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Certain Hermite-Hadamard type inequalities via generalized $k$-fractional integrals

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Abstract
Some Hermite-Hadamard type inequalities for generalized $k$-fractional integrals (which are also named $(k, s)$-Riemann-Liouville fractional integrals) are obtained for a fractional integral, and an important identity is established. Also, by using the obtained identity, we get a Hermite-Hadamard type inequality.

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1 Introduction
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$  \hspace{1cm} (1.1)

holds. This double inequality is known in the literature as a Hermite-Hadamard integral inequality for convex functions [1].

Sarikaya et al. established the following results for Riemann-Liouville fractional integrals.

Theorem 1.1 (see Theorem 2 in [2]) Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If $f$ is a convex function on $[a, b]$, then the following inequality for fractional integrals holds:

$$f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(1 + \alpha)}{2(b - a)^\alpha} \left[ f_a^b f(\cdot) + f^b_a f(\cdot) \right] \leq \frac{f(a) + f(b)}{2}$$  \hspace{1cm} (1.2)

with $\alpha > 0$, where the symbols $f_a^b$ and $f^b_a$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$ that are defined by

$$f_a^b f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x - t)^{\alpha - 1} \, dt \quad (0 \leq a < x \leq b)$$

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and

\[ f_{\alpha}^a \left( x \right) = \frac{1}{\Gamma \left( \alpha \right)} \int_{x}^{b} f(t)(t - x)^{\alpha - 1} \, dt \quad (0 \leq a \leq x < b) \]

respectively. Here \( \Gamma \left( \cdot \right) \) denotes the classical gamma function [3], Chapter 6.

**Theorem 1.2** (see Theorem 3 in [2]) Let \( f : \left[ a, b \right] \rightarrow \mathbb{R} \) be a differentiable mapping on \( (a, b) \) with \( a < b \). If \( f' \in L[a, b] \), then the following inequality for Riemann-Liouville fractional integrals holds:

\[
\left| f(a) + f(b) \right| \leq 2^{\alpha} \left( b - a \right) \left( 1 - \frac{1}{2^\alpha} \right) \left( |f'(a)| + |f'(b)| \right)
\]

(1.3)

with \( \alpha > 0 \).

The Pochhammer \( k \)-symbol \( (x)_{n,k} \) and the \( k \)-gamma function \( \Gamma_k \) are defined as follows (see [4]):

\[
(x)_{n,k} := x(x + k)(x + 2k) \cdots (x + (n - 1)k) \quad (n \in \mathbb{N}; k > 0)
\]

(1.4)

and

\[
\Gamma_k(x) := \lim_{n \to \infty} \frac{n!k^n(nx)^{\frac{1}{k} - 1}}{(x)_{n,k}} \quad (k > 0; x \in \mathbb{C} \setminus k\mathbb{Z}_0),
\]

(1.5)

where \( k\mathbb{Z}_0 := \{ kn : n \in \mathbb{Z}_0 \} \). It is noted that the case \( k = 1 \) of (1.4) and (1.5) reduces to the familiar Pochhammer symbol \( (x)_n \) and the gamma function \( \Gamma \). The function \( \Gamma_k \) is given by the following integral:

\[
\Gamma_k(x) = \int_{0}^{\infty} e^{-t} \frac{t^{x/k}}{\Gamma(x)} \, dt \quad (\Re(x) > 0).
\]

(1.6)

The function \( \Gamma_k \) defined on \( \mathbb{R}^+ \) is characterized by the following three properties: (i) \( \Gamma_k(x + k) = x\Gamma_k(x) \); (ii) \( \Gamma_k(1) = 1 \); (iii) \( \Gamma_k(x) \) is logarithmically convex. It is easy to see that

\[
\Gamma_k(x) = k^{\frac{1}{k} - 1} \Gamma \left( \frac{x}{k} \right) \quad (\Re(x) > 0; k > 0).
\]

(1.7)

We want to recall the preliminaries and notations of some well-known fractional integral operators that will be used to obtain some remarks and corollaries.

The \((k,s)\)-Riemann-Liouville fractional integral operator \( ^{\kappa}L_s^a \) of order \( \alpha \) > 0 for a real-valued continuous function \( f(t) \) is defined as (see [5], p.79, 2.1. Definition):

\[
^{\kappa}L_s^a f(t) = \frac{(s + 1)^{\frac{1}{k} - \frac{\beta}{\Gamma_\kappa(\alpha)}}}{k^{\frac{1}{k} - \frac{\beta}{\Gamma_\kappa(\alpha)}}} \int_{a}^{t} \left( x^{s+1} - t^{s+1} \right) \frac{1}{x^{\frac{1}{k} - 1}} f(t) \, dt,
\]

(1.8)

where \( k > 0, \beta > 0 \) and \( s \in \mathbb{R} \setminus \{-1\} \).
The most important feature of \((k,s)\)-fractional integrals is that they generalize some types of fractional integrals (Riemann-Liouville fractional integral, \(k\)-Riemann-Liouville fractional integral, generalized fractional integral and Hadamard fractional integral). These important special cases of the integral operator \(\mathcal{J}^\alpha_a\) are mentioned below.

1. For \(k = 1\), the operator in (1.8) yields the following generalized fractional integrals defined by Katugompola in [6]:

\[
\mathcal{J}^\alpha_a f(x) = \frac{(r + 1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^r}{(x^{r+1} - t^{r+1})^{1-\alpha}} f(t) \, dt.
\]

(1.9)

2. Firstly by taking \(k = 1\), after that by taking limit \(r \to -1^+\) and using L'Hôpital's rule, the operator in (1.8) leads to the Hadamard fractional integral operator [1, 7]. That is,

\[
\lim_{r \to -1^+} \mathcal{J}^\alpha_a f(x) = \lim_{r \to -1^+} \frac{(r + 1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)t^r}{(x^{r+1} - t^{r+1})^{1-\alpha}} \, dt
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_a^x f(t) \lim_{r \to -1^+} \left( \frac{r + 1}{x^{r+1} - t^{r+1}} \right)^{1-\alpha} \, dt
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_a^x \left( \log \frac{x}{t} \right) f(t) \, dt
\]

\[
= \mathcal{H} \mathcal{J}^\alpha_a [f(t)]
\]

(1.10)

(see [8], p.569, eq. (3.13)).

3. If we take \(s = 0\) in (1.8), operator (1.8), reduces to the \(k\)-Riemann-Liouville fractional integral operator, which has been firstly defined by Mubeen and Habibullah in [9]. This relation is as follows:

\[
\mathcal{J}^\alpha_{a,k} f(x) = \frac{1}{k \Gamma(\alpha)} \int_a^x (x - t)^{\frac{1}{k} - 1} f(t) \, dt.
\]

(1.11)

4. Again, taking \(s = 0\) and \(k = 1\), operator (1.8) gives us the Riemann-Liouville fractional integration operator

\[
\mathcal{J}^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) \, dt.
\]

(1.12)

In recent years, these fractional operators have been studied and used to extend especially Grüss, Chebychev-Grüss and Pólya-Szegö type inequalities. For more details, one may refer to the recent works and books [7, 10–21].

2 Main results
Let \(f : I^0 \to \mathbb{R}\) be a given function, where \(a, b \in I^0\) and \(0 < a < b < \infty\). We suppose that \(f \in L_\infty(a,b)\) such that \(\tilde{j}_a \mathcal{J}^\alpha_a f(x)\) and \(\tilde{j}_b \mathcal{J}^\alpha_a f(x)\) are well defined. We define functions

\[
\tilde{j}(x) := f(a + b - x), \quad x \in [a, b]
\]
and

\[ F(x) := f(x) + \tilde{f}(x), \quad x \in [a, b]. \]

Hermite-Hadamard’s inequality for convex functions can be represented in a \((k, s)\)-fractional integral form as follows by using the change of variables \(u = \frac{x - a}{x - a} ; \) we have from (1.8)

\[
\frac{1}{k} J_k^a f(x) = (x - a) \frac{(s + 1) \Gamma_k(\alpha + k)}{4(b^{s+1} - a^{s+1})^{s+1}} \int_{0}^{1} \frac{(ux + (1 - u)a)^s}{(ux + (1 - u)a)^{s+1} - t^{s+1}}^{s+1} \times f(ux + (1 - u)a) \, ds,
\]

where \(x > a.\)

**Theorem 2.1** Let \(\alpha, k > 0\) and \(s \in \mathbb{R} \setminus \{-1\}.\) If \(f\) is a convex function on \([a, b],\) then we have

\[
f \left( \frac{a + b}{2} \right) \leq \frac{(s + 1) \Gamma_k(\alpha + k)}{4(b^{s+1} - a^{s+1})^{s+1}} \left[ \frac{k f_a^s F(b) + k f_b^s F(a)}{2} \right]
\]

\[
\leq \frac{f(a) + f(b)}{2}. \tag{2.2}
\]

**Proof** For \(u \in [0, 1],\) let \(\xi = au + (1 - u)b\) and \(\eta = (1 - u)a + bu.\) Using the convexity of \(f,\) we get

\[
f \left( \frac{a + b}{2} \right) = f \left( \frac{\xi + \eta}{2} \right) \leq \frac{1}{2} f(\xi) + \frac{1}{2} f(\eta).
\]

That is,

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{2} f(au + (1 - u)b) + \frac{1}{2} f((1 - u)a + bu). \tag{2.3}
\]

Now, multiplying both sides of (2.3) by

\[
\frac{(b - a)(s + 1)^{\frac{s}{s+1}} \Gamma_k(\alpha)}{k \Gamma_k(\alpha)} \frac{(ub + (1 - u)a)^s}{[b^{s+1} - (ub + (1 - u)a)^{s+1}]^{\frac{1}{s+1}}}
\]

and integrating over \((0, 1)\) with respect to \(u,\) we get

\[
(b - a)(s + 1)^{\frac{s}{s+1}} \frac{f \left( \frac{a + b}{2} \right)}{k \Gamma_k(\alpha)} \int_{0}^{1} \frac{(ub + (1 - u)a)^s}{[b^{s+1} - (ub + (1 - u)a)^{s+1}]^{\frac{1}{s+1}}} \, du
\]

\[
\leq \frac{1}{2} (b - a)(s + 1)^{\frac{s}{s+1}} \frac{f \left( \frac{a + b}{2} \right)}{k \Gamma_k(\alpha)} \int_{0}^{1} \frac{(ub + (1 - u)a)^s f(au + (1 - u)b)}{[b^{s+1} - (ub + (1 - u)a)^{s+1}]^{\frac{1}{s+1}}} \, du
\]

\[
+ \frac{1}{2} (b - a)(s + 1)^{\frac{s}{s+1}} \frac{f \left( \frac{a + b}{2} \right)}{k \Gamma_k(\alpha)} \int_{0}^{1} \frac{(ub + (1 - u)a)^s f((1 - u)a + bu)}{[b^{s+1} - (ub + (1 - u)a)^{s+1}]^{\frac{1}{s+1}}} \, du.
\]
Note that we have
\[
\int_0^1 \frac{(ub + (1-u)a)^s}{[b^{s+1} - (ub + (1-u)a)^{s+1}]^{1 - \frac{s}{k}}} \, du = \frac{k(b^{s+1} - a^{s+1})^{\frac{s}{k}}}{\alpha(s+1)(b-a)}
\]
Using the identity
\[
\tilde{f}(1-u)a + bu = f(au + (1-u)b),
\]
and from (2.1), we obtain
\[
(b-a) \frac{(s+1)^{1 - \frac{s}{k}}}{k \Gamma_k(\alpha)} \int_0^1 \frac{(ub + (1-u)a)^s f(au + (1-u)b) \, du}{[b^{s+1} - (ub + (1-u)a)^{s+1}]^{1 - \frac{s}{k}}} = \tilde{f}_a^\alpha \tilde{f}(b)
\]
and
\[
(b-a) \frac{(s+1)^{1 - \frac{s}{k}}}{k \Gamma_k(\alpha)} \int_0^1 \frac{(ub + (1-u)a)^s f((1-u)a + bu) \, du}{[b^{s+1} - (ub + (1-u)a)^{s+1}]^{1 - \frac{s}{k}}} = \tilde{f}_b^\alpha \tilde{f}(b).
\]
Accordingly, we have
\[
\frac{(b^{s+1} - a^{s+1})^{\frac{s}{k}}}{(s+1)^{\frac{s}{k}} \Gamma_k(\alpha + k)} \left( \frac{a + b}{2} \right) \leq \frac{\tilde{f}_a^\alpha F(b)}{2}.
\]
(2.4)
Similarly, multiplying both sides of (2.3) by
\[
(b-a) \frac{(s+1)^{1 - \frac{s}{k}}}{k \Gamma_k(\alpha)} \left( \frac{ub + (1-u)a)^s}{[b^{s+1} - (ub + (1-u)a)^{s+1} - a^{s+1}]^{1 - \frac{s}{k}}} \right)
\]
integrating over (0,1) with respect to \(u\), and from (2.1), we also get
\[
\frac{(b^{s+1} - a^{s+1})^{\frac{s}{k}}}{(s+1)^{\frac{s}{k}} \Gamma_k(\alpha + k)} \left( \frac{a + b}{2} \right) \leq \frac{\tilde{f}_b^\alpha F(a)}{2}.
\]
(2.5)
By adding inequalities (2.4) and (2.5), we get
\[
f \left( \frac{a + b}{2} \right) \leq \frac{(s+1)^{\frac{s}{k}} \Gamma_k(\alpha + k)}{4(b^{s+1} - a^{s+1})^{\frac{s}{k}}} \left[ \tilde{f}_a^\alpha F(b) + \tilde{f}_b^\alpha F(a) \right],
\]
which is the left-hand side of inequality (2.2).
Since \(f\) is convex, for \(u \in [0,1]\), we have
\[
f(au + (1-u)b) + f((1-u)a + bu) \leq f(a) + f(b).
\]
(2.6)
Multiplying both sides of (2.6) by
\[
(b-a) \frac{(s+1)^{1 - \frac{s}{k}}}{k \Gamma_k(\alpha)} \frac{(ub + (1-u)a)^s}{[b^{s+1} - (ub + (1-u)a)^{s+1}]^{1 - \frac{s}{k}}}
\]
and integrating over \((0,1)\) with respect to \(u\), we get

\[
(b - a) \frac{(s + 1)^{1 - \frac{\alpha}{\Gamma_s + 1}}}{k \Gamma_k(\alpha)} \int_0^1 \frac{(ub + (1 - u)a)^y (fa + (1 - u)b) \ du}{[b^{\alpha + 1} - (ub + (1 - u)a)^{\alpha + 1}]} + (b - a) \frac{(s + 1)^{1 - \frac{\alpha}{\Gamma_b(\alpha)}}}{k \Gamma_k(\alpha)} \int_0^1 \frac{(ub + (1 - u)a)^y (fa + (1 - u)b) \ du}{[b^{\alpha + 1} - (ub + (1 - u)a)^{\alpha + 1}]} \leq (b - a) \frac{(s + 1)^{1 - \frac{\alpha}{\Gamma_k(\alpha)}}}{k \Gamma_k(\alpha)} [f(a) + f(b)] \int_0^1 \frac{(ub + (1 - u)a)^y \ du}{[b^{\alpha + 1} - (ub + (1 - u)a)^{\alpha + 1}]}.
\]

That is,

\[
\int_0^1 F(a) \leq \frac{(b^{\alpha + 1} - a^{\alpha + 1})^{\frac{\alpha}{\Gamma_k(\alpha)}}}{(s + 1)^{\frac{\alpha}{\Gamma_k(\alpha)}}} (f(a) + f(b)).
\] (2.7)

Similarly, multiplying both sides of (2.6) by

\[
(b - a) \frac{(s + 1)^{1 - \frac{\alpha}{\Gamma_b(\alpha)}}}{k \Gamma_k(\alpha)} \frac{(ub + (1 - u)a)^y}{(ub + (1 - u)a)^{\alpha + 1} - a^{\alpha + 1}}
\]

and integrating over \((0,1)\) with respect to \(u\), we also get

\[
\int_0^1 F(a) \leq \frac{(b^{\alpha + 1} - a^{\alpha + 1})^{\frac{\alpha}{\Gamma_b(\alpha)}}}{(s + 1)^{\frac{\alpha}{\Gamma_b(\alpha)}}} (f(a) + f(b)).
\] (2.8)

Adding inequalities (2.7) and (2.8), we obtain

\[
\frac{(s + 1)^{\frac{\alpha}{\Gamma_k(\alpha)}}}{4(b^{\alpha + 1} - a^{\alpha + 1})^{\frac{\alpha}{\Gamma_k(\alpha)}}} \left[ \int_0^1 F(a) + \int_0^1 F(b) \right] \leq \frac{f(a) + f(b)}{2},
\]

which is the right-hand side of inequality (2.2). So the proof is complete. \(\square\)

We want to give the following function that we will use later: For \(\alpha, k > 0\) and \(s \in \mathbb{R} \setminus \{-1\}\), let \(\nabla_{\alpha,k} : [0,1] \to \mathbb{R}\) be the function defined by

\[
\nabla_{\alpha,k}(t) := \left( (ta + (1 - t)b)^{\alpha + 1} - a^{\alpha + 1} \right)^{\frac{\alpha}{\Gamma_k(\alpha)}} - \left( (tb + (1 - t)a)^{\alpha + 1} - a^{\alpha + 1} \right)^{\frac{\alpha}{\Gamma_k(\alpha)}} + (b^{\alpha + 1} - (tb + (1 - t)a)^{\alpha + 1})^{\frac{\alpha}{\Gamma_k(\alpha)}} - (b^{\alpha + 1} - (ta + (1 - t)b)^{\alpha + 1})^{\frac{\alpha}{\Gamma_k(\alpha)}}.
\]

In order to prove our main result, we need the following identity.

**Lemma 2.1** Let \(\alpha, k > 0\) and \(s \in \mathbb{R}^+\). If \(f\) is a differentiable function on \(I^\circ\) such that \(f' \in L[a,b]\) with \(a < b\), then we have the following identity:

\[
\frac{f(a) + f(b)}{2} - \frac{(s + 1)^{\frac{\alpha}{\Gamma_k(\alpha)}}}{4(b^{\alpha + 1} - a^{\alpha + 1})^{\frac{\alpha}{\Gamma_k(\alpha)}}} \left[ \int_0^1 F(a) + \int_0^1 F(b) \right] = \frac{(b - a)}{4(b^{\alpha + 1} - a^{\alpha + 1})^{\frac{\alpha}{\Gamma_k(\alpha)}}} \int_0^1 \nabla_{\alpha,k}(t)^{\alpha} (ta + (1 - t)b) \ dt.
\] (2.9)
Proof Using integration by parts, we obtain
\[
\int_a^b s^{\alpha} F(b) = \frac{(b^{\alpha+1} - a^{\alpha+1})^\alpha}{(s+1)^\alpha \Gamma_k(\alpha + k)} F(a) + \frac{(b - a)}{(s+1)^\alpha \Gamma_k(\alpha + k)} \\
\times \int_0^1 \left[ (b^{\alpha+1} - (b + (1 - u)a)^{\alpha+1}) \right]^\alpha F'(b + (1 - u)a) du.
\]
(2.10)

Similarly, we get
\[
\int_b^a s^{\alpha} F(a) = \frac{(b^{\alpha+1} - a^{\alpha+1})^\alpha}{(s+1)^\alpha \Gamma_k(\alpha + k)} F(b) - \frac{(b - a)}{(s+1)^\alpha \Gamma_k(\alpha + k)} \\
\times \int_0^1 \left[ (b + (1 - u)a)^{\alpha+1} - a^{\alpha+1} \right]^\alpha F'(b + (1 - u)a) du.
\]
(2.11)

Using the fact that \( F(x) = f(x) + \tilde{f}(x) \) and by simple computation, from equalities (2.10) and (2.11), we get
\[
\frac{4(b^{\alpha+1} - a^{\alpha+1})^\alpha}{(b - a)} \left( \frac{f(a) + f(b)}{2} - \frac{(s+1)^\alpha \Gamma_k(\alpha + k)}{4(b^{\alpha+1} - a^{\alpha+1})^\alpha} \left[ \int_a^b s^{\alpha} F(b) + \int_b^a s^{\alpha} F(a) \right] \right)
\]
\[
= \int_0^1 \left[ (b + (1 - u)a)^{\alpha+1} - (a^{\alpha+1}) \right]^\alpha F'(b + (1 - u)a) \\
\times F'(b + (1 - u)a) du.
\]
(2.12)

Note that we have
\[
F'(b + (1 - u)a) = f'(b + (1 - u)a) - f'(a + (1 - u)b), \quad u \in [0,1].
\]

Then we can easily obtain
\[
\int_0^1 \left( (b + (1 - u)a)^{\alpha+1} - a^{\alpha+1} \right)^\alpha F'(b + (1 - u)a) du \\
= \int_0^1 \left( (ta + (1 - t)b)^{\alpha+1} - a^{\alpha+1} \right)^\alpha f'(ta + (1 - t)b) dt \\
- \int_0^1 \left( (bt + (1 - t)a)^{\alpha+1} - a^{\alpha+1} \right)^\alpha f'(ta + (1 - t)b) dt
\]
(2.13)

and
\[
\int_0^1 \left( b^{\alpha+1} - (b + (1 - u)a)^{\alpha+1} \right)^\alpha F'(b + (1 - u)a) du \\
= \int_0^1 \left( b^{\alpha+1} - (ta + (1 - t)b)^{\alpha+1} \right)^\alpha f'(ta + (1 - t)b) dt \\
- \int_0^1 \left( b^{\alpha+1} - (bt + (1 - t)a)^{\alpha+1} \right)^\alpha f'(ta + (1 - t)b) dt
\]
(2.14)

Thus, the desired inequality (2.9) follows from inequalities (2.12), (2.13) and (2.14). □
For $\alpha, k > 0$, we introduce the following operator:

$$
\mathcal{S}(s, x, y) := \int_{a}^{b} \frac{x^{a} + b^{a}}{u^{a}} \left| x - u \right|^{|y^{x+1} - u^{x+1}|^{\frac{1}{x}}} du - \int_{a}^{b} \frac{x^{a} + b^{a}}{u^{a}} \left| x - u \right|^{|y^{x+1} - u^{x+1}|^{\frac{1}{x}}} du,
$$

$s \in \mathbb{R} \setminus \{-1\}, x, y \in [a, b]$.

Using Lemma 2.1, we can obtain the following $(k, s)$-fractional integral inequality.

**Theorem 2.2** Let $\alpha, k > 0$ and $s \in \mathbb{R} \setminus \{-1\}$. If $f$ is a differentiable function on $I^c$ such that $f' \in L[a, b]$ with $a < b$ and $|f'|$ is convex on $[a, b]$, then

$$
\frac{|f(a) + f(b)|}{2} - \frac{(s + 1)^{\frac{1}{s}}}{4(b^{s+1} - a^{s+1})^{\frac{1}{s}}} \left[ |f'_b^{[a]} F(b) + f'_a^{[a]} F(a)| \right] \leq \frac{\Psi(s, \alpha, a, b)}{4(b^{s+1} - a^{s+1})^{\frac{1}{s}} (b - a)} \left( |f'(a)| + |f'(b)| \right),
$$

where

$$
\Psi(s, \alpha, a, b) = \mathcal{S}(s, b, b) + \mathcal{S}(s, a, b) - \mathcal{S}(s, b, a) - \mathcal{S}(s, a, a).
$$

**Proof** Using Lemma 2.1 and the convexity of $|f'|$, we obtain

$$
\frac{|f(a) + f(b)|}{2} - \frac{(s + 1)^{\frac{1}{s}}}{4(b^{s+1} - a^{s+1})^{\frac{1}{s}}} \left[ |f'_b^{[a]} F(b) + f'_a^{[a]} F(a)| \right] \leq \frac{(b - a)}{4(b^{s+1} - a^{s+1})^{\frac{1}{s}}} \int_{0}^{1} \left| \nabla_{(a)} (t) \right| \left| f'(ta + (1 - t)b) \right| dt
$$

$$
\leq \frac{(b - a)}{4(b^{s+1} - a^{s+1})^{\frac{1}{s}}} \left( |f'(a)| \int_{0}^{1} t \left| \nabla_{(a)} (t) \right| dt + |f'(b)| \int_{0}^{1} (1 - t) \left| \nabla_{(a)} (t) \right| dt \right).
$$

Note that

$$
\int_{0}^{1} t \left| \nabla_{(a)} (t) \right| dt = \frac{1}{(b - a)^{2}} \int_{a}^{b} \varphi(u) \left| (b - u) \right| du,
$$

where

$$
\varphi(u) = \left( u^{s+1} - a^{s+1} \right)^{\frac{1}{s}} - \left( b + a - u \right)^{s+1} - a^{s+1}\right)^{\frac{1}{s}}
$$

$$
+ \left( b^{s+1} - (b + a - u)^{s+1} \right)^{\frac{1}{s}} - \left( b^{s+1} - u^{s+1} \right)^{\frac{1}{s}}, \quad u \in [a, b].
$$

Observe that $\varphi$ is a non-decreasing function on $[a, b]$. Moreover, we have $\varphi(a) = -2(b^{s+1} - a^{s+1})^{\frac{1}{s}} < 0$ and $\varphi(a + \frac{b - a}{2}) = 0$. Thus, we have

$$
\left\{ \begin{array}{ll}
\varphi(u) \leq 0 & \text{if } a \leq u \leq \frac{a + b}{2}, \\
\varphi(u) > 0 & \text{if } \frac{a + b}{2} < u \leq b.
\end{array} \right.
$$
So, we obtain
\[(b - a)^2 \int_0^1 t |\nabla_{a,t}(t)| \, dt = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4,
\]
where
\[
\zeta_1 = \int_a^b (b - u)(b^{r+1} - u^{r+1})^{\frac{1}{r}} \, du - \int_{a+b}^b (b - u)(b^{r+1} - u^{r+1})^{\frac{1}{r}} \, du,
\]
\[
\zeta_2 = -\int_a^b (b - u)(u^{r+1} - a^{r+1})^{\frac{1}{r}} \, du + \int_{a+b}^b (b - u)(u^{r+1} - a^{r+1})^{\frac{1}{r}} \, du,
\]
\[
\zeta_3 = \int_a^b (b - u)((b + a - u)^{r+1} - a^{r+1})^{\frac{1}{r}} \, du - \int_{a+b}^b (b - u)((b + a - u)^{r+1} - a^{r+1})^{\frac{1}{r}} \, du,
\]
\[
\zeta_4 = -\int_a^b (b - u)(b^{r+1} - (b + a - u)^{r+1})^{\frac{1}{r}} \, du + \int_{a+b}^b (b - u)(b^{r+1} - (b + a - u)^{r+1})^{\frac{1}{r}} \, du.
\]

Observe that \(\zeta_1 = \mathcal{I}(s, b, b)\) and \(\zeta_2 = -\mathcal{I}(s, b, a)\). Using the change of variable \(v = a + b - u\), we get \(\zeta_3 = -\mathcal{I}(s, a, a)\) and \(\zeta_4 = \mathcal{I}(s, a, b)\). Thus, we obtain
\[
\int_0^1 t |\nabla_{a,t}(t)| \, dt = \frac{\mathcal{I}(s, b, b) + \mathcal{I}(s, a, b) - \mathcal{I}(s, b, a) - \mathcal{I}(s, a, a)}{(b - a)^2}. \tag{2.17}
\]

Similarly,
\[
\int_0^1 (1 - t) |\nabla_{a,t}(t)| \, dt = \frac{\mathcal{I}(s, b, b) + \mathcal{I}(s, a, b) - \mathcal{I}(s, b, a) - \mathcal{I}(s, a, a)}{(b - a)^2}. \tag{2.18}
\]

So, the desired inequality (2.15) follows from inequalities (2.16), (2.17) and (2.18). \(\square\)

### 3 Conclusions

Lastly, we conclude this paper by remarking that we have obtained a Hermite-Hadamard inequality, an identity and a Hermite-Hadamard type inequality for a generalized \(k\)-fractional integral operator. Therefore, by suitably choosing the parameters, one can further easily obtain additional integral inequalities involving the various types of fractional integral operators from our main results.

#### Competing interests
The authors declare that they have no competing interests.

#### Authors’ contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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