A SEQUENTIAL RPF THEOREM AND ITS APPLICATIONS TO LIMIT THEOREMS FOR TIME DEPENDENT DYNAMICAL SYSTEMS AND INHOMOGENEOUS MARKOV CHAINS

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Abstract. In this paper we will prove various probabilistic limit theorems for some classes of sequential dynamical systems (SDS) and inhomogeneous Markov chains. Our proofs utilize a certain sequential Ruelle-Perron-Frobenius theorem for complex operators, which, as in [16], is proved using contraction properties of a complex version of the projective Hilbert metric that was developed in [35] and [14]. We will also prove a certain type of stability theorem for the corresponding Ruelle-Perron-Frobenius triplets with respect to complex perturbation of the transfer and Markov operators, which leads to natural conditions for linear growth of the corresponding variances. Some of our general results mostly have applications for dynamical systems and Markov chains in random non-stationary environments, while the conditions of the other results hold true for general type of SDS and inhomogeneous Markov chains. This paper is the first time that finer limit theorems such the local central limit theorem and the Berry-Esseen theorem are proved in the SDS setup.

1. Introduction

Probabilistic limit theorems for dynamical systems and Markov chains is a well studied topic. One way to derive such results is relying on some quasi-compactness (or spectral gap) of an appropriate transfer or Markov operator, together with a suitable perturbation theorem (see [31], [32], [15] and [20]). This quasi-compactness can often be verified only via an appropriate Ruelle-Perron-Frobenius (RPF) theorem, which is the main key for thermodynamic formalism type constructions. Probabilistic limit theorems for random dynamical systems and Markov chains in random dynamical environments were also studied in literature (see, for instance, [24], [25], [16], [1], [13], [19] and references therein). In these circumstances, the probabilistic behaviour of the appropriate process is determined by compositions of random operators, and not of a single operator, so no spectral theory can be exploit, and instead, many of these results rely on an appropriate version of the RPF theorem for random operators. Relying on certain contraction properties of...
random complex transfer and Markov operators, with respect to a complex version of the Hilbert protective metric due to H.H. Rugh \[35\] (see also \[14\]), we proved in \[16\] an RPF theorem for random complex operators and presented the appropriate random complex thermodynamic formalism type constructions, which was one of the main keys in the proof of versions of the Berry-Esseen theorem and the local central limit theorem (LCLT) for certain processes in random dynamical environment, in the proof of some nonconventional LCLT for dynamical systems (see Chapters 2 and 7 of \[16\]) and in the proof of the results in \[19\].

In recent years (see, for instance, \[2\], \[8\], \[22\], \[27\] and \[33\]) there has been a growing interest in proving limit theorems for sequences \(X_n = T^n_0x_0\) of random variables generated by an appropriate random variable \(x_0\) and compositions \(T^n_0 = T_{n−1} \circ T_{n−2} \circ \cdots \circ T_0\) of different maps \(T_0, T_1, T_2, \ldots\). Except for the random dynamical system case, in which the maps \(T_j = T_0\) are chosen at random according to a stationary process \(\{\xi_j\}\), the results obtained so far are certain versions of the central limit theorem (CLT), without close to optimal convergence rate and corresponding local CLT’s. Limit theorems for some classes of inhomogeneous Markov chains, namely Markov chains with time dependent transition probabilities, were also studied. The central limit theorem was studied in \[39\] and \[34\], and the local central limit theorem was studied by D. Dologpyat and O. Sarig, and M. Peligrad (both papers are forthcoming). In this paper we will prove several limit theorems for some classes of maps \(T_j\) and inhomogeneous Markov chains, as described in the following paragraphs.

Let \(T_j\) be a sequence of distance expanding maps of finite bounded in \(j\) degree, and \(f_j, u_j\) sequences of uniformly in \(j\) Hölder continuous functions. Let \(\mathcal{L}^{(j)}_0\) be the transfer operator generated by \(T_j\) and the function \(e^{f_j}\) and let \(\mathcal{L}^{(j)}_z(g) = \mathcal{L}^{(0)}_0(g \cdot e^{zu_j})\), \(z \in \mathbb{C}\) be the perturbations of \(\mathcal{L}^{(j)}_0\) corresponding to \(u_j\). In the Markov chain case, let \(\{R^{(j)}_0\}\) be a family of Markov operators, which satisfy a certain sequential version of the two sided Doeblin condition, and let \(\{\xi_j\}\) be the inhomogeneous Markov chain whose \(j\)-th step transition density is given by \(R^{(j)}_0\), taken with some special initial distribution \(\mu_0\) which will be specified in the body of this paper. Let \(u_j\) be a family of bounded functions (uniformly in \(j\)) and consider the perturbations \(R^{(j)}_z\) of \(R^{(j)}_0\) given by \(R^{(j)}_z(g) = R^{(j)}_z(ge^{zu_j})\). In this paper we will apply contraction properties of complex Hilbert metrics with the above parametric families \(\mathcal{L}^{(j)}_z, j \in \mathbb{Z}\) and \(R^{(j)}_z, j \in \mathbb{Z}\), and obtain certain sequential RPF theorems for these operators. In both cases, several limit theorems (finer than the CLT) for sequence of random variables of either the form \(S_{0,n}u(x_0) = \sum_{j=0}^{n−1} u_j \circ T_j \circ (x_0)\) or the form \(S_{0,n} = \sum_{j=1}^{n} u_j(\xi_j)\) will follow (such as Berry-Esseen type theorems, local central limit theorems, exponential deviation inequalities, moderate and large deviations theorems and results concerning the behavior of the moments). We will also study certain stability properties of the RPF triplets, which will yield that the variance of the above random variables grow linearly fast in \(n\), when the \(\mathcal{L}^{(j)}_z\)'s (or the \(R^{(j)}_z\)'s) lie in some neighborhood of a single Transfer (or Markov) operator, when \(z\) lies in some neighborhood of 0.

Several difficulties arise when the operators do not have the form \(\mathcal{L}^{(j)}_z = \mathcal{L}^{\theta^j} \omega\) or \(R^{(j)}_z = R^{\theta^j} \omega\) for an appropriate measure preserving system \((\Omega, \mathcal{F}, P, \theta)\), namely when we are not in the setup of random dynamical environment. For instance, the study of the growth of the variances of \(S_{0,n}u(x_0)\) and \(S_{0,n}\) in the latter case is
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3 reduced to the study of certain some ergodic averages (see [25]), and more generally existence of the asymptotic moment \( \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(S_{0,n} - \mathbb{E}S_{0,n})^k \) (and the relation between them, see [19]), and of the logarithmic moment generating functions

\[
\Gamma(z) = \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}e^{zS_{0,n}}
\]

around \( z_0 = 0 \), follow in the random dynamical environment case from the mean ergodic theorem, applied with some random pressure functions \( \Pi_\omega(z_j) \). It also follows that \( \Gamma(\cdot) \) is analytic, and as a consequence of the Gartner-Ellis theorem (see [10]), a (local) large deviations principle and a moderate deviations principle with optimal scale follow. In our case we have a sequence of pressure functions \( \Pi_j(z), j \in \mathbb{Z} \) and the limits

\[
\Pi(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Pi_j(z)
\]

do not necessarily exist. When \( T_j = T_{\xi_j}, f_j = f_{\xi_j} \) and \( u_j = u_{\xi_j} \) are chosen at random according to several classes of non-stationary processes with certain mixing properties, we will show that the limits \( \Pi(z) \) exist and are analytic functions of \( z \), and then use that in order to apply some of our general results.

This paper is organized as follow. In Section 2 we will describe our SDS setup and formulate our general results. In Section 3 we will prove the RPF theorem discussed above, a stability theorem and a theorem which concerns the case when all of the \( T_j \)'s are non-singular. Section 4 is devoted to the proof of the limit theorems stated in Section 2, while Section 5 is devoted to special applications in the case when \( L^{(j)}_z = L^{(\xi_j)}_z \) is chosen at random according to some classes of non-stationary sequences \( \xi_0, \xi_1, \xi_2, \ldots \) of random variables. In Section 6 we will describe our results concerning inhomogeneous Markov chains.

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2. Sequential dynamical systems: preliminaries and main results

Our setup consists of a compact metric space \( (X,d) \), a family \( \{\mathcal{E}_j : j \in \mathbb{Z}\} \) of compact subsets of \( X \), and a family \( T_j : \mathcal{E}_j \to \mathcal{E}_{j+1}, j \in \mathbb{Z} \) of continuous and surjective maps. For any \( j \in \mathbb{Z} \) and \( n \geq 1 \) put

\[
T_j^n = T_{j+n-1} \circ \cdots \circ T_{j+1} \circ T_j
\]

Our additional requirements from these maps are collected in the following

2.1. Assumption. There exist constants \( \xi > 0, \gamma > 1, L, n_0 \in \mathbb{N}, \) and \( D > 0 \) so that for any \( j \in \mathbb{Z} \):

(i) For any \( x \in \mathcal{E}_j \),

\[
T_j^{n_0} B_j(x, \xi) = \mathcal{E}_{j+n_0}
\]

where \( B_j(x, \xi) := \{w \in \mathcal{E}_j : d(w, x) < \xi\} \).

(ii) For any \( x, x' \in \mathcal{E}_{j+1} \) so that \( d(x, x') < \xi \) we can write

\[
T_j^{-1} \{x\} = \{y_1, ..., y_d\} \quad \text{and} \quad T_j^{-1} \{x'\} = \{y'_1, ..., y'_d\}
\]
where $d \leq D$ and for each $i = 1, 2, 3, \ldots, d$, 
\[ d(g_i, g_i') \leq \gamma^{-1}d(x, x'). \]

(iii) There are $x_{1,j}, x_{2,j}, \ldots, x_{L,j}, L \leq L$ in $E_j$ so that
\[ E_j = \bigcup_{s=1}^{L} B_j(x_{s,j}, \xi). \]

Note that the arguments in [30] show that Assumption 2.1 is satisfied when the maps $T_j$ are locally distances expanding, uniformly in $j$.

Next, for each integer $j$, let $f_j, u_j : E_j \to \mathbb{R}$ be Hölder continuous functions with exponent $\alpha$ which does not depend on $j$. For each integer $j$ and a complex number $z$, let $L_z^{(j)}$ be the linear operator which maps complex valued functions $g$ on $E_j$ to complex valued functions $L_z^{(j)}g$ on $E_{j+1}$ by the formula
\[ L_z^{(j)}g(x) = \sum_{y \in T_j^{-1}(x)} e^{f_j(y) + zu_j(y)}g(y). \]

For each $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ set
\[ L_z^{j,n} = L_z^{(j+n-1)} \circ \cdots \circ L_z^{(j+1)} \circ L_z^{(j)}. \]

Let $H_j$ be the (Banach) space of all Hölder continuous functions $g : E_j \to \mathbb{C}$ with exponent $\alpha$, equipped with the norm $\|g\| = \|g\|_\alpha = \|g\|_\infty + \alpha(g)$, where $\|g\|_\infty = \sup |g|$ and
\[ \alpha(g) = \alpha_{\xi}(g) = \sup \left\{ \frac{|g(x) - g(x')|}{d^\alpha(x, x')} : 0 < d(x, x') \leq \xi \right\}. \]

We will denote here by $H_j^*$ the dual of the Banach space $H_j$. Finally, for any family of functions $\{g_j : E_j \to \mathbb{C} : j \in \mathbb{Z}\}$ we set for each integer $j$ and $n \geq 1$,
\[ S_{n,j}g = \sum_{k=j}^{j+n-1} g_k \circ T_j^k. \]

We will work in this paper under the following

2.2. Assumption. The norms $\|f_j\|_\alpha$ and $\|u_j\|_\alpha$ are bounded in $j$ by some constant $B$.

From now on, we will refer to the constants $B$, $L$, $\xi$, $\gamma$, $n_0$, $D$ and $\alpha$ appearing in Assumptions 2.1 and 2.2 as the “initial parameters”.

The following result summarises several important properties of the operators $L_z^{(j)}$, and its proof proceeds exactly as in Chapter 2.2 of [16].

2.3. Theorem. Suppose that Assumptions 2.1 and 2.2 hold true. Then the following Lasota-Yorke type inequality holds true: for any $n \in \mathbb{N}$, $j \in \mathbb{Z}$, $z \in \mathbb{C}$ and $g \in H_j$,
\[ \alpha_n(L_z^{j,n}g) \leq \|L_z^{j,n}1\|_\infty e^{\|z\|_1 \|S_j, u\|_\infty} \cdot (\alpha_n(g) \gamma^{-\alpha n} + A(1 + \|z\|_1) \|g\|_\infty) \]
where $A = \frac{-2B}{\|z\|_1}$, $\|z\|_1 = \|\Re(z)\| + \|\Im(z)\|$ and $\Re(z)$ ($\Im(z)$) is the real (imaginary) part of $z$. As a consequence, $L_z^{j,n}H_j \subset H_{j+n}$ and the corresponding operator norm satisfies
\[ \|L_z^{j,n}g\|_\alpha \leq \|L_z^{j,n}1\|_\infty e^{\|z\|_1 \|S_j, u\|_\infty} \cdot (\alpha_n(g) \gamma^{-\alpha n} + (1 + A)(1 + \|z\|_1) \|g\|_\infty). \]

In, particular, $L_z^{j,n}|_{H_j} : H_j \to H_{j+n}$ is a continuous linear map.
Moreover, $\mathcal{L}_z^{(j)}$'s are analytic in $z$ when considered as maps from $\mathbb{C}$ to the Banach space of all bounded linear maps between $\mathcal{H}_j$ and $\mathcal{H}_{j+1}$, equipped with the operator norm $\| \cdot \|$. Furthermore, for any $g \in \mathcal{H}_j$, $z \in \mathbb{C}$ and $k \geq 1$,

\[
\frac{d^k \mathcal{L}_z^{(j)}}{dz^k}(g) = \mathcal{L}_z^{(j)}(u^k_j g).
\]

where $u^k_j(y) = (u_j(y))^k$.

Under Assumptions 2.1 and 2.2 we will denote here the dual of the map $\mathcal{L}_z^{(j)}|_{\mathcal{H}_j} : \mathcal{H}_j \to \mathcal{H}_{j+n}$ by $(\mathcal{L}_z^{(j)})^*$. Our next result is the following sequential Ruelle-Perron-Frobenius theorem:

2.4. **Theorem.** Suppose that Assumptions 2.1 and 2.2 hold true. Then there exists a neighborhood $U$ of 0, which depends only on the initial parameters, so that for any $z \in U$ there exist families $\{\lambda_j(z) : j \in \mathbb{Z}\}$, $\{h_j^{(z)} : j \in \mathbb{Z}\}$ and $\{\nu_j^{(z)} : j \in \mathbb{Z}\}$ consisting of a nonzero complex number $\lambda_j(z)$, a complex function $h_j^{(z)} \in \mathcal{H}_j$ and a complex continuous linear functional $\nu_j^{(z)} \in \mathcal{H}_j^*$ such that:

(i) For any $j \in \mathbb{Z}$,

\[
\mathcal{L}_z^{(j)}h_j^{(z)} = \lambda_j(z)h_{j+1}^{(z)}, \quad (\mathcal{L}_z^{(j)})^*\nu_j^{(z)} = \lambda_j(z)\nu_j^{(z)} \quad \text{and} \quad \nu_j^{(z)}(h_j^{(z)}) = \nu_j^{(z)}(1) = 1
\]

where 1 is the function which takes the constant value 1. When $z = t \in \mathbb{R}$ then $\lambda_j(t) > a$ and the function $h_j(t)$ takes values at some interval $[c, d]$, where $a > 0$ and $0 < c < d < \infty$ depend only on the initial parameters. Moreover, $\nu_j^{(t)}$ is a probability measure which assigns positive mass to open subsets of $E_j$ and the equality $\nu_{j+1}(t)(\mathcal{L}_z^{(j)} g) = \lambda_j(t)\nu_j^{(t)}(g)$ holds true for any bounded Borel function $g : E_j \to \mathbb{C}$.

(ii) The maps

$\lambda_j(\cdot) : U \to \mathbb{C}$, $h_j^{(\cdot)} : U \to \mathcal{H}_j$ and $\nu_j^{(\cdot)} : U \to \mathcal{H}_j^*$

are analytic and there exists a constant $C > 0$, which depends only on the initial parameters such that

\[
\max \left( \sup_{z \in U} |\lambda_j(z)|, \sup_{z \in U} \|h_j^{(z)}\|, \sup_{z \in U} \|\nu_j^{(z)}\| \right) \leq C,
\]

where $\|\nu\|_\alpha$ is the operator norm of a linear functional $\nu : \mathcal{H}_j \to \mathbb{C}$. Moreover, there exist a constant $c > 0$, which depends only on the initial parameters, so that $|\lambda_j(z)| \geq c$ and $\min_{x \in \mathcal{H}_j} |h_j^{(z)}(x)| \geq c$ for any integer $j$ and $z \in U$.

(iii) There exist constants $A > 0$ and $\delta \in (0, 1)$, which depend only on the initial parameters, so that for any $j \in \mathbb{Z}$, $g \in \mathcal{H}_j$ and $n \geq 1$,

\[
\left\| \frac{\mathcal{L}_z^{(j)} g}{\lambda_j^{(n)}(z)} - \nu_j^{(z)}(g)h_j^{(n)} \right\|_\alpha \leq A\|g\|_\alpha \delta^n
\]

where $\lambda_j^{(n)}(z) = \lambda_j(z) \cdot \lambda_{j+1}(z) \cdots \lambda_{j+n-1}(z)$. Moreover, the probability measures $\mu_j$, $j \in \mathbb{Z}$ given by $d\mu_j = h_j^{(0)} d\nu_j^{(0)}$, satisfy that $(T_j)_* \mu_j = \mu_{j+1}$ and that for any $n \geq 1$ and $f \in \mathcal{H}_{j+n}$,

\[
|\mu_j(g \cdot T^n_j) - \mu_j(g)\mu_{j+n}(f)| \leq A\|g\|_\alpha \mu_{j+n}(|f|) \delta^n.
\]
The proof of Theorem 2.4 proceeds essentially as in Chapters 4 and 5 of [10], as explained in Section 3. We note that the measures $\mu_j$ and $\nu_j^{(l)}$ satisfy a certain type of Gibbs property, which provides explicit positive lower bounds for the $\mu_j$ and $\nu_j^{(l)}$ measures of open sets. This result is described in Section 3.6.

When the $\{T_j\}$ are "sequentially non-singular" the measures $\mu_j$ are absolutely continuous, as stated in the following

2.5. Theorem. Let $m_j$, $j \in \mathbb{Z}$ be a family of probability measures on $\mathcal{E}_j$, which assign positive mass to open sets, so that for each $j$ we have $(T_j)_* m_j \ll m_{j+1}$ and that $e^{-f_j} = \frac{d(T_j)_* m_j}{d m_{j+1}}$. Then for any $j$ we have $\lambda_j(0) = 1$ and $\nu_j^{(0)} = m_j$.

When $\mathcal{E}_j = X$ we can always take $m_j = m$ for some fixed $m$ (e.g. a volume measure when $X$ is a Riemannian manifold).

The fact that the constants appearing in Theorem 2.4 depend only on the initial parameters $B$, $L$, $\xi$, $\gamma$, $n_0$, $D$ and $\alpha$ yields the following sequential stability:

2.6. Theorem. Let $r > 0$ be so that $B(0, 2r) = \{ z \in \mathbb{Z} : |z| \leq 2r \} \subset U$ and set $K = B(0, r)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $T_{1,j}, j \in \mathbb{Z}$ is a family of maps satisfying Assumption 2.1 (with the same constants) and $f_{1,j}, u_{1,j}$, $j \in \mathbb{Z}$ are families of $\alpha$-Hölder function whose $\|\cdot\|_\alpha$ norms are bounded by $B$, and for any $z \in K$ and $j \in \mathbb{Z}$,

$$\| L_z^{(j)} - L_{1,z}^{(j)} \|_\alpha < \delta$$

where $L_z^{(j)}$ is the operator defined similarly to $L_z^{(j)}$ but with $T_{1,j}, f_{1,j}$ and $u_{1,j}$ in place of $T_j, f_j$ and $u_j$, then for any integer $j$ and $z \in K$ we have

$$\max (|\lambda_j(z) - \lambda_{1,j}(z)|, \| h_j^{(z)} - h_{1,j}^{(z)} \|_\alpha, \| \nu_j^{(z)} - \nu_{1,j}^{(z)} \|_\alpha) < \varepsilon$$

where $\{\lambda_{1,j}(z) : j \in \mathbb{Z}\}, \{h_{1,j}^{(z)} : j \in \mathbb{Z}\}$ and $\{\nu_{1,j}^{(z)} : j \in \mathbb{Z}\}$, $z \in U$ are the RPF families corresponding to the operators $L_z^{(j)}$.

2.1. Probabilistic limit theorems. We begin with the variances:

2.7. Theorem. (i) The variances $\text{var}_\mu(S_{k,n}u), k \in \mathbb{Z}$ do not converge to $\infty$ as $n \to \infty$ if and only if there exists a family of functions $Y_s : \mathcal{E}_s \to \mathbb{R}$, $s \in \mathbb{Z}$ and a constant $C > 0$ so that for any $s \in \mathbb{Z}$ we have

$$\int Y^2_s(x) d\mu_s(x) < C \quad \text{and} \quad u_s - \mu_s(u_s) = Y_{s+1} \circ T_s - Y_s, \mu_s - a.s.$$  

When viewed as a random variable, the function $Y_k$ is a member of the subspace of $L^2(\mathcal{E}_k, \mu_k)$ generated by the functions $\{u_k \circ T_k^d - \mu_k(u_k) : d \geq 0\}$ where $T_k^0 := \text{Id}$. Moreover, the functions $Y_k$ can be chosen to be Hölder continuous so that the norms $\|Y_k\|_\alpha$ are bounded in $k$, and in this case the equalities $u_s(x) - \mu_s(u_s) = Y_{s+1}(T_s x) - Y_s(x)$ hold true for any $x \in \mathcal{E}_s$.

(ii) Suppose that $\mathcal{E}_k = X$ for each $k$. Let $T : X \to X$ be a map so that Assumption 2.1 is satisfied with $T_j = T$, and let $u, f : X \to \mathbb{R}$ be Hölder continuous functions with exponent $\alpha$, so that $\|f\|_\alpha$ and $\|u\|_\alpha$ do not exceed $B$, and $u$ does not admit a co-boundary representation with respect to $T$. Then there exists $\varepsilon_0 > 0$, which depends only on the initial parameters, so that the following holds true: if

$$\sup_{k \in \mathbb{Z}} \| L_z^{(k)} - L_z \| \leq \varepsilon_0$$
for any $z$ in some neighborhood of 0, where $L_z$ is the transfer operator generated by $T$ and $f + zu$, then
\[
\inf_k \text{var}_{\mu_k} (S_{k,n} u) \geq \delta_0 n
\]
for some $\delta_0 > 0$ and all sufficiently large $n$.

Theorem 2.7 (ii) is a consequence of Theorem 2.6 together with the analyticity of the RPF triplets and Theorem 2.9 (iii) below, applied with $k = 2$.

Next, for any $n \geq 1$ set
\[
\sigma_{0,n} = \sqrt{\text{var}_{\mu_0} (S_{0,n} u)}
\]
and let $x_0$ be a $\mathcal{E}_0$-valued random variable which is distributed according to $\mu_0$. Then, relying on (2.5) and the arguments in [8], it follows that $S_{0,n} u(x_0) - \mu_0(S_{0,n} u)$ converges in distribution towards the standard normal law, when $\sigma_{0,n}^2$ converges to $\infty$ as $n \to \infty$ (this essentially means that the quadratic variations of the martingales constructed in the proof of Theorem 2.12 converge as $n \to \infty$, after a proper normalization). When the variances grow faster than $n^{\frac{3}{4}}$ then we are able to prove a self normalized Berry-Esseen theorem:

2.8. **Theorem.** Suppose that
\[
\lim_{n \to \infty} \sigma_{0,n} n^{-\frac{3}{4}} = \infty
\]
and set $S_{0,n} u = S_{0,n} u - \mu_0(S_{0,n} u)$. Then there exists a constant $C > 0$ so that for any $n \geq 1$ and $r \in \mathbb{R}$,
\[
(2.7) \quad \left| \mu_0 \{ x \in \mathcal{E}_0 : S_{0,n} u(x) \leq r \sigma_{0,n} \} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} e^{-t^2/2} dt \right| \leq C n \sigma_{0,n}^{-3}.
\]
In particular, when $\sigma_{0,n}^2$ grows linearly fast in $n$ the above left hand side does not exceed $C_1 n^{-\frac{1}{2}}$ for some constant $C_1$.

Note that we obtain here optimal convergence rate (i.e. rate of order $n^{-\frac{1}{2}}$) in the circumstances of Theorem 2.7 (ii).

Our next result is a theorem which describes the behavior of the moments of $S_{j,n} u$:

2.9. **Theorem.** (i) By possibly decreasing $r$, where $r$ comes from Theorem 2.6, we can define analytic functions $\Pi_j : B(0,r) \to \mathbb{C}$, $j \in \mathbb{Z}$ so that
\[
\Pi_j(0) = 0, \quad \lambda_j(z)/\lambda_j(0) = e^{\Pi_j(z)} \quad \text{and} \quad |\Pi_j(z)| \leq c_0
\]
for any $z \in B(0,r)$, for some constant $c_0$ which does not depend on $j$ and $z$.

(ii) There exists a constant $R_1$ so that for any integer $j$ and $n \geq 1$,
\[
|\mu_j(S_{j,n} u) - \sum_{m=0}^{n-1} \Pi_{j+m}'(0)| \leq R_1
\]
where $\Pi_{j+m}'(0)$ is the derivative of $\Pi_{j+m}$ at $z = 0$.

(iii) Suppose that $\mu_j(u_j) = 0$ for any integer $j$. For any $k \geq 2$, $s \in \mathbb{Z}$ and $n \geq 1$, set
\[
\gamma_{j,k,n} = n^{-\frac{|k|}{2}} \int_{\mathcal{E}_j} (S_{j,n} u(x))^k d\mu_j(x) \quad \text{and} \quad \Pi_{j,k,n} = n^{-1} \sum_{m=0}^{n-1} \Pi_{j+m}'(0)
\]
where $\Pi^{(k)}_{c+n}(0)$ is the $k$-th derivative of the function $\Pi_{c+n}$ at $z = 0$. Then there exist constants $R_k > 0$, $k \geq 2$, so that for any even $k \geq 2$,

$$
\max \left( |\gamma_{j,k,n} - C_k(\gamma_{j,2,n})|, |\gamma_{j,k,n} - C_k(\Pi_{j,2,n})| \right) \leq \frac{R_k}{n}
$$

where $C_k = 2^{-\frac{k}{2}(k-3)!}k!$, while with $D_k = \frac{4k}{\pi^2}2^{-\frac{k}{2}(k-3)!}(k-3)!^{-1}$ for any odd $k \geq 3$,

$$
\max \left( |\gamma_{j,k,n} - D_k(\gamma_{j,2,n})|, |\gamma_{j,k,n} - D_k(\Pi_{j,2,n})| \right) \leq \frac{R_k}{n}.
$$

When $T_j, f_j, u_j$ are chosen at random according to some type of (not necessarily stationary) sequence of random variables, then, in Section 5 we obtain almost sure converges rate of the form

$$
\left| \frac{1}{n} \mu_j(S_{j,n}u) - p \right| \leq \frac{R_{1,\omega}}{n}
$$

where $p$ is some constant and $R_{1,\omega}$ is some random variable. When $\mu_j(u_j) = 0$ we also derive that for any $k \geq 2$,

$$
(2.8) \quad |\gamma_{j,k,n} - \gamma_k| \leq R_{\omega,k}(\ln n)^\frac{3}{2}n^{-\frac{1}{2}}
$$

where $R_{\omega,k}$ is some random variable and $\gamma_k$ is some constant. Using (2.8) with $k = 2$, when $\sigma^2 = \gamma_2 > 0$ we also derive in Section 5.3.4 almost optimal convergence rate in the central limit theorem of the form

$$
(2.9) \quad \sup_{s \in \mathbb{R}} |\mu_0\{x \in \mathcal{E}_0 : S_{j,n}u(x) \leq s\sqrt{n}\} - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt| \leq c_{\omega}(\ln n)^{3n^{-\frac{1}{2}}}.\]

Next, as usual, in order to present the local central limit theorem we will distinguish between two cases. We will call the case a lattice one if the functions $u_j$ take values at some lattice of the form $h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$ for some $h > 0$. We will call the case a non-lattice if there exist no $h$ which satisfy the latter lattice condition. In the non-lattice case set $h = 0$ and $I_h = \mathbb{R} \setminus \{0\}$, while in the lattice case set $I_h = \{-\frac{2\pi}{h}, \frac{2\pi}{h}\} \setminus \{0\}$.

2.10. Theorem. Suppose that for any compact interval $J \subset I_h$ we have

$$
(2.10) \quad \lim_{n \to \infty} \sqrt{n} \sup_{t \in J} \|L_{t,n}^{0,0}\|_a = 0
$$

and that there exists $c_0 > 0$ so that $\sigma_{\alpha,0}^2 = \text{var}_{\mu_0}(S_{0,n}u) \geq c_0 n$ for any sufficiently large $n$. Then for any continuous function $g : \mathbb{R} \to \mathbb{R}$ with compact support we have

$$
\lim_{n \to \infty} \sup_{r \in R_h} \left| \sqrt{2\pi\sigma_{\alpha,0}^2} \int g(S_{0,n}u(x) - \mu_0(S_{0,n}u)) - r) d\mu_0(x) - \left( \int_{-\infty}^{\infty} g(t) dm_h(t) \right) e^{-\frac{t^2}{2\sigma_{\alpha,0}^2}} \right| = 0
$$

where in the non-lattice case $R_h = \mathbb{R}$ and $m_h$ is the Lebesgue measure, while in the lattice case $R_h = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ and $m_h$ is the measure assigning unit mass to each one of the members of $R_h$. When the limit $\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \sigma_{\alpha,0}^2$ exists then we can replace $\sigma_{\alpha,0}^2$ with $\sigma^2$.

Next, let $\mathcal{Y}_1, ..., \mathcal{Y}_{m_0}$ be compact metric spaces, set $\mathcal{Y}_{m_0+1} := \mathcal{Y}_1$ and let $S_{j_1} : \mathcal{Y}_1 \to \mathcal{Y}_{j_1+1}$, $j = 1, 2, ..., m_0$ be maps so that Assumption 2.1 holds true with $\mathcal{E}_j = \mathcal{Y}_{j_1}$ and $T_j = S_{j_1}$ in place of $E_j$ and $T_j$, respectively, where $k = k_j = j \mod m_0$. Let $r_i, v_i : \mathcal{Y}_i \to \mathbb{R}$ be Hölder continuous functions whose $\| \cdot \|_a$ norms are bounded.
by $B$. Set $S = S_{m_0} \circ \cdots \circ S_2 \circ S_1$ and for each $t$ let the transfer operator $L_{it}$ be defined by 

$$L_{it}g(x) = \sum_{y \in S^{-1}(x)} e^{\sum_{j=1}^{m_0} v_j(y) + it \sum_{j=1}^{m_0} v_j(y) g(y)}.$$ 

In Section 4.4 we will show that (2.10) holds true under the following exponential concentration (deviation) inequalities:

$$\epsilon > h = \max \alpha 
\text{to this function (see [15] and [20]).}$$

The following moderate deviations principle also holds true:

$$\lim_{n \to \infty} \frac{|\{0 \leq m < n : B_J(L^{m,m_0}_t - L^t_m) > \alpha < 1 - \delta_0 \ \forall t \in J\}|}{\ln n} = \infty$$

where $|\Gamma|$ is the cardinality of a finite set $\Gamma$, $B_J = 6(1 + \frac{2B}{1-\gamma^2})(1 + M_J)$ and $M_J = \max \{|t| : t \in J\}$. 

Note that in Assumption 2.11 there is an underlying assumption that $\mathcal{E}_m = \mathcal{V}_1$ for any $m$ so that $B_J(L^{m,m_0}_t - L^t_m) > \alpha < 1 - \delta_0$. In the non-lattice case, the condition about the spectral radius of $L_{it}$ means that the function $\sum_{j=1}^{m_0} v_j \circ S_{j-1} \circ \cdots \circ S_2 \circ S_1$ is non-arithmetic (or aperiodic) with respect to the map $S$ defined above, while in the lattice case it means that $h$ satisfies a certain maximality condition with respect to this function (see [15] and [20]).

Assumption 2.11 holds true when $T_k, f_k, u_k$ are chosen at random according to some (not necessarily stationary) sequence of random variables, see Section 5 and a discussion at the end of this Section. Non random examples can be constructed as follows: assume that

$$\|L^{a_k+m_0,m_0}_t - L^t_m\| \leq \delta_k \alpha(t)$$

for any sufficiently large $k$ and $0 \leq i \leq m_0 (a_{k+1} - a_k - m_0)$, for some sequence $\delta_k$ which converges to 0 as $k \to \infty$ and a continuous function $\alpha(t)$, where $(a_k)_{k=1}^\infty$ is a sequence of natural numbers so that $\lim_{k \to \infty} (a_{k+1} - a_k) = \infty$ and $a_k \leq c_1 e^{c_2 k^r}$ for some $r \in (0, 1)$ and $c_1, c_2 > 0$ and all natural $k$'s.

Now we will discuss several large deviations type results. We begin with the following exponential concentration (deviation) inequalities:

2.12. Theorem. There exist constants $C, C_1 > 0$, which depend only on the initial parameters, so that for any natural $n$ there is a martingale $\{M^{(n)}_j = W^{(n)}_1 + \cdots + W^{(n)}_j : j \geq 1\}$ whose differences $W^{(n)}_j$ are bounded by $C$, and

$$\|S_{0,n}u(x_0) - \mathbb{E}[S_{0,n}u(x_0)] - M^{(n)}_n\|_{L^\infty} \leq C_1$$

where $x_0$ is a random member of $\mathcal{E}_0$ which is distributed according to $\mu_0$. Moreover, for any $t \geq 0$ we have

$$\mu_0\{x \in \mathcal{X} : |S_{0,n}u(x) - \mu_0(S_{0,n}u)| \geq t + C_1\} \leq 2e^{-\frac{t^2}{2C_1}}.$$ 

By taking $t = \epsilon n$ we obtain estimates of the form

$$\mu_0\{x \in \mathcal{X} : |S_{0,n}u(x) - \mu_0(S_{0,n}u)| \geq \epsilon n\} \leq 2e^{-\frac{\epsilon^2 n^2}{2C_1}}$$

for any $\epsilon > 0$ and $n \geq 1$ so that $\epsilon n \geq 2C_1$.

The following moderate deviations principle also holds true:
2.13. **Theorem.** Set $\sigma_n^2 := \text{var}_{\mu_n}(S_{0,n,u})$ and suppose that

$$\lim_{n \to \infty} \frac{\sigma_n}{\hat{n}^{1/2} + \varepsilon_0} = \infty$$

for some $0 < \varepsilon_0 < \frac{1}{6}$. Let $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of real numbers so that $\lim_{n \to \infty} a_n = \infty$ and $\lim_{n \to \infty} n^{-\varepsilon_0} a_n = 0$, and set $W_n = \frac{S_{0,n,u} - \mu_0(S_{0,n,u})}{\sigma_n a_n}$. Then for any Borel set $\Gamma \subset \mathbb{R}$,

$$- \inf_{x \in \Gamma} I(x) \leq \liminf_{n \to \infty} \frac{1}{a_n^2} \mu_0 \{ x : W_n(x) \in \Gamma \}$$

and

$$\limsup_{n \to \infty} \frac{1}{a_n^2} \mu_0 \{ x : W_n(x) \in \Gamma \} \leq - \inf_{x \in \Gamma} I(x)$$

where $I(x) = - \frac{x^2}{2}$, $\Gamma^o$ is the interior of $\Gamma$ and $\bar{\Gamma}$ is its closer.

The scaling sequence $(a_n)_{n=1}^{\infty}$ in theorem 2.13 is not optimal even when $\sigma_n^2$ grows linearly fast in $n$. In the following circumstances we can also derive more accurate moderate deviations principle together with a local large deviations principle:

2.14. **Theorem.** Suppose that for some $\delta > 0$ for any $z \in B(0, \delta)$ the following limit

$$\Pi(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Pi_j(z)$$

exists. Then $\Pi(z)$ is analytic on $z \in B(0, 2\delta)$ and we have

$$\Pi(z) = \lim_{n \to \infty} \frac{1}{n} \ln \mu_0(e^{zS_{0,n,u}}) = \lim_{n \to \infty} \frac{1}{n} \ln l_n(e^{z_0,n})$$

where $l_n(g) = \sum_{i=1}^{L_n} g(x_{i,n})$ for any $g : \mathcal{E}_n \to \mathbb{C}$, and the $x_{i,n}$’s come from Assumption 2.7 (iii). Moreover:

(i) The limit

$$p = \lim_{n \to \infty} \frac{1}{n} \int S_{j,n}(x) \mu_j(x)$$

exists, and it does not depend on $j$, and when $\mu_j(u_j) = 0$ for each $j$ the centralized asymptotic moments (defined in Theorem 2.9) $\gamma_k = \lim_{n \to \infty} \gamma_{j,k,n}$ exist and they do not depend on $j$. Furthermore, with $\sigma^2 = \gamma_2$ we have $\gamma_k = C_k \sigma^k$ for even $k$’s, while for odd $k$’s we have $\gamma_k = D_k \sigma^{k-1} \gamma_3$, where the $C_k$’s the $D_k$’s are defined in Theorem 2.6. In addition,

$$p = \Pi'(0), \gamma_2 = \sigma^2 = \Pi''(0) \text{ and } \gamma_3 = \Pi'''(0).$$

In particular, in the circumstances of Theorem 2.7 (ii) we have $\sigma^2 > 0$.

(ii) Suppose that $\sigma^2 > 0$. Then for any strictly increasing sequence $(b_n)_{n=1}^{\infty}$ of real numbers so that $\lim_{n \to \infty} \frac{b_n}{n} = 0$ and $\lim_{n \to \infty} \frac{b_n}{\sqrt{n}} = \infty$ and a Borel set $\Gamma \subset \mathbb{R}$ 2.13 holds true with

$$W_n = \frac{S_{0,n,u} - \mu_0(S_{0,n,u})}{b_n}$$

and $I(x) = - \frac{x^2}{2}$.

(iii) Let $L(t)$ be the Legendre transform of $\Pi(t)$. Then, 2.13 holds true for any Borel set $\Gamma \subset [\Pi'(-\delta), \Pi'(\delta)]$ with $W_n = \frac{S_{0,n,u} - \mu_0(S_{0,n,u})}{n}$ and $I(t) = L(t)$. 

Note that $\Pi'(-\delta) < \Pi'(\delta)$ when $\sigma^2 > 0$ since then the function $t \to \Pi(t)$ is strictly convex in some real neighbourhood of the origin. Theorem 2.14 (ii) is a moderate deviations principle (i.e. with quadratic rate function $I(x)$) which allows scaling sequences $(b_n)_{n=1}^\infty$ of optimal order, as can be viewed from Theorem 2.14 (iii), which is a local large deviations principle. Note also that the equality

$$\Pi(z) = \lim_{n \to \infty} \frac{1}{n} \ln l_n(C_{x,n}^0)$$

can be interpreted for real $z$'s as a sequential analogy of the usual pressure function of the potential $f + zu = f_j + u_j$, in the case of a single map $T$ and functions $f$ and $u$ (see [3] in the subshift case). Remark also that, in fact, our proof shows that if one of the limits in (2.13) exists, then all of them exists and they are equal.

The limits $\Pi(z)$ does not seem to exist in general, not even for a single $z$. When $T_k, f_k, u_k$ are chosen at random according to some (not necessarily stationary) sequence of random variables, we provide in Section 5 quite general conditions guaranteeing that Assumption 2.11 holds true and that the limits $\Pi(z)$ exist. In particular, in the circumstances considered in Section 5 the limit $\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \text{var}_n, (S_j, n, u)$ exist and it does not depend on $j$. In the circumstances of Theorem 2.7 (ii), we will derive that $\sigma^2 > 0$. Under certain circumstances, we will also show that (2.8) holds true, and then derive (2.9).

3. Sequential RPF theorem

The proof of Theorem 2.4 goes exactly as the proof of Corollary 5.4.2 in [16], if we replace $L^p(\omega, n)$ from there by $L^p_{\infty}$. Still, in the proofs of some of the results stated in Section 2 we will rely on the explicit limiting expressions of the RPF triplets $\lambda_j(z), h_j^{(z)}$ and $\nu_j^{(z)}$. Therefore, for readers’ convenience we will repeat here some of the arguments appearing in [16]. The arguments in [16] rely on the theory of complex cones and their contraction properties with respect to certain complex (Hilbert) projective metrics, which was developed in [35] and [14]. We refer the readers to Appendix A in [16] for a collection of all the results that will be used in the proof of Theorem 2.4.

3.1. Real and complex cones and protective contraction. In what follows, when it is more convenient, we will denote the norm $|| \cdot ||_\alpha$ also by $|| \cdot ||$ (i.e. we will omit the subscript $\alpha$). Consider the real cones $C_{j,R} \subset H_j$ given by

$$C_{j,R} = \{ g \in H_j : g \geq 0 \text{ and } g(x) \leq e^{Q \varphi(x, x')} \text{ if } d(x, x') < \xi \}$$

where $Q = \frac{B}{1 - \gamma - \tau}$ and $s > 1$ is an arbitrary number. Let $C_j$ the canonical complexification of $C_{j,R}$ which is given by

$$C_j = \{ g \in H_j : \mu(g) \nu(g), \forall \mu, \nu \in C^*_{j,R} \}$$

where $C^*_{j,R} = \{ \mu \in H^*_j : \mu(g) > 0, \forall g \in C_{j,R}, g \neq 0 \}$ is the dual of the real cone $C_{j,R}$. The cone $C_j$ (see [35] and [14]) can also be written as

$$C_j = C'(C_j + iC_j) = C' \{ x + iy : x \pm y \in C_{j,R} \}$$

where $C' = C \setminus \{0\}$. The dual $C^*_j$ of the complex cone $C_j$ is defined by

$$C^*_j = \{ \mu \in H^*_j : \mu(g) \neq 0; \forall g \in C^*_j \}$$

where $C'_j = C_j \setminus \{0\}$. We refer the readers to Appendix A in [16] for a collection of additional definitions, properties and results concerning complex cones (which
come from [35] and [14]). Next, let the linear functional \( l_j \) be defined on the space of all complex functions on \( \mathcal{E}_j \) by the formula
\[
 l_j(g) = \sum_{i=1}^{L_j} g(x_{i,j})
\]
where \( x_{1,j}, \ldots, x_{L_j,j} \) are specified in Assumption 2.1 (iii). Note that \( L_j = \|l_j\|_\alpha = \|l_j\|_\infty \), where \( \|L_j\|_\infty \) is the operator norm of \( l_j \) with respect to the supremum norm, and \( \|l_j\|_\alpha \) is its norm with respect to the norm \( \| \cdot \|_\alpha \).

The following theorems summarize the properties of the cones \( C_{j,R}, C_j \) and the relations between them and the operators \( \mathcal{L}^{(l)}_z \), and their proofs proceed exactly as in Chapter 5 of [16].

3.1. **Theorem.** Suppose that Assumptions 2.1 and 2.2 hold true. Then for each integer \( j \):

(i) The cone \( C_{j,R} \) is a closed subset of the Banach space of real valued functions in \( \mathcal{H}_j \). Moreover, the complex cone \( C_j \) is linearly convex.

(ii) Set \( K = 2\sqrt{2}L(sQe^{sQ\xi^\alpha} + 1)e^{sQ\xi^\alpha} \) and \( M = 8(1 - e^{-sQ\xi^\alpha})^{-2} \). Then
\[
\|g\|_\alpha \|l_j\|_\alpha \leq K |l_j(g)| \quad \text{for any } g \in C_j
\]
and
\[
\|\mu\|_\alpha \leq M |\mu(1)| \quad \text{for any } \mu \in C_j^*.
\]

(iii) For any \( f \in \mathcal{H}_j \) there exist \( f_1, \ldots, f_8 \in C_j \) so that \( f = f_1 + f_2 + \ldots + f_8 \) and
\[
\|f_1\|_\alpha + \|f_2\|_\alpha + \ldots + \|f_8\|_\alpha \leq r \|f\|_\alpha
\]
where \( r = 4(1 + \frac{2}{sQ}) \).

Next, we recall the definition of the following complex Hilbert metric \( \delta_C \), associated with a complex cone \( C \) in a complex Banach space \( X \): for each \( p_1, p_2 \) in the cone \( C \)
\[
\delta_C(p_1, p_2) = \ln \left( \frac{b}{a} \right)
\]
where for any \( x, y \in C \),
\[
E_C(x, y) = \{ z \in C : zx - y \notin C \}
\]
and
\[
a = \inf |E_C(p_1, p_2)| \in [0, \infty] \quad \text{and} \quad b = \sup |E_C(p_1, p_2)| \in [0, \infty].
\]
We refer again the readers to Appendix A in [16] for more background on these complex projective metrics. For any subset \( Q \) of \( C' = C \setminus \{0\} \) denote by \( \Delta_C(Q) \) its (Hilbert) diameter with respect to \( \delta_C \) which is given by
\[
\Delta_C(Q) = \sup_{q_1, q_2 \in Q} \delta_C(q_1, q_2).
\]

3.2. **Theorem.** There exist a neighborhood \( U \) of 0 and constants \( j_0 \in \mathbb{N} \) and \( d_0 > 0 \) so that for any integer \( j \), \( z \in U \) and \( j_0 \leq m \leq 2j_0 \),
\[
\mathcal{L}^{j,m}_z C_j^{'} \subset C_{j+m}^{'} \quad \text{and} \quad \Delta_{C_{j+m}}^{(L^{j,m}_z)} \subset C_{j+m} < d_0.
\]
The above \( U, j_0 \) and \( d_0 \) depend only on the initial parameters, and this dependence can be easily recovered from the proof.
3.2. Forward block partition and dual operators. Let \( j \in \mathbb{Z}, \ n \geq j_0 \) and write \( n = k_{j_0} + r \) for some natural \( k \geq 1 \) and \( 0 \leq r < j_0 \). For any \( z \in U \) consider the following \enquote{block partition},

\begin{equation}
\mathcal{L}_z^{j,n} = \mathcal{L}_z^{j+(k-1)j_0,j_0+r} \circ \mathcal{L}_z^{j+(k-2)j_0,j_0} \circ \cdots \circ \mathcal{L}_z^{j+2j_0,j_0} \circ \mathcal{L}_z^{j+j_0,j_0} \circ \mathcal{L}_z^{0,j_0}.
\end{equation}

By Theorem 3.2 for any \( 0 \leq m \leq k-2 \),

\begin{equation}
\mathcal{L}_z^{j+m_{j_0}j_0} C_{j+m_{j_0}}' \subset C_{j+(m+1)j_0}, \ \Delta C_{j+(m+1)j_0} \subset (\mathcal{L}_z^{j+m_{j_0}j_0} C_{j+m_{j_0}}') < d_0,
\end{equation}

\begin{equation}
\mathcal{L}_z^{j+(k-1)j_0,j_0+r} C_{j+(k-1)j_0} \subset C_{j+n} \quad \text{and} \quad \Delta C_{j+n} \subset (\mathcal{L}_z^{j+(k-1)j_0,j_0+r} C_{j+(k-1)j_0}) < d_0.
\end{equation}

In particular, it follows that

\begin{equation}
\mathcal{L}_z^{j,n} C_j' \subset C_{j+n}
\end{equation}

for any \( z \in U \) and \( n \geq j_0 \).

Next, for the sake of convenience, we denote sometimes the dual operator \((\mathcal{L}_z^{j,n})^*\) also by \(\mathcal{L}_z^{j,n,*}\). Consider \( P \)-a.s. the dual block partition of \((3.4)\),

\begin{equation}
(\mathcal{L}_z^{j,n})^* = \mathcal{L}_z^{j,j_0} \circ \mathcal{L}_z^{j,k_{j_0}j_0+r} \circ \mathcal{L}_z^{j+(k-2)j_0,j_0} \circ \cdots \circ \mathcal{L}_z^{j+2j_0,j_0} \circ \mathcal{L}_z^{j+j_0,j_0} \circ \mathcal{L}_z^{0,j_0}
\end{equation}

where \( n = k_{j_0} + r, \ k \in \mathbb{N} \) and \( 0 \leq r < j_0 \). Now we begin with construction of \( \nu_j(z) \) and \( \lambda_j(z) \). Note that this construction will not depend on the results from Section 3.3.

First, by (A.2.6) in Appendix A of [10], taking into account \((3.5)\) and \((3.6)\), it follows that for any \( z \in U \) and \( n \geq j_0 \),

\begin{equation}
\mathcal{L}_z^{j+(k-1)j_0,j_0+r} C_{j+n}^* \subset C_{j+(k-1)j_0}^* \quad \text{and} \quad \Delta C_{j+(k-1)j_0} \subset (\mathcal{L}_z^{j+(k-1)j_0,j_0+r} C_{j+(k-1)j_0})^* < d_0.
\end{equation}

and the corresponding \( \delta \)-diameters of the appropriate images (with respect to the appropriate dual cones) do not exceed \( d_0 \). Note that in this application of (A.2.6) we have used that the cones \( C_j' \) are proper and linearly convex. In particular, it follows that

\begin{equation}
\mathcal{L}_z^{j,n,*} C_j' \subset C_j^*
\end{equation}

for any \( n \geq j_0 \) and \( z \in U \).

Next, let \( s \geq n \geq j_0, \ \mu_1 \in C_{j+s}^* \) and \( \mu_2 \in C_{j+n}^* \). We claim that

\begin{equation}
\delta_{C_j'}(\mathcal{L}_z^{j,s,*} \mu_1, \mathcal{L}_z^{j,n,*} \mu_2) \leq d_0 e^{\frac{|z|^2}{2}} - 2.
\end{equation}

Indeed, write \( n = k_{j_0} + r \) and \( s = q_{j_0} + p \) where \( 1 \leq k \leq q \) and \( 0 \leq r, p < j_0 \). Consider the following block partition

\begin{equation}
(\mathcal{L}_z^{j,s})^* = (\mathcal{L}_z^{j,j_0})^* \circ (\mathcal{L}_z^{j,k_{j_0}j_0})^* \circ \cdots \circ (\mathcal{L}_z^{j+(k-2)j_0,j_0})^* \circ B_s
\end{equation}

where

\begin{equation}
B_s = B_s(n, j, z) = (\mathcal{L}_z^{j+(k-1)j_0, j_0+n-(k-1)j_0})^*.
\end{equation}

Using \((3.9)\) with \( n \) and \( s \) we see that \( B_n \mu_1, B_s \mu_2 \in C_{j+(k-1)j_0}^* \) for any \( z \in U \). Therefore, by \((3.9)\) and the diameter estimates following it,

\begin{equation}
\delta_{C_{j+(k-2)j_0}}(\mathcal{L}_z^{j+(k-2)j_0,j_0,*}(B_n \mu_1), \mathcal{L}_z^{j+(k-1)j_0,j_0,*}(B_n \mu_2)) < d_0.
\end{equation}
Similarly to (3.23), (3.11) follows by a repetitive use of Theorem A.2.3 (iii) in Appendix A of [16], taking into account (3.9) and the diameter estimates following it. Next, we derive from Theorem A.2.3 (ii) in [16] and (3.11) that

$$\|\bar{L}_j^{j,n,*} \mu_1 - \bar{L}_j^{j,n,*} \mu_2\| \leq \frac{1}{2} M_j d_0 c_{\frac{M_j}{c_0}}$$

where $M_j$ is specified in Assumption 3.1 and for any $\nu \in C^*_m$ and $m \geq j_0$,

$$\bar{L}_j^{j,m,*} \nu = \frac{L_j^{j,m,*} \nu}{\kappa_j(L_j^{j,m,*} \nu)} = \frac{L_j^{j,m,*} \nu}{\nu(L_j^{j,m} 1)}$$

where $\kappa_j$ is defined by $\kappa_j(\nu) = \nu(1)$. Similarly to (3.28), we conclude that for any choice of $\{\mu_n : n \geq 1\}$ with $\mu_n \in C^*_n$ the limits

$$\nu_j^{(z)} := \lim_{n \to \infty} \bar{L}_j^{j,n,*} \mu_n, \ z \in U$$

exist and are independent of the choice of $\{\mu_n : n \geq 1\}$. The operators $L_j^j$ are analytic and so $\nu_j^{(z)}$ is a uniform limit of analytic functions which makes it analytic in $z$. Note that we have used here that $(L_j^j)^*$ is analytic in $z$ which holds true since the duality map $A \to A^*$ is an isometry.

Next, applying Theorem A.2.3 (i) in Appendix A of [16] and that any dual cone is linearly convex (see Appendix A in [16]) it follows that $\nu_j^{(z)} \in C^*_j$. Moreover, by (3.12) and (3.13),

$$\|L_j^{j,n,*} \mu_n - \nu_j^{(z)}\| \leq \frac{1}{2} M_j d_0 c_{\frac{M_j}{c_0}}$$

for any $n \geq j_0$. Furthermore, since $\nu_j^{(z)} \in C^*_j$, $\kappa_j(\nu_j^{(z)}) = 1$ and $\|\kappa_j\| = 1$, we deduce from (3.12) that

$$1 \leq \|\nu_j^{(z)}\| \leq M.$$

Replacing $j$ with $j + 1$, making the choice $\mu_n = l_{(j+1)+n}$ and then plugging in both sides of (3.13) into the continuous operator $(L_j^j)^*$, we derive that the limits

$$\lambda_j(z) := \lim_{n \to \infty} \frac{\kappa_j(L_j^{j,n+1,*} l_{j+n+1})}{\kappa_{j+1}(L_j^{j+1,n,*} l_{j+n+1})}, \ z \in U$$

exist and satisfy

$$(L_j^j)^* \nu_j^{(z)}_{j+1} = \lambda_j(z) \nu_j^{(z)}.$$

The operators $L_j^j$ are analytic in $z$ and therefore the map $z \to \lambda_j(z)$ is analytic since it is a uniform limit of analytic functions. Substituting both sides of (5.17) in $\kappa_j$ we deduce that

$$\lambda_j(z) = \lambda_j(z) \kappa_j(\nu_j^{(z)}) = \kappa_j((L_j^j)^* \nu_j^{(z)}_{j+1})$$

which, in particular, provides a different proof that $\lambda_j(z)$ is analytic in $z$. Next, applying (5.17) repeatedly with $j' = j + n - 1, \ldots, j + 1, j$ it follows that for any $n \in \mathbb{N},$

$$L_j^{j,n,*} \nu_j^{(z)}_{j+n} = \lambda_j^{(z)}(L_j^{j,n,*} \nu_j^{(z)}_{j+n}).$$
where
\[ \lambda^{(z)}_{j,n} := \prod_{i=0}^{n-1} \lambda^{(z)}_{j+i} = \kappa_j \left( \mathcal{L}^{(n)*}_{z,j+n} \right) \]

Since \( \nu^{(z)}_{j+n} \in C^*_j \) we deduce from (3.10) that \( \mathcal{L}^{(n)*}_{z,j+n} \in C^*_j \) for any \( n \geq j_0 \). This together with the fact that \( \kappa_j \in (C^*_j)^* \) implies that \( \lambda^{(z)}_{j,n} \neq 0 \) for any sufficiently large \( n \), and as a consequence we have \( \lambda_j(z) \neq 0 \) for any \( z \in U \) and \( j \in \mathbb{Z} \).

### 3.3. Reverse block partitions

For any \( j \in \mathbb{Z} \), \( z \in U \) and a natural \( n \geq j_0 \) consider the following “block partition”,

\[ \mathcal{L}^{j-n,n}_{z,j} = \mathcal{L}^{j-j_0}_z \circ \mathcal{L}^{j-2j_0}_z \circ \cdots \circ \mathcal{L}^{j-(k-1)j_0}_z \circ \mathcal{L}^{j-j_0+r}_{z,j-1} \]

where \( n = k j_0 + r \) for some integers \( k \geq 1 \) and \( 0 \leq r < j_0 \). By Assumption 3.2, we have

\[ \mathcal{L}^{j-n,n}_{z,j} \in C^*_{j-n} \quad \text{and} \quad \Delta_{C^*_{j-n}} \left( \mathcal{L}^{j-n,n}_{z,j} \circ \mathcal{L}^{j-j_0+r}_{z,j-1} \right) < d_0 \]

and for any \( 1 \leq m \leq k - 1, \)

\[ \mathcal{L}^{j-mj_0,j_0}_{z,j} \in C^*_{j-mj_0} \quad \text{and} \quad \Delta_{C^*_{j-mj_0}} \left( \mathcal{L}^{j-mj_0,j_0}_{z,j} \circ \mathcal{L}^{j-mj_0}_{z,j} \right) < d_0 \]

In particular, it follows that

\[ \mathcal{L}^{j-n,n}_{z,j} \in C^*_{j} \]

Next, set

\[ c = \tanh \left( \frac{1}{4} d_0 \right) \in (0, 1). \]

We claim that P-a.s.,

\[ \delta_{C^*_{j-n}} \left( \mathcal{L}^{j-n,n}_{z,j} f_n, \mathcal{L}^{j-s,s}_{z,j} g_s \right) \leq d_0 e^{\left( \frac{1}{4} \right) - 2} \]

for any \( s \geq n \geq j_0, z \in U \) and sequences \( \{f_m : m \geq 1\} \) and \( \{g_m : m \geq 1\} \) with \( f_m, g_m \in C^*_{j-m} \) for any \( m \geq 1 \). Before proving (3.23) we need the following notations. For any finite sequence \( G_1, \ldots, G_m \) of linear operators we write \( \prod_{i=1}^m G_i = G_1 \circ G_2 \circ \cdots \circ G_m \). In order to prove (3.23), we first write \( n = k j_0 + r \) and \( s = q j_0 + p \), for some \( 1 \leq k \leq q \) and \( 0 \leq r, p < j_0 \). Then,

\[ \mathcal{L}^{j-n,n}_{z,j} = \left( \prod_{m=1}^{k-2} \mathcal{L}^{j-mj_0,j_0}_{z,j} \right) \circ \mathcal{L}^{j-(k-1)j_0,j_0}_{z,j} \circ B_1 \quad \text{and} \quad \mathcal{L}^{j-s,s}_{z,j} = \left( \prod_{m=1}^{k-2} \mathcal{L}^{j-mj_0,j_0}_{z,j} \right) \circ \mathcal{L}^{j-(k-1)j_0,j_0}_{z,j} \circ B_2 \]

where \( B_1 = B_1(n, j, z) \) and \( B_2 = B_2(n, s, j, z) \) are given by

\[ B_1 \mathcal{L}^{j-n,n}_{z,j} \circ \mathcal{L}^{j-s,s}_{z,j} \quad \text{and} \quad B_2 \mathcal{L}^{j-s,s}_{z,j} \circ \mathcal{L}^{j-j_0+r}_{z,j} \]

Applying repeatedly Theorem A.2.3 from Appendix A in [10], and taking into account [3.21] and [3.22], we derive that

\[ \delta_{C^*_{j-2}} \left( \left( \prod_{m=1}^{k-2} \mathcal{L}^{j-mj_0,j_0}_{z,j} \right) h_1, \left( \prod_{m=1}^{k-2} \mathcal{L}^{j-mj_0,j_0}_{z,j} \right) h_2 \right) \leq c^{k-2} \delta_{C^*_{j-2}} \left( h_1, h_2 \right) \]
for any $h_1, h_2 \in C'_{j-(k-2)j_0}$. Let $f_n \in C'_{j-n}$ and $g_s \in C'_{j-s}$. It follows from the inclusions in (3.21) and (3.22) that $B_1 f_n, B_2 g_s \in C'_{j-(k-1)j_0}$ and that the functions $h_1$ and $h_2$ given by

$$h_1 = \mathcal{L}_z^{j-(k-1)j_0} B_1 f_n \quad \text{and} \quad h_2 = \mathcal{L}_z^{j-(k-1)j_0} B_2 g_s$$

are members of $C'_{j-(k-2)j_0}$. Applying (3.22) we derive that

$$\delta_{C'_{j-(k-2)j_0}} (h_1, h_2) < d_0$$

and (3.23) follows from (3.24), (3.25) and the above estimates.

Since $l_j \in C'_\omega$, we conclude from (3.23), Theorem A.2.3 (ii) in [16] and from (3.1) that

$$\|\mathcal{L}_z^{j-s} g_s - \mathcal{L}_z^{j-n} f_n\| \leq \frac{1}{2} K \rho_0 c [\frac{d_0}{c}]^{-2} \leq R(j) c \frac{d_0}{c}$$

for any $s \geq n \geq j_0$, $z \in U$ and sequences $\{f_m : m \geq 1\}$ and $\{g_m : m \geq 1\}$ with $f_m, g_m \in C'_{j-m}$ for any $m \geq 1$. Here $R(j) = \frac{1}{2} c^{-3} d_0 K_j$ and

$$\mathcal{L}_z^{j-j'} h = \frac{\mathcal{L}_z^{j-j'} h}{l_j(\mathcal{L}_z^{j-j'} h)}$$

for any $j' \geq j_0$, $h \in C'_{j-j'}$, and $z \in U$. By considering the case when $f_m = g_m$ for any $m \in \mathbb{N}$, we deduce that the sequence

$$\{\mathcal{L}_z^{j-n} f_n : n \in \mathbb{N}\} \subset B_j$$

is a Cauchy sequence, and therefore $P$-a.s. the limits

$$\hat{h}_j(z) := \lim_{n \to \infty} \mathcal{L}_z^{j-n} f_n, \quad z \in U$$

exist in the Banach space $B_j$. Considering now the situation when $n = s$ and then letting $n \to \infty$ in (3.20), we deduce that the limits $\hat{h}_j(z)$, $z \in U$ do not depend on the choice of the sequence $\{f_m : m \geq 1\}$. Moreover, fixing $n$ and letting $s \to \infty$ it follows from (3.26) that for any choice of $f_n \in C'_{j-n}$ and $n \geq j_0$,

$$\|\mathcal{L}_z^{j-n} f_n - \hat{h}_j^{(z)}\| \leq R(j) c \frac{d_0}{c}.$$

Furthermore, by (3.23) and Theorem A.2.3 (i) in [16] we have $\hat{h}_j^{(z)} \in C'_j = C_j \setminus \{0\}$ and the limits $\hat{h}_j^{(z)}$ exist also in the projective metric $\delta_{C_j}$. Since $l_j$ is continuous, for any $z \in U$ we have that

$$l_j(\hat{h}_j(z)) = 1.$$

Therefore by (3.1), for any $z \in U$,

$$\frac{1}{\|l_j\|} \leq \|\hat{h}_j(z)\| \leq \frac{K_j}{\|l_j\|}$$

using that $\hat{h}_j(z) \in C_j$.

Finally since the operators $\mathcal{L}_z^{(m)}$, $m \in \mathbb{Z}$ are analytic in $z$ the limit $\hat{h}_j(z)$ is analytic in $z \in U$ as it is a uniform limit of analytic functions.
3.4. RPF triplets. For any $z \in U$ and $j \in \mathbb{Z}$ set

\[ h_j(z) = \frac{\hat{h}_j(z)}{\nu_j(z) \nu_j(z)} \]  

(3.31)

The denominator does not vanish since $\hat{h}_j(z) \in C_j'$ and $\nu_j(z) \in C_j^*$, and so $h_j(z)$ is well defined,

\[ \nu_j(z) \nu_j(z) = 1 \text{ and } \hat{h}_j(z) \in C_j' \]  

(3.32)

where the inclusion follows since $C_j$ is invariant under multiplication of nonzero complex numbers (i.e. it is a complex cone). Since $\nu_j(z)$ and $\hat{h}_j(z)$ are nonzero complex numbers (i.e. it is a complex cone). Since $\nu_j(z)$ and $\hat{h}_j(z)$ are analytic in $z$ so is $h_j(z)$. We claim that the triplets $(\lambda_j(z), h_j(z), \nu_j(z))$, $j \in \mathbb{Z}$ are the RPF triplet from Theorem 2.4. The missing ingredient is to show that for any $h \in H$ the denominator does not vanish since $\hat{h}_j(z)$ is given by (3.33). By (3.34) the numerator converges to $\hat{h}_j(z)$ and therefore by the definition of $h_j(z)$,

\[ \lim_{n \to \infty} \frac{\hat{L}_j^{j-n,n} q_n}{\nu_j(z) \nu_j(z)} = h_j(z) \]  

(3.35)

where $\hat{L}_j^{j-n,n}$ is given by (3.27). By (3.28) the numerator converges to $\hat{h}_j(z)$ while the denominator converges to $\nu_j(z) \nu_j(z)$ and therefore by the definition of $h_j(z)$,

\[ \lim_{n \to \infty} \frac{\hat{L}_j^{j-n,n} q_n}{\nu_j(z) \nu_j(z)} = h_j(z) \]  

(3.36)

which completes the proof of the claim.

The proof of the rest of Theorem 2.4 proceeds exactly as in [16], as we have explained at the beginning of Section 3.\]
3.5. Sequential stability and non-singular maps. We begin with the proof of Theorem 2.6. Let $m_j, j \in \mathbb{Z}$ be a family of probability measures on $\mathcal{E}_j$ so that for each $j$ we have $(T_j)^*m_j \ll m_{j+1}$ and $e^{-f_j} = \frac{d(T_j)^*m_j}{dm_{j+1}}$. Notice that any probability measure $\theta$ which assigns positive mass to open set belongs to the cone $\mathcal{C}_j^*$. Indeed, any non-zero member $g$ of $\mathcal{C}_j$ has the form $g = \beta(g_1 + ig_2)$, where $g_1, g_2 \in \mathcal{C}_{j,R}$ and $\beta \in \mathbb{C} \setminus \{0\}$, and so $\theta(g) \neq 0$ when $g$ is not identically zero. Therefore, by taking $\theta_{n+j} = m_{n+j}$ in (3.35) we derive that for any integer $j$ and $g \in \mathcal{H}_j$,

$$
\nu_j^{(0)}(g) = \lim_{n \to \infty} \frac{m_{j+n}(L_j^{j,n}g)}{m_{j+n}(L_j^{j,n}1)} = m_j(g),
$$

where we used that $(L_j^{j,n})^*m_{j+n} = m_j$.

Now we will prove Theorem 2.6. Let $K$, $T_{1,j}$, $f_{1,j}$ and $u_{1,j}$ be as in the statement of Theorem 2.6. Let $\varepsilon > 0$, and denote by $\nu_{j,m}$ and $\nu_{1,j,m}$ the $m$-th derivative at $z = 0$ of the maps $z \to \nu_j^{(z)}$ and $z \to \nu_{1,j}^{(z)}$, respectively. Since $\|\nu_j(z)\|_\alpha$ and $\|\nu_{1,j}(z)\|_\alpha$ are both bounded by some constant $C$, which does not depend on $z$ and $j$, it follows, by Lemma 2.8.2 in [10] that for any $z \in B(0, r)$ and $k \geq 1$,

$$
\|\nu_j(z) - \sum_{m=0}^{k} \frac{\nu_{j,m}(z)}{m!} z^m\|_\alpha < (k+2)C2^{-k-1}
$$

and the same inequality holds true with $\nu_{1,j}$ in place of $\nu_j$. Let $k = k_\varepsilon$ be the smallest positive integer so that $(k+2)C2^{-k-1} < \frac{1}{2}\varepsilon$. Then it is sufficient to show that there exists $\delta_0 > 0$ so that $\|\nu_j(z) - \nu_{1,j}^{(z)}\|_\alpha < \delta_0$ for any $j$ and $z \in K$ then for any $m \in \{0, 1, ..., k_\varepsilon\}$ we have

$$
(3.36) \quad \|\nu_{j,m} - \nu_{1,j,m}\|_\alpha < \frac{1}{4}r^{-m}k_\varepsilon^{-1}\varepsilon.
$$

First, applying (3.14) with $\theta_{n+j} = L_{n+j}^{-1}l_{n+j}$ we obtain for any $z \in U$ and $j \in \mathbb{Z}$,

$$
\max (\|\nu_j^{(z)} - F_j(j, n, z)\|_\alpha, \|\nu_{1,j}^{(z)} - F_1(j, n, z)\|_\alpha) \leq C_1\delta^n
$$

where the linear functionals $F_j(j, n, z)$ and $F_1(j, n, z)$ are given by

$$
F_j(j, n, z) = \frac{\theta_{n+j}(L_j^{j,n}(\cdot))}{\theta_{n+j}(L_j^{j,n}(1))} \quad \text{and} \quad F_1(j, n, z) = \frac{\theta_{n+j}(L_1^{j,n}(\cdot))}{\theta_{n+j}(L_1^{j,n}(1))}.
$$

Here $C_1$ and $\delta \in (0, 1)$ depend only on the initial parameters $B, L, \alpha, \gamma, n_0, \xi$ and $D$. Let $n_1 = n_1(\varepsilon)$ be the smallest positive integer so that $C_1\delta^n r^{-m}m! < \frac{1}{8}$ for any $0 \leq m \leq k_\varepsilon$. Then by the Cauchy integral formula, for any $0 \leq m \leq k_\varepsilon$ we have

$$
\max (\|\nu_{j,m} - F(j, n_1, 1)\|_\alpha, \|\nu_{1,j,m} - F_1(j, n_1, 1)\|_\alpha) \leq r^{-m}m!C_1\delta^n < \frac{1}{8}\varepsilon
$$

where $F(j, n, 0) = \frac{d^m F(j, n, z)}{z^m}|_{z=0}$ and $F_1(j, n, 0)$ is defined similarly with $F_1(j, n, z)$ in place of $F_1(j, n, z)$. Observe next that

$$
L_j^{-1}l_{j+n_1}(L_j^{j,n_1}(1)) \geq e^{-\|S_j(n_1)\|_{\infty}} \geq e^{-Bn_1}
$$

and the same inequality holds true with $L_1^{j,n_1}$ in place of $L_1^{j,n_1}$. Therefore, the denominators in $F(j, n_1, 0)$ and $F_1(j, n_1, 0)$ are bounded from below by $e^{-Bn_12^n}$. 


which depends only on \( \varepsilon \) and \( m \). Using that for any families of operators \( A_1, ..., A_n \) and \( B_1, ..., B_n \), we have

\[
A_1 \circ A_2 \circ \cdots \circ A_n - B_1 \circ B_2 \circ \cdots \circ B_n = \sum_{i=1}^{n-1} A_1 \circ \cdots \circ A_{i-1}(A_i - B_i)B_{i+1} \circ \cdots \circ B_n
\]

and that \( L_{n_1}^j \) are analytic in \( z \) and uniformly bounded in \( j \) and \( z \in K \) in the operator norm \( \| \cdot \|_\alpha \) (by some constant which depends only on \( n = n_1(\varepsilon) \) and the initial parameters—see Theorem 2.3), we find that if

\[
\sup_{j \in \mathbb{Z}} \sup_{z \in K} \| L_{n_1}^j(z) - L_{n_1}^j(z) \|_\alpha < \delta_0
\]

for some \( \delta_0 > 0 \), then for any \( j \) and \( m \geq 0 \),

\[
\| F^{(m)}(j, n_1, 0) - F^{(m)}(j, n_1, 0) \|_\alpha < C(m, r, \varepsilon) \delta_0
\]

where \( C(m, r, \varepsilon) \) depends only on \( m, \varepsilon, r \) and the initial parameters. Taking a sufficiently small \( \delta_0 \) completes the proof of the claim about the stability of \( \{ \nu_j(z) : j \in \mathbb{Z} \} \) (which was stated in Theorem 2.6). Since \( \lambda_j(z) = \nu_j(z) \) \( (L_{n_1}^j(z) 1) \) and \( \lambda_{n, j}(z) = \nu_1(z) \) \( (L_{1, j}^j(z) 1) \), we drive that for any \( \varepsilon > 0 \) there exists \( \delta_1 > 0 \) so that

\[
\sup_{j \in \mathbb{Z}} \sup_{z \in K} | \lambda_j(z) - \lambda_{n, j}(z) | < \varepsilon \quad \text{if} \quad \sup_{j \in \mathbb{Z}} \sup_{z \in K} \| L_{n_1}^j(z) - L_{n_1}^j(z) \|_\alpha < \delta_1.
\]

Finally, by 2.5 for any \( j \in \mathbb{Z}, n \geq 1 \) and \( z \in U \) we have

\[
\left\| h_j(z) \right\|_{\lambda_j, n, n(z)} \leq A \delta^n
\]

and similar inequality holds true with \( h_j^{(i)} \),

\[
\lambda_{n, j}(z) := \prod_{i=0}^{n-1} \lambda_{n, j}(z)
\]

and \( L_{n, n}^j \). Let \( n_2 \) be so that \( A \delta^{n_2} < \frac{1}{2} \varepsilon \). By taking fixing a sufficiently large \( n_2 \), we can also assume that \( | \lambda_{n, n_2, n_2}(0) | \geq C_2 \| L_{n_1}^j \|_\alpha \geq C_2 e^{-Bn_2} \) for some constant \( C_2 \) which does not depend on \( j \). Using now the stability of \( \lambda_j(z) \), for any \( q > 0 \) and \( m \geq 0 \) we can find \( \delta_2 = \delta_2(q, m, n_2) \) so that

\[
\left\| \frac{d^m L_j(z)}{dz^m} \right\|_{z=0} < q
\]

if \( \| L_{n_1}^j(z) - L_{n_1}^j(z) \|_\alpha < \delta_2 \) for any integer \( j \) and \( z \in K \), where

\[
L_j(z) = \frac{L_{n_1}^j \nu_j(z)}{\lambda_{n_1, n_2}(z)}
\]

and \( L_{j, n} \) is defined similarly but with \( L_{n_1}^j \nu_j(z) \) and \( \lambda_{n, n_2, n_2}(z) \). Using this we can approximate each one of the derivatives of \( h_j(z) \) at \( z = 0 \) by the corresponding derivative of \( h_j(z) \), which, as in the proof of the stability of \( \{ \nu_j(z) : j \in \mathbb{Z} \} \), is enough to drive the stability of \( \{ h_j(z) : j \in \mathbb{Z} \} \) (for \( z \in K \)).
3.6. The Gibbs property and non-stationary subshifts of finite type.

3.4. Theorem. Let \( t \in U \cap \mathbb{R} \). Then the following Gibbs property holds true: for any ball \( B = B_j(y, r) \) of radius \( 0 < r < \xi \) in \( \mathcal{E}_j \) we have

\[
e^{-S^{(t)}(y)} \leq \frac{\nu_j^{(t)}(B)}{e^{S_{j,n(r)} - \ln \lambda_{j,n(r)}(t)}} \leq e^{S^{(t)}(y)R_j^{(t)}(y)}\]

where \( n(r) = n_0 + n_1(r) \), \( n_1(r) \) is the smallest positive integer so that \( \xi < r^{\gamma n_1(r)} \), \( S_{j,n}f_t = S_{j,n}f + tS_{j,n}u \),

\[
S_r^{(t)}(j, y) = \sup_{y' \in B(y, r)} \left| S_{j,n(r)}f_t(y) - S_{j,n(r)}f_t(y') \right|
\]

and

\[
R_r^{(t)}(y) = \sup_{x \in \mathcal{E}_{j+n(r)}} \left| B(y, r) \cap (T_j^{n(r)})^{-1}\{x\} \right|
\]

is the least upper bound on the number of preimages of a point \( x \) by \( T_j^{n(r)} \) in the set \( B(y, r) \). In particular, \( \mu_j^{(t)}(B) := \int_B h_j^{(t)}(x) d\nu_j^{(t)}(x) \geq c(\lambda_{j,n(r)}(t))^{-1}r^q \) where \( q > 0 \) and \( c > 0 \) are some constants which depend only on the initial parameters.

Proof. First, exactly as in (5.10.3) from the proof of Lemma 5.10.3 in [16], for any ball \( B \) and \( \lambda \) we have

\[
\left( 1 \right) \text{There exist constants } C > 0 \text{ and } \mu_j^{(t)}(B) \text{ of radius } 1 \text{ with } C \text{ are positive, for some } j \text{ with } \mu \geq 1 \text{ and } \lambda \text{ are some constants which depend only on the initial parameters.}
\]

The upper bound in the statement of the theorem follows from the above equality, and the proof of the theorem is complete. \( \Box \)

Non-stationary subshifts of finite type. Let \( d_j, j \in \mathbb{Z} \) be a family of positive integers so that \( d_j \leq d \) for some \( d \in \mathbb{N} \) and all \( j \)'s. Let \( A_j = A_j(a, b) \) be a family of matrices of sizes \( d_j \times d_{j+1} \) whose entries are either 0 or 1, so that all the entries of \( A_j + 0 \) \( A_0 \) \( A_{j+1} \) are positive, for some \( n_0 \geq 1 \) and all \( j \)'s. Let the compact space \( \mathcal{X} \) be given by

\[
\mathcal{X} = \{1, 2, \ldots, d\}^{\mathbb{N}\cup\{0\}}
\]

and for each \( j \), and let \( d(x, y) = 2^{-\min\{n \geq 0 : x_n \neq y_n\}} \). For each integer \( j \) set

\[
\mathcal{E}_j = \{(x_0, x_1, \ldots) \in \mathcal{X} : x_j + 1 \leq d_j + m \text{ and } A_j(x_j, x_{j+1}) = 1, \forall m \geq 0\}
\]

and define \( T_j : \mathcal{E}_j \to \mathcal{E}_{j+1} \) by

\[
T_j(x_j, x_{j+1}, \ldots) = (x_{j+1}, x_{j+2}, \ldots)
\]

We also set \( \gamma = 2 \) and \( \xi = 1 \), so the inequality \( d(x, y) < \xi \) means that \( x_0 = y_0 \). In this nonstationary subshift case we have the following result:

3.5. Theorem. (ii) The above \( S^{(t)}(j, y) \) and \( R^{(t)}(y) \) are bounded in \( j, y, r \) and \( t \).

(iii) There exist constants \( C > 0 \) and \( \delta \in (0, 1) \) so that for any integer \( j, r, s, n \geq 1 \), symbols \( a_j, \ldots, a_j \), \( b_j + r + n \), \( b_j + r + n \) and cylinder sets

\[
A = \{(x_0, x_1, \ldots) : x_i = a_i \forall i \in [j, j + r]\} \subset \mathcal{E}_j
\]
and

\[ B = \{(x_m)_{m=j}^{\infty} : x_i = b_i \ \forall \ i \in [j + r + n, j + r + n + s]\} \subseteq \mathcal{E}_j \]

we have

\[ |\mu_j(A \cap B) - \mu_j(A)\mu_j(B)| \leq C\mu_j(A)\mu_j(B)\delta^n. \tag{3.38} \]

Namely, the σ-algebras generated by the cylinder sets are exponentially fast \(\psi\)-mixing (uniformly in \(j\)).

**Proof.** The proof of (i) proceeds exactly as in [3]. In order to prove (ii), we observe first that for any function \(q\) which depends only on the coordinates \(x_j, x_{j+1}, ..., x_{j+r}\) and for any \(g \in \mathcal{C}_{j,\mathbb{R}}\) we have \(\mathcal{L}^{q}(g) \in \mathcal{C}_{j+r,\mathbb{R}}\), where the cones \(\mathcal{C}_{j,\mathbb{R}}\) were defined in Section 3. This means that an appropriate version of Lemma 1.11 in [3] holds true. Relying on (2.5) instead of Lemma 1.10 in [3], the proof of (3.38) proceeds exactly as in the proof of Proposition 1.14 from there. \(\square\)

4. LIMIT THEOREMS: PROOFS

First, since \(\lambda_j(\cdot)\) are analytic function and \(|\lambda_j(z)|\) is bounded uniformly in \(j\) and \(z\), and \(a \leq \lambda_j(0) \leq b\) for some positive constants \(a\) and \(b\) which do not depend on \(j\), it is indeed possible to construct functions \(\Pi_j(\cdot)\) which satisfies the conditions stated in Theorem 2.3 (i).

Next, set

\[ \mathring{\mathcal{L}}^{(j)}(g) = \frac{\mathcal{L}(g\mathring{h}_j(0))}{\lambda_j(0)\mathring{h}_{j+1}(0)} \]

Then \(\mathring{\mathcal{L}}^{(j)}\) is bounded uniformly in \(j\) and \(z\). Put

\[ f_j = f_j + \ln \mathring{h}_j(0) - \ln \mathring{h}_{j+1}(0) \circ T_j - \ln \lambda_j(0). \]

Then the operators \(\mathring{\mathcal{L}}^{(j)}\) are generated by the maps \(T_j\) and the functions \(f_j\) and \(u_j\). As noted in Chapter 2.2 of [10], since \(\mathring{h}_j(0) \in \mathcal{C}_{j,\mathbb{R}}\) and \(\lambda_j(0) \in [a, b]\) for some constants \(0 < a < b < \infty\) which do not depend on \(j\), all the results stated in Theorem 2.3 and in the rest of Section 2 can be applied with the \(\mathring{\mathcal{L}}^{(j)}\), even though the function \(\mathring{f}_j\) is not necessarily Hölder continuous. Note that the triplets

\[ \tilde{\lambda}_j(z) = \frac{a_j(z)\lambda_j(z)}{a_{j+1}(z)\lambda_j(0)}, \quad \tilde{h}_j(z) = \frac{a_j(z)\mathring{h}_j(0)}{\mathring{h}_{j+1}(0)}, \quad \text{and} \quad \tilde{\nu}_j(z) = (a_j(z))^{-1}\mathring{h}_j(0)\nu_j(z) \]

where \(a_j(z) = \nu_j(z)(h_j(0))\) (which is nonzero since \(\mathring{h}_j(0) \in \mathcal{C}_j\), satisfy all the required properties, and so we can always consider them as the RPF triplets corresponding to the operators \(\mathring{\mathcal{L}}^{(j)}\). Hence, there is no real need in applying directly complex cones techniques with \(\mathring{\mathcal{L}}^{(j)}\). Notice that \(\mathring{\nu}_j(0) = \mu_j\), that \(\mathring{h}_j(0) \equiv 1\) and \(\lambda_j(0) = 1\). Moreover, there exists a constant \(C\) so that for any \(j\) and \(z\) we have \(|a_j(z)| \leq c\). Since \(a_j(0) = 0\) it follows that that exist constant \(a, b > 0\) so that for any \(z \in \mathcal{B}(0,a)\) we have \(b \leq |a_j(z)| \leq c\). Therefore, there exist positive constants \(a_1\) \(c_1\) so that for any \(z \in \mathcal{B}(0,a_1)\) and \(n \geq 1\),

\[ |\sum_{m=0}^{n-1} \Pi_{j+m}(z) - \sum_{m=0}^{n-1} \tilde{\Pi}_{j+m}(z)| = |\ln a_j(z) - \ln a_{j+n}(z)| \leq c_1 \]

where \(\tilde{\Pi}_j(z), j \in \mathbb{Z}\) are analytic functions which are defined (simultaneously) in some neighborhood \(V\) of the origin, which depends only on the initial parameters,
so that \( e^{\bar{\Pi}_j(z)} = \bar{\lambda}_j(z) \) for each \( z \) in \( V \). By the Cauchy integral formula we obtain that for any \( k \) there exists a constant \( E_k \) so that

\[
| \sum_{m=0}^{n-1} \Pi_{j+m}^{(k)}(0) - \sum_{m=0}^{n-1} \bar{\Pi}_{j+m}^{(k)}(0) | \leq E_k.
\]

Note that the moments \( \mu_j(S_{j,n}^k, u) \), \( k \geq 1 \) do not depend on whether we have replaced \( f_j \) with \( \bar{f}_j \). We conclude that in the proofs of Theorems 2.7, 2.8 and 2.9 we can assume that \( L_z^{(j)}1 = 1 \), \( \lambda_j(0) = 1 \) and \( \nu_j^{(0)} = \mu_j \).

4.1. The moments of \( S_{0,n,u} \). Here we will prove the rest of Theorem 2.9. In what follows we will assume that \( L_0^{(j)}1 = 1 \) which implies that \( h_j^{(0)} \equiv 1 \) and \( \lambda_j(0) = 1 \).

For each integer \( j \) and \( k \geq 0 \) denote by \( \lambda_j^{(k)}(z) \), \( h_j^{(z)}(z) \) and \( \nu_j(z) \) \( k \)-th derivative at the point \( z \) of \( \lambda_j(\cdot) \), \( h_j^{(z)}(\cdot) \) and \( \nu_j^{(z)} \), respectively. First, it follows from Theorem 2.4 that \( \lambda_{j,n}(z) = \nu_j^{(z)}(L_z^{(j)}1) \), and so, since \( \nu_j^{(z)}(1) = 1 \) and \( \mu_j(e^{\bar{S}_{j,n}}) = \mu_{j+n}(L_z^{(j)}1) \),

\[
\lambda'_{j,n}(0) = \nu_{j,n+1}^{(0)}(1) + \frac{d\nu_j^{(z)}(L_z^{(j)}1)}{dz} \bigg|_{z=0} = \frac{d\mu_j(e^{\bar{S}_{j,n}})}{dz} \bigg|_{z=0} = \mu_j(S_{j,n,u}).
\]

Next, suppose that \( \mu_j(u_j) = 0 \) for any \( j \). Then \( \mu_j(S_{j,n,u}) = 0 \) for any \( n \), since \( (T_j)_{\ast} \mu_j = \mu_{j+1} \). The following arguments are almost identical to the ones in 19. First, by (2.5) for any \( n \geq 1 \) we can write

\[
(3.3) \quad \mu_j(z^{S_{j,n,u}}) = \mu_{j+n}(L_z^{(j)}1) = \lambda_{j,n}(z)(\mu_{j+n}(h_{j+n}^{(z)} + \delta_{j,n}(z))
\]

where \( |\delta_{j,n}(z)| \leq C\lambda^n \). Since \( \delta_{j,n}(\cdot) \) is an analytic function, it follows from the Cauchy integral formula that for any \( k \geq 1 \) there exists a constant \( Q_k \) so that

\[
|\delta_j^{(k)}(z)| \leq Q_k \delta^n \text{ for any } z \in B(0, \frac{r}{2}), \text{ where } r > 0 \text{ a number so that } B(0, r) \subset U.
\]

Since \( \lambda_{j,n}(0) = 1 \) and \( sup_{z \in B(0, r)} |\lambda_j(z)| \) is a bounded random variable, the analyticity of the \( \lambda_j(\cdot) \)'s implies that for any \( k \geq 1 \) we have \( |\lambda_j^{(k)}(0)| \leq R_k \lambda^n \), for some constant \( R_k \) which does not depend on \( n \) and \( \omega \). Therefore, differentiating \( k \) times both sides of (4.3) at \( z = 0 \) yields that

\[
(4.4) \quad \mu_j(S_{j,n,u}) = \sum_{q=0}^{k} \binom{k}{q} \lambda_{j,n}^{(k-q)} \mu_{j+n}(h_{j+n}^{(0)}) + d_{j,n}
\]

where \( |d_{j,n}| \leq Ac^n \) for some \( A \geq 1 \) and \( c \in (0, 1) \), which do not depend on \( j \).

Since \( \lambda_{j,n}(z) = e^{\sum_{m=0}^{n-1} \Pi_{j+m}(z)} \) and \( \Pi_{j+m}(0) = 1 \), it follows from the Faà di Bruno formula that for any \( q \),

\[
(4.5) \quad \lambda_{j,n}^{(q)}(0) = q! \sum_{s=1}^{[\frac{q}{2}]} n^s \prod_{(m_2, \ldots, m_q) \in \Gamma_{q,s}} (\prod_{l=2}^{q} (l!)^{m_l} - 1) \prod_{l=2}^{q} \left( \sum_{i=0}^{n-1} \Pi_{j+l}^{(0)}(0) \right)^{m_l}
\]

where \( \Gamma_{q,s} \) is the set of all \( q - 1 \)-tuples \( (m_2, \ldots, m_q) \) of nonnegative integers so that \( \sum_{l} lm_l = q \) and \( \sum_{l} m_l = s \), and we took into account that

\[
\sum_{q=0}^{n-1} \Pi_{j+q}^{(0)}(0) = \lambda'_{j,n}(0) = \mu_j(S_{j,n,u}) = 0.
\]

Observe that when \( s = [\frac{q}{2}] \) and \( q \) is even we have \( \Gamma_{q,s} = \{(\frac{q}{2}, 0, 0, \ldots, 0)\} \), while for odd \( q \)'s we have \( \Gamma_{q,s} = \{(\frac{q-1}{2}, 1, 0, \ldots, 0)\} \). Since \( \Pi_j(z) \), \( z \in B(0, r) \) is analytic in \( z \)
and uniformly bounded in \( j \) and \( z \), for each \( w \) there exists a constant \( M_w \) so that 
\[
\sup_{z \in B(0, r)} |\Pi_j^{(w)}(z)| \leq M_w \quad \text{for any } j.
\]
By considering the case when \( q = k \), we conclude that there exist constants \( R_k, k \geq 2 \) so that for even \( k \)'s we have
\[
(4.6) \quad \left| \frac{\mu_j(S,j,n,u)^k}{n^{[\frac{k}{2}]}_j} - C_k (n^{-\frac{k+1}{2}} \sum_{i=0}^{n-1} \Pi_j^{(2)}(0)) \right| \leq R_k n^{-1}
\]
where \( C_k = 2^{-\frac{k}{2}} (\frac{k!}{2})^{-1} k! \), while with \( D_k = \frac{k!}{3!} 2^{-\frac{k}{2}} (k-3)! (\frac{k-3}{2})^{-1} \), for odd \( k \)'s larger than 2, we have
\[
(4.7) \quad \left| \frac{\mu_j(S,j,n,u)^k}{n^{[\frac{k}{2}]}_j} - D_k (n^{-\frac{k+3}{2}} \sum_{i=0}^{n-1} \Pi_j^{(3)}(0)) \right| \leq R_k n^{-1}.
\]

4.2. The growth of the variance of \( S_{0,n,u} \). We begin with the proof of Theorem 2.7 (i). It is sufficient to prove this part in the case when \( \mu_s(u_s) = 0 \) for any \( s \), for otherwise we will just replace \( u_s \) with \( u_s - \mu_s(u_s) \). First, if there exists a family of functions \( \{Y_k : k \in \mathbb{Z}\} \) as in the statement of Theorem 2.7 (i), then clearly \( \text{var}_{\mu_k}(S_{k,n,u}) \) is bounded in \( n \), for each integer \( k \). On the other hand, suppose that
\[
\lim_{n \to \infty} \text{var}_{\mu_0}(S_{0,n,u}) < \infty.
\]
Since \((T_j)_*\mu_j = \mu_{j+1} \) and \( S_{j+1,m,u} \circ T_j = S_{j,m+1,u} - u_j \) for any \( j \) and \( m \geq 1 \), it follows from (2.6) that
\[
\lim_{n \to \infty} \text{var}_{\mu_k}(S_{k,n,u}) < \infty
\]
for any integer \( k \). Let \( L_0^{(k)} \) be the closed linear subspace of \( L^2(\mathcal{E}_k, \mu_k) \) generated by the functions \( u_k, u_{k+1} \circ T_k^1, u_{k+2} \circ T_k^2, u_{k+3} \circ T_k^3, \ldots \). We conclude that for any \( k \) there exists a subsequence of \( \text{var}_{\mu_k}(S_{k,n,u}) \) which converges weakly in \( L^2(\mathcal{E}_k, \mu_k) \) to a member \(-Y_k \) of \( L^2(\mathcal{E}_k, \mu_k) \). Using a diagonal argument we can assume that the indexes \( \{n_m\}_{m=1}^\infty \) of the above subsequences do not depend on \( k \).

Next, for any \( \zeta \in L^2(\mathcal{E}_k, \mu_k) \) we have
\[
\lim_{n \to \infty} \mu_k(S_{k,n,m} \cdot \zeta) = -\mu_k(Y_k \zeta).
\]
Therefore, since \( \hat{L}_0^{(k)} \) is the dual of the map \( g \to g \circ T_k \) with respect to the spaces \( L^2(\mathcal{E}_k, \mu_k) \) and \( L^2(\mathcal{E}_{k+1}, \mu_{k+1}) \), we have
\[
\mu_k(Y_{k+1} \circ T_k \cdot \zeta) - \mu_k(Y_k \cdot \zeta) = \mu_{k+1}(Y_{k+1} \cdot \hat{L}_0^{(k)} \cdot \zeta) - \mu_k(Y_k \cdot \zeta)
\]
\[
= - \lim_{m \to \infty} \left( \mu_{k+1}(S_{k+1,n,m} u \cdot \hat{L}_0^{(k)} \cdot \zeta) - \mu_k(S_{k,n,m} u \cdot \zeta) \right)
\]
\[
= - \lim_{m \to \infty} \left( \mu_k (S_{k+1,n,m} u \circ T_k \cdot \zeta) - \mu_k(S_{k,n,m} u \cdot \zeta) \right)
\]
\[
= \mu_k(u_k \cdot \zeta) - \lim_{m \to \infty} \mu_k((u_k, n+1) \circ T_k^{n+1} \cdot \zeta)
\]
where we used that \( S_{k+1,n,m} u \circ T_k = S_{k,n,m+1} u - u_k \). Since \( \zeta \in L_0^{(k)} \) and \( \mu_s(u_s) = 0 \) for any \( s \), it follows from (2.6) that
\[
\lim_{m \to \infty} \mu_k((u_k, n+1) \circ T_k^{n+1} \cdot \zeta) = 0
\]
and hence
\[
Y_{k+1} \circ T_k - Y_k = u_k
\]
as members of $L^2(\mathcal{E}_k, \mu_k)$. Now we will show that $Y_k$ can be chosen to be Hölder continuous so that the norms $\|Y_k\|_\alpha$ are bounded in $k$. Let the function $W_k$ be given by

$$W_k = \sum_{j=1}^{\infty} \tilde{\lambda}_0^{k-j} u_{k-j}.$$ 

This function is defined, and is a member of $\mathcal{H}_k$ because of (2.5), and our assumption that $\mu_j(u_j) = 0$ for any $j$. In fact, the exponential convergence (2.6) implies that $\|W_k\|_\alpha$ is bounded in $k$. Moreover, for each $k$ we have

$$W_k - \tilde{\lambda}_0^{(k-1)} W_{k-1} = \tilde{\lambda}_0^{(k-1)} u_{k-1}.$$ 

Next, substituting both sides of the equality $Y_k \circ T_{k-1} - Y_{k-1} = u_k$ into $\tilde{\lambda}_0^{(k-1)}$, it follows that

$$Y_k - \tilde{\lambda}_0^{(k-1)} Y_{k-1} = \tilde{\lambda}_0^{(k-1)} u_{k-1}.$$ 

Since the functions $\{W_k : k \in \mathbb{Z}\}$ also satisfy the above relations, the family of functions $\{d_k : k \in \mathbb{Z}\}$ given by $d_k = Y_k - W_k$ satisfies that for any $k \in \mathbb{Z}$,

$$d_{k+1} = \tilde{\lambda}_0^{(k)} d_k, \quad \mu_k \text{ a.s..}$$

Since $\tilde{\lambda}_0^{(k)}$ is the dual of $g \to g \circ T_k$ with respect to the spaces $L^2(\mathcal{E}_k, \mu_k)$ and $L^2(\mathcal{E}_{k+1}, \mu_{k+1})$, $\tilde{\lambda}_0^{(k)} (v \circ T_k) = v$ for any $v : \mathcal{E}_{k+1} \to \mathbb{C}$ and $(T_k)_* \mu_k = \mu_{k+1}$, it follows that for any integer $k$,

$$d_{k+1} \circ T_k = d_k$$

or, equivalently, since $Y_{k+1} \circ T_k = Y_k + u_k$,

$$Y_k - W_k = d_k = Y_{k+1} \circ T_k - W_{k+1} \circ T_k = Y_k + u_k - W_{k+1} \circ T_k.$$ 

Therefore, $u_k = W_{k+1} \circ T_k - W_k$. This equality holds true $\mu_k$-almost surely, but since both sides are continuous and $\mu_k$ assigns positive mass to open sets we derive that it holds true for any point in $\mathcal{E}_k$. The proof of Theorem 2.7 (i) is complete. □

Now we will prove Theorem 2.7 (ii). By Theorem 2.10 there exists a constant $R_2$ so that

$$\frac{1}{n} \text{var}_{\mu_0}(S_{0,n} u) - \frac{1}{n} \sum_{j=0}^{n-1} \lambda''_{j+n}(0) \leq \frac{R_2}{n}$$

where we used that $\lambda''_{j}(0) = \Pi''_{j}(0)$ (recall our assumption that $\mu_j(u_j) = 0$). Let $T, f, u$ satisfy the conditions stated in Theorem 2.7 (i), and let $(\lambda(z), h(z), \nu(z))$ be the RPF triplet corresponding to the operators $L_z$ given by

$$L_z g(x) = \sum_{y \in T^{-1}(x)} e^{f(y) + z u(y)} g(y).$$

Then (see 2.2), since $u$ does not admit a co-boundary representation,

$$\sigma^2 = \lambda''(0) = \lim_{n \to \infty} \frac{1}{n} \text{var}_{\mu}(S_n u) > 0.$$ 

Note that $\sigma^2 = \lambda''(0) \geq c > 0$ for some constant $c$ which depends only on the initial parameters (as Theorem 2.4 holds true also when $T_j, f_j$ and $u_j$ do not depend on $j$). Let $\varepsilon > 0$ and let $\delta$ as in Theorem 2.6. Then $|\lambda_j(z) - \lambda(z)| \leq \varepsilon$ for any $j$, and so by the Cauchy integral formula,

$$|\lambda''_j(0) - \lambda''(0)| = |\lambda''_j(0) - \sigma^2| \leq c \varepsilon.$$
for some constant $c$ which depends only on the initial parameters. We conclude that
\[ \left| \frac{1}{n} \sum_{j=0}^{n-1} \lambda''_{j+n}(0) - \sigma^2 \right| \leq c \varepsilon \]
and therefore when $c \varepsilon < \sigma^2$ we obtain that
\[ \frac{1}{n} \text{var}_{\mu_0}(S_{0,n}u) \geq \sigma^2 - c \varepsilon - \frac{R_2}{n} \]
which completes the proof of Theorem 2.7 (ii).

4.3. **Self normalized Berry-Essen theorem.** Set $\sigma_{0,n} = \sqrt{\text{var}_{\mu_0}(S_{0,n}u)}$, and suppose that
\[ \lim_{n \to \infty} \sigma_{0,n} n^{-\frac{1}{3}} = \infty. \]
By replacing $u_j$ with $u_j - \mu_j(u_j)$, we can assume without a loss of generality that $\mu_j(u_j) = 0$, which implies that $\mu_j(S_{0,n}) = 0$ for any $n \geq 1$, since $(T_j)_{\ast} \mu_j = \mu_{j+1}$ for any $j$. By Theorem 2.9,
\[ |\text{var}_{\mu_0}(S_{0,n}u) - \sum_{j=0}^{n-1} \Pi''_j(0)| \leq R_2 \]
for some constant $R_2$, and therefore,
\[ \lim_{n \to \infty} n^{-\frac{1}{3}} \sum_{j=0}^{n-1} \Pi''_j(0) = \infty. \]
Next, since $\lambda_j(0) = 1$ and $\mu_j = \nu_j^{(0)}$ for any $n \geq 1$ and $j$ we have $(L_{0,n}^{0,n})_{\ast} \mu_n = \mu_0$ and therefore for any $z \in \mathbb{C}$,
\[ \mu_0(e^{zS_{0,n}u}) = \mu_n(L_{0,n}^{0,n} e^{zS_{0,n}}) = \mu_n(L_{z}^{0,n} 1). \]
Consider now the analytic function $\varphi_{0,n} : U \to \mathbb{C}$ given by
\[ \varphi_{0,n}(z) = \frac{\mu_n(L_{0,n}^{0,n} 1)}{\lambda_{0,n}(z)} = \int \frac{L_{0,n}^{0,n} 1(x)}{\lambda_{0,n}(z)} d\mu_n(x) \]
where $U$ is the neighborhood of 0 specified in Theorem 2.4. Then by (4.10) for any $z \in U$, $j \in \mathbb{Z}$ and $n \geq 1$,
\[ \mu_j(e^{zS_{0,n}u}) = \lambda_{0,n}(z) \varphi_{n}(z) = e^{\sum_{j=0}^{n-1} \Pi_j(z)} \varphi_{0,n}(z). \]
Next, by Theorem 2.9 (ii) we have $\lambda'_{0,n}(0) = \sum_{j=0}^{n-1} \Pi''_j(0) = \mathbb{E}_{\mu_j} S_{j,n} u$, and therefore by (4.11),
\[ \varphi'_{0,n}(0) = 0. \]
Now, by taking a ball which is contained in the neighborhood $U$ specified in Theorem 2.4 we can always assume that $U = B(0, r_0)$ is a ball around 0 with radius $r_0 > 0$. We claim that there exists a constant $A > 0$ such that for any $n \in \mathbb{N}$ and $z \in U_0$,
\[ |\varphi_{0,n}(z)| \leq A. \]
Indeed, by (25), there exist constants $A_1, k_1 > 0$ and $c \in (0, 1)$ such that for any $z \in U_0$ and $n \geq k_1$,

\begin{equation}
\left\| \frac{L_{z,n} 1}{\lambda_{0,n}(z)} - h_n(z) \right\|_\alpha \leq A_1 \delta^n
\end{equation}

By Theorem 2.4, for any $n \geq 0$, $j \in \mathbb{Z}$ and $z \in U_0$ we have $\|h_n(z)\|_\alpha \leq C_1$, and (4.14) follows.

Next, applying Lemma 2.8.2 in [16] with $k = 1$ we deduce from (4.13) and (4.14) that there exists a constant $B_1 > 0$ such that

\begin{equation}
|\varphi_{0,n}(z) - \varphi_{0,n}(0)| = |\varphi_{0,n}(z) - 1| \leq B_1 |z|^2
\end{equation}

for any $z \in U_1 = B(0, \frac{1}{2}r_0) = \frac{1}{2}U$. Moreover, using (4.15) and the above Lemma 2.8.2 there exist constants $t_0, c_0 > 0$ such that for any $s \in [-t_0, t_0]$ and a sufficiently large $n$,

\begin{equation}
\left| \sum_{j=0}^{n-1} \Pi_j(is) - \frac{s^2}{2} \sigma_{0,n}^2 \right| \leq c_0 |s|^3 n + \frac{1}{2} R_2 s^2
\end{equation}

where we also used that $\sum_{j=0}^{n-1} \Pi_j(0) = \mathbb{E}_{\mu_0} S_{0,n} u = 0$ and that $|\Pi_j(z)| \leq C$ for some $C$ which does not depend on $j$ and $z$. Set $s_n = \frac{s^2 \sigma_{0,n}^2}{n^2}$ (which is bounded in $n$). Then, by (4.17), there exist constants $q, q_0 > 0$ so that $q_0 s_n \leq \min(t_0, \frac{1}{2}r_0)$ and that for any sufficiently large $n$ and $s \in [-q_0 s_n, q_0 s_n],

\begin{equation}
\Re \left( \sum_{j=0}^{n-1} \Pi_j(is) \right) \leq -q s^2 \sigma_{0,n}^2.
\end{equation}

Next, by the Esseen-inequality (see [35]) for any two distribution functions $F_1 : \mathbb{R} \rightarrow [0, 1]$ and $F_2 : \mathbb{R} \rightarrow [0, 1]$ with characteristic functions $\psi_1, \psi_2$, respectively, and $T > 0$,

\begin{equation}
\sup_{x \in \mathbb{R}} |F_1(x) - F_2(x)| \leq \frac{2}{\pi} \int_0^T \left| \frac{\psi_1(t) - \psi_2(t)}{t} \right| dt + \frac{24}{\pi T} \sup_{x \in \mathbb{R}} |F_2'(x)|
\end{equation}

assuming that $F_2$ is a function with a bounded first derivative. Set $T_n = \frac{q_0 s_n^3}{2 c_0 n^2}$, which converges to $\infty$ as $n \rightarrow \infty$. For any real $t$ set $t_n = \sigma_{0,n}^{-1} t$. Let $t \in [-T_n, T_n]$. Then by (4.12),

\begin{equation}
|\mu_0(e^{it \sigma_{0,n}^{-1} S_{0,n} u}) - e^{-\frac{1}{2}t^2}| \leq e^{\sum_{j=0}^{n-1} \Re(\Pi_j(it_n))} |\varphi_{0,n}(it_n) - 1| + e^{\sum_{j=0}^{n-1} \Re(\Pi_j(it_n))} - e^{-\frac{1}{2}t^2} := I_1(n, t) + I_2(n, t).
\end{equation}

By (4.18) we have $e^{\sum_{j=0}^{n-1} \Re(\Pi_j(it_n))} \leq e^{-q t^2}$ and therefore by (4.16),

$$I_1(n, t) \leq B_1 e^{-q t^2} t^2 \sigma_{0,n}^2.$$  

Using the mean value theorem, together with (4.17) applied with $s = t_n$, taking into account that $e^{\sum_{j=0}^{n-1} \Re(\Pi_j(it_u))} \leq e^{-q t^2}$, we derive that

$$I_2(n, t) \leq c_1 n q_0^{-3} (|t|^3 + t^2) e^{-c_2 t^2}$$

for some constants $c_1, c_2 > 0$. Let $F_1$ be the distribution function of $S_{0,n} u(X_0)$, where $X_0$ is distributed according to $\mu_0$, and let $F_2$ be the standard normal distribution. Since the functions $|t|e^{-qt^2}$ and $(|t|^3 + t^2)e^{-c_2 t^2}$ are integrable, applying
with these functions and the above $T = T_n$ we complete the proof of Theorem 2.8 taking into account that $\sigma_{i,n}^2$ converges faster to 0 than $T_n$. □

4.4. Local limit theorem and Edgeworth expansion of order three. Theorem 2.10 follows from the arguments in the proof of Theorem 2.2.3 in [16], which provides conditions for a central local limit theorem to hold true in the case when the asymptotic variance $\lim_{n \to \infty} \text{var}(S_{0,n}u)$ exists, and it is positive (using Fourier transforms). Still, when there exist constants $c_1, c_2 > 0$ so that for any sufficiently large $n$,

$$\text{var}(S_{0,n}u) \geq c_1 n$$

then, by replacing any appearance of $\sigma \sqrt{n}$ with $\text{var}(S_{0,n}u)$, we derive that in order to prove Theorem 2.10 it is sufficient to show that there exists constants $a, b > 0$ and $\delta (0, 1)$ so that for any $n \geq 1$ and $t \in [-\delta, \delta]$,

$$|\mu_0(e^{i t S_{0,n}u})| \leq ae^{-bn t^2}. \quad (4.21)$$

When $\sigma_{i,n}^2$ grows linearly fast in $n$ then by (4.17), there exists a constant $t_1 > 0$ so that for any $t \in [-t_1, t_1]$ and a sufficiently large $n$ we have

$$\sum_{j=0}^{n-1} \Re(\Pi_j(it)) \leq -qns^2 \quad (4.22)$$

which together with (4.12) yields (4.21).

Now we will show that Assumption 2.11 implies that (2.10) holds true. Indeed, as in (2.10) it follows that for any $J \subset I_k$ and $s \geq 1$,

$$\sup_{t \in J} \|L^s_{it}\|_{\alpha} \leq cr^s$$

for some $c = c(J)$ and $r = r(J) \in (0, 1)$ which may depend on $J$ but not on $s$.

Under our assumptions we have $\|L^s_{it}\|_{\alpha} \leq 6(1 + \frac{2\delta_0}{1 + \delta_0}) \leq B_J$ (see Theorem 2.3). Write

$$\{0 \leq m : B_J \|L^m_{it} - L^s_{it}\|_{\alpha} < 1 - \delta_0 \ \forall t \in J\} = \{m_1 \leq m_2 < \ldots\}.$$

for some strictly increasing infinite sequence of nonnegative integers $(m_i)_{i=1}^{\infty}$. Fix some $s$ so that $B_Jcr^s < \frac{\delta_0}{2}$, and for each $k$ set $i_k = sm_0k$. Then $m_{i_{k+1}} + sm_0 \leq m_{i_{k+1}}$ and by (2.11) we have

$$\lim_{n \to \infty} \frac{k_n}{\ln n} = \infty$$

where $k_n = \max\{k : m_{i_k} \leq n - m_{0}s\}$. Set $l_k = m_{i_k}$ and write:

$$L^0_{it} = L_{it}^{l_k} + sm_0, n - l_k - sm_0 \circ L_{it}^{l_{k-1}} + sm_0 \circ \cdots \circ L_{it}^{l_2} + sm_0 \circ L_{it}^{l_1} + sm_0 \circ 0_{it}.$$ 

The blocks of the form $L^0_{it} + sm_0$ satisfy

$$\|L^0_{it} + sm_0\|_{\alpha} < B_J^{-1}(1 - \delta_0) + cr^s < B_J^{-1}(1 - \frac{1}{2}\delta_0)$$

and by Theorem 2.3 the norm of the other block does not exceed $B_J$. Therefore, for any $n \geq 1$ we have

$$\sup_{t \in J} \|L^0_{it} + sm_0\|_{\alpha} \leq (1 - \frac{1}{2}\delta_0)^{k_n}$$

and since $\mu_0(e^{it S_{0,n}u}) = \mu_n(e^{it S_{0,n}1})$ and $k_n$ grows faster than logarithmically in $n$, we conclude that (2.10) holds true.
Next, when (2.10) holds true, then the following theorem is proved exactly as in [19]:

4.1. Theorem. There exists a sequence of polynomials

$$P_{j,n}(s) = \sum_{k=0}^{m_1} a_{j,n,k}s^k, \ n \geq 1$$

with random coefficients, whose degree $m_1$ does not depend on $j$ and $n$, so that for any $n \geq 1$ with $\Pi_{j,n,2} = \frac{1}{n} \sum_{m=0}^{n-1} \Pi_{j,m+1}(0) > 0$ we have

$$\sup_{s \in \mathbb{R}} \left| \sqrt{2\pi \mu_j} \{ x \in \mathcal{E}_j : S_{j,n}(x) \leq \sqrt{ns} \} - \frac{1}{\sqrt{\Pi_{j,n,2}}} \int_{-\infty}^{s} e^{-\frac{t^2}{2\Pi_{j,n,2}}} dt - n^{-\frac{1}{2}} P_{j,n}(s) e^{-\frac{s^2}{2}} \right| = o(n^{-\frac{1}{2}}).$$

In the case when $T_j, f_j$ and $u_j$ do not depend on $j$, such (and high-order) expansions were obtained in [7] and [28]. In [19] we obtained high order Edgeworth expansions for random dynamical systems, and, in concrete examples, we have managed to show that the additional required condition (Assumption 2.4 there) holds true, using some sub-additivity arguments (relying on Kingman’s theorem), together with the arguments in [4], which is impossible to generalize for general sequential dynamical systems (as there is no subadditivity of any kind). Therefore we will not formulate here an appropriate theorem, even though it is possible to obtain high-order Edgeworth expansions under appropriate version of the latter Assumption 2.4 (plus additional assumptions in the spirit of Assumption 2.11).

4.5. Martingale approximation, exponential concentration inequalities and moderate deviations. As we have explained at the beginning of Section 4, we can assume in course of the proof of Theorem 2.12 that $\mathcal{L}^{(j)}_0 1 = 1$ for each $j$, which means that $\lambda_j(0) = 1$ and $h_j^{(j)} \equiv 1$. Moreover, we can clearly assume that $\mu_j(u_j) = 0$ for each $j$. Fix $n \geq 1$, and let $x_0$ be a random member of $\mathcal{E}_0$ which is distributed according to $\mu_0$. Consider the random variables $X_1, X_2, ..., X_n$ given by

$$X_m = X_m^{(n)} = u_{n-m}(T_0^{n-m}x_0).$$

For each $m$, consider the $\mathcal{E}_m$-valued random variable $Z_m = T_0^{n-m}x_0$. Then $Z_{m+1}$ is a function of $Z_m$ and so the family of $\sigma$-algebras $\mathcal{F}_k = \mathcal{F}^{(n)}_k$, $k = 1, 2, ..., n$, which is given by

$$\mathcal{F}_k = \sigma\{Z_n, Z_{n-1}, ..., Z_{n-k}\} = \sigma\{Z_{n-k}\}$$

is increasing in $k$. For the sake of convenience, set $\mathcal{F}_m = \mathcal{F}_n$ when $m > n$. Next, we claim that

$$E[X_k | \mathcal{F}_{k-l}] = \mathcal{L}^{n-k,l}_0(u_{n-k}) \circ T_0^{n-k+l}.$$
Indeed, since for any \( j \) we have \( (L_j)^* \mu_{j+1} = \mu_j \) and \( (T_j)^* \mu_j = \mu_{j+1} \), for any measurable and bounded function \( g : \mathcal{E}_{n-k+l} \to \mathbb{R} \) we have

\[
\mathbb{E}[g(Z_{n-k+l})L_0^{n-k,l}(u_{n-k}) \circ T_0^{n-k+l}] = \]

\[
\int g(T_0^{n-k+l}x)(L_0^{n-k,l}(u_{n-k}) \circ T_0^{n-k+l})(x) \, d\mu_0(x) \]

\[
= \int g(x)(L_0^{n-k,l}(u_{n-k}))(x) \, d\mu_{n-k+l}(x) = \mu_{n-k+l}(L_0^{n-k,l}(g \circ T_{n-k} \cdot u_{n-k}))
\]

\[
= \mu_{n-k}(g \circ T_{n-k} \cdot u_{n-k}) = \mu_0(g \circ T_{n-k} \cdot u_{n-k} \circ T_{n-k}) = \mathbb{E}[g(Z_{n-k+l})X_k].
\]

By (2.5) and since

\[
\text{By (2.5) and since}
\]

\[
\mu \geq \mu_0 \leq \mu_n \leq \mu_{n-k} \leq \mu_{n-k+l} \leq \mu_{n+1},
\]

we have

\[
\|\mathbb{E}[X_k|\mathcal{F}_{k-l}]\|_{L^\infty} \leq C_1 \delta
\]

for some constants \( C_1 > 0 \) and \( \delta \in (0, 1) \), which depend only on the initial parameters. Set \( X_m = 0 \) for any \( m > n \) and then for any \( j \geq 1 \) set

\[
W_j = X_j + \sum_{s \geq j+1} \mathbb{E}[X_s|\mathcal{F}_{j+1}] - \sum_{s \geq j} \mathbb{E}[X_s|\mathcal{F}_j].
\]

Then \( \{W_j : j \geq 1\} \) is a martingale difference with respect to the filtration \( \{\mathcal{F}_j : j \geq 1\} \), whose differences are bounded by some constant \( C > 0 \). Observe that

\[
\|S_{0,n} u(x_0) - \sum_{j=1}^n W_j\|_{L^\infty} = \|\sum_{j=1}^n X_j - \sum_{j=1}^n W_j\|_{L^\infty} \leq C_2
\]

for some other constant \( C_2 \). Set \( M_n = \sum_{j=1}^n W_j \). Let \( (\Omega, \mathcal{F}, P) \) be a probability measure so large so that all the random variables defined above are defined on \( (\Omega, \mathcal{F}, P) \), and denote by \( \mathbb{E}_P \) with expectation with respect to \( P \). Then by the Hoeffding-Azuma inequality (see \[29\]), for any \( \lambda > 0 \) we have

\[
\max(\mathbb{E}_P[e^{\lambda M_n}], \mathbb{E}_P[e^{-\lambda M_n}]) \leq e^{\lambda^2 n C^2}
\]

and so, by the Markov inequality, for any \( t \geq 0 \),

\[
\mu_0\{x : |S_{0,n} u(x)| \geq C_1 + t\} = P(|\sum_{j=1}^n X_j| \geq C_1 + t) \leq P(|M_n| \geq t)
\]

\[
\leq P(M_n \geq t) + P(-M_n \geq t) \leq P(e^{\lambda_1 M_n} \geq e^{\lambda_1 t}) + P(e^{-\lambda_1 M_n} \geq e^{\lambda_1 t})
\]

\[
\leq 2e^{-t + \lambda_1^2 n C^2} = 2e^{-t + \frac{t^2}{2n C^2}}
\]

where \( \lambda_1 = \frac{t}{2C^2} \), which together with the previous estimates completes the proof of Theorem \[21\].

### 4.5.1. Moderate deviations theorems via the method of cumulants

Relying on \[20\] and using that \( (T_j)^* \mu_j = \mu_{j+1} \), we derive (by induction on \( s \)) exactly as in Section 2.1 of \[17\] that for any \( s \) and functions \( f_i : \mathcal{E}_{j+m_i} \), where \( i = 0, 1, ..., s \) and \( 0 \leq m_0 < m_1 < m_2 < ... < m_s \), the following multiple correlation estimate holds true:

\[
\left| \mu_j(\prod_{i=0}^s f_i \circ T_{j+m_i}) - \prod_{j=0}^s \mu_{j+m_i}(f_i) \right| \leq d M^s \sum_{i=1}^s \delta^{m_i-m_{i-1}}
\]

where \( d \) is some constant and \( M = \max\{\|f_i\|_\alpha : 0 \leq i \leq s\} \).
Recall that the $k$-th cumulant of a (bounded) random variable $W$ is given by
\[ \Gamma_k(W) = \frac{1}{i^k} \frac{d^k}{dt^k} \left( \ln \mathbb{E} e^{itW} \right) \Big|_{t=0}. \]
Relying on (4.23), we can apply Lemma 14 in [12], we derive that
\[ |\Gamma_k(S_{j,n}u(x_j) - \mu_j(S_{j,n}u))| \leq n(k!)^2 c_k^k \]
where $x_j$ is distributed according to $\mu_j$ and $c_k$ is some constant. Suppose that
\[ \lim_{n \to \infty} \sigma_{0,n} \frac{\alpha_0}{\alpha + \varepsilon_0} = \infty \]
for some $0 < \varepsilon_0 < \frac{1}{4}$, where $\sigma_{0,n} := \text{var}_{\mu_0}(S_{0,n}u)$. Then, Theorem 2.13 follows from Theorem 1.1 in [11], applied with $Z_n = S_{j,n}u - \mu_j(S_{j,n}u)$, $\Delta_n = n^{-3\varepsilon_0}$ and $\gamma = 2$, taking into account that for any $0 < \varepsilon_0 < \frac{1}{4}$ and $k \geq 3$ we have $n^{-\varepsilon_0} \leq n^{-3\varepsilon_0}(k-2)$. Note that several other types of moderate deviations type results follow from the above estimates of the cumulants, see [11]. Remark also that by Corollary 2.1 in [37] for any $n \geq 1$ we have
\[ (4.25) \quad \left| \mu_0 \left\{ x \in \mathcal{E}_0 : S_{0,n}u(x) - \mu_0(S_{0,n}u) \leq r \sigma_{0,n} \right\} \right| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} e^{-\frac{1}{2}t^2} dt \leq cn^{-\varepsilon_0} \]
for some constant $c$ which depends only on the initial parameters. This provides another proof of the CLT when the variances grow sufficiently fast (note: the rate $n^{-\varepsilon_0}$ is not optimal).

4.6. Logarithmic moment generating functions. In this section we will prove Theorem 2.14. Suppose that the limits
\[ \Pi(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Pi_j(z) \]
exist in some open disk $B(0, \delta)$ around 0 in the complex plane. Then $\Pi(z)$ is analytic in $z$ since it is a pointwise limit of a sequence of analytic functions which is uniformly bounded in $n$ (such limits are indeed analytic, as a consequence of the Cauchy integral formula). Next, we claim that we can construct a branch of the logarithm of $\mu_0(e^{zS_{0,n}u})$ on $B(0, \delta)$ so that for any $z \in B(0, \delta)$,
\[ (4.26) \quad \Pi(z) = \lim_{n \to \infty} \frac{1}{n} \ln \mu_0(e^{zS_{0,n}u}). \]
In view of (4.12), we can prove the above in the case when $L_z^{(j)}1 = 1$. In this case, we have
\[ \mu_0(e^{zS_{0,n}u}) = \mu_n(L_0^n e^{zS_{0,n}}) = \mu_n(L_0^n 1) \]
and so by (2.3) we have
\[ \lim_{n \to \infty} \left| \frac{\mu_0(e^{zS_{0,n}u})}{\lambda_{0,n}(z)} - \mu_n(h_n(z)) \right| = 0. \]
Since $h_n^{(0)} = 1$ and the norms $\|h_n(z)\|_n$ are uniformly bounded in $n$ and $z$, there exist positive constants $\delta_1, c_1$ and $c_2$ so that for any $z \in B(0, \delta_1)$ and $n \in \mathbb{N}$ we have $c_1 \leq |\mu_n(h_n(z))| \leq c_2$, which implies that for any sufficiently large $n$,
\[ C_1 \leq \left| \frac{\mu_0(e^{zS_{0,n}u})}{\lambda_{0,n}(z)} \right| \leq C_2. \]
where $C_1$ and $C_2$ are some positive constants. Therefore, a branch of the logarithm of $\mu_0(e^{z_0},n)$ can be defined so that (1.26) holds true. Note that we also used that $\sum_{j=0}^{n-1} \Pi_j(z)$ is a branch of $\lambda_{0,j}(z)$.

In order to prove that

$$\Pi(z) = \lim_{n \to \infty} \frac{1}{n} \ln l_n(L_z^{0,n})$$

we first observe that since $\lambda_j = \nu_{j+1}^{(z)}(L_z^{j+1})$, then by (3.14) we have

$$|\lambda_j(z) - g_{j,n}(z)| \leq C \delta^n$$

where

$$g_{j,n}(z,j) = \frac{l_{n+j+1}(L_z^{j+1})}{l_{n+j+1}(L_z^{j+1})}$$

and we also used that the norms $\|L_z^{j+1}\|_\alpha$ are bounded in $j$ and $z \in U$. Replacing $n$ with $n - j - 1$ we obtain that

$$\left| \sum_{j=0}^{n-1} \Pi_j(z) - \sum_{j=0}^{n-1} \ln g_{j,n-j-1}(z,j) \right| = \left| \sum_{j=0}^{n-1} \Pi_j(z) - (\ln l_n(L_z^{0,n}) - \ln l_0(L_z^{0})) \right|$$

is bounded in $n$.

All the statements from Theorem 2.14(i) follow from Theorem 2.9. The statements in parts (ii) and (iii) of Theorem 2.14 follow from the Gartner-Ellis theorem (see [10]), as noted in [43].

5. Non-stationary random environments

5.1. Random non-stationary mixing environemnts: LCLT. Let \( \{\xi_n : n \in \mathbb{Z}\} \) be a family of random variables taking values at some measurable space \( \mathcal{Y} \), which are defined on the same probability space \((\Omega, \mathcal{F}, P)\). Let \((\mathcal{X}, d)\) be a compact set and let \( \mathcal{E} \subset \mathcal{Y} \times \mathcal{X} \) a set measurable with respect to the product \( \sigma \)-algebra, so that the fibers \( \mathcal{E}_y = \{x \in \mathcal{X} : (y, x) \in \mathcal{E}\}, \omega \in \Omega \) are compact. The latter yields (see [9] Chapter III) that the mapping \( y \to \mathcal{E}_y \) is measurable with respect to the Borel \( \sigma \)-algebra induced by the Hausdorff topology on the space \( K(\mathcal{X}) \) of compact subspaces of \( \mathcal{X} \) and the distance function \( \text{dist}(x, \mathcal{E}_y) \) is measurable in \( y \) for each \( x \in \mathcal{X} \). Furthermore, the projection map \( \pi_{\mathcal{Y}}(y, x) = y \) is measurable and it maps any \( \mathcal{F} \times \mathcal{B} \)-measurable set to a \( \mathcal{F} \)-measurable set (see “measurable projection” Theorem III.23 in [9]). For each \( y \in \mathcal{Y} \), let \( f_y, u_y : \mathcal{E}_y \to \mathbb{R} \) be functions so that the norms \( \|f_y\|_\alpha \) and \( \|u_y\|_\alpha \) are bounded in \( y \). Let \( T_y \) be a family of maps from \( \mathcal{E}_y \) to \( \mathcal{X} \) so that \( P \)-a.s. we have

\[ T_{\xi_j} : \mathcal{E}_{\xi_j} \to \mathcal{E}_{\xi_{j+1}} \]

and the family \( \{T_j = T_{\xi_j} : j \in \mathbb{Z}\} \) satisfies Assumption 2.1 (with the same constants) with the spaces \( \mathcal{E}_j = \mathcal{E}_{\xi_j} \). Note that for (5.1) to hold true, we can just assume that \( \mathcal{E}_y = \mathcal{X} \) for each \( y \). We also assume here that the maps \( (y, x) \to f_y(x), u_y(x) \) and \((y, x) \to T_y(x)\) is measurable with respect to the \( \sigma \)-algebra induced on \( \{y, x) : y \in \mathcal{Y}, x \in \mathcal{E}_y\} \) from the product \( \sigma \)-algebra on \( \mathcal{Y} \times \mathcal{X} \). Then by Lemma 5.1.3 in [16], the norms \( \|f_y\|_\alpha \) and \( \|u_y\|_\alpha \) are measurable functions
of $y$. Let the transfer operators $L_z^{(y)}$ which maps functions $g$ on $E_y$ to functions on $T_yE_y$ by the formula

$$L_z^{(y)}g(x) = \sum_{a \in T_y^{-1}(x)} e^{f_y(a) + z a_y(a)} g(a)$$

By Lemma 4.11 in [30] there exist $L(y)$ and $x_i,y = x_i,y$ which are measurable $y$, so that

$$E_y = \bigcup_{j=1}^{L(y)} B_{y}(x_i,y,\xi)$$

where $B_{y}(x,\xi)$ is an open open around $x \in E_y$ with radius $\xi$. In the above circumstance we also assume that $L(y)$ is bounded in $y$ (e.g. when $E_y = X$). Consider the random operators $L_{z}^{(j)}$ given by $L_{z}^{(j)} = L_{z}^{(\xi_j)}$.

5.1. **Theorem.** For $P$-almost any $\omega$ the families $T_j = T_{\xi_j(\omega)}$, $f_j = f_{\xi_j(\omega)}$ and $u_j = u_{\xi_j(\omega)}$, where $j \in \mathbb{Z}$, satisfy Assumptions 2.2 and 2.3 and so all the results in Sections 3 and 2 hold true. Moreover, the RPF triplets $\lambda_j(z)$, $h_j^{(z)}$ and $\nu_j^{(z)}$ from Theorem 2.7 are measurable in $\omega$.

**Proof.** The first part of this theorem is trivial, and the main point behind stating it here is the measurability of the RPF triplets. In view of the limiting expressions of $\lambda_j(z)$, $h_j^{(z)}$ and $\nu_j^{(z)}$ from Section 3. In order to show that the RPF triplets are indeed measurable, it is sufficient to show that functional $l_j$ given by

$$l_{j,\omega}(g) = \sum_{i=1}^{L_j(\xi_j)} g(x_i,\xi_j)$$

are measurable in $\omega$, which is a consequence of the measurability of $L(y)$ and the $x_i,y$’s.

In the rest of this section, we will impose restrictions on the process $\{\xi_j\}$, which will guarantee that the local central limit theorem described in [2.10] holds true, where it is sufficient to to derive that Assumption 3.1 holds true. In order to achieve that, we will rely on the following

5.2. **Assumption.** The sequence $\{\xi_j : j \in \mathbb{Z}\}$ satisfies the following $\phi$-mixing type condition: there exists a sequence $\phi(n)$, $n \geq 1$ so that $\sum_{n=1}^{\infty} \phi(n) < \infty$ and for any $j \in \mathbb{Z}$, $n \geq 1$, $A \in \sigma\{\xi_m : m \leq j\}$ and $B \in \sigma\{\xi_m : m \geq j + n\}$,

$$|P(A \cap B) - P(A)P(B)| \leq P(A)\phi(n)$$

where $\sigma\{X_i : i \in \mathcal{I}\}$ is the $\sigma$-algebra generated by a family of random variables $\{X_i : i \in \mathcal{I}\}$.

(ii) There exist points $y_1, y_2, ..., y_{m_0} \in \mathcal{Y}$ so that for any sufficiently large $s \in \mathbb{N}$ and all sufficiently small open neighborhoods $U_i$ of $y_i, i = 1, 2, ..., m_0$ and $s \geq 1$ we have

$$\lim_{n \to \infty} \frac{\sum_{m=1}^{n} P\{\xi_{m}^{sm_0+i} \in U_i ; \forall 1 \leq i \leq m_0s\}}{\sqrt{n \ln n}} = \infty$$

where we set $U_i = U_{k_i}$, for any $i > m_0$, if $i = m_0k_i + r$ some $0 \leq r < m_0$.

Under Assumption 5.2 set

$$L_z = L_z^{(y_{m_0})} \circ \cdots \circ L_z^{(y_2)} \circ L_z^{(y_1)}.$$
5.3. Theorem. Suppose that Assumption 2.2 holds true and that the following two conditions hold true:

(A1) For any compact $J \subset \mathbb{R}$, the family of maps $\{y \to \mathcal{L}_{it}^{(y)}\}$, where $t$ over $J$, is equicontinuous at the points $y_1, y_2, \ldots, y_{m_0}$, and $\mathcal{E}_y$ does not depend on $y$, when $y$ lies in some open neighborhood of one of the $y_i$’s.

(A2) The spectral radius of $L_{it}$ is strictly less than 1 for any $t \in I_h$, where in the non-lattice case we set $I_h = \mathbb{R} \setminus \{0\}$, while in the lattice case we set $I_h = (-\frac{2\pi}{h}, \frac{2\pi}{h})$.

Then Assumption 2.11 holds true.

After showing that Assumption 2.11 holds true we can apply Theorem 2.10 with $T_j = T_{\xi_j}, f_j = f_{\xi_j}$ and $u_j = u_{\xi_j}$, when, $P$-a.s. the variance $\text{var}_{\mu_0}(S_{0,n}\nu)$ grows linearly fast in $n$, which holds true when $\mathcal{E}_y = X$ and $\|\mathcal{L}_y^{(y)} - \mathcal{L}_z^{(y)}\|_a < \varepsilon_0$ for any $y \in \mathcal{Y}$ and complex $z$ in some neighborhood of 0, where $\mathcal{L}_y$ and $\varepsilon_0$ are specified in Theorem 2.7 (ii).

Proof of Theorem 5.3 Let $J \subset \mathbb{R}$ be a compact set and let $B_J$ be as define in Assumption 2.11. Let $s \in \mathbb{N}$ and $\delta_0 \in (0, 1)$. Then, in view of our Assumption (A1), there exist open neighborhoods $U_j$ of $y_j$, $j = 1, 2, \ldots, m_0$ so that $\|\mathcal{L}_y^{(y_j)} - \mathcal{L}_y^{(y_j)}\|_a < \varepsilon_0$ for any $t \in J$, where $\varepsilon_0$ satisfies that $s\varepsilon_0B_j^3 < 1 - \delta_0$. Using that $\|\mathcal{L}_y^{(y_j)}\|_{\delta_0} \leq B_J$ (see Theorem 2.3), and that for any families of operators $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ we have

$$A_1 \circ A_2 \circ \cdots \circ A_m - B_1 \circ B_2 \circ \cdots \circ B_m = \sum_{j=1}^{m} A_1 \circ A_2 \circ \cdots \circ A_{j-1} (A_j - B_j) B_{j+1} \circ B_{j+2} \circ \cdots \circ B_m$$

we obtain that for any $t \in J$ and $j \in \mathbb{Z}$ so that $\xi_{i+k} \in U_j$ for any $1 \leq k \leq m_0s$, we have

$$B_J\|\mathcal{L}_{it}^{j+m_0s} - \mathcal{L}_{it}^j\|_a \leq s\varepsilon_0B_j^3 < 1 - \delta_0$$

where

$$\mathcal{L}_{it}^{j+m_0s} = \mathcal{L}_{it}^{(\xi_{j+m_0s})} \circ \cdots \circ \mathcal{L}_{it}^{(\xi_{j+2})} \circ \mathcal{L}_{it}^{(\xi_{j+1})}.$$  

Next, set $\Gamma_m = \{\omega : \xi_{m+m_0i} \in U_i; \forall 1 \leq i \leq m_0s\}$ and

$$S_n = \sum_{m=1}^{m_0s} \mathbb{I}_{\Gamma_m}$$

where $\mathbb{I}_\Gamma$ is the indicator of a set $\Gamma$. Then $S_n$ does not exceed the number of $j$’s between 1 and $n$ so that 5.3 holds true. In the above circumstances, we can use Theorem 2.4 in 17 with the random vectors $\{\mathbb{I}_{\Gamma_m} : 1 \leq j \leq m_0s\}, m = 1, 2, \ldots$ and derive from (2.11) there that for any $t \geq 0$,

$$P\{|S_n - ES_n| \geq t + c\} \leq 2e^{-c_1t^2}$$

where $c$ and $c_1$ are positive constants which may depend only on $m_0$ and $s$. Taking $t$ of the form $t = t_n = \theta \sqrt{n \ln n}$ for an appropriate $\theta$, we derive from the Borel-Cantelli lemma that $P$-a.s. for any sufficiently large $n$ we have

$$|S_n - ES_n| \leq t_n + c, P - \text{a.s.}$$
which together with (5.3) yields that

\[ \lim_{n \to \infty} \frac{S_n}{\sqrt{n \ln n}} = \infty \]

and so (2.11) holds true P-a.s. with the operators \( \mathcal{L}_{U_i}^{(j)} \), where, in fact, in our circumstances the numerator inside the limit expression in (2.11) grows faster than \( \sqrt{n \ln n} \).

### 5.2. Examples

We will provide here several examples in which Assumption 5.2 is satisfied. First, when considering the simple case when \( \mathcal{Y} = \mathbb{Z} \), then for any \( U_1 \subset (-1,1) \) which contains 0 we have

\[
\sum_{m=1}^{n} P\{\xi_{mm0s+i} \in U_i; \forall 1 \leq i \leq m0s\} = \sum_{m=1}^{n} P\{\xi_{mm0s+i} = y_i; \forall 1 \leq i \leq m0s\}
\]

where we set \( y_i = y_r \) if \( i = m0k + r \) for some integers \( k \geq 0 \) and \( 0 \leq r < m0 \). When \( \xi_i \)'s form an inhomogenous Markov chain (e.g. when they are independent) with \( n \)-th step transition probabilities \( p^{(i,n)}_{j,k} = P(\xi_{i+n} = k|\xi_i = j) \) (so that (5.2) holds true), then we only require that

\[
\sum_{m=1}^{n} \prod_{i=1}^{m0s} P(\xi_{mm0s} = y_i)p^{(m0s+i,1)}_{y_i, y_{i+1}}
\]

grows faster than \( \sqrt{n \ln n} \) in \( n \), for any \( s \). For instance, we could require that \( p^{(j,1)}_{y_i, y_{i+1}} \geq \delta_j \) and \( P(\xi_j = y_1) \geq r_j \) for some \( \delta_j, r_j > 0 \) and all \( i \)'s, which will give us linear growth if \( \delta_i \)'s and \( r_i \)'s are bounded from below by some positive constant, while in general we can impose certain restrictions on the \( \delta_i \) and \( r_i \)'s to obtain the desired growth rate (e.g. when \( \frac{1}{\delta_j} \) is at least of logarithmic order in \( j \) and \( r_j \) decays sufficiently slow to 0 as \( j \to \infty \)). Note that in these circumstances condition (i) from Theorem 5.3 trivially holds true since \( \mathcal{L}_{U}^{(y)} \) are locally constant in \( y \) around \( y_1, \ldots, y_{m0} \).

A close but more general situation is the case when the maps \( y \to T_{y, f_y, u_y} \) are locally constant around \( y_1, y_2, \ldots, y_{m0} \) and \( \{\xi_j\} \) is an inhomogenous Markov chain so that,

\[
P(\xi_{i+1} \in U|\xi_i = x) = \int_U p_i(x, y)d\eta(y), \ \forall x \in V
\]

for any sufficiently small neighborhoods \( U \) and \( V \) some \( y_i \) and \( y_j \), respectively. Here \( \eta \) is some probability measure on \( \mathcal{Y} \) which assigns positive mass to open sets, and \( p_i(x, y) \) are functions which are bounded from below by some positive constants \( \delta_i \). In this case we have

\[
\sum_{m=1}^{n} P\{\xi_{mm0s+i} \in U_i; \forall 1 \leq i \leq m0s\} \geq \left( \prod_{j=1}^{m0} \eta(U_j) \right)^s \sum_{m=1}^{n} \prod_{i=1}^{m0s} \delta_{mm0s+i}.
\]

Imposing some restrictions on the \( \delta_i \)'s we will get that the above right hand side grows faster than \( \sqrt{n \ln n} \). For instance, when \( \frac{1}{\delta_j} \leq c \ln j \) then, for any \( s \),

\[
\prod_{i=1}^{m0s} \delta_{mm0s+i}
\]
is of order $\frac{1}{\ln m}$ in $m$, where $\theta = m_0 s$, and so
\[ \sum_{i=1}^{n} \prod_{j=1}^{m_0 s} \delta_{nm_0 s + i} \]
is at least of order $\frac{n}{\ln n}$ in $n$. When $Y$ is compact and the densities $p_i(x,y)$ are bounded from below and above by some positive constant then condition (5.2) holds true with $\delta_i$'s bounded from below. Still, relying on (5.2) our arguments allow that $\lim \inf \delta_i = 0$, as $i \to \infty$, as described above.

Finally, let $(Q,T,m)$ be a mixing Young tower (see [41] and [42]) whose tails $\nu\{R \geq n\}$ are of order $n^{-a}$ for some $a > 0$ (here $\nu$ is the original measure on the tower, $R$ is the return time function and $m$ is the invariant mixing probability measure). Consider the case when $\xi_j(q) = H_j(T^j q)$, where $q$ is distributed according to $m$ and $H_j$ is a function that is constant on elements of the partition defining the tower. Then by Lemma 5 in [21] the inequality (5.3) holds true with a summable sequence $\phi(n)$. Next, suppose that $T$ has a periodic point $q_0$ with period $m_0$ and that $H_j(T^m q_0) := y_m$ does not depend on $j$ for each $m = 0, 1, \ldots, m_0 - 1$. We will show now that (5.3) holds true with the above $y_i$'s. Let $U_i$ be a neighborhood of $y_i$, where $i = 0, 1, \ldots, m_0 - 1$, and let $s \geq 1$. Observe that $(\xi_{ms_{m_0} + i}(q))_{i=1}^{s_{m_0}}$ is a fixed point of $q$ when $q$ lies in a set of the form $T^{-ms_{m_0}} A_s$, for some open neighborhood $A_s$ of the periodic point $q_0$. Here the power $s$ stands for concatenation: $a^{s_1 s_2} = a a a a a \ldots a$. Since $m$ is $T$-invariant we derive that
\[ \sum_{i=1}^{n} P\{\xi_{ms_{m_0} + i} \in U_i; \forall 1 \leq i \leq m_0 s\} \geq n m(A_s) \]
and so (5.3) holds true (as $m(A_s) > 0$). We note that the case when all of the $H_j$'s are Hölder continuous uniformly in $j$ can be considered, since then we can approximate (in the proof of Theorem 5.3) the $H_j$'s by functions which are constant on the above partitions.

5.3. Random sequential dynamical environments.

5.3.1. Random sequential distance expanding environments. Let $Y$, $E_y, T_y, f_y$ and $u_y$ be as in the beginning of Section 5.1. We assume here that $Y$ is a metric space and let $B$ be its Borel $\sigma$-algebra. Let $Y_j \subset Y$, $j \geq 0$ be a family of closed sets, $\theta_j : Y_j \to Y_{j+1}$, $j \in \mathbb{Z}$ be family of measurable maps and $P_j$, $j \in \mathbb{Z}$ be family of probability measures on $Y$ which are supported on $Y_j$, respectively, so that $(\theta_j)_* P_j = P_{j+1}$ for each $j$. For each $j$ and $m \geq 1$ set $\theta_{jm} = \theta_{jm_1} \circ \cdots \circ \theta_{j+1} \circ \theta_j$ and consider the case when $\xi_j = \theta_j y_0$, where $y_0$ is distributed according to $P_0$. In this section we will start with a certain type of one sided sequences and consider iterates of the form
\[ L_0^{(\xi, n)} := L_{0}^{(\xi_{j-1})} \circ \cdots \circ L_{0}^{(\xi_1)} \circ L_{0}^{(\xi_0)}. \]
Namely, we view here the $\xi_j$'s as a sequential dynamical random environment and consider (random) one sided sequences of maps $T_j$ and functions $f_j$ and $u_j$ of the form
\[ T_j = T_{\xi_j}, f_j = f_{\xi_j} \quad \text{and} \quad u_j = u_{\xi_j}. \]
Note that we can not apply directly Theorem 2.4 and all the other results from Sections 2 and 3 since we only have one sided sequences. In order to overcome
this difficulty we will need the following. Let \( \hat{Y} \) be the space of all sequences 
\( \hat{y} = (y_k)_{k=-\infty}^{\infty} \in Y^\mathbb{Z} \) so that \( y_{k+1} = S_k y_k \) for each \( k \), let \( \sigma : \hat{Y} \to \hat{Y} \) be the shift map 
given by \( \sigma x = (x_{k+1})_{k=-\infty}^{\infty} \) and set \( \xi_k = \sigma^k \xi_0 \) where \( \xi_0 \) is distributed according 
to the measure \( \hat{P} \) induced on \( \hat{Y} \) by the sequence of finite dimensional distributions 
given by 
\[
P_k \{ y : y_i \in A_i; \forall -k \leq i \leq k \} = P_{-k} \{ \bigcap_{i=-k}^{k} (\theta_{-i})^{-1} A_i \}.
\]
Note that the Kolmogorov extension theorem indeed can be applied (i.e. the family 
\( \{P_k\} \) is consistent) since \( (\theta_j)_* P_j = P_{j+1} \) for each \( j \). Henceforth, we will refer to 
the process \( \{\xi_j : j \in \mathbb{Z}\} \) as the “invertible extension” of the process \( \{\xi_j : j \geq 0\} \). 
Now, we can view \( T_j, f_j \) and \( u_j \) as functions of \( \xi_j \): they depend only on the 0-th 
coordinate of \( \xi_j \) (so now \( T_j, f_j \) and \( u_j \) are defined also for negative \( j \)’s). Henceforth, 
\( \{\lambda_j(z) : j \in \mathbb{Z}\}, \{h_j^{(z)} : j \in \mathbb{Z}\} \) and \( \{\nu_j^{(z)} : j \in \mathbb{Z}\} \) will denote the RPF triplets 
corresponding to the (random) family of operators \( L^{(j)}_z = L^{(z)}_{\xi_j} = L^{(\pi_0 \xi_j)} \), where \( \pi_0 \xi_j = y_0 \).

In the following section we will provide general conditions under which the results 
stated in Section 2 hold true. Note that formally, we will show that the limit 
theorems stated in Section 2 hold true with the extension \( \{\xi_j : j \in \mathbb{Z} \} \) as the random 
environment, but when the random Gibbs measure \( \mu_j \) given by \( d\mu_j = h_j^{(0)} d\nu_j^{(0)} \) 
depend only on \( \xi = \{\xi_j : j \geq 0\} \) then we can formulate all the limit theorems 
without passing to the invertible extension. Note that by taking \( \mu_n = l_n \) in (3.13), 
we see that the functional \( \nu_j^{(z)} \) depend only on \( \xi_j \) (\( l_n \) was defined before Theorem 3.1). 
Therefore, \( \mu_j \) depends only on \( \xi_j \) if the function \( h_j^{(0)} \) is deterministic. We 
refer the readers to Theorem 5.7 (and its proof) for conditions which guarantee 
that \( h_j^{(0)} = h \) for any \( j \), for some deterministic function \( h \) (take there \( T_j \) in place of \( S_j \)).

5.3.2. The LCLT. We assume here that for any Lipschitz function \( g \) on \( Y \), an 
integer \( s \geq 1 \) and \( t \geq 0 \) we have 
\[
(5.5) \quad P_0 \left\{ \sum_{j=0}^{n-1} g \circ \theta_0^{js} - \sum_{j=0}^{n-1} \mathbb{E}_{P_0} g \circ \theta_0^{js} \geq t + c_1 \right\} \leq c_2 e^{-c_3 t^2}
\]
where \( c_1, c_2 \) and \( c_3 \) are some positive constants which may depend on \( g \). The 
inequality (5.5) holds true when \( \theta_j \)'s are maps satisfying Assumption 2.1 and \( P_j \)'s 
are the appropriate (sequential) Gibbs measures corresponding to these maps.

In this section we will prove the following:

5.4. Theorem. Suppose that (5.5) holds true, that there exists a point \( y_0 \in \bigcap_{j=0}^{\infty} Y_j \) 
so that \( \theta_j y_0 = y_0 \) for each \( j \) and that the maps \( \theta_j \) are H"older 
continuous with exponent \( \beta \in (0,1] \) and H"older constant less or equal to \( K \), for some constants \( \beta \) and 
\( K \) which do not depend on \( j \). Assume, in addition, that for any open neighbourhood 
\( V \) of \( y_0 \) we have
\[
(5.6) \quad \lim_{n \to \infty} \frac{\sum_{j=1}^{n} P_j(V)}{\sqrt{n \ln n}} = \infty
\]
and that conditions (A1) and (A2) from Theorem 5.3 hold true with \( L_{it} = \mathcal{L}_{it}^{(y_0)} \). Then Assumption 2.11 holds true \( P_0 \)-a.s. with \( m_0 = 1 \) and the above \( L_{it} \).

Note that when verifying Assumption 2.11, we can use the one sided sequence \( \{c_j\} \) without passing to its invertible extension. When \( \theta_j, j \geq 0 \) are distance expanding maps which satisfy the first two parts of Assumption 2.1 assuming that all of them have the same fixed point, then we can take \( P_j \) to be the appropriate \( j \)-th Gibbs measure. In this case, by Theorem 5.4, we have \( P_j B_j(y_0, r) \geq C r^q \) for some \( q > 0 \), which implies that (5.6) holds true, since in this case the numerator grows linearly fast in \( n \). Note that a common fixed point exists, for instance when all \( \mathcal{J}_j \) are, the same torus and all \( \mathcal{S}_j \)’s vanish at the origin, and when \( \mathcal{S}_j \)’s form a non-stationary subshift of finite type (see Section 3.6) so that the matrices \( A_j, j \geq 0 \) defining the shift satisfy that \( A_j(a, a) = 1 \) for some \( a \in \mathbb{N} \) (and then we can take \( y_0 = (a, a, a, \ldots) \)).

Proof of Theorem 5.4. Let \( s \geq 1 \). As in the proof of Theorem 5.3 it is sufficient to show that for any neighborhood \( U \) of \( y_0 \) in \( \mathcal{Y} \) we have

\[
\lim_{n \to \infty} \frac{\sum_{m=0}^{n-s} \mathbb{I}_{\mathcal{Y}_m}(\omega)}{\sqrt{n \ln n}} = \infty, \quad \text{\( P_0 \)-a.s.}
\]

where \( \mathcal{Y}_m = \{ \omega : (\zeta_{ms}(\omega), \zeta_{ms+1}(\omega), \ldots, \zeta_{ms+s-1}(\omega)) \in U \times U \times \cdots \times U \} \), and \( \mathbb{I}_A \) is the indicator function of a set \( A \). Equivalently, we need to show that for \( P_0 \)-almost any \( y \) we have

\[
\lim_{n \to \infty} \frac{\sum_{m=0}^{n-s} \mathbb{I}_{\Delta_m}(y)}{\sqrt{n \ln n}} = \infty
\]

where

\[
\Delta_m = \bigcap_{j=0}^{s-1} \left( \theta_0^{ms+j} \right)^{-1} U = \left( \theta_0^{ms} \right)^{-1} U_{m,s}
\]

and \( U_{m,s} = U \cap \theta_0^{-1} U \cap \left( \theta_0^{ms} \right)^{-1} U \cap \cdots \cap \left( \theta_0^{ms} \right)^{-1} U \). Since \( y_0 \) is a common fixed point, for any \( r > 0 \) we have

\[
B_m(y_0, \frac{1}{k} K^{-\frac{1}{k}}) \subset \theta_0^{-1} B_{m+1}(y_0, r)
\]

where for each \( m, x \in \mathcal{Y}_m \) and \( \delta > 0 \) the set \( B_m(x, \delta) \) denotes an open ball in \( \mathcal{Y}_m \) around \( x \) with radius \( \delta \). Therefore, \( U_{m,s} \) contains an open ball \( V_{m,s} = B_m(y_0, 2 r_s) = B(y_0, 2 r_s) \cap \mathcal{Y} := V_s \) around \( y_0 \) in \( \mathcal{Y}_m \), whose radius does not depend on \( m \) (here \( B(y_0, 2 r_s) \) is the corresponding ball in \( \mathcal{Y} \)). Hence, it is sufficient to show that

\[
\lim_{n \to \infty} \frac{\sum_{m=0}^{n-s} \mathbb{I}_{V_s} \circ \theta_0^{ms}}{\sqrt{n \ln n}} = \infty, \quad \text{\( P_0 \)-a.s.}
\]

Let \( f \) be Lipschitz function so that

\[
\mathbb{I}_{B(0,r_s)} \leq f \leq \mathbb{I}_{V_s} = \mathbb{I}_{B(0,2r_s)}.
\]

Then

\[
\sum_{m=0}^{n-s} \mathbb{I}_{V_s} \circ \theta_0^{ms} \geq \sum_{m=0}^{n-s} f \circ \theta_0^{ms}.
\]

Note that \( \mathbb{E}_P f \circ \theta_0^{ms} = \mathbb{E}_{P_m} f = \int f dP_m \geq P_m(B(0, r_s)) \). Taking \( t \) of the form \( t = t_n = c \sqrt{n \ln n} \) in (5.5), for an appropriate \( c \), and using the Borel-Cantelli lemma
we derive that \( P_0 \text{-a.s.} \) for any sufficiently large \( n \),
\[
\left| \sum_{m=0}^{n-s} f \circ \theta_0^{ms} - \sum_{m=0}^{n-s} \mathbb{E}_{P_0} f \circ \theta_0^{ms} \right| \leq C \sqrt{n \ln n}
\]
where \( C \) is some constant. The latter inequality together with the former inequalities and (5.6) imply (5.7).

5.5. **Remark.** The proof of Theorem 5.4 proceeds similarly if we assume that there exists \( y_0, y_1, \ldots, y_{m_0-1} \in \mathcal{Y} \) so that for any \( j \) and \( i = 0, 1, \ldots, m_0 - 1 \) we have \( S_j \circ y_i = y_{i+1} \), where \( y_{m_0} := y_0 \), that (5.3) holds true, with \( y_i \) in place of \( y_i \) appearing there and with \( \zeta_{mm_0+i} \) in place of \( \zeta_{mm_0+i} \), and that for any family of Lipschitz functions \( g_j \), \( j \geq 0 \) with Lipschitz constant less or equal to 1 so that \( \sup_j |g_j| \leq 1 \) we have
\[
P_0 \left\{ \left| \sum_{j=0}^{n-1} g_j \circ \theta_0^j - \sum_{j=0}^{n-1} \mathbb{E}_{P_0} g_j \circ \theta_0^j \right| \geq t + c_1 \right\} \leq c_2 e^{-c_3 t^2 / n}
\]
for some positive constants \( c_1 \) and \( c_2 \) (which do not depend on the \( g_j \)'s). In this case we require that the spectral radius of
\[
\Lambda_{it} = \Lambda_{it}^{(y_{m_0-1})} \circ \cdots \circ \Lambda_{it}^{(y_1)} \circ \Lambda_{it}^{(y_0)}
\]
is less than 1 for any \( t \in I_{\mathcal{Y}} \). In particular we can consider non-stationary subshifts of finite type, with the property that for some \( a_1, a_2, \ldots, a_{m_0} \in \mathbb{N} \) and all \( j \)'s we have \( A_j(a_1, a_{i+1}) = 1 \), where \( a_{m_0+1} := a_1 \), which means that the periodic word \( (a_1, a_2, a_3, \ldots, a_{m_0}, a_1, a_2, \ldots, a_{m_0}, \ldots) = (a_1, a_2, \ldots, a_{m_0}) \in \mathbb{N} \) belongs to all of the \( E_j \)'s.

5.3.3. **Existence of limiting logarithmic moment generating functions.** We assume here that \( \mathcal{Y} \) is a compact metric space. Let \( S_j : \mathcal{Y} \to \mathcal{Y} \) be a family of maps satisfying all the conditions specified in Assumption 2.A, with \( E_j = \mathcal{Y} \), and consider the case when \( \theta_j = S_j \) for each \( j \). Let \( r_j : \mathcal{Y} \to \mathbb{R} \) be a family of maps so that the Hölder norms \( \|r_j\|_\alpha \) are bounded in \( j \), and let \( (\lambda_j(0), h_j^{(0)}, \nu_j^{(0)}) \) be the RPF triplet corresponding to the operators \( \mathcal{L}_0^{(j)} \) given by
\[
\mathcal{L}_0^{(j)} g(x) = \sum_{y \in S_j^{-1}(x)} e^{r_j(y)} g(y).
\]
In these circumstances, we take \( P_j = \mu_j \), where \( \mu_j \) is given by \( d\mu_j = h_j^{(0)} \, d\nu_j^{(0)} \).
Namely, we consider here a random sequential environment generated by a two sequence of distance expanding maps \( S_j \).

We will first need the following

5.6. **Theorem.** (i) The random pressure function \( \Pi_j(z) \) depends only on \( \zeta_j \).
(ii) When the maps \( S_j \) are Hölder continuous with the same exponent and Hölder constants which are bounded in \( j \) then \( P \text{-a.s.} \) we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Pi_j(z) - \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E}_P \Pi_j(z) = 0.
\]
In particular, when the distribution of \( \zeta_j \) does not depend on \( j \) (i.e., when \( P_j = \mu_j \) does not depend on \( j \)) then the limit \( \Pi(z) \) from Theorem 2.14 exists and \( \Pi(z) = \mathbb{E}_P \Pi_0(z) \).
The distribution of $\zeta_j$ does not depend on $j$ when, for instance, each one of the $S_j$’s has the form $S_j(x) = (m_1^{(j)}x_1, ..., m_d^{(j)}x_d) \mod 1$ for some positive integers $m_i^{(j)}$, where $x = (x_1, ..., x_d) \in E_j = T^d$ belongs to some $d$-dimensional Torus (here $\nu_j^{(0)} = \text{Lebesgue}$ and $h_j^{(0)} \equiv 1$). More generally, when there exists a probably measure $\mu$ on $\mathcal{X}$ which assigns positive mass to open sets, so that $(S_j, \mu), \nu \ll \mu$ for any integer $j$ and $r_j$ be defined by $e^{-r_j} = \frac{d(S_j)\mu}{\nu_j^{(0)}}$, then by Theorem 2.5 we have $\lambda_j(0) = 1$ and $\nu_j^{(0)} = \mu$. In this case, the claim that the distribution of $\zeta_j$ does not depend on $j$ means that $h_j^{(0)}$, $j \geq 0$ does not depend on $j$, and in Theorem 5.7 we will show that $h_j^{(0)}$ does not depend on $j$ and $\omega$ when the $S_j$’s are drawn at random according to some, not necessarily stationary, classes of processes (so we will have ”random random” non-stationary neighborhoods).

Proof of Theorem 5.6

Set

$$g_{j,n}(z, \zeta_j) = \frac{l_{n+j+1} \left( \mathcal{L}_{z,j,n}^j \right) \nu_{j,n}} {l_{n+j+1} \left( \mathcal{L}_{z,j,n+1}^j \right) \nu_{j,n}}.$$ 

Since $\lambda_j = \nu_j^{(z)} \nu_{j+1}^{(z)} \mathbf{1}$ and the norms $\|\mathcal{L}_j\|_{\mathcal{A}}$ are bounded in $j$ and $z \in U$, applying (5.13) we obtain that for any $n \geq 1$ and $j \in \mathbb{Z}$,

$$|\lambda_j(z) - g_{j,n}(z, \zeta_j)| \leq C\delta^n$$

where $C > 0$ and $\delta \in (0,1)$ are constants, and therefore $\lambda_j(z)$ and $\Pi_j(z)$ depend only on $\zeta_j$. We also derive from the above that there exists a constant $r_2 > 0$ so that for any $j$ and complex $z$ so that $|z| < r_2$ we have

$$|\Pi_j(z) - \ln g_{j,n}(z, \zeta_j)| \leq C_1 \delta^n$$

where $C_1 > 0$ is some constant we used that $\lambda_j(0) \geq a$ for some constant $a > 0$. Set

$$\hat{g}_{j,n}(z, \zeta_j) = \ln g_{j,n}(z, \zeta_j).$$

Let $\varepsilon > 0$ and let $k$ be so that $C\delta^k < \varepsilon$. In the circumstances of Theorem 5.6 (i), the maps $g_{j,k}(z, \zeta_j)$ are Hölder continuous functions of $\zeta_j$ with the same exponent and with Hölder constants which are bounded in $j$. Therefore, by Theorem 2.12 there exist constants $c_1, c_2$ and $c_3$, which may depend on $k$, so that for any $\varepsilon > 0$ we have

$$P \left\{ \left| \sum_{j=0}^{n-1} \hat{g}_{j,k}(z, \zeta_j) - \sum_{j=0}^{n-1} \mathbb{E}\hat{g}_{j,k}(z, \zeta_j) \right| \geq c_1 + t \right\} \leq c_2 e^{-c_3 \varepsilon^2}$$. 

By taking $t = t_n$ which grows faster than $\sqrt{n}$ but slower linearly in $n$, we derive from Borel Cantelli lemma that

$$\lim_{n \to \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \hat{g}_{j,k}(z, \zeta_j) - \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E}\hat{g}_{j,k}(z, \zeta_j) \right| = 0, \quad P - \text{a.s.}$$

and so for any $\varepsilon$,

$$\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \Pi_j(z) - \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E}\Pi_j(z) \right| < \varepsilon, \quad P - \text{a.s.}$$

which completes the proof of the theorem. □
Next, let $Q$ be a compact metric space and let $\hat{S}_j : X \to X$ and $\hat{r}_j : X \to \mathbb{R}$ satisfy the conditions in Assumptions 2.1 and 2.2 with $E_j = Q$. Let $(\tilde{\lambda}_j(0), \tilde{r}_j(0), \hat{\nu}_j(0))$ be the RPF triplets corresponding to the transfer operator generated by $S_j$ and $\hat{r}_j$ and let $\overline{\mu}_j = \hat{h}_j \nu_j(0)$ be the appropriate Gibbs measure. Let $\eta_j = \tilde{S}_j(0)\eta_0 = \tilde{S}_{j-1} \circ \cdots \circ \tilde{S}_1 \circ \tilde{S}_0 \eta_0$ be a sequence of random variables, where $\eta_0$ is distributed according to $\overline{\mu}_0$.

5.7. Theorem. Suppose that there exists a probability measure $n$ on $Y$ which assigns positive mass to open sets, so that $(S_j), n \ll n$ for any integer $j$ and $r_j$ is defined by

$$e^{-r_j} = \frac{d(S_j)n}{dn}.$$ 

Assume also that $S_j, j \geq 0$ are random, and that they have form $S_j = S_{\eta_j}$. Then there exists a function $h : X \to \mathbb{R}$ so that, $\hat{P}$-almost surely, the distribution $P_j = \mu_j$ of all the $\zeta_j$’s is $\kappa := hdn$ (i.e. in the extension we have $\mu_j = \mu_{\zeta_j} = \kappa$ for each $j$).

Proof. Note first that the assumptions in the statement of the theorem mean that $S_j$ and $r_j$ are chosen at random by the invertible extension $\hat{\eta}_j, j \in \mathbb{Z}$ of $\eta_j, j \geq 0$ and that they depend only on the $0$-the coordinate. We can assume that $S_j, j < 0$ are chosen at random according to the $j$-the coordinate in this extension. In this case, we only need to show that there exists a function $h : X \to \mathbb{R}$ so that for any $j \geq 0$ we have $h(0) = h$ ($\hat{P}$-a.s.).

Let $\beta_j, j \geq 0$ be independent copies of $\eta_0$, and consider the setup of i.i.d. maps $S_{\beta_j}, j \geq 0$. Suppose that the above i.i.d. process is defined on a probability space $(\Omega, \mathcal{F}, P)$ so that $\beta_j = \theta^j \beta_0$ for some $P$ preserving map $\theta$. Consider the skew product map $S(\omega, x) = (\theta \omega, S_{\beta_0}(\omega) x)$. Then, as in [11], there exists an $S$-invariant measure of the form $P \times (hdn)$ for some strictly positive continuous function $h : X \to \mathbb{R}$ so that $n(h) = 1$. Exactly as in Section 4.1 in [18], it follows that for $\overline{\mu}_0$-almost any $q \in Q$,

$$\mathcal{L}_0^{(q)} \tilde{h} = \tilde{h}.$$ 

Fix some $y \in Q$ and set

$$\Gamma_y = \{q \in Q : \mathcal{L}_0^{(q)} \tilde{h}(y) = h(y)\}.$$ 

Then $\overline{\mu}_0(\mathcal{Q} \setminus \Gamma_y) = 0$ for each $y$. Since $\tilde{\nu}_j(0) = n$ we have

$$\overline{\mu}_j(\mathcal{Q} \setminus \Gamma_y) = \int_{\mathcal{Q} \setminus \Gamma_y} \tilde{h}_j(t) d\overline{\mu}_0(t) = 0$$

and therefore

$$\overline{\mu}_0 \left\{ \bigcap_{j=0}^{\infty} (\tilde{S}_j(t))^{-1} \Gamma_y \right\} = 1$$

and therefore for any $y$ and $j \geq 0$ we have

$$\mathcal{L}_0^{(\tilde{\eta}_j)} h(y) = h(y), \text{ a.s..}$$

Since both sides are continuous in $y$ and $Y$ is compact we conclude that

$$\mathcal{L}_0^{(\tilde{\eta}_j)} h = h, \hat{P} - \text{a.s.}$$

for any $j \geq 0$. Replacing $\tilde{\eta}_0$ with $\tilde{\eta}_k$ for any integer $k$, and using that $\hat{h}_{k+j} = \sigma^j \tilde{\eta}_k$ for any $j \geq 0$ we derive that

$$\mathcal{L}_0^{(\tilde{\eta}_k)} h = h.$$
for any $k$. Since $\lambda_j(0) = 1$ for any $j$, using (2.3) we derive that
\[ \lim_{n \to \infty} \mathcal{L}_0^{(n_k-1)} \circ \mathcal{L}_0^{(n_k-2)} \circ \cdots \circ \mathcal{L}_0^{(n_k-n)} h = n(h) h_k = h_k \]
and so $h_k = h$ for any $k$. Note that (2.4) holds true for Hölder continuous functions, but for real $z$’s the converges itself, without rates, holds true for continuous functions by monotonicity arguments due to Walters (see the proof of Proposition 3.19 in [30] and [40]).

5.3.4. **Converges rate towards the moments.** Suppose that $S_j, j \in \mathbb{Z}$ is a nonstationary subshift of finite type so that $h_j^{(0)}$ does not depend on $j$, i.e. that $(S_j)_{\mu} = \mu$ for some probability measure $\mu$ (e.g. when $S_j$ is random). Moreover, assume that $T_j, f_j$ and $u_j$ depend only on the $j$-the coordinate $X_j$. Then by (5.10) the random variables $\Pi_j(z)$ can be approximated exponentially fast in the $L^\infty$ norm by functions of the coordinates at places $j, j+1, \ldots, j+n$. Taking into account Theorem 3.3 (ii), we conclude that all the conditions of Theorem 2.4 in [17] hold true with $\ell = 1$, with $\Pi_j(z)$ in place of $\xi_j$ (from there) and with any bounded function $F$ which identifies with the function $G(x) = x$ on a compact set which contains all the possible values of all of the $\Pi_j(z)$’s. In particular, (2.11) from [17] holds true, and so for any $t \geq 0$, $r \geq 1$ and $n \geq 1$,

\begin{equation}
(5.12) \quad P\{ \sum_{j=0}^{n-1} \Pi_j(z) - n E \Pi_0(z) \geq t + C n \delta + r \} \leq 2 e^{-C \frac{r^2}{n}}
\end{equation}

where $C$ is some positive constant. Taking $r$ of the form $r = a \ln n$ for an appropriate $a$ we derive that for some constant $C_1 > 0$, for any $t \geq 0$ and $n \geq 1$ we have

\[ P\{ \sum_{j=0}^{n-1} \Pi_j(z) - n E \Pi_0(z) \geq t + C_1 \ln n \} \leq 2 e^{-C_1 \frac{a^2}{\ln n}}. \]

Taking $t$ of the form $t = t_n = a_n \frac{1}{n} \ln \frac{1}{a_n} n$ and using the Borel Cantelli lemma we derive that $P$-a.s. for any sufficiently large $n$,

\[ \frac{1}{n} \sum_{j=0}^{n-1} \Pi_j(z) - E \Pi_0(z) \leq (1 + C_1) n^{-\frac{1}{2}} \ln^{\frac{3}{2}} n. \]

Since $\lambda_j(z)$ is analytic in $z$ and uniformly bounded in $z$ and $j$, it follows that for any $k \geq 1$ there exists a constant $b_k$ so that for any sufficiently large $n$,

\[ \frac{1}{n} \sum_{j=0}^{n-1} \Pi_j(z) - E \Pi_0(z) \leq b_k n^{-\frac{1}{2}} \ln^{\frac{3}{2}} n \]

and (2.3) follows from Theorem 2.9. The almost optimal Berry-Esseen inequality (2.9) follows by arguments similar to the ones in the proof of Theorem 2.8.

6. **Inhomogeneous Markov chains with transition densities**

Let $(\mathcal{X}, d)$ and $\mathcal{E}_j \subset \mathcal{X}$ be as at the beginning of Section 2. For any integer $j$ let $B_j$ the Banach space of all bounded Borel functions $g : \mathcal{E}_j \to \mathbb{C}$ together with the supremum norm $\| \cdot \|_\infty$. Next, let $r_j = r_j(x, y) : \mathcal{E}_j \times \mathcal{E}_{j+1} \to [0, \infty)$ be a family of integrable in $y$ Borel-measurable functions, $m_j, j \in \mathbb{Z}$ be a family of Borel probability measures on $\mathcal{E}_j$ and $\{ u_j : \mathcal{E}_j \to \mathbb{R} : j \in \mathbb{Z} \}$ be a family of functions so that $u_j \in B_j$ for each $j$. Consider the family of random operators $R_z^t, z \in \mathbb{C}$ which
map (bounded) Borel-functions $g$ on $E_{j+1}$ to Borel-measurable functions on $E_j$ by the formula

\[
R_z^{(j)} g(x) = \int_{E_{j+1}} r_j(x, y) e^{\alpha_j z} g(y) dm_{j+1}(y).
\]

### 6.1. Assumption.

For any $j \in \mathbb{Z}$,

\[
\|R_0^{(j)} 1\|_{\infty} < \infty
\]

where $1$ is the function which takes the constant value $1$ on $E_{j+1}$.

Observe that

\[
\|R_0^{(j)} 1\|_{\infty} = \sup_{g \in B_{j+1}, \|g\|_{\infty} \leq 1} \|R_0^{(j)} g\|_{\infty} := \|R_0^{(j)} \|_{\infty}
\]

and therefore under Assumption 6.1 for any integer $j$ we have $\|R_z^{(j)} \|_{\infty} < \infty$ for any $z \in \mathbb{C}$, namely, $R_z^{(j)}$ is a continuous linear operator between the Banach spaces $B_{j+1}$ and $B_j$. Moreover, under this assumption $R_z^{(j)}$ is analytic in $z$ when considered as a map between $B_{j+1}$ to $B_j$ since the series $\sum_{k=0}^{\infty} \frac{(u_{j+1})^k}{k!} z^k$ converges in $B_{j+1}$ for any $z \in \mathbb{C}$.

Under Assumption 6.1 for any $j \in \mathbb{Z}$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$ consider the $n$-th order iterates $R_z^{(j,n)}: B_{j+n} \to B_j$ given by

\[
R_z^{(j,n)} = R_z^{(j)} \circ R_z^{(j+1)} \circ \cdots \circ R_z^{(j+n-1)}.
\]

Then we can write

\[
R_z^{(j,n)} g(x) = \int_{E_{j+n}} r_j(n, x, y) g(y) dm_{j+n}(y)
\]

for some family $r_j(n, \cdot, \cdot) = r_j(n, x, y) : E_j \times E_{j+n} \to [0, \infty)$ of integrable in $y$ Borel measurable functions. We will assume that the following two sided random Doeblin condition holds true.

### 6.2. Assumption.

There exist a bounded family $\{n_j : j \in \mathbb{Z}\} \subset \mathbb{N}$ and positive numbers $\alpha_m(j) < 1$, $m \in \mathbb{N}$ such that for any integer $j$ and $m \geq n_j$,

\[
\alpha_m(j) \leq r_j(m, x, y) \leq (\alpha_m(j))^{-1},
\]

for any $x \in E_j$ and $y \in E_{j+m}$.

Next, in the case of integral operators we say that a family of triplets $(\lambda_j(z), h_j(z), \nu_j(z))$, $j \in \mathbb{Z}$ consisting of a complex number $\lambda_j(z) \neq 0$, a bounded Borel function $h_j(z)$ on $E_j$ and a complex measure $\nu_j(z)$ on $E_j$ is an RPF triplet for the family $R_z^{(j)}$ if for any $j \in \mathbb{Z}$,

\[
R_z^{(j)} h_j(z)_{j+1} = \lambda_j(z) h_j^{(z)}_{j+1}, \quad (R_z^{(j)})^* \nu_j^{(z)} = \lambda_j(z) \nu_j^{(z)}_{j+1}
\]

and \[
\nu_j^{(z)}(1) = \nu_j^{(z)}(h_j^{(z)}) = 1.
\]
6.1. Real and complex cones. Denote by $B_{j,+}$ the set of all nonnegative members of $B_j$. For any integer $j$ and $L > 1$ consider the real proper closed convex cone (see again Appendix A in [16]),

$$
K_{j,L,R} = \{ g \in B_{j,+} : g(x) \leq Lg(y) \forall x, y \in E_j \} \subset B_j.
$$

Then any nonzero member of $K_{j,L,R}$ is strictly positive. Let $K_{j,L} \subset B_j$ be the canonical complexification of $K_{j,L,R}$ (see Appendix A in [16]).

The following theorems summarize the properties of the cones $K_{j,R,L}, K_{j,L}$ and the relations between them and the operators $R^{(j)}_z$, and their proofs proceed exactly as in Chapter 6 of [16].

6.3. Theorem. Let $j \in \mathbb{Z}$ and $L > 1$.

(i) The complex cone $K_{j,L}$ is linearly convex, and

$$
\|g\|_{\infty} \leq 2\sqrt{2L}|m_j(g)| \quad \text{for any } g \in K_{j,L}
$$

and

$$
\|\mu\|_{\infty} \leq 12(1 - L^{-1})^{-1}|\mu(I)| \quad \text{for any } \mu \in K_{j,L}^*.
$$

(ii) For any $f \in B_j$ there exist $f_1, \ldots, f_8 \in K_{j,L}$ so that $f = f_1 + f_2 + \ldots + f_8$ and

$$
\|f_1\|_{\infty} + \|f_2\|_{\infty} + \ldots + \|f_8\|_{\infty} \leq r(L)\|f\|_{\infty}
$$

where $r(L) = \frac{16L(L+1)}{L-1}$.

6.4. Theorem. Suppose that there exist $j_0$ and $\alpha \in (0, 1)$ so that $\alpha_m(n) \geq \alpha$, for any $j_0 \leq n \leq 2j_0$. Then there exist a neighborhood $U_1$ of 0 and $d_0 > 0$ so that for any integer $j$, $z \in U_1$ and $j_0 \leq m \leq 2j_0$,

$$
R^{(j)}_z K'_{j+m,L} \subset K'_{j}
$$

and

$$
\Delta_{K_{j,L}}(R^{(j)}_z K'_{j+m,L}) < d_0
$$

where $L = 2\alpha^{-1}$.

Using the operators $L^{(j)}_z := R^{(-j+1)}_z$, the following result is proved almost exactly as in Chapter 6 of [16], using the arguments from Section [3].

6.5. Theorem. Suppose that Assumptions [6.7] holds true, that the norms $\|u_j\|_{\infty}$ are bounded in $j$ and that there exist $m_0$ and $\alpha \in (0, 1)$ so that $\alpha_m(n) \geq \alpha$, for any $m_0 \leq n \leq 2m_0$. Then, there exists a neighborhood $U$ of 0 which depends only on the initial parameters so that for any $z \in U$ and $j \in \mathbb{Z}$ there exists a triplet $\lambda_j(z)$, $h_j^{(z)}$ and $\nu_j^{(z)}$ consisting of a nonzero complex number $\lambda_j(z)$, a complex function $h_j^{(z)} \in K'_{\omega, 2\alpha^{-2}}$ and a complex continuous linear functional $\nu_j^{(z)} \in K'_{\omega, 2\alpha^{-2}}$ satisfying [6.4]. When $z = t \in \mathbb{R}$ then $h_j^{(t)} \in K'_{\omega, 2\alpha^{-2}, \mathbb{R}}$, $\nu_j^{(t)}$ is a probability measure and there exists constants $a, b > 0$, which depend only on the initial parameters, do that $\lambda_j(t) \in [a, b]$ and $h_j^{(t)} \geq a$.

Moreover, this triplet is analytic and uniformly bounded. Namely, the maps

$$
\lambda_j(\cdot) : U \to \mathbb{C}, \quad h_j^{(\cdot)} : U \to B_j, \quad \text{and} \quad \nu_j^{(\cdot)} : U \to B_j^*
$$

are analytic, where $B_j^*$ is the dual space of $B_j$, and there exists a constant $C > 0$ so that

$$
\max \left( \sup_{z \in U} |\lambda_j(z)|, \sup_{z \in U} \|h_j^{(z)}\|_{\infty}, \sup_{z \in U} \|\nu_j^{(z)}\|_{\infty} \right) \leq C
$$

where $\|\nu\|_{\infty}$ is the operator norm of a linear functional $\nu : B_j \to \mathbb{C}$. 

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Furthermore, there exists constants $C > 0$ and $\delta \in (0, 1)$ such that $P$-a.s. for any $n \geq 1$, $j \in \mathbb{Z}$, $z \in U_0$ and $q \in B_{j+n}$,

\begin{equation}
\left\| R^n_{j,n} q \frac{\lambda_j(z)}{\lambda_{j+n}(z)} - (\nu_{j+n}(q))_j \right\|_\infty \leq C\|q\|_\infty \cdot \delta^n
\end{equation}

where $\lambda_{j+n}(z) = \prod_{k=0}^{n-1} \lambda_{j+k}(z)$.

6.2. Limit theorems. Suppose that $R_n(\phi) 1 = 1$ for any $j$, namely that each $R_n(\phi)$ is a Markov operator (which means that $\lambda_j(0) = 1$ and $h_j(0) \equiv 1$). Note also that in this case, we obtain from (6.10) that $\nu_j(0)$ is absolutely continuous with respect to $m_j$. Let $\xi_0, \xi_1, \ldots$ be a Markov chain whose initial distribution is $\mu_0$ and whose $j$-th step transition operator is $R_n(j)$, which means that

$$E[g(\xi_{j+1}) | \xi_j, \xi_{j-1}, \ldots, \xi_1] = E[g(\xi_{j+1}) | \xi_j] = R_{j+1}^0(g(\xi_j))$$

for any bounded measurable function $g$. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which the above Markov chain is defined. We begin with the following exponential $\phi$-mixing type result

6.6. Theorem. For any integers $k \geq 0$, $n \geq 1$ and sets $A \in \sigma\{\xi_0, \xi_1, \ldots, \xi_k\}$ and $B \in \sigma\{\xi_m : m \geq k + n\}$ we have

$$|P(A \cap B) - P(A)P(B)| \leq CP(A)\delta^n.$$

Here $\sigma\{X_i : i \in \mathcal{I}\}$ is the $\sigma$-algebra generated by a family $\{X_i : i \in \mathcal{I}\}$ of random variables, and the constants $C > 0$ and $\delta \in (0, 1)$ coming from (6.10).

Proof. First, for any two sub-$\sigma$–algebras $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$, set

\begin{equation}
\phi(\mathcal{G}, \mathcal{H}) = \sup \left\{ \left| \frac{P(A \cap B)}{P(A)} - P(B) \right| : A \in \mathcal{G}, B \in \mathcal{H}, P(A) > 0 \right\}.
\end{equation}

Then (see [5], Ch. 4),

\begin{equation}
\phi(\mathcal{G}, \mathcal{H}) = \frac{1}{2} \sup \{ \|E[h|\mathcal{G}] - E[h]\|_\infty : h \in L^\infty(\Omega, \mathcal{H}, P), \|h\|_\infty \leq 1 \}.
\end{equation}

Let $k$ and $n$ be as in the statement of the theorem, and set

$$\mathcal{G} = \sigma\{\xi_0, \xi_1, \ldots, \xi_k\}, \quad \text{and} \quad \mathcal{H} = \sigma\{\xi_m : m \geq k + n\}.$$

Let $h$ be a $\mathcal{H}$-measurable random variable which is bounded $P$-a.s. by 1. Then we can write $h = H(\xi_{k+n}, \xi_{k+n+1}, \ldots)$ for some measurable function $H$ so that $|H| \leq 1$. Set $H_{n+k} = E[h|\xi_0, \xi_1, \ldots, \xi_{n+k}]$. Then $H_{n+k} = H_{n+k}(\xi_{n+k})$ is a function of $\xi_{n+k}$, and

$$E[h|\mathcal{G}] = E[H_{n+k}(\xi_{n+k})|\mathcal{G}] = R_{0}^{k,n+k} H_{n+k}(\xi_k)$$

where we used that $\sigma\{\xi_0, \xi_1, \ldots, \xi_{n+k}\}$ is finer than $\mathcal{G}$ (and the tower property of conditional expectations). Using (6.10) with $j = k$ and $z = 0$, taking into account that $|H_{n+k}| \leq 1$ and that $E[h] = E[H_{n+k}(\xi_{n+k})] = \mu_n(H_{n+k})$ we obtain that

$$\|E[h|\mathcal{G}] - E[h]\|_\infty \leq C\delta^n$$

and the proof of Theorem 6.6 is complete. 

$\square$
Consider the sequence of random variables given by $S_n = \sum_{j=1}^{n} u_j(\xi_j)$, $n \geq 1$. When the variance of $S_n$ grows linearly fast in $n$, then using Theorem 6.6, Stein's method (see Chapter 1 in [16]) or the so-called, Stein-Tikhomirov method (see [38]) yield almost optimal convergence rate in the (self normalized) CLT for the sequence $S_n = \sum_{j=1}^{n} u_j(\xi_j)$, but our analogue of Theorem 2.8 (discussed below) guarantees that we have optimal rate. Note also that several results such as the corresponding versions of Theorem 2.12 and Theorem 2.12 follow from Theorem 6.6 (see Theorem 2.9 in [17], applied in the case when $\beta_{\infty}(0) = 0$ and $r = 0$).

The proofs of all the other theorems stated in Section 2 and the applications in Section 5 proceed exactly as in the previous sections, expect from Theorem 2.7 (i) below) guarantees that we have optimal rate. Note also that several results such as $\beta_{\infty}(0) = 0$ and $r = 0$.

In particular, the version of Theorem 2.7 (ii) holds true, which means that the above variances grow linearly in $n$ when all of the $R_j^{(i)}$ are close to the same operators $R_j$, of the form $R_j g(x) = \int r(x, y) e^{u(y)} g(y) dm(y)$, where $r(x, y)$ and $u$ satisfy the conditions of this section with $r_j(x, y) = r(x, y)$ and $u_j(y) = u(y)$, and the variance of the partial sums $\sum_{j=1}^{n} u_j(\xi_j)$ formed by the stationary Markov chain $\{\xi_j\}$ generated by $R_0$ and the function $u$ grows linearly fast in $n$. Still, the conditions guaranteeing that a local CLT holds true are somehow not natural when considering Markov chains with transition densities (we consider here as "natural" arguments and conditions in the spirit of [31] and [32]). In what follows we will discuss different approaches to prove the limit theorems stated in Section 2 in the case of inhomogeneous Markov chains.

### 6.3. Convergence of the variances to $\infty$: Known results.

First, the following result was proved by D. Dologpyat and O. Sarig in a forthcoming paper under weaker conditions than we have in this paper (see Section 6.4): 6.7. **Theorem.** The variance $\text{var}(S_n)$ converges to $\infty$ as $n \to \infty$ if and only if there exists a sequence of random variables $Y_k$, $k \geq 1$, whose $L^2$-norms are bounded in $k$, so that for each $k$ we have $u_k(\xi_k) - \mathbb{E}u_k(\xi_k) = Y_{k+1} - Y_k$.

Our Theorem 2.7 (i) is an analogue of the above results in the case of sequential dynamical system.

Recall next that the contraction coefficient (due to Dobrushin) of $R_0^{(j)}$ is given by

$$\delta_j = \delta(R_0^{(j)}) = \sup_{g \in L_1} \sup_{x_1, x_2} |R_0^{(j)} g(x_1) - R_0^{(j)} g(x_2)|$$

where $L_1$ is the space of all Lipschitz functions with constant 1. In circumstances of Proposition 13 in [34], when $\sup_j \delta_j < 1$ then there exist constants $a, b > 0$ so that for any $n \geq 1$ we have

$$a \sum_{k=1}^{n} \text{var}(u_k(\xi_k)) \leq \text{var}(S_n) \leq b \sum_{k=1}^{n} \text{var}(u_k(\xi_k))$$

and therefore the variance converges to $\infty$ if and only if

$$\lim_{n \to \infty} \sum_{k=1}^{n} \text{var}(u_k(\xi_k)) = \infty$$

and it grows linearly fast in $n$ if

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \text{var}(u_k(\xi_k)) > 0.$$
6.4. A local central limit theorem. As we have mentioned in the introduction, D. Dologpyat and O. Sarig, and M. Peligrad have proved (in forthcoming papers) local central limit theorems for certain classes of inhomogeneous Markov chains (which do not necessarily have transition densities). More precisely, Dologpyat and Sarig (see https://mathematics.huji.ac.il/event/omri-sarig-weizmann) considered Markov chains whose transition probabilities satisfy a certain type of one-sided Doeblin condition, while Peligrad considered lower \( \psi \)-mixing inhomogeneous Markov chains (see http://homepages.uc.edu/~wangyz/seminar/seminar18FS.html). In this section we will present conditions for the local limit theorem to hold true in the presence of transition densities \( r_j(x, y) \). These conditions are somehow more natural in the Markov chain case than the conditions we had in Section 2 and are in the spirit of [31] and [32].

For any positive integer \( k_0 \) set \( \Gamma_{k_0} = \{ k \geq 0 : n_k = n_{k+k_0} = k_0 \} \), and for any \( k \in \Gamma_{k_0} \) set

\[
d_k(t) = \sup_{y_0} \int \Gamma_k(y_0, z_0, t) dm_{k_0+1}(z_0)
\]

where with \( x_0 = y_0 \) and \( x_{k_0+1} = y_{k_0+1} = z_1 \),

\[
\Gamma_k(y_0, z_0, t) = 1 - (r_k(k_0 + 1, y_0, z_0))^{-2} \int \int |e^{it \sum_{j=1}^{k_0} u_{k+j}(y_j)} - e^{it \sum_{j=1}^{k_0} u_{k+j}(y_j)}|^2 \times \prod_{j=0}^{k_0} (r_{k+j}(y_j, y_{j+1}) r_{k+j}(y_j, y_{j+1})) \prod_{j=1}^{k_0} dm_{j+k_0}(y_j) dm_{j+k_0}(x_j).
\]

Here the \( n_j \)'s are the ones coming from Assumption 6.2. Note that

\[
\Gamma_k(y_0, z_0, t) \leq 1 - c_{k_0} \int \int |e^{it \sum_{j=1}^{k_0} u_{k+j}(y_j)} - e^{it \sum_{j=1}^{k_0} u_{k+j}(y_j)}|^2 \times \prod_{j=1}^{k_0} dm_{j+k_0}(y_j) dm_{j+k_0}(x_j)
\]

for some constant \( c_{k_0} > 0 \), when all the above densities \( r_j(x, y) \) are bounded from above and below by positive constants which do not depend on \( j \) and \( k \), where \( j = k, k + 1, \ldots, k + k_0 \).

6.8. Theorem. Suppose that there exists a positive integer \( k_0 \) so that

\[
\lim_{n \to \infty} \frac{|\{1 \leq k \leq n : k \in \Gamma_{k_0}\}|}{\ln n} = \infty
\]

and that the variance of \( S_n \) grows linearly fast in \( n \). In both lattice case and non-lattice case, assume that for any compact \( J \subset \Omega \) we have

(6.13) \[
\liminf_{n \to \infty} \frac{1}{n} \{ \{1 \leq k \leq n : k \in \Gamma_{k_0}, \sup_{t \in J} d_k(t) < c(J)\} \} > 0
\]

for some \( c(J) < 1 \), where \( \Omega \) was defined in Theorem 6.4. Then for each such \( J \) we have

\[
\lim_{n \to \infty} \sqrt{n} \sup_{t \in J} |e^{it S_n}| = 0.
\]

The proof of Theorem is carried out essentially as described in Section 7.4.4 of [16], replacing the hitting times to the set defined there by the members of the sets appearing in the denominator on the left hand side of (6.13).
Similarly to Section 6.10 condition (6.13) holds true, when the $R_{it}^{(j)}$ are drawn at random according to some (non-stationary) process which satisfies certain mixing conditions. In fact, this condition also hold true when the function $\bar{y} = (y_1, y_2, \ldots, y_k) \rightarrow e^{it\sum_{j=1}^{k_0} u_{k+j}(y_j)}$ is not constant uniformly in $k \in \Gamma_{k_0}$, in the sense that for any $t \in I_k$,

\[(6.14) \inf_{(\bar{y}, \bar{z}) \in \Delta_k} |e^{it\sum_{j=1}^{k_0} u_{k+j}(y_j)} - e^{it\sum_{j=1}^{k_0} u_{k+j}(x_j)}|^2 \geq a(t) > 0\]

where $a(t)$ is some continuous function of $t$, and $\Delta_k$ is a measurable set so that

\[\int_{\Delta_k} \prod_{j=0}^{k_0} (r_{k+j}(y_j, y_{j+1}) r_{k+j}(x_j, x_{j+1})) \prod_{j=1}^{k_0} dm_{j+k_0}(y_j) dm_{j+k_0}(x_j) \geq b\]

for some constant $b$, which does not depend on $k$. When the transition densities are bounded from below and the measures $m_j$ are equivalent, with bounded Radon-Nikodym derivatives, then a constant $b$ with the above conditions will always exist for some sets $\Delta_k = \Gamma$ which do not depend on $k$. In this case we only require that (6.14) will be satisfied.

6.5. **Edgeworth expansions.** For each $\delta \in (0, 1)$, let $\Gamma_0$ be the set of all positive integers $j$ so that

\[(6.15) \limsup_{|t| \rightarrow \infty} \|R_{it}^{(j)}\|_\infty < 1 - \delta.\]

6.9. **Assumption.** There exists $\delta > 0$ so that

\[\lim_{n \rightarrow \infty} \frac{|1, n] \cap \Gamma_\delta}{\ln n} = \infty.\]

The following result is proved exactly as in [28].

6.10. **Lemma.** Let $k \in \mathbb{Z}$ be so that $\mathcal{E}_k = \mathcal{E}_{k+1} = \mathcal{M}$ for some compact connected manifold $\mathcal{M}$, and assume that $u_{k+1}$ and $u_{k+2}$ are piecewise smooth. Moreover, suppose that $n_k = n_{k+1} = 1$ and that the function $U_k(y_1, y_2)$ given by $(y_1, y_2) \rightarrow u_{k+1}(y_1) + u_{k+2}(y_2)$ is not piecewise constant. Then condition (6.15) holds true with the above $k$, for some $\delta = \delta_k$.

In view of this lemma, assumption 6.9 holds true when the $R_{it}^{(j)}$ are drawn at random according to some (non-stationary) process which satisfies certain mixing conditions, but, in fact, it holds true when the norms of some directional derivative of the $U_k$’s are bounded from below by some positive constant $c$ which does not depend on $k$, on sets $V_k$ whose volume measure is bounded from below by a positive constant (which also does not depend on $k$). We refer the readers’ to the arguments proceeding (6.5) in [28].

The following result is proved exactly as in [19]:

6.11. **Theorem.** (i) In the circumstances of Theorem 6.8 there exists a sequence of polynomials $P_{n,1}(s) = \sum_{j=0}^{n_1} a_{n,1,j} s^j$, $n \geq 1$, whose degrees $m_1$ do not depend on $n$, so that for any $n \geq 1$ with $\Pi_{n,2} := \sum_{j=1}^{n} \Pi_j(0) > 0$,

\[\sup_{s \in \mathbb{R}} \sqrt{2\pi} P(S_n \leq \sqrt{ns}) - \frac{1}{\sqrt{\Pi_{n,2}} \int_{-\infty}^{s} e^{-\frac{t^2}{2\Pi_{n,2}}} dt - n^{-\frac{1}{2}} P_{n,1}(s)e^{-\frac{s^2}{2}}} = o(n^{-\frac{1}{2}}).\]
(ii) If in addition Assumption 6.9 holds true then for each $k \geq 2$ there exists a sequence of polynomials $P_{n,k}(x) = \sum_{j=0}^{m_k} a_{n,j,k} s^j$, $n \geq 1$ whose degrees $m_k$ depend only on $k$, so that for any $d \geq 1$ and $n \geq 1$ with $\Pi_{n,2} > 0$,

$$\sup_{s \in \mathbb{R}} \left| \sqrt{2\pi} P(S_n \leq \sqrt{n}s) - \frac{1}{\sqrt{4\pi n^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2n}} dt - \sum_{j=1}^{d} n^{-\frac{j}{2}} P_{n,j}(s) e^{-\frac{s^2}{2}} \right| = o(n^{-\frac{j}{2}}).$$

Note that when the limits $\Pi(z)$ exist then we can also show that the above coefficients $a_{n,j,k}$ converge as $n \to \infty$ to certain limits. Note also that $|\Pi_{n,2} - \text{var}(S_n)|$ is bounded in $n$, and so it also grows linearly fast in $n$.

REFERENCES

[1] R. Aïmino, M. Nicol and S. Vaienti. Annealed and quenched limit theorems for random expanding dynamical systems, Probab. Th. Rel. Fields 162, 233-274, (2015).
[2] P. Arnoux and A. Fisher, Anosov families, renormalization and non-stationary subshifts, Erg. Th. Dyn. Syst., 25 (2005), 661-709
[3] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, second revised edition, Lecture Notes in Mathematics, Springer Verlag, 2008.
[4] O. Butterley, and E. Peyman Exponential mixing for skew products with discontinuities, Trans. Amer. Math. Soc. 369 (2017), no. 2, 783-803.
[5] R.C. Bradley, Introduction to Strong Mixing Conditions, Volume 1, Kendrick Press, Heber City, 2007.
[6] J.R. Chazottes and S. Guédev, Optimal concentration inequalities for dynamical systems, Comm. Math. Phys. 316 (2012), 843-889.
[7] Z. Coelho, W. Parry Central limit asymptotics for shifts of finite type, Israel J. Math. 69, (1990), no. 2, 235-249.
[8] J-P Conze and A. Raugi, Limit theorems for sequential expanding dynamical systems, AMS 2007.
[9] C. Castaing and M. Valadier, Convex analysis and measurable multifunctions, Lecture Notes Math., vol. 580, Springer, New York, 1977.
[10] A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications, 2nd edn. Applications of Mathematics, vol. 38. Springer, New York (1998).
[11] H. Döring and P. Eichelsbacher, Moderate deviations via cumulants, J. Theor. Probab. 26 (2013), 360-385.
[12] P. Doukhan and M. Neumann, Probability and moment inequalities for sums of weakly dependent random variables, with applications, Stochastic Process. Appl. 117 (2007), 878-903.
[13] D. Dragičević, G. Froyland, C. González-Tokman and S. Vaienti, A spectral approach for quenched limit theorems for random expanding dynamical systems, Commun. Math. Phys. 360, 1121-1187 (2018).
[14] L. Dubois, Projective metrics and contraction principles for complex cones, J. London Math. Soc. 79 (2009), 719-737.
[15] Y. Guivarc’h and J. Hardy, Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d’Anosov, Ann. Inst. H. Poincaré Probab. Statist. 24 (1988), no. 1, 73-98.
[16] Y. Hafouta and Yu. Kifer, Nonconventional limit theorems and random dynamics, World Scientific, Singapore, 2018.
[17] Y. Hafouta, Nonconventional moderate deviations theorems and exponential concentration inequalities, to appear in Ann. Inst. H. Poincaré Probab. Statist.
[18] Y. Hafouta, Limit theorems for some skew products with mixing base maps, preprint, arXiv:1808.00936.
[19] Y. Hafouta, Asymptotic moments and Edgeworth expansions for some processes in random dynamical environment, arXiv preprint 1812.06924.
Sequential RPF theorem and apps.

[20] H. Hennion and L. Hervé, Limit Theorems for Markov Chains and Stochastic Properties of Dynamical Systems by Quasi-Compactness, Lecture Notes in Mathematics vol. 1766, Springer, Berlin, 2001.

[21] N.T.A. Haydn and F. Yang, Local escape rates for $\phi$-mixing dynamical systems, arXiv preprint 1806.07148.

[22] Nicolai Haydn, Matthew Nicol, Andrew Trk and Sandro Vaienti, Almost sure invariance principle for sequential and non-stationary dynamical systems, Trans. Amer. Math. Soc. 369 (2017), 5293-5316.

[23] I.A. Ibragimov and Y.V. Linnik, Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Groningen, 1971

[24] Yu. Kifer, Perron-Frobenius theorem, large deviations, and random perturbations in random environments, Math. Z. 222(4) (1996), 677-698.

[25] Yu. Kifer, Limit theorems for random transformations and processes in random environments, Trans. Amer. Math. Soc. 350 (1998), 1481-1518.

[26] Yu. Kifer, Thermodynamic formalism for random transformations revisited, Stoch. Dyn. 8 (2008), 77-102.

[27] A Korepanov, Z Kosloff, I Melbourne Martingale-coboundary decomposition for families of dynamical systems Annales de lInstitut Henri Poincaré C, Analyse non linéaire 35, no. 4 (2018) 859-885.

[28] K. Fernando and C. Liverani, Edgeworth expansions for weakly dependent random variables, preprint, arXiv 1803.07667, 2018.

[29] V.D. Milman and G. Schechtman, Asymptotic theory of finite-dimensional normed spaces, Lecture Notes in Mathematics, Vol. 1200, Berlin: Springer-Verlag, 1986, (With an appendix by M. Gromov).

[30] V. Mayer, B. Skorulski and M. Urbański, Distance expanding random mappings, thermodynamical formalism, Gibbs measures and fractal geometry, Lecture Notes in Mathematics, vol. 2036 (2011), Springer.

[31] S.V. Nagaev, Some limit theorems for stationary Markov chains, Theory Probab. Appl. 2 (1957), 378-406.

[32] S.V. Nagaev, More exact statements of limit theorems for homogeneous Markov chains, Theory Probab. Appl. 6 (1961), 62-81.

[33] M. Nicol, A. Torok, S. Vaienti, Central limit theorems for sequential and random intermittent dynamical systems, Ergodic Theory Dynam. Systems, 38, pp. 1127-1153, 2016.

[34] M. Peligrad, Central limit theorem for triangular arrays of non-homogeneous Markov chains, Prob. Theor. Rel. Field.

[35] H.H. Rugh, Cones and gauges in complex spaces: Spectral gaps and complex Perron-Frobenius theory, Ann. Math. 171 (2010), 1707-1752.

[36] A.N. Shiryaev, Probability, Springer-Verlag, Berlin, 1995.

[37] L. Saulis and V.A. Statulevicius, Limit Theorems for Large Deviations, Kluwer Academic, Dordrecht, Boston, 1991.

[38] A.N. Tikhomirov, On the convergence rate in the central limit theorem for weakly dependent random variables, Theory Probab. Appl. 25 (1980), 790-809.

[39] S. Sethuraman and S.R.S Varadhan, A martingale proof of Dobrushins theorem for non-homogeneous Markov chains, Electron. J. Probab. 10, 12211235 (2005)

[40] P. Walters, Invariant measures and equilibrium states for some mappings which expand distances. Trans. Amer. Math. Soc. 236, 121-153 (1978).

[41] L.S. Young, Statistical properties of dynamical systems with some hyperbolicity, Ann. Math. 7 (1998) 585-650.

[42] L.S. Young, Recurrence time and rate of mixing, Israel J. Math. 110 (1999) 153-88.

[43] L. Rey-Bellet and L.S. Young, Large deviations in non-uniformly hyperbolic dynamical systems, Ergodic Theory Dynam. Systems, 28, 587612, 2008.

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