Distributed Finite-time Least Squares Solver for Network Linear Equations

Tao Yang a, Jemin George b, Jiahu Qin c,*, Xinlei Yi d, Junfeng Wu e

aDepartment of Electrical Engineering, University of North Texas, Denton, TX 76203, USA
bU.S. Army Research Laboratory, Adelphi, MD 20783, USA
cDepartment of Automation, University of Science and Technology of China, Hefei 230027, China
dDepartment of Automatic Control, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, 100 44, Stockholm, Sweden
eCollege of Control Science and Engineering, Zhejiang University, Hangzhou 310027, China

Abstract

In this paper, we study the problem of finding the least square solutions of over-determined linear algebraic equations over networks in a distributed manner. Each node has access to one of the linear equations and holds a dynamic state. We first propose a distributed least square solver over undirected interaction graphs, for which we establish the conditions on the interaction graphs and the step-size under which the proposed algorithm exponentially converges to the least square solution. Next, we develop a finite-time least square solver by equipping the proposed algorithm with a finite-time decentralized computation mechanism. The proposed finite-time computation mechanism enables an arbitrarily chosen node to compute the least square solution in a finite number of time steps, by using its own local successive state values obtained from the underlying algorithm. Finally, we discuss how to extend the proposed algorithm to directed interaction graphs and how the finite-time computation mechanism computes the least square solution even when the underlying algorithms diverge. The theoretical findings are validated and illustrated by numerical simulation examples.

Key words: Distributed Algorithms; Dynamical Systems; Finite-time Computation; Least Squares; Linear Equations

1 Introduction

In recent years, the development of distributed algorithms to solve linear algebraic equations over multi-agent networks has attracted much attention due to the fact that linear algebraic equations are fundamental for various practical engineering applications [1–9]. In these algorithms, each node has access to one equation and holds a dynamic state, which is an estimate of the solution. By exchanging their states with neighboring nodes over an underlying interaction graph, all nodes collaboratively solve the linear equations. Various distributed algorithms based on distributed control and optimization have been developed for solving the linear equations which have exact solutions, among which discrete-time algorithms are given by [1, 4, 5, 7, 8, 10] and continuous-time algorithms are presented in [3, 6]. However, most of these existing algorithms can only produce least square solutions for over-determined linear equations in the approximate sense [1] or for limited graph structures [6, 11].

By reformulating the least square problem as a distributed optimization problem, various distributed optimization algorithms, such as those in the recent survey [12] and references therein, can be applied to solve the problem. For example, a continuous-time version of distributed algorithms proposed in [13, 14] has been applied to solve the exact least square problem in [6]. However, the drawback is the slow convergence rate due to the diminishing step-size. With a fixed step-size, it can only find approximated least square solutions with a bounded error. The recent work focus on developing distributed algorithms with a faster convergence rate to find the exact least square solutions, see, e.g., continuous-time algorithms [15–17] based on the classical Arrow-Hurwicz-Uzawa flow [18], and discrete-time algorithms [11, 17, 19]. In order to implement these continuous-time algorithms, discretization methods are needed. By the Euler’s discretization method, discrete-time algorithms based on these continuous-time algorithms are developed and the exponential convergence is established provided that the discretization step-size is sufficiently small [11, 17]. In [19], a discrete-time algorithm which does not require a step-size is developed. This clever algorithm however requires a matrix inverse, and thus does not scale well with the dimension of the unknown variables of the linear equations.

Due to the exponential convergence of these existing algorithms, all nodes need to constantly perform local computation and communicate with their neighboring nodes, which results in a waste of computation and communication resources. This is not desirable in multi-agent networks since each node is usually equipped with limited communication resources. Therefore, the fundamental problem is how to find the exact least square solution in a finite number of iterations, and hence terminate further communication and computation to save energy and resources. This motivates our study of this paper.
The contributions of this paper are summarized as follows.

- Motivated by the distributed optimization algorithms with fixed step-sizes developed in [20–26], we first develop a distributed algorithm, which on the one hand, converges faster compared to the algorithms based on the Euler discretization [11,17], and on the other hand, scales well with the dimension of the unknown variables of the linear equations compared to [19]. We then establish sufficient conditions on the step-size and the mixing weight matrix for exponential convergence to the exact least square solution.

- More importantly, we further develop a finite-time least square solver by equipping the proposed distributed algorithm with a decentralized computation mechanism based on the finite-time technique for distributed consensus and optimization [23,27–30]. This finite-time computation mechanism enables an arbitrarily chosen node to compute the exact least square solution within a finite number of time steps, by using its local successive state values obtained from the earlier proposed distributed algorithm. This result is among the first distributed algorithms which compute the exact least square solutions in finite number of iterations, while the existing algorithms can only compute the approximated or the least square solutions in finite number of iterations, while the existing algorithms have only applicable to undirected interaction graphs.

- Compared to the existing distributed algorithms for computing the exact least square solutions [11,15–17,19] which are only applicable to undirected interaction graphs, our proposed algorithm can be easily extended to directed graphs.

The remainder of the paper is organized as follows: In Section 2, we first formulate the network linear equations, motivate our study, and present our proposed distributed algorithm. In Section 3, we establish sufficient conditions for the algorithm to exponentially converge to the exact least square solution. In Section 4, we develop a finite-time least square solver by equipping the proposed algorithm with a decentralized finite-time computation mechanism, which enables an individual node to compute the exact least square solution within a finite number of time steps. In Section 5, we extend the proposed algorithm to directed graphs which are strongly connected and discuss how the finite-time computation mechanism computes the exact least square solution even when the underlying algorithm fail to converge. In Section 6, we provide the detailed proofs. Section 7 presents numerical simulation examples. Finally, concluding remarks are offered in Section 8.

2 Problem Formulation and Proposed Algorithm

2.1 Linear Equation

Consider the following linear algebraic equation with unknown $y \in \mathbb{R}^m$:

$$z = H y,$$  \hspace{1cm} (1)

where $z \in \mathbb{R}^N$ and $H \in \mathbb{R}^{N \times m}$ are known. It is well known that if $z \notin \text{span}(H)$, then the linear equation (1) always has one or many exact solutions. If $z \notin \text{span}(H)$, the above equation (1) has no exact solution and the least square solution of (1) is defined by the solution of the following optimization problem:

$$\min_{y \in \mathbb{R}^m} \frac{1}{2} \|z - H y\|^2.$$

(2)

We make the following assumption.

**Assumption 1** There hold that $N > m$ and $\text{rank}(H) = m$.

It is well known that under Assumption 1, the problem (2) has a unique solution and is given by

$$y^* = (H^\top H)^{-1} H^\top z.$$  \hspace{1cm} (3)

2.2 Network Communication Structure

Denote

$$H = \begin{bmatrix} h_1^\top \\ h_2^\top \\ \vdots \\ h_N^\top \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix},$$

(4)

where $h_i^\top$ is the $i$-th row vector of the matrix $H$. With these notations, we can rewrite the linear equation (1) as

$$h_i^\top y = z_i, \quad i = 1, 2, \ldots, N.$$

Let $G = (V, E)$ be a constant, undirected and connected graph with the set of nodes $V = \{1, 2, \ldots, N\}$ and the set of edges $E \subseteq V \times V$. Let $\mathcal{N}_i$ be the set of neighbor nodes that are connected to node $i$, i.e., $\mathcal{N}_i = \{ j : (i, j) \in E \}$. In this paper, we aim to develop distributed algorithms over $G$ to compute the least square solution of (2), where each node only has access to the value of $h_i$ and $z_i$ without knowledge of $h_j$ or $z_j$ from other nodes.

2.3 Distributed Optimization Perspective

Under Assumption 1, we know that unique least square solution exists and can be obtained by using distributed optimization algorithms.

Define $x = [x_1^\top \ldots x_N^\top]^\top \in \mathbb{R}^{Nm}$. Consider a cost function $F(\cdot) : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}$:

$$F(x) = \sum_{i=1}^N f_i(x_i),$$

(5)

where

$$f_i(x_i) = \frac{1}{2} \|h_i^\top x_i - z_i\|^2.$$  \hspace{1cm} (6)

The least square problem (2) is equivalent to the following distributed optimization problem

$$\min_{x \in \mathbb{R}^{Nm}} \sum_{i=1}^N f_i(x_i)$$

s.t. $x_1 = \cdots = x_N.$

(7b)
Therefore, under Assumption 1, the solution to the above optimization problem is given by \( x_1 = \cdots = x_N = y^* \), where \( y^* \) is given by (3).

By reformulating the least square problem (2) as the distributed optimization problem (7), various distributed algorithms have been developed [11, 15–17, 19]. Among these algorithm, algorithms proposed [15, 16] are in continuous-time. The discrete-time implementations of these algorithms [11, 17] require the Euler discretization step-size to be sufficiently small in order to ensure exponential convergence, and the algorithm proposed in [19] although converge fast but involves all nodes to compute a matrix inverse at each time step, and thus does not scale well with the dimension of the unknown variables of linear equations.

Existing algorithms only achieve asymptotic or exponential convergence [11, 15–17, 19]. Due to the exponential convergence, all nodes need to constantly perform local computation and communicate with their neighboring nodes, which results in a waste of computation and communication resources. Therefore, in order to reduce communication overhead and to save computation resources, it is desirable to develop a finite-time least square solver. This motivates our study of this paper.

2.4 Proposed Distributed Algorithm

We are now ready to present a distributed algorithm for solving least square problem (2) based on recently developed distributed optimization algorithms with fixed step-sizes [20–26].

In the proposed algorithm, each node \( i \) maintains two state vectors \( x_i(t) = [x_{i1}(t) \ldots x_{im}(t)]^\top \in \mathbb{R}^m \) and \( v_i(t) = [v_{i1}(t) \ldots v_{im}(t)]^\top \in \mathbb{R}^m \), where \( x_i(t) \) is node \( i \)'s estimate of the least square solution, and \( v_i(t) \) is node \( i \)'s estimate of the average gradient \( \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x_i(t)) \). At each time step \( t \), each node \( i \) updates its state vectors as

\[
\begin{align*}
    x_i(t+1) & = \sum_{j \in \mathcal{N}_i \cup \{i\}} W_{ij}x_j(t) - \alpha v_i(t), \\
    v_i(t+1) & = \sum_{j \in \mathcal{N}_i \cup \{i\}} W_{ij}v_j(t) \\
    & \quad + \nabla f_i(x_i(t+1)) - \nabla f_i(x_i(t)),
\end{align*}
\]

where the initial condition \( v_i(0) = \nabla f_i(x_i(0)) \), \( \alpha > 0 \) is the step-size,

\[
\nabla f_i(x_i(t)) = h_i h_i^\top x_i(t) - z_i h_i,
\]

and \( W_{ij} \) is the non-negative weight that node \( i \) assigns to its neighboring node \( j \) in the undirected connected graph \( \mathcal{G} \).

**Assumption 2** The mixing weight matrix \( \mathbf{W} = [W_{ij}] \in \mathbb{R}^{N \times N} \) satisfies the following properties

(i) For any \( (i, j) \in \mathcal{E} \), \( W_{ij} > 0 \). Moreover, \( W_{ii} > 0 \) for all \( i \in \mathcal{V} \). For other \( (i, j) \), \( W_{ij} = 0 \).

(ii) The matrix \( \mathbf{W} \) is positive definite, that is, \( \mathbf{W} = \mathbf{W}^\top \) and \( \mathbf{W} > 0 \).

(iii) The matrix \( \mathbf{W} \) is doubly stochastic, that is, \( \mathbf{W} \mathbf{1}_N = \mathbf{1}_N \) and \( \mathbf{1}_N \mathbf{W} = \mathbf{1}_N \).

**Remark 1** The selection of mixing weight matrix has been extensively studied, see, e.g., [31–33]. It is known that if the network is undirected and connected, then the matrix \( \mathbf{W} \) satisfying Assumption 2 has one simple eigenvalue at 1 and all other eigenvalues lie in \((0, 1)\), and the null space of the matrix \( \mathbf{I}_N - \mathbf{W} \) is spanned by \( \mathbf{1}_N \), i.e., \( \text{null}(\mathbf{I}_N - \mathbf{W}) = \mathbf{1}_N \).

Let \( x(t) = [x_1^\top \ldots x_N^\top] \in \mathbb{R}^N \), \( v(t) = [v_1^\top \ldots v_N^\top] \in \mathbb{R}^N \), and

\[
\tilde{H} = \text{diag}(h_1 h_1^\top, \ldots, h_N h_N^\top),
\]

\[
z_H = [z_1 h_1^\top \ldots z_N h_N^\top]^\top.
\]

Then, the algorithm (8) can be rewritten in a compact form (network flow):

\[
\begin{align*}
    x(t+1) & = (\mathbf{W} \otimes \mathbf{I}_m) x(t) - \alpha v(t), \\
    v(t+1) & = (\mathbf{W} \otimes \mathbf{I}_m) v(t) + \nabla F(x(t+1)) - \nabla F(x(t)),
\end{align*}
\]

where the initial condition \( v(0) = \nabla F(x(0)) \), and

\[
\nabla F(x(t)) = \tilde{H} x(t) - z_H.
\]

**Remark 2** Note that the algorithm (8) or its compact form (11) is essentially the same as the algorithms in [20, 22–25]. However, these studies either require all the local cost functions \( f_i(\cdot) \), \( i = 1, \ldots, N \) to be strongly convex or at least one local cost function is strongly convex. In our case, local cost functions given in (6) are quadratic, however, they are only convex but not strongly convex since the matrix \( h_i h_i^\top \) is only positive semi-definite. Therefore, the convergence results in these studies cannot be applied. In order to establish the convergence, we need to seek other means as will be shown in Section 3.

3 Convergence Results

In this section, we investigate the proposed algorithm (11) and establish sufficient conditions under which it exponentially converges to the least square solution. To improve the readability, all the proofs will be presented in later subsections.

Before presenting the main convergence result, we present the following lemma.

**Lemma 1** Assume that Assumptions 1 and 2 are satisfied. Then, the distributed algorithm or network flow (11) has a unique equilibrium point \( (\bar{x}, \bar{v}) \), where \( \bar{x} = [\bar{x}_1^\top \ldots \bar{x}_N^\top]^\top \) and \( \bar{v} = [\bar{v}_1^\top \ldots \bar{v}_N^\top]^\top \), which is given by

\[
\bar{x} = \mathbf{1}_N \otimes y^*, \quad \bar{v} = \mathbf{0}_{Nm},
\]

where \( y^* \) is the least square solution given by (3).

We are ready to present our first main result, which states that there is a critical value on the step-size, below which the algorithm exponentially converges to the least square solution, and above which the algorithm diverges.
Theorem 1 Assume that Assumptions 1 and 2 are satisfied. Then, along the distributed algorithm (11), there exists a constant $\bar{\alpha} > 0$, such that

(i) If $0 < \alpha < \bar{\alpha}$, then the algorithm (11) globally exponentially converges to the least square solution, i.e., $\mathbf{x}_i(t) \to \mathbf{y}^*$ exponentially fast as $t \to \infty$ for all $i = 1, \ldots, N$, where $\mathbf{y}^*$ is the least square solution given by (3). Moreover, $\mathbf{v}_i(t) \to 0$ exponentially fast as $t \to \infty$ for all $i = 1, \ldots, N$ when $0 < \alpha < \bar{\alpha}$.

(ii) If $\alpha > \bar{\alpha}$, then there exist infinitely many initial values under which the algorithm (11) diverges.

Remark 3 Note that Theorems 1 shows that there exists a critical value $\bar{\alpha} > 0$, which is the value of $\alpha$ such that $\rho_{\alpha+1}(\mathbf{M}) = 1$, where $\rho_{\alpha+1}(\mathbf{M})$ is the absolute value of the largest eigenvalue of the matrix $\mathbf{M}$ defined in (29) that is closest to 1, below which the algorithm (11) exponentially converges to the least square solution, and above which the algorithm diverges. This critical value depends on the mixing weight parameter $\alpha$. Therefore, the fundamental problem is how to compute the least square solution in a finite number of iterations, and hence termination further communication and computation to save energy and resources.

Remark 4 In view of Theorem 1, if the step-size $\alpha$ is sufficiently small, the algorithm (11) exponentially converges to the least square solution with a low convergence rate. Note that the convergence rate of the algorithm (11) when $\alpha = \bar{\alpha}$ is given by $\rho_{\alpha+1}(\mathbf{M})$. Thus one can aim to increase the convergence rate by solving the optimization problem $\min_{0 < \alpha < \bar{\alpha}} \rho_{\alpha+1}(\mathbf{M})$. However, as discussed in [20], this optimization problem is rather difficult to solve by analytic methods.

We are now ready to present our next main result, which provides an explicit characterization to the phase-transition value $\bar{\alpha}$ defined in Theorem 1.

Theorem 2 Assume that Assumptions 1 and 2 are satisfied. Then

$$\bar{\alpha} \geq \sup \left\{ \alpha : \mathbf{R} = - (\mathbf{I}_N - \mathbf{W})^2 \otimes \mathbf{I}_m - (\mathbf{W}^2 \otimes \mathbf{I}_m - \alpha \mathbf{H})^2 + \mathbf{I}_{Nm} > 0 \right\}. $$

(14)

Remark 5 As discussed in Remark 3, it is challenging to obtain an explicit relation of the critical value, the mixing matrix $\mathbf{W}$ and the matrix $\mathbf{H}$. Compared to Theorem 1 where the critical value is given implicitly, the bound given by (14) is more explicit and will also be illustrated in the Section 7 for numerical Example 1.

Note that it can be verified if $\alpha < \frac{\lambda^2_{\min}(\mathbf{W})}{\lambda_{\max}(\mathbf{H})}$, then $\mathbf{R} > 0$, where the matrix $\mathbf{R}$ is defined in (14). In view of Theorems 1 and 2, a more explicit bound on the critical value $\bar{\alpha}$ can be given, as stated in the following corollary.

Corollary 1 Assume that Assumptions 1 and 2 are satisfied. Then,

$$\bar{\alpha} \geq \frac{\lambda^2_{\min}(\mathbf{W})}{\lambda_{\max}(\mathbf{H})}. $$

(16)

4 Decentralized Finite-time Computation

In Section 3, we have established the sufficient conditions under which the algorithm (11) exponentially converges to the least square solution. Due to the exponential convergence, all nodes need to constantly perform local computation and communicate with their neighboring nodes, which results in a waste of computation and communication resources. Therefore, the fundamental problem is how to compute the least square solution in a finite number of iterations, and hence termination further communication and computation to save energy and resources.

To solve this problem, we will develop a finite-time least square solver by applying the algorithm (11) with a decentralized computation mechanism, which enables an arbitrarily chosen node to compute the exact least square solution in a finite number of time steps, by using the successive values of its local states. The proposed finite-time computation mechanism is based on letting each node runs the distributed algorithm (8) to get a successive local states with which each node is able to compute the exact least square solution in a finite number of time steps.

Consider the distributed algorithm (11). Note that it can be written as the following linear system:

$$
\begin{bmatrix}
\mathbf{x}(t + 1) \\
\mathbf{v}(t + 1)
\end{bmatrix} = \mathbf{M}
\begin{bmatrix}
\mathbf{x}(t) \\
\mathbf{v}(t)
\end{bmatrix},
\quad
\mathbf{v}(0) = \nabla F(\mathbf{x}(0)),
$$

(15)

where the system matrix $\mathbf{M}$ is given by (29). Assume that at time step $t$, an arbitrarily chosen node $r \in \mathcal{V}$ has observations about its state $\mathbf{x}_r(t) \in \mathbb{R}^m$. That is,

$$
\begin{bmatrix}
y_{r,1}(t) \\
\vdots \\
y_{r,m}(t)
\end{bmatrix} = \begin{bmatrix}
x_{r,1}(t) \\
\vdots \\
x_{r,m}(t)
\end{bmatrix} = \mathbf{C}_r \begin{bmatrix}
\mathbf{x}(t) \\
\mathbf{v}(t)
\end{bmatrix},
$$

(16)

with

$$
\mathbf{C}_r = \begin{bmatrix}
\mathbf{C}_{r,1} \\
\vdots \\
\mathbf{C}_{r,m}
\end{bmatrix} = \left[ \mathbf{e}_r^\top \otimes \mathbf{I}_m \ 0_{m \times Nm} \right],
$$

(17)

where $\mathbf{e}_r \in \mathbb{R}^{1 \times N}$ is the row vector whose $r$-th entry is 1, and the remaining entries are all zeros.

Based on these local successive observations, we will propose a decentralized computation algorithm which enables an arbitrarily chosen node $r$ to compute the exact least square solution $\mathbf{y}^* = [y^*_1 \ldots y^*_m]^\top \in \mathbb{R}^m$ in a finite number of iterations. The algorithm is motivated by the finite-time technique which is originally proposed in [27, 28] for distributed consensus, and extended to ratio consensus [29] and distributed optimization [23, 30]. Here, we extend it for computing the exact least square solution in a finite number of iterations. To our best knowledge, the finite-time least square solvers are not available in the literature.
The result hinges on the use of the minimal polynomial associated with a matrix pair. For this reason, we first provide the following definitions.

**Definition 1 (Minimal polynomial of a matrix)** The minimal polynomial associated with the square matrix $A$ is the monic polynomial $q(t) \triangleq t^{D+1} + \sum_{i=0}^{D} \alpha_i t^i$, where $\alpha_i \in \mathbb{R}$, with the smallest degree $D + 1$ that satisfies $q(A) = 0$.

**Definition 2 (Minimal polynomial of a matrix pair)** The minimal polynomial associated with the matrix pair $(A, C)$ denoted by $q_{ij}(t) \triangleq t^{D_{r,j} + 1} + \sum_{i=0}^{D_{r,j}} \alpha_i^{(r)} t^i$, where $\alpha_i^{(r)} \in \mathbb{R}$, is the monic polynomial of the smallest degree $D_{r,j} + 1 \leq D + 1$ that satisfies $C q_{ij}(A) = 0$.

In order for an arbitrarily chosen node $r \in V$ to compute the $j$-th component of the least square solution $y^*_{r,j}$, node $r$ needs to store successive observations for a few time steps. Consider the $2k+2$ successive observations $y_{r,j}(t) = x_{r,j}(t)$ at node $r$, that is, $y_{r,j}(0), y_{r,j}(1), \ldots, y_{r,j}(2k), y_{r,j}(2k+1)$. With these observations, node $r$ then calculates the difference between successive values of $x_{r,j}(t) = y_{r,j}(t)$ according to

$$
\overline{y}_{r,j}(t) \triangleq y_{r,j}(t) - y_{r,j}(t-1), \quad t = 1, \ldots, 2k + 1. 
$$

and constructs a $(k + 1) \times (k + 1)$ square Hankel matrix as

$$
H_{r,j}^{(r)} = \begin{bmatrix} \overline{y}_{r,j}(1) & \overline{y}_{r,j}(2) & \cdots & \overline{y}_{r,j}(k + 1) \\ \overline{y}_{r,j}(2) & \overline{y}_{r,j}(3) & \cdots & \overline{y}_{r,j}(k + 2) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{y}_{r,j}(k + 1) & \overline{y}_{r,j}(k + 2) & \cdots & \overline{y}_{r,j}(2k + 1) \end{bmatrix}.
$$

The next theorem shows that any node is able to compute the exact least square solution by using its own local successive observations in a finite number of iterations.

**Theorem 3** Consider the algorithm (11) with $\alpha < \frac{\lambda^2}{\lambda_{\min}(W)}$.

Then, any node $r \in V$ is able to compute the exact least square solution from a finite number of consecutive states $x_r(t)$.

The proof of Theorem 3 will be presented in Section 6.4. In particular, in the proof, we will give an explicit procedure for an arbitrarily chosen node to compute the $j$-th component of the exact least square solution, by using its own local successive observations obtained from the underlying algorithm (11) in a finite number of time steps. Thus, the proof naturally leads to a decentralized finite-time computation mechanism which is summarized in Algorithm 1.

**Remark 6** Note that Algorithm 1 relies on the analysis of the rank of a square Hankel matrix. As shown in Theorem 3, for node $r$ to compute the $j$-th component of the least square solution $y^*_{r,j}$, the $2k \times 2k$ square Hankel matrix is guaranteed to lose rank when $k = D_{r,j}$. That is, an arbitrarily chosen node $r$ with successive observations over $2(D_{r,j} + 1)$

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**Algorithm 1** Decentralized Finite-time Computation for the Least Square Solution

**Data:** Successive observations of $y_r(t) = x_r(t), t \in \mathbb{Z}_+$.  
**Result:** The least square solution $y^*$.

**Step 1** Compute the vector of differences $\overline{y}_{r,j}(t)$ by (18).

**Step 2** Construct the square Hankel matrix $H_{r,j}^{(r)}$ by (19).

**Step 3** Increase the dimension $k$ of the square Hankel matrix $H_{r,j}^{(r)}$.

**Step 4** When the square Hankel matrix is singular, compute the normalized kernel $\beta_j^{(r)} = [\beta_{j,0}^{(r)}, \ldots, \beta_{j,D_{r,j}}^{(r)}]$.  

**Step 5** Compute the $j$-th component of the least square solution $y^*_{r,j}$ using (43).

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**5 Further Discussions**

In this section, we provide further results regarding the directed interaction graphs and the finite-time computation even when the underlying algorithms fail to converge.

**5.1 Directed Interaction Graphs**

In case of directed graphs, the construction of doubly stochastic mixing weight matrix $W$ in (11) is prohibitive [16]. As shown in [34–36], the algorithm (11) can be extended to directed graphs which are strongly connected. These algorithms use a row stochastic matrix for the mixing of estimates of the minimizer in the update (11a), and employ a column stochastic matrix for tracking the average gradient in the update (11b). More specifically, at time step $t$, each node $i$ performs the following updates:

$$
x_i(t+1) = \sum_{j \in N^{in}_{i}} p_{ij} x_j(t) - \alpha v_i(t), \quad (20a)
$$

$$
v_i(t+1) = \sum_{j \in N^{in}_{i}} q_{ij} v_j(t) + \nabla f_i(x_i(t+1)) - \nabla f_i(x_i(k)), \quad (20b)
$$

where the initialization condition $v_i(0) = \nabla f_i(x_i(0))$, and $N^{in}_{i} = \{j \in V \mid (j,i) \in \mathcal{E}\}$ is the in-neighbor set of node $i$.

Note that the algorithm (20) can be written in a compact form as

$$
x(t+1) = (P \otimes I_m) x(t) - \alpha v(t), \quad (21a)
$$

$$
v(t+1) = (Q \otimes I_m) v(t) + \nabla F(x(t+1)) - \nabla F(x(t)), \quad (21b)
$$

where $v(0) = \nabla F(x(0))$, the mixing weight matrices $P = [p_{ij}] \in \mathbb{R}^{N \times N}$ and $Q = [q_{ij}] \in \mathbb{R}^{N \times N}$ are row stochastic.
and column stochastic, respectively, and are chosen as:

\[ p_{ij} = \begin{cases} \frac{1}{d_i^{\text{in}}+1} & \text{if } j \in \mathcal{N}_i^{\text{in}} \cup \{i\} \\ 0 & \text{otherwise} \end{cases}, \]

\[ q_{ij} = \begin{cases} \frac{1}{d_i^{\text{out}}+1} & \text{if } i \in \mathcal{N}_j^{\text{out}} \cup \{i\} \\ 0 & \text{otherwise} \end{cases}, \] (22)

where \( \mathcal{N}_i^{\text{out}} = \{ j \in \mathcal{V} | (i, j) \in \mathcal{E} \} \) is the out-neighbor set of node \( i \), and \( d_i^{\text{in}} \) and \( d_i^{\text{out}} \) are in-degree and out-degree of node \( i \), respectively.

The next theorem studies the behavior of the algorithm (21), analogous to Theorem 1.

**Theorem 4** Assume that Assumption 1 and the underlying directed interaction graph is strongly connected. Consider the algorithm (21). Then, there exists a \( \bar{\alpha} > 0 \), such that

(i) If \( 0 < \alpha < \bar{\alpha} \), then the algorithm (21) globally exponentially converges to the least square solution.

(ii) If \( \alpha > \bar{\alpha} \), then the algorithm (21) diverges.

**Remark 7** Note that Theorems 4 shows that there exists a critical value \( \bar{\alpha} > 0 \), which is the value of \( \alpha \) such that \( \rho_{\alpha+1}(\mathbf{M}) = 1 \), where \( \rho_{\alpha+1}(\mathbf{M}) \) is the absolute value of the eigenvalue of the matrix \( \mathbf{M} \) defined in (47) that is closest to 1. This critical value depends on the mixing weight matrices \( \mathbf{P} \), \( \mathbf{Q} \) and the matrix \( \mathbf{H} \). However, it is challenging to obtain an explicit expression.

### 5.2 Finite-time Computation

Note that for the algorithm (21) for directed graphs, the finite-time computation mechanism proposed in Algorithm 1 still works since the algorithm can be applied to both undirected and directed graphs as shown in [28, 37]. What is needed for Algorithm 1 is the local successive states obtained from the underlying algorithm. As shown in [27, 37], the finite-time mechanism works as long as the underlying algorithm converges. However, it is interesting to numerically find that the computation mechanism proposed in Algorithm 1 can still enable agents to compute the least square solution in a finite number of iterations, even when the underlying algorithm (11) for undirected interaction graphs or the algorithm (21) for directed interaction graphs fail to converge. The intuition why it still works is that the information about the least square solutions are somehow embedded in the state vectors \( x_i \) and \( v_i \) of these two algorithms. By communicating these state vectors with its neighbors and by storing its own state vector \( x_i \) over a finite number of time steps, each node can use the finite-time computation mechanism proposed in Algorithm 1 to find the exact least square solutions. We shall present in Section 7 a case study to illustrate this observation, though a theoretical explanation is technically rather challenging at this stage.

### 6 Proofs of Statements

We now provide the proofs of earlier presented lemmas and main theorems.

#### 6.1 Proof of Lemma 1

Since \( \mathbf{1}_N^\top \mathbf{W} = \mathbf{1}_N^\top \mathbf{I}_m \), left multiplying (11b) by \( \mathbf{1}_N^\top \otimes \mathbf{I}_m \) yields

\[
(\mathbf{1}_N^\top \otimes \mathbf{I}_m)v(t+1) = (\mathbf{1}_N^\top \otimes \mathbf{I}_m)v(t) + (\mathbf{1}_N^\top \otimes \mathbf{I}_m)(\nabla F(x(t+1)) - \nabla F(x(t))).
\]

This implies that the term \( (\mathbf{1}_N^\top \otimes \mathbf{I}_m)(v(t) - \nabla F(x(t))) \) remains unchanged with respect to the time instant. Note that the algorithm is initialized with \( v(0) = \nabla F(x(0)) \). Therefore,

\[
(\mathbf{1}_N^\top \otimes \mathbf{I}_m)(v(t) - \nabla F(x(t))) = (\mathbf{1}_N^\top \otimes \mathbf{I}_m)(v(0) - \nabla F(x(0)))
\]

\[
= \mathbf{0}_m^\top, \quad \forall t = 0, 1, \ldots
\]

Next, we note that the equilibrium point of the algorithm (11) \( (\bar{x}, \bar{v}) \) must satisfy

\[
\bar{x} = (\mathbf{W} \otimes \mathbf{I}_m)\bar{x} - \alpha \bar{v}
\]

\[
\bar{v} = (\mathbf{W} \otimes \mathbf{I}_m)\bar{v}.
\]

It follows from (24b), \( \bar{v} \) belongs to the null space of the matrix \( (\mathbf{I}_N - \mathbf{W}) \otimes \mathbf{I}_m \). Since \( \text{null}(\mathbf{I}_N - \mathbf{W}) = \mathbf{1}_N \), we have

\[
\bar{v} = \mathbf{1}_N \otimes \mathbf{0}^m,
\]

where \( \mathbf{0}^m \in \mathbb{R}^m \).

Next, left multiplying (24a) by \( \mathbf{1}_N^\top \otimes \mathbf{I}_m \) and using the property \( \mathbf{1}_N^\top \mathbf{W} = \mathbf{1}_N^\top \mathbf{I}_m \) yields

\[
(\mathbf{1}_N^\top \otimes \mathbf{I}_m)\bar{x} = (\mathbf{1}_N^\top \otimes \mathbf{I}_m)\bar{x} - \alpha(\mathbf{1}_N^\top \otimes \mathbf{I}_m)\bar{v}.
\]

This implies that \( (\mathbf{1}_N^\top \otimes \mathbf{I}_m)\bar{v} = \mathbf{0}_m^\top \) since \( \alpha > 0 \). It then follows from (25) that \( \mathbf{0}^m = \mathbf{0}_m \). Therefore, \( \bar{v} = \mathbf{0}_N \).

Substituting this into (24a) yields that \( ((\mathbf{I}_N - \mathbf{W}) \otimes \mathbf{I}_m)\bar{x} = \mathbf{0}_N \). Since \( \text{null}(\mathbf{I}_N - \mathbf{W}) = \mathbf{1}_N \), we have \( \bar{x} = \mathbf{1}_N \otimes \mathbf{x}^0 \), where \( \mathbf{x}^0 \in \mathbb{R}^m \). Therefore, \( \bar{x} \) satisfies the constraint (7b) of the optimization problem (7).

Note that it follows from (23) that

\[
(\mathbf{1}_N^\top \otimes \mathbf{I}_m)\nabla F(\bar{x}) = \sum_{i=1}^{N} \nabla f_i(\mathbf{x}^0) = \mathbf{0}_m.
\]

The above condition is the necessary and sufficient optimality condition of the optimization problem (7), which is equivalent to the least square problem (2). Thus \( \mathbf{x}^0 = \mathbf{y}^* \). Therefore, the unique equilibrium point of the algorithm (11) is \( \bar{x} = \mathbf{1}_N \otimes \mathbf{y}^* \) and \( \bar{v} = \mathbf{0}_N \). This completes the proof.

#### 6.2 Proof of Theorem 1

In view of Lemma 1, to study the behavior of the the algorithm (11), it is equivalent to study the behavior of the unique equilibrium point \( (\bar{x}, \bar{v}) \) of the algorithm (11).
For this purpose, we transfer the unique the equilibrium point to the origin by defining the relative dynamics \( \tilde{x}(t) = x(t) - x \) and \( \tilde{v}(t) = v(t) - \bar{v} \). Note that it follows from (11), (12), (13), and the property that \( W1_N = 1_N \) that these relative dynamics satisfy the following equation

\[
\begin{align*}
\tilde{x}(t + 1) &= (W \otimes I_m)\tilde{x}(t) - \alpha \tilde{v}(t), \quad (27a) \\
\tilde{v}(t + 1) &= -\tilde{H}(1_N - W) \otimes I_m)\tilde{x}(t) \\
&\quad + (W \otimes I_m - \alpha \tilde{H})\tilde{v}(t). \quad (27b)
\end{align*}
\]

Note that the above relative dynamics (27) can also be written as the following linear system:

\[
\begin{bmatrix}
\tilde{x}(t + 1) \\
\tilde{v}(t + 1)
\end{bmatrix} = M \begin{bmatrix}
\tilde{x}(t) \\
\tilde{v}(t)
\end{bmatrix}, \quad (28)
\]

where

\[
M = \begin{bmatrix}
W \otimes I_m & -\alpha I_{Nm} \\
-\tilde{H}(1_N - W) \otimes I_m) W \otimes I_m - \alpha \tilde{H}
\end{bmatrix}, \quad (29)
\]

and the initialization is given by

\[
\begin{align*}
\tilde{x}(0) &= x(0) - \bar{x} = x(0) - 1_N \otimes y^*, \\
\tilde{v}(0) &= v(0) - \bar{v} = \nabla F(x(0)).
\end{align*} \quad (30)
\]

Therefore, to study the behavior of the relative dynamics (28), we focus on the locations of the eigenvalues of the matrix \( M \). The remaining proof is carried out in two steps.

**Step 1:** In this step, we show that there exists a \( \bar{\alpha} > 0 \), such that if \( \alpha > \bar{\alpha} \), the matrix \( M \) has at least one eigenvalues outside the unit circle, thus the algorithm (11) diverges.

We first note that \( M = M_0 + \alpha \Delta \), where

\[
M_0 = \begin{bmatrix}
W \otimes I_m & 0_{Nm \times Nm} \\
-\tilde{H}(1_N - W) \otimes I_m) W \otimes I_m & 0_{Nm \times Nm}
\end{bmatrix},
\]

\[
\Delta = \begin{bmatrix}
0_{Nm \times Nm} & -I_{Nm \times Nm} \\
0_{Nm \times Nm} & -\tilde{H}
\end{bmatrix}.
\]

Thus the matrix \( M \) can be viewed as the matrix \( M_0 \) perturbed by \( \alpha \Delta \). Since Assumption 2 is satisfied, it follows from the block triangular structure of the matrix \( M_0 \) that it has \( 2m \) eigenvalues at 1 and all other eigenvalues lie in \((0, 1)\). Let us denote the \( 2m \) eigenvalues of the matrix \( M_0 \) at 1 as \( \lambda_1 = \ldots = \lambda_{2m} = 1 \). It is easy to verify that the corresponding right eigenvectors associated with these eigenvalues at 1 are

\[
U = \begin{bmatrix}
1_N \otimes I_m & 0_N \otimes I_m \\
0_N \otimes I_m & 1_N \otimes I_m
\end{bmatrix},
\]

and the corresponding normalized left eigenvectors are

\[
\Omega^T = \begin{bmatrix}
\omega_1^T \\
\vdots \\
\omega_{2m}^T
\end{bmatrix} = \frac{1}{N} \begin{bmatrix}
-(1_N \otimes I_m)\tilde{H} & 1_N \otimes I_m \\
1_N \otimes I_m & 0_N \otimes I_m
\end{bmatrix}.
\]

It then follows from the eigenvalue perturbation theory [20, 38] that when \( \alpha \) is small, the variation of \( \lambda_1 = \ldots = \lambda_{2m} = 1 \) perturbed by \( \alpha \Delta \) are quantified by the the eigenvalues of the matrix

\[
\Omega^T \Delta U = \begin{bmatrix}
0_{m \times m} & 0_{m \times m} \\
-I_{m \times m} & \frac{1}{N}(1_N \otimes I_m)\tilde{H}(1_N \otimes I_m)
\end{bmatrix}.
\]

Note that due to the block triangular structure of the above matrix, it has \( m \) eigenvalues are 0, and other \( m \) eigenvalues are the eigenvalues of the matrix \( \frac{1}{N}(1_N \otimes I_m)\tilde{H}(1_N \otimes I_m) \). Given the structures of \( H \) in (4) and \( \tilde{H} \) in (10), we obtain that

\[
(1_N \otimes I_m)\tilde{H}(1_N \otimes I_m) = H^T H, \quad (31)
\]

which is positive definite since \( \text{rank}(H) = m \) from Assumption 1. Therefore, all eigenvalues of the matrix \( \frac{1}{N}(1_N \otimes I_m)\tilde{H}(1_N \otimes I_m) \) are negative.

Thus, \( \frac{d}{d\alpha}\lambda_i(\alpha) = 0 \) for \( i = 1, \ldots, m \) and \( \frac{d}{d\alpha}\lambda_i(\alpha) < 0 \) for \( i = m + 1, \ldots, 2m \). This implies that when \( \alpha \) is small, the eigenvalues \( \lambda_1(\alpha), \ldots, \lambda_m(\alpha) \) stay at 1, while \( \lambda_{m+1}(\alpha), \ldots, \lambda_{2m}(\alpha) \) move to the left along the real axis. However, as \( \alpha \) increases, eigenvalues of the matrix \( M = M_0 + \alpha \Delta \) may move outside of the unit circle.

Since the eigenvalues of the matrix \( M = M_0 + \alpha \Delta \) continuously depend on \( \alpha \), there exists a constant \( \bar{\alpha} > 0 \), which is the maximum value of \( \alpha \) such that \( \rho(M) = 1 \), where \( \rho(M) \) is the spectral radius of the matrix \( M \). Therefore, if \( \alpha > \bar{\alpha} \), then the matrix \( M \) would have at least one eigenvalue outside the unit circle, which implies that the relative dynamics (28) is unstable, and thus the algorithm (11) diverges. On the other hand, if \( \alpha < \bar{\alpha} \), then all eigenvalues of the matrix \( M \) locate either on the unit cycle or strictly within the unit cycle, which needs further analysis in Step 2.

**Step 2:** In this step, we investigate the behavior of the system (28) when \( \alpha < \bar{\alpha} \).

Note that it is easy to verify that matrix \( M \) always has \( m \) simple eigenvalues at 1, and the corresponding right eigenvectors and the normalized left eigenvectors are

\[
v_{M, 1} = \begin{bmatrix}
h_{1_N \otimes I_m} \\
h_{0_N \otimes I_m}
\end{bmatrix},
\]

\[
\omega_{M, 1}^T = (H^T H)^{-1} \begin{bmatrix}
(1_N \otimes I_m)\tilde{H} & -1_N \otimes I_m
\end{bmatrix}.
\]

If \( \alpha < \bar{\alpha} \), then the matrix \( M \) has exactly \( m \) simple eigenvalues at 1 and all other eigenvalues are strictly within
the unit cycle. It then follows from (28) that
\[
\lim_{t \to \infty} \begin{bmatrix} \tilde{x}(t) \\ \tilde{v}(t) \end{bmatrix} = 0_{2N_m},
\]
where the second equality follows the initialization given by (30) and (12), the third equality follows from (31), (3), and the fact that \((1_N \otimes I_m)z_H = H^T z\) due to (4) and (10).

Therefore, for all \(i = 1, \ldots, N\), \(x_i(t) \to y^*\) and \(v_i(t) \to 0_m\) asymptotically as \(t \to \infty\). Note that for linear systems, asymptotic convergence and exponential convergence are equivalent. Hence, the result follows.

6.3 Proof of Theorem 2

In view of Theorem 1, in order to prove the theorem, we need to show that the algorithm (11) exponentially converges to the least square solution if the step-size \(\alpha\) is chosen to satisfy that \(R > 0\), where the matrix \(R\) is given in (14). In view of Lemma 1, it is equivalent to show that the unique equilibrium point \((\bar{x}, \bar{v})\) of the algorithm (11) is globally exponentially stable for such a step-size. Recall that the relative dynamics \(\tilde{x}(t) = x(t) - \bar{x}\) and \(\tilde{v}(t) = v(t) - \bar{v}\) satisfy the equation (27). It is equivalent to show that the origin of the relative dynamics (27) is globally exponentially stable.

Note that it follows from (12), (23), (26) and the fact that \(\bar{v} = 0_{Nm}\) that all trajectories of the relative dynamics (27) satisfy
\[
(1_N \otimes I_m)(\tilde{v} - \tilde{H}x) = 0_m. \tag{32}
\]
Therefore, we focus on the subspace
\[
\left\{ \begin{bmatrix} \tilde{x}^T \\ \tilde{v}^T \end{bmatrix} : (1_N \otimes I_m)(\tilde{v} - \tilde{H}x) = 0 \right\}. \tag{33}
\]
Consider the following Lyapunov candidate function
\[
V(\tilde{x}(t), \tilde{v}(t)) = \eta^T(t)\eta(t) + 2\eta^T(t)(I_N - W)^2 \otimes I_m)\tilde{x}(t) + \tilde{x}^T(t)(I_N - W)^2 \otimes I_m)\tilde{x}(t),
\]
where
\[
\eta(t) = -(I_N - W) \otimes I_m)\tilde{x}(t) - \alpha\tilde{v}(t). \tag{34}
\]
We sometimes drop the dependency of \(V(\tilde{x}(t), \tilde{v}(t))\) on \(\tilde{x}(t)\) and \(\tilde{v}(t)\) for notational simplification when it is clear from the context. We first note that \(V\) is radially unbounded in the considered subspace (33).

Next, we show that \(V \geq 0\). Note that the Lyapunov candidate function (34) can be rewritten as
\[
V = \begin{bmatrix} \tilde{x}^T \\ \eta \end{bmatrix} \hat{P} \begin{bmatrix} \tilde{x} \\ \eta \end{bmatrix}, \tag{35}
\]
where
\[
\hat{P} = \begin{bmatrix} (I_N - W)^2 \otimes I_m & (I_N - W)^2 \otimes I_m \\ (I_N - W)^2 \otimes I_m & I_{Nm} \end{bmatrix}. \tag{36}
\]
Since \(I_{Nm} > 0\), it follows from the Schur complement theory [39, pp. 651] that \(\hat{P} \geq 0\) if and only if
\[
(I_N - W)^2 \otimes I_m - (I_N - W)^4 \otimes I_m \geq 0,
\]
which holds due to the fact that \(W > 0\) from Assumption 2. Thus \(V \geq 0\).

We then show that \(V = 0\) if and only if \(\tilde{x} = 0_{Nm}\) and \(\tilde{v} = 0_{Nm}\). We first note that it is easy to verify that the matrix \(\hat{P}\) has \(m\) single eigenvalues at 0 and the corresponding right eigenvectors are
\[
\begin{bmatrix} 1_N \otimes I_m \\ 0_O \otimes I_m \end{bmatrix}.
\]
Therefore, \(V = 0\) if and only if \(\tilde{x} = 1_N \otimes c\) for some \(c \in \mathbb{R}^m\) and \(\eta = 0\). It then follows from (34) and the property that \(W1_N = 1_N\) that \(\tilde{v} = 0_{Nm}\), since \(\alpha > 0\). This together with the considered subspace (33) implies that
\[
(1_N \otimes I_m)\tilde{H}(1_N \otimes I_m)c = 0_N.
\]
Recall from (31) that
\[
(1_N \otimes I_m)\tilde{H}(1_N \otimes I_m) = H^T H.
\]
Therefore, we know that \(c = 0_m\) since \(\text{rank}(H) = m\) from Assumption 1. Thus \(V = 0\) if and only if \(\tilde{x} = 0_{Nm}\) and \(\tilde{v} = 0_{Nm}\).

Finally, we show that \(\Delta V(\tilde{x}(t), \tilde{v}(t)) = V(\tilde{x}(t + 1), \tilde{v}(t + 1)) - V(\tilde{x}(t), \tilde{v}(t)) \leq 0\). Again, we sometimes drop the dependency of \(\Delta V(\tilde{x}(t), \tilde{v}(t))\) on \(\tilde{x}(t)\) and \(\tilde{v}(t)\) for notational simplification when it is clear from the context.

Let us first compute \(V(t + 1) := V(\tilde{x}(t + 1), \tilde{v}(t + 1))\). We first note that it follows from (27a) and (34) that
\[
\tilde{x}(t + 1) = \tilde{x}(t) + \eta(t). \tag{37}
\]
Also from (34) and (27), we obtain that

\[ \eta(t+1) = \frac{\eta(t + 1)}{-((I_N - W) \otimes I_m) \dot{x}(t + 1) - \alpha \dot{v}(t + 1) + (I_N - W) \otimes I_m) \dot{v}(t)} - \alpha \left[ -H((I_N - W) \otimes I_m) \dot{x}(t) + (W \otimes I_m) \dot{v}(t) \right] = -((I_N - W) \otimes I_m) \dot{x}(t) = -((I_N - W) \otimes I_m) \dot{v}(t) \]

For the notational simplification, let us denote \( \eta(t+1) \) as \( \zeta \). From (34), (37) and (38), we obtain that

\[ V(t+1) = \zeta^T \zeta + 2\zeta^T ((I_N - W)^2 \otimes I_m) \dot{x}(t + 1) + \dot{x}^T (t + 1) \]

\[ = \zeta^T \zeta + 2\zeta^T ((I_N - W)^2 \otimes I_m) \dot{x}(t + 1) + (I_N - W)^2 \otimes I_m) \dot{v}(t) \]

Note that \( \Delta V \leq 0 \) due to the condition (14). Moreover, we note that \( \Delta V = 0 \) if and only if \( \zeta = (W^2 \otimes I_m - \alpha H)x \) and \( \eta = 0_{N_m} \), which implies that \( \eta = 0_{N_m} \) and \( \zeta = 0_{N_m} \).

It then follows from (38) that \( \dot{x} = 0_N \) for some \( c \in \mathbb{R}^{m} \). This together with (34) the property that \( W1_N = 1_N \) implies that \( \dot{x} = 0_N \) since \( \alpha > 0 \). It then follows from the considered subspace (33), the property (31), and the fact that rank(\( H \)) = \( m \) from Assumption 1 that \( c = 0_m \), which implies that \( \dot{x} = 0_{N_m} \). Therefore, \( \Delta V = 0 \) if and only if \( \dot{x} = 0_{N_m} \) and \( \dot{v} = 0_{N_m} \).

Hence, the origin of the relative dynamics (27) is globally asymptotically stable. Finally, the result follows by noting that the relative dynamics (27) is a linear system, for which the asymptotic stability and the exponential stability are equivalent.

6.4 Proof of Theorem 3

In the proof, we will present a procedure which enables an arbitrarily chosen node to compute the least square solution by using the local successive state observations obtained from the underlying algorithm (11).

For each \( j = 1, \ldots, m \), let us denote

\[ q_{r,j}(t) = \sum_{i=0}^{D_{r,j}+1} \alpha_{j,i}^{(r)} t^i \]

as the minimal polynomial associated with the matrix pair \((M, C_{r,j})\), where the matrices \( M \) and \( C_{r,j} \) are given by (29) and (17), respectively. It then follows from Definition 2 that

\[ \sum_{i=0}^{D_{r,j}+1} \alpha_{j,i}^{(r)} y_{r,j}(t + i) = 0, \quad \forall t \in \mathbb{Z}_+, \]

where \( \alpha_{j,i}^{(r)} = 1 \). Next, let us denote the \( z \)-transform of \( y_{r,j}(i) \) as \( Y_{r,j}(z) = \mathbb{Z}(y_{r,j}(k)) \).

By applying the time-shift property of the \( z \)-transform into (40), we obtain that

\[ Y_{r,j}(z) = \sum_{i=0}^{D_{r,j}+1} \alpha_{j,i}^{(r)} \sum_{\ell=0}^{i-1} y_{r,j}(\ell) z^{i-\ell} \]

\[ = \mathbb{Z}(y_{r,j}(k)). \]

Since \( \alpha < \lambda_{\min}(W) / \lambda_{\max}(H) \), it follows from Theorem 1 and Corollary 1 that the matrix \( M \) given by (29) has \( m \) simple eigenvalues at 1 and all other eigenvalues are strictly within the unit circle. These properties together with the structure of the matrix \( C_{r,j} \) given by (17) imply that \( q_{r,j}(t) = 0 \) does not possess any unstable root apart from a single one at 1.

Next, define the polynomial \( p_{r,j}(z) = \mathbb{Z}(y_{r,j}(k)) \). It is then follows from (39) that \( p_{r,j}(z) = \sum_{\ell=0}^{D_{r,j}} \beta_{j,\ell}^{(r)} z^{\ell}, \)

where

\[ \beta_{j,\ell}^{(r)} = - \sum_{i=0}^{\ell} \alpha_{j,i}^{(r)}, \quad \ell = 0, \ldots, D_{r,j}. \]
If these coefficients $\beta^{(r)}_{j,i}$ are known, we can then obtain the $j$-th component of the least square solution $y^*_j$ by applying the final-value theorem and using (41) as

$$y^*_j = \lim_{t \to \infty} y_{r,j}(t) = \lim_{z \to 1} (z-1)Y_{r,j}(z) = \frac{H_{r,j}(1)}{p_{r,j}(1)} = \frac{y_{r,j}^T(I_{D_{r,j}} - \beta^{(r)}_j)}{1^T\beta^{(r)}_j}, \quad (43)$$

where

$$y_{j,D_{r,j}} = \begin{bmatrix} y_{r,j}(0) & y_{r,j}(1) & \cdots & y_{r,j}(D_{r,j}) \end{bmatrix}^T, \quad (44)$$

and

$$\beta^{(r)}_j = \begin{bmatrix} \beta^{(r)}_{j,0} & \cdots & \beta^{(r)}_{j,D_{r,j}-1} & 1 \end{bmatrix}^T. \quad (45)$$

Note that if agent $r$ knows the vector $\beta^{(r)}_j$, then it can compute the $j$-th component of the least square solution by using (43). Next, we show how node $r$ can compute the vector $\beta^{(r)}_j$.

Note that from (42), we obtain

$$\alpha^{(r)}_{j,i} = \beta^{(r)}_{j,i-1} - \beta^{(r)}_{j,i}, \quad i = 0, \ldots, D_{r,j} + 1, \quad (46)$$

where $\beta^{(r)}_{j,-1} = 0$ and $\beta^{(r)}_{j,D_{r,j}+1} = 0$ for the convention.

It then follows from (40), (46), and the fact that $\alpha^{(r)}_{j,D_{r,j}+1} = 1$ that for any $t \in \mathbb{Z}_+$,

$$\beta^{(r)}_{j,0}(y_{r,j}(t+1) - y_{r,j}(t)) + \cdots + \beta^{(r)}_{j,D_{r,j}-1}(y_{r,j}(t+D_{r,j}) - y_{r,j}(t+D_{r,j}-1)) + (y_{r,j}(t+D_{r,j}+1) - y_{r,j}(t+D_{r,j})) = 0.$$ 

This implies that the square Hankel matrix $H^{(r)}_{j;2k}$ given by (19) will lose its rank when $k = D_{r,j}$ and the vector $\beta^{(r)}_{j}$ given by (45) is its kernal. Note that in order to construct this Hankel matrix, node 1 needs $2D_{r,j} + 2$ successive state values. Therefore, node $r$ is able to compute the $j$-th component of the least square solution $y^*_j$ by (43) using $2D_{r,j} + 2$ successive states. Note that the above analysis holds for all $j = 1, \ldots, m$. Hence, node $r$ is able to compute the least square solution $y^*$, which completes the proof.

6.5 Proof of Theorem 4

The proof follows by checking that all the analysis of the proof of Theorem 1 still holds except that the system matrix $M$ in (29) of the relative dynamics (28) now becomes

$$M = \begin{bmatrix} P \otimes I_m & -\alpha I_{Nm} \\ -\tilde{H}((I_N - P) \otimes I_m) Q \otimes I_m - \alpha \tilde{H} \end{bmatrix}. \quad (47)$$

Note that the matrix $M$ can be written as $M = M_0 + \Delta$, where

$$M_0 = \begin{bmatrix} P \otimes I_m \\ -\tilde{H}((I_N - P) \otimes I_m) Q \otimes I_m \end{bmatrix},$$

$$\Delta = \begin{bmatrix} 0 & -I_{Nm} \\ 0 & -\tilde{H} \end{bmatrix}.$$ 

The remainder analysis is similar to that for the proof of Theorem 1 with some minor modifications and thus omitted.

7 Numerical Examples

In this section, we provide numerical examples to validate and illustrate our results. For the purpose of comparison, we have adopted the numerical examples from [17].

Example 1. In this example, we will illustrate the results stated in Theorems 1 and 2. Consider a linear equation in the form of (1) where $y \in \mathbb{R}^2$, and

$$H = \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 2 \end{bmatrix}.$$ 

Since Assumption 1 is satisfied, the linear equation has a unique least squares solution $y^* = [-0.1429, -1]^T$. The underlying interaction graph is given in Fig. 1. Consider the proposed distributed algorithm (11). Choose the mixing weight matrix

$$W = \begin{bmatrix} 0.7 & 0.15 & 0.15 & 0 \\ 0.15 & 0.85 & 0 & 0 \\ 0.15 & 0 & 0.7 & 0.15 \\ 0 & 0 & 0.15 & 0.85 \end{bmatrix}.$$ 

It is easy to verify that this matrix satisfies Assumption 2. Next, choose the step-size $\alpha = 0.18$. Note that it is easy to check that $\rho_{m+1}(M) = 0.9901$, which is less than 1. According to Theorem 1 and Remark 3, we know that $\alpha = 0.18 < \alpha$ and the algorithm (11) exponentially converges to the least square solution. Also note that $R > 0$ where the matrix $R$ is given in (14) when $\alpha = 0.18$.

For the initial condition $x(0) = [4, 1, 2, -2, 1, -2, -1]^T$, the initial condition $v(0)$ is computed as $v(0) = Hx(0) - z_H = [0, 2, 18, 0, 0, -4, 0]^T$. Fig. 2 plots the trajectories of the $x_i(t), i = 1, 2, 3, 4$ in $\mathbb{R}^2$, which shows that all $x_i(t)$ converge to $y^*$. To further confirm this, we have plotted the state evolutions of $x_i(t)$ for $i = 1, 2, 3, 4$ in Fig. 3(a), which shows that all $x_i(t)$ converge to the exact least square solution $y^* = [-0.1429, -1]^T$. The numerical result is consistent with the results in Theorems 1 and 2.
Fig. 1. An undirected and connected graph with four nodes.

Fig. 2. The trajectories of $x_i$.

For the purpose of comparison, the state evolutions obtained by the Euler discretization based algorithm proposed in [17] with the discretization step-size equal to 0.03 are plotted in Fig. 3(b). As can be seen, both algorithms converge to the least square solution, however, we note that our algorithm converges roughly in 300 time steps, while the one proposed in [17] takes about 5000 time steps to converge. Therefore, the convergence of our algorithm is much faster.

Next, we numerically obtain the critical value $\hat{\alpha}$. To do so, let us change the step-size from $\alpha = 0.18$ to $\alpha = 0.19$. The state evolutions of $x_{i,1}$ and $x_{i,2}$ for $i = 1, 2, 3, 4$ are plotted in Fig. 4. As can be seen, although all $x_{i,2}$ converge to the 2-nd component of the least square solution $y^*_2 = -1$, all $x_{i,1}$ fail to converge to the 1-st component of the least square solution. This is because there is an unstable eigenvalue of the matrix $M$ given by (29) at $1.2048$, which lead to the divergence of the algorithm according to Theorem 1. This implies that the critical value $\hat{\alpha}$ in Theorem 1 lies in $(0.18, 0.19)$. More specifically, the relationship between $\rho_{m+1}(M)$, which is the absolute value of the eigenvalue of the matrix $M$ that is closest to 1 and $\alpha$ is plotted in Fig. 5(a), which shows that $\hat{\alpha} = 0.1858$. Also, note that when $\alpha = 0.19$, the matrix $R$ defined in (14) has a negative eigenvalue at $-0.0814$. More specifically, the relationship between $\lambda_{\text{min}}(R)$, which is the minimum eigenvalue of the matrix $R$ and $\alpha$ is plotted in Fig. 5(b), which shows that $R > 0$ is satisfied if $\alpha < 0.1851$. Thus, from Theorem 2, we conclude that $\hat{\alpha} \geq 0.1851$, which is rather tight for this numerical example.

Example 2. In this example, we will illustrate the results stated in Theorem 3. The parameters are chosen the same as those in Example 1. Now we also equip our proposed algorithm (11) with the decentralized finite-time computation mechanism given by Algorithm 1, which enables nodes to compute the least square solution in a finite number of time steps. The state evolutions of $x_i$ for $i = 1, 2, 3, 4$ are shown in Fig. 6(a). By equipping the algorithm (11) with the finite-time mechanism proposed in Algorithm 1, all nodes compute the exact least square solution within 16 time steps, which is indicated by the
vertical blue line in the plot, and hence terminate further computation and communication. However, we observe that, at this time step, even approximated least square solution is not achieved by running the algorithm (11) alone, and it will take much larger time steps to converge to the least square solution with a reasonable accuracy.

To better illustrate how the decentralized finite-time computation mechanism proposed in Algorithm 1 enables nodes to compute the least square solution in a finite number of time steps, let us focus on the case that how node 1 computes the 1-st component of the least square solution $y^*_1 = -0.1429$ by using Algorithm 1. Following Algorithm 1, node 1 stores local states $x_{i,1}$ obtained from the underlying algorithm (11) for a few time steps, computes the state differences $\mathcal{P}_{i,1}$, constructs the square Hankel matrix $H_{1,2k}$, and examines the rank of the Hankel matrix. When $k = 7$ the Hankel matrix becomes singular and loses rank. Note that in order to construct this Hankel matrix, node 1 needs $2k + 2 = 16$ successive state values. Then node 1 computes its normalized kernel as

$$\beta^{(1)}_i = \left[ -0.1036 \quad 0.2450 \quad 0.6113 \quad -2.4246 \quad 1.8605 \quad 1.4916 \quad -2.68 \right]^T.$$ (48)

Finally, node 1 uses $\beta^{(1)}_i$ and its successive observation

$$y_{1,7} = \left[ -2 -1.13 -0.5651 -0.2932 -0.1976 -0.1544 -0.1909 -0.2094 \right]^T$$

to computes the 1-st component of the least square solution according to (43), that is, $y^*_1 = \frac{\mathcal{P}_{i,1}\beta^{(1)}_i}{1^T\beta^{(1)}_i} = -0.1429$.

Therefore, by using the finite-time computation mechanism proposed in Algorithm 1, node 1 is able to compute the exact 1-st component of the least square solution in 16 time steps. At this step, as shown more explicitly in Fig. 6(b) that the algorithm (11) alone is not convergent. It roughly takes 300 time steps for all nodes to converge to the 1-st component of the least square solution $y^*_1 = -0.1429$, by running the algorithm (11) alone. By equipping the algorithm (11) with the proposed finite-time computation mechanism, the time steps to obtain the exact least square solution are greatly reduced, and thus significantly reduce the communication overhead and save the computation resource.

Example 3. In this example, we will illustrate the results stated in Section 5. Consider a linear equation in the form

\[ \begin{pmatrix} -8 & -20 & -8 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} + \begin{pmatrix} 20 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \]
of (1) where \( y \in \mathbb{R}^2 \), and

\[
H = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 2 \end{bmatrix}.
\]

Since Assumption 1 is satisfied, the linear equation has a unique least squares solution \( y^* = [-0.1429 \quad 0.6514]^T \).

The directed interaction graph is given in Fig. 7. Consider the proposed distributed algorithm (21). Choose the mixing weight matrices according to (22) as

\[
P = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} \end{bmatrix},
\]

and the step-size \( \alpha = 0.12 \). It is easy to check that \( \rho_{\text{max}}(M) = 0.9372 \), where the matrix \( M \) is given by (47). Thus according to Theorem 4, the algorithm (21) exponentially converges to the exact least square solution. The simulation results for the state evolutions of \( x_{i,1} \) and \( x_{i,2} \) for \( i = 1, 2, 3, 4 \) are given in Fig. 8(a) and Fig. 8(b), respectively. As can be seen, all \( x_{i,1} \) converge to \( y_1^* = -0.1429 \) and \( x_{i,2} \) converge to \( y_2^* = -1 \), which is consistent with the result of Theorem 4.

Moreover, by equipping the algorithm (21) with the finite-time computation mechanism proposed in Algorithm 1, all nodes compute the exact least square solution within 16 time steps, which are indicated by the vertical blue line in the plots. However, we observe that, at this time step, even approximated least square solution is not achieved by running the algorithm (21) alone.

Finally, to illustrate our finding presented in Section 5.2 that the finite-time computation mechanism proposed in Algorithm 1 still works even when the underlying algorithm fails to converge. Let us now change the step-size to \( \alpha = 0.2 \). It is easy to check that the matrix \( M \) given by (47) now has an unstable eigenvalue at 1.4660. Thus according to Theorem 4, the algorithm (21) actually diverges and thus fails to find the exact least square solution.

However, by equipping the algorithm (21) with the finite-time mechanism proposed in Algorithm 1, all nodes compute the exact least square solution within a finite number of time steps by using its local successive states. For example, in order to compute the 1-st component of the least square solution \( y_1^* = -0.1429 \), node 1 stores its local successive observations \( y_{1,1} \) obtained from the algorithm (21) for a few time steps. Then following the procedures in Algorithm 1, node 1 computes the state differences \( \overline{y}_{i,1} \), constructs the square Hankel matrix \( H_{1,2k}^{(1)} \), and examines the rank of the Hankel matrix. When \( k = 14 \) the Hankel matrix loses its rank. Note that in order to construct this Hankel matrix, node 1 needs \( 2k + 2 = 30 \) successive state values. Then node 1 computes its normalized kernel \( \beta_{1}^{(1)} \) and then uses this together with its successive observations according to (43) to find the 1-st component of the least square solution \( y_1^* = -0.1429 \). Therefore, Algorithm 1 enables node 1 to compute the exact 1-st component of the least square solution in 30 time steps even though the underlying algorithm (21) diverges. It is worth to note that it takes longer time steps for nodes to compute the exact least square solution compared to the case when the underlying algorithm (21) converges.

8 Conclusions

In this paper, we studied the problem of distributed computing the exact least square solution of over-determined linear algebraic equations over multi-agent networks, where each node only has the information of one scalar linear
equation. We first proposed a distributed algorithm as the exact least square solver for linear equations, where nodes exchange information with neighboring nodes over an undirected connected network. We established sufficient conditions on the mixing weight matrix and the step-size under which the proposed algorithm exponentially converges to the exact least square solution. We then developed a finite-time exact least square solver for linear equations, by equipping the proposed algorithm with a decentralized computation mechanism. With the proposed decentralized computation mechanism, an arbitrarily chosen node is able to compute the exact least square solution within a finite number of time steps, by using its local successive observations. We also discussed how to extend the proposed algorithm to directed graphs and how the proposed finite-time computation mechanism computes the exact least square solution even when the underlying algorithms fail to converge. The future direction is to extend the proposed distributed algorithm to networks with communication time-delays.

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