REFINEMENTS TO MUMFORD’S THETA AND ADELIC THETA GROUPS

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Abstract. Let $X$ be an abelian variety defined over an algebraically closed field $k$. We consider theta groups associated to simple semi-homogenous vector bundles of separable type on $X$. We determine the structure and representation theory of these groups. In doing so we relate work of Mumford, Mukai, and Umemura. We also consider adelic theta groups associated to line bundles on $X$. After reviewing Mumford’s construction of these groups we determine functorial properties which they enjoy and then realize the Neron-Severi group of $X$ as a subgroup of the cohomology group $H^2(V(X); k^*)$.

1. Introduction

Let $X$ be an abelian variety defined over an algebraically closed field $k$. The purpose of this paper is to refine and generalize the theory of theta and adelic theta groups associated to line bundles on $X$. Such groups were invented by Mumford and played a fundamental role in his study of syzygies of abelian varieties, moduli of abelian varieties, and theta functions, see [9], [10], [11], and [12].

H. Umemura extended Mumford’s theory of theta groups. He considered theta groups associated to vector bundles on $X$ and determined the weight 1 representation theory of theta groups associated to simple vector bundles on $X$. In addition he posed the problem of determining the structure of theta groups associated to vector bundles on $X$ in general and considered theta groups associated to Brauer-Severi varieties over $X$, see [18 §5], [19 §1 and §2].

More recently theta groups and their higher weight representation theory play a role in work of Gross-Popescu, see [7] Example 2.10, p. 349, while theta groups and their relation to semi-homogeneous vector bundles play a role in work of D. Oprea, see [13 §2]. Even more recently M. Brion has considered theta groups associated to Brauer-Severi varieties over abelian varieties, see [3], while S. Shin has extended Mumford’s work by constructing theta and adelic theta groups associated to line bundles on abelian schemes, see [17].

Let us now briefly describe the results of this paper; we give precise statements of these results in §2.

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We first consider theta groups associated to a class of vector bundles on $X$ which we refer to as simple semi-homogeneous vector bundles of separable type. Our first result, Theorem 2.1, concerns the structure of these groups. This theorem generalizes [9, §1, Theorem 1] and answers the above mentioned problem of Umemura for this class of vector bundles on $X$. (See [18, p. 120] for a precise statement of this problem.)

Our second result, Theorem 2.2 and Corollary 2.3, determines the representation theory of these groups. This theorem generalizes [9, §1, Proposition 3 and Theorem 2] and also makes the inequality asserted in [11 Exercise 6.10.4, p. 175] an equality. In §5.2 we also give a conceptual interpretation of these irreducible representations in terms of induced modules. More specifically we prove that they are induced by 1-dimensional representations of an appropriate subgroup.

The final result of this paper concerns adelic theta groups associated to line bundles on $X$. In §6 we review the construction of these groups as it is important to the proof of our Theorem 2.4.

To describe this result let $I$ denote the set of positive integers which are not divisible by the characteristic of $k$. Let 

$$\text{tor}(X) := \{x \in X(k) : nx = 0 \text{ for some } n \in I\},$$

and let $V(X) := \lim \rightarrow \text{tor}(X)$, where the limit is indexed by $I$ and where the maps are given by multiplication by $n/m$ whenever $m$ divides $n$.

In this notation Theorem 2.4 gives a functorial realization of the Neron-Severi group of $X$ as a subgroup of the cohomology group $H^2(V(X); k^\times)$.

To place our results concerning theta groups into the proper context it is important to emphasize that our results build on work of Mukai, especially [8], and Mumford, namely [9, §1]. We should also point out that there is some overlap amongst our Theorem 5.1 and [4, Appendix]. In addition a special case of Theorem 2.1 is implicit in [15, §2.2]. On the other hand all of the results of this paper were obtained independently in my dissertation [6] and I am not aware of any other reference which states and proves these results explicitly.

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2. Statement of results

In §2.1 we fix some terminology and recall some statements, found in [8], which allow us to state our results concerning theta groups. These results are stated in §2.2. In §2.3 we formulate our results concerning adelic theta groups.

2.1. Preliminaries concerning simple semi-homogeneous vector bundles. Let $X$ be an abelian variety, defined over an algebraically closed field $k$, and let $T_x : X \to X$
denote translation by \( x \in X \). The results of §2.2 concern theta groups associated to *simple semi-homogeneous vector bundles of separable type* on \( X \).

We make this concept precise as follows.

**Definitions.** Let \( E \) be a vector bundle on an abelian variety \( X \). We say that:

- \( E \) is **non-degenerate** if \( \chi(E) \neq 0 \);
- \( E \) is **simple** if \( \dim_k \text{End}_{\mathcal{O}_X}(E) = 1 \);
- \( E \) is **semi-homogeneous** if for all \( x \in X \) there exists a line bundle \( L \) on \( X \) with the property that \( T^*_x E \cong E \otimes L \);
- \( E \) is of **separable type** if \( E \) is non-degenerate and if \( \chi(E) \) is not divisible by the characteristic of \( k \).

Let \( E \) be a simple semi-homogeneous vector bundle of separable type on \( X \). If \( x \in X \), then we let \( \text{Aut}_x(E) \) denote the set of isomorphisms \( E \to T^*_x E \) of \( \mathcal{O}_X \)-modules.

In [8, §6, §7, and Corollary 7.9, p. 271] Mukai has shown that the group

\[ K(E) := \{ x \in X(k) : \text{Aut}_x(E) \neq \emptyset \} \]

is finite and that \( \chi(E)^2 = \#K(E) \).

The theta group of \( E \), which we denote by \( G(E) \) and define in §3.2, is a central extension of \( k^\times \) by \( K(E) \).

### 2.2. Results concerning theta groups.

Our first result, Theorem 2.1, generalizes [9, Theorem 1], sheds light on a problem posed by Umemura [18, p. 120], and concerns the structure of the group \( G(E) \). It is stated as follows.

**Theorem 2.1.** Let \( E \) be a simple semi-homogeneous vector bundle of separable type on \( X \). The theta group \( G(E) \) is a non-degenerate central extension of \( k^\times \) by \( K(E) \).

A consequence of Theorem 2.1 is that every simple semi-homogeneous vector bundle of separable type \( E \) on \( X \) determines a sequence of integers \( d = (d_1, \ldots, d_p) \), \( d_i \mid d_{i+1} \), which we refer to as the type of its theta group \( G(E) \). This fact, together with Theorem 2.1, allows us to determine the representation theory of \( G(E) \).

**Theorem 2.2.** Let \( E \) be a simple semi-homogeneous vector bundle of separable type on \( X \). Let \( d = (d_1, \ldots, d_p) \) be the type of \( G(E) \). The following assertions hold:

1. The theta group \( G(E) \) admits exactly \( \gcd(n, d_1)^2 \times \cdots \times \gcd(n, d_p)^2 \) non-isomorphic irreducible weight \( n \) \( G(E) \)-module(s);
2. A weight \( n \) representation is irreducible if and only if it has dimension \( \frac{d_1 \times \cdots \times d_p}{\gcd(n, d_1) \times \cdots \times \gcd(n, d_p)} \);
3. Every weight \( n \) \( G(E) \)-module decomposes into a direct sum of irreducible weight \( n \) \( G(E) \)-modules. Every \( G(E) \)-module decomposes into a direct sum of weight \( n \) \( G(E) \)-modules.
A consequence of [5, Proposition 2.1] is that each simple semi-homogeneous vector bundle of separable type on $X$ admits exactly one nonzero cohomology group. Combining this fact with Theorems 2.1 and 2.2 we obtain:

**Corollary 2.3.** Let $E$ be a simple semi-homogenous vector bundle of separable type on $X$. The unique nonzero cohomology group $H^i(E)(X, E)$ is the unique irreducible weight 1 representation of its theta group $G(E)$.

We prove Theorem 2.1 in §4.2, while we prove Theorem 2.2 and Corollary 2.3 in §5.3

### 2.3. Results concerning adelic theta groups.

Let $X$ be an abelian variety defined over an algebraically closed field $k$. Let $I$ denote the set of positive integers which are not divisible by the characteristic of $k$. Let

$$\text{tor}(X) := \{x \in X(k) : nx = 0 \text{ for some } n \in I\},$$

and let $V(X) := \varprojlim \text{tor}(X)$, where the limit is indexed by $I$ and where the maps are given by

$$(n/m)_X \mid \text{tor}(X) : \text{tor}(X) \to \text{tor}(X)$$

whenever $m$ divides $n$.

Let $L$ be the total space of a line bundle on $X$. Mumford has constructed a group $\hat{G}(L)$, the *adelic theta group* of $L$, see [10, §7] and [13, Chapter 4]; this group is a central extension of $k^\times$ by $V(X)$. We recall some aspects of Mumford’s construction of $\hat{G}(L)$ in §6.

In §7 we prove that the Neron-Severi group of $X$, which we denote by $\text{NS}(X)$, can be canonically identified with a subgroup of the cohomology group $H^2(V(X); k^\times)$. More specifically we establish:

**Theorem 2.4.** The map $\hat{G} : \text{NS}(X) \to H^2(V(X); k^\times)$, defined by sending the class of a line bundle $L$ to that of its adelic theta group $\hat{G}(L)$, is an injective group homomorphism. It satisfies the following functorial property: if $f : X \to Y$ is a homomorphism of abelian varieties then the diagram

$$\begin{array}{ccc}
\text{NS}(X) & \xrightarrow{\hat{G}} & H^2(V(X); k^\times) \\
\downarrow f^* & & \downarrow f^* \\
\text{NS}(Y) & \xrightarrow{\hat{G}} & H^2(V(Y); k^\times)
\end{array}$$

commutes.

### 3. Theta groups and quasi-coherent sheaves

Let $X$ be an abelian variety defined over an algebraically closed field $k$. In this section we construct theta groups associated to quasi-coherent sheaves on $X$. We determine, in this generality, some of their basic properties.
3.1. Preliminaries from descent theory. Let $K \subseteq X$ be a finite subgroup. We assume that the order of $K$ is not divisible by the characteristic of $k$. We consider descent of quasi-coherent sheaves with respect to the quotient map $f: X \to Y = X/K$.

We first fix some notation. Then, to keep this paper reasonably self contained, we recall some standard definitions from descent theory. The definitions we give here can be extracted from [2 §6.1].

Let $F$ be a quasi-coherent sheaf on $X$. If $x \in X(k)$ then let

$\text{Aut}_x(F) := \{ \phi \in \text{Hom}_{\mathcal{O}_X}(F, T_x^*F) : \phi \text{ is an isomorphism} \}$.

If $\text{Aut}_x(F) \neq \emptyset$ then it is an $\text{Aut}(F)$-torsor (or principal homogeneous space). More specifically:

- the group $\text{Aut}(F)$ acts on $\text{Aut}_x(F)$, for all $x \in X$;
- if $x \in X$ and $\phi \in \text{Aut}_x(F)$ then the group $\{ \alpha \in \text{Aut}(F) : \alpha \cdot \phi = \phi \}$ is trivial;
- if $x \in X$ then $\text{Aut}_x(F) = \{ \alpha \cdot \phi : \alpha \in \text{Aut}(F) \}$ for some (and hence every) element $\phi$ of $\text{Aut}_x(F)$.

More succinctly $\text{Aut}(F)$ acts simply transitively on $\text{Aut}_x(F)$, for all $x \in X$, whenever $\text{Aut}_x(F) \neq \emptyset$.

Covering datum for $F$, with respect to $f$, is a pair $(F, \phi)$ where $\phi$ is a set consisting of exactly one $\phi_x \in \text{Aut}_x(F)$, for all $x \in K$.

Covering datum $(F, \phi)$ is descent datum if the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\phi_x} & T_x^*F \\
\downarrow{\phi_y} & & \downarrow{\phi_{x+y}} \\
T_x^*T_y^* & \xrightarrow{T_y^*\phi_x} & T_x^*(T_y^*F) \\
\end{array}
\]

commutes for all $x, y \in K$.

Let $\text{Descent}(\text{Qcoh}(X), f)$ denote the category whose objects are descent data $(F, \phi)$, with respect to $f$, and where a morphism between pairs $(F, \phi)$ and $(G, \psi)$ is given by an element $\alpha \in \text{Hom}_{\mathcal{O}_X}(F, G)$ which has the property that the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\alpha} & G \\
\downarrow{\phi_x} & & \downarrow{\psi_x} \\
T_x^*F & \xrightarrow{T_x^*\alpha} & T_x^*G \\
\end{array}
\]

commutes for all $x \in K$.

Let $H$ be a quasi-coherent sheaf on $Y$. If $x \in K$ then let $\text{can}_x(H)$ be the isomorphism

$f^*H = (f \circ T_x)^*H = T_x^*(f^*H)$,
and let
\[ \text{can}(H) := \{ \text{can}_x(H) : x \in K \}. \]
The pair \((f^*H, \text{can}(H))\) is an object of \(\text{Descent}(\text{QCoh}(X), f)\).

Descent data \((F, \phi)\) is effective if
\[ (F, \phi) \cong (f^*H, \text{can}(H)) \]
for some \(H \in \text{QCoh}(Y)\). (Here \(\text{QCoh}(Y)\) denotes the category of quasi-coherent sheaves on \(Y\).) In this situation we also say that \(F\) descends, via \(f\), to \(H\).

The maps
\[ H \mapsto f^*H, \phi \in \text{Hom}_{\mathcal{O}_Y}(H, J) \mapsto f^*\phi \in \text{Hom}(f^*H, \text{can}(H)), (f^*J, \text{can}(J))) \]
determine a functor \(f^*: \text{QCoh}(Y) \to \text{Descent}(\text{QCoh}(X), f)\). Grothendieck has proven that this functor is an equivalence of categories. See [2, §6.1 Theorem 4, p. 134], for instance, as well as [14, §7 Proposition 2, p. 66 and §12 Theorem 1, p. 104].

In §4.2 we apply the following lemma in our proof of Lemma 4.4.

**Lemma 3.1.** If a coherent sheaf \(E\) on \(X\) descends, via a separable isogeny \(f: X \to Y\) to a coherent sheaf \(F\) on \(Y\), then
\[ \dim_k \text{End}_{\mathcal{O}_X}(E) \geq \dim_k \text{End}_{\mathcal{O}_Y}(F). \]

**Proof.** Since the descent functor is fully faithful the \(k\)-vector space \(\text{End}_{\mathcal{O}_Y}(F)\) is identified with the subspace of \(\text{End}_{\mathcal{O}_X}(E)\) which commutes with the descent data. \(\square\)

### 3.2. Construction of theta groups.

Let \(F\) be a quasi-coherent sheaf on \(X\). Let
\[ K(F) := \{ x \in X(k) : \text{Aut}_x(F) \neq \emptyset \} \]
and observe that \(K(F)\) is a subgroup of \(X(k)\). Let
\[ G(F) := \{ (x, \phi) : x \in K(F) \text{ and } \phi \in \text{Aut}_x(F) \}. \]

If \((x, \phi)\) and \((y, \psi)\) are elements of \(G(F)\), then \((x, \phi) \cdot (y, \psi) := (x + y, \phi \ast \psi)\), where \(\phi \ast \psi\) is the element of \(\text{Aut}_{x+y}(F)\) determined by the composition
\[ F \xrightarrow{\psi} T_y^*F \xrightarrow{T_y^*\phi} T_y^*(T_x^*F) = T_{x+y}^*(F). \]

The pair \((G(F), \cdot)\) is a group and the homorphisms
\[ \iota_F: \text{Aut}(F) \to G(F), \text{ and } \pi_F: G(F) \to K(F), \]
defined respectively by \(\alpha \mapsto (0, \alpha)\) and \((x, \phi) \mapsto x\), determine a short exact sequence of groups
\[ 1 \to \text{Aut}(F) \xrightarrow{\iota_F} G(F) \xrightarrow{\pi_F} K(F) \to 0. \]
3.3. Level subgroups and descent. Let $F$ be a quasi-coherent sheaf on $X$. In this subsection we relate certain subgroups of $G(F)$ with descent data for $F$ with respect to suitably defined isogenies.

Definitions.

- A subgroup $\mathcal{K} \subseteq G(F)$ is a level subgroup if it is finite, has order not divisible by the characteristic of $k$, and if the homomorphism $\pi_F | \mathcal{K} : \mathcal{K} \to \pi_F(\mathcal{K})$ is injective.
- Let $K \subseteq K(F)$ be a subgroup and let $\mathcal{K} \subseteq G(F)$ be a level subgroup. We say that $K$ lies over $K$ if $\pi_F(\mathcal{K}) = K$.

Proposition 3.2. Let $F$ be a quasi-coherent sheaf on $X$ and let $K \subseteq K(F)$ be a subgroup. There exists a one to one correspondence between level subgroups $\mathcal{K} \subseteq G(F)$ lying over $K$ and (effective) descent datum $(F, \phi)$ with respect to the quotient map $f : X \to X/K$.

Proof. Let $\sigma$ be the inverse of $\pi_F | \mathcal{K}$. If $x \in K$, then let $\phi_x$ be the element of $\text{Aut}_x(F)$ determined by $\sigma$. Let $\phi$ be the set consisting of these isomorphisms. Since $\sigma$ is a group homomorphism the diagram (3.1) commutes for all $x, y \in K$. Hence the pair $(F, \phi)$ is descent datum.

Conversely if $(F, \phi)$ is descent datum then define

$$\mathcal{K} := \{(x, \phi_x) : \phi_x \in \phi\}.$$

The fact that the diagram (3.1) commutes, for all $x, y \in K$, implies that $\mathcal{K}$ is a subgroup. In addition the map $(x, \phi_x) \mapsto x$ is an isomorphism of groups.

It is clear, by construction, that the correspondences just defined are mutual inverses. $\square$

3.4. Theta groups and isogenies. Let $F$ be a quasi-coherent sheaf on $X$, $\mathcal{K} \subseteq G(F)$ a level subgroup, and $(F, \phi)$ the descent datum determined by $\mathcal{K}$. Let $K := \pi_F(\mathcal{K})$, $Y := X/K$, and $f : X \to Y$ the quotient map.

Since $(F, \phi)$ is effective there exists a quasi-coherent sheaf $H$ on $Y$ with the property that $(F, \phi) \cong (f^*H, \text{can}(H))$. In particular there exists an isomorphism $\alpha : f^*H \to F$ of $\mathcal{O}_X$-modules.

We now use $\alpha$ to relate $G(H)$, $G(F)$, and the centralizer $C_K(G(F))$ of $K$ in $G(F)$. This is the content of Proposition 3.3.

Before stating this result first observe that every $x \in X$ determines a morphism

$$\text{Aut}_{f(x)}(H) \xrightarrow{f^*} \text{Aut}_x(f^*H) \xrightarrow{T_x(\alpha) \sigma_{f(x)}^{-1}} \text{Aut}_x(F)$$

of $\text{Aut}(H)$-sets. In particular, we have $f^{-1}(K(H)) \subseteq K(F)$. 

Also if $x$ and $y$ are elements of $X$ then the diagram
\begin{equation}
\begin{array}{ccc}
\text{Aut}_{f(y)}(H) \times \text{Aut}_{f(x)}(H) & \xrightarrow{f^* \times f^*} & \text{Aut}_y(f^* H) \times \text{Aut}_x(f^* H) \\
\downarrow & & \downarrow \\
\text{Aut}_{x+y}(f^* H) & \xrightarrow{T_{x+y}^*(\alpha) \circ \circ \circ \circ \circ \circ \circ} & \text{Aut}_y(F) \times \text{Aut}_x(F)
\end{array}
\end{equation}

of $\text{Aut}(H)$-sets commutes.

Proposition 3.3 can now be stated.

**Proposition 3.3.** Let $F$ be a quasi-coherent sheaf on an abelian variety $X$. Let $\mathcal{K} \subseteq G(F)$ be a level subgroup, and let $(F, \phi)$ be the descent datum determined by $\mathcal{K}$. In addition, let $K := \pi_F(\mathcal{K})$, let $Y := X/K$, and let $f : X \to Y$ be the quotient map. Let $H$ be a quasi-coherent sheaf on $Y$ with the property that $(F, \phi) \cong (f^* H, \text{can}(H))$. The following assertions hold:

(a) if $C_{\mathcal{K}}(G(F))$ denotes the centralizer of $\mathcal{K}$ in $G(F)$ then $C_{\mathcal{K}}(G(F))$ coincides with the set

$$\{(x, \eta) : f(x) \in K(H) \text{ and } \eta = T_x^*(\alpha) \circ f^* \circ \psi \circ \alpha^{-1} \text{ for some } \psi \in \text{Aut}_{f(x)}(H)\};$$

(b) the map $C_{\mathcal{K}}(G(F)) \to G(H)$ defined by $(x, \eta) \mapsto (f(x), \psi)$, where $\psi$ is the (unique) element of $\text{Aut}_{f(x)}(H)$ whose pullback is $\eta$, is a surjective group homomorphism with kernel equal to $\mathcal{K}$.

**Proof.** We first prove (a). If $w \in K$ then let $\phi_w \in \text{Aut}_w(F)$ be the unique isomorphism $F \to T^*_w F$ having the property that $(w, \phi_w) \in \mathcal{K}$.

We know that $(F, \phi)$ descends to $H$ and that if $x \in X$ and $y = f(x)$ then $(T^*_x F, T^*_x \phi)$ descends to $T^*_y H$.

The centralizer of $\mathcal{K}$ in $G(F)$ consists exactly of those $(x, \psi) \in G(F)$ with the property that $\psi * \phi_w = \phi_w * \psi$ for all $w \in K$. Considering this condition we conclude that $C_{\mathcal{K}}(G(F))$ consists exactly of those $(x, \psi) \in G(F)$ such that $\psi$ determines an isomorphism amongst the pairs $(F, \phi)$ and $(T^*_x F, T^*_x \phi)$.

Thus, to determine $C_{\mathcal{K}}(G(F))$, we need to examine, for a fixed $x \in K(F)$, those isomorphisms $\psi : F \to T^*_x F$ which commute with the descent data.

Let $x \in X$ and let $y = f(x)$. The map
\begin{equation}
\text{Hom}_{\mathcal{O}_Y}(H, T^*_y H) \to \text{Hom}((F, \phi), (T^*_x F, T^*_x \phi))
\end{equation}
is given by $\eta \mapsto T^*_x(\alpha) \circ f^* \eta \circ \alpha^{-1}$ and is an isomorphism (since the descent functor is fully faithful).
Furthermore under the map (3.4) isomorphisms carry over to isomorphisms. In particular if \( \psi : F \to T^*_x F \) is an isomorphism which commutes with descent data then \( f(x) \) is an element of \( K(H) \). Considering the discussion above we conclude that (a) holds.

To prove (b), using the diagram (3.3), we check that the asserted map is a group homomorphism. To see that the asserted map is surjective let \( y \in K(H) \). Then \( y = f(x) \) for some \( x \in K(F) \). Since the map (3.4) is an isomorphism, every element of \( \text{Aut}_y(H) \) is in the image. Using the definition of the map we check that its kernel is \( K \).

**Remark.** Using the fact that \( K \) is a level subgroup of \( G(F) \) we can check that the centralizer of \( K \) in \( G(F) \) equals its normalizer.

### 4. Non-degenerate theta groups

Let \( X \) be an abelian variety defined over an algebraically closed field \( k \). Let \( E \) be a simple semi-homogeneous vector bundle of separable type on \( X \). In §4.2 we prove that its theta group \( G(E) \) is a non-degenerate central extension of \( k^\times \) by \( K(E) \).

#### 4.1. Preliminaries on central extensions of \( k^\times \) by a finite abelian group

Let \( K \) be a finite abelian group and assume that the order of \( K \) is not divisible by the characteristic of \( k \). We consider a central extension

\[
1 \to k^\times \xrightarrow{\iota} G \xrightarrow{\pi} K \to 0
\]

of \( k^\times \) by \( K \).

**Definitions.**

- The extension (4.1) is **non-degenerate** if \( \iota(k^\times) \) equals the center of \( G \).
- A subgroup \( K \subseteq G \) is a **level subgroup** if the homomorphism \( \pi | K : K \to \pi(K) \) is injective.

The extension (4.1) determines a bilinear form

\[
[-, -]_G : K \times K \to k^\times, \text{ defined by } [x, y]_G := aba^{-1}b^{-1},
\]

where \( a \) and \( b \) are any elements of \( G \) lying over \( x \) and \( y \) respectively.

The form (4.2) is skew symmetric, that is, \([x, x]_G = 1\), hence \([x, y]_G = [y, x]_G^{-1}\), for all \( x, y \in G \). If, in addition, the extension (4.1) is non-degenerate, then form (4.2) is also non-degenerate, that is for all \( 0 \neq x \in K \) there exists \( y \in K \) such that \([x, y]_G \neq 1\).

In Proposition 4.3 we determine the structure of non-degenerate central extensions of the form (4.1). To achieve this we first prove two auxiliary results namely Proposition 4.1 and Lemma 4.2.

**Proposition 4.1.** If \([-, -] : K \times K \to k^\times\) is a non-degenerate skew symmetric bilinear form, then \( K \) admits subgroups \( K_1 \) and \( K_2 \) with the properties that:
(a) $K = K_1 \oplus K_2$, $[x_i, y_i] = 1$ for all $x_i, y_i \in K_i$, $i = 1, 2$;
(b) the bilinear form $\langle -, - \rangle: K_1 \times K_2 \to k^\times$, defined by $(x_1, x_2) \mapsto [x_1, x_2]$, is non-degenerate.

Proof. We prove (a) and (b) simultaneously by inducting on the number of elementary divisors of $K$. The base case is when $K$ has no elementary divisors in which case statements (a) and (b) hold. Suppose now that $d_1$ is the largest elementary divisor of $K$. Let $x_1 \in K$ be an element of order $d_1$. Since $d_1$ is the largest elementary divisor of $K$, and since $[-, -]$ is non-degenerate and skew-symmetric, there exists $y_1 \in K$, of order $d_1$ and not contained in $\langle x_1 \rangle$, with the property that $[x_1, y_1] = \zeta_{d_1}$, where $\zeta_{d_1}$ is a primitive $d_1$th root of unity in $k^\times$.

Let $W$ be the subgroup of $K$ generated by $x_1$ and $y_1$ and let

$$W^\perp := \{ z \in K : [x_1, z] = 1 \text{ and } [y_1, z] = 1 \}.$$ 

Then $W = \langle x_1 \rangle \oplus \langle y_1 \rangle$ and $W \cap W^\perp = \langle 1_K \rangle$. On the other hand if $z \in K$ then $\zeta_{d_1}^{\parallel} = [x_1, z]$ and $\zeta_{d_1}^{\perp} = [y_1, z]$ for some integers $n$ and $m$. As a consequence if $w = mx_1 + ny_1$ then $w \in W$, $z = w + z - w$, and $z - w \in W^\perp$. Hence $K = W \oplus W^\perp$. Also the restriction of $[-, -]$ to $W^\perp$ is a non-degenerate skew-symmetric bilinear form and $W^\perp$ is a proper subgroup of $K$. By induction the statement holds for $W^\perp$ and we deduce that the statement holds for $K$. \quad \square

The following lemma gives a sufficient condition for producing level subgroups of $G$.

Lemma 4.2. Let $G$ be a central extension of $k^\times$ by $K$. If $K \subseteq K$ is a subgroup, having the property that $[x, y]_G = 1$ for all $x, y \in K$, then $G$ admits a level subgroup $\mathcal{K}$ with the property that $\pi(\mathcal{K}) = K$.

Proof. Write $K$ as an internal direct sum $K = \bigoplus_{i=1}^n \langle g_i \rangle$ and let $d_i$ be the order of $g_i$. Let $h_i$ be an element of $G$ lying over $g_i$, for all $i = 1, \ldots, n$. Then $h_i^{d_i}$ corresponds to a unique $\alpha_i \in k^\times$. Since $k^\times$ is divisible there exists $\beta_i \in k^\times$ such that $\beta_i^{d_i} = \alpha_i$. Set $z_i = h_i \beta_i^{-1}$. Then $z_i$ is an element of $G$ and has order $d_i$. Let $\mathcal{K} := \langle z_1, \ldots, z_n \rangle$. Then $\mathcal{K} = \bigoplus_{i=1}^n \langle z_i \rangle$ and $\mathcal{K}$ is a subgroup of $G$ having the property that $K = \pi(\mathcal{K})$. Since $[x, y]_G = 1$, for all $x, y \in K$, we deduce that $\pi \mid \mathcal{K}: \mathcal{K} \to K$ is an isomorphism. \quad \square

Our next proposition shows that every non-degenerate central extension of $k^\times$ by $K$ takes a more desirable form.

Proposition 4.3. If $G$ is a non-degenerate central extension of $k^\times$ by $K$ then $K$ admits subgroups $K_1, K_2 \subseteq K$ with the properties that

(a) $K = K_1 \oplus K_2$ and $[x_i, y_i]_G = 1$ if $x_i, y_i \in K_i$, for $i = 1, 2$;
(b) the bilinear form

$$\langle -, - \rangle: K_1 \times K_2 \to k^\times,$$ 

defined by $(x_1, x_2) \mapsto [x_1, x_2]_G$

is non-degenerate;
(c) the extension $G$ is equivalent to the extension $G(\langle-,-\rangle)$, where $G(\langle-,-\rangle) := k^\times \times K_1 \oplus K_2$ and where multiplication is defined by

$$(\alpha, x_1, x_2) \cdot (\beta, y_1, y_2) = (\alpha \beta(x_1, y_2), x_1 + y_1, x_2 + y_2).$$

Proof. Statements (a) and (b) are immediate consequences of Proposition 4.1. Let $K$ be elements of $K$. To prove that $G$ is equivalent to $G(\langle-\rangle)$

This defines a normalized set-theoretic section $\sigma$.

By definition of $\langle-\rangle$ we have that

$$\langle x, y \rangle := \sigma(x) \cdot \sigma(y) \cdot \sigma(x+y)^{-1}$$

equals

$$(4.5) \quad \sigma_2(x_2)(\sigma_1(x_1)\sigma_2(y_2))\sigma_1(x_1)^{-1}\sigma_2(y_2)^{-1}\sigma_2(x_2)^{-1}.$$

Since $\sigma_1(x_1)\sigma_2(y_2)\sigma_1(x_1)^{-1}\sigma_2(y_2)^{-1} \in k^\times$, (4.5) equals (4.4). Hence (4.3) holds. \qed

4.2. Non-degenerate theta groups and vector bundles. Let $X$ be an abelian variety defined over an algebraically closed field $k$. Let $E$ be a simple vector bundle on $X$.

We have homomorphisms $\iota_E : k^\times \rightarrow G(E)$ and $\pi_E : G(E) \rightarrow K(E)$ defined, respectively, by $\alpha \mapsto (0, \alpha \text{id}_E)$ and $(x, \phi) \mapsto x$. These homomorphisms determine a short exact sequence of groups

$$1 \rightarrow k^\times \xrightarrow{\iota_E} G(E) \xrightarrow{\pi_E} K(E) \rightarrow 0.$$ 

Since image $\iota_E \subseteq \text{center} G(E)$, the group $G(E)$ is a central extension of $k^\times$ by $K(E)$.

Let $E$ be a simple semi-homogeneous vector bundle of separable type on $X$. Then $K(E)$ is a finite group and $\chi(E)^2 = \#K(E)$, [8, §6, 7, and Corollary 7.9, p. 271].

We are almost able to prove Theorem 2.1 which implies that $G(E)$ is a non-degenerate theta group. Before doing so we must establish one additional lemma.

Lemma 4.4. Let $f : X \rightarrow Y$ be a separable isogeny and let $E$ be a simple semi-homogeneous vector bundle of separable type on $X$. If $E \cong f^*F$, for some vector bundle $F$ on $Y$, then $F$ is a simple semi-homogeneous vector bundle of separable type on $X$. 

Proof. Mukai’s theory implies that $F$ is semi-homogeneous, see [8, Proposition 5.4, p. 259] for instance. To prove that $F$ is simple, using Lemma 3.1, we obtain the inequalities

$$1 = \dim_k \text{End}_{\mathcal{O}_X}(E) \geq \dim_k \text{End}_{\mathcal{O}_Y}(F) \geq 1$$

and conclude that $F$ is simple.

On the other hand, we have that $\chi(E) = (\# \ker f) \chi(F)$ while $\chi(E)^2 = \# K(E)$ and $\chi(F)^2 = \# K(F)$. We conclude that $\chi(F) \neq 0$ and that $\chi(F)$ is not divisible by the characteristic of $k$. \qed

Proof of Theorem 2.1. Let $K$ be a maximal level subgroup of $G(E)$. (Lemma 4.2 ensures that such a subgroup exists.) Let $K := \pi_E(K)$. Then $K$ is a maximal subgroup of $K(E)$ on which $[-, -]_{G(E)} \equiv 1$.

Let $Y := X/K$, let $f: X \to Y$ denote the quotient map, and let $F$ be a vector bundle on $Y$ with the property that $f^*F \cong E$. Then, by Lemma 4.4, $F$ is a simple semi-homogeneous vector bundle of separable type on $X$.

Using the fact that $K$ is maximal, together with Proposition 3.3, we check that $G(F) = (k^\times \cdot K)/K$ and conclude that $K(F)$ is trivial.

Since $\chi(F)^2 = \# K(F)$ we conclude that $\chi(F)^2 = 1$. Since $\chi(E)^2 = (\# K)^2 \chi(F)^2$ we conclude that $\chi(E)^2 = (\# K)^2$. Since $K$ was an arbitrary maximal level subgroup, we conclude that $\chi(E)^2 = (\# K)^2$ for every maximal level subgroup $K$ of $G(E)$.

Let $K_0 := \{ x \in K(E) : [x, y]_{G(E)} = 1 \text{ for all } y \in K(E) \}$. Then $[-, -]_{G(E)}$ induces a non-degenerate skew-symmetric bilinear form

$$K(E)/K_0 \times K(E)/K_0 \to k^\times.$$

Hence, $\# K(E)/K_0 = \ell^2$, for some $\ell$, and, by Proposition 4.1, there exists a maximal subgroup $K'$ of $K(E)/K_0$, of order $\ell$, on which $[-, -]_{G(E)} \equiv 1$.

Let $K$ be the inverse image of $K'$ in $K(E)$. Then, $K$ is the image of a maximal level subgroup of $G(E)$. We conclude that

$$\chi(E)^2 = (\# K)^2 = (\# K_0)^2 \cdot \ell^2$$

and hence that

$$\chi(E)^2 = (\# K_0)^2 \cdot \ell^2 = (\# K_0)^2 \cdot (\# K(E)/K_0) = (\# K_0)(\# (K(E)) = (\# K_0) \cdot \chi(E)^2.$$

Hence $\# K_0 = 1$ which implies that $K_0$ is trivial. \qed
5. The representation theory of non-degenerate theta groups

Let $K$ be a finite abelian group and assume that the characteristic of $k$ does not divide the order of $K$. Let $G$ be a non-degenerate central extension of $k^\times$ by $K$. We determine the representation theory of $G$.

Before proceeding we fix some terminology. A $G$-module $(V, \rho)$ is always a finite dimensional vector space which admits a basis $B$ for which there exists Laurent polynomials $F_{i,j} \in k[t, t^{-1}]$ with the property that the matrix representation of $\rho(\alpha)$, $\alpha \in k^\times$, with respect to $B$ is given by evaluating $F_{i,j}$ at $\alpha$. We say that a $G$-module $(V, \rho)$ is a weight $n$ $G$-module, for $n \in \mathbb{Z}$, if $\rho(\alpha) \cdot v = \alpha^n v$ for all $v \in V$ and all $\alpha \in k^\times$.

In §5.1 we prove the following theorem which we use to establish Theorem 2.2 and Corollary 2.3, see §5.3 for more details.

**Theorem 5.1.** Let $K$ be finite abelian group and assume that the characteristic of $k$ does not divide the order of $K$. Let $G$ be a non-degenerate central extension of $k^\times$ by $K$ of type $(d_1, \ldots, d_p)$. There exists exactly $\gcd(n, d_1) \times \cdots \times \gcd(n, d_p)$ non-isomorphic irreducible weight $n$ $G$-module(s). A weight $n$ representation is irreducible if and only if it has dimension $\frac{d_1 \times \cdots \times d_p}{\gcd(n, d_1) \times \cdots \times \gcd(n, d_p)}$. Every weight $n$ $G$-module decomposes into a direct sum of irreducible weight $n$ $G$-modules. Every $G$-module decomposes into a direct sum of weight $n$ $G$-modules.

To determine the representation theory of $G$, considering Proposition 4.3, it suffices to determine the representation theory of the group $G_{\langle -,- \rangle}$. In what follows we omit the subscript $\langle -,- \rangle$ and denote $G_{\langle -,- \rangle}$ simply by $G$.

Let $(d_1, \ldots, d_p)$ be the type of $G$ and define $D_n := \frac{d_1 \times \cdots \times d_p}{\gcd(n, d_1) \times \cdots \times \gcd(n, d_p)}$, for each $n \in \mathbb{Z}$.

### 5.1. Auxiliary results and proof of Theorem 5.1

Central to our proof of Theorem 5.1 is:

**Proposition 5.2.** If $(V, \rho)$ is a nonzero weight $n$ $G$-module, then $(V, \rho)$ admits a $D_n$-dimensional submodule.

**Proof.** Let $(V, \rho)$ be a weight $n$ $G$-module. An element $x$ of $K_1$ acts on a vector $v \in V$ by the rule $x \cdot v := \rho((1, x, 0))(v)$. We denote the resulting $K_1$-module by $\text{Res}^G_{K_1}(V)$.

Since $K_2 = \text{Hom}_\mathbb{Z}(K_1, k^\times)$ the $K_1$-module $\text{Res}^G_{K_1}(V)$ admits a decomposition into weight spaces. Explicitly, we have

\[
\text{Res}^G_{K_1}(V) = \bigoplus_{y \in K_2} V_y
\]

and if $x \in K_1$, $y \in K_2$, and $v \in V_y$, then $x \cdot v = \langle x, y \rangle v$.

Observe now that, if $y \in K_2$, $v \in V_y$, and $(\alpha, x, w) \in G$, then

\[
\rho((\alpha, x, w))(v) \in V_{y+nw}.
\]
Let \( \pi_n : K_2 \to K_2 \) denote the group homomorphism defined by \( y \mapsto ny \). The image of \( \pi_n \) has order \( D_n \).

Using (5.2), we see that every \( V_y, y \in K_2 \), appearing in the decomposition (5.1), is stable under the (evident) action of \( \ker \pi_n \). As a consequence if \( y \in K_2 \) then the \( \ker \pi_n \)-module \( V_y \) decomposes into weight spaces

\[
(5.3) \quad V_y = \bigoplus_{\chi \in \text{Hom}_Z(\ker \pi_n, k^\times)} V_{y,\chi}.
\]

Let \( \sigma \) be a set-theoretic section of the surjective homomorphism \( K_2 \to \text{image} \pi_n \) induced by \( \pi_n \). If \( z \in \text{image} \pi_n \), \( y \in K_2 \), \( \chi \in \text{Hom}_Z(\ker \pi_n, k^\times) \), and \( s_{y,\chi} \in V_{y,\chi} \), then define

\[
s_{y,\chi}(z) := \rho((1, 0, \sigma(z)))(s_{y,\chi}).
\]

Observe now that

\[
(5.4) \quad \rho((\alpha, x, w))(s_{y,\chi}(z)) = \alpha^n(x, y + z)\chi(w + \sigma(z) - \sigma(nw + z))s_{y,\chi}(nw + z)
\]

for all \( (\alpha, x, w) \in G \).

Since \((V, \rho)\) is nonzero, there exists \( y \in K_2 \) and \( \chi \in \text{Hom}_Z(\ker \pi_n, k^\times) \) such that \( V_{y,\chi} \neq 0 \). Fix such a pair \((y, \chi)\), choose a nonzero vector \( s_{y,\chi} \in V_{y,\chi} \), and define

\[
W_{y,\chi}^\sigma := \text{span}_k \{ s_{y,\chi}(z) \}_{z \in \text{image} \pi_n}.
\]

Using equation (5.4), we see that \( W_{y,\chi}^\sigma \) is a \( D_n \)-dimensional \( G \)-submodule of \( V \).

\[\square\]

**Corollary 5.3.**

(a) A weight \( n \) \( G \)-module is irreducible if and only if it has dimension \( D_n \).

(b) Every weight \( n \) \( G \)-module decomposes into weight \( n \) irreducible \( G \)-modules.

**Proof.** To prove (a) note that if \( V \) is an irreducible weight \( n \) \( G \)-module then it is nonzero and hence, by Proposition 5.2, admits a submodule of dimension \( D_n \). Since \( V \) is irreducible this submodule must equal \( V \) whence the dimension of \( V \) equals \( D_n \).

Conversely let \( V \) be a weight \( n \) \( G \)-module of dimension \( D_n \). Let \( W \) be a nonzero submodule. By Proposition 5.2 \( W \) admits a submodule of dimension \( D_n \). Consequently we have that

\[
D_n \leq \dim W \leq \dim V = D_n.
\]

Hence \( W \) has dimension \( D_n \) so \( W = V \). We conclude that \( V \) is irreducible.

To prove (b) let \( \mu_{d_1} \leq k^\times \) denote the multiplicative group of \( d_1 \)th roots of unity. Let \( G' \) denote the subgroup

\[
G' := \{ (\alpha, x, y) : \alpha \in \mu_{d_1}, x \in K_1, y \in K_2 \}
\]

of \( G \).
To finish the proof of Corollary 5.3, we induct on the dimension of $V$. The base case is $\dim V = 0$ in which case the assertion holds. If $\dim V = N$, then combining Proposition 5.2 and part (a), which we just proved, we see that $V$ admits an irreducible weight $n$ submodule $W$. If $W = V$ then we are done. Otherwise choose a projection $p_0 : V \to W$ and let $p : V \to W$ be the projection defined by

$$v \mapsto \frac{1}{|G'|} \sum_{g \in G'} g \cdot p_0(g^{-1} \cdot v).$$

Then $\ker p$ is a $G'$-submodule of $V$ and $V = W \oplus \ker p$. Let $s \in \ker p$. Then, if $(\alpha, x, y) \in G$, we obtain

$$\rho((\alpha, x, y))(s) = \rho((\alpha, 0, 0))(\rho((1, x, y))(s)) = \alpha^n(\rho((1, x, y))(s)).$$

Since $(1, x, y) \in G'$ and since $\ker p$ is $G'$-stable we conclude that $\alpha^n(\rho((1, x, y))(s)) \in \ker p$.

Hence $\ker p$ is a weight $n$ $G$-submodule of $V$ and, by induction, $\ker p$ decomposes into irreducible weight $n$ $G$-modules. □

To construct irreducible weight $n$ $G$-modules first let

$$y \in K_2, \chi \in \text{Hom}_\mathbb{Z}(\ker \pi_n, k^\times),$$

and

$$W_{y,\chi} := \text{span}_k \{ e_{y+z,\chi} \}_{z \in \text{image} \pi_n}.$$

As in the proof of Proposition 5.2 we fix a set-theoretic section $\sigma$ of the surjective homomorphism $K_2 \to \text{image} \pi_n$ induced by $\pi_n$. For every $(\alpha, x, w) \in G$ define an automorphism

$$\rho_{y,\chi}^\pi((\alpha, x, w)) : W_{y,\chi} \to W_{y,\chi}$$

by

$$e_{y+z,\chi} \mapsto \alpha^n(\chi(y + z)\chi(w + \sigma(z) - \sigma(nw + z))e_{y+z+nw,\chi}$$

and extending linearly.

**Proposition 5.4.** The pair $(W_{y,\chi}, \rho_{y,\chi}^\pi)$ is an irreducible weight $n$ $G$-module.

**Proof.** It is clear from the definition that the identity of $G$ acts trivially. Let $(\alpha, a, b)$ and $(\beta, c, d)$ be elements of $G$. We then have that

$$(\alpha, a, b) \cdot (\beta, c, d) = (\alpha \beta (a, d), a + c, b + d).$$

By linearity it suffices to check that

$$(\alpha \beta (a, d), a + c, b + d) \cdot e_{y+z,\chi} = (\alpha, a, b) \cdot ((\beta, c, d) \cdot e_{y+z,\chi}).$$

By definition of $\rho_{y,\chi}^\pi$ the left hand side of equation (5.5) equals

$$(\alpha \beta^n (a, d) \chi(a + c, y + z) \chi(b + d + \sigma(z) - \sigma(nb + nd + z)) e_{y+z+nb+nd,\chi}.$$

$$e_{y+z,\chi} \mapsto \alpha^n(\chi(y + z)\chi(w + \sigma(z) - \sigma(nw + z))e_{y+z+nw,\chi}$$
On the other hand

\[(\beta, c, d) \cdot e_{y+z,\chi} = \beta^{n}(c, y + z)\chi(d + \sigma(z) - \sigma(nd + z))e_{y+z+nd,\chi}\]

and

\[(\alpha, a, b) \cdot e_{y+z+nd,\chi} = \alpha^{n}(a, y + z + nd)\chi(b + \sigma(z + nd) - \sigma(nb + z + nd))e_{y+z+nd+nb,\chi}.

Hence the right hand side of equation (5.5) equals \(W\), that the pair \((W)\) simplifies to equation (5.6). Hence equation (5.5) holds.

A weight \(\alpha\) is isomorphic to \((W,\rho)\) to \(\text{Hom}_{V,\rho}\pi\) modules and as \(\ker\pi\) isomorphic as \(\ker\chi\) for some \(W\in\pi\) isomorphic as \(\ker\chi\).

For the final assertion let \((V,\rho)\) be a \(G\)-module. Then the \(k^{\chi}\)-module \(\text{Res}_{k^{\chi}}^{G}(V)\) admits a decomposition

\[
\text{Res}_{k^{\chi}}^{G}(V) = \bigoplus_{n\in\mathbb{Z}} V_{n}
\]
into weight spaces. Since the image of $k^\times$ in $G$ is contained in the centre of $G$ each weight space $V_n$ is $G$-stable and, hence, a weight $n$ $G$-module. □

Remark. Let $n$ be an integer and let $r$ be the remainder obtained by dividing $n$ by $d_1$. Let $G'$ be the subgroup

$$G' := \{(\alpha, x, y) : \alpha \in \mu_{d_1}, x \in K_1, \text{ and } y \in K_2\}$$

defined in the proof of Corollary 5.3. There is a one-to-one correspondence between irreducible weight $n$ representations of $G$ and irreducible weight $r$ representations of $G'$.

As a consequence, for a fixed $r$, $0 \leqslant r \leqslant d_1 - 1$, there exists gcd$(r, d_1)^2 \times \cdots \times$ gcd$(r, d_p)^2$ non-isomorphic irreducible $G'$-modules each of which has dimension $d_1 \times \cdots \times d_p \frac{\gcd(r, d_1)^2 \times \cdots \times \gcd(r, d_p)^2}{\gcd(r, d_1) \times \cdots \times \gcd(r, d_p)}$. Since $|G'| = d_1(d_1 \times \cdots \times d_p)^2 = \sum_{r=0}^{d_1-1} \gcd(r, d_1)^2 \times \cdots \times \gcd(r, d_p)^2$ we conclude that $G'$ has exactly $\sum_{r=0}^{d_1-1} \gcd(r, d_1)^2 \times \cdots \times \gcd(r, d_p)^2$ non-isomorphic irreducible $G'$-modules and, also, exactly this number of conjugacy classes. See for example [16, §2.4].

Remark. If $n = 1$, then $\pi_n$ is an isomorphism and we may take the set-theoretic section $\sigma$ to be the identity map. Also, when $n = 1$, every $\chi \in \text{Hom}_Z(\ker \pi_n, k^\times)$ is trivial. The resulting representation $(W_{y,\chi}, \rho_{y,\chi}^\sigma, 1_{y,\chi})$ takes the form $W_{y,\chi} := \text{span}_k \{e_z\}_{z \in K_2}$ and an element $(\alpha, x, w)$ acts by $(\alpha, x, w) \cdot e_z := \alpha(\langle x, z \rangle) e_{z+w}$.

5.2. Induced representations. We now prove that every irreducible representation is induced by a 1-dimensional representation of a suitable subgroup. In light of Theorem 5.1 and its proof, it suffices to prove that every $(W_{y,\chi}, \rho_{y,\chi}^\sigma, n_{y,\chi})$, where $y \in K_2$, $\chi \in \text{Hom}_Z(\ker \pi_n, k^\times)$, and $\sigma$ is a set-theoretic section of $\pi_n$, is induced by such a representation.

Fix $y \in K_2$, fix $\chi \in \text{Hom}_Z(\ker \pi_n, k^\times)$, and define

$$V_{y,\chi} := \text{span}_k \{e_{y,\chi}\}.$$ Let $G(\ker \pi_n)$ denote the subgroup of $G$ defined by

$$G(\ker \pi_n) := \{(\alpha, x, w) : w \in \ker \pi_n\}.$$ We regard $V_{y,\chi}$ as a $G(\ker \pi_n)$-module by defining

$$(\alpha, x, w) \cdot e_{y,\chi} := \alpha^n(\langle x, y \rangle) \chi(w) e_{y,\chi}$$

for $(\alpha, x, w) \in G(\ker \pi_n)$. We let

$$\iota : V_{y,\chi} \rightarrow (W_{y,\chi}, \rho_{y,\chi}^\sigma, n_{y,\chi})$$

denote the $G(\ker \pi_n)$-homomorphism defined by $e_{y,\chi} \mapsto e_{y,\chi}$.

Proposition 5.6. In the above notation, $(W_{y,\chi}, \rho_{y,\chi}^\sigma, n_{y,\chi})$ is isomorphic to $\text{Ind}^G_{G(\ker \pi_n)}(V_{y,\chi})$. 
Proof. We verify the universal property which characterizes induced representations. Let $(V, \rho)$ be a $G$-module and let $\Phi : V_{y, \chi} \to (V, \rho)$ be a homomorphism of $G(\ker \pi_n)$-modules. Define a map $\Psi : W_{y, \chi} \to V$ by

$$e_{y+z, \chi} \mapsto \rho^{x, n}_{y, \chi}((1, 0, \sigma(z))(\Phi(e_{y, \chi}))$$

and extending linearly. Clearly, $\Psi$ is unique and $\Phi = \Psi \circ \iota$. It remains to check that $\Psi$ is a $G$-homomorphism.

Let $z \in \text{image } \pi_n$ and let $(\alpha, x, w) \in G$. Using the definitions of $\rho^{x, n}_{y, \chi}$ and $\Psi$, we obtain

$$\Psi((\alpha, x, w) \cdot e_{y+z, \chi}) \text{ equals} (5.7) \quad \alpha^n \langle x, y + z \rangle \chi(w + \sigma(z) - \sigma(nw + z))(1, 0, \sigma(z + nw)) \cdot \Phi(e_{y, \chi}).$$

On the other hand, since $\Phi$ is a $G(\ker \pi_n)$-homomorphism, $\Phi(e_{y, \chi}) \in V_{y, n}$, where $V_{y, n}$ is the $(y, n)$ weight space of the $k^x \times K_1$-module $\text{Res}^G_{k^x \times K_1}(V)$.

Now, observe

$$(\alpha, x, w) \cdot (1, 0, \sigma(z)) = (1, 0, w + \sigma(z)) \cdot (\alpha \langle x, \sigma(z) \rangle, x, 0)$$

so $(\alpha, x, w) \cdot \Psi(e_{y+z, \chi})$ equals

$$(1, 0, w + \sigma(z)) \cdot (\alpha^n \langle x, \sigma(z) \rangle^n \langle x, y \rangle) \Phi(e_{y, \chi}))$$

which simplifies to

$$\alpha^n \langle x, z + y \rangle ((1, 0, w + \sigma(z)) \cdot \Phi(e_{y, \chi}).$$

Finally, to see that (5.8) equals (5.7) note that

$$w + \sigma(z) = w + \sigma(z) - \sigma(nw + z) + \sigma(z + nw),$$

that $w + \sigma(z) - \sigma(nw + z) \in \ker \pi_n$, and use the fact that $\Phi$ is $G(\ker \pi_n)$-linear. \qed

5.3. Proof of Theorem 2.2 and Corollary 2.3. Combining everything we are able to complete the proof of the results stated in §2.2.

Proof of Theorem 2.2. If $E$ is a simple semi-homogeneous vector bundle of separable type then $G(E)$ is a non-degenerate theta group. As a consequence, Theorem 2.2 is a special case of Theorem 5.1. \qed

Proof of Corollary 2.3. We know that $G(E)$ has a unique irreducible weight 1 representation. This representation has dimension equal to $\# \sqrt{K(E)}$. On the other hand $H^0(E)(X, E)$ is a weight 1 representation of dimension $|\chi(E)| = \# \sqrt{K(E)}$. \qed
6. Adelic theta groups and line bundles

In this section we explain how to construct adelic theta groups associated to (total spaces of) line bundles on abelian varieties. We also determine some first properties of these groups and introduce some notation which we find helpful for proving Theorem 2.4. The proof of this theorem is the subject of §7.

6.1. Conventions about line bundles. Let \( X \) be a projective variety and let \( L \) be the total space of a line bundle on \( X \). If \( \sigma \) is an automorphism of \( X \) then an automorphism of \( L \) covering \( \sigma \) is a linear isomorphism \( \tau : L \to L \) with the property that the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\tau} & L \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & X
\end{array}
\]

commutes [14, p. 208].

If \( f : Y \to X \) is a morphism of projective varieties then \( f^*L := Y \times_X L \) is defined by the fibered product square

\[
\begin{array}{ccc}
f^*L := Y \times_X L & \to & L \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X
\end{array}
\]

It is the total space of a line bundle on \( Y \).

Let \( X \) be an abelian variety and let \( L \) be the total space of a line bundle on \( X \). If \( x \in X \) then let \( \text{Aut}_x(L) \) denote the set of automorphisms of \( L \) which cover \( T_x \).

Let \( K(L) := \{ x \in X(k) : \text{Aut}_x(L) \neq \emptyset \} \) and let \( G(L) \) denote the group consisting of pairs \( (x, \phi) \) where \( x \in K(L) \) and where \( \phi \) is an automorphism of \( L \) covering \( T_x \).

6.2. Conventions about torsion points. We first recall some of the notation introduced in §2.3. In particular \( I \) denotes the set of positive integers which are not divisible by the characteristic of \( k \), \( \text{tor}(X) := \{ x \in X(k) : nx = 0 \text{ for some } n \in I \} \), and \( V(X) := \varprojlim \text{tor}(X) \), where the limit is indexed by \( I \) and where the maps are given by

\[
(n/m)_X : \text{tor}(X) \to \text{tor}(X)
\]

whenever \( m \) divides \( n \).

We identify \( V(X) \) with the set

\[
\{ x = (x_i)_{i \in I} : x_i \in \text{tor}(X) \text{ and } (n/m)x_n = x_m \text{ if } n, m \in I \text{ and } m \text{ divides } n \}.
\]

We let \( T(X) := \{ x \in V(X) : x_1 = 0 \} \). This is a subgroup of \( V(X) \).
We now introduce some additional notation which we find helpful for what follows. Let $L$ be the total space of a line bundle on $X$. If $x \in V(X)$, then let
\[ \text{supp}^L(x) := \{ n \in I : \text{Aut}_{x_n}^*(n^*_X L) \neq \emptyset \}. \]

A homomorphism of abelian varieties $f : Y \to X$ induces a homomorphism
\[ V(f) : V(Y) \to V(X), \text{ defined by } y = (y_i)_{i \in I} \mapsto (f(y_i))_{i \in I}. \]
We denote $V(f)(y) \in V(X)$ simply by $f(y)$ in what follows.

**Proposition 6.1.** Let $f : X \to Y$ be a homomorphism of abelian varieties. Let $L$ be a line bundle on $Y$. The following assertions hold

(a) if $y \in V(Y)$ and $m$ is the order of $y_1$, then $m \in \text{supp}^L(y)$;
(b) if $y \in V(Y)$, $m \in \text{supp}^L(y)$ and $m \mid n$, then $n \in \text{supp}^L(y)$;
(c) if $y, z \in V(Y)$, then $\text{supp}^L(z) \cap \text{supp}^L(y) \neq \emptyset$;
(d) if $x \in V(X)$, then $\text{supp}^L(x) \cap \text{supp}^L(f(x)) \neq \emptyset$.

**Proof.** To prove (a), let $y \in V(Y)$ and let $m$ be the order of $y_1$. Then $y_m \in Y_{m^2}$ and so $\text{Aut}_{y_m}^*(m^*_Y L) \neq \emptyset$ because $Y_{m^2} \subseteq K(m^*_Y L)$. Hence $m \in \text{supp}^L(y)$.

To prove (b), let $q = n/m$ and note that
\[ n^*_Y L = q^*_Y m^*_Y L \cong q^*_Y T^*_{y_m} m^*_Y L = T^*_{y_m} (q^*_Y m^*_Y L) = T^*_{y_m} n^*_Y L. \]

To prove (c), let $y, z \in V(Y)$ and let $m$ be the least common multiple of the order of $y_1$ and $z_1$. Then, by parts (a) and (b), $m \in \text{supp}^L(y) \cap \text{supp}^L(z)$.

To prove (d), if $x \in V(X)$ and if $m$ is the order of $x_1$ then $m \in \text{supp}^L(x)$ by part (a) applied to $X$ and $f^*L$. On the other hand, $f(x_m) \in Y_{m^2}$ so that $m$ is an element of $\text{supp}^L(f(x))$ as well. \[ \square \]

6.3. **Diagrams of $k^\times$-torsors.** Let $L$ be the total space of a line bundle on an abelian variety $X$. Important in the construction of $\hat{G}(L)$ are various commutative diagrams of $k^\times$-torsors which we now describe.

6.3.1. **The pull-back morphisms** $f^* : \text{Aut}_{f(x)}(L) \to \text{Aut}_x(f^*L)$.

Let $f : X \to Y$ be a homomorphism of abelian varieties, let $L$ be a line bundle on $Y$, and let $x \in X$. We construct pull-back maps
\[ f^* : \text{Aut}_{f(x)}(L) \to \text{Aut}_x(f^*L) \]
which are morphisms of $k^\times$-torsors.

There are two cases to consider. The first is when $\text{Aut}_{f(x)}(L) = \emptyset$. In this case, $f^*$ is the empty morphism. The second is when $\text{Aut}_{f(x)}(L) \neq \emptyset$. In this case, let $\phi \in \text{Aut}_{f(x)}(L)$. By
the universal property of fibered products there exists a unique morphism $\Phi: L \to T_{f(x)}^*L$ fitting into the commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\phi} & T_{f(x)}^*L \\
\downarrow & & \downarrow \\
Y & \xrightarrow{T_{f(x)}} & Y,
\end{array}
$$

where the square is a fibered product square and where the unlabelled arrows are the projection morphisms. Since $f \circ T = T_{f(x)} \circ f$, we obtain an isomorphism of $f^*L$ covering $T_x$ via the composition

$$
f^*L \xrightarrow{f^*\phi} f^*T_{f(x)}^*L = T_{f(x)}^*f^*L \to f^*L,
$$

where the rightmost arrow is the projection map. We denote this isomorphism by $f^*\phi$. This defines a map

$$
f^*: \text{Aut}_{f(x)}(L) \to \text{Aut}_x(f^*L),
$$

which is a morphism of $k^\times$-torsors.

Note that if $\text{Aut}_{f(x)}(L) \neq \emptyset$ then the pull-back map

$$
\text{Aut}_{f(x)}(L) \xrightarrow{f^*} \text{Aut}_x(f^*L)
$$

is an isomorphism of $k^\times$-torsors.

6.3.2. The morphisms $a_{n,m}^{L,X}: \text{Aut}_{x_m}(m_X^*L) \to \text{Aut}_{x_n}(n_X^*L)$. Let $x \in V(X)$, let $m \in \text{supp}^L(x)$, suppose that $m \mid n$, and let $q := n/m$. We now describe morphisms

$$
a_{n,m}^{L,X}: \text{Aut}_{x_m}(m_X^*L) \to \text{Aut}_{x_n}(n_X^*L)
$$

which are used in the construction of $\hat{G}(L)$. Observe first that we have isomorphisms of $k^\times$-torsors

$$
(6.1) \quad \text{Aut}_{x_n}(q_X^*m_X^*L) = \text{Aut}_{x_n}(n_X^*L),
$$

defined by sending an element $\phi \in \text{Aut}_{x_n}(q_X^*m_X^*L)$ to the element of $\text{Aut}_{x_n}(n_X^*L)$ determined by the composition

$$
n_X^*L = q_X^*m_X^*L \xrightarrow{\phi} q_X^*m_X^*L = n_X^*L.
$$

Using the isomorphism (6.1) we obtain pull-back morphisms of $k^\times$-torsors

$$
\begin{array}{ccc}
\text{Aut}_{x_m}(m_X^*L) & \xrightarrow{q_X^*} & \text{Aut}_{x_n}(q_X^*m_X^*L) \\
\xrightarrow{a_{n,m}^{L,X}} & & \xrightarrow{a_{n,m}^{L,X}} \text{Aut}_{x_n}(n_X^*L)
\end{array}
$$

which we denote by $a_{n,m}^{L,X}$.

These morphisms have the properties that:
(a) if \( x \in V(X) \), then \( a^{L,x}_{n,m} \circ a^{L,x}_{m,p} = a^{L,x}_{n,p} \), for all \( n, m, p \in \text{supp}^L(x) \) with the property that \( p \mid m \mid n \);

(b) if \( x \in V(X) \), then \( a^{L,x}_{n,n} = \text{id}_{\text{Aut}_{x}(n_{X}^{*}L)} \);

(c) if \( x \) and \( y \) are elements of \( V(X) \), then

\[
 a^{L,x+y}_{n,m}(\phi \circ \psi) = a^{L,x}_{n,m}(\phi) \circ a^{L,y}_{n,m}(\psi),
\]

for all \( \phi \in \text{Aut}_{x}(m_{X}^{*}L) \) and all \( \psi \in \text{Aut}_{y}(m_{X}^{*}L) \) whenever

\[
 m \in \text{supp}^L(x) \cap \text{supp}^L(y).
\]

6.3.3. The morphisms \( b^{f,L}_{x,m} : \text{Aut}_{x}(m_{X}^{*}f^{*}L) \to \text{Aut}_{f(x)}(m_{Y}^{*}L) \). Let \( f : X \to Y \) be a homomorphism of abelian varieties. Let \( L \) be the total space of a line bundle on \( Y \), let \( x \in V(X) \), and let \( m \in \text{supp}^L(f(x)) \). Since \( m \in \text{supp}^L(f(x)) \) we have that \( \text{Aut}_{f(x)}(m_{Y}^{*}L) \neq \emptyset \).

We indicate how to construct isomorphisms

\[
 (6.2) \quad \text{Aut}_{f(x)}(m_{Y}^{*}L) \xrightarrow{b^{f,L}_{x,m}} \text{Aut}_{x}(m_{X}^{*}f^{*}L)
\]

of \( k^* \)-torsors.

Since \( m_{Y} \circ f = f \circ m_{X} \), we have isomorphisms

\[
 \text{Aut}_{x}(f^{*}m_{Y}^{*}L) = \text{Aut}_{x}(m_{X}^{*}f^{*}L).
\]

Inverting the pullback morphism

\[
 \text{Aut}_{f(x)}(m_{Y}^{*}L) \xrightarrow{f^{*}} \text{Aut}_{x}(f^{*}m_{Y}^{*}L)
\]

yields the isomorphism \( (6.2) \).

The morphisms \( b^{f,L}_{x,m} \) have the properties that:

(a) if \( x \in V(X) \) and \( m \mid n \) then the diagram

\[
 \begin{array}{ccc}
 \text{Aut}_{x}(m_{X}^{*}(f^{*}L)) & \xrightarrow{a^{f,L,x}_{n,m}} & \text{Aut}_{x}(n_{X}^{*}(f^{*}L)) \\
 b^{f,L}_{x,m} & \parallel & b^{f,L}_{n,m} \\
 \text{Aut}_{f(x)}(m_{Y}^{*}L) & \xrightarrow{a^{L,f(x)}_{n,m}} & \text{Aut}_{f(x)}(n_{Y}^{*}L)
 \end{array}
\]

commutes;

(b) if \( x, z \in V(X) \), \( m \in \text{supp}^{f^{*}L}(x) \cap \text{supp}^{f^{*}L}(z) \), \( \phi \in \text{Aut}_{x}(m_{X}^{*}(f^{*}L)) \), and \( \psi \in \text{Aut}_{z}(m_{X}^{*}(f^{*}L)) \), then

\[
 b^{f,L}_{x+z,m}(\phi \circ \psi) = b^{f,L}_{x,m}(\phi) \circ b^{f,L}_{z,m}(\psi).
\]
6.4. Construction and first properties of adelic theta groups. Let $X$ be an abelian variety and let $L$ be the total space of a line bundle on $X$. We indicate how the adelic theta group of $L$, which we denote by $\hat{G}(L)$, is constructed and discuss some first properties.

Let $\hat{G}(L)$ denote the set of pairs $(x, \{\alpha_n\}_{n \in \text{supp}^L(x)})$, where $x \in V(X)$, $\alpha_n \in \text{Aut}_{x_n}(n^*_X L)$, and $\alpha_{n,m}^x L(\alpha_m) = \alpha_n$ whenever $m \in \text{supp}^L(x)$ and $m \mid n$.

Observe first that $\hat{G}(L)$ is nonempty. Indeed, if $x \in V(X)$, then choose some $p \in \text{supp}^L(x)$, and let $\alpha_p$ be an element of $\text{Aut}_{x_p}(p^*_X L)$. Then for $\ell \in \text{supp}^L(x)$, define

$$(6.3) \quad \alpha_\ell := \begin{cases} a_{L,x}^\ell(\alpha_p) & \text{if } \ell \mid \ell \\ a_{L,x}^\ell \cdot (a_{L,p}^\ell(\alpha_p)) & \text{if } \ell \nmid \ell. \end{cases}$$

Then, using the properties of the morphisms $a_{m,n}^x L$, we check that $(x, \{\alpha_n\}_{n \in \text{supp}^L(x)})$ is an element of $\hat{G}(L)$. The group operation is defined as follows.

If $(x, \{\alpha_n\}_{n \in \text{supp}^L(x)})$ and $(y, \{\beta_m\}_{m \in \text{supp}^L(y)}) \in \hat{G}(L)$ then let

$$(6.4) \quad (x, \{\alpha_n\}_{n \in \text{supp}^L(x)}) \cdot (y, \{\beta_m\}_{m \in \text{supp}^L(y)}) := (x + y, \{\gamma_\ell\}_{\ell \in \text{supp}^L(x + y)})$$

where $\{\gamma_\ell\}_{\ell \in \text{supp}^L(x + y)}$ is defined by choosing some element $p$ of $\text{supp}^L(x) \cap \text{supp}^L(y)$, which is nonempty and contained in $\text{supp}^L(x + y)$, defining $\gamma_p := \alpha_p \circ \beta_p$, which is an element of $\text{Aut}_{x_p + y_p}(p^*_X L)$, and defining, for all $\ell \in \text{supp}^L(x + y)$,

$${\gamma_\ell} := \begin{cases} a_{L,x+y}^\ell(\gamma_p) & \text{if } \ell \mid \ell \\ a_{L,x+y}^\ell \cdot (a_{L,p}^\ell(\gamma_p)) & \text{if } \ell \nmid \ell. \end{cases}$$

The right hand side of (6.4) is a well defined element of $\hat{G}(L)$, the pair $(\hat{G}(L), \cdot)$ is a group, is a central extension of $k^\times$ by $V(X)$, and contains an isomorphic copy of $T(X)$.

6.4.1. The skew-symmetric bilinear form $[-, -]_{\hat{G}(L)}$. Suppose that $(x, \{\alpha_n\}_{n \in \text{supp}^L(x)})$ and $(y, \{\beta_n\}_{n \in \text{supp}^L(y)})$ are elements of $\hat{G}(L)$. If $p$ is an element of $\text{supp}^L(x) \cap \text{supp}^L(y)$ then $\alpha_p \circ \beta_p \circ \alpha_p^{-1} \circ \beta_p^{-1}$ corresponds to a unique $\gamma \in k^\times$ which is independent of our choice of $p$.

Also if $[-, -]_{\hat{G}(L)}$ denotes the commutator of $\hat{G}(L)$ then

$$[(x, \{\alpha_n\}_{n \in \text{supp}^L(x)}), (y, \{\beta_n\}_{n \in \text{supp}^L(y)})]_{\hat{G}(L)} = (0, \{\gamma \text{id}_{n^*_X L}\}_{n \in \text{supp}^L(0)}).$$

As a consequence we obtain a skew-symmetric bilinear form

$$[-, -]_{\hat{G}(L)} : V(X) \times V(X) \to k^\times,$$ defined by $[x, y]_{\hat{G}(L)} = \gamma$.

Observe also that if $x, y \in V(X)$, if $p \in \text{supp}^L(x) \cap \text{supp}^L(y)$, and if

$$[-, -]_{\hat{G}(p^*_X L)} : K(p^*_X L) \times K(p^*_X L) \to k^\times$$
denotes the skew-symmetric bilinear form determined by the commutator of the theta group $G(p^*_X L)$, then

$$[x, y]_{\hat{G}(L)} = [x_p, y_p]_{G(p^*_X L)}.$$  

6.4.2. The group homomorphism $\hat{G}(f)$. Let $f: Y \to X$ be a homomorphism of abelian varieties. We construct a group homomorphism $\hat{G}(f): \hat{G}(f^* L) \to \hat{G}(L)$ which fits into the commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & k^\times & \longrightarrow & \hat{G}(f^* L) & \longrightarrow & V(Y) & \longrightarrow & 0 \\
& & \downarrow & & \hat{G}(f) & & \downarrow V(f) & & \\
1 & \longrightarrow & k^\times & \longrightarrow & \hat{G}(L) & \longrightarrow & V(X) & \longrightarrow & 0.
\end{array}
$$

In other words the extension determined by $\hat{G}(f^* L)$ is equivalent to the pull-back, with respect to the group homomorphism $V(f): V(Y) \to V(X)$, of the extension determined by $\hat{G}(L)$.

Let $(x, \{\alpha_n\}_{n \in \text{supp} f^* L (x)}) \in \hat{G}(f^* L)$. Let $m$ be the order of $x_1$ and let $y := f(x)$. Then $m \in \text{supp}^f L (x) \cap \text{supp}^L (y)$.

Let $\beta_m := b_{x,m}^{f,L}(\alpha_m)$ and, for $p \in \text{supp}^L (y)$, define

$$\beta_p := \begin{cases} 
  a_{p,m}^{L,Y}(\beta_m) & \text{if } m \mid p \\
  a_{p,m}^{L,Y} - 1 a_{p,m}^{L,Y}(\beta_m) & \text{if } m \nmid p.
\end{cases}$$

Let

$$\hat{G}(f)((x, \{\alpha_n\}_{n \in \text{supp}^f L (x)})) := (y, \{\beta_n\}_{n \in \text{supp}^L (y)}).$$

Using the properties of $b_{x,m}^{f,L}$, we check that the above definition defines a group homomorphism $\hat{G}(f): \hat{G}(f^* L) \to \hat{G}(L)$ fitting into the commutative diagram (6.5).

7. The group homomorphism $\text{NS}(X) \hookrightarrow H^2(\text{V}(X); k^\times)$

**Proof of Theorem 2.4.** If $L$ is a line bundle on $X$ then we let $[\hat{G}(L)]$ denote the class of its adelic theta group in $H^2(V(X); k^\times)$ the group of normalized two cocycles modulo coboundaries.

Step 1. Let $L$ and $M$ be line bundles on $X$. The relation

$$[\hat{G}(L \otimes M)] = [\hat{G}(L)] + [\hat{G}(M)]$$

holds in $H^2(V(X); k^\times)$.

To prove Step 1 we define normalized set-theoretic sections

$$\sigma_L: \text{V}(X) \to \hat{G}(L), \sigma_M: \text{V}(X) \to \hat{G}(M), \text{ and } \sigma_{L \otimes M}: \text{V}(X) \to \hat{G}(L \otimes M)$$
and check that the relation
\[-[-,[-\sigma_{L\otimes M} = [-,[-\sigma_L + [-,[-\sigma_M\]
holds amongst the corresponding factor sets. To establish this we must show that if \(x\) and \(y\) are elements of \(V(X)\), if
\[
\sigma_L(x) \cdot \sigma_L(y) \cdot \sigma_L(x + y) = (0, \{\alpha \id_{n_x^* M}\}_{n \in \supp L(0)})\], \(\alpha \in k^\times\),
\]
if
\[
\sigma_M(x) \cdot \sigma_M(y) \cdot \sigma_M(x + y) = (0, \{\beta \id_{n_y^* M}\}_{n \in \supp^* M(0)})\], \(\beta \in k^\times\),
and if
\[
\sigma_{L\otimes M}(x) \cdot \sigma_{L\otimes M}(y) \cdot \sigma_{L\otimes M}(x + y) = (0, \{\gamma \id_{n_{xy}^* M(L \otimes M)}\}_{n \in \supp L \otimes M(0)})\], \(\gamma \in k^\times\)
then \(\gamma = \alpha \beta\). Observe that this holds if \(\gamma \id_{n_{xy}^* M(L \otimes M)} = \alpha \id_{n_x^* M} \otimes \beta \id_{n_y^* M}\) for some \(n\).

We now define sections \(\sigma_L\), \(\sigma_M\) and \(\sigma_{L\otimes M}\) with the desired properties. First of all for any element \(x\) of \(V(X)\) let \(m_x\) be the order of \(x_1\).

Now let \(x \in V(X)\). If \(x = 0\) then define \(\alpha_{m_x}^x := \id_L\) and \(\beta_{m_x}^x := \id_M\). Otherwise choose \(\alpha_{m_x}^x \in \Aut_{x_m}(m_x^* L)\), choose \(\beta_{m_x}^x \in \Aut_{x_m}(m_x^* M)\) and set
\[
\gamma_{m_x}^x := \alpha_{m_x}^x \otimes \beta_{m_x}^x \in \Aut_{x_m}(m_x^* x(L \otimes M))\]
Then, \(\alpha_{m_x}^x\), \(\beta_{m_x}^x\) and \(\gamma_{m_x}^x\) determine (unique) elements of \(\hat{G}(L), \hat{G}(M)\), and \(\hat{G}(L \otimes M)\) and hence allow us to define normalized sections \(\sigma_L\), \(\sigma_M\) and \(\sigma_{L\otimes M}\).

Now let \(x\) and \(y\) be elements of \(V(X)\) and let \(p := \lcm(m_x, m_y)\). Let
\[
\phi_p \in \Aut_0(p_x^* L), \psi_p \in \Aut_0(p_x^* M)\] and \(\eta_p \in \Aut_0(p_x^* (L \otimes M))\)
be such that
\[
(0, \phi_p) = (x, \alpha_p^x) \cdot (y, \beta_p^y) \cdot (x + y, \alpha_p^{x+y})^{-1} \in G(p_x^* L),
\]
\[
(0, \psi_p) = (x, \alpha_p^x) \cdot (y, \beta_p^y) \cdot (x + y, \beta_p^{x+y})^{-1} \in G(p_x^* M),
\]
\[
(0, \eta_p) = (x, \gamma_p^x) \cdot (y, \gamma_p^y) \cdot (x + y, \gamma_p^{x+y})^{-1} \in G(p_x^* (L \otimes M)).
\]
(These are the “\(p\)th components” of \([x, y]_{\sigma_L}, [x, y]_{\sigma_M}\) and \([x, y]_{\sigma_{L\otimes M}}\).) Since
\[
\gamma_p^x = \alpha_p^x \otimes \beta_p^x, \gamma_p^y = \alpha_p^y \otimes \beta_p^y \quad \text{and} \quad \gamma_p^{x+y} = \alpha_p^{x+y} \otimes \beta_p^{x+y},
\]
computing the above multiplications we conclude that \(\eta_p = \phi_p \otimes \psi_p\).  

Step 2: If \(L\) is a line bundle on \(X\) then \(\hat{G}(L)\) is abelian if and only if \(L \in \Pic^0(X)\).

Assume that \(\hat{G}(L)\) is abelian. To prove that \(L\) is an element of \(\Pic^0(L)\) we relate \([-[-\hat{G}(L)\] to the Weil-pairing. Let \(n\) be a positive integer not divisible by the characteristic of \(k\). There exists a non-degenerate pairing
\[
\bar{c}_n : X_n \times \hat{X}_n \rightarrow \mu_n,
\]
where $\mu_n$ is the group of $n$-th roots of unity of $k$. (See for instance §20, p. 170 of [14].) In addition, if $x \in X_n$, $y \in n^{-1}_X(K(L))$, and $z \in X$ is such that $nz = y$, then

$$\overline{e}_n(x, \phi_L(y)) = [x, z]_{G(n^*L)},$$

by [14] p. 212.

We now prove that if $[x, y]_{\hat{G}(L)} = 1$, for all $x, y \in \mathcal{V}(X)$, then $\phi_L(y) = \mathcal{O}_X$ for all $y \in X$.

Since tor($X$) is Zariski dense in $X$ it suffices to show that $\phi_L(y) = \mathcal{O}_X$ for all $y \in$ tor($X$).

Let $n$ be the order of $y$. Then $y \in K(n^*_X L)$, and so $y \in n^{-1}_X(K(L))$. Choose $z \in \mathcal{V}(X)$ such that $z_1 = y$. Then $nz_n = y$ and also $n^2z_n = ny = 0$ so $z_n \in K(n^*_X L)$. Hence $n \in \text{supp}^L(z)$.

Let $x \in X_n$ and choose $x \in T(X)$ with $x_n = x$. We then have

$$(7.1) \quad \overline{e}_n(x, \phi_L(y)) = [x, z_n]_{G(n^*L)} = [x, z]_{\hat{G}(L)} = 1,$$

where the second rightmost equality follows because $n \in \text{supp}(x) \cap \text{supp}(z)$, and where the rightmost equality follows because $\hat{G}(L)$ is abelian.

Since $x$ is an arbitrary element of $X_n$ the relation (7.1) holds for all $x \in X_n$. Since $\overline{e}_n$ is non-degenerate this means that $\phi_L(y) = \mathcal{O}_X$ which is what we wanted to show.

The above implies that if $\hat{G}(L)$ is abelian then $L \in \text{Pic}^0(X)$. Indeed if $\hat{G}(L)$ is abelian then $[x, y]_{\hat{G}(L)} = 1$ for all $x, y \in \mathcal{V}(X)$. This implies that $\phi_L(y) = \mathcal{O}_X$ for all $y \in X$. Hence $L \in \text{Pic}^0(X)$.

Conversely if $L \in \text{Pic}^0(X)$ then $G(L)$ is abelian which implies that $\hat{G}(L)$ is abelian.

Step 3. The homomorphism $\hat{G} : \text{NS}(X) \hookrightarrow H^2(V(X); k^\times)$ is functorial in $X$.

Let $f : X \to Y$ be a homomorphism of abelian varieties. Using the definition of $\hat{G}(f)$ (see §6.4.2) we check that the diagram (which has exact rows)

$$
\begin{array}{ccc}
1 & \to & \text{Pic}^0(X) \\
\downarrow f^* & & \downarrow f^* \\
1 & \to & \text{Pic}^0(Y)
\end{array}
\quad
\begin{array}{ccc}
\text{Pic}(X) & \xrightarrow{\hat{G}} & H^2(V(X); k^\times) \\
\uparrow f^* & & \uparrow f^* \\
\text{Pic}(Y) & \xrightarrow{\hat{G}} & H^2(V(Y); k^\times)
\end{array}
$$

commutes. \hfill \Box

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