Response statistics of single degree of nonlinear random structure with nonlinear damping characteristic and nonlinear elastic characteristic under white-noise excitations

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Abstract. The paper investigates the applicability of the path integral solution method for calculating the response statistics of nonlinear dynamic systems whose equations of motion can be modelled by the use linearization differential equations. The present paper consists of discussion on dynamic response of structures under random load. They are random processes and commonly described by spectral density functions. An identification technique is proposed for a nonlinear oscillator excited by response-dependent white noise. Stiffness, damping and excitation are estimated from records of the stationary stochastic response. Assume that a single-degree of freedom structure is excited by a force which is a random process described by the spectral density function.

1. Introduction
Nonlinear, dynamic systems subject to random excitations are frequently met in engineering practice. The research goals are, firstly, the computation of stochastic, nonlinear response characteristics (with accuracy and efficiency as important criteria) and, secondly, the investigation and thorough understanding of stochastic, nonlinear response phenomena. The source of randomness can vary from surface randomness in vehicle motion and environmental changes, such as earthquakes or wind exciting high rise buildings or wave motions at sea exciting offshore structures or ships, to electric or acoustic noise exciting mechanical structures. The desire to compute response characteristics, such as the statistical moments and the power spectral density of the response of these systems, leads to the development of methods that can be used to approximate this response. The excitations, that will be studied, are stationary, Gaussian processes [1,2]. They are random processes and commonly described by spectral density functions. Assume that a single-degree of freedom structure is excited by a force $F$ which is a random process described by the spectral density function $S_F(\omega)$ [2]. If Gaussian assumption is adopted for the force and the structure is assumed to be nonlinear, the response is expected to be Gaussian distribution as well. As the force is not deterministic, the response of the structure is expected to be random.

2. System model
As a result, to obtain the standard deviation of the response, the velocity variance of the single-degree of freedom system the relative acceleration of this structure and the spectral density of the response.
A single-story building is modeled by four identical columns of Young’s modulus $E$ and height $h$ and a rigid floor of weight $m$. The damping can be approximated by an equivalent damping constant $c$. The ground acceleration due to an earthquake is assumed to be a Gaussian white noise with a constant spectrum $S_0$. The columns have cylinder sections of diameter $D$. 
The moment of inertia $I$ for the columns and the total stiffness of the four columns $k$ are:

$$I = \frac{\pi D^4}{64},$$  

$$k = \frac{3\pi EI}{h^3}.$$  

Note that as the ground acceleration is assumed to be a Gaussian white noise of constant spectral density $S_0^{\text{G}}$ [3,4] the spectral density of the earthquake force that acts on the structure can be found to be $(0.5...1.1)m^2S_0^{\text{G}}$.

The equation of motion for this single-degree of freedom structure under earthquake excitation can be written as

$$m\ddot{x}(t) + c\dot{x}(t) + g(x(t)) = -m\ddot{x}_g(t),$$  

where $m$ is the mass, $c$ is the viscous damping coefficient, $m\ddot{x}_g(t)$ is the external excitation signal with zero mean and $x(t)$ is the displacement response of the system.

Dividing the equation by $m$ the equation of motion for this single-degree of freedom structure under earthquake excitation can be written as

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{g(x(t))}{m} = -\ddot{x}_g(t).$$  

where $\frac{c}{m}$ is the viscous damping factor, $p$ is the undamped natural frequency, for the linear system, $\alpha$ is the nonlinear factor to control the type and degree of nonlinearity in the system and $x(t)$ is the displacement response of the system. A positive value of $\alpha$ represents a hardening system while a negative value represents a softening system behavior.

In the general form, the previous equation is written

$$\ddot{x}(t) + h(x(t),\dot{x}(t)) = w(t).$$  

Determination of solutions is done considering the following linear equation

$$\ddot{x}(t) + \beta_{\text{ech}}\dot{x}(t) + \gamma_{\text{ech}}x(t) = w(t).$$  

The difference between the nonlinear stiffness and linear stiffness terms is

$$\varepsilon = h(x(t),\dot{x}(t)) - \beta_{\text{ech}}\dot{x}(t) - \gamma_{\text{ech}}x(t).$$
It is required that the error is minimal, that is
\[ \frac{\partial}{\partial \beta_{eh}} E[\varepsilon^2]=0, \] respectively
\[ \frac{\partial}{\partial \gamma_{eh}} E[\varepsilon^2]=0. \]

Because
\[ E[\varepsilon^2]=E[h^2]+\beta_{eh}^2 E[x^2]+2\beta_{eh} E[xh]+2\gamma_{eh} E[xx]-2\gamma_{eh} E[xh], \]
we get a system of two equations with two unknowns
\[ E[\dot{x}h]-\beta_{eh} E[x]-\gamma_{eh} E[\dot{xx}]=0 \]
\[ E[\dot{x}h]-\beta_{eh} E[\dot{xx}]-\gamma_{eh} E[xx^2]=0, \]
with solutions
\[ \beta_{eh} = \frac{E[\dot{x}h] - E[x] E[\dot{x}h]}{E[\dot{x}^2] E[x] - (E[x])^2} \]
\[ \gamma_{eh} = \frac{E[\dot{\eta}h] - E[\dot{\eta}] E[\dot{\eta}h]}{E[\dot{\eta}^2] E[\dot{\eta}] - (E[\dot{\eta}])^2}. \]

Since the average is zero, the solutions are written simplified
\[ \beta_{eh} = \frac{E[\dot{x}h]}{E[x^2]} \]
\[ \gamma_{eh} = \frac{E[\dot{x}^2]}{E[x^2]}. \]

In case of equation (3), by linearization we get
\[ x(t)+2\dot{\xi} p \left[ 1+\varepsilon \left( P_{e1}\frac{E[x]}{2} + P_{e2}\frac{E[x]}{2} \right) \right] + p^2 \frac{E[\dot{x}x]}{E[x]} \dot{\xi}(t) + \frac{E[\dot{x}h]}{E[x]} x(t) = w(t) \]
The damping feature is
\[ \beta_{eh} = 2\dot{\xi} P_e \frac{E[\dot{x}h]}{E[x]} = 2\dot{\xi} p \left[ 1+\varepsilon \left( P_{e1}\frac{E[x]}{2} + P_{e2}\frac{E[x]}{2} \right) \right] + p^2 \frac{E[\dot{x}x]}{E[x]} \dot{\xi}(t) + \frac{E[\dot{x}h]}{E[x]} x(t) = w(t) \]

which can still be written
\[ \beta_{eh} = 2\dot{\xi} P_e \frac{E[\dot{x}h]}{E[x]} = 2\dot{\xi} p \left[ 1+\varepsilon \left( \frac{63}{4} \sigma^2 + \frac{2835}{16} - \frac{r_3}{4} \frac{\dot{\xi}(t)}{E[x]} \right) \right] \]

Because
\[ E[\dot{x}x]=0, \quad E[\dot{x}x^3]=0, \quad \frac{E[x]}{4} \frac{63}{4} \sigma^2, \quad \frac{E[x]}{4} \frac{45}{4} \sigma^2. \]

The elastic characteristic is given by
\[ \gamma_{eh} = p_c^2 \frac{E \{x h \}}{E \{x^2 \}} = 2 \xi p \left\{ \frac{E \{x x \}}{E \{x^2 \}} \right\} + p^2 \frac{E \{x^2 \}}{E \{x^2 \}} + \alpha p^2 \frac{E \{x^4 \}}{E \{x^2 \}} \] (21)

Because

\[ E \{x x \} = 0, \quad \frac{E \{x^4 \}}{E \{x^2 \}} = 3 \sigma^2, \quad \frac{3}{E \{x^2 \}} = 0, \quad \frac{5}{E \{x^2 \}} = 0, \] (22)

We can write

\[ \gamma_{eh} = p_c^2 = p^2 (1 + 3 \alpha \sigma^2). \] (23)

Therefore, the equivalent linear equation is written in the form

\[ \ddot{x}(t) + 2 \xi p \left[ 1 + \epsilon \left( \frac{63}{4} \frac{r_1 \sigma^2}{x} + \frac{2835}{16} \frac{r_2 \sigma^4}{x} \right) \right] \dot{x}(t) + p^2 (1 + 3 \alpha \sigma^2) x(t) = w(t) \] (24)

If the Fourier transform is applied, the transfer function associated with the oscillator system is

\[ H(\omega) = \frac{1}{m \left\{ -\omega^2 + i \omega \frac{E \{x h \}}{E \{x^2 \}} \right\} \left\{ \frac{E \{x^2 \}}{E \{x \}} \right\} } \] (25)

where from

\[ |H(\omega)| = \frac{1}{m \left\{ \frac{E \{x h \}}{E \{x^2 \}} - \omega^2 \right\} + \omega^2 \left\{ \frac{E \{x \}}{E \{x \}} \right\} } \] (26)

The frequency response function, in this case, of the single degree of freedom system is

\[ |H(\omega)| = \frac{1}{m \left\{ p^2 + 3 \alpha p^2 \sigma^2 - \omega^2 \right\} + 4 \xi^2 p^2 \omega^2 \left[ 1 + \epsilon \left( \frac{63}{4} \frac{r_1 \sigma^2}{x} + \frac{2835}{16} \frac{r_2 \sigma^4}{x} \right) \right]^2 } \] (27)

The displacement variance [5] of the single-degree of freedom system under Gaussian white noise excitation can be expressed as,

\[ \sigma^2_x = R_x(0) = \int_{-\infty}^{\infty} |H(\omega)|^2 S_F(\omega) d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2 m S_{x_x} d\omega \] (28)

Using the transfer function, the response as a function of frequency has expression

\[ \bar{X}(\omega) = H(\omega) \bar{F}(\omega), \] (29)

where

\[ \bar{x}(\omega) = F(\dot{x}(t)), \quad \bar{F}(\omega) = F(F(t)). \] (30)

The power spectral density of the response is

\[ S_x(\omega) = \frac{S_F(\omega)}{m^2 \left\{ \frac{E \{x h \}}{E \{x^2 \}} - \omega^2 \right\} + \omega^2 \left\{ \frac{E \{x^2 \}}{E \{x \}} \right\}^2 } \] (31)

or
The most used spectrum to describe the earthquake ground acceleration is earthquake spectrum which is expressed as

\[ S_x = mS'_0, \quad S'_0 = (0, 5...1, 1) \frac{m^2}{s^2}. \]

We obtain:

\[
\sigma^2_x = \int_{-\infty}^{\infty} |H(\omega)|^2 m^2 S'_0 d\omega = \int_{-\infty}^{\infty} \frac{1}{m^2} \left( p^2 + 3\alpha p^2 \sigma_x^2 - \omega^2 \right)^2 + 4\xi^2 p^2 \omega^2 \left[ 1 + \varepsilon \left( \frac{63}{4} \sigma_x^2 + \frac{2835}{16} r_s \sigma_x^4 \right) \right] d\omega = \int_{-\infty}^{\infty} \frac{1}{m^2} \left( p^2 + 3\alpha p^2 \sigma_x^2 - \omega^2 \right)^2 + 4\xi^2 p^2 \omega^2 \left[ 1 + \varepsilon \left( \frac{63}{4} \sigma_x^2 + \frac{2835}{16} r_s \sigma_x^4 \right) \right] d\omega.
\]

The displacement variance is given by:

\[
\sigma^2_x = \frac{S'_0 \pi m^2}{ck} - \alpha E \{ xG(x) \}.
\]

Similarly, the velocity variance of the system [5,6] can be expressed as

\[
R_x(\tau) = -\frac{d^2 R_x(\tau)}{d\tau^2} = \int_{-\infty}^{\infty} \omega \omega^2 S_x(\omega)e^{i\omega \tau} d\omega.
\]

Obtain for the velocity variance

\[
\sigma^2_x = E \{ x' \} = R_x(0) = \int_{-\infty}^{\infty} \omega^2 S_x(\omega) d\omega = S'_0 m^2 \int_{-\infty}^{\infty} \omega^2 \left[ \frac{p^2}{m(p^2 - \omega^2)^2 + 4\xi^2 p^2 \omega^2} \right] d\omega = \frac{S'_0 \pi}{2\xi p} = \frac{\pi S'_0 m}{c}.
\]

The relative acceleration of this structure is unbounded. On the other hand, the absolute acceleration \( \ddot{x}_{abs} \) is bounded and can be obtained as follows:

\[
\ddot{x}_{abs} = \dot{x}(t) + \ddot{x}(t) = 2\xi p_e x(t) + p_e^2 x(t).
\]

3. Numerical results:

In this example, \( S'_0 = 0.52 \frac{m^2}{s^3}, \quad m = 2.2 \times 10^3 \text{kg}, \quad n = 4 \text{ columns}, \quad d = 0.5m, \quad h = 2m, \quad E_0 = 0.2 \times 10^{11} \text{Pa}, \quad \xi = 0.25, \quad \alpha = 7 \text{m}^{-2}, \quad k = 29.1 \times 10^6 \frac{N}{m}; \quad p = 12.04 \text{s}^{-1}; \quad r_1 = 3.1 \times 10^2 \frac{s^2}{m^2}, \quad r_3 = 12.14 \times 10^2 \frac{s^4}{m^4}, \quad c = 1204 \times 10^3 \frac{N \cdot s}{m}, \quad \varepsilon = 0.01.

The power spectral density for excitation \( S_\phi(\omega) \) is \( S_\phi(\omega) = m^2 S'_0 = 2.08 \times 10^{10} N^2 \cdot s \).

Obtain in this case for the displacement variance:

\[
\sigma^2_x = 105 \times 10^{-5} m^2.
\]

In this case, the coefficient \( p_e \) can be expressed

\[
p_e^2 = p_e^2 \left( 1 + 15\alpha \sigma_x^4 \right) = 15.21 s^{-2}.
\]
4. Remarks
This suggests that the variance of absolute acceleration decreases as the damping increases when the damping ratio is smaller than 0.5. The variance increases as the damping increases when the damping ratio is bigger than 0.5. It can be seen that stiffening structure (increase stiffness) can reduce displacement but would result in the increase of absolute acceleration. A positive value of $\alpha$ represents a hardening system while a negative value represents a softening system behavior. On the other hand, increasing mass can reduce absolute acceleration but increase displacement. It seems that the only damping increase (when $0.5 < \xi$) can result in the simultaneous reduction of displacement and absolute acceleration. These conclusions are very useful when designing structure under seismic condition.

References
[1] Zhao L and Chen Q 1997 An equivalent non-linearization method for analyzing response of nonlinear systems to random excitations, Applied Mathematics and Mechanics
[2] Stan P 2009 Analysis of single-degree of freedom non-linear structure under Gaussian white noise ground excitation, The Scientific Book, University of Pitesti
[3] Stan P and Stan M 2017 The study spectral analysis to random vibrations for nonlinear oscillators, International Congress of Automotive and Transport Engineering CAR 2017, Pitesti, Automotive Series 27
[4] Stan P and Stan M 2015 Nonlinear vibrations in automotive suspensions, International Conference Computational Mechanics and Virtual Engineering, Brasov Romania
[5] Stan P and Stan M 2015 Random vibration of Houdaille shock absorber, International Conference Computational Mechanics and Virtual Engineering, Brasov Romania
[6] Stan M and Stan P 2017 Numerical solution of the proces of hydroplaning at wheels with envelope, International Congress of Automotive and Transport Engineering, Mobility Engineering and Environment, CAR 2017, 017 IOP Conf. Ser. Mater. Sci. Eng. 252