Higher Derivatives and Canonical Formalism

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Abstract

A canonical formalism for higher-derivative theories is presented on the basis of Dirac’s method for constrained systems. It is shown that this formalism shares a path integral expression with Ostrogradski’s canonical formalism.
1 Introduction

Higher-derivative theories appear in various scenes of physics: higher derivative terms occur as quantum corrections; nonlocal theories, e.g. string theories, are essentially higher-derivative theories; Einstein gravity supplemented by curvature-squared terms has attracted attention because of its renormalizability.

A canonical formalism for higher-derivative theories was developed by Ostrogradski about one and a half centuries ago. Though being self-consistent, his formulation looks different from the conventional canonical formalism. The purpose of the present paper is to interpret the Ostrogradski’s formulation in the framework of the ordinary constrained canonical formalism. We show that the path integral expression for the ordinary constrained canonical formalism is the same as that for the Ostrogradski formalism.

In §2 we review the Ostrogradski’s formulation. In §3 we give another canonical formulation for higher-derivative theories based on the usual method of Dirac for constrained systems. In §4 it is shown that the two formulations give the same path integral formulae. Section 5 gives summary.

2 Ostrogradski’s canonical formalism

We consider a generic Lagrangian which contains up to $N$th derivative of coordinate $q(t)$:

$$L = L \left( q^{(0)}, q^{(1)}, \ldots, q^{(N)} \right),$$

where

$$q^{(i)} \equiv \frac{d^i}{dt^i} q \quad (i = 0, 1, \ldots, N).$$

The Euler-Lagrange equation is

$$\delta S \equiv \sum_{i=0}^{N} \left( -\frac{d}{dt} \right)^i \frac{\partial L}{\partial q^{(i)}} = 0.$$

The canonical formalism of Ostrogradski is the following. For $i = 0, \ldots, N - 1$ we regard $q^{(i)}$ as independent coordinates $q^i$:

$$q^{(i)} \rightarrow q^i \quad (i = 0, 1, \ldots, N - 1),$$

$$L \left( q^{(0)}, q^{(1)}, \ldots, q^{(N-1)}, q^{(N)} \right) \rightarrow L \left( q^0, q^1, \ldots, q^{N-1}, \dot{q}^{N-1} \right).$$

The momentum $p_{N-1}$ conjugate to $q^{N-1}$ is defined as usual by

$$p_{N-1} \equiv \frac{\partial L}{\partial \dot{q}^{N-1}} \left( q^0, q^1, \ldots, q^{N-1}, \dot{q}^{N-1} \right).$$

Here and hereafter we assume that the Lagrangian is nondegenerate. This means that the relation (6) can be inverted to give $\dot{q}^{N-1}$ as a function of $q^i$ ($i = 0, \ldots, N - 1$) and $p_{N-1}$:

$$\dot{q}^{N-1} = \dot{q}^{N-1} \left( q^0, q^1, \ldots, q^{N-1}, p_{N-1} \right).$$
The Hamiltonian is defined by
\[
H_O = H_O \left( q^0, q^1, \cdots, q^{N-1}; p_0, p_1, \cdots, p_{N-1} \right)
\]
\[
\equiv \sum_{i=0}^{N-2} p_i q^{i+1} + p_{N-1} q^{N-1} - L \left( q^0, q^1, \cdots, q^{N-1}, \dot{q}^{N-1} \right).
\]
This definition is different from the usual Legendre transformation. Note that for \( i = 0, \cdots, N - 2 \), the momenta \( p_i \) are not defined through relations like (6), but introduced just as independent canonical variables. The canonical equations of motion are
\[
\dot{q}^i = \frac{\partial H_O}{\partial p_i},
\]
\[
\dot{p}_i = -\frac{\partial H_O}{\partial q^i}.
\]
Eq.(9) with \( i = 0, \cdots, N - 2 \) gives
\[
\dot{q}^i = q^{i+1} \quad (i = 0, 1, \cdots, N - 2).
\]
Under the assumption of nondegeneracy, Eq.(9) with \( i = N - 1 \) reproduces the definition (6). Eq.(10) gives the following equations:
\[
\begin{cases}
\dot{p}_0 = \frac{\partial L}{\partial q^0}, \\
\dot{p}_i = -p_{i-1} + \frac{\partial L}{\partial q^i} \quad (i = 1, \cdots, N - 1).
\end{cases}
\]
From Eqs.(12), (11) and (6) we regain the Euler-Lagrange equation (3).

3 Constrained canonical formalism

It has been seen that the Ostrogradski formalism gives special treatment to the highest derivative \( q^{(N)} \). Is it possible to treat the highest derivative and the lower derivatives on an equal footing? This is the subject of the present section.

To treat all the derivatives equally, we introduce Lagrange multipliers \( \lambda_i \) \( (i = 0, \cdots, N - 1) \) and start from the following equivalent Lagrangian:
\[
L^* \equiv L \left( q^0, q^1, \cdots, q^N \right) + \sum_{i=0}^{N-1} \lambda_i \left( \dot{q}^i - q^{i+1} \right).
\]
The Euler-Lagrange equations
\[
\begin{cases}
\frac{\delta L^*}{\delta q^i} = 0 \quad (i = 0, \cdots, N), \\
\frac{\delta L^*}{\delta \lambda_i} = 0 \quad (i = 0, \cdots, N - 1)
\end{cases}
\]
which are equivalent to Eq. (3).

Since the Lagrangian \( L^* \) describes a constrained system, we follow a way of Dirac.\(^5\) The conjugate momenta defined by

\[
\begin{align*}
    p_i &\equiv \frac{dL^*}{dq^i} \quad (i = 0, \cdots, N), \\
    \pi^i &\equiv \frac{dL^*}{d\lambda_i} \quad (i = 0, \cdots, N - 1)
\end{align*}
\]

provide the following primary constraints:

\[
\begin{align*}
    \phi_i &\equiv p_i - \lambda_i \approx 0 \quad (i = 0, \cdots, N - 1), \\
    \phi_N &\equiv p_N \approx 0, \\
    \psi^i &\equiv \pi^i \approx 0 \quad (i = 0, \cdots, N - 1).
\end{align*}
\]

The consistency of these constraints under their time developments requires a secondary constraint

\[
\psi^N \equiv \frac{\partial L}{\partial q^N} - \lambda_{N-1} \approx 0.
\]

When the system is nondegenerate, all these constraints

\[
\Phi_\alpha \equiv (\phi_0, \cdots, \phi_N; \psi^0, \cdots, \psi^N)
\]

form a set of second-class constraints: the determinant of the Poisson brackets between these constraints

\[
\det ([\Phi_\alpha, \Phi_\beta])_P = \left( \frac{\partial^2 L}{\partial q^N} \right)^2
\]

is not zero when

\[
\frac{\partial^2 L}{\partial q^N} \neq 0.
\]

The Dirac brackets between the canonical variables \((q^i, \lambda_i; p_i, \pi^i)\) are calculated to be

\[
\begin{align*}
[q^i, p_i]_D &= [q^i, \lambda_i]_D = 1, \\
[q^N, p_i]_D &= [q^N, \lambda_i]_D = -\left( \frac{\partial^2 L}{\partial q^N \partial q^N} \right)^{-1} \frac{\partial^2 L}{\partial q^i \partial q^N}, \\
[q^{N-1}, q^N]_D &= \left( \frac{\partial^2 L}{\partial q^N} \right)^{-1}, \\
\text{The others} &= 0,
\end{align*}
\]
The Hamiltonian is given by
\[
H_D = -L \left( q^0, \cdots, q^N \right) + \sum_{i=0}^{N-1} \lambda_i q^{i+1}.
\] (25)

This is obtained by performing the ordinary Legendre transformation on the Lagrangian \( L^* \) of Eq.(13) and by regarding all the constraints \( \Phi_\alpha \approx 0 \) as strong equalities \( \Phi_\alpha = 0 \). Eqs.(25) and (24) allow us to obtain the canonical equations of motion. The independent equations are
\[
\begin{align*}
\dot{q}^i &= q^{i+1} \quad (i = 0, \cdots, N - 1), \\
\dot{\lambda}_0 &= \frac{\partial L}{\partial q^0}, \\
\dot{\lambda}_i &= \frac{\partial L}{\partial q^i} - \lambda_{i-1} \quad (i = 1, \cdots, N - 1), \\
0 &= \frac{\partial L}{\partial q^N} - \lambda_{N-1},
\end{align*}
\] (26)

which are seen to be equivalent to the Euler-Lagrange equations (15).

4 Path integrals

We have presented two canonical formalisms for higher-derivative theories, the Ostrogradski’s one and the constrained one. It has been seen that though looking different from each other, they give an equivalent set of equations of motion. In this section we show that the two formalisms are completely equivalent to each other by comparing path integral expressions for them.

The Ostrogradski formalism of §2 gives the following path integral expression:
\[
Z_O = \int \prod_{i=0}^{N-1} \left( Dq^i Dp_i \right) \exp \left\{ i \int dt \left[ \sum_{i=0}^{N-1} p_i \dot{q}^i - H_O \left( q^0, \cdots, q^{N-1} ; p_0, \cdots, p_{N-1} \right) \right] \right\},
\] (27)

where the Hamiltonian \( H_O \) is given by Eq.(8). Integrations with respect to \( p_i \) \((i = 0, \cdots, N - 2)\) offer a factor \( \prod_{i=0}^{N-2} \delta (\dot{q}^i - q^{i+1}) \). We can further integrate with respect to \( q^i \) \((i = 1, \cdots, N - 1)\), obtaining
\[
Z_O = \int Dq^N Dp_{N-1} \exp \left\{ i \int dt \left[ p_{N-1} \dot{q}^{(N)} - \hat{H} \left( q, \dot{q}, \cdots, q^{(N-1)}, p_{N-1} \right) \right] \right\},
\] (28)

where
\[
\hat{H} \triangleq p_{N-1} \dot{q}^{(N)} - L \left( q, \dot{q}, \cdots, q^{(N-1)}, \dot{q}^{(N-1)} \right).
\] (29)

In Eq.(29), \( \dot{q}^{(i)} \) is a function of \( q^{(i)} \) \((i = 0, \cdots, N - 1)\) and \( p_{N-1} \), which is obtained by replacing \( q^i \) with \( q^{(i)} \) in Eq.(7). Path integral expression for the case of the
constrained canonical formalism of §3 is

\[
Z_D = \int \prod_{i=0}^{N} (\mathcal{D}q^i \mathcal{D}p_i) \prod_{i=0}^{N-1} \left( \mathcal{D}\lambda_i \mathcal{D}\pi^i \right) \prod_{i=0}^{N} \left( \delta(\phi_i)\delta(\psi^i) \right) \left| \frac{\partial^2 L}{\partial q^{N2}} \right| \\
\times \exp \left\{ i \int dt \left( \sum_{i=0}^{N} p_i \dot{q}^i + \sum_{i=0}^{N-1} \pi_i \dot{\lambda}_i - H_D \left( q^0, \cdots, q^N; \lambda_0, \cdots, \lambda_{N-1} \right) \right) \right\},
\]

where the constraints \((\phi_i, \psi^i) (i = 0, \cdots, N)\) are given by Eqs.(17) – (20), and the Hamiltonian \(H_D\) by Eq.(25). Integrations with respect to \(p_N, \pi^i (i = 0, \cdots, N - 2)\), \(\lambda_i (i = 0, \cdots, N - 2)\), \(p_i (i = 0, \cdots, N - 2)\) and \(q^i (i = 1, \cdots, N)\) give

\[
Z_D = \int \prod_{i=0}^{N} \mathcal{D}q^i \prod_{i=0}^{N-1} \mathcal{D}p_i \delta \left( \frac{\partial L}{\partial q^N} - p_{N-1} \right) \left| \frac{\partial^2 L}{\partial q^{N2}} \right| \\
\times \exp \left\{ i \int dt \left[ \sum_{i=0}^{N-1} p_i \left( \dot{q}^i - q^{i+1} \right) + L \left( q^0, \cdots, q^N \right) \right] \right\} \\
= \int \prod_{i=0}^{N} \mathcal{D}q^i \mathcal{D}p_{N-1} \delta \left( \frac{\partial L}{\partial q^N} - p_{N-1} \right) \left| \frac{\partial^2 L}{\partial q^{N2}} \right| \prod_{i=0}^{N-2} \delta \left( \dot{q}^i - q^{i+1} \right) \\
\times \exp \left\{ i \int dt \left[ p_{N-1} \left( \dot{q}^{N-1} - q^N \right) + L \left( q^0, \cdots, q^N \right) \right] \right\} \\
= \int \mathcal{D}q \mathcal{D}p_{N-1} \exp \left\{ i \int dt \left[ p_{N-1} \left( \dot{q}^{(N)} - q^N \right) + L \left( q, \dot{q}, \cdots, q^{(N-1)}, q^N \right) \right] \right\}.
\]

In the last line of Eq.(31), \(q^N\) is a function of \(q^{(i)} (i = 0, \cdots, N - 1)\) and \(p_{N-1}\) defined by inverting the relation

\[
p_{N-1} = \frac{\partial L}{\partial q^N} \left( q, \dot{q}, \cdots, q^{(N-1)}, q^N \right). \tag{32}
\]

That means \(q^N\) is nothing but \(\dot{q}^{N-1}\) in Eq.(29). Putting \(q^N = \dot{q}^{N-1}\) in Eq.(31) shows that the path integral \(Z_{rmD}\) is the same as \(Z_O\) given by Eq.(28)

\[
Z_D \equiv Z_O. \tag{33}
\]

5 Summary

We have presented a canonical formalism for higher-derivative theories based on the usual method of Dirac for constrained systems. It has been shown that this formalism shares a path integral expression with the Ostrogradski’s one. We thus have laid the foundation of the ordinary canonical formalism for the Ostrogradski’s formulation.

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