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Mathematical modeling of the spread of the COVID’19 with optimal control strategies

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Abstract

In this work, we formulated an SEIR mathematical model with four control variables, which describes the spread of the coronavirus in a community. An optimal control problem was well introduced in order to verify the existence of optimal controls that minimize susceptible, exposed, and infected individuals. Also, we use Pontryagin’s Maximum Principle to characterize the optimal control strategies to minimize the spread dynamics of the COVID’19.

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1. Introduction

The emerging coronavirus disease 2019 (COVID’19) is a new strain of coronavirus that appeared in early December 2019 in the Chinese city of Wuhan. It is known that this strain causes diseases similar to colds and can reach severity into more serious diseases such as Middle East Syndrome. Respiratory Syndrome (MERS) and Severe Respiratory Disease (SARS). Due to the rapid spread of infectious disease, the sharp increase in the number of infected and dead, and the magnitude of social damage and economic loss, it has been considered a global pandemic. There are some cases infected with the Coronavirus that recover without the need for special treatment, while there are many cases that need to follow the health protocol recommended by specialists. The degree of risk is high in the elderly and those who suffer from cardiovascular disease, chronic respiratory disease or disease diabetes, or cancer, and anyone at any age can get severely infected with COVID’19 disease and cause death. Mathematical modeling of infectious diseases has become an essential and effective tool for understanding how infections spread, making decisions about controlling transmission dynamics between individuals, and identifying effective strategies for disease elimination.

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Among the researches concerned with the study and analysis of mathematical models for some infectious diseases, we find [? ? ? ]. With regard to COVID’19, many researchers have relied on optimal control theory to study and analyze mathematical models that describe the spread of the pandemic, and thus find optimal control strategies that would allow it to be controlled. In [? ] we find that the authors have applied the theory of optimal control to the transmission model of COVID-19 in order to obtain optimal control strategies by reducing the number of exposed and infected populations thanks to the presence of optimal controls such as prevention, medical care, and surface disinfection. While, Seda [? ] uses the following control: lockdown, quarantine, and self-isolation strategies in its proposed model to minimize symptomatic, asymptomatic, and unreported infectious individuals. You can also see the following studies [? ? ? ? ? ? ]. In this work, we have taken into account the cases that are re-infected with Coronavirus virus after being cured from it, we have also used the control strategies which are health education and awareness, vaccination, measures preventives (quarantine and isolation), and intensive medical care. The remaining of this paper is organized in this way: This paper is organized as follows: In Section ??, we formulate an SEQIRS model with four control measures. In Section ??, the optimal control problem is given in order to prove the existence of the optimal control and establish its characterization, by applying Pontryagin’s Maximum Principle. Finally, we give a brief conclusion in Section ??.

2. COVID’19 model with controls

In this paper, we consider a mathematical model SEQIRS that describes the dynamic of transmission of the COVID’19 virus. We divide the population into the following compartment:

- Compartment Susceptible (S): Represents people vulnerable to COVID’19 virus infection, whether they have been infected before or not.
- Compartment Exposed (E): Represents people in an incubation period, who have contracted the virus without showing any symptoms and can transmit it to others.
- Compartment Quarantined (Q): Represents isolated and quarantined people.
- Compartment Infected (I): Represents infected people.
- Compartment Recovered (R): Represents people both cured and protected from Coronavirus infection through vaccination in specified doses.

We denote the total population by \( N(t) = S(t) + E(t) + Q(t) + I(t) + R(t) \).

To study the transmission of COVID’19 between individuals, we formulated a pandemic model with the integration of four controls. The first control \( u_1 \) is the public health awareness and education on how to prevent infection, the control \( u_2 \) represents vaccination for susceptible individuals, while the control \( u_3 \) represents the effort to quarantine and isolate either susceptible or exposed individuals and the control \( u_4 \) represents intensive medical care for confirmed cases.

The SEQIRS pandemic model is given by the following system of nonlinear differential equations:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \Theta - (1 - u_1(t))[\alpha E(t) + \beta I(t)]S(t) - (u_3(t) + u_2(t) + \mu)S(t) + \gamma R(t), \\
\frac{dE(t)}{dt} &= (1 - u_1(t))[\alpha E(t) + \beta I(t)]S(t) - (u_3(t) + \sigma + \mu)E(t), \\
\frac{dQ(t)}{dt} &= u_3(t)(S(t) + E(t)) - (r_1 + \delta + \mu)Q(t), \\
\frac{dI(t)}{dt} &= \sigma E(t) + \delta Q(t) - (u_4(t) + \eta + r_2 + \mu)I(t), \\
\frac{dR(t)}{dt} &= u_2(t)S(t) + r_1 Q(t) + (r_2 + u_4(t))I(t) - (\gamma + \mu)R(t),
\end{align*}
\]
3. Optimal control problem

The objective of this section is to verify the existence of the optimal control \( u^* = (u_1^*, u_2^*, u_3^*, u_4^*) \) of the controls \( u = (u_1, u_2, u_3, u_4) \), where the state trajectories \( S^*, E^*, Q^*, I^* \) and \( R^* \) are solutions of the system (1) in the time interval \([0, t_f]\), with nonnegative initial conditions \( S^*(0), E^*(0), Q^*(0), I^*(0) \) and \( R^*(0) \), by minimizing the objective functional that takes into account the number of susceptible, exposed, infected individuals during the course of a pandemic. Also, we aim to determine this optimal control using the famous Pontryagin’s Maximum Principle.

The objective functional \( J \) is defined as follows:

\[
J(u_1, u_2, u_3, u_4) = \int_0^{t_f} \left[ C_1 S(t) + C_2 E(t) + C_3 I(t) + \frac{1}{2} \sum_{i=1}^{4} \theta_i u_i^2(t) \right] dt,
\]

where \( C_i \geq 0, \) for \( i = 1, \cdots, 4 \) denote weights that balance the sizes of the \( S(t), E(t), \) and \( I(t) \). The balancing factors related to public health awareness and education, vaccination, preventative measures (quarantine and isolation), and intensive care, respectively, are represented by the parameters \( \theta_i \), for \( i = 1, \cdots, 4 \).

The optimal control problem can be formulated more exactly as follows:

\[
J(u_1^*, u_2^*, u_3^*, u_4^*) = \min_{\Omega} J(u_1, u_2, u_3, u_4),
\]

where the set of admissible controls is given by

\[
\Omega = \{u_1, u_2, u_3, u_4 : u_i(t) \text{ is lebesgue measurable, } 0 \leq u_i(t) < 1, \ t \in [0, t_f], \ \text{for } i = 1, 2, 3, 4\}
\]

**Theorem 1.** Consider the optimal control problem (3) subject to the state system (1) with nonnegative initial conditions. There exists an optimal control quadruple \( u^* = (u_1^*, u_2^*, u_3^*, u_4^*) \) in \( \Omega \) such that

\[
J(u_1^*, u_2^*, u_3^*, u_4^*) = \min_{\Omega} J(u_1, u_2, u_3, u_4).
\]

We use the Pontryagin Maximum Principle to find the necessary conditions that the optimal control quadruple of the model (1) must satisfy. Let \( X = (S, E, Q, I, R) \), \( u = (u_1, u_2, u_3, u_4) \in \Omega \) and \( \Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \) the adjoint variable. The Hamiltonian function is defined as

\[
H(X, u, \Lambda, t) = C_1 S(t) + C_2 E(t) + C_3 I(t) + \frac{1}{2} \sum_{i=1}^{4} \theta_i u_i^2(t) + \lambda_1(t) \frac{dS(t)}{dt} + \lambda_2(t) \frac{dE(t)}{dt} + \lambda_3(t) \frac{dI(t)}{dt} + \lambda_4(t) \frac{dQ(t)}{dt} + \lambda_5(t) \frac{dR(t)}{dt}.
\]

If \( (X^*(t), u^*(t)) \) be the optimal solution of an optimal control problem (1)-(3), then there exists a non trivial vector function \( \Lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t)) \) such that
equations awareness and education, vaccination, preventive measures, and intensive care. Using the well-known Pontryagin's

By differentiating the Hamiltonian function (5) at the respective solutions of the state system (1), Pontryagin’s Maximum Principle yields the form of the adjoint system (7) endowed with terminal conditions. We also use the optimality conditions to obtain the characterization of the optimal control given by (8). After solving the following partial differential equations on the interior of the control set \( \Omega \):

\[
\frac{\partial H}{\partial u_1} = 0 \quad \text{for} \quad u_1^*, \quad \frac{\partial H}{\partial u_2} = 0 \quad \text{for} \quad u_2^*, \quad \frac{\partial H}{\partial u_3} = 0 \quad \text{for} \quad u_3^*, \quad \frac{\partial H}{\partial u_4} = 0 \quad \text{for} \quad u_4^*.
\]

We obtain directly (8) by taking into consideration the boundedness condition given in (4).

4. Conclusion

In this article, an optimal control model was formulated by considering four control variables such as public health awareness and education, vaccination, preventive measures, and intensive care. Using the well-known Pontryagin’s

\[
\begin{align*}
\frac{dX}{dt} &= \frac{\partial H(X^*(t), u^*(t), \Lambda(t), t)}{\partial \Lambda} \\
0 &= \frac{\partial H(X^*(t), u^*(t), \Lambda(t), t)}{\partial u} \\
\frac{d\Lambda(t)}{dt} &= -\frac{\partial H(X^*(t), u^*(t), \Lambda(t), t)}{\partial X}.
\end{align*}
\]
Maximum Principle, we were able to prove the existence of the optimal control quadruple for the SEQIRS model and characterize it. The results obtained here will have an important role in our future work, which will focus on a careful analytical study of the results of the numerical simulations of the optimality system, in order to propose the effective strategy to stop the spread of the Coronavirus.

Conflicts of Interest
The authors declare that they have no competing interests in this paper’s publication.

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