A TWO-STEP METROPOLIS HASTINGS METHOD FOR BAYESIAN EMPIRICAL LIKELIHOOD COMPUTATION WITH APPLICATION TO BAYESIAN MODEL SELECTION

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ABSTRACT. In recent times empirical likelihood has been widely applied under Bayesian framework. Markov chain Monte Carlo (MCMC) methods are frequently employed to sample from the posterior distribution of the parameters of interest. However, complex, especially non-convex nature of the likelihood support erects enormous hindrances in choosing an appropriate MCMC algorithm. Such difficulties have restricted the use of Bayesian empirical likelihood (BayesEL) based methods in many applications. In this article, we propose a two-step Metropolis Hastings algorithm to sample from the BayesEL posteriors. Our proposal is specified hierarchically, where the estimating equations determining the empirical likelihood are used to propose values of a set of parameters depending on the proposed values of the remaining parameters. Furthermore, we discuss Bayesian model selection using empirical likelihood and extend our two-step Metropolis Hastings algorithm to a reversible jump Markov chain Monte Carlo procedure to sample from the resulting posterior. Finally, several applications of our proposed methods are presented.

Keywords: Bayesian empirical likelihood; Markov chain Monte Carlo; Bayesian model selection; Reversible jump Markov chain Monte Carlo.

1. INTRODUCTION

In recent years, empirical likelihood (Owen, 1988; Qin & Lawless, 1994) based procedures have been frequently used under Bayesian framework. Such procedures specify a statistical model through unbiased estimating equations, without requiring a declaration of the data distribution. The likelihood is estimated from the empirical distribution function computed under constraints imposed by these estimating equations. The estimated likelihood is then used to define a posterior. The validity of empirical and similar likelihoods for Bayesian inference has been a topic of extensive discussion (Monahan & Boos, 1992; Lazar, 2003; Fang & Mukerjee, 2006; Corcoran, 1998). Alternative likelihoods like Bayesian exponential tilted empirical likelihood (BETEL) (Schennach, 2005) have been proposed and justified using basic probabilistic arguments. In recent times, many authors (Chib et al., 2016; Zhong & Ghosh, 2016) have considered asymptotic properties of the posteriors and parameter estimates obtained from such likelihoods. Due to its convenience in statistical modelling, in recent times, the Bayesian empirical likelihood (BayesEL) procedures have seen many applications, such as in analysing complex survey data (Rao & Wu, 2010), small area estimation (Chaudhuri & Ghosh, 2011; Porter et al., 2015), quantile regression (Yang & He, 2012), among others.

The likelihood as well as the posterior in BayesEL procedure is computed numerically for each value of the parameter. Inferences are drawn using the samples generated from the posterior. However, efficient sampling of BayesEL posteriors require bespoke procedures. The cost of Gibbs sampling (Geman & Geman, 1984; Smith & Roberts, 1993) is prohibitive since numerical determination of the so called full conditionals are too expensive. Metropolis Hastings (Hastings, 1970) or similar Markov chain Monte Carlo (MCMC) methods need to be used. These extensively studied methods (Shao & Ibrahim, 2000; Chib & Greenberg, 1995; Tierney, 1994) however is not easily implemented in BayesEL posterior sampling. The support of the empirical likelihood is data dependent, does not usually cover the whole parameter space and is usually non-convex (Chaudhuri et al., 2016). Proposals which can cover such supports, are not easily constructed. Simple random walks would mix very slowly, since they would often get stuck near the boundary of the...
support. Methods like the metropolis algorithm in [Haario et al. (2001)] or parallel tempering ([Geyer, 1992; Liu, 2008]) do not adapt to the support non-convexity satisfactorily. These difficulties in drawing samples from a BayesEL posterior have been a major impediment to its wider use in statistical modelling.

Some authors (e.g. [Porter et al. (2015)]) have designed specific MCMC algorithms to sample from a BayesEL posterior. For smooth estimating equations, a more general method was proposed by [Chaudhuri et al. (2016)]. They show that the gradient of the log-empirical likelihood diverges at the boundaries of its support, and then use this gradient to propose a Hamiltonian Monte Carlo method to sample from a BayesEL posterior. The diverging gradient ensures that the chain always reflects towards the centre of the support from its boundaries.

If one wishes to explore the possibility of Bayesian model selection using BayesEL procedure, the problem becomes much more acute. In order to sample from the posterior arising in Bayesian model selection problems, reversible jump Markov chain Monte Carlo (RJMCMC) ([Green, 1995]) sampler which can jump between models is generally used (see [Fan & Sisson (2010); Dellaportas et al. (2002); Robert et al. (2002)]). The main challenge in RJMCMC is efficient construction of the cross-model proposals. The usual notions that can guide the sampler in the fixed dimensional state space now appear useless. Inefficient proposal makes the reversible jump sampler explore the parameter space slowly or even fail entirely. Consequently, the Markov chain takes a long time to converge. The construction of an efficient proposal is a topic of extensively discussion even in fully parametric setup ([Brooks et al. 2003; Sisson, 2005; Green & Hastie, 2009]). Many variants of RJMCMC have also been proposed ([Al-Awadhi et al., 2004; Jasra et al., 2007]).

Sampling from a BayesEL posterior becomes more difficult when the dimension of the parameter space is allowed to vary between iterations. When the RJMCMC is used in the BayesEL procedures, constructing an efficient proposal becomes even harder. This is because even for the parameters common between the current and proposed models, the posterior supports may be entirely different. Therefore, for RJMCMC in the BayesEL procedure, it is a great challenge to design a cross-model proposal which can ensure the proposed candidates to be in their new marginal supports.

In this work, we propose a two-step Metropolis Hastings algorithm for sampling from a BayesEL posterior. Under our setup, (see Section 3) the parameters of interest are split into two sets. New values of the first set of parameters are proposed first. Next, by using the estimating equations, we find a trial value of the remaining parameters for which the empirical likelihood is the highest. The new values of the remaining parameters are then proposed depending on this trial value. The two-step procedure ensures that given the proposed value of the first set of parameters is in its marginal support, the chance of the remaining parameters to be in their marginal support is high. This avoids the non-convexity of the support and improves the acceptance rate of the resulting chain. Our method does not require the estimating equation to be smooth in the parameters. We extend it to a reversible jump Markov chain Monte Carlo scheme which allows us to successfully implement empirical likelihood in Bayesian model selection.

2. A MOTIVATING EXAMPLE

We start with a simple illustration of the difficulties in using traditional Metropolis Hastings methods for sampling from the posterior derived from an empirical likelihood. Consider 10 independently and identically distributed univariate observations \(x_i, i = 1, \ldots, n = 10\), generated from a normal distribution with mean \(\mu = 0\) and variance \(\sigma^2 = 1\). Our goal here is to estimate \(\mu\) and \(\sigma^2\) from the data.

Suppose we assume diffused but proper prior distributions \(\pi(\mu) \sim \mathcal{N}(0, 100)\) and \(\pi(\sigma^2) \sim \text{Inverse Gamma } IG(.001, .001)\) for \(\mu\) and \(\sigma^2\) respectively. From Bayes Theorem, the full parametric joint posterior distribution of \((\mu, \sigma^2)\) is given by,

\[
\Pi_N(\mu, \sigma^2 | x) = \frac{(2\pi\sigma^2)^{-n/2} \exp \left\{ -\frac{1}{2} (\sigma^2)^{-1} \sum_{i=1}^{n} (x_i - \mu)^2 \right\} \pi(\mu) \pi(\sigma^2)}{\int (2\pi\sigma^2)^{-n/2} \exp \left\{ -\frac{1}{2} (\sigma^2)^{-1} \sum_{i=1}^{n} (x_i - \mu)^2 \right\} \pi(\mu) \pi(\sigma^2) d\mu d\sigma^2}.
\]

The posterior \(\Pi_N\) can be expressed in an analytic form and has been studied extensively.
Now, suppose we assume that the parametric form of the distribution of $x_i$ is not known. We only know $x_i$ has some distribution $F^0$ with mean $\mu_0$ and variance $\sigma^2_0$. In this case the full parametric likelihood is not available and we turn to empirical likelihood instead.

By the definition of mean and variance it is clear that,

$$E_{F^0} [x_i - \mu_0] = 0 \text{ and } E_{F^0} [(x_i - \mu_0)^2 - \sigma^2_0] = 0.$$  \hfill (2)

In order to compute the empirical likelihood, we first assign unknown weight $\omega_i$, to each observation $x_i$, $i = 1, \ldots, n$. The weight vector $\omega = (\omega_1, \ldots, \omega_n)$ is assumed to be in $n - 1$ dimensional simplex, $\Delta_{n-1}$. Additionally, $\omega$ is forced to satisfy certain constraints inherited from (2).

For any $\mu$ and $\sigma^2$, the empirical likelihood $L(\mu, \sigma^2)$ is given by,

$$L(\mu, \sigma^2) = \max_{\omega \in W_{\mu, \sigma^2}} \prod_{i=1}^{n} \omega_i,$$

where the constraints on $\omega$ is defined by

$$W_{\mu, \sigma^2} = \left\{ \omega : \sum_{i=1}^{n} \omega_i (x_i - \mu) = 0, \sum_{i=1}^{n} \omega_i \{(x_i - \mu)^2 - \sigma^2\} = 0 \right\} \cap \Delta_{n-1}. \hfill (3)$$

We define $\mathcal{L}(\mu, \sigma^2) = 0$ if the problem in (3) is infeasible.

By following Lazar (2003), Rao & Wu (2010), Chaudhuri & Ghosh (2011), we can define an empirical likelihood based BayesEL posterior as,

$$\Pi_{EL}(\mu, \sigma^2 | x) = \frac{\mathcal{L}(\mu, \sigma^2) \pi(\mu) \pi(\sigma^2)}{\int \mathcal{L}(\mu, \sigma^2) \pi(\mu) \pi(\sigma^2) d\mu d\sigma^2}. \hfill (4)$$

Clearly, $\Pi_{EL}$ cannot be expressed in an analytic form in most cases. The numerator of (4) can only be computed numerically by solving problem in (3). The denominator requires numerical integration of the numerator over its support, which is too costly to compute. However, since $\omega \in \Delta_{n-1}$, it is clear that for any choice of proper $\pi(\mu)$ and $\pi(\sigma^2)$, the posterior would be proper.

In Figure 1, we present the contour plots of the logarithm of the numerators of the expressions in (1) and (4) for different values of $\mu$ and $\sigma^2$. Our motivation here is to compare the supports of $\Pi_N(\mu, \sigma^2 | x)$ and $\Pi_{EL}(\mu, \sigma^2 | x)$. Since the normal distribution is supported over the whole real line, $\Pi_N$ is supported over the

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**Figure 1.** The contour plots of the logarithms of the numerators of (a) (1) ie. $\log \Pi_N(\mu, \sigma^2 | x)$ and (b) (4) ie. $\log \Pi_{EL}(\mu, \sigma^2 | x)$ for different values of $\mu$ and $\sigma^2$. 
whole half plane. However, for many values of $\mu$ and $\sigma^2$, the problem in (5) may be infeasible. Thus for those values by definition, $\mathcal{L}(\mu, \sigma^2) = 0$. That is, $\Pi_{EL}$ may not be supported on the whole half plane.

This phenomenon is illustrated in Figure 1(b). The support of the BayesEL posterior is restricted and non-convex. Moreover, it is clear that the posterior decreases very sharply near the support boundary. The true value of the parameter however has a relatively high posterior value. This means that the use of BayesEL posterior instead the true one, would be competitive in terms of efficiency. In fact, if the likelihood is mis-specified (which is often the case) the BayesEL procedures may be more efficient.

The non-convexity and the boundedness of the support poses a potential problem in using BayesEL methods. A lack of analytic form implies, that in order to make any inference one has to resort to Markov chain Monte Carlo based techniques to directly sample from the posterior. A non-convex support would often confine many such chains near the boundary for a considerable amount of time. Furthermore, an efficient proposal for $\sigma^2$ must take into account of its marginal range in the support determined by the proposed value of $\mu$. The support would depend on the data and in any real life situation would be too costly to determine. As a result one has to be judicious in choosing a proposal distribution. Otherwise, the resulting chain would mix very slowly.

The simple example presented above demonstrates the difficulty in using traditional random-walk MCMC to sample from BayesEL posterior. A lack of easy and fast mixing sampler is a huge impediment to the utilisation of BayesEL methods in many real life applications.

Our goal here is to design an adaptive scheme which will use the estimating equations and the proposed value of $\mu$, to propose a value of $\sigma^2$ ensuring a relatively large value of $\mathcal{L}(\mu, \sigma^2)$. Such a procedure would ensure that the chain jumps from the boundary of the support to its centre. The utility of such a method cannot be underestimated. It would improve the acceptance rate of the chain which in turn would accelerate its mixing. In addition, the proposed scheme should be applicable to a variety of estimating equations, such as smooth, non-smooth or discontinuous functions of the parameters. Even the dimension of the parameter space is allowed to change in each iteration. A situation encountered in BayesEL model selection, where reversible jump Markov chain Monte Carlo (RJMCMC) methods need to be employed.

In what follows, we describe our proposed adaptive method in details. The procedure is first described for problems with a fixed parameter dimension. An adaptive RJMCMC method for BayesEL model selection is described later.

3. A maximum conditional empirical likelihood estimator with applications to Bayesian empirical likelihood computation

3.1 Setup of the Bayesian Empirical Likelihood. We begin with a description of the setup which will be used in the two-step Metropolis Hastings methods, we introduce later. The Bayesian empirical likelihood method is discussed for our setup.

We consider independent observations $x_1, \ldots, x_n$ from an unknown distribution $F^0$, which depends on a parameter vector $\theta = (\theta_1, \theta_2) \in \Theta = \Theta_1 \times \Theta_2$ of length $p+q$. Assume that the prior knowledge on $(\theta_1, \theta_2)$ is specified by $\pi(\theta_1, \theta_2)$. Suppose the unknown distribution $F^0$ satisfies certain estimating equations. Some of these estimating equations do not explicitly depend on $\theta_2$ while the others involve all parameters. More specifically, let

$$E_{F^0}[g(x, \theta_1^0)] = 0 \quad \text{and} \quad E_{F^0}[h(x, \theta_1^0, \theta_2^0)] = 0,$$

where $g(x, \theta_1) \in \mathbb{R}^l$ and $h(x, \theta_1, \theta_2) \in \mathbb{R}^d$ and $\theta^0$ is the true value of $\theta$.

Estimating equations with such structures appear in traditional linear models, graphical Markov models (Lauritzen, 1996), and models with patterned missing data eg. two-phase designs, models with surrogate variables (Qin et al., 2009) etc.
Let \( \mathcal{F}_\theta \) be a set of distribution functions depending on parameter \( \theta \in \Theta \). For any \( F \in \mathcal{F}_\theta \), its empirical likelihood is derived by maximising the so-called “non-parametric likelihood”

\[
\mathcal{L}(F) = \prod_{i=1}^{n} \{ F(x_i) - \lim_{h \to 0} F(x_i - h) \} = \prod_{i=1}^{n} \{ F(x_i) - F(x_i-) \},
\]

over \( \mathcal{F}_\theta \) under constraints, depending on \( g(x, \theta_1) \) and \( h(x, \theta_1, \theta_2) \).

Suppose, \( \omega_i = F(x_i) - F(x_i-) \) and \( \omega = (\omega_1, \ldots, \omega_n) \) is the vector of weights on the components of \( x = (x_1, \ldots, x_n) \). Given \( \theta_1 \in \Theta_1 \) and \( \theta_2 \in \Theta_2 \), we define a constrained set of \( \omega \), which depends on \( g(x, \theta_1) \) and \( h(x, \theta_1, \theta_2) \) as

\[
\mathcal{W}_G(\theta_1) = \left\{ \omega : \sum_{i=1}^{n} \omega_i g(x_i, \theta_1) = 0 \right\} \cap \Delta_{n-1},
\]

\[
\mathcal{W}_H(\theta_1, \theta_2) = \left\{ \omega : \sum_{i=1}^{n} \omega_i h(x_i, \theta_1, \theta_2) = 0 \right\} \cap \Delta_{n-1},
\]

and

\[
\mathcal{W}(\theta_1, \theta_2) = \mathcal{W}_G(\theta_1) \cap \mathcal{W}_H(\theta_1, \theta_2).
\]

The empirical likelihood of \( F \), which can also be expressed as a likelihood of \( \theta \) is given by

\[
\mathcal{L}(\theta_1, \theta_2) = \mathcal{L}(F) = \max_{\omega \in \mathcal{W}(\theta_1, \theta_2)} \prod_{i=1}^{n} \omega_i(\theta_1, \theta_2).
\]

We define, \( \mathcal{L}(\theta_1, \theta_2) = \mathcal{L}(F) = 0 \), when the problem (10) is infeasible.

Now by using \( \mathcal{L}(\theta_1, \theta_2) \) as a likelihood, one defines Bayesian empirical likelihood (BayesEL) posterior as

\[
\Pi_{EL}(\theta_1, \theta_2 | x) = \frac{\mathcal{L}(\theta_1, \theta_2)\pi(\theta_1, \theta_2)}{\int \mathcal{L}(\theta_1, \theta_2)\pi(\theta_1, \theta_2)d\theta_1d\theta_2}.
\]

The empirical likelihood is based on an empirical estimate of \( F_0 \) which satisfies the estimating equations in (5). Clearly, for any continuous \( F \), the likelihood \( \mathcal{L}(F) = 0 \). Thus, by construction this estimate is discrete. Furthermore, by definition, \( \omega \in \Delta_{n-1} \), that is \( \mathcal{L}(\theta_1, \theta_2) \) is bounded by one. This implies the posterior \( \Pi_{EL} \) is proper for any proper prior.

The BayesEL posterior in (11) uses an plugin estimate of the likelihood obtained under model based constraints. Validity of empirical and similar such likelihoods in Bayesian inference has been a topic with extensive discussion. Using a criterion proposed by Monahan & Boos (1992), Lazar (2003) examined the validity of BayesEL procedures by Monte Carlo simulations. Fang & Mukerjee (2006), Corcoran (1998) considered a general empirical likelihood formulation and computed the asymptotic frequentist coverages of Bayesian credible sets. Higher order asymptotic properties of BayesEL posterior have been studied by Zhong & Ghosh (2016).

By formulation, a BayesEL posterior only needs a specification of the model in terms of estimating equations which are unbiased under the truth. Thus, by using BayesEL procedure, one can avoid making non-testable assumptions about the data distribution. This has been found to be useful and the BayesEL procedures have seen many applications on different problems in recent times. Examples include complex surveys (Rao & Wu, 2010), small area estimation (Chaudhuri & Ghosh, 2011; Porter et al., 2015), quantile regression (Yang & He, 2012) among others.
The computation of BayesEL posterior, however, is often a big challenge. The likelihood is computed numerically. Absence of any analytic form of the posterior prevents direct computation. To make any inference one needs to draw samples from the posterior distribution. So techniques like Gibbs sampling (Geman & Geman, 1984; Smith & Roberts, 1993) cannot be employed. The only way forward is to use a carefully designed Markov chain Monte Carlo (MCMC) sampler.

For BayesEL, however, efficient adaptive proposals for MCMC sampling are not easily constructed. This is primarily due to the complex nature of the support of the empirical likelihood.

Provided the prior is supported over whole $\Theta$, the BayesEL posterior is positive, if the maximisation problem in (10) is feasible for the particular $\theta_1$ and $\theta_2$. This happens when $\hat{\omega}_i(\theta_1, \theta_2) > 0$, for all $i = 1, \ldots, n$, which in turn happens if and only if the origin is in the convex hull of the points $(g(x_i, \theta_1), h(x_i, \theta_1, \theta_2))$, $i = 1, \ldots, n$. However, as seen in Figure 1(b) usually, this condition is not satisfied over the whole $\Theta$. For certain values of $\theta \in \theta$, the problem in (10) is infeasible and the corresponding set $W(\theta_1, \theta_2)$ would be empty.

Even though numerous authors have discussed such empty-set problems and its remedies in the frequentist paradigm (Chen et al., 2008; Emerson et al., 2009; Liu et al., 2010; Tsao et al., 2013; Tsao, 2013), relatively little is known about the properties of empirical likelihood support. Under certain conditions Chaudhuri et al. (2016) show that the support is an open set. Even for simple models though, the posterior support would be non-convex. The marginal support of $\theta_2$ may depend very much on the values of $\theta_1$ and may vary dramatically with new proposed value of the latter. In such cases, usual random walk proposals will not be efficient. Some adaptive MCMC procedure would be required.

The random walk based adaptive procedures e.g Haario et al. (2001) do not take into account the non-convexity of the support and as a result is not very efficient. Methods like parallel tempering (Geyer, 1992; Liu, 2008) too mixes very slowly.

One adaptive procedure have been discussed by Chaudhuri et al. (2016), who show that the gradient of the log-empirical likelihood diverges at the boundary of support and use this information to design an Hamiltonian Monte Carlo procedure to sample from a BayesEL posterior. The diverging derivative ensures that when the chain approaches the boundary it reflects back towards the centre of the support and almost never steps out.

Even though the HMC procedure is useful in many problems, it requires the estimating equations to be smooth in terms of the parameters. In many applications this condition does not hold. In fact, in many cases (eg. estimating quantiles) the estimating equations may not even be continuous function of the parameters.

We introduce a basic two-step Metropolis Hastings procedure to sample from BayesEL posteriors below. This method does not put any smoothness conditions on the estimating equations and thus can be applied to all the situations described above. Furthermore, we develop a BayesEL model selection procedure for which we propose a RJMCMC procedure which provides the only known efficient way to sample from the resulting posterior.

3.2. A Two-step Maximum Conditional Empirical Likelihood Estimator. Under our setup, efficient moves for Metropolis Hastings procedure can be achieved if for a given value of $\theta_1 \in \Theta_1$ we can propose a value of $\theta_2$ such that the empirical likelihood $L(\theta_1, \theta_2)$ is relatively large. This can be ensured from the estimating equations in (7) and (8) above.

Suppose $\theta_1 = a \in \Theta_1$ is fixed. We define the maximum conditional empirical likelihood estimator (MCELE) of $\theta_2$ as

$$
\hat{\theta}_2^{EL}(a) = \arg\max_{\theta_2 \in \Theta_2} \prod_{i=1}^n \hat{\omega}_i(a, \theta_2) = \arg\max_{\theta_2 \in \Theta_2, \omega \in W(a, \theta_2)} \left\{ \prod_{i=1}^n \omega_i \right\}.
$$
Clearly, by definition, for a given \( \theta_1 = a \), \( \mathcal{L}(a, \hat{\theta}_2^{EL}(a)) \) is the highest possible value of the empirical likelihood. We use \( \hat{\theta}_2^{EL} \) in our proposed procedure.

There are several ways to compute \( \hat{\theta}_2^{EL}(a) \). Equation (12) indicates that it can be obtained from a two-stage maximisation. Our setup allows an alternative characterisation of \( \hat{\theta}_2^{EL}(a) \) which reduces the cost of its computation. This characterisation is also key to the two-step Markov chain Monte Carlo method for drawing sample from the BayesEL posterior in (11).

With the definition of \( \mathcal{W}_G(a) \) is (7), suppose we define

\[
\hat{\nu}(a) = \arg\max \sum_{\nu \in \mathcal{W}_G(a)} \log \nu_i.
\]

By substituting \( \hat{\nu}(a) \) for \( \omega \) in (8), we get the following equations in \( \theta_2 \)

\[
\sum_{i=1}^{n} \hat{\nu}_i(a)h(x_i, a, \theta_2) = 0.
\]

It is easily seen that the following result holds.

**Theorem 1.** Suppose \( \hat{\theta}_2 \) solves (14). Then \( \hat{\theta}_2 = \hat{\theta}_2^{EL}(a) \).

Our proposed maximum conditional estimator of \( \theta_2 \) given \( \theta_1 = a \) is motivated partly by Chaudhuri et al. (2008), who used similar procedure in a frequentist setting. We shall see later that computationally, the characterisation in Theorem 1 is very convenient. For a known value \( a \), maximising the product of weights over \( \mathcal{W}_G(a) \) is a convex problem. Thus, it would have an unique solution. Thereafter, given the optimal weights \( \hat{\nu} \), one has to solve (14) at the worst numerically, which is often easy. In many cases, analytic solutions can be found.

For simplicity, we assume that (14) has a unique solution. Multiple solutions can often be avoided by making judicious choices of \( \theta_1 \) and \( \theta_2 \). In the situations, where multiple solutions exist, the proposed procedure often extends as described. We discuss this issue in more details below.

4. A TWO-STEP METROPOLIS HASTINGS METHOD FOR FIXED DIMENSIONAL STATE SPACE

We now describe the two-step Metropolis Hastings method to sample from the BayesEL posterior when the dimension of the parameter \( \theta \in \Theta \) remains fixed. We shall assume the setup in Section 3 and for any value of \( \theta_1 \in \Theta_1 \) such that (13) is feasible, there is an unique \( \hat{\theta}_2^{EL} \) which solves (14).

Our proposed two-step method is based on the following intuition. First of all notice that if \( \theta_1 = a \), from (13) and (14) we get,

\[
\mathcal{L} \left( a, \hat{\theta}_2^{EL}(a) \right) = \prod_{i=1}^{n} \hat{\nu}_i(a),
\]

provided, the problem in (13) is feasible. For any \( \theta_2 \) sufficiently close to \( \hat{\theta}_2^{EL}(a) \), by continuity of the weights and the likelihood, \( \mathcal{L}(a, \theta_2) \) would be large and thus in a Metropolis Hastings sampler, the move to the proposed point \( (a, \theta_2) \) would have a higher probability of getting accepted.

More specifically (see Algorithm 1), suppose at iteration \( t \), the chain is at \( \theta^{(t)} = (\theta_1^{(t)}, \theta_2^{(t)}) \). We first propose a value of \( \theta_1 \), denoted as \( \theta_1^{(t+1)} \) from a proposal distribution \( q_1 \) possibly depending on \( \theta_1^{(t)} \). In the second step, we compute the corresponding MCELE \( \hat{\theta}_2^{EL} \left( \theta_1^{(t+1)} \right) \).

A new value of \( \theta_2 \), denoted by \( \theta_2^{(t+1)} \), is then proposed from a proposal distribution \( q_2 \), possibly depending on \( \theta_1^{(t+1)} \) and \( \hat{\theta}_2^{EL} \left( \theta_1^{(t+1)} \right) \). Finally, similar to usual the Metropolis Hastings algorithm, the proposed point \( (\theta_1^{(t+1)}, \theta_2^{(t+1)}) \) is accepted with probability in (15) of Algorithm 1.
Algorithm 1. The two-step Metropolis Hastings algorithm for fixed dimension

Require: $\theta^{(1)} = \left(\theta_1^{(1)}, \theta_2^{(1)}\right)$.

for $t = 1$ to $L$ do
\begin{enumerate}
\item Propose $\theta_1^{(t+1)}$ following density $q_1(\cdot | \theta_1^{(t)})$;
\item if the problem (13) is infeasible for $\theta_1^{(t+1)}$ then
\item $\theta^{(t+1)} \leftarrow \theta^{(t)}$;
\else
\item Compute $\tilde{\nu}(\theta_1^{(t+1)})$ by solving problem in (13);
\item Compute $\tilde{\theta}_2^{(t+1)} = \tilde{\theta}_2^{EL}(\theta_1^{(t+1)})$ by solving equation (14) with weights $\tilde{\nu}(\theta_1^{(t+1)})$;
\item Propose $\theta_2^{(t+1)}$ following density $q_2(\cdot | \theta_2^{(t+1)}, \tilde{\theta}_2^{(t+1)})$;
\item Calculate the empirical likelihood $L(\theta_1^{(t+1)}, \theta_2^{(t+1)})$;
\item Accept $\theta^{(t+1)} = (\theta^{(t+1)}, \theta_2^{(t+1)})$ with probability $\alpha$, where
\begin{equation}
\alpha = \min\left\{1, \frac{L(\theta_1^{(t+1)}, \theta_2^{(t+1)}) \pi(\theta_1^{(t+1)}, \theta_2^{(t+1)}) q_2(\theta_2^{(t+1)} | \theta_1^{(t+1)}) q_1(\theta_1^{(t+1)} | \theta_1^{(t)})}{L(\theta_1^{(t)}, \theta_2^{(t)}) \pi(\theta_1^{(t)}, \theta_2^{(t)}) q_2(\theta_2^{(t+1)} | \theta_1^{(t+1)}, \theta_2^{(t+1)}) q_1(\theta_1^{(t+1)} | \theta_1^{(t)})}\right\};
\end{equation}
\end{enumerate}
end if
end for

return $(\theta_1^{(1:L)}, \theta_2^{(1:L)})$.

Suppose $L$ is the length of the sequence and for any $t$, we denote $(\theta_1^{(1:t)}, \theta_2^{(1:t)}) = \left\{(\theta_1^{(1)}, \theta_2^{(1)}), \ldots, (\theta_1^{(t)}, \theta_2^{(t)})\right\}$ to be the sequence of observations up to time $t$ obtained from Algorithm 1 from the beginning (see also Figure 2). By construction, the sequence $(\theta_1^{(1:L)}, \theta_2^{(1:L)})$ has following properties.

1. $\theta_1^{(1:L)}$ is a Markov chain, i.e. for each $t = 1, 2, \ldots, T-1$, $\theta_1^{(t+1)} \perp \perp \theta_1^{(1:t-1)} | \theta_1^{(t)}$.
2. Given $\theta_1^{(t)}$, $\theta_1^{(t+1)}$ is independent of $\theta_2^{(1:t)}$, i.e. $\theta_1^{(t+1)} \perp \perp \theta_2^{(1:t)} | \theta_1^{(t)}$.
3. For all $t$, $\hat{\theta}_2^{EL}(\theta_1^{(t)})$ is a deterministic function of $\theta_1^{(t)}$.
4. $\theta_2^{(t+1)}$ is independent of $\theta_1^{(1:t)}$ given $\theta_1^{(t+1)}$, i.e. $\theta_2^{(t+1)} \perp \perp \left(\theta_1^{(1:t)}, \theta_2^{(1:t)}\right) | \theta_1^{(t+1)}$.

By construction, $(\theta_1^{(1:L)}, \theta_2^{(1:L)})$ is jointly a Markov chain and its transition probability can be determined.

Theorem 2. Suppose the sequence $(\theta_1^{(1:L)}, \theta_2^{(1:L)})$ is generated from the Algorithm 1. Assume that for each $\theta_1^{(t+1)}$, $\hat{\theta}_2^{EL}(\theta_1^{(t+1)})$ is the unique solution of the equation (14). Then $(\theta_1^{(1:L)}, \theta_2^{(1:L)})$ is a Markov chain and its transition probability is given by

\begin{equation}
P\left(\theta_1^{(t+1)}, \theta_2^{(t+1)} | \theta_1^{(t)}, \theta_2^{(t)}\right) = P\left(\theta_2^{(t+1)} | \theta_1^{(t+1)}, \hat{\theta}_2^{EL}(\theta_1^{(t+1)})\right) P\left(\theta_1^{(t+1)} | \theta_1^{(t)}\right).
\end{equation}

Theorem 2 shows that jointly $(\theta_1^{(1:L)}, \theta_2^{(1:L)})$ is a Markov chain and the transition probability is the product of $q_1$ and $q_2$. However, marginally, $\theta_2^{(1:L)}$ itself is not a Markov chain (see Figure 2). The chain $(\theta_1^{(1:L)}, \theta_2^{(1:L)})$ is reversible because the chain $\theta_1^{(1:L)}$ is. The irreducibility and aperiodicity of the chain can be ensured by judicious choice of the proposal distributions $q_1$ and $q_2$. A choice such that the support of the joint proposal $q_1 \otimes q_2$ covers the whole of $\Theta_1 \times \Theta_2$ would generally suffice. Under such choices, the chain will converge to its stationary distribution $\Pi_{EL}(\theta_1, \theta_2 | x)$.

In practical implementation of the two step method, $\theta_1$ can be proposed in many ways. One can use traditional variations of random walk proposals or, provided the estimating equations are smooth, Hamiltonian
Monte Carlo (Chaudhuri et al., 2016) or similar methods. The proposal distribution in the second step has to be independent of the previous steps and depend on the MCELE. In fact, in most cases, \( q_2 \) can be chosen only to depend on \( \hat{\theta}_2^{EL}(\theta_1^{t+1}) \).

5. **Bayesian Empirical Likelihood Model Selection for Linear Models**

We now turn to Bayesian variable selection using empirical likelihood. Such problems can now be attempted because our two-step Metropolis Hastings algorithm can be extended to a reversible jump Markov chain Monte Carlo (RJMCMC) procedure, which can efficiently draw samples from the posterior over discrete model space. We first discuss the constraints under which the empirical likelihood is computed. We will concentrate on the linear models, however, the formulation would easily extend to other models as well.

Suppose the response variable is denoted by \( y \) and there are \( s \) potential covariates, where \( s \) is assumed to be strictly smaller than the sample size \( n \). A model is specified by a binary vector \( \gamma = (\gamma_1, \ldots, \gamma_s) \), where \( \gamma_i \) is 1 if the \( i \)th covariate is included in the model and 0 otherwise. For a given \( \gamma \), \( x_\gamma = (x_{\gamma j} : \gamma_j = 1, j \in \{1, \ldots, s\}) \), a subset of columns of \( n \times s \) data matrix \( x \), is the matrix of covariates under model \( \gamma \). The corresponding coefficient vector is denoted by \( \beta_\gamma = (\beta_i : \gamma_i = 1, i \in \{1, \ldots, s\}) \).

For a given \( \gamma \), the linear model can be be described by,

\[
E[y \mid x_\gamma] = x_\gamma \beta_\gamma \quad \text{and} \quad \text{Var}[y \mid x_\gamma] = \sigma^2_\gamma,
\]

where, the expectations are taken with respect to the unknown true distribution.

In order to define the BayesEL posterior, as before, we assign unknown weights \( \omega_i \) to observation \( x_i \). For a model \( \gamma \), parameters \( \beta_\gamma \) and \( \sigma^2_\gamma \), the empirical likelihood is defined as:

\[
L(\gamma, \beta_\gamma, \sigma^2_\gamma) = \max_{\omega \in \mathcal{W}(\gamma, \beta_\gamma, \sigma^2_\gamma)} \prod_{i=1}^{n} \omega_i,
\]

where

\[
\mathcal{W}_0(\beta_\gamma) = \{ \omega : \omega^T(y - x_\gamma \beta_\gamma) = 0 \} \cap \Delta_{n-1},
\]

\[
\mathcal{W}_\gamma(\beta_\gamma) = \bigcap_{j: \gamma_j = 1} \{ \omega : x_{\gamma j}^T D_\omega (y - x_\gamma \beta_\gamma) = 0 \} \cap \Delta_{n-1},
\]

\[
\mathcal{W}_{c\gamma}(\beta_\gamma) = \bigcap_{j: \gamma_j = 0} \{ \omega : x_{\gamma j}^T D_\omega (y - x_\gamma \beta_\gamma) = 0 \} \cap \Delta_{n-1},
\]

\[
\mathcal{W}_\sigma(\beta_\gamma, \sigma^2_\gamma) = \{ \omega : (y - x_\gamma \beta_\gamma)^T D_\omega (y - x_\gamma^T \beta_\gamma) - \sigma^2_\gamma = 0 \} \cap \Delta_{n-1},
\]

\[
\mathcal{W}(\gamma, \beta_\gamma, \sigma^2_\gamma) = \mathcal{W}_0(\beta_\gamma) \cap \mathcal{W}_\gamma(\beta_\gamma) \cap \mathcal{W}_{c\gamma}(\beta_\gamma) \cap \mathcal{W}_\sigma(\beta_\gamma, \sigma^2_\gamma).
\]
Here $D_\omega$ is the $n \times n$ diagonal matrix with $\omega$ as the diagonal. As before, $L(\gamma, \beta_\gamma, \sigma_\gamma^2) = 0$ if the problem in (18) is infeasible.

The set of constraints described above goes beyond the model specification in (17). It is readily seen that $\mathcal{W}_0$ and $\mathcal{W}_\nu$ ensure that the expectation and variance of the residuals under estimated empirical distribution are zero and $\sigma_\gamma^2$ respectively. The set $\mathcal{W}_\gamma$ implies that the residuals from model $\gamma$ are uncorrelated to the covariates in the model. These constraints follow from the score equations of the model. The constraints in $\mathcal{W}_{\gamma \nu}$ demand some explanation. Here we impose that the residual from model $\gamma$ is uncorrelated to all the available variables absent from the model. This constraints do not follow directly from the linear model in (17). However, these constraints can be justified from a predictive modelling consideration. Clearly, if a covariate not in the current model is correlated to the residuals, its inclusion in the model is likely to improve prediction. Several authors (eg. Variyath et al. (2010), Kolaczyk (1995)) have used the same setup in context of frequentist model selection in generalised linear and moment condition models.

Once the likelihood $L(\gamma, \beta_\gamma, \sigma_\gamma^2)$, the priors $\pi(\gamma)$ and $\pi(\beta_\gamma, \sigma_\gamma^2)$ has been determined the BayesEL posterior can be defined as:

$$\Pi_{EL}(\gamma, \beta_\gamma, \sigma_\gamma^2 | y, x) = \frac{L(\gamma, \beta_\gamma, \sigma_\gamma^2)\pi(\beta_\gamma, \sigma_\gamma^2)\pi(\gamma)}{\int L(\gamma, \beta_\gamma, \sigma_\gamma^2)\pi(\beta_\gamma, \sigma_\gamma^2)\pi(\gamma) d\gamma d\beta_\gamma d\sigma_\gamma^2} \propto \left\{ \max_{\omega \in \mathcal{W}(\gamma, \beta_\gamma, \sigma_\gamma^2)} \prod_{i=1}^{n} \omega_i \right\} \pi(\beta_\gamma, \sigma_\gamma^2)\pi(\gamma).$$

(24)

Sampling from a BayesEL posterior is difficult when the dimension of the parameter space can vary between iterations. In fully parametric setups reversible jump Markov chain Monte Carlo (RJMCMC) samplers (Green 1995) are generally used in similar situations. Many efficient RJMCMC procedures have been studied (see Fan & Sisson (2010), Dellaportas et al. (2002), Robert et al. (2002)) in such settings. For BayesEL model selection sampling from the posterior in (24) could be more challenging. Under different models, the posterior supports could be quite different. Proposing parameter values in the new marginal supports specially in a cross-model move would not be easy. The HMC procedure in Chaudhuri et al. (2016) cannot be applied here. The model space is discrete and none of the estimating equations could be a smooth functions of the models as parameters. However, our two-step Metropolis Hastings algorithm can be extended to an efficient reversible jump Markov chain Monte Carlo procedure.

6. Two-Step Reversible Jump Markov Chain Monte Carlo for BayesEL Linear Model Selection

A reversible jump Markov chain Monte Carlo (RJMCMC) algorithm has two stages. First, it updates parameters within a model. Second, it updates parameters from one model to another. This stage requires a cross-model proposal that can propose values, which are more likely to be accepted, in the new model. The main problem is therefore to construct such cross-model proposal on which the efficiency of the RJMCMC algorithm depends.

Suppose the dimension of model is defined by $\sum_{i=1}^{n} \gamma_i$. Let the current model be $(\gamma, \beta_\gamma, \sigma_\gamma^2)$ with dimension $k$ and we want to propose a jump to a model $(\gamma', \beta_{\gamma'}, \sigma_{\gamma'}^2)$ with dimension $k' = k + 1$, where the binary vector $\gamma' - \gamma$ has 0 at all components except the $j$th one (i.e. the $j$th covariate is added to the model). The cross-model proposal is constructed in two steps. First, for the purpose of dimensional matching, a random variable $u$ is generated from a proposal distribution $q(u)$. This ensures $\text{dim}(\beta_\gamma) + \text{dim}(u) = \text{dim}(\beta_{\gamma'})$. Second, we map

$$(\beta_{\gamma', -j}, \beta_{\gamma', j}, \sigma_{\gamma'}^2) = g_{\gamma \rightarrow \gamma'}(\beta_\gamma, u, \sigma_\gamma^2)$$

$$= (\beta_\gamma + (\hat{\beta}_{\gamma', -j} - \hat{\beta}_\gamma), u + \hat{\beta}_{\gamma', j}, \sigma_\gamma^2 + (\tilde{\sigma}_{\gamma'}^2 - \tilde{\sigma}_\gamma^2)).$$

(25)
obtained by only imposing the score constraints in (20) on the weights defining the empirical likelihood. Note that, one can view the parameter for the model $g$ can be proposed based on either its current value or the ordinary least squares estimator of the regression Jacobian in Theorem 3. A new value of $\hat{\beta}_\gamma$ is then proposed by directly applying Theorem 1 to obtain the MCELE $\hat{\sigma}_\gamma^2$ and the maximised weights appropriately. Note that, by construction, $\sigma_\gamma^2$ is a function of both $\beta_\gamma$ and $u$.

With these values of $(\gamma', \beta_\gamma', \sigma_\gamma^2)$, using (24), we compute the BayesEL posterior $\Pi_{EL}(\gamma, \beta_\gamma, \sigma_\gamma^2 | y, x)$ and accept this proposed state with probability

\[
\alpha = \min \left\{ 1, \frac{\Pi_{EL}(\gamma', \beta_\gamma', \sigma_\gamma^2 | y, x) q_\gamma(\gamma' | \gamma) q_\gamma(\gamma') q_\gamma(u)}{\Pi_{EL}(\gamma, \beta_\gamma, \sigma_\gamma^2 | y, x) q_\gamma(\gamma | \gamma') q_\gamma(u)} \frac{\partial (\beta_\gamma', \sigma_\gamma^2)}{\partial (\beta_\gamma, \sigma_\gamma^2)} \right\}
\]

where $q_\gamma(\gamma' | \gamma)$ is the probability of proposing the jump from model $\gamma$ to model $\gamma'$ and the last factor in (26) is the determinant of the Jacobian of the transformation from $(\beta_\gamma, u, \sigma_\gamma^2)$ to $(\beta_\gamma', \sigma_\gamma^2)$.

In order to jump from a $\gamma$ of dimension $k$ to a model $\gamma'$ of dimension $k' = k - 1$, we note that the transformation $g_{\gamma \rightarrow \gamma'}$ is injective, so its inverse can be used to map the parameters. The new model and the parameters are then accepted with probability equal to the minimum of 1 and $\alpha^{-1}$.

By construction it follows that $g_{\gamma \rightarrow \gamma'}$ is a linear mapping and the determinant of Jacobian in (26) does not depend on the value of the parameters.

**Theorem 3.** Consider the one-to-one mapping $g_{\gamma \rightarrow \gamma'}$ as defined in equation (25). Then the determinant of Jacobian in (26) is given by:

\[
\left| \frac{\partial (\beta_\gamma', \sigma_\gamma^2)}{\partial (\beta_\gamma, \sigma_\gamma^2)} \right| = 1.
\]

The proposed RJMCMC algorithm is described more formally in Algorithm 2. Both stages of the algorithm are extensions of the two-stage Metropolis Hastings described in Section 4.

In the first stage, where we update the parameters within a model $\gamma$, we first propose a new value of $\beta_\gamma$. A new value of $\sigma_\gamma^2$ is then proposed by directly applying Theorem 1 to obtain the MCELE $\hat{\sigma}_\gamma^2$ and generating a random value from a pre-specified proposal distribution depending on this MCELE. The new value of $\beta_\gamma$ can be proposed based on either its current value or the ordinary least squares estimator of the regression parameter for the model $\gamma$ i.e. $\hat{\beta}_\gamma$. In Algorithm 2 (see Figure 3) we use the current value of $\beta_\gamma$ for this purpose. Note that, one can view $\hat{\beta}_\gamma$ as a MCELE given the model $\gamma$ as well. This MCELE however, is obtained by only imposing the score constraints in (20) on the weights defining the empirical likelihood.
The second stage of the algorithm, the injective mapping between the parameters of two models $\gamma$ and $\gamma'$ depends on the above MCELEs of the respective models. It should be noted that, for any index $j$ such that $\gamma_j = \gamma'_j = 1$, the relation, $\beta_{\gamma,j} - \hat{\beta}_{\gamma,j} = \beta_{\gamma',j} - \hat{\beta}_{\gamma',j}$ holds. Furthermore, by construction, $\sigma^2_{\gamma,j} - \hat{\sigma}^2_{\gamma,j} = \sigma^2_{\gamma',j} - \hat{\sigma}^2_{\gamma',j}$. Such constructions are intentional. They ensure that the proposed parameters in the two models are equidistant from their MCELEs. Therefore, the values of BayesEL posteriors under the two models would be close. Thus even though supports of $\Pi_{EL}(\gamma, \beta_{\gamma}, \sigma^2_{\gamma} \mid y, x)$ and $\Pi_{EL}(\gamma', \beta_{\gamma'}, \sigma^2_{\gamma'} \mid y, x)$ may be quite different, the proposed model would have a better chance of getting accepted.
7. Illustrative Applications

7.1. Rat population growth data. In this section, we are going to apply our two-step Metropolis Hastings method on the rat population growth data studied in [Gelfand et al. (1990)]. In the reported study, mass of thirty rats each in an experimental and a control group as determined by their diet were measured on the 8th, 15th, 22nd and the 36th weeks. In this illustration, we consider the growth of the control group.

In [Gelfand et al. (1990)], each rat was assumed to have its own growth curve and a parametric hierarchical Gaussian Bayes model was postulated. We consider a similar model here. Specifically, suppose \( t_j \) is the \( j \)th week and \( y_{ij} \) is the mass of the \( j \)th rat in the \( j \)th week. The model for the mass of the rat is given by

\[
y_{ij} = \theta_{1,i} + \theta_{2,i}(t_j - \bar{t}) + \epsilon_{ij}, \quad (i = 1, \ldots, 30; j = 1, \ldots, 5),
\]

where \( \bar{t} = 22 \). The errors \( \epsilon \) is not assumed to be normally distributed. We use empirical likelihood as a likelihood for the data.

The priors on the parameters are specified hierarchically. In particular, it is assumed that \( \theta_{1,i} \mid (\theta_{1c}, \sigma_1^2) \sim N(\theta_{1c}, \sigma_1^2); \theta_{2,i} \mid (\theta_{2c}, \sigma_2^2) \sim N(\theta_{2c}, \sigma_2^2); \sigma_1^2 \sim IG(5/2, 10/2); \theta_{1c} \sim N(0, 100^2); \theta_{2c} \sim N(0, 100^2); \sigma_2^2 \sim IG(5/2, 10/2) \). Note that, here we assume that the slope and intercept of the individual rats are uncorrelated. This is different from [Gelfand et al. (1990)], who assign a Wishart prior on them. However (see Win bugs Example vol. 1 and 2) this simplification makes little difference in practice.

Now suppose \( \omega_{ij} \) be the weight in the empirical likelihood for the mass of the \( j \)th rat in \( j \)th week. Let \( \theta_1 = (\theta_{1,1}, \ldots, \theta_{1,30}), \theta_2 = (\theta_{2,1}, \ldots, \theta_{2,30}) \). The set of feasible weights in our empirical likelihood based formulation is given by

\[
W \left( \theta_1, \theta_2, \sigma_e^2 \right) = \bigcap_{i=1}^{30} \left\{ \omega : \sum_{j=1}^{5} \omega_{ij} \left\{ y_{ij} - \theta_{1,i} - \theta_{2,i}(t_j - \bar{t}) \right\} = 0, \sum_{j=1}^{5} \omega_{ij} t_j \left\{ y_{ij} - \theta_{1,i} - \theta_{2,i}(t_j - \bar{t}) \right\} = 0 \right\}
\]

(27)

\[
\bigcap_{i=1}^{30} \bigg\{ \sum_{j=1}^{5} \omega_{ij} \left[ \left( y_{ij} - \theta_{1,i} - \theta_{2,i}(t_j - \bar{t}) \right)^2 - \sigma_e^2 \right] = 0 \bigg\} \bigcap \Delta_{149}.
\]

The empirical likelihood \( \mathcal{L}(\alpha, \beta, \sigma_e^2) \) is given by

\[
\mathcal{L}(\theta_1, \theta_2, \sigma_e^2) = \arg\max_{\omega \in W(\theta_1, \theta_2, \sigma_e^2)} \prod_{i=1}^{30} \prod_{j=1}^{5} \omega_{ij}.
\]

(28)

\( \mathcal{L}(\theta_1, \theta_2, \sigma_e^2) = 0 \) if the problem is infeasible. The BayesEL posterior can be defined in the same way as in (11).

In (27), there are 61 constraints for 61 parameters. Each rat has more than two constraints with only five observations. Due to the constraints there could be a big change in the marginal support of the posterior of \( \sigma_e^2 \) for a slight change in the values of \( \theta_1 \) and \( \theta_2 \). That is the support of the BayesEL posterior would be non-convex. Designing an efficient simple random walk would by no means be easy. Most likely, such a procedure would have a very low acceptance rate. Other MCMC methods like parallel tempering may work, but it is expected to take a long time to converge.

Our two-step method is a perfect solution for this case. Unlike simple random walk, our method ensures the proposed value for \( \sigma_e^2 \) to be in its marginal support when \( \theta_1 \) and \( \theta_2 \) are updated. The resulting chain could converge faster than parallel tempering.

Further acceleration is possible in this case. Conditional on the parameters \( \theta_1, \theta_2 \) and \( \sigma_e^2 \), the hyper-parameters \( \theta_{1c}, \theta_{2c}, \sigma_1^2 \) and \( \sigma_2^2 \) do not depend on the data or the likelihood and have analytic conditional posteriors. Thus, one can compute the their full conditional distributions and use Gibbs sampling to directly update \( \theta_{1c}, \theta_{2c}, \sigma_1^2 \) and \( \sigma_2^2 \). Thus we end up using Metropolis within Gibbs ([Givens & Hoeting 2003]) procedure, where the hyperparameters are sampled using Gibbs sampling while the proposed two-step Metropolis Hastings are used to sample the rest.
**Table 1.** Posterior means, standard deviations, 2.5% quantile, median and 97.5% quantile of $\theta_0$, $\theta_{2c}$ and $\sigma_\epsilon$ sampled from the BayesEL posterior by two-step Metropolis Hastings (TMH) and full parametric formulation using WinBug.

|       | Mean  | SD    | 2.5%  | Median | 97.5% |
|-------|-------|-------|-------|--------|-------|
|       | TMH   | WB    | TMH   | WB     | TMH   | WB    | TMH   | WB    |
| $\theta_0$ | 106.9 | 106.6 | 3.604 | 3.655  | 99.80 | 99.44 | 106.1 | 106.5 |
| $\theta_{2c}$ | 6.190 | 6.185 | 0.106 | 0.106  | 5.975 | 5.975 | 6.183 | 6.185 |
| $\sigma_\epsilon$ | 4.251 | 6.136 | 0.318 | 0.478  | 3.676 | 5.283 | 4.231 | 6.100 |

In our implementation of the two-step procedure, each $\theta_{1,i}$ and $\theta_{2,i}$, were proposed from a Gaussian distribution with their current value as the mean, 0.3 and 0.03 as standard deviation respectively. For the proposal distribution of $\sigma_\epsilon^2$, we use truncated normal with its MCELE as the mean and 5 as the standard deviation. These choices lead to a respectable acceptance rate with a good exploration of the support. A sample of size 150,000 with 50,000 as burn-in were drawn from the posteriors. Convergences of all the chains were tested using Heidelberger & Welch (1983) diagnostic.

We compare the posteriors of $\theta_0 = \theta_{1c} - \theta_{2c}t$, $\theta_{2c}$ and $\sigma_\epsilon$ obtained from the proposed two-step Metropolis Hastings method with those obtained from the bivariate hierarchical Gaussian model used in Gelfand et al. (1990). The posterior of the latter was obtained using Gibbs sampling (see WinBugs Example vol. 2). The mean, standard deviation and three quartiles of the posterior of the hyperparameters are presented in Table 1. It appears that there is little difference between the posteriors of $\theta_0$ and $\theta_{2c}$ obtained from the two models. The posterior of $\sigma_\epsilon$ in the bivariate hierarchical Gaussian model stochastically dominates the one from the BayesEL formulation. The posterior from the proposed method also matches closely with the one obtained using the HMC method (Chaudhuri et al., 2016, see supplement). However, our method is much simpler to use than the latter.

### 7.2 Gene Expression Data

We now turn to BayesEL model selection and present an example in Graphical Markov model selection. Expression of 40 genes in the MVA and MEP pathways of *Arabidopsis thaliana* were collected from 118 microarray assays. Among them the pathway structure between 13 genes on the MEP pathway was studied by Wille et al. (2004). They used a modified Gaussian graphical modelling approach to select the interaction structure. Drton & Perlman (2007) employed a multiple testing based graphical model selection to select a directed acyclic graph among the genes. However, they also assumed that the expressions are normally distributed. In this example we examine the same data-set, but use our BayesEL approach for model selection and employ the proposed RJMCMC algorithm to sample from the resulting posterior.

Any viable strategy of directed acyclic graph selection requires one to first specify an order among the variables. The selection procedure primarily chooses the parents (i.e. the parenthood) of a particular node from the nodes which precedes it in the ordering. Justification for this procedure can be derived from the theory of Graphical Markov models for directed acyclic graphs (Lauritzen, 1996). Furthermore, it can be shown that the joint data likelihood factors according to the graph, which means each that the parents of each node can be selected independent of the others.

In this example, we assume the same order in the genes as in Drton & Perlman (2007). Like them, we also assume that the first three nodes don’t have any parents. So we select parenthoods of rest of the ten nodes.

For $k \in \{4, 5, \ldots, 13\}$ suppose $g_k$ denotes the $k^{th}$ gene. Given $g_k$, the model $\gamma_{(k)} = (\gamma_1, \ldots, \gamma_{k-1})$, is fit to the gene. Here $\gamma_i$ is 1 when the gene $g_i$ is in the model and 0 otherwise. Let $g_{(k)}$ be the matrix of
covariates with columns \( g_i \) such that \( \gamma(k) = 1, \beta_{\gamma(k)} \) be the corresponding vector of regression coefficients. As in (17), the model \( \gamma(k) \) for \( g_k \) in terms of its parents is then given by:

\[
E \left[ g_k \mid g_{\gamma(k)} \right] = g_{\gamma(k)} \beta_{\gamma(k)} \quad \text{and} \quad \text{Var} \left[ g_k \mid g_{\gamma(k)} \right] = \sigma_{\gamma(k)}^2,
\]

We assume a double exponential \((0, \lambda)\) and an inverse gamma \((0.1, 0.1)\) prior for \( \beta_{\gamma(k)} \) and \( \sigma_{\gamma(k)}^2 \) respectively. The hyper-parameter \( \lambda \) is assumed to follow an inverse gamma \((5, 5)\) prior. Our choice of double exponential priors for the regression parameters mimics the \( L_1 \) penalisation in LASSO (Tibshirani, 1994). The parameter \( \lambda \) controls the stringency of this penalty. In our setup, the amount of shrinkage differs from one parenthood to another. Each \( \gamma_i \) is assumed to be a Bernoulli random variable, where the success probability follows a Beta \((2, 7)\) distribution. Such priors would prefer sparser models.

The constraints imposed on (18) to compute the empirical likelihood of the parenthood of the \( k \)th gene can be easily specified from equations (19), (20), (21), (22) and (23). The observation for the \( k \)th gene is given by:

\[
\mathcal{W} \left( \gamma(k), \beta_{\gamma(k)}, \sigma_{\gamma(k)}^2 \right) = \mathcal{W}_0 \left( \beta_{\gamma(k)} \right) \cap \mathcal{W}_{\gamma(k)} \left( \beta_{\gamma(k)} \right) \cap \mathcal{W}_{\gamma_{\gamma(k)}} \left( \beta_{\gamma(k)} \right) \cap \mathcal{W}_{\sigma} \left( \beta_{\gamma(k)}, \sigma_{\gamma(k)}^2 \right).
\]

Now given \( \gamma(k), \beta_{\gamma(k)}, \sigma_{\gamma(k)}^2 \) the empirical likelihood is given by

\[
\mathcal{L}(\gamma(k), \beta_{\gamma(k)}, \sigma_{\gamma(k)}^2) = \max_{\omega \in \mathcal{W}(\gamma(k), \beta_{\gamma(k)}, \sigma_{\gamma(k)}^2)} \prod_{i=1}^{n} \omega_{\gamma_i}.
\]

The likelihood is zero if the maximisation problem is infeasible. From the likelihood the BayesEL posterior for \( \gamma(k), \beta_{\gamma(k)}, \sigma_{\gamma(k)}^2 \) can be computed.

Given model \( \gamma(k) \), the proposal distribution for \( \beta_{\gamma(k)} \) is Gaussian distribution with its current value as mean and 0.03 as standard deviation. For proposing \( \sigma_{\gamma(k)}^2 \), truncated normal with its MCELE as mean and 1 as standard deviation is used. In the case that a covariate is added, the value of \( \beta_{\gamma(k)} \) is proposed from \( \mathcal{N}(0, 0.0025) \). With these proposals, samples of size 150, 000 are drawn from the derived BayesEL posterior and the first one third is discarded as burn-in.

The parenthood of each node is taken to be the model with highest in the above sample. The selected directed acyclic graph is shown in Figure 4. We compare our results (Figure 4(b)) with the graph selected by Drton & Perlman (2007) (Figure 4(a)), using a step-down Sidak procedure where the family-wise error rate (FWER) is controlled at 0.1. The step-down method chooses a graph with 19 edges, which is sparser than the graph we obtained with 27 edges. The structure of metabolic network obtained from the two approaches are similar. It can be seen that, the parenthoods of DXR, CMK, IPPI1 and PPDS1, are same in for both methods. All but one arrows selected by the step-down method are selected by our proposed method as well.

The proposed RJMCMC sampler appears to move between the models well. The acceptances rates of the cross model moves were quite high for all nodes. In fact, for all but two nodes, these rates were higher than 10%. For some it was even higher than 20%.

8. DISCUSSION

In this article we present a novel method of sampling from a BayesEL posterior with a possible non-convex support. In a BayesEL procedure, instead of specifying a parametric likelihood of the data, one uses a likelihood obtained from a constrained estimate of the distribution function. The constraints depend on the model and its parameters. A non-convex support often makes sampling from a BayesEL posterior hard,
which has prevented its wider use in statistical analysis. The proposed method can be described as a two-step Metropolis Hastings algorithm, where new values of an appropriate subset of parameters are proposed first. Next, using these values and the estimating equations we compute a maximum conditional empirical likelihood estimator of the rest of the parameters. New values of these parameters are then proposed close to their maximum conditional empirical likelihood estimates. The proposed method does not require any smoothness of the estimating equations and can be used for non smooth and even discontinuous estimating functions. We show that the proposed method can easily be extended to an appropriate reversible jump Markov chain Monte Carlo which would allow efficient implementation of Bayesian model selection using BayesEL procedure. As far as we know, this is the first implementation of RJMCMC procedure on BayesEL posteriors. Without the proposed two step Metropolis Hastings method, it would be almost impossible to use empirical likelihood in Bayesian model selection problems.

Assumptions made in the article can potentially be relaxed. First, the assumption that equation (14) has a unique solution is not required in many situations. In presence of multiple solutions, our method would work as described, if one of the solutions could be chosen deterministically. This would apply to the proposed RJMCMC when fewer samples than the number of covariates are available. The ordinary least squared estimator can be deterministically specified by a specified choice of generalised inverse. If all the solutions can be computed either analytically or numerically. We can even proceed by randomly choosing one of the solutions as our MCELE. However in this case, the transition probability would depend on all solutions. The problem intensifies if the solutions of (14) could only be found numerically. The numerical algorithms are deterministic, however their fixed points depend on the initial values, which are usually chosen randomly. If all other solutions are ignored, it is not clear if the resulting Markov chain would converge to the correct posterior.

**Figure 4.** Directed acyclic graphic selected by (a) the step-down Sidak procedure when family-wise error rate is controlled at 0.1 and (b) by two-step reversible jump MCMC with Bayesian empirical likelihood.
In Section 3 we use the structure of the estimating equations to propose values of one subset of parameters (i.e. $\theta_2$) based on the proposed values of its complement (i.e. $\theta_1$). The opposite that is proposing a value of $\theta_1$ based on the proposed value of $\theta_2$ can also be done. However, Theorem 1 will no longer be applicable. Direct maximisation in (12) would be required to obtain the MCELE. This MCELE however may not be unique.

Finally, it should be noted that the structure of the estimating equations in (5) have been made merely for convenience and can be easily relaxed. If both $g$ and $h$ depend on $\theta_1$ and $\theta_2$, the proposed method could be applied. Theorem 1 will not apply to this case either. The problem posed by possible multiple solutions could also be quite prominent in this case as well.

APPENDIX A. PROOFS

Proof of Theorem 2 Since $\hat{\nu}(a)$ maximises over $\mathcal{W}_G(a)$ and $\hat{\nu}(a) \in \mathcal{W}(a, \hat{\theta}_2)$, by the uniqueness of the solution, we get $\hat{\omega}(a, \hat{\theta}_2) = \hat{\nu}(a)$ and $L(a, \hat{\theta}_2) = \prod_{i=1}^{n} \hat{\nu}_i(a)$. By definition of $\hat{\theta}_2^{EL}(\theta_1)$ it follows that,

$$\prod_{i=1}^{n} \hat{\nu}_i(a) = L(a, \hat{\theta}_2) \leq L(a, \hat{\theta}_2^{EL}(\theta_1)) = \prod_{i=1}^{n} \hat{\omega}_i(a, \hat{\theta}_2^{EL}(\theta_1)) \quad (say).$$

However, $\hat{\omega}(a, \hat{\theta}_2^{EL}(\theta_1)) \in \mathcal{W}_G(a)$. That is

$$\prod_{i=1}^{n} \hat{\omega}_i(a, \hat{\theta}_2^{EL}(\theta_1)) = L(a, \hat{\theta}_2^{EL}(\theta_1)) \leq \prod_{i=1}^{n} \hat{\nu}_i(a) = L(a, \hat{\theta}_2).$$

The theorem statement holds. $\square$

Proof of Theorem 2 From $\theta_1^{(t+1)} \perp \perp \left(\theta_1^{(1:t-1)}, \theta_2^{(1:t)}\right) | \theta_1^{(t)}$, using the properties of the conditional independence (Dawid, 1979), it follows that

$$\theta_1^{(t+1)} \perp \perp \left(\theta_1^{(1:t-1)}, \theta_2^{(1:t-1)}\right) | \left(\theta_1^{(t)}, \theta_2^{(t)}\right).$$

From $\theta_2^{(t+1)} \perp \perp \left(\theta_1^{(1:t)}, \theta_1^{(1:t)}\right) | \theta_1^{(t+1)}$, similarly, it follows that

$$\theta_2^{(t+1)} \perp \perp \left(\theta_1^{(1:t-1)}, \theta_1^{(1:t-1)}\right) | \left(\theta_1^{(t)}, \theta_2^{(t)}\right).$$

From (31) and (30), using the conditional independence again, we get

$$\left(\theta_1^{(1:t)}, \theta_2^{(1:t)}\right) \perp \perp \left(\theta_1^{(1:t-1)}, \theta_2^{(1:t-1)}\right) | \left(\theta_1^{(t)}, \theta_2^{(t)}\right).$$

Thus $\left(\theta_1^{(1:L)}, \theta_2^{(1:L)}\right)$ is a Markov chain.

The transition probability is

$$P\left(\theta_1^{(t+1)}, \theta_2^{(t+1)} | \theta_1^{(t)}, \theta_2^{(t)}\right) = P\left(\theta_2^{(t+1)} | \theta_1^{(t+1)}, \theta_2^{(t)}\right) P\left(\theta_1^{(t+1)} | \theta_1^{(t)}, \theta_2^{(t)}\right).$$

By construction, $\theta_1^{(t+1)} \perp \perp \theta_2^{(t)} | \theta_1^{(t)}$. Thus $P\left(\theta_1^{(t+1)} | \theta_1^{(t)}, \theta_2^{(t)}\right) = P\left(\theta_1^{(t+1)} | \theta_1^{(t)}\right)$. Furthermore

$$P\left(\theta_2^{(t+1)} | \theta_1^{(t+1)}, \theta_1^{(t)}, \theta_2^{(t)}\right) = P\left(\theta_2^{(t+1)} | \theta_1^{(t+1)}\right) = \int P\left(\theta_2^{(t+1)}, \tilde{\theta}_2 | \theta_1^{(t+1)}\right) d\tilde{\theta}_2$$

$$= \int P\left(\theta_2^{(t+1)} | \tilde{\theta}_2, \theta_1^{(t+1)}\right) P\left(\tilde{\theta}_2 | \theta_1^{(t+1)}\right) d\tilde{\theta}_2 = P\left(\theta_2^{(t+1)} | \theta_1^{(t+1)}, \hat{\theta}_2^{EL} (\theta_1^{(t+1)})\right).$$
The last equality holds since \( P\left(\tilde{\theta}_2 \mid \theta_1^{(t+1)}\right) = 1 \) if \( \tilde{\theta}_2 = \hat{\theta}^{EL}_2\left(\theta_1^{(t+1)}\right) \) and 0 otherwise. Therefore
\[
P\left(\theta_1^{(t+1)}, \theta_2^{(t+1)} \mid \theta_1^{(t)}, \theta_2^{(t)}\right) = P\left(\theta_1^{(t+1)} \mid \theta_1^{(t)}\right) P\left(\theta_2^{(t+1)} \mid \theta_1^{(t+1)}, \hat{\theta}^{EL}_2\left(\theta_1^{(t+1)}\right)\right).
\]

Proof of Theorem 3. By construction \( \beta_{\gamma'} \) does not depend on \( \sigma_\gamma^2 \). However, the later depends on \( \beta_\gamma \) and \( u \) through \( \hat{\sigma}_\gamma^2 \). Thus,
\[
\begin{vmatrix}
\frac{\partial (\beta_{\gamma'}, \sigma_\gamma^2)}{\partial (\beta_\gamma, u, \sigma_\gamma^2)} & \frac{\partial g_{\gamma \rightarrow \gamma'}(\beta_\gamma, u, \sigma_\gamma^2)}{\partial (\beta_\gamma, u, \sigma_\gamma^2)} \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{\partial \hat{\sigma}_\gamma^2}{\partial \beta_\gamma} & \frac{\partial \hat{\sigma}_\gamma^2}{\partial u} & 1
\end{vmatrix} = 1.
\]

\[\square\]

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