Note on the Clebsch-Gordan coefficients of SU(3)

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Abstract

It is argued that several papers where SU(3) Clebsch-Gordan coefficients used for describing properties of hadronic systems are, up to a phase convention, particular cases of analytic formulae derived by Hecht in 1965 in the context of nuclear physics. This is valid for irreducible representations with multiplicity one in the corresponding Clebsch-Gordan series. For multiplicity two Hecht has proposed an alternative which leads to a large $N_c$ behaviour different from those derived so far.

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I. INTRODUCTION

Since the 1963 classical paper of de Swart \[1\] where Clebsch-Gordan (CG) coefficients of SU(3) were derived for the most important direct (or Kronecker) products of irreducible representations needed in particle physics at that time, namely $8 \times 8$, $8 \times 10$, $8 \times 27$, $10 \times 10$ and $10 \times \bar{10}$, many authors devoted their papers or parts of them to the derivation of CG coefficients which were missing in de Swart’s paper. As recalled in the next section this amounts to derive the corresponding isoscalar factor for each CG coefficient. In 1963 as well, numerical values for the SU(3) isoscalar factors were published by Edmonds [2]. More tables were given in 1964 by McNamee and Chilton [3].

In recent years the SU(3) flavor group was frequently used to study new hadronic properties and quark systems involving an arbitrary number of quarks as for example in large $N_c$ QCD studies. The existing results seemed to be insufficient so that several authors derived their own tables. Here we show that some of them are particular cases of the analytic expressions obtained by Hecht in 1965 in the context of nuclear physics [4].

II. REMINDER OF SOME SU(3) CG PROPERTIES

In the chain $SU(3) \supset SU(2)_I \times U(I)_Y$ each SU(3) CG coefficient factorizes into an SU(2)-isospin CG coefficient and an SU(3) isoscalar factor \[1\]

$$
\left( \begin{array}{c} \lambda \mu \\ Y^a I^a I^a_3 \end{array} \right) \rho = \left( \begin{array}{c} I \\ I^a \end{array} \right) \left( \begin{array}{c} \lambda \mu \\ Y^a I^a \end{array} \right) \rho,
$$

where $(\lambda\mu)$ labels an SU(3) irreducible representation (irrep) and the index $\rho$ distinguishes between identical representations occurring in the decomposition of a given direct product where the multiplicity of a representation is larger than one. The highest multiplicity considered here is two and in this case a typical example of direct product representations is when one takes $(\lambda^a\mu^a) = (11)$, which is the adjoint representation of SU(3), also denoted by its dimension 8. The CG series reads

$$
(\lambda\mu) \times (11) = (\lambda + 1, \mu + 1) + (\lambda + 2, \mu - 1) + (\lambda\mu)_1 + (\lambda\mu)_2
$$

$$
+ (\lambda - 1, \mu + 2) + (\lambda - 2, \mu + 1) + (\lambda + 1, \mu - 2) + (\lambda - 1, \mu - 1).
$$
The isoscalar factors of SU(3) satisfy an orthogonality relation resulting from the orthogonality relations of SU(3) and SU(2) CG coefficients. This is

\[ \sum_{Y''I''} \left( \begin{array}{c} Y'' \mu'' \\ Y'' I'' \ Y^a I^a \end{array} \right) \left( \begin{array}{c} Y' \mu' \\ Y' I' \ Y^a I^a \end{array} \right) \rho = \delta_{\lambda'\lambda} \delta_{\mu'\mu} \delta_{Y''Y'} \delta_{I''I'} \tag{3} \]

and

\[ \sum_{(\lambda\mu)\rho} \left( \begin{array}{c} Y'' \mu'' \\ Y'' I'' \ Y^a I^a \end{array} \right) \left( \begin{array}{c} \lambda \mu \\ Y I \ Y^a I^a \end{array} \right) \left( \begin{array}{c} Y'' \mu'' \\ Y'' I'' \ Y^a I^a \end{array} \right) \left( \begin{array}{c} \lambda \mu \\ Y I \ Y^a I^a \end{array} \right) \rho = \delta_{Y''Y_1''} \delta_{I''I_1''} \delta_{Y^a Y^a_1} \delta_{I^a I^a_1} \tag{4} \]

For completeness, we also recall that the isoscalar factors obey the following symmetry properties [4]

\[ \left( \begin{array}{c} \lambda \mu \\ Y I \ Y^a I^a \end{array} \right) \left( \begin{array}{c} \lambda' \mu' \\ Y' I' \ Y^a I^a \end{array} \right) = (-)^{(\lambda-\mu+\lambda^a-\mu^a-\lambda'+\mu'+I^a-I')} \left( \begin{array}{c} \lambda \mu \\ Y^a I^a \ Y I \end{array} \right) \left( \begin{array}{c} \lambda' \mu' \\ Y' I' \ Y^a I^a \end{array} \right) \tag{5} \]

and

\[ \left( \begin{array}{c} \lambda \mu \\ Y I \ Y^a I^a \end{array} \right) \left( \begin{array}{c} \lambda' \mu' \\ Y' I' \ Y^a I^a \end{array} \right) = (-)^{\frac{1}{3}(\mu'-\mu+\lambda+\frac{3}{2}Y^a)+I'-I} \frac{\dim(\lambda'\mu')(2I'+1)}{\dim(\lambda\mu)(2I+1)} \left( \begin{array}{c} \lambda' \mu' \\ Y' I' \ Y^a I^a \end{array} \right) \tag{6} \]

where \( \dim(\lambda\mu) = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2) \) is the dimension of the irrep \((\lambda\mu)\) of SU(3). An alternative notation of the isoscalar factors is \( \langle (\lambda\mu)Y; (\lambda^a\mu^a)Y^a \rangle \), see Hecht’s paper.

III. CALCULATION OF SU(3) CLEBSCH-GORDAN COEFFICIENTS

The usual procedure to calculate CG coefficients is to start from the highest weight basis vector of a representation and use ladder operators, which in SU(3) are \( U_\pm, V_\pm \) and \( I_\pm \).
Their matrix elements are linear combinations of isoscalar factors. The coefficients of the linear combinations are matrix elements of $U_\pm$ and $V_\pm$ and $I_\pm$. Their analytic forms were first determined by Biedenharn [5].

To define uniquely these matrix elements some phase conventions must be made. For the states in the same isomultiplet the standard Condon and Shortley has been chosen. Accordingly the non-vanishing matrix elements of $I_\pm$ are positive. The relative phases between different isomultiplets were defined by the requirement that the non-vanishing matrix elements of $V_\pm$ are real and positive [1] (for the phase convention of de Swart see [1], Section 10).

This procedure has been followed by Kaeding [6] who provided a large number of tables for $(\lambda^a \mu^a) = (10), (01), (20), (11), (30)$ and (21) or in dimensional notation $3, \bar{3}, 6, 8, 10$ and $15'$.

More recently Hong [7] has derived the isoscalar factors of the direct product of $35 \times 8$, with the purpose of using them to the calculation of baryon magnetic moments and decuplet-to-octet transition magnetic moments. Up to a phase convention, all the isoscalar factors are particular cases of the formulae derived by Hecht [4] in his Table 4.

In large $N_c$ QCD Cohen and Lebed [8] derived $N_c$ dependent SU(3) CG coefficients relevant for the coupling of large $N_c$ baryons to mesons. They provided extended tables for the direct products for

$$(\lambda \mu) = (1, \frac{N_c - 1}{2}); \quad (3, \frac{N_c - 3}{2})$$

(7) denoted by "8" and "10" respectively and $(\lambda^a \mu^a) = (11)$ denoted by 8. Their results, at multiplicity one, up to an overall phase, can directly be reproduced from Hecht’s Table 4. For multiplicity two, for example, "10"$_a \times 8 \rightarrow "10"_a$ they are different at arbitrary $N_c$, but identical at $N_c = 3$, as compared to those derived here.

For the same direct products as those of Cohen and Lebed [8] partial tables were previously provided in Ref. 9.

The explicit algebraic expressions derived by Hecht [4] for SU(3) isoscalar factors were intended to nuclear physics applications, in particular to describing rotational states of deformed light nuclei from the $2s - 1d$ shell. The deformed nuclei possess collective states described by Elliott [10, 11] in a model where the SU(3) group is used. Thus the application of SU(3) in nuclear physics in 1958 predates the SU(3) classification of elementary particles of Gell-Mann [13] and Ne’eman [14] in 1961. The basic reason of using SU(3) in nuclear models
is that intrinsic levels of nuclei can be described by the harmonic oscillator and SU(3) is the symmetry group of the harmonic oscillator in three dimensions (see, for example, Ref. [12] chapter 8). The physical states of a given angular momentum can be obtained by a projection technique [15]. In addition to the isoscalar factors needed for the $2s - 1d$ shell, Hecht also derived explicit expressions for the direct product $(\lambda\mu) \times (11)$, considering such results as being of interest. He used the standard technique of generating CG coefficients through recursion formulae containing matrix elements of the SU(3) generators, but introduced a phase convention different from that of de Swart. The difference is clearly explained in a footnote of Ref. [4]. In addition, when the irrep $(\lambda\mu)$ appears twice in the decomposition of the direct product $(\lambda\mu) \times (11)$, see Eq. (2), he introduced the quantum number $\rho$ to label the independent modes of coupling, such as to have non-zero matrix elements of the SU(3) generators for only one state $\rho$. Then, according to the Wigner-Eckart theorem, the matrix elements of the generators $T^a$ of SU(3) are

$$
\langle (\lambda'\mu')|Y'I'_3; S'S'_3|T^a|(\lambda\mu)YII_3; SS_3 \rangle =
\delta_{SS'}\delta_{S'_3S_3}\delta_{\lambda\lambda'}\delta_{\mu\mu'} \sum_{\rho=1,2} \langle (\lambda'\mu')||T^{(11)}||(\lambda\mu) \rangle_{\rho} \left( \begin{array}{c}
(\lambda\mu) \\
YII_3 \\
Y'a I_3^a \\
Y'I_3'
\end{array} \right) \left( \begin{array}{c}
(\lambda'\mu') \\
Y'Y_a I_3 a \\
Y'I'_3'
\end{array} \right)_{\rho},
$$

where the reduced matrix elements are defined as [4]

$$
\langle (\lambda\mu)||T^{(11)}||(\lambda\mu) \rangle_{\rho} = \left\{ \begin{array}{ll}
\sqrt{C(SU(3))} & \text{for } \rho = 1 \\
0 & \text{for } \rho = 2
\end{array} \right.,
$$

in terms of the eigenvalue of the Casimir operator $C(SU(3)) = \frac{1}{3}g_{\lambda\mu}$ where

$$
g_{\lambda\mu} = \lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu.
$$

Such a definition is useful for extending the method of calculation of isoscalar factors to other SU(N) groups. It has been applied to the calculation of the matrix elements of SU(6) generators, where one takes into account that SU(3) is a subgroup of SU(6) [16, 17].

The correspondence with other notations is

$$
\rho = 1 \iff (\lambda\mu)_2 \iff (\lambda\mu)_a,
$$

$$
\rho = 2 \iff (\lambda\mu)_1 \iff (\lambda\mu)_s.
$$

where $s$ and $a$ stand for symmetric and antisymmetric respectively [18, 19]. Historically, following Gell-Mann, in Eq. (III), it is customary to call the symmetric combinations $D$
coupling and the antisymmetric $a$ combinations $F$ coupling (the $F$ and $D$ notation is used in Ref. 9, for example).

Ambiguities in distinguishing the representations at multiplicity larger than one are typical for all groups, including the permutation group 20.

Another way to derive Clebsch-Gordan coefficients for SU(3) is based on the tensor method (for an introduction see, for example, Ref. 12, Sec. 8.10). This method has been used for the Clebsch-Gordan series "8" $\times$ 8 and "10" $\times$ 8 in the systematic analysis of large $N_c$ baryons 21.

IV. EXAMPLES

Here we present results for two examples obtained by using Table 4 of Ref. 4 to be compared with previous literature. We also use the same table format as that of de Swart because it helps in comparing with previous results and moreover, in easily checking the orthogonality relations (3) and (4).

The first example, shown in Table I, corresponds to one table obtained by Hong in Ref. 7. It contains the isoscalar factors for all irreducible representations with $Y = 2$, $I = 2$ from the decomposition of the direct product $35 \times 8$. These are 81, 64, 35$s$ and 35$s$ in this case.

Note that one must use the symmetry property (5) to recover the results for 8 $\times$ 35 as in Ref. 7, instead of 35 $\times$ 8 considered here. For the columns 81 and 64 the absolute values are the same as those of Hong. Incidentally column 81 also has the same phases as Hong and column 64 has an overall opposite phase. The results for 35$s$ and 35$s$ are entirely different from those of [7] because the definition is different. They were obtained from the table given in the Appendix.

The second example is exhibited in Table II and corresponds to a table of Cohen and Lebed 8, containing isoscalar factors with $Y = N_c/3$, $I = 3/2$ from the decomposition of the direct product "10" $\times$ 8. Cohen and Lebed obtained analytic expressions of the isoscalar factors as a function of $N_c$ needed for large $N_c$ baryons. Our table was obtained as a direct application of Hecht’s Table 4 part of which is reproduced in Table III of the Appendix, referring to the irrep "10" with multiplicity 2, denoted here by "10"$^a$ and "10"$^s$. For completeness, to the three rows listed by Cohen and Lebed we have added a fourth
TABLE I: Isoscalar factors for the irreducible representations with $Y = 2$, $I = 2$ from the decomposition of the direct product $35 \times 8$. The first two columns indicate the hypercharge and isospin of $35$ and $8$ respectively. The phase convention is that of Hecht [4].

| $Y_1I_1; Y_2I_2$ | $81$ | $64$ | $35_s$ | $35_a$ |
|------------------|-----|-----|-------|-------|
| $1, \frac{5}{2}; 1, \frac{1}{2}$ | $-\sqrt{1}$ | $\frac{8}{25}$ | $\frac{7}{8}$ | $\frac{1}{20}$ |
| $1, \frac{3}{2}; 1, \frac{1}{2}$ | $\sqrt{144}$ | $\frac{2}{25}$ | $-\sqrt{2}$ | $\frac{10}{20}$ |
| $2, 2; 0, 1$ | $\sqrt{10}$ | $\frac{5}{25}$ | $\frac{7}{8}$ | $\frac{10}{20}$ |
| $2, 2; 0, 0$ | $\sqrt{45}$ | $-\sqrt{10}$ | $\frac{1}{8}$ | $\frac{5}{20}$ |

one, corresponding to $Y_1 = N_c/3 - 1, I_1 = 2$ and $Y_2 = 1, I_2 = 1/2$, in order to check the orthogonality of columns, given by Eq. (3), valid at every $N_c$. Column ”$35$” has the same phase for all entries as that of Cohen and Lebed and column ”$27$” has opposite phase for all entries. It may happen that the phase conventions of de Swart and Hecht coincide sometimes.

The column ”$10_a$” is entirely different, inasmuch as we use the definition of Hecht to define the representations with multiplicity 2. We have also added the column ”$10_s$” where the first three entries vanish at $N_c = 3$, as observed in Ref. [8], but the last entry does not. Such a result may be important for large $N_c$ baryon studies [22].

We note that our isoscalar factors for ”$10_a$” have different limits at $N_c \to \infty$ as compared to those Cohen and Lebed, which may also be important for large $N_c$ studies. However, at $N_c = 3$ the values of these isoscalar factors are the same in both cases.

Appendix A

In Table III we reproduce part of Table 4 of Hecht’s paper [4] which contains the analytic expressions of the isoscalar factors $\langle (\lambda \mu)Y_1I_1; (11)Y_2I_2|(\lambda \mu)YI \rangle$, often used in quark physics. Note that the entry in the column $\rho = 2$ for $Y_2 = 1, I_2 = \frac{1}{2}, I_1 = I + 1/2$ contains a misprint in Hecht’s paper which has been here corrected. This means that in the numerator the bracket $(\lambda + \mu + 2 - q + 1)$ has been replaced by $(\lambda + \mu + 2 - q)$ and in the denominator
TABLE II: Isoscalar factors for the irreducible representations with \( Y = \frac{N_c}{3}, I = 3/2 \) from the decomposition of the direct product “10” \( \times 8 \) obtained from Table III.

| \( Y_1 I_1; \ Y_2 I_2 \) | “35”       | “27”       | “10”\(_a\) | “10”\(_s\) |
|--------------------------|------------|------------|------------|------------|
| \( \frac{N_c}{3}; \frac{3}{2}; 0, 1 \) | \( \sqrt{\frac{12}{16(N_c + 9)}} \) | \( \sqrt{\frac{5}{4(N_c + 1)}} \) | \( \sqrt{\frac{45}{N_c^2 + 6N_c + 45}} \) | \( -\sqrt{\frac{1}{16(N_c + 9)} (N_c - 3)(N_c + 5)(N_c + 6)^2} \) |
| \( \frac{N_c}{3}; \frac{3}{2}; 0, 0 \) | \( \sqrt{\frac{60}{16(N_c + 9)}} \) | \( -\sqrt{\frac{9}{4(N_c + 1)}} \) | \( \sqrt{\frac{N^2}{N_c^2 + 6N_c + 45}} \) | \( \sqrt{\frac{45(N_c - 3)(N_c + 5)}{(N_c + 1)(N_c + 9)(N_c^2 + 6N_c + 45)}} \) |
| \( \frac{N_c}{3}; -1, 1; \frac{1}{2}, 1 \) | \( \sqrt{\frac{15(N_c + 5)}{16(N_c + 9)}} \) | \( \sqrt{\frac{N_c + 5}{16(N_c + 1)}} \) | \( -\sqrt{\frac{9(N_c + 5)}{4(N_c^2 + 6N_c + 45)}} \) | \( \sqrt{\frac{5(N_c - 3)^2}{4(N_c + 1)(N_c + 9)(N_c^2 + 6N_c + 45)}} \) |
| \( \frac{N_c}{3}; -1, 2; \frac{1}{2}, 1 \) | \( -\sqrt{\frac{(N_c - 3)}{16(N_c + 9)}} \) | \( \sqrt{\frac{15(N_c - 3)}{16(N_c + 1)}} \) | \( \sqrt{\frac{15(N_c - 3)}{4(N_c^2 + 6N_c + 45)}} \) | \( \sqrt{\frac{3(N_c + 5)(N_c + 21)^2}{4(N_c + 1)(N_c + 9)(N_c^2 + 6N_c + 45)}} \) |
TABLE III: Isoscalar factors $\langle (\lambda \mu) Y_1 I_1; (11) Y_2 I_2 \rangle | (\lambda \mu) Y I \rangle$ of Hecht’s Table 4, p.31 with corrections for the row $Y_2 = 1$, $I_2 = 1/2$, $I_1 = I + 1/2$.

| $Y_2 I_2$ | $I_1$ | $(\lambda' \mu') = (\lambda \mu)$ | $(\lambda' \mu') = (\lambda \mu)$ |
|-----------|-------|---------------------------------|---------------------------------|
| $-\frac{1}{2}$ | $I + 1/2$ | $\sqrt{2} \frac{3(p+1)(\lambda - p)(\mu + 2 + p)}{4g_{\lambda \mu}}^{1/2}$ | $\frac{2g_{\lambda \mu} q - \mu(\lambda + \mu + 1)(\lambda + 2\mu + 6)[(p + 1)(\lambda - p)(\mu + 2 + p)]^{1/2}}{\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)2g_{\lambda \mu}(\mu + p - q + 1)^{1/2}}$ |
| $-\frac{1}{2}$ | $I - 1/2$ | $\sqrt{2} \frac{3(q+1)(\mu - q)(\lambda + \mu + 1 - q)}{4g_{\lambda \mu}}^{1/2}$ | $\frac{2g_{\lambda \mu} p + \lambda(\mu + 2)(\lambda - \mu + 3)[(q + 1)(\mu - q)(\lambda + \mu + 1 - q)]^{1/2}}{\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)2g_{\lambda \mu}(\mu + p - q + 1)^{1/2}}$ |
| 00 | $I$ | $\frac{-2\lambda + \mu - 3p - 3q}{\sqrt{2}4g_{\lambda \mu}^{1/2}}$ | $\frac{\sqrt{3} \lambda \mu(\mu + 2)(\lambda + \mu + 1 + q)}{\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)2g_{\lambda \mu}^{1/2}}$ |
| 01 | $I + 1$ | 0 | $\frac{2(p + 1)(\lambda - p)(\mu + 2 + p)q(\mu + 1 - q)(\lambda + \mu + 2 - q)g_{\lambda \mu}}{\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)(\mu + p - q + 1)(\mu + p - q + 2)^{1/2}}$ |
| 01 | $I - 1$ | 0 | $\frac{-2p(\lambda + 1 - p)(\mu + 1 + p)(q + 1)(\mu - q)(\lambda + \mu + 1 - q)g_{\lambda \mu}}{\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)(\mu + p - q + 1)(\mu + p - q + 2)^{1/2}}$ |
| 01 | $I$ | $\frac{3(\mu + p - q)(\mu + p - q + 2)}{\sqrt{2}4g_{\lambda \mu}^{1/2}}$ | $\frac{E}{2[\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)g_{\lambda \mu}(\mu + p - q)(\mu + p - q + 2)^{1/2}}$ |
| $\frac{1}{2}$ | $I + 1/2$ | $\sqrt{2} \frac{3q(\mu + 1 - q)(\lambda + \mu + 2 - q)}{4g_{\lambda \mu}(\mu + p - q + 1)^{1/2}}$ | $\frac{2g_{\lambda \mu} p + \lambda(\mu + 2)(\lambda - \mu + 3)[q(\mu + 1 - q)(\lambda + \mu + 2 - q)]^{1/2}}{\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)2g_{\lambda \mu}(\mu + p - q + 1)^{1/2}}$ |
| $\frac{1}{2}$ | $I - 1/2$ | $\sqrt{2} \frac{3p(\lambda + 1 - p)(\mu + 1 + p)}{4g_{\lambda \mu}(\mu + p - q + 1)^{1/2}}$ | $\frac{-2g_{\lambda \mu} q - \mu(\lambda + \mu + 1)(\lambda + 2\mu + 6)[p(\lambda + 1 - p)(\mu + 1 + p)]^{1/2}}{\lambda(\lambda + 2\mu)(\mu + 2)(\lambda + \mu + 1)(\lambda + \mu + 3)2g_{\lambda \mu}(\mu + p - q + 1)^{1/2}}$ |
TABLE IV: Values of $\lambda'$, $\mu'$, $p$ and $q$ needed for $Y = N_c/3$, $I = 3/2$ to calculate the isoscalar factors of "$10 \times 8$" using Table [II]. The label $(\lambda'\mu')$ identifies the irreps of the Clebsch-Gordan series [2] for a given $(\lambda\mu)$ in the left hand side. The isoscalar factors are presented in Table [II].

|   | $\lambda'$ | $\mu'$ | $p$   | $q$   | $(\lambda'\mu')$ |
|---|------------|--------|-------|-------|------------------|
| 35 | $N_c - 1$  | $N_c - 1$ | 3     | $\frac{3}{2}$ (\lambda + 1, \mu + 1) |
| 27 | $N_c + 1$  | $N_c - 1$ | 2     | $\frac{1}{2}$ (\lambda - 1, \mu + 2) |
| 10 | $N_c - 3$  | $N_c - 3$ | 3     | $\frac{1}{2}$ (\lambda) |

the bracket $(\mu + p - q)$ has been replaced by $(\mu + p - q + 1)$. In Table [II] we have used $g_{\lambda\mu}$ defined by Eq. (10) and $E$ defined by

$$E = \lambda(\lambda + \mu + 1)\mu(\mu + 2)(2\lambda + \mu + 6) + 2(\lambda + \mu + 1)\mu \\
\times [\lambda(\lambda + 2) - (\mu + 2)(\mu + 3)]p - \mu(\lambda + \mu + 1)(\lambda + 2\mu + 6)p^2 \\
-2\lambda[(\mu + 1)(\lambda + \mu + 1)(2\lambda + \mu + 6) - \mu g_{\lambda\mu}]q + \lambda(\mu + 2)(\lambda - \mu + 3)q^2 \\
-2[\lambda(\lambda + \mu + 1)(2\lambda + \mu + 6) - g_{\lambda\mu}]pq + 2g_{\lambda\mu}(p^2q + pq^2). \quad (A1)$$

Table [II] and the rest Table 4 of Hecht can straightforwardly be applied to a given $(\lambda'\mu')$ with definite values of $Y$ and $I$, from which one can obtain the integers $p$ and $q$ defined as

$$Y = p + q - \frac{2\lambda' + \mu'}{2}, \quad I = \frac{\mu' + p - q}{2} \quad (A2)$$

given by Hecht where $Y$ is related to the a quantity called $\epsilon$ by

$$\epsilon = -3Y. \quad (A3)$$

For $\lambda = 3$ and $\mu = \frac{N_c - 3}{2}$ the values of $\lambda'$, $\mu'$ together with $p$ and $q$ defined by Eqs. (A2) are listed in Table [IV].

We believe there is no reason to reproduce the full Table 4 of Hecht which contains four distinct tables.

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