Hom-Yang-Baxter equations and Hom-Yang-Baxter systems

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ABSTRACT

In this paper, we mainly present some new solutions of the Hom-Yang-Baxter equation from Hom-algebras, Hom-coalgebras and Hom-Lie algebras, respectively. Also, we prove that these solutions are all invertible and give some examples. Finally, we introduce the notion of Hom-Yang-Baxter systems and obtain two kinds of Hom-Yang-Baxter systems.

1. Introduction

The Yang-Baxter equation (YBE) originated in the work of Yang [35] and Baxter [1]. In 1967, Yang introduced a matrix equation concerning the many-body problem in one dimension with repulsive delta-function interaction. In 1972, Baxter obtained the same equation in his work concerning statistical model with six-vertices. YBE has various forms in physics and plays an important role in many topics in mathematical physics, including quantum groups, quantum integrable systems, braided categories and invariants of knots and links.

Recently, a twisted Hom-type generalization of the YBE called Hom-Yang-Baxter equation (HYBE) was introduced in [36–38] by Yau. The HYBE leads to the solutions for the equation

\[(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha),\]

where \(\alpha\) is an endomorphism of the vector space \(V\), and \(B : V \otimes V \to V \otimes V\) is a bilinear map that commutes with \(\alpha \otimes \alpha\). The compatibility of \(B\) with the twist map \(\alpha\) is related to enhanced Yang-Baxter operators. So, Yau constructed several classes of solutions of the HYBE, generalizing the solutions of the YBE from Lie algebras and quasitriangular bialgebras. Later, Yau [39] extended the classical Yang-Baxter equation to the classical Hom-Yang-Baxter equation (CHYBE) in a Hom-Lie algebra and studied the related algebraic structure.

The study of Hom-algebras can be traced back to Hartwig, Larsson and Silvestrov’s work in [11], where the notion of Hom-Lie algebra in the context of q-deformation theory of Witt and Virasoro algebras [13] was introduced, which plays an important role in physics, mainly in conformal field theory. Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov [20] as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication is replaced by the Hom-associativity and similar for Hom-coassociativity. They also defined the structures
of Hom-bialgebras and Hom-Hopf algebras, and described some of their properties extending properties of ordinary bialgebras and Hopf algebras in [21, 22]. Many more properties and structures of Hom-Hopf algebras have been developed, see [7, 9, 17] and references cited therein.

In [36, 38] Yau proposed the definition of quasitriangular Hom-Hopf algebras and showed that each quasitriangular Hom-Yang-Baxter system yields a solution of the Hom-Yang-Baxter equation. Meanwhile, several classes of solutions of the Hom-Yang-Baxter equation were constructed from different respects, including those associated to Hom-Lie algebras [8, 32, 36, 37], Drinfeld (co)doubles [5, 41, 42] and Hom-Yetter-Drinfeld modules [6, 15, 19, 23, 33, 34, 40].

As generalizations of the Yang-Baxter equation related to nonultralocal models, Hlavaty and Snobl [12] introduced the notion of Yang-Baxter systems from the study of quantum integrable systems. Yang-Baxter system plays a crucial role in many fields like integrable systems, quantum groups, quantum field, etc, and has become an important topic in both mathematics and mathematical physics [2–4].

In particular, a Yang-Baxter system yields a Yang-Baxter operator. In [25–28], Nichita, Parashar and Popovici presented some solutions of the YBE from associative algebras, coassociative coalgebras and Lie algebras respectively. Because on the above working, the motivation of constructing new solutions of HYBE is natural. The purpose of the present paper is to investigate how to construct solutions of HYBE from Hom-algebras, Hom-coalgebras and Hom-Lie algebras, and show two kinds of Hom-Yang-Baxter systems from Hom-algebras and Hom-coalgebras.

This paper is organized as follows. In Section 2, we recall some basic definitions about Hom-algebras, Hom-coalgebras and Hom-Lie algebras. In Section 3, we construct two solutions of HYBE from Hom-algebras and prove that they are invertible (see Theorems 3.1 and 3.4). In Section 4, we present two solutions of HYBE from Hom-coalgebras and prove that they are invertible (see Theorems 4.1 and 4.4). In Section 5, we obtain a solution of HYBE from Hom-Lie algebras and prove that it is invertible (see Theorem 5.1). In Section 6, we introduce the notion of Hom-Yang-Baxter systems and present two kinds of Hom-Yang-Baxter systems (see Theorems 6.2 and 6.4).

2. Preliminaries

Throughout this paper, \( k \) is a fixed field. Unless otherwise stated, all vector spaces, algebras, modules, maps and unadorned tensor products are over \( k \). For a coalgebra \( C \), the coproduct will be denoted by \( \Delta \). We adopt a Sweedler’s notation \( \Delta(c) = c_1 \otimes c_2 \), for any \( c \in C \), where the summation is understood. We refer to [29] for the Hopf algebra theory and terminology.

We now recall some useful definitions in [14, 20–22].

**Definition 2.1.** A Hom-algebra is a quadruple \((A, \mu, 1_A, \alpha)\) (abbr. \((A, \alpha)\)), where \( A \) is a \( k \)-linear space, \( \mu : A \otimes A \rightarrow A \) is a \( k \)-linear map, \( 1_A \in A \) and \( \alpha \) is an endomorphism of \( A \), such that

\[
\begin{align*}
(HA1) \quad \alpha(aa') &= \alpha(a)\alpha(a'); \\
(HA2) \quad \alpha(a)(a'a'') &= (aa')\alpha(a''); \\
1_A &= 1_Aa = \alpha(a)
\end{align*}
\]

are satisfied for \( a, a', a'' \in A \). Here we use the notation \( \mu(a \otimes a') = aa' \).

**Definition 2.2.** A Hom-coalgebra is a quadruple \((C, \Delta, \varepsilon, \alpha)\) (abbr. \((C, \alpha)\)), where \( C \) is a \( k \)-linear space, \( \Delta : C \rightarrow C \otimes C, \varepsilon : C \rightarrow k \) are \( k \)-linear maps, and \( \alpha \) is an endomorphism of \( C \), such that

\[
\begin{align*}
(HC1) \quad \alpha(c_1) \otimes \alpha(c_2) &= \alpha(c_1) \otimes \alpha(c_2); \\
(HC2) \quad \alpha(c_1) \otimes c_{21} \otimes c_{22} &= c_{11} \otimes c_{12} \otimes \alpha(c_2); \\
\varepsilon(c_1)c_2 &= \varepsilon(c_2) \alpha(c) = \alpha(c)
\end{align*}
\]

are satisfied for \( c \in C \).

**Definition 2.3.** A Hom-Lie algebra is a triple \((L, [\cdot, \cdot], \alpha)\) consisting of a linear space \( L \), a bilinear map \([\cdot, \cdot] : L \otimes L \rightarrow L \) and an endomorphism \( \alpha : L \rightarrow L \), such that
are satisfied for all \( l, l', l'' \in L \).

3. Solutions of the HYBE from Hom-algebras

In this section, we will give two kinds of solutions of the HYBE from Hom-algebras and prove that these solutions are both invertible.

**Theorem 3.1.** Let \((A, \mu, 1_A, \alpha)\) be a Hom-algebra and \(\lambda, \nu \in k\). Then

\[
B : A \otimes A \to A \otimes A, \ a \otimes b \mapsto \lambda ab \otimes 1_A + \nu 1_A \otimes ab - \lambda \alpha(a) \otimes \alpha(b)
\]

is a solution for HYBE.

**Proof.** We first show that \(B\) is compatible with the twist map \(\alpha\). For this, we take any \(a, b \in A\) and calculate

\[
(\alpha \otimes \alpha) \circ B(a \otimes b) = (\alpha \otimes \alpha)(\lambda ab \otimes 1_A + \nu 1_A \otimes ab - \lambda \alpha(a) \otimes \alpha(b))
\]

\[
= \lambda \alpha(ab) \otimes 1_A + \nu 1_A \otimes \alpha(ab) - \lambda \alpha^2(a) \otimes \alpha^2(b),
\]

\[
B \circ (\alpha \otimes \alpha)(a \otimes b) = \lambda \alpha(a)\alpha(b) \otimes 1_A + \nu 1_A \otimes \alpha(a)\alpha(b) - \lambda \alpha^2(a) \otimes \alpha^2(b).
\]

It follows that \((\alpha \otimes \alpha) \circ B = B \circ (\alpha \otimes \alpha)\), as desired.

Next we will verify that \(B\) satisfies the HYBE. In fact, for any \(a, b, c \in A\), one may directly check that \(B(a \otimes 1_A) = \nu 1_A \otimes \alpha(a)\). On the one hand, we have

\[
(\alpha \otimes B)(a \otimes b \otimes c) = \lambda ab \otimes 1_A + \nu 1_A \otimes bc - \lambda \alpha(b) \otimes \alpha(c),
\]

\[
(\alpha \otimes B) \circ (B \otimes \alpha)(a \otimes b \otimes c) = \lambda \nu 1_A \otimes \alpha(ab \otimes 1_A) - \lambda^2 \alpha^2(a) \otimes B(1_A \otimes \alpha(b)\otimes 1_A)
\]

\[
+ \nu^2 1_A \otimes \alpha^2(a) \otimes \alpha(b) - \lambda^2 \alpha(ab) \otimes B(1_A \otimes \alpha^2(c)) - \nu^2 1_A \otimes \alpha(ab) \otimes \alpha^2(c)
\]

\[
+ \lambda^2 \alpha^2(a) \otimes \alpha^2(b) \otimes \alpha^2(c),
\]

\[
(\alpha \otimes B) \circ (B \otimes \alpha)(a \otimes b \otimes c) = \lambda \nu 1_A \otimes \alpha(ab \otimes 1_A) - \lambda^2 \alpha^2(a) \otimes B(1_A \otimes \alpha(b)\otimes 1_A)
\]

\[
+ \nu^2 1_A \otimes \alpha^2(a) \otimes \alpha(b) - \lambda^2 \alpha(ab) \otimes B(1_A \otimes \alpha^2(c)) - \nu^2 1_A \otimes \alpha(ab) \otimes \alpha^2(c)
\]

\[
+ \lambda^2 \alpha^2(a) \otimes \alpha^2(b) \otimes \alpha^2(c),
\]

The last equality holds since \((2) = (11), (3) = (14) and (4) = (10)\). On the other hand, we have

\[
(\nu \otimes \alpha)(a \otimes b \otimes c) = \lambda ab \otimes 1_A \otimes \alpha(c) + \nu 1_A \otimes ab \otimes \alpha(c) - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c),
\]
Corollary 3.2. Let \((A, \mu, 1_A, \alpha)\) be a Hom-algebra and \(\lambda, \nu \in k^*\). Assume that \(\alpha\) is involutive, then the solution \(B\) in Theorems 3.1 is invertible, where the inverse is given by

\[
B^{-1} : A \otimes A \to A \otimes A, \quad a \otimes b \mapsto \frac{1}{\nu} ab \otimes 1_A + \frac{1}{\lambda} 1_A \otimes ab - \frac{1}{\lambda} \alpha(a) \otimes \alpha(b).
\]

**Proof.** We first show that \(B \circ B^{-1} = id_{A \otimes A}\). In fact, for any \(a, b, c \in A\), we have

\[
B \circ B^{-1}(a \otimes b) = \frac{1}{\nu} B(ab \otimes 1_A) + \frac{1}{\lambda} B(1_A \otimes ab) - \frac{1}{\nu B} (\lambda \alpha(a) \otimes \alpha(b))
\]

\[
= 1_A \otimes \alpha(ab) + \frac{1}{\lambda} (\lambda \alpha(ab) \otimes 1_A + \nu 1_A \otimes \alpha(ab) - \nu 1_A \otimes \alpha(ab) - \lambda 1_A \otimes \alpha(ab))
\]

\[
= \frac{1}{\lambda} (\nu \alpha(ab) \otimes 1_A + \nu 1_A \otimes \alpha(ab) - \lambda \alpha^2(ab) \otimes 1_A - \lambda \alpha(ab) \otimes \alpha^2(b))
\]

\[
= a \otimes b.
\]

It follows that \(B \circ B^{-1} = id_{A \otimes A}\). Similarly, one may check that \(B^{-1} \circ B = id_{A \otimes A}\). So \(B^{-1}\) is the inverse of \(B\). \(\Box\)
Example 3.3. Let \( \{x_1, x_2, x_3\} \) be a basis of a 3-dimensional linear space \( A \). The following multiplication \( \mu \) and the twist map \( \alpha \) on \( A \) define a Hom-algebra:

\[
\begin{align*}
\mu(x_1, x_1) &= x_1, \quad \mu(x_1, x_2) = x_2, \quad \mu(x_1, x_3) = lx_3, \\
\mu(x_2, x_1) &= x_2, \quad \mu(x_2, x_2) = x_2, \quad \mu(x_2, x_3) = lx_3, \\
\mu(x_3, x_1) &= lx_3, \quad \mu(x_3, x_2) = 0, \quad \mu(x_3, x_3) = 0, \\
\alpha(x_1) &= x_1, \quad \alpha(x_2) = x_2, \quad \alpha(x_3) = lx_3,
\end{align*}
\]

where \( l \) is a parameter in \( k \) ([18]). It is easy to see that \( 1_A = x_1 \). Therefore, by Theorems 3.1, the solution \( B \) of the HYBE for the Hom-algebra \( A \) satisfies

\[
\begin{align*}
B(x_1 \otimes x_1) &= vx_1 \otimes x_1, \quad B(x_1 \otimes x_2) = \lambda x_2 \otimes x_1 + (v - \lambda)x_1 \otimes x_2, \\
B(x_1 \otimes x_3) &= \lambda lx_3 \otimes x_1 + l(v - \lambda)x_1 \otimes x_3, \quad B(x_2 \otimes x_1) = vx_1 \otimes x_2, \\
B(x_2 \otimes x_2) &= \lambda x_2 \otimes x_1 + vx_1 \otimes x_2 - \lambda x_2 \otimes x_2, \\
B(x_2 \otimes x_3) &= \lambda lx_3 \otimes x_1 + vlx_1 \otimes x_3 - \lambda lx_2 \otimes x_3, \\
B(x_3 \otimes x_1) &= vlx_1 \otimes x_3, \quad B(x_3 \otimes x_2) = -\lambda lx_3 \otimes x_2, \quad B(x_3 \otimes x_3) = -\lambda^2 x_3 \otimes x_3,
\end{align*}
\]

where \( \lambda, v \) are two parameters in \( k \).

The following theorem can be proved directly by using Theorems 3.1 and Corollary 3.2.

Theorem 3.4. Let \( (A, \mu, 1_A, \alpha) \) be a Hom-algebra and \( \lambda, \nu \in k \). Then

\[
B : A \otimes A \to A \otimes A, \quad a \otimes b \mapsto \lambda ab \otimes 1_A + \nu 1_A \otimes ab - \nu \alpha(a) \otimes \alpha(b)
\]

is also a solution for HYBE. If, in addition, \( \lambda, \nu \in k^* \) and \( \alpha \) is involutive, then \( B \) is invertible, where the inverse is given by

\[
B^{-1} : A \otimes A \to A \otimes A, \quad a \otimes b \mapsto \frac{1}{\nu} ab \otimes 1_A + \frac{1}{\lambda} 1_A \otimes ab - \frac{1}{\nu} \alpha(a) \otimes \alpha(b).
\]

Example 3.5. Let \( \{1, g, x, y\} \) be a basis of a 4-dimensional linear space \( H_4 \). The following multiplication \( \mu \) and the twist map \( \alpha \) on \( H_4 \) define a Hom-algebra:

\[
\begin{align*}
\mu(1, 1) &= 1, \quad \mu(1, g) = g, \quad \mu(1, x) = kx, \quad \mu(1, y) = ky, \\
\mu(g, 1) &= g, \quad \mu(g, g) = g, \quad \mu(g, x) = ky, \quad \mu(g, y) = kx, \\
\mu(x, 1) &= kx, \quad \mu(x, g) = ky, \quad \mu(x, x) = 0, \quad \mu(x, y) = 0, \\
\mu(y, 1) &= ky, \quad \mu(y, g) = -kx, \quad \mu(y, x) = 0, \quad \mu(y, y) = 0, \\
\alpha(1) &= 1, \quad \alpha(g) = g, \quad \alpha(x) = kx, \quad \alpha(y) = ky,
\end{align*}
\]

where \( k \) is a parameter in \( k \) ([16]). By Theorem 3.4, the solution \( B \) of the HYBE for the Hom-algebra \( H_4 \) satisfies

\[
\begin{align*}
B(1 \otimes 1) &= \lambda 1 \otimes 1, \quad B(1 \otimes g) = \lambda g \otimes 1, \quad B(1 \otimes x) = \lambda kx \otimes 1, \quad B(1 \otimes y) = \lambda k^2 y \otimes 1, \\
B(g \otimes 1) &= (\lambda - \nu)g \otimes 1 + \nu 1 \otimes g, \quad B(g \otimes g) = (\lambda + \nu)1 \otimes 1 - \nu g \otimes g, \\
B(g \otimes x) &= \lambda ky \otimes 1 + v k \otimes x - \nu kg \otimes x, \quad B(g \otimes y) = \lambda kx \otimes 1 + v k \otimes x - \nu kg \otimes y, \\
B(x \otimes 1) &= k(\lambda - \nu) x \otimes 1 + v k \otimes x, \quad B(x \otimes g) = -\lambda ky \otimes 1 - v k \otimes y - \lambda kx \otimes g, \\
B(x \otimes x) &= -k^2 x \otimes x, \quad B(x \otimes y) = -k^2 x \otimes y, \quad B(y \otimes 1) = k(\lambda - \nu) y \otimes 1 + v k \otimes y, \\
B(y \otimes g) &= -\lambda kx \otimes 1 - v k \otimes x - \lambda ky \otimes g, \quad B(y \otimes x) = -k^2 y \otimes x, \quad B(y \otimes y) = -k^2 y \otimes y,
\end{align*}
\]

where \( \lambda, \nu \) are two parameters in \( k \).
4. Solutions of the HYBE from Hom-coalgebras

In this section, we will show two kinds of solutions of the HYBE from Hom-coalgebras and prove that these solutions are both invertible.

**Theorem 4.1.** Let \((C, \Delta, \varepsilon, \alpha)\) be a Hom-coalgebra and \(\lambda, \nu \in k\). Then

\[
B : C \otimes C \rightarrow C \otimes C, \ a \otimes b \mapsto \lambda \varepsilon(a) b_1 \otimes b_2 + \nu \varepsilon(b) a_1 \otimes a_2 - \lambda \alpha(a) \otimes \alpha(b)
\]

is a solution for HYBE.

**Proof.** We first show that \(B\) is compatible with the twist map \(\alpha\). For this, we take any \(a, b \in C\) and calculate

\[
(\alpha \otimes \alpha) \circ B(a \otimes b) = \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c),
\]

\[
(\alpha \otimes \alpha) \circ (B \otimes \alpha)(a \otimes b \otimes c) = \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c) + \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]

It follows that \((\alpha \otimes \alpha) \circ B = B \circ (\alpha \otimes \alpha)\), as desired.

Next we will verify that \(B\) satisfies the HYBE. For this, we take any \(a, b, c \in C\) and calculate

\[
(\alpha \otimes B)(a \otimes b \otimes c) = \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]

\[
(\alpha \otimes B)(B \otimes \alpha)(a \otimes b \otimes c) = \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c) + \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]

\[
(\alpha \otimes B) \circ (B \otimes \alpha)(a \otimes b \otimes c) = \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c) + \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]

\[
\lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c) + \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]

\[
\lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c) + \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]

\[
\lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c) + \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]

\[
\lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c) + \lambda \varepsilon(b) \alpha(a) \otimes c_1 \otimes c_2 + \nu \varepsilon(c) \alpha(a) \otimes b_1 \otimes b_2 - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c).
\]
\[
\begin{align*}
&+ \lambda^3 \epsilon(b) \alpha^3(a) \otimes \alpha^2(c_1) \otimes \alpha^2(c_2),  \\
&\quad - \lambda^3 \alpha^3(a) \otimes \alpha^2(b) \otimes \alpha^3(c),  \\
&= \lambda^3 \epsilon(a) \epsilon(b) \alpha^2(c_1) \otimes \alpha(c_2),  \\
&\quad \lambda^3 \epsilon(a) \epsilon(b) \alpha^2(c_1) \otimes \alpha(c_2),  \\
&\quad - \lambda^3 \epsilon(b) \alpha^3(a) \otimes \alpha^2(c_1) \otimes \alpha^2(c_2),  \\
&\quad + \lambda^3 \epsilon(b) \alpha^3(a) \otimes \alpha^2(c_1) \otimes \alpha^2(c_2),  \\
&\quad + \lambda^3 \epsilon(b) \epsilon(c) \alpha^2(a_1) \otimes \alpha(a_2_1) \otimes \alpha(a_2_2),  \\
&\quad + \lambda^3 \epsilon(b) \epsilon(c) \alpha^2(a_1) \otimes \alpha(a_2_1) \otimes \alpha(a_2_2),  \\
&\quad - \lambda^3 \epsilon(c) \alpha^3(a_1) \otimes \alpha^2(a_2) \otimes \alpha^2(b),  \\
&\quad + \lambda^3 \epsilon(c) \alpha^3(a_1) \otimes \alpha^2(a_2) \otimes \alpha^2(b),  \\
&\quad - \lambda^3 \epsilon(c) \epsilon(a) \alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(c_1),  \\
&\quad - \lambda^3 \epsilon(c) \epsilon(a) \alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(c_1).
\end{align*}
\]

The last equality holds since (2)=(3), (4)=(22), (5)=(23), (6)=(24), (7)=(9), (10)=(12), (13)=(17) and (16)=(18).

On the other hand, we have

\[
\begin{align*}
(B \otimes \alpha) &\circ (\alpha \otimes B) \circ (\alpha \otimes c) \\
&= \lambda \epsilon(a) b_1 \otimes b_2 \otimes \alpha(c) + \nu \epsilon(b) a_1 \otimes a_2 \otimes \alpha(c) - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c),  \\
&= \lambda \epsilon(a) a_1 \otimes a_2 \otimes \alpha(c) + \nu \epsilon(b) b_1 \otimes b_2 \otimes \alpha(c) - \lambda \alpha(a) \otimes \alpha(b) \otimes \alpha(c)
\end{align*}
\]
Therefore, by Theorems 4.1, the solution $B$ of the HYBE for the Hom-coalgebra $C$ satisfies:

$$B(1 \otimes 1) = v 1 \otimes 1, \quad B(1 \otimes a) = \lambda b \otimes b + v 1 \otimes 1 - \lambda 1 \otimes b,$$

$$B(1 \otimes b) = \lambda a \otimes a + v 1 \otimes 1 - \lambda 1 \otimes a, \quad B(a \otimes 1) = \lambda 1 \otimes 1 + v b \otimes b - \lambda b \otimes 1,$$

$$B(a \otimes a) = v b \otimes b, \quad B(a \otimes b) = \lambda a \otimes a + v b \otimes b - \lambda b \otimes a.$$
Proof. Let \( (C, \Delta, \varepsilon, \alpha) \) be a Hom-coalgebra and \( \lambda, \nu \in k \). Then
\[
B : C \otimes C \to C \otimes C, \quad a \otimes b \mapsto \lambda \varepsilon(a) b_1 \otimes b_2 + \nu \varepsilon(b) a_1 \otimes a_2 - \nu \alpha(a) \otimes \alpha(b)
\]
is also a solution for HYBE. If, in addition, \( \lambda, \nu \in k^* \) and \( \alpha \) is involutive, then \( B \) is invertible, where the inverse is given by
\[
B^{-1} : C \otimes C \to C \otimes C, \quad a \otimes b \mapsto \frac{1}{\nu} \varepsilon(a) b_1 \otimes b_2 + \frac{1}{\lambda} \varepsilon(b) a_1 \otimes a_2 - \frac{1}{\nu} \alpha(a) \otimes \alpha(b).
\]

Example 4.5. Let \( \{1, x, y \} \) be a basis of a 4-dimensional linear space \( H_4 \). The following comultiplication \( \Delta \), counit \( \varepsilon \) and the twist map \( \alpha \) on \( H_4 \) define a Hom-coalgebra ([16]):
\[
\Delta(1) = 1 \otimes 1, \quad \Delta(g) = g \otimes g, \quad \Delta(x) = lx \otimes 1 + g \otimes lx, \quad \Delta(y) = ly \otimes g + 1 \otimes ly,
\]
\[
\varepsilon(1) = 1, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 0,
\]
\[
\alpha(1) = 1, \quad \alpha(g) = g, \quad \alpha(x) = lx, \quad \alpha(y) = ly,
\]
where \( l \) is a parameter in \( k \). By Theorem 4.4, the solution \( B \) of the HYBE for the Hom-algebra \( H_4 \) satisfies:
\[
B(1 \otimes 1) = \lambda 1 \otimes 1, \quad B(1 \otimes g) = \lambda g \otimes g + v1 \otimes 1 - v1 \otimes g,
\]
\[
B(1 \otimes x) = \lambda lx \otimes 1 + \lambda lg \otimes x - vlll \otimes x, \quad B(1 \otimes y) = \lambda ly \otimes g + \lambda ll1 \otimes y - vll1 \otimes y,
\]
\[
B(g \otimes 1) = \lambda 1 \otimes 1 + \nu g \otimes g - vgl \otimes 1, \quad B(g \otimes g) = \lambda g \otimes g,
\]
\[
B(g \otimes x) = \lambda lx \otimes 1 + \lambda lg \otimes x - vlg \otimes x, \quad B(g \otimes y) = \lambda ly \otimes g + vll \otimes y - vlg \otimes y,
\]
\[
B(x \otimes 1) = vlg \otimes x, \quad B(x \otimes g) = vlx \otimes 1 + vlg \otimes x - vlx \otimes g,
\]
\[
B(x \otimes x) = -l^2 x \otimes x, \quad B(x \otimes y) = -l^2 x \otimes y, \quad B(y \otimes 1) = vly \otimes g + vll \otimes y - vly \otimes 1,
\]
\[
B(y \otimes g) = vly \otimes g + vll \otimes y - vly \otimes g, \quad B(y \otimes x) = -l^2 y \otimes x, \quad B(y \otimes y) = -l^2 y \otimes y,
\]
where \( \lambda, v \) are two parameters in \( k \).

5. Solutions of the HYBE from Hom-Lie algebras

In this section, we obtain a new solution of the HYBE from Hom-Lie algebras and prove that this solution is invertible.

Theorem 5.1. Let \( (L, [\cdot, \cdot], \alpha) \) be a Hom-Lie algebra, \( u \) is an \( \alpha \)-invariant element in \( Z(L) \) and \( \lambda, \nu \in k \). Then
\[
B : L \otimes L \to L \otimes L, \quad x \otimes y \mapsto \lambda [x, y] \otimes u - \nu \alpha(y) \otimes \alpha(x)
\]
is a solution for HYBE, where \( Z(L) = \{ u \in L | [u, x] = 0, \forall x \in L \} \).

Proof. Obviously, \( B \) is compatible with the twist map \( \alpha \) since \( z \) is \( \alpha \)-invariant. Now we verify that \( B \) satisfy the HYBE. In fact, for any \( x, y, z \in L \), one may directly check that
\[
B(x \otimes u) = -\nu u \otimes \alpha(x), \quad B(u \otimes x) = -\nu \alpha(x) \otimes u.
\]
On the one hand, we have
\[
(\alpha \otimes B)(x \otimes y \otimes z) = \lambda \alpha(x) \otimes [y, z] \otimes u - \nu \alpha(x) \otimes \alpha(z) \otimes \alpha(y),
\]
Furthermore, it follows that $B$ is a solution for $HYBE$. If, in addition, $\alpha$ is a solution for $HYBE$.

\[
\lambda \beta B(x, y, [y, z]) \otimes u - \lambda u \beta B(x, y, \alpha(z)) \otimes \alpha^2(y) \]

\[
= \lambda^2 \beta [\alpha(x), [y, z]] \otimes u \otimes u - \lambda u \beta [\alpha(y), \alpha(z)] \otimes \alpha^2(x) \otimes u
\]

\[
- \lambda u \beta [\alpha(x), \alpha(z)] \otimes u \otimes \alpha^2(y) + \lambda^2 u \beta [\alpha(y), \alpha(z)] \otimes \alpha^2(x) \otimes u
\]

\[
\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B)(x \otimes y \otimes z)
\]

\[
= \lambda^2 [\alpha^2(x), [\alpha(y), \alpha(z)] \otimes B(u \otimes u) - \lambda u \beta [\alpha^2(y), \alpha^2(z)] \otimes B(u \otimes \alpha^2(y)) + v^2 \alpha^2(z) \otimes B(u \otimes \alpha^2(y))
\]

\[
= -\lambda^2 u [\alpha^2(y), \alpha^2(z)] \otimes u \otimes u + \lambda^2 u [\alpha^2(y), \alpha^2(z)] \otimes u \otimes \alpha^2(x)
\]

\[
+ \lambda^2 u [\alpha^2(y), \alpha^2(z)] \otimes u \otimes \alpha^2(y) + \lambda^2 u [\alpha^2(y), \alpha^2(z)] \otimes \alpha^2(y) \otimes u
\]

\[
- \lambda^2 \beta [\alpha^2(x), [\alpha(y), \alpha(z)] \otimes \alpha^2(z) \otimes \alpha^2(y) \otimes \alpha^2(x).
\]

On the other hand, we have

\[
(B \otimes \alpha)(x \otimes y \otimes z) = \lambda [x, y] \otimes u \otimes \alpha(z) - \nu \beta(\alpha(y) \otimes \alpha(x) \otimes \alpha(z),
\]

\[
\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B)(x \otimes y \otimes z)
\]

\[
= \lambda \beta \beta B([\alpha(x), \alpha(y)] \otimes \alpha^2(z)) \otimes u - \lambda u \beta B(\alpha^2(y) \otimes [\alpha(x), \alpha(z)]) \otimes u
\]

\[
+ \nu^2 \beta(\alpha^2(y) \otimes \alpha^2(z)) \otimes \alpha^2(x)
\]

\[
= -\lambda \beta \beta [\alpha^2(x), [\alpha(y), \alpha(z)] \otimes u \otimes u + \nu^2 \beta [\alpha^2(y), \alpha^2(z)] \otimes \alpha^2(x) \otimes u
\]

\[
- \lambda \beta \beta [\alpha^2(y), \alpha^2(z)] \otimes u \otimes \alpha^2(y) + \nu^2 \beta [\alpha^2(y), \alpha^2(z)] \otimes \alpha^2(y) \otimes u
\]

\[
+ \nu^2 \beta [\alpha^2(y), \alpha^2(z)] \otimes \alpha^2(z) \otimes \alpha^2(y) \otimes \alpha^2(x).
\]

According to the anti-symmetry and the Hom-Jacobi identity, we have

\[
[\alpha^2(x), [\alpha(y), \alpha(z)] = [[\alpha(x), \alpha(y)], \alpha^2(z)] + [\alpha^2(y), [\alpha(x), \alpha(z)]].
\]

It follows that

\[
(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B)(x \otimes y \otimes z) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha)(x \otimes y \otimes z).
\]

That is, $B$ is a solution for $HYBE$.

**Corollary 5.2.** Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra, $u$ is an $\alpha$-invariant element in $Z(L)$ and $\lambda \in k$. Then $B : L \otimes L \to L \otimes L, x \otimes y \mapsto \lambda [x, y] \otimes u - \alpha(y) \otimes \alpha(x)$ is a solution for $HYBE$. If, in addition, $\alpha$ is involutive, then $B$ is also invertible, where the inverse is given by $B^{-1} : L \otimes L \to L \otimes L, x \otimes y \mapsto \lambda u \otimes [x, y] - \alpha(y) \otimes \alpha(x)$.

Furthermore, $B^{-1}$ is also a solution for $HYBE$.

**Proof.** First, it is easy to see that $B$ is a solution for $HYBE$ by setting $\nu = 1$.

Next, show that $B \circ B^{-1} = id_{L \otimes L}$. In fact, for any $x, y, z \in L$, we have

\[
B \circ B^{-1}(x \otimes y) = \lambda \beta B(u \otimes [x, y]) - B(\alpha(y) \otimes \alpha(x))
\]

\[
= -\lambda [\alpha(x), \alpha(y)] \otimes u - \lambda [\alpha(y), \alpha(x)] \otimes u + \alpha^2(y) \otimes \alpha^2(x)
\]

\[
= x \otimes y.
\]

It follows that $B \circ B^{-1} = id_{L \otimes L}$. Similarly, one may check that $B^{-1} \circ B = id_{L \otimes L}$. So $B^{-1}$ is the inverse of $B$. Further, similar to the proof of Theorem 5.1, one may calculate
\[(\alpha \otimes B^{-1}) \circ (B^{-1} \otimes \alpha) \circ (\alpha \otimes B^{-1})(x \otimes y \otimes z)\]
\[= -\lambda^2 u \otimes u \otimes [\alpha(x), \alpha(y)], \alpha^2(z)] + \lambda \alpha^3(z) \otimes u \otimes [\alpha^2(x), \alpha^2(y)]\]
\[+ \lambda \alpha \otimes \alpha^2(y) \otimes [\alpha^2(x), \alpha^2(z)] + \lambda \alpha \otimes [\alpha^2(y), \alpha^2(z)] \otimes \alpha^3(x)\]
\[- \alpha^3(z) \otimes \alpha^3(y) \otimes \alpha^3(x)\]
\[= (B^{-1} \otimes \alpha) \circ (\alpha \otimes B^{-1}) \circ (B^{-1} \otimes \alpha)(x \otimes y \otimes z).\]

So $B^{-1}$ is a solution for HYBE.

**Example 5.3.** Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra on 3-dimensional Euclidean $\mathbb{E}^3$ with basis elements $\{e_1, e_2, e_3\}$, whose bracket $[\cdot, \cdot]$ is given by
\[[e_1, e_2] = e_1, [e_1, e_3] = 0, [e_2, e_3] = 0.\]

The twist map $\alpha$ is given by
\[\alpha(e_1) = e_1, \alpha(e_2) = e_2, \alpha(e_3) = -e_3.\]

Obviously, $e_3 \in Z(L)$. Therefore, by Theorem 5.1, the solution $B$ of the HYBE for the Hom-Lie algebra $L$ satisfies:
\[B(e_1 \otimes e_1) = -ve_1 \otimes e_1, B(e_1 \otimes e_2) = \lambda e_1 \otimes e_3 - ve_2 \otimes e_1, B(e_1 \otimes e_3) = ve_3 \otimes e_1, \]
\[B(e_2 \otimes e_1) = -\lambda e_1 \otimes e_3 - ve_1 \otimes e_2, B(e_2 \otimes e_2) = -ve_2 \otimes e_2, B(e_2 \otimes e_3) = ve_3 \otimes e_2, \]
\[B(e_3 \otimes e_1) = ve_1 \otimes e_3, B(e_3 \otimes e_2) = ve_2 \otimes e_3, B(e_3 \otimes e_3) = -ve_3 \otimes e_3,\]

where $\lambda, v \in k$.

In the final part of this section, we present a kind of solutions of CHYBE from Hom-Lie algebras. Recall from [39], Yau defined the CHYBE in a Hom-Lie algebra $(L, [\cdot, \cdot], \alpha)$ as
\[[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,\]
for $r \in L^\otimes 2$. Here the three brackets above are defined as
\[[r^{12}, r^{13}] = [a_i, a_j] \otimes \alpha(b_i) \otimes \alpha(b_j),\]
\[[r^{12}, r^{23}] = \alpha(a_i) \otimes [b_i, a_k] \otimes \alpha(a_k),\]
\[[r^{13}, r^{23}] = \alpha(a_j) \otimes \alpha(a_k) \otimes [b_j, b_k],\]
where $r^{12} = r \otimes 1 = a_i \otimes b_i \otimes 1, r^{13} = (r \otimes id)(1 \otimes r) = a_j \otimes 1 \otimes b_j, r^{23} = 1 \otimes r = 1 \otimes a_k \otimes b_k$.

**Theorem 5.4.** Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and $u$ be an element in $Z(L)$. Then for any $x, y \in L$ and $m, n \in \mathbb{Z}$,
\[r = \alpha^m([x, y]) \otimes \alpha^n(u)\]
is a solution for CHYBE.

**Proof.** It is easy to see that $[r^{12}, r^{13}] = [r^{12}, r^{23}] = [r^{13}, r^{23}] = 0$ since $u$ is an element in $Z(L)$. 

**Example 5.5.** Let $(L, [\cdot, \cdot], \alpha)$ be the Hom-Lie algebra in Example 5.3, then $e_3 \in Z(L)$. By Theorem 5.4, $r = \pm e_1 \otimes e_3$ is a solution for CHYBE.

In Theorem 5.4, we show a kind of solutions of CHYBE from Hom-Lie algebras in the tensor form. Different to our work, Mishra and Naolekar studied the CHYBE by $O$-operators on Hom-Lie algebras in [24]. As a generalization of Lie algebra, Harathi, Mabrouk, Ncib and Silvestrov studied $O$-operators on Malcev algebras and discussed solutions of Malcev Yang-Baxter equation by $O$-operators in [10].
6. Hom-Yang-Baxter systems

In this section, we extend the notion of Yang-Baxter systems to Hom-Yang-Baxter systems and present two kinds of Hom-Yang-Baxter systems.

Consider three vector spaces \( V, V', V'' \), let \( \alpha_V, \alpha_{V'}, \alpha_{V''} \) be three endomorphisms on \( V, V', V'' \) and \( R: V \otimes V' \rightarrow V \otimes V', S: V \otimes V' \rightarrow V \otimes V'', T: V' \otimes V'' \rightarrow V' \otimes V'' \) be three linear maps. Then a Hom-Yang-Baxter commutator is a map \( [R, S, T]: V \otimes V' \otimes V'' \rightarrow V \otimes V' \otimes V'' \) defined by

\[
[R, S, T] = R^{12} \circ S^{13} \circ T^{23} - T^{23} \circ S^{13} \circ R^{12},
\]
where \( R^{12} = R \otimes \alpha_{V''}, S^{13} = (\tau \otimes \text{id}) \circ (\alpha_{V'} \otimes T) \circ (\tau \otimes \text{id}), T^{23} = \alpha_V \otimes T. \]

**Definition 6.1.** Let \( V, V' \) be two vector spaces and \( \alpha_V, \alpha_{V'} \) be two endomorphisms. A system of linear maps

\[
W: V \otimes V \rightarrow V \otimes V, \quad Z: V' \otimes V' \rightarrow V' \otimes V', \quad X: V \otimes V' \rightarrow V \otimes V'
\]
is called a Hom-Yang-Baxter system, if the following conditions are satisfied:

\[
[W, W, W] = 0, \quad [Z, Z, Z] = 0, \quad [W, X, X] = 0, \quad [X, X, Z] = 0. \tag{6.1}
\]

**Theorem 6.2.** Let \((A, \mu, 1_A, \alpha)\) be a Hom-algebra and \( \lambda, \nu \in k \). Then the following is a Hom-Yang-Baxter system:

\[
W: A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto ab \otimes 1_A + \lambda 1_A \otimes ab - \alpha(b) \otimes \alpha(a),
\]

\[
Z: A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto \nu ab \otimes 1_A + 1_A \otimes ab - \alpha(b) \otimes \alpha(a),
\]

\[
X: A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto ab \otimes 1_A + 1_A \otimes ab - \alpha(b) \otimes \alpha(a).
\]

**Proof.** It is sufficient to prove that the four equalities in Eq. (5.1) hold. Here we only verify the equality \([W, X, X] = 0\) and similar for other three equalities. In fact, for any \( a, b, c \in A \), on the one side, we have

\[
W^{12} \circ X^{13} \circ W^{23}(a \otimes b \otimes c)
\]

\[
= W^{12} \circ X^{13}[\alpha(a) \otimes bc \otimes 1_A + \alpha(\alpha) \otimes 1_A \otimes bc - \alpha(a) \otimes \alpha(b) \otimes \alpha(c)]
\]

\[
= W^{12}[\alpha^2(a) \otimes \alpha(bc) \otimes 1_A + \alpha(\alpha(bc)) \otimes 1_A \otimes 1_A + 1_A \otimes 1_A \otimes \alpha(a)(bc)
\]

\[
- \alpha(bc) \otimes 1_A \otimes \alpha^2(a) - \alpha(ab) \otimes \alpha^2(c) \otimes 1_A - 1_A \otimes \alpha^2(c) \otimes \alpha(ab)
\]

\[
+ \alpha^2(b) \otimes \alpha^2(c) \otimes \alpha^2(a)]
\]

\[
\]

\[
= \alpha^2(a)\alpha(ab)(1_A \otimes 1_A) + \lambda \alpha^2(\alpha(ab)) \otimes 1_A - \alpha^2(bc) \otimes \alpha^2(a)(1_A)
\]

\[
+ \lambda \alpha^2(b) \otimes \alpha^2(a)(1_A)
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

The last equality holds since \((4) = (11), (5) = (12), (8) = (15)\) and \((9) = (16)\).

On the other side, we have

\[
X^{23} \circ X^{13} \circ W^{12}(a \otimes b \otimes c)
\]

\[
= X^{23} \circ X^{13}[\alpha(a) \otimes 1_A \otimes \alpha(c) + \lambda 1_A \otimes ab \otimes \alpha(c) - \alpha(b) \otimes \alpha(\alpha(a) \otimes \alpha(c))]
\]

\[
= X^{23}[\alpha(ab) \otimes 1_A \otimes 1_A + 1_A \otimes 1_A \otimes \alpha(ab) - \alpha^2(c) \otimes 1_A \otimes \alpha(ab)]
\]
Example 6.3. Let \( A = k\{1, a\} \) be a vector space. Define the multiplication \( \mu \) and the endomorphism \( \alpha \) by

\[ 1a = a1 = la, \quad aa = 0; \quad \alpha(1) = 1, \quad \alpha(a) = la, \]

where \( 0 \neq l \in k \). Then \((A, \alpha)\) is a Hom-algebra. Therefore, by Theorems 6.2, there is a Hom-Yang-Baxter system for the Hom-algebra \((A, \alpha)\) as follows:

\[
W : A \otimes A \rightarrow A \otimes A, \quad 1 \otimes 1 \mapsto \lambda 1 \otimes 1, \quad 1 \otimes a \mapsto \lambda a \otimes a,
\]

\[
a \otimes 1 \mapsto la \otimes 1 + l(1 - \lambda)1 \otimes a, \quad a \otimes a \mapsto -l^2 a \otimes a;
\]

\[
Z : A \otimes A \rightarrow A \otimes A, \quad 1 \otimes 1 \mapsto v1 \otimes 1, \quad 1 \otimes a \mapsto l(v - 1)a \otimes 1 + l1 \otimes a,
\]

\[
a \otimes 1 \mapsto vla \otimes 1, \quad a \otimes a \mapsto -l^2 a \otimes a;
\]

\[
X : A \otimes A \rightarrow A \otimes A, \quad 1 \otimes 1 \mapsto 1 \otimes 1, \quad 1 \otimes a \mapsto l1 \otimes a,\]

\[
 a \otimes 1 \mapsto la \otimes 1, \quad a \otimes a \mapsto -l^2 a \otimes a,
\]

where \( \lambda, v \in k \).

Theorem 6.4. Let \((C, \Delta, \varepsilon, \alpha)\) be a Hom-coalgebra and \( \lambda, v \in k \). Then the following is a Hom-Yang-Baxter system:

\[
W : C \otimes C \rightarrow C \otimes C, \quad a \otimes b \mapsto \lambda \varepsilon(a)b_1 \otimes b_2 + \varepsilon(b)a_1 \otimes a_2 - \alpha(b) \otimes \alpha(a),
\]

\[
Z : C \otimes C \rightarrow C \otimes C, \quad a \otimes b \mapsto \varepsilon(a)b_1 \otimes b_2 + v \varepsilon(b)a_1 \otimes a_2 - \alpha(b) \otimes \alpha(a),
\]

\[
X : C \otimes C \rightarrow C \otimes C, \quad a \otimes b \mapsto \varepsilon(a)b_1 \otimes b_2 + \varepsilon(b)a_1 \otimes a_2 - \alpha(b) \otimes \alpha(a).
\]

Proof. We only prove \([X, Y, Z] = 0\) and similar for other three equalities. In fact, for any \(a, b, c \in C\), on the one side, we have

\[
Z^{23} \circ X^{13} \circ X^{12}(a \otimes b \otimes c)
\]

\[
= Z^{23} \circ X^{13}[(\varepsilon(a)b_1 \otimes b_2 \otimes \alpha(c) + \varepsilon(b)a_1 \otimes a_2 \otimes \alpha(c)) - \alpha(b) \otimes \alpha(a) \otimes \alpha(c)]
\]

\[
= Z^{23}[(\varepsilon(a)\alpha(c_1) \otimes \alpha^2(b) \otimes \alpha(c_2) + \varepsilon(a)\varepsilon(c)b_1 \otimes \alpha(b_2) \otimes b_2
\]

\[
- \varepsilon(c)\alpha^2(c) \otimes \alpha(b_2) \otimes \alpha(b_1) + \varepsilon(b)\varepsilon(b_1) \otimes \alpha^2(a) \otimes \alpha(c)]
\]

\[
+ \varepsilon(c)\varepsilon(a_1) \otimes \alpha(a_2) \otimes a_{12} - \varepsilon(b)\alpha^2(c) \otimes \alpha(a_2) \otimes \alpha(a_1)
\]
\(- \varepsilon(b)\alpha(c_1) \otimes \alpha^2(a) \otimes \alpha(c_2) - \varepsilon(c)\alpha(b_1) \otimes \alpha^2(a) \otimes \alpha(b_2) + \alpha^2(c) \otimes \alpha^2(a) \otimes \alpha^2(b)\)
\[= \varepsilon(a)\varepsilon(b)\alpha^2(c_1) \otimes \alpha(c_2) \otimes \alpha(c_22), (1) + \varepsilon(a)\alpha^3(c) \otimes \alpha^2(b_1) \otimes \alpha^2(b_2)\]
\[- \varepsilon(a)\alpha^2(c_1) \otimes \alpha^2(c_2) \otimes \alpha^3(b)_{(3)} + \varepsilon(a)\varepsilon(c)\alpha^2(b_1) \otimes \alpha(b_21) \otimes \alpha(b_22), (4) + \varepsilon(a)\varepsilon(c)\alpha^2(b_1) \otimes \alpha(b_21) \otimes \alpha(b_22), (5) - \varepsilon(a)\varepsilon(c)\alpha(b_11) \otimes \alpha(b_12) \otimes \alpha^2(b_2), (6)\]
\[- \varepsilon(a)\alpha^3(c) \otimes \alpha^2(b_1) \otimes \alpha^2(b_2)_{(7)} - \varepsilon(a)\alpha^3(c) \otimes \alpha^2(b_1) \otimes \alpha^2(b_2), (8)\]
\[- \varepsilon(a)\alpha^3(c) \otimes \alpha^2(b_1) \otimes \alpha^2(b_2)_{(9)} + \varepsilon(b)\varepsilon(c)\alpha^2(a_1) \otimes \alpha(a_21) \otimes \alpha(a_22), (10)\]
\[- \varepsilon(b)\alpha^2(c_1) \otimes \alpha(a_21) \otimes \alpha(a_22), (11) - \varepsilon(b)\varepsilon(c)\alpha(a_11) \otimes \alpha(a_12) \otimes \alpha^2(a_2), (12)\]
\[- \varepsilon(b)\alpha^3(c) \otimes \alpha^2(a_1) \otimes \alpha^2(a_2), (13) - \varepsilon(b)\alpha^3(c) \otimes \alpha^2(a_1) \otimes \alpha^2(a_2), (14)\]
\[- \varepsilon(b)\alpha^3(c) \otimes \alpha^2(a_1) \otimes \alpha^2(a_2), (15) - \varepsilon(a)\alpha^3(c) \otimes \alpha^2(b_1) \otimes \alpha(b_21) \otimes \alpha(b_22), (16)\]
\[- \varepsilon(c)\alpha^3(b) \otimes \alpha^2(a_1) \otimes \alpha^2(a_2), (17) + \varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (18)\]
\[- \varepsilon(a)\alpha^3(c) \otimes \alpha^2(b_1) \otimes \alpha^2(b_2), (19) + \varepsilon(b)\alpha^3(c) \otimes \alpha^2(a_1) \otimes \alpha^2(a_2), (20)\]
\[- \alpha^3(c) \otimes \alpha^2(b) \otimes \alpha^3(a), (21)\]

The last equality holds since (2)=(8), (4)=(6), (7)=(9), (10)=(12), (13)=(15) and (14)=(20).

On the other side, we have

\[X^{12} \circ Z^{13} \circ (a \otimes b \otimes c)\]
\[= \varepsilon(a)\varepsilon(b)\alpha^2(c_1) \otimes \alpha(c_2) \otimes \alpha(c_22), (22) + \varepsilon(a)\varepsilon(b)\alpha(c_11) \otimes \alpha(c_12) \otimes \alpha(c_22), (23)\]
\[- \varepsilon(b)\alpha(c_2) \otimes \alpha(c_1) \otimes \alpha^2(a) + \varepsilon(a)\varepsilon(c)\alpha^2(b_1) \otimes \alpha(b_1) \otimes \alpha^2(b_2), (25) + \varepsilon(a)\varepsilon(c)\alpha^2(b_1) \otimes \alpha(b_1) \otimes \alpha^2(b_2), (26) - \varepsilon(a)\alpha(b_1) \otimes \alpha^2(c) \otimes \alpha^2(a) - \varepsilon(b)\alpha(a_1) \otimes \alpha^2(c) \otimes \alpha^2(a)\]
\[- \alpha^2(b) \otimes \alpha^2(c) \otimes \alpha^2(a)\]
\[- \varepsilon(a)\varepsilon(b)\alpha(c_11) \otimes \alpha(c_12) \otimes \alpha^2(c_2), (28) + \varepsilon(a)\varepsilon(b)\alpha(c_11) \otimes \alpha(c_12) \otimes \alpha(c_22), (29)\]
\[- \varepsilon(b)\alpha^2(c_1) \otimes \alpha(c_12) \otimes \alpha^2(c_2), (30) - \varepsilon(b)\alpha^2(c_1) \otimes \alpha^2(c_2) \otimes \alpha^3(a), (31)\]
\[- \varepsilon(a)\varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (32) + \varepsilon(a)\varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (33)\]
\[- \varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (34) - \varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (35)\]
\[- \varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (36) - \varepsilon(a)\alpha^2(c_1) \otimes \alpha^2(c_2) \otimes \alpha^3(b_2), (37) + \varepsilon(a)\alpha^3(c) \otimes \alpha^2(b_1) \otimes \alpha^2(b_2), (38)\]
\[- \varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (39) + \varepsilon(c)\alpha^2(b_1) \otimes \alpha^2(b_2) \otimes \alpha^3(a), (40)\]
\[- \alpha^3(c) \otimes \alpha^2(b) \otimes \alpha^3(a), (41)\]
\[= \varepsilon(a)\varepsilon(b)\alpha(c_11) \otimes \alpha(c_12) \otimes \alpha^2(c_2), (22) + \varepsilon(a)\varepsilon(c)\alpha(b_1) \otimes \alpha(b_12) \otimes \alpha^2(b_2), (28)\]
Example 6.5. Let $C = k[1, a]$ be a vector space. Define the comultiplication $\Delta$, counit $\varepsilon$ and the endomorphism $\alpha$ by

$$
\Delta(1) = 1 \otimes 1, \quad \Delta(a) = la \otimes 1 + l1 \otimes a; \quad \varepsilon(1) = 1, \quad \varepsilon(a) = 0; \quad \alpha(1) = 1, \quad \alpha(a) = la,
$$

where $l \in k$. Then $(A, \alpha)$ is a Hom-coalgebra. Therefore, by Theorem 6.4, there is a Hom-Yang-Baxter system for the Hom-coalgebra $(C, \alpha)$ as follows:

$$
W : C \otimes C \rightarrow C \otimes C, \quad 1 \otimes 1 \mapsto \lambda 1 \otimes 1, \quad 1 \otimes a \mapsto \lambda l1 \otimes a + l(\lambda - 1)a \otimes 1,
$$

$$
a \otimes 1 \mapsto la \otimes 1, \quad a \otimes a \mapsto -l^2 a \otimes a;
$$

$$
Z : C \otimes C \rightarrow C \otimes C, \quad 1 \otimes 1 \mapsto v1 \otimes 1, \quad 1 \otimes a \mapsto l1 \otimes a,
$$

$$
a \otimes 1 \mapsto l(v - 1)1 \otimes a + vl a \otimes 1, \quad a \otimes a \mapsto -l^2 a \otimes a;
$$

$$
X : C \otimes C \rightarrow C \otimes C, \quad 1 \otimes 1 \mapsto 1 \otimes 1, \quad 1 \otimes a \mapsto l1 \otimes a,
$$

$$
a \otimes 1 \mapsto la \otimes 1, \quad a \otimes a \mapsto -l^2 a \otimes a,
$$

where $\lambda, v \in k$.

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Data availability statement

Our manuscripts does not include a data availability statement.

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