A NOTE ON THE WEAK LEFSCHETZ PROPERTY OF MONOMIAL COMPLETE INTERSECTIONS IN POSITIVE CHARACTERISTIC

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Abstract. Let $K$ be an algebraically closed field of characteristic $p > 0$. We apply a theorem of C. Han to give an explicit description for the weak Lefschetz property of the monomial Artinian complete intersection $A = K[X,Y,Z]/(X^d, Y^d, Z^d)$ in terms of $d$ and $p$. This answers a question of J. Migliore, R. M. Miró-Roig and U. Nagel and, equivalently, characterizes for which characteristics the rank-2 syzygy bundle $\text{Syz}(X^d, Y^d, Z^d)$ on $\mathbb{F}^2$ satisfies the Grauert-Mülich theorem. As a corollary we obtain that for $p = 2$ the algebra $A$ has the weak Lefschetz property if and only if $d = \left\lfloor \frac{2t+1}{3} \right\rfloor$ for some positive integer $t$. This was recently conjectured by J. Li and F. Zanello.

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1. Introduction

Let $R = K[X_0, \ldots, X_N]$ be the polynomial ring in $N + 1$ variables over an algebraically closed field $K$ and let $f_1, \ldots, f_n$ denote $R_+$-primary homogeneous polynomials in $R$ (i.e., $\sqrt{(f_1, \ldots, f_n)} = R_+$). Then the quotient $A := R/(f_1, \ldots, f_n)$ is an Artinian graded $K$-algebra, i.e., $A$ is of the form $A = K \oplus A_1 \oplus \ldots \oplus A_s$ for some integer $s \geq 0$. The algebra $A$ has the weak Lefschetz property (abbreviated by WLP) if for every general linear form $\ell \in R_1$ the multiplication maps $A_m \xrightarrow{\ell} A_{m+1}$ have maximal rank for $m = 0, \ldots, s - 1$.

We also associate to the polynomials $f_1, \ldots, f_n$ the syzygy bundle on $\mathbb{P}^N = \text{Proj} R$. This vector bundle is given by the short exact sequence

$$0 \rightarrow \text{Syz}(f_1, \ldots, f_n) \rightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(-d_i) \rightarrow \mathcal{O}_{\mathbb{P}^N} \rightarrow 0,$$
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where \( d_i := \deg(f_i) \). If \( N = 2 \) and \( \text{char}(K) = 0 \), we gave in our article [2] a characterization for the weak Lefschetz property of the Artinian algebra \( A \) in terms of the generic splitting type of the syzygy bundle \( \text{Syz}(f_1, \ldots, f_n) \) (see [2, Theorem 2.2]). As a consequence we obtained, using the theorem of Grauert-Mülich (see [6, Theorem 3.0.1]), the result of Harima-Migliore-Nagel-Watanabe saying that every Artinian complete intersection in \( K[X, Y, Z] \) has the weak Lefschetz property (see [5, Theorem 2.4] and [2, Corollary 2.4]). The easy examples of the stable syzygy bundles \( S = \text{Syz}(X^p, Y^p, Z^p) \) over a field of characteristic \( p \) show that neither Grauert-Mülich (\( S \) splits on every line \( L \subset \mathbb{P}^2 \) as \( S|_L \cong O_L(-p) \oplus O_L(-2p) \); cf. also the example of L. Ein in [3, Section 4]) nor the theorem of Harima et al. holds in positive characteristic (cf. [8, Example 7.10]).

The aim of this paper is to give a numerical characterization of the WLP for monomial Artinian complete intersections \( K[X, Y, Z]/(X^d, Y^d, Z^d) \) in positive characteristic. This answers [8, Question 7.12] of Migliore-Miró-Roig-Nagel and, equivalently, characterizes for which characteristics the rank-2 syzygy bundle \( \text{Syz}(X^d, Y^d, Z^d) \) on \( \mathbb{P}^2 \) satisfies the Grauert-Mülich theorem. As a consequence we obtain a proof for the recent conjecture [7, Conjecture 3.9] of J. Li and F. Zanello.

Besides our geometric approach, the key ingredient for our investigation is a theorem of C. Han which computes the syzygy gap introduced in [9] by P. Monsky. Let \( K \) be an algebraically closed field and consider the ideal \( I := (X^{d_1}, Y^{d_2}, (X + Y)^{d_3}) \) in \( S := K[X, Y] \). The minimal graded free resolution of the quotient \( S/I \) is given by

\[
0 \longrightarrow S(a) \oplus S(b) \longrightarrow S(-d_1) \oplus S(-d_2) \oplus S(-d_3) \longrightarrow S \longrightarrow S/I \longrightarrow 0,
\]
with integers \(a, b, a \geq b\). The difference \(\delta(d_1, d_2, d_3) := a - b\) is called the \textit{syzygy gap} and constitutes a function \(\delta : \mathbb{N}^3 \to \mathbb{N}\). It is easy to see that \(a + b = -(d_1 + d_2 + d_3)\) and hence \(\delta(d_1, d_2, d_3) \equiv d_1 + d_2 + d_3 \mod 2\).

\textbf{Corollary 2.2.} Let \(K\) be an algebraically closed field (of any characteristic), \(A = K[X, Y, Z]/(X^d, Y^d, Z^d)\) and denote by \(\mathcal{S} = \text{Syz}(X^d, Y^d, Z^d)\) the corresponding syzygy bundle. Then the following conditions are equivalent.

1. The algebra \(A\) has the weak Lefschetz property.
2. The bundle \(\mathcal{S}\) splits on a generic line \(L\) as \(\mathcal{S}|_L \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)\) with \(a \geq b\) and \(0 \leq a - b \leq 1\) (i.e., the theorem of Grauert-Mülich holds).
3. We have \(\delta(d, d, d) \leq 1\).

\textit{Proof.} The equivalence (1) \(?\) (2) is proved in Lemma 2.1.

(2) \(?\) (3). If we want to compute the splitting type of \(\mathcal{S}\) on a line \(L\) given by the equation \(Z = uX + vY\) with coefficients \(u, v \in K\), \(u, v \neq 0\) (in particular this holds for a generic line), we can assume without loss of generality that \(u = v = 1\). Hence computing the generic splitting type of \(\mathcal{S}\) is the same as computing the syzygy gap \(\delta(d, d, d) = a - b\).

We denote by \(\delta^* : [0, \infty)^3 \to [0, \infty)\) the continuous continuation of \(\delta\); see [9, Definition 19 and the following] for this function and some of its properties.

We set \(L_{\text{odd}} := \{(u_1, u_2, u_3) \in \mathbb{Z}^3 : \sum_{i=1}^3 u_i \text{ odd} \} \subset \mathbb{Z}^3\).

An element \(u = (u_1, u_2, u_3) \in \mathbb{Z}^3\) belongs to \(L_{\text{odd}}\) if and only if all entries of \(u\) are odd or if there is only one odd entry \(u_i\), \(i \in \{1, 2, 3\}\). Further, we denote by \(\text{td}\) the \textit{taxi-cab distance} in \(\mathbb{R}^3\) defined as \(\text{td}(v, w) := \sum_{i=1}^3 |v_i - w_i|\) for triples \(v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3\).

The following theorem due to C. Han yields an effective way to compute \(\delta^*\) for a given triple \(v = (v_1, v_2, v_3) \in [0, \infty)^3\).

\textbf{Theorem 2.3 (Han).} Let \(K\) be an algebraically closed field of characteristic \(p > 0\) and assume the entries of \(v = (v_1, v_2, v_3) \in [0, \infty)^3\) satisfy \(v_1 \leq v_2 \leq v_3\) and \(v_3 < v_1 + v_2\). If there exists \(s \in \mathbb{Z}\) and a triple \(u = (u_1, u_2, u_3) \in L_{\text{odd}}\) such that \(m := \text{td}(p^s v, u) < 1\), then there exists such a pair \(s, u\) with minimal \(s\). With these data \(s, u\) and \(m\) we have

\[\delta^*(v) = p^{-s}(1 - m).\]

If no such pair exists, then \(\delta^*(v) = 0\).

\textit{Proof.} See [4, Theorems 2.25 and 2.29] or [9, Corollary 23] for an easier proof.

\textbf{Lemma 2.4.} Let \(d \in \mathbb{N}_+\) and \(p\) be a prime number. Then the following conditions are equivalent.
(1) There exists \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \) such that
\[
\frac{3d}{6k + 2} > p^n > \frac{3d}{6k + 4}.
\]

(2) There exists an odd number \( u \in \mathbb{N} \) and \( s \in \mathbb{Z} \), \( s \leq 0 \), such that
\[
u - \frac{1}{3} < dp^s < u + \frac{1}{3}.
\]

(3) There exists an integer \( s \), \( s \leq 0 \), such that the taxi-cab distance of \((dp^s, dp^s, dp^s)\) to some point in \( L_{\text{odd}} \) is < 1.

Proof. To proof the equivalence between (1) and (2) we set \( s = -n \) and \( u = 2k + 1 \). The condition in (1) is equivalent with
\[
\frac{3}{3u - 1} > \frac{p^n}{d} > \frac{3}{3u + 1}
\]
and by inverting it is equivalent with
\[
u - \frac{1}{3} < dp^s < u + \frac{1}{3}.
\]

If (2) is true, then we have \((u, u, u) \in L_{\text{odd}}\) and the taxi-cab distance between \((dp^s, dp^s, dp^s)\) and \((u, u, u)\) is < 1. On the other hand, the distance of a point on the diagonal to any point in \( L_{\text{odd}} \) outside the diagonal is at least 1, so we only have to consider points on the diagonal. \(\square\)

Lemma 2.5. Let \( d \in \mathbb{N}_+ \) and \( p \) be a prime number. Suppose that there exists \( 0 \leq n' < n \) and \( k', k \in \mathbb{N} \) such that
\[
\frac{3d - 1}{6k' + 2} > p^{n'} > \frac{3d + 1}{6k' + 4}
\]

and
\[
\frac{3d}{6k + 2} > p^n > \frac{3d}{6k + 4}.
\]

Then
\[
\frac{3d - 1}{6k + 2} > p^n > \frac{3d + 1}{6k + 4}.
\]

Proof. Otherwise we would have either
\[
\frac{3d + 1}{6k + 4} \geq p^n > \frac{3d}{6k + 4}
\]
or
\[
\frac{3d}{6k + 2} > p^n \geq \frac{3d - 1}{6k + 2}.
\]
This gives either
\[p^n(6k + 4) = 3d + 1\]
or
\[p^n(6k + 2) = 3d - 1.\]
We plug this in the first inequality and get in the first case
\[
\frac{p^n(6k + 4) - 2}{6k' + 2} > p^{n'} > \frac{p^n(6k + 4)}{6k' + 4}
\]
and by dividing through \( p^n \) we get
\[
\frac{3k + 2 - \frac{1}{p^n}}{3k' + 1} > \frac{p^{n'-n}}{3k' + 2} > \frac{3k + 2}{3k' + 2}.
\]
By inverting we obtain
\[
\frac{3k' + 1}{3k + 2 - \frac{1}{p^n}} < \frac{p^{n-n'}}{3k' + 2} < \frac{3k' + 1}{3k + 2}.
\]
From the right hand side we get \( p^{n-n'} \leq \frac{3k' + 1}{3k + 2} \) which yields the contradiction
\[
\frac{3k' + 1}{3k + 2 - \frac{1}{p^n}} < \frac{3k' + 1}{3k + 2}.
\]
In the second case we obtain
\[
\frac{p^n(6k + 2)}{6k' + 2} > p^{n'} > \frac{p^n(6k + 2) + 2}{6k' + 4}
\]
and similar manipulations yield a contradiction. \( \square \)

The following theorem gives an explicit answer to [8, Question 7.12]. This question was also answered in [7, Corollary 3.6] but in a less explicit way.

**Theorem 2.6.** Let \( K \) be a field of characteristic \( p > 0 \) and consider the monomial Artinian complete intersection \( A := K[X, Y, Z]/(X^d, Y^d, Z^d) \). Then the following holds:

1. If \( d \) is even, then \( A \) does not have the weak Lefschetz property if and only if there exists a \( k \in \mathbb{N} \) and an \( n \in \mathbb{N}_+ \) such that
   \[
   \frac{3d}{6k + 2} > p^n > \frac{3d}{6k + 4}.
   \]
2. If \( d \) is odd, then \( A \) does not have the weak Lefschetz property if and only if there exists a \( k \in \mathbb{N} \) and an \( n \in \mathbb{N}_+ \) such that
   \[
   \frac{3d - 1}{6k + 2} > p^n > \frac{3d + 1}{6k + 4}.
   \]

**Proof.** We prove (1). Assume that we have
\[
\frac{3d}{6k + 2} > p^n > \frac{3d}{6k + 4}
\]
for some \( k \in \mathbb{N} \) and \( n \in \mathbb{N}_+ \). We set \( s := -n \), \( u := 2k + 1 \). Then we have \( m := \text{td}(p^s(d, d, d), (u, u, u)) < 1 \) by Lemma 2.4 and hence \( \delta^s(d, d, d) = p^{-s}(1 - m) > 0 \). Since \( \delta^s(d, d, d) = a - b \) and \( a + b = -3d \) we must have \( \delta^s(d, d, d) \geq 2 \). We apply Corollary 2.2 and see that \( A \) does not have the WLP.
Now we assume that the numerical condition does not hold. Then by Lemma 2.4 there is no \( s \leq 0 \) such that the taxi-cab distance from \( p^s(d, d, d) \) to an element \((u, u, u) \in L_{\text{odd}}\) is < 1. This is also true for \( s > 0 \) since \( d \) is even. Hence it follows from Han’s Theorem 2.3 that \( \delta^*(d, d, d) = 0 \) which implies by Corollary 2.2 the WLP for the algebra \( A \).

Next we prove (2). First we remark that, since \( d \) is odd, the condition

\[
\frac{3d}{6k + 2} > p^n > \frac{3d}{6k + 4}
\]

is always fulfilled for \( n = 0 \) and \( k \) such that \( d = 2k + 1 \). We choose \( n > 0 \) maximal such that

\[
\frac{3d}{6k + 2} > p^n > \frac{3d}{6k + 4}
\]

holds for some \( k \). Hence we can apply Han’s Theorem 2.3 with \( s := -n \) (minimal) and \( u := 2k + 1 \) to compute the syzygy gap.

Suppose that the numerical condition of part (2) is fulfilled for some \( k' \in \mathbb{N} \) and \( n' \in \mathbb{N}_+ \). According to Lemma 2.5 we may assume that this condition also holds for the chosen (maximal) \( n \), hence

\[
\frac{3d - 1}{6k + 2} > p^n > \frac{3d + 1}{6k + 4}.
\]

Then we have in particular

\[
u - \frac{1}{3} = \frac{6k + 2}{3} < dp^s < \frac{6k + 4}{3} = u + \frac{1}{3}
\]

by Lemma 2.4. Now we distinguish two cases.

**Case 1:** Let \( u > dp^s \). Then the taxi-cab distance from \( p^s(d, d, d) \) to the element \((u, u, u) \in L_{\text{odd}}\) equals

\[
m := \text{td}(p^s(d, d, d), (u, u, u)) = 3(u - dp^s)
\]

and we have \( m < 1 \) (by Lemma 2.4). So we obtain for the syzygy gap:

\[
\delta^*(d, d, d) = p^{-s}(1 - m) = p^{-s}(1 - 3u + 3dp^s)
\]

\[
= p^{-s}(1 - 3u) + 3d = -p^n(6k + 2) + 3d
\]

\[
> -(3d - 1) + 3d = 1.
\]

Therefore the syzygy gap is indeed \( \geq 3 \). Hence it follows from Corollary 2.2 that \( A \) does not have the WLP.

**Case 2:** Let \( u \leq dp^s \). Then we obtain

\[
m := \text{td}(p^s(d, d, d), (u, u, u)) = 3(dp^s - u)
\]
which is again < 1. So we can estimate the syzygy gap as follows:

$$\delta^*(d,d,d) = p^{-s}(1 - m)$$

$$= p^{-s}(1 + 3u - 3dp^s)$$

$$= (1 + 3u)p^{-s} - 3d$$

$$= (6k + 4)p^n - 3d$$

$$> 3d + 1 - 3d$$

$$= 1.$$ 

Again we conclude that $A$ does not have the WLP.

Next suppose that the numerical condition of part (2) does not hold. Then we have either

$$\frac{3d + 1}{6k + 4} \geq p^n > \frac{3d}{6k + 4} \text{ or } \frac{3d}{6k + 2} > p^n \geq \frac{3d - 1}{6k + 2},$$

where $n$ and $k$ are chosen as in the beginning of the proof of part (2).

**Case 1:** Let $\frac{3d + 1}{6k + 4} \geq p^n > \frac{3d}{6k + 4}$. Then we even have

$$p^n(6k + 4) = p^n(3u + 1) = 3d + 1.$$

Since

$$\frac{d}{u} = \frac{3d}{3u} > \frac{3d + 1}{3u + 1} = \frac{3d + 1}{6k + 4} = p^n,$$

we have $dp^s > u$. So we obtain

$$m := td(p^s(d,d,d),(u,u,u)) = 3(dp^s - u)$$

which is < 1. This gives:

$$\delta^*(d,d,d) = p^{-s}(1 - m)$$

$$= p^{-s}(1 + 3u - 3dp^s)$$

$$= (1 + 3u)p^{-s} - 3d$$

$$= p^n(3u + 1) - 3d$$

$$= 1.$$ 

Hence $A$ has the WLP by Corollary 2.2.

**Case 2:** Let $\frac{3d}{6k + 2} > p^n \geq \frac{3d - 1}{6k + 2}$. This implies

$$p^n(6k + 2) = -p^n(1 - 3u) = 3d - 1.$$

Since

$$p^n = \frac{3d - 1}{6k + 2} = \frac{3d - 1}{3u - 1} < \frac{d}{u},$$

we now have $u > dp^s$. Hence

$$m := td(p^s(d,d,d),(u,u,u)) = 3(u - dp^s) < 1.$$
Once again we get
\[
\delta^*(d, d, d) = p^{-s}(1 - m) = p^{-s}(1 - 3u - 3dp^s) = p^n(1 - 3u) + 3d = -(3d - 1) + 3d = 1.
\]

We conclude as above that \( A \) has the WLP. \( \square \)

As a corollary we obtain [7, Conjecture 3.9].

**Corollary 2.7.** Let \( K \) be a field of characteristic 2. Then the Artinian complete intersection \( A := K[X, Y, Z]/(X^d, Y^d, Z^d) \) has the weak Lefschetz property if and only if \( d = \left\lfloor \frac{2^t+1}{3} \right\rfloor \) for some positive integer \( t \).

**Proof.** Let \( n \in \mathbb{N} \) such that
\[
\frac{3d}{2} > 2^n > \frac{3d}{4}
\]
(note that there is only one such \( n \) since \( \frac{3d}{4} \) is the half of \( \frac{3d}{2} \)). This \( n \) corresponds to \( k = 0 \) and is the exponent we have to consider by Theorem 2.6. So it follows from part (1) of Theorem 2.6 that the algebra \( A \) never enjoys the WLP for \( d \) even. So we may assume that \( d \) is odd. If
\[
\frac{3d - 1}{2} > 2^n > \frac{3d + 1}{4}
\]
holds then again \( A \) does not have the WLP. So \( A \) does have the WLP if either
\[
\frac{3d - 1}{2} \leq 2^n < \frac{3d}{2} \text{ or } \frac{3d}{2} < 2^{n+1} \leq \frac{3d + 1}{2}
\]
holds, i.e., if we have either \( 3d - 1 = 2^{n+1} \) or \( 3d + 1 = 2^{n+2} \). This gives the assertion of the corollary. \( \square \)

**Remark 2.8.** As remarked in [7] and indicated in our proof, Corollary 2.7 implies that the monomial complete intersection \( K[X, Y, Z]/(X^d, Y^d, Z^d) \) does not have the WLP in characteristic 2 if \( d \) is even.

Theorem 2.6 implies in particular that for given \( d \) the weak Lefschetz property might only fail in characteristic \( p \leq \frac{3d}{2} \). It is easy to generate the list of exceptional characteristics with the help of this numerical criterion.

**Corollary 2.9.** Let \( d \) be odd and let \( p \) be a prime factor of \( d \). Then the Artinian algebra \( K[X, Y, Z]/(X^d, Y^d, Z^d) \) does not have the weak Lefschetz property in characteristic \( p \).
Proof. We write \( d = p^n u \) with \( u = 2k + 1 \) odd, \( n \geq 1 \). Then \[ p^n = \frac{d}{2k + 1} = \frac{3d}{6k + 3}. \]
Since the numerator is larger than the denominator, this number is strictly between \( \frac{d + 1}{6k + 3} \) and \( \frac{3d + 1}{6k + 3} \), so this fulfills the condition of Theorem 2.6(2). \( \square \)

Remark 2.8 and Corollary 2.9 imply that only for \( d = 1 \) the WLP holds in all characteristics. We will see in the examples below that for \( d \) even the weak Lefschetz property can hold in characteristics dividing \( d \) (but not in characteristic 2).

**Example 2.10.** We consider \( d \) even and determine the exceptional prime numbers (here we mean by exceptional that the Artinian complete intersection \( A = K[X,Y,Z]/(X^d,Y^d,Z^d) \) does not enjoy the WLP in these characteristics).

- \( d = 2 \). The only exceptional prime number is 2.
- \( d = 4 \). The condition for \( k = 0 \) is \( 12/2 = 6 > p^n > 12/4 = 3 \), hence the exceptional prime numbers are 2 and 5 (no larger \( k \) have to be considered).
- \( d = 6 \). For \( k = 0 \) we get \( 9 > p^n > 4.5 \), which yields the exceptional primes 2, 5, 7 (no larger \( k \)). The prime number 3 divides \( d \), but the weak Lefschetz property does hold in characteristic 3.
- \( d = 8 \). For \( k = 0 \) we get \( 12 > p^n > 6 \), which yields the exceptional primes 2, 3, 7, 11 (no larger \( k \)).
- \( d = 10 \). For \( k = 0 \) we get the exceptional primes 2, 3, 11, 13 (no larger \( k \)).
- \( d = 12 \). For \( k = 0 \) we get the exceptional primes 2, 11, 13, 17 (no larger \( k \)).
- \( d = 14 \). For \( k = 0 \) we get the condition \( 21 > p^n > 10.5 \), which yields the exceptional primes 2, 11, 13, 17, 19. For \( k = 1 \) we get the condition \( \frac{42}{8} > p^n > \frac{42}{8} \), which yields \( p = 5 \).
- \( d = 20 \). For \( k = 0 \) we get the exceptional primes 2, 3, 5, 17, 19, 23, 29 and for \( k = 1 \) we also get 7. Note that 5 does divide \( d \) and the algebra does not have the weak Lefschetz property in characteristic 5.

**Example 2.11.** We consider \( d \) odd and determine the exceptional prime numbers.

- \( d = 1 \). For \( k = 0 \) we get the condition \( 1 > p^n > 1 \), which has no solution, hence \( K[X,Y,Z]/(X,Y,Z) \cong K \) has the weak Lefschetz property in every characteristic, which is clear anyway.
- \( d = 3 \). The condition for \( k = 0 \) is \( 8/2 = 4 > p^n > 10/4 = 2.5 \), hence the only exceptional prime number is 3 (no larger \( k \) have to be considered).
- \( d = 5 \). For \( k = 0 \) we get \( 7 > p^n > 4 \), which yields the only exceptional prime 5 (no larger \( k \)). The prime number 7 fulfills \( 7 = \frac{14}{2} = \frac{3d - 1}{2} \), which corresponds to the second case in the proof of Lemma 2.5. For \( p = 7 \) the Han number is \( s = -1 \), but the syzygy gap is 1 and not 3.
For $k = 0$ we get $10 > p^n > 5.5$, which yields the exceptional primes 2, 3, 7 (no larger $k$).

For $k = 0$ we get the exceptional primes 2, 3 and 11 (no larger $k$).

For $k = 0$ we get the condition $46 > p^n > 23.5$, which yields the exceptional primes 2, 3, 5, 29, 31, 37, 41, 43. For $k = 1$ we get the condition $\frac{92}{7} > p^n > \frac{44}{10}$, which yields also $p = 11$.

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