CORRIGENDA TO
“FLAT RING EPIMORPHISMS OF COUNTABLE TYPE”

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Abstract. We identify, discuss, and correct two mistakes in [1]. The first one is located in [1, Remark 3.3] and slightly affects [1, Lemma 3.6]. The second mistake is in the proofs of [1, Proposition 5.1] and [1, Theorem 5.3] (all the assertions of the proposition and the theorem remain true, but the proofs need to be modified). We also clarify a confusion in [1, Remark 11.3], leading to an improvement of [1, Theorem 11.2]. Finally, we admit that we do not know whether a claim made at the end of [1, Section 2.4] (in the preliminary material) is true in full generality; we prove it under additional assumptions. No other results of [1] are affected.

1. Directed Unions of Gabriel Topologies

There are two assertions in the first paragraph of [1, Remark 3.3] concerning directed unions of topologies of right ideals in an associative ring $R$:

(1) for any nonempty set $\Xi$ of right linear topologies $F$ on $R$ such that for every $F_1, F_2 \in \Xi$ there exists $F \in \Xi$ with $F_1 \cup F_2 \subset F$, the directed union $\bigcup_{F \in \Xi} F$ is a right linear topology on $R$;

(2) in the same context, if $F$ is a right Gabriel topology for every $F \in \Xi$, then $\bigcup_{F \in \Xi} F$ is also a right Gabriel topology on $R$.

The first assertion, concerning right linear topologies, is correct. The second one, concerning Gabriel topologies, is wrong. Nevertheless, the following version of (2) is correct:

(2') in the context of (1), if $F$ is a right Gabriel topology with a base of finitely generated right ideals for every $F \in \Xi$, then $\bigcup_{F \in \Xi} F$ is also a right Gabriel topology with a base of finitely generated right ideals.

Accordingly, the problem with (2) does not affect the second paragraph of [1, Remark 3.3], which remains valid as stated:

(3) in the context of (1), if $F$ is a perfect right Gabriel topology for every $F \in \Xi$, then $\bigcup_{F \in \Xi} F$ is also a perfect right Gabriel topology on $R$.

A discussion of the problem with (2) follows below.

Counterexample 1.1. Let $R$ be a commutative ring and $I \subset R$ be an ideal with a set of generators $s_i \in R$. Denote by $G_I$ the collection of all ideals $J \subset R$ satisfying the following condition: for every $s \in I$ there exists $m \geq 1$ such that $s^m \in J$, or equivalently, for every index $i$ there exists $m \geq 1$ such that $s_i^m \in J$. Then $G_I$ is a
Gabriel topology on $R$. In fact, the Gabriel topology $G_I$ corresponds to the following
torsion class $T_I$ in $R \text{-mod}$: an $R$-module $M$ belongs to $T_I$ if for every $b \in M$ and
$s \in I$ there exists $m \geq 1$ such that $s^m b = 0$ in $M$. Such $R$-modules are called
"$I$-torsion" in [3] Sections 6-7.

Now let $R = k[x_1, x_2, \ldots, y_1, y_2, \ldots]$ denote the ring of polynomials in a countably
infinite number of variables (separated into two countably infinite sorts) over a field $k$.
Let $J_0 = (x_1, x_2, \ldots) \subset R$ denote the ideal generated by the variables $x_i$ in $R$. For
every $n \geq 1$, let $J_n \subset R$ denote the ideal generated by the sequence of elements
$y_1 y_2 \cdots y_n x_i$, $i \geq 1$. Clearly, one has $J_0 \supset J_1 \supset J_2 \supset \cdots$, hence $T_{J_0} \subset T_{J_1} \subset T_{J_2} \subset \cdots \subset R \text{-mod}$ and $G_{J_0} \subset G_{J_1} \subset G_{J_2} \subset \cdots$.

Let $\Xi$ denote the directed set of Gabriel topologies $G_{J_n}$, $n \geq 1$, on the ring $R$, and
let $\mathbb{H} = \bigcup_{n=0}^\infty G_{J_n}$ be the union of $\Xi$. We observe that $\mathbb{H}$ is not a Gabriel topology
on $R$. Indeed, let $I \subset R$ be the ideal generated by the sequence of elements $x_i y_i$, $i \geq 1$. Then $I \notin \mathbb{H}$, since for every $n \geq 0$ there exists $k = n + 1$ such that for every
$m \geq 1$ the element $(y_1 y_2 \cdots y_n x_k)^m$ does not belong to $I$.

Still, we have $J_0 \in G_0 \subset \mathbb{H}$, and for every element $s \in J_0$ the colon ideal $(I : s)$
belongs to $\mathbb{H}$. To check the latter assertion, pick an integer $n \geq 1$ such that $s$ belongs
to the ideal $(x_1, \ldots, x_n) \subset R$. Then $y_1 y_2 \cdots y_n \in (I : s)$ and $(y_1 y_2 \cdots y_n) \supset J_n \in G_{J_n} \subset \mathbb{H}$. So the filter of ideals $\mathbb{H}$ in $R$ does not satisfy (T4).

The following lemma is to be compared with [3] Lemma 3.1.

Lemma 1.2. Let $F$ be a right linear topology on an associative ring $R$, let $I$ and $J \subset R$ be two right ideals, and let $s_j \in R$ be a set of generators of the ideal $J$. Then
one has $(I : s) \in F$ for all $s \in J$ if and only if $(I : s_j) \in F$ for every generator $s_j$.

Proof. Suppose $s = s_1 r_1 + \cdots + s_m r_m$ with $r_i \in R$ and $s_i \in J$. Set $K_i = (I : s_i)$ and $H = (K_1 : r_1) \cap \cdots \cap (K_m : r_m) \subset R$. Then $s H \subset s_1 K_1 + \cdots + s_m K_m \subset I$. Assume that $K_i \in F$ for every $i = 1, \ldots, m$; then $(K_i : r_i) \in F$ by (T3), hence $H \in F$ by (T2). Since $H \subset (I : s)$, it follows that $(I : s) \in F$ by (T1).

Let $\lambda$ be an infinite cardinal. A poset $\Xi$ is said to be $\lambda$-directed if for any its subset $\Upsilon \subset \Xi$ of the cardinality less than $\lambda$ there exists an element $\xi \in \Xi$ such that $\xi \geq \upsilon$ for all $\upsilon \in \Upsilon$.

Corollary 1.3. Let $R$ be an associative ring, $\lambda$ be an infinite cardinal, and $\Xi$ be
a $\lambda$-directed (by inclusion) set of right Gabriel topologies on $R$. Assume that every
$G \in \Xi$ has a base consisting of right ideals with less than $\lambda$ generators. Then $\mathbb{H} = \bigcup_{G \in \Xi} G$ is a right Gabriel topology on $R$ (with a base consisting of right ideals with
less than $\lambda$ generators).

Proof. To check that $\mathbb{H}$ satisfies (T4), consider a right ideal $I \subset R$ and a right ideal
$J \subset \mathbb{H}$. Then there exist $G_0 \in \Xi$ such that $J \in G_0$, and $J' \subset J$ such that $J'$ has less
than $\lambda$ generators $s_j$ and $J' \in G_0$. Assume that $(I : s) \in \mathbb{H}$ for every $s \in J$. Then
there exist $G_j \in \Xi$ such that $(I : s_j) \in G_j$ for every $j$. Since $\Xi$ is $\lambda$-directed, there is
$G \in \Xi$ such that $G_0 \subset G$ and $G_j \subset G$ for every $j$. Hence $J' \in G$ and $(I : s_j) \in G$
for every $j$. By Lemma 1.2, it follows that $(I : s) \in G$ for every $s \in J'$. Since $G$ is a Gabriel topology, we can conclude that $I \in G \subset H$. □

Specializing to the case of the countable cardinal $\lambda = \omega$, we see from Corollary 1.3 that the assertion $(2_f)$ is correct.

The problem with $(2)$ (demonstrated in Counterexample 1.1) slightly affects [1, Lemma 3.6], which should be restated as follows.

**Lemma 1.4** (corrected version of [1, Lemma 3.6]). Let $R$ be an associative ring, $\Xi$ be a directed set of right Gabriel topologies on $R$, and $G = \bigcup_{\mathbb{H} \in \Xi} \mathbb{H}$ be their union. Assume that $G$ is a tight Gabriel topology on $R$ (e.g., this always holds when every Gabriel topology $\mathbb{H} \in \Xi$ has a base of finitely generated right ideals, or more generally when $\Xi$ is $\lambda$-directed and every $\mathbb{H} \in \Xi$ has a base consisting of right ideals with less than $\lambda$ generators).

Let $N$ be a right $R$-module such that $t_{\mathbb{H}}(N) = t_{G}(N)$ for all $\mathbb{H} \in \Xi$. Then there is a natural isomorphism of right $R$-modules $N_{G} \simeq \lim_{\mathbb{H} \in \Xi} N_{\mathbb{H}}$. □

Accordingly, the assumption that $G = \bigcup_{\mathbb{H} \in \Xi} \mathbb{H}$ is a Gabriel topology is also needed in [1, Remark 3.7].

In the proof of [1, Proposition 3.9], we sometimes refrained from mentioning that the right Gabriel topologies we were dealing with had to have a base of finitely generated right ideals. In view of the problem with $(2)$, this needs to be mentioned throughout the proof of [1, Proposition 3.9]. With this correction in mind, the proof is valid, and the assertion of [1, Proposition 3.9] remains unaffected. All the other results of [1, Section 3] are likewise unaffected.

### 2. Additive Kan Extensions

Let $R$ be an associative ring and $F$ be a right linear topology on $R$. Let $Q_{F}$ be the full subcategory of cyclic discrete right modules $R/I$, $I \in F$, in the category of discrete right $R$-modules $\text{discr}-R$. Then

1. any left exact additive functor $M : Q_{F}^{\text{op}} \to \text{Ab}$ (in the sense of [1, Section 5]) can be extended to a functor $G_{M} : (\text{discr}-R)^{\text{op}} \to \text{Ab}$ taking colimits in $\text{discr}-R$ to limits in $\text{Ab}$;

2. any right exact additive functor $C : Q_{F} \to \text{Ab}$ (in the sense of [1, Section 5]) can be extended to a functor $F_{C} : \text{discr}-R \to \text{Ab}$ preserving colimits.

The assertion (1) is used in the proof (or rather, in one of the proofs) of [1, Proposition 5.1]. The assertion (2) is used in one of the proofs of [1, Theorem 5.3].

Both the assertions (1) and (2) are correct. However, the constructions of the functors $G_{M}$ and $F_{C}$ given in [1, proofs of Proposition 5.1 and Theorem 5.3] are wrong. They need to be modified as explained below. This problem is closely related to [2, Example 1.24 (4)].

For any category $A$, a small full subcategory $Q \subset A$, a complete category $B$, and a functor $F : Q \to B$, the functor $F$ can be extended to a functor $\bar{F} : A \to B$ using
the construction of the Kan extension \[5\] Section X. By the definition, for any object \(N \in A\), we put
\[
\widetilde{F}(N) = \lim_{Q \to N} F(Q)
\]
where the colimit is taken over the diagram whose vertices are all the morphisms \(Q \to N\) in \(A\) with \(Q \in Q\) and arrows are the commutative triangles \(Q' \to Q'' \to N\) in \(A\) with \(Q', Q'' \in Q\).

Now suppose that \(A\) and \(B\) are additive categories (so \(Q\) is a preadditive category), and \(F\) is an additive functor. Then the functor \(\widetilde{F}\) does not need to be additive, as the following example demonstrates.

**Counterexample 2.1.** Let \(R = k\) be a field endowed with the discrete topology \(F = \{(0), (1)\}\). Let \(A = \text{discr} - R\) be the category of \(k\)-vector spaces, \(B = \text{Ab}\) be the category of abelian groups, and \(Q = Q_k \subset A\) be the full subcategory of vector spaces of dimension \(\leq 1\) (cf. \[2\] Example 1.24 (4)).

Let \(F: Q \to B\) be the forgetful functor assigning to a vector space its underlying abelian group. Then the functor \(\widetilde{F}: A \to B\) assigns to a vector space \(V\) the underlying abelian group of the \(k\)-vector space with a basis indexed by the set of all one-dimensional vector subspaces in \(V\). So the functor \(\widetilde{F}\) is not additive (and not isomorphic to the forgetful functor \(A \to B\)).

The above example shows that the construction of the functor \(F_{\mathcal{C}}\) in \[1\] proof of Theorem 5.3] is wrong (in that it does not produce a coproduct-preserving or right exact functor, contrary to what is claimed). The construction of the functor \(G_M\) in \[1\] proof of Proposition 5.1] is wrong for the same reason. The correct constructions are explained below.

**Lemma 2.2.** Let \(A\) and \(B\) be additive categories and \(Q \subset A\) be a full subcategory. Assume for simplicity that \(0 \in Q\), and denote by \(Q^+ \subset A\) the full subcategory consisting of all the objects \(Q_1 \oplus Q_2\), where \(Q_1, Q_2 \in Q\). Let \(F: Q \to B\) be an additive functor. Then there exists a unique additive functor \(F^+: Q^+ \to B\) such that \(F^+|_Q = F\), defined by the rule \(F^+(Q_1 \oplus Q_2) = F(Q_1) \oplus F(Q_2)\).

**Proposition 2.3.** Let \(A\) be an additive category, \(0 \in Q \subset A\) be a small full subcategory, \(B\) be a cocomplete additive category, and \(F: Q \to B\) be an additive functor. Let \(\widetilde{F}^+: A \to B\) be the Kan extension of the functor \(F^+: Q^+ \to B\). Then the functor \(\widetilde{F}^+\) is additive.

**Proof.** Let \(M\) and \(N\) be two objects, and let \(f, g: M \to N\) be a pair of parallel morphisms. We have to check that \(\widetilde{F}^+(f + g) = \widetilde{F}^+(f) + \widetilde{F}^+(g)\).

The object \(\widetilde{F}^+(M) \in B\) is the colimit of the objects \(F(Q_1) \oplus F(Q_2)\) taken over the diagram whose vertices are all the morphisms \(Q_1 \oplus Q_2 \to M\) with \(Q_1, Q_2 \in Q\) and whose arrows are all the commutative triangles \(Q_1' \oplus Q_2' \to Q_1'' \oplus Q_2'' \to M\), where \(Q_1' \oplus Q_2' \to Q_1'' \oplus Q_2''\) is a \((2 \times 2)\)-matrix of morphisms between objects of \(Q\). Therefore, it suffices to check that for every morphism \(h: Q_1 \oplus Q_2 \to M\) with \(Q_1, Q_2 \in Q\) one has \(\widetilde{F}^+((f + g) \circ h) = \widetilde{F}^+(f \circ h) + \widetilde{F}^+(g \circ h)\).
In turn, it suffices to check the latter condition in the case of a pair of objects \((Q_1, Q_2) = (Q, 0)\), i.e., for a morphism \(h: Q \to M\) with \(Q \in Q\). Now we consider the object \((Q, Q) \in Q^+\) and the morphism \((f \circ h, g \circ h): (Q, Q) \to N\) in \(A\). There are three natural morphisms \((1, 0), (0, 1),\) and \((1, 1): Q \to (Q, Q)\) in \(Q^+\). One has \(f \circ h = (f \circ h, g \circ h) \circ (1, 0)\), and similarly \(g \circ h = (f \circ h, g \circ h) \circ (0, 1)\) and \((f + g) \circ h = (f \circ h, g \circ h) \circ (1, 1)\).

By construction of the object \(\bar{F}^+(N)\), it follows that \(\bar{F}^+(f \circ h) = \bar{F}^+(f \circ h, g \circ h) \circ F^+(1, 0)\), and similarly \(\bar{F}^+(g \circ h) = \bar{F}^+(f \circ h, g \circ h) \circ F^+(0, 1)\) and \(\bar{F}^+((f + g) \circ h) = \bar{F}^+(f \circ h, g \circ h) \circ F^+(1, 1)\). It remains to observe that \(F^+(1, 1) = F^+(1, 0) + F^+(0, 1)\), since the functor \(F^+\) is additive.

The following lemma does not depend on any additivity assumptions.

**Lemma 2.4.** Let \(A\) be a category, \(Q \subset A\) be a small full subcategory, \(B\) be a cocomplete category, \(F: Q \to B\) be a functor, \(X\) be a small category, and \(D: X \to A\) be a diagram which has a colimit in \(A\). Assume that for every object \(Q \in Q\) the functor \(\text{Hom}_A(Q, \_): A \to \text{Sets}\) preserves the colimit of the diagram \(D\). Then the functor \(\bar{F}: A \to B\) also preserves the colimit of the diagram \(D\).

**Proof.** Let \(B \in B\) be an object. Then, for every object \(A \in A\), the set of all morphisms \(\bar{F}(A) \to B\) in \(B\) is naturally bijective to the set of all rules assigning to every morphism \(Q \to A\) in \(A\) with an object \(Q \in Q\) a morphism \(F(Q) \to B\) in \(B\) in a way compatible with all the morphisms in \(Q\). It follows that, under the assumptions of the lemma, both the sets of morphisms \(\text{Hom}_B(\lim_{x \in X} F(D(x)), B)\) and \(\text{Hom}_B(F(\lim_{x \in X} D(x)), B)\) are naturally bijective to the set of all rules assigning to every morphism \(Q \to D(x)\) in \(A\) with objects \(Q \in Q\) and \(x \in X\) a morphism \(F(Q) \to B\) in \(B\) in a way compatible with all the morphisms in \(Q\) and \(X\).

Alternatively, one can argue as follows. Consider the category \(C = \text{Sets}^{\text{op}}\) of presheaves of sets on the small category \(Q\). By the Yoneda lemma, \(Q\) is naturally a full subcategory in \(C\). There is a natural functor \(H: A \to C\) assigning to an object \(A \in A\) the presheaf \(H(A) = \text{Hom}_A(\_, A)|_Q\). The functor \(H\) forms a commutative triangle diagram with the fully faithful functors \(Q \to A\) and \(Q \to C\). By assumption, the functor \(H\) preserves the colimit of the diagram \(D\).

Let \(G: C \to B\) be the Kan extension of the functor \(F: Q \to B\) with respect to the Yoneda embedding \(Q \to C\). The functor \(G\) has a right adjoint functor \(R\) assigning to an object \(B \in B\) the presheaf \(R(B) = \text{Hom}_B(F(\_), B)\). It follows that the functor \(G\) preserves all colimits. Finally, it remains to observe that \(\bar{F} = G \circ H\).

Now we can return to the situation at hand. Let \(F\) be a right linear topology on an associative ring \(R\). Set \(A = \text{discr} \cdot R\), \(Q = Q_0 \subset A\), and \(B = \text{Ab}\) or \(\text{Ab}^{\text{op}}\).

Given an additive functor \(M: Q_0 \to \text{Ab}^{\text{op}}\), we set

\[
G_M(N) = \lim_{R/I_1 \oplus R/I_2 \to N} M(R/I_1) \oplus M(R/I_2) \quad \text{for every } N \in \text{discr} \cdot R,
\]

where the projective limit is taken over the diagram formed by all the morphisms of discrete right \(R\)-modules \(R/I_1 \oplus R/I_2 \to N\) (indexing the vertices of the diagram).
and all the commutative triangles $R/I_1 \oplus R/I_2 \rightarrow R/J_1 \oplus R/J_2 \rightarrow N$ (indexing the arrows), where $I_1, I_2, J_1, J_2 \in \mathcal{F}$ and $R/I_1 \oplus R/I_2 \rightarrow R/J_1 \oplus R/J_2$ ranges over all the $(2 \times 2)$-matrices of morphisms in $Q_{\mathcal{F}}$.

By Proposition 2.3, the functor $G_M = M^+: \text{discr}-R \rightarrow \text{Ab}^{\text{op}}$ is additive; hence it preserves finite (co)products. Applying Lemma 2.4 for the diagram representing a coproduct in $\text{discr}-R$ as the filtered colimit of its finite subcoproducts and recalling that the right $R$-modules $R/I_1 \oplus R/I_2$ are finitely generated, we conclude that the functor $G_M$ takes coproducts in $\text{discr}-R$ to products in $\text{Ab}$.

Checking that the functor $G_M$ takes cokernels in $\text{discr}-R$ to kernels in $\text{Ab}$ if and only if the functor $M$ is left exact in the sense of [1, Section 5] is a straightforward diagram-chasing exercise. This proves the assertion (1).

Given an additive functor $C: Q_{\mathcal{F}} \rightarrow \text{Ab}$, we set $F_C(N) = \lim_{\rightarrow R/I_1 \oplus R/I_2 \rightarrow N} C(R/I_1) \oplus C(R/I_2)$ for every $N \in \text{discr}-R$, where the inductive limit is taken over the same diagram as in the previous construction. By Proposition 2.3, the functor $F_C = C^+: \text{discr}-R \rightarrow \text{Ab}$ is additive; hence it preserves infinite coproducts. The same argument based on Lemma 2.4 as above shows that the functor $F_C$ preserves infinite coproducts, too. Finally, a diagram-chasing exercise shown that the functor $F_C$ preserves cokernels if and only if the functor $C$ is right exact in the sense of [1, Section 5]. This proves the assertion (2).

3. Injective Flat Ring Epimorphisms of Projective Dimension 1

The exposition in [1, Remark 11.3] is correct, but confused. Particularly confused is the fourth, last paragraph of that remark (while the first two paragraphs are OK). Let us discuss the situation anew and with updated references.

Let $u: R \rightarrow U$ be an injective ring epimorphism. Following [1, first paragraph of Remark 11.3], we consider the $R$-$R$-bimodule $K = U/R$, and denote by $\mathcal{G} = \text{Hom}_R(K, K)^{\text{op}}$ the opposite ring to the ring of endomorphisms of the left $R$-module $K$. So the ring $\mathcal{G}$ acts in $K$ on the right, making $K$ an $R$-$\mathcal{G}$-bimodule; and the right action of $R$ in $K$ induces a ring homomorphism $R \rightarrow \mathcal{G}$. We endow the ring $\mathcal{G}$ with the right linear topology $\mathfrak{G}$ with a base $\mathfrak{B}$ formed by the annihilators of finitely generated left $R$-submodules in $K$. Then $\mathcal{G}$ is a complete, separated topological ring [8, Theorem 7.1] and $K$ is a discrete right $\mathcal{G}$-module [8, Proposition 7.3] (see also [7, Section 1.13]). The topological ring $\mathcal{G}$ is discussed at length in [3, Section 4] and [4, Sections 14–15] (where it is denoted by $\mathfrak{N}$).

Assume that $U$ is a flat left $R$-module. Following [1, second paragraph of Remark 11.3], let $\mathcal{G}$ be the perfect Gabriel topology of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$ (or equivalently, $U = IU$). Let $\mathfrak{R}$ denote the completion of $R$ with respect to $\mathcal{G}$, viewed as a complete, separated topological ring in its projective limit topology $\mathcal{G}$; and let $\rho: R \rightarrow \mathfrak{R}$ be the completion map. Then $K = U/R$ is a discrete right $R$-module (with respect to the topology $\mathcal{G}$ on $R$), since $U/R \otimes_R U = 0$. Consequently, the right action of $R$ in $K$ extends uniquely to a discrete right action
of $\mathcal{R}$. Hence the ring homomorphism $R \to \mathcal{G}$ factorizes as $R \to \mathcal{R} \to \mathcal{G}$. Since the annihilator of any finite set of elements in $K$ is an open right ideal in $\mathcal{R}$, the ring homomorphism $\sigma: \mathcal{R} \to \mathcal{G}$ is continuous.

**Lemma 3.1.** Any open right ideal $I \in \mathcal{G}$ of the ring $R$ contains the preimage $(\sigma \rho)^{-1}(\mathfrak{J}) \subset R$ of some open right ideal $\mathfrak{J} \in \mathfrak{F}$ of the ring $\mathcal{G}$.

**Proof.** We know that $U = IU$, that is, there exist some elements $s_1, \ldots, s_n \in I$ and $v_1, \ldots, v_n \in U$ such that $\sum_{k=1}^{n} s_k v_k = 1$ in the ring $U$. Denote by $\mathfrak{J} \subset \mathcal{G}$ the annihilator of the finite collection of cosets $v_1 + R, \ldots, v_n + R \in U/R = K$. Then $\mathfrak{J}$ is an open right ideal in $\mathcal{G}$ and the definition of the topology $\mathfrak{F}$ on $\mathcal{G}$.

The ring homomorphism $\sigma \rho: R \to \mathcal{G}$ is induced by the right action of $R$ on $U/R$. Let $r \in R$ be an element such that $\sigma \rho(r) \in \mathfrak{J}$. This means that the cosets $v_1 + R, \ldots, v_n + R$ are annihilated by the right action of $r$, that is, the elements $v_1 r, \ldots, v_n r$ belong to $R \subset U$. Hence we have $r = \sum_{k=1}^{n} s_k v_k r \in \sum_{k=1}^{n} s_k R \subset I$, as desired. □

**Corollary 3.2.** For any injective left flat ring epimorphism $u: R \to U$, the ring homomorphism $\sigma: \mathcal{R} \to \mathcal{G}$ is injective, and the topology $\mathcal{G}$ on $\mathcal{R}$ coincides with the topology induced on the ring $\mathcal{R}$ as a subring in $\mathcal{G}$ via $\sigma$.

**Proof.** The preimages of open right ideals $\mathfrak{J} \in \mathfrak{F}$ of the ring $\mathcal{G}$ under the map $\sigma \rho: R \to \mathcal{G}$ are open right ideals in $R$, because the ring homomorphism $\sigma \rho$ is continuous. The open ideals $(\sigma \rho)^{-1}(\mathfrak{J}) \subset R$ also form a base of the topology $\mathcal{G}$ on the ring $R$ by Lemma 3.1. Passing to the projective limit of the directed diagram of injective maps $R/(\sigma \rho)^{-1}(\mathfrak{J}) \to \mathcal{G}/\mathfrak{J}$ indexed by $\mathfrak{J} \in \mathfrak{F}$, we obtain the map $\sigma: \mathcal{R} \to \mathcal{G}$. Thus the map $\sigma$ is injective.

The topology $\mathcal{G}$ on $\mathcal{R}$ is the topology of projective limit of discrete abelian groups $\lim_{\mathfrak{J} \in \mathfrak{F}} R/(\sigma \rho)^{-1}(\mathfrak{J})$, while the topology $\mathfrak{F}$ on $\mathcal{G}$ is the topology of projective limit of discrete abelian groups $\lim_{\mathfrak{J} \in \mathfrak{F}} \mathcal{G}/\mathfrak{J}$. The map $\sigma$ is the projective limit of an injective morphism from the former diagram of abelian groups to the latter one. By the definition of the topology of projective limit, it follows that the topology on $\mathcal{R}$ is induced from its embedding into $\mathcal{G}$ via $\sigma$. □

The following theorem is a clarified version of [1, Remark 11.3], and it also includes an improved version of [1, Theorem 11.2].

**Theorem 3.3.** Let $u: R \to U$ be an injective ring epimorphism such that $U$ is a flat left $R$-module of projective dimension not exceeding 1. Then the map $\sigma: \mathcal{R} \to \mathcal{G}$ is an isomorphism of topological rings. Furthermore, the left $R$-module morphism

$$\beta_{u, X}: \Delta_u (R[X]) \to \mathcal{R}[[X]]$$

is an isomorphism for any set $X$. The forgetful functor $\mathcal{R}\textendash\text{contra} \to R\textendash\text{mod}$ is fully faithful, and its essential image coincides with the full subcategory $R\textendash\text{mod}_{u\text{extra}} \subset R\textendash\text{mod}$; so there is an equivalence of abelian categories $\mathcal{R}\textendash\text{contra} \simeq R\textendash\text{mod}_{u\text{extra}}$.

**Proof.** We follow [1, second and third paragraphs of Remark 11.3] taking into account the discussion above. The injective topological ring homomorphism $\sigma: \mathcal{R} \to \mathcal{G}$ (see
Corollary 3.2 induces, for every set \( X \), an injective map of sets \( \sigma[[X]] : \mathcal{R}[[X]] \rightarrow \mathcal{S}[[X]] \). In fact, we have a commutative triangle diagram of ring homomorphisms \( R \rightarrow \mathcal{R} \rightarrow \mathcal{S} \), so \( \sigma[[X]] \) is an \( R \)-module morphism.

Every left \( \mathcal{S} \)-contamodule has an underlying left \( \mathcal{R} \)-contramodule structure. By [1, Proposition 10.2], the underlying left \( \mathcal{R} \)-module of any left \( \mathcal{R} \)-contramodule is a \( u \)-contramodule. In particular, \( \mathcal{S}[[X]] \) is a \( u \)-contramodule left \( \mathcal{R} \)-module. Hence, by [1, Lemma 10.1], there exists a unique \( \mathcal{R} \)-module morphism \( \Delta_u(R[[X]]) \rightarrow \mathcal{S}[[X]] \) forming a commutative triangle diagram with the adjunction map \( \delta_{u,R[[X]]} : R[[X]] \rightarrow \Delta_u(R[[X]]) \) and the map \( R[[X]] \rightarrow \mathcal{S}[[X]] \) induced by the ring homomorphism \( \sigma \rho : R \rightarrow \mathcal{S} \). The composition of two maps \( \sigma[[X]] \circ \beta_{u,X} : \Delta_u(R[[X]]) \rightarrow \mathcal{S}[[X]] \) has this diagram commutativity property. So does the isomorphism \( \Delta_u(R[[X]]) \simeq \mathcal{S}[[X]] \) constructed in [4, direct proof of Theorem 14.2] (we remind the reader that the topological ring denoted by \( \mathcal{S} \) here is denoted by \( \mathcal{R} \) in [4]). Thus the composition \( \sigma[[X]] \circ \beta_{u,X} \) is isomorphism.

Since the map \( \sigma[[X]] \) is injective, it follows that both the maps \( \beta_{u,X} \) and \( \sigma[[X]] \) are isomorphisms. In particular, the map \( \sigma \) is an isomorphism of rings, and in view of Corollary 3.2 we can conclude that \( \sigma \) is an isomorphism of topological rings. We have proved the first two assertions of the theorem. The remaining assertions follow in view of [1, Lemma 11.1] or [4, Theorem 14.2].

4. Completions of Gabriel Topologies are Gabriel?

Let \( R \) be a topological ring with a right linear topology \( \mathcal{G} \), and let \( \mathcal{R} = \lim_{\longleftarrow I \in \mathcal{G}} R/I \) be the completion of \( R \), viewed as a topological ring in the projective limit topology \( \mathcal{G} \). It is claimed in [1, final paragraph of Section 2.4] (in the preliminary material to the paper [1]) that if \( \mathcal{G} \) is a Gabriel topology on \( R \), then \( \mathcal{G} \) is a Gabriel topology on \( \mathcal{R} \).

This claim is never used in the main part of the paper, and was only included in the preliminaries for the sake of completeness of the exposition. In fact, we do not know whether this claim is true.

Let us have a little discussion of what this claim entails. A right linear topology on a ring is Gabriel if and only if the full subcategory of discrete right modules is closed under extensions in the category of all right modules. Let \( 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \) be a short exact sequence of right \( \mathcal{R} \)-modules, where the \( \mathcal{R} \)-modules \( N' \) and \( N'' \) are discrete. Restricting the scalars, we can view this sequence as a short exact sequence of right \( R \)-modules. Then \( N' \) and \( N'' \) are discrete right \( R \)-modules. Since \( \mathcal{G} \) is a Gabriel topology by assumption, it follows that \( N \) is a discrete right \( R \)-module. But how can one conclude that \( N \) is a discrete right \( \mathcal{R} \)-module?

Let \( b \in N \) be an element, and let \( \rho : R \rightarrow \mathcal{R} \) denote the completion map. The annihilator of \( b \) in \( \mathcal{R} \) is a right ideal \( \mathcal{I} \) whose full preimage \( I = \rho^{-1}(\mathcal{I}) \) is an open right ideal in \( R \). If we knew that \( \mathcal{I} \) is closed in the topology of \( \mathcal{R} \), it would follow that \( \mathcal{I} \) is open in \( \mathcal{R} \). But does \( \mathcal{I} \) need to be closed?

Alternatively, in order to show that \( \mathcal{G} \) is Gabriel it suffices to check condition (T4) for the ring \( \mathcal{R} \). Let \( \mathcal{I} \subset \mathcal{R} \) be a right ideal and \( \mathcal{J} \subset \mathcal{G} \) be an open right ideal in \( \mathcal{R} \).
Suppose that \((J : s) \in \mathfrak{G}\) for all \(s \in \mathfrak{J}\). How does one show that \(\mathfrak{J} \in \mathfrak{G}\)? What is the technology for proving openness or closedness of a given right ideal in \(\mathfrak{R}\), based on whatever knowledge of right ideals in \(R\)?

The converse implication is also interesting and unclear. Let \(\mathfrak{R}\) be the completion of \(R\) with respect to a right linear topology \(\mathfrak{G}\) (as above). Suppose that \(\mathfrak{G}\) is a Gabriel topology on \(\mathfrak{R}\). Does it follow that \(\mathfrak{G}\) is a Gabriel topology on \(R\)?

Let \(0 \to N' \to N \to N'' \to 0\) be a short exact sequence of right \(R\)-modules, where the \(R\)-modules \(N', N'', R\) are discrete. Is the \(R\)-module \(N\) discrete in this case? The discrete \(R\)-module structures on \(N', N''\) can be uniquely extended to structures of discrete \(\mathfrak{R}\)-modules. If one could extend the action of \(R\) on \(N\) by a finite set of elements \(t \in \mathfrak{G}\), then it would follow that \(N\) is a discrete right \(R\)-module (since the topology \(\mathfrak{G}\) is assumed to be Gabriel); hence \(N\) is also a discrete right \(R\)-module. But how does one extend the action of \(R\) on \(N\) to an action of \(\mathfrak{R}\)?

For example, if \(N = R/I\) is a cyclic right \(R\)-module, one could consider the extension \(J = \rho(I)R \subset R\) of the right ideal \(I \subset R\) to the ring \(\mathfrak{R}\). But the equality \(\rho^{-1}(J) = I\) need not hold for an arbitrary right ideal \(I \subset R\), generally speaking. For example, if \(R = \mathbb{Z}\) is the ring of integers and \(\mathfrak{G}\) is the \(p\)-adic topology on \(R\) (where \(p\) is a prime number), so \(\mathfrak{R} = \mathbb{Z}_p\) is the ring of \(p\)-adic integers, then for any integer \(m\) not divisible by \(p\) and for the ideal \(I = (m)\) one has \(J = \mathfrak{R}\).

Here is a result which we can actually prove.

**Lemma 4.1.** Let \(R\) be a topological ring with a right Gabriel topology \(\mathfrak{G}\) having a countable base consisting of finitely generated right ideals. Let \(\mathfrak{R} = \varprojlim_{i \in \mathbb{G}} R/I\) be the completion of \(R\) with respect to \(\mathfrak{G}\), and let \(\mathfrak{G}\) be the projective limit topology on \(\mathfrak{R}\). Then \(\mathfrak{G}\) is a right Gabriel topology with a countable base consisting of finitely generated right ideals in \(\mathfrak{R}\).

**Proof.** First of all, there is a natural order isomorphism between the posets \(\mathfrak{G}\) and \(\mathfrak{G}\) (with respect to the inverse inclusion), as per the discussion in [1 Section 4]. In particular, this bijection takes any countable base \(\mathfrak{B}\) of the topology \(\mathfrak{G}\) to a countable base \(\mathfrak{B}\) of the topology \(\mathfrak{G}\). Furthermore, let \(I \in \mathfrak{G}\) be an open right ideal generated by a finite set of elements \(s_1, \ldots, s_m \in R\). Let \(J \in \mathfrak{B}\) be the open right ideal corresponding to \(I\). By [1 Theorem 6.6], the right ideal \(J\) is strongly finitely generated. This means, in particular (taking the singleton set of indices \(X = \{s\}\)), that \(I\mathfrak{R} = \mathfrak{J}\); so the ideal \(J\) is generated by the finite set of elements \(\rho(s_1), \ldots, \rho(s_m) \in \mathfrak{R}\).

In order to show that \(\mathfrak{G}\) is a Gabriel topology, let us use [1 Lemma 3.1]. We have to check that the condition \(\text{T4}'\) holds for the topology \(\mathfrak{G}\) on \(\mathfrak{R}\), that is, for any open right ideal \(\mathfrak{J} \in \mathfrak{B}\) the right ideal \(\rho(s_1)\mathfrak{J} + \cdots + \rho(s_m)\mathfrak{J}\) is open in \(\mathfrak{R}\). Let \(J \in \mathfrak{B}\) be the open right ideal in \(R\) corresponding to \(\mathfrak{J}\), and let \(t_1, \ldots, t_n \in R\) be its finite set of generators. Since \(\mathfrak{G}\) is assumed to be a right Gabriel topology on \(R\), by the other implication in [1 Lemma 3.1] we know that \(K = s_1J + \cdots + s_mJ\) is an open right ideal in \(R\). Clearly, \(K\) is also a finitely generated right ideal, with the set of generators \((s_1t_j)_{1 \leq j \leq m, 1 \leq i \leq n}\). Then the argument in the previous paragraph tells that the right ideal generated by the finite set of elements \((\rho(s_jt_i))_{1 \leq j \leq m, 1 \leq i \leq n}\) is open in \(\mathfrak{R}\). Thus \(\rho(s_1)\mathfrak{J} + \cdots + \rho(s_m)\mathfrak{J}\) is an open right ideal in \(\mathfrak{R}\). \(\square\)
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FLAT RING EPIMORPHISMS OF COUNTABLE TYPE
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Abstract. Let \( R \rightarrow U \) be an associative ring epimorphism such that \( U \) is a flat left \( R \)-module. Assume that the related Gabriel topology \( \mathcal{G} \) of right ideals in \( R \) has a countable base. Then we show that the left \( R \)-module \( U \) has projective dimension at most 1. Furthermore, the abelian category of left contramodules over the completion of \( R \) at \( \mathcal{G} \) fully faithfully embeds into the Geigle–Lenzing right perpendicular subcategory to \( U \) in the category of left \( R \)-modules, and every object of the latter abelian category is an extension of two objects of the former one. We discuss conditions under which the two abelian categories are equivalent. Given a right linear topology on an associative ring \( R \), we consider the induced topology on every left \( R \)-module, and for a perfect Gabriel topology \( \mathcal{G} \) compare the completion of a module with an appropriate \( \text{Ext} \) module. Finally, we characterize the \( U \)-strongly flat left \( R \)-modules by the two conditions of left positive-degree \( \text{Ext} \)-orthogonality to all left \( U \)-modules and all \( \mathcal{G} \)-separated \( \mathcal{G} \)-complete left \( R \)-modules.

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1. Introduction

1.0. Ring epimorphisms and Gabriel topologies are a popular subject of contemporary research in associative and commutative ring theory, where nontrivial work is being done \([3, 15, 1, 10, 2]\). To be more precise, one has to say that these are two overlapping, but different subjects: \textit{perfect} right Gabriel topologies correspond
bijectively to left flat ring epimorphisms. In this paper, we are mostly dealing with this overlap, imposing additional conditions as the need arises.

The most important of such additional conditions is the one of countable type. A left flat ring epimorphism $u: R \rightarrow U$ is said to be of countable type if the related Gabriel topology $G$ of right ideals in $R$ has a countable base. In Section 3, we discuss how ubiquitous (perfect) Gabriel topologies of countable type are among (perfect) Gabriel topologies in general; in Sections 6 and 8–10 we show what one can do with (mostly perfect) Gabriel topologies of countable type; and in the final Sections 10–11 we combine these results, proving theorems about completions and contramodules related to some perfect Gabriel topologies of uncountable type.

Another condition on which many of our results depend is that, even when a right Gabriel topology on an associative ring $R$ is not perfect, we often need it to have a base consisting of finitely generated right ideals. For perfect Gabriel topologies, this holds automatically.

Concerning ubiquity of topologies of countable type, the main obstacle to that appears to be unrelated to the topologies being perfect or even Gabriel. The right linear topology axiom (T3) from the book [30], characterizing filters of right ideals defining a topological ring structure (with continuous multiplication) on an associative ring, allows to produce an open ideal from every pair (open ideal, element of the ring). When the ring is commutative, this axiom is trivial; but for rings that are noncommutative enough and uncountable enough, it potentially may prevent existence of countably based right linear topologies.

In Section 3 of this paper, we impose an additional axiom $(T_{\lambda})$ to control this problem. Given an associative ring $R$, an infinite cardinal $\lambda$, and a right Gabriel topology $G$ on $R$ satisfying $(T_{\lambda})$ and having a base consisting of finitely generated right ideals, we show that $G$ is the union of Gabriel topologies $H \subset G$ having bases of the cardinality not exceeding $\lambda$ consisting of finitely generated right ideals. When $G$ is a perfect Gabriel topology, we assume additionally that the $G$-torsion in $R$ is $\lambda$-bounded (e.g., $G$ is faithful) and then show that $G$ is the union of perfect Gabriel topologies $P \subset G$ with bases of the cardinality not exceeding $\lambda$.

1.1. The main heroes of this paper are, of course, the contramodules. In the previous papers [4, 27, 28] of the present author with collaborators, we applied contramodule techniques in order to describe flat modules over certain commutative rings as direct summands of transfinitley iterated extensions of flat modules of special type (namely, the localizations of the ring with respect to various multiplicative subsets). In these approaches, going back to the work of Trlifaj [31, 32] and Bazzoni–Salce [6, 7], one starts from defining certain subclasses of flat modules, and then proceeds to show that, under certain assumptions, the whole class of flat modules coincides with such a subclass.

In the present paper, we make the first steps towards the important goal of extending the results of these papers, and particularly of [28], to noncommutative rings. In Section 3, we define the class of $U$-strongly flat left $R$-modules for a left flat ring epimorphism $u: R \rightarrow U$. Assuming that the related right Gabriel topology $G$ on $R$
has a countable base, we characterize $U$-strongly flat left $R$-modules by a pair of conditions which, in the additional assumption that $G$ has a base consisting of two-sided ideals, reduces to a direct generalization of what was suggested, for commutative case, in [28, Optimistic Conjecture 1.1]. More precisely, a flat left $R$-module $F$ is $U$-strongly flat if and only if the left $U$-module $U \otimes_R F$ is projective and, for every two-sided ideal $H \subset R$ belonging to $G$, the left $R/H$-module $F/HF$ is projective.

1.2. The most important results of this paper are proved in Section 8. First of all, it is the theorem that, for any left flat ring epimorphism $u: R \to U$ of countable type, the flat left $R$-module $U$ has projective dimension at most 1. The proof uses contramodules over the completion $\mathfrak{R}$ of the topological ring $R$ with respect to its perfect right Gabriel topology $G$. Here $\mathfrak{R}$ is viewed as a complete, separated topological ring in the completion (“projective limit”) topology.

Here it should be noted that such a result is easily provable for rings $R$ in which every left ideal has a countable set of generators. Indeed, in these assumptions the left $R$-module $U$ is countably presented (as one can see from its explicit construction as the ring/bimodule of quotients $U = R_G$), and a countably presented flat module over an associative ring always has projective dimension at most 1 (see [14, Corollary 2.23]). Our contramodule-based approach, while much more technical, allows to obtain an extra generality.

Furthermore, in Section 8 we prove that, for any right Gabriel topology $G$ on an associative ring $R$ having a countable base of finitely generated right ideals, the forgetful functor from the abelian category of left $\mathfrak{R}$-contramodules $\mathfrak{R}$–contra to the abelian category of left $R$-modules $R$–mod is fully faithful. In Section 8 we compare the category of left $\mathfrak{R}$-contramodules for the completion $\mathfrak{R}$ of an associative ring $R$ with respect to its perfect Gabriel topology $G$ of countable type with the Geigle–Lenzing abelian perpendicular subcategory $U_{1+1} \subset R$–mod. We show that the Geigle–Lenzing perpendicular subcategory (which we also denote by $R$–mod$_{u$–contra} and call “the full subcategory of $u$-contramodule left $R$-modules”) is, generally speaking, a wider full subcategory in $R$–mod than $\mathfrak{R}$–contra: a left $R$-module belongs to $R$–mod$_{u$–contra} $\subset R$–mod if and only if it is an extension of two left $R$-modules belonging to $\mathfrak{R}$–contra $\subset R$–mod (i.e., admitting left $\mathfrak{R}$-contramodule structures).

When $u: R \to U$ is an injective left flat ring epimorphism of countable type (or, in other words, the corresponding perfect right Gabriel topology $G$ on $R$ is faithful), we show in Section 8 that the two full subcategories $R$–mod$_{u$–contra} and $\mathfrak{R}$–contra $\subset R$–mod actually coincide. In Section 11 we extend this result to left flat ring epimorphisms of uncountable type, showing that, under certain assumption, there is an equivalence of abelian categories $\mathfrak{R}$–contra $\simeq R$–mod$_{u$–contra} for an injective ring epimorphism $u: R \to U$ such that $U$ is a flat left $R$-module of projective dimension at most 1. The additional assumption here is that of the condition $(T_\omega)$, which is needed in order to use the results of Section 8 in particular, it is satisfied automatically for right linear topology with a base of two-sided ideals.

The similar questions for multiplicative subsets and finitely generated ideals in commutative rings (or central multiplicative subsets/centrally finitely generated ideals in
associative rings) were discussed at length in the examples in [25, Sections 2, 3, and 5]. In this paper, we extend this discussion to the realm of left flat ring epimorphisms and perfect Gabriel topologies.

In Section 10 we offer a treatment, in the generality of left flat ring epimorphisms and the related right Gabriel topologies, of a classical topic, going back to Nunke [18] and Matlis [17]. This concerns connections between the completion of an $R$-module $M$ in what was classically called the “$R$-topology” and the Ext module $\text{Ext}^1_R(K, M)$, where $K$ was classically defined as the quotient module $Q/R$ of the field of fractions $Q$ of the ring $R$ by its subring $R \subset Q$. A natural morphism from the Ext module to the completion was constructed and a sufficient condition for it to be an isomorphism was established in Matlis’ memoir [17, Proposition 2.4 and Theorem 6.10].

An original idea of how to produce, under certain assumption, a map in the opposite direction (i.e., from the completion to the Ext module) was suggested in a recent preprint by Facchini and Nazemian [10, Sections 3–4]. We seize on their idea and use it, in combination with Matlis’ classical approach, in order to obtain an isomorphism of left $R$-modules between the completion $\Lambda_G(M)$ of a $u$-torsion-free left $R$-module $M$ with respect to a perfect right Gabriel topology $G$ on $R$, and an Ext (or rather, derived category Hom) module $\text{Ext}^1_R(K_{*R}, U, M)$. Here $u: R \rightarrow U$ is the left flat ring epimorphism related to $G$ and $K_{*R}$ is the two-term complex $R \rightarrow U$.

Some of the technical assumptions on the Gabriel topology $G$ mentioned above in this introduction are required for our proof of this result (which we first prove for perfect Gabriel topologies with a countable base, and then extend to the uncountable case using the results of Section 5).

1.3. Section 2 contains preliminary material on linear topologies, Gabriel topologies, completions, discrete/torsion modules, rings of quotients, and contramodules. The reader may wish to consult [5, Section 1] for further preliminaries on contramodules with further references, and the overview [22] for a more leisurely introduction.

In Sections 4–5, we spell out and develop the technique of $F$-systems, which first appeared, in a slightly disguised form, in the paper [26, Section 6]. This is, in fact, an important technique for working with contramodules over noncommutative topological rings, without which even very simple constructions, such that the completion of a left module with respect to a topology of right ideals in a ring, cannot be confidently performed. In fact, we discuss such a completion construction in Section 9 and its (more complicated) contramodule version in Section 5. The observation that separated contramodules, particularly over topological rings with countably based right linear topologies, can be described in terms of covariant $F$-systems was instrumental in [26, Section 6], and is also important in this paper.

In Section 6, we discuss the important question when the forgetful functor $\mathcal{R}\text{-contra} \rightarrow R\text{-mod}$ is fully faithful, building upon the argument that first appeared in [24, Theorem 1.1] and was subsequently developed in [25, Section 3]. In this paper, we improve upon these results of the papers [24, 25], providing an equivalent characterization of countably based right linear topologies $F$ on associative rings $R$ for which such full-and-faithfulness property holds. We also show that it always
holds for Gabriel topologies $G$ with a countable base of finitely generated right ideals. It is worth mentioning that the contramodule Nakayama lemma was a key technical tool for proving the full-and-faithfulness theorem in [24] and remains so in the present paper.

In Section 7 we discuss the following question, which sheds some light on the theorem that every $u$-contramodule $R$-module is an extension of two $\mathfrak{R}$-contramodules. Given an abelian category $A$ and a full subcategory $C \subset A$ closed under kernels, cokernels, and finite direct sums, it is clear that $C$ is an abelian category with an exact embedding functor $C \rightarrow A$. Consider the full subcategory $C_A^{(2)} \subset A$ of all objects in $A$ that can be presented as extensions of two objects from $A$. Is it true that $C_A^{(2)}$ is necessarily also an abelian category with an exact embedding functor $C_A^{(2)} \rightarrow A$? We show that the answer is “yes”.

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2. Preliminaries on Topological Rings

The material of Sections 2.1–2.6 below is fairly standard; we use the book [30] as the main reference (see also the original sources [12, 19] and the Bourbaki exercises [8, Exercices II.2.16–22]). The material of Sections 2.7–2.8 was developed by the present author [20, 21, 22, 26].

2.1. Linear topologies. A topological abelian group $A$ is said to have a linear topology if open subgroups form a base of neighborhoods of zero in $A$. A linear topology on a topological group $A$ is uniquely determined by the set $F$ of all open subgroups of $A$. Conversely, a set $F$ of subgroups of an abelian group $A$ is the set of all open subgroups in some linear topology on $A$ if and only if it is a filter, i. e., the following three conditions are satisfied [30, Section VI.4]:

(T0) $A \in F$;
(T1) if $V \in F$ and $V \subset U \subset A$, then $U \in F$;
(T2) if $U \in F$ and $V \in F$, then $U \cap V \in F$. 

All topologies considered in this paper will be linear. Abusing terminology, we will call $\mathbb{F}$ “a topology on $A$”.

A set of subgroups $\mathcal{B}$ in an abelian group $A$ is said to be a base of (neighborhoods of zero in) a topology $\mathbb{F}$ on $A$ if $\mathbb{F}$ consists precisely of all the subgroups $U \in \mathcal{B}$ for which there exists $V \in \mathcal{B}$ such that $V \subseteq U$. A set of subgroups $\mathcal{B}$ in an abelian group $A$ is a base of a topology if and only if $\mathcal{B}$ is nonempty and for any two subgroups $U$, $V \in \mathcal{B}$ there exists a subgroup $W \in \mathcal{B}$ such that $W \subseteq U \cap V$.

A topological abelian group $A$ is said to be separated if the intersection of all its open subgroups is the zero subgroup, that is $\bigcap_{U \in \mathcal{F}} U = 0$. In other words, it means that the natural abelian group homomorphism $\lambda_{A,\mathcal{F}}: A \to \varprojlim_{U \in \mathcal{F}} A/U$ is injective. A topological abelian group $A$ is said to be complete if the map $\lambda_{A,\mathcal{F}}$ is surjective.

The abelian group $\mathfrak{A} = \varprojlim_{U \in \mathcal{F}} A/U$ is called the completion of the topological abelian group $A$. The abelian group $\mathfrak{A}$ is endowed with the projective limit topology $\mathfrak{F}$ consisting of all the subgroups $\mathfrak{A}_U \subseteq \mathfrak{A}$ of the form $\mathfrak{A}_U = \ker(\mathfrak{A} \to A/U)$, where $U \in \mathcal{F}$ and $\mathfrak{A} \to A/U$ is the natural projection map.

A topological abelian group $A$ is separated and complete if and only if the natural homomorphism of topological groups $A \to \mathfrak{A}$ from $A$ to its completion $\mathfrak{A}$ is an isomorphism of topological groups. For any topological abelian group $A$, its completion $\mathfrak{A}$ is separated and complete in the projective limit topology.

2.2. Right linear topologies. A topological ring $R$ is said to have a right linear topology if open right ideals form a base of neighborhoods of zero in $R$. A set $\mathcal{F}$ of right ideals in an associative ring $R$ is the set of open right ideals in a right linear topology of $R$ if and only if, in addition to the conditions (T0–T2), it also satisfies the following condition [30, Section VI.4]:

(T3) if a right ideal $I \subseteq R$ belongs to $\mathcal{F}$ and $s \in R$, then the right ideal $(I : s) = \{r \in R \mid sr \in I\} \subseteq R$

belongs to $\mathcal{F}$.

For any topological ring $R$ with a right linear topology $\mathcal{F}$, the topological abelian group $\mathfrak{R} = \varprojlim_{I \in \mathcal{F}} R/I$ with its projective limit topology $\mathfrak{F}$ has a unique topological ring structure such that the natural map $R \to \mathfrak{R}$ is a ring homomorphism. Given two elements $s = (s_I \in R/I)_{I \in \mathcal{F}}$ and $r = (r_I \in R/I)_{I \in \mathcal{F}}$ in $\mathfrak{R}$, in order to compute the $I$-component $t_I$ of their product $t = sr \in \mathfrak{R}$, one chooses preimages $\tilde{s}_I \in R$ and $\tilde{r}_I \in R$ of the elements $s_I$ and $r_I \in R/I$, and considers the ideal $J = (I : \tilde{s}_I) \subseteq \mathbb{F}$. Then one puts $t_I = \tilde{s}_I \tilde{r}_I + I$. The topology $\mathfrak{F}$ on the topological ring $\mathfrak{R}$ is right linear.

2.3. Discrete modules. Let $R$ be a topological ring with a right linear topology $\mathcal{F}$. Then a right $R$-module $N$ is said to be discrete if for every element $b \in N$ its annihilator $I_b = \{r \in R \mid br = 0\} \subseteq R$ belongs to $\mathcal{F}$. Equivalently, this means that the right action map $N \times R \to N$ is continuous as a function of two variables with respect to the given topology on $R$ and the discrete topology on $N$. The full subcategory of all discrete right $R$-modules $\text{discr}-R$ is closed under subobjects, quotient objects, and
infinite direct sums in the abelian category of right \( R \)-modules \( \text{mod} - R \). It follows that \( \text{discr} - R \), just like \( \text{mod} - R \), is a Grothendieck abelian category.

The \( R \)-module structure of any discrete right \( R \)-module \( N \) can be extended in a unique way to a discrete right module structure over the completion \( \mathfrak{R} = \lim_{\leftarrow I \in F} R/I \) of the ring \( R \). Hence the categories of discrete right \( R \)-modules and discrete right \( \mathfrak{R} \)-modules are naturally equivalent (in fact, isomorphic), \( \text{discr} - R \simeq \text{discr} - \mathfrak{R} \).

2.4. Gabriel topologies. Let \( A \) be an abelian category with infinite products and coproducts in which subobjects of any given object form a set. A pair of classes of objects \( T, F \subset A \) is called a torsion pair if an object \( T \in A \) belongs to \( T \) if and only if \( \text{Hom}_A(T, F) = 0 \) for all \( F \in F \), and an object \( F \in A \) belongs to \( F \) if and only if \( \text{Hom}_A(T, F) = 0 \) for all \( T \in T \). The class \( T \) in a torsion pair \((T,F)\) is called the torsion class, and the class \( F \) is called the torsion-free class.

A class of objects \( T \subset A \) is the torsion class of some torsion pair in \( A \) if and only if it is closed under quotients, extensions, and coproducts. Similarly, a class of objects \( F \subset A \) is the torsion-free class of some torsion pair in \( A \) if and only if it is closed under subobjects, extensions, and products [30 Section VI.2].

Given a torsion pair \((T,F)\) in \( A \), for every object \( A \in A \) there exists a (unique and functorial) short exact sequence \( 0 \to T \to A \to F \to 0 \) in \( A \) with \( T \in T \) and \( F \in F \). The object \( T \) is the maximal torsion subobject of \( A \), and the object \( F \) is the maximal torsion-free quotient of \( A \).

A class of objects \( P \subset A \) is said to be a pretorsion class if it is closed under quotients and coproducts. A (pre)torsion class \( P \) is said to be hereditary if it is closed under subobjects. A torsion pair \((T,F)\) is said to be hereditary if its torsion class \( T \) is hereditary.

Let \( R \) be an associative ring. For any right linear topology \( F \) on \( R \), the class of all discrete right \( R \)-modules \( \text{discr} - R \) with respect to the topology \( F \) is a hereditary pretorsion class in the abelian category of right \( R \)-modules \( \text{mod} - R \). Conversely, for any hereditary pretorsion class \( P \subset \text{mod} - R \) there exists a unique right linear topology \( F \) on \( R \) such that \( P = \text{discr} - R \). Given a hereditary pretorsion class \( P \) in \( \text{mod} - R \), the topology \( F \) can be recovered as the set of all right ideals \( I \subset R \) such that \( R/I \in P \), while given a right linear topology \( F \) on \( R \), the hereditary pretorsion class \( P \) is defined as the class of all right \( R \)-modules \( N \) such for every element \( b \in N \) the annihilator of \( b \) in \( R \) belongs to \( F \) [30 Section IV.4].

A right linear topology \( G \) on \( R \) is called a (right) Gabriel topology if the related hereditary pretorsion class \( T = \text{discr} - R \) is a torsion class (i.e., it is closed under extensions in \( \text{mod} - R \)). Thus right Gabriel topologies on \( R \) correspond bijectively to hereditary torsion classes in \( \text{mod} - R \). A right linear topology \( G \) on \( R \) is a Gabriel topology if and only it satisfies the following condition [30 Section IV.5]:

(T4) if \( I \subset R \) is a right ideal and there exists a right ideal \( J \in G \) such that \( (I : s) \in G \) for all \( s \in J \), then \( I \in G \).

The right \( R \)-modules belonging to the hereditary torsion class \( T \) corresponding to a Gabriel topology \( G \) on \( R \) are said to be \( G \)-torsion, and the right \( R \)-modules
belonging to the related torsion-free class \( F \subseteq \text{mod--}R \) are called \( \mathbb{G}\text{-torsion-free} \). So the words “\( \mathbb{G}\text{-torsion right } R\text{-module} \)” are synonymous with “discrete right \( R\text{-module} \) with respect to the \( \mathbb{G}\text{-topology} \) on \( R' \).

Given a topological ring \( R \) with a right linear topology \( \mathbb{G} \), one can consider its completion \( \mathbb{R} = \lim_{\leftarrow l \in \mathbb{G}} R/I \) and view it as a topological ring in the projective limit topology \( \mathbb{G} \). If \( \mathbb{G} \) is a Gabriel topology on \( R \), then \( \mathbb{G} \) is a Gabriel topology on \( \mathbb{R} \).

2.5. Localization. Another name for a hereditary torsion class is a localizing subcategory. Let \( A \) be an abelian category with exact coproduct functors in which subobjects of any given object form a set, and let \( T \subseteq A \) be a full subcategory closed under subobjects, quotients, extensions, and coproducts. The quotient category \( A/T \) is the category whose objects are the objects of \( A \) and morphisms \( A \to B \) can be equivalently described as the equivalence classes of

(a) morphisms \( A' \to B' \) in \( A \), where \( A' \) is a subobject in \( A \) with \( A/A' \in T \) and \( B' = B/T \) is a quotient object of \( B \) by a subobject \( T \in T \); or
(b) fractions \( A \to B' \leftarrow B \) in \( A \), where both the kernel and cokernel of the morphism \( B \to B' \) belong to \( T \); or
(c) fractions \( A \leftarrow A' \to B \) in \( A \), where both the kernel and cokernel of the morphism \( A' \to A \) belong to \( T \).

The quotient category \( A/T \) is an abelian category with (exact) coproducts. The natural functor \( A \to A/T \) is exact and preserves coproducts. Hence, assuming that the category \( A \) has a set of generators, the functor \( A \to A/T \) has a right adjoint functor \( A/T \to A \). The functor \( A/T \to A \) is fully faithful, so it allows to view the quotient category \( A/T \) as a full subcategory in \( A \).

We denote by \( L \) the composition of functors \( A \to A/T \to A \). The functor \( L \) is the reflector onto the full subcategory \( A/T \subseteq A \). For every object \( A \in A \), there is a natural adjunction morphism \( A \to L(A) \) inducing an isomorphism of the Hom groups \( \text{Hom}_A(L(A), B) \cong \text{Hom}_A(A, B) \) for every object \( B \in A/T \subseteq A \).

Let us now consider the particular case when \( A = \text{mod--}R \) is the category of right modules over an associative ring \( R \). Then the hereditary torsion class \( T \) corresponds to a right Gabriel topology \( \mathbb{G} \) on \( R \). Let \( L_G : \text{mod--}R \to \text{mod--}R \) denote the related localization functor.

The right \( R \)-modules belonging to the full subcategory \((\text{mod--}R)/T \subseteq \text{mod--}R \) (that is, to the essential image of the functor \( L_G \)) are said to be \( \mathbb{G}\text{-closed} \). A right \( R \)-module \( B \) is \( \mathbb{G}\text{-closed} \) if and only if any right \( R \)-module morphism \( A' \to A'' \) with \( \mathbb{G}\text{-torsion} \) kernel and cokernel induces an isomorphism \( \text{Hom}_R(A'', B) \cong \text{Hom}_R(A', B) \). So any \( \mathbb{G}\text{-closed} \) right \( R \)-module is \( \mathbb{G}\text{-torsion-free} \).

Consider the free right \( R \)-module \( R \) with one generator. Applying the functor \( L_G \), we obtain a right \( R \)-module \( L_G(R) \). The left action of \( R \) by right \( R \)-module endomorphisms of the right \( R \)-module \( R \) induces a left action of \( R \) in \( L_G(R) \), making it an \( R\text{-}R\text{-bimodule} \). The adjunction map \( R \to L_G(R) \) is a morphism of \( R\text{-}R\text{-bimodules} \).

There exists a unique associative ring structure on \( L_G(R) \) making the map \( R \to L_G(R) \) an associative ring homomorphism in a way compatible with the
\(R\)-\(R\)-bimodule structure on \(L_G(R)\). Moreover, for any right \(R\)-module \(N\), the right \(R\)-module structure on \(L_G(N)\) extends uniquely to a right \(L_G(R)\)-module structure.

The ring \(L_G(R)\) is the universal ring acting on the right on (the underlying abelian groups of) all the \(G\)-closed right \(R\)-modules, and any right \(R\)-linear morphism between \(G\)-closed right \(R\)-modules is right \(L_G(R)\)-linear.

### 2.6. Sheafification construction.

Let \(R\) be an associative ring and \(G\) be a Gabriel topology of right ideals in \(R\). The localization functor \(L_G\) has the following explicit construction, which is essentially a particular case of the “additive sheaf theory” \([30\text{, Section IX.1}]\).

Let \(N\) be a right \(R\)-module. Consider the inductive limit of abelian groups

\[N_{(G)} = \lim_{\text{I} \in G} \text{Hom}_{R^{op}}(I, N),\]

where \(R^{op}\) is the ring opposite to \(R\) and \(\text{Hom}_{R^{op}}(K, N)\) denotes the group of morphisms \(K \rightarrow N\) in the category of right \(R\)-modules \(\text{mod}-R\). The diagram is indexed by the partially ordered set of all right ideals \(I \in G\) with respect to the inverse inclusion, and for any two right ideals \(J \subseteq I\), \(I, J \in G\), the map \(\text{Hom}_{R^{op}}(I, N) \rightarrow \text{Hom}_{R^{op}}(J, N)\) is induced by the inclusion morphism \(J \rightarrow I\). According to the condition \((T2)\), this diagram is indexed by a directed poset \(G\).

By the definition of the inductive limit, the construction above means that elements of \(N_{(G)}\) are represented by right \(R\)-module morphisms \(I \rightarrow N\), where \(I \in G\). Two such morphisms \(I \xrightarrow{\alpha} N\) and \(I \xrightarrow{\beta} N\), where \(I, J \in G\), represent the same element of \(N_{(G)}\) if and only if there exists a right ideal \(K \in G\) such that \(K \subseteq I \cap J\) and \(\alpha|_K = \beta|_K\).

To endow the abelian group \(N_{(G)}\) with a natural right \(R\)-module structure, consider an element \(\beta \in N_{(G)}\) represented by a right \(R\)-module morphism \(I \rightarrow N\), and an element \(s \in R\). According to the condition \((T3)\), we have \((I : s) \in G\). Let \(\beta s \in N_{(G)}\) be the element represented by the composition

\[(I : s) \xrightarrow{s} I \xrightarrow{\beta} N,\]

where \((I : s) \xrightarrow{s} I\) is the right \(R\)-module morphism of left multiplication with \(s\). There is a natural morphism of right \(R\)-modules \(\tau_N : N \rightarrow N_{(G)}\) assigning to an element \(b \in N\) the element of \(N_{(G)}\) represented by the right \(R\)-module morphism \(R \xrightarrow{b} N\) (where \(R \in G\) by the condition \((T0)\)).

The kernel of the morphism \(N \rightarrow N_{(G)}\) coincides with the maximal \(G\)-torsion submodule \(t_G(N)\) of the right \(R\)-module \(N\). The cokernel \(N_{(G)}/\text{im } \tau_N\) is \(G\)-torsion, since for any element \(\beta \in N_{(G)}\) represented by a right \(R\)-module morphism \(I \rightarrow N\) one has \(\beta s \in \text{im } \tau_N\) for all \(s \in I\). Using the condition \((T4)\), one shows that the right \(R\)-module \(N_{(G)}\) is \(G\)-torsion-free for any right \(R\)-module \(N\).

For any right \(R\)-module \(N\) the two maps \(\tau_{N_{(G)}}\) and \((\tau_N)_{(G)} : N_{(G)} \rightarrow N_{(G)}\) coincide, since so do their compositions with the map \(\tau_N\), whose cokernel is \(G\)-torsion, while \(N_{(G)}/\text{im } \tau_N\) is \(G\)-torsion-free. The functor \(N \mapsto N_{(G)}\) is left exact by construction, hence the map \(N_{(G)} \rightarrow N_{(G)}/G\) is an isomorphism for any \(G\)-torsion-free right
For any $\mathbb{G}$-torsion-free right $R$-module $N$, the right $R$-module $N_{(\mathbb{G})}$ is $\mathbb{G}$-closed. The functor $N \mapsto L_{\mathbb{G}}(N)$ can be computed in two alternative ways as

$$L_{\mathbb{G}}(N) = (N/t_{\mathbb{G}}(N))_{(\mathbb{G})} = N_{(\mathbb{G})}(\mathbb{G})$$

for any right $R$-module $N$.

Following the notation in [30], when the $R$-$R$-bimodule $L_{\mathbb{G}}(R)$ is viewed as a ring, it is denoted by $R_{\mathbb{G}}$, and when the right $R$-module $L_{\mathbb{G}}(R)$ is viewed as a right $R_{\mathbb{G}}$-module, it is denoted by $N_{\mathbb{G}}$. So one has $N_{\mathbb{G}} = N_{(\mathbb{G})}(\mathbb{G})$. The ring $R_{\mathbb{G}}$ is called the ring of quotients of the ring $R$ with respect to the Gabriel topology $\mathbb{G}$. The right $R_{\mathbb{G}}$-module $N_{\mathbb{G}}$ is called the module of quotients of a right $R$-module $N$.

The direct limit/sheafification construction above can be used to describe explicitly the multiplication in $R_{\mathbb{G}}$ and the right action of $R_{\mathbb{G}}$ in $N_{\mathbb{G}}$. Let us discuss the ring structure (the construction of the right action being similar).

Let $\sigma: R \rightarrow S$ be a ring homomorphism such that both the kernel and the cokernel of $\sigma$ are $\mathbb{G}$-torsion right $R$-modules. Assume that either the map $\sigma$ is injective, or otherwise $S$ is a $\mathbb{G}$-torsion-free right $R$-module. Our aim is to construct an associative ring structure on the right $R$-module (or $S$-$R$-bimodule) $S_{(\mathbb{G})}$ making $S: S \rightarrow S_{(\mathbb{G})}$ a ring homomorphism.

Let $\alpha$ and $\beta \in S_{(\mathbb{G})}$ be two elements represented by right $R$-module morphisms $I \xrightarrow{\alpha} S$ and $J \xrightarrow{\beta} S$, where $I$ and $J \subset R$ are two right ideals belonging to $\mathbb{G}$. Denote by $\iota = \iota_I: I \rightarrow R$ and $\iota_J: J \rightarrow R$ the identity inclusion maps. Consider the composition of the inclusion $\iota_I: I \rightarrow R$ with the map $\sigma: R \rightarrow S$, and denote by $\tilde{K}$ the fibered product of the right $R$-modules $I$ and $J$ over $S$ with respect to the morphisms $I \xrightarrow{\iota_I} R \xrightarrow{\sigma} S$ and $J \xrightarrow{\iota_J} \beta$. Let $\gamma: \tilde{K} \rightarrow I$ and $\tilde{\gamma}: \tilde{K} \rightarrow J$ be the two related morphisms of right $R$-modules.

The cokernel of the morphism $\tilde{\gamma}: \tilde{K} \rightarrow J$ is a submodule of the cokernel of the composition $I \rightarrow R \rightarrow S$, which is an extension of the cokernel of the morphism $\sigma: R \rightarrow S$ and of a certain quotient module of $R/I$. Since the class of all $\mathbb{G}$-torsion right $R$-modules is closed under submodules, quotients, and extensions, the cokernel of the morphism $\tilde{\gamma}: \tilde{K} \rightarrow J$ is $\mathbb{G}$-torsion. The cokernel of the composition $K \rightarrow J \rightarrow R$ of the morphism $\tilde{\gamma}$ with the inclusion $\iota_J: J \rightarrow R$ is an extension of $R/J$ and the cokernel of $\tilde{\gamma}$; hence it is $\mathbb{G}$-torsion, too.

The kernel of the morphism $\tilde{\gamma}: \tilde{K} \rightarrow J$ is a submodule of the kernel of the composition $I \rightarrow R \rightarrow S$, which is a submodule of ker $\sigma$. Hence the kernel of $\tilde{\gamma}$ is also $\mathbb{G}$-torsion. Consider the composition

$$\tilde{K} \xrightarrow{\tilde{\gamma}} I \xrightarrow{\alpha} S.$$
once again, $K \subset J$ is a right ideal in $R$. In both cases, the quotient module $R/K$, which coincides with the cokernel of the morphism $\tilde{\eta}$, is $G$-torsion; hence $K \in G$.

In both cases, we have a right $R$-module morphism $K \to S$ induced by $\alpha \gamma$. This morphism represents the desired element $\alpha \beta \in S(G)$.

Applying this construction for the first time to the ring $R = S$ and the identity morphism $\sigma = \text{id}_R$, we construct a ring structure on the $R$-$R$-bimodule $R(G)$. Applying the same construction for the second time to the ring $S = R(G)$ and the ring homomorphism $\sigma = \kappa_R$, we obtain a ring structure on the $R$-$R$-bimodule $R_G = R(G)(G)$, which was our aim.

Alternatively, notice that the maximal $G$-torsion submodule $t_G(R) \subset R$ of the right $R$-module $R$ is a two-sided ideal in the ring $R$. Applying the construction above to the ring $S = R/t_G(R)$ and the natural surjective ring homomorphism $\sigma: R \to S$, one obtains the same ring structure on the $R$-$R$-bimodule $R_G = (R/t_G(R))(G)$.

2.7. Contramodules. The following constructions and definitions go back to the book [20, Remark A.3], the memoir [21, Section 1.2], the overview [22, Sections 2.1 and 2.3], and the paper [26, Sections 1.2 and 5].

For any abelian group $A$ and a set $X$, we will use the notation $A[X] = A^X$ for the direct sum of $X$ copies of $A$. Alternatively, one can view $A[X]$ as the group of all finite formal linear combinations $\sum_{x \in X} a_x x$ of elements of $X$ with the coefficients in $A$. For any map of sets $X \to Y$, one has the obvious induced (“pushforward”) map $A[X] \to A[Y]$; so $X \mapsto A[X]$ is a covariant functor $A[-]: \text{Sets} \to \text{Ab}$ from the category of sets to the category of abelian groups.

In particular, if $R$ is an associative ring, then $R[X]$ is a notation for the free left $R$-module with $X$ generators. One can also consider the underlying set of the abelian group $R[X]$ and view it as an abstract set. Then, for any associative ring $R$ and a set $X$, there are natural maps of sets

$$\epsilon_{R,X}: X \to R[X] \quad \text{and} \quad \phi_{R,X}: R[R[X]] \to R[X].$$

Here $\epsilon_{R,X}$ is the “point measure” map assigning to an element $x_0 \in X$ the corresponding generator of the free left $R$-module $R[X]$, or in other words, the formal linear combination $\sum_{x \in X} r_x x$ with $r_{x_0} = 1$ and $r_x = 0$ for $x \neq x_0$. The map $\phi_{R,X}$ is the “opening of parentheses” map assigning to a formal linear combination of formal linear combinations of elements of $X$ with the coefficients in $R$ a formal linear combination of elements of $X$ with the coefficients in $R$ (using both the additive and the multiplicative structures on $R$).

The natural transformations $\epsilon_R$ and $\phi_R$ endow the functor

$$\mathbb{M}_R = R[-]: \text{Sets} \to \text{Sets}$$

with the structure of a monad on the category of sets. The category of left $R$-modules can be defined as the category of algebras (or, in the additive language that we prefer as more suitable to our context, “modules”) over the monad $\mathbb{M}_R$. In other words, the datum of a left $R$-module structure on a set $M$ is equivalent to the datum of a
map of sets $\mu_M : R[M] \rightarrow M$ satisfying the associativity and unity equations of an algebra/module over the monad $M_R$.

Now let $\mathfrak{A}$ be a complete, separated topological abelian group (with a linear topology $\mathfrak{F}$). Then we denote by $\mathfrak{A}[[X]]$ the abelian group

$$\mathfrak{A}[[X]] = \lim_{\leftarrow U \in \mathfrak{F}} (\mathfrak{A}/U)[X],$$

where the projective limit is taken over all the open subgroups $U \subseteq \mathfrak{A}$. Alternatively, $\mathfrak{A}[[X]]$ can be defined as the group of all infinite formal linear combinations $\{\sum_{x \in X} a_x x\}$ with the coefficients $a_x \in \mathfrak{A}$ forming a family of elements of $\mathfrak{A}$ converging to zero in the topology $\mathfrak{F}$ of $\mathfrak{A}$. The latter condition means that for every $U \in \mathfrak{F}$ the set of all $x \in X$ for which $a_x \notin U$ must be finite.

For any map of sets $f : X \rightarrow Y$, there is the induced “push-forward” homomorphism

$$\mathfrak{A}[[f]] : \mathfrak{A}[[X]] \longrightarrow \mathfrak{A}[[Y]]$$

taking an infinite formal linear combination $\sum_{x \in X} a_x x$ to the infinite formal linear combination $\sum_{y \in Y} b_y y$ with the coefficients

$$b_y = \sum_{x : f(x) = y} a_x \in \mathfrak{A}.$$

Here the infinite sum of elements of $\mathfrak{A}$ is understood as the limit of finite partial sums in the (complete and separated, by assumption) topology $\mathfrak{F}$ of $\mathfrak{A}$. Thus we have a functor $\mathfrak{A}[[\!\!\!\!\cdot\!\!\!\!] : \text{Sets} \longrightarrow \text{Ab}$ from the category of sets to the category of abelian groups corresponding to any complete, separated topological abelian group $\mathfrak{A}$.

Let $\mathfrak{R}$ be a complete, separated topological associative ring with a right linear topology. For any set $X$, we have a natural “point measure” map

$$\epsilon_{\mathfrak{R},X} : X \longrightarrow \mathfrak{R}[[X]],$$

which can be constructed as the composition $X \rightarrow \mathfrak{R}[X] \rightarrow \mathfrak{R}[[X]]$. Furthermore, an “opening of parentheses” map

$$\phi_{\mathfrak{R},X} : \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

can be constructed using the multiplication of pairs of elements in $\mathfrak{R}$ and the formation of infinite sums, computed as the limits of finite partial sums in the topology of $\mathfrak{R}$. The assumption that the topology on $\mathfrak{R}$ is right linear guarantees convergence [22, Section 2.1], [26, Section 5].

As in the case of a discrete ring $R$ above, the natural transformations $\epsilon_R$ and $\phi_R$ defined for a complete, separated topological ring $\mathfrak{R}$ with a right linear topology, endow the functor

$$M_\mathfrak{R} = \mathfrak{R}[[\!\!\!\!\cdot\!\!\!\!]] : \text{Sets} \longrightarrow \text{Sets}$$

with the structure of a monad on the category of sets. By the definition, a left $\mathfrak{R}$-contramodule is an algebra/module over this monad. In other words, a left $\mathfrak{R}$-contramodule is a set endowed with a left contraaction map

$$\pi_c : \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C},$$
which must be a map of sets satisfying the following equations. The two composition
\[ R[[R[[C]]]] \to R[[C]] \to C \]
of the maps \( \phi_{R,C} \) and \( R[[\pi_C]] : R[[R[[C]]]] \to R[[C]] \) with the map \( \pi_C : R[[C]] \to C \) should be equal to each other; and the composition
\[ C \to R[[C]] \to C \]
of the map \( \epsilon_{R,C} : C \to R[[C]] \) with the map \( \pi_C : R[[C]] \to C \) should be equal to the identity map \( \text{id}_C \). A morphism of left \( R \)-contramodules \( f : C \to D \) is a morphism of algebras/modules over the monad \( M_R \); in other words, it is a map of sets forming a commutative square diagram with the contraaction maps \( \pi_C, \pi_D \) and the push-forward map \( R[[f]] : R[[C]] \to R[[D]] \).

The category of left \( R \)-contramodules \( R \text{-contra} \) is abelian. For any left \( R \)-contra-module \( C \), the composition \( R[C] \to R[[C]] \to C \) of the natural inclusion \( R[C] \to R[[C]] \) with the contraaction map \( \pi_C : R[[C]] \to C \) defines a map \( \mu_C : R[C] \to C \) endowing \( C \) with a left \( R \)-module structure. This defines a natural forgetful functor \( R \text{-contra} \to R \text{-mod} \), which is exact and preserves infinite products [21, Section 1.2], [26, Section 1.1], [25, Lemma 1.1].

For any set \( X \), the set \( R[[X]] \) has a natural left \( R \)-contra-module structure with the contraaction map \( \pi_{R[[X]]} = \phi_X \). The left \( R \)-contra-module \( R[[X]] \) is called the free left \( R \)-contra-module generated by a set \( X \). For any left \( R \)-contra-module \( D \), there is a natural isomorphism of abelian groups
\[ \text{Hom}^R(R[[X]], D) \cong \text{Hom}_{\text{Sets}}(X, D) = D^X, \]
where \( \text{Hom}^R(C, D) \) denotes the abelian group of morphisms \( C \to D \) in the category \( R \text{-contra} \). It follows that free left \( R \)-contra-modules are projective objects in \( R \text{-contra} \). There are also enough of them; so there are enough projectives in \( R \text{-contra} \) and a left \( R \)-contra-module is projective if and only if it is a direct summand of a free one. The free left \( R \)-contra-module with one generator \( R = R[[\ast]] \) is a projective generator of \( R \text{-contra} \). If \( \lambda \) denotes the cardinality of a base \( \mathfrak{B} \) of the topology \( \mathfrak{T} \) on \( R \), then the category \( R \text{-contra} \) is locally \( \lambda^+ \)-presentable and its projective generator \( R \) is \( \lambda^+ \)-presentable [26, Sections 1.2 and 5].

2.8. Contratensor product. Let \( R \) be a complete, separated topological associative ring with a right linear topology. Let \( A \) be an associative ring, and let \( N \) be an \( A-R \)-bimodule which is discrete as a right \( R \)-module. Let \( V \) be a left \( A \)-module. Then the Hom group \( D = \text{Hom}_A(N, V) \) has a natural left \( R \)-contra-module structure (extending its familiar left \( R \)-module structure). Given an element \( r = \sum_{d \in D} r_d d \in R[[D]] \), one defines its image \( \pi_D(r) \in D \) under the contraaction map \( \pi_D \) by the rule
\[ \pi_D(r)(b) = \sum_{d \in D} d(br_d) \in V \]
for all \( b \in N \), where the sum in the right-hand side is finite because the annihilator of \( b \) is open in \( R \) and the family of elements \( (r_d \in R)_{d \in D} \) converges to zero in \( R \).
The contratensor product $N \otimes_{\mathcal{R}} \mathcal{C}$ of a discrete right $\mathcal{R}$-module $N$ and a left $\mathcal{R}$-contramodule $\mathcal{C}$ is an abelian group constructed as the cokernel of (the difference of) the following pair of maps

$$N \otimes_{\mathcal{Z}} \mathcal{R}[[\mathcal{C}]] \rightarrow N \otimes_{\mathcal{Z}} \mathcal{C}.$$ 

The first map $N \otimes_{\mathcal{Z}} \mathcal{R}[[\mathcal{C}]] \rightarrow N \otimes_{\mathcal{Z}} \mathcal{C}$ is obtained by applying the functor $N \otimes_{\mathcal{Z}} -$ to the contraaction map $\pi_{\mathcal{C}}: \mathcal{R}[[\mathcal{C}]] \rightarrow \mathcal{C}$. The second map $N \otimes_{\mathcal{Z}} \mathcal{R}[[\mathcal{C}]] \rightarrow N \otimes_{\mathcal{Z}} \mathcal{C}$ is defined by the rule

$$b \otimes \sum_{c \in \mathcal{C}} r_c c \mapsto \sum_{c \in \mathcal{C}} b r_c c \in N \otimes_{\mathcal{Z}} \mathcal{C}$$

for any $b \in N$ and $r = \sum_{c \in \mathcal{C}} r_c c \in \mathcal{R}[[\mathcal{C}]]$. Here, once again, the sum in the right-hand side is finite because the annihilator of $b$ is open in $\mathcal{R}$ and the family of elements $(r_c \in \mathcal{R})_{c \in \mathcal{C}}$ converges to zero in $\mathcal{R}$.

The conventional tensor product $N \otimes_{\mathcal{R}} \mathcal{C}$ of the right $\mathcal{R}$-module $N$ and the underlying left $\mathcal{R}$-module of the left $\mathcal{R}$-contramodule $\mathcal{C}$ can be constructed as the cokernel of the pair of maps $N \otimes_{\mathcal{Z}} \mathcal{R}[[\mathcal{C}]] \rightarrow N \otimes_{\mathcal{Z}} \mathcal{R}[[\mathcal{C}]] \rightarrow N \otimes_{\mathcal{Z}} \mathcal{C}$. Hence there is a natural surjective map of abelian groups

$$N \otimes_{\mathcal{R}} \mathcal{C} \twoheadrightarrow N \otimes_{\mathcal{R}} \mathcal{C}.$$ 

For any left $A$-$\mathcal{R}$-bimodule $N$ which is discrete as a right $\mathcal{R}$-module, left $\mathcal{R}$-contramodule $\mathcal{C}$, and a left $A$-module $V$ there is a natural isomorphism of abelian groups

$$\text{Hom}_A(N \otimes_{\mathcal{R}} \mathcal{C}, V) \simeq \text{Hom}^{\mathcal{R}}(\mathcal{C}, \text{Hom}_A(N, V)),$$

where the left $A$-module structure on the contratensor product $N \otimes_{\mathcal{R}} \mathcal{C}$ is induced by the left $A$-module structure on $N$. For any discrete right $\mathcal{R}$-module $N$ and any set $X$, there is a natural isomorphism of abelian groups

$$N \otimes_{\mathcal{R}} \mathcal{R}[[X]] \simeq N[X].$$

For any associative ring $R$, any subgroup $A \subset R$, and any left $R$-module $M$, we denote, as usual, by $AM = A \cdot M$ the subgroup in $M$ spanned by all the elements $am$ with $a \in A$ and $m \in M$. Alternatively, the subgroup $AM \subset M$ can be defined as the image of the composition $A[M] \rightarrow R[M] \rightarrow M$ of the natural inclusion $A[M] \rightarrow R[M]$ with the (monad) action map $\mu_M: R[M] \rightarrow M$.

For any complete, separated topological associative ring $\mathcal{R}$ with a right linear topology, any closed subgroup $\mathfrak{A} \subset \mathcal{R}$, and any left $\mathcal{R}$-contramodule $\mathcal{C}$, we denote by $\mathfrak{A} \times \mathcal{C} \subset \mathcal{C}$ the image of the composition

$$\mathfrak{A}[[\mathcal{C}]] \rightarrow \mathcal{R}[[\mathcal{C}]] \rightarrow \mathcal{C}$$

of the natural inclusion $\mathfrak{A}[[\mathcal{C}]] \rightarrow \mathcal{R}[[\mathcal{C}]]$ with the contraaction map $\pi_{\mathcal{C}}: \mathcal{R}[[\mathcal{C}]] \rightarrow \mathcal{C}$. So $\mathfrak{A} \times \mathcal{C}$ is a subgroup in $\mathcal{C}$. Clearly, one has $\mathfrak{A} \mathcal{C} \subset \mathfrak{A} \times \mathcal{C}$.

Let $\mathfrak{J} \subset \mathcal{R}$ be a closed right ideal. Then for any set $X$ one has

$$\mathfrak{J} \times \mathcal{R}[[X]] \subseteq \mathfrak{J}[[X]] \subset \mathcal{R}[[X]].$$
Let \( \mathcal{I} \subset \mathcal{R} \) be an open right ideal. Then \( \mathcal{R}/\mathcal{I} \) is a discrete right \( \mathcal{R} \)-module and, for any left \( \mathcal{R} \)-contramodule \( \mathcal{C} \), the quotient group \( \mathcal{C}/(\mathcal{I} \bowtie \mathcal{C}) \) can be computed as the contratensor product
\[
\mathcal{C}/(\mathcal{I} \bowtie \mathcal{C}) \simeq (\mathcal{R}/\mathcal{I}) \otimes_{\mathcal{R}} \mathcal{C}.
\]

3. Perfect Gabriel Topologies of Type \( \lambda \)

A homomorphism of associative rings \( u: R \rightarrow U \) is called a ring epimorphism if it is an epimorphism in the category of associative rings, i.e., for any two homomorphisms of associative rings \( v', v'': U \rightarrow V \) the equation \( v'u = v''u \) implies \( v' = v'' \). Equivalently, this means that the multiplication map \( U \otimes_R U \rightarrow U \) is an isomorphism of \( R \)-\( R \)-bimodules [30 Proposition XI.1.2]. If this is the case, then the two \( R \)-\( R \)-bimodule morphisms \( U \rightarrow U \otimes_R U \) induced by \( u \) are also isomorphisms and, in addition, are equal to each other. An associative ring homomorphism \( u: R \rightarrow U \) is a ring epimorphism if and only if the induced functor of restriction of scalars for left modules \( u_*: U \mod \rightarrow R \mod \) is fully faithful, or equivalently, the similar functor for right modules \( u_*: \mod-U \rightarrow \mod-R \) is fully faithful.

A homological ring epimorphism is a ring epimorphism \( u \) such that \( \text{Tor}^R_{i}(U, U) = 0 \) for all \( i > 0 \). A ring epimorphism \( u \) is said to be left flat if \( U \) is a flat left \( R \)-module. Obviously, any (left or right) flat ring epimorphism is homological.

Let \( u: R \rightarrow U \) be a left flat ring epimorphism. Then the functor of extension of scalars \( u^*: \mod-R \rightarrow \mod-U \) taking a right \( R \)-module \( N \) to the right \( U \)-module \( u^*(N) = N \otimes_R U \) is an exact functor left adjoint to the fully faithful functor \( u_*: \mod-U \rightarrow \mod-R \). It follows that \( \mod-U \) is the quotient category of \( \mod-R \) by the localizing subcategory (hereditary torsion class) \( \mathcal{T} \) of all right \( R \)-modules annihilated by \( u^* \), that is, \( \mod-U \simeq (\mod-R)/\mathcal{T} \).

The corresponding Gabriel topology \( \mathcal{G} \) on \( R \) is the set of all right ideals \( I \subset R \) such that \( R/I \otimes_R U = 0 \). The localization functor \( L_G: \mod-R \rightarrow \mod-R \) is isomorphic to the functor of tensor product \( - \otimes_R U \). Hence the ring \( U \) together with the ring homomorphism \( u \) can be recovered from the Gabriel topology \( \mathcal{G} \) on \( R \) using the construction of the ring of quotients \( R_G \) of an associative ring \( R \) with respect to its right Gabriel topology \( \mathcal{G} \) (see Sections 2.5, 2.6). So there is a natural isomorphism of associative rings \( U \simeq R_G \) forming a commutative triangle diagram with the natural homomorphisms \( R \rightarrow R_G \) and \( u: R \rightarrow U \) [30 Theorem XI.2.1].

Following the terminology in [30], right Gabriel topologies \( \mathcal{G} \) on \( R \) corresponding to left flat ring epimorphisms \( u: R \rightarrow U \) in this way are called perfect. A left \( R \)-module \( M \) is said to be \( \mathcal{G} \)-divisible if \( T \otimes_R M = 0 \) for all \( T \in \mathcal{T} \), or equivalently, \( R/I \otimes_R M = 0 \) for all \( I \in \mathcal{G} \) (which means, in other words, that \( IM = M \) for all \( I \in \mathcal{G} \)). A right Gabriel topology \( \mathcal{G} \) on \( R \) is perfect if and only if the left \( R \)-module \( R_G \) is \( \mathcal{G} \)-divisible [30 Proposition XI.3.4]. Any perfect Gabriel topology has a base consisting of finitely generated right ideals.

Let \( \lambda \) be a cardinal. We will say that a right linear topology \( \mathcal{F} \) on an associative ring \( R \) is of type \( \lambda \) if it has a base \( \mathcal{B} \) of cardinality not exceeding \( \lambda \). A poset \( \Xi \) is said
to be $\lambda^+$-directed if for any its subset $\Upsilon \subset \Xi$ of the cardinality not exceeding $\lambda$ there exists an element $\xi \in \Xi$ such that $\xi \geq v$ for all $v \in \Upsilon$. Our aim in this section is to describe perfect Gabriel topologies on associative (and particularly, commutative) rings $R$ as $\lambda^+$-directed unions of perfect Gabriel topologies of type $\lambda$.

The following lemma is a noncommutative generalization of [15, Lemma 2.3] (see also the classical [30, Lemma VI.5.3]).

**Lemma 3.1.** Let $R$ be an associative ring and $\mathcal{G}$ be a right linear topology on $R$ with a base of finitely generated right ideals (i.e., $\mathcal{G}$ satisfies the conditions (T0–T3) of Sections 2.1–2.2). Then $\mathcal{G}$ is a Gabriel topology on $R$ (i.e., $\mathcal{G}$ satisfies the condition (T4) of Section 2.4) if and only if the following condition (T4′) is satisfied:

(T4′) if $J \in \mathcal{G}$ is a finitely generated right ideal, then for some (or equivalently, for any) finite set of generators $s_1, \ldots, s_m$ of $J$ and any right ideal $K \in \mathcal{G}$ the right ideal $s_1K + \cdots + s_mK \subset R$ belongs to $\mathcal{G}$.

**Proof.** As one can see from the arguments below, for any fixed right ideal $J \in \mathcal{G}$, the condition (T4′) remains unchanged when one varies its finite set of generators $s_1, \ldots, s_m$, provided that one is allowed to vary simultaneously the ideal $K$ as well.

“Only if”: set $I = s_1K + \cdots + s_mK$. Let $s = s_1r_1 + \cdots + s_mr_m$ be an element of $J$ (where $r_1, \ldots, r_m \in R$). By the axioms (T2) and (T3), the right ideal $H = (K : r_1) \cap \cdots \cap (K : r_m)$ belongs to $\mathcal{G}$. For any element $h \in H$, we have $sh = s_1r_1h + \cdots + s_mr_mh \in s_1K + \cdots + s_mK = I$, so $H \subset (I : s)$. By the axiom (T1), it follows that the right ideal $(I : s)$ belongs to $\mathcal{G}$. It remains to apply (T4) in order to conclude that $I \in \mathcal{G}$.

“If”: since $\mathcal{G}$ has a base of finitely generated ideals, without loss of generality we can assume the right ideal $J \in \mathcal{G}$ in (T4) to be finitely generated. By assumption, we have $(I : s) \in \mathcal{G}$ for all $s \in J$, and in particular for $s = s_1, s = s_2, \ldots, s = s_m$. Hence, by the axiom (T2), the right ideal $K = (I : s_1) \cap \cdots \cap (I : s_m)$ belongs to $\mathcal{G}$. By (T4′), we have $s_1K + \cdots + s_mK \in \mathcal{G}$. Now $s_1K + \cdots + s_mK \subset I$ by construction, hence by (T1) it follows that $I \in \mathcal{G}$. \hfill $\square$

Let $\mathcal{F}$ be a right linear topology on an associative ring $R$ and $\lambda$ be an infinite cardinal. Consider the following condition:

(T$_\lambda$) for every right ideal $I \in \mathcal{F}$, there exists a subset of right ideals $F_I \subset \mathcal{F}$ of the cardinality not exceeding $\lambda$ such that for every element $s \in R$ there exists a right ideal $J \in F_I$ for which $sJ \subset I$.

**Examples 3.2.** (1) Any linear topology of ideals in a commutative ring $R$ satisfies the condition (T$_\omega$) for the countable cardinal $\omega$ (hence also (T$_\lambda$) for any infinite cardinal $\lambda$). It suffices to take $F_I = \{I\}$ for every ideal $I \in \mathcal{F}$.

(2) More generally, any right linear topology $\mathcal{F}$ with a base $\mathcal{B}$ consisting of two-sided ideals in an associative ring $R$ satisfies (T$_\omega$). For every $I \in \mathcal{F}$, choose $J \in \mathcal{B}$ such that $J \subset I$, and set $F_I = \{J\}$.

(3) If the cardinality of (the underlying set of) an associative ring $R$ does not exceed $\lambda$, then any right linear topology on $R$ satisfies (T$_\lambda$).
More generally, let $Z \subset R$ denote the center of an associative ring $R$. Let $F$ be a right linear topology on $R$. Assume that the $Z$-module $R/I$ has a set of generators of the cardinality not exceeding $\lambda$ for every right ideal $I \in F$. Then $F$ satisfies (T$_\lambda$).

Indeed, let $(s_\gamma \in R)_{\gamma \in \Gamma}$ be a set of preimages in $R$ of a set of generators $s_\gamma \in R/I$ of the $Z$-module $R/I$. For every finite subset $\Delta \subset \Gamma$ in the set of indices, consider the right ideal $J_\Delta = \bigcap_{\delta \in \Delta} (I : s_\delta) \in F$. Then the set $F_\lambda = \{ J_\Delta \}$ all such right ideals in $R$ has the desired property, because for any $s \in \sum_{\delta \in \Delta} z_\delta s_\delta + I \subset R$, where $z_\delta \in Z$ for all $\delta \in \Delta$, one has $s J_\Delta \subset I$.

Remark 3.3. It is clear from the form of the conditions (T0–T3) and (T4) that the set of all right linear topologies on an associative ring $R$ is closed under directed unions (over nonempty sets of indices), and so is the set of all right Gabriel topologies on $R$. In other words, if $\Xi$ is a nonempty set of right linear (resp., right Gabriel) topologies on $R$ such that for any two topologies $F_1, F_2 \in \Xi$ there exists a topology $F \in \Xi$ with $F_1 \cup F_2 \subset F$, then $F = \bigcup_{F \in F_\lambda} F$ is a right linear (resp., Gabriel) topology on $R$. Moreover, if every $F \in \Xi$ has a base of finitely generated right ideals, then so does $F = \bigcup_{F \in F_\lambda} F$. Moreover, if $\Xi$ is a nonempty directed set of perfect Gabriel topologies on $R$, then the Gabriel topology $G = \bigcup_{H \in \Xi} H$ is also perfect. Indeed, by [30 Proposition XI.3.4], a Gabriel topology $G$ on $R$ is perfect if and only if the ring of quotients $R_G$ is a $G$-divisible left $R$-module, that is $R/I \otimes_R R_G = 0$ for every $I \in G$. For any two embedded Gabriel topologies $H \subset G$ on $R$, there is a natural ring homomorphism $R_H \rightarrow R_G$ forming a commutative triangle diagram with the ring homomorphisms $R \rightarrow R_H$ and $R \rightarrow R_G$. Now if $I \subset R$ is a right ideal belonging to $G$ and $G = \bigcup_{H \in \Xi} H$, then there exists $H \in \Xi$ such that $I \subset H$. Hence, if $H$ is a perfect Gabriel topology, we have $R/I \otimes_R R_G = (R/I \otimes_R R_H) \otimes_{R_H} R_G = 0$.

Proposition 3.4. Let $R$ be an associative ring, $\lambda$ be an infinite cardinal, $G$ be a right Gabriel topology on $R$ satisfying (T$_\lambda$) and having a base consisting of finitely generated right ideals, and $I \subset G$ be a subset of the cardinality not exceeding $\lambda$. Then there exists a subset $I' \subset H \subset G$ such that $H$ is a Gabriel topology on $R$ having a base of the cardinality not exceeding $\lambda$ consisting of finitely generated right ideals.

Proof. It suffices to iterate the following procedure over the set of all nonnegative integers $\omega = \mathbb{Z}_{\geq 0}$. For every right ideal $I \in I$, choose a finitely generated right ideal $I' \subset I$, $I' \in G$, and denote by $I_0 \subset G$ the set of all right ideals $I'$ so obtained. To make sure that $I_0$ is nonempty, one can also include the unit right ideal $R$ into $I_0$. Given a subset $I_n \subset G$ consisting of finitely generated right ideals, for every $I \in I_n$ consider a subset $F_I \subset G$ provided by the condition (T$_\lambda$), and choose a finitely generated right ideal $J' \subset J$, $J' \in G$, for every $J \in F_I$. Denote by $I'_n \subset G$ the union of $I_n$ with the set of all right ideals $J'$ so obtained. For every pair of right ideals $J$ and $K \in I'_n$, choose a finite set of generators $s_1, \ldots, s_m$ of the right ideal $J$, and consider the right ideal $I = s_1 K + \cdots + s_m K \subset G$. Denote by $I''_n \subset G$ the union of $I'_n$ with the set of all right ideals $J$ so obtained. For every pair of right ideals $I$ and $J \in I''_n$, choose a finitely generated right ideal $K \subset I \cap J$, $K \in G$. Denote by
I_{n+1}$ the union of $I_n'$ with the set of all right ideals $K$ so obtained. Set $B = \bigcup_{n<\omega} I_n$. Then $B \subseteq G$ is a set of finitely generated right ideals, the cardinality of $B$ does not exceed $\lambda$, and $B$ is a base of a Gabriel topology $H$ on $R$ such that $I \subseteq H \subseteq G$. □

Corollary 3.5. Let $R$ be an associative ring, $\lambda$ be an infinite cardinal, and $G$ be a right Gabriel topology on $R$ satisfying $(T_\lambda)$ and having a base of consisting of finitely generated right ideals. Let $\Xi$ denote the set of all Gabriel topologies $H \subseteq G$ having a base of the cardinality not exceeding $\lambda$ consisting of finitely generated right ideals. Then the set $\Xi$ is $\lambda^+$-directed by inclusion and one has $G = \bigcup_{H \in \Xi} H$.

Proof. Follows immediately from Proposition 3.4. □

We recall from Section 2.6 that for any right Gabriel topology $G$ on an associative ring $R$ the maximal $G$-torsion submodule $t_G(R) \subset R$ of the right $R$-module $R$ is a two-sided ideal, which can be alternatively constructed as the kernel of the ring homomorphism $R \rightarrow R_G$. A Gabriel topology $G$ on $R$ is said to be faithful if $t_G(R) = 0$. More generally, we will say that a right $R$-module $N$ has $\lambda$-bounded $G$-torsion if there exists a set of right ideals $J_0 \subset G$ of the cardinality not exceeding $\lambda$ such that for every element $b \in t_G(N)$ there is a right ideal $J \in J_0$ for which $bJ = 0$.

Clearly, if $\Xi$ is a nonempty directed set of faithful Gabriel topologies on $R$, then the Gabriel topology $\bigcup_{H \in \Xi} H$ is also faithful. If $\Xi$ is a nonempty directed set of faithful Gabriel topologies on $R$ such that the cardinality of $\Xi$ does not exceed $\lambda$ and the right $R$-module $R$ has $\lambda$-bounded $H$-torsion for every $H \in \Xi$, then $R$ has $\lambda$-bounded $G$-torsion for the Gabriel topology $G = \bigcup_{H \in \Xi} H$ as well.

Lemma 3.6. Let $R$ be an associative ring, $\Xi$ be a directed set of right Gabriel topologies on $R$, and $G = \bigcup_{H \in \Xi} H$ be their union. Let $N$ be a right $R$-module such that $t_H(N) = t_G(N)$ for all $H \in \Xi$. Then there is a natural isomorphism of right $R$-modules $N_G \cong \varprojlim_{H \in \Xi} N_H$.

Proof. For any directed set of right Gabriel topologies $\Xi$ on $R$ and any right $R$-module $N$, one has a natural diagram of right $R$-modules $(N_H)_{H \in \Xi}$ indexed by the poset $\Xi$, which is commutative together with the natural right $R$-module morphisms $N \rightarrow N_H \rightarrow N_G$. These right $R$-modules can be described in terms of the sheafification construction of Section 2.6 as $N_H = (N/t_H(N))_{(H)}$ and $N_G = (N/t_G(N))_{(G)}$.

Now when $t_H(N) = t_G(N)$ for all $H \in \Xi$, it is clear from this construction that $N_G$ is the direct limit of $N_H$ over $H \in \Xi$. □

Remark 3.7. When the ring $R$ is right $\lambda^+$-Noetherian (i.e., every right ideal in $R$ has a set of generators of the cardinality $\leq \lambda$), or the Gabriel topology $G = \bigcup_{H \in \Xi} H$ has a base consisting of right ideals with at most $\lambda$ generators and the ring $R$ is right $\lambda^+$-coherent (i.e., every right ideal in $R$ with at most $\lambda$ generators can be defined, as a right $R$-module, by a set of relations of the cardinality $\leq \lambda$), one can drop the assumption $t_H(N) = t_G(N)$ in Lemma 3.6 for a $\lambda^+$-directed set of Gabriel
Lemma 3.8. Let $R$ be an associative ring, $\Xi$ be a directed set of perfect right Gabriel topologies on $R$, and $G = \bigcup_{H \in \Xi} H$ be their union. Then there is a natural isomorphism of associative rings $R_G \simeq \lim_{H \in \Xi} R_H$.

Proof. As in Lemma 3.6, for any directed set of right Gabriel topologies $\Xi$ on $R$ one has a diagram of associative rings ($\Xi$-directed posets of indices in this case, so one has

$$N_G = (N/t_G(N))_{(G)} = (\lim_{H' \in \Xi} N/t_{H'}(N))_{(G)} = \lim_{H' \in \Xi} (N/t_{H'}(N))_{(H')} = \lim_{H \in \Xi} (N/t_H(N)) = \lim_{H \in \Xi} N_H.$$ 

Hence the $\lambda$-bounded $G$-torsion assumption can be dropped in Proposition 3.9 and Corollary 3.10 below when the ring $R$ is right $\lambda^+$-coherent.

The next lemma shows that one does not need the condition on torsion when dealing with directed unions of perfect Gabriel topologies.

Lemma 3.8. Let $R$ be an associative ring, $\Xi$ be a directed set of perfect right Gabriel topologies on $R$, and $G = \bigcup_{H \in \Xi} H$ be their union. Then there is a natural isomorphism of associative rings $R_G \simeq \lim_{H \in \Xi} R_H$.

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Proof. As in Lemma 3.6, for any directed set of right Gabriel topologies $\Xi$ on $R$ one has a diagram of associative rings ($\Xi$-directed posets of indices in this case, so one has

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Hence the $\lambda$-bounded $G$-torsion assumption can be dropped in Proposition 3.9 and Corollary 3.10 below when the ring $R$ is right $\lambda^+$-coherent.

The next lemma shows that one does not need the condition on torsion when dealing with directed unions of perfect Gabriel topologies.
in $R$ of the cardinality not exceeding $\lambda$ such that for every $r \in t_G(R)$ there exists $J \in \mathcal{J}_0$ for which $rJ = 0$. By Proposition 3.4, one can choose a right Gabriel topology $H_0$ on $R$ with a base of the cardinality not exceeding $\lambda$ such that $I \cup \mathcal{J}_0 \subset H_0 \subset G$.

Let $H_n$ be another right Gabriel topology with a base of $\mathcal{B}_n$ the cardinality not exceeding $\lambda$ such that $H_0 \subset H_n \subset G$. By Corollary 3.5, the set $\Xi$ of all right Gabriel topologies $H$ on $R$ having a base of the cardinality not exceeding $\lambda$ consisting of finitely generated right ideals and such that $H_0 \subset H \subset G$ is $\lambda^+$-directed by inclusion, and one has $G = \bigcup_{H \in \Xi} H$.

As in the proof of Lemma 3.8, we have a diagram of associative rings and ring homomorphisms $(R_H)_{H \in \Xi}$ indexed by the poset $\Xi$, which is commutative together with the ring homomorphisms $R \to R_H \to R_G$. Notice that $t_G(R) = t_{H_n}(R)$ for all $H \in \Xi$, as $\mathcal{J}_0 \subset H$. By Lemma 3.6, we have $R_G = \varinjlim_{H \in \Xi} R_H$.

Since $G$ is a perfect Gabriel topology, for any right ideal $I \in G$ one has $R/I \otimes_R R_G = 0$. In particular, this holds for any right ideal $I$ belonging to our base $\mathcal{B}_n$ of the Gabriel topology $H_n$. Since the ring $R_G$ is a direct limit of the rings $R_{H_0}$, there exists a Gabriel topology $H(I) \in \Xi$ such that $R/I \otimes_R R_{H(I)} = 0$. Choose a Gabriel topology $H_{n+1}$ with a base of the cardinality not exceeding $\lambda$ such that $H_n \subset H_{n+1} \subset G$ and $H(I) \subset H_{n+1}$ for all $I \in \mathcal{B}_n$. Then $\mathcal{P} = \bigcup_{n<\omega} H_n$ is a perfect Gabriel topology on $R$ with a base of the cardinality not exceeding $\lambda$, and one has $1 \subset \mathcal{P} \subset G$.

**Corollary 3.10.** Let $R$ be an associative ring, $\lambda$ be an infinite cardinal, and $G$ be a perfect right Gabriel topology on $R$ satisfying $(T_\lambda)$ and such that the right $R$-module $R$ has $\lambda$-bounded $G$-torsion. Let $\Upsilon$ denote the set of all perfect Gabriel topologies $\mathcal{P} \subset G$ having a base of the cardinality not exceeding $\lambda$. Then the set $\Upsilon$ is $\lambda^+$-directed by inclusion and one has $G = \bigcup_{\mathcal{P} \in \Upsilon} \mathcal{P}$. Moreover, the ring of quotients of $R$ with respect to $G$ is the direct limit over $\Upsilon$ of the rings of quotients of $R$ with respect to $\mathcal{P}$,

$$R_G = \varinjlim_{\mathcal{P} \in \Upsilon} R_\mathcal{P}.$$  

**Proof.** Follows from Proposition 3.9 and Lemma 3.8

4. **F-Systems and Pseudo-F-Systems**

In this section we discuss some techniques of working with contramodules over topological associative rings with right linear topology, which were developed originally in [26, Section 6]. Here we rephrase them in a more intuitively accessible language. These techniques become particularly powerful when the topology on the ring has a countable base, as we will see in the next Section 5.

Let $R$ be an associative ring, $\mathcal{F}$ be a right linear topology on $R$, and $\mathcal{F} = \varprojlim_{I \in \mathcal{J}} R/I$ be the completion of $R$ with respect to $\mathcal{F}$, viewed as a topological ring in the projective limit topology $\mathcal{F}$. There is a natural continuous (completion) homomorphism of topological rings $\rho: R \to \mathcal{F}$.

Notice that there is a natural bijection between the set $\mathcal{F}$ of all open right ideals $I \subset R$ and the set $\mathcal{F}$ of all open right ideals $J \subset \mathcal{F}$, given by the rules $I = \rho^{-1}(J)$ and
\[ \mathcal{I} = \lim_{\leftarrow j \in \mathbb{F}, j \subseteq I} I/J = \ker(\mathfrak{R} \to R/I). \] The ring homomorphism \( \rho: R \to \mathfrak{R} \) induces an isomorphism of quotient groups/modules \( R/I \cong \mathfrak{R}/\mathcal{I} \).

Denote the full subcategory of cyclic discrete right modules \( R/I \) in \( \text{discr}-R \) by \( Q_\mathbb{F} \).

The objects of \( R \) are indexed by the open right ideals \( I \in \mathbb{F} \).

According to (T2), the poset of open right ideals \( \mathbb{F} \) is downwards directed by inclusion. When the need arises to distinguish this poset from the topology \( \mathbb{F} \) on the ring \( R \), we will denote the poset \( \mathbb{F} \), and the category associated with it, by \( \Pi_\mathbb{F} \). For any pair of open right ideals \( J \subseteq I \subseteq R \), one of which is contained in the other one, there is the natural surjective morphism of right \( R \)-modules \( R/J \to R/I \). Hence \( \Pi_\mathbb{F} \) is naturally a subcategory in \( Q_\mathbb{F} \) (with the same set of objects). The morphisms in \( \Pi_\mathbb{F} \) are exactly those morphisms of right \( R \)-modules \( R/J \to R/I \) which form a commutative triangle diagram with the projections \( R \to R/I \) and \( R \to R/J \).

According to Section 2.3 the categories of discrete right modules over the topological rings \( R \) and \( \mathfrak{R} \) are naturally equivalent (in fact, isomorphic), \( \text{discr}-R \cong \text{discr}-\mathfrak{R} \).

The restriction of this equivalence of additive categories to the full subcategories \( Q_\mathbb{F} \subset \text{discr}-R \) and \( Q_\mathbb{R} \subset \text{discr}-\mathfrak{R} \) provides an equivalence of preadditive categories \( Q_\mathbb{F} \cong Q_\mathbb{R} \), which further restricts to an equivalence of categories \( \Pi_\mathbb{F} \cong \Pi_\mathbb{R} \).

Furthermore, given an additive category \( A \), let \( \text{Com}(A) \) denote the additive category of \( (\mathbb{Z}\text{-graded}) \) complexes in \( A \). Given an associative ring \( R \) with a right linear topology \( \mathbb{F} \), let us consider the full subcategory in the category of complexes of right \( R \)-modules \( \text{Com}(\text{mod}-R) \) formed by the two-term complexes \( (I \to R) \), where \( I \in \mathbb{F} \) ranges over the open right ideals in \( R \) and the map \( I \to R \) is the identity inclusion. Here the term \( I \) of the complex \( (I \to R) \) sits in the cohomological degree 0 and the term \( R \) in the cohomological degree 1. We denote this full subcategory by \( P_\mathbb{F} \subset \text{Com}(\text{mod}-R) \).

The abelian groups of morphisms in the preadditive category \( P_\mathbb{F} \) can be easily described. For any two open right ideals \( I \) and \( J \in \mathbb{F} \), a morphism of complexes \( (J \to R) \to (I \to R) \) is uniquely determined by its action on the terms \( R \) in degree 1 (because the map \( I \to R \) is injective). The group of all such morphisms is naturally isomorphic to the subgroup in \( R \) consisting of all the elements \( s \in R \) such that \( sJ \subseteq I \). An element \( s \) satisfying this condition corresponds to the morphism \( (J \to R) \overset{s}{\to} (I \to R) \) of left multiplication with \( s \).

There is a natural additive functor \( H^1: P_\mathbb{F} \to Q_\mathbb{F} \) induced by the passage to the discrete right \( R \)-modules of cohomology \( H^1(I \to R) = R/I \) of the two-term complexes of right \( R \)-modules \( (I \to R) \). This functor is bijective on objects and surjective on morphisms. The kernel of the map \( \text{Hom}_{P_\mathbb{F}}(J \to R, I \to R) \to \text{Hom}_{Q_\mathbb{F}}(R/J, R/I) \) consists of all the morphisms \( (J \to R) \overset{s}{\to} (I \to R) \) for which the element \( s \) belongs to the right ideal \( I \).

The inclusion \( \Pi_\mathbb{F} \to Q_\mathbb{F} \) of the category corresponding to the poset \( \mathbb{F} \) into the category \( Q_\mathbb{F} \) lifts naturally to an inclusion \( \Pi_\mathbb{F} \to P_\mathbb{F} \). For any two open right ideals \( J \subseteq I \subseteq R \), the only morphism \( J \to I \) in \( \Pi_\mathbb{F} \) corresponds to the morphism \( (J \to R) \overset{1}{\to} (I \to R) \) in \( P_\mathbb{F} \) acting by the identity inclusion \( J \to I \) in degree 0 and by the identity isomorphism \( \text{id}_R \) in degree 1.
Notice that, unlike the categories $\Pi_{\mathcal{F}}$ and $Q_{\mathcal{F}}$, the category $P_{\mathcal{F}}$ does change when one passes from a topological ring $R$ with a right linear topology $\mathcal{F}$ to its completion $\mathcal{R}$ with its projective limit topology $\mathcal{G}$. The natural functors $\Pi_{\mathcal{F}} \to \Pi_{\mathcal{G}}$ and $Q_{\mathcal{F}} \to Q_{\mathcal{G}}$ are isomorphisms of categories, but the natural additive functor $P_{\mathcal{F}} \to P_{\mathcal{G}}$ is not an equivalence (even though it is bijective on objects).

More specifically, let $I \in \mathcal{F}$ be an open right ideal in $R$ and $J \in \mathcal{G}$ be the corresponding open right ideal in $\mathcal{R}$. Then the functor $P_{\mathcal{F}} \to P_{\mathcal{G}}$ assigns the two-term complex $(J \to R)$ to the two-term complex $(I \to R)$. To construct the action of this functor on morphisms, one can use the explicit description of morphisms in the category $P_{\mathcal{F}}$ given above. For any morphism $(J \to R) \xrightarrow{s} (I \to R)$ in $P_{\mathcal{F}}$, the image $\rho(s) \in \mathcal{R}$ of the element $s \in R$ defines the corresponding morphism $(J \to \mathcal{R}) \xrightarrow{\rho(s)} (J \to \mathcal{R})$.

In particular, one can observe that the functor $P_{\mathcal{F}} \to P_{\mathcal{G}}$ takes the two-term complex of right $R$-modules $(R \to R)$ to the two-term complex of right $\mathcal{R}$-modules $(\mathcal{R} \to \mathcal{R})$. One has $\text{Hom}_{P_{\mathcal{F}}}(R \to R, R \to R) = R \not= \mathcal{R} = \text{Hom}_{P_{\mathcal{G}}}(\mathcal{R} \to \mathcal{R}, \mathcal{R} \to \mathcal{R})$. For comparison, the images of these two-term complexes in the categories $Q_{\mathcal{F}}$ and $Q_{\mathcal{G}}$ are, of course, zero objects.

Let $A$ be an additive category. By a covariant pseudo-$\mathcal{F}$-system in $A$ we will mean a covariant additive functor $P_{\mathcal{F}} \to A$. Similarly, a contravariant pseudo-$\mathcal{F}$-system in $A$ is a contravariant additive functor $P_{\mathcal{F}} \to A$. We will denote the category of covariant pseudo-$\mathcal{F}$-systems in $A$ by $\mathcal{F}A = \text{Add}(P_{\mathcal{F}}, A)$ and the category of contravariant pseudo-$\mathcal{F}$-systems in $A$ by $A_{\{\mathcal{F}\}} = \text{Add}(P_{\mathcal{F}}^{\text{op}}, A)$ (where $\text{Add}(C, A)$ denotes the category of additive functors $C \to A$ and $C^{\text{op}}$ is the opposite category to $C$). When the category $A$ is abelian, so are the categories $\mathcal{F}A$ and $A_{\{\mathcal{F}\}}$.

Composing an additive functor $P_{\mathcal{F}}^{\text{op}} \to A$ or $P_{\mathcal{G}} \to A$ with the natural additive functor $P_{\mathcal{F}} \to P_{\mathcal{G}}$ discussed above, one can construct the underlying pseudo-$\mathcal{F}$-system of a (contravariant or covariant) pseudo-$\mathcal{G}$-system.

A left $R$-module object $M$ in $A$ is an object endowed with a left action of $R$, i.e., with a ring homomorphism $R \to \text{Hom}_A(M, M)$. Similarly, a right $R$-module object $N$ in $A$ is an object endowed with a right action of $R$, i.e., with a ring homomorphism $R^{\text{op}} \to \text{Hom}_A(N, N)$. Denote the category of left $R$-module objects in $A$ by $_{R}A$ and the category of right $R$-module objects in $A$ by $A_{R}$.

**Lemma 4.1.** (a) Let $A$ be an additive category with direct limits and $M : P_{\mathcal{F}}^{\text{op}} \to A$ be a contravariant pseudo-$\mathcal{F}$-system in $A$. Then the object

$$\text{il}(M) = \lim_{\leftarrow I \in \mathcal{F}} M(I \to R) \in A,$$

where the direct limit is taken over the poset $\Pi_{\mathcal{F}}^{\text{op}}$, carries a natural right action of the ring $R$.

(b) Let $A$ be an additive category with inverse limits and $D : P_{\mathcal{F}} \to A$ be a covariant pseudo-$\mathcal{F}$-system in $A$. Then the object

$$\text{pl}(D) = \lim_{\rightarrow I \in \mathcal{F}} D(I \to R) \in A,$$

where the inverse limit is taken over the poset $\Pi_{\mathcal{F}}$, carries a natural left action of the ring $R$. 

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The abbreviation “il” stands for “inductive limit”, while the abbreviation “pl” means “projective limit”. Let us construct the left action of $R$ in the object $pl(D)$ in part (b) (part (a) being dual). Given an element $r \in R$, for every open right ideal $I \in F$ we have the morphism $((I : r) \to R) \to (I \to R)$ in the category $P_F$. Consider the natural projection $pl(D) \to D((I : r) \to R)$ from the inverse limit to one of the objects in the diagram, and its composition

$$pl(D) \to D((I : r) \to R) \xrightarrow{D(r)} D(I \to R)$$

with the morphism $D(r)$. The collection of such morphisms $pl(D) \to D(I \to R)$, defined for all the open right ideals $I \in F$, forms a compatible cone (i.e., a commutative diagram) with all the morphisms in the diagram $D|_{P_F}: \Pi_F \to A$. Hence we obtain the induced morphism

$$pl(D) \to \lim_{I \in F} D(I \to R) = pl(D),$$

providing the desired left action of the element $r \in R$ in the object $pl(D) \in A$. It is straightforward to check that the map $R \to \text{Hom}_A(pl(D), pl(D))$ constructed in this way is a ring homomorphism, using the assumption that $D: P_F \to A$ is an additive functor.

The functor $il: A_{\{F\}} \to A_R$ is left adjoint to the “constant contravariant pseudo-$F$-system” functor $A_R \to A_{\{F\}}$. The latter functors assigns to every right $R$-module object $N$ in $A$ the contravariant pseudo-$F$-system taking an object $(I \to R) \in P_F$ to the object $N \in A$ for every $I \in F$ and a morphism $(J \to R) \to (I \to R)$ in $P_F$ to the right multiplication morphism $s: N \to N$. Similarly, the functor $pl: \{F\}A \to RA$ is right adjoint to the “constant covariant pseudo-$F$-system” functor $RA \to \{F\}A$, which assigns to every left $R$-module object $C$ in $A$ the covariant pseudo-$F$-system taking an object $(I \to R) \in P_F$ to the object $C \in A$ for every $I \in F$ and a morphism $(J \to R) \to (I \to R)$ in $P_F$ to the left multiplication morphism $s: C \to C$.

Let $A$ be an additive category. By the definition, a contravariant $F$-system in $A$ is a contravariant additive functor $Q_F \to A$. A covariant $F$-system in $A$ is a covariant additive functor $Q_F \to A$. We will denote the category of contravariant $F$-systems in $A$ by $A_F$ and the category of covariant $F$-systems in $A$ by $F_A$. When the category $A$ is abelian, so are the categories $A_F$ and $F_A$. Composing an additive functor $Q_F^{op} \to A$ or $Q_F \to A$ with the additive functor $H^1: P_F \to Q_F$, one can view any (covariant or contravariant) $F$-system in $A$ as a (respectively, covariant or contravariant) pseudo-$F$-system. This makes the category of $F$-systems a full subcategory in the category of pseudo-$F$-systems, $F_A \subset \{F\}A$ and $A_F \subset A_{\{F\}}$.

We recall the notation $\text{Ab}$ for the category of abelian groups. We also recall that every left $\mathcal{R}$-contramodule $\mathcal{C}$ has the underlying left $\mathcal{R}$-module structure. The restriction of scalars with respect to the ring homomorphism $\rho: R \to \mathcal{R}$ makes $\mathcal{C}$ a left $R$-module. In this context, we will speak of “the underlying left $R$-module structure” of a left $\mathcal{R}$-contramodule $\mathcal{C}$.
 Proposition 4.2. (a) The functor \( il : \text{Ab}_F \to \text{mod}_R \) constructed in Lemma 4.1(a) takes contravariant \( F \)-systems of abelian groups to discrete right \( R \)-modules. The resulting functor \( IL : \text{Ab}_F \to \text{discr}_R \) has a right adjoint functor \( DH : \text{discr}_R \to \text{Ab}_F \). The functor \( DH \) assigns to every discrete right \( R \)-module \( N \) the restriction of the contravariant functor of discrete right \( R \)-module homomorphisms \( \text{Hom}_{\text{R}^\text{op}}(-, N) : \text{discr}_R \to \text{Ab} \) to the full subcategory \( Q_F \subset \text{discr}_R \).

(b) The left \( R \)-module \( \text{pl}(D) \) assigned by the functor \( \text{pl} : \{ F \} \text{Ab} \to \text{R-mod} \) of Lemma 4.1(b) to a covariant \( F \)-system of abelian groups \( D : \text{Q}_F \to \text{Ab} \) is the underlying \( R \)-module of a naturally defined left \( \mathcal{R} \)-contramodule \( \text{PL}(D) \). The resulting functor \( \text{PL} : \mathcal{F} \text{Ab} \to \mathcal{R}\text{-contra} \) has a left adjoint functor \( \text{CT} : \mathcal{R}\text{-contra} \to \mathcal{F}\text{Ab} \). The functor \( \text{CT} \) assigns to every \( \mathcal{R} \)-contramodule \( \mathcal{C} \) the restriction of the contravariant functor of contratensor product \( - \odot_{\mathcal{R}} \) to the full subcategory \( Q_F = Q_S \subset \text{discr}_R \).

Proof. The abbreviation \( DH \) stands for “discrete module \( \text{Hom} \)”, while the abbreviation \( \text{CT} \) means “contratensor product”. Part (a): let \( M : \text{Q}_F^{\text{op}} \to \text{Ab} \) be a contravariant \( F \)-system of abelian groups. Let \( b \in \text{il}(M) \) be an element represented by an element \( \bar{b} \in M(R/I) \) for some open right ideal \( I \subset R \), and let \( r \in R \) be an element. Consider the discrete right \( R \)-module morphism \( R/(I : r) \overset{r}{\to} R/I \) of left multiplication with \( r \). Applying the contravariant functor \( M \) to this morphism, we obtain an abelian group homomorphism \( M(r) : M(R/I) \to M(R/(I : r)) \). By the definition, the element \( br \in \text{il}(N) \) is represented by the element \( M(r)(b) \in M(R/(I : r)) \).

To check that \( IL(M) = \text{il}(M) \) is a discrete right \( R \)-module, one observes that \( bl = 0 \) in \( \text{il}(M) \) whenever an element \( b \in \text{il}(M) \) is represented by an element \( \bar{b} \in M(R/I) \). Indeed, for any \( r \in I \) one has \( (I : r) = R \), so \( R/(I : r) = 0 \) is a zero object of the preadditive category \( \text{Q}_F \), and \( M(0) = 0 \) for an additive functor \( M \).

The adjunction isomorphism \( \text{Hom}_{\text{discr}_R}(IL(M), N) \simeq \text{Hom}_{\text{Ab}_F}(M, DH(N)) \) holds for any contravariant \( F \)-system of abelian groups \( M \) and any discrete right \( R \)-module \( N \), because the datum of a right \( R \)-module morphism

\[
\lim_{I \in \mathcal{F}} M(R/I) \longrightarrow N
\]

is equivalent to the datum of an \( F \)-indexed family of abelian group homomorphisms

\[
M(R/I) \longrightarrow \text{Hom}_{\text{R}^\text{op}}(R/I, N) \subset N
\]
satisfying the compatibility equation for all the morphisms in the category \( \text{Q}_F \). It is helpful to keep in mind that every morphism in \( \text{Q}_F \) has the form of a composition \( R/J \to R/(I : s) \overset{s}{\to} R/I \), where \( I \subset (I : s) \) are open right ideals, \( s \in R \) is an element, \( R/J \to R/(I : s) \) is the morphism in \( \Pi_F \), and \( R/(I : s) \overset{s}{\to} R/I \) is the morphism of left multiplication with \( s \).

Part (b): this is [26, Lemma 6.2(a,c)] (notice that the existence of a countable base of the topology on \( \mathcal{R} \), which is the running assumption in [26, Section 6], is not yet used in [26, Lemma 6.2]). It will be convenient for us to work with covariant \( \mathcal{F} \)-systems of abelian groups instead of the covariant \( F \)-systems here. Then one has \( \text{CT}(\mathcal{C})(\mathcal{R}/J) = \mathcal{R}/J \odot_{\mathcal{R}} \mathcal{C} = \mathcal{C}/(J \prec \mathcal{C}) \) for any left \( \mathcal{R} \)-contramodule \( \mathcal{C} \) and open
right ideal \( \mathcal{J} \subset \mathcal{R} \). Given an element \( s \in \mathcal{R} \) and a pair of open right ideals \( \mathcal{J} \) and \( \mathcal{J} \subset \mathcal{R} \) such that \( s \mathcal{J} \subset \mathcal{J} \), applying the functor \( \text{CT}(\mathcal{E}) : \mathbb{Q}_{\mathcal{R}} \to \text{Ab} \) to the morphism \( \mathcal{R}/\mathcal{J} \to \mathcal{R}/\mathcal{J} \) in the category \( \mathbb{Q}_{\mathcal{R}} \) produces the map \( \mathcal{E}/(\mathcal{J} \times \mathcal{E}) \to \mathcal{E}/(\mathcal{J} \times \mathcal{E}) \) induced by the abelian group homomorphism \( s : \mathcal{E} \to \mathcal{E} \) of left multiplication with \( s \).

The functor \( \text{PL} : \mathbb{Q}_{\mathcal{R}} \text{Ab} \to \mathcal{R}-\text{contra} \) assigns to a covariant \( \mathcal{F} \)-system of abelian groups \( D : \mathbb{Q}_{\mathcal{R}} \to \text{Ab} \) the abelian group

\[
\text{PL}(D) = \varprojlim_{\mathcal{J} \in \mathcal{F}} D(\mathcal{R}/\mathcal{J}),
\]

where the projective limit is taken over the directed poset \( \Pi_{\mathcal{F}} \), endowed with the following left \( \mathcal{R} \)-contramodule structure. Denote by \( \psi_{2} \) the natural projection map \( \text{PL}(D) \to D(\mathcal{R}/\mathcal{J}) \). For any open right ideal \( \mathcal{J} \subset \mathcal{R} \) and an element \( r \in \mathcal{R} \), we have the related morphism \( \mathcal{R}/(\mathcal{J} : r) \to \mathcal{R}/\mathcal{J} \) in the category \( \mathbb{Q}_{\mathcal{R}} \). Set \( \mathcal{E} = \text{PL}(D) \); and suppose that we are given an element \( r = \sum_{e \in \mathcal{E}} r_{e} \in \mathcal{R}[[\mathcal{E}]] \). Then the element \( \pi_{e}(r) \in \mathcal{E} = \text{PL}(D) \) is defined by the rule

\[
\psi_{2}(\pi_{e}(r)) = \sum_{e \in \mathcal{E}} D(r_{e})(\psi_{2}(r_{e}))(e),
\]

where \( \psi_{2}(r_{e})(e) \in D(\mathcal{R}/(\mathcal{J} : r_{e})) \) and \( D(r_{e}) : D(\mathcal{R}/(\mathcal{J} : r_{e})) \to D(\mathcal{R}/\mathcal{J}) \). The sum in the right-hand side is finite, because one has \( r_{e} \in \mathcal{J} \) for all but a finite subset of elements \( e \in \mathcal{E} \), and \( r_{e} \in \mathcal{J} \) implies \( (\mathcal{J} : r_{e}) = \mathcal{R} \), so \( \mathcal{R}/(\mathcal{J} : r_{e}) = 0 \) is a zero object in \( \mathbb{Q}_{\mathcal{R}} \), and \( D(0) = 0 \) for an additive functor \( D : \mathbb{Q}_{\mathcal{R}} \to \text{Ab} \).

The adjunction isomorphism \( \text{Hom}_{\mathcal{R}}(\mathcal{E}, \text{PL}(D)) \simeq \text{Hom}_{\mathcal{R}}(\text{CT}(\mathcal{E}), D) \) holds for any left \( \mathcal{R} \)-contramodule \( \mathcal{E} \) and any covariant \( \mathcal{F} \)-system of abelian groups \( D \), because the datum of a left \( \mathcal{R} \)-contramodule morphism

\[
\mathcal{E} \longrightarrow \lim_{\mathcal{J} \in \mathcal{F}} D(\mathcal{R}/\mathcal{J})
\]

is equivalent to the datum of an \( \mathcal{F} \)-indexed family of abelian group homomorphisms

\[
\mathcal{E}/(\mathcal{J} \times \mathcal{E}) \longrightarrow D(\mathcal{R}/\mathcal{J})
\]

satisfying the compatibility equations for all the morphisms in the category \( \mathbb{Q}_{\mathcal{R}} \). The argument is similar to the one in part (a), and it is helpful to observe, from the construction of the left \( \mathcal{R} \)-contramodule structure on \( \mathcal{E} = \text{PL}(D) \) above, that one has \( \psi_{2}(e) = 0 \) for any \( e \in \mathcal{J} \times \mathcal{E} \).

\[\square\]

5. SEPARATED CONTRAMODULES

We keep the notation of the previous Section 4. So \( R \) is a topological associative ring with a right linear topology \( \mathbb{F} \) and \( \mathcal{R} \) is the completion of \( R \) with respect to \( \mathbb{F} \), viewed as a topological ring in the projective limit topology \( \mathcal{F} \).

Let \( J \subset I \subset R \) be a pair of open right ideals in \( R \), one of which is contained in the other one. The following exact sequence of discrete right \( R \)-modules will be important for us:

\[
(1) \quad \bigoplus_{s \in I} R/(J : s) \xrightarrow{(s)} R/J \longrightarrow R/I \longrightarrow 0,
\]
where the \( s \)-indexed component of the map \((s)_{s \in I}: \bigoplus_{s \in I} R/(J : s) \to R/J \) is the right \( R \)-module morphism \( R/(J : s) \to R/J \) acting by the left multiplication with \( s \).

Whenever \( A \) is a complete abelian category, we will say that a contravariant \( F \)-system \( M: Q_{\text{op}}^F \to A \) is left exact if for any two open right ideals \( I \subset J \subset R \) the short sequence of objects of \( A \)

\[
0 \to M(R/I) \to M(R/J) \to \prod_{s \in I} M(R/(J : s))
\]

obtained by applying the contravariant functor \( M \) to the sequence (1) (and replacing formally the undefined image of the direct sum with the product of the images) is left exact in \( A \).

Similarly, whenever \( A \) is a cocomplete abelian category, we will say that a covariant \( F \)-system \( D: Q_{\text{op}} \to A \) is right exact if for any two open right ideals \( I \subset J \subset R \) the short sequence of objects of \( A \)

\[
\prod_{s \in I} D(R/(J : s)) \to D(R/J) \to D(R/I) \to 0
\]

obtained by applying the covariant functor \( D \) to the sequence (1) (and replacing formally the undefined image of the direct sum with the coproduct of the images) is right exact in \( A \).

**Proposition 5.1.** In the context of Proposition 4.2(a), the composition \( \text{IL} \circ \text{DH} \) of the two adjoint functors \( \text{DH}: \text{discr}-R \to \text{Ab}_F \) and \( \text{IL}: \text{Ab}_F \to \text{discr}-R \) is the identity functor \( \text{discr}-R \to \text{discr}-R \). A contravariant \( F \)-system of abelian groups \( Q_{\text{op}}^{\text{op}} \to \text{Ab} \) belongs to the image of the functor \( \text{DH} \) if and only if it is left exact. So the functor \( \text{DH} \) is fully faithful, and it provides an equivalence between the category of discrete right \( R \)-modules \( \text{discr}-R \) and the full subcategory in \( \text{Ab}_F \) formed by all the left exact contravariant \( F \)-systems of abelian groups.

**Proof.** For every discrete right \( R \)-module \( N \), the contravariant \( F \)-system of abelian groups \( \text{DH}(N) \) is left exact, because the functor \( \text{Hom}_R(-, N): \text{discr}-R \to \text{Ab} \) is left exact. The adjunction morphism \( \text{IL}(\text{DH}(N)) \to N \) is an isomorphism representing \( N \) as the direct limit/union \( N = \varinjlim_{I \in \mathcal{F}} N_I \) of its subgroups \( \text{Hom}_R(R/I, N) = N_I \subset N \) of all elements annihilated by a given open right ideal \( I \subset R \).

To prove that all the left exact contravariant \( F \)-systems of abelian groups belong to the essential image of the functor \( \text{DH} \), one can argue as follows. By the Special Adjoint Functor Theorem \([11, \text{Corollary 5.57}]\), every contravariant functor \( \text{discr}-R \to \text{Ab} \) taking colimits to limits is representable by an object of \( \text{discr}-R \), since \( \text{discr}-R \) is a cocomplete abelian category with a set of generators (formed, e. g., by the cyclic discrete right \( R \)-modules \( R/I \)). It remains to show that any left exact contravariant \( F \)-system of abelian groups \( M \) can be extended to such a functor. For this purpose, one can use the construction of the Kan extension, setting

\[
G_M(N) = \varprojlim_{R/I \to N} M(R/I) \quad \text{for every } N \in \text{discr}-R,
\]

where the projective limit is taken over the diagram formed by all the morphisms of discrete right \( R \)-modules \( R/I \to N \) (indexing the vertices of the diagram) and
all the commutative triangles $R/I \to R/J \to N$ (indexing the arrows), where $R/I \to R/J$ ranges over the morphisms in $Q_\mathbb{R}$. For any contravariant $\mathbb{F}$-system $M$, the functor $G_M$ takes coproducts to products (since the right $R$-modules $R/I$ are finitely generated); and it is straightforward to check that the contravariant functor $G_M$ is left exact whenever the contravariant $\mathbb{F}$-system $M$ is.

Alternatively, one can prove explicitly from the construction that the adjunction morphism $M \to \text{DH}(\text{IL}(M))$ is an isomorphism for every left exact contravariant $\mathbb{F}$-system of abelian groups $M$. For this purpose, one needs to check that the subgroup of all elements annihilated by an open right ideal $I \subset R$ in the discrete right $R$-module $\text{IL}(M) = \lim_{J \in \mathcal{F}} M(R/J)$ coincides with the image of the (injective) map $M(R/I) \to \lim_{J \in \mathcal{F}} M(R/J)$. This is but a restatement of the definition of left exactness of a contravariant $\mathbb{F}$-system.

Let $\mathcal{C}$ be a left $\mathcal{R}$-contramodule. Denote by $\mathbb{B}_\mathcal{C}$ the set of all subgroups $\mathcal{I} \triangleleft \mathcal{C} \subset \mathcal{C}$ of the underlying abelian group of $\mathcal{C}$, where $\mathcal{I}$ ranges over the open right ideals in $\mathcal{R}$. Then $\mathbb{B}_\mathcal{C}$ is a linear topology base on the underlying abelian group of $\mathcal{C}$, as $\mathbb{B}_\mathcal{C}$ is nonempty and

$$(\mathcal{I} \triangleleft \mathcal{C}) \cap (\mathcal{J} \triangleleft \mathcal{C}) \supset (\mathcal{I} \cap \mathcal{J}) \triangleleft \mathcal{C}$$

for any two open right ideals $\mathcal{I}$ and $\mathcal{J} \subset \mathcal{R}$. Notice a base of neighborhoods of zero in this topology on $\mathcal{C}$ is formed by subgroups and not submodules; so this is not a linear topology on a module in the sense of [30] Section VI.4 (it is instructive to observe that topologies on right modules over a right linearly topological ring are considered in [30] Section VI.4), while our $\mathcal{C}$ is a left $\mathcal{R}$-module).

A left $\mathcal{R}$-contramodule $\mathcal{C}$ is said to be separated (respectively, complete) if it is a separated (resp., complete) abelian group in the topology with a base $\mathbb{B}_\mathcal{C}$. In other words, $\mathcal{C}$ is called separated if the natural map of abelian groups $\lambda_{\mathcal{R},\mathcal{C}}: \mathcal{C} \to \varprojlim_{\mathcal{I} \in \mathbb{B}_\mathcal{C}} \mathcal{C}/(\mathcal{I} \triangleleft \mathcal{C})$ is injective, and $\mathcal{C}$ is complete if this map is surjective. Clearly, $\mathcal{C}$ is separated if and only if $\bigcap_{\mathcal{I} \in \mathbb{B}_\mathcal{C}} \mathcal{I} \triangleleft \mathcal{C} = 0$. Any $\mathcal{R}$-subcontramodule of a separated left $\mathcal{R}$-contramodule is separated.

**Proposition 5.2.** In the context of Proposition 4.2(b), for any covariant $\mathcal{F}$-system of abelian groups $D$ the left $\mathcal{R}$-contramodule $\text{PL}(D)$ is separated. For any left $\mathcal{R}$-contramodule $\mathcal{C}$, the covariant $\mathcal{F}$-system of abelian groups $\text{CT}(\mathcal{C})$ is right exact. For any left $\mathcal{R}$-contramodule $\mathcal{C}$, the abelian group $\varprojlim_{\mathcal{I} \in \mathbb{B}_\mathcal{C}} \mathcal{C}/(\mathcal{I} \triangleleft \mathcal{C})$ is the underlying abelian group of the left $\mathcal{R}$-contramodule $\text{PL}(\text{CT}(\mathcal{C}))$, and $\lambda_{\mathcal{R},\mathcal{C}}$ is the adjunction morphism. In particular, $\lambda_{\mathcal{R},\mathcal{C}}$ is a morphism of left $\mathcal{R}$-contramodules, and its kernel $\bigcap_{\mathcal{I} \in \mathbb{B}_\mathcal{C}} \mathcal{I} \triangleleft \mathcal{C}$ is an $\mathcal{R}$-subcontramodule in $\mathcal{C}$.

**Proof.** The first assertion is [26] Lemma 6.2(b)], while the second one is explained in [26] paragraphs preceding Lemma 6.2]. The covariant $\mathcal{F}$-system $\text{CT}(\mathcal{C})$ is right exact for every left $\mathcal{R}$-contramodule $\mathcal{C}$, because the functor $- \circ_\mathcal{R} \mathcal{C}: \text{discr} \mathcal{R} \to \text{Ab}$ is right exact. The left $\mathcal{R}$-contramodule $\mathcal{C} = \text{PL}(D)$ is separated for any covariant $\mathcal{F}$-system $D$, because, in the notation of the proof of Proposition 4.2(b), one has $\psi_\mathcal{D}(e) = 0$ for any $e \in \mathcal{I} \triangleleft \mathcal{C}$, so $e \in \bigcap_{\mathcal{I} \in \mathbb{B}_\mathcal{C}} \mathcal{I} \triangleleft \mathcal{C}$ implies $e = 0$. The remaining assertions
follow from the constructions of the functors CT and PL and the construction of the adjunction between them in Proposition 4.2(b).

□

Theorem 5.3. Assume that a complete, separated right linear topology \( \mathcal{F} \) on a ring \( R \) has a countable base. Then all left \( R \)-contramodules are complete, so the adjunction/completion morphism \( \lambda_{R,C} : C \to PL(CT(C)) \) is surjective for any left \( R \)-contramodule \( C \). A left \( R \)-contramodule is separated if and only if it belongs to the essential image of the functor PL. A covariant \( \mathcal{F} \)-system of abelian groups is right exact if and only if it belongs to the essential image of the functor CT. The restrictions of the adjoint functors CT and PL are mutually inverse equivalences between the full subcategory of separated left \( R \)-contramodules in \( R \)-contra and the full subcategory of right exact covariant \( \mathcal{F} \)-systems of abelian groups in \( \mathbb{A}b \).

Proof. This is [26, Lemma 6.3]. Notice that the exposition in [26, Section 6] uses direct-limit preserving covariant additive functors \( discr-R \to \mathbb{A}b \) in lieu of our covariant \( \mathcal{F} \)-systems \( Q_{\mathcal{F}} \to \mathbb{A}b \). One readily observes that the two approaches are essentially the same, as no need to apply such functors to any but cyclic discrete right \( R \)-modules ever arises in [26]. Alternatively, one can notice that, as in the proof of Proposition 5.1, the construction of the Kan extension

\[
F_C(N) = \lim_{\oplus_{I \in \mathcal{F}} R/I \to N} C(R/I)
\]

for every \( N \in discr-R \) can be used to extend a covariant additive functor \( C : Q_{\mathcal{F}} \to \mathbb{A}b \) to a coproduct-preserving covariant additive functor \( F_C : discr-R \to \mathbb{A}b \). The functor \( F_C \) is right exact whenever the covariant \( \mathcal{F} \)-system \( C \) is. □

The following lemma is quite basic. It will be useful in the proofs of Proposition 8.4 and Theorem 9.3.

Lemma 5.4. Let \( R \) be a complete, separated topological ring in a right linear topology \( \mathcal{F} \). Then any left \( R \)-contramodule \( C \) is the cokernel of an injective morphism of separated left \( R \)-contramodules \( E \to D \).

Proof. For any left \( R \)-contramodule \( C \), there exists a set \( X \) and a surjective left \( R \)-contramodule morphism \( R[[X]] \to C \) onto \( C \) from the free left \( R \)-contramodule \( R[[X]] \) (e.g., one can take \( X = C \) and the contraction map \( \pi_C : R[[C]] \to C \)). Any free left \( R \)-contramodule \( D = R[[X]] \) is separated, because \( J \otimes R[[X]] = J[[X]] \) for all \( J \in \mathcal{F} \), hence \( \bigcap_{J \in \mathcal{F}} J \otimes D = \bigcap_{J \in \mathcal{F}} J[[X]] = 0 \) in \( R[[X]] \) (since the topology \( \mathcal{F} \) on \( R \) is separated by assumption, so \( \bigcap_{J \in \mathcal{F}} J = 0 \) in \( R \)). The kernel \( E \) of the surjective morphism \( D \to C \) is an \( R \)-subcontramodule in a separated left \( R \)-contramodule \( D \), hence \( E \) is also a separated left \( R \)-contramodule. □

For any discrete right \( R \)-module \( N \) and any open right ideal \( J \subset R \), we denote by \( N_J \subset N \) the subgroup in (the underlying abelian group of) \( N \) consisting of all the elements annihilated by \( J \). So one has \( N = \lim_{\oplus_{J \in \mathcal{F}} N_J} \).

The next lemma will be useful in the proof of Theorem 6.2.
Lemma 5.5. Let $R$ be a complete, separated topological ring in a right linear topology $\mathfrak{F}$. Then a left $R$-contramodule $C$ is separated if and only if it can be embedded as a subcontramodule into a left $R$-contramodule of the form $\text{Hom}_Z(N, Q/Z)$, where $N$ is a discrete right $R$-module.

Proof. “If”: for any associative ring $A$, an $R$-discrete $A$-$R$-bimodule $N$, and a left $A$-module $V$, the left $R$-contramodule $\mathcal{D} = \text{Hom}_A(N, V)$ is separated. Indeed, for any open right ideal $I \subset R$, the subgroup $I \cap \mathcal{D} \subset \mathcal{D}$ consists of (some) $A$-linear maps $N \to V$ annihilating $N_1$. Hence $\bigcap_{\mathcal{D} \in \mathfrak{F}} N \cap \mathcal{D} = 0$.

“Only if” (cf. [26, first proof of Corollary 7.8]): notice that the class of all left $R$-contramodules of the form $\text{Hom}_Z(N, Q/Z)$, where $N \in \text{discr-} R$, is closed under products, as for any family of discrete right $R$-modules $N_\alpha$ one has $\prod_\alpha \text{Hom}_Z(N_\alpha, Q/Z) \simeq \text{Hom}_Z(\bigoplus_\alpha N_\alpha, Q/Z)$. Thus, given a separated left $R$-contramodule $C$, it suffices to show that for every element $c \in C$ there exists a discrete right $R$-module $N_c$ and a left $R$-contramodule morphism $g_c : C \to \text{Hom}_Z(N_c, Q/Z)$ such that $g_c(c) \neq 0$.

Choose an open right ideal $I \subset R$ such that $c \notin I \times C$ and an abelian group homomorphism $f_c : C/(I \times C) \to Q/Z$ such that $f_c(c) \neq 0$ (where $\bar{c}$ is the image of $c$ in $C/(I \times C)$). The natural isomorphism $\text{Hom}^{\text{ab}}(C, \text{Hom}_Z(N_c, Q/Z)) \simeq \text{Hom}_Z(N_c \circ_R C, Q/Z)$ from Section 2.8 allows to assign to the abelian group homomorphism $f_c : N_c \circ_R C \simeq C/(I \times C) \to Q/Z$ a left $R$-contramodule morphism $g_c : C \to \text{Hom}_Z(N_c, Q/Z)$. By construction, the element $g_c(c)$ is an abelian group homomorphism $N_c \to Q/Z$ taking the element $1 = 1 + I \in N_c$ to the element $f_c(c) \in Q/Z$. Hence $g_c(c) \neq 0$ in $\text{Hom}_Z(N_c, Q/Z)$. $\square$

The following proposition is a stronger version of Lemma 5.5 (holding in the narrower generality of a countable topology base). We will use it in the proof of Proposition 8.4.

Proposition 5.6. Let $R$ be a complete, separated topological ring in a right linear topology $\mathfrak{F}$ with a countable base. Then any separated left $R$-contramodule can be presented as the kernel of an $R$-contramodule morphism between two left $R$-contramodules of the form $\text{Hom}_Z(N', Q/Z)$ and $\text{Hom}_Z(N'', Q/Z)$, where $N'$ and $N''$ are discrete right $R$-modules.

First proof. It suffices to show that any separated left $R$-contramodule $C$ can be embedded into a left $R$-contramodule $\mathcal{D}$ of the form $\mathcal{D} = \text{Hom}_Z(N, Q/Z)$, where $N \in \text{discr-} R$, in such a way that the quotient $R$-contramodule $\mathcal{D}/C$ is also separated. (Then it remains to apply Lemma 5.5 to the left $R$-contramodule $\mathcal{D}/C$.)

The argument is based on the following simple lemma.

Lemma 5.7. Let $R$ be a complete, separated topological ring in a right linear topology $\mathfrak{F}$ with a countable base $\mathfrak{B}$. Let $\mathcal{D}$ be a separated left $R$-contramodule and $\mathcal{C} \subset \mathcal{D}$ be an $R$-subcontramodule. Suppose that for every open right ideal $I \in \mathfrak{B}$ the intersection $\mathcal{C} \cap (I \times \mathcal{D}) \subset \mathcal{D}$ is equal to $I \times \mathcal{C}$. Then the quotient $R$-contramodule $\mathcal{D}/\mathcal{C}$ is separated.
Proof. For any complete, separated topological ring \( \mathcal{R} \) with a right linear topology, any short exact sequence of left \( \mathcal{R} \)-contramodules \( 0 \rightarrow \mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E} \rightarrow 0 \), and any open right ideal \( \mathfrak{I} \subset \mathcal{R} \), the short sequence of abelian groups

\[
\mathcal{E}/(\mathfrak{I} \times \mathcal{C}) \rightarrow \mathcal{D}/(\mathfrak{I} \times \mathcal{D}) \rightarrow \mathcal{E}/(\mathfrak{I} \times \mathcal{E}) \rightarrow 0
\]

is right exact. When one has \( \mathfrak{I} \times \mathcal{C} = \mathcal{C} \cap (\mathfrak{I} \times \mathcal{D}) \subseteq \mathcal{D} \), this short sequence is exact at is leftmost term, too.

In the situation at hand, set \( \mathcal{E} = \mathcal{D}/\mathcal{C} \) and consider the projective system of short exact sequences

\[
0 \rightarrow \mathcal{E}/(\mathfrak{I} \times \mathcal{C}) \rightarrow \mathcal{D}/(\mathfrak{I} \times \mathcal{D}) \rightarrow \mathcal{E}/(\mathfrak{I} \times \mathcal{E}) \rightarrow 0
\]

indexed by the poset \( \mathfrak{B} \) of open right ideals \( \mathfrak{I} \). These are projective systems of abelian groups and surjective morphisms between them, indexed by a countable directed poset. Hence the passage to the projective limits preserves exactness, and we have a short exact sequence of abelian groups

\[
0 \rightarrow \lim_{\mathfrak{I} \in \mathfrak{B}} \mathcal{E}/(\mathfrak{I} \times \mathcal{C}) \rightarrow \lim_{\mathfrak{I} \in \mathfrak{B}} \mathcal{D}/(\mathfrak{I} \times \mathcal{D}) \rightarrow \lim_{\mathfrak{I} \in \mathfrak{B}} \mathcal{E}/(\mathfrak{I} \times \mathcal{E}) \rightarrow 0
\]

The natural morphism to the latter short exact sequence from the short exact sequence \( 0 \rightarrow \mathcal{E} \rightarrow \mathcal{D} \rightarrow \mathcal{E} \rightarrow 0 \) is an isomorphism at the middle term (since \( \mathcal{D} \) is separated by assumption and all left \( \mathcal{R} \)-contramodules are complete by Theorem 5.3) and at the leftmost term (since \( \mathcal{E} \) is separated as a subcontramodule of a separated left \( \mathcal{R} \)-contramodule \( \mathcal{D} \)). It follows that this map of short exact sequences is also an isomorphism at the rightmost terms, that is \( \mathcal{E} \) is a separated left \( \mathcal{R} \)-contramodule. \( \square \)

Now we can finish the first proof of Proposition 5.6. It suffices to embed our separated left \( \mathcal{R} \)-contramodule \( \mathcal{C} \) into a left \( \mathcal{R} \)-contramodule \( \mathcal{D} \) of the form \( \mathcal{D} = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \) in such a way that \( \mathcal{C} \cap (\mathfrak{I} \times \mathcal{D}) = \mathfrak{I} \times \mathcal{C} \) for every open right ideal \( \mathfrak{I} \subset \mathcal{R} \). Recall from the proof of Lemma 5.5 that the subgroup \( \mathfrak{I} \times \mathcal{D} \) is contained in the subgroup all abelian group homomorphisms \( N \rightarrow \mathbb{Q}/\mathbb{Z} \) annihilating the subgroup \( N_\mathfrak{I} \subset N \). Thus it suffices to construct a discrete right \( \mathcal{R} \)-module \( N \) and a left \( \mathcal{R} \)-contramodule morphism \( g: \mathcal{C} \rightarrow \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \) in such a way that for every open right ideal \( \mathfrak{I} \subset \mathcal{R} \) and every element \( c \in \mathcal{C} \), \( c \notin \mathfrak{I} \times \mathcal{C} \) the abelian group homomorphism \( g(c): N \rightarrow \mathbb{Q}/\mathbb{Z} \) does not annihilate \( N_\mathfrak{I} \), i.e., there exists an element \( b \in N \) for which \( b\mathfrak{I} = 0 \) in \( N \) and \( g(c)(b) \neq 0 \) in \( \mathbb{Q}/\mathbb{Z} \).

For this purpose, for every open right ideal \( \mathfrak{I} \subset \mathcal{R} \) and every element \( c \in \mathcal{C} \setminus \mathfrak{I} \times \mathcal{C} \) we choose an abelian group homomorphism \( f_{\mathfrak{I},c}: \mathcal{C}/(\mathfrak{I} \times \mathcal{C}) \rightarrow \mathbb{Q}/\mathbb{Z} \) such that \( f_{\mathfrak{I},c}(c + \mathfrak{I} \times \mathcal{C}) \neq 0 \), and consider the related left \( \mathcal{R} \)-contramodule morphism \( g_{\mathfrak{I},c}: \mathcal{C} \rightarrow \text{Hom}_{\mathbb{Z}}(N_{\mathfrak{I},c}, \mathbb{Q}/\mathbb{Z}) \), where \( N_{\mathfrak{I},c} = \mathcal{R}/\mathfrak{I} \) (as in the proof of Lemma 5.5). Then the element \( 1 + \mathfrak{I} \in \mathcal{R}/\mathfrak{I} = N_{\mathfrak{I},c} \) is annihilated by \( \mathfrak{I} \) and one has \( g_{\mathfrak{I},c}(c)(1 + \mathfrak{I}) = f_{\mathfrak{I},c}(c + \mathfrak{I} \times \mathcal{C}) \neq 0 \) in \( \mathbb{Q}/\mathbb{Z} \). It remains to take \( N \) to be the direct sum of the discrete right \( \mathcal{R} \)-modules \( N_{\mathfrak{I},c} \) over all the pairs \( (\mathfrak{I}, c) \) as above, and \( g: \mathcal{C} \rightarrow \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \) to be the morphism with the components \( g_{\mathfrak{I},c} \). \( \square \)

Second proof of Proposition 5.6. We will prove the following more general result, not depending on the countability assumption on the topology of \( \mathcal{R} \): for any complete,
separated topological ring $\mathfrak{R}$ in a right linear topology $\mathfrak{F}$, and any covariant $\mathfrak{F}$-system of abelian groups $D$, the left $\mathfrak{R}$-contramodule $\text{PL}(D)$ can be presented as the kernel of an $\mathfrak{R}$-contramodule morphism between two left $\mathfrak{R}$-contramodules of the form $\text{Hom}_\mathbb{Z}(N', \mathbb{Q}/\mathbb{Z})$ and $\text{Hom}_\mathbb{Z}(N'', \mathbb{Q}/\mathbb{Z})$ in a certain functorial way. In particular, if $\mathcal{C}$ is a separated and complete left $\mathfrak{R}$-contramodule, then the left $\mathfrak{R}$-contramodule $\mathcal{C}$ is isomorphic to $\text{PL}(\text{CT}(\mathcal{C}))$ by Proposition 5.2, so it will follow that $\mathcal{C}$ can be presented as the kernel of a morphism between two left $\mathfrak{R}$-contramodules of the desired form. Then it will remain to recall that, when the topology $\mathfrak{F}$ on $\mathfrak{R}$ has a countable base, all left $\mathfrak{R}$-contramodules are complete by Theorem 5.3. Indeed, for any abelian group $V$ let us denote by $V^+$ the abelian group $V^+ = \text{Hom}_\mathbb{Z}(V, \mathbb{Q}/\mathbb{Z})$. In particular, if $D : \mathfrak{F} \to \text{Ab}$ is a covariant $\mathfrak{F}$-system of abelian groups, then $D^+ : (\mathfrak{F}_0)^{\text{op}} \to \text{Ab}$ is a contravariant $\mathfrak{F}$-system, and vice versa. If $M : (\mathfrak{F}_0)^{\text{op}} \to \text{Ab}$ is a contravariant $\mathfrak{F}$-system of abelian groups and $M^+ : \mathfrak{F} \to \text{Ab}$ is the dual covariant $\mathfrak{F}$-system, then the left $\mathfrak{R}$-contramodule $\text{PL}(M^+)$ can be obtained by applying the functor $\text{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z})$ to the discrete right $\mathfrak{R}$-module $\text{IL}(M)$, i.e., $\text{PL}(M^+) = \text{IL}(M)^+$ (since the functor $V \mapsto V^+$ takes direct limits of abelian groups to inverse limits of abelian groups).

Now for any covariant $\mathfrak{F}$-system of abelian groups $D$ we have a natural short, left exact sequence of covariant $\mathfrak{F}$-systems of abelian groups

$$0 \to D \to D^{++} \to (D^{++}/D)^{++}.$$  

The functor $\text{PL}$ is left exact (because it is a right adjoint, or since the functor of inverse limit is left exact); hence we obtain a left exact sequence of left $\mathfrak{R}$-contramodules

$$0 \to \text{PL}(D) \to \text{PL}(D^{++}) \to \text{PL}((D^{++}/D)^{++}).$$

It remains to recall that, according to the above, we have $\text{PL}(D^{++}) = \text{IL}(D^+)^+$ and $\text{PL}((D^{++}/D)^{++}) = \text{IL}((D^{++}/D)^{++})$. □

6. Fully Faithful Contramodule Forgetful Functors

Let $R$ be an associative ring, $\mathbb{F}$ be a right linear topology on $R$, and $\mathfrak{R} = \lim_{\overset{\leftarrow}{I \in \mathbb{F}}} R/I$ be the completion of $R$ with respect to $\mathbb{F}$, viewed as a topological ring in the projective limit topology $\mathfrak{F}$. Then we have the abelian category of left $\mathfrak{R}$-contramodules $\mathfrak{R}-\text{contra}$ (see Section 2.7), which is endowed with an exact forgetful functor $\mathfrak{R}-\text{contra} \to \mathfrak{R}-\text{mod}$.

We also have the functor of restriction of scalars $\rho_* : \mathfrak{R}-\text{mod} \to R-\text{mod}$ induced by the natural morphism of (topological) rings $\rho : R \to \mathfrak{R}$. In this section we discuss conditions under which the composition of these two forgetful functors

$$\mathfrak{R}-\text{contra} \to \mathfrak{R}-\text{mod} \to R-\text{mod}$$

is a fully faithful functor $\mathfrak{R}-\text{contra} \to R-\text{mod}$. We follow the approach of [24, Theorem 1.1] and [25, Section 3], improving upon the results obtained there.
As it was discussed in Section 2.8 there is a natural bijection between the set \( F \) of all open right ideals \( I \subset R \) and the set \( \mathfrak{I} \) of all open right ideals \( \mathfrak{I} \subset \mathfrak{R} \). The ring homomorphism \( \rho: R \rightarrow \mathfrak{R} \) induces an isomorphism of right \( R \)-modules \( R/I \simeq \mathfrak{R}/\mathfrak{I} \).

We will say that an open right ideal \( I \subset R \) is strongly generated (or, in a different language, the corresponding open right ideal \( \mathfrak{I} \subset \mathfrak{R} \) is strongly generated by elements coming from \( R \)) if for every set \( X \) the subgroups \( \mathfrak{I} \triangleleft \mathfrak{R}[[X]] = \mathfrak{I}[[X]] \subset \mathfrak{R}[[X]] \) and \( \mathfrak{I} \mathfrak{R}[[X]] = I \cdot \mathfrak{R}[[X]] \subset \mathfrak{R}[[X]] \) coincide in the group \( \mathfrak{R}[[X]] \), that is

\[
\mathfrak{I} \mathfrak{R}[[X]] = \mathfrak{I}[[X]] \subset \mathfrak{R}[[X]]
\]

(see Section 2.8 for the notation). The inclusion \( \mathfrak{I} \mathfrak{R}[[X]] \subset \mathfrak{I}[[X]] \) always holds for any open right ideal \( I \subset R \); the right ideal \( I \) is said to be strongly generated if the inverse inclusion holds as well.

**Lemma 6.1.** Let \( R \) be a topological ring with a right linear topology and \( I \subset R \) be an open right ideal, and \( \mathfrak{I} \subset \mathfrak{R} \) be the related open right ideal in the completion \( \mathfrak{R} \) of the topological ring \( R \). Then the following conditions are equivalent:

(i) the ideal \( I \subset R \) is strongly generated;

(ii) for every left \( \mathfrak{R} \)-contramodule \( \mathfrak{C} \), one has \( I \mathfrak{C} = \mathfrak{I} \triangleleft \mathfrak{C} \subset \mathfrak{C} \).

**Proof.** (ii) \( \Rightarrow \) (i) Given a set \( X \), take \( \mathfrak{C} = \mathfrak{R}[[X]] \).

(i) \( \Rightarrow \) (ii) For any left \( \mathfrak{R} \)-contramodule \( \mathfrak{C} \), there exists a set \( X \) such that there is a surjective left \( \mathfrak{R} \)-contramodule morphism \( \mathfrak{R}[[X]] \rightarrow \mathfrak{C} \) (e. g., one can take \( X = \mathfrak{C} \) and the contraction morphism \( \pi_{\mathfrak{C}}: \mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathfrak{C} \)). It remains to observe that for any surjective left \( \mathfrak{R} \)-contramodule morphism \( f: \mathfrak{D} \rightarrow \mathfrak{C} \) one has \( I \mathfrak{C} = f(I \mathfrak{D}) \) and \( \mathfrak{I} \triangleleft \mathfrak{C} = f(\mathfrak{I} \triangleleft \mathfrak{D}) \), so the equation \( I \mathfrak{D} = \mathfrak{I} \triangleleft \mathfrak{D} \) implies \( I \mathfrak{C} = \mathfrak{I} \triangleleft \mathfrak{C} \). \( \square \)

**Theorem 6.2.** Let \( R \) be a topological ring with a right linear topology \( F \) having a countable base \( \mathcal{B} \). Then the following conditions are equivalent:

(i) all the open right ideals \( I \subset R \) are strongly generated;

(ii) all the open right ideals \( J \in \mathcal{B} \) are strongly generated in \( R \);

(iii) the forgetful functor \( \mathfrak{R} \text{-contra} \rightarrow R \text{-mod} \) is fully faithful.

**Proof.** The implications (i) \( \iff \) (ii) \( \iff \) (iii) do not depend on the assumption of countability of \( \mathcal{B} \); the proof of the implication (ii) \( \Rightarrow \) (iii) below does.

(i) \( \Rightarrow \) (ii) Obvious.

(ii) \( \Rightarrow \) (i) Let \( J \subset I \) be two embedded open right ideals in \( R \), and let \( \mathfrak{J} \subset \mathfrak{I} \) be the two related open right ideals in \( \mathfrak{R} \). Then any \( X \)-indexed family of elements in \( \mathfrak{J} \) converging to zero in the topology of \( \mathfrak{R} \) has all but a finite subset of its elements belonging to \( \mathfrak{J} \). Furthermore, \( \mathfrak{J} = \rho(I) + \mathfrak{J} \subset \mathfrak{R} \), hence \( \mathfrak{J}[[X]] = \rho(I)[X] + \mathfrak{J}[[X]] \subset \mathfrak{R}[[X]] \). Therefore, the equation \( \mathfrak{J} \mathfrak{R}[[X]] = \mathfrak{J}[[X]] \) implies \( \mathfrak{J} \mathfrak{R}[[X]] = \mathfrak{J}[[X]] \).

(iii) \( \Rightarrow \) (i) As in Section 2.7 we denote by \( \text{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{D}) \) the group of all morphisms \( \mathfrak{C} \rightarrow \mathfrak{D} \) in the category of left \( \mathfrak{R} \)-contramodules \( \mathfrak{R} \text{-contra} \); while, as usual, \( \text{Hom}_{R}(C, D) \) denotes the group of all morphisms \( C \rightarrow D \) in \( R \text{-mod} \). Then for any left \( R \)-module \( C \) and right \( R \)-module \( N \) one has

\[
\text{Hom}_{R}(C, \text{Hom}_{Z}(N, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_{Z}(N \otimes_{R} C, \mathbb{Q}/\mathbb{Z}),
\]
while for any left $\mathcal{R}$-contramodule $\mathcal{C}$ and discrete right $\mathcal{R}$-module $N$ we have

$$\text{Hom}^{\mathcal{R}}(\mathcal{C}, \text{Hom}_Z(N, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_Z(N \otimes_{\mathcal{R}} \mathcal{C}, \mathbb{Q}/\mathbb{Z})$$

(see Section 2.8). Hence the natural surjective map of abelian groups $N \otimes_{\mathcal{R}} \mathcal{C} \twoheadrightarrow N \otimes_{\mathcal{R}} \mathcal{C}$ is an isomorphism for any left $\mathcal{R}$-contramodule $\mathcal{C}$ and any discrete right $\mathcal{R}$-module $N$ whenever the forgetful functor $\mathcal{R}\text{-contra} \rightarrow R\text{-mod}$ is fully faithful. In particular, one has

$$\mathcal{C}/I\mathcal{C} = R/I \otimes_{\mathcal{R}} \mathcal{C} = \mathcal{R}/I \otimes_{\mathcal{R}} \mathcal{C} \simeq \mathcal{R}/I \otimes_{\mathcal{R}} \mathcal{C} = \mathcal{C}/(I \mathcal{C}),$$

and it remains to take $\mathcal{C} = \mathcal{R}[[X]]$.

(ii) $\implies$ (iii) Let $\mathcal{C}$ and $\mathcal{D}$ be two left $\mathcal{R}$-contramodules, and let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a left $R$-module morphism between them. Choose a set $X$ together with a surjective left $\mathcal{R}$-contramodule morphism $\mathcal{R}[[X]] \rightarrow \mathcal{C}$. In order to show that the map $f$ is a left $\mathcal{R}$-contramodule morphism, it suffices to check that so is the composition $\mathcal{R}[[X]] \rightarrow \mathcal{C} \rightarrow \mathcal{D}$. Let us denote this composition by $f' : \mathcal{R}[[X]] \rightarrow \mathcal{D}$.

Restricting $f'$ to the subset $X \subset \mathcal{R}[[X]]$, we obtain a map of sets $f'' : X \rightarrow \mathcal{D}$. The group of all left $\mathcal{R}$-contramodule morphisms $\mathcal{R}[[X]] \rightarrow \mathcal{D}$ is bijective to the group of all maps of sets $X \rightarrow \mathcal{D}$; hence there exists a unique left $\mathcal{R}$-contramodule morphism $f'' : \mathcal{R}[[X]] \rightarrow \mathcal{D}$ such that $f''|_X = f''$. Consider the difference $g = f'' - f''$; then $g : \mathcal{R}[[X]] \rightarrow \mathcal{D}$ is a left $R$-module morphism such that $g|_X = 0$, and it remains to check that $g = 0$.

Let $\mathcal{E} \subset \mathcal{D}$ denote the $\mathcal{R}$-subcontramodule in $\mathcal{D}$ generated by $\text{im } g$ (that is, the intersection of all $\mathcal{R}$-subcontramodules in $\mathcal{D}$ containing $\text{im } g$). Then $\mathcal{E}$ is a left $\mathcal{R}$-contramodule, $g : \mathcal{R}[[X]] \rightarrow \mathcal{E}$ is a left $R$-module morphism such that $g|_X = 0$, there are no proper $\mathcal{R}$-subcontramodules in $\mathcal{E}$ containing $\text{im } g$, and it remains to prove that $\mathcal{E} = 0$.

According to Proposition 5.2, the kernel $\mathcal{E}' = \ker \lambda_{\mathcal{R}, \mathcal{E}} = \cap_{I \in \mathcal{I}} I \mathcal{C} \cap \mathcal{E}$ of the completion morphism $\lambda_{\mathcal{R}, \mathcal{E}} : \mathcal{C} \rightarrow \varprojlim_{I \in \mathcal{I}} \mathcal{C}/(I \mathcal{C})$ is an $\mathcal{R}$-subcontramodule in $\mathcal{E}$, and the quotient $\mathcal{R}$-contramodule $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$ is separated. Moreover, by the contramodule Nakayama lemma [26, Lemma 6.14] the left $\mathcal{R}$-contramodule $\mathcal{E}$ vanishes whenever its (maximal separated) quotient contramodule $\mathcal{E}''$ does. (This is where the assumption of a countable base of the topology of $\mathcal{R}$ is needed.)

Hence if $\mathcal{E} \neq 0$, then $\mathcal{E}'' \neq 0$, $\mathcal{E}' \neq \mathcal{E}$, and by construction the image of the left $R$-module morphism $g : \mathcal{R}[[X]] \rightarrow \mathcal{E}$ is not contained in $\mathcal{E}'$. The composition $g'' : \mathcal{R}[[X]] \rightarrow \mathcal{E}''$ of the map $g$ with the natural surjection $\mathcal{C} \rightarrow \mathcal{E}''$ is a left $R$-module morphism such that $g''|_X = 0$ and $\text{im } g''$ is not contained in any proper $\mathcal{R}$-subcontramodule of $\mathcal{E}''$. It remains to show that $\mathcal{E}'' = 0$, and for this purpose it suffices to check that $g'' = 0$, or even that $g''$ is actually an $\mathcal{R}$-contramodule morphism (as any $\mathcal{R}$-contramodule morphism from a free left $\mathcal{R}$-contramodule taking the generators to zero vanishes as a whole).

By Lemma 5.5 the left $\mathcal{R}$-contramodule $\mathcal{E}''$ can be embedded into a left $\mathcal{R}$-contramodule of the form $\text{Hom}_Z(N, \mathbb{Q}/\mathbb{Z})$, where $N$ is a discrete right $\mathcal{R}$-module. In fact, following the proof of Lemma 5.5 one take $N$ to be a direct sum of (sufficiently many) copies of the discrete right $\mathcal{R}$-modules $\mathcal{R}/I$, where $I$ ranges over any
topology base $\mathfrak{B}$ of $\mathfrak{R}$. We take $\mathfrak{B}$ to be the topology base of $\mathfrak{R}$ corresponding to our given topology base $\mathfrak{B}$ of $R$.

It remains to show that the composition $\mathfrak{R}[[X]] \to \text{Hom}_Z(N, Q/Z)$ of the morphism $g'' : \mathfrak{R}[[X]] \to C''$ with the embedding $C'' \to \text{Hom}_Z(N, Q/Z)$ is a left $\mathfrak{R}$-contramodule morphism, and for this purpose it suffices to check that every left $R$-module map $\mathfrak{R}[[X]] \to \text{Hom}_Z(N, Q/Z)$ is actually an $\mathfrak{R}$-contramodule morphism. Indeed, for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$ we have

$$\text{Hom}_R(\mathfrak{C}, \text{Hom}_Z(\mathfrak{R}/I, Q/Z)) = \text{Hom}_R(\mathfrak{C}, \text{Hom}_Z(R/I, Q/Z)) \simeq \text{Hom}_Z(\mathfrak{C}/IC, Q/Z)$$

and

$$\text{Hom}^{\mathfrak{R}}(\mathfrak{C}, \text{Hom}_Z(\mathfrak{R}/I, Q/Z)) \simeq \text{Hom}_Z(\mathfrak{R}/I \odot_{\mathfrak{R}} \mathfrak{C}, Q/Z) = \text{Hom}_Z(\mathfrak{C}/(I \odot \mathfrak{C}), Q/Z),$$

so the equation $IC = I \odot \mathfrak{C}$ for all $I \in \mathfrak{B}$ implies $\text{Hom}_R(\mathfrak{C}, \text{Hom}_Z(N, Q/Z)) = \text{Hom}^{\mathfrak{R}}(\mathfrak{C}, \text{Hom}_Z(N, Q/Z)).$ $\square$

**Remark 6.3.** We do not know whether the implication $(ii) \implies (iii)$ in Theorem 6.2 holds true without the assumption of countability of $\mathfrak{B}$.

Following the above proof, the countability assumption was only used in order to invoke the contramodule Nakayama lemma, claiming that any nonzero $\mathfrak{R}$-contramodule has a nonzero separated quotient $\mathfrak{R}$-contramodule. This form of the contramodule Nakayama lemma is not true without a countable topology base, generally speaking. Indeed, let $\mathfrak{R} = R$ be the ring of (commutative) polynomials in an uncountable set of variables $x_i$ over a field $k$, and let $S \subset R$ be the multiplicative subset generated by the elements $x_i$. Endow $\mathfrak{R}$ with the $S$-topology, in which the ideals $s\mathfrak{R}$, $s \in S$, form a base of neighborhoods of zero. By [14 Proposition 1.16], $\mathfrak{R}$ is a complete, separated topological ring. One easily observes that no infinite family of nonzero elements in $\mathfrak{R}$ converges to zero in the $S$-topology; so an $\mathfrak{R}$-contramodule is the same thing as an $\mathfrak{R}$-module and the forgetful functor $\mathfrak{R}\text{-contra} \to \mathfrak{R}\text{-mod}$ is an equivalence of categories. In particular, the localization $\mathfrak{C} = \mathfrak{R}[S^{-1}]$ is an $\mathfrak{R}$-contramodule for which $\bigcap_{s \in S} s\mathfrak{C} = \mathfrak{C}$; so $\mathfrak{C}$ has no nonzero separated quotient $\mathfrak{R}$-contramodules. Still, the implication $(ii) \implies (iii)$ remains true in this case (both (ii) and (iii) are true).

In some special situations, the contramodule Nakayama lemma is provable without the countability assumption; notably for the topological algebras dual to coalgebras over fields [20 Corollary A.2]. Hence Theorem 6.2 holds for all such topological algebras (cf. [24 Theorem 1.1]). Another such special situation is that of contramodules over the completion $\mathfrak{R}$ of an associative ring $R$ with respect to the $S$-topology for some central multiplicative subsets $S \subset R$. In this context, a version of Theorem 6.2 holds under a certain assumption not unrelated to countability but weaker than that of a countable topology base; see [25 Example 3.7 (1)].

An example of topological ring/algebra $\mathfrak{R}$ with a countable topology base (dual to a certain coalgebra of countable dimension over a field) for which the forgetful functor $\mathfrak{R}\text{-contra} \to \mathfrak{R}\text{-mod}$ is not fully faithful (so none of the conditions (i–iii) is true) can be found in [20 Section A.1.2].
The following lemma explains the intuition behind the “strongly generated ideal” terminology. Extending the terminology of Section 2.7, even for a possibly nonseparated and noncomplete topological group $A$ we will say that an $X$-indexed family of elements $a_x \in A$ converges to zero in the topology of $A$ if for every open subgroup $U \subset A$ the set of all $x \in X$ such that $a_x \notin U$ is finite.

**Lemma 6.4.** Let $R$ be a topological ring with a right linear topology with a countable base, and let $I \subset R$ be an open right ideal. Then the following two conditions are equivalent:

(i) the ideal $I \subset R$ is strongly generated;

(ii) for any set $X$ and any $X$-indexed family of elements $r_x \in I$ converging to zero in the topology of $R$ there exists a finite set of elements $s_1, \ldots, s_m \in I$ and a $(\{1, \ldots, m\} \times X)$-indexed family of elements $t_{j,x} \in R$ converging to zero in the topology of $R$ such that

$$\rho(r_x) = \sum_{j=1}^{m} \rho(s_j) t_{j,x} \in R$$

for every $x \in X$.

**Proof.** Let us start from discussing the particular case when the topological ring $R$ is separated and complete, so $R = R$, $I = I$, and $\rho$ is the identity map. In this case, the assertion of the lemma is completely tautological and no countability assumption on the topology of $R$ is needed. The datum of an $X$-indexed family of elements $r_x \in I$ converging to zero in the topology of $R$ is equivalent to the datum of an element $r = \sum_{x \in X} r_x x \in I[[X]] \subset R[[X]]$. By the definition, the element $r$ belongs to $\text{I} R[[X]] \subset I[[X]]$ if and only if there exists a finite set of elements $s_1, \ldots, s_m \in I$ and a finite set of elements $t_1, \ldots, t_m \in R[[X]]$ such that $r = \sum_{j=1}^{m} s_j t_j$ in $R[[X]]$. The datum of an element $t_j = \sum_{x \in X} t_{j,x} x \in R[[X]]$ is equivalent to the datum of an $X$-indexed family of elements $t_{j,x} \in R$ converging to zero in the topology of $R[[X]]$; and the equation $r = \sum_{j=1}^{m} s_j t_j$ in $R[[X]]$ is equivalent to the $X$-indexed family of equations $r_x = \sum_{j=1}^{m} s_j t_{j,x}$ for all $x \in X$.

In the general case, the implication (i) $\implies$ (ii) still does not require the countability assumption on the topology. Given an $X$-indexed family of elements $r_x \in I$ converging to zero in $R$, consider the element $r = \sum_{x \in X} \rho(r_x) x \in R[[X]]$. According to (i), there exist finite sets of elements $s_1, \ldots, s_m \in I$ and $t_1, \ldots, t_m \in R[[X]]$ such that $r = \sum_{j=1}^{m} s_j t_j$ in $R[[X]]$, which is a restatement of (ii).

To prove (ii) $\implies$ (i), one observes that, whenever the topology of $R$ has a countable base, any element $p \in R$ can be presented as a sum $p = \sum_{i=1}^{\infty} \rho(t_i)$ for some elements $t_1, t_2, t_3, \ldots \in R$ (where the infinite sum is understood as the limit of finite partial sums in the topology of $R$). Similarly, any element $q \in I = \varinjlim_{J \in \mathcal{F}, J \subset I} I/J$ can be presented as a countably infinite sum $q = \sum_{i=1}^{\infty} \rho(r_i)$ for some sequence of elements $r_1, r_2, r_3, \ldots \in I$. Moreover, if $q = \sum_{x \in X} q_x x$ is an element of $I[[X]]$ (so the $X$-indexed family of elements $q_x \in I$ converges to zero in the topology of $R$), then one can choose for every $x \in X$ a sequence of elements $(r_{x,i} \in I)_{i=1}^{\infty}$ such that $q_x = \sum_{i=1}^{\infty} \rho(r_{x,i}) \in I$. 

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for every $x \in X$ and the whole doubly indexed family of elements $r_{x,i} \in I$ converges to zero in the topology of $R$.

By (ii), there exists a finite set of elements $s_1, \ldots, s_m \in I$ and a family of elements $t_{j,x,i} \in \mathcal{R}$, indexed by the triples of indices $1 \leq j \leq m, \ x \in X,$ and $i \in \mathbb{Z}_{\geq 1}$ and converging to zero in the topology of $\mathcal{R}$, such that

$$\rho(r_{x,i}) = \sum_{j=1}^{m} \rho(s_j) t_{j,x,i} \quad \text{for all } x \in X \text{ and } i \in \mathbb{Z}_{\geq 1}.$$ 

Passing to the sum over $i$ (understood as the limit of finite partial sums in the topology of $R$), we obtain the equations

$$q_x = \sum_{j=1}^{m} \rho(s_j) \sum_{i=1}^{\infty} t_{j,x,i} \quad \text{for all } x \in X,$$

that is, setting $p_{j,x} = \sum_{i=1}^{\infty} t_{j,x,i} \in \mathcal{R}$,

$$q_x = \sum_{j=1}^{m} \rho(s_j) p_{j,x} \quad \text{for all } x \in X.$$

It remains to observe that the family of elements $(p_{j,x})_{j,x}$ converges to zero in the topology of $\mathcal{R}$ (because the family of elements $(t_{j,x,i})_{j,x,i}$ does) and conclude that

$$q = \sum_{x \in X} q_x x = \sum_{j=1}^{\infty} \sum_{x \in X} \rho(s_j) p_{j,x} x = \sum_{j=1}^{m} \rho(s_j) p_{j},$$

where $p_j = \sum_{x \in X} p_{j,x} x \in \mathcal{R}[[X]]$. Thus $q \in I\mathcal{R}[[X]]$. □

**Remark 6.5.** One can say that an open right ideal $I$ in a topological ring $R$ with a right linear topology is **strongly finitely generated** if there exists a finite set of elements $s_1, \ldots, s_m \in I$ that can be used in the condition of Lemma 6.4(ii) for all zero-convergent families of elements $r_x \in I$. Notice that a strongly generated open right ideal does not need to be strongly finitely generated (e. g., a right ideal in a discrete associative ring does not need to be finitely generated). On the other hand, one easily observes that a strongly generated open right ideal in a topological ring $R$ is strongly finitely generated whenever it is finitely generated as a right ideal in an abstract associative ring $R$.

Moreover, let us say that a subset of elements $Z \subset I$ of an open right ideal $I \subset R$ is a **set of strong generators** of $I$ if for any zero-convergent family of elements $(r_x \in I)_{x \in X}$ there exists a finite set of elements $s_1, \ldots, s_m \in Z$ that can be used in the condition of Lemma 6.4(ii) for the family of elements $r_x \in I$. Then any set of generators of a strongly generated right ideal is a set of strong generators. Indeed, the equality $I = \sum_{x \in Z} zR$ implies $I\mathcal{R}[[X]] = \sum_{x \in Z} z\mathcal{R}[[X]]$.

**Theorem 6.6.** Let $R$ be an associative ring and $G$ be a right Gabriel topology on $R$ having a countable base $\mathcal{B}$ consisting of finitely generated right ideals. Then all the right ideals $I \in \mathcal{B}$ are strongly (finitely) generated in the topological ring $R$. 36
Proof. Let \( s_1, \ldots, s_m \) be a finite set of generators of an open right ideal \( I \subset R \), and let \( r_x \in I \) be an \( X \)-indexed family of elements converging to zero in the Gabriel topology \( \mathcal{G} \) of \( R \). One can assume the set \( X \) to be countable (as any family of elements converging to zero in \( R \) vanishes outside of a countable subset of indices) and presume an identification \( X = \mathbb{Z}_{\geq 1} \) to be chosen.

One can also choose a (nonstrictly) decreasing sequence of open right ideals \( J_1 \supset J_2 \supset J_3 \supset \cdots \) in \( R \) such that \( J_i \in \mathcal{B} \) for all \( i \geq 1 \) and the set of open right ideals \( \mathcal{J} = \{ J_1, J_2, J_2, \ldots \} \) is a base of the topology \( \mathcal{G} \). By Lemma 3.1 for every \( i \geq 1 \) the right ideal \( H_i = s_1 J_i + \cdots + s_m J_i \subset R \) belongs to \( \mathcal{G} \).

Choose an integer \( x_1 \geq 1 \) such that \( r_x \in H_i \) for all \( x > x_1 \). Proceeding by induction, for every \( i \geq 2 \) choose an integer \( x_i > x_{i-1} \) such that for every \( x > x_i \) one has \( r_x \in H_i \). For every \( x \leq x_i \), choose elements \( u_{1x}, \ldots, u_{mx} \in R \) such that \( r_x = \sum_{j=1}^{m} s_j u_{jx} \). For every \( x_i < x \leq x_{i+1} \), choose elements \( u_{1x}, \ldots, u_{nx} \in J_i \) such that \( r_x = \sum_{j=1}^{n} s_j u_{jx} \).

Set \( t_{jx} = \rho(u_{jx}) \in \mathcal{R} \). Now the family of elements \( (t_{jx})_{j,x} \) converges to zero in the projective limit topology \( \mathcal{G} \) of \( \mathcal{R} \) (since the family of elements \( (u_{jx})_{j,x} \) converges to zero in the topology \( \mathcal{G} \) of \( R \), because \( \mathcal{J} \) is a base of the topology \( \mathcal{G} \)), and one has \( \rho(r_x) = \sum_{j=1}^{\infty} \rho(s_j) t_{jx} \) for every \( x \in X \), as desired in Lemma 6.4(ii). \( \square \)

**Corollary 6.7.** Let \( R \) be an associative ring, \( \mathcal{G} \) be a right Gabriel topology on \( R \) having a countable base consisting of finitely generated right ideals, and let \( \mathcal{R} = \varprojlim_{I \in \mathcal{G}} R/I \) be the completion of \( R \) with respect to \( \mathcal{G} \), viewed as a topological ring in the projective limit topology \( \mathcal{G} \). Then the forgetful functor

\[
\mathcal{R}-\text{contra} \longrightarrow R\text{-mod}
\]

is fully faithful. Consequently, the natural surjective map of abelian groups

\[
N \otimes_R \mathcal{C} \longrightarrow N \otimes_{\mathcal{R}} \mathcal{C}
\]

is an isomorphism for any left \( \mathcal{R} \)-contramodule \( \mathcal{C} \) and any discrete right \( \mathcal{R} \)-module \( N \). In particular, one has

\[
I \mathcal{C} = J \lhd \mathcal{C} \subset \mathcal{C}
\]

for any left \( \mathcal{R} \)-contramodule \( \mathcal{C} \), any open right ideal \( I \in \mathcal{G} \) in \( R \), and the related open right ideal \( J \in \mathcal{G} \) in \( \mathcal{R} \).

**Proof.** The first assertion follows from Theorem 6.2(ii) \( \Longrightarrow \) (iii) and Theorem 6.6. The second assertion follows from the first one (see the proof of Theorem 6.2(iii) \( \Longrightarrow \) (i)). The third assertion is provided by Lemma 6.1 (it is also a particular case of the second one corresponding to the choice of the discrete right \( \mathcal{R} \)-module \( N = \mathcal{R}/J = R/I \)). \( \square \)

7. **Extending Full Abelian Subcategories**

Given a topological ring \( R \) with a right linear topology \( \mathcal{F} \) having a countable base consisting of strongly generated open right ideals, and denoting by \( \mathcal{R} \) the completion of \( R \) with respect to \( \mathcal{F} \), Theorem 6.2 claims the forgetful functor \( \mathcal{R}-\text{contra} \longrightarrow R\text{-mod} \) is fully faithful. This functor is also always exact (and it preserves infinite products).
It follows that the essential image of the forgetful functor $\mathcal{R}-\text{contra} \rightarrow R-\text{mod}$ is an exactly embedded abelian full subcategory in $R-\text{mod}$.

In other words, the full subcategory of left $\mathcal{R}$-contramodules in the abelian category of left $R$-modules is closed under the kernels and cokernels of morphisms (and also under infinite products). Generally speaking, it is not closed under extensions, though (see [25, Examples 5.2 (6–8)]). The aim of this section is to develop a simple technique for producing a sequence of bigger kernel- and cokernel-closed full subcategories in a fixed abelian category out of a smaller one, by adding extensions.

Let $A$ be an abelian category and $C \subset A$ be a full subcategory containing the zero object. Denote by $C^{(2)}_A \subset A$ the full subcategory formed by all the objects $X \in A$ for which there exists a short exact sequence $0 \rightarrow K \rightarrow X \rightarrow C \rightarrow 0$ in $A$ with the objects $K$ and $C$ belonging to $C$. Obviously, one has $C \subset C^{(2)}_A \subset A$.

More generally, for every integer $m \geq 0$ let us denote by $C^{(m)}_A \subset A$ the full subcategory consisting of all the objects $X \in A$ admitting a filtration by subobjects $0 = X_0 \subset X_1 \subset \cdots \subset X_{m-1} \subset X_m = X$ such that $X_j/X_{j-1} \in C$ for all $1 \leq j \leq m$. One has $C^{(0)}_A = \{0\}$, $C^{(1)}_A = C$, $C^{(m)}_A \subset C^{(m+1)}_A$, and $(C^{(m)}_A)^{(n)} = C^{(mn)}_A$ for all $m, n \geq 0$.

**Lemma 7.1.** Let $C$ be a full subcategory closed under kernels and cokernels in an abelian category $A$, and let $m \geq 0$ be an integer. Then the cokernel of any morphism in $A$ from an object of $C^{(m)}_A$ to an object of $C$ belongs to $C$. The kernel of any morphism in $A$ from an object of $C$ to an object of $C^{(m)}_A$ belongs to $C$.

**Proof.** For any short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in $A$, an object $A$, and a morphism $f: Y \rightarrow A$, one can compute the cokernel $C = \text{coker } f$ in two steps: consider the cokernel $D$ of the composition $X \rightarrow Y \rightarrow A$; then $C$ is the cokernel of the induced morphism $Z \rightarrow D$. $\square$

**Proposition 7.2.** Let $A$ be an abelian category and $C \subset A$ be a full subcategory closed under finite direct sums of objects and the kernels and cokernels of morphisms. Let $m \geq 0$ be an integer. Then the full subcategory $C^{(m)}_A \subset A$ is also closed under finite direct sums, kernels, and cokernels. So, from any two abelian categories $C$ and $A$ with a fully faithful exact functor $C \rightarrow A$ one can produce a sequence of abelian categories $C^{(m)}_A$ with fully faithful exact functors $C^{(m)}_A \rightarrow A$ in this way.

**Proof.** We proceed by induction on $m \geq 1$ (the cases $m = 0$ and 1 being obvious). Since the class of all short exact sequences in $A$ is closed under finite direct sums, and so is the full subcategory $C \subset A$, it follows that the full subcategory $C^{(m)}_A \subset A$ is also closed under finite direct sums.

To check that the full subcategory $C^{(m)}_A \subset A$ is closed under kernels and cokernels, we consider a morphism $f: X \rightarrow Y$ between two objects from $C^{(m)}_A$, $m \geq 2$. By construction, there exist two short exact sequences $0 \rightarrow K \rightarrow X \rightarrow C \rightarrow 0$ and
0 \rightarrow L \rightarrow Y \rightarrow D \rightarrow 0 \text{ in } A \text{ with } K, L \in C_A^{(m-1)} \text{ and } C, D \in C.

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & X & \rightarrow & C & \rightarrow & 0 \\
& & \downarrow f & & & & & & \\
0 & \rightarrow & L & \rightarrow & Y & \rightarrow & D & \rightarrow & 0
\end{array}
\]

Denote by \( g: K \rightarrow D \) the composition of morphisms \( K \rightarrow X \rightarrow Y \rightarrow D \), and denote by \( L' \) the sum of the subobject \( L \) in \( Y \) with the image of the composition \( K \rightarrow X \rightarrow Y \). Then there is a short exact sequence

\[
0 \rightarrow \ker g \rightarrow L \oplus K \rightarrow L' \rightarrow 0
\]

in \( A \). By the induction assumption, we have \( \ker g \in C_A^{(m-1)} \) (since \( D \in C \subset C_A^{(m-1)} \)) and \( L \oplus K \in C_A^{(m-1)} \), hence \( L' \in C_A^{(m-1)} \).

Set \( D' = \coker g \in A \); by Lemma 7.1, we have \( D' \in C \). Then there is a short exact sequence \( 0 \rightarrow L' \rightarrow Y \rightarrow D' \rightarrow 0 \) in \( A \) and a morphism of short exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & X & \rightarrow & C & \rightarrow & 0 \\
& & \downarrow k & & \downarrow f & & \downarrow c & & \\
0 & \rightarrow & L' & \rightarrow & Y & \rightarrow & D' & \rightarrow & 0
\end{array}
\]

Now the Snake lemma provides a six-term exact sequence

\[
0 \rightarrow \ker k \rightarrow \ker f \rightarrow \ker c \rightarrow \coker k \rightarrow \coker f \rightarrow \coker c \rightarrow 0,
\]

where the kernel and cokernel of the morphism \( k \) belong to \( C_A^{(m-1)} \), while the kernel and cokernel of the morphism \( c \) belong to \( C \). It follows that for the boundary morphism \( \partial: \ker c \rightarrow \coker k \) we have \( \ker \partial \in C \) and \( \coker \partial \in C_A^{(m-1)} \).

Finally, from the short exact sequences \( 0 \rightarrow \ker k \rightarrow \ker f \rightarrow \ker \partial \rightarrow 0 \) and \( 0 \rightarrow \coker \partial \rightarrow \coker f \rightarrow \coker c \rightarrow 0 \) we conclude that the objects \( \ker f \) and \( \coker f \) belong to \( C_A^{(m)} \), as desired. \( \square \)

**Corollary 7.3.** In the assumptions of Proposition 7.2, for any two objects \( X \in C_A^{(m)} \) and \( Y \in C_A^{(n)} \), where \( m, n \geq 0 \), and any morphism \( f: X \rightarrow Y \) in \( A \), one has \( \ker f \in C_A^{(m)} \) and \( \coker f \in C_A^{(n)} \).

**Proof.** By Proposition 7.2, the assertion of Lemma 7.1 is applicable to the full subcategories \( 'C = C_A^{(m)} \subset A \) and \( 'C = C_A^{(n)} \subset A \). So it remains to observe that both \( C_A^{(m)} \) and \( C_A^{(n)} \) are contained in \( C_A^{(mn)} = C_A^{(m)} \) (assuming \( m, n \geq 1 \)). \( \square \)

Obviously, if the infinite product functors are (everywhere defined and) exact in \( A \) and the full subcategory \( C \subset A \) is closed under infinite products, then the full subcategories \( C_A^{(m)} \subset A \) are closed under infinite products, too.
8. Projective Dimension at Most 1

In this section we prove the result announced in the abstract, namely, that any left flat ring epimorphism \( u: R \to U \) of countable type has left projective dimension at most 1. We also describe the Geigle–Lenzing perpendicular subcategory \( U^{\perp_{0,1}} \subset \text{mod } R \) of the left \( R \)-module \( U \) in terms of \( \mathcal{R} \)-contramodules.

We start in the somewhat greater generality of a right linear topology \( \mathbb{F} \) on an associative ring \( R \). Let \( \mathcal{R} = \varprojlim_{I \in \mathbb{F}} R/I \) be the completion of \( R \) with respect to \( \mathbb{F} \), viewed as a complete, separated topological ring in the projective limit topology \( \mathfrak{F} \).

**Lemma 8.1.** Let \( A \) be an associative ring, \( N \) be an \( A \)-\( R \)-bimodule that is discrete as a right \( R \)-module, and \( V \) be a left \( A \)-module. Then for every \( n \geq 0 \) the abelian group \( \text{Ext}_A^n(N, V) \) carries a natural left \( \mathcal{R} \)-contramodule structure whose underlying left \( R \)-module structure is induced by the right \( R \)-module structure on \( N \).

**Proof.** Let us emphasize that the abelian groups \( \text{Ext}_A^n(N, V) \) have natural left \( R \)-module structures for any \( A \)-\( R \)-bimodule \( N \). It is claimed that this \( R \)-module structure on \( \text{Ext}_A^n(N, V) \) underlies a naturally defined left \( \mathcal{R} \)-contramodule structure whenever the \( A \)-\( R \)-bimodule \( N \) is discrete as a right \( R \)-module.

Indeed, according to Section 2.8 there is a natural left \( \mathcal{R} \)-contramodule structure on the abelian group \( \text{Hom}_A(N, V) \). Now let \( 0 \to V \to E^0 \to E^1 \to E^2 \to \cdots \) be an injective resolution of the left \( A \)-module \( V \). Then \( \text{Hom}_A(N, E^i) \) is naturally a complex of left \( \mathcal{R} \)-contramodules, hence its cohomology modules \( \text{Ext}_A^n(N, V) = H^n \text{Hom}_A(N, E^i) \) are left \( \mathcal{R} \)-contramodules, too.

Given a complex of \( A \)-modules \( M^\bullet \) and a left \( A \)-module \( B \), we will use the simplified notation \( \text{Ext}_A^n(M^\bullet, B) \) for the abelian groups of morphisms in the derived category of left \( A \)-modules

\[
\text{Ext}_A^n(M^\bullet, B) = \text{Hom}_{\text{D}(A \text{-mod})}(M^\bullet, B[n]).
\]

When \( M^\bullet \) is a complex of \( A \)-\( R \)-bimodules, the right action of \( R \) in the derived category object \( M^\bullet \in \text{D}(A \text{-mod}) \) induces left \( R \)-module structures on the groups \( \text{Ext}_A^n(M^\bullet, B) \).

Assume that the complex \( M^\bullet \) is concentrated in the nonpositive cohomological degrees, that is \( M^i = 0 \) for \( i > 0 \). Then \( \text{Ext}_A^n(M^\bullet, B) = 0 \) for \( n < 0 \). Moreover, denoting by \( L \) the cokernel of the map \( M^{-1} \to M^0 \), one has \( \text{Ext}_A^0(M^\bullet, B) = \text{Hom}_A(L, B) \). However, one may well have \( \text{Ext}_A^n(M^\bullet, E) \neq 0 \) for an injective left \( A \)-module \( E \) and some \( n > 0 \), if \( H^i(M^\bullet) \neq 0 \) for some \( i < 0 \). (In fact, for \( E \) injective, one has \( \text{Ext}_A^n(M^\bullet, E) = \text{Hom}_A(H^{-n}(M^\bullet), E) \) for all \( n \in \mathbb{Z} \), so \( \text{Ext}_A^n(M^\bullet, E) \neq 0 \) whenever \( E \) is an injective cogenerator of \( A \text{-mod} \) and \( H^{-n}(M^\bullet) \neq 0 \).)

Now let us assume that the forgetful functor \( \mathcal{R} \text{-contra} \to \text{mod } R \) is fully faithful, so the abelian category \( \mathcal{C} = \mathcal{R} \text{-contra} \) can be viewed as a full subcategory in the abelian category \( A = \text{mod } R \). Then the construction of Section 7 provides a sequence of abelian full subcategories \( \mathcal{C}_A = \mathcal{C}_A(1) \subset \mathcal{C}_A(2) \subset \mathcal{C}_A(3) \subset \mathcal{C}_A(4) \subset \mathcal{C}_A(5) \subset \cdots \subset \mathcal{A} \), with exact embedding functors \( \mathcal{C}_A(m) \to \mathcal{A} \), indexed by the integers \( m \geq 0 \).
Lemma 8.2. Let $M^\bullet = (M^{-1} \rightarrow M^0)$ be a two-term complex of $A$-$R$-bimodules whose cohomology bimodules $H^{-1}(M^\bullet) = \ker(M^{-1} \rightarrow M^0)$ and $H^0(M^\bullet) = \coker(M^{-1} \rightarrow M^0)$ are discrete as right $R$-modules. Assume that the forgetful functor $\mathcal{R} \text{- contra} \to R\text{-mod}$ is fully faithful. Let $B$ be a left $A$-module. Then for every $n \geq 0$ the left $R$-module $\Ext^n_A(M^\bullet, B)$ belongs to the full subcategory

$$\mathcal{R} \text{- contra}^{(2)}_{R\text{-mod}} = C^{(2)}_A \subset A = R\text{-mod}.$$ 

Proof. Applying the contravariant cohomological functor $\Hom_{D(A\text{-mod})}(-, B[s])$ to the distinguished triangle

$$H^{-1}(M^\bullet)[1] \longrightarrow M^\bullet \longrightarrow H^0(M^\bullet) \longrightarrow H^{-1}(M^\bullet)[2]$$

in the derived category of $A$-$R$-bimodules $D(A\text{-mod} - R)$, we obtain a long exact sequence of left $R$-modules

$$\cdots \longrightarrow \Ext^n_A(H^{-1}(M^\bullet), B) \longrightarrow \Ext^n_A(H^0(M^\bullet), B) \longrightarrow \Ext^n_A(M^\bullet, B)$$

$$\longrightarrow \Ext^{n+1}_A(H^{-1}(M^\bullet), B) \longrightarrow \Ext^{n+1}_A(H^0(M^\bullet), B) \longrightarrow \cdots$$

By Lemma 8.1 the left $R$-modules $\Ext^n_A(H^i(M^\bullet), B)$ are the underlying left $R$-modules of certain left $\mathcal{R}$-contramodules. Since the forgetful functor $\mathcal{R} \text{- contra} \to R\text{-mod}$ is fully faithful, the left $R$-module morphisms $\Ext^{n-2}_A(H^{-1}(M^\bullet), B) \longrightarrow \Ext^{n}_A(H^0(M^\bullet), B)$ are, in fact, left $\mathcal{R}$-contramodule morphisms. So the kernels and cokernels of these morphisms belong to the full subcategory $\mathcal{R} \text{- contra} \subset R\text{-mod}$, and it follows that the left $R$-modules $\Ext^n_A(M^\bullet, B)$ belong to the full subcategory $\mathcal{R} \text{- contra}^{(2)}_{R\text{-mod}} \subset R\text{-mod}$. \hfill \Box

Let $G$ be a right Gabriel topology on an associative ring $R$, and let $U = RG$ be the ring of quotients of $R$ with respect to $G$. The ring $U$ or, which is easier, the $R$-$R$-bimodule $U$ can be obtained from $R$ by applying the sheafification functor $N \mapsto N_{(G)}$ twice: one has $RG = R(G(G))$ (see Section 2.6). Alternatively, one can first pass to the quotient ring $R/\ell_G(R)$ of the ring $R$ by its maximal $G$-torsion right $R$-submodule (which is a two-sided ideal), and then set $RG = (R/\ell_G(R))(G)$.

There is a natural morphism of $R$-$R$-bimodules (in fact, even of associative rings) $R \to RG = U$. It is important for us that both the kernel and the cokernel of the map $R \to U$ are $G$-torsion right $R$-modules.

Consider the two-term complex of $R$-$R$-bimodules

$$K^\bullet_{R,U} = (R \to U)$$

with the term $R$ sitting in the cohomological degree $-1$ and the term $U$ sitting in the cohomological degree $0$. There is a natural distinguished triangle in $D(R\text{-mod} - R)$

$$R \longrightarrow U \longrightarrow K^\bullet_{R,U} \longrightarrow R[1].$$

Set $A = R$, and suppose that we are given a left $R$-module $B$. Applying the contravariant cohomological functor $\Hom_{D(R\text{-mod})}(-, B)$ to the distinguished triangle (2),
we obtain a natural five-term exact sequence of left $R$-modules

\[ 0 \longrightarrow \text{Ext}^0_R(K^\bullet_{R,U}, B) \longrightarrow \text{Hom}_R(U, B) \longrightarrow B \]

\[ \longrightarrow \text{Ext}^1_R(K^\bullet_{R,U}, B) \longrightarrow \text{Ext}^1_R(U, B) \longrightarrow 0, \]

as well as natural isomorphisms of left $R$-modules

\[ \text{Ext}^n_R(K^\bullet_{R,U}, B) \simeq \text{Ext}^n_R(U, B), \quad n \geq 2. \]

Notice that the right $R$-module structure on the ring $U$ is obtained by restricting scalars from a (free) right $U$-module structure. So, for every $n \geq 0$, the left $R$-module structure on $\text{Ext}^n_R(U, B)$ can be obtained by restricting scalars from a natural left $U$-module structure.

Let $R = \lim_{\leftarrow I \in G} R/I$ be the completion of the topological ring $R$ with respect to its topology $G$, viewed as a complete, separated topological ring in the projective limit topology $G$. Let us assume that a right Gabriel topology $G$ on $R$ has a countable base consisting of finitely generated right ideals. Then, by Corollary 6.7, the forgetful functor $R\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful.

**Corollary 8.3.** Let $G$ be a right Gabriel topology on an associative ring $R$ having a countable base consisting of finitely generated right ideals. Then, in the above notation, one has

\[ \text{Ext}^n_R(K^\bullet_{R,U}, B) \in R\text{-contra}^{(2)}_{R\text{-mod}} \]

for all left $R$-modules $B$ and all integers $n \geq 0$.

**Proof.** This is a particular case of Lemma 8.2. \qed

At this point we restrict the generality level even further in order to pass to the situation we are interested in. Let $u: R \longrightarrow U$ be an epimorphism of associative rings such that $U$ is a flat left $R$-module, and let $G$ be the related perfect Gabriel topology on $R$, consisting of all the open right ideals $I \subset R$ such that $R/I \otimes_R U = 0$ (see the discussion in the beginning of Section 3).

For any left $R$-module $M$, we denote by $\text{pd}_R M$ the projective dimension of $M$ (as an object of $R\text{-mod}$). In particular, $\text{pd}_R U$ denotes the projective dimension of the left $R$-module $U$. As usual, we denote by $\text{Hom}_R$ and $\text{Ext}_R^\bullet$ the Hom and Ext groups computed in the abelian category of left $R$-modules $R\text{-mod}$.

Given a left $R$-module $M$, one can consider the full subcategory $M^{\perp 0,1} \subset R\text{-mod}$ of all left $R$-modules $C$ satisfying $\text{Hom}_R(M, C) = 0 = \text{Ext}^1_R(M, C)$. According to [13 Proposition 1.1], the full subcategory $M^{\perp 0,1}$, which we call the Geigle–Lenzing perpendicular subcategory, is closed under kernels and extensions in $R\text{-mod}$ (it is also closed under infinite products). When $\text{pd}_R M \leq 1$, this full subcategory is closed under cokernels as well; so the category $M^{\perp 0,1}$ is abelian in this case, and its fully faithful identity inclusion functor $M^{\perp 0,1} \longrightarrow R\text{-mod}$ is exact.

**Proposition 8.4.** Assume that the perfect Gabriel topology $G$ on the ring $R$ associated with a left flat ring epimorphism $u: R \longrightarrow U$ has a countable base or, which is equivalent, the topology $G$ on the ring $R$ has a countable base. Then
Since the next Theorem 8.5 we will see that the projective dimension of the left module is never exceeds 1 in the assumptions of this proposition, so the conclusion of part (c) always holds (but we do not know this yet).

Part (a): for any discrete right $R$-module $N$, we have $\text{Hom}_R(N, \mathbb{Q}/\mathbb{Z}) \ni U, B$ where $U$ is a flat $R$-module.

By Proposition 8.4(b), the full subcategory of left $R$-modules consisting of all the left $R$-modules $M$ such that $\text{Hom}_R(U, M) = 0$. By Proposition 8.5(b), the full subcategory of left $R$-modules $\langle R \rangle$ is contained in $U_{\geq 1} - \text{mod}$.

Finally, we can prove the main three theorems of this section (which are also among the main results of this paper).

**Theorem 8.5.** Let $u: R \to U$ be an epimorphism of associative rings such that $U$ is a flat right $R$-module. Let $G$ be the related perfect Gabriel topology of right ideals in $R$. Assume that $G$ has a countable base. Then the projective dimension of the left $R$-module $U$ does not exceed 1.

**Proof.** Let $U_{\leq 1} \subset R - \text{mod}$ denote the full subcategory in the category of left $R$-modules consisting of all the left $R$-modules $M$ such that $\text{Hom}_R(U, M) = 0$. By Proposition 8.4(b), the full subcategory of left $R$-modules $\langle R \rangle$ is contained in $U_{\leq 1} - \text{mod}$. Hence it follows that $\langle R \rangle \subset R - \text{mod}.

Applying Corollary 8.3 we see that $\text{Ext}_R^n(K^\bullet_R U, B) \in U_{\leq 1}$ for all left $R$-modules $B$ and all $n \geq 0$. On the other hand, by (1) we have $\text{Ext}_R^n(U, B) \simeq \text{Ext}_R^n(K^\bullet_R U, B)$ for $n \geq 2$. Thus the left $R$-module $\text{Ext}_R^n(U, B)$ belongs to $U_{\leq 1}$ when $n \geq 2$.

Since the left $R$-module structure on $D = \text{Ext}_R^n(U, B)$ underlies a left $U$-module structure, every element $d \in D$ is the image of the unit $1 \in U$ under a certain left $R$-module morphism (actually, a unique left $U$-module morphism) $U \to D$. So
Hom$_R(U, D) = 0$ implies $D = 0$, and we can conclude that Ext$^n_R(U, B) = 0$ for all left $R$-modules $B$ and all $n \geq 2$. □

**Remark 8.6.** There is a much easier alternative proof of Theorem 8.5 applicable in the case of a left $\omega^+$-Noetherian ring $R$ (i.e., when every left ideal in $R$ has a countable set of generators). The argument is based on the sheafification construction of Section 2.6. Indeed, if $R$ is left $\omega^+$-Noetherian, then any submodule of a countably generated left $R$-module is countably generated, and any countably generated left $R$-module is countably presented. In this case, for any $R$-$R$-bimodule $N$ that is countably generated as a left $R$-module and every finitely generated right ideal $I \subset R$, the left $R$-module Hom$_{R^{op}}(I, N)$ is countably generated (being a submodule of a finite direct sum of copies of the left $R$-module $N$). Since the class of countably generated left $R$-modules is closed under countable direct limits, for any right Gabriel topology $G$ on $R$ with a countable base consisting of finitely generated right ideals the $R$-$R$-bimodule $N_G$ is countably generated as a left $R$-module. Hence the $R$-$R$-bimodule $N_G$ is countably generated as a left $R$-module, too. In the context of Theorem 8.5, we can then conclude that the left $R$-module $U$ is countably generated. Hence, in our assumptions, it is countably presented. By [14, Corollary 2.23], any countably presented flat module has projective dimension at most 1.

**Corollary 8.7.** Let $u: R \to U$ be an epimorphism of associative rings such that $U$ is a flat left $R$-module. Assume that the related perfect Gabriel topology $G$ of right ideals in $R$ satisfies the condition (T$_\omega$) of Section 3 (e.g., this holds if $R$ is commutative; see Examples 3.2 for further cases when (T$_\omega$) is satisfied). Assume further that the right $R$-module $R$ has $\omega$-bounded $G$-torsion. Then there exists a diagram of epimorphisms of associative rings $R \to U_v$, indexed by an $\omega^+$-directed poset $\Upsilon$, such that $U_v$ is a flat left $R$-module of projective dimension not exceeding 1 for every $v \in \Upsilon$ and the ring homomorphism $R \to U$ is the direct limit of the ring homomorphisms $R \to U_v$, that is $U = \lim_{\to v \in \Upsilon} U_v$.

**Proof.** Follows from Corollary 3.10 and Theorem 8.5. □

**Remark 8.8.** The following converse assertion to Corollary 8.7 holds for commutative rings $R$: if $R \to U$ is a ring epimorphism such that Tor$^R_1(U, U) = 0$ and pd$_R U \leq 1$, then $U$ is a flat $R$-module. This is [5, Remark 16.9].

The next theorem is a generalization of [25] Examples 5.4(2) and 5.5(2).

**Theorem 8.9.** Let $u: R \to U$ be an epimorphism of associative rings such that $U$ is a flat left $R$-module. Let $G$ be the related perfect Gabriel topology of right ideals in $R$. Assume that $G$ has a countable base. Then the Geigle–Lenzing perpendicular subcategory $\mathcal{R}^{\omega,1} \subset R-\text{mod}$ coincides with the full subcategory of two-object extensions $\mathcal{R}^{(2)} \subset R-\text{mod}$. In particular, it follows that the full subcategory $\mathcal{R}^{(2)} \subset R-\text{mod}$ is closed under extensions in $R-\text{mod}$. So one has $\mathcal{R}^{(2)} = \bigcap_{m \geq 2} \mathcal{R}^{(m)}$, for all the integers $m \geq 2$. 44
Proof. Let us emphasize that the full subcategory $U^\perp \subset \mod R$ is abelian and its embedding functor $U^\perp \to \mod R$ is exact by Theorem [8.3] and [13] Proposition 1.1; while the full subcategory $\contra_{\mod R} \subset \mod R$ is abelian and its embedding functor $\contra_{\mod R} \to \mod R$ is exact by Proposition [7.2] (applied to the exact forgetful functor $\contra_{\mod R} \to \mod R$, which is fully faithful by Corollary [6.7]).

Furthermore, the Geigle–Lenzing perpendicular subcategory $U^\perp \subset \mod R$ is obviously closed under extensions in $\mod R$. By Proposition [8.3](c), we know that $\contra \subset U^\perp \subset \mod R$; hence $\contra_{\mod R} \subset U^\perp$. It remains to prove that the inclusion in the opposite direction holds as well.

Let $B$ be a left $R$-module belonging to $U^\perp$; so $\text{Hom}_R(U, B) = 0 = \text{Ext}^1_R(U, B)$. Then from the exact sequence (3) we see that the natural left $R$-module morphism $B \to \text{Ext}^1_R(K_{RU}, B)$ is an isomorphism. Applying Corollary [8.3] we can conclude that $B \in \contra_{\mod R}$. □

A second, alternative proof of the following theorem will be given in Section [11]; and an uncountable generalization will be obtained in Theorem [11.2]

**Theorem 8.10.** Let $u: R \to U$ be an injective epimorphism of associative rings such that $U$ is a flat left $R$-module. Let $G$ be the related faithful perfect Gabriel topology of right ideals in $R$. Assume that $G$ has a countable base. Then the Geigle–Lenzing perpendicular subcategory $U^\perp \subset \mod R$ coincides with the full subcategory of left $\contra$-contramodules $\contra \subset \mod R$. In particular, it follows that the full subcategory $\contra$ is closed under extensions in $\mod R$.

**First proof.** We already know that $\contra \subset U^\perp$. To prove the inverse inclusion, consider a left $R$-module $B \in U^\perp$. As in the previous proof, we have an isomorphism $B \simeq \text{Ext}^1_R(K_{RU}^*, B)$. Now the morphism $u: R \to U$ is injective by assumption, so the two-term complex $K_{RU}^*$ is quasi-isomorphic to the quotient bimodule $K_{RU} = U/R = \text{coker} u$. It remains to apply Lemma [8.3] in order to conclude that $\text{Ext}^1_R(K_{RU}^*, B) = \text{Ext}^1_R(K_{RU}, B) \in \contra$. □

9. **$U$-Strongly Flat and $U$-Weakly Cotorsion $R$-Modules**

Given a flat left ring epimorphism $u: R \to U$, a left $R$-module $C$ is said to be $U$-weakly cotorsion if $\text{Ext}^1_R(U, C) = 0$. A left $R$-module $F$ is said to be $U$-strongly flat if $\text{Ext}^1_R(F, C) = 0$ for all $U$-weakly cotorsion left $R$-modules $C$. Since $U$ is a flat left $R$-module, it follows from [33] Lemma 3.4.1] that all $U$-strongly flat left $R$-modules are flat. Moreover, a left $R$-module $F$ is $U$-strongly flat if and only if it is a direct summand of a left $R$-module $G$ that can be included into a short exact sequence of left $R$-modules $0 \to V \to G \to W \to 0$, where $V$ is a free left $R$-module and $W$ is a free left $U$-module (by [14] Corollary 6.13] and since $\text{Ext}^1_R(U, W) = 0$ for any free left $U$-module $W$). In this section we apply the techniques developed above in...
Proposition 9.1. (a) For any right $R$-module $N$, there is a contravariant $\mathbb{F}$-system of abelian groups $\text{mh}(N) \in \mathbb{F}\text{-mod}$ assigning to every cyclic discrete right $R$-module $R/I \in Q_{\mathbb{F}}$ the abelian group $N_I = \text{Hom}_{\mathbb{F}\text{-mod}}(R/I, N)$. The functor $\text{mh}: \mathbb{F}\text{-mod} \rightarrow \mathbb{F}\text{-mod}$ is right adjoint to the composition $\mathbb{F}\text{-mod} \rightarrow \mathbb{F}\text{-mod} \rightarrow \mathbb{F}\text{-mod}$ of the fully faithful functor $\mathbb{F}\text{-mod} \rightarrow \mathbb{F}\text{-mod}$ with the functor pl.

(b) For any left $R$-module $C$, there is a covariant $\mathbb{F}$-system of abelian groups $\text{tp}(C) \in \mathbb{F}\text{-mod}$ assigning to every cyclic discrete right $R$-module $R/I \in Q_{\mathbb{F}}$ the abelian group $C/IC = R/I \otimes_R C$. The functor $\text{tp}: \mathbb{F}\text{-mod} \rightarrow \mathbb{F}\text{-mod}$ is left adjoint to the composition $\mathbb{F}\text{-mod} \rightarrow \mathbb{F}\text{-mod} \rightarrow \mathbb{F}\text{-mod}$.

Proof. The abbreviation “$\text{mh}$” means “module Hom”, while “$\text{tp}$” stands for “tensor product”. Given a right $R$-module $N$, the contravariant $\mathbb{F}$-system $\text{mh}(N): Q_{\mathbb{F}} \rightarrow \mathbb{F}\text{-mod}$ is constructed by restricting the contravariant functor $\text{Hom}_{\mathbb{F}\text{-mod}}(-, N)$ to the full subcategory $Q_{\mathbb{F}} \subset \mathbb{F}\text{-mod} \subset \mathbb{F}\text{-mod}$. Given a left $R$-module $C$, the covariant $\mathbb{F}$-system $\text{tp}(C): Q_{\mathbb{F}} \rightarrow \mathbb{F}\text{-mod}$ is constructed by restricting the covariant functor $- \otimes_R C$ to the same full subcategory of cyclic discrete right $R$-modules in $\mathbb{F}\text{-mod}$.

Let us explain why the two functors are adjoint in part (b). This is similar to, though different from (and simper than) the corresponding argument in the proof of Proposition 4.2(b). The isomorphism of Hom groups $\text{Hom}_R(C, \text{pl}(D)) \simeq \text{Hom}_\mathbb{F}(\text{tp}(C), D)$ holds for any left $R$-module $C$ and any covariant $\mathbb{F}$-system of abelian groups $D$, because the datum of a left $R$-module morphism

$$C \rightarrow \lim_{I \in \mathbb{F}} D(R/I)$$

is equivalent to the datum of an $\mathbb{F}$-indexed family of abelian group homomorphisms

$$C/IC \rightarrow D(R/I),$$

defined for all the open right ideals $I \in \mathbb{F}$ and satisfying the compatibility equations for all the morphisms in the category $Q_{\mathbb{F}}$. \qed
Using the construction of Proposition 9.1(b), we now develop a (similar, but different and simpler) module version of the theory of contramodule completion from Section 5. Let \( C \) be a left \( R \)-module. Denote by \( B_C \) the set of all subgroups \( IC \subset C \) of the underlying abelian group of \( C \), where \( I \in \mathbb{F} \) ranges over the open right ideals in \( R \). Then \( B_C \) is a linear topology on the underlying abelian group of \( C \), as one has \( IC \cap JC \supset (I \cap J)C \) for any two open right ideals \( I \) and \( J \subset R \). As in Section 5 we notice that the open subgroups \( IC \subset C \) are not submodules; so this is not a linear topology on a module in the sense of [30, Section VI.4].

The completion \( \lim_{I \in \mathbb{E}} C/IC \) of a left \( R \)-module \( C \) in the topology with the base \( B_C \) can be described in terms of Proposition 9.1(b) as the left \( R \)-module \( \text{pl}(\text{tp}(C)) \). The natural map \( \lambda_{\mathbb{E},C} : C \rightarrow \lim_{I \in \mathbb{E}} C/IC \) is the adjunction morphism for the pair of adjoint functors in part (b) of the proposition. So the abelian group \( \lim_{I \in \mathbb{E}} C/IC \) is, in fact, a left \( R \)-module (even though the abelian groups \( C/IC \) have no such module structures), and the completion map \( \lambda_{\mathbb{E},C} \) is an \( R \)-module morphism.

Moreover, \( \text{tp}(C) \) is a covariant \( \mathbb{F} \)-system of abelian groups, so one can apply the functor \( \text{PL} \) and obtain a left \( \mathcal{R} \)-contramodule \( \text{PL}(\text{tp}(C)) \in \mathcal{R} \text{-contra} \). In other words, according to Proposition 1.2(b), \( \text{pl}(\text{tp}(C)) \) is the underlying left \( R \)-module of the left \( \mathcal{R} \)-contramodule \( \text{PL}(\text{tp}(C)) \). We can conclude that the left \( R \)-module structure of the abelian group \( \lim_{I \in \mathbb{E}} C/IC \) underlies a naturally defined left \( \mathcal{R} \)-contramodule structure. We will sometimes use the notation \( \Lambda_{\mathbb{E}}(C) = \text{PL}(\text{tp}(C)) = \lim_{I \in \mathbb{E}} C/IC \) for this left \( \mathcal{R} \)-contramodule.

A left \( R \)-module \( C \) is said to be \( \mathbb{F} \)-separated (respectively, \( \mathbb{F} \)-complete) if it is a separated (resp., complete) abelian group in the topology with a base \( B_C \). In other words, \( C \) is called \( \mathbb{F} \)-separated if the map \( \lambda_{\mathbb{F},C} \) is injective and \( \mathbb{F} \)-complete if this map is surjective. Clearly, a left \( R \)-module \( C \) is \( \mathbb{F} \)-separated if and only if \( \bigcap_{I \in \mathbb{F}} IC = 0 \).

One easily observes that the left \( R \)-module \( \text{pl}(D) \) is \( \mathbb{F} \)-separated for any covariant \( \mathbb{F} \)-system of abelian groups \( D \); but this is not always true for a covariant pseudo-\( \mathbb{F} \)-system \( D \) as, indeed, any left \( R \)-module can be obtained by applying the functor \( \text{pl} \) to a constant covariant pseudo-\( \mathbb{F} \)-system; see Section 4.1. In particular, for any left \( R \)-module \( C \), the left \( R \)-module \( \lim_{I \in \mathbb{F}} C/IC \) is \( \mathbb{F} \)-separated.

The construction of the \( \mathbb{F} \)-completion of a left \( R \)-module and the related notions of \( \mathbb{F} \)-separatedness and \( \mathbb{F} \)-completeness of a module, as defined in the previous several paragraphs, do not necessarily agree with the similar construction and notions for left \( \mathcal{R} \)-contramodules, as defined in Section 5. However, they do agree in the case of an associative ring \( R \) endowed with a right Gabriel topology \( \mathcal{G} \) with a countable base of finitely generated right ideals. Indeed, by Corollary 6.7, one has \( I \mathcal{C} = J \prec \mathcal{C} \) for any left \( \mathcal{R} \)-contramodule \( \mathcal{C} \), any open right ideal \( I \subset R \), and the corresponding open right ideal \( J \subset \mathcal{R} \) in this case.

So, in the case of a right Gabriel topology \( \mathcal{G} \) with a countable base of finitely generated right ideals, a left \( \mathcal{R} \)-contramodule is separated (resp., complete) if and only if it is \( \mathbb{G} \)-separated (resp., \( \mathbb{G} \)-complete) as a left \( R \)-module. In fact, an \( \mathcal{R} \)-contramodule does not need to be separated, but it is always complete by Theorem 5.3; hence all left \( \mathcal{R} \)-contramodules are \( \mathbb{G} \)-complete as left \( R \)-modules.
The following lemma collects some assertions which may help the reader feel more comfortable.

**Lemma 9.2.** (a) For any ring \( R \) with a right linear topology \( \mathcal{F} \) with a countable base and any left \( R \)-module \( C \), the left \( R \)-contramodule \( C = \lim_{\leftarrow I \in \mathcal{F}} C/IC \) is separated and complete. For any open right ideal \( I \subset R \) and the related open right ideal \( \mathfrak{I} \subset \mathfrak{R} \), the completion map \( \lambda_{\mathcal{F},C} : C \rightarrow C \) and the projection map \( C \rightarrow C/IC \) induce an isomorphism \( C/IC \cong C/\mathfrak{I} \times C \).

(b) For any ring \( R \) with a right Gabriel topology \( \mathcal{G} \) with a countable base of finitely generated ideals and any left \( R \)-module \( C \), the left \( R \)-module \( C = \lim_{\leftarrow I \in \mathcal{G}} C/IC \) is \( \mathcal{G} \)-separated and \( \mathcal{G} \)-complete. For any open right ideal \( I \subset R \), the completion map \( \lambda_{\mathcal{G},C} : C \rightarrow C \) and the projection map \( C \rightarrow C/IC \) induce an isomorphism \( C/IC \cong C/I \times C \).

(c) For any ring \( R \) with a right Gabriel topology \( \mathcal{G} \) with a countable base of finitely generated ideals, a left \( R \)-module \( C \) is \( \mathcal{G} \)-separated and \( \mathcal{G} \)-complete if and only if it is the underlying left \( R \)-module of a separated left \( \mathfrak{R} \)-contramodule.

**Proof.** Part (a): even without the assumption of a countable topology base, all the left \( \mathfrak{R} \)-contramodules belonging to the image of the functor \( \text{PL} \) are separated by Proposition 5.2; so the left \( \mathfrak{R} \)-contramodule \( C = \text{PL}(\text{tp}(C)) \) is separated. On the other hand, in the assumption of a countable topology base, any left \( \mathfrak{R} \)-contramodule is complete by Theorem 5.3 (this is \([26, \text{Lemma 6.3(b)}]\)).

The last assertion in part (a) holds, because the covariant \( \mathcal{F} \)-system of abelian groups \( \text{tp}(C) : R/I \rightarrow C/IC \) is right exact, and for any right exact covariant \( \mathcal{F} \)-system of abelian groups \( D \) the adjunction map \( \text{CT}(\text{PL}(D)) \rightarrow D \) is an isomorphism. This is the result of \([26, \text{Lemma 6.3(a)}]\); see Theorem 5.3.

Part (b) follows from part (a) and Corollary 6.7.

In part (c), if \( C \) is \( \mathcal{G} \)-separated and \( \mathcal{G} \)-complete, then \( \lambda_{\mathcal{G},C} : C \rightarrow \lim_{\leftarrow I \in \mathcal{G}} C/IC \) is an isomorphism; so \( C \) acquires the left \( \mathfrak{R} \)-contramodule structure of \( \text{PL}(\text{tp}(C)) \). Conversely, any separated left \( \mathfrak{R} \)-contramodule is a \( \mathcal{G} \)-separated \( \mathcal{G} \)-complete left \( R \)-module, as it was explained in the paragraph preceding the lemma.

Notice also that, in the assumptions of parts (b-c), there is always at most one way to extend a given left \( R \)-module structure on \( C \) to a left \( \mathfrak{R} \)-contramodule structure (also by Corollary 6.7).

Now that we are finished with the preparatory work, let us proceed to describe \( U \)-weakly cotorsion and \( U \)-strongly flat left \( R \)-modules. We refer to \([27, \text{Section 3}] \) and \([28, \text{Section 2}] \) for the background discussion of simply right obtainable modules.

**Theorem 9.3.** Let \( u : R \rightarrow U \) be a ring epimorphism such that \( U \) is a flat left \( R \)-module and the related right Gabriel topology \( \mathcal{G} \) on \( R \) has a countable base. Then a left \( R \)-module is \( U \)-weakly cotorsion if and only if it can be obtained from left \( R \)-modules belonging to the union of the following two classes:
• the underlying left $R$-modules of left $U$-modules
• $\mathcal{G}$-separated $\mathcal{G}$-complete left $R$-modules

using the operations of the passage to an extension of two $R$-modules and to the cokernel of an injective $R$-module morphism.

Proof. Notice that all the projective left $U$-modules are $U$-strongly flat left $R$-modules; but other left $U$-modules do not have to be $U$-strongly flat as left $R$-modules. On the other hand, any left $U$-module $D$ is a $U$-weakly cotorsion left $R$-module. Indeed, one has $\text{Ext}^1_R(U,D) = \text{Ext}^1_U(U,D) = 0$, because $U \otimes_R U = U$ and $\text{Tor}_1^R(U,U) = 0$. Furthermore, for any $\mathcal{G}$-separated $\mathcal{G}$-complete left $R$-module $C$ one has $\text{Hom}_R(U,C) = 0 = \text{Ext}^1_R(U,C)$ by Lemma 9.2(c) and Proposition 8.4(a).

To prove the if-part, it remains to show that the class of all $U$-weakly cotorsion left $R$-modules is closed under extensions and the cokernels of injective morphisms. The former is obvious from the definition, and the latter means that the two classes ($U$-strongly flat left $R$-modules, $U$-weakly cotorsion left $R$-modules) form a hereditary cotorsion pair in $R$-$\text{mod}$ (see, e. g., [14, Section 5.2]). This is so because one has $\text{pd}_R U \leq 1$ by Theorem 8.5. In fact, any cotorsion pair generated by a module of projective dimension at most 1 is hereditary.

Conversely, let $B$ be a $U$-weakly cotorsion left $R$-module, i. e., $\text{Ext}^1_R(U,B) = 0$. Then from the five-term exact sequence (3) (which reduces to a four-term exact sequence in this case) we see that the left $R$-module $B$ can be obtained using the cokernel of an injective morphism and an extension from three left $R$-modules, namely, $\text{Ext}^0_R(K_{R,U}^*, B)$, $\text{Hom}_R(U,B)$, and $\text{Ext}^1_R(K_{R,U}^*, B)$.

Now $\text{Hom}_R(U,B)$ is a left $U$-module. As to the left $R$-modules $\text{Ext}^0_R(K_{R,U}^*, B)$, by Corollary 8.3 each of them is an extension of at most two left $\mathcal{R}$-contramodules. Finally, by Lemma 5.4 each left $\mathcal{R}$-contramodule is the cokernel of an injective morphism of separated left $\mathcal{R}$-contramodules. The latter are $\mathcal{G}$-separated $\mathcal{G}$-complete left $R$-modules by Lemma 9.2(c).

Remark 9.4. The above suffices to prove the theorem, but in fact one can say more about the left $R$-module $\text{Ext}^0_R(K_{R,U}^*, B)$, which is $\mathcal{G}$-separated and $\mathcal{G}$-complete for any left $R$-module $B$. Indeed, let $L_{R,U} = H^0(K_{R,U}^*)$ denote the cokernel of the $R$-$R$-bimodule morphism $R \rightarrow U$; then one has $\text{Ext}^0_R(K_{R,U}^*, B) = \text{Hom}_R(L_{R,U}, B)$. Now $L_{R,U} = \text{coker}(R \rightarrow R_G)$ is a $\mathcal{G}$-torsion right $R$-module; in other words, $L_{R,U}$ is an $R$-$R$-bimodule that is discrete as a right $R$-module. For any ring $A$, left $A$-module $V$, and $R$-discrete $A$-$R$-bimodule $N$, the left $R$-module $\text{Hom}_A(N,V)$ is a separated left $\mathcal{R}$-contramodule (see Section 2.3 and the proof of Lemma 5.5).

Moreover, let $H_{R,U} = H^{-1}(K_{R,U}^*)$ be the kernel of the $R$-$R$-bimodule morphism $R \rightarrow U$; then $H_{R,U}$ is also a $\mathcal{G}$-torsion right $R$-module. Looking into the proof of Lemma 8.2 (on which Corollary 8.3 is based; cf. Remark 10.5 below), we see that the left $R$-module $\text{Ext}^1_R(K_{R,U}^*, B)$ is an extension of the left $R$-module $\text{Ext}^1_R(L_{R,U}, B)$ and the kernel of the left $R$-module morphism $\text{Hom}_R(H_{R,U}, B) \rightarrow \text{Ext}^2_R(L_{R,U}, B)$. The former one is a left $\mathcal{R}$-contramodule; while the latter one is a subcontramodule of a separated left $\mathcal{R}$-contramodule $\text{Hom}_R(H_{R,U}, B)$, hence a separated left
\( R \)-contramodule itself. So the left \( R \)-module \( \text{Ext}^1_R(K^*_{R,U}, B) \) is an extension of one left \( R \)-contramodule and one separated left \( R \)-contramodule.

Summarizing the arguments above, one observes that any \( U \)-weakly cotorsion left \( R \)-module \( B \) can be obtained from one underlying left \( R \)-module of a left \( U \)-module and four \( G \)-separated \( G \)-complete left \( R \)-modules using two passages to the cokernel of an injective morphism and two passages to an extension of \( R \)-modules. Thus, in total, one needs to apply our operations four times.

**Corollary 9.5.** Let \( u: R \rightarrow U \) be a ring epimorphism such that \( U \) is a flat left \( R \)-module and the related right Gabriel topology \( G \) on \( R \) has a countable base. Then a left \( R \)-module \( F \) is \( U \)-strongly flat if and only if it satisfies the following two conditions:

1. \( \text{Ext}^1_R(F, D) = 0 \) for all left \( U \)-modules \( D \); and
2. \( \text{Ext}^1_R(F, C) = 0 = \text{Ext}^2_R(F, C) \) for all \( G \)-separated \( G \)-complete left \( R \)-modules \( C \).

**Proof.** The if-assertion of Theorem 9.3 implies the necessity of the condition (i) and of the \( \text{Ext}^1 \)-part of the condition (ii). Furthermore, since \( \text{pd}_R U \leq 1 \), it follows easily from the Eklof lemma and (the proof of) the Eklof–Trlifaj theorem [9 Lemma 1 and Theorem 10] (cf. [14, Corollary 6.14]) that \( \text{pd}_R F \leq 1 \) for every \( U \)-strongly flat left \( R \)-module \( F \). Thus the \( \text{Ext}^2 \)-part of the condition (ii) is also necessary, and in fact one has \( \text{Ext}^1_R(F, D) = 0 = \text{Ext}^2_R(F, C) \) for all \( U \)-strongly flat left \( R \)-modules \( F \), left \( R \)-modules \( D \) and \( C \) as in (i) and (ii), and all \( n \geq 1 \).

Conversely, for any given left \( R \)-module \( F \), the class of all left \( R \)-modules \( B \) satisfying \( \text{Ext}^1_R(F, B) = 0 \) for \( n \geq 1 \) is closed under extensions and cokernels of injective morphisms. Hence, whenever \( \text{Ext}^1_R(F, D) = 0 = \text{Ext}^2_R(F, C) \) for all \( D \) and \( C \) as in (i) and (ii) and all \( n \geq 1 \), one has \( \text{Ext}^1_R(F, B) = 0 \) for all left \( R \)-modules \( B \) that can be obtained from such \( R \)-modules as \( D \) and \( C \) using extensions and cokernels of injections. Since all \( U \)-weakly cotorsion left \( R \)-modules can be so obtained by Theorem 9.3, we can conclude that \( F \) is \( U \)-strongly flat.

A slightly more careful analysis of the specific procedure for producing \( U \)-weakly cotorsion left \( R \)-modules out of the left \( U \)-modules and the \( G \)-separated \( G \)-complete \( R \)-modules used in the proof of the only if-part of Theorem 9.3 reveals that the \( \text{Ext}^1 \) vanishing in (i) and the \( \text{Ext}^{1,2} \) vanishing in (ii) are enough. \( \square \)

Let \( F \) be a flat left \( R \)-module. Then for any left \( U \)-module \( D \) there are natural isomorphisms of abelian groups \( \text{Ext}^n_R(F, D) \cong \text{Ext}^n_u(U \otimes_R F, D) \) for all \( n \geq 0 \). (Indeed, given a projective resolution \( P_u \) of the left \( R \)-module \( F \), the complex \( U \otimes_R P_u \) is a projective resolution of the left \( U \)-module \( U \otimes_R F \), and the two complexes of abelian groups \( \text{Hom}_R(P_u, D) \) and \( \text{Hom}_U(U \otimes_R P_u, D) \) are naturally isomorphic by the tensor-Hom adjunction.) Therefore, the condition (i) in Corollary 9.5 holds if and only if the left \( U \)-module \( U \otimes_R F \) is projective. Assuming that the right Gabriel topology \( G \) has a countable base consisting of two-sided ideals, the condition (ii) can be also reformulated in a similar way, as we will now show.
Given an associative ring \( R \) and a sequence of left \( R \)-modules \( C_0, C_1, C_2, \ldots \) indexed by nonnegative integers, we will say that a left \( R \)-module \( C \) is an infinitely iterated extension of the left \( R \)-modules \( C_i \) in the sense of the projective limit if there exists a decreasing filtration \( C = G^0 \supset G^1 \supset G^2 \supset \cdots \) of \( C \) by its \( R \)-submodules \( G^i \) such that the natural \( R \)-module morphism \( C \to \lim_{\leftarrow i} C/G^i \) is an isomorphism and the quotient module \( G^i/G^{i+1} \) is isomorphic to \( C_i \) for every \( i \geq 0 \). The dual Eklof lemma [9, Proposition 18] tells that if a left \( R \)-module \( C \) is an infinitely iterated extension of left \( R \)-modules \( C_i \) in the sense of the projective limit and \( F \) is a left \( R \)-module such that \( \text{Ext}^1_R(F, C_i) = 0 \) for all \( i > 0 \), then \( \text{Ext}^1_R(F, C) = 0 \).

**Theorem 9.6.** Let \( u: R \to U \) be a ring epimorphism such that \( U \) is a flat left \( R \)-module and the related Gabriel topology of right ideals \( \mathcal{G} \) on \( R \) has a countable base consisting of two-sided ideals. Then a left \( R \)-module is \( U \)-weakly cotorsion if and only if it can be obtained from left \( R \)-modules belonging to the union of the following classes:

- left \( U \)-modules
- left modules over quotient rings \( R/H \) of the ring \( R \) by its two-sided ideals \( H \subset R \) belonging to \( \mathcal{G} \)

using the operations of the passage to an extension of two \( R \)-modules, to an infinitely iterated extension of a sequence of \( R \)-modules, in the sense of the projective limit, and to the cokernel of an injective \( R \)-module morphism.

**Proof.** By the dual Eklof lemma [9, Proposition 18], the class of all \( U \)-weakly cotorsion left \( R \)-modules is closed under infinitely iterated extensions in the sense of the projective limit. Since all left \( R/H \)-modules are \( \mathcal{G} \)-separated \( \mathcal{G} \)-complete left \( R \)-modules, the if-part follows from Theorem 9.3.

To prove the “only if”, it remains to observe that every \( \mathcal{G} \)-separated \( \mathcal{G} \)-complete left \( R \)-module is an infinitely iterated extension of a sequence of \( R \)-modules. Indeed, let \( R \supset H_1 \supset H_2 \supset \cdots \) be a decreasing sequence of open two-sided ideals belonging to \( \mathcal{G} \) such that the collection \( \mathcal{H} \) of all the ideals \( H_i, i \geq 1 \), is a base of the topology \( \mathcal{G} \). Given a \( \mathcal{G} \)-separated \( \mathcal{G} \)-complete left \( R \)-module \( C \), set \( G^i = H_i/C \subset C \) for every \( i \geq 1 \). This is a decreasing filtration of \( C \) by its \( R \)-submodules, the natural map \( C \to \lim_{\leftarrow i} C/G^i \) is an isomorphism, and the quotient module \( G^i/G^{i+1} \) is an \((R/H_{i+1})\)-module for every \( i \geq 1 \).

The following corollary can be thought of as confirming a version of [28 Optimistic Conjecture 1.1] for perfect Gabriel topologies with a countable base of two-sided ideals. This corollary is also a generalization of [28 Theorem 1.3] (while the previous theorem is a generalization of [28 Proposition 1.6]). We refer to the introduction to [28] (see [28 Sections 1.1–1.3]) for a discussion.

**Corollary 9.7.** Let \( u: R \to U \) be a ring epimorphism such that \( U \) is a flat left \( R \)-module and the related Gabriel topology of right ideals \( \mathcal{G} \) on \( R \) has a countable base \( \mathcal{B} \) consisting of two-sided ideals. Then a flat left \( R \)-module \( F \) is \( U \)-strongly flat if and only if it satisfies the following two conditions:
(i) the left $U$-module $U \otimes_R F$ is projective;
(ii) for every open two-sided ideal $H \subset R$, $H \in \mathcal{B}$, the left $R/H$-module $F/HF$ is projective.

Proof. The only if-part is easy to prove, and it does not depend on the assumption of countability of $\mathcal{B}$. Indeed, suppose that a left $R$-module $F$ is a direct summand of a left $R$-module $G$ included in a short exact sequence of (flat) left $R$-modules $0 \to V \to G \to W \to 0$, where $V$ is a free left $R$-module and $W$ is a free left $U$-module. Then the left $U$-module $U \otimes_R G$ is free, since the left $U$-modules $U \otimes_R V$ and $U \otimes_R W \cong W$ are; and the left $R/H$-module $G/HG$ is free for any two-sided ideal $H \in \mathcal{G}$, since the left $R/H$-modules $V/HV$ and $W/HW = 0$ are. Hence the left $U$-module $U \otimes_R F$ and the left $R/H$-module $F/HF$ are projective.

Now we return to the assumption of a countable base of two-sided ideals $\mathcal{B}$ in $\mathcal{G}$ and prove both the if- and only if-parts. The same argument as in the proof of Corollary [9.5] but based on Theorem [9.6] instead of Theorem [9.3] and using also the dual Eklof lemma, shows that an $R$-module $F$ is $U$-strongly flat if and only if $\text{Ext}_R^n(F, D) = 0 = \text{Ext}_R^n(F, C)$ for all left $U$-modules $D$, all left $R/H$-modules $C$ with $H \in \mathcal{B}$, and all $n \geq 1$. Assume that $F$ is flat left $R$-module. We have already seen above that the condition $\text{Ext}_R^n(F, D) = 0$ for all $D \in \text{U-mod}$ and $n \geq 1$ (or just $n = 1$) holds if and only if the left $U$-module $U \otimes_R F$ is projective. Similarly, given any fixed $H \in \mathcal{B}$, one has $\text{Ext}_R^n(F, C) \cong \text{Ext}_{R/H}^n(F/HF, C)$ for all $C \in R/H-\text{mod}$ and all $n \geq 0$. Thus the condition that $\text{Ext}_R^n(F, C) = 0$ for all such $C$ and $n \geq 1$ (or just $n = 1$) holds if and only if the left $R/H$-module $F/HF$ is projective. □

10. When $\Delta$ equals $\Lambda$?

Let $u: R \to U$ be an epimorphism of associative rings such that $U$ is a left $R$-module of projective dimension at most 1. Let us introduce the name left $u$-contramodules (or $u$-contramodule left $R$-modules) for objects of the Geigle–Lenzing perpendicular subcategory $U^{1,0,1}_r \subset R$-mod. Let us also introduce the notation $R$-mod$_{u, \text{tra}} = U^{1,0,1}_r$ for the full subcategory of $u$-contramodules in $R$-mod. We recall that, according to [15] Proposition 1.1], $R$-mod$_{u, \text{tra}}$ is an abelian category and its embedding $R$-mod$_{u, \text{tra}} \to R$-mod is an exact functor.

Given an associative ring homomorphism $u: R \to U$, we consider the two-term complex of $R$-$R$-bimodules $K^\bullet_{R,U} = (R \to U)$, as in Section 8 and for any left $R$-module $M$ set

$$\Delta_u(M) = \text{Ext}_R^1(K^\bullet_{R,U}, M).$$

This defines a functor $\Delta_u: R$-mod $\to R$-mod. For every left $R$-module $M$, there is a natural left $R$-module morphism $\delta_{a,M}: M \to \Delta_u(M)$ appearing in the exact sequence [3] (cf. [5] exact sequence (9) in Section 16]).

The following result goes back to [17], Proposition 2.4] (see also [23] Theorem 3.4(b))).
Lemma 10.1. Let \( u: R \rightarrow U \) be an epimorphism of associative rings such that \( \text{pd}_R U \leq 1 \). Then, for every left \( R \)-module \( M \), the left \( R \)-module \( \Delta_u(M) \) is a \( u \)-contramodule. The functor \( \Delta_u: \text{R-mod} \rightarrow \text{R-mod}_{u\text{-contra}} \) is left adjoint to the embedding \( \text{R-mod}_{u\text{-contra}} \rightarrow \text{R-mod} \). The natural map \( \delta_{u,M}: M \rightarrow \Delta_u(M) \) is the adjunction morphism.

Proof. This is [3] Proposition 17.2(b)]. □

The aim of this section is to establish a sufficient condition for an isomorphism between the \( u \)-contramodule left \( R \)-module \( \Delta_u(M) \) and (the underlying \( R \)-module of) the left \( \mathcal{R} \)-contramodule \( \Lambda_G(M) = \lim_{I \in \mathcal{G}} M/IM \) constructed in the first half of Section \ref{section9} (see also [23, Theorem 6.10]) to our setting.

This will provide a generalization of the classical result of Matlis [17, Theorem 6.10] to our setting.

In fact, we will even obtain a sufficient condition for an isomorphism \( \Delta_u(M) \cong \Lambda_G(M) \) applicable for any left flat ring epimorphism \( u: R \rightarrow U \) (without the assumption on the projective dimension of the left \( R \)-module \( U \)). But we will need some technical assumptions on the right Gabriel topology \( \mathcal{G} \) related to \( u \), which we inherit from Section \ref{section9}.

The next proposition is a generalization of Proposition 8.4(c). It is also a generalization of [25] Examples 2.4(2) and 2.5(2)].

Proposition 10.2. Let \( u: R \rightarrow U \) be an epimorphism of associative rings such that \( U \) is a flat left \( R \)-module of projective dimension not exceeding \( 1 \), let \( \mathcal{G} \) be the perfect Gabriel topology of right ideals in \( R \) associated with the left flat epimorphism \( u \), and let \( \mathcal{R} \) be the completion of \( R \) with respect to \( \mathcal{G} \), viewed as a topological ring in the projective limit topology \( \mathcal{G} \). Then the image of the forgetful functor \( \mathcal{R}\text{-contra} \rightarrow \text{R-mod} \) is contained in the full subcategory \( \text{R-mod}_{u\text{-contra}} \subset \text{R-mod} \).

Proof. The full subcategory of \( u \)-contramodule left \( R \)-modules \( \text{R-mod}_{u\text{-contra}} \) is closed under kernels, cokernels, extensions, and infinite products in \( \text{R-mod} \). The forgetful functor \( \mathcal{R}\text{-contra} \rightarrow \text{R-mod} \) preserves the kernels, cokernels, and infinite products.

Any left \( \mathcal{R} \)-contramodule is the cokernel of a morphism of free left \( \mathcal{R} \)-contramodules; so it suffices to show that the underlying left \( R \)-module of the free left \( \mathcal{R} \)-contramodule \( \mathcal{R}[X] \) belongs to \( \text{R-mod}_{u\text{-contra}} \) for every set \( X \).

For any complete, separated topological ring \( \mathcal{R} \) in a right linear topology \( \mathfrak{F} \), the free left \( \mathcal{R} \)-contramodules \( \mathcal{R}[X] \) are separated and complete, since

\[
\mathcal{R}[X] = \lim_{\mathfrak{F} \in \mathfrak{F}} (\mathcal{R}/\mathfrak{I})[X] = \lim_{\mathfrak{F} \in \mathfrak{F}} \mathcal{R}[X]/\mathfrak{I}[X] = \lim_{\mathfrak{F} \in \mathfrak{F}} \mathcal{R}[X]/(\mathfrak{I} \cap \mathcal{R}[X]).
\]

One can also observe that \( \mathcal{R}[X] = \text{PL}(F(X)) \), where \( F(X): \mathcal{R}/\mathfrak{I} \rightarrow (\mathcal{R}/\mathfrak{I})^X \) is the coproduct of \( X \) copies of the identity/forgetful covariant \( \mathfrak{F} \)-system of abelian groups \( F: \mathcal{R}/\mathfrak{I} \rightarrow \mathcal{R}/\mathfrak{I} \) assigning to a cyclic discrete right \( \mathcal{R} \)-module \( \mathcal{R}/\mathfrak{I} \) the abelian group \( \mathcal{R}/\mathfrak{I} \) for every \( \mathfrak{I} \in \mathfrak{F} \). Following the second proof of Proposition 5.6 for every covariant \( \mathfrak{F} \)-system \( D: \mathcal{Q} \rightarrow \text{Ab} \) the left \( \mathcal{R} \)-contramodule \( \text{PL}(D) \) is the kernel of a morphism between two left \( \mathcal{R} \)-contramodules of the form \( \text{Hom}_\mathbb{Z}(N', \mathbb{Q}/\mathbb{Z}) \) and \( \text{Hom}_\mathbb{Z}(N'', \mathbb{Q}/\mathbb{Z}) \), where \( N' \) and \( N'' \) are discrete right \( \mathcal{R} \)-modules.

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Thus, returning to the situation at hand, it remains to show that \(\text{Hom}_Z(N, \mathbb{Q}/\mathbb{Z}) \in R\text{-mod}_{\text{u,ctra}}\) for every discrete right \(\mathcal{R}\)-module \(N\). The latter assertion is provable by the explicit argument from the first half of the proof of Proposition 8.4(a). \(\square\)

Let \(u: R \rightarrow U\) be a ring epimorphism such \(U\) is a flat left \(R\)-module of projective dimension at most 1, and let \(\mathcal{G}\) be the related perfect right Gabriel topology on \(R\). Let \(M\) be a left \(R\)-module. The left \(R\)-module morphism \(\lambda_{G,M}: M \rightarrow \Lambda_G(M)\) has a left \(\mathcal{R}\)-contramodule as its target; by Proposition 10.2, it follows that \(\Lambda_G(M)\) is a \(u\)-contramodule left \(R\)-module. By Lemma 10.1 the left \(R\)-module map \(\delta_{u,M}: M \rightarrow \Delta_u(M)\) is the universal morphism from \(M\) to a \(u\)-contramodule left \(R\)-module. Thus there exists a unique left \(R\)-module morphism

\[
\beta_{u,M}: \Delta_u(M) \longrightarrow \Lambda_G(M)
\]

making the triangle diagram of \(R\)-module morphisms \(M \rightarrow \Delta_u(M) \rightarrow \Lambda_G(M)\) commutative.

**Remark 10.3.** When \(M = R[X]\) is a free left \(R\)-module, our arguments allow to deduce the existence and uniqueness of a left \(R\)-module morphism

\[
\beta_{u,X} = \beta_{u,R[X]}: \Delta_u(R[X]) \rightarrow \Lambda_G(R[X])
\]

forming commutative triangle diagram with the maps \(\delta_{u,R[X]}\) and \(\lambda_{G,R[X]}\) for any left flat ring epimorphism \(u: R \rightarrow U\) (without the assumption that \(\text{pd}_R U \leq 1\)).

Indeed, even more generally, let \(\mathcal{F}\) be a right linear topology on an associative ring \(R\), and let \(\mathcal{R}\) be the completion of \(R\) with respect to \(\mathcal{F}\), viewed as a topological ring in its projective limit topology \(\mathcal{F}\). Then, for any set \(X\), the \(\mathcal{F}\)-completion of the free left \(R\)-module \(R[X]\) is the free left \(\mathcal{R}\)-contramodule \(\mathcal{R}[[X]]\),

\[
\mathcal{R}[[X]] = \lim_{\mathcal{F}} R[[X]] = \lim_{\mathcal{I} \in \mathcal{F}} R[X]/IR[X] = \Lambda_{\mathcal{F}}(R[X]).
\]

In fact, one has \(\mathcal{R}[[X]] = \text{PL}(F[X])\), where \(F[X] = \text{tp}(R[X])\) is the coproduct of \(X\) copies of the identity/forgetful covariant \(\mathcal{F}\)-system of abelian groups \(F: R/I \rightarrow R/I\) (as in the proof of Proposition 10.2). The completion map \(\lambda_{\mathcal{F},R[X]}\) can be described as the unique left \(R\)-module morphism \(R[X] \rightarrow \mathcal{R}[[X]]\) whose restriction to the subset of generators \(X \subset R[X]\) is the identity inclusion \(X \rightarrow \mathcal{R}[[X]]\).

Following the proof of Proposition 10.2 even without the assumption on the projective dimension of the left \(R\)-module \(U\) one has \(\mathcal{R}[[X]] \in U^{\perp_{0,1}} \subset R\text{-mod}\) for any set \(X\). This makes the existence and uniqueness of the map \(\beta_{u,X}\) provable by the classical Matlis’ argument [17, Proposition 2.4] (see [23, Lemma 2.1(b)] for a recent exposition), which can be easily adapted to the situation at hand. (The very same argument is also used in the proof of Lemma 10.1.)

Using an idea of Facchini and Nazemian [10, Sections 3–4], we will now construct a map in the opposite direction, under certain assumption. Namely, let \(u: R \rightarrow U\) be a left flat ring epimorphism and \(\mathcal{G}\) be the related perfect Gabriel topology of right ideals in \(R\). Let \(M\) be a left \(R\)-module. Assume that the map \(M \rightarrow U \otimes_R M\) induced by the ring homomorphism \(u: R \rightarrow U\) is injective. In the terminology of [5, Section 16], this means that the left \(R\)-module \(M\) is \(u\)-torsion-free. (Let the reader
be warned that the class of all $u$-torsion-free left $R$-modules does not need to be a torsion-free class in $R\text{-mod}$ in our assumptions; see the discussion in loc. cit.)

Then we have a short exact sequence of left $R$-modules $0 \rightarrow M \rightarrow U \otimes_R M \rightarrow \text{coker } u \rightarrow 0$, where $U/R$ is a shorthand notation for the $R$-$R$-bimodule $\text{coker } u$. Applying the functor $\text{Ext}^1_R(K_{R,U}^*, -)$, we obtain the induced morphism (the connecting homomorphism in the long exact sequence) of left $R$-modules

$$\text{Hom}_R(U/R, U/R \otimes_R M) = \text{Ext}^0_R(K_{R,U}^*, U/R \otimes_R M) \rightarrow \text{Ext}^1_R(K_{R,U}^*, M) = \Delta_u(M).$$

The same morphism can be also constructed as the composition

(5) \quad \text{Hom}_R(U/R, U/R \otimes_R M) \rightarrow \text{Ext}^1_R(U/R, M) \rightarrow \text{Ext}^1_R(K_{R,U}^*, M) = \Delta_u(M),

where the left $R$-module morphism $\text{Hom}_R(U/R, U/R \otimes_R M) \rightarrow \text{Ext}^1_R(U/R, M)$ is the connecting homomorphism in the long exact sequence obtained by applying the functor $\text{Ext}^1_R(U/R, -)$ to the same short exact sequence of left $R$-modules, and the morphism $\text{Ext}^1_R(U/R, M) \rightarrow \text{Ext}^1_R(K_{R,U}^*, M)$ is induced by the natural morphism of complexes of $R$-$R$-bimodules $K_{R,U}^* \rightarrow U/R$.

For any associative ring $A$, any $A$-$R$-bimodule $N$ with a $G$-torsion underlying right $R$-module, any left $A$-module $V$, and any right ideal $I \in G$, the subgroup $I \text{Hom}_A(N, V) \subset \text{Hom}_A(N, V)$ is contained in the kernel of the restriction map $\text{Hom}_A(N, V) \rightarrow \text{Hom}_A(N_I, V)$, where $N_I \subset N$ denotes the left $A$-submodule of all elements annihilated by the right action of $I$ in $N$. Hence we have a natural morphism of abelian groups

(6) \quad \Lambda_G(\text{Hom}_A(N, V)) = \lim_{I \in G} \text{Hom}_A(N, V)/I \text{Hom}_A(N, V)

$$\longrightarrow \lim_{I \in G} \text{Hom}_A(N_I, V) = \text{Hom}_A(N, V).$$

In fact, this is a left $R$-contramodule morphism, which can be obtained by applying the functor $\text{PL}$ to a natural morphism from the covariant $G$-system of abelian groups $\text{tp}(\text{Hom}_A(N, V))$ to the covariant $G$-system of abelian groups $R/I \rightarrow \text{Hom}_A(DH(N)/(R/I), V)$. The composition $\text{Hom}_A(N, V) \rightarrow \Lambda_G(\text{Hom}_A(N, V)) \rightarrow \text{Hom}_A(N, V)$ of the map $\lambda_{G,Hom_A(N,V)}$ with our map (6) is the identity map (so the left $R$-module $\text{Hom}_A(N, V)$ is a direct summand of the left $R$-module $\Lambda_G(\text{Hom}_A(N, V))$).

Now we set $A = R, \ N = U/R$, and $V = U/R \otimes_R M$; and consider the composition

(7) \quad \Lambda_G(M) \longrightarrow \Lambda_G(\text{Hom}_R(U/R, U/R \otimes_R M))

$$\longrightarrow \text{Hom}_R(U/R, U/R \otimes_R M) \longrightarrow \Delta_u(M)$$

of the $R$-module morphism obtained by applying $\Lambda_G$ to the natural morphism $M \rightarrow \text{Hom}_R(U/R, U/R \otimes_R M)$, the $R$-module morphism (6), and the $R$-module morphism (5). We will denote the left $R$-module morphism (7) so constructed by $\theta_{G,M}: \Lambda_G(M) \longrightarrow \Delta_u(M)$.

The morphism $\theta_{G,M}$ is well-defined for any $u$-torsion-free left $R$-module $M$.

**Lemma 10.4.** Let $u: R \rightarrow U$ be an epimorphism of associative rings such that $U$ is a flat left $R$-module, and let $G$ be the related perfect Gabriel topology of right
ideals in \( R \). Then for any \( u \)-torsion-free left \( R \)-module \( M \) the triangle diagram formed by the left \( R \)-module morphisms \( \lambda_{G,M} : M \to \Lambda_G(M) \), \( \delta_{u,M} : M \to \Delta_u(M) \), and \( \theta_{G,M} : \Lambda_G(M) \to \Delta_u(M) \) is commutative.

**Proof.** Applying the functor \( \Lambda_G \) and the natural transformation \( \lambda_G \) to the natural morphism \( M \to \text{Hom}_R(U/R, U/R \otimes_R M) \) produces a commutative square diagram. The composition \( \text{Hom}_R(U/R, U/R \otimes_R M) \to \Lambda_G(\text{Hom}_R(U/R, U/R \otimes_R M)) \to \text{Hom}_R(U/R, U/R \otimes_R M) \) is the identity map, as we mentioned above.

It remains to check commutativity of the triangle diagram \( M \to \text{Hom}_R(U/R, U/R \otimes_R M) \to \Delta_u(M) \) formed by the natural map \( M \to \text{Hom}_R(U/R, U/R \otimes_R M) \), the map \( \text{Hom}_R(U/R, U/R \otimes_R M) \to \Delta_u(M) \) \( \Phi \), and the map \( \delta_{u,M} : M \to \Delta_u(M) \). For this purpose, it suffices to show that the triangle diagram \( M \to \text{Hom}_R(U/R, U/R \otimes_R M) \to \text{Ext}_R^1(U/R, M) \) is commutative, as the morphism \( \delta_{u,M} \) is the composition \( M \to \text{Ext}_R^1(U/R, M) \to \text{Ext}_R^1(K^{*,U}, M) \).

Here the map \( \text{Hom}_R(U/R, U/R \otimes_R M) \to \text{Ext}_R^1(U/R, M) \) is obtained by applying the functor \( \text{Ext}_R^*(U/R, -) \) to the short exact sequence of left \( R \)-modules \( 0 \to M \to U \otimes_R M \to U/R \otimes_R M \to 0 \). The map \( M \to \text{Ext}_R^1(U/R, M) \) is obtained by applying the functor \( \text{Ext}_R^*(-, M) \) to the short exact sequence of left \( R \)-modules \( 0 \to \overline{\Phi} \to U \to U/R \to 0 \), where \( \overline{\Phi} \) is the image of the map \( u \). Any left \( R \)-module morphism \( R \to M \) factorizes through the surjection \( R \to \overline{R} \), since \( M \subset U \otimes_R M \); so one has \( M = \text{Hom}_R(R, M) = \text{Hom}_R(\overline{R}, M) \).

Given an element \( m \in M \), one can explicitly check that the two related Yoneda extension classes in \( \text{Ext}_R^1(U/R, M) \) indeed coincide, by constructing an isomorphism between the two related short exact sequences of left \( R \)-modules. \( \square \)

**Remark 10.5.** For any associative ring homomorphism \( u : R \to U \) and any left \( R \)-module \( M \) there is a natural short exact sequence of left \( R \)-modules

\[
0 \to \text{Ext}_R^1(U/R, M) \to \text{Ext}_R^1(K^{*,U}, M) \to \text{Hom}_R(H, M) \to \text{Ext}_R^2(U/R, M) \to \cdots ,
\]

where \( H = \ker u \subset R \) (cf. the proof of Lemma 8.2). In particular, the map \( \text{Ext}_R^1(U/R, M) \to \text{Ext}_R^1(K^{*,U}, M) = \Delta_u(M) \) is always injective. Hence it follows from the construction of the map \( \theta_{G,M} \) and from the next lemma that, in the assumptions of the latter, the natural map \( \text{Ext}_R^1(U/R, M) \to \Delta_u(M) \) is an isomorphism.

**Lemma 10.6.** Let \( u : R \to U \) be an epimorphism of associative rings such that \( U \) is a flat left \( R \)-module of projective dimension not exceeding \( 1 \), and let \( G \) be the related perfect Gabriel topology of right ideals in \( R \). Let \( M \) be a \( u \)-torsion-free left \( R \)-module. Then the composition \( \xi = \theta \beta : \Delta_u(M) \to \Delta_u(M) \) of the left \( R \)-module morphisms \( \beta_{u,M} : \Delta_u(M) \to \Lambda_G(M) \) and \( \theta_{G,M} : \Lambda_G(M) \to \Delta_u(M) \) is the identity map, \( \xi = \text{id} \).

**Proof.** By the definition, the left \( R \)-module morphism \( \beta_{u,M} \) forms a commutative triangle diagram with the morphisms \( \delta_{u,M} \) and \( \lambda_{G,M} \). By Lemma 10.3, the left \( R \)-module morphism \( \theta_{G,M} \) forms a commutative triangle diagram with the morphisms \( \delta_{u,M} \) and \( \lambda_{G,M} \) as well. Hence it follows that the composition \( \xi = \theta \beta \) is
a left $R$-module morphism forming a commutative triangle diagram with the morphism $\delta_{u,M}: M \to \Delta_u(M)$, that is $\xi \delta = \delta$. In view of Lemma 10.1, one can conclude that $\xi = \text{id}$ is the identity morphism. 

Similarly, it also follows from Lemma 10.4 that the composition $\Lambda_G(M) \to \Delta_u(M)$ is a left $R$-module morphism $\zeta = \beta \theta: \Lambda_G(M) \to \Lambda_G(M)$ forming a commutative triangle diagram with the morphism $\lambda_{G,M}: M \to \Lambda_G(M)$, that is $\zeta \lambda = \lambda$. The next lemma allows to prove that $\zeta$ is the identity morphism, too, under certain assumptions.

**Lemma 10.7.** Let $R$ be an associative ring with a right linear topology $\mathbb{F}$ having a countable base, and let $\mathcal{R}$ be the completion of $R$ with respect to $\mathbb{F}$, viewed as a topological ring in its projective limit topology $\mathcal{F}$. Let $M$ be a left $R$-module, let $\lambda_{R,M}: R \to \Lambda_R(M)$ be the natural left $R$-module morphism from $M$ into its $\mathbb{F}$-completion $\Lambda_R(M)$, and let $\zeta: \Lambda_R(M) \to \Lambda_R(M)$ be a left $\mathcal{R}$-contramodule morphism such that $\zeta \lambda_{R,M} = \lambda_{R,M}$. Then $\zeta$ is the identity map, $\zeta = \text{id}$.

**Proof.** By the definition, we have $\Lambda_R(M) = \text{PL}(\text{tp}(M))$. Set $\mathcal{M} = \Lambda_R(M)$; then, by Lemma 9.2(a), we have $M/IM \cong \mathcal{M}/\mathcal{I} \times \mathcal{M}$ for any open right ideal $I \in \mathbb{F}$ in $R$ and the corresponding open right ideal $\mathcal{I} \in \mathcal{F}$ in $\mathcal{R}$. So the map $M \to \mathcal{M}/\mathcal{I} \times \mathcal{M}$ is surjective. Hence for any element $m \in \Lambda_R(M)$ there exists an element $m' \in M$ such that $m \in \lambda(m') + \mathcal{I} \times \mathcal{M}$. Applying the map $\zeta$, we conclude that

$$\zeta(m) = \zeta(m') + \zeta(\mathcal{I} \times \mathcal{M}) = \lambda(m') + \zeta(\mathcal{I} \times \mathcal{M}) \subset \lambda(m') + \mathcal{I} \times \mathcal{M},$$

as $\zeta$ is a left $\mathcal{R}$-contramodule morphism by assumption. Thus the difference $m - \zeta(m)$ belongs to $\mathcal{I} \times \mathcal{M} \subset \mathcal{M}$ for every $\mathcal{I} \in \mathcal{F}$. Since $\mathcal{M}$ is a separated $\mathcal{R}$-contramodule (again by Lemma 9.2(a)), it follows that $m - \zeta(m) = 0$. 

**Corollary 10.8.** Let $u: R \to U$ be an associative ring epimorphism such that $U$ is a flat left $R$-module, and let $\mathcal{G}$ be the related perfect right Gabriel topology on $R$. Assume that $\mathcal{G}$ has a countable base. Let $M$ be a $u$-torsion-free left $R$-module. Then $\beta_{u,M}: \Delta_u(M) \to \Lambda_G(M)$ and $\theta_{G,M}: \Lambda_G(M) \to \Delta_u(M)$ are mutually inverse isomorphisms of left $R$-modules,

$$\beta_{u,M}: \Delta_u(M) \cong \Lambda_G(M) : \theta_{G,M}.$$

**Proof.** By Theorem 8.5, we have $\text{pd}_R U \leq 1$, so our construction of the map $\beta_{u,M}$ is applicable. By Lemma 10.6, $\xi = \theta \beta$ is the identity map. Concerning $\zeta = \beta \theta$, it follows from Lemma 10.4 and from the definition of $\beta_{u,M}$ that $\zeta \lambda = \lambda$, as we have already mentioned. By construction, $\zeta$ is a left $R$-module morphism. In our assumptions on $\mathcal{G}$, it follows by virtue of Corollary 10.7 that $\zeta$ is a left $\mathcal{R}$-contramodule morphism, too. Applying Lemma 10.7, we see that $\zeta = \text{id}$. 

In the rest of this section, we use the results of Section 8 in order to extend the result of Corollary 10.8 to the case of uncountable type.

Assume that $u': R \to U'$ and $u'': R \to U''$ are two epimorphisms of associative rings such that both $U'$ and $U''$ are flat left $R$-modules of projective dimension at most 1. Let $\mathcal{G}'$ and $\mathcal{G}''$ be the related right Gabriel topologies on $R$. Suppose that
the morphism $u''$ factorizes through $u'$, i.e., there is a ring homomorphism $U' \longrightarrow U''$ making the triangle diagram $R \longrightarrow U' \longrightarrow U''$ commutative. Then one has $G' \subset G''$.

Let $R'$ and $R''$ be the completions of the ring $R$ with respect to its right linear (Gabriel) topologies $G'$ and $G''$. As usual, we view the rings $R'$ and $R''$ as complete, separated topological rings in their respective projective limit topologies $G'$ and $G''$. Then there is a natural continuous homomorphism of topological rings $R'' \longrightarrow R'$; so any left $R'$-contramodule can be also considered as a left $R''$-contramodule. For any left $R$-module $M$, there is a natural morphism of left $R$-modules (in fact, of left $R''$-contramodules) $\Lambda_{G''}(M) \longrightarrow \Lambda_{G'}(M)$.

Furthermore, one easily observes that $\text{Ext}^*_{R}(E, C) = 0$ for every left $U'$-module $E$ and every $u'$-contramodule left $R$-module $C$. In particular, this holds for $E = U''$; hence $R\text{-mod}_{u''\text{-contra}} \subset R\text{-mod}_{u'\text{-contra}}$ (cf. [23] Lemma 1.2 and [5] Lemma 1.1(2)). Hence, for every left $R$-module $M$, there is a unique left $R$-module morphism $\Delta_{u''}(M) \longrightarrow \Delta_{u'}(M)$ forming a commutative triangle diagram with the morphisms $\delta_{u'', M} : M \longrightarrow \Delta_{u''}(M)$ and $\delta_{u', M} : M \longrightarrow \Delta_{u'}(M)$. This map $\Delta_{u''}(M) \longrightarrow \Delta_{u'}(M)$ is induced by the morphism of complexes of $R$-$R$-bimodules $K_{R, U''U} \longrightarrow K_{R, U'U''}$.

Finally, for any every left $R$-module $M$ there is a commutative diagram of left $R$-module morphisms

$$
\begin{array}{ccc}
\Delta_{u''}(M) & \xrightarrow{\delta_{u'', M}} & \Lambda_{G''}(M) \\
\downarrow & & \downarrow \\
\Delta_{u'}(M) & \xrightarrow{\delta_{u', M}} & \Lambda_{G'}(M)
\end{array}
$$

(8)

The square diagram is commutative, since $\Lambda_{G'}(M) \in R\text{-mod}_{u''\text{-contra}}$, so there exists a unique left $R$-module morphism $\Delta_{u''}(M) \longrightarrow \Lambda_{G'}(M)$ forming a commutative triangle diagram with the maps $\delta_{u'', M}$ and $\lambda_{G', M}$.

Suppose that we are given a directed set $\Xi$ of right linear topologies $\mathbb{H}$ on an associative ring $R$. Then their union $\mathbb{F} = \bigcup_{\mathbb{H} \in \Xi} \mathbb{H}$ is also a right linear topology (see Section 3). Denote by $R\mathbb{E}$ and $R\mathbb{F}$ the completions of the ring $R$ with respect to these topologies (viewed as complete, separated topological rings in their projective limit topologies $R\mathbb{E}$ and $R\mathbb{F}$). For any left $R$-module $M$, one has a natural isomorphism of left $R$-modules (in fact, of left $R\mathbb{F}$-contramodules)

$$
\Lambda_{\mathbb{F}}(M) = \lim_{\mathbb{H} \in \Xi} \Lambda_{\mathbb{H}}(M).
$$

(9)

Lemma 10.9. Let $(U_{v})_{v \in \Upsilon}$ be a diagram of associative rings indexed by a directed poset $\Upsilon$ and commutative together with associative ring homomorphisms $R \longrightarrow U_{v}$ given for all $v \in \Upsilon$. Set $U = \lim_{v \in \Upsilon} U_{v}$, and let $M$ be a left $R$-module. Assume that for every $v \in \Upsilon$ there exists $v' \in \Upsilon$, $v \leq v'$ such that all left $R$-module morphisms $U_{v'}/R \longrightarrow M$ vanish. Then the natural morphism of left $R$-modules

$$
\text{Ext}^1_{R}(K_{R, U}, M) \longrightarrow \lim_{\mathbb{V} \in \Upsilon} \text{Ext}^1_{R}(K_{R, U_{v}}, M)
$$

is an isomorphism.
Proof. There is a spectral sequence of left $R$-modules

$$E_2^{p,q} = \lim_{\leftarrow \upsilon \in \Upsilon} \Ext_R^{q}(K_{R,U, \upsilon}, M) \implies E_\infty^{n} = \Ext_R^n(K_{R,U}, M)$$

with the differentials $d^{p,q}_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ and the limit term $E_\infty^{p,q} = \gr^p E_\infty^{p,q}$. One has $E_2^{p,q} = 0$ whenever $p < 0$ or $q < 0$, so in low degrees this spectral sequence reduces to an exact sequence

$$0 \to E_2^{1,0} \to E_\infty^{1,0} \to E_2^{0,1} \to E_\infty^{2,0} \to E_\infty^{2,1}.$$

Now the assumption that $\Ext_R^0(K_{R,U, \upsilon'}, M) = \Hom_R(U_{\upsilon'}/R, M) = 0$ for a cofinal subset $\Upsilon' \subset \Upsilon$ of indices $\upsilon'$ implies $E_2^{p,0} = 0$ for all $p \in \mathbb{Z}$. Hence the map $E_\infty^{1,0} \to E_\infty^{2,1}$ is an isomorphism, as desired. □

The following theorem, generalizing Corollary 10.8, is the main result of this section.

**Theorem 10.10.** Let $u: R \to U$ be an epimorphism of associative rings such that $U$ is a flat left $R$-module of projective dimension at most 1. Let $\mathcal{G}$ be the related perfect Gabriel topology on $R$; assume that $\mathcal{G}$ satisfies the condition $(T_\omega)$ of Section 3 and the right $R$-module $R$ has $\omega$-bounded $\mathcal{G}$-torsion. Let $M$ be a $u$-torsion-free left $R$-module.

Then $\beta_u, M: \Delta_u(M) \to \Lambda_\mathcal{G}(M)$ and $\theta_{\mathcal{G}, M}: \Lambda_\mathcal{G}(M) \to \Delta_u(M)$ are mutually inverse isomorphisms of left $R$-modules,

$$\beta_u, M: \Delta_u(M) \simeq \Lambda_\mathcal{G}(M) : \theta_{\mathcal{G}, M}.$$ 

**Proof.** We will prove that $\beta_u, M$ is an isomorphism. Let $\Upsilon$ denote the set of all perfect Gabriel topologies $P \subset \mathcal{G}$ having a countable base. According to Corollary 3.10 the set $\Upsilon$ is $\omega^+$-directed by inclusion and we have $\mathcal{G} = \bigcup_{P \in \Upsilon} P$. For every perfect Gabriel topology $P \in \Upsilon$, denote by $u_P: R \to U_P$ the related left flat ring epimorphism. Then Corollary 3.10 also claims that $U = \lim_{\leftarrow P \in \Upsilon} U_P$.

By Theorem 8.5, for any $P \in \Upsilon$ the projective dimension of the left $R$-module $U_P$ does not exceed 1. So we have a commutative square diagram of left $R$-module morphisms (cf. (8))

$$\begin{array}{ccc}
\Delta_u(M) & \xrightarrow{\beta_u, M} & \Lambda_\mathcal{G}(M) \\
| & & | \\
\Delta_{u_P}(M) & \xrightarrow{\beta_{u_P, M}} & \Lambda_P(M)
\end{array}$$

and a similar diagram for any two perfect Gabriel topologies $P' \subset P''$ belonging to $\Upsilon$. Passing to the projective limit, we obtain a diagram of left $R$-module morphisms

$$\begin{array}{ccc}
\Delta_u(M) & \xrightarrow{\beta_u, M} & \Lambda_\mathcal{G}(M) \\
| & & | \\
\lim_{\leftarrow P \in \Upsilon} \Delta_{u_P}(M) & \xrightarrow{\lim_{\leftarrow P \in \Upsilon} \beta_{u_P, M}} & \lim_{\leftarrow P \in \Upsilon} \Lambda_P(M)
\end{array}$$

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The natural map $M \rightarrow U \otimes_R M$, which is injective by assumption, decomposes as $M \rightarrow U \otimes_R M \rightarrow U \otimes_R M$. So the map $M \rightarrow U \otimes_R M$ is injective as well. Hence the map $\beta_{u, M} : \Delta_u(M) \rightarrow \Lambda_G(M)$ is an isomorphism by Corollary 10.8. Passing to the projective limit, we conclude that the lower horizontal arrow $\lim_{\leftarrow \mathcal{P} \in \mathcal{Y}} \Delta_u(M) \rightarrow \lim_{\leftarrow \mathcal{P} \in \mathcal{Y}} \Lambda_G(M)$ in our diagram (10) is an isomorphism.

According to the discussion above in this section (see (9)), the rightmost vertical map $\Lambda_G(M) \rightarrow \lim_{\leftarrow \mathcal{P} \in \mathcal{Y}} \Lambda_G(M)$ is an isomorphism, since $\mathcal{G} = \bigcup_{\mathcal{P} \in \mathcal{Y}} \mathcal{P}$. To prove that the leftmost vertical map $\Delta_u(M) \rightarrow \lim_{\leftarrow \mathcal{P} \in \mathcal{Y}} \Delta_u(M)$ is an isomorphism, we will check that the assumptions of Lemma 10.9 hold for the diagram of associative ring homomorphisms $R \rightarrow U$ indexed by the poset $\mathcal{Y}$.

Indeed, let us show that all left $R$-module morphisms $U \otimes_R U \rightarrow M$ vanish for all $\mathcal{P} \in \mathcal{Y}$. Since $R \rightarrow U$ is a ring epimorphism, we have $U \otimes_R U \simeq U \otimes U \simeq U$. In other words, applying the functor $U \otimes_R -$ to the morphism $R \rightarrow U$ produces an isomorphism. Hence $U \otimes_R (U \otimes_R R) = 0$ and therefore $\text{Hom}(U \otimes_R R, M) = 0$. By Lemma 10.9 the map $\Delta_u(M) \rightarrow \lim_{\leftarrow \mathcal{P} \in \mathcal{Y}} \Delta_u(M)$ is an isomorphism.

It follows that the upper horizontal arrow $\Delta_u(M) \rightarrow \Lambda_G(M)$ in (10) is an isomorphism, too, as desired. Concerning the map $\theta_{G, M} : \Lambda_G(M) \rightarrow \Delta_u(M)$, we know from Lemma 10.6 that the composition $\xi = \theta \beta$ is the identity map. Hence $\theta = \beta^{-1}$ is the inverse isomorphism. □

Remark 10.11. Dropping the assumption that $\text{pd}_R U \leq 1$ in Theorem 10.10, one can still prove existence of a natural isomorphism of left $R$-modules $\Delta_u(M) \simeq \Lambda_G(M)$ forming a commutative triangle diagram with the morphisms $\delta_u(M)$ and $\lambda_{G, M}$. The argument works in the same way as above, except that the map $\beta_{u, M}$ is not defined from the outset (but the maps $\beta_{u, \mathcal{P}, M}$ are). So the diagram (10) takes the form

$$
\begin{array}{ccc}
\Delta_u(M) & \quad & \Lambda_G(M) \\
\downarrow & & \downarrow \\
\lim_{\leftarrow \mathcal{P} \in \mathcal{Y}} \Delta_u(M) & \longrightarrow & \lim_{\leftarrow \mathcal{P} \in \mathcal{Y}} \Lambda_G(M)
\end{array}
$$

One shows that the leftmost vertical, rightmost vertical, and lower horizontal arrows are isomorphisms, and deduces the existence of an upper horizontal isomorphism.

11. Faithful Perfect Gabriel Topologies

According to Theorem 8.10 if $\mathcal{G}$ is a faithful perfect Gabriel topology with a countable base on an associative ring $R$ and $R \rightarrow U$ is the related injective left flat ring epimorphism, then the Geigle–Lenzing abelian perpendicular subcategory $U^{-0,1} \subset R\text{-mod}$ coincides with the abelian full subcategory of left $\mathcal{R}$-contramodules.
\( \mathfrak{R} \text{-contra} \subset R \text{-mod} \). The aim of this section is to replace the assumption that \( G \) has a countable base with the weaker assumption that \( U \) is a left \( R \)-module of projective dimension 1 in this result (cf. Theorem 8.5). The argument is based on the results of Section 10 and [25, Proposition 2.1]. We will also give a second, alternative proof of Theorem 8.10 as promised in Section 8.

Now let us return to the context (of the formulation) of Proposition 10.2. Let \( u : R \rightarrow U \) be an epimorphism of associative rings such that \( U \) is a flat left \( R \)-module of projective dimension at most 1, let \( G \) be the related Gabriel topology of right ideals in \( R \), and let \( \mathfrak{R} \) be the completion of \( R \) with respect to \( G \), viewed as a topological ring in the projective limit topology \( \mathfrak{G} \).

Then, as a particular case of the construction of the map \( \beta_{u,M} \) in Section 10 (cf. Remark 10.3), we have a unique left \( R \)-module morphism
\[ (11) \beta_{u,X} = \beta_{u,R[X]} : \Delta_u(R[X]) \rightarrow \mathfrak{R}[[X]] \]
forming a commutative triangle diagram with the left \( R \)-module morphisms \( \lambda_{G,R[X]} : R[X] \rightarrow \mathfrak{N}[[X]] \) and \( \delta_{u,R[X]} : R[X] \rightarrow \Delta_u(R[X]) \).

**Lemma 11.1.** Let \( u : R \rightarrow U \) be an associative ring epimorphism such that \( U \) is a flat left \( R \)-module of projective dimension not exceeding 1, let \( G \) be the related perfect Gabriel topology of right ideals in \( R \), and let \( \mathfrak{R} \) be the completion of \( R \) with respect to \( G \). Then the exact functor \( \mathfrak{R} \text{-contra} \rightarrow R \text{-mod} \), as defined in Proposition 10.2, is an equivalence of categories if and only if the map \( \beta_{u,X} \) (11) is an isomorphism for every set \( X \).

**Proof.** This is a particular case of [25, Proposition 2.1]. \( \square \)

**Second proof of Theorem 8.10.** Let \( u : R \rightarrow U \) be an injective ring epimorphism such that \( U \) is a flat left \( R \)-module of projective dimension not exceeding 1, let \( G \) be the related perfect Gabriel topology of right ideals in \( R \), and let \( \mathfrak{R} \) be the completion of \( R \) with respect to \( G \). Then the exact functor \( \mathfrak{R} \text{-contra} \rightarrow R \text{-mod} \), as defined in Proposition 10.2, is an equivalence of categories if and only if the map \( \beta_{u,X} \) (11) is an isomorphism for every set \( X \).

The following theorem is our uncountable generalization of Theorem 8.10. It is also a partial generalization of [25, Examples 2.4(3) and 2.5(3)].

**Theorem 11.2.** Let \( u : R \rightarrow U \) be an injective ring epimorphism such that \( U \) is a flat left \( R \)-module of projective dimension not exceeding 1. Let \( G \) be the related faithful perfect Gabriel topology of right ideals in \( R \). Assume that \( G \) satisfies the condition \( (T_u) \) of Section 3 (e.g., \( G \) has a base consisting of two-sided ideals; see Examples 3.2 for further cases when \( (T_u) \) is satisfied). Let \( \mathfrak{R} \) be the completion of \( R \) with respect to the topology \( \mathfrak{G} \), viewed as a complete, separated topological ring in the projective limit topology \( \mathfrak{G} \). Then the left \( R \)-module morphism
\[ \beta_{u,X} : \Delta_u(R[X]) \rightarrow \mathfrak{R}[[X]] \]
is an isomorphism for any set $X$. Furthermore, the forgetful functor $\mathcal{R}_{\text{contra}} \to R\mod$ is fully faithful, and its essential image coincides with the full subcategory $R\mod_{u,\text{contra}} \subset R\mod$, so there is an equivalence of abelian categories

$$\mathcal{R}_{\text{contra}} \simeq R\mod_{u,\text{contra}}.$$  

Proof. Since $u$ is injective, the free left $R$-module $R[X]$ is $u$-torsion-free for every set $X$, and the right $R$-module $R$ has $\omega$-bounded (in fact, zero) $G$-torsion. Applying Theorem 10.10 for $M = R[X]$, we obtain the first assertion of the theorem. The remaining assertions follow from the first one by virtue of Lemma 11.1. \qed

Remark 11.3. Let $u : R \to U$ be an injective ring epimorphism. Set $K = U/R$, so $K$ is an $R$-$R$-bimodule; and denote by $\mathcal{G} = \text{Hom}_R(K, K)^{\text{op}}$ the opposite ring to the ring of endomorphisms of the left $R$-module $K$. So the ring $\mathcal{G}$ acts in $K$ on the right, making $K$ an $R$-$\mathcal{G}$-bimodule; while the right action of $R$ in $K$ induces a ring homomorphism $R \to \mathcal{G}$. We endow the ring $\mathcal{G}$ with the right linear topology $\mathcal{G}$ with a base $\mathcal{B}$ formed by the annihilators of finitely generated left $R$-submodules in $K$. Then $\mathcal{G}$ is a complete, separated topological ring [29, Theorem 7.1] and $K$ is a discrete right $\mathcal{G}$-module [29, Lemma 7.5] (see also [5, Section 1.13]). The topological ring $\mathcal{G}$ is discussed at length in [5, Sections 17 and 19] (where it is denoted by $\mathcal{R}$).

Assume that $U$ is a flat left $R$-module. Let $\mathcal{G}$ be the perfect Gabriel topology of right ideals in $R$ related to the left flat ring epimorphism $u$, and let $\mathcal{R}$ be the completion of $R$ with respect to $\mathcal{G}$, viewed as a complete, separated topological ring in its projective limit topology $\mathcal{G}$. Then $U/R$ is a discrete right $R$-module (since it is a $\mathcal{G}$-torsion right $R$-module, because $U/R \otimes_R U = 0$), and consequently $U/R$ also has a discrete right $\mathcal{R}$-module structure (see Sections 2.3 and 2.4). It follows that the right action of $\mathcal{R}$ in $U/R$ induces a continuous homomorphism of topological rings $\mathcal{R} \to \mathcal{G}$. Hence for every set $X$ we have the induced map of sets $\mathcal{R}[[X]] \to \mathcal{G}[[X]]$. In fact, we have a commutative triangle diagram of ring homomorphisms $R \to \mathcal{R} \to \mathcal{G}$; so the map $\mathcal{R}[[X]] \to \mathcal{G}[[X]]$ is a left $R$-module morphism.

Now let us assume additionally that $U$ is a left $R$-module of projective dimension not exceeding 1. Then we also have the left $R$-module morphism $\beta_{u,X} : \Delta_u(R[X]) \to \mathcal{R}[[X]]$. Every left $\mathcal{G}$-contramodule, and in particular $\mathcal{G}[[X]]$, has the underlying left $\mathcal{R}$-contramodule structure. By Proposition 10.2, $\mathcal{G}[[X]]$ is a $u$-contramodule left $R$-module. Hence, by Lemma 10.1, there is a unique left $R$-module morphism $\Delta_u(R[X]) \to \mathcal{G}[[X]]$ forming a commutative triangle diagram with the map $\beta_{u,X}$ and the map $R[X] \to \mathcal{G}[[X]]$ induced by the ring homomorphism $R \to \mathcal{G}$. The composition of our two maps $\Delta_u(R[X]) \to \mathcal{R}[[X]] \to \mathcal{G}[[X]]$ has this diagram commutativity property. The isomorphism $\Delta_u(R[X]) \simeq \mathcal{G}[[X]]$ constructed in [5, direct proof of Theorem 19.2] also has the same commutativity property. Thus the composition $\Delta_u(R[X]) \to \mathcal{R}[[X]] \to \mathcal{G}[[X]]$ is an isomorphism.

According to Theorem 11.2, the map $\beta_{u,X}$ is an isomorphism provided that the Gabriel topology $\mathcal{G}$ on $R$ satisfies the condition $(T_u)$. Then it follows that the map $\mathcal{R}[[X]] \to \mathcal{G}[[X]]$ is bijective for every set $X$. In particular, the associative ring homomorphism $\mathcal{R} \to \mathcal{G}$ is an isomorphism. It still does not seem to follow from
anything that it is an isomorphism of topological rings (i.e., that the topologies $\mathcal{G}$ and $\mathcal{F}$ on $\mathcal{R} = \mathcal{G}$ are the same); but it is a bijective continuous ring homomorphism (so $\mathcal{F} \subset \mathcal{G}$) inducing a bijective map $\mathcal{R}[X] \to \mathcal{G}[X]$ for every set $X$.

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