THE MODEL FOR QCD RUNNING COUPLING
CONSTANT WITH DYNAMICALLY GENERATED
MASS AND ENHANCEMENT IN THE INFRARED
REGION

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Abstract

Nonperturbative studies of the strong running coupling constant in the infrared region are discussed. Starting from the analyses of the Dyson – Schwinger equations in the gauge sector of QCD, the conclusion is made on an incomplete fixing of the perturbation theory summation ambiguity within "(forced) analytization procedure" (called also a dispersive approach). A minimal model for $\bar{\alpha}_s(q^2)$ is proposed so that the perturbative time-like discontinuity is preserved and nonperturbative terms not only remove the Landau singularity but also provide the ultraviolet convergence of the gluon condensate. Within this model, on the one hand, the gluon zero modes are enhanced (the dual superconductor property of the QCD vacuum) and, on the other hand, dynamical gluon mass generation is realized, with $m_g$ estimated as 0.6 GeV. The uncertainty connected with the division into perturbative and nonperturbative contributions is discussed with the gluon condensate taken as an example.

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1 Introduction

The infrared region corresponding to large distances is of exceptional interest by virtue of its responsibility for the confinement mechanism and its inaccessibility to perturbative methods. A great number of papers are devoted to the infrared behaviour of gluon Green functions but a commonly accepted opinion is absent in the literature.

Recently a possibility of power infrared behaviour of the gluon propagator in the covariant Landau gauge has been discussed, [1, 2] with a partial account for ghosts. It was found that in the approximation considered, the gluon propagator at small values of Euclidean momenta vanished (in this case, according to gauge identities, all the gluon vertexes turn out to be singular), the ghost propagator was singular, and a running coupling constant had a large but finite value at zero.

In the axial gauge in the framework of nonperturbative Baker – Ball – Zachariasen (BBZ) approach, [3] which seems to be adequate to discuss the possibility of the infrared behaviour not being too singular, the problem of consistency of the behaviour $D(q) \sim (q^2)^{-c}$, $q^2 \to 0$ was considered. It has been shown [4, 5] that for quite a wide interval of non-integer (non-half-integer) values of $c$, $-1 < c < 3$, there are no solutions. The possibility of "frozen" interaction in the infrared region was also considered in the framework of the above approach and the result was negative as well.

On the other hand, the enhanced infrared behaviour of the gluon propagator, of the form $D(q) \sim 1/(q^2)^2$, $q^2 \to 0$ is physically motivated, useful in applications (e.g. in the quark sector), and it was obtained in a number of different approaches. [7, 8] This behaviour asymptotically solves the Dyson – Schwinger equation in the axial gauge, but with this we should decline one of the basic assumptions of the BBZ approach and take into account a transverse part of the triple gluon vertex. [8] Accordingly, we should decline an appealing iteration scheme [3] to find solutions of the Dyson – Schwinger equations for the higher Green functions.

Versions of the infrared behaviour of the gluon propagator mentioned above do not exhaust all the possibilities that can be realized beyond perturbation theory.
2 Model for the Running Coupling Constant with Nonperturbative Contributions Suppressed in the Ultraviolet Region

Let us consider the model for $\bar{\alpha}_s$ introduced in Refs. [6] and discussed further in Ref. [9]. We begin with "analytized" expression for the QCD running coupling constant [10]

$$\bar{\alpha}_s^{(1)}(q^2) = \frac{4\pi}{b_0} \left[ \frac{1}{\ln(q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - q^2} \right],$$

obtained with the use of the idea [11] on the cancellation of the ghost singularity by nonperturbative contributions. We see that this expression has a nonperturbative tail, with the behaviour $1/q^2$ as $q^2 \to \infty$. To answer the question whether this behaviour is admissible, let us consider an important physical quantity, namely, the gluon condensate $K = \langle \text{vac} | \frac{\alpha_s}{\pi} : F^a_{\mu\nu} F^a_{\mu\nu} : | \text{vac} \rangle$.

Up to the quadratic approximation in the gluon fields, the gluon condensate $K$ is defined by nonperturbative contributions in the transverse part of the gluon propagator. Normal ordering of the operators product is defined with respect to perturbative vacuum $|pert \rangle$, as the averaging in the expression for the condensate is carried out with true physical vacuum state $|\text{vac} \rangle$.

The following chain of equations describes the method by which we obtain closed expression for the gluon condensate:

$$K = \lim_{x \to y} \langle \text{vac} | \frac{\alpha_s}{\pi} : F^a_{\mu\nu}(x)F^a_{\mu\nu}(y) : | \text{vac} \rangle \approx$$

$$\approx 2\delta^{ab}\alpha_s \lim_{x \to y} \langle \text{vac} | \frac{\alpha_s}{\pi} T[(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x))\partial_\mu A^b_\nu(y)]|\text{vac} \rangle - \text{pert} =$$

$$= 2\delta^{ab}\alpha_s \lim_{x \to y} (\delta_{\mu\nu}\partial_\lambda^\mu \partial_\lambda^\nu - \partial_\mu^\nu \partial_\nu^\lambda) N^{-1} \int dA A^a_\mu(x)A^b_\nu(y)e^{S[A]+JA}|_{J=0} - \text{pert} =$$

$$= 2\delta^{ab}\alpha_s \lim_{x \to y} (\delta_{\mu\nu}\partial_\lambda^\mu \partial_\lambda^\nu - \partial_\mu^\nu \partial_\nu^\lambda) [D_{\mu\nu}^{ab}(x, y) - D_{\mu\nu}^{\text{pert}ab}(x, y)] =$$

$$= 2\delta^{ab}\alpha_s \lim_{x \to y} (\delta_{\mu\nu}\partial_\lambda^\mu \partial_\lambda^\nu - \partial_\mu^\nu \partial_\nu^\lambda) \int \frac{dk}{(2\pi)^4} e^{-ik(x-y)} [D_{\mu\nu}^{ab}(k) - D_{\mu\nu}^{\text{pert}ab}(k)] =$$

$$= 2\delta^{aa}\alpha_s \int \frac{dk}{(2\pi)^4} (k^2 \delta_{\mu\nu} - k_\mu k_\nu) [D_{\mu\nu}(k) - D_{\mu\nu}^{\text{pert}}(k)] =$$
\[ = \frac{\alpha_s}{\pi^5} \int dk \left( k^2 \delta_{\mu\nu} - k_\mu k_\nu \right) D^{(0)}_{\mu\nu}(k)[Z(k) - Z^{\text{pert}}(k)] = \quad (8) \]

\[ = \frac{3\alpha_s}{\pi^5} \int dk [Z(k) - Z^{\text{pert}}(k)] = \quad (9) \]

\[ = \frac{3}{\pi^5} \int dk [\bar{\alpha}_s(k^2) - \bar{\alpha}_s^{\text{pert}}(k^2)] = \frac{3}{\pi^3} \int dy y \bar{\alpha}_s^{\text{nonpert}}(y). \quad (10) \]

We shall use Eq. (10) to evaluate the gluon condensate (8). The transition from Eq. (7) to (8) is made with the assumption of proportionality of the nonperturbative part of the propagator to the free one. The projector in Eq. (8) removes the terms of the free axial gauge propagator \( D^{(0)}_{\mu\nu}(k) \) which depend on the gauge vector. Assuming \( Z \) is independent of gauge parameter \( y = (k\eta)^2/k^2\eta^2 \), the employment of the axial gauge lets one obtain Eq. (10) from (9), thereby expressing the gluon condensate in terms of nonperturbative part of running coupling constant \( \bar{\alpha}_s(k^2) \). The one-loop ”analytized” behaviour of Eq. (1) leads to a quadratic divergence in Eq. (10) at infinity and this is true for the two- and three-loop expressions [10] of the analytization approach. According to the results of Refs. [7, 3, 8] let us add in Eq. (1) the isolated infrared singular term of the form \( 1/q^2 \). This term is harmless at zero and it can improve the behaviour of the integrand at infinity and make the integral logarithmical divergent. To make the integral (10) convergent at infinity, it is sufficient to add one more isolated singular term of a pole type with parameters chosen appropriately. In this sense the model we come to is minimal. The expression, we obtain for the running coupling constant, is the following:

\[ \bar{\alpha}_s(q^2) = \frac{4\pi}{b_0} \left( \frac{1}{\ln(q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - q^2} + \frac{c\Lambda^2}{q^2} + \frac{(1-c)\Lambda^2}{q^2 + m_g^2} \right), \quad (11) \]

with mass parameter \( m_g = \Lambda/\sqrt{c-1} \). It is worth noting that an account of nonperturbative contributions in Eq. (11) preserves a perturbative time-like cut of Eq. (1). With the given value of the QCD scale parameter \( \Lambda \), the parameter \( c \) can be fixed by the string tension \( \kappa \), assuming the linear confinement \( V(r) \approx \kappa r = a^2 r \) at \( r \to \infty \). We define the potential \( V(r) \) of static \( \bar{q}q \) interaction by means of three-dimensional Fourier transform of \( \bar{\alpha}_s(q^2)/q^2 \) with the contributions of only one dressed gluon exchange taken into account. This gives the relation \( c\Lambda^2 = (3b_0/8\pi)a^2 \). Taking \( a \approx 0.42 \text{ GeV} \), one obtains
\[ c = \frac{\Lambda_1^2}{\Lambda^2} \text{ where } \Lambda_1^2 = \frac{3b_0\kappa}{8\pi} \simeq (0.434 \text{ GeV})^2 \] (b_0 = 9 in the case of 3 light flavours). For \( m_g \) one obtains

\[ m_g = \frac{\Lambda}{\sqrt{\Lambda_1^2 - \Lambda^2}}, \] (12)

and the tachyon absence condition limits the parameter \( \Lambda \) to \( \Lambda < 434 \text{ MeV} \).

It is seen from Eq. (11) that the pole singularities are situated at two points \( q^2 = 0 \) and \( q^2 = -m_g^2 \). This corresponds to the two effective gluon masses, 0 and \( m_g \). Therefore, the physical meaning of the parameter \( m_g \) is not the constituent gluon mass, but rather the mass of the exited state of the gluon.

### 3 Gluon Condensate and Nonperturbative Scale

The acceptance of the cancellation mechanism for the nonphysical perturbation theory singularity by the nonperturbative contributions leads to the necessity of a supplementary definition of integral (11) near point \( k^2 = \Lambda^2 \). This problem can be reformulated as a problem of dividing the perturbative and nonperturbative contributions in \( \bar{\alpha}_s \) resulting in the introduction of a parameter \( k_0 \approx 1 \text{ GeV} \). The following procedure is our definition of the regularized perturbative and nonperturbative parts of \( \bar{\alpha}_s(k^2) \):

\[ \bar{\alpha}_s(k^2) = \frac{4\pi}{b_0} \left( \frac{1}{\ln(k^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - k^2} + \frac{c\Lambda^2}{k^2} + \frac{(1-c)\Lambda^2}{k^2 + m_g^2} \right) \] (13)

\[ = \bar{\alpha}_s^{\text{pert}}(k^2) + \bar{\alpha}_s^{\text{nonpert}}(k^2) = \bar{\alpha}_{s,\text{reg}}^{\text{pert}}(k^2) + \bar{\alpha}_{s,\text{reg}}^{\text{nonpert}}(k^2), \] (14)

where

\[ \bar{\alpha}_{s,\text{reg}}^{\text{pert}}(k^2) = \frac{4\pi}{b_0} \left( \frac{1}{\ln(k^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - k^2} \theta(k_0^2 - k^2) \right) \] (15)

has no power corrections at \( k^2 \to \infty \) and

\[ \bar{\alpha}_{s,\text{reg}}^{\text{nonpert}}(k^2) = \frac{4\pi}{b_0} \left( \frac{\Lambda^2}{\Lambda^2 - k^2} \theta(k_0^2 - k^2) + \frac{c\Lambda^2}{k^2} + \frac{(1-c)\Lambda^2}{k^2 + m_g^2} \right) \] (16)

at \( k^2 \to \infty \) has the same power corrections as \( \bar{\alpha}_s^{\text{nonpert}}(k^2) \), namely \( \sim \Lambda^6/k^6 \).

Thus, we regularize\(^2\) the usual perturbation theory in the infrared region.

\(^2\)See also Ref. [12] where the problem of perturbative and nonperturbative contributions to \( \bar{\alpha}_s \) is discussed and the definition of infrared finite regularized perturbative part of \( \bar{\alpha}_s \) is suggested.
The parameter $k_0$ varies from 0.5 GeV (the lowest curve) up to 3.0 GeV (the highest one) at an interval of 0.25 GeV. The dashed line corresponds to the exceptional value $k_0 = 0.777$ GeV. The "standard" level $K^{1/4} = 0.33$ GeV is indicated.

The arbitrariness of the procedure is parametrized by a scale parameter $k_0$. With account for the stated above, let us calculate the gluon condensate (10). We obtain

$$K(\Lambda, k_0) = \frac{4}{3\pi^2} \left\{ \Lambda^4 \ln \left[ \left( \frac{\Lambda^2}{\Lambda^2} - 1 \right) \left( \frac{k_0^2}{\Lambda^2} - 1 \right) \right] + k_0^2 \Lambda^2 \right\} . \quad (17)$$

In Fig. 1 the dependence of $K^{1/4}$ on $\Lambda$ is shown for different values of $k_0$ in the interval (0.5 — 3.0) GeV. For our estimates we shall use the "standard" value [13] of the gluon condensate (0.33 GeV)$^4$. We can see that at $k_0 < \bar{k}_0 \approx 0.777$ GeV gluon condensate [17] is smaller than its standard value. It means that there exists a lower limit for the value of parameter $k_0$. The value $k_0 = \bar{k}_0$ turns out to be exceptional because with this choice only one value $\Lambda = \bar{\Lambda} \approx 375$ MeV provides a necessary value of gluon condensate. In this case, according to [12], $m_g = \bar{m}_g \approx 0.6$ GeV. At $k_0 > \bar{k}_0$ the two values of
Λ give the needed value of the gluon condensate. When \( k_0 \) increases one of them increases and tends to \( \Lambda_1 \) and the other value of \( \Lambda \) decreases and, at \( k_0 > 2 \) GeV, becomes less than 150 MeV.

4 Gluon Condensate and Vacuum Energy Density

Let us represent expression (11) in an explicitly renormalization invariant form. It can be done without solving the differential renormalization group equations. In this order we write \( \tilde{\alpha}_s(q^2) = \frac{\tilde{g}^2(q^2/\mu^2, g)}{4\pi} \) and use the normalization condition \( \tilde{g}^2(1, g) = g^2 \). Then we obtain the equation for the wanted dependence of the parameter \( \Lambda^2 \) on \( g^2 \) and \( \mu^2 \):

\[
g^2/4\pi = \frac{4\pi}{b_0} \left[ \frac{1}{\ln(\mu^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - \mu^2} + c\frac{\Lambda^2}{\mu^2} + \frac{(1 - c)\Lambda^2}{\mu^2 + m_g^2} \right].
\]

For dimensional reasons \( \Lambda^2 = \mu^2\exp\{-\varphi(x)\} \), where \( x = b_0g^2/16\pi^2 = b_0\alpha_s/4\pi \), and for the function \( \varphi(x) \) we obtain the equation:

\[
x = \frac{1}{\varphi(x)} + \frac{1}{1 - e^{\varphi(x)}} + ce^{-\varphi(x)} - \frac{(c - 1)^2}{(c - 1)e^{\varphi(x)} + 1}.
\]

The solution of this equation at \( c > 1 \) is function \( \varphi(x) \), which has the behaviour \( \varphi(x) \approx 1/x \) at \( x \to 0 \) and \( \varphi(x) \approx -\ln(x/c) \) at \( x \to +\infty \). The relation obtained ensures the renormalization invariance of \( \tilde{\alpha}_s(q^2) \). At low \( g^2 \), we obtain \( \Lambda^2 = \mu^2\exp\{-4\pi/(b_0\alpha_s)\} \), which indicates the essentially non-perturbative character of three last terms in Eq. (11), and these terms are absent in the usual perturbation theory. We will need further the \( \beta \)-function which can be found by the equation

\[
u \frac{\partial \tilde{g}^2(u, g)}{\partial u} = \beta(g^2).
\]  

(18)

Here \( u = q^2/\mu^2 \). Differentiating the running coupling (11) in \( u \) and assuming \( u = 1 \), we obtain

\[
\beta(g^2) = \frac{16\pi^2}{b_0} \left\{ -x + \frac{1}{\varphi(x)} - \frac{1}{\varphi^2(x)} + \frac{1}{(e^{\varphi(x)} - 1)^2} - \frac{(c - 1)^2}{((c - 1)e^{\varphi(x)} + 1)^2} \right\},
\]

(19)
which is simplified using equation for $\varphi(x)$. Then, knowing the behaviour of $\varphi(x)$ at $x \to 0, \infty$, we can find

$$\beta(g^2) \simeq -\frac{b_0}{16\pi^2} g^4 + \ldots, \quad g^2 \to 0,$$

(20)

$$\beta(g^2) \simeq -g^2 - \frac{16\pi^2}{b_0} c(c - 2) + O(1/\ln g^2), \quad g^2 \to \infty.$$  

(21)

Fig. 2 illustrates the dependence $\beta(g^2)$ for $c = 1, 2, \ldots, 5$. For all $g^2 > 0$ the $\beta$-function is negative definite. Let us consider the trace anomaly [14] for the energy - momentum tensor of the gluon field

$$\Theta_{\mu\nu} = \frac{\beta(g^2)}{2g^2} : F_{\mu\nu}^a F_{\mu\nu}^a : .$$

(22)

According to the definition the vacuum is relativistically invariant. So, for the vacuum energy density we have $\epsilon_{\text{vac}} = (1/4) < \text{vac}|\Theta_{\mu\nu}|\text{vac}>$. For $\beta(g^2) = -b_0 g^4/(16\pi^2)$, $b_0 = 9$ and $K = (0.33 \text{ GeV})^4$, the vacuum energy density is $\epsilon_{\text{vac}} \simeq -(240 \text{ MeV})^4$. If we introduce quarks, we destroy the
nonperturbative vacuum in some region (bag). The vacuum inside the bag is perturbative and its energy density is zero. The difference of the vacuum energy densities inside and outside the bag is the reason of external pressure on the bag.

It is important for us that because of \( \beta < 0 \), the maximum of gluon condensate corresponds to the minimum of nonperturbative vacuum energy density. For our model \( \bar{\alpha}_s(q^2) \) it means that the values

\[
\Lambda = \bar{\Lambda} \simeq 375\,\text{MeV}, \quad k_0 = \bar{k}_0 \simeq 777\,\text{MeV}, \quad m_g = \bar{m}_g \simeq 600\,\text{MeV}
\]  

are advantageous from the viewpoint of the energy argument. Let us clarify what the dependence of the gluon condensate on renormalization parameter \( \mu \) is. As a physical quantity, the vacuum energy density is independent of \( \mu \). The renormalized coupling constant \( g \) depends on \( \mu \), \( g^2/4\pi = \bar{g}^2(1, g)/4\pi = \bar{\alpha}_s(\mu^2) \), so as \( \mu \to \infty \) \( g^2 \to 0 \) and at \( \mu \to 0 \) \( g^2 \to \infty \). Using (20), (21), we can write

\[
\epsilon_{\text{vac}} = \frac{\pi^2 \beta(g^2)}{2g^4} < \text{vac} | \frac{\alpha_s}{\pi} : F_{\mu\nu}^a F_{\mu\nu}^a : | \text{vac} >= \quad (24)
\]
\begin{align*}
&= -\frac{b_0}{32} < \text{vac} | \frac{\alpha_s}{\pi} : F_{\mu
u}^a F_{\mu
u}^a : | \text{vac} > |_{\mu \to \infty} = \quad (25) \\
&= -\frac{1}{8} < \text{vac} | : F_{\mu
u}^a F_{\mu
u}^a : | \text{vac} > |_{\mu \to 0} . \quad (26)
\end{align*}

From (24), (23) at $b_0 = 9$, we have the ratio

\begin{equation}
\frac{K(\mu)}{K(\mu \to \infty)} = -\frac{9}{16\pi^2 \beta(g^2)}, \quad (27)
\end{equation}

which describes the dependence of gluon condensate on $\mu$. For the $\beta$-function in Fig. 3 by a solid line we show the ratio $K(\mu)/K(\mu \to \infty)$ as function of $\mu$. The parameter $c = 1,3476$ corresponds to the choice $\Lambda_1 = 434$ MeV, $\Lambda = \bar{\Lambda} = 375$ MeV. We can see that for $\mu > 1$ GeV ratio (27) is practically unity. In the same figure the dependence $(K(\mu)/K(\mu \to \infty))^{1/4}$ on $\mu$ is shown by dots.

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