Exchange options and spread options 
with stochastic interest rates

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Abstract

In this work, we consider the issue of pricing exchange options and spread options with stochastic interest rates. We provide the closed form solution for the exchange option price when interest rate is stochastic. Our result holds when interest rate is modeled with a stochastic term structure of general form, which includes Vasicek model, CIR term structure, and other well-known term structure models as special cases. In particular, we have discussed the possibility of using our closed form solution as a control variate in pricing spread options with stochastic interest rate.
1 Introduction

Spread options have become increasingly important. They give the holders the right to call or put the spread value of two underlying assets against a predetermined parameter $K$ as the strike price. In particular, the spread options reduce to the so-called exchange options when the predetermined strike price $K$ is set to zero. Spread options and exchange options can be viewed as options to exchange one underlying asset for another with respect to the strike price. They are used in many situations. One typical example is that the option holder is interested in exchanging one commodity for another commodity. For instance, in oil industries, the prices of crude oil and refined oil differ from each other, and both prices are fluctuating considerably in response to the weather, regional stabilities of world oil production centers, and other human and natural parameters. Oil companies may deal with the situations of price fluctuations using the spread options or exchange options. Spread and exchange options have been of considerable interests to both practitioners and theoretical researchers[1, 2, 3].

For the exchange option, a closed form solution for its price is available. The valuation of exchange option was first studied by Margrabe[1], based on the option pricing theory of Black and Scholes[4], and Merton[5, 6]. The derivation of Margrabe is a PDE approach[1]. However, in Margrabe’s derivation, the risk-free rate $r$ is assumed to be a constant, which is far from reality. It is of great interest to both practitioners and theoretical researchers to investigate whether closed form solution exists when the interest rate is modeled with stochastic term structure.

This paper investigates how stochastic interest rate will affect the exchange option pricing. The closed form solution for the exchange option’s price is given when we assume a very general stochastic process for the interest rate, which includes Vasicek model[12], CIR model[13], affine term structure models[14] and other interest rate models as special cases. To our knowledge, this is the first time to provide the closed form solution for exchange
option pricing while stochastic interest rate is taken into account.

We also argue that to price a European style spread option of general strike price $K$, one may use our closed form solution as a control variate to reduce variance of simulation when doing Monte Carlo pricing\cite{11} for stochastic interest rate. The closed form result presented here shall be of interest to both theoretician and practitioners.

In our discussion below, we assume an exchange economy populated by risk-averse agents with increasing preferences, and all economic activities take place in the time interval $[0, T]$. All the possible outcomes of this economy is denoted by a measurable space $(\Omega, \mathcal{F})$ where $\Omega$ is the set of all possible states and $\mathcal{F}$ is a sigma algebra of subsets of $\Omega$. Information arrival in this economy is described by a filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$ with $\mathcal{F}_T = \mathcal{F}$, and the agents belief is modelled by a probability measure $P$ defined on $(\Omega, \mathcal{F})$.

In following sections, we will first rederive the closed form solution of exchange option using the risk-neutral martingale approach. It is shown that this provides identical result to the one given by Margrabe using partial differential equation. Then, we discuss, within the framework of martingale measure, how to price the exchange option when interest rate is stochastic. It is shown that our result is valid for most general term structure, only the correlation coefficients between the interest rate and the underlying assets will affect the option price.

2 Exchange Options

Exchange options can be defined by using two underlying assets or commodities which are closely related. This correlation between the two assets or commodities results from demand substitution or the potential for transformation. In general, an exchange option has the following payoff:

$$\max\{\lambda(S_1(T) - S_2(T)), 0\} = [\lambda(S_1(T) - S_2(T))]^+$$

(1)
where $\lambda = 1$ for a call and $\lambda = -1$ for a put, and $S_1(T)$ and $S_2(T)$ are the underlying asset prices at maturity $T$.

2.1 Constant interest rate

Let us first review the case of constant interest rate. The pricing closed form was given by Margrabe via PDE method. In the following, we give a short review within the framework of risk neutral martingales\cite{8,7,9,10}. Part of the results will be used in the section of discussing stochastic interest rate case.

Suppose that in the physical probability space $(\Omega, \mathcal{F}, P)$, the prices of two underlying assets for an exchange option follow the geometric Brownian motion $s$, that is,

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t), \quad i = 1, 2 \tag{2}$$

where $\mu_1$, $\mu_1$, $\sigma_1$, $\sigma_2$, and $\rho_{12} \equiv \text{Corr}[dW_1, dW_2]$ are all constants. The price $C(t, S_1(t), S_2(t))$ of a (European) call exchange option at time $t$ is then given by

$$C(t, S_1(t), S_2(t)) = E_t^Q \left[ e^{-\int_t^T r(s) ds} \left[ S_1(T) - S_2(T) \right]^+ \right] \tag{3}$$

where $Q$ is the corresponding equivalent martingale measure. We should note here that these two processes in (2) are defined in the physical probability space $(\Omega, \mathcal{F}, P)$, however, the general valuation formula in (3) is derived in the risk-neutral probability space $(\Omega, \mathcal{F}, Q)$. The relationship between the risk-neutral probability space and the physical probability space is the standard one, which is described by the Girsanov transformation\cite{7}.

If the risk-free rate $r$ is constant, then the closed form formula for pricing the exchange option was first derived by Margrabe (1978) using the partial differential equation approach. However, we can also compute the expectation value in the risk neutral space, and the exchange option price will follow. It is shown that the approach gives the pricing formula identical to the one derived by Margrabe. In this case, we assume that there is no dividend
paying during the option’s life. Since the price processes of the two underlying assets are governed by (2), under the equivalent martingale measure \( Q \) with constant risk-free rate \( r \), we then have

\[
\left[ \begin{array}{c}
\log[S_1(T)] \\
\log[S_2(T)]
\end{array} \right] \mid \mathcal{F}_t \overset{\mathcal{D}}{\sim} \mathcal{N}\left( \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}, \begin{bmatrix}
\nu_1^2 & \rho_{12}\nu_1\nu_2 \\
\rho_{12}\nu_1\nu_2 & \nu_2^2
\end{bmatrix}\right)
\]

(4)

where

\[
A_i = \log[S_i(t)] + \left( r - \frac{\sigma_i^2}{2} \right)(T - t), \quad \nu_i^2 = \sigma_i^2(T - t), \quad i = 1, 2.
\]

(5)

Note that we can write

\[
S_1(T) - S_2(T) = e^{A_1 + \sigma_1\sqrt{T-t}Z_1} \left\{ 1 - e^{A_2 - A_1 + \sigma_2\sqrt{T-t}Z_2 - \sigma_1\sqrt{T-t}Z_1} \right\}
\]

where

\[
\begin{bmatrix}
Z_1 \\
Z_2 
\end{bmatrix} \overset{\mathcal{D}}{\sim} \mathcal{N}\left( \begin{bmatrix}
0 \\
0
\end{bmatrix}, \begin{bmatrix}
1 & \rho_{12} \\
\rho_{12} & 1
\end{bmatrix}\right).
\]

Hence,

\[
S_1(T) \geq S_2(T) \iff Z_3 \geq m
\]

where

\[
Z_3 \equiv \frac{\sigma_1 Z_1 - \sigma_2 Z_2}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}}
\]

\[
m \equiv \frac{A_2 - A_1}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}(T - t)},
\]

and

\[
\begin{bmatrix}
Z_1 \\
Z_3 
\end{bmatrix} \overset{\mathcal{D}}{\sim} \mathcal{N}\left( \begin{bmatrix}
0 \\
0
\end{bmatrix}, \begin{bmatrix}
1 & \eta \\
\eta & 1
\end{bmatrix}\right)
\]

with

\[
\eta \equiv \frac{\sigma_1 - \sigma_2 \rho_{12}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}}.
\]

So,

\[
E_t^Q[\left[S_1(T) - S_2(T)\right]^+] = \int_{-\infty}^{+\infty} \int_m^{+\infty} e^{A_1 + \sigma_1\sqrt{T-t}x} p(x, y, \eta) dy dx - \int_{-\infty}^{+\infty} \int_m^{+\infty} e^{A_2 + \sigma_1\sqrt{T-t}x + \eta y} p(x, y, \eta) dy dx
\]

\[
= I_1 - I_2
\]

(6)
where
\[ b \equiv \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2)(T - t)}, \]
\[ p(x, y, \eta) \equiv \frac{1}{2\pi \sqrt{1 - \eta^2}} e^{-\frac{1}{2(1-\eta^2)}(x^2 - 2\eta xy + y^2)}. \]

Note that
\[ I_1 = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} e^{A_1 + \sigma_1 \sqrt{T-t}x} p(x, y, \eta) \, dy \right\} \, dx \]
\[ = \int_{m}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{A_1 - \frac{1}{2(1-\eta^2)}(y^2 - (\eta y + (1-\eta^2)\sigma_1 \sqrt{T-t})^2)} \, dy \]
\[ = e^{A_1 + \frac{1}{2} \rho_{12}(T-t) + \eta^2 \sigma_1^2 (T-t)} \int_{m}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - \eta \sigma_1 \sqrt{T-t})^2} \, dy \]
\[ = e^{A_1 + \frac{1}{2} \sigma_1^2 (T-t)} \Phi(\eta \sigma_1 \sqrt{T-t} - m), \quad (7) \]

and
\[ I_2 = \int_{-\infty}^{+\infty} \left\{ \int_{m}^{+\infty} e^{A_2 - \sqrt{T-t}y} p(x, y, \eta) \, dx \right\} \, dy \]
\[ = \int_{m}^{+\infty} e^{A_2 - \eta y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2(1-\eta^2)}(y^2 - (\eta y + (1-\eta^2)\sigma_1 \sqrt{T-t})^2)} \, dy \]
\[ = e^{A_2 + \frac{1}{2} \sigma_1^2 (T-t) - \eta b \sigma_1 \sqrt{T-t} - \frac{b^2}{2}} \Phi(\eta \sigma_1 \sqrt{T-t} - m - b), \quad (8) \]

where \( \Phi(\cdot) \) is the standard Gaussian distribution function. Therefore, by (3), (7), and (8), we obtain
\[ E^Q_t \left[ e^{-r(T-t)} [S_1(T) - S_2(T)]^+] \right] \]
\[ = S_1(t) \Phi(\eta \sigma_1 \sqrt{T-t} - m) - S_2(t) \Phi(\eta \sigma_1 \sqrt{T-t} - m - b). \]

To sum up, for constant interest rate \( r \), the price of a call exchange option at time \( t \), is given by
\[ E^Q_t \left[ e^{-r(T-t)} [S_1(T) - S_2(T)]^+] \right] = S_1(t) \Phi(d_1) - S_2(t) \Phi(d_2), \quad (9) \]
where
\[
    d_1 = \frac{\log[S_1(t)/S_2(t)]}{\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2)(T-t)}} + \frac{1}{2}\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2)(T-t)},
\]

\[
    d_2 = d_1 - \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2)(T-t)},
\]

and \(\Phi(\cdot)\) is the standard Gaussian distribution function. This is consistent with the result of Margrabe derived with partial differential equation approach [1].

As shown above, the exchange option pricing can also be obtained within the framework of the risk neutral measure, consistent with the result obtained by Margrabe, which was derived with partial differential equation approach. In this case of constant interest rate, we wish to note that the interest rate \(r\) does not enter the pricing formula explicitly. This special feature motivates us to look into the issue of pricing exchange option when interest rate is stochastic in a general form. The next subsection discusses this in details within the framework of risk-neutral measures.

### 2.2 Stochastic interest rate

In this subsection, we discuss the issue of pricing exchange options when interest rate is stochastic. It will be shown below that the option price closed form can be found for most general one-factor stochastic interest rate processes (i.e. one Wiener process). Our result applies to the cases where one describes the interest rate such as Vasicek term structure, CIR term structure. These are special cases of our consideration.

In the following, it is assumed that we are always working in the risk-neutral probability space \((\Omega, \mathcal{F}, Q)\). Each process below is referred to this risk-neutral probability measure \(Q\). Now assume that the short rate \(r\) also follows a Markov diffusion process, that is,

\[
    dr(t) = \mu(r(t), t)dt + \sigma(r(t), t)dW_0(t).
\]

Here, we do not specify a concrete interest rate model. All we need to assume is that
the interest rate is a Markov diffusion process. The interest rate is correlated with the two underlying assets of the exchange option being considered. Assume further that the correlation matrix of \([dW_0(t), dW_1(t), dW_2(t)]\) is
\[
\begin{bmatrix}
1 & \rho_{01} & \rho_{02} \\
\rho_{01} & 1 & \rho_{12} \\
\rho_{02} & \rho_{12} & 1
\end{bmatrix}
\]
where \(\rho_{01}, \rho_{02},\) and \(\rho_{12}\) are constants. Using Cholesky decomposition, we can obtain the above correlation structure by setting
\[
\begin{align*}
dW_0(t) &= dB_0(t) \\
dW_1(t) &= \rho_{01} dB_0(t) + \sqrt{1 - \rho_{01}^2} dB_1(t) \\
dW_2(t) &= \rho_{02} dB_0(t) + \rho_{12} - \rho_{02} \rho_{01} \sqrt{1 - \rho_{01}^2} dB_1(t) + \sqrt{1 - \rho_{02}^2 - \left(\rho_{12} - \rho_{02} \rho_{01}\right)^2} dB_2(t)
\end{align*}
\]
where \(B_0(t), B_1(t),\) and \(B_2(t)\) are three independent standard Brownian motions. For the underlying assets, their prices will follow the processes below:
\[
\begin{align*}
\log[S_1(T)] &= \tilde{A}_1 + \tilde{\sigma}_1 \int_t^T dB_1(s) \\
\log[S_2(T)] &= \tilde{A}_2 + \tilde{\sigma}_2 \left[\tilde{\rho}_{12} \int_t^T dB_1(s) + \sqrt{1 - \tilde{\rho}_{12}^2} \int_t^T dB_2(s)\right]
\end{align*}
\]
where
\[
\begin{align*}
\tilde{A}_1 &= \log[S_1(t)] + \int_t^T r(s) ds - \frac{1}{2} \sigma_1^2 (T - t) + \sigma_1 \rho_{01} \int_t^T dB_0(s), \\
\tilde{A}_2 &= \log[S_2(t)] + \int_t^T r(s) ds - \frac{1}{2} \sigma_2^2 (T - t) + \sigma_2 \rho_{02} \int_t^T dB_0(s), \\
\tilde{\sigma}_1 &= \sigma_1 \sqrt{1 - \rho_{01}^2}, \\
\tilde{\sigma}_2 &= \sigma_2 \sqrt{1 - \rho_{02}^2}, \\
\tilde{\rho}_{12} &= \frac{\rho_{12} - \rho_{01} \rho_{02}}{\sqrt{1 - \rho_{01}^2} \sqrt{1 - \rho_{02}^2}}.
\end{align*}
\]
And hence, by conditional expectation, we can price a call exchange option as
\[
E_t^Q \left[ e^{-\int_t^T r(s) ds} \left[ S_1(T) - S_2(T) \right]^+ \right] = E_t^Q \left[ e^{-\int_t^T r(s) ds} E_t^Q \left[ \left[ S_1(T) - S_2(T) \right]^+ \mid \{ B_0(s) : t \leq s \leq T \} \right] \right].
\]
Note that given a sample path of \( \{B_0(s) : t \leq s \leq T\} \), using the results in (7) and (8), we obtain the following:

\[
E_t^Q \left( [S_1(T) - S_2(T)]^+ | \{B_0(s) : t \leq s \leq T\} \right) = e^{\tilde{A}_1 + \frac{1}{2} \tilde{\sigma}_1^2 (T-t)} \Phi(\tilde{\eta}\tilde{\sigma}_1 \sqrt{T-t} - \tilde{m}) - e^{\tilde{A}_2 + \frac{1}{2} \tilde{\sigma}_2^2 (T-t) - \tilde{b}} \Phi(\tilde{\eta}\tilde{\sigma}_1 \sqrt{T-t} - \tilde{m} - \tilde{b})
\]

where

\[
\tilde{\eta} \equiv \frac{\tilde{\sigma}_1 - \tilde{\sigma}_2 \tilde{\rho}_{12}}{\sqrt{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\tilde{\rho}_{12} \tilde{\sigma}_1 \tilde{\sigma}_2}}.
\]

\[
\tilde{m} \equiv \frac{\tilde{A}_2 - \tilde{A}_1}{\sqrt{(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\tilde{\rho}_{12} \tilde{\sigma}_1 \tilde{\sigma}_2)(T-t)}}.
\]

\[
\tilde{b} \equiv \sqrt{(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\tilde{\rho}_{12} \tilde{\sigma}_1 \tilde{\sigma}_2)(T-t)}.
\]

And it is straightforward to check

\[
\tilde{\eta}\tilde{\sigma}_1 \sqrt{T-t} - \tilde{m} = \log[S_1(t)/S_2(t)] + \frac{T-t}{2}(\sigma_2^2 \rho_{02} - \sigma_1^2 \rho_{01}) + (\sigma_1 \rho_{01} - \sigma_2 \rho_{02}) \int_t^T dB_0(s)
\]

\[
+ \frac{1}{2} \sqrt{(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\tilde{\rho}_{12} \tilde{\sigma}_1 \tilde{\sigma}_2)(T-t)},
\]

\[
\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\tilde{\rho}_{12} \tilde{\sigma}_1 \tilde{\sigma}_2 = (\sigma_1^2 + \sigma_2^2 - 2\rho_{12} \sigma_1 \sigma_2) - (\sigma_1 \rho_{01} - \sigma_2 \rho_{02})^2.
\]

\[
e^{\tilde{A}_1 + \frac{1}{2} \tilde{\sigma}_1^2 (T-t)} = S_1(t) e^{\sigma_1 \rho_{01} x - \frac{1}{2} \sigma_1^2 \rho_{01}^2 (T-t)},
\]

\[
e^{\tilde{A}_2 + \frac{1}{2} \tilde{\sigma}_2^2 (T-t) - \tilde{\eta}\tilde{\sigma}_1 \sqrt{T-t} + \frac{\tilde{b}^2}{2}} = S_2(t) e^{\sigma_2 \rho_{02} x - \frac{1}{2} \sigma_2^2 \rho_{02}^2 (T-t)}.
\]

Since \( \int_t^T dB_0(s) \mid F_t \sim \mathcal{N}(0, T-t) \), by (16), \( \cdots \), (21), we then obtain exchange option price with stochastic interest rates: Under the conditions (C1) the prices of two underlying assets follow the geometric Brownian motions (4), and there is no dividend paying during the option’s life; (C2) If the risk-free rate \( r \) is stochastic, then the price of a call exchange option at time \( t \), defined by (3), is given by

\[
E_t^Q \left[ e^{-\int_t^T r(s) ds} \left[ S_1(T) - S_2(T) \right]^+ \right] = S_1(t) \int_{-\infty}^{+\infty} \phi(x; \sigma_1 \rho_{01}, T-t) \Phi(d_1(x)) dx - S_2(t) \int_{-\infty}^{+\infty} \phi(x; \sigma_2 \rho_{02}, T-t) \Phi(d_2(x)) dx
\]
where

\[
\begin{align*}
    d_1(x) &= \frac{\log [S_1(t)/S_2(t)] + \frac{T-t}{2} (\sigma_2^2 - \sigma_1^2) + (\sigma_1 \rho_{01} - \sigma_2 \rho_{02})x}{\sqrt{[(\sigma_1^2 + 2 \sigma_2^2 - 2 \rho_{12} \sigma_1 \sigma_2) - (\sigma_1 \rho_{01} - \sigma_2 \rho_{02})^2] (T-t)}} \\
    d_2(x) &= d_1(x) - \frac{1}{2} \sqrt{[(\sigma_1^2 + 2 \sigma_2^2 - 2 \rho_{12} \sigma_1 \sigma_2) - (\sigma_1 \rho_{01} - \sigma_2 \rho_{02})^2] (T-t)},
\end{align*}
\]

\(\Phi(\cdot)\) is the standard Gaussian distribution function and \(\phi(\cdot; \mu, \nu)\) denote a Gaussian density function with mean \(\mu\) and variance \(\nu\).

Clearly, if \(\sigma_1 \rho_{01} = \sigma_2 \rho_{02}\) (that is, the covariance between processes \(dr(t)\) and \(dS_1(t)\) is the same as the covariance between processes \(dr(t)\) and \(dS_2(t)\)), then the pricing formulae for exchange options are the same for both the stochastic and deterministic term structures. Therefore, it is attempting to argue that we could use this solution as a control variate if one wants to do Monte-Carlo simulation to price spread option with nonzero strike price \(K\).

## 2.3 When underlying assets pay dividends

The pricing formula for exchange options above when interest rates are stochastic is derived with the assumption that the two underlying assets, such as stocks, pay no dividends during the options’ life. However, in case of the underlying assets also pay constant or known dividends, the question will become the general spread options pricing problem.

Assume the amount of dividends \(d_1(t_1), \ldots, d_1(t_m)\) for the first asset and \(d_2(s_1), \ldots, d_2(s_k)\) for the second asset to be paid at the dates \(0 < t_1 < \cdots < t_m < T\) and \(0 < s_1 < \cdots < s_k < T\) respectively are known in advance. Without loss of generality, we can write the dividend streams for both assets by

\[
d_i(t_1), \ldots, d_i(t_n), \quad i = 1, 2
\]

where \(n\) is the number of dividends payment dates for both assets. Therefore, at time \(t\), the
present values of all future dividends will be
\[ \sum_{j=1}^{n} d_i(t_j) e^{-\int_{t_j}^{t} r(\tau)d\tau} I_{[t,T]}(t_j), \quad i = 1, 2, \]
and the values of all dividends paid after time \( t \) and compounded at the risk-free rate till the option’s maturity date \( T \) is given by
\[ \sum_{j=1}^{n} d_i(t_j) e^{\int_{t_j}^{T} r(\tau)d\tau} I_{[t,T]}(t_j), \quad i = 1, 2. \]

By the same argument of Heath and Jarrow (1988), we decompose the capital gain processes \( G_i(t) \) into the asset price process \( S_i(t) \) and the dividends streams. Assume further that the \( G_i(t) \) also follows the geometric Brownian motions, that is,
\[ \frac{dG_i(t)}{G_i(t)} = \mu_i dt + \sigma_i dW_i(t), \quad i = 1, 2. \]

Then the capital gain \( G_i(t) \) may be written by
\[ G_i(t) = S_i(t) + \sum_{j=1}^{n} d_i(t_j) e^{\int_{t_j}^{T} r(\tau)d\tau} I_{[t_j,T]}(t) \]
\[ = S_i(t) + D_i(t) \]
where
\[ D_i(t) = \sum_{j=1}^{n} d_i(t_j) e^{\int_{t_j}^{T} r(\tau)d\tau} I_{[t_j,T]}(t). \]

Since the dynamics of the capital gains processes \( G_i(t) \), \( i = 1, 2 \), under the martingale measure \( Q \), is
\[ \frac{dG_i(t)}{G_i(t)} = r(t)dt + \sigma_i dW_i(t), \]
and \( G_i(0) = S_i(0) \) and \( G_i(T) = S_i(T) + D_i(T), \) \( i = 1, 2 \). Therefore, the risk-neutral pricing formula will be
\[ E_t^Q \left[ e^{-\int_t^T r(\tau)d\tau} \left[ S_1(T) - S_2(T) \right]^+ \right] = E_t^Q \left[ e^{-\int_t^T r(\tau)d\tau} \left[ G_1(T) - G_2(T) - (D_1(T) - D_2(T)) \right]^+ \right]. \]

In this case, we will have to employ numerical method to value the option price.
3 Conclusion

In this work, we have discussed the issue of pricing exchange options and spread options. Closed form for the exchange option price is provided explicitly when the interest rate is stochastic. Our result is valid for most general term structure model of one factor, which includes Vasicek model, CIR model, and well-known models as special cases. Our result indicates that only the correlation coefficients between the interest rate and the underlying assets will affect the exchange option price. In one special case, a completely explicit form of the option pricing can be obtained. We have argued that it is possible to use this solution as a control variate when doing Monte-Carlo simulation to price spread options for nonzero strike price $K$ and stochastic interest rate.

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