QUASI-PERIODIC MOTIONS ON SYMPLECTIC TORI.

Nesrine Yousfi 1, Mauricio Garay 2 and Arezki Kessi 3.

1 yousfi.nesrine@gmail.com
2 garay91@gmail.com
3 arkessi@yahoo.com

May 24, 2018

ABSTRACT. The KAM (Kolmogorov-Arnold-Moser) theorem guarantees the stability of quasi-periodic invariant tori by perturbation in some Hamiltonian systems. Michel Herman proved a similar result for quasi-periodic motions, with $k$-dimensional involutive manifolds in Hamiltonian systems with $n$ degrees of freedom $n \leq k < 2n$, [9]. In this paper, we extend this result to the case of a quasi-periodic motion on symplectic tori $k = 2n$.

1. Statement of the theorem

The KAM (Kolmogorov-Arnold-Moser) theorem guarantees the stability of quasi-periodic motions on Lagrangian invariant tori. In 1991, Herman exhibited Hamiltonian systems with stable higher dimensional quasi-periodic motions [6](see also [9]). In this paper, we consider the case of symplectic quasi-periodic motions on symplectic tori. Such flows are dense, non-Hamiltonian and have no integral of motion.

We start with the construction of a universal family for quasi-periodic motions on symplectic tori. First observe that there is a natural notion of deformation of a (real analytic) symplectic manifold $M_0$. This is simply an usual deformation $\pi : M \rightarrow S$ of $M_0$ with the additional condition that the fibres have symplectic structures induced by a two form on $M$. According to the general terminology, we say that $M$ is a symplectic manifold over $S$.

There is a notion of symplectic versality and it follows from Moser’s homotopy method [7], that the family $\pi$ defined by

$$\mathbb{T}^{2n} \times (\mathbb{R}^*)^n \rightarrow (\mathbb{R}^*)^n, \ (x, \delta) \mapsto \delta$$

with the two-form

$$\omega := \sum_{i=1}^{n} \delta_i dx_i \wedge dx_{i+n},$$

defines a symplectically versal deformation of its fibres. Here

$$\mathbb{T}^{2n} = \mathbb{R}^{2n} / \mathbb{Z}^{2n} := \{(x_1, \ldots, x_{2n})\}$$

is the standard torus.

Next, we define quasi-periodic motions on $M \rightarrow S$. We denote respectively by

$$\Omega_S^\bullet := \Omega_M^\bullet / \pi^* \Omega_S^\bullet.$$
2 QUASI-PERIODIC MOTIONS ON SYMPLECTIC TORI.

and by $\Theta_\pi$ the sheaves of relative forms and vector fields tangent to the fibres. In this text, we consider only trivial fibrations and these are simply sheaves of differential forms and vector fields on the fibres depending analytically on a parameter $s \in S$.

The interior product with the symplectic form $\omega$ induces a sheaf isomorphism

$$\Theta_\pi \rightarrow \Omega^1_{\pi}, \ v \mapsto i_v := \omega(v, \cdot).$$

It maps the "coordinate" vector fields in the following way

$$\partial_{x_i} \mapsto -\delta_i dx_{i+n}$$

$$\partial_{x_{i+n}} \mapsto \delta_i dx_i.$$

The symplectic vector fields over $M$ are the preimages of closed 1-forms over $M_0$. In the particular case where the form is exact, we say that the field is Hamiltonian. We denote $\text{Symp}$ and $\text{Ham}$, respectively, the sheaves of symplectic and Hamiltonian vector fields over $M$. In the case where the torus is standard symplectic, the sheaf $\text{Symp}$ is the direct sum of the subsheaf $\text{Ham} \subset \text{Symp}$ and its complement given by the Casimir vector fields

$$\text{Symp} = \text{Ham} \oplus \text{Cas}$$

with

$$\text{Cas} := \sum_{i=1}^n \pi^{-1} \mathcal{O}_S \partial_{x_i},$$

where $\mathcal{O}_S$ is the sheaf of real analytic functions on $S$.

The symplectic vector field associated to the closed relative 1-form

$$\alpha := \sum_{i=1}^{2n} \tau_i dx_i,$$

defines a quasi-periodic (or periodic) motion on the symplectic torus $T^{2n}$ with frequency

$$(\tau_1/\delta_1, \ldots, \tau_{2n}/\delta_n, -\tau_1/\delta_1, \ldots, -\tau_n/\delta_n).$$

We get the full family of quasi-periodic motions on symplectic tori that we denote by

$$(\pi, \alpha) : M \rightarrow S, \ (x, \delta, \tau) \mapsto (\delta, \tau)$$

and call it the standard family. Here

$$S = (\mathbb{R}^*)^n \times \mathbb{R}^{2n}, \ M = T^{2n} \times S.$$  

The fibres of the standard family carry quasi-periodic motions. In the sequel, we denote by $\phi$ the frequency map

$$\phi : S \rightarrow \mathbb{R}^{2n}, \ (\tau, \delta) \mapsto (\tau_{n+1}/\delta_1, \ldots, \tau_{2n}/\delta_n, -\tau_1/\delta_1, \ldots, -\tau_n/\delta_n).$$

We have a natural notion of deformation of a symplectic vector field (or equivalently of a relative closed one-form) on

$$(\pi, \alpha) : M \rightarrow (S, s)$$

over a smooth base $T$. This means that we give ourselves a trivial symplectic fibration

$$\mathcal{M} \rightarrow (S \times T, s)$$

and a closed relative form

$$\beta \in \Omega^1_{\mathcal{M}/S \times T, s}$$

such that the fibre in $t = 0 \in T$ coincide with the initial fibration and one-form.

Among the deformations, one distinguishes the trivial deformation which neither modifies the vector field nor the symplectic structure.

*These are also called locally Hamiltonian vector field.
We shall say that two vector fields $v_1, v_2$ defined on symplectic manifolds
$$\pi_i : M_i \to S_i, \quad i = 1, 2$$
are isomorphic over $K_i \subset S_i$ if there exists a commuting diagram
$$\begin{array}{ccc}
\pi_1^{-1}(K_1) & \xrightarrow{\varphi} & K_1 \\
\downarrow \varphi & & \downarrow \psi \\
\pi_2^{-1}(K_2) & \xrightarrow{\psi} & K_2
\end{array}$$
where $\varphi$ is a symplectomorphism which maps $v_1$ to $v_2$ and $\psi$ is a homeomorphism. In fact, due to Pöschel's regularity principle the maps $\psi$ that we construct are also $C^\infty$ in Whitney's sense (see [8] [3, Appendix]). We may consider isomorphisms for germs along some set as well, in this case this means that there is such an isomorphism in sufficiently small neighbourhoods of the given set.

As usual in KAM theory, stability of quasi-periodic motion holds for frequencies that satisfy an arithmetical condition that we shall now formulate.

We denote $(\cdot, \cdot)$ the Euclidean scalar product of $\mathbb{R}^{2n}$
$$\langle x, y \rangle \mapsto \sum_{i=1}^{2n} x_i y_i.$$  

For each vector $\omega \in \mathbb{R}^{2n}$, we consider the sequence $\sigma(\omega) := (\sigma_k(\omega))_{k \in \mathbb{N}}$:
$$\sigma_k(\omega) := \min\{ |\langle \omega, i \rangle| : i \in \mathbb{Z}^{2n} \setminus \{0\}, \|i\| \leq 2^k \}.$$  

Definition 1. [1] We say that a real decreasing positive sequence $a := (a_i)$ satisfies Bruno’s condition if
$$\sum_{i \in \mathbb{N}} \frac{\log a_i}{2^i} > -\infty.$$  

The set $\mathcal{C}(a)$ of vectors $\omega$ such that $\sigma(\omega) \geq a$ is called the arithmetic class associated to the sequence $a$.

**Theorem 1.** Let $a = (a_i)$ be a sequence satisfying Bruno’s condition and $s_0 \in S$ such that $\phi(s_0) \in \mathcal{C}(a)$. Any deformation
$$(\xi, \beta) : \mathcal{M} \to (S \times T, s_0)$$
of the standard family
$$(\pi, \alpha) : M \to S$$
is isomorphic to the trivial deformation over the set $\phi^{-1}(\mathcal{C}(a))$ of frequencies which belong to $\mathcal{C}(a)$.

By taking $a = (a_i)$ sufficiently small, we may assume that the intersection of $\mathcal{C}(a)$ with any neighbourhood of $\phi(s_0)$ is a positive measure set (see [4]).

As we shall see, Theorem 1 is a consequence of the abstract KAM formalism developed in [3] and [2], that we shall now recall.
2. Kolmogorov spaces and Arnold spaces

A set $X$ over a base $B$ is simply a map

$$X \rightarrow B.$$  

If the fibres have Banach space structure

$$E \rightarrow B$$

we say that $E$ is a Banach space over $B$. The Banach spaces over $B$ form in the natural way a category $\text{Ban}(B)$, whose morphisms are continuous linear mappings over $B$.

\[
\begin{array}{ccc}
E & \rightarrow & F \\
\downarrow & & \downarrow \\
B & \rightarrow & B
\end{array}
\]

The unit ball $B_E \rightarrow B$ of $E$ is the collection of unit balls. It is a set over $B$.

Suppose now that $B$ is an ordered set. It is then a small category whose objects are elements of $B$ and the space of morphisms $\text{Mor}(i,j)$ contains one element if $j > i$ and is empty otherwise. Then a set over $B$ is a covariant functor from $B$ to the category of sets

$$F : B \rightarrow \text{Sets}.$$  

Concretely, a set $X \rightarrow B$ is an increasing collection of sets. In practise, we consider open sets over a base.

We say that $E$ is a Kolmogorov space if it is a contravariant functor from $B$ to the category of Banach spaces

$$F : B \rightarrow \text{Ban}$$

such that the morphisms

$$E_t \rightarrow E_s, \ s < t$$

have norm at most 1.

Denote by $\mathbb{N}^{op}$ the set of natural numbers, as a category, equipped with opposite order of $\mathbb{N}$. An $S$-Kolmogorov space is a Kolmogorov space over $[0,S]$. An Arnold space is a Kolmogorov space over a set $B$ of the form

$$B = B' \times \tilde{\mathbb{N}},$$

with $\tilde{\mathbb{N}} = \mathbb{N}^{op} \cup \{+\infty\}$.

In an Arnold space, there are special restriction mappings

$$E_{n,t} \rightarrow E_{3c,s}$$

that we simply denote by $i_{\infty}$ without specifying $n,t,s$.

We call An $S$-Huygens open set $U$ in $\mathbb{C}^n$ a collection of strictly increasing relatively compact open sets

$$U_s \subset \mathbb{C}^n, s \in [0,S]$$

such that for any $t > s$ and any $x \in U_s$ the polydisc $x + D_{t-s}$ is contained in $U_t$.

It is clear that many $S$-Huygens sets exist. For example, besides polydiscs $D_s \subset \mathbb{C}^n$, the neighborhoods of the real torus

$$U_s = \{ z \in (\mathbb{C}^*)^n : 1 - s < |z| < 1 + s \} \subset (\mathbb{C}^*)^n$$

form a Huygens set.
The examples that we will use are the following:
1) Let $U$ a relatively compact subset of $\mathbb{C}^n$, considering the space
$$\mathcal{O}^c(U) := C^0(\overline{U}) \cap \mathcal{O}(U)$$
of holomorphic functions with $C^0$-extension to the boundary. The $C^0$-norm
$$|f| := \sup_{z \in U} |f(z)|,$$
endows $\mathcal{O}^c(U)$ of a Banach space structure. Let now $U = (U_s)$ be an open set over $\mathbb{R}$. We have a Kolmogorov space
$$\mathcal{O}^c(U) \to \mathbb{R}$$
whose fibre at $t$ is $\mathcal{O}^c(U_t)$ with restriction mappings
$$e_{st} : \mathcal{O}^c(U_t) \to \mathcal{O}^c(U_s).$$
We construct in the same way, the Kolmogorov space $\mathcal{O}^h(U)$ whose fibre above $t$ is
$$\mathcal{O}^h(U_t) := L^2(U_t) \cap \mathcal{O}(\overline{U_t}),$$
equipped with the norm $L^2 :
$$f \mapsto \int_{D_t} (|f(z)|^2 dV)^{1/2},$$
where $dV$ is the volume form on $\mathbb{C}^n$.
2) An important special case of the previous construction is given by the open set
$$\mathcal{C}(a) \to \overline{\mathbb{N}}$$
defined by
$$\mathcal{C}(a)_n := \{ \omega \in \mathbb{C}, \sigma_j(\omega) > a_j, \forall j \leq 2^n \}, \quad n \in \overline{\mathbb{N}}.$$with $a = (a_n)$ is a positive sequence. There is an associated Arnold space $\mathcal{O}^h(\mathcal{C}(a))$. If $U = (U_b), b \in B$ is an increasing family of open sets, the spaces $\mathcal{O}^h(\mathcal{C}(a) \cap U)$ form an Arnold space over the base $B \times \overline{\mathbb{N}}$.
3) We define $\Omega^{k,c}(U_s)$ as the $\mathcal{O}^c(U_s)$-module of differential k-forms over $U_s$, $\Omega^{k,c}(U_s)$ which is freely generated by $\langle dx_I \rangle$, where $I := (i_1, \ldots, i_k) \subset \{1, \ldots, n\}^k$. The spaces $(\Omega^{k,c}(U_s))$ equipped with the norms
$$\| \sum a_I dx^I \|_s := \max_{I, z \in U_s} |a_I(z)|,$$
define the Kolmogorov space $\Omega^{k,c}(U)$.
4) Let $U$ an $S$-Huygens set and $I := (I_s)$, a sequence of intervals. Consider the symplectic fibration
$$\pi_s : U_s \times I_s \to I_s, \quad \text{avec } s \in [0, S].$$The sheaf of relative differential 1-forms to the fibration $\pi_s$, whose coefficients are holomorphic inside $U_s \times I_s$ and continuous on its boundary
$$\Omega^1_{\pi_s} := \Omega^1_{U_s}/\pi_{st} \Omega^1_{I_s},$$equipped with the norm (1) generate the Kolmogorov space $\Omega^1_s := (\Omega^1_{\pi_s})_s, s \in [0, S]$. 
In the same way, we define the Kolmogorov space of the symplectic vector field tangents to the fibres
\[ \Theta_\tau := \bigoplus_{s \in [0, S]} \Theta_{\tau_s}. \]

**3. Local morphisms**

Let \( E, F \) be \( S \)-Kolmogorov spaces, and \( A \) a subset of \([0, S] \times [0, S] \). A morphism from \( E \) to \( F \) over \( A \) is a family of linear continuous maps
\[ u_{st} : E_t \to F_s, \]
for \((t, s) \in A\) compatible with the structures on \( E, F \). The space \( \text{Hom}_A(E, F) \) of such morphisms has the structure of a vector space over \( A \).

Let \( \Delta_\tau := \{(t, s) \in [0, S]^2 : s < t, t \leq \tau \} \).
Morphisms over \( \Delta_\tau \) are called almost complete.

**Definition 2.** A morphism \( u \in \text{Hom}_{\Delta_\tau}(E, F) \) is called \( k \)-local of index \( \tau \), if there exists a real number \( C > 0 \) such that
\[ |u(x)|_s \leq \frac{C}{(t-s)^k} |x|_t, \]
for all \( s < t \leq \tau, x \in E_t \).

The smallest constant \( C \) which satisfies the inequality (2) defines a norm \( \| \cdot \| \) on the space of \( k \)-local morphism.

We denote by \( L^k(E, F) \to [0, S] \) the vector space of the \( k \)-local morphisms from \( E \) to \( F \) of index \( t \). It is the fibre of a Kolmogorov space over \([0, S] \) [5]:
\[ L^k(E, F) \to [0, S]. \]

Due to Cauchy inequalities, differential operators are local on Huygens set [5]. Comparison between the sheaves \( O^h \) and \( O^c \) also give rise to local operator.

**Proposition 1.** [3, proposition 5.1] Let \( U \) an \( S \)-Huygens set. The morphisms
\[ I : \theta^c(U) \to \theta^h(U), (t, f) \mapsto (s, f|_{U_s}), \]
\[ J : \theta^h(U) \to \theta^c(U), (t, f) \mapsto (s, f|_{U_s}) \]
are local.

**4. Tamed morphisms**

An Arnold space
\[ E \to \mathbb{N} \times B \]
defines a family over \( \mathbb{N} \) of Kolmogorov spaces. So we may extend the notion of local morphism to morphisms defined over sets
\[ A \subset \{(n, t, s) \in \mathbb{N} \times [0, S]^2 : s < t, t \leq \tau \}, \]
these are families of linear continuous maps
\[ u_{stn} : E_{tn} \to F_{sn}, \]
for \((t, s) \in A\) compatible with the structures on \( E_{tn}, F_{sn} \) but in general not with the restrictions \( E_n \to E_{n+1}, F_n \to F_{n+1} \).

The norm of a morphism over \( A \) is a map
\[ \nu : A \to \mathbb{R}_+, u(t, s, n) \mapsto |u(t, s, n)|. \]
The norm of a local morphism becomes a sequence of norms \(|u_n|\) indexed by \(N\).

**Definition 3.** A \(k\)-local morphism of Arnold spaces
\[ u_* := (u_n) : E \rightarrow F \]
is \(k\)-tamed if
\[ \sum_{n \geq 0} \frac{\log |u_n|}{2^n} < \infty. \]

We denote by \(\mathcal{M}^k(E,F)\) the vector space of the \(k\)-tamed morphisms from \(E\) to \(F\). We also need the notion of quasi-inverse.

**Definition 4.** A right quasi-inverse to an Arnold space almost complete morphism \(u : E \rightarrow F\) is an almost complete morphism \(v : E \rightarrow F\) such that
\[ |(\text{id} - uv)(t, s, n)| \leq \frac{1}{(t-s)^2} \left( \frac{s}{t} \right)^{2n}. \]

5. **The abstract KAM theorem**

We formulate a simplified version of the abstract KAM theorem given in [2], suited to our needs. Suppose that \(a \in E\) and \(M \subset E\), we say that the Arnold space \(\mathfrak{g} \subset L^1(E)\) acts on \(a + M\), if \(\forall u \in \mathfrak{g}\) we have
i) \(u(M) \subset M\),
ii) \(e^u\) exists and \(e^u(a + M) \subset a + M\).

**Theorem 2 ([2], [3]).** Let \(E\) an Arnold space, \(M\) a closed subspace of \(E\) and \(\mathfrak{g} \subset \mathcal{M}(E)\) an Arnold subspace acting on \(H + M\). Suppose that the map \(\rho : \mathfrak{g} \rightarrow M, v \mapsto v(H)\) admits a tamed quasi inverse \(j\), then there exists \(r > 0\) such that for any \(R \in M \cap rB_E\), there exists a morphism \(u = (u_n) \in \mathfrak{g}\) such that
1) the sequence \(i_* e^{u_n} \ldots e^{u_0}\) converges to a limit \(\varphi\);
2) \(\varphi(H + R) = i_* (H)\).

6. **The complex symplectic torus**

We complexify the situation and consider the complex symplectic torus of dimension \(2n\)
\[ (\mathbb{C}^*)^{2n} = \{z = (z_1, \ldots, z_{2n}) : z_i \neq 0\}. \]
The fixed points of the antiholomorphic involution
\[ z_i \mapsto \frac{1}{\bar{z}_i} \]
form a real symplectic torus \(\mathbb{T}^{2n}\) isomorphic to the standard torus \((\mathbb{R}/2\pi\mathbb{Z})^{2n}\) via the exponential map
\[ (\mathbb{R}/2\pi\mathbb{Z})^{2n} \rightarrow (\mathbb{C}^*)^{2n}, \ x_k \mapsto e^{ix_k}. \]
The form \(dz_k\) is identifies with \(dz_k/(iz_k)\) via this isomorphism, therefore, the symplectic form takes the form
\[ \sum_{k=1}^n \delta_k \frac{dz_k \wedge d\bar{z}_{k+n}}{z_k \bar{z}_{k+n}}. \]
The Casimir’s fields are constant $\sum_{i=1}^{2n} a_i \partial_{x_i}$, they are dual of constant forms which generate the first De Rham’s cohomology space $H^{1}_{DR}(\mathbb{T}^n) = H^{1}_{DR}(\mathbb{C}^*^n)$.

The interior product with the symplectic form gives a sheaf isomorphism

$$\text{Symp} = \text{Ham} \oplus \text{Cas} \longrightarrow Z = d\theta \oplus \mathcal{H},$$

with

$$\mathcal{H} := \oplus_{i=1}^{2n} C \, dx_i.$$

Let us now look at quasi-periodic motions. We consider the one-form

$$\alpha := \sum_{k=1}^{2n} -i \tau_k \frac{dz_k}{z_k}.$$  

The interior product with the symplectic form maps

$$dz_k/z_k \mapsto -\delta^{-1} z_{k+n} \partial_{z_{k+n}},$$

$$dz_{k+n}/z_{k+n} \mapsto \delta^{-1} z_k \partial_z.$$

The $\mathbb{R}$-neighbourhood of the real torus $U := (U_x)$ defined by

$$U_x := \{ z \in \mathbb{C}^{2n} : 1 - s < |z_1| < 1 + s, \ldots, 1 - s < |z_{2n}| < 1 + s \},$$

will be called the standard neighbourhood and we write $U \subset (\mathbb{C}^*)^{2n} \times \mathbb{R}$

**Lemma 1.** Let $U \subset (\mathbb{C}^*)^{2n} \times \mathbb{R}$ be the standard neighbourhood of the real torus. The map

$$g \mapsto \alpha(X_g)$$

is the convolution operator or Hadamard product : $\ast_h : \mathcal{O}^h(U) \longrightarrow \mathcal{O}^h(U)$, with the formal power series

$$\sum_{I \in \mathbb{Z}^{2n} \setminus 0} (\phi, I) z^I.$$  

**Proof.** The Hamiltonian vector field $X_g$ of an analytic function $g$ on an open subset of the complex symplectic torus is defined by

$$X_g = \sum_{i=1}^{n} \left( \frac{-z_i z_{i+n}}{\delta_i} \partial_{z_i} g \partial_{z_i} + \frac{z_i z_{i+n}}{\delta_i} \partial_{z_i} g \partial_{z_{i+n}} \right).$$

In particular, if $g(z) = z^I$ with $I \in \mathbb{Z}^{2n}$, the interior product of $X_g$ with $\alpha$ gives

$$\alpha(X_g) := -i(\phi, I) z^I,$$

with

$$\phi := (\tau_{1+n}/\delta_1, \ldots, \tau_{2n}/\delta_n, -\tau_1/\delta_1, \ldots, -\tau_n/\delta_n).$$

In general the inverse map, that is the convolution with the formal power series

$$\sum_{I \in \mathbb{Z}^{2n} \setminus 0} (\phi, I)^{-1} z^I,$$

is well-defined. Therefore we consider the family of quasi-inverse;

$$h_k(z) := \sum_{I \in \mathbb{Z}^{2n} \setminus 0, |I| \leq 2^k} (\phi, I)^{-1} z^I.$$
Proposition 2. [3, 8.1] If $U$ is the standard neighbourhood of the torus and if $\phi$ satisfies the Bruno’s condition then the convolution with the functions

$$h_k(z) := \sum_{I \in \mathbb{Z}^n \setminus \{0\}, |I| \leq 2k} (\phi, I)^{-1} z^I$$

is a tamed quasi-inverse to the convolution product with

$$\sum_{I \in \mathbb{Z}^n \setminus \{0\}} (\phi, I) z^I.$$

If $U$ is an $S$-Huygens set then, according to the Cauchy’s inequalities, the exterior derivative $d : \Omega^{k,c}(U) \to \Omega^{k+1,c}(U)$, is 1-local. Its kernel defines the Kolmogorov space of closed forms, denoted $Z^{k,c}$.

The kernel of the application

$$Z^{1,c} \to \mathbb{C}^n,$$

$$\alpha = \sum_{i=1}^n a_i dx_i \mapsto (\int_{S^1} \alpha) = (a_1(0), \ldots, a_n(0)),$$

consists of exact forms and therefore is a Kolmogorov space that we denote $d\Theta^c$.

7. Proof of Theorem 1

Consider the standard neighbourhood of the real torus $U$ and of the origin $V$ in $\mathbb{C}^{3n+1}$ with coordinates $(\delta, \tau, t)$ (the parameter of the symplectic form, of the quasi-periodic motion and of the deformation) restricted over $[0, S]$. We denote by $K$ the preimage of $\mathcal{C}(a)$ under the frequency map

$$\phi : S \to \mathbb{R}^{2n}, (\tau, \delta) \mapsto (\tau_{n+1}/\delta_1, \ldots, \tau_{2n}/\delta_n, -\tau_1/\delta_1, \ldots, -\tau_n/\delta_n).$$

We consider the standard family

$$\pi : M \to S$$

to which we add the deformation parameter as a 'dummy variable':

$$\tilde{\pi} : M \times \mathbb{C} \to S \times \mathbb{C} \times (x, t) \mapsto (\pi(x), t).$$

Consider the Arnold space $\Omega^{1,c}_\pi$ of the relative differential 1-forms to the fibration $\tilde{\pi}$, and put

1. $E := \Omega^{1,c}_\pi(U \times (V \cap K))$,
2. $M := t Z^c(U \times (V \cap K))$ the subspace of $E$ of closed forms,
3. $\mathfrak{g} := t S\text{sym}^c(U \times (V \cap K))$
4. $H := \alpha := \sum_{i=1}^{2n} \tau_i dx_i$.

Assume for a moment that the operator

$$\rho : t S\text{sym}^c(U \times (V \cap K)) \to t Z^c(U \times (V \cap K)), \quad u \mapsto u(\alpha_0)$$

admits a tamed quasi-inverse $j$. As the exponential of a symplectic field is a symplectomorphism, applying the abstract KAM theorem for any closed one form $\beta$ we get a symplectomorphism

$$\Phi : E_0 \to E_\infty$$

which maps the deformed closed one form $\alpha + t \beta$ to $\alpha$.

Thus to prove the theorem, it remains to show the existence of a tamed quasi-inverse $j$.

For this, we use the decomposition:

$$S\text{ym}^c(U \times (V \cap K)) \approx H\text{am}^c(U \times (V \cap K)) \oplus C\text{as}^c(U \times (V \cap K)).$$
As the corresponding decomposition

\[ \text{Sym}^h(U \times (V \cap K)) \cong \text{Ham}^h(U \times (V \cap K)) \oplus \text{Cas}^h(U \times (V \cap K)) \]

is orthogonal, by local equivalence (Proposition 1) the projection operators are local.

**Lemma 2.** Let \( \alpha := \sum_{k=1}^{2n} -i\tau_k \frac{dz_k}{2_k} \). The Lie derivative map

\[ \text{Ham}^c(U \times (V \cap K)) \rightarrow d\co^c(U \times (V \cap K)), \quad \eta \mapsto \eta(\alpha) \]

admits a tamed right inverse.

**Proof.** First we check that the map is well defined. Since \( \alpha \) is a closed form, according to Cartan’s formula

\[ \eta(\alpha) = di_\eta \alpha + i_\eta d\alpha = di_\eta \alpha, \]

for every derivative \( \eta \). Therefore, \( \eta(\alpha) \) is an exact form.

\[ \eta(\alpha) = d(\sum_{I \in Z^n} a_I z^I), \quad a_I \in \co(V \cap K). \]

On the other hand, as \( \eta \) is Hamiltonian, it is symplectically dual to the Hamiltonian vector field of a function

\[ g(z) := \sum_{I \in Z^n} b_I z^I \]

and

\[ i_\eta(\alpha) := \alpha(X_g) := \sum_{k=1}^{2n} -i\tau_k \frac{dz_k}{2_k}(X_g). \]

Using Lemma 1, we get that

\[ \alpha(X_g) := \sum_{I \in Z^n} -i(\phi, I) b_I z^I. \]

The convolution \( \star h \)

\[ \star h : \co^c(U \times (V \cap K)) \rightarrow \co^c(U \times (V \cap K)), \]

with the functions

\[ h_k(z) := \sum_{I \in Z^n \setminus \{0\}, |I| \leq 2\tau_k} (\phi, I)^{-1} z^I, \]

is tamed (Proposition 2) and therefore defines a quasi-inverse to the Lie derivative map

\[ \text{Ham}^c(U \times (V \cap K)) \rightarrow d\co^c(U \times (V \cap K)), \quad \eta \mapsto \eta(\alpha_0). \]

We define then a right quasi-inverse \( j \) using the commutative diagram

\[ \begin{array}{ccc}
\co^c(U \times (V \cap K)) & \xrightarrow{\star h} & \co^c(U \times (V \cap K)) \\
\downarrow j & & \downarrow j \\
d\co^c(U \times (V \cap K)) & \xrightarrow{\co^c} & d\co^c(U \times (V \cap K))
\end{array} \]

where the application \( j \) is defined by

\[ d(\sum_{I \in Z^n} a_I z^I) \mapsto \sum_{I \in Z^n \setminus \{0\}} a_I z^I. \]

\( \square \)
In the basis \((\partial_{\omega_i}), (dx_i)\), the restriction of the map \(\rho\) to \(\text{Cas}^c(U \times (V \cap K))\) defines an isomorphism on \(\mathcal{H}\)

\[\text{Cas}^c(U \times (V \cap K)) \to \mathcal{H}, \sum_{i=1}^n a_i \partial_{\omega_i} \mapsto \sum_{i=1}^n a_i dx_i.\]

The subspaces \(dO^c\) and \(\mathcal{H}\) are stable by \(\rho\) and admit a tamed right inverse on each component. Thus the map \(j\) defined componentwise by \(\circ h\) and

\[\sum_{i=1}^n a_i dx_i \mapsto \sum_{i=1}^n a_i \partial_{\omega_i},\]

is a tamed quasi-inverse. This concludes the proof of the theorem.

References

[1] A. Bruno. Analytic form of differential equations I. *Trans. Moscow Math. Obs.*, 25:119–262, 1971.
[2] M.D. Garay. An abstract KAM theorem. *Moscow Math. J.*, 14(4):745–772, 2014.
[3] M.D. Garay. Degenerations of invariant Lagrangian manifolds. *Journal of Singularities*, 8:50–68, 2014.
[4] M.D. Garay. Arithmetic density. *Proceedings of the Edinburgh Mathematical Society*, 59(3):691–700, 2016.
[5] M.D. Garay and D. Van Straten. KAM theory. *In preparation*.
[6] M. Herman. Exemples de flots Hamiltoniens dont aucune perturbation en topologie \(C^\infty\) n’a d’orbites périodiques sur un ouvert de surfaces d’énergies. *C. R. Acad. Sci. Paris. Série I*, 312(13):989–994, 1991.
[7] J. Moser. On the volume elements on a manifold. *Trans. American Math. Soc.*, 120(2):286–294, 1965.
[8] J. Pöschel. Integrability of Hamiltonian systems on Cantor sets. *Communications on Pure and Applied Mathematics*, 35(5):653–696, 1982.
[9] J.C. Yoccoz. Travaux de Herman sur les tores invariants. *Séminaire Bourbaki*, 754:311–344, 1991–1992.