CONNECTION BLOCKING IN NILMANIFOLDS AND OTHER HOMOGENEOUS SPACES

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ABSTRACT. Let \( G \) be a connected Lie group acting locally simply transitively on a manifold \( M \). By connecting curves in \( M \) we mean the orbits of one-parameter subgroups of \( G \). To block a pair of points \( m_1, m_2 \in M \) is to find a finite set \( B \subset M \setminus \{m_1, m_2\} \) such that every connecting curve joining \( m_1 \) and \( m_2 \) intersects \( B \). The homogeneous space \( M \) is blockable if every pair of points in \( M \) can be blocked. Motivated by the geodesic security \([4]\), we conjecture that the only blockable homogeneous spaces of finite volume are the tori \( \mathbb{R}^n/\mathbb{Z}^n \). Here we establish the conjecture for nilmanifolds.

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1. INTRODUCTION

The theme of finite blocking has its genesis in a Leningrad Mathematical Olympiad problem \([7, 29]\) worded as follows. The president and a terrorist are moving in a rectangular room. The terrorist intends to shoot the president with his ‘magic gun’ whose bullets bounce of the walls perfectly elastically: The angles of incidence and reflection are equal. Presidential protection detail consists of superhuman body
guards. They are not allowed to be where the president or the terrorist are located, but they can be anywhere else, changing their locations instantaneously, as the president and the terrorist are moving about the room. Their task is to put themselves in the way of terrorist’s bullets shielding the president. The problem asks how many body guards suffice.

To translate this into mathematical setting, let \( \Omega \) be a bounded plane domain. For arbitrary points \( p, t \in \Omega \) let \( \Gamma(p, t) \) be the family of billiard orbits in \( \Omega \) connecting these points. Body guards correspond to \( b_1, \ldots, b_N \in \Omega \setminus \{p, t\} \) such that every \( \gamma \in \Gamma(p, t) \) passes through one of these points. If for any \( p, t \in \Omega \) there is a blocking set \( B = B(p, t) = \{b_1, \ldots, b_N\} \) then the domain is uniformly secure. The minimal possible \( N \) is then the blocking number of \( \Omega \). The Olympiad problem is to show that a rectangle is uniformly secure and to find its blocking number.

The solution is an exercise in plane geometry based on two facts: i) A rectangle tiles the euclidean plane under reflections; ii) The torus \( \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) is uniformly secure, where the role of billiard orbits is played by the images of straight lines under the projection \( \mathbb{R}^2 \to \mathbb{T}^2 \). A blocking set in the torus is the set of midpoints of all joining segments: It comprises at most 4 points. A blocking set in the rectangle is also the set of midpoints of all joining billiard orbits. It comprises at most \( 16 = 4 \times 4 \) points where the factor 4 is due to the 4 copies of the rectangle needed to tile the torus.

The billiard orbits in the rectangle and the straight lines in the torus are examples of geodesics in riemannian manifolds. The bizarre olympiad problem grew into the subject of riemannian security. Namely, for a pair of (not necessarily distinct) points \( m_1, m_2 \) in a riemannian manifold \( M \) let \( \Gamma(m_1, m_2) \) be the set of geodesic segments joining these points. A set \( B \subset M \setminus \{m_1, m_2\} \) is blocking if every \( \gamma \in \Gamma(m_1, m_2) \) intersects \( B \). The pair \( m_1, m_2 \) is secure if there is a finite blocking set \( B = B(m_1, m_2) \). A manifold is secure if all pairs of points are secure. If there is a uniform bound on the cardinalities of blocking sets, the manifold is uniformly secure and the best possible bound is the blocking number.

i) What closed riemannian manifolds are secure? ii) What plane polygons are secure? It was the latter question that first got into the literature \cite{21}. A polygon is rational if its corners have \( \pi \)-rational angles. By \cite{21}, all rational polygons are secure. The author studied the security of translation surfaces \cite{16,17} and proved that the regular \( n \)-gon is secure if and only if \( n = 3, 4, 6 \) \cite{12}. Since all regular polygons are rational, this disproves the claim in \cite{21}. The work \cite{13} contains related results on the security of rational polygons, but question ii)
remains wide open [15]. Question i) has been studied in [1, 3, 5, 6, 8, 9, 14, 18, 19, 20, 24, 26]. The following conjecture is widely accepted:

**Conjecture 1.** A closed riemannian manifold is secure if and only if it is flat.

Flat manifolds are uniformly secure, and the blocking number depends only on their dimension [18, 4, 12]. They are also midpoint secure, i.e., the midpoints of connecting geodesics yield a finite blocking set for any pair of points [18, 4, 12]. Conjecture 1 says that flat manifolds are the only secure manifolds. This was verified for several special cases: A manifold without conjugate points is uniformly secure if and only if it is flat [6, 24]; a compact locally symmetric space is secure if and only if it is flat [18]. The generic manifold is insecure [8, 9, 19]. Generic two-dimensional tori are totally insecure, i.e., have no secure pairs of points [4]. Any riemannian metric has an arbitrarily close, insecure metric in the same conformal class [19]. Riemannian surfaces of genus greater than one are totally insecure [4].

This paper adds evidence to the validity of Conjecture 1 albeit indirectly. Integral curves of a *spray* on a differentiable manifold play the role of geodesics on a riemannian manifold [27]. In particular, they yield the set of *connecting curves* for any pair of points in $M$. This allows us to speak of (in)security for sprays the same way we did for riemannian manifolds.

In this work we study this question for *Lie sprays* on homogeneous spaces $M = G/\Gamma$ where $G$ is a Lie group and $\Gamma \subset G$ is a lattice. Connection curves are the orbits of one-parameter subgroups of $G$. To avoid confusion, we do not use the term “security” in this setting. We speak of finite blocking instead. The counterpart of “secure” in this context is the term blockable. See Section 2. The Lie spray analog of Conjecture 1 is as follows:

**Conjecture 2.** Let $M = G/\Gamma$ where $G$ is a connected Lie group and $\Gamma \subset G$ is a lattice. Then $M$ is blockable if and only if $G = \mathbb{R}^n$, i.e., $M$ is a torus.

Our main result, Theorem 2, establishes Conjecture 2 for nilmanifolds. Minimal geodesics proved to be a useful tool in riemannian security [2, 22, 3]. The main tool in the present study is the geometry of Lie groups [25, 1, 28, 10]. Section 2 recalls some properties of spaces $G/\Gamma$. Section 3 characterizes blockable pairs of points in nilmanifolds of the classical heisenberg group. In Section 4 we prove Conjecture 2 for nilmanifolds. In Section 5 we characterize blockable pairs of points in arbitrary heisenberg manifolds. Section 6 reduces midpoint blocking
in $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$ to a study of square roots of $\text{SL}(n, \mathbb{Z})$-cosets. We conclude with a conjecture about midpoint blocking in $G/\Gamma$ for simple noncompact Lie groups.

2. Connection blocking in homogeneous spaces

We will study homogeneous spaces $M = G/\Gamma$, where $G$ is a connected Lie group, and $\Gamma \subset G$ is a lattice. For $g \in G, m \in M$ we denote by $g \cdot m$ the action of $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\exp : \mathfrak{g} \to G$ be the exponential map. For $m_1, m_2 \in M$ let $C_{m_1,m_2}$ be the set of parameterized curves $c(t) = \exp(tx) \cdot m, 0 \leq t \leq 1$, such that $c(0) = m_1, c(1) = m_2$. We say that $C_{m_1,m_2}$ is the collection of connecting curves for the pair $m_1, m_2$. Let $I \subset \mathbb{R}$ be any interval. If $c(t), t \in I$, is a curve, we denote by $c(I) \subset M$ the set \{c(t) : t \in I\}. A finite set $B \subset M \setminus \{m_1, m_2\}$ is a blocking set for the pair $m_1, m_2$ if for any curve $c$ in $C_{m_1,m_2}$ we have $c([0, 1]) \cap B \neq \emptyset$.

If a blocking set exists, we will say that the pair $m_1, m_2$ is connection blockable, often suppressing the adjective ‘connection’. We will also say that $m_1$ is blockable (resp. not blockable) away from $m_2$.

**Definition 1.** A homogeneous space $M = G/\Gamma$ is connection blockable if every pair of its points is blockable. If there exists at least one non-blockable pair of points in $M$, then $M$ is non-blockable.

The analogy with riemannian security [12, 24, 20, 5] suggests the following:

**Definition 2.** 1. A homogeneous space $M = G/\Gamma$ is uniformly blockable if there exists $N \in \mathbb{N}$ such that every pair of its points can be blocked with a set $B$ of cardinality at most $N$. The smallest such $N$ is the blocking number for $M$.

2. A pair $m_1, m_2 \in M$ is midpoint blockable if the set \{c(1/2) : c \in C_{m_1,m_2}\} is finite. A homogeneous space is midpoint blockable if all pairs of its points are midpoint blockable.

3. A homogeneous space is totally non-blockable if no pair of its points is blockable.

**Proposition 1.** Let $M = G/\Gamma$ where $\Gamma \subset G$ is a lattice, and let $m_0 = \Gamma/\Gamma \in M$. Then the following holds:

1. The homogeneous space $M$ is blockable (resp. uniformly blockable, midpoint blockable) if and only if all pairs $m_0, m$ are blockable (resp.}

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1. Our framework is valid for uniform and nonuniform lattices.

2. Some authors prefer to use the terms ‘blocking’ or ‘light blocking’ in the riemannian setting [8, 9, 20, 24, 26].
uniformly blockable, midpoint blockable). The space \( M \) is totally non-blockable if and only if no pair \( m_0, m \) is blockable;

2. Let \( \tilde{\Gamma} \subset \Gamma \) be lattices in \( G \), let \( M = G/\Gamma, \tilde{M} = G/\tilde{\Gamma} \), and let \( p : \tilde{M} \to M \) be the covering. Let \( m_1, m_2 \in M \) and let \( \tilde{m}_1, \tilde{m}_2 \in \tilde{M} \) be such that \( m_1 = p(\tilde{m}_1), m_2 = p(\tilde{m}_2) \). If \( B \subset M \) is a blocking set for \( m_1, m_2 \) then \( p^{-1}(B) \) is a blocking set for \( \tilde{m}_1, \tilde{m}_2 \).

3. Let \( G', G'' \) be connected Lie groups with lattices \( \Gamma' \subset G', \Gamma'' \subset G'' \), and let \( M' = G'/\Gamma', M'' = G''/\Gamma'' \). Set \( G = G' \times G'', M = M' \times M'' \). Then a pair \((m'_1, m''_1), (m'_2, m''_2)\) in \( M \) is connection blockable if and only if both pairs \( m'_1, m'_2 \in M' \) and \( m''_1, m''_2 \in M'' \) are connection blockable.

Proof. Claim 1 is immediate from the definitions. The proofs of claim 2 and claim 3 are analogous to the proofs of their counterparts for riemannian security. See Proposition 1 in [18] for claim 2, and Lemma 5.1 and Proposition 5.2 in [9] for claim 3.

Let \( M_1, M_2 \) be homogeneous spaces. We will use the following terminology. Suppose that one of them is blockable (or not), midpoint blockable (or not), totally non-blockable (or not), etc if and only if the other one is. We will then say that both spaces have identical blocking properties.

Recall that two subgroups \( \Gamma_1, \Gamma_2 \subset G \) are commensurable, \( \Gamma_1 \sim \Gamma_2 \), if there exists \( g \in G \) such that the group \( \Gamma_1 \cap g\Gamma_2g^{-1} \) has finite index in both \( \Gamma_1 \) and \( g\Gamma_2g^{-1} \). Commensurability yields an equivalence relation in the set of lattices in \( G \). We will use the following immediate Corollary of Proposition 1.

**Corollary 1.**

1. If lattices \( \Gamma_1, \Gamma_2 \subset G \) are commensurable, then the homogeneous spaces \( M_i = G/\Gamma_i : i = 1, 2 \) have identical blocking properties.

2. Let \( M_1 = G_1/\Gamma_1, M_2 = G_2/\Gamma_2 \) be homogeneous spaces. Let \( M = M_1 \times M_2 = (G_1 \times G_2)/(\Gamma_1 \times \Gamma_2) \). Then \( M \) is blockable (resp. midpoint blockable, uniformly blockable) if and only if both \( M_1 \) and \( M_2 \) are blockable (resp. midpoint blockable, uniformly blockable).

Let \( \exp : \mathfrak{g} \to G \) be the exponential map. For \( \Gamma \subset G \) denote by \( p_\Gamma : G \to G/\Gamma \) the projection, and set \( \exp_\Gamma = p_\Gamma \circ \exp : \mathfrak{g} \to G/\Gamma \). We will say that a pair \((G, \Gamma)\) is of exponential type if the map \( \exp_\Gamma \) is surjective. Let \( M = G/\Gamma \). For \( m \in M \) set \( \Log(m) = \exp_\Gamma^{-1}(m) \).

**Proposition 2.** Let \( G \) be a Lie group, let \( \Gamma \subset G \) be a lattice such that \((G, \Gamma)\) is of exponential type, and let \( M = G/\Gamma \).
Then \( m \in M \) is blockable away from \( m_0 \) if and only if there is a map \( x \mapsto t_x \) of \( \text{Log}(m) \) to \((0,1)\) such that the set \( \{ \exp(t_x) : x \in \text{Log}(m) \} \) is contained in a finite union of \( \Gamma \)-cosets.

**Proof.** Connecting curves are \( c_x(t) = \exp(tx)\Gamma/\Gamma \) for some \( x \in \text{Log}(m) \). Since \( c(1) = m \), there is \( \gamma \in \Gamma \) such that \( \exp(x) = g\gamma \). Thus
\[
(1) \quad c(t) = \exp(t \log(g\gamma)) \cdot m_0
\]
for some \( \gamma \in \Gamma \), and every such curve is connecting \( m_0 \) with \( m \).

Suppose \( m \) is blockable away from \( m_0 \), and let \( B \subset G/\Gamma \) be a blocking set. Let \( t_x \in (0,1) \) be such that \( c_x(t_x) \in B \). Set \( A = \{ \exp(t_x) : x \in \text{Log}(m) \} \subset G \). Then \( (\Lambda\Gamma/\Gamma) \subset B, \) hence finite. Thus \( A \) is contained in a finite union of \( \Gamma \)-cosets.

On the other hand, if for any collection \( \{ t_x \in (0,1) : x \in \text{Log}(m) \} \) the set \( A = \{ \exp(t_x) : x \in \text{Log}(m) \} \) is contained in a finite union of \( \Gamma \)-cosets, then \( (\Lambda\Gamma/\Gamma) \subset M \) is a finite blocking set. \( \square \)

If \( A \subset G \) any subset, we will say that
\[
(2) \quad \text{Sqrt}(A) = \{ g \in G : g^2 \in A \}.
\]
is the square root of \( A \). We will say that a pair \((G,\Gamma)\) is of *virtually exponential type* if there exists \( \tilde{\Gamma} \sim \Gamma \) such that \((G,\tilde{\Gamma})\) is of exponential type.

**Corollary 2.** Let \( \Gamma \subset G \) be a lattice such that \((G,\Gamma)\) is of virtually exponential type. Then:

1. The homogeneous space \( M = G/\Gamma \) is midpoint blockable if and only if the square root of any coset \( g\Gamma \) is contained in a finite union of \( \Gamma \)-cosets.
2. Any point in \( M \) is midpoint blockable away from itself if and only if the square root of \( \Gamma \) is contained in a finite union of \( \Gamma \)-cosets.

**Proof.** By Corollary [1], we can assume that \((G,\Gamma)\) is of exponential type. Set \( t_x \equiv 1/2 \) in Proposition [2] \( \square \)

3. **Connection blocking in three-dimensional heisenberg manifolds and some other two-step nilmanifolds**

For readers’ convenience, we will recall basic facts about connected, simply connected, real nilpotent Lie groups [23, 1]. Let \( G \) be as above.

Its Lie algebra \( \mathfrak{g} \) has an ascending tower of ideals \( \{ 0 \} \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{p-1} \subset \mathfrak{g}_p = \mathfrak{g} \) such that \( \mathfrak{g}_i/\mathfrak{g}_{i-1} \) is the center of \( \mathfrak{g}/\mathfrak{g}_{i-1} \). We will say that \( \mathfrak{g} \) (resp. \( G \)) is a \( p \)-step nilpotent Lie algebra (resp. group). When \( p = 2 \), the above decomposition becomes \( \mathfrak{c} \subset \mathfrak{g} \) where \( \mathfrak{c} \) is the center of \( \mathfrak{g} \) and \( \mathfrak{g}/\mathfrak{c} \) is abelian.
The map \( \exp : G \rightarrow G \) is a diffeomorphism. Set \( \log = \exp^{-1} \). For \( t \in \mathbb{R} \) we define the diffeomorphism \( g \mapsto g^t \) of \( G \) by \( g^t = \exp(t \log g) \). The haar measure in \( G \) is both left and right invariant. All lattices \( \Gamma \subset G \) are uniform [25, 1]. Referring to a measure on a nilmanifold \( M = G/\Gamma \), we will always mean the unique invariant probability measure \( \mu \). By a measure on the set of pairs \( m_1, m_2 \in M \) we will mean the measure \( \mu \times \mu \).

3.1. Blocking in the classical heisenberg manifold.

The unique nonabelian nilpotent Lie algebra of 3 dimensions is \( \mathfrak{h} = \mathbb{R}X + \mathbb{R}Y + \mathbb{R}Z \), where \([X,Y] = Z\) and \([X,Z] = [Y,Z] = 0\). For historical reasons, \( \mathfrak{h} \) is usually called the heisenberg Lie algebra. In the modern terminology, there is an infinite sequence of heisenberg Lie algebras \( \mathfrak{h}_n, n \geq 1 \), where \( \mathfrak{h}_n \) is a two-step nilpotent Lie algebra of \( 2n + 1 \) dimensions. The corresponding simply connected nilpotent groups \( H_n, n \geq 1 \), are the heisenberg groups; the nilmanifolds \( H_n/\Gamma \) are the heisenberg manifolds. In this subsection we study blocking in a special heisenberg manifold.

We will denote \( \mathfrak{h}_1 \) and \( H_1 \) by \( \mathfrak{h} \) and \( H \) respectively. In order to avoid confusion with the material in section 5, we will speak of the classical heisenberg group and the classical heisenberg manifold. It is standard to represent \( \mathfrak{h} \) and \( H \) by \( 3 \times 3 \) matrices:

\[
\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, \quad H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.
\]

We will use the notation

\[
h(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.
\]

The classical Heisenberg manifold is \( M = H/\Gamma \) where \( \Gamma = \{h(p,q,r) : p,q,r \in \mathbb{Z}\} \). Using the unique decomposition \( h = h(a,b,c)h(p,q,r) \) where \( 0 \leq a, b, c < 1, p, q, r \in \mathbb{Z} \), we identify \( M \) as a set with the unit cube \( Q = [0,1)^3 \). For \((a,b,c) \in Q\) we denote by \( m(a,b,c) \in M \) the corresponding element. Then \( m_0 = m(0,0,0) \).

For \( h = h(x, y, z) \) set \( \pi_x(h) = x \mod 1, \pi_y(h) = y \mod 1, \pi_z(h) = z \mod 1 \). Thus \( \pi_x, \pi_y, \pi_z : H \rightarrow \mathbb{R}/\mathbb{Z} \). We will denote by \( + \) the addition in \( \mathbb{R}/\mathbb{Z} \), i.e., \( x + y = x + y \mod 1 \). We will need a criterion for a subset of \( H \) to be contained in a finite union of \( \Gamma \)-cosets.

**Lemma 1.** Let \( W \subset H \). Set \( A = \pi_x(W), B = \pi_y(W), C = \pi_z(W) \).
Then $|WT/G| < \infty$ if and only if the sets $A, B$ are finite and $C \subseteq \bigcup_{i=1}^{N} \{c_i \oplus Za_i : a_i \in A\}$.

Proof. The identity

\[ h(a, b, c)h(p, q, r) = h(a + p, b + q, c + qa + r) \]

implies that $W \subset h\Gamma$ if and only if there exist $a, b, c \in \mathbb{R}/\mathbb{Z}$ such that $A = \{a\}, B = \{b\}$, and $C \subset (c \oplus Za)$.

If $a, c \in \mathbb{R}/\mathbb{Z}$, we will refer to the set $c \oplus Za \subset \mathbb{R}/\mathbb{Z}$ as a rotation orbit.

**Proposition 3.** An element $m = m(a, b, c) \in M$ is blockable away from $m_0$ if and only if $b \in \mathbb{Q}a + \mathbb{Q}$.

Proof. By a straightforward calculation

\[ (h(x, y, z))^t = h(tx, ty, tz + \frac{t(t-1)}{2}xy). \]

Let $(a, b, c) \in \mathbb{Q}, (p, q, r) \in \mathbb{Z}^3$. Set $h = h(a, b, c), \gamma = h(p, q, r)$, and let $0 < t \leq 1$. Equation (4) implies

\[ (h\gamma)^t = h(t(a + p), t(b + q), t(c + r + qa) + \frac{t(t-1)}{2}(a + p)(b + q)). \]

By Proposition 2, $m(a, b, c)$ is blockable away from $m_0$ if and only if for each $(p, q, r) \in \mathbb{Z}^3$ there exist $0 < t_{pqr} < 1$ such that the set

\[ W = \{(h(a, b, c)h(p, q, r))^t_{pqr} : p, q, r \in \mathbb{Z}\} \]

is contained in a finite union of $\Gamma$-cosets. Set $A = \pi_x(W), B = \pi_y(W), C = \pi_z(W)$.

Then

\[ A = \bigcup_{p,q,r \in \mathbb{Z}} t_{pqr}a \oplus t_{pqr}b \oplus t_{pqr}q, \quad B = \bigcup_{p,q,r \in \mathbb{Z}} t_{pqr}b \oplus t_{pqr}q \]

and

\[ C = \{t_{pqr}c \oplus t_{pqr}r \oplus t_{pqr}qa \oplus \frac{t_{pqr}(t_{pqr}-1)}{2}qa \oplus \frac{t_{pqr}(t_{pqr}-1)}{2}pb \oplus \frac{t_{pqr}(t_{pqr}-1)}{2}ab \oplus \frac{t_{pqr}(t_{pqr}-1)}{2}pq : p, q, r \in \mathbb{Z}\}. \]

The sets $A$ and $B$ are finite if and only if $T = \bigcup_{p,q,r \in \mathbb{Z}} \{t_{pqr}\}$ is a finite subset of $\mathbb{Q}$. Then $C$ is contained in a finite union of rotation orbits by elements in $A$ if and only if $b \in \mathbb{Q}a + \mathbb{Q}$. The claim now follows from Lemma 1.

**Corollary 3.** Let $m_1 = m(a_1, b_1, c_1), m_2 = m(a_2, b_2, c_2)$ be arbitrary points in the classical heisenberg manifold. Then the pair $m_1, m_2$ is blockable if and only if $b_1 - b_2 \in \mathbb{Q}(a_1 - a_2) + \mathbb{Q}$.  


Proof. Set \( h = h(-a_1, -b_1, -c_1 - a_1 b_1) \). Then \( h \cdot m_1 = m_0, h \cdot m_2 = m(a_2 - a_1, b_2 - b_1, c_2 - c_1 - a_1(b_2 + b_1)) \). The claim follows from the invariance of blockability under the group action and Proposition 3. \( \square \)

We will now study connection blocking in nilmanifolds \( \tilde{M} = H/\tilde{\Gamma} \), where \( \tilde{\Gamma} \subset H \) is an arbitrary lattice. If the lattices \( \Gamma', \Gamma'' \subset H \) are isomorphic by an automorphism of \( H \), the nilmanifolds \( H/\Gamma', H/\Gamma'' \) have identical blocking properties. Any lattice in \( H \) is isomorphic by an automorphism of \( H \) to \( \Gamma(\delta) = \{ g(\delta p, q, r) : p, q, r \in \mathbb{Z} \} \) where \( \delta \in \mathbb{N} \). Thus, it suffices to analyze connection blocking in nilmanifolds \( M_\delta = H/\Gamma(\delta) \).

Set \( Q_\delta = [0, \delta) \times [0, 1) \times [0, 1) \). The decomposition \( h = h(a, b, c)h(\delta p, q, r) \), where \( 0 \leq a < \delta, 0 \leq b, c < 1, p, q, r \in \mathbb{Z} \), identifies \( M_\delta \) as a set with \( Q_\delta \). For \( (a, b, c) \in Q_\delta \) we denote by \( m(\delta)(a, b, c) \in M_\delta \) the corresponding element.

**Proposition 4.** Let \( m_1 = m(\delta)(a_1, b_1, c_1), m_2 = m(\delta)(a_2, b_2, c_2) \) be arbitrary points in \( M_\delta \). Then the pair \( m_1, m_2 \) is blockable if and only if \( b_1 - b_2 \in \mathbb{Q}(a_1 - a_2) + \mathbb{Q} \).

Proof. By Corollary 3 the claim holds for \( \delta = 1 \). Let \( \delta > 1 \). The inclusion \( \Gamma_\delta \subset \Gamma \) yields the \( \delta \)-to-1 covering \( p_\delta : M_\delta \to M \) which, under the identifications \( M = \tilde{M}, M_\delta = Q_\delta \), becomes \( p_\delta(a, b, c) = (a \mod 1, b, c) \).

Let \( m_1, m_2 \in M_\delta \) be any pair. By Proposition 1 it is blockable if and only if the pair \( p_\delta(m_1), p_\delta(m_2) \in M \) is blockable. The claim now follows from Corollary 3. \( \square \)

**Theorem 1.** Let \( M \) be any three-dimensional heisenberg manifold. Then the following claims hold.
1. A pair of points in \( M \) is blockable if and only if it is midpoint blockable.
2. Every point in \( M \) is blockable away from itself.
3. The set of blockable pairs of points is a dense countable union of closed submanifolds of positive codimension.

Proof. We assume without loss of generality that \( M = M_\delta \) for some \( \delta \in \mathbb{N} \). Let \( m_i = m(\delta)(a_i, b_i, c_i), i = 1, 2 \) be any pair of points in \( M \). By Proposition 4 the pair \( m_1, m_2 \) is blockable if and only if \( b_1 - b_2 \in \mathbb{Q}(a_1 - a_2) + \mathbb{Q} \). But then, by Proposition 1 and Proposition 3 it is midpoint blockable. This proves claim 1. Claim 2 is immediate from Proposition 4.

Let now \( m_1 = m(\delta)(a_1, b_1, c_1) \in M \). For any \( (a_2, c_2) \in [0, \delta) \times [0, 1) \) the set of numbers \( b_2 \in [0, 1) \) such that \( b_1 - b_2 \in \mathbb{Q}(a_1 - a_2) + \mathbb{Q} \) is countably dense. By Proposition 4 the set of elements \( m_2 = m(\delta)(a_2, b_2, c_2) \)
such that the pair $m_1, m_2$ is blockable, is a dense countable union of two-dimensional manifolds. Claim 3 follows. \hfill \Box

Let $M = M_\delta$ be a three-dimensional heisenberg manifold. The identification $M_\delta = Q_{\delta}$ by $m = m^{(\delta)}(a, b, c)$, sends $\mu$ to the normalized lebesgue measure.

\textbf{Corollary 4.} Almost all pairs of points in a three-dimensional heisenberg manifold are not blockable.

\textbf{Proof.} Let $(M \times M)_{\text{block}} \subset M \times M$ be the set of blockable pairs. By the proof of Theorem 1, for any $m_1 \in M$ the set \{ $m_2 \in M : (m_1, m_2) \in (M \times M)_{\text{block}}$ \} is a countable union of subsets of positive codimension. Hence $\mu (\{ m_2 \in M : (m_1, m_2) \in (M \times M)_{\text{block}} \}) = 0$. The claim follows, by the Fubini theorem. \hfill \Box

3.2. Blocking in a family of two-step nilmanifolds.

Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra with the center $\mathfrak{c}$ such that $\text{dim}(\mathfrak{g}/\mathfrak{c}) = 2$. Let $G$ be the connected, simply connected Lie group with the Lie algebra $\mathfrak{g}$.

\textbf{Proposition 5.} Let $C \subset G$ be the center, and let $d + 1 = \text{dim}(C)$, where $d \geq 0$. Let $\Gamma \subset G$ be a lattice. Set $N = G/\Gamma$.

Then $N = M \times T^d$ where $M$ is a three-dimensional heisenberg manifold and $T^d = \mathbb{R}^d/\mathbb{Z}^d$ is the $d$-dimensional torus. A pair of points $n_1 = (m_1, t_1), n_2 = (m_2, t_2)$ in $N$ is blockable if and only if the pair $m_1, m_2 \in M$ is blockable.

\textbf{Proof.} If $d = 0$ then $G = H_1$, and there is nothing to prove. Thus, we assume from now on that $d \geq 1$. Let $Z \in \mathfrak{c}$ be the unique, up to scalar multiple, element such that $Z = [X, Y]$ for some $X, Y \in \mathfrak{g}$. By [25], we can choose elements $X, Y, Z$ so that $\exp X, \exp Y, \exp Z \in \Gamma$. Let $\mathfrak{v} \subset \mathfrak{c}$ be the $d$-dimensional subspace complementary to $\mathbb{R}Z$, spanned by elements $v_1, \ldots, v_d$ such that $\exp v_1, \ldots, \exp v_d \in \Gamma$. Let $V = \exp \mathfrak{v} \subset C$ and let $L = V \cap \Gamma$. Then $\exp : \mathfrak{g} \to G$ induces the isomorphisms $V = \mathbb{R}^d, L = \mathbb{Z}^d$.

Set $\mathfrak{h} = \mathbb{R}X + \mathbb{R}Y + \mathbb{R}Z$ and $H = \exp \mathfrak{h} \subset G$. Then $H$ is the classical heisenberg group, and $\Gamma \cap H \subset H$ is a lattice. Let $M = H/(\Gamma \cap H)$ and $T^d = V/L$. The decompositions $G = H \times V, \Gamma = (\Gamma \cap H) \times L$ yield the first claim.

By Corollary 1, the pair $n_1, n_2$ is blockable if and only if both pairs $m_1, m_2 \in M$ and $t_1, t_2 \in T^d$ are blockable. Connection curves in $T^d$ are the geodesics for a flat metric. Since a flat torus is secure [18], the claim follows. \hfill \Box
Corollary 5. Let $G$ be a two-step nilpotent Lie group with the center $C$ satisfying $\dim(G/C) = 2$. Let $M$ be a $G$-nilmanifold. Then the following claims hold.
1. A pair of points in $M$ is blockable if and only if it is midpoint blockable.
2. Every point in $M$ is blockable away from itself.
3. The set of blockable pairs of points is a dense countable union of closed submanifolds of positive codimension.

Proof. Mimic the proof of Theorem 1, using Proposition 5 instead of Proposition 4. □

4. Connection blocking in arbitrary nilmanifolds

Let $G$ be a connected, simply connected, nilpotent Lie group and let $\Gamma \subset G$ be a lattice. Let $M = G/\Gamma$ be the corresponding nilmanifold.

Proposition 6. If $G$ is not abelian, then there exist nonblockable pairs of points in $M$.

Proof. Let $C \subset G$ be the center of $G$ and let $\mathfrak{C} \subset \mathfrak{G}$ be its Lie algebra. Then $\mathfrak{C} \subset \mathfrak{G}$ is a proper inclusion, and $\dim(\mathfrak{G}) - \dim(\mathfrak{C}) \geq 2$ [23]. There are $X, Y \in \mathfrak{G}$ and $Z \in \mathfrak{C}, Z \neq 0$ such that $[X, Y] = Z$ [23]. Moreover, we can choose these elements so that $\exp X, \exp Y, \exp Z \in \Gamma$ [25]. Set $\mathfrak{G}_1 = \mathbb{R}X + \mathbb{R}Y + \mathfrak{C}$, and let $G_1 \subset G$ be the corresponding subgroup. Then $G_1$ is a two-step nilpotent Lie group with the center $C$ and $\dim(G_1/C) = 2$. The group $\Gamma_1 = \Gamma \cap G$ is a lattice in $G_1$. Let $M_1 = G_1/\Gamma_1$ be the corresponding nilmanifold.

Let $m_0 \in M$ be the base point. Then $G_1 \cdot m_0 = M_1 \subset M$. If a pair $m_1, m_2 \in M_1$ is not blockable in $M_1$, then it is not blockable in $M$. The claim now follows from Corollary 5. □

Theorem 2. Let $M$ be a nilmanifold of $n$ dimensions. Then the following statements are equivalent:
1. The manifold is connection blockable;
2. The manifold is midpoint blockable;
3. We have $\pi_1(M) = \mathbb{Z}^n$;
4. It is a topological torus;
5. It is uniformly blockable; the blocking number depends only on its dimension.

Proof. Let $M = G/\Gamma$. By Proposition 6, if $G$ is not abelian, then $M$ is not blockable. If $G$ is abelian, then $M = \mathbb{R}^n/\mathbb{Z}^n$; connecting curves are the geodesics in any flat riemannian metric. Since flat tori are secure
$M$ is connection blockable. Thus, $M$ is connection blockable if and only if $M = \mathbb{R}^n / \mathbb{Z}^n$.

Malcev [25] proved that compact nilmanifolds are isomorphic if and only if they are homeomorphic if and only if they have the same fundamental group. Thus, statements 1, 3, and 4 are equivalent. Flat manifolds, in particular, flat tori are midpoint secure [12, 4, 5]. The canonical parameter for connecting curves is proportional to the arc length parameter. Thus, tori are midpoint blockable, proving the equivalence of statements 1 and 2. The implication $1 \rightarrow 5$ is a consequence of the observation that the blocking number for flat tori of $n$ dimensions is $2^n$ [12] and the Bieberbach theorem [4].

**Corollary 6.** Each point in a nilmanifold $M^n$ is blockable away from itself. Moreover, the blocking is uniform, and the optimal bound depends only on $n$.

**Proof.** Let $M = G/\Gamma$ be any homogeneous manifold. By Corollary 2 either all points in $M$ are blockable away from themselves or no point in $M$ is blockable away from itself. The former happens if and only if $|\text{Sqrt}(\Gamma)\Gamma/\Gamma| < \infty$. For lattices in nilpotent Lie groups this property holds [25]. Moreover, there exist $c_n \in \mathbb{N}$ such that for any lattice $\Gamma$ in a nilpotent Lie group $G$ of $n$ dimensions, $|\text{Sqrt}(\Gamma)\Gamma/\Gamma| < c_n$ [25]. The second claim follows, by Proposition 2. □

**5. Blocking in arbitrary Heisenberg manifolds**

Let $n \geq 1$. For $\vec{x}, \vec{y} \in \mathbb{R}^n, z \in \mathbb{R}$ set

$$h_n(\vec{x}, \vec{y}, z) = \begin{bmatrix} 1 & \vec{x} & z \\ 0 & \text{Id}_n & \vec{y} \\ 0 & 0 & 1 \end{bmatrix},$$

where $\vec{x}$ (resp. $\vec{y}$) is the row (resp. column) vector. The group $H_n = \{h_n(\vec{x}, \vec{y}, z) : \vec{x}, \vec{y} \in \mathbb{R}^n, z \in \mathbb{R}\}$ is the $(2n + 1)$-dimensional heisenberg group. Heisenberg manifolds are the nilmanifolds $H_n/\Gamma$ where $\Gamma \subset H_n$ is a lattice. For $\vec{d} = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}_+^n$ let $\vec{d}\mathbb{Z}^n = \{(\delta_1 k_1, \ldots, \delta_n k_n) : \vec{k} \in \mathbb{Z}^n\}$. The group $\Gamma_n(\vec{d}) = \{h_n(\vec{x}, \vec{y}, z) : \vec{x} \in \vec{d}\mathbb{Z}^n, \vec{y} \in \mathbb{Z}^n, z \in \mathbb{Z}\} \subset H_n$ is a lattice. Set $M_n(\vec{d}) = H_n/\Gamma_n(\vec{d})$. We will first study blocking in nilmanifolds $M_n = M_n(\vec{d})$ for $\vec{d} = \vec{1}$. Using the unique decomposition $h = h(\vec{a}, \vec{b}, c)h(\vec{p}, \vec{q}, r)$ where $\vec{a}, \vec{b} \in [0, 1)^n, c \in [0, 1), \vec{p}, \vec{q} \in \mathbb{Z}^n, r \in \mathbb{Z}$, we identify $M_n$ as a set with the $(2n + 1)$-dimensional cube $Q_{2n+1} = [0, 1)^n \times [0, 1)^n \times [0, 1)$. For $(\vec{a}, \vec{b}, c) \in Q_{2n+1}$ we denote by $m(\vec{a}, \vec{b}, c) \in M_n$ the corresponding element. Then $m_0 = m(\vec{0}, \vec{0}, 0)$. 

For \( h = h(\vec{x}, \vec{y}, z) \) set \( \pi_x(h) = \vec{x} \mod I, \pi_y(h) = \vec{y} \mod I, \pi_z(h) = z \mod 1 \). Thus \( \pi_x, \pi_y : H \to \mathbb{R}^n / \mathbb{Z}^n, \pi_z : H \to \mathbb{R} / \mathbb{Z} \). By \( \oplus \) we will denote the addition in \( \mathbb{R}^k / \mathbb{Z}^k \) for any \( k \).

**Lemma 2.** Let \( W \subset H_n \). Set \( A = \pi_x(W), B = \pi_y(W), C = \pi_z(W) \). Then \( |WT/G| < \infty \) if and only if the sets \( A, B \) are finite and \( C \subset \bigcup_i [c_i \oplus \mathbb{Z}(\vec{q}_i, \vec{a}_i) : \vec{a}_i \in A, \vec{q}_i \in \mathbb{Z}^n] \).

The proof of Lemma 2 is the obvious modification of the argument in Lemma 1 and we leave it to the reader. By Lemma 2 a set \( W \subset G \) satisfies \( |WT/G| < \infty \) if and only if the sets \( A, B \) are finite, and the set \( C \) is contained in a finite union of orbits of rotation by \( \langle \vec{q}, \vec{a} \rangle, \vec{a} \in A \). We denote by \( L(n, \mathbb{R}), L(n, \mathbb{Q}), L(n, \mathbb{Z}) \) the sets of \( n \times n \) matrices with entries in \( \mathbb{R}, \mathbb{Q}, \mathbb{Z} \) respectively.

**Proposition 7.** A pair \( m_1 = m(\vec{a}_1, \vec{b}_1, c_1), m_2 = m(\vec{a}_2, \vec{b}_2, c_2) \) in \( M \) is blockable if and only if there exist a matrix \( L \in L(n, \mathbb{Q}) \) and a vector \( \vec{l} \in \mathbb{Q}^n \) such that \( \vec{b}_1 - \vec{b}_2 = L(\vec{a}_1 - \vec{a}_2) + \vec{l} \).

**Proof.** Mimic the proofs of Corollary 3, replacing Lemma 1 by Lemma 2.

We will now study connection blocking in nilmanifolds \( M_n(\vec{\delta}) \) for arbitrary \( \vec{\delta} \). Set \( [0, \vec{\delta}] = \prod_{i=1}^n [0, \delta_i] \subset \mathbb{R}_+^n \) and \( Q_{2n+1}(\vec{\delta}) = [0, \vec{\delta}] \times [0, 1)^n \times [0, 1) \). The unique decomposition \( h = h(\vec{a}, \vec{b}, c)\gamma \) where \( (\vec{a}, \vec{b}, c) \in Q_{2n+1}(\vec{\delta}), \gamma \in \Gamma_n(\vec{\delta}) \), identifies \( M_n(\vec{\delta}) \) as a set with \( Q_{2n+1}(\vec{\delta}) \). For \( (\vec{a}, \vec{b}, c) \in Q_{2n+1}(\vec{\delta}) \) we denote by \( m(\vec{\delta})(\vec{a}, \vec{b}, c) \in M_n(\vec{\delta}) \) the corresponding element. Then \( m_0(\vec{\delta}) = m(\vec{\delta})(0, 0, 0) \).

**Proposition 8.** Let \( m_1 = m(\vec{a}_1, \vec{b}_1, c_1), m_2 = m(\vec{a}_2, \vec{b}_2, c_2) \) be arbitrary points in \( M_n(\vec{\delta}) \). Then the pair \( m_1, m_2 \) is blockable if and only if there exist a matrix \( L \in L(n, \mathbb{Q}) \) and a vector \( \vec{l} \in \mathbb{Q}^n \) such that \( \vec{b}_1 - \vec{b}_2 = L(\vec{a}_1 - \vec{a}_2) + \vec{l} \).

**Proof.** Set \( |\vec{\delta}| = \delta_1 \cdots \delta_n \). Let \( \Gamma_n \) be the standard integer lattice in \( H_n \). The inclusion \( \Gamma_n(\vec{\delta}) \subset \Gamma_n \) yields the \(|\vec{\delta}|\)-to-1 covering \( M_n(\vec{\delta}) \to M_n \). Now mimic the proof of Proposition 4 replacing Corollary 3 by Proposition 7.

The following theorem summarizes the properties of connection blocking in heisenberg manifolds.

**Theorem 3.** Let \( M \) be any heisenberg manifold. Then the following claims hold.
1. A pair of points in $M$ is blockable if and only if it is midpoint blockable.
2. Every point in $M$ is blockable away from itself.
3. The set of blockable pairs of points is a dense countable union of closed submanifolds of positive codimension.
4. Almost all pairs of points are not blockable.

Proof. By [11], any Lattice $\Gamma \subset H_n$ is isomorphic to $\Gamma_n(\vec{\delta})$ by an automorphism of $H_n$. Hence, it suffices to prove the claims for the nilmanifolds $M_n(\vec{\delta})$. The arguments are the multidimensional versions of those used to prove Theorem [1] and Corollary [1] with Proposition [1] replaced by Proposition [8]. Details are left to the reader. □

6. Blocking in semi-simple homogeneous manifolds: Examples and conjectures

We will illustrate connection blocking in homogeneous spaces $G/\Gamma$ which are not nilmanifolds with an example and will formulate a conjecture. We will need the following Lemma. The proof is straightforward, and we leave it to the reader.

Lemma 3. Let $G$ be a Lie group, and let $\Gamma \subset G$ be a lattice. Let $H \subset G$ be a closed subgroup such that $\Gamma \cap H$ is a lattice in $H$. Let $X = G/\Gamma, Y = H/(\Gamma \cap H)$ be the homogeneous spaces, and let $Y \subset X$ be the natural inclusion.

1. If $Y$ is not connection blockable (resp. not midpoint blockable, etc) then $X$ is not connection blockable (resp. not midpoint blockable, etc).
2. If $Y$ contains a point which is not blockable (resp. midpoint blockable) away from itself, then no point in $X$ is blockable (resp. midpoint blockable) away from itself.

The following Lemma will be used in the proof of Proposition [9].

Lemma 4. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in SL(2, \mathbb{R})$. If $a + 1, d + 1, a + d + 2, c \neq 0$, then

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if and only if

$$z^2 = \frac{c^2}{a + d + 2}, \quad x = \frac{a + 1}{c}z, \quad y = \frac{b}{c}z, \quad w = \frac{d + 1}{c}z.$$
Proof. Equation (6) is equivalent to
\[
\begin{bmatrix}
  x & y \\
  z & w
\end{bmatrix} = \begin{bmatrix}
  w & -y \\
  -z & x
\end{bmatrix} \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}.
\]
Solve for \(x, y, w\) in terms of \(z\) and use \(xw - yz = 1\). \qed

We will now study connection blocking in the homogeneous space \(\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})\).

**Proposition 9.** The space \(\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})\) contains pairs of points which are not midpoint blockable. In particular, no point in \(m \in \text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})\) is midpoint blockable away from itself.

**Proof.** Set \(M_n = \text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})\). Although the exponential map for \(\text{SL}(n, \mathbb{R})\) is not surjective, the pair \((\text{SL}(n, \mathbb{R}), \text{SL}(n, \mathbb{Z}))\) is of exponential type. Let \(m_1, m_2 \in M_n\), let \(g \in \text{SL}(n, \mathbb{R})\) satisfy \(m_2 = g \cdot m_1\). By Proposition 2 and Corollary 2, the pair \(m_1, m_2\) is midpoint blockable if and only if \(\sqrt{SL(n, \mathbb{Z})}\) is contained in a finite union of \(\text{SL}(n, \mathbb{Z})\)-cosets.

Let \(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \text{SL}(2, \mathbb{R}), \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \text{SL}(2, \mathbb{Z})\). By Lemma 4, if \(X \in \sqrt{gSL(2, \mathbb{Z})}\) then \(z^2 = (pc + rd)^2(pa + rb + qc + sd + 2)^{-1}\). Let \(K\) be the smallest field containing \(a, b, c, d\). By the above, the entries of matrices \(X\) in \(\sqrt{gSL(2, \mathbb{Z})}\) contain the square roots of infinitely many \(\mathbb{Z}\)-independent elements in \(K\). Hence, the \(\mathbb{Z}\)-module generated by these matrix entries has infinite \(\mathbb{Z}\)-rank. On the other hand, the \(\mathbb{Z}\)-module generated by the entries of matrices in a finite union of \(\text{SL}(2, \mathbb{Z})\)-cosets has finite \(\mathbb{Z}\)-rank. Therefore, \(\sqrt{g \cdot SL(2, \mathbb{Z})}\) is not contained in a finite union of \(\text{SL}(2, \mathbb{Z})\)-cosets.

For \(n > 2, 1 \leq i \leq n - 1\) let \(G_i \subset \text{SL}(n, \mathbb{R})\) be the group \(\text{SL}(2, \mathbb{R})\) imbedded in \(\text{SL}(n, \mathbb{R})\) via the rows and columns \(i, i + 1\). Then \(G_i \cap \text{SL}(n, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})\), and hence \(G_i \text{SL}(n, \mathbb{Z})/\text{SL}(n, \mathbb{Z}) = \text{SL}(2, \mathbb{R})/\text{SL}(2, \mathbb{Z})\). Set \(M_n^{(i)} = G_i \text{SL}(n, \mathbb{Z})/\text{SL}(n, \mathbb{Z}) \subset M_n\). By Lemma 3, no pair \(m_1, m_2\) in \(M_n^{(i)}\) is midpoint blockable, yielding the former part of the claim.

By Proposition 2 and Corollary 2, elements in \(M_n\) are midpoint blockable away from themselves if and only if \(\sqrt{\text{SL}(n, \mathbb{Z})}\) is contained in a finite union of \(\text{SL}(n, \mathbb{Z})\)-cosets. Since the identity element belongs to all \(G_i\), the set \(\sqrt{\text{SL}(n, \mathbb{Z})}\) is not contained in a finite union of \(\text{SL}(n, \mathbb{Z})\)-cosets, yielding the claim. \qed

The preceding argument shows that no pair \(m_1, m_2\) in \(\text{SL}(2, \mathbb{R})/\text{SL}(2, \mathbb{Z})\) is midpoint blockable. We have not shown this for \(\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})\) if \(n > 2\). However, the above proof and related considerations suggest the following.
Conjecture 3. Let $G$ be a simple, connected, noncompact Lie group, and let $\Gamma \subset G$ be a lattice. Then no pair in $G/\Gamma$ is midpoint blockable.

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