SYMBOLIC DYNAMICS FOR THE GEODESIC FLOW
ON LOCALLY SYMMETRIC ORBIFOLDS OF RANK ONE

J. HILGER T AND A. D. POHL

Abstract. We present a strategy for a geometric construction of cross sections for the geodesic flow on locally symmetric orbifolds of rank one. We work it out in detail for $\Gamma \backslash \mathcal{H}$, where $\mathcal{H}$ is the upper half plane and $\Gamma = \text{PGL}(p)$, $p$ prime. Its associated discrete dynamical system naturally induces a symbolic dynamics on $\mathbb{R}$. The transfer operator produced from this symbolic dynamics has a particularly simple structure.

1. Introduction

We consider the upper half plane $\mathcal{H} := \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \}$ with the Riemannian metric given by the line element $ds^2 = y^{-2}(dx^2 + dy^2)$ as a model for two-dimensional real hyperbolic space. The group of orientation-preserving isometries can be identified with $\text{PSL}(2, \mathbb{R})$ via the action

$$g \cdot z = \frac{az + b}{cz + d}$$

for $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$ and $z \in \mathcal{H}$. The geodesic compactification $\overline{\mathcal{H}}$ of $\mathcal{H}$ will be identified with the one-point compactification of the closure of $\mathcal{H}$ in $\mathbb{C}$, hence $\overline{\mathcal{H}} = \{ z \in \mathbb{C} \mid \text{Im} \, z \geq 0 \} \cup \{ \infty \}$. The action of $\text{PSL}(2, \mathbb{R})$ extends continuously to $\partial \mathcal{H} = \mathbb{R} \cup \{ \infty \}$.

The geodesics on $\mathcal{H}$ are the semicircles centered on the real line and the vertical lines. All geodesics shall be oriented and parametrized by arc length. The (unit speed) geodesic flow on $\mathcal{H}$ is the dynamical system

$$\Phi: \begin{cases} \mathbb{R} \times \mathcal{H} & \to \mathcal{H} \\ (t, v) & \mapsto \gamma'_v(t), \end{cases}$$

2000 Mathematics Subject Classification. Primary: 37D40, Secondary: 11J70.
Key words and phrases. cross section, symbolic dynamics, transfer operator.
where $SH$ denotes the unit tangent bundle (sphere bundle) of $H$ and $\gamma_v : \mathbb{R} \to H$ is the unique (unit speed) geodesic that satisfies $\gamma'_v(0) = v$.

Let $\Gamma$ be a properly discontinuous subgroup of $\text{PSL}(2, \mathbb{R})$. The unit tangent bundle $SY$ of the locally symmetric orbifold $Y := \Gamma \backslash H$ shall be identified with $\Gamma \backslash SH$, where we consider the induced $\Gamma$-action on $SH$.

Let $\pi : H \to Y$ and $\pi : SH \to SY$ denote the canonical projection maps. The geodesics on $H$ are in bijection with the $\Gamma$-equivalence classes of geodesics on $Y$, and the geodesic flow on $Y$ is given by

$$\Phi := \pi \circ \Phi \circ (\text{id} \times \pi^{-1}) : \mathbb{R} \times SY \to SY.$$ 

Here, $\pi^{-1}$ shall be read as an arbitrary section of $\pi$. The definition of $\Phi$ is independent of this choice. For the modular group $\text{PSL}(2, \mathbb{Z})$, Series [6] geometrically constructed an amazingly simple cross section for the geodesic flow on the modular surface $\text{PSL}(2, \mathbb{Z}) \backslash H$. Its associated discrete dynamical system is naturally related to a symbolic dynamics on $\mathbb{R}$, namely by continued fraction expansions of the limit points of the geodesics. The Gauss map is a generating function for the future part of this symbolic dynamics. In [5], Mayer investigated the transfer operator $L_\beta$ with parameter $\beta$ of the Gauss map. His work and that of Lewis and Zagier [4] have shown that there is an isomorphism between the space of Maass cusp forms for $\text{PSL}(2, \mathbb{Z})$ with eigenvalue $\beta(1 - \beta)$ and the space of real-analytic eigenfunctions of $L_\beta$ that have eigenvalue $\pm 1$ and satisfy certain growth conditions.

In this article we present, for the examples $\Gamma = \text{PGL}_0(p)$, $p$ prime, a geometric construction of a cross section $\hat{C}_{\text{red}}$ for the geodesic flow on $\Gamma \backslash H$ such that, as in Series’ work, its associated discrete dynamical system $(\hat{C}_{\text{red}}, R)$ naturally gives rise to a symbolic dynamics on $\mathbb{R}$. More precisely, $(\hat{C}_{\text{red}}, R)$ is conjugate to a discrete dynamical system $(\hat{D}, \hat{F})$, where $\hat{D}$ is an open subset of $\mathbb{R} \times \mathbb{R}$. The domain $\hat{D}$ will be seen to naturally decompose into a finite disjoint union $\bigcup_{\alpha \in \mathbb{A}} \hat{D}_{\alpha}$ of open subsets $\hat{D}_{\alpha}$ such that $\hat{F}|_{\hat{D}_{\alpha}} : \hat{D}_{\alpha} \to \hat{F}(\hat{D}_{\alpha})$ is a diffeomorphism of the form $(x, y) \mapsto (h_{\alpha}^{-1}x, h_{\alpha}^{-1}y)$ for some $h_{\alpha} \in \Gamma$. Hence we get a canonical symbolic dynamics for $\Phi$ on the alphabet $\mathbb{A}$ or $\{h_{\alpha} \mid \alpha \in \mathbb{A}\}$. If $\text{pr}_x$ denotes the projection onto the first component, then the sets $D_{\alpha} := \text{pr}_x(\hat{D}_{\alpha})$ are pairwise disjoint. This means that $\hat{F}$ is effectively determined by the map $F : D \to D$, where $D := \text{pr}_x(\hat{D})$ and $F|_{D_{\alpha}} := h_{\alpha}^{-1}$. In addition, $F$ is a generating function for the future part of the symbolic dynamics.

---

1The concepts from symbolic dynamics are explained in Section 2.
Because of the particular structure of $F$, its associated transfer operator can be described completely by a finite number of group elements in $\Gamma$.

For the modular surface, this construction gives a two-term transfer operator from which one can easily read off the three-term functional equation used in the proof of the Lewis-Zagier isomorphism mentioned above and whose eigenfunctions with eigenvalue 1 that satisfy some growth conditions are in bijection with the Maass cusp forms.

There are two main methods to construct cross sections and symbolic dynamics. The geometric coding consists in taking a fundamental domain for $\Gamma$ in $\mathbb{H}$ with side pairing and the sequences of sides cut by a geodesic as coding sequences. The cross section is a set of unit tangent vectors based at the boundary of the fundamental domain. In general, it is very difficult, if not impossible, to find a conjugate dynamical system on the boundary. In contrast, the arithmetic coding starts with a symbolic dynamics on (parts of) the boundary which has a generating function for the future part similar to $F$ from above and asks for a cross section that reproduces this function. Usually, writing down such a cross section is a non-trivial task. A good overview of geometric and arithmetic coding is the survey article [3].

Our method, which we call cusp expansion, is sketched in the following. The description applies to $\Gamma = \text{PGL}_0(p)$, $\Gamma = \text{PSL}(2, \mathbb{Z})$ and some other groups as will be discussed in Section 4. We fix a so-called Ford fundamental domain $F$ for $\Gamma$ in $\mathbb{H}$. To each vertex $v$ (that is, an inner vertex or a representative other than $\infty$ of a cusp) of $F$ we assign a set $A(v)$, which we refer to as a precell. Two precells overlap at most at boundaries, and the union of all precells coincides with $F$. Then we glue together some $\Gamma$-translates of precells to obtain a cell $B(v)$ assigned to $v$. The purpose of these cells is to construct a family of finite-sided $n$-gons with all vertices in $\partial \mathbb{H}$ such that each cell has two vertical boundary components (in other words, $\infty$ is a vertex of each cell) and each non-vertical boundary component of a cell is a $\Gamma$-translate of a vertical boundary component of some cell. Further, the family of all cells shall give a tiling of $\mathbb{H}$ in the sense that the $\Gamma$-translates of all cells cover $\mathbb{H}$ and two $\Gamma$-translates of cells are either disjoint or identical or coincide at exactly one boundary component. Let $\tilde{C}$ denote the set of unit tangent vectors based at the boundary of some cell but that are not tangent to this boundary. We will see that $\tilde{C} := \pi(\tilde{C})$ is a cross section. In general, $(\tilde{C}, R)$ is not conjugate to a discrete dynamical system $(\tilde{D}, \tilde{F})$ with $\tilde{D} \subseteq \mathbb{R} \times \mathbb{R}$. But $\tilde{C}$ contains a subset $\tilde{C}_{\text{red}}$ for which $(\tilde{C}_{\text{red}}, R)$ is naturally conjugate to a dynamical system on some subset of $\mathbb{R} \times \mathbb{R}$. The set $\tilde{C}_{\text{red}}$
is itself a cross section and can be constructed effectively from \( \hat{C} \). For the proofs of these properties we extend the notions of precells and cells to \( SH \). Each precell \( A(v) \) induces a precell \( \tilde{A}(v) \) in \( SH \) in a geometric way. From these precells we construct (in an effective way which, however, involves choices) a finite family \( \{ \tilde{B}(v_k) \}_{k \in A} \) of cells in \( SH \) such that each \( \tilde{B}(v_k) \) projects to a \( \Gamma \)-translate of a cell in \( H \). The union of all \( \tilde{B}(v_k) \) turns out to be a fundamental set for \( \Gamma \) in \( SH \). This property and the interplay of the cells in \( SH \) and \( H \) imply the properties of \( \hat{C} \) and \( \hat{C}_{\text{red}} \).

The decomposition of \( F \) into precells and the construction of cells from precells is based on [7]. Our constructions differ from Vulkh’s in two aspects: one is the way in which cusps are treated, the other difference is that we extend the considerations to cells in the unit tangent bundle.

Section 2 contains the necessary background on symbolic dynamics. In Section 3 we develop the cusp expansion for \( \operatorname{PSL}(2, \mathbb{Z}) \). We end, in Section 4, with a brief discussion on potential extensions of this method to other locally symmetric orbifolds of rank one.

2. Symbolic dynamics

Let \( \hat{C} \) be a subset of \( SY \). A geodesic \( \hat{\gamma} \) on \( Y \) is said to intersect \( \hat{C} \) infinitely often in future if there is a sequence \( (t_n)_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} t_n = \infty \) and \( \hat{\gamma}'(t_n) \in \hat{C} \) for all \( n \in \mathbb{N} \). Analogously, \( \hat{\gamma} \) is said to intersect \( \hat{C} \) infinitely often in past if we find a sequence \( (t_n)_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} t_n = -\infty \) and \( \hat{\gamma}'(t_n) \in \hat{C} \) for all \( n \in \mathbb{N} \). Let \( \mu \) be a measure on the space of geodesics on \( Y \). A cross section \( \hat{C} \) (w.r.t. \( \mu \)) for the geodesic flow \( \hat{\Phi} \) is a subset of \( SY \) such that

\[(C1) \ \text{\( \mu \)-almost every geodesic } \hat{\gamma} \text{ on } Y \text{ intersects } \hat{C} \text{ infinitely often in past and future,}\]
\[(C2) \text{ each intersection of } \hat{\gamma} \text{ and } \hat{C} \text{ is discrete in time: if } \hat{\gamma}'(t) \in \hat{C}, \text{ then there is } \epsilon > 0 \text{ such that } \hat{\gamma}'((t - \epsilon, t + \epsilon)) \cap \hat{C} = \{ \hat{\gamma}'(t) \}.\]

Each Borel probability measure on \( SY \) which is invariant under \( \{ \hat{\Phi}_t \}_{t \in \mathbb{R}} \) induces a measure on the space of geodesics. We shall see that the cross sections constructed in Section 3 are indeed cross sections w.r.t. any choice of a measure of this type.

Let \( \text{pr}: SY \to Y \) denote the canonical projection on base points. If \( \text{pr}(\hat{C}) \) is a totally geodesic submanifold of \( Y \) and \( \hat{C} \) does not contain elements tangent to \( \text{pr}(\hat{C}) \), then \( \hat{C} \) automatically satisfies \((C2)\).
Suppose that $\hat{C}$ is a cross section for $\hat{\Phi}$. If $\hat{C}$ in addition satisfies the property that each geodesic intersecting $\hat{C}$ at all intersects it infinitely often in past and future, then $\hat{C}$ will be called a strong cross section, otherwise a weak cross section. Clearly, every weak cross section contains a strong cross section.

The first return map of $\hat{\Phi}$ w.r.t. a strong cross section $\hat{C}$ is the map

$$R: \begin{cases} \hat{C} \to \hat{C} \\ \hat{v} \mapsto \hat{\gamma}_v(t_0) \end{cases}$$

where $\pi(v) = \hat{v}$, $\pi(\gamma_v) = \hat{\gamma}_v$, the geodesic $\gamma_v$ is chosen as in the definition of $\Phi$, and $\hat{\gamma}_v(t) \notin \hat{C}$ for all $t \in (0, t_0)$. In this case, $t_0$ is the first return time of $\hat{v}$ resp. of $\hat{\gamma}_v$. For a weak cross section $\hat{C}$, the first return map can only be defined on a subset of $\hat{C}$. In general, this subset is larger than the maximal strong cross section contained in $\hat{C}$.

Suppose that $\hat{C}$ is a strong cross section and let $A$ be an at most countable set. Decompose $\hat{C}$ into a disjoint union $\bigcup_{\alpha \in A} \hat{C}_\alpha$. To each $\hat{v} \in \hat{C}$ we assign the (two-sided infinite) coding sequence $(a_n)_{n \in \mathbb{Z}} \in A^\mathbb{Z}$ defined by

$$a_n := \alpha \quad \text{iff} \quad R^n(\hat{v}) \in \hat{C}_\alpha.$$ 

Note that $R$ is invertible and let $\Lambda$ be the set of all sequences that arise in this way. Then $\Lambda$ is invariant under the left shift $\sigma: A^\mathbb{Z} \to A^\mathbb{Z}$, $(\sigma((a_n)_{n \in \mathbb{Z}}))_k := a_{k+1}$. Suppose that the map $\hat{C} \to \Lambda$ is also injective, which it will be in our case. Then we have the natural map $\text{Cod}: \Lambda \to \hat{C}$ which maps a coding sequence to the element in $\hat{C}$ it was assigned to. Obviously, $R \circ \text{Cod} = \text{Cod} \circ \sigma$. The pair $(\Lambda, \sigma)$ is called a symbolic dynamics for $\hat{\Phi}$. If $\hat{C}$ is only a weak cross section and hence $R$ is only partially defined, then $\Lambda$ also contains one- or two-sided finite coding sequences.

Let $C'$ be a set of representatives for the cross section $\hat{C}$, that is, $C' \subseteq SH$ and $\pi|_{C'}$ is a bijection $C' \to \hat{C}$. For each $v \in C'$ let $\gamma_v$ be the geodesic determined by $v$. Define $\tau: \hat{C} \to \partial H \times \partial H$ by $\tau(\hat{v}) := (\gamma_v(\infty), \gamma_v(-\infty))$ where $v = (\pi|_{C'})^{-1}(\hat{v})$. For some cross sections $\hat{C}$ it is possible to choose $C'$ in such a way that $\tau$ is a bijection between $\hat{C}$ and some subset $\hat{D}$ of $\mathbb{R} \times \mathbb{R}$. In this case the dynamical system $(\hat{C}, R)$ is conjugate to $(\hat{D}, \hat{F})$, where $\hat{F} := \tau \circ R \circ \tau^{-1}$ is the induced selfmap on $\hat{D}$ (partially defined if $\hat{C}$ is only a weak cross section). Moreover, to construct a symbolic dynamics for $\hat{\Phi}$, one can start with a decomposition of $\hat{D}$ into pairwise disjoint subsets $\hat{D}_\alpha$, $\alpha \in A$. 


Finally, let \((\Lambda, \sigma)\) be a symbolic dynamics with alphabet \(A\). Suppose that we have a map \(i: \Lambda \to D\) for some \(D \subseteq \mathbb{R}\) such that \(i(\{a_n\}_{n \in \mathbb{Z}})\) depends only on \((a_n)_{n \in \mathbb{N}_0}\), a (partial) selfmap \(F: D \to D\), and a decomposition of \(D\) into a disjoint union \(\bigcup_{a \in A} D_a\) such that
\[
F(i(\{a_n\}_{n \in \mathbb{Z}})) \in D_a \iff a_1 = a
\]
for all \((a_n)_{n \in \mathbb{Z}} \in \Lambda\). Then \(F\), more precisely the triple \((F, i, (D_a)_{a \in A})\), is called a generating function for the future part of \((\Lambda, \sigma)\). If such a generating function exists, then the future part of a coding sequence is independent of the past part.

3. Cusp expansions

We consider the groups
\[
\Gamma := P\Gamma_0(p) := \{ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{PSL}(2, \mathbb{Z}) \mid c \equiv 0 \mod p \}
\]
for \(p\) prime. A subset \(F\) of \(H\) is a fundamental region for \(\Gamma\) in \(H\) if \(F\) is open, \(gF \cap F = \emptyset\) for all \(g \in \Gamma \setminus \{\text{id}\}\) and \(H = \bigcup_{g \in \Gamma} gF\). If, in addition, \(F\) is connected, it is a fundamental domain. A fundamental set for \(\Gamma\) in \(H\) is a subset of \(H\) which contains precisely one representative of each \(\Gamma\)-orbit. Clearly, each fundamental region is contained in a fundamental set. Changing \(H\) into \(SH\) in these definitions, one gets the notions of fundamental region, domain and set for \(\Gamma\) in \(SH\). The isometric sphere of an element \(g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma\) is the set \(I(g) := \{ z \in H \mid |cz + d| = 1 \}\), its exterior is defined by \(\text{ext} I(g) := \{ z \in H \mid |cz + d| > 1 \}\). Note that \(I(g)\) is a (complete) geodesic arc if \(g\) does not fix \(\infty\). The stabilizer of \(\infty\) in \(\Gamma\) is \(\Gamma_\infty = \{ (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) \mid b \in \mathbb{Z} \}\). Hence \(F_\infty := \{ z \in H \mid 0 < \text{Re} z < 1 \}\) is a fundamental domain for \(\Gamma_\infty\) in \(H\). Following through the proofs of Thm. 22 in [1], we find that
\[
F := F_\infty \cap \bigcap_{g \in \Gamma \setminus F_\infty} \text{ext} I(g)
\]
is a fundamental domain for \(\Gamma\) in \(H\). To construct \(F\) if suffices to consider the isometric spheres \(I_q := \{ z \in H \mid |pz - q| = 1 \}\) for \(q = 1, \ldots, p - 1\). Then \(F = F_\infty \cap \bigcap_{q=1}^{p-1} \text{ext} I_q\). From \(F\) we can read off that \(\Gamma\) has two cusps, represented by \(\infty\) and \(v_0 := 0\), see Fig. 1. The points \(v_k := \frac{2k+1}{2p} + \frac{i\sqrt{3}}{2p}\) for \(k = 1, \ldots, p - 2\) will be called the inner vertices of \(F\). Note that \(v_k\) is the intersection point of \(I_k\) and \(I_{k+1}\). Let \(v_{p-1} := 1\) denote the other representative of the cusp \(0\) that is seen by \(F\). Further let \(m_k\) denote the element of \(I_k\) with maximal imaginary part, that is, \(m_k := \frac{k}{p} + \frac{i}{p}\).
For \( a, b \in \mathbb{H} \) let \([a, b]\) denote the geodesic arc in \( \mathbb{H} \) from \( a \) to \( b \) containing the points \( a \) and \( b \), let \([a, b)\) be the geodesic arc from \( a \) to \( b \) containing \( a \) but not \( b \), and define \((a, b]\) and \((a, b)\) analogously. For \( a, b \in \mathbb{R} \cup \{\infty\} \) the context will always clarify whether \((a, b]\) refers to a geodesic arc or a real interval. Then the boundary components of \( \mathcal{F} \) are the geodesic arcs \((v_0, \infty)\), \((v_{p-1}, \infty)\) and \([v_0, v_k+1]\) for \( k = 0, \ldots, p-2 \).

For each vertex \( v_k \) (\( k = 0, \ldots, p-1 \)) the set
\[
A(v_k) := \left\{ z \in \mathcal{F} \mid \frac{k}{p} \leq \text{Re} \ z \leq \frac{k+1}{p} \right\}
\]
is the precell attached to \( v_k \). In other words, \( A(v_0) \) and \( A(v_{p-1}) \) are the hyperbolic triangles with vertices \( v_0, m_1, \infty \) resp. \( v_{p-1}, m_{p-1}, \infty \). The precell \( A(v_k) \) for an inner vertex \( v_k \) is the hyperbolic quadrangle with vertices \( m_k, v_k, m_{k+1}, \infty \). Obviously, the precells overlap only at boundaries and
\[
\mathcal{F} = \bigcup_{k=0}^{p-1} A(v_k).
\]

Let \( D \) be a subset of \( \mathbb{H} \) and \( x \in \mathcal{D} \). A unit tangent vector \( v \) at \( x \) is said to point into \( D \) if the geodesic \( \gamma \), determined by \( \gamma(0) = x \) and \( \gamma'(0) = v \) runs into \( D \), i.e., if there exists \( \varepsilon > 0 \) such that \( \gamma((0, \varepsilon)) \subseteq D \). The unit tangent vector \( v \) is said to point along the boundary of \( D \) if there is \( \varepsilon > 0 \) such that \( \gamma((0, \varepsilon)) \subseteq \partial D \). It is said to point out of \( D \) if it points into \( \mathbb{H} \setminus D \).

For each \( A(v_k) \) let \( \bar{A}(v_k) \) be the set of unit tangent vectors that point into its interior \( A(v_k)^\circ \). Let \( \text{vb}(\bar{A}(v_k)) \) be the set of unit tangent vectors based on \( \partial A(v_k) \) that point along \( \partial A(v_k) \). The set \( \text{vb}(\bar{A}(v_k)) \) is called the visual boundary of \( A(v_k) \). Further, \( \text{vc}(\bar{A}(v_k)) := \bar{A}(v_k) \cup \text{vb}(\bar{A}(v_k)) \) is said to be the visual closure of \( \bar{A}(v_k) \). Then \( \text{vc}(\bar{A}(v_k)) \) is the set of unit tangent vectors that point into \( A(v_k) \).

**Figure 1.** Decomposition of \( \mathcal{F} \) into precells for \( p = 5 \).
Lemma 3.1. There is a fundamental set $E$ for $\Gamma$ in $SH$ such that

$$\bigcup_{k=0}^{p-1} \widetilde{A}(v_k) \subseteq E \subseteq \bigcup_{k=0}^{p-1} \text{vc}(\widetilde{A}(v_k)).$$

One easily checks that the determining element $g \in \Gamma$ of the isometric sphere $I(g)$ is unique up to left multiplication by $\Gamma_\infty$ and that $gI(g) = I(g^{-1})$. This implies that for each $k \in \{1, \ldots, p-1\}$ there is a unique $l \in \{1, \ldots, p-1\}$ such that $I_k$ is determined by

$$g_{kl} := \left( \frac{l - 1 + kl}{p} \right) \in \Gamma.$$

E.g., for $k = 1$ we have $l = p - 1$ and $g_{1,p-1} = \left( \frac{p-1}{p}, -1 \right)$. We now define for each $v_k, k = 0, \ldots, p-1$, a set $B(v_k)$, which we call a cell, as follows. For $1 \leq k \leq p - 2$ we set

$$B(v_k) := \bigcup \{ gA(v_l) \mid l = 1, \ldots, p-1, \ g \in \Gamma, \ gv_l = v_k \},$$

and further $B(v_0) := A(v_0) \cup g_{p-1,1}A(v_{p-1})$ resp. $B(v_{p-1}) := A(v_{p-1}) \cup g_{1,p-1}A(v_0)$. The $\Gamma$-translates of the family $\{B(v_k)\}_{0 \leq k \leq p-1}$ cover $H$, since the $\Gamma$-translates of $\{A(v_k)\}_{0 \leq k \leq p-1}$ do so. Lemma 3.2 shows that the family of cells meets the other demands from Sec. II as well.

Lemma 3.2. For $k = 0, \ldots, p-1$, the cell $B(v_k)$ is the hyperbolic triangle with vertices $\frac{k}{p}, \frac{k+1}{p}$ and $\infty$. If $k \in \{0, p-1\}$, then the non-vertical boundary component of $B(v_k)$ is

$$g_{p-1,1} \cdot (v_{p-1}, \infty) = (v_0, \frac{k}{p}) \quad \text{resp.} \quad g_{1,p-1} \cdot (\infty, v_0) = (\frac{p-1}{p}, v_1).$$

If $1 \leq k \leq p - 2$, then there are unique $a, b \in \{1, \ldots, p-2\}$ such that

$$B(v_k) = A(v_k) \cup g_{a,k+1}A(v_a) \cup g_{b+1,k}A(v_b).$$

The non-vertical boundary component of $B(v_k)$ is

$$g_{a,k+1} \cdot (\frac{a+1}{p}, \infty) = (\frac{k}{p}, \frac{k+1}{p}) = g_{b+1,k} \cdot (\infty, \frac{k}{p}).$$

Moreover, if $g\partial B(v_k) \cap B(v_l) \neq \emptyset$ for some $g \in \Gamma$ and $k, l \in \{0, \ldots, p-1\}$, then $g\partial B(v_k) \cap B(v_l) \subseteq \partial B(v_l)$.

For $k = 0, \ldots, p-1$ let $C'_k$ denote the set of unit tangent vectors based on $\frac{k}{p} + i \mathbb{R}^+$ that point into $B(v_k)^\circ$, hence

$$C'_k = \{ X \in SH \mid X = a \frac{\partial}{\partial x} |_{\frac{k}{p} + iy} + b \frac{\partial}{\partial y} |_{\frac{k}{p} + iy}, \ a > 0, \ b \in \mathbb{R}, \ y > 0 \}.$$

Further set

$$C'_p := \{ X \in SH \mid X = a \frac{\partial}{\partial x} |_{1 + iy} + b \frac{\partial}{\partial y} |_{1 + iy}, \ a < 0, \ b \in \mathbb{R}, \ y > 0 \},$$
which is the set of unit tangent vectors based on $1 + i\mathbb{R}^+$ that point into $B(v_{p-1})^\circ$. Let $\text{pr}: SH \to H$ denote the canonical projection onto base points.

**Proposition 3.3.** There are pairwise disjoint subsets $\tilde{B}(v_k)$ of $SH$, $k = 0, \ldots, p$, such that

(i) $C_k' \subseteq \tilde{B}(v_k)$,

(ii) the disjoint union $\bigcup_{k=0}^p \tilde{B}(v_k)$ is a fundamental set for $\Gamma$ in $SH$,

(iii) $\text{pr}(\tilde{B}(v_p)) = B(v_{p-1})$ and $\text{pr}(\tilde{B}(v_k)) = B(v_k)$ for $k = 0, \ldots, p-1$.

**Proof.** We pick a fundamental set $E$ for $\Gamma$ in $SH$ of the form as in Lemma 3.1. For each $z \in F$ let $E_z$ denote the set of unit tangent vectors in $E$ that are based at $z$. For each $k = 0, \ldots, p-1$ and each $z \in A(v_k)^\circ$ pick a partition of $E_z$ into three non-empty disjoint subsets $W_{1,k,z}$, $W_{2,k,z}$, $W_{3,k,z}$. For $z$ in a non-vertical boundary component of $A(v_k)$, $k \neq p-1$ and $\text{Re}(z) \not\in \{\frac{k}{p}, \frac{k+1}{p}\}$ let $W_{1,k,z} := E_z$ and $W_{2,k,z} = W_{3,k,z} := \emptyset$. For $z$ contained in a non-vertical boundary component of $A(v_{p-1})$ and $\text{Re}(z) \not\in \{\frac{p-1}{p}, 1\}$ divide $E_z$ into two non-empty disjoint subsets $W_{1,k,z}$, $W_{3,k,z}$ and let $W_{2,k,z} := \emptyset$. For $z$ with $\text{Re}(z) = k/p$ for some $k \in \{0, \ldots, p-1\}$ let $W_{k,z}$ be the subset of $E_z$ of all unit tangent vectors that point into $A(v_k)$ and let $W_{1,k,z} := E_z \setminus W_{2,k,z}$ and $W_{2,k,z} := \emptyset$. For $z$ with $\text{Re}(z) = 1$ let $W_{1,k-1,z} := E_z$ and $W_{2,k-1,z} = W_{3,k-1,z} := \emptyset$. Now we define $\tilde{B}(v_k)$ as follows. For $1 \leq k \leq p-2$ let $a, b$ be as in Lemma 3.2. Then

$$\tilde{B}(v_k) := \left( \bigcup_{z \in A(v_k)} W_{1,k,z} \right) \cup \left( g_{a,k+1} \cdot \bigcup_{z \in A(v_a)} W_{2,a,z} \right) \cup \left( g_{b+1,k} \cdot \bigcup_{z \in A(v_b)} W_{3,b,z} \right).$$
For $k \in \{0, p-1, p\}$ we set
\[
\tilde{B}(v_0) := \left( \bigcup_{z \in A(v_0)} W_{0,z}^1 \right) \cup \left( g_{p-1,1} \cdot \bigcup_{z \in A(v_{p-1})} W_{p-1,z}^2 \right)
\]
\[
\tilde{B}(v_{p-1}) := \left( \bigcup_{z \in A(v_{p-1})} W_{p-1,z}^1 \right) \cup \left( g_{1,1} \cdot \bigcup_{z \in A(v_0)} W_{0,z}^2 \right)
\]
\[
\tilde{B}(v_p) := \left( \bigcup_{z \in A(v_{p-1})} W_{p-1,z}^3 \right) \cup \left( g_{1,1} \cdot \bigcup_{z \in A(v_0)} W_{0,z}^3 \right).
\]

By Lemma 3.2 one sees that the $\tilde{B}(v_k)$ satisfy all the claims of the proposition. □

For reasons that will become clear later (see Prop. 3.6) we shift $\tilde{B}(v_p)$ and $C_p$ one unit to the left, hence let
\[
\tilde{B}(v_{-1}) := \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right) \cdot \tilde{B}(v_p) \quad \text{and} \quad C_{-1} := \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right) \cdot C_p.
\]

Further we set $C' := \bigcup_{k=-1}^{p-1} C_p$ and
\[
\hat{C} := \bigcup_{k=0}^{p-1} \bigcup_{y>0} S_{\pi\left(\frac{1}{2}+iy\right)} Y \setminus \left\{ \pi(\pm y^2 \frac{\partial}{\partial y}|_{\frac{1}{2}+iy}) \right\}.
\]

The set $\hat{C}$ is the union of all unit tangent vectors in $SY$ which are based at the boundary of some $\pi(B(v_k))$, $k = 0, \ldots, p-1$, but are not tangent to it. We have the following lemma which shows that $C'$ is a set of representatives for $\hat{C}$.

**Lemma 3.4.** $\pi|_{C'} : C' \to \hat{C}$ is a bijection.

The next major goal is the following theorem.

**Theorem 3.5.** $\hat{C}$ is a cross section w. r. t. any measure induced by a Borel probability measure on $SY$ invariant under $\{\tilde{\Phi}_t\}_{t \in \mathbb{R}}$.

Let $C := \Gamma \cdot C' = \pi^{-1}(\hat{C})$ and $B := \Gamma \cdot \bigcup_{k=0}^{p-1} \partial B(v_k) = \text{pr}(C)$. A geodesic $\hat{\gamma}$ on $Y$ intersects $\hat{C}$ iff some (and hence any) representative of $\hat{\gamma}$ intersects $C$. Since $C$ is the set of all unit tangent vectors based in $B$ that are not tangent to $B$, and since $B$ is totally geodesic, a geodesic $\gamma$ on $H$ intersects $C$ if and only if it intersects $B$ transversely. Hence $\hat{C}$ satisfies (C2).

Suppose that the geodesic $\hat{\gamma}$ intersects $\hat{C}$ in $\hat{\gamma}'(t_0)$. By Lemma 3.4 there is a unique geodesic $\gamma$ on $H$ such that $\gamma'(t_0) \in C'$ and $\pi(\gamma'(t_0)) = \hat{\gamma}'(t_0)$. If
\[
s := \min\{t > t_0 \mid \gamma'(t) \in C\} = \min\{t > t_0 \mid \gamma(t) \in B\}
\]
exists, then \( s \) is the first return time of \( \gamma'(t_0) \) and \( R(\gamma'(t_0)) = \pi(\gamma'(s)) \). We call \( \gamma'(s) \), resp. \( \gamma'(s) \), the next point of intersection of \( \gamma \) with \( C \), resp. of \( \gamma \) with \( \hat{C} \). It might happen that \( \gamma'(s) \in C' \). Then we say that \( \gamma'(t_0) \) and \( \gamma'(t_0) \) are elements with interior intersection in future. The next point of exterior intersection of \( \gamma \) and \( C \) shall be \( \gamma'(r) \), where

\[
 r := \min\{t > t_0 \mid \exists g \in \Gamma \setminus \{\text{id}\} : \gamma'(t) \in gC' \}
\]

if \( r \) exists. Analogously, we define previous point of intersection, previous point of exterior intersection and elements with interior intersection in past.

Prop. 3.6 below discusses which geodesics intersect \( \hat{C} \) at all, which ones intersect it infinitely often, and, moreover, if there is a next point, resp. was a previous point, of exterior intersection of the corresponding geodesic on \( H \) and on which copy of \( C' \) this point lies. Figs. 3 and 4 show the relevant \( \Gamma \)-translates of \( C' \). To simplify the statement let \( \text{NC}(\Gamma) \) denote the (finite) set of geodesics on \( Y \) which have a representative on \( H \) which is tangent to \( B \), and let

\[
 h_{1,-\infty} := \begin{pmatrix} -1 & 0 \\ p & -1 \end{pmatrix} \quad h_{c,k} := g_{c-1,k+1} \quad \text{for} \quad 1 \leq k \leq p-2 \\
 h_{1,-1} := \begin{pmatrix} 1 & 0 \\ p & -1 \end{pmatrix} \quad h_{-1,p-1} := g_{1,p-1} \\
 h_{0,0} := \begin{pmatrix} 0 & 1 \\ p & 1 \end{pmatrix} \quad h_{0,p} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Finally, let \( A := \{-\infty, -1, \ldots, p\} \).

**Proposition 3.6.** Let \( \gamma \) be a geodesic in \( Y \).

(i) \( \gamma \) intersects \( \hat{C} \) iff \( \gamma \notin \text{NC}(\Gamma) \).

(ii) \( \gamma \) intersects \( \hat{C} \) infinitely often in future iff \( \gamma(\infty) \notin \Gamma\{0, \infty\} \).

(iii) \( \gamma \) intersects \( \hat{C} \) infinitely often in past iff \( \gamma(-\infty) \notin \Gamma\{0, \infty\} \).

(iv) Suppose that \( \gamma \) intersects \( \hat{C} \) in \( \gamma'(t_0) \). Let \( \gamma \) be the unique lifted geodesic on \( H \) such that \( \gamma'(t_0) \in C' \) and \( \pi(\gamma'(t_0)) = \gamma'(t_0) \).

(a) There is a next point of exterior intersection of \( \gamma \) and \( C \) iff \( \gamma(\infty) \notin \left\{ \frac{k}{p} \mid k = -1, \ldots, p \right\} \).

(b) If \( \frac{k}{p} < \gamma(\infty) < h_{c,k}, \) \( k \in A \setminus \{-1, p-1\} \), resp. \( h_{-1,-1} < \gamma(\infty) < 0 \) or \( h_{-1,p-1} < \gamma(\infty) < 1 \), then the next point of exterior intersection is on \( h_{c,k} \cdot C'_\gamma \).

(c) There was a previous point of exterior intersection of \( \gamma \) and \( C \) iff \( \gamma(\infty) < 0 \) and \( \gamma(-\infty) \notin \left\{ \frac{k}{p} \mid k = -1, \ldots, p-2 \right\} \), or \( \gamma(\infty) > 0 \) and \( \gamma(-\infty) \notin \left\{ \frac{k}{p} \mid k = -1, \ldots, p-2 \right\} \).
We have the following cases:

| $\gamma(-\infty)$ | $\gamma(\infty)$ | prev pt of ext intsec on |
|-------------------|-------------------|--------------------------|
| $(-\infty, -\frac{1}{p})$ | $(0, \infty)$ | $h_{0,p}^{-1} \cdot C_{p-1}'$ |
| $(-\frac{1}{p}, 0)$ | $(0, \infty)$ | $h_{0,0}^{-1} \cdot C_0'$ |
| $(0, \frac{1}{p})$ | $(\frac{1}{p}, \infty)$ | $h_{-1,-1}^{-1} \cdot C_{-1}'$ |
| $(0, -\frac{1}{p})$ | $(-\infty, 0)$ | $h_{-1,-1}^{-1} \cdot C_{-1}'$ |
| $(\frac{1}{p}, \infty)$ | $(\frac{1}{p}, \infty)$ | $h_{k+1,b}^{-1} \cdot C_b'$ |

Figure 3. The light gray part is the set $C'$, the dark gray parts are $\Gamma$-translates of the $C_k'$ as indicated. This figure shows the translates important for determining the next point of exterior intersection.

Figure 4. Here one can read off the previous point of exterior intersection.

Prop. 3.6 shows that exactly the geodesics on $Y$ with one or two limit points in the cusps do not intersect $\hat{C}$ infinitely often. Let $E$ denote the set of unit tangent vectors to these geodesics. Adapting the proof in [2], p. 59, to our situation, we get that $E$ is a null set w.r.t. any Borel probability measure on $SY$ which is invariant under $\{\hat{\Phi}_t\}_{t \in \mathbb{R}}$. Therefore $\hat{C}$ also satisfies (C1) and hence is a cross section. The maximal strong cross section contained in $\hat{C}$ is $\hat{C}_s := \hat{C} \setminus E$. 
Recall the map $\tau$ from Section 2. If $\hat{v} \in \hat{C}$ is an element with interior intersection in future, then $\tau(\hat{v}) = \tau(R(\hat{v}))$ since the geodesics on $H$ corresponding to $\hat{v}$ resp. to $R(\hat{v})$ only differ by their parametrization. Likewise, if $\hat{v} \in \hat{C}$ is an element with interior intersection in past, then $\tau(\hat{v}) = \tau(R^{-1}(\hat{v}))$. Figs. 3 and 4 clearly show that $\hat{C}$ contains both, elements with interior intersection in future and in past. Hence it is impossible to find a dynamical system on $\partial H$ that is conjugate to $(\hat{C}, R)$.

To overcome this problem we reduce $\hat{C}$ to another cross section. Let $P$ denote the set of elements in $\hat{C}$ that have interior intersection in future or past and set

$$\hat{C}_{\text{red}} := \hat{C} \setminus P \quad \text{resp.} \quad \hat{C}_{\text{red,s}} := \hat{C}_s \setminus P.$$ 

Prop. 3.6 implies that each geodesic on $Y$ that intersects $\hat{C}$ infinitely often, also intersects $\hat{C}_{\text{red,a}}$ infinitely often. Therefore both $\hat{C}_{\text{red}}$ and $\hat{C}_{\text{red,s}}$ are cross sections and $\hat{C}_{\text{red,s}}$ is a strong cross section. For $0 \leq k \leq p - 1$ let

$$\tilde{D}_k := \left(\frac{k}{p}, \frac{k+1}{p}\right) \times (-\infty, \frac{k}{p}).$$

Further set

$$\tilde{D}_{-\infty} := (-\infty, -\frac{1}{p}) \times (0, \infty), \quad \tilde{D}_{-1} := (-\frac{1}{p}, 0) \times (0, \infty),$$
$$\tilde{D}_p := (1, \infty) \times (-\infty, 1), \quad \tilde{D}_{k,s} := \tilde{D}_k \setminus (\Gamma\{\infty\} \times \Gamma\{0, \infty\})$$

and $h_k := h_{c,k}$ for $k \in A$. Define the selfmap $\tilde{F}$ on $\tilde{D} := \bigcup_{k \in A} \tilde{D}_k$ resp. $\tilde{F}_s$ on $\tilde{D}_s := \bigcup_{k \in A} \tilde{D}_{k,s}$ by

$$\tilde{F}|_{\tilde{D}_k} := h_k^{-1} \quad \text{resp.} \quad \tilde{F}_s|_{\tilde{D}_{k,s}} := h_k^{-1}.$$ 

Then Prop. 3.6 implies that the two diagrams

$$\begin{array}{ccc}
\hat{C}_{\text{red}} \xrightarrow{R} & \hat{C}_{\text{red}} \\
\tau \downarrow & \downarrow \tau \\
\tilde{D} \rightarrow & \rightarrow \tilde{D}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\hat{C}_{\text{red,s}} \xrightarrow{R} & \hat{C}_{\text{red,s}} \\
\tau \downarrow & \downarrow \tau \\
\tilde{D}_s \rightarrow & \rightarrow \tilde{D}_s
\end{array}$$

commute. The sets $\tilde{D}_k$ resp. $\tilde{D}_{k,s}$ satisfy all the properties announced in Sec. 1. Therefore we naturally get a symbolic dynamics with alphabet $A$ and a natural generating function for its future part.
3.1. Associated transfer operators. We restrict ourselves to the strong cross section $\hat{C}_{\text{red},s}$. Let

$$D := \text{pr}_x(\widetilde{D}_s) = \mathbb{R} \setminus \Gamma\{0, \infty\}$$

and $D_k := \text{pr}_x(\widetilde{D}_{k,s})$ for $k \in A$. The discrete dynamical system $(\widetilde{D}_s, \widetilde{F}_s)$ induces a symbolic dynamics with generating function $F: D \to D$ given by $F|_{D_k} := h_k^{-1}$ for $k \in A$. Its local inverses are

$$F_k := (F|_{D_k})^{-1}: F(D_k) \to D_k, \; x \mapsto h_k x.$$

Then the transfer operator with parameter $\beta$ is given by

$$(L_{\beta}\varphi)(x) = \sum_{k \in A} F'_k(x)^\beta \varphi(F_k(x)) \chi_{F(D_k)}(x),$$

where $\chi_{F(D_k)}$ is the characteristic function of $F(D_k)$, and the maps $F_k$ and $F'_k$ are extended arbitrarily on $D \setminus F(D_k)$.

3.2. The case of the modular surface. We apply the cusp expansion to the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$. All proofs are omitted as they are analogous to those for $\text{PGL}_0(p)$. The set $F := \{z \in H \mid 0 < \text{Re} z < 1, \; |z| > 1, \; |z-1| > 1\}$ is a fundamental domain for $\Gamma$ in $H$. Note that

$$F = F_{\infty} \cap \bigcap_{g \in \Gamma \setminus F_{\infty}} \text{ext} \; I(g)$$

with $F_{\infty} := \{z \in H \mid 0 < \text{Re} z < 1\}$. The point $v := \frac{1}{2}(1 + i\sqrt{3})$ is the only (inner) vertex of $F$. Its associated precell $A(v)$ coincides with $F$. The cell $B(v)$ is the hyperbolic triangle with vertices $0, 1, \infty$. As before, set $Y := \Gamma \setminus H$ and let $\pi: \text{SH} \to \text{SY}$ denote the canonical projection. Further, let $\hat{C}$ be the set of unit tangent vectors in $\text{SY}$ which are based on $\pi(B(v))$, but that are not tangent to it. Then $\hat{C}$ is a weak cross section with set of representatives

$$C' := \{X \in \text{SH} \mid X = a \frac{\partial}{\partial x}|_y + b \frac{\partial}{\partial y}|_y, \; y > 0, \; a > 0, \; b \in \mathbb{R}\}.$$

The elements of $C'$ are the unit tangent vectors based on $i\mathbb{R}^+$ that point into $B(v)^0$. One easily sees that $\hat{C}$ does not contain elements with interior
intersection. Let \( \hat{C}_s \) denote the maximal strong cross section contained in \( \hat{C} \). Then \((\hat{C}_s, R)\) is conjugate to \((\hat{D}_s, \tilde{F}_s)\), where \( \hat{D}_s := \tilde{D}_{0,s} \cup \tilde{D}_{1,s} \) and

\[
\begin{align*}
\tilde{D}_{0,s} & := \left((0, 1) \setminus \mathbb{Q}\right) \times \left((-\infty, 0) \setminus \mathbb{Q}\right), & \tilde{F}_{s|\tilde{D}_{0,s}} & := \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right), \\
\tilde{D}_{1,s} & := \left((1, \infty) \setminus \mathbb{Q}\right) \times \left((-\infty, 0) \setminus \mathbb{Q}\right), & \tilde{F}_{s|\tilde{D}_{1,s}} & := \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right).
\end{align*}
\]

Note that \( \mathbb{Q} = \Gamma\infty \). Set \( D := \mathbb{R}^+ \setminus \mathbb{Q}, D_0 := (0, 1) \setminus \mathbb{Q} \) and \( D_1 := (1, \infty) \setminus \mathbb{Q} \). Then the function \( F: D \to D \), given by \( F|_{D_0} := \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right), F|_{D_1} := \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \), is the first component of \( \tilde{F}_s \) restricted to the first variable. Its associated transfer operator with parameter \( \beta \) is

\[
(\mathcal{L}_\beta \varphi)(x) = \varphi(x+1) + (x+1)^{-2\beta} \varphi\left(\frac{x}{x+1}\right), \quad x \in D.
\]

If we extend \( \mathcal{L}_\beta \) to act on densities on \( \mathbb{R}^+ \) by the obvious formula, then the eigenfunctions of \( \mathcal{L}_\beta \) with eigenvalue 1 are precisely the solutions of

\[
\varphi(x) = \varphi(x+1) + (x+1)^{-2\beta} \varphi\left(\frac{x}{x+1}\right), \quad x \in \mathbb{R}^+.
\]

This is exactly the functional equation used in [4]. We do not think that it is a coincidence that eigenfunctions of Mayer’s transfer operator are related to those of \( \mathcal{L}_\beta \), since \( F \) is closely related to the Farey process or slow continued fractions, whereas the Gauss map is the generating function for (ordinary) continued fractions.

4. Outlook

The construction of a symbolic dynamics for the geodesic flow on \( \Gamma \setminus H \) via the method of cusp expansion seems to work for a wide class of subgroups \( \Gamma \) of \( \text{PSL}(2, \mathbb{R}) \). Some necessary properties of such a group \( \Gamma \) are that \( \infty \) is a cusp and that the boundary of

\[
\bigcap_{g \in \Gamma \backslash \Gamma_\infty} \text{ext} \ I(g)
\]

contains the maxima of the relevant isometric spheres. For example, cusp expansion is applicable to Hecke triangle groups, thus also to certain non-arithmetic groups. The precells in \( H \) can be defined independently of the choice of a fundamental domain for \( \Gamma \) in \( H \). But if there is no close relation between the union of precells and a fundamental domain for \( \Gamma \), we do not expect to be able to construct symbolic dynamics with this method. On the positive side, we do expect it to work for certain groups acting on higher-dimensional real hyperbolic spaces and, more generally, on rank one symmetric spaces of non-compact type.
References

1. L. Ford, Automorphic functions, Chelsea publishing company, New York, 1972.
2. S. Katok and I. Ugarcovici, Arithmetic coding of geodesics on the modular surface via continued fractions, European women in mathematics—Marseille 2003, CWI Tract, vol. 135, Centrum Wisk. Inform., Amsterdam, 2005, pp. 59–77.
3. ______, Symbolic dynamics for the modular surface and beyond, Bull. Amer. Math. Soc. (N.S.) 44 (2007), no. 1, 87–132 (electronic).
4. J. Lewis and D. Zagier, Period functions for Maass wave forms. I, Ann. of Math. (2) 153 (2001), no. 1, 191–258.
5. D. Mayer, The thermodynamic formalism approach to Selberg’s zeta function for $\text{PSL}(2, \mathbb{Z})$, Bull. Amer. Math. Soc. (N.S.) 25 (1991), no. 1, 55–60.
6. C. Series, The modular surface and continued fractions, J. London Math. Soc. (2) 31 (1985), no. 1, 69–80.
7. L. Vulkah, Farey polytopes and continued fractions associated with discrete hyperbolic groups, Trans. Amer. Math. Soc. 351 (1999), no. 6, 2295–2323.

Institut für Mathematik, Universität Paderborn, 33095 Paderborn, Germany
E-mail address: {hilgert,pohl}@math.upb.de