MULTIQUADRATIC FIELDS GENERATED BY CHARACTERS OF $A_n$

MADELINE LOCUS DAWSEY*, KEN ONO, AND IAN WAGNER

Abstract. For a finite group $G$, let $K(G)$ denote the field generated over $\mathbb{Q}$ by its character values. For $n > 24$, G. R. Robinson and J. G. Thompson [6] proved that

$$K(A_n) = \mathbb{Q}\left(\sqrt{m^*} : p \leq n \text{ an odd prime with } p \neq n - 2\right),$$

where $p^* := (-1)^{\frac{m^* - 1}{2}} m$. Confirming a speculation of Thompson [7], we show that arbitrary suitable multiquadratic fields are similarly generated by the values of $A_n$-characters restricted to elements whose orders are only divisible by ramified primes. To be more precise, we say that a $\pi$-number is a positive integer whose prime factors belong to a set of odd primes $\pi := \{p_1, p_2, \ldots, p_t\}$. Let $K_\pi(A_n)$ be the field generated by the values of $A_n$-characters for even permutations whose orders are $\pi$-numbers. If $t \geq 2$, then we determine a constant $N_\pi$ with the property that for all $n > N_\pi$, we have

$$K_\pi(A_n) = \mathbb{Q}\left(\sqrt{p_1^*}, \sqrt{p_2^*}, \ldots, \sqrt{p_t^*}\right).$$

1. Introduction and Statement of Results

If $G$ is a finite group, then let $K(G)$ denote the field generated over $\mathbb{Q}$ by all of the $G$-character values. In stark contrast to the case of symmetric groups, where $K(S_n) = \mathbb{Q}$, G. R. Robinson and J. G. Thompson [6] proved for alternating groups that the $K(A_n)$ are generally large multiquadratic extensions. In particular, for $n > 24$, they proved that

$$K(A_n) = \mathbb{Q}\left(\sqrt{m^*} : p \leq n \text{ an odd prime with } p \neq n - 2\right),$$

where $m^* := (-1)^{\frac{m - 1}{2}} m$ for any odd integer $m$.

In a letter [7] to the second author in 1994, Thompson asked for a refinement of this result that mirrors the Kronecker-Weber Theorem and the theory of complex multiplication, where abelian extensions are generated by the values of $e^{2\pi i x}$ and modular functions respectively, at arguments that determine the ramified primes. Instead of employing special analytic functions at designated arguments, which is the gist of Hilbert’s 12th Problem [4], Thompson offered the characters of $A_n$ evaluated at elements whose orders are only divisible by ramified primes.

To formulate this problem, we let $\pi := \{p_1, p_2, \ldots, p_t\}$ denote a set of $t \geq 2$ distinct odd primes listed in increasing order. A $\pi$-number is a positive integer whose prime factors belong to $\pi$. The speculation is that $K_\pi(A_n)$, the field generated by the values of $A_n$-characters restricted to elements $\sigma \in A_n$ with $\pi$-number order, generally generates $\mathbb{Q}(\sqrt{p_1^*}, \sqrt{p_2^*}, \ldots, \sqrt{p_t^*})$. However, an inspection of the character tables for the first few $A_n$ casts doubt on this speculation. For example, one easily finds that

$$\mathbb{Q} = K_{\{3,5\}}(A_{22}) = K_{\{5,7\}}(A_{1738}) = K_{\{7,11\}}(A_{159557})$$

2010 Mathematics Subject Classification. 20B30, 20C30, 11R20.

Key words and phrases. Hilbert’s 12th Problem, Multiquadratic fields, group characters.

*This author was previously known as Madeline Locus. The second author thanks the generous support of the Asa Griggs Candler Fund, and the NSF (DMS-1601306).

1The phenomenon cannot hold when $t = 1$ for any $n \not\equiv 0, 1 \pmod{p}$. 

by checking that the $A_n$-character values for even permutations with $\pi$-number order are in $\mathbb{Z}$.

Despite these discouraging examples, Thompson’s speculation is indeed true for sufficiently large $A_n$. We let $\Omega_\pi := \frac{3}{4} ( p_t - 1 )^2$, and in turn we define

\[(1.1) \quad N^{+}_\pi := p_t + 2^{2\Omega_\pi}.\]

The formula for $N^{+}_\pi$ provides a bound for this phenomenon. It can generally be improved when $t \geq 3$. To make this precise, we let $\Omega^{\pi}_\pi := \frac{p}{t} ( p_t - 1 )$ and in turn we define

\[N^{+}_\pi := \frac{p}{t} + \max \left\{ C^{10}_\pi, 2^{400} \right\}. \]

We obtain the following confirmation of Thompson’s speculation in terms of the bound $N^{+}_\pi := \min \{ N^{+}_\pi, N^{-}_\pi \}$.

**Theorem 1.1.** Assuming the notation above, if $n > N^{+}_\pi$, then

\[K^{\pi}(A_n) = \mathbb{Q} \left( \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_t} \right).\]

**Remark.** The proof of Theorem 1.1 uses the fact that $n - p_i$, for each $1 \leq i \leq t$, is the sum of distinct squares of $\pi$-numbers provided $n > N^{+}_\pi$. These Diophantine conditions guarantee that $\mathbb{Q}(\sqrt{p}) \subset K^{\pi}(A_n)$. However, these conditions are not necessary for these inclusions.

**Remark.** Although Theorem 1.1 requires that $t \geq 2$, it is simple to generate $\mathbb{Q}(\sqrt{p})$ for an odd prime $p$ using permutations with cycle lengths that are $\pi$-numbers with $\pi = \{ p, q \}$, where $q \neq p$ is an odd prime. As mentioned above, if $n > N^{+}_\pi$, then $n - p$ is a sum of distinct squares of $\pi$-numbers. Then $K^{\pi}(A_n)$, the field generated by the values of the $A_n$-characters restricted to permutations with a single cycle of length $p$ and other cycle lengths of such odd squares, satisfies $K^{\pi}(A_n) = \mathbb{Q}(\sqrt{p})$.

To prove Theorem 1.1, we follow the straightforward approach of Robinson and Thompson in [6]. In Section 2.1, we recall standard facts about the characters of $A_n$. Theorem 2.1 is the key device for relating cycle types to the surds $\sqrt{p}$. Then, in Section 2.2, we recall a classical result of J. W. Cassels [2] on partitions. We also recall recent work by J.-H. Fang and Y.-G. Chen [3] on a problem of Erdős which implies that every large positive integer is the sum of distinct squares of $\pi$-numbers when $\pi = \{ p_1, p_2 \}$. Theorem 1.1 follows immediately from these results.

**Acknowledgements**

The second author thanks John Thompson for sharing his speculation 25 years ago. The authors thank Geoff Robinson for comments on the first draft of this paper.
2. Nuts and Bolts and the Proof of Theorem 2.3

2.1. Character theory for $A_n$. It is well known that the representation theory of $S_n$ and $A_n$ can be completely described using the partitions of $n$. In particular, a permutation $\sigma \in S_n$ has a cycle type that can be viewed as a partition of $n$, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. The cycle type determines the conjugacy class of the permutation, and so the irreducible representations of $S_n$ can be indexed by (and constructed from) the partitions of $n$. It is also well known that the only conjugacy classes which split in $A_n$ are those corresponding to partitions into distinct odd parts. There is a bijection between the set of partitions $\lambda$ of $n$ into distinct odd parts and the set of self-conjugate partitions $\gamma$ of $n$ which is realized by identifying the parts of each $\lambda$ with the main hook lengths of some $\gamma$. Theorem 2.5.13 of [5] characterizes those $A_n$-character values that are not $\mathbb{Z}$-integral.

**Theorem 2.1.** Let $\sigma \in A_n$ be a permutation with cycle type given by a partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$, and let $d_\lambda := \prod_{i=1}^k \lambda_i$.

1. If $\lambda$ does not have distinct odd parts, then the $A_n$-character values of $\sigma$ are all in $\mathbb{Z}$.
2. If $\lambda$ has distinct odd parts, then let $\gamma$ be the self-conjugate partition of $n$ with main hook lengths $\lambda_1, \lambda_2, \ldots, \lambda_k$, and let $\chi_\gamma$ be the $A_n$-character associated to $\gamma$. We have that

$$\chi_\gamma(\sigma) = \frac{1}{2} \left( (-1)^{d_\lambda - 1} \pm \sqrt{d_\lambda^*} \right),$$

where $d_\lambda^* := (-1)^{d_\lambda - 1} \prod_{i=1}^k \lambda_i$. Moreover, every $A_n$-character $\chi$ which is not algebraically conjugate to $\chi_\gamma$ has $\chi(\sigma) \in \mathbb{Z}$.

2.2. Some facts about partitions into distinct parts. In 1959, Birch [1] proved a conjecture of Erdős on representations of sufficiently large integers by sums of distinct numbers of the form $p^aq^b$, where $p, q$ are coprime and $a, b$ are positive integers. In 2017, this result was quantified by Fang and Chen (see Theorem 1.1 of [3]).

**Theorem 2.2** (Fang-Chen). For any coprime integers $p, q > 1$, there exist positive integers $K$ and $B$ with

$$K < 2^{2p^2}, \quad B < 2^{2q^2}$$

such that every integer $n \geq B$ can be expressed as the sum of distinct terms taken from

$$\{p^aq^b \mid a \geq 0, 0 \leq b \leq K, a + b > 0, a, b \in \mathbb{Z}\}.$$

Motivated by Birch’s earlier work, Cassels (see Theorem 1 of [2]) studied the problem of representing sufficiently large integers as sums of distinct elements from suitable integer sequences. A careful inspection of his paper gives the following theorem, which was proved using the “circle method”.

**Theorem 2.3** (Cassels). Suppose that $\mathbb{N}_T = \{a_1, a_2, \ldots\}$ is a subset of positive integers in increasing order. Let $S_T(n)$ denote the number of integers in $\mathbb{N}_T$ not exceeding $n$. Choose $B_T \geq 2^{2^{40}}$ to be a positive real number for which

$$S_T(2n) - S_T(n) > 2^{20} \log_2 \log_2 n$$

for all $n \geq B_T$, and choose $C_T \geq B_T$ to be a positive real number for which

$$\sum_{s=1}^{S_T(C_T)} ||a_s \theta||^2 > 2 \log_2 B_T + 50$$
for all $\frac{1}{4B_T} \leq \theta \leq 1 - \frac{1}{4B_T}$. If such $B_T$ and $C_T$ exist, then every positive integer $n > \max\left\{C_T^{10}, 2^{2^{4400}}\right\}$ is expressible as a sum of distinct elements of $\mathbb{N}_T$.

2.3. Proof of Theorem 1.1. Let $\pi := \{p_1, \ldots, p_t\}$ be a set of distinct odd primes in increasing order. For $n > N_\pi$, we will establish, for each $1 \leq i \leq t$, that $n - p_i$ is a sum of distinct squares of $\pi$-numbers. These representations imply that $n = p_i + M_i$, where $M_i$ is a sum of distinct squares of $\pi$-numbers. Theorem 2.1 then implies the theorem.

By Theorem 2.2, for each pair $1 \leq i < j \leq t$, we have that every integer $n \geq N_{i,j} := 2^{2^{\Omega_\pi(i,j)}}$, where $\Omega_\pi(i,j) := p_i^4p_j^2$, is a sum of distinct terms taken from the set
\[
\left\{p_i^{2a}p_j^{3b} : a, b \geq 0, a + b > 0, a, b \in \mathbb{Z}\right\}.
\]
Obviously, the maximum bound occurs with $p_{t-1}$ and $p_t$, and so the theorem follows for $n > N^+_{\pi}$. Finally, Cassels’ result, where $\mathbb{N}_T$ denotes the set of squares of $\pi$-numbers, guarantees that Theorem 1.1 holds for all $n > N^-_{\pi}$.

References

[1] B. J. Birch, Note on a problem of Erdős, Math. Proc. Cambridge Phil. Soc. 55 (1959), 370-373.
[2] J. W. Cassels, On the representation of integers as sums of distinct summands taken from a fixed set, Acta Sci. Math. 21 (1960), 111-124.
[3] J.-H. Fang and Y.-G. Chen, A quantitative form of the Erdős-Birch theorem, Acta Arith. 178 (2017), 301-311.
[4] D. Hilbert, Mathematical problems, Bull. Amer. Math. Soc. 8 (1902), 437-479.
[5] G. James and A. Kerber, The representation theory of the symmetric group, Addison-Wesley, Reading, 1981.
[6] G. R. Robinson and J. G. Thompson, Sums of squares and the fields $\mathbb{Q}_n$, J. Algebra 174 (1995), 225-228.
[7] J. G. Thompson, Personal letter to the second author, February 11, 1994.