The complex Monge–Ampère equation on weakly pseudoconvex domains

L’équation de Monge–Ampère complexe sur les domaines faiblement pseudo-convexes

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A B S T R A C T

We show here a “weak” Hölder regularity up to the boundary of the solution to the Dirichlet problem for the complex Monge–Ampère equation with data in the $L^p$ space and $\Omega$ satisfying an $f$-property. The $f$-property is a potential-theoretical condition that holds for all pseudoconvex domains of finite type and many examples of infinite-type ones.

R É S U M É

Nous montrons ici une régularité de Hölder « faible » jusqu’au bord d’une solution du problème de Dirichlet pour l’équation de Monge–Ampère complexe, de données dans l’espace $L^p$, sur un domaine satisfaisant une $f$-propriété. Cette $f$-propriété est une condition de théorie du potentiel qui est satisfaite par tous les domaines pseudo-convexes de type fini et de nombreux exemples de type infini.

1. Introduction

For a $C^2$, bounded, pseudoconvex domain $\Omega \subset \subset \mathbb{C}^n$, the Dirichlet problem for the Monge–Ampère equation consists of

\[
\begin{align*}
    u & \in PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega), \\
    (dd^c u)^n &= \psi \, dV \quad \text{in } \Omega, \\
    u &= \varphi \quad \text{on } \partial \Omega.
\end{align*}
\]

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A great deal of work has been done for the case where $\Omega$ is strongly pseudoconvex. Within this domain, we can divide the literature into three kinds of data $\psi$.

- **The Hölder data**: Bedford–Taylor prove in [2] that if $u \in C^2(\bar{\Omega})$ then $|u|_{L^p_b(\Omega)} \leq C\|u\|_{L^p(\Omega)}$, for $0 < \alpha \leq 1$.
- **The smooth data**: Caffarelli, Kohn and Nirenberg prove in [4] that if $u \in C^\infty(\Omega)$, for $\varphi \in C^\infty(\Omega)$ and $\psi \in C^\infty(\Omega)$, in case $\psi > 0$ in $\Omega$ and $b\Omega$ is smooth.
- **The $L^p$ data**: Guedj, Kolodziej and Zeriahi prove in [6] that if $u \in L^p(\Omega)$ with $p > 1$ and $\varphi \in C^{1,1}(b\Omega)$ then $u \in C^\gamma(\bar{\Omega})$ for any $\gamma < \gamma_p := \frac{2}{2+\frac{1}{p}}$ where $\frac{1}{q} + \frac{1}{p} = 1$.

When $\Omega$ is no longer strongly pseudoconvex but has a certain “finite type”, there are some known results for this problem due to Blocki [3], Coman [5], and Li [11]. Recently, Ha and the second author gave a general related result to a Hölder data under the hypothesis that $\Omega$ satisfies an $f$-property (see Definition 2.1 below). The $f$-property is a consequence of the geometric “type” of the boundary. All pseudoconvex domains of finite type satisfy the $f$-property as well as many classes of domains of infinite type (see [9,7,8] for discussion on the $f$-property). Using the $f$-property, a “weak” Hölder regularity for the solution to the Dirichlet problem of the complex Monge–Ampère equation is obtained in [9]. Coming back to the case of $\Omega$ of finite type, in a recent paper with Zampieri [1], we prove the Hölder regularity for $\psi \in L^p$, with $p > 1$. The purpose of the present paper is to generalize the result in [1] to a pseudoconvex domain satisfying an $f$-property. For this purpose, we recall the definition of a weak Hölder space in [9,7]. Let $f$ be an increasing function such that $\lim_{t \to +\infty} f(t) = +\infty$, $f(t) \lesssim t$.

For a subset $A$ of $\mathbb{C}^n$, define the $f$-Hölder space $A$ by

$$\Lambda^f(A) = \{ u : u \|_{L^\infty(A)} + \sup_{z,w \in A, z \neq w} f(|z - w|^{-1}) \cdot |u(z) - u(w)| < \infty \}$$

and set

$$\| u \|_{\Lambda^f(A)} = \| u \|_{L^\infty(A)} + \sup_{z,w \in A, z \neq w} f(|z - w|^{-1}) \cdot |u(z) - u(w)|.$$  

Note that the notion of the $f$-Hölder space includes the standard Hölder space $\Lambda_{\alpha}$ by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha \leq 1$. Here is our result

**Theorem 1.1.** Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain admitting the $f$-property. Suppose that $\int_1^\infty \frac{\alpha}{\alpha f(a)} < \infty$ and denote by

$$g(t) := \left( \int_t^\infty \frac{\alpha}{\alpha f(a)} \right)^{-1}$$

for $t \geq 1$. If $0 < \alpha \leq 2$, $\varphi \in \Lambda^\omega_{\beta}(b\Omega)$, and $\psi \geq 0$ on $\Omega$ with $\psi \in L^p$ with $p > 1$, then the Dirichlet problem for the complex Monge–Ampère equation (1.1) has a unique plurisubharmonic solution $u \in \Lambda^\delta(\bar{\Omega})$. Here $\beta = \min(\alpha, \gamma)$, for any $\gamma < \gamma_p = \frac{2}{1 + \frac{1}{p}}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

The proof follows immediately from Theorem 2.2 and 2.5 below. Throughout the paper we use $\lesssim$ and $\gtrsim$ to denote an estimate up to a positive constant, and $\approx$ when both of them hold simultaneously. Finally, the indices $p$, $\alpha$, $\beta$, $\gamma$ and $\gamma_p$ only take ranges as in Theorem 1.1.

**2. Hölder regularity of the solution**

We start this section by defining the $f$-property as in [7,8].

**Definition 2.1.** For a smooth, monotonic, increasing function $f : [1, +\infty) \to [1, +\infty]$ with $f(t) t^{-1/2}$ decreasing, we say that $\Omega$ has the $f$-property if there exist a neighborhood $U$ of $b\Omega$ and a family of functions $\{\varphi_\theta\}$ such that

(i) the functions $\varphi_\theta$ are plurisubharmonic, $C^2$ on $U$, and satisfy $-1 \leq \varphi_\theta \leq 0$.
(ii) $i\partial \bar{\partial} \varphi_\theta \gtrsim f(\delta^{-1})^2 Id$ and $|D \varphi_\theta| \lesssim \delta^{-1}$ for any $z \in U \cap \{ z \in \Omega : -\delta < r(z) < 0 \}$, where $r$ is a $C^2$-defining function of $\Omega$.

In [7], using the $f$-property, the second author constructed a family of plurisubharmonic peak functions with good estimates. This family of plurisubharmonic peak functions yields the existence of a defining function $\rho$ which is uniformly strictly plurisubharmonic and weakly Hölder (see [9]).

**Theorem 2.2** (Khanh [7] and Ha–Khanh [9]). Assume that $\Omega$ is a bounded, pseudoconvex domain admitting the $f$-property as in Theorem 1.1. Then there exists a uniformly strictly-plurisubharmonic defining function of $\Omega$ that belongs to the $g^2$-Hölder space of $\Omega$, which means that
\[ \rho \in \Lambda^b(\tilde{\Omega}), \quad \Omega = \{ \rho < 0 \} \quad \text{and} \quad i\partial \bar{\partial} \rho \geq 1d. \quad (2.1) \]

The existence and uniqueness of the solution \( u \in L^\infty(\Omega) \) to the equation (1.1) need a weaker condition, in particular, one only need \( \rho \in C^0(\tilde{\Omega}) \), as shown by [10].

Theorem 2.3 (Kolodziej [10]). Let \( \Omega \) be a bounded domain in \( C^n \). Assume that there exists a function \( \rho \) such that
\[ \rho \in C^0(\tilde{\Omega}), \quad \Omega = \{ \rho < 0 \} \quad \text{and} \quad i\partial \bar{\partial} \rho \geq 1d. \]
Then, for any \( \psi \in C^0(\partial \Omega) \), \( \psi \in L^p(\Omega) \), there is a unique plurisubharmonic solution \( u(\Omega, \varphi, \psi) \in C^0(\tilde{\Omega}) \).

To improve the smoothness of \( u \), we increase the smoothness of \( \rho \) and \( \psi \).

Theorem 2.4 (Ha–Khanh [9]). Let \( \rho \) satisfy (2.1). If \( \psi \in \Lambda^\psi(\partial \Omega) \) and \( \psi \) \( \tilde{\in} \) \( \Lambda^b(\tilde{\Omega}) \), then the Dirichlet problem for the complex Monge–Ampère equation (1.1) has a unique plurisubharmonic solution \( u(\Omega, \varphi, \psi) \in \Lambda^b(\tilde{\Omega}) \).

Now we focus on lowering the smoothness of \( \psi \) and prove the following theorem.

Theorem 2.5. Let \( \rho \) satisfy (2.1). If \( \psi \in \Lambda^\psi(\partial \Omega) \) and \( \psi \in L^p(\Omega) \), then the Dirichlet problem for the complex Monge–Ampère equation (1.1) has a unique plurisubharmonic solution \( u(\Omega, \varphi, \psi) \in \Lambda^b(\tilde{\Omega}) \).

In order to prove this theorem, we need to construct a subsolution with \( L^p \) data. Here, \( v \) is a subsolution to (1.1) in the sense that \( v \) is plurisubharmonic, \( v|_{\partial \Omega} = \varphi \) and \( (dd^c v)^n \geq \psi \) \( dV \) in \( \Omega \).

Proposition 2.6. Let \( \rho \) satisfy (2.1). Then there is a subsolution \( v \in \Lambda^b(\tilde{\Omega}) \) to (1.1) for \( \varphi \in C^0(\partial \Omega) \) and \( \psi \in L^p(\Omega) \).

Proof. For a large ball \( \mathbb{B} \) containing \( \Omega \), we set \( \tilde{\psi}(z) := \begin{cases} \psi(z) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{B} \setminus \Omega. \end{cases} \) First, we apply Theorem 1 in [6] on \( \mathbb{B} \) with \( \tilde{\psi} \in L^p(\mathbb{B}) \) and zero-valued boundary condition; it follows \( u_1 := u(\mathbb{B}, 0, \tilde{\psi}) \in \Lambda^\psi(\tilde{\Omega}) \) (\( \tilde{\Omega} \)). Second, we apply Theorem 2.4 on \( \Omega \) twice: first for \( u_2 := u(\Omega, \varphi, 0) \in \Lambda^b(\tilde{\Omega}) \), since \( u_1|_{\partial \Omega} \in \Lambda^\psi(\partial \Omega) \), and second for \( u_3 := u(\Omega, \varphi, 0) \in \Lambda^b(\tilde{\Omega}) \) by the hypothesis \( \varphi \in \Lambda^\psi(\partial \Omega) \). Finally, taking the summation \( v := u_1 + u_2 + u_3 \), we have the conclusion. \( \square \)

Proof of Theorem 2.5. Keeping the notation of Theorem 2.3, let \( u(\Omega, \varphi, \psi) \in C^0(\tilde{\Omega}) \) be the solution to (1.1). What follows is dedicated to showing that this \( C^0 \) plurisubharmonic solution \( u(\Omega, \varphi, \psi) \) is in fact in \( \Lambda^b(\tilde{\Omega}) \). By Theorem 2.4 we have that \( w := u(\Omega, \varphi, 0) \) is in \( \Lambda^b(\tilde{\Omega}) \). Let \( v \) be as in Proposition 2.6 then the comparison principle yields at once
\[ v \leq u(\Omega, \varphi, \psi) \leq w. \quad (2.2) \]

By (2.2) and the \( g^\beta \)-Hölder regularity of \( v \) and \( w \), we get
\[ |u(z) − u(\zeta)| \lesssim |g(z − \zeta)|^{-\beta} \quad z \in \tilde{\Omega}, \quad \zeta \in \partial \Omega, \]
and therefore for \( \delta \) suitably small
\[ |u(z) − u(z')| \lesssim |g(\delta^{-1})|^{-\beta} \quad z, z' \in \Omega \setminus \Omega_\delta \quad \text{and} \quad |z − z'| < \delta \quad (2.3) \]
where \( \Omega_\delta := \{ z \in \mathbb{C}^n : r(z) < -\delta \} \) and \( r \) is the \( C^2 \) defining function for \( \Omega \) with \( |\nabla r| = 1 \) on \( \partial \Omega \). We have to prove that (2.3) also holds for \( z, z' \in \Omega_\delta \). For \( z \in \Omega_\delta \), we use the notation
\[ u_\frac{1}{2}(z) := \sup_{|\zeta| < \frac{1}{2}} u(z + \zeta), \quad \tilde{u}_\frac{1}{2}(z) := \frac{1}{\sigma_{2n-1}(\frac{1}{2})^{2n-1}} \int_{\mathbb{B}(z, \frac{1}{2})} u(\zeta) \, dS(\zeta), \]
and
\[ \hat{u}_\frac{1}{2}(z) := \frac{1}{\sigma_{2n-1}(\frac{1}{2})^{2n}} \int_{\mathbb{B}(z, \frac{1}{2})} u(\zeta) \, dV(\zeta), \]
where \( \sigma_{2n-1}(\frac{1}{2})^{2n-1} = \text{Vol}(b\mathbb{B}(z, \frac{1}{2})) \) and \( \sigma_{2n}(\frac{1}{2})^{2n} = \text{Vol}(\mathbb{B}(z, \frac{1}{2})) \). It is obvious that
\[ \hat{u}_\frac{1}{2} \leq u_\frac{1}{2} \leq u_{\frac{1}{2}} \quad \text{in} \quad \Omega_\delta. \quad (2.4) \]
Furthermore, we have an \( L^1 \) estimate of the difference between \( u \) and \( \hat{u}_\frac{1}{2} \) and of the stability estimate in the following theorems (2.7 and 2.8).
Theorem 2.7 (Baracco–Khanh–Pinton–Zampieri [1]). For any $0 < \epsilon < 1$, we have
\[
\| \tilde{u}_\delta \|_{L^1(\Omega_\delta)} \lesssim \delta^{1-\epsilon}.
\] (2.5)

Theorem 2.8 (Guedj–Kolodziej–Zeriahi [6]). Fix $0 \leq f \in L^p(\Omega)$, $p > 1$. Let $U$, $W$ be two bounded plurisubharmonic functions in $\Omega$ such that $(dd^c U)^p = f dV$ in $\Omega$ and let $U \geq W$ on $\partial \Omega$. Fix $s \geq 1$ and $0 \leq \eta < \frac{1}{sq+2}$. Set $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a uniform constant $C = C(\eta, n, \| f \|_{L^p(\Omega)}) > 0$ such that
\[
\sup_{\Omega} (W - U) \leq C \|(W - U)\|_{L^s(\Omega)}^\eta,
\]
where $(W - U)_{+} := \max(W - U, 0)$.

By (2.3), we have
\[
\tilde{u}_\delta \leq u_\delta \leq u + c [g(\delta^{-1})]^{-\beta}, \text{ on } b\Omega_\delta \text{ for suitable constant } c.
\]
Thus, we can apply Theorem 2.8 for $\Omega_\delta$ with $U := u + c [g(\delta^{-1})]^{-\beta}$, $W := \tilde{u}_\delta$ and $s := 1$; thus we get
\[
\sup_{\Omega_\delta} \left( \tilde{u}_\delta - u + c [g(\delta^{-1})]^{-\beta} \right) \lesssim \left( \tilde{u}_\delta - (u + c [g(\delta^{-1})]^{-\beta}) \right)_{+} \lesssim \| u \|_{L^1(\Omega_\delta)} \lesssim \| \tilde{u}_\delta - u \|_{L^1(\Omega_\delta)} \lesssim \delta^{(1-\epsilon)\eta},
\] (2.6)
for any $\eta < \frac{1}{2} \gamma_p = \frac{1}{sq+2}$ where $\frac{1}{q} + \frac{1}{p} = 1$. Taking $\gamma < \gamma_p$, $\beta = \min(\alpha, \gamma)$, $\epsilon = \frac{2q-1}{2q+1} > 0$ and $\eta = \frac{1}{4}(\gamma_p + \gamma) < \frac{1}{2} \gamma_p$ so that $(1-\epsilon)\eta = \frac{\gamma}{2}$, it follows
\[
\sup_{\Omega_\delta} \left( \tilde{u}_\delta - u \right) \lesssim \delta^{(1-\epsilon)\eta} + [g(\delta^{-1})]^{-\beta} \lesssim \delta^{\frac{\gamma}{2}} + [g(\delta^{-1})]^{-\beta} \lesssim [g(\delta^{-1})]^{-\beta},
\] (2.7)
where the last inequality of (2.7) follows by $g(\delta^{-1}) \lesssim \delta^{-\frac{1}{q}}$ (by the conditions on $f$ in the $f$-property).

Similarly to [6, Lemma 4.2] by using the fact that $g(\delta^{-1}) \approx g(\delta^{-1})$ for any constant $c > 0$, one can state the equivalence between
\[
\sup_{\Omega_\delta} (u_\delta - u) \lesssim [g(\delta^{-1})]^{-\beta} \quad \text{and} \quad \sup_{\Omega_\delta} (\tilde{u}_\delta - u) \lesssim [g(\delta^{-1})]^{-\beta}.
\]
Using this equivalence together with the inequalities in (2.4), it follows that (2.7) is equivalent to
\[
\sup_{\Omega_\delta} (u_\delta - u) \lesssim [g(\delta^{-1})]^{-\beta}. \quad (2.8)
\]
From (2.3) and (2.8), it is easy to prove that
\[
|u(z) - u(z')| \lesssim [g(|z - z'|^{-1})]^{-\beta} \quad \text{for any } z, z' \in \hat{\Omega}. \quad \square
\]

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