Mathematical modeling of Avian Influenza epidemic with bird vaccination in constant population

M Kharis* and Amidi

1 Department of Mathematics, Universitas Negeri Semarang, Indonesia

*Corresponding author: kharis.mat@mail.unnes.ac.id

Abstract. The development of the industrial world and human life is increasingly modern and less attention to environmental sustainability causes the virus causes the epidemic has a high tendency to mutate so that the virus that initially only attack animals, is also found to have the ability to attack humans. The epidemics that lasted some time were bird flu epidemics and swine flu epidemics. The flu epidemic led to several deaths and many people admitted to the hospital. Strain (derivatives) of H5N1 virus was identified as the cause of the bird flu epidemic while the H1N1 strain of the virus was identified as the cause of the swine flu epidemic. The symptoms are similar to seasonal flu caused by H3N2 strain of the virus. Outbreaks of bird flu and swine flu initially only attacked animals, but over time some people were found to be infected with the virus.

1. Introduction

Perdue [1], Sedyaningsih [2], and Scoones & Forster [3] suggested that bird flu virus can be transmitted to humans and can cause death so that an outbreak occurs. In Yang et al [4] published in 2009, bird flu epidemic then erupted swine flu epidemic. The flu epidemic led to several deaths and many people admitted to the hospital. Strain (derived) H5N1 virus preserved as the cause of the epidemic of bird flu virus H1N1 virus as a cause of swine flu epidemic. Symptoms caused by seasonal flu are caused by H3N2 strain of the virus. Jansen et al [5] mentioned influenza viruses responsible for the number of deaths and people who are sick in the hospital. In Widiasih [6] the sense of presence in poultry infected with this virus with a very large number of economic terms very.

de Jong et al [7] mentioned that the influenza A subtype H5N1 virus with substitution of an amino acid in neuramidase isolated from 2 patients undergoing therapy/treatment and known the virus is immune to a given drug. Both of these patients died because of this viral infection. Gooskens [8] also mentioned that there is a mutation of influenza A virus that produces a virus that is immune to oseltamivir. The mutated virus is contagious pathogenic and lethal for high-risk patients. The ability of the H5N1 virus to mutate is so high that it is necessary to watch out for the spread of this virus in the poultry population so that some precautions have been taken such as the destruction of infected poultry and quarantine for infected humans.

2. Methods

The first step to do this research was literacy study. In this step, we study the fact and some assumptions from various scientific literacies. After that, we complete the facts with some assumptions to build the model. The second step was building and analyzing mathematics model. In this step, we build the mathematics model and then analyze it to determine the equilibrium points and
their stability. The third step was making simulation with parameters value which was gotten from other paper.

3. Mathematical Model

From literature review we got: in Tuncer & Martcheva [9], it were stated that the avian influenza virus of subtype H5N1 can infect humans and cause death in human and bird population. In Tuncer & Martcheva [9] and Bourouiba [10], there was stated that vaccination in poultry are still being implemented. In Bourouiba [10], it was stated that vaccinated poultry which is even free of clinical signs, should not be traded to avoid all risk of silent shedding and transmission. Vemula et al [11] used several different approaches that are currently available for diagnosis of influenza infections in humans. These are used to diagnosis of influenza virus infections following natural infection and vaccination in humans.

In this paper, we assume that the population is constant so the death rate of infected human and infected bird were assumed have same value with natural death rate in every population. We also assumed that death in infected humans and infected birds only occurs due to viral infection and the probability of infectious contacts of bird-human and human-human are same. Transfer diagram of AI epidemic is given at Figure 1.

![Transfer diagram of AI epidemic](image)

**Figure 1.** Transfer diagram of AI epidemic with vaccination on susceptible bird

| Parameter | The Meaning |
|-----------|-------------|
| \( \mu \) | Birth rate in humans is assumed same with death rate |
| \( \mu_b \) | Birth rate in birds is assumed same with death rate |
| \( \beta \) | The probability of infectious contact was happen in humans |
| \( \beta_b \) | The probability of infectious contact was happen in birds |
| \( m \) | Death rate of infected human (assumed equal to \( \mu \)) |
| \( m_b \) | Death rate of infected bird (assumed equal to \( \mu_b \)) |
| \( \gamma \) | Recovery rate of infected human |
| \( \theta \) | Immunity loss rate |
| \( \delta \) | The proportion of susceptible bird to be vaccinated |
where \( N \) is the total human population, \( S \) is total number of susceptible person, \( I \) is total number of infected person, \( R \) is total number of recovered person, \( N_b \) is the total bird population, \( S_b \) is total number of susceptible bird, \( I_b \) is total number of infected bird, and \( V_b \) is total number of vaccinated bird. The meaning of parameter in model were given in Table 1.

From Fig. 1 we construct te system of ordinary differential equation as System (1).

\[
\begin{align*}
\frac{dS}{dt} &= \mu N + \theta R - S \left( \beta \frac{I_b}{N} + 1 \right) - (\mu + \gamma)I \\
\frac{dI}{dt} &= \beta \frac{S}{N} (I_b + 1) - (\mu + \gamma)I \\
\frac{dR}{dt} &= \gamma I - (\theta + \mu)R \\
\frac{dS_b}{dt} &= \mu_b N_b - (\beta_b \frac{I_b}{N_b} + \delta + \mu_b)S_b \\
\frac{dI_b}{dt} &= \beta_b \frac{S_b}{N_b} I_b - m_b I_b \\
\frac{dV_b}{dt} &= \delta S_b - \mu_b V_b \\
S + I + R &= N \\
S_b + I_b + V_b &= N_b
\end{align*}
\]

(1)

We assumed that \( m = \mu \) and \( m_b = \mu_b \) then we get System (2).

\[
\begin{align*}
\frac{dS}{dt} &= \mu N + \theta R - S \left( \beta \frac{I_b}{N} + 1 \right) - (\mu + \gamma)I \\
\frac{dI}{dt} &= \beta \frac{S}{N} (I_b + 1) - (\mu + \gamma)I \\
\frac{dS_b}{dt} &= \mu_b N_b - (\beta_b \frac{I_b}{N_b} + \delta + \mu_b)S_b \\
\frac{dI_b}{dt} &= \beta_b \frac{S_b}{N_b} I_b - m_b I_b \\
\frac{dV_b}{dt} &= \delta S_b - \mu_b V_b \\
S + I + R &= N \\
S_b + I_b + V_b &= N_b
\end{align*}
\]

(2)

Clear that \( \frac{dN}{dt} = 0 \iff N = K > 0, k \in R \) and \( \frac{dN_b}{dt} = 0 \iff N = L > 0, L \in R \).

Hence, we get System (3)

\[
\begin{align*}
\frac{dS}{dt} &= (\mu + \theta)K - \theta I - S \left( \beta \frac{I_b}{K} + 1 \right) - (\mu + \gamma)I \\
\frac{dI}{dt} &= \beta \frac{S}{K} (I_b + 1) - (\mu + \gamma)I \\
\frac{dS_b}{dt} &= \mu_b L - (\beta_b \frac{I_b}{L} + \delta + \mu_b)S_b \\
\frac{dI_b}{dt} &= \beta_b \frac{S_b}{L} I_b - m_b I_b \\
S + I + R &= N \\
S_b + I_b + V_b &= N_b
\end{align*}
\]

(3)

Domain of System (3) is defined

\[ \Gamma = \{ P \in R^4 | P = (S,I,S_b,I_b) \text{ where } 0 \leq S,I < K \text{ and } 0 \leq S_b,I_b < L \} \]

The existence of equilibrium points of System (3) is given in Theorem 1.
Theorem 1.

Let $r_0 = \frac{\beta_b}{\mu_b + \delta}$ and $R_0 = \frac{\beta}{\mu + \gamma}$.

If $r_0 < 1$ and $R_0 < 1$ then System (3) has only one equilibrium point i.e. non endemic equilibrium point $P_0 = (S, I, S_b, I_b) = P = \left( K, 0, \frac{\mu_b L}{\delta + \mu_b}, 0 \right)$.

1. If $r_0 < 1$ and $R_0 > 11$ then System (3) has two equilibrium i.e $P_0$ and $P_1 = (S, I, S_b, I_b) = \left( \frac{K(\mu + \gamma)}{\beta}, \frac{K(\mu + \theta)(\beta - (\mu + \gamma))}{\beta(\mu + \gamma + \theta)} \mu_b L, \frac{\delta + \mu_b}{\delta + \mu_b}, 0 \right)$

2. If $r_0 > 1$ and $R_0 > 1$ then System (3) has Three equilibrium i.e $P_0, P_1, P_2 = (S, I, S_b, I_b) = \left( \frac{K(\mu + \gamma)I^*}{\beta(I_b^* + I')}, \frac{\mu_b L}{\beta b}, I_b^* \right)$

where $I_b^* = \frac{L[\beta b - (\mu_b + \delta)]}{\beta b}, I^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, A = \beta(\theta + \mu + \gamma), B = \beta(\theta + \mu + \gamma)I_b^* - (\mu + \gamma)K[\beta - (\mu + \gamma)],$ and $C = -\beta(\mu + \theta)K I_b^*.$

Proof:

The equilibrium points were solutions of System (4).

$(\mu + \theta)K - \theta I^* - S^* \left( \frac{I_b^* + I^*}{K} + \mu + \theta \right) = 0$

$\beta S^* \left( \frac{I_b^* + I^*}{K} \right) - (\mu + \gamma)I^* = 0$

$\mu_b L - \left( \beta_b \frac{I_b^*}{L} + \delta + \mu_b \right) S_b^* = 0$

$\beta_b \frac{S_b^*}{L} I_b^* - \mu_b I_b^* = 0$

From the fourth equation of System (4), we get

$\left[ \beta_b \frac{S_b^*}{L} - \mu_b \right] I_b^* = 0 \Leftrightarrow I_b^* = 0 \lor \beta_b \frac{S_b^*}{L} - \mu_b = 0 \Leftrightarrow I_b = 0 \lor S_b^* = \frac{\mu_b L}{\beta_b}$.

The case of $I_b^* = 0$:

Substitute the value of $I_b^*$ to the third equation, we get $S_b^* = \frac{\mu_b L}{\delta + \mu_b}.$

Substitute the value of $I_b^*$ to the second equation, we get $I^* = 0 \lor S^* = \frac{K(\mu + \gamma)}{\beta}.$

The case of $I^* = 0$:

For this case, we get $P_0 = (S, I, S_b, I_b) = \left( K, 0, \frac{\mu_b L}{\delta + \mu_b}, 0 \right).$

The case of $I^* \neq 0$:

Clear that $S^* = \frac{K(\mu + \gamma)}{\beta},$ Substitute to the first equation then we get $I^* = \frac{K(\mu + \theta)[\beta - (\mu + \gamma)]}{\beta(\mu + \gamma + \theta)}.$

Clear that if $R_0 = \frac{\beta}{\mu + \gamma} > 1$ then $I^* > 0.$ Hence, we get if $R_0 > 1$ and $\frac{\theta}{\mu + \gamma} < 1$ then

$P_1 = (S, I, S_b, I_b) = \left( \frac{K(\mu + \gamma)}{\beta}, \frac{K(\mu + \theta)[\beta - (\mu + \gamma)]}{\beta(\mu + \gamma + \theta)} \mu_b L, \frac{\delta + \mu_b}{\delta + \mu_b}, 0 \right)$.
The case of \( I_b^* \neq 0 \):

Clear that \( S_b^* = \frac{\mu_b \cdot \beta_b}{\beta_b} \). Substitute the value of \( S_b^* \) to the third equation, we get \( I_b^* = \frac{\mu_b \cdot \beta_b - (\mu_b + \delta)}{\beta_b} \).

Clear that \( I_b^* > 0 \) if \( r_0 = \frac{\beta_b}{\mu_b + \delta} > 1 \). From the second equation, we get \( S^* = \frac{\kappa (\mu + \gamma) I^*}{\beta (I_b^* + I^*)} \).

Substitute to the first equation, then we get

\[
\beta (\theta + \mu + \gamma) I^{*2} + \left[ \beta (\theta + \mu + \gamma) I_b^* - (\mu + \theta) K [\beta - (\mu + \gamma)] \right] I^* - \beta (\mu + \theta) K I_b^* = 0.
\]

Let \( A = \beta (\theta + \mu + \gamma), B = \beta (\theta + \mu + \gamma) I_b^* - (\mu + \theta) K [\beta - (\mu + \gamma)], \) and \( C = -\beta (\mu + \theta) K I_b^* \).

Clear that \( A > 0 \) and \( C < 0 \). Hence \( B^2 - 4AC > B^2 > 0 \).

Hence \( I^*_1 = -\frac{B - \sqrt{B^2 - 4AC}}{2A} < 0 \) and \( I^*_2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} > 0 \).

Hence, \( S^* = \frac{\kappa (\mu + \gamma) I^*}{\beta (I_b^* + I^*)} > 0 \). Hence, if \( r_0 > 1, R_0 > 1 \) and \( \frac{\theta}{\mu + \gamma} > 1 \) then there are exists

\[
P_2 = (S, I, S_b, I_b) = \left( \frac{K (\mu + \gamma) I^*}{\beta (I_b^* + I^*)}, I^*, \frac{\mu_b L}{I_b^*}, I_b^* \right).
\]

where \( I_b^* = \frac{\mu_b \cdot \beta_b - (\mu_b + \delta)}{\beta_b}, I^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, A = \beta (\theta + \mu + \gamma), B = \beta (\theta + \mu + \gamma) I_b^* - (\mu + \theta) K [\beta - (\mu + \gamma)], \) and \( C = -\beta (\mu + \theta) K I_b^* \).

The Stability of equilibrium points of System (3) is given in Theorem 2.

**Theorem 2.**

Let \( r_0 = \frac{\beta_b}{\mu_b + \delta} \) and \( R_0 = \frac{\beta}{\mu + \gamma} \),

1. If \( r_0 < 1 \) and \( R_0 < 1 \) then \( P_0 \) is locally asymptotically stable
2. If \( r_0 < 1 \) and \( R_0 > 1 \) then \( P_0 \) is unstable and \( P_1 \) is locally asymptotically stable.
3. If \( r_0 > 1 \) and \( R_0 > 1 \) then \( P_0 \) and \( P_1 \) are unstable, and \( P_2 \) is locally asymptotically stable.

**Proof:**

The Jacobian matrix of System (4) was given below

\[
Jb(P) = \begin{bmatrix}
\frac{-\beta (I_b + I_b^*)}{K} & -\theta & -\frac{\beta S}{K} & 0 \\
\frac{\beta (I_b + I_b^*)}{K} & \frac{\beta S}{K} & -\frac{(\mu + \gamma)}{K} & 0 \\
0 & 0 & -\frac{\beta S}{L} & -\frac{(\delta + \mu_b)}{L} \\
0 & 0 & -\frac{\beta_b S}{L} & -\frac{(\delta + \mu_b)}{L}
\end{bmatrix}
\]

For \( P_0 = (S, I, S_b, I_b) = \left( K, 0, \frac{\mu_b L}{\delta + \mu_b}, 0 \right) \):

The eigen values of \( Jb(P_0) \) are \( \lambda_1 = -(\mu + \theta), \lambda_2 = \beta - (\mu + \gamma), \lambda_3 = -\left( (\delta + \mu_b) \right), \), and \( \lambda_4 = \frac{\mu_b \cdot \beta_b - (\delta + \mu_b)}{\delta + \mu_b} \).

Clear that \( \lambda_1 \) and \( \lambda_3 \) are negative, \( \lambda_2 < 0 \) if \( R_0 < 1 \), and \( \lambda_4 < 0 \) if \( r_0 < 1 \). Hence, (1) \( P_0 \) is locally asymptotically stable if \( R_0 < 1 \) and \( r_0 < 1 \) and (2) \( P_0 \) is unstable if \( R_0 > 1 \).
For $P_1 = (S, I, S_b, I_b) = \left( \frac{K(\mu + \gamma)}{\beta}, \frac{K(\mu + \theta)(\beta - (\mu + \gamma))}{\beta(\mu + \gamma + \theta)}, \frac{\mu_b L}{\delta + \mu_b}, 0 \right)$:

This analysis was only done at $R_0 > 1$. The characteristics polynomial of $J_b(P_1)$ is

$$\frac{1}{(\mu + \gamma + \theta)(\delta + \mu_b)}(\lambda + \delta + \mu_b)(\delta + \mu_b)\lambda + \mu_b(\delta + \mu_b - \beta_b)\left[(\mu + \gamma + \theta)\lambda^2 + (\beta + \theta)(\mu + \theta)\lambda + (\mu + \theta)(\mu + \gamma - \theta)(\beta - \mu - \gamma)\right] = 0$$

From the characteristic polynomial of $J_b(P_1)$, we got two first eigen values i.e

$$\lambda_1 = -\left(\delta + \mu_b\right)\text{ and } \lambda_2 = \frac{\mu_b(\beta_b - \delta - \mu_b)}{\delta + \mu_b}. \text{ Clear that } \lambda_1 < 0 \text{ and } \lambda_2 < 0 \text{ if } r_0 < 1.$$ 

From $(\mu + \gamma + \theta)\lambda^2 + (\beta + \theta)(\mu + \theta)\lambda + (\mu + \theta)(\mu + \gamma - \theta)(\beta - \mu - \gamma) = 0$, we got the simpler equation $\lambda^2 + \frac{(\beta + \theta)\mu + (\mu + \theta)\beta}{(\mu + \gamma + \theta)}\lambda + (\mu + \theta)(\beta - \mu - \gamma) = 0$.

Define $A = 1, B = \frac{(%(\beta + \theta)\mu + (\mu + \theta)\beta}{(\mu + \gamma + \theta)}$ and $C = (\mu + \theta)(\beta - \mu - \gamma)$.

Clear that $B > 0$ and $C > 0$ if $R_0 > 1$. Hence, $\lambda_3 = \frac{-B - \sqrt{B^2 - 4AC}}{2}$ and $\lambda_4 = \frac{-B + \sqrt{B^2 - 4AC}}{2}$.

where $D = B^2 - 4C$. Because $C > 0$ then $D < B^2$. Hence, $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$ if $R_0 > 1$. Hence, (1) $P_1$ is locally asymptotically stable if $r_0 < 1$ and $R_0 > 1$; (2) $P_1$ is unstable if $r_0 > 1$.

For $P_2 = (S, I, S_b, I_b) = \left( \frac{K(\mu + \gamma)I^\ast}{\beta(I^\ast + b^\ast)}, \frac{I^\ast}{\beta}, \frac{\mu_b L}{\delta + \mu_b}, \frac{I^\ast_b}{b^\ast} \right)$:

We have $I^\ast_b = \frac{I_b^\ast(\mu + \gamma + \theta)}{\beta_b}$, $I^\ast = \frac{B + \sqrt{B^2 - 4AC}}{2A}$, where $A = \beta(\theta + \mu + \gamma)$, $B = \beta(\theta + \mu + \gamma)I^\ast_b - (\mu + \theta)K[\beta - (\mu + \gamma)]$, and $C = -\beta(\mu + \theta)K[I^\ast_b]$.

This analysis was only done at $r_0 > 1$ and $R_0 > 1$. The characteristics polynomial of $J_b(P_2)$ is

$$\frac{1}{K(I^\ast_b + I^\ast) L}\left[\left[L^2 + (L(\delta + \mu_b) + \beta_b I^\ast_b)\lambda + \mu_b \beta_b I^\ast_b\right][A_1 L^2 + B_1 \lambda + C_1] = 0\right]$$

where $A_1 = K(I^\ast_b + I^\ast), B_1 = \beta I^\ast + (2\beta I^\ast_b + K(\mu + \mu))(I^\ast_b + I^\ast_b)I^\ast_b + I^\ast_b[\beta_b I^\ast_b + K(2\mu + \theta + \gamma)]$,

$C_1 = \beta(\mu + \gamma + \theta)I^\ast_b + 2\beta I^\ast_b(\mu + \gamma + \theta)I^\ast_b + I^\ast_b[\beta_b I^\ast_b(\mu + \gamma + \theta) + K(\mu + \theta)(\mu + \gamma)]$.

From $L^2 + (L(\delta + \mu_b) + \beta_b I^\ast_b)\lambda + \mu_b \beta_b I^\ast_b = 0$ we got

$$\lambda_1 = \frac{-(L(\delta + \mu_b) + \beta_b I^\ast_b) - \sqrt{[L(\delta + \mu_b) + \beta_b I^\ast_b]^2 - 4L\mu_b \beta_b I^\ast_b}}{2L} \text{ and }$$

$$\lambda_2 = \frac{-(L(\delta + \mu_b) + \beta_b I^\ast_b) + \sqrt{[L(\delta + \mu_b) + \beta_b I^\ast_b]^2 - 4L\mu_b \beta_b I^\ast_b}}{2L}.$$ 

Clear that $[L(\delta + \mu_b) + \beta_b I^\ast_b]^2 - 4L\mu_b \beta_b I^\ast_b = [(\beta_b I^\ast_b - I\mu_b)^2 + 2L\delta \beta_b I^\ast_b] > 0$ and $[L(\delta + \mu_b) + \beta_b I^\ast_b]^2 - 4L\mu_b \beta_b I^\ast_b > 0$.

Hence, $\lambda_1$ and $\lambda_2$ are negative. From $A_1 \lambda^2 + B_1 \lambda + C_1 = 0$ where $A_1 = K(I^\ast_b + I^\ast)$,

$B_1 = \beta I^\ast_b + (2\beta I^\ast_b + K(\mu + \mu))(I^\ast_b + I^\ast_b)I^\ast_b + I^\ast_b[\beta_b I^\ast_b + K(2\mu + \theta + \gamma)]$,

$C_1 = \beta(\mu + \gamma + \theta)I^\ast_b + 2\beta I^\ast_b(\mu + \gamma + \theta)I^\ast_b + I^\ast_b[\beta_b I^\ast_b(\mu + \gamma + \theta) + K(\mu + \theta)(\mu + \gamma)]$.
we got $\lambda_3 = \frac{-B_1 - \sqrt{B_1^2 - 4A_1C_1}}{2A}$ and $\lambda_3 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$. Clear that $A$, $B$, and $C$ are all positive if $r_0 > 1$ and $R_0 > 1$. Hence, $B_1^2 - 4A_1C_1 < B_1^2$. It caused $\lambda_3$ and $\lambda_4$ have real part.

Hence, $P_2$ is locally asymptotically stable if $r_0 > 1$ and $R_0 > 1$.

4. Simulation

Simulation was done for three cases like three conditions in Theorem 2. Value of some parameter followed from Kharis & Arifudin [12]. Value of parameter were given in Table 2.

| Parameter | Value     | Parameter | Value     |
|-----------|-----------|-----------|-----------|
| $\mu$     | 0.00004   | $\beta_b$ | 0 to 1    |
| $\beta$   | 0 to 1    | $m_b$     | 0.00137   |
| $m$       | 0.00004   | $\delta$  | 0.7       |
| $\gamma$  | 0.098     | $K$       | 6000      |
| $\theta$  | 0.037     | $L$       | 20000     |
| $\mu_b$   | 0.00137   |           |           |

4.1. Simulation for $r_0 < 1$ and $R_0 < 1$

In this case, we used the value $\beta = 0.08$ and $\beta_b = 0.5$. From the formula $r_0$ and $R_0$ in Theorem 1, we got $r_0 = 0.713 < 1$ and $R_0 = 0.816 < 1$. From Theorem 1, There is only one equilibrium point i.e. $P_0 = (S, I, S_b, I_b) = (6000, 0, 39.06, 0)$. The graphs for this simulation were given on Figure 2.

![Figure 2](image_url)

(a) Plane $I_b$ vs $I_h$  (b) Plane $S$ vs $I_h$  (c) Plane $S_b$ vs $I_b$

From Figure 2, it can be seen that the solutions that is near from $P_0$ converge to $P_0$. These simulations were similar with Theorem 2.

4.2. Simulation for $r_0 < 1$ and $R_0 > 1$.

In this case, we used the value $\beta = 0.4$ and $\beta_b = 0.5$. From the formula $r_0$ and $R_0$ in Theorem 1, we got $r_0 = 0.71 < 1$ and $R_0 = 4.08 > 1$. From Theorem 1, There are two equilibrium points i.e. $P_0 = (S, I, S_b, I_b) = (6000, 0, 39.06, 0)$ and $P_1 = (S, I, S_b, I_b) = (1470.6, 1242.36, 39.06, 0)$. The graphs for this simulation were given on Figure 3. From Figure 3, it can be seen that the solutions that is near from $P_1$ converge to $P_1$. These simulations were similar with Theorem 2.
Figure 3. Phase portrait projection of solution at \( r_0 < 1 \) and \( R_0 > 1 \)

Figure 4. Phase portrait projection of solution at \( r_0 > 1 \) and \( R_0 > 1 \)
4.3. Simulation for $r_0 > 1$ and $R_0 > 1$
In this case, we used the value $\beta = 0.4$ and $\beta_b = 0.8$. From the formula $r_0$ and $R_0$ in Theorem 1, we got $r_0 = 1.14 > 1$, $R_0 = 4.08 > 1$. From Theorem 1, There are three equilibrium points i.e. $P_0 = (S, I, S_b, I_b) = (6000,0,39.06,0)$, $P_1 = (S, I, S_b, I_b) = (1470.6,1242.36,39.06,0)$, and $P_2 = (S, I, S_b, I_b) = (554.75,1493.57,34.25,2465.75)$. The graphs for this simulation were given on Figure 4. From Figure 4, it can be seen that the solutions that is near from $P_2$ converge to $P_2$. These simulations were similar with Theorem 2.

5. Conclusion
From analysis above, we get the dynamic of mathematics model of AI-epidemic with vaccination on bird population especially for constant population. We also got the formula of reproduction number ($r_0$ and $R_0$) which can be used to determine whether the epidemic spread widely or vanish. For the formula of $r_0$, we got that the proportion of vaccinated susceptible bird can change the value of $r_0$. If this proportion increase then $r_0$ decrease. It means we can prevent the spreading of this epidemic in bird population by increasing the proportion of vaccinated susceptible bird. For human population, we can prevent the spreading of this epidemic by reducing the probability of infectious contact between infected people and susceptible people. It can be done by quarantine infected people. For the next research, we propose to make the mathematics model for non-constant population.

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