Analysis of Sparse Representations Using Bi-Orthogonal Dictionaries

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Abstract—The sparse representation problem of recovering an N dimensional sparse vector x from M < N linear observations y = Dx given dictionary D is considered. The standard approach is to let the elements of the dictionary be independent and identically distributed (IID) zero-mean Gaussian and minimize the l1-norm of x under the constraint y = Dx. In this paper, the performance of l1-reconstruction is analyzed, when the dictionary is bi-orthogonal D = [O1, O2], where O1, O2 are independent and drawn uniformly according to the Haar measure on the group of orthogonal M × M matrices. By an application of the replica method, we obtain the critical conditions under which perfect l1-recovery is possible with bi-orthogonal dictionaries.

I. INTRODUCTION

The sparse representation (SR) problem has wide applicability, for example, in communications [1], [2], multimedia [3], and compressive sampling (CS) [4], [5]. The standard SR problem is to find the sparsest x ∈ ℜN that is the solution to the set of M < N linear equations

\[ y = Dx, \]  

for a given dictionary or sensing matrix D ∈ ℜM×N and observation y. Finding such x is, however, non-polynomial (NP) hard. Thus, a variety of practical algorithms have been developed that solve the SR problem sub-optimally. The topic of the current paper is the convex relaxation approach where, instead of searching for the x having the minimum l0-norm, the goal is to find the minimum l1-norm solution of (1).

Let K be the number of non-zero elements in x and assume that the convex relaxation method is used for recovery. The trade-off between two parameters ρ = K/N and α = M/N is then of special interest since it tells how much the sparse signal can be compressed under l1-reconstruction. An interesting question then arises: How does the sparsity-undersampling (ρ vs. α) trade-off depend on the choice of dictionary D?

The empirical study in [6] Sec. 15 in SI gave evidence that the worst case ρ vs. α trade-off is quite universal w.r.t different random matrix ensembles. Analysis in [7] further revealed that the typical conditions for perfect l1-recovery are the same for all sensing matrices that are sampled from the rotationally invariant matrix ensembles. Dictionaries with independent identically distributed (IID) zero-mean Gaussian elements is one example of this. But correlations in D can degrade the performance of l1-recovery [8], so it is not fully clear how the choice of D affects the ρ vs. α trade-off.

Besides the random / unstructured dictionaries mentioned above, the information theoretic approach in [9] encompasses more general matrix ensembles but does not consider the l1-reconstruction limit. Several studies in the literature have also considered the specific construction where D is formed by concatenating two orthogonal matrices [10]–[14]. Such bi-orthogonal dictionaries are easy to implement and can give elegant theoretical insights. Unfortunately, the “mutual coherence” based methods used in these papers provide pessimistic, or worst case, thresholds. Furthermore, the result are not easy to compare between the unstructured and bi-orthogonal cases.

We consider the analysis of the bi-orthogonal SR setup

\[ y = Dx = [O_1 \ | \ O_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O_1 x_1 + O_2 x_2, \]  

where the dictionary is constructed by concatenating two independent matrices O1 and O2, that are drawn uniformly according to the Haar measure on the group of all orthogonal M × M matrices. We use the non-rigorous replica method (see, e.g., [2], [15]–[17] for related works) to assess ρ for a given α, up to which the l1-recovery is successful. This allows a direct comparison between the random and bi-orthogonal dictionaries in average or typical sense. The main result of the paper is the sparsity-undersampling trade-off for the bi-orthogonal SR setup (2). We find that this matches the unstructured IID Gaussian dictionary when the non-zero components are uniformly distributed between the two blocks. Surprisingly, when the non-zero components are concentrated more on one block than the other, the bi-orthogonal dictionaries can cope with higher overall densities than the unstructured case. This extends to the case of general T-concatenated orthogonal dictionaries as reported elsewhere [18].

II. PROBLEM SETTING

Consider the SR problem of finding the sparsest vector x = [x1 T x2 T] T ∈ ℜN, given the dense vector y ∈ ℜM and the dictionary D = [O1 \ | \ O2] ∈ ℜM×N. By definition M/N = 1/2 and O1 T O1 = I_M for this setup. Let K1 and K2 be the number of non-zero elements in x1 and x2, respectively, so that K = K1 + K2 is the total number of non-zero elements in x. Denote ρ = K/(2M) for the overall sparsity of the source while ρ_1 = K_1/M and ρ_2 = K_2/M represent the signal densities of the two blocks.
It is important to note that $D$ in (2) does not belong to the rotationally invariant matrix ensembles [7], and there are complex dependencies between the elements due to the orthogonality constraints. The fact that $O_1 O_2 \neq 0$ makes the analysis of the setup highly non-trivial (for a sketch, see Appendices A and B). Thus, only the bi-orthogonal case is considered here and the analysis of general $T$-concatenated orthogonal dictionaries is reported elsewhere [8].

The system is assumed to approach the large system limit $M, K_1, K_2 \to \infty$ where the signal densities $\rho_1, \rho_2$ are finite and fixed. We let $\{x_i\}_{i=1}^t$ be independent sparse random vectors whose components are IID according to

$$p_i(x) = (1 - \rho_i)\delta(x) + \rho_i e^{-x^2/2}/\sqrt{2\pi}, \quad i = 1, 2. \quad (3)$$

The convex relaxation of the original problem is considered and the goal is to find $x = [x_1^T \cdots x_t^T]^T$ that is the solution to

$$\min_{x_1, \ldots, x_t} \|x_1\|^1 + \|x_2\|^1 \text{ s.t. } y = O_1 x_1 + O_2 x_2. \quad (4)$$

Note that we do not consider the weighted $l_1$-reconstruction analyzed for the rotationally invariant $D$ in [15]. This corresponds to the scenario where the user has no prior knowledge about the relative statistics of the data blocks. In the next section we find the typical density $\rho = (\rho_1 + \rho_2)/2$ for which perfect $l_1$-reconstruction is possible under the constraint [7].

III. ANALYSIS

Let the postulated prior of the sparse vector $x_i$ be

$$q_\beta(\tilde{x}_i) = e^{-\beta \|\tilde{x}_i\|_1}, \quad i = 1, 2, \quad (5)$$

where the components of $\tilde{x}_i \in \mathbb{R}^M$ are IID. The inverse temperature $\beta$ is a non-negative parameter. Let $q_\beta(\hat{x}) = q_\beta(\tilde{x}_1) q_\beta(\tilde{x}_2)$ be the postulated prior of $x$ in [7], and define a mismatched posterior mean estimator

$$\langle \hat{x} \rangle_{\beta} = Z_{\beta}(y, D)^{-1} \int \hat{x} \delta(y - D \hat{x}) q_\beta(\hat{x}) d\hat{x}. \quad (6)$$

Here $Z_{\beta}(y, D) = \int \delta(y - D \hat{x}) q_\beta(\hat{x}) d\hat{x}$, acts as the partition function of the system. Then, the zero-temperature estimate $\langle \hat{x} \rangle_{\beta \to \infty}$ is a solution (if at least one exists) to the original $l_1$-minimization problem [7].

Utilizing one of the standard tools from statistical physics, namely the non-rigorous replica method, we study next the behavior of the estimator [6]. We accomplish this by examining the so-called free energy density $f$ of the system in the thermodynamic limit $N \to \infty$. As a corollary, we obtain the critical compression threshold for the original optimization problem (4) when $\beta \to \infty$.

A. Free Energy

As sketched in Appendix A the free energy density related to (6) reads under the replica symmetric (RS) ansatz

$$f_{\text{RS}} = -\frac{1}{2} \lim_{\beta \to \infty} \frac{1}{M} \lim_{M \to \infty} \frac{1}{M} \lim_{u \to 0} \frac{\partial}{\partial u} \log E_y D \{Z_{\beta}^u(y, D)\} = \frac{1}{2} \text{ extr}_{\{\Theta_1, \Theta_2\}} \sum_{i=1}^2 T(\Theta_i), \quad (7)$$

where

$$T(\Theta_i) = \frac{\rho_i - 2m_i + Q_i}{4\chi_i} - \frac{Q_i \hat{Q}_i}{2} + \frac{\chi_i \hat{x}_i}{2} + m_i \hat{m}_i + \int (1 - \rho_i) \phi(z \sqrt{\chi_i}; \hat{Q}_i) + \rho_i \phi(z \sqrt{\hat{m}_i^2 + \hat{x}_i}; \hat{Q}_i) Dz, \quad (8)$$

$$\Theta_i = \{Q_i, \chi_i, m_i, \hat{Q}_i, \hat{x}_i, \hat{m}_i\} \text{ is a set of parameters that take values on the extended real line, } Dz = (2\pi)^{-1/2} e^{-z^2/2} dz \text{ is the Gaussian measure and }$$

$$\phi(h; \hat{Q}) = \min_{x \in \mathbb{R}} \{Q x/2 - h x + |x|\}. \quad (9)$$

In contrast to, e.g., [7], [15], here $\text{extr}_{\Theta_i} g(\Theta)$ is constrained extremization over the function $g(\Theta)$ when $\chi_1 = \chi_2$, needs to be satisfied.

Remark 1. If the dictionary is sampled from the rotationally invariant matrix ensembles, the RS free energy density reads

$$f_{\text{RS}} = \frac{1}{2} \text{ extr}_{\{\Theta_1, \Theta_2\}} \sum_{i=1}^2 \left( \frac{\rho_i - 2m_i + Q_i}{2} - \frac{Q_i \hat{Q}_i}{2} + \frac{\chi_i \hat{x}_i}{2} + m_i \hat{m}_i \right. \right)$$

$$\left. + \int (1 - \rho_i) \phi(z \sqrt{\chi_i}; \hat{Q}_i) + \rho_i \phi(z \sqrt{\hat{m}_i^2 + \hat{x}_i}; \hat{Q}_i) Dz \right) \quad (10)$$

where extr is an unconstrained extremization w.r.t $\{\Theta_1, \Theta_2\}$.

B. Constrained Extremization

Let us denote $Q(x) = \int_x^\infty Dz$ for the Q-function and define

$$r(h) = \sqrt{\frac{2}{\pi}} e^{-\frac{h^2}{2}} - (1 + h) Q \left( \frac{1}{\sqrt{\pi}} \right). \quad (11)$$

After solving the integrals and the optimization problem in [9], the function (8) becomes

$$T(\Theta_i) = \frac{\rho_i - 2m_i + Q_i}{4\chi_i} - \frac{Q_i \hat{Q}_i}{2} + \frac{\chi_i \hat{x}_i}{2} + m_i \hat{m}_i$$

$$+ \frac{1}{2} \frac{\rho_i}{Q_i} r(\hat{x}_i) + \frac{\rho_i}{Q_i} r(\hat{m}_i^2 + \hat{x}_i). \quad (12)$$

Introducing the Lagrange multiplier $\eta$ for the constraint $\chi_1 = \chi_2$, an alternative formulation for the free energy density reads

$$f_{\text{RS}} = \frac{1}{2} \text{ extr}_{\{\Theta_1, \Theta_2, \eta\}} \{\eta(\chi_1 - \chi_2) + T(\Theta_1) + T(\Theta_2)\}, \quad (13)$$

where the extremization is now an unconstrained problem.

Taking partial derivatives w.r.t all optimization variables and setting the results to zero yields the identities

$$\hat{Q}_i = \hat{m}_i \quad \text{and} \quad \chi_i = \frac{1}{2\hat{m}_i}, \quad i = 1, 2. \quad (14)$$

We also find that the expressions

$$\frac{1}{\hat{m}_i} = \frac{2}{\rho_i} \left[ 2(1 - \rho_i) Q \left( \frac{1}{\sqrt{\chi_i}} \right) + 2 \rho_i Q \left( \frac{1}{\sqrt{\hat{m}_i^2 + \chi_i}} \right) \right],$$

$$\hat{x}_i = \frac{\rho_i - 2m_i + Q_i}{2\hat{m}_i^2 + \hat{x}_i} - \eta \frac{\partial}{\partial \chi_i} (\chi_1 - \chi_2), \quad (15)$$

are satisfied by the extremum of (13). Under perfect reconstruction in mean square error (MSE) sense (see, e.g., [7].
Let $x \in \mathbb{R}^{2M}$, $D \in \mathbb{R}^{M \times 2M}$ and $y = D x$ as in (2). Given the parameter $\mu \in [0, 1]$, the typical density $\rho(\mu)$ of the solution to the optimization problem

$$
\arg \min_{x: ||x_1|| + ||x_2|| = 1} \| y - D x \|,
$$

is determined in the large system limit by the solutions of the following set of coupled equations

$$
\hat{\chi}_1 = \left[ Q^{-1} \left( \frac{1}{4} - \frac{2 \mu \rho}{1 + \mu} \left( \frac{1}{2} - Q \left( \frac{1}{\sqrt{X_1}} \right) \right) \right) \right]^{-2},
$$

$$
\eta = \frac{4 \mu \rho}{1 + \mu} \left[ 1 + \hat{\chi}_1 + 2r(\hat{\chi}_1) \right] - 4r(\hat{\chi}_1) - \hat{\chi}_1,
$$

$$
\hat{\chi}_2 = \frac{4 \rho}{1 + \mu} \left[ 1 + \hat{\chi}_2 + 2r(\hat{\chi}_2) \right] - 4r(\hat{\chi}_2) + \eta,
$$

$$
\rho = (1 + \mu) \left[ \frac{1}{2} - 2Q \left( \frac{1}{\sqrt{X_2}} \right) \right] \left[ 2 - 4Q \left( \frac{1}{\sqrt{X_2}} \right) \right],
$$

where $Q^{-1}$ is the functional inverse of the $Q$-function. For uniform sparsity, that is, $\mu = 1$ and $\rho_1 = \rho_2$, we have $\eta = 0$, $\hat{\chi}_1 = \hat{\chi}_2$ and $\chi_1 = \chi_2$ always. The critical density is thus the same as for the dictionary that is drawn from the ensemble of rotationally invariant matrices.

C. Numerical Examples

Given the dictionary $D$ is drawn from the ensemble of rotationally invariant matrices, the critical density for $l_1$-recovery is known to be independent of the block densities $\{\rho_1, \rho_2\}$ and given by $\rho = 0.19284483309074016$... for all $\mu \in [0, 1]$. For the bi-orthogonal $D$, the threshold is the same for details), we have $\rho_i = Q_i = m_i$ and $\hat{m}_i \to \infty \implies \chi_i \to 0$. Hence, (15) simplifies to the condition

$$
2(1 - \rho_i)Q \left( \frac{1}{\sqrt{\chi_i}} \right) + \rho_i = \frac{1}{2},
$$

On the other hand, omitting the terms of the order $O(1/\hat{m}_i^3)$, we have from the partial derivatives of $Q_i$ and $\hat{m}_i$

$$
Q_i = \rho_i - \frac{2\rho_i}{\hat{m}_i \sqrt{2\pi}} = \frac{2(1 - \rho_i)}{\hat{m}_i^3} r(\hat{\chi}_i) + \frac{\rho_i}{\hat{m}_i^3} (1 + \hat{\chi}_i),
$$

$$
m_i = \rho_i - \frac{\rho_i}{\hat{m}_i \sqrt{2\pi}}.
$$

respectively, where we used (14) to simplify the expressions. Plugging the above to (16) and using again (14) yields

$$
\hat{\chi}_i = (-1)^i \eta + 2\rho_i (1 + \hat{\chi}_i) - 4(1 - \rho_i) r(\hat{\chi}_i).
$$

Before stating the final result, let us introduce a real parameter $\mu \in [0, 1]$ and assume without loss of generality that $\rho_1 = \mu \rho_2$. Then the per-block densities can be written as

$$
\rho_i = \frac{2\mu \rho}{1 + \mu} \rho \quad \text{and} \quad \rho_2 = \frac{2}{1 + \mu} \rho,
$$

where $\rho = \rho(\mu)$ is the overall density of the source. The parameter $\mu$ determines thus how uniformly the non-zero components are distributed between the two blocks: $\mu = 1$ means fully uniformly, $\mu = 0$ implies that all non-zero components are in the second block.

Main Result. Let $x \in \mathbb{R}^{2M}$, $D \in \mathbb{R}^{M \times 2M}$ and $y = D x$ as in (2). Given the parameter $\mu \in [0, 1]$, the typical density $\rho(\mu)$ of the solution to the optimization problem

$$
\arg \min_{x: ||x_1|| + ||x_2|| = 1} \| y - D x \|, \quad \text{s.t.} \quad y = D x,
$$

is determined in the large system limit by the solutions of the following set of coupled equations

$$
\hat{\chi}_1 = \left[ Q^{-1} \left( \frac{1}{4} - \frac{2 \mu \rho}{1 + \mu} \left( \frac{1}{2} - Q \left( \frac{1}{\sqrt{X_1}} \right) \right) \right) \right]^{-2},
$$

$$
\eta = \frac{4 \mu \rho}{1 + \mu} \left[ 1 + \hat{\chi}_1 + 2r(\hat{\chi}_1) \right] - 4r(\hat{\chi}_1) - \hat{\chi}_1,
$$

$$
\hat{\chi}_2 = \frac{4 \rho}{1 + \mu} \left[ 1 + \hat{\chi}_2 + 2r(\hat{\chi}_2) \right] - 4r(\hat{\chi}_2) + \eta,
$$

$$
\rho = (1 + \mu) \left[ \frac{1}{2} - 2Q \left( \frac{1}{\sqrt{X_2}} \right) \right] \left[ 2 - 4Q \left( \frac{1}{\sqrt{X_2}} \right) \right],
$$

where $Q^{-1}$ is the functional inverse of the $Q$-function. For uniform sparsity, that is, $\mu = 1$ and $\rho_1 = \rho_2$, we have $\eta = 0$, $\hat{\chi}_1 = \hat{\chi}_2$ and $\chi_1 = \chi_2$ always. The critical density is thus the same as for the dictionary that is drawn from the ensemble of rotationally invariant matrices.

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IV. CONCLUSIONS AND DISCUSSION

The sparsity-undersampling trade-off for the bi-orthogonal SR setup (2) was studied. For uniformly distributed non-zero components, there is no difference in compression ratio if we replace the rotationally invariant dictionary $D \in \mathbb{R}^{M \times 2M}$ by a concatenated matrix $D = [O_1 \ O_2] \in \mathbb{R}^{M \times 2M}$, where $O_1, O_2$ are independent and drawn uniformly according to the Haar measure on the group of all orthogonal $M \times M$.
matrices. For non-uniform block sparsities, however, the bi-orthogonal dictionaries were found to be beneficial compared to the unstructured random dictionaries.

**Appendix A**

**Free Energy**

Following [7, 13], we use the replica trick and write the free energy density as

\[ f = -\frac{1}{2} \lim_{\beta \to \infty} \frac{1}{\beta} \lim_{u \to 0} \frac{\partial}{\partial u} \lim_{M \to \infty} \frac{1}{M} \log \Xi_{\beta,M}^{(u)}, \]

where denoting \( \Delta x_i^{[a]} = x_i^{[a]} - x_i, a = 0, 1, \ldots, u, \)

\[ \Xi_{\beta,M}^{(u)} = \mathbb{E} \lim_{\tau \to 0^+} \frac{1}{\tau^2} \mathbb{E} \left\{ e^{-\frac{1}{\beta} \sum_{i=1}^u \left[ O_i \Delta x_i^{[a]} + O_i \Delta x_i^{[b]} \right]^2} \right\}. \]  

(27)

For \( i = 1, 2, \) the vectors \( \{ x_i^{[a]} \}_{a=1}^u \) are IID conditioned on \( D \) and have the same density \( \mathcal{N} \) as \( x_i \). Furthermore, the elements of the vectors \( x_i^{[0]} \) and \( x_i^{[2]} \) are independently drawn according to \( p_1 \) and \( p_2 \) as given in (4), and \( \mathcal{X} = \{ x_i^{[0]}, x_i^{[2]} \} \). Let us concentrate on \( \Xi_{\beta,M}^{(u)} \) and the inner expectation in (27), which is over the orthogonal matrices \( O_1 \) and \( O_2 \) given \( \mathcal{X} \). Since \( O_i \) are orthogonal, the average affects only the cross-terms of the form \( \langle u_i^{[0]} \rangle \langle u_i^{[b]} \rangle \), where \( u_i^{[a]} = O_i \Delta x_i^{[a]} \). Define matrices \( S_i \in \mathbb{R}^{n \times u} \) for \( i = 1, 2 \), whose \((a,b)\)th element

\[ S_i^{[a,b]} = Q_i^{(a,0)} - Q_i^{(b,0)} - Q_i^{(0,b)} + Q_i^{(a,b)}, \]

(28)

is the empirical covariance between the elements of \( \Delta x_i^{[a]} \) and \( \Delta x_i^{[b]} \), written in terms of the empirical covariances

\[ Q_i^{[a,b]} = M^{-1} \langle x_i^{[a]} \rangle^{\top} x_i^{[b]}, \]

(29)

between the components of the \( a \)th and \( b \)th replicas of \( x_i \). For analytical tractability, we make the standard replica symmetry (RS) assumption on the correlations (29), i.e., \( \tau_i = Q_i^{(0,0)} \), \( m_i = Q_i^{(0,b)} = Q_i^{(0,0)} \forall a, b \geq 1 \), \( Q_i^{(a,b)} \) \( \forall a \neq b \geq 1 \) and \( q_i = Q_i^{(a,b)} \forall a \neq b \geq 1 \). The RS free energy density is denoted \( f_{RS} \) and we remark that it does not match \( f \) if the system is replica symmetry breaking. Under the RS assumption, the \( S_i = S_i^{[1,2]} \mathbf{1}_u \mathbf{1}_u^{\top} + (S_i^{[1,1]} - S_i^{[1,2]}) I_u, \)

(30)

where \( \mathbf{1}_u \in \mathbb{R}^u \) is the vector of all-ones, and we may write the inner expectation in (27) as

\[ e^{-\frac{1}{\beta} \sum_{i=1}^u \langle u_i^{[a]} \rangle^{\top} u_i^{[b]} \mathcal{X}}. \]

(31)

Using Lemma 2 and taking the limit \( \tau \to 0^+ \) leads to

\[ \Xi_{\beta,M}^{(u)} = \int e^{-M G^{(u)}} \prod_{u=1}^u e^{-\beta \| x_i^{[0]} \|_1 + \| x_i^{[2]} \|_1} \ dx_i^{[0]} \ dx_i^{[2]}, \]

(32)

where \( G^{(u)} = \lim_{\tau \to 0^+} G^{(u)}_\tau \). The function \( G^{(u)}_\tau \) given in (33) at the top of the next page is implicitly a function of both \( S_1 \) and \( S_2 \). To obtain (33) we first used (45), then applied (39). Finally, some algebraic manipulations give the reported result.

The problem with the limit \( G^{(u)} = \lim_{\tau \to 0^+} G^{(u)}_\tau \) is that it diverges and the free energy density grows without bound which is an undesired result. To keep \( G^{(u)} \) and the free energy density finite as \( \tau \to 0^+ \), we pose the constraints

\[ S_1^{[1,1]} - S_1^{[1,2]} + u S_1^{[2,2]} = S_2^{[1,1]} - S_2^{[1,2]} + u S_2^{[2,2]}, \]

(34)

\[ S_1^{[1,1]} - S_1^{[1,2]} = S_2^{[1,1]} - S_2^{[1,2]}, \]

(35)

on the elements of the replica symmetric matrices \( S_1, S_2 \). Given (34) and (35) are satisfied, we get in the limit \( \tau \to 0^+ \) the expression for \( G^{(u)} = G_1^{(u)} + G_2^{(u)} \) in terms of

\[ G_i^{(u)} = \frac{1}{4} \log \left( q_i + u (r_i - 2 m_i + q_i) \right) \]

\[ + \frac{u-1}{4} \log (q_i - q_i), \quad i = 1, 2. \]

(36)

Comparing (36) to (7) eq. (A.4) reveals that the corresponding terms for rotationally invariant and bi-orthogonal \( D \) match up to vanishing constants. Furthermore, in the limit \( u \to 0 \) the equalities (34) and (35) are equivalent to the condition \( \chi_1 = \chi_2 \), where we denoted \( \chi_i = \beta (Q_i - q_i) \) for notational convenience. This provides the relevant constraint for the evaluation of the RS free energy, as stated in Section III-A.

The next task would be to average (32) over the correlations (29) using the theory of large deviations and saddle-point integration. But since the effect of the bi-orthogonal sensing matrix \( D \) has been reduced to the above constraint, we omit the calculations here due to space constraints. For details, see [7, Appendix A] and [18].

**Appendix B**

**Matrix Integration**

**Lemma 1.** Let \( O_1 \) and \( O_2 \) be independent and drawn uniformly according to the Haar measure on the group of all orthogonal \( M \times M \) matrices as in (2). Given vectors \( x_1, x_2 \in \mathbb{R}^M \), denote \( \| x_i \|^2 = M r_i \), for \( i = 1, 2 \). Then

\[ I_M(r_1, r_2; c) = \frac{\log I_M(r_1; r_2; c)}{2} = \log \left( \frac{1 + \sqrt{1 + 4c^2r_1r_2}}{2} \right) - \frac{1}{2} \]

\[ \approx \sqrt{c^2r_1r_2} - \log(c^2r_1r_2)/4, \quad \text{for } c^2 r_1 r_2 \gg 1. \]

(37)

(38)

(39)

**Proof:** Let \( u_i = O_i x_i \) where \( \{ x_i \}_{i=1}^2 \) are fixed and \( \{ O_i \}_{i=1}^2 \) independent and drawn uniformly according to the Haar measure on the group of all orthogonal \( M \times M \) matrices. Since \( \| u_i \|^2 = M r_i \) and \( O_i \) rotate the vectors \( u_i \) uniformly in all directions, \( u_i \) is uniformly distributed on the hyper-sphere at the boundaries of an \( M \) dimensional ball having radius \( R_i = \sqrt{M} r_i \), providing the second equality in (37).

To assess the second part of the lemma, the joint measure of \( (u_1, u_2) \) reads

\[ p(u_1; r_1) p(u_2; r_2) d u_1 d u_2, \text{ where } p(u; r) = Z(r)^{-1} \delta(\| u \|^2 - M). \]

(40)
\[
G_r^{(u)} = \frac{1}{2\pi} \left\{ \sqrt{S_1^{[1,1]} - S_1^{[1,2]} + uS_1^{[1,2]}} - \sqrt{S_2^{[1,1]} - S_2^{[1,2]} + uS_2^{[1,2]}} \right\}^2 + \frac{u - 1}{2\pi} \left( \sqrt{S_1^{[1,1]} - S_1^{[1,2]} - S_2^{[1,1]} - S_2^{[1,2]} + uS_2^{[1,2]}} \right)^2
+ \frac{1}{4} \log \left[ (S_1^{[1,1]} - S_1^{[1,2]} + uS_1^{[1,2]})(S_2^{[1,1]} - S_2^{[1,2]} + uS_2^{[1,2]}) \right],
\]

(33)

The normalization constant \(Z(r)\) in (40) is the volume of the hypersphere in which \(u\) is constrained to. Using Stirling’s formula for large \(M\), we get up to a vanishing term \(O(1/M)\)

\[
Z(r) = (2\pi e r)^{M/2}/\sqrt{\pi r}.
\]

(41)

With the help of Laplace transform, we write

\[
\delta(x-a) = \frac{1}{4\pi i} \int_{-i\infty}^{+i\infty} e^{\frac{s}{4}(x-a)} ds, \quad \gamma \in \mathbb{R},
\]

so that using (40) – (42), the latter expectation in (37) becomes

\[
\frac{(4\pi i)^{-2}}{\pi e^r M(r_1 r_2)^{M/2}} \int e^{\frac{s}{2}[r_1 - r_2 - 1]} \sum_{i=1}^{2} \prod_{i=1}^{2} d\nu_i d\nu_i
\]

(43)

where we used Gaussian integration to obtain (43). Since \(M \to \infty\), we next apply saddle-point integration to solve the integrals w.r.t \(s_1\) and \(s_2\). After canceling the vanishing terms,

\[
\lim_{M \to \infty} M^{-1} \log I_M(r_1, r_2; c) = -1 - \frac{1}{2} \sum_{i=1}^{2} \log r_i + \frac{1}{2} \sum_{s_1, s_2} \log(s_1 s_2 - c^2),
\]

(44)

and (38) follows by solving the extremization, and (39) by neglecting the terms that are of the order unity.

Lemma 2. Let \(\{O_i\}_{i=1}^{2} = \{\chi\}\) and \(\Delta x_i^{[a]}\) for \(i = 1, 2\) and \(a = 1, 2, \ldots, u\) as in (27). Then, under RS ansatz;

\[
\lim_{M \to \infty} M^{-1} \log E_{\Delta x_i^{[a]}} \left( e^{\sum_{h=1}^{2} (O_i)^{[a]} \Delta x_i^{[h]}} \right) | \chi \}
\]

\[
= F(S_1^{[1,1]} - S_1^{[1,2]} + uS_1^{[1,2]}, S_2^{[1,1]} - S_2^{[1,2]} + uS_2^{[1,2]}; c) + (u - 1) F(S_1^{[1,1]} - S_1^{[1,2]}, S_2^{[1,1]} - S_2^{[1,2]}; c),
\]

(45)

where \(c \in R \) and \(F(r_1, r_2; c)\) is given in (38).

Proof: Denote \(u_i^{[a]} = O_i \Delta x_i^{[a]}\) for all \(i = 1, 2\) and \(a = 1, \ldots, u\). Given \(\chi\), \(u_i^{[a]}\) lie on the surfaces of hyperspheres as in the proof of Lemma 1. The RS ansatz guarantees that \(u_i^{[a]}\) can be expressed as \(u_i^{[a]} = u_i^{[a]}[1] \hat{u}_i^{[a]} \cdots \hat{u}_i^{[a]} E\), where \(\{u_i^{[a]}\}\) is a set of vectors that satisfies \(M^{-1} \hat{u}_i^{[a]} \cdot \hat{u}_i^{[b]} = 0\) if \(a \neq b\) and

\[
1/M \hat{u}_i^{[a]} \cdot \hat{u}_i^{[b]} = \begin{cases} u_i^{[a]} + S_1^{[1,1]} - S_1^{[1,2]} & \text{if } a = b = 1; \\ u_i^{[a]} + S_1^{[1,1]} - S_1^{[1,2]} & \text{if } a = b \geq 2. \end{cases}
\]

(46)

The matrix \(E = \left[ u_i^{-1/2}e_2 \cdots e_u \right]\) provides an orthonormal basis that is independent of index \(i\). This indicates that the expectation in (45) can be assessed w.r.t. \(\{u_i^{[a]}\}\) instead of the original non-orthogonal set \(\{u_i^{[a]}\}\). The orthogonality allows us to independently evaluate the expectation for each replica index \(a\) when \(u \ll M\). Using Lemma 1 and (46) completes the proof.

Acknowledgment

This work was supported in part by the Swedish Research Council under VR Grant 621-2011-1024 and by grants from the ISPS (KAKENHI Nos. 2230003 and 22300098).

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