Genericity under parahoric restriction

Manish Mishra∗ Mirko Rösner†

Abstract

We study the preservation of genericity under parahoric restriction of depth zero representations.

1 Introduction

Let $G$ be a connected reductive group defined over a non-archimedean local field $k$. Let $B$ be a $k$-Borel subgroup of $G$ with unipotent radical $U$ and let $T$ be a maximal $k$-torus in $B$. The corresponding groups of $k$-rational points are $G$, $B$, $U$, $T$. A character $\psi : U \to \mathbb{C}^\times$ is called generic if the stabilizer of $\psi$ in $T$ is exactly the center $Z$ of $G$. An admissible representation $\pi$ of $G$ is called generic (more specifically $\psi$-generic) if there exists a generic character $\psi$ of $U$ such that $\text{Hom}_G(\pi, \text{Ind}_G^B \psi) \neq 0$.

A basic result due to Rodier [4, Thm. 2] states that genericity is preserved under the Jacquet functor. On the category of depth zero representations of $G$ (see [2] for the notion of depth), there is a functor analogous to the Jacquet functor called the parahoric restriction functor [7]. It is defined as follows. Let $G_x$ be a parahoric subgroup of $G$ with pro-unipotent radical $G_x^+$. The quotient $G_x/G_x^+$ is the $\mathbb{F}_q$-points of a connected reductive group $M$ defined over $\mathbb{F}_q$, where $\mathbb{F}_q$ is the residue field of $k$. The parahoric restriction functor sends a representation $(\pi, V)$ of $G$ to the representation of $M(\mathbb{F}_q)$ obtained by restricting $\pi$ to $G_x$ and then taking the $G_x^+$-invariants of it.

We say $G$ is unramified, if it is quasisplit and splits over an unramified extension. Suppose that $G$ is unramified, that $G_x$ is contained in a hyperspecial parahoric subgroup of $G$ and that $\pi$ is parabolically induced from a supercuspidal representation of a Levi subgroup of

∗University of Heidelberg, Germany, electronic address: manish.mishra@gmail.com
†University of Heidelberg, Germany, electronic address: mirko_rosner@hotmail.com
G. Then we show in Theorem 2 that the parahoric restriction functor preserves genericity of \( \pi \). This does not generalize to arbitrary parahorics and admissible representations, see Section 5.

2 Notations

Fix a non-archimedean local field \( k \) and an unramified connected reductive group \( G \) over \( k \). Let \( \mathbf{B} = \mathbf{T} \ltimes \mathbf{U} \) be a \( k \)-Borel subgroup of \( G \) with a maximal \( k \)-torus \( \mathbf{T} \) and a unipotent radical \( \mathbf{U} \). Let \( \mathbf{Z} \) be the center of \( G \) and let \( \mathbf{T}_{\text{ad}} := \mathbf{T}/\mathbf{Z} \). Their groups of \( k \)-rational points are \( G, B, T, U, Z \) and \( T_{\text{ad}} \), respectively. We will follow the standard abuse of notation to write “parabolic subgroups of \( G \)” instead of “\( k \)-points of \( k \)-parabolic subgroups of \( G \)”.

For a point \( x \) in the Bruhat-Tits building of \( G \), the associated parahoric subgroup will be denoted by \( G_x \). Its Levi quotient is \( G_x^+ = G_x/G_x^+ \) with the pro-unipotent radical \( G_x^+ \). We will denote by \( \text{Rep}(G) \) the category of admissible complex representations of \( G \) and likewise for the other groups.

3 Parahoric restriction functor

Fix a point \( x \) in the apartment attached to \( T \). Restricting an admissible representation \( (\pi, V) \) of \( G \) to the parahoric \( G_x \) and taking invariants with respect to the pro-unipotent radical \( G_x^+ \) gives rise to a representation of \( G_x = G_x/G_x^+ \). This defines the parahoric restriction functor

\[
\mathbf{r}_{G_x} : \text{Rep}(G) \to \text{Rep}(G_x), \quad \begin{cases} (\pi, V) \mapsto \left( G_x \to \text{Aut}(V^{G_x^+}) \right), \\ (V_1 \to V_2) \mapsto \left( V_1^{G_x^+} \to V_2^{G_x^+} \right) \end{cases},
\]

where \( V_1 \to V_2 \) is a morphism between admissible representations \( (\pi_1, V_1) \) and \( (\pi_2, V_2) \). This functor is exact and defines a homomorphism between the corresponding Grothendieck groups, analogous to Jacquet’s functor of parabolic restriction.

For hyperspecial parahorics, parahoric restriction commutes with parabolic induction in the following sense:

**Theorem 1.** Let \( x \) be a hyperspecial point in the apartment attached to \( T \) with corresponding hyperspecial parahoric subgroup \( G_x \subseteq G \). Fix a standard parabolic subgroup \( P \supseteq B \) with Levi decomposition \( P = M \ltimes N \) such that \( T \subseteq M \). Then the following diagram is commutative
The parabolic subgroup $P = (G_x \cap P)/(G_x^+ \cap P)$ of $G_x$ is generated by the same roots as $P$ and has Levi subgroup $M_x$.

**Proof.** The parahoric subgroup of $M$ attached to $x$ is $M_x \cong G_x \cap M$, cp. [2].

Fix an admissible representation $(\sigma, V)$ of $M$ and denote its inflation to $P$ by the same symbol. We construct a natural equivalence $(\text{Ind}^G_P(\sigma))_{G_x} \rightarrow \text{Ind}^{G_x}_{G} \{r_{M_x}(\sigma)\}$. Then $\text{Ind}^G_P(\sigma)$ has a canonical model by right multiplication on the space of functions $f : G \rightarrow V$ with

$$f(pg) = \delta_P^{1/2}(p)\sigma(p)f(g) \quad \forall p \in P, \; g \in G,$$

where $\delta_P$ is the modulus character. By the Iwasawa decomposition $G = PG_x$, every such $f$ is uniquely defined by its restriction $\tilde{f} = f|_{G_x}$ to $G_x$. The linear map $f \mapsto \tilde{f}$ is thus a $G_x$-equivariant isomorphism from $\text{Ind}^G_P(\sigma)$ to the space of $\tilde{f} : G_x \rightarrow V$ with

$$\tilde{f}(pg) = \sigma(p)\tilde{f}(g) \quad \forall p \in G_x \cap P, \; g \in G_x.$$

Such an $\tilde{f} : G_x \rightarrow V$ is invariant under the right action of $G_x^+$ if and only if $\tilde{f}(xu) = \tilde{f}(x)$ for every $u \in G_x^+$ and $x \in G_x$. In that case we have

$$\sigma(p)\tilde{f}(g) = \tilde{f}(pg) = \tilde{f}(gg^{-1}pg) = \tilde{f}(g) \quad \forall p \in P \cap G_x^+, \; g \in G_x,$$

since $g^{-1}pg \in G_x^+$. Hence $\tilde{f}$ actually maps into the invariant space $V^{P \cap G_x^+}$.

Now $\tilde{f}$ factors over a unique function $h_f : G_x \rightarrow V^{P \cap G_x^+}$ with the property

$$h_f(pg) = r_{M_x}(\sigma)(p)h_f(g) \quad \forall p \in P, \; g \in G.$$

The action of $G_x$ by right-multiplication on the space of these $h_f$ is a model of the induced representation $\text{Ind}^{G_x}_{P \cap G_x}(\sigma)$ of $G_x$. The family of isomorphisms $\{f \mapsto h_f\}_\sigma$ provides the natural equivalence. 

$\square$
4 Generic depth zero

A character $\psi: U \to \mathbb{C}^\times$ is called *generic* if its stabilizer in $T_{ad}$ is trivial. Let $z$ be a hyperspecial vertex of $G$ in the apartment attached to $T$. A generic character $\psi$ of $U$ is called *depth-zero* at $z$ if its restriction to $U \cap G_z$ factors through a generic character $\psi_z$ of $\overline{U} := (U \cap G_z)/(U \cap G_z^+)$. 

**Theorem 2.** Let $\pi = \text{Ind}^G_{P,M}(\sigma)$ be an irreducible admissible representation of $G$ that is parabolically induced from a supercuspidal irreducible representation $\sigma$ of a Levi subgroup $M$ of a parabolic $P \subseteq G$. Let $G_x$ be a parahoric subgroup of $G$, contained in a hyperspecial parahoric subgroup, such that $\overline{\pi} := r_{G_x}(\pi) \neq 0$. Then $\pi$ being generic implies that $\overline{\pi}$ is generic.

**Proof.** By conjugating $x$ and $P$ if necessary, we can assume without loss of generality that $P \supseteq B$, that $M \supseteq T$, that $x$ is a point in the apartment associated to $T$ and that $\pi$ is generic with respect to a generic character $\psi$ of $U$. We can assume without loss that $G_x$ is a hyperspecial maximal parahoric subgroup of $G$ because of transitivity of parahoric restriction [7, 4.1.3] and Rodier’s result [4, Thm. 2].

If $w_o$ is an element in the normalizer of $T$ such that $B \cap w_o B w_o^{-1} = T$, then $Q := M \cap w_o U w_o^{-1}$ is a maximal unipotent subgroup of $M$ [4, Thm. 2]. Define a generic character of $Q$ by $\psi_M(q) = \psi(w_o^{-1} q w_o)$ for $q \in Q$. Then by [4, Thm. 2],

\[ \text{Hom}_U(\pi, \psi) \cong \text{Hom}_Q(\sigma, \psi_M). \] (4.1)

Since $\overline{\pi} \neq 0$, Theorem 1 implies that $\sigma$ is a depth zero supercuspidal representation of $M$. By [1, Lemma 6.1.2], there is a hyperspecial point $y$ of $M$ and a cuspidal representation $\tau^o$ of $M_y$ such that

a) $\psi_M$ is depth zero at $y$ and $\tau^o$ is $(\psi_M)_y$-generic.

b) There is an extension of $\tau^o$ to a representation $\tau$ of the normalizer $[M_y]$ of $M_y$ in $M$ such that $\sigma = \text{c-Ind}^{M_y}_{[M_y]} \tau$. Note that since $y$ is a hyperspecial point of $M$, $[M_y] = Z_M M_y$, where $Z_M$ denotes the center of $M$.

Since $\sigma$ has depth zero at $x$ by Thm. 1, we can assume without loss of generality that $x = y$ (see proof of [8, Lemma 3.3(ii)]). We have therefore

\[ \text{Hom}_Q(\tau^o, (\psi_M)_x) \neq 0, \] (4.2)

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where \( Q \) denotes a maximal unipotent subgroup of \( M_x \) defined in the same way as \( Q \). Theorem 1 and a result of Vignéras [6, §7] imply that \( \pi \) is isomorphic to \( \text{Ind}_{G}^{G_x} \tau^o \). Then again by [4, Thm. 2],

\[
\text{Hom}_{U}(\pi, \psi_x) \cong \text{Hom}_{Q}(\tau^o, (\psi_M)_x) \neq 0. \tag{4.3}
\]

This completes the proof.

\[\square\]

5 Non-special parahoric restriction

In Theorem 2 we make two technical assumptions: we assume that \( \pi \) is parabolically induced and that \( G_x \) is contained in a hyperspecial maximal parahoric subgroup. However, if we drop these assumptions, there is a counterexample.

Counterexample: There is a generic irreducible admissible representation \( \pi \) of \( G = \text{GSp}(4, k) \) and a parahoric subgroup \( G_x \subseteq G \) such that the parahoric restriction \( r_{G_x}(\pi) \) is non-zero, but not generic.

Proof. Let \( \xi \) be the non-trivial unramified quadratic character of \( k^\times \). Let \( G = \text{GSp}(4, k) \) be the group of symplectic similitudes with respect to the symplectic form \((-w^w)\) for \( w = (1^1) \). Fix the Borel pair \((B, T)\) where \( B = T \lt U \subseteq G \) is the subgroup of upper triangular matrices and \( T \subseteq B \) is the maximal torus of diagonal matrices. Fix a character

\[
\alpha : T \to \mathbb{C}^\times, \quad \text{diag}(a, b, c/a, c/b) \mapsto \xi(ab) |a| |c|^{-1/2}.
\]

Inflating \( \alpha \) to \( B \) gives rise to the induced representation \( \text{Ind}_B^G \alpha \), which admits a unique irreducible subrepresentation\(^1\) \( \pi \). The representation \( \pi \) is generic for arbitrary with respect to every generic character \( \psi \) of \( G \).

The standard paramodular subgroup \( G_x \subseteq G \) is a non-special maximal parahoric subgroup. The second author has shown [5, Thm. 3.7] that the parahoric restriction \( r_{G_x}(\pi) \) is isomorphic to the restriction of \( 1 \otimes \text{St} \oplus \text{St} \otimes 1 \) to

\[
\{(\xi_1,\xi_2) \in (\text{GL}(2, \mathbb{F}_q))^2 \mid \det \xi_1 = \det \xi_2 \} \cong G_x.
\]

where \( 1 \) is the trivial and \( \text{St} \) is the Steinberg representation of \( \text{GL}(2, \mathbb{F}_q) \). Thus \( r_{G_x}(\pi) \) is not generic.

\[\square\]

\(^1\)This is type Va in the notation of Roberts and Schmidt [3].
Acknowledgment

The authors are thankful to Sandeep Varma for pointing out a mistake in the earlier draft of this article.

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