_0e_0v_1$. The head of $T_m$ is the flag $v_0e_0$, the tail of $T_m$ is the flag $v_{m+1}e_m$, and the feet of $T_m$ are the flags $w_kf_k$ ($1 \leq k \leq m$). A function $\pi : F(T_m) \to \{0, 1, 2\}$ is called a prestar labelling of $T_m$ if the restricted function $\pi_v := \pi|_{\{v_k e_{k-1}, v_k e_k, v_k, f_k\}}$ is a star labelling of $v_k$, for $1 \leq k \leq m$. The sign of a prestar labelling $\pi$ is defined to be $\text{sgn}(\pi) = \prod_{k=1}^{m} \text{sgn}(\pi_v)$, and the exponent of $\pi$ is the edge weighting $w$ where $w(e) = \pi(ue) + \pi(ve)$ for each $e = uv \in E(T_m)$. A prestar labelling $\pi$ is 1-footed if $\pi(w_k f_k) = 1$, for $1 \leq k \leq m$. We say that $\pi$ has type $(i, j)$ if $\pi(v_0 e_0) = i$ and $\pi(v_{m+1} e_m) = j$. We are interested in classifying, according to type, the set of 1-footed prestar labellings of $T_m$ which have a prespecified exponent.

For each $m \geq 0$ we define four special edge weightings of $T_m$, which we denote by $w_2$, $w_{11}$, $w_{02}$ and $w_{20}$. These are illustrated in Figure 1. The weighting $w_2$ is just the constant function $w(e) = 2$. The next three weightings are defined only for $m \geq 1$. The weighting $w_{11}$ is obtained from $w_2$ by transferring one unit of weight from $e_0$ to $e_m$. That is, we have $w_{11}(e_0) = 1$, $w_{11}(e_m) = 3$, and $w_{11}(e) = 2$ for $e \in E(T_m) - \{e_0, e_m\}$. The weighting $w_{02}$ is obtained from $w_2$ by transferring one unit of weight from $e_0$ to $f_1$. The weighting $w_{20}$ is obtained from $w_2$ by transferring one unit of weight from $e_1$ to $e_0$, and then transferring one unit of weight from $e_1$ to $f_1$.

As shown in Figure 1, each special weighting is associated with one or more 1-footed prestar labellings of $T_m$. Each labelling is denoted by either $\rho_{ij}$, $\pi_{ij}$ or $\pi'_{ij}$ where $(i, j)$ is its type.

**Lemma 3.1.** There are three prestar labellings $\rho_{20}$, $\rho_{02}$ and $\rho_{11}$ of $T_0$ with exponent $w_2$. For $m \geq 1$, $\rho_{20}$ and $\rho_{02}$ are the only 1-footed prestar labellings of $T_m$ having exponent $w_2$. Let $m \geq 1$ and let $(i, j) \in \{(1, 1), (2, 0), (0, 2)\}$. If $(m, i, j) \neq (1, 0, 2)$, then $\pi_{ij}$ is the unique 1-footed prestar labelling of $T_m$ having exponent $w_{ij}$. If $(m, i, j) = (1, 0, 2)$, then $T_1$ has exactly two 1-footed prestar labellings with exponent $w_{02}$, namely $\pi_{02}$ and $\pi'_{11}$.

**Proof.** We prove only the statement regarding $w_{02}$ since the arguments are easy and mechanical. Let $\pi$ be a 1-footed prestar labelling of $T_m$ whose exponent equals $w_{02}$. Since $\pi$ is 1-footed, we have $\pi(v_1 f_1) = w_{02}(f_1) - \pi(w_1 f_1) = 3 - 1 = 2$, and $\pi(v_k f_k) = 2 - 1 = 1$, for $2 \leq k \leq m$. Since $\pi_v$ is a star labelling we have $\{\pi(v_1 e_0), \pi(v_1 e_1)\} = \{0, 1\}$. In case $\pi(v_1 e_1) = 0$, we have $\pi(v_1 e_0) = 1$ and $\pi(v_0 e_0) = 1 - 1 = 0$. We now apply the facts $\pi(v_{k+1} e_k) = 2 - \pi(v_k e_k)$ ($k = 1, 2, \ldots, m$), and $\{\pi(v_k e_{k-1}), \pi(v_k e_k)\} =\{0, 2\}$ ($k = 2, 3, \ldots, m$) to find $\pi(v_k e_k) = 0$ and $\pi(v_{k+1} e_k) = 2$ for $k = 1, 2, \ldots, m$. Thus $\pi = \pi_{02}$. In case $\pi(v_1 e_1) = 1$, we find that $\pi(v_1 e_0) = 0$, $\pi(v_0 e_0) = 1 - 0 = 1$ and $\pi(v_2 e_1) = 2 - 1 = 1$. If $m \geq 2$, then we
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Figure 1. The thread \( T_m \), four edge weightings \( w_2 \) and \( w_{ij} \), and all 1-footed prestar labellings, \( \rho_{ij}, \pi_{ij} \) and \( \pi'_{11} \), whose exponents are one of those weights.

have contradicted the fact \( \pi_{v_2} \) is a star labelling, since \( \pi(v_2 f_2) = 1 \). Therefore \( m = 1 \), and \( \pi \)

is the exceptional prestar labelling \( \pi'_{11} \).

Proposition 3.2. For each odd integer \( m \geq 1 \), the prestar labellings of \( T_m \) defined above satisfy \( \text{sgn}(\pi_{20}) = \text{sgn}(\pi_{02}) \). For \( m = 1 \), we have that \( \text{sgn}(\pi'_{11}) = -\text{sgn}(\pi_{02}) \). For each even integer \( m \geq 2 \) we have that \( \text{sgn}(\rho_{02}) = \text{sgn}(\rho_{20}) \).

Proof. Consider the embedding of \( T_m \) in the plane shown in Figure 1. The sign of a star labelling at \( v_k \) depends only on whether the three labels 0, 1, 2 appear in clockwise or anticlockwise order around \( v_k \). When \( m = 1 \), the star labelling at \( v_1 \) is clockwise under \( \pi'_{11} \), and is anticlockwise under \( \pi_{02} \). Therefore \( \text{sgn}(\pi'_{11}) = -\text{sgn}(\pi_{02}) \) for any embedding of \( T_1 \).

Evidently, every vertex \( v_k \) (\( 1 \leq k \leq m \)) is anticlockwise under \( \pi_{20} \), whereas \( v_1 \) is the unique anticlockwise vertex under \( \pi_{02} \). Therefore \( \text{sgn}(\pi_{20}) = \text{sgn}(\pi_{02}) \) provided that \( m \) is odd. Each vertex \( v_k \) is clockwise under \( \rho_{02} \) and anticlockwise under \( \rho_{20} \), so \( \text{sgn}(\rho_{02}) = \text{sgn}(\rho_{20}) \) when \( m \) is even.

We define two variations of an edge-weighted thread. For \( m \geq 1 \), the closed thread of order \( m \) is the graph \( T_m^* \) obtained from \( T_m \) by identifying the vertices \( v_0 \) and \( v_{m+1} \). We
define the edge weightings \( w_{02}, w_2 \), and the prestar labellings \( \pi_{02}, \rho_{20} \) and \( \rho_{02} \) exactly as they were defined for \( T_m \). The reader will easily verify the following lemma.

**Lemma 3.3.** Let \( m \geq 1 \). Then \( \rho_{20} \) and \( \rho_{02} \) are the only 1-footed prestar labellings of \( T_m^o \) having exponent \( w_2 \). Furthermore, \( \pi_{02} \) is the unique 1-footed prestar labelling of \( T_m^o \) having exponent \( w_{02} \), for which the head and tail (that is, the two flags incident with \( v_0 \)) receive different labels.

An injured thread of order \( m \) is any graph \( T_m^− \) that is obtained from \( T_m \) by deleting any one of its \( m \) “feet” \( w_k, 1 \leq k \leq m \). The edge weighting \( w_{11}^− \) of \( T_m^− \) is the restriction of \( w_{11} \) to the edge set of \( T_m^− \). A 1-footed prestar labelling of \( T_m^− \) is the restriction of a 1-footed prestar labelling of \( T_m \) to the flags in \( T_m^− \). In particular, we define \( \pi_{11}^− = \pi_{11} \upharpoonright_{F(T_m^−)} \). To ease notation, we shall write \( w_{11} \) instead of \( w_{11}^− \), and write \( \pi_{11} \) instead of \( \pi_{11}^− \), where no confusion results.

**Lemma 3.4.** For \( m \geq 1 \), the prestar labelling \( \pi_{11} \) is the unique 1-footed prestar labelling of \( T_m^− \) whose exponent equals \( w_{11} \).

4. **A Set of Edge Weightings**

In this section, we define a set of edge weightings \( W \) of a planar cubic graph, to which we will apply Corollary 2.3. Let \( G \) be a connected loopless cubic graph and let \( B(G) \) be the set of cut-edges in \( G \). A block of \( G \) is any connected component of \( G − B(G) \) (this differs from the standard definition of “block”). Each block, \( H \), is either a vertex block, a cycle block or a proper block, depending on whether \( H \) is a single vertex, a cycle or a subdivision of 2-connected cubic graph. The block tree of \( G \) is the tree obtained by contracting each block \( H \) to a single vertex, which we also denote by \( H \) where no confusion results. Since \( G \) is finite, at least one block of \( G \) is a proper block. We designate one proper block to be the root block \( H_0 \) of \( G \). Every other block of \( G \) is called a nonroot block of \( G \). We define \( B(H_0) \) to be the set of edges in \( B(G) \) which have exactly one end in \( H_0 \). Every nonroot block \( H \) is incident to a unique cut-edge, denoted by \( e_H \), which lies on the path from \( H \) to \( H_0 \) in the block tree of \( G \). For nonroot blocks \( H \), we define \( B(H) \) to be the set of edges in \( B(G) − \{e_H\} \) which have an endpoint in \( H \). For any block \( H \) of \( G \), let \( H^+ \) be the subgraph of \( G \) obtained from \( H \) by adding all the edges in \( B(H) \) and their endpoints. Each subgraph \( H^+ \) is called an extended block of \( G \). The extended blocks of \( G \) depend on the choice of \( H_0 \).

The edge sets of the extended blocks of \( G \) form a partition of \( E(G) \). We further refine the extended blocks into pieces that are each isomorphic to one of the threads, \( T_m, T_m^o \) or \( T_m^− \),
Each extended vertex block is a path of length 2, which we regard to be a copy of the injured thread \( T_1^- \). We define the family

\[ T^1 = \{ H^+ | H \text{ is a vertex block of } G \}. \]

Each extended cycle block is isomorphic to a closed thread \( T_m^0 \), for some \( m \geq 1 \). We group these into two families.

\[ T^\text{odd} = \{ H^+ | H \text{ is a cycle block of } G, \text{ and } H^+ \cong T_m^0, \text{ where } m \text{ is odd} \} \]

\[ T^\text{even} = \{ H^+ | H \text{ is a cycle block of } G, \text{ and } H^+ \cong T_m^0, \text{ where } m \text{ is even} \}. \]

The reader should notice that if \( H^+ \cong T_m^0 \), then the length of the cycle \( H \) has opposite parity to \( m \).

Each extended proper block \( H^+ \) of \( G \) decomposes into copies of threads \( T_m \) and injured threads \( T_m^- \) as follows. By suppressing every vertex of degree 2 in \( H \) we obtain a 2-connected cubic graph homeomorphic to \( H \), which is denoted \( \bar{H} \) and called the derived graph of \( H \). Each edge \( \bar{e} \in E(\bar{H}) \) corresponds to a maximal induced path \( P \) of positive length in \( H \). By adding to \( P \) those edges in \( B(H) \) (and their endpoints) which are incident to \( P \), we obtain a subgraph \( T_\bar{e} \subseteq H^+ \) which is isomorphic to either a thread \( T_m, m \geq 0 \), or injured thread \( T_m^-, m \geq 1 \). The edge sets of the subgraphs in \( \{ T_\bar{e} | \bar{e} \in E(\bar{H}) \} \) form a partition of \( E(H^+) \).

Summarizing, we have decomposed \( G \) into a family \( T = T^1 \cup T^\text{odd} \cup T^\text{even} \cup \{ T_\bar{e} | \bar{e} \in E(\bar{G}) \} \) of copies of threads, injured threads and closed threads where

\[ \bar{G} = \cup \{ \bar{H} | H \text{ is a proper block of } G \}. \]

The members of \( T \) are called general threads of \( G \), and \( \bar{G} \) is the derived graph of \( G \). An edge \( \bar{e} \in E(\bar{G}) \) is a base edge of \( \bar{G} \) if \( T_\bar{e} \) is isomorphic to an injured thread. Thus each connected component of \( \bar{G} \) other than \( \bar{H}_0 \) contains exactly one base edge. Each \( T \in T \) has zero or more well defined feet, but there are two ways to select which end is the head of \( T \).

Let \( G \) be a connected planar cubic graph where a base block \( H_0 \) has been selected. Let the \( T \) and \( \bar{G} \) be the general thread decomposition and reduced graph as defined above. To describe a weighting of \( G \) it suffices to specify, for each \( T \in T \), which end of \( T \) is the head, and which of the weightings described in Section \[ \] to assign to \( T \). This specification will make reference to a particular perfect matching in the reduced cubic graph \( \bar{G} \). If \( M \subseteq E(\bar{G}) \) is a perfect matching in \( \bar{G} \), then the edge set \( D = E(\bar{G}) - M \) is a 2-factor of \( \bar{G} \). We say that \( D \) is bipartite if every cycle of \( \bar{G} - M \) has even length. Let \( M \subseteq E(\bar{G}) \) be a perfect matching in \( \bar{G} \) satisfying the following properties.
(1) every base edge of $\bar{G}$ is an edge in $M$,
(2) the 2-factor $D = E(\bar{G}) - M$ is bipartite,
(3) subject to conditions (1) and (2), $M$ contains the maximum possible number of edges $\bar{e}$ for which the general thread $T_{\bar{e}}$ is nontrivial (that is, $T_{\bar{e}} \not\sim T_0$).

The matching $M$ exists because every component of $\bar{G}$ has a proper 3-edge colouring (by the Four Colour Theorem), and has at most one base edge. We partition the set of threads $\{T_{\bar{e}} | \bar{e} \in E(\bar{G})\}$ into five classes ($T^M_0$, $T^M_{\geq 1}$, $T^{D}_{\text{odd}}$, $T^{D}_{\text{even}}$, $T^D_0$) where

$T^M_0 = \{T_{\bar{e}} \in T | e_T \in M, T_{\bar{e}} \cong T_0 \text{ is trivial}\}$,
$T^M_{\geq 1} = \{T_{\bar{e}} \in T | e_T \in M, T_{\bar{e}} \text{ is nontrivial}\}$,
$T^{D}_{\text{odd}} = \{T_{\bar{e}} \in T | e_T \in D, \text{ and } T_{\bar{e}} \text{ has odd order}\}$
$T^{D}_{\text{even}} = \{T_{\bar{e}} \in T | e_T \in D, \text{ and } T_{\bar{e}} \text{ has even order at least 2}\}$
$T^D_0 = \{T_{\bar{e}} \in T | e_T \in D, \text{ and } T_{\bar{e}} \cong T_0 \text{ is trivial}\}$

We have defined the following partition of the general threads of $G$ into eight classes.

(4) $T = T^1 \cup T^0_{\text{odd}} \cup T^0_{\text{even}} \cup T^M_0 \cup T^M_{\geq 1} \cup T^D_{\text{odd}} \cup T^D_{\text{even}} \cup T^D_0$.

By the choice of $M$, every injured thread in $T$ belongs to $T^M_{\geq 1} \cup T^1$.

Let $\bar{D}$ be any fixed cyclic orientation of the 2-factor $D = \bar{G} - M$. For each general thread in (4) we arbitrarily designate one of two possible flags to be its head, subject to the following condition.

(5) For every $T = T_{\bar{e}} \in T^D_{\text{odd}}$, the head of $T$ equals the head of $\bar{e} \in \bar{D}$.

We now refer to the thread weightings defined in Section 3. For every subset $S \subseteq T^D_{\text{odd}}$, we define $w_S : E(G) \rightarrow \{0, 1, 2, 3\}$ to be the edge weighting which restricts to every general thread $T \in T$, as follows.

(6) $w_S \mid_{E(T)} = \begin{cases} w_{11} & \text{if } T \in T^M_{\geq 1} \cup T^1 \\ w_{20} & \text{if } T \in S, \\ w_{02} & \text{if } T \in (T^D_{\text{odd}} - S) \cup T^0_{\text{even}} \\ w_2 & \text{if } T \in T^M_0 \cup T^D_0 \cup T^D_{\text{even}} \cup T^D_{\text{odd}}. \end{cases}$

Finally, we define the following set of edge weightings of $G$,

$W = \{w_S : S \subseteq T^D_{\text{odd}}\}$. 
5. Star labellings of $G$

Let $G$ be a planar cubic graph. We designate a proper block of $G$ to be the root block of $G$. We define the extended blocks, and $\bar{G}$, $M$, $\bar{D}$ and $\mathcal{W} = \{w_S : S \subseteq T^D_{odd}\}$ as in Section 4. Let $\Pi_S$ be the set of star labellings of $G$ whose exponent is $w_S$, and let

$$\Pi = \cup \{\Pi_S : S \subseteq T^D_{odd}\}.$$

A base flag of $G$ is any flag that is a foot of some general thread in $G$. Thus every base flag takes the form $v_H e_H$ where $v_H$ is the unique vertex of degree 2 in some nonroot extended block $H^+$, and $e_H$ is the unique cut-edge of $G$ which is incident to $v_H$, and is not an edge of $H^+$.

**Proposition 5.1.** For every star labelling $\pi \in \Pi$ we have $\pi(v_H e_H) = 1$, for every base flag $v_H e_H$ of $G$.

**Proof.** Let $w_S \in \mathcal{W}$ be the exponent of $\pi$. Let $v_H e_H$ be a base flag in $G$, and let $\mathcal{K}$ be the set of blocks $K$ of $G$ for which $e_H$ lies on the unique path from $K$ to the root block of $G$ in the block tree of $G$. Let $L = \bigcup_{K \in \mathcal{K}} K^+$. Every vertex in the subgraph $L$ has degree 3 except for $v_H$, which has degree 2. For every extended block $K^+$ of $G$, the average value of $w_S(e)$ among the edges $e \in E(K^+)$ equals 2. This is because each of the weightings $w_{11}, w_{20}, w_{02}, w_{2}$ has average value 2 in the definition of $w_S$. Since $\{E(K^+) : K \in \mathcal{K}\}$ is a partition of $E(L)$, the average value of $w_S(e)$ among the edges of $L$ equals 2. Therefore the average value of $\pi(ve)$ among the flags in $F(L)$ equals 1. On the other hand, each vertex of $L$ is incident to three flags in $F(L) \cup \{v_H e_H\}$ and contributes $0 + 1 + 2 = 3$ to the value of $\pi(v_H e_H) + \sum \{\pi(ve) : ve \in F(L)\}$. Thus the average value of $\pi(ve)$ among the flags in $F(L) \cup \{v_H e_H\}$ equals 1, and $\pi(v_H e_H) = 1$. \qed

**Corollary 5.2.** Let $S \subseteq T^D_{odd}$. Then every star labelling $\pi \in \Pi_S$, satisfies the following.

1. For every $T \in T^M_{\geq 1} \cup T^1$, we have $\pi \mid_{F(T)} = \pi_{11}$.
2. For every $T \in S$, we have $\pi \mid_{F(T)} = \pi_{20}$.
3. For every $T \in T^D_{odd} - S$, we have $\pi \mid_{F(T)} \in \{\pi_{02}, \pi'_{11}\}$.
4. For every $T \in T^o_{even}$, we have $\pi \mid_{F(T)} = \pi_{02}$.
5. For every $T \in T^D_{even} \cup T^o_{odd}$, we have $\pi \mid_{F(T)} \in \{\rho_{02}, \rho_{20}\}$.
6. For every $T \in T^o_{even} \cup T^D_{odd}$, we have $\pi \mid_{F(T)} \in \{\rho_{02}, \rho_{20}, \rho_{11}\}$.

**Proof.** For any general thread $T \in \mathcal{T}$, the restriction $\pi \mid_{F(T)}$ is 1-footed, by Proposition 5.1. Now all the statements except (4) follow immediately from the definition of $w_S$ and the three
lemmas in Section 3. For the statement (4), we observe that the head and tail of a circular thread in must receive distinct labels in any star labelling of $G$, and the claim follows from Lemma 3.3. □

For every star labelling $\pi$ of $G$, we define a corresponding star labelling $\bar{\pi}$ of $\bar{G}$ called the derived star labelling. Informally, $\bar{\pi}$ is the restriction of $\pi$ to the heads and the tails of the general threads of $G$. More precisely, for each $\bar{e} \in E(\bar{G})$, let $T = T_{\bar{e}}$ be the corresponding general thread of $G$. Let $u$ and $v$ be the endpoints of $\bar{e}$ that correspond to the head and tail of $T$, respectively. The restriction $\pi \upharpoonright F(T)$ corresponds a 1-footed prestar labelling of a thread or injured thread having type $(i, j)$, for some $(i, j) \in \{(1, 1), (2, 0), (0, 2)\}$. We define $\bar{\pi}(ue) = i$ and $\bar{\pi}(ve) = j$.

Let $\pi \in \Pi$. By the definition of $W$, the exponent of $\bar{\pi}$ is the constant function $\bar{w} = 2$, $\bar{w} : E(\bar{G}) \rightarrow \{2\}$. Let $M_{\pi}$ be the set of edges $\bar{e} = uv \in E(\bar{G})$ such that $\bar{\pi}(ue) = \bar{\pi}(ve) = 1$. Let $D_{\pi} = E(\bar{G}) - M_{\pi}$ and let $\vec{D}_{\pi}$ be the orientation of $D_{\pi}$ where, for $e = uv \in \vec{D}_{\pi}$ we have $\bar{\pi}(ue) = 0$ and $\bar{\pi}(ve) = 2$. Then $M_{\pi}$ is a perfect matching of $\bar{G}$, and $\vec{D}_{\pi}$ is an oriented 2-factor of $G$. The correspondence between the star labellings of $\bar{G}$ with exponent 2 and the pairs $(M', \vec{D}')$ where $\vec{D}'$ is an oriented 2-factor of $\bar{G}$ is bijective.

The following will help us later deal with the exceptional star labelling $\pi^{'11}$ that arises in part (3) of Corollary 5.2.

**Lemma 5.3.** Let $S \subseteq T^D_{\text{odd}}$ and let $\pi \in \Pi_S$. Let $D_{\pi}$ be the 2-factor of $\bar{G}$ as defined above. If $G$ has a general thread $T \in T$ for which $\pi \upharpoonright F(T) = \pi^{'11}$, then some connected component of $D_{\pi}$ is an odd cycle in $\bar{G}$.

**Proof.** Let $T$ be as in the statement. By Corollary 5.2 we necessarily have $T \in T^D_{\text{odd}} - S$. Let $T^\pi_{\geq 1}$ be the set of general threads in $T^M \cup T^D$ which are nontrivial and receive a type $(1, 1)$ prestar labelling under $\pi$. By part (1) of Corollary 5.2 we have $T^\pi_{\geq 1} \subseteq T_{\geq 1}$. By the hypothesis, we also have $T \in T^\pi_{\geq 1} \setminus T^M_{\geq 1}$ so $|T^\pi_{\geq 1}| > |T^M_{\geq 1}|$. If the 2-factor $D_{\pi}$ of $\bar{G}$ were bipartite, then the perfect matching $M_{\pi}$ would contradict our choice of $M$. Therefore some component of $D_{\pi}$ is an odd cycle of $\bar{G}$. □

Let $(M, \vec{D})$ be the perfect matching in $\bar{G}$ and the orientation of the complementary 2-factor used in the definition of the set of weightings $W$. Let $\pi^0$ be the unique star labelling of $G$ satisfying

- $\pi^0 \in \Pi_\emptyset$,
- $M_{\pi^0} = M$,
• every closed thread $T \in \mathcal{T}_{\text{odd}}$ receives the labelling $\rho_{02}$, as in (5) of Corollary 5.2.

The star labelling $\pi^0$ is called the reference star labelling of $G$. Accordingly, the reduced star labelling $\pi^0$ is called the reference star labelling of $\bar{G}$. These star labellings will be used for sign computations. It convenient to assume that $\text{sgn}(\pi^0) = \text{sgn}(\bar{\pi}^0) = 1$.

Let $\Pi_0$ be the set of star labellings $\pi \in \Pi$ for which $D_\pi$ is a bipartite 2-factor of $\bar{G}$. In particular, $\pi^0 \in \Pi_0$ because $D_{\pi^0} = D$ is bipartite. The following was proved by Ellingham and Goddyn [4].

**Lemma 5.4.** Let $\bar{\pi}$ be a star labelling of a planar cubic graph $\bar{G}$ with exponent $w \equiv 2$. If $D_{\bar{\pi}}$ is bipartite, then $\text{sgn}(\bar{\pi}) = 1$.

**Corollary 5.5.** Let $\pi \in \Pi_0$. Then $\text{sgn}(\pi) = (-1)^t$, where $t$ is the number of threads $T \in \mathcal{T}^D$ for which $\pi \upharpoonright F(T) = \pi'_1$.

**Proof.** We have $\text{sgn}(\pi) = \text{sgn}(\bar{\pi}) \cdot \prod \left\{ \text{sgn} \left( \pi \upharpoonright F(T) \right) \mid T \in \mathcal{T} \right\}$. The set of prestar labellings of $T \in \mathcal{T}$ whose exponents are determined by a weighting in $\mathcal{W}$ are restricted according to statements (1) to (6) of Corollary 5.2. By Proposition 3.2, both pre star labellings listed in statement (5) have the same sign, whereas the two labellings in statement (3) have opposite sign. In statements (1), (2) and (4), the prestar labelling is fixed, and in (6) the sign is 1 since the threads there are trivial. By Lemma 5.4, $\text{sgn}(\bar{\pi}) = 1$ for $\pi \in \Pi_0$. The result follows from the facts that $t = 0$ for $\pi = \pi^0$, and that $\text{sgn}(\pi^0) = 1$. \hfill \square

Let $\Pi_1 = \Pi - \Pi_0$. We now define a particular function $f$ which maps each member of $\Pi_1$ to another star labelling of $G$. We fix an arbitrary total ordering of the set of odd cycles in $\bar{G}$. For $\pi \in \Pi_1$, let $C$ be the first odd cycle which is a component of $\bar{G}[D_\pi]$. Every general thread $T \in \{ T_e \mid e \in D_\pi \}$ has type $(0,2)$ or type $(2,0)$, so we have $\pi \upharpoonright F(T) \in \{ \pi_{20}, \pi_{02}, \rho_{20}, \rho_{02} \}$, and one of the cases (2), (3), (5) or (6) of Corollary 5.2 applies to $T$. (We used the fact $\pi'_{11}$ has type $(1,1)$.) Let $f(\pi)$ be the star labelling in $\Pi_1$ obtained from $\pi$ as follows. For every $\bar{e} \in E(C)$ we do the following. If $T = T_{\bar{e}}$ is trivial, then we interchange the labels 0 and 2 on its two flags. Otherwise, we relabel the flags of $T$ in a way that interchanges either the prestar labellings $\pi_{20}$ and $\pi_{02}$ (if $T \in \mathcal{T}_{\text{odd}}^D$), or the prestar labellings $\rho_{20}$ and $\rho_{02}$ (if $T \in \mathcal{T}_{\text{even}}^D$). More precisely, for $\{i,j\} = \{0,2\}$, if $\pi \upharpoonright F(T) = \pi_{ij}$, then $f(\pi) \upharpoonright F(T) = \pi_{ji}$, and if $\pi \upharpoonright F(T) = \rho_{ij}$, then $f(\pi) \upharpoonright F(T) = \rho_{ji}$.

**Proposition 5.6.** The map $f$ is a fixed-point free involution $f : \Pi_1 \rightarrow \Pi_1$ which satisfies $\text{sgn}(f(\pi)) = -\text{sgn}(\pi)$. 

Proof. Let \( \pi \in \Pi_1 \) and let \( w_S \) be the exponent of \( \pi \). Then the exponent of \( f(\pi) \) is the weighting \( w_S \in \mathcal{W} \) where \( S' \) is the symmetric difference of \( S \) and \( \{ \bar{e} \in E(C) \mid T_{\bar{e}} \in T^{D}_{\text{odd}} \} \). Therefore we have \( f(\pi) \in \Pi_1 \). Clearly \( f(\pi) \neq \pi \) and \( f(f(\pi)) = \pi \) so \( f \) is a fixed-point free involution on \( \Pi_1 \). For \( v \in V(C) \), the star labellings \( \pi_v \) and \( f(\pi)_v \) differ by the transposition \((02)\), whereas \( \pi_v = f(\pi)_v \) for every \( v \in V(G) \setminus V(C) \). Since \( C \) has odd length, the derived star labellings therefore satisfy \( \text{sgn}(f(\pi)) = -\text{sgn}(\pi) \). The result now follows from Corollaries 5.5 and 3.2.

We note that the oriented 2-factor \( \tilde{D}_{f(\pi)} \) is obtained from \( \tilde{D}_\pi \) by reversing all the arcs in the odd cycle \( C \).

6. The Main Theorem

Proof of Theorem 1.1. Let the set of edge weights \( \mathcal{W} \) be defined as in Section 4 and let \( \Pi = \Pi_0 \cup \Pi_1 \) be the star labellings of \( G \) with exponent in \( \mathcal{W} \), as defined in Section 5. Let \( \pi \in \Pi_0 \). Then \( D_\pi \) is a bipartite 2-factor of \( \tilde{G} \). Applying Lemma 5.3 we conclude that no general thread \( T \) satisfies \( \pi \mid_{F(T)} = \pi'_{11} \). It follows from Corollary 5.5 that \( \text{sgn}(\pi) = 1 \) for every \( \pi \in \Pi_0 \). We have that \( \Pi \neq \emptyset \), since \( \Pi \) contains the reference star labelling \( \pi^0 \). Therefore \( \sum_{\pi \in \Pi_0} \text{sgn}(\pi) > 0 \). We have by Proposition 5.6 that \( \sum_{\pi \in \Pi_1} \text{sgn}(\pi) = 0 \). Thus we have shown that \( \sum_{\pi \in \Pi} \text{sgn}(\pi) > 0 \).

Applying Corollary 2.3 we have that \( G \) is \((w + 1)\)-edge choosable for some \( w = w_S \in \mathcal{W} \). We are interested in the upper bound \( s(G, 3) \leq \lfloor w^{-1}(3) \rfloor \). Suppose \( G \) has \( b \) cut-edges. For any general thread \( T \in \mathcal{T} \), let

\[
m(T) = \begin{cases} 
m & \text{if } T \cong T_m \\
m + 1 & \text{if } T \cong \bar{T}_m \\
3 & \text{if } T = H^+ \text{ and } H \text{ is a vertex block of } G.\end{cases}
\]

Let \((i, j) \in \{(1, 1), (2, 0), (0, 2)\})\). Define \( \mathcal{T}_{ij} = \{ T \in \mathcal{T} \mid w \upharpoonright_{E(T)} = w_{ij} \} \), let \( n_{ij} = |\mathcal{T}_{ij}| \), and let \( m_{ij} = \sum \{ m(T) \mid T \in \mathcal{T}_{ij} \} \). Since every thread in \( \mathcal{T}_{ij} \) has positive length we have \( m_{ij} \geq n_{ij} \).

Let \( e \) be a cut-edge of \( G \). There are exactly two general threads \( T, T' \in \mathcal{T}_{11} \cup \mathcal{T}_{02} \cup \mathcal{T}_{20} \) such that \( e \) joins a vertex of degree \( \geq 2 \) in \( T \) to a vertex of degree \( \geq 2 \) in \( T' \). Therefore each cut-edge \( e \) contributes exactly twice to the quantity \( m_{11} + m_{02} + m_{20} \), so

\[
n_{11} + n_{02} + n_{20} \leq m_{11} + m_{02} + m_{20} = 2b.
\]
Furthermore, at least one of the two contributions of $e$ goes toward $m_{11}$, because the thread in \{T, T'\} that lies farther from the root block $H_0$ is always a member of $\mathcal{T}_{11}$. Therefore $m_{11} \geq m_{02} + m_{20}$.

By examining Figure \[1\] we find that
\[
|w^{-1}(3)| = n_{11} + n_{02} + 2n_{20}.
\]
By comparing (7) and (8), we deduce that $s(G, 3) \leq 2n_{11} + 2n_{02} + 2n_{20} \leq 4b$. To obtain the claimed upper bound of $\frac{5}{2}b$, we must argue more carefully.

Let $w'$ be the edge weighting of $G$ obtained from $w = w_S$ by interchanging the head and tail of every thread $T \in \mathcal{T}^o$ (see \[5\] in Section \[4\], and then swapping the roles of "S" and "(\mathcal{T}^o - S)" in the definition \[6\] of $w_S$. More precisely, the weightings $w$ and $w'$ are identical, except that for every thread $T \in \mathcal{T}^o$, exactly one of the restricted weightings in \{w'|_{E(T)}, w'|_{E(T)}\} coincides with $w_{02}$, and the other coincides with $w_{20}$ after exchanging the head and tail of $T$. Then the coefficients of $x^w$ and $x^{w'}$ in $\varepsilon(G)$ are equal in absolute value.

This is because there is a natural bijection from the star labellings of $G$ with exponent $w$ to those with exponent $w'$. In the bijection, each star labelling $\pi$ with exponent $w$ maps to the unique star labelling $\pi'$ with exponent $w'$ which is identical on all flags outside of any thread in $\mathcal{T}^o$, and for which the reduced star labellings of $\bar{G}$ satisfy $\bar{\pi} = \bar{\pi}'$.

We now compute $s(G, 3) \leq \min(|w^{-1}(3)|, |(w')^{-1}(3)|) \leq \frac{1}{2}(|w^{-1}(3)| + |(w')^{-1}(3)|)$. The analogue of (8) is that $|(w')^{-1}(3)| = n_{11} + 2n_{02} + n_{20}$. We sum these two equations.

$$2s(G, 3) \leq 2n_{11} + 3n_{02} + 3n_{20} \leq 4b + (n_{02} + n_{20}).$$

We apply the inequality just before (8).

$$2(n_{02} + n_{20}) \leq 2(m_{02} + m_{20}) \leq m_{11} + m_{02} + m_{20} = 2b.$$

So $2s(G, 3) \leq 5b$. That is to say, that one of the two functions, $f = w + 1$ or $f = w' + 1$, satisfies the statement of Theorem 1.1.

7. Comments

Our use of the weightings $w_S$ in the definition \[1\] of $w_S$ was included as a mild attempt to minimize the number of edges of $G$ receiving weight 3. In particular, extended cycle blocks of even length, (open) threads of even order, and edges in trivial threads do not require lists of length 4. By inspecting the above inequalities, it is apparent that the upper bound $s(G, 3) \leq \frac{5}{2}b$ can be improved if many threads of $G$ have length > 1, or if $\bar{G}$ has a proper
3-edge colouring in which most of the edges coming from nontrivial threads have the same colour. It would be very interesting to improve the bounds $2b \leq s(\text{planar cubic}, 3) \leq \frac{5}{2}b$, where $s(\text{planar cubic}, 3) = \sup\{ s(G, 3) \mid G \text{ is planar and cubic} \}$. Similar questions can be asked regarding regular planar graphs of higher degree.

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Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada

E-mail address: goddyn@sfu.ca

Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada

E-mail address: ams33@sfu.ca