ANTI-PALINDROMIC COMPOSITIONS

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Abstract. A palindromic composition of \( n \) is a composition of \( n \) which can be read the same way forwards and backwards. In this paper we define an anti-palindromic composition of \( n \) to be a composition of \( n \) which has no mirror symmetry amongst its parts. We then give a surprising connection between the number of anti-palindromic compositions of \( n \) and the so-called tribonacci sequence, a generalization of the Fibonacci sequence. We conclude by defining a new \( q \)-analogue of the Fibonacci sequence, which is related to certain equivalence classes of anti-palindromic compositions.

1. Introduction

Let \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s) \) be a sequence of positive integers such that \( \sum \sigma_i = n \). The sequence \( \sigma \) is called a composition of \( n \) of length \( s \). The numbers \( \sigma_i \) are called the parts of the composition. The number of compositions of \( n \) equals \( 2^{n-1} \), while the number of compositions of \( n \) into \( s \) parts equals \( \binom{n-1}{s-1} \). The empty composition is often considered the only composition of 0, having length equal to 0.

1.1. Palindromic and anti-palindromic compositions. If \( \sigma_i = \sigma_{s-i+1} \) for all \( i \), then \( \sigma \) is called a palindromic composition. It is well known [11] that if \( pc(n) \) is the number of palindromic compositions, then \( pc(n) = 2^\lfloor n/2 \rfloor \). For instance, the \( pc(5) = 4 \) palindromic compositions of 5 are

\( (5), \ (1, 3, 1), \ (2, 1, 2), \ \text{and} \ (1, 1, 1, 1, 1). \)

Recent work of the authors [3, 12] generalize this result to compositions that are palindromic modulo \( m \), where the condition \( \sigma_i = \sigma_{s-i+1} \) is replaced with the weaker condition \( \sigma_i \equiv \sigma_{s-i+1} \pmod{m} \).

If \( \sigma_i \neq \sigma_{s-i+1} \) for all \( i \neq \frac{s+1}{2} \), then we say \( \sigma \) is an anti-palindromic composition. Let \( ac(n) \) be the number of anti-palindromic compositions of \( n \). Then the \( ac(4) = 5 \) anti-palindromic compositions of 4 are

\( (4), \ (1, 3), \ (3, 1), \ (1, 1, 2), \ \text{and} \ (2, 1, 1). \)

Furthermore, let \( ac(n, s) \) be the number of anti-palindromic compositions of \( n \) of length \( s \), \( ac_0(n) \) be the number of anti-palindromic compositions of \( n \) of even length, and \( ac_1(n) \) be the number of anti-palindromic compositions of odd length (thus \( ac(n) = ac_0(n) + ac_1(n) \)). Notice that for each anti-palindromic composition of \( n \) of length \( s \), we can form \( 2^\lfloor s/2 \rfloor \) flip-equivalent anti-palindromic compositions of \( n \) of length \( s \) by switching any number of the pairs \( \sigma_i \) and \( \sigma_{s-i+1} \) (\( i \neq \frac{s+1}{2} \)). For instance, the anti-palindromic compositions

\( (1, 3, 3, 2, 4), \ (1, 2, 3, 3, 4), \ (4, 3, 3, 2, 1), \ \text{and} \ (4, 2, 3, 3, 1) \)
Table 1. Values of $ac_0(n)$, $ac_1(n)$, $ac(n)$, $rac_0(n)$, $rac_1(n)$, and $rac(n)$ for $n \leq 10$.

| $n$ | $ac_0(n)$ | $ac_1(n)$ | $ac(n)$ | $rac_0(n)$ | $rac_1(n)$ | $rac(n)$ |
|-----|-----------|-----------|---------|------------|------------|----------|
| 0   | 1         | 0         | 1       | 0          | 1          | 1        |
| 1   | 0         | 1         | 1       | 0          | 1          | 1        |
| 2   | 0         | 1         | 1       | 0          | 1          | 1        |
| 3   | 2         | 1         | 3       | 1          | 1          | 2        |
| 4   | 2         | 3         | 5       | 1          | 2          | 3        |
| 5   | 4         | 5         | 9       | 2          | 3          | 5        |
| 6   | 8         | 9         | 17      | 3          | 5          | 8        |
| 7   | 14        | 17        | 31      | 5          | 8          | 13       |
| 8   | 26        | 31        | 57      | 8          | 13         | 21       |
| 9   | 48        | 57        | 105     | 13         | 21         | 34       |
| 10  | 88        | 105       | 193     | 21         | 34         | 55       |
formulae for the $k$-binocci are developed. For instance, we have

$$f_3(n + 1) = \sum_{j=0}^{\lfloor n/4 \rfloor} (-1)^j \binom{n - 3j}{j} \frac{n - 2j}{n - 3j^2} q^{n - 4j - 1}.$$  

The $k$-binocci numbers also play a role in computing the probability of flipping exactly $k$ consecutive heads in $n$ flips of a fair coin.

1.3. Formulae for anti-palindromic compositions. Our first result gives a surprising connection between the tribonacci numbers and anti-palindromic compositions of even length.

**Theorem 1.** For all $n \geq 1$ we have $ac_0(n) = 2 \cdot f_3(n - 2).$

This theorem can be deduced by a careful inspection of the identity

$$\frac{(\frac{q}{1-q})^2 - \frac{q^2}{1-q^2}}{1 - \left(\frac{q}{1-q} - \frac{q^2}{1-q^2}\right)} = \frac{2q^3}{1 - (q + q^2 + q^3)}.$$  

Indeed, the left hand side is

$$\sum_{n \geq 1} ac_0(n) q^n$$
since every even-length anti-palindromic composition is a sequence of pairs of distinct positive integers, and the right hand side is

$$\sum_{n \geq 1} 2 \cdot f_3(n)q^{n+2}.$$ 

We will give an algebraic (Section 2.1) and combinatorial (Section 2.2) proof of this result. Note that $ac(0) = ac_0(0) = 1$, as the empty composition is vacuously anti-palindromic. Our next result gives the number of anti-palindromic compositions of $n$.

**Theorem 2.** For all $n \geq 1$, 

$$ac(n) = f_3(n) + f_3(n - 2).$$ 

We will prove Theorem 2 in Section 2.3, and also observe that (for $n \geq 2$) 

$$ac_1(n) = f_3(n - 1) + f_3(n - 3).$$ 

In Section 2.4, we prove the following results which give the formulae for $ac(n, s)$.

**Theorem 3.** Let $s \geq 0$ be a fixed integer, and 

$$G(q, s) = \sum_{n \geq 0} ac(n, s)q^n.$$ 

Then for $|q| < 1$ 

$$G(q, s) = \frac{2^{s/2}|q|^{3s/2}}{(1 - q)^s(1 + q)^{3s/2}}.$$ 

For instance, $G(q, 0) = 1$, 

$$G(q, 1) = \frac{q}{1 - q} = q + q^2 + q^3 + \ldots,$$ 

and 

$$G(q, 2) = \frac{2q^3}{1 - q - q^2 + q^3} = 2q^3 + 2q^4 + 4q^5 + \ldots,$$ 

which give the (verifiable) formulae $ac(0, 0) = 1$, $ac(n, 0) = 0$ for $n > 0$, $ac(n, 1) = 1$ for $n > 0$, and $ac(n, 2) = 2 \cdot \left\lfloor \frac{n - 1}{2} \right\rfloor$ for $n > 1$. For $s \geq 2$, we have the following corollary.

**Corollary 1.** Let $a$ be a positive integer. If $s = 2a$, then 

$$ac(n, s) = \sum_{r+2t=n-3a} 2^a \binom{a + r - 1}{r} \binom{a + t - 1}{t},$$ 

and if $s = 2a + 1$, then 

$$ac(n, s) = \sum_{r+2t=n-3a} 2^a \binom{a + r}{r} \binom{a + t - 1}{t}.$$ 

By the observation in Section 1.1 regarding $rac(n, s)$ and $ac(n, s)$, we also have a formula for $rac(n, s)$ by dividing by the appropriate power of 2.
1.4. An observation regarding the Fibonacci numbers. Recall the following two \(q\)-analogues of the Fibonacci numbers,

\[
F_n(q) = \begin{cases} 
0 & n = 0, \\
1 & n = 1, \\
F_{n-1}(q) + q^{n-2}F_{n-2}(q) & n > 1,
\end{cases}
\]

and

\[
\hat{F}_n(q) = \begin{cases} 
0 & n = 0, \\
1 & n = 1, \\
\hat{F}_{n-1}(q) + q^{n-1}\hat{F}_{n-2}(q) & n > 1.
\end{cases}
\]

These are referred to as \(q\)-analogues due to the property that \(F_n(q) \to f_2(n)\) and \(\hat{F}_n(q) \to f_2(n)\) as \(q \to 1^-\). Properties of these two sequences of polynomials have been studied extensively, see for instance \([11, 9, 7, 13]\).

We define a new \(q\)-analogue of the Fibonacci numbers, which will have a connection to the anti-palindromic compositions. Define

\[
\phi_n(q) = \begin{cases} 
q & n = 1, \\
q & n = 2, \\
q + q^2 & n = 3, \\
\phi_{n-1}(q) + \phi_{n-2}(q) + (q^2 - 1)\phi_{n-3}(q) & n > 3.
\end{cases}
\]

Clearly \(\phi_n(q) \to f_2(n)\) as \(q \to 1^-\) for all \(n \geq 1\), and our final result gives a combinatorial description of the coefficients of these polynomials. For convenience, we set \(\phi_0(q) = 1\).

**Theorem 4.** The coefficient of \(q^n\) in the polynomial \(\phi_n(q)\) equals \(\text{rac}(n, s)\).

We will give a proof of Theorem 4 in Section 2.5. The first few polynomials \(\phi_n(q)\) are given below, where the coefficients can be compared with Table \(2\).

\[
\begin{align*}
\phi_0(q) &= 1, & \phi_5(q) &= q + 2q^2 + 2q^3 \\
\phi_1(q) &= q, & \phi_6(q) &= q + 2q^2 + 4q^3 + q^4 \\
\phi_2(q) &= q, & \phi_7(q) &= q + 3q^2 + 6q^3 + 2q^4 + q^5 \\
\phi_3(q) &= q + q^2, & \phi_8(q) &= q + 3q^2 + 9q^3 + 5q^4 + 3q^5 \\
\phi_4(q) &= q + q^2 + q^3, & \phi_9(q) &= q + 4q^2 + 12q^3 + 8q^4 + 8q^5 + q^6.
\end{align*}
\]

Also in Section 2.5, we deduce the following corollary.

**Corollary 2.** For \(n \geq 1\) we have \(\text{rac}_0(n) = f_2(n - 2)\), \(\text{rac}_1(n) = f_2(n - 1)\), and \(\text{rac}(n) = f_2(n)\).

We summarize our results regarding \(ac(n)\) and \(\text{rac}(n)\) for sufficiently large \(n\) below, illustrating the elegance of the formulae.

\[
\begin{align*}
ac_0(n) &= 2 \cdot f_3(n - 2), & \text{rac}_0(n) &= f_2(n - 2) \\
ac_1(n) &= f_3(n - 1) + f_3(n - 3), & \text{rac}_1(n) &= f_2(n - 1) \\
ac(n) &= f_3(n) + f_3(n - 2), & \text{rac}(n) &= f_2(n).
\end{align*}
\]
2. Proofs of theorems

2.1. Algebraic proof of Theorem 1. In light of the fact that \( a_0(1) = a_0(2) = 0, a_0(3) = a_0(4) = 2, \) and \( a_0(5) = 4, \) we see that the theorem is true for \( n < 6. \) Assume now \( n \geq 6. \) Clearly we can construct an anti-palindromic composition of \( n \) from one of two fewer parts by inserting \( j \) at the beginning and \( k \) at the end (making sure \( j \neq k \)), where if the inner composition is a composition of \( m, \) then \( j + k \) must equal \( m - n. \) Hence

\[
a_0(n) = \sum_{m=0}^{n-3} (n-m-1-\chi(n-m)) a_0(m),
\]

where \( \chi(j) = 1 \) if \( j \) is even and 0 if \( j \) is odd. The term \( (n-m-1-\chi(n-m)) \) accounts for the number of \( j \) and \( k. \) Hence

\[
a_0(n) - a_0(n-1) = \sum_{m=0}^{n-3} (n-m-1-\chi(n-m)) a_0(m)
- \sum_{m=0}^{n-4} (n-1-m-1-\chi(n-1-m)) a_0(m)
= (2-\chi(3)) a_0(n-3) + \sum_{m=0}^{n-4} (n-m-1-\chi(n-m)) a_0(m)
- \sum_{m=0}^{n-4} (n-1-m-1-\chi(n-1-m)) a_0(m)
= 2a_0(n-3) + 2 \sum_{m=0}^{n-4} \chi(n-m-1)a_0(m).
\]

Thus

\[
a_0(n) - a_0(n-1) - 2a_0(n-3) = 2 \sum_{m=0}^{n-4} \chi(n-m-1)a_0(m).
\]

Let \( r(n) = a_0(n) - a_0(n-1) - 2a_0(n-3). \) Then

\[
r(n) + r(n-1) = 2 \sum_{m=0}^{n-4} \chi(n-m-1)a_0(m)
+ 2 \sum_{m=0}^{n-5} \chi(n-m-2)a_0(m)
= 2 \sum_{m=0}^{n-5} a_0(m)
\]

since \( \chi(n) + \chi(n-1) = 1 \) and \( \chi(3) = 0. \) Therefore,

\[r(n) + r(n-1) - (r(n-1) + r(n-2)) = 2a_0(n-5),\]

and simplifying we obtain

\[a_0(n) - m_1(n-1) - a_0(n-2) - a_0(n-3) = 0.\]
This is the defining recurrence for \( f_3(n) \), and since \( 2 \cdot f_3(n-2) = ac_0(n) \) for \( n > 6 \), we see that by induction \( ac_0(n) = 2 \cdot f_3(n-2) \) for all \( n \geq 1 \).

2.2. Combinatorial proof of Theorem 1. We begin with a lemma regarding the tribonacci numbers.

**Lemma 1.** For \( n \geq 2 \), the tribonacci number \( f_3(n) \) equals number of compositions of \( n-1 \) with parts equal to 1, 2, or 3.

**Proof.** First note that \( f_3(2) = 1 \), \( f_3(3) = 2 \), and \( f_3(4) = 4 \). Since the compositions of 1, 2, and 3 only consist of parts equal to 1, 2, or 3, and the number of compositions of \( n \) is equal to \( 2^{n-1} \), the lemma holds for \( n \leq 4 \). Now for \( n > 4 \), each composition of \( n-1 \) into parts equal to 1, 2, or 3 is formed by taking a composition of \( n-4 \), \( n-3 \), or \( n-2 \) and adjoining a 3, 2, or 1, respectively. Thus the number of compositions of \( n-1 \) into parts equal to 1, 2, or 3 is equal to \( f_3(n-3) + f_3(n-2) + f_3(n-1) = f_3(n) \). \( \square \)

We will now show that for \( n \geq 3 \) the number of compositions of \( n-3 \) into parts equal to 1, 2, or 3 equals the number of anti-palindromic compositions of \( n \). Since \( ac_0(1) = 0 = 2 \cdot f_3(-1) \) and \( ac_0(2) = 0 = 2 \cdot f_3(0) \) this will establish the theorem.

**Proof of Theorem 1.** For \( n = 2 \) we see that \( ac(2) = 2 \cdot f_3(0) = 0 \), so for any \( n \geq 3 \) start with a composition \( \sigma \) of \( n-3 \) into parts equal to 1, 2, or 3. The key will be to use \( \sigma \) to construct a sequence of pairs of distinct positive integers with sum equal to \( n \).

Now recall a partition of \( n \) is a composition of \( n \) where the parts are written in non-increasing order. Let \( \sigma + \tau \) denote sequence concatenation, as in \((1,2) + (4,5) = (1,2,4,5)\). For our choice of \( \sigma \), we can find partitions \( \lambda_1, \lambda_2, \ldots, \lambda_r \) with parts equal to 1 or 2 (or the empty partition, \( \emptyset \)) such that

\[
\sigma = \lambda_1 + \sigma_2 + \lambda_2 + \ldots + \sigma_r + \lambda_r,
\]

where each \( \sigma_j \) is either equal to the composition (3) or equal to the composition (1,2).

For example, take the composition

\[
\sigma = (2,3,1,1,2,2,1,1,2,1,3)
\]

of 20. Then we can decompose \( \sigma \) as

\[
\begin{align*}
\lambda_1 &= (2) \\
\sigma_2 &= (3) \\
\lambda_2 &= (1) \\
\sigma_3 &= (1,2) \\
\lambda_3 &= (2,1,1) \\
\sigma_4 &= (1,2) \\
\lambda_4 &= (1) \\
\sigma_5 &= (3) \\
\lambda_5 &= \emptyset.
\end{align*}
\]

It is not difficult to see that this decomposition is unique; the only way a segment in the composition that is a partition with parts equal to 1 or 2 terminates is with the segment (3) or the segment (1,2).
Now given the decomposition \( \sigma = \lambda_1 + \sigma_2 + \lambda_2 + \ldots + \sigma_r + \lambda_r \), form a sequence of pairs \((s_1, \lambda_1), (s_2, \lambda_2), \ldots, (s_r, \lambda_r)\) where \(s_1 = +3\), \(s_j = +3\) if \(\sigma_j = (3)\), and \(s_j = -3\) if \(\sigma_j = (1, 2)\). For our example shown above, we have the pairs \((+3, (2)), (+3, (1)), (-3, (2, 1, 1)), (-3, (1)), (+3, \emptyset)\).

For each pair \((s_j, \lambda_j)\) we now form a new pair \((b_j, c_j)\) in the following way. Start with \(b_j = 2\) and \(c_j = 1\). For each 2 in the partition \(\lambda_j\) increase both \(b_j\) and \(c_j\) by one. For each 1 in the partition \(\lambda_j\) increase \(b_j\) by one. We now have pairs \((b_j, c_j)\) of positive integers such that \(b_j > c_j\). Now if \(s_j = +3\) we are done. If \(s_j = -3\), we switch the numerical values of \(b_j\) and \(c_j\) so that \(b_j < c_j\), and then we are done.

Finally form the anti-palindromic composition \(\tau = (\tau_1, \tau_2, \ldots, \tau_{2r})\) by setting \(\tau_j = b_j\) and \(\tau_{2r-j+1} = c_j\). Notice that though we started with a composition of \(n - 3\) this is a composition of \(n\); the addition of 3 came from inserting \(s_1 = +3\). In our toy example we have \(\tau = (3, 3, 2, 1, 2, 1, 3, 5, 1, 2)\).

We have now embedded the compositions of \(n - 3\) made up of parts equal to 1, 2, or 3 into the anti-palindromic compositions of \(n\). We still need to embed a second, disjoint copy. To do this we return to the pairs \((s_j, \lambda_j)\) and make a new collection of pairs \((s_j', \lambda_j)\) by setting \(s_j' = -s_j\). Now following the same procedure as before we construct an anti-palindromic word \(\tau'\) that is, in fact, the reverse of \(\tau\). Again looking at our example from before we have \(\tau' = (2, 1, 5, 3, 1, 2, 1, 2, 3, 3)\).

To show that these two embedded sets are disjoint, notice that for a composition \(\tau\) formed by using \(s_1 = +3\) we have \(\tau_1 > \tau_{2r}\), and that for a word \(\tau'\) formed by using \(s_1 = -3\) we have \(\tau_1 < \tau_{2r}\).

Showing this process reverses and that we can send the pairs \(\{\tau, \tau'\}\) of an anti-palindromic composition of \(n\) and its reverse back to a composition of \(n - 3\) with parts equal to 1, 2, or 3 is straightforward, which the reader can verify.

2.3. Proof of Theorem 2. In this section we develop the formula for \(ac(n)\). We start by proving some initial observations regarding \(ac_0(n), ac_1(n), ac(n),\) and \(ac(n, s)\).

**Proposition 1.** For all \(n \geq 3\), we have
\[
ac_0(n) = f_3(n - 1) + f_3(n - 5).
\]

**Proof.** This is just two applications of the defining recurrence for \(f_3(n)\), recalling that \(f_3(n) = 0\) for \(n < 1\).
\[
f_3(n - 1) + f_3(n - 5) = f_3(n - 2) + f_3(n - 3) + f_3(n - 4) + f_3(n - 5)
= f_3(n - 2) + f_3(n - 2)
= 2 \cdot f_3(n - 2)
= ac_0(n) \quad \square
\]

**Proposition 2.** We have \(ac(0, 0) = 1, ac(0, 1) = 0,\) and for all \(n \geq 0\) and \(s \geq 0\)
\[
ac(n, 2s) + ac(n, 2s + 1) = ac(n + 1, 2s + 1).
\]
Proof. When \( n = 0 \), there is only one composition (the empty composition) which has length 0.

Now any anti-palindromic composition \( \sigma \) of \( n + 1 \) of length \( 2s + 1 \) has a central part \( \sigma_{s+1} \). If \( \sigma_{s+1} = 1 \), this composition can be formed from an anti-palindromic composition of \( n \) of length \( 2s \) by adding a central part equal of 1. If \( \sigma_{s+1} > 1 \), this composition can be formed from an anti-palindromic composition of \( n \) of length \( 2s + 1 \) by adding 1 to the central part. Therefore, \( ac(n, 2s) + ac(n, 2s + 1) = ac(n + 1, 2s + 1) \).

Proposition 3. For \( n \geq 0 \) and \( s \geq 0 \)
\[
ac(n, 2s + 1) = \sum_{j=0}^{n-1} ac(j, 2s),
\]
where in the case \( n = 0 \) we take the empty sum to be 0.

Proof. Let \( n > 0 \). Then by applying Proposition 2 \( n \) times, we have
\[
ac(n, 2s + 1) = ac(n - 1, 2s + 1) + ac(n - 1, 2s) = ac(n - 2, 2s + 1) + ac(n - 2, 2s) + ac(n - 1, 2s)
\]
\[\vdots\]
\[
= ac(0, 2s + 1) + \sum_{j=0}^{n-1} ac(j, 2s).
\]
Since \( ac(0, 2s + 1) = 0 \) for all \( s \geq 0 \), the result follows.

Proposition 4. For all \( n \geq 0 \)
\[
ac(n) = ac_{1}(n + 1).
\]

Proof. If \( n = 0 \), we see that \( ac(0) = ac_{1}(1) = 1 \). If \( n > 0 \), by definition we have
\[
ac(n) = \sum_{s \geq 0} ac(n, s) = \sum_{j \geq 0} (ac(n, 2j) + ac(n, 2j + 1)) = \sum_{j \geq 0} ac(n + 1, 2j + 1)
\]
by Proposition 2. But this last expression is equal to \( ac_{1}(n + 1) \).

Proposition 5. For \( n \geq 0 \)
\[
ac_{1}(n) = \sum_{j=0}^{n-1} ac_{0}(j),
\]
where in the case \( n = 0 \) we take the empty sum to be 0.

Proof. For \( n > 0 \), we have by Proposition 4
\[
ac_{1}(n) = ac(n - 1) = ac_{0}(n - 1) + ac_{1}(n - 1).
\]
Now if \( n = 1 \), we are done since \( ac_1(0) = 0 \). If \( n > 1 \), we can again apply Proposition 4 to get
\[
ac_1(n) = ac_0(n - 1) + ac(n - 2).
\]
Repeating the same argument \( n - 2 \) more times gives the result. \( \square \)

**Proposition 6.** For all \( n \geq 0 \), we have
\[
\sum_{j=0}^n f_3(j) = \frac{f_3(n) + f_3(n + 2) - 1}{2}.
\]

**Proof.** We give a proof by mathematical induction. For \( n = 0 \),
\[
f_3(0) = 0 = \frac{f_3(0) + f_3(2) - 1}{2}.
\]
Now for \( n > 0 \), suppose the proposition holds for all \( k < n \). Then
\[
\sum_{j=0}^n f_3(j) = \sum_{j=0}^{n-1} f_3(j) + f_3(n)
= \frac{f_3(n-1) + f_3(n+1) - 1}{2} + f_3(n)
= \frac{f_3(n+2) - f_3(n-1) - 1}{2} + f_3(n)
= \frac{f_3(n) + f_3(n + 2) - 1}{2}.
\]
\( \square \)

**Proposition 7.** For all \( n \geq 2 \),
\[
ac_1(n) = f_3(n - 3) + f_3(n - 1).
\]

**Proof.** By Proposition 4, Theorem 1, and Proposition 6 we have
\[
ac_1(n) = \sum_{j=0}^{n-1} ac_0(j)
= 2 \sum_{j=0}^{n-1} f_3(j - 2) + ac_0(0)
= 2 \sum_{j=0}^{n-3} f_3(j) + ac_0(0)
= f_3(n - 3) + f_3(n - 1) - 1 + ac_0(0).
\]
Since \( ac_0(0) = 1 \), the result follows. \( \square \)

**Proof of Theorem 2.** Theorem 2 now immediately follows from Proposition 7 since \( ac(1) = 1 = f_3(1) + f_3(-1) \), and for \( n \geq 2 \)
\[
ac(n) = ac_0(n) + ac_1(n)
= 2 \cdot f_3(n - 2) + f_3(n - 3) + f_3(n - 1)
= f_3(n) + f_3(n - 2).
\]
2.4. Proof of Theorem 3 and Corollary 1. In this section we develop the formulae for $ac(n, s)$ by deriving the ordinary generating function $G(q, s)$ for a fixed $s > 0$. It is easier to split into the cases when $s$ is even and odd.

Suppose $s = 2a$, where $a \geq 0$. An anti-palindromic composition of $n$ of length $2a$ consists of a sequence of $a$ ordered pairs of distinct positive integers. If $d(n)$ is the number of distinct pairs of positive integers that sum to $n$, then

$$D(q) := \sum_{n \geq 0} d(n)q^n = \left(\frac{q}{1-q}\right)^2 - \frac{q^2}{1-q^2} = \frac{2q^3}{(1-q^2)(1-q)}.$$  

To see why this is the case, notice that

$$\left(\frac{q}{1-q}\right)^2 = (q^{1+1}) + (q^{1+2} + q^{2+1}) + (q^{1+3} + q^{2+2} + q^{3+1}) + \ldots$$

and

$$\frac{q^2}{1-q^2} = q^{1+1} + q^{2+2} + q^{3+3} + \ldots,$$

so we are taking all pairs of positive integers and subtracting the repeated pairs.

To form a sequence of $a$ such pairs, we multiple $D(q)$ by itself $a$ times, showing that

$$G(q, 2a) = [D(q)]^a = \frac{2^aq^{3a}}{(1-q^2)^a(1-q)^a}.$$  

To prove the first half of Corollary 1, recall that for $a > 0$

$$\frac{1}{(1-q^2)^a} = \sum_{n \geq 0} \binom{a-1+n}{n}q^{2n}$$

and

$$\frac{1}{(1-q)^a} = \sum_{n \geq 0} \binom{a-1+n}{n}q^n.$$  

Multiplying these two series and reindexing gives the result.

Now suppose $s = 2a + 1$, where $a \geq 0$. An anti-palindromic composition of $n$ of length $2a + 1$ still consists of $a$ ordered pairs of distinct positive integers, with an additional central part. Therefore,

$$G(q, 2a + 1) = G(q, 2a) \cdot q = \frac{2^aq^{3a}}{(1-q^2)^a(1-q)^a}.$$  

The second half of Corollary 1 follows the same way as the first half once we observe

$$\frac{1}{(1-q)^{a+1}} = \sum_{n \geq 0} \binom{a+n}{n}q^n.$$  

2.5. Proof of Theorem 4 and Corollary 2. We begin with a lemma.

**Lemma 2.** For $n \geq 3$ and $s \geq 2$ we have

$$ac(n, s) = ac(n-1, s) + ac(n-2, s) + 2 \cdot ac(n-3, s-2) - ac(n-3, s).$$

**Proof.** If $\sigma$ is an anti-palindromic composition of $n \geq 3$ of length $s \geq 2$, let $m_\sigma := \sigma_1 + \sigma_s$. Observe that $m_\sigma \geq 3$ and

$$\delta(m_\sigma) \leq |\sigma_1 - \sigma_s| \leq m_\sigma - 2,$$

where $\delta(m_\sigma) = 1$ if $m_\sigma$ is odd and $\delta(m_\sigma) = 2$ if $m_\sigma$ is even.
Let us first count the number of anti-palindromic compositions of \( n \) of length \( s \) with \( m_\sigma = 3 \). Each one of these compositions can be formed by taking an anti-palindromic composition of \( n - 3 \) of length \( s - 2 \) and adjoining a 1 at the beginning and a 2 at the end, or a 2 at the beginning and a 1 at the end. Therefore, the number of anti-palindromic compositions of \( n \) of length \( s \) with \( m_\sigma = 3 \) equals 
\[
2 \cdot ac(n-3,s-2).
\]

Next we count the number of anti-palindromic compositions of \( n \) of length \( s \) with \( m_\sigma > 3 \). Now for any anti-palindromic composition \( \tau \) of \( n - 1 \) of length \( s \), we can form an anti-palindromic composition of \( n \) of length \( s \) by adding 1 to \( \tau_1 \) if \( \tau_1 > \tau_s \), or adding 1 to \( \tau_s \) if \( \tau_s > \tau_1 \). Now in this way, we have constructed all the anti-palindromic compositions of \( n \) of length \( s \) with \( m_\sigma > 3 \) and \( |\sigma_1 - \sigma_s| > \delta(m_\sigma) \).

For any composition \( \gamma \) of \( n - 2 \) of length \( s \), form an anti-palindromic composition of \( n \) of length \( s \) by adding 1 to \( \gamma_1 \) and 1 to \( \gamma_s \). In this way, we have constructed all the anti-palindromic compositions of \( n \) of length \( s \) with \( m_\sigma > 3 \) and \( |\sigma_1 - \sigma_s| \leq m_\sigma - 4 \).

Therefore, the total number of anti-palindromic compositions of \( n \) of length \( s \) with \( m_\sigma > 3 \) and \( \delta(m_\sigma) \leq |\sigma_1 - \sigma_s| \leq m_\sigma - 2 \) equals \( apc(n-1,s) + apc(n-2,s) \) minus the anti-palindromic compositions of \( n \) of length \( s \) with \( m_\sigma > 3 \) and \( \delta(m_\sigma) < |\sigma_1 - \sigma_s| \leq m_\sigma - 4 \), as we have counted these compositions exactly twice. To prove the lemma, we now must show that the number of compositions that we counted twice equals \( apc(n-3,s) \).

Let \( \rho \) be an anti-palindromic composition of \( n - 3 \) of length \( s \). Form an anti-palindromic composition of \( n \) of length \( s \) by adding 2 to \( \rho_1 \) and 1 to \( \rho_s \) if \( \rho_1 > \rho_s \), or 1 to \( \rho_1 \) and 2 to \( \rho_s \) if \( \rho_s > \rho_1 \). In this way, we have constructed all the anti-palindromic compositions of \( n \) of length \( s \) with \( \delta(m_\sigma) < |\sigma_1 - \sigma_s| \leq m_\sigma - 4 \). \( \square \)

**Proof of Theorem 4.** The theorem can be verified for all \( n \) and \( s \) with \( n + s < 5 \):

\[
\begin{align*}
\phi_0(q) &= rac(0,0) \cdot q^0 + rac(0,1) \cdot q^1 + rac(0,2) \cdot q^2 = 1 \cdot q^0 + 0 \cdot q^1 + 0 \cdot q^2 \\
\phi_1(q) &= rac(1,0) \cdot q^0 + rac(1,1) \cdot q^1 + rac(1,2) \cdot q^2 = 0 \cdot q^0 + 1 \cdot q^1 + 0 \cdot q^2 \\
\phi_2(q) &= rac(2,0) \cdot q^0 + rac(2,1) \cdot q^1 + rac(2,2) \cdot q^2 = 0 \cdot q^0 + 0 \cdot q^1 + 0 \cdot q^2 \\
\phi_3(q) &= rac(3,0) \cdot q^0 + rac(3,1) \cdot q^1 + rac(3,2) \cdot q^2 = 0 \cdot q^0 + 0 \cdot q^1 + 1 \cdot q^2.
\end{align*}
\]

Let \([q^s]\phi_n(s)\) be the coefficient of \( q^s \) in the polynomial \( \phi_n(s) \). Now for \( n \geq 3 \) and \( s \geq 2 \), using the defining recurrence for \( \phi_n(q) \) we have

\[
[q^s]\phi_n(q) = [q^s]\phi_{n-1}(q) + [q^{s-2}]\phi_{n-3}(q) - [q^s]\phi_{n-3}(q)
\]

\[
= rac(n-1,s) + rac(n-2,s) + rac(n-3,s-2) - rac(n-3,s)
\]

by induction. Using the relationship between \( rac(n,s) \) and \( ac(n,s) \),

\[
[q^s]\phi_n(q) = \frac{ac(n-1,s)}{2|\mathbb{F}|} + \frac{ac(n-2,s)}{2|\mathbb{F}|} + \frac{ac(n-3,s-2)}{2|\mathbb{F}|} - \frac{ac(n-3,s)}{2|\mathbb{F}|}
\]

\[
= \frac{ac(n,s)}{2|\mathbb{F}|}
\]

by Lemma 2. Therefore, \([q^s]\phi_n(s) = rac(n,s)\). \( \square \)
Proof of Corollary 2. Notice that by Theorem 4 we have
\[ rac(n) = \sum_{s \geq 0} rac(n, s) = \phi_n(1) = f_2(n). \]
As for \( rac_0(n) \), we have \( rac_0(1) = rac_0(2) = 0 \), \( rac_0(3) = 1 \), and for \( n \geq 4 \)
\[ rac_0(n) = \sum_{s \geq 0} rac(n, 2s) = \frac{\phi_n(1) + \phi_n(-1)}{2}, \]
again using Theorem 4. By the definition of \( \phi_n(q) \), this equals
\[ \frac{\phi_{n-1}(1) + \phi_{n-1}(-1)}{2} + \frac{\phi_{n-2}(1) + \phi_{n-2}(-1)}{2} = rac_0(n-1) + rac_0(n-2). \]
This is the defining recurrence relation for the Fibonacci numbers, thus we conclude that \( rac_0(n) = f_2(n-2) \).
Similarly for \( rac_1(n) \), we have \( rac_1(0) = 1 \), \( rac_1(2) = rac_1(3) = 1 \), and for \( n \geq 4 \)
\[ rac_1(n) = \sum_{s \geq 0} rac(n, 2s + 1) = \frac{\phi_n(1) - \phi_n(-1)}{2} \]
by Theorem 4. By the definition of \( \phi_n(q) \), this equals
\[ \frac{\phi_{n-1}(1) - \phi_{n-1}(-1)}{2} + \frac{\phi_{n-2}(1) - \phi_{n-2}(-1)}{2} = rac_1(n-1) + rac_1(n-2). \]
This is the defining recurrence relation for the Fibonacci numbers, thus we conclude that \( rac_1(n) = f_2(n-1) \). □

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