AN IMPROVED ALGORITHM FOR GENERAL POSITION SUBSET SELECTION

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Abstract. In the General Position Subset Selection (GPSS) problem, the goal is to find the largest possible subset of a set of points, such that no three of its members are collinear. If $s_{\text{GPSS}}$ is the size the optimal solution, $\sqrt{s_{\text{GPSS}}}$ is the current best guarantee for the size of the solution obtained using a polynomial time algorithm. In this paper we present an algorithm for GPSS to improve this bound based on the number of collinear pairs of points.

Keywords: General Position Subset Selection, Collinearity testing, Computational geometry

1. Introduction

A subset of a set of $n$ points in the plain is in general position if no three of its points are on the same line. The NP-complete General Position Subset Selection (GPSS) problem asks for the largest possible such subset. This problem, the fame of which is partly due to the fact that several algorithms in computational geometry assume that their input points are in general position, has received relatively little attention in its general setting. The well-known No-Three-In-Line problem, which is a special case of GPSS, asks for the maximum number of points, no three of which are collinear in an $n \times n$ grid. A lower bound of $(3/2 - \epsilon)n$ was proved for this problem [1] and it is conjectured that the best lower bound for this problem for large $n$ is $cn$ [2], in which $c$ is $\pi \sqrt{3}$. This problem has also been extended to three dimensions [3].

Lower bounds for GPSS were proved by Payne and Wood, for the case in which the number of collinear points are bounded [4]. More precisely, if no more than $l \leq \sqrt{n}$ of input points are collinear, they showed that the size of the largest subset of points in general position is $\Omega(\sqrt{n \log(n)})$. More recently Froese et al. proved that GPSS is NP-complete and APX-hard [5]. They also presented several fixed-parameter tractability results for this problem, including a $15k^3$ kernel.

Point Line Cover (PLC) seeks the minimum number of lines required to cover a set of points. This problem has been minutely studied and approximation algorithms with performance ratio $\log(s_{\text{PLC}})$ has been presented for this problem [6], in which $s_{\text{PLC}}$ is the size of optimal solution to PLC. PLC can be used to prove bounds for GPSS [7]: given that at most two points
can be selected from each line of a line cover, clearly \( s_{\text{GPSS}} \leq 2 \cdot s_{\text{PLC}} \). Also, since \( \binom{s_{\text{GPSS}}}{2} \) lines are defined for a set of \( s_{\text{GPSS}} \) points in general position and since all points outside the optimal solution to \( \text{GPSS} \) should be on at least one such line (due to its maximality), we have \( s_{\text{PLC}} \leq s_{\text{GPSS}}^2 \).

Cao presented a greedy algorithm for \( \text{GPSS} \) as follows [7]. Let \( S \) be an empty set initially. For each point \( p \) in the set of input points \( P \) in some arbitrary order, add \( p \) to \( S \), unless it is on a line formed by the points present in \( S \). It is easy to see that in \( S \) no three points can be collinear. If such a triple is introduced while adding the \( i \)-th point, then the \( i \)-th point itself should be one of the collinear points, which is impossible due the condition for adding vertices to \( S \). On the other hand, due to its incremental construction, \( S \) is maximal and no point in \( P \setminus S \) can be added to \( S \). This algorithm achieves the best known approximation ratio for \( \text{GPSS} \) [5]. Since each point in an optimal solution \( Q \) outside \( S \) cannot be added to \( S \), it should be on a line defined by the points in \( S \) and since there are \( \binom{|S|}{2} \) such lines and on each of these lines at most two points of \( Q \) can appear, \( |Q| \leq |S| + 2\binom{|S|}{2} \). Therefore, this algorithm finds a subset of size at least \( \sqrt{s_{\text{GPSS}}} \).

In this paper, we try to improve this bound by reformulating the problem using graphs and finding maximal independent sets in them. Given a set \( P \) of \( n \) points, the algorithm presented in this paper finds a subset in general position with max\{\( 2n^2 / (\text{coll}(P) + 2n) \), \( \sqrt{s_{\text{GPSS}}} \)\} points, in which \( \text{coll}(P) \) is the number of collinear pairs in lines with at least three points in \( P \) (Theorem 3.3). The paper is organized as follows: in Section 2 we define the notation used in this paper, in Section 3 we describe our algorithm, and finally in Section 4 we conclude this paper.

2. Preliminaries and Notation

Let \( P \) be a set of \( n \) points in the plane. Three or more points of \( P \) are \textit{collinear}, if a line intersects all of them. Let \( L \) be the set of all maximal subsets of collinear points in \( P \). For each point \( p \) in \( P \), let \( L(p) \) be a subset of \( L \) containing those members that contain \( p \). Also, let \( N(p) \) denote the union of all members of \( L(p) \), excluding \( p \) itself. We define \( \text{coll}(p) \) as the size of \( N(p) \), \( \text{coll}(Q) \) for a subset \( Q \) of \( P \) as the sum of \( \text{coll}(q) \) for every \( q \) in \( Q \), and \( \overline{\text{coll}}(Q) \) as the average value of \( \text{coll}(q) \) for every point \( q \) in \( Q \).

A subset \( Q \) of \( P \) is \textit{noncollinear}, if for every \( q \) in \( Q \), no point in \( L(q) \) is present in \( Q \). The \textit{collinearity graph} \( G \) of a set of \( n \) points \( P \), is the graph that has a vertex for each point in \( P \); in this paper we use the same symbol to represent a point in \( P \) and its corresponding vertex in \( G \). Two vertices \( p \) and \( q \) are adjacent in \( G \), if and only if \( p \) is in \( N(q) \). It can be observed that the degree of each vertex \( p \) in \( G \) equals \( \text{coll}(p) \).

3. Main Result

Before describing our algorithm, we present two lemmas as follows.
Lemma 3.1. From a set of \( n \) points \( P \), two disjoint noncollinear subsets \( R \) and \( T \) can be selected such that \( |R| + |T| \geq 2n / (\text{coll}(P) + 2) \).

Proof. Let \( G \) be the collinearity graph of \( P \). The vertices of \( G \) can be decomposed into two subgraphs \( P_1 \) and \( P_2 \) such that at least half of the edges of \( G \) have one endpoint in \( P_1 \) and one in \( P_2 \). Let \( P_1 \) and \( P_2 \) be such a decomposition and \( H \) be the graph obtained from \( G \) by removing all edges between \( P_1 \) and \( P_2 \). Clearly, since the number of the edges of \( H \) is at most half of that of \( G \), the average degree of \( H \) is also at most half of the average degree of \( G \).

Let \( S \) be an independent set in \( H \) and let \( R = S \cap P_1 \) and \( T = S \cap P_2 \). Both \( R \) and \( T \) are independent sets in \( H \). Since no edge between the vertices of \( P_1 \) (and hence \( R \)) is removed in \( H \), \( R \) is also an independent set in \( G \). A similar argument shows that \( T \) is also an independent set in \( G \).

Turan’s lower bound of \( n / (\overline{d} + 1) \) for the size of the independent in a graph with \( n \) vertices, in which \( \overline{d} \) is the average degree of the graph, can be attained using the greedy algorithm that iteratively selects vertices ordered increasingly by their degrees and removes the selected vertex and its neighbours [8]. Since \( \overline{d} \leq \text{coll}(P) / 2 \), implying that \( |S| \geq 2n / (\text{coll}(P) + 2) \), as required. □

The lower bound for the greedy algorithm used in Lemma 3.1 is not the best possible; it is actually a weaker form of the celebrated Caro-Wei lower bound, which has been improved by several authors (see, for instance, [8] and [9]). However, Turan’s bound, which is tight for graphs consisting of disjoint cliques, depends only on the average degree and yields a cleaner bound for the size of noncollinear sets in Lemma 3.1.

Lemma 3.2. Let \( R \) and \( T \) be two noncollinear subsets of a set of points \( P \). Then, the points in \( R \cup T \) are in general position.

Proof. Since, for each line with at least three points in \( P \), each of \( R \) and \( T \) can include at most one point, in their union there are at most one point on each line. □

We now present our main algorithm for finding a subset of a set of points \( P \) in general position: i) The set \( L \) of all maximal collinear subsets of \( P \) can be identified using a polynomial time algorithm. ii) Based on Lemma 3.1, two disjoint noncollinear subsets \( R \) and \( T \) of \( P \) can be obtained; Define \( S \) as their union. By Lemma 3.2, \( S \) is in general position. iii) For each point \( p \) in \( P \setminus S \), add \( p \) to \( S \), unless its addition to \( S \) results in three collinear points in \( S \).

Theorem 3.3. For a set of \( n \) points in the plane, it is possible to find a subset in general position, the size of which is at least \( \max\{n^2 / (\text{coll}(P) + 2), \sqrt{s_{GPSS}}\} \).

Proof. In the second step of the above algorithm, \( S \) has at least \( 2n / (\text{coll}(P) + 2) \) points, based on Lemma 3.2. Since \( \text{coll}(P) = n \cdot \overline{\text{coll}}(P) \), the minimum
size of $S$ can be rewritten as $2n^2 / (\text{coll}(P) + 2n)$. In the third step, $S$ is made maximal. By the same argument mentioned in the introduction for the greedy GPSS algorithm, it can be shown that after this step the size of $S$ is at least $\sqrt{s_{GPSS}}$. □

4. Concluding Remarks

GPSS can also be formulated in terms of Hypergraph strong independence. However, it is very surprising that the extensive studies on independence number of hypergraphs are mostly focused on weak independence (see [10] for a summary), in which the independent set can include any but not all of the vertices of each edge. Nevertheless, it would be interesting to investigate the following question.

**Question 4.1.** Given a linear hypergraph (in which every pair of edges share at most one vertex), is it possible to map its vertices into points on the two-dimensional plane, such that three of these points are collinear if and only if their corresponding vertices are on the same edge of the hypergraph?

Also note that the algorithm presented here can be extended for the problem of excluding four points on a circle by modifying Lemma 3.1 to extract three independent set.

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