Dynamic Team Theory of Stochastic Differential Decision Systems with Decentralized Noiseless Feedback Information Structures via Girsanov’s Measure Transformation

Charalambos D. Charalambous

Abstract

In this paper we generalized static team theory to dynamic team theory, in the context of stochastic differential decision system with decentralized noiseless feedback information structures.

We apply Girsanov’s theorem to transformed the initial stochastic dynamic team problem to an equivalent team problem, under a reference probability space, with state process and information structures independent of any of the team decisions. Subsequently, we show, under certain conditions, that continuous-time and discrete-time stochastic dynamic team problems, can be transformed to equivalent static team problems, although computing the optimal team strategies using this method might be computational intensive. Therefore, we propose an alternative method, by deriving team and Person-by-Person (PbP) optimality conditions, via the stochastic Pontryagin’s maximum principle, consisting of forward and backward stochastic differential equations, and a set of conditional variational Hamiltonians with respect to the information structures of the team members.

Finally, we relate the backward stochastic differential equation to the value process of the stochastic team problem.

C.D. Charalambous is with the Department of Electrical and Computer Engineering, University of Cyprus, Nicosia 1678 (E-mail: chadcha@ucy.ac.cy).
I. INTRODUCTION

In classical stochastic control or decision theory the control actions or decisions applied by the multiple controllers or Decision Makers (DM) are based on the same information. The underlying assumption is that the acquisition of information is centralized, or the information collected at different observation posts is communicated to each controller or DMs. Classical stochastic control problems are often classified, based on the information available for control actions, into fully observable [1]–[7] and partially observable [2], [5], [8]. Fully, observable refers to the case when the information structure or pattern available for control actions is generated by the state process (also called feedback information) or the exogenous state noise process (also called nonanticipative information), and partially observable refers to the case when the information structure available for control actions is a nonlinear function of the state process corrupted by exogenous observation noise process.

In this paper, we deviate from the classical stochastic control or decision formulation by consider a system operating over a finite time period \([0, T]\), with the following features.

1) There are \(N\) observation posts or stations collecting information;
2) There are \(N\) control stations, each having direct access to information collected by at most one observation post, without delay;
3) The observation stations may not communicate their information to the other control stations, or they may communicate their information to the other control stations by signaling part or all of their information to some of the control stations with delay;
4) The \(N\) control stations may not have perfect recall, that is, information which is available at any of the control stations at time \(t \in [0, T]\) may not be available at any future time \(\tau \geq t, \tau \in (0, T]\);
5) The control strategies applied at the \(N\) control stations have to be coordinated to optimize a common pay-off or reward.

In the above formulation we have assumed that one observation post is serving one control station without delay, and we allowed the possibility that a subset of the other observation posts signal their information to any of the control stations they are not serving subject to delay. Such signaling among the observation posts and control stations is called information sharing [9]–[12].
The elements of the proposed system of study are the following.

\[ Z_N \triangleq \{1, 2, \ldots, N\} : \text{Set of observation posts/control stations}; \]

\[ x : [0, T] \times \Omega \rightarrow \mathbb{R}^n : \text{Unobserved state process}; \]

\[ W : [0, T] \times \Omega \rightarrow \mathbb{R}^n : \text{State exogenous Brownian Motion (BM) process}; \]

For \( i = 1, \ldots, N, \)

\[ u^i : [0, T] \rightarrow \mathbb{A}^i : \text{Control process with action space } \mathbb{A}^i \subseteq \mathbb{R}^{d_i} \text{ applied at the } i\text{th control station}; \]

\[ z^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{k_i} : \text{Distributed observation process collected at the } i\text{th observation post}; \]

\[ h^i : [0, T] \times C([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}^{k_i} : \text{Information functional generating } z^i \text{ at the } i\text{th observation post}; \]

\[ \mathbb{U}^{z^i}[0, T] : \text{Admissible strategies generating the control actions at the } i\text{th control station based on } \{z^i(t) : t \in [0, T]\}; \]

\[ J : \mathbb{A}^{(N)} \rightarrow (-\infty, \infty] : \text{Team pay-off or reward.} \]

We call as usual the information available as arguments of the control laws, which generate the control actions applied at the \( N \) control stations, “information Structure or Pattern”.

Suppose, for now, there is no signaling of information from the observation posts to any of the control station they are not serving, and let \( \{z^i(t) : 0 \leq t \leq T\} \) denote the observation available to the \( i\)th control station to generate the control actions \( \{u^i_t : 0 \leq t \leq T\} \) for \( i = 1, \ldots, N \). Denote the corresponding control strategies by \( \mathbb{U}^{z^i}[0, T] \), for \( i = 1, \ldots, N \). Given the control strategies, the performance of the collective decisions or control actions applied by the control stations, the stochastic differential decision system is formulated using dynamic team theory, as follows.

\[
\inf \left\{ J(u^1, \ldots, u^N) : (u^1, \ldots, u^N) \in \times_{i=1}^N \mathbb{U}^{z^i}[0, T] \right\},
\]

(1)

\[
J(u^1, \ldots, u^N) = \mathbb{E}_{\mathbb{P}_0}\left\{ \int_0^T \ell(t, x(t), u^1_t(z^1), \ldots, u^N_t(z^N))dt + \varphi(x(T)) \right\},
\]

(2)

subject to stochastic Itô differential dynamics and distributed noiseless observations

\[
dx(t) = f(t, x(t), u^1_t(z^1), \ldots, u^N_t(z^N))dt + \sigma(t, x(t))dW(t), \quad x(0) = x_0, \quad t \in (0, T],
\]

(3)

\[
z^i(t) = h^i(t, x), \quad t \in [0, T], \quad i = 1, \ldots, N,
\]

(4)
where \( \mathbb{E}^\mathbb{P}_\Omega \) denotes expectation with respect to an underlying probability space \( (\Omega, \mathcal{F}, \mathbb{P}_\Omega) \). The stochastic system (3) may be a compact representation of many interconnected subsystems with states \( \{x^i \in \mathbb{R}^{n_i} : i = 1, \ldots, N\} \), \( n \triangleq \sum_{i=1}^{N} n_i \), aggregated into a single state representation \( x \in \mathbb{R}^n \), where \( x^i \) represents the state of the local subsystem, \( \{z^i(t) : 0 \leq t \leq T\} \) its local distributed observation process collected at the \( i \)th local observation post, and \( \{u^i_t(z^i) : 0 \leq t \leq T\} \) its local control process applied at the \( i \)th local control station, such that for each \( t \in [0, T] \), the control law is \( u^i_t(z^i) \equiv \mu^i_t(\{z^i(s) : 0 \leq s \leq t\}) \), a nonanticipative measurable function \( \mu^i_t(\cdot) \) of the \( i \)th control station information structure \( \{z^i(t) : 0 \leq t \leq T\} \), for \( i = 1, \ldots, N \).

We often call the stochastic differential decision system (1)-(4) with decentralized noiseless feedback information structures, \( \{z^1(t), z^2(t), \ldots, z^N(t) : 0 \leq t \leq T\} \), a stochastic dynamic team problem, and a strategy \( u^o \triangleq (u^1^o, u^2^o, \ldots, u^N^o) \in \prod_{i=1}^{N} \mathbb{U}^{z^i}[0, T] \) which achieves the infimum in (1) a team optimal strategy.

Moreover, we call \( u^o \triangleq (u^1^o, u^2^o, \ldots, u^N^o) \in \prod_{i=1}^{N} \mathbb{U}^{z^i}[0, T] \) a PbP optimal strategy if

\[
J(u^1^o, \ldots, u^N^o) \leq J(u^{1, -1}_i, u^{i-1, -1}_i, u^i, u^{i+1, -1}_i, \ldots, u^N^o), \forall u^i \in \mathbb{U}^{z^i}[0, T], \forall i = 1, \ldots, N,
\]

and the infimum subject to constraints (3), (4) is achieved. In team theory terminology \( \{u^1, \ldots, u^N\} \) are called the DMs, agents or members of the team game.

In this paper, we investigate the stochastic dynamic team problem (1)-(4), and its generalization when, there is information sharing from the observation posts to any of the control stations, and there is no perfect recall of information at the control stations.

Recall that a stochastic team problem is called a “Static Team Problem” if the information structures available for decisions are not affected by any of the team decisions. Optimality conditions for static team problems are developed by Marschak and Radner \[13\]–\[15\], and subsequently generalized in \[16\]. Clearly, since the information structures \( \{z^i(t) : 0 \leq t \leq T\} \), \( i = 1, \ldots, N \) generated by (4) are affected by the team decisions via the state process \( \{x(t) : 0 \leq t \leq T\} \) generated by (3), the static team theory optimality conditions given in \[13\]–\[16\] do not apply.

On the other hand, stochastic optimal control theory with full information is developed under a centralized assumption on the information structures. Therefore, a natural question is whether any of these techniques developed over the last 60 years for centralized stochastic control problems and dynamic games, such as, dynamic programming, stochastic Pontryagin’s maximum principle,
and martingale methods are applicable to stochastic dynamic team problems, and if so how.

In this paper we apply techniques from classical stochastic control theory to generalize Marschak’s and Radner’s static team theory \[13\]–\[15\] to continuous-time stochastic differential decision systems with decentralized noiseless feedback information structures, defined by (1)-(4). Moreover, we discuss generalizations of (1)-(4), when there is information sharing from the observation posts to any of the control stations, and there is no perfect recall of information at the control stations. Our methodology is based on deriving team and PbP optimality conditions, using stochastic Pontryagin’s maximum principle, by utilizing the semi martingale representation method due to Bismut [3], under a weak formulation of the probability space by invoking Girsanov’s theorem [17]. First, we apply Girsanov’s theorem to transform the original stochastic dynamic team problem to an equivalent team problem, under a reference probability space in which the state process and the information structures are not affected by any the team decisions. Subsequently, we show the precise connection between Girsanov’s measure transformation and Witsenhausen’s notion of “Common Denominator Condition” and “Change of Variables” introduced in [18] to establish equivalence between static and dynamic team problems. We elaborate on this connection for both continuous-time and discrete-time stochastic systems, and we state certain results from static team theory which are directly applicable. However, since the computation of the optimal team strategies via static team theory might be computationally intensive, we proceed further to derive optimality conditions based on stochastic variational methods, by taking advantage of the fact that under the reference measure, the state process and the information structures do not react to any perturbations of the team decisions. The optimality conditions are given by a “Hamiltonian System” consisting of a backward and forward stochastic differential equations, while the optimal team actions of the \(i\)th team member are determined by a conditional variational Hamiltonian, conditioned on the information structure of the \(i\)th team member, while the rest are fixed to their optimal values, for \(i = 1, \ldots, N\). Finally, we show the connection between the backward stochastic differential equation and the value process of the stochastic dynamic team problem.

We point out that the approach we pursued in this paper is different from the various approaches pursued over the years to address stochastic dynamic decentralized decision systems, formulated using team theory in [10]–[12], [16], [18]–[45], and our recent treatment in [46]. Compared to [46], in the current paper we apply Girsanov’s measure transformation, which allows...
us to derive the stochastic Pontryagin’s maximum principle, for decentralized noiseless feedback information structures, instead of nonanticipative (open loop) information structures adapted to a sub-filtration of the fixed filtration generated by the Brownian motion \( \{ W(t) : t \in [0, T] \} \) (e.g., \( u_t = \mu(t, W) \)) considered in [46]. Since feedback strategies are more desirable compared to nonanticipative strategies, this paper is an improvement to [46]. The only disadvantage is that, unlike [46], we cannot allow dependence of the diffusion coefficient \( \sigma \) on the team decisions. The current paper also generalizes some of the results on centralized partial information optimality conditions derived in [47] to centralized partial feedback information. We note that the case of decentralized noisy information structures is treated in [48], and therefore by combining the results of this paper with those in [48], we can handle any combination of decentralized noiseless and noisy information structures. However, when applying the optimality conditions to determine the optimal team strategies, the main challenge is the computation of conditional expectations with respect to the information structures. The procedure is similar to [49], where various examples from the communication and control areas are presented, using decentralized nonanticipative strategies.

The rest of the paper is organized as follows. In Section II we introduce the stochastic differential team problem and its equivalent re-formulations using the weak Girsanov measure transformation approach. Here, we also establish the connection between Girsanov’s theorem and Witsenhausen’s “Common Denominator Condition” and “Change of Variables” [18] for stochastic continuous-time and discrete-time dynamical systems. Further, we derive the variational equation which we invoke in Section III to derive the optimality conditions, both under the reference probability measure and under the initial probability measure. In Section IV we provide concluding remarks and comments on future work.

**II. EQUIVALENT STOCHASTIC DYNAMIC TEAM PROBLEMS**

In this section, we consider the stochastic dynamic team problem (1)-(4), and we apply Girsanov’s theorem, to transformed it to an equivalent team problem under a reference probability measure, in which the information structures are functionals of Brownian motion, and hence independent of any of the team decisions. We will also briefly discuss the discrete-time counterpart of Girsanov’s theorem, and we will show its equivalence to Witsenhausen’s so-called
“Common Denominator Condition” and “Change of Variables” discussed in [18].

Let \( \left( \Omega, \mathcal{F}, \{ \mathcal{F}_{0,t} : t \in [0,T] \}, \mathbb{P} \right) \) denote a complete filtered probability space satisfying the usual conditions, that is, \((\Omega, \mathcal{F}, \mathbb{P})\) is complete, \(\mathcal{F}_{0,0}\) contains all \(\mathbb{P}\)-null sets in \(\mathcal{F}\). Note that filtrations \(\{ \mathcal{F}_{0,t} : t \in [0,T] \}\) are monotone in the sense that \(\mathcal{F}_{0,s} \subseteq \mathcal{F}_{0,t}, \forall 0 \leq s \leq t \leq T\). Moreover, we assume throughout that filtrations \(\{ \mathcal{F}_{0,t} : t \in [0,T] \}\) are right continuous, i.e., \(\mathcal{F}_{0,t} = \mathcal{F}_{0,t+} \triangleq \bigcap_{s>t} \mathcal{F}_{0,s}, \forall t \in [0,T]\). We define \(\mathcal{F}_T \triangleq \{ \mathcal{F}_{0,t} : t \in [0,T]\}\). Consider a random process \(\{z(t) : t \in [0,T]\}\) taking values in \((\mathbb{Z}, \mathcal{B}(\mathbb{Z}))\), where \((\mathbb{Z}, d)\) is a metric space, defined on the filtered probability space \((\Omega, \mathcal{F}, \{ \mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P})\). The process \(\{z(t) : t \in [0,T]\}\) is said to be (a) measurable, if the map \(t, \omega \rightarrow z(t, \omega)\) is \(\mathcal{B}([0,T]) \times \mathbb{P}/\mathcal{B}(\mathbb{Z})\)-measurable, (b) \(\{ \mathcal{F}_{0,t} : t \in [0,T]\}\)-adapted, if for all \(t \in [0,T]\), the map \(\omega \rightarrow z(t, \omega)\) is \(\mathcal{F}_{0,t}/\mathcal{B}(\mathbb{Z})\)-measurable, (c) \(\{ \mathcal{F}_{0,t} : t \in [0,T]\}\)-progressively measurable if for all \(t \in [0,T]\), the map \((s, \omega) \rightarrow z(s, \omega)\) is \(\mathcal{B}([0,t]) \otimes \mathcal{F}_{0,t}/\mathcal{B}(\mathbb{Z})\)-measurable. It can be shown that any stochastic process \(\{z(t) : t \in [0,T]\}\) on a filtered probability space \((\Omega, \mathcal{F}, \{ \mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P})\) which is measurable and adapted has a progressively measurable modification. Unless otherwise specified, we shall say a process \(\{z(t) : t \in [0,T]\}\) is \(\{ \mathcal{F}_{0,t} : t \in [0,T]\}\)-adapted if the processes is \(\{ \mathcal{F}_{0,t} : t \in [0,T]\}\)-progressively measurable [7].

We use the following notation.

**TABLE I**

**Table of Notation**

\[
\begin{align*}
\mathbb{Z}_N & \triangleq \{1, 2, \ldots, N\}: \text{Subset of natural numbers.} \\
s & \triangleq \{s^1, s^2, \ldots, s^N\}: \text{Set consisting of } N \text{ elements.} \\
s^{-i} &= s \setminus \{s^i\}, \quad s = (s^{-i}, s^i): \text{Set } s \text{ minus } \{s^i\}. \\
\mathcal{L}(\mathcal{X}, \mathcal{Y}) &= \text{Linear transformation mapping a vector space } \mathcal{X} \text{ into a vector space } \mathcal{Y}. \\
A^{(i)} : i\text{th column of a map } A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), i = 1, \ldots, n. \\
\mathcal{A}^i \subseteq \mathbb{R}^d : \text{Action spaces of controls applied at the } i\text{th control station}, i = 1, \ldots, N.
\end{align*}
\]

Let \(C([0,T], \mathbb{R}^n)\) denote the space of continuous real-valued \(n\)-dimensional functions defined on the time interval \([0,T]\), and \(\mathcal{B}(\mathbb{R}^n)\) its canonical Borel filtration.

Let \(L^2_{\mathcal{F}_T}([0,T], \mathbb{R}^n) \subseteq L^2(\Omega \times [0,T], d\mathbb{P} \times dt, \mathbb{R}^n) \equiv L^2([0,T], L^2(\Omega, \mathbb{R}^n))\) denote the space...
of $\mathbb{F}_T$-adapted random processes \{\(z(t) : t \in [0, T]\)\} such that
\[
\mathbb{E} \int_{[0, T]} |z(t)|^2_{\mathbb{R}^n} dt < \infty,
\]
which is a Hilbert subspace of \(L^2([0, T], L^2(\Omega, \mathbb{R}^n))\).

Similarly, let \(L^2_{\mathbb{F}_T}([0, T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \subset L^2([0, T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))\) denote the space of $\mathbb{F}_T$-adapted matrix valued random processes \{\(\Sigma(t) : t \in [0, T]\)\} such that
\[
\mathbb{E} \int_{[0, T]} |\Sigma(t)|^2_{\mathbb{L}(\mathbb{R}^m, \mathbb{R}^n)} dt \triangleq \mathbb{E} \int_{[0, T]} tr(\Sigma^*(t)\Sigma(t)) dt < \infty.
\]

Let \(B^\infty_{\mathbb{F}_T}([0, T], L^2(\Omega, \mathbb{R}^n))\) denote the space of $\mathbb{F}_T$-adapted \(\mathbb{R}^n\)-valued second order random processes endowed with the norm topology \(\| \cdot \|\) defined by
\[
\| \phi \|^2 \triangleq \sup_{t \in [0, T]} \mathbb{E}|\phi(t)|^2_{\mathbb{R}^n}.
\]

Next, we introduce conditions on the coefficients \(\{f, \sigma, h^i, i = 1, \ldots, N\}\), which are partly used to derive the results of this section.

**Assumptions 1. (Main assumptions)** The drift \(f\), diffusion coefficients \(\sigma\), and information functional \(h^i\) are Borel measurable maps:
\[
f : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}^n, \quad \sigma : [0, T] \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n),
\]
\[
h^i : [0, T] \times C([0, T], \mathbb{R}^n) \longrightarrow \mathbb{R}^{k_i}, \quad \forall i \in \mathbb{Z}_N.
\]

Moreover,

(A0) \(\mathbb{A}^i \subseteq \mathbb{R}^{d_i}\) is nonempty, \(\forall i \in \mathbb{Z}_N\).

There exists a \(K > 0\) such that

(A1) \(|f(t, x, u)|_{\mathbb{R}^n} \leq K(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}), \forall t \in [0, T];\)

(A2) \(|\sigma(t, x)|_{\mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)} \leq K(1 + |x|_{\mathbb{R}^n}), \forall t \in [0, T];\)

(A3) \(|\sigma(t, x) - \sigma(t, y)|_{\mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)} \leq K|x - y|_{\mathbb{R}^n}, \forall t \in [0, T];\)

(A4) \(\sigma(t, x)\) is invertible \(\forall (t, x) \in [0, T] \times \mathbb{R}^n;\)

(A5) \(|\sigma^{-1}(t, x)f(t, x, u)|_{\mathbb{R}^n}^2 < K, uniformly in (t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)};\)

(A6) \(|\sigma(t, x)|_{\mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)} \geq K(1 + |x|_{\mathbb{R}^n}^q), \forall t \in [0, T], \quad q \geq 1;\)

(A7) \(|\sigma^{-1}(t, x)f(t, x, u)|_{\mathbb{R}^n} \leq K(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}), \forall t \in [0, T];\)

(A8) \(|\sigma^{-1}(t, x)f(t, x, u) - \sigma^{-1}(t, z)f(t, z, v)|_{\mathbb{R}^n} \leq (1 + |x - z|_{\mathbb{R}^n} + |u - v|_{\mathbb{R}^d}).\)
A. Equivalent Stochastic Team Problems via Girsanov’s

Next, we define the dynamic team problem (2) via Girsanov’s measure change using the weak Girsanov’s change of measure approach.

We start with a canonical space \((\Omega, \mathcal{F}, \mathbb{P})\) on which \((x_0, \{W(t) : t \in [0, T]\})\) are defined by

\[\text{(WP1)}\; x(0) = x_0: \text{ an } \mathbb{R}^n\text{-valued Random Variable with distribution } \Pi_0(dx);\]

\[\text{(WP2)}\; \{W(t) : t \in [0, T]\}: \text{ an } \mathbb{R}^m\text{-valued standard Brownian motion, independent of } x(0);\]

We introduce the Borel \(\sigma\)-algebra \(\mathcal{B}(C([0, T], \mathbb{R}^n))\) on \(C([0, T], \mathbb{R}^n)\) generated by \(\{W(t) : 0 \leq t \leq T\}\), and let \(\mathbb{P}^W\) its Wiener mesure on it. Further, we introduce the filtration \(\mathcal{F}_t^W\) generated by truncations of \(W \in C([0, T], \mathbb{R}^n)\). That is, \(\mathcal{F}_{0,t}^W\) is the sub-\(\sigma\)-algebra generated by the family of sets

\[
\mathcal{F}_{0,t}^W = \{\{F : 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^n)\} : \text{ } t \in [0, T],
\]

which implies \(\mathcal{F}_{0,t}^W\) is the canonical Borel filtration, \(\mathcal{F}_t = \mathcal{B}(C([0, T], \mathbb{R}^n))\), and \(\mathcal{F}_{0,t} = \mathcal{B}(C([0, T], \mathbb{R}^n))\) are the truncations for \(t \in [0, T]\). Next, we define

\[
\Omega \triangleq \mathbb{R}^n \times C([0, T], \mathbb{R}^n), \quad \mathcal{F} \triangleq \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(C([0, T], \mathbb{R}^n));
\]

\[
\mathbb{P}_{0,t} \triangleq \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}_{0,t}, \quad \mathbb{P} \triangleq \Pi_0 \otimes \mathbb{P}^W.
\]

On the probability space \((\Omega, \mathcal{F}, \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P})\) we define the stochastic differential equation

\[
\frac{dx(t)}{dt} = \sigma(t, x(t))dW(t), \quad x(0) = x_0, \quad t \in (0, T).
\]

Then by Assumptions \(\Pi\) (A2), (A3), and for any initial condition satisfying \(\mathbb{E}|x(0)|_q^q, q \geq 1, \quad (7)\) has a unique strong solution \(\Pi_1\), \(x(\cdot) \in C([0, T], \mathbb{R}^n)\) satisfies (7) and \(x(\cdot) \in \mathbb{B}_{\mathbb{P}^x}(\mathbb{F}_{0,t})\).

We also introduce the \(\sigma\)-algebra \(\mathcal{F}_{0,t}\) defined by

\[
\mathcal{F}_{0,t}^x \triangleq \{\{x \in C([0, T], \mathbb{R}^n) : x(s) \in A\} : 0 \leq s \leq t, \quad A \in \mathcal{B}(\mathbb{R}^n)\} = \mathcal{B}_t(C([0, T], \mathbb{R}^n)). \quad (8)
\]

Hence, \(\mathcal{F}_t^x \triangleq \{\mathcal{F}_{0,t}^x : t \in [0, T]\}\) is the canonical Borel filtration generated by \(x(\cdot) \in C([0, T], \mathbb{R}^n)\) satisfies (7), and the additional Assumptions \(\Pi\) (A4) on \(\sigma\), it can be shown that \(\mathcal{F}_{0,t}^x = \mathbb{F}_{0,t} \equiv \mathcal{F}_{x(0)}^W \vee \mathcal{F}_{0,t}^W, \forall t \in [0, T]\), and this \(\sigma\)-algebra is independent of any of the team decisions \(u\). Unlike \(\Pi_1\), where we utilize open loop or nonanticipative information structures, here we use feedback information structures. Note that for the feedback information structures to
be independent of any of the team decisions \( u \), it is necessary that under the reference probability measure, \( \mathbb{P} \) the state process \( x(\cdot) \) is independent of \( u \), which is indeed the case because we have restricted the class of diffusion coefficients \( \sigma \) to those which are independent of \( u \).

Next we prepare to define three sets of admissible team strategies. We define the Borel \( \sigma \)-algebras generated by projections of \( x \in \mathbb{R}^n \) on any of its subspaces say, \( x^i = \Pi^i(x) \), and the distributed observations process \( \{ z^i(t) \triangleq h^i(t, x) : t \in [0, T] \} \) as follows.

\[
\mathcal{G}^{z_i(t)} \triangleq \sigma \left\{ \{ x^i \in C([0, T], \mathbb{R}^{n_i}) : x^i(t) \in A \} : A \in \mathcal{B}(\mathbb{R}^{n_i}) \right\}, \quad t \in [0, T], \quad \forall i \in \mathbb{Z}_N
\]

(9)

Further, define \( \mathcal{G}^{z_i}_{0,t} \triangleq \{ \mathcal{G}^{z_i} : t \in [0, T] \} \), \( \mathcal{G}^{z_i}_{0,t} \subseteq \mathcal{F}^z_{0,t}, \forall t \in [0, T] \), the canonical Borel filtration generated by \( \{ z^i(t) : 0 \leq t \leq T \} \), for \( i = 1, \ldots, N \). Define the delayed sharing information structure at the \( i \)th control station by \( \mathcal{G}^{z_i}_{t} \triangleq \{ \mathcal{G}^{z_i}_{0,t} : t \in [0, T] \} \), which is the minimum filtration generated by the Borel \( \sigma \)-algebra at the \( i \)th observation post \( \{ z^i(s) : 0 \leq s \leq t \} \), and the delayed sharing information signaling, \( \{ z^j(s - \epsilon_j) : \epsilon_j > 0, j \in \mathcal{O}(i), 0 \leq s \leq t \} \), \( t \in [0, T] \), from the observation posts \( \mathcal{O}(i) \subset \{1, 2, \ldots, i - 1, i + 1, \ldots, N\} \), to the control station \( i \), for \( i = 1, \ldots, N \).

Next, we define the three classes of information structures we consider in this paper.

**Definition 1. (Noiseless Feedback Admissible Strategies)**

**Without Signaling:** If there is no signaling from the observation posts to any of the other control stations, the set of admissible strategies at the \( i \)th control station is defined by

\[
\mathbb{U}^{z_i}[0, T] \triangleq \left\{ u^i : [0, T] \times \Omega \rightarrow A^i \subseteq \mathbb{R}^{d_i} : u^i \in \{ \mathcal{G}^{z_i}_{0,t} : t \in [0, T] \} - \text{Progressively Measurable (PM)} \right\}
\]

and

\[
\mathbb{E} \int_0^T \Lambda^u(t)|u_t|^2 \mathrm{d}t < \infty, \quad \forall i \in \mathbb{Z}_N.
\]

(11)

A team strategy is an \( N \) tuple defined by \( (u^1, u^2, \ldots, u^N) \in \mathbb{U}^{(N), z}[0, T] \triangleq \times_{i=1}^N \mathbb{U}^{z_i}[0, T] \).

**With Signaling:** If there is delayed sharing information signaling from the other observation
posts, the set of admissible strategies at the \( i \)th control station is defined by

\[
\mathbb{U}^i[0, T] \triangleq \left\{ u^i : [0, T] \times \Omega \to A^i \subseteq \mathbb{R}^{d_i} : \right. \\
\text{\( u^i \) is } \{ G^i_{0,t} : t \in [0, T] \} - \text{PM and } \mathbb{E} \int_0^T \Lambda^u(t) |u_t|^2 dt < \infty \right\}, \forall i \in \mathbb{Z}_N. \tag{12}
\]

A team strategy is an \( N \) defined by \( (u^1, u^2, \ldots, u^N) \in \mathbb{U}^{(N)}[0, T] \triangleq \times_{i=1}^{N} \mathbb{U}^i[0, T] \).

**Without Perfect Recall \sim \text{Markov}:** If the distributed observation process collected at the \( i \)th observation post is \( z^i = x^i \), and there is no perfect recall, the set of admissible strategies at the \( i \)th control station is defined by

\[
\mathbb{U}^{x^i}[0, T] \triangleq \left\{ u^i : [0, T] \times \mathbb{R}^{n_i} \to A^i \subseteq \mathbb{R}^{d_i} : \right. \\
\text{\( for any } t \in [0, T], \text{ \( u^i_t \) is } G^{x^i(t)} - \text{measurable and } \mathbb{E} \int_0^T \Lambda^u(t) |u_t|^2 dt < \infty \right\}, \forall i \in \mathbb{Z}_N. \tag{13}
\]

A team strategy is an \( N \) defined by \( (u^1, u^2, \ldots, u^N) \in \mathbb{U}^{(N),x}[0, T] \triangleq \times_{i=1}^{N} \mathbb{U}^{x^i}[0, T] \).

The results derived in this paper hold for other variations of the information structures, such as, control stations without perfect recall based on delayed information \( G^{x^i(t-\delta_i)} \), \( \delta_i \geq 0, i = 1, \ldots, N \), etc.

The reason for imposing the condition \( \mathbb{E} \int_0^T \Lambda^u(t) |u_t|^2 dt < \infty \) will be clarified shortly. Thus, an admissible strategy, say, \( u \equiv (u^1, \ldots, u^N) \in \mathbb{U}^{(N)}[0, T] \) is a family of \( N \) functions, say, \( (\mu^1_t(\cdot), \mu^2_t(\cdot), \ldots, \mu^N_t(\cdot)) \), \( t \in [0, T] \), which are progressively measurable (nonanticipative) with respect to the delayed sharing noiseless feedback information structure \( \{ G^i_{0,t} : t \in [0, T] \} \), \( i = 1, 2, \ldots, N \).

Next, for any \( u \in \mathbb{U}^{(N)}[0, T] \) (we can also consider \( \mathbb{U}^{(N),z}[0, T], \mathbb{U}^{(N),x}[0, T] \)) we define on \( \left( \Omega, \mathbb{F}, \{ \mathbb{F}_{0,t} : t \in [0, T] \}, \mathbb{P} \right) \) the exponential function

\[
\Lambda^u(t) \triangleq \exp \left\{ \int_0^t f^*(s, x(s), u_s) a^{-1}(s, x(s)) dx(s) \right. \\
\left. - \frac{1}{2} \int_0^t f^*(s, x(s)) a^{-1}(s, x(s)) f(s, x(s)) ds \right\}, \quad a(t, x) = \sigma(t, x) \sigma^*(t, x), \quad t \in [0, T]. \tag{14}
\]

Under the additional Assumptions \( \square \) (A5), by Itô’s differential rule \{ \( \Lambda^u(t) : t \in [0, T] \} \) it is the unique \{ \( \mathbb{F}_{0,t} : t \in [0, T] \}\)-adapted, \( \mathbb{P} \)-a.s. continuous solution \( \square \) of the stochastic differential
Given any \( u \in \mathcal{U}^{(N)}[0, T]\) we define the reward of the team game under \( \left( \Omega, \mathbb{F}, \{ \mathbb{F}_t : t \in [0, T] \}, \mathbb{P} \) \) by

\[
J(u^*) \triangleq \inf_{u \in \mathcal{U}^{(N)}[0, T]} \mathbb{E} \left\{ \int_0^T \Lambda^u(t) \ell(t, x(t), u_t) dt + \Lambda^u(T) \varphi(x(T)) \right\},
\]

where \( \ell : [0, T] \times \mathbb{R}^n \times \mathcal{A}^{(N)} \rightarrow (-\infty, \infty), \varphi : [0, T] \times \mathbb{R}^n \rightarrow (-\infty, \infty) \) will be such that (16) is finite.

For any admissible strategy \( u \in \mathcal{U}^{(N)}[0, T]\), by Assumptions [A1] (A5), Novikov condition [17]

\[
\mathbb{E} \exp \left\{ \frac{1}{2} \int_0^T |\sigma^{-1}(s, x(s)) f(s, x(s), u_s)|^2 ds \right\} < \infty,
\]

which is sufficient for \( \{ \Lambda^u(t) : 0 \leq t \leq T \} \) defined by (14) to be an \( \{ \mathbb{F}_t : t \in [0, T] \}, \mathbb{P} \)-martingale, \( \forall t \in [0, T] \). Thus, by the martingale property, \( \Lambda^u(\cdot) \) has constant expectation,

\[
\int_0^T \Lambda^u(t, \omega) d\mathbb{P}(\omega) = 1, \forall t \in [0, T], \quad \text{and therefore, we can utilize } \Lambda^u(\cdot) \text{ which represents a version of the Radon-Nikodym derivative, to define a probability measure } \mathbb{P}^u \text{ on } \left( \Omega, \mathbb{F}, \{ \mathbb{F}_t : t \in [0, T] \} \right)
\]

by setting

\[
\frac{d\mathbb{P}^u}{d\mathbb{P}}|_{\mathbb{F}_0} = \Lambda^u(t), \quad t \in [0, T].
\]

Moreover, by Girsanov’s theorem under the probability space \( \left( \Omega, \mathbb{F}, \{ \mathbb{F}_t : t \in [0, T] \}, \mathbb{P}^u \right) \), the process \( \{ W^u(t) : t \in [0, T] \} \) is a standard Brownian motion and it is defined by

\[
W^u(t) \triangleq W(t) - \int_0^t \sigma^{-1}(s, x(s)) f(s, x(s), u_s) ds,
\]

t \in [(0, T], and the distribution of \( x(0) \) is unchanged.

Therefore, under Assumptions [A1] (A1)-(A5) we have constructed the probability space \( \left( \Omega, \mathbb{F}, \{ \mathbb{F}_t : t \in [0, T] \}, \mathbb{P}^u \right) \), the Brownian motion \( \{ W^u(t) : t \in [0, T] \} \) defined on it, and the state process \( x(\cdot) \) which is a weak solution of

\[
dx(t) = f(t, x(t), u_t) dt + \sigma(t, x(t)) dW^u(t), \quad x(0) = x_0,
\]

t \in (0, T], unique in probability law defined via (18), having the properies \( x(\cdot) \in C([0, T], \mathbb{R}^n) - \mathbb{P}^u - a.s., \) it is \( \{ \mathbb{F}_t : t \in [0, T] \} - \text{adapted, and } x(\cdot) \in B_{\mathbb{F}^T}^2([0, T], L^2(\Omega, \mathbb{R}^n)) \).
By substituting (18) into (16), under the probability measure $\mathbb{P}^u$, the team game reward is given by

$$J(u^*) = \inf_{u \in \mathcal{U}^{(N)}[0,T]} \mathbb{E}^u \left\{ \int_0^T \ell(t,x(t),u_t) dt + \varphi(x(T)) \right\}. \quad (21)$$

From the definition of the Radon-Nikodym derivative (18), for any admissible strategy, say, $u \in \mathcal{U}^{(N)}[0,T]$ we also have $\mathbb{E} \int_0^T \Lambda^u(t) u_t^2 dt = \mathbb{E} \int_0^T |u_t|^2 dt < \infty$.

Note that if we start with the stochastic dynamic team problem (20), (21), the reverse change of measure is obtained as follows. Define

$$\rho^u(T) \overset{\triangle}{=} \Lambda^{u,-1}(t) = \exp \left\{ - \int_0^t f^*(s,x(s),u_s) \sigma^{-1}(s,x(s)) dW^u(s) \right. \right.$$

$$\left. - \frac{1}{2} \int_0^t \sigma(s,x(s))^{-1} f(s,x(s),u_i)|_{\mathbb{R}^n} ds \right\}. \quad (22)$$

Then $\mathbb{E} \Lambda^u(t) = \mathbb{E}^u \rho^u(t) \Lambda^u(t) = 1, \ \forall t \in [0,T]$, and $\frac{\partial \rho^u}{\partial x}|_{\mathbb{F}_{0,t}} = \rho^u(t), \ t \in [0,T]$. Consequently, under the reference measure $\left( \Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P} \right)$ the stochastic dynamic team problem is (7), (15), (16).

**Remark 1.** We have shown that under the Assumptions [7] (A1)-(A5), and $E|x(0)|_{\mathbb{R}^n} < \infty$, for any $u \in \mathcal{U}^{(N)}[0,T]$ then $\mathbb{E} \left( \Lambda^u(t) \right) = 1, \forall t \in [0,T]$, and that we have two equivalent formulations of the stochastic dynamic team problem.

1. Under the original probability space $\left( \Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P}^u \right)$ the dynamic team problem is described by the pay-off (27), and the $\{\mathcal{F}_{0,t} : t \in [0,T]\}$—adapted continuous strong solution $x(\cdot)$ satisfying (20), where the distributed observations collected at the observation posts $\{z^i(t) = h^i(t,x) : t \in [0,T]\}$ are affected by the team decisions via $\{x(t) : t \in [0,T]\}$.

2. Under the reference probability space $\left( \Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P} \right)$ the dynamic team problem is described by the pay-off (16), and the $\{\mathcal{F}_{0,t} : t \in [0,T]\}$—adapted continuous pathwise solution of $(x(\cdot), \Lambda(\cdot))$, satisfying (7), (15), where $\{z^i(t) = h^i(t,x) : t \in [0,T]\}$ is not affected by any of the team decisions. Note that strong uniqueness holds for solutions of (7), (15) because both satisfy the Lipschitz conditions (i.e. Assumptions [7] (A3), (A5) hold).

**Remark 2.** The Assumptions [7] (A5) is satisfied if the following alternative conditions hold.

(A5)(a) (A4), (A6) holds and either (i) (A1) is replaced by $|f(t,x,u)|_{\mathbb{R}^n} \leq K(1+|x|_{\mathbb{R}^n}), K > 0, \forall t \in [0,T]$, or (ii) $\Lambda^{(N)}$ is bounded;
Remark 3. The Girsanov’s measure transformation is precisely the continuous-time counterpart of so called “Common Denominator Condition and Change of Variables” (i.e. Sections 4, 5, [18]), of Witsenhausen’s discrete-time stochastic control problems with finite decisions. Witsenhausen in [18] called any discrete-time stochastic dynamical decentralized decision problem which can be transformed via the “Common Denominator Condition and Change of Variables” to observations which are not affected by any of the team decisions “Static”. The main point we wish to make regarding [18] is the following.

Contrary to the belief in [18], and although the distributed observations and information structures of the equivalent stochastic team decision problem, under the reference probability measure \( P \), are not affected by any of the team decisions, this does not mean than Marschak’s and Radner’s [13]–[15] static team theory optimality conditions can be easily applied to compute the optimal team strategies of the equivalent team problem. We further elaborate on this point in Section II-B.

The main problem with developing the team and PbP optimality conditions based on variational methods, under the original probability space \( \big( \Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P} \big) \), is the definition of admissible strategies, which states that \( \{u^i_t : t \in [0,T]\} \) is adapted to feedback information \( \{\mathcal{G}^i_{0,t} : t \in [0,T]\} \subset \{\mathcal{F}^x_{0,t} : t \in [0,T]\}, i = 1, \ldots, N \), and hence affected by the team decisions. Therefore, if one invokes weak or needle variations of \( u \in \mathbb{U}^{(N)}[0,T] \), to compute the Gateaux derivative of the pay-off, then one needs the variational equation of the unobserved state \( x(\cdot) \) satisfying (21), which implies that one should differentiate \( \{u^i_t \equiv \mu(t,I^i) : t \in [0,T]\}, i = 1, \ldots, N \) with respect to \( x \), because \( \{I^1(t), \ldots, I^N(t) : t \in [0,T]\} \) are affected by the decisions. Therefore, the classical methods which assume nonanticipative strategies adapted to \( \{\mathcal{F}^V_{0,t} : t \in [0,T]\} \) or any sub-\( \sigma \)-algebra of this \( \mathcal{G}_{V} \), in general do not apply. One approach to circumvent this technicality is to show that feedback strategies are dense in nonanticipative or open loop strategies, and the pay-off is continuously dependent on \( u \in \mathbb{U}^{(N)}[0,T] \) as in [46]. Another approach is to use Girsanov’s theorem.

Before we proceed we show, in the next theorem, that Girsanov’s change of probability measure, which is based on identifying sufficient conditions so that \( \{\Lambda^u(t) : 0 \leq t \leq T\} \) is an \( \left( \{\mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P} \right) \)-martingale, holds under more general conditions than the uniform bounded condition given by Assumptions 1 (A5).
Theorem 1. (Equivalence of Dynamic Team Problems)

Suppose $\mathbb{E}|x(0)|_{\mathbb{R}^n} < \infty$, Assumptions 7 (A1), (A2), (A7), hold, and consider any of the admissible strategies of Definition 7.

Then $\mathbb{E}\left(\Lambda^u(t)\right) = 1, \forall t \in [0, T]$, and the dynamic team problem with pay-off (21) subject to $x(\cdot)$ satisfying (20) is equivalent to the dynamic team problem with pay-off (16) with $(x(\cdot), \Lambda(\cdot))$ satisfying (7), (15).

Proof: See Appendix.

Thus, Theorem 1 is a significant generalization of the equivalence between the two stochastic team problems.

B. Function Space Integration: Equivalence of Static and Dynamic Team Problems

In this section, we show the precise connection between Girsanov’s measure transformation and Witsenhausen’s “Common Denominator Condition and Change of Variables” [18], for the continuous-time stochastic dynamic team problem (1)-(4), and for general discrete-time stochastic dynamic team problems.

Continuous-Time Stochastic Dynamic Team Problems.

For simplicity we introduce the following assumptions.

Assumptions 2. The Borel measurable diffusion coefficient $\sigma$ in (20) is replaced by

(A9) $G : [0, T] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ (i.e. it is independent of $(x, u) \in \mathbb{R}^n \times \mathcal{A}^{(N)}$, $G^{-1}$ exists and both are uniformly bounded;

(A10) $\mathbb{E}|x(0)|_{\mathbb{R}^n} < \infty$ and Assumptions 7 (A1) hold.

Under Assumptions 2 by Theorem 1 we can apply Girsanov’s theorem to obtain the equivalent stochastic dynamic team problem under the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P})$, such

\footnote{$G$ can be allowed to depend on $x$.}
that \( \{(x(t), \Lambda^u(t) : t \in [0, T]\) \) is defined by
\[
    x(t) = x(0) + \int_0^t G(s)dW(s) \equiv x(0) + \hat{W}(t),
\]
\[
    \Lambda^u(t) = 1 + \int_0^t f^*(s, x(s), u_s) \left( G(s)G^*(s) \right)^{-1} dx(s),
\]
where \( \{W(t) : t \in [0, T]\} \) is a standard Brownian motions, and for any \( u \in \mathbb{U}^N[0, T] \) the the pay-off givan is
\[
    J(u) \triangleq \mathbb{E} \left\{ \int_0^T \Lambda^u(t)\ell(t, x(t), u_t) dt + \Lambda^u(T)\varphi(x(T)) \right\}. \tag{24}
\]
Note that in (24), \( \mathbb{E}\{\cdot\} \) denotes expectation with respect to the product measure \( \mathbb{P}(d\xi, dw) \triangleq \Pi_0(d\xi) \times \mathcal{W}_W(dw) \), where \( \mathcal{W}_W(\cdot) \) is the Wiener measure on the Brownian motion sample paths \( \{\hat{W}(t) : t \in [0, T]\} \in C([0, T], \mathbb{R}^n) \).

Now, we consider the equivalent transformed pay-off (24), and integrate by parts the stochastic integral term appearing in \( \Lambda^u(\cdot) \), and then we define
\[
    \overline{\ell}(t, \xi, \hat{W}, u) \triangleq \Lambda^u\ell(t, x, u), \quad \overline{\varphi}(T, \xi, \hat{W}, u) \triangleq \Lambda^u\varphi(x).
\]
Then the transformed pay-off (24) is given by
\[
    J(u) \triangleq \mathbb{E} \left\{ \int_0^T \overline{\ell}(t, \xi, \hat{W}(s), u_s : 0 \leq s \leq t) dt + \overline{\varphi}(T, \xi, \hat{W}(t), u_t : 0 \leq t \leq T) \right\} \mathcal{W}_W(d\hat{W}) \times \Pi_0(d\xi). \tag{26}
\]
Note that the equivalent pay-off (26) is expressed as a function space integral with respect to a Wiener measure. Such function space integrations are discussed in nonlinear mean-square error nonlinear filtering problems in [50], [51]. Moreover, expression (26) is precisely the continuous-time analog of Witsenhausen’s main theorem [Theorem 6.1, [18]], which is easily verified by comparing (26) and [equation (6.4), [18]]. In fact, \( \Lambda^u(\cdot) \) is the common denominator condition, and the representation of the pay-off as a functional of \( \omega(t) = (x(0), W(t) \triangleq x(0) + \hat{W}(t)), t \in [0, T] \), is the change of variables. Since (26) is a functional of \( (x(0), \hat{W}(\cdot), u^1, \ldots, u^N) \) and \( u^i \)'s are not affected by any of the team decisions, because \( u^i_t = \mu^i_t(I^i) \), and \( I^i(\cdot) \) are functionals of \( (x(0), \hat{W}(\cdot)) \), then we can proceed further to derive team optimality conditions using static team theory, by computing the Gateaux derivative of (26) at \( u^{i,o} \) in the direction of \( u^i - u^{i,o}, i = 1, \ldots \), as in [14], [16]. However, for complicated problems this procedure might not be tractable even
for the simplified case $\ell = 0$, because it will involve function space integrations with respect to the Wiener measure.

**Discrete-Time Stochastic Dynamic Team Problems.**

Next, we consider a discrete-time generalized version of the stochastic differential decentralized decision problem (23), (24), under the reference probability measure $\mathbb{P}$. Let $\mathbb{N}_0 \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}_1 \triangleq \{1, 2, \ldots\}$ denote time-index sets.

We start with a reference probability space $\left( \Omega, \mathbb{F}, \{\mathbb{F}_n : n \in \mathbb{N}_0\}, \mathbb{P} \right)$, under which $\{x(n) : n \in \mathbb{N}_0\}$ is a sequences of independent RVs, having Normal densities denoted by $\lambda_n(\cdot) \sim \mathcal{N}(0, G(n)G^*(n))$, for $n \in \mathbb{N}_1$, $x(0) \sim \Pi_0(dx)$, $\{\mathbb{F}_n : n \in \mathbb{N}_0\}$ is the filtration generated by the completion of the $\sigma-$algebra $\sigma\{x(k) : k \leq n\}, n \in \mathbb{N}_0$, and $\{\mathbb{G}_n^i : n \in \mathbb{N}_0\}$ is the filtration generated by the completion of the $\sigma-$algebra $\sigma\{z^i(k) \triangleq h^i(k, x(0), x(1), \ldots, x(k)) : k \leq n\}, n \in \mathbb{N}_0, i = 1, \ldots, N$.

Next, we define the team strategies which donot assume “Perfect Recall”. Suppose for each $n \in \mathbb{N}_0$, $u^i(n) \in A^i_n$, and that $u^i(n)$ is measurable with respect to $\mathcal{E}^i_n \subset \bigvee_{j=1}^N \mathbb{G}_0^i$, where $\mathcal{E}^i_n$ is not nested, for $i = 1, \ldots, N$, that is, $\mathcal{E}^i_n \not\subset \mathcal{E}^i_{n+1}, n = 0, 1, \ldots$, and hence all control station donot have perfect recall. For each $n \in \mathbb{N}_0$, each information structures can be generated by $\mathcal{E}^i_n \triangleq \sigma\{\Pi^i_n\{\{z^j(0), z^j(1), \ldots, z^j(n) : j = 1, 1, \ldots, N\}\}\}$, where $\Pi^i_n(\cdot)$ is the projection to a subset of $\{z^j(0), z^j(1), \ldots, z^j(n) : j = 1, 1, \ldots, N\}$, for $i = 1, \ldots, N$.

We denote the set of admissible strategies at the $i$th control station at time $n \in \mathbb{N}_0$, by $\gamma^i_n(\cdot) \in \mathcal{U}^i[n]$, their $T-$tuple by $\gamma^{j_{[0, T-1]}(\cdot)} \triangleq (\gamma^0_{0, T-1}(\cdot), \ldots, \gamma^j_{0, T-1}(\cdot)) \in \mathcal{U}^j[0, T-1] \triangleq \times_{j=0}^{T-1} \mathcal{U}^j[j], i = 1, \ldots, N$, and $\gamma^{[0, T-1]}(\cdot) \triangleq (\gamma^{[0, T-1]}_0, \ldots, \gamma^{[0, T-1]}_N(\cdot)) \in \mathcal{U}^{(N)}[0, T-1] \triangleq \times_{j=1}^{N} \mathcal{U}^j[0, T-1]$.

Consider the following measurable functions.

$$f(k, \cdot, \cdot) : \times_{i=0}^{k} (\mathbb{R}^n) \times_{i=1, j=0}^{N, k} (A^i_j) \longrightarrow \mathbb{R}^n, \ \ k \in \mathbb{N}_0,$$

$$h^i(k, \cdot) : \times_{i=0}^{k} (\mathbb{R}^n) \longrightarrow \mathbb{R}^{k^i}, \ \ k \in \mathbb{N}_0, \ \ i = 1, \ldots, N.$$

For any admissible decentralized strategy $u \equiv \gamma^{[0, T-1]} \in \mathcal{U}^{(N)}[0, T-1]$, we define the following quantity.

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\[ \Lambda_{0,n+1}^u \triangleq \prod_{k=0}^n \frac{\lambda_{k+1}(x(k+1) - f(k, x(0), x(1), \ldots, x(k), u(k)))}{\lambda_{k+1}(x(k+1))}, \]

\[ \Lambda_{0,0}^u = 1, \quad n \in \mathbb{Z}_+. \quad (27) \]

Under the reference probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_{0,n} : n \in \mathbb{N}_0\}, \mathbb{P}) \), define the team pay-off

\[ J(u) = \mathbb{E}\left\{ \Lambda_{0,T}(x(0), u(0), \ldots, x(T-1), u(T-1), x(T)) \left( \sum_{k=0}^{T-1} \ell(k, x(k), u(k)) + \varphi(x(T)) \right) \right\} \]

\[ = \int \left\{ \Lambda_{0,T}(x(0), u(0), \ldots, x(T-1), u(T-1), x(T)) \left( \sum_{k=0}^{T-1} \ell(k, x(k), u(k)) + \varphi(x(T)) \right) \right\} \]

\[ \bigg[ \prod_{k=0}^{T-1} \lambda_{k+1}(x(k+1)) dx(k+1) \Pi_0(dx(0)) \bigg] \]

\[ \equiv \int L(\gamma_{[0,T-1]}, x(0), x(1), \ldots, x(T)) \prod_{k=0}^{T-1} \lambda_{k+1}(x(k+1)) dx(k+1) \Pi_0(dx(0)) \equiv J(\gamma_{[0,T-1]}). \quad (28) \]

Clearly, the problem

\[ \inf \left\{ J(\gamma_{[0,T-1]} ) : \gamma_{[0,T-1]} \in \mathbb{U}^{(N)}[0, T-1] \right\}, \quad (31) \]

is a static team problem, because the information structure available for decisions are nonlinear measurable functions of \((x(0), \ldots, x(T))\), which are not affected by any of the team decisions. This is the transformed equivalent stochastic team problem of a certain stochastic dynamic team problem which we introduce next.

Since it can be shown that \( \{\Lambda_{0,n}^u : n \in \mathbb{N}_0\} \) is an \( (\Omega, \mathcal{F}, \{\mathcal{F}_{0,n} : n \in \mathbb{N}_0\}, \mathbb{P}) \)-martingale, with \( \int \Lambda_{0,n}^u(\omega) d\mathbb{P}(\omega) = 1 \), then we can define the probability measure \( \mathbb{P}^u \) on \( (\Omega, \{\mathcal{F}_{0,n} : n \in \mathbb{N}_0\}) \) by setting

\[ \frac{d\mathbb{P}^u}{d\mathbb{P} \bigg| \mathcal{F}_{0,n}} = \Lambda^u(n), \quad \forall n \in \mathbb{N}_0. \quad (32) \]

Then under this probability measure \( \mathbb{P}^u \), the process defined by

\[ w^u(n+1) \triangleq x(n+1) - f(n, x(0), \ldots, x(n), u(n)), \quad n \in \mathbb{N}_0, \quad (33) \]
is a sequence of independent normally distributed RVs with densities, $\lambda_n(\cdot), n \in \mathbb{N}_1$. Therefore, under the probability space $\left( \Omega, \mathcal{F}, \{\mathbb{F}_0, n \in \mathbb{Z}_+ \}, \mathbb{P}^u \right)$, we have

$$x(n + 1) = f(n, x(0), \ldots, x(n), u(n)) + w^u(n + 1),$$  \hspace{1cm} (34)

$$z^i(n) = h^i(n, x(0), \ldots, x(n)), \quad n \in \mathbb{N}_0, i = 1, \ldots, N.$$  \hspace{1cm} (35)

Then, for any admissible discrete-time team strategy $u \equiv \gamma_{[0,T-1]} \in \mathcal{U}^N[0,T-1]$, under measure $\mathbb{P}^u$ the team pay-off is

$$J(u) = \mathbb{E}^u \left\{ \sum_{k=0}^{T-1} \ell(k, x(k), u(k)) + \varphi(x(T)) \right\}.$$  \hspace{1cm} (36)

Thus, we have shown that the dynamic team problem of minimizing pay-off (36), subject to (34), (35) can be transformed to the equivalent static team problem defined by pay-off (31), where $\{x(n) : n \in \mathbb{N}_0\}$ is an independent sequence, distributed according to $x(0) \sim \Pi_0(\cdot), \{\lambda_{n+1}(\cdot) : n \in \mathbb{N}_0\}$, and the information structures are functions of the independent sequence $\{x(n) : n \in \mathbb{N}_0\}$.

Consequently, we have identified the precise connection between the so-called “Common Denominator Conditions and Change of Variables” described by Witsenhausen in [18], for a discrete-time stochastic dynamic team problem to be equivalent to a static team problem.

Therefore, we conclude that the static team theory by Marschak and Radner [13], [14] and its generalization in [16], are directly applicable to the transformed problem (31), in a higher dimension consisting of $TN$ decision strategies (because we have assumed no perfect recall of information at all control stations). Thus, any of the theorems found in [16] are applicable.

Finally, we make the following observations.

Remark 4. The Girsanov theorem is also applicable to more general models that (34) such as,

$$x(n + 1) = f(n, x(0), \ldots, x(n), u(n), w(n + 1)), \quad \text{where} \quad \{w(n) : n \in \mathbb{N}_1\} \text{ is arbitrary distributed, and to finite and countable state Markov decision models. It is also applicable to stochastic dynamic team games with noisy information such as,}$$

$$z^i(n) = h^i(n, x(0), \ldots, x(n), u(n), v^i(n)), \quad \text{where} \quad \{v^i(n) : n \in \mathbb{N}_0\}, i = 1, \ldots, N \text{ are arbitrary distributed.}$$

C. Continuous Dependence of Solutions and Semi Martingale Representation

In this section, we show under appropriate conditions, that (15) has unique continuous solutions (in the strong sense) with finite second moments, and that any solution is continuously dependent
Lemma 1. (Existence and Differentiability) Suppose Assumptions 7 (A2)-(A5) hold. Then for any \( \mathbb{F}_{0,0} \)-measurable initial state \( x_0 \) having finite second moment, and any \( u \in \mathbb{U}^{(N)}[0, T] \), the following hold.

1. (7), (15) have unique solutions \( x \in B_{\mathbb{F}_T}^{\infty}([0, T], L^2(\Omega, \mathbb{R}^n)) \), \( \Lambda^u \in B_{\mathbb{F}_T}^{\infty}([0, T], L^2(\Omega, \mathbb{R})) \) having a continuous modifications, that is, \( \Lambda^u \in C([0, T], \mathbb{R}), \mathbb{P}-a.s. \) Moreover, \( \Lambda^u \in L^p(\Omega, \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P}; \mathbb{R}) \) for any finite \( p \) and also \( \Lambda^u \in L^\infty(\Omega, \{\mathbb{F}_{0,t} : t \in [0, T]\}, \mathbb{P}; \mathbb{R}) \).

2. Under the additional Assumptions 7 (A8), the solution of (15) is continuously dependent on the decisions, in the sense that, as \( u^{i,\alpha} \rightarrow u^{i,\varrho} \) in \( \mathbb{U}^i[0, T] \), \( \forall i \in \mathbb{Z}_N \), \( \Lambda^\alpha \rightarrow \Lambda^\varrho \) in \( B_{\mathbb{F}_T}^{\infty}([0, T], L^2(\Omega, \mathbb{R})) \).

Proof: (1) The uniqueness of solution having a continuous modification is already discussed in Section II-A and it is based on Assumptions 1, (A2)-(A5). The rest of the claims are shown by following the method in [48].

(2) Next, we consider the second part asserting the continuity of \( u \) to solution map \( u \rightarrow \Lambda^u \). Let \( \{u^{i,\alpha} : i = 1, 2, \ldots, N\}, u^\varrho \} \) be any pair of strategies from \( \mathbb{U}^{(N)}[0, T] \times \mathbb{U}^{(N)}[0, T] \) and \( \{\Lambda^\alpha, \Lambda^\varrho\} \) denote the corresponding pair of solutions of (15). Let \( u^{i,\alpha} \rightarrow u^{i,\varrho} \) in \( \mathbb{U}^i[0, T] \), \( i = 1, 2, \ldots, N \). We must show that \( \Lambda^\alpha \rightarrow \Lambda^\varrho \) in \( B_{\mathbb{F}_T}^{\infty}([0, T], L^2(\Omega, \mathbb{R})) \). By the definition of solution to (15), we have

\[
\Lambda^\alpha(t) - \Lambda^\varrho(t) = \int_0^t \left\{ \Lambda^\alpha(s) - \Lambda^\varrho(s) \right\} f^*(s, x(s), u^\alpha(s))a^{-1}(s, x(s))dx(s) \\
+ \int_0^t \Lambda^\alpha(s) \left\{ f^*(s, x(s), u^\alpha(s)) - f^*(s, x(s), u^\varrho(s)) \right\} a^{-1}(s, x(s))dx(s),
\]

where \( a \triangleq \sigma \sigma^* \), and \( \{x(t) : t \in [0, T]\} \) is the solution of (7). From (37) using Doob's martingale inequality we obtain

\[
\mathbb{E}|\Lambda^\alpha(t) - \Lambda^\varrho(t)|^2 \leq 4\mathbb{E} \int_0^t |\Lambda^\alpha(s) - \Lambda^\varrho(s)|^2 |\sigma^{-1}(s, x(s))f(s, x(s), u^\alpha(s))|^2_{\mathbb{R}^n} ds \\
+ 4\mathbb{E} \int_0^t |\Lambda^\alpha(s)|^2 |\sigma^{-1}(s, x(s))(f(s, x(s), u^\alpha(s)) - f(s, x(s), u^\varrho(s)))|^2_{\mathbb{R}^n} ds \\
\leq 4K_1 \mathbb{E} \int_0^t |\Lambda^\alpha(s) - \Lambda^\varrho(s)|^2 ds + 4K_2 \mathbb{E} \int_0^t |\Lambda^\alpha(s)|^2 |u^\alpha(s) - u^\varrho(s)|^2_{\mathbb{R}^n} ds, \quad t \in [0, T],
\]

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where (39) follows from (A5), (A8), with $K_1, K_2 > 0$.

Since by part (1), $\mathbb{P} - ess \sup_{\omega \in \Omega} \sup_{t \in [0, T]} |\Lambda^\alpha(t, \omega)|^2 < M$ for some finite $M > 0$, applying this in (39) we deduce the following bound.

$$E|\Lambda^\alpha(t) - \Lambda^0(t)|^2 \leq 4K_1 \mathbb{E} \int_0^t |\Lambda^\alpha(s) - \Lambda^0(s)|^2 ds + 4MK_2 \mathbb{E} \int_0^t |u_s^\alpha - u_s^\alpha|_E^2 ds, \quad t \in [0, T].$$

(40)

Now, letting $\alpha \to \infty$ and recalling that $u^{i, \alpha} \to u^{i, \alpha}$ in $\mathbb{U}^i[0, T]$ the integrand in the second right hand side of (40) converges to zero for almost all $s \in [0, T], \mathbb{P}$-a.s. Since by our assumptions the integrands are dominated by integrable functions, we obtain by Gronwall’s inequality that

$$\lim_{\alpha \to \infty} \sup_{t \in [0, T]} E|\Lambda^\alpha(t) - \Lambda^0(t)|^2 = 0.$$ This completes the derivation.

The derivation of stochastic minimum principle will be based on certain fundamental properties of semi martingales on Hilbert spaces, which we describe below.

**Definition 2.** An $\mathbb{R}^n$—valued random process $\{m(t) : t \in [0, T]\}$ is said to be a square integrable continuous $\{\mathbb{F}_{0, t} : t \in [0, T]\}$—semi martingale if and only if it has a representation

$$m(t) = m(0) + \int_0^t v(s)ds + \int_0^t \Sigma(s)dW(s), \quad t \in [0, T],$$

(41)

for some $v \in L^2_{\mathbb{F}_T}([0, T], \mathbb{R}^n)$ and $\Sigma \in L^2_{\mathbb{F}_T}([0, T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$ and for some $\mathbb{R}^n$—valued $\mathbb{F}_{0, 0}$—measurable random variable $m(0)$ having finite second moment. The set of all such semi martingales is denoted by $\mathcal{S}M_2^2[0, T]$.

Introduce the following class of $\{\mathbb{F}_{0, t} : t \in [0, T]\}$—semi martingales:

$$\mathcal{S}M_0^2[0, T] \triangleq \left\{ m : m(t) = \int_0^t v(s)ds + \int_0^t \Sigma(s)dW(s), \quad t \in [0, T], \text{ for } v \in L^2_{\mathbb{F}_T}([0, T], \mathbb{R}^n), \Sigma \in L^2_{\mathbb{F}_T}([0, T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \right\}.$$

Now we present the fundamental result which is utilized in the maximum principle derivation.

**Theorem 2.** (Semi Martingale Representation) The class of semi martingales $\mathcal{S}M_0^2[0, T]$ is a real linear vector space and it is a Hilbert space with respect to the norm topology $\| m \|_{\mathcal{S}M_0^2[0, T]}$ arising from

$$\| m \|_{\mathcal{S}M_0^2[0, T]}^2 \triangleq \mathbb{E} \int_{[0, T]} |v(t)|_{\mathbb{R}^n}^2 dt + \mathbb{E} \int_{[0, T]} tr(\Sigma^*(t)\Sigma(t))dt.$$
Moreover, the space $\mathcal{SM}_0^2[0,T]$ is isometrically isomorphic to the space $L_{F_T}^2([0,T],\mathbb{R}^n) \times L_{F_T}^2([0,T],\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))$.

**Proof:** This is found in many books.

### III. Dynamic Team Optimality Conditions

In this section we derive the team and PbP optimality conditions, under the reference probability measure $\mathbb{P}$, and then we translate the results under the original probability measure $\mathbb{P}^u$. For the derivation of stochastic optimality conditions we shall require stronger regularity conditions. These are given below.

**Assumptions 3.** $A_i$ is a closed, bounded and convex subset of $\mathbb{R}^{d_i}, \forall i \in \mathbb{Z}_N$, $\mathbb{E}|x(0)|^2_{\mathbb{R}^n} < \infty$, the maps $\{f, \sigma, \ell, \varphi\}$ are Borel measurable, $\{h^i : i = 1, \ldots, N\}$ are progressively measurable, defined by

$$
\begin{align*}
&f : [0,T] \times \mathbb{R}^n \times A^{(N)} \to \mathbb{R}^n, \quad \sigma : [0,T] \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n), \\
&\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \ell : [0,T] \times \mathbb{R}^n \times A^{(N)} \to \mathbb{R}, \quad h^i : [0,T] \times C([0,T],\mathbb{R}^n) \to \mathbb{R}_{+}^{k_i},
\end{align*}
$$

and they satisfy the following conditions.

- (C1) The map $\sigma$ satisfies (A2), (A3), (A4) and the map $\sigma^{-1}f$ satisfies (A5);

- (C2) The map $f$ is once continuously differentiable with respect to $u \in A^{(N)}$, and the first derivative of $\sigma^{-1}f$ with respect to $u$ is bounded uniformly in $(t, x, u) \in [0,T] \times \mathbb{R}^n \times A^{(N)}$;

- (C3) The maps $\ell$ is once continuously differentiable with respect to $u \in A^{(N)}$, and there exists a $K > 0$ such that

$$
\begin{align*}
&\left(1 + |x|_{\mathbb{R}^n}^2 + |u|_{\mathbb{R}^{d'}}^2 \right)^{-1}|\ell(t, x, u)|_{\mathbb{R}} + \left(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^{d'}} \right)^{-1}|\ell_u(t, x, u)|_{\mathbb{R}^{d'}} \leq K, \\
&\left(1 + |x|_{\mathbb{R}^n}^2 \right)^{-1}||\varphi(x)||_{\mathbb{R}} \leq K;
\end{align*}
$$

A. Necessary Conditions for Team Optimality

Next, we prepare to give the variational equation under the reference measure probability $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P})$. We define the Gateaux derivative of any matrix valued function
Suppose component wise is given by $G : [0,T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with respect to the variable at the point $(t,x,u) \in [0,T] \times \mathbb{R}^n \times \mathbb{A}^{(N)}$ in the direction $v \in \mathbb{A}^{(N)}$ by

$$G_u(t, x, u; v) \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ G(t, x, u + \varepsilon v) - G(t, x, u) \right\}, \quad t \in [0,T].$$

Clearly, for each column of $G$ denoted by $G^{(j)}, j = 1, \ldots, n$, the Gateaux derivative of $G^{(j)}$ component wise is given by $G_u^{(j)}(t, x, u; v) = G_u^{(j)}(t, x, u)v, t \in [0,T]$.

Suppose $u^0 \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in \mathcal{U}^{(N)}[0,T]$ denotes the optimal decision and $u \triangleq (u^1, u^2, \ldots, u^N) \in \mathcal{U}^{(N)}[0,T]$ any other decision. Since $\mathcal{U}^{(N)}[0,T]$ is convex $\forall i \in \mathbb{Z}_N$, it is clear that for any $\varepsilon \in [0,1]$,

$$u_i^{\varepsilon} \triangleq u_i^o + \varepsilon (u_i^o - u^i \varepsilon) \in \mathcal{U}^{(i)}[0,T], \quad \forall i \in \mathbb{Z}_N.$$

Let $\Lambda^\varepsilon(\cdot) \equiv \Lambda^\varepsilon(\cdot; u^\varepsilon(\cdot))$ and $\Lambda^o(\cdot) \equiv \Lambda^o(\cdot; u^o(\cdot)) \in \mathcal{B}_{\mathcal{F}_T}^\infty([0,T], L^2(\Omega, \mathbb{R}))$ denote the solutions of the differential system (15) corresponding to $u^\varepsilon(\cdot)$ and $u^o(\cdot)$, respectively. Consider the limit

$$Z(t) \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \Lambda^\varepsilon(t) - \Lambda^o(t) \right\}, \quad t \in [0,T].$$

Since under the reference probability measure, $dx(t) = \sigma(t,x(t))dW(t), x(0) = x_0$ then its solution is not affected by $u^\varepsilon$, hence we do not consider any variations of $x(\cdot)$ with respect to $u^\varepsilon$.

We have the following characterization of the variational process $\{Z(t) : t \in [0,T]\}$.

**Lemma 2.** (Variational Equation) Suppose Assumptions [8] hold. The process $\{Z(t) : t \in [0,T]\}$ is an element of the Banach space $\mathcal{B}_{\mathcal{F}_T}^\infty([0,T], L^2(\Omega, \mathbb{R}))$ and it is the unique solution of the variational stochastic differential equation

$$dZ(t) = f^*(t, x(t), u^o_0)\sigma^{*,-1}(t, x(t))Z(t)dW(t)$$

$$+ \sum_{i=1}^N (f^*\sigma^{*,-1})_o(t, x(t), u^o_i; u^o_i - u^o_i) dW(t), \quad Z(0) = 0. \quad (42)$$

having a continuous modification.

**Proof:** This follows directly from Lemma [1] and the fact that $f^*\sigma^{*-1}$ and its derivative with respect to $u$ are uniformly bounded.

**Minimum Principle Under Reference Probability Space** $\left(\Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P}\right)$

Next, we state the necessary conditions for team and PbP optimality under the reference probability measure.
Define the Hamiltonian of the augmented system (7), (15), (16).

\[ H : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \times \mathbb{A}^{(N)} \rightarrow \mathbb{R} \]

\[ H(t, x, \Lambda, \Psi, Q, u) = \Delta \Lambda Q \sigma^{-1}(t, x) f(t, x, u) + \Lambda f(t, x, u), \quad t \in [0, T]. \]  

(43)

For any \( u \in \mathcal{U}^{(N)}[0, T] \), the adjoint process \((\Psi, Q) \in L^2_{\mathbb{F}_T}([0, T], \mathbb{R}) \times L^2_{\mathbb{F}_T}([0, T], \mathcal{L}(\mathbb{R}^n, \mathbb{R}))\) satisfies the following backward stochastic differential equations

\[
d\Psi(t) = -\mathcal{L}(t, x(t), \Lambda(t), \Psi(t), Q(t), u_t)dt + Q(t)dW(t), \quad t \in [0, T], \quad \Psi(T) = \varphi(x(T)),
\]

(44)

The state process satisfies the stochastic differential equation (15) expressed in terms of the Hamiltonian as follows.

\[
d\Lambda(t) = \Lambda(t) f^*(t, x(t), u_t) \sigma^*(t, x(t)) dW(t),
\]

\[= \Lambda(t) f^*(t, x(t), u_t) \sigma^*(t, x(t)) dW(t), \quad t \in (0, T], \quad \Lambda(0) = 1. \]  

(45)

Moreover, under measure \( \mathbb{P} \), the process \( \{x(t) : t \in [0, T]\} \) is not affected by \( u \in \mathcal{U}^{(N)}[0, T] \) and satisfies

\[ dx(t) = \sigma(t, x(t)) dW(t), \quad t \in (0, T], \quad x(0) = x_0 \]  

(46)

Next, we state the the necessary conditions for an element \( u^o \in \mathcal{U}^{(N)}[0, T] \) to be team optimal.

**Theorem 3. (Necessary Conditions for Team Optimality under Reference Measure)**

Suppose Assumptions 2 hold. Then we have the following.

**Necessary Conditions.** For an element \( u^o \in \mathcal{U}^{(N)}[0, T] \) with the corresponding solution \( \Lambda^o \in B_{\mathbb{F}_T}^{\infty}([0, T], L^2(\Omega, \mathbb{R})) \) to be team optimal, it is necessary that the following hold.

1. There exists a semi martingale \( m^o \in \mathcal{S}\mathcal{M}_{0}^{2}[0, T] \) (1-dimensional) with the intensity process \((\Psi^o, Q^o) \in L^2_{\mathbb{F}_T}([0, T], \mathbb{R}) \times L^2_{\mathbb{F}_T}([0, T], \mathcal{L}(\mathbb{R}^n, \mathbb{R}))\).

2. The variational inequalities are satisfied:

\[
\sum_{i=1}^{N} \mathbb{E}\left\{ \int_{0}^{T} \mathcal{H}(t, x(t), \Lambda^o(t), \Psi^o(t), Q^o(t), u_t^{-i,o}, u_t^{i,o}) dt \right\} \\
\geq \sum_{i=1}^{N} \mathbb{E}\left\{ \int_{0}^{T} \mathcal{H}(t, x(t), \Lambda^o(t), \Psi^o(t), Q^o(t), u_t^{-i,o}, u_t^{i,o}) dt \right\}, \quad \forall u \in \mathcal{U}^{(N)}[0, T],
\]

(47)
\[ \mathbb{E}\left\{ \int_0^T \mathcal{H}(t, x(t), \Lambda^\circ(t), \Psi^\circ(t), Q^\circ(t), u_{t-i}^0, u_i^1)dt \right\} \]
\[ \geq \mathbb{E}\left\{ \int_0^T \mathcal{H}(t, x(t), \Lambda^\circ(t), \Psi^\circ(t), Q^\circ(t), u_{t-i}^0, u_i^1)dt \right\}, \forall u^i \in \mathbb{U}^i[0, T], i \in \mathbb{Z}_N. \tag{48} \]

(3) The process \((\Psi^\circ, Q^\circ) \in L^2_{\mathbb{P}_T}([0, T], \mathbb{R}) \times L^2_{\mathbb{F}_T}([0, T], \mathcal{L}(\mathbb{R}^n, \mathbb{R}))\) is a unique solution of the backward stochastic differential equation (44) such that \(u^\circ \in \mathbb{U}^{(N)}[0, T]\) satisfies the pointwise \(\mathbb{P}\) almost sure inequalities with respect to the \(\sigma\)-algebras \(\mathcal{G}_{0,t}^i \subset \mathcal{F}_{0,t}, t \in [0, T], i = 1, 2, \ldots, N\):

\[ \mathbb{E}\left\{ \mathcal{H}(t, x(t), \Lambda^\circ(t), \Psi^\circ(t), Q^\circ(t), u_{t-i}^0, u_i^1)|\mathcal{G}_{0,t}^i \right\} \]
\[ \geq \mathbb{E}\left\{ \mathcal{H}(t, x(t), \Lambda^\circ(t), \Psi^\circ(t), Q^\circ(t), u_{t-i}^0, u_i^1)|\mathcal{G}_{0,t}^i \right\}, \forall u^i \in \mathcal{A}^i, a.e., t \in [0, T], \mathbb{P}|_{\mathcal{G}_{0,t}^i} - a.s., \forall i \in \mathbb{Z}_N. \tag{49} \]

(4) For admissible strategies \(\mathbb{U}^{(N),z}[0, T], \mathbb{U}^{(N),x}[0, T]\) the conditional expectation in (49) is taken with respect to the information structures \(\mathcal{G}_{0,t}^z = \mathcal{G}_{0,t}^x\), respectively.

**Proof:** The derivation is based on the variation equation of Lemma 2, the semi-martingale representation theorem, and the Riesz representation theorem. We outline the steps. By Assumptions 3, it can be shown that the Gateaux derivative of \(J(\cdot)\) at \(u^\circ\) in the direction \(u - u^\circ\) exists, and it is computed via \(\frac{d}{d\epsilon} J(u^\circ + \epsilon(u - u^\circ))|_{\epsilon=0}\). Using the semi-martingale representation and Riesz representation theorem for Hilbert space processes, we can show (1) and (2), (47), following the steps in [47] or [Section V, [46]], for regular strategies. Moreover, (48) is obtained by contradiction. Finally, (3) is obtained precisely as in [46]. Finally, (4) follows from the fact that the derivations of (1)-(3) do not depend on the form of the information structures generated via \(z^i, i = 1, \ldots, N\).

The important point to be made regarding Theorem 3 is that its derivation is based on applying, under the new (reference) probability space \((\Omega, \mathbb{F}, \mathbb{F}_T, \mathbb{P})\), any method based on strong formulation (in our case [46], [47]), but with \(u\) adapted to feedback information.

We also point out that the necessary conditions for a \(u^\circ \in \mathbb{U}^{(N)}[0, T]\) to be a PbP optimal can be derived following the procedure described in Theorem 3 and that these necessary conditions are equivalent to the necessary conditions for team optimality, as expected. These results are stated as a Corollary.
Corollary 1. (Necessary Conditions for PbP Optimality under Reference Measure)
Suppose Assumptions 3 hold. Then we have following.

Necessary Conditions. For an element $u^o \in \mathbb{U}^{(N)}[0, T]$ with the corresponding solution $\Lambda^o \in B^\infty_{F_T}([0, T], L^2(\Omega, \mathbb{R}))$ to be a PbP optimal strategy, it is necessary that the statements of Theorem 3 (1)-(4) hold, and statement (2) corresponding to (48).

Proof: The derivation is based on the procedure of Theorem 3, but we only vary in the direction $u^i - u^{i,o}$, while the rest of the strategies are optimal, $u^{-i} = u^{-i,o}$.

Minimum Principle Under Original Probability Space $\left(\Omega, \mathbb{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}^u\right)$

Next, we express the optimality conditions with respect to the original probability space $\left(\Omega, \mathbb{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}^u\right)$.

Since the Hamiltonian under the reference probability measure (43), appearing in Theorem 3 is multiplied by $\Lambda(\cdot)$, then we can write

$$H(t, x, \Lambda, \Psi, Q, u) = \Lambda \left\{ Q\sigma^{-1}(t, x)f(t, x, u) + \ell(t, x, u) \right\} \quad (50)$$

Define the Hamiltonian under the original probability measure $\mathbb{P}^u$ by

$$\mathbb{H} : [0, T] \times \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}$$

$$\mathbb{H}(t, x, Q, u) \overset{\triangle}{=} \ell(t, x, u) + Q\sigma^{-1}(t, x)f(t, x, u). \quad (51)$$

Since $\Lambda(T) = \frac{d\mathbb{P}^u}{d\mathbb{P}}|_{F_T}$, then we can express the Hamiltonian system of equations (44), (46) under the original measure $\mathbb{P}^u$, by translating the martingale term, using the fact that

$$W^u(t) \overset{\triangle}{=} W(t) - \int_0^t \sigma^{-1}(s, x(s))f(s, x(s), u_s)ds, \, \text{is an } \left(\mathbb{F}_{0,t}, \mathbb{P}^u\right) - \text{martingale.} \quad (52)$$

Thus, under the original probability measure $\left(\Omega, \mathbb{F}, \{\mathbb{F}_t : t \in [0, T]\}, \mathbb{P}^u\right)$, by substituting (52) into (44), (46) the adjoint process $\{\Psi(t), Q(t) : t \in [0, T]\}$ is a solution of the backward and forward stochastic differential equation

$$d\Psi(t) = -\ell(t, x(t), u_t)dt + Q(t)dW^u(t), \, \Psi(T) = \varphi(x(T)), \, t \in [0, T), \quad (53)$$

and the process $\{x(t) : t \in [0, T]\}$ is a solution of the following forward equation.

$$dx(t) = f(t, x(t), u_t)dt + \sigma(t, x(t))dW^u(t), \, x(0) = x_0. \quad (54)$$
Moreover, the conditional variational Hamiltonian is given by

\[
\mathbb{E}^u \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u_t^{-i, o}, u_t^i) | \mathcal{G}_{0,t}^i \right\}
\geq \mathbb{E}^u \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u_t^{-i, o}, u_t^i) | \mathcal{G}_{0,t}^i \right\}, \quad \forall u^i \in \mathcal{A}^i \text{ a.e.t.}, \quad \mathbb{P}^u |_{\mathcal{G}_{0,t}^i} - a.s., \forall i \in \mathbb{Z}_N. \quad (55)
\]

Hence, under the original probability space \( \mathbb{P}^u \), we have the following necessary conditions for team optimality.

**Theorem 4.** *(Necessary Conditions for Team Optimality under Original Measure)*

Suppose Assumptions \( 3 \) hold. Then we have the following.

**Necessary Conditions.** For an element \( u^o \in \mathbb{U}^{(N)}[0, T] \) with the corresponding solution \( x^o \in \mathcal{B}_{2,T}^{\infty}([0, T], L^2(\Omega, \mathbb{R}^n)) \) to be team optimal, it is necessary that the following hold.

1. *There exists a semi martingale \( m^o \in \mathcal{SM}_{2,T}^{\infty}[0, T] \) (1-dimensional) with the intensity process \( \{(\Psi^o, Q^o)\} \in L_{2,T}^2([0, T], \mathbb{R}) \times L_{2,T}^2([0, T], \mathcal{L}(\mathbb{R}^n, \mathbb{R})) \).*

2. *The variational inequalities are satisfied:*

\[
\sum_{i=1}^{N} \mathbb{E}^u \left\{ \int_{0}^{T} \mathbb{H}(t, x^o(t), Q^o(t), u_t^{-i, o}, u_t^i) dt \right\}
\geq \sum_{i=1}^{N} \mathbb{E}^u \left\{ \int_{0}^{T} \mathbb{H}(t, x^o(t), Q^o(t), u_t^{-i, o}, u_t^i) dt \right\}, \quad \forall u \in \mathbb{U}^{(N)}[0, T]. \quad (56)
\]

\[
\mathbb{E}^u \left\{ \int_{0}^{T} \mathbb{H}(t, x^o(t), Q^o(t), u_t^{-i, o}, u_t^i) dt \right\}
\geq \mathbb{E}^u \left\{ \int_{0}^{T} \mathbb{H}(t, x^o(t), Q^o(t), u_t^{-i, o}, u_t^i) dt \right\}, \quad \forall u^i \in \mathbb{U}^{(i)}[0, T]. \quad (57)
\]

3. *The process \( \{(\Psi^o, Q^o)\} \in L_{2,T}^2([0, T], \mathbb{R}) \times L_{2,T}^2([0, T], \mathcal{L}(\mathbb{R}^n, \mathbb{R})) \) is a unique weak solution of the backward stochastic differential equations (B) such that \( u^o \in \mathbb{U}^{(N)}[0, T] \) satisfies the point wise almost sure inequalities with respect to the \( \sigma \)-algebras \( \mathcal{G}_{0,t}^i \subset \mathbb{F}_{0,t}, \ t \in [0, T], i = 1, 2, \ldots, N :*

\[
\mathbb{E}^u \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u_t^{-i, o}, u_t^i) | \mathcal{G}_{0,t}^i \right\}
\geq \mathbb{E}^u \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u_t^o) | \mathcal{G}_{0,t}^i \right\}, \quad \forall u^i \in \mathcal{A}^i \text{ a.e.t. in } [0, T], \quad \mathbb{P}^u|_{\mathcal{G}_{0,t}^i} \text{ a.s.}, \forall i \in \mathbb{Z}_N. \quad (58)
\]

4. *For admissible strategies \( \mathbb{U}^{(N),z}[0, T], \mathbb{U}^{(i),z}[0, T] \) the conditional expectation in (58) is taken with respect to the information structures \( \mathcal{G}_{0,t}^i, \mathcal{G}^{x^o(t)} \), respectively.*

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Proof: The statements follow from Theorem 3 and the discussion prior to the Theorem.

For PbP optimality we have the following Corollary.

**Corollary 2. (Necessary Conditions for PbP Optimality under Original Probability Measure)**

Suppose Assumptions 3 hold. Then we have the following.

**Necessary Conditions.** For an element \( u^o \in \mathbb{U}^{(N)}[0, T] \) with the corresponding solution \( x^o \in B_{\mathbb{P}_u}^\infty ([0, T], L^2(\Omega, \mathbb{R}^n)) \) to be a PbP optimal strategy, it is necessary that the statements of Theorem 4 (1), (3), (4) hold and statement (2) corresponding to (57).

Proof: Following from the change of probability measure and Corollary 1.

Therefore, we can apply the necessary conditions for team optimality, either under the reference measure \( \mathbb{P} \) or under original measure \( \mathbb{P}_u \).

In the next remark, we discuss the connection of Theorem 4 to the necessary conditions of optimality of stochastic control problems with centralized feedback information structures.

**Remark 5.** From Theorem 4 one can deduce the optimality conditions of classical stochastic control problems with centralized noiseless feedback information structures, that is, when for any \( t \in [0, T] \), \( u_t \) is measurable with respect to the feedback information structure \( F^x_{t,0} \equiv \sigma \{ x(s) : 0 \leq s \leq t \} \), and to partial noiseless feedback information structure \( G^x_{t,0} \subset F^x_{t,0} \). Indeed, the necessary conditions for such a \( u^o \) to be optimal with say, centralized partial noiseless feedback information structure, under the original probability measure \( \mathbb{P}_u \) are

\[
E^{u^o} \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u)|G^x_{0,t} \right\} \geq E^{u^o} \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u^o)|G^x_{0,t} \right\}, \quad \forall u \in \mathbb{A}^{(N)}, \text{a.e.} t \in [0, T], \mathbb{P}|^{u^o}_{G^x_{0,t}} - a.s., \quad (59)
\]

where \( \{ x^o(t), \Psi^o(t), Q^o(t) : t \in [0, T] \} \) are the solutions of (53), (54). Note that the Hamiltonian \( \mathbb{H}(t, x, Q, u) \equiv Q \sigma^{-1}(t, x) f(t, x, u) + \ell(t, x, u) \) is precisely the one derived in [2], using martingale methods. Clearly, (59) generalizes the necessary conditions of optimality of classical stochastic control problems with partial nonanticipative information structures [17], [52] and references therein,
where for any \( t \in [0, T] \), \( u_t \) is measurable with respect to any Brownian motion generated, sub-\( \sigma \)-algebras of \( \mathcal{G}^W_{0,t} \subset \mathcal{F}^W_{0,t} \triangleq \sigma\{W(s) : 0 \leq s \leq t\}, t \in [0, T] \).

B. Value Processes of Team Problems

In this section, we first show that the solution of the Backward stochastic differential equation is the value process of the stochastic dynamic team problem, lifted to a conditioning with respect to the centralized information structure. Then we use the lifted value process to show that the necessary conditions (i.e. (58)) for PbP optimality are also sufficient.

Define the sample pay-off over the interval \([t, T]\) by

\[
\mathcal{J}_{t,T}(u^1, \ldots, u^N) \triangleq \int_t^T \ell(s, x(s), u_s)ds + \varphi(x(T)).
\] (60)

and its conditional expectation with respect to \( \mathcal{G}^I_{0,t}, i = 1, \ldots, N \), by

\[
\mathcal{J}_{t,T}^i(u) \triangleq \mathbb{E}^u \left\{ \mathcal{J}_{t,T}(u^1, \ldots, u^N) | \mathcal{G}^I_{0,t} \right\}, \ u \in \mathbb{U}^{(N)}[t, T],
\] (61)

where \( \mathbb{U}^{(N)}[t, T] \) is the restriction of the strategies \( \mathbb{U}^{(N)}[0, T] \), to the interval \([t, T]\).

PbP optimality seeks admissible strategies \( u^i \in \mathbb{U}^i[t, T], i = 1, \ldots, N \) to minimize the pay-off, in the sense,

\[
\mathbb{E}^{u^{-i,o},u^{i,o}} \left\{ \mathcal{J}_{t,T}(u^{-i,o}, u^{i,o}) | \mathcal{G}^I_{0,t} \right\} \leq \mathbb{E}^{u^{-i,o},u^i} \left\{ \mathcal{J}_{t,T}(u^{-i,o}, u^i) | \mathcal{G}^I_{0,t} \right\}, \ \forall u^i \in \mathbb{U}^i[t, T], i = 1, \ldots, N.
\]

This means that when team members employ strategies, \( u^{-i,o} \in \times_{j=1,j \neq i}^N \mathbb{U}^j[t, T] \), team member \( u^i \) minimizes the reward \( \mathbb{E}^{u^{-i,o},u^i} \left\{ \mathcal{J}_{t,T}(u^{-i,o}, u^i) | \mathcal{G}^I_{0,t} \right\} \) over all strategies \( \mathbb{U}^i[0, T] \). The set of all such strategies \( (u^{1,0}, \ldots, u^{N,0}) \in \times_{i=1}^N \mathbb{U}^i[t, T] \) is called PbP optimal.

We denote the value processes of the team game for each team member by

\[
V^i(t) \triangleq \mathbb{E}^{u^{-i,o},u^{i,o}} \left\{ \mathcal{J}_{t,T}(u^{-i,o}, u^{i,o}) | \mathcal{G}^I_{0,t} \right\}, \ i = 1, \ldots, N.
\] (62)

Consider the solution of the backward stochastic differential equation (44)

\[
\Psi^u(t) = \Psi^u(T) + \int_t^T \mathbb{H}(s, x(s), Q^u(s), u_s)ds - \int_t^T Q^u(s)dW(s), \ \ t \in [0, T].
\] (63)

For \( u = u^o \) this is the lifted value process of the team pay-off with respect to the information \( \mathbb{E}_{0,t}, t \in [0, T] \). From (63) we have

\[
\Psi^u(t) = \Psi^u(T) + \int_t^T \ell((s, x(s), u_s)ds - \int_t^T Q^u(s)dW^u(s), \ \ t \in [0, T],
\] (64)
Hence, and by taking conditional expectation \( \mathbb{E}^u \{ \cdot \mid F_{0,t} \} \) of both sides of (64), and using \( \Psi^u(T) = \varphi(x(T)) \), we obtain

\[
\Psi^u(t) = \mathbb{E}^u \left\{ \int_t^T \ell(s, x(s), u_s) \, ds + \varphi(x(T)) \mid F_{0,t} \right\}.
\]  

(65)

Hence,

\[
J^i_{t,T}(u) = \mathbb{E}^u \left\{ \Psi^u(t) \mid G_{0,t}^i \right\}, \quad u \in \mathcal{U}^{(N)}[0, T], \forall i \in \mathbb{Z}_N.
\]  

(66)

Now, we state the main theorem.

**Theorem 5.** *(Sufficient Conditions for PbP Optimality)* Let

\[
(\Psi^u, Q^u) \in L^2([0, T], L^2(\Omega, \mathbb{R})) \times L^2([0, T], L^2(\Omega, \mathcal{L}(\mathbb{R}^n, \mathbb{R})))
\]

be a solution of the backward stochastic differential equation (63).

If \( u^{i,o} \in \mathcal{U}^{(i)}[t, T] \) satisfy the conditional variational inequalities (58), then \( (u^{1,o}, \ldots, u^{N,o}) \in \times_{i=1}^N \mathcal{U}^{(i)}[t, T] \) is a PbP optimal.

Moreover, a.e.t \( \in [0, T] \), \( \mathbb{E}^u \mid_{G^i_{0,t}} \) a.s. we have

\[
V^i(t) = \mathbb{E}^{u^{i,o}} \left\{ \mathbb{E}^{u^{i,o}} \left\{ \Psi^{u^{i,o}}(t) \mid F_{0,t} \right\} \mid G_{0,t}^i \right\} = \mathbb{E}^{u^{i,o}} \left\{ \Psi^{u^{i,o}}(t) \mid G_{0,t}^i \right\}, \quad i = 1, \ldots, N.
\]  

(67)

**Proof:** Since (58) holds, by taking expectation on both sides we deduce

\[
\mathbb{E}^u \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u^{-i,o}_t, u^i_t) \right\} \geq \mathbb{E}^{u^o} \left\{ \mathbb{H}(t, x^o(t), Q^o(t), u^o_t) \right\}, \forall u^i \in \mathcal{A}^i, \forall i \in \mathbb{Z}_N.
\]

This implies that for almost every \( (t, x) \in [0, T] \times C([0, T], \mathbb{R}^n) \),

\[
\mathbb{H}(t, x^o(t), Q^o(t), u^{-i,o}_t, u^i_t) \geq \mathbb{H}(t, x^o(t), Q^o(t), u^o_t), \quad \forall u^i \in \mathcal{A}^i, \forall i \in \mathbb{Z}_N.
\]  

(68)

Therefore, an application of the comparison theorem of stochastic differential equations (53) to the second right hand side term of (63) yields \( \Psi^{u^{i,o}, u^i}(-) \geq \Psi^{u^{i,o}, u^i}(-) \), a.e. on \( [0, T] \times C([0, T], \mathbb{R}^n) \). From (65), we have

\[
\Psi^{u^{i,o}, u^i}(t) = \mathbb{E}^{u^{i,o}, u^i} \left\{ \int_t^T \ell(s, x(s), u_s^{-i,o}, u_s^{i,o}) \, ds + \varphi(x(T)) \mid F_{0,t} \right\}
\]  

(69)

\[
\leq \Psi^{u^{i,o}, u^i}(t) \geq \mathbb{E}^{u^{i,o}, u^i} \left\{ \int_t^T \ell(s, x(s), u_s^{-i,o}, u_s^{i,o}) \, ds + \varphi(x(T)) \mid F_{0,t} \right\}, \quad \forall u^i \in \mathcal{U}^{(i)}[0, T].
\]  

(70)
By taking conditional expectation of both sides of last inequality with respect to $G_{0,t}^{I_i}$, we obtain
\[
E_{u^{-i,o},u_t^i}\left\{ \int_t^T \ell(s, x(s), u_s^{-i,o}, u_s^i)ds + \varphi(x(T)) | G_{0,t}^{I_i} \right\} \leq E_{u^{-i,o},u_t^i}\left\{ \int_t^T \ell(s, x(s), u_s^{-i,o}, u_s^i)ds + \varphi(x(T)) | G_{0,t}^{I_i} \right\}, \quad \forall u^i \in U^I_{[0,T]}.
\] (71)

Since this holds for all $i = 1, \ldots, N$ we deduce PBP optimality.

Finally, by taking conditional expectation of both sides of (69) with respect to $G_{0,t}^{I_i}$ we deduce (67). This completes the derivation.

Finally, we note that if we consider the extended state $(x, \Lambda)$ and corresponding Hamiltonian system of equations, under certain global convexity conditions (see [48]), we can show that PBP optimality implies team optimality.

IV. CONCLUSIONS AND FUTURE WORK

This paper generalizes static team theory to stochastic differential decision system with decentralized noiseless feedback information structures. We have applied Girsanov’s theorem to transformed the initial dynamic team problem to an equivalent team problem, under a reference probability space, with state process independent of any of the team decisions. Then, we described the connection to static team theory discussed by Witsenhausen in [18], and we proceeded further to derive team and PBP optimality conditions, using the stochastic Pontryagin’s maximum principle. We also discussed the connection between the backward stochastic differential equation and the value process of the team problem.

In future work we will apply the optimality conditions to problems from the communication and control areas, as in [49], but instead of decentralized nonanticipative strategies, we will use decentralized feedback strategies.

V. APPENDIX

Proof of Theorem [1]

First, we show that $E\left\{ \Lambda^{\alpha}(t)|x(t)|_2^2 \right\} < K$. By applying the Itô differential rule
\[ d|x(t)|_{\mathbb{R}^n}^2 = 2\langle x(t), \sigma(t, x(t))dW(t) \rangle dt + \text{tr} \left( a(t, x(t)) \right) dt; \quad a(t, x) \triangleq \sigma(t, x)\sigma^*(t, x) \]  

(72)

\[
\begin{align*}
\frac{d}{dt} \Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2 &= \Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2 f^*(t, x(t), u_t) \left( a(t, x(t)) \right)^{-1} dx(t) + 2\Lambda^u(t)\langle x(t), \sigma(t, x(t))dW(t) \rangle \\
&\quad + 2\Lambda^u(t)\langle x(t), f(t, x(t), u_t) \rangle + \Lambda^u(t)\text{tr} \left( a(t, x(t)) \right) dt.
\end{align*}
\]  

(73)

Then by applying the Itô differential rule once more we have

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2}{1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2} \right) &= \frac{1}{\left(1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2\right)^2} \left\{ \Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2 f^*(t, x(t)) \left( a(t, x(t)) \right)^{-1} dx(t) \\
&\quad + 2\Lambda^u(t)\langle x(t), \sigma(t, x(t))dW(t) \rangle + 2\Lambda^u(t)\langle x(t), f(t, x(t), u_t) \rangle dt + \Lambda^u(t)\text{tr} \left( a(t, x(t)) \right) dt \right\}
\end{align*}
\]

\[
- \frac{\epsilon(\Lambda^u(t))^2}{\left(1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2\right)^3} \left\{ \left( \left( a(t, x(t)) \right)^{-1} f(t, x(t), u_t) \right)|x(t)|_{\mathbb{R}^n}^2 \right. \\
&\quad \left. + 2x(t, a(t, x(t)) \left( \left( a(t, x(t)) \right)^{-1} f(t, x(t), u_t) \right)|x(t)|_{\mathbb{R}^n}^2 + 2x(t) \right) \right\} dt
\]

Integrating over \([0, T]\) and taking the expectation with respect to \(\mathbb{P}\), and using the fact that \(\mathbb{E}\left( \Lambda^u(t) \right) \leq 1, \forall t \in [0, T]\), yields

\[
\frac{d}{dt} \mathbb{E}\left\{ \frac{\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2}{1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2} \right\} \leq \mathbb{E}\left\{ \frac{\Lambda^u(t)\left[ 2\langle x(t), f(t, x(t), u_t) \rangle + \text{tr} \left( a(t, x(t)) \right) \right]}{1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2} \right\}.
\]  

(74)

By Assumptions \([1] (A1), (A2)\), there exists \(K > 0\) such that

\[
\frac{d}{dt} \mathbb{E}\left\{ \frac{\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2}{1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2} \right\} \leq K \left( \mathbb{E}\left\{ \frac{\Lambda^u(t)\left[ |x(t)|_{\mathbb{R}^n}^2 + |u_t|_{\mathbb{R}^d}^2 \right]}{1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2} \right\} + 1 \right)
\]

Since for any \(u \in \mathcal{U}(\mathbb{N})[0, T]\), we have \(\mathbb{E} \int_0^T \Lambda^u(t)|u_t|_{\mathbb{R}^d}^2 dt\) is finite, then it follows from Gronwall inequality that

\[
\mathbb{E}\left\{ \frac{\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2}{1 + \epsilon\Lambda^u(t)|x(t)|_{\mathbb{R}^n}^2} \right\} \leq C, \quad \forall t \in [0, T].
\]  

(75)
By Fatou’s lemma we obtain $E\left\{ \Lambda^u(t)|x(t)|^2_{\mathbb{R}^n} \right\} < C$, $\forall t \in [0,T]$. Consider

$$
\frac{d}{dt} \frac{\Lambda^u(t)}{1 + \epsilon \Lambda^u(t)} = \frac{\Lambda^u(t)f^*(t, x(t))\left( a(s, x(s)) \right)^{-1} dx(t)}{\left(1 + \epsilon \Lambda^u(t)\right)^2} - \frac{\epsilon \left( \Lambda^u(t) \right)^2 f^*(t, x(t), u_t)\left( a(t, x(t)) \right)^{-1} f(t, x(t), u_t)}{\left(1 + \epsilon \Lambda^u(t)\right)^3},
$$

then

$$
E\left\{ \Lambda^u(t) \right\} = 1 - E \int_0^T \frac{\epsilon \left( \Lambda^u(s) \right)^2 f^*(s, x(s), u_s)\left( a(t, x(t)) \right)^{-1} f(s, x(s), u_s)}{\left(1 + \epsilon \Lambda^u(s)\right)^3} ds. \tag{76}
$$

Since

$$
\frac{\epsilon \left( \Lambda^u(t) \right)^2 f^*(t, x(t), u_t)\left( a(t, x(t)) \right)^{-1} f(t, x(t), u_t)}{\left(1 + \epsilon \Lambda^u(t)\right)^3} \rightarrow 0, \ a.e. t \in [0,T], \ \mathbb{P} - a.s. \ \text{as} \ \epsilon \rightarrow 0,
$$

and by (A7) there exists a constant $C > 0$ such that it is bounded by $C \Lambda^u(t) \left(1 + |x(t)|^2_{\mathbb{R}^n} + |u_t|^2_{\mathbb{R}^d}\right)$, then by the Lebesgue’s dominated convergence theorem we have

$$
E \int_0^T \frac{\epsilon \left( \Lambda^u(s) \right)^2 f^*(s, x(s), u_s)\left( a(s, x(s)) \right)^{-1} f(s, x(s), u_s)}{\left(1 + \epsilon \Lambda^u(s)\right)^3} ds \rightarrow 0 \ \text{as} \ \epsilon \rightarrow 0. \tag{77}
$$

Since $E\left\{ \Lambda^u(t) \right\} \leq 1, \forall t \in [0,T]$ then by using (76) into (77), we obtain $E\left\{ \Lambda^u(t) \right\} \rightarrow E\left\{ \Lambda^u(t) \right\}$, as $\epsilon \rightarrow 0$. Hence, we must have $E\left\{ \Lambda^u(t) \right\} = 1, \forall t \in [0,T]$. Consequently, we have equivalence of the two dynamic team problems.

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