On integral representations of $q$-gamma and $q$–beta functions

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Abstract
We study $q$–integral representations of the $q$–gamma and the $q$–beta functions. This study leads to a very interesting $q$–constant. As an application of these integral representations, we obtain a simple conceptual proof of a family of identities for Jacobi triple product, including Jacobi’s identity, and of Ramanujan’s formula for the bilateral hypergeometric series.

1. Introduction

The $q$–gamma function $\Gamma_q(t)$, a $q$–analogue of Euler’s gamma function, was introduced by Thomae [9] and later by Jackson [4] as the infinite product

$$\Gamma_q(t) = \frac{(1-q)^{t-1}}{(1-q)^{t-1}}, \quad t > 0,$$

(1.1)

where $q$ is a fixed real number $0 < q < 1$. Here and further we use the following notation:

$$(a + b)_q^n = \prod_{j=0}^{n-1} (a + q^j b), \quad \text{if } n \in \mathbb{Z}_+,$$

(1.2)

$$(1 + a)_q^\infty = \prod_{j=0}^{\infty} (1 + q^j a),$$

(1.3)

$$(1 + a)_q^t = \frac{(1 + a)^{\infty}_q}{(1 + q^t a)^{\infty}_q}, \quad \text{if } t \in \mathbb{C}. $$

(1.4)

Notice that, under our assumptions on $q$, the infinite product $\Gamma_q(t)$ is convergent. Moreover, the definitions (1.2) and (1.4) are consistent.

Though the literature on the $q$–gamma function and its applications is rather extensive, [2], [3], [4], the authors usually avoided the use of its $q$–integral representation. In fact, each time when a $q$–integral representation was discussed, it was, as a rule, not quite right. The first correct integral representation of $\Gamma_q(t)$ that we know of is in reference [7]:

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q^{-q} d_q x.$$
Here \( E^x_q \) is one of the two \( q \)-analogues of the exponential function:

\[
E^x_q = \sum_{n=0}^{\infty} q^{n(n-1)/2} x^n/n! = (1 + (1 - q)x)_q^\infty, \quad (1.6)
\]

\[
e^x_q = \sum_{n=0}^{\infty} x^n/n! = \frac{1}{(1 - (1 - q)x)_q^\infty}, \quad (1.7)
\]

and the \( q \)-integral (introduced by Thomae \[9\] and Jackson \[5\]) is defined by

\[
\int_0^a f(x) d_q x = (1 - q) \sum_{j=0}^{\infty} a q^j f(aq^j). \quad (1.8)
\]

The \( q \)-beta function was more fortunate in this respect. Already in the mentioned papers by Thomae and Jackson it was shown that the \( q \)-beta function defined by the usual formula

\[
B_q(t, s) = \frac{\Gamma_q(s) \Gamma_q(t)}{\Gamma_q(s + t)}, \quad (1.9)
\]

has the \( q \)-integral representation, which is a \( q \)-analogue of Euler’s formula:

\[
B_q(t, s) = \int_1^0 x^{t-1} (1 - qx)_q^{s-1} d_q x, \quad t, s > 0. \quad (1.10)
\]

Jackson \[5\] made an attempt to give a \( q \)-analogue of another Euler’s integral representation of the beta function:

\[
B(t, s) = \int_0^\infty \frac{x^{t-1}}{(1 + x)^{t+s}} dx. \quad (1.11)
\]

However, his definition is not quite right, since it is not quite equal to \( B_q(t, s) \), as will be explained in Remark \[4.4\]. A correct \( q \)-analogue of \[1.11\] is the famous Ramanujan’s formula for the bilateral hypergeometric series, see \[1\] pp 502–505 (in fact, Ramanujan’s formula was known already to Kronecker).

In the present paper we give a \( q \)-integral representation of \( \Gamma_q(t) \) based on the \( q \)-exponential function \( e^x_q \), and give a \( q \)-integral representation of \( B_q(t, s) \) which is a \( q \)-analogue of \[1.11\]. Both representations are based on the following remarkable function:

\[
K(x, t) = \frac{x^t}{1 + x} \left( 1 + \frac{1}{x} \right)^t q(1 + x)_q^{1-t}. \quad (1.12)
\]

This function is a \( q \)-constant in \( x \), i.e.

\[
K(qx, t) = K(x, t),
\]

and for \( t \) an integer it is indeed independent on \( x \), and is equal to \( q^{t(t-1)/2} \). However, for \( t \in (0, 1) \) this function does depend on \( x \), since for these \( t \) one has

\[
\lim_{q \to 0} K(x; t) = x^t + x^{t-1}.
\]
Our integral representations are as follows:

\[ \Gamma_q(t) = K(A, t) \int_0^{\infty/A(1-q)} x^{t-1} e^{-x} d_q x, \quad (1.13) \]

\[ B_q(t, s) = K(A, t) \int_0^{\infty/A} \frac{x^{t-1}}{(1 + x) q^{t+s}} d_q x, \quad (1.14) \]

where the improper integral, following \[5\] and \[8\], is defined by

\[ \int_0^{\infty/A} f(x) d_q x = (1 - q) \sum_{n \in \mathbb{Z}} \frac{q^n}{A} f\left( \frac{q^n}{A} \right). \quad (1.15) \]

Since \( K(A, t) \) depends on \( A \), we conclude that integrals in both formulas do depend on \( A \). In his formulas Jackson used the factor \( q^{t(t-1)/2} \) in place of \( K(A, t) \), which is correct only for an integer \( t \).

Note also that formula \( (1.5) \) for \( \Gamma_q(t) \) can be written via an improper integral too (since \( E_q^{-a} = 0 \) for \( n \leq 0 \)):

\[ \Gamma_q(t) = \int_0^{\infty/(1-q)} x^{t-1} E_q^{-ax} d_q x. \quad (1.16) \]

We will refer to the book \[6\] for notation and basic facts on \( q \)-calculus. Unfortunately the authors of the book didn’t know about the reference \[7\] and made the same mistake as their predecessors in the definition \( (1.16) \) (taking \( \infty \) in place of \( \infty/(1-q) \), which gives a divergent series). However all their arguments hold verbatim for the definition \( (1.5) \) (or \( (1.16) \)) and are used in the present paper to derive \( (1.13) \) and \( (1.14) \).

In Sections 5 and 6 we will apply equation \( (1.14) \) to find an integral representation of the \( q \)-beta function which is manifestly symmetric under the exchange of \( t \) and \( s \), and to find a \( q \)-analogue of translation invariance of certain improper integrals. Finally, in Section 7 we will show that equation \( (1.13) \) is equivalent to a family of triple product identities, a special case of which is the Jacobi triple product identity:

\[ (1 - q)^\infty (1 - x)^\infty (1 - q/x)^\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n, \quad (1.17) \]

equation \( (1.13) \) is equivalent to the famous Ramanujan’s identity, see \[11\] pp 501–505:

\[ \sum_{n=-\infty}^{\infty} \frac{(1-a)^n}{(1-b)^n} q^n = \frac{(1-q)^\infty (1 - b/a)^\infty (1 - ax)^\infty (1 - q/a)^\infty (1 - b/a)^\infty}{(1 - b)^\infty (1 - q/a)^\infty (1 - x)^\infty (1 - b/a)^\infty}, \quad (1.18) \]

and the symmetric integral representation of the \( q \)-beta function is equivalent to the following identity:

\[ \sum_{n=-\infty}^{\infty} \frac{(1-a)^n (1 - q/a)^n q^n}{(1-b)^n (1-c)^n} = \frac{(1-a)^\infty (1 - q/a)^\infty (1 - bc/q)^\infty}{(1-b)^\infty (1-c)^\infty} \frac{(1-q)^\infty (1 - bc/q)^\infty}{(1-b/a)^\infty (1 - ac/q)^\infty}. \quad (1.19) \]
2. Notation and preliminary results

Throughout this paper we will assume $q$ to be a fixed number between 0 and 1. We denote by $D_q$ the $q$–derivative of a function

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$  

The Jackson definite integral of the function $f$ is defined to be \[5\]

$$\int_{0}^{a} f(x) d_q x = (1 - q) \sum_{n \geq 0} q^n a f(q^n a).$$

Notice that the series on the right–hand side is guaranteed to be convergent as soon as the function $f$ is such that, for some $C > 0$, $\alpha > -1$, $|f(x)| < C x^\alpha$ in a right neighborhood of $x = 0$. Jackson integral and $q$–derivative are related by the “fundamental theorem of quantum calculus”, \[6\] p 73

**Theorem 2.1.** (a) If $F$ is any anti $q$–derivative of the function $f$, namely $D_q F = f$, continuous at $x = 0$, then

$$\int_{0}^{a} f(x) d_q x = F(a) - F(0).$$

(b) For any function $f$ one has

$$D_q \int_{0}^{x} f(t) d_q t = f(x).$$

**Remark 2.2.** It is easy to check the $q$–analogue of the Leibniz rule:

$$D_q (f(x)g(x)) = (D_q f(x))g(x) + f(qx) D_q g(x).$$

An immediate consequence is the $q$–analogue of the rule of integration by parts:

$$\int_{0}^{a} g(x) D_q f(x) d_q x = f(x)g(x) \bigg|_{0}^{a} - \int_{0}^{a} f(x) D_q g(x) d_q x.$$  

One defines the Jackson integral in a generic interval $[a, b]$ by \[5\]:

$$\int_{a}^{b} f(x) d_q x = \int_{0}^{b} f(x) d_q x - \int_{0}^{a} f(x) d_q x.$$  

One also defines improper integrals in the following way \[5\], \[8\]:

$$\int_{0}^{\infty/A} f(x) d_q x = (1 - q) \sum_{n \in \mathbb{Z}} \frac{q^n}{A} f \left( \frac{q^n}{A} \right).$$  

(2.1)
Remark 2.3. Notice that in order the series on the right–hand side to be convergent, it suffices that the function $f$ satisfies the conditions: $|f(x)| < Cx^\alpha$, $\forall x \in [0, \epsilon)$, for some $C > 0$, $\alpha > -1$, $\epsilon > 0$, and $|f(x)| < Dx^\beta$, $\forall x \in [N, \infty)$, for some $D > 0$, $\beta < -1$, $N > 0$. In general though, even when these conditions are satisfied, the value of the sum will be dependent on the constant $A$. In order the integral to be independent of $A$, the anti $q$–derivative of $f$ needs to have limits for $x \to 0$ and $x \to +\infty$.

One has the following reciprocity relations:

$$\int_0^1 f(x)d_qx = \int_0^{\infty/A} \frac{1}{x^2} f\left(\frac{1}{x}\right) d_qx ,$$

$$\int_0^{\infty/A} f(x)d_qx = \int_0^0 \frac{1}{x^2} f\left(\frac{1}{x}\right) d_qx .$$

This is a special case of the following more general change of variable formula, [6, p 107]. If $u(x) = ax^2$, then

$$\int_{u(a)}^{u(b)} f(u)d_qu = \int_a^b f(u(x))D_{q1/\beta}u(x) d_{q1/\beta}x .$$

The $q$–analogue of an integer number $n$ is

$$[n] = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1} .$$

In general we will denote $[t] = \frac{1 - a^t}{1 - a}$ even for a non integer $t$.

The following lemma follows immediately from the definitions of $(a + b)_q^t$ and $(a + b)_q$. The proof is left as an exercise to the reader. See also [6, pp 106-107].

Lemma 2.4. Let $n \in \mathbb{Z}_+$ and $t, s, a, b, A, B \in \mathbb{R}$.

1. $D_qx^t = [t]x^{t-1}$
2. $D_q(Ax + b)_q^n = [n]A(Ax + b)^n_q^{-1}$
3. $D_q(Ax + Bx)_q^n = [n]B(a + Bqx)_q^{n-1}$
4. $D_q(1 + Bx)_q^t = [t]B(1 + Bqx)_q^{t-1}$
5. $D_q\frac{Ax^s}{(1 + Bx)_q^t} = [s]A\frac{Ax^{s-1}}{(1 + Bx)_q^{t-s}} - B([t] - [s])\frac{Ax^s}{(1 + Bx)_q^{t+s}}$
6. $D_q\frac{1 + Ax}{(1 + Bx)_q^t} = [s]A\frac{1 + Aqx}{(1 + Bx)_q^{t-s}} - B[t]\frac{1 + Ax}{(1 + Bx)_q^{t+s}}$
7. $(1 + x)_q^{s+t} = (1 + x)_q^s(1 + q^sx)_q^t$
8. $(1 + x)_q^{-t} = \frac{1}{(1 + q^{-t}x)_q^t}$
9. $(1 + q^sx)_q^t = \frac{(1 + x)^{s+t}}{(1 + x)_q^s} = \frac{(1 + q^sx)_q^s}{(1 + x)_q^t} (1 + x)_q^t$
10. $(1 + q^{-n}x)_q^t = \frac{(x + q^n)_q^n}{(q^nx + q)_q^n} (1 + x)_q^t$
The \(q\)-analogues of the exponential function are given by (1.6) and (1.7). The equality between the series expansion and the infinite product expansion of \(e_q^x\) and \(E_q^x\) (in the domain where both expansions converge) is proved by taking the limit for \(n \to \infty\) in the Gauss and Heine’s \(q\)-binomial formulas, [6, pp 29–32].

Remark 2.5. Notice that for \(q \in (0, 1)\) the series expansion of \(e_q^x\) has radius of convergence \(1/(1-q)\). This corresponds to the fact that the infinite product \(1/(1-(1-q)x)_q^\infty\) has a pole at \(x = 1/(1-q)\). On the contrary, the series expansion of \(E_q^x\) converges for every \(x\). Both product expansions (1.6) and (1.7) converge for all \(x\).

Lemma 2.6. The \(q\)-exponential functions satisfy the following properties:

(a) \(D_q e_q^x = e_q^x\), \(D_q E_q^x = E_q^{qx}\).

(b) \(e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1\).

The proof is straightforward and it is left as an exercise to the reader. See also [4, pp 29–32].

3. Definition of \(q\)-gamma and \(q\)-beta functions

The Euler’s gamma and beta functions are defined as the following definite integrals \((s, t > 0)\):

\[
\Gamma(t) = \int_0^\infty x^{t-1}e^{-x} \, dx , \tag{3.1}
\]

\[
B(t, s) = \int_0^1 x^{t-1}(1-x)^{s-1} \, dx , \tag{3.2}
\]

\[
= \int_0^\infty \frac{x^{t-1}}{(1+x)^{t+s}} \, dx . \tag{3.3}
\]

Notice that (3.3) follows immediately from (3.2) after a change of variable \(x = 1/(1+y)\). From the expression (3.2) it is clear that \(B(t, s)\) is symmetric in \(t\) and \(s\). Recall some of the main properties of the gamma and beta functions:

\[
\Gamma(t+1) = t\Gamma(t) , \quad \Gamma(1) = 1 , \tag{3.4}
\]

\[
B(t, s) = \Gamma(t)\Gamma(s)/\Gamma(t+s) . \tag{3.5}
\]

In this paper we are interested in the \(q\)-analogue of the gamma and beta functions. They are defined in the following way.

**Definition 3.1.** (a) For \(t > 0\), the \(q\)-gamma function is defined to be

\[
\Gamma_q(t) = \int_0^{1/(1-q)} x^{t-1}E_q^{-qx} \, dq , \tag{3.6}
\]

(b) For \(s, t > 0\), the \(q\)-beta function is

\[
B_q(t, s) = \int_0^1 x^{t-1}(1-qx)^{s-1} \, dq . \tag{3.7}
\]
\( \Gamma_q(t) \) and \( B_q(t, s) \) are the “correct” \( q \)-analogues of the gamma and beta functions, since they reduce to \( \Gamma(t) \) and \( B(t, s) \) respectively in the limit \( q \to 1 \), and they satisfy properties analogues to \( \text{(3.4)} \) and \( \text{(3.5)} \). This is stated in the following

**Theorem 3.2.** (a) \( \Gamma_q(t) \) can be equivalently expressed as

\[
\Gamma_q(t) = \frac{(1 - q)^{t - 1}}{(1 - q)^{t - 1}}. \tag{3.8}
\]

In particular one has

\[
\Gamma_q(t + 1) = [t] \Gamma_q(t), \quad \forall t > 0, \quad \Gamma_q(1) = 1.
\]

(b) The \( q \)-gamma and \( q \)-beta functions are related to each other by the following two equations

\[
\Gamma_q(t) = \frac{B_q(t, \infty)}{(1 - q)^t}, \tag{3.9}
\]

\[
B_q(t, s) = \frac{\Gamma_q(t) \Gamma_q(s)}{\Gamma_q(t + s)}. \tag{3.10}
\]

**Proof.** We reproduce here the proof of Kac and Cheung \[6, pp 76–79\], because similar arguments will be used to prove the results in the next section. If we put \( s = \infty \) in the definition of the \( q \)-beta function, use \( \text{(1.6)} \) and the change of variable \( x = (1 - q)y \), we get

\[
B_q(t, \infty) = \int_0^1 x^{t - 1} E_q^y \frac{d_q x}{x} = \frac{1}{(1 - q)^t} \int_0^{1/(1 - q)} y^{t - 1} E_q^{-qy} d_q y = \frac{1}{(1 - q)^t} \Gamma_q(t),
\]

which proves \( \text{(3.9)} \). It follows from \( q \)-integration by parts and Lemma \( \text{2.4} \) (parts \( \text{(3)} \) and \( \text{(4)} \)) that \( B_q(t, s) \) satisfies the following recurrence relations \( (t, s > 0) \):

\[
B_q(t + 1, s) = \frac{[t]}{[s]} B_q(t, s + 1),
\]

\[
B_q(t, s + 1) = B_q(t, s) - q B_q(t + 1, s).
\]

Putting these two conditions together we get

\[
B_q(t, s + 1) = \frac{[s]}{[s + t]} B_q(t, s).
\]

Since clearly \( B_q(t, 1) = \frac{1}{[t]} \), we get, for \( t > 0 \) and any positive integer \( n \),

\[
B_q(t, n) = \frac{[n - 1] \ldots [1]}{[t + n - 1] \ldots [t]} = \frac{(1 - q)^{n - 1}}{(1 - q)^{t - 1}} = \frac{(1 - q)^{n - 1}(1 - q)^{t - 1}}{(1 - q)^{t + n - 1}}. \tag{3.11}
\]
Taking the limit for $n \to \infty$ in this expression we get

$$B_q(t, \infty) = (1 - q)(1 - q)^{t-1}. \,$$

This together with (3.11) proves (3.10). We are left to prove (3.10). By comparing (3.11) and (3.10) we have that (3.10) is true for any positive integer value of $s$. To conclude that (3.10) holds for non integer values of $s$ we will use the following simple argument. If we substitute $a = q^s$ and $b = q^t$ in (3.10) we can write the left–hand side as

$$(1 - q)\sum_{n=0}^{\infty} b^n\frac{(1 - q^n)^{\infty}}{(1 - aq^{n-1})^{\infty}},$$

and the right–hand side as

$$(1 - q)^{\infty}\frac{(1 - q)^{\infty}(1 - ab)^{\infty}}{(1 - a)^{\infty}(1 - b)^{\infty}}.$$  

Both these expressions can be viewed as formal power series in $q$ with coefficients rational functions in $a$ and $b$. Since we already know that they coincide, for any given $b$, for infinitely many values of $a$ (of the form $a = q^n$, with positive integer $n$), it follows that they must be equal for every value of $a$ and $b$. This concludes the proof of the Theorem.

4. An equivalent definition of $q$–gamma and $q$–beta functions

In the previous section the definition of $\Gamma_q(t)$ was obtained from the integral expression (3.1) of the Euler’s gamma function, simply by replacing the integral with a Jackson integral and the exponential function $e^{-x}$ with its $q$–analogue $E_q^{-x}$. It is natural to ask what happens if we use the other $q$–exponential function. In other words, we want to study the following function ($A > 0$):

$$\tilde{\gamma}^{(A)}_q(t) = \int_0^{\infty/A(1-q)} x^{t-1} e_q^{-x}d_qx. \quad (4.1)$$

Similarly, the function $B_q(t, s)$ was obtained by taking the $q$–analogue of the integral expression (3.2) of the Euler’s beta function. We now want to study the $q$–analogue of the integral expression (3.3). We thus define

$$\tilde{\beta}^{(A)}_q(t, s) = \int_0^{\infty/A} x^{t-1}\frac{d_qx}{(1 + x)^{ts}}. \quad (4.2)$$

In this section we will show how the functions $\tilde{\gamma}^{(A)}_q(t)$ and $\tilde{\beta}^{(A)}_q(t, s)$ are related to the $q$–gamma and $q$–beta function respectively.

We want to adapt the arguments in the proof of Theorem 3.2 to the functions $\tilde{\gamma}^{(A)}_q(t)$ and $\tilde{\beta}^{(A)}_q(t, s)$. First, by taking the limit $s \to \infty$ in the definition of
\( \tilde{\beta}_q^{(A)}(t, s) \), using the infinite product expansion of \( e_q^x \) and making the change of variables \( x = (1-q)y \), we get

\[
\tilde{\beta}_q^{(A)}(t, \infty) = \int_0^{\infty} x^{t-1} e_q^{-x} \, dq = \int_0^{\infty} x^{t-1} e_q^{-y} dq \, dy = (1-q)^t \tilde{\gamma}_q^{(A)}(t, \infty).
\]

We therefore proved

\[
\tilde{\gamma}_q^{(A)}(t) = \frac{1}{1-q} \tilde{\beta}_q^{(A)}(t, \infty). \tag{4.3}
\]

We now want to find recursive relations for \( \tilde{\gamma}_q^{(A)}(t) \) and \( \tilde{\beta}_q^{(A)}(t, s) \). By integration by parts we get

\[
\tilde{\gamma}_q^{(A)}(t+1) = q^{-t}[t] \tilde{\gamma}_q^{(A)}(t).
\]

Here we used the fact that \( x^t e_q^{-x} \) tends to zero as \( x \to 0 \) and \( x \to +\infty \) (The second fact follows from Lemma 2.6 (b)). Since obviously \( \tilde{\gamma}_q^{(A)}(1) = 1 \), we conclude that for every positive integer \( n \) (and any value of \( A > 0 \)),

\[
q^{n(n-1)/2} \tilde{\gamma}_q^{(A)}(n) = \Gamma_q(n). \tag{4.4}
\]

Let us now consider the function \( \tilde{\beta}_q^{(A)}(t, s) \). From integration by parts and the results in Lemma 2.4 we get \( t, s > 0 \)

\[
\tilde{\beta}_q^{(A)}(t+1, s) = -\frac{1}{[t+s]} q^{-t} \int_0^{\infty} (qx)^t D_q \frac{1}{(1+x)^{t+s}} \, dq = \frac{1}{[t+s]} \tilde{\beta}_q^{(A)}(t, s) \tag{4.5}
\]

For \( t = 1 \) we have

\[
\tilde{\beta}_q^{(A)}(1, s) = \int_0^{\infty} \frac{1}{(1+x)^{s+1}} \, dq = \frac{1}{[s]} \tag{4.6}
\]

Formulas (4.5) and (4.6) imply \( s > 0, n \in \mathbb{Z}_+ \)

\[
q^{n(n-1)/2} \tilde{\beta}_q^{(A)}(n, s) = (1-q)^{1-q^{-1}(1-q)^n-1} = B_q(n, s). \tag{4.7}
\]

Similarly we have

\[
\tilde{\beta}_q^{(A)}(t+1, s) = \frac{1}{[t+s]} q^s \int_0^{\infty} (qx)^s D_q \frac{x^{t+s}}{(1+x)^{t+s}} \, dq = -q^s \frac{1}{[t+s]} \int_0^{\infty} x^{t+s} \frac{1}{(1+x)^{t+s}} \, dq = \frac{[s]}{[t+s]} \tilde{\beta}_q^{(A)}(t, s). \tag{4.8}
\]

We now need to compute \( \tilde{\beta}_q^{(A)}(t, 1) \). By definition and Lemma 2.4 (5)

\[
\tilde{\beta}_q^{(A)}(t, 1) = \frac{1}{[t]} \int_0^{\infty} D_q \frac{x^t}{(1+x)^t} \, dq. \tag{4.9}
\]
When using the fundamental theorem of $q$-calculus to compute the right-hand side, we have to be careful, since the limit for $x \to +\infty$ of the function $F(x) = \frac{x^t}{(1+x)^t_q}$ does not exist. On the other hand, by definition of $q$-derivative and Jackson integral, we have

$$\int_0^{\infty/A} D_q F(x) d_q x = \lim_{N \to \infty} F\left(\frac{1}{A q^N}\right) - \lim_{N \to \infty} F\left(\frac{q^N}{A}\right),$$

where the limits on the right-hand side are taken over the sequence of integer numbers $N$. We then have from (4.9)

$$\tilde{\beta}_q^{(A)}(t, 1) = \frac{1}{[t]} \left( \lim_{N \to \infty} (A q^N)^t(1 + \frac{1}{A q^N})\right)^{-1}. \quad (4.10)$$

If we denote by $K(A; t)$ the limit in parenthesis in the right-hand side of (4.10), we can use Lemma 2.4 (10) to get

$$K(A; t) = A^t \lim_{N \to \infty} q^{N t} \left(1 + \frac{q^{-N}}{A}\right)^t_q\left(1 + \frac{A}{q} + q\right)^N_q\left(\frac{q^t}{A} + q\right)^N_q^t\left(\frac{1}{A} + q\right)^N_q\left(1 + q^t A\right)\left(1 + q^{1-t} A\right)_{q}^N$$

$$= \frac{1}{1 + A} A^t \left(1 + \frac{1}{A}\right)^t_q \left(1 + q\right)_{q}^t(1 + A)_{q}^{1-t}.$$

From (4.8) and (4.10) we conclude that for any $t > 0$ and positive integer $n$

$$K(A; t)\tilde{\beta}_q^{(A)}(t, n) = (1 - q)\frac{(1 - q)^{n-1}(1 - q)^{t-1}}{(1 - q)^{n+t-1}} = B_q(t, n). \quad (4.11)$$

In the following lemma we enumerate some interesting properties of the function

$$K(x; t) = \frac{1}{1 + x} x^t \left(1 + \frac{1}{x}\right)_q^t(1 + x)_{q}^{1-t}.$$

**Lemma 4.1.** (a) In the limit $q \to 1$ and 0 we have

$$\lim_{q \to 1} K(x; t) = 1, \quad x, t \in \mathbb{R}$$

$$\lim_{q \to 0} K(x; t) = x^t + x^{t-1}, \quad x \in (0, 1), x \in \mathbb{R}.$$

In particular $K(x, t)$ is not constant in $x$.

(b) Viewed as a function of $t$, $K(x; t)$ satisfies the following recurrence relation:

$$K(x; t + 1) = q^t K(x; t).$$

Since obviously $K(x; 0) = K(x; 1) = 1$, we have in particular that for any positive integer $n$

$$K(x; n) = q^{n(n-1)/2}.$$
(c) As function of \(x\), \(K(x; t)\) is a \("q\--constant\", namely

\[ D_q K(x, t) = 0 \quad \forall \ t, x \in \mathbb{R} . \]

In other words \(K(q^n x; t) = K(x; t)\) for every integer \(n\).

**Proof.** The limit for \(q \to 1\) of \(K(x; t)\) is obviously 1. In the limit \(q \to 0\) we have, for any \(\alpha > 0\),

\[
(1 + x)^\alpha = (1 + x)^\alpha \to (1 + x) .
\]

We therefore have, for \(t \in (0, 1): \lim_{q \to 0} K(x; t) = x^t \left(1 + \frac{1}{q} \right) .\) For part (b), it follows from the definition of \(K(x; t)\) and Lemma 4.1 that

\[
K(x; t + 1) = \frac{1}{1 + x} x^{t+1} \left(1 + \frac{1}{x} \right)^{t+1} (1 + x)^{-t}
\]

\[
= x \left(1 + \frac{q^t}{x} \right) (1 + q^{-t} x) K(x; t) = q^t K(x; t) .
\]

For part (c) it suffices to prove that \(K(qx; t) = K(x; t)\). By definition

\[
K(qx; t) = \frac{1}{1 + qx(qx)^t} \left(1 + \frac{1}{qx} \right)^t (1 + qx)^{1-t} .
\]

We can replace in the right–hand side

\[
\left(1 + \frac{1}{qx} \right)^t = \frac{1+q^t}{1+qx} \left(1 + \frac{1}{x} \right)^t ,
\]

\[
(1 + qx)^{1-t} = \frac{1+q^{-1-t}x}{1+x} (1 + x)^{1-t} .
\]

The claim follows from the following trivial identity

\[
\frac{q^t \left(1 + \frac{1}{qx} \right) (1 + q^{-1-t}x)}{(1 + qx) \left(1 + \frac{1}{1+q^{-t}x} \right)} = 1 .
\]

This concludes the proof of the lemma. 

**Remark 4.2.** The function \(K(x; t)\) is an interesting example of a function which is not constant in \(x\) and with \(q\--derivative\) identically zero.

It follows from (4.7), (4.11) and Lemma 4.1 that the functions \(K(A; t)^{\beta_q(A)}(t, s)\) and \(B_q(t, s)\) coincide for any \(A > 0\) as soon as either \(t\) or \(s\) is a positive integer.

We want to prove that they actually coincide for every \(t, s > 0\).

**Theorem 4.3.** For every \(A, t, s > 0\) one has:

\[
K(A; t)^{\gamma_q(A)}(t) = \Gamma_q(t) , \quad (4.12)
\]

\[
K(A; t)^{\beta_q(A)}(t, s) = B_q(t, s) . \quad (4.13)
\]

**Remark 4.4.** This result corrects and generalizes a similar statement of Jackson [5]. There (4.13) is proved in the special case in which \(s + t\) is a positive integer, but, due to a computational mistake, the factor \(K(A; t)\) is missing.
Proof. (4.12) is an immediate corollary of (3.9), (4.3) and (4.11). As in the proof of Theorem 3.2, in order to prove (4.13) it suffices to prove that \( K(A; t) \tilde{\beta}_q^q(A; t, s) \) can be written as formal power series in \( q \) with coefficients rational functions in \( a = q^s \) and \( b = q^t \). After performing a change of variable \( y = Ax \), we get

\[
K(A; t) \tilde{\beta}_q^q(A; t, s) = \frac{1}{1 + A} \left( \frac{1 + \frac{1}{A}}{q} \right)^t (1 + A)^{1-t} \int_0^{\infty/1} \frac{y^{t-1}d_qy}{(1 + y/A)^{1+s}}.
\]

(4.14)

Fix \( A > 0 \). After letting \( b = q^t \), we can rewrite the factor in front of the integral as

\[
\frac{1}{1 + A} \left( \frac{1 + \frac{1}{A}}{q} \right)^\infty (1 + A)^\infty \left( \frac{1 + \frac{2A}{b}}{q} \right)^\infty,
\]

and this is manifestly a formal power series in \( q \) with coefficients rational functions in \( b \). We then only need to study the integral term in (4.14), which we decompose as

\[
\int_0^1 \frac{y^{t-1}}{(1 + y/A)_q^{1+s}}d_qy + \int_1^{\infty/1} \frac{y^{t-1}}{(1 + y/A)_q}d_qy.
\]

(4.15)

After letting \( a = q^s \) and \( b = q^t \) the first term in (4.15) can be written as

\[
(1 - q) \sum_{n \geq 0} b^n \left( \frac{1 + abq^n}{1 + \frac{q^n}{A}} \right),
\]

and this is a formal power series in \( q \) with coefficients rational functions in \( a \) and \( b \). Consider now the second term of (4.15). By relation (2.2) we can rewrite it as

\[
\int_0^q x^{n-1} \frac{1}{x^{1+s}}d_qx.
\]

(4.16)

Recalling the definition of \( K(x; t) \), we have the identity

\[
\frac{1}{x^{1+s}} \left( \frac{1 + \frac{1}{A}}{x} \right)^{x+s} = \frac{1}{1 + Ax} (1 + Ax)^{1-t-s} A^{t+s} K(Ax; t + s).
\]

The main observation is that, even though \( K(Ax; t + s) \) is not constant in \( x \), by Lemma 4.1 \( K(Aq^n; t + s) = K(A; t + s) \), \( \forall n \in \mathbb{Z} \), therefore inside the Jackson integral it can be treated as a constant. Using this fact, we can rewrite (4.16) as

\[
\frac{A^{t+s}}{K(A; t + s)} \int_0^q \frac{1}{1 + Ax} x^{s-1} (1 + Ax)^{1-t-s}d_qx.
\]

(4.17)

After letting \( a = q^s \) and \( b = q^t \) we can finally rewrite the first factor in (4.14) as

\[
\frac{(1 + A)^{\infty} (1 + \frac{2A}{b})^{\infty}}{(1 + \frac{1}{A})^\infty (1 + A)^\infty}.
\]

(4.18)
and the integral term in (4.17) as
\[(1 - q) \sum_{n \geq 0} a^{n+1} \frac{(1 + Aq^{n+2})_q^\infty}{(1 + Aq^{n+2})_q^\infty}. \tag{4.19}\]

Clearly both expression (4.18) and (4.19) are formal power series in \(q\) with coefficients rational functions in \(a\) and \(b\). This concludes the proof of the theorem.

5. Application 1: an integral expression of the \(q\)-beta function manifestly symmetric in \(t\) and \(s\)

Theorem 3.2 implies that \(B_q(t, s)\) is a symmetric function in \(t\) and \(s\). This is not obvious from its integral expression (3.7). We now want to use Theorem 4.3 to find an integral expression for \(B_q(t, s)\) which is manifestly symmetric under the exchange of \(t\) and \(s\). By Theorem 4.3 we have that, for any \(A > 0\)
\[B_q(t, s) = K(A; t) \int_0^\infty \frac{x^{t-1}}{(1 + x)^q} d_q x. \tag{5.1}\]

By definition of \(K(x, t)\) and using the results of Lemma 2.4 we get, after simple algebraic manipulations
\[\frac{1}{x^t}(1 + x)^{t+s} = K\left(\frac{1}{x}; t\right)\left(1 + \frac{q}{q^t x}\right)^t (1 + q^t x)^s. \tag{5.2}\]

Since by Lemma 4.1 we have
\[K\left(\frac{1}{x}; t\right) = K(A; t), \quad \forall x = \frac{q^n}{A}, \; n \in \mathbb{Z}, \]
when we substitute (5.2) back into (5.1) we get, after a change of variable \(y = q^t x\),
\[B_q(t, s) = \int_0^\infty \frac{1}{y^{1/\alpha}} \frac{d_q y}{(1 + \frac{y}{q^{t+s}})(1 + y)^s_q}, \; \forall \alpha > 0. \tag{5.3}\]

To conclude, we just notice that this integral expression of \(B_q(t, s)\) is manifestly symmetric in \(t\) and \(s\), since performing the change of variable \(x = \frac{y}{q}\) (namely applying the reciprocity relation (2.2) gives the same integral with \(\alpha\) replaced by \(1/\alpha\) and \(s\) replaced by \(t\).

6. Application 2: translation invariance of a certain type of improper integrals

One of the main failures of the Jackson integral is that there is no analogue of the translation invariance identity
\[\int_0^a f(x) dx = \int_c^{a+c} f(x-c) dx, \]

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obviously true for “classical” integrals. By using Theorem 4.3 we are able to write a $q$-analogue of translation invariance for improper integrals of a special class of function, namely of the form $x^\alpha/(1 + x)^\beta_q$. More precisely we want to prove the following

**Corollary 6.1.** For $\alpha > 0$ and $\beta > \alpha + 1$ we have

$$
\int_0^{\infty/q} \frac{x^\alpha}{(1 + x)^\beta_q} dx = q^{\alpha} K(\alpha, \alpha) \int_1^{\infty/q} \frac{x^\alpha}{x^{\beta_q}} d_q x .
$$

(6.1)

**Remark 6.2.** In the “classical” limit $q = 1$, the right-hand side is obtained from the left-hand side by translating $x \rightarrow x - 1$.

**Proof.** From the definition of $B_q(t, s)$ we have

$$
B_q(t, s) = \int_0^1 x^{s-1} (1 - qx)_q^{t-1} d_q x = \int_q^1 \frac{1}{x^{s+t+1}} \left(1 - \frac{q}{x}\right)_q^{t-1} d_q x = \frac{1}{q^s} \int_1^{\infty/q} x^{t-1} \left(1 - \frac{1}{x}\right)_q^{t-1} x^{s+t} d_q x .
$$

(6.2)

The first identity was obtained by applying (2.2) and the second by a change of variable $y = x/q$. From Theorem 4.3 we also have

$$
B_q(t, s) = K(A; t) \int_0^{\infty/A} x^{t-1} (1 + x)^{t\beta} d_q x .
$$

(6.3)

Equation (6.1) is obtained by comparing (6.2) and (6.3), after letting $\alpha = t - 1$, $\beta = t + s$ and using the fact that $K(A; \alpha + 1) = q^{-\alpha} K(A; \alpha)$.

7. Application 3: identities

If we rewrite equations (4.12), (4.13) and (5.3) using the definition of improper integrals, we get some interesting identities involving $q$-bilateral series.

After using the infinite product expansion (1.7) of the $q$-exponential function $e^x_q$, the expression (1.8) of the $q$-gamma function, the definition (1.15) of the improper Jackson integral and simple algebraic manipulations, we can rewrite equation (4.12) as

$$
(1 - q)_q (1 + qt/A)_q (1 + qA/qt)_q = (1 + qA)_q (1 - qt)_q \sum_{n=-\infty}^{\infty} q^n (1 + 1/A)_q^n .
$$

(7.1)

If we let $x = -qt/A$ in equation (4.12) we get

$$
(1 - q)_q (1 - x)_q (1 - qx)_q = (1 + qA)_q (1 + Ax)_q \sum_{n=-\infty}^{\infty} (-x)^n A^n (1 + 1/A)_q^n .
$$

(7.2)
This is a 1-parameter family of identities for the Jacobi triple product \((1 - q)_q^\infty (1 - x)_q^\infty (1 - q/x)_q^\infty\), parametrized by \(A\). Notice that
\[
\lim_{A \to 0} A^n (1 + 1/A)_q^n = q^{n(n-1)/2}.
\]
This implies that, in the special case \(A = 0\), equation (7.2) reduces to the famous Jacobi triple product identity (1.17).

Let’s consider now equation (4.13). After using the definition (1.15) of improper \(q\)-integral, the expression (1.9) for the \(q\)-beta function and simple algebraic manipulations, we can rewrite it as
\[
\sum_{n = -\infty}^{\infty} \frac{(1 + 1/A)_q^n}{(1 + q^{1+s}/A)_q^n} q^{tn} = \frac{(1 - q)_q^\infty (1 - q^{1+s})_q^\infty (1 + q^t/A)_q^\infty (1 + qA/q^t)_q^\infty}{(1 + q^{1+s}/A)_q^\infty (1 + qA)_q^\infty (1 - q^t)_q^\infty (1 - q^s)_q^\infty}.
\] (7.3)
Notice that, after letting \(a = -1/A, b = -q^{1+s}/A, x = q^t\), equation (7.3) is equivalent to the famous Ramanujan’s identity (1.18). In other words, the proof of Theorem 4.3 in Section 4 can be viewed as a new, more conceptual proof of Ramanujan’s identity.

Finally we can rewrite equation (5.3) as
\[
\sum_{n = -\infty}^{\infty} \frac{(1 + 1/\alpha)_q^n (1 + q\alpha)^{-n}}{(1 + q^s/\alpha)_q^n (1 + q^{1+s}\alpha)^{-n}} = \frac{(1 + 1/\alpha)_q^\infty (1 + q\alpha)_q^\infty (1 - q)_q^\infty (1 - q^{1+s})_q^\infty}{(1 + q^s/\alpha)_q^\infty (1 + q^{1+s}\alpha)_q^\infty (1 - q^s)_q^\infty (1 - q^t)_q^\infty}.
\] (7.4)
After letting \(a = -1/\alpha, b = -q^s/\alpha, c = -q^{1+s}\alpha\), equation (7.4) reduces to (1.19).

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