\textbf{Z}_3\text{-graded Symmetries in Quantum Mechanics}

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\textbf{Abstract}

In this paper we consider \(\text{Z}_3\)-graded topological symmetries (TSs) \cite{2,3} in one dimensional quantum mechanics. We give a classification of one dimensional quantum systems possessing these symmetries and show that different classes correspond to a positive integer \(N\).

\section{Introduction}

The notion of \(\text{Z}_3\)-grading and \(\text{Z}_3\)-graded structures have been widely studied in recent years \cite{1}. In most of these studies, the \(\text{Z}_3\)-graded structure is a generalization of a \(\text{Z}_2\)-graded structure from a specific point of view, for example geometrical, group theoretical and algebraical.

In a recent series of papers \cite{2,3} we have studied some \(\text{Z}_n\)-graded structures called topological symmetries in quantum mechanics and explored the algebra of quantum systems possessing certain topological symmetries. Topological symmetries are generalizations of supersymmetry from a topological point of view, i.e. they share the topological properties of supersymmetry.

A quantum system is said to possess a \(\text{Z}_n\)-graded (uniform) topological symmetry (UTS) of type \((m_1, m_2, \cdots, m_n)\) iff the following conditions are satisfied.

1. The quantum system is \(\text{Z}_n\)-graded. This means that the Hilbert space \(\mathcal{H}\) of the quantum system is the direct sum of \(n\) of its (nontrivial) subspaces \(\mathcal{H}_\ell\), and its Hamiltonian has a complete set of eigenvectors with definite color or grading. (A state is said to have a definite color \(c_\ell\) iff it belongs to \(\mathcal{H}_\ell\));

2. The energy spectrum is nonnegative;

3. Every positive energy eigenvalue \(E\) is \(m\)-fold degenerate, and the corresponding eigenspaces are spanned by \(m_1\) vectors of color \(c_1\), \(m_2\) vectors of color \(c_2\), \cdots, and \(m_n\) vectors of color \(c_n\).  

For a system with these properties we can introduce a set of integer-valued topological invariant, namely

\begin{equation}
\Delta_{ij} := m_i n_j^{(0)} - m_j n_i^{(0)},
\end{equation}

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where \( i, j = 1, \ldots, n \) and \( n^{(0)}_\ell \) denotes the number of zero-energy states of color \( c_\ell \). Note that the TS of type \((1,1)\) coincides with supersymmetry and \( \Delta_{11} \) yields the Witten index.

Various aspects of topological symmetries are discussed and the relationship between topological symmetries and parasupersymmetry, orthosupersymmetry and fractional supersymmetry is investigated in some recent articles \([3, 7, 8]\). Also in Ref. \([3, 4]\) some examples of quantum systems possessing topological symmetries are given, but there is no classification of all quantum systems possessing topological symmetry until now.

In this article we try to find the most general form of one dimensional quantum systems with \( \mathbb{Z}_3 \)-graded topological symmetries of type \((1,1,1)\) and give a classification scheme for such systems.

The organization of the paper is as follows. In section 2 we review the algebra of \( \mathbb{Z}_3 \)-graded topological symmetries of type \((1,1,1)\) and find some general conditions on the operator in the algebra. In section 3 we find the general solutions of algebraic relations in one dimensional quantum systems. Section 4 is devoted to conclusion and remarks.

### 2 \( \mathbb{Z}_3 \)-graded topological symmetries

The algebra of a \( \mathbb{Z}_3 \)-graded topological symmetry of type \((1,1,1)\) is given by the following equations

\[
\begin{align*}
[Q, H] &= 0, \\
Q^3 &= K, \\
Q_1^3 + MQ_1 &= \frac{1}{2\sqrt{2}}(K + K^\dagger), \\
Q_2^3 + MQ_2 &= -\frac{i}{2\sqrt{2}}(K - K^\dagger), \\
[\tau, Q]_q &= 0,
\end{align*}
\]

where \( H \) is the Hamiltonian of the system, \( Q \) is the symmetry generator, \( K \) and \( M \) are operators which commute with any other operator in the algebra (furthermore \( M \) is hermitian), \( \tau \) is the grading operator with the properties \( \tau^3 = 1, \tau^\dagger = \tau^{-1} \), and \([.,.]_q \) stands for \( q \)-commutator, i.e. \( [O_1, O_2]_q = O_1O_2 - qO_2O_1 \) with \( q = e^{2\pi i/3} \). The operators \( Q_1 \) and \( Q_2 \) are related to \( Q \) and \( Q^\dagger \) by the following equations

\[
Q_1 := \frac{1}{\sqrt{2}} (Q + Q^\dagger) \quad \text{and} \quad Q_2 := \frac{-i}{\sqrt{2}} (Q - Q^\dagger).
\]

In view of properties of grading operator of a topological symmetry the Hilbert space may be expressed as direct sum of three of its (nontrivial) subspaces \( \mathcal{H}_1, \mathcal{H}_2 \) and \( \mathcal{H}_3 \). Each \( \mathcal{H}_i \) is characterized by one of eigenvalues of \( \tau \). Then one can take the following matrix forms for \( Q, M \) and \( H \).

\[
Q = \begin{pmatrix} 0 & 0 & D_3 \\ D_1 & 0 & 0 \\ 0 & D_2 & 0 \end{pmatrix}, \quad M = \frac{1}{2} \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix},
\]

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where $D_i : \mathcal{H}_i \rightarrow \mathcal{H}_{i+1}$ and $M_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$ are some operators which we are going to determine their explicit forms ($M_i$’s are hermitian).

First of all we note that Eq. (5) is not an independent equation. This means that using the matrix forms given by Eq. (8) one can see that if Eq. (4) is satisfied then Eq. (5) is fulfilled trivially. Using the matrix forms of $Q$ and $M$ given by Eq. (8) and the fact that $Q$ commutes with $H$ and $M$ one gets the following conditions on $D_i$’s, $H_i$’s and $M_i$’s

$$H_{i+1}D_i = D_iH_i, \quad i = 1, 2, 3 \quad (9)$$

$$M_{i+1}D_i = D_iM_i, \quad i = 1, 2, 3 \quad (10)$$

Here and throughout the paper the summation and subtraction in subscripts of $D_i$’s, $H_i$’s and $M_i$’s is summation and subtraction modulo 3 respectively; e. g. we identify $D_4$ with $D_1$ and $D_0$ with $D_3$. In the same manner Eq. (4) results in

$$D_iM_i + D_i(D_i^\dagger D_i + D_{i-1}D_{i-1}^\dagger) + D_{i+1}^\dagger D_{i+1}D_i = 0, \quad i = 1, 2, 3 \quad (11)$$

If we multiply Eq. (11) by $D_i^\dagger$ from the right we get

$$D_iM_iD_i^\dagger + D_i(D_i^\dagger D_i + D_{i-1}D_{i-1}^\dagger)D_i^\dagger + D_{i+1}^\dagger D_{i+1}D_i D_i^\dagger = 0, \quad i = 1, 2, 3 \quad (12)$$

The first two terms of Eq. (12) are hermitian so the last term should be hermitian as well, i. e.

$$D_{i+1}^\dagger D_{i+1}D_i D_i^\dagger = D_iD_i^\dagger D_{i+1} D_{i+1}^\dagger, \quad (13)$$

or equivalently

$$[D_{i+1}^\dagger D_{i+1}, D_i D_i^\dagger] = 0, \quad i = 1, 2, 3 \quad (14)$$

In the same way we can get the following relations from Eq. (10)

$$[M_{i+1}, D_i D_i^\dagger] = [M_i, D_i^\dagger D_i] = 0, \quad i = 1, 2, 3 \quad (15)$$

So there are four mutually commuting operators, in $\mathcal{H}_i$, $i = 1, 2, 3$ namely $H_i$, $M_i$, $D_{i-1}^\dagger D_{i-1}$, $D_i^\dagger D_i$. This fact will help us to find the relation of these operators in next section.

### 3 One dimensional $\mathbb{Z}_3$-graded TSs

By “one dimensional” we mean that Hilbert spaces $\mathcal{H}_i$, $i = 1, 2, 3$ are $L^2(\mathbb{R})$. We assume that all $D_i$’s are nonzero, because if one of $D_i$’s is zero identically then the algebra given by Eqs. (2)-(6) reduces to the algebra of ($p = 2$) parasupersymmetry [10] and it is well known that a ($p = 2$) parasypersymmetric quantum system is $\mathbb{Z}_2$-graded not $\mathbb{Z}_3$-graded [9]. With the above assumptions we are going to find the solutions of Eqs. (9) – (11). To this end we use the following lemmas:
Lemma 1: Let $M$ and $N$ be differential operators of order $m$ and $n$ respectively acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$,

\begin{align*}
M &= M_m(x) \frac{d^m}{dx^m} + \cdots + M_1(x) \frac{d}{dx} + M_0(x) , \\
N &= N_n(x) \frac{d^n}{dx^n} + \cdots + N_1(x) \frac{d}{dx} + N_0(x) ,
\end{align*}

where $M_i(x)$ and $N_i(x)$ are complex valued functions of real variable $x$. Then $[M, N] = 0$ iff an operator $D$ exists such that

\begin{align*}
M &= a_r D^r + \cdots + a_1 D + a_0 , \\
N &= b_s D^s + \cdots + b_1 D + b_0 ,
\end{align*}

in which $m = rd$ and $n = sd$ and $d$ is a common divisor of $m$ and $n$ and $a_i$'s and $b_i$'s are some complex parameters.

Proof: According to the theory of integrability in Quantum Mechanics [5, 6] we know that for a quantum Mechanical system described in $N$-dimensional Euclidean space there are at most $N$ algebraically independent linear operators commuting among each other (in such a situation we call the system integrable). In our system the Hilbert space is $L^2(\mathbb{R})$ and this means that the set of linear algebraically independent operators commuting among each other includes only one element. Therefore if there are two commuting operators in this system they are not linearly independent. In other words there must exist a linear operator $D$ such that Eqs. (18) and (19) hold. □

Lemma 2: Let $M$ and $N$ be differential operators of order $m$ and $n$ respectively acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$,

\begin{align*}
M &= M_m(x) \frac{d^m}{dx^m} + \cdots + M_1(x) \frac{d}{dx} + M_0(x) , \\
N &= N_n(x) \frac{d^n}{dx^n} + \cdots + N_1(x) \frac{d}{dx} + N_0(x) ,
\end{align*}

where $M_i(x)$ and $N_i(x)$ are complex valued functions of real variable $x$. If $MN = 0$, then $M = 0$ or $N = 0$.

Proof: Suppose that $N$ is a non zero differential operator of order $n$ given by Eq. (21) (by this assumption we mean that $N_n(x)$ is a non zero function of $x$). By direct calculating of $MN$ from Eqs. (20) and (21) it is easily seen that the highest order term is $M_m(x)N_n(x) \frac{d^{m+n}}{dx^{m+n}}$. As $N_n(x) \neq 0$ and $MN = 0$ we conclude that $M_m(x) = 0$. By repeating this procedure we get $M = 0$. □

Lemma 3: Let $H$ be a second order differential operator of the form

\[ H = -\frac{d^2}{dx^2} + V(x) , \]

\[ (22) \]
where \( V(x) \) is a real valued (nonconstant) function of \( x \). Then there is no first order differential operator \( D := f(x) \frac{d}{dx} + g(x) \) such that

\[
H = aD^2 + bD + c, \quad (23)
\]

Here \( f(x) \) and \( g(x) \) are real valued functions of \( x \) and \( a, b, \) and \( c \) are real constants.

**Proof:** Substituting \( D = f(x) \frac{d}{dx} + g(x) \) and \( H = -\frac{d^2}{dx^2} + V(x) \) in Eq. (23) we arrive at the following equations

\[
a f^2(x) + 1 = 0, \quad (24)
\]
\[
a (f(x)f'(x) + 2f(x)g(x)) + bf(x) = 0, \quad (25)
\]
\[
a (f(x)g'(x) + g^2(x)) + bg(x) + c - V(x) = 0. \quad (26)
\]

Eqs. (24) and (25) imply that \( f(x) \) and \( g(x) \) are constants and therefore can not satisfy Eq. (26) unless \( V(x) \) is constant. □

Now we are ready to find the solutions of Eqs. (9) – (11). In previous section we found out that the operators \( H_i, M_i, D_{i-1}D_{i-1}^\dagger \) and \( D_i^\dagger D_i \) are mutually commuting in Hilbert space \( \mathcal{H}_i = L^2(\mathbb{R}) \), \( i = 1, 2, 3 \). At this stage we put an important restriction on the form of Hamiltonian. We take the following form for \( H_i \)'s

\[
H_i = -\frac{1}{2m_i} \frac{d^2}{dx^2} + V_i(x), \quad i = 1, 2, 3 \quad (27)
\]

where \( V_i(x) \) is real valued well defined potential such that the spectrum of \( H_i \) is non negative. Although by this assumption we may lose some of mathematical solutions of Eqs. (9) – (11), but it seems that we will not lose any physical solution.

If we confine ourselves to \( H_i \) given by Eq. (27), then according to Lemma 1 and Lemma 3 \( M_i, D_{i-1}D_{i-1}^\dagger \) and \( D_i^\dagger D_i \) should have the following forms

\[
D_i^\dagger D_i = \sum_{n=0}^{N} a_n^{(i)} H_i^n, \quad (28)
\]
\[
D_{i-1}D_{i-1}^\dagger = \sum_{n=0}^{N} b_n^{(i)} H_i^n, \quad (29)
\]
\[
M_i = \sum_{n=0}^{N} c_n^{(i)} H_i^n, \quad (30)
\]

where \( a_n^{(i)}, b_n^{(i)} \) and \( c_n^{(i)} \) are real constants. It should be noted that the above equations does not mean that \( D_i^\dagger D_i, D_{i-1}D_{i-1}^\dagger \) and \( M_i \) are of the same order because \( a_n^{(i)}, b_n^{(i)} \) and \( c_n^{(i)} \) can be zero for some values of \( n \). Therefore \( N \) is the greatest \( n \) for which at least one of \( a_n^{(i)}, b_n^{(i)} \) and \( c_n^{(i)} \) is nonzero. In fact as \( H_i \)'s are second order differential operators we have \( N = \text{Max}\{O(D_i), \frac{1}{2}O(M_i), i = 1, 2, 3\} \) where \( O(.) \) stands for the order of the operator. Substituting Eq. (30) in Eq. (10) and using Eq. (9) one gets \( c_n^{(i+1)} = c_n^{(i)}, i = 1, 2, 3 \). This means that \( c_n^{(1)} = c_n^{(2)} = c_n^{(3)} =: c_n \).
Next we multiply both sides of Eq. (28) by \(D_i\) from the left and use Eq. (9) and Lemma 2 and the fact that none of \(D_i\)s are zero identically to obtain

\[
D_iD_i^\dagger = \sum_{n=0}^{N} a_n^{(i)} H_{i+1}^n .
\]

(31)

Comparing this equation with Eq. (29) one finds that \(b_n^{(i+1)} = a_n^{(i)} , i = 1, 2, 3\).

Finally using Eq. (28) – (30) in Eq. (11) one gets \(c_n = -\sum_{i=1}^{N} a_n^{(i)}\). Therefore we can rewrite Eq. (28) – (30) as

\[
D_iD_i^\dagger = \sum_{n=0}^{N} a_n^{(i)} H_{i+1}^n ,
\]

(32)

\[
D_i^\dagger D_i = \sum_{n=0}^{N} a_n^{(i)} H_i^n ,
\]

(33)

\[
M_i = -\sum_{n=0}^{N} \left( \sum_{j=1}^{3} a_n^{(j)} \right) H_i^n ,
\]

(34)

These equations show that the order of \(M_i\), \(i = 1, 2, 3\) can not be greater than the greatest order of \(D_iD_i\) or \(D_i^\dagger D_i\), \(i = 1, 2, 3\) . Furthermore Eqs. (32) and (33) imply that \(m_1 = m_2 = m_3 =: m\). This arises from the following fact. Suppose that we write \(D_i\) as \(D_i = \sum_{i=1}^{N} d_i(x) \frac{d}{dx}\) and calculate \(D_i^\dagger D_i\) and \(D_i D_i^\dagger\). Obviously coefficients of the greatest powers of \(\frac{d}{dx}\) in \(D_i^\dagger D_i\) and \(D_i D_i^\dagger\) are the same. This means that in the left hand side of Eqs. (32) and (33) the coefficients of the greatest powers of \(\frac{d}{dx}\) are the same. Therefore this is also true for the right hand side of these equations. This, in view of Eq. (27), means that \(m_1 = m_2 = m_3 =: m\).

From the above calculations we conclude that the solutions are classified by the value of \(N\). Now we analyze the solutions for various values of \(N\). \(N = 0\) gives a trivial solution in which \(D_i\)s and \(M_i\)s are constants.

- **Class \(N = 1\):**

  The first nontrivial solution is given by \(N = 1\). For \(N = 1\) we have

\[
D_iD_i = a_0^{(i)} + a_1^{(i)} H_i ,
\]

(35)

\[
D_{i-1}D_i^\dagger = a_0^{(i-1)} + a_1^{(i-1)} H_i ,
\]

(36)

\[
M_i = \left( -\sum_{j=1}^{3} a_0^{(j)} \right) - \left( \sum_{j=1}^{3} a_1^{(j)} \right) H_i ,
\]

(37)

As \(H_i\)’s are second order differential operators \(D_i\)’s must be at most first order differential operators. So we take the following form for \(D_i\)’s

\[
D_i = f_i(x) \frac{d}{dx} + g_i(x) ,
\]

(38)

where \(f_i(x)\) and \(g_i(x)\) are real valued functions of \(x\) (one can also adopt complex functions of \(x\), but without loss of generality we take \(f_i(x)\) and \(g_i(x)\) to be real functions of \(x\) reduce the amount of
where $D = 1$ this case the general solution is the following

$$f_i'(x) = 0, \quad (39)$$

$$-\frac{1}{2m}(f_i''(x) + 2g_i(x)) + (V_{i+1}(x) - V_i(x))f_i(x) = 0, \quad (40)$$

$$-\frac{1}{2m}g_i''(x) + (V_{i+1}(x) - V_i(x))g_i(x) - f_i(x)V_i'(x) = 0, \quad (41)$$

Eq. (39) implies $f_i = const., i = 1, 2, 3$. Thus Eq. (40) takes the following form

$$-\frac{1}{m}g_i'(x) + (V_{i+1}(x) - V_i(x))f_i = 0, \quad (42)$$

Using this equation in Eq. (41) yields

$$(V_{i+1}(x) - V_i(x))g_i(x) = \frac{1}{2}[(V'_{i+1}(x) + V'_i(x))f_i, \quad (43)$$

Now we come back to Eqs. (35) – (37). Substituting $D_i = f_i(x)\frac{d^2}{dx^2} + g_i(x)$ and $H_i = -\frac{1}{2m}\frac{d^2}{dx^2} + V_i(x)$ in these equations and using the fact that $f_i(x) = f_i = const.$ one arrives at the following equations

$$f_i^2 = \frac{a_i^{(i)}}{2m}, \quad (44)$$

$$a_0^{(i)} + a_1^{(i)}V_i(x) = g_i^2(x) - f_i g_i'(x), \quad (45)$$

$$a_0^{(i)} + a_1^{(i)}V_{i+1}(x) = g_i^2(x) + f_i g_i'(x), \quad (46)$$

It is easily verified that in fact Eqs. (12) and (13) can be derived from Eqs. (44) – (46). Therefore in this case the general solution is the following

$$Q = \begin{pmatrix} 0 & 0 & D_3 \\ D_1 & 0 & 0 \\ 0 & D_2 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}, \quad (47)$$

where $D_i = f_i\frac{d^2}{dx^2} + g_i(x), H_i = -\frac{1}{2m}\frac{d^2}{dx^2} + V_i(x)$ and $f_i$s are real constants. Also $g_i(x), V_i(x),$ and $f_i, i = 1, 2, 3$ should satisfy Eqs. (44) – (46). A simple example is given in the following

$$Q = \begin{pmatrix} 0 & 0 & D \\ g & 0 & 0 \\ 0 & D^\dagger & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}, \quad (48)$$

where $D = f\left(\frac{d}{dx} - \alpha x - \beta\right), H_1 = H_2 = \frac{1}{2mf^2}DD^\dagger - a_0,$ and $H_3 = \frac{1}{2mf^2}D^\dagger D - a_0.$ The parameters $\alpha, \beta, a_0, f$ and $g$ are real constants. Furthermore $M$ and $K$ are given by the following equations

$$M = -\frac{1}{2}g^2 - 2mf^2(H + a_0), \quad (49)$$

$$K = 2mf^2g(H + a_0). \quad (50)$$
This completes the study of $\mathbb{Z}_3$-graded symmetries corresponding to $N = 1$. For $N > 1$, solutions are more complicated, but with some special choices for the operators one can find the solutions easily.

Now we wish to comment on a class of solutions which correspond to a special choice of operators. In Ref. [7] a solution is given which corresponds to the choice $D_3 = (D_2 D_1)^\dagger$. One can easily verify that this choice satisfies Eqs. (57) – (60) and $M_2$ is explicitly given by

$$M_2 = D_2^T D_2 + D_1 D_1^\dagger + D_1 D_1^T D_2 D_2.$$

(51)

We use the above choice to find a solution of class $N = 2$.

• Class $N = 2$:

One can easily verify that for $D_3 = (D_2 D_1)^\dagger$ the supercharge $Q$ and the Hamiltonian $H$ which satisfy the algebra of $\mathbb{Z}_3$-graded topological symmetries of type $(1, 1, 1)$ are given by

$$Q = \begin{pmatrix} 0 & 0 & (D_2 D_1)^\dagger \\ D_1 & 0 & 0 \\ 0 & D_2 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix},$$

(52)

with

$$D_1 = f_1 \frac{d}{dx} + g_1(x), \quad D_2 = f_2 \frac{d}{dx} + g_2(x)$$

(53)

$$H_1 = \frac{1}{2m f_1^2} (D_1^T D_1 - a_0) = -\frac{1}{2m} \frac{d^2}{dx^2} + V_1(x),$$

(54)

$$H_2 = \frac{1}{2m f_1^2} (D_1 D_1^\dagger - a_0) = \frac{1}{2m f_2^2} D_2 D_2^\dagger = -\frac{1}{2m} \frac{d^2}{dx^2} + V_2(x),$$

(55)

$$H_3 = \frac{1}{2m f_2^2} D_3 D_3^\dagger = -\frac{1}{2m} \frac{d^2}{dx^2} + V_3(x);$$

(56)

provided that the following conditions are fulfilled

$$g_1^2(x) - f_1 g_1'(x) = 2m f_1^2 V_1(x) + a_0,$$

(57)

$$g_2^2(x) + f_1 g_1'(x) = 2m f_1^2 V_2(x) + a_0,$$

(58)

$$g_2^2(x) - f_2 g_2'(x) = 2m f_2^2 V_3(x),$$

(59)

$$g_2^2(x) + f_2 g_1'(x) = 2m f_2^2 V_3(x).$$

(60)

Here $f_1$, $f_2$ and $a_0$ are real constants. The operators $M$ and $K$ are given in terms of the Hamiltonian through the following equations

$$M = -\frac{1}{2} a_0 - m(f_1^2 + f_1^2 (1 + a_0)) H - 2m^2 f_1^2 f_2^2 H^2,$$

(61)

$$K = 2m f_2^2 H (2m f_2^2 H + a_0).$$

(62)

It is remarkable that in this class the solution is not unique because Eqs. (57) – (60) may have various solutions for $g_1(x)$ and $g_2(x)$. In fact $V_1(x)$ and $V_2(x)$ are superpartner potentials and so are $V_2(x)$ and $V_3(x)$. This result is more or less similar to the result of some attempts to find an application for $p = 2$ parasupersymmetry in the construction of shape invariant potentials [11].
4 Concluding Remarks

In this paper we tried to solve the algebraic equations of \( \mathbb{Z}_3 \)-graded topological symmetries of type (1,1,1). Working in the framework of one dimensional quantum mechanics and taking a special form for the Hamiltonian of the system (Eq. (27)) we obtained a classification for quantum mechanical systems possessing these types of symmetries which are defined in Refs. [2, 3]. It is shown that the classification is given by a positive integer, \( N \). For \( N = 1 \) and \( N = 2 \) the solutions obtained explicitly.

One may apply the method of this article to two dimensional Quantum mechanical systems as well. In this case the Hilbert spaces \( \mathcal{H}_i \)s are \( L^2(\mathbb{R}^2) \). This implies that there are two linear algebraically independent commuting operators in \( \mathcal{H}_i \) which one of them is Hamiltonian. Classification of solutions for two dimensional Quantum systems is the subject of further investigation.

Acknowledgment

We wish to thank A. Mostafazadeh and F. Loran for reading the first draft of the paper and giving valuable comments. Financial supports of Isfahan University of Technology is acknowledged.

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