Degrees of Freedom of the MIMO $2 \times 2$ Interference Network with General Message Sets

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Abstract

We establish the degrees of freedom (DoF) region for the multiple-input multiple-output (MIMO) two-transmitter, two-receiver ($2 \times 2$) interference network with a general message set, consisting of nine messages, one for each pair of a subset of transmitters at which that message is known and a subset of receivers where that message is desired. An outer bound on the general nine-message $2 \times 2$ interference network is obtained and then it is shown to be tight, establishing the DoF region for the most general antenna setting wherein all four nodes have an arbitrary number of antennas each. The DoF-optimal scheme is applicable to the MIMO $2 \times 2$ interference network with constant channel coefficients, and hence, a fortiori, to time/frequency varying channel scenarios.

In particular, a linear precoding scheme is proposed that can achieve all the DoF tuples in the DoF region. In it, the precise roles played by transmit zero-forcing, interference alignment, random beamforming, symbol extensions and asymmetric complex signaling (ACS) are delineated. For instance, we identify a class of antenna settings in which ACS is required to achieve the fractional-valued corner points.

Evidently, the DoF regions of all previously unknown cases of the $2 \times 2$ interference network with a subset of the nine-messages are established as special cases of the general result of this paper. In particular, the DoF region of the well-known four-message (and even three-message) MIMO $X$ channel is established. This problem had remained open despite previous studies which had found inner and outer bounds that were not tight in general. Hence, the DoF regions of all special cases obtained from the general DoF region of the nine-message $2 \times 2$ interference network of this work that include at least three of the four $X$ channel messages are new, among many others. Our work sheds light on how the same physical $2 \times 2$ interference network could be used by a suitable choice of message sets to take most advantage of the channel resource in a flexible and efficient manner.

Index Terms

Beamforming, degrees of freedom, interference network, MIMO, general message sets, interference alignment, asymmetric complex signaling.

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I. INTRODUCTION

In order to design communication systems that can flexibly and efficiently handle the complex signaling requirements of modern applications, such as in the delivery phase of caching systems over wireless interference channels, it may be necessary to offer multiple physical layer modes that allow for the transmission of some or all of multiple unicast, multiple multicast, multiple broadcast (i.e., $X$-channel), and/or cooperative/cognitive/common messages. In this paper, rather than considering each such transmission mode in isolation, we study the unified setting in which any subset (including all) such messages can be transmitted simultaneously over the MIMO $2 \times 2$ interference network. For this simple network, depending on the subset of the two transmitters at which a message is known, and the subset of the two receivers where it is desired, there are nine possible messages in the general message set. For this fully general message set, the associated nine-dimensional DoF region of the MIMO $2 \times 2$ interference network is established herein.

The most studied and also the best understood setting of the $2 \times 2$ interference network is the two-unicast setting, referred to in the literature as the interference channel [2], in which each transmitter has a private message for its single distinct intended receiver (cf. [3], [4], [5] and the references therein). In particular, the DoF region of the two-user MIMO interference channel was found in [3] and more refined characterizations in terms of generalized degrees of freedom and constant bit-gap to capacity were found in [4] and [5], respectively.

The four private message case, which can be thought of as a two-broadcast network, more commonly known as the $X$ channel, allows for the transmission of a private message to each of the two receivers from each transmitter. The now well-known, and more broadly applicable, linear precoding technique known as interference alignment is needed to achieve the DoF in some cases. With its use, the MIMO $X$ channel was shown in [6], [7] to achieve higher sum DoF than the MIMO interference channel. For example, when all transmitters and receivers are equipped with the same number, $M$, of antennas, the two-user MIMO interference channel has a sum DoF of $M$, while the MIMO $X$ channel can achieve a sum DoF of $\frac{4}{3}M$ for $M > 1$, achievable with interference alignment. The key idea (when $M$ is a multiple of 3) is that by aligning undesired signals (i.e., interference) from the two transmitters into the same subspace at a receiver, one can maximize the desired signal dimensions at that receiver. In [7], an outer bound on the DoF region of the MIMO $X$ channel is given based on the sum rate outer bound of the embedded MAC, BC and $Z$ channels in the $X$ channel. Moreover, [7] gives an achievability scheme based on interference alignment and presents an achievable DoF region that is given as the convex hull of all integer-valued degrees of freedom within that outer bound region. But these inner and the outer bounds of [7] are not identical. However, using interference alignment over multi-letter extensions of the MIMO $X$ channel, it was shown that the outer bound is tight (including non-integer corner points) when all nodes have equal number of antennas $M$, when $M > 1$. In the context of the general MIMO $X$ channel with an arbitrary numbers of antennas at the four terminals, [7] claims that the DoF outer bound region obtained therein is tight in “most cases”, but a precise statement and proof of this claim is not provided. Later, the authors of [8] introduced a novel technique named asymmetric complex signaling (ACS). By allowing the inputs to be complex but not circularly symmetric and using an alternative representation
of the channel models in terms of only real quantities, the problem is transformed to delivering real messages over channels with real-valued coefficients. Consequently, it was shown that the 2-user single-input, single-output (SISO) X channel with constant channel coefficients achieves the outer bound of $\frac{4}{3}$ DoF. However, it remained an open problem as to whether the outer bound of [7] is tight for any of the multiple antenna cases. For instance, the problem remained open as to whether there are other scenarios in which ACS is required in addition to multi-letter extensions, as did the problem of identifying cases in which just multi-letter extensions suffice to achieve the outer bound. More recently, it was shown in [9] that the outer bound on the sum DoF for the MIMO X channel (with generic channel coefficients) derived in [7] is tight for any antenna configuration. The work in [9] proposes a linear precoding method based on the generalized singular value decomposition (GSVD), and with the aid of computational experiments, the authors of [9] offer a conjecture that the outer bound region obtained in [7] is also tight. The general DoF region result of this paper for the MIMO $2 \times 2$ interference network with nine distinct messages, when specialized to the four private-message MIMO X channel, settles this conjecture in the affirmative. It therefore also expands on, and makes precise, the claim in [7]. The outer bound on the DoF region of [7] is indeed tight.

Besides the aforementioned MIMO interference and X channels (and its embedded MAC and/or BC), in which only private messages are considered (see also [4], [5], [10]), the $2 \times 2$ interference network can work in various other modes if common messages, multicast messages and/or transmitter cognition are allowed. For example, if both transmitters share the same three messages, and each receiver demands one of the first two messages while both demand the third, we have what is known as the broadcast channel with private and common messages (BC-CM) [11], [12]. On the other hand, if each transmitter has a private message, and both receivers demand both of the messages, the system works as a compound multiple access channel (C-MAC) [13]. If there are two private messages as in the interference channel and there is a common message known by both transmitters and also demanded by both receivers, the network is known as the interference channel with common message (IC-CM) [14], [15]. The network is referred to as a cognitive X channel in [7] if there are four independent messages to be sent as in the X channel, but with one of the four messages known at both transmitters. A new three-message setting could be defined in which one transmitter has 2 messages, each intended for a distinct receiver, and a third shared message that is known to both transmitters and desired at one of the receivers. Interpreting the second transmitter as a relay, such a setting could be described as a broadcast channel with a partially cognitive relay (BC-PCR). A six-message cognitive X channel could be defined as having the four private messages as in the X channel as well as two more messages that are known to both transmitters with each desired at a distinct receiver. Evidently, based on different message sets, the $2 \times 2$ interference network can represent many different settings and potential applications.

**Notation:** $\text{co}(A)$ is the convex hull of set A, $\mathbb{R}^+_n$ and $\mathbb{Z}^+_n$ denote the set of non-negative $n$-tuples of real numbers and integers, respectively. $(x)^+$ represents the larger of the two numbers, $x$ and 0. $A \otimes B$ denotes the Kronecker product of matrix $A$ and $B$. $[A \ B]$ means the horizontal concatenation of matrix $A$ and $B$, and $[A; B]$ means the vertical concatenation of matrix $A$ and $B$. $\text{Re}(A)$ and $\text{Im}(A)$ denote the real part and imaginary part of complex matrix $A$, respectively. $\mathcal{N}(A)$ denotes the null space of the linear transformation $A$. $\text{Span}(V)$ denotes the subspace.
spanned by the column vectors of matrix $V$.

II. SYSTEM MODEL

We consider the complex Gaussian network with two transmitters and two receivers, as it is shown in Figure 1. The two transmitters are equipped with $M_1, M_2$ antennas respectively, and the two receivers are equipped with $N_1, N_2$ antennas respectively. We denote the channel between transmitter $t$ and receiver $r$ as the $N_r \times M_t$ complex matrix $H_{rt}$ and assume all channels to be generic, i.e., all the channel coefficient values are drawn independently from a continuous probability distribution. The channel is assumed to be constant over the duration of communication and all channel coefficients are perfectly known at all transmitters and receivers. The received signal at receiver $r$ is given by

$$Y_r = H_{r1}X_1 + H_{r2}X_2 + Z_r,$$

where $X_t (t = 1, 2)$ is the $M_t \times 1$ input vector at transmitter $t$, $Z_r$ is the $N_r \times 1$ additive white Gaussian noise (AWGN) vector at receiver $r$.

General message sets are considered in this paper. For $2 \times 2$ interference network, there are at most nine possible messages classified by different sources and destinations. We index them as $W_{11}, W_{12}, W_{21}, W_{22}, W_{01}, W_{02}, W_1, W_0$, as shown in Figure 1 $W_{rt} (r, t = 1, 2)$ is a private message sent from transmitter $t$ to receiver $r$; $W_0r (r = 1, 2)$ is a common message transmitted cooperatively from both transmitters to receiver $r$; $W_t (t = 1, 2)$ is a multicast message transmitted from transmitter $t$ and demanded by both receivers simultaneously; $W_0$ is a common multicast message transmitted cooperatively from both transmitters and demanded by both receivers.

Assume the total power across all transmitters to be equal to $\rho$ and indicate the message set size by $|W(\rho)|$. For codewords occupying $t_0$ channel uses, the rates $R(\rho) = \frac{\log|W(\rho)|}{t_0}$ are achievable if the probability of error for all nine messages can simultaneously be made arbitrarily small by choosing appropriately large $t_0$. The capacity region $C(\rho)$ of the MIMO $2 \times 2$ interference network with general message sets is the set of all achievable rate-tuples $R(\rho)=\langle R_{11}(\rho), R_{12}(\rho), ..., R_0(\rho) \rangle$. Define the degrees of freedom region $\mathbb{D}$ for MIMO $2 \times 2$ interference network with general message sets as

$$\mathbb{D} \triangleq \left\{ (d_{11}, d_{12}, ..., d_0) \in \mathbb{R}_+^E : \forall (\omega_{11}, \omega_{12}, ..., \omega_0) \in \mathbb{R}_+^E \right\}$$

$$\sum_{x \in E} \omega_x d_x \leq \limsup_{\rho \to \infty} \left[ \sup_{R(\rho) \in C(\rho)} \frac{\sum_{x \in E} \omega_x R_x(\rho)}{\log(\rho)} \right].$$
where \( E = \{11, 12, 21, 22, 01, 02, 1, 2, 0\} \).

This definition is the general message set counterpart of the one provided in [7] for the MIMO X channel. Note that \( \mathbb{D} \) is a closed convex set.

In the following section, we consider first the previously studied MIMO X channel, for which the best inner and outer bounds of [7], [8], [9] known to date are not coincident in general. The MIMO X channel provides the context in which to introduce the notation used in this paper and all the relevant linear precoding techniques, namely, zero-forcing, interference alignment, symbol extension, and ACS. We provide a class of antenna configurations for which, among linear schemes, ACS is required and is sufficient, along with multi-letter extensions and the other linear precoding techniques, to achieve all fractional DoF corner points for those antenna configurations. More generally, we show that the use of linear precoding techniques including symbol extensions and ACS, whether ACS is required or not, are sufficient to achieve any corner point of the DoF region regardless of the antenna configuration. The DoF region of the general nine-message problem is established in [V].

III. THE MIMO X CHANNEL

The MIMO X channel is an important special case of the 2 \( \times \) 2 interference network in which only the four private messages, namely, \( W_{11} \), \( W_{12} \), \( W_{21} \), \( W_{22} \), are present. Hence the message index set in this case is \( E = \{11, 12, 21, 22\} \). Each of these four messages is intended for one of the two receivers and is a source of interference to the other receiver.

We start by stating the DoF region of the MIMO X channel.

**Theorem 1.** The DoF region of the MIMO X channel with constant generic channel coefficients is (with probability one)

\[
\mathbb{D}_X = \{(d_{11}, d_{21}, d_{12}, d_{22}) \in \mathbb{R}_+^4 : d_{11} + d_{12} + d_{21} \leq \max(M_1, N_1),
\]

\[
d_{11} + d_{12} + d_{22} \leq \max(M_2, N_1),
\]

\[
d_{21} + d_{22} + d_{11} \leq \max(M_1, N_2),
\]

\[
d_{21} + d_{22} + d_{12} \leq \max(M_2, N_2),
\]

\[
d_{11} + d_{12} \leq N_1, \ d_{21} + d_{22} \leq N_2,
\]

\[
d_{11} + d_{21} \leq M_1, \ d_{12} + d_{22} \leq M_2\}.
\]

That the above DoF region is an outer bound for the DoF region of the MIMO X channel is proved in Theorem 2 of [7]. The outer bounding inequalities result, respectively, from the embedded multiple-access channel, broadcast channel and \( Z \) channels, in the MIMO X channel. The readers can refer to [7] for details. Moreover, these outer bounds are generalized to the general nine-message problem in Section [V].

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The authors of [7] also provide a constructive achievability proof to show that the convex hull of all the integer-valued DoF-tuples in $\mathbb{D}_X$ is achievable. The techniques used in the achievable scheme are zero-forcing, interference alignment and random beamforming. Since these techniques are among the techniques used in our $2 \times 2$ interference network with general message sets problem, we provide a succinct account of them in Section III-A, describing in the process, the notation used in this paper as well. The techniques of symbol extension and ACS are described in Sections III-B and III-C to follow.

A. Zero-forcing and interference alignment

Consider message $W_{11}$ as an example. If $M_1 > N_2$, the null space of channel $H_{21}$ is not empty. By transmitting some symbols of message $W_{11}$ using the beamformers chosen from the null space $\mathcal{N}(H_{21})$, we can zero-force these symbols at receiver $R_2$ and thus introduce no interference to it. The maximum number of such symbols that can be zero-forced is $(M_1 - N_2)^+$, which is equal to the rank of $\mathcal{N}(H_{21})$. Similarly, we can transmit, at most, $(M_2 - N_2)^+$ symbols of message $W_{12}$ via the nullspace of channel $H_{22}$ and zero-force them all at their unintended receiver, $R_1$. Note that the null space $\mathcal{N}(H_{21})$ and $\mathcal{N}(H_{22})$ are both subspaces of the null space of the concatenated channel $[H_{21} \; H_{22}]$. The remaining dimension of $\mathcal{N}([H_{21} \; H_{22}])$ is equal to $A = (M_1 + M_2 - N_2)^+ - (M_1 - N_2)^+ - (M_2 - N_2)^+$. By choosing beamformers for message $W_{11}$ and $W_{12}$ jointly from the rest of the subspace of $\mathcal{N}([H_{21} \; H_{22}])$, it is possible to align this part of message $W_{11}$ and $W_{12}$ into the same subspace and thus reserve more dimensions for the desired messages at receiver $R_2$, and the maximum number of such pairs of streams is equal to $A$. If there are more symbols of message $W_{11}$ left, they can be transmitted using random beamforming, which would create unavoidable interference at its unintended receiver. Since the technique of zero-forcing is the more efficient in terms of reducing interference than interference alignment, it is given the highest priority when constructing precoding beamformers. Following that, interference alignment is used to the extent possible, and following which all of the remaining symbols are sent using random beamforming. The beamformers for each private message is hence divided into three linearly independent parts based on the precoding technique used. Here we use superscript 'Z' to indicate a message is zero-forced at its unintended receiver, 'A' to indicate a message is aligned with another interference at their commonly unintended receiver, and 'R' to indicate the remainder of a certain message that is transmitted using random beamforming. Hence a message $W_x$ for $x \in E$ (recall $E = \{11, 12, 21, 22\}$ for the MIMO $X$ channel) is split in general into three components or sub-messages, denoted $W_Z^x$, $W_A^x$ and $W_R^x$, with the number of symbols (dimensions) in each denoted as $d_Z^x$, $d_A^x$ and $d_R^x$, respectively. In general, we use the notation $W_y^x$ and $d_y^x$ with $x \in E$ and $y \in \{Z, A, R\}$ for the component messages and dimensions, respectively. Similarly, the precoding matrix for any sub-message $W_y^x$ is denoted as $V_y^x$. Thus we have that $d_{ij} = d_Z^x + d_A^x + d_R^x$ and let $V_{ij}$ denote the horizontally concatenated matrix $V_{ij} = [V_{ij}^Z \; V_{ij}^A \; V_{ij}^R]$, where $i, j = 1, 2$. It was shown in [7] that any integer-valued DoF-tuple within the outer bound can be divided into three such parts within the decoding ability of the channels. It is thus achievable.
B. Symbol extensions

When a corner point of $\mathbb{D}_X$ is not integer-valued, it is rational-valued. It is therefore natural to consider a multi-letter extension of the channels to obtain a larger but equivalent system with the corresponding corner point of the DoF region being integer-valued. The length of symbol extensions can be chosen to be the least common multiple of the denominators of all the fractional values. To this time-extended channel, the techniques of zero-forcing, alignment and random beamforming can be applied as described in the previous section. This was proposed in [7].

Consider $T$ symbol extensions of the $X$ channel with complex and constant (across time) channel coefficients. We have the equivalent $\tilde{N}_i \times \tilde{M}_j$ channel matrix $\tilde{H}_{ij}$, in which $i,j = 1, 2$, $\tilde{M}_i = T \cdot M_i$, $\tilde{N}_i = T \cdot N_i$, and

$$\tilde{H}_{ij} = \begin{bmatrix} H_{ij} & 0 & \ldots & 0 \\ 0 & H_{ij} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{ij} \end{bmatrix}. \quad (1)$$

Hence, we effectively have an $X$ channel with $\tilde{M}_j$ antennas at the $j$th transmitter and $\tilde{N}_i$ antennas at the $i$th receiver and channel matrices $\tilde{H}_{ij} \in \mathbb{C}^{\tilde{N}_i \times \tilde{M}_j}$. To achieve a degrees of freedom tuple $\vec{d}$ for the original system, we need to achieve $T \cdot \vec{d}$ for this equivalent system, and we can use the exact same precoding scheme designed for integer-valued corner points.

However, the equivalent channel matrices after symbol-extension are unlike those for their original counterparts (with $T = 1$) in that they are block-diagonal. The primary question that arises is whether the channel matrices of the time-extended channel continue to yield the linear independence results of the single-letter generic unstructured channels in spite of their special non-generic structure. If they do, then it can be asserted that multi-letter extensions are sufficient to achieve all fractional DoF tuples of $\mathbb{D}_X$.

However, this is not the case in general. Indeed, as it was observed in [7] the symbol extension technique is not sufficient even for the SISO $X$ channel. Interestingly, on the other hand, it is shown in [7] that, in the symmetric MIMO case, where all nodes have equal number, $M$, of antennas, and $M > 1$, the same idea works.

Nevertheless, the authors of [7] claim, based on a few examples, that the DoF outer bound region obtained therein is tight in “most cases”, and give the SISO case as an exception. But it is not clear if there are other cases that are also such exceptions, and if so, whether they can indeed be seen as exceptions, i.e., it is unclear as to how commonly these exceptions arise, in which just symbol extensions are not enough to achieve all the fractional corner points of $\mathbb{D}_X$. This brings us to the next section.

C. Asymmetric Complex Signaling

As stated previously, since the equivalent channel matrices after symbol-extension will be block-diagonal, many nice properties of the original generic channels can be lost. It is shown in [7] that, in the SISO case, the precoding
scheme provided previously (with three symbol extensions) fails to achieve the important integer-valued corner point 
\((1, 1, 1, 1)\) which achieves sum-DoF, because of the block diagonal structure in the extended channel matrices.

In response to this phenomenon, the authors of [8] introduced a new technique named asymmetric complex 
signaling (ACS). The key idea of ACS is to allow the inputs to be complex but not circularly symmetric and use 
an alternative representation of the channel models in terms of only real quantities. All dimensions of the new 
system will be doubled and all channel coefficients, beamformers, inputs and outputs will be real-valued. Let \(H_{ij}\) 
\((i, j = 1, 2)\) be the original complex channel matrices, their alternative real representations will have the following 
forms

\[
\hat{H}_{ij} = \begin{bmatrix}
  \text{Re}(H_{ij}) & -\text{Im}(H_{ij}) \\
  \text{Im}(H_{ij}) & \text{Re}(H_{ij}) 
\end{bmatrix}.
\]  

(2)

In order to transmit \(T \cdot d\) complex-valued streams over the original system, we need to transmit \(2T \cdot d\) real-valued 
streams over the equivalent real channels.

It is shown in [8] that using ACS, the outer bound of \(4/3\) degrees of freedom is achievable for the SISO X channel. In 
particular, with a three-symbol extension and ACS, all equivalent channel matrices are of size \(6 \times 6\), and using 
the same precoding scheme as used in the other MIMO cases, two real-valued symbols can be transmitted via 
the real channels. The missing independence requirement in the previous complex-valued transmission disappears 
almost surely in this new model. Thus the sum-DoF of \(4/3\) is achievable (and hence also the DoF region). The 
readers are referred to [8] for further details.

D. Closing the gap

The important question as to whether there are MIMO antenna configurations for which, among linear schemes 
including symbol extensions, ACS is necessary, remains open. The question is also open about whether ACS, along 
with the other linear techniques, is sufficient for MIMO antenna configurations to achieve all fractional DoF-tuples 
in \(\mathbb{D}_X\). If so, for what antennas configurations is it sufficient? Are there DoF-tuples and antenna configurations for 
which linear precoding schemes including time extensions and ACS are not sufficient?

In this section, all of the above questions are definitively answered. In particular, a class of antenna configurations 
(that includes the SISO case) are identified that require ACS among linear schemes; i.e., in which just employing 
symbol extensions alone doesn’t suffice. More generally, it is shown that ACS along with the other linear schemes 
is sufficient to achieve any fractional corner points of the DoF region \(\mathbb{D}_X\) of the MIMO X channel for any antenna 
configuration.

**Lemma 1.** In the case that \(M_1 + M_2 = N_1 + N_2\) and \(\min(M_1, M_2, N_1, N_2) = 1\), if interference alignment is 
needed to achieve any fractional DoF-tuple in \(\mathbb{D}_X\), then the achievability scheme in [3, A] applied to the \(T\)-symbol 
extended \(2 \times 2\) interference network, fails to make the corresponding symbols distinguishable at the receiver where 
they are desired. In particular, if \(M_1 = 1\) or \(N_2 = 1\), then \(\text{span} \left( \hat{H}_{21} \bar{V}_{21} \right) \subseteq \text{span} \left( \hat{H}_{22} \left[ \bar{V}_{21} \bar{V}_{22} \right] \right)\); if \(M_2 = 1\), \(N_1 = 1\), 
then \(\text{span} \left( \hat{H}_{12} \bar{V}_{12} \right) \subseteq \text{span} \left( \hat{H}_{11} \left[ \bar{V}_{12} \bar{V}_{11} \right] \right)\).
Proof: We give the proof of Lemma 1 in the case that $M_1$ or $N_2 = 1$, and the validity for the case that $M_2$ or $N_1 = 1$ follows in the same way.

First, consider the situation when $N_2 = 1$, and we have that $N_1 = M_1 + M_2 - 1 \geq \max(M_1, M_2)$. Consequently, zero-forcing any symbol of message $W_{21}$ and $W_{22}$ at receiver $R_1$ is not possible, i.e., $\tilde{V}_Z \tilde{V}_Z^H = \tilde{V}_1 \tilde{V}_1^H = 0$. However, since $M_1 + M_2 - N_1 = 1$, there exists a one dimensional null space of the concatenated channel $[H_{11} \ H_{12}]$. Thus, it is possible to align one symbol of message $W_{21}$ with one symbol of message $W_{22}$ at receiver $R_1$. When $T$ channel extensions are used, the available dimension for interference alignment is equal to $T$. Suppose the basis vector of the null space of $\mathcal{N}([H_{11} \ H_{12}])$ is given by

$$\begin{bmatrix}
V_{a, M_1 \times 1} \\
V_{b, M_2 \times 1}
\end{bmatrix}_{(M_1 + M_2) \times 1}$$

Then one set of basis vectors of $T$-dimensional subspace after symbol extension will be the column vectors of matrix

$$\begin{bmatrix}
V_{a, M_1 \times 1} & 0 & \ldots & 0 \\
0 & V_{a, M_1 \times 1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & V_{a, M_1 \times 1} \\
V_{b, M_2 \times 1} & 0 & \ldots & 0 \\
0 & V_{b, M_2 \times 1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & V_{b, M_2 \times 1}
\end{bmatrix}_{(M_1 T + M_2 T) \times T}$$

All beamformers generated from this basis should be of the form

$$[\alpha_1 V_a; \alpha_2 V_a; \ldots; \alpha_T V_a; \alpha_1 V_b; \alpha_2 V_b; \ldots; \alpha_T V_b],$$

where $\alpha_1, \alpha_2, \ldots, \alpha_T \in \mathbb{C}$ are $T$ random scalars. Since $N_2 = 1$, $H_{21} V_a$ and $H_{22} V_b$ will be scalars. $\tilde{H}_{21} \tilde{V}_{21}^A$ and $\tilde{H}_{22} \tilde{V}_{22}^A$ will have the following form

$$\begin{align*}
\tilde{H}_{21} \tilde{V}_{21}^A &= \begin{bmatrix}
H_{21} & 0 & \ldots & 0 \\
0 & H_{21} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & H_{21}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 V_a \\
\alpha_2 V_a \\
\vdots \\
\alpha_T V_a
\end{bmatrix} \\
&= (H_{21} V_a) \cdot \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix}.
\end{align*}$$

1The dimensions of matrices will be specified in a subscript when such dimensions have to be emphasized or defined for the first time.
and

\[
\begin{bmatrix}
H_{22} & 0 & \ldots & 0 \\
0 & H_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & H_{22}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 V_b \\
\alpha_2 V_b \\
\vdots \\
\alpha_T V_b
\end{bmatrix} = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix}
\]

\[= (H_{22} V_b) \cdot \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix}, \quad (7)
\]

Hence \(\tilde{H}_{21} \tilde{V}_{21}^A\) is also aligned with \(\tilde{H}_{22} \tilde{V}_{22}^A\) at their commonly desired destination, i.e., \(\text{span} (\tilde{H}_{21} \tilde{V}_{21}^A) = \text{span} (\tilde{H}_{22} \tilde{V}_{22}^A)\). This makes \(\tilde{W}_{21}^A\) and \(\tilde{W}_{22}^A\) indistinguishable at receiver \(R_2\).

Next, consider the situation when \(M_1 = 1\). In this case, \(M_1 \leq N_1\) and \(M_2 = N_1 + N_2 - 1 \geq N_1\). In other words, the null space of channel \(H_{11}\) does not exist, and the null space of channel \(H_{12}\) may exist. Recall that we only do interference alignment after zero-forcing of more symbols is not possible. Thus, when interference alignment is used, \(M_2 - N_1\) streams of the message \(W_{22}\) have already been zero-forced at receiver \(R_1\). Since \(\max(d_{22}) = N_2 > M_2 - N_1\), it is still possible to transmit another symbol of \(W_{22}\). The dimension of the null-space of the concatenated channel \([H_{11} \, H_{12}]\) is equal to \(M_1 + M_2 - N_1\). Letting vector \(v_{21}\) be in the subspace of null space \(N(\tilde{H}_{11})\), we have that \(\begin{bmatrix} 0 \\ v_{21} \end{bmatrix}\) will belong to the null space of \(N([H_{11} \, H_{12}])\). In other words, \(M_2 - N_1\) dimensions of the null space \(N([H_{11} \, H_{12}])\) are already occupied when doing zero-forcing of message \(W_{22}\). The remaining dimension of null space \(N([H_{11} \, H_{12}])\) is equal to \((M_1 + M_2 - N_1) - (M_2 - N_1) = M_1 = 1\). Thus, 1 dimension of interference alignment is possible at receiver \(R_1\) for messages \(W_{21}\) and \(W_{22}\). When \(T\) channel extensions are applied, the available dimension for interference alignment is equal to \(T\), and the dimension of zero-forcing subspace is equal to \(T \cdot (M_2 - N_1)\). The beamformers \(\tilde{V}_{21}^A\) and \(\tilde{V}_{22}^A\) are also in the form of (3)-(5). However, since \(N_2\) can be greater than 1 in this case, \(H_{21} V_a\) and \(H_{22} V_b\) are no longer scalars, and we don’t have the desirable result that \(\tilde{H}_{21} \tilde{V}_{21}^A\) is aligned with \(\tilde{H}_{22} \tilde{V}_{22}^A\) any more.

To prove that \(\text{span} (\tilde{H}_{21} \tilde{V}_{21}^A) \subseteq \text{span} (\tilde{H}_{22} \tilde{V}_{22}^A)\), we instead prove that \(\tilde{H}_{22} \tilde{V}_{22,1}^A \in \text{span} (\tilde{H}_{22} \tilde{V}_{22}^A)\), where \((\tilde{v}_{21,i}, \tilde{v}_{22,i})\) are any pair of alignment vectors drawn from the same beamformer from the null space of \(N([H_{11} \, H_{12}])\). Then, we will have that \(\bigcup \text{span} (\tilde{H}_{21} \tilde{v}_{21,1}^A) \subseteq \bigcup \text{span} (\tilde{H}_{22} \tilde{V}_{22}^A)\), which is the desired result. Let \(\tilde{v}_{21,i} = [\alpha_1 V_a; \alpha_2 V_a; \ldots; \alpha_T V_a]\) and \(\tilde{v}_{22,i} = [\alpha_1 V_b; \alpha_2 V_b; \ldots; \alpha_T V_b]\), we have that

\[
\tilde{H}_{21} \tilde{V}_{21}^A = \begin{bmatrix}
H_{21} V_a & 0 & \ldots & 0 \\
0 & H_{21} V_a & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & H_{21} V_a
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix}
\]
and

$$
\tilde{H}_{22} \bar{V}_{22}^A = \begin{bmatrix}
H_{22}V_b & 0 & \cdots & 0 \\
0 & H_{22}V_b & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_{22}V_b
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix}.
$$

Let column vectors of $V_{22}^Z$ be a basis of the nullspace $\mathcal{N}(H_{12})$. We have that

$$
\tilde{H}_{22} \bar{V}_{22}^Z = \begin{bmatrix}
H_{22}V_{22}^Z & 0 & \cdots & 0 \\
0 & H_{22}V_{22}^Z & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_{22}V_{22}^Z
\end{bmatrix}.
$$

Note that the $T : N_2$ dimensional space at receiver $R_2$ can be partitioned into $T$ linearly independent subspaces according to different symbol extension slot index. In order to prove that $\tilde{H}_{21} \bar{v}_{21,i} \in \text{span} \left( \tilde{H}_{22} \left[ \bar{V}_{22}^Z \bar{v}_{22,i}^A \right] \right)$, it is sufficient to prove that $H_{21}V_a \alpha_i \in \text{span} \left( H_{22}[V_{22}^Z V_b \alpha_i] \right)$ for all $i = 1, \ldots, T$. Since $\alpha_i$ here are all scalars, we only need to show that $H_{21}V_a \in \text{span} \left( H_{22}[V_{22}^Z V_b] \right)$.

Because $V_{22}^Z$ is generated from the nullspace of channel $H_{12}$, it is independent with channel matrix $H_{22}$. Consequently, $H_{22}V_{22}^Z$ will almost surely reserve the column rank of $V_{22}^Z$, since $H_{22}$ is a generic full matrix whose rank is greater than $V_{22}^Z$'s. In other words, $\text{rank}(H_{22}[V_{22}^Z V_b]) = M_2 - N_1 = N_2 - 1$ almost surely. Since $V_b$ is linearly independent with the column vectors of $V_{22}^Z$, $H_{22}V_b$ will also be linear independent with the column vectors of $H_{22}V_{22}^Z$ almost surely. Thus, $\text{rank}(H_{22}[V_{22}^Z V_b]) = (N_2 - 1) + 1 = N_2$. In other words, the column vectors of $H_{22}[V_{22}^Z V_b]$ would span the entire $N_2$-dimensional subspace at receiver $R_2$. Since vector $H_{21}V_a$ also belongs to the same subspace, we have that $H_{21}V_a \in \text{span} \left( H_{22}[V_{22}^Z V_b] \right)$, which leads to that $\tilde{H}_{21} \bar{v}_{21,i}^A \in \text{span} \left( \tilde{H}_{22} \left[ \bar{V}_{22}^Z \bar{v}_{22,i}^A \right] \right)$. Thus, message $\tilde{W}_{21}^A$ and $\tilde{W}_{22}^A$ are indistinguishable at receiver $R_2$. 

**Lemma 2.** By using the technique of ACS together with symbol extensions, the problem of unexpected alignment of desired messages is avoided.

**Proof:** The equivalent channel matrices, when doing $T$-symbol extension and ACS, are given as $\tilde{H}_{ij} = I_{T \times T} \otimes \hat{H}_{ij}$, where $\hat{H}_{ij}$ is given in equation [2]. We need to transmit $2T \cdot \tilde{d}$ real-valued streams over the equivalent real channels.

Consider again the independence of $\tilde{H}_{21} \bar{V}_{21}^A$ and $\tilde{H}_{22} \bar{V}_{22}^A$ for the cases in Lemma [1]. If $N_2 = 1$, when doing asymmetric complex signaling, the dimension of $V_a$ and $V_b$ in [3] will be $2M_1 \times 2$ and $2M_2 \times 2$, respectively. $\tilde{H}_{21} \bar{V}_{21}^A$ and $\tilde{H}_{22} \bar{V}_{22}^A$ will instead have the following form

$$
\tilde{H}_{21} \bar{V}_{21}^A = \begin{bmatrix}
\hat{H}_{21}V_a & 0 & \cdots & 0 \\
0 & \hat{H}_{21}V_a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{H}_{21}V_a
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{2T}
\end{bmatrix}.
$$
and

\[ \hat{H}_{22} \hat{V}_{22}^A = \begin{bmatrix} \hat{H}_{22}V_b & 0 & \ldots & 0 \\ 0 & \hat{H}_{22}V_b & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \hat{H}_{22}V_b \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{2T} \end{bmatrix} \]

where \( \alpha_1, \alpha_2, \ldots, \alpha_{2T} \in \mathbb{R}^1 \) are \( 2T \) random real scalars. Now, \( \hat{H}_{21}V_a \) and \( \hat{H}_{22}V_b \) are both \( 2 \times 2 \) real matrices rather than scalars as in (6) and (7). Each diagonal block of \( \hat{H}_{21}V_a \) or \( \hat{H}_{22}V_b \) works as if it is to rotate a random \( 2 \times 1 \) real vector with a certain degree. However, the randomness of \( [\alpha_1; \ldots; \alpha_{2T}] \) makes the projections in different symbol extension slots independent with each other. Thus, \( \hat{H}_{21}V_A^a \) and \( \hat{H}_{22}V_A^b \) will be linearly independent almost surely.

Consider again the case that \( M_1 = 1 \), the column rank of \( \hat{H}_{22}[V_{22}^Z V_b] \) will be equal to \( 2(N_2 - 1) + 1 = N_2 - 1 \). In other words, there is still 1 dimension left in the receiver subspace. Thus, \( \hat{H}_{21}V_a \) is independent with \( \hat{H}_{22}[V_{22}^Z V_b] \) almost surely. In the situation that the column dimension of \( V_a \) and \( V_b \) are \( n \), which is greater than 1, i.e., there are multiple pairs of symbols to be aligned, the column size of \( \begin{bmatrix} \hat{H}_{22}[V_{22}^Z V_b] & \hat{H}_{21}V_a \end{bmatrix} \) will be equal to \( 2(N_2 - 1) + n + n > N_2 \) and thus the columns of \( \hat{H}_{22}[V_{22}^Z V_b] \) are linear dependent with columns of \( \hat{H}_{21}V_a \).

However, since the coefficients required for dependence for \( \hat{H}_{21}V_a \) are different almost surely in different time slots, the \( 2 \times T \) dimensional \( \hat{H}_{22}[\hat{V}_{22}^Z \hat{V}_a^A] \) will still be linearly independent with \( \hat{H}_{12}V_{12}^A \) almost surely, so long as \( d_{22}^Z + d_{22}^A + d_{12}^A \leq T \cdot N_2 \).

In summary, the desired messages are still linearly independent with each other at both receiver.

\[ \square \]

Remark 1. The authors of [8] introduced ACS in the context of the SISO X channel and showed that the total DoF of \( \frac{3}{2} \) can be achieved in that channel. In this paper, we provided a new and simplified perspective on how ACS works. In particular, it transforms the previous scalar multiplication to a local vector rotation, thus obviating the unexpected linear dependences among all the beamformers. Using this we broaden its applicability to MIMO X channel, and more generally, in a later section, to the nine-message MIMO 2×2 interference network.

Remark 2. For all the other antenna settings not included in the cases given in Lemma 1 there is no unexpected loss of independence of desired messages when doing symbol extensions. Thus, ACS is not necessary in those cases.

E. Further results on the MIMO X channel

In this section, we discuss several other observations/results about the MIMO X channel.

Lemma 3. In the symmetric \((M, M, N, N)\) antenna setting, the maximum sum DoF of the MIMO X channel is
given by

\[
\begin{cases}
2M, & \text{if } 0 < \frac{M}{N} \leq \frac{2}{3} \\
\frac{4N}{3}, & \text{if } \frac{2}{3} < \frac{M}{N} \leq 1 \\
\frac{4M}{3}, & \text{if } 1 < \frac{M}{N} \leq \frac{3}{2} \\
2N, & \text{if } \frac{3}{2} < \frac{M}{N}
\end{cases}
\]

Lemma 3 is a special case of Theorem 1. In terms of sum-DoF performance, there are hence redundant antennas at the transmitters if \( \frac{2}{3} < \frac{M}{N} \leq 1 \) or \( \frac{M}{N} > \frac{3}{2} \), and there are redundant antennas at the receivers if \( 0 < \frac{M}{N} < \frac{2}{3} \) or \( 1 \leq \frac{M}{N} < \frac{3}{2} \). In the case that \( M = N \), the redundancy exists both at the transmitters and at the receivers. For example, the three antenna settings of \((3,3,3,3)\), \((3,3,2,2)\) and \((2,2,3,3)\) all have the same maximum sum-DoF of 4.

Interestingly, for the equal-antenna case of \((3,3,3,3)\), one can easily achieve the DoF-tuple of \((1,1,1,1)\) by turning off one antenna at each receiver and then transmitting all four symbols of the private messages using zero-forcing beamforming in each of the one-dimensional null space of the remaining channel matrices. No explicit interference alignment is actually needed to achieve the optimal sum-DoF. Given that explicit interference alignment was first discovered in the context of the symmetric three-antenna MIMO X channel as being the key ingredient needed to achieve DoF-optimality, this observation is surprising. To the best of the authors’ knowledge, this is the first time this simple result has been noted. Shutting down the redundant antenna at each receiver could however be seen as implicitly aligning interference in a subspace that would only be seen by that antenna and then discarding that subspace.

**Lemma 4.** For the special cases given in Lemma 7 in which ACS is required to achieve the maximum sum-DoF, the maximum sum-DoF is equal to \( C - \frac{2}{3} \), where \( C = M_1 + M_2 = N_1 + N_2 \). The DoF tuple to achieve the maximum sum-DoF is given by \( d_{ij} = \min(M_j, N_i) - \frac{2}{3} \).

**Proof:** We give the proof for the case that \( M_1 = \min(M_1, M_2, N_1, N_2) = 1 \) here. The other cases follow in the same way.

Since \( M_1 = 1 \) and \( M_1 + M_2 = N_1 + N_2 = C \), we have that \( \max(M_1, N_1) = N_1, \max(M_1, N_2) = N_2, \max(M_2, N_1) = M_2 \) and \( \max(M_2, N_1) = M_2 \). Adding the first 4 inequalities in \( D_X \) together, we have that

\[
3(d_{11} + d_{12} + d_{21} + d_{22}) \leq N_1 + N_2 + 2M_2 = 3C - 2
\]

Thus, the sum-DoF is bounded by \( C - \frac{2}{3} \). It is easy to verify that the DoF tuple \( d_{ij} = \min(M_j, N_i) - \frac{2}{3} \) achieves the optimal sum DoF and is within the DoF region \( D_X \). Thus, the maximum sum DoF is equal to \( C - \frac{2}{3} \).

A symbol extension of length 3, together with ACS, is required to achieve this corner point.

**Lemma 5.** In the case that only three private messages are transmitted in the channel, all the corner points will be integer-valued. Thus, neither symbol extension nor ACS is necessary.
Proof: Since the channel is isotropic with respect to any message, we can assume without loss of generality that the three private messages are $W_{11}$, $W_{12}$ and $W_{21}$. By deleting $d_{22}$ from $D_X$ and removing the redundant inequalities, we obtain the following 3-dimensional DoF region.

$$D' = \{(d_{11}, d_{21}, d_{12}) \in \mathbb{R}_+^3 :$$

- $$d_{11} + d_{12} + d_{21} \leq \max(M_1, N_1),$$
- $$d_{11} + d_{12} \leq N_1,$$
- $$d_{21} + d_{11} \leq M_1,$$
- $$d_{21} + d_{12} \leq \max(M_2, N_2),$$
- $$d_{21} \leq N_2,$$
- $$d_{12} \leq M_2\}.$$ (8)

Each corner point of this 3-D region will be the intersection of three of the nine facets describing the polytope. Observing the constraints, it is easy to verify that the only possible combination of facets that can have a fractional intersection are (9), (10) and (11), and the corresponding vertex is

$$d_{11} = \frac{M_1 + N_1 - \max(M_2, N_2)}{2},$$
$$d_{12} = \frac{N_1 + \max(M_2, N_2) - M_1}{2},$$
$$d_{21} = \frac{M_1 + \max(M_2, N_2) - N_1}{2}.$$ (12)

These three values will be all integers or all non-integer fractions which are an odd-multiple of $\frac{1}{2}$.

From constraint (8), we have that

$$\frac{M_1 + N_1 + \max(M_2, N_2)}{2} \leq \max(M_1, N_1).$$

Otherwise, this corner point will be outside the DoF region. Consequently, one of $d_{12}$ and $d_{21}$ will be $\leq 0$. If it is less than zero, this corner point is outside the DoF region and therefore irrelevant; if it is equal to 0, then the other two values will be integers.

The intersection of all other combinations of facets will be integer-valued, thus, all the corner points of $D'$ are integer-valued, and neither symbol extension nor ACS are necessary to achieve them.

Lemma 6. For the MIMO $X$ channel of an arbitrary antenna setting, if there are fractional-valued corner points and symbol extension is required to achieve this corner point, the length of symbol extension will be at most 3.

Proof: Again, each corner point of the 4-dimensional DoF region is the intersection of four of the facets describing the polytope. Since the coefficients of any facet are either 0 or 1, any selected 4-by-4 coefficient matrix will be a binary matrix. According to the Hadamard maximal determinant problem [16], the determinant of an order 4 binary matrix can at most be 3. Consequently, the inverse of any 4-by-4 coefficient matrix, if it exists, can
at most have a denominator of 3. Thus, for any non-integer valued corner points, the denominator will be at most 3. Thus, the length of symbol extension will be at most 3.

More specifically in this problem, it is shown that there is only one corner point whose denominator is 3, and this corner point is the intersection of the four facets corresponding to the first four constraints in $\mathbb{D}_X$.

\section*{F. Cognitive MIMO $X$ channel}

If one of the four private messages in the MIMO $X$ channel, for example $W_{11}$, is made available non-causally at the other transmitter, the channel is named cognitive MIMO $X$ channel. It is shown in [7] that the sum DoF of the cognitive MIMO $X$ channel with equal number, $M$, of antennas at each terminal is equal to $\frac{3}{2}M$, which is greater than the sum DoF of $\frac{4}{3}M$ of the symmetric $X$ channel. So, cognitive message sharing helps increase sum DoF in this case. We discuss more general properties of the cognitive MIMO $X$ channel here.

\section*{Theorem 2.}

The degrees of freedom region of the cognitive MIMO $X$ channel with message $W_{21}$, $W_{12}$, $W_{22}$ and $W_{01}$ is given by

$$\mathbb{D}_{co-X} = \{(d_{01}, d_{21}, d_{12}, d_{22}) \in \mathbb{R}_+^4 :$$

$$d_{01} + d_{12} + d_{21} \leq \max(M_1, N_1),$$

$$d_{01} + d_{12} + d_{22} \leq \max(M_2, N_1),$$

$$d_{21} + d_{22} + d_{12} \leq \max(M_2, N_2),$$

$$d_{01} + d_{12} \leq N_1, \quad d_{21} + d_{22} \leq N_2,$$

$$d_{21} \leq M_1, \quad d_{12} + d_{22} \leq M_2,$$

$$d_{01} + d_{21} + d_{12} + d_{22} \leq M_1 + M_2\}$$

Theorem 2 follows directly from our main result of the 9-dimensional DoF region of the MIMO $2 \times 2$ Gaussian interference network with general message sets given in Section IV. When the above DoF region is specialized to the symmetric, equal-antenna case, all but the first three bounds are redundant, and it is easy to see that the DoF-tuple $(d_{01} = M/2, d_{12} = 0, d_{21} = M/2, d_{22} = M/2)$, the achievability of which was shown in [7] for $M > 1$ (using two-symbol extensions), is a maximum sum-DoF corner point of $\mathbb{D}_{co-X}$ for any $M \geq 1$.

More generally, the DoF region of cognitive MIMO $X$ channel is in general greater than that of the MIMO $X$ channel. For example, consider the case of $M_1 = 3$, $M_2 = 4$, $N_1 = 5$, $N_2 = 6$. When $d_{12}$, $d_{21}$ and $d_{22}$ are all set to be equal to 1, $d_{11}$ can be at most 2 in the MIMO $X$ channel, whereas $d_{01}$ can be up to 3 in the cognitive MIMO $X$ channel. Even the cognition of one message among the transmitters can significantly improve the maximum achievable DoF.

\section*{Lemma 7.}

In the symmetric $(M, M, N, N)$ antenna setting, the maximum sum DoF of the cognitive MIMO $X$
channel is given by

\[
\begin{cases}
2M, & \text{if } 0 < \frac{M}{N} \leq \frac{1}{4} \\
\frac{3N}{2}, & \text{if } \frac{1}{4} < \frac{M}{N} \leq 1 \\
M + \frac{N}{2}, & \text{if } 1 < \frac{M}{N} \leq \frac{3}{2} \\
2N, & \text{if } \frac{3}{2} < \frac{M}{N} \\
\end{cases}
\]

Lemma 7 is a special case of Theorem 2. Comparing with the result of MIMO X channel, the sum DoF of the cognitive MIMO X is strictly greater than that of the MIMO X when \( \frac{2}{3} < \frac{M}{N} < \frac{3}{2} \). When \( \frac{M}{N} \leq \frac{2}{3} \) or \( \frac{M}{N} \geq \frac{3}{2} \), there are redundant antennas at the transmitters or the receivers, and message cognition does not help in improving the sum DoF of the system.

Lemma 8. In the case that \( M_1 + M_2 = N_1 + N_2 \) and \( \min(M_1, M_2, N_1, N_2) = 1 \), among linear strategies, ACS is required to achieve the DoF region of the cognitive MIMO X channel.

Proof: The reason that ACS is necessary for the cognitive MIMO X channel is the same as that for the MIMO X channel in lemma 1. We omit the details for brevity.

Lemma 9. For the special cases given in Lemma 8 in which ACS is required to achieve the maximum sum-DoF of the cognitive MIMO X channel, the maximum sum-DoF is equal to \( C - \frac{1}{2} \), where \( C = M_1 + M_2 = N_1 + N_2 \). The DoF tuple to achieve the maximum sum-DoF is given by \( (d_{01}, d_{21}, d_{12}, d_{22}) = (\min(M_1, N_1) - \frac{1}{2}, \min(M_1, N_2) - \frac{1}{2}, \min(M_1 + M_2, N_1) - \min(M_1, N_1), \min(M_2, N_2) - \frac{1}{2}) \text{ or } (\min(M_1 + M_2, N_1) - \frac{1}{2}, \min(M_1, N_2) - \frac{1}{2}, 0, \min(M_2, N_2) - \frac{1}{2}) \).

Proof: Adding the 1st, 2nd and 5th inequalities in \( D_{co-X} \) together, we have that \( 2d_{sum} \leq \max(M_1, N_1) + \max(M_2, N_1) + N_2 \), which is always equal to \( 2C - 1 \). Thus, the sum DoF is upper bounded by \( C - \frac{1}{2} \). One can easily verify that the two given DoF tuples are both within \( D_{co-X} \) and achieve the maximum sum-DoF.

A symbol extension of length 2, together with ACS, is required to achieve this corner point.

There can be two non-integer-valued corner points which achieve the maximum sum-DoF. However, when \( \min(M_1 + M_2, N_1) = \min(M_1, N_1) \), or equivalently \( M_1 \geq N_1 \), these two corner points are the same. If these two corner points are different, we can get one of them from the other by just regarding the non-zero \( d_{12} \) symbols of message \( W_{12} \) as part of message \( W_{01} \).

Lemma 10. For the cognitive MIMO X channel of arbitrary antenna setting, if there are any fractional-valued corner point and symbol extensions are required to achieve this corner point, the length of symbol extension will be at most 2.

Proof: Although the determinant of an arbitrary 4-by-4 binary matrix can be at most 3, it is easy to verify that the maximum determinant of any 4-by-4 coefficient matrix generating from any four facets given in \( D_{co-X} \) is equal to 2. Thus, the length of symbol extension will be at most 2.
IV. Main Result

Now, let us consider the general MIMO 2 × 2 interference network with nine messages.

The following theorem gives the nine-dimensional DoF region of the MIMO 2 × 2 Gaussian interference network with general message sets.

**Theorem 3.** The degrees of freedom region of the MIMO 2 × 2 Gaussian interference network with the general message set is

\[
D = \{ (d_{11}, d_{21}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0) \in \mathbb{R}^9_+ : \\
d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{21} \leq \max(M_1, N_1) \\
d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{22} \leq \max(M_2, N_1) \\
d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{11} \leq \max(M_1, N_2) \\
d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{12} \leq \max(M_2, N_2) \\
d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} \leq N_1 \\
d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} \leq N_2 \\
d_1 + d_{11} + d_{21} \leq M_1 \\
d_2 + d_{12} + d_{22} \leq M_2 \\
d_1 + d_2 + d_0 + d_{01} + d_{02} + d_{11} + d_{21} + d_{12} + d_{22} \leq \min(M_1 + M_2, N_1 + N_2) \}
\]

*Proof:* The proof of \( D \) being an outer bound is given in Section V. The inner bound is given in the Lemmas 11 and 12 in this section.

**Lemma 11.** An inner bound to the degrees of freedom region of the MIMO 2 × 2 interference network with general message set is \( D_{\text{in}} = \text{co} (D \cap \mathbb{Z}^9_+) \), i.e., all the integer-valued degrees of freedom in \( D \) as well as their convex hull are achievable.

*Outline of Proof:* In this outline, we will describe a method to construct the transmit beamformers for various messages. It will be shown later in Section VII that using this scheme the DoF region \( D_{\text{in}} \) can be achieved.

To achieve any integer-valued nine-dimensional DoF tuple \( \vec{d} = (d_{11}, d_{21}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0) \) within \( D \), we use the following precoding scheme.

Consider linear beamforming. Expressing received signals at receive \( r \) (\( r=1,2 \)) in the form of different messages,
Table I

MESSAGE GROUPING AND CORRESPONDING PRECODING METHODS

| Group 1 | (W_{11}, W_{12}, W_{21} and W_{22}) |
|---------|-----------------------------------|
|         | In W_{11}, for example, there are d_{11} independent symbols. d_{11}^Z of them are zero-forced at receiver R_2, d_{11}^A of them are aligned with part of W_{12} at receiver R_2. The remaining d_{11}^R = d_{11} - d_{11}^Z - d_{11}^A symbols are transmitted using random beamforming. |

| Group 2 | (W_{01} and W_{02}) |
|---------|---------------------|
|         | In W_{01} for example, there are d_{01} independent symbols. d_{01}^Z of them are zero-forced at receiver R_2, and the remaining d_{01}^R = d_{01} - d_{01}^Z symbols are transmitted using random beamforming. |

| Group 3 | (W_1, W_2 and W_0) |
|---------|-------------------|
|         | Random beamforming is used for all symbols of this group. |

we have

\[
Y_r = H_{r1} \cdot (V_{11}S_{11} + V_{21}S_{21} + V_1S_1) \\
+ H_{r2} \cdot (V_{12}S_{12} + V_{22}S_{22} + V_2S_2) \\
+ [H_{r1} H_{r2}] \cdot (V_{01}S_{01} + V_{02}S_{02} + V_0S_0) + Z_r,
\]

where S_x and V_x denote the symbols and the corresponding precoding matrices for the message with index x ∈ E. Let the column size of V_x is equal to d_x.

The techniques used here are transmit zero-forcing, interference alignment and random beamforming.

The nine messages are divided into three groups as shown in Table I. Group 1 consists of the four point-to-point private or X-channel messages \{W_{11}, W_{12}, W_{21}, W_{22}\}, Group 2 consists of the cognitive and common messages which are known to both transmitters, namely, \{W_{01}, W_{02}\}. Group 3 consists of the remaining three multicast messages \{W_1, W_2, W_0\}. The transmission of Group 1 messages is done in the exact same way as in the MIMO X channel. Then, the other two groups are transmitted via the channel resources still available. Recall that, for Group 1, message \(W_{ij}\) (i, j = 1, 2) is partitioned into three linearly independent parts, i.e., \(W_{ij}^Z, W_{ij}^A\) and \(W_{ij}^R\). Here, for Group 2, message \(W_{0i}\) (i = 1, 2) is partitioned into two linearly independent parts, namely, \(W_{0i}^Z\) and \(W_{0i}^R\). For Group 3, all messages are transmitted using random beamforming, since no interference elimination is necessary for them. Thus, message \(W_k\) (k = 0, 1, 2) is all classified as \(W_k^R\). We have that

\[
d_{ij} = d_{ij}^Z + d_{ij}^A + d_{ij}^R \\
d_{0i} = d_{0i}^Z + d_{0i}^R \\
d_k = d_k^R
\]
and
\[
V_{ij} = [V_{ij}^Z V_{ij}^A V_{ij}^R]
\]
\[
V_{0i} = [V_{0i}^Z V_{0i}^R]
\]
\[
V_k = V_k^R
\]
where \(i, j = 1, 2\) and \(k = 0, 1, 2\). The dimensions of different parts of each message are given as follows
\[
d_{ij}^Z = \min(d_{ij}, (M_j - N_j)^+)
\]  
(23)
\[
d_{i1}^A = d_{i2}^A = \min(d_{i1}, d_{i2} - d_{i2}^Z, (M_1 + M_2 - N_i - d_{i1}^Z - d_{i2}^Z)^+)
\]  
(24)
\[
d_{ij}^R = d_{ij} - d_{ij}^Z - d_{ij}^A
\]  
(25)
\[
d_{0i}^Z = \min(d_{0i}, (M_1 + M_2 - N_i - d_{i1}^Z - d_{i2}^Z - d_{i1}^A)^+)
\]  
(26)
\[
d_{0i}^R = d_{0i} - d_{0i}^Z
\]  
(27)
\[
d_k^R = d_k
\]  
(28)
where \(i, j = 1, 2, \hat{i} = 3 - i\), and \(k = 0, 1, 2\). To make the expressions more succinct, we define following auxiliary variables:

\[
Z_{ij} \equiv d_{ij}^Z
\]  
(29)
\[
A_i \equiv d_{i1}^A = d_{i2}^A
\]  
(30)
\[
Z_{0i} \equiv d_{0i}^Z
\]  
(31)
These values are pre-determined according to the value of the DoF tuple and the system antenna setting. They naturally follow from the fact that the numbers of beamformers transmitted using zero-forcing or interference alignment cannot exceed the corresponding available null space dimensions. For the four private messages, if zero-forcing is possible, use zero-forcing first. If there are more streams that must be send, use interference alignment next. If there are still more streams after running out of the possibility of doing alignment, use random beamforming. For the two cognitive and common messages, if there are residual available null space dimensions, transmit using zero-forcing; otherwise, just use random beamforming. For three multicast messages, all streams are transmitted using random beamforming.

The key to using zero-forcing or interference alignment is to appropriately utilize the beamformers picking from the null space of corresponding channels. For a generic channel matrix \(H_{n \times m}\) \((n < m)\), the dimension of its nullspace is equal to \(m - n\). To obtain a basis of \(\mathcal{N}(H)\), we can do a singular value decomposition (SVD) of matrix \(H\) while arranging the singular values in non-increasing order. Then, the last \(m - n\) right-singular column vectors, which are corresponding to singular value 0, will form a basis of \(\mathcal{N}(H)\). We construct matrix \(\Phi(H)\)
such that its column vectors are equal to these basis vectors of $\mathcal{N}(H)$. Let matrix $X_{(m-n)\times a}$ denote a randomly $(m - n) \times a$ matrix, whose column vectors are generated independently from a uniform distribution on a $m - n$ dimensional sphere of radius 1. Then, $\Phi(H) \cdot X_{(m-n)\times i}$ will generate $b$ random combinations of these basis vectors. If $b \leq m - n$, these $b$ vectors will be linearly independent of each other almost surely.

Now, construct the beamformers for all 9 messages according to the equations (32)-(38) listed below.

\[
\begin{align*}
V^Z_{ij} & = \Phi(H_{ij}) \cdot X^Z_{ij,(M_j-N_i)\times d^Z_{ij}} \\
V^A_{i1} & = \Phi([H_{i1} 0]) \cdot X^A_{i,(M_1+M_2-N_i)\times d^A_{i1}} \\
V^A_{i2} & = \Phi([H_{i2} H_{i2}]) \cdot X^A_{i,(M_1+M_2-N_i)\times d^A_{i2}} \\
V^R_{ij} & = X^R_{ij,M_j\times d^R_{ij}} \\
V^Z_{0i} & = \Phi([H_{0i} 0]) \cdot X^Z_{0i,(M_1+M_2-N_i)\times d^Z_{0i}} \\
V^R_{0i} & = X^R_{0i,(M_1+M_2)\times d^R_{0i}} \\
V^R_i & = X^R_{i,M_i\times d^R_i} \\
V^R_0 & = X^R_{0,(M_1+M_2)\times d^R_0}
\end{align*}
\]

where $i, j = 1, 2$ and $\hat{i} = 3 - i$. The beamformers used for $V^Z_{ij}, V^A_{ij}$ and $V^Z_{0i}$ come from the nullspace of the channels or concatenated channels, and the beamformers used for $V^R_{ij}, V^R_{0i}, V^R_i$ and $V^R_0$ are just generated randomly as described previously. It’s shown later in Section VI that, if the DoF tuple $\vec{d}$ is in the region of $\mathbb{D}_{in}$, then using the above preceding beamformers, all messages are decodable at their intended receivers with probability one. Hence, the DoF tuple $\vec{d}$ is achievable and $\mathbb{D}_{in}$ is an achievable DoF region.

Remark 3. In linear beamforming, to achieve $\mathbb{D}_{in}$, only the techniques of zero-forcing, interference alignment and random beamforming are required. Furthermore, as is shown later in Section VI interference alignment is needed only among the four private $X$ channel messages, i.e., aligning $W_{11}$ with $W_{12}$ at receiver $R_2$ or aligning $W_{21}$ with $W_{22}$ at receiver $R_1$. Somewhat surprisingly perhaps, it is not necessary to align interference due to any part of $W_{01}$ with that due to $W_{11}$ or $W_{12}$ at receiver $R_2$, or to align interference due to any part of $W_{02}$ with that due to $W_{21}$ or $W_{22}$ at receiver $R_1$.

Remark 4. In the construction of $V^Z_{ij}, V^A_{ij}$ and $V^Z_{0i}$, we use the random linear combinations of the basis vectors of the nullspace of corresponding channels, instead of directly picking beamformers from those basis vectors obtained through an SVD. The advantage is that it avoids picking a same basis vector repetitively in following procedures and potentially leading to unexpected dependence among the beamformers.

Lemma 12. The fractional numbers at the boundary of $\mathbb{D}$, i.e., the gap between $\mathbb{D}_{in}$ and $\mathbb{D}$, can be achieved using appropriate length of symbol extension. In the case that $M_1 + M_2 = N_1 + N_2$ and $\min(M_1, M_2, N_1, N_2) = 1$, ACS is required in addition to symbol extension.

Proof: To achieve a DoF tuple $\vec{d}$ with fractional values, we use a $T$ symbol extensions of the channel such that
Figure 2. MIMO $Z_{21}^*$ channel with general message sets (a) complete (b) reduced (c) only private

$T \cdot \overrightarrow{d}$ is integer-valued. The problem of unexpected dependencies, which is brought on by the structured channel matrices after symbol extensions, also exists here in the nine-message problem. The random beamforming part of messages in Groups 2 and 3, i.e., $W_{0i}^R$ and $W_k^R$ ($i = 1, 2$, $k = 0, 1, 2$), cause no problem; they behave the same as do $W_{0i}^R(i, j = 1, 2)$ from Group 1 in terms of independence results. Since the beamformers for the zero-forcing part of Group 2 messages, i.e., $W_{0i}^E$ ($i = 1, 2$), are generated from the null space of corresponding concatenated channels, they face the same situation as the interference alignment beamformer pairs (of the private messages) do. Since all the zero-forcing and interference alignment beamformers are derived from the same source but belong to different messages, their behaviors are actually equivalent when considering independence results. The analyses of when ACS is necessary and how ACS works which were detailed in Section III for the the MIMO X channel are the same as in the MIMO X channel problem as well.

In summary, Lemmas 11 and 12 establish that $\mathbb{D}$ is an inner bound to the DoF region of the $2 \times 2$ interference network. Together with the proof of the outer bound in Section V, this completes the proof of Theorem 3.

V. OUTERBOUND ON THE DEGREES OF FREEDOM REGION

In this section, we prove the converse part of Theorem 3, i.e., that the region $\mathbb{D}$ is an outer bound for the DoF region of the $2 \times 2$ interference network.

First, the outer bound (22) comes from the MIMO point-to-point channel outer bound when cooperation between transmitters and receivers are both allowed.

Second, consider the embedded multiple-access channel which only contains transmitters $T_1$ and $T_2$ and receiver $R_1$. In this situation, message $W_{02}$, $W_{21}$, $W_{22}$ are irrelevant and set to $\emptyset$ to avoid interference. The original message $W_1$ will degenerate to $W_{11}$, since we don’t require $W_1$ to be decoded by receiver $R_2$. Similarly, $W_2$ will degenerate to $W_{12}$, and $W_0$ will degenerate to $W_{01}$. The cut-set bound for multiple-access channel with common message is $d_{01} + d_{11} + d_{12} \leq N_1$. Hence in this scenario, we get the equivalent outer bound $(d_0 + d_{01}) + (d_1 + d_{11}) + (d_2 + d_{12}) \leq$
$N_1$, which is outer bound (18). In the same way, we get outer bound (19) by considering the embedded multiple-access channel which only contains transmitter $T_1$ and $T_2$ and receiver $R_2$.

Third, consider the embedded broadcast channel which only contains transmitter $T_1$ and receivers $R_1$ and $R_2$. In this situation, message $W_{12}$, $W_{22}$ and $W_2$ are irrelevant and set to $0$ to avoid interference. We also set $W_0$, $W_{01}$, $W_{02}$ to $0$ and loosen the requirement for transmitter $T_1$ by not requiring it to help in transmitting $W_0$, $W_{01}$ and $W_{02}$. Then we get the outer bound from the result of broadcast channel with common message $d_1 + d_{11} + d_{21} \leq M_1$, which is outer bound (20). Similarly, by considering the embedded broadcast channel with transmitter $T_2$, we get outer bound (21).

Next we prove outer bound (15). Outer bounds (14), (16) and (17) can be similarly inferred. Consider the channel depicted in Figure 2(a), in which there is no communication link between transmitter $T_1$ and receiver $R_1$ or $R_2$. This is equivalent to $H_{21} = 0$. Since receiver $R_2$ already knows $W_1$, transmitter $T_1$ only needs to make sure that receiver $R_1$ can successfully decode $W_1$, so that $W_1$ degenerates to $W_{11}$. Similarly, $W_0$ degenerates to $W_{01}$. The resulting $2 \times 2$ interference network becomes identical to the $Z^*$ channel with the general message set as depicted in Figure 2(a). Since neither setting $W_{21}$ and $W_{02}$ to known sequences nor the assistance of genie to receiver $R_2$ can deteriorate the performance of the coding scheme, the same degrees of freedom $d_{01,Z_{21}} = d_{01} + d_0$, $d_{11,Z_{21}} = d_{11} + d_{11}$, $d_{12,Z_{21}} = d_{21}$, $d_{22,Z_{21}} = d_2$, $d_{22,Z_{21}} = d_{22}$ are achievable on the $Z^*$ channel as well. This proves inequality (39). The argument here is similar to the proof of Lemma 1 in (7), in which $Z_{21}$ channel with only private messages is considered.

In the $Z_{21}$ channel depicted in Figure 2(a), message $W_2$ is sent out from transmitter 2 and desired at both receivers, $R_1$ and $R_2$. If we loosen this requirement and only demand receiver $R_2$ to be able to decode this message, the degrees of freedom of the new system will be no less than that of the original system, since reducing decoding requirement cannot hurt. In this case, $W_2$ actually plays the same role as $W_{22}$ does. As a result, we can combine them together and the system reduces to Figure 2(b).

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The system in Figure 2.(c) is the ordinary MIMO $Z$ channel, which only contains private messages $W_{11}$, $W_{12}$, $W_{22}$. An outer bound of ordinary MIMO $Z$ channel is given in Corollary 1 of [7], which is

$$\max(d_{11} + d_{12} + d_{22}) \leq \max(N_1, M_2).$$

The idea of the proof therein is to show the sum capacity of $Z$ channel (Figure 2.(c)) is bounded above by the MAC with $M_2$ receive antennas if $N_1 < M_2$ and bounded above by the MAC with $N_1$ receive antennas if $N_1 \geq M_2$. The multiplexing gain of a MAC cannot be greater than the total number of receive antennas. Therefore, we have $\max(d_{11} + d_{12} + d_{22}) \leq \max(N_1, M_2)$ for Figure 2.(c). Now, consider the $Z$ channel in Figure 2.(b), in which one additional common message $W_{01}$ is applied. Following the exact same argument as in [7], we get that the sum capacity of $Z$ channel (Figure 2.(b)) is bounded above by corresponding MAC with common message, whose multiplexing gain is also no greater than its total number of receive antennas, i.e.,

$$\max(d_{01} + d_{11} + d_{12} + d_{22}) \leq \max(N_1, M_2).$$

Including common message or not doesn’t affect the relationship and transformation between $Z$ channel and corresponding MAC channel in the proof. The reader can refer to [7] for more details.

So far we obtained an outer bound for the MIMO $Z^*$ channel with general message sets in Figure 2.(a), which is

$$\max(d_0 + d_{01} + d_1 + d_{11} + d_{12} + d_2 + d_{22}) \leq \max(N_1, M_2),$$

According to inequality (39), we have that an outer bound for the MIMO $2 \times 2$ interference network with general message sets is

$$d_0 + d_{01} + d_1 + d_{11} + d_{12} + d_2 + d_{22} \leq \max(N_1, M_2),$$

which is the outer bound (15).

Similarly, we obtain outer bounds (14), (16) and (17) from the MIMO $Z^*_{22}$, $Z^*_{12}$, $Z^*_{11}$ channel respectively. The general message set for the MIMO $Z^*_i$ ($i, j \in \{1, 2\}$) channel consists of message $W_{\hat{i}i}$, $W_{\hat{i}j}$, $W_{\hat{j}i}$, $W_{\hat{j}j}$, $W_{\widehat{0i}}$ and $W_{\widehat{0j}}$, where $\hat{i} = 3 - i$, $\hat{j} = 3 - j$.

VI. ACHIEVABILITY OF THE INNER BOUND

We have already described the precoding scheme and given the expressions for all the beamformers for all nine messages in the outline of proof of Lemma 11. In this section, we continue the proof and show that, using this scheme, the inner bound $D_{\text{in}} = \text{co}(D \cap Z^9_+)$ is achievable.

First, it is shown that all the desired messages are distinguishable, at their intended receivers; and then, we show that the region achievable is identical to $D_{\text{in}}$.


A. Independence requirements

The messages received by each receiver can be divided into two groups based on whether they are desired or undesired messages. The undesired messages are also potentially sources of interference. For receiver $R_1$, desired messages contain $W_{D1}=(W_{11}, W_{12}, W_{01}, W_1, W_2, W_0)$, and undesired messages contain $W_{U1}=(W_{21}, W_{22}, W_0)$. For receiver $R_2$, desired messages contain $W_{D2}=(W_{21}, W_{22}, W_{02}, W_1, W_2, W_0)$, and undesired messages contain $W_{U2}=(W_{11}, W_{12}, W_{01})$. Let $D_i$ denote the matrix of received vectors associated with the desired messages at receiver $i$, and $U_i$ denote the matrix of directions of the receive beamformers associated with the undesired messages at receiver $i$. We thus have

$$D_1 = egin{bmatrix} H_{11}V_{11} | H_{12}V_{12} | [H_{11} H_{12}]V_{01} | \cdots \end{bmatrix}$$

$$D_2 = egin{bmatrix} H_{21}V_{21} | H_{22}V_{22} | [H_{21} H_{22}]V_{02} | \cdots \end{bmatrix}$$

$$U_1 = egin{bmatrix} H_{11}V_{21} | H_{12}V_{22} | [H_{11} H_{12}]V_{02} \end{bmatrix}$$

$$U_2 = egin{bmatrix} H_{21}V_{11} | H_{22}V_{12} | [H_{21} H_{22}]V_{01} \end{bmatrix}.$$

For successful communication, each receiver needs to be able to decode all its own desired messages. In order to take the most advantage of channel resource, we allocate as much resource as possible to desired messages to minimize the resource consumed by undesired messages, i.e., by interference.

**Lemma 13.** If all the channels are generic and the following constraints are satisfied

$$d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{21} + d_{22} + d_{02} - Z_{21} - Z_{22} - A_1 - Z_{02} \leq N_1 \quad (40)$$

$$d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{11} + d_{12} + d_{01} - Z_{11} - Z_{12} - A_2 - Z_{01} \leq N_2 \quad (41)$$

$$d_1 + d_{11} + d_{21} \leq M_1 \quad (42)$$

$$d_2 + d_{12} + d_{22} \leq M_2 \quad (43)$$

$$d_1 + d_2 + d_0 + d_{01} + d_{02} + d_{11} + d_{21} + d_{12} + d_{22} \leq \min(M_1 + M_2, N_1 + N_2), \quad (44)$$

using the precoding scheme described in the outline of proof of Lemma 11 in Section IV, we have the following
**Independence results**

\[
\text{rank}(U_1) = (d_{21} - Z_{21}) + (d_{22} - Z_{22}) - A_1 + (d_{02} - Z_{02}) \quad (45)
\]

\[
\text{rank}(U_2) = (d_{11} - Z_{11}) + (d_{12} - Z_{12}) - A_2 + (d_{01} - Z_{01}) \quad (46)
\]

\[
\text{rank}(D_1) = d_{11} + d_{12} + d_{01} + d_1 + d_2 + d_0 \quad (47)
\]

\[
\text{rank}(D_2) = d_{21} + d_{22} + d_{02} + d_1 + d_2 + d_0, \quad (48)
\]

\[
\text{rank}([D_1 \ U_1]) = \text{rank}(D_1) + \text{rank}(U_1) \quad (49)
\]

\[
\text{rank}([D_2 \ U_2]) = \text{rank}(D_2) + \text{rank}(U_2). \quad (50)
\]

These independence results together ensure that all desired messages are distinguishable, and thus decodable, at their intended receivers.

Note that, constraints (40) and (41) imply that the total independent number of received beamformers at receiver \( R_i \) will be no greater than \( N_i \), the number of its antennas; constraints (42) and (43) imply that the number of independent streams sent out by transmitter \( T_i \) are restricted to be no greater than \( M_i \); constraint (44) implies that the number of all independent streams two transmitters sent out together will be no greater than their total number of antennas.

Regarding the independence result, equations (45) and (46) give the dimension of the subspace spanned by the received beamformers associated with the undesired messages, i.e., interference; equations (47) and (48) show that the directions of the received beamformers associated with the desired messages at each receiver are linearly independent of each other; equations (49) and (50) indicate that the subspace occupied by the desired messages is linearly independent of that of the interference.

**Proof:** We only give the proof of (45), (48) and (49), since the other three follow in the same way.

First consider equation (45). According to the expressions of beamformers provided in equations (32), (34) and (36), we have that \( V_{21}^Z \) and \( V_{22}^Z \) are drawn from the nullspace of \( H_{11} \) and \( H_{22} \), respectively, and \( V_{02}^Z \) is generated from the nullspace \( \mathcal{N}([H_{11} \ H_{12}]) \). Thus, they will all be zero-forced at receiver \( R_1 \), i.e.,

\[
H_{11}V_{21}^Z = 0
\]

\[
H_{12}V_{22}^Z = 0
\]

\[
[H_{11} \ H_{12}]V_{02}^Z = 0.
\]
Consequently, we have

\[
\text{rank}(H_{11}V_{21}) = \text{rank}(H_{11}[V_{21}^Z V_{21}^A]) = \text{rank}(H_{11}[V_{21}^A])
\]

\[
\text{rank}(H_{12}V_{22}) = \text{rank}(H_{12}[V_{22}^Z V_{22}^A]) = \text{rank}(H_{12}[V_{22}^A])
\]

\[
\text{rank}([H_{11} H_{12}]V_{02}) = \text{rank}([H_{11} H_{12}][V_{02}^Z V_{02}^R]) = \text{rank}([H_{11} H_{12}]V_{02}^R).
\]

Furthermore, from equation (33), we have

\[
[H_{11} H_{12}]
\begin{bmatrix}
V_{21}^A \\
V_{22}^A
\end{bmatrix} = 0
\Rightarrow
H_{11}V_{21}^A + H_{12}V_{22}^A = 0,
\]

which indicates that the subspace spanned by \(H_{11}V_{21}^A\) is aligned with the subspace spanned by \(H_{12}V_{22}^A\) at receiver \(R_1\). So, we have

\[
\text{rank}([H_{11}V_{21}^A H_{12}V_{22}^A]) = \text{rank}(H_{11}V_{21}^A) = \text{rank}(H_{12}V_{22}^A).
\]

One can observe that the nullspace of \(H_{11}\) and \(H_{12}\) is closely related to the nullspace of \([H_{11} H_{12}]\). In particular, since \(H_{11}\Phi(H_{11}) = 0\) and \(H_{12}\Phi(H_{12}) = 0\), we have that

\[
[H_{11} H_{12}]
\begin{bmatrix}
\Phi(H_{11}) \\
0
\end{bmatrix} = 0,
\]

\[
[H_{11} H_{12}]
\begin{bmatrix}
0 \\
\Phi(H_{12})
\end{bmatrix} = 0,
\]

which means that the column vectors of \(\Phi(H_{11})\) and \(\Phi(H_{12})\) are both in \(\mathcal{N}([H_{11} H_{12}])\). Since beamformer \(\begin{bmatrix} V_{21}^A \\ V_{22}^A \end{bmatrix}\) is obtained as random linear combinations of the null space basis vectors \(\Phi([H_{11} H_{12}])\), the probability that it belongs to the subspace spanned only by column vectors of \(\begin{bmatrix} \Phi(H_{11}) \\ 0 \end{bmatrix}\) and \(\begin{bmatrix} 0 \\ \Phi(H_{12}) \end{bmatrix}\) is zero. In other words, \([H_{11}V_{21}^A]\) and \([H_{12}V_{22}^A]\) will have full column rank almost surely, since none of the column vectors of \(V_{21}^A\) or \(V_{22}^A\) will be accidentally zero-forced at receiver \(R_1\). This is one benefit of using random linear combinations, as mentioned in Remark 4.

Beamformers \(V_{21}^R, V_{22}^R\) and \(V_{02}^R\) are generated randomly, they will all have full column rank almost surely. Their projections at the receivers will be linearly independent of each other unless they can’t be. According to constraint
The total number of beamformers transmitted in any channel is always no greater than the channel dimension, so there will be no loss of column ranks. As a result, we have

$$\text{rank}(U_1) = \text{rank}([H_{11} V_{21}^A \ H_{11} V_{21}^R \ H_{12} V_{22}^R \ [H_{11} \ H_{12}] V_{02}^R])$$

$$= A_1 + (d_{21} - Z_{21} - A_1) + (d_{22} - Z_{22} - A_1) + (d_{02} - Z_{02}),$$

which proves equation (45). Similarly, we have equation (46).

Next, consider equation (48). We have just shown that sending a symbol of $W_{21}^Z$ or $W_{22}^Z$ or $W_{02}^Z$, or a pair of symbols of $W_{21}^A$ and $W_{22}^A$ will consume 1 dimension of the subspace of $[H_{11} \ H_{12}]$. From the dimension of each part given in equations (23), (24) and (26), we have that

$$Z_{21} \leq (M_1 - N_1)^+$$

$$Z_{22} \leq (M_2 - N_1)^+$$

$$Z_{21} + Z_{22} + A_1 + Z_{02} \leq (M_1 + M_2 - N_1)^+,$$

which means the total numbers of beamformers do not exceed the dimensions of corresponding nullspaces. Since we generate all the beamformers as random linear combinations of the entire basis of the respective nullspaces, column vectors of $V_A = \begin{bmatrix} V_{21}^Z & 0 & V_{21}^A & V_{02}^Z \\ V_{22}^Z & V_{22}^A & V_{22}^R & V_{02}^R \end{bmatrix}$ will be linearly independent of each other almost surely. Meanwhile, they will also be linearly independent of the random column vectors of $V_B = \begin{bmatrix} V_{21}^R & 0 & V_{02}^R & V_{12}^R \\ V_{22}^R & V_{22}^A & V_{22}^R & V_{02}^R \end{bmatrix}$.

Since all of these beamformers in $V_A$ and $V_B$ are derived from $[H_{11} \ H_{12}]$ or generated randomly, they are independent of channel matrix $[H_{21} \ H_{22}]$. Since $H_{21}$ and $H_{22}$ are both full rank matrices with generic elements, the column vectors of $[H_{21} \ H_{22}] [V_A \ V_B]$ will be linearly dependent only if they have to be linearly dependent. Because we have the constraint (41), which indicates $d_{21} + d_{22} + d_{02} + d_1 + d_2 + d_0 \leq N_2$, $[H_{21} \ H_{22}] [V_A \ V_B]$ will have rank $d_{21} + d_{22} + d_{02} + d_1 + d_2 + d_0$ almost surely. So, we have equation (48). Similarly, we have equation (47).

Finally, consider equation (49). Since the beamformers associated with $D_1$ are independent of the beamformers associated with $U_1$, the subspace spanned by $D_1$ and the subspace spanned by $U_1$ will be linearly dependent only if they have to be linearly dependent. According to constraint (40), $\text{rank}(D_1) + \text{rank}(U_1) \leq N_1$. Consequently, $\text{rank}([D_1 \ U_1])$ will be equal to $\text{rank}(D_1) + \text{rank}(U_1)$ almost surely. So we have equation (49). Similarly, we have equation (50).

In Lemma 13, we show that if inequalities (40)-(44) are satisfied, all desired messages will be distinguishable at their respectively intended receivers. In other words, DoF tuples that satisfy (40)-(44) are achievable. In the next section, we explicitly characterize this achievable DoF region.
B. The achievability of inner bound

According to the analysis in Lemma [13] of the precoding scheme described in Section IV, we have shown the achievability of the integer-valued points in $\mathbb{D}_{eq}$, which is defined as

$$\mathbb{D}_{eq} \triangleq \left\{ (d_{11}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0) \in \mathbb{R}_{+}^E : ight.$$ 

$$d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{21} + d_{22} + d_0$$ 

$$- Z_{21} - Z_{22} - A_1 - Z_{02} \leq N_1$$  

$$d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{11} + d_{12} + d_0$$ 

$$- Z_{11} - Z_{12} - A_2 - Z_{01} \leq N_2$$  

$$d_1 + d_{11} + d_{21} \leq M_1$$  

$$d_2 + d_{12} + d_{22} \leq M_2$$  

$$d_1 + d_2 + d_0 + d_{01} + d_{02} + d_{11} + d_{12} + d_{22}$$ 

$$\leq \min(M_1 + M_2, N_1 + N_2)$$

are satisfied for some

$$\{ (Z_{11}, Z_{12}, Z_{21}, Z_{22}, A_1, A_2, Z_{01}, Z_{02}) \in \mathbb{R}_{+}^A :$$ 

$$Z_{21} + Z_{22} + A_1 + Z_{02} \leq (M_1 + M_2 - N_1)^+$$  

$$Z_{21} \leq (M_1 - N_1)^+$$  

$$Z_{22} \leq (M_2 - N_1)^+$$  

$$Z_{21} + A_1 \leq d_{21}$$  

$$Z_{22} + A_1 \leq d_{22}$$  

$$Z_{02} \leq d_{02}$$  

$$Z_{11} + Z_{12} + A_2 + Z_{01} \leq (M_1 + M_2 - N_2)^+$$  

$$Z_{11} \leq (M_1 - N_2)^+$$  

$$Z_{12} \leq (M_2 - N_2)^+$$  

$$Z_{11} + A_2 \leq d_{11}$$  

$$Z_{12} + A_2 \leq d_{12}$$  

$$Z_{01} \leq d_{01} \}$$

where set $A$ contains all the auxiliary variables. Inequalities (59)-(70) on the auxiliary variables are obtained from equations (23)-(28).

To prove the inner bound, we need to find the connection between $\mathbb{D}$ and $\mathbb{D}_{eq}$. Interestingly, it is shown that...
these two regions are identical. However, note that $D_{eq}$ is obtained from a 17-dimensional polyhedron in $\mathbb{R}_+^E \times \mathbb{R}_+^A$ defined via 17 inequalities which include eight auxiliary variables. The problem is to project this polyhedron onto the nine dimensional positive orthant $\mathbb{R}_+^E$. The standard technique to perform this projection is via the Fourier-Motzkin Elimination wherein the auxiliary variables are eliminated one at a time but by creating a large number of inequalities of $O(m^2)$ starting with $m$ inequalities and then eliminating redundant inequalities $[2]$. Such a technique is clearly infeasible for the size of the problem at hand here. Instead, we use the special structure of the inequalities that define $D_{eq}$ to prove that it is equivalent to $D$ in the following lemma.

**Lemma 14.** The 9-dimensional region $D$ is equal to $D_{eq}$.

**Proof:** First show any vector in $D$ is also in $D_{eq}$, and then show any vector in $D_{eq}$ is also in $D$. The detailed proof is given in Appendix A.

Thus, we prove that the inner bound $D_{in} = \text{co} \left( D \cap \mathbb{Z}_+^9 \right)$ is achievable.

**C. No interference alignment is needed for $W_{01}$ and $W_{02}$**

In our precoding scheme, interference alignment is used only among the four private messages. Only zero-forcing is used for the cognitive and common messages $W_{01}$ and $W_{02}$. In this section, we demonstrate why.

Consider $W_{02}$, for instance. If $M_1 + M_2 > N_1$, transmit zero-forcing of $W_{02}$ is possible. We can choose beamformers for $W_{02}$ from the null space $\mathcal{N}( [H_{11} \ H_{12}] )$. It is worth noting that $\mathcal{N}( [H_{11} \ H_{12}] )$ has already been used to generate $V_{Z_{21}}^Z$, $V_{Z_{22}}^Z$ and $(V_{A_{21}}^A, V_{A_{22}}^A)$ pairs. To transmit a data symbol in $W_{02}$, we cannot choose a vector in the span of the column vectors in $\begin{bmatrix} V_{Z_{21}}^Z \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ V_{Z_{22}}^Z \end{bmatrix}$ and $\begin{bmatrix} V_{A_{21}}^A \\ V_{A_{22}}^A \end{bmatrix}$, otherwise the data symbol of $W_{02}$ will not be distinguishable with part of $W_{Z_{21}}^Z$, $W_{Z_{22}}^Z$ and $(W_{A_{21}}^A, W_{A_{22}}^A)$ at receiver $R_2$. As a result, $V_{Z_{02}}^Z$ can be only chosen from
the unoccupied subspace of $\mathcal{N}([H_{11} H_{12}])$. This is also why the dimension available for transmit zero-forcing of $W_{02}^A$ is at most $M_1 + M_2 - N_1 - d_{21}^Z - d_{22}^Z - A_1$ in equation (26).

Next, consider the possibility of aligning the beamformer, denoted as $V_{02}^A$, of data symbol in $W_{02}$ with the existing interference due to $W_{21}^R$, $W_{22}^R$ or $(W_{21}^A, W_{22}^A)$. Take $(W_{21}^A, W_{22}^A)$ for example. If vector $[H_{11} H_{12}]v_{02}^A$ aligns with $(H_{11}v_{21}^A, H_{12}v_{22}^A)$, where $v_{21}^A$ and $v_{22}^A$ are some column vectors lie in $\text{span}(V_{21}^A)$ and $\text{span}(V_{22}^A)$, respectively, it is easy to see that

$$v_{02}^A = \begin{bmatrix} \alpha v_{21}^A \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta v_{22}^A \end{bmatrix} + \gamma v_0$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $v_0$ is a column vector in the null space $\mathcal{N}([H_{11} H_{12}])$. To make $W_{02}^A$ distinguishable at receiver $R_2$, $v_0$ must be linearly independent of the already used subspace of $\mathcal{N}([H_{11} H_{12}])$. Hence, if we transmit a data symbol in $W_{02}^A$ by having its direction lie in the subspace spanned by the directions associated with data symbols in $W_{21}^A$ and $W_{22}^A$, we consume one dimension in $\mathcal{N}([H_{11} H_{12}])$. A similar result holds in attempting to align with existing interference $H_{11}V_{21}^R$ or $H_{12}V_{22}^R$ at receiver $R_1$.

In summary, for each $W_{02}$ stream, both transmit zero-forcing and interference alignment consume one more available dimension of $\mathcal{N}([H_{11} H_{12}])$. In other words, either strategy costs the same in terms of using the remaining subspace (if any) of $\mathcal{N}([H_{11} H_{12}])$. As a practical matter, one might choose transmit zero-forcing since it easier to compute the corresponding beamformer.

### VII. Special Cases

In this section, we specify the DoF regions for small special cases of Theorem 3.
A. Known Results as Special Cases

Case 1. IC (Figure 3a)

There are only two messages in the interference channel, i.e., $W_{11}, W_{22}$. By eliminating all absent variables in $D$, we get the degrees of freedom region for two-user interference channel as

$$D_{IC} = \left\{ (d_{11}, d_{22}) \in \mathbb{R}_+^2 : \right. $$
$$d_{11} \leq \min(M_1, N_1), \quad d_{22} \leq \min(M_2, N_2), $$
$$d_{11} + d_{22} \leq \min(\max(M_2, N_1), \max(M_1, N_2)) \left. \right\}$$

Hence, Theorem 3 reduces to the well-known result in [3]. We can follow the precoding scheme shown in Section IV and skip the parts that are not applicable. In this case, we only need to consider $[V_{11}^Z V_{11}^R]$ and $[V_{22}^Z V_{22}^R]$. Only transmit zero-forcing is possible here. Interference alignment is not applicable since there is only one source of interference at each receiver.

Case 2. IC-CM (Figure 3b)

Specializing Theorem 3 to the case where only messages $W_{11}, W_{22}$ and $W_0$ are present as depicted in Fig. 3b (and eliminating absent variables), we have

$$D_{IC-CM} = \left\{ (d_{11}, d_{22}, d_{01}) \in \mathbb{R}_+^3 : \right. $$
$$d_{11} \leq M_1, \quad d_{22} \leq M_2, $$
$$d_0 + d_{11} \leq N_1, \quad d_0 + d_{22} \leq N_2, $$
$$d_0 + d_{11} + d_{22} \leq \min(M_1 + M_2, \max(M_2, N_1), \max(M_1, N_2)) \left. \right\}.$$ 

In this case, $W_{11}$ and $W_{22}$ are transmitted using the same scheme as in IC along with random beamforming for $W_0$ in the remaining channel dimensions that are still available.

Case 3. Cognitive IC (Figure 3c)

Theorem 3 when specialized to the degraded message set depicted in Fig. 3c, results in the DoF region of the Cognitive IC, which is

$$D_{co-IC} = \left\{ (d_{01}, d_{22}) \in \mathbb{R}_+^2 : \right. $$
$$d_{01} \leq N_1, \quad d_{22} \leq \min(M_2, N_2), $$
$$d_{01} + d_{22} \leq \min(M_1 + M_2, \max(M_2, N_1)) \left. \right\}.$$ 

This DoF region matches with the result of [17] in the same cognitive message sharing scenario. In this case, we only need zero-forcing and random beamforming to achieve any vertex of the DoF region. The dimensions of symbols of
$W_0$ and $W_2$ that are transmitted using zero-forcing are $\min (d_{01}, (M_1 + M_2 - N_2)^+) \text{ and } \min (d_{22}, (M_2 - N_1)^+)$, respectively.

B. Examples of New Results

Case 4. Generalized Cognitive IC (Figure 4a)

Consider the generalized cognitive IC, in which there are three messages $W_{21}$, $W_{01}$ and $W_{22}$. In this model, the two transmitters send one message each, i.e., $W_{21}$ and $W_{22}$, respectively, to Receiver 2 along with another message, i.e., $W_{01}$, cooperatively to the Receiver 1. Specializing Theorem 3 to this model, we have the following DoF region result

$$D_{g-co-IC} = \{ (d_{21}, d_{22}, d_{01}) \in \mathbb{R}^3_+ :$$

- $d_{01} \leq N_1, \quad d_{21} \leq M_1, \quad d_{22} \leq M_2,$
- $d_{21} + d_{22} \leq N_2,$
- $d_{01} + d_{21} \leq \max(M_1, N_1),$  
- $d_{01} + d_{22} \leq \max(M_2, N_1),$
- $d_{01} + d_{21} + d_{22} \leq M_1 + M_2 \}.$

Both zero-forcing and interference alignment, if possible, are used to mitigate the impact of two private messages $W_{21}$ and $W_{22}$ on their common unintended receiver, i.e., receiver $R_1$; while zero-forcing, if possible, is used to reduce the interference received by receiver $R_2$ due to message $W_{01}$.

Case 5. Broadcast Channel with Partially Cognitive Relay (BC-PCR) (Figure 4b)

Consider the model depicted in Figure 4b. Transmitter 1 broadcasts two private messages $W_{11}$ and $W_{21}$ to two receivers, respectively, while it simultaneously cooperates with transmitter 2 (the PCR) to send another message $W_{01}$ to receiver $R_1$. From Theorem 3, we can deduce the DoF region of BC-PCR as

$$D_{BC-PCR} = \{ (d_{21}, d_{11}, d_{01}) \in \mathbb{R}^3_+ :$$

- $d_{21} \leq N_2, \quad d_{01} + d_{11} \leq N_1, \quad d_{11} + d_{21} \leq M_1,$
- $d_{01} + d_{11} + d_{21} \leq \min (M_1 + M_2, \max(M_1, N_1)) \}.$

From the analysis in Section VI-C.B.(2), we know that using zero-forcing, if possible, is enough for transmitting message $W_{01}$. There is no need to additionally attempt to align the symbols of $W_{11}$ and $W_{01}$ together at receiver $R_2$, since interference alignment and zero-forcing costs the same in terms of using the null space of $[H_{21} H_{22}]$.

VIII. Conclusion

The degrees of freedom region for the nine-message MIMO $2 \times 2$ interference network is established. Each of the nine messages is uniquely identified based on the transmitter(s) it is known to and the receiver(s) at which it is
desired and therefore include broadcast/multiple-access/multicast/cognitive/common messages. The DoF region for a setting that involves any subset of the nine messages can thus be derived as a special case. In particular, the DoF region of the MIMO X channel, a problem that remained open despite previous studies, is completely settled.

The achievability scheme uses (a) transmit zero-forcing, a well-known technique known to be sufficient for the MIMO IC [2], interference alignment and symbol extensions the necessity (but not sufficiency) for which was discovered in the context of the constant-coefficient MIMO X channel in [7], and finally, asymmetric complex signaling which was discovered in the context of the constant-coefficient SISO X channel in [8], but whose benefit (necessity or sufficiency) in the MIMO (i.e., non-SISO) X channel remained unclear despite [7], [8]. The achievability scheme in this paper combines the principles of transmit zero-forcing, interference alignment, symbol extensions and ACS in a novel way that allows not only the complete characterization of the DoF of the four-message MIMO X channel – thereby proving that they are both necessary and sufficient in general for the constant-coefficient MIMO X channel – but also the precise DoF region of the much more general nine-message, constant-coefficient MIMO 2×2 network considered in this paper.

In considering some interesting subsets of the general message set (including the 9-message case) for the 2×2 MIMO interference network, and making simplifying assumptions on the channel models if needed, future work could include the discovery of new encoding and decoding principles inspired by the goal of characterizing information theoretic metrics that are finer than the degrees of freedom, such as, for instance, the generalized degrees of freedom, as was done for the two-user MIMO interference channel in [4]. There is also the potential for the discovery of hitherto unknown encoding schemes tailored for various models of channel uncertainty, as has been done for the MIMO interference and the MIMO X channels in [18], [19] under delayed CSIT.

APPENDIX A

EQUIVALENCE OF D AND Deq

Proof: To make the expressions more concise, we define

\[ d_{\text{sum},1} = d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} \]
\[ d_{\text{sum},2} = d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22}. \]

Let \( \vec{d} = (d_{11}, d_{21}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0) \). First, prove if \( \vec{d} \in \mathcal{D}_{\text{eq}} \), then \( \vec{d} \in \mathcal{D} \).

Since \( \vec{d} \in \mathcal{D}_{\text{eq}} \), there exists at least a tuple \((Z_{11}, Z_{12}, Z_{21}, Z_{22}, A_1, A_2, Z_{01}, Z_{02}) \in \mathbb{R}^A_+ \) which satisfies the conditions in (59)-(70), such that inequalities (54)-(58) are all satisfied. Then, from inequalities (64), (60), (63) and (64), we get

\[ d_{\text{sum},1} + d_{21} + d_{22} + d_{02} \leq N_1 + (M_1 - N_1)^+ + d_{22} + d_{02}. \]

Hence,

\[ d_{\text{sum},1} + d_{21} \leq N_1 + (M_1 - N_1)^+ = \max(M_1, N_1), \]
which is inequality (14) in the definition of $D$. Similarly, it can be shown that
\[ d_{\text{sum},1} + d_{22} \leq N_1 + (M_2 - N_1)^+ = \max(M_2, N_1) \]
\[ d_{\text{sum},2} + d_{11} \leq N_2 + (M_1 - N_2)^+ = \max(M_1, N_2) \]
\[ d_{\text{sum},2} + d_{12} \leq N_2 + (M_2 - N_2)^+ = \max(M_2, N_2) \]
which are inequalities (15)-(17) in the definition of $D$.

Again, from inequalities (54), (62), (63) and (64), we get
\[ d_{\text{sum},1} + d_{21} + d_{22} + d_{02} \leq N_1 + d_{21} + d_{22} + d_{02} - A_1, \]
hence,
\[ d_{\text{sum},1} \leq N_1 - A_1 \leq N_1, \]
which is inequality (18) in the definition of $D$. Similarly, we have
\[ d_{\text{sum},2} \leq N_2 - A_2 \leq N_2, \]
which is inequality (19) in the definition of $D$.

Furthermore, inequalities (20)-(22) hold for $\vec{d}$ since they are also contained in the definition of $D_{eq}$. Consequently, all inequalities in the definition of $D$ are satisfied and we have that $\vec{d}$ also belongs to $D$. Thus,
\[ D_{eq} \subseteq D. \] (71)

Next, we prove that if $\vec{d} \in D$, then $\vec{d} \in D_{eq}$. For each $\vec{d} \in D$, we choose the value for $(Z_{11}, Z_{12}, Z_{21}, Z_{22}, A_1, A_2, Z_{01}, Z_{02})$ according to equations (23)-(31). It is straightforward to verify the above choices satisfy the constraints (59)-(70). Also, by exhaustively enumerating all possible relations among $M_1, M_2, N_1, N_2, d$ and removing the $(\cdot)^+$ and $\min(\cdot, \cdot)$ operators, and substituting the values of the 8 auxiliary variables, we can verify that if inequalities (14)-(19) hold, then inequalities (54) and (55) also hold. Inequalities (56)-(58) automatically hold since they are contained in the definition of $D$.

and hence $\vec{d}$ also belongs to $D_{eq}$. Thus
\[ D \subseteq D_{eq}. \] (72)
Together with (71), we have $D = D_{eq}$.

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