Abstract. Let $P$ be a polyhedron, defined by a system $Ax \leq b$, where $A \in \mathbb{Z}^{m \times n}$, $\text{rank}(A) = n$, and $b \in \mathbb{Z}^m$. In the Integer Feasibility Problem, we need to decide whether $P \cap \mathbb{Z}^n = \emptyset$ or to find some $x \in P \cap \mathbb{Z}^n$ in the opposite case. Currently, its state of the art algorithm, due to [9,11] (see also [7,18,34] for more general formulations), has the complexity bound $O(n^\omega \cdot \text{poly}(\phi))$, where $\phi = \text{size}(A, b)$. It is a long-standing open problem to break the $O(n^\omega)$ dimension-dependence in the complexity of ILP algorithms.

We show that if the matrix $A$ has a small $l_1$ or $l_\infty$ norm, or $A$ is sparse and has bounded elements, then the integer feasibility problem can be solved faster. More precisely, we give the following complexity bounds

$$\min\{\|A\|_\infty, \|A\|_1\}^{n^5} \cdot 2^n \cdot \text{poly}(\phi),$$
$$\left(\|A\|_{\max}\right)^{5n} \cdot \min\{\text{cs}(A), \text{rs}(A)\}^{3n} \cdot 2^n \cdot \text{poly}(\phi).$$

Here $\|A\|_{\max}$ denotes the maximal absolute value of elements of $A$, $\text{cs}(A)$ and $\text{rs}(A)$ denote the maximal number of nonzero elements in columns and rows of $A$, respectively. We present similar results for the integer linear counting and optimization problems.

Additionally, we apply the last result for multipacking and multicover problems on graphs and hypergraphs, where we need to choose a minimal/maximal multiset of vertices to cover/pack the edges by a prescribed number of times. For example, we show that the stable multiset and vertex multicover problems on simple graphs admit FPT-algorithms with the complexity bound $2^{O(|V|)} \cdot \text{poly}(\phi)$, where $V$ is the vertex set of a given graph.
Keywords: Integer Linear Programming · Counting Problem · Parameterized Complexity · Multipacking · Multicover · Stable Set · Vertex Cover · Dominating Set · Multiset Multicover · Fixed Dimension · Sparse Matrix

1 Introduction

1.1 Basic Definitions and Notations

Let $A \in \mathbb{Z}^{m \times n}$ be an integer matrix. We denote by $A_{ij}$ the $ij$-th element of the matrix, by $A_{i*}$ its $i$-th row, and by $A_{*j}$ its $j$-th column. For subsets $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$, the symbol $A_{I,J}$ denotes the sub-matrix of $A$, which is generated by all the rows with indices in $I$ and all the columns with indices in $J$. If $I$ or $J$ is replaced by $*$, then all the rows or columns are selected, respectively. Sometimes, we simply write $A_I$ instead of $A_{I,*}$ and $A_J$ instead of $A_{*,J}$, if this does not lead to confusion.

We denote by $I_{n \times n}$, $1_{m \times n}$, and $0_{m \times n}$ the $n \times n$ identity matrix, the $m \times n$ all-one matrix, and the $m \times n$ all-zero matrix, respectively. Sometimes, we will ignore subscripts, if this does not lead to confusion. The same notation is used for all-one and all-zero vectors. The $i$-th column of the identity matrix $I_{n \times n}$ is denoted by $e_i$. It represents the $i$-th vector of the standard basis in $\mathbb{R}^n$.

The maximum absolute value of entries of a matrix $A$ (also known as the matrix max-norm) is denoted by $\|A\|_{\max} = \max_{i,j} |A_{ij}|$. The $l_p$-norm of a vector $x$ is denoted by $\|x\|_p$. The number of non-zero components of a vector $x$ is denoted by $\|x\|_0 = |\{i: x_i \neq 0\}|$. For a matrix $A$, by $\|A\|_p$ we denote the standard matrix norm, induced by the $l_p$ vector norm. It is known that

$$
\|A\|_1 = \max_i \|A_{i*}\|_1 = \max_i \sum_j |A_{ij}| \quad \text{and} \quad \|A\|_\infty = \max_j \|A_{*j}\|_1 = \max_j \sum_i |A_{ij}|.
$$

The unit ball with respect to the $l_p$-norm is denoted by

$$
\mathbb{B}_p = \{x \in \mathbb{R}^n: \|x\|_p \leq 1\}.
$$

The maximal number of non-zero elements in rows and columns of the matrix $A$ is denoted by

$$
\text{rs}(A) := \max_i \|A_{i*}\|_0 \quad \text{and} \quad \text{cs}(A) := \max_j \|A_{*j}\|_0.
$$

Definition 1. For a matrix $A \in \mathbb{Z}^{m \times n}$, by

$$
\Delta_k(A) = \max \{\det(A_{I,J}) : I \subseteq \{1, \ldots, m\}, J \subseteq \{1, \ldots, n\}, |I| = |J| = k\},
$$

itself is a key computational problem.
we denote the maximum absolute value of determinants of all the $k \times k$ submatrices of $A$. The matrix $A$ with $\Delta(A) \leq \Delta$, for some $\Delta > 0$, is called $\Delta$-modular. Note that $\Delta_1(A) = \|A\|_{\text{max}}$.

**Definition 2.** For a matrix $B \in \mathbb{R}^{m \times n}$, $\text{cone}(B) = \{Bt : t \in \mathbb{R}^n_+\}$ is the cone, spanned by columns of $B$.

### 1.2 The Problems Under Consideration and Motivation of This Paper

Let $P$ be a polyhedron, defined by a system $Ax \leq b$, where $A \in \mathbb{Z}^{m \times n}$, $\text{rank}(A) = n$, and $b \in \mathbb{Z}^m$. Let, additionally, $\Delta = \Delta(A)$. We consider the following problems:

**Problem 1 (Feasibility).** Find a point $x$ inside $P \cap \mathbb{Z}^n$ or declare that $P \cap \mathbb{Z}^n = \emptyset$.

**Problem 2 (Counting).** Compute the value of $|P \cap \mathbb{Z}^n|$ or declare that $|P \cap \mathbb{Z}^n| = +\infty$.

**Problem 3 (Optimization and Counting).** Given $c \in \mathbb{Z}^n$, compute some $x^* \in P \cap \mathbb{Z}^n$, such that $c^\top x^* = \max\{c^\top x : x \in P \cap \mathbb{Z}^n\}$. Declare, if $P \cap \mathbb{Z}^n = \emptyset$ or if the maximization problem is unbounded. If some $x^*$ exists, compute the number of such $x^*$.

The state of the art algorithm, due to \cite{9,11} (see also \cite{7,18,34} for more general formulations), for the Feasibility and Optimization Problems 1, 3 (optimization without counting) has the complexity bound $O(n^n) \cdot \text{poly}(\phi)$, where $\phi = \text{size}(A, b)$ is the bit-encoding length of the system $Ax \leq b$. It is a long-standing open problem to break the $O(n^n)$ dimension-dependence in the complexity of ILP algorithms. Useful discussion and new ideas on this problem could be found, for example, in \cite{4,9,10}.

The long-standing, asymptotically fastest algorithm for the Counting Problem 2 in fixed dimension is due to A. Barvinok \cite{3,14} (see also \cite{1,2,12}). Another variant of the Barvinok’s algorithm that is more efficient in practise is given in \cite{27}. Different computational experiments and a good survey could be found in \cite{13}. The Barvinok’s algorithm computational complexity can be estimated as

$$\nu \cdot \left(\log_2(\Delta)\right)^{O(n \log n)},$$  \hspace{1cm} (1)

where $\nu$ is the maximal number of vertices in a class of polytopes under consideration. Due to the seminal work of P. McMullen \cite{31}, the value of $\nu$ attains its maximum on the class of polytopes that is dual to the class of cyclic polytopes. Consequently, due to \cite{23} Section 4.7, $\nu = O(m^{n/2})$. Let $\phi$ be the encoding length of the matrix $A$. Due to \cite{22} Chapter 3.2, Theorem 3.2, we have $\Delta \leq 2^{\phi}$. In notation with $\phi$, the bound (2) becomes

$$\left(\frac{m}{n}\right)^{n/2} \cdot \phi^{O(n \log n)},$$  \hspace{1cm} (2)
which gives a polynomial-time algorithm in a fixed dimension for the Counting Problem. The papers \cite{19,20} try to deal with the parameter $\Delta$ to give pseudo-polynomial algorithms, which will be more effective in varying dimension. We emphasise the main result of \cite{19} as the following theorem

\textbf{Theorem 1.} Assume that $\mathcal{P}$ is bounded and $\dim(\mathcal{P}) = n$. Then, the Counting Problem can be solved by an algorithm with the complexity bound

$$O(\nu \cdot n^3 \cdot \Delta^4 \cdot \log_2(\Delta)).$$

Using different ways to estimate $\nu$, the paper \cite{19} gives following complexity bounds for the Counting Problem:

- The bound

$$O\left(\left(\frac{m}{n}\right)^{\frac{n}{\Delta^2}} \cdot n^3 \cdot \Delta^4 \cdot \log(\Delta)\right)$$

that is polynomial on $m$ and $\Delta$, for any fixed $n$. In comparison with the bound \cite{2}, this bound has much better dependence on $n$, considering $\Delta$ as a parameter. For example, taking $m = O(n)$ and $\Delta = 2^{O(n)}$, the above bound becomes $2^{O(n)}$, which is even faster, than the state of the art algorithm for the Feasibility Problem due to \cite{9,11}, with the complexity bound $O(n)^n \cdot \text{poly}(\phi)$.

- The bound

$$O\left(n^{m-n+3} \cdot \Delta^4 \cdot \log(\Delta)\right)$$

that is polynomial on $n$ and $\Delta$, for $m = n + O(1)$. Taking $m = n + 1$, it gives an $O(n^4 \cdot \Delta^4 \cdot \log(\Delta))$-algorithm to compute the number of integer points in a simplex. The last result can be used to count solutions of the Unbounded Subset-Sum problem, which is formulated as follows. Given numbers $c_1, \ldots, c_n$ and $C$, we need to count the number of ways to exchange the value $C$ by an unlimited number of coins with costs $c_1, \ldots, c_n$. Since the problem’s polyhedron is simplex, it can be done by an algorithms with the arithmetical complexity bound

$$O\left(n^4 \cdot c_{\text{max}}^4 \cdot \log(c_{\text{max}})\right).$$

For the best of our knowledge, it is the first FPT-algorithm for this problem parameterised by $c_{\text{max}}$, which is the maximal coin cost.

- The bound

$$O(n)^{3+\frac{4}{\Delta^2}} \cdot \Delta^{4+n} \cdot \log(\Delta)$$

that is polynomial on $\Delta$, for any fixed $n$, and depends only from two parameters. Taking $\Delta = O(1)$, the last bound becomes $O(n)^{3+\frac{4}{\Delta^2}}$, which again gives a faster algorithm for the ILP feasibility problem, than the state of the art algorithm, due to \cite{9,11}. 


In the current work, we try to estimate the value of $\nu$ by a different way, to handle ILP problems with sparse matrices. Next, we apply the result for sparse matrices to combinatorial packing and cover problems on graphs and hypergraphs. For example, we give $2^{O(|V|)} \cdot poly(\phi)$-complexity FPT-algorithms for the Stable Multiset and Vertex Multicover Problems in simple graphs, which are natural generalisations of the original Stable Set and Vertex Cover problems, where $V$ is a set of vertices. The Stable Multiset Problem was introduced in [28]. Some properties of the Stable Multiset Problem polyhedron were investigated in [29,30], which had given a way to construct effective branch & bound algorithms for this problem. We could not find a reference on the paper that introduces the Vertex Multicover problem, but this problem can be interpreted as a blocking problem for the Stable Multiset problem (introduction to the theory of blocking and antiblocking can be found in [15,16], see also [22, p. 225]).

To handle both cover and packing problems on hypergraphs, it is convenient to introduce the following hybrid problem:

**Problem 4 (Hypergraph Packing/Cover).** Let $\mathcal{H} = (\mathcal{E}, \mathcal{F})$ be a set system. Given numbers $c_F, p_F \in \mathbb{N}$, for $F \in \mathcal{F}$, and numbers $u_e \in \mathbb{N}$, for $e \in \mathcal{E}$, compute a multisubset of $\mathcal{E}$, represented by natural numbers $x_e \leq u_e$, for $e \in \mathcal{E}$, such that

(i) $c_F \leq x(F) \leq p_F$, for any $F \in \mathcal{F}$;

(ii) $x(\mathcal{E}) \to \max/\min$.

Here, $x(\mathcal{M}) = \sum_{e \in \mathcal{M}} x_e$, for any $\mathcal{M} \subseteq \mathcal{E}$.

In other words, we need to solve the following ILP:

$$
\begin{align*}
\mathbf{1}^\top x \to & \max/\min \\
& \begin{cases} 
    c \leq A(\mathcal{H})^\top x \leq p \\
    0 \leq x \leq u \\
    x \in \mathbb{Z}^{|\mathcal{E}|},
\end{cases} \\
\end{align*}
$$

where $A(\mathcal{H})$ denotes the vertex-edge incidents matrix of $\mathcal{H}$, and the vectors $c$, $p$ and $u$ are composed of the values $p_F, c_F$, and $u_e$, respectively.

If $c_F = -\infty$, for all $F \in \mathcal{F}$, and $x(\mathcal{E})$ is maximized, it can be considered as the **Stable Multiset Problem on Hypergraphs**, when we need to find a multiset of vertices of the maximum size, such that each hyperedge $F \in \mathcal{F}$ is packed at most $p_F$ times.

Analogously, if $p_F = +\infty$, for all $F \in \mathcal{F}$, and $x(\mathcal{E})$ is minimized, it can be considered as the **Vertex Multicover Problem on Hypergraphs**, when we need to find a multiset of vertices of the minimum size, such that each hyperedge $F \in \mathcal{F}$ is covered at least $c_F$ times.

For the case, when $\mathcal{H}$ presents simple graphs, these problems can be considered as very natural generalizations of the classical Stable Set and Vertex Cover problems. Following to [28], the first one is called the **Stable Multiset Problem**. As it was previously discussed, it is natural to call the second problem as the **Vertex Multicover Problem**.
Additionally, let us consider the *Multiset Multicover Problem*. This problem has received quite a lot of attention in recent papers [4,6,17,24,25,26]. An exact \((c_{\text{max}} + 1)^n \cdot \text{poly}(\phi)\)-complexity algorithm for this problem, parameterized by the universe size \(n\) and the maximum coverage constraint number \(c_{\text{max}}\), is given in [24,25]. A double exponential \(2^{2O(n \log n)} \cdot \text{poly}(\phi)\)-complexity FPT-algorithm, parameterized by \(n\), is given in [5]. The last algorithm was improved to an \(n^{O(n^2)} \cdot \text{poly}(\phi)\)-complexity algorithm in [26]. A polynomial-time approximate algorithm could be found in [17]. The papers [33,6] give good surveys and contain new ideas to use ILP theory in combinatorial optimization setting.

**Problem 5 (Multiset Multicover).** Let \(H = (E,F)\) be a set system. Given numbers \(c_e \in \mathbb{N}\), for \(e \in E\), and numbers \(u_F \in \mathbb{N}\), for \(F \in F\), compute a multisubset of \(F\), represented by the natural numbers \(x_F \leq u_F\), for \(F \in F\), such that

(i) \(x(\delta(e)) \leq c_e\), for any \(e \in E\);

(ii) \(x(F) \to \min\).

Here, \(x(M) = \sum_{e \in M} x_e\), for any \(M \subseteq E\), and \(\delta(e) = \{F \in F : e \in F\}\) denotes the set of hyperedges that are incident to the vertex \(e\).

In other words, we need to solve the following ILP:

\[
\begin{aligned}
\mathbf{1}^\top x &\to \min \\
A(H)x &\geq c \\
0 &\leq x \leq u \\
x &\in \mathbb{Z}^{\left|F\right|},
\end{aligned}
\]

where the vectors \(c\) and \(u\) are composed of the values \(c_e\) and \(u_F\), respectively.

By analogous way, we can introduce the *Dominating Multiset Problem* on simple graphs, which is a natural generalization of the classical *Dominating Set Problem*. In this problem, we need to find a multiset of vertices of the minimal size, such that all vertices of a given graph will be covered given number of times by neighbors of the constructed multiset. The Dominating Multiset Problem can be straightforwardly reduced to the Multiset Multicover Problem. To do that, we just need to construct the set system \(H = (E,F)\), where \(E\) coincides with the set of vertices of a given graph, and \(F\) is constituted by neighbors of its vertices.

Let us explain our motivation with respect to combinatorial problems. The classical Stable Set and Vertex Cover Problems on graphs and hypergraphs admit trivial \(2^{O(n)}\)-arithmetical complexity algorithms. However, the Stable Multiset and Vertex Multicover Problems does not admit such a trivial algorithm. But, both problems can be modeled as ILP problem (4) with \(n\) variables. Consequently, both problems can be solved by the previously mentioned \(O(n)^n \cdot \text{poly}(\phi)\)-complexity general ILP algorithm.

We ask, is it possible to give a faster algorithm? And give a positive answer to this question. We show that these problems on hypergraphs can be solved by a \(\min\{d, r\}^{O(n)} \cdot \text{poly}(\phi)\)-complexity algorithm, where \(d\) is the maximal vertex
degree and $r$ is the maximal hyperedge cardinality. Consequently, the Stable Multiset and Vertex Multicover Problems on simple graphs can be solved by $2^{O(n)} \cdot \text{poly}(\phi)$-complexity algorithms.

Additionally, we give similar results for the Multiset Multicover and Dominating Multiset Problems.

2 Main Results

2.1 Integer Programming Algorithms

Our main result for ILP problems is the following

**Theorem 2.** Let $A \in \mathbb{Z}^{m \times n}$, $\text{rank}(A) = n$, $b \in \mathbb{Z}^m$. Let $\mathcal{P}$ be a polyhedron, defined by the system $Ax \leq b$.

Then, the Feasibility and Counting Problems can be solved by an algorithm with the complexity bounds

$$\min\{\|A\|_{\infty}, \|A\|_1\}^5 2^n \cdot \text{poly}(\phi);$$

$$\left(\|A\|_{\max}\right)^5 n \cdot \min\{\text{cs}(A), \text{rs}(A)\}^3 2^n \cdot \text{poly}(\phi).$$

The Optimization and Counting Problem can be solved by an algorithm with the complexity bounds

$$\min\{\|A\|_{\infty}, \|A\|_1\}^6 n \cdot (\|c\|_{\infty})^4 \cdot 4^n \cdot \text{poly}(\phi);$$

$$\left(\|A\|_{\max}\right)^6 (\|c\|_{\infty})^4 \cdot \min\{\text{cs}(A), \text{rs}(A)\}^4 4^n \cdot \text{poly}(\phi).$$

The proof is given in Section 3.

The following corollary is a straightforward consequence of Theorem 2.

**Corollary 1.** In notation of Theorem, assuming that $\|A\|_{\max} = n^{O(1)}$ and $\|c\|_{\infty} = n^{O(n)}$, the Feasibility, Counting and Optimization and Counting Problems can be solved by algorithms with the complexity bound

$$n^{O(n)} \cdot \text{poly}(\phi).$$

For $\|A\|_{\max} = n^{O(1)}$ the last bound outperforms the state of the art bound.

2.2 Application for the Packing and Covering Problems

As a consequence of Theorem 2, we have the following result:

**Theorem 3.** The Stable Multiset and Vertex Multicover Problems on hypergraphs can be solved by algorithms with the complexity bound

$$\min\{d, r\}^4n \cdot 4^n \cdot \text{poly}(\phi).$$

Here $n$ denotes the number of vertices, $d$ and $r$ denote the maximal vertex degree and maximal hyperedge cardinality, respectively.
The Stable Multiset and Vertex Multicover Problems on simple graphs can be solved by algorithms with the complexity bound
\[ 2^{5^{4n} \cdot \text{poly}(\phi)} \cdot 2^{O(n) \cdot \text{poly}(\phi)}. \]
Here \( n \) denotes the number of vertices.

The Multiset Multicover Problem can be solved by an algorithm with the complexity bound
\[ \min\{d, r\}^{4m} \cdot 4^{m} \cdot \text{poly}(\phi). \]
Here \( m \) denotes the number of hyperedges, \( d \) and \( r \) denote the maximal vertex degree and maximal hyperedge cardinality, respectively.

The Dominating Multiset Problem can be solved by an algorithm with the complexity bound
\[ d^{4n} \cdot 4^{n} \cdot \text{poly}(\phi). \]
Here \( n \) denotes the number of vertices and \( d \) denotes the maximal vertex degree.

The proof is given in Section 3.

Remark 1. Note that for the Multiset Multicover Problem and \( m = O(n^{1+\varepsilon}) \), where \( \varepsilon < 1 \), our algorithm outperforms the \( n^{O(n^{2}) \cdot \text{poly}(\phi)} \)-complexity algorithm due to the paper [26]. However, the algorithm from [26] can deal with an arbitrary goal function \( w^\top x \to \min \) (not just linear, but any separable-convex), while we need to pay an additional factor of \( (\|w\|_{\infty})^{4} \) to deal with a linear goal function \( w^\top x \to \min \).

Remark 2. Note that by the same algorithm we compute not only optimal solutions of the above problems, but, additionally, counts of such optimal solutions.

3 Proofs of Theorem 2 and Theorem 3

First of all, let us prove some auxiliary lemmas.

Lemma 1. Let \( A \in \mathbb{Z}^{n \times n} \), \( \det(A) \neq 0 \), and \( \|\cdot\|: \mathbb{R}^{n} \to \mathbb{R}_{\geq 0} \) be any vector norm, which is symmetric with respect to any coordinate, i.e. \( \|x\| = \|x - 2x_{i} \cdot e_{i}\| \), for any \( x \in \mathbb{R}^{n} \) and \( i \in \{1, \ldots, n\} \).

Let us consider a sector \( U = B_{\|\cdot\|} \cap \text{cone}(A) \), where \( B_{\|\cdot\|} = \{x \in \mathbb{R}^{n}: \|x\| \leq 1\} \) is the unit ball with respect to the \( \|\cdot\|\)-norm. Then,
\[ \text{vol}(U) \geq \frac{\det(A)}{2^{n}} \cdot \text{vol}(r \cdot B_{\|\cdot\|}), \]
where \( r \cdot B_{\|\cdot\|} \) is the \( \|\cdot\|\)-ball of the maximum radius \( r \), inscribed into the set \( \{x \in \mathbb{R}^{n}: \|Ax\| \leq 1\} \).

Consequently, let \( U_{1} = B_{1} \cap \text{cone}(A) \) and \( U_{\infty} = B_{\infty} \cap \text{cone}(A) \). Then,
\[ \text{vol}(U_{1}) \geq \frac{\det(A)}{(2\|A\|_{\infty})^{n}} \cdot \text{vol}(B_{1}) \geq \frac{\det(A)}{(2\|A\|_{\text{max}} \cdot \text{cs}(A))^{n}} \cdot \text{vol}(B_{1}); \]
\[ \text{vol}(U_{\infty}) \geq \frac{\det(A)}{(2\|A\|_{1})^{n}} \cdot \text{vol}(B_{\infty}) \geq \frac{\det(A)}{(2\|A\|_{\text{max}} \cdot \text{rs}(A))^{n}} \cdot \text{vol}(B_{\infty}). \]
Proof. Let us proof the inequality (5). Clearly, 

$$\text{vol}(U) = |\text{det}(A)| \cdot \text{vol}(K \cap \text{cone}(I_{n \times n})),$$

where $K = \{ x \in \mathbb{R}^n : \|Ax\| \leq 1 \}$.

By definition of $r$, we have $K \supseteq r \cdot B_\|\cdot\|$. Consequently,

$$\text{vol}(U) \geq |\text{det}(A)| \cdot \text{vol}(r \cdot B_\|\cdot\| \cap \text{cone}(I_{n \times n})) \geq \frac{|\text{det}(A)|}{2^n} \cdot \text{vol}(r \cdot B_\|\cdot\|).$$

Now, let us prove the inequality (6). To do that, we just need to prove the inequality $r \geq \frac{1}{\|A\|_\infty}$ with respect to the $l_1$-norm. Definitely, let us consider the set $K$. In the current case, it can be represented as the set of solutions of the following inequality:

$$\sum_{i=1}^{n} |A_{i,*}x| \leq 1. \quad (8)$$

Let us consider the $2n$ points $\pm p_i = \pm \frac{1}{\|A\|_\infty} \cdot e_i$, for $i \in \{1, \ldots, n\}$. Substituting $\pm p_j$ to the inequality (8), we have

$$\sum_{i=1}^{n} |A_{i,*}p_j| = \frac{1}{\|A\|_\infty} \cdot \sum_{i=1}^{n} |A_{i,*}e_j| = \frac{1}{\|A\|_\infty} \cdot \sum_{i=1}^{n} |A_{ij}| \leq 1.$$

Hence, all the points $\pm p_i$, for $i \in \{1, \ldots, n\}$, satisfy the inequality (8). Since $K$ is convex, we have $\frac{1}{\|A\|_\infty} \cdot B_1 \subseteq K$, and, consequently, $r \geq \frac{1}{\|A\|_\infty}$.

Finally, let us prove the inequality (7). Again, we need to show that $r \geq \frac{1}{\|A\|_\infty}$ with respect to the $l_\infty$-norm. In the current case, the set $K$ can be represented as the set of solutions of the following system:

$$\forall i \in \{1, \ldots, n\}, \ |A_{i,*}x| \leq 1. \quad (9)$$

Let us consider the set $M = \{ \frac{1}{\|A\|_1} \cdot (\pm 1, \pm 1, \ldots, \pm 1)^T \}$ of $2^n$ points. Substituting any point $p \in M$ to $j$-th inequality of the system (9), we have

$$|A_{j,*}p| \leq \sum_{i=1}^{n} |A_{ji}| |p_i| = \frac{1}{\|A\|_1} \cdot \sum_{i=1}^{n} |A_{ji}| \leq 1.$$

Hence, all the points $p \in M$ satisfy the inequality (9). Since $K$ is convex, we have $\frac{1}{\|A\|_1} \cdot B_\infty \subseteq K$, and, consequently, $r \geq \frac{1}{\|A\|_1}$.

Lemma 2. Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Q}^m$, and $\text{rank}(A) = n$.

Let us consider an $n$-dimensional polyhedron $P$, defined by a system $Ax \leq b$. Then,

$$|\text{vert}(P)| \leq 2^n \cdot \min\{\|A\|_\infty, \|A\|_1\}^n \leq (2\|A\|_{\text{max}})^n \cdot \min\{\text{cs}(A), \text{rs}(A)\}^n.$$
Proof. Let $\mathcal{N}(v) = \text{cone}(A^T_{\mathcal{J}(v)})$ be the normal cone of a vertex $v \in \text{vert}(\mathcal{P})$, where $\mathcal{J}(v) = \{j \in \{1, \ldots, m\} : A_{jv} = b_j\}$. Since $\mathcal{P}$ is full-dimensional, we have $\dim(\mathcal{N}(v)) = n$, for any $v \in \text{vert}(\mathcal{P})$. It is a known fact that $\dim(\mathcal{N}(v_1) \cap \mathcal{N}(v_2)) < n$, for different $v_1, v_2 \in \text{vert}(\mathcal{P})$. Next, we will use the following trivial inclusion
\[ \bigcup_{v \in \text{vert}(\mathcal{P})} \mathcal{N}(v) \cap \mathcal{B} \subseteq \mathcal{B}, \quad (10) \]
where $\mathcal{B}$ is the unit ball with respect to any vector norm $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

Again, since $\mathcal{P}$ is full-dimensional and since $\text{rank}(A) = n$, each matrix $A^T_{\mathcal{J}(v)}$ contains a non-degenerate $n \times n$ sub-matrix. Taking $\mathcal{B} := \mathcal{B}_1$ or $\mathcal{B} := \mathcal{B}_\infty$, by Lemma\,[1] we have $\text{vol}(\mathcal{N}(v) \cap \mathcal{B}) \geq \frac{\text{vol}(\mathcal{B})}{(2\gamma)^n}$, where $\gamma = \min\{\|A\|_\infty, \|A\|_1\}$. Finally, due to (10),
\[ \frac{\text{vol}(\mathcal{B})}{(2\gamma)^n} \cdot |\text{vert}(\mathcal{P})| \leq \text{vol}(\mathcal{B}). \]

Proof of Theorem 2

Proof. Let us assume that $\mathcal{P}$ is bounded. In the opposite case, we need to distinguish between two possibilities: $|\mathcal{P} \cap \mathbb{Z}^n| = 0$ and $|\mathcal{P} \cap \mathbb{Z}^n| = +\infty$. Let $v$ be some vertex of $\mathcal{P}$ and $B$ be the corresponding base, i.e. $A_Bv = b_B$. Due to [8], if $\mathcal{P} \cap \mathbb{Z}^n \neq \emptyset$, then there exists a point $z \in \mathcal{P} \cap \mathbb{Z}^n$, such that $b_B - n\Delta \cdot 1 \leq A_Bz \leq b_B$. Consequently, to transform the unbounded case to the bounded case, we just need to add the inequalities $b_B - n\Delta \cdot 1 \leq A_Bz \leq b_B$ to the original system $Ax \leq b$.

Additionally, we assume that that $\dim(\mathcal{P}) = n$. Indeed, the inequality $\dim(\mathcal{P}) < n$ is equivalent to the fact that there exists $j \in \{1, \ldots, m\}$, such that $A_{jx}x = b_j$, for any $x \in \mathcal{P}$. Note that such $j$ can be found by a polynomial-time algorithm. Next, we replace the original inequality $A_{jx}x \leq b_j$ by the inequality $A_{jx}x \leq b_j + 1/2$. Clearly, this transformation does not change the set of integer solutions, and the inequality $A_{jx}x \leq b_j$ can not hold as equality, for all $x \in \mathcal{P}$, because the new polyhedron contains the old one. After eliminating all such $j$, we finally achieve a polyhedron of dimension $n$.

Due to Theorem 1 the counting problem can be solved by an algorithm with the arithmetical complexity bound
\[ O(\nu \cdot n^3 \cdot \Delta^4 \cdot \log(\Delta)), \quad (11) \]
where $\nu$ is the maximal number of vertices in a class of polyhedra under consideration.

In our case, the value of $\nu$ can be estimated by Lemma 2. To estimate the value of $\Delta(A)$, we use the Hadamard’s bound and trivial quantities $\text{det}(B) = \text{det}(B^T)$ and $\|v\|_2 \leq \|v\|_1$, for any $v \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$. Therefore, we have the following bounds on $\Delta(A)$:
\[ \Delta(A) \leq \min\{\|A\|_\infty, \|A\|_1\}^n; \]
\[ \Delta(A) \leq (\|A\|_{\text{max}})^n \cdot \min\{\text{cs}(A), \text{rs}(A)\}^{n/2}. \]
The inequalities for $\nu$ and $\Delta(A)$, together with the bound (11), give the desired complexity bound for the Counting Problem 2.

Let us show how to find some point $z$ inside of $\mathcal{P} \cap \mathbb{Z}^n$ in the case $|\mathcal{P} \cap \mathbb{Z}^n| > 0$, to handle the Feasibility Problem 1. For $\alpha, \beta \in \mathbb{Z}$, let us consider the polyhedron $\mathcal{P}'(\alpha, \beta)$, defined by the system $Ax \leq b$ with an additional inequality $\alpha \leq x_1 \leq \beta - 1/2$. Note that the polyhedron $\mathcal{P}'(\alpha, \beta)$ has dimension $n$ if and only if the values $\alpha, \beta \in \mathbb{Z}$ satisfy the inequalities $\min_{x \in \mathcal{P}} \{x_1\} \leq \alpha < \beta \leq \max_{x \in \mathcal{P}} \{x_1\}$. Consequently, for appropriate values of $\alpha$ and $\beta$, the polyhedron $\mathcal{P}'(\alpha, \beta)$ is $n$-dimensional.

Let $v$ be some vertex of $\mathcal{P}$. Due to [8], we know that there exists a point $z \in \mathcal{P} \cap \mathbb{Z}^n$, such that $\|v - z\|_\infty \leq n \cdot \Delta$. So, the value of $z_1$ can be found, using the binary search with questions to the $\mathcal{P}'(\alpha, \beta) \cap \mathbb{Z}^n$-feasibility oracle. Clearly, we need $O(\log(n\Delta))$ calls to the oracle.

After the moment, when we already know the value of $z_1$, we add the inequality $z_1 \leq x_1 \leq z_1 + 1/2$ to the system $Ax \leq b$ and start a similar search procedure for the value of $z_2$. The total number of calls to the binary search oracle to compute all components of $z$ is $O(n \cdot \log(n\Delta))$.

Finally, let us explain how to deal with the Optimization and Counting Problem 3. Let $\alpha, \beta$ be integer values. Let us consider the polyhedron $\mathcal{P}'(\alpha, \beta)$, defined by the system $Ax \leq b$ with an additional inequality $\alpha \leq c^T x \leq \beta - 1/2$. Note that the polyhedron $\mathcal{P}'(\alpha, \beta)$ has dimension $n$ if and only if the values $\alpha, \beta \in \mathbb{Z}$ satisfy to the inequalities

$$\min_{x \in \mathcal{P}} \{c^T x\} \leq \alpha < \beta \leq \max_{x \in \mathcal{P}} \{c^T x\}.$$

Let $A' \in \mathbb{Z}^{(n+m+2)\times n}$ be the matrix that defines $\mathcal{P}'(\alpha, \beta)$, i.e. $A' = \begin{pmatrix} c^T \\ -c^T \\ A \end{pmatrix}$. Expanding sub-determinants of $A'$ by the $c^T$-row, we have $\Delta(A') \leq \|c\|_1 \cdot \Delta_{n-1}(A)$. The value of $\Delta_{n-1}(A)$ can be estimated by the same way, as it was done for $\Delta(A)$.

Let us estimate the number of vertices in $\mathcal{P}'(\alpha, \beta)$. The polyhedron $\mathcal{P}'(\alpha, \beta)$ is the intersection of the polyhedron $\mathcal{P}$ with the slab $\{x \in \mathbb{R}^n : \alpha \leq c^T x \leq \beta - 1/2\}$. Clearly, the new vertices may appear only on edges of $\mathcal{P}$, by at most 2 new vertices per edge. The number of edges in $\mathcal{P}$ is bounded by $|\text{vert}(\mathcal{P})|^2/4$. In turn, the value of $|\text{vert}(\mathcal{P})|^2/4$ can be estimated using Lemma 2.

Since $\mathcal{P}'(\alpha, \beta)$ has dimension $n$ for appropriate values of $\alpha, \beta$, due to Theorem 4, the value $|\mathcal{P}'(\alpha, \beta) \cap \mathbb{Z}^n|$ can be computed by an algorithm with the same complexity bounds, as it was stated in the end of Theorem 2.

To complete the proof we note that, using the binary search method, the original optimization problem can be reduced to a polynomial number of feasibility questions in the set $\mathcal{P}'(\alpha, \beta) \cap \mathbb{Z}^n$ for different $\alpha, \beta$.

**Proof of Theorem 3**

Let $Mx \leq b, x \in \mathbb{Z}^n$ be a system that defines one of the problems under consideration. Since, for any of the problems, the matrix $M$ contains $-I_{n \times n}$ as...
a sub-matrix, \( \text{rank}(M) = n \). Hence, the results for all of the problems, except for the Stable Multiset and Vertex Multicover Problems, straightforwardly follow from Theorem \(2\). Now, let us give a more accurate complexity bound for the Stable Multiset and Vertex Multicover Problems, than can be given by a straightforward application of Theorem \(2\).

We just need to follow to the proof of Theorem \(2\) with an exception to use a more accurate bound for \( \Delta_{n-1}(M) \). Let \( A = A(\mathcal{G}) \) be the incidence matrix of the corresponding graph \( \mathcal{G} \). Due to [21], it is known that \( \Delta_i(A) \leq 2^\tau_0 \), where \( \tau_0 = \tau_0(\mathcal{G}) \) is the odd tulgeity of \( \mathcal{G} \) that is defined to be the maximum number of vertex-disjoint odd cycles of \( \mathcal{G} \). Clearly, \( \tau_0 \leq n/3 \), so, \( \Delta_{n-1}(M) \leq 2^{n/3} \). Using this bound for \( \Delta_{n-1}(A) \) in the proof of Theorem \(2\) it gives the desired complexity bounds for the Stable Multiset and Vertex Multicover Problems.

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