Iterative Decoding of Trellis-Constrained Codes
inspired by Amplitude Amplification
(Preliminary Version)

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Abstract—We propose a decoder for Trellis-Constrained Codes, a super-class of Turbo- and LDPC codes. Inspired by amplitude amplification from quantum computing, we attempt to amplify the relative likelihood of the most likely codeword until it stands out from all other codewords.

I. INTRODUCTION

The surprising discovery of Turbo-codes [1] in the early 90’s was a major breakthrough in the field of digital communication. Two simple codes combined with an interleaver can be decoded in a nearly optimal way with loopy belief-propagation (BP) [10], [9] so that they operate close to Shannon’s channel capacity [11]. This lead to the rediscovery of LDPC codes [6] and to the investigation of more general constructions like Trellis-Constrained Codes (TCCs) [5], [4]. However, it turns out that near optimal decoding with BP only works for some specific classes of TCCs, but not in general.

In this paper we describe a method for the probabilistic computation of the most likely codeword in a TCC w.r.t. a vector of symbol likelihoods. We iteratively update the symbol likelihoods so that the relative likelihood of the most likely codeword continually increases until it hopefully stands out from all other codewords. The algorithm is inspired by amplitude amplification [2], [3] which is used in quantum algorithms like Grover search [7]. Our algorithm converges in a more controlled way than BP.

II. PRELIMINARIES

An intersection code $C\cap$ is defined as

$$C\cap := \{ c : c \in C_1 \cap C_2 \},$$

where $C_1, C_2 \subseteq S = \{-1, +1\}^n$ are chosen such that the code $C_1$ and the interleaved code $C_2$ have a low trellis complexity. Some examples of TCCs are represented in Fig. 1.

For a memoryless binary channel defined by $\gamma$, a received word $r = (r_1, ..., r_n) \in \mathbb{R}^n$ and a word $s = (s_1, ..., s_n) \in S$, we define the log-likelihood ratio

$$L(r) := \frac{1}{2} \ln \frac{P(r|+1)}{P(r|-1)}$$

with $P(r|s) := \gamma^{rs}$, and we use Iverson brackets $\langle$[false]$\rangle := 0$ and $\langle$[true]$\rangle := 1$ to define the code-constrained likelihoods

$$P_r(r|s) := \gamma^{rs} \langle s \in C_1 \rangle \langle s \in C_2 \rangle = \gamma^{rs} \langle s \in C\cap \rangle.$$

More details on the channels can be found in the Appendix.

III. LIKELIHOOD AMPLIFICATION

The objective of an ML decoder is to determine

$$\hat{c} = \arg \max_{s \in S} P_r(r|s) = \arg \max_{s \in C\cap} \gamma^{rs}.$$

(a) A TCC constructed from convolutional codes.

(b) A TCC representation of a LDPC code.

Figure 1: Examples of Trellis-Constrained Codes.
To reflect the structure of $\mathbb{C}_r$, with the two constituent codes and the symbol constraints, we can equivalently write

$$(e, c) = \arg \max_{(c, s) \in C_1 \times C_2} \gamma_{w_1 s^T + w_2 s' T} \cdot \prod_{j=1}^{n} \langle s_j = s'_j \rangle$$

where $w_1 + w_2 = r$ with $w_1, w_2 \in \mathbb{R}^n$.

A. Overview

During the decoding process we iteratively update $w_1$ and $w_2$, and in this description we denote the corresponding values in iteration $i$ as $w_1^{(i)}$ and $w_2^{(i)}$. Further we consider

- the likelihood of the most likely codeword $p_c^{(i)} := \gamma_{w_1^{(i)} e^T + w_2^{(i)} e^T}$, and
- the cumulated likelihood of all words in $C_1 \times C_2$

$$\Xi^{(i)} := \sum_{(s, s') \in C_1 \times C_2} \gamma_{w_1^{(i)} s^T + w_2^{(i)} s' T}.$$

Initially, we set

$$w_1^{(0)} \leftarrow r/2$$
$$w_2^{(0)} \leftarrow r/2$$

and we estimate

$$p_c^{(0)} = \gamma_{w_1^{(0)} e^T + w_2^{(0)} e^T} = \gamma_{re^T}.$$

Note that for the BEC we have $p_c^{(0)} = \sum_i \langle r_i \rangle \neq 0$. Then, in iterations where $i$ is even we compute

$$w_1^{(i+1)} \leftarrow w_1^{(i)} + \Delta^{(i)}$$
$$w_2^{(i+1)} \leftarrow w_2^{(i)} - \Delta^{(i)}$$

and in iterations where $i$ is odd we compute

$$w_1^{(i+1)} \leftarrow \rho^{(i)}, \ w_1^{(i+1)}$$
$$w_2^{(i+1)} \leftarrow \rho^{(i)}, \ w_2^{(i+1)}.$$

The corresponding $\Delta^{(i)} \in \mathbb{R}^n$ and $\rho^{(i)} \in \mathbb{R}$ are chosen so that

$$p_c^{(i+1)} \Xi^{(i+1)} \geq p_c^{(i)} \Xi^{(i)}$$

which means that the relative likelihood of the most likely codeword stays the same or increases in every step. Details and stopping criteria are explained in the following sections.

B. Choice of $\Delta^{(i)}$

In order to choose $\Delta^{(i)}$ so that (3) holds, let us investigate the relation between $p_c^{(i+1)}$ and $p_c^{(i)}$, and between $\Xi^{(i+1)}$ and $\Xi^{(i)}$ in (4).

First, we have

$$p_c^{(i+1)} = \gamma_{w_1^{(i)} + \Delta^{(i)} + w_2^{(i)} - \Delta^{(i)} e^T} = p_c^{(i)}$$

for any $\Delta^{(i)} \in \mathbb{R}^n$. The same holds for all codewords in $C_r$, so that the most likely word in $C_r$ under $r$ also remains the most likely word under $w_1$ and $w_2$.

Then, to understand the relation between $\Xi^{(i)}$ and $\Xi^{(i+1)}$ let us assume

$$\Delta^{(i)} = (0, \ldots, 0, \delta_j, 0, \ldots, 0)$$

with a single possibly non zero value $\delta_j$ at position $j$ and

$$\Xi^{(i)} = \Xi_{-1}^{(i)} + \Xi_{0}^{(i)} + \Xi_{+1}^{(i)}$$

where

$$\Xi_{-1}^{(i)} := \sum_{(s, s') \in C_1 \times C_2} \gamma_{w_1^{(i)} s^T + w_2^{(i)} s' T} \cdot \langle s_j = -1 \rangle \cdot \langle s'_j = +1 \rangle,$$

$$\Xi_{0}^{(i)} := \sum_{(s, s') \in C_1 \times C_2} \gamma_{w_1^{(i)} s^T + w_2^{(i)} s' T} \cdot \langle s_j = s'_j \rangle,$$

$$\Xi_{+1}^{(i)} := \sum_{(s, s') \in C_1 \times C_2} \gamma_{w_1^{(i)} s^T + w_2^{(i)} s' T} \cdot \langle s_j = +1 \rangle \cdot \langle s'_j = -1 \rangle.$$

It follows from (1) that

$$\Xi^{(i+1)} = \gamma^{-2\delta_j} \cdot \Xi_{-1}^{(i)} + \Xi_{0}^{(i)} + \gamma^{2\delta_j} \cdot \Xi_{+1}^{(i)}.$$

Thus, for $\delta_j = 0$ we have $\Xi^{(i+1)} = \Xi^{(i)}$, and for $\delta_j$ equal to

$$\delta_{j_{\text{min}}} = \arg \min_{\delta_j} (\Xi^{(i+1)}) = (\log \gamma \Xi_{-1}^{(i)} - \log \gamma \Xi_{+1}^{(i)})/4$$

we obtain a minimal $\Xi^{(i+1)}$ for which

$$\Xi^{(i+1)} \leq \Xi^{(i)}.$$

Hence, we can pick a position $j$ and compute $\Delta^{(i)}$ so that (4) and (5) hold. This implies that (3) must also hold. Like in quantum computing the decoder does not care whether the symbols $s^{(1)} = s^{(2)}$ are both 0 or both 1, the relative likelihood of both states are increased. The effect of a such an optimization is illustrated in Figure 2.

In practice it can be more efficient to compute $\delta_1, \ldots, \delta_n$ for all symbols at once, according to (5), and to use $\Delta^{(i)} = (\kappa \cdot \delta_1, \ldots, \kappa \cdot \delta_n)$, where $\kappa$ is a scaling factor that is used to prevent a too big step that could result from correlations.
C. Choice of $\rho^{(i)}$

In order to choose $\rho^{(i)}$ so that (3) holds, let us investigate the relation between $p^{(i+1)}_e$ and $\Xi^{(i)}$ in (5).

First, we have

$$p^{(i+1)}_e = \gamma_p w_1^{(i)} + \gamma w_2^{(i)} e^T = (p^{(i)}_e)^{\rho}.$$  

The same holds for all words in $C\cap$ and as exponentiation is monotonous, the most likely codeword for $w_1, w_2$ remains also the most likely codeword for $\rho \cdot w_1, \rho \cdot w_2$.

Concerning the relation between $\Xi^{(i+1)}$ and $\Xi^{(i)}$, it is obvious that for $\rho = 1$ we have $\Xi^{(i+1)} = \Xi^{(i)}$, but there is no simple expression for $\rho \neq 1$. However, as one can always compute $\Xi^{(i+1)}$ for a given $\rho$, one can try to optimize $\rho$ using e.g. a gradient technique. Most importantly, one can always ensure that (3) holds, by computing $p^{(i+1)}$ for a given $\rho$. We propose and investigate two simple approaches in the context of our experiments in Section IV.

D. Stopping Criteria

Decoding is successful when $\hat{e}^{(i)} = (\hat{e}_1^{(i)}, ..., \hat{e}_n^{(i)})$ with

$$\hat{e}_j^{(i)} := \text{sign} \left( \frac{\sum_{(s,s') \in C_1 \times C_2} \gamma^{w_1^{(i)} x^T + w_2^{(i)} x'^T} \langle s_j = s'_j \rangle + 1}{\sum_{(s,s') \in C_1 \times C_2} \gamma^{w_1^{(i)} x^T + w_2^{(i)} x'^T} \langle s_j = s'_j \rangle - 1} \right)$$

is contained in $C_1$ and $C_2$.

IV. EXPERIMENTS

TBD

V. CONCLUSIONS

TBD

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APPENDIX

We consider channels for which $P(r|s) \propto \gamma^s$, where the value $\gamma$ is dependent on the channel. For example,

- the Binary Symmetric Channel (BSC) where

$$P(r|s) := \begin{cases} 1-p & \text{if } r = s \\ p & \text{if } r \neq s \end{cases}$$

so that

$$L(r) = \begin{cases} (r = +1) \cdot \frac{1}{2} \ln \frac{1-p}{p} + \\ (r = -1) \cdot \frac{1}{2} \ln \frac{p}{1-p} \end{cases} = \left( \frac{1}{2} \ln \frac{1-p}{p} \right) \cdot r$$

and

$$\gamma_{\text{BSC}} = \exp \left( \frac{1}{2} \ln \frac{1-p}{p} \right) = \sqrt{\frac{1-p}{p}}$$

- the Binary Erasure Channel (BEC) where

$$P(r|s) := \begin{cases} 1-p & \text{if } r = s \\ p & \text{if } r = 0 \\ 0 & \text{if } r = -s \end{cases}$$

so that

$$L(r) = \begin{cases} (r = +1) \cdot \frac{1}{2} \ln \frac{1-p}{0} + \\ (r = 0) \cdot \frac{1}{2} \ln \frac{p}{0} + \\ (r = -1) \cdot \frac{1}{2} \ln \frac{0}{1-p} \end{cases} = \infty \cdot r$$

and (with $\infty \cdot 0 := 0$ and $\infty^0 := 1$)

$$\gamma_{\text{BEC}} = \exp (\infty) = \infty;$$

- the Additive White Gaussian Noise Channel where

$$P(r|s) := \frac{1}{\sqrt{2\pi\sigma}} \exp \left( \frac{(r-s)^2}{2\sigma^2} \right)$$

so that

$$L(r) = \frac{1}{2} \ln \left( \exp \left( \frac{(r-1)^2}{2\sigma^2} \right) - \frac{1}{2} \ln \left( \exp \left( \frac{(r+1)^2}{2\sigma^2} \right) \right) \right) = -\sigma^2 \cdot r$$

and

$$\gamma_{\text{AWGN}} = \exp (-\sigma^{-2})$$.