Hamiltonian renormalisation VI: 
Parametrised field theory on the cylinder

T. Thiemann\textsuperscript{1,*}, E.-A. Zwicknagel\textsuperscript{1†}

\textsuperscript{1} Inst. for Quantum Gravity, FAU Erlangen – Nürnberg, 
Staudtstr. 7, 91058 Erlangen, Germany

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Abstract

Hamiltonian Renormalisation, as defined within this series of works, was derived from covariant Wilson renormalisation via Osterwalder-Schrader reconstruction. As such it directly applies to QFT with a true (physical) Hamiltonian bounded from below. The validity of the scheme was positively tested for free QFT in any dimension with or without Abelian gauge symmetries of Yang-Mills type.

The aim of this Hamiltonian renormalisation scheme is to remove quantisation ambiguities of Hamiltonians in interacting QFT that remain even after UV and IR regulators are removed as it happens in highly non-linear QFT such as quantum gravity. Also, while not derived for that case, the renormalisation flow formulae can without change also be applied to QFT without a single true Hamiltonian but rather an infinite number of Hamiltonian constraints. In that case a number of interesting questions arise: 1. Does the flow reach the correct fixed point also for an infinite number of “Hamiltonians” simultaneously? 2. As the constraints are labelled by test functions, which in presence of a regulator are typically regularised (discretised and of compact support), how do those test functions react to the flow? 3. Does the quantum constraint algebra, which in presence of a regulator is expected to be anomalous, close at the fixed point?

These questions should ultimately be addressed in quantum gravity. Before one considers this interacting, constrained QFT it is well motivated to consider a free, constrained QFT where the fixed point is explicitly known. In this paper we therefore address the case of parametrised field theory for which the quantum constraint algebra coincides simultaneously with the hypersurface deformation algebra of quantum gravity (or any other generally covariant theory) and the Virasoro algebra of free, closed, bosonic string theory or other CFT to which the results of this paper apply verbatim.

The central result of our investigation is that finite resolution (discretised) constraint algebras must not close and that anomaly freeness of the continuum algebra is encoded in the convergence behaviour of the renormalisation flow.

1 Introduction

Interacting QFT typically have to be constructed: One first defines a regulated theory (with both UV and IR regulator present) and then tries to remove the regulator thereby renormalising the bare parameters (i.e. redefining them in terms of measured parameters and regulators). That procedure of constructive QFT, even if successful in the sense that the unregulated, non-perturbative theory is well defined, may yet be ambiguous, i.e. it may keep a memory of which regularisation procedure was applied. We will refer to such ambiguities as quantisation ambiguities. One expects this problem the more likely to occur the more non-linear the theory is. An extreme case is quantum gravity whose Einstein-Hilbert action depends non-polyomially on the metric field.

Such ambiguities are not severe if they can be encoded by a finite number of (so called relevant) parameters. They could be fixed by a finite number of experiments and thus lead to a predictive theory. However, if that parameter space is infinite dimensional, the theory is not predictive. To make it predictive, the number of free

\textsuperscript{*}thomas.thiemann@gravity.fau.de
\textsuperscript{†}ernst-albrecht.zwicknagel@gravity.fau.de
parameters must be down sized to a finite dimensional manifold. To achieve this, one imposes a restriction on the family of regulated theories: they must qualify, at finite regulator, as the coarse grained versions of a continuum theory at a resolution defined by that regulator. For instance a Euclidian QFT maybe defined by a family of measures \( \mu_r \), where \( r \) denotes the regulator. The measure \( \mu_r \) knows how to integrate functionals of the Euclidian quantum field smeared with test functions that are restricted up to resolution \( r \). Thus, in order to produce unambiguous results, for any finer resolution \( r' < r \) we must have \( [\mu_r]_{r'} = \mu_r \), i.e. since the quantum field tested at resolution \( r \) can be written in terms of the quantum field at resolution \( r' \) we can use \( \mu_r' \) instead of \( \mu_r \) to integrate functions restricted to resolution \( r \).

This is so called cylindrical consistency is basically the idea of Wilson renormalisation \([1]\). A cylindrically consistent family of measures \( \mu_r \), in turn defines a continuum measure \( \mu \) that can integrate the quantum field at any resolution under rather mild assumptions \([2]\). From a practical viewpoint, the cylindrical family is then sufficient because in reality one never considers physical processes at infinite resolution, thus the explicit construction of \( \mu \) is not needed. Now in constructive QFT one typically starts with an initial family \( \{\mu_r\} \) where \( R \) is the regulator manifold. It typically comes with an in principle infinite number of parameters \( p \in P \) that enter via the discretisation freedom of the classical theory (action) that one starts from (e.g. next neighbour, next to next neighbour ..., terms in the Laplacian). Even if the limit \( \mu_r^{(0)} := \lim_{r \to 0} \mu_r \) exists as a measure, it will typically retain a non-trivial dependence on all “directions” of the parameter manifold \( P \). Therefore it is natural to improve the initial family and define a sequence of families \( n \in \mathbb{N}_0 \mapsto \{\mu_r^{(n)}\} \) by \( \mu_r^{(n+1)} := [\mu_r^{(n)}]_{\kappa(r)} \) where \( \kappa(r) < r \) maps to a finer resolution. This defines a flow of measure families which may have a fixed point \( \{\mu_r^*\} \) which by construction is consistent at least with respect to the coarse grainings \( \kappa(r) \to r \). Experience shows that this usually also makes the fixed point family consistent with respect to all pairs \( r' < r \). In the course of this process, it may happen that all but a finite (relevant) directions in \( P \) have been fixed to a fixed value. In that case we say that the QFT has been non-perturbatively renormalised to a predictive QFT.

These ideas were first formulated in quantum statistical field theory (i.e. Euclidian field theory \([3]\)) using path integral methods. Using Osterwalder-Schrader (OS) reconstruction one can also translate them into the Hamiltonian language \([4]\) (see also \([5]\) for closely related earlier Hamiltonian renormalisation schemes and references therein). The validity of \([4]\) has been tested in free field theories without \([6]\) and with \([7]\) Abelian gauge symmetry of Yang-Mills type. The motivation for \([4]\) is actually its application in Hamiltonian quantum gravity, specifically in its Loop Quantum Gravity (LQG) incarnation \([8]\). Since the classical Einstein-Hilbert action is non-polynomial in the metric field, the quantisation ambiguity problem is expected to be especially severe in this case. Indeed, quantum gravity is not perturbatively renormalisable which motivates the non-perturbative path integral renormalisation programme known as asymptotic safety \([9]\). In the Hamiltonian setting, while it is possible to rigorously define the Hamiltonian constraint operators \([10]\) they suffer from quantisation ambiguities so that a Hamiltonian renormalisation thereof is well motivated \([11,12]\).

At first it may look strange why an OS motivated Hamiltonian renormalisation scheme should apply at all to quantum gravity: OS reconstruction delivers a Hamiltonian operator \( H \) bounded from below on a Hilbert space \( \mathcal{H} \) and a ground state \( \Omega \in \mathcal{H} \). However, canonical quantum gravity does not come with a Hamiltonian but rather an infinite number of Hamiltonian constraints \( C(N) \) on a Hilbert space \( \mathcal{H}' \) where \( N \) are test functions (called lapse functions). For no choice of \( N \) are these bounded from below and rather than the spectrum of \( H \) on \( \mathcal{H} \) one is interested in the joint kernel of the \( C(N) \) on \( \mathcal{H}' \) defining the physical Hilbert space \( \mathcal{H} \) which does not coincide with \( \mathcal{H} \) and is typically not a subspace thereof (typically it is a space of distributions on a dense subspace of \( \mathcal{H}' \)). However, on the one hand it is possible to cast quantum gravity into the framework of an ordinary quantum Hamiltonian system by using Hamiltonian constraint gauge fixings \([14]\). In this reduced phase space approach one then retains a physical Hamiltonian directly on the physical Hilbert space \( \mathcal{H} \).

On the other hand, it turns out that the Hamiltonian renormalisation flow, while derived from the OS renormalisation scheme, can formally be applied also to more than one operator and in particular also those which are not bounded from below, certainly but not necessarily when they share a common ground state \( \Omega \). This observation allows the attractive perspective to monitor the fate of the commutator algebra of the \( C(N) \) during the renormalisation process which is not possible in the reduced phase approach where the \( C(N) \) are solved classically. Classically we have the closed hypersurface deformation algebra \([15]\) \( \{C(M), C(N)\} = C(f(M,N)) \) where \( f(M,N) \) are new test functions which in more than two spacetime dimensions or with density weight different from two also depend on the metric. This fact makes it especially difficult to turn
this into an anomaly free constraint operator commutator \([C(M), C(N)] = i\hbar \, \text{commutator of } C(M), C(N)\) because of the ordering problem involved in \(C(f(M, N))\) \([16]\). Indeed, the development of \([10]\) can be interpreted as saying that \([C(M), C(N)] = i\hbar C(\tilde{f}(M, N))\) closes with the correct ordering (i.e. the kernel of the \(C(f(M, N))\) is contained in that of the \(C(N)\)) but with the wrong “structure functions”, that is, the operators \(\tilde{f}(M, N)\) do not qualify as the quantisation of \(f(M, N)\). To improve on this state of affairs, one may modify the quantisation of the \(C(N)\) without resorting to renormalisation methods, an ambitious very interesting programme that is now in motion \([17]\) and to which the developments of the current paper maybe viewed as complimentary, see especially the parametrised field theory application of that programme \([18]\) (and also \([19]\) where qualitatively similar results were obtained without changing the notion of convergence of regulated operators as defined in \([10]\)).

More in detail, the Hamiltonian renormalisation flow works with a family of triples \((\mathcal{H}_r, \mathcal{H}_r, \Omega_r)\) where \(\mathcal{H}_r\) is a Hilbert space, \(H_r\) a self-adjoint operator on \(\mathcal{H}_r\) (bounded from below if coming from an OS measure) and \(\Omega_r\) is a ground state of \(H_r\), i.e. \(H_r \, \Omega_r = 0\). The regulator labels \(r\) belong to partially ordered and directed set \(R\). Given isometric embeddings \(J_{rr'} : \mathcal{H}_r \to \mathcal{H}_{r'}; r < r'\) to be constructed subject to the consistency condition \(J_{rr''} \circ J_{rr'} = J_{rr''}; r < r' < r''\) and that ensure \(J_{rr'} \, \Omega_r = \Omega_{r'}\) one defines the inductive limit Hilbert space \(\mathcal{H}\) by a standard construction \([22]\). Moreover, at the fixed point, the \(\mathcal{H}_r\) form a consistently defined family of quadratic forms \(\mathcal{H}_r = J^r_{rr} \, \mathcal{H}_{rr} \, J_{rr'}, r < r'\) defining a continuum form \(\mathcal{H}\). That form may or may not define a self-adjoint operator on \(\mathcal{H}\) and in particular is in general not to be confused with the inductive limit of the \(H_r\), which is not granted to exist.

In extending this framework to more than one (in field theory, even an infinite number of) operators, we face several new questions:

1. We start with an initial family \(C_r(0)(N)\) of operators on an initial family of Hilbert spaces \(\mathcal{H}_r(0)\), one for each resolution scale \(r\) and one for each continuum smearing function \(N\). The origin of \(r\) typically comes from a discretisation of the continuum field \(\phi\) and conjugate momentum \(\pi\) in terms of coarse grained variables \(\phi_r, \pi_r\) and substituting them for \(\phi, \pi\) in the expression for \(C(N)\). Does this automatically induce a discretisation \(N_r(0)\) of \(N\) as well? If not, should one supply one by hand or leave \(N\) in its continuum form?

2. Is it possible or necessary to find a common zero eigenvector \(\Omega_r(0) \in \mathcal{H}_r(0)\) of the \(C_r(0)(N)\) or \(C_r(0)(N_r(0))\) independent of \(N\) or \(N_r(0)?\) This is far from trivial: while the classical continuum constraints form a closed Poisson algebra of real functions, there is no reason to take it for granted that the algebra of the discretised \(C_r(0)(N)\) or \(C_r(0)(N_r(0))\) closes under taking commutators. In fact this is most likely not the case because typically the classical constraint algebra rests on the validity of the Leibniz rule for partial derivatives. However, discretised derivatives do not obey the Leibniz rule \([23]\). Thus not only can these constraints not be simultaneously diagonalised, it may even be that their joint kernel just consists of the zero vector. In that case, we have to assume that there exists at least a cyclic vector \(\Omega_r(0)\) for the algebra of operators under consideration in the common dense domain of all constraints.

3. Given that \(\Omega_r(0)\) can be found, one can proceed as in the case of just one Hamiltonian operator and construct a sequence of families of Hilbert spaces \(\mathcal{H}_r(n)\) and isometric injections \(J_{rr'}(n) : \mathcal{H}_r(n+1) \to \mathcal{H}_{r'}(n)\) for \(r' < r\) such that \(J_{rr'}(n) \, \Omega_r(n+1) = \Omega_{r'}(n)\). The isometry requirement translates into flow equations for the Hilbert space measures \(\nu_r(n)\) underlying \(\mathcal{H}_r(n) = L_2(Q_r, d\nu_r(n))\) where \(Q_r\) is a flow invariant model configuration space. Assuming that a fixed point \(J_{rr''}\) of this flow of isometric injections can be found (equivalently, a cylindrically consistent measure family \(\nu_r\) one can construct a continuum Hilbert space \(\mathcal{H}\) as the inductive limit of the \(\mathcal{H}_r = L_2(Q_r, d\nu_r)\). In tandem, one constructs a flow of families quadratic forms \(C_r(n+1)(N) := \{J_{rr'}(n)\}^\dagger C_r(n)(N) \, J_{rr'}(n)\) one for each \(N\). Can one arrange that all of them flow into a fixed point whatever choice of \(N\) is made? Or should one rather also let the discretised smearing functions flow according to \(C_r(n+1)(N(n+1)) := \{J_{rr'}(n)\}^\dagger C_r(n)(N(n)) \, J_{rr'}(n)\)?

4. Suppose that a simultaneous fixed point family \(C_r(N)\) or \(C_r(N_r)\) can be found. Then by construction \(C_r(N) = J^r_0 C(N) J_r\) or \(C_r(N_r) = J^r_0 C(N) J_r\) where \(J_r : \mathcal{H}_r \to \mathcal{H}\) is the isometric embedding granted to exist by the inductive limit construction. Is it true that \(C(N)\) is no longer plagued by an infinite number of quantisation
ambiguities? Is it true that the algebra of commutators of \( C(N) \) is non-anomalous? Note that it is not clear that the commutators can even be computed because \( C(N) \) is just a quadratic form.

5. Assuming that these questions can be answered in the affirmative, how does one recognise anomaly freeness at finite resolution? Note that the \( C_r(N) \) will most certainly not close under forming commutators even if the \( C(N) \) do because

\[
[C_r(M), C_r(N)] = J_r^*[C(M)P_r C(N) - C(N)P_r C(M)]J_r
\]

where \( P_r := J_r J_r^\dagger \) is a projection in \( \mathcal{H} \). It is therefore generically not expected that the finite resolution projections of the constraints form a closed algebra. However, given closure in the continuum, we may rewrite (1.1) as

\[
[C_r(M), C_r(N)] = i\hbar C_r(f(M, N)) - J_r^*[C(M) (1_{\mathcal{H}} - P_r) C(N) - C(N) (1_{\mathcal{H}} - P_r) C(M)]J_r
\]

and the anomalous term naively vanishes as \( r \) is removed and \( P_r \) becomes \( 1_{\mathcal{H}} \). This, when supplied by a suitable operator topology of convergence, may serve as a pratical guide towards proving anomaly freeness even if one cannot determine the continuum operator \( C(N) \) in closed form.

It would be very interesting to find necessary and sufficient conditions under which the above questions can be answered in the affirmative. In this paper we confine ourselves to the much easier task to illustrate and work out the catalog of questions and answers for the case of parametrised massless Klein-Gordon field theory in 1+1 spacetime dimensions.

The architecture of this paper is as follows:

In section 2 we briefly review 1+1 dimensional PFT following the notation of [19]. We treat both the classical and quantum theory.

In section 3 we specialise the general framework of [4, 11] to PFT. We choose as regulator space a nested system of square lattices. Here we learn the first important lesson from the present work: The constraint operators are ill defined on the dense domain of finite resolution subspace generated by the discretised Weyl algebra unless the test functions that enter that Weyl algebra and which define the renormalisation flow display at least a minimal amount of smoothness. This issue did not arise in the works [6] because there the renormalisation could be phrased in terms of the covariance of the Gaussian measure which decays sufficiently fast at infinity in momentum space even when smeared against the discontinuous test functions used. However in PFT we also need inverse powers of that covariance. This obervation triggered the work [13] where we generalise [11] in a natural way to a generalised Multi-Resolution Analysis (MRA) based renormalisation flows of which there are even smooth candidates thus removing the afore mentioned obstacle. In fact, [6] turns out to be a special case of [13] as [6] is based on the so-called Haar MRA. On the other hand, as the convergence to the continuum via sequences of discontinuous or smooth functions should not affect the continuum fixed point theory, we also offer an equivalent solution to the just mentioned smoothness problem within the Haar MRA class based on zeta function regularisation which is a common tool in conformal field theories (CFT) such as PFT.

In section 4 we show that there exists a well motivated discretisation of the PFT constraints. Clearly, due to the central term in the Virasoro algebra there does not exist a single vector in the joint point kernel of all constraints, not even in the continuum. However, there does exist a preferred cyclic vector in the common dense domain of all constraints which serves as a substitute, both in the continuum and at finite resolution. We can then proceed similar as in [6, 7] and compute the flow and fixed point of the corresponding Hilbert space measures.

In section 5 we compute the flow of the constraint operators. We show that the first option of leaving the smearing functions \( N \) untouched (not discretised by hand) does not induce a canonical discretisation of the smearing functions of the constraints. On the other hand, using the coarse graining map that is used to compute the flow of measures, vacua and constraints to discretise their smearing functions by hand does lead to a cylindrically consistent system (under change of resolution) of constraints.

In sections 6 and 7 we compute the algebra of constraints at finite resolution and illustrate the behaviour (1.1), (1.2). It is at this point that we learn the second most important lesson from the present work when trying
to show that the discrete algebra converges to the continuum algebra in the weak operator topology:

i. When working with non-discretised constraint smearing functions, there is just one correction to the continuum algebra at finite resolution indicated in (1.1) and (1.2). However, when additionally discretising the constraint smearing function by hand, an additional correction arises.

ii. Convergence to zero of the first correction requires a minimal amount of smoothness of the test functions of the Weyl algebra for reasons similar as mentioned before concerning the domain of definition of the constraints.

iii. Convergence to zero of the second correction requires sufficient smoothness of the discretised smearing function $N$ of the constraints, which is of course not surprising because the Virasoro algebra depends on third order (Schwartzian) derivatives of those smearing functions.

We establish convergence using for instance the Dirichlet flow of [13] rather than the Haar flow of [6].

In section 8 we summarise and conclude our findings for this model which presents the next logical step in the research programme started in [4, 6, 7, 11].

The most important lessons learnt from the present work are as follows:

A. Finite resolution constraints typically do not close.

B. This is no problem at all, in fact it would be physically wrong: It just displays the mathematical fact that the constraints typically are not block diagonal w.r.t. different resolution Hilbert subspaces. The failure to close is no anomaly but a finite resolution artefact.

C. Whether the continuum algebra closes, i.e. is free of anomalies can be checked using finite resolution analysis: The finite resolution artefact should converge to zero. This is of practical importance because in more complicated theories one will hopefully be able to construct the theory at finite resolution but perhaps computing the infinite resolution (continuum) theory may be too hard but also unnecessary as measurements always have finite resolution.

2 Brief review of PFT

This section mainly serves to introduce our notation and follows [19]. See [19] for more information and references therein. See also [20] for more details on the quantisation of PFT using classically equivalent constraints for which the quantum anomaly is formally a co-boundary so that it can be (formally - i.e. modulo showing existence of corresponding Hilbert space representations) absorbed into a non-central quantum correction of the constraints. See [21] for renormalisation of closely related (fermionic) CFT's.

2.1 Classical Theory

The spacetime is the infinite cylinder $Z_R = \mathbb{R} \times C_R$ where $C_R$ is the circle of radius $R$ with Minkowski metric $\eta=\text{diag}(-1,1)$ and Cartesian coordinates $T := X^0 \in \mathbb{R}$, $X := X^1 \in [0, 2\pi R)$. We introduce another cylinder $Z$ of unit radius $Z = \mathbb{R} \times S^1$ with coordinates $(x^0 = t, x^1 = x)$ and consider the diffeomorphism $\varphi : Z \to Z_R; (t, x) \mapsto (T(t, x), X(t, x))$ upon which $T, X$ become fields on $Z$. Note that $T$ is periodic $T(t, x+1) = T(t, x)$ while $X$ is an angular variable $X(t, x + 1) = X(t, x) + 2\pi R$.

The action of the massless Klein-Gordon field $\phi$ on $Z_R$

$$S = -\frac{1}{2} \int_{Z_R} d^2X \eta^{AB} \Phi_{\alpha A} \Phi_{\beta B} \tag{2.1}$$

is pulled back by above diffeomorphism and yields via $\phi = \varphi^* \Phi$ the PFT action

$$S = -\frac{1}{2} \int_Z d^2x \sqrt{\det(g)}^{1/2} g^{\alpha \beta} \Phi_{\alpha \beta}; \quad g = \phi^* \eta \tag{2.2}$$

which by construction is invariant under reparametrisations (diffeomorphisms) of $Z$. It is thus an example of a generally covariant field theory and thus its canonical formulation in terms of Hamiltonian $C$ and spatial diffeomorphism constraints $D$ must yield a representation of the abstract hypersurface deformation algebra of the one parameter family of hypersurfaces $t \mapsto \Sigma_t = \varphi(t, [0, 1])$ discovered in [15]. Using standard methods one finds

$$H = P X' + Y T' + \frac{1}{2}[\Pi^2 + (\Phi')^2], \quad D = P T' + Y X' + \Pi \Phi' \tag{2.3}$$
where \( (\cdot) = \partial_t (\cdot), (\cdot)' = \partial_x (\cdot) \) and \( (P,Y,\Pi) \) are the momenta conjugate to \( (T,X,\Phi) \) respectively, i.e. the non-trivial equal \( t \) Poisson brackets are
\[
\{P(u), T(v)\} = \{Y(u), X(v)\} = \{\Pi(u), \Phi(v)\} = \delta(u,v)
\] (2.4)
with the \( \delta \) distribution on \( S^1 \)
\[
\delta(u,v) = \sum_{n \in \mathbb{Z}} e^{i 2\pi (u-v) n} 
\] (2.5)

One quickly verifies the hypersurface deformation algebra \( \mathfrak{H} \) relations
\[
\{D(f), D(g)\} = D([f,g]), \{D(f), H(g)\} = H([f,g]), \{H(f), H(g)\} = D([f,g]); \{f, g\} := f' - f g' 
\] (2.6)
where \( f, g \) are periodic, real valued smearing functions on \( S^1 \) and e.g. \( D(f) = \int_{S^1} dx f \, D \). Geometrically, \( C, D \) are scalar densities of weight two, \( f,g \) are scalar densities of weight minus one which is why \( [f,g] \) is independent of the spatial metric \( g = g_{xx} \), an effect that can happen only in one spatial dimension.

We note that the constraints depend only on the derivatives of \( X, T, \Phi \) and thus do not contain information about their respective zero modes. We denote them by \( \Phi_0, X_0, T_0 \). Also, since \( X \) is not periodic in contrast to \( Y, P, \Pi, T, \Phi, X' \) has a phase space independent zero mode given by \( 2\pi R \). We thus write
\[
X(x) = 2\pi R x + \tilde{X}(x)
\] (2.7)
where \( \tilde{X} \) has the same zero mode as \( X \) and is still conjugate to \( P \). We can thus write the constraints as
\[
D = 2\pi R Y + \tilde{D}, \quad H = 2\pi R P + \tilde{H}
\] (2.8)
where \( \tilde{D}, \tilde{H} \) differ from \( D, H \) upon replacing \( X \) by \( \tilde{X} \). The zero modes of \( Y, P, \Pi \) can be extracted as
\[
Y_0 = Q_\perp \cdot Y := \int_0^1 dx Y(x), \quad Q := 1_L - Q_\perp
\] (2.9)
and similar for \( P_0, \Pi_0 \). Note \( Q \) is an orthogonal projection on \( L := L_2([0,1), dx) \) extracting the non-zero modes of a function.

It is convenient to introduce the field combinations
\[
X_\pm := \tilde{X} \pm T, \quad P_\pm := \frac{1}{2} (Y \pm P), \quad A_\pm := P_+ \pm X'_+, \quad B_\pm := P_- \pm X'_-, \quad C_\pm := \Pi \pm \Phi'
\] (2.10)
in terms of which we can write the constraints as
\[
D_\pm := \frac{1}{2} (\tilde{D} \pm \tilde{H}), \quad \tilde{D}_+ = \frac{1}{4} [(A_+)^2 - (A_-)^2 + (C_+)^2], \quad \tilde{D}_- = \frac{1}{4} [(B_+)^2 - (B_-)^2 - (C_-)^2]
\] (2.11)
One checks
\[
\{A_\pm(u), A_\pm(v)\} = \pm 2 \partial_v \delta(u,v) \{A_\pm(u), A_\pm(v)\} = 0
\] (2.12)
and similar for \( B, C \), all other brackets vanishing, so that
\[
\{D_\pm(f), D_\pm(g)\} = D_\pm([f,g]), \quad \{D_\pm(f), D_\mp(g)\} = 0
\] (2.13)
The original variables can be recovered from \( A_\pm, B_\pm, C_\pm \) except for the zero modes of the configuration variables
\[
\Pi = \frac{1}{2} (C_+ + C_-), \quad Y = P_+ + P_- = \frac{1}{2} [A_+ + A_- + B_+ + B_-], \quad P = P_+ - P_- = \frac{1}{2} [A_+ + A_- - B_+ - B_-]
\]
\[
\Phi' = \frac{1}{2} (C_+ - C_-), \quad \tilde{X}' = X'_+ + X'_- = \frac{1}{2} [A_+ - A_- + B_+ - B_-], \quad T' = X'_+ - X'_- = \frac{1}{2} [A_+ - A_- - B_+ + B_-]
\] (2.14)
so that the zero modes of \( Y, P, \Pi \) but not those of \( \tilde{X}, T, \Phi \) are available from \( A_\pm, B_\pm, C_\pm \). For the original constraints we find
\[
\tilde{D}_\pm = \frac{1}{2} [D \pm H] = D_\pm + 2\pi R P_\pm
\] (2.15)
with \( P_+ = \frac{1}{2} [A_+ + A_-], \quad P_- = \frac{1}{2} [B_+ + B_-] \). Therefore also
\[
\{\tilde{D}_\pm(f), \tilde{D}_\pm(g)\} = \tilde{D}_\pm([f,g]), \quad \{\tilde{D}_\pm(f), \tilde{D}_\mp(g)\} = 0
\] (2.16)
In what follows we will only consider the algebra of the \( D_\pm(f) \). The algebra of the \( \tilde{D}_\pm \) can be treated by identical methods.
2.2 Quantum Theory

The classical system consists of three independent scalar fields \( X, T, \Phi \) which are coupled via the constraints which are only quadratic in the fields and their momenta. We thus use a Fock representation. In most approaches to PFT and also the closed bosonic string \[25\] one constructs a Fock space using the mode functions \( e_n(x) := \exp(i \cdot 2\pi n \cdot x) \) which form an orthonormal basis of the “one particle Hilbert space” \( L = L_2([0,1), dx) \) and defines \( A_\pm(n) := A_\pm(e_n) = \int dx \, e_n(x)^* A_\pm(x) \) etc. from which one finds \( A_\pm(n)^* = A_\pm(-n) \)

\[
\{A_\pm(n_1), A_\pm(n_2)\} = \pm 2 \delta_{n_1+n_2,0}
\]  

(2.17)

or in terms of commutators

\[
[A_\pm(n_1), A_\pm(n_2)] = \pm 2 n_1 \delta_{n_1+n_2,0}
\]  

(2.18)

This allows to interpret \( A_\pm(n) \) as an annihilation operator and \( A_\pm(n)^* \) as a creation operator for \( n > 0 \), \( A_\pm(n) \) as an annihilation operator and \( A_\pm(n)^* \) as a creation operator for \( n < 0 \), while \( A_\pm(0) = A_\pm(0) = (P_\pm)_0 \) (zero mode). Similar remarks hold for \( B_\pm, C_\pm \) where \( B_\pm(0) = B_\pm(0) = (P_\pm)_0 \) and \( C_\pm(0) = C_\pm(0) = (\Pi)_0 \). This split with respect to the sign of \( n \) makes the discussion somewhat cumbersome as it requires to introduce six different Fock spaces and a separate discussion of the zero mode sector.

Let us therefore introduce the quantities

\[
A := \frac{1}{\sqrt{2}} [\omega^{1/2} \left( Q X_+ - i\omega^{-1/2} Q P_+ \right)], \quad B := \frac{1}{\sqrt{2}} [\omega^{1/2} \left( Q X_- - i\omega^{-1/2} Q P_- \right)], \quad C := \frac{1}{\sqrt{2}} [\omega^{1/2} \left( \Phi - i\omega^{-1/2} Q \Pi \right)]
\]  

(2.19)

where

\[
\omega^2(.) = -(.)'' = -\Delta
\]  

(2.20)

is minus the Laplacian on \( S^1 \). The quantities (2.19) are the standard annihilation operators of three massless Klein-Gordon fields where we have been careful to remove the zero mode on which the Laplacian is not invertible (if there would be a mass term, we would have \( \omega^2 = m^2 - \Delta \) and in this case a separate discussion of the zero mode is not necessary).

For the zero modes we set

\[
A_0 := \frac{1}{\sqrt{2}} [\omega_0^{1/2} \left( Q X_+ - i\omega_0^{-1/2} Q P_+ \right)], \quad B_0 := \frac{1}{\sqrt{2}} [\omega_0^{1/2} \left( Q X_- - i\omega_0^{-1/2} Q P_- \right)],
\]

\[
C_0 := \frac{1}{\sqrt{2}} [\omega_0^{1/2} \left( \Phi - i\omega_0^{-1/2} Q \Pi \right)]
\]  

(2.21)

where \( \omega_0 > 0 \) is an arbitrary parameter of dimension of inverse length. It is therefore natural to set it equal to \( 1/R \) but we will keep it unfixed for the moment.

For any operator valued distribution \( O \) and and any smearing function \( f \) we set

\[
< f, O >= \int dx \, f^*(x) \, O(x) =: O(f^*)
\]  

(2.22)

Then, by promoting the Poisson brackets to commutators

\[
< f, A_0 >, < g, A_0 >^* = < f, Q_+ g >, \quad [ < f, A >, < g, A >^* ] = < f, Qg >
\]  

(2.23)

and similar for the \( B, C \) sectors, all other commutators vanishing. Here * is the respective complex conjugate of \( f \), \( g \) extended to an involution on linear combinations of products.

The relation among these annihilators is as follows

\[
A_\pm = P_+ X_\mp = Q_\pm P_+ + Q(P_+ X_\pm')
\]

\[
= i \sqrt{\frac{\omega_0}{2}} [A_0 - A_0^*] + i\sqrt{\frac{\omega}{2}} [A - A] + \sqrt{\frac{1}{2\omega}} [A + A^*]
\]

\[
= i \sqrt{\frac{\omega_0}{2}} [A_0 - A_0^*] + i\sqrt{2\omega} \{[Q_\pm A] - [Q_\pm A]^*\}
\]  

(2.24)
where
\[ Q_\pm = \frac{1}{2} \left[ 1_L \mp i \frac{\partial}{\omega} \right] Q \]  
\hspace{1cm} (2.25)

projects onto the positive/negative Fourier modes: \( Q_\pm e_n = e_n \) if \( n > 0 < 0 \) and zero otherwise. Note that \( Q_\pm \) is an orthogonal (i.e. self-adjoint) projection on the 1-particle Hilbert space \( L \) which commutes with \( Q, \partial, \omega \) which can be seen by using the common eigenbasis \( e_n \). As \( [Q_\pm A]^\dagger = Q_\mp A^\dagger \) it follows
\[ i\sqrt{2\omega} A = Q_+ A_+ + Q_- A_- \]  
\hspace{1cm} (2.26)

which demonstrates that the Fock space defined by declaring \( A \) as annihilation operators is the same as the tensor product of Fock spaces defined by declaring \( Q_+, A_+, Q_-, A_- \) as annihilators which is exactly relation (2.18). Similar statements hold for the \( B, C \) sectors. It is thus equivalent but more economic to work with \( A_\pm \) and we consider the Fock space \( \mathcal{H} \) with Fock vacuum \( \Omega \) annihilated by \( A_0, A \).

We compute the commutators corresponding to (2.13). We introduc the building blocks
\[ E_0 := \sqrt{\omega_0}|A_0 - A_0^\dagger|, \quad E_\pm := \sqrt{\omega} Q_\pm A \]  
\hspace{1cm} (2.27)

so that
\[ A_\pm = i\left( \frac{1}{\sqrt{2}} E_0 + \sqrt{2}[E_\pm - E_\pm^*] \right) \]  
\hspace{1cm} (2.28)

Since we need \( A_{\pm}^2 \), there is an ordering ambiguity w.r.t. the term \((E_\pm - E_\pm^*)^2\). We pick normal ordering wrt the annihilators \( A \) and leave a possible normal ordering constant proportional to the algebraic unit 1 open for the moment, that is we set
\[ A_{\pm}^2(f) = -\frac{1}{2} E_0^2(Q_\pm f) + 2 E_0(1) (E_\pm(f) - E_\pm(f)^*)(f) + 2 : (E_\pm - E_\pm^*)^2 : (f) \]  
\hspace{1cm} (2.29)

where \( : (\cdot) : \) denotes normal ordering. We have used in (2.29) that \( f \) is real valued. As \([E_0, E_\pm] = 0\) we find with \( s, s' = \pm \)
\[ [A_s^2(f), A_{s'}^2(g)] = [T_{s,1}^1(f), T_{s',1}^1(g)] + [T_s^1(f), T_{s'}^1(g)] - [T_{s,1}^1(g), T_{s'}^1(f)] + [T_s^2(f), T_{s'}^2(g)] \]  
\hspace{1cm} (2.30)

We have with
\[ E_s(f) = < \omega^{1/2} Q_s f, A >, \]  
\hspace{1cm} (2.31)

that
\[ [T_s^1(f), T_{s'}^1(g)] = 4 E_0(1)^2 \left[ E_s(f) - E_s(f)^*, E_{s'}(g) - E_{s'}(g)^* \right] \]
\[ = -4 E_0(1)^2 \left\{ \left[ E_s(f), E_{s'}(g)^* \right] - \left[ E_s(g), E_{s'}(f)^* \right] \right\} \]
\[ = -4 E_0(1)^2 \left\{ < \omega^{1/2} Q_s f, \omega^{1/2} Q_{s'} g > - < \omega^{1/2} Q_{s'} g, \omega^{1/2} Q_s f > \right\} \]
\[ = -4 \delta_{s,s'} E_0(1)^2 \left\{ < f, g > - < g, f > \right\} \]
\[ = 4 i s \delta_{s,s'} T_0^0([f,g]) \]  
\hspace{1cm} (2.32)

where we used that \( 2 \omega Q_s = 1 - is\partial \). Next
\[ [T_s^1(f), T_{s'}^1(g)] = 4 E_0(1)^2 \left[ E_s(f) - E_s(f)^*, [E_{s'}]^2(g) + [E_{s'}^*]^2(g) - 2[E_{s'}^* E_{s'}](g) \right] \]
\[ = 4 E_0(1)^2 \left\{ \left[ E_s(f), [E_{s'}]^2(g) - 2[E_{s'}^* E_{s'}](g) \right] - [E_s(f)^*, [E_{s'}]^2(g) - 2[E_{s'}^* E_{s'}](g) \right\} \]
\[ = 8 E_0(1) \int dx dy f(x) g(y) \left[ K_{s,s'}(x,y) \{ E_{s'}^*(y) - E_{s'}(y) \} - K_{s',s}(y,x) \{ E_{s}^*(y) - E_{s}(y) \} \right] \]  
\hspace{1cm} (2.33)

with the kernel
\[ K_{s,s'}(x,y) = [E_s(x), E_{s'}^*(y)] \Rightarrow K_{s,s'}(x,y)^* 1_\mathcal{H} = [K_{s,s'}(x,y) 1_\mathcal{H}]^* = K_{s',s}(y,x) \]  
\hspace{1cm} (2.34)
Explicitly
\[ < f, K_{ss'} \cdot g > = < f, E_s, g, E_{s'} >^* = \delta_{ss'} < f, \omega Q_s g > =: \delta_{ss'} < f, K_s \cdot g > \] (2.35)

Abbreviating \( E_{s'}^g(y) = g(y) E_{s'}^g(y) \) we obtain
\[
[T^1_s(f), T^2_s(g)] = -8 E_0(1) \{ < f, K_{ss'} \cdot [E_{s'}^g - [E_{s'}^g]^*] > - < f, K_{ss'} \cdot [E_{s'}^g - [E_{s'}^g]^*] > \}
\]
\[ = -8 E_0(1) \{ K_{ss'} - K_{ss'}^* \} \cdot f, \{ E_{s'}^g - [E_{s'}^g]^* \} > \]
\[ = -8 E_0(1) \delta_{ss'} \{ < f, \omega (Q_s - Q_{-s}) \{ E_{s'}^g - [E_{s'}^g]^* \} > \}
\]
\[ = 8 E_0(1) \delta_{ss'} (i s) \{ < f, \{ E_{s'}^g - [E_{s'}^g]^* \} > \} \]
whence
\[
[T^1_s(f), T^2_s(g)] - [T^1_s(g), T^2_s(f)] = 8 E_0(1) \delta_{ss'} (i s) \{ < f, \{ E_{s'}^g - [E_{s'}^g]^* \} > - < g, \{ E_{s'}^g - [E_{s'}^g]^* \} > \}
\]
\[ = -8 E_0(1) \delta_{ss'} (i s) \{ E_s - (E_{s})^*(f g' - f g) \}
\]
\[ = 4 i s \delta_{ss'} T^1_s([f, g]) \] (2.37)

Finally
\[
[T^2_s(f), T^2_s(g)] = 4 \int dx \int dy f(x) g(y) \times
\]
\[ \{ [E_s(x)^2, [E_{s'}^g(y)]^2 - 2 E_{s'}^g(y) E_{s'}^g(y)] + [E_s(x)^*]^2, [E_{s'}^g(y)]^2 - 2 E_{s'}^g(y) E_{s'}^g(y)]
\]
\[ -2 \{ E_s(x)^* E_s(x), [E_{s'}^g(y)]^2 + [E_{s'}^g(y)]^2 - 2 E_{s'}^g(y) E_{s'}^g(y) \} \]
\[ = 4 \int dx \int dy f(x) g(y) \times
\]
\[ \{ K_{ss'}(x, y) [2 E_s(x) E_{s'}(y)^* + 2 E_{s'}(y) E_s(x) - 4 E_s(x) E_{s'}(y)]
\]
\[ - K_{ss'}(x, y) [2 E_s(x)^* E_{s'}(y) 2 E_{s'}(y) E_s(x)^* - 4 E_s(x)^* E_{s'}(y)^*]
\]
\[ -2(K_{ss'}(x, y) E_s(x)^* [2 E_s(x)^* - 2 E_{s'}(y)] - K_{ss'}(x, y) [2 E_s(y) - 2 E_{s'}(y)^*] E_s(x)) \}
\[ = 4 \int dx \int dy f(x) g(y) \times
\]
\[ \{ K_{ss'}(x, y) [2 K_{ss'}(x, y) + 4 [E_{s'}(y)^* - E_{s'}(y)] E_s(x) - K_{ss'}(x, y) [2 K_{ss'}(x, y) - 4 E_s(x)^* [E_{s'}(y)^* - E_{s'}(y)]]
\]
\[ -4(K_{ss'}(x, y) E_s(x)^* [E_{s'}(y)^* - E_{s'}(y)] - K_{ss'}(x, y) [E_{s'}(y) - E_{s'}(y)^*] E_s(x)) \}
\] (2.38)

Since \( K_{ss'} = \delta_{ss'} K_s \) we can simplify (5.31) using \( F_s(x) := E_s(x) - E_s(x)^\dagger \) and \( f_x = f(x), g_y = g(y) \)
\[
[T^2_s(f), T^2_s(g)] = 8 \delta_{ss'} \int dx \int dy f(x) g(y) \times \{ K_s(x, y) [K_s(x, y) - 2 F_s(y)] E_s(x) \]
\[ - K_s(x, y) [K_s(x, y) + 2 E_s(x)^* F_s(y)]
\]
\[ + 2 K_s(x, y) E_s(x)^* F_s(y) + 2 K_s(x, y) F_s(y) E_s(x) \}
\[ = 8 \delta_{ss'} \int dx \int dy K_s(x, y) \times \{ f_x g_y [K_s(x, y) - 2 : E_s(x) F_s(y)] : \]
\[ + 2 : E_s(x)^* F_s(y) : ] - f_y g_x [K_{s}(x, y) + 2 : E_{s}(y)^* F_{s}(x) : - 2 : E_{s}(y) F_{s}(x) : ]
\]
\[ = 8 \delta_{ss'} \int dx \int dy K_s(x, y) \times
\]
\[ \{ f_x g_y - f_y g_x \} \{ K_s(x, y) - 2 : F_s(x) F_s(y) : \}
\]
\[ = 8 \delta_{ss'} \int dx \int dy K_s(x, y) \times
\]
\[ \{ f_x g_y - f_y g_x \} \{ K_s(x, y) - 2 : F_s(x) F_s(y) : \} \] (2.39)

where we used that within the normal ordering symbol operator valued distributions commute. Using \( F^g_s(x) =
\[ f(x) F_s(x) \) the second term in (2.39) can be written
\[
-16 \delta_{ss'} : [F_s^f(\omega Q_s \cdot F_s^g) - F_s^g(\omega Q_s \cdot F_s^f)] : \\
= -8 \delta_{ss'} (-i s) : [F_s^f((F_s^g)' - F_s^g((F_s^f)')] : \\
= -8 \delta_{ss'} (-i s) : [F_s^2] : (g' - f') \\
= 4 i s \delta_{ss'} T_s^2([f, g])
\] (2.40)

where we used that the operator \( \omega \) is symmetric.

The first term in (2.39) can be evaluated as follows: Let \( \mathbb{Z}_s = \{ n \in \mathbb{Z}; sn \geq 0 \} \) then
\[ (K_s \cdot f)(x) = 2\pi \sum_{n \in \mathbb{Z}_s} |n| e_n(x) < e_n, f > \Rightarrow K_s(x, y) = 2\pi \sum_{n \in \mathbb{Z}_s} |n| e_n(x) e_n(y)^* \] (2.41)

Thus
\[
\int dx \int dy f(x) g(y) K_s(x, y)^2 = (2\pi)^2 \sum_{n_1, n_2 \in \mathbb{Z}_s} |n_1, n_2| < f, e_{n_1+n_2} > < e_{n_1+n_2}, g >
\]
\[
= (2\pi)^2 \sum_{n_1, n_2 \in \mathbb{Z}_+} n_1 n_2 < f, e_{s(n_1+n_2)} > < e_{s(n_1+n_2)}, g >
\]
\[
= (2\pi)^2 \sum_{n \in \mathbb{Z}_+} < f, e_{sn} > < e_{sn}, g > [\sum_{n_1=0}^n n_1(n-n_2)]
\]
\[
= (2\pi)^2 \sum_{n \in \mathbb{Z}_+} < f, e_{sn} > < e_{sn}, g > [n^2 + n - 1] (n+1)
\]
\[
= (2\pi)^2 \frac{1}{6} \sum_{n \in \mathbb{Z}_+} < f, e_{sn} > < e_{sn}, g > [n^3 - n]
\]
\[
= (2\pi)^2 \frac{s}{6} \sum_{n \in \mathbb{Z}_+} < f, e_{sn} > < e_{sn}, g > [(sn)^3 - (sn)]
\]
\[
= (2\pi)^2 \frac{s}{6} \sum_{n \in \mathbb{Z}} < Q_s f, e_n > < e_n, g > [n^3 - n]
\]
\[
= (2\pi)^2 \frac{s}{6} \sum_{n \in \mathbb{Z}} < Q_s f, e_n > < e_n, [(\frac{-i \partial}{2\pi})^3 - \frac{i \partial}{2\pi}] g >
\]
\[
= (2\pi)^2 \frac{s}{6} \sum_{n \in \mathbb{Z}} < Q_s f, e_n > < e_n, [\frac{-i \partial}{2\pi}]^3 - \frac{i \partial}{2\pi} g >
\]
\[
= (2\pi)^2 \frac{s}{6} < Q_s f, [(\frac{-i \partial}{2\pi})^3 - \frac{i \partial}{2\pi}] g >
\]
\[
= (2\pi)^2 \frac{i s}{6} < Q_s f, \frac{1}{(2\pi)^3} g'' + \frac{1}{2\pi} f' >
\]
(2.42)

Thus the first term in (2.39) can be written
\[
8 i s \delta_{ss'} \frac{1}{6} (2\pi)^2 \langle Q_s f, [\frac{1}{(2\pi)^3} g'' - \frac{1}{2\pi} f'] \rangle - \langle Q_s g, [\frac{1}{(2\pi)^3} g'' - \frac{1}{2\pi} f'] \rangle
\]
\[
= 4 i s \delta_{ss'} \frac{1}{6} (2\pi)^2 \langle f, [\frac{1}{(2\pi)^3} g'' + \frac{1}{2\pi} f'] \rangle - \langle g, [\frac{1}{(2\pi)^3} f''' + \frac{1}{2\pi} f''] \rangle
\]
\[
=: 4 i s \delta_{ss'} c S(f, g)
\]
(2.43)

where the term proportional \( i s \partial / \omega \) in \( Q_s \) has dropped out as \( \partial^2 / \omega = -\omega, \partial^4 / \omega = \omega^3 \) are symmetric operators on \( L \). The term (2.43) displays the anomaly of the classical hypersurface deformation algebra or equivalently its central extension with central charge \( c = \frac{1}{6} \) which is called the Virasoro algebra with that central charge.
A combination of the \( T \) etc. (thereby replacing algebraic adjoint \( \ast \)) constraints (and thus their adjoints as they are manifestly symmetric) are densely defined. Since the chosen (normal) ordering. It is not necessary to assume a Fock representation, we just used the

by construction but no 2-coboundary, i.e. there is no linear functional \( F \) on the space of test functions \( f \) such that \( S(f,g) = F([f,g]) \). Thus, the \( D_s(f) \) cannot be modified by adding \( 3F(f) \cdot 1 \) to obtain a proper Lie algebra.

It should be noted that the result (2.46) is purely algebraic, it just follows from \( \ast \)-algebraic relations and the chosen (normal) ordering. It is not necessary to assume a Fock representation, we just used the \( \ast \)-algebra generated by \( A_0, A \) and their algebraic adjoints. In order that our intended Fock representation defined by \( A_0, A \) etc. (thereby replacing algebraic adjoint \( \ast \) by Hilbert space adjoint \( \dag \)) we must therefore check whether the constraints (and thus their adjoints as they are manifestly symmetric) are densely defined. Since \( D_s \) is a linear combination of the \( A_s^2, B_s^2, C_s^2 \), it will be sufficient to show that \( A_s^2(f) \) is densely defined. Since \( A_s^2(f) \) is a linear combination of the \( T_s^2(f) \), \( j = 0, 1, 2 \) (see 2.29) it will be sufficient to consider those. Consider first the action of \( A_s^2(f) \) on the Fock vacuum

\[
\begin{align*}
\|T_0^s(F)\Omega\|^2 &= \|\frac{i\omega_0}{2} (Q_{-} F) A_0^4 \Omega\|^2 = \|\frac{i\omega_0}{2} (Q_{-} F)\|^2 \\
\|T_1^s(F)\Omega\|^2 &= \|2\sqrt{\omega_0} A_0^4 E_1^s(F) \Omega\|^2 = 4\omega_0 < F, \omega Q_s F > \\
\|T_2^s(F)\Omega\|^2 &= 4 \| : (E_{-} - E_{+})^2 : (f) \Omega\|^2 = 4 < (E_{-})^2(F)\Omega\|^2 \\
&= 8 \int dx \, dy \, F(x) \, F(y) \, K_s(x,y)^2 \\
&= (2\pi)^2 \frac{i \, s}{6} < Q_s f, \frac{1}{(2\pi)^3} F''' + \frac{1}{2\pi} F' > \\
&= (2\pi)^2 \frac{(i \, s)^2}{12} < \omega^{-1} F', \frac{1}{(2\pi)^3} F'''' + \frac{1}{2\pi} f' > \\
&= (2\pi)^2 \frac{1}{12} \frac{1}{(2\pi)^3} < f, \omega^{-1} F''' > + \frac{1}{2\pi} < f, \omega^{-1} F'' > \\
&= (2\pi)^2 \frac{1}{12} \frac{1}{(2\pi)^3} < F, \omega^3 F > - \frac{1}{2\pi} < F, \omega F > 
\end{align*}
\]

where we used that for smooth, real valued, periodic functions \( F \)

\[
<F, F''' > = -\frac{1}{2} < (F')^2 > = 0, \quad <F, F'' > = -\frac{1}{2} < (F^2)' > = 0
\]

Note that \( [\omega/(2\pi)]^3 - [\omega/(2\pi)] \) has spectrum in \( \mathbb{N} \).

To show that the hypersurface deformation generators are indeed densely defined and symmetric in the chosen Fock representation we should check that they map Fock states into normalisable states. It is convenient not to work with Fock states directly but rather with the states

\[
w[f] \Omega, \quad w[f] = \exp(i < f, \Phi >)
\]

for the \( C \) sector and similar for the \( A, B \) sector. By choosing \( f = \sum_{n=0}^{\infty} s_n b_n \) for some real valued ONB of \( L \) one can generate all Fock states from the corresponding Weyl element \( w[f] \) by taking suitable derivatives of (2.49) at \( s_n = 0, \, n \in \mathbb{N} \). This shows that the \( w[f] \Omega \) with \( f \) real valued span a dense subset. A short standard calculation reveals

\[
E_s(x) \, w[f] \Omega = i \, g_s(x) \, w[f] \Omega; \quad g_s(x) := [Q_s f](x)
\]
We establish the finiteness of the constraint operators on the Fock states only for the most difficult piece $T_s^2(F)$, the other pieces are left to the reader. We have

$$ - T_s^2(F) \, w[f] \Omega = \int dx \, F(x) \, \{(E_s(x) + ig_s(x)^*)^\dagger\}^2 \, w[f] \, \Omega \quad (2.51) $$

Thus using the creation/annihilation algebra as in (2.47) and (2.50) a straightforward calculation reveals

$$ ||T_s^2(F) \, w[f]\omega||^2 = ||w[f]\omega||^2 \int dx \, F(x) \int dy \, F(y) \left[ 2 \, K_s(x,y)^2 + 4 \, K_s(x,y) \, g(x) \, g(y) + g(x)^2 \, g(y)^2 \right] $$

where $g = [Q_s + Q_{-s}]f = Qf$. (2.52)

We now discuss the finiteness of (2.52). To be sure, if $f$ is smooth, then finiteness is immediate. therefore, with respect to the smooth and quasi-local wavelet like functions introduced in [13] for the purpose of renormalisation, the following complications do not arise. However, the particular set of functions that were used for renormalisation in [2, 3, 4] are only piecewise smooth (in fact constant) and display finitely many discontinuities. We therefore consider these functions in what follows in order to pin point which convergence issues arise, why passing to smoother coarse graining functions to define the renormalisation flow is more convenient and how one can still work with only piecewise smooth coarse graining functions using zeta function regularisation. Readers not interested in these issues can safely skip the rest of the following paragraph.

**Zeta function regularisation**

The first term in (2.52) is of course the vacuum contribution (2.47) and thus independent of $f$. We already showed that it is finite in (2.47) for smooth $F$. The third term can be estimated by $||F||_\infty \, ||f||_\infty$ where $||.||_\infty$ denotes the supremum norm. Thus it is finite even if $f$ is a discontinuous but bounded function on $[0, 1)$. The second term is given by (up to the factor 4)

$$ < F(Qf), [2wQ_s] \, F(Qf) > = < F(Qf), Q[\omega - is\bar{\partial}]Q \, F(Qf) > \quad (2.53) $$

If $f$ is at least $C^1$ then the piece $-is\bar{\partial}$ vanishes by a similar calculation as in (2.70). If $f$ has discontinuities but is periodic and together with $F$ is real valued as is the case here then this piece still vanishes if we define for a step function with $0 \leq a < b < 1, \, x \in [0, 1)$

$$ \chi_{[a,b)}^l(x) = \delta(x,a) - \delta(x,b), \, \chi_{[a,b)}(x) = \begin{cases} 1 & a < x < b \\ \frac{1}{2} & x = a \land x = b \\ 0 & x < a \land b < x \end{cases} \quad (2.54) $$

The boundary values of the step function are uniquely selected by requiring

$$ < \chi_{[a,b)}, \chi_{[c,d]}^l > + < \chi_{[a,b)}, \chi_{[c,d]} > = 0 \quad (2.55) $$

for all possible (namely thirteen) orderings of $a, b, c, d$. These values also ensure that the sum of step functions for a partition of $[0, 1)$ equals unity at every point. Thus even in the case of discontinuities (2.53) simplifies to

$$ < F(Qf), Q\omega Q \, F(Qf) > = < F(Qf)', Q\omega^{-1} Q \, F(Qf) > \quad (2.56) $$

$$ = < F'(Qf), Q\omega^{-1} Q \, F'(Qf) > + 2 < (Qf)', F; Q\omega^{-1} Q \, F(Qf) > + < F(Qf)', Q \omega^{-1} Q[Q(Qf)] > $$

We have explicitly, using the spectral theorem

$$ 2\pi G := 2\pi \, Q\omega^{-1} Q \, [F'(Qf)] = \sum_{n=0}^{\infty} \, n^{-1} \, [e_n < e_n, F(Qf)] + e_n < e_n, F(Qf)] \quad (2.57) $$

pointwise in $[0, 1)$, thus the modulus squared of (2.57) can be estimated from above by the Cauchy-schwartz inequality and using $|e_n| = 1$ pointwise

$$ \sum_{n=1}^{\infty} \, n^{-2} \, \sum_{n \neq 0} \, | < e_n, F(Qf) | \leq c \, ||F(Qf)||_L^2 \leq c \, (||F||_\infty ||(F)\|_\infty)^2 \quad (2.58) $$
where $c > 0$ is a constant. Thus since for bounded $L_2([0, 1), dx)$ functions we have $||.||_L \leq ||.||_\infty$

\[
||Q^{-1} Q [F'(Qf)]||_L \leq ||Q^{-1} Q [F'(Qf)]||_\infty \leq c \ ||F||_\infty \ |(Qf)||_\infty
\]  

(2.59)

which shows that the first term in (2.56) is finite due to $||F'|| < \infty$ and the CS inequality. The second term is also finite if $Qf$ has finitely many discontinuities because the contributions of these discontinuities to the integral involving $(Qf)'$ amounts to a finite linear combination of evaluations of $F \ G$ at those points and both functions have finite supremum norm. The only potentially troublesome term is the last one which involves products of $\delta$ distributions. We evaluate it explicitly for the case encountered in the next sections namely

\[
Qf = \sum_{m=0}^\infty f(m) \ \chi_m(x), \ \chi_m(x) = \chi_{[x_m, x_{m+1})}(x)
\]  

(2.60)

with real valued $f(m)$ and characteristic functions $\chi_m$ of an interval where $M < \infty$ and $1 \equiv 0 = x_0 < x_1 < .. < x_{M-1} < 1$ is a partition of $[0, 1)$. We find

\[
<F (Qf)', Q\omega^{-1} [F(Qf)'] > = \sum_{m_1, m_2} f(m_1) f(m_2) \{(F (Q\omega^{-1}Q[\chi_{m_2}' F]))(x_{m_1}) - (F (Q\omega^{-1}Q[\chi_{m_2} F]))(x_{m_2})\}
\]

\[
= -M^{-1} \sum_{m_1, m_2} [\partial_M f](m_1) f(m_2) \{F (Q\omega^{-1}Q[\chi_{m_2}' F]))(x_{m_1}) - (F (Q\omega^{-1}Q[\chi_{m_2} F]))(x_{m_2})\}
\]

\[
= -M^{-1} \sum_{m_1, m_2} [\partial_M f](m_1) f(m_2) F(x_{m_1}) \sum_{n=1}^{\infty} \frac{1}{n} \{e_n(x_{m_1}) < e_n, \chi_{m_2}' F > + e_n(x_{m_1}) < e_n, \chi_{m_2} F >\}
\]

\[
= -M^{-1} \sum_{m_1, m_2} [\partial_M f](m_1) f(m_2) F(x_{m_1}) \sum_{n=1}^{\infty} \frac{1}{n} \{e_n(x_{m_1}) [(e_n F)(x_{m_2}) - (e_n F)(x_{m_2+1})] + c.c.\}
\]

\[
= -M^{-2} \sum_{m_1, m_2} [\partial_M f](m_1) [\partial_M f](m_2) F(x_{m_1}) F(x_{m_2}) \sum_{n=1}^{\infty} \frac{1}{n} \{e_n(x_{m_1} - x_{m_2}) + e_n(x_{m_1} - x_{m_2})\}
\]  

(2.61)

with $(\partial_M f)(m) = M[f(m + 1) - f(m)]$. It is the sum over $n \in \mathbb{N}$ in (2.61) which is problematic. We isolate and manipulate it as follows

\[
\sum_{n=1}^{\infty} \frac{1}{n} \{e_n(x_{m_1} - x_{m_2}) + e_n(x_{m_1} - x_{m_2})\} = 2 \sum_{n=1}^{\infty} \frac{1}{n} \cos(k_M(m_1 - m_2)n)
\]

\[
= 2 \sum_{l=1}^{M-1} \frac{1}{l} \cos(k_M(m_1 - m_2)l) + 2 \sum_{l=0}^{M-1} \frac{1}{l} \cos(k_M(m_1 - m_2)l) \sum_{n=1}^{\infty} \frac{1}{l + nM}
\]  

(2.62)

where we considered an equidistant partition, set $k_M = 2\pi/M$ and and exploited periodicity modulo $M$. Then for any $0 \leq l \leq M - 1$ we consider

\[
\sum_{n=1}^{\infty} \frac{1}{l + nM} = \sum_{n=1}^{\infty} \left[\frac{1}{l + nM} - \frac{1}{nM}\right] + \frac{1}{M} \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n}
\]  

(2.63)

The first infinite sum in (2.63) converges absolutely for each $l$. The limit of the second sum marginally diverges to the simple pole (with residue unity) value of the Riemann zeta function. Consider

\[
\gamma(N, \epsilon, \delta) := \frac{1}{2} \sum_{n=1}^{N} \left[\frac{1}{n^{1+\delta+\epsilon}} + \frac{1}{n^{1+\delta-\epsilon}}\right]
\]  

(2.64)

If we take the limits $\epsilon \to 0^+, \delta \to 0^+, N \to \infty$ in exactly this order, then we return to (2.63). As usual, regularisation of infinities consists in interchanging limits that would be allowed if the sums involved would
converge absolutely. We take the limits in the order $N \to \infty, \delta \to 0, \epsilon \to 0$. After $N \to \infty$ we obtain for $\delta > \epsilon > 0$ the finite result

$$\gamma(\epsilon, \delta) := \lim_{N \to \infty} \gamma(N, \epsilon, \delta) \frac{1}{2} \left[ \zeta(1 + \delta + \epsilon) + \zeta(1 + \delta - \epsilon) \right]$$ (2.65)

where $\zeta$ is the Riemann zeta function. It has an analytic extension to the whole complex plane except for its simple pole $z = 1$. With this analytic extension being understood in (2.65) we can now take $\delta \to 0$

$$\gamma(\epsilon) := \lim_{\delta \to 0} \gamma(\epsilon, \delta) \frac{1}{2} \left[ \zeta(1 + \epsilon) + \zeta(1 - \epsilon) \right]$$ (2.66)

Finally we take $\epsilon \to 0$ which results in the principal value of the zeta function at unity

$$\gamma := \lim_{\epsilon \to 0} \gamma(\epsilon) = [\text{pv} \, \zeta](1) = \lim_{\epsilon \to 0} \frac{1}{2} \left[ \zeta(1 + \epsilon) + \zeta(1 - \epsilon) \right]$$ (2.67)

which turns out to be finite and equal to the Euler-Mascheroni constant $\gamma$.

$$\gamma = \lim_{N \to \infty} \left[ -\ln(N) + \sum_{n=1}^{N} \frac{1}{n} \right]$$ (2.68)

which is numerically 0.58 in the second decimal precision.

This kind of regularisation is of course standard in conformal field theory. It would not be necessary if the functions $f$ were smooth. In the smooth case exactly the same infinite sum of $1/n$ would occur but the difference would be that it is multiplied by n-dependent coefficients that either have compact support in $n$ or lead to stronger decay rendering the sum absolutely convergent. Thus in the smooth case the result of the calculation would be dominated by the respective and corresponding first term in (2.62), (2.63). Note also that the proposed regularisation can be considered as the regularisation

$$\omega^{-1} \to \frac{1}{2} \left[ \omega^{1+\delta+\epsilon} + \omega^{1+\delta-\epsilon} \right]$$ (2.69)

with $\delta > \epsilon > 0$ and then taking the limits in the order described. This regularisation is the price to pay when working with bounded discontinuous functions $f$ but it extracts exactly the dominating terms that would arise if $f$ was smooth. The motivation for using non smooth step functions is that they result in coarse graining maps for purposes of renormalisation with almost perfect properties as we will see in the next section. In we introduce smooth coarse graining maps which come very close to those step functions, for which above regularisation is not necessary and for which the finite result obtained here after regularisation is exact. As these step functions are finite position resolution approximants of smooth continuum functions, our manipulation is physically justified. This can also be seen as follows: The absolute value of both terms in (2.70) can be bounded from above (after above regularisation) by $c/M$ where $c = \gamma + \sum_{N=1}^{\infty} N^{-2}$ (observing that $n \leq M-1$). If $f(m) = M < \chi_{m}^M, f >$ for smooth $f$ as we assume in the next section, with $\chi_{m}^M$ the characteristic function of the interval $[m/M, (m+1)/M)$ then the first term in (2.61) converges to the smooth continuum value $< Ff', Q\omega^{-1}QFf' >$ as $M \to \infty$ while the two remaining terms can be bounded by $c \|Ff'\|^2/M$ which converges to zero. Accordingly our zeta function regularisation (only necessary for non smooth finite resolution approximants) ensures that the continuum limit (taking the finite resolution regulator $M \to \infty$) agrees with the direct continuum result.

With this understanding, the hypersurface generators are densely defined on the span of Fock states.

**Comments on the space of solutions to the constraints**

For completeness, we close this section with a few remarks on the actual solution of the quantum constraints which are mostly standard. These will not be of any relevance for the rest of the paper and the reader not interested in these remarks can safely jump to the next section.
Not even the Fock vacuum is in the kernel of any of them not to speak of the joint kernel. Indeed, there can be no joint zero eigenvector $v$ of all the constraints except the zero vector due to the anomaly

$$0 = [D_s(f), D_s(g)] v = i s 3 S(f, g) v$$

(2.70)

In solving the constraints, we thus look not for joint zero eigenvectors (zero is not in the joint point spectrum) but for generalised joint eigenvectors (distributions), i.e. linear functionals $l$ on a dense and invariant (under the action of the $D_s(f)$) domain $\mathcal{D}$ such that

$$l[D_s(F) v] = 0 \quad \forall \ f, \ s, \ v \in \mathcal{D}$$

(2.71)

Note that the finite linear span of Fock states is dense but not invariant. However, this also does not work for any such choice of domain, because if $\mathcal{D}$ is invariant then any such $l$ also satisfies $l[[D_s(f), D_s(g)] v]] = i s c S(f, g) l[v] = 0$ i.e. $l$ vanishes identically on $\mathcal{D}$. We thus resort, as it is common practice, to solving the equations $D_s(f) v = 0$ not in the strong operator topology but in the weak operator topology. That is, we look for a proper subspace $\mathcal{D} \subset \mathcal{H}$ in the domain of the $D_s(f)$ such that the $D_s(f) \mathcal{D} \subset \mathcal{D} \perp$, i.e. the image of $\mathcal{D}$ under any $D_s(f)$ lies in the orthogonal complement of (the completion of) $\mathcal{D}$. That is, for any $v, v' \in \mathcal{D}$ we impose for all $s, f$

$$< v, [D_s(F) - a < f > 1_H] v' >= 0$$

(2.72)

where a possible normal ordering constant $a$ was introduced. In other words, w.r.t. the split $\mathcal{H} = \mathcal{D} \oplus \mathcal{D} \perp$ all operators $D_s(f)$ contain no diagonal block corresponding to $\mathcal{D}$. A well known choice of $\mathcal{D}$ consists in the solution to the system of equations

$$[D_s(e_n) - a \delta_n 1_H] v = 0; \ \forall \ s, \ n \geq 0$$

(2.73)

Since $D_s(e_n)^\dagger = D_s(e_{-n})$ it follows that (2.73) implies (2.72) for all $F$. The system (2.73) does not suffer from the anomaly because for $m, n \geq 0$

$$[D_s(e_m) - a \delta_n 1_H, D_s'(e_n) - a \delta_n 1_H] = i \delta_{ss'} [D_s([e_m, e_n]) + 3 c S(e_m, e_n)]$$

$$= i \delta_{ss'} (2\pi i (m - n)) D_s(e_{m+n}) - 3 i c (m^3 - m - n^3 + n) \delta_{m+n,0} = -2\pi \delta_{ss'} (m - n) D_s(m + n)$$

(2.74)

as the second term only contributes for $m = n = 0$ if $m, n \geq 0$ but then the prefactor vanishes. Thus the r.h.s. of (2.75) is non-vanishing iff $m + n > 0$ so that the system of conditions (2.73) is consistent. Of course other choices of $\mathcal{D}$ are equally valid such as imposing (2.74) for $n \leq 0$ only for both values of $s$ or using (2.73) with $n \geq 0$ for $s = +$ and with $n \leq 0$ for $s = -$.

Alternatively, to actually solve (2.73) we could use master constraint methods [26], i.e. we set

$$M := \sum_{s,n \geq 0} m_n D_s(e_n)^\dagger D_s(e_n)$$

(2.75)

where $m_n > 0$ are coefficients that decay sufficiently fast in order that $M$ be densely defined in the Fock space. Then any solution $v$ of (2.73) solves $M v = 0$ and conversely any solution of $M v = 0$ solves $< v, M v >= 0$ and therefore (2.73). The task is now to solve for the ground states of the master constraint $M$. One will look for them in the form

$$v = v_{AB} \otimes v_c, \ v_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B, \ v_c \in \mathcal{H}_C$$

(2.76)

where $v_c$ is any Fock state and $v_{AB}$ is to be determined in dependence on $v_c$. In this way, the physical Hilbert space is isomorphic to $\mathcal{H}_C$.

This is of course expected as the PFT should be equivalent to the massless KG field on the cylinder. Indeed, the natural gauge fixing conditions $T = t, \ X = x$ reproduce this theory which one immediately arrives at using the corresponding reduced phase space quantisation. The actual solution of PFT is beyond the scope of the present work in which we are just interested in studying how the system behaves under renormalisation.
3 Hamiltonian Renormalisation of Hamiltonian systems

This section is to recall the essential elements from \[4,11\] to which the reader is referred to for more information.

We introduce some coordinate \( x \in [0,1) \) and equidistant lattices \( \Lambda_M \) on \([0,1]\) with \( M \) points \( x_m = \frac{m}{M} \), \( m \in \mathbb{Z}_M := \{0,1,2,\ldots,M-1\} \). Among the numbers \( M \in \mathbb{N} \) we introduce the relation \( M < M' \) iff \( \frac{M'}{M} \in \mathbb{N} \) which means that \( \Lambda_M \) is a sublattice of \( \Lambda_{M'} \). It is not difficult to see that this defines a partial order and that \( \mathbb{N} \) is directed with respect to it.

The space of complex valued sequences \( \{f_M(m)\}_{m \in \mathbb{Z}_M} \) is denoted by \( L_M \) and given a Hilbert space structure by

\[
<f_M,f'_M>_{L_M} := \frac{1}{M} \sum_{m \in \mathbb{Z}_M} f_M(m)^* f'_M(m) \tag{3.1}
\]

Let \( \chi_{(a,b)} \) be the characteristic function of the left closed, right open interval \((a,b) \subset [0,1)\) and for \( x \in [0,1) \)

\[
\chi^M_m(x) := \chi_{\left[\frac{m}{M},\frac{m+1}{M}\right]}(x) \tag{3.2}
\]

Consider the embedding (recall \( L = L_2([0,1), dx) \))

\[
I_M : L_M \to L;
(I_M f_M)(x) := \sum_{m \in \mathbb{Z}_M} f_M(m) \chi^M_m(x) \tag{3.3}
\]

which is in fact an isometry

\[
<f_M,f'_M>_{L_M} = <f_M,f'_M>_{L_M} \tag{3.4}
\]

and thus allows the interpretation of (3.1) as the Riemann sum approximation of \( <f,f'>_L \) with \( f_M(m) := f(m/M), f'_M(m) = f'(m/M) \).

For \( M < M' \) we construct the embeddings

\[
I_{M,M'} : L_M \to L_{M'}; \quad I_{M,M'} := I^1_{M',M} \quad I_M \tag{3.5}
\]

The operator \( I^1_{M,M} \) can be worked out explicitly

\[
[I^1_{M,M} f](m) = M < \chi^M_m,f>_L \tag{3.6}
\]

It is also an isometry

\[
< I_{M,M'} f_M, I_{M,M'} f'_M >_{L_{M'}} = < f_M,f'_M >_{L_M} \tag{3.7}
\]

and these embeddings automatically obey the consistency conditions for all \( M < M' < M'' \)

\[
I_{M',M''} \circ I_{M,M'} = I_{M,M''} \tag{3.8}
\]

This follows from the identity

\[
I_{M',M} I^1_{M',M} I_M = I_M \tag{3.9}
\]

which in turn is due to the property of the \( \chi^M_m \) to define partitions of \([0,1)\) which are nested for \( M < M' \), that is

\[
\chi^M_m = \sum_{l=0}^{k-1} \chi^M_{km+l}, \quad k = \frac{M'}{M} \tag{3.10}
\]

We can also work out \( I_{M,M'} \) explicitly \( (k := M'/M) \)

\[
(I_{M,M'} f_M)(m') = M' < \chi^M_m, I_M f_M >_L = M' \sum_{m \in \mathbb{Z}_M} f_M(m) < \chi^M_{m'}, \chi^M_m >_L =
\]

\[
= M' \sum_{m \in \mathbb{Z}_M} f_M(m) \left( \frac{1}{M'} \sum_{l=0}^{k-1} \delta_{m',mk+l} \right) = f_M\left(\left\lfloor \frac{m'}{M'} \right\rfloor \right) \tag{3.11}
\]
where \([\cdot]\) denotes the floor function (Gauss bracket).

Consider a scalar field \(\phi\) on \([0,1]\) with conjugate momentum \(\pi\). Note that geometrically \(\pi\) is a scalar density of weight one on \([0,1]\) as one can see from the Poisson bracket

\[
\{\pi(x), \phi(y)\} = \delta(x,y) \tag{3.12}
\]

We consider real density one valued test functions \(f\) and real density zero valued test functions \(F\) on \([0,1]\). Then the real numbers

\[
\phi(f) := \langle f, \phi \rangle, \quad \pi(F) := \langle F, \pi \rangle \tag{3.13}
\]

are invariant under diffeomorphisms of \([0,1]\) and we have

\[
\{\pi(F), \phi(f)\} = F(f) \tag{3.14}
\]

One can construct the abstract \(*-\)algebra (even \(C^*\)-algebra) \(\mathfrak{A}\) generated by the Weyl elements

\[
w(f, F) = \exp(i[\phi(f) + \pi(F)]) \tag{3.15}
\]

and the corresponding Weyl relations that follow from the reality of \((3.13)\) and \((3.14)\).

Representations of \(\mathfrak{A}\) can be constructed from a state (positive, normalised, linear functional) \(\omega\) on it via the GNS construction \cite{22}. This delivers a Hilbert space \(\mathcal{H}\), a representation \(\rho\) of \(\mathfrak{A}\) by bounded operators on \(\mathcal{H}\) and a vector \(\Omega \in \mathcal{A}\) cyclic for \(\rho(\mathfrak{A})\). If \(\mathcal{H}\) is separable, we always find an Abelian sub-\(*\)-algebra \(\mathfrak{B}\) of \(\mathfrak{A}\) for which \(\Omega\) is still cyclic. For instance, we can pick an ONB \(e_I\), \(I \in \mathbb{Z}\) with \(\delta_{00} := \Omega\) of \(\mathcal{H}\) and consider the Abelian group of unitary operators \(U_I\), \(I \in \mathbb{Z}\), \(U_I^\dagger = U_{-I}\) such that \(U_I e_J = e_{I+J}\). Then we find \(a_I \in \mathfrak{A}\) such that \(\rho(a_I) = U_I\) and \(\mathfrak{B}\) is generated by those \(a_I\). See \cite{11} for more details and more general cases. Of course the \(a_I\) may in general be a very complicated (in general infinite) linear combinations of the Weyl elements \((3.15)\). Still it follows that \(\mathcal{H}\) can be thought of as \(L^2(\Delta(\mathfrak{B}),d\nu)\) where \(\Delta(\mathfrak{B})\) is the Gel’fand spectrum (space of "characters" i.e. homomorphisms \(\chi : \mathfrak{B} \to \mathbb{C}\) equipped with the Gel’fand topology) of \(\mathfrak{B}\) and \(\nu\) a probability measure thereon. More precisely, there is a unitary map \(U : \mathcal{H} \to L^2(\Delta(\mathfrak{B}),d\nu)\) with \([U \rho(b)\Omega]\chi) := \hat{b}(\chi) := \chi(b)\) which is essentially the Gel’fand isomorphism.

We will assume that \(\mathfrak{B}\) can be generated by the \(w(f) := f(f, F = 0)\) so that we can identify the space of characters with the space of fields \(\phi\) and \(\nu\) as a probability measure on that space. Indeed this is the case in Fock representations \(\hat{\omega} = \langle ., . \rangle_{\mathcal{H}}\) in which \(w(f)\Omega\) is essentially \(\exp(i [\pi, f]_{L})\Omega\) up to a phase where \(A = (\sqrt{\omega\phi} - i \sqrt{\omega^{-1} \pi})/\sqrt{2}\) is the annihilator. Thus arbitrary linear combinations of Fock states \(\cdots < f_n, A > \cdots\) can be obtained by taking derivatives at \(s_1 = \cdots = s_n = 0\) of \(w(\sum_k s_k [2\omega^{1/2} f_k])\Omega\) determining that the span of the \(w(f)\Omega\) is dense. Then \(\nu\) is the Gaussian measure with covariance \(1/(2\omega)\)

\[
\nu(w(f)) := \langle \Omega, w(f)\Omega \rangle_{\mathcal{H}} = \exp(-\frac{1}{4} < f, \omega^{-1} f >_{L}) \tag{3.16}
\]

Given the injections \(I_M; L_M \to L\) we may restrict \(\phi\) to the subspace \(I_M L_M\) i.e. we define a scalar field \(\phi_M\) on the lattice \(\Lambda_M\) by

\[
\phi_M(f_M) := \phi(I_M f_M), \quad \phi_M = I_M^\dagger \phi \tag{3.17}
\]

which provides a natural “discretisation”. Here \(\phi(f) := \langle f, \phi \rangle\) for real valued \(f\). As \((I_M^{1/2} \phi)(m) = M < \chi^M_m, \phi >\) approaches \(\phi(x)\) in the limit \(M \to \infty\) for \(m = xM\) we see that the density zero valued \(\phi\) is smeared against the density one valued discretised \(\delta\) distribution \(M \chi^M\) which is diffeomorphism covariant. We may likewise define a discretised momentum \(\pi_M = M^{-1} I_M^{1/2} \pi = \langle \chi^M, \pi \rangle\) which smears the density one valued \(\pi\) against the density zero valued \(\chi^M\) which is also covariant. Together this ensures that \(\phi_M, \pi_M\) are conjugate on \(\Lambda_M\)

\[
\{\pi_M(m), \phi_M(m')\} = M \{\pi(\chi^M_m, \phi(\chi^M_m))\} = M < \chi^M_m, \chi^M_{m'} >_{L} = \delta_{m,m'} \tag{3.18}
\]

Although this is geometrically more natural, we will instead use

\[
\pi_M(m) := [I_M^{1/2} \pi](m), \quad \{\pi_M(m), \phi_M(m')\} = M \delta_{m,m'} \tag{3.19}
\]

so that \(\phi_M, \pi_M\) are conjugate not in the sense of a Kronecker \(\delta\) but rather a discrete \(\delta\) distribution.
Given a function $H[\phi, \pi]$ on the continuum phase space coordinatised by the variables $\phi, \pi$ we may try to define a discretised function

$$H_M[\phi_M, \pi_M] := H[I_M \phi_M, I_M \pi_M]$$  \hspace{1cm} (3.20)

where the approximation $I_M I_M^\dagger \to I_L$ as $M \to \infty$ was used. This indeed works as long as $H$ depends on $\pi, \phi$ only algebraically. However, when derivatives are involved, the simple prescription (3.20) may cause trouble because the functions $\chi_M$ are not differentiable. This can be improved by passing to alternative, smoother coarse graining maps $I_M$ which lead to coarse graining maps satisfying the consistency conditions (3.5) which are essential for the renormalisation scheme. For the examples discussed in [13] it turns out that the natural discretisation $\partial_M := I_M^\dagger \partial I_M$ is a well defined and antisymmetric discrete derivative operator on $L_M$.

To keep the presentation simple and to see into which problems one may run using step functions, we take the usual point of view that the prescription (3.20) is as good as any other as long as $\nu_M$" are automatically consistently defined: For any $M < M'$

$$\nu_M'(w_M'[f_M]) = \nu_M(w_M[f_M])$$  \hspace{1cm} (3.24)

i.e. integrating the excess degrees of freedom in artificially writing the function $w_M$ of $\phi_M$ as the function $w_M'$ of $\phi_M'$ which however depends on $\phi_M'$ only in terms of the blocked variables $I_M \phi_M$ does not change the result. Conversely, under relatively mild technical assumptions [2], a cylindrically consistent family of measures $\nu_M$ on quantum configuration spaces $K_M$ can be extended to a measure $\nu$ on a space $K$ called the projective limit of the $K_M$. In that sense, a cylindrically consistent family is as good as the continuum definition but the practical advantage of the family is that the $\nu_M$ are easier to compute.

Consider the Hilbert spaces $H_M = L_2(K_M, d\nu_M)$ and the embeddings

$$J_M : H_M \to H = L_2(\Phi, d\nu), \quad w_M[f_M]|_{\Omega_M} \to w[I_M f_M]|_{\Omega}$$  \hspace{1cm} (3.25)

which by construction are isometries. Here $\nu_M(\cdot) = <\Omega_M, \cdot >_{H_M}$ and $\nu(\cdot) = <\Omega, \cdot >_H$. It is also not difficult to see that the $J_M$ inherit from the $I_M$ the consistency properties

$$J_{M' M''} : J_{M' M''} = J_{M'' M'} \forall M < M' < M''$$  \hspace{1cm} (3.26)

where $J_{M M'} = J_{M M'}^\dagger$, $J_M$, $M < M'$. It follows that $H$ is the inductive limit of the $H_M$ [22]. Given a symmetric quadratic form $H$ on $H$ with dense domain $D$ spanned by the $w[f]\Omega$ we may construct the symmetric quadratic forms $H_M := J_M^\dagger H J_M$ which are automatically consistently defined: For any $M < M'$ we have

$$J_M^\dagger H_M' J_M' = H_M$$  \hspace{1cm} (3.27)

Moreover, given $J_M \psi_M$, $J_M \psi_M' \in H$ with $\psi_M \in D_M$, $\psi_M' \in D_M'$ in the dense set of the span of vectors $w_M[f_M]\Omega_M$ etc. we find $M'' > M, M'$ and can compute

$$< J_M \psi_M, H J_M' \psi_M' >_H = < J_{M'' M''} \psi_M, H_{M''} J_{M'' M'} \psi_M' >_{H_{M''}}$$  \hspace{1cm} (3.28)
i.e. for all practical purposes the family of quadratic forms $H_M$ is as good as $H$ but easier to compute. Note that $H$ is not the inductive limit of the $H_M$ \cite{22} for two reasons: First, while $H_M$ are actually operators and not only quadratic forms (as the systems labelled by $M$ only depend on finitely many degrees of freedom) the object $H$ is in general not. Second, for $H$ to be the inductive limit of the $H_M$ we require the much stronger intertwiner property $J_M H_M = H J_M$ which implies $H_M = J_M H J_M$ but not vice versa.

The problem that one encounters in quantising a classical Hamiltonian system with canonical variables $\phi, \pi$ and Hamiltonian $H$ is this: Provide a representation $\rho$ of the $*-\$ algebra generated by the $\phi(f), \pi(F)$ (or the $C^*$-algebra generated by the $w(f,F)$) that supports “the” Hamiltonian $H$ as a self-adjoint operator. We have used inverted commas as this task is ill-defined as it stands: The classical function $H$ typically is ill-defined when naively substituting the classical $\phi, \pi$ by their corresponding operator valued distributions. The strategy of constructive QFT is to come up with quantisations of the simpler, well-defined (since finite dimensional - if both properties then restrict the discretisation ambiguities inherent in these systems by inverting the logic: the automatic consistency conditions indeed hold for all $M < M'$, if this is true for the initial data $\Omega_M(0), H_M(0)$ respectively. Thus one usually picks a fixed $M'(M)$ satisfying $M'(M) > M$, a popular choice being $M'(M) = 2 M$. Then, relying on the intuition of universality, the fixed point is hoped for not to depend on the choice $M'(M)$, so that at the fixed point the consistency conditions indeed hold for all $M < M'$.

An automatic feature of this renormalisation scheme is that for all $M$ the fixed point vacuum $\Omega_M$ is a ground state of the fixed point Hamiltonian $H_M$ if this is true for the initial data $\Omega_M(0), H_M(0)$: This follows inductively from

\[
H_M^{(n+1)} \Omega_M^{(n+1)} = \left[ J_M^{(n)} \right]_{M' M'(M)}^\dagger H_M^{(n)} J_M^{(n)} \Omega_M^{(n+1)} = \left[ J_M^{(n)} \right]_{M' M'(M)}^\dagger H_M^{(n)} \Omega_M^{(n)} = 0
\]

This condition is necessary in order to make the renormalisation scheme compatible with Wilsonian renormalisation of the Euclidian (path integral) formulation from which the present scheme was derived via Osterwalder-Schrader (OS) reconstruction \cite{4,11}.

4 Hamiltonian renormalisation of constrained systems

As mentioned, the scheme reviewed in the previous section was motivated using the Euclidian formulation of a QFT which needs as a minimal input a self-adjoint Hamiltonian $H$ on a Hilbert space $H$ bounded from below
with vacuum $\Omega$. From these one can attempt to construct the associated Gibbs measure $\mu$ on the space of field histories and when this exists, it satisfies a minimal set of Euclidian axioms (in particular reflection positivity) ensuring that $(H, \mathcal{H}, \Omega)$ can be recovered from $\mu$.

When we consider constrained Hamiltonian systems, in particular when there is no Hamiltonian but just a set of Hamiltonian constraints, we are strictly speaking leaving that framework. One can return to it by using the reduced phase space formulation in which one gauge fixes the Hamiltonian constraints thereby ending up with a true Hamiltonian again which just acts on the gauge invariant (or true) degrees of freedom [27] and this is the strategy followed so far [11]. However, in this paper we want to explore a different route:

The observation is that the two renormalisation steps (3.30) and (3.31) actually do not rely on $H$ being bounded from below or that $\Omega$ is the vacuum of $H$. Thus we propose to “abuse” (3.30) and (3.31) and use them also for constrained Hamiltonian systems. In other words, we keep (3.30) as it is and apply (3.31) to each constraint operator separately.

This proposal raises two immediate questions:

1. The classical continuum constraints are of the form $H(F) = \int dx \, F(x) \, H(x)$ where $F$ is a smearing function and $H(x)$ is the Hamiltonian constraint density. Thus the essential difference between a true Hamiltonian system and a constrained Hamiltonian system (apart from the fact that true Hamiltonian densities are typically bounded from below at least classically) is that for the true Hamiltonian the only allowed smearing function is $F = 1$ while for the constrained case the space of smearing function is infinite dimensional. The question is now how $F$ should be treated when we discretise $H(F)$. There are two extreme and equally natural points of view:

i. The first is that for each $F$ the function $H(F)$ is simply an independent object and should be treated just as a true Hamiltonian. That is, the function $F$ remains as it is, it is not discretised.

ii. The second is that $F$ should be treated on equal footing with the phase space variables $\phi, \pi$ and thus should be discretised, perhaps by the same map $I_M^i$, perhaps by another. This of course introduces yet more discretisation ambiguities into the quantisation and also requires to invent a flow equation on the space of discretised smearing functions $F_M$ when stating (3.31).

Note that the second point of view is often taken for granted in lattice inspired approaches to constrained systems [28]. One may think that the first point of view in fact provides a natural choice of discretisation of $F$ as follows:

Suppose that we actually have the continuum theory, i.e. the Hilbert space $\mathcal{H}$ and the constraints $H(F)$ at our disposal. Then the idea is to define a map $E_M : L \to L_M$ via the identity

$$H_M(E_M F) := \sum_{m \in \mathbb{Z}_M} (E_M F)(m) \, H_M(m) := J_M^i \, H(F) \, J_M$$

which assumes that the r.h.s. can actually be written in this local form. This is unfortunately already not the case even for the PFT considered here. The reason for this to happen is that $H$ when written in terms of polynomials of annihilation and creation operators involves non-local integral kernels. While these do get discretised by means of $J_M$ this leads to an effective $E_M$ which maps $L \to L_M$ where $N \geq 2$ is the polynomial degree. We will demonstrate this explicitly below for PFT.

This establishes that viewpoints i. and ii. are drastically different, i.e. a map $E_M : L \to L_M$ generically cannot be induced via (4.1). Instead, according to viewpoint ii. we consider as an extra structure maps $\tilde{I}_M L_M \to L$ and $\tilde{I}_{MM'} : L_M \to L_{M'}$ and define

$$H_M(F_M) := \sum_{m \in \mathbb{Z}_M} F_M(m) \, H_M(m) := J_M^i \, H(\tilde{I}_M F_M) \, J_M$$

This is consistently defined

$$H_M(F_M) = J_{MM'}^i \, H_{M'}(\tilde{I}_{MM'} F_M) \, J_{MM'}$$

due to $J_{MM'} J_{MM'} = J_M$ and provided that $\tilde{I}_M^i, I_{MM'}^i = I_M$. We may reduce the ambiguity and actually consider $\tilde{I}_M = I_M, \tilde{I}_{MM'} = I_{MM'}$, however, this choice is inconvenient for the following reason: While we
can certainly compute the commutator $[H_{M}(F_{M}), H_{M}(G_{M})]$ directly which is well defined, one would like to see the deviation from the continuum computation by using the identity

$$\begin{align*}
[H_{M}(F_{M}), H_{M}(G_{M})] &= \tilde{J}_{M}^\dagger \{[H(I_{M}F_{M}), H(I_{M}G_{M})] - H(I_{M}F_{M}) (1_{\mathcal{H}} - P_{M}) H(I_{M}G_{M}) \}
\quad + H(I_{M}G_{M}) (1_{\mathcal{H}} - P_{M}) H(I_{M}F_{M}) \tilde{J}_{M}^\dagger \\
&= \tilde{J}_{M}^\dagger \{H(I_{M}F_{M}), H(I_{M}G_{M})\} - H(I_{M}F_{M}) (1_{\mathcal{H}} - P_{M}) H(I_{M}G_{M}) \tilde{J}_{M}^\dagger \\
&+ H(I_{M}G_{M}) (1_{\mathcal{H}} - P_{M}) H(I_{M}F_{M}) \tilde{J}_{M}^\dagger \\
&= \tilde{J}_{M}^\dagger \{[H(F_{M}), H(G_{M})] - H(F_{M}) (1_{\mathcal{H}} - P_{M}) H(G_{M}) + H(G_{M}) (1_{\mathcal{H}} - P_{M}) H(F_{M})\} \tilde{J}_{M}^\dagger
\end{align*}$$

(4.4)

where we defined $P_{M} = J_{M}^\dagger J_{M}$ which is a projection in $\mathcal{H}$ due to the isometry of $J_{M}$. The first term gives the cylindrical projection of the continuum algebra which in our case is the Virasoro algebra. The second and third term should vanish as $M \to \infty$ because $J_{M}$ becomes the identity in $\mathcal{H}$. Therefore (4.4) appears to be an appropriate way to monitor how the cylindrically projected theories approach the correct continuum. The catch is that we know that in PFT the commutator $[H(I_{M}F_{M}), H(I_{M}G_{M})]$ depends on first and third derivatives of the $I_{M}F_{M}, I_{M}G_{M}$ which are, however, not even continuous. Accordingly, if we want to use (4.4) we should instead use $I_{M}, I_{MM'}$ which are at least $C^{3}$ and which share all the properties of $\tilde{I}_{M}, \tilde{I}_{MM'}$. Thus such maps constructed from wavelets [24] suggest themselves, we will give more details below.

To summarise this part of the discussion, for the purpose of this paper we take viewpoint i. and leave $F, G$ un-discretised and then with $H_{M}(F) = \tilde{J}_{M}^\dagger H(F) J_{M}$ the computation

$$\begin{align*}
[H_{M}(F), H_{M}(G)] &= \tilde{J}_{M}^\dagger \{[H(F), H(G)] - H(F) (1_{\mathcal{H}} - P_{M}) H(G) + H(G) (1_{\mathcal{H}} - P_{M}) H(F)\} \tilde{J}_{M}^\dagger \\
&= \tilde{J}_{M}^\dagger \{[H(F), H(G)] - H(F) (1_{\mathcal{H}} - P_{M}) H(G) + H(G) (1_{\mathcal{H}} - P_{M}) H(F)\} \tilde{J}_{M}^\dagger
\end{align*}$$

(4.5)

is unproblematic. To avoid confusion note that (4.5) is supposed to yield the Virasoro algebra, as $M \to \infty$, including the central term, i.e. the anomaly as compared to the classical computation (Witt algebra) should be present. We thus want to check that the Virasoro algebra is recovered without anomaly, not the Witt algebra.

2. As noted in the previous section, due to the central term in the Virasoro algebra, there cannot be a joint vacuum $\Omega$ for all the constraints $H(F)$. This is even more the case for the $H_{M}(F)$ at finite resolution because they typically do not close as it is plain to see from (4.5), hence the states $\Omega_{M}$ that arise at the fixed point cannot be joint vacua for the $H_{M}(F)$.

This is no obstacle for the renormalisation scheme when applied separately to the $H(F)$ because the $H_{M}(F)$ are operators (and not only quadratic forms) of systems with finitely many degrees of freedom and thus one does not expect the usual problems in finding a domain that is typical for QFT (infinitely many degrees of freedom) especially if $H(F)$, even when normal ordered, contains terms that are monomials made solely from creation operators. Thus we expect to find dense domains $D_{M}(F)$ for $H_{M}(F)$ and by construction $J_{MM'}D_{M}(F) \subset D_{M'}(F)$. However, a problem may occur when we compute commutators such as (4.5) because the domains $D_{M}(F)$ may depend on $F$ and it may be the case that $H_{M}(F) D_{M}(F) \not\subset D_{M(F)}$. [29]. At least it is true that at finite $M$ the domains are invariant $H_{M}(F) D_{M}(F) \subset D_{M}(F)$ because they are just finite linear combinations of monomials (and not infinite linear combinations as in case of $H(F)$) of creation and annihilation operators. Thus a minimal requirement for (4.5) to be meaningful is that the $H_{M}(F)$ have a dense, invariant domain $D_{M}$ independent of $F$ and then by construction $J_{MM'} D_{M} \subset D_{M'}$.

Since the span $D$ of the $J_{M} D_{M}$ is dense in the inductive limit $\mathcal{H}$ on which by construction is a form domain of $H(F)$, this then also makes the fixed point $H(F)$ densely defined as a quadratic form. However, this does not ensure that the commutators of the $H(F)$ are well defined because matrix elements of the formal expression $H(F) H(F')$, which can be formally computed by invoking resolutions of the identity in terms of an ONB made from vectors in $D$, may diverge, which is a potential danger even if $H(F)$ can be promoted to an operator especially if $D$ is not invariant for $H(F)$. It is here where a joint cyclic vacuum would be very convenient to build a common dense operator domain upon. In absence of it, the construction of such a domain may be very difficult, if it exists at all. In PFT we know that this problem does not occur, despite the non-existence of such a joint vacuum, as a common dense (but not invariant) operator domain is given explicitly by the span of the chosen Fock states. However, it may be in more complicated theories, especially if the domains depend on $F$ which in unfortunate cases can have non-dense intersections [28].
3. Note that our renormalisation scheme constructs a single Hilbert space \( \mathcal{H} \) (or measure \( \nu \)) but an infinite number of quadratic forms \( H(F) \) if a simultaneous fixed point of the respective flow equations exists at all. While the flow equations for \( \nu \) and \( H(F) \) are tightly coupled, the flow equations for the various \( H(F) \) are treated as independent for each choice of \( F \). Now it could happen that these latter equations have several different fixed points for each choice of \( F \) that are reached depending on the choice of initial discretisation \( H_M^{(0)}(F) \). Then the corresponding fixed point family \( H_M(F) \) may depend rather dis-continuously on \( F \) and thus would probably not coincide with the result of blocking from the continuum \( H_M(F) := J_M H(F) J_M \).

In the next section we examine whether these issues arise in the Hamiltonian renormalisation of PFT.

5 Hamiltonian renormalisation of PFT

Since the constraint operators are of the form

\[
D_+ = [A^2 - A_0^2] \otimes 1_B \otimes 1_C + 1_A \otimes 1_B \otimes C_+^2, \quad D_+ = 1_A \otimes [B^2 - B_0^2] \otimes 1_C - 1_A \otimes 1_B \otimes C_-^2
\]

it will be sufficient to consider one of the sectors \( A, B, C \) only, say \( C \). Our first task is to pick initial discretisations of the \( C^{(0)}_{\pm, M} \) and corresponding Hilbert space measures \( \nu^{(0)}_M \) on \( K_M = \mathbb{R}^M \). As suggested by the considerations of section 2 we build \( C^{(0)}_{\pm, M} \) out of \( C^{(0)}_{0, M}, C^{(0)}_M \). We define in parallel to the continuum (see (2.24), (2.25))

\[
\begin{align*}
\Phi_M &= \int_{M} \Phi \\
\Pi_M &= \int_{M} \Pi \\
Q_{M}^f M &= \langle m, m \rangle \Pi_M \\
Q_{M} C_{0, M}^{(0)} &= \frac{1}{\sqrt{2}} \left[ \sqrt{\omega_0} Q_{M} (\Phi_M - i \frac{1}{\sqrt{\omega_0}} Q_{M} M \Pi_M) \right] \\
C_{M}^{(0)} &= \frac{1}{\sqrt{2}} \left[ \sqrt{\omega_M} Q_{M} \Phi_M - i \frac{1}{\sqrt{\omega_M}} Q_{M} M \Pi_M \right] \\
C_{s, M}^{(0)} &= i \frac{\sqrt{\omega_0}}{2} [C_{0, M}^{(0)} - (C_{0, M}^{(0)})^\dagger] + i \sqrt{2} \frac{\omega_0}{\omega_M} [Q_{s} M^{(0)} C_{M}^{(0)} - (Q_{s} M^{(0)} C_{M}^{(0)})^\dagger] \\
(\omega_M^{(0)})^2 &= - (\partial^{(0)} M)^2 \\
(\partial^{(0)} M f_M)(m) &= (2M)^{-1} [f_M(m + 1) - f_M(m - 1)] \\
Q_{s} M^{(0)} &= \frac{1}{2} \left[ 1_{L_{M}} - i \frac{\partial^{(0)} M}{\omega_M} \right] Q_{M} \\
D_{s, M}^{(0)} &= [C_{s, M}^{(0)}]^2 
\end{align*}
\]

Here the adjoint operation and normal ordering is with respect to the Fock Hilbert space structure \( \mathcal{H}_{K_M}^{(0)} \) defined by the annihilation operators \( C_{0, M}^{(0)}, C_{M}^{(0)} \) with Fock vacuum \( \Omega_{K_M}^{(0)} \). Note that \( Q_{M}^+, Q_{M}^-, i \partial_{M}, \omega_{M} \) are self-adjoint on \( L_{M} \) and that \( Q_{M}^+, Q_{M}^-, Q_{s} M^{(0)} \) are orthogonal projections in \( L_{M} \) with \( Q_{M}^+ Q_{M}^- = Q_{M}^- Q_{M}^+ = 0 \) and \( 1_{L_{M}} = Q_{M}^+ + Q_{M}^- \). Q_{M} = Q_{M}^+ + Q_{M}^-.

An immediate observation is that

\[
Q_{M}^+ \Phi_M = \langle 1, \Phi_M \rangle_{L_{M}} = \frac{1}{M} \sum_m 1(m) \Phi_M(m) = \frac{1}{M} \sum_m (I_{M}^+ \Phi)(m) = \sum_m \langle \chi_m^M, \Phi \rangle_{L} = \langle 1, \Phi \rangle = Q_{M}^+ \Phi
\]

and similarly for \( Q_{M}^- \Pi_M = Q_{M}^- \Pi \) so that in fact

\[
C_{0, M}^{(0)} = C_0
\]
is actually the same as in the continuum in the initial discretisation. We will see that this property is preserved
by the renormalisation flow so that the zero modes remain un-renormalised.

We proceed to the flow equation for the Fock measure. We have

\[
< f_M, \Phi_M >_{L_M} = < Q^M_{L} f_M, \Phi_M >_{L_M} + < Q^M_H f_M, \Phi_M >_{L_M} \\
= < [2\omega_0]^{-1/2} Q^M_{L} f_M, C^{(0)}_{L,M} >_{L_M} + < [2\omega_0]^{-1/2} Q^M_{H} f_M, C^{(0)}_{L,M} >_{L_M} \\
+ < [2\omega_M^{(0)}]^{-1/2} Q^M_{L} f_M, C^{(0)}_{L,M} >_{L_M} + < [2\omega_M^{(0)}]^{-1/2} Q^M_{H} f_M, C^{(0)}_{L,M} >_{L_M} 
\]  
(5.5)

Thus the initial measure family has generating functional of moments

\[
\nu_M^{(0)}(w_M(f_M)) = < \Omega_M^{(0)}, \exp(i < f_M, \Phi_M >) \Omega_M^{(0)} >_{H(0)} \\
= \exp(-\frac{1}{4} < Q^M_{L} f_M, \omega_0^{-1} Q^M_{L} f_M >_{L_M} + Q^M_{H} f_M, [\omega_M^{(0)}]^{-1} Q^M_{H} f_M >_{L_M} 
\]  
(5.6)

It is a family of Gaussian measures with covariances (kernels on \(L_M\))

\[
K^{(0)}_M = \frac{1}{2} [Q^M_{L} \omega_0^{-1} Q^M_{L} + Q^M_{H} [\omega_M^{(0)}]^{-1} Q^M_{H}] 
\]  
(5.7)

This is exactly as for the 1+1 Klein Gordon field treated in the first reference of [6] except that there we assumed
a non-vanishing mass \(p\) so that the projections \(Q^M_{L}, Q^M_{H}\) are not not necessary and the initial covariance is just
\([2\omega_M^{(0)}(p)]^{-1} = [\omega_0^{(0)}]^{-1} \omega_M^{(0)^2 + p^2} \).

To study the flow of (5.7) we can borrow the results of [6] as follows:

In [6] we used the spectral theorem to write

\[
[2\omega_M(p)]^{-1} = \int_\mathbb{R} \frac{dk}{2\pi} [k^2 + (\omega_M^{(0)}(p))^2]^{-1}
\]  
(5.8)

by the residue theorem where due to \(p \neq 0\) there is no real pole of the holomorphic integrand. Here, instead of
integrating over the real line, we consider the path

\[
c_\rho : \mathbb{R} \to \mathbb{C}; c_\rho(k) = \begin{cases} k & |k| > \rho \\
-\rho e^{i \frac{\pi}{2}(\frac{1}{\rho} + 1)} & |k| = \rho 
\end{cases}
\]  
(5.9)

where \(\rho > 0\) is arbitrarily small thus avoiding the real pole \(k = 0\). Then

\[
Q^M [2\omega_M^{(0)}]^{-1} Q^M = \lim_{\rho \to 0+} \int_{c_\rho} \frac{dk}{2\pi} [k^2 + (\omega_M^{(0)})^2]^{-1}
\]  
(5.10)

By the flow equation

\[
\nu_M^{(n+1)}(w_M(f_M)) = \nu_M^{(n)}(w_{M'_M}(I_{MM'}(M) f_M))
\]  
(5.11)

the measure family stays always inside the Gaussian class and (5.12) translates into a flow of covariances

\[
K^{(n+1)} = I_{MM'}(M) K_M^{(n)}(M) I_{MM'}(M)
\]  
(5.12)

where \(M'_M > M\) is the fixed higher resolution that enters the concrete implementation of the blocking
equations. As in [6] we will choose \(M'_M = 2M\) for simplicity.

We note that

\[
Q^M_{L} I_{MM'} f_M = 1, I_{MM'} f_M >_{L_{M'}} = \frac{1}{M'} \sum_{m' \in \mathbb{Z}_{M'}} f_M([M \to M']) \\
= \frac{1}{M'} \sum_{m \in \mathbb{Z}_M} f_M(m)[\sum_{l=0}^{M'/M-1} 1] = \frac{1}{M} \sum_{m \in \mathbb{Z}_M} f_M(m) = < 1, f_M >_{L_M} \\
= Q^M f_M = I_{MM'} Q^M_{L} f_M
\]  
(5.13)
where in the last step we used that $I_{MM'}c = c$ if $c$ is a constant. Thus

$$Q_{\perp}^{M'} I_{MM'} = I_{MM'} Q_{\perp}^{M}$$  \hspace{1cm} (5.14)

i.e. the family of projections $Q_{\perp}^{M}$ is equivariant w.r.t. the coarse graining maps $I_{MM'}$. Similarly

$$Q^{M'} I_{MM'} = (1_{L^M} - Q_{\perp}') I_{MM'} = I_{MM'} - I_{MM'} Q_{\perp}' = I_{MM'} (1_{L^M} - Q_{\perp}') = I_{MM'} Q^{M}$$  \hspace{1cm} (5.15)

It follows from (5.7) and (5.12) that the covariance always takes the form

$$K_{(n)} = \frac{1}{2}[Q_{\perp}^{M} [\omega_{0,M}^{(n)}]^{-1} Q_{\perp}^{M} + Q_{\perp}^{M} [\omega_{M}^{(n)}]^{-1} Q_{\perp}^{M}]$$  \hspace{1cm} (5.16)

in particular the projections $Q^{M}, Q_{\perp}^{M}$ are not changed under the flow. Moreover we have separated the flow

$$[\omega_{0,M}^{(n+1)}]^{-1} = I_{MM'}^{(M)} [\omega_{0,M'}^{(M)}]^{-1} I_{MM'}^{(M)}, \quad [\omega_{M}^{(n+1)}]^{-1} = I_{MM'}^{(M)} [\omega_{M'}^{(M)}]^{-1} I_{MM'}^{(M)},$$  \hspace{1cm} (5.17)

The obvious fixed point of the first equation in (5.17) is

$$[\omega_{0,M}^{(n)}]^{-1} = \omega_{0}^{-1} Q_{\perp}^{M}$$  \hspace{1cm} (5.18)

i.e. the zero modes remain unrenormalised as promised. As for the second equation, we can in view of (5.10) immediately copy the results of [6]: Instead of the parameter $q^2 := k^2 + p^2$ used there we just use $q^2 = k^2$. All other relations remain literally identical. As the flow equations in [6] depend analytically on $q^2$ we infer that the fixed point covariance $\omega_{M}$ is the same as in [6] except that $p = 0$ and that it appears sandwiched between $Q^{M}$

$$K_{M} = \frac{1}{2}[Q_{\perp}^{M} \omega_0^{-1} Q_{\perp}^{M} + Q_{\perp}^{M} \omega_{M}^{-1} Q_{\perp}^{M}]$$  \hspace{1cm} (5.19)

and moreover $K_{M}$ agrees with the covariance obtained by blocking from the continuum.

Next we turn to the smeared constraints. Here we enter new territory as compared to [6], first due to the presence of the projections $Q_{\perp}^{M(0)}$ and second because the constraints do not annihilate the Fock vacuum. We focus just on the part of $D_{s}(f)$ quadratic in the non-zero mode fields as this term by itself also satisfies the Virasoro algebra, see section 2 where this term was denoted by $T_{s}^{2}(f)$, and it is also this term alone which leads to the anomaly. The other terms denoted $T_{s}^{0}(f), T_{s}^{1}(f)$ can be treated by similar methods. We start with the continuum expression and write it in terms of integral kernels

$$D_{s}(F) = \int dx F(x) \int dy \int dz \left[ \kappa_{s}(x; y, z) C(y)^\dagger C(z) + \kappa_{s}^{2}(x; y, z) C(y) C(z) + \kappa_{s}^{2}(x; y, z)^* C(y)^\dagger C(z) \right]$$  \hspace{1cm} (5.20)

where $\kappa_{s}(x; y, z)^* = \kappa_{s}^{1}(x; z, y)$ and $\kappa_{s}^{2}(x; y, z) = \kappa_{s}^{2}(x; z, y)$. We block from the continuum and compute

$$[D_{s}(f)]_{M} := J_{M}^{\dagger} D_{s}(f) J_{M}$$

$$< w_{M}[f_{M}]\Omega_{M}, [D_{s}(F)]_{M} w_{M}[g_{M}]\Omega_{M} >_{\mathcal{H}_{M}} = < w[I_{M} f_{M}]\Omega, D_{s}(F) w[I_{M} g_{M}]\Omega >_{\mathcal{H}}$$  \hspace{1cm} (5.21)

We have for any $f, g$

$$< w[f]\Omega, D_{s}(F) w[g]\Omega >_{\mathcal{H}} = \int dx F(x) \int dy \int dz \left[ \kappa_{s}(x; y, z) < C(y w[f] \Omega, C(z) w[g] \Omega > + \kappa_{s}^{2}(x; y, z) < C(y) C(z) w[f] \Omega, w[g] \Omega > \right]$$  \hspace{1cm} (5.22)

and

$$C(y) w[f] \Omega = w[f] w[f]^{-1} C(y) w[f] \Omega = w[f] (C(x) - i[\phi(f), C(y)]) \Omega = i[C(x), \phi(f)] w[f] \Omega$$
$$= [(2\omega)^{-1/2} Q f](y) w[f] \Omega$$

$$C(y) C(z) w[f] \Omega = [(2\omega)^{-1/2} Q f](z) C(y) w[f] \Omega = [(2\omega)^{-1/2} Q f](z) [(2\omega)^{-1/2} Q f](y) w[f] \Omega$$  \hspace{1cm} (5.23)
Abbreviating $\sigma = (2\omega)^{-1/2}Q$ we thus find
\[< w[f]\Omega, D_s(F) w[g]\Omega > = < w[f]\Omega, w[g]\Omega > \times \]
\[\int dx F(x) \int dy \int dz (\sigma f)(y) (\sigma g)(z) \left[ \kappa_1^1(x; y, z) + \kappa_2^2(x; y, z) + \kappa_3^3(x; y, z) \right] \]
(5.24)

Applied to $f = I_M f_m, g = I_M g_M$ we obtain due to $J^+_M J_M = 1_{\mathcal{H}_M}$
\[< w_M[f_M]\Omega_M, [D_s(F)]_M w_M[g_M]\Omega_M >
= < w_M[f_M]\Omega_M, w_M[g_M]\Omega_M > \times \]
\[\int dx F(x) \int dy \int dz (\sigma I_M f_M)(y) (\sigma I_M g_M)(z) \left[ \kappa_1^1(x; y, z) + \kappa_2^2(x; y, z) + \kappa_3^3(x; y, z) \right] \]
(5.25)
\[= < w_M[f_M]\Omega_M, w_M[g_M]\Omega_M > \times \sum_{m_1, m_2 \in \mathcal{Z}_M} f_M(m_1) g_M(m_2) \int dx F(x) \int dy \int dz \sigma_M(y, m_1) \sigma_M(z, m_2) \times \]
\[\left[ \kappa_1^1(x; y, z) + \kappa_2^2(x; y, z) + \kappa_3^3(x; y, z) \right] \]
\[=: < w_M[f_M]\Omega_M, w_M[g_M]\Omega_M > \times \sum_{m_1, m_2 \in \mathcal{Z}_M} f_M(m_1) g_M(m_2) \int dx F(x) \left[ \kappa_1^1(x; m_1, m_2) + \kappa_2^2(x; m_1, m_2) + \kappa_3^3(x; m_1, m_2) \right] \]
with $\sigma_M(x, m) := (\sigma \chi_m^M(x))$. Now in terms of
\[C_M = \frac{1}{\sqrt{2}}[\sqrt{\omega} M \Phi_M - i \sqrt{\omega}^{-1} Q \Pi_M] \]
(5.26)
where $\omega^{-1}_M$ is the fixed point covariance that we obtained from the flow of the measures and which annihilates $\Omega_M$. We find with the abbreviation $\hat{\sigma}_M = [2\omega_M]^{-1/2}Q^M$ and the Ansatz
\[ [D_s(F)]_M = \sum_{m_1, m_2 \in \mathcal{Z}_M} \int dx F(x) \left[ \hat{\kappa}_1^1(x; m_1, m_2) C_M(m_1) \hat{C}^\dagger_M(m_2) + \hat{\kappa}_2^2(x; m_1, m_2) C_M(m_1) \hat{C}^\dagger_M(m_2) \right] \]
(5.27)
with
\[\hat{\kappa}_1^1(x; \hat{m}_1, \hat{m}_2) = \hat{\kappa}_2^2(x; \hat{m}_1, \hat{m}_1), \quad \hat{\kappa}_2^2(x; \hat{m}_1, \hat{m}_2) = \hat{\kappa}_3^3(x; \hat{m}_2, \hat{m}_1) \]
(5.28)
by the exactly the same calculation
\[< w_M[f_M]\Omega_M, [D_s(F)]_M w_M[g_M]\Omega_M > \mathcal{H}_M \]
\[= < w_M[f_M]\Omega_M, w_M[g_M]\Omega_M > \times \sum_{m_1, m_2 \in \mathcal{Z}_M} f_M(m_1) g_M(m_2) \int dx F(x) \sum_{m_1, m_2} \hat{\sigma}_M(m_1, m_1) \hat{\sigma}_M(m_2, m_2) \left[ \hat{\kappa}_1^1(x; \hat{m}_1, \hat{m}_2) + \hat{\kappa}_2^2(x; \hat{m}_1, \hat{m}_2) + \hat{\kappa}_3^3(x; \hat{m}_1, \hat{m}_2) \right] \]
\[=: < w_M[f_M]\Omega_M, w_M[g_M]\Omega_M > \times \sum_{m_1, m_2 \in \mathcal{Z}_M} f_M(m_1) g_M(m_2) \int dx F(x) \left[ \hat{\kappa}_1^1(x; m_1, m_2) + \hat{\kappa}_2^2(x; m_1, m_2) + \hat{\kappa}_3^3(x; m_1, m_2) \right] \]
(5.29)
Comparing (5.29) and (5.25) we obtain exact match iff for $j = 1, 2$
\[\hat{\kappa}_j^j(x; m_1, m_2) = \kappa_j^j(x; m_1, m_2) \iff \int dy \int dz \hat{\kappa}_j^j(x; y, z) \sigma_M(y, m_1) \sigma_M(z, m_2) \]
\[= \sum_{\hat{m}_1, \hat{m}_2} \hat{\kappa}_j^j(x; \hat{m}_1, \hat{m}_2) \hat{\sigma}_M(\hat{m}_1, m_1) \hat{\sigma}_M(\hat{m}_2, m_2) \]
(5.30)
which determines the discrete kernels \( \hat{\kappa}^j_{s,M}(x; \hat{m}_1, \hat{m}_2) \) in terms of the continuum kernels \( \kappa^j(x; y, z) \).

The question is, whether the flow \( n \mapsto [D^{(n)}(F)]_M \) starting from \( (5.2) \) actually yields this fixed point. Before we answer this question we note that \( (5.2) \) is simply not of the form

\[
\int dx \ F(x) \sum_m E_M(x;m) \ D_{s,M}(m)
\]

which would yield a natural map (kernel) \( E_M; L \mapsto L_M \), see the discussion of item 1., viewpoint i. in section 4.

It is not even of the form

\[
\int dx \ F(x) \sum_{m_1,m_2} E_M(x;m_1,m_2) \ D_{s,M}(m_1,m_2)
\]

in terms of a bi-kernel \( E_M: L \mapsto L_M \times L_M \) because there are three independent monomials of annihilation and creation operators involved, not only one. Thus, blocking from the continuum does not give rise to such a natural kernel or bi-kernel which would allow us to consider the discretised constraints as \( [C(F)]_M \) as smeared with a discretised function or bi-function. However, one may introduce such an interpretation by hand by restricting \( F \) to be of the form \( \hat{I}_M F_M \) where \( \hat{I}_M \) should be sufficiently differentiable and has all the properties of \( I_M \), see again the discussion of item 1., viewpoint ii. in section 4. Such \( \hat{I}_M \) will be indeed be provided in [13].

To study the actual flow of the constraints we note that

\[
\kappa^1_s(x; y, z) = \kappa_s(x, y)^* \kappa_s(x, z), \quad \kappa^2_s(x; y, z) = \kappa_s(x, y) \kappa_s(x, z), \quad \kappa_s(x, y) = [Q_s \sqrt{2\omega}](x, y)
\]

while \( \sigma = (2\omega)^{-1/2} Q \) and \( \sigma_M = \sigma \circ I_M \) so that

\[
\kappa^1_{s,M}(x; m_1, m_2) = \kappa_{s,M}(x, m_1)^* \kappa_{s,M}(x, m_2), \quad \kappa^2_{s,M}(x; m_1, m_2) = \kappa_{s,M}(x, m_1) \kappa_{s,M}(x, m_2), \quad \kappa_{s,M}(x, m) = \sigma_M \circ I_M(x, m)
\]

Accordingly we conclude that

\[
\hat{\kappa}^1_{s,M}(x; m_1, m_2) = \hat{\kappa}_{s,M}(x, m_1)^* \hat{\kappa}_{s,M}(x, m_2),
\]

\[
\hat{\kappa}^2_{s,M}(x; m_1, m_2) = \hat{\kappa}_{s,M}(x, m_1) \hat{\kappa}_{s,M}(x, m_2), \quad \hat{\kappa}_{s,M}(x, m) = [Q_s \sqrt{2\omega_M}](x, m)
\]

because with \( \hat{\sigma}_M = (2\omega_M)^{-1/2} Q_M \) we have

\[
[\hat{\kappa}_{s,M} \circ \hat{\sigma}_M](x, m) = [Q_s I_M Q_M](x, m) = [Q_s I_M](x, m) = [Q_s I_M](x, m) = \kappa_{s,M}(x, m)
\]

To see whether these fixed point values of the kernels are reached from the initial discretisation we write

\[
\hat{\kappa}^{(n)}_{s,M}(x; m_1, m_2) = \hat{\kappa}^{(n)}_{s,M}(x, m_1)^* \hat{\kappa}^{(n)}_{s,M}(x, m_2),
\]

\[
\hat{\kappa}^{(n)}_{s,M}(x; m_1, m_2) = \hat{\kappa}^{(n)}_{s,M}(x, m_1) \hat{\kappa}^{(n)}_{s,M}(x, m_2), \quad \hat{\kappa}^{(n)}_{s,M}(x, m) = [Q_s \sqrt{2\omega_M^{(n)}}](x, m)
\]

and by the literally identical calculation we obtain

\[
\hat{\sigma}^{(n)}_M(x, m) = [2\omega_M^{(n)}]^{-1/2} Q_M
\]

in terms of which the flow equation reads

\[
\sum_{\hat{m}_1, \hat{m}_2} \hat{\kappa}^{j(n+1)}_{s,M}(x; \hat{m}_1, \hat{m}_2) \hat{\sigma}^{j(n+1)}_M(\hat{m}_1, \hat{m}_1) \hat{\sigma}^{j(n+1)}_M(\hat{m}_2, \hat{m}_2)
\]

\[
= \sum_{\hat{m}_1', \hat{m}_2'} \hat{\kappa}^{j(n)}_{s,M'}(x; \hat{m}_1', \hat{m}_2') (\hat{\sigma}^{j(n)}_M \circ I_{MM'})(\hat{m}_1', \hat{m}_1) (\hat{\sigma}^{j(n)}_M \circ I_{MM'})(\hat{m}_2', \hat{m}_2)
\]

which is equivalent to

\[
\hat{\kappa}^{j(n+1)}_{s,M} \circ \hat{\sigma}^{j(n+1)}_M = \hat{\kappa}^{j(n)}_{s,M'} \circ \hat{\sigma}^{j(n)}_M \circ I_{MM'}
\]
or
\[
\hat{k}^{(n+1)}_{s,M} \circ Q_M = \hat{k}^{(n)}_{s,M'} \circ [\omega_{M'}^{(n)}]^{-1/2} \circ I_{MM'}[\omega_{M'}^{(n)}]^{1/2} \circ Q_M
\]  
(5.41)

where the sequence \( n \mapsto \omega_{M}^{(n)} \) was constructed explicitly from the measure flow and satisfies for \( M'(M) = 2M \)
\[
J_{MM'}^{\dagger} [\omega_{M'}^{(n)}]^{-1} I_{MM'}(M) = [\omega_{M}^{(n+1)}]^{-1}
\]
(5.42)

Starting with
\[
\hat{k}^{(0)}_{s,M} = I_M Q_{s,M}^{(0)}[\omega_{M}^{(0)}]^{1/2}
\]
(5.43)

one finds from (5.41) using the consistency of the maps \( I_{M_2M_3} I_{M_1M_2} = I_{M_1M_3} \) for \( M_1 < M_2 < M_3 \)
\[
\hat{k}^{(n)}_{s,M} = I_{2^nM} Q_{s,2^nM}^{(0)} I_{M,2^nM}[\omega_{M}^{(n)}]^{1/2}
\]
(5.44)

Taking the limit \( n \to \infty \) we get, due to limit values \( I_\infty = 1_L, Q_{s,\infty}^{(0)} = Q_s, I_{M,\infty} = I_M, \omega_{M}^{(\infty)} \) formally
\[
\hat{k}^{(\infty)}_{s,M} = \hat{k}_{s,M}
\]
(5.45)

However, it must be shown if and in what sense the sequence (5.44) actually runs into the limit (5.45) which coincides with that blocked from the continuum. This will be done in the next section.

6 Discrete Virasoro Algebra

The current section is the most important one of the present paper as it answers the question whether the continuum algebra is visible at finite resolution, how large its finite resolution anomaly is and in what sense that anomaly is simply a finite resolution artefact and converges to zero as we increase the resolution.

We thus consider the finite resolution \( M \) constraint operators on \( \mathcal{H}_M \)
\[
D_{sM}(F) := J_M^{\dagger} D_s(F) J_M
\]
(6.1)

and compute the finite resolution anomaly
\[
\alpha_M(F; s; G, t) := [D_{sM}(F), D_{tM}(G)] - J_M^{\dagger} [D_s(F), D_t(G)] J_M = -J_M^{\dagger} [D_s(F), P_M^{\perp} D_t(G) - D_t(G) P_M^{\perp} D_s(F)] J_M
\]
(6.2)

where
\[
P_M^{\perp} = 1_\mathcal{H} - P_M, \quad P_M = J_M^{\dagger} J_M = P_M^{\perp} = P_M
\]
(6.3)

is an orthogonal projection thanks to the isometry \( J_M^{\dagger} J_M = 1_{\mathcal{H}_M} \). The finite resolution anomaly vanishes only when the constraint operators preserve the subspaces \( P_M \mathcal{H} \) of \( \mathcal{H} \) which is generically not the case and certainly for PFT it is not.

Heuristically the anomaly vanishes as we increase the resolution \( M \to \infty \) as we expect that \( P_M^{\perp} \to 0 \). The rest of this section is devoted to showing that this is the case rigorously in a suitable operator topology. In fact showing that \( \alpha_M(s, F; t, G) \) as \( M \to \infty \) is a delicate issue and must be defined appropriately. This is because we change the Hilbert space \( \mathcal{H}_M \) on which \( \alpha_M \) is defined. Hence we cannot simply probe the anomaly, say with respect to the weak operator topology on \( \mathcal{H}_M \), that is, fixing \( \psi_M, \psi'_M \in \mathcal{H}_M \), considering the matrix elements
\[
< \psi_M, \alpha_M(s, F; t, G) \psi'_M >_{\mathcal{H}_M}
\]
(6.4)

and taking \( M \to \infty \) at fixed \( \psi_M, \psi'_M \) as these depend themselves on \( M \). However, what we can do is to consider fixed \( \psi, \psi' \in \mathcal{H} \) independent of \( M \) and probe the anomaly with \( \psi_M := J_M^{\dagger} \psi, \psi'_M := J_M^{\dagger} \psi' \). Accordingly we study the large \( M \) behaviour of
\[
< J_M^{\dagger} \psi, \alpha_M(s, F; t, G) J_M^{\dagger} \psi' >_{\mathcal{H}_M}
\]
(6.5)
It will be sufficient to study one of the two terms in \((6.2)\) i.e. the matrix element
\[
< \psi, P_M D_s(F) P_M^\dagger D_t(G) P_M \psi' >_H = < D_s(F) P_M \psi, P_M^\dagger D_t(G) P_M \psi' >_H \tag{6.6}
\]
where used the symmetry of all operators involved.

There are several issues with \((6.6)\) that require clarification: First of all, one would like to take \(\psi, \psi'\) from the dense domain \(\mathcal{D}\) given by the span of the Weyl vectors \(w[f]\Omega\), however, to be useful we need an explicit formula for \(J^\dagger_M \psi\), \(P_M \psi\) for \(\psi \in \mathcal{D}\) which is not available from \([4, 5, 7]\). We derive this formula below. Next, as expected, the range of \(J^\dagger_M \mathcal{D}\) is in \(\mathcal{D}_M\) which is the span of the \(w[I_M f_M]\Omega\) which is dense in \(P_M \mathcal{H}\). However, as \(I_M f_M\) is a step function, it is not clear that \(D_s(F) w[I_M f_M] \Omega\) is well defined, i.e. a normalisable element of \(\mathcal{H}\). It is for this reason that we considered also the case of discontinuous functions \(f\) such as \(I_M f_M\) as the domain of the constraint operators in section 2 and we showed that after suitable regularisation we have indeed \(D_s(F) w[I_M f_M] \Omega \in \mathcal{H}\). Finally, the image of \(\mathcal{D}\) or \(P_M \mathcal{D}\) is not invariant under the constraints so that evaluation of the matrix elements of \(P_M^\dagger \) between vectors in \(\mathcal{D}\) is again not directly possible. In fact, in order to evaluate \(P_M^\dagger\) on \(D_s(F) P_M w[f]\Omega\) one would need to know how to write it as a linear combination of the \(w[g]\Omega\), a task which has no obvious solution. One could think that one can avoid this complication and use the fact that \(\tilde{\psi}\) does not help as \(s,F,\psi,\epsilon\) from \((6.6)\) converges to zero for all \(\psi, \psi'\) with \(\psi, \psi' \in \mathcal{D}\) because \(|P_M^\dagger| = 1\) is bounded. Unfortunately, such \(\tilde{\psi}\) does depend on \(M\) and without explicitly knowing how it does so, it is not possible to estimate the limit of \((6.6)\). The fact that also \(|P_M^\dagger| = 1\) does not help as \(P_M\) stands between \(D_s(F)\) and \(\psi\).

We are therefore forced to have a detailed look at \((6.6)\). A simplification can be obtained by observing that
\[
|< D_s(F) P_M \psi, P_M^\dagger D_t(G) P_M \psi' >| \leq < D_s(F) P_M \psi, P_M^\dagger D_s(F) P_M \psi >^{1/2}
\]
thanks to the CS inequality and the projector property \((P_M^\dagger)^2 = (P_M^\dagger)^\dagger = P_M^\dagger\). Thus \((6.6)\) converges to zero as \(M \to \infty\) for all \(s,F,\psi,\epsilon\) if and only if
\[
< D_s(F) P_M \psi, P_M^\dagger D_s(F) P_M \psi > \tag{6.8}
\]
converges to zero for all \(s,F,\psi \in \mathcal{D}\): That convergence of \((6.6)\) implies convergence of \((6.8)\) follows by choosing \(t = s,G = F,\psi' = \psi\). Next convergence of \((6.8)\) for all \(\psi \in \mathcal{D}\) implies in particular convergence of
\[
< D_s(F) P_M w[f] \Omega, P_M^\dagger D_s(F) P_M w[f] \Omega > \tag{6.9}
\]
for the choice \(\psi = w[f] \Omega\) and conversely convergence of \((6.9)\) implies convergence of \((6.8)\) for finite linear combinations of the \(w[f] \Omega\), that is, general \(\psi \in \mathcal{D}\) again by the CS inequality.

Accordingly we will prove that \((6.9)\) converges to zero. Our first task is to compute \(P_M w[f] \Omega\). We begin by computing \(J^\dagger_M w[f] \Omega\)
\[
< w_M[g_M] \Omega_M, J^\dagger_M w[f] \Omega >_{\mathcal{H}_M} < J_M w_M[g_M] \Omega_M, w[f] \Omega >_{\mathcal{H}}
\]
\[
= < w[I_M g_M] \Omega, w[f] \Omega >_{\mathcal{H}} < \Omega, w[f - I_M g_M] \Omega >_{\mathcal{H}} = \exp(-\frac{1}{2} C(f - I_M g_M, f - I_M g_M)) \tag{6.10}
\]
where we have written out the continuum covariance
\[
2 C = Q^\perp \omega^{-1}_0 Q^\perp + Q \omega^{-1} Q \tag{6.11}
\]
as a symmetric bilinear form on \(L \times L\). We can also consider it as an operator defined by
\[
< f, C g >_{L} = C(f,g) \tag{6.12}
\]
We will make use of these two meanings of \(C\) as appropriate, it is clear from the context which meaning is used respectively. We also remind of the covariance at resolution \(M\)
\[
2 C_M = I_M^\dagger 2 C I_M = Q_M^\perp \omega^{-1}_0 Q^\perp_M + Q_M \omega^{-1}_M Q_M \tag{6.13}
\]
28
where equivariance $Q I_M = I_M Q_M$ was used. Note that both $C, C_M$ considered as operators on $L, L_M$ respectively have, in contrast to $\omega, \omega_M$ an inverse, explicitly

$$\frac{1}{2} C^{-1} = I_M^\dagger 2 C I_M = Q_M^\dagger \omega_0 Q_M^\dagger + Q_M \omega_M Q_M$$

(6.14)

and similar for $C^{-1}$.

We make the Ansatz

$$J_M^\dagger, w[f] \Omega = \kappa_M(f) w_M[f_M(f)] \Omega_M$$

(6.15)

for numbers $\kappa_M(f)$ and vectors $f_M(f) \in L_M$ to be determined. Plugging (6.15) into (6.10) we find

$$\exp(-\frac{1}{2} C(f - I_M g_M, f - I_M g_M)) = \kappa_M(f) \exp(-\frac{1}{2} C_M(f_M(f) - g_M, f_M(f) - I_M g_M))$$

(6.16)

which is uniquely solved by

$$f_M(f) = C_M^{-1} I_M^\dagger C f, \kappa_M(f) = \exp(-\frac{1}{2} C(f, f) - C_M(f_M(f), f_M(f)))$$

(6.17)

Note that $\kappa_M(f)$ can be simplified

$$C_M(f_M(f), f_M(f)) = C(I_M f_M(f), I_M f_M(f)) = \exp(-\frac{1}{2} C_M(f_M(f) - g, f_M(f) - I_M g))$$

(6.18)

It follows

$$P_M w[f] \Omega = \kappa_M(f) w[f_M(f)] \Omega, \ f_M(f) = I_M f_M(f) = (I_M C_M^{-1} I_M^\dagger) C f$$

(6.19)

It is instructive to verify the projection property $P_M^2 = P_M$ and the isometry property $J_M^\dagger J_M = 1_M$ which relies on $\kappa_M(I_M f_M) = 1$ and $f_M(I_M f_M) = I_M f_M$ for any $f_M \in L_M$.

The next task is to compute $D_\delta(F) P_M w[f] \Omega$ which given (6.19) can be done of course using the explicit expression of $D_\delta(F)$ in terms of creation and annihilation operators. However, to be useful, we must write $D_\delta(F) P_M w[f] \Omega$ in the form of linear combinations of $w[h] \Omega$ again because in order to apply $P_M^\dagger$ to it, whose action follows from (6.19), its action is only known in closed form on vectors in $D$ and not on Fock states. The other option would be to expand $P_M w[f] \Omega$ into of Fock states. While this is possible, it leads to very complex expressions. We therefore choose the former route which also has the advantage to maximally benefit from the identity $P_M^\dagger P_M = 0$.

We note that (we pick the $C$ sector for definiteness and focus only on the corresponding contribution to the constraints)

$$w[h] \Omega = \exp(i < h, \Phi >) \Omega = \exp(i < C^{-1/2} h, A_C^+ + A_C >_L) \Omega$$

$$= \exp(-\frac{1}{2} < h, C h > \exp(i < C^{1/2} h, A_C^+ >_L) \Omega$$

(6.20)

using well known Fock space techniques (BCH formula). Here we have denoted the annihilation operator of the $C$ sector by $A_C$ in order not to confuse it with the covariance $C$. Thus we find the functional derivatives

$$\frac{\delta}{\delta h(y)} w[C^{-1} h + g] \Omega = [-g(y) - (C^{-1} h)(y) + i(C^{-1/2} A_C^\dagger)(y)] w[C^{-1} h + g] \Omega$$

$$\frac{\delta^2}{[\delta h(y)] [\delta h(z)]} w[C^{-1} h + g] \Omega = \{ -C^{-1}(y, z) + [-g(y) - (C^{-1} h)(y) + i(C^{-1/2} A_C^\dagger)(y)] [-g(z) - (C^{-1} h)(z) + i(C^{-1/2} A_C^\dagger)(z)] w[C^{-1} h + g] \Omega$$

(6.21)

i.e. at $h = 0$

$$\frac{\delta}{\delta h(y)} w[C^{-1} h + g] \Omega)_{h=0} = [-g(y) + i(C^{-1/2} A_C^\dagger)(y)] w[g] \Omega$$

(6.22)

$$\frac{\delta^2}{[\delta h(y)] [\delta h(z)]} w[C^{-1} h + g] \Omega)_{h=0} = \{ -C^{-1}(y, z) + [-g(y) + i(C^{-1/2} A_C^\dagger)(y)] [-g(z) + i(C^{-1/2} A_C^\dagger)(z)] \} w[g] \Omega$$
Here we used that all expressions just depend on creation operators which mutually commute. Recall the constraint operator

\[-D_s(F) = \int dx \, F(x) \int dy \int dz \left\{ Q_s(x, y) Q_s(x, z) \left( C^{-1/2} A_C \right)(y) \left( C^{-1/2} A_C \right)(z) \right\} \]

(6.23)

\[+ Q_s(x, y) Q_s^*(x, z) \left( C^{-1/2} A_C^\dagger \right)(y) \left( C^{-1/2} A_C^\dagger \right)(z) - 2 Q_s^*(x, y) Q_s(x, z) \left( C^{-1/2} A_C^\dagger \right)(y) \left( C^{-1/2} A_C \right)(z) \}

where \( Q_s(x, y) \) is the integral kernel of the projection \( Q_s \). We have explicitly

\[[C^{-1/2} A_C](y) \, w[g] \Omega = w[g] \, \left( [C^{-1/2} A_C](y) \, w[g] \right) \Omega = w[g] \, \left( [C^{-1/2} A_C](y) - i[\phi[g], (C^{-1/2} A_C)(y)] \right) \Omega \]

(6.24)

whence

\[-D_s(F) \, w[g] \Omega = \int dx \, F(x) \int dy \int dz \left\{ -Q_s(x, y) Q_s(x, z) \, g(y) \, g(z) \right\} \]

(6.25)

\[+ Q_s^*(x, y) Q_s^*(x, z) \left( C^{-1/2} A_C^\dagger \right)(y) \left( C^{-1/2} A_C^\dagger \right)(z) - 2 Q_s^*(x, y) Q_s(x, z) \left( C^{-1/2} A_C^\dagger \right)(y) \left( C^{-1/2} A_C \right)(z) \]

and can be evaluated. Let

\[ \Omega = \int dx \int dy \int dz \left\{ \delta [\delta h(y)] - 2 \delta [\delta h(z)] \right\} \Omega \]

(6.26)

where the terms in (6.25) that do not involve creation operators could be dropped because at \( h = 0 \) we get

\[ P_M^+ \, \delta \, (f_M) \, \Omega = M \, \delta \, \Omega \]

Formula (6.26) is the desired expression because \( P_M^+ \) can be pulled past the functional derivatives where it hits \( w[C^{-1} h + f_M(f)] \Omega \) and can be evaluated. Let \( h' = C^{-1} h \), \( g = f_M(f) \). Then due to the projector property

\[ f_M(g) = g \] and \( \kappa_M(g) = 1 \) whence

\[ P_M^+ \, w[h' + g] \Omega = w[h' + g] \Omega - \kappa_M(h' + g) \, w[f_M(h' + g)] \Omega \]

(6.27)

\[ f_M(h' + g) = f_M(h') + g \]

\[ \kappa_M(h' + g) = \kappa_M(h') \] and \( \kappa_M(g) \) exp\( (<h', C(1 - I_M C^{-1} M^1_M)> \) \( g \) \( \Omega = \kappa_M(h') \)

We can now evaluate (6.8)

\[ \left\| P_M^+ \, D_s(F) \, P_M \, w[f] \Omega \right\|^2 = \int dx \int dy \int dz \int dx' \int dy' \int dz' \left\{ Q_s(x, y) \frac{\delta^2}{\delta h(y)} Q_s(x, z) \frac{\delta^2}{\delta h(z)} \right\} \]

(6.29)

\[ \left\{ Q_s^*(x', y') \frac{\delta^2}{\delta h(y')} Q_s^*(x', z') \frac{\delta^2}{\delta h(z')} \right\} \]

\[
< [w[h'] - \kappa_M(h')] \Omega \, w[f_M(h')] P_M \, w[f] \Omega \Omega >_H \}
\]
with \( g = f^M(h) \), \( h' = C^{-1} h \), \( \hat{h}' = C^{-1} \hat{h} \). We have

\[
< \{ w[h'] - \kappa_M(h') \} \{ w[f^M(h')] \} P_M \{ w[f] \} \Omega > = \kappa_M(f^2) < \{ w[h'] - \kappa_M(h') \} \{ w[f^M(h')] \} w[g] \Omega >
\]

Before evaluating the functional derivatives we can simplify (6.30)

\[
\kappa_M(f^2) < \{ w[h'] - \kappa_M(h') \} \{ w[f^M(h')] \} \{ w[f^M(h')] \} \Omega >
\]

With \( \kappa = \exp(\lambda_\kappa) \), \( \kappa_M = \exp(\lambda_M) \), \( \kappa_M^\dagger = \exp(\lambda_M^\dagger) \) we set

\[
\kappa_M(f^2) \exp(-\frac{1}{2} \lambda_\kappa) < h' - f^M(h') > + < f^M(h') - h' >
\]

Accordingly, (6.30) can be rewritten as (reintroducing \( h = C h' \), \( \hat{h} = C \hat{h}' \))

\[
\kappa_M(f^2) \exp(-\frac{1}{2} \lambda_\kappa) < h - h' > + < f^M(h') - h' >
\]

It will be convenient to define the symmetric kernels \( K = C^{-1}, \Delta K = C^{-1} - I_M C^{-1} I_M \). In carrying out the double, triple and four-fold functional derivative of (6.32) at \( h = \hat{h} = 0 \) we use arguments familiar from Wick’s theorem in perturbative QFT: as (6.32) is a linear combination of two exponentials \( E(H) = \exp(B(H,H)/2) \) of a quadratic polynomial \( B \) in \( H = (h, h') \), their derivatives are schematically

\[
E' = (BH) E, \ E'' = [B + (BH)^2] E, \ E''' = [3 B^2 H + (BH)^3] E,
\]

so that at \( H = 0 \) only second and fourth derivatives survive. To simplify the notation we set

\[
E_{1} := \exp(-\frac{1}{2} \lambda_\kappa) < h - h' >, \ E_{2} := \exp(-\frac{1}{2} \lambda_\kappa) < f^M(h') - h' >, \ E_{j y} := \frac{\delta}{\delta h(y)}, \ E_{j y'} := \frac{\delta}{\delta h(y')},
\]

with \( j = 1, 2 \) and similar for \( z, z' \). Then

\[
(E_{1} E_{2})_{y y'} = E_{1, y y'} E_{2} + E_{1} E_{2, y y'} + E_{1, y} E_{2, y'} + E_{1, y'} E_{2, y}
\]

\[
(E_{1} E_{2})_{yy' z z'} = [E_{1, y y'} E_{2} + E_{1, y} E_{2, y'} + E_{1, y'} E_{2, y}]
\]

where ... denotes odd order derivatives which vanish at \( H = 0 \). We have at \( H = 0 \)

\[
E_{1, y z} = -K(y, z), \ E_{1, y' z'} = -K(y', z'), \ E_{1, y z'} = K(y, z'), \ E_{2, y z} = 0, \ E_{2, y' z'} = 0, \ E_{2, y z'} = [\Delta K](y, z'),
\]

\[
E_{1, y y' z z'} = K(y, z) K(y', z') + K(y, y') K(z, z') + K(y, z') K(z, y') \quad \text{and}
\]

\[
E_{2, y y' z z'} = [\Delta K](y, y') [\Delta K](z, z') + [\Delta K](y, z') [\Delta K](z, y')
\]
Collecting all terms we find at $H = 0$

$$[E_1(1 - E_2)]_{yy'} = [\Delta K](y, y')$$

(6.37)

$$[E_1(1 - E_2)]_{yy'zz'} = K(y, y') \Delta K(z, z') + K(y, z') \Delta K(y, y') + \Delta K(y, z')$$

$$- [\Delta K](y, y') \Delta K(z, z') - [\Delta K](y, z') \Delta K(z, y')$$

(6.38)

where importantly both terms proportional to $E_{1,yy'zz'}$ have cancelled so that all functional derivatives contain at least one factor of $\Delta K$ which we expect to imply the convergence to zero of (6.39) which now can be vastly simplified to

$$\kappa_M(f)^2 \int dx \int dx' F(x) \{ 3 K_s(x, x') \Delta K_s(x, x') - 2 (\Delta K_s(x, x')^2 + 4 g'(x)g'(x') \Delta K_s(x, x')) \}$$

(6.39)

where using $Q_s^*(y, z) = Q_{-s}(y, z) = Q_s(z, y)$

$$K_s(x, x') = \int dy \int dz Q_s(x, y) Q_s(x', z) K(y, z) = [Q_s K Q_s](x, x')$$

(6.40)

and similar for $\Delta K_s(x, x')$. Here

$$g'(x) = [Q I_M C_M^{-1} I_M^1 C f](x) = (Q(K - \Delta K)C f)(x), \kappa_M(x) = \exp(-\frac{1}{2} < Cf, [\Delta K]C f >)$$

(6.41)

Since $P_M$ is a projection we have $||P_M|| = 1$ thus

$$||P_M w[f]\Omega|| = \kappa_M(f) ||w[f^M(f)]\Omega|| = \kappa_M(f) ||P_M|| \leq ||P_M|| ||w[f]\Omega|| = 1$$

(6.42)

and it will be sufficient to show that the integral term in (6.39) converges. Also we focus on $s = +$ the case $s = -$ being completely analogous. Obviously then, the convergence or not of (6.39) rests on the properties of $\Delta K$ and $g'$. We begin with the term

$$\int dx \int dx' F(x) F(x') K_+(x, x') \Delta K_+ (x, x') = \int dx \int dx' F(x) F(x') K_+(x, x') \Delta K_- (x, x')$$

(6.43)

where in the second step we used that $\Delta K(y, z) = [\Delta K](z, y)$ and $Q_s(y, z) = Q^*_s(z, y) = Q_{-s}(z, y)$. We expand into the Fourier basis

$$K_+(x, x') = \sum_{n, n' \in \mathbb{Z}} e_n(x) < e_n, K Q_+ e_{n'} > e_{-n'}(x') = \sum_{n, n' > 0} e_n(x) < e_n, K e_{n'} > e_{-n'}(x')$$

$$[\Delta K]_-(x', x) = \sum_{n, n' < 0} e_n(x') < e_n, [\Delta K] e_{n'} > e_{-n'}(x)$$

$$F(x) = \sum_{|n| < n_0} \hat{F}(n) e_n(x) = F^*(x), \hat{F}^*(n) = \hat{F}(-n)$$

(6.44)

where we assume that $F$ has compact momentum support $|n| < n_0$. Presumably what follows can also be shown under milder decay assumptions on the Fourier modes $\hat{F}(n)$ (e.g. rapid decrease in $n \in \mathbb{Z}$) but we will be satisfied if convergence can be proved for this class of smearing functions of the constraint. Then (6.43) turns into

$$\sum_{|n_1|, |n_2| < n_0} \hat{F}(n_1); \hat{F}^*(n_2) \sum_{m,n > 0} < e_m, K e_n > \sum_{m',n' < 0} [\Delta K]_{m',n'} \delta_{n_1+m-n'} \delta_{-n_2-n+m'}$$

(6.45)

This implies the constraints on the range of $m, n, m', n'$

$$n' = n_1 + m < 0, m' = n_2 + n < 0, m = n' - n_1 > 0, n = m' - n_2 > 0$$

$$\Rightarrow 0 < m < -n_1 < n_0, 0 < n < -n_2 < n_0, 0 > n' > n_1 > -n_0, 0 > m' > n_2 > -n_0$$

(6.46)
thus the compact momentum support propagates to the \( m, n, m', n' \) modes. For bounded values of \( m, n \) the
modulus of the matrix element \(| < e_n, K e_n | \) is uniformly bounded and we are left to study the behaviour of
\(< e_n, [\Delta K] e_{n'} > \) at fixed values of \( n, n' \neq 0 \) (of equal sign). We have

\[
< e_n, [\Delta K] e_{n'} > = 2[\omega(n) \delta_{n,n'} - \sum_{\hat{n} \in Z_M} \omega_M(\hat{n}) < e_n, I_M e_{\hat{n}}^M > < I_M e_{\hat{n}}^M, e_{n'} >
\]

(6.47)

where in the second step we expanded into the spectral basis \( e_{\hat{n}}^M \in L_M \) of \( \omega_M \) given by \( e_{\hat{n}}^M(m) = e_{\hat{n}}(x_m^M), x_m^M = \frac{m}{M}, m \in Z_M \). The eigenvalues \( \omega_M(\hat{n}) \) follow from the definition \( C_M = \hat{I}_M^* C I_M \) i.e.

\[
Q_M^{-1} \omega_M^{-1} Q_M e_{\hat{n}}^M = \sum_{0 \neq n, n' \in Z} \hat{I}_M e_n < e_{n'} > < \omega^{-1}(n) < e_n, I_M e_{n'}^M >
\]

(6.48)

from which

\[
Q_M^{-1} \omega_M^{-1} Q_M e_{\hat{n}}^M = \sum_{l = 0, n, n' \in Z} \hat{I}_M e_n < e_{n'} > < \omega^{-1}(n) < e_n, I_M e_{n'}^M >
\]

(6.49)

Here we need the Fourier modes of the characteristic functions \( \chi_n^M \) of the interval \([x_m^M, x_{m+1}^M)\)

\[
(\hat{I}_M e_n)(m) = M < \chi_n^M, e_n >= M e_n(x_m^M) \frac{e^{ikMn} - 1}{2\pi i n}, k_M = \frac{2\pi}{M}
\]

(6.50)

We note that (6.50) does not have compact momentum support and also does not decay rapidly. This has some
bearing further below. It follows

\[
< e_{\hat{n}}^M, I_M e_n > = < I_M e_{\hat{n}}^M, e_n >
\]

\[
= \sum_{m \in Z_M} \left[ e_{\hat{n}}^M(m) \right]^* e_{\hat{n}}^M(m) \frac{e^{ikMn} - 1}{2\pi i n} = \left[ \sum_{m} e_{\hat{n}-\hat{n}}^M(m) \right] \frac{e^{ikMn} - 1}{2\pi i n}
\]

(6.51)

where \( \hat{n} \in Z_M \) and \( n = \hat{n} + lM, l \in Z \) uniquely decomposes a general integer \( n \) into a multiple \( l \) of \( M \) and a
remainder \( \hat{n} \in Z_M = \{0, 1, \ldots, M - 1\} \). Accordingly

\[
Q_M^{-1} \omega_M^{-1} Q_M e_{\hat{n}}^M = \sum_{\hat{n}'} \hat{n} \in Z_M \sum_{n \neq 0} \omega^{-1}(n) \frac{2M^2 \left[ 1 - \cos(k_M \hat{n}) \right]}{2\pi n^2}
\]

\[
\omega_M(\hat{n})^{-1} = \sum_{l} \omega(n+lM)^{-1} \frac{2\left[1 - \cos(k_M n)\right]}{[k_M(n+lM)]^2} = \omega(n)^{-1} \frac{2\left[1 - \cos(k_M n)\right]}{[k_M n]^2} + \sum_{l \neq 0} \omega(n+lM)^{-1} \frac{2\left[1 - \cos(k_M n)\right]}{[k_M(n+lM)]^2}
\]

(6.52)

whence for \( M > n > 0 \)

\[
\omega_M(n)^{-1} = \sum_{l} \omega(n+lM)^{-1} \frac{2\left[1 - \cos(k_M n)\right]}{[k_M(n+lM)]^2} = \omega(n)^{-1} \frac{2\left[1 - \cos(k_M n)\right]}{[k_M n]^2} + \sum_{l \neq 0} \omega(n+lM)^{-1} \frac{2\left[1 - \cos(k_M n)\right]}{[k_M(n+lM)]^2}
\]

(6.53)

Since \( \omega(n) = 2\pi |n| \), at fixed \( n \) the first term in (6.53) converges to \( \omega^{-1}(n) \) as \( M \to \infty \) while the modulus of
the second is bounded by the series

\[
\frac{4}{(2\pi)^3 M} \sum_{l=1}^{\infty} \frac{1}{[l + \frac{1}{M}]^3} + \frac{1}{[l - \frac{1}{M}]^3} < \frac{4}{(2\pi)^3 M} \sum_{l=1}^{\infty} \frac{1}{l^3} + \frac{1}{[l - \frac{1}{M}]^3}
\]

(6.54)
for \( n < M/2 \) and thus converges to zero as \( M^{-1} \). Accordingly \( \omega_M(n) - \omega(n) = O(1/M) \) at fixed \( n \). Then (6.47) becomes

\[
< e_n, [\Delta K] e_{n'} > = 2[\omega(n) - \omega(n')] \delta_{n,n'} - \sum_{\tilde{n} \in \mathbb{Z}_M} \omega_M(\tilde{n}) \sum_{m_1,m_2} e^{ikMn'-1} \frac{e^{ikMn}}{2\pi in'} \left( \frac{e^{ikMn}}{2\pi in} \right)^* e^{M}_{\tilde{n}-n}(m_1)e^{M}_{m'-\tilde{n}}(m_2)
\]

\[
= 2M^2[\omega(n) - \omega(n')] \delta_{n,n'} - \sum_{\tilde{n} \in \mathbb{Z}_M} \omega_M(\tilde{n}) \delta_{\tilde{n},n} \delta_{\tilde{n},n'} \frac{2[1 - \cos(k_M\tilde{n})]}{(2\pi)^2 nn'}
\]

(6.55)

where \( n = \tilde{n} + lM, \ n' = \tilde{n}' + l'M \) and \( \tilde{n}, \tilde{n}' \in \mathbb{Z}_M \). Since \( 0 < n, n' < -n_0 \) and eventually \( M > n_0 \) we have \( l = l' = -1 \) and \( \tilde{n} = M + n = \tilde{n}', n' = n' + n = \tilde{n} \) therefore \( n = n' \) in the second term of (6.56) and \( \tilde{n} = n + M \)

\[
< e_n, [\Delta K] e_{n'} > = 2\delta_{n,n'} \omega(n) - \omega(M + n) \frac{2[1 - \cos(k_M(n))]}{(k_Mn)^2}
\]

(6.56)

Note that for \( -M < -n_0 < n < 0 \) we have

\[
\omega_M(M+n)^{-1} = \sum_{l \in \mathbb{Z}} \omega(M+n+lM)^{-1} \frac{2[1 - \cos(k_M(M+n))]}{k_M(M+n+lM)^2}
\]

\[
= \sum_{l \in \mathbb{Z}} \omega(n+lM)^{-1} \frac{2[1 - \cos(k_Mn)]}{k_M(n+lM)^2} \to \omega(n)^{-1} = \omega(-n)^{-1}
\]

(6.57)

as \( M \to \infty \). Thus indeed (6.43) converges to zero.

Next consider

\[
\int dx \int dx' F(x) F(x') [\Delta K]_+(x,x') [\Delta K]_+(x',x) = \int dx \int dx' F(x) F(x') [\Delta K]_+(x,x') [\Delta K]_-(x',x)
\]

(6.58)

By the same argument as above, if \( F \) has compact momentum support, then (6.53) is a quadratic polynomial in the \( < e_n, [\Delta K] e_{n'} > \) with \( M \) independent coefficients where either \( n_0 > n, n' > 0 \) or \( -n_0 < n, n' < 0 \) and hence converges to zero.

Finally consider

\[
\int dx \int dx' F(x) g(x)g(x') [\Delta K]_+(x,x')
\]

(6.59)

where

\[
g(x) = [QI_M C^{-1}_M I^*_M C f](x)
\]

(6.60)

We note that \( QI_M = I_M Q_M, \ [C_M, Q_M] = [C, Q] = 0 \) implies that \( I^*_M Q = q - m I^*_M \) whence by the now familiar argument

\[
g(x) = [QI_M C^{-1}_M I^*_M C f](x)
\]

(6.61)

so that

\[
g' = \sum_{n,n' \neq 0} \omega_M(\tilde{n}) e_{n'} < e_n, I_M e^M_{\tilde{n}} > < I_M e^M_{\tilde{n}}, e_n > \omega(n)^{-1} \tilde{f}(n)
\]

\[
= \sum_{l,l' \in \mathbb{Z}} \omega_M(\tilde{n}) e_{\tilde{n}+lM} \frac{2[1 - \cos(k_M\tilde{n})]}{k_M^2(\tilde{n}+lM)(\tilde{n}+l'M)} \omega(\tilde{n}+lM)^{-1} \tilde{f}(\tilde{n}+lM)
\]

(6.62)

It follows that \( g' \) does not have compact momentum support \( n' \) even if \( f \) does. Therefore \( F(x)g'(x) \) also does not have compact momentum support even if \( F \) does. It is not even clear that (6.62) converges. This feature of \( f' \) is again due to the fact that the functions \( \chi^M_{n,0} \) are discontinuous. If one would replace them by \( \chi^M_{n,0} \) where \( \chi^M_{n,0} \) is the Fourier expansion of \( \chi^M_n \) restricted to modes \( |n| < n_0 \) then \( \chi^M_{n,0} \to \chi^M_n \) in the \( L \) norm and if we define \( I^M_{n,0}, [I^M_{n,0}]^\dagger \) like \( I_M, I^*_M \) with \( \chi^M_n \) replaced by \( \chi^M_{n,0} \) and first take the limit \( M \to \infty \) in (6.59) and then \( n_0 \to \infty \) then (6.59) vanishes as \( M \to \infty \). This regularisation using the momentum cut-off \( n_0 \) is similar to the zeta function regularisation of section 3 and is justified by the following argument: while the \( \chi^M_n \) have all
the necessary features in order to define a renormalisation flow, they are not the only choice. There are other, smoother choices \([13]\) satisfying the same necessary requirements but those have a built in compact momentum support of order \(M\). In that case the sum over \(l,l'\) in (6.62) disappears and the compact momentum support of \(f\) propagates to that of \(g\) and then e.g. \(g = Qf\) even exactly for sufficiently large \(M\). Then also \(Fg'\) have compact momentum support and the same argument as was made for (6.43) and (6.53) can be used to show that (6.59) converges to zero without any regulator. Since the choice of the \(\chi^M\) is quite arbitrary subject to a minimal set of requirements and since one wants to probe functions \(f\) of compact momentum support using their \(I_M f\) approximants, such a smooth choice of \(\chi^M\) is simply more convenient. With respect to any choice we have convergence of \(I_M I_M' \rightarrow 1_L\) in the \(L_2\) sense but the finite resolution approximants have additional smoothness or momentum compactness properties while others do not and those additional properties turn out to be important in the present convergence analysis. The strict proof that with the choice of \(I_M\) made in \([13]\) expression (6.59) converges to zero is given in section 5 of \([13]\) and also provides the argument that was missing at the end of the previous section to establish convergence of the flow of constraints.

We conclude this section with the remark that the functions \(\chi^M\) used in \([13]\) are smooth with compact momentum support and that smooth smearing functions \(F, f\) of constraints and Weyl elements respectively are of rapid decrease in the momentum mode label \(n\). Thus with respect to those functions all estimates of this section pass through without any regularisation and convergence is established.

7 Discretised Smearing Functions of the constraints

As we have seen, the embeddings \(J_M\) do not induce a canonical map \(E_M L \rightarrow L_M\) such that (we drop the index \(s\) for the purpose of this section)

\[
D_M(F) := J_M^\dagger D(F) J_M =: \tilde{D}_M(E_M F)
\]

However, we may use the map \(E_M := I_M^\dagger\) to define the family of discretised smearing functions \(F_M := I_M^\dagger F\)

\[
\tilde{D}_M(F_M) := J_M^\dagger D(I_M F_M) J_M = D_M(p_M F)
\]

where

\[
p_M = I_M I_M^\dagger : L \rightarrow L
\]

is a projection due to isometry \(I_M I_M^\dagger = 1_{L_M}\). This defines a consistent family of quadratic forms in the sense that for any \(M < M'\)

\[
J^\dagger_{M,M'} \tilde{D}_M'(I_{M,M'}F_M) J_{M,M'} = \tilde{D}_M(F_M)
\]

with \(I_{M,M'} = I_M^\dagger I_M\) thanks to \(I_{M'} I_{M'}^\dagger = I_M\) and \(J_{M'} J_{M',M'} = J_M\). We can therefore compute

\[
[D_M(F_M), \tilde{D}_M(G_M)] = J_M^\dagger ([D(p_M F), D(p_M G)] - D(p_M F) (1 - P_M) D(p_M G) + D(p_M G) (1 - P_M) D(p_M F)) J_M
\]

and modulo the central term we have in our case

\[
[D(p_M F), D(p_M G)] = D([p_M F, p_M G])
\]

The new semaring function in (7.6) is given by

\[
[p_M F, p_M G] := [p_M F]' [p_M G] - [p_M F][p_M G]' = p_M([p_M F, p_M G]) + (1 - p_M)([p_M F, p_M G])
\]

Thus (7.5) becomes

\[
[D_M(F_M), \tilde{D}_M(G_M)] = \tilde{D}_M(\kappa_M(F_M, G_M))
\]

modulo the central term and the corrections involving \(1_H - P_M\) and \(1_L - P_M\). Here the discretised structure functions are defined by

\[
\kappa_M(F_M, G_M) := I_M^\dagger \kappa(I_M F_M, I_M G_M), \kappa(F, G) = [F, G]
\]
which are well defined if the functions \(\chi_{m}^{M}\) defining \(I_{M}\) are sufficiently differentiable. We have already seen in the previous section that the correction involving \(1_{H} - P_{M}\) converges to zero if \(F\) has compact momentum support. That is no longer the case for \(F\) replaced by \(p_{M}F\) if the functions \(\chi_{m}^{M}\) are step functions but it is the case when those functions themselves have compact momentum support as those in [13]. The functions \(\chi_{m}^{M}\) in general span a closed, finite dimensional subspace \(V_{M} \subset L\) and their derivatives \([\chi_{m}^{M}]^{n}\) may or may not lie in \(V_{M}\) (for the case [13] they actually do). However, the products \(\chi_{m}^{M} [\chi_{m}^{M}]^{n}\) are no longer in \(V_{M}\) so that the term proportional to \(1_{L} - p_{M}\) does not vanish automatically. If however \(F, G\) have compact momentum support then the projections \(p_{M}F\) coincide with \(F\) for sufficiently large \(M\) because \(V_{M}\) roughly involves all Fourier modes up to order \(|n| \leq M\) and thus also \([F, G]\) eventually lies in \(V_{M}\) and the correction involving \(1 - p_{M}\) eventually vanishes.

If \(F, G\) do not have compact momentum support but are smooth then their Fourier transforms are of rapid decrease in the mode label \(n\). In this case the terms involving \(1_{L} - p_{M}\) are not exactly zero for sufficiently large \(M\) but do converge to zero rapidly. Thus we see that with respect to the coarse graining maps of [13] the correction terms of type \(1_{H} - P_{M}, 1_{L} - p_{M}\) of the discrete Virasoro algebra converge to zero in the weak operator topology of \(\mathcal{H}\) and that in particular the central term of the Virasoro algebra is correctly reproduced.

8 Conclusion and outlook

In the present work we have investigated the question whether Hamiltonian renormalisation in the sense of [4, 6, 7, 11], while derived in the context of ordinary Hamiltonian systems, can be “abused” to study also generally covariant Hamiltonian systems with an infinite number of Hamiltonian constraints rather than a single Hamiltonian. We have chosen parametrised field theory on the 1+1 cylinder to test related questions where the exact quantum theory is known.

We have explicitly demonstrated that indeed the general framework of [11] can be applied, although the system does not exhibit a common vacuum vector \(\Omega\) for all constraint operators due to the central term in the Virasoro algebra. The renormalisation flow indeed finds the correct fixed point theory. This enabled us to study the constraint algebra at finite resolution. That finite resolution algebra generically does not close (not even when including the central term). However, it does not close for a simple mathematical reason: The constraints at finite resolution are forced to map states in the Hilbert space of given finite resolution to themselves. However, to achieve closure, matrix elements with states at higher resolution are needed. These are restored as we increase the resolution and explains why the failure of closure is parametrised by the projection \(1_{H} - P_{M}\) where \(P_{M}\) projects on the given finite resolution subspace. In that sense the failure to close does not represent an anomaly but just a finite size artefact. In QFT’s which are not exactly solvable one can distinguish between true anomalies and these artefacts by studying whether their size decreases as we increase the resolution.

In addition we could address the question if and in what sense smearing functions of constraint operators can or should also be discretised when probing them at finite resolution. Namely, while it is not necessary or even natural to do so, one can use the coarse graining map that was employed for reasons of renormalisation also for those smearing functions. This leads to an additional finite size artefact in the finite resolution constraint algebra parametrised by \(1_{L} - p_{M}\) where now \(p_{M}\) projects on smearing functions (rather than Hilbert space states) of finite resolution. This is because the commutator of constraints is smeared by a bilinear expression in two smearing functions and typically derivatives thereof of finite order and those aggregates generically leave the subspace \(p_{M}L\). However, again these corrections converge to zero as we increase the resolution for coarse graining maps with sufficient smoothness.

In the convergence proofs that we supplied it was important that the functions that define the coarse graining maps of the renormalisation flow display sufficient smoothness as otherwise the estimates that were needed do not hold: the Fourier transform of a merely piecewise smooth function is not of rapid decrease and displays the Gibbs phenomenon at the discontinuities [32], i.e. the partial Fourier transform of the function at finite resolution has points within the resolution size away from the discontinuity which differ from the function by a size independent of the resolution.

We will use the lessons learnt for more complicated and physically more interesting constrained QFT such as PFT in higher dimensions and the \(U(1)^{3}\) model for quantum gravity [33] which present the next logical step in the degree of complexity as in these models the constraint algebra (hypersurface deformation algebra) no longer closes with structure constants but only structure functions.
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