The goal of the present paper is primarily to exhibit the effectiveness of using Lie algebras to compute explicit perturbation eigenvalues for quantum anharmonic oscillators in one dimension. There are however, several goals secondary in stature, but which merit discussion. The first of these is to enable the reader to work with Weyl algebras in the abstract. Two presentations of it arise readily in quantum physics. In particular the first is the algebra of position and momentum operators in nonrelativistic mechanics wherein \([x, p] = i\hbar\). The second is the algebra of ladder operators \([a, a^\dagger] = 1\). It is the second presentation with which we will be primarily concerned in this paper. Of course, the first presentation may be made to look like the second by considering not \(p\), but instead the simple derivative \(\frac{d}{dx}\) whereby one has \([\frac{d}{dx}, x] = 1\).

1.1. **Normal Ordering and Weyl Binomial Coefficients.** For any abstract Weyl algebra determined by two elements \(A\) and \(B\) obeying \([A, B] = 1\), an ordering of a polynomial in \(A\) and \(B\) will be said to be *normally ordered* if all powers of \(B\) appear to the left of powers of \(A\). For example \(A^3B^2\) is not normally ordered, but \(B^2A^3\) is. In our case \(a^\dagger\) will always be placed to the left of \(a\). As it is well known in
elementary quantum mechanics one may move back and forth between presentations of problems in position-momentum coordinates and annihilation-creation coordinates with the following equivalences

\[
x = \frac{a + a^\dagger}{\sqrt{2}} \\
p = \frac{a - a^\dagger}{i\sqrt{2}}.
\]

Since this paper is concerned with anharmonic oscillators we will be concerned with \(x^n\) in the potential. Thus, we need an efficient way of normally ordering \((a + a^\dagger)^n\).

**Lemma 1.** Let \(A\) and \(B\) determine a Weyl algebra so that \([A, B] = 1\). The normal ordering of \((A + B)^n\) is given by

\[
(A + B)^n = \sum_{m=0}^{n} \sum_{k=0}^{\min\{n, n-m\}} \binom{n}{m}_k B^{m-k} A^{n-m-k},
\]

where

\[
\binom{n}{m}_k = \frac{n!}{2^k k!(m-k)!(n-m-k)!}
\]

is the Weyl binomial coefficient.

The proof of this lemma involves nothing more than counting commutations.

**Example 2.** We will use the fourth order relation explicitly later, so here is an example of how the Weyl coefficients factor in.

\[(a + a^\dagger)^4 = a^4 + 4a^3a + 6a^2a^2 + 4a^1a^3 + a^4 + 6a^1a + 6a^2 + 3.
\]

**Remark 3.** Notice that

\[
\binom{n}{m}_k = \binom{n}{n-m}_k.
\]

If one wishes to attempt calculations within a Weyl algebra it may be useful to compute with abstract elements \(A, B\) first and then plug into a specific situation one has in mind. One other useful tip is that if one has Weyl variables \(A, B\) then it can be convenient to consider representing the algebra as \(\frac{d}{dB}, B\) or \(A, -\frac{d}{dA}\). This becomes consistent with the first presentation considered. For example, one peculiar formula which is arguably easier to compute with abstract Weyl variables is

\[
\mu^{x\partial_x} f(x) = f(\mu x).
\]

1.2. Baker-Campbell-Hausdorff and the Hadamard Lemma. We will be concerned throughout much of this paper with exponentiating noncommuting variables. We run into a stopping block in trying to compute the exponentials explicitly. The main issue is that for noncommuting variables \(X, Y\) we see

\[e^Y e^X \neq e^{Y+X} = e^X e^Y \neq e^Y e^X.
\]

The Baker-Campbell-Hausdorff formula is the solution to \(Z = \log(e^X e^Y)\). The explicit solution is formally given as symmetric sums and differences of nested commutators in \(X\) and \(Y\). One may find this expression in nearly any textbook on
advanced quantum mechanics. We shall not be concerned, however, with isolated exponentials, but rather expressions of the form

\[ e^X e^{-X}. \]

Using elementary combinatorics and the Baker-Campbell-Hausdorff formula one can arrive at the Hadamard lemma.

**Lemma 4.** Let \( X, Y \) be noncommuting variables then one has

\begin{equation}
(1.2.1) \quad e^X e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]] + \cdots
\end{equation}

If we allow the notation \( [X^{(n)}, Y] = [X, \ldots, [X, Y]] \) then we may write more succinctly

\begin{equation}
(1.2.2) \quad e^X e^{-X} = \sum_{k=0}^{\infty} \frac{1}{k!} [X^{(k)}, Y].
\end{equation}

1.3. The Formula Often Desired and Rarely Known. One final assertion about Weyl variables in the algebraic preliminaries must be the formula

\begin{equation}
(1.3.1) \quad [A^n, B^m] = \sum_{k=1}^{\min\{n, m\}} k! \binom{m}{k} \binom{n}{k} B^{m-k} A^{n-k}.
\end{equation}

This formula is often left as an exercise in quantum mechanics texts and sometimes in homological algebra, but rarely is it completed. One might jokingly say it is similar to the snake lemma in that no one knows if it’s really true since the only persons who have ever proven it are graduate students. All kidding aside, this is indeed the correct formula for normally ordering variables obeying the Weyl relation.

2. The Method of [JA]

The main impetus for this research comes from the paper [JA]. The goal of this section is to explicate in a reasonably clear manner the content of that paper.

The premise upon which [JA] begins is the idea that we can create a new Lie algebra from simply taking commutators of the unperturbed Hamiltonian \( H_0 \) and the new anharmonic Hamiltonian \( H_n \). For the sake of mathematical simplicity the Hamiltonians in question are given as essentially unitless operators:

\begin{align*}
(2.0.2) \quad H_0 &= \frac{1}{2} (p^2 + x^2) = a^\dagger a + \frac{1}{2}, \\
H_n &= H_0 + \lambda x^n = a^\dagger a + \frac{1}{2} + \frac{\lambda}{\sqrt{2}} (a + a^\dagger)^n.
\end{align*}

As one may infer, we have made the following assumptions and simplifications:

1. \( \hbar = \omega = m = 1 \),
2. \( a = \frac{x + ip}{\sqrt{2}}, a^\dagger = \frac{x - ip}{\sqrt{2}} \),
3. \( x = \frac{a^\dagger a}{\sqrt{2}}, p = \frac{1}{\sqrt{2}} (a - a^\dagger) \).

Let us now give the formulation of the Lie algebras.
Definition 5. The Lie Algebra $\mathcal{A}_n^{(k)} = \{ L_m \}_{m \in I}$ is generated by the elements

$$L_1 = H_0, \quad L_2 = H_n$$

and other $L_m$ satisfying

$$[L_i, L_j] = \sum c_{ijm} L_m.$$  

(2.0.3)

for some structure constants $c_{ijm} \in \mathbb{C}$. Furthermore, this Lie algebra should be closed under commutators up to order $\lambda^k$. In other words no $L_m$ should be of the form $\lambda^{k+1}(a^\dagger s a^t - a^\dagger t a^s)$ for any $s, t$. That is, formally we require $\lambda^{k+1} = 0$ within the Lie algebra.

In the case of this paper we will consider $\mathcal{A}_n^{(1)}$ unless otherwise explicitly stated. In fact, [JA] only considers Lie algebras up to order one in $\lambda$ with the exceptions of $n = 1$ and $n = 2$ because these determine harmonic oscillators and their solutions are already known. We deal with the special technique for solving harmonic oscillators in the appendix.

Once the algebra $\mathcal{A}_n^{(1)}$ is determined we proceed in the following way. Suppose

$$[L_1, L_2] = \sum_{k=3}^j c_{12k} L_k$$

(2.0.4)

where each $L_k$ is of the form

$$\lambda(a^\dagger m a^t - a^\dagger t a^m).$$

(2.0.5)

The symmetry of these $L_k$ is important and comes back in an important way due to the normal ordering procedures we have adopted. We will see this explicitly in the computations.

We then construct a unitary element of the associated Lie group by

$$U = \exp(\sum_{k=3}^j \alpha_k L_k).$$

(2.0.6)

This says that the only $L_k$ allowed in our unitary are those arising directly from the commutator $[L_1, L_2]$. The $\alpha_k$ are real constants which we will tune as necessary.

Once we produce such a unitary we make a transformation from $H_0$ to $H_n$ by

$$U^\dagger H_0 U = H_n - \Lambda_n.$$  

(2.0.7)

In each case $\Lambda_n$ is an operator which simply controls the perturbations of eigenvalues. Furthermore, by the clever choice of $U$ we will have $[U, \Lambda_n] = 0 + O(\lambda^2)$. Due to the Hadamard lemma we can produce $\Lambda_n$ by computing simple commutators.

At this stage one may write the new eigenvectors as $U^\dagger |j\rangle$ where $|j\rangle$ are the eigenvectors for the harmonic Hamiltonian with eigenvalues $j + \frac{1}{2}$. Therefore, up to order $\lambda^2$ our equation now reads

$$H_n U^\dagger |j\rangle = (U^\dagger H_0 U + \Lambda_n) U^\dagger |j\rangle$$

(2.0.8)

$$= U^\dagger H_0 |j\rangle + \Lambda_n U^\dagger |j\rangle$$

$$= U^\dagger (j + \frac{1}{2}) |j\rangle + U^\dagger \Lambda_n |j\rangle$$

$$= (j + \frac{1}{2} + \lambda_n) U^\dagger |j\rangle.$$
In essence, depending on the form of $\Lambda_n$, we will be able to read off the first order perturbation eigenvalues ($\lambda_n$) of $H_n$ with relative ease.

A natural question arises as to when we can solve this system explicitly. [JA] makes a passing statement which we will now state as a formal theorem.

**Theorem 6.** If the Lie Algebra $A_n$ is closed (in all orders of $\lambda$) then we can solve the $n^{th}$ order anharmonic oscillator in closed form.

**Proof.** Let $A_n = \{L_k\}_{k=1}^N$ be a closed Lie algebra corresponding to the Hamiltonian $H_n$. Then consider the general Lie group element given by

$$U = \exp(\sum_{k=1}^N \alpha_k L_k) =: \exp(L).$$

From the Hadamard lemma we obtain

$$(2.0.9) \quad U^\dagger H_0 U = \sum_{k=1}^\infty \frac{1}{k!} [L^{(k)}, H_0].$$

Since $A_n$ is closed, the commutators $[L^{(k)}, H_0]$ either vanish or give Lie algebra elements with some periodicity. In this way we can formally sum them in a power series. Setting our parameters to appropriate values we obtain

$$U^\dagger H_0 U = H_n + \text{perturbations}. \quad \Box$$

### 3. Lie Algebras up to Order One in $\lambda$

We begin the computation of the Lie algebras by giving an important commutator relation.

$$(3.0.10) \quad [a^\dagger a, a^\dagger a^k \pm a^\dagger a^k] = (k - \ell)(a^\dagger a^\ell \mp a^\dagger a^\ell)$$

This equation paired with the symmetry of Weyl binomial coefficients points us to assigning $a^\dagger a^m - a^\dagger a^m$ as our Lie algebra elements. Let us begin by computing $[H_0, H_n]$.

$$[H_0, H_n] = [H_0, H_0 + \frac{\lambda}{\sqrt{2}}(a^\dagger a + a^\dagger a)] =$$

$$[H_0, \lambda \sqrt{2}(a^\dagger a)^n] = \frac{\lambda}{\sqrt{2}}[a^\dagger a, (a^\dagger + a)^n] =$$

$$= \frac{\lambda}{\sqrt{2}} [a^\dagger a, \sum_{k,m} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_k (a^{tm-k} a^{n-m-k} + a^{tm-m-k} a^{n-k})]$$

$$(3.0.11) \quad = \frac{\lambda}{\sqrt{2}} \sum_{k,m} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_k (2m-n)(a^{tm-k} a^{n-m-k} - a^{tm-m-k} a^{n-k})$$

We will throw away the multiplicative constants in favor of rescaling them by $\alpha_k$ in our general Lie group element. Therefore, the first batch of elements revealed to us are those of the form $\lambda(a^{tm} a^m - a^{tm} a^m)$. Once we realize these, we begin commuting again with $H_0$ to find more elements of the form $\lambda(a^{tm} a^m + a^{tm} a^m)$. In order to close $A_n^{(1)}$ we also need to add a central element $I$ to our Lie algebra. All other commutators will involve terms with $\lambda^2$ and therefore we disregard them in $A_n^{(1)}$.  

Remark 7. Notice that no elements of the form \(a^{\dagger n}a^n\) appear anywhere. This is because they can be written in terms of the number operator \(N = a^{\dagger}a\) which commutes with \(H_0\).

Given a general Hamiltonian \(H_n\), with special exceptions, by simple combinatorial formulae one infers the number of generators for \(A_6^{(1)}\) by

\[
|A_{2k}^{(1)}| = (k+1)k + 3
\]

\[
|A_{2k+1}^{(1)}| = (k+1)k + 3
\]

Example 8. Let’s take a quick look at \(A_6^{(1)}\). Of course, we let \(L_1 = H_0\) and \(L_2 = H_6\). By our computation we know the generators arising from \([L_1, L_2]\) are as follows:

\[
\lambda(a^{16} - a^6), \hspace{1cm} \lambda(a^{15}a - a^{\dagger}a^5), \hspace{1cm} \lambda(a^{14}a^2 - a^{\dagger}a^4), \\
\lambda(a^{14} - a^4), \hspace{1cm} \lambda(a^{13}a - a^{\dagger}a^3), \hspace{1cm} \lambda(a^{12} - a^2).
\]

Furthermore, commuting these with \(L_1\) we obtain similarly symmetric elements with plus signs. Finally we add in \(\mathcal{I}\) to account for commuting elements. Notice if we commute any other elements we obtain an element in \(A_6^{(2)}\) which we have formally disallowed for now. Therefore \(|A_6^{(1)}| = 15 = (3+1)3 + 3\) as previously stated.

4. Explicit Computations

In this section we will derive the first order perturbation for all anharmonic oscillators with Hamiltonians of the form

\[
H_n = \frac{1}{2}(p^2 + x^2) + \lambda x^n.
\]

There are two distinct cases for computing first order perturbations; odd and even. We will treat the odd case first.

For the sake of uniformity in our calculations we will consider Hamiltonians of the form \(H_{2k-1}\) and \(H_{2k}\).

4.1. Odd Powered Potentials. From our earlier computations of \([H_0, H_{2k-1}]\) and our prescribed form of \(U\) we have

\[
U = \exp(\lambda \sum_{\ell=0}^{m-1} \sum_{m=1}^{k} \alpha_{m,\ell} a^{2m-1-\ell} a^{\dagger}\ell a^{2m-1-\ell}).
\]

By requiring \(\alpha_{m,\ell} \in \mathbb{R}\) we obtain \(U^\dagger = U^{-1}\) and we may now apply the Hadamard lemma. Letting

\[
X := \lambda \sum_{\ell=0}^{m-1} \sum_{m=1}^{k} \alpha_{m,\ell} (a^{2m-1-\ell} a^{\dagger}\ell a^{2m-1-\ell})
\]

we have:

\[
U^\dagger H_0 U = H_0 + [-X, H_0] + \frac{1}{2!}[-X, [-X, H_0]] + \cdots
\]
We notice immediately that $X$ contains a multiplicative factor of $\lambda$ and since we have $\lambda^2 = 0$ we may ignore all terms past $[-X, H_0]$.

Using previous computations and elementary properties of derivations we have

$$[H_0, X] = \lambda \sum_{\ell=0}^{m-1} \sum_{m=1}^{k} \alpha_{m,\ell}(2m - 1 - 2\ell)(a^{2m-1-\ell}a^{\ell} + a^{\ell}a^{2m-1-\ell}).$$

If we recognize that

$$x^{2k-1} = \frac{(a + a^\dagger)^{2k-1}}{\sqrt{2^{2k-1}}} = \frac{1}{\sqrt{2^{2k-1}}} \sum_{m=0}^{2k-1} \sum_{j=0}^{\min\{2k-1, 2k-1-m\}} \left\{ \begin{array}{c} 2k-1 \\ m \end{array} \right\} a^{m-j}a^{2k-1-m-j},$$

we see that in order to produce $H_{2k-1} = H_0 + \lambda x^{2k-1}$ we need to set

$$\alpha_{m,\ell}(2m - 2\ell - 1) = \frac{1}{\sqrt{2^{2k-1}}} \left\{ \begin{array}{c} 2k-1 \\ k - m + \ell \end{array} \right\}_{k-m}.$$

Notice what we have done. We have transformed $H_0$ into $H_{2k-1} + O(\lambda^2)$. Hence, there is no perturbation term up to first order.

Example 9. Let us compute the example of $H_1 = H_0 + \lambda x$ explicitly. We already know that this is a shifted harmonic oscillator, where a simple change of variables reveals the energy eigenvalues are $n + \frac{1}{2} - \frac{\lambda^2}{2}$.

In our case the Lie algebra simplifies slightly and is fully closed as

$$H_0, H_1, \lambda(a^\dagger - a), I.$$

Our appropriate unitary transformation $U$ is therefore given as

$$U = \exp(\alpha \lambda (a^\dagger - a)).$$

We can compute this even more explicitly than before given that $[a^\dagger - a, a^\dagger] = [a^\dagger - a, a] = -1$.

In this case we know

$$[A, B] = \beta \in \mathbb{C} \implies [A, e^B] = \beta e^B$$

for any abstract operators $A, B$.

Therefore $[a, U] = \alpha U$ and $[a^\dagger, U] = \alpha U$

$$U^\dagger H_0 U = \frac{1}{2} U^\dagger a^\dagger a U + \frac{1}{2}$$

$$= U^\dagger a^\dagger U(\alpha \lambda + a) + \frac{1}{2}$$

$$= U^\dagger U(a^\dagger + \alpha \lambda)(a + \alpha \lambda) + \frac{1}{2}$$

$$= a^\dagger a + \frac{1}{2} + \alpha \lambda(a^\dagger + a) + \alpha^2 \lambda^2.$$

Setting $\alpha = \frac{1}{\sqrt{2}}$ we derive

$$U^\dagger H_0 U = H_1 + \frac{\lambda^2}{2}.$$
This is exactly the result we previously knew. Notice however, there is no $\lambda$ term. The first perturbation term is order $\lambda^2$.

**Example 10.** To better see the odd powered result more explicitly, let us compute the result for $H_5$.

Our Lie algebra up to order one is given by

$$
\begin{align*}
\lambda(a^{13} - a^3), & \quad \lambda(a^{14}a - a^4), & \quad \lambda(a^{13}a^2 - a^{12}a^3), \\
\lambda(a^{13} - a^3), & \quad \lambda(a^{12}a - a^4), & \quad \lambda(a^1 - a), \\
\lambda(a^{15} + a^3), & \quad \lambda(a^{14}a + a^4), & \quad \lambda(a^{13}a^2 + a^{12}a^3), \\
\lambda(a^{13} + a^3), & \quad \lambda(a^{12}a + a^4), & \quad \lambda(a^1 + a), \\
H_0, & \quad H_5, & \quad I. 
\end{align*}
$$

Therefore $U$ is

$$
U = \exp(\lambda(\alpha_{3,0}(a^{15} - a^5) + \alpha_{3,1}(a^{14}a - a^4) + \cdots + \alpha_{1,0}(a^1 - a))).
$$

Our Hamiltonian transforms as

$$
U^\dagger H_0 U = H_0 + \lambda[H_0, \alpha_{3,0}(a^{15} - a^5) + \alpha_{3,1}(a^{14}a - a^4) + \cdots + \alpha_{1,0}(a^1 - a)] + O(\lambda^2)
$$

$$
= H_0 + \lambda(5\alpha_{3,0}(a^{15} + a^5) + 3\alpha_{3,1}(a^{14}a + a^4) + \cdots + \alpha_{1,0}(a^1 + a)).
$$

Setting

$$
\begin{align*}
\alpha_{3,0} & = 2^{-5/2}/5, \\
\alpha_{3,1} & = 2^{-5/25}/3, \\
\alpha_{3,2} & = 2^{-5/210}, \\
\alpha_{2,0} & = 2^{-5/210}/3, \\
\alpha_{2,1} & = 2^{-5/230}, \\
\alpha_{1,0} & = 2^{-5/215},
\end{align*}
$$

we obtain

$$
(4.1.7) \quad U^\dagger H_0 U = H_0 + \lambda x^5 + O(\lambda^2) = H_5 + O(\lambda^2).
$$

4.2. **Even Powered Potentials.** It is the goal of this section to show the first order perturbation energies for oscillators corresponding to $H_{2k}$ are

$$
(4.2.1) \quad n + \frac{1}{2} + \frac{\lambda}{2^k} \sum_{j=0}^{k} j! \left\{ \begin{array}{c} 2k \\ k \end{array} \right\} \sum_{j-k}^{n} \left( \begin{array}{c} n \\ j \end{array} \right).
$$

This result is obtained rather easily utilizing the technology we have developed for odd powered potentials. We only need to realize that the operators which are not Lie algebra elements are of the form $a^m a^n$. This comes from the required symmetry of our generators. Hence our unitary $U$ will appear exactly as in the odd powered case and the Hadamard lemma yields:

$$
(4.2.2) \quad U^\dagger H_0 U = H_0 + \lambda x^{2k} - \frac{\lambda}{2^k} \sum_{j=0}^{k} \left\{ \begin{array}{c} 2k \\ k \end{array} \right\} a^{j} a^{j} + O(\lambda^2).
$$

The only thing left to pretty up our example is changing $a^{ik} a^k$ into an expression of number operators.
Recall $N = a^\dagger a$ has nonnegative integer eigenvalues given by $N|n\rangle = n|n\rangle$. In this way any expression $f(N)$ in our perturbation expansion will give eigenvalues $f(n)$ by the functional calculus.

**Proposition 11.**

\[(4.2.3)\]

$a^\dagger k a^k = k! \left( \begin{array}{c} N \\ k \end{array} \right).$

**Proof.** We refer back to our commutation relation $[A^n, B^m]$ from §1.3. Thus we have

\begin{align*}
a^\dagger k a^k &= a^\dagger (a^\dagger k-1) a^{k-1} \\
&= a^\dagger (aa^\dagger k-1 - (k-1)a^\dagger k-2) a^{k-1} \\
&= (a^\dagger a)^{k-1} a^{k-1} - (k-1)a^\dagger a^{k-1-a^{k-1}} \\
&= (N - (k-1))a^\dagger a^{k-1-a^{k-1}}.
\end{align*}

Repeating this we see

\[(4.2.4)\]

$a^\dagger k a^k = N(N-1) \cdots (N-(k-1)) = k! \left( \begin{array}{c} N \\ k \end{array} \right).$

\[\square\]

It is merely a matter of rearranging terms to see

\[(4.2.5)\]

$U^\dagger H_0 U = H_{2k} - \frac{\lambda}{2k} \sum_{j=0}^{k} j! \left\{ \frac{2k}{k} \right\}_{k-j} \left( \begin{array}{c} N \\ j \end{array} \right) + O(\lambda^2).$

**Example 12.** Let us look briefly at the Hamiltonian

$H_4 = a^\dagger a + \frac{1}{2} + \frac{\lambda}{4}(a^\dagger + a)^4.$

This is the famous quartic which has received much attention in texts and papers. We can check our results against those of standard perturbation theory.

Our Lie algebra $A_4^{(\dagger)}$ is given by

$\lambda(a^{14} - a^4), \lambda(a^{13}a - a^4a^3), \lambda(a^{12} - a^2), \lambda(a^{14} + a^4), \lambda(a^{13}a + a^4a^3), \lambda(a^{12} + a^2), H_0, H_4, I.$

Our unitary is given by

$U = \exp(\lambda(\alpha_{2,0}(a^{14} - a^4) + \alpha_{2,1}(a^{13}a - a^4a^3) + \alpha_{1,0}(a^\dagger - a))).$

From here we must simply crank the handle for our machine and we realize

$U^\dagger H_0 U = H_0 + [H_0, \lambda(\alpha_{2,0}(a^{14} - a^4) + \alpha_{2,1}(a^{13}a - a^4a^3) + \alpha_{1,0}(a^\dagger - a))]$

\[(4.2.6)\]

$= H_0 + \lambda(4\alpha_{2,0}(a^{14} + a^4) + 2\alpha_{2,1}(a^{13}a + a^4a^3) + \alpha_{1,0}(a^\dagger + a)).$

Setting

$\alpha_{2,0} = 1/16$
$\alpha_{2,1} = 1/2$
$\alpha_{1,0} = 3/4$

we arrive at
(4.2.7) \[ U^\dagger H_0 U = H_4 - \frac{\lambda}{4}(6a^2a^2 + 12a^4a + 3) = H_4 - \frac{3\lambda}{2}(N(N + 1)) - \frac{3\lambda}{4}. \]

If we look to the ground state we see that

\[ E_0 = \frac{1}{2} + \frac{3\lambda}{4} + O(\lambda^2) \]

which agrees with the standard perturbation theory.

In particular the perturbed ground state of the anharmonic oscillator corresponding to \( H_{2k} \) is given by

(4.2.8) \[ E_0 = \frac{1}{2} + \frac{\lambda}{2^k}\left\{ \frac{2k}{k} \right\} + O(\lambda^2) = \frac{1}{2} + \frac{\lambda(2k)!}{2^{2k}k!} + O(\lambda^2). \]

5. Extending the Method

The second subsidiary goal of this paper is to show several extensions to this method and invite research into even more applications of Lie algebras into physics.

5.1. Simple One Dimensional Corollaries. Now that we have given explicit formulae for computing perturbation eigenvalues for potentials of the form \( \lambda x^n \) we can extend by linearity (up to order one) and immediately recover eigenvalues for polynomial potentials. In fact we can extend this further to convergent power series.

**Example 13.** Let us consider

\[ H = a^\dagger a + \frac{1}{2} + \lambda e^x. \]

Of course this can be rewritten

\[ H = a^\dagger a + \frac{1}{2} + \lambda \left( \sum_k \frac{x^k}{k!} \right). \]

If we notice that only the even powered potentials contribute perturbations up to first order then we will also compute perturbations for \( H = H_0 + \lambda \cosh(x) \) as well.

Let us compute only the ground state energy. We have

\[ E_0 = \frac{1}{2} + \lambda \left( \sum_k \frac{(2k)!}{2^{2k}k!(2k)!} \right) = \frac{1}{2} + \lambda \left( \sum_k \frac{4^{-k}}{k!} \right) = \frac{1}{2} + \lambda \exp(1/4). \]

Moreover, we can add any number of perturbation parameters and solve the system accordingly. In particular we can essentially read off first order perturbations for Hamiltonians of the form

\[ H = H_0 + \sum_{j=1}^{n} \lambda_j x^j. \]
5.2. Simple N Dimensional Corollaries. In extending this method, it is natural to ask whether one can tackle higher dimensional systems with a similar approach. In our case, we certainly can attack higher dimensional problems similarly, but the construction of the Lie algebra is different. For the simple \( N \)-dimensional corollaries, we will assume our oscillator potential is not coupled (i.e., no terms of the form \( \lambda x^j y^k \) appear). For the sake of simplicity let us go through the construction of the Lie algebras for a two-dimensional oscillator.

Consider
\[
H_{n,m} = a_x^\dagger a_x + \frac{1}{2} + \lambda_1 x^n + a_y^\dagger a_y + \frac{1}{2} + \lambda_2 y^m.
\]

We will take four elements as given in our Lie algebra
\[
\begin{align*}
H_{0,0} &= a_x^\dagger a_x + \frac{1}{2} + a_y^\dagger a_y + \frac{1}{2}, \\
H_{0,m} &= H_{0,0} + \lambda_2 y^m, \\
H_{n,0} &= H_{0,0} + \lambda_1 x^n, \\
H_{n,m} &= H_{0,0} + \lambda_1 x^n + \lambda_2 y^m.
\end{align*}
\]

In this way we will set up our Lie algebra as two independent oscillator Lie algebras and solve our problems from before. Consider for example
\[
H_{1,4} = H_{0,0} + \lambda_1 x + \lambda_2 y^4.
\]

Our Lie algebra \( \mathcal{A}_{1,4}^{(1,1)} \) will have the following elements
\[
\begin{align*}
H_{0,0}, & \quad H_{1,0}, & \quad \lambda_1 (a_x^\dagger - a_x), \\
H_{0,4}, & \quad H_{1,4}, & \quad I, \\
\lambda_2 (a_y^4 - a_y^4), & \quad \lambda_2 (a_y^4 a_y - a_y^4), & \quad \lambda_2 (a_y^4 a_y + a_y^4), \\
\lambda_2 (a_y^4 - a_y^4), & \quad \lambda_2 (a_y^4 a_y - a_y^4), & \quad \lambda_2 (a_y^4 a_y + a_y^4).
\end{align*}
\]

Our unitary takes the form
\[
U = \exp(\alpha \lambda_1 (a_x^\dagger - a_x) + \beta_1 \lambda_2 (a_y^4 - a_y^4) + \beta_2 \lambda_2 (a_y^4 a_y - a_y^4) + \beta_3 \lambda_2 (a_y^4 a_y + a_y^4)).
\]

Now we use the Hadamard lemma again, but taking advantage of the relations
\[
[a_x^\dagger, a_k] = \delta_{jk}
\]

we can completely separate \( x \) variables from \( y \) variables and our calculation plays out exactly as before.

For the Hamiltonian \( H_{1,4} \) our perturbed ground state is
\[
E_0 = \frac{1}{2} + O(\lambda_1^2) + \frac{1}{2} + \frac{3\lambda_2}{4} + O(\lambda_2^2).
\]

Now we can use all the simple one dimensional corollaries in turn as well.

5.3. Higher Order Perturbations. Since perturbation theory is meant to compute more than first order terms we seek to use this Lie algebraic method to compute higher order terms. Certainly, one can see that using the transformations \( U \) we have set up thus far will produce higher order terms. One can see this if we set
\[
U = \exp(\lambda L).
\]

Our transformation becomes
\[
(5.3.1) \quad U^\dagger H_0 U = H_0 + \lambda [H_0, L] - \frac{\lambda^2}{2}[L, [H_0, L]] \ldots
\]
This approach, however, changes our Hamiltonian fundamentally. In fact, we end up not solving any problems, but instead creating more. A quick trial calculation with any Hamiltonian carrying term \( x^3 \) or higher will reveal that we cannot cancel certain terms arising from \([L, [L, H_0]]\). To remove this difficulty we must expand our Lie algebra to include terms carrying \( \lambda^k \) for whichever \( k \) we should choose. It is therefore convenient to write our unitary transformation as

\[
U = \exp(\sum_{j=1}^{k} \lambda^j L_{(j)}),
\]

where \( L_{(j)} \) are Lie algebra elements arising from \( j^{th} \) order commutators.

**Example 14.** Let us return briefly to the quartic oscillator and calculate its second order perturbation. Since it is well studied we may verify our results easily.

Computing commutators and commutators of commutators one will arrive at the following Lie algebra up to order 2.

\[
\begin{align*}
H_0 & = \lambda (a^{14} + a^4), & H_4 & = \lambda (a^{13} a + a^1 a^3), & I & = \lambda (a^{12} + a^2), \\
\lambda^2 (a^{16} + a^6) & = \lambda^2 (a^{14} a^2 + a^{12} a^4), & \lambda^3 (a^{14} + a^4) & = \lambda^3 (a^{13} a + a^1 a^3), & \lambda^2 (a^{12} + a^2).
\end{align*}
\]

Knowing the form of our necessary first order transformation, we add four terms to the exponential by

\[
(5.3.3) \quad U = \exp(\lambda (a^{14} - a^4) + \frac{1}{2} (a^{13} a - a^1 a^3) + \frac{3}{4} (a^{12} - a^2)) \\
+ \lambda^2 \beta_1 (a^{16} - a^6) + \lambda^2 \beta_2 (a^{14} a^2 - a^{12} a^4) \\
+ \lambda^3 \beta_3 (a^{13} a - a^1 a^3) + \lambda^2 \beta_4 (a^{12} - a^2))
\]

For simplicity, let us compute only the ground state energy given by \( U^\dagger H_0 U \). We have

\[
U^\dagger H_0 U = \exp(-\lambda L_{(1)} - \lambda^2 L_{(2)}) H_0 \exp(\lambda L_{(1)} + \lambda^2 L_{(2)})
\]

\[
= H_0 + \lambda [H_0, L_{(1)}] + \lambda^2 [H_0, L_{(2)}] - \frac{\lambda^2}{2} [L_{(1)}, [H_0, L_{(1)}]] + O(\lambda^3)
\]

From here it is a matter of computing commutators and adjusting \( \beta_1, \beta_2, \beta_3, \beta_4 \) to cancel higher order terms not given as functions of number operators.

When we compute the ground state energy we are concerned only with constant terms. Therefore, looking to our commutators we have the \( \lambda^2 \) term

\[
\left[ \frac{1}{16} (a^{14} - a^4) + \frac{1}{2} (a^{13} a - a^1 a^3) + \frac{3}{4} (a^{12} - a^2), \frac{1}{4} (a^{14} + a^4) + (a^{13} a + a^1 a^3) + \frac{3}{2} (a^{12} + a^2) \right].
\]

Expanding this we are left with two terms giving constants

\[
\frac{1}{64} [a^{14} - a^4, a^{14} + a^4] \quad \text{and} \quad \frac{9}{8} [a^{12} - a^2, a^{12} + a^2].
\]

Our constant terms turn out to be \( -\frac{2(4)}{64} \) and \( -\frac{2(259)}{8} \) yielding \( \frac{-21}{4} \).

Finally, our ground state energy up to second order will be given by
(5.3.4) \((U\dagger H_0 U)\dagger|0\rangle = (H_4 - \frac{3\lambda}{4} - \frac{-21\lambda^2}{4})U\dagger|0\rangle\),
yielding

\[E_0 = \frac{1}{2} + \frac{3\lambda}{4} - \frac{21\lambda^2}{8} + O(\lambda^3)\].

Indeed, this ground state energy agrees with the standard perturbation theory.

For the interested reader, the correct \(\beta\) parameter values are

\[\beta_1 = \frac{1}{48}, \quad \beta_2 = \frac{-9}{16}, \quad \beta_3 = \frac{-9}{4}, \quad \beta_4 = \frac{-63}{32}\],

and the second order equation appears as

\[U\dagger H_0 U = H_4 - \frac{3\lambda}{2} N(N+1) - \frac{3\lambda}{4} +
51\lambda^2 \left(\frac{N}{3}\right) + \frac{117\lambda^2}{2} \left(\frac{N}{2}\right) + 36\lambda^2 N + \frac{21\lambda^2}{8} + O(\lambda^3)\]

6. Discussion

One important problem among many arising from this paper and which we have
yet neglected to mention is the representation theory the Lie algebras. For example
if we are dealing with the cases \(\lambda x, \lambda x^2\) in the one dimensional case or any qua-
dratic term in higher dimensional cases, we have a closed Lie algebra. This may
not be terribly surprising as a closed Lie algebra offers an exact solution, and these
particular potentials are simply shifted or coupled harmonic oscillators. However,
the Lie algebras we have constructed all contain central elements. In the cases of
\(A_1\) and \(A_2\) we have 4 dimensional closed Lie algebras with center. Repre-
sentation theory tells us that these are isomorphic to \(gl_2\). It remains to be seen exactly the
relationship between these Lie algebras and the symmetries they describe. In this
paper we have only used them to calculate perturbed energy levels. It is entirely
possible there is a simpler approach to this problem from an entirely representation
theoretic standpoint.

As it stands, we have given explicit constructions for Lie algebras up to any
order and the method by which we may construct a unitary operator to make the
transformation

\[H_0 \mapsto H_n + \text{perturbations up to } O(\lambda^k)\].

By taking advantage of our symmetric construction of these Lie algebras, the
Hadamard lemma, and several formulae concerning abstract Weyl algebras we have
managed to give eigenvalues in agreement with standard methods.

Another issue which we have neglected to resolve it how to deal with coupled
oscillators in general. In the appendix we briefly mention the way to deal with
potentials of the form \(\lambda xy\). The Lie algebra computations for coupling terms of the
form \(\lambda x^n y^m\) are more taxing and trickier. This method, however, should be able
to deal with situation, but some coordinate change may be required first.
It is the hope of the author that anharmonic oscillators are simply a useful class of examples for the propitiation of this method. Furthermore, it is hoped that this method will help to give rise to additional representation theoretic methods in physics.

**Appendix: Dealing with Harmonic Oscillators**

Two important cases we haven’t touched upon are those of actual harmonic oscillators in one and multiple dimensions. Consider for example the two Hamiltonians

\[
H_2 = a^\dagger a + \frac{1}{2} + \lambda \left( \frac{a^\dagger + a}{\sqrt{2}} \right)^2
\]

\[
H_c = a^\dagger x a_x + a^\dagger y a_y + 1 + \frac{\lambda}{2} (a^\dagger_a + a_x)(a^\dagger_y + a_y).
\]

These correspond to the shifted frequency oscillator in one dimension with new frequency \( \sqrt{1 + 2\lambda} \) and a coupled oscillator in two dimensions with quadratic coupling term. These cases have been well studied and so we have neglected them thus far. However, the techniques to compute the perturbations are special because these Hamiltonians along with \( H_0 \) and \( H_{0,0} \) produce closed Lie algebras. By our theorem earlier we know that we can solve these exactly and not concern ourselves with \( k^{th} \) order perturbations.

The main technique we employ is to transform our ladder operators via the so called Bogoliubov transforms. In one dimension we have

\[
\begin{align*}
b^\dagger &= U^\dagger a^\dagger U = \sigma a^\dagger + \tau a \\
b &= U^\dagger a U = \sigma a + \tau a^\dagger.
\end{align*}
\]

In this way we produce the new Hamiltonian

\[
\sqrt{1 + 2\lambda (b^\dagger b + \frac{1}{2})} = a^\dagger a + \frac{1}{2} + \frac{\lambda}{2} (a^\dagger + a)^2.
\]

This algebra to move from \( \sigma, \tau \) to this clean form of the new Hamiltonian is tedious to be sure. The interested reader should confer with [JA] or email the author for a small set of notes.

A similar technique can be used for the quadratic coupling, but the transformation must take into account much more coupling. Our transformation should look something like

\[
\begin{pmatrix} b_x^\dagger \\ b_x \\ b_y^\dagger \\ b_y \end{pmatrix} =
\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \delta & \gamma \\ \sigma & \tau & \mu & \nu \\ \tau & \sigma & \nu & \mu \end{pmatrix}
\begin{pmatrix} a_x^\dagger \\ a_x \\ a_y^\dagger \\ a_y \end{pmatrix}
\]

This is simply a coordinate change which decouples the coordinate variables. The matrix, however, will take a very special form so that \( [b_i, b_j^\dagger] = \delta_{ij} \) as did the initial coordinates.

**Remark 15.** Notice here that we can couple our ladder operators in many more ways. For example we can tackle problems such as dynamic coupling

\[
H = x^2 - \partial_x^2 + y^2 - \partial_y^2 + \lambda \partial_x \partial_y,
\]
or oscillators in a magnetic field

\[ H = x^2 - \partial_x^2 + y^2 - \partial_y^2 + \lambda(y\partial_x - x\partial_y). \]

So long as our coupling term contains terms of order two or less in each of the ladder operators we can tackle these problems with a simple coordinate change.

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