Conformal string operators and evolution of skewed parton distributions

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Abstract

We have investigated skewed parton distributions in coordinate space. We found that their evolution can be described in a simple manner in terms of non-local, conformal operators introduced by Balitsky and Braun. The resulting formula is given by a Neumann series expansion. Its structure resembles, for all values of the asymmetry parameter, the well-known solution of the ERBL equation in the momentum space. Performing Fourier transformation we have reproduced known results for evolution of momentum-space distributions.

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1 Introduction

Recently, a lot of effort has been invested in exploring deeply-virtual Compton scattering and hard, exclusive meson production processes, see \[1, 2\] and \[3, 4, 5, 6\] for recent reviews of theoretical and experimental situation. The celebrated factorization theorems \[7, 8, 9\] assure that when photon virtuality is large the underlying photon-parton subprocesses are indeed dominated by short distances and can be calculated perturbatively. The necessary information about long-distance, non-perturbative dynamics enters in the form of distribution amplitudes of the produces mesons, and new, generalized \[10, 8\] parton distribution functions of the hadronic target. Factorizable hard, exclusive processes provide therefore a possibility to study new aspects of nucleon structure, which cannot be accessed in inclusive measurements.

More detailed studies of hard, exclusive meson production \[11, 12, 13, 14, 15, 16\] have demonstrated that cross-section of these processes should be sufficiently large, so they can be investigated in the next round of dedicated experiments. These estimates are based on leading order (LO) QCD calculations which typically neglect the scale-dependence of generalized parton distributions, except for a diffractive region \[17, 18\]. There, one can argue \[12, 19\] that generalized parton distributions which determine the dominant, imaginary part of the amplitude are proportional to the corresponding forward parton distributions. On the other hand, kinematical conditions for future measurements are such that a lot of data will be collected outside the diffractive region. Once data points achieve sufficient accuracy, a practical discussion of scale dependence of amplitudes of hard, exclusive processes will become unavoidable. Evolution equations for generalized parton distributions can be found in \[20, 21, 11\]. Their general properties have been discussed at length in the literature \[10, 8, 22\], and by now various numerical algorithms \[23, 24, 12, 25, 26\] and theoretical methods \[27\] have been proposed and tested numerically. Most of these methods make more or less explicit use of the local conformal operators which are multiplicatively renormalized at the one-loop level \[28, 29\]. Note also that so far the main effort has been devoted to study evolution of skewed parton distributions themselves, while away from the diffractive region the real issue is the scale dependence of the QCD amplitude.

The goal of this paper is twofold. First, we aim at a new, practically applicable method of treating the problem of scale-dependence of skewed parton distributions and corresponding amplitudes of hard, exclusive processes. Second, we want to explore the explicit operator solution to one-loop QCD evolution equations found by Balitsky and Braun, which involves non-local conformal operators. Although it has been found about a decade ago, we are not aware about any previous attempt to make phenomenological use of it. As the Balitsky-Braun solution was formulated in terms of coordinate-space quark string operators it is natural to introduce here a concept of coordinate-space skewed parton distributions which are generalization of coordinate-space forward parton distributions \[30\]. As we shall argue below, hard exclusive processes involve momentum- and coordinate-space parton distributions in a rather symmetric way, so both languages are
equally convenient from the point of view of practical calculations.

It turns out that the language of coordinate-space string operators can indeed be applied successfully to the problem of finding a solution of the evolution equations for coordinate-space distributions. The result can be interpreted as a Neumann series expansion of the latter in terms of matrix elements of multiplicatively renormalizable, non-local conformal operators. It can be understood also as an operator identity - an expansion of QCD string operator in terms of non-local conformal operators.

Having found a solution of the evolution equations in coordinate-space, we have investigated also a corresponding scheme for evolution of momentum-space distributions. Here, we have found that coordinate-space considerations suggest splitting of initial conditions into two pieces, corresponding to initial parton distributions with average momentum larger and smaller than the asymmetry parameter. Because of the linear character of the evolution equations, both terms evolve independently. The resulting algorithm is a combination of expansion in terms of local conformal operators of one part, and general expansion in terms of orthogonal polynomials of the other. Both techniques have been already discussed in the literature \[31, 24, 12, 25\]. Here we contribute to this discussion by showing how the corresponding formulae obey general self-consistency constraints.

Our presentation is organized as follows. In the following section we introduce a basic framework for coordinate-space skewed parton distributions. In the next section we present a solution to the problem of scale dependence of the coordinate-space distributions as an expansion in terms of Bessel functions. Armed with this result, we show that it can be interpreted as an expansion in terms of matrix elements of non-local, conformal operators and deduce the corresponding operator identity. Our result is closely related to the explicit operator solution found in \[20\]. The equivalence of both approaches is explicitly demonstrated in the Appendix A.

In the next section we turn our attention to momentum-space quark distributions. We will argue that coordinate-space considerations suggest a particular algorithm for treating evolution in the momentum space, and discuss general constraints which any such solution has to fulfill. The next section contains a summary. Finally, two appendices are devoted to mathematical details which are not discussed in the main body of the paper.

2 Coordinate-Space Skewed Parton Distributions

In this section, we introduce the basic theoretical description of coordinate-space skewed parton distributions. Consider the twist-2, gauge invariant, light-cone quark string operator normalized at a scale \(\mu^2\):

\[
O(\alpha, \beta) = \bar{q}(\frac{\alpha + \beta}{2} z) \hat{z} \left[\frac{\alpha + \beta}{2} z, \frac{\alpha - \beta}{2} z\right] q(\frac{\alpha - \beta}{2} z)_{z^2 = 0}.
\]

Here, \([a, b]\) denotes the path-ordered exponential

\[
[a, b] = \mathcal{P} \exp[-ig \int_b^a A^\mu(x) dx]_{\mu}
\]
which reduces to 1 in the Schwinger gauge \( z \cdot A = 0 \) (\( g \) stands for the strong coupling constant and \( A^\mu \) denotes the gluon field). Coordinates \( \alpha \) and \( \beta \) are defined in such a way that \( \alpha z \) defines the center of the string composed from quark fields and the gluon line between them while \( \beta z \) corresponds to its 'length', understood here simply as the difference between coordinates of the quark operators.

Charge conjugation odd and even quark string operators are obtained from \( O(\alpha, \beta) \) by taking its components symmetric, respectively antisymmetric in \( \beta \). Denoting the former and the latter by \( O^+(\alpha, \beta) \), respectively \( Q^-(\alpha, \beta) \) one has therefore

\[
O^\pm(\alpha, \beta) = \frac{1}{2} \left( O(\alpha, \beta) \pm O(\alpha, -\beta) \right). \tag{2}
\]

Factorizable hard exclusive processes are described by matrix elements of \( O(\alpha, \beta) \) between nucleon states \( \langle n(P', S') | \, O(n(P, S)) \rangle \) with the corresponding four-momenta \( P, P' \) and spins \( S, S' \). Here, the momentum space skewed parton distributions enter naturally \( [11] [8] \). In the following, we have chosen to use convention introduced by Ji \([11]\). In this notation, the matrix element \([1]\) can be represented as:

\[
\langle n(P', S') | \, O(\alpha, \beta) \, | n(P, S) \rangle = \frac{1}{2} \left( \int_{-1}^{1} du F(u, \xi; \mu^2) e^{iu\beta(P_z)} + \int_{-1}^{1} du K(u, \xi; \mu^2) e^{iu\beta(P_z)} \right) \tag{3}
\]

Here \( N(P, S) \) and \( \bar{N}(P', S') \) denote initial and final nucleon spinors, respectively. The average nucleon momentum is denoted by \( \bar{P} = (P + P')/2 \), and the momentum transfer is \( r = P - P' \). The asymmetry parameter \( \xi \) is defined by \( r \cdot z = 2\xi \bar{P} \cdot z \).

From now on, we will explicitly consider the skewed quark distribution \( F(u, \xi; \mu^2) \) only. Evolution equations for \( K(u, \xi; \mu^2) \) are exactly the same, so one can replace \( F(u, \xi; \mu^2) \) by \( K(u, \xi; \mu^2) \) at any stage of the following presentation. To proceed further, we separate the symmetric and antisymmetric part of \( F(u, \xi; \mu^2) \), such that

\[
\int_{-1}^{1} du F(u, \xi; \mu^2) e^{iu\beta(P_z)} = \int_{0}^{1} du F^S(u, \xi; \mu^2) \cos \left[ u\beta(P \cdot z) \right] + i \int_{0}^{1} du F^A(u, \xi; \mu^2) \sin \left[ u\beta(P \cdot z) \right] \tag{4}
\]

Obviously, matrix elements of \( O^+(\alpha, \beta) \) are parametrized by \( F^S(u, \xi; \mu^2) \), or 'valence' (quark minus antiquark) skewed quark distributions, while matrix elements of \( O^-(\alpha, \beta) \) are given by \( F^A(u, \xi; \mu^2) \), or 'quark plus antiquark' distributions.

Note that due to charge conjugation and time-reversal invariance, \( F^S(u, \xi; \mu^2) \) and \( F^A(u, \xi; \mu^2) \) contribute to the corresponding amplitudes \( M^S(\xi; \mu^2) \) and \( M^A(\xi; \mu^2) \) of hard, exclusive processes through

\[
M^S(\xi; \mu^2) \propto \int_{0}^{1} du F^S(u, \xi; \mu^2) \left[ \frac{1}{\xi - u - i\epsilon} + \frac{1}{\xi + u - i\epsilon} \right]
\]
\[ M^A(\xi; \mu^2) \propto \int_0^1 du F^A(u, \xi; \mu^2) \left[ \frac{1}{\xi - u - i\epsilon} - \frac{1}{\xi + u - i\epsilon} \right]. \] (5)

Now we introduce, in analogy to the forward case, \textit{coordinate-space} skewed quark distributions \( F^S(\beta, \xi; \mu^2) \) and \( F^A(\beta, \xi; \mu^2) \) as

\[
F^S(\beta, \xi; \mu^2) = \frac{2}{\pi} \int_0^1 d\omega F^S(\omega, \xi; \mu^2) \cos(\omega \beta),
\]

\[
F^A(\beta, \xi; \mu^2) = \frac{2}{\pi} \int_0^1 d\omega F^A(\omega, \xi; \mu^2) \sin(\omega \beta). \] (6)

The inverse transformation reads

\[
F^S(u, \xi; \mu^2) = \int_0^\infty d\beta F^S(\beta, \xi; \mu^2) \cos(u \beta),
\]

\[
F^A(u, \xi; \mu^2) = \int_0^\infty d\beta F^A(\beta, \xi; \mu^2) \sin(u \beta). \] (7)

Inserting these formulae into (5) one finds that the amplitudes \( M^S(\xi; \mu^2) \) and \( M^A(\xi; \mu^2) \) can be calculated directly in terms of coordinate-space skewed quark distributions as

\[
M^S(\xi; \mu^2) \propto i\pi \int_0^\infty d\beta e^{-i\beta \xi} \frac{1}{\xi - u - i\epsilon} - \frac{1}{\xi + u - i\epsilon} F^A(\beta, \xi; \mu^2),
\]

\[
M^A(\xi; \mu^2) \propto i\pi \int_0^\infty d\beta e^{-i\beta \xi} F^A(\beta, \xi; \mu^2). \] (8)

Note that everywhere above we have used the notation \( \beta \) for a variable which arises from the string length \( \beta \), introduced in (1), by a rescaling \( \beta \rightarrow \beta(P \cdot z) \).

Equations (5) and (8) show that in general momentum- and coordinate-space distributions play a symmetric role the in description of hard, exclusive processes. This is a new situation; in the case of DIS only the former have a direct physical interpretation in terms of observables. In particular, equations (8) can be reinterpreted as

\[
M^S(\xi; \mu^2) \propto \langle n(P', S')| \int_0^\infty d\beta e^{-i\beta \xi} O^+(0, \beta)|n(P, S)\rangle
\]

\[
M^A(\xi; \mu^2) \propto \langle n(P', S')| \int_0^\infty d\beta e^{-i\beta \xi} O^-(0, \beta)|n(P, S)\rangle \] (9)

We have found it interesting to note that it is as convenient to discuss hard, exclusive processes in terms of the correlation function (3) at various longitudinal distances, as it is in terms of momentum-space parton distribution functions.

For numerical calculations it is often more convenient to rewrite the amplitude \( M^A(\xi; \mu^2) \) as

\[
M^A(\xi; \mu^2) \propto \frac{1}{\xi} \int_0^1 du F^A(u, \xi; \mu^2) \left[ \frac{1}{\xi - u - i\epsilon} + \frac{1}{\xi + u - i\epsilon} \right]. \] (10)

The corresponding expression in terms of coordinate-space distribution reads:

\[
M(\xi; \mu^2) \propto i\pi \int_0^\infty d\beta e^{-i\beta \xi} F^A(\beta, \xi; \mu^2), \] (11)

Note that everywhere above we have used the notation \( \beta \) for a variable which arises from the string length \( \beta \), introduced in (1), by a rescaling \( \beta \rightarrow \beta(P \cdot z) \).
where we have introduced a special notation

\[ \mathcal{F}^A(\beta, \xi; \mu^2) \equiv \frac{d}{\xi d\beta} \mathcal{F}^A(\beta, \xi; \mu^2). \]

It is easy to see from (6) and (7) that coordinate-space skewed distributions introduced here are generalizations of coordinate-space parton distributions known from the forward case. On the other hand, the physical interpretation of the coordinate-space skewed quark parton distributions is twofold. First, they describe, according to (3), non-forward matrix elements of the non-local string operator \( Q(\alpha, \beta) \). Second, through (8), they directly determine amplitudes \( M^S(\xi; \mu^2) \) and \( M^A(\xi; \mu^2) \). Comparing with the forward case one finds that the way nucleon structure can be probed in hard exclusive processes is indeed complementary to what happens in deep-inelastic scattering experiments, as the latter are sensitive only to the imaginary part of the amplitude \( M^A \). We observe that real and imaginary parts of amplitudes (8) indeed probe in different ways, respectively through cosine and sine Fourier transformations, twist-2 quark correlation function in a nucleon target at various longitudinal distances.

3 Evolution of coordinate-space skewed parton distributions

In this section we discuss how the explicit scale dependence of coordinate-space skewed parton distributions can be derived using arguments based on the conformal symmetry [29]. Here we consider flavour non-singlet quark distributions only. A generalization to the flavour-singlet case, including effects of mixing between quark and gluon operators in the flavor-singlet sector, will be given in a separate paper.

Explicit solutions of evolution equations for momentum-space skewed parton distributions rely on the fact that, due to the conformal symmetry, one can find basis of local operators which are multiplicatively renormalized at the one-loop level. Our aim here is to construct similar expansion in the coordinate space, but in terms of matrix elements of multiplicatively renormalizable non-local string operators. As we shall see, the corresponding coefficient functions are given by Bessel functions \( J_{\nu}(\lambda) \), and the whole procedure can be understood mathematically as the Neumann series expansion of coordinate-space distributions.

Recall that a function \( f(x) \) can be expanded in a Neumann series according to [32]

\[ f(x) = \sum_{n=0}^{\infty} (2\nu + 2 + 4n)J_{\nu+1+2n}(x) \int_0^{\infty} d\lambda \frac{\lambda}{f(\lambda)J_{\nu+1+2n}(\lambda)}. \] (12)

\footnote{In the forward, DIS case, the skewed quark distributions become the usual quark densities i.e., \( F^A(u, \xi; \mu^2) \rightarrow F^A(u, 0; \mu^2) = q(u; \mu^2) \).}
In particular, one finds that \((\beta \xi)^\nu e^{i\omega \beta}\) can be decomposed according to the Sonine’s formula \([32]\):

\[
(\beta \xi)^\nu e^{i\omega \beta} = 2^\nu \Gamma [\nu] \sum_{n=0}^{\infty} i^n (\nu + n) C_n^{\nu} (\omega/\xi) J_{\nu+n}(\beta \xi)
\]  

(13)

where \(\Gamma\) is the Euler gamma function and \(C_n^{\nu}\) denotes the Gegenbauer polynomial, respectively. Choosing \(\nu = 3/2\), and taking the real part of (13) one obtains:

\[
\cos (\omega \beta) = \left(\frac{2}{\beta \xi}\right)^{3/2} \Gamma \left[\frac{3}{2}\right] \sum_{n=0}^{\infty} (-1)^n (3/2 + 2n) C_{2n}^{3/2} (\omega/\xi) J_{3/2+2n}(\beta \xi)
\]

(14)

Inserting this expansion in the definition of the coordinate-space skewed quark distribution \([7]\) and interchanging summation and integration one obtains:

\[
F_S^{\beta, \xi}(\beta, \xi; \mu^2) = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta \xi}\right)^{3/2} \sum_{n=0}^{\infty} (-1)^n (3/2 + 2n) J_{3/2+2n}(\beta \xi) \int_0^1 du F_S^{\omega, \xi}(\omega, \xi; \mu^2) C_{2n}^{3/2} (\omega/\xi)
\]

(15)

Now, note that a Gegenbauer moment \([8]\)

\[
\xi^{2n} \int_0^1 d\omega F_S^{\omega, \xi}(\omega, \xi; \mu^2) C_{2n}^{3/2} (\omega/\xi)
\]

is proportional to the matrix element of multiplicatively renormalizable local conformal operator \([28, 29]\). Its scale dependence is therefore given by

\[
\xi^{2n} \int_0^1 d\omega F_S^{\omega, \xi}(\omega, \xi; Q^2) C_{2n}^{3/2} (\omega/\xi) = L_{2n+1} \xi^{2n} \int_0^1 d\omega F_S^{\omega, \xi}(\omega, \xi; \mu^2) C_{2n}^{3/2} (\omega/\xi)
\]

(16)

where

\[
L_k = \left(\frac{\log (Q^2/\Lambda^2)}{\log (\mu^2/\Lambda^2)}\right)^{-\gamma(k) / b_0}
\]

(17)

Here, \(\Lambda\) is the QCD scale. \(\gamma(k)\) is the anomalous dimension,

\[
\gamma(k) = 4/3 \left(3 + \frac{2}{k(k+1)} - 4(\Psi(k+1) + \gamma_E)\right),
\]

with \(\Psi(x) = \frac{d \log(\Gamma(x))}{dx}\), \(\gamma_E\) being the Euler constant, and \(b_0 = 11 - 2/3N_F\).

As it follows, the scale dependence of the coordinate space distribution \(F_S^{\beta, \xi}(\beta, \xi; \mu^2)\) is given simply by

\[
F_S^{\beta, \xi}(\beta, \xi; Q^2) = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta \xi}\right)^{3/2} \sum_{n=0}^{\infty} (-1)^n (3/2+2n) L_{2n+1} J_{3/2+2n}(\beta \xi) \int_0^1 d\omega F_S^{\omega, \xi}(\omega, \xi; \mu^2) C_{2n}^{3/2} (\omega/\xi)
\]

(18)
In the case of \( \mathcal{F}^A(\beta, \xi; \mu^2) \) one can apply a decomposition of \( \sin(u\beta) \) analogous to (14):

\[
\sin(\omega/\xi) = \left(\frac{2}{\beta \xi}\right)^{\frac{3}{2}} \Gamma\left[\frac{3}{2}\right] \sum_{n=0}^{\infty} (-1)^n (5/2 + 2n) C_{2n+1}^{3/2}(\omega/\xi) J_{5/2+2n}(\beta \xi)
\]

(19)

to obtain

\[
\mathcal{F}^A(\beta, \xi; \mu^2) = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta \xi}\right)^{\frac{3}{2}} \sum_{n=0}^{\infty} (-1)^n (5/2 + 2n) J_{5/2+2n}(\beta \xi) \int_0^1 d\omega \mathcal{F}^A(\omega; \xi; \mu^2) C_{2n+1}^{3/2}(\omega/\xi).
\]

(20)

If \( \mathcal{F}^A(\beta, \xi; \mu^2) \) is associated with a flavor non-singlet quark distribution,

\[
\xi^{2n+1} \int_0^1 du \mathcal{F}^A(\omega; \xi; \mu^2) C_{2n+1}^{3/2}(\omega/\xi)
\]
is again multiplicatively renormalizable, but instead of (14) we have

\[
\xi^{2n+1} \int_0^1 d\omega \mathcal{F}^A(\omega; \xi; Q^2) C_{2n+1}^{3/2}(\omega/\xi) = L_{2n+2} \xi^{2n+1} \int_0^1 d\omega \mathcal{F}^A(\omega; \xi; \mu^2) C_{2n+1}^{3/2}(\omega/\xi).
\]

(21)

The resulting scale dependence of \( \mathcal{F}^A(\beta, \xi; Q^2) \) is governed simply by

\[
\mathcal{F}^A(\beta, \xi; Q^2) = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta \xi}\right)^{\frac{3}{2}} \sum_{n=0}^{\infty} (-1)^n (5/2 + 2n) L_{2n+2} J_{5/2+2n}(\beta \xi) \int_0^1 d\omega \mathcal{F}^A(\omega; \xi; \mu^2) C_{2n+1}^{3/2}(\omega/\xi).
\]

(22)

Now we show that equations (18) and (22), can be naturally understood as expansions of coordinate-space skewed quark distributions in terms of matrix elements of non-local, multiplicatively renormalizable, conformal operators. Indeed, applying (12) one can rewrite (18) and (22) as a Neumann-type series:

\[
\mathcal{F}^S(\beta, \xi; Q^2) = \beta^{-\frac{3}{2}} \sum_{n=0}^{\infty} (3 + 4n) J_{3/2+2n}(\beta \xi) L_{2n+1} \int_0^{\infty} d\lambda \sqrt{\lambda} \mathcal{F}^S(\lambda; \xi; \mu^2) J_{3/2+2n}(\lambda \xi)
\]

\[
\mathcal{F}^A(\beta, \xi; Q^2) = \beta^{-\frac{3}{2}} \sum_{n=0}^{\infty} (5 + 4n) J_{5/2+2n}(\beta \xi) L_{2n+2} \int_0^{\infty} d\lambda \sqrt{\lambda} \mathcal{F}^A(\lambda; \xi; \mu^2) J_{5/2+2n}(\lambda \xi)
\]

(23)

We are now in position to make relation to conformal symmetry explicit. Let us start from the obvious identity

\[
O(\alpha, \beta) = \int_{-\infty}^{\infty} d\alpha' \int_0^{\infty} d\beta' \delta(\alpha - \alpha')\delta(\beta - \beta') O(\alpha', \beta').
\]

(24)

Applying (14) one finds a representation of a \( \delta \)-function in terms of a Neumann series

\[
\beta \delta(\beta - \beta') = \sum_{n=0}^{\infty} (2n + 2 + 4n) J_{n+1+2n}(\beta) J_{n+1+2n}(\beta').
\]

(25)
Inserting this expansion into (24) one finds that the string operator \( O(\alpha, \beta) \) can be decomposed as

\[
O(\alpha, \beta) = \beta^{-\frac{3}{2}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ika} \sum_{n=0}^{\infty} (1 + 2j)J_{1/2+j}(|k|\beta)S(1/2 + j, k; \mu^2)
\]  

(26)
in terms of conformal string operators \( S(1/2+j, k; \mu^2) \). Here we have introduced a variable \( j = 1/2 + \nu + 2n \) which will be identified below with the conformal spin. Operators

\[
S(1/2+j, k; \mu^2) = \int_{-\infty}^{\infty} d\alpha e^{ika} \int_{0}^{\infty} d\beta \sqrt{\beta} J_{1/2+j}(|k|\beta) O(\alpha, \beta),
\]

(27)
introduced first in [24], form a representation of a conformal group and are therefore multiplicatively renormalizable at a one-loop level [20], i.e.

\[
S(1/2+j, k; Q^2) = L_j \ S(1/2+j, k; \mu^2).
\]

(28)
Because of symmetry properties in the variable \( \beta \), one finds that for charge-conjugation odd and even components of \( O(\alpha, \beta) \), see equation (2), \( \nu \) has to be equal to 1/2 and 3/2, respectively. Equivalently, one finds that corresponding expansions run over operators \( S(1/2+j, k; \mu^2) \) with the conformal spin \( j \) which assumes values \( j = 2n + 1 \), respectively \( j = 2n + 2 \). Combining this observation with (26) and (28) one finds

\[
O^+(\alpha, \beta; Q^2) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ika} \beta^{-\frac{3}{2}} \sum_{n=0}^{\infty} (3 + 4n)L_{2n+1}J_{3/2+2n}(|k|\beta)S(3/2 + 2n, k; \mu^2),
\]

\[
O^-(\alpha, \beta; Q^2) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ika} \beta^{-\frac{3}{2}} \sum_{n=0}^{\infty} (5 + 4n)L_{2n+2}J_{5/2+2n}(|k|\beta)S(5/2 + 2n, k; \mu^2).
\]

(29)
Taking matrix elements of both sides of the above equations one immediately reproduces equations (18) and (22).

Finally, let us note that equation (23) can be rewritten in the form

\[
\mathcal{F}^S(\beta, \xi; Q^2) = \int_{0}^{\infty} d\lambda \mathcal{F}^S(\lambda, \xi; \mu^2) \sum_{n=0}^{\infty} L_{2n+1}(3 + 4n)\Psi_n(\beta, \xi)\bar{\Psi}_n(\lambda, \xi)
\]

(30)
where \( \Psi_n(\beta, \xi) = \beta^{-3/2}J_{2n+1}(\beta \xi) \) and \( \bar{\Psi}_n(\lambda, \xi) = \lambda^{1/2}J_{2n+1}(\lambda \xi) \). It corresponds to a standard representation of solution of a non-stationary evolution equation

\[
\frac{\partial}{\partial \tau} \mathcal{F}^S(\beta, \xi, \tau) = \int_{0}^{\infty} d\beta' \hat{\mathcal{H}}(\beta, \beta'; \xi)\mathcal{F}^S(\beta', \xi, \tau).
\]

(31)
Here, ‘time’ \( \tau \) is given by \( \frac{1}{b_0} \log \left( \frac{\log(Q^2/\Lambda^2)}{\log(\mu^2/\Lambda^2)} \right) \). Functions \( \Psi_n(\beta, \xi) \) play a role of eigenfunctions of the corresponding stationary equation

\[
\hat{\mathcal{H}} \Psi_n = E_n \Psi_n.
\]

(32)
Functions $\bar{\Psi}_n(\beta, \xi)$ are eigenfunctions of the conjugated equation

$$\bar{\Psi}_n \hat{H} = E_n \bar{\Psi}_n.$$  \hfill (33)

Energy levels are given by anomalous dimensions $\gamma(2n + 1)$. The explicit form of the hamiltonian $\hat{H}$ can be found in [20]. Functions $\bar{\Psi}_n$ and $\Psi_m$ are orthogonal with the scalar product given by the integral

$$\langle \bar{\Psi}_n | \Psi_m \rangle = \int_0^\infty d\beta \bar{\Psi}_n(\beta, \xi) \Psi_m(\beta, \xi) = \frac{\delta_{nm}}{3 + 4n}. \hfill (34)$$

The sum

$$R^S(\beta, \beta'; \xi) = \sum_{n=0}^{\infty} L_{2n+1}(3 + 4n) \bar{\Psi}_n(\beta, \xi) \bar{\Psi}_n(\beta', \xi) \hfill (35)$$

is the standard representation of the resolvent of the evolution equation (31). Evolution of the charge-conjugation even quark distribution $F^A(\beta, \xi; Q^2)$ can be interpreted in an analogous way.

Note that coordinate-space conformal expansions (23) and (29) have a form which closely resembles a general solution to the ERBL evolution equation [28, 33]. In momentum space, however, such an expansion can be written only for distribution amplitudes i.e., in the case $\xi = 1$. In analogy with the latter, a phenomenological model for coordinate-space distributions and, ultimately, physical amplitudes can be obtained by choosing expansion coefficients

$$a^S_n(\xi, \mu^2) = \int_0^\infty d\lambda \sqrt{\lambda} F^S(\lambda, \xi; \mu^2) J_{3/2+2n}(\lambda \xi)$$

$$= \frac{1}{2\sqrt{\pi}} \left( \frac{2}{\xi} \right)^{\frac{1}{2}} (-1)^n \int_0^1 d\omega F^S(\omega, \xi; \mu^2) C_{2n}^{3/2}(\omega/\xi)$$

$$a^A_n(\xi, \mu^2) = \int_0^\infty d\lambda \sqrt{\lambda} F^A(\lambda, \xi; \mu^2) J_{5/2+2n}(\lambda \xi)$$

$$= \frac{1}{2\sqrt{\pi}} \left( \frac{2}{\xi} \right)^{\frac{1}{2}} (-1)^n \int_0^1 d\omega F^A(\omega, \xi; \mu^2) C_{2n+1}^{3/2}(\omega/\xi) \hfill (36)$$

according to some phenomenological model of low-energy nucleon structure. In particular, it would be interesting to try a model in which $a_n$’s are constructed from linear combinations of their values corresponding to the forward, $\xi \to 0$, and totally exclusive, $\xi \to 1$, limits. The resulting momentum-space distributions can be obtained by applying formalism developed in the next section.

Equations (23) and (29), which provide a natural solution to the problem of LO scale-dependence of coordinate-space skewed quark distributions, belong to the main results of the present paper. Their form corresponds to an expansion in terms of a set of orthogonal eigenfunctions which are solutions of the stationary equations (32) and (33). Note that a different representation of the same solution has been found in [20]. The latter should
be understood as a representation of the Neumann series by an integral over complex values of conformal spin $j$. In the Appendix A we shall show in details how the Neumann expansion (23) can be obtained starting from the solution found in [20].

Although we intend to discuss numerical aspects of the above algorithm in a separate publication, a short comment on its practical applicability is in order here. We have

\begin{equation}
|\tilde{M}S(\xi)|^2 = 1.1641u^{-1/2}(1-u)^{3.5}.
\end{equation}

The solid line denotes $|\tilde{M}S(\xi)|^2$ at the initial scale $\mu_0 = 1.777$ GeV, the dashed line represents $|\tilde{M}S(\xi)|^2$ evolved to $\mu = 10$ GeV.

checked, using various models of skewed quark distributions, that numerical algorithm for evaluation of physical amplitudes, based on equations (23) and (8), gives accurate and stable results, see Figure 1 for an example, except for a case where the variable $\xi$ becomes small. This is related to an observation that a non-zero $\xi$ provides a natural cut-off for large $\beta$ behavior of coordinate-space distributions, which significantly improves convergence of the Fourier integral (8), as compared to the forward case [34].

4 Momentum-space skewed quark distributions

So far, we have discussed skewed parton distributions in coordinate space. In this section we turn our attention to the momentum-space distributions. Let us concentrate, for example, on $F^S(u, \xi; Q^2)$. It is related to $F^S(\beta, \xi; Q^2)$ by the cosine Fourier transformation...
For simplicity of notation, we denote in this section $F_S(u, \xi; Q^2)$ by $F(u, \xi)$, and the corresponding distribution normalized at the initial scale $\mu^2$ by $F_0(u, \xi)$.

Thanks to the conformal symmetry, the Gegenbauer moments of momentum-space skewed parton distributions are renormalized multiplicatively, but the corresponding Gegenbauer polynomials do not form a complete set of functions on the interval $-1 \leq u \leq 1$.

A solution to the problem of scale dependence of $F(u, \xi)$ based on an expansion in terms of orthogonal polynomials has been proposed in [24, 12] and further developed, up to the NLO, in [25]. Recently, another approach has been proposed in [27]. In the following, we shall discuss an alternative scheme, which originates from coordinate-space considerations presented in the previous section.

In order to perform the Fourier transformation let us first split $F(\beta, \xi)$ into two pieces, $F_<(\beta, \xi)$ and $F_>(\beta, \xi)$:

$$F_<(\beta, \xi) = 1/\sqrt{\pi} \left( \frac{2}{\beta \xi} \right)^{3/2} \sum_{k=0}^{\infty} L_{2k+1} (-1)^k (3/2 + 2k) J_{3/2+2k}(\beta \xi) \int_0^\xi d\omega F_0(\omega, \xi) C_{3/2}^{3/2} (\omega/\xi)$$

$$F_>(\beta, \xi) = 1/\sqrt{\pi} \left( \frac{2}{\beta \xi} \right)^{3/2} \sum_{k=0}^{\infty} L_{2k+1} (-1)^k (3/2 + 2k) J_{3/2+2k}(\beta \xi) \int_1^\xi d\omega F_0(\omega, \xi) C_{3/2}^{3/2} (\omega/\xi)$$

(37)

Obviously, $F(\beta, \xi) = F_<(\beta, \xi) + F_>(\beta, \xi)$.

It can be shown that both series expansions in (37) are convergent for all $\xi$ and $\beta$. We denote their Fourier transforms as $F_< (u, \xi)$ and $F_>(u, \xi)$, such that

$$F(u, \xi) = F_< (u, \xi) + F_>(u, \xi).$$

(39)

At this stage, splitting of $F$ into $F_<$ and $F_>$ may seem arbitrary, but we shall demonstrate below that the associated decomposition of $F$ into $F_<$ and $F_>$ offers new insights into the evolution of skewed quark distributions in momentum space.

The next step is to insert the decomposition (38) into the cosine Fourier integral (11). In the case of $F_< (u, \xi)$ calculation is simple. Integrating corresponding series in (37) term by term one obtains the known result [31]

$$F_< (u, \xi) = \theta(u < \xi) \frac{1}{\xi} \left( 1 - \frac{u^2}{\xi^2} \right) \sum_{k=0}^{\infty} L_{2k+1} \frac{(3/2 + 2k)}{(k+1)(1+2k)} C_{3/2}^{3/2} (u/\xi) \times$$

$$\times \int_0^\xi d\omega F_0(\omega, \xi) C_{3/2}^{3/2} (\omega/\xi) .$$

(40)

The situation is, however, more complicated for $F_>(u, \xi)$ as the series obtained by performing Fourier integral diverges term by term. Instead, one can rewrite $\cos[u \beta]$ as a
series expansion in Bessel functions, which follows from the Sonine’s formula (13):

\[ \cos[u\beta] = \cos[\beta] + \sqrt{2\pi\beta} (1 - u^2) \sum_{k=0}^{\infty} (-1)^k \frac{(3/2 + 2k)}{2(k + 1)(1 + 2k)} C_{2k}^{3/2}(u) J_{3/2 + 2k}(\beta). \] (41)

The detailed derivation of this equation is given in the Appendix. Now, inserting the above expansion into (7) and interchanging the integration with summation one obtains a convergent series:

\[ F_\gamma(u, \xi) = (1 - u^2) \sum_{n=0}^{\infty} \frac{(3/2 + 2n)}{(n+1)(1+2n)} C_{2n}^{3/2}(u) \sum_k L_{2k+1} R_{k}^n(\xi^2) \int_\xi^1 d\omega F_0(\omega, \xi) C_{2k}^{3/2}(w/\xi). \] (42)

Here we have introduced the notation

\[ R_{k}^n(\xi^2) = \frac{(-1)^{(k+n)} \Gamma[3/2 + n + k]}{(n - k)! \Gamma[3/2 + 2k]} \xi^{2k} \frac{\Gamma[2k]}{\Gamma[3k/2 + k]} \int_\xi^1 d\omega F_0(\omega, \xi) C_{2k}^{3/2}(w/\xi). \] (43)

with \( _2F_1 \) being the hypergeometric function. It is easy to see that \( R_{k}^n(\xi^2) \) are polynomials in the variable \( \xi^2 \) of order \( n \) for any \( k \leq n \).

As it follows, \( F(u, \xi) \) can be split naturally into two pieces, each of them being represented by a series expansion in Gegenbauer polynomials. The first part, \( F_\gamma(u, \xi) \), is defined only in the region \( u < \xi \). It has been noted [10, 8, 24] that the evolution of \( F_\gamma(u, \xi) \) is governed by ERBL-type evolution equation [28, 33]. The form of \( F_\gamma(u, \xi) \) is identical with a distribution amplitude \( \phi(X) \) with \( X = u/\xi \). The second part \( F_(u, \xi) \) can contribute for all \( 0 \leq u \leq 1 \). This part is therefore similar to to the solution based on the expansion of \( F(u, \xi) \) in terms of polynomials orthogonal on a segment \(-1 \leq u \leq 1\).

Formula (41) can also be used to rewrite \( F_\gamma(u, \xi) \) as an expansion in terms of Gegenbauer polynomials \( C_{2k}^{3/2}(u) \). In this case, the sum of \( F_\gamma(u, \xi) \) and \( F_(u, \xi) \) reproduces solution found in [24, 14].

Let us now demonstrate that \( F(u, \xi) \), given by the sum of \( F_\gamma(u, \xi) \) and \( F_(u, \xi) \), and represented by series expansions (40) and (42), respectively, have all properties of the correct solution. For the sake of clarity we have omitted in the following all details of underlying calculations. Reader interested in mathematical aspects of this discussion will find the corresponding material in the Appendix B.

1. Expansion of initial conditions.

In the limit \( Q^2 = \mu^2, L_{2k+1} \equiv 1 \), one obtains \( F(u, \xi) = F_0(u, \xi) \). In this limit, two pieces in (39) give the following contributions:

\[ F_\gamma(u, \xi)_{Q^2=\mu^2} = \theta(u \leq \xi) F_0(u, \xi), \]
\[ F_(u, \xi)_{Q^2=\mu^2} = \theta(u > \xi) F_0(u, \xi). \] (44)

As \( F_0(u, \xi) \) is continuous and typically non-zero for \( u = \xi \), it follows immediately that the function \( F_\gamma(\xi, \xi) \) is nonzero despite the factor \( 1 - \frac{u^2}{\xi^2} \) in front of the
series in $[11]$! This means that, in general, the corresponding series expansion in Gegenbauer polynomials represents a function with the singularity at the end-point $u = \xi$. It belongs therefore to the functional space $L^p$ with $1 < p < 2$. Further details related to the convergence of series expansion in terms of Gegenbauer polynomials can be found in $[32]$.

Note that the presence of the theta-function in $[11]$ implies that partons which initially belonged to the segment $0 \leq u \leq \xi$ stay there in the course of the evolution. On the other hand, partons described by $F_>(u, \xi)$, which belonged initially to the segment $\xi < u \leq 1$, diffuse into the segment $0 \leq u \leq \xi$. An example of evolution of $F_>(u, \xi)$ starting from $\xi$-independent initial conditions is shown in Figure 2. It has been obtained taking 130 terms in the expansion $[12]$.

Moreover, considering evolution of $F_0(u, \xi)$ to infinitesimally larger scale $\mu^2 \rightarrow \mu^2 + \delta \mu^2$ and interpreting the result as a new initial condition for the next infinitesimal step, one finds that upon evolution partons cross the border point $u = \xi$ from right to left side only - once they enter the segment $0 \leq u \leq \xi$, they never come back. These properties are in agreement with discussion in $[8, 10]$, based on explicit form of evolution equations.

Figure 2. Evolution of $F_>(u, \xi = 0.1)$ starting from a $\xi$-independent initial conditions at the scale $\mu_0 = 1.777$ GeV, solid line. Figures 2a and 2b show $F_>(u, \xi = 0.1)$ evolved to scales 10, respectively 1000 GeV, dashed line. Area under the dashed curve in the region $u \leq \xi$ represents number of partons which diffused from the region $u > \xi$. As the number of partons which initially occupied the region $u > \xi$ is conserved, this area is always equal to the area between dashed and solid lines for $u > \xi$.

Finally, both $F_<(u, \xi)$ and $F_>(u, \xi)$ can be treated as two independent solutions
corresponding to boundary conditions \(^{(14)}\). Hence, they should separately obey all constraints imposed by symmetry etc. on a valid solution.

2. Asymptotic limit.
   In the limit \(Q^2 \to \infty\) one finds:
   \[
   F(u, \xi) \to \frac{2}{3} \frac{1}{\xi^2} \left(1 - \frac{u^2}{\xi^2}\right) \int_0^1 d\omega F_0(\omega, \xi). \tag{45}
   \]
   In particular, the contribution of \(F_>(u, \xi)\) reads
   \[
   F_>(u, \xi) \to \frac{2}{3} \frac{1}{\xi^2} \left(1 - \frac{u^2}{\xi^2}\right) \int_0^1 d\omega F_0(\omega, \xi), \tag{46}
   \]
   so in this limit all partons from the segment \(\xi < u \leq 1\) have diffused into the segment \(0 \leq u \leq \xi\).

3. Relation to solutions of ERBL and DGLAP evolution equations.
   In the limit \(\xi \to 1\) function \(F(u, \xi = 1)\) is identical with the solution of the ERBL evolution equation for distribution amplitude:
   \[
   F(u, 1) = F_<(u, 1) = \left(1 - u^2\right) \sum_{k=0}^{\infty} L_{2k+1} \frac{(3/2 + 2k)}{(k+1)(1+2k)} C_{2k}^{3/2}(u) \int_0^1 d\omega F_0(\omega, 1) C_{2k}^{3/2}(\omega). \tag{47}
   \]
   In the limit \(\xi \to 0\) function \(F(u, \xi = 0)\) is a solution of the DGLAP-evolution equation, represented as an expansion in terms of Gegenbauer polynomials \(C_{2n}^{3/2}(u)\):\(^2\)
   \[
   F(u, 0) = F_>(u, 0) = \left(1 - u^2\right) \sum_{n=0}^{\infty} L_{2k+1} \frac{3/2 + 2n}{(2n+1)(n+1)} C_{2n}^{3/2}(u) \times
   \times \sum_{k=0}^{n} C_{2k}^{3/2}(u) \int_0^1 d\omega \omega^{2k} F_0(\omega, 0). \tag{48}
   \]
   In particular one finds, as expected, that
   \[
   \int_0^1 du u^{2k} F(u, 0) = L_{2k+1} \int_0^1 du u^{2k} F_0(u, 0), \quad F(u, 0) \to Q^2 \to \infty \delta(u) \int_0^1 d\omega F_0(\omega, 0). \tag{49}
   \]

4. Conformal constraints.
   As required by conformal symmetry, Gegenbauer moments of the \(F(u, \xi)\) are multiplicatively renormalizable \(^{[8, 28, 29]}\):
   \[
   \int_0^1 du F(u, \xi) C_{2k}^{3/2}(u/\xi) = L_{2k+1} \int_0^1 du F_0(u, \xi) C_{2k}^{3/2}(u/\xi). \tag{50}
   \]

\(^2\)This formula have also been obtained in \([24]\)
As discussed above, the same is true separately for $F_<(u, \xi)$ and $F_>(u, \xi)$:

$$\int_0^1 du F_<(u, \xi) C_{2p}^{3/2} (u/\xi) = L_{2p+1} \int_0^\xi d\omega F_0(\omega, \xi) C_{2p}^{3/2} (\omega/\xi)$$
$$\int_0^1 du F_>(u, \xi) C_{2p}^{3/2} (u/\xi) = L_{2p+1} \int_\xi^1 d\omega F_0(\omega, \xi) C_{2p}^{3/2} (\omega/\xi). \quad (51)$$

As $L_1 = 1$, in the case of valence quarks the integral from $F(u, \xi)$ over the whole domain is conserved and equal to the number of valence quarks in the target:

$$\int_0^1 du F(u, \xi) = \int_0^1 du F_0(u, \xi) = N_q. \quad (52)$$

In addition, the integrals involving $F_<(u, \xi)$ and $F_>(u, \xi)$ are separately conserved as well.

5. Polynomiality.

Moments of $F(u, \xi)$,

$$M^n(\xi) = \int_0^1 du u^{2n} F(u, \xi) \quad (53)$$

can be related to matrix elements of usual, twist-2 local operators of dimension $3 + 2n$. Combining this observation with time-reversal invariance and hermicity one finds that they have to be polynomials in $\xi^2$ of a degree at most $n$. Note that this is a property of $F(u, \xi)$, but not of $F_<(u, \xi)$ and $F_>(u, \xi)$ separately. It is preserved by evolution.

5 Summary

The main goal of this paper has been to prepare a theoretical framework for a quantitative discussion of scale dependence of amplitudes of hard, exclusive processes with a general hadronic target. Factorizable amplitudes can be obtained either as a Mellin convolution, or as a Fourier integral, of momentum-space, respectively coordinate-space skewed parton distributions with corresponding Wilson coefficients. The explicit solution to the problem of the scale-dependence of coordinate-space skewed parton distributions, obtained in section 3, has a structure of a Neumann-series expansion of coordinate-space distributions in terms of matrix elements of non-local, conformal operators. Alternatively, it can be understood as the expansion of a QCD string operator in terms of non-local operators with a definite conformal spin. The particular form of both expansions explicitly reflect the conformal symmetry of one-loop QCD evolution equations.

As discussed in [20], the evolution equations for skewed parton distributions in the coordinate-space representation have a simple form, analogous to the ERBL evolution equation for distribution amplitudes. Accordingly, the solution found here represents a series expansion in term of a set of orthogonal eigenfunctions of the evolution kernel in
Performing Fourier transformation between coordinate- and momentum-space parton distributions naturally leads to a scheme in which the momentum-space initial conditions are split into two parts, involving partons with average momentum fractions smaller and larger than the asymmetry parameter. The former never leave their initial domain, while the latter 'diffuse' in the course of the evolution from one domain to the other, spreading over the whole interval. We have used our explicit solution to investigate this process numerically and found that the spreading occurs rather slowly.

Preliminary numerical studies demonstrate that the coordinate-space approach leads to stable numerical results for scale-dependence of physical amplitudes away from the diffractive region.

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Appendix A.

In what follows we will demonstrate how the Neumann expansion (18) of the charge-conjugation odd skewed quark distribution follows from the operator solution to one-loop QCD evolution equations found in [20]. As in Section 4, we will denote in the following $F^S_{\beta, \xi}(\beta, \xi; \mu^2)$ simply by $F(\beta, \xi)$. The authors of [20] argued that conformal string operators $S(1/2 + j, k; \mu^2)$ form a representation of the collinear conformal group $SO(2,1)$ and are therefore renormcovariant. Moreover, applying the Lebedev-Kantorovich transformation [36], they have shown that a flavour non-singlet QCD quark string operator $O(\alpha, \beta; Q^2)$ can be decomposed in terms of $S(1/2 + j, k; \mu^2)$ as:

$$O(\alpha, \beta; Q^2) = -\beta^{-3/2} \int \frac{dk}{4\pi} e^{-ik\alpha} \int_0^\infty d\beta \sqrt{\beta} Z_{1/2+j}(|k|\beta)O(\alpha, \beta; \mu^2), \quad (54)$$

where $Z_\nu$ is a solution of Bessel differential equation, form a representation of the collinear conformal group $SO(2,1)$ and are therefore renormcovariant. Moreover, applying the Lebedev-Kantorovich transformation [38], they have shown that a flavour non-singlet QCD quark string operator $O(\alpha, \beta; Q^2)$ can be decomposed in terms of $S(1/2 + j, k; \mu^2)$ as:

$$O(\alpha, \beta; Q^2) = -\beta^{-3/2} \int \frac{dk}{4\pi} e^{-ik\alpha} \int_{\delta^+}^{\delta^-} dj (j + \frac{1}{2}) J_{j+\frac{1}{2}}(|k|\beta)L_j S(1/2 + j, k; \mu^2). \quad (55)$$

The Lebedev-Kantorovich transformation requires that $S(1/2 + j, k; \mu^2)$, the multiplicatively renormalized conformal string operator [37], is defined by taking $Z_\nu = H^{(2)}_\nu$, the Bessel function of the second kind [32]. The contour of integration over $j$ runs along the imaginary axis. $\delta$ is a real number chosen in such a way that the contour lies inside the stripe parallel to the imaginary axis, where the integrand is an analytic function of $j$. 

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Matrix elements of the conformal string operator $S(1/2 + j, k; \mu^2)$ can be easily found with the help of (3):

\[
\langle n(P', S') | S(\frac{1}{2} + j, k; \mu^2) | n(P, S) \rangle = \tilde{N}(P', S') \hat{z} N(P, S) \times \\
\times 2\pi \delta(k - \frac{r}{2} \cdot z) \int_0^\infty d\beta \sqrt{\beta H^2_{\frac{j}{2} + j}(k\beta)}(F^S(\beta, \xi; \mu^2) + i F^A(\beta, \xi; \mu^2)).
\]

Finally, taking matrix elements of the both sides of the decomposition (55) we arrive at the following representation for the skewed quark distribution $F(\beta, \xi)$:

\[
F(\beta, \xi; Q^2) = \frac{1}{\pi} \beta^{-3/2} \int_{\delta - i\infty}^{\delta + i\infty} dj(j + \frac{1}{2})J_{j + \frac{1}{2}}(\xi\beta)L_j\tilde{S}(\frac{1}{2} + j, \xi; \mu^2),
\]

with

\[
\tilde{S}(\frac{1}{2} + j, \xi; \mu^2) = \frac{1}{\pi}(-1)^{j/2}(\frac{2}{\xi})^{1/2}\Gamma[1 + j/2, 1/2 - j/2]I_0(j, \xi) + \\
\frac{i}{\sqrt{2\pi\xi}} \frac{\Gamma[1/2 + j, 1/2 - j/2, 1/2 - j]}{\Gamma[1 + j/2]}I_1(j, \xi) + \\
\frac{i(-1)^{j+3/2}}{\sqrt{2\pi\xi}} \frac{\Gamma[1/2 + j, 1/2 - j]}{(1/2 + j)\Gamma[1/2 - j/2]}I_2(j, \xi).
\]

To proceed further, we first integrate by parts over $\omega$. The boundary terms vanish because $\lim_{\omega \to 0} u F_0(u, \xi) = F_0(1, \xi) = 0$. The resulting integrand has a large-$\beta'$ behavior, and therefore it is possible to interchange integrations over $\omega$ and $\beta'$ to obtain:

\[
\tilde{S}(\frac{1}{2} + j, \xi; \mu^2) = \frac{1}{\pi}(-1)^{j/2}(\frac{2}{\xi})^{1/2}\Gamma[1 + j/2, 1/2 - j/2]I_0(j, \xi) + \\
\frac{i}{\sqrt{2\pi\xi}} \frac{\Gamma[1/2 + j, 1/2 - j/2, 1/2 - j]}{\Gamma[1 + j/2]}I_1(j, \xi) + \\
\frac{i(-1)^{j+3/2}}{\sqrt{2\pi\xi}} \frac{\Gamma[1/2 + j, 1/2 - j]}{(1/2 + j)\Gamma[1/2 - j/2]}I_2(j, \xi).
\]

To simplify the above expression we have introduced a special notation for functions $I_i(j, \xi)$, $i = 1, 2, 3$, which are analytic in the region $Rej > 0$:

\[
I_0(j, \xi) = \int_0^\xi d\omega(\omega/\xi)F'_0(\omega, \xi)2F_1\left[1 + j/2, 1/2 - j/2 \left| \frac{\omega^2}{\xi^2} \right. \right] \\
I_1(j, \xi) = \int_\xi^1 d\omega \left(\frac{\omega}{\xi}\right)^jF'_0(\omega, \xi)\frac{1}{\Gamma[1/2 - j]^2}2F_1\left[1/2 - j/2, -j/2 \left| \frac{\xi^2}{\omega^2} \right. \right] \\
I_2(j, \xi) = \int_\xi^1 d\omega \left(\frac{\xi}{\omega}\right)^{j+1}F'_0(\omega, \xi)2F_1\left[1/2 + j/2, 1/2 + j/2 \left| \frac{\xi^2}{\omega^2} \right. \right].
\]

Here $F'_0(\omega, \xi) \equiv \frac{d\omega}{d\omega}F_0(\omega, \xi)$. The integration contour over $j$ can be wrapped around the real positive axis, yielding the sum of residues in poles at $j = 1 + 2k$ and $j = 1/2 + k$ with $k = 0, 1, 2, \ldots$. The series of poles at $j = 1 + 2k$ originates from the first two terms in the rhs of (53). The second series of poles at $j = 1/2 + k$ arises from the second and
third terms in the rhs of (33). We have found that residues of poles at \( j = 1/2 + k \) cancel exactly each other, so only poles at \( j = 1 + 2k \) give a non-zero contribution to the final result. The remaining calculations are tedious but simple. Finally, after integrating by parts over \( \omega \) in the opposite direction and rewriting the hypergeometric functions in terms of the Gegenbauer polynomials, the Neumann series (18) arise. From a purely mathematical point of view this result illustrates a relation between the Lebedev-Kantorovich transformation and the Neumann series expansion, as discussed in [36].

Appendix B.

Here we shall give the proof of equation (41) and provide necessary details of calculations related to discussion in section 4.

In order to obtain formula (41) we first expand \( \cos(u\beta) \) in the Neumann series using Sonine’s formula (13) with \( \nu = -1/2 \) and \( \xi = 1 \):

\[
\cos(u\beta) = -\frac{1}{2} (\frac{\beta}{2})^{1/2} \Gamma[-\frac{1}{2}] J_{1/2}(\beta) + (\frac{\beta}{2})^{1/2} \Gamma[-\frac{1}{2}] \sum_{k=1}^{\infty} (-1)^k (-1/2 + 2k) C_{2k}^{-1/2}(u) J_{1/2+2k}(\beta) = \\
= \cos(\beta) + \sqrt{2\pi\beta} \sum_{k=0}^{\infty} (-1)^{k+1} (3/2 + 2k) C_{2(k+1)}^{-1/2}(u) J_{3/2+2k}(\beta).
\]

(61)

Now, using known properties of the Gegenbauer and Legendre polynomials [33] one can rewrite \( C_{-1/2}^{-1/2}(u) \) as:

\[
C_{2(k+1)}^{-1/2}(u) = \frac{(-1)}{2(k+1)} \left[ u C_{2k+1}^{1/2}(u) - C_{2k}^{1/2}(u) \right] = \frac{(-1)}{2(k+1)} \left[ u P_{2k+1}(u) - P_{2k}(u) \right] = \\
= \frac{(1 - u^2)}{2(k+1)(2k+1)} \frac{d}{du} P_{2k+1}(u) = \frac{(1 - u^2)}{2(k+1)(2k+1)} \frac{d}{du} C_{2k+1}^{1/2}(u) = \\
= \frac{1}{2(k+1)(2k+1)} (1 - u^2) C_{2k}^{3/2}(u).
\]

(62)

Inserting this formula into (61) one immediately obtains equation (11).

Next, let us discuss properties of matrices \( R_p^k(\xi^2) \), introduced in equation (13). In our approach they originate from the cosine Fourier transformation of \( F_{>}(\beta, \xi) \), performed with the help of the decomposition (11). They can interpreted as expansion coefficients of a Gegenbauer polynomial \( C_{2p}^{3/2}(u) \) in terms of Gegenbauer polynomials \( C_{2p}^{3/2}(u/\xi) \) for arbitrary \( \xi \), which for definiteness will be considered here to be greater than 0:

\[
C_{2p}^{3/2}(u) = \sum_{k=0}^{p} R_p^k(\xi^2) \frac{u}{\xi} C_{2k}^{3/2}(u/\xi),
\]

\[
R_p^k(\xi^2) = \frac{(-1)^{k+p}}{(p-k)!} \Gamma[3/2+p+k] \frac{\xi^{2k}}{\Gamma[3/2+2k]} \sum_{k=0}^{p} \frac{\xi^2}{\xi^2} \left( \begin{array}{c} k-p, 3/2+p+k \\ 5/2+2k \end{array} \right).
\]

(63)
Note that similar matrices appeared also in [24] as an element of a solution to the evolution equations for momentum-space skewed parton distributions based on the expansion of the whole skewed quark distribution in terms of polynomials orthogonal on the segment $0 \leq u \leq 1$. A straightforward way to obtain $[33]$ is to take into account that polynomials $C_{2p}^{3/2}(u/\xi)$ form an orthogonal basis for even functions of $u$ on the segment $-\xi \leq u \leq \xi$, with the weight function $(1-u^2/\xi^2)$. On the other hand, as $C_{2p}^{3/2}(u)$ is a polynomial in $u$ of a degree $2p$, equation (63) ensures that left- and right-hand sides are identical independently whether $u$ is smaller or larger then $\xi$.

Note that the hypergeometric function $\binom{2}{1}[a; b; c|\xi^2]$ with $a = k - p$, $b = 3/2 + p + k$ and $c = 5/2 + 2k$ is itself a polynomial in $\xi^2$ of order $p$ as long as $a = k - p \leq 0$, i.e. for all $0 \leq k \leq p$. In addition, the expansion inverse to $[33]$ is given by the same matrix function, but evaluated at $1/\xi$, i.e. $R_k^p(\xi^2)$ has the following remarkable property:

$$
\sum_{k=l}^p R_k^p(1/\xi^2) R_l^p(\xi^2) = \delta_{lp}.
$$

Unfortunately, we could not find a representation of $R_k^p(\xi^2)$ in terms of standard orthogonal polynomials.

Finally, let us concentrate on mathematical aspects which have not been discussed in details in section 4.

Asymptotic limit: here one finds that as $Q^2 \to \infty$, all $L_{2k+1} \to 0$ for $k > 1$ and $L_1 = 1$. For $F_<(u, \xi)$ only the first term in the expansion survives, and one obtains:

$$
F_<(u, \xi)_{Q^2=\infty} = \theta(u < \xi) \frac{2}{3} \frac{1}{\xi} \left( 1 - \frac{u^2}{\xi^2} \right) \int_0^\xi d\omega F_0(\omega, \xi).
$$

Analogously, for $F_>(u, \xi)$ one obtains:

$$
F_>(u, \xi)_{Q^2=\infty} = (1-u^2) \int_\xi^1 d\omega F_0(\omega, \xi) \times \sum_{n=0}^{\infty} \frac{(3/2 + 2n)}{(n + 1)(1 + 2n)} C_{2n}^{3/2}(u) R_n^0(\xi^2).
$$

To prove that the above series indeed equals to $\theta(u < \xi) \frac{2}{3} \frac{1}{\xi} (1 - u^2/\xi^2)$ one should determine coefficients $a_p$ of the following expansion:

$$
\theta(u < \xi) \frac{3}{2} \frac{1}{\xi} (1 - u^2/\xi^2) = (1-u^2) \sum_{n=0}^{\infty} a_n C_{2n}^{3/2}(u)
$$

and compare them with the coefficients of the series in (65). To this end, we multiply both sides by $C_{2p}^{3/2}(u)$ and integrate over $u$ to obtain:

$$
a_p = \frac{(3/2 + 2p)}{(p + 1)(2p + 1)} \frac{2}{3} \frac{1}{\xi} \int_0^\xi du (1-u^2/\xi^2) C_{2p}^{3/2}(u).
$$

The last integral can be easily calculated in terms of matrix $R_k^p(\xi^2)$:

$$
\int_0^\xi (1-u^2/\xi^2) C_{2p}^{3/2}(u) du/\xi = \sum_{k=0}^p R_k^p(\xi^2) \int_0^\xi du/\xi (1-u^2/\xi^2) C_{2k}^{3/2}(u/\xi) = R_0^p(\xi^2),
$$
so one indeed obtains

\[ a_p = \frac{(3/2 + 2p)}{(p + 1)(2p + 1)} R_0^p(\xi^2) . \]  

(70)

Expansion of initial conditions: when \( Q^2 = \mu^2 \), one should reproduce the initial distribution \( F_0(u, \xi) \) as the evolution operator is equal to unity. Consider first \( F_< (u, \xi) \). Setting \( L_{2k+1} \equiv 1 \) and interchanging summation and integration one obtains from (40):

\[
F_<(u, \xi)_{Q^2=\mu^2} = \theta(u < \xi) \frac{1}{\xi} \int_0^\xi d\omega F_0(\omega, \xi) \left( 1 - \frac{u^2}{\xi^2} \right) \sum_{k=0}^\infty \frac{(3/2 + 2k)}{(k + 1)(1 + 2k)} C_{2k}^{3/2}(u/\xi) C_{2k}^{3/2}(\omega/\xi) .
\]

(71)

Considering expansion analogous to (67) it is easy to see that the series above represents a \( \delta \)-function:

\[
\delta(\omega/\xi - u/\xi) = \left( 1 - \frac{u^2}{\xi^2} \right) \sum_{k=0}^\infty \frac{(3/2 + 2k)}{(k + 1)(1 + 2k)} C_{2k}^{3/2}(u/\xi) C_{2k}^{3/2}(\omega/\xi) ,
\]

(72)

and therefore one obtains

\[
F_<(u, \xi)_{Q^2=\mu^2} = \int_0^1 d\omega F_0(\omega, \xi) \delta(\omega - u/\xi) = \theta(u < \xi) F_0(u, \xi) .
\]

(73)

Consider now \( F_>(u, \xi) \), as given by (42). In the case \( L_{2k+1} \equiv 1 \) the internal sum can be evaluated with the help of (13):

\[
F_>(u, \xi)_{Q^2=\mu^2} = \left( 1 - u^2/\xi^2 \right) \sum_{n=0}^\infty \frac{(3/2 + 2n)}{(n + 1)(1 + 2n)} C_{2n}^{3/2}(u) \int_0^\xi d\omega F_0(\omega, \xi) C_{2n}^{3/2}(\omega) =
\]

\[
= \int_0^\xi d\omega F_0(\omega, \xi) \delta(\omega - u) = \theta(u > \xi) F_0(u, \xi)
\]

(74)

Combining (73) and (74) one immediately finds the desired answer:

\[
F(u, \xi)_{Q^2=\mu^2} = F_0(u, \xi) .
\]

(75)

Conformal constraints: here we check explicitly that conformal moments of \( F(u, \xi) \) are multiplicatively renormalized:

\[
\int_0^1 du F(u, \xi) C_{2k}^{3/2}(u/\xi) = L_{2k+1} \int_0^1 d\omega F_0(\omega, \xi) C_{2k}^{3/2}(u/\xi)
\]

(76)

The case of \( F_<(u, \xi) \) is easy. Integrating (42) term by term one gets

\[
\int_0^1 du F_<(u, \xi) C_{2p}^{3/2}(u/\xi) = L_{2p+1} \int_0^\xi d\omega F_0(\omega, \xi) C_{2p}^{3/2}(\omega/\xi)
\]

(77)
In case of $F_>(u, \xi)$ one can apply (63) and (64) to obtain:

$$
\int_0^1 du F_>(u, \xi)C_{2p}^{3/2}(u/\xi) = \sum_{l=0}^{p} R_l^p (1/\xi^2) \int_0^1 du F_>(u, \xi)C_{2l}^{3/2}(u) =
\sum_{n=0}^{p} R_n^p (1/\xi^2) \sum_{k=0}^{n} L_{2k+1} L_k^n (\xi^2) \int_\xi^1 dw F_0(w, \xi)C_{2k}^{3/2}(w/\xi) =
\sum_{k=0}^{L_{2p+1}} L_{2k+1} \int_\xi^1 dw F_0(w, \xi)C_{2k}^{3/2}(w/\xi).
$$

(78)

Sum of (77) and (78) gives explicitly (76).

Polynomiality: Assume that moments $M_n^q(\xi)$ are polynomials in $\xi^2$ of a degree $n$. Then, expanding $u^{2n}$ in terms of $C_{2p}^{3/2}(u/\xi)$ and using (76) one finds

$$
\int_0^1 du u^{2n} F(u, \xi) = \frac{(2n)!}{2^{2n}} \Gamma \left[ \frac{3}{2} \right] \sum_{k=0}^{\infty} \frac{(2n - 2k + 3/2)}{k! \Gamma[2n - k + 5/2]} L_{2n-2k+1} \xi^{2n} \int_0^1 d\omega F_0(\omega, \xi)C_{2p}^{3/2}(\omega/\xi).
$$

(79)

The last integral can be expressed in terms of $M_0^n(\xi)$ as

$$
\xi^{2n} \int_0^1 d\omega F_0(\omega, \xi)C_{2p}^{3/2}(\omega/\xi) = \frac{1}{\Gamma(\frac{3}{2})} \sum_{p=0}^{2n-2k-2p} \frac{2n - 2k - p + 3/2}{p!(2n - 2k - 2p)!} \xi^{2k+2p} M_0^{2n-2k-2p}
$$

(80)

Collecting all terms, one immediately finds that polynomiality is preserved by evolution.

Finally, let us give some arguments in support of the convergence of the Gegenbauer series (40) and (42). Strictly speaking, we have shown that expansions are convergent for the case $Q^2 = \mu^2$, or $L_{2k+1} \equiv 1$ only. We expect, however, that evolution to higher scales will make the convergence only better because of the known fact that for large $k$, $L_{2k+1} \sim k^{-c}$ with $c > 0$:

$$
L_{2k+1} \sim_{k \to \infty} \text{const} \cdot k^{-4C_F/b_0 \ln \left[ \frac{\ln(Q/\Lambda)}{\ln(\mu/\Lambda)} \right]}.
$$

(81)

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