Exotic Instantons in Eight Dimensions

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Abstract. In this paper, we study the (anti-)self-duality equations \( *F \wedge F = \pm F \wedge F \) in the eight-dimensional Euclidean space. Using properties of the Clifford algebra \( Cl_{0,8}(\mathbb{R}) \), we find a new solution to these equations.

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1. Introduction

The discovery of instanton solutions to the Yang–Mills field equations in the four-dimensional Euclidean space has led to an intensive study of such theory and the search for multidimensional generalizations of the self-duality equations. In Refs. [5,24], such equations were found and classified. These were first-order equations that satisfy the Yang–Mills field equations as a consequence of the Bianchi identity. Later, solutions to these equations were found, see Refs. [6,8–10,14,16,17], and then used to construct classical solitonic solutions of the low energy effective theory of the heterotic string.

Another approach to the construction of self-duality equations was proposed in Ref. [23]. In this work, it was considered self-duality relations between higher-order terms of the field strength. An example of instantons satisfying such self-duality relations was obtained in Ref. [11], see also Refs. [18,20]. As it turned out, these instantons play a role in smoothing out the singularity of heterotic string soliton solutions by incorporating one-loop corrections. Therefore, these exotic solutions were used to construct various string and membrane solutions, see Refs. [2–4,7,19,21,22] and to study the higher dimensional quantum Hall effect, see Refs. [1,12,13].

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In this paper, we study the (anti-)self-duality equations in the eight-dimensional Euclidean space with the flat metric. We use properties of the Clifford algebra $Cl_{0,8}(\mathbb{R})$ to find new solutions of the equations.

2. The Self-Duality Equations

In this section, we give a brief summary of Clifford algebras and related constructions. We list the features of the mathematical structure as far as they are of relevance to our work.

We recall that the Clifford algebra $Cl_{0,8}(\mathbb{R})$ is a real associative algebra generated by the elements $\Gamma_1, \Gamma_2, \ldots, \Gamma_8$ and defined by the relations
\[ \Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2\delta_{ij}. \tag{1} \]
Its subalgebra $Cl_{0,0,8}(\mathbb{R})$ generated by the elements $\Gamma_{ij} = (\Gamma_i \Gamma_j - \Gamma_j \Gamma_i)/2$ is called its even subalgebra. It can be shown that the even subalgebra of $Cl_{0,8}(\mathbb{R})$ is isomorphic to $Cl_{0,7}(\mathbb{R})$. The element $\Gamma_9 = \Gamma_1 \Gamma_2 \cdots \Gamma_7$ commutes with all other elements of $Cl_{0,8}(\mathbb{R})$, and its square $\Gamma_9 \Gamma_9 = 1$. Therefore the pair $\Gamma^{\pm} = \frac{1}{2}(1 \pm \Gamma_9)$ are a complete system of mutually orthogonal central idempotents, and hence the subalgebra decomposes into the direct sum of two ideals. It can be shown that these ideals are isomorphic to the algebra $M_8(\mathbb{R})$ of all real matrices of size $8 \times 8$.

Suppose $\phi : Cl_{0,8}(\mathbb{R}) \to M_8(\mathbb{R})$, $\Gamma_{ij} \to R_{ij}$ is the homomorphism of the algebras with the kernel $\text{Ker} \phi = \{\alpha(1 - \Gamma_9) \mid \alpha \in \mathbb{R}\}$. In turn, the homomorphism of the algebras induces the homomorphism $\text{Spin}(8) \to SO(8)$ of the group. Therefore the matrices $R_{ij}$ are generators of $SO(8)$. Now, note (see, e.g., [15]) that the identity
\[ \Gamma_{i_1 \ldots i_k} = \frac{1}{(8 - k)!} \varepsilon_{i_1 \ldots i_k} \Gamma_9 \Gamma_{i_8 \ldots i_{k+1}}, \tag{2} \]
holds in the Clifford algebra $Cl_{0,8}(\mathbb{R})$. Here $\varepsilon_{i_1 \ldots i_8}$ is the Levi-Civita symbol in eight dimensions and $\Gamma_{i_1 \ldots i_k} = \Gamma_{[i_1 \ldots i_k]}$, where the square bracket stands for the anti-symmetrization of indices with the weight $1/k!$. Choosing $k = 4$ in (2), we obtain the self-duality equations
\[ R_{[mnR_{ps}]} = \frac{1}{24} \varepsilon_{mnpsijkl} R_{[ijR_{kl}]} . \tag{3} \]

Any totally antisymmetric eight-dimensional tensor of fourth rank can be written as the sum of the self-dual and the anti-self-dual parts $F_{mnps} = F^+_{mnps} + F^{-}_{mnps}$, where
\[ F^\pm_{mnps} = \left( \delta_{[m}^i \delta_{n}^j \delta_{p}^k \delta_{s}]^l \pm \frac{1}{24} \varepsilon_{mnpsijkl} \right) F_{ijkl}. \tag{4} \]
If we now use the identity
\[ \Gamma_p \Gamma_{s_1 \ldots s_k} = \Gamma_{ps_1 \ldots s_k} + \sum_{i=1}^{k} (-1)^i \delta_{ps_i} \Gamma_{ps_1 \ldots s_i \ldots s_k}, \tag{5} \]
then we obtain the following expression for the self-dual tensor

\[ F^+_{mnps} = \frac{1}{24} \text{Tr} (R_{mn} R_{ps} R_{ij} R_{kl}) F_{ijkl}. \]  

(6)

Thus, the tensor \( F_{mnps} \) is anti-self-dual if \( R_{mn} R_{ps} F_{mnps} = 0 \). In particular, the tensor \( F_{mnps} = F_{[mn} F_{ps]} \) is anti-self-dual if \( R_{mn} F_{mn} = 0 \). Note that the last equality is a sufficient condition for the anti-self-duality, but not necessary. Note also that previously known solutions do not satisfy that condition.

### 3. Instantons in Eight Dimensions

In this section, we find solutions of the anti-self-duality equations in the eight-dimensional Euclidean space. This equation is given by the formula

\[ F_{[mn} F_{ps]} = -\frac{1}{24} \varepsilon_{mnpsijkl} F_{ijkl}, \]  

(7)

where the gauge field strength \( F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] \)  

(8)

and the potential \( A_m \) takes values in the Lie algebra \( \text{so}(8) \).

We choose the ansatz

\[ A_m = -\frac{1}{2} R_{mp} \partial_p \varphi, \]  

(9)

where \( \varphi \) is a function of \( x^2 = x_n x^n \). To find the gauge field strength, we substitute the potential (9) into (12) and use the identity

\[ R_{mn} R_{sn} = \delta_{nn} (R_{ms} - \delta_{ms}), \quad R_{mn} R_{mn} = -\delta_{mm} \delta_{nn}, \]  

(10)

which is a consequence of (5). As a result, we get

\[ F_{mn} = \frac{1}{2} [R_{ms} (\partial_n \partial_s \varphi - \partial_n \varphi \partial_s) - R_{ns} (\partial_m \partial_s \varphi - \partial_m \varphi \partial_s) + R_{mn} (\partial_s \varphi)^2]. \]  

(11)

Now we impose the anti-self-duality condition \( R_{mn} F_{mn} = 0 \) and use the identities

\[ R_{mn} R_{sn} = \delta_{nn} (R_{ms} - \delta_{ms}), \quad R_{mn} R_{mn} = -\delta_{mm} \delta_{nn}, \]  

(12)

which are consequences of (10). As a result, we obtain the equation

\[ \partial_s \partial_s \varphi + 3 \partial_s \varphi \partial_s \varphi = 0. \]  

(13)

Since \( \varphi = \varphi(x^2) \), this equation is equivalent to the ordinary differential equation

\[ x^2 \varphi'' + 4 \varphi' + 3 x^2 (\varphi')^2 = 0. \]  

(14)

where \( \varphi' = \partial \varphi / \partial (x^2) \). Solving this equation, we find

\[ \varphi = \frac{1}{3} \ln \left( c_1 + \frac{c_2}{x^6} \right). \]  

(15)

This is a solution of the anti-self-duality equations (7).
Interestingly, the ansatz (9) is a solution to the self-duality equations as \( \varphi = \ln(\lambda^2 + x^2) \). To show this, we represent the matrices \( R_{mn} \) in the following form

\[
R_{mn} = \frac{1}{2} (e^t_m e_n - e^t_n e_m),
\]

where \( e_8 \) is the unit 8 \( \times \) 8 matrix, \( e_m \) is an image of \( \Gamma_m \in \text{Cl}_{0,7}(\mathbb{R}) \) as \( m \neq 8 \), and \( e^t_m \) signifies the transposition of the matrix \( e_m \). In the case, the potential

\[
\tilde{A}_m = -\frac{R_{mn} x_n}{\lambda^2 + x^2}
\]

and the gauge field strength

\[
\tilde{F}_{mn} = \frac{2\lambda^2 R_{mn}}{\lambda^2 + x^2}.
\]

This is exactly a solution of the self-duality equations that was obtained in Ref. [11].

Let us return again to the obtained solution of the anti-self-duality equations. If we substitute the solution (15) into (11), then we get the gauge field strength

\[
F_{mn} = \frac{2\lambda^2 x^2}{(\lambda^2 + x^2)^2} (4R_{mp}x_n x_p - 4R_{np}x_m x_p - R_{mn}x^2),
\]

where \( \lambda^2 = c_2/c_1 \). It is not difficult to see that the solutions (18) and (19) retain their form when \( x_n \) is replaced by \( x_n - b_n \), where \( b_n \in \mathbb{R} \). Further, the gauge transformations of \( A(x) \) and \( \tilde{A}(x) \) induce the transformations \( R_{mn} \rightarrow U^{-1} R_{mn} U \), there \( U \in \text{Spin}(8) \), which only lead to a change in the basis of \( \text{so}(8) \) and therefore leaves the solutions unchanged. Consequently, the solutions (18) and (19) have the same number of free parameters and the same gauge group. At the same time, the potentials \( A(x) \) and \( \tilde{A}(x) \) are gauge nonequivalent. To show this, it suffices to note that

\[
\text{tr} F_{mn}^2 = 56^2 \frac{4\lambda^4 x^8}{(\lambda^2 + x^2)^4} \neq \text{tr} \tilde{F}_{mn}^2.
\]

Note that the field strength (19) is not a function of \( x^2 \) and therefore the found solution is not rotationally invariant. This fundamentally distinguishes it from the known solutions to the anti-auto-duality equations in eight dimensions.

Thus, the formula (15) indeed gives a new solution of the anti-self-duality equations (7). Note also that the resulting anti-self-dual solution becomes self-dual and the self-dual solution be anti-self-dual if, instead of \( \phi \), we will use the homomorphism \( \phi' : \text{Cl}_{0,8}^0(\mathbb{R}) \rightarrow \text{M}_8(\mathbb{R}) \) with the kernel \( \text{Ker} \phi' = \{ \alpha(1 + \Gamma_9) \mid \alpha \in \mathbb{R} \} \).

4. D7-Brane Effective Action

We now consider the Euclidean D7-brane in Type IIB string theory. On the world-volume of this D7-brane, there is an eight-dimensional Yang–Mills theory which is naturally realized as low-energy effective field theory. In order to see the (anti-)self-dual instanton effects of the obtained solutions, we consider
the $\alpha'$ corrections to the gauge theory. The gauge part of the effective action can be written as

$$S_D = S_2 + S_4 + \cdots ,$$

(21)

where the first term is the quadratic Yang–Mills action in eight dimensions

$$S_2 = \frac{1}{2 g_Y^2} \int d^8x \text{tr}(F^2),$$

(22)

while the second part is a quartic action of the form

$$S_4 = \frac{(4\pi\alpha')^2}{4!g_Y^2} \int d^8x \text{tr}(t_8 F^4) - 2\pi i C_0 k.$$  

(23)

Here $t_8$ is the ten-dimensional extension of the eight-dimensional light-cone gauge “zero-mode” tensor, i.e.

$$t_8 F^4 = F^{MN} F_{PN} F_{MS} F^{PS} + \frac{1}{2} F^{MN} F_{PN} F^{PS} F_{MS} - \frac{1}{4} F^{MN} F_{MN} F^{PS} F_{PS} - \frac{1}{8} F^{MN} F^{PS} F_{MN} F_{PS}.$$  

(24)

Moreover, $k$ is the fourth Chern number

$$k = \frac{1}{4!(2\pi)^4} \int \text{tr}(F \wedge F \wedge F \wedge F)$$

(25)

and $C_0$ is a scalar field of the closed string RR sector.

Following [2], we will interpret the eight-dimensional instantons as the $D$-instantons, i.e. as instantons embedded in $D7$-branes. Such instantons are sources for RR 0-form $C_0$. When the (anti-)self-duality condition holds, the trace

$$\text{tr}(t_8 F^4) = \pm \frac{1}{2} \text{tr}(F \wedge F \wedge F \wedge F),$$

(26)

and hence the quartic action $S_4$ becomes

$$S_4 = -2\pi i \left( C_0 \pm \frac{i}{g_s} \right) k.$$  

(27)

This precisely matches the action of the action of $k$ $D$-instantons. Thus, the eight-dimensional (anti-)self-dual instantons become the $D$-instantons when $S_2 = 0$, $S_4 \neq 0$, and all the $O(\alpha'^4/g_Y^2)$ terms vanish. The second condition is fulfilled in the zero-slope limit $\alpha' \to 0$ with fixed $\alpha'^2/g_Y^2$.

On the other side, it follows from the identities (12) that $\text{tr} F_{mnps}^2 = 0$ and therefore

$$\frac{1}{12} \text{tr}(t_8 F^4) = \frac{1}{4!} \cdot \frac{2^4}{2^4} \text{tr}(\epsilon^{mnpsijkl} F_{mn} F_{ps} F_{ij} F_{kl}) = 0.$$  

(28)

Hence the fourth Chern number and the quartic action (23) are equal to zero. Thus, the solution to the anti-self-duality equations found in the previous section is not the $D$-instanton embedded in the $D7$-brane.

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References

[1] Bernevig, B.A., Hu, J.P., Toumbas, N., Zhang, S.C.: The eight dimensional quantum Hall effect and the octonions. Phys. Rev. Lett. 91, 236803 (2003)
[2] Billo, M., Frau, M., Gallot, L., Lerda, A., Pesando, I.: Classical solutions for exotic instantons? JHEP 0903, 056 (2009)
[3] Billo, M., Ferro, L., Frau, M., Gallot, L., Lerda, A.: Exotic instanton counting and heterotic/type I-prime duality. JHEP 07, 092 (2009)
[4] Billo, M., Frau, M., Fucito, F., Gallot, L., Lerda, A., Morales, J.F.: On the D(−1)/D7-brane systems. JHEP 04, 096 (2021)
[5] Corrigan, E., Devchand, C., Fairlie, D.B., Nuyts, J.: First-order equations for gauge fields in spaces of dimension greater than four. Nucl. Phys. B 214, 452 (1983)
[6] Corrigan, E., Goddard, P., Kent, A.: Some comments on the ADHM construction in 4k dimensions. Commun. Math. Phys. 100, 1 (1985)
[7] Duff, M., Lu, J.: Strings from five-brans. Phys. Rev. Lett. 66, 1402 (1991)
[8] Dunajski, M.: SU(2) solutions to self-duality equations in eight dimensions. J. Geom. Phys. 62, 1747 (2012)
[9] Fairlie, D.B., Nuyts, J.: Spherically symmetric solutions of gauge theories in eight dimensions. J. Phys. A 17, 2867 (1984)
[10] Fubini, S., Nicolai, H.: The octonionic instanton. Phys. Lett. B 155, 369 (1985)
[11] Grossman, B., Kephart, T.W., Stasheff, J.D.: Solutions to Yang–Mills field equations in eight dimensions and the last Hopf map. Commun. Math. Phys. 96, 431 (1984)
[12] Hasebe, K.: Higher dimensional quantum Hall effect as A-class topological insulator. Nucl. Phys. B 886, 952 (2014)
[13] Inoue, T., Sakamoto, M., Ueba, I.: Instantons and Berry’s connections on quantum graph. J. Phys. A 54, 355301 (2021)
[14] Ivanova, T.A., Popov, A.D.: Self-dual Yang–Mills fields in d = 7, 8, octonions and Ward equations. Lett. Math. Phys. 24, 85 (1992)
[15] Kennedy, A.D.: Clifford algebras in 2ω dimensions. J. Math. Phys. 22, 1330 (1981)
[16] Loginov, E.K.: Self-dual Yang–Mills fields in pseudo-Euclidean spaces. J. Phys. A 37, 6599 (2004)
[17] Loginov, E.K.: Multi-instantons in seven dimensions. J. Math. Phys. 46, 063506 (2005)
[18] Loginov, E.K.: Octonionic instantons in eight dimensions. Phys. Lett. B 816, 136244 (2021)
[19] Minasian, K., Shatashvili, S.L., Vanhove, P.: Closed strings from SO(8) Yang–Mills instantons. Nucl. Phys. B 613, 87 (2001)
[20] Nakamula, A., Sasaki, S., Takesue, K.: ADHM construction of (anti-)self-dual instantons in eight dimensions. Nucl. Phys. B 910, 199 (2016)
[21] Olsen, K., Szabo, R.J.: Brane descent relations in K-theory. Nucl. Phys. B 566, 562 (2000)
[22] Pedder, C., Sonner, J., Tong, D.: The berry phase of D0-branes. JHEP 0803, 065 (2008)
[23] Tchrakian, D.H.: N-dimensional instantons and monopoles. J. Math. Phys. 21, 166 (1980)

[24] Ward, R.S.: Completely solvable gauge-field equations in dimension greater than four. Nucl. Phys. B 236, 381 (1984)

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