NEUTRAL POINTS OF OSCILLATION MODES ALONG EQUILIBRIUM SEQUENCES OF RAPIDLY ROTATING POLYTROPES IN GENERAL RELATIVITY:
APPLICATION OF THE COWLING APPROXIMATION

SHIN'ICHIRO YOSHIDA\(^1\) AND YOSHIHARU ERIGUCHI

Department of Earth Science and Astronomy, Graduate School of Arts and Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan; yoshida@valis.c.u-tokyo.ac.jp

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ABSTRACT

The relativistic Cowling approximation in which all metric perturbations are omitted is applied to nonaxisymmetric infinitesimal oscillations of uniformly rotating general relativistic polytropes.

Frequencies of lower order \(f\)-modes, which are important in analysis of secular instability driven by gravitational radiation, are investigated, and neutral points of the mode along equilibrium sequences of rotating polytropes are determined. Since this approximation becomes more accurate as stars are more relativistic and/or as they rotate more rapidly, we will be able to analyze how a rotation period of a neutron star may be limited by this instability.

Possible errors in determining neutral points caused by omitting metric perturbations are also estimated.

Subject headings: relativity — stars: oscillations — stars: rotation

1. INTRODUCTION

Since the discovery of the nonaxisymmetric secular instability of rotating stars driven by gravitational radiation (Chandrasekhar 1970; Friedman & Schutz 1978; Friedman 1978), several authors have pointed out the possibility that it could limit the rotational period of a neutron star. As the instability sets in at neutral points in the absence of viscosity, it would become essential to determine these neutral points for rapidly rotating stars in general relativity. However, thus far, only several investigations in Newtonian gravity (Managan 1985; Imamura, Friedman & Durisen 1985) and one post-Newtonian investigation (Cutler & Lindblom 1992) have been made because it has been very difficult to manage general relativistic perturbations of rotating stars completely.

In such a situation, Yoshida & Kojima (1997) have examined the accuracy of the relativistic version of the Cowling approximation for oscillations of slowly rotating relativistic stars. Comparing the results by the relativistic Cowling approximation (hereafter RCA) (for earlier results, see, e.g., McDermott, Van Horn, & Scholl 1983; Finn 1988; Lindblom & Splinter 1990; Ipser & Lindblom 1992) with “exact” eigenfrequencies obtained by fully relativistic analysis, they argued that the approximation works well for slowly rotating relativistic stars. This suggests that we can determine, with fairly good accuracy, the neutral points by this approximation.

We here investigate the “counterrotating” \(f\)-mode oscillations of rapidly rotating relativistic stellar models by the RCA. Although, as seen in the Newtonian results (Robe 1968), errors of \(f\)-mode eigenfrequencies are larger than those of \(p\)- or \(g\)-modes, absolute values of errors arising from the RCA are likely to become smaller for relativistic and rotating stars compared to the Newtonian values (see below). Furthermore we will estimate roughly the size of the corrections to the results of the RCA.

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in Newtonian gravity. First, our frequencies for relativistic spherical stars agree well with those of other RCA calculations (Yoshida & Kojima 1997). As for rapidly rotating Newtonian stars, by comparing our results with those of the Newtonian Cowling approximation that are obtained by the modified code of Yoshida & Eriguchi (1995), we have found that our RCA code can reproduce the neutral points of $f$-modes with sufficient accuracy, i.e., within errors of 2%.

### 3.2. Neutral Points along Sequences

From the intersection of $T/W$-$\sigma$ curves of equilibrium sequences and $\sigma = 0$ we can obtain neutral points, which are tabulated in Table 1.

In the framework of Newtonian gravity, critical values of $T/W$ for the $m = 2$ “bar” mode have been considered to be universal, i.e., $T/W \sim 0.14$ irrespective of compressibility and/or the rotation laws (Tassoul 1978). However, even for $N = 0.5$ polytropes, we could not find a neutral point against this bar mode in the weak gravity limit, i.e., for small $\kappa$. This may result from the crudeness of the Cowling approximation for lower order modes. In the Newtonian Cowling approximation, smaller frequencies are obtained than those calculated from the full treatment. As a consequence, the critical values of $T/W$ at neutral points tend to be raised in this approximation. Thus, even if $N < 0.808$, neutral points for the bar mode may not be reached before the termination of equilibrium sequences owing to mass shedding.

On the other hand, neutral points for the bar mode could be found for stronger gravity, although the value of $T/W$ does not coincide with the Newtonian universal value of 0.14. It is remarkable that even for soft equations of state ($N > 0.808$) with sufficiently strong gravity, neutral points for the bar mode do appear. This is not, however, so surprising because the general relativistic effect on equilibrium states becomes significant.\(^3\)

Our numerical computational results can be summarized as follows: (1) for sequences with $N$ and $\kappa$ held constant, values of $T/W$ at neutral points become smaller as the value of $m$ increases; (2) for sequences with $N$ and $m$ held constant, values of $T/W$ at neutral points become smaller as the value of $\kappa$ increases; (3) for sequences with $m$ and $\kappa$ held constant, values of $T/W$ at neutral points becomes smaller as the value of $N$ increases.

In Newtonian gravity, the normalized eigenfrequency scales as $\sigma(4\pi\rho_0)^{1/2} \sim \sqrt{\langle \tilde{\rho} / \rho_0 \rangle}$, where $\rho_0$ is the mass density at the center, $\tilde{\rho}$ is the mean density of the star, and $l$ is the zenithal quantum number. From this relation, normalized frequencies become smaller for models with higher mass concentration toward the center. Because of strong gravity, matter distributions are concentrated toward the central region for relativistic models. In other words, relativistic stars can be considered to have effectively become softer. Consequently, frequencies become lower for larger values of $\kappa$.

### 4. Discussion: How Far Is the RCA Reliable?

Before applying the RCA to realistic neutron stars, it is important to note how far this approximation can be reliably applied. In particular we need to estimate how the strength of relativity and the amount of rotation affects the results.

It should be noted that eigenfunctions of energy density perturbation for $f$-modes are peaked near the surface. This implies that surface regions are important for $f$-modes. On the other hand, as is known in Newtonian models, the relative importance of gravitational perturbations declines as the amount of mass participating in oscillations decreases.

Consequently, relativistic gravity can be improved to approximate the effect of the matter gravity by having the role of metric perturbations less important. Furthermore, perturbed density distribution of rotating models is more sharply peaked toward the surface of the star as the stars are spun up. Therefore, the accuracy of the Cowling approximation is greater for rotating models than for spherical ones. Thus these two factors can be said qualitatively to have advantages in deciding neutral points.

Our proposal is as follows. We assume that the qualitative dependence of the mode on stellar rotation in general relativity is similar to that of the Newtonian counterpart. In Newtonian theory, differences between the eigenfrequencies obtained by the Cowling approximation and those obtained by the full analysis including gravitational perturbations decrease monotonically as stars are spun up. If this tendency is the same for general relativistic rotating stars, the maximal error of the RCA in determining neutral points can be estimated.

Let us select one $T/W$-$\sigma$ curve for a certain equilibrium sequence computed by the RCA. Since the maximum error of this curve is assumed to be that of the spherical model, the amount of the maximum error can be calculated by comparing our results with exact values of spherical models in general relativity. A parallel displacement of this $T/W$-$\sigma$ curve by this amount of the maximum error yields a curve.

\(^{3}\) After submitting this paper, we received a preprint by Stergioulias & Friedman (1997), where they also report having obtained neutral points of the $m = 2$ mode for softer equations of state while taking metric perturbations into consideration.
Therefore if the error of the eigenfrequency for the spherical model and the inclination of $T/W$-$\sigma$ curve of the RCA at the approximated neutral point are known, the maximal error in determining a neutral point will be determined by

$$\text{maximal error of } \frac{T}{|W|} = \Delta\sigma_0 \cdot \frac{d(T/|W|)}{d\sigma},$$

where $\Delta\sigma_0$ is the maximum error of the frequency.

For $N = 1$ polytropes we have obtained the maximal errors of $T/W$ at neutral points for several modes (Fig. 2). We also display the post-Newtonian values of $T/W$ at neutral points computed by Cutler & Lindblom (1992). The two sets of results agree within the maximal errors estimated here except for $m = 2$ modes, which are not available for the post-Newtonian calculations.

5. CONCLUDING REMARKS

Here we have shown that the RCA works rather well in evaluating eigenfrequencies of rotating stars in general relativity, despite its naivety. As eigenmode analysis in the RCA is much easier than in the post-Newtonian treatment, this approximation can be extensively used in determining oscillatory frequencies of realistic neutron stars and/or determining their neutral points.

The main drawback of this approximation is that it is hard to include the effect of gravitational radiation that is responsible for damping or growing of corresponding modes. The post-Newtonian treatment can, in principle, systematically include this effect by increasing further the order of approximation. In contrast, the RCA in itself cannot include its effect consistently. Some modification such as an estimation of effective damping due to gravita-
tional radiation by using eigenfunctions obtained by the RCA may be needed.

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APPENDIX

BASIC EQUATIONS OF THE RCA

A1. ADIABATIC PERTURBATIONS TO STATIONARY AXISYMMETRIC CONFIGURATIONS

The background spacetime is expressed by the following metric that is used for stationary axisymmetric spacetime:

\[ ds^2 = -e^{2\sigma(r,h)}dt^2 + e^{2\tau(r,h)}(dr^2 + r^2d\theta^2) + e^{2\phi(r,h)}r^2 \sin^2\theta [d\varphi - \alpha(r,h)dt]^2. \]  (A1)

As we adopt the Cowling approximation here, perturbations to this metric are totally omitted.

As for the matter variables, perfect fluid is assumed:

\[ T_{ab} = (p + \epsilon)u_a u_b + \rho g_{ab}, \]  (A2)

where \( u^a, \epsilon, \) and \( p \) are the 4-velocity, the energy density, and the pressure of the fluid, respectively. The matter in the equilibrium state is assumed to be polytropic, i.e., \( p = K \epsilon^{1+1/N} \), where \( K \) is a constant. Perturbations for the matter are assumed to be adiabatic so that the following Lagrangian relation holds for the energy density perturbation, \( \Delta \epsilon \), and the pressure perturbation, \( \Delta p \):

\[ \frac{\Delta p}{p} = \gamma \frac{\Delta \epsilon}{\epsilon}, \]  (A3)

where \( \gamma \) denotes the adiabatic exponent. Though this exponent needs not, in general, coincide with the equilibrium polytropic exponent, we assume in this paper that they coincide with each other for simplicity. Under this assumption, this Lagrangian relation is reduced to the Eulerian one for the Euler perturbations for the energy density, \( \delta \epsilon \), and the pressure, \( \delta p \), as follows:

\[ \frac{\delta p}{p} = \gamma \frac{\delta \epsilon}{\epsilon}. \]  (A4)

Since the background spacetime has timelike and spacelike killing vectors, \( \partial/\partial t \) and \( \partial/\partial \varphi \), all coefficients that appear in the perturbed equations do not depend on either \( t \) or \( \varphi \). Consequently, solutions of the perturbed equations have a dependence of \( \sim \exp(-i\sigma t + im\varphi) \) on \( t \) and \( \varphi \) variables where \( m \) is an azimuthal eigenvalue and integer.

A2. INTRODUCTION OF DIMENSIONLESS VARIABLES

We introduce the following dimensionless variables by using \( \epsilon_c \) and \( \kappa \) (underlined variables are dimensionless):

\[ \epsilon \equiv \epsilon/\epsilon_c, \]  (A5)

\[ p \equiv p/\epsilon_c = K \epsilon_c^{1/N} \cdot \epsilon^{1+1/N} \equiv \kappa \epsilon^{1+1/N}, \]  (A6)

\[ v' \equiv u'(u_0 \sqrt{\kappa}), \]  (A7)

\[ \sigma \equiv \sigma/\Omega_c, \]  (A8)

\[ \Omega \equiv \Omega/\Omega_c, \]  (A9)

\[ r = r/r_c, \]  (A10)

where \( \Omega \) is the angular velocity. Parameters \( r_c \) and \( \Omega_c \) are defined by

\[ \Omega_c \equiv \sqrt{4\pi \epsilon_c}, \]  (A11)

\[ r_c \equiv \sqrt{\kappa/\Omega_c}. \]  (A12)

The quantity \( \sqrt{\kappa} \) is a dimensionless sound velocity at the center (normalized by speed of light, which is now chosen as \( c = 1 \)). This parameter measures the system's strength of gravity and corresponds to the expansion parameter in the weak gravity limit.

Furthermore, we introduce the Emden function, \( \psi \), for it will help us to impose the numerical boundary condition at the stellar surface properly:

\[ \epsilon = \psi^N, \quad p = \kappa \psi^{N+1} \]  (A13)
A3. SURFACE-FITTED COORDINATES

Since a rapidly rotating star deforms from a spherical configuration, grid points do not, in general, fall on the surface in the ordinary polar coordinates. This makes it difficult to impose the boundary condition at the stellar surface. To avoid this difficulty, we introduce the surface-fitted coordinates as in the Newtonian analysis (see, e.g., Yoshida & Eriguchi 1995). For a configuration with the stellar surface \( r = R_s(\theta) \) in the polar coordinates \((r, \theta)\), new coordinates \((\tilde{r}, \tilde{\theta})\) are defined by

\[
\tilde{r} \equiv r/R_s(\theta), \quad \tilde{\theta} = \theta.
\]

Although, owing to this coordinate transformation, the basic equations become slightly complicated, the boundary condition at the surface, i.e., the vanishing of the Lagrangian perturbation of the pressure,

\[
\Delta p = 0,
\]

can be imposed easily.

Hereafter, for simplicity, tilde and underlines will be omitted in the equations expressed by this new coordinate.

A4. LINEARIZED HYDRODYNAMIC EQUATIONS IN THE COWLING APPROXIMATION

Basic equations of relativistic hydrodynamics are

\[
\nabla \cdot (nu) = 0, \quad \text{(baryon number conservation)}
\]

\[
\nabla_b T^{ab} = 0, \quad \text{(energy-momentum conservation)}
\]

\[
d\epsilon = \frac{C}{n} dn, \quad \text{(the first law of thermodynamics)}
\]

where \( n \) is the baryon number density, and differential operators \( \nabla \) and \( \partial \) are the covariant derivative and the exterior derivative, respectively.

By introducing 3-velocity components on an orthogonal basis,

\[
U \equiv -ie^{\gamma-z} \delta v^r, \quad (A19)
\]

\[
V \equiv ire^{\gamma-z} \delta v^\theta, \quad (A20)
\]

\[
W \equiv e^{\gamma-z} r \sin \theta \delta v^\phi, \quad (A21)
\]

and rewriting the variable, \( \delta \psi \), as \( \Psi = \delta \psi \), the basic equations for the perturbed quantities are written as follows. For the linearized baryon number conservation, we have,

\[
(\sigma - m\Omega)\Psi + \left( -e^{\gamma-z} \partial_r \Psi - \partial_t \right) U + \frac{\partial_t \Psi}{R_s(\theta)} + \frac{\Psi(1 + \kappa\psi)e^{\gamma-z} \partial_r \psi}{2NAR_s(\theta)} + \frac{\Psi(1 + \kappa\psi)}{R_s(\theta)} - \Psi(1 + \kappa \psi) e^{\gamma-z} \frac{N\sigma}{R_s(\theta)} = 0.
\]

A relativistic version of the linearized Euler equation is obtained by applying the projection tensor, \( P_{ab} = u_a u_b + g_{ab} \), onto the equation of energy-momentum conservation as follows:

\[
(\sigma - m\Omega)U + \left[ r \partial_r \omega + 2(\omega - \Omega) \partial_t \partial_r \right] \Psi + \partial_t \Psi = 0,
\]

\[
(\sigma - m\Omega)V + \left[ (\partial_r - rS \partial_r) \omega + 2(\omega - \Omega) (\partial_r - rS \partial_r) \partial_t \right] \Psi = 0,
\]

\[
(\sigma - m\Omega)W + \left[ (\partial_r - rS \partial_r) \omega + 2(\omega - \Omega) (\partial_r - rS \partial_r) \partial_t \right] \Psi = 0.
\]
\[(\sigma - m\Omega)W = \left[ 2(\omega - \Omega)\dot{\theta}v - \kappa(\omega - \Omega)^2e^{2\beta - 2\nu}R_{i}^2r^2 \sin^2 \theta - \dot{\theta}\omega - 2(\omega - \Omega)\left(\dot{\theta} + \frac{1}{r}\right)e^{\beta + r} \sin \theta \Psi \right] = 0 , \quad (A25)\]

where functions \(A\), \(B\), and \(S\) are defined as

\[A \equiv g_{00} + 2g_{ij}v^j + g_{ik}v^jv^k = -e^{2\nu} + e^{2\beta}r^2 \sin^2 \theta(\omega - \Omega)^2 , \quad (A26)\]

\[B \equiv \frac{1}{1 + \kappa\psi} , \quad (A27)\]

\[S(\theta) = \frac{1}{R_{i}} \frac{dR_{i}}{d\theta} . \quad (A28)\]

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