Scale Invariance + Unitarity $\implies$ Conformal Invariance?

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Abstract

We revisit the long-standing conjecture that in unitary field theories, scale invariance implies conformality. We explain why the Zamolodchikov-Polchinski proof in $D = 2$ does not work in higher dimensions. We speculate which new ideas might be helpful in a future proof. We also search for possible counterexamples. We consider a general multifield scalar-fermion theory with quartic and Yukawa interactions. We show that there are no counterexamples among fixed points of such models in $4 - \varepsilon$ dimensions. We also discuss fake counterexamples, which exist among theories without a stress tensor.

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1 Introduction

Conformal invariance is a fundamental concept in statistical mechanics, high energy physics, and string theory. It is usually considered to be a natural consequence of scale invariance. However, the precise relation between these two spacetime symmetries is more subtle.

In $D = 2$ spacetime dimensions the situation is well understood, as we have a theorem: any scale invariant, unitary 2D QFT is conformally invariant. The assumption of unitarity is essential, as the example of the free vector field without gauge invariance shows.

It $D \geq 3$, it is not known if the above theorem is valid. A proof has never been given, and there is no known reason why it should be generally true. At the same time, there is no known embarrassing counterexample. Given the importance of conformal invariance, the situation is rather embarrassing. It is also in contradiction to Gell-Mann’s “Everything which is not forbidden, is compulsory.”

As our title shows, we would like to reopen the discussion of this interesting problem. The paper is organized as follows.

In Section 2 we revisit the proof in $D = 2$. Very little dynamical information is used in this well-known proof. It is enough to study the 2-point function of the symmetric stress tensor $T_{\mu\nu}$, which is fixed by Lorentz and scale invariance up to a few numerical coefficients. The stress tensor conservation provides a strong constraint on these coefficients, to the extent that a particular linear combination which enters the 2-point function of the trace $T_{\mu}^{\mu}$ is constrained to be zero. It is at this point that the unitarity is invoked to conclude that $T_{\mu}^{\mu} = 0$, and thus the theory is conformally invariant.

We then discuss why this $D = 2$ proof does not generalize to higher dimensions. The conclusions of Section 2 is that if a proof in $D \geq 3$ exists, it must be based on ideas essentially different from $D = 2$. We speculate that perhaps the stress tensor Ward identities may turn out useful.

Having failed to find a proof, we proceed to look for counterexamples. For any $D$, the scale current of the theory is related to $T_{\mu\nu}$ via

$$ S_{\mu}(x) = x^\nu T_{\nu\mu}(x) - K_{\mu}(x), \quad (1.1) $$

where $K_{\mu}(x)$ is a local operator without explicit dependence on the coordinates. In a scale

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1 There are two mild technical assumptions, discussed in Section 2.1 below.

2 See also [3], [4], [5] for recent mentionings of the issue. Ref. [6] has shown that, at the level of string/M-theory low energy effective actions, it is impossible to deform AdS/CFT by breaking conformal invariance while preserving scale invariance.
invariant theory the current (1.1) is conserved, which means:

\[ T_\mu^\mu = \partial_\mu K^\mu. \]  

(1.2)

Now, if \( K_\mu \) has the special form

\[ K_\mu = \partial_\nu L^{\nu\mu}, \]  

(1.3)

where \( L_{\mu\nu} \) is another local field, then a conserved conformal current can be constructed. At the same time, one can construct an ‘improved’ symmetric stress tensor \( T'_{\mu\nu} \) which is traceless:

\[ T'_{\mu}^\mu = 0. \]  

(1.4)

\( T'_{\mu\nu} \) is obtained from \( T_{\mu\nu} \) by adding a local operator which is a total divergence and identically conserved:

\[ T'_{\mu\nu} = T_{\mu\nu} + \partial_\rho \partial_\sigma Y_{\mu\rho\sigma}. \]  

(1.5)

Here \( Y_{\mu\rho\sigma} \) is antisymmetric in \( \mu\rho \) and \( \nu\sigma \) and symmetric under \( \mu\rho \leftrightarrow \nu\sigma \); such an operator can be constructed in terms of \( L_{\mu\nu} \) [2]. Thus \( T'_{\mu\nu} \) is physically equivalent to \( T_{\mu\nu} \), and the condition (1.4) makes conformal symmetry manifest.

As stressed by Polchinski [2], this analysis narrows significantly the circle of possible counterexamples. Namely, a theory must have nontrivial candidates for a dimension 3 vector operator \( K_\mu \) which is a) not conserved and b) is not of the form (1.3). Several better-known perturbative fixed points, such as the Belavin-Migdal-Banks-Zaks fixed points for non-abelian gauge theories coupled to fermions in \( D = 4 \) [8], or the Wilson-Fisher \( \lambda\phi^4 \) fixed point in \( D = 4 - \varepsilon \) [9] do not contain such a candidate, and thus are automatically conformally invariant.

The simplest class of theories with \( K_\mu \) candidates are the multi-field generalizations of \( \lambda\phi^4 \):

\[ \mathcal{L} = \frac{1}{2} (\partial s_i)^2 + \frac{1}{4!} \lambda_{ijkl} s_is_js ks_l. \]  

(1.6)

The \( K_\mu \) could be given by

\[ K_\mu = Q_{[ij]} s_i \partial_\mu s_j. \]  

(1.7)

where \( Q_{[ij]} \) is a real antisymmetric matrix. These theories can have perturbative fixed points in \( D = 4 - \varepsilon \). The usual fixed points, obtained by setting the one-loop \( \beta \)-functions to zero, have \( T'_{\mu}^\mu = 0 \) and are conformally invariant. One could hope that more general fixed points of the type (1.2) exist. However, this turns out not to be the case [2]. Namely, one can show that if (1.2) holds, then necessarily \( K_\mu = 0 \), and we are back to the usual case.
To our knowledge, this was the first and the last systematic attempt to look for counterexamples, and it did not succeed. We don’t really know why: the only known proof is by inspection. Nor do we know what happens in higher orders of perturbation theory. In our opinion, it is important to continue the search. Clearly, (1.6) is not the most general unitary field theory which can have fixed points in $D = 4 - \varepsilon$.

Thus, in Section 3 we consider a more general model which contains an arbitrary number of scalars and Weyl fermions, with quartic and Yukawa interactions. The full Lagrangian consists of (1.6) and of

$$L' = i\bar{\psi}_a \sigma_\mu \partial_\mu \psi_a + \frac{1}{2!} (y_i |ab s_i \psi_a \psi_b + y_i |ab s_i \bar{\psi}_a \bar{\psi}_b).$$

We thus allow both for possible parity and CP breaking. It’s not clear if these discrete symmetries have anything to do with the relation between scale and conformal invariance, but we would like to explore without prejudice.

We then look for one-loop fixed points with scale but without conformal invariance. Once again, it turns out that there are none. The proof of this fact is however more involved than in the case without fermions. We find a way to represent it graphically: contractions of the $\beta$-function Feynman graphs with graphs representing $K_\mu$ candidates all vanish by (anti)symmetry. Unfortunately, this is still a proof by inspection: we don’t explain why all contracted graphs had the needed symmetry properties. This is a problem for the future.

In all of the above discussion we were assuming, for good reasons indeed, that a stress tensor exists. If one drops this physical requirement, there are theories which are unitary and scale invariant but not conformal. We mention a couple of such fake counterexamples in Section 3.

We conclude in Section 4 with comments about the possible impact of the eventual resolution of this problem on the TeV-scale phenomenology.

## 2 Looking for a proof

Zamolodchikov had an idea to take the stress tensor 2-point function, impose conservation, and see what comes out of it. What comes out is the 2D $c$-theorem [1], and the fact that any 2D scale invariant theory is conformally invariant [2]. Unfortunately, the trick works only in $D = 2$. In this section we would like to discuss why this is so. If the theorem is valid in $D \geq 3$, some extra ideas should appear in the proof, and we outline one such preliminary idea.
2.1 Theorem in D = 2

We will present here a slightly different version of the proof \cite{1,2} that in D = 2 scale invariance plus unitarity implies conformal invariance.

Consider the 2-point function of the symmetric stress tensor

\[
\langle T_{\mu\nu}(x)T_{\lambda\sigma}(0) \rangle = [t_1 (\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\nu\lambda}\eta_{\mu\sigma}) + t_2 \eta_{\mu\nu}\eta_{\lambda\sigma}]/(x^2)^2 \\
+ t_3 (\eta_{\mu\lambda}x_{\nu}x_{\sigma} + \eta_{\nu\lambda}x_{\mu}x_{\sigma} + \eta_{\nu\sigma}x_{\mu}x_{\lambda})/(x^2)^3 \\
+ t_4 (\eta_{\mu\sigma}x_{\lambda}x_{\nu} + \eta_{\lambda\sigma}x_{\mu}x_{\nu})/(x^2)^3 + t_5 x_{\mu}x_{\nu}x_{\lambda}x_{\sigma}/(x^2)^4. \tag{2.1}
\]

This is the most general tensor structure consistent with Lorentz invariance and three permutation symmetries 1) \( \mu \leftrightarrow \nu \); 2) \( \lambda \leftrightarrow \sigma \); 3) \( \mu\nu \leftrightarrow \lambda\sigma \), \( x \to -\vec{x} \). \(^3\)

Scaling is fixed from the canonical dimension \( [T_{\mu\nu}] = D \). We assume that there are no logarithms, see below. It will be convenient to use an alternative parametrization in terms of derivatives:

\[
\langle T_{\mu\nu}(x)T_{\lambda\sigma}(0) \rangle = [a_1 (\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\nu\lambda}\eta_{\mu\sigma}) + a_2 \eta_{\mu\nu}\eta_{\lambda\sigma}]/(x^2)^2 \\
+ a_3 (\eta_{\mu\lambda}\partial_\nu \partial_\sigma \frac{1}{x^2} + 3 \text{ perms}) + a_4 (\eta_{\mu\nu}\partial_\lambda \partial_\sigma \frac{1}{x^2} + \eta_{\lambda\sigma}\partial_\mu \partial_\nu \frac{1}{x^2}) \\
+ a_5 \partial_\mu \partial_\lambda \partial_\sigma \partial_\nu \log(x^2), \tag{2.2}
\]

where the constants \( a_i \) are certain linear combinations of \( t_i \). Passing to Fourier transform we get:

\[
\langle T_{\mu\nu}(k)T_{\lambda\sigma}(-k) \rangle = \log k^2 \left[ A_1 k^2(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\nu\lambda}\eta_{\mu\sigma}) + A_2 k^2 \eta_{\mu\nu}\eta_{\lambda\sigma} \right. \tag{2.3}
+ A_3(\eta_{\mu\lambda}k_{\nu}k_{\sigma} + 3 \text{ perms}) + A_4(\eta_{\mu\nu}k_{\lambda}k_{\sigma} + \eta_{\lambda\sigma}k_{\mu}k_{\nu}) \bigg] + A_5 k_{\mu}k_{\nu}k_{\lambda}k_{\sigma}/k^2,
\]

where \( A_i \propto a_i \). Notice that even though \( \log k^2 \) is present, the correlator changes only by a local quantity if the scale of the logarithm is changed.

Now let us impose the conservation:

\[
0 = k^\mu \langle T_{\mu\nu}T_{\lambda\sigma} \rangle = \log k^2 \left[ (A_1 + A_3) k^2(\eta_{\nu\sigma} + k_{\nu}\eta_{\sigma}) + (A_2 + A_4) k^2 \eta_{\nu} \eta_{\lambda\sigma} \right.
+ (2A_3 + A_4) k_{\nu}k_{\lambda}k_{\sigma} \bigg] + A_5 k_{\mu}k_{\nu}k_{\lambda}k_{\sigma}. \tag{2.4}
\]

This equation does not constrain \( A_5 \) since its contribution to \( \partial^\mu \langle T_{\mu\nu}(x)T_{\lambda\sigma}(0) \rangle \) is purely local. On the other hand, the coefficient multiplying \( \log k^2 \) must be set to zero. The three terms making it

\(^3\)Eq. (4.72) in Section 4.3.3 of \cite{10} is missing the \( t_3 \) term.

\(^4\)This form is also valid in presence of parity breaking, since in \( D = 2 \) it turns out impossible to write down a parity breaking term consistent with these symmetries.
up have different tensor structure; their linear independence can be checked by going to the frame $k^\mu = (1, 0)$. We conclude that

$$A_1 + A_3 = A_2 + A_4 = 2A_3 + A_4 = 0,$$

from where all the coefficients can be fixed in terms of, say, $A_1$, apart from $A_5$ which is left undetermined.

This is all that can be derived on the basis of conservation alone. Now let us see what this tells us about the 2-point function of the trace. From (2.3), we have

$$\langle T_\mu^\mu T_\lambda^\lambda \rangle = 4(A_1 + A_2 + A_3 + A_4) k^2 \log k^2 + A_5 k^2 = 0 + \text{local}. \quad (2.6)$$

Thus, the 2-point function of $T_\mu^\mu$ is zero up to local terms. In a unitary theory (or reflection-positive, if we are working in the Euclidean), this implies that $T_\mu^\mu \equiv 0$, and thus the theory is conformal.

Let us list here two implicit assumptions of the given proof: 1) the stress tensor 2-point function exists; 2) the stress tensor has canonical scaling, without logarithms.

Assumption 2) can be expressed formally as the requirement that the commutator with the scale generator $S$ take its canonical form:

$$i [S, T_{\mu\nu}(x)] = x^\rho \partial_\rho T_{\mu\nu}(x) + D T_{\mu\nu}(x). \quad (2.7)$$

In general, one can add to the right-hand side a term $\partial_\sigma \partial_\rho \tilde{Y}_{\mu\sigma\nu\rho}$, where $\tilde{Y}_{\mu\sigma\nu\rho}$ has the same symmetry as $Y_{\mu\sigma\nu\rho}$ in (1.5). This would be consistent with the integrated relation $i[S, H] = H$, where $H$ is the Hamiltonian. However, the correlators of such $T_{\mu\nu}$ would in general contain logarithms.

In this case one looks for a redefined, equivalent stress tensor with canonical scaling. In [2], it was shown that such a redefinition can always be achieved provided that the theory has a discrete spectrum of scaling dimensions. The redefined stress tensor 2-point function is free of logarithms, and the above argument is applicable. A nice example of this phenomenon can be found in Ref. [11]. One of several equivalent stress tensors considered in that paper, Eq. (14), does not scale canonically, and its trace 2-point function is nonzero. However, an appropriate improvement exists which restores the canonical scaling, and leads to the vanishing 2-point function of the trace. [The last step of the proof, concluding that $T_\mu^\mu \equiv 0$, cannot be carried out since the theory of Ref. [11] is not reflection-positive.]

Let us come back to the implicit assumption 1). Hull and Townsend [12] have shown that scale-invariant but not conformally-invariant unitary theories exist among 2D sigma-models with non-compact target space. These models likely violate assumption 1), and perhaps also 2) [2].
2.2 $D \geq 3$: extra ideas needed

In Appendix A, we repeat the $D = 2$ argument in higher dimensions and see that it does not go through. In other words, in $D \geq 3$ it is impossible to conclude from scaling, conservation, and unitarity alone that the stress tensor is traceless. In fact, we can understand this by an explicit example. Consider the free massless scalar theory. Its non-improved stress tensor,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial \phi)^2,$$

is conserved, the 2-point function is unitary, and does not contain logarithms. Yet $T^\mu_\mu = (1 - D/2)(\partial \phi)^2$ is not vanishing for $D \geq 3$. Of course we know that in this case an improved traceless tensor exists. But if you are given a random stress tensor you cannot expect to be able to prove that it’s traceless.

What this means is that in $D \geq 3$ we should start from (1.2) and aim for proving (1.3). This is weaker than tracelessness, but is enough to show that the theory is conformal.

Why is there such a difference between $D \geq 3$ and $D = 2$? One explanation is as follows. In $D = 2$, among all the equivalent stress tensors, only one will have the 2-point function which scales canonically, without logarithms. This is because $Y_{\mu\rho\nu\sigma}$ is dimensionless in 2D, and its 2-point function is logarithmic. Then, it turns out that this very special canonically-scaling $T_{\mu\nu}$ is traceless. In $D \geq 3$ the situation is different, since $Y_{\mu\rho\nu\sigma}$ has positive mass dimension. Thus the transformation (1.5) does not have to introduce logarithms.

At present, one can only hypothesize on how (1.3) could be derived. For example, one could start with correlators containing one $T_{\mu\nu}$ insertion and some other fields of the theory. These correlators are constrained via Ward identities. Another constraint is provided by (1.2). Now, perhaps one could show that correlators of $K_{\mu}$ satisfy an integrability condition which allows to define a local field $L_{\mu \nu}$, consistently with (1.3). Then one would have to extend the definition of $L_{\mu \nu}$ to correlators with two stress tensor insertions etc. This is not an easy plan to carry out.

Also, we know that unitarity must enter the scene, and it is not clear how this will happen.

3 Looking for a counterexample

It makes sense to attack the problem from both sides. If the theorem is not true, a counterexample must exist. In this section we will discuss what is known in this direction. We will explain that one should restrict attention to theories which have stress tensor. Actually, if this physical requirement
is dropped, the problem trivializes and simple counterexamples exist. We will also rule out the presence of counterexamples within a new large class of models in \(4 - \varepsilon\) dimensions.

### 3.1 Fake counterexamples without stress tensor

First a general remark about theories which should not be considered as counterexamples. We are referring to field theories which, while being unitary and scale invariant, do not have a stress tensor. As is well known, Wightman axioms require existence of the full energy and momentum, but not of their density, which is the stress tensor. However, it seems reasonable to adopt existence of a stress tensor as an additional axiom that a full-fledged field theory must satisfy. Otherwise we would not know how to couple the theory to (classical) gravity.

Consider a theory which contains a vector field \(A_\mu\) of scaling dimension \(\Delta\), whose 2-point function is given by:

\[
\langle A_\mu(x)A_\nu(0) \rangle = (x^2)^{-\Delta}(\alpha \eta_{\mu\nu} - 2x_\mu x_\nu/x^2).
\]  

(3.1)

We assume that the theory is free (Gaussian). In practice this means that all higher-order correlators are computed from (3.1) via Wick’s theorem. Composite fields can be defined via OPE. In essence, this defines a field theory, which is unitary if and only if the two-point function (3.1) is unitary.

One can analyze the unitarity of (3.1) by the same method we used to study the stress tensor 2-point function in Appendix A, see also [3]. The answer is that it is unitary if and only if

\[
\Delta \geq 2, \quad 1/\Delta \leq \alpha \leq 2 - 3/\Delta \quad (D = 4).
\]  

(3.2)

Now, let us analyze when the theory based on (3.1) is conformal. This can happen in only two ways: 1) \(A_\mu\) is a descendant of a primary scalar field, \(A_\mu = \partial_\mu \phi\); 2) \(A_\mu\) is a primary vector field. Case 1) is realized for \(\alpha = 1/\Delta\), at the lower bound of the interval allowed by unitarity. Case 2) requires \(\alpha = 1\), which is consistent with unitarity provided \(\Delta \geq 3\), a well-known result [13].

Any other value of \(\alpha\) will give a theory which is scale invariant, unitary, but not conformal.

As mentioned above, we consider this a fake counterexample because this theory does not have a stress tensor. In the conformal case, one could introduce a stress tensor in the \(1/N\) expansion realizing this model via AdS/CFT. In the non-conformal case, we do not know how to do even that, without introducing pathologies or lowering cutoff in the dual gravitational theory.

Another fake counterexample is linearized gravity\(^5\). Again, there is no (gauge-invariant) stress tensor in this theory. This can be viewed as a consequence of the Weinberg-Witten theorem [15].

\(^5\)See [14] for a discussion; we are grateful to Damiano Anselmi for bringing this example to our attention.
3.2 Systematic search in $4 - \varepsilon$ dimensions

As explained in the introduction, a putative counterexample theory must, to begin with, contain a non-conserved, dimension 3, hermitean vector field $K_\mu$ which can appear in (1.2). Nontrivial $K_\mu$ candidates should not be expressible as in (1.3), since in the latter case the theory is conformal.

The simplest model with $K_\mu$ candidates was considered in [2]; it is a theory of $N$ massless real scalars with quartic self-interaction, Eq. (1.6). The nontrivial $K_\mu$ candidates are given by (1.7). The $Q_{[ij]}$ is a real matrix, assumed antisymmetric since otherwise $K_\mu$ is a total derivative, which is a partial case of (1.3).

À la Wilson-Fisher, the model can have fixed points in $4 - \varepsilon$ dimensions. The one-loop $\beta$-function is given by

$$\beta_{ijkl} = -\varepsilon \lambda_{ijkl} + \frac{1}{16\pi^2} (\lambda_{ijmn}\lambda_{mnkl} + 2 \text{ perms}). \quad (3.3)$$

We assume that $\lambda_{ijkl}$ is symmetrized; it is also real as required by unitarity. The condition

$$\beta = 0 \quad (3.4)$$

gives a scale invariant theory which is also conformally invariant, since at one-loop the trace anomaly is given by

$$T_\mu^\mu = \frac{1}{4!} \beta_{ijkl}s_is_js_ks_l. \quad (3.5)$$

However, Eq. (3.4) is not the most general condition for scale invariance. The $\beta$-function encodes a change in the couplings when we integrate out a momentum shell. The theory will remain scale invariant if this change, though nonzero, can be compensated by adding to the Lagrangian a total derivative term $\partial_\mu K^\mu$. Using equations of motion (EOM) at leading order in the coupling, this term can be rewritten as

$$\partial_\mu K^\mu \approx \frac{1}{4!} Q_{ijkl}s_is_js_ks_l, \quad Q_{ijkl} = Q_{[im]}\lambda_{mjkl} + 3 \text{ perms}. \quad (3.6)$$

Thus the most general condition for a scale invariant fixed point:

$$T_\mu^\mu = \partial_\mu K^\mu \iff \beta = Q. \quad (3.7)$$

A fixed point (3.7) would break conformal invariance if $Q \neq 0$. However, as noticed in [2], this never happens. To see this, we contract with $Q$ and get:

$$Q \cdot Q = \beta \cdot Q \equiv 0, \quad (3.8)$$

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where the RHS vanishes identically by the symmetry properties of $\lambda_{ijkl}$ and $Q_{[ij]}$:

$$
\lambda_{ijkl} Q_{[IM]} \lambda_{Mjkl} = 0,
$$

$$
(\lambda_{jmn}\lambda_{mnkl} + \lambda_{kmn}\lambda_{mnjl} + \lambda_{lmn}\lambda_{mnjk}) Q_{[IM]} \lambda_{Mjkl} = 3 Q_{[IM]} \lambda_{jmn}\lambda_{Mjkl}\lambda_{mnkl} = 0. \tag{3.9}
$$

[In both cases, antisymmetric $Q_{[IM]}$ is contracted with a symmetric tensor.] Eq. (3.8) implies that $Q \equiv 0$ and we are back to the conformally-invariant case (3.4).

It is not immediately clear what to make out of this result. On the one hand, the considered class of models was rather large. On the other hand, it is by far not the most general unitary field theory which can have perturbative fixed points in $D = 4 - \varepsilon$. It seems reasonable to try to complicate the model, in the hope that a new qualitative effect may appear. For example, reality of couplings seems to have played a role in the above argument, and one could think that breaking CP may help in constructing a counterexample.

For this reason, we extend the field content of the model, by adding an arbitrary number of Weyl fermions, and making them interact with the scalars via Yukawa couplings. The full Lagrangian is now the sum of (1.6) and (1.8). In general, it breaks both CP and P. If these discrete symmetries have anything to do with the relation between scale and conformal invariance, we can hope to detect this.

Let us now repeat the steps of the above analysis for the new theory. The standard $\beta$-functions are given by:

$$
\beta^{(\lambda)}_{ijkl} = -\varepsilon \lambda_{ijkl} + \frac{1}{16\pi^2} [\lambda_{ijmn}\lambda_{mnkl} + 2 \text{ perm}] + \frac{1}{4\pi^2} [\text{Tr}(y_i^* y_m + y_i y_m^*) \lambda_{mjkl} + 3 \text{ perms}]
$$

$$
- \frac{1}{4\pi^2} \left[ \text{Tr}(y_i^* y_j^* y_k y_l + y_i^* y_j^* y_k^*) + 5 \text{ other perms (jkl)} \right],
$$

$$
\beta^{(y)}_{ij|ab} = -\frac{\varepsilon}{2} y_{ij|ab} + \frac{1}{4\pi^2} \text{Tr}(y_i^* y_j + y_i y_j^*) y_{j|ab} + \frac{1}{8\pi^2} (y_i y_j^*)_{ab} + \frac{1}{8\pi^2} \left[ (y_i^* y_j^*)_{ab} + (y_j y_i^*)_{ab} \right].
$$

The matrix $y_{ij|ab} \equiv (y_i)_{ab}$ is assumed symmetrized in $ab$. Matrix multiplication and trace are in the fermion flavor space.

Now, the usual fixed points satisfy

$$
\beta^{(\lambda)} = 0, \quad \beta^{(y)} = 0, \tag{3.11}
$$

will have both scale and conformal invariance. More general fixed points are associated with nontrivial $K_\mu$ candidates, which in this model have the form

$$
K_\mu = Q_{[ij]} s_i \partial_\mu s_j + i P_{ab} \bar{\psi}_a \sigma_\mu \psi_b . \tag{3.12}
$$
Here $P$ must be antihermitean to have a hermitean $K_{\mu}$.

As before, we try to compensate the renormalization group transformation by adding the total derivative $\partial_{\mu}K^{\mu}$ and re-expressing via the EOM. We have

$$\partial_{\mu}K^{\mu} \cong \frac{1}{4!}Q_{ijkl}s_is_js ks_l + \frac{1}{2!}(P_{i\alpha\beta}\psi_a\psi_b + \text{h.c.}),$$

where $Q$ is the same as before, and

$$P_{i\alpha\beta} = Q_{ij}y_{j\alpha\beta} + [(y_iP)_{\alpha\beta} + a \leftrightarrow b].$$

Thus, fixed points without conformal invariance are the solutions of

$$\beta(\lambda) = Q, \quad \beta(y) = P$$

with nonzero $P$ and/or $Q$. As we will show now, no such solutions exist.

**Theorem** All solutions of this equation have zero $P$ and $Q$, and thus do not break conformal symmetry.

The proof is based on the same idea. We contract (3.15) with $Q$ and $P^*$ and show that $\beta(\lambda) \cdot Q$ and $\beta(y) \cdot P^*$ vanish. This way we conclude first that $P = 0$, and then that $Q = 0$. It turns out crucial to proceed in this order because, as we will see, $\beta(\lambda) \cdot Q$ does not vanish identically but only modulo terms proportional to $P$.

These statements can be and were verified by tensor manipulations analogous to (3.9), only more tedious. We will now show an alternative, diagrammatic, way of organizing this computation. The $\beta$-functions (3.10) are the sum of the following Feynman graphs:

$$\beta(\lambda) = \ldots + \ldots + \ldots + \ldots + \text{perms},$$

$$\beta(y) = \ldots + \ldots + \ldots + \ldots + \text{perm}.$$  

with the vertices

$$\kappa_{ijkl} = \ldots, \quad y_{i\alpha\beta} = \ldots, \quad y^*_{i\alpha\beta} = \ldots.$$  

The arrows on fermion lines show the flow of chirality. If no arrows are shown in a fermion loop, a sum over both ways to distribute the arrows is presumed.
The precise numerical values of the loop integrals, appearing as prefactors in (3.10), will not be important below. Thus we strip Feynman graphs from all spacetime dependence (propagators, σ-matrices). The only thing that counts is that these graphs correctly encode the tensor structure of various terms in (3.10).

The \( Q \) and \( P \) tensors can be encoded in the same language, introducing new vertices to denote contractions with \( Q \) and \( P \):

\[
Q = \text{\includegraphics{Q}} + \text{perms}, \quad (3.19)
\]

\[
P = \text{\includegraphics{P}} + \text{perm}. \quad (3.20)
\]

Let us use this formalism to give a diagrammatic proof of (3.8). We have to contract the \( Q \) graph (3.19) with the purely scalar diagrams in the graphical representation of \( \beta^{(\lambda)} \), Eq. (3.16). This contraction gives diagrams of two types:

\[
\text{\includegraphics{Q_contraction}} \quad \text{and} \quad \text{\includegraphics{P_contraction}}. \quad (3.21)
\]

The thin wavy lines cutting the diagrams show where indices were contracted.

Both these diagrams look like a contraction of (antisymmetric) \( Q \) with a left-right symmetric graph. Clearly, these contractions are zero. Two graphs (3.21) neatly visualize the content of Eqs. (3.9).

Armed with this formalism, we will now prove the theorem without danger of getting lost in the forest of indices. We begin by contracting the second Eq. (3.15) with \( P^* \):

\[
P \cdot P^* = \beta^{(y)} \cdot P^*. \quad (3.22)
\]

Various terms appearing in the RHS correspond to contractions of \( Q \) and \( P \) and are as follows:

\[
\text{\includegraphics{Q_contraction_1}} \quad \text{\includegraphics{Q_contraction_2}} \quad \text{\includegraphics{Q_contraction_3}} \quad \text{\includegraphics{Q_contraction_4}} \quad \text{\includegraphics{Q_contraction_5}} \quad \text{\includegraphics{Q_contraction_6}} \quad \text{\includegraphics{Q_contraction_7}}. \quad (3.23)
\]

\[
\text{\includegraphics{P_contraction_1}} \quad \text{\includegraphics{P_contraction_2}} \quad \text{\includegraphics{P_contraction_3}} \quad \text{\includegraphics{P_contraction_4}} \quad \text{\includegraphics{P_contraction_5}} \quad \text{\includegraphics{P_contraction_6}}. \quad (3.24)
\]
All these graphs are symmetric, up to inversion of the arrow directions, which corresponds to complex conjugation. This means that the real antisymmetric $Q$ and antihermitean $P$ are contracted with hermitean matrices. Such contractions are pure imaginary, and do not contribute to the manifestly real LHS of (3.22).

We conclude that $\mathcal{P} \cdot \mathcal{P}^* = 0$, and thus $\mathcal{P} = 0$. Let us proceed with $Q$. From the first Eq. (3.15) we have:

$$Q \cdot Q = \beta^{(\Lambda)} \cdot Q. \quad (3.25)$$

Purely scalar terms in the RHS were already shown above to vanish identically. The terms involving fermion loops are:

$$Q \cdot Q = \beta^{(\Lambda)} \cdot Q, \quad (3.26)$$

The first graph is symmetric and vanishes for the usual reason. The other two do not exhibit any obvious symmetry. However, let us invoke the already shown property $\mathcal{P} = 0$. Graphically:

$$Q \quad = \quad - \left( \left\langle \ldots \right\rangle \mathcal{P} + \text{perm} \right). \quad (3.27)$$

Using this identity, the two non-symmetric graphs are transformed into:

$$, \quad (3.28)$$

which are symmetric and vanish. The proof is complete.

4 Conclusions

In this paper we gave a fresh look at the relation of scale and conformal invariance in unitary field theories in $D > 2$. As we discussed at length, the situation is frustrating: we don’t know if one implies the other, although this holds in all known examples. We added to the list a new large class of models: multi-field theories of massless scalars and Weyl fermions with arbitrary quartic
and Yukawa interactions. What makes these models interesting is that they contain nontrivial candidates for a non-conserved dimension 3 vector which could in principle appear as a correction term in the scale current, thus making a scale-invariant theory non-conformal. However, we showed that this never happens for the one-loop fixed points in $4 - \varepsilon$ dimensions. Our current proof proceeds by inspecting symmetry properties of certain Feynman graphs. We suspect that a deeper explanation must exist and hope that it may one day be found.

In conclusion, we would like to point out that if scale-invariant but non-conformal unitary theories exist, this may have not only theoretical but also phenomenological consequences:

1. In the unparticle physics scenario \[17\] one assumes the existence of a scale-invariant hidden sector weakly coupled to the Standard Model via non-renormalizable operators. To study phenomenology of such a scenario, one would like to know if dimensions and propagators of vector operators in the hidden sector must follow the rules of conformal theories, or can be more general \[3\].

2. In the conformal technicolor scenario of the electroweak symmetry breaking \[18\], one assumes that the Higgs field $H$ belongs to a strongly interacting sector with conformal invariance above a TeV. Furthermore, one assumes an unusual pattern of operator dimensions: $H$ should have dimension close to 1 (to avoid problems with flavor), while the composite operator $H^\dagger H$ must have dimension close to 4 (to solve the hierarchy problem). It is not known if such large deviation from the naive relation $[H^\dagger H] \simeq 2[H]$ can be realized in a conformal field theory. In fact, recent work \[19\] uses conformal symmetry to show that there is at least one scalar in the OPE $H^\dagger \times H$, whose dimension is close to 2 if $[H] \simeq 1$. While it is not yet known if this scalar is a singlet $H^\dagger H$ or a triplet $H^\dagger \sigma^a H$, one can well imagine that future studies may rule out conformal technicolor. In this case it would be interesting to know if the scenario could be saved by assuming that the Higgs sector is scale-invariant but not conformal, in which case the bounds of \[19\] do not apply.

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13
A Stress tensor two-point function in \( D \geq 3 \)

In this section, we will explore how conservation and unitarity constrain the stress tensor 2-point function in \( D \geq 3 \). As explained in Section [2.2](#), we cannot expect to derive tracelessness. It is instructive to see where exactly the \( D = 2 \) argument breaks down.

Analogously to (2.2), we have

\[
\langle T_{\mu\nu}(x)T_{\lambda\sigma}(0) \rangle = \left[ a_1 (\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\nu\lambda}\eta_{\mu\sigma}) + a_2 \eta_{\mu\nu}\eta_{\lambda\sigma} \right] / (x^2)^D \\
+ a_3 (\eta_{\mu\lambda}\partial_{\nu}\partial_{\sigma} - \eta_{\nu\lambda}\partial_{\mu}\partial_{\sigma} + 3 \text{ perms}) + a_4 (\eta_{\mu\nu}\partial_{\lambda}\partial_{\sigma} - \eta_{\lambda\sigma}\partial_{\mu}\partial_{\nu} + 1) \\
+ a_5 \partial_{\mu}\partial_{\lambda}\partial_{\sigma}\partial_{\nu} - \frac{1}{(x^2)^{D-2}}. \\
\tag{A.1}
\]

This is the most general form in \( D = 4 \). In \( D = 3 \) one could also consider a parity-breaking term \( \propto \epsilon_{\mu\lambda\nu}\cdot x^\rho\eta_{\rho\sigma} + \text{symmetrizations} \); we do not analyze this possibility.

Passing to the momentum space,

\[
\langle T_{\mu\nu}(k)T_{\lambda\sigma}(-k) \rangle = f_D(k^2) \left[ A_1 k^2(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\nu\lambda}\eta_{\mu\sigma}) + A_2 k^2\eta_{\mu\nu}\eta_{\lambda\sigma} \\
+ A_3 (\eta_{\mu\lambda}k_{\nu}k_{\sigma} + 3 \text{ perms}) + A_4 (\eta_{\mu\nu}k_{\lambda}k_{\sigma} + \eta_{\lambda\sigma}k_{\mu}k_{\nu}) + A_5 k_{\mu}k_{\nu}k_{\lambda}k_{\sigma}/k^2 \right], \\
\tag{A.2}
\]

where

\[
f_3 \propto (k^2)^{3/2}, \quad f_4 \propto (k^2)^2 \log k^2. \\
\tag{A.3}
\]

We now see a crucial difference with \( D = 2 \). In 2D, the \( A_5 \) 4-derivative term in (2.2) had an analytic structure in the momentum space, Eq. (2.3), different from the other terms. That’s why it dropped out from the conservation constraint and contributed only a local term to the 2-point function of the trace. In retrospect, this was essentially due to an accident, that in 2D the number of indices of \( T_{\mu\nu} \) becomes equal to its scaling dimension, which led to the appearance of the logarithm in Eq. (2.2). This does not happen in \( D \geq 3 \), where all 5 terms in (A.1) are on equal footing. Thus, we can already foresee that conservation will not be as constraining.

To see this explicitly, we have:

\[
0 = k^\mu \langle T_{\mu\nu}T_{\lambda\sigma} \rangle = f_D(k^2)[(A_1 + A_3)(k_{\lambda}\eta_{\nu\sigma} + k_{\sigma}\eta_{\nu\lambda}) + (A_2 + A_4)k_{\nu}g_{\lambda\sigma} \\
+ (2A_3 + A_4 + A_5)k_{\nu}k_{\lambda}k_{\sigma}/(k^2)], \\
\tag{A.4}
\]

and we conclude

\[
A_1 + A_3 = A_2 + A_4 = 2A_3 + A_4 + A_5 = 0, \\
\tag{A.5}
\]

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which is weaker than \[(2.5)\].

Let us now see which further constraints on \(A_i\) come out by imposing unitarity. The unitarity constraint is obtained by considering the non time-ordered 2-point function in Minkowski space \[(A.2)\], and is concentrated in the forward lightcone \(k^0 > 0, \ k^2 \equiv -k_0^2 + \vec{k}^2 < 0\). In a unitary theory, the tensorial spectral density must be positive definite, i.e. for any fixed tensor \(Y_{\mu\nu}\) the operator \(Y_{\mu\nu}T_{\mu\nu}\) must have positive spectral density.

An important partial case is \(Y_{\mu\nu} = \eta_{\mu\nu}\), which corresponds to studying the 2-point function of the trace. From \[(A.2), (A.5)\], we have

\[
\langle T^\mu(k)T^{\nu}_{\chi}(-k)\rangle = (D - 1)[2A_1 + (D - 1)A_2] f_D(k^2), \tag{A.6}
\]

where we expressed all \(A_i\) in terms of \(A_1, A_2\). Thus

\[
T_{\mu}^\mu \text{ unitarity } \iff 2A_1 + (D - 1)A_2 \geq 0. \tag{A.7}
\]

Passing to general \(Y_{\mu\nu}\), by Lorentz invariance it is enough to examine the spectral density for \(\vec{k} = 0\). Multiplying \[(A.2)\] with a symmetric \(Y_{\mu\nu}\) and its conjugate, and extracting the imaginary part, we get the following constraint on the coefficients:

\[
2A_1 Y_{\mu\nu}^* Y_{\mu\nu} + A_2 |Y_{\mu\nu}|^2 - 4A_3 Y_{\mu0}^* Y_{\mu0} - A_4 (Y_{\mu\nu}^* Y_{\nu00} + h.c.) + A_5 |Y_{00}|^2 \geq 0. \tag{A.8}
\]

Separating the spatial and temporal coordinates, this condition can be rewritten as:

\[
(2A_1 + A_2 + 4A_3 + 2A_4 + A_5) |Y_{00}|^2 - 4(A_1 + A_3) |Y_{0i}|^2 - (2A_2 + A_4) (Y_{ii}^* Y_{00} + h.c.) + 2A_1 Y_{ij}^* Y_{ij} + A_2 |Y_{ii}|^2 \geq 0, \tag{A.9}
\]

The first three terms drop out thanks to the conservation constraints \[(A.5)\], and we are left with

\[
2A_1 Y_{ij}^* Y_{ij} + A_2 |Y_{ii}|^2 \geq 0. \tag{A.10}
\]

The necessary and sufficient conditions for this to be true are derived by considering separately off-diagonal and diagonal \(Y_{ij}\). We obtain:

\[
T_{\mu\nu} \text{ unitarity } \iff A_1 \geq 0, \ 2A_1 + (D - 1)A_2 \geq 0. \tag{A.11}
\]

This is not much extra mileage compared to the partial case \[(A.7)\]. As expected, we cannot conclude that \(T_{\mu}^\mu = 0\).

---

\(^6\)Cardy \[16\] considers reflection positivity for the \(T_{\mu\nu}\) components in \(D \geq 3\) and states that the trace 2-point function gives the strongest condition. However, we do find an extra condition \(A_1 \geq 0\).
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