On space-times admitting shear-free, irrotational, geodesic null congruences

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Abstract
Space-times admitting a shear-free, irrotational, geodesic null congruence are studied. Attention is focused on those space-times in which the gravitational field is a combination of a perfect fluid and null radiation.

1 Introduction
In this article we wish to extend earlier work on shear-free, irrotational and geodesic (SIG) timelike and spacelike congruences \[1, 2\] to SIG null congruences. The fact that we are dealing with null congruences means that we have to approach the problem in a completely different way; we must make extensive use of the Newman-Penrose formalism.

Thus, we wish to study a congruence of curves whose tangent vector \( k \) is null and geodesic. Hence, we have a family of null geodesics \( x^a = x^a(y^\alpha, v) \), where \( y^\alpha \) distinguishes the different geodesics, and \( v \) is the affine parameter along a fixed geodesic. The null tangent vector is \( k^a = \frac{\partial x_a}{\partial v} \), and satisfies \( k^a_k b = 0 \). The spin coefficients are defined in \[3\], where \( \rho = -\theta + i\omega \) is called the complex divergence and \( \sigma \) is the complex shear. The geodesic condition implies that the spin coefficient \( \kappa \) vanishes and \( \epsilon + \bar{\epsilon} = 0 \) follows from the choice of an affine parameter along the congruence. The congruence is said to be shear-free if \( \sigma = 0 \). Also, from the relation \( k_{[a;b;]c} = (\bar{\rho} - \rho)\bar{m}_{(a}k_{b)c} \), it follows that \( w = 0 \) (i.e., zero twist) is a necessary and sufficient condition for \( k \) to be hypersurface orthogonal.

First we shall briefly review some of the results of relevance to this work. Goldberg and Sachs \[5\] proved that if a gravitational field contains a shear-free, geodesic null congruence \( k \), then \( \kappa = \sigma = 0 \), and if

\[
R_{ab}k^ak^b = R_{ab}k^am^b = R_{ab}m^am^b = 0 ,
\]

then the field is algebraically special (i.e., \( \Psi_0 = \Psi_1 = 0 \)), and \( k \) is a degenerate eigendirection. In addition, a vacuum metric is algebraically special if and only if it contains a shear-free geodesic null congruence.

A space-time admits a geodesic, shear-free, twist-free \( (\kappa = \sigma = \omega = 0) \) and diverging \( (\rho = \bar{\rho} = \theta = -1/r) \) null congruence \( k \), and satisfies \[1\], if and only if the metric can be written in the form

\[
ds^2 = 2r^2P^{-2}(z, \bar{z}, u)dzd\bar{z} - 2du - 2H(z, \bar{z}, r, u)du^2 .
\]

Robinson-Trautman models \[6\] with this metric have been found for vacuum, Einstein-Maxwell and pure radiation fields with or without a cosmological constant \[3\].
For geodesic null vector fields we have that $(\theta + i \omega)k^a + (\theta + i \omega)^2 + \sigma \bar{\sigma} = -R_{ab}k^a k^b / 2$. Therefore, in the non-diverging case (i.e., $\rho = -(\theta + i \omega) = 0$), if the energy condition $T_{ab}k^a k^b \geq 0$ is satisfied, it follows that $\sigma = 0 = R_{ab}k^a k^b$. Thus, non-twisting (and therefore geodesic) and non-expanding null congruences must be shear-free. Hence, the space-time is algebraically special, and it corresponds to vacuum, Einstein-Maxwell, and pure radiation field. Perfect fluid solutions violate $R_{ab}k^a k^b = 0$ unless $\mu + p = 0$. This class of solutions has been studied by Kundt [8].

Another important case corresponds to the Kerr-Schild metric, which is given by $g_{ab} = \eta_{ab} - 2\phi k_a k_b$. The null vector $k$ of a Kerr-Schild metric is geodesic if and only if the energy-momentum tensor obeys the condition $T_{ab}k^a k^b = 0$, and then $k$ is a multiple principal null direction of the Weyl tensor and the space-time is algebraically special. The general properties of the Kerr-Schild metrics and their applications to vacuum, Einstein-Maxwell, and pure radiation space-times can be found in [8].

Finally, we note the algebraically special perfect fluid space-times corresponding to the generalized Robinson-Trautman solutions investigated by Wainwright [3]. They are characterized by a multiple null eigenvector $k$ of the Weyl tensor which is geodesic, shear-free, and twist-free but expanding (i.e., $\Psi_0 = \Psi_1 = 0$, $\kappa = \sigma = \omega = 0$, $\rho = \bar{\rho} \neq 0$), and the four-velocity obeys $u_{[a,b]} = 0$, $k_{(u,a)kb} = 0$. The line-element of the space-time can be written in the form

\[
\frac{ds^2}{P} = -\frac{1}{2} \chi^2 (r,u) P^{-2} (z, \bar{z}, u) dz d\bar{z} + 2du (dr - U du),
\]

with

\[
U = r (\ln P)_u + U^0 (z, \bar{z}, u) + S (r, u), \quad \chi, r \neq 0, \quad \frac{\chi_{rr}}{\chi} \leq 0.
\]

In this case no dust solutions nor solutions of Petrov types $III$ and $N$ are possible.

2 Analysis

Let us consider space-times admitting a shear-free, irrotational, geodesic null congruence in which the source of the gravitational field is a combination of a perfect fluid and null radiation, so that the energy-momentum tensor has the form

\[
T_{ab} = (\mu + p)u_a u_b - pg_{ab} + \phi^2 k_a k_b,
\]

where $u^a$ is the four-velocity of the fluid, $\mu$ and $p$ are the density and the pressure of the fluid, respectively, and $k$ is a null vector. The null radiation is geodesic, shear-free, and twist-free, and defines the null congruence. Wainwright [3] proved that for a space-time in which there exists a SIG null congruence, coordinates can be chosen so that the metric takes on the simplified form [8] with $u = x^1$, $r = x^2$, $z = x^3 + ix^4$, the tangent field of the null congruence is given by $k^a = \delta^a_2$, $k_a = \delta^1_a$, and we can introduce the null tetrad

\[
k^a = \delta^a_r, \quad l^a = \delta^a_u + U \delta^a_r, \quad m^a = P \chi^{-1} (\delta^a_3 + i \delta^a_4),
\]

\[
k_a = \delta^a_u, \quad l_a = -U \delta^a_u + \delta^a_r, \quad m_a = P^{-1} \chi (\delta^a_3 + i \delta^a_4)/2.
\]

With the sign convention used here we have that $u^a u_a = k^a l_a = 1 = -m^a m_a$. Note that the null radiation is everywhere tangent to the repeated null congruence of the space-time.

First, since $\Phi_0 \equiv -4R_{ab}k^a m^b = 0$, we conclude that the four-velocity satisfies $u^a m_a = 0$, and hence it can be expressed in terms of the null tetrad by

\[
u^a = \frac{1}{\sqrt{2} B} (B^2 k^a + l^a) \quad \text{and} \quad u_a = \frac{1}{\sqrt{2} B} [(B^2 - U) \delta^a_3 + \delta^a_4],
\]
for some function $B$. The conditions $\Phi_{02} \equiv -\frac{1}{2} R_{ab} m^a m^b = 0$ and $\Phi_{12} \equiv -\frac{1}{2} R_{ab} n^a n^b = 0$ are satisfied identically. The non-zero components of the Ricci tensor are

$$
\Phi_{00} \equiv -\frac{1}{2} (R_{ab} - \frac{1}{4} R g_{ab}) k^a k^b = \frac{1}{2} (\mu + p) (k \cdot u)^2 ,
$$

$$\Phi_{11} \equiv -\frac{1}{4} (R_{ab} - \frac{1}{4} R g_{ab}) (k^a p^b + m^a m^b) = \frac{1}{4} (\mu + p) (k \cdot u)(1 \cdot u) ,$$

$$\Phi_{22} \equiv -\frac{1}{2} (R_{ab} - \frac{1}{4} R g_{ab}) l^a l^b = \frac{1}{2} (\mu + p)(l \cdot u)^2 + \frac{1}{2} \phi^2 .$$

In addition, since $k \cdot u = \frac{1}{\sqrt{2B}}$ and $l \cdot u = \frac{1}{\sqrt{2}} B$ implies $l \cdot u = B^2 (k \cdot u)$, we obtain

$$B^2 \Phi_{00} = 2 \Phi_{11} ,$$

$$B^4 \Phi_{00} = \Phi_{22} - \frac{1}{2} \phi^2 .$$

If we now assume that the fluid is non-rotating, then $B^2 = U + F(r,u)$, and the compatibility condition (12) can be written as

$$(U + F) \Phi_{00} = 2 \Phi_{11} .$$

On differentiating this equation successively with respect to $z$ and $r$, we obtain the restriction

$$(\chi^2)_{rr}(U^{0},z + r(\ln P),_{uz}) = 0 .$$

There are consequently two different cases to consider.

In the first case $U^{0},z + r(\ln P),_{uz} = 0$, which is equivalent to $U^{0},z = (\ln P),_{uz} = 0$, so that $P = P(z,\bar{z})$ and $U^{0} = U^{0}(u)$. Obviously, the solutions admit a multiply transitive group of motions, $G_3$, acting on the 2-spaces $r =$ const, $u =$ const, of constant curvature, and belong to class II of Stewart and Ellis [9]. The metric (3) can then be rewritten as

$$ds^2 = -\chi^2(r,u) \frac{2dzd\bar{z}}{(1 + \frac{1}{2}z\bar{z})^2} + 2du(dr - U(r,u)du) .$$

The non-zero Ricci components are given by

$$\Phi_{00} = -\frac{\chi_{,rr}}{\chi} ,$$

$$\Phi_{11} = \frac{\chi_{,r}\chi_{,u}}{2\chi^2} + \frac{(\chi_{,r})^2 U}{2\chi^2} - \frac{U_{,rr}}{4} + \frac{k}{4\chi^2} ,$$

$$\Phi_{22} = \frac{\chi_{,u}U_{,r}}{\chi} - \frac{\chi_{,uu}}{\chi} - \frac{2\chi_{,ur}U}{\chi} - \frac{\chi_{,ru}U_{,u}}{\chi} - \frac{\chi_{,rr}U^2}{\chi} ,$$

and the Ricci scalar is given by

$$\frac{R}{2} = 12 \Lambda = 4 \frac{\chi_{,r}U_{,r}}{\chi} + 2 \frac{\chi_{,r}\chi_{,u}}{\chi^2} + \frac{(\chi_{,r})^2 U}{\chi^2} + 4 \frac{\chi_{,ur}U}{\chi} + U_{,rr} + 4 \frac{\chi_{,rr}U}{\chi} + \frac{k}{\chi^2} .$$

Hence, the metric (16) can be interpreted as pure radiation plus a perfect fluid where $\mu$ and $p$ are given by

$$\mu = R_{4} + 6 \Phi_{11} , \quad p = -\frac{R}{4} + 2 \Phi_{11} ,$$

$u_a$ is determined by (9) with $B^2 = 2 \Phi_{11}/\Phi_{00}$, and $\phi^2$ is given by

$$\phi^2 = 2 \left( \Phi_{22} - \frac{\Phi_{11}^2}{\Phi_{00}} \right) .$$
In the second case (i.e., $\chi^2,_{rr} = 0$) two possibilities arise:

(i) $\chi^2 = cr, \quad c = \pm 1 \quad (23)$

(ii) $\chi^2 = c(r^2 - k^2), \quad k = \text{const.} \quad (24)$

In both subcases $\chi = \chi(r)$, and they can be written together as $\chi^2 = ar^2 + 2br + c$, with $a$, $b$, $c$ taken to be appropriate constants. From equation (14) we obtain

$$a U^0 - b \ln P,_{u} + K = G(u),$$

and

$$\frac{1}{2} \left[ \chi^2 S,_{r} - S(\chi^2),_{r} \right]_{,r} + \frac{F \Sigma}{\chi^2} = G(u), \quad (25)$$

where $K \equiv 4P^2(\ln P)_{z\bar{z}}$, $\Sigma \equiv b^2 - ac$, and $G(u)$ is an arbitrary function of $u$.

Subcase (i): $a = c = 0$, $b = \epsilon/2$. Integrating equation (26) we see that $S$ can be written in the form

$$S = rh(u) + 2\epsilon G(u)r \ln r - f(u) - \frac{1}{2} \int \frac{dr}{r^2} \int d\hat{r} F(\hat{r}, u), \quad (27)$$

where $h(u)$ and $f(u)$ are arbitrary functions of $u$.

Subcase (ii): $a = c, b = 0, c = -\epsilon k^2, \Sigma = k^2$. We obtain

$$S = -\epsilon G(u) + f(u)\chi^2 \int \frac{dr}{\chi^4} + h(u)\chi^2 - 2k^2\chi^2 \int \frac{dr}{\chi^4} \int d\hat{r} F(\hat{r}, u). \quad (28)$$

Therefore, the metric (3) with $\chi(r)$ given by (23) or (24), $S(r, u)$ given by (27) or (28), and $P(z, \bar{z}, u)$ satisfying (25) can be interpreted as pure radiation plus a perfect fluid, in which the four-velocity is determined by (8) and $\phi^2$, $\mu$ and $p$ are determined by (21) and (22), respectively.

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References

[1] A.A. Coley and D.J. McManus, Class. Quantum Grav., 11(1994)1261.
[2] D.J. McManus and A.A. Coley, Class. Quantum Grav., 11(1994)2045.
[3] D. Kramer, H. Stephani, M.A.H. MacCallum and E. Herlt, Exact Solutions of Einstein’s Field Equations, Deutscher Verlag der Wissenschaften, Berlin (1980).
[4] F.A.E. Pirani, in Lectures on General Relativity, 1964 Brandeis Summer Institute in Theoretical Physics, Vol. 1. Prentice-Hall, Englewood Cliffs, NJ (1965).
[5] J.N. Goldberg and R.K. Sachs, Acta. Phys. Polon., Suppl. 22(1962)13.
[6] I. Robinson and A. Trautman, Proc. Roy. Soc. Lond., A265(1962)463.
[7] W. Kundt, Z. Phys., 163(1961)77.
[8] J. Wainwright, Int. J. Theor. Phys., 10(1974)39.
[9] J.M. Stewart and G.F.R. Ellis, J. Math. Phys., 9(1968)1072.