Special Functions of the Isomonodromy Type, Rational Transformations of Spectral Parameter, and Algebraic Solutions of the Sixth Painlevé Equation

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Abstract
We discuss relations which exist between analytic functions belonging to the recently introduced class of special functions of the isomonodromy type (SFITs). These relations can be obtained by application of some simple transformations to auxiliary ODEs with respect to a spectral parameter which associated with each SFIT. We consider two applications of rational transformations of the spectral parameter in the theory of SFITs. One of the most striking applications which is considered here is an explicit construction of algebraic solutions of the sixth Painlevé equation.

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1 Introduction
The general notion of special functions of the isomonodromy type (SFITs) is introduced in the work [1]. It was shown there that many classical special functions, e.g., the Gamma function, Gauss hypergeometric functions, Painlevé functions, etc., belong to the class of SFITs. It was argued in [1] that a unique definition of such functions as the functions describing isomonodromy deformations of the matrix ODEs of the form,

\[ \frac{d\Psi}{d\lambda} = A(\lambda)\Psi, \]  \tag{1.1}

where \( \lambda \) is called throughout this paper the spectral parameter and \( A(\lambda) \) is a \( n \times n \) matrix-valued rational function of \( \lambda \), allows one to find many relations between these special functions, and suggests a very natural and unique approach to study their properties by using techniques developed for the matrix Riemann-Hilbert problem. More precisely, define class of ODEs of the form (1.1) under the following equivalence

\[ A \rightarrow G^{-1}AG, \quad A \rightarrow A + \partial_\lambda \log f(\lambda), \]  \tag{1.2}

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where $G$ is $n \times n$ matrix independent of $\lambda$, and $f(\lambda)$ is an arbitrary scalar function of $\lambda$ with a rational logarithmic derivative. We always suppose throughout the paper, that the function $f$ is chosen so that $A(\lambda) \in sl_n(\mathbb{C})$. Isomonodromy deformations are considered with respect to pole parameters; to each pole $t^0_k$, $k = 1, \ldots, m$, of order $p_k$ one can associate $(n-1)(p_k-1)$ "continuous" pole parameters, $t^{ij}_k$, $i = 1, \ldots, p_k-1$, $j = 1, \ldots, n-1$ via asymptotic expansion of the function $\Psi(\lambda)$ at $\lambda = t^0_k$ and $n-1$ "discrete" parameters $\theta^i_k$. With respect to the variables $t^{ij}_k$ and $t^0_k$ isomonodromy deformations of the coefficients of Eq. (1.1), i.e., the corresponding SFITs, satisfy an overdetermined system of PDEs. With respect to the variables $\theta^i_k$ the SFIT solves a system of difference equations. One can write for SFITs also differential-difference systems of equation. Due to the fractional-linear transformation of the spectral parameter $\lambda$, SFIT effectively depends on $(n-1) \sum_{k=1}^{m} (p_k-1) + m - 3$ continuous variables and $m(n-1)$ discrete ones. Moreover, each SFIT depends on the monodromy variables, a point $M$ of the so-called monodromy manifold (an algebraic variety of the data characterizing monodromy group of Eq. (1.1)). For those SFITs which solve linear equations, monodromy manifold is just $\mathbb{C}^N$ for some natural $N$, and the dependence on the monodromy variables can be excluded by a specification of the linear independent solutions. Instead of the set of the coefficients of Eq. (1.1), one may think of SFIT as about one function of the variables $t^{ij}_k$, $t^0_k$, and $\theta^i_k$. This can be done in a variety of ways, e.g., by means of the Jimbo-Miwa $\tau$-function.

It is clear, that in this general setting, when we fix only the matrix dimension and functional dependence of $A(\lambda)$ as a rational function of $\lambda$, different equations of the type (1.1) define different SFITs, namely, they have either different number of continuous, discrete, or monodromy variables. Nevertheless, for special values of the variables, SFITs defined by Eq. (1.1) with different $A(\lambda)$ may be related to each other by some explicit transformations. Many of these transformations can be found by a direct group-theoretical analysis of the corresponding overdetermined systems of PDEs; however, due to the large number of variables, this analysis is quite complicated even with the help of a computer. An interesting question, therefore, is to consider transformations acting on the set of classes of equations of the type of Eq. (1.1), which generate relations between corresponding SFITs. In Section 3, we consider some of these transformations; in particular, rational transformations of the spectral parameter are introduced.

A combination of rational and Schlesinger transformations (we call them RS-transformations) is a simple but powerful method which allows one to get many non-trivial results in the theory of SFITs. In Section 4, on the example of the sixth Painlevé equation ($P_6$), we explain how one can construct algebraic SFITs. In Section 5, two particular examples of new algebraic solutions of $P_6$ are constructed by this method. In Section 6, another application of RS-transformations for the sixth Painlevé equation is considered. Namely, it is shown that there are some special points, in the complex plane of independent variable of $P_6$, which are different from 0, 1, and $\infty$ and have the following property: for each point there are, at least, two transcendental solutions of $P_6$ (with coefficients satisfying some simple restriction) such that at this point the first solution has the pole, the second has the zero, and the monodromy data of both solutions can be explicitly calculated in terms of their Laurent or Taylor expansions at this point, respectively. These special transcendental solutions are similar to the so-called symmetric solutions of the first and second Painlevé equations. The latter property allows one to solve the so-called connection problem for these solutions in terms of their expansions in the special points mentioned above.

Constructions which are based on the RS-transformations for the multivariable SFITs
will lead to the functions which are algebraic with respect to a given subset of their continuous variables, whilst the special points which are discussed in the previous paragraph will be some special hypersurfaces in the space of the continuous variables.

It is also interesting to discuss relations between different SFITs from the point of view of transcendency of SFITs. In fact, from the point of view of transcendental functions existence of these relations means that the SFITs which can be expressed in terms of other SFITs don’t define new transcendental functions. Recently, this question for the Painlevé equations has been intensively discussed in the literature due to the approach developed by H. Umemura [4, 5]. A key notion in the Umemura’s approach is the notion of classical functions. Actually, SFITs in many respects are “not worse” than Umemura’s classical functions; say, properties of the Painlevé functions can be studied to a much greater extent than properties of the most classical functions. Moreover, the study of the transcendency of the Painlevé equations in the Umemura’s setting has shown that among all classical functions only those which are SFITs solve Painlevé equations. From this point of view it would be interesting to have a kind of generalized Umemura’s theory which would include “reducibility” of the Painlevé transcendentals. Therefore, it would be natural to introduce a notion of transcendency of a SFIT (denote it \( \tau \)) with respect to a given set of SFITs \( G = \{ \tau_1, \ldots, \tau_r \} \), where each SFIT of the set \( G \) depends on the same or fewer number of continuous variables than \( \tau \). In this setting, the problem is to determine the set of discrete and monodromy variables of \( \tau \) such that \( \tau \) belongs to the \( G \)-extension of the differential field (more precisely its multivariable generalization) of classical functions.

A particular case of the RS-transformations, namely, a combination of quadratic transformations of the spectral parameter with the Schlesinger transformations, has already been used in the study of the Painlevé equations. It is shown that, they allow to get quite non-trivial results: quadratic transformations for the sixth Painlevé equation [6], a quadratic transformation between the third and fifth Painlevé equations (reducibility of the fifth Painlevé equation to the third one for the special values of the discrete parameters), and quadratic transformations between different Lax representations for the fourth and third Painlevé equations, which follows from the ”gauge equivalence” between AKNS and KN hierarchies of soliton equations. The last fact can be reformulated as an example of relations between different SFITs which are discussed in the previous paragraph.

2 Transformations of SFITs

As explained in the Introduction any transformation of the solution \( \Psi \) corresponding to some isomonodromy deformation of Eq. (1.1), which maps it into a function \( \Phi \) solving an ODE of the same type, i.e., Eq. (1.1) with \( \hat{A}(\lambda) \rightarrow \hat{A}(\lambda) \) where \( \hat{A}(\lambda) \) is a rational function of \( \lambda \) whose coefficients depend isomonodromically on its pole parameters, generates transformations of the corresponding SFITs. Here, we list some transformations of this kind.

1. P-Transformations. If \( \Psi_k \) satisfies \( \frac{d\Psi_k}{d\lambda} = A_k(\lambda)\Psi_k \) for \( k = 1, 2 \), then \( \Phi = \Psi_1 \oplus \Psi_2 \) solves

\[
\frac{d\Phi}{d\lambda} = \begin{pmatrix}
A_1 & \hat{\Theta} \\
\hat{\Theta} & A_2
\end{pmatrix} \Phi,
\]

where \( \hat{\Theta} \) are the matrices each of whose elements are equal to zero;
2. T-transformations. Let vector-functions $\psi_k(\lambda)$ solve $\frac{d\psi_k}{d\lambda} = A_k(\lambda)\psi_k$ for $k = 1, 2$, then vector $\psi = \psi_1 \otimes \psi_2$ solves

$$\frac{d\psi}{d\lambda} = (A_1 \otimes I_2 + I_1 \otimes A_2) \psi,$$

where $I_k$ are identical matrices of the sizes $A_k$.

3. $D_\kappa$-Transformations. If $\Psi$ is a solution of Eq. (1.1) and $\kappa$ any parameter, say, $\lambda$, $i_k^l$, or $\theta^l_k$, then $\Phi = \frac{d\Psi}{d\kappa} \oplus \Psi$ solves

$$\frac{d\Phi}{d\lambda} = \left( A \frac{dA}{d\kappa} \hat{0} \right) \Phi.$$

This transformation is easy to generalize for any vector parameter $\kappa$.

4. L-Transformations. The Laplace transformation,

$$\Psi = \int_C \chi(\mu)e^{\lambda\mu} d\mu, \quad \Phi = (\chi^{(N)}, \ldots, \chi)^T,$$

where $T$ means transposition and $(N)$ denotes $N$th derivative with respect to $\mu$; $N + 1$ is the sum of orders of poles of matrix $A(\lambda)$. More information concerning L-transformations can be found in [7].

5. S-Transformations. The Schlesinger transformations of Eq. (1.1). These transformations are generated by elementary transformations of the following form,

$$\Phi = R(\nu)\Psi, \quad \nu = \left( \frac{\lambda - a}{\lambda - b} \right)^{1/n} \text{ or } \nu = (\lambda - a)^{1/n},$$

where $R(\nu)$ is a rational function of $\nu$ with poles at $\nu = 0$ and $\infty$, and $a$ and $b$ are parameters which may coincide with the poles of $A(\lambda)$;

6. R-Transformations. Rational transformations of the spectral parameter $\lambda \rightarrow \mu$,

$$\lambda = R(\mu), \quad \Psi(\lambda) = \Phi(\mu), \quad (2.3)$$

evidently transforms Eq. (1.1) into ODE with respect to $\mu$ of the same type,

$$\frac{d\Phi}{d\mu} = B(\mu)\Phi, \quad (2.4)$$

moreover, generators of the monodromy group of Eq. (2.4) can be calculated as multiplications of the corresponding monodromy matrices and their inverses for Eq. (1.1). Thus, transformation (2.3) preserve the isomonodromic property.

In the next two sections we consider applications of R- and S-transformations to the theory of the sixth Painlevé equation.

\[1\] I would like to thank the referee for this comment
3 Construction of Algebraic SFITs.

The Sixth Painlevé Equation

In general SFITs are transcendental functions of several variables, but for some special values of the discrete and monodromy parameters, these functions can be algebraic functions of a subset of continuous variables. The idea which is presented in this section can be applied for construction of algebraic SFITs defined by Eq. (3.1) of Fuchsian type. We explain this idea taking as an example the sixth Painlevé transcendent. This function is related with one of the simplest equation of the type (1.1) in $2 \times 2$ matrices. Constructions of algebraic SFITs related with the isomonodromy deformations of ODEs in matrix dimension higher than 2 can be obtained via application of the transformations given in Section 2. A modification of this method which allows to get more general construction of algebraic SFITs defined by Fuchsian equations in the matrix dimension higher than 2 will be given elsewhere.

Let us recall basic facts concerning definition of the sixth Painlevé equation as a SFIT $SF_6$. Consider $2 \times 2$ matrix Fuchsian ODE with four singular points,

$$\frac{d\Psi}{d\lambda} = \left( \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t} \right) \Psi, \quad (3.5)$$

where we suppose usual conditions, $A_k \in sl_2(\mathbb{C})$ for $k = 0, 1, t, \infty$, $A_\infty \equiv A_0 + A_1 + A_t = -\frac{\theta_0}{2} \sigma_3$, $\theta_\infty \neq 0$, which is in fact (excluding one exceptional solvable case $\mathbb{H}$) also a normalization rather than a condition on $A_k$. Consider the system of Schlesinger equations,

$$\frac{dA_0}{dt} = \frac{1}{t}[A_t, A_0], \quad \frac{dA_1}{dt} = \frac{1}{t-1}[A_t, A_1], \quad \frac{dA_t}{dt} = \frac{1}{t}A_0 + \frac{1}{t-1}A_1, A_t. \quad (3.6)$$

This system is the compatibility condition of Eq. (3.5) with

$$\frac{d\Psi}{dt} = -\frac{A_t}{\lambda - t} \Psi. \quad (3.7)$$

We call system (3.6) the Schlesinger deformations of Eq. (3.5) Any solution of system (3.6) define an isomonodromy deformation of Eq. (3.5). The general solution (the set of all solutions) of the system (3.6) we call special function of the isomonodromy type and denote it as $SF_6^2$. This function depends on one continuous variable $t$ and four discrete variables $\theta_0, \theta_1, \theta_t, \theta_\infty$, the latter are nothing but the eigenvalues of the matrices $2A_k$, $k = 0, 1, t, \infty$. It follows from Eq. (3.6) that $\theta_k$ are independent of $t$. Thus the complete notation for this function is $SF_6^2(t; \theta_0, \theta_1, \theta_t, \theta_\infty)$. As the function of variable $t$ $SF_6^2$ is known to be closely related with the classical sixth Painlevé equation, $P_6$,

$$\frac{d^2y}{dt^2} = \frac{1}{2} \left( \frac{1}{t} + \frac{1}{y} + \frac{1}{t-1} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha_6 + \beta_6 \frac{t}{y} + \gamma_6 \frac{t-1}{(y-t)^2} + \delta_6 \frac{t(t-1)}{(y-t)^3} \right), \quad (3.8)$$

where $\alpha_6, \beta_6, \gamma_6, \delta_6 \in \mathbb{C}$ are parameters. We need explicit relation $SF_6^2 \longrightarrow P_6$. Suppose that a set of matrices $\{A_k\}$ solves system (3.6) and denote $A_i^{ij}$ corresponding matrix elements of $A_k$. Note, that due to the normalization

$$A_0^{12} + A_1^{12} + A_t^{12} = A_0^{21} + A_1^{21} + A_t^{21} = 0,$$
therefore equations,
\[ \frac{A^0_{ik}}{y_{ik}} + \frac{A^i_{ik}}{y_{ik} - 1} + \frac{A^t_{ik}}{y_{ik} - t} = 0, \]
for \( \{ik\} = \{12\} \) and \( \{ik\} = \{21\} \) have in general situation \( (A^i_{ik} + tA^t_{ik} \neq 0) \) unique solutions \( y_{ik} \). These functions solve Eq. (3.8) with the following values of the parameters,
\[
y_{12}(t) : \quad \alpha_6 = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta_6 = -\frac{\theta_0^2}{2}, \quad \gamma_6 = \frac{\theta_1^2}{2}, \quad \delta_6 = \frac{1 - \theta_1^2}{2}, \quad (3.9)
y_{21}(t) : \quad \alpha_6 = \frac{(\theta_\infty + 1)^2}{2}, \quad \beta_6 = -\frac{\theta_0^2}{2}, \quad \gamma_6 = \frac{\theta_1^2}{2}, \quad \delta_6 = \frac{1 - \theta_1^2}{2}. \quad (3.10)
\]
Instead of Eq. (3.8) one can associate with \( SF^2 \) the so-called \( \tau \)-function, which plays a very important role in applications. This function \[9\] is defined via the function \( \sigma \),
\[
\sigma(t) = \text{tr}((t - 1)A_0 + tA_1)A_\tau + t\kappa_1\kappa_2 - \frac{1}{2}(\kappa_4\kappa_1 + \kappa_1\kappa_2),
\]
where
\[
\kappa_1 = \frac{\theta_1 + \theta_\infty}{2}, \quad \kappa_2 = \frac{\theta_1 - \theta_\infty}{2}, \quad \kappa_3 = -\frac{\theta_1 + \theta_0}{2}, \quad \kappa_4 = \frac{\theta_1 - \theta_0}{2}.
\]
The function \( \sigma \) solves the following ODE,
\[
t^2(t - 1)^2\sigma''\sigma' + \left(2\sigma'(t\sigma' - \sigma) - \sigma'^2 - \kappa_1\kappa_2\kappa_3\kappa_4\right)^2 = \left(\sigma' + \kappa_1\kappa_2\kappa_3\kappa_4\right)\left(\sigma' + \kappa_1\kappa_2\kappa_3\kappa_4\right).
\]
where the prime is differentiation by \( t \). The \( \tau \)-function is defined up to a multiplicative constant as the solution of the following ODE,
\[
t(t - 1)\frac{d}{dt}\ln \tau = \sigma(t).
\]
Now, we are ready to explain our construction of algebraic solutions for \( P_6 \). Consider the following matrix form of hypergeometric equation,
\[
\frac{d\Phi}{d\mu} = \left(\frac{\hat{A}}{\mu} + \frac{\hat{B}}{\mu - 1}\right)\Phi, \quad (3.11)
\]
where we, following \[2\], parameterize the matrices \( A \) and \( B \) by three complex numbers, \( \alpha, \beta, \) and \( \delta \),
\[
\hat{A} = \left(\begin{array}{cc}
-\frac{(\alpha+\beta)(1-\delta)+2\alpha\beta}{2\beta-2\alpha} & \frac{\beta(\beta+1-\delta)}{\beta-\alpha} \\
-\frac{\alpha(\alpha+1-\delta)}{2\beta-2\alpha} & \frac{\alpha(\alpha+1-\delta)(1+2\alpha\beta)}{2\beta-2\alpha}
\end{array}\right), \quad \hat{B} = -\hat{A} + \frac{\beta - \alpha}{2}\sigma_3.
\]
Explicit formula for the fundamental solution \( \Phi \) in terms of the Gauß hypergeometric functions can be found in \[3\]. Here we don’t use it, however this formula is very important in applications. Now, we make \( R \)-transformation, \( \mu = P(\lambda)/Q(\lambda), \Phi(\mu) = \Psi(\lambda), \) where polynomials \( P(\mu) \) and \( Q(\mu) \) have no common roots. Define the rank of \( R \)-transformation,
\[
\text{rank}(R) = \max\{\text{deg}(P), \text{deg}(Q)\}.
\]
The number of singular points of the function \( \Psi(\lambda) \) is not greater than \( 3 \cdot \text{rank}(R) \). The function \( R \) depends on \( \text{deg}(P) + \text{deg}(Q) + 1 \) parameters. These parameters can be used to reduce a number of singular points of \( \Psi \), moreover, one of them should
remain free to play the role of the deformation parameter $t$. If rank($R$) $\geq 3$, then the number of the parameters cannot be chosen such that the function $\Psi(\lambda)$ has four singular points. In this case we can further specify some of the discrete parameters $\theta_k$ such that, after a set of four singular points is chosen, all extra "unwanted" singular points can be removed by Schlesinger transformation. Construction of algebraic solutions by this method requires therefore classification of all rational functions for which this program can be fulfilled. A suitable parameter for this classification is rank($R$). At the moment all transformations of rank($R$) $\leq 4$ are classified \[10\]. Transformations with rank($R$) $\geq 5$ are under classification.

Another method to construct algebraic solutions of the sixth Painlevé equation is to classify all cases when one-parameter families of solutions of the sixth Painlevé equations, which are known to be expressible in terms of logarithmic derivatives of the general solution of the hypergeometric equation, are algebraic due to the special choice of the latter solution.

Based on the results concerning algebraic solutions, which are known by the time this paper is written, it seems reasonable to make the following

**Conjecture.** The methods explained in the last two paragraphs allows one to construct all algebraic solutions of the sixth Painlevé equation, perhaps with the help of the certain transformations given in the Okamoto’s work \[11\].

### 4 Examples of Algebraic Solutions of the Sixth Painlevé Equation

My interest in the construction of algebraic solutions of the sixth Painlevé equation is related with the works of Hitchin \[12, 13\], Umemura \[4\], and Dubrovin and Mazzocco \[14\]. Note, that one of the steps in the Umernura’s approach to the notion of transcendency of the Painlevé functions is to classify algebraic solutions. The work \[14\] shows that such classification in the case of the sixth Painlevé equation is much more complicated than for other Painlevé equations.

Here we use notation introduced in the previous section, say, $\Psi(\lambda)$ is a fundamental solution of Eq. (3.5) and $\Phi(\mu)$ is the fundamental solutions of Eq. (3.11).

It is convenient to use the following notation for $R$-transformations,

$$R(a_1 + \ldots + a_{n_1}|b_1 + \ldots + b_{n_2}|c_1 + \ldots + c_{n_3}),$$

where $\{a_p\}_{p=1}^{n_1}$, $\{b_q\}_{q=1}^{n_2}$, and $\{c_r\}_{r=1}^{n_3}$ are the sets of integers denoting multiplicities of images of the points $\lambda = 0$, 1, and $\infty$ respectively. It is clear that

$$\text{rank}(R) = \sum_{1}^{n_1} a_p = \sum_{1}^{n_2} b_q = \sum_{1}^{n_3} c_r. \quad (4.12)$$

We denote compositions of $R$ and $S$ transformations as $RS_k(m)$; this symbol stands as a general notation for transformations from equation with $m$ singular points into equation with $k$ singular points. For particular transformations of this kind we use notation $RS_k(\ldots|\ldots|\ldots)$ where the space inside the brackets is separated by the vertical lines on $m$ boxes. Each box contains a partition of the rank($R$) into the sum of integers as it is explained for Eq. (4.12). We define $\text{rank}(RS_k) = \text{rank}(R)$.

Consider first a very simple example, $RS_4(3|2 + 1|1 + 1 + 1)$. It means that

$$\mu = \frac{\rho(\lambda - a)^3}{\lambda(\lambda - 1)}, \quad \mu - 1 = \frac{\rho(\lambda - b)^2(\lambda - t)}{\lambda(\lambda - 1)},$$

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where 
\[ \rho = \frac{(2b - 1)^3}{27b^2(b - 1)^2}, \quad a = \frac{-b(b - 2)}{2b - 1}, \quad t = \frac{-b(b - 2)^3}{(2b - 1)^3}. \]

Solution of the system (3.5), (3.7) reads,
\[ \Psi(\lambda) = \left( J_a \sqrt{\frac{\lambda - a}{\lambda - b}} + J_b \sqrt{\frac{\lambda - b}{\lambda - a}} \right) \Phi(\mu), \]
where
\[ J_a = \begin{pmatrix} \beta & -\beta \\ \alpha & -\alpha \end{pmatrix}, \quad J_b = \begin{pmatrix} -\alpha & \beta \\ -\alpha & \beta \end{pmatrix}, \]
and \( \Phi(\mu) \) solves Eq. (3.11) with
\[ \delta = \frac{2}{3}, \quad \alpha = \frac{1}{6} - \beta. \]

Note that
\[ J_a^2 = J_a, \quad J_b^2 = J_b, \quad J_aJ_b = J_bJ_a = 0, \quad J_a + J_b = I. \]

Eigenvalues of the matrices \( 2A_k \) in Eq. (3.5), denoted \( \pm \theta_k \), are
\[ \theta_t = \frac{1}{2}, \quad \theta_0 = \theta_1 = -\theta_\infty = 2\beta - \frac{1}{6}. \]

Algebraic solutions of the sixth Painlevé equation (3.9) and (3.10) are as follows,
\[ y_{12}(t) = \frac{(b - 2)^2(6\beta(b + 1) + 1 - 2b)}{(2b - 1)(6\beta(b + 1)(b - 2) - 5 + 2b - 2b^2)}, \quad (4.13) \]
\[ y_{21}(t) = \frac{(b - 2)^2(6\beta(b + 1) + b - 2)}{(2b - 1)(6\beta(b + 1)(b - 2) + 7 - b + b^2)}, \quad (4.14) \]
\[ \sigma(t) = \frac{2(12\beta - 1)^2b^3(b - 2) + (6\beta^2 - \beta)(180b^2 - 132b + 30) + 12b^2 - 10b - 1}{72(2b - 1)^3}, \quad (4.15) \]
\[ \tau(t) = C \left( \frac{2b - 1}{b(b - 1)} \right)^{\frac{\beta - \frac{1}{12}}{2\beta - \frac{1}{12}} - \frac{1}{2\beta}}, \quad (4.16) \]
where \( C \) is independent of \( b \). In the derivation of formulae (1.13) – (1.16) it is supposed that \( \beta \neq \frac{1}{12} \); nevertheless, only minor modifications are required for \( \beta = \frac{1}{12} \) (\( \theta_\infty = 0 \)), in particular, the functions (1.13) – (1.16) satisfy corresponding ODEs for all complex \( \beta \).

Another transformation we consider here is a bit more complicated, \( RS_4(2+1+1|3+1|2+2) \). In this case,
\[ \mu = \frac{\rho\lambda_{1}(\lambda_{1} - 1)(\lambda_{1} - a)^2}{(\lambda_{1} - b)^2(\lambda_{1} - c)^2}, \quad \mu - 1 = \frac{(\rho - 1)(\lambda_{1} - d)^3(\lambda_{1} - c)}{(\lambda_{1} - b)^2(\lambda_{1} - c)^2}, \]
where \( \lambda_1 = \frac{\epsilon \lambda}{\alpha + \epsilon - 1} \), \( a, c, e \) and \( \rho \) are the following functions of complex parameters \( b \) and \( d \):

\[
\begin{align*}
a &= -\frac{d(-d^2 - 4bd^2 + 8b^2d^2 + 6bd - 12b^2d + 3b^2)}{4d^3 - 8bd^3 - 3d^2 + 12bd^2 - 6bd + b^2}, \\
c &= \frac{(4bd - b - 3d)d}{8bd^2 - 8bd + 3b - 4d^2 + d^3}, \\
e &= -\frac{b^2(d - 1)(4bd - b - 3d)^2}{9b^2d + b^4 - 12b^5d - 27bd^2d^2 + d^3 - 10bd^3 - 16b^3d^3 + 24b^3d^2 + 24b^2d^3 + 6bd^2}, \\
\rho &= \frac{(4d^3 - 8bd^3 - 3d^2 + 12bd^2 - 6bd + b^2)^2}{d(d - 1)(8bd^2 - 8bd + 3b - 4d^2 + d)^2}.
\end{align*}
\]

Now we construct solution of the system (3.5), (3.7) with the parameter \( \Phi(\mu) \):

\[
t = -\frac{(b - 1)^2d^2(4bd - d - 3b)^2(-d^2 - 4bd^2 + 8b^2d^2 + 6bd - 12b^2d + 3b^2)}{(2bd - b - d)^3(b - d)^3},
\]

as follows,

\[
\Psi(\lambda) = D^{-1}S^{-1}(d, c)S^{-1}(d, b)\Phi(\mu),
\]

where: \( \Phi(\mu) \) solves Eq. (3.11) with

\[
\delta = \frac{\alpha}{6} + 2\alpha, \quad \beta = \frac{1}{2} + \alpha;
\]

\[
S(x, y) = J_+ \sqrt{\frac{\lambda_1 - y}{\lambda_1 - x}} + J_- \sqrt{\frac{\lambda_1 - x}{\lambda_1 - y}}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad J_- = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix};
\]

\[
D = \begin{pmatrix} 1 + \det D & 1 \\ 1 & 1 \end{pmatrix},
\]

\[
\det D = -\frac{2b(b - 1)(b - d)(2bd - b - d)(4bd - 3b - d)(4bd - b - 3d)}{\alpha(6\alpha - 1)(8bd^2 - 8bd + 3b - 4d^2 + d)(8b^2d^2 - 8b^2d - 8bd^2 + 6bd + b^2 + d^2)^2}.
\]

Note, that \( J_\pm \) are the orthogonal projectors,

\[
J_+^2 = J_+, \quad J_-^2 = J_-, \quad J_+J_- = J_-J_+ = 0, \quad J_+ + J_- = I.
\]

Introducing parameter

\[
s = \frac{d - b}{d(b - 1)} + 1,
\]

one rewrites Eq. (4.17) as

\[
t = \frac{(3s + 1)^2(3s^2 + 6s - 1)}{(s^2 - 1)^3}.
\]

Solutions (3.9) and (3.10) of the sixth Painlevé equation corresponding to the parameters:

\[
\theta_\infty = \frac{2}{3}, \quad \theta_0 = \theta_1 = \frac{1}{2}, \quad \theta_t = 2\alpha - \frac{1}{6},
\]

(4.19)
are as follows
\[ y_{12}(t) = \frac{(3s + 1)(3s^2 + 6s - 1)}{(s^2 - 1)(s^2 + 6s + 1)}, \]

\begin{align*}
y_{21}(t) &= \frac{y_{12}(t)}{Q(s, \alpha)} (3 + 6\alpha + 72\alpha s + (228\alpha - 42)s^2 + (72\alpha - 24)s^3 + (6\alpha - 1)s^4) \\
&\times (-2 + 3\alpha + (36\alpha - 6)s + (114\alpha + 2)s^2 + (36\alpha + 6)s^3 + 3\alpha s^4), \quad (4.20) \\
Q(s, \alpha) &= (31212\alpha^2 - 5202\alpha - 2164)s^4 + (16848\alpha^2 - 2808\alpha - 996)(s^5 + s^3) \quad (4.21) \\
&+ (3960\alpha^2 - 660\alpha)(s^6 + s^2) + (432\alpha^2 - 72\alpha + 36)(s^7 + s) \\
&+ (18\alpha^2 - 3\alpha - 6)(s^8 + 1).
\end{align*}

Note, that solution \( y_{12}(t) \) solves Eq. (3.8) for arbitrary complex \( \alpha \), whilst it does not depend on \( \alpha \)! This means that \( y_{12}(t) \) solves the following algebraic equation,

\[ \frac{t - 1}{(y_{12} - 1)^2} = \frac{t}{y_{12}^2} + \frac{4(t - 1)}{(y_{12} - t)^2}. \]

The functions \( \sigma(t) \) and \( \tau(t) \) corresponding to the parameters given by Eq. (4.13) are as follows,

\[ \sigma(t) = \frac{s^2 + 6s + 1}{72(s^2 - 1)^3}((1 - 36\alpha + 216\alpha^2)(1 + s^4) - 6(s + s^3) - (22 - 72\alpha + 432\alpha^2)s^2), \]

\[ \tau(t) = C(s^2 - 1)^{\frac{1}{12}} \frac{(3s^2 + 6s - 1)(s^2 - 6s - 3)}{(s(s + 3)(3s + 1)}(\frac{(12\alpha - 1)^2}{24} - \frac{1}{18}). \]

We would like to note that whilst the construction of the function \( \Psi(\lambda) \) should be modified for some special values of \( \alpha \) (see, e.g. Eq. (4.18)), the formulae obtained for the functions \( y_{12}(t), y_{21}(t), \sigma(t) \) and \( \tau(t) \) remain valid for all \( \alpha \in \mathbb{C} \). Since the formula (4.20) is quite complicated, we write below the function \( y_{21}(t) \) for some particular values of \( \alpha \):

\[ \alpha = 0 \text{ or } \alpha = \frac{1}{6}, \quad y_{21}(t) = \frac{(3s + 1)^2(3s^2 + 6s - 1)(s^4 + 24s^3 + 42s^2 - 3)}{(s^2 + 6s + 1)(1082s^4 + 498(s^5 + s^3) - 18(s^7 + s) + 3(s^8 + 1))}, \]

\[ \alpha = -\frac{1}{12}, \quad y_{21}(t) = \frac{(3s + 1)(s^2 + 18s + 5)}{5(s^2 + 6s + 1)(s^2 - 1)}, \]

\[ \alpha = \frac{1}{12}, \quad y_{21}(t) = \frac{(3s + 1)(3s^2 + 6s - 1)(s^4 + 36s^3 + 46s^2 - 12s - 7)^2}{(s^2 - 1)(s^2 + 6s + 1)Q(s, 1/12)}, \]

\[ Q(s, 1/12) = 19046s^4 + 8904(s^5 + s^3) + 220(s^6 + s^2) - 264(s^7 + s) + 49(s^8 + 1). \]

5 \( RS_4(3) \) of Rank 2

There are only two (modulo fractional-linear transformations \( RS_3(3) \) and \( RS_4(3) \) of rank 1) transformations \( RS_4(3) \) of rank 2. One of them generates an algebraic solution of the sixth Painlevé equation, whilst another gives a simplest example of the points on the complex \( t \) plane, different from 0, 1, and \( \infty \), for which there exist transcendental solutions of the sixth Painlevé equation whose monodromy data of the associated Eq. (3.3) can
be calculated explicitly in terms of their expansions at these points. More sophisticated
examples (generated by RS-transformations of the ranks 3 and 4) of the points with such
property are given in [10].

It is noticed by many authors that \( y(t) = \pm \sqrt{t} \) is a solution of Eq. (3.8) provided the
parameters satisfy the following relations:

\[
(\theta_{\infty} - 1)^2 = \theta_0^2, \quad \theta_1^2 = \theta_t^2.
\]

Here we show that this algebraic solution is generated by transformation \( RS_4(2|1+1|1+1) \); the first transformation which is mentioned in the previous paragraph. Though this
transformation does not lead to any new solutions the construction given below can be
valuable for applications.

\[
\mu = \frac{(\lambda - \sqrt{t})^2}{(1 - \sqrt{t})^2 \lambda}, \quad \mu - 1 = \frac{(\lambda - 1)(\lambda - t)}{(1 - \sqrt{t})^2 \lambda}.
\]

The solution of the system (3.5), (3.7) reads:

\[
\Psi(\lambda) = \begin{pmatrix} J_0 \sqrt{\frac{\lambda}{\lambda - \sqrt{t}}} + J_{\sqrt{t}} \sqrt{\frac{\lambda - \sqrt{t}}{\lambda}} \end{pmatrix} \Phi(\mu),
\]

where \( \Phi(\mu) \) solves Eq. (3.11) with \( \delta = 1/2 \) and

\[
J_0 = \begin{pmatrix} 0 & 0 \\ -\alpha/\beta & 1 \end{pmatrix}, \quad J_{\sqrt{t}} = \begin{pmatrix} 1 & 0 \\ \alpha/\beta & 0 \end{pmatrix}
\]

are the orthogonal projectors:

\[
J_0^2 = J_0, \quad J_{\sqrt{t}}^2 = J_{\sqrt{t}}, \quad J_0 J_{\sqrt{t}} = J_{\sqrt{t}} J_0 = 0, \quad J_0 + J_{\sqrt{t}} = I.
\]

Solutions (3.9) and (3.10) of Eq. (3.8) corresponding to the parameters:

\[
\theta_{\infty} = \alpha - \beta, \quad \theta_0 = \alpha - \beta - 1, \quad \theta_1 = \theta_t = \alpha + \beta + \frac{1}{2}, \quad (5.22)
\]

are as follows:

\[
y_{12}(t) = -\sqrt{t}, \quad y_{21}(t) = -\sqrt{t} \frac{\alpha(1 + 2\beta)(1 + t) - 2\sqrt{t}(\alpha^2 + \beta^2 + \beta)}{\beta(1 + 2\alpha)(1 + t) - 2\sqrt{t}(\alpha^2 + \beta^2 + \alpha)}.
\]

The functions \( \sigma(t) \) and \( \tau(t) \) corresponding to the parameters (5.22) read:

\[
\sigma(t) = \frac{1}{2} (\beta^2 + \alpha^2 + \beta + \frac{1}{3} - \alpha(1 + 2\beta)) - \alpha(1 + 2\beta) \sqrt{t} - \frac{t}{16},
\]

\[
\tau(t) = C t^{-\frac{1}{16}} \left( 1 - \frac{1}{t} \right)^{\frac{1}{2}(\alpha^2 + \beta^2 + \beta + \frac{1}{3})} \left( 1 + \frac{1}{t} \right)^{\alpha(1 + 2\beta)}.
\]

The second transformation mentioned in the beginning of this section is \( RS_4(2|1+1|2) \). Actually, this transformation maps three points into four ones; therefore there is no need
to use S-transformations, and it coincides with \( R_4(2|1+1|2) \). The transformation reads,

\[
\Psi(\lambda) = \Phi(\mu), \quad \mu = \lambda^2.
\]
The function $\Psi$ solves the following equation:

$$
\frac{d\Psi}{d\lambda} = \left( \frac{2\hat{A}}{\lambda} + \frac{\hat{B}}{\lambda - 1} + \frac{\hat{B}}{\lambda + 1} \right) \Psi, \quad (5.23)
$$

The monodromy data of Eq. (5.23) can be calculated in terms of the monodromy data
of the hypergeometric equation (3.11), i.e., in terms of matrix elements of $\hat{A}$ and $\hat{B}$. The matrix elements of Eq. (5.23) can be viewed as the initial data at $t = -1$ for the
system of Schlesinger equations (3.6). In the general case, $\theta_\infty \neq \pm 1$, both solutions
of Eq. (3.8), $y_{12}(t)$ and $y_{21}(t)$, which corresponds to this deformation, has a pole at
$t = -1$. By using parameterization of the Schlesinger equations in terms of the solutions
of the sixth Painlevé equation given in [2] one can completely determine corresponding
Laurent expansions of $y_{12}(t)$ and $y_{21}(t)$ at $t = -1$. Note, that the parameters of formal
monodromy corresponding to this deformation are

$$
\theta_\infty = \beta - \alpha \neq \pm 1, \quad \theta_0 = 1 - \delta, \quad \theta_1 = \theta_t = \beta + \alpha + 1 - \delta. \quad (5.24)
$$

Now, using transformation for the solutions of Eq. (3.8):

$$
y(t) = 1/\hat{y}(\hat{t}), \quad t = 1/\hat{t}, \quad \alpha_6 = -\hat{\alpha}_6, \quad \beta_6 = -\hat{\beta}_6, \quad \gamma_6 = \hat{\gamma}_6, \quad \delta_6 = \hat{\delta}_6, \quad (5.25)
$$

we get two solutions of Eq. (3.8), which are holomorphic at $t = -1$, actually $\hat{y}_{12}(-1) = \hat{y}_{21}(-1) = 0$, and due to the explicit formula (5.25) relating them with solutions $y_{12}(t)$ and $y_{21}(t)$ we can by means of the work [3] find asymptotics of $\hat{y}_{12}(\hat{t})$ as $\hat{t} \to 0, 1, \infty$. It is also easy to find an action of the involution (5.25) on the $\Psi$-function. Therefore, one can find explicitly initial (and monodromy) data of the Schlesinger system corresponding
to both solutions $\hat{y}_{12}(\hat{t})$ and $\hat{y}_{21}(\hat{t})$. By fractional-linear transformation of $\lambda$ preserving
the set of points $\{0, 1, \infty\}$ ($R_4(4)$ in our notation) the point $-1$ can be mapped to the
points $1/2$ and $2$. So, that these points have the same property as $-1$, i.e., for each
point there exist at least two transcendental solutions of Eq. (3.8), with the coefficients
defined by Eq. (5.24) with the poles and two solutions with the zeros at this point, such
that their monodromy data can be calculated explicitly in terms of their Laurent or,
respectively, Taylor expansions at this point. The word transcendental here means that
these solutions are neither algebraic nor classical (in the Umemura sense) functions.

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