Energy of weak gravitational waves in spacetimes with a positive cosmological constant *

P.T. Chruściel†, Sk Jahanur Hoque‡, Tomasz Smołka§

March 24, 2020

Abstract

We derive a formula for the total energy, and its flux, of weak gravitational waves on a de Sitter background.

In view of the recent gravitational waves detections, and of the measurement of a positive cosmological constant, there arises an urgent need of a thorough understanding of gravitational waves in spacetimes with a positive cosmological constant. An important starting point for such studies is the understanding of weak gravitational waves in this context, as these are the ones that are seen by the detectors. One of the key issues is to determine how much energy is carried away by these gravitational waves.

Indeed, consider an astrophysical system emitting gravitational waves, e.g. a collection of clusters of galaxies in a binary or in a localised many-body system; here “localised” should be understood in terms of cosmological scales. At the scale of the universe the resulting gravitational waves can be well approximated by linearised fields on a cosmological background except on a cosmologically-negligible region in an immediate neighborhood of the emitting system. And without any doubt, for such systems the field in the radiation zone is well described by the linearised theory. As a first approximation to the problem at hand it appears reasonable to consider solutions of the linearised equations throughout. The aim of this note is to derive a formula for the energy emitted in such a setting in the simplest

---

*Preprint UWThPh-2020-12
†Faculty of Physics, University of Vienna
‡Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, Prague, and Chennai Mathematical Institute, India
§Department of Mathematical Methods in Physics, University of Warsaw
cosmological model with a positive cosmological constant, namely de Sitter spacetime.

More precisely, we will calculate the canonical energy of weak gravitational waves on light cones in a de Sitter universe, and their flux.

Now, the worldline of an isolated system as above is well described by a timelike geodesic in de Sitter spacetime. Such geodesics are orbits of a Killing vector field, say $X$, which is timelike along this geodesic and tangent to it. A first guess would be that at any moment of time along the geodesic, the total energy contained in the gravitational wave emitted equals the integral, over the light-cone with vertex on the geodesic, of the canonical energy-momentum tensor of the gravitational waves contracted with the vector field describing the motion of the light-cone, which in this case is the Killing vector field $X$. However, linearised gravitational waves are defined up to gauge transformations, and the question of a physically meaningful choice of the gauge arises. Whatever the gauge, the canonical energy-momentum tensor contracted with a Killing vector field of the de Sitter background provides a current with vanishing divergence, with the integral of this current over a three-dimensional hypersurface providing an associated charge. This is an integral over a region of infinite extent, and will not even be finite when a random gauge is chosen. So a minimal requirement is to use gauges which allow one to obtain, or at least identify (as will be done below), finite integrals. The gauges should be restrictive enough so that the residual gauge freedom left does not affect the total energy calculated with the method.

In the case of vanishing cosmological constant, a good gauge for the purpose has been found by Bondi and collaborators [4,8]. In the nonlinear theory it leads to the Bondi energy, as well as the associated Trautman-Bondi mass loss formula [9]. For linearised gravitational waves and $\Lambda = 0$ the canonical energy, when calculated in the Bondi gauge, reproduces the Bondi energy and the Trautman-Bondi mass loss formula. It is therefore reasonable to expect that the use of the Bondi gauge will also provide a meaningful definition of total energy of weak gravitational waves in the presence of a positive cosmological constant. This is the approach taken here. After isolating terms which would lead to infinite energy and which have a dynamics of their own, one is thus led to our formula (20) below for the total energy of weak gravitational waves contained in a light cone, together with the formula (23) for the flux of energy when the cones are dragged along a timelike geodesic.

This work is related to that in [1,2], where a definition of energy is used which differs from the one used here by boundary terms. This difference is irrelevant when integrating over compact boundaryless surfaces. However,
as made clear below, a proper inclusion of boundary terms is important to obtain the energy flux formulae that we are about to derive. We emphasise that keeping track of these terms is crucial already in the case $\Lambda = 0$, whether in the linearised case addressed here or in the full nonlinear setting.

We note that an approximate solution describing a gravitational wave emitted by a gravitating system has been derived in [3]. Unfortunately, this solution is singular near the vertex of the light cone. This not an issue for the analysis there, since the authors of [3] are concerned with the large-distance behaviour of the solution. However, such solutions are not suitable in our context: while the knowledge of the asymptotics of the field suffices to obtain the flux formula for energy, the solution needs to be put into the Bondi gauge to calculate our energy flux. This requires regularity everywhere, including the vertex. It would be of interest to extend the analysis in [3] to obtain globally regular solutions, we hope to address this in the future.

Recall that a metric in Bondi coordinates takes the form

$$g_{\alpha\beta} dx^\alpha dx^\beta = -V e^{2\beta} du^2 - 2e^{2\beta} dudr$$

$$+ r^2 \gamma_{AB} \left( dx^A - U^A du \right) \left( dx^B - U^B du \right),$$

(1)

where $(x^A) \equiv (x^2, x^3) \equiv (\theta, \varphi)$, together with the condition

$$\det \gamma_{AB} = \sin^2 \theta.$$ 

(2)

The de Sitter metric can be written in this form (cf., e.g., [5])

$$g \equiv g_{\alpha\beta} dx^\alpha dx^\beta = -(1 - \Lambda r^2 / 3) du^2 - 2du^2 dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(3)

$$\epsilon \in \{\pm 1\},$$

(4)

with $\epsilon$ determined by the sign of $g_{uu}$. We consider linearised solutions $h_{\mu\nu}$ of the vacuum Einstein equations which, in the coordinate system of (4), satisfy the gauge conditions resulting from (1)-(2):

$$h_{rr} = 0 = h_{rA}, \quad \gamma^{AB} h_{AB} = 0.$$

(5)

The equations satisfied by $h_{\mu\nu}$ can be derived from the Lagrangian obtained by taking the quadratic part of $\sqrt{\left|\det g\right|} / 16\pi \left( R - 2\Lambda \right)$ and discarding a divergence,

$$\mathcal{L}[h] = \frac{1}{32\pi} \sqrt{\left|\det g\right|} \left( P^{\alpha\beta\gamma\delta\varepsilon\sigma} \nabla_\alpha h_{\beta\gamma} \nabla_\delta h_{\varepsilon\sigma} + Q(h) \right),$$

(6)
where $Q$ is a quadratic polynomial in $h$,

$$Q(h) = \frac{2\Lambda}{(d-2)} \left[ g^{\alpha\rho} g^{\beta\sigma} h_{\alpha\beta} h_{\rho\sigma} - \frac{1}{2} (g^{\alpha\beta} h_{\alpha\beta})^2 \right], \quad (7)$$

and

$$P^{\alpha\beta\gamma\delta\epsilon\sigma} = \frac{1}{2} \left( g^{\alpha\epsilon} g^{\delta\beta} g^{\gamma\sigma} + g^{\alpha\epsilon} g^{\gamma\beta} g^{\delta\sigma} - g^{\alpha\delta} g^{\beta\epsilon} g^{\gamma\sigma} - g^{\alpha\beta} g^{\gamma\delta} g^{\epsilon\sigma} - g^{\beta\gamma} g^{\alpha\epsilon} g^{\delta\sigma} + g^{\beta\gamma} g^{\alpha\delta} g^{\epsilon\sigma} \right). \quad (8)$$

Given a Lagrangian field theory of fields $\phi^A$ with Lagrangian $\mathcal{L}$, the canonical energy associated with a vector field $X$ and a hypersurface $\mathcal{S}$ is defined as the integral

$$\mathcal{H}[\mathcal{S}, X, \phi] := \int_{\mathcal{S}} \left( \frac{\partial \mathcal{L}}{\partial \phi^A} \mathcal{L}_X \phi^A - X^\mu \mathcal{L} \right) dS_\mu, \quad (9)$$

where $\phi^A_{\mu} := \partial_\mu \phi^A$.

Let $\mathcal{C}_u$ denote a level set of the coordinate $u$ as in (4), this is a light cone emanating from $r = 0$. Let $\mathcal{C}_{u,R}$ denote this light cone truncated at coordinate-radius $r = R$. Given a solution $h_{\mu\nu}$ of the linearised vacuum Einstein equations in Bondi gauge, a somewhat lengthy calculation gives the following formula for the canonical energy $E_c[h, \mathcal{C}_{u,R}]$:

$$E_c[h, \mathcal{C}_{u,R}] := \mathcal{H} \left[ \mathcal{C}_{u,R}, \partial_u, h \right] = \frac{1}{64\pi} \int_{\mathcal{C}_{u,R}} \left( \partial_\mu \mathcal{L}_X \phi^A - X^\mu \mathcal{L} \right) dS_\mu,$$

$$- \frac{1}{32\pi} \int_{S(R)} P^{r(\beta\gamma)\delta(\epsilon\sigma)} \nabla_\delta h_{\epsilon\sigma} h_{\beta\gamma} r^2 \sin \theta \, d\theta \, d\varphi, \quad (10)$$

where $S(R)$ denotes a sphere of radius $R$.

To continue one needs to understand the asymptotic behaviour of the fields for large $R$. It is convenient to define the rescaled fields

$$\tilde{h}_{AB} := r^{-2} h_{AB},$$

keeping in mind that the symmetric trace-free tensor field $h_{AB}$ on a light cone is freely prescribable. It follows e.g. from [6] that there exists a dynamically
consistent class of fields $\hat{h}_{AB}$ which have an asymptotic expansion of the form, for large $r$,

$$\hat{h}_{AB} = \frac{(1)}{r} \hat{h}_{AB} + \frac{(2)}{r^2} \hat{h}_{AB} + \ldots ,$$

(11)

where the coefficients of the expansion $(1) \hat{h}_{AB}$, etc., do not depend upon $r$. The remaining $h_{\mu \nu}$’s are determined by the linearised version of the characteristic constraint equations in Bondi coordinates [7], this proceeds as follows. First, these equations give, in vacuum, $\partial_r h_{ru} = 0$, and regularity at the tip of the light-cone leads to $h_{ru} \equiv 0$. Next we have

$$\partial_r [r^4 \partial_r (r^{-2} h_{uA})] = -r^2 \mathcal{D}_E (\hat{\gamma}^{EF} \partial_r (r^{-2} h_{AF})) ,$$

(12)

where $\mathcal{D}_A$ is the covariant derivative of the round-sphere metric $\hat{\gamma}_{AB}$. Integrating in $r$ twice and using regularity of the metric at the vertex one obtains

$$h_{uA} = r^2 \mu_A (u, x^B) + r^2 \int_0^r \psi_A (s, x^B) \left( \frac{1}{3 r^3} - \frac{1}{3 s^3} \right) ds ,$$

(13)

where $\mu_A$ is chosen to cancel the leading order $r^2$-term which could arise at the right-hand side. This leads to the asymptotics

$$h_{uA} = h_{uA}^{(0)} + h_{uA}^{(1)} / r + \ldots$$

(14)

(Strictly speaking, a non-trivial choice of $\mu_A$ might prevent the metric perturbation to be smooth at the vertex, but one checks that the field $\mu_A$ has a gauge character and that the associated gauge transformation preserves finiteness of the volume integral.) One then finds the following form of the boundary term in (10):

$$-\frac{R}{64 \pi} \int_{S^2} \left( \frac{\Lambda}{3} \hat{\gamma}^{AB} \hat{\gamma}^{CD} \hat{h}_{AC} \hat{h}_{BD} \right) d^2 \mu_{\hat{\gamma}}$$

$$-\frac{1}{64 \pi} \int_{S^2} \hat{\gamma}^{AB} \hat{\gamma}^{CD} \left( \hat{h}_{AC} \partial_u \hat{h}_{BD} + \frac{\Lambda}{3} \hat{h}_{AC} \hat{h}_{BD} \right) d^2 \mu_{\hat{\gamma}}$$

$$+ o(1) ,$$

(15)

where $o(1)$ denotes terms which tend to zero as $R$ tends to infinity. This tends to minus infinity in the limit $R \rightarrow \infty$ if $(1) \hat{h}_{AC}$ is not identically zero, and begs the questions whether
1. the divergence of the boundary integral is compensated by that of the volume integral and, if not,

2. whether the boundary integral is needed at all in the definition of energy and, if so,

3. can one obtain consistent solutions by restricting oneself to a set of fields with \( \hat{h}_{AC} \equiv 0 \).

The answer to the second question is yes: if \( \Lambda = 0 \), one will not obtain the Trautman-Bondi mass loss formula without this term.

To answer the remaining questions one needs to make use the linearisation of the evolution equation for \( g_{AB} \) [7, Equation (32)]: Denoting by

\[ TS[\cdot] \]

the traceless symmetric part of a tensor, we have in vacuum

\[ r \partial_r [r (\partial_u \hat{h}_{AB})] - \frac{1}{2} \partial_r [N^2 (\partial_r \hat{h}_{AB})] + TS [\hat{D}_A (\partial_r (r^2 \hat{h}_B))] = 0. \]  

(16)

Integrating, we find

\[ \partial_u \hat{h}_{AB}(r, \cdot) = \frac{1}{r} \int_0^r \frac{1}{s} \left( \frac{1}{2} \partial_r [N^2 (\partial_r \hat{h}_{AB})] - TS [\hat{D}_A (\partial_r (r^2 \hat{h}_B))] \right)(s, \cdot) ds \]

\[ = \frac{\partial_u \hat{h}_{AB}(\cdot)}{r} + \frac{\partial_u \hat{h}_{AB}(\cdot)}{r^3} + o(r^{-3}), \]  

(17)

where

\[ \partial_u \hat{h}_{AB}(\cdot) = \int_0^\infty \frac{1}{s} \left( \frac{1}{2} \partial_r [N^2 (\partial_r \hat{h}_{AB})] - \hat{\gamma}_{CA} \hat{D}_B \partial_r (r^2 \hat{h}_C) \right)(s, \cdot) ds, \]  

(18)

and note that the integral exists and is finite under the current conditions.

Equation (18) shows that there is no reason for \( \partial_u \hat{h}_{AB} \) to vanish in general. Hence the field \( \hat{h}_{AB} \) will be nonzero at later times, even if the fields \( \hat{h}_{AB} \) are compactly supported on the initial light cone, unless the initial data are very special. Therefore the answer to question 3. is clearly negative.

Next, using (17) one finds a finite volume contribution to the canonical energy. So the volume integral cannot be used to compensate the divergence
of the boundary integral. This answers the first question in the negative, and is rather worrisome.

A way out is provided by the flux formula satisfied by the energy. Indeed, if \( X \) is a Killing vector of the background metric \( g \) and if the field \( h_{\mu\nu} \) satisfies the linearised Einstein equations, then the divergence of the field \( \mathcal{H}^\mu \) vanishes. This leads to the following flux equation

\[
\frac{dE_c[h, \xi]}{du} = -\frac{R}{32\pi} \int_{S^2} \frac{\Lambda}{3} \tilde{\gamma}^{AB} \tilde{\gamma}^{CD} \tilde{h}_{AC} \partial_u \tilde{h}_{BD} \mu^\gamma + \frac{1}{32\pi} \int_{S^2} \tilde{\gamma}^{AB} \tilde{\gamma}^{CD} \left( \partial_u \tilde{h}_{AC} \partial_u \tilde{h}_{BD} + \frac{\Lambda}{3} \tilde{h}_{BD} \partial_u \tilde{h}_{AC} \right) d^2\mu^\gamma.
\]

This equation shows that the divergent term in \( E_c \) has a dynamics of its own, evolving separately from the remaining part of the canonical energy. It is therefore natural to introduce a renormalised canonical energy, say \( \hat{E}_c[h, \xi] \), by removing the divergent term in (15). After having done this, we can pass to the limit \( R \to \infty \) to obtain:

\[
\hat{E}_c[h, \xi] := \frac{1}{64\pi} \int_{\mathcal{C}_u} \left( \tilde{g}^{BE} \tilde{g}^{FC} \left( \partial_u \tilde{h}_{BC} \partial_r \tilde{h}_{EF} - \tilde{h}_{BC} \partial_r \partial_u \tilde{h}_{EF} \right) - \frac{1}{32\pi} \int_{S^2} \tilde{\gamma}^{AB} \tilde{\gamma}^{CD} \left( \partial_u \tilde{h}_{AC} \partial_u \tilde{h}_{BD} + \frac{\Lambda}{3} \tilde{h}_{BD} \partial_u \tilde{h}_{AC} \right) d^2\mu^\gamma \right).
\]

This is our first main result here, and is our proposal how to calculate the total energy contained in a light cone of a weak gravitational wave on a de Sitter background.

One checks that vector fields \( \xi \) generating the gauge transformations

\[
h_{\mu\nu} \mapsto h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu
\]

and preserving the Bondi coordinate conditions take the form

\[
\xi = \left( \frac{\tilde{D}_B \tilde{\xi}^B(x^A)}{2} + \tilde{\xi}^A(x^A) \right) \partial_u + \tilde{\xi}^B(x^A) \partial_B + \frac{\Delta_u \xi^u}{2} - \frac{(r + u)\tilde{D}_B \tilde{\xi}^B}{2} \partial_r,
\]

where \( \tilde{\xi}^B(x^A) \partial_B \) is a conformal Killing vector field of \( \tilde{\gamma} \), and where \( \tilde{\xi}^u \) is a linear combination of \( \ell = 0 \) and \( \ell = 1 \) spherical harmonics. (This is
a key difference compared to the case $\Lambda = 0$, where $\xi^u$ is an arbitrary function of angles: there are no supertranslations in our setting.) A simple calculation then shows that our formula for energy is invariant under the transformations (22) with $D_B \xi^B = 0$. The remaining transformations, involving generators of non-trivial conformal transformations of $S^2$, correspond to Lorentz boosts, leading to the usual Lorentz transformation of the energy-momentum vector.

The flux equation for the renormalised energy $\hat{E}_c$ is obtained by dropping the term linear in $R$ in (19) and passing again to the limit $R \to \infty$:

$$\frac{d\hat{E}_c[h, \xi_u]}{du} = -\frac{1}{32\pi} \int_{S^2} \gamma^{AB} \gamma^{CD} \left( \partial_u \hat{h}_{AC} \partial_u \hat{h}_{BD} + \frac{\Lambda}{3} \hat{h}_{AC} \partial_u \hat{h}_{BD} \right) d^2\mu_+ .$$

(23)

This is our key new formula. When $\Lambda = 0$, equivalently $\alpha = 0$, we recover the weak-field version of the usual Trautman-Bondi mass loss formula. Hence the last term in (23) shows how the cosmological constant affects the flux of energy emitted by a gravitating astrophysical system.

Our mass loss formula has a $\Lambda$-dependent correction which can be both positive or negative, which is expected. Indeed, the de Sitter spacetime contains Cauchy hypersurfaces which are three-dimensional spheres, which implies that the total energy of gravitational waves in the full non-linear theory vanishes. Equivalently, in spatially closed universes the kinetic energy of the waves is exactly compensated by the negative potential energy arising from the self-interaction of the gravitational field. This implies in turn that any flux formula for energy must contain terms with indeterminate sign.

**ACKNOWLEDGEMENTS:** JH is grateful to the Erwin Schrödinger Institute, and University of Vienna for hospitality during part of work on this paper. He thanks Ghanashyam Date and Pavel Krtouš for discussions. His research was supported in part by the Czech Science Foundation Grant 19-01850S, and by the DST Max-Planck partner group project “Quantum Black Holes” between CMI, Chennai and AEI, Golm. TS thanks Jacek Jezierski for helpful remarks. He acknowledges the hospitality of the University of Vienna during part of work on this project and financial support from the COST Action CA16104 GWverse. The research of PTC was supported by the Austrian Research Fund (FWF), Project P 29517-N27 and by the Polish National Center of Science (NCN) under grant 2016/21/B/ST1/00940.
References

[1] A. Ashtekar, B. Bonga, and A. Kesavan, Asymptotics with a positive cosmological constant: I. Basic framework, Class. Quantum Grav. 32 (2015), 025004, 41 pp., arXiv:1409.3816 [gr-qc]. MR 3291776

[2] ———, Asymptotics with a positive cosmological constant: II. Linear fields on de Sitter spacetime, Phys. Rev. D92 (2015), 044011, 14, arXiv:1506.06152 [gr-qc]. MR 3441014

[3] ———, Asymptotics with a positive cosmological constant. III. The quadrupole formula, Phys. Rev. D 92 (2015), 104032, 21, arXiv:1510.05593 [gr-qc]. MR 3465140

[4] H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner, Gravitational waves in general relativity VII: Waves from axi-symmetric isolated systems, Proc. Roy. Soc. London A 269 (1962), 21–52. MR MR0147276 (26 #4793)

[5] K. Fischer, Interpretation of Einstein’s theory of gravitation including the cosmological term as a de Sitter-invariant field theory on the de Sitter space, Z. Physik 229 (1969), 33–43. MR 0255216

[6] H. Friedrich, On the existence of n-geodesically complete or future complete solutions of Einstein’s field equations with smooth asymptotic structure, Commun. Math. Phys. 107 (1986), 587–609.

[7] T. Mäddler and J. Winicour, Bondi-Sachs Formalism, Scholarpedia 11 (2016), 33528, arXiv:1609.01731 [gr-qc].

[8] R.K. Sachs, Gravitational waves in general relativity VIII. Waves in asymptotically flat spacetime, Proc. Roy. Soc. London A 270 (1962), 103–126. MR MR0149908 (26 #7393)

[9] A. Trautman, Radiation and boundary conditions in the theory of gravitation, Bull. Acad. Pol. Sci., Série sci. math., astr. et phys. VI (1958), 407–412.