Elliptic and parabolic problems in thin domains with doubly weak oscillatory boundary

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Dedicated to Tomás Caraballo on the occasion of his 60-th birthday

Abstract: In this work we consider higher dimensional thin domains with the property that both boundaries, bottom and top, present oscillations of weak type. We consider the Laplace operator with Neumann boundary conditions and analyze the behavior of the solutions as the thin domains shrinks to a fixed domain \( \omega \subset \mathbb{R}^n \). We obtain the convergence of the resolvent of the elliptic operators in the sense of compact convergence of operators, which in particular implies the convergence of the spectra. This convergence of the resolvent operators will allow us to conclude the global dynamics, in terms of the global attractors of a reaction diffusion equation in the thin domains. In particular, we show the upper semicontinuity of the attractors and stationary states. An important case treated is the case of a quasiperiodic situation, where the bottom and top oscillations are periodic but with period rationally independent.

Keywords: Thin domain; oscillatory boundary; compact convergence; attractors; quasiperiodic oscillations

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1 Introduction

We are interested in analyzing the behavior of the solutions of certain Partial Differential Equations of elliptic and parabolic type which are posed in a varying domain \( R^\epsilon \). This domain is a thin domain in \( \mathbb{R}^{n+1} \) which presents an oscillatory behavior at the boundary and it is given as the region between two oscillatory functions, that is,

\[
R^\epsilon = \left\{ (x,y) \in \mathbb{R}^{n+1} \mid x \in \omega \subset \mathbb{R}^n, -\epsilon k_1(x,\epsilon) < y < \epsilon k_2(x,\epsilon) \right\},
\]

(1.1)

where \( \omega \) is a smooth bounded domain in \( \mathbb{R}^n \) and the functions \( k_1, k_2 \) satisfy certain hypotheses, see (H) below. Observe that the lower part of the boundary is described by the function \( k_1 \) and the upper part of the boundary by \( k_2 \). These two function may present oscillations. See Figure 1 for a two-dimensional example.

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One equation we are interested in is the following linear elliptic equation

\[
\begin{cases}
-\Delta w^\epsilon + w^\epsilon = f^\epsilon & \text{in } \mathbb{R}^\epsilon, \\
\frac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial \mathbb{R}^\epsilon,
\end{cases}
\] (1.2)

where \( f^\epsilon \in L^2(R^\epsilon) \), and \( \nu^\epsilon \) is the unit outward normal to \( \partial R^\epsilon \). But eventually we will be interested in saying something about the behavior of the global dynamics of a reaction diffusion equation of the type

\[
\begin{cases}
w^\epsilon_t - \Delta w^\epsilon + w^\epsilon = f(w^\epsilon) & \text{in } \mathbb{R}^\epsilon, t > 0, \\
\frac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial \mathbb{R}^\epsilon.
\end{cases}
\] (1.3)

As we will see and as it is already well known, see for instance [4, 5, 8, 34], the analysis of the convergence of (1.2) will basically dictate the behavior of the dynamics of (1.8) in terms of the behavior of the global dynamics (continuity of solutions, continuity of equilibria upper and lower semicontinuity of attractors, etc). As a matter of fact, if the solutions of (1.2) approach in certain sense the solutions of the limiting problem

\[-Lw + w = f \text{ in } \omega, \]

where \( L \) will be an elliptic operator then, the solutions of (1.8) will converge in certain sense to the solutions of the equation

\[w_t - Lw^\epsilon + w^\epsilon = f(w^\epsilon) \text{ in } \omega.\]

In order to simplify the notation, we write every point in \( \mathbb{R}^{n+1} \) as follows

\[(x, y) \in \mathbb{R}^{n+1}, \text{ with } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \text{ and } y \in \mathbb{R}.\]

Let \( k^i \) be a function, \( i = 1, 2, \)

\[k^i : \omega \times (0, 1) \rightarrow \mathbb{R}^+ \]

\[(x, \epsilon) \rightarrow k^i(x, \epsilon) = k^i_\epsilon(x), \]

such that

(H.1) \( k^i_\epsilon \) is a \( C^1 \) function in the first variable and

\[\epsilon \left| \frac{\partial k^i_\epsilon}{\partial x_j}(x) \right| \rightarrow 0 \text{ uniformly in } \omega, \quad j = 1, \ldots, n. \] (1.4)

(H.2) There exist two positive constants independent of \( \epsilon \) such that

\[0 < C^i_1 \leq k^i_\epsilon(\cdot) \leq C^i_2. \] (1.5)

(H.3) There exists a function \( K^i \) in \( L^2(\omega) \) such that

\[k^i_\epsilon \overset{w-L^2(\omega)}{\rightarrow} K^i. \]

Indeed, the thickness of the domain has order \( \epsilon \) and we say that the domain presents weak oscillations due to the convergence (1.4). An interesting particular example of the introduced general setting is the case where

\[k^1_\epsilon(x) = h(x/\epsilon^\alpha), \quad k^2_\epsilon(x) = g(x/\epsilon^\beta), \] (1.6)
with $0 < \alpha, \beta < 1$ and the functions $g, h : \mathbb{R}^n \to \mathbb{R}$ are $C^1$ periodic functions.

The variational formulation of (1.2) is the following: find $w^\epsilon \in H^1(R^\epsilon)$ such that

$$
\int_{R^\epsilon} \left\{ \nabla w^\epsilon \nabla \varphi + w^\epsilon \varphi \right\} dx_1 dx_2 = \int_{R^\epsilon} f^\epsilon \varphi dx_1 dx_2, \forall \varphi \in H^1(R^\epsilon).
$$

Observe that, for fixed $\epsilon > 0$, the existence and uniqueness of solution to problem (1.2) is guaranteed by Lax-Milgram Theorem. Then, we will analyze the behavior of the solutions as the parameter $\epsilon$ tends to zero. In particular, we first perform suitable change of variables which allows us to substitute, in some sense, the original problem (1.2) posed in an $(n+1)$-oscillating thin domain into a simpler problem with oscillating coefficients posed in an $n-$dimensional fixed domain. Notice that, this fact is in agreement with the intuitive idea that the family of solutions $u^\epsilon$ should converge to a function of just $n$ variables as $\epsilon$ goes to zero since the domain is thin. Subsequently, by using the previous results and adapting well-known techniques in homogenization we obtain explicitly the homogenized limit problem for the interesting cases where $k_i^\epsilon$, $i = 1, 2$, satisfy (1.6). In this respect, notice that the coupled effect of both oscillating boundaries requires to consider more general settings than classical periodicity, for instance, quasi-periodicity or reiterated homogenization framework.

It is worth observing that we cannot apply a direct homogenization technique to obtain the limit problem of (1.2) since there is not a representative cell which describe the domain and, moreover, the effect of the oscillations at both boundaries, top and bottom, cannot be separated in an easy way since at the same time that the boundaries oscillate, the domain is shrinking and therefore the effect of the oscillations at one boundary is coupled in a nontrivial way with the oscillations at the other boundary.

Finally, we show that the performed analysis of the elliptic linear problem (1.2) together with known results from the theory on nonlinear dynamics of dissipative systems allows to analyze the convergence properties of the solutions and attractors of the following semilinear parabolic evolution equation

$$
\begin{cases}
  w_t^\epsilon - \Delta w^\epsilon + w^\epsilon = f(w^\epsilon) \quad \text{in } R^\epsilon, t > 0, \\
  \frac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 \quad \text{on } \partial R^\epsilon,
\end{cases}
$$

where $\nu^\epsilon$ is the unit outward normal to $\partial R^\epsilon$ and the function $f : \mathbb{R} \to \mathbb{R}$ is a $C^2-$ function with bounded derivatives. Moreover, since we are interested in the behavior of solutions as $t \to \infty$ and
its dependence with respect to the small parameter $\epsilon$, we will require that the solutions of (1.8) be bounded for large values of time. A natural assumption to obtain this boundedness of the solutions is expressed in the following dissipative condition

$$\limsup_{|s| \to \infty} \frac{f(s)}{s} < 0. \quad (1.9)$$

In order to accomplish our goal, we consider here the linear parabolic problems associated to the perturbed equation (1.8) and its limit in the abstract framework given by [25,27]. Then, we show that under an appropriate notion of convergence, the spectral convergence and the convergence of the corresponding linear semigroups may be obtained. Moreover, we are in conditions to transfer this information to the nonlinear dynamics through the variation of constants formula. This way, the continuity of nonlinear semigroups and the upper semicontinuity of the set of stationary states can be established. We refer to the fundamental works [25], [14], [42], [38], [20] to understand the asymptotic dynamics of evolutionary equations and the behavior under perturbations. See also, [28,29] and references therein for related works.

We would like to mention that there are several papers addressing the problem of studying the effect of rough boundaries on the behavior of the solution of partial differential equations. Among others, we can mention [22,30] in the context of flows, [2,18] from the point of view of the elasticity and [15] where complete asymptotic expansions of the solutions were studied. In particular, let us point out that the convergence properties of the solutions of the elliptic equation (1.2) have been discussed in several papers in the literature for different kind of thin domains. It is well known which is the limit if the thin domain does not present oscillations, see for instance, [11,26,36,37]. The case where the thin domain presents weak roughness, $k_1^1(x) = h(x/\epsilon^{\alpha})$, $k_2^2(x) = 0$ with $0 < \alpha < 1$, was treated in [3] using changes of variables and rescaling the thin domain as in classical works in thin domains with no oscillations, see for instance [26,36]. The resonant case, $k_1^1(x) = h(x/\epsilon^{\alpha})$, $k_2^2(x) = 0$ with $\alpha = 1$, was studied in [8,32] using standard techniques in homogenization, see [16,23,24,37] for a general introduction to the homogenization theory. More recently, the homogenized limit problem for the case of thin domains with a fast oscillatory boundary, $k_1^1(x) = h(x/\epsilon^{\alpha})$, $k_2^2(x) = 0$ with $\alpha > 1$, was obtained in [10] by decomposing the domain in two parts separating the oscillatory boundary. Moreover, in [13] the previous cases were analyzed in a unified way adapting the recent unfolding operator method.

Indeed, we think that considering thin domains with doubly oscillatory boundary is a natural way to extend the models studied in previous papers to a more realistic situations where several microscopic scales appear. In fact, understanding how the complicated micro geometry of the thin structures affects the macro properties of the material is a very relevant issue in engineering and applied science. Moreover, let us point out since upper and lower boundary present different orders of frequency and profiles of oscillation it is not possible to analyze this kind of problems using a simple generalization of the methods applied to thin domains with only one oscillatory boundary.

The case where one of the two oscillatory boundary presents an extremely high oscillatory behavior, $k_1^1(x) = h(x/\epsilon^{\alpha})$, $k_2^2(x) = g(x/\epsilon^{\beta})$ with $\alpha > 1$, is well-know in the literature, see [10] for the periodic case and [34] for the locally periodic case where the parabolic problem is also studied. The authors of both papers combine the classical oscillatory test functions method of Tartar together with an adaptation of the method from [10] in order to get the homogenized limit
problem. However the approach introduced in both papers is essentially based on the fact that \( \alpha > 1 \).

Let us point out that despite the works mentioned above, the case introduced in this paper has not been treated previously in the literature. As a main novelty, we approach a problem posed in a general thin domain in \( \mathbb{R}^{n+1} \) with weak oscillations at top and bottom boundary, see assumption (1.4) without extra conditions on the periodicity by a more studied equation with oscillating coefficients in \( \mathbb{R}^n \). The crucial idea is that since the oscillations at the boundary are not too “wild” we can transform the original problem in \( \mathbb{R}^\epsilon \) into a one less dimensional simpler problem posed in a fixed domain through diffeomorphisms which will depend on the parameter \( \epsilon \).

This paper is organized as follows. In Section 2 we set up the notation and state important results which allows to reduce the problem to study the behavior of the solutions of an equation with oscillating coefficients in a fixed domain. In Section 3 and Section 4 we study the convergence properties of the elliptic and parabolic equation combining results of homogenization theory and the theory on nonlinear dynamics of dissipative systems.

We have given detailed proofs of the new results, when the weak oscillating boundaries play an important role, while, for the proofs involving routine procedures of homogenization theory or nonlinear dynamics of dissipative systems theory we refer to well-known results in the literature.

## 2 Reduction to an \( n \)-dimensional problem with oscillating coefficients

In this section we concentrate in the study of the elliptic problem (1.2) and start analyzing the behavior of the solutions as \( \epsilon \to 0 \). As a matter of fact, we will be able to reduce the study of (1.2) in the thin domain \( \mathbb{R}^\epsilon \) to the study of an elliptic problem with oscillating coefficients in the lower dimensional fixed domain \( \omega \). This dimension reduction will be the key point to obtain the correct limiting equation. In order to state the main result of this section, let us first make some definitions. We will denote by

\[
\eta^i(\epsilon) = \max_{i \in \{1, \ldots, n\}} \left\{ \sup_{x \in \omega} \left| \epsilon \frac{\partial k_i^\epsilon}{\partial x_i}(x) \right| \right\} > 0, \quad i = 1, 2 \quad \text{and} \quad \eta(\epsilon) = \eta^1(\epsilon) + \eta^2(\epsilon). \tag{2.1}
\]

Observe that from hypothesis (H.1) we have \( \eta(\epsilon) \xrightarrow{\epsilon \to 0} 0 \).

Also, we denote by \( K_\epsilon(x) = k_1^\epsilon(x) + k_2^\epsilon(x) \) (that is \( \epsilon K_\epsilon(x) \) is the thickness of the thin domain \( R^\epsilon \) at the point \( x \in \omega \)), \( \hat{f}^\epsilon(x) = \frac{1}{\epsilon K_\epsilon(x)} \int_{-\epsilon k_1^\epsilon(x)}^{\epsilon k_2^\epsilon(x)} f^\epsilon(x, y) dy \) and we consider the problem:

\[
\begin{aligned}
- \frac{1}{K_\epsilon} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( K_\epsilon \frac{\partial \hat{w}^\epsilon}{\partial x_i} \right) + \hat{w}^\epsilon &= \hat{f}^\epsilon \quad \text{in} \ \omega, \\
\frac{\partial \hat{w}^\epsilon}{\partial \eta} &= 0 \quad \text{on} \ \partial \omega.
\end{aligned}
\tag{2.2}
\]

Our main result in this section is

**Proposition 2.1.** There exists a constant \( C \) independent of \( \epsilon > 0 \) such that for all \( f^\epsilon \in L^2(R^\epsilon) \), we have

\[
\| \hat{w}^\epsilon - \hat{w}^\epsilon \|_{H^1(R_\epsilon)}^2 \leq C \eta(\epsilon) \| f^\epsilon \|_{L^2(R_\epsilon)}^2, \tag{2.3}
\]
In order to prove this result, we will need to obtain first some preliminary lemmas. We start transforming equation (1.2) into an equation in the modified thin domain

\[ R_\epsilon^a = \{ (x, \bar{y}) \in \mathbb{R}^{n+1} \mid x \in \omega, 0 < \bar{y} < \epsilon k_\epsilon^2(x) + \epsilon k_\epsilon^1(x) \}. \]  

(2.4)

(see Figure 2) where it can be seen that we have transformed the oscillations of both boundaries into oscillations of just the boundary at the top. For this, we considering the following family of diffeomorphisms

\[ L_\epsilon : R_\epsilon^a \rightarrow R^\epsilon \]

\[ (\bar{x}, \bar{y}) \rightarrow (x, y) := (\bar{x}, \bar{y} - \epsilon k_\epsilon^1(\bar{x})). \]

Notice that the inverse of this diffeomorphism is \((L_\epsilon)^{-1}(x, y) = (x, y + \epsilon k_\epsilon^1(x))\). Moreover, from the structure of these diffeomorphisms and hypothesis (H1) we easily get that there exists a constant \(C\) such that the Jacobian Matrix of \(L_\epsilon\) and \((L_\epsilon)^{-1}\) satisfy

\[ \|JL_\epsilon\|_{L^\infty}, \|J(L_\epsilon)^{-1}\|_{L^\infty} \leq C. \]  

(2.5)

Moreover, we also have \(det(JL_\epsilon)(\bar{x}, \bar{y}) = det(J(L_\epsilon)^{-1})(x, y) = 1\).

We will show that the study of the limit behavior of the solutions of (1.2) is equivalent to analyze the behavior of the solutions of the following problem

\[ \begin{cases} -\Delta v_\epsilon + v_\epsilon = f_1^\epsilon \quad \text{in } R_\epsilon^a, \\ \frac{\partial v_\epsilon}{\partial n_\epsilon} = 0 \quad \text{on } \partial R_\epsilon^a, \end{cases} \]  

(2.6)

where \(n_\epsilon\) is the unit outward normal to \(\partial R_\epsilon^a\) and

\[ f_1^\epsilon = f^\epsilon \circ L_\epsilon. \]  

(2.7)

Notice that

\[ \|f_1^\epsilon\|_{L^2(R_\epsilon^a)}^2 = \int_{R_\epsilon^a} |f_\epsilon \circ L_\epsilon(\bar{X})|^2 d\bar{X} = \int_{R^\epsilon} |f_\epsilon(X)|^2 |det(J(L_\epsilon^{-1}))(X)|dX = \|f^\epsilon\|_{L^2(R^\epsilon)}^2, \]

for all \(f_\epsilon \in L^2(R_\epsilon^a)\). In particular, this implies that

\[ \|v_\epsilon\|_{H^1_0(R^\epsilon)} \leq \|f_1^\epsilon\|_{L^2(R_\epsilon^a)} = \|f^\epsilon\|_{L^2(R^\epsilon)}. \]  

(2.8)
Let $w^\epsilon$ and $v^\epsilon$ be the solutions of problems (1.2) and (2.6) respectively. Then, there is a constant $C > 0$, independent of $\epsilon$, such that

$$
\|(w^\epsilon \circ L^\epsilon) - v^\epsilon\|^2_{H^1(R_n^\epsilon)} \leq C \eta^1(\epsilon) \|f^\epsilon\|^2_{L^2(R_n)}.
$$

(2.9)

**Proof.** From the definition of $L^\epsilon$ we have

$$
\frac{\partial (w^\epsilon \circ L^\epsilon)}{\partial x_i} = \frac{\partial w^\epsilon}{\partial x_i} - \epsilon \left( \frac{\partial k^1_\epsilon}{\partial x_i}(x) \right) \frac{\partial w^\epsilon}{\partial y}, \quad i = 1, \ldots, n,
$$

$$
\frac{\partial (w^\epsilon \circ L^\epsilon)}{\partial y} = \frac{\partial w^\epsilon}{\partial y}.
$$

In the new system of variables $(x = x$ and $\bar{y} = y + \epsilon k^1_\epsilon(x)$ ) the variational formulation of (1.2) is given by

$$
\int_{R_n^\epsilon} \left\{ \sum_{i=1}^n \frac{\partial (w^\epsilon \circ L^\epsilon)}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + \frac{\partial (w^\epsilon \circ L^\epsilon)}{\partial y} \frac{\partial \varphi}{\partial y} + (w^\epsilon \circ L^\epsilon) \varphi \right\} dx d\bar{y}
$$

$$
+ \int_{R_n^\epsilon} \sum_{i=1}^n \epsilon \left( \frac{\partial k^1_\epsilon}{\partial x_i}(x) \right) \left( \frac{\partial (w^\epsilon \circ L^\epsilon)}{\partial y} \frac{\partial \varphi}{\partial x_i} + \frac{\partial (w^\epsilon \circ L^\epsilon)}{\partial x_i} \frac{\partial \varphi}{\partial y} \right) dx d\bar{y}
$$

$$
+ \int_{R_n^\epsilon} \left\{ \sum_{i=1}^n \epsilon \left( \frac{\partial k^1_\epsilon}{\partial x_i}(x) \right)^2 \frac{\partial (w^\epsilon \circ L^\epsilon)}{\partial y} \frac{\partial \varphi}{\partial y} \right\} dx d\bar{y}
$$

$$
= \int_{R_n^\epsilon} f^\epsilon \varphi dx d\bar{y}, \quad \forall \varphi \in H^1(R_n^\epsilon).
$$

(2.10)

On the other hand, the weak formulation of (2.6) is: find $v^\epsilon \in H^1(R_n^\epsilon)$ such that

$$
\int_{R_n^\epsilon} \left\{ \sum_{i=1}^n \frac{\partial v^\epsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + \frac{\partial v^\epsilon}{\partial y} \frac{\partial \varphi}{\partial y} + v^\epsilon \varphi \right\} dx d\bar{y} = \int_{R_n^\epsilon} f_1^\epsilon \varphi dx d\bar{y}, \quad \forall \varphi \in H^1(R_n^\epsilon).
$$

(2.11)

Therefore, subtracting (2.11) from (2.10), taking $(v^\epsilon - w^\epsilon \circ L^\epsilon)$ as a test function and after some computations and simplifications, we obtain

$$
\|w^\epsilon \circ L^\epsilon - v^\epsilon\|^2_{H^1(R_n^\epsilon)} \leq \eta^1 \|\nabla (w^\epsilon \circ L^\epsilon)\|_{L^2(R_n^\epsilon)} \|\nabla (w^\epsilon \circ L^\epsilon - v^\epsilon)\|_{L^2(R_n^\epsilon)}.
$$

This implies

$$
\|w^\epsilon \circ L^\epsilon - v^\epsilon\|_{H^1(R_n^\epsilon)} \leq \eta^1 \|w^\epsilon \circ L^\epsilon\|_{H^1(R_n^\epsilon)} \leq \eta^1 \|w^\epsilon \circ L^\epsilon - v^\epsilon\|_{H^1(R_n^\epsilon)} + \eta^1 \|v^\epsilon\|_{H^1(R_n^\epsilon)},
$$

and therefore

$$
\|w^\epsilon \circ L^\epsilon - v^\epsilon\|_{H^1(R_n^\epsilon)} \leq \eta^1 \|w^\epsilon \circ L^\epsilon\|_{H^1(R_n^\epsilon)} \leq \frac{\eta^1}{1 - \eta^1} \|v^\epsilon\|_{H^1(R_n^\epsilon)} \leq \frac{\eta^1}{1 - \eta^1} \|f_1^\epsilon\|_{L^2(R_n^\epsilon)},
$$

where we use that $\|v^\epsilon\|_{H^1(R_n^\epsilon)} \leq \|f_1^\epsilon\|_{L^2(R_n^\epsilon)} = \|f^\epsilon\|_{L^2(R_n)}$ since $v^\epsilon$ is the solution of (2.6). This proves the result. \qed
Corollary 2.3. With the same hypothesis of the previous Lemma, we also have that there exists a constant $C > 0$ (probably different from the one in the previous Lemma) such that

$$||w^\epsilon - v^\epsilon \circ (L^\epsilon)^{-1}||_{H^1(R^\epsilon)}^2 \leq C\eta^1(\epsilon)||f^\epsilon||_{L^2(R^\epsilon)}^2.$$ 

Proof. Just perform the change of variables with the diffeomorphism $(L^\epsilon)^{-1}$ to (2.9) and use (2.5).

Remark 2.4. Notice that from the point of view of the limit behavior of the solutions it is the same to study problem (1.2) defined in the doubly oscillating thin domain as to analyze problem (2.6) posed in a thin domain with just one oscillating boundary. It is important to note that this is true because of (H.1). If that assumption is not satisfied, at least for $k_1^\epsilon$, then the simplification it is not possible. For instance, if we have a domain with oscillations with the same period in both boundaries like this one

$$R^\epsilon = \left\{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1), \ -\epsilon(2 - g(x/\epsilon)) < y < \epsilon g(x/\epsilon) \right\},$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth $L$-periodic function. Observe that in this particular case we have $k_1^\epsilon(x) = 2 - g(x/\epsilon)$, $k_2^\epsilon(x) = g(x/\epsilon)$.

Therefore, it follows straightforward that condition (1.4) is not satisfied. Observe that the original problem (1.2) for this particular thin domain is in the framework of the classical periodic homogenization while the converted problem (2.6) is posed in a rectangle of height $\epsilon$ where homogenization theory is not necessary to analyze the behavior of the solutions. Indeed, if $k_1^\epsilon$ does not satisfy (1.4) the solutions of problems (1.2) and (2.6) are not comparable in general.

Now we define a transformation on the thin domain $R^\epsilon$, which will map $R^\epsilon$ into the fixed cylindrical domain $Q = \omega \times (0, 1)$. This transformation is given by

$$S^\epsilon : Q \to R^\epsilon$$

$$(x, y) \to (\bar{x}, \bar{y}) := (x, y \epsilon K_\epsilon(x)).$$

We recall that $K_\epsilon(x) = k_2^\epsilon(x) + k_1^\epsilon(x)$.

Using the chain rule and standard computations it is not difficult to see that there exists $c, C > 0$ such that if $V^\epsilon \in H^1(R^\epsilon_a)$ and $U^\epsilon = V^\epsilon \circ S^\epsilon \in H^1(Q)$ then the following estimates hold

$$ce^{-1}||V^\epsilon||_{L^2(R^\epsilon_a)}^2 \leq ||U^\epsilon||_{L^2(Q)}^2 \leq C\epsilon^{-1}||V^\epsilon||_{L^2(R^\epsilon_a)}^2,$$  

(2.12)

$$ce^{-1}||\partial V^\epsilon/\partial y||_{L^2(R^\epsilon_a)}^2 \leq 1/\epsilon^2||\partial U^\epsilon/\partial y||_{L^2(Q)}^2 \leq C\epsilon^{-1}||\partial V^\epsilon/\partial y||_{L^2(R^\epsilon_a)}^2,$$  

(2.13)

$$ce^{-1}||\nabla V^\epsilon||_{L^2(Q)}^2 \leq ||\nabla U^\epsilon||_{L^2(Q)}^2 \leq C\epsilon^{-1}||\nabla V^\epsilon||_{L^2(R^\epsilon_a)}^2.$$  

(2.14)

Now, under this change of variables and defining $u^\epsilon = v^\epsilon \circ S^\epsilon$ where $v^\epsilon$ satisfies (2.6) and $f_2^\epsilon = f_1^\epsilon \circ S^\epsilon$, where $f_1^\epsilon$ is defined in (2.7), problem (2.6) becomes

$$\left\{ \begin{array}{ll}
-\frac{1}{K_\epsilon}\text{div}(B^\epsilon(u^\epsilon)) + u^\epsilon = f_2^\epsilon & \text{in } Q, \\
B(u^\epsilon) \cdot \eta = 0 & \text{on } \partial Q, \\
u^\epsilon = v^\epsilon \circ S^\epsilon & \text{in } Q,
\end{array} \right.$$  

(2.15)
where \( \eta \) denotes the unit outward normal vector field to \( \partial Q \) and every coordinate of \( B(u^\epsilon) \) is defined as follows

\[
B(u^\epsilon)_i = K_\epsilon \frac{\partial u^\epsilon}{\partial x_i} - y \frac{\partial K_\epsilon}{\partial x_i} \frac{\partial u^\epsilon}{\partial y}, \quad i = 1, \ldots, n,
\]

\[
B(u^\epsilon)_n = \sum_{i=1}^n \left( -y \frac{\partial K_\epsilon}{\partial x_i} \frac{\partial u^\epsilon}{\partial x_i} + \frac{y^2}{K_\epsilon} \left( \frac{\partial K_\epsilon}{\partial x_i} \right)^2 \frac{\partial u^\epsilon}{\partial y} \right) + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial y}.
\]

Notice that in the new system of coordinates we obtain a domain which is neither thin nor oscillating anymore. In some sense, we have substituted the oscillating thin domain by oscillating coefficients in the differential operator.

In order to analyze the limit behavior of the solutions of \((2.15)\) we establish the relation to the solutions of the following easier problem

\[
\begin{cases}
- \frac{1}{K_\epsilon} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( K_\epsilon \frac{\partial w^\epsilon_1}{\partial x_i} \right) + \frac{1}{\epsilon^2 K_\epsilon} \frac{\partial^2 w^\epsilon_1}{\partial y^2} + w^\epsilon_1 = f^2 \quad \text{in } Q,
\end{cases}
\]

\[
\frac{\partial w^\epsilon_1}{\partial \eta} = 0 \quad \text{on } \partial Q.
\]

Observe that under the assumptions on the functions \( k^1_\epsilon \) and \( k^2_\epsilon \), equation \((2.16)\) admits a unique solution \( w^\epsilon_1 \in H^1(Q) \), which satisfies the a priori estimates

\[
\|w^\epsilon_1\|_{L^2(Q)}, \quad \left\| \frac{\partial w^\epsilon_1}{\partial x_i} \right\|_{L^2(Q)}, \quad \left\| \frac{\partial w^\epsilon_1}{\partial y} \right\|_{L^2(Q)} \leq C\|f^2\|_{L^2(Q)}, \quad i = 1 \cdots n.
\]

**Lemma 2.5.** Let \( u^\epsilon \) and \( w^\epsilon_1 \) be the solution of problems \((2.15)\) and \((2.16)\) respectively. Then, we have

\[
\sum_{i=1}^n \left\| \frac{\partial (u^\epsilon - w^\epsilon_1)}{\partial x_i} \right\|^2_{L^2(Q)} + \frac{1}{\epsilon^2} \left\| \frac{\partial (u^\epsilon - w^\epsilon_1)}{\partial y} \right\|^2_{L^2(Q)} \leq C\eta(\epsilon)\epsilon^{-1}\|f^\epsilon\|^2_{L^2(R_\epsilon)},
\]

where \( \eta(\epsilon) \) is defined in \((2.1)\).

**Proof.** Subtracting the weak formulation of \((2.16)\) from the weak formulation of \((2.15)\) and choosing \( w^\epsilon_1 - u^\epsilon \) as test function we get

\[
\int_Q \left\{ \sum_{i=1}^n K_\epsilon \left( \frac{\partial (u^\epsilon - w^\epsilon_1)}{\partial x_i} \right)^2 + \frac{1}{\epsilon^2 K_\epsilon} \left( \frac{\partial (u^\epsilon - w^\epsilon_1)}{\partial y} \right)^2 + (u^\epsilon - w^\epsilon_1)^2 \right\}dx dy
\]

\[
= \int_Q \left\{ \sum_{i=1}^n y \frac{\partial u^\epsilon}{\partial y} \frac{\partial K_\epsilon}{\partial x_i} \frac{\partial (w^\epsilon_1 - u^\epsilon)}{\partial x_i} + \sum_{i=1}^n y \frac{\partial K_\epsilon}{\partial x_i} \frac{\partial u^\epsilon}{\partial x_i} \frac{\partial (w^\epsilon_1 - u^\epsilon)}{\partial y} - \sum_{i=1}^n \frac{y^2}{K_\epsilon} \left( \frac{\partial K_\epsilon}{\partial x_i} \right)^2 \frac{\partial u^\epsilon}{\partial y} \frac{\partial (w^\epsilon_1 - u^\epsilon)}{\partial y} \right\}dx dy.
\]

Taking into account that \( \epsilon |\frac{\partial K_\epsilon}{\partial x_i}| \leq \eta(\epsilon) \), convergence \((1.4)\), estimates \((2.12), (2.13), (2.14)\) applied to \( u^\epsilon \) and \( v^\epsilon \) and the a priori estimates of \( v^\epsilon \) (see \((2.8)\)), \( u^\epsilon \) and \( w^\epsilon_1 \) (see \((2.17)\)) and following standard computations, similar to the ones in the proof of Lemma 2.2, we obtain the result. \( \square \)
Finally, we compare the behavior of the solution of \((2.16)\) to the solutions of the following problem posed in \(\omega \subset \mathbb{R}^n\)

\[
\begin{aligned}
- \frac{1}{K_\epsilon} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( K_\epsilon \frac{\partial w_1}{\partial x_i} \right) + u_1 = f_3' \quad \text{in } \omega, \\
\frac{\partial u_1}{\partial n} = 0 \quad \text{on } \partial \omega,
\end{aligned}
\]

(2.19)

where \(f_3'(x) = \int_0^1 f_2'(x, y)dy\) for a.e. \(x \in \omega\), which is a function depending only on the \(x\) variable.

Then, considering \(u_1'(x)\) as a function defined in \(Q\) (extending it in a constant way in the \(y\) direction) we prove the following lemma.

**Lemma 2.6.** Let \(u_1'\) and \(w_1'\) be the solution of problems \((2.19)\) and \((2.16)\) respectively. Then, we have

\[
\sum_{i=1}^N \left\| \frac{\partial(u_1' - w_1')}{\partial x_i} \right\|_{L^2(Q)}^2 + \frac{1}{c^2} \left\| \frac{\partial w_1'}{\partial y} \right\|_{L^2(Q)}^2 + \left\| u_1' - w_1' \right\|_{L^2(Q)}^2 \leq C \left\| f' \right\|_{L^2(R)}^2.
\]

Proof. Taking \(w_1' - u_1'\) as test function in the variational formulation of \((2.16)\) and \((2.19)\) and subtracting both weak formulations we obtain

\[
\int_Q \left\{ \sum_{i=1}^n K_\epsilon \left( \frac{\partial(u_1' - w_1')}{\partial x_i} \right)^2 + \frac{1}{c^2} K_\epsilon \left( \frac{\partial w_1'}{\partial y} \right)^2 + K_\epsilon (u_1' - w_1')^2 \right\} dxdy = \int_Q K_\epsilon (f_2' - f_3')u_1' dxdy - \int_Q K_\epsilon (f_2' - f_3')w_1' dxdy.
\]

(2.20)

Now we analyze the two terms in the right hand-side. First, taking into account the definition of \(f_3'\) we have that for any function \(\varphi\) defined in \(\omega\), that is, \(\varphi = \varphi(x)\)

\[
\int_Q K_\epsilon (f_2' - f_3') \varphi dxdy = \int_{\omega} \varphi K_\epsilon \left( \int_0^1 f_2' dy - f_3' \right) dx = 0.
\]

In particular \(\int_Q K_\epsilon (f_2' - f_3')u_1' dxdy = 0\) and \(\int_Q K_\epsilon (f_2' - f_3')w_1'(x, 0) dxdy = 0\). Hence, using Holder inequality and \((2.17)\) we get

\[
\left| \int_Q K_\epsilon (f_2' - f_3')w_1' dxdy \right| = \left| \int_Q K_\epsilon (f_2' - f_3')(w_1'(x, y) - w_1'(x, 0)) dxdy \right| \leq \left\| K_\epsilon (f_2' - f_3') \right\|_{L^2(Q)} \left\| w_1' - w_1'(x, 0) \right\|_{L^2(Q)} \leq C \left\| f_2' \right\|_{L^2(Q)} \left\| \frac{\partial w_1'}{\partial y} \right\|_{L^2(Q)} \leq \left\| f' \right\|_{L^2(R)}.
\]

Therefore, from \((2.20)\) the lemma is proved.

After this lemmas we can provide a proof of the main result of this section.
Proof of Proposition \ref{prop:2.1}. Notice first that $u_1'$, the solution of (2.19) coincides with $\hat{w}$, the solution of (2.2) and the function appearing in the statement of the proposition.

Hence,
\[
\|u^\epsilon - \hat{w}\|_{H^1(R)}^2 = \|u^\epsilon - u_1^\epsilon\|_{H^1(R)}^2 \leq C\|u^\epsilon \circ L^\epsilon - u_1^\epsilon \circ L^\epsilon\|_{H^1(R_\epsilon)}^2 = C\|u^\epsilon \circ L^\epsilon - u_1^\epsilon\|_{H^1(R_\epsilon)}^2,
\]
where we use \ref{lem:2.5} and the fact that $u_1^\epsilon$ does not depend on the $y$ variable.

But,
\[
\|u^\epsilon \circ L^\epsilon - u_1^\epsilon\|_{H^1(R_\epsilon)}^2 \leq 2\|u^\epsilon \circ L^\epsilon - v^\epsilon\|_{H^1(R_\epsilon)}^2 + 2\|v^\epsilon - u_1^\epsilon\|_{H^1(R_\epsilon)}^2 \leq C\eta(\epsilon)\|f^\epsilon\|_{L^2(R)}^2 + 2\|v^\epsilon - u_1^\epsilon\|_{H^1(R_\epsilon)}^2,
\]
where we have used the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ and Lemma \ref{lem:2.2}.

Now, using that $u^\epsilon = v^\epsilon \circ S^\epsilon$ and \ref{lem:2.12}, \ref{lem:2.13}, \ref{lem:2.14} and that $u_1^\epsilon$ is independent of $y$ and therefore $u_1^\epsilon \circ S^\epsilon = u_1^\epsilon$, we get
\[
\|v^\epsilon - u_1^\epsilon\|_{H^1(R_\epsilon)}^2 \leq C(\epsilon\|u^\epsilon - u_1^\epsilon\|_{H^1(Q)}^2 + \frac{1}{\epsilon}\|\partial u^\epsilon\|_{L^2(Q)}^2),
\]
and applying now the triangular inequality,
\[
\leq C(\epsilon\|u^\epsilon - w_\epsilon^1\|_{H^1(Q)}^2 + \frac{1}{\epsilon}\|u^\epsilon - w_\epsilon^1\|_{L^2(Q)}^2 + \epsilon\|w_\epsilon^1 - u_1^\epsilon\|_{H^1(Q)}^2 + \frac{1}{\epsilon}\|w_\epsilon^1 - u_1^\epsilon\|_{L^2(Q)}^2),
\]
and with Lemma \ref{lem:2.5} and Lemma \ref{lem:2.6} we get
\[
\leq C(\epsilon\|f^\epsilon\|_{L^2(R)}^2 + \epsilon\|f^\epsilon\|_{L^2(R)}^2) \leq C\eta(\epsilon)\|f^\epsilon\|_{L^2(R)}^2.
\]

Putting all this inequalities together we prove the result.

3 Limit problem for the elliptic equation

In view of Proposition \ref{prop:2.1} the homogenized limit problem of (1.2) will be obtained passing to the limit in the reduced problem (2.2).

We will obtain explicitly the homogenized limit problem of (2.2) for several interesting cases, in particular when the oscillating boundaries are given by
\[
k_1^\epsilon(x) = h(x/\epsilon^\alpha), \quad k_2^\epsilon(x) = g(x/\epsilon^\beta),
\]
where $0 < \alpha, \beta < 1$ and the functions $g, h : \mathbb{R}^n \to \mathbb{R}$ are $C^1$ periodic functions verifying
\[
0 \leq h_0 \leq h(\cdot) \leq h_1, \quad 0 \leq g_0 \leq g(\cdot) \leq g_1.
\]

Note that these particular functions $k_1^1$ and $k_2^2$ satisfy hypothesis (H) because of the fact that $0 < \alpha, \beta < 1$. Moreover, in this case, problem (2.2) can be written as
\[
\begin{cases}
- \frac{1}{G_\epsilon} \sum_{i=1}^n \frac{\partial}{\partial x_i}(G_\epsilon \frac{\partial \tilde{w}^\epsilon}{\partial x_i}) + \tilde{w}^\epsilon = \hat{f}^\epsilon \quad \text{in } \omega, \\
\frac{\partial \tilde{w}^\epsilon}{\partial \eta} = 0 \quad \text{on } \partial \omega,
\end{cases}
\]
(3.3)
where \( G_\epsilon \) plays the role of \( K_\epsilon \) and it is given by \( G_\epsilon(x) = g(x/\epsilon^\beta) + h(x/\epsilon^\alpha) \).

From now on we will assume that \( \hat{f}_\epsilon \) satisfies the following convergence

\[
G_\epsilon(\cdot) \hat{f}_\epsilon(\cdot) = \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} g(\cdot/\epsilon^\beta) f^\epsilon(\cdot, y) dy \xrightarrow{\epsilon \to 0} f_0(\cdot) \quad \text{w}-L^2(\omega),
\]

for certain \( f_0 \in L^2(\omega) \).

We will study first the two-dimensional case adapting a classical simple argument in the context of periodic homogenization of the one-dimensional problems with oscillating coefficients, see [16,24]. Secondly, we obtain the limit problem for the general and more complicated situation where the domain is \( n \)-dimensional, \( n > 2 \).

**Remark 3.1.** Notice that considering this kind of periodic functions in problems with rough boundaries is very common, see [13, 22, 32] and the references therein. However, it is important to highlight that we contemplate the possibility of situations beyond the classical periodic setting in homogenization. For example, we consider cases where both boundaries oscillate with different rationally independent periods, which amounts to study a quasi-periodic problem.

### 3.1 Two-dimensional case

In this subsection we consider a two-dimensional thin domain \( \mathbb{R}^\epsilon \) which is given as the region between two oscillatory functions, that is,

\[
R^\epsilon = \left\{ (x, y) \in \mathbb{R}^2 \mid x \in (0, 1), -\epsilon h(x/\epsilon^\alpha) < y < \epsilon g(x/\epsilon^\beta) \right\},
\]

where \( 0 < \alpha, \beta < 1 \) and the functions \( g, h : \mathbb{R} \to \mathbb{R} \) are \( C^1 \) periodic functions with period \( L_1 \) and \( L_2 \) respectively. Then, we study the behavior of the solutions of the Neumann problem (1.2).

In this case, problem (3.3) can be written as the following one dimensional problem

\[
\begin{cases}
-1 \frac{\partial}{\partial x} \left( G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x} \right) + \hat{w}_\epsilon = \hat{f}_\epsilon & \text{in } (0, 1), \\
(\hat{w}_\epsilon)'(0) = (\hat{w}_\epsilon)'(1) = 0,
\end{cases}
\]

where \( G_\epsilon(x) = g(x/\epsilon^\beta) + h(x/\epsilon^\alpha) \). We would like to point that (3.5) presents the particularity of having not necessarily periodic coefficients. For instance, if \( \alpha \neq \beta \) the problem is not periodic. Moreover, if \( \alpha = \beta \) and the period of \( g \) and \( h \) are rationally independent periods then we have the situation of a quasi-periodic coefficients.

The weak formulation of (3.5) is given by

\[
\int_0^1 \left\{ G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x} \frac{\partial \phi}{\partial x} + G_\epsilon \hat{w}_\epsilon \phi \right\} dx = \int_0^1 G_\epsilon \hat{f}_\epsilon \phi dx, \quad \text{for all } \phi \in H^1(0, 1).
\]

We start by establishing a priori estimates of \( \hat{w}_\epsilon \). Considering \( \hat{w}_\epsilon \) as a test function in (3.6), we easily get

\[
||\hat{w}_\epsilon||_{H^1(0, 1)} \leq C.
\]

Thus, by weak compactness there exists \( u_0 \in H^1(0, 1) \) such that, up to subsequences

\[
\hat{w}_\epsilon \xrightarrow{\epsilon \to 0} u_0 \quad w - H^1(0, 1).
\]
As in the simplest cases for the periodic homogenization, see for example [16, 24], the key question now is: How is the limit of the product \( G_\epsilon \frac{\partial \hat{w}}{\partial x} \)? To solve this, we first obtain the weak limit of the functions \( G_\epsilon \) and \( \frac{1}{G_\epsilon} \).

On one hand, since \( G_\epsilon(x) = g(x/\epsilon^\beta) + h(x/\epsilon^\alpha) \) is the sum of two periodic functions it is obvious from the Average Convergence for Periodic Functions Theorem (see, e.g., [23, p. xvi]) that \( G_\epsilon(x) \) converges in a weak sense to the sum of the corresponding mean values, that is,

\[
G_\epsilon \xrightarrow{\epsilon \to 0} \frac{1}{L_1} \int_0^{L_1} g(y) \, dy + \frac{1}{L_2} \int_0^{L_2} h(z) \, dz \equiv \mathcal{M}(g) + \mathcal{M}(h) \quad \omega - L^2(0,1). \tag{3.8}
\]

On the other hand, to prove the convergence \( \frac{1}{G_\epsilon} \) we state the following lemma.

**Lemma 3.2.** Let \( G_\epsilon(x) = g(x/\epsilon^\beta) + h(x/\epsilon^\alpha) \). Then the following convergence holds

\[
\frac{1}{G_\epsilon} = \frac{1}{g(\frac{x}{\epsilon^\beta}) + h(\frac{x}{\epsilon^\alpha})} \xrightarrow{\epsilon \to 0} \frac{1}{p_0},
\]

where \( p_0 \) is defined as follows

\[
\frac{1}{p_0} = \begin{cases} 
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{g(y) + h(y)} \, dy, & \text{if } \alpha = \beta, \\
\frac{1}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} \frac{1}{g(y) + h(z)} \, dz \, dy, & \text{if } \alpha \neq \beta.
\end{cases}
\]

**Proof.** We distinguish two different cases:

i) Same order of oscillation \((\alpha = \beta)\).

\[
\frac{1}{g(\frac{x}{\epsilon^\beta}) + h(\frac{x}{\epsilon^\alpha})} \xrightarrow{\epsilon \to 0} \frac{1}{p_0} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{g(y) + h(y)} \, dy \quad \omega - L^2(0,1). \tag{3.9}
\]

We treat here the case where the function \( G_\epsilon \) presents only one small scale, that is,

\[
\frac{1}{G_\epsilon(x)} = \frac{1}{g(x/\epsilon^\alpha) + h(x/\epsilon^\alpha)}, \quad \text{for } x \in (0,1) \text{ and } \alpha \in (0,1).
\]

Note that if the periods \( L_1 \) and \( L_2 \) are rationally dependent, there exist \( p, q \in \mathbb{N} \) such that \( pL_1 = qL_2 \), then we immediately have from the Average Convergence for Periodic Functions (see, e.g., [23, p. xvi]) the weak convergence of \( \frac{1}{G_\epsilon} \)

\[
\frac{1}{G_\epsilon} \xrightarrow{\epsilon \to 0} \frac{1}{pL_1} \int_0^{pL_1} \frac{1}{g(y) + h(y)} \, dy \quad \omega - L^2(0,1).
\]

However, if \( L_1 \) and \( L_2 \) are rationally independent the usual periodicity hypothesis is replaced by a more general behavior: almost periodicity, see for example [17, 19]. Indeed, in this case the function \( \frac{1}{G_\epsilon} = \frac{1}{g(y) + h(y)} \) is not periodic but we show that it is almost periodic which allows us to obtain the weak limit.

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Since $G(y) = g(y) + h(y)$ is the sum of two periodic functions with different period we can ensure that $G$ is an almost periodic function. Then, from the definition of almost periodicity, for every $\epsilon > 0$ there exists $T_0(\epsilon)$ such that every interval of length $T_0(\epsilon)$ contains a number $\tau$ with the following property:

$$|G(y + \tau) - G(y)| \leq m^2 \epsilon, \quad \text{for each } y \in \mathbb{R},$$

where $m$ is a constant such that $0 < m \leq g(y) + h(y), \forall y \in \mathbb{R}$.

So we have,

$$\left| \frac{1}{G(y + \tau)} - \frac{1}{G(y)} \right| = \left| \frac{G(y) - G(y + \tau)}{G(y + \tau)G(y)} \right| \leq \frac{m^2 \epsilon}{m^2} = \epsilon,$$

and hence $\frac{1}{G(y)}$ is almost periodic.

Therefore, note that $\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{g(y) + h(y)} dy$ is well defined since it is the mean value of the almost periodic function $\frac{1}{G(y)}$.

Now, we are in conditions to prove the desired weak convergence (3.9).

To obtain (3.9), since $\| \frac{1}{G(y)} \|_{L^\infty(0,1)} \leq \frac{1}{g(y) + h(y)}$ and the set of all the step functions is dense in $L^p(0,1), 1 \leq p < \infty$, it is enough to prove

$$\lim_{\epsilon \to 0} \int_a^b \frac{1}{g\left(\frac{x}{\epsilon}\right) + h\left(\frac{x}{\epsilon}\right)} \to (b - a) \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{g(y) + h(y)} dy, \quad \text{for any } (a, b) \subset (0,1). \quad (3.10)$$

We can write

$$\int_a^b \frac{1}{g\left(\frac{x}{\epsilon}\right) + h\left(\frac{x}{\epsilon}\right)} dx = \int_0^b \frac{1}{g\left(\frac{x}{\epsilon}\right) + h\left(\frac{x}{\epsilon}\right)} dx - \int_a^0 \frac{1}{g\left(\frac{x}{\epsilon}\right) + h\left(\frac{x}{\epsilon}\right)} dx. \quad (3.11)$$

By a simple change of variables we have

$$\int_0^e \frac{1}{g\left(\frac{x}{\epsilon}\right) + h\left(\frac{x}{\epsilon}\right)} dx = \frac{e}{\epsilon} \int_0^{e\epsilon} \frac{1}{g(y) + h(y)} dy, \quad \forall e \in (0,1).$$

Then, since $\frac{1}{g(y) + h(y)} dy$ is almost periodic we can pass to the limit at the right-hand side of the last equality above to get

$$\lim_{\epsilon \to 0} \int_0^e \frac{1}{g\left(\frac{x}{\epsilon}\right) + h\left(\frac{x}{\epsilon}\right)} dx \to e \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{g(y) + h(y)} dy, \quad \forall e \in (0,1). \quad (3.12)$$

Finally, from (3.11) and (3.12) we get convergence (3.10).

ii) Different order of oscillation ($\alpha \neq \beta$).

$$\frac{1}{p_0} \int_0^{L_1} \int_0^{L_2} \frac{1}{g(z) + h(z)} dzdy w - L^2(0,1). \quad (3.13)$$

Observe that in this case we are dealing with two microscopic scales which is a generalization of the classical result for periodic functions. The result is well-known in the literature, see e.g. [16].
Now we get the convergence of the product $G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x}$.

Observe that $G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x}$ is uniformly bounded in $L^2(0, 1)$ since

$$\left\| \frac{\partial \hat{w}_\epsilon}{\partial x} \right\|_{L^2(0, 1)} \leq C$$

and $0 < G_\epsilon(x) < g_1 + h_1$, for each $x \in (0, 1)$.

Moreover, taking into account that

$$\frac{\partial}{\partial x} (G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x}) = -\hat{f}_\epsilon G_\epsilon + G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x},$$

we deduce that $G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x}$ is uniformly bounded in $H^1(0, 1)$. Then, it follows that there exists a function $\sigma$ such that, up to subsequences,

$$G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x} \rightarrow \sigma \quad \text{strongly in } L^2(0, 1).$$

Thus,

$$\frac{\partial \hat{w}_\epsilon}{\partial x} = \frac{1}{G_\epsilon} \left( G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x} \right) \epsilon \rightarrow 0 \frac{1}{p_0} \sigma \quad \text{in } L^2(0, 1).$$

Consequently, due to convergence (3.7) we have

$$\frac{\partial u_0}{\partial x} = \frac{1}{p_0} \sigma,$$

or equivalently,

$$G_\epsilon \frac{\partial \hat{w}_\epsilon}{\partial x} \epsilon \rightarrow 0 \frac{1}{p_0} \frac{\partial u_0}{\partial x} \quad \text{strongly in } L^2(0, 1). \quad (3.14)$$

Therefore, in view of (3.7), (3.4) and (3.14) we can pass to the limit and we obtain the following weak formulation

$$\int_0^1 \left\{ P_0 \frac{\partial u_0}{\partial x} \frac{\partial \phi}{\partial x} + (M(g) + M(h))u_0 \phi \right\} dx = \int_0^1 f_0 \phi \, dx.$$ 

Now we are in conditions to state the convergence result

**Theorem 3.3.** Let $w_\epsilon$ be the solution of problem (1.2). Then, with the definition of $f_0$ given by (3.4) and denoting by $\hat{f} = \frac{f_0}{M(g) + M(h)}$, then we have

$$\hat{w}_\epsilon \rightarrow \hat{w}, \quad w \in H^1(\omega),$$

$$\epsilon^{-1} ||w_\epsilon - \hat{w}||_{L^2(R^\epsilon)} \rightarrow 0,$$

where $\hat{w} \in H^1(0, 1)$ is the weak solution of the following Neumann problem

$$\begin{cases}
-\frac{P_0}{M(g) + M(h)} \hat{w}_{xx} + \hat{w} = \hat{f}, & x \in (0, 1), \\
\hat{w}'(0) = \hat{w}'(1) = 0,
\end{cases} \quad (3.15)$$

where the constant $p_0$ is such that

$$\frac{1}{h(x/\epsilon^\alpha) + g(x/\epsilon^\beta)} \epsilon \rightarrow 0 \frac{1}{p_0} \quad \text{in } L^2(0, 1). \quad (3.16)$$
Therefore $p_0$ is given by

\[
\frac{1}{p_0} = \begin{cases} 
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{g(y) + h(y)} \, dy, & \text{if } \alpha = \beta, \\
\frac{1}{L_1L_2} \int_0^{L_1} \int_0^{L_2} \frac{1}{g(y) + h(z)} \, dz \, dy, & \text{if } \alpha \neq \beta.
\end{cases}
\]

### 3.2 n-dimensional case

Let $\omega \subset \mathbb{R}^n$ be a smooth domain. Then, we consider the following thin domain

\[
R^\epsilon = \left\{ (x, y) \in \mathbb{R}^{n+1} \mid x \in \omega, \ -\epsilon h(x/\epsilon^\alpha) < y < \epsilon g(x/\epsilon^\beta) \right\},
\]

where $0 < \alpha, \beta < 1$ and the functions $g, h : \mathbb{R}^n \to \mathbb{R}$ are $C^1$ functions periodic in the cells $[0, L_1]^n$ and $[0, L_2]^n$ respectively. We consider problem (3.3) and taking $\hat{w}^\epsilon$ as test function in the variational formulation of (3.3) we immediately get the a priori estimate

\[
||\hat{w}^\epsilon||_{H^1(\omega)} \leq C.
\]

Similarly to the two-dimensional case we consider two different situations:

1) **Same order of oscillation ($\alpha = \beta$) and periods rationally dependent** Note that if both periods, $L_1$ and $L_2$, are rationally dependent, there exist $p, q \in \mathbb{N}$ such that $pL_1 = qL_2$, then (3.3) is the classical problem in homogenization with periodic oscillating coefficients. Therefore, the following convergence result has been proved in the literature by using different techniques in homogenization, see e.g. Chapter 6 in [24] or [16,23].

**Proposition 3.4.** Let $w^\epsilon$ be the solution of problem (1.2). Then, with the definition of $f_0$ given by (3.4) and denoting by $\hat{f} = \frac{f_0}{M(g) + M(h)}$, we have

\[
\hat{w}^\epsilon \to \hat{w}, \ w^\epsilon H^1(\omega),
\]

\[
\epsilon^{-1}||w^\epsilon - \hat{w}||_{L^2(R^\epsilon)} \to 0,
\]

where $\hat{w}$ is the unique solution of the following Neumann problem

\[
\begin{cases} 
-\frac{1}{M(g) + M(h)} \text{div}(A_0 \nabla \hat{w}) + \hat{w} = \hat{f} \quad \text{in } \omega, \\
\frac{\partial \hat{w}}{\partial \eta} = 0 \quad \text{on } \partial \omega.
\end{cases}
\]

The matrix $A_0 = (a_{ij})_{1 \leq i, j \leq n}$ is constant and given by

\[
a_{ii} = M_Y \left( G(z) \left( 1 - \frac{\partial X^i}{\partial z_i} \right) \right),
\]

\[
a_{ij} = M_Y \left( -G(z) \frac{\partial X^j}{\partial z_i} \right),
\]

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where $G(z) = g(z) + h(z)$, $Y^* = [0, pL_1]^n$ and $X^i$ is the unique solution of the following auxiliary problem

\[
\begin{aligned}
-\sum_{j=1}^n \frac{\partial}{\partial y_j} \left( G \frac{\partial (X^i - z_i)}{\partial z_j} \right) &= 0 \quad \text{in } Y^*, \\
G \frac{\partial (X^i - z_i)}{\partial \eta} &= 0 \quad \text{on } \partial Q, \\
\mathcal{M}_{Y^*}(X^i) &= 0, \\
X^i &= Y^* - \text{periodic}.
\end{aligned}
\] (3.18)

**ii) Same order of oscillation ($\alpha = \beta$) and periods rationally independent**

However, if the periods, $L_1$ and $L_2$, are rationally independent the coefficients of (3.3) are rapidly oscillating quasi-periodic functions. Therefore, this specific case can be analyzed in the context of the more general almost periodic homogenization theory, see for instance \[31, 33\]. In fact, Proposition 3.5 can be proved rigorously by applying directly the ideas introduced in [31].

Note that, the main difference respect to periodic homogenization lies in the solvability of the auxiliary problem. To overcome this difficulty Kozlov lifted the auxiliary equation to a sub-elliptic problem on a higher dimensional torus which he solved thanks to a higher-order Poincaré inequality implied by the Diophantine condition.

Therefore, since the auxiliary problem in our particular case can be degenerate, the following “frequency condition” is necessary for the formation of the quasiperiodic solutions of the auxiliary problem

**Diophantine condition** There exists $s_0 > 0$ such that

\[|n_1 L_1 + n_2 L_2| \geq \frac{C}{n_1 + n_2} s_0, \quad \forall (n_1, n_2) \in \mathbb{N}^2.\]

**Proposition 3.5.** Let $w^\epsilon$ be the solution of problem (1.2). Then, with the definition of $f_0$ given by (3.4) and denoting by $\hat{f} = \frac{f_0}{\mathcal{M}(g) + \mathcal{M}(h)}$, we have

\[
\begin{aligned}
\hat{w}^\epsilon &\to \hat{w}, \quad w^{-H^1(\omega)}, \\
\epsilon^{-1} ||w^\epsilon - \hat{w}||_{L^2(R^d)} &\to 0,
\end{aligned}
\]

where $\hat{w}$ is the unique solution of the following Neumann problem

\[
\begin{aligned}
-\frac{1}{\mathcal{M}(g) + \mathcal{M}(h)} \text{div}(A_0 \nabla \hat{w}) + \hat{w} &= \hat{f} \quad \text{in } \omega, \\
\frac{\partial \hat{w}}{\partial \eta} &= 0 \quad \text{on } \partial \omega.
\end{aligned}
\] (3.19)

The matrix $A_0 = (a_{ij})_{1 \leq i, j \leq n}$ is constant and it is given by

\[a_{ij} = \lim_{L \to \infty} \sup \frac{1}{(2L)^n} \int_{[-L,L]^n} \left( -G(z) \frac{\partial X^j}{\partial z_i} \right) dz,
\]
where \( G(z) = g(z) + h(z) \), \( X^j \) and its derivatives are quasiperiodic functions which satisfy

\[
\begin{align*}
- \left( \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \left( G \frac{\partial (X^j - z_j)}{\partial z_i} \right) \right) &= 0, \\
\limsup_{L \to \infty} \frac{1}{(2L)^n} \int_{[-L,L]^n} X^j \, dz &= 0.
\end{align*}
\] (3.20)

**ii) Different order of oscillation \((\alpha \neq \beta)\).**

In this case two microscopic scales appear, \( \epsilon^\alpha \) and \( \epsilon^\beta \) with \( \alpha \neq \beta \). This means that each scale can be distinguished from the other, the frequency of the oscillations in both boundaries are not of the same order. Therefore, this problem must be studied in the framework of reiterated homogenization theory, see [1,16].

By using directly the generalization of two-scale convergence given in [1] or the reiterated unfolding method [35] the homogenized problem for (2.19) is obtained. Then, we get the following homogenization result assuming without loss of generality that \( \alpha \) is less than \( \beta \).

**Proposition 3.6.** Let \( w^\varepsilon \) be the solution of problem (1.2). Then, with the definition of \( f_0 \) given by (3.4) and denoting by \( \hat{f} = \frac{f_0}{\mathcal{M}(g) + \mathcal{M}(h)} \), we have

\[
\hat{w}^\varepsilon \to \hat{w}, \ w-H^1(\omega),
\]

\[
e^{-1}||w^\varepsilon - \hat{w}||_{L^2(\mathbb{R}^n)} \to 0,
\]

where \( \hat{w} \) is the unique solution of the following Neumann problem

\[
\begin{align*}
- \frac{1}{\mathcal{M}(g) + \mathcal{M}(h)} \text{div}(A_0 \nabla \hat{w}) + \hat{w} &= \hat{f} \quad \text{in } \omega, \\
\frac{\partial \hat{w}}{\partial \eta} &= 0 \quad \text{on } \partial \omega,
\end{align*}
\] (3.21)

where \( A_0 \) is a constant matrix defined by the inductive homogenization formula

- \( A_2 \) is a diagonal matrix of order \( n \) with the function \( G(x,z) = g(x) + h(z) \) in the elements of the diagonal.
- \( A_1 \) is obtained by periodic homogenization of \( A_2(x,\frac{z}{\epsilon^\beta}) \).
- \( A_0 \) is obtained by periodic homogenization of \( A_1(\frac{z}{\epsilon^\alpha}) \).

### 4 Convergence properties of the semilinear parabolic equation

In this section we show some convergence properties of the solutions and attractors of the evolutionary equations (1.8) assuming that the functions which define the oscillating functions satisfy (1.6). In particular we analyze the relation between the semilinear parabolic problem defined in (1.8) and its homogenized limit

\[
\begin{align*}
\begin{cases}
\frac{1}{\mathcal{M}(g) + \mathcal{M}(h)} \text{div}(A_0 \nabla u_0) + u_0 = f(u_0) \quad \text{in } \omega, \\
\frac{\partial u_0}{\partial \eta} = 0 \quad \text{on } \partial \omega,
\end{cases}
\end{align*}
\] (4.1)
where $A_0$ depends on the dimension and the relation between $\alpha$ and $\beta$ as we have seen in the previous section. The behavior of the nonlinear dynamics in thin domains with not necessarily oscillating boundaries is not a new topic and we would like to refer to [26], [36], [11] for some works in this respect. Also, [8] deals with the case of thin domains with oscillating boundaries.

**Remark 4.1.** We provide only a sketch of the proofs of the results of this section because they are obtained by using standard arguments of a general approach discussed, for instance, in [4–8,21,34].

The functional setting given by [25,27] allows to obtain the convergence of the linear semigroup given by (1.8) to the one established by (4.1). The concept of compact convergence that we adopt here was introduced in the works [39–41,43,44].

First, we consider a family of Hilbert spaces $\{Z_\epsilon\}_{\epsilon>0}$ defined by

$$Z_\epsilon = L^2(R_\epsilon)$$

with the canonical inner product

$$(u, v)_\epsilon = \frac{1}{\epsilon} \int_{R_\epsilon} u(x, y)v(x, y) \, dxdy,$$

and let $Z_0 = L^2(\omega)$ be the limit Hilbert space with the following inner product

$$(u, v)_0 = (\mathcal{M}(g) + \mathcal{M}(h)) \int_{\omega} u(x, y)v(x, y) \, dx.$$ 

Observe that the inner product in $Z_\epsilon$ has been scaled with a factor of $1/\epsilon$. Hence, the induced norm will be also rescaled by the same factor. We will use the notation:

$$|||u|||_2^{Z_\epsilon} = \frac{1}{\epsilon} ||u||_2^{L^2(R_\epsilon)}.$$ 

Now, we write the elliptic problem (1.2) as an abstract equation $L_\epsilon w^\epsilon = f^\epsilon$ where $L_\epsilon : \mathcal{D}(L_\epsilon) \subset Z_\epsilon \to Z_\epsilon$ is the self-adjoint, positive linear operator with compact resolvent defined as follows

$$\mathcal{D}(L_\epsilon) = \left\{ \omega^\epsilon \in H^2(R_\epsilon) \mid \frac{\partial \omega^\epsilon}{\partial \nu^\epsilon} = 0 \text{ on } \partial R_\epsilon \right\},$$

$$L_\epsilon w^\epsilon = -\Delta w^\epsilon + w^\epsilon, \quad w^\epsilon \in \mathcal{D}(L_\epsilon). \quad (4.2)$$

Moreover, we denote by $Z_\epsilon^{\alpha}$ the fractional power scale associated to operators $L_\epsilon$, with $0 \leq \alpha \leq 1$ and $0 \leq \epsilon \leq 1$. Then, $Z_\epsilon = Z_\epsilon^0$ and $Z_\epsilon^{\frac{1}{2}}$ is the Sobolev Space $H^1(R_\epsilon)$ with norm $|||\cdot|||_2^{Z_\epsilon^{\frac{1}{2}}} = ||\cdot||^2_{H^1(R_\epsilon)}$.

Similarly, we rewrite the limit elliptic problem as $L_0u_0 = f$ where $L_0 : \mathcal{D}(L_0) \subset Z_0 \to Z_0$ is the self-adjoint, positive linear operator with compact resolvent defined as follows

$$\mathcal{D}(L_0) = \left\{ u \in H^2(\omega) \mid \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \omega \right\},$$

$$L_0u = -\frac{1}{\mathcal{M}(g) + \mathcal{M}(h)} \text{div}(A_0 \nabla u) + u, \quad u \in \mathcal{D}(L_0), \quad (4.3)$$

where $A_0$ is the matrix of homogenized coefficients obtained in the previous section. Notice that, $L_0$ is a positive self-adjoint operator with compact resolvent.

In the previous section we have passed to limit in the variational formulation of (1.7). Now, we are in conditions to apply the concept of compact convergence to obtain convergence properties of the eigenvalues, eigenfunctions and the linear semigroups generated by $L_\epsilon$ and $L_0$. 

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Before recalling the main concepts of convergence we consider the family of linear continuous operators \( E_\epsilon : Z_0 \to Z_\epsilon \) given by
\[
E_\epsilon u(x, y) = u(x) \quad \forall u \in Z_0.
\]
Then, it is clear \( |||E_\epsilon u|||_{Z_\epsilon} \to ||u||_{Z_0} \). Analogously, we can consider \( E_\epsilon : Z_0^1 \to Z_\epsilon^1 \) and, taking in \( Z_0^1 \) the equivalent norm \( ||-\Delta u + u||_{Z_0} \) we get
\[
|||E_\epsilon u|||_{Z_\epsilon^1} \to ||u||_{Z_0^1}.
\]
Consequently, since
\[
\sup_{0 \leq \epsilon \leq 1} \{ ||E_\epsilon||_{L(Z_0, Z_\epsilon)}, ||E_\epsilon||_{L(Z_0^1, Z_\epsilon^1)} \} < \infty,
\]
we get by interpolation that
\[
C = \sup_{\epsilon > 0} ||E_\epsilon||_{L(Z_0^\alpha, Z_\epsilon^\alpha)} < \infty \quad \text{for} \quad 0 \leq \alpha \leq 1.
\]
Now we recall the main concept of convergence associated to the operators \( \{E_\epsilon\}_{\epsilon > 0} \).

**Definition 4.2.** We say that a family of compact operators \( \{B_\epsilon \in L(Z_\epsilon) | \epsilon > 0 \} \) converges compactly to a compact operator \( B \in L(Z_0) \), we write \( \{B_\epsilon \}_{\epsilon \to 0} \xrightarrow{CC} B \), if for any family \( \{f^\epsilon\}_{\epsilon > 0} \) with \( |||f^\epsilon|||_{Z_\epsilon} \leq 1 \) we have
- For each subsequence \( \{B_{\epsilon m}^n f_{\epsilon m}^n\} \) of a sequence \( \{B_{\epsilon m} f_{\epsilon m}\} \), \( \epsilon_m \to 0 \), there exits a subsequence \( \{B_{\epsilon m} f_{\epsilon m}^n - E_{\epsilon m} F\} \) and \( F \in Z_0 \) such that \( |||B_{\epsilon m} f_{\epsilon m}^n - E_{\epsilon m} F|||_{Z_\epsilon} \to 0 \).
- There exists \( B \in L(Z_0) \) such that \( ||B f^\epsilon - E_\epsilon B f|||_{Z_\epsilon} \to 0 \) if \( ||f^\epsilon - E_\epsilon f|||_{Z_\epsilon} \to 0 \).

Moreover, the following lemma holds, see Lemma 4.7 in [5].

**Lemma 4.3.** Assume that \( \{B_\epsilon \in L(Z_\epsilon) | \epsilon \in [0, 1] \} \) converges compactly to \( B \) as \( \epsilon \to 0 \). Then,

i) \( ||B_\epsilon||_{L(Z_\epsilon)} \leq C \), for some constant \( C \) independent of \( \epsilon \).

ii) Assume that \( N(I + B) = \{0\} \) then, there exists an \( \epsilon_0 > 0 \) and \( M > 0 \) such that
\[
|| (I + B_\epsilon)^{-1} ||_{L(Z_\epsilon)} \leq M, \quad \forall \epsilon \in [0, \epsilon_0].
\]

Then, the convergence results of previous section can be rewritten according to this framework. In particular, we have easily the following two results.

**Corollary 4.4.** The family of compact operators \( \{L_\epsilon^{-1} \in L(Z_\epsilon) \}_{\epsilon > 0} \) converges compactly to the compact operator \( L_0^{-1} \in L(Z_0) \) as \( \epsilon \to 0 \).

**Corollary 4.5.** Let \( M_\epsilon : L^r(R^\epsilon) \to L^r(\omega) \) be the bounded linear operator given by
\[
M_\epsilon f^\epsilon(x) = \frac{1}{\epsilon} \int_{-\epsilon h(x/\epsilon^2)}^{\epsilon g(x/\epsilon^2)} f^\epsilon(x, y) dy, \quad x \in \omega.
\]
Then, for each \( \{f^\epsilon\} \subset Z_\epsilon \) with \( |||f^\epsilon|||_{Z_\epsilon} \) uniformly bounded in \( \epsilon \) there exists a subsequence such that the following convergence holds
\[
|||L_\epsilon^{-1} f^\epsilon - E_\epsilon L_0^{-1} M_\epsilon f^\epsilon|||_{Z_\epsilon} \xrightarrow{\epsilon \to 0} 0.
\]
Moreover, there exists \( \epsilon_0 > 0 \) and a function \( \nu : (0, \epsilon_0) \to (0, \infty) \), with \( \nu(\epsilon) \xrightarrow{\epsilon \to 0} 0 \), such that
\[
||L_\epsilon^{-1} - E_\epsilon L_0^{-1} M_\epsilon||_{L(Z_\epsilon)} \leq \nu(\epsilon), \quad \forall \epsilon \in (0, \epsilon_0).
\]
Notice that Corollary 4.4 implies that \( L_\epsilon \) is a closed operator, has compact resolvent, zero belongs to its resolvent, denoted by \( \rho(L_\epsilon) \), and \( L_\epsilon^{-1} \xrightarrow{CC} L_0^{-1} \). Moreover, Corollary 4.5 gives the convergence of the resolvent operators.

With this convergence we can show now the spectral convergence of the operators. Observe first that since the operators \( L_\epsilon, L_0 \) are selfadjoint and with compact resolvent, the spectrum is only discrete and real. Hence, let us denote by \( \{ \lambda_n^\epsilon \}_{n=1}^{\infty} \subset \mathbb{R} \) the set of eigenvalues ordered and counting multiplicity of the operator \( L_\epsilon \) for \( 0 \leq \epsilon \leq \epsilon_0 \), for certain \( \epsilon_0 > 0 \) and let us denote by \( \{ \varphi_n^\epsilon \}_{n=1}^{\infty} \) a corresponding complete system of orthonormal eigenfunctions of \( L_\epsilon \). Notice also that the corresponding set of eigenfunctions is not unique since we can always change the sign of an eigenfunction of a simple eigenvalue and in case an eigenvalue is multiple we have different choices to choose a base in the eigenspace associated to this multiple eigenvalue.

Consequently, see Lemma 4.10 in [5] for the details, we have the following result about the spectrum convergence of operator \( L_\epsilon \).

**Theorem 4.6.** Let \( L_\epsilon \) and \( L_0 \) the operators considered in (4.2) and (4.3) respectively. Then we have the spectral convergence of \( L_\epsilon \) to \( L_0 \). That is, for each sequence \( \epsilon_k \to 0 \) there exists a subsequence, that we still denote by \( \epsilon_k \), and a choice for a complete system of eigenfunctions of \( L_0 \), \( \{ \varphi_0^\epsilon \}_{n=1}^{\infty} \), such that the following statements hold:

i) \( \lambda_n^\epsilon \xrightarrow{k \to \infty} \lambda_n^0 \) for each \( n \in \mathbb{N} \),

ii) \( ||| \varphi_n^\epsilon - E_{\epsilon_k} \varphi_0^\epsilon |||_{Z^{1/2}} \xrightarrow{k \to \infty} 0 \), for each \( n \in \mathbb{N} \).

**Proof.** We will give an indication on how to obtain this result. The proof is based on the convergence of the spectral projections. If \( \lambda_0 \in \mathbb{R} \) is an eigenvalue of \( L_0 \) of multiplicity \( m \in \mathbb{N} \), (for instance \( \lambda_n^0 < \lambda_0 = \lambda_{n+1}^0 = \ldots = \lambda_{n+m}^0 < \lambda_{n+m+1}^0 \)), then we have a small \( \delta > 0 \) such that \( \lambda_0 \) is the unique spectral value in the set \( \mathcal{O}(\lambda_0, \delta) = \{ \lambda \in (C) : |\lambda - \lambda_0| \leq \delta \} \) and the projection over the eigenspace generated by \( [\varphi_{n+1}^0, \ldots, \varphi_{n+m}^0] \) is given by

\[
Q_0(\lambda_0) = \frac{1}{2\pi i} \int_{|z-\lambda_0|=\delta} (zI - L_0)^{-1} \, dz,
\]

which can be rewritten as

\[
Q_0(\lambda_0) = L_0^{-1} \frac{1}{2\pi i} \int_{|z-\lambda_0|=\delta} (zL_0^{-1} - I)^{-1} \, dz.
\]

Using now the compact convergence of \( L_\epsilon^{-1} \) to \( L_0^{-1} \) together with Lemma 4.3, allows us to show that the operators

\[
Q_\epsilon(\lambda_0) = \frac{1}{2\pi i} \int_{|z-\lambda_0|=\delta} (zI - L_\epsilon)^{-1} \, dz,
\]

is well defined and converges to the operator \( Q_0 \). Both operators have finite dimensional range. This implies that necessarily for \( \epsilon \) small enough we have eigenvalues of \( L_\epsilon \) in the set \( \mathcal{O}(\lambda_0, \delta) \) and the combined multiplicity of all eigenvalues in this neighborhood is \( m \). Moreover the convergence of the spectral projections imply the convergence of the eigenfunctions stated above. We refer to [5][21] for details.

We can also obtain the convergence of the linear semigroups generated by the operators \( L_\epsilon \) and \( L_0 \), denoted by \( e^{-L_\epsilon t} \) and \( e^{-L_0 t} \).
Theorem 4.7. There exists a function \( \nu : (0, \epsilon_0] \to (0, \infty) \), \( \nu(\epsilon) \xrightarrow{\epsilon \to 0} 0 \), and numbers \( 1/2 < \gamma < 1 \), \( 0 < b < 1 \) such that
\[
||e^{-Lt} - E_0 e^{-Lt} M_{\epsilon}\|_{L^2(Z_\epsilon, Z^{1/2}_\epsilon)} \leq \nu(\epsilon) e^{-bt} t^{-\gamma}, \quad \forall t > 0.
\] (4.5)

Proof. The proof of this result follows the same line of proof as Proposition 2.7 in [4]. Using the spectral decomposition of the operators, we can write
\[
e^{-Lt} u_\epsilon = \sum_{n=1}^{\infty} e^{-\lambda_n t} (u_\epsilon, \varphi_n^\epsilon)_{L^2(R)} \varphi_n^\epsilon,
\]
But the convergence of the eigenvalues and eigenfunctions obtained above in Theorem 4.6 and with some computations as the ones performed in the proof of Proposition 2.7 in [4] we obtain the result.

Let us mention that the proof can also be done from a more functional analytic framework using the representation of the linear semigroups as
\[
e^{L_\epsilon t} = \frac{1}{2\pi i} \int e^{\mu t} (\mu I + L_{\epsilon})^{-1} d\mu, \quad 0 \leq \epsilon \leq \epsilon_0,
\]
where \( \tilde{\Gamma} \) is the border of sector \( \sum_{-1, \phi} = \{ \mu \in \mathbb{C} : |\text{arg}(\mu + 1)| \leq \phi \} \), \( \frac{\pi}{2} < \phi < \pi \), oriented in such a way the imaginary part of \( \mu \) increases as \( \mu \) describes the curve \( \tilde{\Gamma} \). Using the convergence of the resolvent operators, Corollary 4.4, we can also prove the result, see [7] for instance.

Therefore, we have analyzed the behavior of the linear parts of problem (1.8) as \( \epsilon \) goes to zero. Finally, we show the upper semicontinuity of the attractors and of the set of stationary states. This result is a consequence of the relation between the continuity of the linear semigroups with the continuity of the nonlinear semigroups through the Variation of Constants Formula.

Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded \( C^2 \)-function with bounded derivatives up to second order and satisfying condition (1.9). Then, it is known that the solutions of problems (1.8) and (4.1) are globally defined and we can associate to them the nonlinear semigroups \( \{ T_\epsilon(t) \} \) and \( \{ T_0(t) \} \).

Theorem 4.8. For each \( \tau, R > 0 \) there exist a function \( \bar{\nu} : (0, \epsilon_0] \to (0, \infty) \), \( \bar{\nu}(\epsilon) \xrightarrow{\epsilon \to 0} 0 \), such that
\[
||T_\epsilon(t) w^\epsilon - E_\epsilon T_0(t) M_{\epsilon} w^\epsilon||_{Z^{1/2}_\epsilon} \leq \bar{\nu}(\epsilon) t^{-\gamma}, \quad \forall t \in (0, \tau), ||w^\epsilon||_{Z^\epsilon} \leq R.
\] (4.6)

Moreover, the family of attractors \( \{ \mathcal{A}_\epsilon | \epsilon \in [0, \epsilon_0] \} \) of problem (1.8) is upper semicontinuous at \( \epsilon = 0 \) in \( Z^{1/2}_\epsilon \), in the sense that
\[
\sup_{\varphi^\epsilon \in \mathcal{A}_\epsilon} \left\{ \inf_{\varphi \in \mathcal{A}_0} \left\{ ||| \varphi^\epsilon - E_\epsilon \varphi |||_{Z^{1/2}_\epsilon} \right\} \xrightarrow{\epsilon \to 0} 0. \right. \] (4.7)

If \( \mathcal{E}_\epsilon \) and \( \mathcal{E}_0 \) are the set of stationary states of problems (1.8) and (4.1) respectively, then they satisfy the following convergence
\[
\sup_{\varphi^\epsilon \in \mathcal{E}_\epsilon} \left\{ \inf_{\varphi \in \mathcal{E}_0} \left\{ ||| \varphi^\epsilon - E_\epsilon \varphi |||_{Z^{1/2}_\epsilon} \right\} \xrightarrow{\epsilon \to 0} 0. \right. \] (4.8)
Proof. We follow the proof of Proposition 3.1 from [4]. To prove (4.6) we use the Variations of Constant Formula, that is,

\[ T_\epsilon(t)w^\epsilon = e^{L_\epsilon t}w^\epsilon + \int_0^t e^{L_\epsilon (t-s)} f(T_\epsilon(s)w^\epsilon)ds, \quad 0 \leq \epsilon \leq \epsilon_0. \]

Subtractiong \( T_\epsilon(t)w^\epsilon \) from \( E_\epsilon T_0(t)M_\epsilon w_\epsilon \) and taking norms in \( Z_\epsilon^{1/2} \) we obtain

\[ |||T_\epsilon(t)w^\epsilon - E_\epsilon T_0(t)M_\epsilon w_\epsilon|||_{Z_\epsilon^{1/2}} \leq |||e^{-L_\epsilon t}w^\epsilon - E_\epsilon e^{-L_\epsilon t}M_\epsilon w_\epsilon|||_{Z_\epsilon^{1/2}} \]

\[ + \int_0^t |||e^{-L_\epsilon (t-s)} f(T_\epsilon(s)w^\epsilon) - E_\epsilon e^{-L_\epsilon (t-s)} f(T_0(s)M_\epsilon w^\epsilon)|||_{Z_\epsilon^{1/2}} ds. \]

Adding and subtracting appropriate terms in the integral (as in Proposition 3.1 [4]) using estimate (4.5) and Gronwall inequality we get (4.6).

The upper semicontinuity of the attractors \( A_\epsilon \) follows from the continuity of the nonlinear semigroups given by (4.6), the fact that \( \cup_{0 \leq \epsilon \leq \epsilon_0} M_\epsilon A_\epsilon \) is a bounded set in \( Z_0^{1/2} \), that the attractor \( A_0 \) attracts bounded sets (in particular it attracts \( \cup_{0 \leq \epsilon \leq \epsilon_0} M_\epsilon A_\epsilon \)) and the fact that \( A_0 \) is an invariant set for the flow \( T_0(t) \), see for instance [4][25].

we first note that \( \cup_{0 \leq \epsilon \leq \epsilon_0} M_\epsilon A_\epsilon \) is a bounded set in \( L^\infty(0,1) \) and in \( Z_0 \). Then, using the attractivity property of \( A_0 \) in \( Z_0 \) we have that for any \( \eta > 0 \) there exists \( \tau > 0 \) such that

\[ \inf_{\varphi \in A_0} |||E_\epsilon T_0(\tau)M_\epsilon \varphi^\epsilon - E_\epsilon \varphi|||_{Z_0^{1/2}} < \eta/2, \quad \text{for all } \varphi^\epsilon \in A_\epsilon \text{ and } 0 \leq \epsilon \leq \epsilon_0. \]

Consequently, (4.7) is obtained using (4.6) and taking into account \( A_\epsilon \) is an invariant set.

Finally we show the upper semicontinuity of the set of stationary states \( \mathcal{E}_\epsilon \). For this, we will show that for any sequence \( \varphi^\epsilon \in \mathcal{E}_\epsilon \) we can get a subsequence that we still denote by \( \varphi^\epsilon \) and a \( \varphi^0 \in \mathcal{E}_0 \) such that \( |||\varphi^\epsilon - E_\epsilon \varphi^0|||_{Z_0^{1/2}} \to 0 \). But since \( \mathcal{E}_\epsilon \subset A_\epsilon \) and we have already proved the uppersemicontinuity of the attractors, we have the existence of \( \varphi^0 \in A_0 \) such that via a subsequence we have \( |||\varphi^\epsilon - E_\epsilon \varphi^0|||_{Z_\epsilon^{1/2}} \to 0 \). We need to prove that indeed \( \varphi^0 \in \mathcal{E}_0 \). But for this, observe that for any \( t > 0 \) since \( \varphi^\epsilon \) is a stationary state we have

\[ |||\varphi^\epsilon - E_\epsilon T_0(t)\varphi^0|||_{Z_\epsilon^{1/2}} = |||T_\epsilon(t)\varphi^\epsilon - E_\epsilon T_0(t)\varphi^0|||_{Z_\epsilon^{1/2}} \xrightarrow{\epsilon \to 0} 0, \]

where we have used (4.6). But since \( \varphi^\epsilon - E_\epsilon \varphi^0 \to 0 \), we get \( \varphi^0 - T_0(t)\varphi^0 = 0 \). This implies that \( \varphi^0 \in \mathcal{E}_0 \), see Proposition 3.1, [4].

\[ \square \]

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