A Novel Faster All-Pair Shortest Path Algorithm Based on the Matrix Multiplication for GPUs

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Abstract

The shortest path problem is a class of typical problems in graph theory and network science, with a wide range of application scenarios. The state of the art all-pair shortest path algorithm is implemented by the parallel single source shortest path algorithms, since the Floyd algorithm is difficult to accelerate by parallelism. We propose a novel all-pair shortest path algorithm based on block matrix multiplication via GPUs, which transforms shortest path problems into the linear algebra problems and takes advantage of the GPUs’ superior performance in this regard. In the experiments, the novel algorithm achieves average of 41.257× and the maximum of 89.919× over widely used Dijkstra algorithm which implements the priority queue by the binary heap and is optimized via multi-threading.

Keywords: Graph Theory, Network Science, Shortest Path, Parallel Computing, Matrix Multiplication
1 Introduction

The shortest path problem is a basic problem in graph theory, and has attracted the attention of researchers from many disciplines, such as transportation planning, computer science, network science, and applied mathematics[1–5]. It is difficult for serial algorithms to adapt to changes as the scale of the graph increases. Researchers hope to tackle this type of problem through parallel computing, which top conferences in the field of parallel and distributed computing have focused on[6].

The shortest path problem is generally divided into four categories: the point-to-point shortest path problem, single-source shortest path problem, multi-source shortest path problem, and all-pair shortest path problem. The classic algorithms to solve the shortest path problem are the Dijkstra algorithm[7], Bellman–Ford algorithm[8] and Floyd algorithm[9]. In the second half of the last century, artificial intelligence underwent its initial development and spawned new ideas to solve the shortest path problem[10–13]. The development of the genetic algorithm provides a new solution for shortest path problem, and researchers continue to focus on the methods[14–17].

Parallel computing, provides more possibilities for researchers to solve general problems. Co-processing through multiple cores is not widely applicable, when the development of single-core performance slowed. The Floyd algorithm which is based on dynamic programming technology is difficult to accelerate the calculation through parallelism because of the pursuit of the optimal solution at each step. The state of art method is to deal with the all-pair shortest path problem using the parallel single-source shortest path algorithms, which has received extensive attention at academic conferences in the field of parallel computing[18–25].

The key contribution of this paper is to transform the shortest path problem into a linear algebra problem that can fully exploit the huge advantages of GPUs. We propose a novel all-pair shortest path algorithm via an adjacency matrix operation Worked on the graphs Networks, which has been named DAWN.

DAWN is a more efficient algorithm based on adjacency matrix operations for solving the all-pair shortest paths problem, which requires $O(n^2)$ space and $O(d \cdot n^2.387)$ time, where $d$ is the diameter of the graph. DAWN can accelerate computing via a multi-GPU system, and its time complexity depends on the number of nodes and is insensitive to graph density. We do not distinguish between graphs and networks in the paper, which have the same meaning.

The main contributions of this work are as follows:

1. We propose a novel method based on the matrix representation of graphs, which transforms the shortest path problem into the linear algebra problem and can fully exploit the huge advantages of GPUs.
2. We propose an algorithm for parallel shortest path search by matrix multiplications, which avoid read-write conflicts and improves the parallelism of the algorithm.
3. We optimize the novel algorithm by the matrix block multiplications, and find that the matrix can be guaranteed to be reblockable by continuously adding rows and columns with all elements zero, which expands the applicability of the algorithm.

4. We propose a path combination method to deal with the shortest path problem on the large diameter graphs.

In Section 2, we introduce the problem of shortest path and its typical algorithms. In Section 3, we describe the design of the DAWN algorithm and propose optimization methods to make it more widely applicable to various graphs. In Section 4, we demonstrate the efficiency of DAWN through multiple sets of comparative experiments. In Section 5, we conclude the work of this paper.

### Table 1 Definition of notations

| Notation | Definition |
|----------|------------|
| G        | A graph    |
| V, E, W  | Set of nodes, edges and weights |
| v, e, w  | A node, edge and weight of edges |
| A, Aw    | Adjacency matrix of unweighted graphs and weighted graphs |
| a_{i,j}, w_{i,j} | Entries in the adjacency matrix |
| M, E     | Matrix of amount and length of path between all-pair of nodes |
| a^{(k)}_{i,j} | Amount of paths from v_i to v_j with length k |
| k(i,j)   | Length of paths from v_i to v_j |
| P        | Set of paths |
| P(i,j)   | Set of paths from v_i to v_j |
| k        | Length of paths |
| k_{min}(i,j) | Length of the shortest paths from v_i to v_j |
| N        | Set of natural numbers |
| µ        | Constant parameter |

## 2 Related Works

The shortest path problem is a classic problem in graph theory and network science. In this section, we introduce the Dijkstra algorithm, the Bellman–Ford algorithm, the Floyd algorithm, and their extensions, which are typical algorithms for solving the shortest path problems.

### 2.1 Dijkstra algorithm

In 1959, Dijkstra et al. proposed and solved two graph problems: construct the tree of the minimum total length between n nodes, and find the path of the minimum total length between two given nodes, v_i and v_j [7]. The main optimization methods of Dijkstra algorithm are priority queue optimization, binary heap optimization, and Fibonacci heap optimization. The algorithms
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improve the priority queue data structure based on the Dijkstra algorithm[26–31]. Though the Fibonacci heap support it in $O(1)$ amortized time and the pairing heap does not, an experimental study of the PrimDijkstra MST algorithm found the standard binary heap performed better than the rest in most cases [32–34].

Meyer et al. proposed an optimized Dijkstra algorithm, which can be a parallel setting for a large class of graphs. The best parallel version of the $\Delta$-stepping Dijkstra algorithm takes $O(D \cdot Len \cdot \log n + \log^2 n)$ time and $O(n + m + D \cdot Len \cdot \log n)$ work on average, where $Len$ denotes the maximum shortest path weight from the source node ($s$) to any node reachable from $s$, and $D$ represents the maximum node degree [35].

Davidson et al. proposed an adaptation of the $\Delta$-stepping Dijkstra algorithm that makes several simplifications to the worklist, which uses two buckets (Near and Far) and double buffering. The near and far buckets implement an approximate priority queue, and the barrier synchronization and double buffering technology limit the parallelism of the algorithm. A single core in a GPU is not as powerful as in a CPU, but instead is rather mediocre.

In one of the latest works, researchers demonstrated that sophisticated data structures can be used on these mediocre cores. Wang et al. proposed a $\Delta$-stepping Dijkstra algorithm based on the GPU, using a sophisticated work scheduler to improve work efficiency and parallelism, which was named ADDS. They demonstrated that a balance of algorithmic efficiency and GPU fitness can be achieved by an efficient scheduler and showed that sophisticated data structures can be used on the GPUs via careful design.

2.2 Bellman–Ford algorithm

In 1958, Bellman et al. proposed a new shortest path algorithm [8] that is suitable for sparse graphs and can handle negative weighted graphs. The Bellman–Ford algorithm needs to perform $n − 1$ relaxation operations on $m$ edges, and requires $O(nm)$ time. Compared with the Dijkstra algorithm, the Bellman–Ford algorithm is more suitable for parallelism, and researchers have been doing work on the parallel versions [23, 24].

2.3 Floyd algorithm

In 1962, Floyd proposed an algorithm that uses dynamic programming to solve the shortest path problem and is easy to implement [9]. The Floyd algorithm uses only two-dimensional arrays, compared to the Dijkstra algorithm, which uses sophisticated data structures for optimization. However, the pursuit of the optimal solution at each step limits the parallelism of the Floyd algorithm. Researchers have also been working on the optimization of the Floyd algorithm and the development of a parallel version [36–38]. The latest representative work on the Floyd algorithm was published in PPoPP’ 20, in which
Piyush Sao et al. proposed a Floyd algorithm optimized by techniques of fill-in reducing ordering, symbolic analysis, supernodal traversal, and elimination tree parallelism[39].

3 Methods

In this section, we introduce the design of the DAWN algorithm, which is mainly divided into four aspects: the algorithm on the unweighted graphs, the algorithm on the weighted graphs, time complexity of the DAWN, and optimization methods.

3.1 DAWN on the Unweighted Graphs

We represent the graph \( G = (V, E) \) as an adjacency matrix:

\[
A = \begin{bmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n}
\end{bmatrix},
\]

where \( V \) represents the set of nodes and \( E \) represents the set of edges. The adjacency matrix is a square matrix which represents the graphs, in which the entry \( a_{i,j} \) represents the number of paths between \( v_i \) and \( v_j \) if path between \( v_i \) and \( v_j \) is in the graph, while \( a_{i,j} = 0 \) if path between \( v_i \) and \( v_j \) is not in the graph[40].

**Lemma 1** In the matrix \( A^k = (a^{(k)}_{i,j})_{n \times n} \), the element \( a^{(k)}_{i,j} \) represents the number of paths with the length \( k \) from \( v_i \) to \( v_j \)[40].

**Proof** We use the cumulative formula to express the multiplication of matrices:

\[
A \times B = \left( \sum_{k=1}^{n} a_{i,k} b_{k,j} \right)_{n \times n},
\]

and we define:

\[
a^{(2)}_{i,j} = a_{i,1} a^{(1)}_{1,j} + a_{i,2} a^{(1)}_{2,j} + \cdots + a_{i,n} a^{(1)}_{n,j},
\]

and we find that the above formula represents the sum of the path combinations. The entry \( a_{i,k} a^{(1)}_{k,j} \) represents the path combination of \( v_i \) to \( v_k \) and \( v_k \) to \( v_j \), where \( 1 \leq k \leq n \). If both subpaths exist, we will get a new path combination. The entry \( a^{(2)}_{i,j} \) represents the total number of the path combinations.

We promote the descriptions,

\[
a^{(k)}_{i,j} = a_{i,1} a^{(k-1)}_{1,j} + a_{i,2} a^{(k-1)}_{2,j} + \cdots + a_{i,n} a^{(k-1)}_{n,j},
\]

and the above formula represents the path combinations of subpaths with length \( k - 1 \) and 1, which also represents the path with length \( k \) from node \( v_i \) to \( v_j \).  \( \square \)
Lemma 1 gives us ideas for designing shortest path algorithms, which reveals the relationship between paths and matrix multiplication.

In the previous representative works, the methods are based on path search and dynamic programming, which have been detailed in Section 2. To design the shortest path algorithm through the relationship between paths and matrix multiplication, we need to solve the main problem that how to identify the shortest paths.

**Theorem 1** In the unweighted graphs, the length of the shortest paths from $v_i$ to $v_j$ is $k_{min}$, if and only if $a_{i,j}^{(k_{min})} \neq 0 \land i \neq j \land \sum_{k=1}^{k_{min}-1} a_{i,j}^{(k)} = 0$.

**Proof** In the graphs, the arbitrary shortest path of length $k$ can be expressed as the connection of the two shortest paths with the length $k_1$ and $k_2$, where $k = k_1 + k_2$ and $k \geq 2$. Thus, we can get a fact:

**Fact 1** In the unweighted graphs, the arbitrary shortest path of length $k$ can be expressed as the connection of the two shortest paths with the length $k-1$ and $1$, where $k \geq 2$.

Therefore, we can obtain an expression for the all paths from $v_i$ to $v_j$:

$$P(i, j) = [a_{i,j}^{(1)} + a_{i,j}^{(2)} + a_{i,j}^{(3)} + \cdots + a_{i,j}^{(d)}] = \sum_{k=1}^{d} a_{i,j}^{(k)}, \quad (6)$$

where $d$ is the diameter of the graph and $1 \leq k \leq n-1$. The summation result of Formula 6 is not less than 0. $P(i, j)_{\text{min}}$ which is not 0 is the number of shortest paths from $v_i$ to $v_j$.

$$P(i, j)_{\text{min}} = \sum_{k=1}^{d} a_{i,j}^{(k)} = \sum_{k=1}^{k_{min}-1} a_{i,j}^{(k)} + a_{i,j}^{(k_{min})} + \sum_{k=k_{min}+1}^{d} a_{i,j}^{(k)}, \quad (7)$$

where $1 \leq k_{min} \leq d$. If $a_{i,j}^{(k_{min})} \neq 0$, we will get $\sum_{k=k_{min}+1}^{d} a_{i,j}^{(k)} \neq 0$, accroding to Fact 1. Thus, we get,

$$P(i, j)_{\text{min}} = a_{i,j}^{(k_0)}, \quad (8)$$

and holds under the condition $\sum_{k=1}^{k_{min}-1} a_{i,j}^{(k)} = 0 \land a_{i,j}^{(k_{min})} \neq 0$. This condition indicates that the first time the entry $a_{i,j}^{(k)}$ is not 0, which represents the number of shortest paths, in the matrix operations. The cycles would not be counted in the paths, so we add new condition $i \neq j$. \hfill \Box

Theorem 1 states the sufficient condition for the shortest paths and reveals the relationship between the shortest paths and matrix multiplication. Thus, we transform the shortest path problem into the linear algebra problem. The calculation result can be expressed as:

$$M = \left( a_{i,j}^{(k_{min})} \right)_{n \times n}, \quad (9)$$

$$E = \left( k_{min}(i, j) \right)_{n \times n}. \quad (10)$$
We use pseudocode to demonstrate the DAWN, and $q$ represents the number of fillings in the matrix $M$. We use $q_{\text{pre}}$ to store the value of $q$ at the end of the previous loop.

**Algorithm 1** DAWN on the Unweighted Graphs

| Input: $A, B, M, E$ |
|---------------------|
| Output: $M, E$ |

1. for $i = 0$ to $n - 1$, $j = 0$ to $n - 1$ do
   2. if $A[i][j] == 1$ then
      3. $B[i][j] \leftarrow 1$
      4. $M[i][j] \leftarrow 1$
      5. $E[i][j] \leftarrow 1$
   6. else
      7. $B[i][j] \leftarrow 0$
      8. $M[i][j] \leftarrow -1$
      9. $E[i][j] \leftarrow -1$
   10. end if
   11. $q \leftarrow 0$
   12. $q_{\text{pre}} \leftarrow 0$
   13. end for
14. for $k = 2$ to diameter, $i = 0$ to $n - 1$, $j = 0$ to $n - 1$ do
   15. $B \leftarrow B \times A$
   16. if $B[i][j] \neq 0 \land E[i][j] == -1$ then
      17. $M[i][j] \leftarrow B[i][j]$
      18. $E[i][j] \leftarrow k$
      19. if $q > n \times (n - 1) - 1$ then
         20. break
   21. end if
   22. if $q_{\text{pre}} == q$ then
      23. break
   24. end if
   25. $q_{\text{pre}} \leftarrow q$
   26. end for
27. return $M, E$

There are two conditions for breaking the loop and ending it:

1. DAWN already found the shortest paths between all pairs of nodes in the graph;
2. The filling rate of the amount matrix does not change when a loop ends, which means no new paths were found in the loop.

Fact 1 provides the stopping condition 2 of DAWN. If no new shortest path is found in $A^{(k)}$, then no new paths will be found in $A^{(k+1)}$, ending DAWN.
There is no difference in the operation of DAWN on directed and undirected graphs.

3.2 DAWN on the Weighted Graphs

We represent the graph $G = (V, E, W)$ as an adjacency matrix:

$$A_w = \begin{bmatrix}
w_{1,1} & w_{1,2} & \cdots & w_{1,n-1} & w_{1,n} \\
w_{2,1} & w_{2,2} & \cdots & w_{2,n-1} & w_{2,n} \\
\vdots & \vdots & & \vdots & \vdots \\
w_{n-1,1} & w_{n-1,2} & \cdots & w_{n-1,n-1} & w_{n-1,n} \\
w_{n,1} & w_{n,2} & \cdots & w_{n,n-1} & w_{n,n}
\end{bmatrix},$$

where $V$ represents the set of nodes, $E$ represents the set of edges, and $W$ represents the set of weight. The adjacency matrix is a square matrix which represent the graphs, in which the entry $w_{i,j} = k$ if path with weight $k$ between $v_i$ and $v_j$ is in the graph, while $w_{i,j} = 0$ if path between $v_i$ and $v_j$ is not in the graph.

We assume that $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, and use summation to describe the matrix $A_w$:

$$A_w = \sum_{k=1}^{w_0} H^{(k)},$$

where $w_0$ represents the maximum weight of paths in the graphs, and matrix $H^{(k)} = (h_{i,j}^{(k)})_{n \times n}$ represent the number of edges with the weight $k$ from $v_i$ to $v_j$.

We define a new operation for matrix multiplication:

$$A_w^{(k)} = \sum_{i+j=k \land 1 \leq i,j \leq k} \left[ A_w^{(i)} \times H^{(j)} + A_w^{(j)} \times H^{(i)} \right].$$

The above formula includes all possible path combinations, but the computational cost is expensive and the time complexity is about $O(n^4)$. We provide an approximate calculation formula that retains most of the paths and reduces the computational cost.

First, we define the left and right multiplication of the matrix as:

$$L^{(k)} = A_w^{(k-1)} \times H^{(1)} + H^{(k)},$$

$$R^{(k)} = H^{(1)} \times A_w^{(k-1)} + H^{(k)},$$

where $k \geq 2$, and then, we define:

$$A_w^{(k)} = L^{(k)} + R^{(k)} - H^{(k)},$$

$$L^{(1)} = A_w^{(0)} \times H^{(1)},$$

$$R^{(1)} = H^{(1)} \times A_w^{(0)}.$$
and it operation will result in many paths being double counted, but also fewer missing paths. We provide another method to ensure that paths are not counted repeatedly under the condition of as few missing paths as possible:

$$A_w^{(k)} = \max \left[ L^{(k)} - H^{(k)}, R^{(k)} - H^{(k)} \right] + H^{(k)},$$

where $\max[\cdot, \cdot]$ represents the maximum value of the corresponding positions of the two matrices. The DANW is built on this method, which achieves a good balance between accuracy and computational cost. Thus, we get the fact following.

**Fact 2** In the matrix $A_w^{(k)} = (w_{i,j}^{(k)})_{n \times n}$, the element $w_{i,j}^{(k)}$ represents the number of most paths with the weight $k$ from $v_i$ to $v_j$.

It is necessary to calculate both left and right multiplications since DAWN developed based on Formula (17), and the number of matrix multiplications is twice that of the unweighted graphs. If the graphs are undirected, we can simplify the operation. The symmetric matrix satisfies the following,

$$A = A^T,$$

$$(A \times B)^T = B^T \times A^T,$$

which has been described in detail[41]. The matrix $H^{(k)}$ will not affect the matrix relationship in the undirected graph and can be ignored. We transpose the matrix $R^{(k)}$ as:

$$[R^{(k)}]^T = [H^{(1)}] \times A_w^{(k-1)}]^T$$

$$= [A_w^{(k-1)}]^T \times [H^{(1)}]^T$$

$$= A_w^{(k-1)} \times H^{(1)}$$

$$= L^{(k)},$$

and the equation means that we do not need to compute the matrix multiplication twice; instead, we only need to transpose the matrix that can obtain the result of matrix left and right multiplication, in the undirected and weighted graphs.

We give an example to facilitate the understanding of DAWN on the weighted graphs, and use the method of generative function to discover the missing paths[42], which is provided in the appendix. We use pseudocode to demonstrate the DAWN on the weighted graphs.

There is a close relationship between the accuracy of the approximation method and the characteristics of the graph. The approximate method has good accuracy on the graphs which mainly contains edges with smaller weights and a few edges with larger weights. We provide the weight transfer function to accommodate graphs for which the weight distribution does not meet the requirements:

$$w' = \lfloor \frac{3}{2} + \log w \rfloor,$$
Algorithm 2 DAWN on the Weighted Graphs

Input: $A, B, M, E$
Output: $M, E$

1: for $i = 0$ to $n - 1, j = 0$ to $n - 1$ do
2:    if $A[i][j] == 1$ then
3:        $B[i][j] \leftarrow 1$
4:        $M[i][j] \leftarrow 1$
5:        $E[i][j] \leftarrow 1$
6:    else
7:        $B[i][j] \leftarrow 0$
8:        $M[i][j] \leftarrow -1$
9:        $E[i][j] \leftarrow -1$
10:       $map\{i, j\} \leftarrow A[i][j]$
11:    end if
12:    $q \leftarrow 0$
13:    $q_{pre} \leftarrow 0$
14:    $C[i][j] \leftarrow 0$
15: end for
16: for $k = 2$ to diameter, $i = 0$ to $n - 1, j = 0$ to $n - 1$ do
17:    if Graphs is directed then
18:        $C \leftarrow A \times B$
19:        $B \leftarrow B \times A$
20:        $B[i][j] \leftarrow \max(B[i][j], C[i][j])$
21:    else
22:        $B[i][j] \leftarrow \max(B[i][j], B[j][i])$
23:    end if
24:    if $map\{i, j\} == k$ then
25:        $B[i][j] \leftarrow B[i][j] + 1$
26:    end if
27:    if $B[i][j] \neq 0 \land E[i][j] == -1$ then
28:        $M[i][j] \leftarrow B[i][j]$
29:        $E[i][j] \leftarrow k$
30:        $k \leftarrow k + 1$
31:        if $q > n \times (n - 1) - 1$ then
32:            break
33:        end if
34:    end if
35:    if $q_{pre} == q$ then
36:        break
37:    end if
38:    $q_{pre} \leftarrow q$
39: end for
40: return $M, E$
where $w'$ represents the transformed weight, and $\lfloor \cdot \rfloor$ means round down.

We have noticed that if arbitrary path with weight $k$ can be obtained by combining the path with weight $k-1$ and an edge of weight 1, no matter how the weighted edges are distributed, the DANW can accurately calculate the result without losing any paths in the graphs.

### 3.3 Time Complexity of the DAWN

In this section, we discuss the time complexity of DAWN in detail. First, the time complexity of the unoptimized DAWN algorithm is as follows:

$$T(n) = O(d \cdot n^3), \quad (22)$$

where $d$ represents the diameter of the graph. The diameter of the graph can be up to $n-1$ on the unweighted graphs, which will cause the DAWN time complexity to increase to $n^4$. However, there have been many well-established studies showing that the diameter of a graph (network) is $\mu \cdot \log n$[43–48], where $\mu$ is a constant parameter.

Volker Strassen proposed the matrix block multiplication algorithm requiring $O(n^{\log 7})$ time[49], which uses 7 multiplications and 18 additions. In 1987, Coppersmith and Winograd proposed a method accelerating matrix multiplication, which requires $O(n^{2.376})$ time[50]. Williams contributed the latest representative work on the matrix block multiplication algorithm[51], which runs in the time of $O(n^{2.3727})$. We use the Coppersmith–Winograd algorithm to reduce the time complexity of the algorithm and speed up the computation. So we get the time complexity of DAWN as

$$T(n) = O(n^{2.376} \cdot \log n), \quad (23)$$

and we introduce targeted optimization methods in Section 3.4, which reduce the time complexity of DAWN.

### 3.4 Optimization Methods

In this section, we introduce the optimization methods of the DAWN algorithm, which are the path combination and matrix block method.

#### 3.4.1 Path Combination

In 1999, Albert-László Barabási et al. found scale-free properties in real networks[44, 52], and the degree distribution obeys a power-law distribution. We found that there are some graphs that obey scale-free properties with long diameters, and show one of real networks used in the study, which is a collaboration network [53]. The method of path combination can be used to reduce the number of matrix multiplication calculations on the graph.

The time complexity of the path combination method is $O(n)$, and $x, y$ represents the pair of nodes for which the shortest path has not yet occurred
in the previous calculation. When the number of non-occurring node pairs is reduced to $\mu \cdot n \log n$, this method can be used instead of matrix multiplication, which reduces time complexity and is parallelizable. A necessary condition for parallelism is that most pairs of nodes in the graph have already found the shortest path. When the scale of the graph is relatively small, the threshold of the path combination method can be limited to $\mu n$.

Optimized via the path combination method, the time complexity of DAWN is reduced from $O(n^{2.376} \cdot d)$ to $O[n^{2.376} \cdot (d - \eta) + \mu n^2 \cdot \log n]$, where $\eta$ represents the reduced times of matrix multiplications. In the collaboration network, the path combination method reduces matrix multiplications to 9, which originally required 16 multiplications.

### 3.4.2 Matrix Block

The matrix block algorithm is only suitable for even-order matrices, and if the matrix cannot be continuously divided into blocks, it will greatly limit

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**Algorithm 3** Path Combination

**Input:** $E, \{x, y\}$

**Output:** $E$

1. for $i = 0$ to $n - 1$ do
2. if $\text{length}[x][i] \neq -1 \wedge \text{length}[i][y] \neq -1$ then
3. $\text{tmp} \leftarrow \text{length}[x][i] + \text{length}[i][y]$
4. if $\text{length}[x][y] == -1 \vee \text{length}[x][y] > \text{tmp}$ then
5. $\text{length}[x][y] \leftarrow \text{tmp}$
6. end if
7. end if
8. end for
9. return $E$
the performance of the algorithm. The graphs which has $\mu \cdot 2^n$ nodes is very rare, and we propose a novel method to expand the scope of application of the algorithm.

If the dimension of the adjacency matrix $A = (a_{i,j})_{n \times n}$ is odd, we add a isolated node to the graph. It means that we add row and column to the matrix, which all entry in the row and column are set to 0. We define the augmented matrix of $A$:

$$A' = (a'_{i,j})_{(n+1) \times (n+1)},$$

(24)

where $a_{n,k_i} = 0$, $a_{k_j,n} = 0$, $a'_{i,j} = a_{i,j}$ and $k_i, k_j \in [1, n+1]$.

**Theorem 2**

$$c'_{i,j} = \sum_{l=1}^{n+1} a'_{i,l} \times b'_{l,j} = \sum_{l=1}^{n} a_{i,l} \times b_{l,j} = c_{i,j}, \text{ where } i, j \in [1, n].$$

**Proof** First, we confirm about the fact following.

**Fact 3** In the graphs, adding the isolated nodes would not affect the shortest paths in them.

Isolated nodes means that these nodes are not connected to any other nodes in the graphs. Therefore, all elements in the row and column representing the connection relationship of isolated nodes are 0, in the adjacency matrix. Then, we get from Formula (5),

$$c'_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} + a_{i,n_0}b_{n_0,j},$$

(25)

where $n_0$ represents the index of isolated node. Because all elements of the row and column representing the connection relationship of isolated nodes are 0, we get $a_{i,n_0} = 0 \land a_{n_0,j} = 0$, and

$$a_{i,n_0}b_{n_0,j} = 0.$$

(26)

For a matrix that qualifies for partitioning, the results of multiplication and block multiplication are consistent. So, we get that adding a new row and column to the matrix in which all elements are set to 0 would not affect the result of matrix block multiplications.

We can get from Theorem 2,

$$a^{(k)}_{i,j} = a_{i,1}a^{(k-1)}_{1,j} + a_{i,2}a^{(k-1)}_{2,j} + \cdots + a_{i,n}a^{(k-1)}_{n,j} + a_{i,n_0}a^{(k-1)}_{n_0,j} = a^{(k)}_{i,j},$$

(27)

Thus, we get that isolated nodes do not affect the result of matrix multiplication. This operation, which requires at most $\log n$ time, does not affect the result of matrix block multiplication and extends the applicability of the DAWN. The temporal complexity of the algorithm slightly increases to $O((n + \log n)^2.376)$.

Matrix block multiplication can not only reduce the time complexity of the algorithm, but also provide the basic conditions for the DAWN to accelerate by multi-GPU systems. Since the time complexity of matrix multiplication is only determined by the dimension of the matrix, we do not need to use
the sophisticated synchronization mechanism, and can efficiently coordinate multiple GPUs in the computing systems through simple task division. The dependency of the DAWN being accelerated by multi-GPU systems is the data independence of matrix block multiplication, not the segmentation of the graph, and does not need to consider about the characteristics of the graphs.

4 Results

In this section, the real networks come from the Stanford University public dataset[54], and the Barabási–Albert models and Erdős–Rényi model graphs are generated using NetworkX. NetworkX is a Python package for complex network analysis; it has excellent performance and a wide range of application scenarios[55]. Erdos et al. proposed the concept of the random graph in 1960[56].

Table 2  Parameter Table of the Test Machine

| Hardware | Parameters          |
|----------|---------------------|
| CPU      | Intel Xeon Gold 6151|
| RAM      | 512GB               |
| GPU      | NVIDIA TESLA V100   |
| IDE      | Visual Studio 2019  |
| OS       | Windows Server 2019 |
| Toolkit  | CUDA 11.4           |

We noticed that the maximum of matrix orders in multiplications that TESLA V100 can support is about 50000 when DAWN is running. When the scale of the network is above the upper limit, continuously partitioning the matrix becomes a natural choice. Table 1 shows some of the parameters of the test machine.

We show the running time of DAWN on Erdős–Rényi graphs and Barabasi–Albert graphs with 10000 nodes, which are the random graphs generated using NetworkX. The unit for the running time is seconds, with three decimals reserved. The speedup shown in the table is the average value of 5 tests. We compared the original Dijkstra (OD)[7], the version optimized by sequential priority queue(DOQ)[26], the version optimized via binary heap to implement priority queue(DOBH)[31], DAWN and the parallel versions based on OpenMP of the above algorithms. Moret et al. found the standard binary heap performed better than the rest in most cases by an experimental study[34], thus we take the DOBH with parallel accelerated as the baseline.

Katzav et al. proposed the phenomenon of network diameter shrinkage[61]. As the density increases, the diameter of the network shrinks and the calculation time of DAWN on the networks decreases. In these tests, the speedup ratio
can reach up to 709. Compared with the Dijkstra algorithm, which is optimized by binary heap and parallel computing, DAWN can achieve of $352.479 \times$ and $19.272 \times$, respectively.

We tested the performance on the some real networks: General Relativity and Quantum Cosmology collaboration network\cite{53}, Autonomous systems - Oregon-1\cite{57}, Enron email network\cite{58}, High-energy physics

### Table 3 Parameter Table of Real Networks

| Name          | Nodes | Edges     | Diameter | Nodes in largest SCC |
|---------------|-------|-----------|----------|----------------------|
| GRQC\(^1\)   | 5242  | 14496     | 17       | 0.793                |
| ASO1\(^2\)   | 10670 | 22002     | 9        | 1.000                |
| EE\(^3\)     | 36692 | 183831    | 11       | 0.918                |
| HEPCCN\(^4\) | 34546 | 421578    | 12       | 0.368                |
| Brightkite\(^5\) | 58228 | 214078   | 16       | 0.974                |
| GESCFA\(^6\) | 50515 | 819306    | 11       | 1.000                |

Source: This is a parameter table footnote of the real networks.

\(^1\) General Relativity and Quantum Cosmology collaboration network is from the e-print arXiv and covers scientific collaborations between authors papers submitted to General Relativity and Quantum Cosmology category\cite{53}.

\(^2\) Autonomous systems - Oregon-1 is the graph of Autonomous Systems (AS) peering information inferred from Oregon route-views between March 31 2001 and May 26 2001\cite{57}.

\(^3\) Enron email network is originally made public, and posted to the web, by the Federal Energy Regulatory Commission during its investigation\cite{58}.

\(^4\) High-energy physics citation network is from the e-print arXiv and covers all the citations within a dataset of 34,546 papers with 421,578 edges\cite{57}.

\(^5\) Brightkite was once a location-based social networking service provider where users shared their locations by checking-in\cite{59}.

\(^6\) Graph Embedding with Self Clustering: Facebook-Artist represents blue verified Facebook page networks of different categories\cite{60}.
citation network [57], Brightkite [59], Graph Embedding with Self Clustering: Facebook-Artist [60]. The artifact is available on GitHub (https://github.com/lxrzlyr/DAWN-and-Baseline-algorithm.git). It contains source code packages for evaluation and source websites for the public data. We use tables to show the detailed parameters of these networks.

Fig. 4 GRQC is from the e-print arXiv and covers scientific collaborations between authors papers submitted to General Relativity and Quantum Cosmology category, has 5242 nodes and 14496 edges. We take the DOBH with parallel accelerated as the baseline and the vertical axis is the logarithm of the speedup.

Fig. 5 ASO1 is a graph of the Autonomous Systems (AS) peering information inferred from Oregon route-views, has 10670 nodes and 22002 edges. We take the DOBH with parallel accelerated as the baseline and the vertical axis is the logarithm of the speedup.

Fig. 6 Enron email communication network posted to the web by the Federal Energy Regulatory Commission during its investigation, which consists of 36692 nodes and 183831 edges. We take the DOBH with parallel accelerated as the baseline and the vertical axis is the speedup.

Fig. 7 HEPCN is the high energy physics phenomenology citation graph from the e-print arXiv and covers all the citations within a dataset of 34546 nodes with 421578 edges. We take the DOBH with parallel accelerated as the baseline and the vertical axis is the speedup.

On the graph with tens of thousands nodes, the naive Dijkstra algorithm and the serial version of the algorithm are extremely computationally expensive. We replace the baseline algorithm with a parallel version of the binary heap optimized Dijkstra algorithm. On the graph generated by computing,
the diameter of the graph is strongly correlated with the number of nodes and edges in the graph. On the real networks, the diameter is related to the characteristics of itself, and even on a small-scale network, the diameter may be large. GRQC is one such example, and DAWN does not perform as well as we would expect. The results prove that on sparse and small-scale graphs, the transmission overhead greatly limits the performance of DAWN.

The problem of long data transfer time is a problem that has not been solved, and we did not consider at the beginning of the DAWN design. We use the cudastream to asynchronously transfer data to the GPU, and hoping to reduce the waiting time. In the experiment, the average waiting time is still more than 40% of the operation time. We have to admit that in addition to the graph diameter, the performance of the test machine’s memory also affects the operation time of DAWN.

Wang Kai et al. mentioned that many GPU SSSP algorithms use double buffering, which reduces the concurrency of the algorithm. On a experimental graph, the average work count per iteration is only 800, while a RTX 2080 GPU has 68K hardware threads[6]. DAWN does not have the problem of read-write conflicts, and the GPU can utilize almost threads to calculate matrix multiplication.

Compared with the parallel binary heap optimization dijkstra algorithm, the DAWN achieves the speedup of $1.038 \times$, $5.749 \times$, $39.031 \times$, $39.184 \times$, $72.619 \times$, and $89.919 \times$ on the real networks shown by Fig.4-9, respectively.

5 Conclusion

In this paper, we proposed a novel all-pair shortest path algorithm based on matrix operations, which has been named DAWN and requires $O(n^2)$ space.
and $O(d \cdot n^{2.387})$ time. We propose a novel method based on the matrix representation of graphs, which transforms the shortest path problem into the linear algebra problem and can fully exploit the huge advantages of GPUs. This transformation method was first systematically described, and we developed the feasible solution based on mathematical foundations. DAWN can accelerate computing via the multi-GPU systems, and its time complexity depends on the number of nodes and is sensitive to graph diameter. Compared with the Dijkstra algorithm optimized by the binary heap with 32 threads in parallel computing, the DAWN achieves the average of $41.257 \times$, and the maximum of $89.919 \times$. 
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Appendix A  Example

There is a directed and weighted graph $G_8 = (V, E, W)$:

$$G_8 = \begin{bmatrix}
0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (A1)$$

We use the matrix $H$ to represent $G_8$:

$$H^{(1)} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\quad H^{(2)} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\quad H^{(4)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (A2)$$
which has been defined in Formula (12). The weighted edges start to affect the matrix product in $G(3)$:

$$L^{(3)} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}, \quad R^{(3)} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix} \quad (A3)$$

In this example, the missing term of the approximation method is exactly 0, so it shows good accuracy. Let’s discuss this issue more generally. The definitions of matrices $L$ and $R$ can be found in Section 3.2 of the paper by Formula (14) and (15).

Paths can be combined by shorter paths, which is the general case of Fact 1 in the Section 3.1. We provide an equation for this as

$$x_1 + 2x_2 + 3x_3 \cdots + (w_i - 1)x_{w_i-1} + w_ix_{w_i} = w_i, \quad (A4)$$

and $w_i$ represents the maximum weight of the path to be solved in this time. This is a multivariate linear equation with many solutions, and each solution is a combination of paths. George Pólya propose that the method of generative function can be used to solve the problem of the integer split[42].

We define the generating function of the $G_8$:

$$C(x, y, z, w) = (1 + x + x^2 + x^4) \times (1 + y^2 + y^4) \times (1 + z^3) \times (1 + w^4), \quad (A5)$$

where $x$, $y$, $z$ and $w$ represents edges with weight 1, 2, 3 and 4, respectively. The exponents of the terms represent the weights of the path combination, and we keep only terms with exponent 4:

$$C(x, y, z, w)^{(4)} = x \cdot z^3 + x^4 + y^4 + w^4 + x^2 \cdot y^2. \quad (A6)$$

Thus, we get all path combinations of weight 4,

$$Path = x \cdot z + z \cdot x + x \cdot x \cdot x + y \cdot y + w + x \cdot y \cdot x \cdot y \cdot x \cdot x \cdot y, \quad (A7)$$
which has 8 possibilities. Then, we demonstrate the capabilities of DAWN:

\[
L^{(1)} = H^{(1)}, \\
L^{(2)} = L^{(1)} \times H^{(1)} + H^{(2)} \\
    = (H^{(1)})^2 + H^{(2)}, \\
L^{(3)} = L^{(2)} \times H^{(1)} + H^{(3)} \\
    = [(H^{(1)})^2 + H^{(2)}] \times H^{(1)} + H^{(3)} \\
    = (H^{(1)})^3 + H^{(2)} \times H^{(1)} + H^{(3)}, \\
L^{(4)} = L^{(3)} \times H^{(1)} + H^{(4)} \\
    = [(H^{(1)})^3 + H^{(2)} \times H^{(1)} + H^{(3)}] \times H^{(1)} + H^{(4)} \\
    = (H^{(1)})^4 + H^{(2)} \times (H^{(1)})^2 + H^{(3)} \times H^{(1)} + H^{(4)}. \\
R^{(1)} = H^{(1)}, \\
R^{(2)} = H^{(1)} \times R^{(1)} + H^{(2)} \\
    = (H^{(1)})^2 + H^{(2)}, \\
R^{(3)} = H^{(1)} \times R^{(2)} + H^{(3)} \\
    = H^{(1)} \times [(H^{(1)})^2 + H^{(2)}] + H^{(3)} \\
    = (H^{(1)})^3 + H^{(1)} \times H^{(2)} + H^{(3)}, \\
R^{(4)} = H^{(1)} \times R^{(3)} + H^{(4)} \\
    = H^{(1)} \times [(H^{(1)})^3 + H^{(1)} \times H^{(2)} + H^{(3)}] + H^{(4)} \\
    = (H^{(1)})^4 + H^{(1)} \times H^{(2)} \times H^{(1)} + H^{(1)} \times H^{(3)} + H^{(4)}. \\
\]

We get these path combination, \((H^{(1)})^4, H^{(2)} \times (H^{(1)})^2, H^{(3)} \times H^{(1)}, H^{(1)} \times H^{(2)} \times H^{(1)}, H^{(1)} \times H^{(3)}, H^{(4)}, \) and \((H^{(2)})^2\) is not included.

Obviously, the DAWN algorithm can only consider one of weighted edge in a path at most, such as 3 + 1, and would lose the path like 2 + 2. In the graph, if arbitrary path with weight \(k\) can be obtained by combining the path with weight \(k - 1\) and an edge of weight 1, no matter how the weighted edges are distributed, the DANW can accurately calculate the result without losing any paths.

There is a close relationship between the accuracy of the approximation method and the characteristics of the graph. Approximate method is suitable for the graphs, which mainly contains edges with smaller weights and a few edges with larger weights, and has good accuracy on these graphs.

Appendix B  Applicability

In this section, we will discuss in detail the weighted graph model applicable to the DAWN algorithm.
Theorem 3 The DAWN has good accuracy on the graphs which mainly contains edges with smaller weights and a few edges with larger weights.

Proof First, we exemplify the general form of the weighted path:

\[ P = \{ 1 + k_1, k_2 + k_3, k_4 \}, \]  

(B8)

where \( P \) is the set of the paths, \( k_1, k_2, k_3, k_4 \geq 2 \), and \( k_1 + 1 = k_2 + k_3 = k_4 \). And we get:

\[ P(i, j|k_4) = a_{i,j}^{(k_4)} = \sum_{k_i=1}^{n} a_{i,k_i}^{(1)} a_{k_i,j}^{(k_1)} + \sum_{k_j=1}^{n} a_{i,k_j}^{(k_2)} a_{k_j,j}^{(k_3)} + h_{i,j}^{(k_4)}, \]  

(B9)

where \( h_{i,j}^{(k_4)} \) represents the edge with weight \( k_4 \). We found DAWN cannot calculate the combination of two paths whose weight is not 1. If the path with weight \( k_2 \) or \( k_3 \) can be split into the path combinations, the path combinations \( k_2 + k_3 \) will change into the \( k_1 + 1 \), which can be calculated by DAWN. Thus, we obtain the following fact.

Fact 3 DAWN can get the path combinations that contain only one edge with the weight greater than 1 which cannot be replaced by other path combinations.

We can easily find path combinations to replace the edges with weight greater than 1, which ensures the accuracy of the calculation as much as possible, on the graph that is not particularly sparse and whose weight values are generally small. Thus, we get the Theorem 3. \( \square \)

We use a set of experiments to verify the accuracy of the DAWN algorithm on the weighted graphs. The real network data of the experimental group comes from the Stanford open source dataset, which is who-trusts-whom network of Table B1 Parameter Table of Graphs Group

| Graph | Nodes | Weighted Edges | Edges | Proportion$^1$ |
|-------|-------|----------------|-------|---------------|
| A-5573$^2$ | 5573 | 11981 | 32029 | 0.374 |
| B-5509$^3$ | 5509 | 11108 | 31156 | 0.357 |
| C-5484$^4$ | 5484 | 10623 | 30671 | 0.346 |
| D-5399$^5$ | 5399 | 9090 | 29138 | 0.312 |
| E-5228$^6$ | 5228 | 5562 | 25610 | 0.217 |

Source: This is a parameter table footnote of the subnetworks of the real networks.

1Proportion represents the ratio of the weighted edges to the total number of edges.

2Graph A-5573 is the subgraph formed by deleting edges with negative weights in the original graph, and deletes the resulting isolated nodes.

3Graph B-5509 is the subgraph formed by deleting edges with weights 9 and 10 in the Graph A-5573, and deletes the resulting isolated nodes.

4Graph C-5484 is the subgraph formed by deleting edges with weights 7 and 8 in the Graph B-5509, and deletes the resulting isolated nodes.

5Graph D-5399 is the subgraph formed by deleting edges with weights 5 and 6 in the Graph C-5484, and deletes the resulting isolated nodes.

6Graph E-5228 is the subgraph formed by deleting edges with weights 3 and 4 in the Graph D-5399, and deletes the resulting isolated nodes.
people who trade using Bitcoin on a platform called Bitcoin OTC[62, 63]. The weight of the network describes the degree of trust from $v_i$ to $v_j$, in a scale of -10 (total distrust) to +10 (total trust).

Neither the DAWN nor Dijkstra’s algorithm can handle negative weighted edges, so we delete the edges with negative weights in the graph. We gradually reduce the number of weighted edges to obtain a set of graphs and list the parameters of the graphs in the table. We use experiments to show the variation trend of the computational accuracy of DAWN.

![Graph](image)

**Fig. B2** Graph is the who-trusts-whom network of people who trade using Bitcoin on a platform called Bitcoin OTC[62, 63]. We gradually reduce the number of weighted edges to obtain a set of graphs. The x-axis is the graphs, the y-axis is the scale.

The scale of paths found refers to the scale of the node pairs with the shortest paths found by DAWN to the total number of connected node pairs. The scale of paths corrected refers to the scale of the node pairs with the corrected shortest paths found by DAWN to the total number of connected node pairs.