MAXIMIZING THE RATIO OF EIGENVALUES OF 
NON-HOMOGENEOUS PARTIALLY HINGED PLATES

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Abstract. We study the spectrum of non-homogeneous partially hinged plates having structural engineering applications. A possible way to prevent instability phenomena is to maximize the ratio between the frequencies of certain oscillating modes with respect to the density function of the plate; we prove existence of optimal densities and we investigate their analytic expression. This analysis suggests where to locate reinforcing material within the plate; some numerical experiments give further information and support the theoretical results.

1. Introduction

In recent years the trend in bridge design is to replace expensive experiments in wind tunnels with numerical tests; hopefully, these tests should be preceded by a suitable mathematical modelling and, possibly, by analytic arguments. In this respect, a possible way to model the deck of a suspension bridge or a footbridge is by means of a long narrow rectangular thin plate \( \Omega \subset \mathbb{R}^2 \) hinged at short edges and free on the remaining two, see problem (1.1) below.

When the wind comes up against the deck of the bridge, a form of dynamic instability arises, which appears as uncontrolled vortices and it is usually named flutter. The origin of asymmetric vortices generates a forcing lift which launches vertical oscillations of the deck; this phenomenon finds confirmation in wind tunnel tests, see e.g. [29]. In particular, a transition between these vertical oscillations to torsional ones may happen which, in some cases, leads to the collapse of the bridge; we refer to [24] Chapter 1 for a survey of historical events where this phenomenon occurred, among which the celebrated Tacoma Narrows Bridge collapse. Therefore, it becomes extremely important preventing flutter instability to provide a structure strong and safe. Rocard [32] suggested that for common bridge there exists a threshold of wind velocity \( V_c \) at which flutter arises. The computation of \( V_c \) is not an easy task, since it depends on the wind and on the geometric features of the deck; a possible way is to determine \( V_c \) experimentally. On the other hand, in engineering literature there exist some closed formulas for \( V_c \); even if the debate on these formulas is still open, it seems to be accordance in thinking that the critical velocity depends on the frequencies or, equivalently, on the eigenvalues of the normal modes of \( \Omega \), see [18, 26, 32]. More precisely, since \( V_c \) represents the critical threshold at which an energy transfer occurs between the \( j \)-th and the \( i \)-th mode of oscillation, most of the authors propose \( V_c \) directly proportional to the difference between the square of the corresponding eigenvalues \( \lambda_i > \lambda_j \), i.e.

\[
V_c \propto (\lambda_i^2 - \lambda_j^2).
\]

It follows that a way to increase the critical velocity \( V_c \), and in turn to prevent instability, is by increasing the distance between \( \lambda_i^2 \) and \( \lambda_j^2 \); this purpose is achievable moving the ratio \( (\lambda_i/\lambda_j)^2 \) away as much as possible from 1. A theoretical explanation of this fact was given in [10], within the classical stability theory of Mathieu equations, by relating large ratios of eigenvalues to the situation in which the instability resonant tongues of the Mathieu diagram become very thin.

In order to prevent dynamical instability, different strategies to optimize the design of the plate have been proposed in literature; for instance, one may modify its shape, see [6], or rearrange the materials composing it, see [7, 8]. Within the present research, we exploit the latter approach to maximize the

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ratio of selected eigenvalues of a partially hinged non-homogeneous plate. More precisely, by rescaling, we assume that the plate has length \( \pi \) and width \( 2\ell \) with \( 2\ell \ll \pi \) so that
\[
\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2;
\]
then we characterize the non-homogeneity of the plate by a density function \( p = p(x, y) \) and we consider the weighted eigenvalues problem:
\[
\begin{cases}
\Delta^2 u = \lambda p(x,y)u & \text{in } \Omega \\
u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell, \ell) \\
uyy(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = uy_y(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0, \pi).
\end{cases}
\]
The boundary conditions on short edges are of Navier type, see \[30\], and model the situation in which the deck of the bridge is hinged on \( (0, \pi) \times (-\ell, \ell) \). Instead, the boundary conditions on large edges are of Neumann type, see \[17, 31\], they model the fact that the deck is free to move vertically and involve the Poisson ratio \( \sigma \) which, for most of materials, satisfies \( \sigma \in (0,1/2) \). Finally, we focus on densities \( p \) satisfying some natural constraints, i.e. for \( \alpha, \beta \in (0, +\infty) \) with \( \alpha < \beta \) fixed, we assume that \( p \) belongs to the following class of weights
\[
P_{\alpha, \beta} := \left\{ p \in L^\infty(\Omega) : \alpha \leq p \leq \beta, \ p(x,y) = p(x,-y) \ a.e. \ in \ \Omega \right\}.
\]
The integral condition in \[1.2\] represents the preservation of the total mass of the plate, while the symmetry requirement on \( p \) means that we focus on designs which are symmetric with respect to the mid-line of the roadway. From a mathematical point of view, the symmetry of \( p \) produces two classes of eigenfunctions of \[1.1\], respectively, even or odd in the \( y \)-variable, that we named longitudinal and torsional modes. For what discussed above, in order to prevent the energy transfer from longitudinal to torsional modes, one may study the effect of the weight \( p \) on the ratio \( \nu(p)/\mu(p) \), being \( \nu \) and \( \mu \) two selected eigenvalues corresponding, respectively, to a torsional and a longitudinal mode. Since the final goal is to find the best rearrangement of materials in \( \Omega \) which maximizes this ratio, we study, either from a theoretical and a numerical point of view, the optimization problem:
\[
\mathcal{R} = \sup_{\nu(p) \in \mathcal{P}_{\alpha, \beta}} \frac{\nu(p)}{\mu(p)}.
\]
We refer to \[4\] for optimization results on the ratio of eigenvalues of second order operators subject to domain perturbations and to \[27\] for optimization results, with respect to the weight, in 1-dimensional domains; see also \[25\] Chapter 9 and references therein. In particular, in \[27\] the author studied a problem representing an inhomogeneous string with fixed endpoints; he proved that the best weight is of bang-bang type, namely a piecewise constant function, symmetric with respect to the middle of the string and getting the minimum value there. Unfortunately, the techniques exploited in \[27\] are closely related to the 1-dimensional nature of the problem treated and seem not applicable to our situation. Furthermore, here, things are complicated by dealing with a fourth order operator with non standard boundary conditions, for which no general positivity results are known. We refer the interested reader to \[8\] where a partial positivity property result was proved for the operator in \[1.1\].

As a consequence of what remarked, at the current state of art, a complete theoretical solution to problem \[1.3\] is difficult to reach and we proceed by steps. More precisely, we concentrate our efforts in looking for weights increasing \( \nu(p) \) or reducing \( \mu(p) \), separately. The numerical results we collect in Section \[4\] reveal that this apparently not rigorous approach turns out to be effective in increasing the ratio \[1.3\]: indeed, as a matter of fact, weights having strong effect on torsional eigenvalues \( \nu(p) \) produce very confined effects on longitudinal eigenvalues \( \mu(p) \), and vice versa. In this regard, the present paper can be seen as the prosecution of the research done in \[8\], where the goal was minimizing the first eigenvalue of \[1.1\], see Proposition \[3.3\] below. Nevertheless, it is well-known that supremum problems require different methods than infimum ones, hence problem \[1.3\] deserves to be studied independently. About the optimization of the first weighted eigenvalue of \( \Delta^2 \) under Dirichlet or Navier boundary conditions, we mention the papers \[2, 3, 15-21\]. Concerning higher eigenvalues we refer
coming back to problem (1.3), in order to increase its numerator, i.e. the first torsional eigenvalue, we adapt to our situation the approach developed by [20] in the second order case, and partially extended by [21] to optimize the first biharmonic eigenvalue under Navier or Dirichlet boundary conditions. The main novelty of our approach is the exploitation of the explicit information we have from [22] by [21] to optimize the first biharmonic eigenvalue under Navier or Dirichlet boundary conditions. The novelty of our approach is the exploitation of the explicit information we have from [22] by [21] to optimize the first biharmonic eigenvalue under Navier or Dirichlet boundary conditions. The main novelty of our approach is the exploitation of the explicit information we have from [22] by [21] to optimize the first biharmonic eigenvalue under Navier or Dirichlet boundary conditions. The main novelty of our approach is the exploitation of the explicit information we have from [22] by [21] to optimize the first biharmonic eigenvalue under Navier or Dirichlet boundary conditions.

From now onward we fix \( \Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2 \) with \( \ell > 0 \) and \( \sigma \in (0, 1/2) \). We denote by \( \| \cdot \|_q \) the norm related to the Lebesgue spaces \( L^q(\Omega) \) with \( 1 \leq q \leq \infty \) and we shall omit the set \( \Omega \) in the notation of the functional spaces, e.g. \( V := V(\Omega) \). The natural functional space where to set problem (1.1) is

\[
H^2_x = \{ u \in H^2 : u = 0 \text{ on } (0, \pi) \times (-\ell, \ell) \}.
\]

Note that the condition \( u = 0 \) above has to be meant in a classical sense because \( \Omega \) is a planar domain and the energy space \( H^2_x \) embeds into continuous functions. Furthermore, \( H^2_x \) is a Hilbert space when endowed with the scalar product

\[
(u, v)_{H^2_x} := \int_{\Omega} [\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})] \, dx \, dy
\]

and associated norm

\[
\|u\|^2_{H^2_x} = (u, u)_{H^2_x},
\]

which is equivalent to the usual norm in \( H^2 \), see [22] Lemma 4.1. Then, we reformulate problem (1.1) in the following weak sense

\[
(2.1) \quad (u, v)_{H^2_x} = \lambda \int_{\Omega} p(x, y) uv \, dx \, dy \quad \forall v \in H^2_x,
\]

where \( p \) belongs to the family of weights \( P_{\alpha, \beta} \) defined in (1.2) with \( \alpha, \beta \in (0, +\infty) \) and \( \alpha < \beta \) fixed. We underline that condition \( p \in P_{\alpha, \beta} \) implies \( \alpha \leq p \leq \beta \) since \( \int_{\Omega} p \, dx \, dy = |\Omega| \). Moreover, it is not restrictive to assume \( \alpha < 1 < \beta \) when we consider weights that do not coincide a.e. with the constant function \( p \equiv 1 \). In fact, if we assume \( \beta = 1 \), it must be \( p = 1 \) a.e. in \( \Omega \), since otherwise we would have \( \int_{\Omega} p \, dx \, dy < |\Omega| \); similarly, if we put \( \alpha = 1 \). For these reasons, since the aim of our research is to study the effect of a non-constant weight on the eigenvalues of (1.1), in what follows we will always assume

\[
0 < \alpha < 1 < \beta.
\]

The bilinear form \((u, v)_{H^2_x}\) is continuous and coercive and \( p \in L^\infty \) is positive a.e. in \( \Omega \), by standard spectral theory of self-adjoint operators we infer
Proposition 2.1. Let \( p \in P_{\alpha, \beta} \). Then all eigenvalues of \( (1.1) \) have finite multiplicity and can be represented by means of an increasing and divergent sequence \( \lambda_h(p) \) \((h \in \mathbb{N}_+)\), where each eigenvalue is repeated according to its multiplicity. Furthermore, the corresponding eigenfunctions form a complete system in \( H^2_\omega \).

We refer to [28, Lemma 2.1] for a detailed proof of Proposition 2.1 in a more general setting. On the other hand, it is well-known, see [19, 25], that the following variational representation of eigenvalues holds for every \( h \in \mathbb{N}_+ \):

\[
\lambda_h(p) = \inf_{W_h \subset H^2_\omega} \sup_{\dim W_h = h} \frac{\|u\|_{H^2_\omega}^2}{\|\sqrt{\nu}u\|_2^2}.
\]

When \( h = 1 \), \( (2.2) \) includes the well known characterization for the first eigenvalue

\[
\lambda_1(p) = \inf_{u \in H^2_\omega \setminus \{0\}} \frac{\|u\|_{H^2_\omega}^2}{\|\sqrt{\nu}u\|_2^2}.
\]

If \( h \geq 2 \) the minimum in \( (2.2) \) is achieved by the space \( W_h \) spanned by the \( h \)-th first eigenfunctions. Assuming that the first \( h-1 \) eigenfunctions, \( u_1, u_2, \ldots, u_{h-1} \) are known, one also obtains

\[
\lambda_h(p) = \inf_{u \in H^2_\omega \setminus \{0\}} \frac{\|u\|_{H^2_\omega}^2}{\|\sqrt{\nu}u\|_2^2},
\]

\((u,v)_{H^2} = 0 \ \forall i = 1, \ldots, h-1\)

We recall that when \( p = 1 \) the whole spectrum of \( (1.1) \) was determined in [22] (see also [6]); we collect what known in Proposition 2.2 below. First, for \( m, k \in \mathbb{N}_+ \), we need to define the functions:

\[
\phi_{m,1}(y) := \frac{1}{N_{m,1}} \left\{ \frac{\sigma m^2 - c_{m,1}}{\cosh(\ell \tau_{m,1})} \cosh(y \tau_{m,1}) + \frac{\tilde{e}_{m,1} - \sigma m^2}{\cosh(\ell \tau_{m,1})} \cosh(y c_{m,1}) \right\},
\]
\[
\phi_{m,k}(y) := \frac{1}{N_{m,k}} \left\{ \frac{\sigma m^2 + c_{m,k}}{\cosh(\ell \tau_{m,k})} \cosh(y \tau_{m,k}) + \frac{\tilde{e}_{m,k} - \sigma m^2}{\cosh(\ell \tau_{m,k})} \cos(y c_{m,k}) \right\},
\]
\[
\psi_{m,k}(y) := \frac{1}{N_{m,k}} \left\{ \frac{\sigma m^2 + d_{m,k}}{\sinh(\ell \vartheta_{m,k})} \sinh(y \vartheta_{m,k}) + \frac{\tilde{d}_{m,k} - \sigma m^2}{\sinh(\ell \vartheta_{m,k})} \sin(y d_{m,k}) \right\},
\]
\[
\psi_{m,1}(y) := \frac{1}{N_{m,1}} \left\{ \frac{\sigma m^2 - d_{m,1}}{\sinh(\ell \vartheta_{m,1})} \sinh(y \vartheta_{m,1}) + \frac{\tilde{d}_{m,1} - \sigma m^2}{\sinh(\ell \vartheta_{m,1})} \sin(y d_{m,1}) \right\},
\]

where

\[
c_{m,k} := \sqrt{\left| (\Lambda_{m,k} \right|^{1/2} - m^2 \right|}, \quad \tau_{m,k} := \sqrt{\left| (\Lambda_{m,k} \right|^{1/2} + m^2 \right|},
\]
\[
d_{m,k} := \sqrt{\left| (\Lambda_{m,k} \right|^{1/2} - m^2 \right|}, \quad \vartheta_{m,k} := \sqrt{\left| (\Lambda_{m,k} \right|^{1/2} + m^2 \right|}
\]

and the constants \( N_{m,k} \) and \( N_{m,1} \) are fixed in such a way that \( \|\phi_{m,k}(y)\sin(mx)\|_2 = \|\psi_{m,k}(y)\sin(mx)\|_2 \) = 1. Finally, we have

Proposition 2.2. [22] Let the functions \( \phi_{m,k}(y) \) and \( \psi_{m,k}(y) \) be defined in (2.5). Consider problem (1.1) with \( p = 1 \), there holds:

(i) for any \( m \geq 1 \), there exists a unique eigenvalue \( \lambda = \Lambda_{m,1} \in ((1 - \sigma^2)m^4, m^4) \) with corresponding eigenfunction \( \phi_{m,1}(y)\sin(mx) \);

(ii) for any \( m \geq 1 \) and any \( k \geq 2 \) there exists a unique eigenvalue \( \lambda = \Lambda_{m,k} > m^4 \) satisfying \( \left( m^2 + \frac{\sigma^2}{\ell^2} \right) (k - \frac{3}{2})^2 \right| < \mu_{m,k} < \left( m^2 + \frac{\sigma^2}{\ell^2} (k - 1)^2 \right) \), with corresponding eigenfunction \( \phi_{m,k}(y)\sin(mx) \);

(iii) for any \( m \geq 1 \) and any \( k \geq 2 \) there exists a unique eigenvalue \( \lambda = \Lambda_{m,k}^* > m^4 \) with corresponding eigenfunctions \( \psi_{m,k}(y)\sin(mx) \);
we define longitudinal eigenfunctions and results tell that these eigenvalues are simple for usual choices of \( \ell \) that they not appear in realistic situations, we devote our analysis to “low” eigenvalues; our numerical and torsional eigenfunctions. However, since “high” modes are activated by bending energy so large eigenvalues is not know, hence, in principle, the same eigenvalue may correspond either to longitudinal

For actual bridges one usually has \( \nu \) eigenvalues. First, for \( \mu \) and \( \nu \) sequences of the eigenvalues \( \nu \) the sequence of the eigenvalues \( \mu \) readily gives the stability inequality

\[
\frac{\lambda_h(1)}{\beta} \leq \lambda_h(p) \leq \frac{\lambda_h(1)}{\alpha},
\]

for every \( h \in \mathbb{N}_+ \). In applicative terms, if we choose materials having close densities, we obtain eigenvalues close to those of the homogeneous plate.

By Proposition 2.2 we distinguish two classes of eigenfunctions of problem (1.1) with \( p \equiv 1 \):

- \( y \)-even eigenfunctions \( \phi_{m,k}(y) \sin(mx) \) corresponding to the eigenvalues \( \Lambda_{m,k} \);
- \( y \)-odd eigenfunctions \( \psi_{m,k}(y) \sin(mx) \) corresponding to the eigenvalues \( \Lambda^{m,k} \).

As in [11], this suggests to introduce the subspaces of \( H^2 \):

\[
H^2_\xi = \{ u \in H^2 : u(x,-y) = u(x,y) \quad \forall (x,y) \in \Omega \},
\]

\[
H^2_\sigma = \{ u \in H^2 : u(x,-y) = -u(x,y) \quad \forall (x,y) \in \Omega \},
\]

where

\[
H^2_\xi \perp H^2_\sigma, \quad H^2 = H^2_\xi \oplus H^2_\sigma.
\]

By the symmetry assumption on \( p \in P_{\alpha,\beta} \) it is readily verified that all linearly independent eigenfunctions of (1.1) may be thought in the class \( H^2_\xi \) or in the class \( H^2_\sigma \). We call the eigenfunctions belonging to \( H^2_\xi \) longitudinal modes and those belonging to \( H^2_\sigma \) torsional modes. In what follows we order all eigenvalues of (1.1), eventually repeated if not simple, into two increasing and divergent sequences: the sequence of the eigenvalues \( \mu_j(p) \) \( (j \in \mathbb{N}_+) \) corresponding to longitudinal eigenfunctions and the sequence of the eigenvalues \( \nu_j(p) \) \( (j \in \mathbb{N}_+) \) corresponding to torsional eigenfunctions. From Proposition 2.2 we infer that the sequences \( \mu_j(1) \) and \( \nu_j(1) \) can be written explicitly by ordering, respectively, the numbers \( \Lambda_{m,k} \) and \( \Lambda^{m,k} \). In particular, we have

\[
\lambda_1(1) = \mu_1(1) = \Lambda_{1,1} < \nu_1(1) = \min\{\Lambda_{1,1}^1, \Lambda_{1,2}^1\}.
\]

For actual bridges one usually has \( \nu_1(1) = \Lambda_{1,2}^1 \), indeed the inequality required in Proposition 2.2 (iv) is not satisfied for \( \ell \) small, see Table 1 in Section 4. We note that, even in the case \( p \equiv 1 \), simplicity of eigenvalues is not know, hence, in principle, the same eigenvalue may correspond either to longitudinal and torsional eigenfunctions. However, since “high” modes are activated by bending energy so large that they not appear in realistic situations, we devote our analysis to “low” eigenvalues; our numerical results tell that these eigenvalues are simple for usual choices of \( \ell \ll \pi \) and \( \sigma \in (0, 1/2) \).

For future purposes it is convenient to characterize in a variational way also longitudinal and torsional eigenvalues. First, for \( j \in \mathbb{N}_+ \) fixed, we introduce, respectively, the spaces \( U^\xi_j \subset H^2_\xi \) of the first \( (j-1) \) longitudinal eigenfunctions and \( U^\sigma_j \subset H^2_\sigma \) of the first \( (j-1) \) torsional eigenfunctions of (1.1). Then we define

\[
V^\xi_j := \{ u \in H^2_\xi : (u,v)_{H^2} = 0 \quad \forall v \in U^\xi_j \}, \quad V^\sigma_j := \{ u \in H^2_\sigma : (u,v)_{H^2} = 0 \quad \forall v \in U^\sigma_j \},
\]
where if \( j = 1 \) we mean \( V_1^\varepsilon = H_2^\varepsilon \) and \( V_1^O = H_2^O \). Finally, using (2.4), we set

\[
\mu_j(p) = \inf_{u \in V_j^\varepsilon \setminus \{0\}} \frac{\|u\|_{H_2}^2}{\|\sqrt{p}u\|_2^2} \quad \text{and} \quad \nu_j(p) = \inf_{u \in V_j^O \setminus \{0\}} \frac{\|u\|_{H_2}^2}{\|\sqrt{p}u\|_2^2}.
\]

3. Main results

As in Section 2 we always assume

\[ 0 < \sigma < \frac{1}{2} \quad \text{and} \quad \alpha < 1 < \beta \quad (\alpha, \beta \in (0, +\infty)). \]

The final goal of our analysis is to maximize the ratio (1.3) with the family \( P_{\alpha, \beta} \) defined in (1.2). To this aim we need first to clarify which eigenvalues we shall consider in the ratio; the model situation we have in mind is a motion concentrated on a longitudinal mode, with corresponding eigenvalue \( \mu_j \) and we want to prevent the transfer of energy from this mode to the nearest torsional one \( \nu_i \), for suitable \( i, j \in \mathbb{N}_+ \).

Rocard [32, p.169] claims that, for the usual design of bridges, the eigenvalues of the observed longitudinal oscillating modes are larger than those of torsional modes, i.e. \( \mu_j < \nu_i \). For the homogeneous plate this inequality readily follows from (2.7) if \( j = i = 1 \). More in general, we set

\[
j_0 := \max\{j \in \mathbb{N}_+ : \nu_1(1) - \mu_j(1) > 0\}.
\]

Clearly, \( j_0 \geq 1 \) and \( j_0 = j_0(\ell, \sigma) \). Note that in our numerical experiments, for several values of \( \ell \) and \( \sigma \), chosen taking into account real bridges, we always obtain \( j_0 = 10 \). As explained in [9, Section 1] this number is in accordance with what reported in the Federal Report [11], since a moment before the collapse of the Tacoma Narrows Bridge the motion was involving nine or ten longitudinal waves.

Coming back to the choice of the eigenvalues in the ratio (1.3), for what observed, we finally focus on the problem

\[
\mathcal{R} = \sup_{p \in P_{\alpha, \beta}} \frac{\nu_1(p)}{\mu_{j_0}(p)}.
\]

Note that if \( j_0 > 1 \), then \( \nu_1(1)/\mu_{j_0}(p) < \nu_1(p)/\mu_j(p) \) for all \( 1 \leq j < j_0 \); therefore weights \( p \) increasing the value of \( \nu_1(p)/\mu_j(p) \) also increase the value of \( \nu_1(p)/\mu_j(p) \) for all \( 1 \leq j < j_0 \).

First we prove

**Theorem 3.1.** Let \( j_0 \in \mathbb{N}_+ \) be as defined in (3.1). Then, problem (3.2) admits a solution.

As already explained in the introduction, a precise characterization of maximizers to problem (3.2) seems hard to reach at the current state of studies. For this reason, we concentrate our efforts in looking for weights increasing \( \nu_1(p) \) or reducing \( \mu_{j_0}(p) \), separately. Note that, in the following we will always indicate with \( \chi_D \) the characteristic function of a set \( D \subset \mathbb{R}^2 \). We start by facing the problem

\[

\nu_1^{\alpha, \beta} := \sup_{p \in P_{\alpha, \beta}} \nu_1(p),
\]

where \( \nu_1(p) \) is defined in (2.8) taking \( j = 1 \). We call optimal pair for (3.3) a couple \((\rho, \tilde{u})\) such that \( \rho \) achieves the supremum in (3.3) and \( \tilde{u} \) is an eigenfunction of \( \nu_1(\rho) \). In Section 5 we prove

**Theorem 3.2.** Problem (3.3) admits an optimal pair \((\rho, \tilde{u}) \in P_{\alpha, \beta} \times H_2^O \). Furthermore, \( \tilde{u} \) and \( \rho \) are related as follows

\[

\rho(x, y) = \beta \chi_{\Omega}(x, y) + \alpha \chi_{\Omega}(x, y) \quad \text{for a.e. } (x, y) \in \Omega,
\]

where \( \hat{S} = \{(x, y) \in \Omega : \hat{u}^2(x, y) \leq \ell \} \) for some \( \ell > 0 \) such that \( |\hat{S}| = \frac{1 - \alpha}{\beta - \alpha} |\Omega| \).

Next we focus on longitudinal eigenvalues. For \( j \in \mathbb{N}_+ \), we set the minimum problem

\[

\mu_j^{\alpha, \beta} := \inf_{p \in P_{\alpha, \beta}} \mu_j(p),
\]

where \( \mu_j(p) \) is as defined in (2.8). We call optimal pair for (3.4) a couple \((\sigma_j, \pi_j)\) such that \( \sigma_j \) achieves the infimum in (3.4) and \( \pi_j \) is an eigenfunction of \( \mu_j(\sigma_j) \). When \( j = 1 \) the counterpart of Theorem
Proposition 3.3. [8] Set \( j = 1 \), then problem \((3.4)\) admits an optimal pair \((\overline{u}_1, \overline{v}_1) \in P_{\alpha, \beta} \times H^2_\Omega\). Furthermore, \(\overline{u}_1\) and \(\overline{v}_1\) are related as follows
\[
\overline{v}_1(x, y) = \alpha \chi_{S_1}(x, y) + \beta \chi_{\overline{\Omega}} S_1(x, y) \quad \text{for a.e. } (x, y) \in \Omega,
\]
where \(S_1 = \{(x, y) \in \Omega : \overline{v}_1^3(x, y) \leq t_1 \}\) for some \(t_1 > 0\) such that \(|S_1| = \frac{\beta - 1}{\beta - \alpha} |\Omega|\).

Things become more involved for higher longitudinal eigenvalues. Indeed, the proofs of Theorem 3.2 and Proposition 3.3 are based on suitable rearrangement inequalities, see Lemma 5.4 below, involving \(\hat{u}\) and \(\overline{u}_1\), respectively; this approach does not carry over to the case \(j \geq 2\) since, in general, the orthogonality condition in the sets \(V_j^2\) of (2.8) is not preserved when changing weights. For this reason, we proceed differently and we lower \(\mu_j(p)\) “indirectly.” More precisely, we first derive upper bounds for \(\mu_j(p)\), where the eigenfunctions \(\overline{u}_j\) are, in some sense, replaced by functions properly chosen in \(H^2_\pi\); then we look for weights effective in lowering the upper bounds and, in turn, \(\mu_j(p)\).

For \(j \geq 2\) fixed and \(m = 1, \ldots, j\), we introduce the following functions having disjoint supports
\[
w_m(x, y) := \begin{cases} 
\sin^2(jx) & \text{if } (x, y) \in \Omega_m^j \\
0 & \text{if } (x, y) \in \Omega \setminus \Omega_m^j,
\end{cases}
\]
where \(\Omega_m^j := \left(\frac{(m-1)\pi}{j}, \frac{m\pi}{j}\right) \times (-\ell, \ell) \subset \mathbb{R}^2\); it is readily checked that \(w_m \in C^1(\Omega) \cap H^2_\pi\) for all \(m = 1, \ldots, j\). Then, we prove:

Theorem 3.4. Let \(j \geq 2\), then problem \((3.4)\) admits an optimal pair \((\overline{u}_j, \overline{v}_j) \in P_{\alpha, \beta} \times H^2_\Omega\) and there holds
\[
\mu_j^{\alpha, \beta} \leq \inf_{p \in P_{\alpha, \beta}} \left\{ \max_{m = 1, \ldots, j} \left\{ \frac{1}{\sqrt{p} \|w_m\|_2^2} \right\} \right\} j^3 |\Omega|.
\]
In particular, if \(P_{\alpha, \beta}^{\per} := \{p \in P_{\alpha, \beta} : p(x, y) = p(x + \frac{\pi}{j}, y)\}, \) for a.e. \((x, y) \in \Omega\) we have
\[
\mu_j^{\alpha, \beta} \leq \inf_{p \in P_{\alpha, \beta}^{\per}} \left\{ \frac{1}{\sqrt{p} \sin^2(jx) \|j\|_2^2} \right\} j^4 |\Omega|
\]
and the latter infimum is achieved by the functions
\[
p_j(x, y) = \alpha \chi_{S_j}(x, y) + \beta \chi_{\overline{\Omega}} S_j(x, y) \quad \text{for a.e. } (x, y) \in \Omega,
\]
where \(S_j = \{(x, y) \in \Omega : \sin^2(jx) \leq t_j \}\) for \(t_j > 0\) such that \(|S_j| = \frac{\beta - 1}{\beta - \alpha} |\Omega|\).

Remark 3.5. A comment on the choice of the functions \(w_m\) is in order. The idea of taking functions \(\pi/j\)-periodic in the \(x\)-variable comes from the explicit form of the longitudinal eigenfunctions of Proposition 2.2; slightly changes in the analytic expression of functions \(w_m\) will qualitatively produce the same weights \(p_j\), e.g. replacing \(\sin^2(jx)\) with \(\sin^{2n}(jx)\) \((n \geq 2\) integer) or \(\exp \left[-1/(1 - |x|^2)\right]\) properly rescaled and shifted in each \(\Omega_m\). We underline that there is accordance between the optimal weights found numerically in Section 4 and the weights \(p_j\) of Theorem 3.4.

We observe that, while the sets \(\hat{S}\) and \(S_1\) of Theorem 3.2 and Proposition 3.3 depend on the unknown functions \(\hat{u}\) and \(\overline{u}_1\), the set \(S_j\) of Theorem 3.4 is explicitly given once determined \(t_j > 0\). As a matter of example, in Figure 4 we plot the function \(z = \sin^2(5x)\), the corresponding set \(S_5\) and the related weight \(p_5(x, y)\). To conclude, it is worth noting that the statement of Theorem 3.2 combines nicely with those of Proposition 3.3 and Theorem 3.4 in increasing the ratio in (3.2). This is highlighted by the numerical experiments we collect in Section 4.
4. Numerical results

In the previous section we proved that an optimal weight maximizing the ratio $\nu_1(p)/\mu_{j_0}(p)$, with $j_0$ defined in (3.1), exists. Then, in order to find information on its analytic expression, we decided to minimize $\mu_{j_0}(p)$ or maximize $\nu_1(p)$, separately. All the theoretical results obtained tell that the optimal weights $p \in P_{\alpha,\beta}$, either for problem (3.3) and (3.4), must be of the bang-bang type, i.e.

$$p(x,y) = \alpha \chi_S(x,y) + \beta \chi_{\Omega \setminus S}(x,y)$$

for a.e. $(x,y) \in \Omega$.

for a suitable set $S \subset \Omega$. In other words, the plate must be composed by two different materials properly located in $\Omega$; this is useful in engineering terms, since the assemblage of two materials with constant density is simpler than the manufacturing of a material having variable density. Unfortunately, the above mentioned theoretical results, i.e. Theorem (3.2) and Proposition (3.3), give no precise information on the location of the set $S$; nevertheless, through suitable numerical experiments, we are able to suggest what could be the optimal design of the set $S$, in problems (3.3) and (3.4), and to guess a possible maximizer to problem (3.2). Note that we always suppose to strengthen the plate with steel and we consider the weaken material composed by a mixture of steel and concrete, so that, in approximate way, the denser material has triple density with respect to the weaken, i.e. $\beta = 3\alpha$.

4.1. Eigenvalues computation. We propose a numerical method to find approximate solutions of (1.1) which relies on the explicit information we have from Proposition 2.2 about the case $p \equiv 1$. Indeed, we expand the solutions $u$ of (1.1) in Fourier series, adopting the orthonormal basis of $L^2$ made of eigenfunctions of the homogeneous plate. More precisely, denoting by $z_m(x,y) \in H^2_2$ and $\theta_m(x,y) \in H^2_2$, respectively, the (ordered) longitudinal and torsional eigenfunctions of problem (1.1) with $p \equiv 1$, $u$ writes

$$u(x,y) = \sum_{m=1}^{\infty} [a_m z_m(x,y) + b_m \theta_m(x,y)] ,$$

for suitable $a_m, b_m \in \mathbb{R}$. In order to get a numerical approximation, we trunk the series in (4.1) at $N \in \mathbb{N}_+$ and we plug the Fourier sum into (2.1). We recall that, for all $m \in \mathbb{N}_+$, $z_m$ and $\theta_m$ solve:

$$\begin{align*}
(z_m, v)_{H^2_2} &= \mu_m(1) (z_m, v)_2 \quad \forall v \in H^2_2 \\
(\theta_m, v)_{H^2_2} &= \nu_m(1) (\theta_m, v)_2 \quad \forall v \in H^2_2,
\end{align*}$$
where \( \mu_m(1) \) and \( \nu_n(1) \) are defined in (2.8) with \( p = 1 \). Therefore, we obtain the following finite dimensional linear system in the unknowns \( a_n \) and \( b_n \):

\[
\begin{align*}
\left\{ \begin{array}{ll}
a_n \mu_n(1) &= \mu(p) \sum_{m=1}^{N} a_m C_{n,m}^p, \\
b_n \nu_n(1) &= \nu(p) \sum_{m=1}^{N} b_m \mathcal{C}_{n,m}^p,
\end{array} \right. \\
&\text{for } n = 1, \ldots, N
\end{align*}
\]  

(4.2)

where

\[
C_{n,m}^p := \int_{\Omega} p(x, y) z_n(x, y) z_m(x, y) \, dx \, dy
\]

\[
\mathcal{C}_{n,m}^p := \int_{\Omega} p(x, y) \theta_n(x, y) \theta_m(x, y) \, dx \, dy.
\]

In particular, by solving (4.2), it is possible to determine \( N \) approximated longitudinal eigenvalues \( \mu_n(p) \) and \( N \) torsional eigenvalues \( \nu_n(p) \). We observe that the decoupling between the unknowns \( a_n \) and \( b_n \), which produces eigenfunctions even or odd in \( y \), is due to the assumption on \( p \in F_{\alpha,\beta} \), being \( y \)-even.

In order to compute numerically the eigenvalues \( \mu_n(p) \) and \( \nu_n(p) \) for suitable choices of the weight \( p \), we fix from now onward:

\[
\sigma = 0.2 \quad \text{and} \quad \ell = \frac{\pi}{150},
\]

which is a choice consistent with common bridge design. The explicit values of \( \mu_n(1) \) and \( \nu_n(1) \) are computed by exploiting Proposition 2.2 see Table 1. When condition (4.3) holds, we numerically find that the eigenvalues \( \Lambda_{m,1} \) do exist for \( 1 \leq m \leq 2734 \) and that:

\[
\begin{align*}
\mu_m(1) &= \Lambda_{m,1} \quad \text{for} \quad m = 1, \ldots, 113, \\
\nu_m(1) &= \Lambda_{m,2} \quad \text{for} \quad m = 1, \ldots, 174.
\end{align*}
\]

Hence, by Proposition 2.2 we know that the basis of eigenfunctions exploited in (4.1) writes \( z_m(x, y) = \phi_{m,1}(y) \sin(mx) \) and \( \theta_m(x, y) = \psi_{m,2}(y) \sin(mx) \) for \( 1 \leq m \leq 113 \).

4.2. Numerical solution of (3.4). Let us begin by minimizing \( \mu_1(p) \), i.e. the first longitudinal eigenvalue of problem (1.1) as characterized in (2.8) with \( j = 1 \). In order to find the optimal weight given by Proposition 3.3, we adopt a numerical algorithm proposed in [14], based on the variational characterization of the eigenvalues. The numerical algorithm develops through the following passages: we solve numerically (4.2) with a certain weight \( p(i) \) and we determine the corresponding eigenfunction \( u_1^{(i)} \). Then, we find the weight at the next iteration \( p^{(i+1)} \) such that

\[
\| \sqrt{p^{(i+1)}} u_1^{(i)} \|_2 \geq \| \sqrt{p^{(i)}} u_1^{(i)} \|_2,
\]

in order to have

\[
\mu_1^{(i+1)} = \min_{u \in H_2^2 \setminus \{0\}} \frac{\|u\|_{H_2^2}^2}{\| \sqrt{p^{(i+1)}} u \|_2^2} = \frac{\|u_1^{(i+1)}\|_{H_2^2}^2}{\| \sqrt{p^{(i+1)}} u_1^{(i+1)} \|_2^2} \leq \frac{\|u_1^{(i)}\|_{H_2^2}^2}{\| \sqrt{p^{(i)}} u_1^{(i)} \|_2^2} = \mu_1^{(i)}.
\]

Iterating, we obtain a decreasing sequence of eigenvalues; since the infimum in (3.4) is achieved, the sequence is bounded from below so that is convergent. We stop the algorithm when \( |\mu_1^{(i+1)} - \mu_1^{(i)}| < \epsilon \), with \( \epsilon = 10^{-4} \div 10^{-3} \). As pointed out in [13] it is not clear a priori if the sequence converges to \( \mu_{\alpha,\beta} \) or not; to avoid the latter case we repeated the procedure considering different weights at the first iteration and we always obtain the convergence to the same values.

In Figure 2 we plot the set \( S_1 \) of Proposition 3.3 for the obtained optimal weight; the direction is to concentrate the denser material near the maximum of \( u_1^2(x, y) \). Since the set \( \Omega \setminus S_1 \) is similar to a rectangle, we propose the following approximated optimal weight for \( \mu_{\alpha,\beta} \):
| \( m \) | \( \mu_m(1) = \Lambda_m,1 \) | \( \nu_m(1) = \Lambda_m,2 \) |
|---|---|---|
| 1 | 9.6001·10^{-1} | 1.0944·10^4 |
| 2 | 1.5361·10^1 | 4.3786·10^4 |
| 3 | 7.7676·10^1 | 9.8561·10^4 |
| 4 | 2.4580·10^2 | 1.7532·10^5 |
| 5 | 6.015·10^2 | 2.7416·10^5 |
| 6 | 1.2466·10^3 | 3.9516·10^5 |
| 7 | 9.8561·10^3 | 5.3845·10^5 |
| 8 | 1.7532·10^5 | 7.0419·10^5 |
| 9 | 2.7416·10^5 | 8.9254·10^5 |
| 10 | 3.9516·10^5 | 1.1037·10^6 |
| 11 | 1.7532·10^6 | 1.3379·10^6 |
| 12 | 2.7416·10^6 | 1.5953·10^6 |

**Table 1.** On the left the lowest longitudinal eigenvalues \( \mu_m(1) \) and on the right the lowest torsional eigenvalues \( \nu_m(1) \) of (1.1) with \( p \equiv 1 \).

\[
\mathcal{P}_1(x,y) = \mathcal{P}_1(x) := \beta \chi I_1(x) + \alpha \chi(0,\pi)\chi I_1(x) \quad \text{for a.e. } (x,y) \in \Omega,
\]

where \( I_1 := \left( \frac{\pi}{2} - \frac{\pi}{2} \frac{(1-\alpha)}{(\beta-\alpha)} ; \frac{\pi}{2} + \frac{\pi}{2} \frac{(1-\alpha)}{(\beta-\alpha)} \right) \).

**Figure 2.** Plot of the set \( S_1 \), related to the optimal weight \( \mathcal{P}_1 \), and \( z = \overline{u}_1^2(x,y) \) intersected with the plane \( z = t_1 \) (\( \alpha = 0.5, \beta = 1.5, N = 20 \)).

The previous algorithm can be adapted to determine \( \mu_j(\mathcal{P}_j) \) for generic \( j \in \mathbb{N}_+ \); in particular, if \( j > 1 \) we apply the characterization (2.4) of eigenvalues, i.e. we consider the minimum onto the space \( V_j^E \) instead of \( H_j^2 \). The further difficulty is that, now, at the end of every iteration, we have to check that \( u_j^{(i)}(x,y) \) does not belong to the subspace spanned by \( \{ u_1^{(i+1)}, \ldots, u_{j-1}^{(i+1)} \} \), for more details see [14]. For each \( j \in \mathbb{N}_+ \), the obtained optimal weight has the denser material concentrated near to the peaks of the associated eigenfunction which are, approximatively, located at \( \frac{\pi}{2j}(2h-1) \) with \( h = 1, ..., j \); this is aligned with the statement of Theorem 3.4. Therefore, we propose the following **approximated optimal weight** for \( \mu_j^{\alpha,\beta} \):

\[
(4.4) \quad \mathcal{P}_j(x,y) = \mathcal{P}_j(x) := \beta \chi I_j(x) + \alpha \chi(0,\pi)\chi I_j(x), \quad \text{for a.e. } (x,y) \in \Omega,
\]
where \( I_j := \bigcup_{h=1}^{j} \left( \frac{\pi}{2j} (2h-1) - \frac{\pi}{j} \frac{(1-\alpha)}{2(\beta-\alpha)}, \frac{\pi}{2j} (2h-1) + \frac{\pi}{j} \frac{(1-\alpha)}{2(\beta-\alpha)} \right). \)

The numerical results show that the optimal weight changes if we change \( j \). Nevertheless, numerically, we observe that the weight \( \tilde{p}_j(x) \) in (4.4) reduces not only \( \mu_1(\tilde{p}_j) \), but also all the previous longitudinal eigenvalues \( \mu_i(\tilde{p}_j) \) with \( 1 \leq i < j \); it increases \( \mu_i(\tilde{p}_j) \) with \( i > j \). This means that, if it were possible to predict the highest mode of vibration for a plate during its design, then there would be an optimal reinforce for it, reducing at the same time all the previous ones.

### 4.3. Numerical solution to (3.3).

About the maximization of the first torsional eigenvalue, we cannot adapt the algorithm in [14], since it is not an infimum problem. Nevertheless, Theorem 3.2 suggests to put the denser material in the region \( \hat{S} = \{(x,y) \in \Omega : \hat{u}^2(x,y) \leq \hat{t}\} \) for some \( \hat{t} > 0 \), where \( \hat{u} \) is the eigenfunction corresponding to \( \nu_1(\hat{p}) \). Since we do not know explicitly \( \hat{u} \), we proceed by trial and error; we start by replacing \( \hat{u} \) with the first torsional eigenfunction \( \theta_1(x,y) = \psi_{1,2}(y) \sin(x) \) of problem (1.1) with \( p \equiv 1 \) and we define \( S^* := \{(x,y) \in \Omega : \theta_1^2(x,y) \leq t^* \} \), for \( t^* > 0 \) such that \( |S^*| = \frac{1-\alpha}{\beta-\alpha} |\Omega| \), and \( p^*(x,y) := \beta \chi_{S^*}(x,y) + \alpha \chi_{\Omega \setminus S^*}(x,y) \). We proceed by solving (1.1) with \( p^*(x,y) \), obtaining a new first torsional eigenfunction \( u^*(x,y) \) and a new weight \( p^{**}(x,y) \) in a similar manner; iterating the procedure, we always obtain weights very close to \( p^*(x,y) \), so that we conjecture that the theoretical optimal weight \( \hat{p} \) is qualitatively very similar to \( p^* \). In Figure 3 we plot \( S^* \) and in Table 2 we give the corresponding eigenvalues.

\[
\begin{align*}
(p_{1,2}(y) \sin(x))^2 & \quad t^* \\
\end{align*}
\]

**Figure 3.** Plot of the set \( S^* \), related to the weight \( p^* \), and \( z = (\psi_{1,2}(y) \sin(x))^2 \) intersected with the plane \( z = t^* \) (\( \alpha = 0.5, \beta = 1.5 \)).

In order to find a reinforce more suitable for practical reproduction, inspired by Figure 3 we test a second weight depending only on \( y \) and concentrated around the mid-line \( y = 0 \), i.e.

\[
P(x,y) = \hat{p}(y) := \beta \chi_{\hat{l}}(y) + \alpha \chi_{(-\ell,\ell) \setminus \hat{l}}(y)
\]

for a.e. \( (x,y) \in \Omega \), where \( \hat{l} := \left( -\frac{\ell(\beta-1)}{\beta-\alpha}, \frac{\ell(\beta-1)}{\beta-\alpha} \right) \). This choice produces some simplifications in the problem, see also [8], and the coefficients in (4.2) become simpler, since \( p \) loses the dependence on \( x \). Since the weight \( \hat{p}(x,y) \) increases \( \nu_1(p) \) less than \( p^*(x,y) \), we keep as **approximated optimal weight** for \( \nu_1^{\hat{p}} \):

\[
P^*(x,y) = \beta \chi_{S^*}(x,y) + \alpha \chi_{\Omega \setminus S^*}(x,y)
\]

for a.e. \( (x,y) \in \Omega \), where \( S^* = \{(x,y) \in \Omega : (\theta_1)^2(x,y) \leq t^* \} \) for \( t^* > 0 \) such that \( |S^*| = \frac{1-\alpha}{\beta-\alpha} |\Omega| \).

### 4.4. Numerical solution of (3.2).

From the eigenvalues in Table 1 we infer

\[
\mu_1(1) < ... < \mu_{10}(1) < \nu_1(1) < \mu_{11}(1) < ...
\]

This is the reason why, we fix \( j_0 = 10 \) in (3.1), and we focus on the ratio \( \mathcal{R} \) between \( \nu_1(p) \) and \( \mu_{10}(p) \). In order to increase \( \mathcal{R} \) we test weights raising \( \nu_1(p) \) and lowering \( \mu_{10}(p) \). First we consider the optimal weight \( \tilde{p}_{10}(x) \) as defined in (4.4). As we can see from Table 2 it has a limited effect on the variation of \( \nu_1(p) \), so that it makes sense to minimize \( \mu_{10}(p) \) in order to increase the ratio (3.2).
Table 2. The lowest longitudinal eigenvalues \( \mu_j(p) \) with \( j = 1, \ldots, 12 \), the first two torsional \( \nu_1(p) \) with \( i = 1, 2 \) and the ratio \( \mathcal{R} = \frac{\nu_1(p)}{\mu_1(p)} \) of (4.2) with different weights, assuming (4.3), \( \alpha = 0.5 \), \( \beta = 1.5 \) and \( N = 30 \).

| \( p \equiv 1 \) | \( \bar{p}_{10}(x) \) | \( p^*(x, y) \) | \( \bar{p}(y) \) | \( \bar{p}(x) \) | \( \bar{p}(x, y) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \mu_1(p) \)  | 9.6001 \( \times 10^{-1} \) | 9.4472 \( \times 10^{-1} \) | 1.1632          | 9.6162 \( \times 10^{-1} \) | 1.4009          |
| \( \mu_2(p) \)  | 1.5361 \( \times 10^1 \)  | 1.5133 \( \times 10^1 \) | 1.6595 \( \times 10^1 \) | 1.5387 \( \times 10^1 \) | 1.5130 \( \times 10^1 \) |
| \( \mu_3(p) \)  | 7.7676 \( \times 10^1 \)  | 7.6697 \( \times 10^1 \) | 8.0573 \( \times 10^1 \) | 7.7904 \( \times 10^1 \) | 8.0483 \( \times 10^1 \) |
| \( \mu_4(p) \)  | 2.4580 \( \times 10^2 \)  | 2.4246 \( \times 10^2 \) | 2.5071 \( \times 10^2 \) | 2.4625 \( \times 10^2 \) | 2.9536 \( \times 10^2 \) |
| \( \mu_5(p) \)  | 6.0015 \( \times 10^2 \)  | 5.9116 \( \times 10^2 \) | 6.1007 \( \times 10^2 \) | 6.0131 \( \times 10^2 \) | 6.7540 \( \times 10^2 \) |
| \( \mu_6(p) \)  | 1.2446 \( \times 10^3 \)  | 1.2217 \( \times 10^3 \) | 1.2676 \( \times 10^3 \) | 1.2472 \( \times 10^3 \) | 1.3030 \( \times 10^3 \) |
| \( \mu_7(p) \)  | 2.3061 \( \times 10^3 \)  | 2.2482 \( \times 10^3 \) | 2.3575 \( \times 10^3 \) | 2.3111 \( \times 10^3 \) | 2.5904 \( \times 10^3 \) |
| \( \mu_8(p) \)  | 3.9346 \( \times 10^3 \)  | 3.7794 \( \times 10^3 \) | 4.0369 \( \times 10^3 \) | 3.9439 \( \times 10^3 \) | 4.5397 \( \times 10^3 \) |
| \( \mu_9(p) \)  | 6.3034 \( \times 10^3 \)  | 5.7937 \( \times 10^3 \) | 6.4844 \( \times 10^3 \) | 6.3194 \( \times 10^3 \) | 6.8325 \( \times 10^3 \) |
| \( \mu_{10}(p) \)| 9.6091 \( \times 10^3 \)  | 7.2040 \( \times 10^3 \) | 9.8972 \( \times 10^3 \) | 9.6354 \( \times 10^3 \) | 1.0433 \( \times 10^4 \) |
| \( \mu_{11}(p) \)| 1.4071 \( \times 10^4 \)  | 1.6440 \( \times 10^4 \) | 1.4495 \( \times 10^4 \) | 1.4113 \( \times 10^4 \) | 1.6074 \( \times 10^4 \) |
| \( \mu_{12}(p) \)| 1.9933 \( \times 10^4 \)  | 2.2331 \( \times 10^4 \) | 2.0520 \( \times 10^4 \) | 1.9997 \( \times 10^4 \) | 2.2280 \( \times 10^4 \) |
| \( \nu_1(p) \)  | 1.0944 \( \times 10^4 \)  | 1.0769 \( \times 10^4 \) | 1.9826 \( \times 10^4 \) | 1.7545 \( \times 10^4 \) | 1.6014 \( \times 10^4 \) |
| \( \nu_2(p) \)  | 4.3786 \( \times 10^4 \)  | 4.3138 \( \times 10^4 \) | 7.0396 \( \times 10^4 \) | 7.0196 \( \times 10^4 \) | 4.3640 \( \times 10^4 \) |
| \( \mathcal{R} \)  | 1.1389          | 1.4949          | 2.0032          | 1.8209          | 1.5349          |

Next we consider weights having strong effects on \( \nu_1(p) \), such as the weights \( p^* \) and \( \bar{p} \) defined in the previous section. Table 2 highlights that they have a confined effect on longitudinal eigenvalues. Moreover, they increase the ratio \( \mathcal{R} \) much more than the weights optimal for the longitudinal modes.

We complete the numerical experiments by testing other weights which seem to be reasonable in order to increase \( \mathcal{R} \); more precisely, we consider a weight concentrated near the short edges of the plate:

\[
\bar{p}(x, y) = \bar{p}(x) := \alpha \chi_I(x) + \beta \chi_{(0, \pi) \setminus I}(x)
\]

for a.e. \( (x, y) \in \Omega \),

where \( I := \left( \frac{\pi}{2} - \frac{\pi}{2(\beta - 1)}, \frac{\pi}{2(\beta - 1)} \right) \cup \left( \frac{\pi}{2}, \frac{\pi}{2(\beta - 1)} \right) \), and the cross-type weight \( \bar{p}(x, y) \), given in the last column of Table 2 which is obtained by combining \( p_{10}(x) \) and \( \bar{p}(y) \). From Table 2 we observe that these weights have effects both on torsional and on longitudinal eigenvalues, so that they do not seem optimal for the ratio \( \mathcal{R} \). Therefore, by Table 2, we conclude that \( p^* \) is the best weight among our candidates and we propose as **approximated optimal weight** for \( \mathcal{R} \):

\[
p^*(x, y) = \beta \chi_{S^*}(x, y) + \alpha \chi_{\partial \Omega \setminus S^*}(x, y)
\]

for a.e. \( (x, y) \in \Omega \), where \( S^* = \{(x, y) \in \Omega : \theta^*(x, y) \leq t^* \} \) for \( t^* > 0 \) such that \( |S^*| = \frac{1 - \alpha}{\beta - \alpha} |\Omega| \), cfr. Figure 3.

Although the present work is focused on the first torsional eigenvalue, in Table 2 we also collect the results obtained for the second torsional eigenvalue \( \nu_2(p) \). We observe that \( \nu_2(p) \) follows the same trend of \( \nu_1(p) \) with respect to the weight considered, hence we may conjecture that the same reinforcement could be adopted to optimize ratios involving subsequent (low) torsional eigenvalues.
5. Proofs

In what follows we will always assume that
\[ 0 < \sigma < \frac{1}{2} \quad \text{and} \quad \alpha < 1 < \beta \quad (\alpha, \beta \in (0, +\infty)) \]
and the family \( P_{\alpha,\beta} \) is as defined in (1.2).

5.1. Proof of Theorem 3.1. The proof follows by combining three lemmas that we state here below. In this section we will not need to distinguish between longitudinal and torsional eigenvalues.

Given \( p \in P_{\alpha,\beta} \), it is convenient to endow the space \( L^2 \) of the weighted scalar product: \( (p u, v)_{L^2} \), for all \( u, v \in L^2 \), which defines an equivalent norm in \( L^2 \). Then, for \( h \in N_+ \), we introduce the orthogonal projection of \( u \in H^2 \), with respect to the above weighted scalar product, onto the space generated by the first \((h - 1)\) eigenfunctions \( u_1, \ldots, u_{h-1} \) of problem (1.1):
\[
P_{h-1}(p)u := \sum_{i=1}^{h-1} (p u, u_i)_{L^2} u_i;
\]
when \( h = 1 \) we adopt the convention \( P_0(p)u = 0 \). Finally, we recall the Auchmuty’s principle [5] in the following:

**Lemma 5.1.** Let \( p \in P_{\alpha,\beta} \) and \( \lambda_h(p) \) the \( h \)-th eigenvalue of (1.1) with \( h \in N_+ \), then
\[
-\frac{1}{2\lambda_h(p)} = \inf_{u \in H^2} A_h(p, u) \quad \text{where} \quad A_h(p, u) := \frac{1}{2} \| u \|^2_{H^2} - \| \sqrt{p} (u - P_{h-1}(p)u) \|_2^2.
\]
Furthermore, the minimum is achieved at a \( h \)-th eigenfunction normalized according to
\[
\| u_h \|^2_{H^2} = \| \sqrt{p} u_h \| = 1/\lambda_h(p).
\]

**Proof.** The proof follows arguing as in [21, Lemma 3.3] by simply replacing \( H^2 \cap H^1_0 \) with \( H^2 \). In alternative, in [5] one can find the original proof in a general setting. \( \square \)

**Lemma 5.2.** The set \( P_{\alpha,\beta} \) is compact for the weak* topology of \( L^\infty \).

**Proof.** First we prove that \( P_{\alpha,\beta} \) is a strongly closed set in \( L^2 \).

Let \( \{p_m\}_m \subset P_{\alpha,\beta} \) be a sequence such that \( p_m \to q \) in \( L^2 \) (as \( m \to +\infty \)) for some \( q \in L^2 \); then \( p_m \to q \) in \( L^1 \) (as \( m \to +\infty \)) and up to a subsequence (still denoted by \( p_m \)) we infer that \( p_m \to q \) a.e. in \( \Omega \). Therefore, \( \alpha \leq q \leq \beta \) and \( q \) is \( y \)-even a.e. in \( \Omega \); moreover, \( \int_\Omega p_m v \, dx \, dy \to \int_\Omega q v \, dx \, dy \) for all \( v \in L^2 \), so that, choosing \( v \equiv 1 \in L^2 \), we obtain \( |\Omega| = \int_\Omega q \, dx \, dy \). This implies that \( q \in P_{\alpha,\beta} \) and \( P_{\alpha,\beta} \) is strongly closed in \( L^2 \).

Next, we show that any sequence \( \{p_m\}_m \subset P_{\alpha,\beta} \) admits a subsequence converging in the weak* topology of \( L^\infty \) to an element of \( P_{\alpha,\beta} \). By the definition of \( P_{\alpha,\beta} \) we have \( \|p_m\|_\infty \leq \beta \), so that, up to a subsequence, we obtain
\[
p_{m_k} \overset{\ast}{\rightharpoonup} \overline{p} \quad \text{in} \quad L^\infty \quad \text{as} \quad k \to \infty.
\]
Moreover, we have \( \|p_{m_k}\|_2^2 \leq \beta |\Omega| \), so that, up to a subsequence, we infer that \( p_{m_{k_j}} \rightharpoonup \overline{q} \) in \( L^2 \) as \( j \to \infty \). It is easy to check that \( P_{\alpha,\beta} \) is a convex set and, since convex strongly closed space are weakly closed, we readily infer that \( \overline{q} \in P_{\alpha,\beta} \).

Therefore,
\[
\int_\Omega p_{m_{k_j}} v \, dx \, dy \to \int_\Omega \overline{q} v \, dx \, dy \quad \forall v \in L^2 \subset L^1 \quad \text{as} \quad j \to \infty \quad \text{with} \quad \overline{q} \in P_{\alpha,\beta}
\]
and, since \( p_{m_k} \overset{\ast}{\rightharpoonup} \overline{p} \) in \( L^\infty \) yields \( \int_\Omega p_{m_{k_j}} v \, dx \, dy \to \int_\Omega \overline{p} v \, dx \, dy \quad \forall v \in L^1 \), we conclude that \( \overline{p} = \overline{q} \) a.e. in \( \Omega \). Whence, \( \overline{p} \in P_{\alpha,\beta} \) and the proof is complete. \( \square \)

**Lemma 5.3.** Let \( \lambda_h(p) \) the \( h \)-th eigenvalue of (1.1) with \( h \in N_+ \). The map \( p \mapsto \lambda_h(p) \) is continuous on \( P_{\alpha,\beta} \) for the weak* convergence.
Proof. Let \( \{p_m\}_m \subset P_{\alpha, \beta} \) be a sequence converging in the weak* topology of \( L^\infty \) to \( \overline{p} \), i.e.

\[
p_m \xrightarrow{\text{weak*}} \overline{p} \quad \text{in } L^\infty \quad \text{as } m \to \infty;
\]

then \( \overline{p} \in P_{\alpha, \beta} \) by Lemma 5.2.

To \( p_m \) we associate the \( h \)-th eigenvalue \( \lambda_h(p_m) \) of \( (1.1) \) and an eigenfunction \( u_h(p_m) \) normalized with respect to the weighted scalar product, i.e. \( \int_\Omega P_m u_h(p_m) u_r(p_m) \, dx \, dy = \delta_{hr} \), where \( \delta_{hr} \) is the Kronecker delta for all \( h, r \in \mathbb{N}_+ \) and \( \lambda_h(p_m) = \| u_h(p_m) \|_{H^2}^2 \).

By (1.2) and (2.2) we have

\[
\lambda_h(p) \leq \frac{\lambda_h(1)}{\alpha} \quad \forall p \in P_{\alpha, \beta},
\]

where \( \lambda_h(1) \) is the \( h \)-th eigenvalue of \( (1.1) \) with \( p \equiv 1 \), implying that \( \lambda_h(p_m) = \| u_h(p_m) \|_{H^2} \leq \lambda_h(1)/\alpha \). Therefore, we can extract a subsequence, still denoted by \( u_h(p_m) \), such that

\[
\lambda_h(p_m) \to \overline{\lambda}_h \quad \text{in } \mathbb{R},
\]

\[
u_h(p_m) \to \overline{\nu}_h \quad \text{in } H^2 \quad \text{as } m \to \infty.
\]

Moreover, due to the compact embedding \( H^2 \to L^2 \), we obtain that \( u_h(p_m) \) strongly converges to \( \overline{u}_h \) in \( L^2 \) as \( m \to \infty \); this implies, for all \( v \in H^2 \), that

\[
\int_{\Omega} p_m u_h(p_m) v \, dx \, dy \to \int_{\Omega} \overline{p} \overline{u}_h v \, dx \, dy \quad \text{as } m \to \infty
\]

indeed

\[
\left| \int_{\Omega} (p_m u_h(p_m) - \overline{p} \overline{u}_h) v \, dx \, dy \right| \leq \| p_m v \|_2 \| u_h(p_m) - \overline{u}_h \|_2 + \left| \int_{\Omega} p_m \overline{u}_h v \, dx \, dy - \int_{\Omega} \overline{p} \overline{u}_h v \, dx \, dy \right| \to 0,
\]

since \( \overline{u}_h \in H^2 \subset L^1 \). Therefore, we obtain

\[
(u_h(p_m), v)_{H^2} - \lambda_h(p_m)(p_m u_h(p_m), v)_{L^2} \to (\overline{u}_h, v)_{H^2} - \overline{\lambda}_h(p \overline{u}_h, v)_{L^2} \quad \forall v \in H^2 \quad \text{as } m \to \infty,
\]

inferring that \( \overline{\lambda}_h \) is an eigenvalue of \( (1.1) \) and \( \overline{\nu}_h \) is a corresponding eigenfunction.

Arguing as before we also obtain \( \int_{\Omega} p_m u_h(p_m) u_r(p_m) \, dx \, dy \to \int_{\Omega} \overline{p} \overline{u}_h \overline{u}_r \, dx \, dy = \delta_{hr} \) for all \( h, r \in \mathbb{N}_+ \), so that \( \overline{\lambda}_h \) is a diverging sequence for \( h \to \infty \). To prove that \( \overline{\lambda}_h = \lambda_h(\overline{p}) \) for every \( h \in \mathbb{N}_+ \), we assume by contradiction that, for \( p = \overline{p} \), there exists an eigenfunction \( \overline{\pi} \) associated with the eigenvalue \( \overline{\lambda}_h \) such that \( (\overline{p} \overline{\pi}, \overline{\pi})_{L^2} = 0 \) for all \( h \in \mathbb{N}_+ \). We suppose that \( \overline{\pi} \) is normalized in \( H^2 \) so that \( \| \sqrt{\overline{p}} \overline{\pi} \|_{L^2} = 1/\overline{\lambda}_h \) applying Lemma 5.1 we have

\[
(5.1) \quad -\frac{1}{2\overline{\lambda}_h(p_m)} \leq A_h(p_m, \overline{\pi}) = \frac{1}{2} \| \overline{\pi} \|_{H^2}^2 - \| \sqrt{p_m} [\overline{\pi} - P_{h-1}(p_m) \overline{\pi}] \|_2 \to -\frac{1}{2} \| \overline{\pi} \|_{H^2}^2 - \| \sqrt{\overline{p}} \overline{\pi} \|_2 = -\frac{1}{2\overline{\lambda}_h},
\]

where the convergence comes from

\[
P_{h-1}(p_m) \overline{\pi} = \sum_{i=1}^{h-1} (p_m \overline{\pi}, u_i(p_m))_{L^2} u_i(p_m) \to \sum_{i=1}^{h-1} (\overline{p} \overline{\pi}, \overline{u}_i)_{L^2} \overline{u}_i = 0 \quad \text{in } L^2.
\]

Therefore, by (5.1), letting \( m \to \infty \), we obtain

\[
\overline{\lambda}_h \geq \lambda_h(p_m) \to \overline{\lambda}_h \quad \forall h \in \mathbb{N}_+,
\]

giving a contradiction since \( \overline{\lambda}_h \) is an unbounded sequence for \( h \to \infty \). Thus \( \overline{\lambda}_h = \lambda_h(\overline{p}) \), implying the continuity of \( p \mapsto \lambda_h(p) \) for every \( h \in \mathbb{N}_+ \) fixed.

\[\square\]

**Proof of Theorem 3.1 completed.** Let us consider the function \( F : (0, +\infty) \times (0, +\infty) \to \mathbb{R} \) given by \( F(t, s) := \frac{t}{s} \), continuous on its domain. By Lemma 5.3, the maps \( p \mapsto \nu_1(p) \) and \( p \mapsto \mu_{j_0}(p) \) are continuous on \( P_{\alpha, \beta} \) for the weak* convergence; since \( \nu_1(p) > \mu_{j_0}(p) > 0 \), we infer that \( F(\mu_{j_0}(p), \nu_1(p)) = \frac{\nu_1(p)}{\mu_{j_0}(p)} \) is also continuous on \( P_{\alpha, \beta} \) for the weak* convergence. Finally, the existence of a maximum (or
minimum) of \( F(\mu_{j_0}(p), \nu_1(p)) = \frac{\nu_1(p)}{\mu_{j_0}(p)} \) on \( P_{\alpha,\beta} \) follows thanks to the compactness proved in Lemma 5.2 of the set \( P_{\alpha,\beta} \) for the weak* topology of \( L^\infty \).

5.2. Proof of Theorem 3.2. The existence issue follows as in the proof of Theorem 3.1 by considering the continuous function \( F(\mu_{j_0}(p), \nu_1(p)) = \nu_1(p) \). In the sequel we will denote by \((\hat{p}, \hat{u}) \in P_{\alpha,\beta} \times H^2_\Omega\) an optimal pair for (3.3) suitable normalized as follows

\[
\nu_{1,\alpha,\beta} = \nu_1(\hat{p}) = \frac{\|\hat{u}\|_{H^2_\Omega}^2}{\|\sqrt{\hat{p}} \hat{u}\|_2^2} \quad \text{and} \quad \|\hat{u}\|_{H^2_\Omega}^2 = \frac{1}{\nu_1(\hat{p})}.
\]

Next we state a couple of lemmas useful to complete the proof.

Lemma 5.4. Let \( u \in H^2_\Omega \setminus \{0\} \) and let \( J : P_{\alpha,\beta} \to \mathbb{R} \) be defined as follows

\[
J(p) = \int_\Omega p(x,y)u^2 \, dx \, dy.
\]

Then, the problem

\[
I_{\alpha,\beta} := \inf_{p \in P_{\alpha,\beta}} J(p)
\]

admits the solution

\[
p_u(x,y) = \beta \chi_{\tilde{S}}(x,y) + \alpha \chi_{\Omega \setminus \tilde{S}}(x,y) \quad \text{for a.e.} \ (x,y) \in \Omega,
\]

where \( \tilde{S} = \tilde{S}(u) \subset \Omega \) is such that \( |\tilde{S}| = \frac{1-\alpha}{\beta-\alpha} |\Omega| = \tilde{C}_{\alpha,\beta} \) and satisfies one of the following

\[
\tilde{S} \subseteq \{(x,y) \in \Omega : u^2(x,y) = 0\} \quad \text{or} \quad \{(x,y) \in \Omega : u^2(x,y) < t\} \subseteq \tilde{S} \subseteq \{(x,y) \in \Omega : u^2(x,y) \leq t\}
\]

where \( t \) is defined as

\[
t := \sup \left\{ s > 0 : \left|\left\{(x,y) \in \Omega : u^2(x,y) < s\right\}\right| < \tilde{C}_{\alpha,\beta} \right\}.
\]

Similarly, the problem

\[
M_{\alpha,\beta} := \sup_{p \in P_{\alpha,\beta}} J(p)
\]

admits the solution

\[
p_u(x,y) = \alpha \chi_{\hat{S}}(x,y) + \beta \chi_{\Omega \setminus \hat{S}}(x,y) \quad \text{for a.e.} \ (x,y) \in \Omega,
\]

where \( \hat{S} = \hat{S}(u) \subset \Omega \) is such that \( |\hat{S}| = \frac{\beta-1}{\beta-\alpha} |\Omega| = \hat{C}_{\alpha,\beta} \) and satisfies (5.3)-(5.4) with \( \hat{C}_{\alpha,\beta} \) replaced by \( \hat{C}_{\alpha,\beta} \).

Proof. We only prove the statement for \( I_{\alpha,\beta} \) since the statement for \( M_{\alpha,\beta} \) follows basically by reversing all the inequalities below. Since \( p_u \in P_{\alpha,\beta} \) we have

\[
I_{\alpha,\beta} \leq J(p_u).
\]

If we prove that

\[
J(p) \geq J(p_u) \quad \forall p \in P_{\alpha,\beta},
\]

...
the thesis is obtained. To this aim, we observe that for all $t > 0$, we have
\[ \int_{\Omega} u^2(p_u - p) \, dx \, dy \]
which implies
\[ = \int_{\{u^2 < t\}} u^2(\beta - p) \, dx \, dy + \int_{\{u^2 > t\}} u^2(\alpha - p) \, dx \, dy + \int_{\{u^2 = t\}} u^2(p_u - p) \, dx \, dy \]
\[ \leq t \int_{\{u^2 < t\}} (\beta - p) \, dx \, dy + t \int_{\{u^2 > t\}} (\alpha - p) \, dx \, dy + t \int_{\{u^2 = t\}} (p_u - p) \, dx \, dy \]
\[ = t \int_{\Omega} (p_u - p) \, dx \, dy = 0 ; \]
similarly, for $t = 0$
\[ \int_{\Omega} u^2(p_u - p) \, dx \, dy = \int_{\{u^2 > 0\}} u^2(p_u - p) \, dx \, dy \leq 0. \]
In both the cases we conclude that $J(p) \geq J(p_u)$, and in turn that $J(p_u) = I_{\alpha,\beta}$. \hfill \Box

We will also invoke the Aubin-Muty’s principle stated in Lemma 5.1 that, in terms of $\nu_1(p)$, rewrites
\[ (5.5) \quad - \frac{1}{2\nu_1(p)} = \inf_{u \in H^2_D} A(p, u) \quad A(p, u) := \frac{1}{2} \|u\|^2_{H^2} - \|\sqrt{\nu} u\|_2. \]
By this, if $(\tilde{p}, \tilde{u}) \in P_{\alpha,\beta} \times H^2_D$ and (5.2) is satisfied, it is readily deduced that
\[ (5.6) \quad \sup_{p \in P_{\alpha,\beta}} \inf_{u \in H^2_D} A(p, u) = - \frac{1}{2\nu_1(\tilde{p})} = - \frac{1}{2\nu_1(\tilde{p})} = A(\tilde{p}, \tilde{u}). \]
Furthermore, we have
\textbf{Lemma 5.5.} Let $A(p, u)$ be as defined in (5.5), the following equality holds
\[ (5.7) \quad \sup_{p \in P_{\alpha,\beta}} \inf_{u \in H^2_D} A(p, u) = \inf_{u \in H^2_D} \sup_{p \in P_{\alpha,\beta}} A(p, u). \]

The proof of Lemma 5.5 is the same of [21] Lemma 3.7] once replaced the set $H^2 \cap H^1_0$ there with our set $H^2_D$ (strongly and weakly closed subspace of $H^2$), and the $H^2 \cap H^1_0$ norm there with our $H^2_D$ norm. Hence, we omit it and we refer the interested readers to [21] or [20], where the proof was originally given in the second order case.

Finally, we prove
\textbf{Lemma 5.6.} Let $(\tilde{p}, \tilde{u}) \in P_{\alpha,\beta} \times H^2_D$ be as in (5.2), there holds
\[ (5.8) \quad \int_{\Omega} \tilde{p}(x, y) \tilde{u}^2 \, dx \, dy \leq \int_{\Omega} p(x, y) \tilde{u}^2 \, dx \, dy \quad \forall p \in P_{\alpha,\beta}. \]

\textbf{Proof.} The idea of the proof is taken from [21] Proposition 3.8], the main difference here is the use of Lemma 5.3

First we consider the functional $B : H^2_D \rightarrow \mathbb{R}$ defined as follows
\[ B(u) = \sup_{p \in P_{\alpha,\beta}} A(p, u) = \frac{1}{2} \|u\|^2_{H^2} - \inf_{p \in P_{\alpha,\beta}} \|\sqrt{\nu} u\|_2, \]
and we claim that
\[ (5.9) \quad \text{there exists} \ \tilde{u} \in H^2_D \text{ such that } B(\tilde{u}) = \inf_{u \in H^2_D} B(u) = \inf_{u \in H^2_D} \sup_{p \in P_{\alpha,\beta}} A(p, u) =: I. \]
To this aim, let \( \{ u_k \} \) be a minimizing sequence for \( I \), namely
\[
\frac{1}{2} \| u_k \|_{H_2^\ast}^2 - \inf_{p \in P_{\alpha, \beta}} \| \sqrt{p} u_k \|_2 = I + o(1) \quad \text{as } k \to +\infty.
\]
By the boundedness of \( p \) and the continuity of the embedding \( H_2^\ast \subset L^2 \) it is readily deduced that
\[
\| u_k \|_{H_2^\ast} \leq C \| u_k \|_{H_2^\ast} + 2I + o(1) \quad \text{as } k \to +\infty
\]
and, in turn, that
\[
\| u_k \|_{H_2^\ast} \leq C \quad \text{for } k \text{ sufficiently large},
\]
with \( C, C' > 0 \). Then, up to a subsequence, we have
\[
u_k \rightharpoonup \nu \text{ in } H_2^\ast \quad \text{and} \quad u_k \to \nu \text{ in } L^2 \quad \text{as } k \to +\infty.
\]
Therefore,
\[
\| \nu \|_{H_2^\ast} \leq \liminf_{k \to +\infty} \| u_k \|_{H_2^\ast}^2.
\]
Next we take \( p_\nu \) as given in Lemma 5.4, namely such that
\[
\inf_{p \in P_{\alpha, \beta}} \int_{\Omega} p(x, y) \nu^2 \, dx \, dy = \int_{\Omega} p_\nu(x, y) \nu^2 \, dx \, dy.
\]
Clearly,
\[
\lim_{k \to +\infty} \int_{\Omega} p_\nu(x, y) u_k^2 \, dx \, dy = \int_{\Omega} p_\nu(x, y) \nu^2 \, dx \, dy
\]
and
\[
\inf_{p \in P_{\alpha, \beta}} \int_{\Omega} p(x, y) u_k^2 \, dx \, dy \leq \int_{\Omega} p_\nu(x, y) u_k^2 \, dx \, dy.
\]
In particular, we conclude that
\[
\limsup_{k \to +\infty} \inf_{p \in P_{\alpha, \beta}} \int_{\Omega} p(x, y) u_k^2 \, dx \, dy \leq \limsup_{k \to +\infty} \int_{\Omega} p_\nu(x, y) u_k^2 \, dx \, dy = \inf_{p \in P_{\alpha, \beta}} \int_{\Omega} p(x, y) \nu^2 \, dx \, dy.
\]
The above inequalities yield
\[
B(\nu) \leq \liminf_{k \to +\infty} B(u_k) = I
\]
which is the claim (5.9).

By combining (5.6), (5.7) and (5.9), it follows that
\[
\sup_{p \in P_{\alpha, \beta}} A(p, \nu) = \sup_{u \in H_2^\ast} \sup_{p \in P_{\alpha, \beta}} A(p, u) = \sup_{p \in P_{\alpha, \beta}} \sup_{u \in H_2^\ast} A(p, u) = A(\hat{p}, \hat{u}) = \inf_{u \in H_2^\ast} A(p, u).
\]
In particular, this implies that
\[
A(p, \nu) \leq \sup_{p \in P_{\alpha, \beta}} A(p, \nu) = \inf_{u \in H_2^\ast} A(\hat{p}, u) \leq A(\hat{p}, \nu) \quad \forall p \in P_{\alpha, \beta}
\]
and that
\[
A(\hat{p}, \nu) \leq \sup_{p \in P_{\alpha, \beta}} A(p, \nu) = \inf_{u \in H_2^\ast} A(\hat{p}, u) \leq A(\hat{p}, \nu) \quad \forall u \in H_2^\ast.
\]
From the above inequality and (5.6) we infer that \( \nu = \hat{u} \), hence
\[
A(p, \hat{u}) \leq A(\hat{p}, \hat{u}) \quad \forall p \in P_{\alpha, \beta},
\]
which, recalling the definition of \( A(p, u) \), yields (5.8).
Proof of Theorem 3.2 completed.

Step 1. Set \( p_\alpha(x,y) := \beta \chi_{S_\alpha}(x,y) + \alpha \chi_{\Omega \setminus S_\alpha}(x,y) \) with \( S_\alpha = \tilde{S}(\hat{\nu}) \) as defined in Lemma 5.4 we prove that
\[
(5.10) \quad \| \sqrt{p} \hat{u} \|^2 = \| \sqrt{p_\alpha} \hat{u} \|^2.
\]
By Lemma 5.4 on problem \( I_{\alpha,\beta} \) we know that that
\[
\int_\Omega \hat{p}(x,y) \hat{u}^2 \, dx \, dy \geq \int_\Omega p_\alpha(x,y) \hat{u}^2 \, dx \, dy
\]
whence,
\[
(5.11) \quad \nu_1^{\alpha,\beta} = \frac{\| \hat{u} \|^2_{L^2}}{\| \sqrt{p} \hat{u} \|^2} \leq \frac{\| \hat{u} \|^2_{L^2}}{\| \sqrt{p_\alpha} \hat{u} \|^2}.
\]
On the other hand, by (5.8) with \( p = p_\alpha \) we infer
\[
\nu_1^{\alpha,\beta} = \frac{\| \hat{u} \|^2_{L^2}}{\| \sqrt{p} \hat{u} \|^2} \geq \frac{\| \hat{u} \|^2_{L^2}}{\| \sqrt{p_\alpha} \hat{u} \|^2}.
\]
Comparing the above inequality with (5.11), the proof of Step 1 follows.

Step 2. Let \((p_\alpha, \hat{\nu}) \in P_{\alpha,\beta} \times H^2_0\) be as in Step 1 and let \( \hat{\nu} \geq 0 \) be the number corresponding to \( \hat{u} \) in \( S_\alpha \). We prove that \( \hat{p} = p_\alpha \) a.e. in \( \Omega \).

By (5.10), if \( \hat{\nu} \geq 0 \) then we have
\[
0 = \int_\Omega (\hat{p} - p_\alpha) \hat{u}^2 \, dx \, dy = \int_{\{\hat{\nu}^2 = 0\}} (\hat{p} - p_\alpha) \hat{u}^2 \, dx \, dy + \int_{\{\hat{\nu}^2 > 0\}} (\hat{p} - p_\alpha) \hat{u}^2 \, dx \, dy,
\]
implying \( \hat{p} \hat{u} = \alpha \hat{u} \) a.e. in \( \Omega \). On the other hand, since \( \hat{u} \in H^4(\Omega) \) we can write almost everywhere the Euler-Lagrange equation related to the Rayleigh quotient of \( \nu_1^{\alpha,\beta} = \nu_1(\hat{p}) \) and, for what observed above, we infer that
\[
\Delta^2 \hat{u} = \nu_1^{\alpha,\beta} \alpha \hat{u} \quad \text{a.e. in } \Omega.
\]
Recalling that \( \hat{u} \) satisfies the partially hinged boundary conditions, this means that it must be one of the eigenfunctions listed in Proposition 2.2 since the set of zeroes of any of the eigenfunctions of Proposition 2.2 has zero measure, this contradicts the definition of \( S_\alpha \) and forces \( S_\alpha \) to be a set of positive measure. Whence, the above arguments proves that \( \hat{\nu} > 0 \).

For \( \hat{\nu} > 0 \) we set
\[
A_\ell = \{(x,y) \in \Omega : \hat{u}^2(x,y) = \hat{\nu}\}.
\]
By (5.10) we obtain
\[
0 = \int_\Omega (\hat{p} - p_\alpha) \hat{u}^2 \, dx \, dy = \int_{\{\hat{\nu}^2 < \ell\}} (\hat{p} - \beta) \hat{u}^2 \, dx \, dy + \int_{\{\hat{\nu}^2 = \ell\}} (\hat{p} - \alpha) \hat{u}^2 \, dx \, dy + \int_{A_\ell} (\hat{p} - p_\alpha) \hat{u}^2 \, dx \, dy.
\]
Assume by contradiction that \( \hat{p} > \alpha \) in a set \( A \subset \{\hat{u}^2 > \ell\} \) with \(|A| > 0\), then we get
\[
\int_{\{\hat{\nu}^2 > \ell\}} (\hat{p} - \alpha)(\hat{u}^2 - \ell) \, dx \, dy \geq 0
\]
and, in turn, that
\[
\int_{\{\hat{\nu}^2 > \ell\}} (\hat{p} - \alpha)(\hat{u}^2 - \ell) \, dx \, dy > \ell \int_{\{\hat{\nu}^2 > \ell\}} (\hat{p} - \alpha) \, dx \, dy.
\]
Whence,
\[
0 > \int_{\{\hat{\nu}^2 < \ell\}} (\hat{p} - \beta) \hat{u}^2 \, dx \, dy + \ell \int_{\{\hat{\nu}^2 > \ell\}} (\hat{p} - \alpha) \, dx \, dy + \ell \int_{A_\ell} (\hat{p} - p_\alpha) \, dx \, dy = \ell \int_{\Omega} (\hat{p} - p_\alpha) \, dx \, dy = 0.
\]
This contradicts the definition of the set \( A \) and implies \( \hat{p} = \alpha \) a.e. in \( \{\hat{u}^2 > \ell\} \). Proceeding as before, we suppose that \( \hat{p} < \beta \) in a subset of positive measure of \( \{\hat{u}^2 < \ell\} \) and we obtain a further contradiction
\[
0 > \ell \int_{\{\hat{u}^2 < \ell\}} (\hat{p} - \beta) \, dx \, dy + \ell \int_{A_\ell} (\hat{p} - p_\alpha) \, dx \, dy = \ell \int_{\Omega} (\hat{p} - p_\alpha) \, dx \, dy = 0.
\]
It remains to study \( \tilde{p} \) in \( A_{\tilde{r}} \). When \( |A_{\tilde{r}}| > 0 \), we write the Euler-Lagrange equation related to \( \nu_1^{\alpha, \beta} = \nu_1(\tilde{p}) \) obtaining

\[
\Delta^2 \tilde{u} = \nu_1^{\alpha, \beta} \tilde{p} \tilde{u} \quad \text{a.e. in } A_{\tilde{r}}.
\]

Since \( \hat{\nu}^2 = \hat{\ell} \) we infer \( 0 = \nu_1^{\alpha, \beta} \hat{\tilde{p}}, \) that is absurd since \( \hat{\tilde{p}} \geq \alpha > 0 \). This implies that \( A_{\tilde{r}} \) has zero measure, so that \( \hat{\tilde{p}} = p_\alpha \) a.e. in \( \Omega \).

**Step 3.** Since \( |A_{\tilde{r}}| = 0 \), also \( |A_{\tilde{r}} \setminus S_{\alpha} \| = 0 \), therefore it is not restrictive, up to a set of zero measure, to assume that \( A_{\tilde{r}} \setminus S_{\alpha} = \emptyset \) in such way that \( A_{\tilde{r}} \subseteq S_{\alpha} \) and, in turn, that

\[
S_{\alpha} = \{(x, y) \in \Omega : \hat{\nu}^2(x, y) \leq \hat{\ell}\}.
\]

### 5.3. Proof of Theorem 3.4

The existence issue follows as in the proof of Theorem 3.1 by considering the continuous function \( F(\mu_j(p), \nu_1(p)) = \mu_j(p) \).

Fixed \( j \geq 2 \), by the characterization (2.2) of \( \mu_j(p) \), we may choose \( \mathcal{W}_j = \{w_1, \ldots, w_j\} \subset \mathcal{H}_2^2 \), where the functions \( w_m \) for \( m = 1, \ldots, j \) are defined in (3.5). Therefore, we obtain

\[
\mu_j(p) \leq \sup_{u \in \mathcal{W}_j \setminus \{0\}} \frac{\|w_m\|_{\mathcal{H}_2^2}}{\|w_{m}\|_{\mathcal{H}_2^2}} = \max_{m=1,\ldots,j} \left\{ \frac{\|w_m\|_{\mathcal{H}_2^2}}{\|\sqrt{p}w_m\|_{\mathcal{H}_2^2}} \right\},
\]

where we have exploited the fact that the space \( \mathcal{W}_j \) is generated by disjointly supported functions. Since \( \|w_m\|_{\mathcal{H}_2^2} = |\Omega|^{j/2} \), we finally obtain the upper bound (3.6).

In order to reduce this bound and, in turn \( \mu_j^{\alpha, \beta} \), it is convenient to choose a weight \( p \) having the same effect on every \( \|\sqrt{p}w_m\|_{\mathcal{H}_2^2}^2 \); this suggests to take a weight \( \pi/j \)-periodic in \( x \), i.e. \( p \in \mathcal{P}_{\alpha, \beta}^{\text{per}} \). In this way we obtain

\[
\|\sqrt{p}w_m\|_{\mathcal{H}_2^2}^2 = \frac{1}{j}\|\sqrt{p}\sin^2(jx)\|_{\mathcal{H}_2^2}^2,
\]

and the proof of (3.7) readily follows from (3.6).

The last part of the statement follows by slightly modifying the proof of Lemma 5.4 by which we infer that the problem

\[
\sup_{p \in \mathcal{P}_{\alpha, \beta}^{\text{per}}} \int_\Omega p(x, y) \sin^4(jx) \, dx \, dy
\]

admits the solution

\[
p_j(x, y) = \alpha \chi S_j(x, y) + \beta \chi_{\Omega \setminus S_j}(x, y) \quad \text{for a.e.} \ (x, y) \in \Omega,
\]

where \( S_j = \{(x, y) \in \Omega : \sin^4(jx) \leq t_j\} \) for \( t_j > 0 \) such that \( |S_j| = \frac{2^{-1}}{\beta - \alpha} |\Omega| \).

### 6. Appendix

The aim of this section is to give further information on the eigenvalues \( \lambda_h(p) \) (\( h \in \mathbb{N}_+ \)) of (1.1), having its own independent interest; in particular, we compare problem (1.1) with the more studied Dirichlet and Neumann type problems and we derive a Weyl-type asymptotic law for \( \lambda_h(p) \).

We begin by writing the above mentioned problems in our rectangular domain \( \Omega \); the Dirichlet problem reads

\[
\begin{cases}
\Delta^2 u = \lambda p(x, y)u & \text{in } \Omega \\
u(0, y) = u_x(0, y) = u(\pi, y) = u_x(\pi, y) = 0 & \text{for } y \in (-\ell, \ell) \\
u(x, \pm \ell) = u_y(x, \pm \ell) = 0 & \text{for } x \in (0, \pi),
\end{cases}
\]

with weak form

\[
\int_\Omega \Delta u \Delta v = \lambda \int_\Omega puv \quad \forall v \in \mathcal{H}_0^2.
\]
Observing that in (6.1) and (6.2) as $H$(6.1) and (6.2) as $H$

\[\begin{aligned}
&\Delta^2 u = \lambda p(x, y)u \quad \text{in } \Omega \\
&u_{xx}(0, y) + \sigma u_{yy}(0, y) = u_{xx}(0, y) + (2 - \sigma)u_{yy}(0, y) = 0 \quad \text{for } y \in (-\ell, \ell) \\
&u_{xx}(\pi, y) + \sigma u_{yy}(\pi, y) = u_{xx}(\pi, y) + (2 - \sigma)u_{yy}(\pi, y) = 0 \quad \text{for } y \in (-\ell, \ell) \\
&u_{yy}(x, \pm \ell) + \sigma u_{xx}(x, \pm \ell) = u_{yy}(x, \pm \ell) + (2 - \sigma)u_{xx}(x, \pm \ell) = 0 \quad \text{for } x \in (0, \pi),
\end{aligned}\]

with weak form

\[(u, v)_{H^2} = \lambda \int_{\Omega} p(x, y) uv \, dx \, dy \quad \forall v \in H^2,\]

where the scalar product $(\cdot, \cdot)_{H^2}$ is defined in Section 2, see also [31] for a more general setting of (6.2).

By exploiting the inclusions $H^2_0 \subset H^2 \subset H^2$, we derive

**Proposition 6.1.** Let $p \in P_{\alpha, \beta}$ and let $\lambda_h(p)$ be the eigenvalues of (1.1). Furthermore, denote with $\Lambda^\text{Dir}_h(p)$ and $\Lambda^\text{Neu}_h(p)$ the divergent sequences of eigenvalues of (6.1) and (6.2). There holds:

\[(6.3) \quad \Lambda^\text{Neu}_h(p) \leq \lambda_h(p) \leq \Lambda^\text{Dir}_h(p) \quad \text{for all } h \in \mathbb{N}_+.\]

and

\[\lambda_h(p) \sim \frac{h^2 16\pi^2}{\left(\int_{\Omega} \sqrt{p} \, dx \, dy\right)^2} \quad \text{as } h \to +\infty.\]

We refer to [12] for similar comparisons and a sharper asymptotic analysis in the Neumann case for the homogeneous plate.

**Proof.** The proof is based on the variational characterization of the eigenvalues (2.2) and on some general results presented in [23].

To prove (6.3) we observe that, by density arguments, it follows that $\|\Delta u\|_{H^2}^2 = \|u\|_{H^2}^2$ for all $u \in H^2_0(\Omega)$; in this way we may write the variational representation of Dirichlet and Neumann eigenvalues in (6.1) and (6.2) as

\[\begin{aligned}
\Lambda^\text{Dir}_h(p) &= \inf_{W_h \subset H^2_0} \sup_{u \in W_h \setminus \{0\}, u \in W_h \setminus \{0\}} \frac{\|u\|_{H^2}^2}{\sqrt{\|pu\|_{L^2}^2}} \quad \text{and } \\
\Lambda^\text{Neu}_h(p) &= \inf_{W_h \subset H^2_0} \sup_{u \in W_h \setminus \{0\}} \frac{\|u\|_{H^2}^2}{\sqrt{\|pu\|_{L^2}^2}}.
\end{aligned}\]

Observing that $H^2_0 \subset H^2 \subset H^2$, we infer that

\[\begin{aligned}
\inf_{W_h \subset H^2_0} \sup_{u \in W_h \setminus \{0\}} \frac{\|u\|_{H^2}^2}{\sqrt{\|pu\|_{L^2}^2}} \leq \inf_{W_h \subset H^2} \sup_{u \in W_h \setminus \{0\}} \frac{\|u\|_{H^2}^2}{\sqrt{\|pu\|_{L^2}^2}} \leq \inf_{W_h \subset H^2} \sup_{u \in W_h \setminus \{0\}} \frac{\|u\|_{H^2}^2}{\sqrt{\|pu\|_{L^2}^2}}.
\end{aligned}\]

implying inequality (6.3).

Finally, the asymptotic law for $\lambda_h(p)$ follows from [23] Theorem 4.1, 4.2 where the authors prove that

\[\begin{aligned}
\Lambda^\text{Dir}_h(p) \sim 16\pi^2 \frac{h^2}{\left(\int_{\Omega} \sqrt{p} \, dx \, dy\right)^2} \quad \text{and } \\
\Lambda^\text{Neu}_h(p) \sim 16\pi^2 \frac{h^2}{\left(\int_{\Omega} \sqrt{p} \, dx \, dy\right)^2} \quad \text{as } h \to +\infty,
\end{aligned}\]

implying the same asymptotic behavior for $\lambda_h(p)$. \hfill \square

The estimate (6.3) confirms the general principle that, fixed $h \in \mathbb{N}_+$, any additional constraint increases the eigenvalue and, therefore, the vibration frequency. We point out that the Dirichlet problem represents the most constrained situation, while the Neumann the most free. Problem (1.1) has intermediate boundary conditions, reflecting the trend given by (6.3).
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