On Hedetniemi’s conjecture and the Poljak-Rödl function

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Abstract

Hedetniemi conjectured in 1966 that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ for any graphs $G$ and $H$. Here $G \times H$ is the graph with vertex set $V(G) \times V(H)$ defined by putting $(x, y)$ and $(x', y')$ adjacent if and only if $xx' \in E(G)$ and $yy' \in V(H)$. This conjecture received a lot of attention in the past half century. It was disproved recently by Shitov. The Poljak-Rödl function is defined as $f(n) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = n\}$. Hedetniemi’s conjecture is equivalent to saying $f(n) = n$ for all integer $n$. Shitov’s result shows that $f(n) < n$ when $n$ is sufficiently large. Using Shitov’s result, Tardif and Zhu showed that $f(n) \leq n - (\log n)^{1/4}$ for sufficiently large $n$. Using Shitov’s method, He–Wigderson showed that for $\epsilon \approx 10^{-9}$ and $n$ sufficiently large, $f(n) \leq (1 - \epsilon)n$. In this note we show that a slight modification of the proof in the paper of Zhu and Tardif shows that $f(n) \leq (1/2 + o(1))n$ for sufficiently large $n$. On the other hand, it is unknown whether $f(n)$ is bounded by a constant. However, we do know that if $f(n)$ is bounded by a constant, then the smallest such constant is at most 9. This lecture note gives self-contained proofs of the above mentioned results.

1 Introduction

The product $G \times H$ of graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and has $(x, y)$ adjacent to $(x', y')$ if and only if $xx' \in E(G)$ and $yy' \in E(H)$. Many names for this product are used in the literature, including the categorical product, the tensor product and the direct product. It is the most important product in this lecture note. We just call it the product. Edges in a graph $G$ are denoted by $xy \in E(G)$ or $x \sim y$ (in $G$).

∗This note is based on an invited lecture at the MATRIX program “Structural Graph Theory Downunder” held in Creswick, Australia, Nov. 24–Dec. 1st, 2019. The main part of the lecture is to explain Shitov’s proof, which is based on Shitov’s paper, but more explanations are added.

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A proper colouring $\phi$ of $G$ induces a proper colouring $\Phi$ of $G \times H$ defined as $\Phi(x, y) = \phi(x)$. So $\chi(G \times H) \leq \chi(G)$. Symmetrically, we also have $\chi(G \times H) \leq \chi(H)$. Therefore $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. In 1966, Hedetniemi conjectured in [3] that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ for all graphs $G$ and $H$. This conjecture received a lot of attention in the past half century (see [1, 6, 10, 13, 18, 19]). Some special cases are confirmed. In particular, it is known that if $\min\{\chi(G), \chi(H)\} \leq 4$, then the conjecture holds [1]. Also, a fractional version of Hedetniemi’s conjecture is true [19]. However, Shirot recently refuted Hedetniemi’s conjecture [11]. He proved that for sufficiently large $n$, there are $n$-chromatic graphs $G$ and $H$ with $\chi(G \times H) < n$.

The Poljak-Rödl function [9] is defined as

$$f(n) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = n\}.$$ 

Hedetniemi’s conjecture is equivalent to saying $f(n) = n$ for all positive integer $n$. Shirot’s result shows that $f(n) < n$ for sufficiently large $n$. Right after Shirot put his result on arxiv, using his result, Tardif and Zhu [16] showed that the difference $n - f(n)$ can be arbitrarily large. Indeed, they proved that $f(n) \leq n - (\log n)^{1/4}$ for sufficiently large $n$. It is also shown in [16] that if Stahl’s conjecture in [12] on the multi-chromatic number of Kneser graphs is true, then $\lim_{n \to \infty} f(n)/n \leq 1/2$. He–Wigderson, using Shirot’s method, proved that $f(n) \leq (1 - \epsilon)n$ for $\epsilon \approx 10^{-9}$ and sufficiently large $n$. In this note we prove that the conclusion $\lim_{n \to \infty} f(n)/n \leq 1/2$ holds without assuming Stahl’s conjecture.

2 Exponential graph

One of the standard tools used in the study of Hedetniemi’s conjecture is the concept of exponential graphs. Let $c$ be a positive integer. We denote by $[c]$ the set $\{1, 2, \ldots, c\}$. For a graph $G$, the exponential graph $K^G_c$ has vertex set

$$\{f : f \text{ is a mapping from } V(G) \to [c]\},$$

with $f \sim g$ if and only if for any edge $xy \in E(G)$, $f(x) \neq g(y)$. In particular, $f \sim f$ is a loop in $K^G_c$ if and only if $f$ is a proper $c$-colouring of $G$. So if $\chi(G) > c$, then $K^G_c$ has no loop.

For convenience, when we study properties of $K^G_c$, vertices in $K^G_c$ will be called maps. The term “vertices” is reserved for vertices of $G$. We write a map in $K^G_c$ or a map from $G$ to $[c]$ when we refer to a vertex of $K^G_c$. For a map $f \in K^G_c$, the image set of $f$ is $\text{Im}(f) = \{f(v) : v \in V(G)\}$. Note that for $f, g \in K^G_c$, if $\text{Im}(f) \cap \text{Im}(g) = \emptyset$, then $f \sim g$. For $i \in [c]$, we denote by $g_i$ the constant map $g_i(v) = i$ for all $v \in V(G)$. So $\text{Im}(g_i) = \{i\}$. Thus $\{g_i : i \in [c]\}$ induces a $c$-clique in $K^G_c$ for any graph $G$ and $\chi(K^G_c) \geq c$.

For any graph $G$, the mapping $\Psi : V(G \times K^G_c) \to [c]$ defined as $\Psi(x, f) = f(x)$ is a proper colouring of $G \times K^G_c$. So $\chi(G \times K^G_c) \leq c$. If $\chi(G) > c$ and $\chi(K^G_c) > c$, then we would have a counterexample to Hedetniemi’s conjecture (with $H = K^G_c$).
For two graphs $G$ and $H$, a homomorphism from $G$ to $H$ is a mapping $\phi : V(G) \rightarrow V(H)$ that preserves edges, i.e., for every edge $xy$ of $G$, $\phi(x)\phi(y)$ is an edge of $H$. We say $G$ is homomorphic to $H$, and write $G \rightarrow H$, if there is a homomorphism from $G$ to $H$. The “homomorphic” relation “$\rightarrow$” is a quasi-order. It is reflexive and transitive: if $G \rightarrow H$ and $H \rightarrow Q$ then $G \rightarrow Q$. The composition $\psi \circ \phi$ of a homomorphism $\phi$ from $G$ to $H$ and a homomorphism $\psi$ from $H$ to $Q$ is a homomorphism from $G$ to $Q$.

Note that a homomorphism from a graph $G$ to $K_c$ is equivalent to a proper $c$-colouring of $G$. Thus is $G \rightarrow H$, then $\chi(G) \leq \chi(H)$.

For any graph $H$, if $\Psi : V(G \times H) \rightarrow [c]$ is a proper colouring, then the mapping sending $u \in V(H)$ to $f_u \in K_c^G$ defined as $f_u(v) = \Psi(u, v)$ is a homomorphism from $H$ to $K_c^G$. In this sense, $K_c^G$ is the largest graph $H$ in the order of homomorphism with the property that $\chi(G \times H) \leq c$.

Thus if $\chi(K_c^G) = c$, then for any graph $H$ with $\chi(H) > c$, $\chi(G \times H) > c$. So Hedetniemi’s conjecture is equivalent to the following statement:

**If $\chi(G) > c$, then $\chi(K_c^G) = c$.**

The concept of exponential graphs was first used in [11], where it is shown that if $\chi(G) \geq 4$, then $K_3^G$ is 3-colourable. Hence the product of two 4-chromatic graphs has chromatic number 4.

The result of El-Zahar and Sauer is still the best result in the positive direction of Hedetniemi’s conjecture. We do not know whether or not the product of two 5-chromatic graphs equals 5. On the other hand, there is a nice strengthening of this result by Tardif [14] in the study of multiplicative graphs. We say a graph $Q$ is multiplicative if for any two graphs $G, H$, $G \not\rightarrow Q$ and $H \not\rightarrow Q$ implies that $G \times H \not\rightarrow Q$. Hedetniemi’s conjecture is equivalent to say that $K_n$ is multiplicative for any positive integer $n$. El-Zahar and Sauer proved that $K_3$ is multiplicative. Häggkvist, Hell, Miller and Neumann Lara [3] proved that odd cycles are multiplicative and Tardif [14] proved that circular cliques $K_{p/q}$ for $p/q < 4$ are multiplicative, where $K_{p/q}$ has vertex set $[p]$ with $i \sim j$ if and only if $q \leq |i - j| \leq p - q$. (So $K_{p/1} = K_p$ and $K_{(2k+1)/k} = C_{2k+1}$).

This is a little bit astray from the main track. We come back to the proofs.

### 3 Shitov’s Theorem

We denote by $G[K_q]$ the graph obtained from $G$ by blowing up each vertex of $G$ into a $q$-clique. The vertices of $G[K_q]$ are denoted by $(x, i)$, where $x \in V(G)$ and $i \in [q]$. So $(x, i)$ and $(y, j)$ are adjacent in $G[K_q]$ if and only if either $x \sim y$ or $x = y$ and $i \neq j$. For a graph $G$, the independence number $\alpha(G)$ of $G$ is the size of a largest independent set in $G$. This section proves the following result of Shitov:

**Theorem 1 (Shitov)** Assume $G$ is a graph with $|V(G)| = p$, $\alpha(G) \leq \frac{p}{3^4}$ and $\text{girth}(G) \geq 6$. Let $q \geq 2^{p-1}p^2$ and $c = 4q + 2$. Then $\chi(G[K_q]) > c$ and $\chi(K_c^{G[K_q]}) > c$. 


Proof. The above formulation of the theorem is slightly different from the formulation in \[11\]. The proof also seems different. But all the claims and lemmas are either stated in \[11\] or hidden in the text in \[17\].

It is a classical result of Erdős \[2\] that there are graphs of arbitrary large girth and large chromatic number. This result is included in most graph theory textbooks (see \[17\]). The probabilistic proof of this result actually shows that there are graphs $G$ of arbitrary large girth and arbitrary small independence ratio $\alpha(G)/|V(G)|$. What we need here is a graph of girth 6 and with $\alpha(G) \leq |V(G)|/4.1$, As $G[K_q]$ has the same independence number as $G$, we conclude that

$$\chi(G[K_q]) \geq \frac{|V(G[K_q])|}{\alpha(G[K_q])} = \frac{|V(G)|q}{\alpha(G)} \geq 4.1q > c.$$  

Now we shall show that $\chi(K_c^{G[K_q]}) > c$.

Assume to the contrary that $\chi(K_c^{G[K_q]}) = c$ (recall that $K_c^{G[K_q]}$ has a $c$-clique and hence has chromatic number at least $c$), and $\Psi$ is $c$-colouring of $K_c^G$. We may assume that the constant map $g_i$ is coloured by colour $i$. Thus for any map $\phi \in K_c^{G[K_q]}$, if $i \notin Im(\phi)$, then $\phi \sim g_i$ and hence $\Psi(\phi) \neq i$. Thus we have the following observation.

**Observation 2** For any map $\phi \in K_c^{G[K_q]}$, $\Psi(\phi) \in Im(\phi)$.

**Definition 1** A map $\phi \in K_c^{G[K_q]}$ is called simple if $\phi$ is constant on each copy of $K_q$ that is a blow-up of a vertex of $G$, i.e., for any $x \in V(G), i, j \in [q]$, $\phi(x, i) = \phi(x, j)$.

For simplicity, we shall write $\phi(x)$ for $\phi(x, i)$ when $\phi$ is a simple map.

Note that in $K_c^{G[K_q]}$, two simple maps $\phi$ and $\psi$ are adjacent if and only if for each edge $xy$ of $G$, $\phi(x) \neq \psi(y)$, and moreover, for each vertex $x$, $\phi(x) \neq \psi(x)$. This is so, because for $i \neq j \in [q]$, $(x, i)(x, j)$ is an edge of $G[K_q]$ and $\psi(x)$ is a shorthand for $\phi(x, i)$ and $\psi(x)$ is a shorthand for $\psi(x, j)$.

In this sense, the subgraph of $K_c^{G[K_q]}$ induced by simple maps is isomorphic to $K_c^{G_o}$, where $G_o$ is obtained from $G$ by adding a loop to each vertex of $G$. We shall just treat $K_c^{G_o}$ as an induced subgraph of $K_c^{G[K_q]}$ and write $\phi \in K_c^{G_o}$ to mean that $\phi$ is a simple map in $K_c^{G[K_q]}$. Most of our argument is about properties of the subgraph $K_c^{G_o}$ of $K_c^{G[K_q]}$.

The graph $K_c^{G[K_q]}$ is a huge graph. As $G$ has girth 6 and fractional chromatic number at least 4.1, $p = |V(G)|$ is probably about 200. The number in $K_c^{G[K_q]}$ is $c^{pq}$, which is roughly $(2^{200})^{200}$. The subgraph $K_c^{G_o}$ has $c^p$ vertices, which is roughly $(2^{200})^{200}$. So $K_c^{G_o}$ is huge, but it is a very tiny fraction of $K_c^{G[K_q]}$.

**Definition 2** For $v \in V(G)$ and $b \in [c]$, let

$$I(v, b) = \{\phi \in K_c^{G_o} : \Psi(\phi) = b = \phi(v)\}.$$
Since $\Psi(\phi) \in \text{Im}(\phi)$ for any $\phi \in K_c^{G^a}$, we conclude that 
\[ V(K_c^{G^a}) = \bigcup_{v \in V(G), b \in [c]} I(v, b). \]

As $K_c^{G^a}$ has $c^p$ vertices, the average size of $I(v, b)$ is
\[ \frac{c^p}{cp} = \frac{c^{p-1}}{p}. \]

**Definition 3** We say $I(v, b)$ is large if $|I(v, b)| \geq 2pc^{p-2}$. 

Observe that $c$ is much larger than $p$. The powers of $c$ is the dominating factor. So $2pc^{p-2}$ is much smaller than the average size of $I(v, b)$. Thus intuitively, “most” of the $I(v, b)$’s should be large. So the next lemma is not a surprise.

**Lemma 3** There exists a vertex $v$ of $G$ such that 
\[ |\{b \in [c] : I(v, b) \text{ is large} \}| > c/2. \]

**Proof.** For each vertex $v$ of $G$, let $S(v) = \{b : I(v, b) \text{ is small}\}$. Assume to the contrary that for each $v$, $|S(v)| \geq c/2$. Let 
\[ \mathcal{L} = \{\phi \in K_c^{G^a} : \forall v \in V(G), \phi(v) \in S(v)\}. \]

Then 
\[ |\mathcal{L}| = \prod_{v \in V(G)} |S(v)| \geq \left(\frac{c}{2}\right)^p. \]

For any $\phi \in \mathcal{L}$, if $\phi \in I(v, b)$, then $I(v, b)$ is small. Thus 
\[ \mathcal{L} \subset \bigcup_{v \in V(G), b \in [c], I(v, b) \text{ is small}} I(v, b). \]

Therefore $|\mathcal{L}| < p \times c \times 2pc^{p-2} = 2p^2c^{p-1}$. But then 
\[ \left(\frac{c}{2}\right)^p < 2p^2c^{p-1} \]

which implies that $c < 2^{p+1}p^2$. But by our choice of $c$, we have $c = 4q + 2 > 4q \geq 2^{p+1}p^2$, a contradiction. 

Let $v$ be a vertex of $G$ for which $|\{b \in [c] : I(v, b) \text{ is large} \}| > c/2$. For $t \in \{2q + 1, 2q + 2, \ldots, 4q + 2\}$, let $\mu_t \in K_c^{G[K_t]}$ be defined as 
\[ \mu_t(x, i) = \begin{cases} 
  i, & \text{if } d_G(x, v) = 0, 2, \\
  q + i, & \text{if } d_G(x, v) = 1, \\
  t, & \text{if } d_G(x, v) \geq 3.
\end{cases} \]

Observe that $\mu_t$ are not simple maps. These will be the only non-simple maps used in the proof.
Claim 4 The set of maps \( \{ \mu_t : t = 2q + 1, 2q + 2, \ldots, 4q + 2 \} \) induces a clique in \( K^c[G[K_q]] \).

**Proof.** Assume to the contrary that for some \( t \neq t' \), \( \mu_t \neq \mu_{t'} \). Then there is an edge \((x, i)(y, j)\) of \( G[K_q] \) such that \( \mu_t(x, i) = \mu_{t'}(y, j) \). Let \( \alpha = \mu_t(x, i) = \mu_{t'}(y, j) \).

Then \( \alpha \in Im(\mu_t) \cap Im(\mu_{t'}) \subseteq \{i, q + i, t\} \cap \{j, q + j, t'\} \). As \( t \neq t' \), we conclude that \( i = j \) and \( \alpha = i \) or \( q + i \). Since \((x, i), (y, i)\) are distinct adjacent vertices, we conclude that \( x \neq y \) and \( xy \in E(G) \). If \( \alpha = i \), then \( d_G(x, v), d_G(y, v) \in \{0, 2\} \) implies that \( G \) has a 3-cycle or a 5-cycle, contrary to the assumption that \( G \) has girth 6. If \( \alpha = q + i \), then \( d_G(v, x) = d_G(v, y) = 1 \), and \( G \) has a 3-cycle, again a contradiction. This completes the proof of Claim 4. ■

So maps \( \{ \mu_t : t = 2q + 1, 2q + 2, \ldots, 4q + 2 \} \) are coloured by distinct colours, and hence there exists \( t \) such that \( \Psi(\mu_t) \notin \{1, 2, \ldots, 2q\} \). As \( \Psi(\mu_t) \in Im(\mu_t) = \{1, 2, \ldots, q, t\} \), we have \( \Psi(\mu_t) = t \).

Since \( |\{b \in [c] : I(v, b) \text{ is large }\}| > c/2 = 2q + 1 \), there is a colour \( b \in [c] - \{1, 2, \ldots, 2q, t\} \) such that \( I(v, b) \) is large. Let \( \theta \in K^c[G^o] \) be defined as follows:

\[
\theta(x) = \begin{cases} 
  b, & \text{if } d_G(x, v) \geq 2, \\
  t, & \text{if } d_G(x, v) \leq 1.
\end{cases}
\]

Claim 5 \( \theta \sim \mu_t \).

**Proof.** Assume to the contrary that \( \theta \neq \mu_t \). Then there is an edge \((x, i)(y, j)\) of \( E(G[K_q]) \) such that \( \theta(x) = \theta(x, i) = \mu_t(y, j) \). (Note that \( \theta(x, i) = \theta(x) \) as \( \theta \) is a simple map). As \( Im(\theta) \cap Im(\mu_t) = \{t\} \), we conclude that \( \theta(x) = \mu_t(y, j) = t \). But then \( d_G(v, x) \leq 1 \) and \( d_G(v, y) \geq 3 \), and hence \( x \neq y \) and \( xy \notin E(G) \), contrary to the assumption that \((x, i)(y, j) \in E(G[K_q]) \).

Thus \( \Psi(\theta) \neq \Psi(\mu_t) = t \). As \( \Psi(\theta) \in Im(\theta) \), we conclude that \( \Psi(\theta) = b \).

Claim 6 For any \( \phi \in I(v, b) \), there exists a vertex \( x \neq v \) such that \( \phi(x) \in \{b, t\} \).

**Proof.** Assume \( \phi \in I(v, b) \). By definition \( \Psi(\phi) = b = \phi(v) \). So \( \Psi(\phi) = \Psi(\theta) \). Hence \( \phi \neq \theta \).

So there is an edge \( xy \in E(G^o) \) such that \( \phi(x) = \theta(y) \). If \( x = v \), then \( \theta(y) = \phi(v) = b \). By definition of \( \theta \), we have \( d_G(y, v) \geq 2 \). Hence \( xy \) cannot be an edge in \( G^o \), a contradiction. So \( x \neq v \). As \( \phi(x) = \theta(y) \in \{b, t\} \), this completes the proof of the claim. ■

For each \( x \neq v \), let \( J_x = \{ \phi(x) \in I(v, b) : \phi(x) \in \{b, t\} \} \).

As \( \phi(v) = b \) for \( \phi \in I(v, b) \), we conclude that \( |J_x| \leq 2c^{n-2} \). By Claim 6 \( I(v, b) = \cup_{x \in V(G) - \{v\}} J_x \). So \( |I(v, b)| \leq 2(n - 1)c^{n-2} \), contrary to the assumption that \( I(v, b) \) is large. This completes the proof of Theorem. ■
Remark. The key part of the proof of Theorem 1 is to show that \( K_c^{G[K_q]} \) is not \( c \)-colourable. For each vertex \( v \) of \( G \), for \( t \in \{2q+1, 2q+2, \ldots, 4q+2\} \), let

\[
\mu_{v,t}(x) = \begin{cases} 
  i, & \text{if } d_G(x,v) = 0, 2; \\
  q + i, & \text{if } d_G(x,v) = 1; \\
  t, & \text{if } d_G(x,v) \geq 3;
\end{cases}
\]

Let \( H \) be the subgraph of \( K_c^{G[K_q]} \) induced by

\[ V(K_c^{G^n}) \cup \{ \mu_{v,t} : v \in V(G), t \in \{2q+1, 2q+2, \ldots, 4q+2\} \}. \]

What we have proved is that the subgraph \( H \) of \( K_c^{G[K_q]} \) is not \( c \)-colourable. Note that \( H \) is a very tiny fraction of \( K_c^{G[K_q]} \), although \( H \) by itself is a huge graph. Possibly, the chromatic number of \( K_c^{G[K_q]} \) is much larger than \( c \).

4 The Poljak-Rödl function

The Poljak-Rödl function is defined in [9]:

\[ f(n) = \min \{ \chi(G \times H) : \chi(G), \chi(H) \geq n \}. \]

Hedetniemi’s conjecture is equivalent to say that \( f(n) = n \) for all positive integer \( n \). Shitov’s Theorem says that for sufficiently large \( n \), \( f(n) \leq n - 1 \). Using Shitov’s result, Tardif and Zhu [16] proved that \( f(n) \leq n - (\log n)^{1/4} \). Tardif and Zhu asked in [16] if there is a positive constant \( \epsilon \) such that \( f(n) \leq (1 - \epsilon)n \) for sufficiently large \( n \). This question was answered in affirmative by He and Wigderson [4] with \( \epsilon \approx 10^{-9} \). On the other hand, in [16], Tardif and Zhu proved that if a special case of a conjecture of Stahl [12] concerning the multi-chromatic number of Kneser graph is true, then we have \( \lim_{n \to \infty} \frac{f(n)}{n} \leq \frac{1}{2} \).

In this note, we that the conclusion \( \lim_{n \to \infty} \frac{f(n)}{n} \leq \frac{1}{2} \) holds without assuming Stahl’s conjecture.

Theorem 7 For \( d \geq 1 \), let \( G \) be a graph of girth 6 and with \( \chi_f(G) \geq 8.1d \). Let \( p = |V(G)| \), \( q \geq 2^{p-1}p^2 \) and \( c = 4q + 2 \). Then \( \chi(G[K_q]) \geq 2dc - 2c + 2 \) and \( \chi(K_c^{G[K_q]}) \geq 2dc - 2c + 2 \). Consequently, \( f(2dc - 2c + 2) \leq dc \).

Proof. Similarly as in the proof of Theorem 1, \( \chi(G[K_q]) \geq \chi_f(G)q \geq 8.1dq \geq 2dc > 2dc - 2c + 2 \). Now we show that \( \chi(K_c^{G[K_q]}) \geq 2dc - 2c + 2 \).

Assume \( \Psi \) is a \( (dc + t) \)-colouring of \( K_c^{G[K_q]} \) with colour set \([dc + t]\). We shall show that \( dc + t \geq 2dc - 2c + 2 \), i.e., \( t \geq dc - 2c + 2 \). Let \( S = [dc + t] - [dc] \). The colours in \([dc]\) are called primary colours and colours in \( S \) are called secondary colours. So we have \( t = |S| \) secondary colours.
Similarly as before, we may assume that $\Psi(g_i) = i$ for $i \in [dc]$. Then for any map $\phi \in K_{dc}^{G[K_q]}$, if $i \notin Im(\phi)$, then $\phi \sim g_i$ and $\Psi(\phi) \neq i$. Thus for any $\phi \in K_{dc}^{G[K_q]}$, $\Psi(\phi) \in Im(\phi) \cup S$.

For positive integers $m \geq 2k$, $K(m, k)$ is the Kneser graph whose vertices are $k$-subsets of $[m]$, and for two $k$-subsets $A, B$ of $[m]$, $A \sim B$ if $A \cap B = \emptyset$. It was proved by Lovász in [7] that $\chi(K(m, k)) = m - 2k + 2$.

For a $c$-subset $A$ of $[cd]$, let $H_A$ be the subgraph of $K_{cd}^{G[K_q]}$ induced by

$$\{\phi \in V(K_{cd}^{G[K_q]}): Im(\phi) \subseteq A\}.$$ 

Then $H_A$ is isomorphic to $K_c^{G[K_q]}$. By Theorem [1] $|\Psi(H_A)| \geq c + 1$. As $|Im(\phi)| = c$, $\Phi(H_A)$ contains at least one secondary colour. Let $\tau(A)$ be an arbitrary secondary colour contained in $\Psi(H_A)$.

If $A, B$ are $c$-subsets of $[dc]$ and $A \cap B = \emptyset$, then every vertex in $H_A$ is adjacent to every vertex in $H_B$. Hence $\Psi(H_A) \cap \Psi(H_B) = \emptyset$. In particular, $\tau(A) \neq \tau(B)$. Thus $\tau$ is a proper colouring of the Kneser graph $K(dc, c)$. As $\chi(K(dc, c)) = dc - 2c + 2$, we conclude that $t = |S| \geq dc - 2c + 2$. This completes the proof of Theorem [7].

For a positive integer $d$, let $p = p(d)$ be the minimum number of vertices of a graph $G$ with girth 6 and $\chi_f(G) \geq 8.1d$. It follows from Theorem [7] that for any integer $q \geq p^{2p-1}$, $f(2(d-1)(4q+2)+2) \leq (4q+2)d$. As $f(n)$ is non-decreasing, for integers $n$ in the interval $[2(d-1)(4q+2)+2, 2(d-1)(4q+6)+2]$, we have $f(n) \leq (4q+6)d$.

Hence for all integers $n \geq 2(4q+2)(d-1)+2$,

$$\frac{f(n)}{n} \leq \frac{(4q+6)d}{2(4q+2)(d-1)+3} = \frac{1}{2} + \frac{4q+4d+1}{2(d-1)(4q+2)+2}.$$

Note that if $d \to \infty$, then $p = p(d)$ goes to infinity, and hence $q \geq p^{32p}$ goes to infinity. Hence

$$\lim_{n \to \infty} \frac{f(n)}{n} \leq \frac{1}{2}.$$

Theorem [7] improves the result of He and Wigderson [4]. However, He and Wigderson uses a modification of Shitov’s method, which might be of independent interest.

In the proof of Theorem [7], we actually showed that a tiny subgraph of $K_{dc}^{G[K_q]}$ has chromatic number close to $2dc$. It is not clear if the remaining part of the graph $K_{dc}^{G[K_q]}$ can be used to show that this graph actually have a much larger chromatic number. We observe that if one can show that the chromatic number of $K_{dc}^{G[K_q]}$ is more than $kdc$ for some positive integer $k$, then Stahl’s conjecture implies that $\lim_{n \to \infty} \frac{f(n)}{n} \leq \frac{1}{k+1}$.

5 Lower bound for $f(n)$

The breakthrough result of Shitov leads to improvement of the upper bound for the function $f(n)$. On the other hand, the only known lower bound for $f(n)$ is that $f(n) \geq 4$. 
for $n \geq 4$. We do not know if $f(n)$ is bounded by a constant or not. What we do know is that if $f(n)$ is bounded by a constant, then the smallest such constant is at most 9.

To prove this result, we need to consider the product of digraphs. For a digraph $D$, we use $A(D)$ to denote the set of arcs of $D$. An arc in $D$ is either denoted by an ordered pair $(x, y)$, or by an arrow $x \rightarrow y$. Digraphs are allowed to have digons.

Assume $D_1, D_2$ are digraphs. The product $D_1 \times D_2$ has vertex set $V(D_1) \times V(D_2)$, with $(x, y) \rightarrow (x', y')$ be an arc if and only if $(x, x')$ is an arc in $D_1$ and $(y, y')$ is an arc in $D_2$. The chromatic number of a digraph $D$ is defined to be $\chi(D)$, where $D$ is the underline graph of $D$, i.e., obtained from $D$ by replacing each arc $(x, y)$ with an edge $xy$. Given a digraph $D$, let $D^{-1}$ be the digraph obtained from $D$ by reversing the direction of all its arcs. It is easy to see that for any digraphs $D_1, D_2$,

$$D_1 \times D_1 = (D_1 \times D_2) \cup (D_1 \times D_2^{-1}).$$

Hence

$$\chi(D_1 \times D_1) \leq \chi(D_1 \times D_2) \times \chi(D_1 \times D_2^{-1}).$$

Let

$$g(n) = \min\{\chi(D_1 \times D_2) : \chi(D_1), \chi(D_2) \geq n\},$$

$$h(n) = \min\{\max\{\chi(D_1 \times D_2), \chi(D_1 \times D_2^{-1})\} : \chi(D_1), \chi(D_2) \geq n\}.$$  

It follows from the discussion above that

$$g(n) \leq h(n) \leq f(n) \leq h(n)^2.$$  

The following result was proved by Poljak and Rödl in [9].

**Theorem 8** If $g(n)$ (respectively $h(n)$) is bounded by a constant, then the smallest such constant is at most 4. Consequently, if $f(n)$ is bounded by a constant, then the smallest such constant is at most 16.

**Proof.** For a graph $D$, let $\partial(D)$ be the digraph with vertex set $A(D)$, with $(x, y) \rightarrow (x', y')$ be an arc of $\partial(D)$ if and only if $y = x'$. In particular, if $(x, y), (y, x)$ is a digon in $D$, then $(x, y) \rightarrow (y, x)$ and $(y, x) \rightarrow (x, y)$ is a digon in $\partial(D)$.

**Lemma 9** For any digraph $D$,

$$\min\{k : 2^k \geq \chi(D)\} \leq \chi(\partial(D)) \leq \min\{k : \left\lceil \frac{k}{2} \right\rceil \geq \chi(D)\}.$$  

**Proof.** If $\phi : V(\partial(D)) \rightarrow [k]$ is a proper colouring of $\partial(D)$, then for each vertex $v$ of $D$, let $\psi(v) = \{\phi(e) : e \in A^+(v)\}$, where $A^+(v)$ is the set of out-arcs at $v$. Then $\psi$ is a proper colouring of $D$ (with subsets of $[k]$ as colours). Indeed, if $e = (x, y)$ is an arcs of
D, then \( \phi(e) \in \psi(x) - \psi(y) \). So \( \psi(x) \neq \psi(y) \). The number of colours used by \( \psi \) is at most the number of subset of \([k]\), which is \( 2^k \).

If \( \psi : V(D) \to \binom{[k]}{k/2} \) is a proper colouring of \( D \) (where the colours are \([k/2]\)-subsets of \([k]\)), then for any arc \( e = (x, y) \) of \( D \), let \( \phi(e) \) be any integer in \( \psi(y) - \psi(x) \) (as \( \psi(y) \neq \psi(x) \), such an integer exists). Then if \( (x, y) \to (y, z) \) is an arc in \( \partial(D) \), then \( \phi(x, y) \in \psi(y) \) and \( \phi(y, z) \notin \psi(y) \). Hence \( \phi(x, y) \neq \phi(y, z) \). I.e., \( \phi \) is a proper colouring of \( \partial(D) \). This completes the proof of Lemma 9. \( \blacksquare \)

It follows easily from the definition that

\[
\partial(D_1 \times D_2) = \partial(D_1) \times \partial(D_2), \\
\partial(D^{-1}) = (\partial(D))^{-1}.
\]

Suppose \( g(n) \) is bounded and \( C \) is the smallest upper bound. As \( g(n) \) is non-decreasing, there is an integer \( n_0 \) such that \( g(n) = C \) for all \( n \geq n_0 \). Let \( n_1 = 2^{n_0} \), and let \( D_1, D_2 \) be digraphs with \( (\chi(D_1), \chi(D_2)) \geq n_1 \) and \( \chi(D_1 \times D_2) = C \). It follows from Lemma 9 that \( (\chi(\partial(D_1)), \chi(\partial(D_2))) \geq n_0 \) and hence \( \chi(\partial(D_1) \times \partial(D_2)) = \chi(\partial(D_1 \times D_2)) \geq C \). By Lemma 9 again, we have

\[
C \geq \chi(D_1 \times D_2) > \left( \frac{C - 1}{\lceil (C - 1)/2 \rceil} \right).
\]

This implies that \( C \leq 4 \).

The same argument shows that if \( h(n) \) is bounded by a constant, then the smallest such constant is at most 4. Since \( h(n) \leq f(n) \leq h(n)^2 \), if \( f(n) \) is bounded by a constant, then the smallest such a constant is at most 16. \( \blacksquare \)

Next we show that if \( g(n) \) (respectively, \( h(n) \)) is bounded by a constant, then the smallest such constant cannot be 4. Assume to the contrary that the smallest constant bound for \( g(n) \) is 4. Let \( n_0 \) be the integer given above, and let \( n_1 = 2^{n_0}, n_2 = 2^{n_1} \). Then \( g(n_2) = g(n_1) = g(n_0) = 4 \). Let \( D_1, D_2 \) be two digraphs with \( \chi(D_1), \chi(D_2) \geq n_2 \) and \( \chi(D_1 \times D_2) = 4 \). The same argument as above shows that

\[
\chi(\partial(D_1 \times D_2)) = 4.
\]

However, we shall show that if \( \chi(D) \leq 4 \), then \( \chi(\partial(\partial(D))) \leq 3 \). Let \( \tilde{K}_4 \) be the complete digraph with vertex set \( \{1, 2, 3, 4\} \), with \( (i, j) \) be an arc for any distinct \( i, j \in \{1, 2, 3, 4\} \). If \( \chi(D) = 4 \), then \( D \) admits a homomorphism to \( \tilde{K}_4 \). Hence \( \partial(\partial(D)) \) admits a homomorphism to \( \partial(\partial(\tilde{K}_4)) \). So it suffices to show that \( \partial(\partial(\tilde{K}_4)) \leq 3 \). In 1990, I was a Ph.D. student at The University of Calgary. After reading the paper by Poljak and Rödl 9, I found a 3-colouring of \( \partial(\partial(\tilde{K}_4)) \) by brute force. I was happy to tell this to my supervisor Professor Norbert Sauer, who then told the result to Duffus. Then I learned from Duffus the following elegant 3-colouring of \( \partial(\partial(\tilde{K}_4)) \), given earlier by Schelp and was not published.
Each vertex of $\partial(\partial(\tilde{K}_4))$ is a sequence $ijk$ with $i,j,k \in [k]$, $i \neq j$, $j \neq k$ (but $i$ may equal to $k$). Let

$$c(ijk) = \begin{cases} 
 j, & \text{if } j \neq 4, \\
 s, & \text{if } j = 4 \text{ and } s \in \{1,2,3\} - \{i,k\}
\end{cases}$$

Then it is easy to verify that $c$ is a proper 3-colouring of $\partial(\partial(\tilde{K}_4))$. This completes the proof that $g(n)$ is either bounded by 3 or goes to infinity. Similarly, $h(n)$ is either bounded by 3 or goes to infinity, and consequently, $f(n)$ is either bounded by 9 or goes to infinity.

Later I learned from Hell that Poljak also obtained this strengthening independently and that was published in 1992 [8].

Tardif and Wehlau [15] proved that $f(n)$ is bounded if and only if $g(n)$ is bounded. The fractional version of Hedetniemi’s conjecture was proved in [19]: For any two graphs $G$ and $H$, $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$. Thus if $f(n)$ is bounded by 9, and $G$ and $H$ are $n$-chromatic graphs with $\chi(G \times H) \leq 9$, then at least one of $G$ and $H$ has fractional chromatic number at most 9.

In [19], I defined the following Poljak-Rödl type function:

$$\psi(n) = \min\{\chi(G \times H) : \chi_f(G), \chi(H) \geq n\}.$$ 

I proposed a weaker version of Hedetniemi’s conjecture, which is equivalent says that $\psi(n) = n$ for all positive integer $n$. However, Shitov’s proof actually refutes this weaker version of Hedetniemi’s conjecture, as the graph $G$ used in the proof of Theorem [1] have large fractional chromatic number. The proof of Theorem [7] shows that

$$\lim_{n \to \infty} \frac{\psi(n)}{n} \leq \frac{1}{2}.$$ 

On the other hand, it follows from the definition that $f(n) \leq \psi(n)$. A natural question is the following:

**Question 10** Is $\psi(n)$ bounded by a constant? If $\psi(n)$ is bounded by a constant, what could be the smallest such constant?

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