Brownian motion meets Riemann curvature

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Received 7 May 2010
Accepted 12 July 2010
Published 3 August 2010

Online at stacks.iop.org/JSTAT/2010/P08006
doi:10.1088/1742-5468/2010/08/P08006

Abstract. The general covariance of the diffusion equation is exploited in order to explore the curvature effects appearing in Brownian motion over a $d$-dimensional curved manifold. We use the local frame defined by the so-called Riemann normal coordinates to derive a general formula for the mean-square geodesic distance (MSD) at the short-time regime. This formula is written in terms of $O(d)$ invariants that depend on the Riemann curvature tensor. We study the $n$-dimensional sphere case to validate these results. We also show that the diffusion for positive constant curvature is slower than the diffusion in a plane space, while the diffusion for negative constant curvature turns out to be faster. Finally the two-dimensional case is emphasized, as it is relevant for single-particle diffusion on biomembranes.

Keywords: Brownian motion, vesicles and membranes, diffusion

ArXiv ePrint: 1005.0650
1. Introduction

The Brownian motion phenomenon occurs in a wide diversity of physical areas; from colloidal physics to quantum gravity and biophysics (see, for instance, [1]–[3]). In the last decade, motivated by problems in biophysics (see [4]–[7]), intense activity has emerged on the study of diffusion processes on curved manifolds. For instance, the transport phenomena occurring in cell membranes is an interesting and complex problem. In particular, the random motion of a single integral protein or lipid in a cell membrane is difficult to realize, mainly because of the interactions and obstacles with the remaining components of the cell. In addition to this difficulty, the thermal fluctuations produce shape undulations on the curved membrane which have a crucial contribution for these phenomena [8]–[12]. By simplifying this problem to the study of free diffusion on a cell membrane, considered as a regular and continuous two-dimensional surface, Smoluchowski’s diffusion equation has been proposed in [13,14]. Explicit formulae for constant mean and Gaussian curvatures have been presented in [15] and [16], respectively. Nevertheless, the analytical issues that appear on the general surface case have motivated the incorporation of novel computer simulations [17,18] (see also [6] for a related work).

From a theoretical standpoint, the Brownian motion can be used to probe the geometry of the manifold in the spirit of Kac’s famous question: can one hear the shape of a
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drum? [19], and vice versa, the geometry will cause a change in the standard randomness of the particle motion [20]. It is then imminent to analyze a quantitative contribution coming from the geometry, and to be specific, how the curvature of the manifold affects the motion. These questions have arisen in [16], where the local concentration of a diffusing substance is obtained in terms of the local curvature. Furthermore, a detailed study of the way in which the mean values are affected by the curvature for two-dimensional manifolds is presented in [21], with a special emphasis on developable and isotropic surfaces. Also, the study of particular cases presented in [22] suggests that, for negative Gaussian curvature, the diffusion accelerates, whereas the diffusion decelerates for surfaces with positive Gaussian curvature. We should point out that, although with different formalism, these same questions have been posted in [23].

In this work we explore the curvature effects on the Brownian motion when the particle movement takes place on a d-dimensional Riemannian manifold. These effects manifest differently for different physical observables. Here, we will use the geodesic distance as the displacement of the particle, but some other observables can also be defined [14,17,21], where either intrinsic or extrinsic properties of the manifold can be probed [24]. Furthermore, we will take advantage of the general covariance of the diffusion equation to use a special frame defined by the so-called Riemann normal coordinates (RNC) [25]. In this frame we will compute curvature corrections for the mean-square geodesic distance. In particular, we use a technique developed in [26], originally used to compute curvature corrections that appear in effective actions of field theory on curved spaces (see, for instance, [27,28]). Related work concerning the RNC can be found in [29,30]. It is remarkable that, using these coordinates, the geodesic curves on the manifold look like straight lines and therefore the square geodesic distance $s^2$ will have the same structure as the square distance in a Euclidean space: $s^2 = \delta_{ab} y^a y^b$ (where $y^a$ are the RNC). As is shown in this paper, the mean-square geodesic distance is isometric; this is because it only depends on $O(d)$ invariant combinations of the Riemann curvature tensor.

This paper is organized as follows. In section 2 we summarize the geometrical concepts used to approach the Brownian motion. In particular, we introduce the frame defined by the Riemann normal coordinates. In section 3, we present the diffusion equation on curved manifolds and we derive general remarks concerning the short-time regime. In section 4, we focus on the computation of the mean-square geodesic displacement using RNC. The n-dimensional sphere is explored in order to validate our results. In particular, we study the curvature effects for manifolds with constant curvature and, furthermore, the diffusion on curved surfaces. Finally, in section 5 we summarize the main conclusions and perspectives of this work.

2. Geometrical preliminaries and notation

In this section we review the preliminary notions about manifolds and Riemannian geometries (following [31]) needed to describe the Brownian motion. Let us call $M$ a d-dimensional manifold with local coordinates $\varphi: U \subset M \rightarrow \mathbb{R}^d$, where $U$ is a local neighborhood and $\mathbb{R}^d$ is the d-dimensional Euclidean space. As a consequence of the differentiability of the map $\varphi$, the set $U$ is locally diffeomorphic to a piece of Euclidean space. The local coordinates are also denoted by $\varphi(p) = (x^1, \ldots, x^d)$, where $p \in M$. For

doi:10.1088/1742-5468/2010/08/P08006
Each point \( p \) on the manifold we associate a vector space called the tangent space \( T_p\mathcal{M} \), whose elements are denoted by capital letters \( X, Y, Z, \ldots \).

We are interested in manifolds endowed with a Riemannian metric. If \( g_p: T_p\mathcal{M} \times T_p\mathcal{M} \rightarrow \mathbb{R} \) denotes a Riemannian metric, we write

\[
g_p = g_{ab} \, dx^a \otimes dx^b, \tag{1}\]

where \( \{dx^a\} \) constitute a 1-form basis of the dual tangent space \( T_p^\ast\mathcal{M} \) and \( g_{ab} \) is the metric tensor. The Einstein summation rule is adopted for the repeated indexes. The knowledge of the metric tensor components allow us to compute further geometrical quantities. The geometrical meaning of how the manifold is curved is defined in terms of the torsion tensor and the Riemann curvature tensor [26, 29, 30]:

\[
\begin{align*}
T(X, Y) & \equiv \nabla_X Y - \nabla_Y X - [X, Y], \\
R(X, Y, Z) & \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z
\end{align*} \tag{2}
\]

where \( \nabla \) is the affine connection. For our purposes, we will use the Levi-Civita connection, because the Christoffel symbols are symmetric. The quantity \( R_{abcd} \) is a \((1, 3)\)-type one whose elements are denoted by capital letters \( X, Y, Z, \ldots \).

Using the coordinate basis \( \{e_a\} \equiv \{\partial_a\} \) of the tangent space \( T_p\mathcal{M} \), the affine connection defines the components \( \Gamma^a_{\ bc} \) by

\[
\nabla_a e_b \equiv \nabla_{e_a} e_b = \Gamma^c_{\ ab} e_c, \tag{3}
\]

where \( \nabla_a \) stands for the covariant derivative and for the Christoffel symbols \( \Gamma^a_{\ bc} \).\(^1\) The torsion is a \((1, 2)\)-type tensor \( T = T^a_{\ bc} e_a \otimes dx^b \otimes dx^c \), whereas the Riemann curvature is a \((1, 3)\)-type one \( R = R_{abcd} e_a \otimes dx^b \otimes dx^c \otimes dx^d \). The components of these tensors are given in terms of the Christoffel symbols\(^2\). Clearly, the manifold is free torsion for the Levi-Civita connection, because the Christoffel symbols are symmetric. The quantity \( R_{abcd} \equiv g_{af} R^f_{\ bcd} \) satisfies the following useful identities: \( R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \).

Using the Riemann curvature components we can define the Ricci tensor \( R_{ab} \equiv R^c_{\ abc} \) and the scalar curvature \( R_g = g^{ab} R_{ab} \).

There is an important device introduced by Riemann, nowadays called the Riemann normal coordinates [25]. This coordinate system can be defined by mapping a point \( p \) on the manifold to the origin of \( \mathbb{R}^d \) and the following conditions:

\[
g_{ab}(0) = \delta_{ab}, \quad y^a g_{ab}(y) = y^a \delta_{ab}. \tag{4}\]

As is pointed out in [29], the second condition is equivalent to the following gauge condition on the affine connection (see appendix A):

\[
y^a y^b \Gamma^c_{\ ab}(y) = 0. \tag{5}\]

Furthermore, in RNC the Taylor coefficients of the metric tensor can be found in terms of the covariant derivatives of the Riemann curvature tensor [26, 29, 30] as

\[
g_{ab}(y) = \delta_{ab} + \frac{1}{2} R_{acdb}(0) y^c y^d + \frac{1}{6} \nabla_e R_{acdb}(0) y^e y^c y^d
\]

\[
+ \frac{2}{40} R_{acdf}(0) R^f_{\ ghb}(0) y^g y^d y^h + \frac{1}{20} \nabla_e \nabla_f R_{acdb}(0) y^e y^f y^g y^d + \cdots, \tag{6}\]

where \( y \) denotes \( \varphi(q) \), the RNC, and \( q \) belong to the same patch of \( p \). See appendix A for a derivation of this series expansion. Using these coordinates, the geodesic curves

\(^1\) \( \Gamma^a_{\ bc} = \frac{1}{2} g^{ac} (\partial_b g_{ec} + \partial_c g_{eb} - \partial_e g_{bc}) \).

\(^2\) \( T^a_{\ bc} = \Gamma^a_{\ bc} - \Gamma^a_{\ cb} \) and \( R^a_{\ bcd} = \partial_b \Gamma^a_{\ cd} - \partial_c \Gamma^a_{\ bd} + \Gamma^a_{\ de} \Gamma^e_{\ cb} - \Gamma^a_{\ ce} \Gamma^e_{\ db} \).
look like straight lines passing through the point \( p \). Indeed, using the gauge condition (5) and the geodesic equation it is easy to figure that out. Therefore, the geodesic curve can be written as \( y^a(s) = \xi^a s \), where \( s \) is the geodesic distance and \( \xi^a \) are constants [30]. Furthermore, as the geodesic curve is parameterized by the arclength, the coefficients \( \xi^a \) satisfy \( g_{ab} \xi^a \xi^b = 1 \), so the square geodesic distance is given by

\[
s^2 = g_{ab} y^a y^b = \delta_{ab} y^a y^b,
\]

where the last equality comes from the conditions of the RNC (4). This equation is remarkable, because the geodesic distance has the same form as it has in Euclidean geometry.

3. Diffusion and geometry

Here, we introduce the simplest model for the Brownian motion of a free particle, which takes place on a \( d \)-dimensional Riemannian geometry. This is a direct generalization of Brownian motion on Euclidean spaces, which basically consists of replacing the Euclidean Laplacian by the Laplace–Beltrami operator in the diffusion equation [13,14]:

\[
\frac{\partial P(x, x', t)}{\partial t} = D \Delta_g P(x, x', t), \quad P(x, x', 0) = \frac{1}{\sqrt{g}} \delta^{(d)}(x - x').
\]

Here, \( P(x, x', t) \, dv \) is the probability to find the diffusing particle in the volume element \( dv = \sqrt{g} \, dx \) when the particle started to move at \( x' \). The probability distribution \( P(x, x', t) \) is normalized with respect to the volume \( v \) of the manifold and \( D \) is the diffusion coefficient. The operator \( \Delta_g \) is the Laplace–Beltrami operator, which is defined by

\[
\Delta_g = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b),
\]

where \( g = \det(g_{ab}) \). We should point out that, when the manifold is not compact, we will require the probability and all its partial derivatives to vanish at the boundary. The formal solution of the diffusion equation (8) on curved manifolds (see [32,28]) is given in terms of the Minakshisundaram–Pleijel coefficients, which depend on both \( x \) and \( x' \). This solution has already been used in order to describe the concentration of a diffusing substance over curved manifolds in the limit when \( x \to x' \) [16].

Once we have a probability distribution, we want to look at the mean values of physical observables (for example, the mean-square displacement) in order to get information on the Brownian motion. For scalar functions \( \Omega \) (defined on the manifold), the expectation values are defined in the standard fashion

\[
\langle \Omega(x) \rangle_t = \int_M dv \Omega(x) P(x, x', t).
\]

Note that \( \langle \Omega(x) \rangle_t \) also depends on the initial point \( x' \). In principle, it is possible to evaluate the expectation values using the formal solution mentioned above. However, using this procedure it may be very involved because the Minakshisundaram–Pleijel coefficients, as far as we know, are not known for points \( x \) different from \( x' \). Here, we use a different strategy, which will be applied only for physical observables well behaved under actions of \( \Delta_g \).
3.1. General remark on the short-time asymptotic

In the case when \( \Omega(x) \) is well behaved under any number of actions of the Laplace–Beltrami operator, its expectation value can be expanded in Taylor series in the variable \( t \). The \( k \)th derivative of \( \langle \Omega(x) \rangle_t \) at \( t = 0 \) can be computed as follows. First, let us compute the first derivative using the diffusion equation:

\[
\frac{\partial \langle \Omega(x) \rangle_t}{\partial t} = D \int_M dv \Omega(x) \Delta_g P(x, x', t)
\]

\[
= D \int_M dv \Delta_g \Omega(x) P(x, x', t) + \int_M dv \nabla_a J^a,
\]

(11)

where \( J^a \) is a boundary current given by \( J^a = g^{ab} \partial_b (\Omega P) \). Using this procedure it is possible to compute the \( k \)th Taylor coefficient by

\[
\frac{\partial^k \langle \Omega(x) \rangle_t}{\partial t^k} \bigg|_{t=0} = D^k \Delta_g^k \Omega(x'),
\]

(12)

where we dropped all the boundary terms because the probability and its derivatives vanish there. The expectation value for our physical observable is then given by the formal series [33]

\[
\langle \Omega(x) \rangle_t = \left[ 1 - \frac{1}{1!} t \mathcal{H} + \frac{1}{2!} t^2 \mathcal{H}^2 + \cdots \right] \Omega(x) \bigg|_{x=x'}
\]

(13)

where \( \mathcal{H} = -D \Delta_g \). This expression is very useful to access the short-time regime of the Brownian motion. Indeed, given a physical observable which is well behaved under the actions of the Laplace–Beltrami operator, its mean value at the short-time regime can be calculated using (13). In particular, for the plane case \( \mathbb{R}^d \) we want to know the mean-square displacement \( |x|^2 \) (where \( x \in \mathbb{R}^d \) and we have chosen \( x' = 0 \)). In this case, the Laplace–Beltrami reduces simply to the Laplacian \( \partial^a \partial_a \). Applying formula (13) we get the standard kinematical Einstein relation for the mean-square displacement: \( \langle |x|^2 \rangle_t = 2dDt \).

Observe that we cannot compute the mean value for the displacement \( |x| \) by this method because it is not well behaved under the actions of the Laplacian at \( x = 0 \).

4. The mean values of \( s^2 \) and short-time asymptotic

On curved manifolds, there are several quantities that can be used to describe the Brownian motion. For instance, in [17] the particle position is given in terms of the parameterization of a manifold embedded in the ambient space \( \mathbb{R}^3 \). For this case, the displacement is given by the Euclidean norm of the parameterization. However, in [21] the Brownian displacement is defined by the geodesic distance. In addition, using the Monge parameterization for a surface we can also define a projected displacement [14]. Here, nevertheless, we will not discern between these quantities: instead, we will stress the fact that all of them represent different manifestations of the same phenomenon. The analysis between these observables is beyond the scope of this work.

In this paper, we use the geodesic distance as the definition of the displacement of the particle. As in the plane case, this quantity is rotational-and translational-invariant. Furthermore, the geodesic distance is invariant under general coordinate
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transformations. Then, the mean value of $s^2$ will be expected to be invariant under a general coordinate transformation. In what follows we compute the expectation value of the square displacement using the Riemann normal frame centered at the point $p = \varphi^{-1}(0)$. Therefore the particle starts to move at the initial condition and the expectation value of $s^2$ can be computed using (13):

$$\langle s^2 \rangle_t = \sum_{k=1}^{\infty} \frac{G_k}{k!} (Dt)^k$$

(14)

where the terms $G_k = \Delta^k_y s^2|_{y=0}$ ($k = 1, 2, 3, \ldots$) are purely geometric factors. These factors can be computed explicitly using the technology of the RNC. We will compute the first three factors, $G_0$, $G_1$ and $G_3$.

Using the definition of the Laplace–Beltrami operator (9) and $s^2 = \delta_{ab} y^a y^b$, it is not difficult to show that, for every coordinate $y$ on the manifold:

$$\Delta_y s^2 = 2d + y \cdot \partial (\log g),$$

(15)

where $g$ is the determinant of the metric and $y \cdot \partial = y^a \partial_a$. Clearly, when we evaluate at $y = 0$ we get $G_1 = 2d$. The factors $G_2$ and $G_3$ can be found using the result of appendix B, summarized as follows, if $f$ is a differentiable function on the manifold then the Laplace–Beltrami operator acting on $f$ at $y = 0$ is given by $\partial^2 f|_{y=0}$. So, using this result $G_2 = \partial^2 (\Delta_y s^2)|_{y=0}$ and $G_3 = \partial^2 (\Delta^2_y s^2)|_{y=0}$. Then, we need at least the second-order Taylor expansion of $\Delta_y s^2$ and $\Delta^2_y s^2$. Using equation (15) and the Taylor expansion of $\log g$ (see appendix A), we obtain the second order of $\Delta_y s^2$:

$$\Delta_y s^2 = 2d - \frac{2}{3} R_{ab}(0) y^a y^b + O(y^3).$$

(16)

Therefore, the second geometric factor is $G_2 = -\frac{4}{3} R_g$, where the Ricci scalar curvature is evaluated at $y = 0$. For the third geometric factor $G_3$, let $\Delta_y$ act on equation (15):

$$\Delta^2_y s^2 = \frac{1}{2} (\partial_a \log g) g^{ab} \partial_b (y \cdot \partial \log g) + (\partial_a g^{ab}) \partial_b (y \cdot \partial \log g) + g^{ab} \partial_a \partial_b (y \cdot \partial \log g).$$

(17)

Using the perturbative expansion of the inverse metric $g^{ab}$ and the one of $\log g$ it is not difficult to obtain the second order of $\Delta^2_y s^2$ (see appendix A). Therefore, by a straightforward calculation, we get

$$G_3 = \frac{8}{15} R_{ab} R_{ab} - \frac{16}{45} R_{abcd} (R_{abca} + R_{dcba}) - \frac{16}{5} (\nabla^a \nabla^b + \frac{1}{2} g^{ab} \Delta_y) R_{ab}.$$ 

(18)

In general, for the $G_k$ factor we need to compute the second-order perturbative expression for $\Delta^{k-1}_y s^2$.

For the expectation value of $s^2$ at the short-time regime we have considered only the values $k = 1, 2, 3$. Hence, this value can be written as

$$\langle s^2 \rangle_t = 2d Dt - \frac{2}{3} R_g (Dt)^2 + \frac{1}{3!} \left[ \frac{8}{15} R_{ab} R_{ab} - \frac{16}{45} R_{abcd} (R_{abca} + R_{dcba}) - \frac{16}{5} (\nabla^a \nabla^b + \frac{1}{2} g^{ab} \Delta_y) R_{ab} \right] (Dt)^3 + \cdots.$$ 

(19)

As we anticipate in the introduction, the mean-square geodesic distance (i.e. $\langle s^2 \rangle_t$) is deviated from the planar expression by terms which are $O(d)$-invariant as well as invariant under a general coordinate transformation. Clearly, for very short times, the particle
movement is not affected by the curvature of the manifold and the standard mean-square displacement is recovered [20]. This follows from the very nature of the manifold (every local neighborhood looks like a piece of a Euclidean space). However, as the particle explores away from the local boundaries, the movement is affected by the curvature of the manifold. Additionally, the curvature corrections to the planar expression are isometric. Therefore, every manifold which is isometric to a Euclidean space will have null curvature effects for this observable. In particular, this is the case of developable surfaces [21].

4.1. Example: the Brownian motion on the $n$-sphere

Here, we perform two independent calculations in order to validate equation (19). In particular, we compute the mean-square geodesic distance at the short-time regime, when the Brownian motion takes place on an $n$-dimensional sphere. The hypersphere $S^n$ of radius $R$ is defined by

$$S^n = \{ x \in \mathbb{R}^{n+1} : x^2 = R^2 \}. \quad (20)$$

This manifold can be parameterized using the local coordinates $(\theta_0, \theta_1, \ldots, \theta_{n-1})$, where $\theta_0$ takes values in $[0, \pi]$ whereas the remainder are coordinates in $[0, 2\pi)$. In this example, we use a parameterization concerning to the ambient space $\mathbb{R}^{n+1}$. This parameterization is given by $x = (x_0, \ldots, x_n)$,

$$\begin{align*}
x_0 &= R \cos \theta_0 \\
x_1 &= R \sin \theta_0 \cos \theta_1 \\
&\vdots \\
x_n &= R \sin \theta_0 \sin \theta_1 \cdots \sin \theta_{n-1}.
\end{align*} \quad (21)$$

These functions satisfy $x = Rn$, where $n$ is the unit normal vector pointing outwards from the hypersurface of $S^n$. The metric tensor can be computed using $g_{ab} = \partial_a x \cdot \partial_b x$ and it can be written in a matrix form as

$$g_{ab} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\
\sin^2 \theta_0 & \sin^2 \theta_0 \sin^2 \theta_1 & \cdots & 0 \\
0 & \cdots & \sin^2 \theta_0 \cdots \sin^2 \theta_{n-1} \end{pmatrix}. \quad (22)$$

The square root of the metric tensor determinant is $\sqrt{g} = R^n \sin^{n-1} \theta_0 \sin^{n-2} \theta_1 \cdots \sin \theta_{n-1}$. It is remarkable that, using extrinsic geometry, we can easily compute the Riemann curvature tensor. Indeed, using the Gauss–Codazzi equations, $R_{abcd} = K_{ac} K_{bd} - K_{ad} K_{bc}$, where $K_{ab} = \partial_a x \cdot \partial_b n$ are the components of the second fundamental form [34], the Riemann curvature tensor is 

$$R_{abcd} = \frac{1}{R^2} (g_{ac} g_{bd} - g_{ad} g_{bc}). \quad (23)$$

The Ricci tensor and the scalar curvature are then given by

$$R_{ab} = \frac{n-1}{R^2} g_{ab}, \quad R_g = \frac{n(n-1)}{R^2}, \quad (24)$$

doi:10.1088/1742-5468/2010/08/P08006
respectively. Substituting equations (23) and (24) into the expressions of $G_1$, $G_2$ and $G_3$, obtained above, we find

$$G_1 = 2n, \quad G_2 = -\frac{4n(n-1)}{3R^2}, \quad G_3 = \frac{8n(n-1)(n-3)}{15R^4}. \quad (25)$$

Now, in order to perform an independent calculation we compute the mean-square geodesic distance using equation (13). For practical purposes, we use the geodesic curve starting at the north pole defined as the point $x = (1, 0, \ldots, 0)$ of $S^n$. It is not difficult to show that $\theta_0 = (1/R)s$ and $\theta_j = 0$ (with $j = 1, \ldots, n-1$) is a geodesic curve. The geodesic distance is simply given by $s = R\theta_0$.

In this case, the Laplace–Beltrami operator on $S^n$ can be written as

$$\Delta_g = \frac{1}{R^2 \sin^{n-1} \theta_0} \partial_0 \sin^{n-1} \theta_0 \partial_0 + \sum_{i,j=1}^{n-1} \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j, \quad (26)$$

where $\partial_0 \equiv \partial/\partial \theta_0$ and $\partial_j \equiv \partial/\partial \theta_j$. The actions of the Laplace–Beltrami operator on $s^2$ involve only the first term of (26). The first, second and third action of $\Delta_g$ on $s^2$ are given by

$$\Delta_g s^2 = 2(n-1) \theta_0 \cot \theta_0 + 2,$$

$$\Delta_g^2 s^2 = -\frac{2(n-1)}{R^2} \left[ n - 1 + (n-3)(1 + \cot^2 \theta_0)(\theta_0 \cot \theta_0 - 1) \right],$$

$$\Delta_g^3 s^2 = -\frac{2(n-1)(n-3)}{R^4} \left[ (n-1) \cot \theta_0(1 + \cot^2 \theta_0)[\cot \theta_0(3 - \theta_0 \cot \theta_0) \right.
\left. - \theta_0(1 + \cot^2 \theta_0)] + 4(1 + \cot^2 \theta_0)[(\theta_0 \cot \theta_0 - 1)(2 \cot^2 \theta_0 + 1) \right.
\left. - \cot \theta_0(\cot \theta_0 - \theta_0 + 1)] \right]. \quad (27)$$

Therefore taking the limit when $\theta_0 \to 0$ we get (25). Therefore, the mean-square geodesic distance is

$$\langle s^2 \rangle_t = 2nDt - \frac{2(n-1)}{3} \frac{R^2}{R^2} (Dt)^2 + \frac{4n(n-1)(n-3)}{45} \frac{R^4}{R^4} (Dt)^3 + \cdots. \quad (28)$$

This equation is the desired result (19) for the particular case of the $n$-sphere.

### 4.2. The constant curvature spaces

For constant curvature manifolds, the Riemann curvature tensor is given by [34]

$$R_{abcd} = \frac{R_g}{d(d-1)} (g_{ac}g_{bd} - g_{ad}g_{bc}), \quad (29)$$

where $R_g$ is a constant that can be either positive, negative or zero. It is known [34] that the only three solutions for constant curvature are the $d$-dimensional sphere when $R_g > 0$, the $d$-dimensional hyperboloid when $R_g < 0$ and the Euclidean space when $R_g = 0$. For these cases, the mean-square geodesic distance is given by

$$\langle s^2 \rangle_t = 2dDt - \frac{2}{3} R_g (Dt)^2 + \frac{4}{45} \frac{d-3}{d(d-1)} R_g^2 (Dt)^3 + \cdots. \quad (30)$$

doi:10.1088/1742-5468/2010/08/P08006
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Figure 1. The time evolution of the MSD for the spherical $R_g > 0$, hyperbolic $R_g < 0$ and Euclidean spaces $R_g = 0$.

For these cases, the curvature effects depend on the sign of the curvature $R_g$. For the hyperbolic spaces, the MSD deviates from the planar result by an increasing monotonic function (in time) whereas for the spherical space by a decreasing monotonic function. In other words, the geometry of the space affects the Brownian motion in such a way that the particle’s diffusion is accelerated when the curvature is negative $R_g < 0$ whereas it decelerates when the curvature is positive $R_g > 0$; see figure 1. These same results were also observed in the two-dimensional cases in [22].

4.3. The two-dimensional case

The two-dimensional case is the most relevant one for the diffusion on biological membranes. For this case, the Riemann curvature tensor is given by

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

(31)

where $K = R_g/2$ is the Gaussian curvature of the surface [31]. Note that $K$ is not necessarily a constant. The mean-square geodesic distance, at the short-time regime, for an arbitrary surface of Gaussian curvature $K$ is given by

$$\langle s^2 \rangle_t = 4Dt - \frac{4}{3}K(Dt)^2 - \frac{8}{15}[\frac{1}{3}K^2 + 2\Delta gK](Dt)^3 + \cdots.$$  

(32)

This equation captures the curvature effects on Brownian motion over curved surfaces. The diffusion coefficient $D$ depends on specific conditions of the system like the temperature, the viscosity, the thickness of the membrane and the shape of the particle. For instance, the experimental results reported in [5] indicate that the diffusion coefficient is inversely proportional to the radius of the diffusing particle. In principle, the curvature effects shown in equation (32) can be measured experimentally in the limit where thermal fluctuations are not relevant. In what follows, we derive a few consequences of equation (32).

doi:10.1088/1742-5468/2010/08/P08006
Note that the standard result is recovered for those cases of zero Gaussian curvature: the plane and the cylinder. This follows from the fact that the cylinder is isometric to the Euclidean plane. In the case when the diffusion takes place on a circular torus, the metric is given by $ds^2 = r^2 d\theta^2 + (a + r \cos \theta)^2 d\varphi^2$ (with $0 < \theta, \varphi < 2\pi$), where $a$ and $r$ are the two radii. The Gaussian curvature is given by

$$K = \frac{\cos \theta}{r(a + r \cos \theta)}. \quad (33)$$

In figure 2 three regions on the torus are shown. These regions are defined by the conditions $K > 0$, $K < 0$ and $K = 0$, respectively.\(^3\) It is shown in figure 3 how the time evolution of the mean-square displacement depends on the initial position of the particle. Observe that the particle’s diffusion decelerates faster in the region where $K < 0$ than the way it does in the region where $K > 0$. At the parallels ($K = 0$), the particle’s diffusion accelerates; in fact, the MSD in the parallels is given by

$$\langle s^2 \rangle_t = 4Dt + \frac{16}{15a^2r^2}(Dt)^3 + \cdots. \quad (34)$$

In general, the biological membranes have a wide range of morphologies; from discoidal shapes to catenoidal ones \cite{35}. Also, we can consider wavy surface-like membranes with microvilli \cite{13}, elliptic paraboloid, hyperbolic paraboloid \cite{22} and periodic nodal surfaces \cite{17}, where the diffusion will be affected by the geometry. The curvature effects on the diffusion on these surfaces can be quantified using equation (32).

\(^3\) Clearly, the Gaussian curvature vanishes along the parallels ($\theta = \pi/2$ and $3\pi/2$). In the region given by $\pi/2 < \theta < 3\pi/2$, $K$ is negative; whereas in the one given by $0 < \theta < \pi/2$ or $\pi/2 < \theta < 3\pi/2$, $K$ is positive.
5. Conclusions and perspectives

In this paper, we have studied the Brownian motion over \( d \)-dimensional curved manifolds. The model we have considered here is based in the diffusion equation on curved manifolds. Here, the displacement of the particle is defined by the geodesic distance. Using a Riemann normal frame we derived a general formula for the mean-square geodesic distance at the short-time regime. This formula reproduces the standard result \( \langle s^2 \rangle = 2dDt \) for very short times and it is expressed by \( O(d) \) isometric covariant terms, depending upon the Riemann curvature tensor. Indeed, given a Riemannian metric \( g_{ab} \) for a curved manifold (with curvature not necessarily constant), we are able to give an expression for the MSD at the short-time regime. In particular, we explored the diffusion over constant curvature manifolds, where it is shown that the diffusion accelerates for the hyperbolic case whereas it decelerates for the spherical one.

The Brownian motion on two-dimensional surface has been studied as it is relevant for the diffusion on biological membranes. The mean-square displacement can be used to quantify the curvature effects that appear for this case. In particular, the behavior of the diffusion on a circular torus can be classified according to the region (defined by \( K < 0 \), \( K > 0 \) or \( K = 0 \)) where the particle starts to move. In principle, the curvature effects can be measured experimentally by fixing the value of the diffusion coefficient and neglecting the thermal fluctuations of the membrane.

Our approach can be extended in various directions. As we have mentioned, there are several physical observables that describe the motion of a particle diffusing on a curved space. These observables quantify the curvature effects in different ways. For instance, as was discussed in [21] the mean-square geodesic distance has null curvature effects for the

\[ \text{doi:10.1088/1742-5468/2010/08/P08006} \]
diffusion on developable surfaces, whereas using the parameterization displacement \[17\] it is clear that there are curvature effects. Indeed, different physical observables have different manifestations of the same phenomenon. It is then a rather natural question to ask what is the geometrical and physical content of these observables in the context of Brownian motion on curved manifolds \[24\]. Also, we could explore the large-time regime for mean values in the case of compact manifolds. In these cases, the expectation values will have a bounded above limit as a consequence of the compactness of the manifold \[33\]. In particular, it is not difficult to realize that, for the case when diffusion takes place on a sphere, the MSD reaches a constant value for large times. Finally, the phenomenon of mobility of proteins in biomembranes is still an open and complex problem. However, we can highlight several features for this phenomenon. The curvature effects studied above are one of these characteristics. These effects were considered in the limit where the thermal fluctuations of the membrane are not relevant. Nevertheless, as is shown in \[10, 12\], the fluctuations induce an effective value for the diffusion coefficient. Furthermore, the finite size of the protein induces a deformation on the local shape of the membrane, which has a crucial contribution for the effective diffusion constant \[11\]. It would be interesting to derive an effective diffusion equation which includes all these aspects.

Acknowledgments

The author would like to thank Ramon Castañeda Priego and Sendic Estrada Jiménez for many valuable discussions. The support by a PROMEP/1035/08/3291 grant is acknowledged.

Appendix A. Riemann normal coordinates

A.1. The gauge condition

As is mentioned in \[29\] the second condition of (4) is equivalent to the gauge condition (5). In order to prove this we use the Christoffel symbols; therefore we have

\[ y^a y^b \Gamma^{ce}_{ab}(y) = \frac{1}{2} g^{ce}(2 y^a y^a \partial_a g_{eb} - y^b y^b \partial_e g_{ab}). \]  

(A.1)

Now, the first derivative on the second condition of (4) is \( g_{ab} + y^e \partial_a g_{eb} = \delta_{ab} \), then \( y^a y^b \partial_a g_{eb} = 0 \). Similarly, the second term of (A.1) vanishes.

A.2. An affine connection expansion

In order to derive (6) we follow the same procedure presented in \[26\]. Let \( \omega \) be a matrix 1-form connection given by \( \omega = \omega_a \, dx^a \), where the matrix 1-form component \( (\omega_a)^b_c = \Gamma^b_{ac} \) is given by the Christoffel symbols. Let us consider the Lie derivative \( L_X \) along the radial vector \( X = y \cdot \partial \) acting on \( \omega \):

\[ L_X \omega = [(1 + y \cdot \partial)\omega_a] \, dy^a, \]  

(A.2)

where \( y^a \) are the Riemann normal coordinates. The Lie derivative can be written as

\[ L_X = di_X + i_X \, d, \]  

(A.3)
where \( d \) and \( i_X \) are the exterior and interior derivative, respectively [31]. Observe that the gauge condition (5) can be written as \( i_X \omega = 0 \). Therefore, using the Cartan structure equation for the curvature, \( R = d \omega + \omega \wedge \omega \), it is easy to get
\[
\mathcal{L}_X \omega = i_X R - i_X (\omega \wedge \omega) = i_X R - y^a \mathcal{R}_{ab} dy^b.
\]
Equating (A.2) and (A.4), we get the condition \((1 + y \cdot \partial) \omega_a = y^a \mathcal{R}_{ab}, \) where \((\mathcal{R}_{ab})^c_d = R^c_{dab}\) is the Riemann curvature tensor. A Taylor expansion centered at 0 on both sides of this condition gives
\[
\Gamma^a_{bc} = \sum_{k=0}^{\infty} \frac{(y \cdot \partial)^k}{k!(k+2)} y^d R^a_{cdb},
\]
where the Riemann curvature tensor is evaluated at 0.

A.3. A vielbein expansion

As is pointed out in [26], it is sufficient to compute the Taylor expansion of the vielbein \( \theta = \theta_a dx^a \) in order to find (6). The vielbein satisfies the free torsion condition \( d \theta + \omega \wedge \theta = 0 \) and the metric can be rewritten in terms of the vielbein as \( g_{ab} = \theta^i_a \theta^j_b \delta_{ij} \). Using (A.3) and the gauge condition, the Lie derivative on \( \theta \) is given by
\[
\mathcal{L}_X \theta = (i_X \theta) \omega + d i_X \theta.
\]
As \( \theta \) is 1-form, \( i_X \theta \) is scalar. Therefore the second Lie derivative of the last expression is
\[
\mathcal{L}_X \mathcal{L}_X \theta = X [i_X \theta] \omega + (i_X \theta) \mathcal{L}_X \omega + \mathcal{L}_X d i_X \theta.
\]
The additional gauge condition \( y^a g_{ab}(y) = y^a \delta_{ab} \) is equivalent to choosing \( y^a \theta^i_a(y) = y^a \delta^i_a \), then \( X [i_X \theta] = i_X \theta \) and \( \mathcal{L}_X d i_X \theta = d i_X \theta \). Combining equation (A.6) with (A.7), we get
\[
\mathcal{L}_X (\mathcal{L}_X - 1) \theta = (i_X \theta) \mathcal{L}_X \omega.
\]
Now using equation (A.4), we find
\[
\mathcal{L}_X (\mathcal{L}_X - 1) \theta = (i_X \theta) i_X R.
\]
The condition for the vielbein is then given by \((y \cdot \partial + 1)(y \cdot \partial) \theta^i_a = y^j y^b R^i_{jbc} \theta^c_a\). By a Taylor expansion centered at 0 on both sides of this condition we get
\[
\theta^i_a = \delta^i_a + \sum_{k=2}^{\infty} \frac{(y \cdot \partial)^{k-2} [R^i_{cde} \theta^c_a]}{k!(k+1)(k-2)!}, \quad y \cdot \partial \theta^i_a = 0
\]
where \( R^{i}_{cde} \equiv y^a y^b R^{i}_{abc} \). This system of equations can be solved iteratively and they give the following expression for the vielbein until fourth order in \( y^a \):
\[
\theta^i_a = \delta^i_a + \frac{1}{4} y^a R^a_{..a} - \frac{1}{12} y \cdot \partial R^a_{..a} + \frac{1}{720} (y \cdot \partial)^2 R^a_{..a} + \frac{1}{120} R^a_{cde} R^{cde}_{..a} + \cdots.
\]
The metric (6) is then obtained using \( g_{ab} = \theta^i_a \theta^j_b \delta_{ij} \).
A.4. Determinant of the metric

A Taylor expansion of the determinant of the metric \( g \equiv \det g_{ab} \) can be obtained using the identity \( \log \det g_{ab} = \text{tr} \log g_{ab} \). Since the metric (6) can be written as \( g_{ab} = \delta_{ab} + \Lambda_{ab}(y) \), its logarithm is given by \( \log(1 + \Lambda) \approx \Lambda - \frac{1}{2} \Lambda^2 \). The trace of this term is then given by

\[
- \log g = \frac{1}{3} R_{ab}(0) y^a y^b + \frac{1}{6} \nabla_a R_{ab}(0) y^a y^b y^c + \frac{1}{90} R_{cd}(0) R_{\gamma\delta\alpha\beta}(0) R_{\gamma\delta\alpha\beta}(0) y^\alpha y^\beta y^\gamma y^\delta y^\epsilon \nabla_c \nabla_f R_{ab}(0) y^a y^b y^f + \cdots. \tag{A.11}
\]

A.5. Geometric \( G_3 \) factor

In order to get the geometric factor \( G_3 \equiv \Delta_g^2 s^2 \), we need at least the Taylor expansion of \( \Delta_g^2 s^2 \) until second order \( O(y^2) \). By straightforward calculation this expansion is given by

\[
\Delta_g^2 s^2 = -\frac{1}{3} R_g - y^a \partial_a R_g - 2 y^a \nabla_b R_{ab} + \frac{4}{3} R_{cd} R_{\gamma\delta\alpha\beta}(0) y^\alpha y^\beta y^\gamma y^\delta \nabla_c \nabla_f R_{ab}(0) y^a y^b y^f + \cdots.
\tag{A.12}
\]

Appendix B. \( \Delta_g \) on a scalar function

Let \( \Omega : M \to \mathbb{R} \) be a scalar differentiable function on the manifold. Let us take a Riemann normal system of coordinates centered at \( y^a = 0 \), then the Laplace–Beltrami operator acting on \( \Omega \) is given by the second derivative of \( \Omega \), i.e.

\[
\Delta_g \Omega|_{y=0} = \partial^2 \Omega|_{y=0}. \tag{B.1}
\]

In order to show this result we split the inverse metric and the square root of the metric determinant as

\[
g^{ab} = \delta^{ab} + \Lambda^{ab}(y) \quad \sqrt{g} = 1 + \lambda(y), \tag{B.2}
\]

where \( \Lambda^{ab}(y) \) and \( \lambda(y) \) can be found in terms of the covariant derivative of the Riemann curvature as we have seen above. These functions satisfy the following properties: \( \Lambda^{ab}(0) = 0 \), \( \partial_a \Lambda^{ab}(0) = 0 \), \( \lambda(0) = 0 \) and \( \partial_y \lambda(0) = 0 \). Thus the action of \( \Delta_g \) on \( \Omega \) can be written as

\[
\Delta_g \Omega = \{ \partial^2 \Omega + \partial_a \lambda(y) \partial^a \Omega + \lambda(y) \partial^2 \Omega + \partial_a \Lambda^{ab}(y) \partial_b \Omega + \Lambda^{ab}(y) \partial_a \partial_b \Omega \\
+ \partial_a \lambda(y) \Lambda^{ab}(y) \partial_b \Omega + \lambda(y) \partial_a \Lambda^{ab}(y) \partial_b \Omega + \lambda(y) \Lambda^{ab}(y) \partial_a \partial_b \Omega \}_{y=0}. \tag{B.3}
\]

Then, after the substitution \( y = 0 \), the desired result is obtained.

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