LIE DIMENSION SUBRINGS

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Abstract. We compare, for $L$ a Lie ring over the integers, its lower central series $(\gamma_n(L))_{n\geq 1}$ and its dimension series defined by $\delta_n(L) := L \cap \varpi^n(L)$ in the universal enveloping algebra of $L$. We show that $\gamma_n(L) = \delta_n(L)$ for all $n \leq 3$, but give an example showing that they may differ if $n = 4$. We introduce simplicial methods to describe these results, and to serve as a possible tool for further study of the dimension series.

1. Introduction

Let $G$ be a group and let $\mathbb{Z}[G]$ be its integral group ring with augmentation ideal $\varpi(G)$. Two series of subgroups may be considered: the first, purely group-theoretical, is the lower central series $\gamma_n(G)$ defined inductively by setting $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$. The second, more algebraic, is the dimension series $\delta_n(G) = G \cap (1 + \varpi(G))^n$. One always has $\gamma_n(G) \leq \delta_n(G)$, and in fact the quotient $\delta_n(G)/\gamma_n(G)$ is a torsion abelian group [10] whose exponent is bounded by a function of $n$ only [39]. However, its precise structure, despite extensive investigation, is still not completely understood question (see [13, 28, 30]) known as the $\delta$ subgroup problem.

The aim of the present paper is to study an analogous problem for Lie rings and their universal enveloping algebras. On the one hand, the arguments in the case of groups have a strong Lie-theoretical flavour, so it seems desirable to cast them in their natural environment. On the other hand, there is a classical construction of a Lie ring (over $\mathbb{Z}$) out of a group, due to Magnus [26], see also [23]. As an abelian group, it is the direct sum $\text{gr}(G)$ of successive quotients $\gamma_n(G)/\gamma_{n+1}(G)$; the Lie bracket comes from the group commutator on homogeneous elements. We attempt to establish a different link between groups and Lie algebras. Finally, we believe that Lie algebras are important objects to study in their own right.

If $L$ be a Lie ring over a commutative ring $\mathbb{k}$ with identity, $\mathcal{U}(L)$ its universal enveloping algebra and $\varpi(L)$ the augmentation ideal of $\mathcal{U}(L)$, then we have, for every integer $n \geq 1$, a Lie subring $\delta_n(L) := L \cap \varpi^n(L)$ of $L$, called the $n$th Lie dimension subring of $L$. Once again the $n$th lower central Lie subring $\gamma_n(L)$ of $L$ is always contained in $\delta_n(L)$, and there arises the problem of identifying the quotient $\delta_n(L)/\gamma_n(L)$. While a complete answer to this problem is known in case $\mathbb{k}$ is a field (see [22, 19]), the “universal” case $\mathbb{k} = \mathbb{Z}$ does not seem to have been investigated so far.

1.1. Main results. Let $L$ be a Lie ring, and consider the lower central series $(\gamma_n(L))_{n\geq 1}$ and dimension series $(\delta_n(L))_{n\geq 1}$. The terms $\gamma_1(L)$ and $\delta_1(L)$ are by definition equal. We prove in Theorem 1.3 that $\gamma_2(L) = \delta_2(L)$, and in Theorem 1.5 that $\gamma_3(L) = \delta_3(L)$. We show by an example that they can differ for $n = 4$, see Theorem 1.12 in that case, nevertheless, $2\delta_4(L) \subseteq \gamma_4(L)$, see Corollary 1.11.
Given a free presentation $0 \to R \to F \to L \to 0$ of a Lie ring $R$, one may ask, in analogy with the Fox subgroup problem [3,13], for the identification of the Lie subrings $F \cap \varpi^n(F)\tau$, with $\tau$ the two-sided ideal generated by $R$, in the universal enveloping algebra $\mathcal{U}(F)$ of $F$. This too is going to be a challenging problem; for, a simple example shows that, unlike in the case of groups, $(F \cap \varpi^n(F)\tau)/[R, R]$ can be non-zero.

We eschew the problem of identifying $F \cap \varpi^n(F)\tau$, setting $M = F \cap \varpi^n(F)\tau$, and derive some results relating $F/M$ to the universal enveloping algebra $\mathcal{U}(F/R)$, motivated by their group-theoretic counterpart. We show, under the assumption that $R/M$ has trivial annihilator in $\mathcal{U}(F/R)$, that $\bigcap_{n \geq 1} \varpi^n(F/\mathcal{U}) = 0$ if and only if $\bigcap_{n \geq 1} \gamma_n(F/M) = 0$. The assumption on $R/M$ always holds if $F$ is a Lie algebra over a field.

Finally, we develop simplicial methods, analogous to those in [13,28], to investigate Lie dimension subrings. We obtain in this manner a “conceptual” proof of Theorem 4.5, but we also expect these methods to bear more fruits in the future.

2. Notation

The following notation will be used throughout the text:

- $k$ = a commutative ring with identity
- $\mathbb{N}$ = the natural numbers $\{0, 1, \ldots\}$
- $L$ = a Lie ring over $k$
- $\gamma_n(L)$ = the $n$th term in the lower central series of $L$. It is defined inductively by $\gamma_1(L) = L$ and, for $n \geq 1$, by letting $\gamma_{n+1}(L)$ be the $k$-submodule of $L$ generated by all elements of the form $[x, y] = xy - yx$ with $x \in L, y \in \gamma_n(L)$.
- $\Gamma_n(L)$ = the quotient $L/\gamma_n(L)$
- $L''$ = the second derived Lie subring $[[L, L], [L, L]]$
- $\mathcal{T}(A)$ = the tensor algebra over the $k$-module $A$
- $\mathcal{L}(A)$ = the free Lie algebra over the $k$-module $A$; it is a subspace of $\mathcal{T}(A)$
- $\text{Sym}(A)$ = the quotient $\mathcal{T}(L)/\langle x \otimes y - y \otimes x : x, y \in L \rangle$, the symmetric algebra of $L$
- $\mathcal{U}(L) = \mathcal{T}(L)/\langle x \otimes y - y \otimes x - [x, y] : x, y \in L \rangle$, the universal envelope of $L$
- $\mathcal{U}_n(L)$ = the homogeneous component of degree $n$ in $\mathcal{U}(L)$
- $\text{gr}(L) = \bigoplus_{i=1}^{\infty} \gamma_n(L)/\gamma_{n+1}(L)$, the associated graded Lie ring with $[\tilde{x}_i, \tilde{x}_j] = [x_i, x_j] + \gamma_{i+j+1}(L)$, for $\tilde{x}_i = x_i + \gamma_{i+1}(L)$, $\tilde{x}_j = x_j + \gamma_{j+1}(L)$, $x_i \in \gamma_i(L), x_j \in \gamma_j(L)$.
- $\varpi(L) = \gamma_n(L)/\gamma_{n+1}(L)$ for $n \geq 1$
- $\varpi(L)$ = the augmentation ideal of $\mathcal{U}(L)$, namely, the two-sided ideal of $\mathcal{U}(L)$ generated by $\iota(L)$
- $\mathcal{U}(\text{gr}(L)) = \bigoplus_{n=0}^{\infty} \varpi^n(L)/\varpi^{n+1}(L)$, the associated graded ring of $\mathcal{U}(L)$ arising from its $\varpi(L)$-adic filtration
- $\mathcal{U}(\text{gr}(L))$ = the universal envelope of the graded Lie ring $\text{gr}(L)$
- $\delta_n(L) = L \cap \varpi^n(L)$ for $n \geq 1$, the $n$th Lie dimension subring of $L$.

For basic properties of Lie algebras, the reader may refer to the classic [20].

3. Lie Lower Central and Dimension Subrings

We begin by listing some properties of the Lie lower central and dimension subrings which are easily verified.

**Proposition 3.1.** For every $m, n \geq 1$, we have

(i) $[\delta_m(L), \delta_n(L)] \subseteq \delta_{m+n}(L)$;
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(ii) $\delta_n(L)$ is a Lie ideal of $L$;  
(iii) $\gamma_m(L) \subseteq \delta_m(L)$. □

The homomorphism $\iota : L \to \mathcal{U}(L)$ is not, in general, a monomorphism [2, 37]; however, it is known to be so in the following cases:

Theorem 3.2 (Poincaré-Birkhoff-Witt; see [19] for a unified proof). The Lie homomorphism $\iota : L \to \mathcal{U}(L)$ is injective if either

(i) $L$ is a free $k$-module [1, 44];
(ii) $k$ is a Dedekind domain [2, 24];
(iii) $k$ is an algebra over the rationals [4].

In fact, in all these cases the stronger result holds, that the natural map $\text{Sym}(L) \to \text{gr}\mathcal{U}(L)$ is an isomorphism of $L$-modules.

There exists a canonical graded Lie ring homomorphism

$$\text{gr}(\iota) : \text{gr}(L) \to \text{gr}\mathcal{U}(L), \quad \bar{x}_n \mapsto x_n + \mathfrak{m}^{n+1}(L) \text{ for } x_n \in \gamma_n(L).$$

Consequently, we have an induced $k$-algebra homomorphism

$$\iota^* : \mathcal{U}(\text{gr}(L)) \to \text{gr}\mathcal{U}(L).$$

Theorem 3.3 (Riley [33]; Knus [22]). If $k$ be a field of characteristic zero and $L$ be a Lie algebra over $k$, then

(i) $\iota^* : \mathcal{U}(\text{gr}(L)) \to \text{gr}\mathcal{U}(L)$ is an isomorphism;
(ii) $\delta_n(L) = \gamma_n(L)$ for all $n \geq 1$.

Similar results hold, with universal envelope (respectively Lie ring) replaced by restricted universal envelope (respectively restricted Lie ring), if $k$ is a field of prime characteristic; see [34, 35]. We are thus led to the following

Problem 3.4. If $L$ be a Lie ring over a commutative ring $k$ with identity, identify the quotients $\delta_n(L)/\gamma_n(L)$, for all $n \geq 1$.

In this paper we limit ourselves to the consideration of the case when $k$ is $\mathbb{Z}$, the ring of integers.

4. THE DIMENSION PROBLEM FOR LIE RINGS OVER $\mathbb{Z}$

We begin by considering free Lie rings over $\mathbb{Z}$. It may be recalled that any subalgebra of a free Lie algebra over a field is itself free (see [33, 34]). On the other hand, a subring of a free Lie ring over $\mathbb{Z}$ is, in general, not free; see [43] for a counter-example. See [32] for an exhaustive treatise on free Lie algebras, and [6] as a reference for universal enveloping algebras.

If $F$ be a free Lie ring over $\mathbb{Z}$, then $\gamma_n(F)/\gamma_{n+1}(F)$ is a free abelian group (see [17] for an explicit basis). Consequently, we have the counterpart of the fundamental theorem of free group rings ([13, Theorem 3.7]):

Theorem 4.1. If $F$ be a free Lie ring over $\mathbb{Z}$, then $F \cap \mathfrak{w}^n(F) = \gamma_n(F)$ for all $n \geq 1$.

Proof. Let $F$ be free on the subset $X$. The universal enveloping algebra $\mathcal{U}(F)$ is isomorphic to the algebra of non-commuting polynomials over $X$, and the Lie subalgebra of $\mathcal{U}(F)$ generated by $X$ is free, by Theorem [32,4]. It is a graded subalgebra, so the algebras $F, \text{gr}(F) = \bigoplus_{n \geq 1} \gamma_n(F)/\gamma_{n+1}(F)$ and $\bigoplus_{n \geq 1} (F \cap \mathfrak{w}^n(F))/(F \cap \mathfrak{w}^{n+1}(F))$ are all isomorphic. □
We now fix a free presentation of the Lie ring $L$, namely, an exact sequence

$$0 \longrightarrow R \longrightarrow F \overset{p}{\longrightarrow} L \longrightarrow 0$$

of Lie rings with $F$ a free Lie ring and $R$ a Lie ideal in $F$. Then we have the following

**Proposition 4.2** (See [3, §2.2.15]). $\mathcal{U}(L) \cong \mathcal{U}(F)/\mathfrak{t}$, where $\mathfrak{t}$ is the two-sided ideal generated by $R$ in $\mathcal{U}(F)$.

**Proof.** Recall the natural Lie homomorphism $\iota : L \rightarrow \mathcal{U}(L)$. The Lie homomorphism $\iota \circ p : F \rightarrow \mathcal{U}(L)$ induces a homomorphism $\theta : \mathcal{U}(F) \rightarrow \mathcal{U}(L)$ of associative algebras which clearly vanishes on the ideal $\mathfrak{t}$. Thus we have a homomorphism $\overline{\theta} : \mathcal{U}(F)/\mathfrak{t} \rightarrow \mathcal{U}(L)$ of associative algebras.

On the other hand, the map $F \rightarrow \mathcal{U}(F)/\mathfrak{t}$ defined by $f \mapsto f + \mathfrak{t}$ vanishes on $R$, and consequently induces a Lie homomorphism $\varphi : L \rightarrow \mathcal{U}(F)/\mathfrak{t}$. Therefore, we have a homomorphism

$$\overline{\varphi} : \mathcal{U}(L) \rightarrow \mathcal{U}(F)/\mathfrak{t}$$

which maps $w \in L$ to $f + \mathfrak{t}$ whenever $p(f) = w$. Clearly $\overline{\theta}$ and $\overline{\varphi}$ are inverses of each other, so the proof is complete. \hfill $\square$

Let $L$ be a Lie ring and consider $w \in L \cap \varpi^n(L)$ for some $n \geq 1$. Choose $f \in F$ with $p(f) = w$. Then, in view of Proposition 4.2, we have $f \in \varpi^n(F) + \mathfrak{t}$. Observe that

$$\varpi^n(F) + \mathfrak{t} = \varpi^n(F) + \varpi(F)\mathfrak{t} + R.$$ 

Therefore, there exists $r \in R$ with $f + r \in \varpi^n(F) + \varpi(F)\mathfrak{t}$. Consequently, in order to determine $L \cap \varpi^n(L)$, it suffices to determine

$$F \cap (\varpi^n(F) + \varpi(F)\mathfrak{t}).$$

Since $F \cap \varpi^2(F) = \gamma_2(F)$ by Theorem 4.1 and $\mathfrak{t} \subseteq \varpi(F)$, we immediately have the following

**Theorem 4.3.** For every Lie algebra $\mathcal{L}$ over $\mathbb{Z}$, we have $\delta_2(L) = \gamma_2(L)$. \hfill $\square$

The following result parallels Gupta-Kuzmin’s [10].

**Proposition 4.4.** If $L$ be a Lie algebra over $\mathbb{Z}$, then $\delta_n(L)/\gamma_{n+1}(L)$ is abelian for all $n \geq 1$.

**Proof.** Let $L$ be a Lie algebra over $\mathbb{Z}$, and assume that $L$ is nilpotent of class $n$, namely, $\gamma_{n+1}(L) = 0$. Let $A$ be a maximal abelian ideal in $L$.

We first show that $A$ equals its centralizer $Z_L(A)$. Indeed, if $Z_L(A) > A$, choose $\mathfrak{g} \neq 0$ in the centre of $Z_L(A)/A$, and let $x$ denote a lift to $Z_L(A) \setminus A$. Then $A + \mathbb{Z}x$ is a larger abelian ideal, contradicting the maximality of $A$. We naturally view $A$ as a right $\mathcal{U}(L)$-module, writing the action $[-, -]$. Then $[a, x] \in \gamma_{k+1}(L)$ for all $a \in A$, $x \in \varpi^k(L)$. In particular, every $x \in \delta_n(L)$ belongs to $Z_L(A)$, and therefore to $A$, so $\delta_n(L)$ is abelian. \hfill $\square$

4.1. $n = 3$. We next proceed to examine the third and the fourth Lie dimension subrings. Consider a Lie ring

$$L = \langle X_1, X_2, \ldots, X_m | e_1X_1 + \xi_1, e_2X_2 + \xi_2, \ldots, e_mX_m + \xi_m, \xi_{m+1}, \ldots \rangle$$

given by its preabelian presentation. This is a presentation making apparent the elementary divisors $e_1, \ldots, e_m$ of $L/[L, L]$. Let $F$ be the free Lie ring generated by $X_1, \ldots, X_m$, and let $R$ be the ideal of $F$ generated by $e_1X_1 + \xi_1, e_2X_2 + \xi_2, \ldots, e_mX_m + \xi_m, \xi_{m+1}, \ldots$, where $e_1|e_2| \ldots |e_m$ are integers $\geq 0$ and $\xi_1, \ldots, \xi_m, \ldots$,
are certain elements of $\gamma_2(F)$. Thus $L = F/R$. Let $r$ be the two-sided ideal generated by $R$ in $\mathcal{W}(F)$, and let $s$ denote the two-sided ideal of $\mathcal{W}(F)$ generated by $\{e_1X_1, \ldots, e_mX_m\} \cup \gamma_2(F)$; thus $s = r + \mathcal{W}(F)\gamma_2(F)$. For notational brevity, we write $\varpi$ for $\mathcal{W}(F)$.

**Theorem 4.5.** For every Lie algebra $L$ over $\mathbb{Z}$, we have $\delta_3(L) = \gamma_3(L)$.

**Proof.** Consider $w \in (\varpi^n + r) \cap F$ representing an element of $\delta_3(L)$. Then, for some $r \in R$, we have

$$v := w + r \in (\varpi^n + \varpi r) \cap F \subseteq (\varpi^n + \varpi s) \cap F \subseteq \gamma_2(F).$$

We may therefore write

$$v \equiv \sum_{i,j} a_{ij}[X_i, X_j] \mod \gamma_3(F).$$

Since $v \in \varpi^n + \varpi s$, we also have

$$v \equiv \sum_{i,j} c_{ij}e_jX_i \mod \varpi^n.$$

Equating coefficients of $X_iX_j$, we get $a_{ij} = c_{ij}e_j = -c_{ji}e_i$ for all $i > j$. Then, from $e_iX_i \in R + \gamma_2(F)$, we have

$$\sum_{i>j} a_{ij}[X_i, X_j] \in [F, R] + \gamma_3(F),$$

so $w \in \gamma_3(F) + R$. \hfill $\square$

The preceding proof can be extended to yield the following

**Theorem 4.6.** If $w \in (\varpi^n + r) \cap F$, then there exist simple commutators $c_1, \ldots, c_\ell$, all of degree $\geq 2$, and coefficients $a_i \in \mathbb{Z}$, such that

$$w \equiv \sum_{i=1}^{\ell} a_i c_i \mod \gamma_n(F) + F'' + R,$$

and, if $c_i = [X_{i_1}, X_{i_2}, \ldots, X_{i_\ell}]$, then $a_i$ is divisible by $c_{i_1}$.

**Proof.** For some $r \in R$, we have

$$v := w + r \in (\varpi^n + \varpi r) \cap F \subseteq (\varpi^n + \varpi s) \cap F \subseteq \gamma_2(F).$$

We view $F$ as a right $\mathcal{W}(F)$-module, for the adjoint action written $[-,-]$. We claim that $F \cap (\varpi^n + \varpi s)$ is generated, modulo $\gamma_n(F) + F''$, by the elements $e_i[[X_i, X_j], u_{ij}]$ for all $1 \leq i < j \leq m$, with $u_{ij} \in \mathcal{W}(F)$. The conclusion of the Theorem then follows immediately.

First, it is clear that $\gamma_n(F)$, $F''$, and $e_i[[X_i, X_j], u_{ij}]$ all belong to $F \cap (\varpi^n + \varpi s)$. Consider $v \in F \cap (\varpi^n + \varpi r)$. Then, as noted above, $v \in \gamma_2(F)$, so we may write

$$v \equiv \sum_{1 \leq i < j \leq m} [[X_i, X_j], u_{ij}] \mod F''$$

for some $u_{ij} \in \mathcal{W}(F)$. Furthermore, still working modulo $F''$, only the value in $\mathcal{W}(F/F'')$ of the elements $u_{ij}$ is relevant; so that the variables $X_1, \ldots, X_m$ may be arbitrarily permuted in their monomials. For $k < i$, we use the Jacobi identity $[[X_j, X_k], X_i] = [[X_k, X_j], X_i] - [[X_j, X_i], X_k]$ to rewrite the $u_{ij}$ in such a manner that no variable $X_k$ appears in $u_{ij}$; so $u_{ij} \in \mathcal{W}((X_1, \ldots, X_m))$. We thus write

$$v = v_1 + \cdots + v_{m-1}, \text{ with } v_i \equiv \sum_{j>i} [[X_i, X_j], u_{ij}] \mod F''.$$
\( \theta_{m-1}, \ldots, \theta_1 \) successively gives \( v_i \in \mathcal{W}^n + \mathcal{W}^s \) for all \( i \in \{1, \ldots, m-1\} \). Next, modulo \( \mathcal{W}^n + \mathcal{W}^s \), we have
\[
0 \equiv v_i \equiv X_i \sum_{j>i} X_j u_{ij} - \sum_{j<i} X_j X_i u_{ij},
\]
and \( \mathcal{W} \) is a free right \( \mathcal{W}(F) \)-module with basis \( \{X_1, \ldots, X_m\} \), so
\[
X_i u_{ij} \in \mathcal{W}^{n-1} + s \text{ for all } i < j.
\]
Now \( s = \sum_{i=1}^m e_i X_i \mathcal{W}(F) + a \), for the ideal \( a = \mathcal{W}(F) \gamma_2(F) \), so
\[
\mathcal{W}^{n-1} + s = \sum_{i=1}^m X_i (\mathcal{W}^{n-2} + e_i \mathcal{W}(F)) + a.
\]
Since \( \mathcal{W}(F)/a = \mathbb{Z}[X_1, \ldots, X_m] \) is an integral domain, (1) yields \( u_{ij} \in \mathcal{W}^{n-2} + e_i \mathcal{W}(F) + a \). Now \( [X_i, X_j], u] \in \mathcal{W}^n \) when \( u \in a \), and \( [[X_i, X_j]] \in \gamma_n(F) \) when \( u \in \mathcal{W}^{n-2} \); thus the proof is complete. \( \square \)

Note, in particular, that if \( e_i = 0 \) for all \( i \) then we get the following special case of the “Fox problem” (see [5]):

**Corollary 4.7.** For all \( n \in \mathbb{N} \) we have
\[
F \cap (\mathcal{W}^n + \mathcal{W} \gamma_2(F)) = \gamma_n(F) + F''.
\]

**4.2. \( n = 4 \).** We continue with the notation set in the paragraph preceding Theorem 4.8 and proceed to give an identification of the fourth Lie dimension subring.

**Theorem 4.8.** The Lie subalgebra \( (\mathcal{W}^4 + \tau) \cap F \) of \( F \) consists, modulo \( \gamma_4(F) + R \), of all elements
\[
\sum_{i>j} a_{ij} [X_i, X_j],
\]
with integer \( a_{ij} \), such that \( e_i \) divides \( a_{ij} \) and for all \( i \in \{1, \ldots, m\} \)
\[
W_i := \sum_{i>j} a_{ij} X_j - \sum_{i<j} a_{ji} X_j \in e_i \gamma_2(F) + \gamma_3(F) + R.
\]

**Proof.** Consider \( w \in (\mathcal{W}^4 + \tau) \cap F \). Then, for some \( r \in R \), we have
\[
v := w + r \in (\mathcal{W}^4 + \mathcal{W} \tau) \cap F \subseteq (\mathcal{W}^4 + \mathcal{W} s) \cap F \subseteq \gamma_2(F).
\]
We may therefore write
\[
v \equiv \sum_{i>j} a_{ij} [X_i, X_j] + \sum_{i>j \leq k} b_{ijk} [X_i, X_j, X_k] \mod \gamma_4(F); \quad (3)
\]
and, since \( v \in \mathcal{W}^4 + \mathcal{W} s \), we also have
\[
v \equiv \sum_{i,j} c_{ij} e_j X_i X_j + \sum_{i,j,k} d_{ijk} e_k X_i X_j X_k + \sum_{i \text{ any}, j > k} f_{ijk} [X_i, X_j, X_k] \mod \mathcal{W}^4.
\]
Comparing homogeneous terms of degree two, we have \( a_{ij} = c_{ij} e_j = -c_{ji} e_i \). By Theorem 4.6 noting that \( \gamma_4(F) \) contains \( F'' \), we may write \( b_{ijk} = b_{ijk}' e_i \) with \( b_{ijk}' \in \mathbb{Z} \); then
\[
b_{ijk} [X_i, X_j, X_k] = b_{ijk}' [e_i X_i, X_j, X_k] \in [R + \gamma_2(F), F, F] \subseteq \gamma_4(F) + R.
\]
Consequently,
\[
w \equiv \sum_{i>j} a_{ij} [X_i, X_j] \mod \gamma_4(F) + R.
\]
Define next \( Y_i := \sum_{j,k} d_{ijk} e_k X_j X_k + \sum_{j>k} f_{ijk} [X_j, X_k] \); then
\[
v \equiv \sum_{i} X_i W_i + \sum_{i} X_i Y_i \mod \mathcal{W}^4,
\]
and \( v \in \mathbb{R}^4 + \mathbb{R}^r \), so \( W_i + Y_i \in \mathbb{R}^3 + \mathbb{R} \) for all \( i \in \{1, \ldots, m\} \). All degree-3 summands in \( \{3\} \), say involving the variables \( \{X_i, X_j, X_k\} \), are multiples of \( \gcd(e_i, e_j, e_k) \); so we may write \( f_{ijk} = f'_{ijk} e_i + f'_{ijk} e_j + f'_{ijk} e_k \) for some \( f'_{ijk}, f'_{ijk}, f'_{ijk} \in \mathbb{Z} \); then

\[
Y_i = \sum_{j>k} e_i f'_{ijk} [X_j, X_k] + \sum_{j,k} d_{ijk} X_j (e_k X_k)
\]

\[
+ \sum_{j>k} f'_{ijk} [e_j X_j, X_k] + \sum_{j>k} f'_{ijk} [X_j, e_k X_k]
\]

\[
\in e_i \gamma_2(F) + \mathbb{R}^3 + \mathbb{R}.
\]

therefore \( W_i \in e_i \gamma_2(F) + \mathbb{R}^3 + \mathbb{R} \). Noting then that \( W_i \) belongs to \( F \), we get \( W_i \in (e_i \gamma_2(F) + \mathbb{R}^3 + \mathbb{R}) \cap F = e_i \gamma_2(F) + \gamma_3(F) + R \) by invoking Theorem 4.5

Conversely, choose any \( a_{ij} \in \mathbb{Z} \) such that

\[
W_i := \sum_{i>j} a_{ij} X_j - \sum_{i<j} a_{ij} X_j \in e_i \gamma_2(F) + \gamma_3(F) + R;
\]

then

\[
\sum_{i>j} a_{ij} [X_i, X_j] = \sum_i X_i W_i
\]

\[
\in \sum_i e_i X_i \gamma_2(F) + \gamma_3(F) + X_i R \subseteq \mathbb{R}^4 + \mathbb{R}. \quad \square
\]

**Corollary 4.9.** If \( L \) be a Lie algebra over \( \mathbb{Z} \), then

\[
[\delta_4(L), L] \subseteq \gamma_5(L) + L''
\]

with \( L'' \) the second derived subring of \( L \).

In particular, if \( L \) be a metabelian Lie ring, then \( \delta_4(L) / \gamma_5(L) \) is central in \( L / \gamma_5(L) \).

**Proof.** Consider a typical generator \( \sum_{i>j} a_{ij} [X_i, X_j, X_k] \) of \( [\delta_4(L), L] \) modulo \( \delta_5(L) \), resulting from the generators \( \{2\} \) of \( \delta_4(L) \) as in Theorem 4.8. Using the Jacobi identity, rewrite it as

\[
\sum_{i>j} \left( \frac{a_{ij}}{e_i} [e_i X_j, [X_j, X_k]] - \frac{a_{ij}}{e_j} [e_j X_j, [X_i, X_k]] \right);
\] (4)

since \( e_i X_j, e_j X_j \in [L, L] \), the above element belongs to \( L'' \). \( \square \)

**Corollary 4.10.** If \( L \) be a Lie algebra over \( \mathbb{Z} \), then

\[
2 \delta_4(L) \subseteq \gamma_4(L).
\]

**Proof.** Consider a typical element \( a = \sum_{i>j} a_{ij} [X_i, X_j] \) of \( \delta_4(L) \) modulo \( \gamma_4(L) \). Recall our notation

\[
W_i := \sum_{i>j} a_{ij} X_j - \sum_{i<j} a_{ij} X_j \in e_i \gamma_2(L) + \gamma_3(L),
\]

so that \( a = \sum_i X_i W_i \). Write \( W_i = e_i Y_i + Z_i \) with \( Y_i \in \gamma_2(L) \) and \( Z_i \in \gamma_3(L) \). From \( [X_i, X_j] = -[X_j, X_i] \), we also get \( a = -\sum_i X_i W_i \). Therefore,

\[
2a = \sum_i [X_i, W_i] = \sum_i e_i [X_i, Y_i] + [X_i, Z_i] \in \gamma_4(L). \quad \square
\]

We briefly recall Magnus’s construction alluded to in the Introduction. Let \( G \) be a group, and let \( (\gamma_n(G))_{n \geq 1} \) denote its lower central series. The abelian group

\[
\text{gr} \ G = \bigoplus_{n \geq 1} \gamma_n(G) / \gamma_{n+1}(G)
\] (5)
naturally has the structure of a graded Lie ring, for the Lie bracket defined on
homogeneous elements by \([x \gamma_{m+1}(G), y \gamma_{n+1}(G)] = x^{-1}y^{-1}xy\gamma_{m+n+1}(G)\).

We show that, even though the fourth dimension quotient of \(G\) may non-trivial
(namely \(\delta_4(G) \neq \gamma_4(G)\)), the corresponding quotient of \(\text{gr} \, G\) is always trivial:

**Corollary 4.11.** If \(L\) be a graded Lie ring, generated in degree 1, then \(\delta_4(L) = \gamma_4(L)\).

**Proof.** Since \(\delta_3(L) = \gamma_3(L)\) by Theorem 4.5, the generators of \(\delta_4(L)/\gamma_4(L)\) have
degree at least 3; however, by Theorem 4.8, they have degree 2. Therefore, \(\delta_4(L)/\gamma_4(L) = 0\).

**4.3. A counterexample in degree 4.** We now give an example of a Lie ring
for which \(\delta_4(L) \neq \gamma_4(L)\). This is an adaptation of Rips’s counterexample \([36]\)
to the dimension conjecture. Note however that, by Corollary 4.11, the Magnus
Lie algebra \(\text{gr} \, G\) associated with Rips’s counterexample does satisfy \(\delta_4(\text{gr} \, G) = \gamma_4(\text{gr} \, G)\).

**Theorem 4.12.** Consider the Lie ring \(L\) with presentation
\[
\langle x_1, x_2, x_3, x_4 \mid 4x_1 + 2[x_4, x_3] + [x_4, x_2],
16x_2 + 4[x_4, x_1] - [x_4, x_1], 64x_3 - 4[x_4, x_2] - 2[x_1, x_1] \rangle.
\]
Set \(a := 32[x_1, x_2] + 64[x_1, x_3] + 128[x_2, x_3] \in L\). We then have
\(a \in \delta_4(L) \setminus \gamma_4(L)\).

**Proof.** First, observe that \(a \in \delta_4(L)\) by Theorem 4.5.

We may then seek a quotient \(L^r\) of \(L\), nilpotent of class 3, in which \(a\) does not
vanish. We add as relations to \(L\):

- all triple commutators except \([x_i, x_j, x_k]\) for \(i = 1, 2, 3\);
- \(4[x_4, x_1, x_1], 4[x_4, x_2, x_2] − [x_4, x_1, x_1]\) and \(4[x_4, x_3, x_3] − [x_4, x_2, x_2]\);
- all quadruple commutators (namely, \(\gamma_4(L)\)).

It is then easily checked that \(L^r\) is, additively, a \(\mathbb{Z}\)-module of rank 9, of the form
\[
\mathbb{Z} \oplus \mathbb{Z}/256 \oplus \mathbb{Z}/256 \oplus \mathbb{Z}/256 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2,
\]
and that \(a\) is a non-trivial element in \(L^r\). These calculations on finite-rank Lie
algebras were performed using GAP [20] and its package LieRing [24] by Willem
de Graaf and Serena Cicalò.

5. The Fox Problem

Analogous to the Fox problem for free group rings [13, Chapter III], there arises
the following

**Problem 5.1.** Identify the Lie subrings \(F \cap \varpi^n(F)\tau\), for \(n \geq 1\), where \(F\) is a free
Lie ring over a commutative ring \(k\) with identity and \(\tau\) is the two-sided ideal of \(\mathcal{U}(F)\) generated by a Lie ideal \(R\) of \(F\).

Note that the subrings \(F \cap \varpi^n(F)\tau\) and \(F \cap \varpi^n\tau\) are anti-isomorphic; indeed the
map \(x \mapsto −x\) on \(F\) extends to an anti-isomorphism on \(\mathcal{U}(F)\), the antipode (see
e.g. [20, Theorem V.1]).

In case \(k\) is a field, a solution to the above problem is provided by the following

**Theorem 5.2** (Yunus [45]). Let \(F\) be a free Lie algebra over a field \(k\), let \(R\) be an
ideal of \(F\), and let \(\tau\) be the ideal of \(\mathcal{U}(F)\) generated by \(R\). Then, for all \(n \geq 1\),
\[
F \cap \varpi^n(F)\tau = \sum_{2 \leq m \leq n+1} [R_{i_1}, \ldots, R_{i_m}],
\]
where \(i_j \geq n + \max_j i_j\).
with $R_i = R \cap \gamma_i(F)$.

The problem has also been solved in the case of restricted Lie algebras \cite{11,12}; however, the integral case ($k = \mathbb{Z}$) of Problem \ref{Problem1} does not seem to have been investigated so far. The very first case here ($n = 1$) manifests a sharp difference with the corresponding result for group rings. It is well-known that if $N$ is a normal subgroup of a group $G$, then

$$\gamma_1(F) \cap (1 + \omega(G)\omega(N)) = [N, N].$$

In contrast with this result, we note the following:

Example 5.3. Let $F$ be the free $\mathbb{Z}$-free Lie algebra with basis $\{X_1, X_2\}$, let $R = pF$ be the Lie ideal of $F$ generated by $\{pX_1, pX_2\}$, and let $\tau$ be the two-sided ideal of $\mathcal{U}(F)$ generated by $R$. Then

$$F \cap \omega(F)\tau = p[F, F] \neq [R, R] = p^2[F, F].$$

However, with $\sqrt{[R, R]} = \{x \in R : nx \in [R, R] \text{ for some } n \neq 0\}$, we always have $[R, R] \subseteq F \cap \omega(F)\tau \subseteq \sqrt{[R, R]} \subseteq R$.

Proof. The element $p[X_1, X_2] = X_1(pX_2) - X_2(pX_1)$ belongs to $\omega(\tau)$, so the ideal $p[F, F] \tau$ that it generates is contained in $F \cap \omega(\tau)$. Conversely, $F \cap \omega(\tau)$ is contained in $[F, F]$ and in $pF$, so is contained in their intersection $p[F, F]$. Finally, we clearly have $[R, R] = p^2[F, F]$.

The second claim follows from Theorem \ref{Theorem5.2}.

Let us now derive some consequences of the identification of $L \cap \omega(\tau)$. Consider a free presentation $L = F/R$, the associated ideal $\tau = \mathcal{U}(F)R$, and write $\omega := \omega(F)$. To avoid trivial exceptional cases, we assume that $F$ is non-abelian, or equivalently that its rank is at least 2, and that $R$ is non-zero. Set

$$M := F \cap \omega(\tau), \quad \widetilde{L} = F/M.$$

The natural Lie monomorphism $\iota : F \to \mathcal{U}(F)$ induces a Lie monomorphism

$$\iota : F/M \to \omega/\omega(\tau),$$

which restricts to a monomorphism of right $\mathcal{U}(L)$-modules $\iota : R/M \to \omega/\omega(\tau)$. Observe that $\omega/\omega(\tau)$ is a free right $\mathcal{U}(L)$-module with basis $\{X_i + \omega(\tau) : 1 \leq i \leq m\}$. We write as before $[\cdot, -]$ for the adjoint action of $\mathcal{U}(L)$.

By Theorem \ref{Theorem5.2} we have $M = [R, R]$ if $L$ is an algebra over a field. In analogy with the case of groups, we call $R/M$ the relation module of the presentation $L = F/R$.

Proposition 5.4. Let $L$ be a Lie algebra. If $\omega$ is an integral domain (for example, if $L$ is an algebra over a field), then $R/M$ is a faithful $\mathcal{U}(L)$-module.

Proof. We have, by definition of $M$, an embedding $R/M \hookrightarrow \omega/\omega(\tau)$; moreover, there is a natural embedding $\omega/\omega(\tau) \hookrightarrow \omega/\omega(\tau)$, and there is an isomorphism

$$\omega/\omega(\tau) \cong \omega/\omega^2 \otimes \mathcal{U}(L),$$

$$\sum_{i=1}^{m} X_i u_i + \omega(\tau) \mapsto (X_i + \omega(\tau)) \otimes (u_i + \tau) \text{ with } u_i \in \mathcal{U}(F).$$

Composing these maps, we see that $R/M$ embeds in the free $\mathcal{U}(L)$-module $\omega/\omega^2 \otimes \mathcal{U}(L)$. Now the assumption that $\mathcal{U}(L)$ is an integral domain implies that all non-zero submodules of a free $\mathcal{U}(L)$-module are faithful.

Recall that $\mathcal{U}(L)$ is a domain if its associated graded $\text{Sym}(L)$ is a domain; this holds e.g. when $L$ is an algebra over a field. \hfill $\Box$
It may be noted, continuing on Example 5.3, that in that case $M = p[F, F]$ so $R/M \cong \mathbb{Z}^2$ with basis \{pX_1, pX_2\}; so that $R/M$ is annihilated by $\varpi(L)$. Thus, in contrast with the case for groups, $R/M$ is, in general, not a faithful $\mathcal{W}(L)$-module.

Note also, when $k = \mathbb{Z}$, that $\varpi$ is an integral domain if and only if $L$ either is torsion-free or satisfies $pL = 0$ for some prime $p$.

Note finally, as in the classical Gashütz theory for groups, we have an exact sequence

\[
0 \longrightarrow R/M \longrightarrow \varpi/\varpi^2 \otimes \mathcal{W}(L) \longrightarrow \varpi(L) \longrightarrow 0 
\]

**Theorem 5.5.** Let as above $L$ be a Lie algebra presented as $F/R$ with $R \neq 0$ and $F$ non-abelian. Set $M = F \cap \varpi R$ and $\tilde{L} = F/M$.

(i) If $\bigcap_{n \geq 1} \varpi^n(L) = 0$, then $\bigcap_{n \geq 1} \gamma_n(\tilde{L}) = 0$.

(ii) If $\bigcap_{n \geq 1} \gamma_n(\tilde{L}) = 0$ and the annihilator $\text{ann}_{\mathcal{W}(L)}(R/M)$ is trivial, then $\bigcap_{n \geq 1} \varpi^n(L) = 0$.

(iii) In particular, if $\mathcal{W}(L)$ is an integral domain (for example, if $L$ is an algebra over a field), then $\bigcap_{n \geq 1} \varpi^n(L) = 0$ if and only if $\bigcap_{n \geq 1} \gamma_n(\tilde{L}) = 0$.

**Proof.** Suppose first $\bigcap_{n \geq 1} \varpi^n(L) = 0$, and consider $w \in \bigcap_{n \geq 1} \gamma_n(\tilde{L})$. Clearly $\bigcap_{n \geq 1} \gamma_n(\tilde{L}) = 0$, because $\gamma_n(\tilde{L}) \subseteq \varpi^n(L)$, so $w \in R/M$. Writing $\tau = \mathcal{W}(F)R$, note that

\[
\iota(w) \in \bigcap_{n \geq 1} (\varpi^n + \varpi \tau)/\varpi \tau = \bigcap_{n \geq 1} [X_i + \varpi \tau, \bigcap_{n \geq 1} \varpi^n(L)] = 0.
\]

Since $\iota$ is injective, we deduce $w = 0$, which proves (i).

Suppose next $\bigcap_{n \geq 1} \gamma_n(\tilde{L}) = 0$, and consider $z \in \bigcap_{n \geq 1} \varpi^n(L)$. We then have $[R/M, z] \subseteq \bigcap_{n \geq 1} \gamma_n(\tilde{L}) = 0$, so $z$ belongs to the annihilator $\text{ann}(R/M)$. This proves (ii).

(iii) follows immediately from Proposition [54].

We remark that, in the case of groups $G = F/R$, we always have $M = [R, R]$, and $R/M$ is a faithful $\mathbb{Z}[G]$-module, see [29].

6. A simplicial approach

In this section, we formulate an approach to the investigation of the dimension problem using simplicial methods. We adapt to Lie rings, on the one hand, the spectral-sequence methods developed by Grünfelder for group rings and, on the other hand, Keune’s theory of derived functors of the functors $\Gamma_n$.

As a byproduct, we will reprove Theorem 4.5. We hope, however, that these methods will bear more fruits in later work. We follow the notation in [28] Appendix A; see also [7]. General valuable references for simplicial methods include [27] and [10].

Fortunately, Lie rings are abelian groups equipped with the extra structure of a Lie bracket; therefore, all the machinery developed for groups applies readily. We have attempted to give sufficient detail so as to make the text self-sufficient, and refer the reader to the cited literature for details if needed.

Let $\mathfrak{Lie}$ denote the category of Lie rings $L$ over $\mathbb{Z}$, and let $\underline{\mathfrak{Lie}}$ denote the category of simplicial Lie rings $L = \{L_n\}_{n \geq 0}$ over $\mathbb{Z}$. We recall briefly its definition: first, the simplicial category $\Delta$ is the category with object set $\mathbb{N}$, and with $\text{hom}_\Delta(m, n)$ equal to the set of order-preserving maps from $\{0, \ldots, m - 1\}$ to $\{0, \ldots, n - 1\}$. A simplicial Lie ring $L$ is a contravariant functor $\Delta \to \mathfrak{Lie}$; equivalently, for every $n \in \mathbb{N}$, a Lie ring $L_n = L(n)$, and face maps $d_0, \ldots, d_n : L_n \to L_{n-1}$
and degeneracy maps \(s_0, \ldots, s_n : L_n \to L_{n+1}\) between the Lie rings, satisfying appropriate conditions (see e.g. [27, Definition 1.1]).

The associated Moore complex \(\mathcal{M}\) is the complex of Lie rings \(M_n = \bigcap_{i=0}^{n-1} \ker(d_i|L_n)\); its differential is given by \(d_n : M_n \to M_{n-1}\). The fundamental Lie rings \(\pi_n(L)\) are the homology groups of the Moore complex: \(\pi_n(L) = \ker(d_n|M_n)/\text{im}(d_{n+1}|M_{n+1})\).

Since, in particular, Lie rings are abelian groups, we may define alternatively the fundamental Lie rings as follows. Form the total differential

\[
\partial_n : L_n \to L_{n-1}, \quad \partial_n = \sum_{i=0}^{n} (-1)^i d_i.
\]

Then \(\partial^2 = 0\), and \(\pi_n(L)\) is isomorphic to the homology of the resulting complex \((L, \partial)\), via the natural inclusions \(M_n \to L_n\).

A fibration is a morphism of simplicial Lie rings \(L \to Q\) that is surjective in all degrees. Its fibre is the simplicial Lie ring \(K\) whose Lie rings in respective degrees are the kernels of the aforesaid morphisms. As is well known, a fibration gives rise to a long exact homotopy sequence of simplicial Lie rings, namely the long exact sequence

\[
\cdots \to \pi_1(K) \to \pi_1(L) \to \pi_1(Q) \to \pi_0(K) \to \pi_0(L) \to \pi_0(Q) \to 0.
\]

The zeroth homotopy ring is usually readily computable, in contrast with higher homotopy rings \(\pi_n(L)\) for \(n \geq 1\). Again, this follows from the classical result for abelian groups:

**Proposition 6.1.** \(\pi_0(L)\) is the coequalizer of \(d_0, d_1 : L_1 \rightrightarrows L_0\). □

The Lie algebra \(L_0\) acts by derivations on \(L_n\), via

\([y, x] = [s_0^n(y), x]\) for all \(y \in L_0, x \in L_n\).

This action preserves the ideal \(M_n\), and induces an action on \(\pi_n(L)\). Thus

**Proposition 6.2.** The Lie ring \(\pi_n(L)\) has the natural structure of a \(Ψ(\pi_0(L))\)-module. □

Assume that \(L_2\) is generated by degeneracies: \(L_2 = \langle s_0(L_1), s_1(L_1) \rangle\). Then \(d_2(L_2) = [\ker(d_1), \ker(d_2)]\), whence

**Proposition 6.3.** If \(L_2\) is generated by degeneracies, then

\[
\pi_1(L) = \frac{\ker(d_0|L_1) \cap \ker(d_1|L_1)}{[\ker(d_0), \ker(d_1)]}. \tag{□}
\]

### 6.1. Two spectral sequences

We take inspiration from Gr"unenfelder’s approach [31], via spectral sequences, to the dimension subgroup problem. For every integer \(n \geq 1\), we have functors

\[
\Gamma_n : \mathfrak{Lie} \to \mathfrak{Lie}, \quad L \mapsto L/\gamma_n(L)
\]

and

\[
\text{gr}_n : \mathfrak{Lie} \to \mathfrak{Lie}, \quad L \mapsto \gamma_n(L)/\gamma_{n+1}(L).
\]

These functors extend naturally to the category \(s\mathfrak{Lie}\), and we have, for \(L \in s\mathfrak{Lie}\) and all \(n \geq 1\), an exact sequence of simplicial Lie rings

\[
0 \longrightarrow \text{gr}_n(L) \longrightarrow \Gamma_{n+1}(L) \longrightarrow \Gamma_n(L) \longrightarrow 0. \tag{6}
\]

The resulting homotopy exact couple

\[
\pi_*(\Gamma_*(L)) \xrightarrow{\text{ident}} \pi_*(\Gamma_*(L)) \xleftarrow{\text{ident}} \pi_*(\text{gr}_*(L))
\]
yields a first-quadrant spectral sequence \( \{ E^r_{p,q}(L) \} \) having differentials \( d^r \) of bidegree \((r, -1)\), and with

\[
E^1_{p,q}(L) = \pi_q(\text{gr}_p(L)).
\]  

This is the classical sequence associated with the filtration of \( L \) by its lower central series \( \{ \gamma_n(L) \} \); thus

**Proposition 6.4** (Essentially [5] and [11]). Let \( L = F/R \) be a nilpotent Lie algebra, and let \( E^*_{\ast, \ast} \) be the above spectral sequence. Then \( E^*_{\ast, \ast} \Rightarrow \text{gr} \pi_*(L) \). In particular,

\[
E_{\infty,0} = \frac{\gamma_n(L)}{\gamma_{n+1}(L)} = \frac{\gamma_n(F) + R}{\gamma_{n+1}(F) + R} \text{ for all } n \geq 1. \quad \square
\]

We consider now a parallel construction in the context of the universal enveloping algebra of \( L \). Let \( \mathfrak{Ass} \) denote the category of augmented associative algebras over \( \mathbb{Z} \), and let \( s\mathfrak{Ass} \) denote the category of simplicial augmented algebras. If \( L \) be a simplicial Lie ring, then \( \mathcal{U}(L) \) is the simplicial associative algebra with \( \mathcal{U}(L_n) \) in degree \( n \). As above, for every integer \( n \geq 1 \) we have functors

\[
\Gamma_n : \text{Lie} \to \mathfrak{Ass}, \quad L \mapsto \mathcal{U}(L)/\mathfrak{r}_n(L)
\]

and

\[
\text{gr}_n : \text{Lie} \to \mathfrak{Ass}, \quad L \mapsto \mathcal{U}(L)/\mathfrak{r}_{n+1}(L),
\]

where, as usual, \( \mathfrak{r}(L) \) denotes the augmentation ideal of the augmented algebra \( \mathcal{U}(L) \). These functors also extend naturally to the category \( s\mathfrak{Ass} \) of simplicial augmented algebras, and we have for all \( n \geq 1 \) an exact sequence of simplicial algebras

\[
0 \longrightarrow \text{gr}_n(L) \longrightarrow \Gamma_{n+1}(L) \longrightarrow \Gamma_n(L) \longrightarrow 0. \tag{8}
\]

The resulting homotopy exact couple

\[
\pi_*(\Gamma_*(L)) \quad \longrightarrow \quad \pi_*(\Gamma_*(L)) \quad \longrightarrow \quad \pi_*(\text{gr}_*(L))
\]

yields another first-quadrant spectral sequence \( \{ E^r_{p,q}(L) \} \) having differentials \( \partial^r \) of bidegree \((r, -1)\), with

\[
E^\infty_{p,q}(L) = \pi_q(\text{gr}_p(L)). \tag{9}
\]

**Proposition 6.5** (Essentially [5] and [11]). Let \( L = F/R \) be a nilpotent Lie algebra, and let \( E^*_{\ast, \ast} \) be the above spectral sequence. Then \( E^*_{\ast, \ast} \Rightarrow \text{gr} \pi_*(\mathcal{U}(L)) \). In particular,

\[
E_{\infty,0} = \frac{\mathfrak{r}_n(L)}{\mathfrak{r}_{n+1}(L)} = \frac{\mathfrak{r}_n(F) + r}{\mathfrak{r}_{n+1}(F) + r} \text{ for all } n \geq 1. \quad \square
\]

In view of Theorem 3.2, for every \( L \in \mathfrak{slie} \) there exists a canonical embedding \( \iota : L \to \mathcal{U}(L) \). We thus obtain a morphism of spectral sequences

\[
\iota : E^*_{\ast, \ast} \to E^*_{\ast, \ast}.
\]
6.2. Resolutions. Recall that an augmented simplicial object in the category \( \mathfrak{Lie} \) is a simplicial Lie ring \( L \) together with a Lie homomorphism \( d_0 : L_0 \to L_{-1} \) such that \( d_0 d_0 = d_0 d_1 : L_1 \to L_{-1} \). Let \( \mathfrak{asLie} \) denote the category of augmented simplicial Lie rings. For \( L \in \mathfrak{asLie} \) and \( n \geq -1 \), set
\[
Z_n L = \{(a_0, \ldots, a_{n+1}) \in L_n^{n+2} \mid d_i a_j = d_{j-1} a_i \text{ for all } i < j\}.
\]
For each \( n \geq -1 \), there is a homomorphism
\[
d^{(n)} : L_{n+1} \to Z_n L, \quad a \mapsto (d_0 a, \ldots, d_{n+1} a).
\]
An augmented simplicial object \( L \in \mathfrak{asLie} \) is called aspherical if, for each \( n \geq -1 \), the homomorphism \( d^{(n)} \) is surjective. In this case \( L \) is said to be a resolution of \( L_{-1} \). An object \( L \in \mathfrak{asLie} \) is said to be free if, for each \( n \geq 0 \), the Lie ring \( L_n \in \mathfrak{Lie} \) is free and if, moreover, sets \( X_n \subset L_n \) of free generators can be chosen in such a way that they are preserved under the degeneracy maps \( s_i \), namely, \( s_i X_n \subset X_{n+1} \) for all \( 0 \leq i \leq n \).

The following statement follows directly from the fact that \( \mathfrak{Lie} \) is a closed model category in the sense of Quillen [31]; we include its proof because it is informative.

**Proposition 6.6** (F. Keune, [21]). Every Lie ring \( L \in \mathfrak{Lie} \) has a free resolution \( L \in \mathfrak{asLie} \).

**Proof.** Let \( L \) be a Lie ring. Choose a set of generators \( X_0 \) of \( L \), and define \( L_0 = FX_0 \), the free Lie ring over \( X_0 \). The surjection \( d_0 : FX_0 \to L \) is then defined by \( d_0(x) = x \) for all \( x \in X_0 \). Suppose that \( L_n = FX_n \) has been defined, and that we wish to construct \( L_{n+1} = FX_{n+1} \) with homomorphisms \( d_i : L_{n+1} \to L_n \) and \( s_j : L_n \to L_{n+1} \). Set
\[
Z = \{(a_0, \ldots, a_{n+1}) \in L_n^{n+2} \mid d_i a_j = d_{j-1} a_i \text{ for all } i < j\}.
\]
There are \( n + 2 \) homomorphisms \( d_i' : Z \to L_n \), defined by \( d_i'(a_0, \ldots, a_{n+1}) = a_i \) for all \( i = 0, \ldots, n+1 \), and \( n + 1 \) homomorphisms \( s_j' : L_n \to Z \), defined by
\[
s_j'(a) = (s_{i-1} d_0 a, \ldots, s_{i-1} d_i a, a, s_i d_{i+1} a, \ldots, s_j d_n a)
\]
for \( i = 0, \ldots, n \). Complete the subset \( \bigcup_{i=0}^{n} s_i X_n \subset Z \) to a set \( X_{n+1} \) of generators of \( Z \). Define
\[
L_{n+1} = FX_{n+1}, \quad d_i \text{ extending } d_i' : L_{n+1} \to L_n, \quad s_i \text{ extending } s_i' : L_n \to L_{n+1}.
\]
Thus we obtain an augmented simplicial object \( L \) which by its construction is a free resolution of \( L \in \mathfrak{Lie} \).

**Note 6.7.** We extract from the proof above a concrete construction of a free resolution \( L \) for any Lie ring given by a presentation. Let indeed \( L = \langle X \mid \mathcal{R} \rangle \) be a presentation, meaning \( L = F(X)/R \) with \( R = \langle \mathcal{R} \rangle \), the ideal in \( F(X) \) generated by \( \mathcal{R} \).

A free resolution \( L \) of \( L \) may then start as follows. Set \( L_0 = F(X) \) and \( L_1 = F(X \cup \mathcal{R}) \). The map \( L_0 \to L \) is the natural quotient map. The differentials \( d_0, d_1 : L_1 \to L_0 \) are defined on generators by \( d_0(x) = d_1(x) = x \) for \( x \in X \) and \( d_0(r) = 0, d_1(r) = r \) for \( r \in \mathcal{R} \), while the degeneracy \( s_0 : L_0 \to L_1 \) is the natural embedding \( s_0(x) = x \).

6.3. Kan’s condition. We now show that objects in \( \mathfrak{Lie} \) satisfy Kan’s extension condition. Let \( L \in \mathfrak{asLie} \) be a simplicial Lie ring, and let \( y_0, \ldots, y_k, \ldots, y_n \) be a
collection of simplices in \( L_{n-1} \) satisfying \( d_i y_j = d_{j-1} y_i \) for \( k \neq i \neq j \neq k \). Define
\[
\begin{aligned}
w_0 &= s y_0, \\
w_i &= w_{i-1} - s_i d_i w_{i-1} + s_i y_i & \text{ for } 0 < i < k, \\
w_n &= w_{k-1} - s_{n-1} a_n w_{k-1} + s_{n-1} y_n, \\
w_i &= w_{i+1} - s_{i-1} d_i w_{i+1} + s_i y_i & \text{ for } k < i < n.
\end{aligned}
\]
Then \( w_{k+1} \in L_n \) and \( d_i w_{k+1} = y_i \) for \( i \neq k \). Thus \( L \) satisfies the Kan extension condition \([5\text{ Proposition 1.13 and Lemma 3.1}]\). As a consequence, the following comparison theorem holds.

**Theorem 6.8** (F. Keune, [21]). Let \( L \in \text{asLie} \) be free, let \( M \in \text{asLie} \) be aspherical, and let \( \alpha : L_{-1} \to M_{-1} \) be a Lie homomorphism. Then

1. there exists a homomorphism \( \alpha : L \to M \) in \( \text{asLie} \) such that \( \alpha_{-1} = \alpha \);
2. if \( \alpha' \) also extends \( \alpha \), then there exists a simplicial homotopy \( h : \alpha \simeq \alpha' \),

with the \( h_i \) Lie homomorphisms in \( \text{Lie} \).

### 6.4. Derived functors

Let \( F : \text{Lie} \to \text{Lie} \) be a right-exact functor. Its left derived functors are defined as follows, see \([40]\). In view of Theorem 6.8, the homotopy groups \( \pi_n(F L) \) for a free resolution \( L \to L \) depend only on \( L \in \text{Lie} \) and not on the chosen free resolution \( L \) of \( L \). The left derived functors \( \mathcal{D}_n F : \text{Lie} \to \text{Lie} \), for \( n \geq 0 \), are defined by setting
\[
\mathcal{D}_n F(L) = \pi_n(F L).
\]

Let \( R \) be an ideal of a free Lie ring \( F \). We introduce the following notation, for all \( n \geq 0 \):
\[
\begin{align*}
R(0) &= R, & R(n+1) &= [R(n), F] \\
t(0) &= \psi(F) R, & t(n+1) &= \psi(F) t(n) + t(n) \psi(F).
\end{align*}
\]

Recall that, for \( n \geq 1 \), the functor \( \Gamma_n : \text{Lie} \to \text{Lie} \) is given by \( L \mapsto L/\gamma_n(L) \). Let \( L = F/R \) be a free presentation of \( L \), and let \( L \) be a free simplicial resolution of \( L \) with the first two terms \( F_0 \) and \( F_1 \) as in Note 6.7. Then clearly \( \mathcal{D}_0 \Gamma_n(L) = L/\gamma_n(L) \) for all \( n \geq 2 \). Consider the short exact sequence
\[
0 \longrightarrow \gamma_n(L) \longrightarrow L \longrightarrow L/\gamma_n(L) \longrightarrow 0
\]
of simplicial Lie rings. It gives rise to a long exact homotopy sequence, namely
\[
\cdots \to \pi_1(L) \to \pi_1(L/\gamma_n(L)) \to \pi_0(\gamma_n(L)) \to \pi_0(L) \to \pi_0(L/\gamma_n(L)) \to 0.
\]

Now \( \pi_1(L) = 0 \), and the \( \pi_0 \) may be computed via Proposition 6.1 as coequalizers:
\[
\begin{align*}
\pi_0(L) &= F/R = L, \\
\pi_0(L/\gamma_n(L)) &= F/(R + \gamma_n(F)) = L/\gamma_n(L), \\
\pi_0(\gamma_n(L)) &= \gamma_n(F)/\gamma_n(F + R) = \gamma_n(F)/R(n-1).
\end{align*}
\]

Hence
\[
\mathcal{D}_1 \Gamma_n(L) = \pi_1(L/\gamma_n(L)) = \ker [\pi_0(\gamma_n(L)) \to \pi_0(L)] = \frac{R \cap \gamma_n(F)}{R(n-1)}.
\]

We deduce in passing that \( R \cap \gamma_n(F)/R(n-1) \) depends only on \( L \), and not on the choice of presentation \( F/R \); they are known, in the context of groups, as *Bauer invariants*. See [3] for more details.

The sequence (6) yields a long exact homotopy sequence
\[
\cdots \to \pi_1(L/\gamma_{n+1}(L)) \to \pi_1(L/\gamma_n(L)) \to \\
\to \pi_0(\gamma_n(L)/\gamma_{n+1}(L)) \to \pi_0(L/\gamma_{n+1}(L)) \to \pi_0(L/\gamma_n(L)) \to 0;
\]
all the terms above are known, except the middle one which we now proceed to determine. Here is the same sequence, presented as a splice of (diagonal) short exact sequences, in which we abbreviate \( \gamma_m \) for \( \gamma_m(F) \):

whence

\[
E_{n,0}^{1} = D_0 \mathcal{L}(L) = \frac{\gamma_n(F)}{\gamma_n(F) + R(n-1)} \quad \text{for all } n \geq 2. \tag{11}
\]

6.5. Functors to associative algebras. Consider next, for \( n \geq 1 \), the functor \( T_n : \mathfrak{L} \to \mathfrak{As} \) given by \( L \mapsto \mathcal{U}(L)/\mathcal{U}^n(L) \). Let \( L = F/R \) be a free presentation of \( L \) and let \( L \) a free simplicial resolution of \( L \) with the first two terms \( F_0 \) and \( F_1 \) as in Note [6.6]. Then clearly

\[
D_0 T_{n}(L) = \mathcal{U}(L)/\mathcal{U}^n(L) \quad \text{for all } n \geq 2.
\]

Consider the short exact sequence

\[
0 \longrightarrow \mathcal{U}^n(L) \longrightarrow \mathcal{U}(L) \longrightarrow \mathcal{U}(L)/\mathcal{U}^n(L) \longrightarrow 0
\]

of simplicial algebras. It gives rise to a long exact homotopy sequence, namely

\[
\cdots \longrightarrow \pi_1(\mathcal{U}(L)) \longrightarrow \pi_1(\mathcal{U}(L)/\mathcal{U}^n(L)) \longrightarrow \pi_0(\mathcal{U}^n(L)) \longrightarrow \pi_0(\mathcal{U}(L)) \longrightarrow \pi_0(\mathcal{U}(L)/\mathcal{U}^n(L)) \longrightarrow 0.
\]

Now \( \pi_1(\mathcal{U}(L)) = 0 \), and the \( \pi_0 \) can be computed via Proposition [6.1] as coequalizers:

\[
\pi_0(\mathcal{U}(L)) = \mathcal{U}(F)/\mathfrak{v} = \mathcal{U}(L),
\]

\[
\pi_0(\mathcal{U}(L)/\mathcal{U}^n(L)) = \mathcal{U}(F)/(\mathcal{U}^n(F) + \mathfrak{v}) = \mathcal{U}(L)/\mathcal{U}^n(L),
\]

\[
\pi_0(\mathcal{U}^n(L)) = \mathcal{U}^n(F)/\mathfrak{v}(n-1).
\]

Hence

\[
D_1 T_{n}(L) = \pi_1(\mathcal{U}(L)/\mathcal{U}^n(L)) = \ker \left[ \pi_0(\mathcal{U}^n(L)) \to \pi_0(\mathcal{U}(L)) \right] = \frac{\mathfrak{v} \cap \mathcal{U}^n(F)}{\mathfrak{v}(n-1)}. \tag{12}
\]

The sequence (12) yields a long exact homotopy sequence

\[
\cdots \longrightarrow \pi_1(\mathcal{U}(L)/\mathcal{U}^{n+1}(L)) \longrightarrow \pi_1(\mathcal{U}(L)/\mathcal{U}^n(L)) \longrightarrow \pi_0(\mathcal{U}(L)/\mathcal{U}^{n+1}(L)) \longrightarrow \pi_0(\mathcal{U}(L)/\mathcal{U}^n(L)) \longrightarrow \pi_0(\mathcal{U}(L)/\mathcal{U}^n(L)) \longrightarrow 0
\]

and a similar computation as above yields

\[
T_{n,0} = D_0 T_{n}(L) = \frac{\mathcal{U}^n(F)}{\mathcal{U}^{n+1}(F) + \mathfrak{v}(n-1)} \quad \text{for all } n \geq 2. \tag{13}
\]
6.6. The linear part of the dimension problem. Let $L$ be a free resolution of $L$, and write $X = L_{ab} = \Gamma_2(L)$, which we view as a graded module concentrated in degree 1. We recall the “free associative algebra”, “free Lie algebra” and “universal envelope” functors $T, \mathcal{L}, \mathcal{U}$ respectively, with $T = \mathcal{U} \otimes \mathcal{L}$. If $X$ is a graded module, then $T(X)$ and $\mathcal{L}(X)$ are naturally graded.

Proposition 6.9 (Essentially \cite[Theorem 3.3]{11}).

(i) $E_{1,n}^1 = \pi_m(\mathcal{L}_n(X))$ and $E_{1,n}^1 = \pi_m(T_n(X));$

(ii) $\iota_{1,0}^1 : E_{1,0}^1 \to E_{n,0}^1$ is injective.

Proof. Since $L$ is free, we have $L = \mathcal{L}(X)$ and $\mathcal{U}(L) = T(X);$ so (i) follows immediately from (1) follows from \cite{7} and \cite{9}. Now by (i) we have

$$E_{1,0}^1 = \pi_0(\mathcal{L}_n(X)) = \mathcal{L}_n(\pi_0(X))$$

because $'\mathcal{L}'$ preserves coequalizers so commutes with $\pi_0$, see Proposition \cite{11} and similarly, because $'T'$ and $'\mathcal{U}'$ preserve coequalizers,

$$E_{1,0}^1 = \pi_0(T_n(X)) = \mathcal{U}(\pi_0(\mathcal{L}_n(X))) = T_n(\pi_0(X)).$$

The map $\iota_{1,0}^1$ is the degree-$n$ part of the natural map $\iota : \mathcal{L}(\pi_0(X)) \to T(\pi_0(X))$, which is injective by Theorem \cite{11} so (ii) is proven. □

We deduce an analogue of one of Sjogren’s main results from \cite{39}. This is at the time of writing the main outcome of the simplicial approach, and should serve as a stepping-stone for further investigation of dimension quotients of Lie rings:

Theorem 6.10. For every free Lie ring $F$ over $\mathbb{Z}$, and all $n \geq 1$, we have

$$F \cap (z^n + 1(F) + \tau(n - 1)) = \gamma_n + 1(F) + R(n - 1).$$

Proof. The statement is equivalent to the claim

$$\frac{\gamma_{n+1}(F) + R(n-1)}{\gamma_{n+1}(F)} = \gamma_n(F) \cap (z^n + 1(F) + \tau(n - 1)),\gamma_{n+1}$$

an equality that takes place in $\text{gr}_n(F)$, or (by Theorem \cite{11}) equivalently in $\mathcal{R}_n(F)$, via the embedding $\iota : F \to \mathcal{U}(F)$. The statement is therefore equivalent, using \cite{11} and \cite{13}, to injectivity of $\iota_{1,0}^1$. □

6.7. Consequences for the dimension problem. Note that the special case $n = 2$ of Theorem 6.10 already yields the identification of the third Lie dimension subring, namely $\delta_3(L) = \gamma_3(L)$. We now show how this result is naturally interpreted using the two spectral sequences from \cite{6,11} and how they bear on the identification of $\delta_n(L)/\gamma_n(L)$.

Thanks to our computations \cite{11,13} for $E_{1,0}^1, E_{1,0}^1$ and \cite{10,12} for $E_{1,1}^1, E_{1,1}^1$ using $\text{gr}_1 = \Gamma_2$ and $\text{gr}_1 = \Gamma_2$, we obtain the beginning of the first pages of the spectral sequences:

\[
\begin{array}{ccc}
q = 1 & \gamma_n R \\
F & R^n & \gamma + [R,F] \\
E_{1,p,q}^1 & p = 1 & q = 2 \\
\end{array}
\]

and

\[
\begin{array}{ccc}
q = 1 & \gamma_n R \\
F & R^n & \gamma + [R,F] \\
E_{1,p,q}^1 & p = 1 & q = 2 \\
\end{array}
\]
On the last page, we then see

\[ \frac{(\gamma_3 + [R,F]) \cap R}{[R,F]} \]

and

\[ \frac{(m^3 + m^r + m^t) \cap t}{m^t + m^r} \]

In particular,

\[ E_{2,0}^2 = \gamma_2(L)/\gamma_3(L) \text{ and } \overline{E}_{2,0}^2 = \varpi^2(L)/\varpi^3(L). \]

The morphism \( \iota \) induced on spectral sequences by the inclusion \( L \to \mathcal{W}(L) \) yields then the following commutative diagram with exact rows:

\[
\begin{array}{cccc}
E_{1,1}^1 & \xrightarrow{d_1} & E_{0,0}^1 & \xrightarrow{0} \\
\downarrow{\iota_{1,1}} & & \downarrow{d_{1,0}} & \downarrow{0} \\
\overline{E}_{1,1}^1 & \xrightarrow{\overline{d}_1} & \overline{E}_{0,0}^1 & \xrightarrow{0} \\
\end{array}
\]

The homomorphism \( \iota_{1,1} \) is an isomorphism because the dimension problem is true for abelian Lie algebras. The homomorphism \( \iota_{2,0} \) is injective by Proposition 6.9(ii). It follows therefore that \( \iota_{2,0} \) is injective. We deduce a new proof of Theorem 4.5:

For every Lie algebra over \( \mathbb{Z} \), we have \( \delta_3(L) := L \cap \varpi(L) = \gamma_3(L) \).

Similarly, \( \delta_4(L)/\gamma_4(L) \) is an epimorphic image of \( \ker(\iota_{3,0}^2 : E_{3,0}^2 \to \overline{E}_{3,0}^2) \); and, more generally, Propositions 6.4 and 6.5 imply the

**Theorem 6.11.** For every Lie ring \( L \),

\[
\frac{\gamma_n(L) \cap \delta_{n+1}(L)}{\gamma_{n+1}(L)} = \ker \left[ n_{1,0} : E_{n,0}^\infty \to \overline{E}_{n,0}^\infty \right]. \quad \square
\]

7. **Concluding remarks**

We have attempted to show, in this text, how the theories of groups and Lie rings parallel each other; and, in particular, how tools developed to study dimension quotients in one context bear on the other.

It is our general feeling that most aspects become simpler when transposed to the Lie ring setting. The only exception we are aware of is Fox’s problem, see Example 5.3.

We believe that, at least in the Lie algebra setting, an optimal bound on the exponent of the dimension quotients, sharpening Sjogren’s bound, should be achievable.

One particularly tempting direction for further investigations is that of metabelian Lie rings. What is the best that one can say about dimension quotients in that case?

In the corresponding group case, the two series agree from some stage onward, depending only on \( G/[G,G] \), see [15]. Gupta also has obtained in [12, 14] a bound for the exponent of the dimension quotient that is much smaller than the Sjogren bound from [39].

Our Theorem 4.6 shows much more: in the metabelian case, the exponent of the \( n \)th dimension quotient is bounded by the exponent of the torsion of \( L/[L,L] \).

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