Multidimensional Random Polymers: A Renewal Approach

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Abstract In these lecture notes, which are based on the mini-course given at 2013 Prague School on Mathematical Statistical Physics, we discuss ballistic phase of quenched and annealed stretched polymers in random environment on $\mathbb{Z}^d$ with an emphasis on the natural renormalized renewal structures which appear in such models. In the ballistic regime an irreducible decomposition of typical polymers leads to an effective random walk reinterpretation of the latter. In the annealed case the Ornstein-Zernike theory based on this approach paves the way to an essentially complete control on the level of local limit results and invariance principles. In the quenched case, the renewal structure maps the model of stretched polymers into an effective model of directed polymers. As a result one is able to use techniques and ideas developed in the context of directed polymers in order to address issues like strong disorder in low dimensions and weak disorder in higher dimensions. Among the topics addressed: Thermodynamics of quenched and annealed models, multidimensional renewal theory (under Cramer’s condition), renormalization and effective random walk structure of annealed polymers, very weak disorder in dimensions $d \geq 4$ and strong disorder in dimensions $d = 1, 2$.

1 Introduction

Mathematical and probabilistic developments presented here draw inspiration from statistical mechanics of stretched polymers, see for instance [11, 27]. Polymers chains to be discussed in these lecture notes are modeled by paths of finite range random walks on $\mathbb{Z}^d$. We shall always assume that the underlying random walk distribution has zero mean. The word stretched alludes to the situation when the end-point of a polymer is pulled by an external force, or in the random

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walk terminology, by a drift. In the case of random walks this leads to a ballistic
behaviour with limiting spatial extension described in terms of the usual law of
large numbers (LLN) for independent sums. Central limit theorem (CLT) and large
deviations (LD) also hold.

Polymer measures below are non-Markovian objects (see Remark[1]), which gives
rise to a rich morphology. We shall distinguish between ballistic and sub-ballistic
phases and between quenched and annealed polymers. Quenched polymers corre-
spond to pulled random walks in random potentials. Their annealed counterparts
correspond to pulled random walks in a deterministic attractive self-interaction po-
tentials.

Two main themes are the impact of the drift, that is when the model in question,
annealed or quenched, becomes ballistic, and the impact of disorder, that is whether
or not quenched and annealed models behave similarly.

It is instructive to compare models of stretched polymers with those of directed
polymers [4]. In the latter case sub-ballistic to ballistic transition is not an issue.
Furthermore, in the stretched case polymers can bend and return to the same ver-
tices, which makes even the annealed model to be highly non-trivial (in the directed
case the annealed model is a usual random walk). On the other hand it is unlikely
that a study of stretched polymers will shed light on questions which are open in the
directed context.

1.1 Class of models.

**Underlying random walk.** Consider random walk on $\mathbb{Z}^d$ with an irreducible finite
range step distribution. We use the notation $P_d$ both for the random walk path mea-
sure and for the distribution of individual steps. For convenience we shall assume
that nearest neighbour steps $\pm e_k$ are permitted,

$$P_d(\pm e_k) > 0$$

(1)

The size of the range is denoted $R$: $P_d(X = x) > 0 \Rightarrow |x| \leq R$. Without loss of gen-
erality we shall assume that $E_dX = 0$.

**Random environment.** The random environment is modeled by a collection \( \{V_\omega\}_{\omega} \)
i.i.d non-negative random variables. The notation $Q$ and $E$ are reserved for the
corresponding product probability measure and the corresponding expectation. We
shall assume:

(A1 ) $V^\omega$ is non-trivial and $0 \in \text{supp}(V^\omega)$ .

(A2 ) $Q(V^\omega < \infty) > p_c(\mathbb{Z}^d)$, where $p_c$ is the critical Bernoulli site percolation prob-
ability.

**Polymers and polymer weights.** Polymers $\gamma = (\gamma_0, \ldots, \gamma_n)$ are paths of the under-
lying random walk. For each polymer $\gamma$ we define $|\gamma| = n$ as the number of steps,
and $X(\gamma) = \gamma_n - \gamma_0$ as the displacement along the polymer.
The are two type of weights we associate with polymers: quenched random weights
\[ W_\omega^\gamma(\gamma) = \exp\left\{ -\beta \sum_{i=1}^{\gamma} V_{\omega}^{\gamma_i} \right\} P_d(\gamma), \] (2)
and annealed weights
\[ W_d(\gamma) = \mathcal{E}(W_\omega^\gamma(\gamma)) = e^{-\Phi_\beta(\gamma)} P_d(\gamma), \] (3)
where the self-interacting potential
\[ \Phi_\beta(\gamma) = \sum_x \phi_\beta(\ell_\gamma(x)). \] (4)
Above \( \ell_\gamma(x) \) is the local time of \( \gamma \) at \( x \);
\[ \ell_\gamma(x) = \sum_{i=1}^{n} I_{\{\gamma_i = x\}}, \] (5)
and \( \phi_\beta \) is given by:
\[ \phi_\beta(\ell) = -\log \mathcal{E}\left( e^{-\beta \ell V_\omega^\gamma} \right). \] (6)
The inverse temperature \( \beta > 0 \) modulates the strength of disorder.

**Pulling force, partition functions and probability distributions.** For \( h \in \mathbb{R}^d \) we shall consider quenched and annealed partition functions
\[ Z_\omega^n(h) = \sum_{|\gamma| = n} e^{h \cdot X(\gamma)} W_\omega^\gamma(\gamma) \quad \text{and} \quad Z_d^n(h) = \mathcal{E}(Z_\omega^n(h)) = \sum_{|\gamma| = n} e^{h \cdot X(\gamma)} W_d(\gamma), \] (7)
and the corresponding probability distributions,
\[ P_{h,\omega}^n(\gamma) = \frac{1}{Z_\omega^n(h)} e^{h \cdot X(\gamma)} W_\omega^\gamma(\gamma) \quad \text{and} \quad P_{h}^n(\gamma) = \frac{1}{Z_d^n(h)} e^{h \cdot X(\gamma)} W_d(\gamma). \] (8)

**Remark 1.** Annealed measures \( P_{h}^n \) are non-Markovian. Quenched measures \( P_{h,\omega}^n \) are also non-Markovian in the sense that in general \( P_{h,\omega}^n \) is not a marginal of \( P_{h}^m \) for \( m > n \).

### 1.2 Morphology.

We shall distinguish between ballistic and sub-ballistic behaviour of quenched and annealed polymers and between strong an weak impact of disorder on the properties of quenched polymers (as compared to the annealed ones).
**Ballistic phase.** For the purpose of these lecture notes, let us say that a self-interacting random walk (or polymer) is ballistic if there exists $\delta > 0$ and a vector $v \neq 0$ such that

$$\lim_{n \to \infty} \mathbb{P}_n^h (|X| \leq \delta n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_n^h X(\gamma) = v. \quad (9)$$

The model is said to be sub-ballistic, if

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_n^h |X(\gamma)| = 0. \quad (10)$$

At this stage it is unclear whether there are models which comply neither with (9) nor with (10). It is the content of Theorem 2.1 below that for the annealed models the above dichotomy always holds.

Similarly, the quenched model is said to be in the ballistic, respectively sub-ballistic, phase if

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{P}_n^{h,\omega} (|X| \leq \delta n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_n^{h,\omega} X(\gamma) = v, \quad (11)$$

and, respectively,

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_n^{h,\omega} |X(\gamma)| = 0. \quad (12)$$

For quenched models it is in general open question whether the limiting spatial extension (second limit in (11)) always exists. See Theorem 2.2 below for a precise statement.

Ballistic and sub-ballistic phases correspond to very different patterns of behaviour. We focus here on the ballistic phase. In a sense sub-ballistic behaviour is more intricate than the ballistic one. In the continuous context (Brownian motion) results about sub-ballistic phase are summarized in [30]. The theory (so called *enlargement of obstacles*) was adjusted to random walks on $\mathbb{Z}^d$ in [1].

**Strength of disorder.** For each value of the pulling force $h$ and the interaction $\beta$ strength of disorder may be quantified on several levels:

**L1.** $\mathcal{Q}$-a.s $\limsup_{n \to \infty} \frac{1}{n} \log \frac{Z_n^h(\omega)}{Z_n(\omega)} < 0.$

Since by Assumption (A1) the annealed potential $\phi_\beta$ in (6) satisfies $\lim_{\beta \to \infty} \frac{\phi_\beta}{\beta} = 0$, it is not difficult to see that, at least in the case when $\{x : \nu_x^\omega = 0\}$ does not percolate, L1. holds in any dimension whenever the strength of interaction $\beta$ and the pulling force $h$ are large enough.

Furthermore, as we shall see in Section 5 (and as it was originally proved in [35]) the disorder is strong in the sense of L1. in lower dimensions $d = 2, 3$ for any $\beta > 0$ provided that the annealed polymer is in (the interior of) the ballistic phase.

**L2.** $\mathcal{Q}$-a.s $\lim_{n \to \infty} \frac{Z_n^h(\omega)}{Z_n(\omega)} = 0.$

This is presumably always the case when the quenched model is in sub-ballistic phase. The case $h = 0$ is worked out in great detail [30][1].
Remark 2. A characterization of annealed and quenched sets of sub-critical drifts; \( K_0 \) and \( K_0^q \) is given in \([20]\) and \([58]\) below. It always holds that \( K_0 \subseteq K_0^q \). The inclusion is strict in any dimension for \( \beta \) large enough, and it is presumably strict in dimensions \( d = 2,3 \) for any \( \beta > 0 \). On the other hand, it seems to be an open question whether in higher dimensions \( d > 3 \) the two sets of sub-critical drifts coincide at sufficiently small \( \beta \).

L3. Typical polymers under \( \mathbb{P}_n^h \) and \( \mathbb{P}_n^{h,\omega} \) have very different properties. Ballistic phase of annealed polymers in dimensions \( d \geq 2 \) is by now completely understood, and we expose the core of the corresponding (Ornstein-Zernike) theory developed in \([15,13]\) in Section 3. In dimensions \( d \geq 4 \), the disorder happens to be weak in the sense of any of L1-L3 in the following regime (which we shall call very weak disorder): Fix \( h \neq 0 \) and then take \( \mathcal{Q}(V^0 = \infty) \) and \( \beta \) in \([2]\) to be sufficiently small. These results \([10,34,16,19]\) are explained in Section 4.

Zero drift case. At \( h = 0 \) properties of both annealed and quenched measures were described in depth in \([30]\) and references therein, following an earlier analysis of Wiener sausage in \([7,8]\). This is not the case we consider here. However, it is instructive to keep in mind what happens if there is no pulling force, and, accordingly, we give a brief heuristic sketch. To fix ideas consider the case of pure traps structively to keep in mind what happens if there is no pulling force, and, accordingly, we give a brief heuristic sketch. To fix ideas consider the case of pure traps for which \( p = \mathcal{Q}(V^0 = 0) = 1 - \mathcal{Q}(V^0 = \infty) \). If \( 1 - p \) is small, then \( \{x : V^0_x = 0\} \) percolates, and the model is non-trivial. Let us start with a quenched case. Let \( B_r \) be a lattice box \( B_r = \{x : |x|_1 \leq r\} \) and \( B_r(x) = x + B_r \). We say that there is an \((R,r)\)-clearing if

\[ \exists x \in B_R \text{ such that } V^0_y = 0 \text{ for all } y \in B_r(x). \]

The probability

\[ \mathcal{Q} \text{ (there is a } (R,r) \text{ clearing)} \approx 1 - \left(1 - p^{r^d}R^d\right)^{c_2r^d/\ell^d} \approx 1 - e^{-c_3p^{r^d}R^d/\ell^d}. \]

Up to leading terms this is non-negligible if \( p^{r^d}R^d \approx \text{const} \), or if \( r \approx (\log R)^{1/d} \). On the other hand, a probability that a random walk will go ballistically to a box (clearing) \( B_r(x) \) at distance of order \( R \) from the origin is of order \( e^{-cR^d} \), and the probability that afterwards it will spend around \( n \) units of time in \( B_r(x) \) is \( e^{-c_5n/r^d} \).

We, therefore need to find an optimal balance between \( R \) and \( n/r^d \approx n/(\log R)^{2/d} \), terms, which gives, again up to leading terms, \( R \approx n/(\log n)^{2/d} \). This suggests both a survival pattern for typical quenched polymer (see Figure 1), and an asymptotic relation for the quenched partition function

\[ \log Z_n^q \approx -\frac{n}{(\log n)^{2/d}}. \]

As far as the annealed model is considered for \( \ell \geq 1 \) define as before \( \phi_{\ell}(\nu) = -\log \delta(\nu - \beta \nu_0) = -\log \rho \frac{\delta}{\nu} \). Consider random walk which stays all \( n \) units of time inside \( B_R \). The probabilistic price for the latter is \( e^{-c_6n/R^2} \). On the other hand, the self-interaction price is \( e^{-c_7\nu R^2} \). Choosing optimal balance leads to
Fig. 1 On the left: A survival pattern for an \( n \)-step quenched polymer with \( R \approx n/(\log n)^{2/d} \). On the right: \( n \)-step annealed polymer in \( B_R \) with \( R \approx n^{1/(d+2)} \).

\( R \approx n^{1/(d+2)} \). This suggests a behavior pattern for typical annealed polymers (see Figure 1), which is very different from the survival pattern for typical quenched polymer as discussed above. This also suggests the following asymptotics for the annealed partition function:

\[
\log Z_n \approx -n^{d/(d+2)}.
\]  

(14)

The above discussion indicates that in the zero drift case the disorder is strong on levels \( L_2, L_3 \), but not on \( L_1 \).

**Outline of the notes.** We do not attempt to give a comprehensive survey of the existing results on the subject. Neither the notes are self-contained, in many instances below we shall refer to the literature for more details on the corresponding proofs. The emphasis is on the exposition of the renewal structure behind stretched polymers in the ballistic regime, and how this might help to explore and understand various phenomena in question.

Section 2 is devoted to the thermodynamics of annealed and quenched polymers, namely to the facts which can be deduced from sub-additivity arguments and large deviation principles.

Multidimensional renewal theory (under assumption of exponential tails) is discussed in detail in Section 3. In Subsection 3.2 we explain renormalization procedures which lead to a reformulation of annealed models in this renewal context, which is the core of the Ornstein-Zernike theory of the latter. As a result we derive very sharp and essentially complete description of the ballistic phase of the annealed polymers on the level of invariance principles and local limit asymptotics on all deviation scales.

In Section 4 we explain why the annealed renewal structure persists for quenched models in the regime of very weak disorder in dimensions \( d \geq 2 \). More precisely, it happens that in the latter case the disorder is weak on all three levels \( L_1 - L_3 \).
In Section 5 we explain how to check that the disorder is always strong already on level $L_1$ in dimensions $d = 2, 3$. A more or less complete argument is given only in two dimensions.

To facilitate references and the reading some of the back ground material on convex geometry and large deviations is collected in the Appendix.

**Notation conventions.** Values of positive constants $c, \nu, c_1, \nu_1, c_2, \nu_2, \ldots$ may change between different Sections.

In the sequel we shall use the following notation for asymptotic relations: Given a set of indices $\mathcal{A}$ and two positive sequences \{a_\alpha, b_\alpha\}_\alpha \in \mathcal{A}, we say that

- $a_\alpha \sim b_\alpha$ uniformly in $\alpha \in \mathcal{A}$ if there exists a constant $c > 0$ such that $a_\alpha \leq cb_\alpha$ for all $\alpha \in \mathcal{A}$.
- We shall use $a_\alpha \asymp b_\alpha$ if both $a_\alpha \sim b_\alpha$ and $a_\alpha \sim b_\alpha$ hold.

For $1 \leq p \leq \infty$, the $\ell_p$-norms on $\mathbb{R}^d$ are denoted $|x|^p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}}$.

The default notation is for the Euclidean norm $|\cdot| = |\cdot|_2$.

If not explicitly stated otherwise paths $\gamma = (\gamma_0, \ldots, \gamma_n)$ are assumed to have their starting point at the origin; $\gamma_0 = 0$. A concatenation $\gamma \circ \eta$ of two paths $\gamma = (\gamma_0, \ldots, \gamma_n)$ and $\eta = (\eta_0, \ldots, \eta_m)$ is the path

$$\gamma \circ \eta = (\gamma_0, \ldots, \gamma_n, \gamma_n + \eta_1, \ldots, \gamma_n + \eta_m).$$

A union of two paths $\gamma \cup \eta$, with end-points at the origin or not, is a subset of $\mathbb{Z}^d$ with multiplicities counted. In particular, local times satisfy $\ell_{\gamma \cup \eta}(x) = \ell_{\gamma}(x) + \ell_{\eta}(x)$.

### 2 Thermodynamics of Annealed and Quenched Models.

In the sequel we shall employ the following notation for families of polymers:

$$\mathcal{P}_x = \{\gamma : X(\gamma) = x\} \quad \mathcal{P}_n = \{\gamma : |\gamma| = n\} \quad \text{and} \quad \mathcal{P}_{x,n} = \mathcal{P}_x \cap \mathcal{P}_n \quad (15)$$

**Conjugate ensembles.** Let $\lambda \geq 0$. Consider

$$G_\lambda^\omega(x) = \sum_{X(\gamma) = x} e^{-\lambda |\gamma|} W_\omega^\gamma(\gamma) \quad \text{and} \quad G_\lambda(x) = \mathcal{E}\left(G_\lambda^\omega(x)\right) = \sum_{X(\gamma) = x} e^{-\lambda |\gamma|} W_\lambda(\gamma).$$

(16)

**Free energy and inverse correlation length.** One would like to define quenched and annealed free energies via:

$$\lambda^q(h) = \lim_{n \to \infty} \frac{1}{n} \log Z^\omega_n(h) \quad \text{and} \quad \lambda(h) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(h). \quad (17)$$
Similarly one would like to define and the inverse correlation lengths: For $x \in \mathbb{R}^d$ set
\[
\tau^\theta_\lambda(x) = -\lim_{r \to \infty} \frac{1}{r} \log G^\theta_\lambda(|rx|) \quad \text{and} \quad \tau_\lambda(x) = -\lim_{r \to \infty} \frac{1}{r} \log G_\lambda(|rx|). \tag{18}
\]

Depending on the context other names for $\lambda(h)$ are connectivity constant and log-moment generating function, and for $\tau_\lambda(x)$ are Lyapunov exponent and, for some models in two dimensions, surface tension.

By Thermodynamics we mean here statements about existence of limits in (17) and (18), and their relation to Large Deviation asymptotics under quenched and annealed polymer measures (8). Facts about Thermodynamics of annealed and quenched models are collected in Theorem 2.1 and Theorem 2.2 below, and, accordingly, discussed in some detail in Subsections 2.1 and 2.2.

2.1 Annealed Models in dimensions $d \geq 2$.

**Theorem 2.1** A. The free energy $\lambda$ is well defined, non-negative and convex on $\mathbb{R}^d$. Furthermore,
\[
0 = \min_h \lambda(h) = \lambda(0). \tag{19}
\]

The set
\[
K_0 \triangleq \{h : \lambda(h) = 0\} \tag{20}
\]
is a compact convex set with a non-empty interior.

B. The inverse correlation length $\tau_\lambda$ is well defined for any $\lambda \geq 0$, and it can be identified as the support function of the compact convex set
\[
K_\lambda \triangleq \{h : \lambda(h) \leq \lambda\}. \tag{21}
\]

Define
\[
I(\nu) = \sup_h \{h \cdot \nu - \lambda(h)\} = \sup_{\lambda} \{\tau_\lambda(\nu) - \lambda\}. \tag{22}
\]

C. For any $h \in \mathbb{R}^d$ the family of polymer measures $\mathbb{P}^h$ satisfies LD principle with the rate function
\[
I_h(\nu) \triangleq \sup_f \{f \cdot \nu - (\lambda(f + h) - \lambda(h))\} = I(\nu) - (h \cdot \nu - \lambda(h)). \tag{23}
\]

D. For $h \in \text{int}(K_0)$ the model is sub-ballistic, whereas for any $h \notin K_0$ the model is ballistic.

E. Furthermore, at critical drifts $h \in \partial K_0$ the model is still ballistic. In other words, the ballistic to sub-ballistic transition is always of the first order in dimensions $d \geq 2$. 
Proofs of Parts A-C and of Part D for sub-critical drifts \((h \in \text{int}(K_0))\) of Theorem 2.1 are based on sub-additivity arguments. Parts D (namely existence of limiting spatial extension \(v\) in (9) for super-critical drifts \(h \not\in K_0\)) and E require a more refined multidimensional renewal analysis based on Ornstein-Zernike theory.

**Sub-additivity.** The following result is due to Hammersley [13]:

**Proposition 2.1** Let \(\{a_n\}\) and \(\{b_n\}\) be two sequences such that:

(a) For all \(m, n\), \(a_{n+m} \leq a_n + a_m + b_{n+m}\).

(b) The sequence \(b_n\) is non-decreasing and

\[
\sum_n \frac{b_n}{n(n+1)} < \infty.
\]  

Then, there exists the limit

\[
\xi^A \triangleq \lim_{n \to \infty} \frac{a_n}{n} \text{ and } \frac{a_n}{n} \geq \xi + \frac{b_n}{n} - 4 \sum_{k=2n}^{\infty} \frac{b_k}{k(k+1)}.
\]  

Note that \(\xi^A \triangleq \liminf_{n \to \infty} \frac{a_n}{n}\) is always defined. The usual sub-additivity statement is for \(b_n \equiv 0\). In this case \(\xi = \inf_n \frac{a_n}{n}\). The proof (in the case \(b_n \equiv 0\)) is straightforward: Indeed, iterating on the sub-additive property, we infer that for any \(n \geq m\) and any \(k\) (and \(N = kn + m\)),

\[
\frac{a_N}{N} \leq \frac{ka_n}{kn + m} + \frac{am}{kn + m}.
\]

Therefore, for any \(n\) fixed

\[
\limsup_{N \to \infty} \frac{a_N}{N} \leq \frac{a_n}{n}.
\]

**Remark 3.** Note that (25) implies that \(\xi < \infty\), however the case \(\xi = -\infty\) is not excluded by the argument. Note also that even if \(\xi > -\infty\), no upper bounds (apart from \(\lim a_n = \xi\)) on \(\frac{a_n}{n}\) are claimed. In other words the sub-additivity argument above does not give information on the speed of convergence.

**Attractivity of the interaction.** The interaction \(\phi_{\beta}\) in (6) is attractive in the sense that

\[
\phi_{\beta}(\ell + m) \leq \phi_{\beta}(\ell) + \phi_{\beta}(m).
\]  

Indeed, (26) follows from positive association of probability measures on \(\mathbb{R}\). Namely, if \(X \in \mathbb{R}\) is a random variable, and \(f, g\) two bounded functions on \(\mathbb{R}\) which are either both non-increasing or both non-decreasing, then

\[
\mathbb{E} f(X)g(X) \geq \mathbb{E} f(X)\mathbb{E} g(X).
\]  

The universal validity of the latter inequality is related to total ordering of \(\mathbb{R}\). If \(Y\) is an i.i.d. copy of \(X\), then

\[
(f(X) - f(Y))(g(X) - g(Y)) \geq 0.
\]
Taking expectation we deduce (27).

**Exercise 2.1** Show that if \( \phi \) is attractive, then for any \( h \in \mathbb{R}^d \) and for any \( \lambda \) and \( x,y \in \mathbb{Z}^d \)

\[
Z_{n+m}(h) \geq Z_n(h)Z_m(h). \tag{28}
\]

Note that due to a possible over-counting such line of reasoning does not imply that \( G_\lambda(x+y) \geq G_\lambda(x)G_\lambda(y) \).

**Part A of Theorem 2.1** Since the underlying random walk has finite range \( R \), \( Z_n(h) \leq e^{Rn|h|} \). On the other hand, since we assumed that \( P_d(e_1) > 1 \),

\[
Z_n(h) \geq \left(e^{h e_1 - \phi_\beta(1)}P_d(e_1 > 1)\right)^n.
\]

Consequently, \( \lambda_n(h) \triangleq \frac{1}{n} \log Z_n(\cdot) \) is a sequence of convex (by Hölder’s inequality) locally uniformly bounded functions. By Jensen’s inequality, \( \lambda_n(0) = \min \lambda_n(0) \). For each \( h \) fixed, \( \log Z_n(h) \) is, by (28) super-additive in \( n \). By Proposition 2.1, \( \lambda(h) = \lim_{n \to \infty} \lambda_n(h) \) exists, and, by the above, convex and finite on \( \mathbb{R}^d \).

Let us check that \( \lambda(0) = \min_h \lambda(h) = 0 \). Assumption (A1) implies that \( \phi_\beta \) is monotone non-decreasing, \( \phi_\beta(1) = \min_\ell \phi_\beta(\ell) > 0 \), \( \tag{29} \)

and that

\[
\lim_{\ell \to \infty} \frac{\phi_\beta(\ell)}{\ell} = 0. \tag{30}
\]

Next we rely on the following well-known estimate for the underlying finite range random walk:

**Estimate 1.** There exists \( c = c(P_d) < \infty \) and \( L_0 < \infty \), such that for any \( L > L_0 \),

\[
P_d \left( \max_{\ell \leq n} |X(\ell)| \leq L \right) \sim e^{-c \frac{n}{L^2}}. \tag{31}
\]

uniformly in \( n \in \mathbb{N} \).

Let \( A_L = \{ x : |x|_1 \leq L \} \). Then,

\[
Z_n(h) \geq e^{-|h|L} \sum_{\gamma \subset B_L} W_d(\gamma) 1_{\{|\gamma| = n\}}.
\]

If \( |\gamma| = n \) and \( \gamma \subset B_L \), then

\[
\Phi_\beta(\gamma) \leq (2L + 1)^d \max_{\ell \leq n} \phi_\beta(\ell).
\]

Consequently, in view of (31),

\[
\liminf_{n \to \infty} \frac{1}{n} \log Z_n(h) \geq -\frac{c}{L^2} - (2L + 1)^d \liminf_{n \to \infty} \frac{\max_{\ell \leq n} \phi_\beta(\ell)}{n}, \tag{32}
\]
Fig. 2 Paths $\gamma \in \mathcal{P}_x^{(k)}$.

for any $L > L_0$. By (30),

$$\lim_{n \to \infty} \max_{\ell \leq n} \frac{\phi_\beta(\ell)}{n} = 0.$$  

It follows that $\lambda(h) \geq 0$ for any $h \in \mathbb{R}$. On the other hand, since the interaction potential $\phi_\beta$ is non-negative $Z_n = Z_n(0) \leq 1$, and, consequently, $\lambda(0) \leq 0$. Hence $\lambda(0) = 0 = \min_h \lambda(h)$ as claimed.

Since $\lambda \geq 0$, the set $K_0$ in (20) is convex. In order to check that it contains an open neighbourhood of the origin it would be enough to show that there exists $\delta > 0$, such that

$$\sum Z_n(h) = \sum e^{h \cdot x} G_0(x) < \infty,$$  

whenever, $|h| < \delta$. The convergence in (33) will follow as soon as we shall show that the critical two-point function $G_0(x)$ in (16) is exponentially decaying in $x$. We continue to employ notation $\mathcal{P}_x$ for paths $\gamma$ with $X(\gamma) = x$ (and, of course, with $P_d(\gamma) > 0$). Consider the disjoint decomposition

$$\mathcal{P}_x = \bigcup_{k \geq 2} \mathcal{P}_x^{(k)},$$

where (see Figure 2),

$$\mathcal{P}_x^{(k)} = \{ \gamma \in \mathcal{P}_x : \gamma \subset \Lambda_k |x|_1 \} \setminus \{ \gamma \in \mathcal{P}_x : \gamma \subset \Lambda_{(k-1)} |x|_1 \}.$$  

If $\gamma \in \mathcal{P}_x^{(k)}$, then since the range $R$ of the underlying random walk is finite,

$$\Phi_\beta(\gamma) \geq \frac{(k-1) |x|_1}{R} \inf_\ell \phi_\beta(\ell) = \frac{(k-1) |x|_1}{R} \phi_\beta(1).$$

As a result,
\[ G_0^{(k)}(x) \triangleq W_d(\mathcal{H}_x^{(k)}) \leq e^{-(k-1)|x|_1} \phi_{1/R} \sum_{\gamma \in \mathcal{H}_x^{(k)}} P_d(\gamma). \]

At this stage we shall rely on another well known estimate for short range zero-mean random walks:

**Estimate 2.** Let \( \sigma_0 \) be the first hitting time of 0. Then,

\[ E_d \left( \sum_{\ell=0}^{\sigma_0} I_{\{X(\ell)=x\}} |X(0)=x\} \right) \approx A_d(|x|) \triangleq \begin{cases} |x|, & d = 1 \\ \log |x|, & d = 2 \\ 1, & d \geq 3 \end{cases} \]  
\[(34)\]

uniformly in \( x \in \mathbb{Z}^d \). By a crude application of \( (34) \),

\[ \sum_{\gamma \in \mathcal{H}_x^{(k)}} P_d(\gamma) \leq A_d(k|x|_1) \Rightarrow G_0^{(k)}(x) \leq A_d(k|x|_1)e^{-\phi_{1/R}(k-1)|x|_1}. \]  
\[(35)\]

Therefore,

\[ G_0(x) \leq \sum_{k \geq 1} A_d((k+1)|x|_1)e^{-\phi_{1/R}(k+1)|x|_1}, \]

and \( (33) \) follows.

**Part B of Theorem 2.1.** As we have already noted, due to a possible over-counting it is not obvious that \( G_\lambda(x+y) \geq G_\lambda(x)G_\lambda(y) \). However, in view of the attractivity \( (26) \) of \( \phi_{1/R} \), the latter super-multiplicativity property holds for the following first-hitting time version \( H_\lambda \) of \( G_\lambda \):

\[ H_\lambda(x) = \sum_{\gamma \in \mathcal{H}_x} e^{-\lambda|\gamma|W_d(\gamma)} I_{\mathcal{H}_\lambda(\gamma)=1}. \]  
\[(36)\]

In particular, the limit

\[ \tau_\lambda(x) = -\lim_{r \to \infty} \frac{1}{r} \log H_\lambda \left( \langle rx \rangle \right) \]  
\[(37)\]

exists, and, by Proposition 2.1, is a non-negative, convex, homogeneous of order one function on \( \mathbb{Z}^d \). Furthermore, \( H_\lambda(x) \lesssim e^{-\tau_\lambda(x)} \).

**Exercise 2.2** Prove the above statements.

We claim that the second of \( (18) \) holds with the very same \( \tau_\lambda \). Clearly, \( H_\lambda(x) \leq G_\lambda(x) \). The proof, therefore, boils down to a derivation of a complementary upper bound, which would render negligible correction on the logarithmic scale. We shall consider two cases: Fix any \( \lambda_0 > 0 \).

**Case 1.** \( \lambda > \lambda_0 \). Then for any \( x \),

\[ G_\lambda(x) \leq H_\lambda(x)G_\lambda(0) \leq H_\lambda(x) \sum_y G_\lambda(y) \leq H_\lambda(x) \frac{1}{1-e^{-\lambda_0}}. \]  
\[(38)\]
CASE 2. \( \lambda \leq \lambda_0 \). Evidently \( \tau_\lambda \) is non-decreasing in \( \lambda \). Define:

\[
k_0 = 3 \frac{R}{\phi_0(1)} \max_y \frac{\tau_{\lambda_0}(y)}{|y|_1}.
\]

(39)

By (35),

\[
\sum_{k \geq k_0} G_\lambda^{(k)}(x) \leq \sum_{k \geq k_0} A_d(k|x|_1)e^{-\frac{\phi_0(1)}{2}(k-1)|x|_1} \leq e^{-2\tau_\lambda(x)}
\]

is exponentially negligible with respect to \( e^{-\tau_\lambda(x)} \). Consequently,

\[
G_\lambda(x) \sim \sum_{k < k_0} G_\lambda^{(k)}(x) \leq H_\lambda(x) A_d(k_0|x|_1) \leq e^{-\tau_\lambda(x)} A_d(k_0|x|_1),
\]

(40)

and the second of (18) indeed follows.

\( \tau_\lambda \) is the support function of \( K_\lambda \). In order to see this notice that \( K_\lambda = \{ h : \lambda(h) \leq \lambda \} \) is the closure of the domain of convergence

\[
h \mapsto \sum_n e^{-\lambda_n Z_n(h)} = \sum_x e^{h \cdot x} G_\lambda(x).
\]

(41)

Consider

\[
\alpha_\lambda(h) = \max \{ h \cdot x : \tau_\lambda(x) \leq 1 \}.
\]

(42)

The series in (41) diverges if \( \alpha_\lambda(h) > 1 \), whereas, \( h \in \text{int}(K_\lambda) \) if \( \alpha_\lambda(h) < 1 \). Hence,

\[
\partial K_\lambda = \{ h : \alpha_\lambda(h) = 1 \} \quad \text{or} \quad \tau_\lambda(x) = \max_{h \in \partial K_\lambda} h \cdot x.
\]

(43)

Part C of Theorem 2.1. To be precise large deviations are claimed for the distribution of end-points, which we, with a slight abuse of notation, proceed to call \( \mathbb{P}_n^h(x) \):

\[
\mathbb{P}_n^h(x) = \sum_{x(\gamma) = x} \mathbb{P}_n^h(\gamma) \triangleq \frac{e^{h \cdot x} Z_n(x)}{Z_n(h)}.
\]

(44)

Let \( h \in \mathbb{R}^d \). The limiting log-moment generation function under the sequence of measures \( \{ \mathbb{P}_n^h \} \) is

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n^h(e^{fX}) = \lambda(h + f) - \lambda(h).
\]

The function \( I_h \) in (23) is just the Legendre-Fenchel transform of the above. Since the underlying random walk has bounded range, exponential tightness is automatically ensured. By Theorem 6.3 and Exercise 6.10 of the Appendix, upper large deviation bounds hold with \( I_h \).
We still need a matching lower bound. Let us sketch the proof which relies on sub-additivity and Lemma 6.1 of Appendix. Due to a possible over-counting the function \( n \rightarrow \log Z_n(\lfloor nx \rfloor) \) is not necessarily super-additive. However, its first hitting time version (see (15) for the definition of \( \mathcal{P}_{x,n} \)):

\[
Z_n(\lfloor nx \rfloor) = \sum_{\gamma \in \mathcal{P}_{x,n}} W_d(\gamma) I_{\{\ell_\gamma(x) = 1\}},
\]

is super-additive. By Proposition 2.1 the limit

\[
J(x) = -\lim_{n \to \infty} \frac{1}{n} \log Z_n(\lfloor nx \rfloor) \quad \text{and} \quad Z_n(\lfloor nx \rfloor) \leq c e^{-nJ(x)}
\]

exists and is a convex non-negative function on \( \mathbb{R}^d \). Some care is needed to make this statement rigorous. Indeed, \( \hat{Z}_n(x) = 0 \) whenever \( x \) does not belong to the (bounded) range of the underlying \( n \)-step walk, and what happens at boundary points should be explored separately. We shall ignore this issue here.

A slight modification of arguments leading to (19) imply that

\[
J(0) = 0.
\]

By convexity this means that for any \( \alpha \in (0,1) \) and any \( x \),

\[
J(x) \leq \alpha J\left(\frac{x}{\alpha}\right),
\]

Since, \( \phi_\beta \) is non-negative,

\[
\hat{Z}_n(\lfloor nx \rfloor) \leq Z_n(\lfloor nx \rfloor) \leq \sum_{m=1}^n \hat{Z}_m(\lfloor nx \rfloor) \leq \sum_{m=1}^n c e^{-mJ(\frac{x}{m})} \leq cne^{-nJ(x)},
\]

where we used (46) and (47) on the last two steps. We conclude:

\[
J(x) = -\lim_{n \to \infty} \frac{1}{n} \log Z_n(\lfloor nx \rfloor) \quad \text{and} \quad Z_n(\lfloor nx \rfloor) \leq c e^{-nJ(x) + \log n}
\]

Now, (49) means that (213) and (217) are satisfied, the latter uniformly in \( x \). Since the range of the underlying random walk is bounded, (220), and in particular exponential tightness, is trivially satisfied as well. Hence, by Lemma 6.1, \( I = J \). Since

\[
-\frac{1}{n} \log \mathbb{P}_n^h(\lfloor nx \rfloor) = -\frac{1}{n} \log Z_n(\lfloor nx \rfloor) - \left( h \cdot \frac{|nx|}{n} - \frac{\log Z_n(h)}{n} \right),
\]

claim C of Theorem 2.1 follows as well.

**Part D of Theorem 2.1 and limiting spatial extension.** For drifts \( h \in \text{int}(K_0) \),

\[
I_h(v) = \sup_{g} \{ g \cdot v - (\lambda(h + g) - \lambda(h)) \} \geq |v| \text{dist}(h, \partial K_0) > 0,
\]

for any \( v \neq 0 \). Hence, if \( h \in \text{int}(K_0) \), then the model is sub-ballistic in the sense of (10).

On the other hand, if \( h \notin K_0 \) or, equivalently, if \( \lambda(h) > 0 \), then
\[ I_h(0) = \sup_g \{ -\lambda (g + h) + \lambda (h) \} = \lambda (h) > 0. \]

This is a rough expression of ballisticity. It implies that the polymer is pulled away from the origin on the linear scale, but it does not imply that the limit in (9) exists.

More precisely, if \( \lambda \) is differentiable at \( h \notin K_0 \), then (9) holds with \( v = \nabla \lambda (h) \). Indeed in the latter case \( I_h \) is strictly convex at \( v \) and, consequently \( I_h(v) = 0 \) is the unique minimum. Furthermore, in such a case, the following law of large numbers holds: For any \( \epsilon > 0 \),
\[
\sum_n \gamma^n_h \left( \frac{X}{n} - v \geq \epsilon \right) < \infty, \tag{50}
\]
and the series converge exponentially fast.

However, the above sub-additivity based thermodynamics of annealed polymers does not imply that the sub-differential \( \partial \lambda (h) = M_h = \{ v : I_h(v) = 0 \} \) is always a singleton. The general form of (50) is
\[
\sum_n \gamma^n_h \left( \min_{v \in \partial \lambda} \frac{X}{n} - v \geq \epsilon \right) < \infty. \tag{51}
\]
Therefore, in general, large deviations (Part C of Theorem 2.1) imply neither existence of the limit in (9), nor a LLN. The set \( M_h \) could be characterized as follows [9, 17]:

**Lemma 2.1** For any \( h \notin K_0 \) the set \( M_h \) satisfies: Set \( \mu = \lambda (h) > 0 \). Then,
\[
v \in M_h \iff \left\{ \begin{array}{c}
\tau_{\lambda} (v) = h \cdot v \\
\frac{d}{dx} \bigg|_{\lambda=\mu} \tau_{\lambda} (v) \leq 1 \leq \frac{d}{dx} \bigg|_{\lambda=\mu} \tau_{\lambda} (v)
\end{array} \right. \tag{52}
\]

**Proof.** By (23),
\[
v \in M_h \iff \sup_{\lambda} (\tau_{\lambda} (v) - \lambda) + (\mu - h \cdot v) = 0.
\]
The choice \( \lambda = \mu \) implies that \( \tau_{\lambda} (v) \leq h \cdot v \). Since \( h \in \partial K_{\mu} \), the first of (52) follows by (43). As a result,
\[
\tau_{\lambda} (v) - \tau_{\mu} (v) \leq \lambda - \mu, \tag{53}
\]
for any \( \lambda \). Since the function \( \lambda \rightarrow \tau_{\lambda} (v) \) is concave, left and right derivatives are well defined, and the second of (52) follows from (53).

The differentiability (and even analyticity) of \( \lambda \) at super-critical drifts \( h \notin K_0 \) and, in particular, the existence of the limit in (9) and the LLN (50), is established in Subsection 3.2 as a consequence of much sharper asymptotic results based on analysis of renewal structure of ballistic polymers.

**Part E of Theorem 2.1** Finally, \( I_h(0) = 0 \) whenever \( h \in \partial K_0 \), which sheds little light on ballistic properties of the model at critical drifts. The critical case was worked out in [13] via refinement of the renormalization construction of the
Ornstein-Zernike theory (see Subsection 3.2), and it is beyond the scope of these notes to reproduce the corresponding arguments here.

2.2 Thermodynamics of quenched polymers.

The underlying random walk imposes a directed graph structure on \( \mathbb{Z}^d \). Let us say that \( y \) is a neighbour of \( x \); \( x \xrightarrow{} y \) if \( P_d (y - x) > 0 \). Because of (1) and Assumption (A.2) there is a unique infinite component \( C_{\omega_\infty} \) of \( \{ x : V_{\omega_x}^\omega < \infty \} \). Clearly, non-trivial thermodynamic limits may exist only if \( 0 \in C_{\omega_\infty} \). Furthermore, if \( E (V_{\omega_x}^\omega) = \infty \), then
\[
\sum r Q (V_{\omega_x}^\omega \lceil r x \rceil > cr) = \infty \text{ for any } c > 0,
\]
and consequently,
\[
\liminf_{r \to \infty} \frac{1}{r} \log G_{\omega_\lambda}^\omega (\lceil r x \rceil) = -\infty,
\]
\( \mathcal{D} \)-a.s. for any \( x \neq 0 \). Hence, in order to define inverse correlation length \( \tau q^\lambda \) one needs either to impose more stringent requirements on disorder and use (18), or to find a more robust definition of \( \tau q^\lambda \). A more robust definition is in terms of the so called point to hyperplane exponents:

Given \( h \neq 0 \) define \( H^+ h, t = \{ x : h \cdot x \geq t \} \). Let \( \mathcal{P} h, t \) be the set of paths \( \gamma = (\gamma(0), \ldots, \gamma(n)) \) with \( \gamma(n) \in H^+ h, t \). For \( \lambda \geq 0 \) consider,
\[
D_{h,\lambda} (t) = \sum_{\gamma \in \mathcal{P} h, t} e^{-\lambda \vert \gamma \vert W_d (\gamma)}.
\]

Assume that the limit
\[
- \lim_{t \to \infty} \frac{1}{t} \log D_{h,\lambda} (t) = \Delta \frac{1}{\alpha q^\lambda (h)}
\]
exists. Should the inverse correlation length \( \tau q^\lambda \) be also defined (and positive), the following relation should hold:
\[
\frac{1}{\alpha q^\lambda (h)} = \min_{x \in H^+ h, 1} \tau q^\lambda (x).
\]

If \( \tau q^\lambda \) is the support function of a convex set \( K q^\lambda \), then, by (198) of the Appendix, \( \alpha q^\lambda \) should be the support function of the polar set \( K q^\lambda \) or, equivalently, the Minkowski function of \( K q^\lambda \).

Conversely, if the limit \( \alpha q^\lambda \) in (54) exists, then we may define \( \tau q^\lambda \) via
\[
\tau q^\lambda (x) = \max \{ h \cdot x : \alpha q^\lambda (h) \leq 1 \},
\]
even if a direct application of (18) does not make sense.

There is an extensive literature on thermodynamics of quenched models, \([30, 33, 9, 25]\) to mention a few. The paper \([25]\) contains state of the art information on the
matter, and several conditions on the random environment were worked out there in an essentially optimal form. The treatment of $\mathcal{L}(V^\omega = \infty) > 0$ case and, more generally, of $\mathcal{L}(V^\omega) = \infty$ case is based on renormalization techniques for high density site percolation and, eventually, on sub-additive ergodic theorems and large deviation arguments. It is beyond the scope of these lectures to reproduce the corresponding results here. Below we formulate some of the statements from [25] and refer to the latter paper for proofs and detailed discussions.

We assume (A1) and (A2).

**Theorem 2.2** The following happens $\mathcal{D}$-a.s on the event $0 \in \text{Cl}_\infty$:

**A.** The free energy $\lambda^g$ is well defined, deterministic, non-negative and convex on $\mathbb{R}^d$. Furthermore,

$$0 = \min_h \lambda^g(h) = \lambda^g(0).$$

The set

$$K^g_0 \overset{\Delta}{=} \{ h : \lambda^g(h) = 0 \}$$

is a compact convex set with a non-empty interior.

**B.** The point to hyperplane exponent $\alpha^g_\lambda$ in (54) is well defined for any $\lambda \geq 0$. Consequently, the inverse correlation length $\tau^g_\lambda$ is well defined via (56) also for any $\lambda \geq 0$, and, furthermore, it can be identified as the support function of the compact convex set

$$K^g_\lambda \overset{\Delta}{=} \{ h : \lambda^g(h) \leq \lambda \}.$$

Define

$$I^g_h(v) = \sup_h \{ h \cdot v - \lambda^g(h) \} = \sup_{\lambda} \{ \tau^g_\lambda(v) - \lambda \}.$$  

**C.** For any $h \in \mathbb{R}^d$ the family of polymer measures $\mathbb{P}^h_{\omega_n}$ satisfies LD principle with the rate function

$$I^g_h(v) \overset{\Delta}{=} \sup_f \{ f \cdot v - (\lambda^g(f + h) - \lambda^g(h)) \} = I^g(v) - h \cdot v - \lambda^g(h).$$

**D.** For $h \in \text{int}(K^g_0)$ the model is sub-ballistic, whereas for any $h \notin K^g_0$ the model is ballistic in the sense that $I^g_h(0) > 0$.

The above theorem does not imply strong limiting spatial extension form of the ballisticity condition $\{ f \}$ for all $h \notin K^g_0$, exactly for the same reasons as Theorem 2.1 does not imply the corresponding statement for annealed models. Existence of limiting spatial extension for quenched models in the very weak disorder regime is discussed, together with other limit theorems, in Section 4.

In the case of critical drifts $h \in \partial K^g_0$, a form of ballistic behaviour was established in the continuous context in [31].
3 Multidimensional Renewal Theory and Annealed Polymers.

3.1 Multi-dimensional renewal theory

One-dimensional renewals. Let \( \{f(n)\} \) be a probability distribution on \( \mathbb{N} \) (with strictly positive variance). We can think of \( f \) as of a probability distribution for a step \( T \) of the effective one-dimensional random walk

\[
S_N = \sum_{i=1}^{N} T_i.
\]

The distribution of \( \{S_n\} \) is governed by the product measure \( \mathbb{P} \). The renewal array \( \{t(n)\} \) is given by

\[
t(0) = 1 \quad \text{and} \quad t(n) = \sum_{m=1}^{n} f(m)t(n-m).
\]

In probabilistic terms \([62]\) reads as:

\[
t(n) = \mathbb{P}(\exists N : S_N = n) = \sum_{N} \mathbb{P}(S_N = n).
\]

Renewal theory implies that

\[
\lim_{n \to \infty} t(n) = \frac{1}{\mathbb{E}T} = \frac{1}{\mu}.
\]

A proof of \([64]\) is based on an analysis of complex power series

\[
\hat{t}(z) \triangleq \sum_{n=0}^{\infty} t(n)z^n \quad \text{and} \quad \hat{f}(z) \triangleq \sum_{n=0}^{\infty} f(n)z^n
\]

(65)

Exercise 3.1 Show that \( \hat{t} \) is absolutely convergent and hence analytic on the interior of the unit disc \( \mathbb{D}_1 = \{z : |z| < 1\} \). Check that \( \hat{t}(1) = \infty \).

It follows that on \( \mathbb{D}_1 \),

\[
\hat{t}(z) = \frac{1}{1 - \hat{f}(z)}.
\]

(66)

Exercise 3.2 Check that \( |f(z)| \leq \hat{f}(|z|) < 1 \) for any \( z \in \mathbb{D}_1 \). Prove \([66]\).

Consequently, by Cauchy formula,

\[
t(n) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{\hat{t}(z)}{z^{n+1}} \, dz = \frac{1}{2\pi i} \oint_{|z|=r} \frac{dz}{z^{n+1} (1 - \hat{f}(z))},
\]

(67)

for any \( r < 1 \).
Exponential tails. Assume that there exists $\nu > 0$, such that

$$f(n) \lesssim e^{-\nu n}, \quad (68)$$

uniformly in $n \in \mathbb{N}$.

**Lemma 3.1** Under Assumption (68) the convergence in (64) is exponentially fast in $n$.

We start proving Lemma 3.1 by noting that under (68) the function $\hat{f}$ is defined and analytic on $D_1 + \nu$.

**Exercise 3.3** Check that there exists $\varepsilon \in (0, \nu)$ such that $z = 1$ is the only zero of $1 - \hat{f}(z)$ on $\bar{D}_1 + \varepsilon$. Furthermore, $\frac{1 - z}{1 - \hat{f}(z) - 1}$ is analytic on $D_1 + \varepsilon$.

Recall that we defined $\mu = \sum_n nf(n) = \hat{f}'(1)$. Consider the representation,

$$\frac{1}{\mu} = \frac{1}{2\pi i} \int_{|z|=r} \frac{dz}{\mu(1-z)z^{n+1}},$$

which holds for any $r < 1$. By (67),

$$t(n) - \frac{1}{\mu} = \frac{1}{2\pi i} \int_{|z|=r} \frac{\hat{f}(z) - 1 - \mu(z-1)}{(1-\hat{f}(z))(1-z)z^{n+1}} dz = \frac{1}{2\pi i} \int_{|z|=r} \frac{\Delta(z)}{z^{n+1}} dz. \quad (69)$$

On $\bar{D}_1 + \varepsilon$ the denominator in the definition of $\Delta$ vanishes only at $z = 1$. However, since $\mu = \hat{f}'(1)$, expansion of the numerator in a neighbourhood of $z = 1$ gives:

$$\hat{f}(z) - 1 - \mu(z-1) = (z-1)^2 U(z),$$

with some analytic $U$. It follows that

$$\Delta(z) = \frac{U(z)}{(1-\hat{f}(z))/z - 1 - \hat{f}(z)/z - 1}. \quad (70)$$

In view of Exercise 3.3, $\Delta$ is analytic on $\bar{D}_1 + \varepsilon$. As a result,

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{\Delta(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \int_{|z|=1+\varepsilon} \frac{\Delta(z)}{z^{n+1}} dz.$$ 

By (69),

$$\left| t(n) - \frac{1}{\mu} \right| \lesssim (1+\varepsilon)^{-n},$$

uniformly in $n$. This is precisely the claim of Lemma 3.1.

**Complex renewals.** Suppose that (62) holds with complex $\{f(n)\}$ and, accordingly, with complex $\{t(n)\}$. As before, define $\hat{f}(z)$ and $\hat{f}(z)$ as in (65). In the sequel we shall work with complex renewals which satisfy one of the following two assumptions, Assumption 3.1 or Assumption 3.2 below.
Assumption 3.1 There exists $\varepsilon > 0$, such that the function $\hat{f}$ satisfies the following three properties:

(a) $\hat{f}(0) = 0$ and the $\hat{f}(z)$ in (65) is absolutely convergent in a neighbourhood of $\bar{D}_{1+\varepsilon}$.

(b) $z = 1$ is the only zero of $(\hat{f}(z) - 1)$ in $\bar{D}_{1+\varepsilon}$.

(c) $\hat{f}'(1) \neq 0$.

Under Assumption 3.1 the exponential convergence bound (70) still holds. Indeed, the only thing we have to justify is that $\hat{t}(z) = \sum_n t(n)z^n$ is defined and analytic on some neighbourhood of the origin, and that

$$\hat{t}(z) = \frac{1}{1 - \hat{f}(z)},$$

(71)

for all $|z|$ sufficiently small. Indeed, if this is the case, then (67) holds for some $r > 0$, and we may just proceed as before. However, by Assumption 3.1(a), $\sum_n |f(n)| |z|^n < 1$ for all $|z|$ small enough. Hence (71).

Assumption 3.2 There exists $\varepsilon > 0$, such that the function $\hat{f}$ satisfies the following two properties:

(a) The series $\hat{f}(z)$ in (65) is absolutely convergent in a neighbourhood of $\bar{D}_{1+\varepsilon}$.

(b) There exists $\kappa > 0$, such that $\min_{|z| \leq 1+\varepsilon} |1 - \hat{f}(z)| \geq \kappa$.

Under Assumption 3.2, the function $(1 - \hat{f}(z))^{-1}$ is analytic in a neighbourhood of $\bar{D}_{1+\varepsilon}$, and the Cauchy formula (67), which again by absolute convergence of $\hat{f}$ still holds for $r$ sufficiently small, implies:

$$|t(n)| \leq \frac{1}{\kappa(1+\varepsilon)^n}. \quad (72)$$

Multi-dimensional renewals. Let $\{f(x, n)\}$ be a probability distribution on $\mathbb{Z}^d \times N$. As in the one-dimensional case we can think of $f$ as of a probability distribution for a step $U = (X, T)$ of the effective $(d+1)$-dimensional random walk

$$S_N = \sum_1^N U_i.$$

The distribution of $\{S_n\}$ is governed by the product measure $\mathbb{P}$. We assume:

Assumption 3.3 Random vector $U = (X, T)$ has a non-degenerate $(d+1)$-dimensional distribution. The random walk $S_N$ is aperiodic (that is its support is not concentrated on a regular sub-lattice).

The renewal array $\{t(x, n)\}$ is given by

$$t(x, 0) = I_{\{x = 0\}} \quad \text{and} \quad t(x, n) = \sum_{m=1}^n \sum_y f(y, m)t(x - y, n - m). \quad (73)$$
Again, as in the one dimensional case (63), in probabilistic terms (73) reads as:

$$t(x, n) = \mathbb{P}(\exists N : S_N = (x, n)).$$  \hfill (74)

The renewal relation is inherited by one-dimensional marginals: Set

$$f(n) = \sum_x f(x, n) \quad \text{and} \quad t(n) = \sum_x t(x, n).$$

Then, (62) holds.

We are going to explore the implications of the renewal relation (73) for a local limit analysis of conditional measures

$$Q_n(x) = \frac{t(x, n)}{t(n)}. \hfill (75)$$

**Exponential tails.** Assume that there exists $$\nu > 0$$, such that

$$f(x, n) \sim e^{-\nu(|x|+n)}, \hfill (76)$$

uniformly in $$(x, n) \in \mathbb{Z}^d \times \mathbb{N}$$. In particular, (70) holds, and as a result we already have a sharp control over denominators in (75).

Consider the following equation

$$F(\xi, \lambda) \triangleq \log \sum_{x,n} e^{\xi\cdot x - \lambda n}f(x, n) = 0. \hfill (77)$$

Above $$F : \mathbb{C}^d \times \mathbb{C} \rightarrow \mathbb{C}$$.

**Exercise 3.4** Check that under (76) there exists $$\delta > 0$$ such that $$F$$ is well defined and analytic on the disc $$D^{d+1}_{\delta} \subset \mathbb{C}^{d+1}$$.

**Shape theorem.** We shall assume that $$\delta$$ is sufficiently small. Then by the analytic implicit function theorem [20], whose application is secured by Assumption 3.3, there is an analytic function $$\lambda : D^{d+1}_{\delta} \rightarrow \mathbb{C}$$ with such that for $$(\xi, \lambda) \in D^{d+1}_{\delta}$$,

$$F(\xi, \lambda) = 0 \iff \lambda = \lambda(\xi). \hfill (78)$$

For $$\xi \in D^{d}_{\delta}$$ define:

$$f_{\xi}(x, n) = f(x, n)e^{\xi\cdot x - \lambda(\xi)n} \quad \text{and} \quad t_{\xi}(x, n) = t(x, n)e^{\xi\cdot x - \lambda(\xi)n}. \hfill (79)$$

Evidently, the arrays $$\{f_{\xi}(x, n)\}$$ and $$\{t_{\xi}(x, n)\}$$ satisfy (73). Also, under (76), $$f_{\xi}(n) \triangleq \sum_x f_{\xi}(x, n)$$ is well defined for all $$|\xi| < \nu$$.

**Lemma 3.2** There exists $$\delta > 0$$ and $$\varepsilon > 0$$ such that
\[ \hat{f}_\xi(z) = \sum_n f_\xi(n) z^n, \]

satisfies Assumption 3.1 for all \(|\xi| < \delta\).

**Proof.** Conditions (a) and (c) are straightforward. In order to check (b) note that it is trivially satisfied at \(\xi = 0\). Which, by continuity means that we can fix \(\varepsilon > 0\) such that for any \(\nu > 0\) fixed, the equation

\[ \hat{f}_\xi(z) = 1 \]  \hspace{1cm} (80)

has no solutions in \(D_{1+\varepsilon} \setminus D_\nu(1)\) for all \(|\xi| < \delta\). However, the family of analytic functions \(\{\hat{f}_\xi\}_{|\xi| < \delta}\) is uniformly bounded on \(\bar{D}_\nu(1)\). Furthermore, for \(\delta > 0\) small the collection of derivatives.

\[ \left\{ \hat{f}_\xi'(1) \triangleq \mu(\xi) \triangleq \sum_n n f_\xi(n) \right\}_{|\xi| < \delta} \]  \hspace{1cm} (81)

is uniformly bounded away from zero. Therefore, there exist \(\nu > 0\) and \(\delta = \delta(\nu) > 0\), such that \(z = 1\) is the only solution of \(\hat{f}_\xi(z) = 1\) on \(\bar{D}_\nu(1)\) for all \(|\xi| < \delta\).

**Remark 4.** Note that the restriction of \(F\) to \(\mathbb{R}^{d+1} \cap D^d_\delta\) is convex, and it is monotone non-increasing in \(\lambda\). Hence, the restriction of \(\lambda\) to \(\mathbb{R}^d \cap D^d_\delta\) is convex as well. Indeed, let \(\lambda_i = \lambda(\xi_i); i = 1,2\), for two vectors \(\xi_1, \xi_2 \in \mathbb{R}^d \cap D^d_\delta\). From convexity of level set \(\{(\xi, \lambda) : F(\xi, \lambda) < 0\}\), we infer that for any convex combination \(\xi = \alpha \xi_1 + (1 - \alpha) \xi_2\)

\[ F(\xi, \alpha \lambda_1 + (1 - \alpha) \lambda_2) \leq 0 \Rightarrow \lambda(\xi) \leq \alpha \lambda_1 + (1 - \alpha) \lambda_2. \]

The term shape theorem comes from the fact that in applications function \(\lambda\) frequently describes local parametrization of the boundary of the appropriate limiting shape.

**Limit theorems** Consider the canonical measure \(Q_n\) defined in (75). The following Proposition describes ballistic behaviour under \(Q_n\).

**Proposition 3.1** Under assumption on exponential tails (76),

\[ \lim_{n \to \infty} \frac{1}{n} Q_n(X) = \lim_{n \to \infty} \frac{1}{n} \sum x t(x,n) \frac{x}{t(n)} = \frac{E X}{E T} = v. \]  \hspace{1cm} (82)

**Proof.** Note that

\[ Q_n(X) = \sum x t(x,n) \frac{x}{t(n)} = \nabla_\xi \log \left( \sum_x e^{\xi \cdot t(x,n)} \right)(0). \]

For \(|\xi|\) small we can rely on Lemma 3.1 and Lemma 3.2 to conclude that
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\[ e^{-\lambda(\xi) n} \sum_x e^{\xi \cdot x} t(x, n) = \frac{1}{\mu(\xi)} \left( 1 + o \left( (1 + \epsilon)^{-n} \right) \right), \]  

(83)

where \( \mu(\xi) \) was defined in (81). The convergence in (83) is in a sense of analytic functions on \( D^d_\delta \) for \( \delta \) small enough. As a result, the convergence,

\[ e^{-\lambda(\xi) n} Q_n \left( e^{\xi \cdot X} \right) = \frac{\mu(0)}{\mu(\xi)} \left( 1 + o \left( (1 + \epsilon)^{-n} \right) \right), \]  

(84)

is also in a sense of analytic functions on \( D^d_\delta \). Since \( \lambda(0) = 0 \),

\[ v_n \Delta = \frac{1}{n} Q_n(X) = \nabla \lambda(0) - \frac{1}{n} \nabla \log \mu(0) + o \left( (1 + \epsilon)^{-n} \right). \]  

(85)

(82) follows, since by (77)

\[ \nabla \lambda(0) = E X E T = v. \]  

(86)

**Integral central limit theorem.** For \( \xi \in \mathbb{C}^d \) consider

\[ \phi_n(\xi) = Q_n \left( \exp \{ i(X - n v_n) \cdot \xi \} \right). \]  

(87)

For any \( R \) fixed, \( \phi_n \left( \frac{\xi}{\sqrt{n}} \right) \) is well defined on \( D^d_R \). By (84) and (85),

\[ \phi_n \left( \frac{\xi}{\sqrt{n}} \right) = \exp \left\{ n \left( \lambda(\frac{i \xi}{\sqrt{n}}) - \nabla \lambda(0) \cdot \frac{i \xi}{\sqrt{n}} \right) + O \left( \frac{R}{\sqrt{n}} \right) \right\} = e^{-\frac{1}{2} \Xi \xi \cdot \xi} + O \left( \frac{R}{\sqrt{n}} \right), \]  

(88)

in the sense of analytic functions on \( D^d_R \). Above \( \Xi \triangleq \text{Hess}(\lambda) \). We claim:

**Lemma 3.3** \( \Xi \) is a positive definite \( d \times d \) matrix.

**Proof.** Fix \( \xi \in \mathbb{R}^d \setminus 0 \) and consider (78):

\[ \sum_{x, n} f(x, n) e^{\xi \cdot x - n \lambda(\xi)} \equiv 1, \]

which holds for all \( |\epsilon| < \delta / |\xi| \). The second order expansion gives:

\[ \text{Hess} \lambda(0) \xi \cdot \xi = \frac{1}{E T} E \left( \left( X - \frac{E X}{E T} \right) \cdot \xi \right)^2. \]

The claim of the lemma follows from the non-degeneracy Assumption 3.3. In view of Lemma 3.3, asymptotic formula (88) already implies the integral form of the CLT: The family of random vectors \( \frac{1}{\sqrt{n}} (X - n v_n) \) weakly converges (under \( \{ Q_n \} \)) to \( N(0, \Xi) \).

**Local CLT.** \( \phi_n \) is related to the characteristic function of \( X \) in the following way: For any \( \theta \in \mathbb{R}^d \),
\[ \phi_n(\theta) = e^{-inv_\theta}Q_n(e^{i\theta}x) = \frac{e^{-inv_\theta}}{t(n)} \sum_x t(x,n)e^{i\theta}. \]

The complex array \( \{ t(x,n)e^{ix\theta} \} \) is generated via multi-dimensional renewal relation (73) by \( \{ f(x,n)e^{ix\theta} \} \). Since \( \{ f(x,n) \} \) is a non-degenerate probability distribution on \( \mathbb{Z}^d \) with exponentially decaying tails, for any \( \delta > 0 \) one can find \( \kappa = \kappa(\delta) > 0 \) and \( \varepsilon = \varepsilon(\delta) \), such that the array \( \{ f(x,n)e^{ix\theta} \} \) satisfies Assumption 3.2 uniformly in \( \varepsilon \geq \delta \). We conclude:

**Lemma 3.4** For any \( \delta > 0 \) there exists \( c_\delta > 0 \) such that

\[ |\phi_n(\theta)| \leq e^{-c_\delta n}, \quad (89) \]

whenever \( \theta \in \mathbb{R}^d \) satisfies \( |\theta| \geq \delta \).

One applies Lemma 3.4 as follows: By the Fourier inversion formula,

\[ Q_n(x) = \frac{1}{(2\pi)^d} \int_{2\pi^d} e^{-i\theta \cdot (x-n\nu_n)} \phi_n(\theta) d\theta. \quad (90) \]

Choose \( \delta, \varepsilon > 0 \) small. The above integral splits into the sum of three terms:

\[ \int e^{-i\theta \cdot (x-n\nu_n)} \phi_n(\theta) d\theta = \int_{A_n} e^{-i\theta \cdot (x-n\nu_n)} \phi_n(\theta) d\theta + \int_{B_n} e^{-i\theta \cdot (x-n\nu_n)} \phi_n(\theta) d\theta + \int_{|\theta| \geq \delta} e^{-i\theta \cdot (x-n\nu_n)} \phi_n(\theta) d\theta. \quad (91) \]

Above \( A_n = \{ \theta : |\theta| < n^{-1/2+\varepsilon} \} \) and \( B_n = \{ \theta : n^{-1/2+\varepsilon} \leq |\theta| < \delta \} \). The third integral is negligible by Lemma 3.4. In order to control the second integral (over \( B_n \)) note that for \( \delta \) small enough (84) applies, and hence, in view of positive definiteness of \( \Sigma \),

\[ \left| \int_{B_n} e^{-i\theta \cdot (x-n\nu_n)} \phi_n(\theta) d\theta \right| \leq \int_{B_n} e^{-\frac{\varepsilon}{2} \Sigma \cdot \theta} d\theta. \]

The first integral in (91) gives local CLT asymptotics uniformly in \( |x-n\nu_n| = o \left( n^{1/2-(d+1)\varepsilon} \right) \). Namely, for such \( x \)-s

\[ \int_{|\theta| < n^{-1/2+\varepsilon}} e^{-i\theta \cdot (x-n\nu_n)} \phi_n(\theta) d\theta = \int_{|\theta| < n^{-1/2+\varepsilon}} \phi_n(\theta) d\theta + o \left( \frac{1}{n^{d\varepsilon}} \right). \]

As in (88),

\[ \phi_n(\theta) = e^{-\frac{\varepsilon}{2} \Sigma \cdot \theta} + o(|\theta|^2), \]

uniformly in \( |\theta| < n^{-1/2+\varepsilon} \). We have proved:

**Proposition 3.2** For any fixed \( \varepsilon > 0 \) the asymptotic relation:
\[
Q_n(x) = \frac{1}{\sqrt{(2\pi n)^d \det \Sigma}} (1 + o(1)), \quad (92)
\]

holds uniformly in \( |x - nv_n| \lesssim n^{\frac{1}{2} - \varepsilon} \).

In fact, as it will become clear from local large deviations estimates below, it would be enough to state (92) only for \( |x - nv_n| \leq 1 \).

**Local large deviations estimates.** Assumption (76) implies that the family of measures \( \{Q_n\} \) is exponentially tight. Furthermore, by the very definition of \( t \) in (73)
\[
t(x + y, n + m) \geq t(x, n) t(y, m).
\]

Hence, by the sub-additivity argument the function
\[
J(u) = -\lim_{n \to \infty} \frac{1}{n} \log t(n, \lfloor nu \rfloor) \quad (93)
\]
is well defined and convex on \( \mathbb{R}^d \). By the renewal theorem (64),
\[
J(u) = -\lim_{n \to \infty} \frac{1}{n} \log Q_n(\lfloor nu \rfloor).
\]

Consequently, \( \{Q_n\} \) satisfies the large deviation principle with \( J \).

A large deviation result states what it states. Obviously, \( J \) in (93) is non-negative, and \( \min J = J(v) = 0 \), where \( v \) was defined in (82). We shall show that \( J \) has a quadratic minimum on \( B^d_\kappa(v) \), and prove a local LD asymptotic relation for any \( u \in B^d_\kappa(v) \).

Recall that \( \lambda \) is analytic (and convex) on a (real) ball \( B^d_\delta \). Since, as we already know by Lemma 3.3, \( \text{Hess} \lambda(0) = 0 \), there exists \( \kappa > 0 \) such that
\[
B^d_\kappa \subset \nabla \lambda \big|_{B^d_\delta}.
\]

Let \( u \in B^d_\kappa(v) \). Set \( u_n = \lfloor nu \rfloor / n \) and choose \( \xi_n \in B^d_\delta \) such that \( u_n = \nabla \lambda(\xi_n) \). By (94) such \( \xi_n \) exists (at least for all \( n \) sufficiently large), and, for it is unique by the implicit function theorem. Recall how we defined tilted function \( t_{\xi_n} \) in (79). Then,
\[
t(n, \lfloor nu \rfloor) = e^{\lambda(\xi_n) - \xi_n \cdot \lfloor nu \rfloor} t_{\xi_n}(n, \lfloor nu \rfloor) \quad (95)
\]
The term \( t_{\xi_n}(n, \lfloor nu \rfloor) \) obeys uniform sharp CLT asymptotics (92) with \( \Sigma(\xi_n) = \text{Hess} \lambda(\xi_n) \). The term
\[
J(u_n) = \xi_n \cdot u_n - \lambda(\xi_n)
\]
is quadratic. Indeed, for any \( \eta \in B^d_\kappa(v) \) and \( w = \nabla \lambda(\eta) \), one, using \( \lambda(0) = 0 \), can rewrite:
\[
\eta \cdot w - \lambda(\eta) = \lambda(0) - \lambda(\eta) - (-\eta) \cdot \nabla \lambda(\eta)
\]
\[
= \left\{ \int_0^1 \int_0^\tau \text{Hess} \lambda((1 - \tau)\eta) d\tau ds \right\} \eta \cdot \eta, \quad (96)
\]
and rely on non-degeneracy of Hess(λ) on B^d_κ. Incidentally, we have checked that on B^d_κ(v) the function J = λ^* is real analytic with Hess(J)(v) being positive definite.

The local limit estimates we have derived reads as: Recall notation u_n = [nu]/n and ξ_n being defined via u_n = ∇λ(ξ_n). Then,

\[ Q_n([nu]) = \frac{\mu(0)}{\mu(\xi_n) \sqrt{(2\pi)^d \det \Xi(\xi_n)}} e^{-n(J(u_n))}(1 + o(1)), \tag{97} \]

uniformly in u \in B^d_κ(v).

### 3.2 Ballistic phase of annealed polymers

Recall that the reference polymer weights W_d are given by \(\Phi\) is the self-interaction potential which satisfies the attractivity condition (26). Let h \notin K_0 and, accordingly, h \in \partial K_λ with λ = λ(h) > 0. We shall consider the normalized weights

\[ W^{h,\lambda}_d(γ) = e^{h \cdot X(γ) - \lambda |γ|} W_d(γ). \tag{98} \]

These weights are normalized for the following reason: As before define

\[ \mathcal{P}_x = \{ γ : X(γ) = x \}, \quad \mathcal{P}_n = \{ γ : |γ| = n \} \quad \text{and} \quad \mathcal{P}_{x,n} = \mathcal{P}_x \cap \mathcal{P}_n. \tag{99} \]

Then,

\[ \sum_{γ \in \mathcal{P}_x} W^{h,\lambda}_d(γ) = e^{h \cdot x} G_λ(x) \geq |e^{h \cdot x} - \tau_λ(x)| \quad \text{and} \quad \sum_{γ \in \mathcal{P}_n} W^{h,\lambda}_d(γ) = e^{-n\lambda} Z_n(h) \asymp 1. \tag{100} \]

If h \cdot x = \tau_λ(x), then the first term in (100) is also of order 1. More generally, let us define the following crucial notion:

**Surcharge function.** For any x \in \mathbb{R}^d define the surcharge function

\[ s_h(x) = \tau_λ(x) - h \cdot x \geq 0. \tag{101} \]

In view of (40) the first of the estimates in (100) could be upgraded as follows (see (34) for the definition of A_d)

\[ W^{h,\lambda}_d(x) \leq \sum_{γ \in \mathcal{P}_x} W^{h,\lambda}_d(γ) \asymp e^{-s_h(x)} \quad \text{and} \quad W^{h,\lambda}_d(x) \asymp A_d(k_0 |x|_1) e^{-s_h(x)}. \tag{102} \]

**Surcharge cone.** Let us say that \(\mathcal{C}_h\) is a δ_h-surcharge cone with respect to h if:

(a) \(\mathcal{C}_h\) is a positive cone (meaning that its opening is strictly less than \(\pi\)) and it contains a lattice direction \(± e_k\) in its interior.

(b) For any x \notin \mathcal{C}_h the surcharge function s satisfies

\[ s(x) = \tau_λ(x) - h \cdot x > δ_1 \tau_λ(x). \tag{103} \]
For the rest of this section we shall fix $\delta_1 \in (0, 1)$ and a $\delta_1$-surcharge cone $\mathcal{B}_1$ with respect to $h$.

**Factorization bound.** Assume that the path $\gamma$ can be represented as a concatenation,

$$\gamma = \eta_0 \circ \eta_1 \circ \cdots \circ \eta_m \circ \eta_{m+1},$$  \hspace{1cm} (104)

such that paths $\gamma_i = (u_i, \ldots, v_{i+1})$ satisfy the following two properties:

(P1) $\gamma_i$ is disjoint from $\gamma_j$ for all $i > j$.

(P2) For any $i$ the local time $\ell_i(\gamma_{i+1}) = 1$.

By (29) and (P1),

$$\Phi_B(\gamma) \geq \Phi_B(\gamma \cup \cdots \cup \gamma_m) = \sum_{\ell} \Phi_B(\gamma_i).$$

Consequently,

$$W_d(\gamma) e^{-\lambda |\gamma|} \leq \prod_{i=1}^{m} W_d(\gamma_i) e^{-\lambda |\gamma_i|} \prod_{k=1}^{m} e^{-\lambda |\eta_k|}.$$  \hspace{1cm} (105)

Fixing end points $u_1, v_2, u_2, \ldots$ and paths $\eta_i$ in (104), and summing up with respect to all paths $\gamma_1, \ldots, \gamma_m$ (with $\gamma_i = (u_i, \ldots, v_{i+1})$) satisfying properties (P1) and (P2) above we derive the following upper bound:

$$\sum_{\gamma_1, \ldots, \gamma_m} W_d(\gamma) e^{-\lambda |\gamma|} \leq \prod_{i=1}^{m} H_d(v_{i+1} - u_i) e^{-\lambda \sum |\eta_i|} \leq e^{-\lambda \sum \tau_i (v_{i+1} - u_i) - \lambda \sum |\eta_i|}.$$  \hspace{1cm} (106)

Let us proceed with describing our algorithm to construct representation (104) with properties (P1) and (B2) for any path $\gamma \in \mathcal{B}$.

**Construction of skeletons.** Skeletons $\hat{\gamma}_K$ are constructed as a collection $\hat{\gamma}_K = \{t_K, h_K\}$, where $t_K$ is the trunk and $h_K$ is the set of hairs of $\hat{\gamma}_K$. Let $\gamma \in \mathcal{B}_x$ and choose a scale $K$. In the sequel we use $U^K_x = \{u : \tau_x(u) \leq K\}$ denote the ball of radius $K$ with respect to $\tau_x$ (note that since, in general, $\tau_x(y) \neq \tau_x(-y)$, it does not have to be a distance). Recall that $R$ denotes the range of the underlying random walk. Choose $r = r_\lambda = \min \{s : \mathbb{B}^d \subset U^K_x\}$, where as before $\mathbb{B}^d$ is the Euclidean ball of radius $R$. Let us first explain decomposition (104) and construction of trunks (see Figure 3).

**STEP 0.** Set $u_0 = 0$, $\tau_0 = 0$ and $u_0 = \{u_0\}$. Go to **STEP 1**.

**STEP (l+1)** If $(\gamma(\tau_l), \ldots, \gamma(n)) \subseteq U^K_x(u_l)$ then set $\sigma_{l+1} = n$ and stop. Otherwise, define

$$\sigma_{l+1} = \min \{i > \tau_l : \gamma(i) \not\in U^K_x(u_l)\}$$

and

$$\tau_{l+1} = 1 + \max \{i > \tau_l : \gamma(i) \in U^{K+\epsilon}_x(u_l)\}.$$  \hspace{1cm} \square
Clearly the above algorithm leads to a decomposition of $\gamma$ as in (104) with
\[ \gamma_l = (\gamma(\tau_l), \ldots, \gamma(\sigma_l + 1)) \quad \text{and} \quad \eta_l = (\gamma(\sigma_l), \ldots, \gamma(\tau_l)), \]
and with $\gamma_1, \gamma_2, \ldots$ satisfying conditions (P1) and (P2).

The set $t_K$ is called the trunk of the skeleton $\hat{\gamma}_K$ of $\gamma$ on $K$-th scale. The hairs $h_K$ of $\hat{\gamma}_K$ take into account those $\eta_l$-s which are long on $K$-th scale. Recall that $\eta_l : v_l \mapsto u_l$. It is equivalent, but, since eventually we want to keep track of vertices from the trunk $t_K$, more convenient to think about $\eta_l$ as of a reversed path from $u_l$ to $v_l$. Then the $l$-th hair $h_K^l = h_K[\eta_l]$ of $\gamma$ is constructed as follows:

If $\eta_l \subseteq U^K_\lambda(u_l)$ then $h_K^l = \emptyset$. Otherwise, set $u = u_l, v = v_l, \eta = \eta_l, m = |\eta|$, and proceed with the following algorithm (see Figure 4):

\[ \text{Fig. 4 Construction of a hair} \]
\[ h_K = \{w_1, \ldots, w_4\}. \]

**STEP 0.** Set $w_0 = u, \tau_0 = 0$ and $h_K[\eta] = \emptyset$. Go to **STEP 1.**

**STEP (l+1).** If $(\eta(\tau_l), \ldots, \eta(m)) \subseteq U^K_\lambda(u_l)$ then stop. Otherwise set
\[ \tau_{l+1} = \min \{ j > \tau_l : \eta(j) \notin U^K_\lambda(w_l) \}. \]

Define $w_{l+1} = \eta(\tau_{l+1})$, update $h_K = h_K \cup \{w_{l+1}\}$ and go to **STEP (l+2).** \(\square\)

**Control of** $W^{h,\lambda}_{d}(\hat{\gamma}_K)$. In the super-critical case $\lambda > 0$, and hairs could be controlled in a crude fashion via comparison with an underlying walk killed at rate $\lambda$.

**Exercise 3.5** There exists $\varepsilon = \varepsilon(P_d, \lambda) > 0$, such that
the following happens: Let \( \hat{\tau}_U \gamma \) Each realization of \( F \) and \( h \) metrics).

Choosing \( u \) paths is bounded above by the product of \( e^{-\epsilon \tau_2(u)} \) Now, by construction \( v_{t+1} \in U^{K+r_\lambda}(u_t) \setminus U^K(u_t) \), which means that \( \tau_\lambda(v_{t+1} - u_t) \in [K, K + r_\lambda] \). On the other hand if \( u_{t+1} \) is defined, then \( u_{t+1} \in U^{K+2r_\lambda}(u_t) \setminus U^{K+r_\lambda}(u_t) \) Hence, \( \tau_\lambda(u_{t+1} - u_t) \leq \tau_2(v_{t+1} - u_t) + 2r_\lambda \). There are \( \lesssim RK_d \) possible exit points from \( U^K(v_t) \)-balls which are possible candidates for \( v_{t+1} \) vertices. Consequently, (106) and (107) imply that there exists \( c = c(\lambda, \beta) > 0 \) such that the following happens: Let \( \hat{h}_K = (t_K, h_K) \) be a skeleton with trunk \( t_K = (u_0, \ldots, u_N) \), and \( h_K = \{ h_\ell \} \) collection of hairs. Notation \( \hat{h}_K \) means that \( \hat{h}_K \) is the \( K \)-skeleton of \( \gamma \) in the sense of the two algorithms above. Then,

\[
\sum_{\gamma \sim \hat{h}_K} W^h_d \gamma d \sim \exp \left\{ h \cdot x - \sum_{\ell=0}^{N} \tau_\lambda(u_{t+1} - u_t) - \epsilon K \#(h_K) + cN \log K \right\} , \tag{108}
\]

uniformly in \( x \), large enough scales \( K \) and skeletons \( \hat{h}_K \).

**Kesten’s bound on the number of forests.** A forest \( \mathcal{F}_N \) is a collection of \( N \) rooted trees \( \mathcal{F}_N = (T_1, \ldots, T_N) \) of forward branching ratio at most \( b \). The tree \( T_i \) is rooted at \( u_i \). Given \( M \in \mathbb{N} \) we wish to derive an upper bound on \#(\( M, N \))-number of all forests \( \mathcal{F}_N \) satisfying \( |\mathcal{F}_N| = M \). Above \( |\mathcal{F}_N| = M \) is the number of vertices of \( \mathcal{F}_N \) different from the roots \( u_1, \ldots, u_N \). Let \( \mathbb{P}^N \) be the product percolation measure on \( \times T^b \) at the percolation value \( p \), where \( T^b \) is the set of (edge) percolation configurations on the rooted tree of branching ratio \( b \). In this way \( T_i \) is viewed as a connected component of \( u_i \). Clearly,

\[
\mathbb{P}^N_{p}(|\mathcal{F}_N| = M) \leq 1. \tag{109}
\]

Each realization of \( \mathcal{F}_N \) with \( |\mathcal{F}_N| = M \) has probability which is bounded below by \( p^M (1-p)^{(N-M)} \). Therefore, (109) implies:

\[
\#(M, N) \leq \left( \max_{p \in [0,1]} p^M (1-p)^{(N-M)} \right)^{-1} . \tag{110}
\]

For \( x \in [0,1] \),

\[
\log(1-x) = - \int_0^x \frac{dt}{1-t} \geq - \frac{x}{1-x} .
\]

Choosing \( p = \frac{1}{b} \) we, therefore, infer from (110)
implies: There exist \( \varepsilon', c' > 0 \), such that

\[
\sum_{y \sim h_k \atop \#(y_k) \geq M} W^{h, \lambda}(y) \lesssim \exp \left\{ h \cdot x - \sum_{\ell=0}^{N} \tau_k(u_{\ell+1} - u_\ell) - \varepsilon' K M + c' N \log K \right\}. \tag{112}
\]

uniformly in \( x \), scales \( K \), trunks \( t_k \) and \( M \in \mathbb{N} \).

With (112) in mind let us define the surcharge cost of a skeleton \( \gamma_k = [t_k, h_k] \) as follows: Recall the notation \( s_h(u) = \tau(u) - h \cdot u \). Then,

\[
g_h(\gamma_k) = \sum_{\ell=1}^{N} s_h(u_\ell - u_{\ell-1}) + \varepsilon' K \#(h_k). \tag{113}
\]

Above the trunk \( t_k = (u_0, \ldots, u_N) \). We conclude:

**Lemma 3.5** For any \( \varepsilon > 0 \) there exists a scale \( K_0 \), such that

\[
\sum_{s_h(\gamma_k) > 2 \varepsilon |x|} W^{h, \lambda}(\gamma_k) \lesssim e^{-\varepsilon |x|}, \tag{114}
\]

for all \( K \geq K_0 \) fixed and uniformly in \( h \in \partial K_\lambda, x \in \mathbb{Z}^d \). By convention the summation above is with respect to skeletons \( \gamma_k \) of paths \( \gamma \in \mathcal{P}_x \).

**Cone points of skeletons.** Recall the definition of the surcharge cone \( \mathcal{Y}_i \) in (103).

Fix \( \delta_2 \in (\delta_1, 1) \) and \( \delta_3 \in (\delta_2, 1) \), and define enlargements \( \mathcal{Y}_i \) of \( \mathcal{Y}_1 \) as follows: For \( i = 2, 3 \),

\[
\mathcal{Y}_i = \{ x : s(x) \leq \delta_i \tau_2(x) \} \tag{115}
\]

Clearly, \( \mathcal{Y}_i \)-s are still positive cones for \( i = 2, 3 \). Let \( A^0 \overset{A}{=} A \setminus 0 \). Then, by construction, \( \mathcal{Y}_1^0 \subset \text{int}(\mathcal{Y}_2^0) \) and \( \mathcal{Y}_2^0 \subset \text{int}(\mathcal{Y}_3^0) \).

Consider a skeleton \( \gamma_k = [t_k, h_k] \). Let us say that a vertex of the trunk \( u_\ell \in t_k = (u_0, \ldots, u_\ell, \ldots, u_m) \) is a \( \mathcal{Y}_2 \)-cone point of the skeleton \( \gamma_k \) if

\[
\gamma_k \subset (u_\ell - \mathcal{Y}_2) \cup (u_\ell + \mathcal{Y}_2). \tag{116}
\]

Let \( \#_{\text{bc}}(\gamma_k) \) be the total number of vertices of \( \gamma_k \) which are not cone points.

**Proposition 3.3** There exists \( v_2 > 0 \) such that the following happens: For any \( \varepsilon > 0 \) there exists a scale \( K_0 \), such that

\[
\sum_{\#_{\text{bc}}(\gamma_k) > \varepsilon |x|} W^{h, \lambda}(\gamma_k) \lesssim e^{-v_2 |x|}, \tag{117}
\]

for all \( K \geq K_0 \) fixed and uniformly in \( x \in \mathbb{Z}^d \).
Fig. 5 $u$ is a cone point of the path $\gamma = (\gamma_0, \ldots, \gamma_n)$. Black vertices belong to the trunk. Paths leading from white vertices to black vertices give rise to hairs.

A proof of Proposition 3.3 contains several steps and we refer to [15] for more details. First of all we show that, up to exponentially small corrections, most of the vertices of the trunk $t_K$ are $\mathcal{Y}_1$-cone points of the latter. We shall end up with $N' \sim |x|^{\delta_1}$ $\mathcal{Y}_1$-cone points of $t_K$.

Any $\mathcal{Y}_1$-cone point of the trunk $t_K$ is evidently also a $\mathcal{Y}_2$-cone point of the latter. On the other hand, in view of (114), we can restrict attention to $\#(h_K) \leq \frac{2|\epsilon|}{K}$. For $\epsilon \ll \epsilon'$ the total number of leaves $\#(h_K)$ is only a small fraction of $N'$. It is clear that an addition of a leaf is capable of blocking at most $c = c(\delta_1, \delta_2)$ $\mathcal{Y}_1$-cone points of $t_K$ from being a $\mathcal{Y}_2$-cone point of the whole skeleton $\hat{\gamma}_K$. It is important that the above geometric constant $c = c(\delta_1, \delta_2)$ does not depend on the running scale $K$. Consequently, under the reduction we are working with on large enough scales $K$, there are just not enough leaves to block all (and actually a small fraction of) $\mathcal{Y}_1$-cone points of $t_K$ from being a $\mathcal{Y}_2$-cone point of the whole skeleton $\hat{\gamma}_K$.

Cone points of paths $\gamma \in \mathcal{P}_x$. Let us say that $u_{\ell} \in \gamma = (u_0, \ldots, u_{\ell}, \ldots, u_n) \in \mathcal{P}_x$ is a cone point of $\gamma$ if $0 < \ell < n$ and

$$\gamma \subset (u_{\ell} - \mathcal{Y}_1) \cup (u_{\ell} + \mathcal{Y}_1).$$

Let $\#_{\text{cone}}(\gamma)$ be the total number of the cone points of $\gamma$.

**Proposition 3.4** There exist $\epsilon > 0$ and $\nu > 0$ such that:

$$\sum_{\#_{\text{cone}}(\gamma) < \epsilon |x|} \mathcal{W}_d^{h, \lambda}(\gamma) \lesssim e^{-\nu |x|},$$

uniformly in $x \in \mathbb{Z}^d$.

As before, we refer to [15] for details of the proof. Construction of cone points is depicted on Figure 5.

**Irreducible decomposition of paths in $\mathcal{P}_x$, $\mathcal{P}_n$ and $\mathcal{P}_{x,n}$.** In the sequel we set $\mathcal{Y} = \mathcal{Y}_3$, where $\mathcal{Y}_3$ is the positive cone in Proposition 3.4. A path $\gamma = (u_0, \ldots, u_n)$ is
said to be irreducible if it does not contain \( \mathcal{Y} \)-cone points. We shall work with three sub-families \( \mathcal{F}[l], \mathcal{F}[r] \) and \( \mathcal{F} = \mathcal{F}[l] \cap \mathcal{F}[r] \) of irreducible paths. Those are defined as follows:

\[
\mathcal{F}[l] = \{ \gamma \text{ irreducible} : \gamma \subset u_n - \mathcal{Y} \}, \quad \mathcal{F}[r] = \{ \gamma \text{ irreducible} : \gamma \subset u_0 + \mathcal{Y} \}.
\]

Note that any \( \gamma = (u_0, \ldots, u_n) \in \mathcal{F} \) is automatically confined to the diamond shape \( \gamma \subset D(u_0, u_n) \triangleq (u_0 + \mathcal{Y}) \cap (u_n - \mathcal{Y}) \).

Proposition 119 implies that up to corrections of order \( e^{-\nu|x|} \) one can restrict attention to paths \( \gamma \in \mathcal{P}_x \) which have the following decomposition into irreducible pieces (see Figure 6):

\[
\gamma = \gamma[l] \circ \gamma[1] \circ \cdots \circ \gamma[N] \circ \gamma[r].
\]

Define

\[
f[l](x, n) = \sum_{X(\gamma) = x, |\gamma| = n} W^d_{h, \lambda}(\gamma) \quad \text{and} \quad f[r](x, n) = \sum_{X(\gamma) = x, |\gamma| = n} W^d_{h, \lambda}(\gamma).
\]

**Theorem 3.1** The weights \( f[l] \) and \( f[r] \) have exponentially decaying tails: There exists \( \nu > 0 \) and, for every \( \lambda > 0 \), \( \chi_\lambda > 0 \) such that the following mass gap estimate holds uniformly in \( x \) and \( n \):

\[
f[l](x, n), f[r](x, n) \lesssim e^{-\nu|x| - \chi_\lambda n}.
\]

Furthermore, for each \( \lambda = \lambda(h) > 0 \), \( W^d_{h, \lambda} \) is a probability distribution on \( \mathcal{F} \). In particular, the family of weights \( \{f(x, n)\} \).
\[ f(x, n) = \sum_{\gamma \in \mathcal{F}} W_d^{h, \lambda}(\gamma) \overset{\Delta}{=} \sum_{\gamma \in \mathcal{F}, \gamma = n} W_d^{h, \lambda}(\gamma) \]  

is a probability distribution on \( \mathbb{Z}^d \times \mathbb{N} \) with exponentially decaying tails.

**Remark 5.** Note that exponential decay in \( n \) is claimed only if \( \lambda > 0 \). \( \lambda = 0 \) corresponds to the case of critical drifts \( h = \partial K_0 \). For critical drifts, the decay in \( n \) is sub-exponential and, furthermore, the whole coarse-graining (skeleton construction) procedure should be modified. It happens, nevertheless, that the decay in \( x \) is still exponential. We do not discuss critical case in these lecture notes, and refer to [18].

**Proof.** For fixed \( \lambda > 0 \) bounds \( f^{[i]}(x, n), f^{[i]}(x, n) \leq e^{-\nu|x|} \) directly follow from Proposition [3.4]. The case to work out is when \( |x| \) is much smaller than \( n \), say \( |x| < \frac{1}{2} n \). But then \( e^{n|x-\lambda n|} \leq e^{-(\lambda-\varepsilon)n} \). Since by (40) the two-point function \( G_0(\cdot) \) is bounded, the decay in \( n \) indeed comes for free as long as \( \lambda > 0 \).

In order to see that \( \{f(x, n)\} \) is a probability distribution recall that \( K_\lambda \) was characterized as the closure of the domain of convergence of \( h \mapsto \sum G_h^n\gamma W_d(\gamma) \).

Thinking in terms of (121), and in view of (123) this necessarily implies that \( \sum f(x, n) = 1 \).

**Local geometry of \( \partial K_\lambda \) and analyticity of \( \lambda \).** Inspecting construction of the cone \( \mathcal{Y} \) for \( h \in \partial K_\lambda \) we readily infer that the very same \( \mathcal{Y} \) would do for all drifts \( g \in (h + B^d) \cap \partial K_\lambda \), for some \( \varepsilon > 0 \) sufficiently small. Similarly, it would do for all \( |\mu - \lambda| \) sufficiently small. We are in the general renewal framework of (77). The following theorem is a consequence of (78), Lemma 3.3 of Subsection 3.1 and (202) of the Appendix:

**Theorem 3.2** There exists \( \varepsilon = \varepsilon_\lambda > 0 \), such that the following happens: Let \( h \) be a super-critical drift; \( \lambda(h) = \lambda > 0 \). Construct \( \mathcal{Y} \) and, accordingly, \( \{f(x, n)\} \) as in Theorem [3.1]. Then for \( (g, \mu) \in B^d_{\varepsilon} \) \( g(\cdot) \in \mathbb{R}^d \) is non-degenerate.

As a result, \( \lambda(\cdot) \) is real analytic on \( \mathbb{B}_\varepsilon^d(h) \) and \( \mathbb{Z}(h) = \text{Hess}(\lambda)(h) \) is non-degenerate.

In particular,

\[ g \in (h + B^d_{\varepsilon}) \cap \partial K_\lambda \iff \sum_{x,n} e^{(g-h)\cdot x + (\lambda - \mu)n} f(x, n) = 1. \]  

As a result, \( \partial K_\lambda \) is locally analytic and has a uniformly positive Gaussian curvature.

**Ornstein-Zernike Theory.** Theorem [3.1] paves the way for an application of the multidimensional renewal theory, as described in Subsection 3.1 to a study of various limit properties of annealed measures \( P^h_n \); whenever \( h \in K_0 \) is a super-critical drift. For the rest of the section let us fix such \( h \) and \( \lambda = \lambda(h) > 0 \). By the above,
this generates a cone $\mathcal{V}$ and a probability distribution \( f(x,n) \) with exponentially decaying tails. We declare that it is a probability distribution of a random vector $U = (X, T) \in \mathbb{Z}^d \times \mathbb{N}$. In view of our assumptions on the underlying random walk, $U$ satisfies the non-degeneracy Assumption 3.3, which means that it is in the framework of the theory developed therein.

Let us construct the array \( \{t(x,n)\} \) via the renewal relation (7). The number $t(x,n)$ has the following meaning: Recall our definition $D(x,y) = (x+\mathcal{V}) \cap (y-\mathcal{V})$ of diamond shapes, and define the following three families of diamond-confined paths:

\[ \mathcal{F}_x = \{ \gamma \in \mathcal{P}_x : \gamma \subset D(0,x) \}, \quad \mathcal{F}_n = \{ \gamma \in \mathcal{P}_n : \gamma \subset D(0,\gamma_n) \} \text{ and } \mathcal{F}_{x,n} = \mathcal{F}_x \cap \mathcal{F}_n. \]

As before, $t(x) = \sum_n t(x,n)$ and $t(n) = \sum_x t(x,n)$. Then,

\[ t(x,n) = \sum_{\gamma \in \mathcal{F}_{x,n}} W_d^{h,\lambda}(\gamma), \quad t(x) = \sum_{\gamma \in \mathcal{F}_x} W_d^{h,\lambda}(\gamma) \text{ and } t(n) = \sum_{\gamma \in \mathcal{F}_n} W_d^{h,\lambda}(\gamma). \]

**Asymptotics of partition functions.** By Theorem 3.1, the partition function $Z_n(h)$ in (7) satisfies:

\[ e^{-nh} Z_n(h) = W_d^{h,\lambda}(\mathcal{P}_n) = O(e^{-X_n}) + \sum_{k+m+j=n} f^{[i]}(k)t(m)f^{[j]}(j). \]

Define $\kappa(h) = \left( \sum_k f^{[i]}(k) \right) \left( \sum_j f^{[j]}(j) \right)$ and $\mu(h) = \sumnf(n) = \mathbb{E}T$. By Lemma 3.1

\[ \lim_{n \to \infty} e^{-nh} Z_n(h) = \frac{\kappa(h)}{\mu(h)}, \]

exponentially fast.

**Limiting spatial extension and other limit theorems.** Since $\lambda$ is differentiable at any $h \not\in K_0$, (9) and LLN (50) follow with $\nu = \nabla(h)$. However, since for any $h \not\in K_0$ the probability distribution \( \{f(x,n)\} \) in (124) has exponential tails much sharper local limit results follow along the lines of Subsection 3.1. Let $g \not\in K_0$ and $u = \nabla\lambda(g)$. Fix $\delta$ sufficiently small and consider $\mathbb{P}_\delta^d(u)$. For $w \in \mathbb{P}_\delta^d(u)$ define $w_n = \lfloor nw \rfloor / n$ and let $g_n$ being defined via $w_n = \nabla\lambda(g_n)$.

**Theorem 3.3** There exists a positive real analytic function $\psi$ on $\mathbb{P}_\delta^d(u)$ such that

\[ \mathbb{P}_n^h(\lfloor nw \rfloor) = \frac{\psi(w)}{\sqrt{(2\pi)^d\det(\mathbb{E}g_n)}} e^{-nh(w_n)} (1 + o(1)), \]

uniformly in $w \in \mathbb{P}_\delta^d(u)$.

In particular (considering $u = \nu$), under $\mathbb{P}_n^h$ the distribution of the rescaled endpoint $\mathbb{X}_n^\nu / \sqrt{n}$ converges to the $d$-dimensional mean-zero normal distribution with covariance matrix $\mathbb{E}(h) = \text{Hess}(\lambda)(h)$. 

Let us turn to the (Ornstein-Zernike) asymptotics of the two point function $G_{\lambda}$. Let $x \neq 0$ and $h = \nabla \tau_{\lambda}(x)$, that is $h \in \partial K_{\lambda}$ and $\tau_{\lambda}(x) = h \cdot x$. Since, as we already know, $\partial K_{\lambda}$ is strictly convex, such $h$ is unambiguously defined. Under the weights $W_{h}^{\lambda}$, the irreducible decomposition (121) folds in the sense that the $W_{h}^{\lambda}$-weight of all paths $\gamma \in \mathcal{P}_{x}$ which do not comply with it, is exponentially negligible as compared to $G_{\lambda}(x)$. Hence:

$$e^{\tau_{\lambda}(x)} G_{\lambda}(x) (1 + o(1)) = \sum_{n} t(x, n) = \frac{c(h)}{\sqrt{|x|^{d-1}}} (1 + o(1)), \quad (132)$$

asymptotically in $x$ large. This follows from (131) and Gaussian summation formula.

**Invariance principles.** There are two possible setups for formulating invariance principles for annealed polymers. The first is when we consider $P_{h}^{n}$, and accordingly polymers $\gamma$ with fixed number $n$ of steps. In this case one defines

$$x_{n}(t) = \frac{1}{\sqrt{n}} (\gamma_{\lfloor nt \rfloor} - nt v),$$

and concludes from Theorem 3.3 that $x_{n}(\cdot)$ converges to a $d$-dimensional Brownian motion with covariance matrix $\Xi(h)$.

A somewhat different $(d-1)$-dimensional invariance principle holds in the conjugate ensemble of crossing polymers. To define the latter fix $x \neq 0$, $\lambda > 0$ and consider the following probability distribution $P_{x}^{\lambda}$ on the family $\mathcal{P}_{x}$ of all polymers $\gamma$ which have displacement $X(\gamma) = x$:

$$P_{x}^{\lambda}(\gamma) = \frac{1}{G_{\lambda}(x)} e^{-\lambda |\gamma|/W_{d}(\gamma)}. \quad (133)$$

Consider again the irreducible decomposition (121) of paths $\gamma \in \mathcal{P}_{x}$. Let $0, u_{1}, \ldots, u_{N+1}, x$ be the end-points of the corresponding irreducible paths. We can approximate $\gamma$ by a linear interpolation through these vertices. We employ the language of Subsection 6.2 of the Appendix. Let $n(h) = \frac{x}{|x|}$ and $v_{1}, \ldots, v_{d-1}$ are unit vectors in the direction of principal curvatures of $\partial K_{\lambda}$ at $h$. In the orthogonal frame $(v_{1}, \ldots, v_{d-1}, n(h))$, the linear interpolation through the vertices of the irreducible decomposition of $\gamma$ can be represented as a function $Y : [0, |x|] \mapsto T_{h} \partial K_{\lambda}$. Consider the rescaling

$$y_{x}(t) = \frac{1}{\sqrt{x}} Y(t |x|).$$

Then, (132) and the quadratic expansion formula (207) of the Appendix leads to the following conclusion: Let $x_{m}$ be a sequence of points with $|x_{m}| \to \infty$ and $\lim_{m \to \infty} \frac{x_{m}}{|x_{m}|} = n(h)$. Then the distribution of $y_{x_{m}}(\cdot)$ under $P_{x_{m}}^{\lambda}$ converges to the distribution of the $(d-1)$-dimensional Brownian bridge with the diagonal covariance
matrix \( \text{diag} (\chi_1(h), \ldots, \chi_{d-1}(h)) \), where \( \chi_i(h) \) are the principal curvatures of \( \partial K_\lambda \) at \( h \).

4 Very weak disorder in \( d \geq 4 \).

The notion of very weak disorder depends on the dimension \( d \geq 4 \) and on the pulling force \( h \neq 0 \). It is quantified in terms of continuous positive non-decreasing functions \( \zeta_d \) on \( (0, \infty) \): \( \lim_{h \downarrow 0} \zeta_d(h) = 0 \). Function \( \zeta_d \) does not have an independent physical meaning: It is needed to ensure a certain percolation property \((143)\), and to ensure validity of a certain \( L_2 \)-type estimate formulated in Lemma \(4.1\) below.

**Definition 1.** Let us say that the polymer model \((8)\) is in the regime of very weak disorder if \( h \neq 0 \) and

\[
\phi_\beta(1) \leq \zeta_d(|h|).
\]  

(134)

**Remark 6.** \((134)\) is a technical condition, and it has three main implications as it is explained below after formulation of Theorem \(4.2\).

**Quenched Polymers at very weak disorder.** In the regime of very weak disorder quenched polymers behave like their annealed counter-parts. Precisely: For \( h \neq 0 \), continue to use \( v_h = \nabla \lambda(h) \) and \( \Xi_h = \text{Hess}(\lambda)(h) \) for the limiting spatial extension and the diffusivity of the annealed model. Let \( \text{Cl}_\infty \) be the unique infinite connected cluster of \( \{ x : \mathbb{V}_\omega x < \infty \} \).

**Theorem 4.1** Fix \( h \neq 0 \). Then, in the regime of very weak disorder, the following holds \( \mathcal{L}\)-a.s. on the event \( \{ 0 \in \text{Cl}_\infty \} \):

- The limit
  \[
  \lim_{n \to \infty} \frac{Z^{\omega}_n(h)}{Z_n(h)}
  \]  
  exists and is a strictly positive, square-integrable random variable.
- For every \( \varepsilon > 0 \),
  \[
  \sum_n \mathbb{P}^{h,\omega}_n \left( \left| \frac{X(\gamma)}{n} - v_h \right| > \varepsilon \right) < \infty.
  \]  
  (136)
- For every \( \alpha \in \mathbb{R}^d \),
  \[
  \lim_{n \to \infty} \mathbb{P}^{h,\omega}_n \left( \exp \left\{ \frac{i\alpha}{\sqrt{n}} (X(\gamma) - nv_h) \right\} \right) = \exp \left\{ -\frac{1}{2} \Xi_h \alpha \cdot \alpha \right\}.
  \]  
  (137)

**Remark 7.** Since in the regime of very weak disorder \( h \notin K_0 \), the series in \((50)\) converge exponentially fast. Using \( Z^{\omega}_{n_0}(A|h) \) for the restriction of \( Z^{\omega}_n \) to paths from \( A \) we conclude (from exponential Markov inequality) that there exists \( c = c(\varepsilon) > 0 \) such that

\[
\sum_n \mathcal{Q} \left( Z^{\omega}_n \left( \left| \frac{X(\gamma)}{n} - v_h \right| > \varepsilon \right) | h \right) > e^{-cn} Z_n(h) < \infty.
\]
In other words, (136) routinely follows from (135) and exponential bounds on annealed polymers.

**Reformulation in terms of basic partition functions.** For each \( h \neq 0 \) and \( \beta > 0 \) basic annealed partition functions were defined in (128). Here is the corresponding definition of basic quenched partition functions:

\[
t^\omega(x, n) = \sum_{\gamma \in T^\omega(x, n)} e^{h X(\gamma) - \lambda |\gamma|} W^\omega_{\gamma}(\gamma),
\]

and, accordingly \( t^\omega(n) = \sum_x t^\omega(x, n) \) and \( t^\omega(x) = \sum_n t^\omega(x, n) \).

Let \( \mathcal{Y} = \mathcal{Y}_h \) be the cone used to define irreducible paths. Then \( Cl^h_\infty \) is the infinite connected component (unique if exists) of \( \{ x \in \mathcal{Y}_h : V^\omega_x < \infty \} \).

**Theorem 4.2** Fix \( h \neq 0 \). Then, in the regime of very weak disorder, infinite connected cluster \( Cl^h_\infty \) exists \( Q \)-a.s. Furthermore, the following holds \( Q \)-a.s. on the event \( \{ 0 \in Cl^h_\infty \} \):

- The limit
  \[
  s^\omega = \lim_{n \to \infty} \frac{t^\omega(n)}{t(n)},
  \]
  exists and is a strictly positive, square-integrable random variable.
- For every \( \alpha \in \mathbb{R}^d+1 \),
  \[
  \lim_{n \to \infty} \frac{1}{t^\omega(n)} \sum_x \exp \left\{ \frac{i\alpha}{\sqrt{n}} \cdot (x - mv) \right\} t^\omega(x, n) = \exp \left\{ -\frac{1}{2} \Sigma \alpha \cdot \alpha \right\},
  \]

Below we shall explain the proof of (139). The \( Q \)-a.s. CLT follows in a rather similar fashion, albeit with some additional technicalities, and we refer to [19] for the complete proof.

**Three properties of very weak disorder.** The role of (technical) condition \( \phi_\beta(1) \leq \zeta_d(|h|) \) is threefold:

First of all, setting \( p_d = Q(\mathcal{Y}^\omega = \infty) \) and noting that

\[
\zeta_d(|h|) \geq \phi_\beta(1) = -\log E \left[ e^{-\beta V^\omega} \right] \geq -\log(1 - p_d) \geq p_d,
\]

we conclude that (134) implies that:

\[
Q(\mathcal{Y}^\omega = \infty) \leq \zeta_d(|h|).
\]

Thus, in view of (141), condition (134) implies that the infinite cluster \( Cl^h_\infty \) exists: Namely if we choose \( \zeta_d \) such that \( \sup_h \zeta_d(|h|) \) is sufficiently small, then

\[
Q \left( \text{there is an infinite cluster } Cl^h_\infty \right) = 1,
\]

for all situations in question.
The second implication is that for any \( h \neq 0 \) fixed, \( h \not\in K_0(\beta) \) for all \( \beta \) sufficiently small. In other words, in the regime of very weak disorder the annealed model is always in the ballistic phase. Indeed, since \( \phi_\beta(\ell) \leq \ell \phi_\beta(1) \),

\[
Z_n(h) \geq e^{-n \phi_\beta(1)} \left( E_n e^{h X} \right)^n .
\]

Consequently \( \lambda(h) > 0 \) whenever \( \log \left( E_n e^{h X} \right) > \phi_\beta(1) \).

The third implication is an \( L_2 \)-estimate \( \text{[144]} \) below. Recall that for \( h \) fixed, annealed measures \( \mathbb{E}_n^h \) have limiting spatial extension \( \nu_h = \nabla \lambda(h) \), and they satisfy sharp classical local limit asymptotics around this value.

For a subset \( A \subseteq \mathbb{Z}^d \), let \( \mathcal{A} \) be the \( \sigma \)-algebra generated by \( \{ V^\omega_A \}_{x \in A} \). We shall call such \( \sigma \)-algebras cylindrical.

**Lemma 4.1** For any dimension \( d \geq 4 \) there exists a positive non-decreasing function \( \xi_d \) on \( (0, \infty) \) and a number \( \rho < 1/12 \) such that the following holds: if \( \phi_\beta(1) < \xi_d(h) \), then there exist constants \( c_1, c_2 < \infty \) such that

\[
\begin{align*}
&\left| \mathbb{E} \left[ t^\omega(x, \ell) t^\omega(y, \ell) \mathbb{E} \left[ \frac{\left| f^\omega_\theta(z, m) - f(z, m) \right|}{\mathcal{A}} \mathbb{E} \left[ \frac{\left| f^\omega_\theta(w, k) - f(w, k) \right|}{\mathcal{A}} \right] \right] \right] \right| \\
&\leq c_1 e^{-c_2 \left( m + k \right)} \frac{\ell^d - \rho}{\ell^d} \exp \left( -c_2 \left( \frac{|x - y|}{\ell} + \frac{|x - [\nu_h]^2}{\ell} + \frac{|y - \nu_h|^2}{\ell} \right) \right) ,
\end{align*}
\]

(144)

for all \( x, y, z, w, m, k \) and all cylindrical \( \sigma \)-algebras \( \mathcal{A} \) such that both \( t^\omega(x, \ell) \) and \( t^\omega(y, \ell) \) are \( \mathcal{A} \)-measurable.

**Remark 8.** Since the underlying random walk is of bounded range,

\[
\# \{ z : f^\omega(z, m) \neq 0 \} \lesssim m^d .
\]

Hence \( \text{[144]} \) also holds with \( f^\omega_\theta(m) \) and \( f^\omega_\theta(k) \) instead of \( f^\omega_\theta(z, m) \) and \( f^\omega_\theta(w, k) \).

There is nothing sacred about the condition \( \rho < 1/12 \). We just need \( \rho \) to be sufficiently small. In fact, \( \text{[144]} \) holds with \( \rho = 0 \), although a proof of such statement would be a bit more involved.

We refer to [19] (Lemma 2.1 there) for a proof of Lemma 4.1. The claim \( \text{[144]} \) has a transparent meaning: For \( \rho = 0 \), the expression

\[
\frac{c_1}{\ell^d} \exp \left( -c_2 \left( \frac{|x - [\nu_h]^2}{\ell} + \frac{|y - \nu_h|^2}{\ell} \right) \right)
\]

is just the local limit bound on the annealed quantity \( t(x, \ell) t(y, \ell) \). The term \( e^{-c_2 \left( m + k \right)} \) reflects exponential decay of irreducible terms \( f^\omega_\theta(z, m) \) and, accordingly, \( f^\omega_\theta(w, k) \). The term \( e^{-c_2 |x - y|} \) appears for the following reason (see Figure 7): By irreducibility, \( f^\omega_\theta(z, m) - f(z, m) \) depends only on \( V^\omega_u \) with \( u \) belonging to the diamond shape \( D(x, x + z) \). Similarly, \( f^\omega_\theta(w, m) - f(w, m) \) depends
only on variables inside $D(y, y + w)$. All these terms have zero mean. Consequently,
\[
\mathcal{E} \left[ t^{\alpha}(x, \ell) t^{\alpha}(y, \ell) \mathcal{E} \left( f^{\beta, \alpha}(z, m) - f(z, m) \right) \right] = 0,
\]
whenever $D(x, x + z) \cap D(y, y + w) = \emptyset$. The remaining terms satisfy $\max\{z, w\} \sim |x - y|$. In other words, the term $e^{-c|x-y|}$ also reflects exponential decay of irreducible connections.

![Diagram](image)

**Fig. 7** Kites $t^{\alpha}(x, \ell) \mathcal{E} \left( f^{\beta, \alpha}(z, m) - f(z, m) \right)$ and $t^{\alpha}(y, \ell) \mathcal{E} \left( f^{\beta, \alpha}(w, k) - f(w, k) \right)$ may have a non-zero covariance only if their diamond shapes $D(x, x + z)$ and $D(y, y + w)$ intersect.

The disorder imposes an attractive interaction between two replicas. The impact of small $\xi_d(|h|)$ condition in the very weak disorder regime, as formulated in (144), is that this interaction is not strong enough to destroy individual annealed asymptotics.

**Sinai’s decomposition of $t^{\alpha}(x, n)$**. We rely on an expansion similar to the one employed by Sinai [29] in the context of directed polymers. By construction quantities $t^{\alpha}(x, n)$ satisfy the following (random) renewal relation:
\[
t^{\alpha}(x, 0) = 1_{\{x=0\}} \quad \text{and} \quad t^{\alpha}(x, n) = \sum_{m=1}^{n} f^{\alpha}(y, m) t^{\beta, \alpha}(x - y, n - m). \quad (145)
\]

Iterating in (145) we obtain (for $n > 0$):
\[
t^{\alpha}(x, n) = f^{\alpha}(x, n) + \sum_{r=1}^{\infty} \sum_{x_1, n_1 + \cdots + n_r = n} \prod_{i=1}^{r} f^{\beta, \alpha}(x_i - y_{i-1}, n_i),
\]
where $x_0 \overset{\Delta}{=} 0$. Writing,
\[
f^{\alpha}(y, m) = f(y, m) + (f^{\alpha}(y, m) - f(y, m)),
\]
we, after expansion and re-summation, arrive to the following decomposition:
\[ t^\theta(x, n) = t(x, n) + \sum_{\ell=0}^{n-1} \sum_{m=1}^{n-\ell} \sum_{r=0}^{\ell-m} t^\theta(y, \ell) \left( f^\theta_{\omega}(z - y, m) - f(z - y, m) \right) t(x - z, r). \] (146)

In particular, the decomposition of \( t^\theta(n) \) is given by

\[ t^\theta(n) = t(n) + \sum_{\ell=0}^{n-1} \sum_{m=1}^{n-\ell} \sum_{r=0}^{\ell-m} t^\theta(y, \ell) \left( f^\theta_{\omega}(m) - f(m) \right) t(r). \] (147)

In order to prove (140) one needs to consider the full decomposition (146). As it was already mentioned, we shall not do it here and, instead, refer to [19]. From now on, we shall concentrate on proving (139) and, accordingly shall consider the reduced decomposition (147). Nevertheless, modulo additional technicalities, the proof of (139) captures all essential features of the argument.

Recall that for the annealed quantities, \( \lim_{n \to \infty} t(n) = \frac{1}{\mu(n)} = \frac{1}{\mu} \) exponentially fast. Writing \( t(m) = \frac{1}{\mu} + (t(m) - \frac{1}{\mu}) \) in all the corresponding terms in (147), we infer that \( t^\theta(n) \) can be represented as

\[ t^\theta(n) = \frac{1}{\mu} s^\theta(n) + \varepsilon^\theta_n + \left( t(n) - \frac{1}{\mu} \right) \] (148)

where

\[ s^\theta(n) = 1 + \sum_{\ell \leq n} \sum_{x} t^\theta(x, \ell) \left( f^\theta_{\omega} - 1 \right), \] (149)

and the correction term \( \varepsilon^\theta_n \) is given by

\[ \varepsilon^\theta_n = \varepsilon_{n, 1} - \varepsilon_{n, 2} = \sum_{\ell + m + r = n} \sum_{x} t^\theta(x, \ell) \left( f^\theta_{\omega}(m) - f(m) \right) \left( t(r) - \frac{1}{\mu} \right) + \frac{1}{\mu} \sum_{\ell \leq n} \sum_{m > n - \ell} \sum_{x} t^\theta(x, \ell) \left( f^\theta_{\omega}(m) - f(m) \right). \] (150)

The term \( t(n) - \frac{1}{\mu} \) is negligible. Our target claim (139) is a direct consequence of the following proposition:

**Proposition 4.1** In the very weak disorder regime the following happens \( \mathcal{P} \)-a.s.:

\[ \lim_{n \to \infty} s^\theta(n) = s^\theta = 1 + \sum_{x} t^\theta(x) \left( f^\theta_{\omega} - 1 \right) \] and \( \sum_n \varepsilon^\theta_n \) is finite. (151)

Furthermore, \( s^\theta > 0 \) on the set \( \{ 0 \in C_{\theta}^\mu \} \).

**Remark 9.** Note that the formula for \( s^\theta \) is compatible with the common sense if the random walk is trapped (case \( \mathcal{B}(V^\theta = \infty) > 0 \)). Indeed, in such situation \( \lim_{n \to \infty} t^\theta(n) \) should be clearly zero. On the other hand, if the random walk is trapped, then the sum \( 1 + \sum_x t^\theta(x) \left( f^\theta_{\omega} - 1 \right) \) contains only finitely many non-zero terms. Using \( t^\theta(0) = 1 \), let us rewrite it as
However, for $x \neq 0$,
\[
t^\omega(x) = \sum_x t^\omega(x) f^\omega f_0 - \sum_x t^\omega(x) f^\omega f_0 \neq 0.
\]

**Mixingale form of $s^\omega(n)$ and $\varepsilon^\omega_m$.** Let us rewrite $s^\omega(n)$ as
\[
s^\omega(n) = 1 + \sum_{\ell \leq n} Y_\ell \quad \text{where} \quad Y_\ell = \sum_x t^\omega(x, \ell) \left( f^\omega f_0 - 1 \right),
\]
(152)
The variables $Y_\ell$ are mean zero, and it is easy to deduce from the basic L₂-estimate that in the regime of weak disorder, $\sum Y^2 < \infty$. Should $\{Y_\ell\}$ be a martingale difference sequence (as in the case of directed polymers), we would be done. However, since, in principle, same vertices $x$ may appear in different $Y_\ell$-s, there seems to be no natural martingale structure at our disposal. Instead one should make a proper use of mixing properties of $\{Y_\ell\}$. Hence the name mixingale, which was introduced in [24]. In order to prove convergence of $s^\omega(n)$ we shall rely on the mixingale approach developed in [24].

Turning to the correction terms in (150), note that both $\varepsilon^\omega_{n,1}$ and $\varepsilon^\omega_{n,2}$ could be written in the form
\[
\sum_{\ell \leq n} \sum_x t^\omega(x, \ell) \sum_m a^{(n)}(\ell, m) \left( f^\omega f_0(m) - f(m) \right) \Delta \sum_{\ell \leq n} Z^{(n)}_\ell,
\]
(153)
where
\[
a^{(n)}(\ell, m) = \left( t(n - \ell - m) - \frac{1}{\mu} \right) I_{\ell + m \leq n} \quad \text{and} \quad a^{(n)}(\ell, m) = -I_{\ell + m > n}
\]
(154)
respectively in the cases of $\varepsilon^\omega_{n,1}$ and $\varepsilon^\omega_{n,2}$. Again, $\{Z^{(n)}_\ell\}$ is not a martingale difference sequence, and we shall rely on the mixingale approach of [24] for deducing their second convergence statement in (151).

Below we shall formulate a particular case of the maximal inequality for mixingales [24]. To keep relation with quenched polymers, and specifically with (152) and (153), in mind, let us introduce the following filtration $\{\mathcal{A}_m\}$. Recall that the end-point of the $m$-step annealed polymer stays close to $m\nu_h = m\nabla \lambda(h)$. Define half-spaces $\mathcal{H}_m^-$ and the corresponding $\sigma$-algebras $\mathcal{A}_m$ as
\[
\mathcal{H}_m^- = \left\{ x \in \mathbb{Z}^d : x \cdot \nu_h \leq m||\nu_h||^2 \right\} \quad \text{and} \quad \mathcal{A}_m = \sigma \left\{ \mathcal{V}_{x,0}^\omega : x \in \mathcal{H}_m^- \right\}
\]
(155)

**Mixingale maximal inequality and convergence theorem of McLeish.** Let $Y_1, Y_2, \ldots$ be a sequence of zero-mean, square-integrable random variables. Let also $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a filtration of $\sigma$-algebras. Suppose that there exist $\varepsilon > 0$ and numbers $d_1, d_2, \ldots$
in such a way that
\[
\mathcal{E} \left( \mathbb{E} \left( Y_\ell \middle| \mathcal{F}_{\ell-k} \right)^2 \right) \leq \frac{d_\ell^2}{(1+k)^{1+\varepsilon}} \quad \text{and} \quad \mathcal{E} \left( \mathbb{E} (Y_\ell - \mathbb{E} (Y_\ell \middle| \mathcal{F}_{\ell+k}) )^2 \right) \leq \frac{d_\ell^2}{(1+k)^{1+\varepsilon}}
\]  
(156)
for all \( \ell = 1, 2, \ldots \) and \( k \geq 0 \). Then there exists \( K = K(\varepsilon) < \infty \) such that, for all \( n_1 \leq n_2 \),
\[
\mathcal{E} \left\{ \max_{n_1 \leq r \leq n_2} \left( \sum_{n_1}^{r} Y_\ell \right)^2 \right\} \leq K \sum_{n_1}^{n_2} d_\ell^2.
\]
(157)
In particular, if \( \sum d_\ell^2 < \infty \), then \( \sum Y_\ell \) converges \( \mathcal{D} \)-a.s. and in \( L_2 \).

**Convergence of** \( s^\omega(n) \). Consider decomposition (152). Clearly,
\[
\mathcal{E} \left( Y_\ell \middle| \mathcal{F}_{\ell-k} \right) = \sum_{x \in \mathcal{H}_{\ell-k}} t^\omega(x, \ell) \left( f^{\theta, \omega} - 1 \right)
\]
(158)
Applying (144) we conclude that for any \( x, y \in \mathcal{H}_{\ell-k} \),
\[
\left| \mathcal{E} \left[ t^\omega(x, \ell) t^\omega(y, \ell) \mathcal{E} \left( f^{\theta, \omega} - 1 \middle| \mathcal{F}_{\ell-k} \right) \mathcal{E} \left( f^{\theta, \omega} - 1 \middle| \mathcal{F}_{\ell-k} \right) \right] \right| \\
\leq c_3 \frac{d_\ell^2}{d_\ell^{1-\rho}} \exp \left\{ -c_2 \left( |x - y| + \frac{|x - \ell y|}{\ell} + \frac{|y - \ell y|}{\ell} \right) \right\},
\]
(159)
Consequently, summing up with respect to \( x, y \in \mathcal{H}_{\ell-k} \) we infer that for any \( \varepsilon \geq 2\rho \):
\[
\mathcal{E} \left( \mathcal{E} \left( Y_\ell \middle| \mathcal{F}_{\ell-k} \right)^2 \right) \leq c_6 e^{-c_4 \frac{d_\ell^2}{d_\ell^{1-\rho}}} \leq \frac{c_6}{d_\ell^{1-\rho}} \frac{1}{d_\ell^{(d-1)/2-\varepsilon(1+k)^{1+\varepsilon}}} = \frac{d_\ell^2}{(1+k)^{1+\varepsilon}}.
\]
(160)
On the last step we have relied on a trivial asymptotic inequality
\[
e^{-c_4 \frac{d_\ell^2}{d_\ell^{1-\rho}}} \leq \frac{1}{(1+k)^{1+\varepsilon}}.
\]
Note that if \( (d-1)/2 - \varepsilon > 1 \), which is compatible with \( d \geq 4 \) and \( \rho < 1/12 \), then \( \sum d_\ell^2 < \infty \).
Turning to the second condition in (156) note first of all that \( \mathcal{E} \left( t^\omega(x, \ell) \left( f^{\theta, \omega} - 1 \right) \middle| \mathcal{F}_{\ell+k} \right) = 0 \) whenever \( x \in \mathcal{H}_{\ell+k} \), and
\[
\mathcal{E} \left( t^\omega(x, \ell) \left( f^{\theta, \omega} - f(y) \right) \middle| \mathcal{F}_{\ell+k} \right) = t^\omega(x, \ell) \left( f^{\theta, \omega}(y) - f(y) \right)
\]
whenever \( x + y \in \mathcal{H}_{\ell+k} \). Therefore,
\[ Y_{t+s} - \mathcal{L}(Y_t|\mathcal{A}_{t+k}) = \sum_{x \in \mathcal{H}_{t+k}} t^{\omega}(x, t) \left(f^{\theta,\omega} - 1\right) + \sum_{x \in \mathcal{H}_{t+k}} \sum_{y \in \mathcal{H}_{t+k}} t^{\omega}(x, t) \left(f^{\theta,\omega}(y-x) - \mathcal{L}(f^{\theta,\omega}(y-x)|\mathcal{A}_{t+k})\right). \]  

(161)

The first term in (161) has exactly the same structure as (158). The second term in (161) happens to be even more localized (see discussion of (2.14) in [19]). The conclusion is:

\[ \mathcal{L}(Y_{t+s} - \mathcal{L}(Y_t|\mathcal{A}_{t+k}))^2 \leq \frac{d_{t+k}^2}{(1+k)^{1+\epsilon}}. \]  

(162)

where \(d_{t+k}^2 \sim \ell^{-(d-1)/2+\epsilon}\).

Set \(d_t^2 = \max\{d_{t-1}^2, d_{t+1}^2\} \sim \ell^{-(d-1)/2+\epsilon}\), and, in view of the feasible choice \((d-1)/2 - \epsilon > 1\), conclude from (157) that \(s^{\omega}(n)\) is indeed a \(\mathcal{D}\)-a.s. converging sequence.

**Correction terms.** Treatment of correction terms in their mixing angle representation (153) follows a similar pattern. We refer to Section 2.2 in [19] for the proof of the second claim in (151).

**Positivity of \(s^{\omega}\).** As we have already checked the sum \(s^{\omega} = 1 + \sum_{x} t^{\omega}(x) (f^{\theta,\omega} - 1)\) converges \(\mathcal{D}\)-a.s. and in \(L_2\). In particular, \(\mathcal{L}(s^{\omega}) = 1\). We claim that \(s^{\omega} > 0\), \(\mathcal{D}\)-a.s. on the event \(\{0 \in \text{Cl}^+_n\}\). In order to prove this it would be enough to check that

\[ \mathcal{L}(\exists x \in \mathcal{Y} : s^{\theta,\omega} > 0) = 1. \]  

(163)

Let us sketch the argument: If \(s^{\theta,\omega} > 0\), then \(x \in \text{Cl}^+_n\). But there is exactly one infinite cluster in \(\mathcal{Y}\). Hence \(0\) is connected to \(x\) by a finite path \(y \in \text{Cl}^+_n \subset \mathcal{Y}\). Now, by assumption on \(x\), \(\lim_{n \to \infty} t^{\theta,\omega}(n) = \lim_{n \to \infty} \sum_{z} t^{\theta,\omega}(z, n) > 0\). By comparison with annealed quantities (large deviations, for instance (77)) we, at least for large \(n\), may ignore terms \(t^{\theta,\omega}(z, n)\) with \(y \nsubseteq (x+z) - \mathcal{Y}\). Which means that \(\liminf_{n \to \infty} t^{\omega}(n) \geq W_{d}^{h,\lambda,\omega}(y) s^{\theta,\omega} > 0\), where \(W_{d}^{h,\lambda,\omega}(y) = e^{hX(y)-\lambda|y|}W_{d}(y) > 0\).

It remains to check (163). Consider sets

\[ B_n = \partial \mathcal{H}^+_n \cap \mathcal{Y}, \quad |B_n| \approx n^{d-1}. \]

We refer to the last Subsection of [16] for the proof of the following statement:

\[ \left|\text{Cov}(s^{\theta,\omega}, s^{\theta,\omega})\right| \approx \frac{1}{|y-x|^{d-2-\tau}}, \]  

(164)

uniformly in \(n\) and in \(x, y \in B_n\). The quantity \(\frac{1}{|y-x|^{d-2-\tau}}\) in (164) represents an intersection probability for trajectories of two ballistic \(d\)-dimensional random walks (such as the effective random walks with step distribution \(f(w, m)\)) which start at \(x\).
and y. The statement (164) is very similar in spirit to that of Lemma 4.1: it says that possible weak attraction due to disorder does not destroy such asymptotics. With (164) at our disposal it is very easy to finish the proof of (163). Indeed, it implies that
\[ \text{Var} \left( \frac{1}{n^{d-1}} \sum_{x \in B_n} s^{\theta_{x,\omega}} \right) \lesssim \frac{1}{n^{d/2-1}}, \]
and since \( E ( \sum_{x \in B_n} s^{\theta_{x,\omega}} ) = |B_n| \sim n^{d-1} \), the conclusion follows by Chebychev inequality and Borel-Cantelli argument.

5 Strong disorder

In this section, we work only under Assumption A1 of the Introduction, and we do not impose any further assumptions on the environment \( \{ V_{\omega} \} \). The case of traps; \( Q ( V_{\omega} = \infty ) \in (0, 1) \), is not excluded and we even do not need A2 or any other restriction on the size of the latter probability.

The environment is always strong in two dimensions in the following sense (level L1 in the language of the Introduction):

**Theorem 5.1** Let \( d = 2 \) and \( \beta, \lambda > 0 \). There exists \( c = c(\beta, \lambda) > 0 \) such that the following holds: Let \( \lambda(h) = \lambda \) (in particular \( h \notin K_0 \)). Then, \( \mathcal{Q} \)-a.s.

\[ \limsup_{n \to \infty} \frac{1}{n} \log \frac{Z_{\omega}^n(h)}{Z_n(h)} < -c. \]  

(165)

In particular, \( \lambda_{\omega}(h) < \lambda(h) = \lambda \) whenever \( \lambda_{\omega} \) is well defined.

**Remark 10.** As in [22] and, subsequently, [35] proving strong disorder in dimension \( d = 3 \) is a substantially more delicate task.

Let us explain Theorem 5.1: By the exponential Markov inequality (and Borel-Cantelli) it is sufficient to prove that there exist \( c' > 0 \) and \( \alpha > 0 \) such that

\[ \mathcal{E} \left\{ \left( \frac{Z_{\omega}^n(h)}{Z_n(h)} \right)^{\alpha} \right\} \leq e^{-c'n}. \]  

(166)

We shall try to establish (166) with \( \alpha \in (0, 1) \). This is the fractional moment method of [22]. It has a transparent logic: Since \( \mathcal{E} (Z_{\omega}^n(h)) = Z_n(h) \), expecting (165) means that \( Z_{\omega}^n(h) \) takes excessive exponentially high values with exponentially small probabilities. Taking fractional moments in (166) amounts to truncating these high values.

**Reduction to Basic Partition Functions.** Recall the definition of diamond-confined (basic) partition functions \( f(x, n) = \mathcal{E} (f^\omega(x, n)) \), \( t(x, n) = \mathcal{E} (t^\omega(x, n)) \) and, accordingly, \( f(x) \), \( t(x) \), ... in (128). Since \( \lim_{n \to \infty} t(n) = \mu(h)^{-1} \), theorem target statement (165) would follow from
\[
\limsup_{n \to \infty} \frac{1}{n} \log t^\omega(n) = \limsup_{n \to \infty} \frac{1}{n} \log t^\omega(n) < 0. \quad (167)
\]

In its turn, in view of Theorem 3.1, (167) is routinely implied by the following statement ((169) below): Let \( r_N^\omega \) be the partition function of \( N \) irreducible steps:

\[
r_N^\omega = \sum_{u_1, \ldots, u_N} f^\omega(u_1) f^\omega_1(u_2 - u_1) \cdots f^\omega_{N-1}(u_N - u_{N-1}) = \sum_{x} r_{x,N}^\omega. \quad (168)
\]

Then, \( \mathcal{Q} \)-a.s.

\[
\limsup_{N \to \infty} \frac{1}{N} \log r_N^\omega < 0. \quad (169)
\]

Again by Borel-Cantelli and the exponential Markov inequality, (169) would follow as soon as we check that for some \( \alpha > 0 \),

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}(r_N^\alpha) < 0. \quad (170)
\]

**Fractional Moments.** The proof of the fractional moment bound (170) comprises several steps.

**STEP 1** Following (171) is verified once we show that there exist \( N \in \mathbb{N} \) and \( \alpha \in (0, 1) \) such that

\[
\mathcal{E} \left\{ \sum_{x} (r_{x,N}^\alpha) \right\} < 1. \quad (171)
\]

Indeed, first of all if \( a_i \geq 0 \) and \( \alpha \in (0, 1) \), then

\[
(\sum a_i)^\alpha \leq \sum a_i^\alpha. \quad (172)
\]

Equivalently (setting \( p = 1/\alpha > 1 \) and \( b_i = a_i^\alpha \), \( \sum b_i^p \leq (\sum b_i)^p \)). Since

\[
\frac{d}{db_i} (\sum b_i)^p \geq pb_i^{p-1} = \frac{d}{db_i} \sum b_i^p,
\]

the latter form of (172) follows by induction.

We proceed with proving that (171) implies (170). Evidently, by (172),

\[
r_{N+M}^\alpha = \sum_{x} r_{N,x}^\alpha r_{M}^\alpha \Rightarrow (r_{N+M}^\alpha)^\alpha \leq \sum_{x} (r_{N,x}^\alpha)^\alpha \left( r_{M}^\alpha \right)^\alpha,
\]

for any \( \alpha \in (0, 1) \). Since \( r_{N,x}^\alpha \) and \( r_{M}^\alpha \) are independent, and \( r_{M}^\alpha \) is translation invariant, it follows that

\[
\mathcal{E} \{ (r_{N+M}^\alpha) \} \leq \mathcal{E} \left\{ \sum_{x} (r_{N,x}^\alpha)^\alpha \right\} \mathcal{E} \{ (r_{M}^\alpha)^\alpha \}.
\]

Hence (171), implies exponential decay of \( M \mapsto \mathcal{E} \{ (r_{M}^\alpha)^\alpha \} \).
STEP 2 Let $v_h = \sum x f(x)$; mean displacement under probability measure $\{f(x)\}$. By Theorem 3.1, the latter distribution has exponential tails, and classical moderate deviation results apply. For $y \in \mathbb{Z}^d$ define the distance from $y$ to the line in the direction of $v_h$: $d_h(y) = \min_a |y - av_h|$. Pick $K$ sufficiently large and $\varepsilon$ small, and consider

$$A_N = \left\{ y \in \mathbb{Z}^d : 0 \leq y \cdot v_h \leq KN \text{ and } d_h(y) \leq N^{\frac{1}{2}+\varepsilon} \right\}.$$ 

Recall that $r_{x,N}$ is the distribution of the end point of the $N$-step random walk with $\{f(x)\}$ being the one step distribution. With a slight abuse of notation, we can consider $r_N$ as a distribution on the set of all $N$-step trajectories of this random walk:

$$r_N(x_1, \ldots, x_N) = f(x_1) f(x_2 - x_1) \ldots f(x_N - x_{N-1}).$$  \hspace{1cm} (173)

By classical (Gaussian) moderate deviation estimates, there exists $c > 0$ such that

$$\sum_{x \notin A_N} r_{x,N} \leq e^{-cN^{2\varepsilon}} \text{ and } r_N (\{x_1, \ldots, x_N\} \not\subset A_N) \leq e^{-cN^{2\varepsilon}}.$$  \hspace{1cm} (174)

Furthermore, with another slight abuse of notation we can consider $r_N(\cdot)$ as the distribution on the family of all $N$-concatenations $\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_N$ of irreducible paths $\gamma_i \in \mathcal{F}$. In this way,

$$r_N(\gamma) = \prod_{i=1}^N W^{h,\lambda}_{\mathcal{F}}(\gamma_i).$$

Recall from (120) that irreducible paths $\gamma_i$ satisfy the following diamond confinement condition: If $x_{i-1}, x_i$ are the end points of $\gamma_i$, then $\gamma_i \subset D(x_{i-1}, x_i)$.

**Exercise 5.1** Prove the following generalization of the second of (174) (see Figure 8): There exists $c > 0$ such that

$$r_N (\bigcup_{i} D(x_{i-1}, x_i) \not\subset A_N) \leq e^{-cN^{2\varepsilon}}.$$  \hspace{1cm} (175)

---

**Fig. 8** Example: $N = 5$. The path $(0, x_1, \ldots, x_5)$ of the effective random walk, and the union of diamond shapes $\bigcup_i D(x_{i-1}, x_i) \subset A_N$. 

- $D(0, x_1)$ 
- $D(x_1, x_2)$ 
- $D(x_2, x_3)$ 
- $D(x_3, x_4)$ 
- $D(x_4, x_5)$ 
- $D(x_5)$ 
- $0$ 
- $v_h$ 
- $x_1$ 
- $x_2$ 
- $x_3$ 
- $x_4$ 
- $x_5$
Since any concatenation $\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_N$ of irreducible paths $\gamma_i \in \mathcal{F}$ satisfies $\gamma \subseteq \cup_{i} \mathcal{D}(x_{i-1}, x_i)$, we readily infer that under $r_N$ typical annealed paths stay inside $A_N$,

$$r_N (\gamma \nsubseteq A_N) \leq e^{-cN^{2\epsilon}} \Rightarrow \mathcal{E} \left\{ (r_N^N (\gamma \nsubseteq A_N))^\alpha \right\} \leq e^{-\alpha cN^{2\epsilon}}, \quad (176)$$

for any $\alpha \in (0, 1)$ (by Jensen’s inequality).

Since $\mathcal{E} \left\{ (r_N^N)\alpha \right\} \leq r_N^\alpha$, we, in view of the first of (174), may restrict summation in (171) to $x \in A_N$. In view of (176), it would be enough to check that

$$\sum_{x \in A_N} \mathcal{E} \left\{ (r_N^N (\gamma \subseteq A_N))^\alpha \right\} \leq |A_N| \mathcal{E} \left\{ (r_N^N (\gamma \subseteq A_N))^\alpha \right\} < 1. \quad (177)$$

The first inequality above is a crude over-counting, but for $d = 2$ it will do.

**STEP 3** We, therefore, concentrate on proving the second inequality in (177). At this stage, we shall modify the distribution of the environment inside $A_N$ in the following way: The modified law of the environment, which we shall denote $\mathcal{D}_{\delta}$, is still product and, for every $x \in A_N$,

$$\frac{d\mathcal{D}_{\delta}}{d\mathcal{D}} (V_x) \overset{A}{=} e^{\delta (V_x) - \log E e^{\log E (V_x)}}, \quad \text{where} \quad e^{\log E} = \log E \left( e^{\log E (V_x)} \right),$$

and $\delta$ is a bounded non-decreasing function on $\mathbb{R}_+$, for instance $\psi(v) = v \lor 1$.

The annealed potential in the modified environment is $\psi_{\beta}(\ell, \delta) = -\log \mathcal{E} (e^{\beta (V_x)})$.

Note that for any $\ell \geq 1$,

$$\frac{d\psi_{\beta}(\ell, \delta)}{d\delta} \bigg|_{\delta=0} = \mathcal{E} (\psi(0)) \mathcal{E} (e^{-\ell \beta V_x}) - \mathcal{E} (\psi(0) e^{-\ell \beta V_x}) > 0. \quad (178)$$

Indeed, since $\psi$ is non-decreasing and $e^{-\ell \beta}$ is decreasing, the last inequality follows from positive association of one-dimensional probability measures, as described in the beginning of Subsection 2.1.

By (176) we can ignore paths which do not stay inside $A_N$. Thus, (178) implies: There exists $c_{\psi} > 0$ such that for all $\delta$ sufficiently small,

$$\mathcal{E} (f^\alpha (x, n)) \leq 1 - c_{\psi} \delta \Rightarrow \mathcal{E} (r_N^N) \leq e^{-c_{\psi} \delta}. \quad (179)$$

From Hölder’s inequality,

$$\mathcal{E} \left\{ (r_N^N (\gamma \subseteq A_N))^\alpha \right\} \leq \left( \mathcal{E} \left\{ (\frac{d\mathcal{D}}{d\mathcal{D}_{\delta}})^{1/(1-\alpha)} \right\} \right)^{1-\alpha} \left( \mathcal{E} \left\{ (r_N^N (\gamma \subseteq A_N))^\alpha \right\} \right)^\alpha \overset{(179)}{\leq} \left( \mathcal{E} \left\{ (\frac{d\mathcal{D}}{d\mathcal{D}_{\delta}})^{1/(1-\alpha)} \right\} \right)^{1-\alpha} e^{-c_{\psi} \alpha \delta}. \quad (180)$$

Now, the first term on the right hand side of (180) is
\[
\mathcal{E}_\delta \left\{ \left( \frac{d\mathcal{L}}{d\mathcal{L}_\delta} \right)^{1/(1-\alpha)} \right\} = \mathcal{E}_\delta \left\{ \frac{d\mathcal{L}}{d\mathcal{L}_\delta} \left( \frac{d\mathcal{L}}{d\mathcal{L}_\delta} \right)^{1/(1-\alpha)} \right\} \\
= \left( \mathcal{E} \left\{ e^{\frac{\alpha}{\alpha}(g(\delta)-\delta\psi(V^\omega))} \right\} \right)^{|A_N|}. 
\]  

(181)

However, the first order terms in \( \delta \) cancel:

\[
\frac{d}{d\delta} \bigg|_{\delta=0} \log \mathcal{E} \left\{ e^{\frac{\alpha}{\alpha}(g(\delta)-\delta\psi(V^\omega))} \right\} = \frac{\alpha}{1-\alpha} \left( g'(0) - \mathcal{E} \{ \psi(V^\omega) \} \right) = 0. 
\]  

(182)

Consequently, by the second order expansion, there exists \( \nu_\psi < \infty \), such that

\[
\left( \mathcal{E}_\delta \left\{ \left( \frac{d\mathcal{L}}{d\mathcal{L}_\delta} \right)^{1/(1-\alpha)} \right\} \right)^{1-\alpha} \leq e^{\nu_\psi |A_N|}. 
\]  

(183)

A substitution to (180) yields:

\[
\mathcal{E} \left\{ \left( \frac{\alpha}{A_N} (\gamma \subset A_N) \right)^{\alpha} \right\} \leq e^{-N\nu_\psi \alpha \delta + \frac{\nu_\psi}{1-\alpha} \delta^2 |A_N|}. 
\]  

(184)

We are now ready to specify the choice of \( \delta = \delta_N \): In two dimensions; \( d = 2 \), the cardinality \( |A_N| \approx N^{\frac{1}{2}+\epsilon} \). Hence, (177) follows whenever we choose

\[
\frac{C \log N}{N} \ll \delta_N \ll N^{-\frac{1}{2}+\epsilon} 
\]

with \( C = C(\alpha) \) being sufficiently large.

### 6 Appendix: Geometry of convex bodies and Large deviations.

In these notes we shall restrict attention to finite dimensional spaces \( \mathbb{R}^d \). The principal references are [3, 26, 28] for convex geometry and [32, 6, 5, 14] for large deviations.

#### 6.1 Convexity and duality.

**Convex functions.** A function \( \phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \) is said to be convex if

\[
\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y),
\]

for all \( x, y \in \mathbb{R}^d \) and all \( t \in [0,1] \).

**Remark 11.** Note that by definition we permit \( \infty \) values, but not \( -\infty \) values.

Alternatively, \( \phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \) is convex if the set
\[
\text{epi}(\phi) = \{(x, \alpha) : \phi(x) \leq \alpha\} \subset \mathbb{R}^{d+1}
\]  

(185)

is convex. We shall work with convex lower-semicontinuous functions: \(\phi\) is lower-semicontinuous if for any \(x\) and any sequence \(x_n\) converging to \(x\),

\[
\phi(x) \leq \lim_{n \to \infty} \phi(x_n).
\]

Alternatively, \(\phi\) is lower-semicontinuous if the set \(\text{epi}(\phi)\) is closed.

A basic example of a convex and lower-semicontinuous (actually continuous) function is an affine function

\[
\mathcal{L}_{a,h}(x) = a + h \cdot x.
\]

**Theorem 6.1** The following are equivalent:

(a) \(\phi : \mathbb{R}^d \to \mathbb{R} \cup \infty\) is convex and lower-semicontinuous.

(b) \(\text{epi}(\phi)\) is convex and closed.

(c) \(\phi\) can be recovered from its affine minorants:

\[
\phi(x) = \sup_{a,h} \left\{ \mathcal{L}_{a,h}(x) : \mathcal{L}_{a,h} \leq \phi \right\}.
\]

(186)

(d) \(\text{epi}(\phi)\) is the intersection of closed half-spaces

\[
\text{epi}(\phi) = \bigcap_{\mathcal{L}_{a,h} \leq \phi} \text{epi}(\mathcal{L}_{a,h}).
\]

(187)

**Definition 2.** \(\phi\) is sub-differentiable at \(x\) if there exists \(l_{a,h} \leq \phi\) such that \(\phi(x) = \mathcal{L}_{a,h}(x)\). In the latter case we write \(h \in \partial \phi(x)\).

**Exercise 6.1** Check that \(h \in \partial \phi(x)\) iff \(\phi(x) < \infty\) and

\[
\phi(y) - \phi(x) \geq h \cdot (y - x)
\]

(188)

for any \(y \in \mathbb{R}^d\).

Convex functions on \(\mathbb{R}^d\) are always sub-differentiable at interior points of their effective domains. In general, sets \(\partial \phi(x)\) may be empty, may be singletons or they may contain continuum of different slopes \(h\).

**Exercise 6.2** Find an example with \(\partial \phi(x) = \emptyset\). Prove that in general \(\partial \phi(x)\) is closed and convex. Check that a convex \(\phi\) is differentiable at \(x\) with \(\nabla \phi(x) = h\) iff \(\partial \phi(x) = \{h\}\).

**Definition 3.** Let \(\phi : \mathbb{R}^d \to \mathbb{R} \cup \infty\). The Legendre-Fenchel transform, or the convex conjugate, of \(\phi\) is

\[
\phi^*(x) = \sup_h \{ h \cdot x - \phi(h) \}.
\]

(189)
By construction, $\phi^*$ is always convex and lower-semicontinuous: Indeed, 
\[ \text{epi}(\phi^*) = \bigcap_h \text{epi} (-\phi(h), h), \]
which is obviously closed and convex.

**Duality.** Let $\phi$ be a convex and lower-semicontinuous function. Let us say that $h$ and $x$ are a pair of conjugate points if $x \in \partial \phi(h)$.

**Theorem 6.2** If $\phi$ is convex and lower-semicontinuous, then $(\phi^*)^* = \phi$. In the latter situation, the notion of conjugate points is symmetric, namely the following are equivalent:

\[ x \in \partial \phi(h) \iff h \in \partial \phi^*(x) \iff \phi(h) + \phi^*(x) = h \cdot x. \]  

Let $h, x$ be a pair of conjugate points. Then strict convexity of $\phi$ at $h$ is equivalent to differentiability of $\phi^*$ at $x$. Namely,

\[ \forall g \neq h, \quad \phi(g) - \phi(h) > x \cdot (g - h) \iff \nabla \phi^*(x) = h. \]  

**Support and Minkowski functions.** Let $K \subset \mathbb{R}^d$ be a compact convex set with non-empty interior around the origin $0 \in \text{int}(K)$.

**Definition 4.** The function

\[ \chi_K(h) = \begin{cases} 0, & \text{if } h \in K \\ \infty, & \text{otherwise} \end{cases} \]  

is called the characteristic function of $K$.

The function

\[ \tau_K(x) = \sup_{h \in K} h \cdot x = \max_{h \in \partial K} h \cdot x \]  

is called the support function of $K$.

The function

\[ \alpha_K(h) = \inf \{ r > 0 : h \in rK \} \]  

is called the Minkowski function of $K$.

As it will become apparent below functions $\chi_K, \tau_K$ and $\phi_K$ are convex and lower-semicontinuous.

**Duality relation between $\chi_K$ and $\tau_K$.** The characteristic function $\chi_K$ is convex and lower-semicontinuous since $\text{epi}(\chi_K) = K \times [0, \infty)$. The support function $\tau_K$ is the supremum of linear functions. As such it is homogeneous of order one. Also, could be recorded in the form which makes $\tau_K$ to be the convex conjugate:

\[ \tau_K = \chi_K^* \text{ and, by Theorem 6.2 } \chi_K = \tau_K^*. \]  

Since $\tau_K$ is homogeneous, the latter reads as
\[ K = \bigcap_{n \in \mathbb{S}^{d-1}} \{ h : h \cdot n \leq \tau_K(n) \}. \]  

(196)

By \([192]\) if \( h \in \partial K \) then \( x \in \partial \chi_K(h) \) if and only if \( x \) is in the direction of the outward normal to a hyperplane which touches \( \partial K \) at \( h \). Thus, \( \partial \chi_K(h) \) is always a closed convex cone (which can be just a semi-line).

Other way around, \( \partial \tau_K(x) \) contains all boundary points \( h \in \partial K \), such that \( x \) is the direction of the outward normal to a supporting hyperplane at \( h \). In particular,

\[ \partial \tau_K(rx) = \partial \tau_K(x), \]  

(197)

and \( \partial \tau_K(x) \subset \partial K \) is a closed convex facet (which can be just one point).

As a consequence: \( \tau_K \) is differentiable at \( x \neq 0 \) iff the supporting hyperplane with the outward normal direction of \( x \) touches \( \partial K \) at exactly one point \( h \). In particular, \( \tau_K \) is differentiable at any \( y \neq 0 \) iff \( \partial K \) is strictly convex.

**Polarity relation between \( \tau_K \) and \( \alpha_K \).** As for \( \alpha_K(h) \) the assumptions \( K \) is bounded and \( 0 \in \text{int}(K) \) imply that for any \( h \neq 0 \), the value \( \alpha_K(h) \) is positive and finite. Consequently,

\[ \frac{x}{\alpha_K(h)} \in \partial K \text{ and } \exists x \neq 0 \text{ such that } \tau_K(x) = \frac{x \cdot h}{\alpha_K(h)}. \]

On the other hand, again since \( \frac{h}{\alpha_K(h)} \in \partial K \),

\[ \tau_K(y) \geq \frac{y \cdot h}{\alpha_K(h)}, \]

for any \( y \in \mathbb{R}^d \). Since \( \tau_K \) is homogeneous of order one, we, therefore, conclude:

\[ \alpha_K(h) = \max \{ h \cdot y : \tau_K(y) \leq 1 \}. \]  

(198)

In other words, \( \alpha_K \) is the support function of the closed convex set:

\[ K^* = \{ y : \tau_K(y) \leq 1 \}. \]  

(199)

In particular, \( \alpha_K \) is convex and lower-semicontinuous.

**Exercise 6.3** For any \( x, h \neq 0, x \cdot h \leq \tau_K(x) \alpha_K(h) \). Furthermore,

\[ \frac{x}{\tau_K(x) \alpha_K(h)} = 1 \iff \frac{x}{\tau_K(x)} \in \partial \alpha_K(h) \iff \frac{h}{\alpha_K(h)} \in \partial \tau_K(x). \]  

(200)

Actually, by homogeneity it would be enough to establish \((200)\) for \( x \in \partial K^* \) (equivalently \( \tau_K(x) = 1 \)) and \( h \in \partial K \) (equivalently \( \alpha_K(h) = 1 \)). In the latter case, let us say that \( x \in \partial K^* \) and \( h \in \partial K \) are in polar relation if \( x \cdot h = 1 \).

**Exercise 6.4** Let \( x, h \) be in polar relation. Then
\( \partial K \) is strictly convex (smooth) at \( h \) iff \( \partial K^\circ \) is smooth (strictly convex) at \( x \).

\[ (201) \]

Remark 12. Most of the above notions can be defined and effectively studied in much more generality than we do. In particular, one can go beyond assumptions of finite dimensions and non-empty interior.

6.2 Curves and surfaces.

Let \( M \) be a smooth \((d-1)\)-dimensional surface (without boundary) embedded in \( \mathbb{R}^d \). For \( u \in M \) let \( n(u) \in S^{d-1} \) be the normal direction at \( u \). \( T_u M \) is the tangent space to \( M \) at \( u \). Thus \( n \) is a map \( n : M \to S^{d-1} \). It is called the Gauss map, and its differential \( d_n u \) is called the Weingarten map. Since \( T_u M = T_n(u) S^{d-1} \), we may consider \( d_n u \) as a linear map on the tangent space \( T_u M \).

Exercise 6.5 Check that \( d_n u \) is self-adjoint (with respect to the usual Euclidean scalar product on \( \mathbb{R}^d \)). Hence, the eigenvalues \( \chi_1, \ldots, \chi_{d-1} \) of \( d_n u \) are real, and the corresponding normalized eigenvectors \( v_1, \ldots, v_{d-1} \) form an orthonormal basis of \( T_u M \).

Definition 5. Eigenvalues \( \chi_1, \ldots, \chi_{d-1} \geq 0 \) of \( d_n u \) are called principal curvatures of \( M \) at \( u \). The normalized eigenvectors \( v_1, \ldots, v_{d-1} \) are called directions of principal curvature. The product \( \prod \chi_\ell \) is called the Gaussian curvature.

Assume that \( M \) is locally given by a level set of a smooth function \( \lambda(\cdot) \), such that \( \nabla \lambda(u) \neq 0 \). That is, in a neighbourhood of \( u \); \( v \in M \leftrightarrow \lambda(v) = 0 \). Then,

\[ n_v = \frac{\nabla \lambda(v)}{|\nabla \lambda(v)|}. \]

Define \( \Sigma_u = \text{Hess}[\lambda](u) \).

Exercise 6.6 Check that for \( g \in T_u M \),

\[ d_n u g \cdot g = \frac{1}{|\nabla \lambda(v)|} \Sigma_u g \cdot g. \]  

Convex surfaces. Let now \( M = \partial K \), and \( K \) is a bounded convex body with non-empty interior. In the sequel we shall assume that the boundary \( \partial K \) is smooth (at least \( C_2 \)). Let \( \tau = \tau_K \) be the support function of \( K \). Whenever defined the Hessian \( \Sigma_x = \text{Hess}[\tau](x) \) has a natural interpretation in terms of the curvatures of \( K \) at \( h = \nabla \tau(x) \). We are following Chapter 2.5 in \[28\].

General case. We assume that \( M \) is smooth and that the Gaussian curvature of \( M \) is uniformly non-zero. In particular, \( M = \partial K \) is strictly convex, and, by duality relations, its support function \( \tau \) is differentiable. In the sequel \( n(h) \) is understood as the exterior normal to \( M \) at \( h \).
Recall that for every \( x \neq 0 \), the gradient \( \nabla \tau(x) = h_x \in \partial K \), and could be characterized by \( h_x \cdot x = \tau(x) \). Consequently, \( \nabla \tau(rx) = \nabla \tau(x) \). This is a homogeneity relation. It readily implies the following: Let \( \Xi_x \) be the Hessian of \( \tau \) at \( x \). Then,

\[
\Xi_x x = 0. \tag{203}
\]

Let \( h \in \partial K \) and let \( n = n(h) \) be the normal direction to \( \partial K \) at \( h \). Then

\[
\nabla \tau(n(h)) = h.
\]

In other words, the restriction of \( \nabla \tau \) to \( S^{d-1} \) is precisely the inverse of the Gauss map \( n \). Hence the restriction \( \hat{\Xi}_{n(h)} \) of \( \Xi_n \) to \( T_h \partial K \) is the inverse of the Weingarten map \( d n_h \).

**Definition 6.** Let \( n \in S^{d-1} \) and \( h = \nabla \tau(n) \). Eigenvalues \( r_\ell = 1/\chi_\ell \) of \( \hat{\Xi}_n \) are called principal radii of curvature of \( M = \partial K \) at \( h \).

**Example: Smooth convex curves.** Let \( n_\theta = (\cos \theta, \sin \theta) \). Radius of curvature \( r(\theta) = 1/\chi(\theta) \) of the boundary \( \partial K \) at a point \( h_\theta = \nabla \tau(n_\theta) \) is given by

\[
r(\theta) = \frac{d^2}{d\theta^2} \tau(\theta) + \tau(\theta),
\]

where we put \( \tau(\theta) = \tau(n_\theta) \). Indeed, \( v_\theta = n_\theta' = (-\sin \theta, \cos \theta) \) is the unit spanning vector of \( T_{h_\theta} \partial K \). Note that \( \frac{d}{d\theta} v_\theta = -n_\theta \). Hence,

\[
\frac{d^2}{d\theta^2} \tau(\theta) = \frac{d}{d\theta} (\nabla \tau(n_\theta) \cdot v_\theta) = \Xi_{n_\theta} v_\theta \cdot v_\theta - \nabla \tau(n_\theta) \cdot n_\theta = \Xi_{n_\theta} v_\theta \cdot v_\theta - \tau(n_\theta)
\]

**Second order expansion.** Let \( (v_1, \ldots, v_{d-1}, n(h)) \) be orthonormal coordinate frame, where \( (v_1, \ldots, v_{d-1}) \) is a basis of \( T_h M \). Consider matrix elements \( \Xi_n(i, j) \) in this coordinates. Then the homogeneity relation (203) applied at \( x = n(h) \) yields:

\[
\Xi_n(\ell, d) = 0 \text{ for all } \ell = 1, \ldots, d. \tag{204}
\]

Which means that as a quadratic form \( \Xi_n \) satisfies:

\[
\Xi_n u \cdot v = \Xi_n n_h u \cdot n_h v, \tag{205}
\]

where \( n_h \) is the orthogonal projection on \( T_h \partial K \). Furthermore,

**Exercise 6.7** Check that the Hessian \( \Xi_x \triangleq \text{Hess}_x \tau = \frac{1}{|x|} \Xi_n \), where \( n = n_x \in S^{d-1} \) is the unit vector in the direction of \( x \).

Consequently, second order expansion takes the form: For any \( x \neq 0 \) and \( t \in (0, 1) \)

\[
\tau_h (tx + v) + \tau_h ((1-t)x - v) - \tau_h (x) = \frac{\Xi_n v \cdot v}{2t(1-t)|x|} + o\left(\frac{|v|^2}{|x|}\right). \tag{206}
\]
Recording this in the (orthonormal) basis of principal curvatures, we deduce the following Corollary:

**Corollary 1.** Let $x \in \mathbb{R}^d$; $n = n_x = \frac{x}{|x|} \in S^{d-1}$, and let $h = \nabla \tau(x)$. Consider the orthogonal frame $(v_1, \ldots, v_{d-1}, n)$, where $v_i$ are the directions of principal curvature of $\partial K$ at $h$. Then, for any $t \in (0, 1)$ and for any $y_1, \ldots, y_{d-1}$,

$$
\tau \left( tx + \sum_{\ell=1}^{d-1} y_\ell v_\ell \right) + \tau \left( (1-t)x - \sum_{\ell=1}^{d-1} y_\ell v_\ell \right) - \tau(x)
= \sum_{\ell=1}^{d-1} \frac{y_\ell^2}{2(1-t)|x|} \tau + o \left( \sum y_\ell^2 \right). \tag{207}
$$

**Strict triangle inequality.** If principal curvatures of $\partial K$ are uniformly bounded or, equivalently, if quadratic forms $\Xi_n$ are uniformly (in $n \in S^{d-1}$) positive definite, then there exists a constant $c > 0$ such that

$$
\tau(x) + \tau(y) - \tau(x+y) \geq c \left( |x| + |y| - |x+y| \right). \tag{208}
$$

In order to prove (208), note, first of all, that since for any $z \neq 0$, $\nabla \tau(z) = \nabla \tau(n_z)$, one can rewrite the left hand side of (208) as

$$
\tau(x) + \tau(y) - \tau(x+y) = x \cdot (\nabla \tau(n_x) - \nabla \tau(n_{x+y})) + y \cdot (\nabla \tau(n_y) - \nabla \tau(n_{x+y})).
$$

Similarly,

$$
|x| + |y| - |x+y| = x \cdot (n_x - n_{x+y}) + y \cdot (n_y - n_{x+y}).
$$

Therefore, (208) will follow if we show that for any two unit vectors $n, m \in S^{d-1}$,

$$
n \cdot (\nabla \tau(n) - \nabla \tau(m)) \geq c n \cdot (n - m). \tag{209}
$$

Set $\Delta = n - m$. Since $n \cdot (n - m) \approx |\Delta|^2$, and since we are not pushing for the optimal value of $c$ in (209), it would be enough to consider second order expansion in $|\Delta|$. To this end define $\gamma = m + t\Delta$ and $h_t = \nabla \tau(\gamma)$. Then,

$$
n \cdot (\nabla \tau(n) - \nabla \tau(m)) = n \cdot \int_0^1 \frac{d}{dr} \nabla \tau(\gamma) dr = \int_0^1 \Xi_{\gamma} n \cdot \Delta dr \tag{210}
$$

The last equality above is (205). By construction $\gamma$ is orthogonal to $T_h \partial K$. Hence the projection

$$
\pi_0 n = n - \frac{n \cdot \gamma}{|\gamma|^2} \gamma = (1-t)\Delta + \frac{(1-t)\Delta \cdot \gamma}{|\gamma|^2} \gamma = (1-t)\Delta + o(|\Delta|).
$$

On the other hand $\pi_0 \Delta = \Delta + o(|\Delta|)$. Hence, up to higher order terms in $|\Delta|$,
\[ \int_0^t \Xi_t \pi_{n_t} n \cdot \pi_{h_t} \Delta t \geq \frac{1}{2} \min_{h \in \partial K} \min_t r_t(h) |n - m|^2, \tag{211} \]

and (208) follows.

### 6.3 Large deviations.

**The setup.** Although the framework of the theory is much more general we shall restrict attention to probabilities on finite-dimensional spaces. Let \( \{ \mathbb{P}_n \} \) be a family of probability measures on \( \mathbb{R}^d \).

**Definition 7.** A function \( J : \mathbb{R}^d \to [0, \infty] \) is said to be a rate function if it is proper \( \text{Dom}(J) \triangleq \{ x : J(x) < \infty \} \neq \emptyset \) and if it has compact level sets. In particular rate functions are always lower-semicontinuous.

**Definition 8.** A family \( \{ \mathbb{P}_n \} \) satisfies large deviation principle with rate function \( J \) (and speed \( n \)) if:

**Upper Bound** For every closed \( F \subseteq \mathbb{R}^d \)
\[ \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n (F) \leq -\inf_{x \in F} J(x). \tag{212} \]

**Lower Bound** For every open \( O \subseteq \mathbb{R}^d \)
\[ \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n (O) \geq -\inf_{x \in O} J(x). \tag{213} \]

There is an alternative formulation of the lower bound:

**Exercise 6.8** Check that (213) is equivalent to: For every \( x \in \mathbb{R}^d \) the family \( \{ \mathbb{P}_n \} \) satisfies the LD lower bound at \( x \), that is for any open neighbourhood \( O \) of \( x \),
\[ \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n (O) \geq -J(x). \tag{214} \]

All the measures we shall work with are exponentially tight:

**Definition 9.** A family \( \{ \mathbb{P}_n \} \) is exponentially tight if for any \( R \) one can find a compact subset \( K_R \) of \( \mathbb{R}^d \) such that
\[ \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n (K_R^c) \leq -R. \tag{215} \]

If exponential tightness is checked then one needs derive upper bounds only for all compact sets:

**Exercise 6.9** Check that if \( \{ \mathbb{P}_n \} \) is exponentially tight and it satisfies (213) for all open sets and (212) for all compact sets, then it satisfies LDP.
In particular, \( \{P_n\} \) satisfies an upper large deviation bound with \( J \) if
(a) It is exponentially tight.
(b) For every \( x \in \mathbb{R}^d \), the family \( \{P_n\} \) satisfies the following upper large deviation bound at \( x \):
\[
\lim_{n \to \infty} \frac{1}{n} \log P_n \left( \frac{X}{n} - x \leq \delta \right) \leq -J(x). \tag{216}
\]

We shall mostly work with measures on \( \frac{1}{n} \mathbb{Z}^d \) which are generated by scaled random variables \( \frac{1}{n} X \), for instance when \( X = X(\gamma) \) is the spatial extension of a polymer or the end point of a self-interacting random walk. In the latter case we shall modify the notion (216) of point-wise LD upper bound as follows:

**Definition 10.** A family \( \{P_n\} \) of probability measures on \( \mathbb{Z}^d \) satisfies an upper LD bound at \( x \in \mathbb{R}^d \) if for any \( R > 0 \)
\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n (X = \lfloor nx \rfloor) \leq -J(x) \wedge R. \tag{217}
\]

At a first glance constant \( R \) in (217) does not seem to contribute to the statement. However, checking and formulating things this way may be convenient.

**Exercise 6.10** Let \( J \) be a rate function. Check that if \( \{P_n\} \) is exponentially tight, if the lower bound (214) is satisfied, and if (217) is satisfied, for any \( R \in [0, \infty) \), uniformly on compact subsets of \( \mathbb{R}^d \), then \( \{P_n\} \) satisfies the LD principle in the sense of Definition 8.

**Log-moment generating functions and convex conjugates.** Frequently quests after LD rate functions stick to the following pattern: Assume that the (limiting) log-moment generating function
\[
\lambda(h) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_n e^{h X} \tag{218}
\]
is well defined (and not identically \( \infty \)) for all \( h \in \mathbb{R}^d \).

**Exercise 6.11** Check that if \( \lambda(\cdot) \) in (218) is indeed defined, then it is convex and lower-semicontinuous.

Consider the Legendre-Fenchel transform \( I \) of \( \lambda \)
\[
I(x) = \sup_h \{ h \cdot x - \lambda(h) \}. \tag{219}
\]

Here is one of the basic general LD results:

**Theorem 6.3** Assume that \( \lambda \) in (218) is well defined and proper.

**Upper Bound.** For any \( x \in \mathbb{R}^d \) the family \( \{P_n\} \) satisfies upper LD bound (217) with \( I \) at \( x \).

**Lower Bound.** If, in addition, \( I \) is sub-differential and strictly convex at \( x \), then \( \{P_n\} \) satisfies a lower LD bound at \( x \) with \( I \) in (214).
Sub-differentiability and strict convexity over finite-dimensional spaces are studied in great generality (e.g. low-dimensional effective domains, behaviour at the boundary of relative interiors etc.) and detail [26].

Lower LD bounds with \( I \) generically do not hold. In particular true LD rate functions \( J \) are generically non-convex. However, \( I = J \) in many important examples such as sums of i.i.d.-s and Markov chains. Moreover, \( I = J \) for most of polymer models with purely attractive or repulsive interactions. A notable exception is provided by one-dimensional polymers with repulsion \([12][21]\)(which we do not discuss here).

Under minor additional integrability conditions the relation between \( I \) and \( J \) could be described as follows:

Let \( \phi \) be a function on \( \mathbb{R}_+ \) with a super-linear growth at \( \infty \):

\[
\lim_{t \to \infty} \frac{\phi(t)}{t} = \infty.
\]

**Lemma 6.1** Assume that \( \{P_n\} \) satisfies LDP with rate function \( J \), and assume that

\[
\limsup \frac{1}{n} E_n e^{\frac{n\phi(|X|)}{n}} < \infty.
\]

Then \( \lambda(\cdot) \) in (218) is defined and equals to

\[
\lambda(h) = \sup_x \{ x \cdot h - J(x) \}.
\]

Consequently, \( I \) is the convex lower-semicontinuous envelop of \( J \), that is

\[
I(x) = \sup \{ I_{a,h}(x) : I_{a,h} \leq J \}.
\]

In particular, if \( J \) convex, then \( I = J \).

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