Cesàro Summability of Taylor Series in Weighted Dirichlet Spaces

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Received: 24 September 2020 / Accepted: 5 November 2020 / Published online: 18 November 2020
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Abstract
We show that, in every weighted Dirichlet space on the unit disk with superharmonic weight, the Taylor series of a function in the space is \((C, \alpha)\)-summable to the function in the norm of the space, provided that \(\alpha > 1/2\). We further show that the constant \(1/2\) is sharp, in marked contrast with the classical case of the disk algebra.

Keywords Weighted Dirichlet space · Cesàro mean · Riesz mean · Hadamard multiplication

Mathematics Subject Classification 40G05 · 40J05 · 41A10 · 46E20

1 Introduction
Let \(f(z)\) be a formal power series, say \(f(z) = \sum_{k=0}^\infty a_k z^k\). Many holomorphic function spaces on the unit disk \(\mathbb{D}\) have the property that, if \(f\) belongs to the space, then its Taylor partial sums...
\[ s_n[f](z) := \sum_{k=0}^{n} a_k z^k \]

converge to \( f \) in the norm of the space. This is the case, for example, if the space in question is the Hardy space \( H^2 \), the Dirichlet space \( \mathcal{D} \) or the Bergman space \( \mathcal{A}^2 \). It is also true in all the Hardy spaces \( H^p \) for \( 1 < p < \infty \), even though the proof is not as straightforward as in the other cases.

There are also spaces in which convergence may fail. For instance, a classic example of du Bois-Reymond shows that there exists \( f \) in the disk algebra \( A(\mathbb{D}) \) such that \( s_n[f] \) does not converge to \( f \) in the norm of \( A(\mathbb{D}) \). The same phenomenon can occur in the Hardy space \( H^1 \). In both of these cases, however, Fejér’s theorem shows that the Cesàro sums

\[
\sigma_n[f](z) := \frac{1}{n+1} \sum_{k=0}^{n} s_k[f](z) = \sum_{k=0}^{n} \left( 1 - \frac{k}{n+1} \right) a_k z^k
\]

do converge to \( f \) in the norm of the space.

In the case of the disk algebra, there is a refinement of Fejér’s theorem due to M. Riesz [6], who showed that the generalized Cesàro means

\[
\sigma_n^\alpha[f](z) := \frac{(n+\alpha)^{-1}}{\alpha} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} a_k z^k \tag{1}
\]

converge to \( f \) in the norm of \( A(\mathbb{D}) \) for each \( \alpha > 0 \). Here, the binomial coefficients should be interpreted as

\[
\binom{n+\alpha}{\alpha} := \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)},
\]

where \( \Gamma \) denotes the gamma function. An analogous refinement holds in \( H^1 \).

The fact that \( \sigma_n^\alpha[f] \) converges to \( f \) in the space is often described by saying that the Taylor series of \( f \) is \((C, \alpha)\)-summable to \( f \) in the space. It is well known that \((C, \alpha)\)-summability implies \((C, \beta)\)-summability if \( \alpha < \beta \). Thus the \((C, \alpha)\)-summability of \( f \) for \( \alpha > 0 \) improves Fejér’s result on the convergence of \( \sigma_n[f] \) (namely \((C, 1)\)-summability) almost to the point of establishing the convergence of \( s_n[f] \) itself (namely \((C, 0)\)-summability). For background on summability methods, we refer to Hardy’s book [2].

What happens in other spaces? In this article, we consider the family of weighted Dirichlet spaces with superharmonic weights. Dirichlet spaces with harmonic weights were introduced by Richter [4] and further studied by Richter and Sundberg [5]. The generalization to superharmonic weights was treated by Aleman [1].
Let us recall the definition. Given a positive superharmonic function \( \omega \) on \( \mathbb{D} \) and a holomorphic function \( f \) on \( \mathbb{D} \), we define

\[
\mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \, dA(z),
\]

where \( dA \) denotes normalized area measure on \( \mathbb{D} \). The weighted Dirichlet space \( \mathcal{D}_\omega \) is the set of holomorphic \( f \) on \( \mathbb{D} \) with \( \mathcal{D}_\omega(f) < \infty \). Defining

\[
\|f\|^2_{\mathcal{D}_\omega} := |f(0)|^2 + \mathcal{D}_\omega(f) \quad (f \in \mathcal{D}_\omega),
\]

makes \( \mathcal{D}_\omega \) into a Hilbert space containing the polynomials.

It is known that, if \( \omega \) is a superharmonic weight and if \( f \in \mathcal{D}_\omega \), then \( \sigma_n[f] \to f \) in \( \mathcal{D}_\omega \). In particular, polynomials are dense in \( \mathcal{D}_\omega \). On the other hand, there exist a superharmonic weight \( \omega \) and a function \( f \in \mathcal{D}_\omega \) such that \( \sigma_n[f] \not\to f \) in \( \mathcal{D}_\omega \). For proofs of these facts, see [3, Theorem 1.6].

Thus \( \mathcal{D}_\omega \) behaves a bit like the spaces \( A(\mathbb{D}) \) and \( H^1 \). By analogy with what happens in these spaces, we might therefore expect Taylor series to be \((C, \alpha)\)-summable in \( \mathcal{D}_\omega \) for all \( \alpha > 0 \). This turns out not to be the case. We shall establish the following results.

**Theorem 1.1** If \( \omega \) is a superharmonic weight on \( \mathbb{D} \), if \( f \in \mathcal{D}_\omega \) and if \( \alpha > 1/2 \), then \( \sigma_n^{\alpha}[f] \to f \) in \( \mathcal{D}_\omega \).

**Theorem 1.2** Let \( \omega_1 \) be the harmonic weight on \( \mathbb{D} \) defined by the formula \( \omega_1(z) := (1 - |z|^2)/|1 - z|^2 \). Then there exists \( f \in \mathcal{D}_{\omega_1} \) such that \( \sigma_n^{1/2}[f] \not\to f \) in \( \mathcal{D}_{\omega_1} \).

In the terminology of [5], the space \( \mathcal{D}_{\omega_1} \) is a local Dirichlet space. Theorem 1.2 shows that, even though \( \mathcal{D}_{\omega_1} \) is a Hilbert space, Taylor series in the space actually have worse summability behaviour than in \( A(\mathbb{D}) \) or \( H^1 \).

The proofs of Theorems 1.1 and 1.2 make use of the theory of Hadamard multiplication operators of \( \mathcal{D}_\omega \) as developed in [3]. In Sect. 2 we briefly review this theory, before passing to the proofs of the theorems themselves in Sect. 3.

### 2 Hadamard Multiplication Operators

Given formal power series \( h(z) := \sum_{k=0}^{\infty} c_k z^k \) and \( f(z) := \sum_{k=0}^{\infty} a_k z^k \), we define their Hadamard product to be the formal power series given by the formula

\[
(h \ast f)(z) := \sum_{k=0}^{\infty} c_k a_k z^k.
\]

Obviously, if \( h \) is a polynomial, then \( h \ast f \) is a polynomial too. In this case, for each superharmonic weight \( \omega \) on \( \mathbb{D} \), the map \( M_h : f \mapsto h \ast f \) is a bounded linear map from \( \mathcal{D}_\omega \) to itself, sometimes called a Hadamard multiplication operator. We are interested in estimating its operator norm, \( \|M_h : \mathcal{D}_\omega \to \mathcal{D}_\omega\| \).
To state our results, we need a little extra notation. Given a sequence of complex numbers \((c_k)_{k \geq 1}\), we write \(T_c\) for the infinite matrix

\[
T_c := \begin{pmatrix}
    c_1 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \ldots \\
    0 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \ldots \\
    0 & 0 & c_3 - c_2 & c_4 - c_3 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \ddots \\
    0 & 0 & 0 & c_4 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}.
\] (2)

If the \((c_k)\) are the coefficients of a formal power series \(h(z) = \sum_{k=0}^{\infty} c_k z^k\), then we also write \(T_h\) in place of \(T_c\). Note that, in this situation, the coefficient \(c_0\) plays no role.

**Theorem 2.1** Let \(h\) be a polynomial.

(i) For each superharmonic weight \(\omega\) on \(\mathbb{D}\), we have

\[
\| M_h : \mathcal{D}_\omega \to \mathcal{D}_\omega \| \leq \| T_h : \ell^2 \to \ell^2 \|.
\]

(ii) If \(\omega_1\) is the harmonic weight on \(\mathbb{D}\) given by \(\omega_1(z) := (1 - |z|^2)/|1 - z|^2\), then

\[
\| M_h : \mathcal{D}_{\omega_1} \to \mathcal{D}_{\omega_1} \| = \| T_h : \ell^2 \to \ell^2 \|.
\]

**Proof** Part (i) is a special case of [3, Theorem 1.1]. Part (ii) is established in the course of the proof of the same result, see [3, p.52]. \(\square\)

To apply Theorem 2.1, it is helpful to have at our disposal some explicit estimates for \(\| T_c : \ell^2 \to \ell^2 \|\).

**Theorem 2.2** Let \(c := (c_k)_{k \geq 1}\) be a sequence of complex numbers that is eventually zero, and let \(T_c\) be defined by (2).

(i) If \(n\) is an integer such that \(c_k = 0\) for all \(k > n\), then

\[
\| T_c : \ell^2 \to \ell^2 \|^2 \leq (n + 1) \sum_{k=1}^{n} |c_{k+1} - c_k|^2.
\]

(ii) For all integers \(m, n\) with \(1 \leq m \leq n\), we have

\[
\| T_c : \ell^2 \to \ell^2 \|^2 \geq m \sum_{k=m}^{n} |c_{k+1} - c_k|^2.
\]

**Proof** Part (i) was already established in [3, Theorem 1.2(ii)]. For part (ii), we remark that the operator norm of \(T_c\) is bounded below by the norm of any submatrix, in particular that of the \(m \times (n - m + 1)\) submatrix

\[
A := \begin{pmatrix}
    c_{m+1} - c_m & \ldots & c_{n+1} - c_n \\
    \vdots & \ddots & \vdots \\
    c_{m+1} - c_m & \ldots & c_{n+1} - c_n
\end{pmatrix}.
\]
Now $AA^*$ is an $m \times m$ matrix, all of whose entries have the same value, namely $\sum_{k=m}^{n} |c_{k+1} - c_k|^2$. It follows that

$$\|T_c\|^2 \geq \|A\|^2 = \|AA^*\| = m \sum_{k=m}^{n} |c_{k+1} - c_k|^2.$$ 

\[\square\]

### 3 Proofs of Theorems 1.1 and 1.2

Instead of using the Cesàro means $\sigma^n_\alpha$, defined in (1), we prefer to work with the so-called discrete Riesz means, defined as follows. Given a formal power series $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $\alpha > 0$, we let

$$\rho^n_\alpha[f](z) := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^\alpha a_k z^k.$$

The following result allows to us pass between Cesàro means and Riesz means, at least when $0 < \alpha < 1$.

**Proposition 3.1** Let $\omega$ be a superharmonic weight on $\mathbb{D}$, let $f \in \mathcal{D}_\omega$ and let $0 < \alpha < 1$. Then $\sigma^n_\alpha[f] \to f$ in $\mathcal{D}_\omega$ if and only if $\rho^n_\alpha[f] \to f$ in $\mathcal{D}_\omega$.

To prove this proposition we use a theorem due to M. Riesz [7]. Riesz actually proved the result in the scalar case, but a careful reading of Riesz’s proof shows that the theorem easily extends to general Banach spaces.

**Theorem 3.2** Let $X$ be a Banach space, let $(x_k)_{k \geq 0}$ be a sequence in $X$ and let $y$ be an element of $X$. Then, for each $\alpha \in (0, 1)$,

$$\lim_{n \to \infty} \left(\frac{n + \alpha}{\alpha}\right)^{-1} \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^\alpha x_k \equiv \lim_{n \to \infty} \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^\alpha x_k = y.$$ 

**Proof of Proposition 3.1** The result follows directly upon applying Theorem 3.2 with $X := \mathcal{D}_\omega$ and $y := f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $x_k := a_k z^k$. \[\square\]

**Proof of Theorem 1.1** Let $\omega$ be a superharmonic weight on $\mathbb{D}$. To show that Taylor series are $(C, \alpha)$-summable in $\mathcal{D}_\omega$ for all $\alpha > 1/2$, it suffices to do so for $\alpha \in \left(1, \frac{1}{2}\right)$. Fix such an $\alpha$.

By Proposition 3.1, it is enough to show that $\rho^n_\alpha[f] \to f$ in $\mathcal{D}_\omega$ for all $f \in \mathcal{D}_\omega$. It is obvious that $\rho^n_\alpha[f] \to f$ if $f$ is a polynomial, and, as noted in the introduction, polynomials are dense in $\mathcal{D}_\omega$. Therefore the result will follow if we can show that the operator norms of the linear maps $f \mapsto \rho^n_\alpha[f] : \mathcal{D}_\omega \to \mathcal{D}_\omega$ are bounded independently of $n$. 


To do this, we identify these maps as a Hadamard multiplication operators. Indeed, we have $\rho_n^\alpha[f] = M_{h_n}(f)$, where

$$h_n(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^\alpha z^k.$$ 

By Theorem 2.1(i), we have

$$\|M_{h_n} : D_\omega \to D_\omega\| \leq \|T_{h_n} : \ell^2 \to \ell^2\|,$$

and using Theorem 2.2(i), we obtain

$$\|T_{h_n} : \ell^2 \to \ell^2\|^2 \leq (n + 1) \sum_{k=1}^{n} \left(1 - \frac{k + 1}{n+1}\right)^\alpha - \left(1 - \frac{k}{n+1}\right)^\alpha \right|^2$$

$$= \frac{1}{(n + 1)^{2\alpha - 1}} \sum_{k=1}^{n} \left((n + 1 - k)^\alpha - (n - k)^\alpha\right)^2$$

$$= \frac{1}{(n + 1)^{2\alpha - 1}} \sum_{k=1}^{n} \left(\int_{n-k}^{n+1-k} \alpha t^{\alpha - 1} \, dt \right)^2$$

$$\leq \frac{1}{(n + 1)^{2\alpha - 1}} \sum_{k=1}^{n} \int_{n-k}^{n+1-k} \alpha^2 t^{2\alpha - 2} \, dt$$

$$= \frac{1}{(n + 1)^{2\alpha - 1}} \int_{0}^{n} \alpha^2 t^{2\alpha - 2} \, dt$$

$$\leq \frac{\alpha^2}{2\alpha - 1}.$$ 

Thus the operator norms of $f \mapsto \rho_n^\alpha[f] : D_\omega \to D_\omega$ are indeed bounded independently of $n$, and the proof is complete. \hfill \Box

**Proof of Theorem 1.2** Let $\omega_1(z) := (1 - |z|^2)/|1 - z|^2$. By Proposition 3.1, to show that there exists $f \in D_{\omega_1}$ whose Taylor series is not $(C, \frac{1}{2})$-summable to $f$, it is enough to show that there exists $f$ such that $\rho_n^{1/2}[f] \not\to f$ in $D_{\omega_1}$.

We prove that the operator norms of the maps $f \mapsto \rho_n^{1/2}[f] : D_{\omega_1} \to D_{\omega_1}$ tend to infinity as $n \to \infty$. If so, then, by the Banach–Steinhaus theorem, there exists $f \in D_{\omega_1}$ such that the sequence $\rho_n^{1/2}[f]$ is unbounded in $D_{\omega_1}$. In particular, $\rho_n^{1/2}[f] \not\to f$ in $D_{\omega_1}$, as desired.

Once again, to estimate the norm of the map $f \mapsto \rho_n^{1/2}[f]$, we identify it as a Hadamard multiplication operator, namely $\rho_n^{1/2}[f] = M_{h_n}(f)$, where

$$h_n(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{1/2} z^k.$$
By Theorem 2.1(ii), we have

$$\|M_{h_n} : D_{\omega^1} \to D_{\omega^1}\| = \|T_{h_n} : \ell^2 \to \ell^2\|,$$

and, using Theorem 2.2(ii), for each $m$ with $1 \leq m \leq n$, we have

$$\|T_{h_n} : \ell^2 \to \ell^2\|^2 \geq m \sum_{k=m}^{n} \left( 1 - \frac{k + 1}{n + 1} \right)^{1/2} - \left( 1 - \frac{k}{n} \right)^{1/2} \right)^2$$

$$= \frac{m}{n+1} \sum_{k=m}^{n} \left( n + 1 - k \right)^{1/2} - \left( n - k \right)^{1/2} \right)^2$$

$$= \frac{m}{n+1} \sum_{k=m}^{n} \frac{1}{(n + 1 - k)^{1/2} + (n - k)^{1/2}} \tag{2.2}$$

$$\geq \frac{m}{4(n+1)} \sum_{k=m}^{n} \frac{1}{n + 1 - k}$$

$$\geq \frac{m}{4(n+1)} \log(n + 2 - m).$$

In particular, taking $m := \lceil (n + 1)/2 \rceil$, we obtain

$$\|T_{h_n} : \ell^2 \to \ell^2\|^2 \geq \frac{1}{8} \log\left( \frac{n + 1}{2} \right),$$

which tends to infinity with $n$.

Thus the operator norms of $f \mapsto \rho_n^{1/2} [f] : D_{\omega} \to D_{\omega}$ tend to infinity with $n$, as claimed, and the proof is complete. $\square$

Funding JM supported by an NSERC Discovery Grant. POP supported by an NSERC Alexander-Graham-Bell Scholarship. TR supported by grants from NSERC and the Canada Research Chairs program.

Compliance with ethical standards

Conflict of interest None.

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