FREE BOUNDARIES OF CREDIT RATING MIGRATION IN SWITCHING MACRO REGIONS

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Abstract. In this paper, under the structure framework, a valuation model for a corporate bond with credit rating migration risk and in macro regime switch is established. The model turns to a free boundary problem in a partial differential equation (PDE) system. By PDE techniques, the existence, uniqueness and regularity of the solution are obtained. Furthermore, numerical examples are also presented.

1. Introduction. Regime switch was introduced by Hamilton in 1989, [8]. In his paper, he described an autoregressive regime switching process. This topic lighted wide interests as it well described a general phenomenon that the situations might change in switching macro atmosphere. Later, the topic became very popular and was extended to various areas such as energy, economics etc. Many researchers developed the Hamilton’s model in different ways, e.x. [5, 21]. Especially, in financial area, valuations of the financial products, such as stocks, options etc. were also considered in regime switching models (see[2, 3, 7]).

The Financial Crisis in 2008 and later the European Debt Crisis gave a lesson to the financial market that managing the credit risk was important, where the credit risks included both default risk and credit rating migration one. People realized that in different macro atmospheres, the credit risks behaved totally differently. However, the academic researches were behind legs.

There are two traditional ways to study credit risks, known as structural and reduced-form ones, where the first one treats the credit event is an exogenous variant, (e.x. see [11, 14, 6]), while the second one is assumed that the credit event occurs once the firm’s value touches a threshold, (e.x. see [19, 1, 15]).

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Though main research of credit risks is on default, there are still some studies on the credit rating migration, where Markov chain, which is type of reduced-form method, is a main tool. The transferring intensity matrix is adapted, which usually comes from general statistical data, (e.x. see [12, 4, 14, 20]). However reduced form model does not take firm’s value, which is an important factor of the credit rating migrations, into consideration. From this point of the view, Liang et al. ([17, 18]) started to use structural model to study credit rating migration risk based on the Merton’s model. As a first step, they set a predetermined migration threshold to divide firm’s value into high and low rating regions. The firm’s value followed different stochastic processes in different rating regions. Later, Hu et al.([10], 2015) improved this model to treat the migration boundary to be the proportion of the debt and the value of the firm. Thus the model became to a free boundary problem. Liang et al. ([16], 2016) further took some factor so called risk discount of the firm into consideration. And the authors showed that the problem admitted an asymptotic traveling wave solution.

In this paper, the free boundary model for credit rating migration is considered in macro regime switch. So that, the model turns to a free boundary problem in PDE system. That means, in different states of the macro atmosphere, the credit rating migration boundaries are different. Existence, uniqueness and regularities are proved, which let the model make sense. In mathematical theory, the system problem is much harder than the single equation one. We need to overcome the difficulties of the couple variables, which cause maximum principle is not valid in general, while the maximum principle is the main tool to prove our result as the model has discontinuous in leading coefficient. Besides, in our model, the leading coefficients are discontinuous, then the theoretical results of PDE system in ([23]) can not be applied directly.

The paper is organized as follows: in Section 2, a model is established; this model is reduced to a PDE free boundary system with initial and boundary condition in Section 3; in Section 4, an approximated problem is analyzed and some preliminary lemmas are collected; through this approximated problem, the existence and uniqueness of the solution for the free boundary are obtained in Section 5. Numerical results are presented in Section 6 and conclusions are listed in Section 7.

2. Model. In order to establish a pricing model with credit rating migration and macro environment region switching risks, the following assumptions are presented:

2.1. Assumption. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space. We assume that the firm issues a corporate bond, which is a contingent claim of its value on the space.

Assumption 2.1 (Macro Environment Regions). The macro environment has finite regions. At time $t$, the region $M_t$ takes random variable in a finite set $\kappa = \{1, 2, \cdots, N\}$. All stochastic processes introduced below are supposed to be adapted processes in the filtered probability space. Given an initial region $M_0$, the future switching regions are described by a continuous-time Markov chain $M_t$ referred as the region switching process, with the intensity matrix $\Lambda = (\lambda_{ij})_{i, j \in \kappa}$, where $\lambda_{ij} (i \neq j, i \neq N)$ are positive constants, $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$.

Assumption 2.2 (the firm asset with credit rating migration). In the risk neutral world, let $S_t$ denote the firm’s value, which may be in a high rating or a low one.
It satisfies
\[
    dS_t = \begin{cases} 
        r_{H}^{M} S_t dt + \sigma_{H}^{M} S_t dW_t, & \text{in the high rating}, \\
        r_{L}^{M} S_t dt + \sigma_{L}^{M} S_t dW_t, & \text{in the low rating},
    \end{cases}
\]
where \( \gamma_{H}^{M} \) is the real free interest rate in different macro environment regions, and \( \sigma_{H}^{M}, \sigma_{L}^{M} : \kappa \to \mathbb{R}_{+} \) represent the volatilities of the firm asset, \( M_{t} \) is the region of the switching process mentioned above, \( W_{t} \) is the Brownian motion which generates the filtration \( \{ F_{t} \} \). \( W_t \) and \( M_t \) are assumed to be independent. Also
\[
    \sigma_{H}^{M} < \sigma_{L}^{M}.
\]  
(2.1)

It is reasonable to assume (2.1), namely, that the volatility in the high rating region is lower than the one in the low rating region.

**Assumption 2.3** (the corporate bond). The firm issues only one corporate zero-coupon bond with face value \( F \). We focus on the effect of the firm’s value with credit rating migration to the bond, so the discount value of bond is considered. Denote \( \Phi_{t}^{M} \) the discount value of the bond at time \( t \) in the macro environment region \( M_{t} \). Therefore, on the maturity time \( T \), an investor can get
\[
    \Phi_{T}^{M} = \min\{ S_{T}, F \}.
\]

**Assumption 2.4** (the credit rating migration time). The high and low rating regions are determined by the proportion of the debt and value. The credit rating migration time \( \tau_{d}^{M_{t}} \) and \( \tau_{u}^{M_{t}} \) are the first moment when the firm’s grade is downgraded and upgraded respectively as follows:
\[
    \tau_{d}^{M_{t}} = \inf\{ t > 0 | \Phi_{0}^{M_{t}}/S_0 < \gamma, \Phi_{t}^{M_{t}}/S_t \geq \gamma \}, \quad \tau_{u}^{M_{t}} = \inf\{ t > 0 | \Phi_{0}^{M_{t}}/S_0 > \gamma, \Phi_{t}^{M_{t}}/S_t \leq \gamma \},
\]
where \( \Phi_{t} = \Phi_{t}(S_t, t) \) is a contingent claim with respect to \( S_t \) and
\[
    0 < \gamma < 1
\]  
(2.2)

is a positive constant representing the threshold proportion of the debt and value of the firm’s rating.

**Assumption 2.5** (the macro region changing time). The probability that the credit rating and the macro region transfer in the same time is zero. Denote \( \tau^{ij} \) is the macro region changing time, which the state \( i \) turns to \( j \).

2.2. Cash flow. Once the credit rating migrates or macro region switches before the maturity \( T \), though there is no cash flow, a virtual substitute termination happens, i.e., the bond is virtually terminated and substituted by a new one with a new credit rating (new macro region). There is a virtual cash flow of the bond. Denoted by \( \Phi_{H}^{i}(y, t) \) and \( \Phi_{L}^{i}(y, t) \) the values of the bond on the \( M_{t} = i \in \kappa \) state and in high and low grades respectively. Then, they are the conditional expectations as follows:
\[
    \Phi_{H}^{i}(y, t) = E_{y,t} \left[ e^{-r(T-t)} \min(S_T, F) \cdot 1_{\min(\tau_{d}^{M_{t}}, \tau^{ij}) \geq T} \right. \\
    + \sum_{j \neq i} e^{-r(\min(\tau_{d}^{M_{t}}, \tau^{ij}) - t)} \Phi_{O}^{j}(y, t) \cdot 1_{\min(\tau^{ij}) \leq \min(\tau_{d}^{M_{t}}, T)} \\
    \left. + e^{-r(\tau_{d}^{M_{t}} - t)} \Phi_{L}^{i}(S_{\tau_{d}^{M_{t}}}, \tau_{d}^{M_{t}}) \cdot 1_{\tau_{d}^{M_{t}} \leq \min(A, T)} \right| S_t = y > \frac{1}{\gamma} \Phi_{H}^{i}(y, t), M_t = i \right],
\]  
(2.3)
\[
\Phi^i_L(y,t) = E_{y,t}[e^{-r(T-t)} \min(S_T, F) \cdot 1_{\min_j \{\tau^M_{u}, \tau^i\} \geq T} + \sum_{j \neq i} e^{-r(\min_j \tau^i - t)} \Phi^j_O(y,t) \cdot 1_{\min_j \{\tau^i\} \leq \min \{\tau^M_{u}, T\}} + e^{-r(\tau^M_{u} - t)} \Phi^i_H(S_{\tau^M_{u}}, \tau^M_{u}) \cdot 1_{\tau^M_{u} \leq \min_j \{\tau^i, T\}}
\]
\[
|S_t = y < \frac{1}{\gamma} \Phi^i_L(y,t), M_t = i|, \tag{2.4}
\]
where \(1_{\text{event}} = \begin{cases} 1, & \text{if “event” happens}, \\ 0, & \text{otherwise}, \end{cases} \)

\[
\Phi^j_O(y,t) = \begin{cases} \Phi^j_H, & \text{if } \Phi^j_O < \gamma y, \\ \Phi^j_L, & \text{otherwise}, \end{cases} \quad i \in \kappa.
\]

2.3. **PDE problem.** By Feynman-Kac formula, it is not difficult to drive that \(\Phi^i_H\) and \(\Phi^i_L\) are the functions of the firm value \(S\) and time \(t\). They satisfy the following partial differential equations in their regions:

\[
\frac{\partial \Phi^i_H}{\partial t} + \frac{1}{2} \sigma^H_i S^2 \frac{\partial^2 \Phi^i_H}{\partial S^2} + r^i S \frac{\partial \Phi^i_H}{\partial S} - r^i \Phi^i_H + \sum_{j \neq i} \lambda_{ij}(\Phi^j_O - \Phi^i_H) = 0, \\
S > \frac{1}{\gamma} \Phi^i_H, \quad t > 0, \tag{2.5}
\]

\[
\frac{\partial \Phi^i_L}{\partial t} + \frac{1}{2} \sigma^L_i S^2 \frac{\partial^2 \Phi^i_L}{\partial S^2} + r^i S \frac{\partial \Phi^i_L}{\partial S} - r^i \Phi^i_L + \sum_{j \neq i} \lambda_{ij}(\Phi^j_O - \Phi^i_L) = 0, \\
0 < S < \frac{1}{\gamma} \Phi^i_L, \quad t > 0, \tag{2.6}
\]

with the terminal condition:

\[
\Phi^i_H(S,T) = \Phi^i_L(S,T) = \min\{S, F\}. \tag{2.7}
\]

where \(i \in \kappa\).

(2.3) and (2.4) imply that the value of the bond is continuous when it passes the rating threshold, i.e.

\[
\Phi^i_H = \Phi^i_L \quad \text{on the rating migration boundary}. \tag{2.8}
\]

**Remark 2.1.** Equation (2.8) holds on the rating migration boundary in each macro region. Otherwise there would be arbitrage opportunities.

Also, if we construct a risk free portfolio \(\pi\) by longing a bond and shorting \(\Delta\) amount asset value \(S\), i.e., \(\pi^i_t = \Phi^i_t - \Delta^i_t S_t\) and such that \(d\pi_t = r\pi_t\), this portfolio is also continuous when it passes the rating migration boundary, i.e.,

\[
\pi^i_H = \pi^i_L \quad \text{on the rating migration boundary}, \tag{2.9}
\]

or by (2.8),

\[
\Delta^i_H = \Delta^i_L \quad \text{on the rating migration boundary}. \tag{2.10}
\]

By Black-Scholes theory (e.g., [13]), it is equivalent to

\[
\frac{\partial \Phi^i_H}{\partial S} = \frac{\partial \Phi^i_L}{\partial S} \quad \text{on the rating migration boundary}, \tag{2.11}
\]

for \(i \in \kappa\).
3. Free boundary problem. Using the standard change of variables \( x = \log S \) and rename \( T - t \) as \( t \), for \( i \in \kappa \), define

\[
\phi^j(x, t) = \begin{cases} 
\Phi^j_H(e^x, T - t) & \text{in the high rating,} \\
\Phi^j_L(e^x, T - t) & \text{in the low rating,}
\end{cases}
\]

using also (2.8) and (2.11), we then derive the following equation from (2.5), (2.6):

\[
\frac{\partial \phi^j}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 \phi^j}{\partial x^2} - \left(r^i - \frac{1}{2} \sigma^2 \right) \frac{\partial \phi^j}{\partial x} + r^i \phi^j - \sum_{j \in \kappa} \lambda_{ij} (\phi^j - \phi^i) = 0,
\]

\[-\infty < x < \infty, \ t > 0, \ i \in \kappa, \] (3.1)

where \( \sigma^i \) is a function of \( \phi^j \) and \( x \), i.e.,

\[
\sigma^i = \sigma^i(\phi, x) = \begin{cases} 
\sigma^i_H & \text{if } \phi^j < \gamma e^x, \\
\sigma^i_L & \text{if } \phi^j \geq \gamma e^x.
\end{cases}
\] (3.2)

The constants \( \gamma, \sigma^i_H, \sigma^i_L \) are defined in (2.1), (2.2).

Without losing generality, we assume \( F = 1 \). Equation (2.5) is supplemented with the initial condition (derived from (2.7))

\[
\phi^j(x, 0) = \min\{e^x, 1\}, \quad -\infty < x < \infty, \ i \in \kappa.
\] (3.3)

For any \( i \), in \( i \)-macro environment state, the domain is divided into the high rating region \( \Omega^i_H \) where \( \phi^i < \gamma e^x \) and a low rating region \( \Omega^i_L \) where \( \phi^i > \gamma e^x \). We shall prove that these two domains will be separated by a free boundary \( x = s^i(t) \), and

\[
\Omega^i_H = \{x > s^i(t)\}, \quad \Omega^i_L = \{x < s^i(t)\}.
\]

In another word, \( s^i(t) \) is apriorily unknown since it should be solved by the equation

\[
\phi^i(s^i(t), t) = \gamma e^{s^i(t)},
\] (3.4)

where the solution \( \phi^i \) is apriorily unknown.

Since we have assumed that equation (2.5) is valid across the free boundary \( x = s^i(t) \), we can derive from (2.8), (2.11):

\[
\phi^i(s^i(t)^-, t) = \phi^i(s^i(t)^+, t) = \gamma e^{s^i(t)},
\] (3.5)

\[
\frac{\partial \phi^i}{\partial x}(s^i(t)^-, t) = \frac{\partial \phi^i}{\partial x}(s^i(t)^+, t),
\] (3.6)

\( i \in \kappa. \)

4. Preliminaries.

4.1. Approximation. Let \( H(\xi) \) be the Heaviside function, i.e., \( H(\xi) = 0 \) for \( \xi < 0 \) and \( H(\xi) = 1 \) for \( \xi > 0 \). Then we can rewrite (3.2) as

\[
\sigma^i = \sigma^i_H + (\sigma^i_L - \sigma^i_H)H(\phi^i - \gamma e^x), \ i \in \kappa.
\]

We approximate \( H(\xi) \) by a \( C^\infty \) function \( H_\varepsilon \) such that

\[
H_\varepsilon(\xi) = 0 \quad \text{for } \xi < -\varepsilon, \ H_\varepsilon = 1 \quad \text{for } \xi > 0, \ 0 \leq H_\varepsilon(\xi) \leq 2/\varepsilon \quad \text{for } -\infty < \xi < \infty.
\]

Consider the approximated system for \( i \in \kappa \),

\[
\mathcal{L}_\varepsilon[\phi^i_\varepsilon] = \frac{\partial \phi^i_\varepsilon}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 \phi^i_\varepsilon}{\partial x^2} - \left(r^i - \frac{1}{2} \sigma^2 \right) \frac{\partial \phi^i_\varepsilon}{\partial x} + r^i \phi^i_\varepsilon + \sum_{j \neq i} \lambda_{ij} (\phi^j_\varepsilon - \phi^i_\varepsilon) = 0, \ -\infty < x < \infty, \ t > 0,
\] (4.1)
with the initial condition
\[ \phi^\varepsilon(x, 0) = \min\{e^{\varepsilon}, 1\}, \quad -\infty < x < \infty, \quad (4.2) \]
\[ \sigma_i^\varepsilon(\phi^\varepsilon) = \sigma_i^\varepsilon + (\sigma_i^\varepsilon - \sigma_i^\varepsilon)H(\phi^\varepsilon - \gamma e^{\varepsilon}). \quad (4.3) \]
As \( \sigma_i^\varepsilon \) has uniform upper and lower positive bounds, it is not difficult to uncouple the system. Then by a suitable Fixed Point Theorem ([9]), we can prove that the system (4.1)-(4.2) admits a unique classical solution \( \{\phi^\varepsilon_i, i \in \kappa\} \) for any \( \varepsilon > 0 \). We now proceed to derive estimates for \( \phi^\varepsilon_i \).

4.2. Estimates for the approximating system. For \( i \in \kappa \), for the sake of the simplicity of (4.1), denote the operator
\[ \mathcal{L}e_{\varepsilon x} = \frac{\partial}{\partial t} - \frac{1}{2} \sigma_i^\varepsilon(u) \frac{\partial^2}{\partial x^2} - \left( r_i - \frac{1}{2} \sigma_i^\varepsilon(u) \right) \frac{\partial}{\partial x} + \left( r_i + \sum_{j \in \kappa} \lambda_{ij} \right). \]
We prove Lemma 4.1-Lemma 4.3 which are applied on a more general operator \( \mathcal{G} \) first. The lemmas are treated as maximum principles and can be extended as comparison lemmas for a system.

**Lemma 4.1.** For \( i \in \kappa \), \( \mathcal{G}^i = \frac{\partial}{\partial t} - a^i(x, t, u^i) \frac{\partial^2}{\partial x^2} + b^i(x, t, u^i) \frac{\partial}{\partial x} \), where \( a^i > 0 \), \( b^i \) and \( h_{ij} \) are continuous and bounded on \( \bar{Q}_T = \Omega \times [0, T] \), where \( \Omega \) is a bounded domain, and \( u(x, t) = (u^i(x, t)) \), if
1. \( \mathcal{G}^i[u^i] + \sum_{j \in \kappa} h_{ij} u^j < 0(> 0), \quad i \in \kappa; \]
2. \( h_{ij} \leq 0, \quad j \neq i, \quad i, j \in \kappa; \]
3. \( u^i(x, 0) < 0(> 0), \quad i \in \kappa; \]
4. \( u^i(x, t) < 0(> 0), \quad (x, t) \in \partial \Omega \times (0, T), \quad i \in \kappa, \)
then
\[ u(x, t) < 0(> 0), \quad (x, t) \in Q_T. \]

**Proof.** Let \( v^i = v^i e^{\alpha t} \), where \( \alpha > 0 \) is big enough. We want to prove \( v^i < 0 \). In fact, from conditions of the Lemma,
\[ \mathcal{G}^i[v^i] + \sum_{j \in \kappa} h_{ij} v^j < 0, \quad \text{for } \begin{cases} \tilde{h}_{ii} = h_{ii} + \alpha, & i \in \kappa \\ \tilde{h}_{ij} = h_{ij} \leq 0, & i \neq j \end{cases}, \]
and \( v^i(x, 0) = u^i(x, 0) < 0, \quad i \in \kappa. \) Thus there exists \( \delta > 0 \), when \( 0 \leq t \leq \delta, \)
\( v(x, t) < 0 \) for all \( x \). Let
\[ A = \{ t : t \leq T, \text{ for all } x, 0 \leq s \leq t, v(x, s) < 0 \}, \]
then \( \bar{t} = \sup A \) exists and \( 0 \leq \bar{t} < T. \)

If the conclusion is not true, then when \( 0 < t < \bar{t}, \) \( v(x, t) \leq 0 \) and there exists \( \pi \in \bar{\Omega} \), such that \( v^i(\pi, \bar{t}) = 0 \), where \( v^i \) is one of components of \( v \). From condition 4., we conclude that \( x \in \bar{\Omega} \), and \( v^i \) attains maximum at \( (\pi, \bar{t}) \) in \( Q_{\bar{t}} \), then
\[ \left( \frac{\partial v^i}{\partial t} \right)_{(\pi, \bar{t})} \geq 0, \quad \left( \frac{\partial v^i}{\partial x} \right)_{(\pi, \bar{t})} = 0, \quad \left( \frac{\partial^2 v^i}{\partial x^2} \right)_{(\pi, \bar{t})} \leq 0, \]
thus \( \mathcal{G}^i[v^i]_{(\pi, \bar{t})} \geq 0. \) On the other hand, \( v^i(\pi, \bar{t}) \leq 0, \) \( v^i(\pi, 0) = 0, \) then
\[ \left[ \mathcal{G}^i[v^i] + \sum_{j \in \kappa} h_{ij} v^j \right]_{(\pi, \bar{t})} \geq 0, \]
which is a contradiction. \( \square \)
Corollary 4.2. If we change the conditions 1. 3. and 4. in Lemma 4.1 to $\mathcal{G}^i[u^i] + \sum_{j \in \kappa} h_{ij} w^j \leq 0 (\geq 0)$, $u(x,0) \leq 0 (\geq 0)$ and $u(x,t) \leq 0 (\geq 0)$ for $(x,t) \in \partial\Omega \times (0,T)$ respectively, then the conclusion turns to be $u(x,t) \leq 0 (\geq 0)$.

Proof. Since each $h_{ij}$ is bounded, there exists $\beta > 0$, such that $\beta + \sum_{j \in \kappa} h_{ij} > 0$. Let $v^i = u^i - \varepsilon e^{\beta t}$, then

$$\mathcal{G}^i[v^i] + \sum_{j \in \kappa} \tilde{h}_{ij} v^j = \mathcal{G}^i[u^i] + \sum_{j \in \kappa} \tilde{h}_{ij} w^j - \varepsilon e^{\beta t}(\beta + \sum_{j \in \kappa} \tilde{h}_{ij}) < 0,$$

and $v(x,0) < 0, u(x,t)|_{\partial\Omega} < 0$ From Lemma 4.1, it is obtained that $v(x,t) < 0$. Then let $\varepsilon \to 0$, we have $u(x,t) \leq 0$. \hfill $\Box$

We have proved the maximum principle for our problem in bounded domain. Next, we will prove that the conclusion still holds in unbounded domain by using above lemmas.

Lemma 4.3. For $i \in \kappa$, $\mathcal{G}^i = \frac{\partial}{\partial t} - a^i(x,t,u^i) \frac{\partial^2}{\partial x^2} + b^i(x,t,u^i) \frac{\partial}{\partial x}$, where $a^i > 0, b^i, h_{ij}$ and $u^i$ are continuous and bounded on $R \times [0,T]$, and $u(x,t) = (u^i(x,t))$, if

1. $\mathcal{G}^i[u^i] + \sum_{j \in \kappa} h_{ij} w^j \leq 0 (\geq 0)$, $i \in \kappa$;
2. $h_{ij} \leq 0$, $j \neq i, \ k, j \in \kappa$;
3. $u^i(x,0) \leq 0 (\geq 0)$, $i \in \kappa$;

then

$$u(x,t) \leq 0 (\geq 0), (x,t) \in R \times [0,T].$$

Proof. Set $w^i(x,t) = u^i(x,t) - \frac{m}{L^2} x^2 e^{\alpha t}$ for $i \in \kappa$, and $w(x,t) = (w^i(x,t))$, where $m, \alpha > 0$. Then

$$\mathcal{G}^i[w^i] + \sum_{j \in \kappa} h_{ij} w^j = \mathcal{G}^i[w^i + \frac{m}{L^2} x^2 e^{\alpha t}] + \sum_{j \in \kappa} h_{ij} (w^j + \frac{m}{L^2} x^2 e^{\alpha t}) \leq 0.$$ 

By calculation, we could obtain that

$$\mathcal{G}^i[w^i] + \sum_{j \in \kappa} h_{ij} w^j \leq -\frac{m}{L^2} e^{\alpha t}(\alpha x^2 - 2a^i + 2b^i x + \sum_{j \in \kappa} h_{ij} x^2).$$

Since $a^i > 0, b^i$ and $h_{ij}$ are bounded on $R \times [0,T]$, we choose $\alpha$ big enough, such that

$$-\frac{m}{L^2} e^{\alpha t}(\alpha x^2 - 2a^i + 2b^i x + \sum_{j \in \kappa} h_{ij} x^2) \leq 0.$$ 

That is, $\mathcal{G}^i[w^i] + \sum_{j \in \kappa} h_{ij} w^j \leq 0$. Consider $(x,t) \in Q_L = \{|x| \leq L, 0 \leq t \leq T\}$, then we have

$$\mathcal{G}^i[w^i] + \sum_{j \in \kappa} h_{ij} w^j \leq 0, (x,t) \in Q_L$$

$$w^i(x,0) \leq u^i(x,0) \leq 0$$

On the other hand, $u^i$ is bounded on $R \times [0,T]$, so we choose $m$ big enough, such that $|u^i| < m$. Then at $|x| = L$, we have $w^i(x,t) = u^i(x,t) - m e^{\alpha t} \leq u^i(x,t) - m \leq 0$. It follows by Corollary 4.2 that $w \leq 0$, for $(x,t) \in Q_L$.

For any point $(x_0,t_0) \in R \times [0,T]$, we could choose $L$ big enough such that $(x_0,t_0) \in Q_L$. From above we know that $w(x_0,t_0) = u(x_0,t_0) - \frac{m}{L^2} x_0^2 e^{\alpha t_0} \leq 0$. Then the conclusion holds by letting $L \to +\infty$. \hfill $\Box$

Lemma 4.4.

$$0 \leq \phi^i \leq \min\{1, e^x\}, \quad i \in \kappa.$$
Proof. It is not difficult to verify the conclusion for $\phi^i_\epsilon$ satisfying (4.1) by Lemma 4.3 we have proved.

Lemma 4.5.

\[ \frac{\partial \phi^i_\epsilon}{\partial x} \geq 0, \quad -e^{-rt} \leq \frac{\partial \phi^i_\epsilon}{\partial x} - \phi^i_\epsilon < 0, \quad i \in \kappa, \quad \text{for } 0 < t \leq T \]

where $r = \min\{r^i\}.$

Proof. Differentiating equation (4.1) with respect to $x$, we obtain

\[ \hat{\mathcal{L}}^i_\epsilon \left[ \frac{\partial \phi^i_\epsilon}{\partial x} \right] = \left( \frac{\partial^2 \phi^i_\epsilon}{\partial x^2} - \frac{\partial \phi^i_\epsilon}{\partial x} \right) \cdot \sigma^i_\epsilon \cdot (\sigma^i_L - \sigma^i_H) \cdot H^i_\epsilon (\phi^i_\epsilon - \gamma e^x) \cdot \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \gamma e^x \right) - \lambda i j \frac{\partial \phi^j_\epsilon}{\partial x} = 0. \] (4.4)

Treat $\sigma^i_\epsilon \cdot (\sigma^i_L - \sigma^i_H) \cdot H^i_\epsilon (\phi^i_\epsilon - \gamma e^x) \cdot \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \gamma e^x \right)$ to be a given function, which is bounded and $-\lambda i j < 0$, $i \neq j$, we are able to use Lemma 4.3 with checking \( \frac{\partial \phi^i_\epsilon}{\partial x}(x, 0) = e^x > 0 \) for $x < 0$ and \( \frac{\partial \phi^i_\epsilon}{\partial x}(x, 0) = 0 \) for $x > 0$. Therefore, it follows that \( \frac{\partial \phi^i_\epsilon}{\partial x} \geq 0. \)

Using (4.1), we find that $w^i = \frac{\partial \phi^i_\epsilon}{\partial x} - \phi^i_\epsilon$, $i \in \kappa$ satisfies

\[ \hat{\mathcal{L}}^i_\epsilon \left[ w^i \right] = \hat{\mathcal{L}}^i_\epsilon \left[ \phi^i_\epsilon \right] - \frac{\partial w^i}{\partial x} \cdot \sigma^i_\epsilon \cdot \left( \sigma^i_L - \sigma^i_H \right) \cdot H^i_\epsilon (\phi^i_\epsilon - \gamma e^x) \cdot \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \gamma e^x \right) - \lambda i j w^j = 0. \]

It is also clear that initially $w^i = 0$ for $x < 0$ and $w^i = -1$ for $x > 0$. It follows by Lemma 4.1 that $w^i \leq 0$. It is also clear that $\hat{\mathcal{L}}^i_\epsilon [-e^{-rt}] = (r - r^i)e^{-rt} \leq 0$, so that $w^i \geq -e^{-rt}.$

Lemma 4.6.

\[ \frac{\partial^2 \phi^i_\epsilon}{\partial x^2} - \frac{\partial \phi^i_\epsilon}{\partial x} < 0, \quad i \in \kappa. \]

Proof. Differentiating equation (4.1) with respect to $t$, we obtain

\[ \hat{\mathcal{L}}^i_\epsilon \left[ \frac{\partial \phi^i_\epsilon}{\partial t} \right] = \left( \frac{\partial^2 \phi^i_\epsilon}{\partial x^2} - \frac{\partial \phi^i_\epsilon}{\partial x} \right) \cdot \sigma^i_\epsilon \cdot \left( \sigma^i_L - \sigma^i_H \right) \cdot H^i_\epsilon (\phi^i_\epsilon - \gamma e^x) \cdot \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \gamma e^x \right) - \lambda i j \frac{\partial \phi^j_\epsilon}{\partial t} = 0. \] (4.5)

Combining the equations for $\phi^i_\epsilon$, \( \frac{\partial \phi^i_\epsilon}{\partial x} \) and \( \frac{\partial \phi^i_\epsilon}{\partial t} \), we obtain

\[ \hat{\mathcal{L}}^i_\epsilon \left[ \frac{\partial \phi^i_\epsilon}{\partial t} - r^i \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \phi^i_\epsilon \right) \right] = \left( \frac{\partial^2 \phi^i_\epsilon}{\partial x^2} - \frac{\partial \phi^i_\epsilon}{\partial x} \right) \cdot \sigma^i_\epsilon \cdot \left( \sigma^i_L - \sigma^i_H \right) \cdot H^i_\epsilon (\phi^i_\epsilon - \gamma e^x) \cdot \left\{ \frac{\partial \phi^i_\epsilon}{\partial t} - r^i \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \gamma e^x \right) \right\}
+ \lambda i j \left[ \frac{\partial \phi^i_\epsilon}{\partial t} - r^i \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \phi^i_\epsilon \right) \right]. \]

Thus

\[ w^i = \frac{1}{2} (\sigma^i_\epsilon)^2 \left( \frac{\partial^2 \phi^i_\epsilon}{\partial x^2} - \frac{\partial \phi^i_\epsilon}{\partial x} \right) = \frac{\partial \phi^i_\epsilon}{\partial t} - r^i \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \phi^i_\epsilon \right) + \lambda i j (\phi^i_\epsilon - \phi^i_\epsilon) \]

satisfies

\[ \hat{\mathcal{L}}^i_\epsilon \left[ w^i + \lambda i j (\phi^i_\epsilon - \phi^i_\epsilon) \right] = \frac{2w^i}{\sigma^i_\epsilon} \cdot \left( \sigma^i_L - \sigma^i_H \right) \cdot H^i_\epsilon (\phi^i_\epsilon - \gamma e^x) \cdot \left\{ \frac{\partial \phi^i_\epsilon}{\partial t} - r^i \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \gamma e^x \right) \right\}
+ \lambda i j \left[ \frac{\partial \phi^i_\epsilon}{\partial t} - r^i \left( \frac{\partial \phi^i_\epsilon}{\partial x} - \phi^i_\epsilon \right) \right]. \]
It is also clear that initially 
\[ L^i_\varepsilon(\phi^i) = \lambda_i \phi^i_\varepsilon, \]
then we have
\[ L^i_\varepsilon(\phi^i) = \frac{\partial \phi^i}{\partial t} - \frac{1}{2} (\sigma^i_\varepsilon)^2 \frac{\partial^2 \phi^i}{\partial x^2} - \left[ r^i - \frac{1}{2} (\sigma^i_\varepsilon)^2 \right] \frac{\partial \phi^i}{\partial x} + (r^i + \lambda_i) \phi^i_\varepsilon \]
\[ = \frac{\partial \phi^i}{\partial t} - r^i \left( \frac{\partial \phi^i}{\partial x} - \phi^i_\varepsilon \right) - \left( \frac{\sigma^i_\varepsilon}{\varepsilon} \right)^2 w^i + \lambda_{ij} \phi^j_\varepsilon, \]
then we have
\[ L^i_\varepsilon[w^i] = \frac{2w^i}{\sigma^i_\varepsilon} \cdot (\sigma^i_L - \sigma^i_H) \cdot \epsilon_i^i (\phi^i_\varepsilon - \gamma e^x) \cdot \epsilon_i^i \left( \frac{\partial \phi^i}{\partial t} - r^i \left( \frac{\partial \phi^i}{\partial x} - \gamma e^x \right) \right) - \lambda_{ij} \left( \frac{\sigma^j_\varepsilon}{\varepsilon} \right)^2 w^j = 0. \] (4.6)

At \( t = 0 \), \( w^i \) produces a dirac measure of intensity \(-1\) at \( x = 0 \) and by \( w^i(x, 0) = 0 \) for both \( x < 0 \) and \( x > 0 \). By further approximating the initial data with smooth functions if necessary, we derive \( w^i < 0 \). Hence the lemma holds.

Lemma 4.7. There exist constants \( c_1, C_2 \) and \( C_3 \), independent of \( \varepsilon \), such that
\[ -C_3 - \frac{C_2}{\sqrt{t}} \exp \left( -c_1 \frac{x^2}{t} \right) \leq \frac{\partial \phi^i}{\partial t} < 0, \quad i \in \kappa, \quad \text{for } 0 < t \leq T. \]

Proof. From (4.5), \( \frac{\partial \phi^i}{\partial t} \) satisfies
\[ L^i_\varepsilon \left[ \frac{\partial \phi^i}{\partial t} \right] = \left( \frac{\partial^2 \phi^i}{\partial x^2} - \frac{\partial \phi^i}{\partial x} \right) \cdot (\sigma^i_L - \sigma^i_H) \cdot \epsilon_i^i (\phi^i_\varepsilon - \gamma e^x) \cdot \epsilon_i^i \frac{\partial \phi^i}{\partial t} - \lambda_{ij} \frac{\partial \phi^j}{\partial t} = 0. \]
It is also clear that initially
\[ \frac{\partial \phi^i}{\partial t}(x, 0) = -r^i \quad \text{for } x > 0, \quad \frac{\partial \phi^i}{\partial t}(x, 0) = 0 \quad \text{for } x < 0. \]
At \( x = 0 \), \( \frac{\partial^2 \phi^i}{\partial x^2}(x, 0) \) produces a dirac measure of density \(-1\). Thus \( \frac{\partial \phi^i}{\partial t}(x, 0) \leq 0 \) in the distribution sense. By further approximating the initial data with smooth functions and restricting the infinite region if necessary, we conclude by the Maximum Principle
\[ \frac{\partial \phi^i}{\partial t} < 0 \quad \text{for } -\infty < x < \infty, \quad t > 0. \]
Next, since \( \phi^i_0(0, 0) = 1 > \gamma \), and by Hölder continuity of the solution, there exists a \( \rho > 0 \), independent of \( \varepsilon \), such that
\[ \phi^i(x, t) > (1 + \gamma)/2 > \gamma e^x \quad \text{for } |x| \leq \rho, \quad 0 \leq t \leq \rho^2. \]
Thus \( \sigma^i_\varepsilon \equiv \sigma^i_L \) for \( |x| \leq \rho, \quad 0 \leq t \leq \rho^2 \). It follows from the standard parabolic estimates that
\[ \frac{\partial \phi^i}{\partial t} \geq -C_2 - \frac{C_2}{\sqrt{t}} \exp \left( -c_1 \frac{x^2}{t} \right) \quad \text{for } |x| < \frac{\rho}{2}, \quad 0 < t \leq \rho^2. \] (4.7)
In particular, this implies that
\[ \frac{\partial \phi^i}{\partial t} \geq -C_3 \quad \text{on } \{|x| = \frac{\rho}{2}; 0 < t \leq \rho^2\} \cup \{|x| > \frac{\rho}{2}, t = \rho^2\}. \] (4.8)
We now consider \( \rho^2 \leq t \leq T \), and define
\[ \phi^i_\varepsilon[v^i] = L^i_\varepsilon[v^i] - \left( \frac{\partial^2 \phi^i}{\partial x^2} - \frac{\partial \phi^i}{\partial x} \right) \cdot (\sigma^i_L - \sigma^i_H) \cdot \epsilon_i^i (\phi^i_\varepsilon - \gamma e^x) \cdot v^i - \lambda_{ij} v^j. \]
Then, for any $C > 0$,
\[ G_i^\varepsilon \left[ \frac{\partial \phi_i^\varepsilon}{\partial t} \right] = 0, \]
\[ G_i^\varepsilon \left[ -C \right] = -r^i C + \left( \frac{\partial^2 \phi_i^\varepsilon}{\partial x^2} - \frac{\partial \phi_i^\varepsilon}{\partial x} \right) \cdot \sigma_i^\varepsilon \cdot (\sigma_i^L - \sigma_i^H) \cdot H_i^\varepsilon (\phi_i^\varepsilon - \gamma \varepsilon^x) \cdot C < 0, \]
which implies
\[ G_i^\varepsilon \left[ \frac{\partial \phi_i^\varepsilon}{\partial t} + C \right] = r^i C - \left( \frac{\partial^2 \phi_i^\varepsilon}{\partial x^2} - \frac{\partial \phi_i^\varepsilon}{\partial x} \right) \cdot \sigma_i^\varepsilon \cdot (\sigma_i^L - \sigma_i^H) \cdot H_i^\varepsilon (\phi_i^\varepsilon - \gamma \varepsilon^x) \cdot C \geq 0. \]
Then $\frac{\partial \phi_i}{\partial t} \geq -C_3$ in the region $R \times [0, +\infty) \setminus \left( \frac{-\rho}{2}, \frac{\rho}{2} \right) \times (0, \rho^2)$.

As an immediate corollary, we have

**Lemma 4.8.**
\[ -C_4 - C_5 \exp \left( -c_1 \frac{x^2}{T} \right) \leq \frac{\partial^2 \phi_i^\varepsilon}{\partial x^2} \leq C_6 \]

Next, we derive the estimates of free boundaries. Denote $s_i^\varepsilon (t)$ is the approximated free boundary, which is the implied solution of the equation
\[ \phi_i^\varepsilon (s_i^\varepsilon (t), t) = \gamma e^{s_i^\varepsilon (t)}. \tag{4.9} \]

**Lemma 4.9.** The approximated free boundary $s_i^\varepsilon (t)$ is uniquely defined by (4.9).

**Proof.** Let $F_i^\varepsilon (x, t) = \phi_i^\varepsilon (x, t) - \gamma e^x$, $i \in \kappa$. Since $\phi_i^\varepsilon (x, t)$ is smooth enough, then $\frac{\partial \phi_i^\varepsilon}{\partial x} (x, t)$ and $\frac{\partial \phi_i^\varepsilon}{\partial t} (x, t)$ are continuous with respect to both $x$ and $t$ in $R \times [0, T]$.

That is,
\[ F_i^\varepsilon, \frac{\partial \phi_i^\varepsilon}{\partial x}, \frac{\partial \phi_i^\varepsilon}{\partial t} \text{ are continuous with respect to } x \text{ and } t. \tag{4.10} \]

For any $t_0 \in [0, T]$, there exists $K > 0$, such that $\phi_i^\varepsilon (-K, t_0) = \gamma e^{-K}$ and $\phi_i^\varepsilon (K, t_0) = \gamma e^K < 0$ (The proof could be seen in Lemma 4.10 and Lemma 4.11). Then by Zero Point Theorem, there exists $x_0 \in [-K, K]$ such that,
\[ F_i^\varepsilon (x_0, t_0) = \phi_i^\varepsilon (x_0, t_0) - \gamma e^{x_0} = 0. \tag{4.11} \]

Besides, for $0 < t \leq T$, we have $\frac{\partial \phi_i^\varepsilon}{\partial x} (x, t) - \phi_i^\varepsilon (x, t) < 0$ from Lemma 4.5. And at $t = 0$, $\phi_i^\varepsilon (x, 0) = \gamma e^{x_0} = \min \{ e^{x_0}, 1 \} = 1$ and then $\frac{\partial \phi_i^\varepsilon}{\partial x} (x, 0) = 0$, i.e. $\frac{\partial \phi_i^\varepsilon}{\partial x} (x, 0) - \phi_i^\varepsilon (x, 0) < 0$. Then for any $t_0 \in [0, T]$,
\[ \frac{\partial \phi_i^\varepsilon}{\partial x} (x_0, t_0) - \gamma e^{x_0} = \frac{\partial \phi_i^\varepsilon}{\partial x} (x_0, t_0) - \phi_i^\varepsilon (x_0, t_0) < 0. \]

That is,
\[ F_i^\varepsilon (x_0, t_0) = \frac{\partial \phi_i^\varepsilon}{\partial x} (x, t) - \gamma e^x \bigg|_{(x_0, t_0)} < 0, \tag{4.12} \]
and (4.12) holds in a small neighborhood of $(x_0, t_0)$. From (4.10)-(4.12), for any $t_0 \in [0, T]$, there exists $x_0 \in [-K, K]$, such that $\phi_i^\varepsilon (x_0, t_0) = \gamma e^{x_0} = 0$. We claim that the $x_0$ is unique for any fixed $t_0 \in [0, T]$. If not, we denote the two adjacent zero points by $\bar{x}_0$ and $\tilde{x}_0$, which both satisfy (4.11). And from (4.12), we have $F_i^\varepsilon (x, t) < 0$ holds in the neighborhood of $(\bar{x}_0, t_0)$. Since $\bar{x}_0$ is the adjacent zero point of $\tilde{x}_0$, then $F_i^\varepsilon (\bar{x}_0, t_0) \geq 0$, which is a contradiction of (4.12). Therefore, for any $t \in [0, T]$, we could correspondingly find a unique $x$ such that $\phi_i^\varepsilon (x, t) - \gamma e^x = 0$, which means $x$ is a function of $t$. We denote the function by $x = s_i^\varepsilon (t)$ and conclude that $s_i^\varepsilon (t)$ is uniquely defined by (4.9).
Then, we have the following estimates for the free boundary.

**Lemma 4.10.** The approximated free boundary defined in (4.9) satisfies

\[ s^i_\varepsilon(t) \leq \log \frac{1}{\gamma} - rt, \quad i \in \kappa, \quad (4.13) \]

where \( r = \min_{i \in \kappa} \{r^i\} \).

**Proof.** It is also clear that, by Maximum Principle, \( \phi^i_\varepsilon < e^{-rt} \), so that,

\[ \phi^i_\varepsilon < \gamma e^x \quad \text{for} \quad x > \log \frac{1}{\gamma} - rt. \]

This means that the region \( \{x > \log \frac{1}{\gamma} - rt\} \) is in the high rating region and hence (4.13) holds. \( \Box \)

We next derive lower bound for \( s^i_\varepsilon(t) \).

**Lemma 4.11.** The approximated free boundary defined in (4.9) satisfies, for any \( A > 1 \).

\[ s^i_\varepsilon(t) \geq \frac{1}{A-1} \log(1-\gamma) - \left( \frac{1}{2} \sigma^2_L A + \tau \right) t, \quad i \in \kappa, \]

where \( \sigma_L = \max_{i \in \kappa} \{\sigma^i_L\} \), \( \tau = \max_{i \in \kappa} \{r^i\} \)

**Proof.** For any \( A > 1 \), we take \( B = (A-1)\left(\frac{1}{2} \sigma^2_L A + \tau\right) \). Then

\[
\mathcal{L}_\varepsilon[e^{Ax+bt}] = e^{Ax+bt} \left[ B - \frac{1}{2} \sigma^2_L (A^2 - A) - \tau(A-1) \right] = e^{Ax+bt}(A-1)A \left( \frac{1}{2} \sigma^2_L - \frac{1}{2} \sigma^2_L + \tau - r^i \right) \geq 0.
\]

It follows that

\[ \mathcal{L}_\varepsilon[e^x - e^{Ax+bt}] \leq 0. \]

It is obvious that \( \phi_\varepsilon(0,t) > 0 \geq 1 - e^{Bt} \). Thus we can apply Maximum Principle to obtain

\[ \phi^i_\varepsilon > e^x - e^{Ax+bt} \quad \text{for} \quad -\infty < x < 0, \ t > 0. \]

Hence

\[ \phi^i_\varepsilon > \gamma e^x \quad \text{for} \quad x < \frac{1}{A-1} \log(1-\gamma) - \frac{B}{A-1} t, \]

which means that \( \{x < \frac{1}{A-1} \log(1-\gamma) - \frac{B}{A-1} t\} \) is in the low rate region and

\[ s^i_\varepsilon(t) \geq \frac{1}{A-1} \log(1-\gamma) - \frac{B}{A-1} t. \]

Using the definition of \( B \) we conclude. \( \Box \)

**Lemma 4.12.** For any \( T > 0 \), there exists \( C_T > 0 \), independent of \( \varepsilon \), such that the derivative of the approximated free boundary \( s^i_\varepsilon(t) \) is bounded by

\[ -C_T \leq s^i_\varepsilon'(t) < 0 \quad \text{for} \quad 0 < t < T, \quad i \in \kappa. \quad (4.14) \]
Proof. Clearly,
\[ s^{i \iota}_\varepsilon(t) = \frac{\partial \phi^i_\varepsilon(s^{i}_\varepsilon(t),t)}{\phi^i_\varepsilon(s^{i}_\varepsilon(t),t) - \partial \phi^i_\varepsilon(s^i(t),t)}. \]

The estimates \( \frac{\partial \phi^i_\varepsilon}{\partial t} < 0 \) and \( \frac{\partial \phi^i_\varepsilon}{\partial x} - \phi^i_\varepsilon < 0 \) imply that the approximated free boundary is strictly decreasing:
\[ s^{i \iota}_\varepsilon(t) < 0. \tag{4.15} \]

By Lemma 4.7, there is a constant \( \rho > 0 \) (independent of \( \varepsilon \)) such that \( s^{i}_\varepsilon(t) \geq \rho \) for \( 0 \leq t \leq \rho^2 \). It follows from Lemma 4.7 that
\[ -C^* \leq \frac{\partial \phi^i_\varepsilon}{\partial t}(s^{i}_\varepsilon(t),t) \leq 0 \quad 0 \leq t < T \tag{4.16} \]
for some constant \( C^* \) independent of \( \varepsilon \). To finish the proof, it suffices to establish \( \phi^i_\varepsilon(s^{i}_\varepsilon(t),t) - \frac{\partial \phi^i_\varepsilon}{\partial t}(s^{i}_\varepsilon(t),t) \geq c^* \) for some positive \( c^* \) independent of \( \varepsilon \).

Let \( \mathcal{D}^i \) be the operator defined inLemma 4.5. As shown in Lemma 4.5, \( w^i \equiv \psi^i - \frac{\partial \phi^i}{\partial x} \) satisfies \( \mathcal{D}^i[w^i] = 0 \) and \( w^i(x,0) = 1 \) for \( x > 0 \), \( w^i(x,0) = 0 \) for \( x < 0 \). By Lemmas 4.7, 4.10 and 4.11, there exists \( R_T > 0 \), independent of \( \varepsilon \), such that
\[ -R_T + 1 \leq s^i_\varepsilon(t) \leq R_T - 1 \quad 0 < t \leq T, \]
and
\[ s^i_\varepsilon(t) \geq \rho \quad \text{for} \quad 0 \leq t \leq \rho^2. \]

Consider the region
\[ \Omega_1 \equiv \{ \rho/2 < x < R, 0 < t \leq \rho^2 \} \cup \{-R_T \leq x \leq R_T, \rho^2 \leq t \leq T \}. \]
The parabolic boundary of this region \( \Omega_1 \) consists of 5 line segments. On the initial line segment \( \{(x,0), \rho/2 \leq x \leq \rho^2 \} \cup \{(R_T,0), 0 \leq t \leq T \}, w^i(x,0) = 1 \). The remaining 4 parabolic boundaries \( \{(R_T,t), 0 \leq t \leq T \} \cup \{(-R_T,t), 0 \leq t \leq T \} \cup \{(x,\rho^2), 0 \leq x \leq \rho/2 \} \cup \{(x,0), -R_T \leq x \leq R_T \} \) are completely and uniformly within the high or low rating region (independent of \( \varepsilon \)). Thus by compactness and strong maximum principle, on these 4 boundaries, \( w^i \geq \bar{c} > 0 \) for some \( \bar{c} \) independent of \( \varepsilon \). It follows that \( w^i \geq \min(1, \bar{c}) \geq \min(1, \bar{c}) \equiv c^* \) on \( \Omega_1 \) and this establishes in (4.14). \( \square \)

Lemma 4.13. (Comparison principle) If \( \mathcal{L}^i[\phi^i] \geq \mathcal{L}^i[\psi^i] \), \( \phi^i(x,0) \geq \psi^i(x,0) \), and \( \frac{\partial^2 \phi^i}{\partial x^2} - \frac{\partial \phi^i}{\partial x} \leq 0 \) or \( \frac{\partial^2 \psi^i}{\partial x^2} - \frac{\partial \psi^i}{\partial x} \leq 0 \), then \( \phi^i(x,t) \geq \psi^i(x,t), i \in \kappa \).

Proof. Without loss of generality, we suppose \( \frac{\partial^2 \psi^i}{\partial x^2} - \frac{\partial \psi^i}{\partial x} \leq 0 \). Besides, in this paper, two macro environment regions are considered, then \( \kappa = \{1,2\} \). Let \( w^i = \phi^i - \psi^i \), then \( w^i \) satisfies
\[
\frac{\partial w^i}{\partial t} - \frac{1}{2} \sigma^2(\phi^i) \left( \frac{\partial^2 w^i}{\partial x^2} - \frac{\partial w^i}{\partial x} \right) - r^i \frac{\partial w^i}{\partial x} + r^i w^i - \sum_{j \in \kappa} \lambda_{ij}(w^j - w^i) = \frac{1}{2} \left( \sigma^2(\phi^i) - \sigma^2(\psi^i) \right) \left( \frac{\partial^2 \psi^i}{\partial x^2} - \frac{\partial \psi^i}{\partial x} \right) - F^i,
\]
and \( w^i(x, 0) = \phi^i(x, 0) - \psi^i(x, 0) \geq 0 \). Denote \( v = (v^i) \), \( i \in \kappa \) is the solution of

\[
\frac{\partial v^i}{\partial t} - \frac{1}{2} \sigma^2(\phi^i) \left( \frac{\partial^2 v^i}{\partial x^2} - \frac{\partial v^i}{\partial x} \right) - r^i \frac{\partial v^i}{\partial x} + r^i v^i - \sum_{j \neq i} \lambda_{ij} (v^j - v^i) = F^i,
\]

\[ v^i(x, 0) = 0, \]

then \( w^i(x, t) \geq v^i(x, t) \). Since \( F^i \equiv 0 \) for \( |x| > M \), the solution \( v \) decays exponentially fast to 0 as \( x \to \pm \infty \). It follows that

\[
\liminf_{x \to \pm \infty} w^i(x, t) \geq \liminf_{x \to \pm \infty} v^i(x, t) = 0, \quad 0 \leq t \leq T. \tag{4.17}
\]

Therefore if the conclusion is not true, one of components of \( w \), which is denoted by \( w^1 \) must attain a negative minimum at a point \( (x_1^*, t_1^*) \) with \( x_1^* \) finite and \( 0 < t_1^* \leq T \). It is clear that at this point \( \phi^1(x_1^*, t_1^*) < \psi^1(x_1^*, t_1^*) \) and \( \sigma^1(\phi^1) \leq \sigma^1(\psi^1) \) in a small parabolic neighborhood of \( (x_1^*, t_1^*) \). It follows that

\[
\liminf_{(x,t) \to (x_1^*, t_1^*)} \text{ess } F^1(x, t) \geq 0. \tag{4.18}
\]

**Case 1.** If \( w^1(x_1^*, t_1^*) \leq w^2(x_1^*, t_1^*) \), by parabolic version of Bony’s maximum principle,

\[
\limsup_{(x,t) \to (x_1^*, t_1^*)} \text{ess } \left\{ w^1_i - \frac{1}{2} (\sigma(\phi^1))^2 (w^1_{xx} - w^1_x) - r^1 w^1 + r^1 w^1 + \lambda_{12} (w^1 - w^2) \right\} < 0,
\]

which is a contradiction.

**Case 2.** If \( w^1(x_1^*, t_1^*) > w^2(x_1^*, t_1^*) \), that is \( w^2(x_1^*, t_1^*) < 0 \), then \( w^2 \) must attain a negative minimum at a point \( (x_2^*, t_2^*) \), and

\[
\liminf_{(x,t) \to (x_2^*, t_2^*)} \text{ess } F^2(x, t) \geq 0. \tag{4.20}
\]

It follows that

\[
w^2(x_2^*, t_2^*) \leq w^2(x_1^*, t_1^*) < w^1(x_1^*, t_1^*) \leq w^1(x_2^*, t_2^*),
\]

then

\[
\limsup_{(x,t) \to (x_2^*, t_2^*)} \text{ess } \left\{ w^2_i - \frac{1}{2} (\sigma(\phi^2))^2 (w^2_{xx} - w^2_x) - r^2 w^2 + r^2 w^2 + \lambda_{21} (w^2 - w^1) \right\} < 0,
\]

which is also a contradiction.

5. **Existence & uniqueness.** Lemmas 4.4-4.8 provide estimates of approximated solution \( \phi^i_\varepsilon \), \( i \in \kappa \). By taking a limit as \( \varepsilon \to 0 \) (along a subsequence if necessary), we derive the existence of problem (3.1)-(3.6).

Lemmas 4.10-4.12 show that there is a uniform estimate in space \( C^1([0, T]) \) for the approximated free boundary \( s^i_\varepsilon(t), i \in \kappa \). Therefore, the limit of \( s^i_\varepsilon(t) \) as \( \varepsilon \to 0 \) exists, which is denoted by \( s^i(t), i \in \kappa \). This \( s^i(t), i \in \kappa \) is the free boundary of our problem (3.1)-(3.6).
Theorem 5.1. The free boundary problem (3.1)-(3.6) admits a solution \((\phi^i, s^i)\), \(i \in \kappa\) with \(\phi^i\) in \(W^{2,1}_\omega((-\infty, \infty) \times [0, T]) \cap Q_\rho \cap W^{1,0}_\omega(-\infty, \infty) \times [0, T])\) for any \(\rho > 0\), where \(Q_\rho = (-\rho, \rho) \times (0, \rho^2)\), and \(s \in W^{1,\infty}[0, T]\). Furthermore, the solution satisfies
\[
\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial x} \leq 0, \quad -\infty < x < \infty, \quad 0 < t \leq T.
\]
By the classical parabolic theory, it is also clear that the solution is in \(C^\infty(\Omega^i_L) \cap C^\infty(\Omega^i_H)\), where \(\Omega^i_L = \{(x, t); -\infty < x < s^i(t), 0 < t \leq T\}\) and \(\Omega^i_H = \{(x, t); s^i(t) < x < \infty, 0 < t \leq T\}\), \(i \in \kappa\).

Applying Lemma 4.13, we obtain the uniqueness of the solution directly.

Theorem 5.2. The solution \((\phi^i, s^i)\) with \(\phi \in \left\{\bigcap_{\rho > 0} W^{2,1}_\omega((-\infty, \infty) \times [0, T]) \setminus \hat{Q}_\rho\right\} \cap W^{1,0}_\omega(-\infty, \infty) \times [0, T])\), \(s^i \in C[0, T]\), \(i \in \kappa\) satisfying
\[
\frac{\partial^2 \phi^i}{\partial x^2} - \frac{\partial \phi^i}{\partial x} \leq 0, \quad -\infty < x < \infty, \quad 0 < t \leq T, \quad i \in \kappa
\]
is unique.

6. Numerical results. Here some numerical results are presented by explicit finite difference approach. In this section, we just consider two regimes, where bull market denotes good macro-economic situation and bear market represents bad situation.

6.1. Calibration. To apply the finite difference scheme, we need calibration first. The parameters to be calibrated are: \(\gamma_i, r_i, \sigma^i_H, \sigma^i_L, \lambda_{ij}\) for \(i, j = 1, 2, i \neq j\).

1. The method of estimating \(\gamma_i, r_i, \sigma^i_H, \sigma^i_L\), \(i = 1, 2\) could be seen in [22].
2. The intensities \(\lambda_{12}\) and \(\lambda_{21}\) can be calibrated through the market quotes of the stock index. The details are shown as follows:
   (a) Estimating the “Bull-Bear Boundary”. Bull-Bear Boundary which distinguishes bull market and bear market could be estimated by 250-day moving average of stock index. If the index is trading below its 250-day moving average, the market is said to be bear market. Otherwise, it is bull market.
   (b) Estimating the transition intensities. The transition matrix \((P_{ij})\) denotes the probabilities of moving regimes, where \(i, j = 1, 2\) and \(P_{ij} + P_{ii} = 1\) for \(i \neq j\). From [8], the relationship between \(P_{ij}\) and \(T_{ii}\) which denotes the average length of a single run in regime \(i\) is
   \[
P_{ij} = \frac{1}{T_{ii}},
   \]
   where \(T_{ii}\) can be estimated from the above step. On the other hand, we have
   \[
P_{ij}(t) = e^{-\int_0^t \lambda_{ij} ds}.
   \]
   Then transition intensities could be estimated from the above equation.
6.2. **Examples.** In this section, we take MTR Corporation Ltd. for example. This company is listed in Hong Kong and assigned ratings by Moody’s. In order to show the macro situations effect on the credit rating migration, we consider two cases by selecting two periods, and in each single period there are two regimes, where regime 1 denotes bear market and regime 2 denotes bull market.

**Case 1.** Choose the data during 2001.1.1 – 2005.12.31, and calibrate the parameters using above method. We have

\[
    r_1 = 0.03, \quad r_2 = 0.035, \quad \sigma^1_L = 0.2, \quad \sigma^1_H = 0.18, \quad \sigma^2_L = 0.18, \quad \sigma^2_H = 0.15,
\]

\[
    F = 1, \quad \gamma_1 = \gamma_2 = 0.43, \quad \lambda_{12} = 0.34, \quad \lambda_{21} = 0.37, \quad T = 5.
\]

The results are shown in Figure 1 and Figure 2, where the parameters are chosen as above.

**Figure 1.** Value function in different regimes

**Figure 2.** Free boundary
Case 2. Choose the data during 2007.1.1—2011.12.31, and calibrate the parameters using above method. We have 

\[ r_1 = 0.035, r_2 = 0.04, \sigma^1_L = 0.17, \sigma^1_H = 0.15, \sigma^2_L = 0.14, \sigma^2_H = 0.13, \]

\[ F = 1, \gamma_1 = \gamma_2 = 0.43, \lambda_{12} = 0.36, \lambda_{21} = 0.2, T = 5. \]

The differences of value function and free boundary between Case 1 and Case 2 are shown in Figure 3.

(From Figure 1 and Figure 2, the value functions in deferent regime regions are divided into two regions respectively. The value changes quite significantly across the free boundary. The free boundaries are decreasing as expected.

From Figure 3, it could be seen that for fixed \( x \), the value in Case 2 is greater the the one in Case 1. And the high rating region of Case 2 is larger than the one in Case 1. That is because in Case 2 the transition intensity \( \lambda_{21} < \lambda_{12} \), which is opposite of the situation in Case 1. That means, in Case 2, bear market is more likely to move into bull market, which is different with Case 1. In other words,
“good” macro-economic situation prevails during the period in Case 2, which leads to lower volatilities and higher valuation.

7. Conclusion. By establishing a free boundary system model, we have valued a corporate bond with credit rating migration risks in regime switch probability. This is a new model for measuring credit rating migration as well as regime switch. Some theoretical results, such as existence, uniqueness and regularities are obtained. Numerical results are also graphed and discussed.

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