THE ORDER OF THE NON-ABELIAN TENSOR PRODUCT OF GROUPS

R. BASTOS, I. N. NAKAOKA, AND N. R. ROCCO

Abstract. Let $G$ and $H$ be groups that act compatibly on each other. We denote by $[G, H]$ the derivative subgroup of $G$ under $H$. We prove that if the set $\{g^{-1}g^h \mid g, h \in H\}$ has $m$ elements, then the derivative $[G, H]$ is finite with $m$-bounded order. Moreover, we show that if the set of all tensors $T_\otimes(G, H) = \{g \otimes h \mid g, h \in H\}$ has $m$ elements, then the non-abelian tensor product $G \otimes H$ is finite with $m$-bounded order. We also examine some finiteness conditions for the non-abelian tensor square of groups.

1. Introduction

Let $G$ and $H$ be groups each of which acts upon the other (on the right),

$$G \times H \to G, \ (g, h) \mapsto g^h; \quad H \times G \to H, \ (h, g) \mapsto h^g$$

and on itself by conjugation, in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g^{(h^{g_1})} = \left((g^{g_1^{-1}})^h\right)^{g_1} \quad \text{and} \quad h^{(g^{h_1})} = \left((h^{h_1^{-1}})^g\right)^{h_1}.$$  

In this situation we say that $G$ and $H$ act compatibly on each other. The derivative of $G$ under (the action of) $H$, $[G, H]$, is defined to be the subgroup $[G, H] = \langle g^{-1}g^h \mid g \in G, h \in H \rangle$ of $G$. Similarly, the subgroup $[H, G] = \langle h^{-1}h^g \mid h \in H, g \in G \rangle$ of $H$ is called derivative of $H$ under $G$. In particular, if $G = H$ and all actions are conjugations, then the derivative $[G, H]$ becomes the derived subgroup $G'$ of $G$.

Schur [15, 10.1.4] showed that if $G$ is central-by-finite, then the derived subgroup $G'$ is finite and thus, the group $G$ is a BFC-group. Neumann [15, 14.5.11] improved Schur’s theorem in a certain way, showing that the group $G$ is a BFC-group if and only if the derived subgroup $G'$ is finite, and this occurs if and only if $G$ contains only finitely many

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commutators. Latter, Wiegold proved a quantitative version of Neumann’s result: if \( G \) contains exactly \( m \) commutators, then the order of the derived subgroup \( G' \) is finite with \( m \)-bounded order [20, Theorem 4.7]. Now, the next result can be viewed as a version of Wiegold’s result in the context of actions and derivatives subgroups \([G, H]\) and \([H, G]\), where \( G \) and \( H \) are groups acting compatibly on each other.

**Theorem A.** Let \( G \) and \( H \) be groups that act compatibly on each other. Suppose that the set \( \{g^{-1}g^h \mid g \in G, h \in H\} \subseteq [G, H] \) has exactly \( m \) elements. Then \([G, H]\) is finite, with \( m \)-bounded order.

It should be noted that the structure of derivative subgroups provides important information about the structure of the non-abelian tensor product of groups (see for instance [1, 11, 12, 19, 18]). In this direction, we want to describe quantitative results for the non-abelian tensor product of groups (cf. [1]).

Let \( H^\varphi \) be an extra copy of \( H \), isomorphic via \( \varphi : H \to H^\varphi, h \mapsto h^\varphi \), for all \( h \in H \). Consider the group \( \eta(G, H) \) defined in [11] as

\[
\eta(G, H) = \langle G, H^\varphi \mid [g, h^\varphi]^{g_1} = [g^{g_1}, (h^{g_1})^\varphi], [g, h^\varphi]^{h_1^\varphi} = [g^{h_1}, (h^{h_1})^\varphi], \forall g, g_1 \in G, h, h_1 \in H \rangle.
\]

We observe that when \( G = H \) and all actions are conjugations, \( \eta(G, H) \) becomes the group \( \nu(G) \) introduced in [16]:

\[
\nu(G) = \langle G \cup G^\varphi \mid [g_1, g_2^\varphi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\varphi] = [g_1, g_2^\varphi]^{g_3^\varphi}, g_i \in G \rangle.
\]

It is a well known fact (see [11, Proposition 2.2]) that the subgroup \([G, H^\varphi]\) of \( \eta(G, H) \) is canonically isomorphic with the non-abelian tensor product \( G \otimes H \), as defined by Brown and Loday in their seminal paper [5], the isomorphism being induced by \( g \otimes h \mapsto [g, h^\varphi] \) (see also Ellis and Leonard [7]). It is clear that the subgroup \([G, H^\varphi]\) is normal in \( \eta(G, H) \) and one has the decomposition

\[
(2) \quad \eta(G, H) = ([G, H^\varphi] \cdot G) \cdot H^\varphi,
\]

where the dots mean (internal) semidirect products. For a deeper discussion of non-abelian tensor product and related constructions we refer the reader to [8, 13].

An element \( \alpha \in \eta(G, H) \) is called a tensor if \( \alpha = [a, b^\varphi] \) for suitable \( a \in G \) and \( b \in H \). We write \( T_\otimes(G, H) \) to denote the set of all tensors (in \( \eta(G, H) \)). When \( G = H \) and all actions are by conjugation, we simply write \( T_\otimes(G) \) instead of \( T_\otimes(G, G) \). The influence of the set of tensors in the general structure of the non-abelian tensor product and related constructions was considered for instance in [1, 2, 3, 9, 17]. In [1] the authors proved that if the set of all tensors \( T_\otimes(G, H) \) is finite,
then the non-abelian tensor product $[G, H^p]$ is finite. Here we obtain the following quantitative version:

**Theorem B.** Let $G$ and $H$ be groups that act compatibly on each other. Suppose that there exist exactly $m$ tensors in $\eta(G, H)$. Then the non-abelian tensor product $[G, H^p]$ is finite with $m$-bounded order.

An immediate consequence of the above theorem is a quantitative version of the a well known result due to Ellis [6] concerning the finiteness of the non-abelian tensor product of finite groups (cf. [1, 9, 18]). See also Theorem 2.6 and Remark 2.7 below.

It is well known that the finiteness of the non-abelian tensor square $G \otimes G$, does not imply that $G$ is a finite group (and so, the group $\nu(G)$ cannot be finite). A useful result, due to Parvizi and Niroomand [14, Theorem 3.1], provides a sufficient condition: if $G$ is a finitely generated group in which the non-abelian tensor square is finite, then $G$ is finite (see also [17, Remark 5] for more details). The following result is a quantitative version of the above result and is a refinement of Theorem B in the context of the non-abelian tensor square of groups.

**Corollary C.** Let $G$ be a group. Suppose that there exist exactly $m$ tensors in $\nu(G)$. Then,

- (a) The non-abelian tensor square $[G, G^p]$ is finite with $m$-bounded order. More specifically, $|[G, G^p]| \leq m^{m-n}$, where $n$ is the order of the derived subgroup $G'$;
- (b) Additionally, if the abelianization $G^{ab}$ is finitely generated, then the group $G$ is finite, with $m$-bounded order.

Note that the assumption of the abelianization $G^{ab}$ to be finitely generated is necessary. For instance, the Prüfer group $C_p^\infty$ is an infinite group such that $T_\otimes(C_p^\infty) = \{1\} = [C_p^\infty, C_p^\infty]$. We also obtain a list of equivalent conditions related to the finiteness of the non-abelian tensor square and the structure of the group $\nu(G)$ (see Theorem 2.9 below).

2. **Proofs**

The following result is a consequence of [5, Proposition 2.3].

**Proposition 2.1.** Let $G$ and $H$ be groups acting compatibly on each other. The following statements hold in $\eta(G, H)$:

- (a) There exists an action of the free product $G \ast H$ on $[G, H^p]$ so that for all $g \in G, h \in H, p \in G \ast H$:
  
  $[g, h^p]^p = [g^p, (h^p)^p]$;
of generality we may assume that \( G, H \) derived subgroup \( m \) has finite \( \in \delta \). Then \( \ker((g, h)) = g^{-1}h^{-1} \) for each \( g \in G \), \( h \in H \);

(c) The actions of \( G \) on \( \ker(\mu) \) and of \( H \) on \( \ker(\lambda) \) are trivial.

The next lemma is an immediate consequence from the definition of \( \eta(G, H) \) and Proposition 2.1(c).

**Lemma 2.2.** If \( G \) and \( H \) are groups that act compatibly on each other, then \( \ker(\mu) \cap \ker(\lambda) \) is a central subgroup of \( \eta(G, H) \);

For the reader’s convenience we restate Theorem A.

**Theorem A.** Let \( G \) and \( H \) be groups that act compatibly on each other. Suppose that the set \( \{g^{-1}h^x \mid g \in G, \ h \in H\} \subseteq [G, H] \) has exactly \( m \) elements. Then the derivative subgroup \( [G, H] \) is finite with \( m \)-bounded order.

**Proof.** Put \( D = \{g^{-1}h^x \mid g \in G, \ h \in H\} \). For \( g \in G \) and \( h, k \in H \), let us write \( [g, h] = g^{-1}h^g \) and \( [g, h, k] = [[g, h], k] \). The compatibility of the actions gives us that \( [g, h]^x = [g^x, h^x] \), for all \( x, g \in G \) and \( h \in H \). Thus, \( D \) is a normal subset of \( [G, H] \) and, as \( |D| = m \), for each \( \delta \in D \) we have \( [[G, H] : C_{[G, H]}(\delta)] \leq m \). Consequently, \( \bigcap_{\delta \in D} C_{[G, H]}(\delta) \) has finite \( m \)-bounded index in \( [G, H] \) and, by [20] Theorem 4.7, the derived subgroup \( [G, H] \) is finite with \( m \)-bounded order. Without loss of generality we may assume that \( [G, H] \) is abelian. Since for all \( x, g \in G \), \( h, k \in H \), we have \( [[g, h], k]^x = [[g^x, h^x], k^x] \) and

\[
[[g, h], k]^2 = ([g, h]^{-1}[g, h]^k)^2 = [g, h]^{-2}[g, h]^{2k} = [[g, h]^2, k] \in D,
\]

we conclude that the abelian finitely generated subgroup \( [[G, H], H] \) is normal in \( G \) and each generator of this subgroup has \( m \)-bounded order. From this we deduce that \( [[G, H], H] \) is finite with \( m \)-bounded order and we may assume that \( H \) acts trivially on \( [G, H] \). Hence, for all \( g \in G \) and \( h \in H \),

\[
[g, h]^2 = [g, h][g, h]^h = g^{-1}h^{-1}g^{-1}h = [g, h^2] \in D.
\]

Since \( |D| = m \), it follows that every element \( [g, h] \) has finite \( m \)-bounded order. We conclude that the order of the derivative subgroup \( [G, H] \) is \( m \)-bounded. The proof is complete. \( \square \)

**Remark 2.3.** Since \( [G, H] \) and \( [H, G] \) are epimorphic images of the non-abelian tensor product \( [G, H^\varphi] \), the finiteness of \( [G, H^\varphi] \) implies that \( [G, H] \) and \( [H, G] \) are finite. However, the converse does not hold in general. In fact, let \( F_m \) and \( F_n \) be free groups of finite ranks \( m \)
and \( n \), respectively, where \( m, n \geq 1 \) and suppose that these groups act trivially on each other. Thus \([G, H] = \{1\}\) and \([H, G] = \{1\}\) are finite, but by [5, Proposition 2.4], \([F_m, (F_n)^\varphi] \cong (F_m)^{ab} \otimes \mathbb{Z} (F_n)^{ab}\), which is not finite.

Now we will deal with Theorem B: Let \( G \) and \( H \) be groups that act compatibly on each other. Suppose that there exist exactly \( m \) tensors in \( \eta(G, H) \). Then the non-abelian tensor product \([G, H^\varphi]\) is finite with \( m \)-bounded order.

**Corollary 2.4.** Let \( G \) and \( H \) be groups that act compatibly on each other. Suppose that the sets \( \{g^{-1}g^h \mid g \in G, h \in H\} \subseteq [G, H] \) and \( \{h^{-1}h^g \mid g \in G, h \in H\} \subseteq [H, G] \) have at most \( m \) elements. Then the index \( n = [[G, H^\varphi] : \ker(\lambda) \cap \ker(\mu)]\) is finite and \( m \)-bounded.

**Proof.** By Theorem A, both derivative subgroups \([G, H]\) and \([H, G]\) are finite groups with \( n \)-bounded orders. Since \([G, H^\varphi] : \ker(\lambda)] = [[G, H]]\) and \([H, H^\varphi] : \ker(\mu)] = [[H, G]]\), it follows that \( \ker(\lambda) \cap \ker(\mu)\) has index at most \([|G, H|] \cdot |H, G|\). The proof is complete. \(\square\)

**Lemma 2.5.** Let \( G \) and \( H \) be groups that act compatibly on each other. Suppose that there are exactly \( m \) tensors in \( \eta(G, H) \). Then for every \( x \in G \) and \( y \in H \) we can write:

\[
[x, y^\varphi]^{n+1} = [x, (y^2)^\varphi][x^y, y^\varphi]^{n-1},
\]

where \( n = [[G, H^\varphi]/(\ker(\mu) \cap \ker(\lambda))].\)

**Proof.** Since \( |T_{\otimes}(G, H)| = m \), each of the sets \( \{g^{-1}g^h \mid g \in G, h \in H\} \) and \( \{h^{-1}h^g \mid g \in G, h \in H\} \) has at most \( m \) elements. By Theorem A, the derivative subgroups \([G, H]\) and \([H, G]\) are finite with \( m \)-bounded order. Moreover, the index \([[[G, H^\varphi] : \ker(\mu) \cap \ker(\lambda)] = n\) is finite (Corollary 2.4). We conclude that for every \( x, y \in G \) the element \( [x, y^\varphi]^n \in \ker(\mu) \cap \ker(\lambda) \). Thus, by Lemma 2.2, \( [x, y^\varphi]^n \in \mathbb{Z}(\eta(G, H)) \) and so, \( [x, y^\varphi]^{n+1} = x^{-1}(y^{-1})^\varphi x[x, y^\varphi]^n y^\varphi \). Further,

\[
[x, y^\varphi]^{n+1} = x^{-1}(y^{-1})^\varphi x[x, y^\varphi]^n y^\varphi = [x, (y^2)^\varphi](y^{-1})^\varphi[x, y^\varphi]^{n-1} y^\varphi = [x, (y^2)^\varphi]([x, y^\varphi]^{n-1}) y^\varphi = [x, (y^2)^\varphi][x^y, y^\varphi]^{n-1},
\]

by definition of \( \eta(G, H) \), which establishes the formula. \(\square\)

We are now in a position to prove Theorem B.

**Proof of Theorem B.** By Lemma 2.2, the subgroup \( \ker(\mu) \cap \ker(\lambda) \) is a central subgroup of \( \eta(G, H) \). Set \( N = \ker(\mu) \cap \ker(\lambda) \) and \( n =
By Corollary 2.4 the index $n$ is $m$-bounded. We claim that every element in $[G, H^r]$ can be written as a product of at most $m \cdot n$ tensors. Indeed, suppose that an element $\alpha \in [G, H^r]$ can be expressed as a product of $r$ tensors but cannot be written as a product of fewer tensors. If $r > m \cdot n$, then one of the tensors must appear in the product at least $n + 1$ times. In particular, since the set of tensors is normal and by definition of $\eta(G, H)$, $[g, h^r]^r = [g^r, (h^r)^r]$ and $[g, h^r]^{r+1} = [g^r, (h^r)^r]$, for all $g, x \in G$ and $h, y \in H$, we can write

$$\alpha = [a, b^\varphi]^{n+1}[a_{n+2}, b^\varphi_{n+2}] \ldots [a_r, b^\varphi_r],$$

where $a, a_{n+2}, \ldots, a_r \in G$ and $b, b_{n+2}, \ldots, b_r \in H$. By Lemma 2.5,

$$[a, b^\varphi]^{n+1} = [a, (b^2)^\varphi][a^b, b^\varphi]^{n-1}.$$  

It follows that $\alpha$ can be rewritten as a product of $r - 1$ simple tensors, contrary to the minimality of $r$. From this we conclude that $r \leq m \cdot n$. Now, since there exists at most $m$ simple tensors, we conclude that $|[G, H^r]| \leq m^{m-n}$, as well. In particular, $[G, H^r]$ is finite with $m$-bounded order. The proof is complete.  

In [10], Moravec proved that if $G$ and $H$ are locally finite groups of finite exponent acting compatibly on each other, then there is a bound to the exponent of the non-abelian tensor product $G \otimes H$ in terms of the exponent of the involved groups. This bound depends to the positive solution of the restricted Burnside problem (Zel’manov, [21] [22]). Using the general description of the group $\eta(G, H)$ we present an explicit bound to the exponent of the non-abelian tensor product of groups, when $G$ and $H$ are finite groups. Moreover, we present another proof of Ellis’ result [6].

**Theorem 2.6.** Let $G$ and $H$ be finite groups that act compatibly on each other. Then the non-abelian tensor product $[G, H^r]$ is finite. Moreover, the exponent $\exp([G, H^r])$ is finite and $\{|G|, |H|\}$-bounded.

**Proof.** By Lemma 2.5 $\ker(\mu) \cap \ker(\lambda)$ is a central subgroup of $\eta(G, H)$. Set $n = |[G, H^r] : \ker(\mu) \cap \ker(\lambda)|$. Note that $n$ divides $|G| \cdot |H|$, because $[G, H] \leq G$ and $[H, G] \leq H$. Since $|\eta(G, H)/(\ker(\mu) \cap \ker(\lambda))| = |G| \cdot |H| \cdot n$, it follows that the derived subgroup $\eta(G, H)'$ is finite and $\exp(\eta(G, H)')$ divides $|G| \cdot |H| \cdot n$ (Schur’s theorem [15, 10.1.4]). In particular, the non-abelian tensor product $[G, H^r]$ is finite and $\exp([G, H^r])$ divides $|G| \cdot |H| \cdot n$. The proof is complete.  

**Remark 2.7.** Since the proof of the above result is based on the general structure of $\eta(G, H)$ (cf. [11]) and on Schur’s theorem [15, 10.1.4], it becomes evident that it provides only a crude bound to both, the order...
and the exponent of the non-abelian tensor product $[G, H^\varphi]$. However, the advantages of these results are the explicit limits and the elementary proofs (without using homological methods). See [10] for more details. Recently, other proofs of this result which are of non-homological nature have appeared (see for instance [11] [9] [18]).

The remainder of this section will be devoted to obtain finiteness conditions for the non-abelian tensor square of groups.

**Lemma 2.8. [1] Theorem C, (a)]** Let $G$ be a group with finitely generated abelianization. Assume that the diagonal subgroup $\Delta(G)$ is periodic. Then the abelianization $G^{ab}$ is finite. Moreover, $G^{ab}$ is isomorphic to some subgroup of $\Delta(G)$.

For the reader’s convenience we restate Corollary C:

**Corollary C.** Let $G$ be a group. Suppose that there exist exactly $m$ tensors in $\nu(G)$. Then,

(a) The non-abelian tensor square $[G, G^\varphi]$ is finite, with $m$-bounded order. More specifically, $|[G, G^\varphi]| \leq m^{m-n}$, where $n$ is the order of the derived subgroup $G'$;

(b) Additionally, if the abelianization $G^{ab}$ is finitely generated, then the group $G$ is finite, with $m$-bounded order.

**Proof.** (a). Applying Theorem B to $[G, G^\varphi]$ we deduce that the order of the non-abelian tensor square is finite with $m$-bounded order. Arguing as in the proof of Theorem B we conclude that $|[G, G^\varphi]| \leq m^{m-n}$.

(b). By the previous item, the non-abelian tensor square $[G, G^\varphi]$ and the derived subgroup $G'$ are finite with $m$-bounded orders. Now, it suffices to prove that the abelianization is finite with $m$-bounded order. By Lemma 2.8 the abelianization $G^{ab}$ is isomorphic to a subgroup of the diagonal subgroup $\Delta(G)$. Since $\Delta(G) \leq [G, G^\varphi]$, it follows that $\Delta(G)$ is finite with $m$-bounded order. The proof is complete. □

It should be noted that the next result makes evident an interesting relation between the constructions $\nu(G)$ and the non-abelian tensor square $G \otimes G$. More precisely, we collect a list of equivalences which give a relation between the set of commutators of the group $\nu(G)$ and the set of tensors $T_\otimes(G)$.

**Theorem 2.9.** Let $G$ be a group. The following properties are equivalents.

(a) $\nu(G)$ is a BFC-group;
(b) The set of all commutators $\{[\alpha, \beta] \mid \alpha, \beta \in \nu(G)\}$ is finite;
(c) The derived subgroup $\nu(G)'$ is finite;
(d) The non-abelian tensor square $[G, G^\varphi]$ is finite;
(e) $G$ is a BFC-group and $G^{ab} \otimes \mathbb{Z} G^{ab}$ is finite;
(f) The set of tensors $T\otimes (G) = \{[g, h^\varphi] \mid g, h \in G\} \subseteq \nu(G)$ is finite.

Proof. The equivalences $(a) \iff (b) \iff (c)$ are immediate consequences of Newmann’s result [15, 14.5.11]. The equivalences $(d) \iff (e)$ and $(d) \iff (f)$ were proved in [1, Corollary 1.1] and [1, Theorem A], respectively. It is clear that $(b)$ implies $(f)$. Finally, if part $(f)$ holds then, from the decomposition (2) and items $(d)$, $(e)$, we obtain $(a)$. The proof is complete. □

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Departamento de Matemática, Universidade de Brasília, Brasília-DF, 70910-900 Brazil
E-mail address: (Bastos) bastos@mat.unb.br; (Rocco) norai@unb.br

Departamento de Matemática, Universidade Estadual de Maringá, Maringá-PR, 87020-900 Brazil
E-mail address: (Nakaoka) innakaoka@uem.br