Contraction Elimination in Sequent Based Ground Equational Calculus

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Abstract
In [1] we have shown that the cut rule is eliminable in two ground equational sequent calculi to be denoted by $EQ_M$ and $EQ'$. In this note we prove that the contraction rule is not eliminable in $EQ_M$ but it is eliminable in $EQ'$.

0.1 $EQ_M$ and $EQ$

**Definition 0.1** $EQ_M$ is the calculus acting on sequents with one formula in the succedent having atomic logical axioms of the form $A \Rightarrow A$, the reflexivity axioms $t = t$ ($\Rightarrow =$); the left structural rules of weakening, exchange and contraction:

$$
\frac{\Gamma \Rightarrow H}{\Gamma, F \Rightarrow D} \quad \frac{\Gamma_1, F, G, \Gamma_2 \Rightarrow H}{\Gamma_1, G, F, \Gamma_2 \Rightarrow H} \quad \frac{\Gamma, F, F \Rightarrow H}{\Gamma, F \Rightarrow H}
$$

the atomic cut rule:

$$
\frac{\Gamma \Rightarrow A \quad \Lambda, A \Rightarrow H}{\Gamma, \Lambda \Rightarrow H}
$$

and the atomic equality left introduction rules $=_1\Rightarrow$ and $=_2\Rightarrow$, namely:

$$
\frac{\Gamma \Rightarrow D\{v/r\}}{\Gamma, \gamma = \delta \Rightarrow D\{\nu/s\}} \quad \frac{\Gamma \Rightarrow D\{v/r\}}{\Gamma, \gamma = \delta \Rightarrow D\{\nu/s\}}
$$

where by atomic we mean that $A$ and $D$ are required to be atomic formulae.

**Notation** In the following $A$ and $D$ will denote atomic formulae.

1
**DEFINITION 0.2** EQ is the calculus acting on sequents with one formula in the succedent having logical axioms of the form \( A \Rightarrow A \), the reflexivity axioms \( \Rightarrow t = t \ (\Rightarrow) \); the left structural rules of weakening, exchange and contraction, the atomic cut rule and the following atomic congruence rule CNG:

\[
\begin{align*}
\Gamma \Rightarrow D\{v/r\} & \quad \Lambda \Rightarrow r = s \\
\Gamma, \Lambda \Rightarrow D\{v/s\}
\end{align*}
\]

**DEFINITION 0.3** cf.EQ and cf.EQ\_M denote the systems EQ and EQ\_M deprived of the cut rule.

**PROPOSITION 0.1** EQ and EQ\_M are equivalent over the structural rules of weakening, exchange and cut, more precisely the rules \( =_1 \Rightarrow \) and \( =_2 \Rightarrow \) are derivable in EQ, without using the contraction and the cut rule and, conversely, CNG is derivable in EQ\_M without using the contraction rule.

**Proof**

\[
\begin{align*}
\Gamma \Rightarrow D\{v/r\} & \quad r = s \Rightarrow r = s \\
\Gamma, r = s \Rightarrow D\{v/s\}
\end{align*}
\]

\[
\begin{align*}
\Gamma \Rightarrow D\{v/r\} & \quad \Rightarrow s = s \quad s = r \Rightarrow s = r \\
\Gamma, s = r \Rightarrow D\{v/s\}
\end{align*}
\]

\[
\begin{align*}
\Lambda \Rightarrow r = s & \quad \Gamma \Rightarrow D\{v/r\} \\
\Gamma, r = s \Rightarrow D\{v/s\}
\end{align*}
\]

\[\square\]

**0.2 Eliminating the Contraction Rule**

The contraction rule is not eliminable from EQ\_M. For example the sequent \( a = f(a) \Rightarrow a = f(f(a)) \), where \( a \) is an individual parameter, in cf.EQ\_M has the following derivation:

\[
\begin{align*}
a = f(a) & \Rightarrow a = f(a) \\
\begin{align*}
a = f(a), a = f(a) & \Rightarrow a = f(f(a)) \\
a = f(a) & \Rightarrow a = f(f(a))
\end{align*}
\end{align*}
\]

but we can show that there is no derivation in EQ\_M of \( a = f(a) \Rightarrow a = f(f(a)) \), that does not use the contraction rule.

**DEFINITION 0.4** Let EQ\_M\_ be obtained from EQ\_M by suppressing the contraction rule.
**Notation** \(\Gamma_\equiv\) denotes the sequence that is obtained from \(\Gamma\) by suppressing all the formulae that are not equalities.

**Proposition 0.2** \(\Gamma \Rightarrow r = s\) is derivable in \(EQ_M (EQ_M^\equiv)\), if and only if \(\Gamma_\equiv \Rightarrow r = s\) is derivable in \(EQ_M (EQ_M^\equiv)\) with a derivation that contains only equalities.

**Proof** The "if" direction is immediate by the weakening rule.

The "only if " direction is established by induction on the height of a given derivation \(D\) of \(\Gamma \Rightarrow r = s\). If \(h(D) = 0\) then either \(\Gamma = \emptyset\) and \(r \equiv s\) or \(\Gamma\) reduces to \(r = s\). In both case the conclusion is obvious. If \(h(D) > 0\) and the last rule of \(D\) is not a cut, then, if a principal formula is not an equality, the induction hypothesis yields directly the desired derivation of \(\Gamma_\equiv \Rightarrow r = s\). Otherwise it suffices to apply the same rule to the derivation provided by the induction hypothesis. Finally suppose \(D\) ha the form:

\[
\begin{array}{c}
\Gamma \Rightarrow A \\
\Lambda, A \Rightarrow r = s \\
\hline
\Gamma_\equiv \Rightarrow r = s
\end{array}
\]

If \(A\) is not an equality, by induction hypothesis applied to \(D_1\) we have a derivation of \(\Lambda_\equiv \Rightarrow r = s\), from which we can obtain the desired derivation of \(\Gamma_\equiv \Rightarrow r = s\) by weakenings and exchanges. On the other hand if \(A = p = q\), by induction hypothesis applied to both \(D_0\) and \(D_1\), we have derivations of \(\Gamma_\equiv \Rightarrow p = q\) and \(\Lambda_\equiv, p = q \Rightarrow r = s\), from which the desired derived of \(\Gamma_\equiv, \Lambda_\equiv \Rightarrow r = s\) can be obtained by applying the cut rule.

**Notation** \(r \equiv s\) denotes that \(r\) and \(s\) are syntactically identical.

**Proposition 0.3** If \(p_1 = p_1, \ldots, p_n = p_n \Rightarrow r = s\) is derivable in \(EQ_M (EQ_M^\equiv)\), then \(r \equiv s\), in particular if \(\Rightarrow r = s\) is derivable in \(EQ_M (EQ_M^\equiv)\), then \(r \equiv s\).

**Proof** By induction on the height of a given derivation \(D\) of

\[
\begin{array}{c}
p_1 = p_1, \ldots, p_n = p_n \Rightarrow p = q \\
p_{i+1} = p_{i+1}, \ldots, p_n = p_n, p = q \Rightarrow r = s \\
\hline
p_1 = p_1, \ldots, p_n = p_n \Rightarrow r = s
\end{array}
\]

By induction hypothesis applied to \(D_0\) we have that \(p \equiv q\), then, by induction hypothesis applied to \(D_1\) (with \(n + 1 - i\) in place of \(n\)) we conclude that \(r \equiv s\).
PROPOSITION 0.4 If $E$ is an equality, then the following hold:

\begin{itemize}
  \item[a)] If $\ast$ \quad $p_1 = p_1, \ldots, p_j = p_j, E, p_{j+1} = p_{j+1}, \ldots, p_n = p_n \Rightarrow a = f(f(a))$ is derivable in $\text{EQ}_M$, then $E$ coincides with $a = f(f(a))$ or with $f(f(a)) = a$
  \item[b)] If $\ast\ast$ \quad $p_1 = p_1, \ldots, p_j = p_j, E, p_{j+1} = p_{j+1}, \ldots, p_n = p_n \Rightarrow f(f(a)) = a$ is derivable in $\text{EQ}_M$, then $E$ coincides with $a = f(f(a))$ or with $f(f(a)) = a$
\end{itemize}

**Proof** a) and b) are proved simultaneously by induction on the height of derivations.

a) Let $\mathcal{D}$ be a derivation in $\text{EQ}_M$ of $\ast$. If $h(\mathcal{D}) = 0$, then $n = 0$ and $E$ coincides with $a = f(f(a))$. As for the induction step, let us first observe that $\mathcal{D}$ cannot end with a weakening that introduces $E$, since, by the previous Proposition, $p_1 = p_1, \ldots, p_n = p_n \Rightarrow a = f(f(a))$ is not derivable. If $\mathcal{D}$ ends with an exchange the conclusion is immediate by the induction hypothesis. If $\mathcal{D}$ ends with a $=_{1}\Rightarrow$-inference, namely it has the form

\[
\begin{array}{c}
\mathcal{D}_0 \\
p_1 = p_1, \ldots, p_n = p_n \Rightarrow r = s \\
p_1 = p_1, \ldots, p_n = p_n, E \Rightarrow a = f(f(a))
\end{array}
\]

then, by the previous Proposition, $r \equiv s$. The only possibilities of obtaining $a = f(f(a))$ by a substitution applied to $r = r$ is that $r \equiv f(f(a))$ or $r \equiv a$, in which case $E$ is either $f(f(a)) = a$ or $a = f(f(a))$. Similarly if $\mathcal{D}$ ends with a $=_{2}\Rightarrow$-inference we have that $E$ is $a = f(f(a))$ or $f(f(a)) = a$.

If $\mathcal{D}$ ends with a cut, we have two cases.

Case 1. (Assuming for notational simplicity that $j = n$) $\mathcal{D}$ has the form:

\[
\begin{array}{c}
\mathcal{D}_0 \\
p_1 = p_1, \ldots, p_i = p_i \Rightarrow A \\
p_i+1 = p_{i+1}, \ldots, p_n = p_n, E, A \Rightarrow a = f(f(a))
\end{array}
\]

By the previous Proposition, $A$ must be an identity, hence, by induction hypothesis (with $n$ replaced by $n + 1$) applied to $\mathcal{D}_1$, $E$ must coincide either with $a = f(f(a))$ or with $f(f(a)) = a$.

Case 2. $\mathcal{D}$ has the form:

\[
\begin{array}{c}
\mathcal{D}_0 \\
p_1 = p_1, \ldots, p_j = p_j, E \Rightarrow A \\
p_{j+1} = p_{j+1}, \ldots, p_n = p_n, A \Rightarrow a = f(f(a))
\end{array}
\]

By induction hypothesis applied to $\mathcal{D}_1$, $A$ coincides with $a = f(f(a))$ or with $f(f(a)) = a$. We can then apply the induction hypothesis, either case a) or case b), to $\mathcal{D}_0$, to conclude that $E$ coincides with $a = f(f(a))$ or with $f(f(a)) = a$.

The proof of b) is entirely similar. \qed
Thus $a = f(a) \Rightarrow a = f(f(a))$ is not derivable in $EQ^*_M$. Since it is derivable in $EQ_M$, we have that the contraction rule is not eliminable from derivations of $EQ_M$. As a consequence it is not eliminable in $EQ$ either. In fact $a = f(a) \Rightarrow a = f(f(a))$ is derivable in $EQ$, but it cannot have a derivation in $EQ$ without applications of the contraction rule, since, by Proposition 0.1, such a derivation could be translated into a derivation in $EQ^*_M$ of $a = f(a) \Rightarrow a = f(f(a))$ which we know it does not exist.

If we replace in $EQ_M$ or $EQ$ the cut rule by its context sharing version, namely the rule

$$
\begin{array}{c}
\Gamma \Rightarrow A \\
\Gamma, A \Rightarrow H
\end{array}
\Rightarrow

\Gamma \Rightarrow H
$$

then the contraction rule turns out to be derivable, thanks to the weakening and exchange rule, as shown by the following derivation:

$$
\begin{array}{c}
F \Rightarrow F
\end{array}
\Rightarrow

\begin{array}{c}
\Gamma, F \Rightarrow F \\
\Gamma, F, F \Rightarrow H
\end{array}
\Rightarrow

\begin{array}{c}
\Gamma, F \Rightarrow H
\end{array}
$$

In the case of $EQ$, a less trivial way of making the contraction rule eliminable is to replace the CNG rule by its context sharing version, while retaining the context independent cut rule (thus obtaining the system denoted by $EQ$ in [1])

**DEFINITION 0.5** $EQ'$ is obtained by replacing in $EQ$ the rule CNG by its context sharing version CNG', namely:

$$
\begin{array}{c}
\Gamma \Rightarrow D\{v/r\}
\end{array}
\Rightarrow

\begin{array}{c}
\Gamma \Rightarrow r = s
\end{array}
\Rightarrow

\begin{array}{c}
\Gamma \Rightarrow D\{v/s\}
\end{array}
$$

cf.$EQ'$ is $EQ'$ deprived of the cut rule, and $ccf.EQ'$ is $c.f.EQ'$ deprived also of the contraction rule.

**Notation** $\Gamma_F$ will denote the sequence that is obtained from $\Gamma$ by eliminating all the occurrences of $F$ but the last one, provided that there is at least one occurrence of $F$ in $\Gamma$, and $\Gamma$ otherwise.

**LEMMA 0.1**

a) If $\Gamma \Rightarrow H$ is derivable in $ccf.EQ'$, then $\Gamma_F \Rightarrow H$ is derivable in $ccf.EQ'$.

b) If $\Gamma, F, F \Rightarrow H$ is derivable in $ccf.EQ'$ then $\Gamma, F \Rightarrow H$ is derivable in $ccf.EQ'$.

c) If $\Gamma \Rightarrow H$ is derivable in $ccf.EQ'$ and $\Gamma_0$ contains all the formulae occurring in $\Gamma$ then $\Gamma_0 \Rightarrow H$ is derivable in $ccf.EQ'$. 

5
Proof. a) If $F$ has no occurrences in $\Gamma$ the conclusion is trivial. Otherwise we proceed by induction on the height of a derivation $D$ of $\Gamma \Rightarrow H$ in $ccf.EQ$. In the base case the given derivation reduces to the axiom $F \Rightarrow F$. Then $\Gamma_F \Rightarrow H$ also reduces to the axiom $F \Rightarrow F$. As for the induction step, we have the following cases. Case 1. $D$ ends with an exchange. Then it suffices to apply the induction hypothesis to the immediate subderivation of $D$, and then the same exchange by which $D$ ends, unless one of the exchanged formula is $F$ itself, but it is not the last occurrence of $F$ in $\Gamma$. In that case the desired derivation is directly provided by the induction hypothesis. Case 2. $D$ ends with a weakening, i.e. is of the form:

$$\begin{array}{c}
\vdots \\
D_0 \\
\Gamma' \Rightarrow H \\
\hline
\Gamma', G \Rightarrow H
\end{array}$$

By induction hypothesis there is a derivation of $\Gamma'_F \Rightarrow H$. If $F$ is different from $G$, then, since $\Gamma_F$ coincides with $\Gamma'_F, G$, it suffices to apply the same weakening to obtain the desired derivation of $\Gamma_F \Rightarrow H$. If $F$ coincides with $G$ and does not occur in $\Gamma'$ then $D$ is already a derivation of $\Gamma_F \Rightarrow H$. Otherwise, if $F$ occurs last in $\Gamma'$ we are done. If not, the desired derivation is obtained by applying the exchanges needed to bring the unique occurrence of $F$ in $\Gamma_F$ at the end of the sequence.

Case 3. $D$ ends with a $CNG'$-inference. Since a $CNG'$-inference does not modify the antecedent of the premisses, the claim is an immediate consequence of the induction hypothesis.

b) By a), if $\Gamma, F, F \Rightarrow H$ is derivable in $ccf.EQ$ and $\Gamma_0$ is obtained from $\Gamma$ by eliminating all the occurrences of $F$, then $\Gamma_0, F \Rightarrow H$ has a derivation in $ccf.EQ$, from which by means of weakenings, introducing $F$, and exchanges we can obtain a derivation in $ccf.EQ$ of $\Gamma, F \Rightarrow H$.

c) is obtained by applying a) for $k$ times, where $k$ is the number of different formulae occurring in $\Gamma$. □

**Proposition 0.5** The contraction rule is eliminable from derivations in $ccf.EQ'$.

Proof. By the previous Lemma 0.1 b), the contraction rule is admissible in $ccf.EQ'$ and therefore eliminable from derivations in $ccf.EQ'$ □

Thus, taking into account the eliminability of the cut rule from derivations in $EQ'$, established in [1], we have the following:

**Corollary 0.1** Both the cut and the contraction rule are eliminable from derivations in $EQ'$.

To sum up: contraction elimination does not hold for $EQ_M$ and $EQ$, but it does hold if the cut rule, in the case of $EQ_M$, and the cut rule or the rule $CNG$, in the case of $EQ$, are replaced by their context sharing versions.
References

[1] F. Parlamento, F. Previale, Cut elimination for Gentzen’s Sequent Calculus with Equality and Logic of Partial Terms. Lecture Notes in Computer Science 7750, 161-172 (2013)