Coorbit spaces associated to integrably admissible dilation groups

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Abstract

This paper considers coorbit spaces parametrized by mixed, weighted Lebesgue spaces with respect to the quasi-regular representation of the semi-direct product of Euclidean space and a suitable matrix dilation group. We provide conditions on the dilation group for which the associated quasi-regular representation is integrable. The class of matrix groups satisfying these conditions, the so-called integrably admissible dilation groups, contains the class of dilation groups yielding an irreducible, square-integrable quasi-regular representation as a proper subclass. As a consequence, the scale of coorbit spaces associated to integrably admissible dilation groups extends the well-studied wavelet coorbit spaces associated with an integrable, irreducible quasi-regular representation. It is shown that for any integrably admissible dilation group there exists a suitable space of smooth, admissible analyzing vectors that can be used to define a consistent coorbit spaces possessing all the essential features that are known to hold in the setting of integrable discrete series representations. In particular, it is shown that the obtained coorbit spaces can be realized as (Besov-type) decomposition spaces, and that the anisotropic Besov spaces associated to expansive matrices coincide with the coorbit spaces induced by the integrably admissible one-parameter groups.

1 Introduction

Let $H \leq \text{GL}(d, \mathbb{R})$ be a closed subgroup and consider the semi-direct product $G = \mathbb{R}^d \rtimes H$. The group $G$ acts unitarily on $L^2(\mathbb{R}^d)$ through the quasi-regular representation $(\pi, L^2(\mathbb{R}^d))$ by

$$
\pi(x, h)f = |\det(h)|^{-\frac{d}{2}}f(h^{-1}(\cdot - x)).
$$

Given a non-zero $\psi \in L^2(\mathbb{R}^d)$, the associated wavelet transform $W_\psi : L^2(\mathbb{R}^d) \to C_b(G)$ is defined by the representation coefficients

$$
W_\psi f(x, h) = \langle f, \pi(x, h)\psi \rangle.
$$

A vector $\psi \in L^2(\mathbb{R}^d)$ is called admissible if $W_\psi : L^2(\mathbb{R}^d) \to L^2(G)$ is an isometry. An important consequence of admissibility of $\psi \in L^2(\mathbb{R}^d)$ is the reproducing formula

$$
W_\psi f = W_\psi f * W_\psi \psi
$$

valid for any $f \in L^2(\mathbb{R}^d)$. The existence of admissible vectors and associated groups has been studied in numerous papers, including [3, 25, 28, 37]. See also [14, 38, 39] for similar investigations, in which $\mathbb{R}^d$ is replaced by a non-commutative nilpotent Lie group $N$.

2010 Mathematics Subject Classification. 22D10, 42C15, 42C40, 46E15.

Keywords and phrases. Admissible dilation group, coorbit space theory, induced coverings, integrable group representation, smooth admissible vectors.
1.1 Integrable wavelet transforms

This paper focuses on wavelet transforms associated with so-called integrably admissible dilation groups; see Section 2.1 for the precise details. An important property of such a dilation group $H \leq \text{GL}(d, \mathbb{R})$ is that the quasi-regular representation $(\pi, L^2(\mathbb{R}^d))$ of $G = \mathbb{R}^d \rtimes H$ is not merely square-integrable, but also integrable in the sense that there exists an admissible $\psi \in L^2(\mathbb{R}^d)$ such that $W_\psi \psi \in L^1(G)$ for some suitable weighting function $v$ on $G$. The integrability of $(\pi, L^2(\mathbb{R}^d))$ is a property that is of interest for at least two reasons. On the one hand, it is exploited in abstract harmonic analysis for constructing projections in the convolution algebra $L^1(G)$. Indeed, the reproducing formula (1.1) yields that $W_\psi \psi = (W_\psi \psi)^\ast = W_\psi \psi * W_\psi \psi$, with $(W_\psi \psi)^\ast(x) = \overline{W_\psi \psi(x^{-1})}$. Thus, if $F := \Delta^{-1/2}_G W_\psi \psi \in L^1(G)$, then $F = F^\ast = F * F$, where $F^\ast(x) = \Delta_G^{-1}(x) F(x^{-1})$ denotes the usual involution in $L^1(G)$, showing that $F$ is a projection in $L^1(G)$. The existence of projections in $L^1(G)$ is related to the existence of compact open sets in the unitary dual $\hat{G}$, equipped with the so-called Fell topology, and is an area of ongoing research [13, 32, 34, 35]. In particular, we mention the recent paper [13], in which the Kirillov-Bernat correspondence between the dual and coadjoint orbits is used, to obtain sufficient and necessary conditions for integrable wavelet transforms phrased in terms of dual orbit spaces $\mathcal{O} \subset \hat{\mathbb{R}}^d$. Building on [34], the results in [13] paved the way for the notion of an integrably admissible dilations group as used in this paper.

Integrable representations also play an essential role in the coorbit space theory developed by Feichtinger and Gröchenig [19–21] and it is in this context that we investigate the integrable wavelet transform in this paper. The coorbit spaces associated to semi-direct products are defined in terms of decay properties of the (extended) matrix coefficients $W_\pi f(x,h) = \langle f, \pi(x,h) \psi \rangle$ and provide a family of Banach function spaces induced by the action of the group. The main ingredient for defining coorbit spaces is a reproducing formula as in (1.1). The formula (1.1) is well-known to hold for any discrete series representation [15, 33], but also holds for numerous reducible square-integrable representations, including the representations considered here. If, in addition to (1.1), the associated representation $\pi$ is integrable, then discretization results such as the existence of Banach frames and atomic decompositions in the coorbit spaces can be obtained [31].

1.2 Coorbit spaces associated to integrably admissible dilation groups

In this paper we consider coorbit spaces associated to integrably admissible dilation groups. Since this setting contains the class of irreducibly admissible dilation groups as a proper subclass, the quasi-regular representation might be reducible. The scale of coorbit spaces that we obtain extends therefore the wavelet coorbit spaces associated to discrete series representations. A key property of the original coorbit spaces that is guaranteed by the irreducibility is its independence of the choice of the admissible analyzing vector used to define the space. This property might fail dramatically if the assumption of irreducibility of the representation is dropped. Indeed, in $d = 2$, the quasi-regular representation $(\pi, L^2(\mathbb{R}^2))$ of $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$ acts reducibly and [26] Section 2.1 provides an example of two integrably admissible vectors for which the associated coorbit spaces do not coincide. This indicates that for reducible representations additional considerations are required in defining the coorbit spaces. More precisely, to compensate for the reducible representations, the requirements on the space of analyzing vectors need to be more restrictive than in the setting of discrete series representations. It turns out that this adjustment is all that is needed to derive a coorbit space theory with the same features as for the irreducible setup. A key example that can covered in the setup considered here, but not by the discrete series case, is the space induced by the action of $G = \mathbb{R}^d \rtimes H$ with $H = \exp(\mathbb{R} A)$ being the one-parameter subgroup generated by a suitable
matrix $A \in \text{GL}(d, \mathbb{R})$. The obtained space turns out to coincide, up to suitable identifications, with the well-known anisotropic Besov spaces [2, 5].

The representation-theoretic nature of the definition of coorbit spaces, together with a reproducing formula such as (1.1), allows to transfer questions concerning these spaces into related questions concerning functions on the group $G = \mathbb{R}^d \rtimes H$. This has the advantage that questions which do not rely on the geometry underlying $G$ can be approached in a unified manner using general tools from abstract harmonic analysis. On the other hand, questions that rely on the structure of $G = \mathbb{R}^d \rtimes H$, and hence of $H$, might not even be well-posed in the usual realization of the coorbit space. Canonical questions of this type are, for example, the dilation-invariance of the coorbit space under the action of dilation groups other than $H$, and the embeddings between coorbit spaces that are defined using different dilations groups. For the investigation of these questions the realization of the coorbit space as a (Besov-type) decomposition space [17, 18] is highly beneficial as this realization encodes the essential properties of the coorbit space that are determined by the dual action of the dilation group $H$. The isotropic Besov spaces [22, 41, 45] form a key example of decomposition spaces and have been identified as the coorbit space of the affine group $G = \mathbb{R} \rtimes \mathbb{R}^+$ from the very beginning [19, Section 7.2]. The identification of coorbit spaces over general irreducibly admissible dilation groups and suitable decomposition spaces have only been established more recently [30]. In this paper we extend this identification to the class of integrably admissible dilation groups. As a consequence, the powerful embedding theory for Besov-type decomposition spaces developed in [46] can be used to obtain embedding theorems for these coorbit spaces.

1.3 Related work

Wavelet coorbit theory. The papers that are closest in spirit to the work presented here are the predecessors [26, 27, 30] by the first named author. The papers [26, 27] establish explicit and verifiable criteria on continuous wavelet transforms for applying the coorbit space theory for irreducible, integrable representations [20, 21]. In this paper we generalize the results of [26] from the setting of irreducibly admissible dilation groups to the class of integrably admissible dilation groups. In doing so, some auxiliary results in [26, 27] will be used by citation, e.g., the explicit construction of a suitable control weight (see Section 2.3 below). Moreover, some of the proof methods used to generalize results of [26, 27] to the setting of integrably admissible dilations groups are non-trivial adaptations of the ones in [26, 27]. The difficulties arising in these adoptions are caused by the fact that for irreducibly admissible dilation groups, the dual actions have a unique, singly generated open orbit, whereas for the integrably admissible dilations groups considered in this paper, the (possibly non-unique) open orbit might be generated by an arbitrary compact set rather than just a singleton; see Section 2.4 for the details. On the other hand, we like to emphasize that several of the results in this paper are obtained relatively easily, compared to the proofs of analogous results in [26, 27]. In part, this is due to recent progress in decomposition space theory. In any case, given the fact that coorbit theory is generally viewed as somewhat technical and cumbersome, we regard this simplicity as an important asset of our approach.

The paper [30] identifies the coorbit spaces considered in [26, 27] as suitable decomposition spaces. The approach and techniques used in [30] are followed in the identification established in Section 4 and 5 below. Several preliminary results of [30] that do not rely on properties of irreducibly admissible dilation groups are used here by citation. However, the extensions of the results in [30] that rely on properties of an irreducibly admissible dilation group are slightly more involved in general. These additional technicalities are most apparent in the
construction of an induced cover and a suitable partition of unity subordinate to this cover; see the results in Section 4. Again, these technicalities are caused by the fact that the open dual orbits of the dilation groups considered in this paper might be generated by an arbitrary compact set instead of a singleton. Lastly, we mention that the actual identification of suitable decomposition spaces and the coorbit spaces considered in this paper is much simpler than in [30] as the reservoir of the coorbit spaces can be canonically identified with the reservoir used to define decomposition spaces, in contrast to [30]. Moreover, we simplify the proof by using a delicate density result for Besov-type decomposition spaces recently established in [43].

**Coorbit theory for dual pairs.** A framework for coorbit spaces associated with possibly reducible, non-integrable representations have been developed by Christensen and Ólafsson [9, 10]. The spaces in [9, 10] are defined under suitable continuity and smoothness conditions of the representation and its associated matrix coefficients. In particular, several desired discretization results have been obtained for this setting in [8]. The coorbit spaces considered in this paper do formally not fit in the framework for dual pairs since the space of analyzing vectors we consider does not form a Fréchet space, which is a standing assumption in [9, 10]. However, as in [8, 9, 10], the basic properties of the coorbit spaces are also established using smoothness properties.

### 1.4 Aims and contributions

The central motivation of this paper is to extend, unify and synthesize results from [7, 13, 26, 30], using the language of coorbit spaces and decomposition spaces as developed in [18, 20, 21]. In particular, we generalize the explicit criteria for a wavelet coorbit theory obtained in [26] to the setting of integrably admissible dilation groups. The fact that we are able to extend a fairly large portion of the results of the precursor papers in comparatively little space, while developing a somewhat novel version of coorbit space theory in the process, is testament both to the versatility and generality of the arguments in the original sources [20, 21] underlying our adaptation, and to the recent progress in the theory of decomposition spaces, e.g., in [43, 46, 47]. As will be seen below, our simple approach is made possible by the topological conditions underlying the notion of an integrably admissible dilation group, which are weak enough to cover a wide variety of settings, but sufficiently stringent to provide a class of analyzing wavelets that is very convenient to work with.

### 1.5 Organization

The paper is organized as follows. In Section 2 the class of integrably admissible dilation groups is introduced. Moreover, several important properties of their associated wavelet transforms are obtained. Section 3 is devoted to the coorbit spaces associated to integrably admissible dilation groups and their basic properties. In particular, it is demonstrated that the standard results on discretization of convolution operators can be applied in the setting of reducible representations to obtain results on Banach frames and atomic decompositions in the coorbit space. Section 4 consists of preliminary results needed for the realization of a coorbit space as a decomposition space. The actual identification is carried out in Section 5. Examples (of classes of) coorbit spaces associated with integrably admissible dilation groups are provided in 6. In particular, it is shown that the coorbit spaces considered in this paper coincide with the original coorbit spaces in the setting of discrete series representation.
2 Wavelet transforms with semi-direct products

For a linear Lie group \( H \leq \text{GL}(d, \mathbb{R}) \), define \( G = \mathbb{R}^d \times H \). The admissibility of a vector \( \psi \in L^2(\mathbb{R}^d) \) can be conveniently characterized by use of the dual action of \( H \) on the Fourier domain \( \hat{\mathbb{R}}^d \cong \mathbb{R}^d \), defined by \( H \times \hat{\mathbb{R}}^d \ni (h, \xi) \mapsto h^T \xi \), where \( h^T \) denotes the transpose of \( h \in H \).

For a proof of the following, cf. [28, 37].

**Lemma 2.1.** A vector \( \psi \in L^2(\mathbb{R}^d) \) is admissible if, and only if,

\[
C_\psi := \int_{H} |\hat{\psi}(h^T \xi)|^2 \, d\mu(h) = 1
\]

for a.e. \( \xi \in \hat{\mathbb{R}}^d \).

Given two vectors \( \psi_1, \psi_2 \in L^2(\mathbb{R}^d) \) satisfying the admissibility condition \((2.1)\), the property that the wavelet transform \( W_{\psi_i} : L^2(\mathbb{R}^d) \rightarrow L^2(G) \), with \( i \in \{1, 2\} \), isometrically intertwines \((\pi, L^2(\mathbb{R}^d))\) and the regular representation \((\lambda_G, L^2(G))\), yields the reproducing formula

\[
W_{\psi_1} f * W_{\psi_2} \psi_1 = W_{\psi_2} f
\]

for all \( f \in L^2(\mathbb{R}^d) \). The identity \((2.2)\) plays a key role in the theory developed below.

2.1 Integrably admissible dilation groups

The following definition introduces the class of dilation groups that will be treat in this paper.

**Definition 2.2.** A closed subgroup \( H \leq \text{GL}(d, \mathbb{R}) \) is called integrably admissible with essential frequency support \( \mathcal{O} \) if \( \mathcal{O} \subset \hat{\mathbb{R}}^d \) is an open set satisfying:

(i) The set \( \mathcal{O} \) is co-null and \( H^T \)-invariant;

(ii) The dual action of \( H \) on \( \mathcal{O} \) is proper, i.e., for all compact subsets \( K \subset \mathcal{O} \), the set

\[
\{(h, \xi) \in H \times \mathcal{O} : (h^T \xi, \xi) \in K \times K\} \subset K \times \mathcal{O}
\]

is compact;

(iii) There exists a compact subset \( K \subset \mathcal{O} \) such that \( \mathcal{O} = H^T K \).

**Remark 2.3.** (a) The compactness criterion \((iii)\) in Definition 2.2 is equivalent to the orbit space \( \mathcal{O}/H^T \) being compact with respect to the quotient topology [34, Corollary 4.2].

Condition (ii) therefore entails, in particular, that the isotropy subgroups \( H_\xi := \{h \in H : h^T \xi = \xi\} \) are compact for all \( \xi \in \mathcal{O} \). Moreover, the orbit maps \( p_\xi : H \rightarrow \mathcal{O}, \ h \mapsto h^T \xi \) are proper for \( \xi \in \mathcal{O} \).

(b) The condition that the dilation group \( H \) is closed is rather natural, and is necessary for the existence of an admissible vector. Confer [24, Proposition 5.1] and [23, Proposition 5].

It is currently not well-understood whether the essential frequency support associated to an integrably admissible matrix group is uniquely defined by the group or not. It is conceivable that given such a group, two different sets \( \mathcal{O} \) and \( \mathcal{O}' \) could serve as essential frequency support, which would necessarily only differ by a nullset. However, note that essential frequency support is described in terms of topological conditions, which cannot be expected to be stable under addition or removal of a set of measure zero. This somewhat subtle point is relevant, since the essential frequency support enters into the decomposition space description of coorbit spaces.
In order to state a convenient characterization of integrably admissible dilation groups, we define, following [10], for given \(Y,Z \subset \mathbb{R}^d\),
\[(Y,Z) := \{ h \in H \mid h^T Y \cap Z \neq \emptyset \}.
\]
The following result is [13, Proposition 12] rephrased in the terminology of the current paper.

**Lemma 2.4.** Let \(H \trianglelefteq \text{GL}(d, \mathbb{R})\) and let \(O \subset \hat{\mathbb{R}}^d\) be open, co-null and \(H^T\)-invariant. Then the following are equivalent:

(i) The group \(H\) is integrably admissible with essential frequency support \(O\).

(ii) There exists some open, relatively compact \(C \subset O\) with \(H^T C = O\) and for any such set \(C\), the set ((\(C,C\)) is relatively compact in \(H\).

**Corollary 2.5.** Let \(H \trianglelefteq \text{GL}(d, \mathbb{R})\) be integrably admissible with essential frequency support \(O\). Then, for all compact sets \(K_1, K_2 \subseteq O\), the set ((\(K_1, K_2\)) is relatively compact in \(H\).

**Proof.** Let \(K \subset O\) be a compact set such that \(H^T K = O\). Then ((\(K,K\)) is relatively compact by Lemma 2.4. For \(i \in \{1,2\}\), the compactness of \(K_i\) yields finite sets \(F_i \subset H\) such that \(K_i \subset F_i K\). But ((\(K_1,K_2\)) \(\subseteq F_2((K,K))F_1^{-1}\), and thus ((\(K_1,K_2\)) is relatively compact.

We next list several classes of integrably admissible dilation groups. The associated coorbit spaces are discussed in more detail in Section 6.

**Example 2.6.** (a) If \(H \trianglelefteq \text{GL}(d, \mathbb{R})\) is (irreducibly) admissible, i.e., there exists a single open orbit \(O = H^T \xi_0\) of full Lebesgue measure for which the associated isotropy group \(H_{\xi_0}\) is compact in \(H\), then \(H\) is readily seen to be integrably admissible. The irreducibly admissible dilation groups are precisely the ones for which the associated quasi-regular representation \((\pi, L^2(\mathbb{R}^d))\) of \(G\) is a discrete series representation [25]. A full classification, up to conjugacy classes, in dimension three can be found in [12]. Explicit necessary and sufficient conditions for abelian dilation groups \(H \trianglelefteq \text{GL}(d, \mathbb{R})\) can be found in [6].

(b) If \(H = \exp(\mathbb{R} A)\), then \(H\) is integrably admissible iff the real parts of all eigenvalues of \(A\) are either strictly negative or strictly positive, see [32,35]. The essential frequency support can then be taken as \(O = \mathbb{R}^d \setminus \{0\}\).

(c) For the case \(d = 3\) and \(H \trianglelefteq \text{GL}(d, \mathbb{R})\) being connected and abelian, a complete classification of the integrably admissible dilation groups is given in [13, Proposition 18]. As a concrete example, we mention \(H = \exp(\mathbb{R} A) \exp(\mathbb{R} B)\), with
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix},
\]
and
\[
B = \begin{bmatrix} 0 & 1 \\ 0 & \beta \end{bmatrix}.
\]

Then \(H\) is an abelian, two-dimensional closed subgroup of \(\text{GL}(3, \mathbb{R})\). By [13, Proposition 4.1], \(H\) is integrably admissible if and only if \(\alpha \beta = 0\) and \(\alpha + \beta > 0\), i.e., one of the two
parameters is zero, and the other one strictly positive. The essential frequency support is given by
\[ \mathcal{O} = \{ (x_1, x_2, x_3) : x_1 \neq 0 \neq x_2^2 + x_3^2 \} \text{ for } \alpha = 0 \]
and
\[ \mathcal{O} = \{ (x_1, x_2, x_3) : x_1^2 + x_3^2 \neq 0 \neq x_2 \} \text{ for } \beta = 0. \]

Picking parameters \( \alpha, \beta \) with \( \alpha \beta < 0 \) provides an intriguing example of semidirect product groups \( \mathbb{R}^3 \rtimes H \) with the property that the support of the quasi-regular representation \( \pi \) is an open compact subset of the dual. However, it is currently unknown whether there exists an admissible vector \( \psi \) such that \( \Delta_G^{-1/2} \mathcal{W}\psi \in L^1(G) \). Since in this case \( H \) is not integrably admissible, the methods for constructing these vectors that this paper relies on are not available.

The significance of an integrably admissible dilation group is that it guarantees the existence of suitable admissible vectors.

**Theorem 2.7.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} = H^dK \). Then there exists a vector \( \psi \in \mathcal{F}^{-1}(C^\infty_c(\mathcal{O})) \), with \( \hat{\psi}^{-1}(\mathbb{C} \setminus \{0\}) \supset K \). Moreover, any such vector \( \psi \) can be chosen to be admissible.

**Proof.** The existence of a \( \psi \in \mathcal{F}^{-1}(C^\infty_c(\mathcal{O})) \) is contained in the proof of [13, Proposition 10]. Any such vector can be made re-normalized to obtain a vector satisfying (2.1).

### 2.2 Extended matrix coefficients

This section is concerned with extending the action of the quasi-regular representation to suitable distribution spaces. In particular, a point-wise estimate for the decay of the extended matrix coefficients will be obtained and the reproducing formula (2.2) will be extended.

**Definition 2.8.** Let \( \mathcal{O} \subseteq \mathbb{R}^d \) be open. The space of analyzing vectors \( \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) is defined as the inverse Fourier image \( \mathcal{S}_\mathcal{O}(\mathbb{R}^d) := \mathcal{F}^{-1}(\mathcal{D}(\mathcal{O})) \), where \( \mathcal{D}(\mathcal{O}) := C^\infty_c(\mathcal{O}) \) is equipped with the usual inductive limit topology. The space \( \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) will be equipped with the unique topology making \( \mathcal{F}^{-1} : \mathcal{D}(\mathcal{O}) \to \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) into a homeomorphism. The anti-dual space of \( \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \), i.e., the space of all continuous conjugate-linear functionals, will be written as \( \overline{\mathcal{S}_\mathcal{O}}(\mathbb{R}^d) \). The space \( \overline{\mathcal{S}_\mathcal{O}}(\mathbb{R}^d) \) is assumed to be equipped with the weak*-topology and the sesquilinear pairing \( \langle \cdot, \cdot \rangle : \overline{\mathcal{S}_\mathcal{O}}(\mathbb{R}^d) \times \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \to \mathbb{C} \).

We mention the following simple properties of \( \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) without proof.

**Lemma 2.9.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} \).

(i) The space \( \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) is \( \pi \)-invariant.

(ii) The space \( \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) is weak*-dense in \( \overline{\mathcal{S}_\mathcal{O}}(\mathbb{R}^d) \)

For \( f \in \overline{\mathcal{S}_\mathcal{O}}(\mathbb{R}^d) \) and \( \psi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \), the (extended) wavelet transform \( W_\psi f \) of \( f \) is defined as the mapping
\[ W_\psi f : G \to \mathbb{C}, \quad (x, h) \mapsto \langle f, \pi(x, h)\psi \rangle. \]

Point-wise estimates of the coefficient decay are provided by the following result. Here, given a compact set \( K \subseteq \mathbb{R}^d \) and an \( N > 0 \), the semi-norm of \( f \in C^\infty(K) \) is as usual denoted by \( \|f\|_{K, N} := \max_{|\alpha| \leq N} \|\partial^\alpha f\| \cdot \mathbf{1}_K \|_{\infty} \).
Lemma 2.10. Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $\mathcal{O} \subset \mathbb{R}^d$. Let $\psi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d)$ with $K_1 := \text{supp} \, \psi$.

(i) Suppose $\varphi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d)$ with $K_2 := \text{supp} \, \hat{\varphi}$. Then, for all $N \in \mathbb{N}$, there exists a $C = C(N) > 0$ such that

$$|W_\psi \varphi(x, h)| \leq C(1 + |x|)^{-N}(1 + \|h\|)^{N+1/2}\|\hat{\varphi}\|_{K_1,N}\|\hat{\varphi}\|_{K_2,N}\mathbf{1}_{((K_1,K_2))}(h).$$

for all $(x, h) \in G$.

(ii) Suppose $f \in \mathcal{S}^c_\mathcal{O}(\mathbb{R}^d)$. Then there exist measurable functions $\alpha, m : H \to \mathbb{R}^+$ that are bounded on compact sets, such that

$$|W_\psi f(x, h)| \leq \alpha(h)(1 + |x|)^m(h)(1 + \|h\|^m(h))\|\hat{\varphi}\|_{K_1,m(h)}$$

for all $(x, h) \in G$.

Proof. The proof of (i) is similar to that of [26, Theorem 3.7]. We provide a sketch of the proof to make the paper self-contained.

We first write

$$W_\psi \varphi(x, h) = \left(\left|\det(h)\right|^{1/2}\hat{\varphi} \cdot (D_h\psi)\right)^\vee(x).$$

where $D_h\psi(\xi) = \hat{\psi}(h^T\xi)$. In particular, whenever $K_1 \cap h^{-T}K_2$ is empty, the right-hand side is zero, and thus

$$\forall h \notin ((K_1, K_2)) : W_\psi \varphi(\cdot, h) \equiv 0.$$  \hspace{1cm} (2.3)

In order to get a pointwise decay estimate with respect to $x \in \mathbb{R}^d$, we next estimate the partial derivatives of the right-hand side as follows: Applying first the Leibniz formula and then the chain rule yields

$$|\partial^\alpha(\hat{\varphi} \cdot D_h\hat{\psi})(\xi)| \leq \sum_{\gamma + \beta = \alpha} \frac{\alpha!}{\beta!\gamma!} \left|\left(\partial^\beta \hat{\varphi}\right)(\xi) \cdot \left(\partial^\gamma(D_h\hat{\psi})\right)(\xi)\right| \leq C(d, \alpha)(1 + \|h\|_{\infty})^{\alpha} \sum_{|\beta|,|\gamma| \leq |\alpha|} \left|\left(\partial^\beta \hat{\varphi}\right)(\xi) \cdot \left(\partial^\gamma\hat{\psi}\right)(h^T\xi)\right| \leq C(d, \alpha)(1 + \|h\|_{\infty})^{\alpha}\|\hat{\varphi}\|_{K_1,\alpha}\|\hat{\psi}\|_{K_2,\alpha}\mathbf{1}_{K_2}(\xi).$$

By integrating this estimate over $\xi \in \mathbb{R}^d$, we obtain

$$|W_\psi \varphi(x, h)| \leq C(d)(1 + |x|)^{-N}\max_{|\alpha| \leq N} \left\|\partial^\alpha \left(\left|\det(h)\right|^{1/2}\hat{\varphi} \cdot (D_h\psi)\right)\right\|_{L^1} \leq C(d, N, K_2)(1 + |x|)^{-N}(1 + \|h\|_{\infty})^{N+1/2}\|\hat{\varphi}\|_{K_1,N}\|\hat{\varphi}\|_{K_2,N}\mathbf{1}_{((K_1,K_2))}(h),$$

where we used Hadamard’s inequality to estimate the determinant against the norm.

To prove (ii), an application of [14, Theorem 6.8] yields that, for every compact $K \subset \mathcal{O}$, there exists an $N(K) \in \mathbb{N}$ and a $C(K) > 0$ such that

$$|\langle f, \varphi \rangle| \leq C(K)\|\hat{\varphi}\|_{K,N(K)}$$ \hspace{1cm} (2.4)

for every $\varphi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d)$ with $\text{supp} \, \hat{\varphi} \subset K$. We want to apply this to $\varphi = \pi(x, h)\psi$. For this purpose, we first estimate, via another application of the Leibniz formula, for all $\xi \in K$,

$$|\langle \partial^\alpha(\pi(x, \text{Id})\psi) \rangle(\xi)| = |\langle \partial^\alpha(\exp(2\pi i(x, \cdot)\psi)) \rangle(\xi)| \leq C(d, \alpha, K)(1 + |x|)^{|\alpha|}\|\hat{\psi}\|_{K,|\alpha|}.$$
By the chain rule argument already employed for part (i), we further get

\[ |(\partial^\alpha (\pi(0,h)\psi))(\xi)| \leq C(d, |\alpha|)(1 + \|h\|)|\alpha|+1/2\|\hat{\psi}\|_{K,|\alpha|} \]

Plugging \( \varphi = \pi(x,\text{Id}) \circ \pi(0,h)\psi \) into (2.4) then yields

\[ |W_\psi f(x,h)| \leq C(d,K)(1 + |x|)^{N(K)}(1 + \|h\|)^{N(K)+1/2}\|\hat{\psi}\|_{K,1,N(K)} \]

provided that \( K \supset \text{supp}(\pi(x,h)\psi) = h^{-T}K_1 \).

Lastly, to define the functions \( \alpha : H \to \mathbb{R}^+ \) and \( m : H \to \mathbb{N}_0 \), take an increasing covering \( (A_\ell)_{\ell \in \mathbb{N}} \) of \( \mathcal{O} \) consisting of open, relatively compact sets \( A_\ell \subset \mathbb{R}^d \). Let \( \ell(h) := \inf\{n \in \mathbb{N} \mid h^{-T}K_1 \subset A_n\} \) for \( h \in H \). Then every compact set \( K \subset \mathcal{O} \) is contained in some \( A_\ell \), hence \( \ell \) is a well-defined measurable function that is bounded on compact sets. The same then holds for \( \alpha, m \), defined by \( \alpha(h) = C(A_\ell(h)) \) and \( m(h) = N(A_\ell(h)) + 1/2 \). These functions yield the desired estimate. \( \square \)

As a first application of Lemma 2.10, the \( L^2 \)-reproducing formula (2.2) will be extended to the whole distribution space \( \mathcal{S}'(\mathbb{R}^d) \).

**Lemma 2.11.** Let \( H \leq \text{GL}(d,\mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} \). Then the following assertions hold:

(i) For \( \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d) \) with \( \psi_1 \in \mathcal{S}(\mathbb{R}^d) \) admissible, the map

\[ \mathcal{S}(\mathbb{R}^d) \ni f \mapsto \int_G \langle f, \pi(x,h)\psi_1 \rangle \langle \pi(x,h)\psi_1, \psi_2 \rangle \, dG(x,h) \in \mathbb{C} \]

is weakly continuous.

(ii) For any two admissible \( \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d) \) and any \( f \in \mathcal{S}(\mathbb{R}^d) \), the reproducing formula

\[ W_{\psi_2}f = W_{\psi_1}f * W_{\psi_2}\psi_1 \]

holds.

Proof. To show (i), it suffices to prove that

\[ \int_G \langle f, \pi(x,h)\psi_1 \rangle \langle \pi(x,h)\psi_1, \psi_2 \rangle \, dG(x,h) = \langle f, \psi_2 \rangle. \]

For this, consider the Fourier-side reproducing formula

\[ \hat{\psi}_2 = \int_G W_{\psi_1}\psi_2(x,h)[\pi(x,h)\psi_1] \, dG(x,h), \]

which converges absolutely by Lemma 2.10(i). Since the action \( H \times \mathbb{R}^d \ni (h,\xi) \mapsto h^T\xi \) is proper and \( \hat{\psi}_1 \in \mathcal{D}(\mathcal{O}) \), there exists a compact set \( K \subset \mathcal{O} \) such that

\[ \text{supp} \left( W_{\psi_1}\psi_2(x,h)[\pi(x,h)\psi_1] \right) \subset K \]

for all \( (x,h) \in G \). Given \( N \in \mathbb{N} \), the decay estimate in Lemma 2.10(i) guarantees that the map

\[ (x,h) \mapsto W_{\psi_1}\psi_2(x,h)[\pi(x,h)\psi_1] \]

from \( G \) into the Banach space \( (C^N(K), \| \cdot \|_{K,N}) \) is Bochner integrable, yielding that

\[ \int_G W_{\psi_1}\psi_2(x,h)[\pi(x,h)\psi_1] \, d\mu_G(x,h) \in C^N(K). \]
is well-defined. An application of [44, Theorem 6.6], together with the Hahn-Banach theorem, entails that \( \hat{f} : D(O) \cap C^N(K) \to \mathbb{C} \) induces a continuous extension to all of \( C^N(K) \) for sufficiently large \( N \in \mathbb{N} \). Applying [48, V.5, Corollary 2] justifies to interchange Bochner integration and the evaluation of continuous functionals, yielding that

\[
\langle f, \psi_2 \rangle = \langle \hat{f}, \hat{\psi}_2 \rangle = \int_G \langle \hat{f}, \mathcal{F}(W_{\psi_1} \psi_2(x, h) \cdot \pi(x, h) \psi_1) \rangle \, dG(x, h) \\
= \int_G \langle f, \pi(x, h) \psi_1 \rangle \langle \pi(x, h) \psi_1, \psi_2 \rangle \, dG(x, h)
\]

which shows (i).

For (ii), let \( f \in \mathcal{S}_G(\mathbb{R}^d) \) be arbitrary. By density, there exists a net \( (\phi_\alpha)_{\alpha \in \Lambda} \) of functions \( \phi_\alpha \in \mathcal{S}_G(\mathbb{R}^d) \) converging weakly to \( f \). Equation (2.2) yields that \( W_{\psi_1} \phi_\alpha * W_{\psi_2} \psi_1 = W_{\psi_2} \phi_\alpha \) for all \( \alpha \in \Lambda \). Since the mapping

\[
f \mapsto \int_G \langle \pi(y^{-1}) f, \pi(x) \psi_1 \rangle \langle \pi(x) \psi_1, \psi_2 \rangle \, d\mu_G(g) = (W_{\psi_1} f * W_{\psi_2} \psi_1)(y)
\]

is weakly continuous by part (i), it follows that

\[
\lim_{\alpha \in \Lambda} (W_{\psi_1} \phi_\alpha * W_{\psi_2} \psi_1)(y) = (W_{\psi_1} f * W_{\psi_2} \psi_1)(y)
\]

for every \( y \in G \). On the other hand, clearly \( \lim_{\alpha \in \Lambda} W_{\psi_2} \phi_\alpha = W_{\psi_2} f \). Combining both identities gives therefore \( W_{\psi_2} f = W_{\psi_1} f * W_{\psi_2} \psi_1 \), as required.

### 2.3 Norm estimates

The left Haar measure \( \mu_G \) on \( G = \mathbb{R}^d \times H \) is given by \( \mu_G(x, y) = |\det h|^{-1} : dx \, dh \), and the modular function \( \Delta_G \) on \( G \) is \( \Delta_G(x, h) = |\det h|^{-1} \Delta_H(h) \). Given a weight \( v : G \to \mathbb{R}^+ \), the associated weighted mixed Lebesgue space is defined as

\[
L^p_{v,q}(G) := \left\{ F \in L^1_{\text{loc}}(G) : \| F \|_{L^p_{v,q}} < \infty \right\}
\]

with

\[
\| F \|_{L^p_{v,q}} := \left( \int_H \left( \| v(\cdot, h) f(\cdot, h) \|_{L^p(\mathbb{R}^d)} \right)^q \, \frac{dh}{|\det h|} \right)^{1/q} < \infty
\]

for \( p \in [1, \infty] \) and \( q \in [1, \infty) \) and \( \| F \|_{L^p_{v,\infty}} := \sup_{h \in H} \left( \| v(\cdot, h) f(\cdot, h) \|_{L^p(\mathbb{R}^d)} \right) \). As usual, the conjugate \( p' \) of \( p \in (1, \infty) \) is defined as \( p' := \frac{p}{p-1} \) and \( 1' := \infty \) and \( \infty' := 1 \).

Throughout this paper, any weighting function on \( G \) will be assumed to be admissible in the following sense.

**Definition 2.12.** A continuous weighting function \( v : G \to (0, \infty) \) is called admissible if

(i) The submultiplicativity condition is satisfied, i.e., \( v((x_1, h_1)(x_2, h_2)) \leq C_v v(x_1, h_1) v(x_2, h_2) \) for all \((x_1, h_1), (x_2, h_2) \in G \) and some \( C_v > 0 \).

(ii) There exists a locally bounded weight \( v_0 : H \to (0, \infty) \) and an \( s \in [0, \infty) \) such that \( v(x, h) \leq (1 + |x|^s v_0(h)) \) for all \((x, h) \in G \).

Norm estimates for convolution operators play an essential role in coorbit theory. To assure such estimates, the notion of a so-called control weight is indispensable. See [20, Section 4] and [31, Section 2.2] for a general discussion.
Definition 2.13. Let $Y = L^p_{v,q}(G)$ for $p,q \in [1, \infty]$ and an admissible weight $v : G \to (0, \infty)$. A weight $v_0 : G \to (0, \infty)$ is called a control weight for $Y$ if it satisfies

$$v_0(x,h) = \Delta_G(x,h)^{-1} v_0((x,h)^{-1})$$

and

$$\max \left( \|L(x,h)\|_{Y \to Y}, \|L(x,h)^{-1}\|_{Y \to Y}, \|R(x,h)\|_{Y \to Y}, \|R(x,h)^{-1}\|_{Y \to Y} \Delta_G(x,h)^{-1} \right) \leq v_0(x,h),$$

where $L(x,h)$ and $R(x,h)$ denote the left and right translation by $(x,h) \in G$, respectively.

In the setting considered in this paper, for any admissible weight, there exists an associated control weight of the same type. The corresponding control weight is explicitly given by the following result, which is [26 Lemma 2.3].

Lemma 2.14. Let $p,q \in [1, \infty]$ and let $v : G \to (0, \infty)$ be an admissible weight with $v(x,h) \leq (1 + |x|)^s w(h)$. Then there exists a control weight $v_0 : G \to (0, \infty)$ for $Y = L^p_{v,q}(G)$ satisfying the estimate

$$v_0(x,h) \leq (1 + |x|)^s w_0(h),$$

where $w_0 : H \to (0, \infty)$ is defined by

$$w_0(h) = (w(h) + w(h^{-1})) \cdot \max \left( \Delta_G(0,h)^{-\frac{1}{q}}, \Delta_G(0,h)^{\frac{1}{q} - 1} \right) \cdot (|\det h|^{\frac{1}{q} - \frac{1}{p}} + |\det h|^{\frac{1}{p} - \frac{1}{q}}) \left(1 + \|h\|_{\infty} + \|h^{-1}\|_{\infty}\right)^s$$

with the convention $1/\infty := 0$.

For the sake of completeness, we mention the following properties.

Proposition 2.15. Let $v : G \to (0, \infty)$ be an admissible weight and let $p,q \in [1, \infty]$. Then

(i) The space $L^p_{v,q}(G)$ is a solid Banach function space. Moreover, left and right translation act strongly continuously on $L^p_{v,q}(G)$.

(ii) There exists an admissible weight $w : G \to (0, \infty)$ such that $L^p_{v,q}(G) * L^1_{w}(G) \hookrightarrow L^p_{v,q}(G)$, with $\|F_1 * F_2\|_{L^p_{v,q}} \leq \|F_1\|_{L^p_{v,q}} \|F_2\|_{L^p_{w}}$.

(iii) Any $F_1 \in L^p_{v,q}(G)$ and $F_2 \in L^{p',q'}_{v}(G)$ satisfy the generalized Hölder inequality

$$\left| \int_G F_1(x,h) F_2(x,h) d_G(x,h) \right| \leq \|F_1\|_{L^p_{v,q}} \|F_2\|_{L^{p',q'}_{v}},$$

where $p',q' \in [1, \infty]$ are the conjugate exponents of $p,q$.

To finish the preliminaries for developing the coorbit theory, it will be shown that the wavelet transform of an analyzing vector belongs to a certain Amalgam space and, in particular, that the quasi-regular representation $(\pi, L^2(\mathbb{R}^d))$ is $v$-integrable.

Definition 2.16. Let $Y = L^p_{v,q}(G)$ for $p,q \in [1, \infty]$ and an admissible weight $v : G \to \mathbb{R}^+$. Let $U \subseteq G$ be a relatively compact unit neighborhood. For $F \in L^\infty_{\text{loc}}(G)$, define

$$F^\sharp_U : G \to [0, \infty], \quad x \mapsto \|1_U x F\|_{L^\infty}.$$ 

The (right-sided) Wiener amalgam space is defined by

$$W^R(L^\infty, Y)(G) := \{ F \in L^\infty_{\text{loc}}(G) \mid F^\sharp_U \in Y \}$$

and equipped with the norm $\|F\|_{W^R(L^\infty, Y)} = \|F^\sharp_U\|_Y$. 

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Applying the estimates

\[ 1 + (1 + sup_{k}\|k\|_{1})^{-N}(1 + \|h\|_{\infty})N^{1/2}1_{(K,K)}(kh). \]

Proposition 2.18. Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( O \). Suppose that \( \psi \in S_{O}(\mathbb{R}^{d}) \). Then \( W_{\psi'}W_{\psi} \in W^{R}(L^{\infty}, L^{p,q}_{\psi'}(G)) \) for all \( p, q \in [1, \infty] \) and any admissible weighting function \( v : G \to \mathbb{R}^{+} \). In particular, \( W_{\psi'}W_{\psi} \in L^{1}_{\psi'}(G) \).

Proof. Let \( K := \text{supp} \, \hat{\psi} \). Define the unit neighborhood \( U := B_{1}(0) \times W \) of \( G \) for some open, relatively compact unit neighborhood \( W \subset H \). This unit neighborhood will be used in the control function. Let \( (x, h) \in G \) be fixed. Then, for arbitrary \( N \in \mathbb{N} \), an application of Lemma 2.10 (i) gives a constant \( C = C(N) > 0 \) such that

\[
(W_{\psi'}W_{\psi})_{U}^{2}(x, h) \leq \sup_{(y,k)\in U} |W_{\psi'}W_{\psi}(y + kh, kh)| \leq C\|\psi\|^2_{K,N} \sup_{(y,k)\in U} (1 + |y + kh|)^{-N}(1 + \|h\|_{\infty})^{N+1/2}1_{(K,K)}(kh).
\]

Applying the estimates \( 1 + |x| \leq (1 + sup_{k}\|k\|_{1})^{-1}(1 + |y + kx|) \) and \( (1 + \|h\|_{\infty}) \leq (1 + sup_{k}\|k\|_{1})(1 + |h|_{\infty}) \), and defining \( C' := C(1 + sup_{k}\|k\|_{1})^{-N}(1 + sup_{k}\|k\|_{1})^{-1} \), therefore gives

\[
(W_{\psi'}W_{\psi})_{U}^{2}(x, h) \leq C' \sup_{(y,k)\in G} (1 + |x|)^{-N}(1 + |h|_{\infty})^{N}1_{(K,K)}^{-1}(h). \tag{2.7}
\]

To estimate the norm \( \|W_{\psi'}W_{\psi}\|_{WR(L^{\infty}, Y)} \), recall that since \( v : G \to (0, \infty) \) is admissible, there exists an \( s \in (0, \infty) \) and a locally bounded weight \( v_{0} : H \to (0, \infty) \) yielding the estimate \( v(x, h) \leq (1 + |x|)^{s}v_{0}(h) \). Using this estimate and (2.7) directly entails

\[
\|W_{\psi'}W_{\psi}\|_{WR(L^{\infty}, Y)} \leq C' \left\| \det(h^{-1})^{1/2}v_{0}(h)1_{\psi^{-1}(K,K)}(h)(1 + |h|_{\infty})^{N}(1 + |x|)^{(s-N)} \right\|_{L^{p,q}} \leq C' \sup_{h\in\psi^{-1}(K,K)} \left( |\det(h^{-1})|^{1/2}v_{0}(h) \right) \left\| (1 + |x|)^{(s-N)} \right\|_{L^{p,q}} \|1_{\psi^{-1}(K,K)}\|_{L^{q}}
\]

for arbitrary \( N \in \mathbb{N} \). Hence, choosing \( N \in \mathbb{N} \) such that \( (1 + |x|)^{(s-N)} \in L^{p}(\mathbb{R}^{d}) \), that is, \( N > d/p + s \), yields the claim.

The \( v \)-integrability follows since \( W^{R}(L^{\infty}, L^{1}_{\psi'}(G)) \hookrightarrow L^{1}_{\psi'}(G) \).

3 Coorbit spaces associated to integrably admissible dilation groups

This section considers coorbit spaces associated with integrably admissible dilation groups. In the first subsection the spaces are formally defined and several of their basic properties are proven. Section 3.2 is devoted to several discretization results.

3.1 Definition and basic properties

Definition 3.1. Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( O \). Let \( v : G \to (0, \infty) \) be an admissible weight and let \( Y = L^{p,q}_{\psi'}(G) \) for \( p, q \in [1, \infty] \). For a fixed admissible vector \( \psi \in S_{O}(\mathbb{R}^{d}) \), the associated coorbit space is defined by

\[
\text{Co}_{\psi}(Y) = \left\{ f \in \mathcal{S}_{O}(\mathbb{R}^{d}) : W_{\psi}f \in Y \right\} \tag{3.1}
\]

and will be equipped with the norm \( \|f\|_{\text{Co}_{\psi}(Y)} = \|W_{\psi}f\|_{Y} \).
In settling the basic properties of coorbit spaces, we first prove that the subspace

\[ Y_\psi := \{ F \in Y : F = F \ast W_\psi \psi \} \]

is a reproducing kernel Banach space in \( Y \), provided that \( \psi \in L^2(\mathbb{R}^d) \) is an analyzing vector.

**Lemma 3.2.** Let \( H \leq GL(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( O \). Let \( v: G \to (0, \infty) \) be an admissible weight and let \( Y = L_{p,q}^{v,p}(G) \) for \( p, q \in [1, \infty] \). Then, for \( \psi \in \mathcal{S}_O(\mathbb{R}^d) \), the space \( Y_\psi \) is closed in \( Y \).

**Proof.** Let \( (F_n)_{n \in \mathbb{N}} \) be a sequence in \( Y_\psi \) converging to some \( F \in Y \). Then there exists a subsequence \( (F_{n_k})_{k \in \mathbb{N}} \) converging to \( F \in Y \) \( \mu_G \)-almost everywhere. Since \( W_\psi \psi \in L_{1/v}^{p,q}(G) \) by Lemma 2.10(i), it follows from Proposition 2.15(iii) that \( (F \ast W_\psi \psi)(g) \) is well-defined for every \( g \in G \). Hence, for \( \mu_G \)-a.e. \( g \in G \),

\[
|F(g) - (F \ast W_\psi \psi)(g)| \\
\leq |F(g) - F_{n_k}(g)| + |F_{n_k}(g) - (F_{n_k} \ast W_\psi \psi)(g)| + |(F_{n_k} \ast W_\psi \psi)(g) - (F \ast W_\psi \psi)(g)| \\
\leq |F(g) - F_{n_k}(g)| + \|W_\psi \psi\|_{L_{p,q}^v} \|F_{n_k} - F\|_{L_{p,q}^v},
\]

since \( F_{n_k} - (F_{n_k} \ast W_\psi \psi) = 0 \) \( \mu_G \)-a.e. on \( G \). From this it follows easily that \( F = F \ast W_\psi \psi \) \( \mu_G \)-a.e. on \( G \), and thus \( F \in Y_\psi \), as required. \( \square \)

**Proposition 3.3.** Let \( \text{Co}_\psi(Y) \) be a coorbit space defined by an admissible vector \( \psi \in \mathcal{S}_O(\mathbb{R}^d) \). Then the following assertions hold:

(i) The space \( \text{Co}_\psi(Y) \) is a \( \pi \)-invariant Banach space.

(ii) The space \( \text{Co}_\psi(Y) \) is independent of the chosen admissible vector, i.e., for any two admissible vectors \( \psi_1, \psi_2 \in \mathcal{S}_O(\mathbb{R}^d) \), it holds \( \text{Co}_{\psi_1}(Y) = \text{Co}_{\psi_2}(Y) \), with equivalent norms.

(iii) The map \( W_\psi : \text{Co}_\psi(Y) \to Y_\psi \) is an isometric isomorphism.

**Proof.** (i). The subadditivity and absolute homogeneity of \( \| \cdot \|_{\text{Co}(Y)} \) are clear. To show that \( \| \cdot \|_{\text{Co}(Y)} \) is positive definite, let \( f \in \text{Co}(Y) \) and suppose that \( \|f\|_{\text{Co}(Y)} = 0 \). Then \( \langle f, \pi(x,h)\psi \rangle = 0 \) for \( \mu_G \)-a.e. \( (x,h) \in G \). Using this, together with the identity (2.6), yields that

\[
\langle f, \phi \rangle = \int_G \langle f, \pi(x,h)\psi \rangle \langle \pi(x,h)\psi, \phi \rangle \, d\mu_G(x,h) = 0
\]

for arbitrary \( \phi \in \mathcal{S}_O(\mathbb{R}^d) \). Thus \( f = 0 \), showing that \( \| \cdot \|_{\text{Co}(Y)} \) is indeed a norm.

For the completeness of \( \text{Co}_\psi(Y) \), let \( (f_n)_{n \in \mathbb{N}} \) be a Cauchy sequence in \( \text{Co}_\psi(Y) \). Then \( (W_\psi f_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( Y_\psi \), and hence converges to some \( F \in Y_\psi \) by Lemma 3.2. The map \( W_\psi : \mathcal{S}_O(\mathbb{R}^d) \to L_{1/v}^{p',q'}(G) \) is bounded by Lemma 2.10(i). Combining this with Proposition 2.15(iii) yields that

\[
f : \mathcal{S}_O(\mathbb{R}^d) \to \mathbb{C}, \quad \phi \mapsto \int_G F(g) \langle \pi(g)\psi, \phi \rangle \, d\mu_G(g)
\]

is a well-defined element of \( \mathcal{S}'_O(\mathbb{R}^d) \). Evaluating \( f \in \mathcal{S}'_O(\mathbb{R}^d) \) on \( \phi = \pi(y)\psi, y \in G \), gives

\[
W_\psi f(y) = \int_G F(g) \langle \pi(g)\psi, \pi(y)\psi \rangle \, d\mu_G(g) = F \ast W_\psi \psi(y) = F(y).
\]
From this it easily follows that Co(Y) is complete.

(ii). Let \( \psi_1, \psi_2 \in S_0(\mathbb{R}^d) \) be admissible. Then \( W_{\psi_2} \psi_1 \in L^1_w(G) \) for any control weight \( w \) by Lemma 2.10(i). Thus, if \( W_{\psi_1} f \in Y \), then \( W_{\psi_2} f = W_{\psi_1} f * W_{\psi_2} \psi_1 \in Y \), with
\[
\| W_{\psi_2} f \|_Y = \| W_{\psi_1} f * W_{\psi_2} \psi_1 \|_Y \leq \| W_{\psi_2} \psi_2 \|_{L^1_w} \| W_{\psi_1} f \|_Y.
\]
by Lemma 2.15(ii). Reversing the role of \( \psi_1 \) and \( \psi_2 \) yields \( \| W_{\psi_1} f \|_Y \leq \| W_{\psi_2} f \|_Y \), as required.

(iii). We prove the surjectivity of \( W_{\psi} : Co(Y) \to Y_{\psi} \). The inclusion \( W_{\psi}(Co(Y)) \subseteq Y_{\psi} \) follows by Lemma 2.11(ii). For the reverse, let \( f \in Y_{\psi} \). Then similar arguments as in (i) show that the map \( \phi \to \int_G F(g)(\pi(g)\psi, \phi) \, d\mu_G(g) \) defines an element \( f \in S_0(\mathbb{R}^d) \) satisfying \( W_{\psi} f = F * W_{\psi} \psi = F \in Y \), and thus \( f \in Co(\psi)(Y) \).

In light of the above independence result, we will omit the dependence of \( Co(\psi)(Y) \) on \( \psi \in S_0(\mathbb{R}^d) \) and simply write \( Co(Y) = Co_\psi(Y) \) in the remainder.

### 3.2 Discretization results

In this section the results on the discretization of convolution operators as developed in [20,31] will be applied to the current setting. For this, we adopt the following terminology.

**Definition 3.4.** Let \( G \) be a locally compact group, let \( X = (x_i)_{i \in I} \subseteq G \) be a countable, discrete set and let \( U \subseteq G \) be a unit neighborhood.

(i) The set \( X = (x_i)_{i \in I} \) is called \( U \)-dense in \( G \) if \( \bigcup_{i \in I} x_i U = G \).

(ii) The set \( X = (x_i)_{i \in I} \) is called \( U \)-separated in \( G \) if \( (x_i U)_{i \in I} \) is pairwise disjoint. The set \( X \) is called well-spread if it is \( U \)-separated for some unit neighborhood \( U \subseteq G \) and is called relatively separated if it is \( U \)-dense and separated.

(iii) The set \( X = (x_i)_{i \in I} \) is relatively separated if it is a finite union of separated sets.

**Remark 3.5.** In \( G = \mathbb{R}^d \rtimes H \), the typical discrete sets \( X \subseteq G \) possess the form
\[
X = \{(h_j x_k, h_j) \mid j \in J, k \in K \}
\]
for uniformly dense and separated sets \( (h_j)_{j \in J} \subseteq H \) and \( (x_k)_{k \in K} \subseteq \mathbb{R}^d \). Any such set \( X \) can be shown to be uniformly dense and separated in \( G \).

**Definition 3.6.** Let \( Y \) be a solid Banach function space on \( G \) and let \( X = (x_i)_{i \in I} \subseteq G \) be a relatively separated set. For a relatively compact neighborhood of the identity \( U \subseteq G \), define
\[
\|(c_i)_{i \in I}\|_{Y_d(U)} := \left\| \sum_{i \in I} |c_i| \mathbb{1}_{x_i U} \right\|_Y
\]
and \( Y_d(X) = \{(c_i)_{i \in I} \in \mathbb{C}^I : \|(c_i)_{i \in I}\|_{Y_d(U)} < \infty\} \).

For further reference, we mention some properties of \( Y_d(X) \) to be used in the sequel.

**Remark 3.7.** (i) The sequence space \( Y_d(X) \) forms a well-defined Banach space, independent of the choice of the defining neighborhood \( U \).

(ii) If \( Y = L^p_w(G) \) for some weight \( v : G \to \mathbb{R}^+ \), then the norm \( \| \cdot \|_{Y_d(X)} \) is equivalent to a weighted \( \ell^p \)-norm. More precisely, the equivalence reads
\[
\|(c_{j, i})_{j \in J, i \in K}\|_{Y_d} \asymp \left( \sum_{j \in J} \left| \det(h_j) \right|^{\frac{p}{p'} \alpha} \left( \sum_{i \in K} \left| c_{j, i} \right| v(h_j x_i, h_j) \right) \left( \det(h_j) \right)^{\frac{1}{p} - \frac{1}{p'}} \right)^{\frac{1}{p'}} \right)
\]

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for \( p, q \in [1, \infty) \) and the usual modifications if \( \max\{p, q\} = \infty \).

In particular, for the Lebesgue space \( Y = L^p(G) \), the associated sequence space is \( Y_d(X) = \ell^p(X) \) for all \( p \in [1, \infty] \).

The following atomic decomposition result is obtained by applying the general discretization procedure of convolution operators \cite[Section 5]{20} to the integrable kernel \( K = W_\psi \). The proof method is similar in spirit to the one used in the setting for discrete series representations, but since the result will be used in the sequel, we provide its proof.

**Theorem 3.8.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} \). Suppose that \( \psi \in \mathcal{S}_O(\mathbb{R}^d) \) is admissible. Then there exists a unit neighborhood \( U \subset G \) such that, for any \( U \)-dense and relatively separated set \( X = (x_i)_{i \in I} \), the following assertions hold:

(i) Every \( f \in \text{Co}(Y) \) possesses an expansion

\[
f = \sum_{i \in I} c_i(f) \pi(x_i) \psi \tag{3.2}
\]

for some coefficients \((c_i(f))_{i \in I} \in Y_d(X)\) satisfying \( \| (c_i(f))_{i \in I} \|_{Y_d} \leq C_1 \| f \|_{\text{Co}(Y)} \), with a constant \( C_1 > 0 \) only depending on \( \psi \in \mathcal{S}_O(\mathbb{R}^d) \).

(ii) Every \((c_i)_{i \in I} \in Y_d(X)\) defines an element

\[
f = \sum_{i \in I} c_i \pi(x_i) \psi \tag{3.3}
\]

in \( \text{Co}(Y) \) with norm estimate \( \| f \|_{\text{Co}(Y)} \leq C_2 \| (c_i)_{i \in I} \|_{Y_d} \) for some \( C_2 > 0 \).

The series (3.2) and (3.3) converge unconditionally in the norm of \( \text{Co}(Y) \) if \( p, q \in [1, \infty) \) and in the weak*-topology, otherwise.

**Proof.** (i). Since \( W_\psi \psi \in W^R(L^\infty, L^1_\psi) \), an application of \cite[Theorem 4.10]{21} yields the invertibility of the operator

\[
T_\Phi : Y_\psi \rightarrow Y_\psi, \quad F \mapsto \sum_{i \in I} \langle \phi_i, F \rangle L(x_i) W_\psi \psi,
\]

for some partition of unity \( \Phi = (\phi_i)_{i \in I} \) subordinate to \( U \subset G \). Applying this to \( F = W_\psi f \in Y_\psi \) for \( f \in \text{Co}(Y) \) yields that

\[
F = T_\Phi T_\Phi^{-1} F = \sum_{i \in I} \langle \phi_i, T_\Phi^{-1} F \rangle L(x_i) W_\psi \psi.
\]

Pulling this back to \( \text{Co}(Y) \) by \( W_\psi^{-1} : Y_\psi \rightarrow \text{Co}(Y) \) gives \( f = \sum_{i \in I} \langle \phi_i, T_\Phi^{-1} W_\psi f \rangle \pi(x_i) \psi \). The fact that \((\langle \phi_i, T_\Phi^{-1} W_\psi f \rangle)_{i \in I} \in Y_d(X)\) follows directly from \cite[Proposition 5.1]{20}.

(ii). Since \( W_\psi \psi \in W^R(L^\infty, L^1_\psi) \), it follows by \cite[Lemma 5.2]{20} that the mapping

\[
(c_i)_{i \in I} \mapsto \sum_{i \in I} c_i L(x_i) W_\psi \psi \tag{3.4}
\]

is well-defined and bounded from \( Y_d(X) \) into \( Y \), with convergence in \( Y \) if \( p, q \in [1, \infty) \) and point-wise, otherwise. Let \((c_i)_{i \in I} \in Y_d(X) \mapsto \ell^\infty(I)\) be arbitrary. Define the function \( F \in \ell^\infty(G) \) by

\[
F(x) = W_\psi \left( \sum_{i \in I} c_i \pi(x_i) \psi \right)(x) = \sum_{i \in I} c_i L(x_i) W_\psi \psi(x).
\]
for \( x \in G \). Choose an enumeration \((i_n)_{n \in \mathbb{N}}\) of the index set \( I \) and define \( f_n \in \mathcal{S}_O(\mathbb{R}^d) \) by

\[
f_n := \sum_{\ell=1}^{n} c_{\ell} \pi(x_{i_{\ell}}) \psi.
\]

Then \( W_{\psi} f_n(x) = \sum_{\ell=1}^{n} c_{\ell} L_{x_{i_{\ell}}} W_{\psi}(x) \), and thus \( |W_{\psi} f_n(x)| \leq |F(x)| \) for all \( x \in G \). This yields that \( f_n \in \text{Co}(Y) \). By the Lebesgue dominated convergence theorem, it follows that, for all \( \phi \in \mathcal{S}_O(\mathbb{R}^d) \),

\[
\lim_{n \to \infty} \langle f_n, \phi \rangle = \lim_{n \to \infty} \int_G \hat{f}_n(x, h) \hat{\phi}(x, h) \, d\mu_G(x, h) = \int_G \hat{f}(x, h) \hat{\phi}(x, h) \, d\mu_G(x, h)
\]

where \( f := \sum_{i \in I} c_i \pi(x_i) \psi \in \mathcal{S}^d(\mathbb{R}^d) \). Thus \( f_n \to f \) in the \( w^s \)-topology of \( \mathcal{S}_O(\mathbb{R}^d) \) as \( n \to \infty \). Moreover, if \( p, q \in [1, \infty) \), then the norm convergence of the series in (3.4) yields that \( f_n \to f \) in \( \text{Co}(Y) \) as \( n \to \infty \).

\[\Box\]

**Corollary 3.9.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( O \). The space \( \mathcal{S}_O(\mathbb{R}^d) \) is norm dense in \( \text{Co}(Y) \) if \( p, q \in [1, \infty) \) and \( \text{w}^s \)-dense, otherwise. Moreover, for any \( f \in \text{Co}(Y) \), there exists a sequence \((f_n)_{n \in \mathbb{N}} \in \mathcal{S}_O(\mathbb{R}^d) \) with \( f_n \in \mathcal{S}_O(\mathbb{R}^d) \) such that \( f_n \to f \) in \( \mathcal{S}_O(\mathbb{R}^d) \) and a constant \( C > 0 \) such that \( \|f_n\|_{\text{Co}(Y)} \leq C\|f\|_{\text{Co}(Y)} \) for all \( n \in \mathbb{N} \).

**Proof.** Let \( f \in \text{Co}(Y) \) and \( \psi \in \mathcal{S}_O(\mathbb{R}^d) \). Then \( f = \sum_{i \in I} c_i(\pi(x_i)) \psi \) for some coefficients \((c_i(\pi))_{i \in I} \in Y_d(X) \) by Theorem 3.8. Let \( f_n \in \mathcal{S}_O(\mathbb{R}^d) \) be as in (3.5). Then \( f_n \to f \) as \( n \to \infty \) in the norm topology of \( \text{Co}(Y) \) if \( p, q \in [1, \infty) \) and in the \( \text{w}^s \)-topology, otherwise.

For the moreover part, note that by Theorem 3.8, there exist constants \( C_1, C_2 > 0 \) such that

\[
\|f_n\|_{\text{Co}(Y)} \leq C_2 \|c_i(\pi)\|_{\ell^1} \|Y_d\| \leq C_1 C_2 \|f\|_{\text{Co}(Y)},
\]

which yields the claim. \( \Box \)

Dual to the existence of atomic decompositions as asserted by Theorem 3.8, the existence of so-called Banach frames in \( \text{Co}(Y) \) can be guaranteed. As above, the proof method is very similar to the discrete series case [31] and now omitted.

**Theorem 3.10.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( O \). Suppose that \( \psi \in \mathcal{S}_O(\mathbb{R}^d) \). Then there exists a unit neighborhood \( U \subset G \) such that, for any \( U \)-dense and relatively separated set \( X = (x_i)_{i \in I} \), the following assertions hold:

(i) There exist two constants \( A, B > 0 \) such that

\[
A \|f\|_{\text{Co}(Y)} \leq \|(f, \pi(x_i) \psi)\|_{\ell^1} \|Y_d\| \leq B \|f\|_{\text{Co}(Y)}
\]

for all \( f \in \text{Co}(Y) \).

(ii) There exists a bounded surjection \( R : Y_d(X) \to \text{Co}(Y) \) such that \( f = R((f, \pi(x_i)) \psi)_{i \in I} \).

Lastly, we show the existence of \( p \)-Riesz sequences [11, 11] in coorbit spaces. To obtain this, we exploit the following interpolation property of the wavelet transform, which is [21, Theorem 8.2] adapted to our current setting.
Proposition 3.11. Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $\mathcal{O}$. Let $\psi \in \mathcal{S}_O(\mathbb{R}^d)$ be normalized. There exists a compact set $K \subseteq G$ and a constant $C > 0$ such that, for every $K$-separated set $X = (x_i)_{i \in I}$ in $G$ and any sequence $c = (c_i)_{i \in I} \in Y_d(X)$, there exists $f \in \text{Co}(Y)$ such that $\|f\|_{\text{Co}(Y)} \leq C\|c\|_{Y_d(X)}$ and $W_\psi f(x_i) = c_i$ for all $i \in I$.

Proof. By [21, Theorem 7.2], there exists a compact set $K \subseteq G$ such that, for every $K$-separated set $X = (x_i)_{i \in I}$, the mapping $F \mapsto (F * W_\psi \psi(x_i))_{i \in I}$ is a bounded surjection from $Y$ onto $Y_d(X)$, i.e., given $c = (c_i)_{i \in I} \in Y_d(X)$, there exists $F \in Y$ such that $(F * W_\psi \psi)(x_i) = c_i$ for all $i \in I$, and $\|F\|_Y \lesssim \|c\|_{Y_d(X)}$. Since $W_\psi : \text{Co}(Y) \to Y_\psi$ is an isometric isomorphism, the claim follows readily. \[ \Box \]

Theorem 3.12. Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $\mathcal{O}$. Suppose that $\psi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d)$. Then there exists a compact set $K \subseteq G$ such that, for every $K$-separated set $X = (x_i)_{i \in I}$ in $G$, there exist constants $A, B > 0$ such that

$$A\|c\|_{Y_d(X)} \leq \left\| \sum_{i \in I} c_i \pi(x_i) \psi \right\|_{\text{Co}(Y)} \leq B\|c\|_{Y_d(X)}$$

for all $c = (c_i)_{i \in I} \in Y_d(X)$.

Proof. Let $K \subseteq G$ be a compact set with the properties asserted in Proposition 3.11. Let $X = (x_i)_{i \in I}$ be $K$-separated. Let $c = (c_i)_{i \in I} \in Y_d(X)$ be arbitrary. Choose a $c' = (c'_i)_{i \in I} \in Y'_d(X)$ satisfying $\|c'\|_{Y'_d(X)} = 1$ and $\langle c', c \rangle = \sum_{i \in I} c'_i c_i = \|c\|_{Y_d(X)}$. Then, by Proposition 3.11, there exists an $f \in \text{Co}(Y')$ such that $\|f\|_{\text{Co}(Y')} \lesssim 1$ and $W_\psi f(x_i) = c'_i$ for all $i \in I$. Hence

$$\|c\|_{Y_d(X)} = \sum_{i \in I} c'_i c_i = \left\langle f, \sum_{i \in I} c_i \pi(x_i) \psi \right\rangle \leq \|f\|_{\text{Co}(Y')} \left\| \sum_{i \in I} c_i \pi(x_i) \psi \right\|_{\text{Co}(Y)}$$

$$\lesssim \left\| \sum_{i \in I} c_i \pi(x_i) \psi \right\|_{\text{Co}(Y)} ,$$

which proves the desired lower bound. The upper bound follows directly by Theorem 3.5. \[ \Box \]

4 Decomposition spaces associated to induced coverings

This section is devoted to decomposition spaces and the main ingredients for defining these spaces, namely admissible coverings and associated subordinate partitions of unity. In particular, we will construct covers and partitions of unity that are suitable for identifying coorbit spaces as decomposition spaces.

4.1 Construction of induced covering and an admissible weight

The definition of a decomposition space is based on the notion of an admissible coverings [18].

Definition 4.1. Let $\mathcal{O} \subset \mathbb{R}^d$ be open. A family $Q = (Q_i)_{i \in I}$ of subsets $Q_i \subset \mathcal{O}$ is called an admissible covering of $\mathcal{O}$, if

(i) $Q$ is a covering of $\mathcal{O}$, i.e., $\mathcal{O} = \bigcup_{i \in I} Q_i$;

(ii) $Q_i \neq \emptyset$ for all $i \in I$;

(iii) $N_Q := \sup_{i \in I} |i^*| < \infty$, where $i^* := \{ \ell \in I : Q_\ell \cap Q_i \neq \emptyset \}.$
The aim of this subsection is to construct a cover of the essential frequency support \( \mathcal{O} \) that is induced by the action of the associated integrably admissible dilation group \( \hat{H} \leq \text{GL}(d, \mathbb{R}) \). The construction relies on the following technical lemma, which extends \cite{30} Lemma 18] to the setting of integrably admissible dilation groups.

**Lemma 4.2.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} \). Let \( K \subseteq \mathcal{O} \) be compact. Then, given a relatively compact unit neighborhood \( V \subseteq H \) and a \( V \)-separated family \( (h_i)_{i \in I} \subseteq H \), there exists a constant \( C = C(K, V) > 0 \) such that, for all \( h \in H \),
\[
|I_h| \leq C,
\]
where \( I_h := \{ i \in I \mid h^{-T}K \cap h_i^{-T}K \neq \emptyset \} \).

**Proof.** Let \( K \subseteq \mathcal{O} \) be compact and let \( g, h \in H \) be such that \( g^{-T}K \cap h^{-T}K \neq \emptyset \). By assumption on \( H \), there exists a compact \( C \subseteq \mathcal{O} \) such that \( \mathcal{O} = H^T C \). Take \( k := h_0^{-T}\xi_0 \in K \), with \( h_0 \in H \) and \( \xi_0 \in C \subseteq \mathcal{O} \), such that \( g^{-T}k = h^{-T}k \). Then \( (h_0 h^{-1} g_0^{-1})^T \xi_0 = \xi_0 \), that is, \( h_0 h^{-1} g_0^{-1} \in H_{\xi_0} \). Hence
\[
g \in h_0^{-1}H_{\xi_0} h_0 \subset h \cdot (p^{-1}_\xi(K))^{-1} \cdot H_{\xi_0} \cdot p_\xi^{-1}(K) =: h \cdot L,
\]
where \( p_\xi : H \rightarrow \mathcal{O} \) denotes the orbit map \( h \mapsto h^T \xi_0 \). Since any stabilizer group \( H_{\xi_0} \), with \( \xi_0 \in \mathcal{O} \), is compact and \( p_\xi : H \rightarrow \mathcal{O} \) is proper, it follows that \( L \subseteq H \) is a compact set.

Take a \( V \)-separated set \( (h_i)_{i \in I} \subseteq H \). If \( i \in I_h \), then by definition, \( h^{-T}K \cap h_i^{-T}K \neq \emptyset \), which entails \( h_i \in h \cdot L \). Hence \( h_i V^o \subseteq h_i V \subseteq h L V \), and \( (h_i V^o)_{i \in I_h} \) is a pairwise disjoint collection of subsets of \( h L V \). Using these observations, it follows that, for a finite subset \( J \subseteq I_h \),
\[
|J| = \frac{1}{\mu_H(V^o)} \sum_{i \in J} \mu_H(h_i V^o) = \frac{1}{\mu_H(V^o)} \mu_H \left( \bigcup_{i \in J} h_i V^o \right) \leq \frac{\mu_H(h L V)}{\mu_H(V^o)} = \frac{\mu_H(L V)}{\mu_H(V^o)}.
\]
Since \( J \subseteq I_h \) can be chosen arbitrary, the result follows. \( \square \)

**Lemma 4.3.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} = H^T C \). Suppose that \( (h_i)_{i \in I} \) is a well-spread family in \( H \). Then there exists a relatively compact set \( Q \subset \mathbb{R}^d \) satisfying \( \overline{Q} \subset \mathcal{O} \) and \( \mathcal{O} = \bigcup_{i \in I} h_i^{-T}Q \).

**Proof.** Let \( (h_i)_{i \in I} \) be a well-spread family and let \( U \subseteq H \) be a relatively compact set such that \( H = \bigcup_{i \in I} h_i U \). Then
\[
\mathcal{O} = H^T C = H^{-T} C = \bigcup_{i \in I} h_i^{-T} U^{-T} C,
\]
which shows that \( Q = (h_i^{-T} U^{-T} C)_{i \in I} \) is a covering of \( \mathcal{O} \). Note that \( Q := U^{-T} C \) has a compact closure \( \overline{Q} = \overline{U^{-T} C} \subseteq \mathcal{O} \) since \( C \subseteq \mathcal{O} \) is compact. \( \square \)

Motivated by the previous result, we make the following formal definition.

**Definition 4.4.** Let \( H \leq \text{GL}(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} \subset \mathbb{R}^d \). Let \( (h_i)_{i \in I} \subseteq H \) be well-spread and let \( Q \subseteq \mathbb{R}^d \) be relatively compact with \( \overline{Q} \subset \mathcal{O} \). A covering \( Q = (Q_i)_{i \in I} \) of \( \mathcal{O} \) possessing the form \( Q_i = h_i^{-T}Q \) is called a covering of \( \mathcal{O} \) induced by \( H \).
Any induced cover is admissible in the sense of Definition 4.1.

**Proposition 4.5.** Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $O$. Any cover $Q = (Q_i)_{i \in I}$ of $O$ induced by $H$ is an admissible cover.

**Proof.** Let $Q = (h_i^{-1}Q_i)_{i \in I}$ be an induced cover of $O$, with $Q \subseteq \hat{\mathbb{R}}^d$ having compact closure $\overline{Q} \subseteq O$. Let $V$ be a relatively compact unit neighborhood such that $(h_i)_{i \in I}$ is $V$-separated. If $i \in I$ and $j \in i^*$, then $h_i^{-1}Q \cap h_j^{-1}Q \neq \emptyset$, whence Lemma 1.2 yields the existence of a compact set $L = L(Q)$ such that $h_j \subset h_i L$. Furthermore, there exists a constant $C = C(Q,V) > 0$ such that $|I_{h_i}| \leq C$ for all $i \in I$, where

$$I_{h_i} := \{ j \in I \mid h_j^{-1}Q \cap h_i^{-1}Q \neq \emptyset \}$$

as in Lemma 4.2. But $i^* = I_{h_i}$, and thus $Q$ is admissible.

To close this subsection, we mention some facts on $Q$-moderate weights that will be needed in the remainder.

**Definition 4.6.** Let $Q = (Q_i)_{i \in I}$ be an admissible covering of an open set $O \subset \hat{\mathbb{R}}^d$.

(i) A function $u : O \to \mathbb{R}^+$ is called $Q$-moderate if there exists a $C_Q > 0$ such that $u(x)/u(y) \leq C$ for all $x, y \in Q_i$ and $i \in I$.

(ii) A sequence $u : I \to \mathbb{R}^+$ is called $Q$-moderate if $\sup_{i \in I} \sup_{x \in \xi} u_i(x) < \infty$.

(iii) A sequence $u' : I \to \mathbb{R}^+$ is called a discretization of $u : O \to \mathbb{R}^+$ if, for every $i \in I$, there exists an $x_i \in Q_i$ such that $u'_i = u(x_i)$.

**Remark 4.7.** A discretization of a $Q$-moderate function is $Q$-moderate as a sequence and, moreover, any two discretizations are equivalent as weighting functions.

In the sequel, we will repeatedly transplant a weight $v : H \to \mathbb{R}^+$ into a function $u : O \to \mathbb{R}^+$ by use of a cross-section. The resulting function $u$ will be called a transplant of $v$ from $H$ onto $O$. By suitable assumptions on the weight $v$, the resulting transplant $u$ is $Q$-moderate.

**Lemma 4.8.** Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $O = H^T K$. Let $v : H \to \mathbb{R}^+$ be a $v_0$-moderate for some locally bounded, submultiplicative weight $v_0 : H \to \mathbb{R}^+$.

For each $\xi \in O$, choose an $h_\xi \in H$ and a $\xi_0 \in K$ and let

$$u : O \to \mathbb{R}^+, \quad \xi \mapsto v(h_\xi).$$

Then $u : O \to \mathbb{R}^+$ is $Q$-moderate for every covering $Q$ of $O$ induced by $H$. Moreover, it is independent of the choices of $h_\xi$ and $\xi_0$.

**Proof.** The proof is essentially the same as the arguments proving [30] Lemma 23.

### 4.2 Construction of a BAPU subordinate to an induced covering

This subsection is devoted to the explicit construction of a BAPU subordinate to a covering of $O$ induced by $H$.

**Definition 4.9.** Let $Q = (Q_i)_{i \in I}$ be an admissible covering of an open subset $O \subset \hat{\mathbb{R}}^d$. A family $\Phi = (\phi_i)_{i \in I}$ is called a bounded admissible partition of unity (BAPU) subordinate to $Q$ if

...
(i) \( \varphi_i \in C_c^\infty(\mathcal{O}) \) for all \( i \in I \);
(ii) \( \sum_{i \in I} \varphi_i(\xi) = 1 \) for all \( \xi \in \mathcal{O} \);
(iii) \( \varphi_i(\xi) = 0 \) for all \( \xi \in \mathcal{O} \setminus Q_i \) for all \( i \in I \);
(iv) \( C_4 := \sup_{i \in I} \|F^{-1}\varphi_i\|_{L^1} < \infty \).

The construction of the BAPU in Theorem 4.11 below is similar in spirit as the construction in [30] Section 6 and relies on the following auxiliary lemma.

**Lemma 4.10.** Let \( H \leq GL(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} \subseteq \mathbb{R}^d \). For \( \psi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) and a measurable, relatively compact set \( \mathcal{U} \subseteq H \), the function
\[
\varphi_\mathcal{U} : \mathbb{R}^d \to [0, \infty), \quad \xi \mapsto \int_\mathcal{U} \left| \hat{\psi}(h^T \xi) \right|^2 d\mu_H(h)
\]
is well-defined and \( \varphi_\mathcal{U} \in \mathcal{D}(\mathcal{O}) \), with \( \text{supp} \varphi_\mathcal{U} \subseteq \overline{\mathcal{U}^{-T}} \text{supp} \hat{\psi} \subseteq \mathcal{O} \).

The inverse Fourier transform \( F^{-1} \varphi_\mathcal{U} \) of \( \varphi_\mathcal{U} \) is given by
\[
(F^{-1} \varphi_\mathcal{U})(x) = \int_\mathcal{U} \frac{F^{-1}(\left| \hat{\psi} \right|^2(h^{-1}x))}{|\text{det}(h)|} d\mu_H(h)
\]
for all \( x \in \mathbb{R}^d \), with \( \|F^{-1} \varphi_\mathcal{U}\|_{L^1} \leq \mu_H(H)\|F^{-1}(\left| \hat{\psi} \right|^2)\|_{L^1} \).

**Proof.** Combine [30] Lemma 27] and [30] Lemma 28].

**Theorem 4.11.** Let \( H \leq GL(d, \mathbb{R}) \) be integrably admissible with essential frequency support \( \mathcal{O} = H^T \mathcal{C} \). Let \( (h_i)_{i \in I} \) be well-spread in \( H \) and let \( \mathcal{U} \subseteq H \) be a measurable, relatively compact set such that \( H = \bigcup_{i \in I} h_i \mathcal{U} \).

(i) Given an enumeration \( (i_n)_{n \in \mathbb{N}} \) of \( I \), define \( U_{i_n} := h_{i_n} \mathcal{U} \setminus \bigcup_{m=1}^{n-1} h_{i_m} \mathcal{U} \) for \( n \in \mathbb{N} \). Then \( (U_i)_{i \in I} \) is a measurable partition of \( H \) satisfying \( U_i \subseteq h_i \mathcal{U} \) for all \( i \in I \).

(ii) For an admissible analyzing vector \( \psi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \), with \( \hat{\psi}^{-1}(\mathcal{C} \setminus \{0\}) \supset C \), define the open set \( Q := U^{-T}(\hat{\psi}^{-1}(\mathcal{C} \setminus \{0\})) \). Then \( Q \subset \mathcal{O} \) is relatively compact with closure \( \overline{Q} \subset \mathcal{O} \) and the collection \( \mathcal{Q} = (h_i^{-T} Q)_{i \in I} \) forms a covering of \( \mathcal{O} \) induced by \( H \). Moreover, the family \( (\varphi_{U_i})_{i \in I} \) of functions
\[
\varphi_{U_i} := \int_{U_i} |\hat{\psi}(h^T \xi)|^2 d\mu_H(h) \in \mathcal{D}(\mathcal{O})
\]
forms a BAPU subordinate to \( \mathcal{Q} \).

**Proof.** Assertion (i) is a standard fact from measure theory.

For (ii), let \( \psi \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \) be admissible, with \( W := \hat{\psi}^{-1}(\mathcal{C} \setminus \{0\}) \supset C \). Define the open set
\[
Q = \bigcup_{h \in \mathcal{U}} h^{-T} W \subseteq \mathcal{O}.
\]
Then \( Q \) is relatively compact since \( \overline{Q} \subseteq \overline{U^{-T}} \text{supp} \hat{\psi} \subseteq \mathcal{O} \) is compact. For the covering property \( \mathcal{O} = \bigcup_{i \in I} h_i^{-T} Q \), simply note that
\[
\bigcup_{i \in I} h_i^{-T} Q = \bigcup_{i \in I} (h_i \mathcal{U})^{-T} W \supset H^T \mathcal{C} = \mathcal{O},
\]
and \( O \supset \bigcup_{i \in I} h_i^{-T} Q \) since \( O \) is \( H^T \)-invariant.

It remains to show that \( (\varphi_{U_i})_{i \in I} \) forms a BAPU subordinate to \( Q \). For this, note that an application of Lemma 4.10 yields that \( \varphi_{U_i} \in \mathcal{D}(O) \) with \( \varphi_{U_i} \equiv 0 \) on \( O \setminus (h_i^{-T} Q) \), and

\[
\left\| \mathcal{F}^{-1} \varphi_{U_i} \right\|_{L^1} \leq \mu(H(U_i)) \left\| \mathcal{F}^{-1} (\hat{\psi}^2) \right\|_{L^1} \leq \mu_H(U) \left\| \mathcal{F}^{-1} (\hat{\psi}^2) \right\|_{L^1},
\]

yielding that \( \sup_{i \in I} \left\| \mathcal{F}^{-1} \varphi_{U_i} \right\|_{L^1} < \infty \). Lastly, using that \( \psi \in \mathcal{S}_O(\mathbb{R}^d) \) is admissible and \( (U_{i,n})_{n \in \mathbb{N}} \) forms a partition of \( I \), it follows that

\[
\sum_{i \in I} \varphi_{U_i}(\xi) = \sum_{n=1}^{\infty} \sum_{i \in I} \varphi_{U_{i,n}}(\xi) = \sum_{n=1}^{\infty} \int_{U_{i,n}} |\hat{\psi}(h^T \xi)| \, d\mu_H(h) = \int_H |\hat{\psi}(h^T \xi)| \, d\mu_H(h) = 1,
\]

which shows that \( (\varphi_{U_i})_{i \in I} \) is a BAPU subordinate to \( Q \). \( \square \)

### 4.3 Definition of decomposition spaces

Following [45][46], we consider the Fourier distributions as a reservoir for the decomposition spaces.

**Definition 4.12.** Let \( O \subset \mathbb{R}^d \) be open and set \( Z(O) := \mathcal{F}(C_c^\infty(O)) \). The space \( Z(O) \) will be equipped with the unique topology making the Fourier transform \( \mathcal{F} : C_c^\infty(O) \to Z(O) \) into a homeomorphism. The topological dual space \( (Z(O))^\prime \) of \( Z(O) \) is denoted by \( Z(O) \). The bilinear duality pairing between \( Z'(O) \) and \( Z(O) \) is denoted by \( \langle \cdot, \cdot \rangle = Z'(O) \times Z(O) \to \mathbb{C} \).

The **Fourier transform** of an \( f \in Z'(O) \) is defined by duality as

\[
\mathcal{F} : Z'(O) \to \mathcal{D}'(O), \quad f \mapsto \hat{f} := f \circ \mathcal{F}.
\]

**Definition 4.13.** Let \( p, q \in [1, \infty] \). Let \( Q = (Q_i)_{i \in I} \) be an admissible covering of an open set \( O \subset \mathbb{R}^d \) with a subordinate BAPU \( (\varphi_i)_{i \in I} \). Let \( u = (u_i)_{i \in I} \) be \( Q \)-moderate. For \( f \in Z'(O) \), set

\[
\left\| f \right\|_{D(Q, L^p, \ell^q_u)} := \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i \cdot \hat{f}) \right\|_{L^p} \right)_{i \in I} \right\|_{\ell^q_u} \in [0, \infty], \tag{4.1}
\]

and define the associated decomposition space \( \mathcal{D}(Q, L^p, \ell^q_u) \) as

\[
\mathcal{D}(Q, L^p, \ell^q_u) := \left\{ f \in Z'(O) : \left\| f \right\|_{D(Q, L^p, \ell^q_u)} < \infty \right\}.
\]

**Remark 4.14.** The decomposition spaces \( \mathcal{D}(Q, L^p, \ell^q_u) \) form Banach spaces for all \( p, q \in [1, \infty] \) and its definition is independent of the chosen BAPU, with equivalent norms for different choices. For proofs of these and other properties, the reader is referred to [46].

The following result, which is specific for the setting considered here, is [30 Corollary 26].

**Lemma 4.15.** Let \( H \leq GL(d, \mathbb{R}) \) be integrably admissible with frequency support \( O \subset \mathbb{R}^d \). Suppose that \( Q = (h_i^{-T} Q)_{i \in I} \) and \( Q' = (h_i^{-T} Q')_{i \in I} \) are coverings of \( O \) induced by \( H \). Then

\[
\mathcal{D}(Q, L^p, \ell^q_u) = \mathcal{D}(Q, L^p, \ell^q_u')
\]

with equivalent norms for every transplant \( u : O \to \mathbb{R}^+ \) obtained from a \( v_0 \)-moderate weight \( v : H \to \mathbb{R}^+ \) relative to a locally bounded, submultiplicative weight \( v_0 : H \to \mathbb{R}^+ \).
In the next section, we will prove our results on the space $S_O(\mathbb{R}^d)$ and then extend it by a suitable density argument. For this, the following notion of a dominated distribution will be useful. See [33].

**Definition 4.16.** Let $I$ be a countable index set, and let $w : I \to \mathbb{R}^+$ be a weight. For a sequence $F = (F_i)_{i \in I}$ of functions $F_i \in L^p(\mathbb{R}^d)$, define

$$\ell^q_w(I; L^p) := \{ F : I \to L^p(\mathbb{R}^d) : \| F_i \|_{L^p} \|_{\ell^q_w(I)} < \infty \}$$

and equip it with the norm $\| F \|_{\ell^q_w(I; L^p)} = \| F_i \|_{L^p} \|_{\ell^q_w(I)}$. Let $Q = (Q_i)_{i \in I}$ be an admissible covering of an open set $O \subset \mathbb{R}^d$ with $Q$-BAPU $(\varphi_i)_{i \in I}$. A distribution $f \in Z'(O)$ is called $F$-dominated by $F \in \ell^q_w(I; L^p)$ if

$$|F^{-1}(\varphi_i : \hat{f})| \leq F_i$$

for all $i \in I$.

**Lemma 4.17.** Let $Q = (Q_i)_{i \in I}$ be an admissible covering of an open set $O \subset \mathbb{R}^d$ with $Q$-BAPU $\Phi = (\varphi_i)_{i \in I}$ and let $w = (w_i)_{i \in I}$ be a $Q$-moderate weight. Then

(i) The inclusion $S_O(\mathbb{R}^d) \subset D(Q, L^p, \ell^q_w)$ holds for all $p, q \in [1, \infty]$.

(ii) If $p, q \in [1, \infty)$, then $S_O(\mathbb{R}^d)$ is norm dense in $D(Q, L^p, \ell^q_w)$.

(iii) If $p, q \in [1, \infty]$ and $f \in D(Q, L^p, \ell^q_w)$, then there is an $F \in \ell^q_w(I; L^p)$ and a constant $C_{\Phi, Q} > 0$ such that

$$\| F \|_{\ell^q_w(I; L^p)} \leq C_{\Phi, Q} \| f \|_{D(Q, L^p, \ell^q_w)}$$

Moreover, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of $F$-dominated functions $f_n \in S_O(\mathbb{R}^d)$ such that $f_n \to f$, with convergence in $Z'(O)$.

For the proof of the following Fatou-like property, cf. [30] Lemma 36.

**Lemma 4.18.** Let $D(Q, L^p, \ell^q_w)$ be a decomposition space. Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence in $D(Q, L^p, \ell^q_w)$ such that $\liminf_{n \to \infty} \| f_n \|_{D(Q, L^p, \ell^q_w)} < \infty$ and $f_n \to f \in Z'(O)$, with convergence in $Z'(O)$. Then $f \in D(Q, L^p, \ell^q_w)$ and $\| f \|_{D(Q, L^p, \ell^q_w)} \leq \liminf_{n \to \infty} \| f_n \|_{D(Q, L^p, \ell^q_w)}$.

5 Identification of coorbit and decomposition spaces

This section is devoted to identifying coorbit spaces associated to integrally admissible dilation groups with suitable decomposition spaces. To obtain this, it will first be shown that the identity map $I : S_O(\mathbb{R}^d) \to S_O(\mathbb{R}^d)$ is bi-continuous from the coorbit space into the decomposition space. The general result will then be obtained by a suitable density argument.

Throughout the section, we will only consider weighting functions $v : G \to \mathbb{R}^+$ of the form $(x, h) \mapsto v_0(h)$ for some weight $v_0 : H \to \mathbb{R}^+$. Clearly, any such weight is admissible with $s = 0$ and we will identify $v_0$ with its trivial extension and simply write $v = v_0$. Given any such weight $v$, denote $\tilde{v} : H \to \mathbb{R}^+$, $h \mapsto |\det(h^{-1})|^{1-s}v(h^{-1})$ and let $u : O \to \mathbb{R}^+$ be a transplant of $\tilde{v}$ with discretization $u : I \to \mathbb{R}^+$ given by $u_i = |\det(h_i)|^{1-s}v(h_i)$.

5.1 Continuity from the coorbit space into the decomposition space

For estimating the decomposition space norm of an $f \in S_O(\mathbb{R}^d)$, we will make use of the following localization result [30] Lemma 34, where we use the notation $D_h f := f(h^2)$. 

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Lemma 5.1. Let $U \subseteq H \leq \text{GL}(d, \mathbb{R})$ be relatively compact and measurable. For $f, \psi \in S(\mathbb{R}^d)$, 
\[
\left( F^{-1}(\varphi_U \cdot \hat{f}) \right)(x) = \int_U |\det(h)|^{-3/2}(W_\psi f(\cdot, h) * D_{h^{-1}}\psi)(x) \, d\mu_H(h)
\]
for any $x \in \mathbb{R}^d$.

Proposition 5.2. Let $p, q \in [1, \infty]$. Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $\mathcal{O} \subseteq \mathbb{R}^d$. Let $\psi \in S_\mathcal{O}(\mathbb{R}^d)$ be admissible and let $(h_i)_{i \in I}$ be well-spread in $H$ with $H = \bigcup_{i \in I} h_iU$ for some relatively compact unit neighborhood $U \subseteq H$. Let $Q = U^{-T}(\hat{\psi}^{-1}(\mathbb{C} \setminus \{0\}))$ and let $\mathcal{Q} = (h_i^{-T}Q)_{i \in I}$ be the corresponding induced covering of $\mathcal{O}$. Then
\[
\|f\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell^q)} \lesssim \|f\|_{C_0(\ell^p, \ell^q)}
\]
for all $f \in S_\mathcal{O}(\mathbb{R}^d)$.

Proof. Let $(\varphi_{U_i})_{i \in I}$ be a $\mathcal{Q}$-BAPU as guaranteed by Theorem 4.1. Then applying Lemma 5.1 and Young’s inequality give
\[
\left\| F^{-1}(\varphi_{U_i} \cdot \hat{f}) \right\|_{L^p} \leq \left( \left\| \psi \right\|_{L^1} \mu_H(U_i) \right)^{-1/2} \int_{U_i} |\det(h)|^{-1/2} ||W_\psi f(\cdot, h)|_{L^p} \, d\mu_H(h)
\]
Assume that $\mu_H(U_i) > 0$. If $q \in [1, \infty)$, then using (5.1) gives
\[
\left\| F^{-1}(\varphi_{U_i} \cdot \hat{f}) \right\|_{L^p}^q \leq \left( \left\| \psi \right\|_{L^1} \mu_H(U_i) \right)^{-q/2} \int_{U_i} |\det(h)|^{-q/2} ||W_\psi f(\cdot, h)|_{L^p} \, d\mu_H(h)
\]
Clearly, the estimate remains valid if $\mu_H(U_i) = 0$. Write $h \in U_i$ as $h = h_iu$ for some $u \in U$ and set $C_1 := \min_{h \in U} |\det(h)|$ and $C_2 := \max_{h \in U} |\det(h)|$. Then
\[
\frac{|\det(h)|^{\frac{q}{2} - \frac{q}{2} - \frac{q}{2}}}{|\det(h_i)|^{\frac{q}{2} - \frac{q}{2} - \frac{q}{2}}} \leq \max \left\{ C_1^{\frac{q}{2} - \frac{q}{2} - \frac{q}{2}}, C_2^{\frac{q}{2} - \frac{q}{2} - \frac{q}{2}} \right\} =: C_3
\]
Using this uniform bound, together with (5.2), we estimate
\[
\sum_{i \in I} \left( u_i \left\| F^{-1}(\varphi_{U_i} \cdot \hat{f}) \right\|_{L^p} \right)^q \leq \left( \left\| \psi \right\|_{L^1} \left( \mu_H(U) \right)^{1-\frac{q}{2}} \right)^{-q} \sum_{i \in I} \left( \int_{U_i} \left( \frac{|\det(h)|^{\frac{q}{2} - \frac{q}{2} - \frac{q}{2}}}{|\det(h)|^{\frac{q}{2} - \frac{q}{2} - \frac{q}{2}}} v(h_i) \left\| W_\psi f(\cdot, h) \right\|_{L^p} \right) \, d\mu_H(h) \right)^q
\]
\[
\leq C_3^q C_2^q \left( \left\| \psi \right\|_{L^1} \left( \mu_H(U) \right)^{1-\frac{q}{2}} \right)^{-q} \sum_{i \in I} \left( \int_{U_i} \left( v(h) \left\| W_\psi f(\cdot, h) \right\|_{L^p} \right) \, d\mu_H(h) \right)^q
\]
\[
= C_3^q C_2^q \left( \left\| \psi \right\|_{L^1} \left( \mu_H(U) \right)^{1-\frac{q}{2}} \right)^{-q} \left\| W_\psi f \right\|_{L^p, \ell^q}^q,
\]
where $C_v > 0$ is such that $v(h_i) \leq C_v v(h)$. This proves the case $q \in [1, \infty[$.
For the remaining case $q = \infty$, note that
\[
\sup_{i \in I} \left( u_i \left\| \mathcal{F}^{-1} (\varphi U_i \cdot \hat{f}) \right\|_{L^p} \right) \leq \left\| \psi \right\|_{L^1} \sup_{i \in I} \int_{U_i} \left( \frac{1}{\left| \det(h_i) \right|} \right)^{\frac{1}{2}} \left\| W_\psi f(\cdot, h) \right\|_{L^p} \ d\mu_H(h) \\
\leq C_1^{-\frac{1}{2}} C_\psi \left\| \psi \right\|_{L^1} \sup_{i \in I} \int_{U_i} v(h) \left\| W_\psi f(\cdot, h) \right\|_{L^p} \ d\mu_H(h) \\
\leq C_1^{-\frac{1}{2}} C_\psi \mu_H(U) \left\| \psi \right\|_{L^1} \left\| W_\psi f \right\|_{L^p}^{\ast, q},
\]
yields the result. \hfill \Box

### 5.2 Continuity from the decomposition space into the coorbit space

The next result provides the continuity from the decomposition space into the coorbit space on the dense subspace $\mathcal{S}_0(\mathbb{R}^d)$.

**Proposition 5.3.** Let $p, q \in [1, \infty]$. Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $\mathcal{O} \subset \mathbb{R}^d$. Let $Q = (h_i^{-T} Q)_{i \in I}$ be any cover of $\mathcal{O}$ induced by $H$ admitting a BAPU $\Phi = (\varphi_i)_{i \in I}$ subordinate to $Q$. Then
\[
\| f \|_{C_{0}(\mathbb{R}^d)} \lesssim \| f \|_{D(\mathcal{O}, L^p, \ell^q_2)}
\]
for all $f \in \mathcal{S}_0(\mathbb{R}^d)$.

**Proof.** Let $(\varphi_i)_{i \in I}$ be a $\mathcal{Q}$-BAPU and let $f, \psi \in \mathcal{S}_0(\mathbb{R}^d)$ with $\psi$ admissible. Set $K := \text{supp} \hat{\psi}$. Then an application of Lemma 4.2 provides a constant $C_1 > 0$ such that $\# I_h \leq C_1$ for all $h \in H$, with $I_h := \{ i \in I \mid h^{-T} K \cap h_i^{-T} Q \neq \emptyset \}$. For fixed $h \in H$, note that
\[
1 = \sum_{i \in I} \varphi_i(x) = \sum_{i \in I_h} \varphi_i(x)
\]
for all $x \in h^{-T} K$. Hence, for fixed $(x, h) \in G$,
\[
W_\psi f(x, h) = \langle \hat{f}, \pi(x, h) \psi \rangle = \sum_{i \in I_h} \langle \hat{f}, \varphi_i, \pi(x, h) \psi \rangle = \sum_{i \in I_h} \langle \mathcal{F}^{-1}(\hat{f} \cdot \varphi_i), \pi(x, h) \psi \rangle
\]
and
\[
\langle \mathcal{F}^{-1}(\hat{f} \cdot \varphi_i), \pi(x, h) \psi \rangle = \left| \det(h) \right|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \mathcal{F}^{-1}(\hat{f} \cdot \varphi_i)(y) D_{h^{-T}} \psi(y-x) \ dy \\
= \left| \det(h) \right|^{-\frac{1}{2}} \left( \mathcal{F}^{-1}(\hat{f} \cdot \psi_i) \ast (D_{h^{-T}} \psi^*) \right)(x),
\]
where $\psi^* := \overline{\psi(\cdot)}$. Therefore, an application of Young’s inequality yields
\[
\left\| W_\psi f(\cdot, h) \right\|_{L^p} \leq \left| \det(h) \right|^{-\frac{1}{2}} \sum_{i \in I_h} \left\| \mathcal{F}^{-1}(\hat{f} \cdot \varphi_i) \ast (D_{h^{-T}} \psi^*) \right\|_{L^p} \\
\leq \left| \det(h) \right|^{\frac{1}{2}} \left\| \psi^* \right\|_{L^1} \sum_{i \in I_h} \left\| \mathcal{F}^{-1}(\hat{f} \cdot \varphi_i) \right\|_{L^p}.
\]
To estimate $(5.3)$ further, we construct a suitable auxiliary covering $Q'$ of $\mathcal{O}$. For this, recall that since $(h_i)_{i \in I}$ is well-spread, there exists a relatively compact, measurable set $U \subseteq H$ such that $H = \bigcup_{i \in I} h_i U$. Setting $K_2 := \overline{U}^{-T} K \cup \overline{\mathcal{Q}}$ yields the induced covering $Q' = (Q'_i)_{i \in I}$, where
Then a direct calculation using (5.3) gives
\[ I^*_{Q'} \subseteq h^{-T}K_2. \]
Hence \( N_{Q'} := \sup_{i \in I} i^*_{Q'} < \infty \) by definition, where \( i^*_{Q'} := \{ \ell \in I : Q'_i \cap Q'_j \neq \emptyset \} \).
For \( i \in I, h \in h_1U \) and \( j \in I_h \), note that
\[ h^{-T}K \cap h_j^{-T}Q' \subset Q'_i \cap Q'_j, \]
and thus \( I_h \subset i^*_{Q'} \) for any \( i \in I \) and \( h \in h_1U \).
To show the case \( q \in [1, \infty[, \) let \( i \in I \) and \( h \in h_1U \). Using the just proven inclusion \( I_h \subset i^*_{Q'} \), we estimate (5.3) further by
\[
| \det(h)^{-1} (v(h) \| W_\psi (\cdot, h) \|_{L^p})^q |
\leq | \det(h)^{-1} (v(h) \| \psi^* \|_{L^1}) \sum_{\ell \in i^*_{Q'}} \| F^{-1} (\hat{f} \cdot \varphi_\ell) \|_{L^p})^q |
\leq \left( \| \psi^* \|_{L^1} | \det(h) |^{\frac{1}{q} - \frac{1}{q}} v(h) \right)^q \left( \| F^{-1} (\hat{f} \cdot \varphi_\ell) \|_{L^p})^q \right)_{\ell \in i^*_{Q'}} \| \ell \|_{\ell^1}
\leq \left( C_{Q'} N_{Q'} \sup_{h \in U} | \det(h) |^{\frac{1}{q} - \frac{1}{q}} C_v \| \psi^* \|_{L^1} \right)^q \sum_{\ell \in i^*_{Q'}} \left( u_\ell \| F^{-1} (\hat{f} \cdot \varphi_\ell) \|_{L^p})^q \right) ;
\]
where \( C_v > 0 \) is such that \( v(h) \leq C_v v(h_i) \). Setting \( C_2 := C_{Q'} N_{Q'} \sup_{h \in U} | \det(h) |^{\frac{1}{q} - \frac{1}{q}} C_v \| \psi^* \|_{L^1} \)
gives
\[
\| W_\psi f \|_{L^p}^q \leq \sum_{i \in I} \int_{h_iU} (v(h) \| W_\psi (\cdot, h) \|_{L^p})^q \frac{d \mu_H(h)}{| \det(h) |}
\leq \mu_H(U) C_2 \sum_{i \in I} \sum_{\ell \in i^*_{Q'}} \left( u_\ell \| F^{-1} (\hat{f} \cdot \varphi_\ell) \|_{L^p})^q \right)
\leq \mu_H(U) C_2 N_{Q'} \| f \|_{D(Q, L^p, \ell^1)^q} ;
\]
which shows the case \( q \in [1, \infty[ \).
For the remaining case \( q = \infty \), let \( h \in H \) be arbitrary and choose \( i \in I \) such that \( h \in h_iU \).
Then a direct calculation using (5.3) gives
\[
v(h) \| W_\psi (\cdot, h) \|_{L^p} \leq C_v \max_{h \in U} | \det(h) |^{\frac{1}{q} - \frac{1}{q}} \| \psi^* \|_{L^1} \sum_{\ell \in I_h} \| F^{-1} (\hat{f} \cdot \varphi_\ell) \|_{L^p}
\leq C_v C_{Q'} N_{Q'} \max_{h \in U} | \det(h) |^{\frac{1}{q} - \frac{1}{q}} \| \psi^* \|_{L^1} \sup_{\ell \in I} \left( u_\ell \| F^{-1} (\hat{f} \cdot \varphi_\ell) \|_{L^p}) \right)
= C_v C_{Q'} N_{Q'} \max_{h \in U} | \det(h) |^{\frac{1}{q} - \frac{1}{q}} \| \psi^* \|_{L^1} \| f \|_{D(Q, L^p, \ell^1)^q} ;
\]
This completes the proof.

5.3 Isomorphism

This section finishes the proof of the identification of a coorbit space with a suitable associated decomposition space. For this, we first relate the reservoirs \( S_\infty(\mathbb{R}^d) \) and \( Z'(\mathcal{O}) \) defining \( \text{Co}(Y) \) and \( D(Q, L^p, \ell^1) \), respectively.
Lemma 5.4. Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set. Then the map $\Phi : \mathcal{S}_\mathcal{O}(\mathbb{R}^d) \to Z'(\mathcal{O})$ defined by $\Phi(f)(\phi) := \langle f, \phi \rangle$ forms a linear homeomorphism.

Proof. The map $\mathcal{S}_\mathcal{O}(\mathbb{R}^d) \ni \phi \mapsto \overline{\phi} \in Z(\mathcal{O})$ forms a homeomorphism. The claim follows therefore by duality. \hfill \Box

In the sequel, we will canonically identify $f \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d)$ and $\Phi(f) \in Z'(\mathcal{O})$ and write, with an abuse of notation, simply $f = \Phi(f)$.

Theorem 5.5. Let $H \leq \text{GL}(d, \mathbb{R})$ be integrably admissible with essential frequency support $\mathcal{O} \subset \mathbb{R}^d$. Let $Q = (h_i^{-1}Q)_{i \in I}$ be any cover of $\mathcal{O}$ induced by $H$ admitting a BAPU $\Phi = (\varphi_i)_{i \in I}$ subordinate to $Q$. Moreover, let $u : \mathcal{O} \to \mathbb{R}^+$ be a transplant of $h \mapsto |\det(h^{-1})|^{1/2 - \frac{d}{2}v(h^{-1})}$. Then, up to canonical identification, the norm equivalence

$$\| \cdot \|_{\text{Co}(L^p, \ell_q^u)} \asymp \| \cdot \|_{D(Q, L^p, \ell_q^u)}$$

holds for all $p, q \in [1, \infty]$. In particular, the spaces $\text{Co}(L^p, \ell_q^u)$ and $D(Q, L^p, \ell_q^u)$ are isomorphic under the Fourier transform.

Proof. Let $f \in \text{Co}(Y)$. By Corollary 3.9 there exists a sequence $(f_n)_{n \in \mathbb{N}}$ with functions $f_n \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d)$ satisfying $f_n \rightarrow f$ in $\mathcal{S}_\mathcal{O}(\mathbb{R}^d)$ and $\|f_n\|_{\text{Co}(Y)} \leq C\|f\|_{\text{Co}(Y)}$ for all $n \in \mathbb{N}$. Proposition 5.2 yields a constant $C' > 0$ such that $\| \cdot \|_{D(Q, L^p, \ell_q^u)} \leq C'\| \cdot \|_{\text{Co}(Y)}$ on $\mathcal{S}_\mathcal{O}(\mathbb{R}^d)$. Combining this, together with Lemma 4.18, yields that

$$\|f\|_{D(Q, L^p, \ell_q^u)} \leq \liminf_{n \to \infty} \|f_n\|_{D(Q, L^p, \ell_q^u)} \leq C' \liminf_{n \to \infty} \|f_n\|_{\text{Co}(Y)} \leq C'\|f\|_{\text{Co}(Y)},$$

which gives the embedding $\text{Co}(Y) \hookrightarrow D(Q, L^p, \ell_q^u)$.

For the reverse, let $f \in D(Q, L^p, \ell_q^u)$. Then, by Lemma 4.17 there exists an $F \in \ell_q^u(I; L^p)$ satisfying $\|F\|_{\ell_q^u(I; L^p)} \leq C_{\Phi, Q}\|f\|_{D(Q, L^p, \ell_q^u)}$. Moreover, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of $F$-dominated functions $f_n \in \mathcal{S}_\mathcal{O}(\mathbb{R}^d)$ such that $f_n \rightarrow f$ in the weak*-topology on $Z'(\mathcal{O})$. Hence $\lim_{n \to \infty} W_{\phi, f_n}(x) = W_{\phi, f}(x)$ for all $x \in G$. Therefore

$$\|f\|_{\text{Co}(Y)} = \lim_{n \to \infty} \|f_n\|_{\text{Co}(Y)} \leq C_0 \lim_{n \to \infty} \|f_n\|_{D(Q, L^p, \ell_q^u)} \leq C_0 \lim_{n \to \infty} \|F\|_{\ell_q^u(I; L^p)} \leq C_0 C_{\Phi, Q}\|f\|_{D(Q, L^p, \ell_q^u)},$$

where $C_0 > 0$ is provided by Proposition 5.3. \hfill \Box

6 Examples and applications

This section provides three examples of (classes of) integrably admissible dilation groups and their associated coorbit spaces.

Example 6.1. Let $H \leq \text{GL}(d, \mathbb{R})$ be (irreducibly) admissible, i.e., there exists a single open orbit $\mathcal{O} = H^\circ \xi_0$, for $\xi_0 \in \mathbb{R}^d$, of full measure for which the stabilizer group $H_{\xi_0}$ is compact. Then the quasi-regular representation $(\pi, L^2(\mathbb{R}^d))$ of $G = \mathbb{R}^d \rtimes H$ is a discrete series representation by [23, Corollary 21]. Moreover, since $H$ is integrably admissible, the representation $(\pi, L^2(\mathbb{R}^d))$ is $\nu$-integrable by Theorem 2.7. Thus the coorbit space theory developed in [20, 21] is applicable in this setting. We will relate these original spaces to the ones defined in (3.1).

For this, fix a non-zero vector $\psi \in L^2(\mathbb{R}^d)$ satisfying $W_{\psi, \psi} \in L^1_v(G)$ and define

$$\mathcal{H}_{1, \psi} = \left\{ f \in L^2(\mathbb{R}^d) : W_{\psi} f \in L^1_v(G) \right\} \neq \emptyset.$$
Denoting by $\overline{H^1_v}$ the anti-dual space of $H_{1,v}$, the coorbit spaces in [20] are defined as

$$\widetilde{\text{Co}}(Y) := \{ f \in \overline{H^1_v} : \| g \mapsto (f, \pi(g)\psi) \| \in L^p_v(G) \}. \quad (6.1)$$

Any non-zero $\psi \in S_O(\mathbb{R}^d)$ can be used to define the space $H_{1,v}$, cf. Proposition 2.18. We next outline the relation between the reservoir $\mathcal{S}_O(\mathbb{R}^d)$ used in (3.1) and $\overline{H^1_v}$ used in (6.1). According to [30, Corollary 11], the mapping $\Theta : H_{1,v}^d \to Z^*(O)$, $f \mapsto \langle \phi \mapsto f(\hat{\phi}) \rangle$ is well-defined, injective, linear and continuous with respect to the weak*-topology on $\overline{H^1_v}$. By use of the mapping $\Phi : \mathcal{S}_O(\mathbb{R}^d) \to Z^*(O)$ defined in Lemma 5.4, it can then be deduced from [30 Theorem 38] that $\Phi^{-1} \circ \Theta : \overline{H^1_v} \to \mathcal{S}_O(\mathbb{R}^d)$ induces an isometric isomorphism

$$\Phi^{-1} \circ \Theta : \widetilde{\text{Co}}(Y) \to \text{Co}(Y).$$

Thus, up to suitable identification, the spaces $\widetilde{\text{Co}}(Y)$ and $\text{Co}(Y)$ coincide, provided that both spaces are defined in terms of some admissible $\psi \in S_O(\mathbb{R}^d)$.

For applications of the realization of a coorbit space as a decomposition space in the setting of irreducibly admissible dilation groups, see [30 Section 9] and [39].

The following example covers the anisotropic Besov spaces [2,5].

**Example 6.2.** Let $H = \langle A \rangle$ be the cyclic subgroup generated by $A \in \text{GL}(d, \mathbb{R})$. Recall that a matrix $A \in \text{GL}(d, \mathbb{R})$ is called *expansive* if all its eigenvalues $\lambda \in \sigma(A)$ satisfy $|\lambda| > 1$. By Proposition 6.3 below, the dilation group $H = \langle A \rangle$ is integrably admissible if, and only if, either $A$ or its inverse $A^{-1}$ is expansive. In this case, the essential frequency support of the quasi-regular representation $(\pi, L^2(\mathbb{R}^d))$ is given by $O = \mathbb{R}^d \setminus \{0\}$.

Given an open set $C \subset \mathbb{R}^d$ such that $C \supset \mathbb{R}^d \setminus \{0\}$ is compact, a cover $Q_A = (Q_j)_{j \in \mathbb{Z}}$ of $\mathbb{R}^d \setminus \{0\}$, with $Q_j := A^{j-1}C$, is called an *homogeneous cover induced by $A$*, see [7]. A homogeneous cover induced by $A^\beta$ is readily seen to be an induced cover of $O$ in the sense of Definition 4.4.

For $\alpha \in \mathbb{Z}$, define the weighting function $u_{\alpha,A} : \mathbb{Z} \to \mathbb{R}^+$ by $u_{\alpha,A}(j) = |\det(A)|^{j\alpha}$. Then, by [7, Theorem 5.6], for $p, q \in [1, \infty]$, the decomposition space $D(\mathcal{Q}_{\alpha A^\beta}, L^p, \ell^q_{\alpha})$ coincides, up to suitable identification, with the (homogeneous) anisotropic Besov space

$$B^{p,q}_{\alpha}(\mathbb{R}^d, A) := \{ f \in S'(\mathbb{R}^d) : \| \langle f \cdot \varphi \rangle_j \|_{L^p(\mathbb{R}^d)} \|_{\ell^q_{\alpha}} < \infty \},$$

where $\varphi_j := |\det(A)|^{j\varphi(A^j)}$ for some $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\text{supp} \hat{\varphi} \subset [-1,1]^d \setminus \{0\}$ and $\sum_{j \in \mathbb{Z}} |\hat{\varphi}(\xi)| > 0$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Thus, by means of Theorem 5.3, the anisotropic Besov space $B^{p,q}_{\alpha}(\mathbb{R}^d, A)$ can be canonically identified as a coorbit space $\text{Co}(L^p_v(\mathbb{R}^d \rtimes \langle A \rangle))$.

We will next detail the classification results obtained in [7] to state similar results for the coorbit spaces. For this, define an *$A$-homogeneous quasi-norm* as a Borel map $\rho_A : \mathbb{R}^d \to [0, \infty)$ that is positive definite, $A$-homogeneous, i.e., $\rho_A(Ax) = |\det(A)|\rho_A(x)$, and satisfies the quasi-triangle inequality $\rho_A(x + y) \lesssim (\rho_A(x) + \rho_A(y))$. As usual, two quasi-norms $\rho_A, \rho_B$ are called *equivalent* if $\rho_A \simeq \rho_B$. In this terminology, [7, Theorem 5.10] asserts that that, given two expansive matrices $A_1, A_2 \in \text{GL}(d, \mathbb{R})$, it holds that $B^{p,q}_{\alpha}(\mathbb{R}^d, A_1) = B^{p,q}_{\alpha}(\mathbb{R}^d, A_2)$ for all $p, q \in [1, \infty]$ if, and only if, the quasinorms $\rho_{A_1^\beta}$ and $\rho_{A_2^\beta}$ are equivalent. As a direct consequence, the corresponding coorbit spaces coincide, i.e.,

$$\text{Co}(L^p_v(\mathbb{R}^d \rtimes \langle A_1 \rangle)) = \text{Co}(L^p_v(\mathbb{R}^d \rtimes \langle A_2 \rangle))$$

The following proposition states similar results for the coorbit spaces.
for all \( p, q \in [1, \infty] \) if, and only if, the equivalence \( \rho_{A^T} \asymp \rho_{A^T} \) holds.

For explicit and checkable criteria for the equivalence of homogeneous quasi-norms, the interested reader is referred to [1] Section 10 and [7] Section 6, with the caveat that the latter source corrects some fallacies contained in the earlier one.

We next want to comment on coorbit spaces associated to one-parameter subgroups in more detail. The following observation formulates a criterion when these groups are integrably admissible, it is [32, Theorem 1.1]

**Proposition 6.3.** Let \( A \in \mathbb{R}^{d \times d} \). Then the associated one-parameter group \( H = \exp(\mathbb{R}A) \) is integrably admissible if and only if either the real parts of all eigenvalues are strictly positive or strictly negative. The essential frequency support associated to \( H \) is given by \( \mathcal{O} = \mathbb{R}^d \setminus \{0\} \).

Note that the condition on \( A \) formulated in the proposition is equivalent to saying that the eigenvalues of \( \exp(A) \) either all have modulus < 1 or all have modulus > 1, which is precisely the condition of expansiveness underlying the construction of anisotropic Besov spaces according to [1]. This raises the very natural question how the coorbit spaces associated to \( \exp(\mathbb{R}A) \) are related to those associated to \( \langle \exp(A) \rangle \), i.e., to (homogeneous) anisotropic Besov spaces. Since \( \langle \exp(A) \rangle \subset \exp(\mathbb{R}A) \), with compact quotient, this question is a special case of the following somewhat more general observation, which gives a criterion when two different matrix groups induce the same scales of coorbit spaces.

**Lemma 6.4.** Let \( H_1 < H_2 < GL(d, \mathbb{R}) \) denote two closed subgroups, with \( H_2/H_1 \) compact. Assume that \( H_2 \) is integrably admissible with essential frequency support \( \mathcal{O} \). Then \( H_1 \) is integrably admissible with essential frequency support \( \mathcal{O} \), and \( H_1 \) and \( H_2 \) have the same coorbit spaces, in the following sense:

Let \( G_i = \mathbb{R}^d \rtimes H_i \), for \( i = 1, 2 \). Then, given any admissible weights \( v_i \) on \( G_i \) with the property that \( v_1 \) is equivalent to the restriction \( v_2|_{G_1} \), one has that \( Co(L^{p,q}_{v_1}(G_1)) = Co(L^{p,q}_{v_2}(G_2)) \), for all \( 1 \leq p, q \leq \infty \).

**Proof.** If \( H_2/H_1 \) is compact, there exists a compact set \( K \subset H_2 \) with \( KH_1 = H_2 \). Then, if \( C \subset \mathcal{O} \) is a relatively compact open subset with \( H_2 C = \mathcal{O} \) and \( ((C, C)) \subset H_2 \) compact, it is easy to see that \( KC \subset \mathcal{O} \) is relatively compact and open as well, with \( ((KC, KC)) \subset H_1 \) compact, and \( H_1 KC = H_2 C = \mathcal{O} \). This proves that \( H_1 \) is integrably admissible.

In order to show that the coorbit spaces coincide, we make use of their decomposition space realization. Furthermore, if \( (h_i)_{i \in I} \subset H_1 \) is well-spread, it is also a well-spread subset of \( H_2 \). In particular, any induced covering of the style \( (h_i^{-1} Q)_{i \in I} \) can be understood of induced both by \( H_1 \) and \( H_2 \).

By assumption on \( v_1 \) and \( v_2 \), transplanting either \( v_1 \) or \( v_2 \) to the induced covering results in equivalent weights on the covering, and therefore the induced decomposition spaces coincide. Theorem 5.5 now yields the desired conclusion.

**Remark 6.5.** In the setting of the previous lemma, it is easy to see that restricting an admissible weight from \( G_2 \) to \( G_1 \) yields an admissible weight on the smaller group. It is less clear whether every admissible weight on the smaller group extends (up to equivalence) to the larger one. For weights only depending on the \( H_i \)-variable, some partial answers are available. For instance, if \( H_2 \) is a abelian, or a direct product of \( H_1 \) and a second (necessarily compact) subgroup, such an extension is easily constructed. In this case, since \( H_2/H_1 \) is assumed compact, there exists a measurable cross-section \( \sigma : H_2/H_1 \rightarrow H_2 \) with relatively compact image and with \( \sigma(h) = e \) for all \( h \in H_1 \). Here one checks immediately \( v_2(h) = v_1(\sigma(h)^{-1}h) \) yields an extension of \( v_1 \) to \( H_2 \) with the desired properties.
We expect that a more direct proof of Lemma 6.4, relating integration over $H_2$ to integration over $H_1$ by Weil’s integral formula, is also available. However, the proof using the decomposition space identification is remarkably effortless. Note also that already the formulation of the lemma is greatly facilitated by the choice of a common reservoir. In the absence of a common reservoir, comparing coorbit spaces associated to different groups requires making potentially cumbersome identifications.

**Corollary 6.6.** Let $H = \exp(\mathbb{R}A)$ denote an integrably admissible one-parameter group. Then the coorbit spaces associated to $H$ are precisely the anisotropic Besov spaces associated to $\exp(A)$.

**Remark 6.7.** There is a converse to Corollary 6.6. Given any expansive matrix $A$, there exists a second expansive matrix $B$ with the property that $A$ and $B$ induce the same scales of anisotropic Besov spaces, and in addition, $B = \exp(C)$ for a suitable matrix $C$; see [7, Lemma 7.8]. Thus the classes of anisotropic Besov spaces on the one hand, and of coorbit spaces associated to one parameter groups on the other, coincide precisely.

The following result is also well-known, and was already stated in [19].

**Corollary 6.8.** Let $H = \mathbb{R}^+ \cdot SO(d)$. Then the coorbit spaces of $H$ and $\mathbb{R}^+ \cdot \text{Id}_{\mathbb{R}^d}$ coincide.

We sketch a further potential benefit of the ability to switch groups in the description of coorbit spaces. One natural problem arising in the study of coorbit spaces is the question which dilation matrices $g \in \text{GL}(d, \mathbb{R})$ leave the coorbit spaces invariant. It is shown in [26] that this set of matrices may well depend on the choice of underlying dilation group $H$, but a deeper understanding of this dependence is currently missing. By construction of the coorbit spaces, it is clear that the elements of $H$ leave coorbit spaces associated to $H$ invariant, and more generally, natural candidates are the elements of the normalizer $N$ of $H$

$$N(H) = \{ g \in \text{GL}(d, \mathbb{R}) : g^{-1}Hg = H \}.$$  

We restrict the following discussion to weights that are independent of the translation variable. Then the arguments showing [30, Lemma 44] can be used to show the following.

**Lemma 6.9.** Let $H \leq \text{GL}(d, \mathbb{R})$ be an integrably admissible dilation group. For $g \in N(H)$, the associated dilation operator $D_g$ is an isomorphism from $\text{Co}(L_{p,q}^{v_1})$ onto $\text{Co}(L_{p,q}^{v_2})$, where $v_2(h) = v(ghg^{-1})$.

Once the role of the normalizer is understood, a benefit of Lemma 6.4 becomes apparent, that is best exemplified with the help of the groups $H_1 = \mathbb{R}^+ \cdot \text{Id}_{\mathbb{R}^d}$ and $H_2 = \mathbb{R}^+ \cdot SO(d)$. Trying to use the normalizer of $H_2$ to find additional dilational symmetries of the associated coorbit spaces yields very little, since $N(H_2)$ turns out to be a finite extension of $H_2$.

By contrast, since $H_1$ is central in $\text{GL}(d, \mathbb{R})$, one has $N(H_1) = \text{GL}(d, \mathbb{R})$, and in addition, $v_g = v$ for every weight on $H_1$ and every $g \in \text{GL}(d, \mathbb{R})$. Thus computing the normalizer of $H_1$ is sufficient to see that the isotropic homogeneous Besov spaces are invariant under arbitrary dilations.

**Remark 6.10.** Consider the group $H$ described in Example 2.6 (c), one has $H_1 \subset H \subset H_3$, where $H_3$ is the diagonal group, and $H_1 = \exp(\mathbb{R}C)$ with

$$C = \begin{bmatrix} 1 & \alpha + \beta \\ 1 & \alpha \end{bmatrix}.$$
Hence, $H_1, H_3$ are both integrably admissible. However, $H/H_1 \cong \mathbb{R} \cong H_3/H$, hence Lemma 6.4 is not applicable, and we expect that the coorbit spaces associated to $H$ are different from those associated to either $H_1$ or $H_3$.

Note also the interesting phenomenon that, unlike for cocompactly contained pairs of groups, the essential frequency supports of $H_1, H_3$ on the one hand, and of $H$ on the other, are in fact different.

Outlook

The decomposition space description of coorbit spaces can be understood as a generalization of the $\varphi$-transform characterization of Besov spaces due to Frazier and Jawerth [22]. As outlined in [30], the chief benefit of this description comes from the fact that it provides a natural common framework for the unified treatment of spaces associated to different groups, and the papers [46, 47] provide tools to tackle the question of embeddings between decomposition spaces, but also of decomposition spaces into Sobolev and BV spaces, in a systematic manner. As one keeps introducing new classes of function spaces, one fundamental question becomes particularly relevant, namely that of classification. By this we mean the problem of deciding whether two different dilation groups (representing two different constructions of wavelet systems) actually induce different scales of coorbit spaces. On the decomposition space level, this question is settled in [46]. This result was used for the classification of anisotropic Besov spaces in [7], and of shearlet coorbit spaces in [36]. Furthermore, the latter develops an observation made in [18] into an interesting connection between decomposition space theory and coarse geometry [42], which will be further pursued in upcoming publications. In this paper, a first glimpse of coarse geometric reasoning can be seen in the discussion in Section 6 in particular in connection with Lemma 6.4 and its applications. We expect that the question whether a converse to this lemma holds, e.g. as conjectured in Remark 6.10, can be ultimately determined using coarse geometry methods. Likewise, it seems plausible to us that the question whether the essential frequency support of an integrably admissible dilation group is unique is ultimately a coarse geometric one.

It should also be emphasized that, while the coorbit theory developed here is ultimately a branch of decomposition space theory, the focus on this class allows the formulation of sharper results. This can be exemplified by the discretization results such as Theorem 3.8 which clearly benefit from the underlying group. Further cases in point are provided by the discussion of dilational symmetries in Section 6 and by the classification results in [30], which also make crucial use of the group structure underlying the construction of the coorbit spaces.

An interesting question for future work regards explicit criteria and existence proofs for atomic decompositions using wavelets that are compactly supported in space rather than in frequency. For the irreducible case, these were obtained in [27, 29]. The techniques developed for proving the existence of such wavelets in the irreducible case are already fairly involved, and they make heavy use of the fact that the dual action has a unique open orbit. We therefore expect that the extension to integrably admissible dilation groups presents a major challenge.

Acknowledgement. The second named author acknowledges support from the Austrian Science Fund (FWF): P 29462 - N35.

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