NUCLEON-NUCLEON SCATTERING IN
THE $1/N_c$ EXPANSION

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The nucleon-nucleon $^3S_1 - ^3D_1$ coupled-channel problem is solved analytically to leading order in a joint expansion in the quark masses and in $1/N_c$. An approximate expression is derived for the $^3S_1$ scattering length in the large-$N_c$ limit, and the large-$N_c$ behavior of the deuteron is discussed.

1 Introduction

When the number of colors, $N_c$, of the QCD gauge group is taken large, QCD simplifies. One may then hope to learn about the real world with $N_c = 3$, in an expansion in $1/N_c$. The relevance of the large-$N_c$ expansion for nuclear physics was first discussed in the seminal paper by Witten, where it was argued that the nucleon-nucleon (NN) potential grows as $N_c$ in the large-$N_c$ limit, and Hartree theory becomes exact. In a recent revival of interest, it was shown that large-$N_c$ nuclear interactions are spin-flavor $SU(4)$ symmetric, with Wigner’s phenomenologically successful supermultiplet symmetry following as an accidental symmetry. Subsequently, the large-$N_c$ scaling of the NN potential was analyzed in a general way and large-$N_c$ expectations were shown to explain generic features of the NN interaction. Given the levels of complexity that must be unravelled in going from QCD to a theory of nuclear forces, it is quite remarkable that large-$N_c$ counting rules work so well. Hence large-$N_c$ methods offer an important link between QCD and nuclear physics. The establishment of such links is, of course, an important goal for nuclear physics. In this paper, basic aspects of NN scattering in the large-$N_c$ approximation will be reviewed and the $^3S_1 - ^3D_1$ NN coupled-channel system will be analyzed in the large-$N_c$ limit. I will also say some words about the deuteron and the bound-state spectrum in the large-$N_c$ limit.

Assuming confinement, in the large-$N_c$ limit the QCD chiral symmetry is spontaneously broken. With two light flavors the chiral symmetry breaking pattern, $U(2) \otimes U(2) \rightarrow U(2)$, gives rise to four Goldstone bosons, the pions and $\eta$. While the chiral symmetry breaking scale, $\Lambda_{\chi}$, and the meson masses scale as $N_c^0$, the baryon masses scale as $N_c$. In the large-$N_c$ limit the nucleons and $\Delta$s are degenerate and fall into an infinite-dimensional representation of a contracted $SU(4)$ spin-flavor symmetry. In this paper I will assume that $\eta$
and the $\Delta$s can be integrated out of the low-energy effective field theory (EFT) of large-$N_c$ QCD in a manner consistent with the large-$N_c$ expansion. This is, of course, a highly non-trivial assumption. Consider the resulting EFT involving nucleons and pions below $\Lambda_\chi$ (for reviews, see Ref. 8). This EFT consists of nucleons emitting and absorbing pions with an interaction highly constrained by chiral symmetry. The most general Lagrange density consistent with QCD symmetries is

$$\mathcal{L} = \frac{N^1}{4} \left[ i \partial_t + \nabla^2 / (2M) \right] N \left[ C_0^S (N^\dagger N)^2 + C_0^T (N^\dagger \sigma N)^2 \right] + \frac{F_\pi^2}{4} Tr \partial_\mu \Sigma^\dagger \partial^\mu \Sigma + \frac{\omega F_\pi^2}{2} Tr M_q \left( \Sigma^\dagger + \Sigma \right) + g_A N^\dagger \vec{A} \cdot \vec{\sigma} N + \ldots$$

where $\vec{A}$ is the axial-vector current and the ellipses represent operators with more derivatives, more powers of the quark mass matrix, and more nucleon fields. In the large-$N_c$ limit, $F_\pi \propto \sqrt{N_c}$, $g_A \propto N_c$ and $\omega \propto N_c^0$. The four-nucleon operators scale as $C_0^T \propto 1/N_c$ and $C_0^S \propto N_c^3$. When considering the interactions of more than one nucleon, the ordering of the operators in Eq. (1) by way of a power-counting scheme is nontrivial. This is especially so in the $^3S_1 - ^3D_1$ coupled-channel of NN scattering in an EFT with pions, which I will focus on in this paper. The presence of the deuteron, a near-threshold bound state, and the highly-singular tensor force imply complicated nonperturbative physics.

2 NN Scattering at Large-$N_c$

Defining a sensible large-$N_c$ limit for NN scattering is nontrivial because the nucleon mass grows as $N_c$. There are several limits we may consider. The momentum-space Schrödinger equation is

$$\left( \frac{\vec{p}^2}{M} + \hat{V} \right) |\Psi\rangle = E |\Psi\rangle,$$

where $M$ is the nucleon mass. At fixed velocity, the momentum transfer scales as $N_c$. Since the potential also scales as $N_c$, an overall factor of $N_c$ scales out of the Hamiltonian and Hartree theory becomes exact. Evidently there is no EFT description in this limit as characteristic momenta quickly overcome $\Lambda_\chi$ and a description in terms of quarks and gluons becomes appropriate. At fixed momentum transfer, which is the limit relevant to EFT, the nucleon

\[ \text{In principle, the momentum transfer can scale as an inverse power of } N_c \text{ in the EFT, but then one is constrained to threshold.} \]
kinetic energy is suppressed by $1/N_c$ while the potential energy grows as $N_c$. In this limit, the nucleons are always below threshold for scattering. The static nucleons settle at the bottom of the potential well and nuclear matter forms a classical crystal. The absence of a suitable large-$N_c$ limit to describe scattering at fixed momentum transfer poses something of a dilemma. One way out is to focus on the large-$N_c$ scaling properties of the NN potential and then to compare the resulting predictions with modern phenomenological NN potentials that fit phase shifts to high precision. This has proved highly successful. The question of whether the fixed momentum transfer limit exists has been called into question in interesting recent work which points out that there would appear to be violations of the counting rules for the potential arising from diagrams with multiple meson exchanges. This is an important unsolved puzzle.

3 The (Un)Coupled Channel Problem at Large-$N_c$

The most economical power-counting scheme for the $^3S_1-^3D_1$ coupled channels is an expansion about the chiral limit. The pion-exchange part of the NN force in the chiral limit is purely tensor,

$$V_T(r) = -\frac{3\alpha_\pi}{r^4} + \mathcal{O}(m_\pi^2), \quad (3)$$

where

$$\alpha_\pi = \frac{g^2_A}{16\pi F^2_\pi}. \quad (4)$$

At leading order in the EFT expansion there is a local four-nucleon operator containing the effect of non-pionic short-distance physics which renormalizes the tensor force at short distances. The coefficient of this operator is $C_0 \equiv C_0^S + C_0^T = C_0^S + \mathcal{O}(1/N_c)$. I choose to regulate this coordinate-space delta-function using a square well,

$$C_0 \delta^{(3)}(r) \rightarrow \frac{3C_0}{4\pi R^3} \theta(R-r) \equiv V_0 \theta(R-r), \quad (5)$$

with strength $V_0$ and width $R$. The $S$-wave and $D$-wave components of the $J=1$ system, $u(r)$ and $w(r)$ respectively, are coupled by the tensor potential, $V_T$, defined in Eq. (3). The long-distance potential outside the square well is

$$V_L(r) = \begin{pmatrix} 0 & -2\sqrt{2}M V_T(r) \\ -2\sqrt{2}M V_T(r) & 2MV_T(r) - 6/r^2 \end{pmatrix}, \quad (6)$$
while the short-distance potential inside the square well is 

\[ V_s(r) = \begin{pmatrix} -MV_0 & 0 \\ 0 & -MV_0 - 6/r^2 \end{pmatrix}. \] (7)

Defining \( \Psi \) to be

\[ \Psi(r) = \begin{pmatrix} u(r) \\ w(r) \end{pmatrix}, \] (8)

the regulated Schrödinger equation takes the compact form

\[ \Psi''(r) + \left( k^2 + V_L(r)\theta(r - R) + V_S(r)\theta(R - r) \right) \Psi = 0. \] (9)

One may ask whether there is a well-defined large-\( N_c \) limit. In Eq. (9), \( MV_T \) and \( MV_0 \) scale as \( N_c^2 \), while the angular-momentum barrier scales as \( N_c^0 \). In the EFT we treat the momentum transfer as fixed (\( N_c \)-independent). The total center-of-mass-energy is then suppressed by \( 1/N_c^2 \) compared to the interaction energy. Here we will focus on this rather peculiar large-\( N_c \) limit. At leading order in the \( 1/N_c \) expansion, the coupled channel problem can be diagonalized via the similarity transformation \( \Psi \rightarrow S^{-1}\Psi \) where

\[ S = \begin{pmatrix} \sqrt{2}/1 & -1/\sqrt{2} \\ 1 & 1 \end{pmatrix}. \] (10)

Clearly \( ^3S_1 - ^3D_1 \) mixing is suppressed in the large-\( N_c \) limit. The general solution of Eq. (9) in the large-\( N_c \) limit is a linear combination of Bessel functions. Keeping the leading contributions I find

\[ u(r) = \sqrt{2}A r^{3/4} \cos \left( 2\sqrt{\frac{6\alpha M}{r}} + \phi_0 \right) - \frac{B}{\sqrt{2}} r^{3/4} \cosh \left( 2\sqrt{\frac{12\alpha M}{r}} + \delta_0 \right) \]

\[ w(r) = A r^{3/4} \cos \left( 2\sqrt{\frac{6\alpha M}{r}} + \phi_0 \right) + B r^{3/4} \cosh \left( 2\sqrt{\frac{12\alpha M}{r}} + \delta_0 \right), \] (11)

where \( \phi_0, \delta_0, A \) and \( B \) are determined from boundary conditions. Note that the repulsive hyperbolic solutions grow exponentially with \( N_c \). Since these solutions cannot match to a power law in \( N_c \), we find \( B = 0 \) in the large-\( N_c \) limit. We then have

\[ u(r) = \sqrt{2}A r^{3/4} \cos \left( 2\sqrt{\frac{6\alpha M}{r}} + \phi_0 \right) = \sqrt{2} w(r), \] (12)
or

\[ \frac{u(r)}{w(r)} = \sqrt{2} + O \left( \frac{1}{N_c^2} \right) \quad (13) \]

and the S-wave and D-wave wavefunctions oscillate in phase, with a ratio of \( \sqrt{2} \). If there are nodes in the wavefunctions, then they must overlap. It is remarkable that such detailed information about nuclear wavefunctions can be deduced directly from QCD. Notice that the \( 1/N_c \) expansion is equivalent to the short-distance expansion. Eq. (13) is a remarkably accurate representation of the short-distance part of deuteron wavefunctions in conventional NN models with both hard- and soft-cores whose long-range part is governed by pion exchange \([13]\). At large distances in the real world, the angular momentum barrier is not negligible. With \( M\alpha = 1/\Lambda_{NN} \), the tensor force dominates the angular momentum barrier for distances \( r \ll 1/\Lambda_{NN} \), which are outside the validity of the EFT description. It might prove useful to assign a more realistic large-\( N_c \) scaling to the angular momentum barrier and treat it as a perturbation using the WKB approximation, which becomes arbitrarily accurate in the large-\( N_c \) limit.

The phase \( \phi_0 \) is determined by matching logarithmic derivatives of the interior and exterior wavefunctions,

\[ \sqrt{-MV_0} R \cot \left( \sqrt{-MV_0} R \right) = \frac{3}{4} + \sqrt{\frac{6M\alpha_\pi R}{\chi}} \tan \left( 2\sqrt{\frac{6M\alpha_\pi R}{\chi}} + \phi_0 \right). \quad (14) \]

This equation determines the renormalization-group flow of the four-nucleon contact operator \([14][15]\). The multi-branch structure of the right side of Eq. (14) is a consequence of the nonperturbative treatment of the tensor force. Treating \( R \equiv 1/\Lambda \) as a Wilsonian cutoff, we see that the tensor force becomes perturbative when \( \Lambda/\Lambda_{NN} \ll 1 \). In the large-\( N_c \) limit, \( \Lambda/\Lambda_{NN} \propto N_c^2 \gg 1 \) and the tensor force becomes nonperturbative at arbitrarily small momentum scales. This suggests an interesting bound-state spectrum in the large-\( N_c \) limit.

4 A Deuteron in the Large-\( N_c \) Limit?

It is an intriguing mystery that QCD has a single scale, \( \Lambda_{QCD} \), which is \( O(100 \text{ MeV}) \), while characteristic nuclear binding energies are \( O(1 \text{ MeV}) \). In particular, the deuteron binding energy, \( -B \), is \( B = 2.224575 \text{ MeV} \). The problem might not actually be this severe. The tensor force in the chiral limit is governed by a single scale, \( F_\pi \). If we neglect geometrical factors and assume that the momenta of the bound nucleons is \( \sim F_\pi \), then we would expect a
binding energy of order $F_\pi^2/M \sim 10$ MeV. Of course this argument breaks down as we vary $N_c$, since the axial coupling grows with $N_c$. Before considering the bound state spectrum in the pionful EFT, we will consider a simplified scenario.

At distance scales much larger than the pion Compton wavelength, the pion can be integrated out and the EFT simplifies considerably\cite{15}. In EFT($\pi/\Lambda$) observables are determined purely by four-nucleon contact operators\cite{8}. In the $^3S_1-^3D_1$ coupled-channel there is a single momentum-independent interaction, $C_0 \propto N_c$. We will assume that this operator is dominant at low energies in the large-$N_c$ limit. The $^3S_1$ scattering length is then

$$ a = R \left( 1 - \frac{\tan \left( \sqrt{-M V_0 R} \right)}{\sqrt{-M V_0 R}} \right) \propto \frac{MC_0}{4\pi}, \quad (15) $$

where we use a square-well regulator and the arrow indicates that renormalization has been performed and $C_0$ is a renormalized quantity. Hence $a \propto N_c^2$. If the sign of $C_0$ is positive, then there is a bound state with binding energy

$$ B = \frac{16\pi^2}{M C_0^2}. \quad (16) $$

Hence, $B \propto 1/N_c^5$. What do we expect if we include pions in the EFT? We have seen that the tensor force dominates the potential at long distances. Hence we can write the Schrödinger equation in coordinate space as

$$ \left( \frac{\nabla^2}{N_c M} - \frac{3N_c \alpha_\pi}{r^3} \right) |\Psi> = E |\Psi>, \quad (17) $$

where the $N_c$ dependence has been scaled out and the barred quantities are $N_c$-independent. We can scale out the $N_c$ dependence through a rescaling of the coordinate, $r \rightarrow N_c^2 r$. The eigenvalue equation then becomes

$$ \frac{1}{N_c^5} \left( \frac{\nabla^2}{M} - \frac{3\alpha_\pi}{r^3} \right) |\Psi_B> = B |\Psi_B>, \quad (18) $$

and we again find $B \propto 1/N_c^5$. This argument can be formulated in a more physical way as follows\cite{16}. We can write the potential as

$$ V_T \sim \frac{N_c}{r_B^3}, \quad (19) $$

where $r_B$ can be loosely interpreted as the size of the bound state and I have assumed that all dimensional parameters are given by powers of $\Lambda_{QCD}$, which
is defined to be unity. Now assume that there is a bound state in the $^3S_1$ channel at large-$N_c$. The presence of a bound state requires a balance between the kinetic energy, $N_cv^2_B/2$, and the potential energy, and therefore we expect $V_T \sim N_cv^2_B$. Here $v_B$ is the velocity of the bound state. As required for consistency, $N_c$ drops out of this relation and we arrive at $v_B^2 \sim r_B^{-3}$. This is simply a statement of the virial theorem. Since the nucleon momentum is $N_cv_B$, it follows from the uncertainty principle that $v_B \sim (Ncr_B)^{-1}$. Again using the virial theorem, we find

$$r_B \propto N_c^2, \quad v_B \propto 1/N_c^3, \quad B \propto 1/N_c^5,$$

and we have a bound state that is large, slow, and shallow in the large-$N_c$ limit. Hence our naive scaling expectations in the pionful EFT agree with those from EFT($\pi$). Unfortunately both of these arguments would appear to be too naive as they ultimately rely on the virial theorem constraint that the coordinate scales as $N_c^2$, or, equivalently, that the momentum transfer scales as $1/N_c^2$. As we will see below, a more detailed study in the pionful theory suggests that there are an infinite number of bound states in the large-$N_c$ limit, with the ground state binding energy increasing as a complicated function of $N_c$. This last point is not surprising, given the large-$N_c$ wavefunctions found in the previous section. If one acts with the kinetic energy operator on the wavefunction of Eq. (12), one finds a complicated (harmonic) dependence on $N_c$.

5 The Bound-State Spectrum

One way of studying the bound-state spectrum is via the scattering length. Although a scattering length is strictly speaking not defined for a $1/r^3$ potential, we can impose an infrared cutoff, $\mathcal{X}$, on $V_T$ and match to a wavefunction at $r > \mathcal{X}$ of the form $r - a^{^3S_1}$. This gives

$$a^{^3S_1} = \mathcal{X} \left( -1 + 4\sqrt{6M\alpha_\pi/\mathcal{X}} \tan \left( 2\sqrt{6M\alpha_\pi/\mathcal{X}} \phi_0 \right) \right) / \left( 3 + 4\sqrt{6M\alpha_\pi/\mathcal{X}} \tan \left( 2\sqrt{6M\alpha_\pi/\mathcal{X}} \phi_0 \right) \right).$$

(21)

The purpose of this formula is to give analytical understanding of the bound state spectrum in the large-$N_c$ limit. We would expect the infrared cutoff to take a value around $1/m_\pi$. Choosing an ultraviolet cutoff $R = 0.375$ fm and a square-well depth of $MV_0 = -2.07 \times 10^6$ MeV$^2$ which reproduce the scattering length for the physical value of $N_c = 3$ in the full problem, gives $\phi_0 = -7.6$. I then choose the infrared cutoff $\mathcal{X} = 0.97$ fm to reproduce the scattering length in the approximate formula. In order to explore the
Figure 1: The $^3S_1$ scattering length as a function of $N_c$. The solid line is a numerical solution of the Schrödinger equation with a small but finite pion mass and with $R = 0.375$ fm and a square-well depth of $MV_0 = -2.07 \times 10^6$ MeV$^2$ tuned to the physical scattering length at $N_c = 3$ ($\phi_0 = -7.6$). The dashed line is the approximate analytic solution, Eq. (21) with $\phi_0 = -7.6$ and $X = 0.97$ fm which reproduce the physical scattering length at $N_c = 3$. Bound state spectrum as a function of $N_c$, I naively scale all quantities in the approximate formula and in the exact numerical result by the expected large-$N_c$ scaling. The approximate formula for the scattering length and the exact numerical result are plotted as a function of $N_c$ in Fig. (1). The approximate formula is reasonably accurate in the vicinity of $N_c = 3$ (for several branches of the tangent). The corresponding plot of the binding energy is given in Fig. (3). The plotted expression is derived from the approximate formula for the scattering length using effective range theory, and has been verified numerically along the plateaus using the exact solution. With the given choice of cutoff and square-well depth, there are no nodes in the wavefunction and the deuteron (at $N_c = 3$) is, by construction, the ground state. As $N_c$ is increased from its physical value, the deuteron binding energy grows as a complicated function of $N_c$ and quickly passes beyond $\Lambda_{\chi}$ and out of the EFT. There is then no bound state in the EFT. As $N_c$ approaches 4, a new bound state, the first excited state, appears at threshold and then undergoes the same decoupling as $N_c$ is increased. This cycle repeats itself an infinite number of times in the large-$N_c$ limit. The excited levels can be studied in more detail through the singularities of the scattering length, using the approximate formula, Eq. (21). One can estimate the density of eigenstates [13] for the $1/r^3$ potential using WKB; with excitation number, $n$, one finds $|E_n| \sim n^6 \Lambda_{QCD}$ with $n = 1, 2, \ldots$. Therefore as $N_c \to \infty$ there are an infinite number of bound states, infinitely
dispersed in energy. But there would appear never to be more than one bound state present in the EFT.

6 Caveats

There are many caveats to the large-$N_c$ picture of NN scattering presented in this paper. Among them is the absence of the $\Delta$ degrees of freedom. NN scattering becomes a significantly more complicated coupled channel problem when the $\Delta$s are degenerate with the nucleons. It would be interesting to repeat the analysis presented here with the complete contracted spin-flavor $SU(4)$ multiplet. A more serious problem lies in the fact that the large-$N_c$ limit in which the momentum transfer is held fixed does not experience scattering, and potentially does not exist as a sensible limit of QCD. One may also seriously question the procedure of varying $N_c$ slightly away from its physical value to deduce information about the large-$N_c$ limit. It is conceivable that the true large-$N_c$ Schrödinger equation behaves quite differently. On a related note, there are several issues of noncommutativity of limits. First there is the issue of whether to take the large-$N_c$ limit at the level of the Schrödinger equation, or at the level of its solution. There is also the issue of whether the chiral limit commutes with the large-$N_c$ limit; note that the tensor force in the chiral limit is highly singular at all distances, and this has an important effect on the bound state spectrum. Anthropic arguments suggest that the fine-tuned nature of the deuteron binding energy is essential to the emergence of life. The results presented in this paper suggest that the large-$N_c$ limit can
be relevant to nature only at discrete values of $N_c$ where there continues to be a bound state with deuteron quantum numbers and binding energy within anthropic bounds. Whether this be the ground state or the $N_c$th excited state is irrelevant in the EFT. Other (large) values of $N_c$—away from the plateaus in Fig. (2)—would in all likelihood not support the intelligent life necessary to think about large-$N_c$ QCD.

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