TOPOLOGICAL STRUCTURE ON ABEL-GRASSMANN’S
GROUPOIDS

QAI SER MUSHTAQ AND MADAD KHAN

ABSTRACT. In this paper we have discussed the ideals in Abel Grassmann’s
groupoids and construct their topologies.

1. INTRODUCTION

An Abel-Grassmann’s groupoid (AG-groupoid) [3] is a groupoid S with left in-
vertive law

\[(ab)c = (cb)a, \text{ for all } a, b, c \in S.\]

Every AG-groupoid S satisfy the medial law [2]

\[(ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in S.\]

In every AG-groupoid with left identity the following law [3] holds

\[(ab)(cd) = (db)(ca), \text{ for all } a, b, c, d \in S.\]

Many characteristics of several non-associative AG-groupoids similar to a com-
mutative semigroup.

The aim of this note is to define the topological spaces using ideal theory. Several
ideals concerning the number of occurrence of topological spaces in AG-groupoids.
The topological spaces formation guarantee for the preservation of finite intersection
and arbitrary union between the set of ideals and the open subsets of resultant
topologies.

A subset I of an AG-groupoid S is called a right (left) ideal if

\[IS \subseteq I \text{ (SI \subseteq I),\}

and is called an ideal if it is two sided ideal, if I is a left ideal of S then \(I^2\) becomes
an ideal of S. By a bi-ideal of an AG-groupoid S, we mean a sub AG-groupoid
B of S such that \((BS)B \subseteq B.\) It is easy to note that each right ideal is a bi-
ideal. If S has a left identity then it is not hard to show that \(B^2\) is a bi-ideal
of S and \(B^2 \subseteq SB^2 = B^2S.\) If \(E(B_S)\) denote the set of all idempotents subsets
of S with left identity e, then \(E(B_S)\) form a semilattice structure also if \(C = C^2\)
then \((CS)C \in E(B_S).\) The intersection of any set of bi-ideals of an AG-groupoid
S is either empty or a bi-ideal of S. Also the intersection of prime bi-ideals of an
AG-groupoid S is a semiprime bi-ideal of S.

If S is an AG-groupoid with left identity e, and assume that \(a^3 = a^2a\) then

\[(x^n_1 x^n_2) (x^q_3 x^r_4) = (x^{m p(1)}_n x^{m p(2)}_n) (x^{q p(3)}_r x^{q p(4)}_r), \text{ for } m, n, q, r \geq 2,\]

\[2000 \text{ Mathematics Subject Classification. 20M10 and 20N99.}\]

\[Key \ words \ and \ phrases. \ AG-groupoid, \ Anti-rectangular \ band, \ Medial \ law, \ bi-ideals \ and \ prime \ bi-ideals.\]
where \{p(1), p(2), p(3), p(4)\} means any permutation on the set \{1, 2, 3, 4\}. As a consequence \( (x_{i_{p(1)}}x_{i_{p(2)}}x_{i_{p(3)}}x_{i_{p(4)}})^k = (x_{i_1}x_{i_2}x_{i_3}x_{i_4})^k \), for \( k \geq 2 \). The result can be generalized for finite numbers of elements of \( S \). If \( 0 \in S \), then \( 0s = s0 = 0 \), for all \( s \) in \( S \).

**Proposition 1.** Let \( T \) be a left ideal and \( B \) is a bi-ideal of an AG-groupoid \( S \) with left identity, then \( BT \) and \( T^2B \) are bi-ideals of \( S \).

**Proof.** Using (2), we get

\[
((BT)S)(BT) = ((BT)B)(ST) \subseteq ((BS)B)T \subseteq BT,
\]
also \( (BT)(BT) = (BB)(TT) \subseteq BT. \)

Hence \( BT \) is a bi-ideal of \( S \). Now using (2), we obtain

\[
(T^2B)S(T^2B) = ((T^2S)(BS))(T^2B) \subseteq (T^2(BS))(T^2B) = (T^2T^2)((BS)B) \subseteq T^2B,
\]
also \( (T^2B)(T^2B) = (T^2T^2)(BB) \subseteq T^2B. \)

Hence \( T^2B \) is a bi-ideal of \( S \).

**Proposition 2.** The product of two bi-ideals of an AG-groupoid \( S \) with left identity is a bi-ideal of \( S \).

**Proof.** Using (2), we get

\[
((B_1B_2)S)(B_1B_2) = ((B_1B_2)(SS))(B_1B_2) = ((B_1S)(B_2S))(B_1B_2)
\]
\[
= ((B_1S)B_1)((B_2S)B_2) \subseteq B_1B_2.
\]

If \( B_1 \) and \( B_2 \) are non-empty, then \( B_1B_2 \) and \( B_2B_1 \) are connected bi-ideals. Also the above Proposition leads us to easy generalizations that is, if \( B_1, B_2, B_3, \ldots \) and \( B_n \) are bi-ideals of an AG-groupoid \( S \) with left identity, then

\[
((B_1B_2)B_3)B_n = (...((B_1B_2)B_3)B_4)...B_n
\]

are bi-ideals of \( S \), consequently the set \( \mathfrak{C}(S_B) \) of bi-ideals form an AG-groupoid.

If \( S \) an AG-groupoid with left identity \( e \) then \( (a)_L = Sa, (a)_R = aS \) and \( (a)_S = (Sa)_S \) are bi-ideals of \( S \). Now it is not hard to show that \( (ab)_L = (a)_L(b)_L, (ab)_R = (a)_R(b)_R \) and \( (ab)_R = (b)_L(a)_L \), from these it can be deduce that \( (a)_{R}^{2}R = (b)_{L}^{2}(a)_L \) and \( (a)_{L}^{2}L = (b)_{R}^{2}(a)_R \). Also \( (a)_{L}^{2}L = (b)_{R}^{2}(a)_R, (a)_{L}^{2}L = (a)_{L}^{2}, (a)_{R}^{2}R = (a)_{R}^{2}, (a)_{L}^{2}L = (a)_{R}^{2} \) and \( (a)_{L} = (a)_{R} \) (if \( a \) is an idempotent), consequently \( (a)_{L}^{2}L = (a)_{R}^{2} \). It is easy to show that \( (a)_{R}^{2}a^{2} = a^{2}(a)_L. \)

**Lemma 1.** If \( B \) is an idempotent bi-ideal of an AG-groupoid \( S \) with left identity, then \( B \) is an ideal of \( S \).

**Proof.** Using (1),

\[
BS = (BB)S = (SB)B = (SB^2)B = (B^2S)B = (BS)B,
\]
and every right ideal in \( S \) with left identity is left.

**Lemma 2.** If \( B \) is a proper bi-ideal of an AG-groupoid \( S \) with left identity \( e \), then \( e \notin B. \)

**Proof.** Let \( e \in B, \) since \( sb = (es)b \in B, \) now using (1), we get \( s = (ee)s = (se)e \in (SB)B \subseteq B. \)
It is easy to note that \( \{ x \in S : (xa)x = e \} \not\subseteq B \).

**Proposition 3.** If \( A, B \) are bi-ideals of an AG-groupoid \( S \) with left identity, then the following assertions are equivalent.

(i) Every bi-ideal of \( S \) is idempotent,
(ii) \( A \cap B = AB \), and
(iii) the ideals of \( S \) form a semilattice \( (L_S, \land) \) where \( A \land B = AB \).

**Proof.** (i) \( \Rightarrow \) (ii): Using Lemma 1, it is easy to note that \( AB \subseteq A \cap B \). Since \( A \cap B \subseteq A, B \) implies \( (A \cap B)^2 \subseteq AB \), hence \( A \cap B \subseteq AB \).

(ii) \( \Rightarrow \) (iii): \( A \land B = AB = A \cap B = B \land A \) and \( A \land A = AA = A \cap A = A \).

Similarly, associativity follows. Hence \( (L_S, \land) \) is a semilattice.

(iii) \( \Rightarrow \) (i):

\[
A = A \land A = AA.
\]

\( \square \)

A bi-ideal \( B \) of an AG-groupoid \( S \) is called a prime bi-ideal if \( B_1B_2 \subseteq B \) implies either \( B_1 \subseteq B \) or \( B_2 \subseteq B \) for every bi-ideal \( B_1 \) and \( B_2 \) of \( S \). The set of bi-ideals of \( S \) is totally ordered under inclusion if for all bi-ideals \( I, J \) either \( I \subseteq J \) or \( J \subseteq I \).

**Theorem 1.** Each bi-ideal of an AG-groupoid \( S \) with left identity is prime if and only if it is idempotent and the set of bi-ideals of \( S \) is totally ordered under inclusion.

**Proof.** Assume that each bi-ideal of \( S \) is prime. Since \( B^2 \) is an ideal and so is prime which implies that \( B \subseteq B^2 \), hence \( B \) is idempotent. Since \( B_1 \cap B_2 \) is a bi-ideal of \( S \) (where \( B_1 \) and \( B_2 \) are bi-ideals of \( S \)) and so is prime, now by Lemma 1, either \( B_1 \subseteq B_1 \cap B_2 \) or \( B_2 \subseteq B_1 \cap B_2 \) which further implies that either \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). Hence the set of bi-ideals of \( S \) is totally ordered under inclusion.

Conversely, assume that every bi-ideal of \( S \) is idempotent and the set of bi-ideals of \( S \) is totally ordered under inclusion. Let \( B_1, B_2 \) and \( B \) be the bi-ideals of \( S \) with \( B_1B_2 \subseteq B \) and without loss of generality assume that \( B_1 \subseteq B_2 \). Since \( B_1 \) is an idempotent, so \( B_1 = B_1B_1 \subseteq B_1B_2 \subseteq B \) implies that \( B_1 \subseteq B \) and hence each bi-ideal of \( S \) is prime. \( \square \)

A bi-ideal \( B \) of an AG-groupoid \( S \) is called strongly irreducible bi-ideal if \( B_1 \cap B_2 \subseteq B \) implies either \( B_1 \subseteq B \) or \( B_2 \subseteq B \) for every bi-ideal \( B_1 \) and \( B_2 \) of \( S \).

**Theorem 2.** Let \( D \) be the set of all bi-ideals of an AG-groupoid \( S \) with zero and \( \Omega \) be the set of all strongly irreducible proper bi-ideals of \( S \), then \( \Gamma(\Omega) = \{ O_B : B \in D \} \), form a topology on the set \( \Omega \), where \( O_B = \{ J \in \Omega : B \not\subseteq J \} \) and \( \phi : \text{bi-ideal}(S) \rightarrow \Gamma(\Omega) \) preserves finite intersection and arbitrary union between the set of bi-ideals of \( S \) and open subsets of \( \Omega \).

**Proof.** As \( \{ 0 \} \) is a bi-ideal of \( S \), and 0 belongs to every bi-ideal of \( S \), then \( O_{\{ 0 \}} = \{ J \in \Omega , \{ 0 \} \not\subseteq J \} = \{ \} \), also \( O_S = \{ J \in \Omega , S \not\subseteq J \} = \Omega \) which is the first axiom for the topology. Let \( \{ O_{B_\alpha} : \alpha \in I \} \subseteq \Gamma(\Omega) \), then \( \cup O_{B_\alpha} = \{ J \in \Omega , B_\alpha \not\subseteq J \} \), for some \( \alpha \in I \). Consider \( \cup B_\alpha \) in \( \Omega \), \( \cup B_\alpha \not\subseteq J \) and \( \cup B_\alpha \not\subseteq J \) is a bi-ideal of \( S \) generated by \( \cup B_\alpha \). Let \( O_{B_1} \) and \( O_{B_2} \in \Gamma(\Omega) \), if \( J \in O_{B_1} \cap O_{B_2} \), then \( J \in \Omega \) and \( B_1 \not\subseteq J \) or \( B_2 \not\subseteq J \). Suppose \( B_1 \cap B_2 \subseteq J \), this implies that either \( B_1 \subseteq J \) or \( B_2 \subseteq J \), which leads us to a contradiction. Hence \( B_1 \cap B_2 \not\subseteq J \) which further implies that \( J \in O_{B_1} \cup O_{B_2} \). Thus \( O_{B_1} \cup O_{B_2} \subseteq O_{B_1} \cap O_{B_2} \). Now if \( J \in O_{B_1} \cap O_{B_2} \), then \( J \in \Omega \) and \( B_1 \cap B_2 \not\subseteq J \). Thus \( J \in O_{B_1} \) and \( J \in O_{B_2} \), therefore \( J \in O_{B_1} \cap O_{B_2} \).
which implies that $O_{B_1 \cap B_2} \subseteq O_{B_1} \cap O_{B_2}$. Hence $\Gamma(\Omega)$ is the topology on $\Omega$. Define $\phi : \text{bi-ideal}(S) \rightarrow \Gamma(\Omega)$ by $\phi(B) = O_B$, then it is easy to note that $\phi$ preserves finite intersection and arbitrary union.

An ideal $P$ of an AG-groupoid $S$ is called prime if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$ for all ideals $A$ and $B$ in $S$.

Let $P_S$ denote the set of proper prime ideals of an AG-groupoid $S$ absorbing 0. For an ideal $I$ of $S$ define the set $\Theta_I = \{ J \in P_S : I \not\subseteq J \}$ and $\Gamma(P_S) = \{ \Theta_I, I \text{ is an ideal of } S \}$.

**Theorem 3.** Let $S$ be an AG-groupoid with 0. The set $\Gamma(P_S)$ constitutes a topology on the set $P_S$.

**Proof.** Let $\Theta_{I_1}, \Theta_{I_2} \in \Gamma(P_S)$, if $J \in \Theta_{I_1} \cap \Theta_{I_2}$, then $J \in P_S$ and $I_1 \not\subseteq J$ and $I_2 \not\subseteq J$. Let $I_1 \cap I_2 \subseteq J$ which implies that either $I_1 \subseteq J$ or $I_2 \subseteq J$, which is contradiction. Hence $J \in \Theta_{I_1} \cap \Theta_{I_2}$. Similarly $\Theta_{I_1} \cap \Theta_{I_2} \subseteq \Theta_{I_1} \cap \Theta_{I_2}$. The remaining proof follows from Theorem 2.

The assignment $I \rightarrow \Theta_I$ preserves finite intersection and arbitrary union between the ideal $I$ and their corresponding open subsets of $\Theta_I$.

Let $P$ be a left ideal of an AG-groupoid $S$. $P$ is called quasi-prime if for left ideals $A$, $B$ of $S$ such that $AB \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$.

**Theorem 4.** If $S$ is an AG-groupoid with left identity $e$, then a left ideal $P$ of $S$ is quasi-prime if and only if $(Sa)b \subseteq P$ implies that either $a \in P$ or $b \in P$.

**Proof.** Let $P$ be a left ideal of an AG-groupoid $S$ with left identity $e$. Now assume that $(Sa)b \subseteq P$, then

$$S((Sa)b) \subseteq SP \subseteq P,$$

that is

$$S((Sa)b) = (Sa)(Sb)$$

Hence, either $a \in P$ or $b \in P$.

Conversely, assume that $AB \subseteq P$ where $A$ and $B$ are left of $S$ such that $A \not\subseteq P$. Then there exists $x \in A$ such that $x \not\in P$. Now using the hypothesis we get $(Sx)y \subseteq (SA)B \subseteq AB \subseteq P$ for all $y \in B$. Since $x \not\in P$, so by hypothesis, $y \in P$ for all $y \in B$, we obtain $B \subseteq P$. This shows that $P$ is quasi-prime.

An AG-groupoid $S$ is called an anti-rectangular if $a = (ba)b$, for all $a,b$ in $S$. It is easy to see that $S = S^2$. In the following results for an anti-rectangular AG-groupoid $S$, $e \notin S$.

**Proposition 4.** If $A$ and $B$ are the ideals of an anti-rectangular AG-groupoid $S$, then $AB$ is an ideal.

**Proof.** Using (2), we get

$$(AB)S = (AB)(SS) = (AS)(BS) \subseteq AB,$$

also

$$S(AB) = (SS)(AB) = (SA)(SB) \subseteq AB$$

which shows that $AB$ is an ideal.

Consequently, if $I_1, I_2, I_3,...$ and $I_n$ are ideals of $S$, then

$$((I_1 I_2 I_3)...I_n)$$

are ideals of $S$ and the set $\otimes_I$ of ideals of $S$ form an anti-rectangular AG-groupoid.
Lemma 3. Any subset of an anti-rectangular AG-groupoid $S$ is left ideal if and only if it is right.

Proof. Let $I$ be a right ideal of $S$, then using (1), we get, $si = ((xs)x)i = (ix)(xs) \in I$.

Conversely, suppose that $I$ be a left ideal of $S$, then using (1), we get, $is = ((yi)y)s = (sy)(yi) \in I$.

It is fact that $SI = IS$. From above Lemma we remark that, each quasi prime ideals becomes prime in an anti-rectangular AG-groupoid.

Lemma 4. If $I$ is an ideal of an anti-rectangular AG-groupoid $S$ then, $H(a) = \{ x \in S : (xa)x = a, \text{ for } a \in I \} \subseteq I$.

Proof. Let $y \in H(a)$, then $y = (ya)y \in (SI)S \subseteq I$. Hence $H(a) \subseteq I$.

Also $H(a) = \{ x \in S : (xa)x = x, \text{ for } a \in I \} \subseteq I$. □

An ideal $I$ of an AG-groupoid $S$ is called an idempotent if $I^2 = I$. An AG-groupoid $S$ is said to be fully idempotent if every ideal of $S$ is idempotent.

Proposition 5. If $S$ is an anti-rectangular AG-groupoid and $A, B$ are ideals of $S$, then the following assertions are equivalent.

(i) $S$ is fully idempotent,
(ii) $A \cap B = AB$, and
(iii) the ideals of $S$ form a semilattice $(LS, \wedge)$ where $A \wedge B = AB$.

The proof follows from Proposition 3.

The set of ideals of $S$ is totally ordered under inclusion if for all ideals $I, J$ either $I \subseteq J$ or $J \subseteq I$ and denoted by ideal $(S)$.

Theorem 5. Every ideal of an anti-rectangular AG-groupoid $S$ is prime if and only if it is idempotent and ideal $(S)$ is totally ordered under inclusion.

The proof follows from Theorem 1.

REFERENCES

[1] Ahsen Javed and Liu Zhonghui, Strongly idempotent seminearrings and their prime ideals paces,G. Saad and M. J. Thomson (eds), Nearrings and K-Loops, 1997, 151 – 166.
[2] Kazim, M. A and M. Naseeruddin, On almost-semigroups, The Alig Bull. Math., 2 (1972), 1 - 7.
[3] Protic, P. V and M. Bozinovic, Some congruences on an AG**-groupoid, Algebra Logic and Discrete Mathematics, 14 – 16 (1995), 879 – 886.

DEPARTMENT OF MATHEMATICS, QUAD-I-AZAM UNIVERSITY, ISLAMABAD, PAKISTAN.

E-mail address: qmuhaq@isb.apollo.net.pk
E-mail address: madadmath@yahoo.com