The Spectrum of the turbulence based on theory of stochastic equations and equivalence of measures

A V Dmitrenko\textsuperscript{1,2}

\textsuperscript{1}Department of Thermal Physics, National Research Nuclear University «MEPhI», Kashirskoye shosse 31, Moscow, 115409, Russian Federation
\textsuperscript{2}Department of Thermal Engineering, Russian University of Transport «MIIT», Obraztsova Street 9, Moscow, 127994, Russian Federation

E-mail: AVDmitrenko@mephi.ru, ammssv@yandex.ru

Abstract. The formation of the spectrum of turbulence in the inertial interval on the basis of the new theory of stochastic hydrodynamics is presented. This theory is based on the theory of stochastic equations of continuum laws and equivalence of measures between random and deterministic movements. The purpose of the article is to present a solution based on these stochastic equations for the formation of the turbulence spectrum in the inertial interval in the form of the spectral function \( E(k) \) depending on wave numbers \( k \) in form \( E(k) \sim k^n \). The results of analytical solutions showed a satisfactory correspondence of the obtained dependence with the classical Kolmogorov’s dependence in the form of \( E(k) \sim k^{5/3} \).

1. Introduction
The development of the theory of turbulence with the use of different ideas is presented in [1-41]. Special attention was focused on the theoretical description of the spectral density. It should be noted that the most well-known ratio based on the theory of dimension was, as is known, determined by Kolmogorov [37-41], who wrote the formula for the spectral density in the wave number function as \(-5/3\). But on the basis of these works, it was not possible to obtain a single mathematical apparatus that would allow to determine analytically all the important characteristics of turbulence. At the same time, stochastic turbulence theory based on stochastic equations and the theory of equivalent measures makes it possible to derive analytical dependences for the first and second critical Reynolds numbers, profiles of averaged velocity and temperature fields, friction and heat transfer coefficients, and second-order correlations [42-61].

In [46,47,53] analytical formula of the spectral density were presented, which are consistent with the classical dependencies of the statistical theory. Vortexes has a formula of spectrum \( E(k) \) depending on wave numbers \( k \) for interval of generation of turbulence in form \( E(k) \sim k^n \), \( n=-1.2\sim-1.5, -1.66<n<-1 \). This ratio was named as the ratio of uncertainty in turbulence generation.

2. Stochastic equations of conservation
Equations were derived in [42-54] take the form:

the equation of mass (continuity)
The set of equations of mass, momentum, and energy (1)-(3) for the area1), referring the pair (N, M) = (1,0) is:

\[
\frac{d (\rho)}{d \tau} = \frac{\rho_a}{\tau_{cor}},
\]

(4)

the momentum equation

\[
\frac{d \left( \rho U \right)_{col, st}}{d \tau} = \text{div}(\tau_{i,j} U_{col, st}) - \frac{(\rho U)_{st}}{\tau_{cor}} \frac{d (\rho U)_{st}}{d \tau} + \frac{d E}{d \tau} + F_{col, st} + F_{st},
\]

(2)

and the energy equation

\[
\frac{d E_{col, st}}{d \tau} = \text{div}(\lambda \frac{\partial T}{\partial x_j} + u_i \tau_{i,j})_{col, st} + \text{div}(\frac{\partial T}{\partial x_j} + u_i T_{i,j})_{st} - \left( \frac{E_{st}}{\tau_{cor}} \right) \left( \frac{d E}{d \tau} \right) + \left( u_i F_{col, st} + (u_i F)_{st} \right).
\]

(3)

Here, \( E, \rho, U, u_i, u_j, \mu, \tau_{i,j} \) are the energy, the density, the velocity vector; the velocity components in directions \( x_i, x_j, x_k (i, j, k = 1, 2, 3) \); the dynamic viscosity; the time; and stress tensor \( \tau_{i,j} \) are the components of liquid or gas; \( \lambda \) is the thermal conductivity; \( c_p \) and \( c_v \) are the specific heat at constant pressure and volume, respectively; \( F \) is the external force, and the scale of turbulence. Indexes \((u, \rho)\) and \((1)\) refer to the velocity field and index \((T)\) refers to the temperature field. \( L \) on \( x_2 \) and \( L \) in \( x_3 \) are coordinates along and normal to the wall. Index “col st” refers to components, which are actually the deterministic. Index “st” refers to component, which are actually the stochastic. Then for the non-isothermal motion of the medium, using the definition of equivalency measures between deterministic and random process in the critical point, the sets of stochastic equations of energy, momentum, and mass are defined for the next space-time areas: 1) the beginning of the generation (index 1,0 or 1); 2) generation (index 1,1); 3) diffusion (1,1,1) and 4) the dissipation of the turbulent fields.

3. Sets of stochastic equations

The set of equations of mass, momentum, and energy (1)-(3) for the area1), referring the pair (N, M) = (1,0) is:

\[
\frac{d (\rho)_{cor, st}}{d \tau} = \frac{\rho_a}{\tau_{cor}},
\]

(4)

\[
\frac{d \left( \rho U \right)_{cor, st}}{d \tau} = \left( \frac{\rho U}{\tau_{cor}} \right)_{cor, st},
\]

\[
\frac{d \left( \frac{\partial T}{\partial x_j} + u_i \tau_{i,j} \right)_{cor, st}}{d \tau} = \left( \frac{\partial T}{\partial x_j} + u_i \tau_{i,j} \right)_{cor, st}.
\]

Further, \( L = L_{u}, \rho = L_{U} \) is the scale of turbulence. Indexes \((u, \rho)\) and \((1)\) refer to the velocity field and index \((T)\) refers to the temperature field. \( L \) on \( x_2 \) and \( L \) in \( x_3 \) are coordinates along and normal to the wall. Index “col st” refers to components, which are actually the deterministic. Index “st” refers to component, which are actually the stochastic. Then for the non-isothermal motion of the medium, using the definition of equivalency measures between deterministic and random process in the critical point, the sets of stochastic equations of energy, momentum, and mass are defined for the next space-time areas: 1) the beginning of the generation (index 1,0 or 1); 2) generation (index 1,1); 3) diffusion (1,1,1) and 4) the dissipation of the turbulent fields.
The set of equations of mass, momentum, and energy (1)-(3) for the area 2), referring the pair (N, M) = (1,0), was written as:

\[
\begin{align*}
\left( \frac{d(\rho)_{col,nt}}{d\tau} \right)_{1,1} &= - \left( \frac{d\rho_{st}}{d\tau_{st}} \right) ; \\
\left( \frac{d(\rho \dot{U})_{col,nt}}{d\tau} \right)_{1,1} &= - \left( \frac{d(\rho \dot{U})_{st}}{d\tau} \right) ; \\
\text{div}(\tau_{i,j})_{col,nt2} &= \frac{d(\rho \dot{U})_{st}}{d\tau} \\
\left( \frac{d(E)_{col,nt}}{d\tau} \right)_{1,1} &= - \left( \frac{d(E)_{st}}{d\tau} \right)_{1,1} ; \\
\text{div}(\lambda \frac{\partial T}{\partial x_j} + u_i \tau_{i,j})_{col,nt2} &= \left( \frac{d(E)_{st}}{d\tau} \right)_{1,1}.
\end{align*}
\]

Using the sets (4),(5) formulas for the velocity and temperature fields may be obtained [45,54].

The area 3) is the diffusion of the turbulence. So, for the area of 3) diffusion, in accordance with [41-52] we have two fractal equations. The first equation is written as

\[
\frac{d(E_{st})}{d\tau} = -(R_{st})_{(1,1,1)} \frac{(E_{st})}{\tau_{cor1}}.
\]

Here \((E_{st})\) is the field energy component, which is actually the stochastic one (index “st”), index \(j=1\) refers to the space-time area of the diffusion of turbulence 3). The the solution may be written for \((R_{st2})_{(1,1,1)} = 1\) and for \(\rho=\text{constant}\) as an expression for the correlation function in statistic theory. The initial turbulence energy in critical point for the wave number \(k_{cr}\) is

\[
(E_{st}/\rho)_{cr} = \int_{k_{cor1}}^{k_{cr}} \delta(k-k_{cr})E(k) dk = \frac{k_{cr}}{2\pi} \sqrt{\frac{2}{\pi}} \frac{k}{k_{cor1}} \exp \left[ -\frac{1}{2} \left( \frac{k}{k_{cor1}} \right)^2 \right] d \left( \frac{k}{k_{cor1}} \right).
\]

The value \(k\) is the wave number and index \(cor1\) determines the characteristics of the single or main perturbation generating turbulence. The value \(k_{cr}\) is the wave number of the beginning of the interaction between the deterministic field and random field that leads to turbulence in critical point. \((E_{st}/\rho)_{cr}\) - is turbulence energy, which corresponds to the wave number \(k_{cr}\). Note that the written expression determines the turbulence energy in the case of a continuous spectrum for the fully turbulence flow.

4. The spectral function in the inertial interval

The specificity of the importance of the inertial interval as shown by the experience of experimental study of turbulence is the independence of this interval from the molecular viscosity. In this regard, to find a solution, equation (8) can be transformed to a stochastic equation of the following form

\[
\frac{d(E_{st})}{d\tau} = \frac{(E_{st})}{\delta \tau_{cor1}}.
\]
\[ (E_s(r))_j = \cos \tan t \cdot r^{2/3} e^{2/3} \] \hspace{1cm} (9)

So we have the law \(-5/3\)

\[ E(k)_j = \cos \tan t \cdot k^{-5/3} e^{2/3} \] \hspace{1cm} (10)

5. Conclusions

The results of analytical solutions showed a satisfactory correspondence of the obtained dependence with the classical Kolmogorov’s dependence in the form of \(E(k)\sim k^{-5/3}\) or the law of 2/3.

References

[1] Kolmogorov A N 1940 *Dokl. Akad. Nauk* 26 (1) 6
[2] Kolmogorov A N 1958 *Dokl. Akad. Nauk* 119 No. 5 861
[3] Kolmogorov A N 1959 *Dokl. Akad. Nauk* 124 No. 4 754
[4] Kolmogorov A N 2004 *Usp. Mat. Nauk* 59 (1) (355) 5
[5] Landau L D 1944 *Dokl. Akad. Nauk* 44 (8) 339
[6] Lorenz E N 1963 *J. Atmos. Sci.* 20 130
[7] Ruelle D and Takens F 1971 *Communs. Math. Phys.* 20 167p
[8] Klimontovich Yu L 1989 *Usp. Fiz. Nauk* 158 B1 59p
[9] Arnol’d V I 1990 *Theory of Catastrophes* (Nauka: Moscow)
[10] Haller G 1999 *Chaos Near Resonance* (Springer: Berlin)
[11] Orzag S A and Kells L C 1980 *J. Fluid Mech.* 96 (1) 159p
[12] Babin A V and Vishik M I 1982 *Russian Math. Surveys*, 37:3 195
[13] Vishik M I and Chepyzhov V V 2011 *Russian Math. Surveys*, 66:4 637
[14] Ladyzhenskaya O A 1975 *J. Soviet Math.*, 3 458
[15] Vishik M I, Zelik S V and Chepyzhov V V 2013 *Sb. Math.*, 204:1 1
[16] Landau L D , Lifshits E F 1959 Fluid mechanics (Perg. Press Oxford London)
[17] Constantin P, Foais C, Temam R 1988 *Physica D* 30 284
[18] Babin A V and Vishik M I 1983 Russian Math. Surveys, 38:4, 151
[19] Vishik M I, Komech A I 1983 *Tr. Mosk. Mat. Obs.*, 46 3
[20] Packard N H, Crutchfield J P, Farmer J D, Shaw R S 1980 *Phys.Rev. Lett.* 45 N9 712
[21] Malraison B, Berge P, Dubois M 1983 *J. Physique-Lett.* 44 , P.L897
[22] Procaccia I, Grassberger P 1983 *Phys.Rev. Lett.* 50 346
[23] Procaccia I, Grassberger P 1983 *Phys. Rev. A.* 28 N4, 2591
[24] Gromov P R, Zobin F B, Rabinovich M I, Reiman AM, Sushchik M M 1987 *DAN* 292 N2, 284
[25] Rabinovich M I, Reiman AM, Sushchik M M and etc. 1987 *JETP Letters.* 13 (16) 987
[26] Brandstater A., Swift J . Harry L. Swinney and etc. 1983 *Phys. Rev. Let. 51* N16 1442
[27] Priymak V G 2013 *Dok. Phys.* 58:10 457
[28] Davidson P A 2004 Turbulence (Oxford Univ. Press) 678p
[29] Millionshchikov M D 1969 Turbulent flow in boundary layers and in pipes (Nauka: Moscow)
[30] Schlichting H 1968 *Boundary-Layer Theory* (6th Edition McGraw-Hill)
[31] Monin A S and Yaglom A M 1971 *Statistical Fluid Mechanics* (M.I.T. Press)
[32] Hinze J O 1975 *Turbulence* (McGraw-Hill)
[33] Dmitrenko A V 2008 *Fundamentals of heat and mass transfer and hydrodynamics of single-phase and two-phase media*. Criterion integral statistical methods and direct numerical simulation. (Galleya print: Moscow) 398p http://search.rsl.ru/ru/catalog/record/6633402
[34] Dmitrenko A V 1997 *33th AIAA/ASME/SAE/ASEE AIAA Paper*97-2911; doi: 10.2514/6.1997-2911; http://arc.aiaa.org/doi/abs/10.2514/6.1997-2911.
[35] Dmitrenko A V 1998 *34thAIAA/ASME/SAE/ASEE AIAA Paper*98-3444; doi:10.2514/6.1998-3444; http://arc.aiaa.org/doi/abs/10.2514/6.1998-3444
[36] Dmitrenko A V 1993 *Aviats. Tekh.* No. 1 39p
[37] Dmitrenko A V 1986 *Proc.11th Conf. Young Scientists* Moscow Physicotechnical Institute Part 2 Moscow (1986) pp 48–52. Deposited at VINITI 08.08.86 No. 5698-B86
[38] Kolmogorov A N 1932 *Dokl. Akad. Nauk* 32 (1) 19
[39] Kolmogorov A N 1941 *Dokl. Akad. Nauk* 30 (4) 299
[40] Kolmogorov A N 1962 *Coll. Intern. du CNRS a Marseille. Paris, Ed. CNRS*, 447
[41] Kolmogorov A N 1962 *J. Fluid Mech.* 13 No. 1 82
[42] Dmitrenko AV 2013 *Dokl. Phys.* 58 (6) 228 doi.org/10.1134/s1028335813060098
[43] Dmitrenko A V 2007 *Dokl. Phys.* 52 (7) 384 doi.org/10.1134/s1028335807120166
[44] Dmitrenko A V 2014 *Adv. Stud. in Theoret. Phys* 8 (25) 1101 doi.org/10.12988/astp.2014.49131
[45] Dmitrenko A V 2015 *J.of Eng. Phys.and Thermophys.* 88(6)1569 doi:10.1007/s10891-015-1344-x
[46] Dmitrenko A V 2016 *Heat TransferResearch* 47 (1) 338 doi:10.1615/HeatTransRes.014191
[47] Dmitrenko A V 2016 *Int. J. Fluid Mech. Res.* 43 (3) 82 doi:10.1615/InterJFluidMechRes.v43.i3.
[48] Dmitrenko A V 2016 *Int. J. Fluid Mech. Res.* 43 (2) 182 doi:10.1615/InterJFluidMechRes.v43.i2
[49] Dmitrenko A V 2017 *Contin. Mechan. and Thermod.* 29 (1) 1 doi:10.1007/s00161-016-0514-1
[50] Dmitrenko A V 2017 *Contin.Mechan. and Thermod.* 29 (6) 1197 doi.org/10.1007/s00161-017-0566x
[51] Dmitrenko A V 2017 *Heat Trans. Res.* 49(15) 1195 doi:10.1615/HeatTransRes.2017018750.
[52] Dmitrenko AV2017 *J.of Eng. Phys.and Thermophys.* 90 (6) 1288 doi.org/10.1007/s10891-017-1685-8
[53] Dmitrenko A V 2013 *Regular Coupling between Deterministic (Laminar) and Random (Turbulent) Motions-Equivalence of Measures* Scientific Discovery Diploma No. 458 registration No. 583 of December 2
[54] Dmitrenko A V 2013 *Theory of Equivalent Measures and Sets with Repeating DenumerableFractal Elements. Stochastic Thermodynamics and Turbulence. Determinacy–Randomness Correlator* [in Russian] (Galleya-Print: Moscow) 226p https://search.rsl.ru/ru/record/01006633402
[55] Dmitrenko A V 2018 *J. of Phys.:Conf. Series* 1009 doi:10.1088/1742-6596/1009/1/012017
[56] Dmitrenko A V 2018 *J. Materials Sci. and Engin.:Conf.Series* 468 (1), doi:10.1088/1757-899X/468/1/012021
[57] Dmitrenko A V 2019 *J. of Phys.:Conf. Series* 1250 doi:10.1088/1742-6596/1250/1/012001
[58] Dmitrenko A V 2019 *J. of Phys.:Conf. Series* 1291 doi:10.1088/1742-6596/1291/1/012001
[59] Dmitrenko AV 2019 *JP J. of HMT.* 18 (2) 463 http://dx.doi.org/10.17654/HM018020463
[60] Dmitrenko A V 2020 *Contin. Mechan. and Thermod.* 32(1) 63 doi.org/10.1007/s00161-019-00784-0
[61] Dmitrenko AV 2020 *Contin.Mechan. and Thermod.* 32(1) 161 doi.org/10.1007/s00161-019-00792-0

**Acknowledgements**

This work was supported by the program of increasing the competitive ability of National Research Nuclear University MEPhI (agreement with the Ministry of Education and Science of the Russian Federation of August 27, 2013, Project No.02.a03.21.0005).

This article is dedicated to the memory of Academician N.A. Anfimov