Quantum coordinate ring in WZW model and affine vertex algebra extensions

Yuto Moriwaki

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Abstract
In this paper, we construct various simple vertex superalgebras which are extensions of affine vertex algebras, by using abelian cocycle twists of representation categories of quantum groups. This solves the Creutzig and Gaiotto conjectures (Creutzig and Gaiotto in Comm Math Phys 379:785–845, 2020, Conjecture 1.1 and 1.4) in the case of type ABC. If the twist is trivial, the resulting algebras correspond to chiral differential operators in the chiral case, and to WZW models in the non-chiral case.

Keywords  Vertex algebra · Quantum coordinate ring · Conformal field theory

Mathematics Subject Classification  17B69 · 17B81

Introduction
Let \( g \) be a finite dimensional simple Lie algebra, \( h^\vee \) the dual Coxeter number and \( r^\vee \) the lacing number, that is, \( r^\vee = 1 \) (resp. \( r^\vee = 2 \) and \( r^\vee = 3 \)) if the simple Lie algebra \( g \) is simply-laced (resp. of type BCF and of type G). Let \( k, k' \in \mathbb{C} \setminus \mathbb{Q} \) and \( N \in \mathbb{Z} \) satisfy
\[
\frac{1}{r^\vee(k + h^\vee)} + \frac{1}{r^\vee(k' + h^\vee)} = N. \tag{0.1}
\]
Let \( P \) be the weight lattice and \( Q \) the root lattice and set
\[
V^N_{g,k,k'}(P) = \bigoplus_{\lambda \in P^+} L_{g,k}(\lambda) \otimes L_{g,k'}(\lambda^\ast),
\]
with \( P^+ \) the set of dominant integral weights. Here, \( L_{g,k}(\lambda) \) is a module of the affine Lie algebra \( \hat{g} \) at level \( k \) induced from \( L(\lambda) \), the irreducible finite dimensional representation

\( \mathbb{C} \)  Yuto Moriwaki
moriwaki.yuto@gmail.com

1 Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan
of \(\mathfrak{g}\) with highest weight \(\lambda\). The module induced from the dual representation \(L(\lambda)^*\) at level \(k'\) is denoted by \(L_{\mathfrak{g},k}(\lambda^*)\).

Creutzig and Gaiotto conjectured that \(V_{\mathfrak{g},k,k'}^N(P)\) inherits a vertex superalgebra structure based on gauge theory [6, Conjecture 1.1]. The construction of vertex superalgebras of this form has been an important problem also in some other closely related areas, say, the quantum geometric Langlands program [20] and the duality among W-superalgebras as conjectured by [9, 10].

For \(N = 0\), the condition \((0.1)\) can be written as \(k + k' = -2h^\vee\). Such a pair \((k, k')\) is called dual and \(V_{\mathfrak{g},k,k'}^0(P)\) is realized as the algebra of chiral differential operators, which has been studied in various contents [1, 23, 25, 26, 45]. In [8, Corollary 1.4] and [19, Proposition 5.3], it was shown that \(V_{\mathfrak{g},k,k'}^N(P)\) is a vertex algebra when \(N \in 2n_\mathfrak{g}\mathbb{Z}\). Here \(n_\mathfrak{g}\) is the smallest positive integer such that \(n_\mathfrak{g}P\) is an integral lattice with respect to the bilinear form \(\langle\langle - , - \rangle\rangle : P \times P \to \mathbb{Q}\) which is normalized as \((\alpha, \alpha) = 2\) for short roots \(\alpha\).

In this paper, we will do two things: One is to reduce the problem of affine VOA extensions to a problem in the theory of quantum groups, which we call Conjecture A, and the other one is to prove Conjecture A in many cases.

More precisely, if Conjecture A is true for \((\mathfrak{g}, N)\), then we show that \(V_{\mathfrak{g},k,k'}^N(P)\) is an abelian intertwining algebra with abelian cocycle \(EM^{-1}(Q^N_{\mathfrak{g}}) \in H^3_{ab}(P/Q, \mathbb{C}^\times)\).

Conjecture A is proved for any \(N \in 2\mathbb{Z}\) (Proposition 2.14) and for any \(N \in \mathbb{Z}\) in the case that \(\mathfrak{g}\) is of type ABC (see Main Theorem A below). This in particular solves the conjecture by Creutzig and Gaiotto for type ABC.

The approach using the representation theory of quantum groups enables us to construct a more general family of vertex superalgebras, extending

\[
\left( \bigotimes_{i=1}^r L_{\mathfrak{g},k_i}(0) \otimes L_{\mathfrak{g},k_i'}(0) \right) \otimes V_M.
\]

Here \(V_M\) is the lattice vertex (super)algebra associated with an integral lattice \(M\) (for the precise statement, see Theorem B below). This allows us to construct, for example, the following vertex superalgebra for any \(n \geq 2\) (Proposition 3.12 and 3.15):

\[
\bigoplus_{\lambda \in P_{\mathfrak{sl}_n}} L_{\mathfrak{sl}_n,k}(\lambda) \otimes L_{\mathfrak{sl}_n,k}(\lambda^*) \otimes V_{\frac{\omega(\lambda)}{n\sqrt{\pi} + /\sqrt{n\mathbb{Z}}}} \quad \text{for} \quad \frac{1}{k+n} + \frac{1}{k'+n} = 1, \tag{0.2}
\]

\[
\bigotimes_{\lambda \in P_{\mathfrak{sl}_n}} L_{\mathfrak{sl}_n,k_i}(\lambda) \otimes L_{\mathfrak{sl}_n,k_i'}(\lambda^*) \quad \text{for} \quad \frac{1}{k_i+n} + \frac{1}{k_i'+n} = 1 \quad \text{and} \quad i = 1, \ldots, n, \tag{0.3}
\]

where \(V_{\sqrt{n\mathbb{Z}}}\) is a rank one lattice vertex (super)algebra and \(V_{\frac{\omega(\lambda)}{n\sqrt{\pi} + /\sqrt{n\mathbb{Z}}}}\) is its module, \(i\) is a map given by \(i : P_{\mathfrak{sl}_n} \to P_{\mathfrak{sl}_n} / Q_{\mathfrak{sl}_n} \cong \mathbb{Z}/n\mathbb{Z}\) (The extension \((0.2)\) is conjectured in [6, Conjecture 1.4]).

In the following, we will explain that quantum groups and quantum coordinate rings appear naturally when considering extensions of affine vertex algebras, and then we will discuss the conjecture about the representation categories of quantum groups and the main results derived from it.
0.1 Quantum coordinate rings and WZW models

There is a natural correspondence between commutative algebra objects in the representation category of a vertex (operator) algebra and extensions of the vertex (operator) algebra [22, 30]. The representation category of the affine vertex algebra $L_{g,k}(0)$ at level $k \in \mathbb{C}\setminus \mathbb{Q}$ is called the Drinfeld category $D(g, k)$ [17, 24, 43] and is equivalent to the representation category of the quantum group $(U_q(g), R(\rho))$-mod as braided tensor categories [14, 15, 34, 39]. Here, $q = \exp(\pi i \rho), \rho = \frac{1}{r^\vee(k + h^\vee)}$, and $R(\rho)$ is the R-matrix of $U_q(g)$ which gives the braiding. This equivalence is called the Kazhdan–Lusztig correspondence.

The dual Hopf algebra of $U_q(g)$ is called a quantum coordinate ring and denoted by $O_q(G)$. The algebra $O_q(G)$ inherits a natural $U_q(g)$-bimodule structure and is a commutative algebra object in the braided tensor category $(U_q(g), R(\rho))$-mod $\otimes (U_q(g)^{\text{cop}}, R(\rho)^{-1})$-mod (Proposition 1.11 and Proposition 3.1).

Using an isomorphism of quasi-triangular Hopf algebras $(U_q(g), R(\rho)^{-1})$ and $(U_q(g)^{\text{cop}}, R(\rho)^{-1})$ (Proposition 2.11) and the Kazhdan–Lusztig correspondence, $O_q(G)$ is regarded as a commutative algebra object of $D(g, k) \otimes D(g, k)^{\text{rev}}$, where $D(g, k)^{\text{rev}}$ is the braided tensor category with the reverse braiding. Hence, $O_q(G)$ defines an extension of the tensor product of the “holomorphic” affine vertex algebra $L_{g,k}(0)$ and an “anti-holomorphic” affine vertex algebra $L_{g,k}(0)$,

$$F_{g,k} = \bigoplus_{\lambda \in P^+} L_{g,k}(\lambda) \otimes L_{g,k}(\lambda^*), \quad Y : F_{g,k} \to \text{End} F_{g,k}[|z, \bar{z}, |z|^C]].$$

This algebra satisfies the axiom of a full vertex algebra\(^1\) (one of the mathematical formulations [40] of the non-chiral CFTs in physics) and is nothing but the analytic continuation of the WZW model in physics (Proposition 3.2).

On the other hand, using an isomorphism of quasi-triangular Hopf algebras $(U_q(g)^{\text{cop}}, R(\rho)^{-1}) \cong (U_q^{-1}(g), R(-\rho))$ (Proposition 2.13), $O_q(G)$ gives a commutative algebra object in $D(g, \bar{k}) \otimes D(g, \bar{k})$, which corresponds to the chiral differential operator $V^0_{g,k,k}(P)$.

It is worth mentioning here that from this quantum group approach the non-chiral vertex algebra (the WZW model) appears first naturally, and the chiral vertex algebra (the chiral differential operator) appears second by applying the quantum group isomorphism. These are the stories for the $N = 0$ case.

0.2 Graded twisting

Next, we discuss affine vertex algebra extensions for (0.1) with $N \in \mathbb{Z}$ in general. The category $(U_q(g), R(\rho))$-mod is naturally graded by $P/Q$. In general, for an abelian group $A$ and an $A$-graded braided tensor category, we can “twist” the braided tensor category structure by an abelian cocycle $(\alpha, c) \in H^3_{ab}(A, \mathbb{C}^\times)$ (see [36, 41]). It is

\(^1\) $F_{g,k}$ also satisfies the axiom of a full field algebra introduced by Huang and Kong in [29].
well-known that $H^3_{\text{ab}}(A, \mathbb{C}^\times)$ is isomorphic as a group to the space of all quadratic forms on $A$.

For $N \in \mathbb{Z}$, let us define the quadratic form $Q^N_B : P/Q \to \mathbb{C}^\times$ on $P/Q$ by

$$Q^N_B(\lambda) = \exp(N\pi i (\langle \lambda, \lambda \rangle))$$

for $\lambda \in P/Q$.

Denote by $(U_q(g), R(\rho))$-mod$^{Q^N_B}$ the twist of $(U_q(g), R(\rho))$-mod by the abelian cocycle associated with $Q^N_B$ (for more precise definitions, see Sect. 1.3 and Sect. 2.4). Then, we expect the following:

**Conjecture A** For any $N \in \mathbb{Z}$, $(U_q(g), R(\rho))$-mod$^{Q^N_B}$ and $(U_q(g), R(\rho + N))$-mod are equivalent as braided tensor categories.

We will show this conjecture in the following cases (Proposition 2.14 and Theorem 2.15):

**Main Theorem A** Conjecture A is true in the following cases:

1. $N \in 2\mathbb{Z}$;
2. $N \in \mathbb{Z}$, for $g$ of type ABC.

Furthermore, for $g$ of type D, $(U_q(so_{2n}), R(\rho), \Lambda_v)$-mod$^{Q^{so_{2n}}_B}$ and $(U_q(so_{2n}), R(\rho + N), \Lambda_v)$-mod are equivalent as braided tensor categories, where $(U_q(so_{2n}), R(\rho), \Lambda_v)$-mod is the full subcategory of $(U_q(so_{2n}), R(\rho))$-mod generated by the vector representation of $U_q(so_{2n})$.

We briefly explain the proof of Theorem A. Theorem A follows from the definition when $N$ is even. In the case of type ABCD, Theorem A can be proved by using the characterization of the braided tensor category of type $A$ (resp. type BCD) by [36] (resp. [44]). The result for the braided tensor category of type BD by [44] is a characterization of subcategories, and hence the claim for type BD is weaker than Conjecture A. We may give a direct proof for type B by using a Hopf algebra isomorphism $\phi : U_q(so_{2n+1}) \to U_{-q}(so_{2n+1})$. In this case, the pullback by $\phi$ does not transfer the type 1 representation of $U_{-q}(so_{2n+1})$ to the type 1 representation $U_q(so_{2n+1})$. This is why “the twisted category” appears. This will be discussed in detail in Appendix.

### 0.3 Lax monoidal functors and main theorem

We will now state the result and the proof of the construction of vertex superalgebras (Main Theorem B) using Conjecture A. Let $A$ be an abelian group and let $\text{Vec}_A$ denote the braided tensor category of $A$-graded vector spaces equipped with trivial braiding. Using the fact that $O_q(G)$ gives a commutative algebra object in $(U_q(g), R(\rho))$-mod $\otimes (U_{-q}(g), R(-\rho))$-mod and that $O_q(G)$ is a $P/Q$-graded algebra, we can construct a lax monoidal functor

$$O_B : \text{Vec}_{P/Q} \to (U_q(g), R(\rho))$-mod $\otimes (U_{-q}(g), R(-\rho))$-mod$$

which preserves the braiding (we call it a lax braided monoidal functor, see Sect. 1.2).
Using Conjecture A with simultaneous grading twists of $\textbf{Vec}_{P/Q}$ and $(U_q(\mathfrak{g}), R(\rho))$-mod, we obtain a lax braided monoidal functor (Lemma 3.4 and Proposition 3.5)

$$O^N_\mathfrak{g} : \textbf{Vec}^{Q^N_\mathfrak{g}}_{P/Q} \to (U_{(-1)^N_q}(\mathfrak{g}), R(\rho + N))$-mod $\otimes (U_{q^{-1}}(\mathfrak{g}), R(-\rho))$-mod.$$

Since a lax braided monoidal functor preserves supercommutative algebra objects, it suffices to find them in $\textbf{Vec}^{Q^N_\mathfrak{g}}_{P/Q}$. Although we can find various nontrivial supercommutative algebra objects in $\textbf{Vec}^{Q^N_\mathfrak{g}}_{P/Q}$, we can make more examples by considering extensions of the tensor product of the affine vertex algebra and a lattice vertex algebra.

Let $M$ be an even lattice and $M^\vee$ denote the dual lattice of $M$. By $[12]$, $\textbf{Vec}^{Q^N_\mathfrak{g}}_{P/Q} \oplus M^\vee$ is equivalent to the representation category of the lattice vertex algebra $V_{M}$ as a braided tensor category. Thus, from supercommutative algebra objects in $\textbf{Vec}^{Q^N_\mathfrak{g}}_{P/Q} \oplus M^\vee$, we can construct extensions of $L_{\mathfrak{g},k}(0) \otimes L_{\mathfrak{g},k'}(0) \otimes V_{M}$. Here, $k, k'$ satisfy

$$\rho + N = \frac{1}{r^\vee(k + h^\vee)}, \quad -\rho = \frac{1}{r^\vee(k' + h^\vee)},$$

and thus (0.1). Supercommutative algebra objects in $\textbf{Vec}^{Q^N_\mathfrak{g}}_{P/Q} \oplus M^\vee$ can be constructed from a super isotropic subspace of quadratic space $(P/Q \oplus M^\vee, Q^N_\mathfrak{g} \oplus Q_M)$ (for the definition of a super isotropic subspace, see Sect. 1.3). Our main result can then be stated as follows:

**Main Theorem B** Let $r$ be a positive integer and $\mathfrak{g}_i$ simple Lie algebras and $k_i, k'_i \in \mathbb{C}\backslash \mathbb{Q}$ and $N_i \in \mathbb{Z}$ satisfy

$$\frac{1}{r^\vee_i(k_i + h^\vee_i)} + \frac{1}{r^\vee_i(k'_i + h^\vee_i)} = N_i$$

for $i = 1, \ldots, r$. Let $M$ be an even lattice and $(A, Q)$ a quadratic space defined by

$$A = \left( \bigoplus_{i=1}^{r} P_i/Q_i \right) \oplus M^\vee/M, \quad Q = \left( \bigoplus_{i=1}^{r} Q^N_{\mathfrak{g}_i}/Q_{\mathfrak{g}_i} \right) \oplus Q_M.$$  

Let $(I, p)$ be a super isotropic subspace of the quadratic space $(A, Q)$. Set

$$V^N_{\mathfrak{g},k,k',M}(I) = \bigoplus_{(\lambda_1, \ldots, \lambda_r, \mu) \in I} \bigotimes_{i=1}^{r} L_{\mathfrak{g}_i,k_i,k'_i}(\lambda_i + Q_i) \otimes V_{\mu+M},$$
for $(g, k, k', M, I)$ with $g = (g_1, \ldots, g_r)$ and $k = (k_1, \ldots, k_r), k' = (k'_1, \ldots, k'_r), N = (N_1, \ldots, N_r)$, where

$$L^N_{g_i, k_i, k'_i}(\lambda_i + Q_i) = \bigoplus_{\lambda \in (\lambda_i + Q_i) \cap P_i^+} L_{g_i, k_i}(\lambda) \otimes L_{g_i, k'_i}(\lambda).$$

Suppose that for each $a = 1, \ldots, r$ one of the following conditions is satisfied:

1) $N_a$ is even;
2) $g_a$ is of type ABC;
3) $g_a$ is of type D and $\text{pr}_a(I) \subset \Lambda_v/Q_i$, where $\text{pr}_a : (\bigoplus_{i=1}^r P_i/Q_i) \oplus M^\vee/M \to P_a/Q_a$ is the projection to the $a$-th component.

Then, there is a simple vertex superalgebra structure on $V^N_{g, k, k', M}(I)$ as an extension of $\left( \bigotimes_{i=1}^r L_{g_i, k_i}(0) \otimes L_{g_i, k'_i}(0) \right) \otimes V_M$. Furthermore, the even part ($s = 0$) and the odd part ($s = 1$) are given by

$$V^N_{g, k, k', M}(I)_s = \bigoplus_{(\lambda_1, \ldots, \lambda_r, \mu) \in I} \bigotimes_{i=1}^r L_{g_i, k_i, k'_i}(\lambda_i + Q) \otimes V_{\mu + M}.$$  

We remark that Conditions (1), (2) and (3) in the theorem are due to the fact that Conjecture A has only been partially proved. The list of all vertex superalgebras (except for type A) that can be constructed from the theorem for $M = 0, r = 1$ is summarized in Table 1 and 2. In this case, super isotropic subspaces $I \subset P/Q$ correspond to lattices $L$ such that $Q \subset L \subset P$.

| Type    | Shift $N$ | Lattice $L$ | Super |
|---------|-----------|-------------|-------|
| $B_{2n}$ | $1 + 2\mathbb{Z}$ | $P$ | Yes |
| $B_{2n}$ | $2\mathbb{Z}$ | $P$ | |
| $B_{2n+1}$ | $2 + 4\mathbb{Z}$ | $P$ | Yes |
| $B_{2n+1}$ | $4\mathbb{Z}$ | $P$ | |
| $C_n$ | $1 + 2\mathbb{Z}$ | $P$ | Yes |
| $C_n$ | $2\mathbb{Z}$ | $P$ | |
| $D_n$ | $1 + 2\mathbb{Z}$ | $\Lambda_v$ | Yes |
| $D_n$ | $2\mathbb{Z}$ | $\Lambda_v$ | |
| $D_{4n+2}$ | $2 + 4\mathbb{Z}$ | $P$ | Yes |
| $D_{4n}$ | $2\mathbb{Z}$ | $P$ | |
| $D_{2n}$ | $4\mathbb{Z}$ | $P$ | |
| $D_{2n+1}$ | $4 + 8\mathbb{Z}$ | $P$ | Yes |
| $D_{2n+1}$ | $8\mathbb{Z}$ | $P$ | |
Table 2 List of type EFG

| Type | Shift $N$ | Lattice $L$ | Super |
|------|-----------|-------------|-------|
| $E_6$ | $6\mathbb{Z}$ | $P$ | |
| $E_6$ | $2\mathbb{Z}$ | $Q$ | |
| $E_7$ | $2 + 4\mathbb{Z}$ | $P$ | Yes |
| $E_7$ | $4\mathbb{Z}$ | $P$ | |
| $E_8$ | $2\mathbb{Z}$ | $P$ | |
| $F_4$ | $2\mathbb{Z}$ | $P$ | |
| $G_2$ | $2\mathbb{Z}$ | $P$ | |

In the case of $A_{n-1}$ type, there are various choices of $N$ and $L$. In fact, for $N \in \mathbb{Z}$ and $L = mP + Q$, $V^N_{sl_n,k,k'}(mP + Q)$ inherits a vertex algebra structure if $\exp\left(\frac{m^2N(n-1)}{n}\pi i\right) = 1$ and vertex superalgebra structure if $\exp\left(\frac{m^2N(n-1)}{n}\pi i\right) = -1$ (see Proposition 3.13 (4)).

This paper is organized as follows: we recall the definition of a braided tensor category, a lax monoidal functor and a supercommutative algebra object in Sect. 1.2 and the fundamental results on abelian cocycles, graded twists and super isotropic subspaces in Sect. 1.3. In Sect. 1.4, we recall some elementary results of quasi-triangular Hopf algebra and in Sect. 1.5, we briefly review the Drinfeld category and the fact that supercommutative algebra objects in the Drinfeld category correspond to vertex super-algebra extensions of the affine vertex algebra. Non-chiral cases are briefly explained in Sect. 1.6, which will be used for the construction of the WZW models.

Sect. 2.1 and 2.2 are devoted to recalling the definition and some results of quantum group, their R-matrices and the Kazhdan–Lusztig correspondence. Then, we show that isomorphisms among quantum groups give equivalences of braided tensor categories in Sect. 2.3. In Sect. 2.4, we state Conjecture A and prove Theorem A.

In Sect. 3.1 and 3.2 , we recall the definition of quantum coordinate rings and construct the lax monoidal functor by using it. Then, Theorem B is proved. Various vertex algebras as applications of Theorem B are constructed in Sect. 3.3. Finally, in Appendix, Theorem A is proved in the case of type B.

1 Preliminary

1.1 Notation

We fix the following notations:

- $\mathfrak{g}$: finite dimensional simple Lie algebra over $\mathbb{C}$
- $\mathfrak{h}$: its Cartan subalgebra
- $\Delta$: the root system of $\mathfrak{g}$
- $\Pi$: $\{\alpha_1, \ldots, \alpha_r\} \subset \Delta$ the set of simple roots
- $( -, -)$: the invariant bilinear form on $\mathfrak{g}$ normalized by $(\alpha, \alpha) = 2$ for long roots $\alpha$
- $\langle\langle-, -\rangle\rangle$: the invariant bilinear form on $\mathfrak{g}$ normalized by $\langle\langle\alpha, \alpha\rangle\rangle = 2$ for short roots $\alpha$
\(h^\vee\): the dual Coxeter number
\(P \subset h^\ast\): the weight lattice
\(P^+ \subset P\): the dominant integer weights
\(Q \subset P\): the root lattice
\(Q^+\): \(= \bigoplus_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha\)
\(\rho \in P\): half the sum of all positive roots

Let \(z\) and \(\bar{z}\) be independent formal variables. We will use the notation \(z\) for the pair \((z, \bar{z})\) and \(|z|\) for \(z\bar{z}\). For a vector space \(V\), we denote by \(V[[z^C, \bar{z}^C]]\) the set of formal sums
\[
\sum_{s, \bar{s} \in C} a_{s, \bar{s}} z^s \bar{z}^{\bar{s}}
\]
such that \(a_{s, \bar{s}} \in V\). The space \(V[[z^C, \bar{z}^C]]\) contains various subspaces:

\[
V[[z^C]] = \left\{ \sum_{s \in C} a_s z^s \mid a_s \in V \right\},
\]
\[
V[[z^\pm]] = \left\{ \sum_{n \in \mathbb{Z}} a_n z^n \mid a_n \in V \right\},
\]
\[
V[[z, \bar{z}, |z|^C]] = \left\{ \sum_{s, \bar{s} \in C} a_{s, \bar{s}} z^s \bar{z}^{\bar{s}} \mid a_{s, \bar{s}} = 0 \text{ unless } s - \bar{s} \in \mathbb{Z} \right\},
\]
\[
V[[z, \bar{z}]] = \left\{ \sum_{n, \bar{n} \in \mathbb{N}} a_{n, \bar{n}} z^n \bar{z}^{\bar{n}} \mid a_{n, \bar{n}} \in V \right\}.
\]

We also denote by \(V((z, \bar{z}, |z|^C))\) the subspace of \(V[[z, \bar{z}, |z|^C]]\) consisting of the series \(\sum_{s, \bar{s} \in \mathbb{R}} a_{s, \bar{s}} z^s \bar{z}^{\bar{s}} \in V[[z, \bar{z}, |z|^C]]\) such that:

1. For any \(H \in \mathbb{R}\), \(\# \{(s, \bar{s}) \in \mathbb{C}^2 \mid a_{s, \bar{s}} \neq 0 \text{ and } \text{Re} \ (s + \bar{s}) \leq H\} \) is finite.
2. There exists \(N \in \mathbb{R}\) such that \(a_{s, \bar{s}} = 0\) unless \(\text{Re} \ s \geq N\) and \(\text{Re} \ \bar{s} \geq N\) and \(V((z))\) the subspace of \(V[[z^\pm]]\) consisting of the series \(\sum_{n \in \mathbb{Z}} a_n z^n \in V[[z^\pm]]\) such that:

1. There exists \(N \in \mathbb{R}\) such that \(a_n = 0\) unless \(n \geq N\).

The space \(V((z))\) is called the space of formal Laurent series. Thus, \(V((z, \bar{z}, |z|^C))\) is a generalization of the Laurent series to two-variables.

### 1.2 Braided tensor category

This section is mainly based on [16] (for a lax monoidal functor, see also [2]). Let \(\mathcal{B}\) be an essentially small, \(\mathbb{C}\)-linear, monoidal category. We write
\[
\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}, \quad (M, N) \mapsto M \otimes N
\]
for the tensor product functor with unit object $1_B$, by

$$l_b : 1_B \otimes \bullet \sim \bullet, \quad (l_M : 1_B \otimes M \sim M), \quad \text{(1.1)}$$

$$r_b : \bullet \otimes 1_B \sim \bullet, \quad (r_M : M \otimes 1_B \sim M), \quad \text{(1.2)}$$

$$\alpha_{\bullet, \bullet, \bullet} : (\bullet \otimes \bullet) \otimes \bullet \sim \bullet \otimes (\bullet \otimes \bullet), \quad \left(\alpha_{M,N,L} : (M \otimes N) \otimes L \sim M \otimes (N \otimes L)\right), \quad \text{(1.3)}$$

the structural natural isomorphisms of functors satisfying the pentagon and triangle axioms, see [16]. The last one is called the associativity isomorphism.

A braided tensor category is a tensor category $\mathcal{B}$ equipped with a natural isomorphism of functors, called a braiding,

$$B_{\bullet, \bullet} : (\bullet \otimes \bullet) \sim (\bullet \otimes \bullet) \circ \sigma, \quad (B_{M,N} : M \otimes N \sim N \otimes M),$$

where $\sigma$ is the functor

$$\sigma : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}, \quad (M, N) \mapsto (N, M).$$

The natural isomorphism $B_{\bullet, \bullet}$ is required to satisfy the hexagon identity,

$$\begin{array}{c}
(M \otimes N) \otimes L \xrightarrow{\alpha_{M,N,L}} M \otimes (N \otimes L) \xrightarrow{B_{M,N \otimes L}} (N \otimes L) \otimes M \\
\downarrow_{B_{M,N \otimes 1}} \quad \downarrow_{\alpha_{N,L,M}} \\
(N \otimes M) \otimes L \xrightarrow{\alpha_{N,M,L}} N \otimes (M \otimes L) \xrightarrow{1 \otimes B_{M,L}} N \otimes (L \otimes M)
\end{array}$$

and

$$\begin{array}{c}
M \otimes (N \otimes L) \xrightarrow{\alpha_{M,N,L}^{-1}} (M \otimes N) \otimes L \xrightarrow{B_{M \otimes L, N}} L \otimes (M \otimes N) \\
\downarrow_{1 \otimes B_{N,L}} \quad \downarrow_{\alpha_{L,M,N}^{-1}} \\
M \otimes (L \otimes N) \xrightarrow{\alpha_{L,N,M}^{-1}} (M \otimes L) \otimes N \xrightarrow{B_{M,L \otimes 1}} (L \otimes M) \otimes N
\end{array}$$

A twist, or a balance, on a braided tensor category $\mathcal{B}$ is a natural isomorphism $\theta$ from the identity functor on $\mathcal{B}$ to itself satisfying

$$B_{M,N}B_{N,M} = \theta_{M \otimes N} \circ (\theta_M^{-1} \otimes \theta_N^{-1}).$$

A balanced braided tensor category is a braided tensor category equipped with such a balance.

Let $(\mathcal{B}, \alpha, B)$ be a braided tensor category and $B_{\bullet, \bullet}^{\text{rev}}$ a natural isomorphism defined by

$$B_{M,N}^{\text{rev}} = B_{N,M}^{-1} : M \otimes N \rightarrow N \otimes M.$$
Then, it is clear that \((\mathcal{B}, \alpha, B^{rev})\) is a braided tensor category, which is denoted by \(\mathcal{B}^{rev}\).

A **supercommutative algebra object** in \(\mathcal{B}\) is a triple \((S = S_0 \oplus S_1, \{m_{i,j}\}_{i,j=0,1}, \eta)\), consisting of objects \(S_0, S_1 \in \text{Ob}(\mathcal{B})\) and morphisms \(m_{i,j} \in \text{Hom}_\mathcal{B}(S_i \otimes S_j, S_{i+j})\) and a nonzero morphism \(\eta \in \text{Hom}_\mathcal{B}(1_B, S_0)\) such that:

\[(SCA1)\] \(m \circ (\eta \otimes \text{id}) \circ l_S^{-1} = \text{id} = m \circ (\text{id} \otimes \eta) \circ r_S^{-1}\) as a map \(S \rightarrow S;\)

\[(SCA2)\] \(m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m) \circ \alpha_{S,S,S}\) as a map \((S \otimes S) \otimes S \rightarrow S,\)

where \(m : S \otimes S \rightarrow S\) is a map defined linearly by \(\{m_{i,j}\}_{i,j=0,1}\), and;

\[(SCA3)\] \(m_{j,i} : B_{S_i,S_j} = (-1)^{ij} m_{i,j}\) as a map \(S_i \otimes S_j \rightarrow S_{i+j}\) for any \(i, j \in \mathbb{Z}_2\).

If a supercommutative algebra object in \(\mathcal{B}\) consists of only the even part, i.e., \(S_1 = 0\), then the triple \((S_0, m_00, \eta)\) is called a **commutative algebra object** in \(\mathcal{B}\).

Let \(\mathcal{C}\) and \(\mathcal{D}\) be braided tensor categories. A **lax monoidal** functor between them is a functor \(F : \mathcal{C} \rightarrow \mathcal{D}\) equipped with a morphism

\[\epsilon : 1_\mathcal{D} \rightarrow F(1_\mathcal{C})\]

and a natural transformation

\[\mu_{M,N} : F(M) \otimes_\mathcal{D} F(N) \rightarrow F(M \otimes_\mathcal{C} N) \quad \text{(for all } M, N \in \mathcal{C})\]

such that

**LM1)** For all objects \(M, N, L \in \mathcal{C}\), the following diagram commutes

\[
\begin{array}{c}
(F(M) \otimes_\mathcal{D} F(N)) \otimes_\mathcal{D} F(L) \\
\downarrow_{\mu_{M,N} \otimes \text{id}} \\
F(M \otimes_\mathcal{C} N) \otimes_\mathcal{D} F(L) \\
\downarrow_{\mu_{M \otimes_\mathcal{C} N,L}} \\
F((M \otimes_\mathcal{C} N) \otimes_\mathcal{C} L) \\
\end{array}
\rightarrow
\begin{array}{c}
F(M) \otimes_\mathcal{D} (F(N) \otimes_\mathcal{D} F(L)) \\
\downarrow_{\text{id} \otimes \mu_{N,L}} \\
F(M) \otimes_\mathcal{D} F(N \otimes_\mathcal{C} L) \\
\downarrow_{\mu_{M,N} \otimes_\mathcal{C} L} \\
F(M \otimes_\mathcal{C} (N \otimes_\mathcal{C} L)). \\
\end{array}
\]

**LM2)** For all \(M \in \mathcal{C}\) the following diagrams commute

\[
\begin{array}{c}
1_\mathcal{D} \otimes_\mathcal{D} F(M) \\
\downarrow_{r_{F(M)}} \\
F(M) \\
\end{array}
\rightarrow
\begin{array}{c}
\epsilon \otimes \text{id} \\
\downarrow_{\mu_{1C,M}} \\
F(1 \otimes_\mathcal{C} M). \\
\end{array}
\]

and

\[
\begin{array}{c}
F(M) \otimes_\mathcal{D} 1_\mathcal{D} \\
\downarrow_{r_{F(M)}} \\
F(M) \\
\end{array}
\rightarrow
\begin{array}{c}
\text{id} \otimes \epsilon \\
\downarrow_{\mu_{M,1C}} \\
F(M \otimes_\mathcal{C} 1). \\
\end{array}
\]
Lemma 1.1 Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a lax braided monoidal functor and $(S = S_0 \oplus S_1, \{m_{i,j}\}_{i,j=0,1}, \eta)$ be a supercommutative algebra object in $\mathcal{C}$. Then, $F(S_0) \oplus F(S_1)$ inherits a structure of a supercommutative algebra object in $\mathcal{D}$ by setting

$$\eta^F : 1_\mathcal{D} \xrightarrow{\xi} F(1_\mathcal{C}) \xrightarrow{F(\eta)} F(S_0)$$

and for $i, j = 0, 1$

$$m^F_{i,j} : F(S_i) \otimes_{\mathcal{D}} F(S_j) \xrightarrow{\mu_{S_i,S_j}} F(S_i \otimes_{\mathcal{C}} S_j) \xrightarrow{F(m_{i,j})} F(S_{i+j}).$$

1.3 Graded twist of braided tensor categories

In this section, we will review how to construct a new braided tensor category using an abelian cocycle and give examples of braided tensor categories and (super) commutative algebra objects, which play an important role in this paper.

Let $A$ be a finite abelian group. An $A$-graded category $\mathcal{C}$ is a $\mathbb{C}$-linear category with full subcategories $\mathcal{C}_a$ for $a \in A$, such that any object $M$ in $\mathcal{C}$ admits a unique (up to isomorphism) decomposition $M \cong \bigoplus_{a \in A} M_a$ with $M_a \in \mathcal{C}_a$ and there are no nonzero morphisms between objects in $\mathcal{C}_a$ and $\mathcal{C}_b$ for any $a \neq b$. We say that $\mathcal{C}$ is an $A$-graded braided tensor category if in addition it is a braided tensor category, such that $1_\mathcal{C} \in \mathcal{C}_0$ and the monoidal structure satisfies $M \otimes N \in \mathcal{C}_{ab}$ for all homogeneous objects $M \in \mathcal{C}_a$ and $N \in \mathcal{C}_b$. For a semisimple braided tensor category $\mathcal{C}$, there is a universal grading (see [5]).

Let $\omega : A \times A \rightarrow \mathbb{C}^\times$ and $c : A \times A \rightarrow \mathbb{C}^\times$ be maps satisfying

$$\omega(\alpha_1 + \alpha_2, \alpha_3, \alpha_4) = \omega(\alpha_1, \alpha_2, \alpha_3 + \alpha_4) = \omega(\alpha_1, \alpha_2, \alpha_3)\omega(\alpha_1, \alpha_2) + \alpha_3, \alpha_4)\omega(\alpha_2, \alpha_3, \alpha_4),$$

$$\omega(\alpha_2, \alpha_3, \alpha_1)c(\alpha_1, \alpha_2 + \alpha_3)\omega(\alpha_1, \alpha_2, \alpha_3) = c(\alpha_1, \alpha_3)\omega(\alpha_2, \alpha_1, \alpha_3)c(\alpha_1, \alpha_2),$$

$$\omega(\alpha_3, \alpha_1, \alpha_2)^{-1}c(\alpha_1 + \alpha_2, \alpha_3)\omega(\alpha_1, \alpha_2, \alpha_3)^{-1} = c(\alpha_1, \alpha_3)\omega(\alpha_1, \alpha_3, \alpha_2)^{-1}c(\alpha_2, \alpha_3)$$

(1.5)
for \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in A \). A pair of functions \((\omega, c)\) satisfying (1.5) is called an abelian cocycles. Denote by \(Z^3_{ab}(A, \mathbb{C}^\times)\) the set of all abelian cocycles, which is an abelian group with respect to the pointwise multiplication. For a map \(k : A \times A \to \mathbb{C}^\times\), we can define an abelian cocycle \((d_2(k), c_k)\), called an abelian coboundaries, by

\[
(d_2(k)(\alpha_1, \alpha_2, \alpha_3)) = k(\alpha_2, \alpha_3)k(\alpha_1 + \alpha_2, \alpha_3)^{-1}k(\alpha_1, \alpha_2 + \alpha_3)k(\alpha_1, \alpha_2)^{-1}, \\
c_k(\alpha_1, \alpha_2) = k(\alpha_1, \alpha_2)k(\alpha_2, \alpha_1)^{-1},
\]

for \( \alpha_1, \alpha_2, \alpha_3 \in A \). Let \(B^3_{ab}(A, \mathbb{C}^\times) \subset Z^3_{ab}(A, \mathbb{C}^\times)\) be the subgroup of all the abelian coboundaries. The group \(H^3_{ab}(A, \mathbb{C}^\times) = Z^3_{ab}(A, \mathbb{C}^\times)/B^3_{ab}(A, \mathbb{C}^\times)\) is called the abelian cohomology group of \(A\) with coefficients in \(\mathbb{C}^\times\) (see [16]), which can be used to construct a new braided tensor category from an \(A\)-graded braided tensor category.

Let \((B, \alpha, B)\) be an \(A\)-graded braided tensor category. Given \((\omega, c) \in Z^3_{ab}(A, \mathbb{C}^\times)\), we may introduce a new associativity isomorphism \(\alpha^{\omega}\), unit morphism \(l^{\omega}, r^{\omega}\) and braiding \(B^c\) by setting:

\[
\alpha_{M_1, M_2, M_3}^{\omega} = \omega(\alpha_1, \alpha_2, \alpha_3)\alpha_{M_1, M_2, M_3} : (M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3), \\
l_{M_1}^{\omega} = \omega(0, 0, \alpha_1)^{-1}l_{M_1} : 1_B \otimes M_1 \to M_1, \\
r_{M_1}^{\omega} = \omega(\alpha_1, 0, 0)r_{M_1} : M_1 \otimes 1_B \to M_1, \\
B^c_{M_1, M_2} = c(\alpha_1, \alpha_2)B_{M_1, M_2} : M_1 \otimes M_2 \to M_2 \otimes M_1
\]

for any \(\alpha_i \in A\) and \(M_i \in B_{\alpha_i} (i = 1, 2, 3)\). We note that the equivalence classes of the twisted braided tensor category only depends on the cohomology classes in \(H^3_{ab}(A, \mathbb{C}^\times)\). We also note that by [16, Remark 2.6.3], we may assume that the abelian cocycle satisfies \(\omega(\alpha, 0, 0) = \omega(0, 0, \beta) = 1\) for any \(\alpha, \beta \in A\), which is called a normalized cocycle.

Let \(\textbf{Vec}_A\) denote the category of \(A\)-graded vector spaces over \(\mathbb{C}\), whose objects are \(A\)-graded vectors spaces \(V = \bigoplus_{\alpha \in A} V_\alpha\) and morphisms are linear maps which preserve the gradings. The category \(\textbf{Vec}_A\) is an \(A\)-graded tensor category by setting the tensor product

\[
(V \otimes W)_\alpha = \bigoplus_{\beta \in A} V_\beta \otimes W_{\alpha - \beta}
\]

and the unit object by \(1_{\textbf{Vec}_A} = \mathbb{C}\delta_0\), where for \(\alpha \in A\) in general, \(\mathbb{C}\delta_\alpha\) is a one-dimensional vector space defined by

\[
(\mathbb{C}\delta_\alpha)_\beta = \begin{cases} 
\mathbb{C} & \text{(if } \alpha = \beta) \\
0 & \text{(otherwise)}
\end{cases}
\]

Then \(\textbf{Vec}_A\) is a braided tensor category with the associativity isomorphism, unit morphisms, and the braiding being the identities. Since \(\textbf{Vec}_A\) has an \(A\)-grading as a braided tensor category, for any \((\omega, c) \in Z^3_{ab}(A, \mathbb{C}^\times)\), \(\textbf{Vec}_A^{\omega, c}\) is a braided tensor category.
A quadratic form on $A$ with values in $\mathbb{C}^\times$ is a map $Q : A \to \mathbb{C}^\times$ such that $Q(\alpha) = Q(-\alpha)$ and the symmetric function

$$b(\alpha, \beta) = \frac{Q(\alpha + \beta)}{Q(\alpha)Q(\beta)}$$

is a bicharacter, i.e., $b(\alpha_1 + \alpha_2, \beta) = b(\alpha_1, \beta)b(\alpha_2, \beta)$ for any $\alpha_1, \alpha_2, \beta \in A$. Let $\text{Quad}(A)$ denote the set of quadratic forms on $A$, which forms a group by the pointwise multiplication.

For any abelian cocycle $(\omega, c) \in Z^3_{ab}(A, \mathbb{C}^\times)$, let $Q_{\omega, c} : A \to \mathbb{C}^\times$ be the function defined by $Q_{\omega, c}(\alpha) = c(\alpha, \alpha)$ for $\alpha \in A$. Then, $Q_{\omega, c}$ is a quadratic form and the map $\text{EM} : H^3_{ab}(A, \mathbb{C}^\times) \to \text{Quad}(A), (\omega, c) \mapsto Q_{\omega, c}$ is well-defined. The following result is due to Eilenberg and Mac Lane (see [16]):

**Theorem 1.2** (Eilenberg and Mac Lane) The map $\text{EM} : H^3_{ab}(A, \mathbb{C}^\times) \to \text{Quad}(A)$ is an isomorphism of abelian groups.

We denote by $\text{Vec}_A^Q$ the braided tensor category associated with $\text{EM}^{-1}(Q) \in H^3_{ab}(A, \mathbb{C}^\times)$ for $Q \in \text{Quad}(A)$. The following lemma in [16, Section 8.4] is useful:

**Lemma 1.3** Let $Q \in \text{Quad}(A)$ and $(\omega, c) \in Z^3_{ab}(A, \mathbb{C}^\times)$ be a corresponding abelian cocycle. The following conditions are equivalent:

1. There exists a bicharacter $B : A \times A \to \mathbb{C}^\times$ such that $Q(x) = B(x, x)$ for all $x \in A$.
2. There exists a map $k : A \times A \to \mathbb{C}^\times$ such that $\omega = d^2(k)$.

We will construct (super) commutative algebra objects in $\text{Vec}_A^Q$. Let $Q$ be a quadratic form on $A$. A pair of a subgroup $I \subset A$ and a group homomorphism $p : I \to \mathbb{Z}_2$ is called a super isotropic subspace of $(A, Q)$ if it satisfies

$$Q(\alpha) = (-1)^{p(\alpha)}, \quad \text{for any } \alpha \in I. \quad (1.7)$$

If a super isotropic subspace $(I, p)$ satisfies $p(\alpha) = 0$ for any $\alpha \in I$, then $I$ is called just an isotropic subspace of $(A, Q)$.

Let $(I, p)$ be a super isotropic subspace of $(A, Q)$ and introduce the following object

$$S(I)_0 = \bigoplus_{\alpha \in A, p(\alpha) = 0} \mathbb{C}\delta_\alpha,$$

$$S(I)_1 = \bigoplus_{\alpha \in A, p(\alpha) = 1} \mathbb{C}\delta_\alpha,$$

in $\text{Vec}_A^Q$. Let $(\omega_Q, c_Q) \in Z^3_{ab}(A, \mathbb{C}^\times)$ be a normalized abelian cocycle associated with the quadratic form $Q$ by Theorem 1.2.
Lemma 1.4 There is a function \( k : I \times I \to \mathbb{C}^\times \) such that:

\[
k(\alpha, 0) = 1 = k(0, \alpha),
\]

\[
\omega_Q(\alpha_1, \alpha_2, \alpha_3) = k(\alpha_2, \alpha_3)k(\alpha_1 + \alpha_2, \alpha_3)^{-1}k(\alpha_1, \alpha_2 + \alpha_3)k(\alpha_1, \alpha_2)^{-1},
\]

\[
c_Q(\alpha_1, \alpha_2) = (-1)^{p(\alpha_1)p(\alpha_2)}k(\alpha_1, \alpha_2)k(\alpha_2, \alpha_1)^{-1},
\]

for any \( \alpha, \alpha_1, \alpha_2, \alpha_3 \in I \).

**Proof** Let \( B_I : I \times I \to \mathbb{C}^\times \) be the bicharacter defined by \( B_I(\alpha, \beta) = (-1)^{p(\alpha)p(\beta)} \) and \( 1_I : I \times I \times I \to \mathbb{C}^\times \) be the map defined by \( 1_I(\alpha, \beta, \gamma) = 1 \) for any \( \alpha, \beta, \gamma \in I \). Then, by (1.7), \( B_I(\alpha, \alpha) = Q(\alpha) \) for any \( \alpha \in I \) and by Lemma 1.3, \( EM(1_I, B_I) = Q|_I \in \text{Quad}(I) \), where \( Q|_I \) is the restriction of the quadratic form \( Q \) on \( I \subset A \). Since the restriction of \( (\omega_Q, c_Q) \in Z^3_{ab}(A, \mathbb{C}^\times) \) on \( I \subset A \) also satisfies \( EM(\omega_Q|_I, c_Q|_I) = Q|_I \), by Theorem 1.2 there exists \( k : I \times I \to \mathbb{C}^\times \) such that:

\[
\omega_Q(\alpha_1, \alpha_2, \alpha_3) = k(\alpha_2, \alpha_3)k(\alpha_1 + \alpha_2, \alpha_3)^{-1}k(\alpha_1, \alpha_2 + \alpha_3)k(\alpha_1, \alpha_2)^{-1},
\]

\[
c_Q(\alpha_1, \alpha_2) = B_I(\alpha_1, \alpha_2)k(\alpha_1, \alpha_2)k(\alpha_2, \alpha_1)^{-1}
\]

for any \( \alpha_1, \alpha_2, \alpha_3 \in I \). Since \( \omega_Q \) is normalized, \( \omega_Q(\alpha_1, 0, \alpha_2) = k(0, \alpha_3)k(\alpha_1, 0)^{-1} \) implies that \( k(-, -) \) satisfies \( k(\alpha, 0) = 1 = k(0, \alpha) \) for any \( \alpha \in I \). \( \square \)

Define maps \( m_{i,j} : S(I)_i \otimes S(I)_j \to S(I)_{i+j} \) by the linear extensions of \( k(\alpha, \beta)id_{\mathbb{C}\delta_{\alpha+\beta}} : \mathbb{C}\delta_{\alpha} \otimes \mathbb{C}\delta_{\beta} \to \mathbb{C}\delta_{\alpha+\beta} \) for \( \alpha \in p^{-1}(i) \) and \( \beta \in p^{-1}(j) \), and \( \eta : \mathbb{C}\delta_0 \to S(I)_0 \) by the inclusion map. Then, we have:

Lemma 1.5 Let \((I, p)\) be a super isotropic subspace of \((A, Q)\). Then, the triple \((S(I) = S(I)_0 \oplus S(I)_1, \{m_{i,j}\}_{i,j=0,1}, \eta)\) is a supercommutative algebra object in \( \text{Vec}_A^Q \).

**Remark 1.6** Let \( k' : I \times I \to \mathbb{C}^\times \) be another map satisfying (1.8). Then, \( f(\alpha, \beta) = k(\alpha, \beta)k'(\alpha, \beta)^{-1} \) satisfies \( d_2(f)(\alpha, \beta, \gamma) = 1 \) for any \( \alpha, \beta, \gamma \in I \). Thus, \( f \) is a normalized 2-cocycle of \( I \) (in the sense of the usual group cohomology). Since \( f(\alpha, \beta) = f(\beta, \alpha) \) by (1.8), \( f \) is coboundary (see for example [21]). Thus, the nontrivial superalgebra structure on \( S(I) \) is unique up to isomorphisms.

Note that the braided tensor category \( \text{Vec}_A^Q \) naturally arises from lattice vertex operator algebras. An even lattice is a finite rank free abelian group \( M \) equipped with a non-degenerate symmetric bilinear form \((-,-) : M \times M \to \mathbb{Z} \) such that \((\alpha, \alpha) \in 2\mathbb{Z} \) for any \( \alpha \in M \). Let \( M \) be an even lattice (not assumed to be positive-definite). We can extend the bilinear form \((-,-) \) linearly on the vector space \( M \otimes \mathbb{R} \). The dual lattice of \( M \) is the subgroup \( M^\vee = \{ \lambda \in M \otimes \mathbb{R} \mid (\lambda, \alpha) \in \mathbb{Z} \text{ for any } \alpha \in M \} \). Define a quadratic form \( Q_M \) on \( M^\vee / M \) by

\[
Q_M(\lambda) = \exp(\pi i(\lambda, \lambda)) \text{ for } \lambda \in M^\vee.
\]

Since \( M \) is an even lattice, \( Q_M \) is well-defined.
Let $V_M$ be the lattice vertex operator algebra associated with $M$. Then, by [12], we have:

**Proposition 1.7** The representation category of the lattice vertex operator algebra $V_M$ is equivalent to $\text{Vec}^Q_{M^\vee/M}$ as braided tensor categories.

### 1.4 Quasi-triangular Hopf algebra

In this section, we recall the definition of a quasi-triangular Hopf algebra and some standard results, following [18, 33, 35]. Let $(H, \eta, \epsilon, m, \Delta, S)$ be a Hopf algebra with $\eta : \mathbb{C} \to H$ the unit morphism, and $\epsilon : H \to \mathbb{C}$ the counit, $m : H \otimes H \to H$ the multiplication, $\Delta : H \to H \otimes H$ the co-multiplication, and $S : H \to H$ the antipode. Throughout this paper, the antipode is assumed to be invertible.

Let $P_{21} : H \otimes H \to H \otimes H$ be the transposition defined by $a \otimes b \mapsto b \otimes a$ for $a, b \in H$ and set

$$m_{\text{op}} = m \circ P_{21} : H \otimes H \to H,$$

$$\Delta_{\text{cop}} = P_{21} \circ \Delta : H \otimes H \to H.$$

Then, $(H, \eta, \epsilon, m_{\text{op}}, \Delta, S^{-1})$, $(H, \eta, \epsilon, m, \Delta_{\text{cop}}, S^{-1})$ and $(H, \eta, \epsilon, m_{\text{op}}, \Delta_{\text{cop}}, S)$ are again Hopf algebras, which are denoted by $H_{\text{op}}$, $H_{\text{cop}}$ and $H_{\text{op}}^{\text{cop}}$, respectively.

Give an invertible element $R \in H \otimes H$, the pair $(H, R)$ is called quasi-triangular if $R$ is an R-matrix, i.e., satisfies

1. $R \Delta(x) R^{-1} = \Delta_{\text{op}}(x)$ for all $x \in H$;
2. $(\Delta \otimes 1)(R) = R_{13} R_{23}$;
3. $(1 \otimes \Delta)(R) = R_{13} R_{12}$.

Let $(H, R)$ be a quasi-triangular Hopf algebra. Then, the category of left $H$-modules, denoted by $(H, R)$-mod, inherits a structure of braided tensor category by setting the braiding $B_{M,N} : M \otimes N \to N \otimes M$ to be

$$B_{M,N} = P_{M,N} \circ (\rho_M \otimes \rho_N)(R),$$

where $\rho_M : H \to \text{End } M$ and $\rho_N : H \to \text{End } N$ are the structure homomorphisms. Quasi-triangular Hopf algebras $(H_1, R_1)$ and $(H_2, R_2)$ are said to be isomorphic if there exits a Hopf algebra isomorphism $f : H_1 \to H_2$ such that $(f \otimes f)(R_1) = R_2$. In this case, their module categories $(H_1, R_1)$-mod and $(H_2, R_2)$-mod are equivalent as braided tensor categories.

Note that an R-matrix of a Hopf algebra $H$ is not unique. In fact, for an R-matrix $R \in H \otimes H$, $R_{21}^{-1} \in H \otimes H$ is again an R-matrix, where $R_{21} = P_{21}(R) = \sum \beta_i \otimes \alpha_i$ for $R = \sum \alpha_i \otimes \beta_i$, and the quasi-triangular Hopf algebras $(H, R)$ and $(H, R_{21}^{-1})$ are not isomorphic in general. The following lemmas are clear from the definition:

**Lemma 1.8** Let $(H, R)$ be a quasi-triangular Hopf algebra. Then, $(H, R_{21}^{-1})$-mod and $((H, R)$-mod)$^{\text{rev}}$ are equivalent as braided tensor categories.
Lemma 1.9  Let \((H, R)\) be a quasi-triangular Hopf algebra. Then,  
1. \(R^{-1}\) and \(R_{21}\) are \(R\)-matrices of \(H_{\text{op}}\);  
2. \(R^{-1}\) and \(R_{21}\) are \(R\)-matrices of \(H^{\text{cop}}\).

We will give an example of a commutative algebra object in a braided tensor category, introduced in the previous section. Let \((H, \eta, \epsilon, m, \Delta)\) be a finite dimensional Hopf algebra and \(H^{\vee}\) is a dual vector space of \(H\). Then, \(H^{\vee}\) is canonically a Hopf algebra, called a dual Hopf algebra. The unit, counit, multiplication and co-multiplication are given by

\[
\begin{align*}
\eta^{\vee} &= \epsilon \in H^{\vee}, \\
\epsilon^{\vee}(f) &= f(\eta(1)), \\
m^{\vee}(f \otimes g) &= (f \otimes g) \circ \Delta \in H^{\vee}, \\
\Delta^{\vee}(f) &= f \circ m \in H^{\vee} \otimes H^{\vee}
\end{align*}
\]  

for \(f, g \in H\).

Furthermore, \(H^{\vee}\) is a left \(H \otimes H\) module where the left module structure is defined by

\[
(a \otimes b) \cdot f = f(S(b) - a)
\]

for \(a, b \in H\) and \(f \in H^{\vee}\).

Let \(\langle - \rangle : H^{\vee} \otimes H \to \mathbb{C}\) be the canonical pairing, which satisfies

\[
\langle m^{\vee}(f \otimes g), a \rangle = \langle f \otimes g, \Delta(a) \rangle, \\
\langle f, m(a \otimes b) \rangle = \langle \Delta^{\vee}(f), a \otimes b \rangle
\]

for \(a, b \in H\) and \(f, g \in H^{\vee}\).

The following lemma implies that it is natural to view \(H^{\vee}\) as an \(H \otimes H^{\text{cop}}\)-module.

Lemma 1.10  The unit \(\eta^{\vee} : \mathbb{C} \to H^{\vee}\) and the multiplication \(m^{\vee} : H^{\vee} \otimes H^{\vee} \to H^{\vee}\) are morphisms in \(H \otimes H^{\text{cop}}\)-mod.

Proof  Let \(a, b \in H\), \(f, g \in H^{\vee}\), and \(x \in H\). It follows from \((a \otimes b)\eta^{\vee}(1) = (a \otimes b)(S(b) - a) = \epsilon(S(b))\epsilon(-)\epsilon(a) = \epsilon(-)\epsilon(a) = \eta^{\vee}((a \otimes b)1)\) that \(\eta^{\vee} : \mathbb{C} \to H^{\vee}\) is an \(H \otimes H^{\text{cop}}\)-module homomorphism. It follows from

\[
\langle (a \otimes b) \cdot m^{\vee}(f \otimes g), x \rangle = \langle m^{\vee}(f \otimes g), S(b)x a \rangle \\
= \langle f \otimes g, \Delta(S(b)x a) \rangle \\
= \langle f \otimes g, \Delta(S(b))\Delta(x)\Delta(a) \rangle \\
= \langle f \otimes g, S(\Delta^{\text{cop}}(b))\Delta(x)\Delta(a) \rangle \\
= \langle (a \otimes b) \cdot (f \otimes g), \Delta(x) \rangle \\
= \langle m^{\vee}((a \otimes b) \cdot (f \otimes g)), x \rangle
\]

that \(m^{\vee}\) is an \(H \otimes H^{\text{cop}}\)-mod-module homomorphism.  \(\square\)
By Lemma 1.9, \((H^\text{cop}, R^{-1})\) is a quasi-triangular Hopf algebra. By expanding \(R = \sum_i \alpha_i \otimes \beta_i\) and \(R^{-1} = \sum_j \alpha'_j \otimes \beta'_j\), we introduce an element \(R^{1,-1} = \sum_{i,j} \alpha_i \otimes \alpha'_j \otimes \beta_i \otimes \beta'_j \in H \otimes H \otimes H \otimes H\), which is an R-matrix of \(H \otimes H^\text{cop}\). The following result is well-known [11, 13]:

**Proposition 1.11** Let \((H, R)\) be a quasi-triangular Hopf algebra. Then, \((H^\vee, m^\vee, \eta^\vee)\) is a commutative algebra object in \((H \otimes H^\text{cop}, R^{1,-1})\)-mod.

**Proof** It suffices to show (CA1)-(CA3). Since \(H^\vee\) is a unital associative algebra, (CA1) and (CA2) hold. To show (CA3), we recall that \((S \otimes S)(R) = R\) holds for any R-matrix. Then, by (1.11), for any \(x \in H\) and \(f, g \in H^\vee\), we have

\[
\langle m^\vee(P \circ R^{1,-1} \cdot (f \otimes g)), x \rangle = \langle P \circ R^{1,-1} \cdot (f \otimes g), \Delta(x) \rangle
= \langle R^{1,-1} \cdot (f \otimes g), \Delta(x)^\text{cop} \rangle
= \langle f \otimes g, S(R^{-1}) \Delta(x)^\text{cop} R \rangle
= \langle f \otimes g, R^{-1} \Delta(x)^\text{cop} R \rangle
= \langle f \otimes g, \Delta(x) \rangle
= \langle m^\vee(f \otimes g), x \rangle.
\]

Hence, (CA3) holds.

\[\square\]

### 1.5 Affine vertex algebra and Drinfeld category

Here we recall Drinfeld categories [14, 18, 37, 43], which are braided tensor categories arising from the representation theory of affine vertex algebras at irrational levels, following [17]. See also [27] for a general theory for vertex algebras.

Let \(k \in \mathbb{C} \setminus \mathbb{Q}\) and \(\mathfrak{g}\) be a simple Lie algebra over \(\mathbb{C}\). We will use the notations in Sect. 1.1. Let \(\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c\) be the affine Lie algebra which is naturally \(\mathbb{Z}\)-graded: \(\deg(a \otimes t^n) = -n\), \(\deg(c) = 0\). Introduce the following subalgebras: \(\widehat{\mathfrak{g}}^+ = \mathfrak{g} \otimes t\mathbb{C}[t], \widehat{\mathfrak{g}}^- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]\), \(\widehat{\mathfrak{g}}^0 = \mathfrak{g} \oplus \mathbb{C}c\), and \(\widehat{\mathfrak{g}}^{\geq 0} = \widehat{\mathfrak{g}}^+ \oplus \widehat{\mathfrak{g}}^0\). Let \(L(\lambda)\) denote the finite dimensional irreducible representation of \(\mathfrak{g}\) with highest weight \(\gamma \in P^+\), which can be regarded as a \(\widehat{\mathfrak{g}}^{\geq 0}\)-module so that \(\widehat{\mathfrak{g}}^+\) act trivially and \(c\) by the scalar \(k\). The induced module \(L_{\mathfrak{g},k}(\lambda) = \text{Ind}_{\widehat{\mathfrak{g}}^{\geq 0}}^{\widehat{\mathfrak{g}}} L(\lambda)\) is called the Weyl module.

It has a natural \(\mathbb{C}\)-grading, \(L_{\mathfrak{g},k}(\lambda) = \bigoplus_{n=0}^{\infty} (L_{\mathfrak{g},k}(\lambda))_{\Delta(\lambda)+n}\) with \(\Delta(\lambda) = \frac{(\lambda+2\rho,\lambda)}{2(k+h^-)}\) and \((L_{\mathfrak{g},k}(\lambda))_{\Delta(\lambda)} = L(\lambda)\), which is compatible with the grading on \(\widehat{\mathfrak{g}}\).

Then, \(L_{\mathfrak{g},k}(0)\) inherits a vertex operator algebra structure, called an affine vertex operator algebra and \(L_{\mathfrak{g},k}(\lambda)\) is a \(L_{\mathfrak{g},k}(0)\)-module. We consider a category of \(L_{\mathfrak{g},k}(0)\)-modules whose object is a direct sum of \(L_{\mathfrak{g},k}(\lambda)\’s\) for \(\lambda \in P^+\) and morphisms are \(L_{\mathfrak{g},k}(0)\)-module homomorphisms. We denote this \(\mathbb{C}\)-linear semisimple abelian category by \(D(\mathfrak{g},k)\).

Let \(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda \in P^+\). It is well-known that for a \(\mathfrak{g}\)-module homomorphisms, \(f \in \text{Hom}_\mathfrak{g}(L(\lambda_1) \otimes L(\lambda_2), L(\lambda_0))\), there exists a unique intertwining operator

\[
I_f(-, z) : L_{\mathfrak{g},k}(\lambda_1) \rightarrow \text{Hom}(L_{\mathfrak{g},k}(\lambda_2), L_{\mathfrak{g},k}(\lambda_0))[[z^C]] \quad (1.12)
\]
satisfying
\[ I_f(a, z) = \sum_{n \in \mathbb{Z}} a(n - \Delta) z^{-n-1+\Delta} \] with \( \Delta = \Delta(\lambda_0) - \Delta(\lambda_1) - \Delta(\lambda_2) \) \hspace{1cm} (1.13)
for any \( a \in L_{g,k}(\lambda_1) \) and
\[ a(-\Delta - 1)b = f(a \otimes b) \]
for any \( a \in L_{g,k}(\lambda_1) \Delta(\lambda_1) = L(\lambda_1) \) and \( b \in L_{g,k}(\lambda_2) \Delta(\lambda_2) = L(\lambda_2) \) (See [17, Theorem 3.1.1] or [24, Theorem 1.5.3]).

We denote the space of \( g \)-module homomorphisms, \( \text{Hom}_g(L(\lambda_1) \otimes L(\lambda_2), L(\lambda_0)) \) by \( V^\lambda_{\lambda_1\lambda_2}(\lambda_3) \) and \( \text{Hom}_g(L(\lambda_1) \otimes L(\lambda_2) \otimes L(\lambda_3), L(\lambda_0)) \) by \( V^\lambda_{\lambda_1\lambda_2\lambda_3} \). Then, introduce a tensor product on \( D(g, k) \) by
\[ L_{g,k}(\lambda_1) \otimes L_{g,k}(\lambda_2) = \bigoplus_{\alpha \in P^+} V^\alpha_{\lambda_1\lambda_2}(\lambda_3) \otimes L_{g,k}(\alpha). \]
Then, by the natural \( \mathbb{C} \)-linear isomorphism \( V^\lambda_{\lambda_1\lambda_2\lambda_3} \cong \bigoplus_{\alpha \in P^+} V^\lambda_{\lambda_1\alpha} \otimes V^\alpha_{\lambda_2\lambda_3} \), we have
\[ L_{g,k}(\lambda_1) \otimes (L_{g,k}(\lambda_2) \otimes L_{g,k}(\lambda_3)) \cong L_{g,k}(\lambda_1) \otimes \left( \bigoplus_{\alpha} V^\alpha_{\lambda_2\lambda_3} \otimes L_{g,k}(\alpha) \right) \]
\[ \cong \bigoplus_{\lambda_0} \bigoplus_{\alpha} V^\lambda_{\lambda_1\alpha} \otimes V^\alpha_{\lambda_2\lambda_3} \otimes L_{g,k}(\lambda_0) \]
\[ \cong \bigoplus_{\lambda_0} V^\lambda_{\lambda_1\lambda_2\lambda_3} \otimes L_{g,k}(\lambda_0). \] \hspace{1cm} (1.14)

Set \( Y_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq z_2, z_1 \neq 0 \text{ and } z_2 \neq 0 \} \), an open subset of \( \mathbb{C}^2 \), \( Y_2^\circ = \{(z_1, z_2) \in Y_2 \mid |z_1| > |z_2|, \text{Arg } z_1 \text{ < } \pi, |\text{Arg } z_1| < \pi \} \) and \( Y_2^\circ = \{(z_1, z_2) \in Y_2 \mid |z_1| < |z_2|, \text{Arg } z_2 \text{ < } \pi, |\text{Arg } z_1| < \pi \} \). Let \( \text{Mul}(Y_2^\circ) \) (resp. \( \text{Mul}(Y_2^\circ) \)) be the space of holomorphic functions on \( Y_2^\circ \) (resp. \( Y_2^\circ \)) such that it has an analytic continuation to a multi-valued holomorphic function on \( Y_2 \).

Let \( f \in V^\lambda_{\lambda_1\alpha} \) and \( g \in V^\alpha_{\lambda_2\lambda_3} \) and \( a_i \in L_{g,k}(\lambda_i) \) \((i = 1, 2, 3)\) and \( u \) be an element of the restricted dual space \( L_{g,k}(\lambda_0)^{\vee} = \bigoplus_{h} L_{g,k}(\lambda_0)^* \), Then, the composition of the intertwining operators
\[ u(I_f(a_1, z_1)I_g(a_2, z_2)a_3) \in \mathbb{C}[[z_1^C, z_2^C]] \] \hspace{1cm} (1.15)
is absolutely convergent to a holomorphic function on \( Y_2^\circ \) and has an analytic continuation to a multivalued holomorphic function on \( Y_2 \). This follows since the formal power series \( u(I_f(a_1, z_1)I_g(a_2, z_2)a_3) \) satisfies the Knizhnik–Zamolodchikov equation, see [18, 37, 43]. Thus, we have a linear map
\[ C(u, a_1, a_2, a_3; z_1, z_2) : V^\lambda_{\lambda_1\alpha} \otimes V^\alpha_{\lambda_2\lambda_3} \to \text{Mul}(Y_2^\circ) \]
for \( a_i \in L_{g,k}(\lambda_i) \) and \( u \in L_{g,k}(\lambda_0)^\vee \). Hence, by \( V_{\lambda_1 \lambda_2 \lambda_3}^{\lambda_0} \approx \bigoplus_{\alpha \in \mathbb{P}^+} V_{\lambda_1 \alpha}^{\lambda_0} \otimes V_{\lambda_2 \lambda_3}^{\alpha} \), we have

\[
C(u, a_1, a_2, a_3; z_1, z_2) : V_{\lambda_1 \lambda_2 \lambda_3}^{\lambda_0} \rightarrow Mul(Y_2^{\gamma})
\]

and similarly

\[
C(u, a_1, a_2, a_3; z_2, z_1) : V_{\lambda_1 \lambda_2 \lambda_3}^{\lambda_0} \rightarrow Mul(Y_2^{\gamma})
\]

by changing the role of \( z_1 \) and \( z_2 \).

Let \( \gamma \) be a path from a point in \( Y_2^{\gamma} \) to a point in \( Y_2^{\gamma} \) and \( A(\gamma) : Mul(Y_2^{\gamma}) \rightarrow Mul(Y_2^{\gamma}) \) a linear map defined by the analytic continuation of a function in \( Mul(Y_2^{\gamma}) \) along the path \( \gamma \). Then, by using the Knizhnik–Zamolodchikov equation, there exists a unique linear isomorphism

\[
M(\gamma) : V_{\lambda_1 \lambda_2 \lambda_3}^{\lambda_0} \rightarrow V_{\lambda_2 \lambda_1 \lambda_3}^{\lambda_0}
\]

such that for any \( a_i \in V_{g,k}(\lambda_i) \) and \( u \in V_{g,k}(\lambda_0)^\vee \) the following diagram commutes:

\[
\begin{array}{ccc}
V_{\lambda_1 \lambda_2 \lambda_3}^{\lambda_0} & \xrightarrow{M(\gamma)} & V_{\lambda_2 \lambda_1 \lambda_3}^{\lambda_0} \\
C(u, a_1, a_2, a_3; z_1, z_2) \downarrow & & \downarrow C(u, a_2, a_1, a_3; z_2, z_1) \\
Mul(Y_2^{\gamma}) & \xrightarrow{A(\gamma)} & Mul(Y_2^{\gamma})
\end{array}
\]

Note that \( M(\gamma) \) is independent of the choice of \( a_i \in L_{g,k}(\lambda_i) \) and \( u \in L_{g,k}(\lambda_0)^\vee \) and thus by (1.14) \( M(\gamma) \) gives an isomorphism in \( D(g,k) \)

\[
M(\gamma) : L_{g,k}(\lambda_1) \otimes (L_{g,k}(\lambda_2) \otimes L_{g,k}(\lambda_3)) \rightarrow L_{g,k}(\lambda_2) \otimes (L_{g,k}(\lambda_1) \otimes L_{g,k}(\lambda_3)).
\]

(1.16)

Then the braiding

\[
B_{\lambda_1, \lambda_2} : L_{g,k}(\lambda_1) \otimes L_{g,k}(\lambda_2) \rightarrow L_{g,k}(\lambda_2) \otimes L_{g,k}(\lambda_1)
\]

on \( D(g,k) \) is given by the isomorphism (1.16) with \( \lambda_3 = 0 \) and the associative isomorphism is given by the composition

\[
\alpha_{\lambda_1, \lambda_2, \lambda_3} : L_{g,k}(\lambda_1) \otimes (L_{g,k}(\lambda_2) \otimes L_{g,k}(\lambda_3))
\]

\[
1 \otimes B_{\lambda_2, \lambda_3}^{\lambda_0} \rightarrow L_{g,k}(\lambda_1) \otimes (L_{g,k}(\lambda_3) \otimes L_{g,k}(\lambda_2))
\]

\[
M(\gamma) \rightarrow L_{g,k}(\lambda_3) \otimes (L_{g,k}(\lambda_1) \otimes L_{g,k}(\lambda_2))
\]

\[
B_{\lambda_3, \lambda_1, \lambda_2} \otimes 1 \rightarrow (L_{g,k}(\lambda_1) \otimes L_{g,k}(\lambda_2)) \otimes L_{g,k}(\lambda_3)
\]
This braided tensor category is introduced by Drinfeld by using the Knizhnik–Zamolodchikov equation and called a Drinfeld category (see for example [17, 33]).

Note that by replacing the formal variables \((z_1, z_2)\) in (1.15) with the anti-holomorphic ones \((\bar{z}_1, \bar{z}_2)\), the formal power series thus obtained gives rise to an antiholomorphic function \(Y^\omega\) and then another braided tensor category, which we denote by \(\overline{D(g, k)}\).

Since the holomorphic and antiholomorphic solutions of the Knizhnik–Zamolodchikov equation are the same on the real subspace \(\mathbb{R}\) \(\cap Y_2\), the associativity isomorphisms of \(D(g,k)\) and \(\overline{D(g, k)}\) are the same. On the other hand, the braidings are inverse to each other since for \(\alpha \in \mathbb{C}\) the monodromy of the holomorphic function \(z^\alpha\) around the origin (counterclockwise) is \(\exp(2\pi i \alpha)\), whereas that of the antiholomorphic function \(\bar{z}^\alpha\) around the origin is \(\exp(-2\pi i \alpha)\). Hence, we have:

**Lemma 1.12** For \(k \in \mathbb{C}\backslash \mathbb{Q}\), \(\overline{D(g, k)}\) and \(D(g, k)^{\text{rev}}\) are equivalent as braided tensor categories.

We note that \(D(g,k)\) admits the balance

\[
\theta_{L_{g,k}(\lambda)} = \exp(2\pi i \Delta(\lambda)) = \exp(\pi i \frac{(\lambda + 2\rho, \lambda)}{k + h^+}) \quad (1.17)
\]

and \(\overline{D(g, k)}\) does

\[
\theta_{\overline{L_{g,k}(\lambda)}} = \exp(-\pi i \frac{(\lambda + 2\rho, \lambda)}{k + h^+}) \quad (1.18)
\]

Finally, we explain that for any commutative algebra objects in \(D(g,k) \otimes D(g,k')\) with \(k,k' \in \mathbb{C}\backslash \mathbb{Q}\), we can construct a vertex algebra as an extension of the affine vertex algebra \(L_{g,k}(0) \otimes L_{g,k'}(0)\) (see [30] for detail).

Let \((V, m, \eta)\) be a commutative algebra object in \(D(g,k) \otimes D(g,k')\) such that the balance \(\theta\) acts on \(V\) by identity, \(\theta_V = \text{id}\). It follows that the multiplicity \(n_{\lambda,\lambda'}\) in the decomposition \(V = \bigoplus_{\lambda,\lambda' \in \mathcal{P}^+} (L_{g,k}(\lambda) \otimes L_{g,k'}(\lambda'))^{n_{\lambda,\lambda'}}\) is zero unless

\[
\frac{(\lambda + 2\rho, \lambda)}{2(k + h^+)} + \frac{(\lambda' + 2\rho, \lambda')}{{2(k' + h^+)} \in \mathbb{Z},
\]

that is, \(V\) is \(\mathbb{Z}\)-graded. By the above discussion (1.12), the multiplication \(m : V \otimes V \rightarrow V\) corresponds to an intertwining operator \(I_m(-, z) : V \rightarrow \text{End}(V)[[z^\pm]]\). For any \(a_1, a_2, a_3 \in V\) and \(u \in V^\gamma = \bigoplus_{n \in \mathbb{Z}} V^\gamma_n\) and any path \(\gamma\) from \(Y_2^\omega\) to \(Y_2^\nu\),

\[
A(\gamma)u(I_m(a_1, z_1)I_m(a_2, z_2)a_3) = A(\gamma)C(u, a_1, a_2, a_3; z_1, z_2)(m(\text{id} \otimes m)) = C(u, a_2, a_1, a_3; z_2, z_1)(M(\gamma)m(\text{id} \otimes m)) = C(u, a_2, a_1, a_3; z_2, z_1)(m(\text{id} \otimes m)) = u(I_m(a_2, z_2)I_m(a_1, z_1)a_3).
\]
Here, we used the assumption that $V$ is a commutative associative algebra object. Hence, $I_m(\cdots z)$ satisfies the locality condition, which implies that $V$ is a vertex algebra (We omit the details here, but refer instead to [30]). The vacuum vector $1$ is defined by $\eta(1_{L_{g,k}(0)} \otimes 1_{L_{g,k'}(0)})$ where $1_{L_{g,k}(0)}$ (resp. $1_{L_{g,k'}(0)}$) is the vacuum vector of $L_{g,k}(0)$ (resp. $L_{g,k'}(0)$). Hence, we have:

**Proposition 1.13** Let $k, k' \in \mathbb{C} \setminus \mathbb{Q}$ and $(V, m, \eta)$ be a commutative algebra object in $D(g, k) \otimes D(g, k')$ such that $\theta_V = \text{id}$. Then, $(V, I_m(\cdots), 1)$ is a $\mathbb{Z}$-graded vertex algebra.

More generally, let $(V = V_0 \oplus V_1, \{ m_{i,j}\}_{i,j=01}, \eta)$ be a supercommutative algebra object in $D(g, k) \otimes D(g, k')$ such that $\theta|_{V_i} = (-1)^i$ for $i = 0, 1$. By the above discussion (1.12), the multiplication $m : V \otimes V \to V$ corresponds to an intertwining operator $I_m(\cdots z) : V \to \text{End} (V, V)[[z^{\pm 1}]]$. Since $m$ preserves the parity, the intertwining operator $I_m(\cdots z) \in \text{End} (V, V)[[z^{\pm 1}]]$. Following the same argument as above, we obtain the following (see [7] for detail):

**Proposition 1.14** Let $k, k' \in \mathbb{C} \setminus \mathbb{Q}$ and $(V = V_0 \oplus V_1, \{ m_{i,j}\}_{i,j=01}, \eta)$ be a supercommutative algebra object in $D(g, k) \otimes D(g, k')$ such that $\theta_{V_i} = (-1)^i$. Then, $(V, I_m(\cdots), 1)$ is a vertex superalgebra.

### 1.6 Non-chiral case

In Sect. 1.5, we have introduced $D(g, k)$ and $D(g, k)$ by using intertwining operators which are holomorphic and antiholomorphic, respectively. Then it is reasonable to expect that commutative algebra objects in the Deligne tensor product $D(g, k) \otimes D(g, k')$ has a nature of “non-chiral” vertex algebra whose vertex operator is of the form

$$Y(a, z) = \sum_{r,s\in \mathbb{C}} a(r, s) z^{-r-1}\bar{z}^{-s-1}.$$  

Such a variant of vertex algebra is introduced by the author in [40] under the name of a full vertex algebra. Here we reproduce the axioms and show the statement analogous to Proposition 1.13.

For a $\mathbb{C}^2$-graded vector space $F = \bigoplus_{h,\bar{h}\in \mathbb{C}^2} F_{h,\bar{h}}$ set $F^\vee = \bigoplus_{h,\bar{h}\in \mathbb{C}^2} F_{h,\bar{h}}^*$, where $F_{h,\bar{h}}^*$ is the dual vector space of $F_{h,\bar{h}}$. A full vertex algebra is a $\mathbb{C}^2$-graded $\mathbb{C}$-vector space $F = \bigoplus_{h,\bar{h}\in \mathbb{C}^2} F_{h,\bar{h}}$ equipped with a linear map

$$Y(\cdots, \bar{z}) : F \to \text{End}(F)[[z^{\pm}, \bar{z}^{\pm}, |z|^C]], \ a \mapsto Y(a, \bar{z}) = \sum_{r,s\in \mathbb{C}} a(r, s) z^{-r-1}\bar{z}^{-s-1}$$

and a non-zero element $1 \in F_{0,0}$ satisfying the following axioms:

- **FV1** For any $a, b \in F$, $Y(a, z)b \in F((z, \bar{z}, |z|^C));$
- **FV2** $F_{h,\bar{h}} = 0$ unless $h - \bar{h} \in \mathbb{Z};$
- **FV3** For any $a \in F$, $Y(a, z)1 \in F[[z, \bar{z}]]$ and $\lim_{z \to 0} Y(a, \bar{z})1 = a(-1, -1)1 = a;$.  

FV4) $Y(1, z) = \text{id} \in \text{End}(F)$;

FV5) For any $a, b, c \in F$ and $u \in F^\vee$, the formal power series $u \left( Y(a, z_1)Y(b, z_2) c \right)$, $u \left( Y(b, z_2)Y(a, z_1) c \right)$ and $u \left( Y(Y(a, z_0)b, z_2)c \right)$ are absolutely convergent in the regions $Y_2^\geq = \{ |z_1| > |z_2| \}$, $Y_2^\leq = \{ |z_1| > |z_2| \}$ and $\{ |z_2| > |z_0| \}$ respectively and have analytic continuations to the same single-valued real analytic function on $Y_2$ under the relation $z_0 = z_1 - z_2$;

FV6) $F_{h, \bar{h}}(r, s) F_{h', \bar{h}'} \subseteq F_{h + h' - r - 1, \bar{h} + \bar{h}' - s - 1}$ for any $r, s, h, h', \bar{h}, \bar{h}' \in \mathbb{C}$.

**Remark 1.15** In the original definition in [40], a full vertex algebra $F$ is assumed to be $\mathbb{R}^2$-graded and $Y(a, z) \in \text{End}(F)[[z, \bar{z}, |z|^2]]$ but there is no difference to develop the theory in this $\mathbb{C}^2$-graded case.

Let $(F, Y, 1)$ be a full vertex algebra. Set $\bar{F} = F$ and $\bar{F}_{h, \bar{h}} = F_{h, \bar{h}}$ for $h, \bar{h} \in \mathbb{C}$. Define $\bar{Y}(\cdot, z) : \bar{F} \to \text{End}(\bar{F})[[z, \bar{z}, |z|^2]]$ by $\bar{Y}(a, z) = \sum_{r, s \in \mathbb{C}} a(r, s) z^{-s-1} \bar{z}^{-r-1}$. Then, $(\bar{F}, \bar{Y}, 1)$ is a full vertex algebra. We call it a conjugate full vertex algebra of $(F, Y, 1)$.

We will consider a full vertex algebra $L_{g, k}(0) \otimes \bar{L}_{g, k'}(0)$ which is the tensor product of the vertex algebra $L_{g, k}(0)$ and the conjugate (full) vertex algebra $\bar{L}_{g, k'}(0)$.

Let $(F, m, \eta)$ be a commutative algebra object in $D(g, k) \otimes \bar{D}(g, k')$ such that $\theta_F = \text{id}$. It follows from (1.13) and (1.18) that the multiplicity $n_{\lambda, \lambda'}$ in the decomposition $F = \bigoplus_{\lambda, \lambda' \in \mathbb{P}_+} (L_{g, k}(\lambda) \otimes \bar{L}_{g, k'}(\lambda'))^{n_{\lambda, \lambda'}}$ is zero unless

$$\frac{(\lambda + 2 \rho, \lambda)}{2(k + h^\vee)} - \frac{(\lambda' + 2 \rho, \lambda')}{2(k' + h^\vee)} \in \mathbb{Z},$$

which implies that $F$ satisfies (FV2). By (1.12), the multiplication $m : F \otimes F \to F$ corresponds to an intertwining operator $I_m(\cdot, z) : F \to \text{End}(F)[[z, \bar{z}, |z|^2]]$. (FV6) follows from the definition of the intertwining operators and (FV2) and (FV6) imply $I_m(a, z) \in \text{End}(F)[[z, \bar{z}, |z|^2]]$ for any $a \in F$. Thus, (FV1), (FV3) and (FV4) follows from the definition of the intertwining operators and the assumption (CA1). (FV5) follows from the same argument in the last section. Hence, we have:

**Proposition 1.16** Let $k, k' \in \mathbb{C} \setminus \mathbb{Q}$ and $(F, m, \eta)$ be a commutative algebra object in $D(g, k) \otimes \bar{D}(g, k')$ such that $\theta_F = \text{id}$. Then, $(F, I_m(\cdot, z), 1)$ is a full vertex algebra.

## 2 Quantum group and isomorphisms

### 2.1 Definition of quantum group

Here we recall the definition of a quantum group, following [31, 35].

Fix an element $q \in \mathbb{C} \setminus \mathbb{Q}$. We use the notation in Sect. 1.1, in particular, the invariant bilinear form $\langle \cdot, \cdot \rangle$ which satisfies $\langle \alpha, \alpha' \rangle = 2$ for short roots $\alpha$. Put $q_{\alpha} := q^{\langle \alpha, \alpha' \rangle/2}$ for $\alpha \in \Pi$. 
The quantum group $U_q(g)$ is the algebra defined by generators $\{E_\alpha, F_\alpha, K_\alpha, K_{-\alpha} \mid \alpha \in \Pi\}$ and relations

$$K_0 = 1, \quad K_\alpha K_\beta = K_\beta K_\alpha, \quad K_\alpha K_{-\lambda} = 1,$$

$$K_\alpha E_\beta K_{-\alpha} = q^{\langle (\alpha,\beta) \rangle} E_\beta, \quad K_\alpha F_\beta K_{-\alpha} = q^{-\langle (\alpha,\beta) \rangle} F_\beta,$$

$$[E_\alpha, F_\beta] = \delta_{\alpha,\beta} \frac{K_\alpha - K_{-\alpha}}{q_\alpha - q_{-\alpha}},$$

$$\sum_{r=0}^{1-a_{\alpha\beta}} (-1)^r \left( 1 - \frac{a_{\alpha\beta}}{r} \right) q_\alpha E_\alpha^r E_\beta E_\alpha^{1-a_{\alpha\beta}-r} = 0,$$

$$\sum_{r=0}^{1-a_{\alpha\beta}} (-1)^r \left( 1 - \frac{a_{\alpha\beta}}{r} \right) q_\alpha F_\alpha^r F_\beta F_\alpha^{1-a_{\alpha\beta}-r} = 0,$$

where $a_{\alpha\beta} = \frac{2 \langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} = \frac{2 \langle (\alpha,\beta) \rangle}{\langle (\alpha,\alpha) \rangle}$, the Cartan matrix, for any $\alpha, \beta \in \Pi$. One may define a Hopf algebra structure on $U_q(g)$ by

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \epsilon(K_\alpha) = 1, \quad S(K_\alpha) = K_{-\alpha}^{-1},$$

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \quad \epsilon(E_\alpha) = 0, \quad S(E_\alpha) = -K_{-\alpha}^{-1} E_\alpha,$$

$$\Delta(F_\alpha) = F_\alpha \otimes K_{-\alpha}^{-1} + 1 \otimes F_\alpha, \quad \epsilon(F_\alpha) = 0, \quad S(F_\alpha) = -F_{-\alpha} K_\alpha.$$

The algebra $U_q(g)$ is graded $U_q(g) = \bigoplus_{\mu \in Q} U_q(g)_\mu$ by the root lattice $Q = \sum_{\alpha \in \Pi} \mathbb{Z} \alpha$ by setting $\deg E_\alpha = \alpha$, $\deg F_\alpha = -\alpha$ and $\deg K_\alpha = 0$. Then, for any $\mu \in Q$,

$$U_q(g)_\mu = \{ u \in U \mid K_\alpha u K_{-\alpha}^{-1} = q^{\langle \mu,\alpha \rangle} u \text{ for any } \alpha \in \Pi \}.$$

Set $Q_+ = \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha$ and let $U_q^+(g)$ (resp. $U_q^0(g)$, $U_q^-(g)$) be the subalgebra of $U_q(g)$ generated by all $E_\alpha$'s (resp. $K_\alpha$'s, $F_\alpha$'s) with $\alpha \in \Pi$. Since $U_q^0(g)$ is a commutative algebra, we can define for each $\lambda \in Q$ an element $K_\lambda$ in $U_q^0(g)$ by

$$K_\lambda = \prod_{\beta \in \Pi} K_{\beta}^{m_{\beta}}, \quad \text{if } \lambda = \sum_{\beta \in \Pi} m_{\beta} \beta.$$
For each $\lambda \in P$, let $L_q(\lambda)$ be the unique irreducible highest weight module of highest weight $\lambda$, which has a nonzero element $v_\lambda \in L_q(\lambda)$ satisfying

$$K_\mu v_\lambda = q^{(\mu, \lambda)} v_\lambda, \quad E_\alpha v_\lambda = 0.$$ 

If $\lambda \in P^+$, then $L_q(\lambda)$ is finite dimensional. We say a $U_q(\mathfrak{g})$-module is of type 1 if it decomposes into a direct sum of $L_q(\lambda)$'s for $\lambda \in P^+$. Notice that any subquotient of type 1 module is also of type 1. Hence, the category of type 1 $U_q(\mathfrak{g})$-modules is a locally finite semisimple abelian category, which is denoted by $U_q(\mathfrak{g})$-mod. We remark that the abelian category structure does not depend on $q \in \mathbb{C}\setminus \mathbb{Q}$, however the tensor category structure does.

For any type 1 module $M$, set for all $\lambda \in P$

$$M_\lambda = \{m \in M \mid K_\mu m = q^{(\mu, \alpha)} m \text{ for all } \alpha \in \Pi\}.$$ 

Then, we have

$$M = \bigoplus_{\lambda \in P} M_\lambda.$$ 

The following Hopf algebra isomorphisms are important in this paper:

**Lemma 2.1** For any simple Lie algebra $\mathfrak{g}$,

1. There exist Hopf algebra isomorphisms $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\text{cop}}$, $\theta : U_q(\mathfrak{g}) \to U_q^{-1}(\mathfrak{g})_{\text{op}}$ and $\psi : U_q(\mathfrak{g}) \to U_q^{-1}(\mathfrak{g})^{\text{cop}}$ satisfying for $\alpha \in \Pi$

   $$\omega(E_\alpha) = F_\alpha, \quad \omega(F_\alpha) = E_\alpha, \quad \omega(K_\alpha) = K_\alpha^{-1},$$

   $$\theta(E_\alpha) = E_\alpha, \quad \theta(F_\alpha) = F_\alpha, \quad \theta(K_\alpha) = K_\alpha,$$

   $$\psi(E_\alpha) = -K_\alpha^{-1} E_\alpha, \quad \psi(F_\alpha) = -F_\alpha K_\alpha, \quad \psi(K_\alpha) = K_\alpha^{-1}.$$ 

2. There exists an algebra isomorphism $\tau : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})_{\text{op}}$ satisfying

   $$\tau(E_\alpha) = E_\alpha, \quad \tau(F_\alpha) = F_\alpha, \quad \tau(K_\alpha) = K_\alpha^{-1}.$$ 

**Proof** It is easy to show that $\theta$ and $\omega$ are Hopf algebra isomorphisms. Since $\psi = S \circ \theta$ and $U_q(\mathfrak{g}) \to U_q^{-1}(\mathfrak{g})_{\text{op}} \overset{S}{\to} U_q^{-1}(\mathfrak{g})^{\text{cop}}$, $\psi$ gives a Hopf algebra isomorphism. \qed

### 2.2 Bilinear form and R-matrix

Here, we recall the definitions of a bilinear form on $U_q(\mathfrak{g})$ and some R-matrices. For simplicity, we abbreviate $U = U_q(\mathfrak{g})$, $U^+ = U_q(\mathfrak{g})^+$ and $U^- = U_q(\mathfrak{g})^-$. This section is mainly based on [31] (see also [35, 38]). The subalgebra of $U$ generated by $U^+$ and $U^0$ (resp. $U^-$ and $U^0$) is denoted by $U_{\geq 0}$ (resp. $U_{\leq 0}$). Set $U_{\mu}^+ = U^+ \cap U_\mu$ and $U_{\mu}^- = U^+ \cap U_{-\mu}$ for $\mu \in Q$ with $\mu \geq 0$. 


Proposition 2.2 ([31, Proposition 6.12 and Corollary 8.30])

There exists a unique bilinear pairing \((-,-): U_{\leq 0} \times U_{\geq 0} \rightarrow \mathbb{C}\) such that for all \(x, x' \in U_{\leq 0}, y, y' \in U_{\geq 0}, \mu, \nu \in Q, \) and \(\alpha, \beta \in \Pi\)
\[
(y, xx') = (\Delta y, x' \otimes x), \quad (yy', x) = (y \otimes y', \Delta x),
\]
\[
(K_\mu, K_\nu) = q^{-\langle \mu, \nu \rangle}, \quad (F_\alpha, E_\beta) = -\delta_{\alpha, \beta}(q_\alpha - q_\beta^{-1})^{-1},
\]
\[
(K_\mu, E_\alpha) = 0 \quad \text{(}F_\alpha, K_\mu) = 0.\]

Furthermore, the restriction of \((-,-)\) to any \(U_{\mu^-} \times U_{\mu^+}\) with \(\mu \in Q_+\) is a non-degenerate paring.

The bilinear form is preserved by the isomorphism \(\omega : U_q(g) \rightarrow U_q(g)^{\text{cop}}:\)

Lemma 2.3 ([31, Lemma 6.16.]) For all \(x \in U^+\) and \(y \in U^-\), \((\omega(x), \omega(y)) = (y, x)\).

We will construct the R-matrix following [31]. Choose for each \(\mu \in Q_+\) a basis \(u_1^\mu, u_2^\mu, \ldots, u_{r(\mu)}^\mu\) of \(U^+_{\mu}\), where \(r(\mu) = \dim U^+_{\mu}\). By Proposition 2.2 we can find a basis \(v_1^\mu, v_2^\mu, \ldots, v_{r(\mu)}^\mu\) of \(U^-_{\mu}\) such that \((v_j^\mu, u_i^\mu) = \delta_{i,j}\) for all \(i\) and \(j\). Set
\[
\Theta_\mu = \sum_{i=1}^{r(\mu)} v_i^\mu \otimes u_i(\mu) \in U^-_{\mu} \otimes U^+_{\mu}.
\]

By linear algebra, \(\Theta_\mu\) does not depend on the choice of the basis \((u_i^\mu)_i\). Thus, by Lemma 2.3, we have:

Lemma 2.4 ([31]) For all \(\mu \in Q_+\),
\[
(\omega \otimes \omega) \Theta_\mu = P_{21}(\Theta_\mu),
\]
where \(P_{21}\) is the transposition.

Let \(U \hat{\otimes} U\) be the completion of the vector space \(U \otimes U\) with respect to the descending sequence of vector spaces
\[
(U^+ U^0 \sum_{\text{wt } \mu \geq N} U^-_{\mu}) \otimes U + U \otimes U^- U^0 \sum_{\text{wt } \mu \geq N} U^+_{\mu}
\]
for \(N \geq 1\), where \(\text{wt } \mu\) is defined by \(\text{wt } \mu = \sum_{\beta \in \Pi} m_\mu \) for \(\mu = \sum_{\beta \in \Pi} m_\beta \beta\), and similarly \(U \hat{\otimes} \text{rev}\) \(U\) the completion with respect to
\[
(U^- U^0 \sum_{\text{wt } \mu \geq N} U^+_{\mu}) \otimes U + U \otimes U^+ U^0 \sum_{\text{wt } \mu \geq N} U^-_{\mu}
\]
for \(N \geq 1\). Then, \(U \hat{\otimes} U\) and \(U \hat{\otimes} \text{rev}\) \(U\) are topological algebras containing \(U \otimes U\) as a subalgebra and \(\Theta = \sum_{\mu \geq 0} \Theta_\mu\) is in \(U \hat{\otimes} U\) and \(P_{21} \Theta\) is in \(U \hat{\otimes} \text{rev}\) \(U\).

Set \(\Delta^\tau = (\tau \otimes \tau) \circ \Delta \circ \tau^{-1}\), c.f. Sect. 2.1. Then, we have:
Proposition 2.5 ([31]) The element $\Theta \in U \hat{\otimes} U$ satisfies
\[ \Delta(u) \circ \Theta = \Theta \circ \Delta^\tau(u) \]
for any $u \in U$.

Furthermore, $\Theta$ is unique in the following sense:

Theorem 2.6 ([38, Theorem 4.1.2.]) Let $\Gamma_\mu \in U_\mu^- \otimes U_\mu^+$ (with $\mu \in Q_+$) be a family satisfying
1. $\Gamma_0 = 1 \otimes 1$;
2. $\Gamma = \sum_{\mu \geq 0} \Gamma_\mu$ satisfies $\Delta(u) \Gamma = \Gamma \Delta^\tau(u)$ for all $u \in U$ in $U \hat{\otimes} U$.

Then, $\Gamma_\mu = \Theta_\mu$ for all $\mu \in Q_+$.

Fix $\rho \in \mathbb{C}$ with $q = \exp(\pi i \rho)$. Note that if $\rho' \in \mathbb{C}$ satisfies $q = \exp(\pi i \rho')$, then $\rho - \rho' \in 2\mathbb{Z}$. For type 1 $U$-modules $M$ and $N$, define a linear isomorphism $f_\rho : M \otimes N \to M \otimes N$ by
\[ f_\rho(m \otimes n) = \exp(-\pi i \rho(\langle \lambda, \mu \rangle)) m \otimes n \quad (m \in M_\lambda, n \in N_\mu) \]
and for all $\lambda, \mu \in P$.

Lemma 2.7 Let $n_\xi$ be the minimal positive integer such that $n_\xi \langle \langle \lambda, \mu \rangle \rangle \in \mathbb{Z}$ for any $\lambda, \mu \in P$. Then, $f_\rho = f_{\rho + 2n_\xi}$ for any $N \in \mathbb{Z}$.

Then $R(\rho)$, as an element in $\text{End}(M \otimes N)$, satisfies
1. $R(\rho) \Delta(x) R(\rho)^{-1} = \Delta^\text{op}(x)$ for all $x \in U$;
2. $(\Delta \otimes 1)(R(\rho)) = R_{13}(\rho) R_{23}(\rho)$;
3. $(1 \otimes \Delta)(R(\rho)) = R_{13}(\rho) R_{12}(\rho)$.

One can show that
\[ B_{M,N} = P_{M,N} \circ R(\rho) : M \otimes N \to N \otimes M \]
defines a braiding on $U$-mod where $P_{M,N}$ denotes the transpose. Let $(U_q(\mathfrak{g}), R(\rho))$-mod denote the braided tensor category on $U_q(\mathfrak{g})$-mod thus obtained.

Set $r^\vee = \frac{\langle \alpha, \alpha \rangle}{2}$ for a long root $\alpha$, which is the ratio of the norm of long roots and short roots. Note that $r^\vee = 1$ when $\mathfrak{g}$ is simply-laced. The following result is due to Drinfeld [14, 15], Kazhdan-Lusztig [34], Lusztig [39] (see also [4]):

Theorem 2.8 ([14, 15, 34, 39]) Let $k \in \mathbb{C} \setminus \mathbb{Q}$ and set $\rho(k) = \frac{1}{r^\vee(k + h^\vee)} \in \mathbb{C}$. Then, there exists an equivalence of braided tensor categories between $D(\mathfrak{g}, k)$ and $(U_{\exp(\pi i \rho(k))}(\mathfrak{g}), R(\rho(k)))$-mod.
We note that $R_{21}(\rho)^{-1} = (P_{21}(\Theta)) \circ f_\rho$ also satisfies (R1)-(R3). Denote by $(U_q(\mathfrak{g}), R_{21}(\rho)^{-1})$-mod the braided tensor category on $U_q(\mathfrak{g})$-mod with the braiding defined via $R_{21}(\rho)^{-1}$. Then, by Lemma 1.8, Theorem 2.8 implies the following:

**Corollary 2.9** There exists an equivalence of braided tensor categories between $D(g, k)^{rev}$ and $(U_q(\mathfrak{g}), R_{21}(\rho(k))^{-1})$-mod.

Although the braiding on $(U_q(\mathfrak{g}), R(\rho))$-mod depends on $\rho \in \mathbb{C}$ by construction, it has a periodicity by Lemma 2.7.

**Corollary 2.10** There exists an equivalence of balanced braided tensor categories between $(U_q(\mathfrak{g}), R(\rho))$-mod and $(U_q(\mathfrak{g}), R(\rho + 2nqN))$-mod for any $N \in \mathbb{Z}$.

In Sect. 2.4, we will come back to the equivalence problem of $(U_q(\mathfrak{g}), R(\rho + N))$-mod for all $N \in \mathbb{Z}$ more precisely.

### 2.3 Isomorphism between $q$ and $q^{-1}$

Here we prove the three braided tensor categories $(U_q(\mathfrak{g}), R_{21}(\rho)^{-1})$, $(U_q(\mathfrak{g})^\text{cop}, R(\rho)^{-1})$ and $(U_q^{-1}(\mathfrak{g}), R(-\rho))$ are all equivalent.

Firstly, we prove the equivalence for the first two by using the isomorphism $\omega : U_q(g) \to U_q(g)^\text{cop}$ in Lemma 2.1. Let $M$ be a type 1 $U_q(\mathfrak{g})^\text{cop}$-module and define a type 1 $U_q(\mathfrak{g})$-module $\omega^* M$ by $M$ as a vector space equipped with the $U_q(\mathfrak{g})$-action

$$a \cdot_\omega m = \omega(a) \cdot m$$

for any $a \in U_q(\mathfrak{g})$ and $m \in M$.

Denote by $\omega^* M \otimes \omega^* N$ (resp. $M \otimes N$) the tensor product of $M$ and $N$ as $U_q(\mathfrak{g})$-modules (resp. $U_q(\mathfrak{g})^\text{cop}$-modules). Since $\omega^* M \otimes \omega^* N$ and $M \otimes N$ have the same underlying vector space, it suffices to compare the R-matrices acting on them to show the equivalence of the categories. Since $K_\alpha \cdot_\omega v = q^{-\langle(\alpha, \lambda)\rangle} v$ for $v \in M_\lambda$ and $v \in (\omega^* M)_{-\lambda}$, $f(\rho)$ acts in the same way. Hence, it suffices to compare $\Theta$ with $\langle(\omega \otimes \omega)(P_{21}(\Theta))$. They are the same by Lemma 2.4. Therefore, we have:

**Proposition 2.11** The Hopf algebra isomorphism $\omega : U_q(g) \to U_q(g)^\text{cop}$ induces an equivalence between $(U_q(\mathfrak{g}), R_{21}(\rho)^{-1})$-mod and $(U_q(\mathfrak{g})^\text{cop}, R(\rho)^{-1})$-mod as balanced braided tensor categories.

Secondly, we prove the equivalence for the first and third ones by using the isomorphism $\psi : U_q(\mathfrak{g}) \to U_q^{-1}(\mathfrak{g})^\text{cop}$ in Lemma 2.1. Let $M'$ and $N'$ be a type 1 $U_q^{-1}(\mathfrak{g})^\text{cop}$-modules. Denote by $\psi^* M' \otimes \psi^* N'$ (resp. $M' \otimes N'$) the tensor product of $M'$ and $N'$ as $U_q(\mathfrak{g})$-modules (resp. $U_q^{-1}(\mathfrak{g})^\text{cop}$-modules). As in the previous case, it suffices to prove the coincidence of the R-matrices to establish the equivalence of categories since $\psi^* M' \otimes \psi^* N' = M' \otimes N'$ as vector spaces. Let $\Theta^q$ (resp. $\Theta^{q^{-1}}$) denote $\Theta$ for $U_q(\mathfrak{g})$ (resp. $U_q^{-1}(\mathfrak{g})$) for distinguish them. It suffices to show that $R(-\rho)^{-1} = \Theta^{q^{-1}} \circ f_{-\rho}$ and $\psi^* R(\rho) = \psi^* (f_{\rho}^{-1} \circ (\Theta^q)^{-1})$ are the same as an linear maps on $M' \otimes N'$. We need the following lemma:
Lemma 2.12 Let $u \in U_q(\mathfrak{g})^-_\mu$ and $u' \in U_q(\mathfrak{g})^+_\mu$ for $\mu \in Q$ with $\mu \in Q_+$.

1. For any type 1 $U_q(\mathfrak{g})$-modules $M$ and $N$,

$$f_\rho^{-1} \circ (u \otimes u') \circ f_\rho = uK_\mu \otimes K_{-\mu}u'$$

as linear maps acting on $M \otimes N$.

2. $\psi(uK_\mu) \in U_q^{-1}(\mathfrak{g})^-_\mu$ and $\psi(K_{-\mu}u') \in U_q^{-1}(\mathfrak{g})^+_\mu$.

**Proof** Let $\lambda, \lambda' \in P$. Since $(u \otimes u') \cdot M_\lambda \otimes N_{\lambda'} \subset M_{\lambda - \mu} \otimes N_{\lambda' + \mu}$, for any $v \in M_\lambda$ and $w \in N_{\lambda'}$,

$$f_\rho^{-1} \circ (u \otimes u') \circ f_\rho(v \otimes w) = \exp(\pi i \rho(\langle \lambda, \mu \rangle) - \langle \lambda', \mu \rangle)$$

$$-\langle \mu, \mu \rangle)$$

$$(u \otimes u')(v \otimes w).$$

Hence, $f_\rho^{-1} \circ (u \otimes u') \circ f_\rho = q^{-(\langle \mu, \mu \rangle)}(u \otimes u')(K_\mu \otimes K_{-\mu}) = (uK_\mu \otimes K_{-\mu}u')$, which implies (1). (2) follows from the definition of $\psi$. □

By decomposing $(\Theta^q)^{-1} = \sum_\mu (\Theta^q)_\mu^{-1} \in U_q(\mathfrak{g})^-_\mu \otimes U_q(\mathfrak{g})^+_\mu$ with $(\Theta^q)_\mu^{-1} \in U_q(\mathfrak{g})^-_\mu \otimes U_q(\mathfrak{g})^+_\mu$, we find that $\Gamma_\mu = (\psi \otimes \psi)(f_\rho^{-1} \circ (\Theta^q)_\mu^{-1}) \circ f_\rho$ lies in $U_q^{-1}(\mathfrak{g})^-_\mu \otimes U_q^{-1}(\mathfrak{g})^+_\mu$ by Lemma 2.12. It is clear that the family $\{\Gamma_\mu\}_{\mu \in Q_+}$ satisfies the assumptions in Theorem 2.6. Thus, $\Theta^q^{-1} = \sum_{\mu \geq 0} \Gamma_\mu$ as an element in $U_q^{-1}(\mathfrak{g}) \otimes U_q^{-1}(\mathfrak{g})$. Hence, we have:

**Proposition 2.13** The Hopf algebra isomorphism $\psi : U_q(\mathfrak{g}) \to U_q^{-1}(\mathfrak{g})^{\text{cop}}$ induces an equivalence between $(U_q(\mathfrak{g}), R(\rho))$-mod and $(U_q^{-1}(\mathfrak{g})^{\text{cop}}, R(-\rho)^{-1})$-mod as braided tensor categories.

### 2.4 Graded twists of $U_q(\mathfrak{g})$-mod

In this section, we will consider a graded twist of $(U_q(\mathfrak{g}), R(\rho))$-mod (see Sect. 1.3).

Let $S$ be a subset of the weight lattice $P$ such that $Q + S \subset S$ (a subset of the coset $P/Q$). Let $(U_q(\mathfrak{g}), R(\rho), S)$-mod be the full subcategory of $(U_q(\mathfrak{g}), R(\rho))$-mod whose object is isomorphic to a direct sum of $L_q(\mu)$ for $\mu \in S \cap P^+$. Since $L_q(\mu) \otimes L_q(\mu') \in (U_q(\mathfrak{g}), R(\rho), \mu + \mu' + Q)$-mod for any $\mu, \mu' \in P$, the subcategories $\{(U_q(\mathfrak{g}), R(\rho), \mu + Q)\text{-mod})\}_{\mu \in P/Q}$ define a $P/Q$-grading on $(U_q(\mathfrak{g}), R(\rho))$-mod.

We note that if $L$ is a subgroup of $P$ with $Q \subset L$, then $(U_q(\mathfrak{g}), R(\rho), L)$-mod is closed under the tensor product and thus a braided tensor subcategory.

Since $Q$ is an even lattice with respect to $\langle \langle - , - \rangle \rangle$, we may introduce the map

$$Q_\mathfrak{g} : P/Q \to \mathbb{C}^\times, \quad Q_\mathfrak{g}(\lambda) = \exp(\pi i \langle \langle \lambda, \lambda \rangle \rangle).$$

Since

$$\frac{Q_\mathfrak{g}(\lambda + \mu)}{Q_\mathfrak{g}(\lambda) Q_\mathfrak{g}(\mu)} = \exp(2\pi i \langle \langle \lambda, \mu \rangle \rangle)$$
is a bicharacter on $P/Q$, $Q_\mathfrak{g}$ is a quadratic form. We note that $Q_\mathfrak{g}^N$ is also a quadratic form for any $N \in \mathbb{Z}$. By Theorem 1.2, we can consider the graded twist of $(U_q(\mathfrak{g}), R(\rho))$-mod associated with these quadratic forms. We first show the following proposition:

**Proposition 2.14** For any $N \in \mathbb{Z}$, the identify functor gives an equivalence between $(U_q(\mathfrak{g}), R(\rho))$-mod$^{Q_\mathfrak{g}^N}$ and $(U_q(\mathfrak{g}), R(\rho + 2N))$-mod as braided tensor categories.

**Proof** Since $(U_q(\mathfrak{g}), R(\rho))$-mod and $(U_q(\mathfrak{g}, R(\rho + 2N))$-mod are the same as tensor categories, it suffices to compare the braiding. From the definition, they differ by $f_\rho$ and $f_{\rho + 2N}$ in the R-matrices.

Let $B_N : P/Q \times P/Q \to \mathbb{C}^\times$ be a bicharacter defined by

$$B_N(\lambda, \mu) = \exp(2\pi i N(\langle \lambda, \mu \rangle))$$

and $1_{P/Q} : P/Q \times P/Q \to \mathbb{C}^\times$ be a trivial 3-cocycle. Then, $\text{EM}(Q_\mathfrak{g}^N)^{-1} = (1_{P/Q}, B_N) \in H^3_{ab}(P/Q, \mathbb{C}^\times)$ by Lemma 1.3. Thus, $(U_q(\mathfrak{g}), R(\rho))$-mod$^{Q_\mathfrak{g}^N}$ is equivalent to $(U_q(\mathfrak{g}), B_N \circ R(\rho))$-mod. Since

$$B_N \circ f_\rho^{-1}
|_{L_q(\lambda) = \sum L_q(\mu) \mu + \beta} = \exp(2\pi i N(\langle \lambda, \mu \rangle)) \exp(\pi i \rho(\lambda + \alpha, \mu + \beta))$$

$$= \exp(\pi i (\rho + 2N)(\lambda + \alpha, \mu + \beta))$$

$$= f_\rho^{-1}$$

the assertion holds. \qed

Hereafter, we will consider the twisted braided tensor category $(U_q(\mathfrak{g}), R(\rho))$-mod$^{Q_\mathfrak{g}^N}$ for odd integers $N \in \mathbb{Z}$. We note that the modification $\rho \to \rho + N$ changes the value $q = \exp(\pi i \rho)$ to $(-1)^N q$. Hence, if $N$ is an odd number, $q$ will change. We conjecture that the following statement holds:

**Conjecture 1** Let $\mathfrak{g}$ be a simple Lie algebra. For any $N \in \mathbb{Z}$, $(U_q(\mathfrak{g}, R(\rho))$-mod$^{Q_\mathfrak{g}^N}$ and $(U_{(-1)^N q}(\mathfrak{g}, R(\rho + N)$-mod are equivalent as braided tensor categories.

The conjecture is true if $N$ is even by Proposition 2.14. We will prove this conjecture for $\mathfrak{g}$ of type ABC and partially for type D. More precisely, let $\{\lambda_i\}_{i=1,...,n-1}$ be the fundamental weights with respect to the labeling of the Dynkin diagram in [28]. We will use this labeling throughout the paper. Then, the vector representation of $\mathfrak{so}_{2n}$ is isomorphic to the highest weight representation $L_q(\lambda_1)$. Let $\Lambda_v$ be the subgroup of $P$ generated by the root lattice $Q$ and $\lambda_1$. Then, the tensor subcategory of $(U_q(\mathfrak{so}_{2n}), R(\rho))$-mod generated by the vector representation is equal to $(U_q(\mathfrak{so}_{2n}), R(\rho), \Lambda_v)$-mod.

Note that $(U_q(\mathfrak{so}_{2n}, R(\rho), \Lambda_v)$-mod is graded by $\Lambda_v/Q = \mathbb{Z}/2\mathbb{Z}$. By abuse of notation, we denote by $Q_{\mathfrak{so}_{2n}}$ the restriction of the quadratic form $Q_{\mathfrak{so}_{2n}} : P/Q \to \mathbb{C}^\times$ to $\Lambda_v/Q$.

**Theorem 2.15** Conjecture 1 is true if $\mathfrak{g}$ is of type ABC. In the case of $\mathfrak{g}$ of type $D_n$, $(U_q(\mathfrak{so}_{2n}), R(\rho), \Lambda_v)$-mod$^{Q_\mathfrak{g}^N}$ and $(U_{(-1)^N q}(\mathfrak{so}_{2n}), R(\rho + N), \Lambda_v)$-mod are equivalent as braided tensor categories for any $N \in \mathbb{Z}$.
If \( \mathfrak{g} \) is of type B, that is \( \mathfrak{g} \cong \mathfrak{so}_{2n+1} \), there is a Hopf algebra isomorphism \( \phi : U_q(\mathfrak{so}_{2n+1}) \rightarrow U_q(\mathfrak{so}_{2n+1}) \). So in this case, the above theorem can be proved as in the previous section. However, this case is a little more complicated because of the existence of twist, and the proof will be given in appendix Theorem A.10. In this section, we prove the theorem for \( \mathfrak{g} \) of type ACD.

We first consider the case of \( \mathfrak{g} \) of type A, i.e., \( \mathfrak{g} = \mathfrak{sl}_n \) and denote \( Q_{\mathfrak{sl}_n} \) by \( Q \) for short. In [36], semisimple rigid tensor categories with fusion rules of type \( \mathfrak{so}_2 \) for \( n \geq 2 \). Remark 2.16 for \( (\mathfrak{so}_2, R(\rho)) \)-mod, the eigenvalues of the twisted braiding \( B_{X,X}^Q \) are

\[
B_{X,X}^Q|_{X_s} = \exp(\pi i(1 - \frac{1}{n})) \exp(\pi i \rho(1 - \frac{1}{n})),
\]

\[
B_{X,X}^Q|_{X_a} = - \exp(\pi i (1 - \frac{1}{n})) \exp(\pi i \rho(-1 + \frac{1}{n})) = - \exp(\pi i ((\rho + 1) - \frac{1}{n} + 2))
\]

which coincide with those on \((U_q(\mathfrak{sl}_n), R(\rho + 1))\). Hence, we have proved Theorem 2.15 for \( \mathfrak{g} = \mathfrak{sl}_n \).

**Remark 2.16** For odd \( n \), we may prove this result by using the generator and relation of the quantum coordinate ring \( O_q(\text{SU}(n)) \) [3, 42], which is carried out for \( n = 2 \) in [5].

There is a partial extension of the work [36] to the case of type BCD [44], where semisimple rigid tensor categories with fusion rules of type \( \mathfrak{sp}_{2n} \) and \( \mathfrak{so}_n \) are classified. Then it follows that for any positive integer \( n \) there are equivalences of tensor categories,

\[
(U_q(\mathfrak{sp}_{2n}), R(\rho))\text{-mod}^{Q_{\mathfrak{sp}_{2n}}} \cong (U_q(\mathfrak{sp}_{2n}), R(\rho + 1))\text{-mod},
\]

\[
(U_q(\mathfrak{so}_n), R(\rho), \Lambda_v)\text{-mod}^{Q_{\mathfrak{so}_n}} \cong (U_q(\mathfrak{so}_n), R(\rho + 1), \Lambda_v)\text{-mod}.
\]
It is important to point out that the classification for type BD is restricted to smaller subcategories which do not contain the spin representations and there is no established result on the classification of the whole category \((U_q(so_n), R(\rho))\)-mod. We will prove the equivalence \((U_q(so_n), R(\rho))\)-mod\(\cong\) \((U_{-q}(so_n), R(\rho + 1))\)-mod more directly in Appendix.

Hereafter, we upgrade the equivalences (2.1) as braided tensor categories.

We will first consider the case of type \(C_n\). Set \(X = L_q(\lambda_n)\) the vector representation. As in the previous case, it suffices to prove the braidings are the same on \(X \otimes X\). The tensor product \(X \otimes X\) decomposes into three simple objects, \(X \otimes^2 \cong X_s \oplus X_a \oplus X_1\), where \(X_s\) is the symmetric tensor and \(X_1\) is the trivial representation. The eigenvalues of the braiding \(B_{X,X} \in \text{End}(X \otimes^2)\) are

\[
B_{X,X}|_{X_s} = q, \quad B_{X,X}|_{X_a} = -q^{-1}, \quad B_{X,X}|_{X_1} = -q^{-(1+2n)}
\]

(see [35, Section 8.4.3]). Since \(Q_{sp_{2n}}(\lambda_n) = \exp(\pi i (\langle \lambda_n, \lambda_n \rangle)) = -1\), the braidings of \((U_q(sp_{2n}), R(\rho))\)-mod\(^Q\) and \((U_{-q}(sp_{2n}), R(\rho + 1))\)-mod are the same on \(X \otimes^2\). Hence, the conjecture is true for \(sp_{2n}\).

Finally, we will consider the case of type \(D_n\). Let \(X\) be the vector representation of \(so_{2n}\), which is isomorphic to the highest weight representation \(L_q(\lambda_1)\). The tensor product \(X \otimes X\) is a direct sum of three simple objects, \(X \otimes^2 \cong X_s \oplus X_a \oplus X_1\), where \(X_s\) is the traceless symmetric tensor, \(X_a\) the antisymmetric tensor, \(X_1\) the trivial representation. The eigenvalues of the braiding \(B_{X,X} \in \text{End}(X \otimes^2)\) are

\[
B_{X,X}|_{X_s} = q, \quad B_{X,X}|_{X_a} = -q^{-1}, \quad B_{X,X}|_{X_1} = q^{1-2n}
\]

(see [35, Section 8.4.3]). Since \(Q(\lambda_1) = \exp(\pi i (\langle \lambda_1, \lambda_1 \rangle)) = -1\), the braidings of \((U_q(so_{2n}), R(\rho))\)-mod\(^Q\) and \((U_{-q}(so_{2n}), R(\rho + 1))\)-mod are the same on \(X \otimes^2\). Hence, we have proved Theorem 2.15 for \(g = so_{2n}\).

### 3 Construction of commutative algebra objects

By Lemma 1.8, Proposition 2.11 and Proposition 2.13, we have equivalences of the balanced braided tensor categories

\[
((U_q(g), R(\rho))\text{-mod})^{\text{rev}} \cong (U_q(g), R_{21}(\rho)^{-1})\text{-mod} \\
\cong (U_q(g)^{\text{cop}}, R(\rho)^{-1})\text{-mod} \\
\cong (U_{-q-1}(g), R(-\rho))\text{-mod}.
\]
We will give commutative algebra objects of \((U_q(\mathfrak{g}), R(\rho)) \otimes (U_q(\mathfrak{g})^\text{cop}, R(\rho)^{-1})\text{-mod}\) in Sect. 3.1. In Sect. 3.2, by using it, we will construct a lax monoidal functor

\[ O_\mathfrak{g} : \text{Vec}_{P/Q} \to (U_q(\mathfrak{g}), R(\rho)) \otimes (U_q^{-1}(\mathfrak{g}), R(-\rho))\text{-mod}. \]

Then, by Theorem 2.15 the grading twist associated with the quadratic form \(Q_\mathfrak{g} : P \to \mathbb{C} \times \mathfrak{g}\) gives us a lax braided monoidal functor

\[ O_N^\mathfrak{g} : \text{Vec}_{P/Q} \to (U_{(-1)^\mathfrak{g}}(\mathfrak{g}), R(\rho + N)) \otimes (U_q^{-1}(\mathfrak{g}), R(-\rho))\text{-mod}. \]

The main Theorem will be proved by using this functor. As an application, we construct many vertex superalgebras. This will be discussed in Sect. 3.3.

### 3.1 Quantum coordinate ring

Let \(U_q(\mathfrak{g})^*\) be the dual vector space of \(U_q(\mathfrak{g})\). In contrast to finite-dimensional Hopf algebras, \(U_q(\mathfrak{g})^*\) is not naturally a Hopf algebra since the product (1.9) is not defined on \(U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^*\). Instead, we introduce the functionals

\[ c^\lambda_{m,f} : U_q(\mathfrak{g}) \to \mathbb{C}, \quad a \mapsto f(a \cdot m) \]

for \(m \in L_q(\lambda)\) and \(f \in L_q(\lambda)^* (\lambda \in P^+)\) and then the subspace \(O_q(G) \subset U_q(\mathfrak{g})^*\) spanned by these elements. Here \(G\) represents the simply-connected simple Lie group whose Lie algebra is \(\mathfrak{g}\). Following the same argument in Sect. 1.4, we can show that \(O_q(G)\) is a Hopf algebra by (1.9) and is a \(U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})^\text{cop}\)-module, which decomposes into

\[ \bigoplus_{\lambda \in P^+} c^\lambda : \bigoplus_{\lambda \in P^+} L_q(\lambda) \otimes L_q(\lambda)^* \cong O_q(G). \quad (3.1) \]

The isomorphism (3.1) is a q-analogue of the Peter-Weyl theorem for \(G\) and indeed \(O_q(G)\) is called the quantum coordinate ring of \(G\) in the literature [35].

By Proposition 1.11 and Proposition 2.11, we have:

**Proposition 3.1** The quantum coordinate ring \(O_q(G)\) is a commutative algebra object in \((U_q(\mathfrak{g}), R(\rho)) \otimes (U_q(\mathfrak{g})^\text{cop}, R(\rho)^{-1})\text{-mod}\) and satisfies \(\theta_{O_q(G)} = \text{id}\). The unit of \(O_q(G)\) is given by a natural injection \(\epsilon : L_q(0) \otimes L_q(0) \to O_q(G)\). Furthermore, if \(I \subset O_q(G)\) satisfies

1. \(a \cdot m \in I\) for any \(m \in I\) and \(a \in O_q(G)\);
2. \((u \otimes v) \cdot a \in I\) for any \(a \in I\) and \(u \otimes v \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})^\text{cop}\),

then \(I = 0\) or \(I = O_q(G)\).
Proposition 3.3

For any $k \in P^+$, set $\lambda^* = -w_0(\lambda)$, where $w_0$ is the longest element in the Weyl group of $\mathfrak{g}$. Then, the dual module $L_q(\lambda)^*$ is isomorphic to $L_q(\lambda^*)$. By Theorem 2.8, we may calculate the value $\theta_{L_q(\lambda) \otimes L_q(\lambda)^*}$ by using (1.17) and (1.18) and find

$$
\theta_{L_q(\lambda) \otimes L_q(\lambda)^*} = \exp \left( \frac{\pi i}{k + h^\vee} (\lambda + 2\rho, \lambda) - (\lambda^* + 2\rho, \lambda^*) \right).
$$

Since

$$(\lambda + 2\rho, \lambda) - (\lambda^* + 2\rho, \lambda^*) = (\lambda + 2\rho, \lambda) - (w_0(\lambda) - 2\rho, w_0(\lambda)),$$

we have $\theta_{O_q(G)} = \text{id}$.

Assume that $I \neq O_q(G)$. Since $I \subset O_q(G)$ is stable under the bimodule action, $I \neq O_q(G)$ implies $I \cap L_q(0) \otimes L_q(0) = 0$. Thus, $\epsilon_\vee(I) = 0$ or more specifically $a(1) = 0$ for any $a \in I$. By (1.10), it follows that $a = 0$ and thus $I = 0$. \hfill \Box

Consequently, by Proposition 1.16 and Theorem 2.8, we have:

**Proposition 3.2** For any $k \in \mathbb{C} \setminus \mathbb{Q}$, there exists a simple full vertex algebra structure on

$$F_{G,k} = \bigoplus_{\lambda \in P^+ \cap L} L_{g,k}(\lambda) \otimes L_{g,k}(\lambda^*),$$

as an extension of the full vertex algebra $L_{g,k}(0) \otimes \overline{L_{g,k}(0)}$.

**Proof** It suffices to show the simplicity. Let $I \subset F_{G,k}$ be a left ideal such that $I \neq F_{G,k}$. Since $F_{G,k}$ contains $L_{g,k}(0) \otimes \overline{L_{g,k}(0)}$ as a subalgebra, $I$ is an $L_{g,k}(0) \otimes \overline{L_{g,k}(0)}$-module. Thus, the embedding $i : I \to F_{G,k}$ is a morphism in $D(g, k) \otimes \overline{D(g, k)}$ and by the equivalence of categories it corresponds to a subobject $i' : I' \to O_q(G)$. Then, $I'$ is a left ideal of $O_q(G)$ and stable under the action of $U_q(g) \otimes U_q(g)^\text{cop}$. By Proposition 3.1, $I' = 0$ and $I = 0$, which implies that $F_{G,k}$ is a simple full vertex algebra. \hfill \Box

We note that $F_{G,k}$ is the underlying algebra of the (analytic continuation of) WZW-model associated with the Lie group $G$ and at level $k$.

By Proposition 2.13, $O_q(G)$ is a commutative algebra object in $(U_q(g), R(\rho)) \otimes (U_q^{-1}(g), R(-\rho))$-mod and satisfies $\theta_{O_q(G)} = \text{id}$. Hence, by Proposition 1.13, we also have:

**Proposition 3.3** For any $k \in \mathbb{C}$ there exists a simple vertex algebra structure on

$$V_{G,k} = \bigoplus_{\lambda \in P^+ \cap L} L_{g,k}(\lambda) \otimes L_{g,k}(\lambda^*),$$

where $\tilde{k} = -k - 2h^\vee$, as an extension of the vertex algebra $L_{g,k}(0) \otimes L_{g,k}(0)$.

The vertex algebra in Proposition 3.3 is known as the algebra of chiral differential operators (see [1, 23, 25, 26, 45] for example).
3.2 Quantum coordinate ring as lax monoidal functor

Recall that in Sect. 1.3 we introduce a category of $P/Q$-graded vector space $\text{Vec}_{V g}$, which is a braided tensor category with the trivial associative isomorphism and the trivial braiding, and for $\lambda, \lambda' \in \text{Vec}_{V g}$ is the one-dimensional vector space with the grading $\lambda$. We will construct a lax braided monoidal functor $O_\lambda : \text{Vec}_{V g} \rightarrow (U_q(g), R(\rho)) \otimes (U_{q^{-1}}(g), R(-\rho))$-mod.

For $\lambda \in P/Q$, set

$$O_q(G)_\lambda = \bigoplus_{\mu \in (\lambda + Q) \cap P^+} L_q(\mu) \otimes L_q(\mu)^*.$$ 

Then, $O_q(G) = \bigoplus_{\lambda \in P/Q} O_q(G)_\lambda$, and $O_q(G)$ is a $P/Q$-graded algebra, that is, $O_q(G)_\lambda \cdot O_q(G)_{\lambda'} \subset O_q(G)_{\lambda + \lambda'}$, for any $\lambda, \lambda' \in P/Q$.

Define a $\mathbb{C}$-linear functor $O_\lambda : \text{Vec}_{V g} \rightarrow (U_q(g), R(\rho)) \otimes (U_{q^{-1}}(g), R(-\rho))$-mod as follows: For an object $V = \bigoplus_{\lambda \in P/Q} V_\lambda \in \text{Vec}_{V g}$,

$$O_\lambda(V) = \bigoplus_{\lambda \in P/Q} O_q(G)_\lambda \otimes_{\mathbb{C}} V_\lambda,$$

where $- \otimes_{\mathbb{C}} -$ is the tensor product of $\mathbb{C}$-vector spaces. For a morphism $\{f_\lambda : V_\lambda \rightarrow W_\lambda\}_{\lambda \in P/Q}$,

$$O_\lambda(f) = \bigoplus_{\lambda \in P/Q} \text{id}O_q(G)_\lambda \otimes f_\lambda : \bigoplus_{\lambda \in P/Q} O_q(G)_\lambda \otimes_{\mathbb{C}} V_\lambda \rightarrow \bigoplus_{\lambda \in P/Q} O_q(G)_\lambda \otimes_{\mathbb{C}} W_\lambda.$$

Since $O_q(G)$ is a $P/Q$-graded algebra, we may restrict the product as $m(\lambda, \lambda') : O_\lambda(\lambda) \otimes O_\lambda(\lambda') \rightarrow O_\lambda(\lambda + \lambda')$, which is a left $U_q(g) \otimes U_{q^{-1}}(g)$-module homomorphism.

Thus, for objects $V, W \in \text{Vec}_{V g}$, we have the natural transformation

$$m_{V, W} : O_\lambda(V) \otimes O_\lambda(W) \rightarrow O_\lambda(V \otimes W)$$

defined component-wise by

$$m_{V, W} : (O_q(G)_\lambda \otimes_{\mathbb{C}} V_\lambda) \otimes (O_q(G)_{\lambda'} \otimes_{\mathbb{C}} W_{\lambda'}) \cong (O_q(G)_\lambda \otimes O_q(G)) \otimes_{\mathbb{C}} (V_\lambda \otimes_{\mathbb{C}} W_{\lambda'}) \xrightarrow{m(\lambda, \lambda') \otimes \text{id}V_\lambda \otimes W_{\lambda'}} O_q(G)_{\lambda + \lambda'} \otimes_{\mathbb{C}} (V_\lambda \otimes_{\mathbb{C}} W_{\lambda'})$$

for any $\lambda, \lambda' \in P/Q$ and the linear extension of it. Let $\epsilon : L_q(0) \otimes L_q(0) \rightarrow O_q(G)_0$ be the natural injection. Then, we have:

**Lemma 3.4** The functor $O_\lambda : \text{Vec}_{V g} \rightarrow (U_q(g), R(\rho)) \otimes (U_{q^{-1}}(g), R(-\rho))$-mod together with a morphism $\epsilon : L_q(0) \otimes L_q(0) \rightarrow O_{\text{Vec}_{V g}}(1)$ and the natural transformation $m_{V, W} : O_\lambda(V) \otimes O_\lambda(W) \rightarrow O_\lambda(V \otimes W)$ is a lax braided monoidal functor.
Proof} Since both $\text{Vec}_{P/Q}$ and $(U_q(g), R(\rho)) \otimes (U_q^{-1}(g), R(-\rho))$-mod are strict monoidal categories, that is, the associativity isomorphism $\alpha$ and unit morphisms $l, r$ are trivial, (LM1) and (LM2) follow from the fact that $O_q(G)$ is an associative algebra. Furthermore, since $O_q(G)$ is a commutative algebra object, (1.4) follows and thus $O_g$ is a lax braided monoidal functor. \hfill $\Box$

We consider the $P/Q$-grading on $(U_q(g), R(\rho)) \otimes (U_q^{-1}(g), R(-\rho))$-mod obtained from the $P/Q$-grading on the left component, $(U_q(g), R(\rho))$-mod. Let us consider the grading twist of $\text{Vec}_{P/Q}$ and $(U_q(g), R(\rho)) \otimes (U_q^{-1}(g), R(-\rho))$-mod by the quadratic form $Q^N_g : P/Q \rightarrow \mathbb{C}^\times$ for $N \in \mathbb{Z}$. Since the grading twist does not change the underlying category structure, $O_g$ still gives a functor $\text{Vec}_{P/Q} \rightarrow (U_q(g), R(\rho))$-mod$^{O_g}_N \otimes (U_q^{-1}(g), R(-\rho))$-mod. We denote it by $O^N_g$. Since $O_g$ preserves the gradings on $\text{Vec}_{P/Q}$ and $(U_q(g), R(\rho)) \otimes (U_q^{-1}(g), R(-\rho))$-mod, Lemma 3.4 immediately implies the following:

**Proposition 3.5** The functor $O^N_g : \text{Vec}_{P/Q} \rightarrow (U_q(g), R(\rho))$-mod$^{O_g}_N \otimes (U_q^{-1}(g), R(-\rho))$-mod is a lax braided monoidal functor.

Now, we can state the main theorem. Recall that given an even lattice $M$, we have associated the dual lattice $M^\vee$ and the quadratic form $Q_M : M^\vee/M \rightarrow \mathbb{C}^\times$ by $Q_M(\lambda) = \exp(\pi i (\lambda, \lambda))$.

Let $r$ be a positive integer and $g_i$ be simple Lie algebras and $k_i, k_i' \in \mathbb{C}\setminus\mathbb{Q}$ and $N_i \in \mathbb{Z}$ satisfy

$$\frac{1}{r^\vee(k_i + h_i^\vee)} + \frac{1}{r^\vee(k_i' + h_i^\vee)} = N_i$$

for $i = 1, \ldots, r$.

Let $(A, Q)$ be a quadratic space defined by

$$A = \bigoplus_{i=1}^{r} P_i/Q_i \oplus M^\vee/M,$$

$$Q = \bigoplus_{i=1}^{r} Q^N_{g_i} \oplus Q_M,$$

where $P_i$ (resp. $Q_i$) is the weight lattice (resp. the root lattice) of $g_i$. For $\lambda_i \in P_i/Q_i$, set

$$L_{g_i,k_i,k_i'}^{N_{g_i}}(\lambda_i + Q_i) = \bigoplus_{\mu \in (\lambda_i + Q_i) \cap P_i^+} L_{g_i,k_i}(\mu) \otimes L_{g_i,k_i'}(\mu)^\ast.$$

Let $(I, p)$ be a super isotropic subspace of the quadratic space $(A, Q)$. Set

$$V^N_{g,k,k',M}(I) = \bigoplus_{(\lambda_1, \ldots, \lambda_r, \mu) \in I} \bigotimes_{i=1}^{r} L_{g_i,k_i,k_i'}^{N_{g_i}}(\lambda_i + Q_i) \otimes V_{\mu+M}$$

(3.3)
for \((g, k, k', M, I)\) with \(g = (g_1, \ldots, g_r)\) and \(k = (k_1, \ldots, k_r), k' = (k'_1, \ldots, k'_r), N = (N_1, \ldots, N_r)\). Then, we have:

**Theorem 3.6** Suppose that for each \(a = 1, \ldots, r\) one of the following conditions is satisfied:

1. \(N_a\) is even;
2. \(g_a\) is of type ABC;
3. \(g_a\) is of type D and \(pr_a(I) \subset \Lambda_v/Q_i\), where \(pr_a : (\bigoplus_{i=1}^{r} P_i/Q_i) \oplus M^\vee/M \to P_a/Q_a\) is the projection to the \(a\)-th component.

Then, there is a simple vertex superalgebra structure on \(V^N_{g,k,k',M}(I)\) as an extension of

\[
\left( \bigotimes_{i=1}^{r} L_{g_i,k_i}(0) \otimes L_{g_i,k'_i}(0) \right) \otimes V_M.
\]

Furthermore, the even part \((s = 0)\) and the odd part \((s = 1)\) are given by

\[
V^N_{g,k,k',M}(I)_s = \bigoplus_{(\lambda_1, \ldots, \lambda_r, \mu) \in I} \bigotimes_{i=1}^{r} L_{g_i,k_i,k'_i}(\lambda_i + Q) \otimes V_{\mu + M}.
\]

**Proof** For the sake of simplicity, we only show the case of \(r = 1\). Fix \(q \in \mathbb{C} \setminus Q\). Then by Proposition 2.14 and Theorem 2.15, the conditions 1) and 2) imply that \(O^N_q(G)\) induces a lax braided monoidal functor

\[
O^N_q : \text{Vec}_{P/Q}^{Q_N} \to C
\]

with \(C = (U_{(-1)^N_q}(g), R(\rho + N))\)-mod \(\otimes (U_{q^{-1}}(g), R(-\rho))\)-mod, and thus

\[
\hat{O}^N_g : \text{Vec}_{P/Q}^{Q_N} \otimes \text{Vec}_{M^\vee/M}^{Q_M} \to C \otimes V_M\text{-mod}
\]

by Proposition 1.7. Now, the super isotropic subspace \((I, p)\) defines a supercommutative algebra object in \(\text{Vec}_{P/Q}^{Q_N} \otimes \text{Vec}_{M^\vee/M}^{Q_M}\) by Lemma 1.5 and thus in \(C \otimes V_M\text{-mod}\) through \(\hat{O}^N_g\) by Lemma 1.1 and, therefore, in \(D(g, k) \otimes D(g, k') \otimes V_M\text{-mod}\) with \((k, k')\) defined by

\[
\rho + N = \frac{1}{r^\vee(k + h^\vee)}, \quad -\rho = \frac{1}{r^\vee(k' + h^\vee)},
\]

through Theorem 2.8. By Proposition 1.14, it follows that \((I, p)\) induces a vertex superalgebra of the form (3.3). Since (3.6) implies (3.2), we obtain the assertion in the case (1) and (2). The remaining case 3) for \(g = so_{2n}\) can be shown in the same way by replacing (3.4) with

\[
O^N_g : \text{Vec}_{\Lambda_v/Q}^{Q_N} \to C
\]

and \(C = (U_{(-1)^N_q}(so_{2n}), R(\rho + N), \Lambda_v)\)-mod \(\otimes (U_{q^{-1}}(so_{2n}), R(-\rho), \Lambda_v)\)-mod.
Table 3  Quadratic space

| Type | \( P/Q \) | Generator | Value of \( Q_\mathfrak{g} \) |
|------|-----------|-----------|-----------------|
| \( A_{n-1} \) | \( \mathbb{Z}/n\mathbb{Z} \) | \( \lambda_1 \) | \( Q(\lambda_1) = \exp(\frac{n-1}{n} \pi i) \) |
| \( B_n \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \lambda_n \) | \( Q(\lambda_n) = i^n \) |
| \( C_n \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \lambda_n \) | \( Q(\lambda_n) = -1 \) |
| \( D_{2n} \) | \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) | \( \lambda_1, \lambda_{2n}, \lambda_{2n-1} \) | \( Q(\lambda_1) = -1, Q(\lambda_{2n-1}) = Q(\lambda_{2n}) = i^n \) |
| \( D_{2n+1} \) | \( \mathbb{Z}/4\mathbb{Z} \) | \( \lambda_{2n+1} \) | \( Q(\lambda_{2n+1}) = \exp(\frac{2n+1}{4} \pi i) \) |
| \( E_6 \) | \( \mathbb{Z}/3\mathbb{Z} \) | \( \lambda_6 \) | \( Q(\lambda_6) = \exp(\frac{4}{3} \pi i) \) |
| \( E_7 \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \lambda_7 \) | \( Q(\lambda_7) = -i \) |
| \( E_8, F_4, G_2 \) | 0 | – | – |

The simplicity of \( V_{\mathfrak{g}, k, k', M}^N(I) \) follow from Proposition 3.1 and the definition of \((I, p)\).

\[ \square \]

**Remark 3.7** For full vertex algebras, one can show that

\[
\tilde{V}_{\mathfrak{g}, k, k', M}^N(I) = \bigoplus_{(\lambda_1, \ldots, \lambda_r, \mu) \in I} \bigotimes_{i=1}^r \tilde{L}_{\mathfrak{g_i}, k_i, k'_i}^N(\lambda_i + Q_i) \otimes V_{\mu + M},
\]

\[
\tilde{L}_{\mathfrak{g_i}, k_i, k'_i}^N(\lambda_i + Q_i) = \bigoplus_{\mu \in (\lambda_i + Q_i) \cap P_i^+} L_{\mathfrak{g_i}, k_i}^N(\mu) \otimes \overline{L_{\mathfrak{g_i}, k'_i}(\mu)}^*,
\]

with (3.2) replaced by

\[
\frac{1}{r_i^N(k_i + h_i^\vee)} - \frac{1}{r_i^N(k'_i + h_i^\vee)} = N_i
\]

has a structure of simple full vertex superalgebra.

**Remark 3.8** The proof of Theorem 3.6 implies that

\[
\bigoplus_{\lambda \in P/Q} L_{\mathfrak{g}, k, k'}^N(\lambda + Q)
\]

is an abelian intertwining algebra with abelian cocycle \( EM^{-1}(Q_\mathfrak{g}^N) \in H_{ab}^3(P/Q, \mathbb{C}^\times) \).

### 3.3 Applications

Here we give some examples of Theorem 3.6. For this purpose, we use the group structure and the values of \( Q_\mathfrak{g} \) of the quadratic space \((P/Q, Q_\mathfrak{g})\) as summarized in Table 3:

We note that for type D, \( \lambda_1 \) corresponds to the vector representation, and \( \lambda_{n-1}, \lambda_n \) correspond to the spin representation and its conjugate representation. Let \( \Lambda_x \subset P \) (resp. \( \Lambda_c, \Lambda_v \)) denote the subgroup generated by \( Q \) and \( \lambda_n \) (resp. \( \lambda_{n-1}, \lambda_1 \)).
Remark 3.9 To check the value of $Q_{\mathfrak{g}}$ for type $E_6$ or $E_7$, it is convenient to use the fact that $\text{Vec}_{\mathcal{A}_k}^{Q_{E_8}}$ and $\text{Vec}_{E_8}^{Q_{E_8-k}}$ are braided reverse equivalent. In fact, the unimodular lattice $E_8$ is an index 2 (resp. index 3) extension of $A_1 \oplus E_7$ (resp. $A_2 \oplus E_6$). Thus, $Q(\lambda_{E_7})Q(\lambda_{A_1})$ (resp. $Q(\lambda_{E_6})Q(\lambda_{A_2})$) is equal to 1.

The following lemma is obvious:

Lemma 3.10 Let $(A, Q)$ be a quadratic space such that $Q(A) \subset \{ \pm 1 \}$. If $I \subset A$ is a cyclic subgroup, then $I$ is a super isotropic subspace.

Remark 3.11 If $I$ is not cyclic, then the above lemma is not true. For example, $(P/Q, Q_{\mathfrak{g}})$ for type $D_{2n}$ satisfies the assumption, but is not a super isotropic.

For $\mathfrak{g} = \mathfrak{sl}_n$, the following is a specialization of Theorem 3.6, which is conjectured in [6, Conjecture 1.4].

Proposition 3.12 Let $n \geq 2$ and $s \in \{ \pm 1 \}$. Then, we have a simple vertex superalgebra of the form

$$\bigoplus_{a=0}^{n-1} L_{\mathfrak{sl}_n, k, k'}^{1+nN}(a\lambda_1 + Q) \otimes V_{\frac{a\sqrt{n}}{n} + \sqrt{sn}Z}$$

at levels $k, k' \in \mathbb{C} \setminus \mathbb{Q}$ satisfying the relation $\frac{k}{k+n} + \frac{k'}{k'+n} = s + nN$ for some $N \in \mathbb{Z}$. Here, $V_{\sqrt{sn}Z}$ is a lattice vertex (super)algebra associated with the rank one lattice $\mathbb{Z} \alpha$ such that $(\alpha, \alpha) = sn$.

Proof First, we show the case when $n$ is even. Then, $\sqrt{sn}Z$ is an even lattice. Let $I \subset P/Q \oplus (\sqrt{sn}Z)^\vee/\sqrt{sn}Z \cong (\mathbb{Z}/n\mathbb{Z})^2$ be the subgroup generated by $(\lambda_1, \sqrt{sn}Z)$. Since

$$Q(\lambda_1, \sqrt{sn}Z) = Q_{\mathfrak{sl}_n}(\lambda_1)^3 Q_{\sqrt{sn}Z}(\sqrt{sn}Z) = \exp(s\pi i \frac{n-1}{n}) \exp(s\pi i \frac{1}{n}) = -1,$$

by Lemma 3.10, $I$ is a super isotropic subspace. Thus, the assertion follows from Theorem 3.6.

Next, we show the case when $n$ is odd. For this we use the subgroup $2\sqrt{sn}Z \subset \sqrt{sn}Z$. Let $I \subset P/Q \oplus (2\sqrt{sn}Z)^\vee/(2\sqrt{sn}Z) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/4n\mathbb{Z}$ be the subgroup generated by $(\lambda_1, \sqrt{sn}Z)$. Then, $I \cong \mathbb{Z}/2n\mathbb{Z}$ and by the above calculus $I$ is a super isotropic. Hence,

$$V = \bigoplus_{b=0}^{2n-1} L_{\mathfrak{sl}_n, k, k'}^{1+nN}(b\lambda_1 + Q) \otimes V_{\frac{b\sqrt{n}}{n} + 2\sqrt{sn}Z}$$ (3.7)

is a vertex superalgebra. Since $n(\lambda_1, \sqrt{sn}Z) = (0, \sqrt{sn}) \in P/Q \oplus (2\sqrt{sn}Z)^\vee/(2\sqrt{sn}Z)$, $V$ contains a subalgebra isomorphic to the lattice vertex superalgebra $V_{\sqrt{sn}Z}$. Rewrite (3.7) as the sum of $V_{\sqrt{sn}Z}$-modules to get the assertion. \qed
Next, we consider the case of \( r = 1 \) and \( M = 0 \) in Theorem 3.6. Then, \((I, p)\) is indeed a subgroup of \( P/Q\) and (3.3) specializes to

\[
V_{B, k, k'}^N(I) = \bigoplus_{\lambda \in I} L_{g, k, k'}(\lambda + Q).
\]

Even when the shift \( N \) is even and \( M = 0 \), various nontrivial vertex superalgebras can be constructed, for example:

**Proposition 3.13** For any \( n \geq 1 \) and \( N \in \mathbb{Z} \),

1. \( V_{\text{so}_{4n+a}, k, k'}^{2+4n} \) is a vertex superalgebra if \( a = 0, 3 \), and a vertex algebra if \( a = 1 \), and an abelian intertwining algebra if \( a = 2 \);
2. \( V_{\text{so}_{4n+2}, k, k'}^{1+2n} \) is a vertex superalgebra if \( a = 2 \), and a vertex algebra if \( a = 0, 1, 3 \);
3. \( V_{E_7, k, k'}^{2+4n} \) is a vertex superalgebra;
4. \( V_{\text{sl}_{n}, k, k'}^{N}(mP + Q) \) is a vertex superalgebra if \( \exp (\frac{m^2N(n-1)}{n} \pi i) = -1 \) and a vertex algebra if \( \exp (\frac{m^2N(n-1)}{n} \pi i) = 1 \).

**Remark 3.14** (1), (2) solves some conjectures in [6, 9, 10] and (3) agrees with [7].

For \( r = 1 \) and \( M = 0 \), Tables 4 and 5 summarize all \((g, N, I)\) for which the vertex superalgebras can be constructed from Theorem 3.6.

Finally, we give examples of Theorem 3.6 with \( r \geq 2\):
Table 5  List for type EFG

| Type  | N       | I   | Super |
|-------|---------|-----|-------|
| $E_6$ | $6\mathbb{Z}$  | $P$ |       |
| $E_6$ | $2\mathbb{Z}$  | $Q$ |       |
| $E_7$ | $2 + 4\mathbb{Z}$ | $P$ | Yes  |
| $E_7$ | $4\mathbb{Z}$  | $P$ |       |
| $E_8$ | $2\mathbb{Z}$  | $P$ |       |
| $F_4$ | $2\mathbb{Z}$  | $P$ |       |
| $G_2$ | $2\mathbb{Z}$  | $P$ |       |

Proposition 3.15  For any $n, m \geq 1$,

1. For any $N_1, \ldots, N_{n-1} \in \mathbb{Z}$,

$$\bigoplus \bigotimes_{\lambda \in P/Q} n L_{s_{n_i}, k, k'}^{1+n \cdot N_i}(\lambda + Q)_{i=1}$$

is a vertex superalgebra if $n$ is even, and a vertex algebra if $n$ is odd.

2. For any $N_A, N_B \in \mathbb{Z}$,

$$\bigoplus n L_{s_{2n}, k, k'}^{n+2n \cdot N_A}(a \lambda_1 + Q) \otimes L_{so_{4m+3}, k, k'}^{1+2n \cdot N_B}(a \lambda_{2m+1} + Q)$$

is a vertex superalgebra if $n + m + N_A + N_B$ is odd, and a vertex algebra if $n + m + N_A + N_B$ is even.

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A Appendix

In Appendix, we will prove Conjecture 1 for the simple Lie algebras of type $B_n$ ($n \geq 1$). Here, we need to consider representations of quantum groups which are not type 1 (see Sect. 2.1 for type 1 representations). We will first review the non-type 1 representations of $U_q(\mathfrak{g})$.

For $\lambda \in P$, let $\mathbb{C} \chi_{\lambda}$ be a one-dimensional representation of $U_q(\mathfrak{g})$ defined by

$$E_\alpha \cdot \chi_{\lambda} = F_\alpha \cdot \chi_{\lambda} = 0, \quad K_\alpha \cdot \chi_{\lambda} = (-1)^{\langle (\alpha, \lambda) \rangle} \chi_{\lambda} \quad \text{for} \ \alpha \in \Pi.$$

Then, for a type 1 module $M$, $M \otimes \mathbb{C} \chi_{\lambda}$ and $\mathbb{C} \chi_{\lambda} \otimes M$ are $U_q(\mathfrak{g})$-modules, which are not of type 1. For example, for $L_q(\lambda) \otimes \mathbb{C} \chi_{\lambda}$, we have

$$K_\alpha \cdot v_\lambda \otimes \chi_{\lambda} = q^{\langle (\alpha, \lambda) \rangle} (-1)^{\langle (\alpha, \lambda) \rangle} v_\lambda \otimes \chi_{\lambda}.$$
for the highest weight vector \( v_{\lambda} \in L_q(\lambda) \).

In order to define a braided tensor category structure on a non-type 1 representation category, the following lemma is very important.

**Lemma A.1** For any type 1 module \( M \in U_q(\mathfrak{g})\)-mod, define a linear map \( h^\gamma_M : M \otimes \mathbb{C}\chi_\gamma \rightarrow \mathbb{C}\chi_\gamma \otimes M \) by

\[
h^\gamma_M(m_\lambda \otimes \chi_\gamma) = \exp(\pi i (\gamma, \lambda))\chi_\gamma \otimes m_\lambda
\]

for any \( \lambda \in P \) and \( m_\lambda \in M_\lambda \). Then, \( h^\gamma_M \) is a \( U_q(\mathfrak{g})\)-module homomorphism. In particular, the family of the maps \( \{h^\gamma_M\}_{M \in U_q(\mathfrak{g})\text{-mod}} \) is a natural transformation of \(- \otimes \mathbb{C}\chi_\gamma \) and \( \mathbb{C}\chi_\gamma \otimes -\).

**Proof** Let \( \lambda \in P \) and \( m_\lambda \in M_\lambda \) and \( \alpha \in \Pi \). Since

\[
E_\alpha \cdot (m_\lambda \otimes \chi_\gamma) = \Delta(E_\alpha) \cdot (m_\lambda \otimes \chi_\gamma) = (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha) \cdot (m_\lambda \otimes \chi_\gamma) = (E_\alpha \cdot m_\lambda) \otimes \chi_\gamma,
\]

and \( E_\alpha \cdot m_\lambda \in M_{\lambda+\alpha} \) we have \( h^\gamma_M(E_\alpha \cdot (m_\lambda \otimes \chi_\gamma)) = \exp(\pi i (\gamma, \lambda+\alpha))\chi_\gamma \otimes E_\alpha \cdot m_\lambda \).

Similarly, since \( E_\alpha \cdot (\chi_\gamma \otimes m_\lambda) = (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha) \cdot (\chi_\gamma \otimes m_\lambda) = (K_\alpha \cdot \chi_\gamma \otimes E_\alpha \cdot m_\lambda) \), and \( K_\alpha \cdot \chi_\gamma = (-1)^{\langle \gamma, \alpha \rangle} \chi_\gamma \), we have

\[
E_\alpha \cdot h^\gamma_M(m_\lambda \otimes \chi_\gamma) = \exp(\pi i (\gamma, \lambda))E_\alpha \cdot (\chi_\gamma \otimes m_\lambda) = \exp(\pi i (\gamma, \lambda+\alpha))\chi_\gamma \otimes E_\alpha \cdot m_\lambda.
\]

Hence, \( h^\gamma_M(E_\alpha \cdot (m_\lambda \otimes \chi_\gamma)) = E_\alpha \cdot h^\gamma_M(m_\lambda \otimes \chi_\gamma) \) for any \( \alpha \in \Pi \). It is easy to check this for \( F_\alpha \) and \( K_\alpha \) and thus \( h^\gamma_M \) is a \( U_q(\mathfrak{g})\)-module homomorphism. The naturality is obvious. \( \square \)

It is easy to show that only for \( \mathfrak{g} \) of type B one-dimensional representations satisfy the following important properties:

**Lemma A.2** The following conditions are equivalent:

1. \( \chi_{\lambda+\alpha} = \chi_\lambda \) for any \( \alpha \in Q \) and \( \lambda \in P \);
2. \( \langle \alpha, \beta \rangle \in 2\mathbb{Z} \) for any \( \alpha, \beta \in Q \);
3. The simple Lie algebra \( \mathfrak{g} \) is of type \( A_1 \) or of type \( B_n \) (\( n \geq 2 \)).

We will now proceed to the case of type B. According to [28], the root system of type \( B_n \) can be written as

\[
\{ \pm e_i \pm e_j, e_i \}_{1 \leq i, j \leq n},
\]

where \( \{e_i\}_{i=1,2,...,n} \) is the standard basis of \( \mathbb{R}^n \), and the simple roots and the fundamental weights as

\[
(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) = (e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n),
\]

\[
(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) = (e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_{n-1}, \frac{e_1 + e_2 + \cdots + e_n}{2}).
\]
The weight lattice is spanned by \( \{ e_i, \lambda_n \}_{i=1,2,...,n} \) and \( P/Q \cong \mathbb{Z}_2 \) is generated by \( \lambda_n \). Note that by the normalization, \( \langle (e_i, e_i) \rangle = 2 \) for any \( i = 1, \ldots, n \) (see Lemma A.2) and \( \langle (\lambda_n, \lambda_n) \rangle = \frac{n}{2} \). Let us denote the generator of \( U_q(\text{so}(2n+1)), E_{\alpha_i}, F_{\alpha_i}, K_{\alpha_i} \), by \( E_i, F_i, K_i \) for short. We remark that among \( \alpha_1, \alpha_2, \ldots, \alpha_n \), only \( \alpha_n \) is a short root.

**Proposition A.3** There exist Hopf algebra isomorphisms \( \phi : U_q(\text{so}(2n+1)) \to U_{-q}(\text{so}(2n+1)) \) such that:

\[
\phi(E_i) = E_i, \quad \phi(F_i) = F_i, \quad \phi(K_i) = K_i \quad \text{for } i = 1, \ldots, n-1
\]

and

\[
\phi(E_n) = -E_n, \quad \phi(F_n) = F_n, \quad \phi(K_n) = K_n.
\]

**Proof** The assertion follows from an easy computation. The point is that \( q_{\alpha_i} = \frac{q}{q^{\frac{1}{2}}_2} = q^2 \) for any \( i = 1, 2, \ldots, n-1 \) since they are long roots and \( \langle (\alpha, \beta) \rangle \in 2\mathbb{Z} \) for any \( \alpha, \beta \in Q \) (see Lemma A.2). In particular, \( q_{\alpha} = (-q)_{\alpha} \) for long roots. The only non-trivial relation is

\[
\sum_{r=0}^{1-a_{\alpha \beta}} (-1)^r \binom{1-a_{\alpha \beta}}{r} \frac{1}{q_{\alpha}} E_{\alpha}^r E_{\beta} E_{\alpha}^{1-a_{\alpha \beta} - r} = 0,
\]

for \( \alpha = \alpha_n \) and \( \beta = \alpha_{n-1} \), which follows from \( \binom{3}{1}_{-q} = 3_{-q} = (-1)^{3+1}3_q = \binom{3}{1}_q \).

\[\square\]

**Remark A.4** We note that the above proposition is also applicable to the case of \( A_1 = B_1 \), that is,

\[
\phi : U_q(\text{sl}_2) \to U_{-q}(\text{sl}_2), \quad \phi(E) = -E, \quad \phi(F) = F, \quad \phi(K) = K.
\]

It is noteworthy that the Hopf algebras \( U_q(\text{sl}_2) \) and \( U_{q'}(\text{sl}_2) \) are isomorphic if and only if \( q' = \pm q^\pm \) (see [35, Proposition 6 in Section 3]).

Let \( M \) be a type 1 \( U_{-q}(\text{so}(2n+1)) \)-module and \( \phi^* M \) an \( U_q(\text{so}(2n+1)) \)-module defined by

\[
a \cdot \phi m = \phi(a) \cdot m \quad \text{for any } a \in U_q(\text{so}(2n+1)) \text{ and } m \in M.
\]

Then, we have

\[
K_i \cdot \phi v = (-q)^{\langle (\alpha_i, \lambda) \rangle} v = (-1)^{\langle (\alpha_i, \lambda) \rangle} q^{\langle (\alpha_i, \lambda) \rangle} v \text{ for } v \in M_\lambda.
\]

Hence, \( \phi^* M \) is not necessarily a type 1 representation.
Based on this observation, we will define a type 2 module of $U_q(\mathfrak{so}_{2n+1})$. We first observe that by Lemma A.2 the one dimensional representation $\mathbb{C}\chi_\gamma$ is only depends on $\gamma \in \mathbb{P}/\mathbb{Q} = \mathbb{Z}/2\mathbb{Z}$. Denote $\chi_{\lambda_n}$ by $\chi$.

For each $\lambda \in \mathbb{P}^+$, let $L^I_q(\lambda)$ be the unique irreducible highest module defined by

$$K_i v_\lambda = (-q)^{\langle \alpha_i, \lambda \rangle} v_\lambda, \quad E_i v_\lambda = 0 \quad \text{for} \ i = 1, \ldots, n.$$  

We say a $U_q(\mathfrak{so}_{2n+1})$-module is of type 2 if it decomposes into a direct sum of $L^I_q(\lambda)$’s for $\lambda \in \mathbb{P}^+$. Denote the category of type 2 (resp. of type 1) $U_q(\mathfrak{so}_{2n+1})$-modules by $C^I$ (resp. $C^I$).

Let $M, N \in C^I$. Since $U_q(\mathfrak{so}_{2n+1})$ is a Hopf algebra, $M \otimes N$ is a $U_q(\mathfrak{so}_{2n+1})$-module and it is easy to show that $M \otimes N \in C^I$. Thus, $C^I$ is naturally a monoidal category.

Let $\rho \in \mathbb{C}$ satisfy $\exp(\pi i \rho) = q$ and denote the braided tensor category $(U_q(\mathfrak{so}_{2n+1}), R(\rho))\text{-mod}$ by $C^I(\rho)$. In this section, we will prove Conjecture 1 for type B in three steps:

1. To give a braided tensor category structure on $C^I$, which depends on the choice of $\rho$. We denote it by $C^I(\rho + 1)$;
2. To show that a Hopf algebra isomorphism $\phi : U_q(\mathfrak{so}_{2n+1}) \to U_{-q}(\mathfrak{so}_{2n+1})$ induces an equivalence of braided tensor categories between $C^I(\rho + 1)$ and $(U_{-q}(\mathfrak{so}_{2n+1}), R(\rho + 1))\text{-mod}$;
3. To construct a functor $F : C^I(\rho)\mathbb{Q}_{\mathfrak{so}_{2n+1}}$ and $C^I(\rho + 1)$ which gives an equivalence of braided tensor categories.

We will first consider Step (1). For any type 2 module $M^I$, set for all $\lambda \in \mathbb{P}$

$$M^I_\lambda = \{ m \in M^I \mid K_i m = (-q)^{\langle \lambda, \alpha_i \rangle} m \text{ for } i = 1, \ldots, n \}.$$  

Then, we have

$$M^I = \bigoplus_{\lambda \in \mathbb{P}} M^I_\lambda.$$  

In order to define the R-matrix for type 1 representations, we consider a linear map $f_\rho$ (see Sect. 2.2). Define for all type 2 $U_q(\mathfrak{so}_{2n+1})$-modules $M^I$ and $N^I$ a bijective linear map $f^I_\rho : M^I \otimes N^I \to M^I \otimes N^I$ by

$$f^I_\rho(m \otimes n) = \exp(-\pi i (\rho + 1)\langle \lambda, \mu \rangle) m \otimes n \text{ for any } m \in M^I_\lambda \text{ and } n \in N^I_\mu$$  

and for all $\mu, \lambda \in \mathbb{P}$. Then, a statement similar to Lemma 2.12 holds for type 2 modules by replacing $f_\rho$ with $f^I_\rho$.

**Lemma A.5** Let $u \in U_q(\mathfrak{so}_{2n+1})_\mu^-$ and $u' \in U_q(\mathfrak{so}_{2n+1})_\mu^+$ for $\mu \in \mathbb{Q}$ with $\mu \geq 0$. For any type 2 $U_q(\mathfrak{so}_{2n+1})$-modules $M^I$ and $N^I$,

$$(f^I_\rho)^{-1} \circ (u \otimes u') \circ f^I_\rho = u K_\mu \otimes K_{-\mu} u'$$
as linear maps acting on \( M^{II} \otimes N^{II} \).

Let us define a linear map \( R(\rho)^{II} \) by

\[
R(\rho)^{II} = (\Theta \circ f_{\rho}^{II})^{-1} : M^{II} \otimes N^{II} \to M^{II} \otimes N^{II}.
\]

Then, by the above lemma, \( R(\rho)^{II} \) satisfies the axiom of an R matrix (R1-R3 in Sect. 2.2) as an operator on \( C^{II} \) (see for example [31, Section 3]). Denote by \( C^{II}(\rho + 1) \) the braided tensor category defined by \( R(\rho)^{II} \).

Then, the following lemma follows from a similar argument in Sect. 2.3:

\textbf{Lemma A.6} The Hopf algebra isomorphism \( \phi : U_q(\text{so}(2n+1)) \to U_q(\text{so}(2n+1)) \) induces an equivalence between \( C^{II}(\rho + 1) \) and \( U_q(\text{so}(2n+1)) \)-mod as braided tensor categories.

This completes Step (1) and Step (2). Finally, we will show the last step. For \( S = I, II \) and \( i \in P/Q = \mathbb{Z}/2\mathbb{Z} \), let \( C^S \) be a full subcategory of \( C^S \) consisting of modules which is isomorphic to a direct sum of \( L^S_q(\lambda) \)'s for \( \lambda \in \mathbb{Z}/2\mathbb{Z} + P \). This grading coincides with the \( P/Q \)-grading introduced in Sect. 2.4. We can define a (grading preserving) functor \( F : C^I \to C^{II} \) by \( F(M) = (\mathbb{C} \chi^{0} \otimes M_0) \oplus (\mathbb{C} \chi \otimes M_1) \) for any \( M = M_0 \oplus M_1 \in C^I = C^0 \oplus C^1 \), where \( \mathbb{C} \chi^{0} \) is the trivial representation. Then, \( F \) gives an equivalence of \( \mathbb{Z}/2\mathbb{Z} \)-graded abelian categories. For any \( M \in C^I \), define a linear map \( h_M : M \otimes \mathbb{C} \chi \to \mathbb{C} \chi \otimes M \) by

\[
h_M(m_{\lambda} \otimes \chi) = \exp(\pi i(\lambda, \lambda)) \chi \otimes m_{\lambda}
\]

for any \( \lambda \in P \) and \( m_{\lambda} \in M_{\lambda} \). Then, \( h^\lambda \) is a natural transformation by Lemma A.1.

Note that \( C^{II} \) is a strict monoidal category since it is a full subcategory of the category of all \( U_q(\text{so}_{2n+1}) \)-modules, which is clearly strict. Hence, we can identify \( \chi^{0} \otimes M = M \) for any \( M \in C^{II} \.

Define a \( U_q(\text{so}_{2n+1}) \)-module isomorphism \( \epsilon_2 : \mathbb{C} \chi \otimes \mathbb{C} \chi \to \mathbb{C} \) by \( \epsilon_2(\chi \otimes \chi) = 1 \).

Let \( M_i \in C^I \) and \( N_j \in C^I \) for \( i, j = 0, 1 \). Define a natural transformation \( g_{M_i, N_j} : F(M_i) \otimes F(N_j) \to F(M_i \otimes N_j) \) by

\[
g_{M_0, N_0} : (\chi^{0} \otimes M_0) \otimes (\chi^{0} \otimes N_0) \to \chi^{0} \otimes M_0 \otimes N_0,
\]

\[
g_{M_1, N_0} : (\chi^{1} \otimes M_1) \otimes (\chi^{0} \otimes N_0) \to \chi^{1} \otimes M_1 \otimes N_0,
\]

\[
g_{M_0, N_1} : (\chi^{0} \otimes M_0) \otimes (\chi^{1} \otimes N_1) \xrightarrow{id_{\chi^{0}} \otimes h_{M_0} \otimes id_{N_1}} \chi^{1} \otimes M_0 \otimes N_1,
\]

\[
g_{M_1, N_1} : (\chi^{1} \otimes M_1) \otimes (\chi^{1} \otimes N_1) \xrightarrow{id_{\chi^{1}} \otimes h_{M_1} \otimes id_{N_1}} \chi^{2} \otimes M_1 \otimes N_1 \xrightarrow{\epsilon_2 \circ id_{M_1} \otimes N_1} \chi^{0} \otimes M_1 \otimes N_1.
\]

Then, we have:

\textbf{Proposition A.7} The functor \( F : C^I \to C^{II} \) together with the natural transformation \( g_{M, N} : F(M) \otimes F(N) \to F(M \otimes N) \) and \( \epsilon : 1 = F(1) \) give an equivalence of braided tensor categories from \( (C^I)^{Q_{so_{2n+1}}} \) to \( C^{II} \).
Before giving the proof, we remark that the value \( \exp(\pi i p (\langle \lambda_n, \lambda_n \rangle)) = \exp(\frac{\pi i p n}{2}) \) is not well-defined for \( p \in \mathbb{Z}/2\mathbb{Z} \), which is the source of the 3-cocycle \( \alpha : (\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{C}^\times \). In fact, let \( \iota : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \) be a map defined by sending \( \{0, 1\} \mapsto \{0, 1\} \). Then, we have:

**Lemma A.8** The explicit form of the abelian cocycle \( (\alpha_n, c_n) \in \mathbb{Z}_a^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) \) such that \( c(a, a) = Q_{so_{2n+1}}(a) \) for \( a \in \mathbb{Z}/2\mathbb{Z} = P/Q \) can be given by

\[
\alpha_n(a, b, c) = \begin{cases} 
(-1)^n & (a = b = c = 1) \\
1 & \text{otherwise} 
\end{cases},
\]

\[
c_n(a, b) = \begin{cases} 
i^n & (a = b = 1) \\
1 & \text{otherwise} 
\end{cases}.
\]

**Proof of Proposition A.7** We will verify the conditions (LM1) and (LM2) in Sect. 1.2. Since both \( C^I \) and \( C^{II} \) are strict monoidal categories, the associative isomorphisms are trivial before twisting. Let \( p_i \in \mathbb{Z}/2\mathbb{Z} \) and \( M_i \in C^I_{p_i}, \beta_i \in P \), and \( v_i \in (M_i)_{\beta_i} \) for \( i = 1, 2, 3 \). Then, we have

\[
g_{M_1 \otimes M_2, M_3} \circ (g_{M_1, M_2} \otimes \text{id}_{M_3}) \left( (\chi^{p_1} \otimes v_1 \otimes \chi^{p_2} \otimes v_2) \otimes \chi^{p_3} \otimes v_3 \right) \\
= g_{M_1, M_2, M_3}(\exp(p_2 \pi i (\langle \lambda_n, \beta_1 \rangle))(\chi^{p_1+p_2} \otimes v_1 \otimes \chi^{p_2} \otimes v_2) \otimes \chi^{p_3} \otimes v_3) \\
= \exp(\pi i (p_2 \langle \lambda_n, \beta_1 \rangle) + p_3 \langle \langle \lambda_n, \beta_1 + \beta_2 \rangle))(\chi^{p_1+p_2+p_3} \\
\otimes (v_1 \otimes v_2) \otimes v_3)
\]

and

\[
g_{M_1, M_2 \otimes M_3} \circ (\text{id}_{M_1} \otimes g_{M_2, M_3}) \left( (\chi^{p_1} \otimes v_1 \otimes (\chi^{p_2} \otimes v_2 \otimes \chi^{p_3} \otimes v_3) \right) \\
= g_{M_1, M_2 \otimes M_3}(\exp(\pi i p_3 (\langle \lambda_n, \beta_2 \rangle))\chi^{p_1} \otimes v_1 \otimes (\chi^{p_2+p_3} \otimes v_2 \otimes v_3) \\
= \exp(\pi i (p_3 \langle \langle \lambda_n, \beta_2 \rangle \rangle + \iota(p_2, p_3, \langle \langle \lambda_n, \beta_1 \rangle \langle \lambda_n, \beta_1 \rangle))\chi^{p_1+p_2+p_3} \otimes v_1 \\
\otimes (v_2 \otimes v_3).
\]

Thus, in order to verify (LM1), it suffices to show that

\[
\alpha(p_1, p_2, p_3) \exp(\pi i (p_2 \langle \lambda_n, \beta_1 \rangle) + p_3 \langle \langle \lambda_n, \beta_1 + \beta_2 \rangle \rangle) \\
= \exp(\pi i (p_3 \langle \langle \lambda_n, \beta_2 \rangle \rangle + \iota(p_2, p_3, \langle \langle \lambda_n, \beta_1 \rangle \rangle)),
\]

which follows from Lemma A.8. (LM2) is obvious. Hence, the assertion holds. \( \square \)

Finally, we will prove that \( F : C^I(\rho)^{Q_{so_{2n+1}}} \to C^{II}(\rho + 1) \) is a braided monoidal functor. Let \( p_i \in \mathbb{Z}/2\mathbb{Z} \) and \( M_i \in C^I_{p_i}, \beta_i \in P \), and \( m_i \in (M_i)_{\beta_i} \) for \( i = 1, 2 \). It suffices to show that the following diagram commutes:
where \( c_n(p_1, p_2) \) is given in Lemma A.8. Recall \( \Theta = \sum_{\mu \geq 0} \Theta_\mu \) and \( \Theta_\mu = \sum_{i=0}^{r(\mu)} v_i^\mu \otimes u_i^\mu \in U_\mu^{-} \otimes U_\mu^{+} \) (see Sect. 2.2). Then, we have:

\[
g_{M_1, M_2} \circ (B_{F(F(M_1), F(M_2))}^{II})^{-1}(\chi^{p_2} \otimes m_2) \otimes (\chi^{p_1} \otimes m_1) = g_{M_1, M_2} \circ \Theta \circ f_\rho^{II} \circ P_21(\chi^{p_2} \otimes v_2) \otimes (\chi^{p_1} \otimes v_1) = \exp(-\pi i (\rho + 1) \langle \langle \beta_1, \beta_2 \rangle \rangle) \sum_{\mu \geq 0} \sum_{i=0}^{r(\mu)} g_{M_1, M_2}(v_i^\mu \cdot (\chi^{p_1} \otimes m_1)) \otimes (u_i^\mu \cdot (\chi^{p_2} \otimes m_2)).
\]

Since \( \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \) and \( \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \), we have

\[
(v_i^\mu \cdot (\chi^{p_1} \otimes m_1)) \otimes (u_i^\mu \cdot (\chi^{p_2} \otimes m_2)) = (\chi^{p_1} \otimes v_i^\mu \cdot m_1) \otimes (K_\mu \cdot \chi^{p_2} \otimes u_i^\mu \cdot m_2) = (-1)^{p_2(\langle \lambda_\mu, \beta_1 \rangle)}(\chi^{p_1} \otimes v_i^\mu \cdot m_1) \otimes (\chi^{p_2} \otimes u_i^\mu \cdot m_2).
\]

Hence, we have:

\[
g_{M_1, M_2} \circ (B_{F(F(M_1), F(M_2))}^{II})^{-1}(\chi^{p_2} \otimes m_2) \otimes (\chi^{p_1} \otimes m_1) = \exp(\pi i (\rho + 1) \langle \langle \beta_1, \beta_2 \rangle \rangle) \sum_{\mu \geq 0} \sum_{i=0}^{r(\mu)} \left( \exp(\pi i (p_2) \langle \langle \lambda_\mu, \beta_1 + \mu \rangle \rangle) \times (-1)^{p_2(\langle \lambda_\mu, \beta_1 \rangle)}(\chi^{p_1} \otimes v_i^\mu \cdot m_1) \otimes (\chi^{p_2} \otimes u_i^\mu \cdot m_2) \right) = \exp(\pi i (\rho + 1) \langle \langle \beta_1, \beta_2 \rangle \rangle) \exp(\pi i (p_2) \langle \langle \lambda_\mu, \beta_1 \rangle \rangle) \sum_{\mu \geq 0} \sum_{i=0}^{r(\mu)} (\chi^{p_1+p_2} \otimes v_i^\mu \cdot m_1 \otimes u_i^\mu \cdot m_2).
\]

Similarly, we have

\[
c_n(p_1, p_2) F(B_{M_1, M_2}^{I})^{-1} \circ g_{M_1, M_2}(\chi^{p_2} \otimes m_2) \otimes (\chi^{p_1} \otimes m_1) = c_n(p_1, p_2) \exp(\pi i (p_1) \langle \langle \lambda_\mu, \beta_2 \rangle \rangle) F(B_{M_1, M_2}^{I})^{-1}(\chi^{p_1+p_2} \otimes m_2 \otimes m_1) = c_n(p_1, p_2) \exp(\pi i (p_1) \langle \langle \lambda_\mu, \beta_2 \rangle \rangle) \chi^{p_1+p_2} \otimes (\Theta \circ f_\rho \circ P_21(m_2 \otimes m_1))
\]
Thus, the proof of the conjecture comes down to the following lemma:

**Lemma A.9** If \((M_i)_{\beta_i} \neq 0\) for \(i = 1, 2\), then
\[
c_n(p_1, p_2) = \exp\left(\pi i \left(\langle \beta_1, \beta_2 \rangle + \iota(p_2) \langle \lambda_n, \beta_1 \rangle - \iota(p_1) \langle \lambda_n, \beta_2 \rangle \right)\).
\]

**Proof** Let \(k : P \times P \to \mathbb{C}^\times\) be a map defined by \(k(\beta_1, \beta_2) = \exp(\pi i (\langle \beta_1, \beta_2 \rangle + \iota(\beta_2)(\langle \lambda_n, \beta_1 \rangle) - \iota(\beta_1)(\langle \lambda_n, \beta_2 \rangle)), \) where \(\iota : P \to P/Q = \{0, 1\}\) is defined by the composition of the projection and the identification.

We claim that \(k(\beta_1 + \alpha, \beta_2) = k(\beta_1, \beta_2 + \alpha) = k(\beta_1, \beta_2)\) for any \(\alpha \in Q\). The difference \(k(\beta_1 + \alpha, \beta_2)k(\beta_1, \beta_2)^{-1}\) is equal to \(\exp(\pi i (\langle \beta_2, \alpha \rangle) + \iota(\beta_2)(\langle \lambda_n, \alpha \rangle))\). Thus, if \(\beta_2 \in Q\) i.e., \(\iota(\beta_2) = 0\), then \(k(\beta_1 + \alpha, \beta_2)k(\beta_1, \beta_2)^{-1}\) is equal to 1 by Lemma A.2. Similarly, if \(\iota(\beta_2) = 1\), then \(k(\beta_1 + \alpha, \beta_2)k(\beta_1, \beta_2)^{-1} = \exp(\pi i (\langle \beta_2, \alpha \rangle + \langle \lambda_n, \alpha \rangle)) = \exp(\pi i (\langle \lambda_n, \alpha \rangle + \langle \lambda_n, \alpha \rangle)) = 1\), thus the claim is proved.

Since \(k(0, 0) = k(\lambda_n, 0) = k(0, \lambda_n) = 1\) and \(k(\lambda_n, \lambda_n) = i^n\), the assertion follows from Lemma A.8. \(\square\)

Hence, we have:

**Theorem A.10** The composition of \(F\) and \(\phi^e\) gives a braided monoidal equivalence between \((U_q(\mathfrak{so}_{2n+1}), R(\rho))\)-mod\(^Q_{\omega_{2n+1}}\) and \((U_{-q}(\mathfrak{so}_{2n+1}), R(\rho))\)-mod for any \(n \geq 1\).

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