Appendix A  Additional Definitions, Lemmas, and Proofs for Section 3

A.1 Unit-sloped paths and Lemma A.1

Definition A.1 ((Recurring) unit-sloped path).
A unit-sloped path of length $2i$ is a path in $\mathbb{R}^2$ from $(0,0)$ to $(2i,s_{2i})$ consisting only of line segments between $(k-1,s_{k-1})$ and $(k,s_k)$ for $k = 1, 2, \ldots, 2i$ where $s_k = s_{k-1} + 1$ or $s_k = s_{k-1} - 1$ and $s_0 = 0$. A recurring unit-sloped path of length $2i$ is a unit-sloped path of length $2i$ that ends in $(2i,0)$, i.e., it has $s_{2i} = 0$. \(\triangle\)

Note that if we restrict $s_k \geq 0$ for all $k = 0, 1, 2, \ldots, 2i$, this definition coincides with that of the well-known Dyck path (see, e.g., Deutsch (1999), Deutsch and Shapiro (2001)). Figure 4 shows an example for a recurring unit-sloped path and a Dyck path, respectively.

![Figure 1: Recurring unit sloped paths. a) General path. and b) Dyck path.](image)

Lemma A.1.

a) The number of recurring unit-sloped paths of length $2i$ which have $s_k \geq 0$ for all $k = 0, 1, 2, \ldots, 2i$ is $C_i$.

b) The number of recurring unit-sloped paths of length $2i$ which have $s_k \geq -1$ for all $k = 0, 1, 2, \ldots, 2i$ is $C_{i+1}$.

Proof.

a) See Michaels and Rosen (1991), chapter 7, pages 115 and 116.

b) The proof is a straightforward consequence of the fact that the number of recurring unit-sloped paths of length $2i$ with $s_k \geq -1$ for $k = 0, 1, 2, \ldots, 2i$ is the number of all recurring unit-sloped paths of length $2i$ minus the number of all recurring unit-sloped paths of length $2i$ which hit the number $-2$ at least once. As a result of the reflection principle, counting the paths from $(0,0)$ to $(2i,0)$ hitting $-2$ at least once is the same as counting the paths from $(0,0)$ to $(2i,-4)$ hitting $-2$ at least once. But any such path must hit $-2$ at some point, i.e., we are computing the total number of paths from $(0,0)$ to $(2k,-4)$. In total, we obtain that the number of recurring unit-sloped paths of length $2i$ with $s_k \geq -1$ for $k = 0, 1, 2, \ldots, 2i$ is equal to the total number of paths
from \((0, 0)\) to \((2i, 0)\) minus the total number of paths from \((0, 0)\) to \((2i, -4)\) which is equal to

\[
\binom{2i}{i} - \binom{2i}{i+2} = \frac{(2i)!}{(i!)^2} - \frac{(2i)!}{(i+2)!(i-2)!} = \frac{(2i)!}{(i!)^2} \left(1 - \frac{i(i-1)}{(i+1)(i+2)}\right)
\]

\[
= \frac{(2i)!}{(i!)^2} \frac{(i+1)(i+2) - i(i-1)}{(i+1)(i+2)} = \frac{(2i)!}{(i!)^2} \frac{4k+2}{(i+1)(i+2)}
\]

\[
= \frac{1}{i+2} \frac{1}{(i+1)!} \frac{(2i)!(4i+2)}{i!} = \frac{1}{i+2} \frac{1}{(i+1)!} \frac{(2i)!(4i+2)(i+1)}{(i+1)!}
\]

\[
= \frac{1}{i+2} \frac{1}{(i+1)!} \frac{(2i)!(4i^2 + 6i + 2)}{(i+1)!} = \frac{1}{i+2} \frac{1}{(i+1)!} \frac{(2i)!(2i+1)(2i+2)}{(i+1)!}
\]

\[
= \frac{1}{i+2} \left(\frac{2i+2}{i+1}\right) = C_{i+1}.
\]
b) A total number of $m$ bins with $m \in \{n, n+1, \ldots, 2n\}$ is obtained when in an item sequence without condensations $m - n$ out of the $n$ pairs of successive items are pairs of large items for which no matching small items can be found afterwards. Each such item sequence corresponds to a unit-sloped path of length $2n$ with $s_k \geq -1$ for $k = 0, 1, 2, \ldots, 2n$ ending at height $2(m - n)$ because each pair of large items contributes an amount of 2 to the total height achieved at the end of the path, and the result follows.

$\square$

A.3 Proof of Lemma 3.4

a) Since for $2n$ items at least $n$ and at most $2n$ bins are needed, $B_f(2n, m) = 0$ for $m < n$ and $m > 2n$. For the remaining $m$, we perform a reverse induction on $m$. The base case $m = 2n$ is valid because the only item sequence which needs $2n$ bins has $2n$ large items and it holds that

$$\sum_{k=2n-n}^{n} a_{n,k} = a_{n,n} = \frac{n+1}{n+1} \binom{2n+2}{0} = 1.$$  

For the inductive step, let $B_f(2n, m) = \sum_{k=m-n}^{n} a_{n,k}$ be valid for some $m$ with $2n \geq m > n$. We show that $B_f(2n, m - 1) = \sum_{k=m-1-n}^{n} a_{n,k}$. Because of $m > n$, there must be a pair of large items starting at an odd position for which no matching small items follow in every item sequence with objective value $m$ since otherwise these large items could be matched with small items and would fit into a bin contradicting $m > n$. Hence, we obtain for any item sequence with objective value $m$ an item sequence with objective value $m - 1$ by replacing the first pair of large items starting at an odd position for which no matching small items follow with a pair of small items which in turn lead to a condensation. As a result, we have $B_f(2n, m - 1) = B_f(2n, m) + |\sum_{add}|$ where $\sum_{add}$ are the additional item sequences leading to objective value $m - 1$ which have not resulted from establishing a condensation in an item sequence with objective value $m$. These item sequences can be mapped to a unit-sloped path of length $2n$ not going below level $-1$ and ending at height $2(m - 1 - n)$. From Lemma A.2, we have that $|\sum_{add}| = a_{n,m-n-1}$. Together with the induction hypothesis we conclude that

$$B_f(2n, m - 1) = B_f(2n, m) + |\sum_{add}| = \sum_{k=m-n}^{n} a_{n,k} + a_{n,m-n-1} = \sum_{k=m-n-1}^{n} a_{n,k}.$$  

b) Since for $2n+1$ items at least $n+1$ and at most $2n+1$ bins are needed, $B_f(2n+1, m) = 0$ for $m < n + 1$ and $m > 2n + 1$. Notice that whenever $m > n + 1$ for an item sequence of length $2n + 1$, we have $m > n$ for the same item sequence where the last item is deleted. Thus, there must be a pair of large items beginning at an odd position in the truncated sequence from the same reasoning as in part a) of the proof. Objective value $m$ with $n + 1 < m \leq 2n + 1$ for an item sequence of length $2n + 1$ can be attained in two ways: First, $B_f$ needed $m - 1$ bins after $2n$ items and the $2n + 1$st item leads to the $m$th bin. Second, $B_f$ needed $m$ bins after $2n$ items and the $2n + 1$st item needs no new
bin. In the first case, we have \( \text{BF}(2n, m - 1) \) item sequences which must incur a new bin upon appending a large item; appending a small item would leave the objective value at \( m \) because there are at least two large items which could be matched with the small item. In the second case, \( \text{BF}(2n, m) \) item sequences will not incur a new bin upon appending a small item as this item can be matched with one of the large items; appending a large item would lead to objective value \( m + 1 \) since after \( 2n \) items there can never be a bin with a small item only. We obtain

\[
\text{BF}(2n + 1, m) = \text{BF}(2n, m - 1) + \text{BF}(2n, m)
\]

\[
= \sum_{k=m-1-n}^{n} a_{n,k} + \sum_{k=m-n}^{n} a_{n,k} = 2 \sum_{k=m-n}^{n} a_{n,k} + a_{n,m-1-n}.
\]

Objective value \( n + 1 \) can be attained in three ways: First, \( \text{BF} \) needed \( n \) bins after \( 2n \) items and the \( 2n + 1 \)st item is large leading to the \( n + 1 \)st bin. Second, \( \text{BF} \) needed \( n \) bins after \( 2n \) items and the \( 2n + 1 \)st item is small leading to the \( n + 1 \)st bin. Third, \( \text{BF} \) needed \( n + 1 \) bins after \( 2n \) items and the \( 2n + 1 \)st item is small, but does not lead to a new bin. The first case is trivial. In the second case, we seek for the same item sequences because neither of them can exhibit a pair of large items starting in an odd position. In the third case, we seek for the item sequences of length \( 2n \) with objective value \( n + 1 \) which have at least one pair of large items beginning at an odd position such that the appended small item does not incur a new bin. These item sequences are counted by \( \text{BF}(2n, n + 1) \). We obtain

\[
\text{BF}(2n + 1, n + 1) = \text{BF}(2n, n) + \text{BF}(2n, n) + \text{BF}(2n, n + 1)
\]

\[
= \sum_{k=0}^{n} a_{n,k} + \sum_{k=0}^{n} a_{n,k} + \sum_{k=1}^{n} a_{n,k} = 3 \sum_{k=0}^{n} a_{n,k} - a_{n,0}.
\]

### A.4 Proof of Theorem 3.5

a) We show by two-dimensional induction on \( n \) and \( m \) that \( \sum_{k=m-n}^{n} a_{n,k} = \binom{2n+1}{m+1} \). Recall that \( n = 1, 2, \ldots \) and \( m = n, n+1, \ldots, 2n \). The base case \( n = 1 \) and \( m = n = 1 \) is valid because it holds that \( \sum_{k=0}^{n} a_{1,k} = a_{1,0} + a_{1,1} = 2 + 1 = 3 = \binom{2+1}{1+1} = \binom{3}{2} = 3 \). In the first inductive step (on \( n \) with fixed \( m = n \)), we show that \( \sum_{k=0}^{n} a_{n,k} = \binom{2n+1}{n+1} \) holds.

From Shapiro (1976), we know that \( 1 \cdot (\binom{2(n+1)}{n+1}) = \sum_{k=0}^{n} a_{n,k} \). The result follows from

\[
\frac{1}{2} \binom{2(n+1)}{n+1} = \frac{1}{2} \cdot \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)!}{2(n+1)n!(n+1)!} = \binom{2n+1}{n+1}.
\]

In the second inductive step (on \( m \) with arbitrary \( n \)), we show that \( \sum_{k=m-n}^{n} a_{n,k} = \binom{2n+1}{m+1} \)
implies \( \sum_{k=m+1-n}^n a_{n,k} = \binom{2n+1}{m+2} \). This can be seen by the following calculations:

\[
\sum_{k=m+1-n}^n a_{n,k} = \sum_{k=m-n}^n a_{n,k} - a_{n,m-n} = \binom{2n+1}{m+1} - \frac{m-n+1}{n+1} \binom{2n+2}{2n-m}
\]

\[
= \frac{(2n+1)!((n+1)(m+2) - (m-n+1)(2n+2))}{(2n-m)!(m+2)!(n+1)}
\]

\[
= \frac{(2n+1)!}{(m+2)!(2n-m-1)!} \cdot \frac{(n+1)(m+2) - (m-n+1)(2n+2)}{(2n-m)(n+1)}
\]

\[
= \frac{(2n+1)!}{(m+2)!(2n-m-1)!} \cdot 1 = \binom{2n+1}{m+2}.
\]

The result now immediately follows from the formula given in part a) of Lemma 3.4.

b) For \( m = n + 1 \), we have

\[
3 \sum_{k=0}^n a_{n,k} - a_{n,0} \overset{a)}{=} 3 \binom{2n+1}{n+1} - \frac{1}{n+1} \binom{2n+2}{n} = \frac{(2n+1)!}{(n+2)!(n+1)!} (3n^2 + 7n + 4)
\]

and

\[
\binom{2n+3}{n+2} - \binom{2n+1}{n+1} = \frac{(2n+1)!}{(n+2)!(n+1)!} (3n^2 + 7n + 4).
\]

which together yields the desired relation for \( m = n + 1 \).

For \( n + 1 < m \leq 2n + 1 \), we have

\[
2 \sum_{k=m-n}^n a_{n,k} + a_{n,m-n-1} \overset{a)}{=} 2 \binom{2n+1}{m+1} + \frac{m-n}{n+1} \binom{2n+2}{2n-m+1}
\]

\[
= \frac{(2n+2)!}{(m+1)!(2n-m+1)!} \left( 2 \frac{(2n-m+1)}{2n+2} + \frac{m-n}{n+1} \right)
\]

\[
= \frac{(2n+2)!}{(m+1)!(2n-m+1)!} \left( 2 \frac{2n+2}{2n+2} \right) = \binom{2n+2}{m+1}.
\]

The result now immediately follows from the formula given in part b) of Lemma 3.4.
A.5 Lemma A.3

Lemma A.3. For \( n \leq m \leq 2n \) it holds that \( \sum_{i \geq 1} C_i \binom{2n-2i}{m-i} = \sum_{i \geq 1} C_{i-1} \binom{2n-2i+1}{m-i+1} \).

Proof. We show that \( \sum_{i \geq 1} C_i \binom{2n-2i}{m-i} - \sum_{i \geq 1} C_{i-1} \binom{2n-2i+1}{m-i+1} = 0 \). Notice that from the definition of the binomial coefficient, \( i \) ranges in \( \{1, 2, \ldots, 2n-m\} \) in both terms. Hence,

\[
\sum_{i \geq 1} C_i \binom{2n-2i}{m-i} - \sum_{i \geq 1} C_{i-1} \binom{2n-2i+1}{m-i+1} =
\]

\[
= C_1 \binom{2n-2}{m-1} + C_2 \binom{2n-4}{m-2} + C_3 \binom{2n-6}{m-3} + \ldots + C_{2n-m} \binom{2m-n}{2m-n}
\]

\[
- \left( C_0 \binom{2n-1}{m} + C_1 \binom{2n-3}{m-1} + C_2 \binom{2n-5}{m-2} + \ldots + C_{2n-m-1} \binom{2m-n+1}{2m-n+1} \right)
\]

\[
= C_1 \binom{2n-3}{m-2} + C_2 \binom{2n-5}{m-3} + \ldots + C_{2n-m-1} \binom{2m-n+1}{2m-2n} + C_{2n-m} - \binom{2n-1}{m}
\]

\[
= \sum_{i=1}^{2n-m-1} C_i \binom{2n-2i-1}{m-i-1} + C_{2n-m} - \binom{2n-1}{m}
\]

\[
= \sum_{i=0}^{2n-m-1} C_i \binom{2n-2i-1}{m-i-1} - \binom{2n-1}{m-1} + C_{2n-m} - \binom{2n-1}{m}
\]

\[
= \sum_{i=0}^{2n-m} C_i \binom{2n-2i-1}{m-i-1} - C_{2n-m} \binom{2m-2n-1}{2m-2n-1} - \binom{2n-1}{m-1} + C_{2n-m} - \binom{2n-1}{m}
\]

\[
= \sum_{i \geq 0} C_i \binom{2n-2i-1}{m-i-1} - C_{2n-m} - \binom{2n-1}{m-1} + C_{2n-m} - \binom{2n-1}{m}
\]

\[
= \binom{2n}{m} - \binom{2n-1}{m-1} - \binom{2n-1}{m} = \binom{2n}{m} - \binom{2n}{m} = 0. \]
A.6 Proof of Corollary 3.7

From Theorem 3.1, we know that $B_f^2[\sigma] \in \{m, m-1\}$ whenever $B_f[\sigma] = m$; from the previous Theorem 3.6, we know that $\left|\{\sigma \mid |\sigma| = n, B_f^2[\sigma] = m, B_f[\sigma] = m\}\right| = \binom{n}{m+1}$ for $m = \lceil \frac{n}{2} \rceil + 1, \ldots, n - 1$. Hence, from Theorem 3.5 it immediately follows for these $m$ that $\left|\{\sigma \mid |\sigma| = n, B_f^2[\sigma] = m\}\right| = \binom{n+1}{m+1} - \binom{n}{m+1} = \binom{n}{m}$. Clearly, $\left|\{\sigma \mid |\sigma| = n, B_f^2[\sigma] = n\}\right| = 1.

A.7 Proof of Corollary 3.10

a) For $v < n$ and $v \geq 2n - 1$, $F_{B_f^2}(v) = F_{B_f}(v)$. For $n \leq v < 2n - 1$

$$F_{B_f^2}(v) - F_{B_f}(v) = \left(\sum_{m=n}^{2n} \left(\binom{2n+1}{m+2} - 2\binom{2n}{m+1}\right) + \binom{2n}{n+1}\right) \cdot (2^{-2n})$$

$$= \left(\sum_{m=n}^{2n} \left(\frac{2n}{m+2} - \frac{2n}{m+1}\right) + \binom{2n}{n+1}\right) \cdot (2^{-2n}) = \binom{2n}{[v]+2} \cdot (2^{-2n}) > 0.$$

The second part follows immediately from Pascal’s triangle as a result of $\binom{2n}{v+2} > \binom{2n}{v+3}$ for $v = n, n+1, \ldots, 2n-3$.

b) Using the formula of Stirling ($n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, Kâ¶ningsberger (2001)), we get for $n \to \infty$ that

$$F_{B_f^2}(n) - F_{B_f}(n) = \binom{2n}{n+2} \cdot 2^{-2n} \approx \frac{\sqrt{4\pi n} \left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi (n+2)} \left(\frac{n+2}{e}\right)^{n+2} \sqrt{2\pi (n-2)} \left(\frac{n-2}{e}\right)^{n-2}} \cdot 2^{-2n}$$

$$= \frac{\sqrt{n} \left\{2n \cdot 2n\right\}^2}{\sqrt{n} \sqrt{n} \left\{n^2 - 4\right\} \left\{n+2\right\} \left\{n-2\right\} \left\{n-2\right\}} \cdot 2^{-2n} \in \Theta\left(\frac{1}{\sqrt{n}}\right).$$

In addition, $\binom{2n}{v} > \binom{2n}{v+1}$ for $v \in \mathbb{N}$ with $v \geq n$, i.e., $F_{B_f^2}(v) - F_{B_f}(v)$ monotonously decreases for $v \geq n$. Since also $F_{B_f^2}(v) = F_{B_f}(v)$ for $v < n$ and $v \geq 2n - 1$, the result follows.