The special values of $L$-functions at $s = 1$ of theta products of weight 3

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Abstract

In this paper, we compute the special values of $L$-functions at $s = 1$ of some theta products of weight 3, and express them in terms of special values of generalized hypergeometric functions.

Keywords: Theta series, $L$-value for theta products, Generalized hypergeometric function

Mathematics Subject Classification: 11F27, 11F67, 33C20

1 Introduction and main results

For a modular form $f$ of weight $k$ with the $q$-expansion $f(q) = \sum_{n=1}^{\infty} a_n q^n$ ($q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$), we define its $L$-function $L(f, s)$ by

$$L(f, s) := \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{Re}(s) > k + 1.$$  

It is well-known (cf. [13]) that $L(f, s)$ has meromorphic continuation to $\mathbb{C}$, and satisfies a functional equation under $s \leftrightarrow k - s$. In this paper, we consider $L$-functions of modular forms which are products of the Jacobi theta series

$$\theta_2(q) := \sum_{n \in \mathbb{Z}} q^{(n+1)/2}^2, \quad \theta_3(q) := \sum_{n \in \mathbb{Z}} q^n, \quad \theta_4(q) := \sum_{n \in \mathbb{Z}} (-1)^n q^n,$$

(these are modular forms of weight 1/2), or the Borweins theta series [5,6]

$$a(q) = \sum_{n,m \in \mathbb{Z}} q^{n^2+nm+m^2}, \quad b(q) = \sum_{n,m \in \mathbb{Z}} \omega^{n-m} q^{n^2+nm+m^2},$$

$$c(q) = \sum_{n,m \in \mathbb{Z}} q^{(n+1/3)^2+(n+1/3)(m+1/3)+(m+1/3)^2},$$

(these are modular forms of weight 1), where $\omega$ denotes a primitive 3rd root of unity.

In 2010s, for some modular forms, it was found that special values of $L$-functions can be expressed in terms of special values of generalized hypergeometric functions

$$\sum_{n=0}^{\infty} \binom{a_1 n \cdots a_{p+1} n}{b_1 n \cdots b_p n} \frac{z^n}{(1)_n}.$$  

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where \((a)_n := \Gamma(a + n)/\Gamma(a)\) denotes the Pochhammer symbol. For example,

1. Otsubo [9] expressed \(L(f, 2)\) for some theta products \(f(q)\) of weight 2 in terms of \(3F_2(1)\) by using his regulator formula and Bloch’s theorem.
2. Rogers [10], Rogers and Zudilin [11], Zudilin [15] and Ito [7] expressed \(L(f, 2)\) for some theta products \(f(q)\) of weight 2 in terms of \(3F_2(1)\) by an analytic method. Moreover, Zudilin [15] expressed \(L(f, 3)\) for the theta product corresponding to the elliptic curve of conductor 32 in terms of \(4F_3(1)\).
3. Rogers et al. [12] expressed \(L(f, 2)\) (resp. \(L(f, 3), L(f, 4)\)) for some quotients \(f(q)\) of the Dedekind eta function \(\eta(q) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\) of weight 3 (resp. 4, 5) in terms of special values of generalized hypergeometric functions or the gamma function \(\Gamma(s)\) by an analytic method.

Note that all of these results are in the case of cusp forms.

In this paper, we consider the case when a theta product \(f(q)\) of weight 3 is not necessarily a cusp form but \(f(q)\) and its image under the Fricke involution vanish at \(q = 0\) (i.e. \(\tau = i\infty\)). In this case, the Mellin transformation of \(f(q)\) converges and defines an entire function (see [13, Theorem 3.2]).

Our main results are the following.

**Theorem 1** We have the following Table 1.

**Theorem 2** We have the following Table 2.

Here \(f(q)\) is normalized so that its leading Fourier coefficient is equal to 1.

Both \(L(f, 1)\) and the hypergeometric evaluations in the tables can be numerically evaluated using MATHEMATICA, and the author verified all the formulas in Theorems 1 and 2.

We remark that for theta products \(f(q)\) of weight 3 which are considered in [12, Theorem 5], the values \(L(f, 2)\) (hence \(L(f, 1)\) by the functional equation) are expressed in terms of special values of the gamma function, not generalized hypergeometric functions. It is new that the values of \(L\)-functions at \(s = 1\) of theta products of weight 3 are expressed in terms of special values of generalized hypergeometric functions.

Our strategy to compute \(L(f, 1)\) is the same as that used in [7,10–12] and [15]. For a modular form \(f(q) = \sum_{n=1}^{\infty} a_n q^n\) and \(m \in \mathbb{Z} \geq 1\), the value \(L(f, m)\) is obtained by the Mellin transformation of \(f(q)\)

\[
L(f, m) = \frac{(-1)^{m-1}}{\Gamma(m)} \int_0^1 f(q)(\log q)^{m-1} \frac{dq}{q}.
\]

The case \(m = 1\) is special since the logarithm in the integral above vanishes:

\[
L(f, 1) = \int_0^1 f(q) \frac{dq}{q}.
\]  \hspace{1cm} (1)

Since the Jacobi theta series and the Borweins theta series have connections with hypergeometric functions (see (2) and (3) below), we can reduce (1) to an integral of the form

\[
\int_0^1 \text{(polynomials in } \alpha(1 - \alpha)^m) _2F_1 (\alpha^n) \frac{d\alpha}{\alpha(1 - \alpha)}.
\]
Then, we obtain a hypergeometric evaluation of \( L(f, 1) \) after some computation.

Finally, we remark that one of the results of Rogers et al. [12] can be recovered from our results. Let \( f(q) = \frac{1}{2} \theta_2(q^4) \theta_3(q^4) \theta_4(q^4) \). By the Jacobi identity (see section 2), we have

\[
2f(q) = \theta_2(q^4) \theta_3(q^4) \theta_4(q^4) = \theta_2(q^4) \theta_4(q^4) + \theta_2^*(q^4) \theta_4(q^4).
\]

Thereupon, by Theorem \( 1(iv), (xiv) \), we obtain

\[
L(f, 1) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{16 \sqrt{\pi}} \left( 3F_2 \left[ \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right] + 3F_2 \left[ \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right] \right).
\]

If we use the following contiguous relation

\[
(b - a)3F_2 \left[ \begin{array}{c} a, b, c \\ e, f \end{array} \right] \left[ \begin{array}{c} z \\ \end{array} \right] + a3F_2 \left[ \begin{array}{c} a + 1, b, c \\ e, f \end{array} \right] \left[ \begin{array}{c} z \\ \end{array} \right] = b3F_2 \left[ \begin{array}{c} a, b + 1, c \\ e, f \end{array} \right] \left[ \begin{array}{c} z \\ \end{array} \right],
\]

\[
\begin{array}{|c|c|}
\hline
l(q) & L(f, 1) \\
\hline
(i) & \frac{1}{2} \theta_2(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\sqrt{2} + \Gamma^2 \left( \frac{1}{3} \right)}{8 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(ii) & \frac{1}{2} \theta_2(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{8 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(iii) & \frac{1}{2} \theta_2(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\pi}{4 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(iv) & \frac{1}{2} \theta_2(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{16 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(v) & \frac{1}{2} \theta_2(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{16 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(vi) & \frac{1}{4} \theta_2^2(q^4) \theta_1(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{8 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(vii) & \frac{1}{4} \theta_2^2(q^4) \theta_1(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{8 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(viii) & \frac{1}{4} \theta_2^2(q^4) \theta_1(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{8 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(ix) & \frac{1}{4} \theta_2^2(q^4) \theta_1(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{8 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(x) & \frac{1}{4} \theta_2^2(q^4) \theta_1(q^4) \theta_4(q^4) = \frac{\Gamma^2 \left( \frac{1}{3} \right)}{8 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(xi) & \frac{1}{8} \theta_2^3(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\pi}{16 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(xii) & \frac{1}{8} \theta_2^3(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\pi}{16 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(xiii) & \frac{1}{16} \theta_2^4(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\pi}{16 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(xiv) & \frac{1}{32} \theta_2^4(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\pi}{256 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
(xv) & \frac{1}{32} \theta_2^4(q^4) \theta_3(q^4) \theta_4(q^4) = \frac{\pi}{256 \sqrt{\pi}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, 1 \\ \frac{1}{2}, 1 \end{array} \right) \\
\hline
\end{array}
Table 2  Borweins theta products

|   | \( f(q) \)                           | \( l(f, 1) \) |
|---|--------------------------------------|---------------|
| (i) | \( \frac{1}{3} a(q^3) c(q^3) b(q^3) \) | \( \Gamma^4 \left( \frac{1}{4} \right) \) |
| (ii) | \( \frac{1}{3} c(q^3) b^2(q^3) \) | \( \frac{2\pi}{9\sqrt{3}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ 1, 1, 1 \end{array} \right) \) |
| (iii) | \( \frac{1}{9} c^2(q^3) b(q^3) \) | \( \frac{2\pi}{27\sqrt{3}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ 1, 1, 1 \end{array} \right) \) |
| (iv) | \( \frac{1}{3} c(q^9) b^2(q^9) \) | \( \frac{2\pi}{3\sqrt{3}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ 1, 1, 1 \end{array} \right) \) |
| (v) | \( \frac{1}{5} c^2(q^9) b(q^9) \) | \( \frac{2\pi}{3\sqrt{3}} \left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ 1, 1, 1 \end{array} \right) \) |

then we have

\[
L(f, 1) = \frac{\Gamma^4 \left( \frac{1}{4} \right)}{8\sqrt{2}\pi^2}.
\]

This is the second last formula in [12, Theorem 5].

2 Proof of Theorem 1

There are many relations between the Jacobi theta series \( \theta_2(q), \theta_3(q) \) and \( \theta_4(q) \). One of the most important relations is the Jacobi identity [4, p.35, (2.1.10)]

\[
\theta_4^3(q) = \theta_2^4(q) + \theta_4^4(q).
\]

Moreover, they have a connection with hypergeometric functions. Let \( \alpha := \theta_2^4(q)/\theta_4^4(q) \). Note that we have \( 1 - \alpha = \theta_2^4(q)/\theta_3^4(q) \) by the Jacobi identity. Then we have

\[
\theta_3^2(q) = {}_2F_1 \left[ \begin{array}{c} \frac{1}{3}, \frac{1}{3} \\ 1 \end{array} \right] \alpha, \quad \frac{dq}{q} = \frac{d\alpha}{\alpha(1 - \alpha){}_2F_1 \left[ \begin{array}{c} \frac{1}{3}, \frac{1}{3} \\ 1 \end{array} \right] \alpha}. \tag{2}
\]

The former is [2, p. 101, Entry 6], and the latter follows from the former and [1, p. 87, Entry 30].

Proof of Theorem 1  Theorem 1 follows from (2). First, we show the formula (xiii). For simplicity, we denote

\[
\Gamma \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right] := \frac{\Gamma(a_1) \cdots \Gamma(a_p)}{\Gamma(b_1) \cdots \Gamma(b_q)}.
\]

By (1), we have

\[
L(f, 1) = \frac{1}{16} \int_0^1 \theta_2^2(q) \theta_3^2(q) \frac{dq}{q} = \frac{1}{16} \int_0^1 \left( \frac{\theta_2(q)}{\theta_3(q)} \right)^4 \left( \frac{\theta_4(q)}{\theta_3(q)} \right)^2 \theta_3^2(q) \frac{dq}{q}.
\]
If we use (2), the integral above is equal to
\[
\frac{1}{16} \int_0^1 \alpha (1 - \alpha)^{1/2} \binom{1}{1/2} \binom{1/2}{1} \frac{da}{\alpha (1 - \alpha)}.
\]

Since generalized hypergeometric functions have the integral representation [14, p. 108, (4.1.2)]
\[
\binom{p+1}{p} \binom{a_1, a_2, \ldots, a_{p+1}}{b_1, b_2, \ldots, b_p} z = \Gamma \left[ \frac{b_1}{a_1, a_2 - a_1} \right] \int_0^1 \frac{t^{a_1 (1 - t)} b_1 - a_1 \binom{a_2, \ldots, a_{p+1}}{b_2, \ldots, b_p} \left[ \alpha \left( 1 - \frac{t}{(1 - t)} \right) \right] dt,
\]
we have
\[
L(f, 1) = \frac{1}{16} \Gamma \left[ \frac{3}{2}, 1 \right] 3F_2 \left[ \frac{1}{2}, \frac{1}{2}, 1 \right] 1.
\]

Note that this $3F_2$ reduces to a $2F_1$. By Gauss’s summation formula [14, p. 28, (1.7.6)]
\[
2F_1 \left[ a, b \right. c \cdot 1 = \Gamma \left[ \frac{c, c - a - b}{c - a, c - b} \right],
\]
we obtain
\[
L(f, 1) = \frac{\Gamma^2 \left( \frac{1}{2} \right)}{16} = \frac{\pi}{16}.
\]

Next we show the formula (v). Since we have $\theta_3(q) \theta_5(q) = \theta_4^2(q^2)$ [4, p. 34, (2.1.7 ii)], we obtain
\[
L(f, 1) = \frac{1}{8} \int_0^1 \frac{\theta_2(q) \theta_3(q) \theta_5(q)}{\theta_4(q)} \frac{dq}{q}.
\]

If we use (2) and the integral representation of generalized hypergeometric functions, we obtain
\[
L(f, 1) = \frac{1}{8} \Gamma \left[ \frac{5}{8}, \frac{1}{2} \right] 3F_2 \left[ \frac{1}{2}, \frac{1}{8}, \frac{1}{2} \right] 1.
\]

By the Watson summation formula [14, p. 54, (2.3.3.13)]
\[
3F_2 \left[ \frac{a, b, c}{1-a+b, 2c} \cdot 1 = \Gamma \left[ \frac{1, 1+2c}{2}, \frac{1+a+b}{2}, \frac{1-a-b+2c}{2} \right]
\]
we have
\[
L(f, 1) = \Gamma \left[ \frac{4/7, 4/3, 1}{3/7, 3/3, 1} \right].
\]
Therefore we obtain the formula if we simplify this $\Gamma$-factor using the reflection formula and the multiplication formula for $\Gamma(s)$ [8, (5.5.3) and (5.5.6)]

$$\Gamma(n)\Gamma(1 - n) = \frac{\pi}{\sin n\pi},$$

$$\Gamma(nz) = \frac{n^{nz - 1/2}}{(2\pi)^{(n-1)/2}} \prod_{k=0}^{n-1} \Gamma \left( z + \frac{k}{n} \right).$$

Similar computations lead to the remaining formulas. Note that we use the Watson summation formula for the formulas (xi), (xii) and (xv).

\[ \Box \]

3 Proof of Theorem 2

The Borweins theta series $a(q)$, $b(q)$ and $c(q)$ have many relations analogous to those of the Jacobi theta series. For example, J.M. Borwein and P.B. Borwein proved [5]

$$a^3(q) = b^3(q) + c^3(q),$$

which is a cubic analogue of the Jacobi identity. Moreover, like the Jacobi theta series, the Borweins theta series have a connection with hypergeometric functions. Let $\alpha := c^3(q)/a^3(q)$. Note that we have $1 - \alpha = b^3(q)/a^3(q)$ by the cubic identity above. Then we have

$$a(q) = {}_2F_1 \left[ \begin{array}{c} 1/3, \ 2/3 \\ 1 \end{array} \right| \alpha \right], \quad \frac{dq}{q} = \frac{d\alpha}{\alpha(1 - \alpha){}_2F_1 \left[ \begin{array}{c} 1/3, \ 2/3 \\ 1 \end{array} \right| \alpha \right]}.$$ (3)

The former is [3, p. 97, (2.26)], and the latter follows from the former and [1, p. 87, Entry 30]).

Proof of Theorem 2 Like the cases of Jacobi products, Theorem 2 follows from (3). For example, the formula (i) follows from the following computations.

By (1), we have

$$L(f, 1) = \frac{1}{9} \int_0^1 a(q) c(q) b(q) \frac{dq}{q}.$$ If we use (3), the integral above is equal to

$$\frac{1}{9} \int_0^1 \alpha^{1/3}(1 - \alpha)^{1/3} {}_2F_1 \left[ \begin{array}{c} 1/3, \ 2/3 \\ 1 \end{array} \right| \alpha \right] \frac{d\alpha}{\alpha(1 - \alpha)}.$$ 

By the integral representation of $\,_3F_2(z)$, we obtain

$$L(f, 1) = \frac{1}{9} \Gamma \left[ \begin{array}{c} 1/3, \ 1/3 \\ 2/3 \end{array} \right] \_3F_2 \left[ \begin{array}{c} 1/3, \ 1/3, \ 2/3 \\ 2/3, \ 1 \end{array} \right] 1.$$ 

Note that this $\,_3F_2$ reduces to a $\,_2F_1$, hence, by Gauss’s theorem, we have

$$L(f, 1) = \frac{1}{9} \Gamma \left[ \begin{array}{c} 1/3, \ 1/3, \ 1/3 \\ 2/3, \ 2/3, \ 2/3 \end{array} \right].$$
If we use the reflection formula to simplify the $\Gamma$-factor, we obtain the formula.

Similar computations lead to the formulas (ii) and (iii).

Next we prove the formula (iv). Since we have $a(q^3) = (a(q) + 2b(q))/3$, $c(q) = (a(q) - b(q))/3$ [6, Lemmas 2.1 (iii) and (2.1)], we obtain

\[
b^3(q^3) = a^3(q^3) - c^3(q^3) = \frac{a^2(q)b(q) + a(q)b^2(q) + b^3(q)}{3}.
\]

Therefore we have

\[
L(f, 1) = \frac{1}{9} \int_0^1 c(q)b^2(q^3) \frac{dq}{q} = \frac{1}{38/3} \int_0^1 \alpha^{1/3}(1 - \alpha)^{1/3}(1 - \alpha)^{2/3}(1 - \alpha)^{2/3} \left( \frac{1}{1 + 2u} \right)^{1/3} 2F1 \left[ \frac{3/3}{1} \right] \frac{d\alpha}{\alpha(1 - \alpha)}.
\]

If we substitute $\alpha \mapsto 1 - \alpha^3$ and $\alpha \mapsto \frac{1 - u}{1 + 2u}$, the integral above is equal to

\[
\frac{1}{35/3} \int_0^1 \left(1 - \frac{u}{1 + 2u}\right)^{2/3} \left( \frac{3u}{1 + 2u} \right)^{1/3} 2F1 \left[ \frac{3/3}{1} \right] \frac{1 - \left(\frac{1 - u}{1 + 2u}\right)^3}{u(1 - u)} du.
\]

We have the cubic transformation [3, p.96, Theorem 2.3]

\[
2F1 \left[ \frac{a}{3}, \frac{a+1}{3} \right] \left( \frac{1 - (1 - \alpha^3)^3}{1 + 2x} \right) = (1 + 2x)^a 2F1 \left[ \frac{a}{3}, \frac{a+1}{3} \right] \left( \frac{1 - \alpha^3}{\alpha} \right),
\]

hence we obtain

\[
L(f, 1) = \frac{1}{34/3} \int_0^1 u^{1/3}(1 - u)^{2/3} 2F1 \left[ \frac{3/3}{1} \right] \frac{u^3}{u(1 - u)} du.
\]

If we integrate term-by-term, we have

\[
L(f, 1) = \frac{1}{34/3} \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)_n \left( \frac{2}{3} \right)_n \Gamma \left[ \frac{3n + 1/3}{3n + 1} \right] 3F1 \left[ \frac{3n + 1/3}{9}, \frac{2}{3}, \frac{2}{3} \right] \frac{\Gamma\left( n + \frac{2}{3}\right)}{\Gamma\left( n + \frac{7}{3}\right)}.
\]

By the multiplication formula, we obtain

\[
\Gamma \left( 3n + \frac{1}{3} \right) = \frac{3^{3n-1/6}}{2\pi} \Gamma \left( n + \frac{1}{9} \right) \Gamma \left( n + \frac{4}{9} \right) \Gamma \left( n + \frac{7}{9} \right),
\]

\[
\Gamma \left( 3n + 1 \right) = \frac{3^{3n+1/2}}{2\pi} \Gamma \left( n + \frac{1}{3} \right) \Gamma \left( n + \frac{2}{3} \right) \Gamma \left( n + 1 \right),
\]

hence we have

\[
L(f, 1) = \frac{1}{9} \Gamma \left[ \frac{1}{9}, \frac{4}{9}, \frac{7}{9} \right] 3F1 \left[ \frac{1}{9}, \frac{4}{9}, \frac{7}{9} \right] \frac{\Gamma\left( n + \frac{2}{3}\right)}{\Gamma\left( n + \frac{7}{3}\right)}.
\]

We can simplify the $\Gamma$-factor by using the multiplication formula again, then we obtain the formula.

We can prove the formula (v) by similar computations. \qed
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References
1. Berndt, B.C.: Ramanujan's Notebooks, Part II. Springer, New York, NY (1989)
2. Berndt, B.C.: Ramanujan's Notebooks, Part III. Springer, New York, NY (1991)
3. Berndt, B.C.: Ramanujan's Notebooks, Part V. Springer, New York, NY (1998)
4. Borwein, J.M., Borwein, P.B.: Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity. Wiley, Hoboken (1987)
5. Borwein, J.M., Borwein, P.B.: A cubic counterpart of Jacobi's identity and the AGM. Trans. Am. Math. Soc. 323(2), 691–701 (1991)
6. Borwein, J.M., Borwein, P.B., Garvan, F.G.: Some cubic modular identities of Ramanujan. Trans. Am. Math. Soc. 343, 35–47 (1994)
7. Ito, R.: The Beilinson conjectures for CM elliptic curves via hypergeometric functions. Ramanujan J. 45, 433–449 (2018)
8. NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/5.5
9. Otsubo, N.: Certain values of Hecke L-functions and generalized hypergeometric functions. J. Number Theory 131, 648–660 (2011)
10. Rogers, M.: Boyd's conjectures for elliptic curves of conductor 11, 19, 39, 48 and 80, unpublished notes (2010)
11. Rogers, M., Zudilin, W.: From L-series of elliptic curves to Mahler measures. Compos. Math. 148, 385–414 (2012)
12. Rogers, M., Wan, J.G., Zucker, I.J.: Moments of elliptic integrals and critical L-values. Ramanujan J. 37, 113–130 (2015)
13. Shimura, G.: Elementary Dirichlet Series and Modular Forms. Springer, New York (2007)
14. Slater, L.J.: Generalized Hypergeometric Functions. Cambridge University Press, Cambridge (1966)
15. Zudilin, W.: Periodicity of L-values. Number theory and related fields. Springer Proc. Math. Stat. 43, 381–395 (2013)

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