Mathematical Principle of the Natural Philosophy of Incompressible Flows: The Principle of Minimum Pressure Gradient

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In this paper, we discover the fundamental quantity that Nature minimizes in every incompressible flow; we call it the principle of minimum pressure gradient (PMPG). We prove mathematically that Navier-Stokes’ equation represents the necessary condition for minimization of the pressure gradient. Hence, the PMPG is to Navier-Stokes’ equations what the principle of least action and Lagrangian mechanics are to Newton’s equations. In fact, the PMPG stands on its own philosophy; in the light of the fact that the pressure force is a constraint force (its role is to ensure continuity), we show that Nature’s minimization of the pressure gradient is deeply rooted in classical mechanics via Gauss’ principle of least constraint. Consequently, the PMPG turns any fluid mechanics problem into a minimization one. We demonstrate this intriguing property by solving three of the classical problems in fluid mechanics using the PMPG without resorting to Navier-Stokes’ equation.

I. INTRODUCTION

The great victories in mechanics achieved by physicists in the past century were mainly enabled by variational approaches. While classical mechanics fails at small scales (e.g., atoms) and large scales (e.g., planets), analytical mechanics (e.g., the principle of least action) has provided a fundamental approach in both Einstein’s general relativity [1] and quantum mechanics [2], in addition to recovering the classical Newtonian mechanics in the ordinary scales. On the other hand, when we focus our attention on the mechanics of fluids, we find little success has been achieved beyond the Newtonian approach which was culminated in the Navier-Stokes equations of motion. Perhaps, the field of theoretical mechanics of fluids is stalling due to the lack of a fundamental variational formulation that is firmly rooted in first principles.

The dominant variational principle in physics is Hamilton’s principle of least action. Because Hamilton’s principle does not directly allow for non-conservative forces (polygenic forces that do not come from a scalar work function), most of the variational principles of fluid mechanics in the literature were developed for ideal fluids (inviscid fluids governed by Euler’s equations) [3–12], which ignores important features (e.g., viscosity, turbulence, and other irreversible phenomena).

There have been several efforts aiming at extending these variational formulations to account for dissipative/viscous forces [13–19]. However, these extensions do not directly follow from first principles; some ingenious mathematical manipulations are required to show the connection with the governing equations. So, in many times, these variational formulations are imbued with a sense of ad hoc and contrived treatments, which detracts from the beauty of analytical and variational formulations. It may be prudent to recall Salmon’s statement: “the existence of a Hamiltonian structure is, by itself, meaningless because any set of evolution equations can be written in canonical form” by adding artificial variables. So, we view the few existing variational formulations of Navier-Stokes (e.g., [13–19]) as aesthetic mathematical constructions that managed to recover the (already known) governing equations (Navier-Stokes), but may not be practically useful. In particular, it is not clear at all how one can use any of these variational formulations to solve (even simple) fluid mechanics problems (or inferring new concepts about the physics of fluids) without invoking the governing equations.

So, the main focus here is not on the mere development of a variational principle. Rather, we aim to discover the fundamental quantity that Nature minimizes in every incompressible flow problem. The result is a variational principle of Navier-Stokes’ equations that naturally stems from one of the first principles of mechanics: Gauss’ principle of least constraint. Such a variational principle may revolutionize fluid mechanics by turning any fluid mechanics problem into an optimization one where fluid mechanicians need not to apply Navier-Stokes’ equations anymore. Rather, they merely need to minimize the objective/cost function.

II. GAUSS’ PRINCIPLE OF LEAST CONSTRAINT

Consider the dynamics of \( N \) constrained particles, each of mass \( m_i \), such that we have a total of \( n \) generalized coordinates (degrees of freedom) \( q \). The dynamics of these \( N \) particles are governed by Newton’s equations

\[
m_i a_i(q, \dot{q}, q) = F_i + R_i, \quad \forall i \in \{1, \ldots, N\},
\]

where \( a_i \) is the inertial acceleration of the \( i \)th particle. The right hand side of the equation represents the total force acting on the particle, which is typically decomposed in analytical mechanics into: (i) impressed forces \( F_i \), which are the directly applied (driving) forces (e.g., gravity, elastic, viscous); and (ii) constraint forces \( R_i \) whose raison d’etre is to enforce kinematical/geometrical constraints; they are passive or workless forces [20]. That is, they do not contribute to the motion abiding by the constraint; their main mission is to preserve the constraint (i.e., prevent any deviation from it).

Inspired by his method of least squares, Gauss asserted...
that the quantity
\[ Z = \frac{1}{2} \sum_{i=1}^{N} m_i \left( a_i - \frac{F_i}{m_i} \right)^2 \] (2)
is minimum with respect to the generalized accelerations \( \dot{q} \) [21, pp. 911-912]. Note that the quantity \( Z \) is nothing but \( Z = \frac{1}{2} \sum_{i=1}^{N} \frac{R_i^2}{m_i} \), i.e., the magnitude of constraint forces must be minimum—hence the name least constraint.

Gauss’ principle is adroitly intuitive. In the absence of constraints, a particle follows the applied acceleration/force \( \frac{F}{m} \). However, if the motion of the particle is constrained, it will deviate from this applied/desired motion to satisfy the hard constraint, but this deviation will be minimum; the particle will deviate from the applied motion only by the amount that satisfies the constraint. Nature will not overdo it.

Several points are worthy of clarification here. First, in Gauss’ principle, \( Z \) is actually a minimum, not just stationary. Second, unlike the time-integral principle of least action, Gauss’ principle is applied instantaneously (at each point in time). Third, in contrast to Hamilton’s principle, Gauss’ explicitly allows for non-conservative forces that do not come from a scalar work function; the impressed forces \( F \) can be arbitrary.

III. APPLICATION TO FLUID DYNAMICS

To the best of our knowledge, Gauss’ principle has never been applied to fluid mechanics, which is the main contribution of this paper. Recall the Navier-Stokes equations for incompressible flows:
\[ \rho a = -\nabla p + \rho \nu \Delta u, \quad \text{in } \Omega \] (3)
subject to continuity:
\[ \nabla \cdot u = 0, \quad \text{in } \Omega \] (4)
and the Dirichlet boundary condition \( u = 0 \) on \( \delta \Omega \), where \( \Omega \subset \mathbb{R}^3 \) is the spatial domain, \( \delta \Omega \) is its boundary, and \( a = u_t + u \cdot \nabla u \) is the total acceleration of the fluid particle.

It is noteworthy to mention that Gauss’ principle is almost useless for unconstrained systems; it reduces to least-squares. Interestingly, for incompressible flows, the pressure force \( \nabla p \) is a constraint force. The sole role of the pressure force in incompressible flows is to enforce the continuity constraint: the divergence-free kinematic constraint on the velocity field \( \nabla \cdot u = 0 \). It is straightforward to show that if \( u \) satisfies Eq. (4) and the no-penetration boundary condition
\[ u \cdot n = 0, \quad \text{on } \delta \Omega \] (5)
where \( n \) is the normal to the boundary, then
\[ \int_{\Omega} (\nabla p \cdot u) d\Omega = 0, \] (6)
which indicates that pressure forces are workless through divergence-free velocity fields! That is, if continuity is already satisfied (the velocity field is divergence-free), the pressure forces will not affect the dynamics of this field. This fact is the main reason behind vanishing the pressure force in the first step in Chorin’s standard projection method for incompressible flows [22], which is based on the Helmholtz-Hodge decomposition (e.g., [23]): a vector \( v \in \mathbb{R}^3 \) can be decomposed into a divergence-free component \( u \) and a curl-free component \( \nabla f \) for some scalar function \( f \) (i.e., \( v = u + \nabla f \)). These two components are orthogonal as shown in Eq. (6) provided that \( u \) satisfies the homogeneous condition Eq. (4). This decomposition can be visualized in the schematic in Fig. 1. This setup provides the basis for Arnold’s seminal result [24].

From the above discussion (and Fig. 1), it is clear that the pressure force is a constraint force. Hence, applying Gauss’ principle of least constraint to the dynamics of incompressible fluids, governed by the Navier-Stokes equation Eq. (3), classifying the pressure force as a constraint force and the viscous force as an impressed force, and labeling fluid parcels with their Lagrangian coordinates \( \zeta \), we write the action (Gauss’ quantity) as
\[ A = \frac{1}{2} \int \rho_0 (a - \nu \Delta u)^2 d\zeta, \]
where \( \rho_0 = \rho_0(\zeta) \) is the initial density. Realizing that \( \rho J = \rho_0 \), where \( J \) is the Jacobian of the flow map \( x = \Phi(\zeta) \) [4], then the action \( A \) is rewritten in Eulerian coordinates as
\[ A = \frac{1}{2} \int_{\Omega} \rho (u_t + u \cdot \nabla u - \nu \Delta u)^2 dx, \] (7)
which must be minimum according to Gauss’ principle.

It is interesting to discuss the meaning of the minimization of \( A \). Note that \( A \) is simply \( \frac{1}{2\rho_0} \int_{\Omega} (\nabla p)^2 dx \): the integral of the norm of the pressure gradient over the field. Since the pressure force is a constraint force (its sole role
is to enforce the continuity constraint), the flow field will deviate from the motion dictated by the inertial $u \cdot \nabla u$ and viscous $\nu \Delta u$ forces only by the amount to satisfy continuity; no larger pressure gradient will be generated than that necessary to maintain continuity. Nature will not overdo it. This new principle is what we call The Principle of Minimum Pressure Gradient (PMPG).

The question then is: What is the independent (free) variable that minimizes $A$? Interestingly, it is every “free” variable! If a fluid mechanic parameterizes a flow field with some free parameters, then $A$ is minimum with respect to these parameters whatever they are, as long as the representation/parameterization is admissible; i.e., it satisfies the kinematical constraint and boundary conditions. The pressure gradient must always be minimum. Otherwise, there will be a larger pressure gradient than necessary, which violates Nature’s laws.

In particular, to recover Navier-Stokes’ equations from Gauss’ principle, we adhere to the philosophy of the principle and minimize the action Eq. (7) with respect to the local acceleration $u_t$:

$$u_t(x,t) = \arg\min_{u_t} \frac{1}{2} \int_{\Omega} \rho (u_t + u \cdot \nabla u - \nu \Delta u)^2 \, dx \quad \forall t.$$  

(8)

The justification of this choice is discussed in detail in the Supplementary Material. Here, we provide the following theorem:

**Theorem** If $u(.)$ is differentiable in $\Omega \subset \mathbb{R}^3$ and minimizes the functional

$$A(u(.)) = \frac{1}{2} \int_{\Omega} \rho (u_1 + u \cdot \nabla u - \nu \Delta u)^2 \, dx$$

for all $t \in \mathbb{R}$ subject to the constraint

$$\nabla \cdot u = 0 \quad \forall \quad x \in \Omega, \quad t \in \mathbb{R},$$

and the Dirichlet boundary condition $u(x,t) = g(x,t)$ for all $x \in \partial \Omega, \ t \in \mathbb{R}$, for some $g$ differentiable in $t$, then $u(.)$ must satisfy

$$\rho (u_t + u \cdot \nabla u) = -\nabla p + \nu \Delta u \quad \forall \quad x \in \Omega, \quad t \in \mathbb{R},$$

for some differentiable function $p$ on $\Omega$.

The proof is given in the Supplemental Material.

The above theorem implies that Navier-Stokes’ equation represents the necessary condition for minimizing the pressure gradient. In fact, the pressure in the proof is the Lagrange multiplier that enforces the continuity constraint imposed on the local acceleration $\nabla \cdot u_t = 0$.

The variational principle [25] represents the sought deterministic variational principle of the unsteady Navier-Stokes that is derived from first principles. The principle is naturally represented in the convenient Eulerian formulation without invoking artificial variables. In fact, the proof is straightforward, applying standard techniques from calculus of variations [25].

**IV. FLUID MECHANICS AS A MINIMIZATION PROBLEM**

In this section, we will apply the developed variational principle of minimum pressure gradient (PMPG) to solve a few classical problems in fluid mechanics—performing pure optimization without resorting to Navier-Stokes’ equations.

**A. Viscous Steady Case: Channel Flow**

Consider the simple laminar flow in a channel $y \in [-h, h]$. The velocity field is parameterized as $u = (u(y), 0)$, which automatically satisfies continuity. The flow dynamics must then dictate a specific shape for the free function $u(y)$. It is straightforward to determine this shape from the proposed PMPG without invoking the Navier-Stokes equations. Substituting by the velocity field into the action $A$ in Eq. (7), we obtain

$$A(u(.)) = \frac{1}{2} \int_{-h}^{h} \left( \frac{\partial^2 u(y)}{\partial y^2} \right)^2 \, dy,$$

(9)

which should be minimum according to the PMPG. Therefore, the fluid mechanics problem is turned into the minimization problem: Find $u(y)$ that minimizes the functional Eq. (9) subject to a specified flow rate $Q$ (i.e., $\int_{-h}^{h} u(y) \, dy = Q$) and $u(-h) = 0$ and $u(h) = 0$. It is a standard calculus of variations problem (Euler’s isoperimetric problem [25]) whose solution yields $\frac{\partial^2 u(y)}{\partial y^2} = 0$, which after satisfying the boundary conditions and the flow rate constraint, results in the well-known quadratic velocity profile $u(y) = \frac{Q}{3h} \left(1 - \frac{y^2}{h^2}\right)$; the non-zero function with a minimum magnitude of second derivative $(\frac{\partial^2 u(y)}{\partial y^2})^2$ is the quadratic function.

**B. Viscous Unsteady Case: Stokes’ Second Problem**

Recall the Stokes’ second problem: the flow above a harmonically-oscillating, infinitely-long plate. The unsteady velocity field is parameterized as $u = (\phi(t) \psi(y), 0)$, which automatically satisfies continuity for any shapes of the free functions $\psi(y)$ and $\phi(t)$; their shapes are dictated by dynamical considerations (e.g., Navier-Stokes or the PMPG). Because there are no changes with $x$, we write the action $A$ over a slice along the $y$-axis:

$$A(\psi(.), \phi(.)) = \frac{1}{2} \int_{0}^{\infty} \left( \psi(y) \dot{\phi}(t) - \nu \psi''(y) \phi(t) \right)^2 \, dy,$$

(10)

which should be minimum for all $t$. With this modal representation $u(y, t) = \psi(y) \phi(t)$, the PMPG is capable of determining both the mode shape $\psi(y)$ and temporal coefficient $\phi(t)$ from the same minimization principle. For a given mode shape, the condition $\frac{\partial A}{\partial \phi} = 0$ yields the differential equation: $\dot{\phi}(t) = \frac{c_1}{\nu} \phi(t)$ whose solution is $\phi(t) = \phi(0) e^{ct}$, where $c_2 = \rho \int_{0}^{\infty} \psi^2(y) \, dy \neq 0$, $c_1 = \nu \rho \int_{0}^{\infty} \psi(y) \psi''(y) \, dy$, and $\gamma = \frac{c_1}{c_2}$ are determined from the mode shape $\psi(y)$. 
Having determined the temporal solution $\phi(t)$ (up to a constant $\gamma$), the action $A$ \[ (10) \) can be rewritten as

$$
A(\psi(.)) = \frac{1}{2} \rho \phi^2 (t) \int_0^\infty (\gamma \psi(y) - \nu \psi''(y))^2 \, dy. \quad (11)
$$

Again, the fluid mechanics problem of computing the mode shape $\psi(y)$ turns, via the PMPG, into a pure minimization problem: Find $\psi(y)$ that makes the functional $A$ in Eq. \[ (11) \) stationary and satisfies the boundary conditions $\psi(0) = 1$, $\psi(\infty) = 0$. The solution is straightforward by applying standard techniques in calculus of variations; the Euler-Lagrange equation results in

$$
\psi''(y) = \frac{\gamma}{\nu} \psi(y)
$$

whose solution, after substituting by the boundary conditions, is given by $\psi(y) = e^{-\sqrt{\frac{\gamma}{\nu}}y}$. As such, the velocity field is then written as

$$
u(y, t) = \phi(0)e^{i\omega t} e^{-\sqrt{\frac{\gamma}{\nu}}y}.
$$

Matching with the boundary condition $u(0, t) = U e^{i\omega t}$ results in $\phi(0) = U$ and $\gamma = i\omega$, which yields the well-known solution of the Stokes' second problem \[26\].

**C. Ideal Fluid Case: The Airfoil Problem**

Consider the potential flow over a two-dimensional object [e.g., \[27\]]. It is straightforward to construct a flow field $u(x; \Gamma)$ that is (i) divergence-free, (ii) irrotational, and (iii) satisfies the no-penetration boundary condition for any value of the circulation $\Gamma \in \mathbb{R}$. That is, Euler's equation does not possess a unique solution for this problem. The only theoretical fix available in the literature is through the so-called *Kutta condition*, which dictates that the flow is bounded everywhere. Therefore, when the body shape possesses some singularity (e.g., a sharp trailing edge in a conventional airfoil), the circulation is set to remove such a singularity. However, for singularity-free shapes (e.g., ellipse, circle), the Kutta condition is not applicable; and there is no theoretical model that can predict circulation and lift over these shapes.

In contrast, even the inviscid version of the developed PMPG is capable of providing a unique solution over arbitrarily smooth shapes. Considering a steady snapshot (i.e., $a = u \cdot \nabla u$), we write the inviscid action $A$ (which reduces to the *Appelian* \[28\]) as

$$
A(\Gamma) = \frac{1}{2} \rho \int_\Omega [u(x; \Gamma) \cdot \nabla u(x; \Gamma)]^2 \, dx. \quad (12)
$$

And the PMPG yields the circulation over the airfoil as

$$
\Gamma^* = \arg \min_{\Gamma} \frac{1}{2} \rho \int_\Omega |u(x; \Gamma) \cdot \nabla u(x; \Gamma)|^2 \, dx. \quad (13)
$$

Equation \[13\] provides a generalization of the Kutta-Zhukovsky condition that is, unlike the latter, derived from first principles. The PMPG allows, for the first time, computation of lift over smooth shapes without sharp edges where the Kutta condition fails.

Consider a family of airfoils parameterized by $D$, which controls smoothness of the trailing edge: $D = 0$ results in the classical Zhukovsky airfoil with a sharp trailing edge, and $D = 1$ results in a circular cylinder. Figure 2 shows the variation of the minimizing circulation $\Gamma^*$ from Eq. \[13\], normalized by Kutta's value $\Gamma_K$, with the parameter $D$. The figure shows that $\Gamma^* \to \Gamma_K$, as $D \to 0$ (i.e., for a sharp-edged airfoil). It also shows that the PMPG recovers the classical result about the non-lifting nature of a circular cylinder in an ideal fluid: $\Gamma^* \to 0$ as $D \to 1$. It is remarkable that the PMPG captures the whole spectrum (from a zero lift over a circular cylinder to the Kutta-Zhukovsky lift over a sharp-edged airfoil) from the same unified principle.

The fact that the minimization principle \[13\] reduces to the Kutta condition in the special case of a sharp-edged airfoil, wedded to the fact that this principle is an inviscid principle challenge the accepted wisdom about the viscous nature of the Kutta condition that prevailed over a century. In contrast, it is found that the Kutta condition is not a manifestation of viscous effects, rather of inviscid momentum effects.

**V. CONCLUDING REMARKS**

In this paper, we developed a deterministic variational principle of Navier-Stokes' equations that is firmly rooted in first principles. The developed principle possesses an interesting physical meaning: the pressure gradient is developed only to the level that satisfies continuity; i.e., it is the least pressure gradient that maintains continuity—hence, we call it the principle of minimum pressure gradient (PMPG). The principle reduces to minimum dissipation \[29\] in the special case of ignorable inertial/convective accelerations (Stokes’ flow). On the other hand, for ignorable viscous actions (Euler’s dynamics), the PMPG reduces to minimum acceleration (i.e., minimum curvature) \[25\]. The PMPG shows how Nature balances between minimizing dissipation and curvature in the general case.

The PMPG is so generic; it is naturally written in the convenient Eulerian formulation and is applicable to 3D, unsteady, viscous flows. It is basically equivalent to Navier-Stokes’ equations; we proved that Navier-Stokes’ equation is the necessary condition for minimizing the pressure gradient subject to the continuity constraint. Consequently, the PMPG may shed light on the Millennium Prize problem of existence of solutions of Navier-Stokes’ equations; variational principles have usually been useful in studying existence of solutions of
partial differential equations [30]. Also, as a minimization principle, it may not suffer from a closure problem, which may be useful in the analysis of turbulence: the most elusive problem in fluid mechanics. For example, one can easily determine the "optimal" parameters in a RANS model by minimizing the action.

The PMPG is expected to be of particular importance to theoretical and reduced-order modeling; a fluid mechanic can utilize his/her experience to parameterize the flow field in any form that satisfies the kinematical constraint (continuity) and boundary conditions. Then, minimizing the action $\mathcal{A}$ with respect to the parameters of the model will provide a natural way of projecting the Navier-Stokes equations on the space of these parameters. It is interesting to recall that, after Euler developed his seminal equations that govern the dynamics of ideal fluids, Lagrange commented [31] "By the discovery of Euler the whole mechanics of fluids was reduced to a matter of [mathematical] analysis alone, .... Unfortunately, they are so difficult that, up to the present, it has only been possible to succeed in very special cases".

Clearly, the situation is exacerbated with Navier-Stokes (after adding viscous forces). Interestingly, the PMPG reinstates the mechanics of fluids from pure mathematical (and computational) analysis back to the theoretical mechanics plane where the focus is not on the numerical solution of the governing equations, rather on the appropriate parameterization/representation of the flow field; it will allow fluid mechanicians to show their prowess.

[1] A. Einstein, *Relativity: the special and the general theory* (General Press, 1916).
[2] R. P. Feynman, The principle of least action in quantum mechanics, in *Feynman’s Thesis—A New Approach To Quantum Theory* (World Scientific, 1942) pp. 1–69.
[3] R. Hargreaves, Xxvii. a pressure-integral as kinetic potential, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 16, 436 (1908).
[4] H. Bateman, Notes on a differential equation which occurs in the two-dimensional motion of a compressible fluid and the associated variational problems, Proc. R. Soc. Lond. A 125, 598 (1929).
[5] J. Serrin, Mathematical principles of classical fluid mechanics, in *Fluid Dynamics I/Strömungsmechanik I* (Springer, 1959) pp. 125–263.
[6] J. W. Herivel, The derivation of the equations of motion of an ideal fluid by hamilton’s principle, in *Mathematical Proceedings of the Cambridge Philosophical Society*, Vol. 51 (Cambridge University Press, 1955) pp. 344–349.
[7] P. Penfield Jr, Hamilton’s principle for fluids, *The Physics of Fluids* 9, 1184 (1966).
[8] R. L. Seigler and G. B. Whitham, Variational principles in continuum mechanics, in *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, Vol. 305 (The Royal Society, 1968) pp. 1–25.
[9] F. P. Bretherton, A note on hamilton’s principle for perfect fluids, *Journal of Fluid Mechanics* 44, 19 (1970).
[10] R. Salmon, Hamiltonian fluid mechanics, Annual review of fluid mechanics 20, 225 (1988).
[11] M. I. Loffredo, Eulerian variational principle for ideal hydrodynamics and two-fluid representation, *Physics Letters A* 135, 294 (1989).
[12] P. J. Morrison, Hamiltonian description of the ideal fluid, *Reviews of modern physics* 70, 467 (1998).
[13] K. Yasue, A variational principle for the navier-stokes equation, *Journal of Functional Analysis* 51, 133 (1983).
[14] R. R. Kerswell, Variational principle for the navier-stokes equations, *Physical Review E* 59, 5482 (1999).
[15] D. A. Gomes, A variational formulation for the navier-stokes equation, *Communications in mathematical physics* 257, 227 (2005).
[16] G. L. Eyink, Stochastic least-action principle for the incompressible navier–stokes equation, *Physica D: Nonlinear Phenomena* 239, 1236 (2010).
[17] H. Fukagawa and Y. Fujitani, A variational principle for dissipative fluid dynamics, *Progress of Theoretical Physics* 127, 921 (2012).
[18] C. R. Galley, D. Tsang, and L. C. Stein, The principle of stationary nonconservative action for classical mechanics and field theories, *arXiv preprint arXiv:1412.3082* (2014).
[19] F. Gay-Balmaz and H. Yoshimura, A lagrangian variational formulation for nonequilibrium thermodynamics. part ii: continuum systems, *Journal of Geometry and Physics* 111, 194 (2017).
[20] C. Lanczos, *The variational principles of mechanics* (Courier Corporation, 1970).
[21] J. Papastavridis, *Analytical mechanics: a comprehensive treatise on the dynamics of constrained systems – Reprint edition* (Word Scientific Publishing Company, 2014).
[22] A. J. Chorin, Numerical solution of the navier-stokes equations, *Mathematics of computation* 22, 745 (1968).
[23] T. Kambe, *Geometrical theory of dynamical systems and fluid flows*, Vol. 23 (World Scientific Publishing Co Inc, 2009).
[24] V. I. Arnold, Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, in *Annales de l’institut Fourier*, Vol. 16 (1966) pp. 319–361.
[25] J. A. Burns, *Introduction to the calculus of variations and control with modern applications* (CRC Press, 2013).
[26] H. Lamb, *Hydrodynamics* (Cambridge university press, 1932).
[27] H. Schlichting and E. Truckenbrodt, *Aerodynamics of the Airplane* (McGraw-Hill, 1979).
[28] C. Gonzalez and H. E. Taha, A variational theory of lift, *arXiv preprint arXiv:2104.13904* (2021).
[29] B. A. Finlayson, Existence of variational principles for the navier-stokes equation, *The physics of fluids* 15, 963 (1972).
[30] M. Reed and B. Simon, *Methods of modern mathematical physics. vol. 1. Functional analysis* (Academic New York, 1980).
[31] R. Dugas, *A History of Mechanics, translated into English by JR Maddox. NY* (Dover Publications, Inc, 1988).