Fractional Josephson Effect in Number-Conserving Systems

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We study fractional Josephson effect in particle-number conserving system consisting of a quasi-one-dimensional superconductor coupled to a nanowire or an edge carrying \(e/m\) fractional charge excitations with \(m\) being an odd integer. We show that, due to the topological ground-state degeneracy in the system, the periodicity of the supercurrent on magnetic flux through the superconducting loop is non-trivial which provides a possibility of detecting topological phases of matter by the dc supercurrent measurement. Using a microscopic model for the nanowire and quasi-one-dimensional superconductor, we derived an effective low-energy theory for the system which takes into account effects of quantum phase fluctuations. We discuss the stability of the fractional Josephson effect with respect to the quantum phase slips induced by the charging energy.

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I. INTRODUCTION

Josephson effect, a hallmark of the macroscopic quantum coherence, has played a crucial role in the research and applications of superconductivity, both conceptually and practically since its discovery\(^1\)\(^-\)\(^5\). Recently, it has been recognized that Josephson effect can also reveal the topological aspects of superconductivity, most notably the \(4\pi\)-periodic ac Josephson effect\(^6\)\(^,\)\(^7\) in one-dimensional class D topological superconductors\(^8\)\(^-\)\(^11\), as well as in time-reversal-invariant generalizations\(^12\)\(^,\)\(^13\). The doubling of the periodicity is tied to the existence of non-trivial excitations - localized Majorana zero-energy modes (MZMs) at the opposite ends of a topological superconductor. The presence of the zero-modes allows for the coherent charge \(e\) tunneling processes between two superconductors. This, in turn, leads to the doubling of the flux periodicity as compared to the conventional Josephson effect involving charge \(2e\) (Cooper-pair) tunneling.

In the last five years there has been a surge of research interest in topological superconductors\(^9\)\(^-\)\(^11\),\(^14\)\(^-\)\(^28\). Apart from the fundamental importance, the search for Majorana zero-modes and other non-Abelian quasiparticles is fueled by the prospects of topological quantum information processing\(^29\)\(^-\)\(^34\). Recent theoretical breakthrough indicating that Majorana-based topological phases can be accessed in heterostructures involving a semiconductor nanowire coupled to a conventional s-wave superconductor\(^10\),\(^11\) has sparked a significant experimental activity on this subject\(^19\)\(^-\)\(^27\),\(^35\)\(^-\)\(^37\). Apart from Ref. \(^20\), most of the aforementioned experiments have been focusing so far on the zero-bias peak anomaly associated with presence of the zero-energy modes\(^18\)\(^-\)\(^48\).

Fractional Josephson effect provides perhaps the simplest setup where the non-Abelian nature of the Majorana zero modes are manifested. The stability of this effect under various realistic situations (i.e. energy splitting, disorder, multiple bands, quasiparticle poisoning) have been extensively studied in literature\(^19\)\(^-\)\(^38\). The standard approach to understand the Josephson effect is based on BCS mean-field theory, where the U(1) particle number conservation is spontaneously broken. However, due to the mesoscopic nature of the experimental setups in engineering topological superconductors, quantum phase fluctuations may play an important role. Indeed, charging energy is at the heart of many mesoscopic superconducting devices such as superconducting qubits. Furthermore, charging energy is important for topological quantum computing schemes with Majorana zero modes\(^38\),\(^34\),\(^59\)\(^-\)\(^64\). There have been several works that take into account quantum phase fluctuations in a phenomenological way\(^34\),\(^55\),\(^65\)\(^-\)\(^68\). However, a microscopic theory of fractional Josephson effect is still lacking. In this work, we start from a microscopic model of helical nanowires coupled to a fluctuating one-dimensional s-wave superconductor introduced in Ref. \(^69\). Although there is no long-range superconducting order, there are still Majorana zero modes and a related topological degeneracy when two or more nanowires are coupled to the same quasi-one-dimensional superconductor (QSC)\(^69\)\(^,\)\(^70\). We study the Josephson effect within this model and find that fractional Josephson effect survives quantum fluctuations. However, the splitting of the ground state degeneracy as well as the hybridization between different topological sectors now becomes power-law dependent on the system size. In order to show that we consider instanton tunneling events between different topological sectors. We find that phase slip events at weak links caused by the spatial inhomogeneities (e.g. impurities) in the quasi-one-dimensional superconductor are responsible for the power-law decay with length of the ground-state energy splitting. Finally, we calculate the periodicity of the Josephson current on magnetic flux through the ring, see Fig.1 for the proposed setup. It has been previously believed that one needs to perform an ac measurement to detect \(4\pi\)-periodicity of the supercurrent which might be quite challenging due to quasiparticle poisoning problem\(^10\),\(^55\). However, we show that by suitably designing the experimental system, one can detect \(4\pi\)-periodicity in the dc experiments.

Our approach based on Luttinger liquid formalism...
allows one to extend our theory to the recently proposed systems hosting parafermionic zero modes\cite{Liu:2011,Regnault:2012,Fradkin:2012}.

In this case, we consider an edge of a two-dimensional Abelian fractional quantum Hall system properly coupled to QSC. Our main conclusions discussed above for the Majorana case remain also valid for parafermions, and we show how to probe topological properties of parafermions in the dc measurements, see Fig. 1.

The paper is organized as follows. We begin by reviewing Josephson effect in a Luttinger liquid coupled to a bulk $s$-wave\cite{Semenoff:1983} and $p$-wave\cite{Son:2009,Nayak:2010} superconductors, and discuss the mechanisms for the change of a periodicity with magnetic flux in these two systems. Next, in Sec. II C we review the fractional Josephson effect in the parafermion systems by considering an edge of a two-dimensional Abelian fractional quantum Hall system at the filling fraction $\nu = 1/m$ coupled to a bulk superconductor. In Sec. III, we present our results for Josephson effect in a number-conserving setup and discuss the dependence of the ground-state energy of the system on magnetic flux. Finally, the effect of quantum phase fluctuations on the flux periodicity of the Josephson current is discussed in Sec. IV. Technical details are relegated to the Appendix A.

\section{II. JOSEPHSON EFFECT IN LUTTINGER LIQUIDS}

In this section we first review Josephson effect in a spinful Luttinger liquid coupled to a conventional $s$-wave superconductor\cite{Semenoff:1983,Nayak:2010} and obtain the spectrum of Andreev states as a function of superconducting phase difference across the junction. Next, we discuss a spinless Luttinger liquid coupled to spinless $p$-wave superconductors\cite{Semenoff:1983}. The latter supports Majorana zero-energy modes which are ultimately responsible for the change of fundamental periodicity of the Josephson current. Finally, we will review Josephson effect in more exotic structures involving parafermions.

\subsection{A. Josephson effect in a Luttinger liquid coupled to an $s$-wave superconductor.}

We now consider Luttinger liquid coupled to two bulk $s$-wave superconductors with the superconducting phase difference $\Phi$. After integrating out the superconducting degrees of freedom, the effective Hamiltonian for the system becomes ($\hbar = e = c = 1$)

\begin{equation}
H = H_{\text{NW}} + \int dx \langle \Delta(x)\psi_\uparrow(x)\psi_\downarrow(x) + \text{H.c.} \rangle \tag{1}
\end{equation}

where $m^*$ and $\mu$ are the effective mass and chemical potential. One can introduce magnetic field into the Hamiltonian (1) via minimal coupling $-i\partial_x \rightarrow -i\partial_x + eA$ and then gauge it away so that it only appears in the superconducting pairing potential. Thus, the induced superconducting pair potential $\Delta(x)$ is given by

\begin{equation}
\Delta(x) = \begin{cases} 
\Delta_0 & x < 0 \\
0 & 0 < x < l \\
\Delta_0 e^{-i\Phi} & x > l 
\end{cases} \tag{2}
\end{equation}

In a ring geometry, see Fig. 1, the phase difference $\Phi$ can be related to the magnetic flux through the loop $\Phi = 2\pi f/f_0$ with $f$ being the flux piercing the ring and $f_0 = \frac{hc}{2e}$ is the flux quanta.

We now perform standard bosonization procedure for spinful fermions using the convention\cite{Semenoff:1983}:

\begin{equation}
\psi_{r,\sigma} = \frac{1}{\sqrt{2\pi a}} e^{-\frac{i}{\sqrt{2}} [(r\phi_{r,\sigma} - \theta_{r,\sigma}) + \sigma (r\phi_{r,\sigma} - \theta_{r,\sigma})]} \tag{3}
\end{equation}

where $r = \pm$ and $\sigma = \pm$ for right/left-moving fermion with $\uparrow / \downarrow$ spin, and $a$ the lattice cutoff. The effective

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Proposed setup to study Josephson effect in particle-number conserving system. The nanowire is coupled to a quasi-one-dimensional superconducting wire forming a loop with the circumference $L$. In the case of parafermion proposals, the Luttinger liquid corresponds, for example, to an edge of a two-dimensional Abelian fractional quantum Hall system. The boundaries at $x = L - l - l'$ and $x' = l$ are determined by the mass terms $\cos(2\pi l')$ at $x$ and $x'$, see Refs.\cite{Liu:2011,Regnault:2012,Fradkin:2012} for more details on the setup. A magnetic flux $f$ pierces through the loop and leads to supercurrent flow in the loop. The periodicity of the supercurrent on flux allows one to probe topological properties of the system.}
\end{figure}
Hamiltonian now reads
\[ H = \int dx \sum_{\mu = \rho, \sigma} \frac{v_\mu}{2\pi} \left[ K_\mu (\partial_x \theta_\mu)^2 + K_\mu^{-1} (\partial_x \phi_\mu)^2 \right] + \int_{x < 0} dx \frac{\Delta_0}{2\pi a} \cos(\sqrt{2}\theta_\rho) \cos \sqrt{2}\phi_\rho + \int_{x > 1} dx \frac{\Delta_0}{2\pi a} \cos(\sqrt{2}\theta_\rho - \Phi) \cos \sqrt{2}\phi_\rho \] (4)
Assuming that \( \Delta_0 \) is large, the proximity-induced terms constrain the values of \( \theta_\rho \) and \( \phi_\sigma \) to the minima of the cosine potential in the corresponding bulk superconductors
\[ \theta_\rho(x < 0) = 0 \text{ and } \theta_\rho(x > l) = \frac{\pi J_\rho + \Phi}{\sqrt{2}} \] (5)
\[ \phi_\sigma(x < 0) = 0 \text{ and } \phi_\sigma(x > l) = \frac{\pi N_\sigma}{\sqrt{2}}, \] (6)
where \( J_\rho = \sum_{r,s} rN_{rr,s} + \hat{N}_\sigma = \sum_{r,s} sN_{rr,s} \) with \( N_{rr,s} \) being the zero momentum component of the density operator with spin \( s \) and chirality \( r \). Thus, the problem has been effectively reduced to solving for the modes of Luttinger liquid subject to the above boundary conditions. Note that the allowed integer values for the operators \( \hat{J}_\rho \) and \( \hat{N}_\sigma \) have to obey certain constraints (superselection rules): \((-1)^I_l = (-1)^{\hat{N}_\sigma} \) which immediately follows from Eq. (4).

We now expand the bosonic fields in terms of the normal modes satisfying the above boundary conditions:
\[ \varphi_\rho(x) = \sqrt{2} \varphi_\rho^{(0)} + \sqrt{K_\rho} \sum_{n > 0} \frac{1}{\sqrt{n}} \cos \frac{\pi nx}{l} (a_{n\rho} + a_{n\rho}^\dagger) \]
\[ \theta_\rho(x) = \frac{\pi J_\rho + \Phi}{\sqrt{2}} x + \frac{i}{\sqrt{K_\rho}} \sum_{n > 0} \frac{1}{\sqrt{n}} \sin \frac{\pi nx}{l} (a_{n\rho} - a_{n\rho}^\dagger). \]
\[ \varphi_\sigma(x) = \frac{\pi N_\sigma}{\sqrt{2}} x + i \sqrt{K_\rho} \sum_{n > 0} \frac{1}{\sqrt{n}} \sin \frac{\pi nx}{l} (a_{n\sigma}^\dagger - a_{n\sigma}). \]
\[ \theta_\sigma(x) = \sqrt{2} \theta_\sigma^{(0)} + \frac{1}{\sqrt{K_\sigma}} \sum_{n > 0} \frac{1}{\sqrt{n}} \cos \frac{\pi nx}{l} (a_{n\sigma} + a_{n\sigma}^\dagger). \]

Here \( a_n \) and \( a_n^\dagger \) are the annihilation and creation operators for particle-hole excitations, satisfying the canonical commutation relation \([a_{n\mu}, a_{m\nu}^\dagger] = \delta_{nm} \delta_{\mu\nu} \). The operators \( \varphi_\rho^{(0)} \) and \( \theta_\sigma^{(0)} \) represent the zero modes of the corresponding fields satisfying the commutation relations \([\varphi_\rho^{(0)}, J_\mu] = i \) and \([\theta_\sigma^{(0)}, N_\sigma] = -i \). Using the above normal-mode expansion, one can find the energy of the system:
\[ \mathcal{E}(N_\sigma, J_\rho) = \frac{\pi v_\sigma}{4 K_\rho} N_\sigma^2 + \frac{\pi v_\rho}{4 K_\rho} \left( J_\rho + \frac{\Phi}{\pi} \right)^2 \]
\[ + \sum_{\mu = \rho, \sigma} \sum_{k > 0} \omega_\mu(k) \left( n_\mu(k) + \frac{1}{2} \right), \] (7)
where \( \omega_\mu(k) = v_\mu \frac{k^2}{2m_\mu} \) and \( n_\mu(k) = \langle a_{k\mu}^\dagger a_{k\mu} \rangle \). One can notice that the partition function for the system factorizes into the product of the zero modes \( Z_0 \) and finite-energy excitations \( Z_n \): \( Z = Z_0 Z_n \) with only \( Z_0 \) being dependant on the flux \( \Phi \)
\[ Z_0 = \sum_{N_\sigma, J_\rho \in \mathbb{Z}} e^{-\beta \mathcal{E}(N_\sigma, J_\rho)}. \] (8)
Clearly, the sector with odd \( N_\sigma \) is gapped out which constraints the values of \( J_\rho \) to be even. Thus, the flux-dependent ground-state energy of the system becomes
\[ E_g(\Phi) = \min_{m \in \mathbb{Z}} \frac{\pi K_\rho v_\rho}{l} \left( m + \frac{\Phi}{2\pi} \right)^2. \] (9)
Finally, one can obtain the expression for Josephson current using
\[ I_J(\Phi) = 2 \frac{\partial E_g(\Phi)}{\partial \Phi} \] (10)
One can see that the ground-state energy and the Josephson current through the junction are \( 2\pi \)-periodic which is consistent with the previous studies of the Josephson effect in s-wave superconductors.

B. Fractional ac Josephson effect in topological p-wave superconductors.

Next, we consider a case of topological p-wave superconductors and study Josephson effect in the presence of Majorana zero modes. Realistically a “spinless nanowire” can be engineered in spin-orbit-coupled spin-1/2 quantum wires subject to an external Zeeman field. The spinless nanowire is then proximity-coupled to a bulk s-wave superconductors at \( x < 0 \) and \( x > l \). The Hamiltonian for the “spinless nanowire” reads:
\[ H_{NW} = \int dx \psi_\uparrow^\dagger(x) \left( -\frac{\partial^2}{2m^*} - \mu + i\alpha \sigma_y \partial_x + V_z \sigma_z \right) \psi_\uparrow(x), \]
\[ H_T = \int dx \left[ \Delta(x) \psi_\uparrow^\dagger \psi_\downarrow + \text{h.c.} \right], \] (11)
where \( \alpha \) is the strength of the spin-orbit Rashba interaction and \( V_z \) is the Zeeman splitting, and superconducting pairing is defined in Eq.(2). When chemical potential \( \mu < V_z \), the nanowire is effectively spinless. The electron tunneling between the NW and the SC leads to the proximity effect described by the Hamiltonian \( H_T \). The superconducting pairing potential \( \Delta_0 \) is assumed to be a static classical field and quantum fluctuations of the superconducting phase are neglected.

Assuming that Zeeman gap is large, one performs standard bosonization procedure to find the effective Hamiltonian to be equivalent to a spinless nanowire coupled to
spinless p-wave superconductors:

\[ H = \int dx \frac{v}{2\pi} \left[ K(\partial_x \theta)^2 + K^{-1}(\partial_x \varphi)^2 \right] \]
\[ - \frac{\Delta_p}{2\pi a} \left( \int_{x<0} dx \cos 2\theta + \int_{x>l} dx \cos (2\theta - \Phi) \right). \]

The superconducting phase difference across the junction is seen to be \( \Phi \). We assume the superconducting pairing potential is large (i.e. \( \Delta_p \) is relevant and flows to strong coupling under RG flow) and gaps out the Luttinger liquid in the region \( x < 0 \) and \( x > l \). In this limit, the values of \( \theta \) are constraint to the minima of the cosine potential imposing the following boundary conditions for the LL in the region \( 0 < x < l \):

\[ \theta(0) = 0 \quad \text{and} \quad \theta(l) = \frac{\Phi + 2\pi \hat{J}}{2}. \]

where \( \hat{J} = N_R - N_L \) with \( N_r \) being the zero momentum component of the density operator with chirality \( r \). One can expand the bosonic fields in terms of normal modes satisfying the boundary conditions:

\[ \varphi(x) = \varphi^{(0)} + \sqrt{K} \sum_{n>0} \frac{1}{\sqrt{n}} \cos \frac{\pi n x}{l} (a_n + a_n^\dagger) \]

\[ \theta(x) = \frac{\Phi + 2\pi \hat{J} x}{2} + \frac{i}{\sqrt{K}} \sum_{n>0} \frac{1}{\sqrt{n}} \sin \frac{\pi n x}{l} (a_n^\dagger - a_n). \]

Here \( a_n \) and \( a_n^\dagger \) are annihilation and creation operators for particle-hole excitations, satisfying the canonical commutation relation \( [a_m, a_n^\dagger] = \delta_{mn} \); \( \varphi^{(0)} \) is the zero mode of the \( \varphi \) field and is conjugate to \( \hat{J} \): \( [\varphi^{(0)}, \hat{J}] = i \).

After substituting (14) into the effective Hamiltonian for the junction, one finds

\[ H = \frac{\pi v K}{2l} \left( \hat{J} + \frac{\Phi}{2\pi} \right)^2 + \sum_{n>0} \frac{v \pi n}{l} \left( a_n^\dagger a_n + \frac{1}{2} \right). \]

One can now easily find the flux dependent part of the ground state energy

\[ E_g(\Phi) = \min_{J \in \mathbb{Z}} \pi v K \left( J + \frac{\Phi}{2\pi} \right)^2. \]

We remind that different parity of \( J = N_R - N_L \) actually corresponds to different parity of electron number operator \( N = N_R + N_L \). If electron number in the junction is conserved, \( J \) should be restricted to either even or odd sectors, i.e. \( J = 2m + \frac{1}{2}(-1)^n \). Thus, the ground state energy as well as the current are 4\( \pi \)-periodic: \( E_g(\Phi) = E_g(\Phi + 4\pi) \). On the other hand, if there are processes allowing to change the fermion parity in the junction, the ground-state energy is 2\( \pi \)-periodic.

In practice, one should define the time scale associated with such processes \( \tau_p \). When \( t \gg \tau_p \) (dc limit), the Josephson current in spinless superconductors is 2\( \pi \)-periodic and, in this sense, is similar to the Josephson effect in conventional s-wave superconductors. However, if measured at \( t \ll \tau_p \) (ac limit), the fundamental periodicity of the Josephson current is \( 4\pi \), and, thus, one could distinguish between the topological (spinless p-wave) and non-topological (spinful s-wave) junctions. This is why this phenomenon in the literature was “coined” as fractional ac Josephson effect.

C. Fractional Josephson effect in parafermion systems.

We now review the Josephson effect in the presence of the parafermionic zero modes. Instead of a spinless Luttinger liquid discussed above in the Majorana context, we now consider gapless edge modes of a fractional topological insulator, or the counter-propagating edge modes along a trench in a \( \nu = \frac{1}{m} \) Laughlin state. The fundamental excitations in this case are fractionalized quasiparticles characterized by the fields \( \varphi \) and \( \theta \) satisfying the following commutation relations:

\[ [\varphi(x), \theta(x')] = \frac{i\pi}{m} \Theta(x - x'). \]

with \( m > 1 \). The effective Hamiltonian for the edge reads

\[ H = \frac{vm}{2\pi} \int dx [K(\partial_x \theta)^2 + K^{-1}(\partial_x \varphi)^2]. \]

The physical electron operators in this effective theory are given by \( \psi_{r/l} = \frac{1}{\sqrt{2\pi a}} e^{-i\pi(\varphi - \theta)} \), and, thus, the proximity-induced superconductivity due to spinless p-wave superconductor for \( x < 0 \) and \( x > l \) can be taken into account using the following effective Hamiltonian:

\[ H_\Delta = -\frac{\Delta_p}{2\pi a} \left( \int_{x<0} dx \cos 2m \theta + \int_{x>l} dx \cos (2m \theta - \Phi) \right) \]

Assuming that \( \Delta_p \) is large, we once again consider Luttinger liquid Hamiltonian confined in the region \( 0 < x < l \).
subject to the boundary conditions:
\[
\theta(0) = 0 \quad \text{and} \quad \theta(l) = \frac{\Phi + 2\pi \hat{J}}{2m}.
\]
(20)

Following similar analysis as in the previous section, we find the normal mode expansion for the fields \( \theta \) and \( \varphi \) is given by
\[
\varphi(x) = \varphi^{(0)} + \sqrt{\frac{K}{m}} \sum_{n>0} \frac{1}{\sqrt{n}} \cos \frac{\pi n x}{l} (a_n + a_n^\dagger)
\]
\[
\theta(x) = \frac{\Phi + 2\pi \hat{J} x}{2m} + i \sqrt{\frac{K}{m}} \sum_{n>0} \frac{1}{\sqrt{n}} \sin \frac{\pi n x}{l} (a_n^\dagger - a_n).
\]
The fields \( \theta(x) \) and \( \varphi(x) \) should satisfy the commutation relations (17) and have fundamental periodicity of \( 2\pi \). The latter imposes a constraint on the values of \( J \) requiring that \( J = 2mk + nJ \). After substituting above expressions for \( \theta(x) \) and \( \varphi(x) \) into Eq.(18), the ground-state energy for the system is given by
\[
E_g(\Phi) = \min_k \frac{2\pi v K}{ml} \left( k + \frac{nJ}{2m} + \frac{\Phi}{4\pi m} \right)^2
\]
(21)
where \( nJ \in \mathbb{Z}_{2m} \) labels different topological sectors. One can see that ground state energy is \( 4\pi m \) periodic \( E_g(\Phi) = E_g(\Phi + 4\pi m) \).

III. FRACTIONAL AC JOSEPHSON EFFECT IN NUMBER-CONSERVING SYSTEMS

So far we have included superconductivity at the mean field level neglecting quantum phase fluctuations, which is appropriate when a nanowire is coupled to bulk superconductors. In other words, this approximation implies that the particle number is not conserved corresponding to the grand canonical ensemble. In mesoscopic structures, however, particle number fluctuations might be suppressed by the charging energy, and, thus, it is interesting to investigate Josephson effect in particle number conserving systems. Given that particle number and superconducting phase are conjugate variables, in particle number conserving systems (canonical ensemble) one needs to take into account strong quantum fluctuations of the superconducting phase. This fact is particularly important in one-dimensional systems, where U(1) symmetry cannot be spontaneously broken due to the Mermin-Wagner theorem. Therefore, in the following we consider the fractional Josephson effect in a model where a nanowire is coupled to the quasi-long-range ordered superconductor (QSC) with strongly fluctuating SC phase.

We now consider an attractive Hubbard model, where the spin backscattering caused by the electron interaction is marginally relevant and flows to strong coupling, resulting in the formation of the Luther-Emery phase.\(^\text{25}\) with a finite spin gap but no charge gap, and therefore can be thought as a one-dimensional analogy of an s-wave superconductor. After the bosonization, the Hamiltonian for the superconducting wire reads
\[
H_{SC} = H_{SC}^{(0)} + H_{SC}^{(s)}
\]
(22)
\[
H_{SC}^{(0)} = \frac{v_p}{2\pi} \int dx \left[ K_p (\partial_x \theta_p)^2 + K_p^{-1} (\partial_x \phi_p)^2 \right]
\]
(23)
\[
H_{SC}^{(s)} = \frac{v}{2\pi} \int dx \left[ K_s (\partial_x \theta_s)^2 + K_s^{-1} (\partial_x \phi_s)^2 \right] - \frac{2|U|}{(2\pi a)^2} \int dx \cos(2\sqrt{2}\varphi_s)
\]
(24)
where \( v_F \), \( a \) and \( U \) are the Fermi velocity, the effective cutoff length and the interparticle interaction potential, respectively. The physics of this quasi-long-range superconducting wire can be understood in terms of fluctuating Cooper pairs having algebraically-decaying correlations. The cosine term in Eq. (24) is marginally relevant so that the spin field \( \varphi_s \) is pinned to the classical minima, opening a spin gap \( \Delta_s \propto e^{-\frac{\pi v_F}{2\Delta_s}} \) in the QSC wire. As shown below, spin gap is crucial for our construction since it prohibits single-electron tunneling to the QSC at low energies. Thus, Cooper pair tunneling is the dominant tunneling process between the nanowire and the superconductor. One can relate the parameters of the above model for the Luther-Emery phase to that of a quasi-1D superconductor: \( K_p = 2\pi \sqrt{A_w \rho_s / \kappa} \) and \( v_p = \sqrt{A_w \rho_s / \kappa} \) with \( A_w \), \( \rho_s \) and \( \kappa \) being the cross-sectional area of the superconductor, the superfluid stiffness and the compressibility, respectively.

Our theoretical model also involves the Hamiltonian for the nanowire \( H_{NW} \) (11), in which we again assume that the Zeeman field is large so that the nanowire is in the “spinless regime” with only the lowest band occupied. The nanowire and the QSC are coupled by the single particle tunneling term:
\[
H_T = t \sum_{\sigma} \int dx \left( \psi_\sigma^\dagger(x) \eta_\sigma + \text{h.c.} \right).
\]
(25)
Here \( \psi_\sigma \) and \( \eta_\sigma \) are electron annihilation operators in the nanowire and QSC, respectively. After the bosonization, we arrive at following Hamiltonian:
\[
H = H_{NW}(\theta, \varphi) + H_{SC}^{(0)}(\theta_p, \varphi_p) + H_{SC}^{(s)}(\theta_s, \varphi_s) + H_T.
\]
(26)

Given that single-electron tunneling into the superconducting wire is suppressed due to the presence of the spin gap \( \Delta_s \), the dominant contribution to the low-energy effective action comes from pair tunneling. The perturbative expansion in \( t \) to second order leads to the following imaginary-time action
\[
S_{PH} = -t^2 \sum_{\sigma} \int dx d\tau \psi_\sigma^\dagger(x, \tau) \psi_\sigma(x', \tau') \eta_\sigma(x, \tau) \eta_{-\sigma}(x', \tau') + \text{h.c.}
\]
(27)
Given that the spin field \( \varphi_x \) orders as a result of the last term in Eq. 24, the dual field \( \theta_x \) is strongly disordered, and its correlation function decays exponentially:

\[
\langle e^{-\frac{1}{2}\theta_x(x,\tau)} e^{\frac{1}{2}\theta_x(0,0)} \rangle \sim \frac{ae^{-\frac{2\pi}{\sqrt{\Delta^2 + v^2 \tau^2}}}}{\sqrt{\Delta^2 + v^2 \tau^2}}.
\]

This allows us to simplify the action (27) and make a local approximation

\[
S_{PT} \approx -\frac{\Delta_P}{\alpha^2} \int d\tau \int dx \cos \left( 2\theta - \sqrt{2} \theta_P \right),
\]

which is valid in the long-time limit \(|\tau - \tau'| \gg \Delta_s^{-1}\).

Here the Cooper pair tunneling amplitude \( \Delta_P \) is given by \( \Delta_P \sim \frac{\Delta^{PF}_P}{\Delta_{\sqrt{(\alpha P)^2 + V^2}}} \). The derivation above can be straightforwardly generalized to the case of fractionalized Luttinger liquid discussed in Sec. II C where the index \( m \) corresponds to a specific edge theory, i.e. \( m = 1 \) represents Majorana case whereas \( m > 1 \) corresponds to a specific parafermion model. After some algebra, one finds that the effective action for the tunneling between QSC and LL now reads

\[
S_{PT} \approx -\frac{\Delta_P}{\alpha^2} \int d\tau \int dx \cos \left( 2m\theta - \sqrt{2} \theta_P \right).
\]

We also notice that for \( m > 1 \), it is not physical to consider a finite fractionalized liquid since it exists on the edge of a 2D system which has no boundaries. We therefore have to induce a distinct gap on the edge to terminate the paired topological regions e.g. by a backscattering term \( \psi_j \psi_L + h.c. \), which becomes \( \cos 2m\varphi \) after bosonization. We assume this is the case for \( m > 1 \) in the following discussion.

Finally, one arrives at the effective Hamiltonian for the system of interest:

\[
H = H_{NW} + H_{SC}^\theta + H_{PT}
\]

\[
H_{NW} = \frac{v}{2\pi} \int dx [K(\partial_x \theta - A)^2 + K^{-1}(\partial_x \varphi)^2]
\]

\[
H_{SC}^\theta = \frac{v}{2\pi} \int dx [K_P(\partial_x \theta_P - \sqrt{2}A)^2 + K_P^{-1}(\partial_x \varphi_P)^2]
\]

\[
H_{PT} = -\frac{\Delta_P}{\alpha^2} \int dx \cos (2m\theta - \sqrt{2} \theta_P).
\]

Here we also introduced the vector potential \( A \) due to the out-of-plane magnetic field, see Fig. 1. Henceforth, we assume that Cooper-pair tunneling term is large so that it “locks” the phase difference between the edge modes/nanowire and the QSC \( 2m\theta - \sqrt{2} \theta_P \).\(^{79}\)

Following Ref. 69, we now review the topological degeneracy in the wires/edge proximity-coupled to QSC. Given that the total number of electrons is conserved, the minimal setup with topologically protected ground-state degeneracy involves four domain walls (i.e. two separated nanowires) coupled to the same QSC, see Fig. 1.

In the topological regions, \( \Theta = \theta - \frac{\theta_P}{\sqrt{2m}} \) are pinned to the classical minima and we denote its value in the first and second wire by \( \Theta_1 \) and \( \Theta_2 \), respectively. Naively, the pair tunneling leads to superficially \((2m)^2\)-fold degenerate ground state manifold: \( (\Theta_1, \Theta_2) = \frac{\pi}{m} (n_1, n_2), n_1, n_2 \in \{0, 1, 2, \ldots, 2m - 1\} \). However, one needs to be more careful and study the moduli space of the phase variables.

In fact, since the two wires are coupled to the same QSC, we are allowed to make a global gauge transformation \( \theta_P \rightarrow \theta_P + 2\pi k \), which leaves the ground state invariant. This shows that we need to make the following identification: \( (n_1, n_2) \sim (n_1 + k, n_2 + k), k \in \mathbb{Z} \). Therefore, we can fix \( \Theta_1 = 0 \), and different ground states are labeled by \( \Theta_2 \) and we have \( 2m \)-fold ground state degeneracy.

We now study Josephson current for the setup shown in Fig. 1. We consider a ring of length \( L \) with \( 0 \leq x < L \), and assume that the nanowire covers \([0, l] \cup [L-l-l', L] \), and the QSC covers \([0, L-l'] \) with \( l' \) being the length of the junction. The vector potential can be chosen to be \( A = \frac{\Phi}{L} \) where \( \Phi \) is the magnetic flux threading the loop. We first perform the following gauge transformation to eliminate the vector potential from the Hamiltonian:

\[
\eta \eta^\dagger \psi \psi \rightarrow \begin{cases} 
\eta \eta^\dagger \psi \psi & L-l-l' \leq x < L \\
\psi \psi \eta \eta^\dagger & 0 \leq x \leq l 
\end{cases}
\]

where \( \Phi = 2\pi f/f_0 \) with \( f \) being the magnetic flux piercing the ring and \( f_0 = \frac{hc}{2e} \). It can be readily checked that this transformation is continuous at \( x = 0 \) and, thus, fermion operators are single-valued everywhere (i.e. this transformation does not introduce a branch cut in the fermionic fields).

However, the pair tunneling terms are affected in a nontrivial way and change under gauge transformation as

\[
\eta^\dagger \eta \psi \psi \rightarrow \begin{cases} 
\eta^\dagger \eta \psi \psi & L-l-l' \leq x \leq L-l' \\
\psi \psi \eta \eta^\dagger & 0 \leq x \leq l 
\end{cases}
\]

Upon standard bosonization, the modified pair tunneling terms are given by

\[
H_{PT} = -\frac{\Delta_P}{\alpha^2} \int_0^l dx \cos (2m\theta - \sqrt{2} \theta_P - \Phi)
\]

\[
-\frac{\Delta_P}{\alpha^2} \int_{L-l-l'}^{L-l} dx \cos (2m\theta - \sqrt{2} \theta_P) \]

Assuming that \( \Delta_P \) term is large, one can approximate the phase difference \( \theta - \frac{\theta_P}{\sqrt{2m}} \) by its values at the minima of the cosine potential:

\[
\begin{align*}
\theta - \frac{\theta_P}{\sqrt{2m}} &= \begin{cases} 
0 & 0 \leq x \leq l \\
\chi & \frac{4\pi + 2\pi j}{2m} \leq x \leq L-l' 
\end{cases} 
\end{align*}
\]

where \( \chi = \frac{4\pi + 2\pi j}{2m} \) and \( J \in \mathbb{Z} \). As discussed in Sec. II C, \( J \mod 2m \) is a conserved quantity, corresponding to different superselection sectors.
We now calculate the ground state energy and its dependence on the magnetic flux $f$. It is convenient to integrate out $\phi$ and $\phi_2$ fields and write the partition function in terms of the imaginary-time effective action.

\[
S = \int_0^\beta \! \! \! \! d\tau \left[ \int_{L-l'}^{L} \! \! \! dx \frac{mK}{2\pi v} \left[ (\partial_x \tilde{\theta})^2 + \tilde{\theta}(\partial_x \tilde{\theta})^2 \right] 
+ \int_{0}^{L-l'} \! \! \! dx \frac{K_{\rho}}{2\pi v_{\rho}} \left[ (\partial_x \rho)^2 + \rho(\partial_x \rho)^2 \right] 
- \frac{\Delta_{\rho}}{2\pi a} \int_{0}^{l'} \! \! \! dx \cos(2\rho \theta - \Phi) 
- \frac{\Delta_{\rho}}{2\pi a} \int_{L-l-l'}^{L} \! \! \! dx \cos(2\rho \theta - \Phi) \right]
\]

(36)

It is easy to see that the system is gapless in particle-number conserving setup contrary to the previous case discussed in Sec.II. Indeed, the combination of the fields $2m\theta - \sqrt{2}\theta_\rho$ is free to fluctuate. Given that the combination of the fields $2m\theta - \sqrt{2}\theta_\rho$ is pinned in the topological regions, one can integrate them out. As a result, we can impose the following constraint:

\[
\frac{1}{\sqrt{2m}} \partial_{\tau \vert x} \theta_\rho = \partial_{\tau \vert x} \tilde{\theta}.
\]

(37)

Thus, the problem reduces to that of an inhomogeneous single-component Luttinger liquid:

\[
S = \int_0^L dx \! \! \! \! d\tau \tilde{K}(x) \left[ (\partial_x \tilde{\theta})^2 + \tilde{\theta}(\partial_x \tilde{\theta})^2 \right]
\]

(38)

where the phase field $\tilde{\theta}$ is defined as

\[
\tilde{\theta} = \begin{cases} \frac{1}{\sqrt{2m}} \theta_\rho & 0 < x < L - l' \\ \theta & L - l' \leq x \leq L \end{cases}
\]

(39)

The Luttinger parameters $\tilde{K}(x)$ and velocity $\tilde{v}(x)$ are given by:

\[
\tilde{v}(x) = \begin{cases} v_+ & 0 < x < l \lor L - l - l' < x < L - l' \\ v & L - l - l' < x < L \\ v_{\rho} & l < x < L - l - l' \end{cases}
\]

(40)

and

\[
\tilde{K}(x) = \begin{cases} K_+ & 0 < x < l \lor L - l - l' < x < L - l' \\ mK & L - l - l' < x < L \\ 2m^2 K_{\rho} & l < x < L - l - l' \end{cases}
\]

(41)

with $v_+, K_+$ being

\[
K_+ = \sqrt{K_{\rho}^2 + \frac{K^2}{4m^2} + \frac{KK_{\rho}}{2m} \left( \frac{v}{v} + \frac{v_{\rho}}{v} \right)},
\]

\[
v_+ = \sqrt{vv_+K_++2mv_+K_{\rho}}.
\]

(42)

In order to calculate partition function, one needs to specify boundary conditions for the field $\theta$ at $x = 0$ and $x = L - l'$. Since QSC terminates at these points, the appropriate boundary conditions are

\[
\partial_x \theta_\rho(x) = 0 \text{ for } x = 0, L - l'.
\]

(44)

The field $\theta$ should be continuous at $x = 0, L - l'$ so that there are no singularities in the effective action.

\[
\theta(x^-) = \theta(x^+) \text{ for } x = 0, L - l'.
\]

(45)

In terms of the field $\tilde{\theta}$, these boundary conditions translate to

\[
\tilde{\theta}(L - l - 0^+, \tau) = \tilde{\theta}(L - l - 0^-, \tau) \quad \tilde{\theta}(0^+, \tau) = \tilde{\theta}(0^-, \tau) - \chi,
\]

(46)

where we used Eq.(5). Thus, the modified field $\tilde{\theta}$ satisfies twisted boundary conditions at $x = 0$. We should also include winding numbers of $\tilde{\theta}$ in the imaginary-time direction:

\[
\tilde{\theta}(x, \beta) = \tilde{\theta}(x, 0) + 2\pi mM.
\]

(47)

However, one needs to be careful about the values of $M$. Because of the pair tunneling term, $M$ must be an integer to preserve the periodicity of $\tilde{\theta}_\rho$. It is easy to see that to minimize the action, we can write

\[
\tilde{\theta}(x, \tau) = \frac{2\pi m M}{\beta} \tau + \tilde{\theta}(x),
\]

(48)

and the action evaluates to

\[
S = \frac{2\pi m^2}{\beta} \sum_i K_i l_i M^2 + \beta \int_0^L dx \frac{\tilde{K}(x) \tilde{v}(x)}{2\pi} (\partial_x \tilde{\theta})^2.
\]

(49)

Now one can minimize the action (49) for a fixed $\chi$. Since $K(x), v(x)$ are piecewise constant, the action is minimized by piecewise-linear $\tilde{\theta}$. To calculate partition function, one needs to find the stationary solution (i.e. $\partial_x \tilde{\theta} = 0$) satisfying the following constraints. To simplify the notations, we define $x_0 = 0, x_1 = l, x_2 = L - l - l', x_3 = L - l', x_4 = L$ as the locations of the interfaces where $v$ and $K$ are discontinuous and the field $\theta(x_i)$ are given by

\[
\tilde{\theta}(x_0) = 0 \equiv \theta_0
\]

\[
\tilde{\theta}(x_i) = \theta_i, 1 \leq i \leq 3
\]

\[
\tilde{\theta}(x_4) = \chi \equiv \theta_N.
\]

(50)

Since we restrict ourselves to the space of piecewise linear functions, $\tilde{\theta}$ is then uniquely specified once all the values at $\{x_i\}_{i=0}^4$ are determined. Thus, the action we want to minimize is given by

\[
S[\{\theta_i\}] = \frac{\beta}{2\pi} \sum_{i=1}^N \frac{v_i K_i}{l_i} (\theta_i - \theta_{i-1})^2.
\]

(51)
where \( l_i = x_i - x_{i-1} \) and \( \beta \) is the inverse temperature.

After minimizing above expression with respect to \( \theta(x_i) = \theta_i, 1 \leq i \leq 3 \), the minimum of the action is given by
\[
S_{\text{min}} = \frac{2\pi m^2 C_{\text{loop}}}{\beta} M^2 + \frac{2\pi \beta}{m^2 L_{\text{loop}}} \left( J + \frac{\Phi}{2\pi} \right)^2 .
\]

where \( J = 2mk + n_J \) with \( k \in \mathbb{Z} \) and \( n_J \) being a number corresponding to different topological sectors, i.e. for \( m = 1 \) this number simply denotes fermion parity. The summing over \( k \), thus, we recover the results of Sec. II C. Since for Eq. (55) is one of the main results of this paper showing that for \( m = 1 \) the ground state energy is still 4\( \pi \) periodic provided fermion parity is conserved. We can now recover the previous results obtained for a bulk superconductor. Indeed, by suppressing quantum fluctuations in the QSC (e.g. \( K_F \rho_s \approx \rho_s \rightarrow \infty \)), the capacitance and inductance for the loop become \( C_{\text{loop}} \rightarrow \infty \) and \( L_{\text{loop}} \approx \frac{\hbar}{e \Delta M} \), and, thus, we recover the results of Sec. II C. Since \( J \) and \( N \) commute, the first term is just an additive constant corresponding to different particle number in the ring which is assumed to be fixed henceforth.

**IV. TOPOLOGICAL DEGENERACY SPLITTING**

An important aspect of a topological phase is its ground-state degeneracy. In a finite-size system, the degeneracy is lifted by certain instanton events which connect different topological sectors. In the case of a bulk superconductor, such events can be understood in term of the overlap of the MZM wave functions. In the interacting system considered here the splitting calculation becomes more subtle. One can identify the processes which are present in the BdG formalism and the ones that are not, i.e. are only present due to quantum fluctuations. Possible sources of the topological degeneracy splitting can be classified into three types: (1) quasiparticle tunneling across the topological region, (2) electron tunneling between the nanowires through the QSC, (3) splitting caused by electron backscattering in the QSC.

The splitting of the ground state degeneracy due to the tunneling of a fundamental topological charge can be obtained by the instanton calculation corresponding to a process that tunnels between two degenerate vacua, for example, \( |\theta - \frac{\theta_p}{\sqrt{2}m} = 0 \rangle \) and \( |\theta - \frac{\theta_p}{\sqrt{2}m} = \frac{\pi}{m} \rangle \). This process can be intuitively understood as the process of a \( \frac{\hbar c}{2\pi} \) vortex encircling just the nanowire region, see Fig. 4 process (b). The instanton corresponding to such an event is homogeneously in space resulting in the splitting \( \delta E_1 \sim \exp \left( -\frac{4\sqrt{K_F} l}{\pi m \xi} \right) \) where \( l \) is the length of the nanowire segment as shown in Fig. 4 and \( \xi = v/\Delta_F \). Such a process essentially shifts \( J \) by 1, and can be incorporated into the low energy Hamiltonian as \( H_{S1} = \delta E_1 |J\rangle \langle J + 1| + h.c. \).

The second type of splitting is due to the virtual tunneling of a single electron through the QSC between the topological wires (depicted in Fig. 4 as the process (a)). We obtain this term in the perturbative expansion of the action in the single-fermion tunneling amplitude \( t \):
\[
H_{S2} = -\frac{2t^2}{\Delta_n} e^{-\Delta_1|x-x'|/v_F} \times \cos \left[ m\theta(x') - \frac{\theta_p(x')}{\sqrt{2}m} + m\theta(x) - \frac{\theta_p(x)}{2\pi} \right] \times \cos \left[ m\varphi(x) + \frac{\varphi_p(x)}{\sqrt{2}m} - m\varphi(x') - \frac{\varphi_p(x')}{\sqrt{2}m} \right]
\]

where \( x, x' \) are the ends of the nanowire at \( x = l \) and \( x' = L - l - l' \), see Fig. 1. Using Eq. (35), one finds that \( H_{S2} \) is given by \( H_{S2} = \delta E_2 \cos(\pi J) \) with
\[
\delta E_2 = \frac{2t^2}{\Delta_n} \exp \left( -\frac{4\Delta_2|x-x'|}{v_F} - 1 \frac{1}{4K_F} \log \left| \frac{x-x'}{a} \right| \right) .
\]

Here we used the fact that \( \varphi(x) \) and \( \varphi(x') \) are pinned by the boundaries. For nanowires this is imposed by the boundary conditions at the ends of the nanowire whereas for parafermion setup the induced backscattering terms
cos 2m\varphi effectively pin fields \varphi(x) and \varphi(x'), see Refs. 71–73 for more details.

We now discuss electron backscattering effects. Consider, for example, electron backscattering in the QSC given by the Hamiltonian

$$H_{\text{imp}} = \sum_i v_i \int dx \delta(x - x_i) \cos 2\varphi_\rho(x).$$

One can see that \(\sqrt{2}\varphi_\rho\) creates a kink of \(\pi\) in the dual field \(\varphi_\rho/\sqrt{2}\), and therefore can be thought as a phase slip event. Indeed, a \(2\pi\) phase slip created in the phase field \(\sqrt{2}\varphi_\rho\) can be intuitively understood as an \(\frac{\pi}{2}\) vortex tunneling across the QSC. Such vortex measures the fermion parity in the topological nanowire plus an underlying QSC\(^9\), causing a splitting of the ground state degeneracy. Since vortex actually measures the charge of the encircled region, see Fig. 4, such a process is associated with the effective charging energy of the enclosed region involving both the nanowire as well as the QSC. We note that impurity scattering in the pair tunneling region is suppressed. Therefore, we consider the effect of impurities outside of this region. The details of the splitting calculation are relegated to the Appendix A. Here we simply present our main results. To simplify the instanton calculation, we consider the system shown in Fig. 4 with two impurities at the positions \(x_1\) and \(x_2\). Impurity backscattering in the QSC leads to the following splitting energy

$$H_3 = \delta E_3 |J\rangle \langle J + 1| + \text{h.c.}$$

where the energy \(\delta E_3\) scales as a power-law of the system size. This is an important consequence of quantum fluctuations: topological ground-state degeneracy does not scale exponentially as in the case of a bulk superconductor but rather as a power-law. In the limit \(K_\rho \gg K\), the splitting energy becomes \(\delta E_3 \propto v_1 v_2 |x_1 - x_2|^{1-K_\rho}\).

If we replace an impurity at \(x_2\), for example, by the hard-wall boundary, the splitting energy is given by \(\delta E_3 \propto v_1 |x_1 - x_2|^{1-K_\rho}/2\) which is simply determined by the scaling dimension of the \(\cos \sqrt{2}\varphi_\rho\) operator, see Ref. 69.

Next, we consider impurity scattering in the nanowire described by the following Hamiltonian:

$$H_{\text{imp}}^{\text{NW}} = \sum_i v_i \int dx \delta(x - x_i) \cos 2m\varphi(x)$$

One can notice that an operator \(e^{i2m\varphi(x)}\) acting on the ground state, characterized by the field \(\theta_- = \theta - \varphi_\rho/\sqrt{2m}\), shifts \(\theta_-\) by \(2\pi m\) and, therefore, does not induce any transitions between degenerate ground states.

In summary, we have derived low energy theory for the superconducting loop structure shown in Fig. 1. The corresponding Hamiltonian can be written as

$$H_{\text{eff}} = \left( E_J \left( J + \frac{\Phi}{2\pi} \right)^2 + \delta E_2 (-1)^J \right) |J\rangle \langle J|$$

$$+ (\delta E_1 + \delta E_3) |J\rangle \langle J + 1| + \text{h.c.},$$

where \(J = 2m k + n_J\) and \(E_J = \frac{\gamma_1^2 J_0^2}{m^2 L_{\text{loop}}}\). The spectrum of the system is shown in Fig. 5. One can see that there are two types of non-commuting splitting terms. The processes described by \(\delta E_1\) and \(\delta E_3\) splitting energies result in the hybridization between different \(J\) and \(J+1\) sectors and, thus, open a gap at the avoided level crossings. This is not surprising since fractional excitations have an associated charge \(e/m\), and thus charging energy inducing vortex tunneling can distinguish between different topological sectors and couple them. On the other hand, we have a process causing the splitting energy \(\delta E_2\) which lifts the degeneracy between even and odd \(J\)-sectors. In particular, in \(m = 1\) case the splitting energy \(\delta E_2\) distinguishes between even and odd parity sectors and restores \(4\pi\) periodicity of the fractional Josephson effect, see Fig. 5. It is instructive to understand this result using the simple model involving bulk superconductor with the long-range order. Using the BCS mean-field theory and neglecting quantum phase fluctuations (i.e. \(K_\rho \to \infty\)), one can write an effective low-energy model in terms of the Majorana zero-energy modes \(\gamma_i\) residing at the ends of the 1D topological superconductors. The effective Hamiltonian can be written as

$$H = i E_J \gamma_2 \gamma_3 \cos \frac{\Phi}{2} + i \delta E_1 (\gamma_1 \gamma_2 + \gamma_3 \gamma_4) + i \delta E_2 \gamma_1 \gamma_4.$$
where $\gamma_{1,2,3,4}$ are labelled from left to right. To understand the physics of the aforementioned even-odd effect, it is sufficient to consider the limit $\delta E_1 = 0$. Next, we need to fix the global fermion parity $P = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ to be constant. Thus, $\gamma_1 \gamma_4 = -P \gamma_2 \gamma_3$, and the effective two-level Hamiltonian can be written in terms of fermion parity at the junction $\gamma_2 \gamma_3 = 1 - 2n$ with $n = 0, 1$:

$$H = -\left(E_J \cos \frac{\Phi}{2} + \delta E_2\right)(2n - 1).$$

(63)

The ground state energy at $\Phi = 0$ and $\Phi = 2\pi$ correspond to different fermion parity (i.e. different $J$ in Eq.(61)) is different. However, ground state energies at $\Phi = 0$ and $\Phi = 2\pi$ are clearly different by $\delta E_2$ and thus supercurrent is not $2\pi$-periodic anymore, see Fig.6. In the previous theoretical\cite{6,7,10} and experimental\cite{20} studies, the splitting energy $\delta E_2$ was designed to be exponentially small and therefore neglected. Thus, in order to observe a non-trivial flux periodicity one has to perform ac measurements since ground-state energy is $2\pi$-periodic. However, in the properly designed experiment where the splitting energy $\delta E_2$ is sufficiently large, e.g. $T < \delta E_2 < E_J \ll \Delta$, we predict that the supercurrent in the ground-state as well as an associated effective inductance of the loop will become $4\pi$-periodic, see Fig.6. We emphasize that given there is now a distinction between even- and odd-parity ground-state energies, the low-temperature dc measurements can capture this nontrivial dependence.

V. CONCLUSIONS

In this paper, we develop a theory for the fractional Josephson effect within particle-number conserving formalism. The system we consider consists of a quasi-one-dimensional superconductor (no long-range order) coupled to a nanowire or an edge carrying $e/m$ fractional charge excitations with $m$ being an odd integer. Using Luttinger liquid formalism and instanton analysis, we analyze various processes (e.g. quantum phase slips) leading to the splitting of the ground state degeneracy. We find that quantum phase slips induced by impurities in the quasi-one-dimensional superconductor lead to a power-law dependence of the splitting energy in contrast to long-range-ordered superconductors where the splitting energy scales exponentially with the system size. We calculate the periodicity of the supercurrent (as well as other observable quantities) on magnetic flux through the superconducting loop (see Fig.1) and find that it nontrivially depends on the topological ground-state degeneracy.

Previously, it was believed that fractional Josephson effect can be accessed in ac transport measurements which might be quite challenging. By properly designing the experimental setup, we discuss how one might be able to detect this anomalous periodicity on flux in dc transport measurements.

Our work provides a theoretical framework for modeling mesoscopic superconducting quantum computing devices involving Majorana zero-energy modes. Given the $4\pi$ periodicity of the ground-state energy on magnetic flux $\Phi$, we believe such devices offer a new playground for investigating the phase slips in superconducting wires.

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Appendix A: Degeneracy splitting due to impurity backscattering

In this Appendix, we calculate the topological degeneracy splitting due to impurity backscattering. For simplicity, we consider the geometry shown in Fig.4 where the impurities are located at the positions $x_1$ and $x_2$, just outside the pair-hopping region. For simplicity, we consider $m = 1$ case. The corresponding Hamiltonian then reads

$$H_{\text{imp}} = v_1 \cos \sqrt{2} \varphi(x_1) + v_2 \cos \sqrt{2} \varphi(x_2).$$

(A1)

We now compute the amplitude $M$ for the instanton tunneling between the minima of $\cos(2\theta - \sqrt{2}\varphi)$ in the interval $x_1 < x < x_2$:

$$M = \left< \theta - \frac{\theta_0}{\sqrt{2}} = \pi \left| e^{-H_{\text{imp}}} \theta - \frac{\theta_0}{\sqrt{2}} = 0 \right| \right>_{T \to \infty},$$

(A2)

where $H$ is the total Hamiltonian including $H_{\text{imp}}$. This amplitude can be calculated using a path integral with the appropriate boundary conditions:

$$Z = \sum_{k_1, k_2 \in \mathbb{Z}} \int D\varphi D\theta \delta(\theta(x, 0) - \pi(2k_1 + 1)) \delta(\theta(x, T) - 2\pi k_2) e^{-\int dx(V + L_{\text{imp}})}}$$

(A3)
where \( \theta_-(x, \tau) = \theta(x, \tau) - \frac{\theta_0(x, \tau)}{\sqrt{2}} \) and

\[
\mathcal{L} = \frac{i}{\pi} \partial_x \varphi_\rho \partial_\tau \varphi_\rho + \frac{i}{\pi} \partial_x \varphi \partial_\tau \theta + \frac{v}{2\pi} [K^{-1}(\partial_x \varphi)^2 + K(\partial_x \theta)^2] + \frac{v_\rho}{2\pi} [K^{-1}(\partial_x \varphi_\rho)^2 + K_\rho(\partial_x \theta_\rho)^2] - \frac{\Delta_\rho}{2\pi a} \Theta(x - x_1) \Theta(x_2 - x) \cos[2\theta(x) - \sqrt{2}\theta_\rho(x)],
\]

(A4)

\[
\mathcal{L}_{\text{imp}} = [v_1 \delta(x - x_1) + v_2 \delta(x - x_2^\top)] \cos \sqrt{2} \varphi_\rho(x, \tau).
\]

(A5)

Here the pairing tunneling term is non-zero in the interval \( x_1 < x < x_2 \) which defines the topological wire segment. In the limit \( K_\rho > 2 \) when backscattering is irrelevant, we can calculate partition function perturbatively in \( v_1 \)

\[
\mathcal{Z} = \int \mathcal{D}\varphi \mathcal{D}' \varphi e^{-\int d\tau dx \mathcal{L}} = \int \mathcal{D}\varphi \mathcal{D}\theta \left( 1 + \frac{v_1 v_2}{2} \int d\tau_1 d\tau_2 \cos \sqrt{2} \varphi_\rho(x_1, \tau_1) \cos \sqrt{2} \varphi_\rho(x_2^\top, \tau_2) + \cdots \right) e^{-\int d\tau dx \mathcal{L}_0}.
\]

(A6)

Here \( \mathcal{D}' \theta \) denotes integration with the \( \delta \) function constraints, see Eq.(A3). Using the identity

\[
\cos \sqrt{2} \varphi_\rho(x_1, \tau_1) \cos \sqrt{2} \varphi_\rho(x_2^\top, \tau_2) = \frac{1}{4} \sum_{s_1,s_2 = \pm 1} e^{i\sqrt{2} s_1 \varphi_\rho(x_1)} e^{i\sqrt{2} s_2 \varphi_\rho(x_2)},
\]

(A7)

one finds that the effective action for \( \varphi_\rho \) is given by

\[
S[\varphi_\rho] = \int dz d\tau \left[ \frac{i}{\pi} \partial_x \varphi_\rho \partial_\tau \varphi_\rho + \frac{v_\rho}{2\pi} K_\rho^{-1}(\partial_x \varphi)^2 \right] + i\sqrt{2} \left[ s_1 \varphi_\rho(x_1, \tau_1) + s_2 \varphi_\rho(x_2^\top, \tau_2) \right].
\]

(A8)

After integrating out \( \varphi \), the effective action becomes

\[
S[\varphi_\rho] = \int \frac{d^2 q}{(2\pi)^2} \left[ \frac{i}{\pi} \omega K_\rho \varphi_\rho(-q) - \frac{v_\rho K_\rho^2}{2\pi^2} \varphi_\rho(-q) + \sqrt{2} i(\pi s_1 e^{iq \cdot x_1} + \pi s_2 e^{iq \cdot x_2}) \varphi_\rho(q) \right]
\]

\[
\approx \int \frac{d^2 q}{(2\pi)^2} \left[ \frac{vk^2}{2\pi^2 K_\rho} \varphi_\rho(-q) + i \left( \frac{\omega k}{\pi} \theta_\rho(-q) + \sqrt{2} s_1 e^{iq \cdot x_1} + \sqrt{2} s_2 e^{iq \cdot x_2} \right) \varphi_\rho(q) \right],
\]

(A9)

where \( q = (k, w), x = (x, \tau) \) and the inner product \( q \cdot x = kx - \omega \tau \). We can now integrate out \( \varphi_\rho \) to find

\[
S[\theta_\rho] = -\int \frac{d^2 q}{(2\pi)^2} \left[ \frac{v K_\rho}{2\pi v_\rho} \right] \left( \frac{\omega}{\pi} \theta_\rho(q) + \sqrt{2} s_1 e^{-iq \cdot x_1} + \sqrt{2} s_2 e^{-iq \cdot x_2} \right)^2
\]

\[
\approx \frac{K_\rho}{2\pi v_\rho} \int \frac{d^2 q}{(2\pi)^2} \left[ \frac{\omega}{\pi} \theta_\rho(q) + \sqrt{2} s_1 e^{-iq \cdot x_1} + \sqrt{2} s_2 e^{-iq \cdot x_2} \right]^2
\]

\[
\approx \frac{K_\rho}{2\pi v_\rho} \int d\tau dx \left[ \partial_\tau \theta_\rho + \sqrt{2} s_1 \Theta(x - x_1) \delta(\tau - \tau_1) + \sqrt{2} s_2 \Theta(x - x_2) \delta(\tau - \tau_2) \right]^2
\]

(A10)

Next, we integrate out \( \varphi \) and obtain the following effective action:

\[
\mathcal{L}_{s_1s_2}[\theta_\rho, \theta] = \frac{K_\rho}{2\pi} \left[ \frac{1}{v_\rho} \left( \partial_\tau \theta_\rho + \sqrt{2} s_1 \Theta(x_1 - x) \delta(\tau - \tau_1) + \sqrt{2} s_2 \Theta(x_2 - x) \delta(\tau - \tau_2) \right)^2 + v_\rho(\partial_x \theta_\rho)^2 \right]
\]

\[
+ \frac{K}{2\pi} \left[ \frac{1}{v} (\partial_\tau \theta)^2 + v(\partial_x \theta)^2 \right] - \frac{\Delta_\rho}{2\pi a} \Theta(x - x_1) \Theta(x_2 - x) \cos[2\theta(x) - \sqrt{2}\theta_\rho(x)].
\]

(A11)

Combining all the terms together, the partition function now reads

\[
\mathcal{Z} = \int \mathcal{D}\theta_\rho \mathcal{D}' \theta \left( e^{-S_0} - \frac{v_1 v_2}{8} \int d\tau_1 d\tau_2 \sum_{s_1s_2} e^{-S_{s_1s_2}} + \cdots \right).
\]

(A12)

The calculation of the first term \( \int \mathcal{D}' \theta \mathcal{D} \theta e^{-S_0} \) reproduces the splitting energy \( \delta E_1 \), see Sec.IV. We will focus here on the second term and calculate the contribution of the classical field configuration minimizing the action \( S_{s_1s_2} \). One can notice that the quadratic action of \( \theta_\rho \) contains \( \delta \) function in \( \tau \). Therefore, in order to get a finite action \( \theta_\rho \) field must have a discontinuity at \( \tau_1 \) and \( \tau_2 \). Indeed, let us write \( \theta_\rho \) as

\[
\theta_\rho = \tilde{\theta}_\rho - A(x), A(x) = \sqrt{2} \pi [s_1 \Theta(x_1 - x) \Theta(\tau - \tau_1) + s_2 \Theta(x_2 - x) \Theta(\tau - \tau_2)],
\]

(A13)
where $\tilde{\theta}(x, \tau)$ is a now smooth field as far as time dependence is concerned. Notice that in doing so we have introduced a jump in the spatial profile of $\theta$ at $x_1$ and $x_2$. The discontinuity at these points has to be carefully taken into account by considering an inhomogeneous problem since the pairing field also has jumps at $x_1$ and $x_2$. However, in the limit when the length of the topological region is large, the bulk energy gives dominant contribution and thus the boundary effects can be ignored.

Next, we rewrite the action using the new fields:

$$\mathcal{L}_{s_1, s_2}[(\theta, \phi)] = \frac{K_\rho}{2\pi} \left[ \frac{1}{v_\rho} (\partial_\tau \tilde{\theta})^2 + v_\rho (\partial_x \tilde{\theta} - \partial_x \phi)^2 \right]$$

$$+ \frac{K}{2\pi} \left[ \frac{1}{v} (\partial_\tau \theta)^2 + v (\partial_x \theta)^2 \right] - \frac{\Delta_\rho}{2\pi} \Theta(x - x_1) \Theta(x_2 - x) \cos \left[ 2(\theta(x) - \sqrt{2}\tilde{\theta}(x)) \right].$$

(14)

In the domain $x_1 < x < x_2$, the combination $\theta - \tilde{\theta}/\sqrt{2}$ is pinned and one can use the relation $\partial_\tau \theta = \partial_x \tilde{\theta}/\sqrt{2}$ to simplify the calculation. Thus, within this space of field configurations, the corresponding partition function can be evaluated exactly

$$Z_{v_1, v_2} = \frac{v_1 v_2}{4} \int d\tau_1 d\tau_2 \int D\tilde{\theta}_\rho e^{-S_{\text{eff}}},$$

(15)

$$S_{\text{eff}}[\theta] = \int d\tau dx \frac{1}{2\pi} \left[ \left( \frac{K_\rho}{v_\rho} + \frac{K}{2v} \right) (\partial_\tau \tilde{\theta})^2 + v_\rho K_\rho (\partial_x \tilde{\theta} - \partial_x \phi)^2 + \frac{v K}{2} (\partial_x \tilde{\theta})^2 \right].$$

(16)

We can now evaluate the path integral and calculate the dependence of this term on $L = x_2 - x_1$. Let us look closer at the term $(\partial_x \tilde{\theta} - \Delta_\rho)^2$:

$$\int d\tau dx (\partial_x \tilde{\theta} - \Delta_\rho)^2 = \int d\tau \int dx \left[ (\partial_x \tilde{\theta})^2 - 2\sqrt{2} \pi (\partial_x \tilde{\theta})(x_1, \tau) \Theta(\tau - \tau_1) + 2\sqrt{2} \pi (\partial_x \tilde{\theta})(x_2, \tau) \Theta(\tau - \tau_2) + (\Delta_\rho)^2 \right].$$

(17)

For our purpose, it is sufficient to consider large $L$ limit and neglect the boundary conditions for $\tilde{\theta}$ which does not affect the scaling of the action with $L$. Therefore, we neglect the divergent boundary term $(\partial_x \tilde{\theta})^2$ in Eq.(17) which is simply an artifact of our approximation. The effective partition function in the momentum space becomes

$$S_{\text{eff}} = \int \frac{dkd\omega}{(2\pi)^2} \left\{ \frac{1}{2\pi} \left[ \left( \frac{K_\rho}{v_\rho} + \frac{K}{2v} \right) \omega^2 + \left( v_\rho K_\rho + \frac{v K}{2} \right) k^2 \right] |\tilde{\theta}_\rho(q)|^2 + \sqrt{2} v_\rho K_\rho k \omega (s_1 e^{iq \cdot x_2} + s_2 e^{iq \cdot x_1}) \tilde{\theta}_\rho(q) \right\}. \quad (18)$$

It is clear that the correlation function is vanishing for $s_1 = s_2$. Therefore, in the following we set $s_1 = 1, s_2 = -1$. After integrating out $\tilde{\theta}$ field, one finds

$$\int D\tilde{\theta}_\rho e^{-S_{\text{eff}}} = \exp \left( - \int \frac{dkd\omega}{4\pi K_+} \frac{-v_\rho^2 \omega^2 + K_\rho^2 k^2}{K_+ \omega^2 (\omega^2 + v_\rho^2 k^2)} e^{iq \cdot x_2} - e^{iq \cdot x_1} \right) = C_1 \exp \left( - \frac{v_\rho^2 K_\rho^2}{2v_\rho^2 K_+} \ln \left[ \frac{v_\rho^2 |\tau_1 - \tau_2|^2 + |x_1 - x_2|^2}{a^2} \right] \right),$$

where $a$ is a UV cutoff and $C_1$ is the numerical prefactor. Finally, the contribution to the partition function reads

$$Z_{s_1, s_2} = \frac{v_1 v_2 C_1}{8} \int d\tau_1 d\tau_2 \frac{a^2}{\sqrt{v_\rho^2 |\tau_1 - \tau_2|^2 + |x_1 - x_2|^2}} \frac{v_\rho^2 K_\rho^2}{2v_\rho^2 K_+} \sim \frac{v_1 v_2 Ta}{v_+} \frac{1}{\sqrt{|x_1 - x_2|^2 + 1}},$$

(20)

Following standard calculation, see Ref.[81], in order to obtain the energy splitting, we need to take into account multiple instanton processes corresponding to multiple insertions of jumps of $\tilde{\theta}_\rho$. In the end, one finds that the energy splitting is given by

$$\delta E \sim \frac{v_1 v_2 a}{v_+} \frac{1}{\sqrt{|x_1 - x_2|^2 + 1}},$$

(21)

and is power-law dependent on the system size. In the limit $K_\rho \gg K$, $v_+ \approx v_\rho$ and $K_+ \approx K_\rho$, so the splitting energy becomes $\delta E \propto L^{1-K_\rho}$. Note that the boundary of a QSC can be represented as a strong impurity, say, at $x_2$ which effectively cuts off the superconductor at this point. In this case, the fluctuations of the phase $\varphi_\rho$ at $x_2$ are suppressed, and the splitting is given by scaling dimension of $\cos \sqrt{2}\varphi_\rho(x_1)$, i.e. $\delta E \propto |x_1 - x_2|^{1-K_\rho/2}$ at $K_\rho \gg K$.

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