SMOOTHING ESTIMATES FOR THE SCHRÖDINGER
EQUATION WITH UNBOUNDED POTENTIALS

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Abstract. We prove a local in time smoothing estimate for a magnetic Schrödinger equation with coefficients growing polynomially at spatial infinity. The assumptions on the magnetic field are gauge invariant and involve only the first two derivatives. The proof is based on the multiplier method and no pseudodifferential techniques are required.

1. Introduction

Smoothing properties of dispersive equations have become a standard tool in the study of nonlinear problems. For the Schrödinger flow on \( \mathbb{R}^n \) the basic smoothing estimate is the following:

\[
\| \langle x \rangle^{-s} |D|^{1/2} e^{it\Delta} f \|_{L^2 L^2} \lesssim \| f \|_{L^2}, \quad s > 1/2.
\]

Here as usual the symbol \( A \lesssim B \) means \( A \leq CB \) for some absolute constant \( C \), \( \langle x \rangle = (1 + |x|^2)^{1/2} \) and \( |D|^r f = \mathcal{F}^{-1}(|\xi|^r \hat{f}(\xi)) \). With \( L^2 L^2 \) we denote the space \( L^2(\mathbb{R}^n; L^2(\mathbb{R}^n)) \).

In the form (1.1) the estimate was proved by Ben-Artzi and Klainerman [3] and Chihara [4], but it can be traced back at least as far as the work of Kato on \( H \)-smoothing [11] and subsequent works of Kato-Yajima, Vega, Sjölin, Constantin-Saut [12, 23, 21, 5]. In view of its importance, especially for the applications to the derivative NLS, it has been extended and improved in a variety of directions (see e.g. [25, 13, 24, 20]). We recall also the close connection of this property with the Morawetz estimates for the wave and Klein-Gordon equation, which play a central role in scattering theory. The gain of 1/2 derivative, at least on a bounded time interval \([-T,T]\), is a quite general phenomenon, extending to Schrödinger equations on manifolds and with variable coefficients. In these general situations, it is well known that smoothing holds as long as the metric has no trapped rays.

A more precise way to express smoothing is using a Morrey-Campanato type norm:

\[
\sup_{R>0} \frac{1}{R^d} \int_{-\infty}^{+\infty} dt \int_{|x| \leq R} |\nabla e^{it\Delta} f|^2 \, dx \leq C \| f \|_{H^{1/2}}
\]

(see [5, 21, 18]). This stronger form of (1.1) can be proved by a variant of Morawetz’ multiplier method; more general pseudodifferential techniques allow only to prove smoothing in the form (1.1).

In the following we shall focus on the variable coefficient problem on \( \mathbb{R}_t \times \mathbb{R}_x^n \)

\[
iu_t(t,x) - (\nabla - iA(t,x))^2 u + V(t,x)u(t,x) = 0
\]

\[u(0,x) = f(x),\]

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for suitable potentials $A(t, x) \in \mathbb{R}^n$ and $V(t, x) \in \mathbb{R}$, $n \geq 3$. For this equation, in general, one can only expect local (in time) smoothing, where the $L^2 L^2$ space is replaced by

$$L^2 T^2 = L^2([-T, T]; L^2(\mathbb{R}^n)), \quad T > 0.$$ 

This was proved by Yajima [26] for smooth potentials $V(t, x)$ with subquadratic growth and magnetic potentials $A(t, x)$ with sublinear growth. This result was further extended by Doi [6] to equations of the form

$$i u_t(t, x) - \sum (D_j - i A_j(t, x)) g^{jk}(x)(D_k - i A_k(t, x)) u + V(t, x) u(t, x) = 0 \quad \text{under suitable assumptions on the metric} \ g.$$ 

It has been known for some time that the quadratic growth represents a critical threshold for potentials. Indeed, the fundamental solution of the Schrödinger propagator corresponding to $-\Delta + V(x)$ with $V(x) \gtrsim \langle x \rangle^{2+\delta}$ is nowhere $C^1$ and can be unbounded at infinity [14]. This reflects in a weaker smoothing property of the solution; Yajima and Zhang ([29], [30]; see also [27]) obtained for the operator $H = -\Delta + V(x)$, with a smooth potential $V(x) \simeq \langle x \rangle^m$, $m \geq 2$, the estimate

$$\left( \int_{-T}^T \int_{|x| \leq R} \left| \langle D \rangle^m e^{i t H} f \right|^2 dx \, dt \right)_{T, R} \leq C_{T, R} \| f \|_{L^2}.$$ 

The result is sharp, in the sense that the analogous estimate with $1/m$ replaced by $s > 1/m$ is false. More recently, Robbiano and Zuily [19] extended (1.5) to general equations of the form (1.4), with $C^\infty$ potentials in suitable symbol classes; as in Doi’s result, the metric must be nontrapping and sufficiently flat at infinity, moreover the electric potential $V$ can grow at most like $\langle x \rangle^m$ and the magnetic potential $A$ can grow at most like $\langle x \rangle^{m/2}$, with corresponding conditions on all derivatives.

All the results mentioned so far are based on pseudodifferential techniques. These allow to handle operators of a very general form, but with some drawbacks:

- The coefficient are required to be $C^\infty$, with conditions involving all the derivatives; this could probably be improved to assumptions involving a large enough number of derivatives.
- A more relevant problem is that these methods hide some important physical aspects; indeed, the assumptions on the magnetic terms are expressed in terms of the vector potential $A(t, x)$, while for example, in dimension $n = 3$, the physically relevant quantity is the vector field $B = \text{curl} A$. In particular, the assumptions are not gauge invariant.
- A precise estimate like (1.2) for the Morrey-Campanato norm of the solution seems difficult to obtain uniquely by pseudodifferential methods.

Our goal here is to follow a different path and adapt the method of multipliers to handle unbounded potentials. Indeed, by elementary methods, we can prove a Morrey-Campanato equivalent of (1.5), and address at the same time some of the problems listed above. In the present work we shall only focus on equations of the form (1.3); note that in order to study a general metric by the multiplier method, it is necessary to exhibit a ‘physical space’ replacement for the nontrapping condition. This is an interesting problem in itself and will be the subject of future work.

We shall express our assumptions on the magnetic field in terms of curl$A$, which has the following standard extension to general space dimension:

**Definition 1.1.** For any $n \geq 2$ the matrix-valued field $B : \mathbb{R}^n \rightarrow \mathcal{M}_{n \times n}(\mathbb{R})$ is defined by

$$B := DA - DA^t, \quad B_{ij} = \frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}.$$
We also define the vector field \( B_\tau : \mathbb{R}^n \to \mathbb{R}^n \) as follows:

\[
B_\tau = \frac{x}{|x|} B.
\]

Of course we can rephrase the definition as \( B = dA \) with \( A = \sum_j A^i dx^j \); in dimension \( n = 3 \), this reduces to \( B = \text{curl} A \), more precisely

\[
Bv = \text{curl} A \wedge v, \quad \forall v \in \mathbb{R}^3.
\]

In particular, we have

\[
(1.6) \quad B_\tau = \frac{x}{|x|} \wedge \text{curl} A, \quad n = 3.
\]

Hence \( B_\tau (x) \) is the projection of \( B = \text{curl} A \) on the tangential space in \( x \) to the sphere of radius \( |x| \), for \( n = 3 \). Observe also that \( B_\tau \cdot x = 0 \) for any \( n \geq 2 \), hence \( B_\tau \) is a tangential vector field in any dimension. Notice that our assumptions on the magnetic field involve \( B_\tau \) exclusively (see \( (1.9) \)) and hence are gauge invariant.

Our main tool will be the following (with the notation \( \nabla_A = \nabla - iA(t, x) \)):

**Magnetic Virial Identity.** Let \( u(t, x) \) be a solution of \( (1.3) \), \( \phi = \phi(|x|) \) a smooth, radial, real valued function and let \( \Theta(t) = \int \phi |u|^2 \, dx \). Denoting with \( V_r \) the radial derivative of \( V \), \( D^2 \phi \) the Hessian matrix and with \( \Delta^2 \phi = \Delta \Delta \phi \) the bilaplacian of \( \phi \), we have

\[
4 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \overline{\nabla_A u} \, dx - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx - 2 \int_{\mathbb{R}^n} |u|^2 \phi' V_r \, dx
\]

\[
+ 4 \int_{\mathbb{R}^n} u \phi' B_\tau \cdot \overline{\nabla_A u} \, dx = \frac{d}{dt} \int_{\mathbb{R}^n} \pi \nabla_A u \cdot \nabla \phi \, dx = \dot{\Theta}(t).
\]

We give a proof of \( (1.7) \) in Section 2 for sufficiently smooth \( (H^{3/2}) \) solutions, by a variant of the classical Morawetz multiplier method. This approach has a long history, starting with [16] for the Klein-Gordon equation, [17], [22], [15]: then the multiplier method was extended to the Helmholtz and wave equations in [18], and for the Schrödinger equation with an electric potential in [1], [2]. In the case of magnetic potentials, a 3D version of the virial identity for Schrödinger first appeared in [9], [10], while in [8] identity \( (1.7) \) is proved for any dimension.

In order to apply the formal identity \( (1.7) \) we shall need the following assumptions: the functions \( V(t, x) \in C^1 \) and \( A(t, x) = (A_1, \ldots, A_n) \in C^2 \) are real valued, and for some constants \( C, c > 0 \) and some \( m \geq 2 \),

\[
(1.8) \quad c|x|^m \leq V(t, x) \leq C|x|^m, \quad m \geq 2;
\]

(but see Remark 1.1 below). Moreover we shall assume that for some \( m/2 \leq \lambda \leq m - 1 \),

\[
(1.9) \quad (\partial_r V)^+ \leq C(x)^{m-1} \quad |\nabla \cdot B_\tau| \leq C(x)^{\lambda}, \quad |B_\tau| \leq C(x)^{\lambda - m/2}
\]

where \( (\partial_r V)^+ \) is the positive part of the radial derivative \( \partial_r V \).

Recall that for superquadratic, time dependent potentials, the existence of the propagator is still partially an open question. Hence we prefer to add an abstract, albeit very natural, assumption concerning the well-posedness of the Cauchy problem \( (1.3) \):

**Assumption (H): well posedness.** For each \( t \in [-T, T] \), the operator

\[
(1.10) \quad H(t) = - (\nabla - iA(t, x))^2 + V(t, x)
\]

is essentially selfadjoint on \( C_0^\infty \), with maximal domain \( D(H(t)) = D(H) \) independent of \( t \); we shall use the notation

\[
(1.11) \quad \mathcal{H}^s = D((H(t))^{s/2}), \quad 0 \leq s \leq 2.
\]
Moreover, we assume that for each \( f \in L^2 \) problem (1.3) has a solution \( u \in C([0,T], L^2) \), which is in \( u \in C([-T,T], H^1) \) for \( f \in H^1 \), and satisfies the estimates

\[
\|u(t)\|_{L^2} \leq C_T \|f\|_{L^2}, \quad \|u(t)\|_{H^1} \leq C_T \|f\|_{H^1}, \quad t \in [-T,T].
\]

Finally, we assume that for \( C_0^\infty \) data the solution is at least in \( C([-T,T], H^{3/2}) \).

Notice that if \( V, A \) do not depend on time, Assumption (H) is trivially satisfied as soon as the operator \( H \) is selfadjoint. As for the general case of superquadratic, time dependent potentials, the optimal conditions for well posedness are not clear. Some partial results in this direction have been obtained by Yajima in [28], where a propagator is constructed under condition slightly more restrictive than (1.8), (1.9) (in particular, quadratic bounds for \( \partial_t V, \partial_t A \) are required).

In the classical Morawetz estimates the tangential component of \( \nabla u \) satisfies better estimates than the full gradient. A similar phenomenon occurs in presence of a magnetic potential; we need to define here the modified radial and tangential derivatives of \( u \) as

\[
\nabla_A^R u = \frac{x}{|x|} \nabla_A u, \quad \nabla_A^T u = \nabla_A u - \frac{x}{|x|} \nabla_A^R u
\]

with \( \nabla_A = \nabla - i A(t, x) \), so that

\[
|\nabla_A^T u|^2 = \sum_{j<k} \frac{x_k}{r} (\partial_k - iA_k)u - \frac{x_j}{r} (\partial_j - iA_j)u|^2.
\]

Notice that

\[
|\nabla u|^2 = |\nabla_A^R u|^2 + |\nabla_A^T u|^2
\]

and indeed \( \nabla_A^T u \) reduces to the usual tangential derivative when \( A \equiv 0 \).

We are in position to state the main result of the paper:

**Theorem 1.2.** Let \( n \geq 3 \), and assume that (1.8), (1.9) and (H) hold for some \( T > 0 \). Then for all data \( f \in H^{1-1/m} \) the solution \( u(t, x) \) of problem (1.3) satisfies for all \( R > 0 \), with a constant \( C \) independent of \( R \), the following smoothing estimates:

when \( n \geq 4 \)

\[
\int_{-T}^{T} \int \left[ \frac{R^{n-1}}{(R \vee r)^n} |\nabla_A^R u|^2 + \frac{|\nabla_A^T u|^2}{r} + \frac{|u|^2}{r^3} \right] dxdt + \frac{1}{R^2} \int_{-T}^{T} \int_{|x|=R} |u|^2 d\sigma dt \leq C \|f\|_{H^{1-1/m}}^2.
\]

while for \( n = 3 \)

\[
\int_{-T}^{T} dt \int \left[ \frac{R^2 |\nabla_A^R u|^2}{(R \vee r)^3} + \frac{|\nabla_A^T u|^2}{r} \right] dx + \frac{1}{R^2} \int_{-T}^{T} dt \int_{|x|=R} |u|^2 d\sigma \leq C \|f\|_{H^{1-1/m}}^2.
\]

If in addition we assume that \( V \) is repulsive, i.e., \( V_\ast \leq 0 \), we can improve the above estimate by replacing the \( H^{1-1/m} \) norm at the right hand side with \( H^{\lambda/m} \).

**Remark 1.1.** In assumption (1.8) we require a growth condition on \( V \) from below; this was one of the original assumptions of Yajima-Zhang [30] for \( V = V(x) \), and was relaxed to

\[
-C(x)^m \leq V(t, x) \leq C(x)^m
\]

(plus the corresponding ones for all derivatives \( \partial_x^\alpha V \)) in Robbiano-Zuily [19]. We prefer to keep here this quite restrictive condition, since it makes it easier to deal with the spaces \( H^s \) used in the statement of our result. Actually, we can reduce any potential satisfying (1.17) to our situation by applying the time-dependent change of gauge

\[
u(t, x) = e^{-ic_0(t-x)^m} u(t, x)\]
which transforms the equation into
\begin{equation}
(1.19) \quad iw_t(t,x) - (\nabla - i\tilde{A}(t,x))^2 w + \tilde{V}(t,x) w(t,x) = 0,
\end{equation}
with
\begin{equation}
(1.20) \quad \tilde{V} = V + c_0(x)^m, \quad \tilde{A} = A + c_0 \nabla(x)^m \cdot t.
\end{equation}
It is easy to check that the other assumptions remain true, with different constants; notice in particular that the field $B$ is unchanged.

2. Proof of the magnetic virial identity

Let $u \in \mathcal{H}^k$ be a solution of (1.3). Recall that the quantity $\Theta_S(t)$ is defined as
\begin{equation}
(2.1) \quad \Theta_S(t) = \int \phi |u(t,x)|^2 dx
\end{equation}
where the radial weight function $\phi$ will be chosen in the following. Writing equation (1.3) in the form
\begin{equation}
(2.2) \quad u_t = -iHu,
\end{equation}
we obtain immediately
\begin{equation}
(2.3) \quad \dot{\Theta}_S(t) = -i \langle u, [H, \phi] u \rangle,
\end{equation}
\begin{equation}
\ddot{\Theta}_S(t) = \langle u, [H, [H, \phi]] u \rangle,
\end{equation}
where the brackets $[,]$ are the commutator and the brackets $\langle,\rangle$ are the hermitian product in $L^2$. In order to simplify the notations, we shall write
\begin{equation}
(2.3) \quad T = -[H, \phi].
\end{equation}

By the Leibnitz formula
\begin{equation}
(2.4) \quad \nabla_A(fg) = g \nabla_A f + f \nabla g,
\end{equation}
which implies
\begin{equation}
(2.5) \quad H(fg) = (Hf)g + 2\nabla_A f \cdot \nabla g + f(\Delta g),
\end{equation}
we can write explicitly
\begin{equation}
(2.6) \quad T = 2\nabla \phi \cdot \nabla_A + \Delta \phi.
\end{equation}
Observe that $T$ is anti-symmetric, namely
\begin{equation}
\langle f, Tg \rangle = -\langle Tf, g \rangle.
\end{equation}
Hence we can rewrite (2.2) in the following form
\begin{equation}
(2.7) \quad \dot{\Theta}_S(t) = \langle u, [H, T] u \rangle,
\end{equation}
where $T$ is given by (2.6).

In the following we shall use the shorthand notations, for a function $f : \mathbb{R}^n \to \mathbb{C}$,
\begin{equation}
\begin{aligned}
f_j &= \frac{\partial f}{\partial x_j}, & f_j^* &= f_j - iA_j f, & f_j^\star &= f_j + iA_j f.
\end{aligned}
\end{equation}
With these notations we have
\begin{equation}
\begin{aligned}
(fg)_j &= f_j^\star g + fg_j
\end{aligned}
\end{equation}
while the integrations by parts formula can be written
\begin{equation}
\int_{\mathbb{R}^n} f_j^\star(x) g(x) dx = -\int_{\mathbb{R}^n} f(x) g_j^\star(x) dx.
\end{equation}

We now compute explicitly the commutator $[H, T]$; by (2.6) we have
\begin{equation}
(2.8) \quad [H, T] = -[\nabla_A^2, 2\nabla \phi \cdot \nabla_A] - [\nabla_A^2, \Delta \phi] + [V, T] =: I + II + III.
\end{equation}
The term $III$ is easy:
\begin{equation}
(2.9) \quad III = [V, T] = 2[V, \nabla_A \cdot \nabla] = -2\nabla \phi \cdot \nabla V = -2\phi'V_r.
\end{equation}
As to $I$, we have

$$-I = 2 \sum_{j,k=1}^{n} \left( \partial_j \partial_j \phi_k \partial_k - \phi_k \partial_j \partial_j \right)$$

(2.10)

$$= \sum_{j,k=1}^{n} \left( 2\phi_{kj} \partial_k - 4\phi_{jk} \partial_j \partial_k + 2\phi_k (\partial_j \partial_j \phi_k - \partial_j \partial_j \partial_k) \right).$$

Notice that

$$\partial_j \partial_k - \partial_k \partial_j = i \left( A_{jk}^l - A_{kj}^l \right),$$

$$\partial_j \partial_j \partial_k - \partial_k \partial_j \partial_j = i \left( A_{jk}^l - A_{kj}^l \right) + 2i \left( A_{jk}^l - A_{kj}^l \right) \partial_k;$$

hence, by (2.10) we obtain

$$-I = \sum_{j,k=1}^{n} \left( 2\phi_{kj} \partial_k + 4\phi_{jk} \partial_j \partial_k + 2\phi_k (\partial_j \partial_j \phi_k - \partial_j \partial_j \partial_k) \right).$$

The term $II$ can be written

$$-II = \sum_{j,k=1}^{n} \left( \partial_k \partial_j \phi_{kj} - \phi_{kj} \partial_k \partial_j \right)$$

(2.11)

$$= \sum_{j,k=1}^{n} \left( \phi_{jk} + 2\phi_{kj} \partial_k \right).$$

By (2.11) and (2.12) we have

$$\langle u, [\nabla_A^2, T]u \rangle = \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2u \phi_{kj} \overline{\partial_k u} + 4u \phi_{jk} \overline{\partial_j \partial_k u} + 2u \overline{\phi_{kj} \partial_k u} \right) dx$$

$$+ \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2i\phi_j \left( A_{jk}^l - A_{kj}^l \right) \partial_k |u|^2 + 4i\phi_j \left( A_{jk}^l - A_{kj}^l \right) \overline{u} \right) dx$$

$$+ \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \ dx.$$

Using the identity

$$\overline{\partial_j \partial_k u} = \partial_j, \partial_k, \overline{u}$$

integrating by parts the first three terms of (2.13) we have

$$\sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2u \phi_{kj} \overline{\partial_k u} + 4u \phi_{jk} \overline{\partial_j \partial_k u} + 2u \overline{\phi_{kj} \partial_k u} \right) dx$$

(2.14)

$$= \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} -4i\phi_j \overline{u} \partial_j u \ dx = -4 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \nabla_A u \ dx.$$

For the 4th and 5th term in (2.13) we notice that

$$\sum_{j,k=1}^{n} \phi_{jk} \left( A_{jk}^l - A_{kj}^l \right) = 0,$$
and integrating by parts we obtain
\[
\sum_{j,k=1}^{n} \int_{\mathbb{R}^n} \left( 2i\phi_j \left( A^k_j - A^k_j \right)_k |u|^2 + 4iu\phi_j \left( A^k_j - A^k_j \right) \overline{u}_k \right) dx
\]
\[
= 4\mathcal{I} \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} u\phi_j \left( A^k_j - A^k_j \right) \overline{u}_k dx
\]
\[
= 4\mathcal{I} \int_{\mathbb{R}^n} u\phi' B_{\tau} \cdot \nabla_A u dx,
\]
with \(B_{\tau}\) as in Definition 1.1.
Collecting (2.9), (2.13), (2.14), (2.15) we conclude that
\[
\langle u, [H,T]u \rangle = 4 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \overline{\nabla_A u} - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi - 2 \int_{\mathbb{R}^n} \phi' V_{\tau} |u|^2 + 4\mathcal{I} \int_{\mathbb{R}^n} u\phi' B_{\tau} \cdot \nabla_A u.
\]
Identities (2.7) and (2.16) imply (1.7).

Remark 2.1. Notice that, in order to justify all of the above computations, it is sufficient to require that the solution \(u\) belongs to \(H^{3/2}\) (recall Assumption (H)); indeed, the highest order term is of the form \(\int \nabla^2 A u \nabla \phi \cdot \nabla A u\).

3. Choice of the multiplier

The precise form of the multiplier \(\phi\) will depend on the space dimension. Writing \(r = |x|\), we introduce the radial function
\[
\phi_0(r) = \int_0^r \phi'_0(\rho) d\rho,
\]
where
\[
\phi'_0(r) = \begin{cases} 
M + \frac{1}{2n} r - \frac{1}{2n(n+2)} r^2, & r \leq 1 \\
M + \frac{1}{2n} - \frac{1}{2n(n+2)} \cdot \frac{1}{r^n}, & r > 1,
\end{cases}
\]
for some constant \(M \geq 1\). Hence we have also
\[
\phi''_0(r) = \begin{cases} 
\frac{1}{2n} - \frac{3}{2n(n+2)} \cdot \frac{1}{r^n}, & r \leq 1 \\
\frac{1}{n-1} \cdot \frac{n-1}{2n(n+2)} \cdot \frac{1}{r^n}, & r > 1.
\end{cases}
\]
Observe that both \(\phi'_0(r)\) and \(\phi''_0(r)\) are positive and continuous on \([0, +\infty)\). In order to compute \(\Delta^2 \phi_0(|x|)\), we start by the laplacian, using the formula
\[
\Delta \phi_0(r) = r^{1-n} \partial_r (r^{n-1} \phi'_0(r)),
\]
which gives
\[
\Delta \phi_0(r) = \begin{cases} 
M(n-1) \cdot \frac{1}{r} + \frac{1}{2} - \frac{1}{2n} r^2, & r \leq 1 \\
M(n-1) \cdot \frac{1}{r} + \frac{n-1}{2n} \cdot \frac{1}{r}, & r > 1;
\end{cases}
\]
also \(\Delta \phi_0(r)\) is continuous on \([0, +\infty)\). Now we can compute the bi-laplacian using the formula
\[
\Delta^2 \phi_0(r) = r^{1-n} \partial_r (r^{n-1} (\Delta \phi_0)'(r)).
\]
Due to the presence of the function \(1/r\) in (3.4), which is the fundamental solution of the laplacian in dimension \(n = 3\), the cases \(n \geq 4\) and \(n = 3\) are slightly different.
Case \( n \geq 4 \). By direct computation, from (3.4) we get
\[
r^{n-1} (\Delta \phi_0)'(r) = \begin{cases} 
-M(n-1)r^{n-3} - \frac{1}{n}r^n, & r \leq 1 \\
-(M + \frac{1}{2n}) (n-1)r^{n-3}, & r > 1.
\end{cases}
\]
Observe that \( r^{n-1} (\Delta \phi_0)'(r) \) is discontinuous at \( r = 1 \), and the jump is given by
\[
(\Delta \phi_0)'(1^+) - (\Delta \phi_0)'(1^-) = -\frac{n-3}{2n}.
\]
As a consequence, (3.5) implies
\[
\Delta_{\phi_0}(r) = -\left(1 + \frac{M(n-1)(n-3)}{r^3}\right) \chi_{[0,1]} - \left(M + \frac{1}{2n}\right) (n-1)(n-3) \cdot \frac{1}{r^3} \chi_{[1,\infty)} - \frac{n-3}{2n} \delta_{r=1}, \quad (n \geq 4),
\]
where \( \delta_{r=1} \) is the Dirac measure supported on the unit sphere of \( \mathbb{R}^n \). Notice that \( \Delta^2 \phi_0 \) is negative.

Case \( n = 3 \). We rewrite (3.4) as
\[
\Delta \phi_0(r) = \varphi(r) + \psi(r),
\]
where
\[
\varphi(r) = 2M \cdot \frac{1}{r},
\]
\[
\psi(r) = \begin{cases} 
\frac{1}{2} - \frac{1}{6} r^2, & r \leq 1 \\
\frac{1}{4} \cdot \frac{1}{r^2}, & r > 1.
\end{cases}
\]
Clearly we have
\[
\Delta \varphi(r) = -8\pi M \delta_{x=0},
\]
\[
\Delta \psi(r) = -\chi_{[0,1]},
\]
where \( \delta_{x=0} \) is the Dirac mass at the origin, and hence
\[
\Delta^2 \phi_0(r) = -\chi_{[0,1]} - 8\pi M \delta_{x=0}, \quad (n = 3).
\]
Notice that also in this case the bilaplacian is negative.

We can now choose the multiplier \( \phi \), which will be defined as a suitable scaling of \( \phi_0 \): for any \( R > 0 \) we set
\[
\phi_R(r) = R \phi_0 \left( \frac{r}{R} \right).
\]
We have explicitly
\[
\phi''_R(r) = \begin{cases} 
M + \frac{1}{2n} \cdot \frac{r}{R} - \frac{1}{2n} (n+2) \cdot \frac{r^2}{R^2}, & r \leq R \\
M + \frac{1}{2n} \cdot \frac{1}{2n(n+2)} \cdot \frac{R^{n-1}}{r}, & r > R,
\end{cases}
\]
\[
\phi''_R(r) = \begin{cases} 
\frac{1}{r} \left( \frac{1}{2n} - \frac{3}{2n(n+2)} \cdot \frac{r^2}{R^2} \right), & r \leq R \\
\frac{1}{r} \left( \frac{n-1}{2n(n+2)} \cdot \frac{R^n}{r^n} \right), & r > R.
\end{cases}
\]
Notice that \( \phi'_R, \phi''_R \) are strictly positive and more precisely
\[
\frac{\phi'}{r} \geq \begin{cases} 
\frac{M}{r} + \frac{n-1}{2n(n+2)} \cdot \frac{1}{r} & \text{if } r \leq R, \\
\frac{M}{r} + \frac{n-1}{2n(n+2)} \cdot \frac{R^{n-1}}{r^n} & \text{if } r > R,
\end{cases}
\]
while
\[
\phi'' \geq \begin{cases} 
\frac{n-1}{2n(n+2)} \frac{1}{r} & \text{if } r \leq R, \\
\frac{n-1}{2n(n+2)} \frac{1}{r^{n-1}} & \text{if } r \geq R.
\end{cases}
\]

Moreover
\[
\sup_{r \geq 0} \phi'_R(r) = M + \frac{1}{2n}, \quad \sup_{r \geq 0} \phi''_R(r) = \frac{1}{2nR}.
\]
The laplacian is given by
\[
\Delta \phi_R(r) = \begin{cases} 
M(n-1) \cdot \frac{1}{r} + \frac{2n}{R^2} - \frac{2n}{R^2}, & r \leq R \\
M(n-1) \cdot \frac{1}{r} + \frac{n-1}{2n} \cdot \frac{1}{r}, & r > R
\end{cases}
\]
whence in particular the estimate
\[
|\Delta \phi_R| \leq \frac{M(n-1)}{r} + \frac{1}{2(r \vee R)}.
\]

Also here the bilaplacian has a different form in the cases \(n \geq 4\) and \(n = 3\). For \(n \geq 4\) we have
\[
\Delta^2 \phi_R(r) = -\left( \frac{1}{R^3} + \frac{M(n-1)(n-3)}{r^3} \right) \chi_{[0,R]}
\]
\[
- \left( M + \frac{1}{2n} \right) (n-1)(n-3) \cdot \frac{1}{r^3} \chi_{[R,\infty)}
\]
\[
- \frac{n-3}{2n} \frac{1}{R^2} \delta_{r=R}, \quad (n \geq 4)
\]
while in dimension \(n = 3\) the bilaplacian is given by
\[
\Delta^2 \phi_R(r) = -\frac{1}{R^3} \chi_{[0,R]} - 8\pi M \delta_{x=0}, \quad (n = 3).
\]
Observe that in both cases the bilaplacian is negative. In the following we shall drop the index \(R\) and write simply \(\phi\) instead of \(\phi_R\).

We can now plug these quantities into the identity (1.7). Let us consider the Hessian term on the L.H.S. of (1.7); using implicit summation over repeated indices, we can write for a generic vector \(v = (v_1, \ldots, v_n)\)
\[
v \cdot D^2 \phi \cdot v = \phi''(r) \left[ \frac{x_i v_i v_j}{r} \right] + \phi'(r) \left[ \frac{v^2 - \frac{x_i v_i}{r}}{r} \right]
\]
with \(v^2 = v_j v_j\). Hence in particular
\[
\nabla_A u \cdot D^2 \phi \cdot \nabla_A u = \phi'' \left[ \frac{x}{|x|} \cdot \nabla_A u \right]^2 + \phi' \left[ |\nabla_A u|^2 - \left\| \frac{x}{|x|} \cdot \nabla_A u \right\|^2 \right].
\]

Then the elementary identity
\[
v^2 u^2 - (v \cdot u)^2 = \sum_{i<j} (v_i w_j - v_j w_i)^2
\]
gives, recalling the notations (1.13), (1.14),
\[
\nabla_A u \cdot D^2 \phi \cdot \nabla_A u = \phi'' |\nabla_A u|^2 + \phi' |\nabla^T_A u|^2.
\]
Now the identity (1.7) can be written
\[
4 \int_{\mathbb{R}^n} \phi'' |\nabla_A u|^2 dx + 4 \int_{\mathbb{R}^n} \phi' |\nabla^T_A u|^2 dx - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi dx - 2 \int_{\mathbb{R}^n} |u|^2 V_r \phi' dx
\]
\[
+ 4 \int_{\mathbb{R}^n} (B_r \cdot \nabla_A u) \phi' u dx = \frac{d}{dt} \int_{\mathbb{R}^n} \nabla_A u \cdot \nabla \phi dx.
\]
Using (3.14), (3.15) and the expressions for $\Delta^2 \phi$, we obtain the following estimates: for $n \geq 4$

\begin{equation}
(n-1)(n-3)M \int \frac{|u|^2}{r^3} dx + \frac{n-3}{2n} \int_{|x|=R} |u|^2 d\sigma + \\
\frac{2(n-1)}{n(n+2)} \int \frac{R^{n-1}}{(R \vee r)^n} |\nabla_A u|^2 dx + 2M \int \frac{|\nabla_A^2 u|^2}{r} dx \leq \\
\leq 2 \int \phi'(V_r) |u|^2 dx + 4 \left| \int \phi' B_r \cdot \nabla_A u \, u \, d\sigma \right| + \frac{d}{dt} \int_{\mathbb{R}^n} \pi \nabla_A u \cdot \nabla \phi \, dx
\end{equation}

while for $n=3$ we have

\begin{equation}
8\pi M |u(t,0)|^2 + \frac{1}{R^3} \int_{|x| \leq R} |u|^2 dx + \frac{4}{15} \int \frac{R^2}{(R \vee r)^3} |\nabla_A u|^2 dx + 2M \int \frac{|\nabla_A^2 u|^2}{r} dx \leq \\
\leq 2 \int \phi'(V_r) |u|^2 dx + 4 \left| \int \phi' B_r \cdot \nabla_A u \, u \, d\sigma \right| + \frac{d}{dt} \int_{\mathbb{R}^n} \pi \nabla_A u \cdot \nabla \phi \, dx.
\end{equation}

4. Proof of Theorem 1.2

By the definition of $H(t)$ we have, for all $|t| \leq T$,

\begin{equation}
\|v\|_{H^1}^2 = \|\nabla_A v\|_{L^2}^2 + \int V|v|^2 \gtrsim \|\langle x \rangle^{m/2} v\|_{L^2}^2
\end{equation}

under our assumptions on $V(t,x)$. Thus by interpolation we get

**Lemma 4.1.** For any $0 \leq \mu \leq m/2$ and any $v \in H^{2\mu/m}$ we have

\begin{equation}
\|\langle x \rangle^\mu v\|_{L^2} \lesssim \|v\|_{H^{2\mu/m}}.
\end{equation}

As a consequence, recalling the energy estimates (1.12), we have for any solution $u(t,x)$

\begin{equation}
\|\langle x \rangle^\mu u\|_{L^2} \lesssim \|u\|_{H^{2\mu/m}} \leq C_T \|u(0)\|_{H^{2\mu/m}}
\end{equation}

provided $0 \leq \lambda \leq m/2$.

Also by interpolation we can prove the following bound which will be used to estimate the left hand side in (3.23), (3.24):

**Lemma 4.2.** For any function $\phi \in C^2(\mathbb{R}^n)$, such that

\begin{equation}
|\nabla \phi| + |x| \cdot |\Delta \phi| \leq K,
\end{equation}

the following inequality holds:

\begin{equation}
\left| \int \mathcal{J} \nabla_A g \cdot \nabla \phi \, dx \right| \leq C(K) \|f\|_{H^{1/2}} \|g\|_{H^{1/2}}.
\end{equation}

Moreover, if $F(t,x)$ satisfies, for some $0 \leq \lambda \leq m$

\begin{equation}
\langle x \rangle^{m/2} |F| + |\nabla F| \leq K(\langle x \rangle)^\lambda
\end{equation}

we have also

\begin{equation}
\left| \int F(t,x) \cdot \mathcal{J} \nabla_A g \cdot \nabla \phi \, dx \right| \leq C(K) \|f\|_{H^{\lambda/m}} \|g\|_{H^{\lambda/m}}.
\end{equation}

**Proof.** Denote by $T(f,g)$ the bilinear operator

\[ T(f,g) = \int \mathcal{J} \nabla_A g \cdot \nabla \phi \, dx. \]
By Cauchy-Schwartz we have immediately
\begin{equation}
|T(f, g)| \leq K \|f\|_{L^2} \|g\|_{H^1}.
\end{equation}
On the other hand, after an integration by parts, we have
\[
T(f, g) = - \int \nabla_A f \cdot g \nabla \phi - \int \bar{f} g \Delta \phi
\]
and again by Cauchy-Schwartz we get
\[
|T(f, g)| \leq K \left\| \frac{f}{|x|} \right\|_{L^2} \|g\|_{L^2} + K \|f\|_{H^1} \|g\|_{L^2}.
\]
Using the magnetic Hardy inequality (Theorem A.1) this implies
\[
|T(f, g)| \leq (K \cdot 2(n-2)^{-1} + K) \|f\|_{H^1} \|g\|_{L^2}.
\]
By interpolation with (4.8) we obtain (4.5).

The proof of (4.7) is similar. Denoting again by $T(g, f)$ the bilinear form at the left hand side of (4.7), we have
\begin{equation}
|T(f, g)| \leq K^2 \|\nabla_A g\|_{L^2} \|\langle x \rangle^{\lambda-m/2} f\|_{L^2} \leq K^2 \|g\|_{H^1} \|f\|_{H^{2\lambda/m-1}}
\end{equation}
by (4.2). Integrating by parts we have instead
\[
T(f, g) = - \int F \cdot \nabla_A f \cdot g \nabla \phi - \int F \bar{g} \Delta \phi - \int \nabla F \cdot \nabla \phi \bar{g} = I + II + III.
\]
The first term is equivalent to $|T(g, f)$ and is estimated as above:
\[
|I| \leq K^2 \|f\|_{H^1} \|g\|_{H^{2\lambda/m-1}}.
\]
Then, using the assumptions on $F, \phi$ we see that
\[
|II| \leq K^2 \|\langle x \rangle^{\lambda-m/2} g\|_{L^2} \|\langle x \rangle^{-1} f\|_{L^2} \leq C(K) \|g\|_{H^{2\lambda/m-1}} \|f\|_{H^1}
\]
where we applied again the magnetic Hardy inequality (A.1). The third term gives
\[
|III| \leq K^2 \int \langle x \rangle^\lambda \|f\|_{L^2} \leq \|\langle x \rangle^{\lambda-m/2} g\|_{L^2} \|\langle x \rangle^{m/2} f\|_{L^2} \leq C(K) \|g\|_{H^{2\lambda/m-1}} \|f\|_{H^1}.
\]
In conclusion we have proved that
\[
|T(f, g)| \leq C(K) \|g\|_{H^{2\lambda/m-1}} \|f\|_{H^1}
\]
and by interpolation with (4.9) we obtain (4.7). \qed

We can conclude the proof of the Theorem. In the case $n \geq 4$, it is clear that the left hand side of (3.23) is larger than a multiple of
\[
\int \left[ \frac{R^{n-1} |\nabla_A u|^2}{(R \vee r)^n} + \frac{|\nabla_T u|^2}{r} + \frac{|u|^2}{r^3} \right] dx + \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma.
\]
Hence, in order to obtain (1.15), it is sufficient (after an integration on $[-T, T]$) to prove the following estimates:
\begin{align}
\int_{-T}^T \int \phi' (V_\gamma)^+ |u|^2 dx dt &\leq C_T \|f\|_{H^{1\lambda/m-1/2}}, \\
\int_{-T}^T \int \phi' B_\gamma \cdot \nabla_A u dx dt &\leq C_T \|f\|_{H^{1\lambda/m}}, \\
\int_{|x|=R} \nabla \phi \cdot d\sigma &\leq C_T \|f\|_{H^{1\lambda/m-1/2}}.
\end{align}
In order to prove the first estimate (4.10), we can write using (3.16), assumption (1.9) on $V$ and the inequality (4.2),
\[
\int_{-T}^{T} \int \phi'(V_{*})|u|^2 dx dt \leq C \int_{-T}^{T} \|u(t)\|_{\mathcal{H}^{1-1/m}}^2 dt \leq C_{T} \|f\|_{\mathcal{H}^{1-1/m}}^2
\]
where in the final step we applied the energy estimate (4.3). To prove the second estimate (4.11), it is sufficient to use (4.7) of Lemma 4.2 with the choice $F = B_{r}$, recalling assumptions (1.9) on $B_{r}$, the bounds (3.16), (3.18) on $\phi$ and using again the energy estimate (4.3). Finally, the third estimate (4.12) is exactly (4.5) of Lemma 4.2. Since $\lambda \leq m - 1$ and $m \geq 2$, this concludes the proof in the case $n \geq 4$.

The proof in the case $n = 3$ is completely analogous.

**Appendix A. Some Technical Lemmas**

**Theorem A.1** (Magnetic Hardy Inequality). Assume $A(x) = (A_{1}, \ldots, A_{n})$ is in $L^{2}_{loc}$, with values in $\mathbb{R}^{n}$, $n \geq 3$. Then for all $u$ in the domain of $\nabla_{A}^{2} = (\nabla - iA)^{2}$ the following inequality holds:
\[
\int \frac{|u|^2}{|x|^2} dx \leq \left( \frac{2}{n - 2} \right)^2 \int |\nabla_{A}u|^2 dx.
\]

**Proof.** The proof is similar to the standard one for $A = 0$. Indeed, for any $\alpha \in \mathbb{R}$ we have
\[
0 \leq \int \left| \nabla_{A}u + \frac{\alpha x}{|x|^2} u \right|^2 \equiv \int \left| \nabla_{A}u \right|^2 + \alpha^2 \int \frac{|u|^2}{|x|^2} + 2\alpha \Re \int \nabla_{A}u \cdot \frac{x}{|x|^2} \text{.}
\]
We notice that
\[
2\alpha \Re \int \nabla_{A}u \cdot \frac{x}{|x|^2} \overline{u} = 2\alpha \Re \int \nabla u \cdot \frac{x}{|x|^2} \overline{u} = \alpha \int \nabla |u|^2 \cdot \frac{x}{|x|^2}
\]
and integrating by parts we get
\[
0 \leq \int \left| \nabla_{A}u \right|^2 + \alpha(\alpha - n + 2) \int \frac{|u|^2}{|x|^2}.
\]
Choosing $\alpha = (n - 2)/2$ we conclude the proof. \hfill \Box

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