Simple nanomagnets execute limit cycle trajectories under ferromagnetic resonance conditions

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Abstract

Nanomagnetic particles respond sensitively and nonlinearly to electromagnetic radiation. Many excitation schemes are now well known. However, nonlinear dynamics determinations have not been examined in detail under FMR conditions. The nonlinear dynamics of a simple nanomagnet is studied and the excitation field, $H_1$, is varied. We solve numerically the Landau–Lifshitz nonlinear dynamics of $M(t)$. There is a special focus on the spherical degrees of freedom: $\theta(t), \phi(t)$. We find that the $\theta(t)$ trajectories converge asymptotically to $\theta_{\text{sym}} = \text{constant}$, while $\phi(t)$ is a linear function of time. The combined dynamics of $\theta'(t)$ and $\phi'(t)$ produce a limit cycle for each value of $H_1$. The systematic numerical calculations and analysis show that the limit cycle $\theta_{\text{sym}}$ follows a fourth-degree polynomial on $H_1$ and an inverse law on frequency $v_1$. It is also found that the limit cycles are established after 12–20 nanoseconds. They cause $M(t)$ to sweep a constant precession cone that lasts for more than 200 ns independent of initial conditions. These results bring significant novel knowledge for fast information technology.

1. Introduction

Fundamental studies of the spin dynamics and the development of nanomagnetic materials and arrays are of great relevance in the present time due to the general trend towards miniaturization of all sort of electronic devices [1, 2], GHz wireless communication, the development of spintronic devices [3, 4], and the potential development of ever faster magnetic memories with increased storage capacities [5, 6]. A better understanding of the full dynamics and the time development of the magnetization, $M(t)$, of individual and structured nanomagnets under different excitation schemes [7–9], and of magnetic stripes and thin films [10–12], is desired in order to develop new nanomagnetic functions and devices. For example there are complete studies of nonlinear response and synchronized magnetization dynamics of spin-transfer nano-oscillators under spin current and weak microwave fields [13–15]. Simpler nanomagnets without demagnetizing fields, anisotropies or thermal fields are also good potential candidates for applications. Here, we focus on the nonlinear dynamics of the magnetization as given by the Landau–Lifshitz equation of motion [9, 16, 17] of an isolated, isotropic nanomagnet, with the magnetic structure of monodomain under excitation conditions of ferromagnetic resonance-FMR. This is to say, a Zeeman magnetic field, $H_0$, and a periodically varying microwave field, $H_1$, are applied orthogonally to each other. The Landau–Lifshitz, nonlinear equation of motion–LL for $M(t)$ has not been analytically solved for any general conditions of excitation, and/or anisotropies [9, 18]. It has been analytically solved in a few simple instances under excitation conditions far from ours. [19, 20]. In this work, the Landau–Lifshitz equation of motion is solved numerically, while Kittel’s resonance condition holds [17, 21]. The thermal field effects are not considered in this model.
The nonlinear dynamics of \( \mathbf{M}(t) \) under FMR conditions are only sketched in standard FMR literature. The focus is on the resonance phenomenon \([17, 21]\). In the linearized-approximation FMR-treatment, \( \mathbf{M}(t) \) precession is supposed to be uniform and stable as long as the system absorbs the microwave energy from the excitation field, \( \mathbf{H}_e(t) \). Since \( \mathbf{H}_e \) is very small (perturbative) compared to the Zeeman field, \( \mathbf{H}_0 \), it is usually stated that the angle of aperture, \( \theta_{\text{FMR}} \), of the precession cone is small, \( \sim 3^\circ \). But it is not known how the magnetization reaches such state; and within itself, what the effect of \( \mathbf{H}_e(2\pi f t) \) in the motion of \( \mathbf{M}(t) \) is. The time scales of these processes are of the order of nanoseconds, but more precise knowledge is lacking. So, the picture of the dynamics of \( \mathbf{M}(t) \) that emerges from these treatments is broad and approximated. Our detailed numerical calculations allow finding dynamic-asymptotic states of \( \mathbf{M}(t) \), which are limit cycles denoted by \( \theta_{\text{asy}} \). \( \theta(t) \) trajectories converge asymptotically to \( \theta_{\text{asy}} = \) constant, while \( \phi(t) \) is a linear function of time. \( \mathbf{H}_1 \), the amplitude of the excitation, is the cause of the existence of the limit cycles. \( \theta_{\text{asy}} \) depends non-linearly on \( \mathbf{H}_1 \) and varies as a power law with the excitation frequency. The limit cycles are stable, circular, with periods in the picosecond regime, and are established non-instantaneously. \( \mathbf{M}(t) \) in the limit cycle state of motion sweeps a constant precession cone with the tip of \( \mathbf{M}(t) \) contained always on the \([\mathbf{M}]\)-sphere, resembling the simplified linear FMR-precession. All these results inform us of a very rich non-linear dynamics of the magnetization while executing ferromagnetic resonance, being the non-instantaneous development of the limit cycle of \( \mathbf{M}(t) \) (after many nanoseconds) perhaps the most significant feature found. Some of these complex and rich nonlinear dynamics could be harvested in order to design new functions and operations in all areas of modern technology.

1.1. Theory of magnetization vector dynamics

The non-linear dynamics of the magnetization, \( \mathbf{M}(t) \), are given by the Landau–Lifshitz equation of motion and damping is included in the right-most term in equation (1), below \([9, 17, 19, 20]\):

\[
\frac{d\mathbf{M}}{dt} = -\gamma \mathbf{M} \times \mathbf{H}_f - \left( \frac{\alpha \gamma}{M} \right) \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_T)
\]  

(1)

Where \( \alpha \) is the damping constant, \( \gamma = (e/2mc)g \) is the gyromagnetic ratio; \( e \) is the electron charge; \( m \) is the electron mass; \( c \) is the light velocity; \( g \) is the gyromagnetic factor; \( \mathbf{M} \) is the vector of magnetization; and \( \mathbf{H}_T \) is the total magnetic field applied to the system. Notice that \( \mathbf{H}_f \) will apply as many torques to \( \mathbf{M} \) as components that it has, provided these components are not parallel to \( \mathbf{M} \).

A fundamental property of the Landau–Lifshitz equation of motion, equation (1), is the conservation of the magnitude of the magnetization vector, \( M = |\mathbf{M}(t)| \), i.e.; \( \mathbf{M} \) is a constant of motion. This brings the important consequence that the tip of the vector magnetization is always constrained to move, at all times, on the surface of a sphere of radius \( M \). This geometric result will be of great help in describing the limit cycles found.

In order to see how this \( M = |\mathbf{M}(t)| \) = constant property arises, we construct the dot product of \( \mathbf{M} \) with its velocity \( d\mathbf{M}/dt \) as given in the equation of motion (1), i.e.,

\[
\mathbf{M} \cdot \frac{d\mathbf{M}}{dt} = -\gamma \mathbf{M} \cdot (\mathbf{M} \times \mathbf{H}_f) - \left( \frac{\alpha \gamma}{M} \right) \mathbf{M} \cdot (\mathbf{M} \times (\mathbf{M} \times \mathbf{H}_T)) = 0.
\]

Recalling that \( \mathbf{M} \) is always perpendicular to the vector \( \mathbf{M} \times \mathbf{H}_f \) then, in general, \( \mathbf{M} \cdot (d\mathbf{M}/dt) = 0 \). Irrespective of the instantaneous direction that takes \( \mathbf{M} \times \mathbf{H}_f \) and \( \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_T) \), \( \mathbf{M} \) is always perpendicular to the corresponding torques.

Since \( M^2 = |\mathbf{M}|^2 = \mathbf{M} \cdot \mathbf{M} \), taking its derivative yields:

\[
d|M|^2/\text{dt} = d(\mathbf{M} \cdot \mathbf{M})/\text{dt} = 2\mathbf{M} \cdot (d\mathbf{M}/dt) = 0.
\]

Hence, \( M^2 \) = constant and so, \( M \) is a constant of motion. In addition to the fact that the tip of the magnetization vector lies always on the surface of a sphere of radius \( M \), this result has also the important consequence that only two angular degrees of freedom (sweeping the surface of a sphere) are necessary to describe completely the vector magnetization, \( \mathbf{M} \), and its three components, \( M_x \), \( M_y \), \( M_z \). The spherical coordinate angles \( \theta \) and \( \phi \) are the most natural choice and relate to \( \mathbf{M} \) as \( M_x(t) = M \sin \theta(t) \cos \phi(t) \), \( M_y(t) = M \sin \theta(t) \sin \phi(t) \), and \( M_z(t) = M \cos \theta(t) \).

1.2. Ferromagnetic resonance

The ferromagnetic resonance condition for any magnetic specimen, simple nanomagnets included, subjected to the following external and internal magnetic fields: a static field \( \mathbf{H}_0 \), an excitation microwave magnetic field, \( \mathbf{H}_e(\omega t) \), of small amplitude and usually orthogonal to \( \mathbf{H}_0 \), and an assortment of anisotropic fields (including demagnetizing fields), is given in general by Kittel’s expression \([17, 21]\):

\[
\omega_r = \gamma H_T
\]

(2)

Where \( H_T \) is a function of \( H_0 \), \( H_\text{dip} \), \( H_\text{D} \), \( H_\text{K} \), \( H_\text{K} \) i.e., \( H_T = f(H_0, H_\text{dip}, H_\text{D}, H_\text{K}) \). \( H_T \) contains the contributions of the external (\( H_0, H_\text{D} \)) and internal fields that the magnetic specimen, in the general shape of an ellipsoid of revolution, experiences. \( H_\text{D} \) represents the demagnetizing fields, \( H_\text{dip} \) and \( H_\text{K} \) the anisotropy fields experienced by the magnetic specimen. \( \omega_r \) is the angular frequency of the resonant absorption of microwave power and \( \gamma \) is again the gyromagnetic ratio.

The nonlinear dynamics of \( \mathbf{M}(t) \) is studied when both equations (1) and (2) are simultaneously satisfied.
For the nanomagnetic specimen, isotropic, isolated and monodomain, all the anisotropy and demagnetizing fields in \( \mathbf{H}_1 \) vanish, leaving only

\[
\mathbf{H}_1 = \mathbf{H}_0 + \mathbf{H}_1(\omega t)
\]

(3)

Just in resonant condition

\[
\omega_1 = \gamma H_1
\]

(4)

To say that our nanomagnet is under ferromagnetic resonance condition is to say that (3) and (4) hold simultaneously.

In the FMR experiment \( \mathbf{H}_1(t) = H_1 \cos(\omega_1 t)\mathbf{e}_r \) is linearly polarized and under easy control at the laboratory with \( H_1 \leq 1 \text{mT} \) and \( \nu_1 = \omega_1/2\pi \) usually taking any value from 2 to 33 GHz and sometimes even 64 up to ~150 GHz [22, 23]. This means that we fix \( \omega_1 \), the frequency of the small, oscillating, excitation field, \( \mathbf{H}_1 \), and \( \mathbf{H}_0 = H_0\mathbf{e}_z \), the external magnetic field (also known as the Zemann field), has to be varied to meet the equality in equation (4), so the resonance frequency is set to \( \omega_1 \), then \( \omega_1 = \omega_1 \), and \( H_0 \) is varied until it meets the value \( (\omega_1/\gamma) \). It is under these resonance conditions that the solutions for the Landau–Lifshitz equation of motion are sought.

Now, the Landau–Lifshitz equation of motion (1) is simply rewritten by using \( \omega = \omega_1 \), \( \mathbf{H}_0 \) and \( \mathbf{H}_1 \) into \( \mathbf{H}_1 \):

\[
\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times (\mathbf{H}_0 + \mathbf{H}_1(\omega t)) - \alpha\gamma^2 \mathbf{m} \times [\mathbf{M} \times (\mathbf{H}_0 + \mathbf{H}_1(\omega t))]\]

(5)

Defining the unitary vector magnetization \( \mathbf{m} = \mathbf{M}/M \), the LL equation of motion becomes:

\[
\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times (\mathbf{H}_0 + \mathbf{H}_1(\omega t)) - \alpha\gamma^2 \mathbf{m} \times [\mathbf{m} \times (\mathbf{H}_0 + \mathbf{H}_1(\omega t))]
\]

(6)

In terms of applied torques we have

\[
\frac{d\mathbf{m}}{dt} = -\tau_0 - \tau_1(\omega t) - \tau_{\text{damp}10} - \tau_{\text{damp}11}.
\]

Where the ‘free’ motion torques are

\[
\tau_0 = \gamma \mathbf{m} \times \mathbf{H}_0, \quad \text{dependent of time}, \quad \tau_1 = \gamma \mathbf{m} \times \mathbf{H}_1, \quad \text{dependent of time and oscillating at } \omega_1,
\]

and the damping torques are

\[
\tau_{\text{damp}10} = \alpha\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_0), \quad \text{dependent of time}, \quad \text{and}
\]

\[
\tau_{\text{damp}11} = \alpha\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_1).
\]

We observe that \( \tau_0/\tau_{\text{damp}10} = \alpha^{-1} \) and \( \tau_1/\tau_{\text{damp}11} = \alpha^{-1} \). The damping torques are smaller than the ‘free’ motion torques by a factor \( \alpha \), the strength of the damping. For nanomagnets with very small damping constant \( \alpha \), the dynamics lose energy and damp very slowly. Notice that the external oscillating torque \( \tau_1 \), in addition to contribute to the nonlinear dynamics, is injecting energy (\( h\nu_1 = g/H_1 \)) because it is the ‘source’ that provides \( \omega_1 = \nu_1/2\pi \), and from quantum mechanics, it provides the photons \( h\nu_1 \), for the FMR transition to occur. It should be noticed that the Landau–Lifshitz equation in the form (6) contains a first term that produces motion of \( \mathbf{M}(t) \) in more than one dimension, on the surface of a sphere, and also contains a term that produces damping (which implies dissipation of energy) in the motion of \( \mathbf{M}(t) \). To get advantage of the fact that \( M \) is a constant of motion, we transform the LL-equation of motion to spherical coordinates. Figure 1 shows the coordinate system, the magnetic vector fields applied, and the acting torques at an arbitrary instant of time: the first torque \( \tau_0 = -\gamma \mathbf{m} \times \mathbf{H}_0 \), going tangent to the red circle; the second torque \( \tau_1(\omega t) = -\gamma \mathbf{m} \times \mathbf{H}_1(\omega t) \), going perpendicular to the \( \mathbf{m} \times \mathbf{H}_1 \) plane in grey and alternating directions every half period of \( H_1 \); the first damping torque \( \tau_{\text{damp}10} = -\alpha\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_0) \), going tangent to the blue circle; the second damping torque \( -\alpha\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_1) \), going parallel to the orange plane \( \mathbf{m} \times \mathbf{H}_1 \) and alternating directions every half period of \( H_1 \).

The Landau–Lifshitz nonautonomous system of equations of motion in spherical coordinates becomes

\[
\theta'(t) = -\alpha\gamma H_0 \sin \theta + \gamma H_1 \cos \omega_1 t [\alpha \cos \theta \cos \phi - \sin \phi]
\]

(7)

\[
\phi'(t) = \gamma H_0 - (\gamma H_1 \cos \omega_1 t / \sin \theta) [\cos \theta \cos \phi + \alpha \sin \phi]
\]

(8)

2. Numerical calculations and experimental parameters

The system of equations (7) and (8) has no analytic solutions, for general cases, so, it is solved numerically and the time evolution of \( \theta(t), \phi(t), M_z = M \cos \theta(t), M_x = M \sin \theta(t) \cos \phi(t), M_y = M \sin \theta(t) \sin \phi(t), \)

\( M_z(t) = (M_x^2 + M_y^2)^{1/2} = M \sin \theta(t) \) are obtained, i.e., the nonlinear dynamics of

\( M(t) = M_z(t)\hat{x} + M_x(t)\hat{y} + M_y(t)\hat{z} \) are determined while the resonance condition of Kittel holds.

In order to solve the system of equations (7) and (8) the Wolfram Mathematics 8 is utilized [24] and its function NDSolve is used extensively. This command is amply used to solve different equations in other areas of research [24, 25]. The experimental parameters of damping \( \alpha = 0.003 \) and frequency of \( \nu_1 = 13.2 \text{ GHz} \) for Fe film are used in all the calculations, as given in [26]. The gyromagnetic ratio \( \gamma = 1.79 \times 10^{11} \text{ rad s}^{-1} \text{T}^{-1} (10.256 \text{ T}^{-1}) \) with \( g = 2.10 \) for Fe is used [21]. The amplitude of \( H_1 = 1 \text{ mT} \) and the frequency \( \nu_1 = \omega_1/2\pi = 13.2 \text{ GHz} \) are fixed, while the initial conditions are varied.
Figure 1. The free body diagram of the vector magnetization, M(t). The tip of the magnetization vector cannot leave the surface of the sphere of radius M. The spherical coordinates, $\theta$, $\phi$, the Zeeman field, $H_0$, and the excitation field, $H_1$, are indicated. Each field produces two torques: $\tau_{\text{Zeeman}}$, $\tau_{\text{dampH}_0}$, and $\tau_{\text{dampH}_1}$, respectively, as shown with the coloured arrows on the tip of the magnetization vector. Two torques produce ‘free’ motion and the other two torques damp these motions.

With the numerical solutions of $\theta(t)$ and $\phi(t)$ in (7)–(8), plots of $\theta(t)$ and $\phi(t)$ versus time are produced, with them, $M_x(t)$, $M_y(t)$, $M_z(t)$, and M(t) are constructed, as mentioned above. The strength and the frequency of the microwave excitation $H_1$ are varied in order to determine how $\theta(t)$ and $\phi(t)$ and M(t) respond to this control parameters.

3. Results and discussion

Figure 2(a) shows the calculated full trajectory of M(t), for 70 ns over the sphere of radius M for a resonant field $H_0=463.341 \text{mT}$, the other parameters are as defined above and initial conditions are $\theta_1=162^\circ$ and $\phi_1=0$. For a better appreciation of the evolution of the dynamics of M(t), the plotted time interval of 70 ns, is divided into four quarters (70/4 = 17.5 ns): black trace for the first quarter, blue trace for the second quarter, green for the third quarter and red for the fourth quarter. Note that the magnetization vector M sweeps – precesses over the whole of the sphere in just the first quarter, ~17.5 ns, then it moves tracing the blue spirals, then the green spirals to end up in the red circle when it reaches the fourth quarter.

Computer runs for more than 200 ns do not change the red circle. The motion is damped with a damping constant $\alpha$, yet, it does not end up at $\theta_{\text{asym}}=0^\circ$, completely aligned with $H_0$. Moreover, the magnetization vector reaches a stationary precession motion, which tip is the red circle, with a large aperture cone-angle. How come, the magnetization vector does not end up at $\theta_{\text{asym}}=0^\circ$? The microwave excitation field $H_1$ is injecting energy in the form of quanta, $h\nu_1=g\mu_BH_0$, with $\nu_1$ being the frequency of the microwave field. After one or two decades of nanosecond dynamics, energy reaches a balance, input compensates exactly the losses.

In Figure 2(b), the time behaviour of the polar angle $\theta(t)$ starting at $t=0$ is shown, initial conditions as: $\theta_1=3^\circ, 8^\circ, 18^\circ, 45^\circ, 60^\circ, 90^\circ, 135^\circ$ and $162^\circ$; and $\phi_1=0$.

The trajectory – precession of M(t) over the sphere, as shown in figure 2(a), is accompanied by changes of its polar angle $\theta$, as shown in figure 2(b). And linear progression of the precession angle $\phi(t)$, as shown in figure 2(c); $\theta(t)$ changes continuously during the first two quarters of time (black and blue) from its initial polar angle $\theta_1=162^\circ$, $\theta(t)$ decreases rapidly until it reaches a $\theta_{\text{MIN}}$, still in the first quarter or interval (black), then it increases slowly and continues to increase during the second quarter (blue), until $\theta(t)$ reaches asymptotically a final value, $\theta_{\text{asym}}=21^\circ$ during the third (green) and fourth (red) quarters. Meanwhile $\phi(t)$ advances linearly in time as $\phi(t) = (4752^\circ/\text{ns})t$, or $\phi(t) = (13.2 \text{turns/\text{ns}})t$, see Figure 2(c). The combination of $\theta_{\text{asym}}$ and $\phi(t)$ produces that the tip of M(t) ends up tracing the red circle on figure 2(a). This red circle on the sphere is the asymptotic limit cycle state that M(t) reaches over time. And this red circle along with the M(t) vector forms a stationary precession cone. This precession cone is the type of precession cone depicted for M(t) in standard FMR literature except that in this work the cone aperture – angle is too wide, $\theta_{\text{asym}}=21^\circ$ when compared to that proposed in the FMR standard literature of the order of $\sim 3^\circ$ [16]. And $M_z(t)=M \cos{\theta(t)}$ varies with time, in as much as $M_x$ and $M_y$ vary with time and $\theta_{\text{asym}}$ is reached in a finite time (slow time-scale-regime) of the order of tens of nanoseconds for the specified FMR conditions in this work. The time spiralling to a limit cycle under
FMR conditions with a precession cone much wider than expected is found for ferromagnetic resonance conditions. This dynamic is seldom described in FMR literature [21, 27–32].

Figure 2(b) also shows how \( \theta(t) \) develops along 70 ns, starting from quite different initial conditions, \( \phi_i = 0 \) and \( \theta_i = 3^\circ, 8^\circ, 18^\circ, 45^\circ, 60^\circ, 90^\circ, 135^\circ \) and \( 162^\circ \) with the same fields, \( H_0 \) and \( H_f(t) \), applied. For initial polar angles \( \theta_i \geq 18^\circ \), the dynamics are similar and a \( \theta(t)_{\text{MIN}} \) is developed. \( \theta(t) \) always begins by decreasing until it reaches a \( \theta_{\text{MIN}} < 21^\circ \), still in the first quarter of motion (black), at that point, \( \theta(t) \) increases slowly in order to asymptotically reach a final value, \( \theta_{\text{asy}} \). By the beginning of the fourth quarter (red), \( M(t) \) has already stationed into \( \theta_{\text{asy}} = 21^\circ \). This means that the same limit cycle is reached independently of the initial conditions, from below \( \theta_i < 21^\circ \), or, from above \( \theta_i > 21^\circ \), and the time needed to reach the limit cycle is shorter while the initial polar angle, \( \theta_i \), is smaller. Moreover, for small initial polar angles such as: \( \theta_i = 3^\circ \), below \( 21^\circ \), \( \theta(t) \) gradually increases, meanwhile \( M(t) \) spirals monotonically, yet non-linearly, opening its precession cone (not closing it as a simplified picture tells) towards \( \theta_{\text{asy}} \) and the time to reach the limit cycle, \( t_{\text{LC}} \), is \( \approx 18.3 \) ns. In contrast, for \( \theta_i = 162^\circ \), \( t_{\text{LC}} \) is \( \approx 26.7 \) ns. For any initial conditions, \( t_{\text{LC}} \) is in the range 18.3–26.7 ns. And the whole of changes of \( \theta(t) \) in the \((0–t_{\text{LC}}) \) time interval and corresponds to the first two quarters of the total time (black, blue and green traces). The fact that \( \theta(t) \)-trajectories, on the sphere, starting with either initial \( \theta \) values larger or smaller than \( \theta_{\text{asy}} = 21^\circ \), in the long run, spiral to the same \( \theta_{\text{asy}} \), \( \phi(t) \) state is clear evidence that this state is a stable limit cycle [9, 33]. Given that \( |M| = \text{constant of motion} \) and \( \theta_{\text{asy}} = \text{constant} \), then \( M_y(t) = (M_x^2 + M_z^2)^{1/2} = M \sin(\theta(t)) \) also reaches asymptotically the constant value \( M_{\text{asy}} = M \sin(\theta_{\text{asy}}) \), which is the radius of a circular limit cycle, the red circle in figure 1(a).

All the above dynamics develop in the slow time scale, nanoseconds, that belongs to the relaxation (dissipative) dynamics controlled by the damping constant \( \alpha \), i.e., the torques \( \tau_{\text{damp}1} \) and \( \tau_{\text{damp}2} \) in figure 1. As shown in figures 2(a) and (b), the evolution of \( \theta(t) \), by itself, is capturing these many details of the rich slow-time-scale \( M(t) \) dynamics from the very beginning of the motion, passing through a minimum, \( \theta_{\text{MIN}} \), then opening the precession cone, slowly approaching the limit cycle, and reaching the limit cycle itself.

Figure 2(c) gives the evolution, \( \phi(t) \), of the magnetization for the different initial conditions, \( \theta_i \) and \( \phi_i = 0 \), as in figure 2(b). All the \( \phi(t) \) curves superpose, and all have the slope = \( 4752^\circ / \text{ns} = \Delta \phi(t) / \Delta t \), which is the same as 13.2turns/\text{ns}. Hence, a \( \phi \)-turn is completed in \( t = 75.75 \) ps, and a limit cycle is established in approximately 18 ns (18000 ps), doing about 237.6 \( \phi \)-turns. The time scale to complete a \( \phi \)-turn, tens of picoseconds, is quite fast compared with the time scale to reach the limit cycles. Two-time scales, one fast and the
other one slow, are common in the dynamics of nonlinear systems [9, 18, 33]. Note that 13.2 turns/ns equates one $\phi$-turn to one cycle of the excitation frequency, $v_1 = 13.2$ GHz, which is another way to say that $\omega_1 = \omega_2 = \gamma H_1 \approx \gamma H_0$. The frequency of the excitation and the resonance condition determine the fast-time-scale dynamics of $M(t)$. In this sense, and in this case, the slow and fast magnetization dynamics are mathematically decoupled ($\theta(t)$-Slow and $\phi(t)$-Fast). And this is a welcome result, since, in general, the fast-time-scale $M(t)$-dynamics is concealed and obscured [9] due to the entangled effects of the acting torques. In order to decouple them, special methods must be developed. Here, the decoupling arises naturally.

Angular velocity behaviours in the plane $(\theta', \phi')$ are shown in figure 3, the phase plots that contain the portrait of a developing limit cycle. A horn-like spiral, $(\theta', \phi')$, forms, starting at the initial values $\phi' = 4752^\circ/\text{ns}$, and $\theta_1 \approx 0$, as shown in figure 3(a). We denote the center of the horn-like spiral as $(\phi_c', \theta_c')$, and it traces, in time, a curve (yellow) that is named the Horn Centre trajectory (HC trajectory). At any time, the $\phi'$, $\theta'$ velocity coordinates spiral around the HC, and the horn spiral ends up in a closed trajectory which is not a point, but it is centred at $(\phi_c', \theta_c')$ of the limit cycle, as it is shown in the successive zooms that capture progressive time windows of the horn spiral motion. The trajectory can start from above, $\theta_1 > 21^\circ$, or below, $\theta_1 < 21^\circ$, they develop, in time, different horn-like trajectories. Yet, these trajectories end up in the red circle centred at $(21^\circ/\text{ns}, 4752^\circ/\text{ns})$.

Given that figure 2(b) also informs us that initial conditions $(\theta_1, \phi_1)$ do not determine changes in the limit cycle, therefore, a question arises to define what dynamic factors would affect the final value, $\theta_{\text{asy}}$, of the limit cycle. The effects of $H_1$ on the limit cycle of $M(t)$ are investigated now. It is maintained as initial conditions $\theta_1 = 162^\circ$, $\phi_1 = 0^\circ$, and other parameters as before, and $H_1$ is varied from a ‘high-value-for FMR’ of $2.7 \text{mT}$, down to nanoteslas and down to $H_1 = 0 \text{ T}$ (excitation-off condition). Figure 4(a) shows the trajectories of $\theta(t)$ for seven different values of $H_1$.

The general shape of the trajectories $\theta(t)$ in figure 4(a) is as it is shown in figure 2(b): they all start a fast decrease in the first 10 ns, so that they also develop a minimum during the first quarter of motion, except for the $H_1 = 0$, and the lowest microtesla excitations. Now the asymptotic horizontal tail levels off at different $\theta_{\text{asy}}$ values; the lower the magnitude of $H_1$, the smaller the limit cycles $\theta_{\text{asy}}$ value, reaching just 2.1° for $H_1 = 100 \mu \text{T}$; 0.0021° for 1mT, and finally no limit cycle, $\theta_{\text{asy}} = 0^\circ$ for exactly zero excitation field, as expected. So, under FMR conditions of excitation, $H_1$ is the field responsible to produce two torques $\tau = -\gamma M \times H_1$ and $\tau_{\text{dampH1}} = -(\alpha \gamma / M) M \times (M \times H_1)$ which, in the long run, equilibrate the damping torques $\tau_{\text{dampH0}}$ and

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Horn-like spiralling dynamics of the magnetization velocities on the representation of $\theta'(t)$ versus $\phi'(t)$ phase portrait, (a) the first 10 nanoseconds of horn-like spiralling of the $\theta'$ versus $\phi'$ velocities starting at $\phi' = 4752^\circ/\text{ns}$, and $\theta_1 \approx 0$. $\theta'$ and $\phi'$ are negative at first, then a minimum is reached, after that they both become positive. The amplitude of the cyclic variations of $\phi'$ decrease from about $15^\circ/\text{ns}$ to $9^\circ/\text{ns}$ as time goes on, while the cyclic variations of $\theta'$ preserve amplitude, around $5.1^\circ/\text{ns}$. The trajectories are divided into four, equal time, intervals in each time frame. Initial conditions: $\theta_1 = 162^\circ$ and $\phi_1 = 0$. (b) For the 10–20 ns dynamics, the horn-like trajectory changes direction, $\phi'$ continues to change its cycle-amplitude, yet $\theta'$ preserves amplitude. (c) For the 20–30 ns dynamics, the center of the horn-like trajectory has practically stationed at $\theta' = 0^\circ$ and its cycle amplitude remains $\approx 5.1^\circ/\text{ns}$. (d) For the 30–40 ns dynamics, both, $\theta'$ and $\phi'$ are almost stationary at $0^\circ$ and $4752^\circ$, respectively, and the horn-like spiralling, giving time, becomes an ellipse. (e) For the 65–70 ns dynamics, both, $\theta'$ and $\phi'$ have reached definite constant values, and the horn-like figure is now a perfect and definite ellipse. (f) For times around 200 ns, the stationary ellipse at $\approx 70$ ns remains the same. This is the limit cycle (red) that is already present at 65 ns.}
\end{figure}
shown in figure 1, leaving a finite \( \theta_{\text{sym}} \) that along with the linear increase of \( \phi(t) \) produce a stationary \( \mathbf{M} \) precession-cone, but only after the time \( t_{LC} \) is reached. The limit cycle dynamics of \( \mathbf{M} \) is not instantly produced; and its aperture, \( \theta_{\text{sym}} \), depends nonlinearly on \( H_1 \). For \( H_1 \) excitation conditions that include FMR conditions, but are not limited to them, the aperture cone follows a quartic equation:

\[
\theta_{\text{sym}} = A H_1 + B H_1^2 + C H_1^3 + D H_1^4
\]

as shown in figure 4(b). The radius of the circular limit cycle, \( M_{\text{sym}}(H_1) = M \sin(\theta_{\text{sym}}(H_1)) \), inherits the dependency on \( H_1 \) through \( \sin(\theta_{\text{sym}}(H_1)) \). The selected range of values for \( H_1 \) covers and goes above and below experimental standard FMR parameters.

We now determine how the frequency of the excitation microwaves, \( \nu_1 = \omega_1 / 2\pi \), affects the limit cycles we have found. For this, we continue to maintain parameters and initial conditions as before and we vary \( \nu_1 \) from around 33 GHz (for Q-band) down to 5 GHz (for S-band approximately, and the Wi-Fi range). Notice the very important fact that now changing \( \nu_1 \) will change \( H_0 \) proportionately, because the resonance condition \( \omega_1 = \gamma H_0 \) must hold. \( H_1(2\pi \nu_1 t) \) changes also and torques \( \tau_0, \tau_{\text{damp11}} \) change in magnitude and torques \( \tau_1 \), and \( \tau_{\text{damp11}} \) change in frequency, magnitude and sign, periodically, and new dynamic equilibria can be reached.

Figure 5(a) shows the trajectories of \( \theta(t) \) for five different values of \( \nu_1 \):

The general shape of the trajectories \( \theta(t) \) is, once again, as shown in figure 2(b), except that now the asymptotic horizontal tail levels off at different \( \theta_{\text{sym}} \) values. The lower the frequency of the excitation \( H_1 \), and
the lower the energy that it carries in its, $h\nu_q$, photons, the larger the limit cycle – $\theta_{\text{asym}}$ value, reaching just 8.25° for $\nu_1 = 33$ GHz, but 71.74° for 5 GHz; all the other parameters remain the same. Moreover, these limit cycles are reached more slowly and longer $t_{\text{LC}}$ times are required for lower frequencies, $\nu_1$. So, under FMR conditions of excitation, $\nu_1$ of the excitation modifies the limit cycles–$\theta_{\text{asym}}$ and therefore the precession-cone dynamics of $\textbf{M}$ follows a power law relation that is given by the equation: $\theta_{\text{asym}} = \left(265.61 \text{ GHz}\right)^{-0.993}$. The radius of the limit cycles, $M_{\text{asy}} = M \sin (\theta_{\text{asy}})$ inherits the dependence on $\nu_1$ through $\sin (\theta_{\text{asy}}(\nu_1))$. The $\phi(t)$ dynamics continues to be linear in time, for example; for $\nu_1 = 5 \text{ GHz}$, the slope is 1800°/ns, which means a period of 200 ps for one $\phi$-turn, while, for $\nu_1 = 30 \text{ GHz}$, the slope is 10 800°/ns, which means a period of 33.33 ps for one $\phi$-turn, and the $\phi$-period remains one-to-one with one cycle of the excitation, as expected. The selected range of values for $\nu_1$ falls within experimental standard FMR parameters. And 5 GHz, which is the lowest frequency limit studied, covers the upper limit that wireless communications have recently reached. These limit cycles and precession-cone dynamics appear very slow for the low end of the frequencies studied. For frequencies smaller than 5 GHz, the establishment of a very wide precession-cone can take 70 ns, or even more than 200 ns with an aperture larger than 71.72°. The precession cones, just described, seem qualitatively quite distant from the FMR-like narrow-precession cones.

4. Conclusions

The limit cycle states of the magnetization nonlinear dynamics are found for isolated and isotropic nanomagnets through numerical solutions of the Landau–Lifshitz equation of motion under FMR conditions. These limit cycles are reached when $\theta(t)$ trajectories, in the long run, converge asymptotically to $\theta_{\text{asy}}$ constant, while $\phi(t)$ varies linearly in time. The solutions $\theta(t)$ are very rich in dynamical changes in which $\textbf{M}(t)$ evolves and characterizes mainly the slow-time-scale motion. The $\phi(t)$ solutions are linear in time with slopes proportional to $\nu_1$, as expected. The excitation $H_1$ is the cause of the existence of limit cycles–$\theta_{\text{asym}}$. If $H_1$ is turned off, the limit cycle disappears, and the magnetization ends its dynamic alignment along the applied field $H_o$. The limit cycle–$\theta_{\text{asy}}$ follows a 4th-degree polynomial function on $H_1$, and a power law on frequency $\nu_1$, $\theta_{\text{asy}}$, and $\phi(t)$ produce the precession cones that $\textbf{M}(t)$ draws on the surface of the $M = \text{constant}$-sphere once the time $t_{\text{LC}}$ is reached. For the parameters used here, the limit cycles are circular, stable with periods in the picoseconds range, and its radius is a nonlinear function of $H_1$, typically $\sim 0.35|M|$, and $\theta_{\text{asy}}$ independent of initial conditions. The frequency, $\nu_1$, of the excitation field $H_1$ modifies the precession-cone dynamics of $\textbf{M}$ following an inverse relation. For the 5 GHz low end of the frequencies, the establishment of a very wide precession-cone can take 188 ns with periods of $\sim 200$ ps each $\phi$-turn. The precession cones described here seem qualitatively quite distant from the FMR-like narrow-precession cones. Then, $H_0(2\pi\nu_1 t)$ plays a crucial role in the non-linear dynamics of $\textbf{M}(t)$ under FMR conditions beyond that of being a simple perturbative excitation. The results are significantly more detailed and go beyond what has been reported previously.

All these results inform a very rich nonlinear dynamics of the magnetization of an isotropic nanomagnet while executing ferromagnetic resonance, being the non-instantaneous development of the limit cycle of $\textbf{M}(t)$ (after many nanoseconds) the most outstanding feature of all. Controllability of limit cycles by an external microwave field, $H_1$, on a simple nanomagnet could prove very valuable in technological applications.

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