Quantum affine wreath algebras

\[ \begin{align*}
\begin{tikzpicture}
\draw[->, thick] (0,0) -- (0,1);
\draw[->, thick] (1,0) -- (1,1);
\draw[->, thick] (0,0) -- (1,1);
\end{tikzpicture}
\end{align*}\]

\[ = \begin{tikzpicture}
\draw[->, thick] (0,0) -- (0,1);
\draw[->, thick] (1,0) -- (1,1);
\draw[->, thick] (0,0) -- (1,1);
\end{tikzpicture} ,
\begin{tikzpicture}
\draw[->, thick] (0,0) -- (0,1);
\draw[->, thick] (1,0) -- (1,1);
\draw[->, thick] (0,0) -- (1,1);
\end{tikzpicture}
- \begin{tikzpicture}
\draw[->, thick] (0,0) -- (0,1);
\draw[->, thick] (1,0) -- (1,1);
\draw[->, thick] (0,0) -- (1,1);
\end{tikzpicture}
= z
\]

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Outline

Goal: Unify and generalize existing algebras by defining families of Hecke-like algebras depending on Frobenius algebras.

Overview:

1. Strict monoidal categories and string diagrams
2. Warm up: symmetric groups, degenerate affine Hecke algebras
3. Frobenius algebras
4. Affine wreath product algebras
5. Quantum affine wreath algebras
Strict monoidal categories

A strict monoidal category is a category $C$ equipped with
- a bifunctor (the tensor product) $\otimes : C \times C \rightarrow C$, and
- a unit object $1$,

such that, for objects $A$, $B$, $C$ and morphisms $f$, $g$, $h$,
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $1 \otimes A = A = A \otimes 1$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_1 \otimes f = f = f \otimes 1_1$.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.
\( \mathbb{k} \)-linear monoidal categories

Fix a commutative ground ring \( \mathbb{k} \).

A strict \( \mathbb{k} \)-linear monoidal category is a strict monoidal category such that
- each morphism space is a \( \mathbb{k} \)-module,
- composition of morphisms is \( \mathbb{k} \)-bilinear,
- tensor product of morphisms is \( \mathbb{k} \)-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For \( A_1 \xrightarrow{f} A_2 \) and \( B_1 \xrightarrow{g} B_2 \), the following diagram commutes:

\[
\begin{array}{ccc}
A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\
\downarrow f \otimes 1 & & \downarrow f \otimes 1 \\
A_2 \otimes B_1 & \xrightarrow{1 \otimes g} & A_2 \otimes B_2
\end{array}
\]
String diagrams

Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f : A \to B$ by:

![Diagram of a morphism]

The identity map $1_A : A \to A$ is a string with no label:

![Diagram of the identity map]

We sometimes omit the object labels when they are clear or unimportant.
String diagrams

Composition is *vertical stacking* and tensor product is *horizontal juxtaposition*:

\[ f \Rightarrow g = fg \quad f \otimes g = fg \]

The *interchange law* then becomes:

\[ f \Rightarrow g = fg = f \Rightarrow g = f \Rightarrow g \]

A morphism \( f : A_1 \otimes A_2 \to B_1 \otimes B_2 \) can be depicted:
Presentations of strict monoidal categories

One can give presentations of some strict $\mathbb{k}$-linear monoidal categories, just as for monoids, groups, algebras, etc.

**Objects:** If the objects are generated by some collection $A_i, i \in I$, then we have all possible tensor products of these objects:

$$1, \ A_i, \ A_i \otimes A_j \otimes A_k \otimes A_\ell, \ \text{etc.}$$

**Morphisms:** If the morphisms are generated by some collection $f_j, j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$1_{A_i}, \ f_j \otimes (f_i f_k) \otimes (f_\ell), \ \text{etc.}$$

We then often impose some relations on these morphism spaces.

**String diagrams:** We can build complex diagrams out of our simple generating diagrams.
The symmetric group category

Define a strict $\mathbb{k}$-linear monoidal category $Sym$ with one generating object $\uparrow$ and denote

$$1_\uparrow = \uparrow$$

We have one generating morphism

$$\begin{tikzpicture}[baseline = 0, center only, every node/.style = {circle, draw}, evaluate = {a = 1; b = 1; c = 1; d = 1;}]$$

$$\begin{tikzpicture}[baseline = 0, center only, every node/.style = {circle, draw}, evaluate = {a = 1; b = 1; c = 1; d = 1;}]$$

We impose the relations:

$$\begin{tikzpicture}[baseline = 0, center only, every node/.style = {circle, draw}, evaluate = {a = 1; b = 1; c = 1; d = 1;}]$$

Then

$$\text{End}_{Sym}(\uparrow \otimes n) = \mathbb{k}S_n$$

is the group algebra of the symmetric group on $n$ letters.
The symmetric group category

This monoidal presentation of $\mathbb{k}S_n$ is very efficient! We only needed

- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the “distant braid relation”

$$s_is_j = s_j s_i, \quad |i - j| > 1$$

for simple transpositions follows for free from the interchange law:
The degenerate affine Hecke algebra \( H_n \) of type \( A \) is
\[
\mathbb{k}[x_1, \ldots, x_n] \otimes \mathbb{k}S_n
\]
as a \( \mathbb{k} \)-module.

The factors \( \mathbb{k}[x_1, \ldots, x_n] \) and \( \mathbb{k}S_n \) are subalgebras, and
\[
s_ix_j = x_js_i, \quad j \neq i, i + 1,
\]
\[
s_ix_i = x_{i+1}s_i - 1.
\]
Obtain $\mathcal{H}$ from $Sym$ by adjoining one additional morphism (a dot)

$$\uparrow : \uparrow \to \uparrow$$

and one additional relation:

$$\begin{array}{c}
\begin{array}{c}
\circ \quad - \quad \circ
\end{array}
\end{array} = \uparrow \uparrow \uparrow .$$

Then

$$\text{End}_{\mathcal{H}}(\uparrow \otimes n) = H_n$$
Frobenius algebras

Definition (symmetric Frobenius algebra)
An associative algebra $A$ together with a linear trace map

$$\text{tr}: A \to k, \quad \text{tr}(ab) = \text{tr}(ba),$$

such that $\ker \text{tr}$ contains no nonzero left ideals.

Example ($k$)
$k$ with $\text{tr} = \text{id}_k$.

Example ($k[x]/(x^k)$)
$k[x]/(x^k)$ with $\text{tr}(x^\ell) = \delta_{\ell,k-1}$.

Example (Matrix algebra)
Matrix algebras with the usual trace.
Frobenius algebras: Examples

Example (Group algebra)
If $G$ is a finite group, then the group algebra $kG$ is a Frobenius algebra with

$$\text{tr}(g) = \delta_{g,1_G}, \quad g \in G.$$
Wreath product algebras

The symmetric group $S_n$ acts on $A^\otimes n$ by permutations:

$$\pi(a_n \otimes \cdots \otimes a_1) = a_{\pi^{-1}(n)} \otimes \cdots \otimes a_{\pi^{-1}(1)},$$

Wreath product algebra

The wreath product algebra is

$$\text{Wr}_n(A) = A^\otimes n \otimes \mathbb{k}S_n$$

as $\mathbb{k}$-modules. Multiplication is determined by

$$(a \otimes \pi)(b \otimes \sigma) = a_{\pi}(b) \otimes \pi\sigma.$$
Example ($A = \mathbb{k}$)

$\text{Wr}_n(\mathbb{k}) \cong \mathbb{k}S_n$

Example ($A = \text{Cl}$)

$\text{Wr}_n(\text{Cl})$ is the Sergeev algebra, which plays an important role in the projective representation theory of the symmetric group.

Example ($A = \mathbb{k}G$, $G = \mathbb{Z}/2\mathbb{Z}$)

$\text{Wr}_n(\mathbb{k}G)$ is the group algebra of the hyperoctahedral group, the Weyl group of type $B$.

Example ($A = \mathbb{k}G$, $G = \mathbb{Z}/r\mathbb{Z}$)

$\text{Wr}_n(\mathbb{k}G)$ is the group algebra of the complex reflection group $G(r, 1, n)$. 
The wreath product category

Define $\mathcal{W}r(A)$ by adjoining to $\text{Sym}$ morphisms (tokens)

$$\uparrow a : \uparrow \rightarrow \uparrow, \quad a \in A,$$

subject to the relations ($\alpha, \beta \in \mathbb{k}, \ a, b \in A$)

$$\uparrow 1 = \uparrow, \quad \uparrow \alpha a + \beta b = \alpha \uparrow a + \beta \uparrow b, \quad \uparrow a = \uparrow ab,$$

(so $A \mapsto \text{End}_{\mathcal{W}r(A)}(\uparrow)$, $a \mapsto \uparrow a$ is an algebra homomorphism) and

$$a \begin{array}{c} \longrightarrow \\ \text{a} \end{array} = \begin{array}{c} \longrightarrow \\ \text{a} \end{array}, \quad a \in A.$$

Then

$$\text{End}_{\mathcal{W}r(A)}(\uparrow \otimes^n) = \mathcal{W}r_n(A).$$
Teleporters

Fix a basis $B$ of $A$. The dual basis is

$$B^\vee = \{ b^\vee | b \in B \} \quad \text{defined by} \quad \text{tr} (b^\vee c) = \delta_{b,c}, \quad b, c \in B.$$

Exercise 1: $\sum_{b \in B} b \otimes b^\vee \in A \otimes A$ is independent of the basis $B$.

Exercise 2: For all $a \in A$, we have

$$\sum_{b \in B} ab \otimes b^\vee = \sum_{b \in B} b \otimes b^\vee a, \quad \sum_{b \in B} ba \otimes b^\vee = \sum_{b \in B} b \otimes ab^\vee.$$

Define the teleporter

$$\begin{array}{c}
\uparrow \quad \uparrow := \sum_{b \in B} b \uparrow \quad b^\vee \uparrow.
\end{array}$$

Then tokens “teleport” across teleporters:

$$\begin{array}{c}
\uparrow \quad \uparrow = a \quad \uparrow \quad \uparrow \quad \quad , \quad a \quad \uparrow \quad \uparrow = \uparrow \quad \uparrow \quad a.
\end{array}$$
Affine wreath product algebras

Define $W_{r, \text{aff}}(A)$ by adjoining to $W_r(A)$ the morphism (dot)

\[ \uparrow : \uparrow \rightarrow \uparrow \]

and relations

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{and relations}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \]

We define the affine wreath product algebra to be

\[ W_{r, \text{aff}}(A) := \text{End}_{W_{r, \text{aff}}(A)}(\uparrow \otimes n). \]
Affine wreath product algebras

Example ($A = \mathbb{k}$)

$\uparrow \uparrow \uparrow = \uparrow \uparrow \uparrow$ and $\text{Wr}_{n}^{\text{aff}}(\mathbb{k})$ is the degenerate affine Hecke algebra.

Example ($A = \text{Cl}$, Clifford algebra)

$\text{Wr}_{n}^{\text{aff}}(\text{Cl})$ is the affine Sergeev algebra, aka the degenerate affine Hecke–Clifford algebra.

Example ($A = \mathbb{k}G$)

$\text{Wr}_{n}^{\text{aff}}(\mathbb{k}G)$ is the wreath Hecke algebra (Wan–Wang).

Example (Affine zigzag algebras)

When $A$ is a certain zigzag algebra, $\text{Wr}_{n}^{\text{aff}}(A)$ is related to imaginary strata for quiver Hecke algebras (Kleshchev–Muth).
Fix $z \in k$. Let $\mathcal{H}(z)$ be the strict $k$-linear monoidal category with one generating object $\uparrow$, generating morphisms

$$\begin{align*}
\begin{array}{cccc}
\xymatrix{\uparrow & \uparrow \\
\downarrow & \downarrow}
\end{array}
, \quad
\begin{array}{cccc}
\xymatrix{\uparrow & \uparrow \\
\downarrow & \downarrow}
\end{array}
: \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow
\end{align*}$$

and relations

$$\begin{align*}
\begin{array}{cccc}
\xymatrix{\uparrow & \uparrow \\
\downarrow & \downarrow}
\end{array}
&= \begin{array}{cccc}
\xymatrix{\uparrow & \\
\downarrow & \\
\uparrow & \\
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\downarrow & \\
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\end{array},
\begin{array}{cccc}
\xymatrix{\uparrow & \uparrow \\
\downarrow & \downarrow}
\end{array}
&= \begin{array}{cccc}
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\end{array},
\begin{array}{cccc}
\xymatrix{\uparrow & \uparrow \\
\downarrow & \downarrow}
\end{array}
&= \begin{array}{cccc}
\xymatrix{\uparrow & \\
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\uparrow & \\
\downarrow & \\
\uparrow & \\
\downarrow & }
\end{array},
\begin{array}{cccc}
\xymatrix{\uparrow & \uparrow \\
\downarrow & \downarrow}
\end{array}
- \begin{array}{cccc}
\xymatrix{\uparrow & \uparrow \\
\downarrow & \downarrow}
\end{array}
&= z \uparrow \uparrow
\end{align*}$$

(skein relation).

Then

$$H_n(z) := \text{End}_{\mathcal{H}(z)}(\uparrow \otimes^n)$$

is the Iwahori–Hecke algebra of type $A_{n-1}$ (often $z = q - q^{-1}$).
Define $\mathcal{H}^{\text{aff}}(z)$ by adjoining to $\mathcal{H}(z)$ the invertible morphism

$$\uparrow : \uparrow \rightarrow \uparrow$$

and relations

$$\begin{array}{ccc}
\begin{array}{c}
\circ \circ \\
\circ \circ \\
\end{array}
\begin{array}{c}
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\end{array}
= \begin{array}{c}
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, \quad
\begin{array}{c}
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\begin{array}{c}
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\end{array}
= \begin{array}{c}
\circ \\
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\end{array}
\begin{array}{c}
\circ \\
\circ \\
\end{array}.
\end{array}$$

Then

$$H_n^{\text{aff}}(z) := \text{End}_{\mathcal{H}^{\text{aff}}} (\uparrow \otimes n)$$

is the affine Hecke algebra of type $A_{n-1}$ (often $z = q - q^{-1}$).
Define $\mathcal{H}(A, z)$ by adjoining to $\mathcal{H}(z)$ morphisms

$$\uparrow a : \uparrow \rightarrow \uparrow, \quad a \in A,$$

subject to the relations ($\alpha, \beta \in \mathbb{k}, \ a, b \in A$)

$$\uparrow 1 = \uparrow, \quad \uparrow \alpha a + \beta b = \alpha \uparrow a + \beta \uparrow b, \quad \uparrow ab = \uparrow a \uparrow b,$$

$$\uparrow a \downarrow a - \uparrow a \downarrow a = z \uparrow \uparrow 1,$$

$$\uparrow a \downarrow a = \uparrow a \downarrow a, \quad \uparrow a \downarrow a = \uparrow a \downarrow a, \quad \uparrow a \downarrow a = \uparrow a \downarrow a.$$

We call

$$H_n(A, z) := \text{End}_{\mathcal{H}(A, z)}(\uparrow \otimes n)$$

a Frobenius Hecke algebra.
Example \((A = \mathbb{k})\)

\[ H_n(\mathbb{k}, z) \text{ is an Iwahori–Hecke algebra.} \]

Example \((A = \mathbb{k}G, \ G \text{ a cyclic group})\)

\[ H_n(\mathbb{k}G, z) \text{ is a Yokonuma–Hecke algebra.} \]

Other choices of \(A\) yield \textit{new} algebras.
Quantum affine wreath algebras

Define $\mathcal{W}_r^{\text{aff}}(A, z)$ by adjoining to $\mathcal{H}(A, z)$ the invertible morphism

\[
\uparrow &: \uparrow \rightarrow \uparrow
\]

and relations

\[
\begin{align*}
\begin{array}{c@{}c@{}c}
\xrightarrow{ullet} & = & \left\langle \begin{array}{c}
\bullet \quad \bullet
\end{array} \right. \\
\left\langle \begin{array}{c}
\bullet \quad \bullet
\end{array} \right. & = & \begin{array}{c@{}c}
\bullet & \bullet
\end{array}
\end{array}
\end{align*}
\]

We call

\[
\mathcal{W}_r^{\text{aff}}(A, z) := \text{End}_{\mathcal{W}_r^{\text{aff}}(A, z)}(\uparrow \otimes n)
\]

a quantum affine wreath algebra.

One could also call it a affine Frobenius Hecke algebra.
Quantum affine wreath algebras

Example \((A = \mathbb{k})\)
\[
\Wr_{n}^{\text{aff}}(\mathbb{k}, z) \text{ is an affine Hecke algebra.}
\]

Example \((A = \mathbb{k}G, \ G \text{ a cyclic group})\)
\[
\Wr_{n}^{\text{aff}}(\mathbb{k}G, z) \text{ is an affine Yokonuma–Hecke algebra.}
\]

Other choices of \(A\) yield new algebras.

Example \((A = \text{zigzag algebra})\)
\[
\Wr_{n}^{\text{aff}}(A, z) \text{ is a quantum analogue of affine zigzag algebras.}
\]
The Jucys–Murphy elements in $\mathbb{C}S_n$ are

\[ J_1 = 0, \quad J_i = (1 \, i) + (2 \, i) + \cdots + (i - 1 \, i), \quad i = 2, \ldots, n. \]

**Useful facts**

- $J_n$ commutes with elements of $\mathbb{C}S_{n-1}$.
- The $J_i$ generate a commutative subalgebra of $\mathbb{C}S_n$.
- The basis elements of Young’s seminormal representation are eigenvectors for the $J_i$.

**Theorem (Jucys)**

The center of $\mathbb{C}S_n$ is generated by symmetric polynomials in the $J_i$.

Jucys–Murphy elements play a central role in the Okounkov–Vershik approach to the representation theory of symmetric groups.
Recall the degenerate affine Hecke algebra:

$$H_n = \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}S_n$$

with relations

$$s_ix_i = x_{i+1}s_i - 1, \quad s_ix_j = x_js_i, \quad j \neq i, i + 1.$$ 

Clearly we have an injection

$$\mathbb{C}S_n \hookrightarrow H_n.$$ 

We also have a surjection

$$H_n \twoheadrightarrow \mathbb{C}S_n, \quad x_i \mapsto J_i.$$ 

This is an example of a **cyclotomic quotient**. Map is uniquely determined by $$x_1 \mapsto 0$$. In general, we can quotient by any polynomial in $$x_1$$. 

Alistair Savage (Ottawa)
Jucys–Murphy elements

For $1 \leq i < j \leq n$, define

$$t_{i,j} = \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$$



In the wreath product algebra, we define the Jucys–Murphy elements

$$J_1 = 0, \quad J_i = t_{1,i}(1i) + t_{2,i}(2i) + \cdots + t_{i-1,i}(i-1i), \quad 1 \leq i \leq n.$$ 

We have a surjection

$$\text{Wr}_{n}^{\text{aff}}(A) \rightarrow \text{Wr}_{n}(A), \quad x_i \mapsto J_i,$$

where $x_i$ is a dot on the $i$-th strand.

Quantum version: Can also define Jucys–Murphy elements in $\text{Wr}_{n}(A, z)$ generalizing usual Jucys–Murphy elements for the Hecke algebra.
Other structure theory results

One can prove many general structure theory results in a uniform way:

- Demazure operators (aka divided difference operators)
- Basis theorem:

  \[ \begin{align*}
  W_{r,n}^{\text{aff}}(A) &\cong A^\otimes n \otimes \mathbb{k}[x_1, \ldots, x_n] \otimes \mathbb{k}S_n \\
  W_{r,n}^{\text{aff}}(A, z) &\cong A^\otimes n \otimes \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \otimes H_n(z)
  \end{align*} \]

  as \( \mathbb{k} \)-modules.

- Center:

  \[ \begin{align*}
  Z(W_{r,n}^{\text{aff}}(A)) &= (Z(A)^\otimes n \otimes \mathbb{k}[x_1, \ldots, x_n])^{S_n} \\
  Z(W_{r,n}^{\text{aff}}(A, z)) &= (Z(A)^\otimes n \otimes \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])^{S_n}
  \end{align*} \]

- Mackey theorem
- Cyclotomic quotients (basis theorem, Mackey theorem, etc.)
Fix $k \in \mathbb{Z}$. The Heisenberg category (Khovanov, Mackaay–S., Brundan) is defined by adjoining to the degenerate affine Hecke category $\mathcal{H}$ an object

\[
\downarrow
\]

and morphisms and relations so that

- $\uparrow$ is right dual to $\downarrow$,
- we have an isomorphism

\[
\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus 1 \oplus k \quad \text{(when } k \geq 0\text{)},
\]

\[
\uparrow \otimes \downarrow \oplus 1 \oplus (-k) \cong \downarrow \otimes \uparrow \quad \text{(when } k \leq 0\text{)}
\]

(the inversion relation).

Acts on modules for degenerate cyclotomic Hecke algebras, categorifies the Heisenberg algebra.
We can now repeat this with our (quantum) affine wreath categories!

We get:

- **quantum Heisenberg category** (Licata–S., Brundan–S.–Webster)
- **Frobenius Heisenberg category** (Rosso-S., S.)
- **quantum Frobenius Heisenberg category** (Brundan–S.–Webster, work in progress)

These act on modules for the corresponding cyclotomic quotients.

Can also define an **odd quantum Frobenius Heisenberg category**...