NECESSARY CONDITIONS FOR TWO WEIGHT INEQUALITIES FOR SINGULAR INTEGRAL OPERATORS

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In memoriam of Benjamin Muckenhoupt and Richard Wheeden, who pioneered the study of two weight inequalities.

Abstract. We prove necessary conditions on pairs of measures \((\mu, \nu)\) for a singular integral operator \(T\) to satisfy weak \((p, p)\) inequalities, \(1 \leq p < \infty\), provided the kernel of \(T\) satisfies a weak non-degeneracy condition first introduced by Stein [12], and the measure \(\mu\) satisfies a weak doubling condition related to the non-degeneracy of the kernel. We also show similar results for pairs of measures \((\mu, \sigma)\) for the operator \(T_\sigma f = T(f d\sigma)\), which has come to play an important role in the study of weighted norm inequalities. Our major tool is a careful analysis of the strong type inequalities for averaging operators; these results are of interest in their own right. Finally, as an application of our techniques, we show that in general a singular operator does not satisfy the endpoint strong type inequality \(T : L^1(\nu) \to L^1(\mu)\). Our results unify and extend a number of known results.

1. Introduction

The goal of this paper is to establish necessary conditions for two weight, weak type inequalities for Calderón-Zygmund operators. This problem has a long history. In the one weight case it is well known that if each of the Riesz transforms is of weak type \((p, p)\) with respect to a weight \(w\), then \(w \in A_p\). See for example [4, Theorem 3.7, p. 417]. Stein [12, p. 210], showed that if any convolution type singular integral operator whose kernel satisfies a weak non-degeneracy condition is bounded on \(L^p(w)\), then \(w \in A_p\). The necessity of two weight \(A_p\) for the weak \((p, p)\) inequality for the Hilbert transform was established by Muckenhoupt and Wheeden [7].

In this paper we consider two versions of this problem. First suppose \((\mu, \nu)\) is a pair of positive regular Borel measures on \(\mathbb{R}^n\), where \(\mu\) satisfies a weak doubling condition (see Definition 2.20) and \(T\) is a Calderón-Zygmund operator whose kernel satisfies

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a weak non-degeneracy condition (see Definition 2.7). We first prove the following result on the necessity of the two-weight \( A_p \) condition (see Definition 2.9).

**Theorem 1.1.** Let \( T \) be a Calderón-Zygmund operator with a non-degenerate kernel in the direction \( u_0 \). Suppose that for some \( 1 \leq p < \infty \), and a pair of positive regular Borel measures \((\mu, \nu)\), with \( \mu \) directionally doubling in the direction \( u_0 \),

\[
\|T f\|_{L^{p,\infty}(\mu)} \leq C\|f\|_{L^p(\nu)}.
\]

Then:

1. \( d\nu = d\nu_s + vdx \) where \( v \in L^1_{\text{loc}} \) and \( \nu_s \) is singular;
2. \( \mu \ll \nu \), and \( \mu \ll dx \) so \( d\mu = udx \) where \( u \in L^1_{\text{loc}} \);
3. \( (u, v) \in A_p \) and \( u(x) \leq C v(x) \) a.e.

The second version of the problem is to let \((\mu, \sigma)\) be a pair of positive regular Borel measures on \( \mathbb{R}^n \), and consider the singular integral operator \( T_\sigma \) defined by

\[
T_\sigma f(x) = T(f d\sigma)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, d\sigma(y).
\]

This approach to weighted norm inequalities first appeared implicitly in [8] for the maximal operator. We establish necessary conditions on \((\mu, \sigma)\) for \( T_\sigma \) to satisfy the weak type inequality

\[
\mu(\{x : |T_\sigma f(x)| > \lambda\}) \leq \frac{C}{\lambda^p} \int |f|^p d\sigma
\]

for \( 1 < p < \infty \). (See Definition 2.4 for a more careful definition of this operator and the meaning of this inequality.) This problem was considered in [6], where they proved the necessity of the \( A_p \) condition for measures (see Definition 2.12) assuming a strong ellipticity condition. More precisely, they assumed that there exists a family of kernels \( \{K_j\}_{j=1}^N \) such that given any unit direction vector \( u \), there exists \( j \) such that \( K_j \) satisfies a non-degeneracy condition in the direction \( u \) (see Definition 2.7). In [9], they were able to prove the stronger \( PA_p \) condition (see Definition 2.17) is necessary with a similar hypothesis assuming a strong \((p, p)\) inequality. Our results are similar but we only assume that we have a single operator \( T_\sigma \) whose kernel is non-degenerate in one direction. We obtain the necessity of \( A_p \) under the additional hypothesis that \( \mu \) satisfies a weak doubling condition related to the non-degeneracy condition (see Definition 2.20). We prove this result for completeness since it is a simple application of the techniques used to prove Theorem 1.1.

**Theorem 1.3.** Let \((\mu, \sigma)\) be a pair of positive regular Borel measures with \( \mu \) directionally doubling in the direction \( u_0 \). Suppose the operator \( T_\sigma \) has a non-degenerate kernel in the direction \( u_0 \), and that for some \( 1 < p < \infty \),

\[
\|T_\sigma f\|_{L^{p,\infty}(\mu)} \leq C\|f\|_{L^p(\sigma)}.
\]
Then \((\mu, \sigma) \in A_p\).

More importantly, we also establish the necessity of the \(PA_p\) condition with the additional assumption that \(\sigma\) is doubling.

**Theorem 1.5.** Let \((\mu, \sigma)\) be a pair of positive regular Borel measures with \(\mu\) directionally doubling in the direction \(u_0\) and \(\sigma\) doubling. Suppose \(T_\sigma\) has a non-degenerate kernel in the direction \(u_0\) and for some \(1 < p < \infty\),

\[
\|T_\sigma f\|_{L^p(\mu)} \leq C\|f\|_{L^p(\sigma)}.
\]

Then \((\mu, \sigma) \in PA_p\).

Finally as an application of our techniques we consider the question of whether a Calderón-Zygmund operator can be bounded from \(L^1(\mu)\) to \(L^1(\nu)\) for a pair of positive Borel measures \((\mu, \nu)\). If the operator under consideration is translation invariant and the measures \((\mu, \nu)\) are regular, it is known that this is impossible. See [4, p. 468]. In [7, Theorem 4] Muckenhoupt and Wheeden derived a necessary condition for the Hilbert transform to be bounded from \(L^1(\nu)\) to \(L^1(\mu)\), where \(u\) and \(v\) are weights. We obtain an analogous estimate (see (5.6) in the proof of Theorem 1.7), but give a more complete characterization in terms of measures.

**Theorem 1.7.** Let \(T\) be a Calderón-Zygmund operator with a non-degenerate kernel in the direction \(u_0\), and let \((\mu, \nu)\) be positive Borel measures on \(\mathbb{R}^n\).

1. If \(\nu\) is singular with respect to Lebesgue measure and \(T : L^1(\nu) \to L^1(\mu)\), then \(\mu = 0\).
2. If \(\mu\) is a regular measure with \(d\mu = d\mu_s + udx\) where \(\mu_s\) is singular with respect to Lebesgue measure, \(u \neq 0\), \(d\nu = d\nu_s + vdx\) where \(\nu_s\) is singular with respect to Lebesgue measure, and \(v\) is a non-negative measurable function such that \(v(x) < \infty\) a.e., then \(T\) is not bounded from \(L^1(\nu)\) to \(L^1(\mu)\).
3. If \(\mu\) is a regular measure that is singular with respect to Lebesgue measure, and directionally doubling in the direction \(u_0\), and \(\nu\) is a positive regular Borel measure, then \(T\) is not bounded from \(L^1(\nu)\) to \(L^1(\mu)\).

**Remark 1.8.** The following example shows that the hypothesis in (2) that \(v(x) < \infty\) a.e. is needed. Let \(d\mu = \chi_{[-1,1]} \, dx\), \(d\nu = \chi_{\mathbb{R}\setminus[-2,2]} \, dx + \infty \cdot \chi_{[-2,2]} \, dx\), and let \(Tf(x) = Hf(x)\). Then \((\mu, \nu)\) satisfies the key estimate (5.6) below, and for \(f\) with \(\text{supp}(f) \subset \{x : |x| > 2\}\) we have that

\[
\int_{\mathbb{R}} |Hf(x)| \, d\mu(x) = \int_{-1}^{1} |Hf(x)| \, dx \leq \int_{-1}^{1} \int_{|y| > 2} \frac{|f(y)|}{|x-y|} \, dy \, dx
\]

\[= \int_{|y| > 2} \int_{|x| > 2} \frac{1}{|x-y|} \, dx \, |f(y)| \, dy \leq 2 \int_{|y| > 2} |f(y)| \, dy = 2 \int_{\mathbb{R}} |f(y)| \, d\nu(y).
\]
Remark 1.9. The following example shows that the hypothesis in (2) that \( \mu \) is not totally singular with respect to Lebesgue measure is needed. Let \( \mu = \delta(0) \), \( \nu = \frac{1}{x} dx \) and let \( T f(x) = H f(x) \). Then for any \( f \in L^1(\nu) \),

\[
\int_{\mathbb{R}} |H f(x)| d\mu(x) = |H f(0)| = \left| \int_{\mathbb{R}} \frac{f(y)}{y} \, dy \right| \leq \int_{\mathbb{R}} |f(y)| d\nu(y).
\]

The main idea in our proofs is to reduce the problem of obtaining necessary conditions for the \( L^p \) boundedness of singular integrals to that of averaging operators (see Definitions 3.1 and 4.1). For singular integrals \( T \), we work with the averaging operator \( A_Q \). When \( \mu = u dx, \nu = v dx \) it is well known that the \( A_p \) condition characterizes the strong type inequality for \( A_Q \); see Jawerth [5]. For completeness we prove this result (see Theorem 3.2). Furthermore, we obtain a characterization of the strong type inequality for \( A_Q \) when \((\mu, \nu)\) are positive regular Borel measures (see Theorem 3.5). This result is new and is interesting in its own right.

To study the singular integrals \( T_{\sigma} \) we introduce the analogous averaging operator \( A_{Q,\sigma} f = A_Q (f d\sigma) \). We show that the \( A_p \) condition for measures is also necessary and sufficient (see Theorem 4.3) for these operators to be bounded, \( 1 < p < \infty \).

The rest of the paper is organized as follows. In Section 2 we give preliminary definitions and notation used in this paper. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.3 and Theorem 1.5. Finally, in Section 5 we prove Theorem 1.7.

2. Preliminaries

Throughout this paper will use the following notation. The symbol \( n \) will denote the dimension of the Euclidean space \( \mathbb{R}^n \). \( Q(x, r) \) denotes the cube with center \( x \in \mathbb{R}^n \) and sidelength \( 2r \), while \( B(x, r) \) denotes the ball with center \( x \in \mathbb{R}^n \) and radius \( r \). For a cube \( Q, rQ \) is the cube with the same center as \( Q \) and with side length \( r \) times the length of \( Q \). Positive constants \( C, c \) may change value at each appearance. Sometimes we will indicate the dependence on certain parameters by writing for instance, \( C(n, p) \) etc. We will work extensively with average integrals and use the notation,

\[
\int_Q u \, dx = \frac{1}{|Q|} \int_Q u \, dx.
\]

We now define the singular integral operators we are interested in. For further, details, see [2].

**Definition 2.1.** We say that an operator \( T \) defined on measurable functions is a Calderón-Zygmund operator if \( T \) is bounded on \( L^2(\mathbb{R}^n) \) and for any \( f \in L^2_c(\mathbb{R}^n) \) we
have the representation
\[ T f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy, \quad x \notin \text{supp}(f). \]

Here \( K(x, y) \) is a kernel defined for all \( x \neq y \) in \( \mathbb{R}^n \times \mathbb{R}^n \), that satisfies the standard estimates
\[ |K(x, y)| \leq \frac{C_0}{|x - y|^n} \]  
and
\[ |K(x + h, y) - K(x, y)| + |K(x, y + h) - K(x, y)| \leq C_0 \frac{|h|^{\delta}}{|x - y|^{n+\delta}} \]
for all \( |h| < \frac{1}{2}|x - y| \) and some fixed \( \delta > 0 \).

We want to define the operators \( T_\sigma \) more carefully; to do so we follow the treatment given in [10]. Let \((\mu, \sigma)\) be a pair of regular Borel measures. Fix a Calderón-Zygmund operator \( T \) with kernel \( K \). Let \( \{\eta_{\epsilon,R}\}_{0<\epsilon<R<\infty} \) be a family of non-negative truncation functions with supports in the annuli \( \epsilon < |x| < R \), and such that \( \eta_{\epsilon,R}(x) = 1 \) if \( 2\epsilon < |x| < \frac{R}{2} \). For example, we can take \( \eta_{\epsilon,R} = \chi_{\{\epsilon<|x|<R\}} \), but other choices are possible. Define the family of truncated kernels \( K_{\epsilon,R}(x, y) = \eta_{\epsilon,R}(x - y)K(x, y) \). These are bounded with compact support for a fixed \( x \) or \( y \). Thus, the truncated operators defined by
\[ T_{\epsilon,R}^\sigma f(x) = \int_{\mathbb{R}^n} K_{\epsilon,R}(x, y) f(y) \, d\sigma(y), \quad x \in \mathbb{R}^n, \]
are pointwise well defined for \( f \in L^1_{\text{loc}} \). Hereafter, we will assume that each of the truncated kernels \( \{K_{\epsilon,R}\}_{0<\epsilon<R<\infty} \) satisfies the standard kernel estimates (2.2) and (2.3) with uniform constants.

**Definition 2.4.** Given a Calderón-Zygmund operator \( T \) with kernel \( K \), we say that \( T_\sigma \) satisfies the weak \((p, p)\) inequality, \( 1 < p < \infty \), provided that there exists a family of truncations \( \{\eta_{\epsilon,R}\}_{0<\epsilon<R<\infty} \) such that for all \( f \in L^p(\sigma) \),
\[ \|T_{\epsilon,R}^\sigma f\|_{L^p,\infty(\mu)} \leq C \|f\|_{L^p(\sigma)} \]
with constant independent of \( \epsilon \) and \( R \). In this case we write
\[ \|T_\sigma f\|_{L^p,\infty(\mu)} \leq C \|f\|_{L^p(\sigma)}. \]

Given this definition, in our proofs below we will need to fix particular values of \( \epsilon \) and \( R \) and apply inequality (2.5). We will, however, generally write \( T_\sigma \) instead of \( T_{\epsilon,R}^\sigma \) when there is no possibility of confusion.
Remark 2.6. While we need to fix a family of truncations to define $T_\sigma$, the choice is less important than it might seem at first. In [10], they showed that if the pair $(\mu, \sigma)$ satisfies the $A_p$ condition for measures, (2.12) below, then the corresponding strong $(2, 2)$ inequality for $T_\sigma$ holds independent of the choice of truncations used.

Definition 2.7. Given a Calderón-Zygmund operator $T$ with kernel $K(x, y)$, we say $T$ has a non-degenerate kernel if there exists $a > 0$, and a unit vector $u_0$ such that for $x, y \in \mathbb{R}^n$, $x - y = tu_0$, $t \in \mathbb{R}$,

$$|K(x, y)| \geq \frac{a}{|x - y|^n}. \quad (2.8)$$

For example, (2.8) holds for the Hilbert transform as well as of any of the Riesz transforms in the direction $e_j$. However, not all singular integrals satisfy this property. See for example [1, Lemma 1.4] where they construct a “one-sided” Calderón-Zygmund kernel with support in $(0, \infty)$; they establish that a sufficient condition for this operator to be bounded is a “one-sided” $A_p$ condition that is strictly weaker than the conditions we consider.

Definition 2.9. Let $u, v$ be non-negative, measurable functions. We say the pair $(u, v) \in A_p$, $1 < p < \infty$, if

$$[u, v]_{A_p} = \sup_Q \left( \int_Q u \, dx \right) \left( \int_Q v^{1 - \frac{1}{p'}} \, dx \right)^{\frac{1}{p - 1}} < \infty, \quad (2.10)$$

and in $A_1$ if

$$\int_Q u \, dx \leq [u, v]_{A_1} \text{ess inf}_{x \in Q} v(x). \quad (2.11)$$

Definition 2.12. If $(\mu, \sigma)$ are positive Borel measures, we say that $(\mu, \sigma) \in A_p$, $1 < p < \infty$, if

$$[\mu, \sigma]_{A_p} = \sup_Q \frac{\mu(Q)}{|Q|} \left( \frac{\sigma(Q)}{|Q|} \right)^{p - 1} < \infty. \quad (2.13)$$

Remark 2.14. If $d\mu = u \, dx$, $d\sigma = v^{1 - \frac{1}{p'}} \, dx$, then (2.13) is equivalent to (2.10).

Remark 2.15. It is straightforward to see that if (2.10), (2.11), or (2.13) hold for any cube $Q \subset \mathbb{R}^n$, then they also hold for any ball $B \subset \mathbb{R}^n$. We will use this fact below.

Remark 2.16. Inequality (2.13) implies that $\mu, \sigma$ do not share a common point mass: if there exists a point $a$ such that $\sigma\{a\} \mu\{a\} > 0$, then the expression in (2.13) blows up as $Q$ shrinks to $\{a\}$. 


Definition 2.17. We say the pair \((\mu, \sigma)\) is in \(PA_p\), \(1 < p < \infty\) if for any cube \(Q(y_0, r)\),

\[
(2.18) \quad \left( \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \right) \left( \int_{\mathbb{R}^n} \left( \frac{r^{p'-1}}{(|x-y_0|+r)^p} \right)^n \, d\sigma(x) \right)^{p-1} \leq C.
\]

This condition first appeared in \([7]\) in one dimension where they proved that it was necessary for the strong type inequality for the Hilbert transform to hold. The \(n\)-dimensional version first appeared in \([11]\) in the context of the fractional integral operator. When \(p = 2\), this condition is sometimes called “Poisson \(A_p\)”. This is because the second term on the left-hand side of (2.18) is approximately the Poisson extension of \(\sigma\) evaluated at a point in the upper half plane given by \(y_0\) and \(r\). It is straightforward to see that the \(PA_p\) condition implies the \(A_p\) condition.

Definition 2.19. A positive measure \(\mu\) is said to be doubling if there exists a constant \(C > 0\) such that for any cube \(Q\), \(\mu(2Q) \leq C\mu(Q)\).

Equivalently \(\mu\) is doubling if \(\mu(P) \leq C\mu(Q)\), whenever \(P, Q\) are adjacent cubes with \(|Q| = |P|\). (We say the cubes \(P, Q\) are adjacent if the boundaries of \(P\) and \(Q\) share a point in common.)

For our results we do not need to assume the full doubling condition, but rather a “directional” doubling condition.

Definition 2.20. Let \(\mu\) be a positive Borel measure, and fix a unit vector \(u_0\). We say \(\mu\) is directionally doubling in direction \(u_0\) if there exists a constant \(C_\mu > 0\) such that given adjacent cubes \(P(x_0, r), Q(y_0, r)\) whose centres satisfy \(x_0 - y_0 = tu_0\), \(t \in \mathbb{R}\),

\[
(2.21) \quad \mu(P(x_0, r)) \leq C_\mu \mu(Q(y_0, r)).
\]

Remark 2.22. Definition 2.20 is weaker than the doubling condition. For example, for \(E \subset \mathbb{R}^2\), define \(\mu(E) = \iint_E e^{-|x|} \, dx \, dy\). Then it is straightforward to show that \(\mu\) is directionally doubling in the direction \(e_2\) but is not doubling.

3. Proof of Theorem 1.1

In order to proceed with the proofs of our first main result we will need to prove some preliminary results about averaging operators.

Definition 3.1. Given a cube \(Q\), define the averaging operator \(A_Q\) on a function \(f \in L^1_{\text{loc}}\) by

\[
A_Qf(x) = \int_Q f(y) \, dy \, \chi_Q(x).
\]

The following result first appeared in \([5]\) but to the best of our knowledge a proof does not appear in the literature. For completeness we sketch the proof.
Theorem 3.2. Given a cube $Q$, $1 \leq p < \infty$, and $(u, v) \in A_p$, for all $f \in L^p(v)$

$$\|A_Q f\|_{L^p(u)} \leq [u, v]_{A_p}^{1/p} \|f\|_{L^p(v)}.$$  

Conversely, given $1 \leq p < \infty$, if $(u, v)$ are a pair of weights such that for every cube $Q$,

$$(3.3) \quad \|A_Q f\|_{L^p(u)} \leq K \|f\|_{L^p(v)},$$

then $(u, v) \in A_p$. Moreover, $[u, v]_{A_p} \leq K^p$.

Proof. Let $Q \subset \mathbb{R}^n$. We first prove the sufficiency of the $A_1$ condition when $p = 1$. Indeed,

$$\|A_Q f\|_{L^1(u)} = \int_{\mathbb{R}^n} \left| \int_Q f \, d\chi_Q(x) \right| u \, dx \leq \int_Q \frac{u(Q)}{|Q|} |f| \, dy \leq [u, v]_{A_1} \int_{\mathbb{R}^n} |f| v \, dy.$$  

If $p > 1$, by Hölder’s inequality,

$$\|A_Q f\|_{L^p(u)}^p = \int_{\mathbb{R}^n} \left( \int_Q f \, d\chi_Q(x) \right)^p u \, dx \leq \left( \int_Q |f|^{\frac{1}{p'}} v^{-\frac{1}{p'}} \, dy \right)^p u(Q) \leq \left( \int_Q |f|^p v \, dy \right) \left( \int_Q u \, dy \right)^{\frac{p}{p'}} \left( \int_Q v^{1-p'} \, dy \right)^{\frac{1}{p'}} \leq [u, v]_{A_p} \int_{\mathbb{R}^n} |f|^p v \, dy.$$

To prove necessity, let $S \subset Q$ be measurable and set $f = \chi_S$. Then (3.3) becomes

$$(3.4) \quad u(Q) \left( \frac{|S|}{|Q|} \right)^p \leq K v(S).$$

In [4, p. 388] they show that if (3.4) holds, then $(u, v) \in A_p$, and that $[u, v]_{A_p} \leq K_p$. This completes the proof. □

We now prove an analogue of Theorem 3.2 for measures.

Theorem 3.5. Let $(\mu, \nu)$ be a pair of positive regular Borel measures. Given $1 \leq p < \infty$, suppose that there exists a constant $C$ such that for every cube $Q$

$$(3.6) \quad \|A_Q f\|_{L^p(\mu)} \leq C \|f\|_{L^p(\nu)}.$$  

Then:

1. $d\nu = d\nu_s + v \, dx$ where $v \in L^1_{\text{loc}}$ and $\nu_s$ is singular;
2. $\mu \ll \nu$, and $\mu \ll dx$, so $d\mu = u \, dx$ where $u \in L^1_{\text{loc}}$;
3. $(u, v) \in A_p$ and $u(x) \leq C v(x)$ a.e.
Proof. Fix a cube $Q \subset \mathbb{R}^n$ and let $S \subset Q$ be measurable. Let $f = \chi_S$ in (3.6); then arguing as before we obtain

\begin{equation}
\left( \frac{|S|}{|Q|} \right)^p \mu(Q) \leq C \nu(S).
\end{equation}

Suppose $\nu$ were singular with respect to Lebesgue measure and $|S| > 0$. Then there exists a set $A \subset \mathbb{R}^n$ such that $|A| = 0$ and $\nu(S) = \nu(S \cap A)$. If we replace $S$ with $S \setminus A$ in (3.7), we find $\mu(Q) = 0$ for any cube $Q \supset S$. This implies that $\mu = 0$. Hence, $d\nu = d\nu_s + v \, dx$ where $v \in L^1_{\text{loc}}$ $v \neq 0$, and $\nu_s$ is singular.

Now fix any set $S \subset Q$ with $\nu(S) = 0$. Since $\nu$ is regular, for any $\epsilon > 0$ there exists an open set $E \supset S$ such that $\nu(E) < \epsilon$. Since $E$ is open, $E = \cup_j Q_j$ where $\{Q_j\}$ is a disjoint collection of dyadic cubes. If we let $Q = S = Q_j$ in (3.7), we have

\[ \mu(S) \leq \mu(E) = \sum_j \mu(Q_j) \leq C \sum_j \nu(Q_j) = C \nu(E) < C \epsilon. \]

Since $\epsilon > 0$ was arbitrary we have $\mu(S) = 0$, and so $\mu \ll \nu$.

We can now write (3.6) as

\begin{equation}
\mu(Q) \left| \int_Q f \, dx \right|^p \leq C \left( \int_{\mathbb{R}^n} |f|^p \, d\nu_s + \int_{\mathbb{R}^n} |f|^p \nu \, dx \right).
\end{equation}

Let $A = \text{supp}(\nu_s)$. Since $|A| = 0$, if we set $f = \chi_{S \setminus A}$, we have

\begin{equation}
\left( \frac{|S|}{|Q|} \right)^p \mu(Q) \leq C \nu(S).
\end{equation}

Using the same argument that showed $\mu \ll \nu$, replacing $\nu$ with $v$, we can see that $\mu \ll vdx \ll dx$. Hence, $d\mu = u \, dx$ for some $u \in L^1_{\text{loc}}$. Let $S = Q$ in (3.9), then by the Lebesgue differentiation theorem we have that $u(x) \leq C \nu(x)$ a.e. Moreover, by (3.9) we have that

\[ \left( \frac{|S|}{|Q|} \right)^p u(Q) \leq C \nu(S). \]

Hence, the fact that $(u, v) \in A_p$ follows as in the proof of Theorem 3.2. \qed

Proof of Theorem 1.1. We will show that given any cube $Q$ the averaging operator satisfies $A_Q : L^p(\nu) \to L^p(\mu)$. The desired conclusion then follows from Theorem 3.5. Choose a constant $t \geq 4$ such that $2C_0(1 + 2^{n+\delta})t^{-\delta} \leq a$. Here, $a$ is the constant in (2.8), and $\delta, C_0$ are as in (2.3). We further require $t = \frac{NC_2}{\sqrt{n}}$, where $\frac{1}{\sqrt{n}} \leq C_2 \lesssim 1$ and $N$ is an integer. The exact choice of the constant $C_2$ will be made clear below. Let $x_0, y_0$ be two points satisfying $x_0 - y_0 = tr\sqrt{n}u_0$, $r > 0$, and consider the cubes $Q(x_0, r), Q(y_0, r)$. Given any point $x \in Q(x_0, r)$ we can write $x = x_0 + h$, where
$|h| < r \sqrt{n}$. Similarly, given $y \in Q(y_0, r)$, $y = y_0 + k$ where $|k| < r \sqrt{n}$. We claim that for such $x$ and $y$,

$$ (3.10) \quad |K(x, y) - K(x_0, y_0)| \leq \frac{1}{2} |K(x_0, y_0)|. $$

To prove this we will apply (2.3) which is possible since $|h| < r \sqrt{n} \leq \frac{1}{2} |x_0 - y_0|$, and

$$ |x_0 + h - y_0| \geq |x_0 - y_0| - |h| \geq t r \sqrt{n} - r \sqrt{n} \geq \frac{t}{2} r \sqrt{n} \geq 2 |k|. $$

Thus, we can estimate as follows:

$$ (3.11) \quad |K(x, y) - K(x_0, y_0)| $$

$$ \leq |K(x_0 + h, y_0 + k) - K(x_0 + h, y_0)| + |K(x_0 + h, y_0) - K(x_0, y_0)| $$

$$ \leq \frac{C_0 |k|^{\delta}}{|x_0 + h - y_0|^{n+\delta}} + \frac{C_0 |h|^{\delta}}{|x_0 - y_0|^{n+\delta}} $$

$$ = I_1 + I_2. $$

We can bound $I_2$ immediately:

$$ I_2 \leq \frac{C_0 (r \sqrt{n})^{\delta}}{(tr \sqrt{n})^{\delta} |x_0 - y_0|^n} = C_0 \frac{t^{-\delta}}{|x_0 - y_0|^n}. $$

To estimate $I_1$, note that

$$ |x_0 + h - y_0| \geq \frac{t}{2} r \sqrt{n} = \frac{1}{2} |x_0 - y_0|. $$

Hence,

$$ I_1 \leq \frac{C_0 2^{n+\delta} (r \sqrt{n})^{\delta}}{(tr \sqrt{n})^{\delta} |x_0 - y_0|^n} = C_0 \frac{2^{n+\delta} t^{-\delta}}{|x_0 - y_0|^n}. $$

If we combine these estimates, by our choice of $t$ and (2.8) we have

$$ I_1 + I_2 \leq \frac{1}{2} |x_0 - y_0|^n \leq \frac{1}{2} |K(x_0, y_0)|, $$

which proves (3.10).

It now follows that for any $x \in Q(x_0, r)$, $y \in Q(y_0, r)$, the kernel $K(x, y)$ always has the same sign. Therefore, if we fix a non-negative function $f$ with supp$(f) \subset Q(x_0, r)$, then

$$ (3.12) \quad |T f(y)| = \left| \int_{Q(x_0, r)} K(x, y) f(x) \, dx \right| $$

$$ = \int_{Q(x_0, r)} |K(x, y)| f(x) \, dx $$

).
\[ \geq \int_{Q(x_0,r)} |K(x_0,y_0)|f(x) \, dx - \int_{Q(x_0,r)} |K(x,y) - K(x_0,y_0)|f(x) \, dx; \]

again by (3.10) and (2.8),

\[ \geq \frac{1}{2} |K(x_0,y_0)| \int_{Q(x_0,r)} f(x) \, dx \]
\[ \geq \frac{a}{2|x_0 - y_0|^n} \int_{Q(x_0,r)} f(x) \, dx \]
\[ \geq \frac{a}{2(t \sqrt{n})^n} \int_{Q(x_0,r)} f(x) \, dx \]
\[ = c(a,t,n) \int_{Q(x_0,r)} f(x) \, dx. \]

Given this inequality and the assumption that $T$ satisfies a weak $(p,p)$ inequality we have for any $0 < \lambda < c(a,t,n)$

\[ \mu(Q(y_0,r)) \leq \mu(\{x : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{Q(x_0,r)} |f|^p \, d\nu. \]

If we take the supremum over all such $\lambda$, we get

\[ (3.13) \quad \mu(Q(y_0,r)) \left( \int_{Q(x_0,r)} f \, dx \right)^p \leq c(a,t,n,p) \int_{Q(x_0,r)} |f|^p \, d\nu. \]

Now fix a value of $C_2$, depending only on $u_0$, so that starting from $Q(y_0,r)$ we can form a chain of adjacent cubes $Q(x_j,r), j = 1, \ldots, N$ in the direction $-u_0$ such that $x_1 = y_0$ and $x_N = x_0$. Each $Q(x_j,r)$ satisfies $\mu(Q(x_{j+1},r)) \leq C_\mu \mu(Q(x_j,r))$, where $C_\mu$ is the directional doubling constant from (2.21). The number of cubes $N$, lying between $Q(y_0,r)$ and $Q(x_0,r)$ depends only on $t$ and $n$. Thus, there exists constant $C = C(C_\mu, t, n)$ such that $\mu(Q(x_0,r)) \leq C \mu(Q(y_0,r))$. Hence,

\[ \mu(Q(x_0,r)) \left( \int_{Q(x_0,r)} f \, dx \right)^p \leq C \mu(Q(y_0,r)) \left( \int_{Q(x_0,r)} f \, dx \right)^p \leq C \int_{Q(x_0,r)} |f|^p \, d\nu. \]

Since the resulting constant depends only on $C_1, p, t, n, a$ and not on $Q(x_0,r)$ we have shown that the averaging operators $A_Q : L^p(\nu) \to L^p(\mu)$ uniformly for all $Q$. Therefore, by Theorem 3.5 we get the desired conclusion. \qed

4. Proofs of Theorems 1.3 and 1.5

Before proceeding with the proof of Theorem 1.3 we first define the related averaging operator.
Definition 4.1. Given a non-negative measure $\sigma$ and a cube $Q$, define the averaging operator $A_{Q,\sigma}$ acting on a function $f \in L^1_{\text{loc}}(\sigma)$ by

\begin{equation}
A_{Q,\sigma}f(x) = \frac{1}{|Q|} \int_Q f(y) \, d\sigma(y) \chi_Q(x).
\end{equation}

The following result characterizes the $L^p$ boundedness of $A_{Q,\sigma}$.

Theorem 4.3. Given a cube $Q$, $1 \leq p < \infty$, and a pair of positive regular Borel measures $(\mu, \sigma)$, suppose that $(\mu, \sigma)$ satisfy the $A_p$ condition (2.13). Then for all $f \in L^p(\sigma)$,

\[ \|A_{Q,\sigma}f\|_{L^p(\mu)} \leq [\mu, \sigma]^{1/p}_{A_p} \|f\|_{L^p(\sigma)}. \]

Conversely given $1 \leq p < \infty$, if $(\mu, \sigma)$ is a pair of positive regular Borel measures such that for every cube $Q$,

\begin{equation}
\|A_{Q,\sigma}f\|_{L^p(\mu)} \leq K \|f\|_{L^p(\sigma)},
\end{equation}

then $(\mu, \sigma) \in A_p$. Moreover $[\mu, \sigma]_{A_p} \leq K^p$.

Proof. We first prove necessity. Fix a cube $Q$ and let $1 \leq p < \infty$. If $\sigma(Q) = 0$, then (2.13) is immediate. If $\sigma(Q) > 0$ let $f = \chi_Q$ in (4.4). Then we obtain

\[ \mu(Q) \left( \frac{\sigma(Q)}{|Q|} \right)^p \leq K^p \sigma(Q). \]

Dividing by $\sigma(Q)$ and taking the supremum over all cubes $Q$ we have $(\mu, \sigma) \in A_p$ and $[\mu, \sigma]_{A_p} \leq K^p$. The proof of sufficiency is similar to Theorem 3.2 so we omit the details. \qed

We can now prove Theorems 1.3 and 1.5.

Proof of Theorem 1.3. The proof is a straightforward modification of the proof of Theorem 1.1. Fix the cubes $Q(x_0, r)$ and $Q(y_0, r)$ as before. Then with the same notation as before, we have that if $x \in Q(x_0, r)$ and $y \in Q(y_0, r)$,

\[ |x - y| = |x_0 - y_0 + h - k| < tr\sqrt{n} + 2r\sqrt{n} < 2tr\sqrt{n}. \]

Similarly, we have $|x - y| > \frac{1}{2}tr\sqrt{n}$. Therefore, if we choose $0 < \epsilon < \frac{1}{4}tr\sqrt{n}$ and $R > 4tr\sqrt{n}$, we have that the kernel $K_{\epsilon,R}(x, y) = K(x, y)$ and so satisfies the non-degeneracy condition (2.8) with a uniform constant. We also have that it satisfies the standard estimates (2.2) and (2.3).

For simplicity, we now write $T_\sigma$ instead of $T_{\sigma}^{\epsilon,R}$ and $K$ for $K_{\epsilon,R}$. If we repeat the previous argument, we have that $K$ satisfies the estimate (3.10). We can then repeat the proof of (3.12), using the fact that $T_\sigma$ satisfies the weak $(p,p)$ inequality with uniform constant, to get

\[ |T_\sigma f(y)| \geq c(a, t, n) |Q(x_0, r)| \int_{Q(x_0, r)} f(x) \, d\sigma(x). \]
Given this inequality we continue to argue as we did in the proof of Theorem 1.1 to get that the averaging operator $A_{Q,\sigma} = A_{Q(x_0, r), \sigma}$ satisfies $A_{Q,\sigma} : L^p(\sigma) \to L^p(\mu)$. This estimate holds for every cube $Q(x_0, r)$ with constants independent of $\epsilon$ and $R$, and so $(\mu, \sigma) \in A_p$ by Theorem 4.3. □

Proof of Theorem 1.5. We adapt the proof of Theorem 1.1, exchanging the roles of $x_0$ and $y_0$. Fix a cube $Q(y_0, r)$. Choose $t \geq 4$ as in the proof of Theorem 1.1. Rather than considering the cube $Q(x_0, r)$ we replace it with a ball. Fix $S > r$ and for each $r \leq s \leq S$ let $B_s = B(x_s, s\sqrt{n})$, where $x_s = y_0 + ts\sqrt{n}u_0$. If we now argue as we did in the proof of Theorem 1.3, if we fix $R > 4tS\sqrt{n}$ and $\epsilon < \frac{1}{4}tr\sqrt{n}$, then for $y \in Q(y_0, r)$ and $x \in B_s$, $K_{\epsilon, R}$ satisfies the non-degeneracy condition (2.8) with a uniform constant. We also have that it satisfies the standard estimates (2.2) and (2.3). Again, we will write $T_\nu$ for $T_\nu^R$ and $K$ for $K_{\epsilon, R}$.

We can now argue as follows: for all $y \in Q(y_0, r)$, $y = y_0 + k$ where $|k| \leq r\sqrt{n}$ and for $x \in B_s$, $x = x_s + h$ where $|h| \leq s\sqrt{n}$. As in the proof of Theorem 1.1 we have $|h| \leq s\sqrt{n} \leq \frac{1}{2}|x_s - y_0|$, and

$$|x_s + h - y_0| = |x_s - y_0| - |h| \geq ts\sqrt{n} - s\sqrt{n} \geq t\frac{s\sqrt{n}}{2} \geq \frac{t}{2}r\sqrt{n} \geq 2|k|.$$  

We can now apply (2.3) as in estimate (3.11) to get that for $y \in Q_r$ and $x \in B_s$,

$$|K(x, y) - K(x_s, y_0)| \leq \frac{1}{2}|K(x_s, y_0)|.  \tag{4.5}$$

This implies that for any $y \in Q(y_0, r)$ and $x \in B_s$ $K(x, y)$ always has the same sign. Moreover, we have

$$|K(x, y)| \geq \frac{1}{2}|K(x_s, y_0)|.  \tag{4.6}$$

Therefore,

$$|K(x, y)| \geq \frac{1}{2}|K(x_s, y_0)| \geq \frac{a}{2} \frac{1}{|x_s - y_0|^n} \geq \frac{a}{2} \frac{1}{(|x - x_s| + |x - y_0|)^n} \geq c(a,n) \frac{1}{(|x - y_0|)^n} \geq c(a,n) \frac{1}{(|x - y_0| + r)^n};  \tag{4.7}$$

the second to last inequality follows since $|x - x_s| \leq s\sqrt{n} \leq ts\sqrt{n} = |x - y_0|$.

Define the truncated cone

$$C_r = \bigcup_{s \geq r} B_s.$$  

Notice $C_r$ has a central axis of $y_0 + su_0$, for $s \geq r$. For $S > 0$ let

$$f_{r,S}(x) = \left(\frac{1}{|x - y_0| + r}\right)^{n(p'-1)} \chi_{C_r \cap B(y_0, S)}.$$
Then for all $y \in Q(x_0, r)$ we have

\begin{equation}
\begin{aligned}
|T_{\sigma} f_{r,S}(y)| &= \int_{C_r \cap B(y_0, S)} |K(x, y)| \left( \frac{1}{(|x - y_0| + r)} \right)^{n(p' - 1)} d\sigma(x) \\
&\geq c(a, n) \int_{C_r \cap B(y_0, S)} \left( \frac{1}{(|x - y_0| + r)^p} \right)^n d\sigma(x).
\end{aligned}
\end{equation}

We have that $T_{\sigma}$ satisfies the weak $(p, p)$ inequality with uniform constant, so we can argue as we did to derive (3.13) in the proof of Theorem 1.1 to get

\begin{equation}
\begin{aligned}
\mu(Q(y_0, r)) \left( \int_{C_r \cap B(y_0, S)} \left( \frac{1}{(|x - y_0| + r)^p} \right)^n d\sigma(x) \right)^p &\leq C \int_{C_r \cap B(y_0, S)} \left( \frac{1}{(|x - y_0| + r)^{n(p' - 1)}} \right)^n d\sigma(x).
\end{aligned}
\end{equation}

Since $p(p' - 1) = p'$ we have

\begin{equation}
\begin{aligned}
\frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \int_{C_r \cap B(y_0, S)} \left( \frac{r^{p' - 1}}{(|x - y_0| + r)^p} \right)^n d\sigma(x) \right)^{p-1} &\leq C.
\end{aligned}
\end{equation}

Since the constant $C$ is independent of $\epsilon$ and $R$, and so of $S$, we can take the limit as $S \to \infty$, and by the monotone convergence theorem we get

\begin{equation}
\begin{aligned}
\frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \int_{C_r} \left( \frac{r^{p' - 1}}{(|x - y_0| + r)^p} \right)^n d\sigma(x) \right)^{p-1} &\leq C.
\end{aligned}
\end{equation}

We will now extend inequality (4.9) to all of $\mathbb{R}^n$. Let

$$A_k = B(y_0, 2^{k+1}tr \sqrt{n}) \setminus B(y_0, 2^ktr \sqrt{n}).$$

Consider the ball $B(x_k, 2^{k+2}tr \sqrt{n})$, where

$$x_k = y_0 + \left( \frac{2^{k+1} + 2}{2} \right) tr \sqrt{n} u_0 = y_0 + \frac{3}{8}(2^{k+2}) tr \sqrt{n} u_0.$$

This is the ball of radius $2^{k+2}tr \sqrt{n}$ centered at the midpoint of the portion of the central axis of $C_r$ that lies inside $A_k$. We claim $A_k \subset B(x_k, 2^{k+2}tr \sqrt{n})$. To see this, fix $x \in A_k$; then

$$|x - x_k| \leq |x - y_0| + |x_k - y_0| \leq 2^{k+1}tr \sqrt{n} + \frac{3}{8}(2^{k+2}) tr \sqrt{n} \leq 2^{k+2}tr \sqrt{n}.$$

Since the ball $B(x_k, \frac{3}{8}(2^{k+2}) r \sqrt{n})$ is one of the balls $B_\epsilon$ that defines $C_r$, it is immediate that

$$\bigcup_{k=0}^{\infty} B(x_k, \frac{3}{8}(2^{k+2}) r \sqrt{n}) \subset C_r.$$
Since $\sigma$ is doubling there exists a constant $C = C(t, n, \sigma)$ such that
\[
\sigma(B(x_k, 2^{k+2}tr\sqrt{n})) \leq C \sigma \left( B(x_k, \frac{3}{8}(2^{k+2})r\sqrt{n}) \right).
\]
Hence, we can estimate as follows:
\[
\frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \int_{\mathbb{R}^n \setminus B(y_0, tr\sqrt{n})} \left( \frac{r^{p'-1}}{(|x - y_0| + r)^{p'}} \right)^n d\sigma(x) \right)^{p-1}
\]
\[
= \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \sum_{k=0}^{\infty} \int_{A_k} \left( \frac{r^{p'-1}}{(|x - y_0| + r)^{p'}} \right)^n d\sigma(x) \right)^{p-1}
\]
\[
\leq \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \sum_{k=0}^{\infty} \left( \frac{r^{p'-1}}{2^{k+2}r\sqrt{n} + r} \right)^n \sigma(A_k) \right)^{p-1}
\]
\[
\leq \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \sum_{k=0}^{\infty} \left( \frac{r^{p'-1}}{2^{k+2}r\sqrt{n} + r} \right)^n \sigma(B(x_k, 2^{k+2}tr\sqrt{n})) \right)^{p-1}
\]
\[
\leq C \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \sum_{k=0}^{\infty} \left( \frac{r^{p'-1}}{2^{k+2}r\sqrt{n} + r} \right)^n \sigma(B(x_k, \frac{3}{8}(2^{k+2})r\sqrt{n})) \right)^{p-1}
\]
\[
\leq C \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \int_{B(y_0, \frac{3}{8}(2^{k+2})r\sqrt{n})} \left( \frac{r^{p'-1}}{(|x - y_0| + r)^{p'}} \right)^n d\sigma(x) \right)^{p-1}
\]
\[
\leq C.
\]

The third to last inequality holds since $2^{k+2}tr\sqrt{n} \geq \frac{1}{2}|x - y_0|$ for any $x \in B(x_k, \frac{3}{8}(2^{k+2})r\sqrt{n})$.

By Remark 2.15, we can apply the result of Theorem 1.3 to the ball $B(y_0, tr\sqrt{n})$ to get
\[
\left( \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \right) \left( \int_{B(y_0, tr\sqrt{n})} \left( \frac{r^{p'-1}}{(|x - y_0| + r)^{p'}} \right)^n d\sigma(x) \right)^{p-1}
\]
\[
\leq \left( \frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \right) \left( \frac{\sigma(B(y_0, tr\sqrt{n}))}{|B(y_0, tr\sqrt{n})|} \right)^{p-1}
\]
\[
\leq \left( \frac{\mu(B(y_0, tr\sqrt{n}))}{|B(y_0, tr\sqrt{n})|} \right) \left( \frac{\sigma(B(y_0, tr\sqrt{n}))}{|B(y_0, tr\sqrt{n})|} \right)^{p-1}
\]
\[
\leq C.
\]
If we combine this inequality with the previous estimate, we get
\[
\frac{\mu(Q(y_0, r))}{|Q(y_0, r)|} \left( \int_{\mathbb{R}^n} \left( \frac{r'}{r} - 1 \right) \left( \frac{r'}{|x - y_0| + r} \right)^n \, d\sigma(x) \right)^{p-1} \leq C.
\]
Since this holds for every cube \(Q(y_0, r)\), it follows that \((\mu, \sigma)\) satisfy the \(PA_p\) condition. \(\square\)

5. **Strong \((1, 1)\) Inequalities**

For the proof of Theorem 1.7 we first give some preliminary lemmas.

**Lemma 5.1.** Let \(v\) be a measurable function. Then for a.e. \(x \in \mathbb{R}^n\),

\[
\lim_{r \to 0^+} \left[ \text{ess inf}_{y \in Q(x, r)} v(y) \right] \leq v(x).
\]

The proof of Lemma 5.1 is implicit in [7, Theorem 4] in one dimension; the proof is the same in higher dimensions.

**Definition 5.3.** A family \(\{E_r\}_{r>0}\) of Borel subsets of \(\mathbb{R}^n\) is said to shrink nicely to \(x \in \mathbb{R}^n\) if

\[E_r \subset B(x, r)\]

for each \(r\), and there exists a constant \(\alpha\) independent of \(r\) such that

\[|E_r| > \alpha|B(x, r)|.\]

**Lemma 5.4.** Let \(\mu\) be a regular Borel measure on \(\mathbb{R}^n\), and let \(d\mu = d\mu_s + u \, dx\) be its Lebesgue Radon-Nikodym decomposition. Then for a.e. \(x \in \mathbb{R}^n\),

\[
\lim_{r \to 0} \frac{\mu(E_r)}{|E_r|} = u(x).
\]

The proof of Lemma 5.4 can be found in [3, Theorem 3.22, p.99].

**Proof of Theorem 1.7.** First suppose that the measure \(\nu\) is singular with respect to Lebesgue measure. As in the proof of Theorem 1.5 fix a cube \(Q(y_0, r)\) and define the truncated cone \(C_r\). Let \(f\) be a non negative function with \(\text{supp}(f) \subset Q(y_0, r)\). Then, if we estimate as in the proof of Theorem 1.5 to get (4.8), we have for all \(x \in C_r\),

\[
|Tf(x)| \geq c(a, n) \int_{Q(y_0, r)} \frac{f(y)}{(r + |x - y_0|)^n} \, dy.
\]

By assumption \(T : L^1(\nu) \to L^1(\mu)\), so we have that

\[
\int_{Q(y_0, r)} f(x) \, d\nu(x) \geq c \int_{\mathbb{R}^n} |Tf(x)| \, d\mu(x)
\]

\[
\geq c \int_{C_r} \int_{Q(y_0, r)} \frac{f(y)}{(r + |x - y_0|)^n} \, dy \, d\mu(x)
\]
If \( \mu \neq 0 \), since \( \mu \) is a Borel measure, there exists a ball \( B \) such that \( \mu(B) > 0 \). Fix a point \( y_0 \) and \( r > 0 \) such that \( B \subset C_r \). Let \( f = \chi_{Q(y_0,r)} \) in inequality (5.5); then the left-hand side equals 0. Since \( \nu \) is singular with respect to Lebesgue measure, \( |\text{supp}(\nu)| = 0 \), so the first term on the right-hand side is positive. Since the integrand in the second term on the right-hand side is bounded away from 0, the second term is positive unless \( \mu(C_r) = 0 \), a contradiction. Hence, \( \mu = 0 \).

Now let \( \mu \) be a regular measure with Lebesgue decomposition \( d\mu = d\mu_s + u \, dx \), where \( u \neq 0 \), and suppose \( dv = dv_s + v \, dx \), where \( v \) is a non-negative function such that \( v(x) < \infty \) a.e. Fix a point \( y_0 \) such that \( 0 < u(y_0) < \infty \). We can further assume that \( y_0 \) is a Lebesgue point for \( \mu \) in the sense of Lemma 5.4, and that the conclusion of Lemma 5.1 holds for the function \( v \) at \( y_0 \). Let \( \alpha = \text{ess inf}_{x \in Q(y_0,r)} v(x) \). Given \( \epsilon > 0 \), let \( E = \{ x \in Q(y_0,r) : v(x) < u + \epsilon \} \), \( A = E \setminus \text{supp}(\nu_s) \), and set \( f = |A|^{-1} \chi_A \) in inequality (5.5). By the definition of the essential infimum, \( |E| > 0 \), and since \( |\text{supp}(\nu_s)| = 0 \), we have that \( |A| > 0 \). Thus,

\[
\int_{C_r} \frac{1}{(r + |x - x_0|)^n} d\mu(x) \leq C \frac{\nu(A)}{|A|} \leq C \frac{v(A)}{|A|} \leq C(a + \epsilon) = C(\text{ess inf}_{x \in Q(y_0,r)} v(x) + \epsilon).
\]

Since \( \epsilon > 0 \) was arbitrary, this inequality holds with \( \epsilon = 0 \). As \( r \to 0 \), \( C_r \) converges to the cone \( C_0 \) with central axis \( y_0 + ts\sqrt{n}u_0, s \geq 0 \). Therefore, by the monotone convergence theorem and Lemma 5.1 we have

\[
\int_{C_0} \frac{1}{|x - y_0|^n} d\mu(x) \leq Cv(y_0).
\]

Let \( B_j = B(y_0, 2^{-j}) \), \( A_j = (C_0 \cap B_j) \setminus B_{j+1} \). Since \( C_0 \) has constant aperture, there exists \( 0 < \alpha < 1 \) such that \( |A_j| = \alpha |B_j| \), so the collection \( \{A_j\} \) shrinks nicely to \( y_0 \). Then we have that

\[
\lim_{j \to \infty} \frac{\mu(A_j)}{|B_j|} = \alpha u(y_0).
\]

Fix \( j_0 \) such that for all \( j \geq j_0 \) we have \( \frac{\mu(A_j)}{|B_j|} \geq \frac{\alpha}{2} u(y_0) \). Hence,

\[
v(y_0) \geq c \sum_{j \geq j_0} \int_{A_j} \frac{1}{|x - y_0|^n} d\mu(x) \geq c \sum_{j \geq j_0} 2^{nj} \mu(A_j) \geq c \sum_{j \geq j_0} \frac{\mu(A_j)}{|B_j|} \geq c \sum_{j \geq j_0} u(y_0) = \infty.
\]

Let \( E = \{ x : 0 < u(x) < \infty \} \) which has positive measure since \( u \neq 0 \). Then we have \( v(x) = \infty \) for a.e \( x \in E \). This contradicts the fact that \( v(x) < \infty \) a.e.
Finally, suppose $\mu$ is a regular measure that is singular with respect to Lebesgue measure, and is directionaly doubling in the direction $u_0$, and $\nu$ is a positive regular Borel measure. If $T : L^1(\nu) \to L^1(\mu)$, then it satisfies a weak $(1, 1)$ inequality, and so by Theorem 1.1 $\mu$ is absolutely continuous; a contradiction. \hfill \square

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