UNIRULED SYMPLECTIC DIVISORS

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1. INTRODUCTION

This is the second in a series of papers devoted to the symplectic birational geometry program. A fundamental problem in symplectic geometry is to generalize birational geometry to symplectic category. Such a generalization should be viewed as the first step towards the classification of symplectic manifolds. Hopefully, it will provide a better understanding of
birational geometry itself as well. In [HLR], the authors set up some general framework for such a symplectic birational geometry programs. Among other things, we proposed to use Guillemin-Sternberg’s birational cobordisms to replace birational maps. Using sophisticated GW-machinery we also settled successfully the fundamental birational cobordism invariance of uniruledness.

In symplectic category uniruledness is defined via Gromov-Witten invariants. More precisely, a symplectic manifold is called uniruled if there is a non-vanishing genus zero GW-invariant \( \langle [pt], \alpha_2, \cdots, \alpha_k \rangle_{0,A} \neq 0 \) for some nonzero class \( A \). Such a class \( A \) is referred as a uniruled class.

Kollár-Ruan showed that all projective uniruled manifolds are symplectic uniruled. There are many reasons to believe that this is a better generalization of projective uniruledness to symplectic category than the geometric notion of “a manifold covered by rational curve”. However, many obvious properties of algebraic uniruledness are no longer obvious in our context. The birational cobordism invariance is such an example. In fact, it is related to a rather difficult problem in Gromov-Witten theory in terms of finding blow-up formula of Gromov-Witten invariants.

The next step of symplectic birational geometry program is to study various surgery operations such as contraction, flip and flop. The main perspective comes from the basic fact in the projective birational program that various birational surgery operations such as contraction and flop have a common feature: the subset being operated on is necessarily uniruled. Therefore in our program we also need to understand uniruled symplectic submanifolds. In this article and its sequel, we focus on symplectic uniruled divisors. Our key observation is that, as in the projective birational program, such a divisor admits a dichotomy depending on the positivity of its normal bundle. If the normal bundle is non-negative in certain sense, it will force the ambient manifold to be uniruled. If the normal bundle is negative in certain sense, we can contract it to obtain a simpler symplectic manifold. In this article, we treat the case of non-negative normal bundles. In [LR] the negative case will be dealt with in dimension six.

To state our main theorem we need the notion of a minimal uniruled class, which is a uniruled class with minimal symplectic area among all uniruled classes. Suppose that \( \iota : D \to (X,\omega) \) is a symplectic submanifold of codimension 2, i.e. a smooth divisor. Let \( N_D \) be the normal bundle of \( D \) in \( X \). Notice that \( N_D \) is a 2–dimensional symplectic vector bundle and hence has a well defined first Chern class. We will often use \( N_D \) to denote the first Chern class.

**Theorem 1.1.** Suppose \( D \) is uniruled and \( A \) is a minimal uniruled class of \( D \) such that

\[
\langle \iota^*\alpha_1, \cdots, \iota^*\alpha_l, [pt], \beta_2, \cdots, \beta_k \rangle_{0,A}^{D,A} \neq 0
\]

for \( k \leq N_D(A) + 1 \). Then \((X,\omega)\) is uniruled.
In particular, we have

**Corollary 1.2.** Suppose $D$ is uniruled and the normal bundle $N_D$ is non-negative on a minimal uniruled class. Then $X$ is uniruled if either

1. $D$ is homologically injective or
2. $D$ is projectively uniruled.

Another consequence of Theorem 1.1 is

**Corollary 1.3.** Suppose that $X$ is a non-uniruled manifold containing a uniruled divisor satisfying (44). Then, $k > N_D(A) + 1$. In particular, $N_D(A) < 0$ if $k = 1$.

The above corollary is the first step towards the constructions of symplectic divisorial contraction in the third paper of the series [LR].

The idea of proof is similar to that of [HLR]. We partition insertions of $D$ into two types, global insertions $\iota^* \beta_i$ and local insertions $\alpha_j$. The relative/absolute correspondence of Maulik-Okounkov-Pandharipande interchanges relative GW-invariants with certain admissible absolute GW-invariants having a similar partition of insertions. We extend the correspondence to include certain super-admissible GW-invariants. In addition, as in [Ga], in this case invariants of the divisor enters the “extended” relative/absolute correspondence in a nontrivial way. In fact, this is our main strategy to lift a minimal uniruled invariant of divisor to a uniruled invariant of ambient manifold.

As a by product our main theorem also gives a rather general from divisor to ambient space inductive construction of uniruled symplectic manifolds. Theorem 1.1 can be easily applied in a variety of situations, giving a comprehensive generalization of some early results of McDuff. For instance, if $D$ is a Fano manifold with $b_2 = 1$ and non-negative normal bundle, then $X$ must be uniruled. The case of $D = P^{n-1}$ is studied by McDuff in [Mc1], [Mc2] (see section 2). We list many more examples in section 6. We should mention that another obvious inductive construction is from fiber to total space. In particular, a uniruled and homologically injective fiber in a symplectic fibration will force the total space to be uniruled (Corollary 2.11). Although it might be possible to derive this result as a consequence of our main theorem, it actually can be established by a simpler and classical argument.

The paper is organized as follows. In section 2, we first review basic properties of uniruled symplectic manifolds. Then using direct geometric arguments we describe the fiber-to-total space approach, as well as some early examples of McDuff which motivate our divisor-to-ambient strategy. In section 3, we sketch a relative-divisor/absolute correspondence to connect absolute invariants with relative invariants and invariants of divisor. The new ingredient is the appearance of invariants of the divisor corresponding to the additional super-admissible absolute invariants. To prove the
main theorem, we need to study a more delicate version of our correspondence involving a point insertion (see section 3). This requires our extensive knowledge of relative invariants of \( \mathbb{P}^1 \)-bundles. We establish it in section 4 via several powerful techniques in GW theory. The main theorem is then proved in section 5. The applications will be studied in section 6.

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2. Uniruled symplectic manifolds

2.1. Definition and basic properties.

**Definition 2.1.** Let \( A \in H_2(X; \mathbb{Z}) \) be a nonzero class. \( A \) is said to be a uniruled class if there is a nonzero GW invariant

\[
\langle [pt], \alpha_2, \cdots, \alpha_k \rangle_A^X
\]

with a point insertion.

Let \( A \) be a uniruled class. \( A \) is said to be a strongly uniruled class if \( k \) can be chosen to be 3. \( A \) is said to be minimal if it has the smallest symplectic area among all uniruled classes.

**Remark 2.2.** Clearly a uniruled class \( A \) is a spherical class with positive symplectic area. Thus, in dimension 6 and higher, it follows from Gromov’s \( h \)-principle (see e.g. [Ltj]) there is an embedded symplectic sphere in the class \( A \) passing through any given point. However, the converse is not true.

For the 4-dimensional case see Proposition [6.7].

**Definition 2.3.** \( X \) is said to be (symplectically) uniruled if there is a uniruled class, and strongly uniruled if there is a strongly uniruled class.

The notion of strongly uniruled is studied e.g. in [Lu1], [Lu2], [Lu3] and [Mc3].

**Remark 2.4.** If the insertions in (2) are all of even degree, then the class is called an evenly uniruled class. This notion is studied in [Mc3] and there is provided a beautiful characterizations of uniruledness in terms of units of the quantum cohomology ring.

It is easy to see that we could well use the more general disconnected GW invariants to define this concept. Moreover, we have the following basic property proved in [HLR].

**Theorem 2.5.** A symplectic manifold \( X \) is uniruled if there is a nonzero disconnected genus zero descendent GW invariant involving a point constraint.

This flexibility is important for the proof of another basic property also proved in [HLR].
Theorem 2.6. Being uniruled is a birational cobordism property. In particular, if $\tilde{X}$ is a blow-up of $X$, then $X$ is uniruled if and only $\tilde{X}$ is uniruled.

As mentioned in the introduction, the notion of symplectically uniruled is a natural extension of the fundamental notion in algebraic geometry. More precisely, for projective manifold, we have the following result of Kollár and the second author, further strengthened by McDuff.

Theorem 2.7. A projectively uniruled manifold is symplectically uniruled. Moreover, a minimal uniruled class with respect to an ample class is strongly uniruled with both additional insertion being powers of a Kähler form.

Proof. Let $A$ be the class of a $\mathbb{P}^1$ of minimal energy through a very general point $x_0 \in X$.

Then (cf. Theorem 4.2 in [HLR]) for some $k$ there is a nonzero invariant of the form

$$< [pt], [\omega^1], \ldots, [\omega^k] >^X_A,$$

where the first $[pt]$ represents the Poincaré dual of the point class of $\overline{\mathcal{M}}_{0,k}$ and $\omega$ is a Kähler form on $X$.

Choose a homogeneous basis of $H^2(X; \mathbb{R})$

$$\Upsilon = \{1, [\omega], \ldots, [\omega^n], e_{n+1}, \ldots \},$$

where $e_i \cdot e_j = 0$ if $i \leq n$ and $j > n$. This is possible as $[\omega^n] \neq 0$.

Apply the splitting axiom to the invariant (3) with respect to the basis $\Upsilon$, (3) is expressed as a sum of products of $3$−point invariants. in curve classes $A_1, \ldots, A_r$. One of the curve classes, say $A_1$, has a $[pt]$ constraint. But then $A_1$ must have a holomorphic representative through $x_0$. Hence $\omega(A_1) = \omega(A)$ and all $A_j, j > 1$ are zero.

By our choice of $\Upsilon$ it is easy to see that the $k$−point invariant (3) collapses to a nonzero invariant of the form

$$I_{p,q} := < [pt], [\omega^p], [\omega^q] >^D_A$$

where $p = \sum_{j \in J_1} i_j, q = \sum_{j \in J_2} i_j$ for some partition $J_1, J_2$ of $\{2, \ldots, k\}$. □

This sharper version due to McDuff is particularly powerful in light of Theorem 1.1 leading immediately to Corollary 1.2.

In dimension 4 it follows from [Mc1], [LL1], [LL2], [LM] that the converse of Theorem 2.7 is essentially true.

Fano manifolds are (projectively) uniruled. The analogue of a Fano manifold in the symplectic category is a monotone symplectic manifold where $C_1 = \lambda \omega$ for $\lambda > 0$. It would be a challenging problem to show that any monotone symplectic manifold is indeed uniruled.

Another important class of examples is provided by the following beautiful result in [Mc3].

Theorem 2.8. Hamiltonian $S^1$−manifolds are uniruled.
Remark 2.9. It would be interesting to see whether there are uniruled manifolds such that every uniruled invariant must have odd degree insertions. Interestingly, such a manifold cannot be projective by Theorem 2.7. We are not aware of such manifolds. It seems that Hamiltonian $S^1$-manifolds are a good case to investigate.

A rich source of uniruled manifolds comes from uniruled fibrations. Suppose that $\pi : X \to B$ is a fibration (with possibly singular fibers) where $X$ and $B$ are symplectic manifolds. We call it an almost complex fibration if there are tamed $J, J'$ for $X, B$ such that $\pi$ is almost complex. For example, by the famous Thurston construction, if a symplectic fiber bundle over a symplectic manifold has fiber $(F, \sigma)$ and $[\sigma]$ is a restriction class, then the total space $X$ has a symplectic form $\Omega$ that restricts to $\sigma$ on the fibers and hence is almost complex. Lefschetz fibrations, or more generally, locally holomorphic fibrations studied in [Go2] are also almost complex.

Let $\iota : \pi^{-1}(b) \to X$ be the embedding for a generic fiber over $b \in B$. We have the following

Proposition 2.10. Suppose that $\pi : X \to B$ is an almost complex fibration between symplectic manifolds $X, B$. Then, for $A \in H_2(\pi^{-1}(b); \mathbb{Z})$ and $\alpha_2, \ldots, \alpha_k \in H^*(X; \mathbb{R})$,

$$< [pt], \iota^* \alpha_2, \ldots, \iota^* \alpha_k >^X_{\pi^{-1}(b)} = < [pt], \alpha_2, \ldots, \alpha_k >^X_A.$$

Proof. First of all, (6) makes sense as both invariants are well-defined at the same time. Choose $J, J'$ such that $\pi$ is almost complex. Suppose that $f : C \to X$ is a genus 0 stable map with homology class $A$. Then, $\pi \circ f$ is holomorphic with zero homology class. Therefore, $im(\pi \circ f)$ is a point. Namely, $im(f)$ is contained in a fiber. Choose a point $pt \in \pi^{-1}(b)$. Then we have the identification of the moduli spaces of $k$-marked genus zero stable curves with the 1st marked point going to $pt$,

$$\overline{M}^{X}_{0,k}(A, pt) = \overline{M}^{\pi^{-1}(b)}_{0,k}(A, pt).$$

Furthermore, they have the same virtual fundamental cycles. As $\pi$ is almost complex we have the splitting

$$f^*TX = f^*T\pi^{-1}(b) \oplus \mathbb{C}^l,$$

where $l$ is the codimension of a fiber and $\mathbb{C}^l$ is the trivial complex bundle of dimension $l$. As $C$ has genus zero, we have

$$H^1(C, \mathbb{C}^l) = 0.$$

It implies that

$$[\overline{M}^{X}_{0,k}(A, pt)]^{vir} = [\overline{M}^{\pi^{-1}(b)}_{0,k}(A, pt)]^{vir}.$$

By integrating $\alpha_2, \cdots, \alpha_k$ against the virtual fundamental cycles, we obtain (6).
Consequently, we have

**Corollary 2.11.** Suppose that \( \pi : X \to B \) is an almost complex fibration between symplectic manifolds \( X, B \). If a smooth fiber is uniruled and homologically injective (over \( \mathbb{R} \)), then \( X \) is uniruled.

The homologically injective assumption could be a strong one. Notice that for a fiber bundle, the Leray-Hirsch theorem asserts that, under the homologically injective assumption, the homology group of the total space is actually isomorphic to the product of the homology group of the fiber and the base. However, Corollary 2.11 can still be applied for all product bundles, and all projective space fibrations (more generally, if the rational cohomology ring of a smooth uniruled fiber is generated by the restriction of \([\omega]\)).

Moreover, we were informed by McDuff that a Hamiltonian bundle is homologically injective (or equivalently, cohomologically split) if (cf. [LM2])

a) the base is \( S^2 \) (Lalonde-McDuff-Polterovich), and more generally, a complex blow up of a product of projective spaces,

b) the fiber satisfies the hard Lefschetz condition (Blanchard), or its real cohomology is generated by \( H^2 \).

Here is another variation. As in the case of a projective space, for a uniruled manifold up to dimension 4, insertions of a uniruled class can all be assumed to be of the form \([\omega]^i\), thus we also have

**Corollary 2.12.** If the general fibers of a possibly singular uniruled fibration are 2-dimensional or 4-dimensional, then the total space is uniruled.

This in particular applies to a 2-dimensional symplectic conic bundle. A symplectic conic bundle is a conic hypersurface bundle in a smooth \( \mathbb{P}^k \) bundle. Holomorphic conic bundles are especially important in the theory of 3-folds. It is conjectured that a projective uniruled 3-fold is either birational to a trivial \( \mathbb{P}^1 \)-bundle or a conic bundle.

### 2.2. Some motivating examples.

In this subsection, we present examples of uniruled manifolds from the divisor-to-ambient construction. These examples motivate Theorem 1.1 and generalize some early results of McDuff in a slightly different context. The common feature is that the geometric situation is simple enough that a direct geometric argument can be applied.

We start from the simplest situation of trivial normal bundles. Let \( \iota : D \to X \) be a symplectic divisor. McDuff treated the case that \( D \) is a standard projective space and \( X \) is semi-positive, see Theorem 2.13. In general we have,

**Theorem 2.13.** Suppose the normal bundle \( N_D := N_{D|X} \) is trivial. If there is a nonzero invariant \( \langle [pt], \iota^*\alpha_2, \ldots, \iota^*\alpha_k \rangle^D_A \), then \( X \) is uniruled and in fact,

\[
\langle [pt], \alpha_2, \cdots, \alpha_k \rangle^X_A = \langle [pt], \iota^*\alpha_2, \ldots, \iota^*\alpha_k \rangle^D_A
\]
Proof. The argument is parallel to that of Proposition 2.10. First notice that the triviality of the normal bundle implies that the
\[ \text{vir} \dim \overline{M}_{0,k}^X(A) = \text{vir} \dim \overline{M}_{0,k}^D(A) + 2. \]
On the other hand, \( \deg_X([pt]) = \deg_D([pt]) + 2 \). Hence \( \langle [pt], \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k \rangle^X \) is also defined.

We choose an \( \omega \)-compatible almost structure \( j \) on \( D \) and extend it to an \( \omega \)-compatible almost complex structure on \( X \). Furthermore, we choose \( J \) in such a fashion that \( D \) has an almost complex product neighborhood.

Choose a point \( pt \in D \). Let \( \overline{M}_{0,k}^X(A, pt) \) and \( \overline{M}_{0,k}^D(A, pt) \) be the moduli spaces of genus zero stable maps of homology class \( A \) such that \( e(x_1) = pt \).

Suppose that \( f : C \to X \) is a genus zero stable map in \( \overline{M}_{0,k}^X(A, pt) \). It is well-known that any component of \( \text{im}(f) \) either lies in \( D \) or intersects \( D \) positively. One the other hand, \( D \cdot A = 0 \) by the assumption and \( f(x_1) \in D \). Therefore, \( \text{im}(f) \) lies completely inside \( D \). Namely, \( \overline{M}_{0,k}^X(A, pt) = \overline{M}_{0,k}^D(A, pt) \).

Furthermore, \( f^*TX = f^*TD \oplus \mathbb{C} \) and \( H^1(C, \mathbb{C}) = 0 \) imply that
\[ \overline{M}_{0,k}^X(A, pt) |^{\text{vir}} = \overline{M}_{0,k}^D(A, pt) |^{\text{vir}}. \]
By integrating \( \alpha_2, \ldots, \alpha_k \) against the virtual fundamental cycles, we obtain
\[ \langle [pt], \alpha_2, \ldots, \alpha_k \rangle^X = \langle [pt], \iota^* \alpha_2, \ldots, \iota^* \alpha_k \rangle^D. \]
Therefore \( X \) is uniruled.

\( \square \)

When the normal bundle is not trivial, the situation is more complicated. But the identification between appropriate GW-invariants of the divisor and the ambient manifold still remains to be valid in some cases. The following is a particular important example in [Mc2] established in the early 90s.

**Theorem 2.14.** Let \( (X, \omega) \) be a semi-positive symplectic \( 2n \)-manifold which contains a submanifold \( P \) symplectomorphic to \( \mathbb{P}^{n-1} \) whose normal Chern number \( m \) is non-negative. Then certain blow-up of \( X \) is uniruled, and if \( 0 \leq m \leq 2 \) or \( n = 2 \), \( X \) itself is uniruled.

As a consequence of Theorem 2.6, \( X \) itself is still uniruled even if \( n > 1 \) and \( m \geq 3 \).

The case of \( D = \mathbb{P}^1 \) was first proved in [Mc1], generalizing a result of [Gr]. Moreover, in that case, \( X \) is shown to be the connected sum of either \( \mathbb{P}^2 \) or an \( S^2 \)-bundle over a Riemann surface with a number of \( \mathbb{P}^2 \). This is a fundamental result in symplectic 4-manifold theory.

It is instructive to examine her argument in the case of \( D = \mathbb{P}^2 \) geometrically.
Remark 2.15. Notice that here we are in the semi-positive territory, thus we can directly compute invariants via cutting down a generic moduli space by generic cycles. We will not mention this explicitly.

Let us first consider the case of trivial normal bundle. We pick a point $x$ in $D$ and a surface $F$ intersecting $D$ at 1 point $y \neq x$. Let $l$ be the line class. By the previous proposition,

$$< [x], F >_X = < [x], [y] >_D = 1.$$

In the case of normal bundle $O(1)$, we choose two points $x$ and $y$ in $M$. Any $A$–curve outside $D$ can intersect $D$ only at one point, and there is a unique line in $D$ through $x$ and $y$. Namely,

$$< [x], [y] >_I = < [x], [y] >_D = 1.$$

In the case of normal bundle $O(2)$, we choose two points $x$ and $y$ and a line $L$ in $D$ away from $x$ and $y$. Namely,

$$< [x], [y], [L] >_X = < [x], [y], [L] >_I = 1.$$

In each of above cases, we show that the GW-invariant of the ambient manifold $X$ is equal to the corresponding invariant of $D$.

The next case of normal bundle $O(3)$ is different. The simple relation between Gromov-Witten invariant is no longer true. McDuff’s strategy is to blow up a line of $\mathbb{P}^2$ in $X$ to reduce it to the previous situation. Here, we give a different argument which motivates the correspondence in the next section.

Now, for the invariant of $X$ to be well defined we need two points and two lines, or 3 points, or one point and four lines. We choose two points $x$ and $y$. We also pick two lines $L_1$ and $L_2$ in $D$. Let $z$ be the intersection point of $L_1$ and $L_2$. We assume that $z$ is not in the unique line through $x$ and $y$. We claim that

$$1 = < [x], [y] >_I = < [x], [y], [L_1], [L_2] >_X - < [x], [y], [z] >_I.$$

The point is that any curve through $x, y, z$ also intersects $(x, y, L_1, L_2)$. Let $C$ be a curve intersecting $(x, y, L_1, L_2)$. If $C$ is not inside $D$, then $C$ has to intersect $z$ because $C$ has at most three intersection points with $D$. If $C$ is inside $D$, then $C$ must be the unique line through $x$ and $y$.

We remark that the above proof is just a sketch. To do this calculation we realize the constraints $x, y, L_1, L_2$ in a very non-generic way, and for a complete proof we would have to prove that this is justified.

It follows that one of the invariants on the right hand side of equation (7) is not zero and hence $X$ is uniruled. We want to emphasize that in this case the invariant of $D$ can not be identified with a single invariant of $X$.

When $O(m)$ increases, we can similarly express the relevant invariant of $D$ as a more and more complicated combination of invariants of the ambient space $X$. Our main idea is that such a process is best cast into the language
of the relative-divisor/absolute correspondence established in the following 3 sections.

3. Degeneration formula and correspondence

A powerful tool in GW theory is the degeneration formula. To explore its power systematically, a very useful "relative/absolute correspondence" was constructed by Maulik-Okounkov-Pandharipande. It has been generalized to the situation of blow-up by the authors and Hu to relate absolute invariants of a manifold and relative invariants of the blow-up manifold. Such an extended relative/absolute correspondence is crucial to prove the birational invariance of uniruledness.

However, only a subset set of colored absolute invariants appears in the relative/absolute correspondence. They are admissible in the sense that the multiplicity of relative insertions is exactly \( D \cdot A \), where \( D \) is the divisor and \( A \) is the curve class. It is natural to consider non-admissible absolute invariants. If the absolute invariant is sub-admissible in the sense that the multiplicity of relative insertions is less than \( D \cdot A \), we can always use the divisor axiom to add more insertions to obtain an equivalent admissible invariants. The interest is on super-admissible invariants where the multiplicity is bigger than \( D \cdot A \).

3.1. Symplectic cut and the degeneration formula.

3.1.1. Symplectic cut. Let \((X, \omega)\) be a closed symplectic manifold. Let \( S \) be a hypersurface having a neighborhood with a free Hamiltonian \( S^1 \)-action. For instance, if there is a symplectic submanifold in \( X \), then hypersurfaces corresponding to sphere bundles of the normal bundle have this property. Let \( Z \) be the symplectic reduction at the level \( S \), then \( Z \) is the \( S^1 \)-quotient of \( S \) and is a symplectic manifold of 2 dimension less.

We can cut \( X \) along \( S \) to obtain two closed symplectic manifolds \((X^+, \omega^+)\) and \((X^-, \omega^-)\) each containing a smooth copy of \( Z \), and satisfying \( \omega^+ \mid_Z = \omega^- \mid_Z \) (Le).

In particular, the pair \((\omega^+, \omega^-)\) defines a cohomology class of \( \overline{X}^+ \cup_Z \overline{X}^- \), denoted by \([\omega^+ \cup_Z \omega^-]\). Let \( p \) be the continuous collapsing map
\[
p : X \to \overline{X}^+ \cup_Z \overline{X}^-.
\]

It is easy to observe that
\[
(8) \quad p^*([\omega^+ \cup_Z \omega^-]) = [\omega].
\]

3.1.2. Degeneration formula. Given a symplectic cut, there is a basic link between absolute invariants of \( X \) and relative invariants of \((\overline{X}^+, Z)\) in [LR] (see also [P], [L2]). We now describe such a formula.

Let \( B \in H_2(X; \mathbb{Z}) \) be in the kernel of
\[
p_* : H_2(X; \mathbb{Z}) \longrightarrow H_2(\overline{X}^+ \cup_Z \overline{X}^-; \mathbb{Z}).
\]
By (8) we have $\omega(B) = 0$. Such a class is called a vanishing cycle. For $A \in H_2(X; \mathbb{Z})$ define $[A] = A + \ker(p_*)$ and

$$
\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_k} \alpha_k \rangle_{g,[A]}^X = \sum_{B \in [A]} \langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_k} \alpha_k \rangle_{g,B}^X.
$$

Notice that $\omega$ has constant pairing with any element in $[A]$. It follows from the Gromov compactness theorem that there are only finitely many such elements in $[A]$ represented by $J$-holomorphic stable maps. Therefore, the summation in (9) is finite.

At this stage we need to assume that each cohomology class $\alpha_i$ is of the form

$$
\alpha_i = p^*(\alpha_i^+ \cup_{Z} \alpha_i^-).
$$

Here $\alpha_i^+ \in H^*(\overline{X}^+; \mathbb{R})$ are classes with $\alpha_i^+|_Z = \alpha_i^-|_Z$ so that they give rise to a class $\alpha_i^+ \cup_{Z} \alpha_i^- \in H^*(\overline{X}^+ \cup_{Z} \overline{X}^-; \mathbb{R})$.

The degeneration formula expresses $\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_k} \alpha_k \rangle_{g,[A]}^X$ as a sum of products of relative invariants of $(\overline{X}^+, Z)$ and $(\overline{X}^-, Z)$, possibly with disconnected domains. In each product of relative invariants, what is relevant for us are the following conditions:

- the union of two domains along relative marked points is a stable genus $g$ curve with $k$ marked points,
- the total curve class is equal to $p_*(A)$,
- the relative insertions are dual to each other,
- if $\alpha_i^+$ appears for $i$ in a subset of $\{1, \cdots, k\}$, then $\alpha_j^-$ appears for $j$ in the complementary subset of $\{1, \cdots, k\}$.

### 3.2. Sup-admissible graphs.

Let $\iota : D \to X$ be a smooth connected symplectic divisor. As mentioned, we can cut along $D$, or precisely, a cut along a small circle bundle $S$ over $D$ inside $X$.

In this case, as a smooth manifold, $\overline{X}^+ = X$, which we will denote by $\tilde{X}$.Denote the symplectic reduction of $S$ in $\tilde{X}$ still by $D$. Notice however, the symplectic structure is different from the original divisor. And $\overline{X}^- = \mathbb{P}(N_D \oplus \mathbb{C})$, the projectivization of $\mathbb{P}(N_D \oplus \mathbb{C})$. We will often denote it simply by $P_D$ or $P$. Notice that $\mathbb{P}(N_D \oplus \mathbb{C})$ has two natural sections,

$$
D_0 = \mathbb{P}(0 \oplus \mathbb{C}), \quad D_{\infty} = \mathbb{P}(N_D \oplus 0).
$$

The symplectic reduction of $S$ in $P_D$ is the section $D_{\infty}$.

In summary, in this case, $X$ degenerates into $(\tilde{X}, D)$ and $(P_D, D_{\infty})$. We also denote $\omega^-$ by $\omega_P$.

We also observe that the section $D_0$ actually has the same symplectic structure and same neighborhood as the original divisor. We denote the inclusion of $D_0$ in $P_D$ still by $\iota$.

---

\[1\text{Notice that our convention here is opposite to that in [HLR]}\]
**Definition 3.1.** A class \( A \in H_2(X; \mathbb{Z}) \) is called effective for the symplectic cut along \( D \) if either

- \( A \) is represented by a pseudo-holomorphic stable map to \( X \) for all \( \omega \)-tamed almost complex structures, or
- \( A \) is represented by a pseudo-holomorphic stable map to \( X^- \) for all \( \omega_P \)-tamed almost complex structures, or
- \( A \) is in the image of \( i_* \) and is represented by a pseudo-holomorphic stable map to \( \tilde{X} \) for all \( \omega^+ \)-tamed almost complex structures.

Notice that the zero class \( A = 0 \) is considered to be effective here as a constant map is pseudo-holomorphic.

**Definition 3.2.** A connected colored graph \( \Gamma \) consists of one vertex decorated by \( (g, A) \) with \( g \) an integer, \( A \) an \( H_2(X; \mathbb{Z}) \)-class, and two sets of colored tails, \( X \)-tails and \( D \)-tails.

We further weight each \( X \)-tail by a class \( \alpha_i \in H^*(X; \mathbb{R}) \), called an \( X \)-insertion. We also weight each \( D \)-tail by a pair \( (\mu_i, \beta_i) \), where \( \mu_i \) is a non-negative integer, and \( \beta_i \) is a class in \( H^*(D; \mathbb{R}) \) called a \( D \)-insertion. We call the resulting graph a connected colored weighted graph and denoted by

\[ \Gamma((\alpha_i)\mid(\mu_i, \beta_i)). \]

The collection of pairs, \( \mu = (\mu_i, \beta_i) \), is called a weighted partition.

There is also the distinguished graph, the empty graph \( \Gamma(\emptyset\mid\emptyset) \).

**Definition 3.3.** The dimension of the empty graph is defined to be zero. For a nonempty graph \( \Gamma((\alpha_i)\mid(\mu_i, \beta_i)) \), its dimension is defined to be

\[
\dim \Gamma((\alpha_i)\mid(\mu_i, \beta_i)) = 2[C_1(A) + (n - 3)(1 - g) + D \cdot A] \\
+ \sum (2 - 2\mu - \deg(\mu)) \\
+ \|\varpi\|_1 \\
+ \sum_{\deg(\alpha_i) \neq 1} (2 - \deg(\alpha_i)),
\]

where \( \|\varpi\|_1 \) is the number of degree 1 insertions in \( \varpi \).

**Remark 3.4.** We can also consider the disjoint union \( \Gamma^* \) of several such graphs and use \( A_{\Gamma^*}, \gamma_{\Gamma^*} \) to denote the total homology class and total arithmetic genus. Here the total arithmetic genus is \( 1 + \sum (g_i - 1) \).

**Definition 3.5.** A connected colored weighted graph with

\[ \varpi = (\alpha_i) \quad \text{and} \quad \mu = ((\mu_i, \beta_i)), \]

written simply as \( \Gamma(\varpi\mid\mu) \), is called

- admissible if \( \sum_j \mu_j = D \cdot A \),
- strictly sup-admissible if

\[
\sum_j \mu_j > D \cdot A \quad \text{and} \quad A \in \text{im}[i_* : H_2(D; \mathbb{Z}) \to H_2(X; \mathbb{Z})],
\]

- strictly sub-admissible if \( \sum_j \mu_j < D \cdot A \).
A possibly disconnected graph is called sup-admissible if it is a connected strictly sup-admissible graphs or the disjoint union of one or more connected admissible graphs.

Notice that every strictly sub-admissible absolute invariant can be made admissible by adding an appropriate number of $D$-insertions.

These graphs will be used to describe the structure of the components appearing in the decomposition formula; cf. equation (17). The strictly sup-admissible graphs are connected because they correspond to curves that lie entirely in $P_D$. The other sup-admissible graphs describe the part of the curve lying in $\tilde{X}$ and hence need not be connected.

**Definition 3.6.** Suppose $X$ is of dimension $2n \geq 4$. Let $\Theta = \{\delta_i\}$ be a self dual basis of $\oplus_{q=0}^{2n-2} H^q(D; \mathbb{R})$ with respect to the cup product of $D$.

Let $\Xi = \{\gamma_i\}$ be a basis of $\oplus_{0 \leq p \leq 2n} H^p(X; \mathbb{R})$.

We will fix $\Theta$ and $\Xi$ in the rest of this paper.

**Remark 3.7.** Notice that we do not require any compatibility of $\Theta$ and $\Xi$.

**Definition 3.8.** An $\Theta$-standard weighted partition $\mu$ is a partition weighted by classes of $D$ from $\Theta$, i.e.

$$\mu = \{(\mu_1, \delta_{K_1}), \cdots, (\mu_{\ell(\mu)}, \delta_{K_{\ell(\mu)}})\}.$$  

$\varpi = (\alpha_i)$ is called $\Xi$-standard if each $\alpha_i \in \Xi$.

Let $c(X, \omega, J)$ be the minimal symplectic area of a connected non-constant $J$-holomorphic curve. $c(X, \omega, J)$ is positive due to Gromov compactness.

Let $c(X, \omega)$ be the maximum of $c(X, \omega, J)$ over $J$.

**Definition 3.9.** $\Gamma(\varpi|\mu)$ is called standard if

- the class of each vertex is a nonzero effective class,
- $g \geq -\frac{\omega(A)}{c(X, \omega)} + 1$,
- $\varpi$ is $\Xi$-standard,
- $\mu$ is $\Theta$-standard,
- its dimension is zero.

3.2.1. **Ordering the graphs.** Let $I$ be the set of possibly disconnected sup-admissible standard colored weighted graphs. We will order $I$ following [MP]. The partial order is defined in terms of several preliminary partial orders.

**Definition 3.10.** The set of pairs $(m, \delta)$ where $m \in \mathbb{Z}_{\geq 0}$ and $\delta \in H^*(D; \mathbb{R})$ is partially ordered by the following size relation

$$(m, \delta) > (m', \delta')$$

if $m > m'$ or if $m = m'$ and $\text{deg}(\delta) > \text{deg}(\delta')$.

We may place the pairs of $\mu$ in decreasing order by size, i.e. by (13).
Definition 3.11. A lexicographic ordering on weighted partitions is then defined as follows:

\[ \mu > \mu' \]

if, after placing \( \mu \) and \( \mu' \) in decreasing order by size, the first pair for which \( \mu \) and \( \mu' \) differ in size is larger for \( \mu \).

Next we introduce a relevant partial order on the effective curve classes of \( X \) (see Definition 3.1).

Definition 3.12. For effective classes \( A \) and \( A' \) in \( H_2(X; \mathbb{Z}) \), we say that \( A' < A \) if \( A - A' \in H_2(X; \mathbb{Z}) \) has positive pairing with the symplectic form on \( X \).

We partially order such weighted graphs in the following way.

Definition 3.13. The empty graph is smaller than any other graph. For any two non-empty admissible graphs \( \Gamma(\varpi'|\mu') \) and \( \Gamma(\varpi|\mu) \),

\[ \Gamma(\varpi'|\mu') \preceq \Gamma(\varpi|\mu) \]

if one of the conditions below holds

1. \( A' < A \),
2. equality in (1) and the arithmetic genus satisfies \( g' < g \),
3. equality in (1-2) and \( \|\varpi'\| < \|\varpi\| \),
4. equality in (1-3) and \( \deg(\mu') > \deg(\mu) \),
5. equality in (1-4) and \( \mu' \parallel \mu \),

where \( \|\varpi\| \) denotes the number of \( X \)-tails, and \( \deg(\mu) \) is the sum of \( \deg(\mu_i) \).

If \( \Gamma(\varpi'|\mu') \) is admissible and \( \Gamma(\varpi|\mu) \) is connected and strictly sup-admissible,

\[ \Gamma(\varpi'|\mu') \preceq \Gamma(\varpi|\mu) \]

if \( A' \leq A \).

The inequalities (3-5) are designed so that the dimension of the moduli space satisfying the larger constraint/condition is larger. This explains the seemingly strange conditions (4) and (5) where the inequalities are reversed.

Remark 3.14. It is easy to observe that this extended partial order \( \preceq \) is preserved under disjoint union of admissible graphs. Notice that we don’t compare strictly sup-admissible graphs.

Remark 3.15. If we are only interested in genus zero invariants, then we can replace \( g' < g \) in (2) by the inequality of the number of connected components, \( n' > n \).

Here is a crucial property of the ordering.

Lemma 3.16. Given a standard colored weighted graph there are only finitely many standard colored weighted lower in the partial ordering. In particular, there is a minimal standard invariant with \( A \neq 0 \) and nonzero value.
Proof. As a strictly sup-admissible graph is not smaller than any other graph, we only need to bound the number of admissible graphs.

First of all, the number of effective classes with area bounded above is finite due to the Gromov compactness.

In particular, there is a minimal area \( \omega(X, \omega) \) among all nonzero effective classes. As each vertex is a nonzero effective class, this implies the number of components of a standard graph with bounded area is bounded by \( g \geq \omega(X, \omega) \).

As the number of \( \sigma \) insertions is bounded from above and the \( \sigma \) insertions are chosen from a finite generating set \( \Sigma \), there is only a finite number of choices of \( \sigma \).

Finally, the number of \( \mu \) insertions and the total multiplicity of \( \mu \) are both bounded by the intersection number \( D \cdot A \). As the \( \mu \) insertions are chosen from a finite generating set \( \Theta \) and the multiplicities are positive, there is only a finite number of choices of \( \mu \).

\( \square \)

A partially order set is called lower bounded if there are only finitely many elements lower than a given element. \( I \) is lower bounded, so is any subset of \( I \).

3.3. Invariants associated to graphs. In this subsection, we associate to a sup-admissible standard colored weighted graph certain GW invariants of the symplectic cut. We just give the definition for connected graphs, the extension to disconnected graphs is straightforward.

3.3.1. Absolute invariants.

Definition 3.17. For a relative insertion \((m, \delta)\), we associate the absolute descendent insertion \( \tau_{m-1}(\tilde{\delta}) \) on \( X \) supported on \( D \), where \( \tilde{\delta}_i = \delta_i[D] \). Given a standard (relative) weighted partition \( \mu \), let

\[
d_i(\mu) = \mu_i - 1,
\]

and

\[
\tilde{\mu} = \{ \tau_{d_1(\mu)}(\tilde{\delta}K_1), \ldots, \tau_{d_l(\mu)}(\tilde{\delta}K_l(\mu)) \}.
\]

It is convenient to view \([D]\) as the class of a Thom form supported near the symplectic divisor \( D \). Then class \( \tilde{\delta} = \delta[D] \) is the represented by the wedge product of the pull back of a form representing \( \delta \) in a neighborhood of \( D \) with the compactly supported Thom form of \( D \). In terms of homology constraints, \( \tilde{\delta} \) and \( \delta \) correspond to the same cycle lying inside \( D \).

Definition 3.18. The absolute descendent invariant associated to a connected standard colored weighted graph \( \Gamma(\varpi; \tilde{\mu}) \) is

\[
\langle \Gamma(\varpi; \tilde{\mu}) \rangle^X.
\]

The invariant associated to the empty graph is called the empty absolute invariant and its value is defined to be 1.
Notice for such an absolute descendent invariant of $X$ all the descendent insertions are supported on $D$. Such an invariant is colored in the sense that the insertions are divided into two collections, the $X$-insertions $\varpi$ and the $D$-insertions $\tilde{\mu}$, with each insertion in $\varpi$ being of the form $\gamma_L$, and each insertion in $\tilde{\mu}$ being of the form $\tau_\delta \tilde{\delta}_K$.

3.3.2. Relative invariants.

**Definition 3.19.** Let $\Gamma(\varpi|\mu)$ be a connected standard colored weighted graph.

If it is admissible, the relative invariant of the symplectic cut associated to it is

$$\langle \Gamma(\varpi|\mu) \rangle_{\bar{X},D}.$$

If it is strictly sup-admissible, the relative invariant of the symplectic cut associated to it is

$$\langle \pi^* i^* \varpi, \tilde{\mu}|\emptyset \rangle_{A^{P,D,\infty}},$$

where we view $P$ as a bundle over its zero section $D_0$ and $\pi : P \to D_0$ is the projection, and $\tilde{\mu}$ here is given by

$$(\tau_{d_1(\mu)}(\delta_{K_1}(D_0)), \cdots, \tau_{d_l(\mu)}(\delta_{K_l(\mu)}(D_0))).$$

Finally, the invariant associated to the empty graph is the empty relative invariant and its value is defined to be 1.

3.4. Sup-admissible correspondence. Consider the infinite dimensional vector space $\mathbb{R}^I$ whose coordinates are ordered in the way compatible with the partial order of $I$. From the relative invariants in Definition 3.19 we can form a vector

$$v_{rel} \in \mathbb{R}^I$$

given by the numerical values. We also have the vector

$$v_{abs} \in \mathbb{R}^I$$

given by the numerical values of the sup-admissible invariants of $X$ relative to $D$ in definition 3.18.

**Theorem 3.20.** There is an invertible lower triangular linear transformation

$$A : \mathbb{R}^I \to \mathbb{R}^I$$

such that (i) the coefficients of $A$ are local in the sense of being dependent on $D$ only; (ii)

$$A(v_{rel}) = v_{abs}.$$ 

In particular, $v_{rel}$ and $v_{abs}$ determine each other.

Finally, if $I_0 \subset I$ denotes the subset of genus zero invariants with $\varpi = \emptyset$, then $A$ further restricts to an invertible lower triangular transformation from $\mathbb{R}^{I_0}$ to $\mathbb{R}^{I_0}$. 
Proof. The idea is as follows. We start with a connected colored weighted graph \( \Gamma(\varpi|\mu) \). The associated absolute invariant is \((\Gamma(\varpi;\tilde{\mu}))^X\).

We apply the degeneration formula to this connected absolute invariant to express it as a linear combination of relative invariants of \((\tilde{X}, D)\) with the coefficients being essentially certain relative invariants of the \(\mathbb{P}^1\)-bundle. In the strictly sup-admissible case there is an additional term being the associated relative invariant of \((P, D_\infty)\).

This is possible because the homomorphism \(p_*\) is obviously injective.

Of course we also need to first split the \(\varpi\) and \(\tilde{\mu}\) insertions as in (10).

Recall that each \(\mu\) insertion is of the form \(\gamma = \tau_d(\delta[D])\) for \(\delta \in \Theta\). Then we set

\[
\gamma^+ = 0, \quad \gamma^- = \tau_d(\delta[D_0]).
\]

In other words we distribute all the \(\tilde{\mu}\) insertions to the \(\mathbb{P}^1\)-bundle side. With this preferred distribution of insertions, the original graph \(\Gamma(\varpi|\mu)\) turns out to be the largest weighted relative graph appearing in the linear combination. For an \(\varpi\) insertion \(\tau\) we can take the \(+\) class to be itself and the \(-\) part to be the class of the \(\mathbb{P}^1\)-bundle over the cycle of intersection, i.e

\[
\tau^+ = \tau, \quad \tau^- = \pi^*\iota^*\tau.
\]

The arguments for \(I_0\) and \(I\) are similar, we just treat the case of \(I_0\), i.e. genus 0 and \(\varpi = \emptyset\).

The absolute invariant associated to \(\Gamma(\emptyset|\mu) \in I_0\) is of the form

\[
(\tau_{d_{i_{1}}}((\tilde{\delta}K_1)), \ldots, \tau_{d_{i_{l}}}((\tilde{\delta}K_{l}))\bigg|_B).
\]

With all insertions distributed to the \(\mathbb{P}^1\)-bundle side, (16) is expressed as the following sum

\[
\sum \langle \Gamma^-(\emptyset|\eta) \rangle^{\tilde{X}, D} \Delta(\eta) \langle \Gamma^+(\tau_{d_{i_{1}}}((\delta K_1), \ldots, \tau_{d_{i_{l}}}((\delta K_{l})) \big| (\tilde{\delta}K_{l})) \rangle^{P, D_\infty}
\]

over appropriate pairs of weighted graphs. Here \(\Delta(\eta)\) is a nonzero combinatorial constant depending on the multiplicities of \(\eta\).

If the graph \(\Gamma(\emptyset|\mu)\) is strictly sup-admissible, i.e. \(\sum \mu_j > D \cdot A\) and \(A\) is in the image of \(\iota_* : H_2(D; \mathbb{Z}) \to H_2(X; \mathbb{Z})\), then there is a term with \(\tilde{\eta} = \eta = \emptyset\) in (17). In this term the relative invariant of \((P, D_\infty)\) is the relative invariant associated to the given graph in \(I_0\), and the relative invariant of \((\tilde{X}, D)\) is associated to the empty graph and hence value equal to 1.

In any other term with \(\eta \neq \emptyset\) we have a relative invariant of \((\tilde{X}, D)\) associated to a possibly disconnected admissible graph \(\Gamma^-(\emptyset|\eta) \in I_0\). Regard the relative invariant

\[
\langle \tilde{\Gamma}^+(\tau_{d_{i_{1}}}((\delta K_1), \ldots, \tau_{d_{i_{l}}}((\delta K_{l})) \big| (\tilde{\delta}K_{l})) \rangle^{P, D_\infty}
\]
of \((P,D_\infty)\) as the coefficient of the graph of \(\Gamma^-(\emptyset|\eta)\). The coefficient is nonzero only if the class of \(\Gamma^-\) is at most \(A\). Thus we have \(\Gamma^-(\emptyset|\eta) < \Gamma(\emptyset|\mu)\), according to our extended order in Definition 3.13.

Suppose the graph \(\Gamma(\emptyset|\mu)\) is admissible. For the term with \(\eta = \emptyset\), the relative invariant of \((P,D_\infty)\) is not associated to any graph in \(I_0\) as \(\Gamma(\emptyset|\mu)\) is not strictly sup-admissible. Instead the relative invariant of \((P,D_\infty)\) is considered to be the coefficient of the empty graph in \(I_0\). But empty graph is certainly smaller than \(\Gamma(\emptyset|\mu)\) according to Definition 3.13. For all other terms, as our order of admissible graphs agrees with that in \(\text{[MP]}\), it follows from \(\text{[MP]}\) that the largest graph \(\Gamma^-\) appearing in (17) with nonzero coefficient is the graph \(\Gamma(\emptyset|\mu)\) itself.

Finally we look at possibly disconnected admissible graphs. Notice that the invariant of the disjoint union of two graphs is the product of invariants. We have also remarked that if \(\Gamma_1\) is bigger than \(\Gamma_1'\) and \(\Gamma_2\) is bigger than \(\Gamma_2'\), then the union of \(\Gamma_1\) and \(\Gamma_2\) is bigger than the union of \(\Gamma_1'\) and \(\Gamma_2'\). Therefore we still have the leading term being the given graph.

Thus the correspondence is lower triangular with nonzero diagonal entries. Such a correspondence is actually invertible as \(I_0\) is lower bounded by Lemma 3.16.

Remark 3.21. When \(\sum_j \mu_j < D \cdot A\), we have \(l(\eta) - l(\mu) > 0\). the largest \(\eta\) is \(\mu\) followed by \(D \cdot A - \sum_j \mu_j\) pairs of \((1,D)\). In the extreme case all \(\mu_j = 0\), the largest invariant has \(\eta\) with \(A \cdot E\) pairs of \((1,D)\). Notice that when \(\sum_j \mu_j < D \cdot A\), then the relative invariant \(\langle [pt], \omega|\mu \rangle_{g,A}^{X,D} \) is zero by definition. What Theorem 3.20 says in this case is that \(\langle [pt], \omega, \mu \rangle_{g,p,(A)}^X \) is expressed as the sum of standard relative invariants whose weighted graph is lower than \(\langle [pt], \omega|\mu \rangle_{g,A}^{X,D} \).

3.5. Correspondence with a point \(D\)-insertion. With the application to uniruledness in the mind, we require graphs have a point \(D\)-insertion. This is different from \(\text{[HLR]}\) where the point insertion is always an \(X\)-insertion.

In this subsection we still use \(P\) to denote \(\mathbb{P}(N_D \oplus \mathbb{C})\).

3.5.1. Statement. Recall \(\iota : D \to X\) is the embedding. Let

\[ V = \min\{0 < \omega_D \cdot A | A \in H_2(D), \iota^* \omega, \tau_1([pt]), \cdots, \tau_k (\beta_k) >_A^D \neq 0\}. \]

Here, \(\iota^* \omega\) is of the form \(\{\iota^* \alpha_1, \cdots, \iota^* \alpha_l\}\) with \(\deg \alpha_j \neq 2\).

Remark 3.22. By linearity we can assume that each \(\beta_i\) is in \(\Theta\) and each \(\alpha_j\) is in \(\Xi\). Moreover, according to \(\text{[HLR]}\) \(V\) is achieved by invariants with no descendants. Finally, \(V\) is finite if and only if \(D\) is uniruled.

Such an invariant determines a standard colored weighted graph \(\Gamma_0(\omega|\mu)\) in \(I\) with \(\mu = ((i_1 + 1, [pt]), (i_2 + 1, \alpha_2), \cdots, (i_k + 1, \alpha_k))\).
Definition 3.23. We consider the following subset $I_{D-\text{pt}} \subset I$ of colored standard graphs of $X$,
1. $g(\Gamma) = 0$,
2. the class $A$ is nonzero and $\omega(A) \leq V$,
3. admissible graphs with a $D$–point insertion,
4. sup-admissible graphs of the form $\Gamma_0$ with $A \in \text{im}[\iota_* : H_2(D;\mathbb{Z}) \to H_2(X;\mathbb{Z})]$,
5. empty graph excluded.

If such a graph is not strictly sup-admissible we call it a restricted graph.

Let $\mathbb{R}_I^{\text{pt}}$ be the vector subspace spanned by the partially ordered set $I_{D-\text{pt}}$ of graphs, and $v^{obs}_{D-\text{pt}}$ be the vector of associated absolute invariants.

Notice that all the associated absolute invariants have a point insertion (possibly descendent).

We also have the vector of associated relative invariants $v^{rel}_{D-\text{pt}}$ of the symplectic cut, including all the relative invariants of $(P,D_\infty)$ with class $A \in \text{im}[\iota_* : H_2(D;\mathbb{Z}) \to H_2(X;\mathbb{Z})]$ satisfying $\omega(A) = V$, and insertions of the form $(\tilde{\nu}|\emptyset)$.

Theorem 3.24. A restricts to an invertible lower triangular linear transformation

$$T : \mathbb{R}_I^{\text{pt}} \to \mathbb{R}_I^{\text{pt}}$$

such that

$$T(v^{rel}_{D-\text{pt}}) = v^{obs}_{D-\text{pt}}.$$ 

Moreover, there is also an $I_{D-\text{pt},0}$ version.

In the remaining we provide the proof using the following vanishing results on the relative invariants of $(P,D_\infty)$.

Theorem 3.25. Suppose $A$ is a non-fiber class, i.e. $0 < \omega_{D_\infty}(\pi_*(A))$.

(i) If $\omega_{D_0}(\pi_*(A)) < V$, then,

$$\langle \varpi, \tau_1([pt]), \tau_2(\beta_2[D_0]), \ldots, \tau_k(\beta_k[D_0])|\mu \rangle^{P,D_\infty}_A = 0.$$ 

(ii) If $\omega_{D_0}(\pi_*(A)) = V$ and

$$m = \sum_{t} (i_t + 1) \leq D_0 \cdot A$$

is admissible or sub-admissible, then,

$$\langle \varpi, \tau_1([pt]), \tau_2(\beta_2[D_0]), \ldots, \tau_k(\beta_k[D_0])|\emptyset \rangle^{P,D_\infty}_A = 0.$$ 

Theorem 3.25 will be proved in the next section.
3.5.2. Proof of Theorem 3.24. As in the proof of Theorem 3.20 we only prove the version with \( \varpi = \emptyset \), i.e. the \( I_{D-\text{pt},0} \) version.

We can assume that the graph \( \Gamma(\emptyset|\mu) \) in \( I_{D-\text{pt},0} \) is connected.

Case I. Let us first look at the case that \( \Gamma(\emptyset|\mu) \) is restricted, or in other words, admissible.

Apply the degeneration formula to it as in the proof of Theorem 3.20.

Notice that in this case \( \delta_K = [pt] \).

Definition 3.26. The \( P- \) graphs \( \Gamma^+(\cdots) \) in (17) are divided into 3 types.

- (i) The special graph \( \Gamma^+(\mu|\emptyset) \). In this case the entire curve lies on the \( \mathbb{P}^1 \)-bundle side.
- (ii) The connected component of \( \Gamma^+ \) containing the point insertion is not a fiber curve.
- (iii) The connected component of \( \Gamma^+ \) containing the point insertion is a fiber curve (possibly multiply covered).

The type of a \( \Gamma^{-}(\emptyset|\eta) \) graph in (17) is the type of the company \( P- \) graph. In particular, the type (i) \( \Gamma^{-}(\emptyset|\eta) \) graph is just the empty graph.

Now let us fix a term in (17).

Neither \( \Gamma^+ \) nor \( \Gamma^- \) is the empty graph. Then there are associated classes \( B^+ \in H_2(P; \mathbb{Z}) \) and \( B^- \in H_2(X; \mathbb{Z}) \) respectively with

\[ B^+ + B^- = p_*(B). \]

It follows from (8) and Definition 3.23,

\[ \omega_X(B^-) + \omega_P(B^+) = \omega(B) \leq V. \]

Since \( B \) is not the zero class, and the graph \( \Gamma(\emptyset|\mu) \) is connected, we have \( B^+ \neq 0 \) and \( B^- \neq 0 \). Hence it follows from (19)

\[ 0 < \omega_P(B^+) < V, \quad 0 < \omega_X(B^-) < V. \]

We will show in this case

Proposition 3.27. If the \( \Gamma^- \) graph contributes then it is restricted.

Proof. This will be proved by a series of lemmas.

Lemma 3.28. For a type (iii) \( P- \) graph each relative insertion on the fiber curve containing the absolute point insertion must be of the form \( (s, [D]) \). Consequently, for each type (iii) \( \Gamma^-(\emptyset | \eta) \) graph there is a point relative insertion, i.e. \( \eta = \{(s, [pt]), \cdots \} \).

Proof. Otherwise the fiber curve cannot meet both the point and the relative cycle. \( \square \)

---

In this case it is tempting to think that the relative invariant \( \langle [pt], \alpha_2, \cdots, \alpha_k | \emptyset \rangle_{A}^{P_D} \) is the same as the absolute invariant \( \langle [pt], \alpha_2, \cdots, \alpha_k \rangle_{A}^{P_D} \). But in general this is not true.
Lemma 3.29. Let \( \pi : P \to D_0 \) be the projection. Then
\[
B^+ = \pi_* B + (B \cdot D_\infty) F.
\]
In particular,
\[
\omega_{D_0}(\pi_*(B^+)) \leq \omega_P(B^+)
\]
if \( B^+ \cdot D_\infty \geq 0 \), and the inequality is strict if \( B^+ \cdot D_\infty \) is positive.

Proof. \( H_2(P; \mathbb{Z}) \) is generated by \( H_2(D_0; \mathbb{Z}) \) and \( F \), so we can write
\[
B^+ = B_0 + mF
\]
for some class \( B_0 \) of \( D_0 \). Since \( \pi_* B_0 = B_0 \) we have
\[
\pi_* B^+ = \pi_* B_0 + 0 = B_0.
\]
On the other hand, since \( B_0 \cdot D_\infty = 0 \), we have \( m = B^+ \cdot D_\infty \geq 0 \).
Since \( \omega_P(F) > 0 \) and \( \omega_{D_0} = \omega_D \), if \( m = B^+ \cdot D_\infty \geq 0 \), then
\[
\omega_P(B^+) = \omega_P(\pi_* B^+) + \omega_P(mF) \geq \omega_P(\pi_* B^+) = \omega_{D_0}(\pi_* B^+) = \omega_D(\pi_* B^+).
\]

Lemma 3.30. Type (ii) \( P- \) graph invariants vanish. Hence there are no contributing type (ii) \( \Gamma^{-}(\emptyset|\eta) \) graphs in (17).

Proof. Observe that in this case the class \( B^+ \) is not a fiber class. And by Lemma 3.29 and (20) we have
\[
\omega_D(\pi_*(B^+)) \leq \omega_P(B^+) < V.
\]
The conclusion then follows from part 1 of Theorem 3.25.

Now it follows Lemmas 3.28, 3.29, 3.30 that the contributing \( \Gamma^{-}(\emptyset|\eta) \) graphs in (17) are still restricted.

\( \Gamma^+ \) is empty. In this case \( \Gamma^- \) is simply the given graph \( \Gamma(\emptyset|\mu) \).
\( \Gamma^- \) is empty. In this case we have

Lemma 3.31. Suppose the given graph \( \Gamma(\emptyset|\mu) \) is admissible. Either the type (i) \( P- \) graph \( \Gamma^+(\bar{\mu}|\emptyset) \) is not allowed or its invariant vanishes. Hence the empty \( \Gamma^- \) graph is not a contributing graph in (17).

Proof. The type (i) \( P- \) graph \( \Gamma(\emptyset|\mu) \) does not appear if the class \( B \) is not in the image \( \iota_* \).
Suppose \( B = B^+ \) is in the image of \( \iota \) and we denote the class of \( D \) still by \( B \). By our assumption, either \( \omega(B) = \omega_D(B) < V \) or \( \omega(B) = \omega_D(B) = V \).
Since \( B^+ \) is not a fiber class, in either case the vanishing is given by Theorem 3.25.

Case II. Now consider the case that \( \Gamma(\emptyset|\mu) \) is of the form \( \Gamma_0 \) satisfying (18).
We again apply the degeneration formula to the corresponding invariant of \( X \) distributing all the insertions to the \( \mathbb{P}^1- \) bundle side. Then as argued in Theorem 3.20 the leading term in (17) is the special graph.
We only need to show that the remaining $\Gamma^-$-graphs in (17) are restricted.

By Lemma 3.31 the type (i) $\Gamma^-$-graph does not appear. Since $A$ is a minimal uniruled class of $D$, by Theorem 3.25 type (ii) $\Gamma^-$-graphs do not appear either. In particular, $\tilde{\eta}$ is constrained by (iii). For the remaining $\Gamma^-$-graphs, the corresponding $P$-graphs are of type (iii), hence they are restricted.

Thus we have completed the proof of Theorem 3.24.

Remark 3.32. In fact we have shown there is a sub-correspondence for the admissible graphs in $I_{D_0}^{\text{pt},0}$.

4. Relative Gromov-Witten invariants of $(P_D, D_\infty)$

Relative Gromov-Witten invariants of $\mathbb{P}^1$-bundle have been studied in [MP] and [Ga]. We will use both their techniques and results extensively. In [MP] it is shown using virtual localization that relative invariants of a projective $\mathbb{P}^1$-bundle are determined by absolute invariants of the base projective manifold. We need to apply the symplectic relative virtual localization theorem of Chen-Li [CL] in our more general setting. However, for $\mathbb{P}^1$-bundle, the theorem is the same as the corresponding algebro-geometric case in [GV]. We conveniently use the notations from the algebro-geometric case. Compared to [MP], a new ingredient of our case is the point insertion for which we have to keep track of it at each induction step.

Let $L$ be a complex line bundle over $D$ and $P_D = \mathbb{P}(L \oplus \mathbb{C})$. Then $\pi : P_D \to D$ is a $\mathbb{P}^1$-bundle with zero section $D_0$ and infinity section $D_\infty$. Clearly, $D_0, D_\infty$ are isomorphic to $D$. In this section, we calculate certain relative Gromov-Witten invariants of $P_D$ relative to the infinity section $D_\infty$.

First of all, there are two kinds of cohomology classes: $\pi^*\alpha$ and $\beta[D_0]$. As previously remarked, the class $\beta[D_0]$, which is the cup product of the pullback of $\beta$ with the Poincaré dual of $D_0$, corresponds to the Poincaré dual of $\beta$ viewed as a homology cycle of $D_0$.

Let $A \in H_2(D_\infty; \mathbb{Z})$. We view it as a homology class of $P_D$. We shall consider relative invariants of the form

$$<\varpi, \tau_{i_1}([pt][D_0]), \tau_{i_2}([\beta_2[D_0]]), \ldots, \tau_{i_k}([\beta_k[D_0]])|\mu >_{0,A,D_\infty},$$

where $\varpi$ consists of insertions of the form $\pi^*(\alpha_1), \ldots, \pi^*(\alpha_l)$.

To evaluate such an invariant we will need to study other types of relative invariants, twisted rubber invariants, as well as twisted invariants of the divisor. We will write $P$ for $P_D$.

4.1. Twisted invariants of $D$. There is an important twisted Gromov-Witten theory treated by Farber and Pandharipande in [FP] and Coates and Givental [CG].

Suppose that $f : \Sigma \to D_0$ is a stable map. For the line bundle $L \to D_0$ we can define a virtual bundle

$$H^1(f^*L) - H^0(f^*L)$$
over the moduli space $\overline{M}_{0, k+l}(A)$ of stable maps along with its Euler class $e$. For example, we use the $S^1$-action on the fiber to define the equivariant Euler class of (22), still denoted by $e$, and then take non-equivariant limit.

Associated to the relative invariant (21) we have the Gromov-Witten invariant of $D_0 = D$ twisted by $e$ defined as

\[
<\iota^* v, \tau_1([pt]), \tau_2(\beta_2), \cdots, \tau_k(\beta_k)>_{0,A}^D = \int_{[\overline{M}_{0, k+l}(A)]^{vir}} \prod_s \iota^* \alpha_s [pt] \psi_1^{\beta_1} \prod_{i=2}^k \beta_i \psi_i^{\beta_i} \wedge e.
\]

The above twisted invariant has been studied in [FP] and [CG]. Their idea is to mark an additional point. We summarize in the following form.

**Proposition 4.1.** A twisted invariant can be expressed in terms of a similar invariant replacing $e$ by a descendent at the additional marked point together with some products of ordinary invariants. The total curve class of each product of invariants is still equal to $A$, each insertion of the original invariant appears as an insertion of a factor invariant, and the factor invariants are linked by dual insertions from the diagonal class.

These products have the stated properties because they correspond to boundary contributions from nodal curves and are obtained via diagonal splitting.

In our situation, we also need to consider a slight variant of the above twisted invariants. Here, we consider the following complex

\[
D_{f,L} : \{v \in \Omega^0(f^*L), v(x_i) = 0, i \leq k\} \rightarrow \Omega^{0,1}(f^*L).
\]

coker$D_{f,L} - \ker D_{f,L}$ defines a virtual bundle of rank

\[
k - c_1(L)(A) - 1.
\]

Let $e_L$ be its Euler class. Then, the Gromov-Witten invariant twisted by $L$ is defined as

\[
<\iota^* v, \tau_1([pt]), \tau_2(\beta_2), \cdots, \tau_k(\beta_k)>_{0,A}^D = \int_{[\overline{M}_{0, k+l}(A)]^{vir}} \prod_s \iota^* \alpha_s [pt] \psi_1^{\beta_1} \prod_{i=2}^k \beta_i \psi_i^{\beta_i} \wedge e_L.
\]

We call it a **generalized $L$-twisted invariant**. There is a short exact sequence

\[
0 \rightarrow \ker D_{f,L} \rightarrow H^0(f^*L) \rightarrow \bigoplus_x L_{f(x)} \rightarrow \text{coker} D_{f,L} \rightarrow H^1(f^*L) \rightarrow 0.
\]

As a virtual bundle,

\[
\text{coker} D_{f,L} - \ker D_{f,L} = H^1(f^*L) - H^0(f^*L) - \bigoplus_i L_{f(x_i)}.
\]

Observe that we can use (27) to express $e_L$ in terms of the Euler class $e$ of the virtual bundle (22) and the insertion $c_1(L)$. It is then not hard to see that Proposition 4.1 applies to generalized $L$-twisted invariants of the form (26) as well.
4.2. Type I/II relative invariants and twisted rubber invariants. Let $Y$, $L$ and $R$ all denote the $\mathbb{P}^1$ bundle $\mathbb{P}(N_D \oplus \mathcal{O})$.

For any non-negative integer $m$, construct $R_m$ by gluing together $m$ copies of $R$, where the infinity section of the $i^{th}$ component is glued to the zero section of the $(i+1)^{th}$ component for $1 \leq i \leq m$. Denote the zero section of the $i^{th}$ component by $D_{i,0}$, and the infinity section by $D_{i,\infty}$, so $\text{Sing}R_m = \bigcup_{i=1}^{m-1} D_{i,\infty}$. Define $Y_m$ by gluing $Y$ along its infinity section denoted by $D_{0,\infty}$ to $R_m$ along $D_{1,0}$. Thus $\text{Sing}Y_m = \bigcup_{i=0}^{m-1} D_{i,\infty}$. $Y_0 = Y$ will be referred to as the level 0 component and the $R_i$ will be called the level $i$ component. We will also sometimes denote $D_{m,\infty}$ by $D_\infty$ if there is no confusion.

Let $\text{Aut}_D R_m$ be the group of automorphisms of $Q_m$ preserving each $D_{i,0}, D_{i,\infty}, 1 \leq i \leq m$, and the morphism to $D_{1,0}$. And let $\text{Aut}_D Y_m$ be the group of automorphisms of $Y_m$ with restriction to $R_m$ being contained in $\text{Aut}_D R_m$. Clearly, $\text{Aut}_D Y_m = \text{Aut}_D R_m \cong (\mathbb{C}^*)^m$, where each factor of $(\mathbb{C}^*)^m$ dilates the fibers of the $\mathbb{P}^1$-bundle $R_i \longrightarrow D_{1,0}$. Denote by $\pi[m] : Y_m \longrightarrow Y$ the map which is the identity on the root component $Y_0$ and contracts all the bubble components to $D_{1,0}$ via the fiber bundle projections.

Similarly we can form $L_n$ and glue it to the left of $Y_m$. We denote the resulting chain of $\mathbb{P}^1$-bundle by $nY_m$. If $m = 0$ we simply write $nY$ for $nY_0$. Of course $Y_m$ is the same as $Q Y_m$. $L_j$ is regarded as the level $-j$ component. The automorphism group for $L_n$ is defined in the same way as for $Y_m$, and the extension to $nY_m$ is obvious.

4.2.1. Type I invariants and Type II invariants. Given a relative insertion $\mu$ at $D_\infty$, the relative moduli space for $(Y, D_\infty)$ consists of the union over $m$ of equivalence classes of marked relative stable maps into $(Y_m, D_{m,\infty})$ satisfying the relative constraint. It comes with a virtual fundamental homology class.

There are also natural cohomology classes associated to marked points: the pull back classes via the evaluation maps at the marked points, as well as descendents classes. Thus we can integrate these classes over the virtual class to define relative invariants, called type I invariants in [MP].

In the same way we can define relative invariants for $(Y, D_0)$ by considering marked relative stable maps into $(nY, D_{-n,0})$ for various. These invariants are also called type I invariants.

Type II invariants are relative invariants for $Y$ relative to both $D_0$ and $D_\infty$. They are defined via moduli spaces of equivalence classes of marked stable maps into $(nY_m, D_{-n,0}, D_{m,\infty})$.

4.2.2. Distinguished type II invariants and their orders. A distinguished invariant of type II is an invariant of $(Y, D_0 \cup D_\infty)$ with a distinguished insertion of the form $[D_\infty]^{\delta}$ (Notice that our definition is different from that in [MP] where $[D_0]$ is in place of $[D_\infty]$).

Distinguished type II invariants are ordered in a similar way as in Definition 4.13. The new features are that parts (3) and (4) of that order are
replaced successively by the number of non-distinguished insertions, the total degree of $D_0$--relative insertions, the total degree of $D_\infty$--relative insertions, the degree of the distinguished insertion.

4.2.3. Non-rigid targets and rubber invariants with $\Psi$ insertions. Let we denote by $nY_m$ the collapsing of $Y_0$ in $nY_m$. $nY_m$ is called a non-rigid target, due to the $\mathbb{C}^*$ action at each level.

Given relative insertions $\mu$ and $\nu$ on the two sides $D_{-n,0}$ and $D_{m,\infty}$, there is the rubber moduli space $\overline{M}_{0,1}^{\nu}(B; \mu|\nu)$ consisting of the union over $m,n$ of equivalence classes of stable pseudo-holomorphic maps from $l$--marked genus 0 curve into $nY_m$ with class $B$. Notice that here the stability is the rubber stability and the equivalence is the rubber equivalence.

As for the ordinary moduli space, a rubber moduli space also comes with the virtual fundamental (homology) class. Evaluation classes and descendant classes are invariant under enlarged rubber automorphisms and hence define classes on the rubber moduli space. Thus we can integrate these classes over the virtual class to define invariants, which are called rubber invariants in [MP].

In fact, there are two additional degree 2 cohomology classes on the rubber moduli space coming from two classes on the relative moduli space, $\Psi_0$ and $\Psi_0$. They are the cotangent classes at $D_0$ and $D_\infty$ respectively. We give the description of $\Psi_0$ here following [Ga], the construction for $\Psi_\infty$ is the same. Given a relative map $f$ with a marked point $x_i$ mapping to $D_0$, if the multiplicity of $f$ at $x_i$ is $\alpha_i$, then define

\[ \Psi_0 = \alpha_i \psi_i + ev_x^* c_1(N_{D_0/Y}) = \alpha_i \psi_i + ev_x^* c_1(L). \]

It is independent of the choice of $x_i$ as along as it is mapped to $D_\infty$.

As $\Psi_0$ and $\Psi_\infty$ are invariant under enlarged rubber automorphisms they descend to degree 2 classes, still called $\Psi_0$ and $\Psi_0$, on rubber moduli spaces. An important new feature is that these classes are not generated by evaluation classes. We call rubber invariants with $\Psi$ insertions twisted rubber invariants. We will see very soon that twisted rubber invariants appear in the relative virtual localization formula. They are generally very difficult to evaluate explicitly.

We can certainly define type II invariants with $\Psi$ insertions. We can also define type I invariants for $(Y,D_\infty)$ with $\Psi_0$ insertions, and type I invariants for $(Y,D_0)$ with $\Psi_0$ insertions. However, we will have no use for these more general invariants. Thus we reserve the name a type I or type II invariant only for one with no $\Psi$ insertions.

4.3. Relative virtual localization on $\mathbb{P}^1$ bundle. In this subsection we follow p. 135-8 in [Ga].

We switch back to the notation $P$, i.e. $P = Y$ and $nP_m = nY_m$. $P$ carries a natural $S^1$ action by rotating the fibers. The fixed point loci are precisely $D_0$ and $D_\infty$. It induces a natural action on the moduli space of relative stable maps $\overline{M}_{0,k+i}^{P,D_{\infty}}(A)$.
4.3.1. **Fixed point components labeled by bipartite graphs.** The set of connected components of fixed point locus is indexed by bi-partite graphs. Each vertex corresponds to either the ordinary (connected) stable maps into $D_0$ or stable maps into rubber over $D_∞$. The first type of vertex is called an ordinary vertex, and the second type is called a rubber vertex.

Each edge corresponds to Galois covers of fibers of $P$ totally ramified over $D_{0,0}$ and $D_{0,∞}$ with no marked points away from $D_{0,0}$ and $D_{0,∞}$.

The connection data of the Galois covers is described by a sum over cohomology weighted partitions specifying the rubber relative conditions on the connecting divisor.

Given each bipartite graph, each component $F$ is then a finite quotient of a fiber product $M_0 \boxtimes M_1$ by a finite group $G$. Here

- $M_0$ is the product over vertices on $D_0$ of ordinary moduli spaces $\overline{M}^D_{0,l}(B)$ to $D_0$, adding the nodes $y_i$ where the fibers are attached;
- $M_1$ is the product of the rubber moduli space $\overline{M}^P_{0,l}(B; \mu|\nu) \sim$ over vertex on $D_∞$, adding the nodes $y_i$ where the fibers are attached;
- $\boxtimes$ denotes a fiber product over evaluation maps to $D_0$, $∞$ at the gluing points $y_i$ (in the intersection of levels 0 and 1);
- $G$ is the group of permutations of the points $y_i$ that preserves the multiplicities.

The virtual fundamental class of $F$ is the one induced by this product structure.

4.3.2. **Equivariant Euler class.** For each component $F \subset \overline{M}^P_{0,k+1}(A)$, the equivariant Euler class of the virtual normal bundle $N^\text{virt}_{F/\overline{M}^P_{0,k+1}(A)}$ is also a fiber product,

$$e(N^\text{virt}_{F/\overline{M}^P_{0,k+1}(A)}) = \frac{1}{G} e_0(N^\text{virt}_{F/\overline{M}^P_{0,k+1}(A)}) \boxtimes \frac{[M_0]^\text{virt}}{e_1(N^\text{virt}_{F/\overline{M}^P_{0,k+1}(A)})}$$.

$e_0$ is a product with one factor being the equivariant Euler class of the virtual bundle

$$(29) \quad H^0(f^*N_{D_0/p}) - H^1(f^*N_{D_0/p}) = H^0(f^*L) - H^1(f^*L)$$

on $M_0$. This is why we need to consider the twisted invariants as in [Ga].

There are two other types of factors of $e_0$. Denote the generator of $H^*_S(pt)$ by $t$. Every node $y_i$ in $D_{0,0}$ connecting to a multiple cover of degree $m$ contributes a product

$$\left(\frac{t + ev_{y_i}^*c_1(L)}{m} - \psi_{y_i}\right)^{m-1} \prod_{k=1}^{m-1} \frac{k(t + ev_{y_i}^*c_1(L))}{m}.$$

Here the first term corresponds to part (iii) on p. 137 in [Ga], and the second term corresponds to (ii)(b) there.
Concerning $e_1$, each marked point $x_i$ in $D_0$ on a multiple cover of degree $m$ contributes to $e_1$ a product

$$(-t - \Psi_0) \prod_{k=1}^{m} \frac{k(t + ev_0^*c_1(L))}{m}.$$ 

Here the first term corresponds to part (iv) on p. 137 in [Ga], and the second term corresponds to (ii)(a) there. This is why we need to consider twisted rubber invariants as in 4.2.3.

4.3.3. Virtual localization formula. The virtual localization formula equates

$$[\mathcal{M}_{0,k+1}^{p,D_\infty}(A)]^{\text{virt}} = \sum_F \frac{[\mathcal{F}]^{\text{virt}}}{e(N_{\text{virt}}^{p,D_\infty})_{/\mathcal{M}_{0,k+1}^{p,D_\infty}(A)}}$$

in the equivariant cohomology of $\mathcal{M}_{0,k+1}^{p,D_\infty}(A)$.

Thus, as mentioned, to apply the virtual relative localization we need to evaluate an ordinary twisted invariants over $\mathcal{M}_0$ and certain twisted rubber invariants. As mentioned in 4.1, ordinary twisted invariants have been treated in [FP]. As for twisted rubber invariants, a nice algorithm, the rubber calculus, has been developed in [MP].

4.4. Rubber calculus. In this subsection we review the rubber calculus and in addition, we keep track how the curve class behaves.

4.4.1. The first reduction–removing $\Psi_0$. The first step is to remove $\Psi$ insertions from rubber invariants, i.e. express twisted rubber invariants in terms of (ordinary) rubber invariants.

It involves the dilaton equation, the divisor equation and the topological recursion relation. We only describe them in the form needed.

**Dilation equation:** if $c = 2g - 2 + n + l(\mu) + l(\nu) \neq 0$ for a rubber invariant, then it is $c^{-1} \times$ the rubber invariant with an extra absolute insertion $\tau_1(1)$. The curve class is again preserved.

**Divisor equation:** if a divisor is nonzero on the curve class, a $\Psi_0$–rubber invariant is the sum of the rubber invariant with the divisor added as an absolute insertion, rubber invariants with smaller descendent powers, and $\Psi_0^{k-1}$–rubber invariants. The curve classes are still preserved.

Finally we describe the application the topological recursion to a rubber invariant with a $\Psi_0$ insertion and at least one absolute insertion. The starting point is that, by (28), $\Psi_0$ over the rubber moduli space can be expressed as $ev_0^*(c_1(L)) + \alpha^* \psi_3$, where $p$ is the marked point carrying one absolute insertion. $\alpha$ is the canonical map to the Artin stack of 3–pointed genus 0 curves, sending a rubber map $f$ to the fiber $C_f$ containing $p$ with marked points $D_0 \cap C_f, f(p), D_0 \cap C_f$.

**Topological recursion:** $\psi_3$ is expressed as a sum of boundary divisor, hence a $\Psi_0^{k-1}$–rubber invariant with at least one absolute insertion is the sum of a $\Psi_0^{k-1}$–invariant, and $\Psi_0^{k-1}$–rubber invariants multiplied by a rubber
invariant with no $\Psi_0$ insertion and at least one absolute insertion. The curve classes are possibly smaller.

**Remark 4.2.** To be able to apply topological recursion, we need at least one absolute insertion. This can be achieved either by the dilaton equation or the divisor equation.

For a fiber class $\Psi^k_0$-rubber invariant, the target stability insures that the relevant $c$ in the dilaton equation is nonzero, hence it can be turned into a $\Psi^k_0$-rubber invariant with an extra dilaton insertion, and hence at least one absolute insertion. Apply the topological recursion to reduce the dilaton rubber invariant to rubber invariants with fewer $\Psi_0$ insertions. Repeating the cycle yields rubber invariants without $\Psi_0$ insertions.

For a non-fiber class $\Psi^k_0$-rubber invariant, first add a divisor insertion $\pi^*\omega_D$. As $\pi^*\omega_D$ pairs positively with a non-fiber class, the divisor equation can be used to express the original invariant in terms of the new invariant which has the divisor insertion and hence at least one absolute insertion, together with either $\Psi^{k-1}_0$-rubber invariants or $\Psi^k_0$-rubber invariants with smaller descendent powers.

Notice that invariants with a negative power descendent insertion is automatically zero. Thus we can employ the topological recursion repeatedly until there are no $\Psi_0$ insertions.

4.4.2. The second reduction—Rubber to type II. This is achieved through the rigidification process. A rubber invariant with at least one absolute insertion is turned into a type II invariant with one absolute insertion replaced by its product with the divisor $D_{\infty}$. The curve class is preserved.

A fiber class rubber integral with no $\Psi_0$ insertions is turned into type II invariants through rigidification after adding a dilaton insertion.

While a non-fiber class rubber integral with no $\Psi_0$ insertions is expressed in terms of type II invariants through rigidification after adding a divisor insertion of the form $\pi^*\omega_D$.

In summary, the following is proved in [MP].

**Proposition 4.3.** Any $\Psi^k_\infty$-rubber invariant can be expressed in terms of type II invariants with the same class, or products of type II invariants. The total curve class of each product of invariants is still equal to $A$, each insertion of the original invariant appears as an insertion of a factor invariant, and the adjacent factor invariants are linked by dual insertions.

Moreover, if the curve class is non-fiber and the domain is connected, the type II invariants with the same curve class are distinguished and their relative insertions are no bigger than those of the rubber invariant, the number of non-distinguished insertions is non bigger than the number of absolute insertion of the rubber invariant.

4.5. The reduction algorithm—relative to divisor. Let us now finally describe the algorithm in [MP] to determine relative invariants of $(P, D_{\infty})$. 
from invariants of $D$. This step involves both virtual localization and degeneration.

For a fiber class, the moduli space of relative stable maps fibers over $D$ with fiber isomorphic to the moduli space of stable maps to $\mathbb{P}^1$ relative to $\infty$. Thus a fiber class invariant can be expressed in terms of the classical cohomology of $D$ and equivariant GW invariant of $(\mathbb{P}^1, \infty)$.

4.5.1. Three relations and the final reduction. Now consider a class $A$ with $\pi_*(A) \neq 0$, i.e. a non-fiber class.

By first applying localization a type I relative invariant of $(P, D_\infty)$ can be expressed in terms of twisted rubber invariants (with $\Psi_0$ insertions) and twisted integrals in the GW theory of $D$.

Then Proposition 4.3 expresses the involved twisted rubber invariants in terms of distinguished type II invariants. And Proposition 4.1 expresses the involved twisted integrals in the GW theory of $D$ in terms of ordinary invariants of $D$.

Thus it suffices to show every distinguished type II invariant can be expressed in terms of invariants of $D$. This is achieved through the following 3 relations.

For relative conditions

$$\mu = \{ (\mu_i, \delta_{\mu_i}) \}, \nu = \{ (\nu_j, \delta_{\nu_j}) \},$$

consider a $(Y, D_\infty)$ type I invariant of the form

$$< \prod_i \tau_{\mu_i-1}([D_0]\delta_{\mu_i}) \omega \tau_0([D_\infty]\delta)|\nu >_{A, D_\infty}^P,$$

and the distinguished type II invariant

$$< \mu | \omega \tau_0([D_\infty]\delta)|\nu >_A.$$

Relation 1, proved by the degeneration formula for (30) relative to $D_0$, expresses the distinguished type II invariant (31) in terms of the type I invariant (30), undistinguished type I invariants of $(P, D_\infty)$ with class $A$, type I invariants of $(P, D_\infty)$ with smaller curve classes, and distinguished type II invariants lower than (31).

Relation 2, proved again by virtual localization, Proposition 4.1 and Proposition 4.3 expresses the type I invariant (30) in terms of distinguished type II invariants with class $A$ but strictly lower than (31), type II invariants with class smaller than $A$ and invariants of $D$. We briefly review the argument.

In the virtual localization formula, there are contributions from $M_0$ and $\mathcal{M}_1$ respectively. The contribution from $M_0$ is a twisted integral in the GW theory of $D$, which can be reduced to ordinary GW invariants of $D$ by Proposition 4.1.

Thus the principal terms of the localization formula come from fixed loci with constant $D_0$ vertices. Apply Proposition 4.3 to the rubber invariants, which have the curve class $A$ and a $\Psi_0$ insertion, we obtain distinguished type
II invariants with curve class $A$, together with products of type II invariants with the same total curve class and including each original insertion.

Similarly, Relation 2’ expresses undistinguished type I invariants with class $A$ in Relation I in terms of distinguished type II invariants with class $A$ which are lower than $[31]$, products of type II invariants with the same total curve class, and products of GW invariants of $D$ the same total curve class.

By Relations 1, 2, and 2’, a non-fiber class distinguished type II invariant can be inductively computed from products of lower order distinguished type II invariants together with invariants of $D$ with the total curve class. Here the primary induction is on the pair $(g, A)$, and the secondary induction is on the ordering.

In summary, the following is proved in [MP].

**Proposition 4.4.** Any type I/II invariant can be expressed in terms of invariants of $D$ with the same class, or products of invariants of $D$. The total curve class of each product of invariants is still equal to $A$, each insertion of the original invariant appears as an insertion of a factor invariant, and the adjacent factor invariants are linked by dual insertions.

**4.6. Two vanishing theorems for invariants with a descendent point insertion.** Our interest here is to relate relative invariants with a descendent point insertion to invariants of $D$ with a descendent point insertion.

**4.6.1. The first vanishing theorem.** By keeping track of the reduction scheme of [MP] we have the following vanishing theorem

**Theorem 4.5.** Suppose that a non-fiber class $B$ has the property that $\pi_*(B)$ is not a sum of a uniruled class and an effective class of $D$. Then,

$$<\omega, \tau_1([pt]), \tau_2(\beta_2[D_0]), \ldots, \tau_k(\beta_k[D_0])>_{B}^P, D_\infty = 0.$$

**Proof.** To simplify the computation, we consider the following invariant subset of the moduli space $\overline{M}_{0,k+l}(A)$ of relative stable maps. Without loss of generality, we choose a point, denoted by $S_1 = pt$, and choose submanifolds $S_i, 2 \leq i \leq k$, of $D_0$ representing $\beta_1 = [pt]$ and $\beta_i[D_0]$. We here assume that $S_i, i \geq 1$ intersect transversely. Let

$$\overline{M}_{0,k+l}(A, pt, S_2, \ldots, S_k) = \{f \in \overline{M}_{0,k+l}(A); f(x_0) = pt, f(x_i) \in S_i, 1 \leq i \leq k\}.$$

We can construct a virtual fundamental cycle for this space which will give the desired relative invariants. Of course we need to modify the deformation-obstruction complex. The linearization of the Cauchy-Riemann operator is the complex

$$D_f : \{v \in \Omega^0(f^*TP), v(x_i) \in TS_i\} \to \Omega^{0,1}(f^*TP).$$

As each $S_i$ is in the fixed locus $D_0$, $\overline{M}_{0,k+l}(A, pt, S_2, \ldots, S_k)$ also carries a natural $S^1$-action. Thus we still can apply the localization formula to
this moduli space. The only difference here is that we need to calculate the fixing-moving part of our new complex to determine the contributions.

Since $\pi$ is an equivariant map, each $\pi^*(\alpha_i)$ is an equivariant class containing no equivariant parameter. Applying the localization formula, the relative invariant is given by

$$
(33) \sum_F \int_{[\mathcal{M}_F]^{vir}} \prod_t \psi^t_{\bar{\nu}} e(N_{vir})
$$

where each $F$ is indexed by a bipartite graph and $\bar{\psi}_i$ is the equivariant extension of $\psi_i$.

Recall that for each bipartite graph the corresponding component of the fixed point loci is the fiber product of a stable map moduli space of $D_0$ and a rubber stable map moduli space. Each vertex of the bipartite graph either corresponds to a connected component of stable maps into $D_0$ or a connected component of rubber stable maps into $D_0$. We call the vertex constant vertex if its homology class is zero.

Choose a self dual basis $\{\alpha_i\}$ of $H^*(D_0; \mathbb{R})$ and let $\{\bar{\alpha}_i\}$ be its dual basis.

Then the diagonal class of $D$ is simply

$$
(34) [\Delta] = \sum_i \alpha_i \otimes \bar{\alpha}_i.
$$

Fix a bipartite graph $\Lambda$ corresponding to a component $F$. Let $E$ be the set of oriented edges of $\Upsilon$. For each edge of $\Lambda$, we insert the diagonal class $[\Delta]$ in (34) to split $[\mathcal{M}_F]^{vir}$ as a disjoint union of products of virtual fundamental cycles, where each product has factors indexed by vertices of $\Upsilon$. The union is over the space $T$ of the maps $E$ to the set of ordered pairs $\{(\alpha_i, \bar{\alpha}_i)\}$ such that two edges with opposite orientations map to opposite pairs. For each $P \in T$ we define $P_\lambda$ to be the set of elements of $\{\alpha_i\}$, each coming from $P$ and an edge originating at $\lambda$.

We thus can write explicitly the contribution of each bipartite graph as a sum

$$
(35) \sum_{P \in T_{\text{vertices}}} \prod_{\lambda} V_{\lambda, P|\lambda},
$$

where $V_{\lambda, P|\lambda}$ is the twisted or rubber invariant at $\lambda$ with added marked points constrained by $P|\lambda$.

Clearly in each summand of (35) there is a vertex $\lambda$ such that $V_{\lambda, P_\lambda}$ contains a descendent point insertion.

For each vertex $\lambda$, we denote its curve class by $B(\lambda)$. It is clear that $\pi_*(B) = \sum_{\lambda} \pi_*(B(\lambda))$. For each $D_0$ vertex, $V_\lambda$ is a generalized twisted Gromov-Witten invariant by normal bundle of $D_0$, of the form (26). For each rubber vertex $\lambda$ with class $B(\lambda)$, $V_\lambda$ is a twisted rubber invariant.

By the construction, one of $V_\lambda$’s contains a point insertion. The remaining argument is to keep track of the point insertion on the invariant with nonzero curve class. The case where the vertex $\lambda$ containing a point insertion could
be a constant vertex is dealt with in the next lemma which we will prove afterwards.

**Lemma 4.6.** If the vertex \( \lambda \) containing a point insertion is a constant vertex, then there is another vertex \( \lambda' \) in the graph such that \( B(\lambda') \neq 0 \) and \( V_{\lambda'} \) contains a point insertion.

**Lemma 4.7.** Each \( V_{\lambda,P|_{\lambda}} \) can be expressed as

\[
\sum_{\pi_*(B(\lambda))=B_1+\cdots+B_k} a <,>_{B_1} \cdots <,>_{B_i} \cdots <,>_{B_k},
\]

where \( a \) is a combinatoric constant.

Suppose

\[
< \varpi, \tau_1([pt]), \tau_2(\beta_2[D_0]), \ldots, \tau_k(\beta_k[D_0])|\mu >_{B}^{P,D,\infty} \neq 0.
\]

Then one summand in (33) is nonzero. Hence one summand in the corresponding (35) is nonzero, and this possible only if one \( V_{\lambda,P|_{\lambda}} \) is nonzero.

By Lemmas 4.6 and 4.7 we can express \( \pi_*(B(\lambda)) \) as an effective decomposition such that one of the summands is a uniruled class. This contradicts to the assumption.

We now prove Lemma 4.6.

**Proof.** We have to show that the point insertion can always be applied to a nonconstant vertex. Suppose that the point insertion is on a constant vertex in \( D_0 \). We claim that this vertex cannot contain any other \( D \)-insertions. Suppose that the constant vertex contains absolute marked points \( j_1, \ldots, j_h \).

There is an additional marked point from the edge connecting it to a rubber vertex. The moduli space is then \( \mathcal{M}_{0,h+1} \).

Via the diagonal insertion process it is easy to see that the contribution of the constant vertex under consideration is the summation of terms

\[
\int_{\mathcal{M}_{0,h+1}} \alpha_i H,
\]

for some equivariant class \( H \). This term is nonzero only if \( \alpha_i = 1 \). Hence, its dual \( \tilde{\alpha}_i = [pt] \) is inserted into the contribution of the rubber component.

If the rubber vertex under consideration is also constant. We can apply the same argument to transport the point insertion into the next vertex along the graph until reaching a nonconstant vertex.

We now prove Lemma 4.7.

**Proof.** There are two types of vertices, \( D_0 \)-vertices and rubber vertices.

If \( \lambda \) is a \( D_0 \)-vertex, \( V_{\lambda,P|_{\lambda}} \) is a generalized \( L \)-twisted Gromov-Witten invariant. We simply apply Farber-Pandharipande’s argument.

As mentioned in 4.1, the basic idea is that by marking an additional point we can express \( V_{\lambda,P|_{\lambda}} \) as an ordinary Gromov-Witten invariant of \( D \) with an
additional marked point plus the boundary contributions. By Proposition 4.1, the boundary contribution can be written as

\[
\sum_{\pi_\ast(B(\lambda)) = B_1 + \cdots + B_k} a <, >_{B_1}, \ldots, >_{B_k}, \ldots, >_{B_k},
\]

When \( \lambda \) is a rubber vertex, the much more complicated argument of Maulik and Pandharipande achieves the same for the twisted rubber invariant \( V_{\lambda, P|\lambda} \).

\( \square \)

Recall

\[ V = \min\{ \omega(A), \ < t^* \varpi, \tau_{i_1}(\{pt\}), \tau_{i_2}(\beta_2), \ldots, \tau_{i_k}(\beta_k) >_{D} \neq 0 \}. \]

The above theorem implies that the first part of Theorem of 3.25

**Corollary 4.8.** Suppose that a non-fiber class \( B \) has the property that \( \omega(B) < V \). Then,

\[ < \varpi, \tau_{i_1}(\{pt\}), \tau_{i_2}(\beta_2[D_0]), \ldots, \tau_{i_k}(\beta_k[D_0]) >_B^P, D = 0. \]

**Proof.** We observe that \( B = B_0 + |\mu|F \) for \( B_0 = \pi_\ast(B) \in H_2(D_0, \mathbb{Z}) \). Therefore, \( \omega(\pi_\ast(B)) \leq \omega(B) < V \). By the definition of \( V \), \( \pi_\ast(B) \) can not be the sum of an uniruled class and an effective class of \( D \).

\( \square \)

### 4.6.2. The second vanishing theorem

Another case of interest is the special graph in the degeneration formula where the degeneration graph lies completely in \( P \). This is an admissible invariant with no relative insertion on \( D_\infty \). The purpose of this subsection is to show that the corresponding relative invariant of \( (P, D_\infty) \) is zero. Namely, we would like to calculate certain admissible invariants with empty relative insertion,

\[ < \varpi, \tau_{i_1}(\{pt\}), \tau_{i_2}(\beta_2[D_0]), \ldots, \tau_{i_k}(\beta_k[D_0]) >_A^P, D_\infty = 0, \]

with \( m = \sum_i (i + 1) \leq D_0 \cdot A \). Since there are no relative insertions at \( D_\infty \), we have \( D_\infty \cdot A = 0 \). It implies that \( A \in H_2(D_0, \mathbb{Z}) \). The dimension condition is

\[
2(C_1(A) + D_0 \cdot A + 2|\nu| + n - 2 + k + l) = \sum_i i + 2k + 2n + \sum_i \deg(\beta_i) + \deg(\varpi).
\]

Using assumption \( m \leq D \cdot A \), we have

\[
2(C_1(A) + n - 2 + k + l) \leq 2n + \sum_i \deg(\beta_i) + \deg(\varpi).
\]

**Theorem 4.9.** Suppose that \( 0 < \omega(\pi_\ast(A)) \leq V \) or \( \pi_\ast(A) \) is not a sum of a uniruled class and nonzero effective class of \( D \) and \( m = \sum_i (i + 1) \leq D_0 \cdot A \). Then,

\[ < \varpi, \tau_{i_1}(\{pt\}), \tau_{i_2}(\beta_2[D_0]), \ldots, \tau_{i_k}(\beta_k[D_0]) >_A^P, D_\infty = 0. \]

Notice that unlike Theorem 4.5, \( \pi_\ast(A) \) could be a uniruled class.
Proof. If a constant vertex is a $D_0$-vertex, it represents constant stable maps into $D_0$. If a constant vertex is a rubber vertex, it represents a multiple of fiber classes in the rubber. Such a rubber stable map necessarily contains relative insertions which contradicts to our assumption. Therefore, rubber constant vertices do not exist in our situation.

There are three cases to consider.

Case 1: Suppose that our bipartite graph consists of no edges and only one $D_0$-vertex. Since there are no rubber constant vertices, there are no other constant vertices either. In this case, the fixed point set is simply $\overline{\mathcal{M}}_{0,k+l}(A,pt,S_2,\cdots,S_k)$.

It is clear that the fixed part of the complex (24) is

$$D_{f,D_0} : \{ v \in \Omega^0(f^*TD_0), v(x_i) \in TS_i, 1 \leq i \leq k \} \to \Omega^{0,1}(f^*TD_0),$$

while the moving part is the complex (24).

Therefore, as the fixed point set, its virtual fundamental agrees with

$$[\overline{\mathcal{M}}_{0,k+l}(A,pt,S_2,\cdots,S_k)]_{\text{vir}}$$

Furthermore, $N_{\text{vir}} = \ker D_{f,L} - \text{coker}D_{f,L}$.

By the localization formula, the contribution of this graph is

$$\int_{[\overline{\mathcal{M}}_{0,k+l}(A,pt,S_2,\cdots,S_k)]_{\text{vir}}} \prod_t \psi_t^{\beta_t} \frac{N_{\text{vir}}}{e(N_{\text{vir}})}.$$

The degree of the virtual fundamental class is

$$2(C_1(A) + n - 3 + k + l) - 2n - \sum_{i=2} \deg(\beta_i).$$

Both absolute and relative insertions have no equivariant parameter. By the dimension condition of the relative invariant, the total degree of insertions is strictly larger than the degree of the virtual class. Therefore, the contribution is zero.

Case 2: There is a only one rubber vertex with homology class $B$, several $D_0$-vertices including some constant $D_0$-vertices. The constant $D_0$-vertices must be connected to the rubber vertex by edges. But there may be other edges not connected to any constant vertex. Suppose that the partition for the rubber vertex is $\mu$. It is easy to see that

$$|\mu| = D_0 \cdot A, \quad A = B + |\mu|F.$$

To simplify the notation, we first assume that there are only relative absolute insertions. Then all the marked points lies either on $M_0$ or on an edge.

Suppose that the vertex under consideration contains only marked points $i_1, \cdots, i_t$ from relative absolute insertions. It has an additional marked point from the edge which is a degree $m$ multiple cover of the fiber. The moduli space is clearly

$$\overline{\mathcal{M}}_{0,t+1} \times (S_{i_1} \cap \cdots \cap S_{i_t}).$$
Suppose that the number of constant vertices is $l'_\mu$. We also need to consider the edge containing a marked point, say $x_i$. Since the image of $x_i$ has to be in $S_i$, it contributes a factor

$$2n - \deg(\beta_i)$$

to the dimension of fixed point loci. We treat it as part of $M_0$. Let $k'$ be the number of such edges. It is clear that

$$l'_\mu + k' \leq l_\mu.$$

The dimension of $M_0$ is

$$2n(l'_\mu + k') - 2n - \sum_t \deg(\beta_t) + 2(k - k') - 4l'_\mu.$$

Let us compute the virtual dimension for the rubber vertex $\mathcal{M}_{0,0}(B,\mu)$. Let us compute the virtual dimension of the rubber moduli space $\mathcal{M}_{0,0}(B,\mu|\nu)$. A rubber stable map can be thought as a $C^*$-equivalence class of stable maps to $P$ relative to both $D_0, D_\infty$. The relative condition determines a unique lift of $B \in H_2(D_\infty, \mathbb{Z})$ to a class of $P$. A moment of thought tells us that the lift is $A$ itself. Then,

$$\dim_{\mathbb{C}}[\mathcal{M}_{0,0}(B,\mu)]^{vir}$$

$$= C_1(A) + D_0 \cdot A + n + 1 - 3 + l_\mu - |\mu| - 1$$

$$= C_1(A) + n - 3 - l_\mu.$$

Adding back $\omega$ and using the fact that $m \leq D \cdot A = |\mu|$, the dimension of fixed loci is

$$2(C_1(A) + n - 3 + l - l_\mu) + 2(k - k') - 4l'_\mu - 2n - \sum_t \deg(\beta_t)$$

$$= 2(C_1(A) + n - 2 + k + l) - 2n - \sum_t \deg(\beta_t) - 2(k' + l_\mu + 2l'_\mu - 1)$$

$$< \deg(\omega).$$

Hence, the contribution is zero.

**General case:** The general case contains at least two vertices with non-fiber homology classes. Let us consider the marked point corresponding to the point class. There are two cases, either on a nonconstant vertex or on a constant vertex.

By Lemma 4.6 we can transport the point $D$-insertion on a constant vertex to one on a nonconstant vertex. Suppose that the homology class of this nonconstant vertex is $B$. Then, $\omega(\pi_*(B)) \leq \omega(\pi_*(A))$. By the argument of Theorem 4.5, the contribution is zero.

\[\Box\]

### 5. The proof of main theorem

In this section, we establish our main theorem. For that purpose, we need an additional result from localization.
5.1. A nonvanishing Theorem. In previous section, we prove a vanishing theorem for the admissible invariant of \((P, D_\infty)\) with empty relative insertion on \(D_0\). In this section, we consider the sup-admissible case with \(m = \sum_t (i_t + 1) > D_0 \cdot A\). The idea is that the invariant of \((P, D_\infty)\) is no longer zero and the invariant of \(D_0\) will contribute in a nontrivial way. Consider relative invariant

\[
<\varpi, \tau_i([pt]), \tau_2(\beta_2[D_0]), \cdots, \tau_k(\beta_k[D_0])|_|_{P, D_\infty}^{A}\n\]

The dimension condition is

\[
2(C_1(A) + D_0 \cdot A + n + 1 + k + l - 3) = 2 \sum_t i_t + 2n + \sum_t \deg(\beta_t) + 2k + \deg(\varpi).
\]

Let \(m = \sum_t (i_t + 1)\) be the multiplicity. Recall that it is (i) admissible if \(m = D_0 \cdot A\); (ii) sup-admissible if \(m > D_0 \cdot A\); (iii) sub-admissible if \(m < D_0 \cdot A\). As mentioned in the introduction, we can always define an equivalent admissible invariant from a sub-admissible invariant. Hence, it is enough to consider admissible and sup-admissible invariants only. The dimension condition of the divisor invariant

\[
<\iota^*\varpi, \tau_i([pt]), \tau_2(\beta_2), \cdots, \tau_k(\beta_k)|_{D_\infty}^{A}\n\]

is

\[
2(C_1(A) + n + k + l - 3) = 2 \sum_t i_t + 2n + \sum_t \deg(\beta_t) + \deg(\varpi).
\]

Both invariants are well-defined only when \(k = D_0 \cdot A + 1\).

However, there is a well-defined twisted Gromov-Witten invariants for all the cases. Our expectation is that in the sup-admissible case, the invariant of \((P, D_\infty)\) is dominated by the twisted Gromov-Witten. We do not know yet how to carry out a correspondence with the twisted Gromov-Witten invariants. We hope to come back to it in the future.

Instead, we use the above idea to study a specific sup-admissible case.

**Theorem 5.1.** Suppose that \(0 < \omega(A) \leq M\) and \(k = D_0 \cdot A + 1\). Then,

\[
<\varpi, [pt][D_0], \beta_2[D_0], \cdots, \beta_k[D_0]|_\emptyset|_{P, D_\infty}^{A}\n\]

\[
te <\iota^*\varpi, [pt], \beta_2, \cdots, \beta_k|_{D_\infty}^{A}\n\]

for \(c \neq 0\).

**Proof.** The proof is similar to that of the second vanishing theorem. Applying localization formula, relative invariant is the non-equivariant limit of

\[
\sum_F \int_{[M_F]^{vir}} e(N^{vir}).
\]

We divide into 3 cases.

**Case 1:** Suppose that our bi-partite graph consists of only one vertex corresponding to the ordinary stable map into \(D_0\) and no edges. In this case,
the fixed point set is simply $\overline{M}_{0,k+1}^D(A,pt,S_2,\cdots,S_k)$. By the localization formula, the contribution of this graph is

$$\int_{[\overline{M}_{0,k+1}^D(A,pt,S_2,\cdots,S_k)]^\text{vir}} e(N_{\text{vir}}).$$

Since $\varpi$ have no equivariant parameter, we only need to take non-equivariant limit of $\frac{1}{e(N_{\text{vir}})}$. The latter is precisely the Euler class $e_L$ appearing in twisted Gromov-Witten invariants. Therefore, the contribution is just the twisted Gromov-Witten invariant

$$<t^*\varpi,[pt],\beta_2,\cdots,\beta_k>_{D,L,A}.$$ 

In the case of $k = D_0 \cdot A + 1$,

$$\dim[\overline{M}_{0,k+1}^D(A,pt,S_2,\cdots,S_k)]^\text{vir} = 2 \sum t + \deg(\varpi).$$

We immediately obtain the contribution as

$$c < t^*\varpi,[pt],\beta_2,\cdots,\beta_k>_{D,A},$$

where $c$ is the constant term of $\frac{1}{e(N_{\text{vir}})}$. Note that $N_{\text{vir}}$ has rank zero and hence $\frac{1}{e(N_{\text{vir}})}$ is a degree zero equivariant class. Therefore, it has an expression

$$c + t^{-1} \gamma_1 + t^{-2} \gamma_4 + \cdots + t^{-m} \gamma_{2m} + \cdots,$$

where $\gamma_{2m}$ has degree $2m$. On the other hand, the equivariant Euler class $e(N_{\text{vir}})$ is an invertible element after inverting equivariant parameter $t$. It implies that $c \neq 0$.

The precise value of $c$ can also be worked out by the method of [FP]. However, we don’t need it in our paper.

**Case 2**: There is only one rubber vertex with homology class $B$, several $D_0-$vertices including some constant vertices. Using the computation in the same $c$ of the second vanishing theorem, the dimension condition and the assumption $k = D_0 \cdot A + 1$, the virtual dimension of fixed loci is seen to be

$$2(C_1(A) + n - 3 + k + l) - 2n - \sum \deg(\beta_i) - 2(k' + l_\mu + l'_\mu)$$

$$< \deg(\varpi) - 2(k' + l_\mu + 2l'_\mu)$$

The contribution in this case is therefore equal to zero.

**General case**: The general case contains at least two vertices with non-fiber homology classes. The proof is identical to that of the second vanishing theorem. We omit it. $\square$

Using the above non-vanishing theorem, we are ready to prove our main theorem.
5.2. Proof. We are finally able to give the proof of Theorem 1.1 which we restate here for the convenience of readers.

Theorem 5.2. Suppose $D$ is uniruled and $A$ is a minimal uniruled class of $D$ such that

\[(\mathcal{44}) \quad <\iota^*\alpha_1, \cdots, \iota^*\alpha_l, [pt], \beta_2, \cdots, \beta_k >^D_A \neq 0\]

for $k \leq D \cdot A + 1$, $\beta_i \in H^*(D; \mathbb{R})$, and $\alpha_j \in H^*(X; \mathbb{R})$. Then $(X, \omega)$ is uniruled.

Proof. Notice that all the insertions of the invariant (44) are non-descendent. By linearity we can assume that each $\beta_i \in \Theta$ and each $\alpha_j \in \Xi$. Recall that $\Theta$ is the chosen self dual basis of $\oplus_{q=0}^{2n-2} H^q(D; \mathbb{R})$, and $\Xi$ is the chosen basis of $\oplus_{p=0}^{2n} H^p(X; \mathbb{R})$.

Assume first that the number $k$ of insertions of (44) satisfies $k = D \cdot A + 1$.

Then the following relative invariant of $(P, D_\infty)$,

\[(\mathcal{45}) \quad \langle \alpha_1, \cdots, \alpha_l, [pt], \tilde{\beta}_2, \cdots, \tilde{\beta}_k \rangle_{P \cdot D_\infty},\]

is well defined, as the dimension difference of the moduli spaces $2D \cdot A + 2$ matches with the difference of the total cohomology degree $2k$. Moreover, by the minimality of the class $A$ and Theorem 5.1, the invariant (45) is nonzero.

Notice that the invariant (45) is the relative invariant associated to the following standard sup-admissible graph $\Gamma(\varpi | \mu)$ of the symplectic cut with

\[(\mathcal{46}) \quad \varpi = (\alpha_1, \cdots, \alpha_l), \quad \mu = ((1, [pt]), (1, \beta_2), \cdots, (1, \beta_k)).\]

Hence the vector $v_{P \cdot pt}^{rel}$ is nonzero.

By Theorem 3.24 the vector $v_{D \cdot pt}^{abs}$ is nonzero as well. Notice that all the relevant invariants of $X$ have a point insertion (possibly descendent). Thus $X$ is uniruled by Theorem 2.5.

For the general case that $k \leq D \cdot A + 1$, notice that we can always increase the number of $D-$insertions by adding divisor insertions of the form $PD(\omega_D)$ to achieve the equality. \qed

Remark 5.3. We think it is possible to weaken the assumption by the existence of a minimal descendent invariant of the form

\[<\iota^*\varpi, \tau_{i_1}([pt]), \tau_{i_2}(\beta_2), \cdots, \tau_{i_k}(\beta_k) >^D_A \neq 0,\]

with $\sum_i (i + 1) \leq D \cdot A + 1$. What is missing is an analogue of Theorem 5.1 in this set up.

We are able to weaken the assumption in a different direction in the following subsection.
5.3. **Weakening the minimal condition.** The minimality condition on the uniruled class $A$ is a symplectic condition and is in agreement with the main theme of the paper conceptually. However, given a tamed almost complex structure, it is possible to weaken somewhat the minimal condition.

If $J$ is a tamed almost complex structure, a uniruled class $A$ is called $J$–indecomposable or just indecomposable if it is not the sum of another uniruled class and a nonzero $J$–effective class. For example a uniruled class which is primitive and lies on an extremal ray of the $J$–effective cone is an indecomposable uniruled class. A minimal uniruled class is obviously indecomposable. Unfortunately, our argument does not extend to the indecomposable situation. Instead, it applies to the following intermediate situation.

We call a uniruled class $A$ of $D$ a **globally indecomposable uniruled class** of $(X, D)$ if $A$ is not a sum of a uniruled class of $D$ and a nonzero effective class of $X$. Certainly a globally indecomposable class uniruled class of $(X, D)$ is an indecomposable uniruled class of $D$. The converse is not true, as a class of $D$ is not necessarily effective in $D$ even if it is effective in $X$.

Notice that a minimal uniruled class is globally indecomposable. Thus the following is a slightly stronger version of Theorem 5.4.

**Theorem 5.4.** Suppose $A$ is a globally indecomposable uniruled class of $D$ such that

$$<i^*\alpha_1, \cdots, i^*\alpha_l, [pt], \beta_2, \cdots, \beta_k>_A^D \neq 0$$

for $k \leq D \cdot A + 1$. Then $X$ is uniruled.

**Proof.** We shall sketch the necessary modification of the proof while leaving the details for the interested readers.

First step is to modify the partial order where we define $B \leq A$ if $A$ is a sum of $B$ and an effective class.

Next, we modify the correspondence.

**Definition 5.5.** We consider the following subset $I_{D-\text{pt},0}$ of colored standard graphs of $X$.

1. $g(\Gamma) = 0$,
2. the fixed class $A$,
3. admissible graphs with a $D$–point insertion,
4. sup-admissible graphs of the form $\Gamma_0$

$$\sum_{t=1}^{k}(i_t + 1) = D \cdot A + 1,$$

5. **empty graph excluded.**

To prove the correspondence, we apply the degeneration formula as before. The point of attention is the nonzero term involving nontrivial graphs for both $\Gamma^+, \Gamma^-$. In this case, we express $A = A_1 + A_2$, where $A_2 = \pi_*(A(\Gamma^+))$ is the effective class of $D$, and $A_1$ is an effective class of $X$. By Theorem 4.1, $A(\Gamma^-)$ is either a fiber class or $A_2$ is a sum of a uniruled class and an
effective class of $D$. If $A_2$ is a sum of a uniruled class and an effective class
of $D$, $A$ itself is a sum of a uniruled class of $D$ and an effective class of $X$.
This contradicts the assumption. Therefore, only the case of $A(\Gamma^-)$ being
a fiber class can appear. But then $A(\Gamma^+) = A$, which is exactly what we
want.

The other situation is of course the special graph. The rest of proof goes
through without change.

\[
\square
\]

6. Applications

As mentioned in the introduction, our main theorem can be applied as an
existence theorem of uniruled manifolds. In this section, we apply our main
theorem to construct uniruled symplectic manifolds inductively, generalizing
several early results of McDuff.

6.1. 4–dimensional uniruled divisors. A deep result in dimension 4 is
that being uniruled is a smooth property. More precisely, a 4–manifold
$(M, \omega)$ is uniruled if and only if $M$ is diffeomorphic to a connected sum
of $P^2$ or a $S^2$–bundle over a surface with a number of $\mathbb{P}^2$. Moreover, the
isotopy class of $\omega$ is determined by $[\omega]$.

We need to analyze uniruled classes and the corresponding in sertions.

**Proposition 6.1.** If $A$ is a uniruled class of a 4–manifold, then

(i) $A$ is represented by an embedded symplectic surface,
(ii) $C_1(A) \geq 2$,
(iii) $A^2 \geq 0$,
(iv) $A \cdot B \geq 0$ for any class $B$ with a non-trivial GW invariant.

**Proof.** The point is that 4–manifolds are semi-positive. Thus, for a generic
tamed almost complex structure $J$ and a generic point, only somewhere
injective $J$–holomorphic curves with a smooth domain $S^2$ contribute to the
relevant GW invariant (see [McS]). Such a curve can be smoothed to an
embedded symplectic surface.

The genus 0 moduli space of a class $A$ has dimension $C_1(A) - 1$. Since
there is at least a point insertion, we have (ii).

Now the adjunction inequality in [Mc4], together with (i) and (ii), implies
that $A \cdot A \geq 0$.

(iv) follows from (i), (iii) and positivity of intersection in [Mc4]. By (i) we
have an embedded $J$–holomorphic sphere $C$ in the class $A$ for some tamed $J$.
The class $B$ is represented by a union $\cup m_i D_i$ of possibly singular irreducible
$J$–holomorphic curves with multiplicities. If an irreducible component $D_i$
is distinct from $C$, then apply the positivity of intersection, if $D_i = C$ then
apply (iii).

\[
\square
\]
6.1.1. **Simple case—proportional.** For $\mathbb{P}^2$, let $H$ be the generator of $H_2$ with positive area. $H$ is a uniruled class and any uniruled class of the form $aH$ with $a > 0$. Obviously, $H$ is the minimal uniruled class. The relevant insertion is $(pt, pt)$. As $pt$ is a restriction class, i.e. an $\alpha$ class, we can take $k = 1$.

Similarly, for the blow-up of an $S^2$—bundle over a surface of positive genus, the fiber class is a uniruled class, and any uniruled class is a positive multiple of the fiber class. The relevant insertion for the fiber class is $pt$. Thus again we can take $k = 1$.

It is easier to apply Theorem 5.4 in this case.

**Corollary 6.2.** Suppose $(X^6, \omega)$ contains a divisor $D$ which is diffeomorphic to $\mathbb{P}^2$ or the blow-up of a $S^2$—bundle over a surface of positive genus. If the normal bundle $N_D$ is non-negative on a uniruled class, then $X$ is uniruled.

6.1.2. **General case.** For other $M^4$, the uniruled classes are not proportional to each other. Thus the minimality condition depends on the class of the symplectic form on $M$.

We first analyze the easier case of an $S^2$—bundles over $S^2$. For $S^2 \times S^2$, by uniqueness of symplectic structures, any symplectic form is of product form. Let $A_1$ and $A_2$ be the classes of the factors with positive area. It is easy to see that any uniruled class is of the form $a_1 A_1 + b_1 A_2$ with $a_1 \geq 0, b_1 \geq 0$. Thus either $A_1$ or $A_2$ has the minimal area.

For the nontrivial bundle $S^2 \times S^2$, let $F_0$ be the class of a fiber with positive area and $E$ be the unique $-1$ section class with positive area. If $aF_0 + bE$ is a uniruled class then $b \geq 0$ by (iv) of Proposition 6.1 since $F_0 \cdot E = 1, F_0 \cdot F_0 = 0$. And if $b > 0$, then, by (ii) of Proposition 6.1, $a \geq 1$ as $C_1(E) = 1$. Thus $F_0$ is always the minimal uniruled class no matter what the symplectic structure is.

Since the relevant insertion for $A_1, A_2$ and $F_0$ is just $pt$, we have

**Corollary 6.3.** Suppose $D = S^2 \times S^2$ and the restriction of the normal bundle $N_D$ to the factor with the least area is non-negative, then $X$ is uniruled.

In the case of the non-trivial bundle, $X$ is uniruled if the restriction of the normal bundle $N_D$ to $F_0$ is non-negative.

**Remark 6.4.** The following restatement is related to the contraction criterion in [LLR]. Suppose $(X^6, \omega)$ is a non-uniruled 6—manifold containing an $S^2$—bundle $D$. Then, if $D = S^2 \times S^2$, the normal bundle is negative along the $S^2$—fibers, and in the case of $S^2 \times S^2$ where there are two $S^2$—directions, the normal bundle is negative along the one with the least area.

The remaining uniruled 4—manifolds are $(\mathbb{P}^2 \# k \mathbb{P}^2, \tau)$ with $k \geq 2$. Let $H, E_i$ be the generators of $H^2$ of the summands with positive area. Then $[\tau]$ is of the form $uH - \sum_{i=1}^{k} v_i E_i$ with $u, v_i > 0$.

$H, E_i$ all have non-trivial GW invariants. Thus, by (iv) of Proposition 6.1, a class $\xi = aH - \sum_{i=1}^{k} b_i E_i$ is uniruled only if

$$ a \geq 0, \quad b_i \geq 0. $$
The condition that $\xi$ has square 0 is
\[(49) \quad a^2 = \sum_i b_i^2.\]
If (49) is satisfied, then the condition that $\xi$ is represented by an embedded sphere of genus 0 is given by the adjunction formula
\[(50) \quad 3a = \sum_i b_i + 2.\]

**Definition 6.5.** Any primitive class $\xi$ satisfying (48), (49), (50) is called a fiber class.

**Theorem 6.6.** Suppose $(D, \tau)$ is a symplectic $\mathbb{P}^2 \# k \mathbb{P}^2$ with $k \geq 1$. Then a fiber class is a uniruled class. In addition, any uniruled class is the sum of a positive multiple of a fiber class and another class with positive symplectic area.

Consequently, if $(D, \tau)$ is a symplectic divisor of a symplectic 6 manifold $(X, \omega)$ and the normal bundle $N_D$ is non-negative on a fiber class with the minimal $\tau -$area, then $X$ is uniruled.

**Proof.** By [LiL], a fiber class is equivalent to the indecomposable uniruled class $H - E_1$ via diffeomorphisms preserving the canonical class.

By Proposition 6.1, a uniruled class is represented by an embedded symplectic surface with non-negative self-intersection. By [LiL] such a class is equivalent to a reduced class via diffeomorphisms preserving the canonical class. A class $\xi = aH - \sum_{i=1}^{k} b_i E_i$ is called reduced if
\[a \geq b_1 + b_2 + b_3, \quad b_i \geq b_{i+1} \geq 0.\]
It is also shown in [LiL] if the surface is actually a sphere, then the class is equivalent to
\[2H, \quad H - E_1, \quad (l+1)H - lE_1, \quad (l+1)H - lE_1 - E_2, \quad l \geq 2.\]
The effective curve cone is generated by $-1$ classes.

Hence it suffices to show that a reduced class with non-negative self-intersection is the sum of a positive multiple of a fiber class and another class with positive symplectic area.

\[\square\]

To enumerate fiber classes, we notice that
\[H - E_1 = (H - E_1 - E_2) + E_2,\]
i.e. it is the sum of two $-1$ classes whose intersection number is equal to 1. Thus a fiber class is the sum of two $-1$ classes.

When $k \leq 8$, there are only finitely many $-1$ classes. So it is straightforward though tedious to list all the fiber classes.

Any class of the form $H - E_i, 1 \leq i \leq k$ is such a class, and when $k \leq 3$, there are no other classes.
When $k = 4$, there is a new class with the coefficient of $H$ being 2,

$$2H - E_1 - E_2 - E_3 - E_4.$$

By choosing any 4 distinct numbers between 1 and $k \geq 4$, we get many such classes for higher $k$. When $k = 4, 5$, there are no other new classes.

When $k = 6$, there are 6 new classes with the coefficient of $H$ being 3,

$$3H - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6,$$

and its permutations in the $E_i$. For higher $k$, there are similar classes.

When $k = 7$, there are additional classes,

$$(4|2, 2, 2, 1, 1, 1, 1), \quad (5|2, 2, 2, 2, 2, 1)$$

and their permutations in the $E_i$.

When $k = 8$, there are additional classes,

$$(4|3, 1, 1, 1, 1, 1, 1, 1), \quad (5|3, 2, 2, 1, 1, 1, 1),$$

$$(6|3, 3, 2, 2, 2, 1, 1, 1), \quad (7|4, 3, 2, 2, 2, 2, 2),$$

and their permutations in the $E_i$.

### 6.2. Higher dimensional case.

We start with the proof of Corollary 1.2.

**Proof.** The homologically injective case, which is equivalent to being cohomology surjective, is clear.

For the case of a projective uniruled divisor, according to Theorem 2.7, there is a nonzero invariant $I_{p,q}$ for a minimal uniruled class $A$. We now only need to observe that $[\omega|_D]^p = [\omega^p|_D]$.

\[\square\]

As we already mentioned that a Fano manifold is projectively uniruled. In particular, a hypersurface of $\mathbb{P}^n$ (for $n \geq 4$) of degree at most $n$ is Fano and hence projectively uniruled.

**Corollary 6.7.** Suppose $n \geq 4$ and $D$ is a Fano hypersurface symplectic divisor of $X$. If $N_D = \lambda[\omega_D]$ with $\lambda \geq 0$ then $X$ is uniruled.

**Proof.** Since $n \geq 4$, by the Lefschetz hyperplane theorem, $D$ has $b_2 = 1$. As $N_D = \lambda[\omega_D]$ for some $\lambda$ for any uniruled class of $D$, and in particular a minimal uniruled class, the statement follows from Corollary 1.2.

\[\square\]

Of course a particular case is $D = \mathbb{P}^{n-1}$ discussed in §2.

In general case we still need to verify the minimal condition. Of course the uniruled divisor needs not to be a projective manifold. For instances, the divisor could be a rather general uniruled fibration discussed in §2. Let us treat the case of a symplectic $\mathbb{P}^k$—bundle. Since the line class in the fiber is uniruled, and the relevant insertions can be taken to be $(pt, \omega^k_D)$, we have

**Corollary 6.8.** Suppose $D$ is a symplectic divisor of $X$. If $D$ is a projective space bundle with the fiber class being the minimal uniruled class and normal bundle $N_D$ is non-negative along the fibers, then $X$ is uniruled.
McDuff also considered the case of product $\mathbb{P}^k$–bundles in \cite{Mc2}. A natural source of such $D$ is from blowing up a ‘non-negative’ $\mathbb{P}^k$ with a large trivial neighborhood. Suppose $\mathbb{P}^k \subset X$ has trivial normal bundle. Then the blow up along $\mathbb{P}^k$ has a divisor $D = \mathbb{P}^k \times \mathbb{P}^{n-k-1}$. The normal bundle of $D$ along a line in $\mathbb{P}^k$ is trivial. Similar to the case of $S^2 \times S^2$, we can argue that the area of this line is minimal among all uniruled class of $D$. In particular, as observed by \cite{Mc2}, a symplectic $\mathbb{P}^1$ with a sufficiently large product symplectic neighborhood can only exist in a uniruled manifold.

In fact we can prove more.

**Corollary 6.9.** Suppose $S$ is a uniruled symplectic submanifold whose minimal uniruled class has area $\eta$ and insertions all being restriction classes. If $S$ has trivial symplectic neighborhood of radius at least $\eta$. Then $X$ is uniruled.

We will treat the more general case of ‘non-negative’ normal bundle in another paper on uniruled submanifolds.

**References**

[CL] B. Chen and A. Li, Symplectic relative virtual localization, in preparation.

[CG] T. Coates and A. Givental, Quantum Riemann - Roch, Lefschetz and Serre, arXiv:math/0110142.

[FP] C. Faber, R. Pandharipande, Hodte integrals and Gomov-Witten theory, Invent. Math. 139 (2000), 173-199.

[Ga] A. Gathmann, Gromov-Witten invariants of hypersurfaces, Habilitation thesis, University of Kaiserslautern

[G] R. Gompf, A new construction of symplectic manifolds, Ann. Math. 142(1995), 527-595.

[Go2] R. Gompf, Locally holomorphic maps yield symplectic structures. Comm. Anal. Geom. 13 (2005), no. 3, 511–525.

[Gr] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, Inventiones Math. 82 (1985), 307-347

[GV] T. Graber, R. Vakil, Relative virtual localization and vanishing of tautological classes on moduli spaces of curves, math.AG/0309227

[HLR] J. Hu, T-J. Li, Y. Ruan, Birational cobordism invariance of symplectic uniruled manifolds, arXiv:math/0611592, to appear in Invent. Math.

[IP] E. Ionel, T. Parker, The symplectic sum formula for Gromov-Witten invariants. Ann. of Math. (2) 159 (2004), no. 3, 935–1025.

[Le] E. Lerman, Symplectic cuts, Math. Research Lett. 2(1995), 247-258.

[Li] B. H. Li, T. J. Li, Symplectic genus, minimal genus and diffeomorphisms. Asian J. Math. 6 (2002), no. 1, 123–144.

[Li2] J. Li, Stable morphisms to singular schemes and relative stable morphisms, J. Diff. Geom. 57(2001),509-578.

[Li2] J. Li, Relative Gromov-Witten invariants and a degeneration formula of Gromov-Witten invariants, J. Diff. Geom. 60(2002), 199-293

[Ltj] T. J. Li, Existence of embedded symplectic surfaces, Geometry and topology of manifolds, 203–217, Fields Inst. Commun., 47, Amer. Math. Soc., Providence, RI, 2005.

[LL1] T. J. Li, A. Liu, Symplectic structures on ruled surfaces and a generalized adjunction inequality, Math. Res. Letters 2 (1995), 453-471.
[LL2] T. J. Li, A. Liu, Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4—manifolds with $b^+ = 1$, J. Differential Geom. Vol. 58 No. 2 (2001) 331-370.

[LM] F. Lalonde, D. McDuff, The classification of ruled symplectic 4-manifolds, Math. Res. Lett. 3 (1996), 769-778.

[LM2] F. Lalonde, D. McDuff, Symplectic structures on fiber bundles, Topology 42 (2003), 309-347.

[LR] A. Li, Y. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds, Invent. Math. 145(2001), 151-218.

[tLR] T-J. Li, Y. Ruan, Symplectic divisorial contractions in dimension 6, in preparation.

[Lu1] G. Lu, Finiteness of the Hofer-Zehnder capacity of neighborhoods of symplectic submanifolds, IMRN vol. 2006, 1-33.

[Lu2] G. Lu, Gromov-Witten invariants pseudo symplectic capacities, Israel Journal of Mathematics, 156(2006), 1-63.

[Lu3] G. Lu, Symplectic capacities of toric manifolds and related results, Nagoya Math. J., 181(2006), 149-184.

[Mc1] D. McDuff, The structure of rational and ruled symplectic 4—manifold, J. AMS.v.1. no.3. (1990), 679-710.

[Mc2] D. McDuff, Symplectic manifolds with contact type boundaries, Invent. Math. 103 (1991), 651-671.

[Mc3] D. McDuff, Hamiltonian $S^1$ manifolds are uniruled, preprint 2007.

[Mc4] D. McDuff, Singularities and positivity of intersections of $J$-holomorphic curves. With an appendix by Gang Liu. Progr. Math., 117, Holomorphic curves in symplectic geometry, 191–215, Birkhuser, Basel, 1994.

[McS] D. McDuff, D. Salamon, $J$—holomorphic curves and symplectic topology,

[MP] D. Maulik, R. Pandharipande, A topological view of Gromov-Witten theory, math.AG/0412503

[R1] Y. Ruan, Virtual neighborhoods and pseudoholomorphic curves, Turkish J. Math., 23(1999), 161-231.

[R2] Y. Ruan, Surgery, quantum cohomology and birational geometry. Northern California Symplectic Geometry Seminar, 183–198, Amer. Math. Soc. Transl. Ser. 2, 196.

[R3] Y. Ruan, Symplectic topology on algebraic 3—folds, J. Diff. Geom. 39 (1994), 215-227.

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