Asymptotic stability of a composite wave for the one-dimensional compressible micropolar fluid model without viscosity

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Abstract

We are concerned with the large time behavior of solutions to the Cauchy problem of the one-dimensional compressible micropolar fluid model without viscosity, where the far-field states of the initial data are prescribed to be different. If the corresponding Riemann problem of the compressible Euler system admits a contact discontinuity and two rarefaction waves solutions, we show that for such a non-viscous model, the combination of the viscous contact wave with two rarefaction waves is time-asymptotically stable provided that the strength of the composite wave and the initial perturbation are sufficiently small. The proof is given by an elementary $L^2$ energy method.

Keywords Compressible micropolar fluid model; Viscous contact wave; Rarefaction waves; Without viscosity; Nonlinear Stability

AMS Subject Classifications: 35Q35, 35L65, 35B40

1 Introduction

The one-dimensional full compressible micropolar fluid model in the Lagrangian coordinates reads as:

$$\begin{cases}
v_t - u_x = 0, \\
_u_t + p_x = \left( \frac{\mu u_x}{v} \right)_x, \\
\left( e + \frac{u^2}{2} \right)_t + (pu)_x = \left( \frac{k\theta_x}{v} \right)_x + \frac{\omega_x^2}{v} + v\omega^2, \quad t > 0, \quad x \in \mathbb{R}, \\
\omega_t = A \left[ \left( \frac{w_x}{v} \right)_x - v\omega \right],
\end{cases}$$

(1.1)

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where the unknown functions are the specific volume $v(t, x)$, the velocity $u(t, x)$, the microrotation velocity $w$, the absolute temperature $\theta(t, x) > 0$, the internal energy $e(t, x)$ and the pressure $p(v, \theta)$ of the fluid, respectively, while the constants $\mu, \kappa$ and $A$ denote the viscosity coefficient, the heat conductivity coefficient and the microviscosity coefficient, respectively.

The model of micropolar fluid was first introduced by Eringen [1] in 1966. This model can be used to describe the motions of a large variety of complex fluids consisting of dipole elements such as the suspensions, animal blood, liquid crystal, etc. For more physical background on this model, we refer to [2, 3]. Note that, if the microstructure of the fluid is neglected, i.e., $w = 0$, then system (1.1) is reduced to the classical compressible Navier-Stokes system.

In this paper, we consider the the Cauchy problem of the system (1.1) without viscosity, i.e.,

$$
\begin{align*}
&v_t - u_x = 0, \\
&u_t + p_x = 0, \\
&(e + \frac{u^2}{2})_t + (pu)_x = \left(\frac{\kappa \theta_x}{v}\right)_x + \frac{\omega_x^2}{v} + v \omega^2, \quad t > 0, \quad x \in \mathbb{R} \\
&\omega_t = A \left[\left(\frac{u_x}{v}\right)_x - v \omega\right],
\end{align*}
$$

with the following initial and far field conditions:

$$
\begin{align*}
&(v, u, \theta, \omega)(x, 0) = (v_0, u_0, \theta_0, \omega_0)(x), \quad x \in \mathbb{R}, \\
&(v, u, \theta, \omega)(\pm \infty, t) = (v_\pm, u_\pm, \theta_\pm, \omega_\pm), \quad t > 0.
\end{align*}
$$

Here $v_\pm > 0, u_\pm, \theta_\pm > 0, \omega_\pm$ are given constants and we assume that $(v_0, u_0, \theta_0, \omega_0)(\pm \infty) = (v_\pm, u_\pm, \theta_\pm, \omega_\pm)$ as compatibility conditions.

Throughout this paper, we assume that the pressure $p(v, \theta)$ and the internal energy $e(t, x)$ are given by

$$
p(v, \theta) = \frac{R \theta}{v} = B v^{-\gamma} \exp\left(\frac{\gamma - 1}{R} s\right), \quad e = \frac{R}{\gamma - 1} \theta,
$$

where $s$ is the entropy of the fluid, and $\gamma > 1, B$ and $R$ are positive constants.

The mathematical theory of the compressible micropolar fluid model has been studied extensively in the last several decades. For the non-isentropic case, Mujaković first analyzed the one-dimensional model and obtained a series of results concerning the local-in-time existence and uniqueness, the global existence and regularity of solutions to an initial-boundary value problem with homogeneous [4, 5, 6] and non-homogeneous [7, 8, 9] boundary conditions. Besides, she also studied the large time behavior of the solutions to initial-boundary value problem [10] and the Cauchy problem [11] of the one-dimensional model. The 1-D compressible viscous micropolar fluid model was also studied by many other authors, such as Qin et al. [19] proved the stabilization and the regularity of solutions with weighted small initial data, Duan [20, 21] investigated the global existence of strong solutions, and the authors in [18, 22, 23] showed the nonlinear stability of some basic waves (such as rarefaction waves, viscous contact wave and viscous shock wave etc.). For the 3-D compressible micropolar fluid model, Dražić and N. Mujaković considered the local and global existence, uniqueness, large time behavior and regularity of spherical symmetry solutions, see [12, 13, 14, 15, 16, 17] and the references therein.

For the isentropic case, Chen [24] proved the global existence of strong solutions to the one-dimensional model with initial vacuum. Later, Chen and his collaborators further obtained the global existence of weak solutions [25] and various blowup criteria of strong solutions [26, 27] for the
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3-D compressible micropolar fluid system. Liu and Zhang [28] established the optimal time decay of strong solutions to the 3-D compressible micropolar fluid model. We also mention that there have been many results on the incompressible micropolar fluid system, see [29, 30, 31] and the references therein.

However, few results have been obtained for the compressible non-viscous micropolar fluid model as far. Especially, there no results available now for the one-dimensional non-viscous micropolar fluid model. This paper is devoted to this problem, and we are concerned with the time-asymptotic nonlinear stability of the combination of viscous contact wave with two rarefaction waves for the Cauchy problem (1.2)-(1.3) when the far-field states of the initial data are different, i.e., \((v_+, u_+, \theta_+, \omega_+) \neq (v_-, u_-, \theta_-, \omega_-)\).

Motivated by the close relationship between the compressible micropolar fluid model and the compressible Navier-Stokes system as mentioned before, it is expect that the large-time asymptotic profiles of solutions of the Cauchy problem (1.2)-(1.3) will be the same as those of the compressible Navier-Stokes system in the case of \(\omega_+ = \omega_- = 0\). It is well-known that the large-time behavior of solutions to the Cauchy problem of the compressible Navier-Stokes system can be described by the Riemann solution of the corresponding Euler system:

\[
\begin{aligned}
&v_t - u_x = 0, \\
&u_t + p(v, \theta)_x = 0, \\
&\frac{R}{\gamma - 1} \theta_t + p(v, \theta)u_x = 0, \quad x \in \mathbb{R}, \ t > 0
\end{aligned}
\]  

(1.5)

with the Riemann initial data

\[
(v, u, \theta)(0, x) = \begin{cases} 
(v_-, u_-, \theta_-), & x < 0, \\
(v_+, u_+, \theta_+), & x > 0.
\end{cases}
\]

(1.6)

Then it is well known [32] that the Euler system (1.5) is a strict hyperbolic system of conservation laws with three distinct eigenvalues:

\[
\lambda_1(v, \theta) = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3(v, \theta) = \sqrt{\frac{\gamma p}{v}},
\]

and the Riemann solutions of the problem (1.5)-(1.6) have the wave patterns: the shock wave, rarefaction wave and contact discontinuity, and certain linear combinations of these three basic waves. Therefore, we hope that the solutions of the Cauchy problem (1.2)-(1.3) will tend to some basic waves of the Euler system in the case of \(w(t, x) = 0\).

In the following two subsections, we shall construct the viscous contact wave and the combination of viscous contact wave with two rarefaction waves for the Cauchy problem (1.2)-(1.3), respectively.

1.1 Viscous contact wave

It is known that the contact discontinuity solution of the Riemann problem (1.5)-(1.6) takes the form [32]

\[
(v^{cd}, u^{cd}, \theta^{cd})_t(t, x) = \begin{cases} 
(v_-, u_-, \theta_-), & x < 0, \ t > 0, \\
(v_+, u_+, \theta_+), & x > 0, \ t < 0.
\end{cases}
\]

(1.7)

provided that

\[
u_- = u_+, \quad p_- \triangleq \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} \triangleq p_+.
\]

(1.8)
The viscous contact wave \( (V^c, U^c, \Theta^c, W^c)(t, x) \) with \( W^c(t, x) = 0 \) corresponding to the contact discontinuity \( (v^d, u^d, \theta^d, w^d)(t, x) \) with \((v^d, u^d, \theta^d)\) defined in (1.7) and \( w^d(t, x) = 0 \) for the compressible micropolar fluid model (1.2) becomes smooth and behaviors as a diffusion waves due to the effect of heat conductivity. As [18], the viscous contact wave \( (V^c, U^c, \Theta^c)(t, x) \) can be defined as follows.

Since the contact discontinuity wave is expected to be almost constant, we set

\[
P^c = \frac{R \Theta^c}{V^c} = p_+ = p_-, \quad |U^c| \ll 1, \quad (1.9)
\]

then the leading part of the energy equation (1.2) is

\[
\frac{R}{\gamma - 1} \Theta^c_t + p_+ U^c_x = \kappa \left( \frac{\Theta^c}{V^c} \right)_x. \quad (1.10)
\]

Using the equations (1.9), \( V^c_t = U^c_x \) and (1.10), we get a nonlinear diffusion equation

\[
\left\{ \begin{array}{l}
\Theta^c_t = a \kappa \left( \frac{\Theta^c}{\Theta^f} \right)_x, \quad a = \frac{p_+ (\gamma - 1)}{\gamma R^2}, \\
\Theta^c(\pm \infty, t) = \theta_{\pm}.
\end{array} \right. \quad (1.11)
\]

Due to [33], (1.11) has a unique self-similar solution \( \Theta^f(t, x) = \Theta^f(\xi) \) with \( \xi = \frac{x}{\sqrt{1+t}} \), which is a monotone function, increasing if \( \theta_+ > \theta_- \) and decreasing if \( \theta_+ < \theta_- \). On the other hand, there exists some positive constant \( \delta \), such that for \( \delta = |\theta_+ - \theta_-| \leq \delta \), \( \Theta(t, x) \) satisfies

\[
(1 + t)^{\frac{k}{2}} \left| \partial_x^k \Theta \right| + |\Theta - \theta_{\pm}| \leq C \delta e^{-C_0 x^2}, \quad k \geq 1, \quad as |x| \to \infty, \quad (1.12)
\]

where \( C_0 \) and \( C \) are two positive constants depending only on \( \theta_- \) and \( \delta \).

Once \( \Theta^f(t, x) \) is determined, the viscous contact wave \( (V^c, U^c, \Theta^c)(t, x) \) is defined by

\[
V^c = \frac{R \Theta^c}{p_+}, \quad U^c = u_- + \frac{\kappa(\gamma - 1)}{\gamma R} \frac{\Theta^c_x}{\Theta^c}, \quad \Theta^c = \Theta^f(t, x), \quad (1.13)
\]

then it is easy to check that the viscous contact wave \( (V^c, U^c, \Theta^c)(t, x) \) satisfies

\[
\left\{ \begin{array}{l}
V^c_t - U^c_x = 0, \\
U^c_t + \left( \frac{R \Theta^c}{V^c} \right)_x = R_1, \\
\frac{R}{\gamma - 1} \Theta^c_t + p(V^c, \Theta^c) U^c_x = \kappa \left( \frac{\Theta^c}{V^c} \right)_x
\end{array} \right. \quad (1.14)
\]

with

\[
R_1 = U_t = O(\delta) (1 + t)^{-\frac{3}{2}} e^{-\frac{C_0 x^2}{(1+t)}}. \quad (1.15)
\]

Our first result is stated as follows.

**Theorem 1.1 (Stability of viscous contact wave).** For any given left end state \((v_-, u_-, \theta_-)\), suppose that the right end state \((v_+, u_+, \theta_+)\) satisfies (1.8). Let \((V^c, U^c, \Theta^c)(t, x)\) be the viscous contact wave \( (V^c, U^c, \Theta^c)(t, x) \) satisfies...
defined in (1.13) with strength \( \delta = |\theta_+ - \theta_-| \). Then there exist two suitably small positive constants \( \varepsilon_1 \) and \( \delta_1 \leq (\min \{\delta, 1\}) \) such that if \( 0 < \delta \leq \delta_1 \) and

\[
\|(v_0(\cdot) - V^c(0, \cdot), u_0(\cdot) - U^c(0, \cdot), \theta_0(\cdot) - \Theta^c(0, \cdot), w_0(\cdot))\|_{H^2(\mathbb{R})} \leq \varepsilon_1,
\]

then the Cauchy problem (1.2)-(1.3) admits a unique global smooth solution \((v, u, \theta, w)(t, x)\) satisfying

\[
\begin{cases}
(v - V^c, u - U^c, \theta - \Theta^c) \in C(0, +\infty; H^2(\mathbb{R})), \\
(v - V^c, u - U^c)_x \in L^2(0, +\infty; H^1(\mathbb{R})), \\
(\theta - \Theta^c)_x \in L^2(0, +\infty; H^2(\mathbb{R})), \ w(t, x) \in L^2(0, +\infty; H^3(\mathbb{R})).
\end{cases}
\tag{1.16}
\]

Moreover, the following large-time behaviors hold:

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \{||(v - V^c, u - U^c, \theta - \Theta^c, w)(t, x)||\} = 0.
\tag{1.17}
\]

### 1.2 Composite waves: combination of viscous contact wave with rarefaction waves

When the relation (1.8) fails, the basic theory of hyperbolic systems of conservation laws [32] implies that for any given constant state \((v_-, u_-, \theta_-)\) with \(v_- > 0, u_- \in \mathbb{R} \) and \(\theta_- > 0\), there exists a neighborhood \(\Omega(v_-, u_-, \theta_-)\) of \((v_-, u_-, \theta_-)\) such that for any \((v_+, u_+, \theta_+)\) in \(\Omega(v_-, u_-, \theta_-)\), the Riemann problem (1.5)-(1.6) has a unique solution. In this paper, we only consider the stability of the combination of the viscous contact wave with rarefaction waves. Consequently, we suppose that

\[
(v_+, u_+, \theta_+) \in R_- C_v R_+(v_-, u_-, \theta_-) \subset \Omega(v_-, u_-, \theta_-), \quad |\theta_+ - \theta_-| \leq \delta_2,
\tag{1.18}
\]

where \(\delta_2(\leq \bar{\delta})\) is a positive constant, \(R_-, R_+, C_v\) denote the 1-rarefaction wave curve, 3-rarefaction wave curve, and the contact wave curve respectively, and

\[
R_- C_v R_+(v_-, u_-, \theta_-) \triangleq \left\{(v, u, \theta) \in \Omega(v_-, u_-, \theta_-) \mid s \neq s_-, \right. \quad u \geq u_- - \int_{v_-}^{\frac{v_+ - v_-}{v_+ - v_m}} \frac{\lambda_-(\eta, s_-)}{v_-} \cdot d\eta,
\quad u \geq u_- - \int_{v_-}^{\frac{v_+ - v_-}{v_+ - v_m}} \frac{\lambda_+(\eta, s)}{v_+} \cdot d\eta \right\}
\]

It is known from [32] that if \(\delta_2\) in (1.18) is suitably small, then there exist a positive constant \(C = C(\theta_-, \delta_1)\) and a pair of points \((v_m^+, u_m^+, \theta_m^+)\) and \((v_m^-, u_m^-, \theta_m^-)\) in \(\Omega(v_-, u_-, \theta_-)\) such that

\[
\frac{R \theta_m^+}{v_m^+} = \frac{R \theta_m^-}{v_m^-} \triangleq p^m, \quad |v_m^+ - v_m^-| + |u_m^+ - u_m^-| + |\theta_m^+ - \theta_m^-| \leq C|\theta_- - \theta_+|.
\tag{1.19}
\]

Moreover, the states \((v_m^-, u_m^-, \theta_m^-)\) and \((v_m^+, u_m^+, \theta_m^+)\) belong to the 1-rarefaction wave curve \(R_-(v_-, u_-, \theta_-)\) and 3-rarefaction wave curve \(R_+(v_+, u_+, \theta_+)\) respectively, where

\[
R_\pm(v_\pm, u_\pm, \theta_\pm) = \left\{ s = s_\pm, u = u_\pm - \int_{v_\pm}^{v} \frac{\lambda_\pm(\eta, s_\pm)}{v_\pm} \cdot d\eta, \quad v > v_\pm \right\}
\tag{1.20}
\]

with

\[
s = \frac{R}{\gamma - 1} \ln \frac{R \theta}{A} + R \ln v, \quad s_\pm = \frac{R}{\gamma - 1} \ln \frac{R \theta_\pm}{A} + R \ln v_\pm,
\]

\[
\lambda_\pm(v, s) = \pm \sqrt{A \gamma v^{-\gamma^{-1}} (\gamma^{-1}) s / R}.
\]
The contact discontinuity wave curve $C_c$ is defined by

$$C_c((v^m, u^m, \theta^m)) = \{(v, u, \theta)(t, x)| u = u^m, \ p = p^m, \ v \neq v^m\}. \quad (1.21)$$

The 1-rarefaction wave $((v^v_r, u^v_r, \theta^v_r))(t, x)$ (respectively the 3-rarefaction wave $((v^v_l, u^v_l, \theta^v_l))(t, x)$) connecting $(v_-, u_-, \theta_-)$ and $(v^m, u^m, \theta^m)$ (respectively $(v^m, u^m, \theta^-)$ and $(v^m, u^m, \theta^m)$) is a weak solution of the Euler system (1.5) with the Riemann initial data:

$$(v^v_\pm, u^v_\pm, \theta^v_\pm)(0, x) = \begin{cases} (v^m, u^m, \theta^m), & \pm x < 0, \\ (v_\pm, u_\pm, \theta_\pm), & \pm x > 0. \end{cases} \quad (1.22)$$

The contact discontinuity wave $((v^{cd}, u^{cd}, \theta^{cd}))((t, x)$ connecting $(v^m, u^m, \theta^-)$ and $(v^m, u^m, \theta^m)$ is defined by

$$(v^{cd}, u^{cd}, \theta^{cd})(t, x) = \begin{cases} (v^m, u^m, \theta^m), & x < 0, \ t > 0, \\ (v^m, u^m, \theta^m), & x > 0, \ t < 0. \end{cases} \quad (1.23)$$

To study the stability problem, we need to construct the smooth approximations of the rarefaction waves $(v^v_\pm, u^v_\pm, \theta^v_\pm)(t, x)$ and $(v^{cd}, u^{cd}, \theta^{cd})(t, x)$ as follows:

$$\begin{align*}
\lambda_\pm(V^v_\pm, s_\pm) &= \hat{w}_\pm(t + 1, x), \\
U^v_\pm(t, x) &= u_\pm - \int_{v^v_\pm}^{v^v_r(x)} \lambda_+(\eta, s_\pm) d\eta, \\
\Theta^v_\pm &= \theta_\pm(v^v_\pm)\gamma^{-1}(V^v_\pm)^{1-\gamma},
\end{align*} \quad (1.24)$$

where $\hat{w}_-(t, x)$ (respectively $\hat{w}_+(t, x)$) is the solution of the Cauchy problem of the Burger equation:

$$\begin{align*}
\hat{w}_t + \hat{w}_x &= 0, \ x \in \mathbb{R}, \ t > 0, \\
\hat{w}(0, x) &= \frac{\hat{w}_r + \hat{w}_l}{2} + \frac{\hat{w}_r - \hat{w}_l}{2} \tanh x
\end{align*} \quad (1.25)$$

with $\hat{w}_l = \lambda_-(v^v_-, s_-)$ and $\hat{w}_r = \lambda_-(v^v_+, s_+)$ (respectively $\hat{w}_l = \lambda_+(v^m_-, s_-)$ and $\hat{w}_r = \lambda_+(v^m_+, s_+)$).

Let $(V^c, U^c, \Theta^c)(t, x)$ be the viscous contact wave defined in (1.12) with $(v_\pm, u_\pm, \theta_\pm)$ replaced by $(v^m, u^m, \theta^m)(t, x)$ respectively. Set

$$(V^c, U^c, \Theta^c)(t, x) = \begin{cases} V^r + V^c + V^r, & (t, x) - \begin{pmatrix} V^c + V^+ \cr U^c + U^+ \cr \Theta^c + \Theta^+ \end{pmatrix}, \\
2u^m & (t, x) - \begin{pmatrix} V^+ \cr U^+ \cr \Theta^+ \end{pmatrix} \end{cases} \quad (1.26)$$

then our second main result is the following:

**Theorem 1.2 (Stability of composite waves).** Suppose that the end states $(v_\pm, u_\pm, \theta_\pm)$ satisfy (1.18) for some small constant $\delta_2 > 0$. Let $(V, U, \Theta)(t, x)$ be the superposition of the viscous contact wave and the approximate rarefaction waves defined in (1.26) with strength $\delta = |\theta_+ - \theta_-|$. Then there exist two small positive constants $\varepsilon_2$ and $\delta_3 (\leq \min\{\delta_2, 1\})$ such that if $0 < \delta \leq \delta_3$ and the initial data $(v_0, u_0, \theta_0, w_0)(x)$ satisfies

$$\|(v_0(\cdot) - V(0, \cdot), u_0(\cdot) - U(0, \cdot), \theta_0(\cdot) - \Theta(0, \cdot), w_0(\cdot))\|_{H^2(\mathbb{R})} \leq \varepsilon_2,$$
then the Cauchy problem (1.4)-(1.5) admits a unique global smooth solution \((v, u, \theta, w)(t, x)\) satisfying

\[
\begin{align*}
(v - V, u - U, \theta - \Theta, w)(t, x) &\in C(0, +\infty; H^2(\mathbb{R})), \\
(v - V, u - U)_{t}(t, x) &\in L^2(0, +\infty; H^1(\mathbb{R})), \\
(\theta - \Theta)_{t}(t, x) &\in L^2(0, +\infty; H^3(\mathbb{R})),
\end{align*}
\]

and the following large-time behaviors:

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left( \left| \left( v - v_{c} - V_{c} - v_{+} + v_{m} + v_{m}^{m} \right)(t, x) \right| \\
+ \left| \left( u - u_{c} - u_{+} + u_{m} \right)(t, x) \right| \\
+ \left| \left( \theta - \Theta_{c} - \Theta_{+} + \theta_{m} + \theta_{m}^{m} \right)(t, x) \right| \right) = 0.
\]

Now we make some comments on the analysis of this paper. Since there is no viscosity in the momentum equation (1.2), the system (1.2) is less dissipative than the viscous ones considered in the literatures before. This is the first main difficulty of this paper. Thus how to derive some suitable momentum equation (1.1) one can not get the a priori estimate of \(H\) in different estimates. If the dependence need to be explicitly pointed out, the notation Lemma 3.3 which play an important role in the energy estimates.

Devoted to the proof of main theorems of this paper. In the final Sections 4, we give the proof of heat kernel and a domain decomposition technique were also presented in this section. Sections 3 is devoted to the proof of main theorems of this paper. In the final Sections 4, we give the proof of Lemma 3.3 which play an important role in the energy estimates.

Notations: Throughout this paper, \(C\) stands for some generic positive constant which may vary in different estimates. If the dependence need to be explicitly pointed out, the notation \(C(\cdot, \cdot, \cdot, \cdot)(i \in \mathbb{N})\) is used. For function spaces, \(L^p(\mathbb{R})(1 \leq p \leq +\infty)\) denotes the standard Lebesgue space with the norm \(\|f\|_{L^p(\mathbb{R})} = \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}\), and \(W^{k,p}(\mathbb{R})\) is the usual \(k\)-th order Sobolev space with its norm \(\|f\|_{W^{k,p}(\mathbb{R})} = \left( \sum_{i=0}^{k} \|\partial_x^i f\|_{L^p}^p \right)^{\frac{1}{p}}\). When \(p = 2\), we simply denote the the space \(W^{k,p}(\mathbb{R})\) by \(H^{k}(\mathbb{R})\), and the norms \(\| \cdot \|_{H^{k}(\mathbb{R})}\) and \(\| \cdot \|_{L^2}\) by \(\| \cdot \|_{k}\) and \(\| \cdot \|\), respectively.
2 Preliminaries

First of all, we give the following lemma on the heat kernel which will play an important role in the analysis of this paper, whose proof can be found in [44].

For \( \alpha > 0 \), we define

\[
h(t, x) = (1 + t)^{-\frac{1}{2}} \exp \left\{ - \frac{\alpha x^2}{1 + t} \right\}, \quad g(t, x) = \int_{-\infty}^{x} h(t, y) dy.
\]  

(2.1)

Then it is easy to check that

\[4\alpha g_t = h_x, \quad \|g(t)\|_{L^\infty} = \sqrt{\pi} \alpha^{-1/2},\]

and we have

Lemma 2.1 ([44]). For any \( 0 < T \leq \infty \), suppose that the function \( F(t, x) \) satisfies

\[F \in L^\infty(0, T; L^2(\mathbb{R})), \quad F_x \in L^2(0, T; L^2(\mathbb{R})), \quad F_t \in L^2(0, T; H^{-1}(\mathbb{R})).\]

Then the following estimate holds:

\[
\int_{0}^{T} \int_{\mathbb{R}} F^2 h^2 dx dt \leq 4 \pi \|F(0)\|^2 + 4 \pi (\gamma - 1) \alpha^{-1} \int_{0}^{T} \|F_x(\tau)\|^2 d\tau + \frac{8 \alpha}{\gamma - 1} \int_{0}^{T} \langle F_t, F g^2 \rangle d\tau,
\]

(2.3)

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( H^{-1} \times H^1 \).

The solution \( \hat{w}(t, x) \) of the Cauchy problem (1.25) has the following properties.

Lemma 2.2. For given \( \hat{w}_t \in \mathbb{R} \) and \( \hat{w} > 0 \), let \( \hat{w}_r \in \{ \hat{w} | 0 < \hat{w} \leq \hat{w} - \hat{w}_r \leq \hat{w} \} \). Then the problem (1.25) has a unique global smooth solution satisfying the following.

(i) \( \hat{w}_t < \hat{w}(t, x) < \hat{w}_r, \hat{w}_x > 0, x \in \mathbb{R}, t > 0 \).

(ii) For any \( p \in [1, +\infty] \), there exists some positive constant \( C = C(p, \hat{w}_t, \hat{w}) \) such that for \( \hat{w} \geq 0 \) and \( t \geq 0 \),

\[
\|\hat{w}_x(t)\|_{L^p} \leq C \min\{\hat{w}, \hat{w}^{\frac{1}{p}}(1 + t)^{-1 + \frac{1}{p}}\}, \quad \|\hat{w}_{xx}(t)\|_{L^p} \leq C \min\{\hat{w}, (1 + t)^{-1}\}.
\]

(iii) If \( \hat{w}_t > 0 \), for any \( (t, x) \in [0, +\infty) \times (-\infty, 0) \),

\[
|\hat{w}(t, x) - \hat{w}_t| \leq \hat{w} e^{-2|x|+\hat{w}_t t}, \quad |\hat{w}_x(t, x)| \leq 2 \hat{w} e^{-2|x|+\hat{w}_t t}.
\]

(iv) If \( \hat{w}_r < 0 \), for any \( (t, x) \in [0, +\infty) \times [0, +\infty) \),

\[
|\hat{w}(t, x) - \hat{w}_r| \leq \hat{w} e^{-2|x|+|\hat{w}_r| t}, \quad |\hat{w}_x(t, x)| \leq 2 \hat{w} e^{-2|x|+|\hat{w}_r| t}.
\]

(v) Let \( w^r(\frac{x}{t}) \) be the Riemann solution of the scalar equation (1.25) with the Riemann initial data

\[
\hat{w}(0, x) = \begin{cases} \hat{w}_t, & x < 0, \\ \hat{w}_r, & x > 0, \end{cases}
\]

then we have

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| \hat{w}(t, x) - w^r \left( \frac{x}{t} \right) \right| = 0.
\]
In order to use Lemma 2.2 to study the properties of the smooth rarefaction waves \((V^r_\pm, U^r_\pm, \Theta^r_\pm)\) constructed in (1.24) and the viscous contact wave \((V^c, U^c, \Theta^c)\)(\(t, x\)), we divided the the domain \(\mathbb{R} \times (0, t)\) into three parts, that is \(\mathbb{R} \times (0, t) = \Omega_+ \cup \Omega_c \cup \Omega_\pm\) with

\[
\Omega_\pm = \{(x, t) | \pm 2x > \pm \lambda_\pm(v^m_\pm, s_\pm)t\}, \quad \Omega_c = \{(x, t) | \lambda_-(v^m_-, s_-)t > 2x < \lambda_+(v^m_+, s_+)t\}.
\]

**Lemma 2.3.** Assume that (1.19) holds with \(\delta = |\theta_+ - \theta_-| \leq \dot{\delta}\). Then the smooth rarefaction waves \((V^r_\pm, U^r_\pm, \Theta^r_\pm)\) constructed in (1.24) and the viscous contact wave \((V^c, U^c, \Theta^c)(t, x)\) satisfy the following

(i) \((U^r_\pm)_x \geq 0, x \in \mathbb{R}, t > 0\).

(ii) For any \(p \in [1, +\infty]\), there exists a positive constant \(C = C(p, v_-, u_-, \theta_-, \delta_1, m_0)\) such that for \(\delta = |\theta_+ - \theta_-| \) and \(t \geq 0\),

\[
\left\| \left(\frac{V^c_\pm}{U^c_\pm}, \left(\Theta^c_\pm\right)_x\right)(t) \right\|_{L^p} \leq C \min\{\delta, \frac{\dot{\delta}}{p}(1 + t)^{-1 + \frac{1}{p}}\},
\]

\[
\left\| \left(\frac{\partial_x^2 V^r_\pm}{U^r_\pm}, \partial_x^k U^r_\pm, \partial_x^k \Theta^r_\pm\right)(t) \right\|_{L^p} \leq C \min\{\delta, (1 + t)^{-1}\}, \quad k = 2, 3.
\]

(iii) There exists some positive constant \(C = C(p, v_-, u_-, \theta_-, \delta_1, \delta_2)\) such that for \(\delta = |\theta_+ - \theta_-| \) and

\[
c_1 = \frac{1}{10} \min \{ |\lambda_-(v^m_-, s_-)|, \lambda_+(v^m_+, s_+), c_0\lambda^2_-(v^m_-, s_-), c_0\lambda^2_+(v^m_+, s_+), 1\}
\]

we have in \(\Omega_c\) that

\[
\left| (V^c_\pm)_x, (U^c_\pm)_x, (\Theta^c_\pm)_x \right| + |V^r_\pm - v^m_\pm| + |\Theta^r_\pm - \theta^m_\pm| \leq C\delta e^{-c_1(|x|+t)},
\]

and in \(\Omega_\pm\),

\[
|V^c_\pm| + |\Theta^c_\pm| + |V^c - v^m_\pm| + |\Theta^c - \theta^m_\pm| + |U^c_\pm| \leq C\delta e^{-c_1(|x|+t)},
\]

\[
\left| (V^r_\pm)_x, (U^r_\pm)_x, (\Theta^r_\pm)_x \right| + |V^r_\pm - v^m_\pm| + |\Theta^r_\pm - \theta^m_\pm| \leq C\delta e^{-c_1(|x|+t)}.
\]

(iv) It holds that

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| (V^r_\pm, U^r_\pm, \Theta^r_\pm)(t, x) - (v^r_\pm, u^r_\pm, \theta^r_\pm) \left(\frac{x}{t}\right) \right| = 0.
\]

### 3 Proof of the main results

In this section, we shall prove the time-asymptotic stability of solutions of the Cauchy problem (1.2)-(1.3). Notice that when \((V^m_\pm, U^m_\pm, \Theta^m_\pm) = (V_\pm, U_\pm, \Theta_\pm)\), Theorem 1.2 will coincide with Theorem 1.1. Since Theorem 1.1 can be proved in the same way as that of Theorem 1.2, we only give here the proof of Theorem 1.2 for brevity.

#### 3.1 Reformulation of the problem

First, note that the viscous contact wave \((V^c, U^c, \Theta^c)\) satisfy (1.14), and the rarefaction waves \((V^r_\pm, U^r_\pm, \Theta^r_\pm)(x, t)\) solve the Euler equations

\[
\begin{cases}
(V^r_\pm)_t - (U^r_\pm)_x = 0, \\
(U^r_\pm)_t + p(V^r_\pm, \Theta^r_\pm)_x = 0, \\
\frac{R}{\gamma - 1}(\Theta^r_\pm)_t + p(V^r_\pm, \Theta^r_\pm)(U^r_\pm)_x = 0,
\end{cases}
\]
it is easy to check that the composite wave \((V, U, \Theta)(x, t)\) defined in (1.26) satisfy

\[
\begin{aligned}
V_t - U_x &= 0, \\
U_t + P_x &= -R_1, \\
\frac{R}{\gamma - 1} \Theta_t + PU_x &= \left(\frac{\kappa \Theta_x}{V}\right)_x - R_2,
\end{aligned}
\]

where

\[
R_1 = -(P - P_+ + P)_{x} + U^c_t := R^1_1 + U^c_t, \quad P = \frac{R \Theta}{V}, \quad P_\pm = \frac{R \Theta_{\pm}}{V_{\pm}},
\]

\[
R_2 = (p'' - P)U^c_x + (P_+ - P) \left( U^r_t \right)_x + (P_+ - P) \left( U^r_t \right)_x + \kappa \left( \frac{\Theta_x}{V} - \frac{\Theta^c_x}{V^c} \right)_x
\]

\[
:= R^1_2 + R^2_2.
\]

Set the perturbation \((\phi, \psi, \zeta)(t, x)\) by

\[
\phi(t, x) = v(t, x) - V(t, x), \quad \psi(t, x) = u(t, x) - U(t, x), \quad \zeta(t, x) = \theta(t, x) - \Theta(t, x),
\]

then it follows from (1.2) and (3.2) that

\[
\begin{aligned}
\phi_t - \psi_x &= 0, \\
\psi_t + \left( \frac{R \zeta - P \phi}{v} \right)_x &= R_1, \\
\frac{R}{\gamma - 1} \zeta_t + p(v, \theta) \psi_x + (p - P) U_x &= \kappa \left( \frac{\Theta_x}{V} - \frac{\Theta^c_x}{V^c} \right)_x + \frac{\omega^2}{v} + v \omega^2 + R_2, \\
\omega_t &= A \left[ \left( \frac{\omega_x}{v} \right)_x - v \omega \right]
\end{aligned}
\]

with the following initial and far-field conditions:

\[
\begin{aligned}
(\phi, \psi, \zeta)(0, x) &= (\phi_0, \psi_0, \zeta_0)(x) = (v - V, u - U, \theta - \Theta)(0, x), \\
(\phi, \psi, \zeta)(t, \pm \infty) &= 0.
\end{aligned}
\]

We seek the solutions of the Cauchy problem (3.3)-(3.4) in the following set of functions:

\[
X(0, t) = \{ \left( \phi, \psi, \zeta, \omega \right) \mid (\phi, \psi, \zeta, \omega) \in C([0, t], H^2(\mathbb{R})), \phi_x \in \text{L}^2([0, t], H^1(\mathbb{R})), \\
\psi, \zeta \in \text{L}^2([0, t], H^2(\mathbb{R})), \omega \in \text{L}^2([0, t], H^3(\mathbb{R})) \},
\]

where \(t > 0\) is a positive constant.

The local existence of the Cauchy problem (3.3)-(3.4) is standard, which can be obtained by the iteration technique (See [4]). Thus we omit its proof here for brevity.

To show the global existence of solutions to the Cauchy problem (3.3)-(3.4), it suffice to prove the following a priori estimates.

**Proposition 3.1 (A priori estimates).** Under the assumptions of Theorem 1.2, suppose that \((\phi, \psi, \zeta, \omega)(t, x) \in X(0, T)\) for some positive constant \(T > 0\), and satisfies the following a priori assumption:

\[
\sup_{0 \leq t \leq T} \| (\phi, \psi, \omega, \zeta)(t) \|_2 \leq \varepsilon
\]
for some small positive constant $\varepsilon$. Then there exist two small positive constants $\varepsilon_3$ and $\delta_3(\leq \min\{\delta, \delta_2, 1\})$ and a constant $C_0$ which are independent of $T$, such that if $0 < \varepsilon \leq \varepsilon_3$ and $0 < \delta = |\theta_+ - \theta_-| \leq \delta_3$, then it holds that

\[
\|\phi, \psi, \zeta, \omega(t)\|_2^2 + \int_0^t (\|\phi_x, \psi_x(\tau)\|_2^2 + \|\zeta_x(\tau)\|_2^2 + \|\omega(\tau)\|_2^2) \, d\tau \leq C_0 \left( \|\phi_0, \psi_0, \zeta_0, \omega_0\|_2^2 + \delta_3^2 \right)
\]  
(3.7)

for all $t \in [0, T]$.

### 3.2 A priori estimates

This subsection is devoted to proving Proposition 2.1, which follows a series of Lemmas below. First of all, we have from the a priori assumption (3.6) and the Sobolev inequality

\[
\|f\|_{L^\infty} \leq \|f\|_{H^1}^{1/2}, \quad \forall f(x) \in H^1(\mathbb{R})
\]  
(3.8)

that

\[
\|(\phi, \psi, \zeta, \phi_x, \psi_x, \zeta_x, w, w_x)(t)\|_{L^\infty(\mathbb{R})} \leq \varepsilon, \quad \forall t \in [0, T].
\]  
(3.9)

Furthermore, by the smallness of $\varepsilon$, we have

\[
0 < \frac{1}{2} \min\{v_-, v_+\} \leq v(t, x) = \phi(t, x) + V(t, x) \leq \frac{3}{2} \max\{v_-, v_+\},
\]  
(3.10)

and

\[
0 < \frac{1}{2} \min\{\theta_-, \theta_+\} \leq \theta(t, x) = \zeta(t, x) + \Theta(t, x) \leq \frac{3}{2} \max\{\theta_-, \theta_+\}.
\]  
(3.11)

For later use, we first give the time-decay estimates of the reminder terms $(R_1, R_2)$.

**Lemma 3.1.** Under the assumption Proposition 3.1, we have

\[
\begin{aligned}
\|R_1(t)\|_{L^\infty} &\leq C\delta(1 + t)^{-\frac{3}{2}}, \\
\|R_2(t)\|_{L^\infty} &\leq C(1 + t)^{-1}, \\
\|R_1(t)\|_{W^{2,1}} &\leq C\delta(1 + t)^{-1}, \\
\|R_2(t)\|_{W^{2,1}} &\leq C\delta^\frac{3}{4}(1 + t)^{-\frac{5}{2}}, \\
\|R_1(t)\|_2 &\leq C\delta(1 + t)^{-\frac{5}{4}}.
\end{aligned}
\]  
(3.12)

**Proof.** First, we have by a direct calculation that

\[
R_1^1 = \left( \frac{\Theta_R}{V^2} + \frac{\Theta^r}{V^2} + \frac{\Theta^c}{V^2} - \frac{\Theta}{V} \right)_x
\]  
(3.13)

\[
= \left( \frac{\Theta_R}{V^2} + \frac{\Theta^r}{V^2} + \frac{\Theta^c}{V^2} - \frac{\Theta^R + \Theta^r + \Theta^c - \Theta^m - \Theta^m}{V} \right)_x
\]  
(3.13)

\[
= (\Theta^R)_x \left( \frac{1}{V^2} - \frac{1}{V} \right) + (\Theta^r)_x \left( \frac{1}{V^2} - \frac{1}{V} \right) + \Theta^c \left( \frac{1}{V^2} - \frac{1}{V} \right)
\]  
(3.13)

\[
+ (V^r)_x \left( \Theta - \frac{\Theta_R}{V^2} \right) + (V^r)_x \left( \Theta^r - \frac{\Theta^r}{V^2} \right) + V^c \left( \Theta - \frac{\Theta^c}{V^2} \right).
\]
It follows from (1.26) and Lemma 2.3 that
\[ |(\Theta^c)_{x} (V^{-1} - (V^c)^{-1})| \leq C |((\Theta^c)_{x} (|V^c_t - v^m| + |V^c - v^m|)) \leq C |((\Theta^c)_{x} (|V^c_t - v^m| + |V^c - v^m|))_{\Omega} + C |((\Theta^c)_{x} (|V^c_t - v^m| + |V^c - v^m|))_{\Omega+\Omega_c} \leq C \delta \left[ (|V^c_t - v^m| + |V^c - v^m|)_{\Omega} + |((\Theta^c)_{x})_{\Omega+\Omega_c} \right] \leq C \delta^2 e^{-c_1(|x|+t)}.
\]

Similarly,
\[ |(\Theta^c)_{x} (V^{-1} - (V^c)^{-1})| + \left| \left( \frac{\Theta^c}{(V^c)^2} - \frac{\Theta^c}{V^2} \right) \right| \leq C \delta^2 e^{-c_1(|x|+t)}, \]
\[ |\Theta^c_x (V^{-1} - (V^c)^{-1})| + \left| \left( \frac{\Theta^c}{(V^c)^2} - \frac{\Theta^c}{V^2} \right) \right| \leq C \delta^2 e^{-c_1(|x|+t)}. \]

Consequently, it follows from (3.13)-(3.15) and (1.12) that
\[ |R_1| \leq |R_1^1| + |U^C_t| \leq C \delta^2 e^{-c_1(|x|+t)} + C \delta (1 + t)^{-\frac{3}{2}} e^{-\frac{ct^2}{t+1}}. \]

By the same argument as above, we have for $R_1^1$ that
\[ |R_2^1| \leq C \delta^2 e^{-c_1(|x|+t)}. \]

For $R_2^2$, we have
\[ R_2^2 = \left( \frac{(\Theta^c)_{x}}{V} + \frac{(\Theta^c)_{x}}{V} \right) + \left( \frac{\Theta^c}{V} - \frac{\Theta^c}{V} \right) \leq C \left\{ |(\Theta^c)_{xx}| + |(\Theta^c)_{xx}| + |(\Theta^c)_{x}|(|V^c|_{x}) + |(\Theta^c)_{x}|(V^c_t)_{x} \\
+ |(\Theta^c)_{x}|(|V^c_t| + |V^c_t| + |V^c_t| + |V^c_t|) \right\} + C \left\{ |(\Theta^c)_{xx}| + |(\Theta^c)_{x}|(|V^c_t| - v^m_t) + |V^c_t - v^m_t| + |(\Theta^c)_{x}|(|V^c_t| + |V^c_t|) \right\} \leq R_2^1 + R_2^2, \]

then Lemma 2.3 implies that
\[ \|R_2^1\|_{L^1} \leq C \delta^{\frac{3}{2}} (1 + t)^{-\frac{3}{2}}, \quad |R_2^1| \leq C (1 + t)^{-1}, \quad |R_2^2| \leq C \delta^2 e^{-c_1(|x|+t)}. \]

Thus we have from (3.16)-(3.19) that
\[ \|R_1(t)\|_{L^\infty} \leq C \delta (1 + t)^{-\frac{3}{2}}, \quad \|R_2(t)\|_{L^\infty} \leq C (1 + t)^{-1}, \]
\[ \|R_1(t)\|_{L^1} \leq C \delta (1 + t)^{-1}, \quad \|R_2(t)\|_{L^1} \leq C \delta^{\frac{3}{2}} (1 + t)^{-\frac{3}{2}}, \quad \|R_1(t)\|_{L^2} \leq C \delta (1 + t)^{-\frac{3}{2}}. \]

Similarly, we can also obtain
\[ \|R_1(t), R_1(t)\|_{L^1} \leq C \delta (1 + t)^{-1}, \quad \|(R_2(t), R_2(t))\|_{L^1} \leq C \delta^{\frac{3}{2}} (1 + t)^{-\frac{3}{2}}, \]
\[ \|(R_1(t), R_1(t))\|_{L^2} \leq C \delta (1 + t)^{-\frac{3}{2}}. \]

(3.12) thus follows from (3.20) and (3.21) immediately. This completes the proof of Lemma 3.1.

For the $L^2$ estimates on $\|\phi, \psi, \zeta, \omega\|_{L^2}$, we have
Lemma 3.2. Under the assumption Proposition 3.1, there exists a positive constant $C$ such that
\[
\|(\phi, \psi, \zeta, \omega)(t)\|^2 + \int_0^t \int_{\mathbb{R}} \left[ ((U^-)_x + (U^+_x)(\phi^2 + \zeta^2 + \omega^2 + \zeta_\zeta + \omega_x + \omega^2) \right] dx d\tau \leq C \left( \|(\phi_0, \psi_0, \zeta_0, \omega_0)\|^2 + \delta_{3}^2 + \delta_{\frac{1}{2}} \int_0^t \|(\phi_\tau, \psi_\tau)(\tau)\|^2 d\tau \right).
\] (3.22)

Proof. Multiplying (3.3) by $-R\Theta(\frac{1}{\theta} - \frac{1}{p})$, (3.3) by $\psi$, (3.3) by $\frac{\zeta}{\theta}$, (3.3) by $\omega$, and adding the resultant equations together, we have
\[
\left\{ \Theta \Phi \left( \frac{v}{V} \right) + \frac{1}{2} \phi^2 + \Theta \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\omega^2}{2} \right\}_t + \frac{\kappa \zeta^2}{\theta v} + Av\omega^2 + A \frac{\omega_x^2}{v} + H_{1x} + Q_1
\]
\[= R_1 \psi + R_2 \frac{\zeta}{\theta} + Q_2 + Q_3, \tag{3.23}
\]
where
\[
H_1 = (p - P)\varphi - \frac{\kappa \zeta}{\theta} \left( \frac{\zeta_x}{v} - \frac{\Theta x}{vV} \right) - \frac{Aww_x}{v},
\]
\[
Q_1 = -R\Theta_1 \Phi \left( \frac{v}{V} \right) + \frac{P U_x}{vV} \phi^2 + \frac{R\Theta_1}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\zeta}{\theta} (p - P) U_x,
\]
\[
Q_2 = \frac{\Theta_x}{\theta vV} \zeta_x + \frac{\kappa \zeta_x \phi}{\theta vV} \Theta_x - \frac{\kappa \zeta_x \phi}{\theta^2 vV} \Theta_x, \quad Q_3 = \frac{\zeta}{\theta} \left( \frac{\omega^2}{v} + v \omega^2 \right),
\]
\[
\Phi(s) = s - 1 - \ln s.
\]

Since (3.2) implies that
\[
-R\Theta_1 = (\gamma - 1) P_-(U^c)_x + (\gamma - 1) P_+(U^c)_x + (\gamma - 1) \left( P^m U^c_x - \left( \frac{\Theta x}{V^c} \right)_x \right)
\]
\[= (\gamma - 1) P((U^-)_x + (U^+_x)_x) + (\gamma - 1)(P_+ - P)(U^-)_x + (\gamma - 1)(P_+ - P)(U^+_x)_x - P^m U^c_x, \tag{3.25}
\]
we have
\[
Q_1 = -R\Theta_1 \left[ \Phi \left( \frac{v}{V} \right) - \frac{1}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) \right] + \left[ \frac{P \phi^2}{vV} + \frac{\zeta}{\theta} (p - P) \right] U_x
\]
\[= ((U^-)_x + (U^+_x)_x) Q_{11} + Q_{12}, \tag{3.26}
\]
where
\[
Q_{11} = (\gamma - 1) P(V, \Theta) \Phi \left( \frac{v}{V} \right) + \frac{P \phi^2}{vV} - P \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\zeta}{\theta} (p - P)
\]
\[= P \left( \Phi \left( \frac{\theta V}{\Theta v} \right) + \gamma \Phi \left( \frac{v}{V} \right) \right) \geq C(\phi^2 + \zeta^2), \tag{3.27}
\]
\[
Q_{12} = -U^c \left[ \frac{P(V, \Theta) \phi^2}{vV} - P^m \Phi \left( \frac{v}{V} \right) + \frac{P^m}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\zeta}{\theta} (p - P) \right]
\]
\[+ (\gamma - 1)(P - P_+)(U^-)_x \left( \Phi \left( \frac{v}{V} \right) - \frac{1}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) \right) \tag{3.28}
\]
\[+ (\gamma - 1)(P - P_+)(U^+_x)_x \left( \Phi \left( \frac{v}{V} \right) - \frac{1}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) \right). \]
It follows from (1.12), (3.9), Lemma 2.3 and the Cauchy inequality that

\[
|Q_{12}| + |Q_2| + |Q_3| \\
\leq \eta \zeta_x^2 + C_\eta \Theta_x^2 (\phi^2 + \zeta^2) + \epsilon (w^2 + w_x^2) + C \left( |U_\omega| + \delta ((U_\omega^+)_x + (U_\omega^+)_x) \right) (\phi^2 + \zeta^2) \\
\leq \eta \zeta_x^2 + \epsilon (w^2 + w_x^2) + C_\eta \delta ((U_\omega^+)_x + (U_\omega^+)_x) (\phi^2 + \zeta^2) + C \delta (1 + t)^{-1} e^{-\frac{\eta t^2}{1+\tau}} (\phi^2 + \zeta^2). \tag{3.29}
\]

Here and hereafter, \( \eta \) is a small positive constant and \( C_\eta > 0 \) is a constant depending on \( \eta \), and in (3.29), we have used the fact that

\[
|((\Theta^-)_x_1| = |\theta_-(v_-)^{\gamma-1}(1 - \gamma)(V^-_\tau)^{-\gamma}(V^-_\tau)_x| \\
= |\theta_-(v_-)^{\gamma-1}(1 - \gamma)(V^-_\tau)^{-\gamma} \frac{(U^-_\tau)_x}{\lambda_-(V^-_\tau, s_-)}| \leq C(U^-_\tau)_x.
\]

Moreover, we derive from the Sobolev inequality, the Young inequality and Lemma 3.1 that

\[
\left| \int_0^t \int_\mathbb{R} \left( R_1 \psi + R_2 \frac{\zeta}{\vartheta} \right) dx \, d\tau \right| \\
\leq C \int_0^t \| (\psi, \zeta)(\tau) \|^2 \| (\psi_x, \zeta_x)(\tau) \|^2 \| (R_1, R_2)(\tau) \|_{L^1} d\tau \\
\leq \delta^{\frac{1}{8}} \int_0^t \| (\psi_x, \zeta_x)(\tau) \|^2 d\tau + C \int_0^t \sup_{0 \leq \tau \leq t} \| (\phi, \zeta)(\tau) \|^2 \delta^{\frac{1}{8}} (1 + \tau)^{-\frac{7}{8}} d\tau \\
\leq \delta^{\frac{1}{8}} \int_0^t \| (\psi_x, \zeta_x)(\tau) \|^2 d\tau + C \delta^{\frac{1}{8}}. \tag{3.30}
\]

Integrating (3.23) in \( t \) and \( x \) over \([0, t] \times \mathbb{R}\), and using (3.29), (3.30), the Lemma 3.3 below and the smallness of \( \eta, \epsilon \) and \( \delta \), we can get (3.22). This completes the proof of Lemma 3.2.

**Lemma 3.3.** Under the assumption Proposition 3.1, there exist positive constants \( c_2 = \frac{C}{4} \) and \( C \) such that

\[
\int_0^t \int_\mathbb{R} (1 + \tau)^{-1} e^{-\frac{\eta t^2}{1+\tau}} (\phi^2 + \psi^2 + \zeta^2) dx \, d\tau \\
\leq C + C \int_0^t \| (\phi_x, \psi_x, \zeta_x, \omega_x, \omega)(\tau) \|^2 d\tau + C \int_0^t \int_\mathbb{R} ((U^-_\tau)_x + (U^+_\tau)_x) (\phi^2 + \zeta^2) dx \, d\tau. \tag{3.31}
\]

**Proof.** The proof of (3.31) is divided into the following two inequalities:

\[
\int_0^t \int_\mathbb{R} h^2 [(R \zeta - P \phi)^2 + \psi^2] \, dx \, d\tau \leq C + C \int_0^t \| (\phi_x, \psi_x, \zeta_x, \omega_x)(\tau) \|^2 + \| \omega(\tau) \|^2 \) \, d\tau \tag{3.32}
\]

and for any \( \eta > 0 \),

\[
\int_0^t \int_\mathbb{R} (R \zeta + (\gamma - 1) P \phi)^2 h^2 dx \, d\tau \leq C + C_\eta \int_0^t \| (\phi_x, \psi_x, \zeta_x, \omega_x)(\tau) \|^2 + \| \omega(\tau) \|^2 \| \omega(\tau) \|^2 \) \, d\tau + C(\delta + \eta) \int_0^t \int_\mathbb{R} h^2 (\psi^2 + \zeta^2) \, dx \, d\tau \tag{3.33}
\]

and

\[
\int_0^t \int_\mathbb{R} h^2 (\phi^2 + \zeta^2) dx \, d\tau \leq C + C_\eta \int_0^t \| (\phi_x, \psi_x, \zeta_x, \omega_x)(\tau) \|^2 + \| \omega(\tau) \|^2 \| \omega(\tau) \|^2 \) \, d\tau + C \int_0^t \int_\mathbb{R} (\phi^2 + \zeta^2)(U^-_\tau)_x + (U^+_\tau)_x dx \, d\tau.
\]
Indeed, if we add (3.33) to (3.32), then taking first \( \eta \) and then \( \delta \) suitably small, and noticing that

\[
[R\zeta + (\gamma - 1)P\phi]^2 + (R\zeta - P\phi)^2 \geq R^2\zeta^2 + 2(\gamma - 1)P^2\phi^2 \geq c_2 (\phi^2 + \zeta^2)
\]

with \( c_2 \) being a positive constant depending only on \( R, \gamma, \theta_+, \theta_-, v_+ \) and \( v_- \), we can get (3.31) immediately.

We first prove (3.32). To do so, denote

\[
f(t, x) = \int_{-\infty}^{x} h^2(t, y) dy,
\]

then we have

\[
\|f(t)\|_{L^\infty} \leq 2\alpha^{-\frac{1}{2}}(1 + t)^{-\frac{1}{2}}, \quad \|f_t(t)\|_{L^\infty} \leq 4\alpha^{-\frac{1}{2}}(1 + t)^{-\frac{3}{2}} \tag{3.34}
\]

Multiplying (3.3) by \((R\zeta - P\phi)vf\), and integrating the resulting equation over \( \mathbb{R} \) leads to

\[
\frac{1}{2} \int_\mathbb{R} (R\zeta - P\phi)^2 h^2 dx = \int_\mathbb{R} \psi_t (R\zeta - P\phi) vf dx - \int_\mathbb{R} R_1 (R\zeta - P\phi) vf dx
\]

\[
= \left( \int_\mathbb{R} \psi_t (R\zeta - P\phi) vf dx \right)_t - \int_\mathbb{R} \psi (R\zeta - P\phi) vf dx - \int_\mathbb{R} \psi (R\zeta - P\phi) vf dx
\]

\[
- \int_\mathbb{R} \psi (R\zeta - P\phi)vf dx - \int_\mathbb{R} v^{-1} (R\zeta - P\phi)^2 v_x f dx - \int_\mathbb{R} R_1 (R\zeta - P\phi) vf dx
\]

\[
:= \left( \int_\mathbb{R} \psi (R\zeta - P\phi) vf dx \right)_t + \sum_{i=1}^{5} L_i.
\]

We now turn to estimate \( L_i (1 \leq i \leq 5) \) term by term. Using equations (3.3)_1 and (3.3)_2, we have

\[
\left( \frac{R}{\gamma - 1} \zeta + P\phi \right)_t = -\frac{R\zeta - P\phi}{v} (\psi_x + U_x) + \kappa \left( \frac{V\zeta_x - \phi \Theta_x}{vV} \right)_x + P_t \phi + v\omega^2 + \frac{\omega_x^2}{v} + R_2, \tag{3.36}
\]

thus it holds that

\[
L_1 = -(\gamma - 1) \int_\mathbb{R} \psi vf \left( \frac{R}{\gamma - 1} \zeta + P\phi \right)_t dx + \gamma \int_\mathbb{R} (P\phi)_t vf \psi dx
\]

\[
= (\gamma - 1) \int_\mathbb{R} \psi f (R\zeta - P\phi)(U_x + \psi_x) dx + \kappa (\gamma - 1) \int_\mathbb{R} \frac{V\zeta_x - \phi \Theta_x}{vV} (\psi f)_x dx
\]

\[
- (\gamma - 1) \int_\mathbb{R} \psi vf \left( v\omega^2 + \frac{\omega_x^2}{v} + R_2 \right) dx + \frac{\gamma}{2} \int_\mathbb{R} rf (\psi^2)_x dx + \int_\mathbb{R} \psi f P_t \phi dx
\]

\[
:= \sum_{i=1}^{5} L_i. \tag{3.37}
\]

To control \( L_i (1 \leq i \leq 5) \) and \( L_i (1 \leq i \leq 5) \), we derive from the Cauchy inequality, the Sobolev inequality (3.8), the Young inequality, (1.12), (3.3), (3.9), (3.12) and (3.34) that

\[
|L_1 + L_2| \leq C \left| \int_\mathbb{R} \psi f (R\zeta - P\phi)(U_x + \psi_x) dx \right|
\]
\[ |L_4^2| \leq C \int_R |\zeta_x| + |\phi|(1+t)^{-\frac{1}{2}} |\psi_x f| + |\psi v f| + |\psi v f_x| \, dx \]
\[ \leq C \left( |\zeta_x(t)| + ||\phi(t)|| (1+t)^{-\frac{1}{2}} \right) \left( ||(\zeta_x, \phi_x)(t)|| (1+t)^{-\frac{1}{2}} + ||\psi(t)|| (1+t)^{-\frac{1}{2}} \right) \]
\[ \leq C ||(\phi_x, \psi_x, \zeta_x)(t)||^2 + C(1+t)^{-\frac{1}{2}}, \quad (3.39) \]

\[ |L_5| \leq C ||(f(t)||_{L^\infty} \int_R (\omega^2 + \omega_x^2 + |R_2|) \, dx \leq C \omega(t)||^2 + C(1+t)^{-\frac{11}{8}}, \quad (3.40) \]

\[ |L_1| \leq C \int_R \frac{1}{2} \phi^2 \psi^2 \, dx + C \int_R \frac{1}{2} P^2 \, dx \]
\[ \leq ||(\psi(t)||_{L^\infty} (1+t)^{-\frac{1}{2}} ||(\psi(t)||^2 + C(1+t)^{-\frac{1}{2}} \|(U^*_x, U^*_x, U^*_x)(t)\|^2 \]
\[ \leq C ||(\psi(t)|| ||(\zeta_x(t)|| (1+t)^{-\frac{1}{2}} + C(1+t)^{-\frac{1}{2}} \]
\[ \leq C ||(\psi(t)||^2 + C(1+t)^{-\frac{1}{2}}, \quad (3.41) \]

\[ \delta \leq C ||(\zeta(t)|| + ||(\theta(t)||) ||f(t)||_{L^\infty} \leq C (1+t)^{-\frac{3}{2}} \]

\[ |L_5| \leq C(1+t)^{-\frac{3}{2}} (||\zeta(t)||_{L^\infty} + ||\phi(t)||_{L^\infty}) R_1 \|_{L^1} \leq C (1+t)^{-\frac{3}{2}} \quad (3.45) \]

Integrating (3.35) over (0, t), and using the estimates (3.37) - (3.45), we have (3.32) holds.

Next we prove (3.33). Let \( \Gamma = R\zeta + (\gamma - 1)P\phi \), then from (3.36), we have

\[ \langle \Gamma_t, \Gamma g^2 \rangle_{H^{-1} \times H^1} = \int_R (R\zeta + (\gamma - 1)P\phi) \Gamma g^2 \, dx \]
Using (1.2), we derive

\[ \gamma - \frac{1}{v} (R\zeta - P\phi)\Gamma\psi_x g^2 dx - \frac{1}{v} (R\zeta - P\phi)\Gamma U_x g^2 dx \]

\[ -\kappa (\gamma - 1) \int_R \frac{V\zeta_x - \phi\Theta_x}{vV} (\Gamma g^2)_x dx + (\gamma - 1) \int_R P_t \phi g^2 dx \]

\[ + (\gamma - 1) \int_R (R_2 + v\omega^2 + \frac{\omega^2}{v}) \Gamma g^2 dx := \sum_{i=6}^{10} L_i. \]

Noticing that

\[ |U_x| \leq C \left| \left( \frac{\Theta_x}{\Theta} \right)_x \right| \leq C (|\Theta_{xx}| + \Theta^2) \leq C \left( \delta h^2 + \left| ((\Theta^+_{xx}, (\Theta^-_{xx}) + |((\Theta^+_{xx}, (\Theta^-_{xx}))|^2 \right) \right], \]

we derive

\[ |L_7| \leq C \int_R \phi^2 + \zeta^2 \delta h^2 dx + C \left( |(\phi, \zeta)(t)| |(\phi_x, \zeta_x)(t)||((\Theta^+_{xx}, (\Theta^-_{xx}))^2) (t)\right) \leq \frac{1}{\eta} \int_R \psi(t) + \phi(t) \right) \leq C(1 + t)^{-2}. \]

Using (1.12), Lemma 2.3 and the Cauchy inequality, we get

\[ |L_8| \leq C \left[ \int_R \left| v^{-1} \zeta_x \Gamma g^2 \right| dx + C \int_R \left| v^{-1} \zeta_x \Gamma g h \right| dx \right] \]

\[ + C \int_R \left| v^{-1} \phi \Theta_x \Gamma g^2 \right| dx + C \int_R \left| v^{-1} \phi \Theta_x \Gamma g h \right| dx \]

\[ \leq C(\delta + \eta) \int_R \left( \phi^2 + \zeta^2 \right) h^2 dx + \frac{C}{\eta} \left( \| \zeta_x(t) \|^2 + \| \phi_x(t) \|^2 \right) + C(1 + t)^{-2}. \]

\[ |L_9| \leq C \int_R \phi \left| \Gamma \left( (U^c_x) + (U^-) + (U^+_x) dx \right) \right| \]

\[ \leq C \int_R \left( \phi^2 + |\phi \zeta| \right) \left( (U^c_x) + (U^-) + (U^+_x) dx \right) \]

\[ \leq C \delta \int_R \left( \phi^2 + \zeta^2 \right) h^2 dx + C \int_R \left( \phi^2 + \zeta^2 \right) ((U^-) + (U^+_x) dx, \]

\[ |L_{10}| \leq C \int_R \left( \omega^2 + \omega_x^2 + |R_2| \right) \left| \Gamma \right| dx \]

\[ \leq C \|\omega(t)\|^2 + C \|\zeta(t)\|^2 + \|\zeta_x(t)\|^2 \right) \leq C(1 + t)^{-\frac{7}{2}}. \]

Finally, we estimate \( L_6. \) Noticing that \( R\zeta - P\phi = H - \gamma P\phi, \) we obtain by (3.3)1, (2.2) and (3.6) that

\[ \frac{-2L_6}{\gamma - 1} = 2 \int_R \left( \Gamma - \gamma P\phi \right) \Gamma \psi_x g^2 dx \]

\[ = \int_R \left( 2v^{-1} \Gamma^2 \phi g^2 - \gamma P v^{-1} (\phi^2 g^2) \right) dx \]

\[ = \left( \int_R \frac{2\Gamma - \gamma P \phi}{v} \Gamma g^2 \phi dx \right) - 2 \int_R \frac{2\Gamma - \gamma P \phi}{v} \Gamma gg_t \phi dx. \]
\[
\begin{align*}
+ \int_{\mathbb{R}} \frac{2\Gamma - \gamma P\phi g^2 u \phi dx - \int_{\mathbb{R}} \frac{4\Gamma - \gamma P\phi g^2 \phi dx}{v} + \int_{\mathbb{R}} v^{-1}\gamma P\phi \Gamma g^2 \phi dx &= \left( \int_{\mathbb{R}} \frac{2\Gamma - \gamma P\phi \Gamma g^2 \phi dx}{v} \right) + \frac{1}{2\alpha} \int_{\mathbb{R}} \frac{2\Gamma - \gamma P\phi \Gamma g h_x \phi dx}{v} \\
&+ \int_{\mathbb{R}} v^{-2} u g^2 \phi [\Gamma(2\Gamma - \gamma P\phi) + (\gamma - 1)(4\Gamma - \gamma P\phi)(R\zeta - P\phi)] \phi dx \\
&+ \kappa(\gamma - 1) \int_{\mathbb{R}} \frac{V\zeta_x - \phi \Theta x}{v\Psi_x} \left( g^2 \phi (4\Gamma - \gamma P\phi) \right) \phi dx \\
&- (\gamma - 1) \int_{\mathbb{R}} \frac{g^2 \phi}{v} (4\Gamma - \gamma P\phi)(v\omega^2 + \frac{\omega^2}{v} + R_2) \phi dx \\
&+ \int_{\mathbb{R}} v^{-1} P\phi g^2 [(4 - 3\gamma)\Gamma + \gamma(\gamma - 1)P\phi] \phi dx
\end{align*}
\]

\[
:= \left( \int_{\mathbb{R}} \frac{2\Gamma - \gamma P\phi \Gamma g^2 \phi dx}{v} \right) + \sum_{i=1}^{5} L_i^6.
\]

Similar to the estimates of \(L_i, i = 8, 9, 10\), we have

\[
|L_i^6| \leq C (1 + t)^{-1} \int_{\mathbb{R}} (|\zeta|^3 + |\phi|^3) dx
\]

\[
\leq C (1 + t)^{-1} \left( \||\zeta(t)||^2 \|\zeta_x(t)|| \|\zeta(t)||^2 \right) + \left( \||\phi(t)||^2 \|\phi_x(t)|| \|\phi(t)||^2 \right)
\]

(3.50)

\[
|L_i^5| + |L_i^6| \leq C \int_{\mathbb{R}} (|U_x| + |(\Theta_x)_x|) + |(\Theta_x^c)| + |(\Theta_x^i)| + |\psi_x|) \left( |\zeta|^3 + |\phi|^3 \right) dx
\]

\[
\leq C \left( (\zeta, \phi) \right) \left( ||\zeta_x, \phi_x(x, t)|| (|U_x| + |\psi_x|) + C \right) \int_{\mathbb{R}} (\phi^2 + \zeta^2) ((U_x^c, x) + (U_x^i, x)) dx
\]

\[
\leq C \left( (\zeta, \phi) \right)^2 + (1 + t)^{-\frac{3}{2}} + C \int_{\mathbb{R}} (\phi^2 + \zeta^2) ((U_x^c, x) + (U_x^i, x)) dx,
\]

(3.51)

\[
|L_i^4| \leq C \int_{\mathbb{R}} (|\zeta| + |\phi|) \left( \left( \|\zeta_x||^2 \right) \left( \|\phi_x||^2 \right) \right) \phi dx
\]

\[
\leq C \int_{\mathbb{R}} (|\zeta| + |\phi|) \left( \left( \|\zeta_x||^2 \right) \left( \|\phi_x||^2 \right) \right) dx
\]

\[
+ \int_{\mathbb{R}} (|\zeta| + |\phi|) \left( \left( \|\zeta_x||^2 \right) \left( \|\phi_x||^2 \right) \right) dx
\]

\[
\leq C \left( (\zeta, \phi) \right) \left( ||\zeta_x|| + ||\phi_x|| \right) + C (1 + t)^{-\frac{1}{2}} \left( \||\phi_x(t)||^2 \right) + C (1 + t)^{-\frac{1}{2}} \left( \||\phi_x(t)||^2 \right)
\]

\[
\leq C \left( (\zeta, \phi) \right) \left( ||\zeta_x|| \right) + C (1 + t)^{-\frac{1}{2}}
\]

(3.52)

\[
|L_i^3| \leq C \int_{\mathbb{R}} |\phi| \left( \left( |\zeta| + |\phi| \right) \right) \left( \left( \|\zeta_x||^2 \right) \left( \|\phi_x||^2 \right) \right) dx
\]

\[
\leq C \int_{\mathbb{R}} |\phi| \left( \left( |\zeta| + |\phi| \right) \right) \left( \left( \|\zeta_x||^2 \right) \left( \|\phi_x||^2 \right) \right) dx
\]

(3.53)

(3.36) thus follows from (3.46)-(3.53) and the smallness of \(\delta\) and \(\eta\). The proof of Lemma 3.3 is completed.

For the estimate of \(\|(\phi, \psi, \zeta, \omega_x)(t)\|\), we have
Lemma 3.4. Under the assumption Proposition 3.1, there exists a positive constant $C$ such that

$$
\|(\phi_x, \psi_x, \zeta_x, \omega_x)(t)\|^2 + \int_0^t \|((\zeta_{xx}, \omega_{xx}, \omega_x))(\tau)\|^2 d\tau \\
\leq C \left( \|(\phi_0, \psi_0, \zeta_0, \omega_0)\|^2 + \delta^\frac{1}{2} + (\delta^\frac{1}{2} + \varepsilon + \eta) \int_0^t \|((\phi_x, \psi_x)(\tau)\|^2 d\tau \right).
$$

(3.54)

Proof. Multiplying $(3.3)_{1x}$ by $\frac{P}{2} \phi_x$, $(3.3)_{2x}$ by $\psi_x$, $(3.3)_{3x}$ by $\frac{\zeta_x}{\theta}$, $(3.3)_{4x}$ by $\omega_x$, and adding the resultant equations together, we have

$$
\begin{align*}
& \left(\frac{P}{2v} \partial_t^2 + \frac{\psi_x^2}{2} + \frac{R \zeta_x^2}{2(\gamma - 1)} + \frac{\omega_x^2}{2}\right) + \frac{\kappa}{v \theta} \zeta_{xx}^2 + \frac{A}{v} \omega_{xx}^2 + \frac{\omega_x^2}{v} \zeta_x - \frac{H_2}{x} + J_1,
\end{align*}
$$

(3.55)

where

$$
H_2 = (p - P)_x \partial_t^2 + \frac{\zeta_x}{\theta} \left( (p - P) \partial_x U_x - \kappa \left( \frac{\zeta_x}{v} - \frac{\Theta_x \phi}{v^2} \right) \right) + \frac{\zeta_x u_x^2}{v \theta} + \frac{\omega_x^2}{v} \zeta_x - A \left( \frac{\omega_x}{v} \right) \omega_x,
$$

$$
J_1 = \left( \frac{\zeta_x}{\theta} \right)_x \left( - \kappa \left( \frac{\zeta_x}{v} - \frac{\Theta_x \phi}{v^2} \right) x + (p - P) U_x \right) + \frac{\kappa \Theta_x}{v \theta} \partial_x^2 \zeta_x + \frac{P}{v} \phi_x^2 + \frac{R \zeta_x^2}{2(\gamma - 1)} \left( \frac{1}{\theta} \right)_x
$$

$$
- p_x \zeta_x \frac{\zeta_x}{\theta} - Av_x \omega_x - Av_x \omega_x \left( \frac{1}{v} \right)_x + \left( \frac{R}{v} \right)_x \zeta_{xx} - \left( \frac{P}{v} \right)_x \phi_{xx}.
$$

Integrating (3.55) over $[0, t] \times \mathbb{R}$ yields

$$
\|(\phi_x, \psi_x, \zeta_x, \omega_x(t))\|^2 + \int_0^t \|((\zeta_{xx}, \omega_{xx}, \omega_x))(\tau)\|^2 d\tau
\leq \|(\phi_0, \psi_0, \zeta_0, \omega_0)\|^2 + \int_0^t \int_R |J_1| dx d\tau + \int_0^t \int_R (|R_{1x}| |\psi_x| + |R_{2x}| |\zeta_x|) dx d\tau.
$$

(3.56)

It follows from the Cauchy inequality and (3.9) that

$$
\int_0^t \int_R |J_1| dx d\tau \leq C(\eta + \varepsilon + \delta) \int_0^t \int_R \left( \phi_x^2 + \zeta_x^2 + \phi_{xx}^2 + \psi_x^2 + \psi_{xx}^2 + u_x^2 + w_{xx}^2 \right) dx d\tau
$$

$$
+ C \int_0^t \int_R ((\Theta_x, \Theta_{xx}))^2 (\phi_x^2 + \zeta_x^2) dx d\tau.
$$

(3.57)

Using (1.12), (3.9) and Lemma 2.3, we obtain

$$
J_1 \leq \int_0^t \int_R \left( \|(\Theta_x^+)_x, (\Theta_x^+_{xx})(\tau)\|^2 + |\Theta_x^+|, |\Theta_x^+_{xx}| \right)^2 (\phi_x^2 + \zeta_x^2) dx d\tau
\leq \delta \int_0^t \int_R \left( 1 + \tau \right)^{-1} e^{-\frac{c^2 t^2}{1 + \tau}} (\phi_x^2 + \zeta_x^2) dx d\tau + \int_0^t \delta^\frac{1}{2} (1 + \tau)^{-\frac{1}{2}} \sup_{0 \leq \tau \leq t} \|(\phi, \zeta)(\tau)\|^2 d\tau
\leq \delta \int_0^t \int_R \left( 1 + \tau \right)^{-1} e^{-\frac{c^2 t^2}{1 + \tau}} (\phi_x^2 + \zeta_x^2) dx d\tau + C \varepsilon^2 \delta^\frac{1}{2}.
$$

(3.58)
Similar to the estimate of (3.30), we have

$$\int_0^t \int_\mathbb{R} \left( |R_{1x}^* \psi_2| + |R_{2x}^* \zeta_2| \right) dx dt \leq \delta \frac{1}{2} \int_0^t \left( (\psi_{xx}^2, \zeta_{xx}) (\tau) \right) ^2 d\tau + C \delta \frac{1}{2}. \quad (3.59)$$

Combining (3.56)-(3.59) and using Lemmas 3.2-3.3 and the smallness of \( \eta, \varepsilon \) and \( \delta \) leads to (3.54) immediately. The proof of Lemma 3.4 is finished.

The next lemma give the estimate on \( \| (\phi_{xx}, \psi_{xx}, \zeta_{xx}, \omega_{xx}) (t) \| \).

**Lemma 3.5.** Under the assumption Proposition 3.1, there exists a positive constant \( C \) such that

$$\| (\phi_{xx}, \psi_{xx}, \zeta_{xx}, \omega_{xx}) (t) \|^2 + \int_0^t \| (\zeta_{xx}, \omega_{xx}, \omega_{xx}) (\tau) \|^2 d\tau \leq C \left( \| (\phi_0, \psi_0, \zeta_0, \omega_0) \|^2_2 + \delta \frac{1}{2} + (\delta \frac{1}{2} + \varepsilon + \eta) \int_0^t \| (\phi, \psi_3) (\tau) \|^2 d\tau \right). \quad (3.60)$$

**Proof.** Multiplying (3.3)_{1xx} by \( \frac{P}{2} \phi_{xx} \), (3.3)_{2xx} by \( \psi_{xx} \), (3.3)_{3xx} by \( \frac{\zeta_{xx}}{\theta} \), (3.3)_{4xx} by \( \omega_{xx} \), and adding the resultant equations together, we obtain

$$\begin{align*}
J_5 &= J_2 + J_3 + J_4 + J_5 + R_{1xx} \psi_{xx} + R_{2xx} \zeta_{xx} \theta,
\end{align*}$$

where

$$H_3 = (p - P)_{xx} \psi_{xx} + \frac{\zeta_{xx}}{\theta} \left( (p - P) U_x - \kappa \left( \frac{\zeta_x}{v} - \frac{\Theta_x \phi}{v V} \right) \right) - A \left( \frac{w_x}{v} \right)_{xx} \omega_{xx},$$

$$J_2 = \left( \frac{P}{2v} \right)_{xx} \phi_{xx}^2 + \left( \frac{R}{2(\gamma - 1) \theta} \right)_{xx} \zeta_{xx}^2 - (2 p_x \psi_{xx} + p _xx \psi_x) \frac{\zeta_{xx}}{\theta},$$

$$J_3 = \frac{\kappa \theta_x \zeta_{xx} \zeta_{xxx}}{\theta^2 v^2} + \kappa \left( \frac{\theta_x \zeta_{xx}}{\theta^2} - \frac{\zeta_{xxx}}{\theta} \right) \left[ \frac{-2 v_x \zeta_{xx}}{v^2} - \zeta_x \left( \frac{-v_x}{v^2} \right) \right] - \left( \frac{\Theta_x \phi}{v V} \right)_{xx} (p - P) U_x, \quad (3.61)$$

$$J_4 = \frac{\zeta_{xx}}{\theta} \left( \frac{w_x^2}{v} + \frac{w_x^2}{v} \right)_{xx} + 2 A w_{xxx} \frac{w_{xxx} v_x}{v^2} + A w_{xxx} \frac{v_x}{v^2} - A w_{xxx} v_{xx} - 2 A w_{xxx} w_{xx} v_x,$$

$$J_5 = \frac{R}{v} \left( \frac{P}{v} \right)_{xx} \zeta_{xx} \psi_{xx} - \left( \frac{P}{v} \right)_{xx} \phi_{xx} \psi_{xx} + 2 \left( \frac{R}{v} \right)_{xx} \zeta_{xx} \psi_{xx} - 2 \left( \frac{P}{v} \right)_{xx} \phi_{xx} \psi_{xx}.$$

The Cauchy inequality, (1.2), (1.12), (3.9) and Lemma 2.3 imply that

$$\int_0^t \int_\mathbb{R} \left( |J_2| + |J_3| + |J_4| + |J_5| \right) dx dt \leq C \varepsilon + \delta \int_0^t \left( \| (\zeta, \omega) (\tau) \|^2_2 + \| (\phi, \psi) (\tau) \|^2_2 \right) d\tau$$

$$+ C \int_0^t \int_\mathbb{R} \left( (\Theta_x^2 + \Theta_x^2) (\phi^2 + \zeta^2) \right) dx dt + \eta \int_0^t \| (w_{xx}, \zeta_x) (\tau) \|^2 d\tau$$

$$+ C \int_0^t \int_\mathbb{R} \left( |\zeta_{xx}| \right) ^2 + |(\zeta_{xx} w_{xxx}^2) | d\tau \int K_1$$

$$+ C \eta \int_0^t \int_\mathbb{R} \left( |W_x|^2 + |\Theta_x^2 \Theta_{xxx}^2| + |\Theta_x^4 \Theta_{xxx}^2| + \omega^2 V_{xxx}^2 \right) d\tau \int K_2. \quad (3.62)$$
Similarly, it holds that

\[ K_1 \leq \int_0^t \sup_{0 \leq \tau \leq t} \{ \| \zeta_{xx}^{(2)}(\tau) \|, \| \zeta_{xx}^{(2)}(\tau) \|, \} \, d\tau \]

\[ + \int_0^t \sup_{0 \leq \tau \leq t} \{ \| w_{xx}(\tau) \|, \| w_{xx}(\tau) \|, \} \, d\tau \]

\[ \leq C \varepsilon \int_0^t \| (\zeta_{xx}, w_{xx}) \|^2 \, d\tau, \]

\[ K_2 \leq \int_0^t \left( \| (V_+^{(2)}_{xx}, V_-^{(2)}_{xx})(\tau) \|_L^2 + \| \Theta_{xx}^{(2)}(\tau) \|_L^2 \right) \sup_{0 \leq \tau \leq t} \{ \| w(\tau) \|^2 \} \, d\tau \]

\[ + \int_0^t \left( \varepsilon^2 \delta^2 (1 + \tau)^{-\delta} + \delta^2 (1 + \tau)^{-\delta} \right) \, d\tau \]

\[ \leq C \int_0^t \left( \varepsilon^2 \delta^2 (1 + \tau)^{-\delta} + \delta^2 (1 + \tau)^{-\delta} \right) \, d\tau \]

\[ \leq C \delta^2. \]

To estimate \( J_5 \), we have from integration by parts and (3.3) that

\[
\int_0^t \int_\mathbb{R} \left( \frac{P}{v} \right)_{xx} \phi \psi_{xxx} \, dx \, d\tau = \int_0^t \int_\mathbb{R} \left\{ -\frac{P\phi}{2v^{2}} \right\}_{t} \phi_{xx} \, dx \, d\tau + \int_0^t \int_\mathbb{R} \left\{ -\frac{P\phi}{2v^{2}} \right\}_{t} \phi_{xx} \, dx \, d\tau \\
- \int_0^t \int_\mathbb{R} \left\{ \phi \left( \frac{P_{xx}}{v} + 2P_{x} \left( \frac{1}{v} \right) x + P \left( \frac{2v^{2}_{x}}{v^{3}} - \frac{V_{xx}}{v^{2}} \right) \right) \right\}_{x} \, dx \, d\tau \\
\leq - \int_\mathbb{R} \frac{P\phi}{2v^{2}} \phi_{xx} \, dx + \int_\mathbb{R} \frac{P(\Theta_{0})\phi_{0}}{2v^{2}} \phi_{0xx} \, dx + \eta \int_0^t \| \psi_{xx}(\tau) \|^2 \, d\tau \\
+ C(\delta + \varepsilon) \int_0^t \int_\mathbb{R} \left( \frac{P_{xx}}{v} + \phi_{x}^{2} + \phi_{xx}^{2} \right) \, dx \, d\tau + C_{\eta} \int_0^t \int_\mathbb{R} \theta_{x}^{2} \phi_{xx} \, dx \, d\tau \\
\leq C \left( \varepsilon \| \phi_{xx}(t) \|^2 + \| \phi_{0xx} \|^2 \right) + C(\delta + \varepsilon) \int_0^t \left( \| \psi_{xx}, \phi_{x}, \phi_{xx} \right(\tau) \|^2 \, d\tau \\
+ \eta \int_0^t \| \psi_{xx}(\tau) \|^2 \, d\tau + C_{\eta} \int_0^t \int_\mathbb{R} \theta_{x}^{2} \phi_{xx} \, dx \, d\tau. \]

Similarly, it holds that

\[
\int_0^t \int_\mathbb{R} \left( \frac{R}{v} \right)_{xx} \zeta \psi_{xxx} \, dx \, d\tau \\
\leq C \left( \varepsilon \| \phi_{xx}(t) \|^2 + \| \phi_{0xx} \|^2 \right) + \eta \int_0^t \left( \| \zeta_{xx}, \zeta_{xxx}, \psi_{xx} \right(\tau) \|^2 \, d\tau \\
+ C(\delta + \varepsilon) \int_0^t \left( \| \psi_{xx}, \zeta_{x}, \phi_{x} \right(\tau) \|^2 \, d\tau + C_{\eta} \int_0^t \int_\mathbb{R} \left( \varepsilon^2 \phi_{xx}^{2} + \Theta_{x}^{2} \phi_{xx}^{2} \right) \, dx \, d\tau,
\]

and

\[
\int_0^t \int_\mathbb{R} \left( \frac{R}{v} \right)_{x} \zeta_{x} \psi_{xxx} - \left( \frac{P}{v} \right)_{x} \phi_{x} \psi_{xxx} \, dx \, d\tau \leq C(\delta + \varepsilon) \int_0^t \left( \| \phi_{x}, \zeta_{x}, \zeta_{xxx}, \psi_{xx}, \phi_{xx} \right(\tau) \|^2 \, d\tau. \]
Therefore, it follows from (3.65)-(3.67) that

$$
\int_0^t \int_\mathbb{R} |J_5| dx \, d\tau \leq C(\delta + \varepsilon) \int_0^t \| (\phi, \zeta, \zeta, \psi, \phi_x) (\tau) \|^2 \, d\tau + \eta \int_0^t \| (\zeta, \zeta, \zeta, \psi_x) (\tau) \|^2 \, d\tau \\
+ C \left( \varepsilon \| \phi_x (t) \|^2 + \| \phi_{0xx} \|^2 \right) + C \eta \int_0^t \int_\mathbb{R} \left( \varepsilon^2 \phi_x^2 + \Theta \left( \zeta^2 + \phi^2 \right) \right) \, dx \, d\tau.
$$

(3.68)

Finally, by using Lemma 3.1, the Sobolev inequality and the Young inequality, we have

$$
\int_0^t \int_\mathbb{R} \left| R_{1xx} \frac{\zeta_{xx}}{\theta} + R_{2xx} \frac{\zeta_{xx}}{\theta} \right| \, dx \, d\tau \\
\leq C \int_0^t \left( \int \frac{\| \psi_{xx}(\tau) \|}{\| R_{1xx}(\tau) \|} \right) + \left[ \int \| \zeta_{xx}(\tau) \|^2 \| \zeta_{xx}(\tau) \| \| \zeta_{xx}(\tau) \| \| R_{2xx}(\tau) \|_{L^1} \right] \, d\tau \\
\leq C \int_0^t \left( \| \psi_{xx}(\tau) \| \delta (1 + \tau)^{-\frac{1}{2}} + \| \zeta_{xx}(\tau) \| \frac{1}{2} \| \zeta_{xx}(\tau) \| \frac{1}{2} \delta \tau \frac{1}{2} (1 + \tau)^{-\frac{3}{2}} \right) \, d\tau \\
\leq C \delta \frac{1}{2} \int_0^t \left( \| \psi_{xx}, \zeta_{xx}, \zeta_{xx} \| \right) \| \zeta_{xx} \| \, d\tau + C \delta \frac{1}{2}.
$$

(3.69)

Integrating (3.61) over [0, t] × \mathbb{R}, and using (3.62)-(3.64), (3.68)-(3.69), Lemmas 3.2-3.4 and the smallness of \( \eta, \varepsilon \) and \( \delta \), we get (3.60) immediately. This completes the proof of Lemma 3.6.

As a direct consequence of Lemmas 3.2-3.5, we have the following corollary.

**Corollary 3.1.** Under the assumption Proposition 3.1, there exists a positive constant \( C \) such that

$$
\| (\phi, \psi, \omega, \zeta) (t) \|^2 + \int_0^t \left( \| \zeta_{xx}(\tau) \|^2 + \| \omega(\tau) \|^2 \right) \, d\tau \\
\leq C \left( \left( \| (\phi, \psi, \omega, \zeta_0) \|^2 + \delta \frac{1}{2} + \left( \delta \frac{1}{2} + \varepsilon + \eta \right) \right) \right) \int_0^t \| (\phi, \psi_x) (\tau) \|^2 \, d\tau.
$$

(3.70)

For the reminder term \( \int_0^t \| (\phi_x, \psi_x) (\tau) \|^2 \, d\tau \) in (3.70), we have

**Lemma 3.6.** Under the assumption Proposition 3.1, there exists a positive constant \( C \) such that

$$
\int_0^t \| (\phi, \psi_x) (\tau) \|^2 \, d\tau \leq C \left( \left( \| (\phi, \psi, \zeta_0) \|^2 + \delta \frac{1}{2} \right) \right) .
$$

(3.71)

**Proof.** Multiplying (3.3) by \(-\frac{P}{2} \phi_x, (3.3)\) by \( \psi_x \), and adding the resultant equations together, we have

$$
\begin{align*}
&\left\{ \frac{R}{\gamma - 1} \zeta \psi_x - \frac{P}{2} \phi_x \psi_x \right\}_t + \left\{ \frac{P}{2} \phi_x \psi - \frac{R}{\gamma - 1} \zeta \psi_x \right\}_x + \frac{P^2}{2} \psi_x^2 + \frac{P}{2} \psi_x^2 \\
&= \frac{P}{2} \psi_x^2 - \frac{R}{2} \phi_x \psi + \frac{P}{2} \phi_x \left( \frac{R \zeta_v}{v} - \frac{R \phi_x v_x}{v^2} - R_1 \right) - \frac{R}{\gamma - 1} \zeta \psi_x \\
&+ \kappa \left( \frac{\zeta}{v} - \frac{\Theta \phi_x}{v^2} \right) \psi_x - (p - P)(U_x + \psi_x) \psi_x + R_2 \psi_x + \frac{\omega}{v} \psi_x + v \omega \psi_x.
\end{align*}
$$

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Integrating (3.72) on $[0, t] \times \mathbb{R}$ yields

$$
\int_0^t \int_{\mathbb{R}} (\phi_x^2 + \psi_x^2) \, dx \, d\tau \leq C \|((\phi, \psi, \zeta, \omega)(t))\|_2^2 + C \|((\phi_0, \psi_0, \zeta_0, \omega_0))\|_2^2 + C \int_0^t \|((\zeta_x, \omega_x)(\tau))\|_2^2 \, d\tau \\
+ \left(\frac{1}{4} + \delta + \varepsilon\right) \int_0^t \int_{\mathbb{R}} (\phi_x^2 + \psi_x^2) \, dx \, d\tau + C \int_0^t \int_{\mathbb{R}} (|\Theta_x| + |\Theta_{xx}|)^2 (\phi^2 + \psi^2) \, dx \, d\tau \\
+ C \int_0^t \int_{\mathbb{R}} |R_1 \phi_x + R_2 \psi_x| \, dx \, d\tau.
$$

(3.73)

Similar to (3.69), we have

$$
\int_0^t \int_{\mathbb{R}} |R_1 \phi_x + R_2 \psi_x| \, dx \, d\tau \leq C \int_0^t \left( \|\phi_x(\tau)\|_2 \|R_1(\tau)\|_2 + \|\psi_x\|_2^\frac{3}{2} \|\psi_x\|_2 \|R_2(\tau)\|_L^2 \right) \, d\tau \\
\leq C \int_0^t \left( \|\phi_x(\tau)\|_2 \|\delta(1 + \tau)^{-\frac{5}{2}} + \|\psi_x(\tau)\|_2 \|\psi_{xx}(\tau)\|_2 \delta^\frac{3}{2} (1 + \tau)^{-\frac{7}{2}} \right) \, d\tau \\
\leq C \delta^\frac{3}{2} \int_0^t \|\phi_x, \psi_x, \psi_{xx}(\tau)\|_2^2 \, d\tau + C \delta^\frac{3}{2}.
$$

(3.74)

Combining (3.73)-(3.74), and using (3.31), (3.58) and (3.70), we get by the smallness of $\varepsilon, \delta$ and $\eta$ that

$$
\int_0^t \int_{\mathbb{R}} (\phi_{xx}^2 + \psi_{xx}^2) \, dx \, d\tau \leq C \left( \|((\phi_0, \psi_0, \zeta_0, \omega_0))\|_2^2 + \delta^\frac{3}{2} \right) + C(\varepsilon + \delta^\frac{3}{2}) \int_0^t \|((\phi_{xx}, \psi_{xx})(\tau))\|_2^2 \, d\tau.
$$

(3.75)

Similarly, by multiplying (3.3)_{2x} by $-\frac{P}{\tau} \phi_{xx}$, (3.3)_{3x} by $\psi_{xx}$, and repeating the same argument as above, we can also obtain

$$
\int_0^t \int_{\mathbb{R}} (\phi_{xx}^2 + \psi_{xx}^2) \, dx \, d\tau \leq C \left( \|((\phi_0, \psi_0, \zeta_0, \omega_0))\|_2^2 + \delta^\frac{3}{2} \right) + C(\varepsilon + \delta^\frac{3}{2}) \int_0^t \|((\phi_x, \psi_x)(\tau))\|_2^2 \, d\tau.
$$

(3.76)

(3.71) thus follows from (3.75)-(3.76) and the smallness of $\varepsilon$ and $\delta$. This completes the proof of Lemma 3.6.

**Proof of Proposition 3.1.** Proposition 3.1 follows from Corollary 3.1, Lemma 3.6, and the smallness of $\varepsilon, \delta$ and $\eta$ immediately.

**Acknowledgement**

This work was supported by the National Natural Science Foundation of China (Grant No. 11501003), and the Cultivation Fund of Young Key Teacher at Anhui University.

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