A Sharp Existence Theorem for Vortices in the Theory of Branes

Xiaosen Han

Institute of Contemporary Mathematics
School of Mathematics
Henan University
Kaifeng, 475004, PR China

Abstract. We investigate the BPS equations arising from the theory of multiply intersec
ting D-branes. By using the direct minimization method, we establish a sharp
existence and uniqueness theorem for multiple vortex solutions of the BPS equations
over a doubly periodic domain and over the full plane, respectively. In particular, we ob-
tain an explicit necessary and sufficient condition for the existence of a unique solution
for the doubly periodic domain case.

Mathematics Subject Classification (2000). 35A05, 58E50.

Keywords: Vortices; D-branes; BPS equations; nonlinear elliptic system; direct mini-
mization.

1 Introduction

Vortices play important roles in many areas of theoretical physics including superconductivity
theory [1][13][27], condensed-matter physics [25][28], optics [5], and cosmology [22][29][45]. It is
Taubes who first obtained the rigorous construction of multiple vortex static solutions for the
Abelian Higgs model in [27][13][44]. Since then a great deal of work has been done on various
vortex equations. See [7][8][16][19][23][26][33][38][39][42][46][49]. Motivated by the seminal work of
Seiberg and Witten [34] on monopoles condensation and confinement, followed by Hanany and
Tong [21], considerable attention has been devoted to the studies on non-Abelian multiple vortices
in supersymmetric gauge field theories [3][4][9][11][15][35][37]. For the rigorous existence of such
vortices, Lieb and Yang [30], Lin and Yang [31][32] developed a series of existence and uniqueness
theories. Han and Tarantello [20] established existence of doubly periodic non-Abelian Chern-
Simons vortices with a general gauge group. For more related work and references we refer to the
monograph [49].

The purpose of this paper is to establish the existence of multiple vortex solutions for the BPS
equations derived in [41] from the theory of multi-intersection of D-branes. By complexifying the
variables, we can formulate the corresponding BPS equations into an \( l \times l \) \((l \geq 2)\) system of nonlinear
elliptic equations. Then, using the direct minimization method developed in [30], we can establish
the existence and uniqueness of solutions to the BPS equations over a doubly periodic domain
\( \Omega \) and over the full plane \( \mathbb{R}^2 \), respectively. It is worth noting that we can establish an explicit
necessary and sufficient condition for the existence of a unique solution for the doubly periodic domain case and such result is very rare in the existing literature.

The rest of our paper is organized as follows. In section 2, following Suyama [40,41] we derive the BPS equation and state our main results on existence and uniqueness of multiple vortex solutions for the BPS equations. Section 3 is devoted to the proof of the existence result in the doubly periodic domain case. In section 4 we prove the existence result for the planar case. In section 5 we summarize our results and draw a conclusion.

2 Vortices in the theory of Branes

To formulate our problem, we follow Suyama [40,41]. We consider the following brane configuration in Type IIA theory compactified on $T^6$. There are $Q_1$ D4-branes and $Q_2$ D4-branes wrapped along different directions of $T^6$. They intersect over a 3-dimensional hyperplane. The low energy effective theory on the D4-D4’ intersection is a 3-dimensional gauge theory, whose gauge group is $U(Q_1) \times U(Q_2)$. For simplicity, we assume that D4(D4’)-branes are separated from each other. Then the gauge group is reduced to $U(1)^{Q_1} \times U(1)^{Q_2}$. The action is the following,

$$S = \int d^3x \left[ \sum_{n=1}^{Q_1} \left( -\frac{1}{4g^2} F_{n,m}^{(1)} F_{n}^{(1)ij} - \frac{g^2}{2} (D_{n})^2 - g^2 |F_{n}^{(1)}|^2 \right) 
+ \sum_{m=1}^{Q_2} \left( -\frac{1}{4g^2} F_{m,n}^{(2)} F_{m}^{(2)ij} - \frac{g^2}{2} (D_{m})^2 - g^2 |F_{m}^{(2)}|^2 \right) 
- \sum_{n=1}^{Q_1} \sum_{m=1}^{Q_2} |D_i q_{nm}|^2 \right], \quad (i,j = 0,1,2), \quad (2.1)$$

where

$$D_i q_{nm} = \partial_i q_{nm} + i (A_{ni}^{(1)} - A_{mi}^{(2)}) q_{nm},$$

$$D_{n}^{(1)} = - \sum_{m=1}^{Q_2} (|q_{nm}|^2 - |\tilde{q}_{nm}|^2 - \zeta),$$

$$D_{m}^{(2)} = + \sum_{n=1}^{Q_1} (|q_{nm}|^2 - |\tilde{q}_{nm}|^2 - \zeta),$$

$$F_{n}^{(1)} = - \sqrt{2} \sum_{m=1}^{Q_2} q_{nm} \tilde{q}_{mn},$$

$$F_{m}^{(2)} = + \sqrt{2} \sum_{n=1}^{Q_1} \tilde{q}_{mn} q_{nm},$$

$A_{ni}^{(1)}$ and $A_{mi}^{(2)}$ are the gauge fields in the Cartan subalgebra of $U(Q_1)$ and $U(Q_2)$.

Assume that $\zeta$ is positive, without potential we have the vacuum configuration

$$|q_{nm}| = \zeta, \quad \tilde{q}_{mn} = 0. \quad (2.2)$$
The vortex solution must satisfy this condition at spatial infinity.

When \( Q_1 = l (l \geq 2), Q_2 = 1 \), by the Bogomol'nyi reduction \([6, 27]\) for the static solutions, Suyama \([41]\) obtained the BPS equations of the following form,

\[
F_{j,12}^{(1)} \pm g^2 (|q_{j1}|^2 - \zeta) = 0, \quad j = 1, \ldots, l, \tag{2.3}
\]

\[
F_{1,12}^{(2)} \equiv g^2 \sum_{i=1}^{l} (|q_{i1}|^2 - \zeta) = 0, \tag{2.4}
\]

\[
[ \partial_1 q_{j1} + i \left( A_{j,1}^{(1)} - A_{1,1}^{(1)} \right) q_{j1} ] \pm i \left[ \partial_2 q_{j1} + i \left( A_{j,2}^{(1)} - A_{1,2}^{(1)} \right) q_{j1} \right] = 0, \quad j = 1, \ldots, l. \tag{2.5}
\]

Redefining the fields as follows

\[
A_{ji} = A_{j,i}^{(1)} - A_{1,i}^{(2)}, \quad i = 1, 2, \quad F_j = \partial_1 A_{j2} - \partial_2 A_{j1}, \quad j = 1, \ldots, l \tag{2.6}
\]

and using suitable re-scaling, the BPS equations \((2.3) - (2.5)\) are transformed into

\[
F_j + |q_{j1}|^2 + \sum_{i=1}^{l} |q_{i1}|^2 - (l + 1) = 0, \quad j = 1, \ldots, l, \tag{2.7}
\]

\[
(\partial_1 q_{j1} + i A_{j1} q_{j1}) - i(\partial_2 q_{j1} + i A_{j2} q_{j1}) = 0, \quad j = 1, \ldots, l, \tag{2.8}
\]

where we take lower sign in \((2.3) - (2.5)\). As in \([27]\), we can see from the equation \((2.5)\) that the zeros of \(q_{11}, \ldots, q_{l1}\) are discrete and of integer multiplicities. Now we denote the zero set of \(q_{j1}\) by \(Z_{q_{j1}}\),

\[
Z_{q_{j1}} = \{ p_{j,1}, \ldots, p_{j,N_j} \}, \quad j = 1, \ldots, l \tag{2.9}
\]

such that the repetitions among the points take account of the multiplicities of these zeroes.

For the equations \((2.7) - (2.8)\), we are interested in two cases. In the first case we study the equations over a doubly periodic domain \(\Omega\), governing multiple vortices hosted in \(\Omega\) such that the field configurations are subject to the ’t Hooft boundary condition \([24, 47, 49]\) under which periodicity is achieved modulo gauge transformations. In the second case we consider the equations over the full plane \(\mathbb{R}^2\) with the natural boundary condition

\[
|q_{j1}| \rightarrow 1 \quad \text{as} \quad x \rightarrow \infty, \quad j = 1, \ldots, l. \tag{2.10}
\]

Now we can state our main result concerning the existence and uniqueness of solutions to the BPS equations \((2.7) - (2.8)\) as the following.

**Theorem 2.1** Consider the BPS system of multiple vortex equation \((2.7) - (2.8)\) for \((q_{11}, \ldots, q_{l1}, A_{11}, \ldots, A_{l1}, A_{12}, \ldots, A_{l2})\) with prescribed sets of zeros given by \((2.9)\) such that \(q_{j1}\) have \(N_j\) arbitrarily distributed zeros, \(j = 1, \ldots, l\).

(i) For the problem over a doubly periodic domain \(\Omega\), a solution exists if and only if

\[
\max_{1 \leq j \leq l} \{ N_j \} < \frac{(l + 1)|\Omega|}{4\pi}. \tag{2.11}
\]
Furthermore, if there exists a solution, it must be unique.

(ii) For the problem over the full plane $\mathbb{R}^2$ subjected to the boundary condition (2.10), there exists a unique solution up to gauge transformations such that the boundary behavior (2.10) is reached exponentially fast

$$|q_{ij}|^2 - 1| \leq C(\varepsilon)e^{-(1-\varepsilon)|x|} \quad \text{as } |x| \to \infty, \quad i = 1, \ldots, l$$

(2.12)

where $\varepsilon \in (0, 1)$ is arbitrarily small, $C(\varepsilon)$ is a positive constant depending on $\varepsilon$.

(iii) In either case, the total vortex fluxes are quantized quantities given by

$$\int F_j dx = 4\pi N_j, \quad j = 1, \ldots, l.$$ (2.13)

Remark 1: For the first part of the theorem, it can also be proved by a constrained minimization method developed in [48] and chapter 4 of [49], where a more general elliptic system was studied. However, in this paper we will prove it by using a direct minimization method developed in [30], which is more direct and powerful. In fact, following the direct minimization method used here, we can prove Theorem 1 in [48].

For convenience we complexify the variables. Let $z = x^1 + ix^2$, $\tilde{A}_j = A_{j1} + iA_{j2}$, $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$, $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. Hence, noting that $\partial\bar{\partial} = \frac{1}{4}\Delta$, the BPS equations (2.7)–(2.8) are transformed into

$$\Delta \ln |q_{j1}|^2 = |q_{j1}|^2 + \sum_{i=1}^l |q_{i1}|^2 - (l + 1), \quad j = 1, \ldots, l,$$ (2.14)

away from the zeros of $q_{11}, \ldots, q_{l1}$. Then the substitutions

$$u_j = \ln |q_{j1}|^2, \quad j = 1, \ldots, l$$

transform the equations (2.14) into the following nonlinear elliptic system

$$\Delta u_j = e^{u_j} + \sum_{i=1}^l e^{u_i} - (l + 1) + 4\pi \sum_{s=1}^{N_j} \delta_{p_{j,s}}(x), \quad j = 1, \ldots, l,$$ (2.15)

defined over the entire domain. The boundary condition (2.10) now reads

$$u_j \to 0 \quad \text{as } |x| \to \infty, \quad j = 1, \ldots, l.$$ (2.16)

Throughout this paper we will use the following notations. Let $u = (u_1, \ldots, u_l)^\tau$, $v = (v_1, \ldots, v_l)^\tau$, $U = (e^{u_1}, \ldots, e^{u_l})^\tau$, and $A = (a_{ij})$ be the $l \times l$ matrix

$$A = \begin{pmatrix}
2 & 1 & 1 & \ldots & 1 \\
1 & 2 & 1 & \ldots & 1 \\
1 & 1 & 2 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & 2 
\end{pmatrix}.$$
For vectors \( a = (a_1, \ldots, a_l)^\tau, \ b = (b_1, \ldots, b_l)^\tau \), we denote \( a > (\geq) b \) if \( a_i > (\geq) b_i, \ i = 1, \ldots, l \).

Now the equations (2.15) can be written in the vector form

\[
\Delta u = A U - c,
\]

where

\[
c = (c_1, \ldots, c_l) \quad \text{with} \quad c_j = (l + 1) - 4\pi \sum_{s=1}^{N_j} \delta_{p_j,s}(x), \quad j = 1, \ldots, l.
\]

To prove Theorem 2.1 it is equivalent to prove the following theorem for the nonlinear elliptic system (2.15) or (2.17).

**Theorem 2.2** Consider the system of nonlinear elliptic equations (2.15) or (2.17).

(i) For the problem over a doubly periodic domain \( \Omega \), there exits a solution if and only if the condition

\[
\max_{1 \leq j \leq l} \{N_j\} < \frac{(l + 1)|\Omega|}{4\pi}
\]

holds. Moreover, if there is a solution, it must be unique.

(ii) For the problem over \( \mathbb{R}^2 \), there exists a unique solution satisfying the boundary condition (2.16). Moreover, this boundary condition is achieved exponentially fast,

\[
\sum_{i=1}^{l} |u_i|^2 \leq C(\varepsilon)e^{-(1-\varepsilon)|x|} \quad \text{as} \quad |x| \to \infty,
\]

where \( \varepsilon \in (0, 1) \) is arbitrarily small, \( C(\varepsilon) \) is a positive constant depending on \( \varepsilon \).

(iii) In both cases, there holds the following quantized integrals

\[
\int \left( e^{u_j} + \sum_{i=1}^{l} e^{u_i} - (l + 1) \right) dx = -4\pi N_j, \quad j = 1, \ldots, l.
\]

In the following sections, we just need to prove Theorem 2.2.

3 Proof of existence for doubly periodic case

In this section we prove Theorem 2.2 for the doubly periodic domain case. We consider the problem (2.15) over a doubly periodic domain \( \Omega \).

Let \( u_j^0 \) be the solution of (see [2])

\[
\Delta u_j^0 = 4\pi \sum_{s=1}^{N_j} \delta_{p_j,s}(x) - \frac{4\pi N_j}{|\Omega|}, \quad x \in \Omega, \quad j = 1, \ldots, l.
\]

Set \( u_j = u_j^0 + v_j, \ j = 1, \ldots, l \). The equations (2.15) are transformed into

\[
\Delta v_j = e^{u_j^0 + v_j} + \sum_{i=1}^{l} e^{u_i^0 + v_i} - (l + 1) + \frac{4\pi N_j}{|\Omega|}, \quad j = 1, \ldots, l.
\]
First we show that the condition (2.18) is necessary. If \( \mathbf{v} = (v_1, \ldots, v_l) \) is a solution to the equations (3.2), integrating the equations over \( \Omega \) and by a direct computation, we obtain

\[
\int_{\Omega} e^{u_0 + v_j} \, dx = |\Omega| - 4\pi N_j + \frac{4\pi}{l+1} \sum_{i=1}^l N_i \equiv K_j > 0, \quad j = 1, \ldots, l.
\]  

(3.3)

Then (2.18) and (2.20) follows.

Next we prove that (2.18) is also sufficient for the existence of a solution to (2.15). In other words, we will prove that under the condition (2.18) the elliptic system (2.15) admits a unique solution.

Let \( \mathbf{a} = (a_1, \ldots, a_l)\tau \) with

\[
\mathbf{a} = \begin{pmatrix}
(l + 1) - \frac{4\pi N_1}{|\Omega|} \\
\vdots \\
(l + 1) - \frac{4\pi N_l}{|\Omega|}
\end{pmatrix}.
\]

We can rewrite the equations (3.2) in a vector form

\[
\Delta \mathbf{v} = A \mathbf{U} - \mathbf{a},
\]

(3.4)

To find a variational principle, we need to use the property of the matrix \( A \). It is easy to check that the matrix \( A \) is positive definite. Then, by Cholesky decomposition theorem [14], we see that the matrix \( A \) can be uniquely expressed as a product of a lower triangular matrix \( L \) and its transpose, \( A = LL^\tau \), \( L = (L_{ij})_{l \times l} \). Indeed, using the iteration scheme in [14]

\[
L_{11} = \sqrt{a_{11}}, \quad L_{jj} = \frac{a_{jj}}{L_{1j}},
\]

\[
L_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} L_{jk}^2} = \sqrt{2 - \sum_{k=1}^{j-1} L_{jk}^2}, \quad j = 1, \ldots, l,
\]

\[
L_{jk} = \frac{a_{jk} - \sum_{k=1}^{j-1} L_{jk} L_{kk'}}{L_{kk}} = \frac{1 - \sum_{k=1}^{j-1} L_{jk} L_{kk'}}{L_{kk}}, \quad j = k + 1, \ldots, l, \quad k = 2, \ldots, l,
\]

we have

\[
l_{kk} = \sqrt{\frac{k+1}{k}}, \quad k = 1, \ldots, l;
\]

\[
l_{jk} = \sqrt{\frac{1}{k(k+1)}}, \quad j = 2, \ldots, l, \quad k = 1, \ldots, j - 1.
\]

More explicitly, we obtain

\[
L = \begin{pmatrix}
\sqrt{2} & 0 & 0 & \cdots & 0 \\
\frac{\sqrt{3}}{2} & \sqrt{3} & 0 & \cdots & 0 \\
\frac{\sqrt{4}}{3} & \frac{\sqrt{4}}{2} & \sqrt{4} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\sqrt{\frac{1}{l+1}} & \sqrt{\frac{1}{l-1}} & \sqrt{\frac{1}{l+1}} & \cdots & \sqrt{\frac{1}{l+1}}
\end{pmatrix}.
\]

(3.5)

6
Denote the inverse of $L$ by
\[ L^{-1} = (\hat{l}_{jk})_{l \times l}. \]

Then, we get
\[ \hat{l}_{kk} = \sqrt{\frac{k}{k+1}}, \quad k = 1, \ldots, l; \]
\[ \hat{l}_{jk} = -\frac{1}{\sqrt{j(j+1)}}, \quad j = 2, \ldots, l, \quad k = 1, \ldots, j - 1, \]
that is,
\[
L^{-1} = \begin{pmatrix}
\sqrt{\frac{1}{2}} & 0 & 0 & \cdots & 0 \\
-\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 & \cdots & 0 \\
-\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{5}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{\sqrt{l(l+1)}} & -\frac{1}{\sqrt{l(l+1)}} & -\frac{1}{\sqrt{l(l+1)}} & \cdots & \sqrt{\frac{l}{l+1}}
\end{pmatrix}. \tag{3.6}
\]

As a result, by $A^{-1} = (LL^\tau)^{-1} = (L^{-1})^\tau L^{-1}$, we have
\[
A^{-1} = \frac{1}{l+1} \begin{pmatrix}
l & -1 & -1 & \cdots & -1 \\
-1 & l & -1 & \cdots & -1 \\
-1 & -1 & l & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & l
\end{pmatrix}. \tag{3.7}
\]

Let $w = (w_1, \ldots, w_l)^\tau$. We introduce the transformation
\[
w = L^{-1}v \quad \text{or} \quad v = Lw, \tag{3.8}
\]
which can be expressed in the component form as (by (3.5), (3.6))
\[
\begin{cases}
w_1 = \frac{1}{\sqrt{2}} v_1, \\
w_j = -\frac{1}{\sqrt{j(j+1)}} \sum_{i=1}^{j-1} v_i + \sqrt{\frac{j}{j+1}} v_j, \quad j = 2, \ldots, l,
\end{cases} \tag{3.9}
\]
or
\[
\begin{cases}
v_1 = \sqrt{2} w_1, \\
v_j = \sum_{i=1}^{j-1} \frac{w_i}{\sqrt{i(i+1)}} + \sqrt{\frac{j+1}{j}} w_j, \quad j = 2, \ldots, l.
\end{cases} \tag{3.10}
\]

Let $b = L^{-1}a$. Then the system (3.4) becomes
\[
\Delta w = L^\tau U - b. \tag{3.11}
\]

Set $K = (K_1, \ldots, K_l)^\tau$. Hence, by (3.11) and (3.3), we have
\[
K = |\Omega|(L^\tau)^{-1} b \quad \text{or} \quad b = \frac{1}{|\Omega|} L^\tau K. \tag{3.12}
\]
We may express (3.11) in the component form

\[
\Delta w_1 = \sqrt{2} \exp \left( u_1^0 + \sqrt{2}w_1 \right) + \frac{1}{\sqrt{2}} \sum_{i=2}^l \exp \left( u_i^0 + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i} w_i} \right) - b_1, \quad (3.13)
\]

\[
\Delta w_j = \sqrt{\frac{j+1}{j}} \exp \left( u_j^0 + \sum_{k=1}^{j-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{j+1}{j} w_j} \right) + \sqrt{\frac{1}{j(j+1)}} \sum_{i=j+1}^l \exp \left( u_i^0 + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i} w_i} \right) - b_j, \quad j = 2, \ldots, l-1 (3.14)
\]

\[
\Delta w_l = \sqrt{\frac{l+1}{l}} \exp \left( u_l^0 + \sum_{k=1}^{l-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{l+1}{l} w_l} \right) - b_l \quad (3.15)
\]

It is easy to check that the above system of equations (3.13)–(3.15) are the Euler–Lagrange equations of the functional

\[
I(w) = I(w_1, \ldots, w_l) = \int_{\Omega} \left\{ \frac{1}{2} \sum_{i=1}^l |\nabla w_i|^2 + \exp \left( u_i^0 + \sqrt{2w_i} \right) - b_i w_i + \sum_{i=2}^l \exp \left( u_i^0 + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i} w_i} \right) - b_i w_i \right\} dx. \quad (3.16)
\]

Let \( W^{1,2}(\Omega) \) be the usual Sobolev space of scalar-valued or vector-valued \( \Omega \)-periodic \( L^2 \) functions with their derivatives also in \( L^2(\Omega) \). For \( W^{1,2}(\Omega) \) in scalar case, we have the decomposition

\[
W^{1,2}(\Omega) = \mathbb{R} \oplus \hat{W}^{1,2}(\Omega)
\]

such that any \( w \in W^{1,2}(\Omega) \) can be expressed as

\[
w = \bar{w} + \dot{w}, \quad \bar{w} \in \mathbb{R}, \quad \dot{w} \in \hat{W}^{1,2}(\Omega), \quad \int_{\Omega} \dot{w} dx = 0. \quad (3.17)
\]

For any function \( w \in \hat{W}^{1,2}(\Omega) \), there holds the Trudinger-Moser inequality \([2, 12]\)

\[
\int_{\Omega} e^w dx \leq C \exp \left( \frac{1}{16\pi} \int_{\Omega} |\nabla w|^2 dx \right), \quad (3.18)
\]

which is important for our estimate.

When \( \bar{w} \in W^{1,2}(\Omega) \), using the above inequality (3.18) we see that the functional defined by (3.16) is a \( C^1 \) functional and weakly lower semi-continuous with respect to the weak topology of \( W^{1,2}(\Omega) \).

For \( \bar{w} \in W^{1,2}(\Omega) \), applying (3.8), (3.12) and the decomposition formula (3.17), we have

\[
I(\bar{w}) = \int_{\Omega} \left\{ \frac{1}{2} \sum_{i=1}^l |\nabla \dot{w}_i|^2 \right\} dx + \int_{\Omega} \sum_{i=1}^l \exp(u_0 + \dot{v}_i + \bar{v}_i) dx - K^T \bar{v}
\]

\[
= \int_{\Omega} \left\{ \frac{1}{2} \sum_{i=1}^l |\nabla \dot{w}_i|^2 \right\} dx + \int_{\Omega} \sum_{i=1}^l \exp(u_0 + \dot{v}_i + \bar{v}_i) dx - \sum_{i=1}^l K_i \bar{v}_i \quad (3.19)
\]
Using Jensen’s inequality, we obtain
\[
\int_{\Omega} \exp(u_i^0 + \dot{v}_i + u_i) \geq |\Omega| \exp \left( \frac{1}{|\Omega|} \int_{\Omega} (u_i^0 + \dot{v}_i + u_i) dx \right) = |\Omega| \exp \left( \frac{1}{|\Omega|} \int_{\Omega} u_i^0 dx \right) e^{\omega_i} \equiv \sigma_i e^{\omega_i}, \quad i = 1, \ldots, l. \quad (3.20)
\]

Using the condition (3.18), we have \( K_i > 0, i = 1, \ldots, l \). Then, combining (3.19) and (3.20), we have
\[
I(\mathbf{w}) - \int_{\Omega} \left\{ \frac{1}{2} \sum_{i=1}^{l} |\nabla \dot{w}_i|^2 \right\} dx \geq \sum_{i=1}^{l} (\sigma_i e^{\omega_i} - K_i u_i) \geq \sum_{i=1}^{l} K_i \ln \frac{\sigma_i}{K_i}. \quad (3.21)
\]

From (3.21) we can see that the functional \( I(\mathbf{w}) \) is bounded from below in \( W^{1,2}(\Omega) \) and the following minimization problem
\[
\eta_0 \equiv \inf \{ I(\mathbf{w}) | \mathbf{w} \in W^{1,2}(\Omega) \} \quad (3.22)
\]
is well-defined.

Let \( \{ w_i^{(k)}, \ldots, w_i^{(k)} \} \) be a minimizing sequence of (3.22). It is easy to see that the function \( F(t) = \sigma e^t - \eta t, \) where \( \sigma, \eta \) are positive constants, satisfies the property that \( F(t) \to +\infty \) as \( t \to \pm \infty \). Then, we infer from (3.21) that \( \{ \mathbf{w}_i^{(k)} \} (i = 1, \ldots, l) \) are bounded. As a result, \( \{ \mathbf{w}_i^{(k)} \} (i = 1, \ldots, l) \) are bounded. Then, the sequences \( \{ \mathbf{w}_i^{(k)} \} (i = 1, \ldots, l) \) admit convergent subsequences, still denoted by \( \{ \mathbf{w}_i^{(k)} \} (i = 1, \ldots, l) \) for convenience. Then, there exist \( l \) real numbers \( w_1^{(\infty)}, \ldots, w_l^{(\infty)} \in \mathbb{R} \) such that \( \mathbf{w}_i^{(k)} \to \mathbf{w}_i^{(\infty)}, i = 1, \ldots, l, \) as \( k \to \infty \).

In addition, using (3.21), we conclude that \( \{ \nabla \dot{w}_i^{(k)} \} (i = 1, \ldots, l) \) are bounded in \( L^2(\Omega) \). Therefore, it follows from the Poincaré inequality that the sequences \( \{ \dot{w}_i^{(k)} \} (i = 1, \ldots, l) \) admit weakly convergent subsequences, still denoted by \( \{ \dot{w}_i^{(k)} \} (i = 1, \ldots, l) \) for convenience. Then, there exist \( l \) functions \( \dot{w}_i^{(\infty)} \in W^{1,2}(\Omega) (i = 1, \ldots, l) \) such that \( \dot{w}_i^{(k)} \to \dot{w}_i^{(\infty)} \) weakly in \( W^{1,2}(\Omega) \) as \( k \to \infty \) (i = 1, \ldots, l) of course, \( \dot{w}_i^{(\infty)} \in W^{1,2}(\Omega) (i = 1, \ldots, l) \).

Set \( w_i^{(\infty)} = \mathbf{w}_i^{(\infty)} + \dot{w}_i^{(\infty)} (i = 1, \ldots, l) \), which are all in \( W^{1,2}(\Omega) \) naturally. Then, the above convergence implies \( w_i^{(k)} \to w_i^{(\infty)} \) (i = 1, \ldots, l) weakly in \( W^{1,2}(\Omega) \) as \( k \to \infty \). Since the functional \( I(\mathbf{w}) \) is weakly lower semi-continuous in \( W^{1,2}(\Omega) \), we conclude that \( \{ w_1^{(\infty)}, \ldots, w_l^{(\infty)} \} \) is a solution of the minimization problem (3.22) and is a critical point of \( I(\mathbf{w}) \). As a critical point of \( I(\mathbf{w}) \), it satisfies the system (3.13).

Noting that the matrix \( A \) is positive definite, it is easy to check that \( I(\mathbf{w}) \) is strictly convex in \( W^{1,2}(\Omega) \). Then, it has at most one critical point in \( W^{1,2}(\Omega) \), which implies the uniqueness of the solution to the equations (3.13).
4 Proof of existence for the planar case

In this section we prove Theorem (2.2) for the full plane case. In other words, we study the nonlinear elliptic system (2.15) or (2.17) over the full plane with the boundary condition (2.16).

As in [27] we introduce the background functions

$$u_j^0 = -\sum_{k=1}^{N_j} \ln(1 + \mu|x - p_{j,k}|^{-2}), \mu > 0, \ j = 1, \ldots, l. \quad (4.1)$$

Then we have

$$\Delta u_j^0 = 4\pi \sum_{k=1}^{N_j} \delta_{p_{j,k}} - g_j, \ g_j = \sum_{k=1}^{N_j} \frac{4\mu}{(\mu + |x - p_{j,k}|^2)^2}, \ j = 1, \ldots, l. \quad (4.2)$$

It is easy to see that

$$g_j \in L(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \ \int_{\mathbb{R}^2} g_j dx = 4\pi N_j, \ j = 1, \ldots, l. \quad (4.3)$$

Let $$u_j = v_j + u_j^0, \ j = 1, \ldots, l$$, then the system (2.15) become

$$\Delta v_j = e^{u_j^0 + v_j} + \sum_{i=1}^l e^{u_i^0 + v_i} - (l + 1) + g_j, \ j = 1, \ldots, l. \quad (4.4)$$

As in the previous section we use the transformation (3.8) to change (4.4) into

$$\Delta w = L^\tau (U - 1) + h \quad (4.5)$$

or in the component form

$$\begin{align*}
\Delta w_1 &= \sqrt{2} \left[ \exp \left( u_1^0 + \sqrt{2} w_1 \right) - 1 \right] \\
&\quad + \frac{1}{\sqrt{2}} \sum_{i=2}^l \left[ \exp \left( u_i^0 + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) - 1 \right] + h_1, \quad (4.6) \\
\Delta w_j &= \sqrt{\frac{j+1}{j}} \left[ \exp \left( u_j^0 + \sum_{k=1}^{j-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{j+1}{j}} w_j \right) - 1 \right] \\
&\quad + \frac{1}{\sqrt{j(j+1)}} \sum_{i=j+1}^l \left[ \exp \left( u_i^0 + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) - 1 \right] + h_j, \quad (4.7) \\
\Delta w_l &= \sqrt{\frac{l+1}{l}} \left[ \exp \left( u_l^0 + \sum_{k=1}^{l-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{l+1}{l}} w_l \right) - 1 \right] + h_l, \quad (4.8)
\end{align*}$$

where

$$\mathbf{1} = (1, \ldots, 1)^\tau, \quad \mathbf{h} = (h_1, \ldots, h_l)^\tau = L^{-1} \mathbf{g}, \quad \mathbf{g} = (g_1, \ldots, g_l)^\tau.$$
It is easy to check that (4.5) are the Euler–Lagrange equations of the following functional

\[
I(\mathbf{w}) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_{i=1}^{l} |\nabla w_i|^2 + \sum_{i=1}^{l} h_i w_i + \exp \left( u^0_1 + \sqrt{2} w_1 \right) - \exp(u^0_1) - \sqrt{2} w_1 \\
+ \sum_{i=2}^{l} \left[ \exp \left( u^0_i + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) - \exp(u^0_i) \right] - \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} - \sqrt{\frac{i+1}{i}} w_i \right\} \, dx.
\]

To proceed further, it is convenient to rewrite the functional \( I(\mathbf{w}) \) as the following form

\[
I(\mathbf{w}) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_{i=1}^{l} |\nabla w_i|^2 dx + e^{u^0_1} \left[ e^{\sqrt{2} w_1} - 1 - \sqrt{2} w_1 \right] \\
+ \sum_{i=2}^{l} e^{u^0_i} \left[ \exp \left( \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) - 1 - \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} - \sqrt{\frac{i+1}{i}} w_i \right] \\
+ \sqrt{2} \left( e^{u^0_i} - 1 + \frac{1}{\sqrt{2}} h_i \right) w_1 + \sqrt{\frac{l+1}{l}} \left( e^{u^0_i} - 1 \right) w_l \\
+ \sum_{i=2}^{l-1} \left[ \sqrt{\frac{i+1}{i}} \left( e^{u^0_i} - 1 \right) + \frac{1}{\sqrt{i(i+1)}} \sum_{k=i+1}^{l} \left( e^{u^0_k} - 1 + h_i \right) w_i \right] \right\} \, dx
\]

Then we obtain

\[
(DI(\mathbf{w}))(\mathbf{w}) = \int_{\mathbb{R}^2} \left\{ \sum_{i=1}^{l} |\nabla w_i|^2 dx + \left[ \sqrt{2} \left( \exp(u^0_1 + \sqrt{2} w_1) - 1 \right) \\
+ \frac{1}{\sqrt{2}} \sum_{i=2}^{l} \left( \exp \left( u^0_i + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) - 1 \right) \right] w_1 \\
+ \sum_{j=2}^{l} \left[ \frac{1}{\sqrt{j(j+1)}} \sum_{i=j+1}^{l} \left( \exp \left( u^0_i + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) - 1 \right) \right] w_j \\
+ \sqrt{\frac{l+1}{l}} \left[ \exp \left( u^0_l + \sum_{k=1}^{l-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{l+1}{l}} w_l \right) - 1 \right] w_l + \sum_{j=1}^{l} h_j w_j \\
+ \sum_{i=2}^{l} \left[ \exp \left( u^0_i + \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) - 1 + \tilde{h}_i \right] \times \left( \sum_{k=1}^{i-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{i+1}{i}} w_i \right) \right\} \, dx,
\]

where

\[
\tilde{h} \equiv (\tilde{h}_1, \ldots, \tilde{h}_l)^T = (L^{-1})^T \mathbf{h} = (L^{-1})^T L^{-1} \mathbf{g} = A^{-1} \mathbf{g},
\]
or in the component form (using (3.7))

$$\tilde{h}_i = \frac{1}{l+1} \left( l g_i - \sum_{j \neq i}^l g_j \right), \quad i = 1, \ldots, l. \quad (4.11)$$

Noting the transformation (3.8), we easily see that

$$c_1 \sum_{i=1}^m v_i^2 \leq \sum_{i=1}^m w_i^2 \leq c_2 \sum_{i=1}^m v_i^2, \quad c_1 \sum_{i=1}^m |\nabla v_i|^2 \leq \sum_{i=1}^m |\nabla w_i|^2 \leq c_2 \sum_{i=1}^m |\nabla v_i|^2 \quad (4.12)$$

holds for some positive constants $c_1$ and $c_2$. Therefore, from (3.10), (4.10) and (4.12), we can obtain

$$(DI(v))(u) \geq C_1 \sum_{i=1}^l \int_{\mathbb{R}^2} |\nabla v_i|^2 \, dx + \sum_{i=1}^l \int_{\mathbb{R}^2} \left( e^{u_0 + v_i} - 1 + \tilde{h}_i \right) v_i \, dx, \quad (4.13)$$

where and in the sequel we use $C_i$ to denote a generic positive constant.

In what follows we estimate the second term on the right hand side of (4.13). To this end we use the approach developed in [27]. It is sufficient to deal with a general term of the following form

$$G(v) = \int_{\mathbb{R}^2} \left( e^{u_0 + v} - 1 + \tilde{h} \right) v \, dx,$$

where we use $u_0$, $v$, and $\tilde{h}$ to denote $u_0$'s, $v$'s and $\tilde{h}_i$'s, respectively. For convenience, we decompose $G(v)$ as $G(v) = G(v_+) + G(-v_-)$ where $v_+ = \max\{v, 0\}$, $v_- = \max\{-v, 0\}$.

Using the inequality $e^t - 1 \geq t$, $t \in \mathbb{R}$ and the fact $u_0, \tilde{h} \in L^2(\mathbb{R}^2)$, we obtain

$$G(v_+) \geq \int_{\mathbb{R}^2} (u_0 + v_+ + \tilde{h}) v_+ \, dx \geq \frac{1}{2} \int_{\mathbb{R}^2} v_+^2 \, dx - C_2. \quad (4.14)$$

In view of the inequality $1 - e^{-t} \geq \frac{t}{1+t}$, $t \geq 0$, $e^{u_0} - 1, \tilde{h} \in L^2(\mathbb{R}^2)$, we estimate $G(-v_-)$ as follows

$$G(-v_-) = \int_{\mathbb{R}^2} \left( 1 - e^{u_0 - v_-} - \tilde{h} \right) v_- \, dx$$

$$= \int_{\mathbb{R}^2} \left( 1 - \tilde{h} - e^{u_0} + e^{u_0} [1 - e^{-v_-}] \right) v_- \, dx$$

$$\geq \int_{\mathbb{R}^2} \left( 1 - \tilde{h} - e^{u_0} + e^{u_0} \frac{v_-}{1+v_-} \right) v_- \, dx$$

$$= \int_{\mathbb{R}^2} \left( 1 - \tilde{h} \right) \frac{v_-^2}{1+v_-} \, dx + \int_{\mathbb{R}^2} \left( 1 - e^{u_0} - \tilde{h} \right) \frac{v_-}{1+v_-} \, dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^2} \frac{v_-^2}{1+v_-} \, dx + \int_{\mathbb{R}^2} \left( 1 - e^{u_0} - \tilde{h} \right) \frac{v_-}{1+v_-} \, dx$$

$$\geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{v_-^2}{(1+v_-)^2} \, dx - C_3, \quad (4.15)$$

where we have used the fact $\tilde{h} \leq \frac{1}{2}$, assured by taking $\mu$ sufficiently large.
Therefore from (4.14) and (4.15) we conclude that
\[ G(v) \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{v^2}{(1 + |v|)^2}dx - C_4. \]  
(4.16)

By the following standard interpolation inequality over \(W^{1,2}(\mathbb{R}^2)\):
\[ \int_{\mathbb{R}^2} v^4 dx \leq 2 \int_{\mathbb{R}^2} v^2 dx \int_{\mathbb{R}^2} |\nabla v|^2 dx, \quad \forall v \in W^{1,2}(\mathbb{R}^2), \]  
(4.17)
we have
\[ \left( \int_{\mathbb{R}^2} |v|^2 dx \right)^2 \]
\[ = \left( \int_{\mathbb{R}^2} \frac{|v|}{1 + |v|} (1 + |v|)|v| dx \right)^2 \]
\[ \leq \int_{\mathbb{R}^2} \frac{|v|^2}{(1 + |v|)^2} dx \int_{\mathbb{R}^2} (|v| + |v|^2)^2 dx \]
\[ \leq 4 \int_{\mathbb{R}^2} \frac{|v|^2}{(1 + |v|)^2} dx \int_{\mathbb{R}^2} |v|^2 dx \left( \int_{\mathbb{R}^2} |\nabla v|^2 dx + 1 \right) \]
\[ \leq \frac{1}{2} \left( \int_{\mathbb{R}^2} |v|^2 dx \right)^2 + C \left( \left[ \frac{|v|^2}{(1 + |v|)^2} dx \right]^4 + \left[ \int_{\mathbb{R}^2} |\nabla v|^2 dx \right]^4 + 1 \right), \]
which implies
\[ \|v\|_2 \leq C_5 \left( \int_{\mathbb{R}^2} \frac{|v|^2}{(1 + |v|)^2} dx + \int_{\mathbb{R}^2} |\nabla v|^2 dx + 1 \right), \]  
(4.18)
where and in the sequel we use \(\| \cdot \|_p\) to denote the norm of the space \(L^p(\mathbb{R}^2)\).

From (4.13) and (4.16), we obtain
\[ (DI(w))(w) \geq C_6 \sum_{j=1}^m \int_{\mathbb{R}^2} \left( |\nabla v_j|^2 + \frac{v_j^2}{(1 + |v_j|)^2} \right) dx - C_7. \]  
(4.19)
Then it follows from (4.12), (4.13) and (4.19) that
\[ (DI(w))(w) \geq C_8 \sum_{j=1}^m \|w_j\|_{W^{1,2}(\mathbb{R}^2)} - C_9. \]  
(4.20)

Now we show that the functional \(I\) has a critical point. Using (4.20), we can choose \(R > 0\) such that
\[ \inf \{ DI(w)(w) \mid \|w\|_{W^{1,2}(\mathbb{R}^2)} = R \} \geq 1. \]  
(4.21)
Noting that functional \(I\) is weakly lower semi-continuous on \(W^{1,2}(\mathbb{R}^2)\), we see that the minimization problem
\[ \eta_0 \equiv \inf \{ I(w) \|w\|_{W^{1,2}(\mathbb{R}^2)} \leq R \} \]  
(4.22)
admits a solution, say, \(\hat{w}\). We may prove that it must be an interior point. Otherwise, we assume that \(\|\hat{w}\|_{W^{1,2}(\mathbb{R}^2)} = R\). Therefore
\[ \lim_{t \to 0} \frac{I((1-t)\hat{w}) - I(\hat{w})}{t} = \frac{d}{dt} I((1-t)\hat{w})|_{t=0} = -(DI(\hat{w}))(\hat{w}) \leq -1 \]
Hence, if \( t > 0 \) is sufficiently small, set \( \hat{w}' = (1 - t)\hat{w} \), we see that
\[
I(\hat{w}') < I(\hat{w}) = \eta_0, \quad \|\hat{w}'\|_{W^{1,2}(\mathbb{R}^2)} = (1 - t)R < R,
\]
which contradicts the definition of \( \eta_0 \). Hence, \( \hat{w} \) must be an interior critical point for the problem (4.22). As a result, it is a critical point of the functional \( I \). Since the functional \( I \) is strictly convex, this critical point must be unique.

Now we investigate the asymptotic behavior of the solution established above. Since \( w \in W^{1,2}(\mathbb{R}^2) \), using the well-known inequality
\[
\|e^v - 1\|^2 \leq d_1 \exp(d_2 \|v\|^2_{W^{1,2}(\mathbb{R}^2)}), \quad \forall v \in W^{1,2}(\mathbb{R}^2),
\]
where \( d_1, d_2 \) are some positive constants, we see that the right-hand sides of the equations (4.6)–(4.8) all belong to \( L^2(\mathbb{R}^2) \). By the standard elliptic \( L^2 \)-estimates, we obtain that \( w_i \in W^{2,2}(\mathbb{R}^2) \), which implies \( w_i \to 0 \) as \( |x| \to \infty, i = 1, \ldots, l \). Using the transformation (3.10), we conclude that \( v_i \to 0 \) as \( |x| \to \infty \), which gives the desired boundary condition \( u_i \to 0 \) as \( |x| \to \infty, i = 1, \ldots, l \).

Now we prove that \( |\nabla w_i| \to 0 \) as \( |x| \to \infty \), \( i = 1, \ldots, l \). We rewrite a typical term of the right hand sides of (4.6)–(4.8) as
\[
\sqrt{\frac{j+1}{j}} \left[ \exp \left( u_j^0 + \sum_{k=1}^{j-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{j+1}{j}} w_j \right) - 1 \right]
= \sqrt{\frac{j+1}{j}} \left[ (e^{a_j^0} - 1) \exp \left( \sum_{k=1}^{j-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{j+1}{j}} w_j \right) \right]
+ \exp \left( \sum_{k=1}^{j-1} \frac{w_k}{\sqrt{k(k+1)}} + \sqrt{\frac{j+1}{j}} w_j \right) - 1 \right] (4.23)
\]
which lies in \( L^p(\mathbb{R}^2) \) for any \( p > 2 \) due to the embedding \( W^{1,2}(\mathbb{R}^2) \subset L^p(\mathbb{R}^2) \) and the definition of \( u_j^0 \). Then we see that all the right-hand-side terms of (4.6)–(4.8) belong to \( L^p(\mathbb{R}^2) \), for any \( p > 2 \). Using the elliptic \( L^p \)-estimates, we see that \( w_i \in W^{2,p}(\mathbb{R}^2) \) for any \( p > 2, i = 1, \ldots, l \). As a result, \( |\nabla w_i| \to 0 \) as \( |x| \to \infty, i = 1, \ldots, l \). In other words, \( |\nabla u_i| \to 0 \) as \( |x| \to \infty, i = 1, \ldots, l \).

Next we establish the exponential decay rate of the solutions at infinity. To do this, we consider the equations (2.15) or (2.17) over an exterior domain
\[
D_R = \{ x \in \mathbb{R}^2 \mid |x| > R \},
\]
where \( R > 0 \) satisfies
\[
R > \max \left\{ |p_{i,s}| \mid i = 1, \ldots, l, s = 1, \ldots, N_1 \right\}. (4.25)
\]
For convenience, we consider the system of equations (2.17) over \( D_R \). We may rewrite (2.17) in \( D_R \) as
\[
\Delta u = A(U - 1) = Au + A(U - 1 - u). (4.26)
\]
Since the matrix \( A \) in (2.17) is positive definite and its eigenvalues are \( \lambda_1 = l + 1, \lambda_2 = \lambda_3 = \cdots = \lambda_l = 1 \), which can be checked easily. Then there exists an orthogonal matrix \( O \) such that
\[
O^T AO = \text{diag}\{l + 1, 1, \ldots, 1\}. (4.27)
\]
Now apply $O^\tau$ in (4.26) and set
\[ \tilde{u} = O^\tau u. \]
Then we have
\[ \Delta \tilde{u} = \text{diag}\{l + 1, 1, \ldots, 1\} \tilde{u} + O^\tau (U - 1 - u). \] (4.28)
Noting that $U \to 1$ as $|x| \to \infty$, we have $U - 1 = E(x)u$, where $E(x)$ is an $l \times l$ diagonal matrix so that $E(x) \to I_l$ (the $l \times l$ identity matrix) as $|x| \to \infty$. Then we can rewrite (4.28) as
\[ \Delta \tilde{u} = \text{diag}\{l + 1, 1, \ldots, 1\} \tilde{u} + Z(x)\tilde{u}, \] (4.29)
where $Z(x)$ is an $l \times l$ matrix which vanishes at infinity. Hence from (4.29) we see that
\[ \Delta |\tilde{u}|^2 \geq 2\tilde{u}^T \Delta \tilde{u} \geq |\tilde{u}|^2 - b(x)|\tilde{u}|^2, \] (4.30)
where $b(x) \to 0$ as $|x| \to \infty$.

Therefore, for any $\varepsilon \in (0, 1)$, we can take a suitably large $R_\varepsilon \geq R$ such that
\[ \Delta |\tilde{u}|^2 \geq \left(1 - \frac{\varepsilon}{2}\right) |\tilde{u}|^2, \quad x \in D_{R_\varepsilon}. \] (4.31)

Taking a comparison function, say $\eta$, of the form
\[ \eta = Ce^{-\sigma|x|}, \quad |x| > 0, \quad C, \sigma \in \mathbb{R}, \quad C, \sigma > 0 \] (4.32)
we have $\Delta \eta = \sigma^2 \eta - \frac{\sigma}{|x|} \eta$. Hence, by (4.31) we obtain
\[ \Delta \left(|\tilde{u}|^2 - \eta\right) \geq \left(1 - \frac{\varepsilon}{2}\right) |\tilde{u}|^2 - \sigma^2 \eta, \quad |x| \geq R_\varepsilon. \] (4.33)
We take the obvious choice $\sigma^2 = \left(1 - \frac{\varepsilon}{2}\right) \lambda \lambda_0$ which gives us $\Delta (|\tilde{u}|^2 - \eta) \geq \sigma^2 (|\tilde{u}|^2 - \eta), \quad |x| \geq R_\varepsilon$. Taking $C$ in (4.32) large such that $|\tilde{u}|^2 - \eta \leq 0$ for $|x| = R_\varepsilon$, using the fact that $|\tilde{u}| \to 0$ as $|x| \to \infty$ and the maximum principle, we conclude that $|\tilde{u}|^2 \leq \eta$ for $|x| \geq R_\varepsilon$. Noting $(1 - \varepsilon)^2 < (1 - \frac{\varepsilon}{2})$ for any $\varepsilon \in (0, 1)$, we get the precise exponential decay estimate
\[ |\tilde{u}|^2 \leq C(\varepsilon)e^{-\left(1-\varepsilon\right)|x|}, \quad |x| \geq R_\varepsilon. \] (4.34)
Thus we get the desired exponential decay rate (2.19).

At last, we aim to prove the quantized integrals. To this end, we need to establish the exponential decay rate for the derivatives.

Let $\partial$ denote any of the two derivatives $\partial_1$ and $\partial_2$. Define
\[ v = (\partial u_1, \ldots, \partial u_m)^T, \quad M = \text{diag}\{e^{a_1} - 1, \ldots, e^{a_l} - 1\}. \] (4.35)
Then differentiating (2.17) in $D_R$, we have
\[ \Delta v = Av + AMv. \] (4.36)
Let $O$ be as before and set

$$v = O \tilde{v}. \quad (4.37)$$

Then by (4.36), we have

$$\Delta \tilde{v} = \text{diag}\{l + 1, 1, \ldots, 1\} \tilde{v} + O^T A M O \tilde{v}. \quad (4.38)$$

Since $e^{u_i} - 1 \to 0$ as $|x| \to \infty$, $i = 1, \ldots, l$, we may rewrite (4.39) as

$$\Delta \tilde{v} = \text{diag}\{l + 1, 1, \ldots, 1\} \tilde{v} + \tilde{Z}(x) \tilde{v}, \quad (4.39)$$

where $\tilde{Z}(x)$ is an $l \times l$ matrix vanishing at infinity. As previously, from (4.39) we have

$$\Delta |\tilde{v}|^2 \geq 2 \tilde{v}^T \Delta \tilde{v} \geq |\tilde{v}|^2 - \tilde{b}(x)|\tilde{v}|^2; \quad (4.40)$$

where $\tilde{b}(x) \to 0$ as $|x| \to \infty$.

Similarly, we may infer that, for any $\varepsilon \in (0, 1)$, there is a positive constant $C(\varepsilon) > 0$, such that

$$|\tilde{v}|^2 \leq C(\varepsilon)e^{-(1-\varepsilon)|x|},$$

when $|x|$ is sufficiently large. Then we obtain the exponential decay rate near infinity

$$\sum_{i=1}^{m} |\nabla u_i(x)|^2 \leq C(\varepsilon)e^{-(1-\varepsilon)|x|}. \quad (4.41)$$

Now we can calculate the quantized integrals (2.20) stated in Theorem 2.2 for the planar case.

Using (4.1), (4.2), and the exponential decay property of $|\nabla u_i|$’s in (4.41), we conclude that $|\nabla v|$’s vanish at infinity at least at the rate $|x|^{-3}$. Then it follows from the divergence theorem that

$$\int_{\mathbb{R}^2} \Delta v_i \, dx = 0, \quad i = 1, \ldots, l. \quad (4.42)$$

Thus, by integrating the equations (4.4) over $\mathbb{R}^2$, and applying (1.3) and (4.42), we obtain the desired results stated in (2.20).

5 Conclusions

We have carried out a rigorous analysis of the BPS equations derived from the theory of multi-intersections of $l$ ($l \geq 2$) D4-branes and 1 D4-brane. The BPS equations are investigated in two situations. In the first situation the equations are studied over a doubly periodic domain. In the second situation the equations are studied over the full plane. Via the direct minimization method we establish a sharp existence and uniqueness theorem for multiple vortex solutions of the BPS equations. We find an explicit necessary and sufficient condition for the existence of a unique solution for the doubly periodic domain case. By the obtained vortex solutions we can interpret them as D0-branes on the intersections.
References

[1] A. A. Abrikosov, On the magnetic properties of superconductors of the second group, *Sov. Phys. JETP* 5 (1957) 1174–1182.

[2] T. Aubin, *Nonlinear Analysis on Manifolds: Monge-Ampère Equations*, Springer, Berlin and New York, 1982.

[3] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nonabelian superconductors: vortices and confinement in N = 2 SQCD, *Nucl. Phys. B* 673 (2003) 187–216.

[4] R. Auzzi and S. P. Kumar, Non-Abelian vortices at weak and strong coupling in mass deformed ABJM theory, *J. High Energy Phys.* 0910 (2009) 071.

[5] A. Bezryadina, E. Eugenieva and Z. Chen, Self-trapping and flipping of double-charged vortices in optically induced photonic lattices, *Optics Lett.* 31 (2006) 2456–2458.

[6] E. B. Bogomol’nyi, The stability of classical solutions, *Sov. J. Nucl. Phys.* 24 (1976) 449–454.

[7] L. Caffarelli and Y. Yang, Vortex condensation in the Chern-Simons Higgs model: an existence theorem, *Comm. Math. Phys.* 168 (1995) 321–336.

[8] R. M. Chen and D. Spirn, Symmetric Chern-Simons-Higgs vortices, *Comm. Math. Phys.* 285 (2009) 1005–1031.

[9] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi and N. Sakai, Multiple layer structure of non-Abelian vortex, *Phys. Lett. B* 678 (2009) 254–258.

[10] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Solitons in the Higgs phase the moduli matrix approach, *J. Phys. A* 39 (2006) R315–R392.

[11] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Moduli space of non-Abelian vortices, *Phys. Rev. Lett.* 96 (2006) 161601.

[12] L. Fontana, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, *Comment. Math. Helv.* 68 (1993) 415-454.

[13] V. L. Ginzburg and L. D. Landau, On the theory of superconductivity, In: Collected Papers of L. D. Landau (edited by D. Ter Haar), New York: Pergamon, 1965, pp. 546–568.

[14] G. H. Golub and J. M. Ortega, *Scientific computing and Differential Equations*, Academic, San Diego, 1992.

[15] S. B. Gudnason, Y. Jiang, and K. Konishi, Non-Abelian vortex dynamics: effective world-sheet action, *J. High Energy Phys.* 012 (2010) 1008.

[16] J. Han, Quantization effects for Maxwell–Chern–Simons vortices, *J. Math. Anal. Appl.* 363 (2010) 265–274.
[17] J. Han and H. Huh, Self-dual vortices in a Maxwell–Chern–Simons model with non-minimal coupling, *Lett. Math. Phys.* **82** (2007) 9–24.

[18] J. Han and J. Jang, Self-dual Chern–Simons vortices on bounded domains, *Lett. Math. Phys.* **64** (2003) 45-56.

[19] J. Han and S. Kim, Self-dual Maxwell–Chern–Simons theory on a cylinder, *J. Phys. A: Math. Theor.* **44** (2011) 135203

[20] X. Han and G. Tarantello, Doubly periodic self-dual vortices in a relativistic non-Abelian Chern–Simons model, Calculus of Variations and Partial Differential Equations, 2013, in press.

[21] A. Hanany and D. Tong, Vortices, instantons and branes, J. High Energy Phys. 037 (2003).

[22] M. B. Hindmarsh and T. W. B. Kibble, Cosmic strings, *Rep. Prog. Phys.* **58** (1995) 477–562.

[23] J. Hong, Y. Kim and P. Y. Pac, Multivortex solutions of the Abelian Chern–Simons–Higgs theory, *Phys. Rev. Lett.* **64** (1990) 2230–2233.

[24] G. ’t Hooft, A property of electric and magnetic flux in nonabelian gauge theories, *Nucl. Phys. B* **153** (1979) 141–160.

[25] S. Inouye, S. Gupta, T. Rosenband, A. P. Chikkatur, A. Grilz, T.L. Gustavson, A.E. Leanhardt, D. E. Pritchard and W. Ketterle, Observation of vortex phase singularities in Bose-Einstein condensates, *Phys. Rev. Lett.* **87** (2001) 080402.

[26] R. Jackiw and E. J. Weinberg, Self-dual Chern–Simons vortices, *Phys. Rev. Lett.* **64** (1990) 2234–2237.

[27] A. Jaffe and C. H. Taubes, *Vortices and Monopoles*, Birkhäuser, Boston, 1980.

[28] Y. Kawaguchi and T. Ohmi, Splitting instability of multiply charged vortex in a Bose–Einstein condensate, *Phys. Rev. A* **70** (2004) 043610.

[29] T. W. B. Kibble, Some implications of a cosmological phase transition, *Phys. Rep.* **67** (1980) 183–199.

[30] E. H. Lieb and Y. Yang, Non-Abelian vortices in supersymmetric gauge field theory via direct methods, *Communications in Mathematical Physics*, (2012) to appear.

[31] C. S Lin and Y. Yang, Sharp existence and uniqueness theorems for non-Abelian multiple vortex solutions, *Nuclear Physics B* **846** (2011) 650–676.

[32] C. S. Lin and Y. Yang, Non-Abelian multiple vortices in supersymmetric field theory, *Commun. Math. Phys.* **304** (2011) 433–457.

[33] T. Ricciard and G. Tarantello, Vortices in the Maxwell–Chern–Simons theory, *Comm. Pure Appl. Math.* **53** (2000) 811–851.
[34] N. Seiberg and E. Witten Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory, *Nucl. Phys. B* 426 (1994) 19–52.

[35] M. Shifman and A. Yung, Non-Abelian string junctions as connected monopoles, *Phys. Rev. D* 70 (2004) 045004.

[36] M. Shifman and A. Yung, Localization of non-Abelian gauge fields on domain walls at weak coupling: D-brane prototypes, *Phys. Rev. D* 70 (2004) 025013.

[37] M. Shifman and A. Yung, Supersymmetric solitons and how they help us understand non-Abelian gauge theories, *Rev. Mod. Phys.* 79 (2007) 1139.

[38] J. Spruck and Y. Yang, On multivortices in the electroweak theory. II. Existence of Bogomol’ny solutions in $\mathbb{R}^2$. *Comm. Math. Phys.* 144 (1992) 215–234.

[39] J. Spruck and Y. Yang, On multivortices in the electroweak theory. I. Existence of periodic solutions, *Comm. Math. Phys.* 144 (1992) 1–16.

[40] T. Suyama, Monopoles and black hole entropy, *Mod. Phys. Lett. A* 15 (2000) 271–280.

[41] T. Suyama, Intersecting branes and generalized vortices, arXiv:hep-th/9912261v1.

[42] G. Tarantello, Multiple condensate solutions for the Chern–Simons–Higgs theory, *J. Math. Phys.* 37 (1996) 3769–3796.

[43] C. H. Taubes, Arbitrary $N$-vortex solutions to the first order Ginzburg-Landau equations, *Commun. Math. Phys.* 72 (1980) 277–292.

[44] C. H. Taubes, On the equivalence of the first and second order equations for gauge theories, *Commun. Math. Phys.* 75 (1980) 207-2-27.

[45] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge U. Press, 1994.

[46] R. Wang, The existence of Chern–Simons vortices, *Comm. Math. Phys.* 137 (1991) 587–597.

[47] S. Wang and Y. Yang, Abrikosov’s vortices in the critical coupling, *SIAM J. Math. Anal.* 23 (1992) 1125–1140.

[48] Y. Yang, On a system of nonlinear elliptic equations arising in theoretical physics, *Journal of Functional Analysis* 170 (2000) 1–36.

[49] Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer, New York, 2001