SUPPLEMENTAL INFORMATION
Far-from-equilibrium universality in the two-dimensional Heisenberg model

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The outline of the Supplemental Information is as follows. In Sec. I, we numerically demonstrate the insensitivity of the scaling exponents (α, β) to variations in the initial conditions by considering incoherent initial states, which are different from the spin spiral initial conditions used in the main text. In Sec. II, we study the time evolution of the spin-spin correlation functions for the transverse and longitudinal components of magnetization and show that they become isotropic even when the initial condition is not. In Sec. III, we show the fitting scheme used to numerically estimate the scaling exponents (α, β). In Sec. IV, we discuss the scaling exponents in the XXZ model. In Sec. V, we discuss the derivation of Eq. (7) in the main text. In Sec. VI, we derive the canonical power counting used to estimate the scaling exponents of the gaussian fixed point and, in Sec. VII, we discuss the differences in the vertex interactions between the Heisenberg model and O(n) models.

I. INDEPENDENCE OF INITIAL CONDITIONS

To illustrate the universal character of our results, we repeat our calculations for incoherent initial conditions and show that the system exhibits the same scaling behavior as the one observed in the main text. Here we use initial conditions of the form

\[ \langle S_i^z \rangle = S \cos \phi_i \sin \theta, \quad \langle S_i^y \rangle = S \sin \phi_i \sin \theta, \quad \langle S_i^x \rangle = S \sin \theta, \quad \phi_i = \arg \left( \sum_k f_k e^{i k \cdot r_i} \right), \]

where \( f_k \) is a Gaussian-distributed complex function \( f_k = e^{i \theta_k - (|k| - q)^2/\sigma_k^2} \) that satisfies \( f_{-k} = f_k^* \), with \( \sigma_q = 0.1q \) and \( \theta_k \) a random phase uniformly sampled \([0, 2\pi]\). Figure S1(a) shows the self-similar scaling for initial conditions with zero net magnetization and with finite magnetization computed through the Truncated Wigner Approximation (TWA). In both cases, we rescaled the datapoints using \((\alpha, \beta) = (2/3, 1/3)\). We find excellent agreement with the results reported in the main text using spin spiral initial conditions. In addition, the scaling function also matches remarkably well with the ones found in the main text, both with zero magnetization (Fig. 2 of main text) and non-zero magnetization (Fig. 3 of main text).

II. ISOTROPIC SPIN-SPIN CORRELATIONS

In the analytical derivation of the scaling exponents in the main text, we assumed that the spin-spin correlation function becomes isotropic in spin space. Although this is a plausible assumption, we numerically check that this is indeed the case. Figures S1(b-c) show the evolution of the (b) xx and (c) zz spin-spin correlation function at short times computed using TWA. We find that in a short timescale \( \approx 5\tau_s \) the spin-spin correlations become isotropic. In addition, both the xx and zz correlations exhibit the same scaling behavior. We recall that \( \tau_s \) is defined from the initial conditions as \( 1/\tau_s = J S^2 \sin^2 \theta [2 - \cos(q_x \ell) - \cos(q_y \ell)] \), with \((q_x, q_y)\) the characteristic wavevector of the initial state, and \( \theta \) defines the magnetization of the initial states, \( S^z = NS \cos \theta \).

III. FITTING OF THE SCALING EXPOENTS

To find the scaling exponents \((\alpha, \beta)\) that best fit the numerical data, we minimize an error function that quantifies the accuracy of the datapoint collapse, as we describe next. We denote as \( C(|\mathbf{k}|, t) = \langle \hat{S}_{-\mathbf{k}}(t) \hat{S}_{\mathbf{k}}(t) \rangle_c \) the connected component of the spin-spin correlation function. The values of \(|\mathbf{k}| = k_i\) take discrete values on a lattice, and we choose...
FIG. S1. (a) Insensitivity of the scaling function and scaling exponents to the initial conditions. We show the rescaled spin-spin correlator $\sum_a \langle \hat{S}_a - k(t) \hat{S}_a k(t) \rangle$ using incoherent initial conditions [see Eq. (S1)] with no net magnetization (blue circles) and with average magnetization $\langle S_z i \rangle = S/2$ (orange triangles). Shown are datapoints in the time range $15\tau_s < t < 40\tau_s$, with decreasing shades of color indicating increasing time. We rescaled the data with scaling exponents $(\alpha, \beta) = (2/3, 1/3)$. (b-c) Evolution of the spin-spin correlation function $\langle \hat{S}_a - k \hat{S}_a k(t) \rangle$ for (b) $a = x$ and (c) $a = z$. Shown with black lines is $\langle \hat{S}_a - k(0) \hat{S}_a k(0) \rangle$, and with blue lines is $\langle \hat{S}_a - k(t) \hat{S}_a k(t) \rangle$ for $t = 2\tau_s$ to $t = 10\tau_s$ in steps of $2\tau_s$ (color shade decreases with time). The plots show that, after a transient time $\sim 5\tau_s$, correlations become isotropic in spin space.

IV. THE XXZ MODEL

The central ingredients in the discussion of the main text are the global SU(2) symmetry of the Heisenberg Hamiltonian and the dimensionality $d = 2$. We checked that the XXZ model exhibits different scaling behavior as we tune the anisotropic exchange $J_z$ from easy-plane to easy-axis across the isotropic point. The XXZ model is given by

$$\hat{H} = -\sum_{\langle i,j \rangle} [J(S_i^+ S_j^- + S_i^- S_j^+) + J_z S_i^z S_j^z],$$

which, in the low spin wave density limit, can be interpreted as a Bose gas in which the $J(S_i^+ S_j^- + h.c.)$ terms give rise to hopping and the $J_z S_i^z S_j^z$ terms give rise to interactions. In the regime $J_z \ll J$, we expect to see the same
Fig. S2. Contour plot of the error $E$ of the self-similar fitting, see Eq. (S3), as a function of the parameters $(\alpha, \beta)$ for (a) short times in the Gaussian regime and (b) long times in the non-Gaussian regime. Lighter colors indicate decreasing errors. In panel (a), we used datapoints of the distribution function in a time window of size $\Delta t = 10\tau_*$ centered at $t_0 = 15\tau_*$. In panel (b), we used datapoints for the distribution function in a time window of size $\Delta t = 10\tau_*$ centered at $t_0 = 25\tau_*$. scaling behavior as that observed in a weakly-coupled Bose gas ($\beta = 1/2$, $\alpha = d\beta$). Our numerical simulations for $J_z/J_\perp = 0.5$ show exactly these scaling exponents, see Fig. S3(a-b).

V. EQUATIONS OF MOTION

In the main text, we used the equations of motion of spin operators to analytically derive the dynamical scaling exponents. This approach is far more general than kinetic theory which may not be legitimate if, for example, magnetization fluctuations are large (as in our case). The microscopic equations of motion (in units of $J$) are given by:

$$\partial_t \hat{S}_a^i = \epsilon_{abc} \sum_j \hat{S}_b^i \hat{S}_c^j,$$  
(S5)

where we sum over repeated indices only if the appear both as subscripts and superscripts. Going to momentum space, we obtain

$$\partial_t \hat{S}_a^k = (\gamma_0 - \gamma_p) \epsilon_{abc} \hat{S}_b^k \hat{S}_c^p + i\gamma_0 \hat{S}_a^k,$$  
(S6)

with $\gamma_k = \sum_\ell e^{ik\cdot\ell}$ ($\ell$ are unit cell vectors). Multiplying on the left with the operator $\hat{S}_{-k}^a$, summing with the complex conjugate of $\hat{S}_{-k}^a\partial_t \hat{S}_k^a$, and taking expectation value, results in

$$\partial_t \langle \hat{S}_{-k}^a \hat{S}_k^a \rangle = 2 \sum_p (\gamma_0 - \gamma_p) \text{Re} \left[ \epsilon_{abc} \langle \hat{S}_{-k}^a \hat{S}_k^b \hat{S}_p^c \rangle \right].$$  
(S7)

We emphasize that correlations of the form $\langle \hat{S}_x^a \hat{S}_y^b \hat{S}_z^q \rangle$ appearing on the right-hand side of Eq. (S7) are not necessarily zero because the three components of magnetization are not independent. Equation (S7) is used in the main text to derive the scaling exponents under the assumption that a single lengthscale $\xi$ (the quasi-condensate correlation length) describes collective dynamics and correlation functions at intermediate timescales.

VI. EFFECTIVE HAMILTONIAN AND GAUSSIAN FIXED POINT

At short times, we observe a scaling regime with exponents $(\alpha, \beta) \approx (1, 1/2)$, which we attribute to a Gaussian fixed point. In addition, we argue in the main text that non-linearities in the Heisenberg model are marginal in $d = 2$. Both observations can be rationalized from a Holstein-Primakoff expansion of the Heisenberg model close to the ferromagnetic ground state. Assuming small deviations from the ferromagnetic ground state $|F\rangle$ (all spins pointing up) and using the Holstein-Primakoff transformation, $\hat{S}_j^+ = \sqrt{2S} \hat{\psi}_j^\dagger \hat{\psi}_j$ and $\hat{S}_j^z = S - \hat{\psi}_j^\dagger \hat{\psi}_j$, to quartic order in the bosonic operators $\hat{\psi}_j$ leads to the long-wavelength Hamiltonian

$$\hat{H} = J S a^2 \int x \left( \nabla \psi_x^\dagger \nabla \psi_x + \frac{1}{4S} \psi_x^\dagger \psi_x \nabla \psi_x^\dagger \nabla \psi_x + h.c. \right).$$  
(S8)
Unlike the usual Bose gas with hard core collisions, here the collision amplitude of two quasiparticles with momentum \( k \) and \( p \) is \( -\mathbf{k} \cdot \mathbf{p} \). This reflects the SU(2) symmetry of the Hamiltonian: collisions become negligible at small momenta because a \( k \to 0 \) magnon state, \( \hat{\psi}_k \dagger (F) \approx \hat{S}_0 \sqrt{2S} \langle F \rangle \), is effectively a global rotation of \( |F\rangle \) that would not affect the dynamics of a second incoming magnon.

The Gaussian exponents discussed in the main text can be derived by dropping non-linearities from (S8), and scaling the Fourier transform of the field with \( \xi \) as in Eq. (7), following \( \hat{\psi}_k \sim \xi^{\alpha/(2\beta)} \) and \( \xi \sim t^{\beta} \) of the main text. We can now take the action associated to the free version of (S8) (equivalent to considering just the kinetic term), and require that this is invariant under a running scale, \( \xi \). It will yield \( \xi^{1/\beta-d-2+\alpha/\beta} \sim \xi^0 \) or, in other words, \( (\alpha + 1)/\beta = d + 2 \). Combining it with relation (6) in the main text, \( \alpha = \beta d \), related to total spin conservation, we find \( \alpha = 1 \) and \( \beta = 1/2 \) in \( d = 2 \). Such exponents are observed at short times in Fig. S3. Note that one can also derive the same Gaussian exponents in a similar way as for the non-thermal fixed point in Eq. (7) by deriving the equations of motion associated to Eq. (S8) without non-linearities, \( \partial_t \hat{\psi}_k \sim |k|^2 \hat{\psi}_k \), and rescaling both sides, which directly yields \( \beta = 1/2 \).

**VII. SOFT VERTICES IN THE HOLSTEIN-PRIMAKOFF FIELD THEORY OF THE HEISENBERG MODEL**

The leading non-linearity in the low-density Holstein-Primakoff expansion of the Heisenberg magnet is a ‘soft’ vertex [1, 2], a scattering term ruled by spatial gradients which would therefore vanish at low momenta in Fourier space. In Keldysh non-equilibrium field theory [3], this appears as a quartic term \( \propto \mathcal{L}_{k,x} \nabla^2 \varphi^3 \hat{\varphi} \). Here, the fields \( \varphi \) and \( \hat{\varphi} \) incorporate classical and quantum fluctuations, see for instance [3–5]. The Laplacian \( \nabla^2 \) is responsible for the soft vertex proportional to momentum squared, and it has been originally derived within spin-wave hydrodynamics [1, 2]. At the level of canonical power counting, the two vertices scale respectively as \( \lambda \sim \xi^{-(2-d)} \) and \( g \sim \xi^{-(4-d)} \) in a regime dominated by classical fluctuations; this regime is justified by the high energy of the initial states we use in this work. As a result, a different power counting is obtained for higher order non-linearities: they become less relevant with higher powers of \( \varphi \) for \( O(n) \) models in \( d = 2 \), while they are all marginal in the Heisenberg magnet. Therefore, the effective field theory for \( O(n) \) models and the Heisenberg magnet differ already at the level of canonical power counting, and this in turn dictates a different set of exponents for dynamical scaling, whenever diagrammatic corrections are included. This hints that the two models belong to different non-equilibrium universality classes.

[1] J. F. Rodriguez-Nieva, D. Podolsky, and E. Demler, arXiv:1810.12333 (2018).
[2] B. Halperin and P. Hohenberg, Physical Review** 188, 898 (1969).
[3] A. Kamenev, *Field theory of non-equilibrium systems* (Cambridge University Press, 2011).
[4] A. Polkovnikov, Annals of Physics** 325, 1790 (2010).
[5] J. Berges, arXiv e-prints, arXiv:1503.02907 (2015).