Remarks on planar edge-chromatic critical graphs

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Abstract

The only open case of Vizing’s conjecture that every planar graph with $\Delta \geq 6$ is a class 1 graph is $\Delta = 6$. We give a short proof of the following statement: there is no 6-critical plane graph $G$, such that every vertex of $G$ is incident to at most three 3-faces. A stronger statement without restriction to critical graphs is stated in [7]. However, the proof given there works only for critical graphs. Furthermore, we show that every 5-critical plane graph has a 3-face which is adjacent to a $k$-face ($k \in \{3, 4\}$).

For $\Delta = 5$ our result gives insights into the structure of planar 5-critical graphs, and the result for $\Delta = 6$ gives support for the truth of Vizing’s planar graph conjecture.

Keywords: planar graph; edge coloring; Vizing’s conjecture; critical graph

1 Introduction

We consider finite simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. The vertex-degree of $v \in V(G)$ is denoted by $d_G(v)$, and $\Delta(G)$ denotes the maximum vertex-degree of $G$. If it is clear from the context, then $\Delta$ is frequently used. A graph is planar if it is embeddable into the Euclidean plane. A plane graph $(G, \Sigma)$ is a planar graph $G$ together with an embedding $\Sigma$ of $G$ into the Euclidean plane. If $(G, \Sigma)$ is a plane graph, then

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$F(G)$ denotes the set of faces of $(G, \Sigma)$. The degree $d_{(G, \Sigma)}(f)$ of a face $f$ is the length of its facial circuit. A face $f$ is a $k$-face if $d_G(f) = k$, and it is a $k^+$-face if $d_G(f) \geq k$.

The edge-chromatic number $\chi'(G)$ of a graph $G$ is the minimum $k$ such that $G$ admits a proper $k$-edge-coloring. Vizing [4] proved that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$. If $\chi'(G) = \Delta(G)$, then $G$ is a class 1 graph, and it is a class 2 graph otherwise. A class 2 graph $H$ is $k$-critical, if $\Delta(H) = k$ and $\chi'(H') < \chi'(H)$ for every proper subgraph $H'$ of $H$.

Vizing [4] showed for each $k \in \{2, 3, 4, 5\}$ that there is a planar class 2 graph $G$ with $\Delta(G) = k$. He proved that every planar graph with $\Delta \geq 8$ is a class 1 graph, and conjectured that every planar graph with $\Delta \in \{6, 7\}$ is a class 1 graph. Vizing’s conjecture is proved for planar graphs with $\Delta = 7$ by Grünwald [1], Sanders, Zhao [3], and Zhang [7] independently. It is still open for the case $\Delta = 6$. The paper provides short proofs for the following statements.

**Theorem 1.1.** There is no 6-critical plane graph $(G, \Sigma)$, such that every vertex of $G$ is incident to at most three 3-faces.

If Vizing’s conjecture is not true, then every 6-critical graph has the following property.

**Corollary 1.2.** Let $(G, \Sigma)$ be a plane graph. If $G$ is 6-critical, then there is a vertex of $G$ which is incident to at least four 3-faces.

**Theorem 1.3.** Let $(G, \Sigma)$ be a plane graph. If $G$ is 5-critical, then $(G, \Sigma)$ has a 3-face which is adjacent to a 3-face or to a 4-face.

A significant longer proof of Theorem 1.1 is given in [5], but the statement is formulated for plane graphs. However, the proof works for critical graphs only. The assumption that a minimal counterexample is critical is wrong. It might be that a subgraph of this minimal counterexample $G$ does not fulfill the pre-condition of the statement. For example, if $G$ has a triangle $[vxyv]$ and a bivalent vertex $u$ such that $u$ is the unique vertex inside $[vxyv]$ and $u$ is adjacent to $x$ and $y$, then the removal of $u$ increases the number of 3-faces containing $v$ (see Figure 1).

### 2 Proofs of Theorems 1.1 and 1.3

We will use the following two lemmas.

**Lemma 2.1** ([2]). If $G$ is a 6-critical graph, then $|E(G)| \geq \frac{1}{2}(5|V(G)| + 3)$.

**Lemma 2.2** ([6]). If $G$ is a 5-critical graph, then $|E(G)| \geq \frac{15}{4}|V(G)|$. 

Proof of Theorem 1.1

Suppose to the contrary that there is a counterexample to the statement. Then there is a 6-critical graph \( G \) which has an embedding \( \Sigma \) such that every \( v \in V(G) \) is incident to at most three 3-faces. With Euler’s formula and Lemma 2.1 we deduce
\[
\sum_{f \in F(G)} (d_G(f) - 4) = 2|E(G)| - 4|F(G)| = 2|E(G)| - 4(|E(G)| + 2 - |V(G)|) \leq -|V(G)| - 11. \]
Therefore, \(|V(G)| + \sum_{f \in F(G)} (d_G(f) - 4) \leq -11\).

Give initial charge 1 to each \( v \in V(G) \) and \( d_G(f) - 4 \) to each \( f \in F(G) \). Discharge the elements of \( V(G) \cup F(G) \) according to the following rule:

\textbf{R1:} Every vertex sends \( \frac{1}{3} \) to its incident 3-faces.

The rule only moves the charge around and does not affect the sum. Furthermore, the final charge of every vertex and face is at least 0. Therefore, \( 0 \leq \sum_{v \in V(G)} 1 + \sum_{f \in F(G)} (d_G(f) - 4) = |V(G)| + \sum_{f \in F(G)} (d_G(f) - 4) \leq -11, \) a contradiction.

Proof of Theorem 1.3

Suppose to the contrary that there is a counterexample to the statement. Then there is a 5-critical graph \( G \) which has an embedding \( \Sigma \) such that every 3-face is adjacent to 5+-faces only. Hence, every vertex of \( G \) is incident to at most two 3-faces, and every vertex which is incident to a 3-face is also incident to a 5+-face. By Lemma 2.2 we have
\[
\sum_{f \in F(G)} (d_G(f) - 4) \leq -\frac{2}{7}|V(G)| - 8. \]
Therefore, \( \frac{2}{7}|V(G)| + \sum_{f \in F(G)} (d_G(f) - 4) \leq -8. \)

Give initial charge of \( \frac{2}{7} \) to each vertex and \( d_G(f) - 4 \) to each face of \( G \). Discharge the elements of \( V(G) \cup F(G) \) according to the following rules:

\textbf{R1:} Every vertex sends \( \frac{1}{3} \) to its incident 3-faces.

\textbf{R2:} Every 5+-face sends \( \frac{d_G(f) - 4}{d_G(f)} \) to its incident vertices.

Denote the final charge by \( ch^* \). Rules R1 and R2 imply that \( ch^*(f) \geq 0 \) for every \( f \in F(G) \). Let \( n \leq 2 \) and \( v \) be a vertex which is incident to \( n \) 3-faces. If \( n = 0 \), then
$\text{ch}^*(v) \geq \frac{2}{7} > 0$. If $n = 1$, then $v$ is incident to at least one $5^+$-face, and therefore, $\text{ch}^*(v) \geq \frac{2}{7} + \frac{1}{5} - \frac{1}{3} > 0$ by rule R2. If $n = 2$, then $v$ is incident to at least two $5^+$-faces, and therefore $\text{ch}^*(v) \geq \frac{2}{7} + 2 \times \frac{1}{5} - 2 \times \frac{1}{3} = \frac{2}{105} > 0$, by rule R2. Hence, $0 \leq \sum_{v \in V(G)} \frac{2}{7} + \sum_{f \in E(G)} (d_G(f) - 4) \leq -8$, a contradiction.

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