Second-order expansions for maxima of dynamic
bivariate normal copulas

*Rui Wang    *Xin Liao*    *Zuoxiang Peng

*a*School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China

*b*Business School, University of Shanghai for Science and Technology, Shanghai, 200093, China

Abstract. In this paper, we establish the second-order distributional expansions of normalized maxima of \( n \) independent observations, where the \( i \)th observation follows from a normal copula with its correlation coefficient being a monotone continuous function. These expansions can be used to deduce the convergence rates of distributions of normalized maxima to their limits.

Keywords. Dynamic bivariate normal copula; Maximum; Second-order expansion.

1 Introduction

Let \( \{(X_i, Y_i) \}, 1 \leq i \leq n, n \geq 1 \) denote independent and identically distributed bivariate random vectors with distribution function \( F(x, y) \) and continuous marginal distributions \( F_1 \) and \( F_2 \). The copula of \( F \) is given by \( F(F_1^-(x), F_2^-(y)) \), where \( F_i^- \) denotes the inverse function of \( F_i, i = 1, 2 \). We say that the copula of \( F \) is a normal copula \( C(x, y; \rho) \), if the density of \( C(x, y; \rho) \) is given by

\[
c(x, y; \rho) = \frac{1}{\sqrt{1 - \rho^2}} \exp \left( \frac{2\rho \Phi^-(x) \Phi^-(y) - \rho^2 (\Phi^-(x))^2 - \rho^2 (\Phi^-(y))^2}{2(1 - \rho^2)} \right), \tag{1.1}
\]

where \( \rho \in (-1, 1) \) and \( \Phi(x) \) is the standard normal distribution function.

Due to its easy to simulation and some attractive properties, the normal copula has received

*Corresponding author. Email address: liaoxin2010@163.com*
many applications. Taylor et al. (2015) proposed causal quantities to evaluate surrogacy based on normal copula; Naldi and D’Acquisto (2008) considered the economic consequences of failures as a figure of merit of reliable communications networks by using normal copula, a few mentioned here. But the biggest weakness of normal copula is its tail asymptotic independence, see Sibuya (1960) and Embrechts et al. (2002). The tail asymptotic independence of normal copula may deduce the under-estimation of extreme probabilities in risk management. To overcome the drawback, Frick and Reiss (2013) showed that

\[
\lim_{n \to \infty} \mathbb{P}\left( n \left( \max_{1 \leq i \leq n} F_1(X_i) - 1 \right) \leq x, n \left( \max_{1 \leq i \leq n} F_2(Y_i) - 1 \right) \leq y \right) = \exp \left( \Phi \left( \sqrt{\lambda + \frac{\log x}{2\sqrt{\lambda}}} \right)x + \Phi \left( \sqrt{\lambda + \frac{\log y}{2\sqrt{\lambda}}} \right)y \right) \tag{1.2}
\]

for \( x < 0 \) and \( y < 0 \), if the correlation coefficient \( \rho = \rho_n \) satisfies the following so-called Hüsler-Reiss condition

\[
(1 - \rho_n) \log n \to \lambda \in [0, \infty] \tag{1.3}
\]
as \( n \to \infty \), see Hüsler and Reiss (1989). Note that (1.2) is a copula version of the limit in Hüsler and Reiss (1989) for the normalized maxima of bivariate normal triangular arrays with correlation coefficients satisfying (1.3). Further, Liao et al. (2016) extended the work of Frick and Reiss (2013) by assuming

\[
\rho_{ni} = 1 - \frac{m(i/n)}{\log n} \tag{1.4}
\]
for some nonnegative function \( m(x) \), which allows \( \rho_{ni} \) depending on both \( i \) and \( n \). Under the condition (1.4), Liao et al. (2016) proved that

\[
\lim_{n \to \infty} \mathbb{P}\left( n \left( \max_{1 \leq i \leq n} F_1(X_i) - 1 \right) \leq x, n \left( \max_{1 \leq i \leq n} F_2(Y_i) - 1 \right) \leq y \right) = G(x, y), \tag{1.5}
\]
with

\[ G(x, y) = \exp \left( x \int_0^1 \Phi \left( \sqrt{m(s)} + \frac{\log \frac{x}{y}}{2\sqrt{m(s)}} \right) ds + y \int_0^1 \Phi \left( \sqrt{m(s)} + \frac{\log \frac{y}{x}}{2\sqrt{m(s)}} \right) ds \right) \]

if \( m(s) \) defined on \([0, 1]\) is continuous and positive, and \( G(x, y) = \exp(x+y) \) as \( \lim_{n \to \infty} \min_{1 \leq i \leq n} m(i/n) = \infty \), and \( G(x, y) = \exp(\min(x,y)) \) if \( \lim_{n \to \infty} \max_{1 \leq i \leq n} m(i/n) = 0 \). The above normal copulas with \( \rho_{ni} \) depending on both \( i \) and \( n \) or only sample size \( n \), can be called dynamic copulas. Recently, various dynamic copulas are receiving more attention in modeling financial time series; see, e.g., Salvatierra and Patton (2015), Wang et al. (2015), Chu (2015).

In this paper, we are interested in the second-order distributional expansions of normalized maxima \((n \max_{1 \leq i \leq n} F_1(X_i) - 1) \leq x, n \max_{1 \leq i \leq n} F_2(Y_i) - 1) \leq y\). For independent bivariate normal triangular arrays satisfying (1.3), Hashorva et al. (2016) imposed second-order Hüsler-Reiss condition, and derived the second-order distributional expansions of maxima. Under the second-order Hüsler-Reiss condition, Liao and Peng (2014, 2015) obtained the uniform convergence rates of maxima, and the second-order expansions of joint distributions of maxima and minima of bivariate normal triangular arrays. For the independent and non-identically distributed bivariate normal triangular arrays satisfying (1.4), the second-order distributional expansions of maxima are given by Liao and Peng (2016), and the second-order expansions of joint distributions of maxima and minima are derived by Lu and Peng (2017). To the best of our knowledge, there are no studies on the second-order expansions of distributions of \((n \max_{1 \leq i \leq n} F_1(X_i) - 1) \leq x, n \max_{1 \leq i \leq n} F_2(Y_i) - 1) \leq y\). The aim of this paper is to fill this gap.

The rest of this paper is organized as follows. In Section 2 we establish the second-order distributional expansions of \((n \max_{1 \leq i \leq n} F_1(X_i) - 1) \leq x, n \max_{1 \leq i \leq n} F_2(Y_i) - 1) \leq y\) by considering three cases: \( m(s) \) is continuous positive function on \([0,1]\), \( \lim_{n \to \infty} \min_{1 \leq i \leq n} m(i/n) = \infty \), and \( \lim_{n \to \infty} \max_{1 \leq i \leq n} m(i/n) = 0 \). Some examples are also given in Section 2. All proofs are deferred in Section 3.
2 Main Results

In this section, the second-order expansions of

\[ G_n(x, y) = \mathbb{P} \left( n \left( \max_{1 \leq i \leq n} F_1(X_i) - 1 \right) \leq x, n \left( \max_{1 \leq i \leq n} F_2(Y_i) - 1 \right) \leq y \right) \]

are established with \( m(s) \) given by (1.4) satisfying some regular conditions. Note that Liao et al. (2016) derived the convergence of \( G_n(x, y) \) for three different cases. In order to derive the second-order expansions of \( G_n(x, y) \), additional conditions are needed in each case. The following Theorem is about the second-order distributional expansion of \( G_n(x, y) \) as \( m(s) \) defined on \([0, 1]\) is monotone, continuous and positive.

**Theorem 2.1.** Assume that (1.4) holds with \( m(s) \) being monotone and continuous on \([0, 1]\). For any \( x < 0 \) and \( y < 0 \), we have

\[
\lim_{n \to \infty} \frac{\log n}{\log \log n} \left[ G_n(x, y) - G(x, y) \right] = 1
\]

\[
\frac{1}{2\sqrt{2\pi}} G(x, y) \int_0^1 \sqrt{m(s)} \exp \left( -\frac{m(s) - \log(xy) + \frac{(\log(x/y))^2}{4m(s)}}{2} \right) ds.
\]

(2.1)

**Example 2.1.** Assume that (1.4) holds with \( m(x) = x + 1, 0 \leq x \leq 1 \). It follows from Theorem 2.1 that

\[
G_n(x, y) = G(x, y) + \frac{\log \log n}{\log n} (-x)G(x, y) \int_1^\sqrt{2} s^2 \exp \left( -\frac{s + \frac{\log(x/y)}{2s}}{2} \right) ds(1 + o(1))
\]

for large \( n \).

For the case of \( \lim_{n \to \infty} \min_{1 \leq i \leq n} m(i/n) = \infty \), with addition condition Theorem 2.2 shows the second-order expansion of \( G_n(x, y) \) as follows.

**Theorem 2.2.** Assume that (1.4) holds with \( \lim_{n \to \infty} \min_{1 \leq i \leq n} m(i/n) = \infty \) and \( \lim_{n \to \infty} \frac{\max_{1 \leq i \leq n} m(i/n)}{\log \log n} = 0 \), then for any \( x < 0 \) and \( y < 0 \) we have

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^n \frac{\exp(-m(i/n)/2)}{\sqrt{m(i/n)}} \right)^{-1} \left[ G_n(x, y) - e^{x+y} \right] = \sqrt{\frac{2}{\pi}} (xy)^{\frac{1}{2}} e^{x+y}.
\]

(2.2)
Example 2.2. Let

\[
m(i/n) = \begin{cases} 
4 \log \log \log \frac{n}{i}, & i \in [1, n^{\frac{1}{2}}], \\
2 \left( \log \log \log n \right), & \text{otherwise}.
\end{cases}
\]

One can check that \(m(i/n)\) given above satisfies the conditions of Theorem 2.2, so by Theorem 2.2, the second-order expansion of \(G_n(x, y)\) is given by

\[
G_n(x, y) = e^{x+y} + \frac{\sqrt{2}(xy)^{\frac{1}{2}} e^{x+y}}{(\log \log n)^{\frac{1}{2}}} \left(1 + o(1)\right)
\]

for large \(n\).

For the last case of \(\lim_{n \to \infty} \max_{1 \leq i \leq n} m(i/n) = 0\), with additional condition we have the following second-order expansion of \(G_n(x, y)\).

**Theorem 2.3.** Assume that (1.4) holds with \(\lim_{n \to \infty} (\log \log n) \min_{1 \leq i \leq n} m(i/n) = \infty\) and \(\lim_{n \to \infty} \max_{1 \leq i \leq n} m(i/n) = 0\). Then for \(x < 0, y < 0\),

(i) if \(x \neq y\), we have

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} (m(i/n))^2 \exp \left( \frac{-\left( \log \min_{\max} (x, y) \right)^{\frac{1}{2}}}{8m(i/n)} \right) \right)^{-1} \left[ G_n(x, y) - e^{\min(x, y)} \right] = -2 \frac{x}{\pi} e^x.
\]

(ii) if \(x = y\), we have

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \sqrt{m(i/n)} \right)^{-1} \left[ G_n(x, y) - e^x \right] = -2 \frac{x}{\pi} e^x.
\]

Example 2.3. One can check that

\[
m(i/n) = \begin{cases} 
\frac{\log \log i}{\log \log \log n}, & i \in [1, \log n], \\
\frac{1}{\log \log \log n}, & \text{otherwise}.
\end{cases}
\]
satisfies the conditions of Theorem 2.3, and the second-order expansion of \( G_n(x, y) \) is given by

\[
G_n(x, y) = \begin{cases} 
  e^{\min(x,y)} - \frac{1}{(\log \log n)^2 (\log \log n)^{\frac{1}{2}} \sqrt{2\pi} \log \frac{\min(x,y)}{\max(x,y)}} \cdot \frac{8(xy)^{\frac{1}{2}} \min(x,y)}{\sqrt{2\pi} \log \frac{\min(x,y)}{\max(x,y)}} (1 + o(1)), & x \neq y, \\
  e^x - \frac{2xe^x}{\pi (\log \log n)^{\frac{1}{2}}} (1 + o(1)), & x = y
\end{cases}
\]

for large \( n \).

**Remark 2.1.** For different cases, Theorems 2.1-2.3 show that the convergence rates of \( G_n(x, y) \) to \( G(x, y) \) are given as follows:

(i) if \( m(s) \) is monotone and continuous, Theorem 2.1 shows that the convergence rate is proportional to \( \frac{\log \log n}{\log n} \).

(ii) if \( m(s) \) satisfies \( \lim_{n \to \infty} \min_{1 \leq i \leq n} m(i/n) = \infty \) and \( \lim_{n \to \infty} \frac{\max_{1 \leq i \leq n} m(i/n)}{\log \log n} = 0 \), Theorem 2.2 shows that the convergence rate is the same order of \( \frac{1}{n} \sum_{i=1}^{n} \exp(-m(i/n)/2) \sqrt{m(i/n)} \).

(iii) if \( m(s) \) satisfies \( \lim_{n \to \infty} \max_{1 \leq i \leq n} m(i/n) = 0 \) and \( \lim_{n \to \infty} (\log \log n) \min_{1 \leq i \leq n} m(i/n) = \infty \), Theorem 2.3 shows that the convergence rate of \( G_n(x, y) \) to its limit \( G(x, y) \) is the same order of \( \frac{1}{n} \sum_{i=1}^{n} \sqrt{m(i/n)} \) for \( x \neq y \), and the same order of \( \frac{1}{n} \sum_{i=1}^{n} \sqrt{m(i/n)} \) for \( x = y \).

### 3 Proofs

The aim of this section is to prove our main results. In order to prove Theorem 2.1, we need the following key lemma, which shows the convergence rate of \( \frac{1}{n} \sum_{i=1}^{n} \int_{y}^{0} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt \).

**Lemma 3.1.** Under the conditions of Theorem 2.1 for \( x < 0 \) and \( y < 0 \) we have

\[
\int_{0}^{1} \int_{y}^{0} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dt ds - \frac{1}{n} \sum_{i=1}^{n} \int_{y}^{0} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt = O \left( \frac{1}{n} \right). \tag{3.1}
\]

**Proof of Lemma 3.1.** Without loss of generality, assume that \( m(s) \) is increasing.
For $x \leq y$, noting that $\int_y^0 \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dt$ is increasing about $s$, we have

$$\int_0^1 \int_y^0 \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds = \sum_{i=1}^n \int_{i-1}^y \int_0^0 \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds$$

$$< \sum_{i=1}^n \int_{i-1}^0 \int_y^0 \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dtds$$

$$= \frac{1}{n} \sum_{i=1}^n \int_y^0 \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt$$  \hspace{1cm} (3.2)

and

$$\int_0^1 \int_y^0 \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds = \sum_{i=0}^{n-1} \int_{i/n}^{i+1/n} \int_y^0 \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds$$

$$< \sum_{i=0}^{n-1} \int_{i/n}^{i+1/n} \int_y^0 \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dtds$$

$$= \frac{1}{n} \sum_{i=1}^n \int_y^0 \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt + O \left( \frac{1}{n} \right),$$

(3.3)

so, (3.2) and (3.3) implies that (3.1) holds for $x \leq y$.

For $x > y$, let

$$\int_0^1 \int_y^0 \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds - \int_y^0 \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt = A_1(n) + A_2(n)$$

with

$$A_1(n) = \int_0^1 \int_x^0 \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds - \frac{1}{n} \sum_{i=1}^n \int_x^0 \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt$$

and

$$A_2(n) = \int_0^1 \int_y^x \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds - \frac{1}{n} \sum_{i=1}^n \int_y^x \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt.  \hspace{1cm} (3.4)$$

By arguments similar to (3.2) and (3.3), we can get $A_1(n) = O(1/n)$. The rest is to show that $A_2(n) = O(1/n)$.
First note that the function \( f(z) = \sqrt{z} + \frac{\log(y/x)}{2\sqrt{z}} \) is increasing for \( z > \frac{\log(y/x)}{2} \) and decreasing as \( z < \frac{\log(y/x)}{2} \). So, we need to deal with (3.4) through the following three cases:

(i). \( y \leq xe^{2m(1)} \); (ii). \( xe^{2m(1)} < y < xe^{2m(0)} \); (iii). \( xe^{2m(0)} \leq y \leq x \).

Arguments similar to that of (3.2) and (3.3), we can show that \( A_2(n) = O(1/n) \) for case (iii). Details are omitted here. So, there are only cases (i) and (ii) left as we estimate the bound of (3.4).

For case (i), note that \( y \leq xe^{m(s)} \leq x < 0 \) for \( s \in [0, 1] \) since \( m(s) \) is increasing and \( \Phi\left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \) is decreasing respect to \( s \) for \( t \in [y, xe^{m(s)}] \). Hence,

\[
\int_0^1 \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds = \sum_{i=1}^{n} \int_{i-1/n}^{i/n} \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds \\
> \sum_{i=1}^{n} \int_{i-1/n}^{i/n} \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dtds \\
= \frac{1}{n} \sum_{i=1}^{n} \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt
\]

and

\[
\int_0^1 \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds = \sum_{i=0}^{n-1} \int_{i/n}^{i+1/n} \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds \\
< \sum_{i=0}^{n-1} \int_{i/n}^{i+1/n} \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dtds \\
< \sum_{i=0}^{n-1} \int_{i/n}^{i+1/n} \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dtds \\
= \frac{1}{n} \sum_{i=1}^{n} \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt + O \left( \frac{1}{n} \right).
\]
Combining (3.5) with (3.6), we have

\[ \int_0^1 \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds - \frac{1}{n} \sum_{i=1}^n \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \, dt = O \left( \frac{1}{n} \right). \]

(3.7)

Similarly, noting that \( \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \) is increasing respect to \( s \) for \( t \in (xe^{2m(s)}, x] \), we have

\[ \int_0^1 \int_{xe^{2m(s)}}^x \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds < \frac{1}{n} \sum_{i=1}^n \int_{xe^{2m(i/n)}}^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \, dt \]

and

\[ \int_0^1 \int_{xe^{2m(s)}}^x \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds > \frac{1}{n} \sum_{i=1}^n \int_{xe^{2m(i/n)}}^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \, dt + O \left( \frac{1}{n} \right), \]

implying

\[ \int_0^1 \int_{xe^{2m(s)}}^x \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds - \frac{1}{n} \sum_{i=1}^n \int_{xe^{2m(i/n)}}^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \, dt = O \left( \frac{1}{n} \right). \]

(3.8)

It follows from (3.7) and (3.8) that (3.4) holds for case (i).

For case (ii), i.e. \( xe^{2m(1)} < y < xe^{2m(0)} \), there exists \( s_0 \in (0,1) \) such that \( y = xe^{2m(s_0)} \) since \( m(s) \) is increasing and continuous. We split the following integral into two parts:

\[ \int_0^1 \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds = \int_0^{s_0} \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds + \int_{s_0}^1 \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds. \]

(3.9)

By arguments similar to (3.5)-(3.8), we have

\[ \int_0^{s_0} \int_y^{xe^{2m(s)}} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) \, dt \, ds = \frac{1}{n} \sum_{i=1}^n \int_y^{xe^{2m(i/n)}} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \, dt + O \left( \frac{1}{n} \right). \]
\[ \int_0^1 \int_{s_0}^{s} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dt ds = \frac{1}{n} \sum_{i=[n\alpha]+1}^{n} \int_{s_0}^{s} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) + O \left( \frac{1}{n} \right), \]

implying that

\[ \int_0^1 \int_{s_0}^{s} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dt ds = \frac{1}{n} \sum_{i=1}^{n} \int_{s_0}^{s} \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) + O \left( \frac{1}{n} \right). \]  

(3.10)

It follows from (3.8) and (3.10) that (3.4) holds for case (ii).

The proof is complete. \[ \square \]

In order to prove Theorems 2.1-2.3, we first give the following definitions:

\[ I_k(x, y; m(i/n)) = \int_{-\infty}^{\log n} (-\log(-t))^k \phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt, k = 0, 1, 2, 3, \]  

(3.11)

where \( \phi(x) \) is the standard normal density. One can check that

\[ I_0(x, y; m(i/n)) = -2x \sqrt{m(i/n)} \left[ \Phi \left( \sqrt{m(i/n)} + \frac{\log(-x \log n)}{2\sqrt{m(i/n)}} \right) - \Phi \left( \sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}} \right) \right], \]  

(3.12)

\[ I_1(x, y; m(i/n)) = 4xm(i/n) \left[ \phi \left( \sqrt{m(i/n)} + \frac{\log(-x \log n)}{2\sqrt{m(i/n)}} \right) - \phi \left( \sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}} \right) \right] \]

+ \( 2x \sqrt{m(i/n)}(\log(-x) + 2m(i/n)) \left[ \Phi \left( \sqrt{m(i/n)} + \frac{\log(-x \log n)}{2\sqrt{m(i/n)}} \right) - \Phi \left( \sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}} \right) \right], \]

(3.13)

\[ I_2(x, y; m(i/n)) = -2x \sqrt{m(i/n)} \left( 4(m(i/n))^2 + 4(m(i/n)) + 4(m(i/n)) \log(-x) + (\log(-x))^2 \right) \]

\[ \times \left[ \Phi \left( \sqrt{m(i/n)} + \frac{\log(-x \log n)}{2\sqrt{m(i/n)}} \right) - \Phi \left( \sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}} \right) \right] \]

- \( 4x (2(m(i/n))^2 + m(i/n) \log(-x / \log n)) \phi \left( \sqrt{m(i/n)} + \frac{\log(-x \log n)}{2\sqrt{m(i/n)}} \right), \]
+4x (2(m(i/n))^2 + m(i/n) log(xy)) \phi \left( \sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}} \right), \quad (3.14)

I_3(x, y; m(i/n)) = 2x \sqrt{m(i/n)} \left( 8(m(i/n))^3 + (24 + 12 \log(-x))(m(i/n))^2 + (6\log(-x) + 12 \log(-x)) (m(i/n)) + (\log(-x))^3 \right)
\times \left[ \Phi \left( \sqrt{m(i/n)} + \frac{\log(-x \log n)}{2\sqrt{m(i/n)}} \right) - \Phi \left( \sqrt{m(i/n)} + \frac{\log(\frac{t}{y})}{2\sqrt{m(i/n)}} \right) \right]
+ 2x \left( 8(m(i/n))^3 + (m(i/n))^2(8 \log(-x) - 4 \log \log n + 16) + 2m(i/n)((\log(-x))^2 - (\log \log n) \log(-x) + (\log \log n)^2) \right)
\times \phi \left( \sqrt{m(i/n)} + \frac{\log(-x \log n)}{2\sqrt{m(i/n)}} \right)
- 2x \left( 8(m(i/n))^3 + (m(i/n))^2(8 \log(-x) + 4 \log(-y) + 16) + 2m(i/n)((\log(-x))^2 + (\log(-y)) \log(-x) + (\log(-y))^2) \right)
\times \phi \left( \sqrt{m(i/n)} + \frac{\log(\frac{t}{y})}{2\sqrt{m(i/n)}} \right). \quad (3.15)

With Lemma 3.1, we can prove Theorem 2.1 as follows.

**Proof of Theorem 2.1.** By the Mill’s ratio of normal distribution, for any fixed \( x < 0 \) we have,

\[
\Phi^{-1}(1 + \frac{x}{n}) = \sqrt{2} \log n \left( 1 - \frac{\log 4\pi + \log \log n}{4 \log n} + \frac{\log 4\pi + \log \log n}{8(\log n)^2} - \frac{(\log 4\pi + \log \log n)^2}{32(\log n)^2} \right)
- \log(-x) \sqrt{2} \log n \left( 1 - \frac{1}{2 \log n} + \frac{(-x) \log \log n}{4 \log n} + (\log 4\pi + \log \log n) \frac{4 \log \log n}{4 \log n} \right) + o \left( (\log n)^{-\frac{3}{2}} \right) . \quad (3.16)
\]

Note that \( o \left( (\log n)^{-\frac{3}{2}} \right) \) also holds uniformly for \( x \in [y, -\frac{1}{\log n}] \) with fixed \( y \). It follows from (1.13), (3.16) and the monotonicity and continuity of \( m(s) \) that for large \( n \) and fixed \( x < 0 \) and \( y < 0 \),

\[
\frac{\Phi^{-1}(1 + \frac{t}{n}) - \rho m \text{\Phi}^{-1}(1 + \frac{t}{n})}{\sqrt{1 - \rho^2 m}}
= \frac{\Phi^{-1}(1 + \frac{t}{n}) - \Phi^{-1}(1 + \frac{t}{n})}{\sqrt{1 - \rho^2 m}} + \sqrt{\frac{1 - \rho m}{1 + \rho m}} \Phi^{-1}(1 + \frac{t}{n})
= \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} - \frac{\log \log n}{4 \log n} \left( \sqrt{m(i/n)} - \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right)
\]

11
\[
\begin{align*}
+ \frac{(\log(t))^2 + (\log 4\pi - 3m(i/n)) - 2\log(-t)}{8\sqrt{m(i/n)} \log n} - \frac{(\log(-x))^2 + (\log 4\pi + m(i/n)) - 2\log(-x)}{8\sqrt{m(i/n)} \log n} \\
+ \frac{(m(i/n))^2 - (\log 4\pi)\sqrt{m(i/n)}}{4\log n} + \frac{1 + \log(-t)}{\log n}o(1)
\end{align*}
\]

holds uniformly for all \(1 \leq i \leq n\) and \(t \in [y, -\frac{1}{\log n}]\). Noting that

\[
\int_{y}^{\frac{1}{\log n}} \frac{1}{x} \left( \sqrt{m(i/n)} - \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \left( \sqrt{m(i/n)} - \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt
\]

we have

\[
\begin{align*}
&\left[ \frac{-1}{n} \sum_{i=1}^{n} \int_{y}^{\frac{1}{\log n}} \phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \left( \Phi^{-1}(1 + \frac{2}{n}) - \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) dt \right] \\
&= \frac{\log \log n}{4\log n} \sum_{i=1}^{n} \sqrt{m(i/n)} I_2(x, y; m(i/n)) + \frac{1}{4n \log n} \sum_{i=1}^{n} \frac{\log 4\pi - 3m(i/n) - 2}{2\sqrt{m(i/n)}} I_1(x, y; m(i/n)) \\
&\quad + \frac{1}{4n \log n} \sum_{i=1}^{n} \frac{(\log(-x))^2 + (\log 4\pi + m(i/n)) - 2\log(-x) + 2(\log 4\pi)m(i/n) - 2(m(i/n))^2}{2\sqrt{m(i/n)}} I_0(x, y; m(i/n)) \\
&\quad + \left( \frac{1}{n} \sum_{i=1}^{n} I_0(x, y; m(i/n)) + \frac{1}{n} \sum_{i=1}^{n} I_1(x, y; m(i/n)) \right) o \left( \frac{1}{\log n} \right) \\
&\sim \frac{\log \log n}{2\sqrt{2\pi \log n}} \int_{0}^{1} \sqrt{m(s)} \exp \left( -\frac{m(s) - \log(xy) + \frac{(\log(xy))^2}{4m(s)}}{2} \right) ds
\end{align*}
\]
as \( n \to \infty \), where \( I_0(x, y; m(i/n)) \), \( I_1(x, y; m(i/n)) \) and \( I_2(x, y; m(i/n)) \) are given by (3.12), (3.13) and (3.14), respectively.

By Taylor expansion with Lagrange reminder term, we have

\[
\Phi \left( \frac{\Phi^{-}(1 + \frac{x}{n}) - \rho_{ni} \Phi^{-}(1 + \frac{1}{n})}{\sqrt{1 - \rho_{ni}^2}} \right) = \Phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) + \phi \left( \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) \left( \frac{\Phi^{-}(1 + \frac{x}{n}) - \rho_{ni} \Phi^{-}(1 + \frac{1}{n})}{\sqrt{1 - \rho_{ni}^2}} - \sqrt{m(i/n)} - \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right)^2, \tag{3.20}
\]

where

\[
\min \left( \frac{\Phi^{-}(1 + \frac{x}{n}) - \rho_{ni} \Phi^{-}(1 + \frac{1}{n})}{\sqrt{1 - \rho_{ni}^2}}, \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) < \theta_i < \max \left( \frac{\Phi^{-}(1 + \frac{x}{n}) - \rho_{ni} \Phi^{-}(1 + \frac{1}{n})}{\sqrt{1 - \rho_{ni}^2}}, \sqrt{m(i/n)} + \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right).
\]

Note that

\[
\frac{1}{n} \sum_{i=1}^{n} \int_{y}^{-\log n} |\theta_i| \cdot \phi(\theta_i) \left( \frac{\Phi^{-}(1 + \frac{x}{n}) - \rho_{ni} \Phi^{-}(1 + \frac{1}{n})}{\sqrt{1 - \rho_{ni}^2}} - \sqrt{m(i/n)} - \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right)^2 dt \leq C \frac{1}{n} \sum_{i=1}^{n} \int_{y}^{-\log n} \left( \log \log n \left( \sqrt{m(i/n)} - \frac{\log(t/x)}{2\sqrt{m(i/n)}} \right) - \frac{(\log(-t))^2 + (\log 4\pi - 3m(i/n) - 2)\log(-t)}{8\sqrt{m(i/n)} \log n} \right) \frac{(\log(-t))^2 + (\log 4\pi + m(i/n) - 2)\log(-t) - (m(i/n))^\frac{3}{2} - (\log 4\pi)\sqrt{m(i/n)} - 1 + \log(-t) o(1)}{8 \sqrt{m(i/n)} \log n} dt \\
= O \left( \left( \frac{\log \log n}{\log n} \right)^2 \right). \tag{3.21}
\]
Combining (3.19), (3.20) and (3.21), we have

\[- \frac{1}{n} \sum_{i=1}^{n} \int_{y}^{1} \Phi \left( \Phi^{-1} \left( 1 + \frac{x}{n} \right) - \rho_m \Phi^{-1} \left( 1 + \frac{t}{n} \right) \right) \Phi \left( \frac{\sqrt{m(i/n)} + \log(t/x)}{\sqrt{1 - \rho_{ni}^2}} \right) \Phi \left( \frac{\sqrt{m(i/n)} + \log(t/x)}{\sqrt{1 - \rho_{ni}^2}} \right) \] \[= - \frac{1}{n} \sum_{i=1}^{n} \int_{y}^{1} \Phi \left( \Phi^{-1} \left( 1 + \frac{x}{n} \right) - \rho_m \Phi^{-1} \left( 1 + \frac{t}{n} \right) \right) \Phi \left( \frac{\sqrt{m(i/n)} + \log(t/x)}{\sqrt{1 - \rho_{ni}^2}} \right) \Phi \left( \frac{\sqrt{m(i/n)} + \log(t/x)}{\sqrt{1 - \rho_{ni}^2}} \right) \] \[\sim \frac{\log \log n}{2\sqrt{2\pi \log n}} \int_{0}^{1} \sqrt{m(s)} \exp \left( - \frac{m(s) - \log(xy) + \frac{(\log(xy))^2}{4m(s)}}{2} \right) ds \] (3.22)

as \( n \to \infty \).

Note that

\[ \frac{1}{n} \sum_{i=1}^{n} \int_{y}^{1} \max \left( \Phi \left( \Phi^{-1} \left( 1 + \frac{x}{n} \right) - \rho_m \Phi^{-1} \left( 1 + \frac{t}{n} \right) \right) \right) \Phi \left( \frac{\sqrt{m(i/n)} + \log(t/x)}{\sqrt{1 - \rho_{ni}^2}} \right) dt = O \left( \frac{1}{\log n} \right). \] (3.23)

Combining (3.22), (3.23) with Lemma 3.1, we can get

\[ \frac{1}{n} \sum_{i=1}^{n} \int_{y}^{1} \Phi \left( \Phi^{-1} \left( 1 + \frac{x}{n} \right) - \rho_m \Phi^{-1} \left( 1 + \frac{t}{n} \right) \right) \Phi \left( \frac{\sqrt{m(s)} + \log(t/x)}{\sqrt{2m(s)}} \right) dt = \int_{0}^{1} \int_{0}^{y} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds \] \[\sim \frac{\log \log n}{2\sqrt{2\pi \log n}} \int_{0}^{1} \sqrt{m(s)} \exp \left( - \frac{m(s) - \log(xy) + \frac{(\log(xy))^2}{4m(s)}}{2} \right) ds \] (3.24)

as \( n \to \infty \).

Since

\[ x \int_{0}^{1} \Phi \left( \sqrt{m(s)} + \frac{\log(x/y)}{2\sqrt{m(s)}} \right) ds + y \int_{0}^{1} \Phi \left( \sqrt{m(s)} + \frac{\log(y/x)}{2\sqrt{m(s)}} \right) ds \] \[= x + \int_{0}^{1} \int_{0}^{y} \Phi \left( \sqrt{m(s)} + \frac{\log(t/x)}{2\sqrt{m(s)}} \right) dtds \] and from Liao et al.(2016) we have

\[ P \left( n \left( \max_{1 \leq i \leq n} F_1(X_i) - 1 \right) \leq x, n \left( \max_{1 \leq i \leq n} F_2(Y_i) - 1 \right) \leq y \right) - G(x, y) \]
= \sum_{i=1}^{n} \log \mathbb{P} \left( F_1(X_i) \leq 1 + \frac{x}{n}, F_2(Y_i) \leq 1 + \frac{y}{n} \right) - \log G(x, y) (1 + o(1)) \\
= G(x, y) \left( - \sum_{i=1}^{n} \left( 1 - \mathbb{P} \left( F_1(X_i) \leq 1 + \frac{x}{n}, F_2(Y_i) \leq 1 + \frac{y}{n} \right) \right) \\
- x \int_{0}^{1} \Phi \left( \frac{\log(x/y)}{2 \sqrt{m(s)}} \right) ds - y \int_{0}^{1} \Phi \left( \frac{\log(y/x)}{2 \sqrt{m(s)}} \right) ds \right) (1 + o(1)) \\
= G(x, y) \left( - \sum_{i=1}^{n} \left( \mathbb{P} \left( F_1(X_i) > 1 + \frac{x}{n} \right) + \mathbb{P} \left( F_2(Y_i) > 1 + \frac{y}{n} \right) - \mathbb{P} \left( F_1(X_i) > 1 + \frac{x}{n}, F_2(Y_i) > 1 + \frac{y}{n} \right) \right) \\
- x - \int_{0}^{y} \int_{0}^{1} \Phi \left( \frac{\log(x/y)}{2 \sqrt{m(s)}} \right) dt ds \right) (1 + o(1)) \\
= G(x, y) \left( \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{y} \Phi \left( \frac{\Phi^{-1} \left( 1 + \frac{x}{n} \right) - \rho_i \Phi^{-1} \left( 1 + \frac{t}{n} \right)}{\sqrt{1 - \rho_i^2}} \right) dt - \int_{0}^{y} \int_{0}^{1} \Phi \left( \frac{\log(t/x)}{2 \sqrt{m(s)}} \right) dt ds \right) \times (1 + o(1)) \\
\sim \frac{\log \log n}{2\sqrt{2\pi \log n}} G(x, y) \int_{0}^{1} \sqrt{m(s)} \exp \left( - \frac{m(s) - \log(xy) + \frac{(\log(xy))^2}{4m(s)}}{2} \right) ds \quad (3.25)
as \( x \to \infty \), c.f., Castro (1987). By using (3.27) and \( \lim_{n \to \infty} \frac{\max_{1 \leq i \leq n} m(i/n)}{\log \log n} = 0 \), we have

\[
\Phi\left( \frac{\Phi^{-}\left(1 + \frac{x}{n}\right) - \rho \Phi^{-}\left(1 - \frac{1}{n \log n}\right)}{\sqrt{1 - \rho^2_{ni}}} \right) = 1 - \Phi\left( \frac{\Phi^{-}\left(1 + \frac{x}{n}\right) - \rho \Phi^{-}\left(1 - \frac{1}{n \log n}\right)}{\sqrt{1 - \rho^2_{ni}}} \right)
\]

\[
= -\frac{2\sqrt{m(i/n) \log n}}{\sqrt{2\pi \log \log n}} \exp \left( -\frac{1}{2} \left( \frac{m(i/n)}{4m(i/n)} - \frac{\log \log n}{8m(i/n)} \right) \right) (1 + o(1))
\]

and

\[
1 - \Phi\left( \frac{\Phi^{-}\left(1 + \frac{x}{n}\right) - \rho \Phi^{-}\left(1 + \frac{y}{n}\right)}{\sqrt{1 - \rho^2_{ni}}} \right) = \frac{(x/y)^{\frac{1}{2}}}{\sqrt{2\pi \sqrt{m(i/n)}}} \exp \left( -\frac{m(i/n)}{2} \right) \left( 1 - \frac{(\log \frac{x}{y})^2 - 4 \log \frac{x}{y} + 8}{8m(i/n)} + o \left( \frac{1}{m(i/n)} \right) \right)
\]

for large \( n \).

From (3.12)-(3.15), \( \lim_{n \to \infty} \min_{1 \leq i \leq n} m(i/n) = \infty \) and \( \lim_{n \to \infty} \frac{\max_{1 \leq i \leq n} m(i/n)}{\log \log n} = 0 \), it follows that

\[
I_0(x, y; m(i/n)) = -2x \phi \left( \sqrt{m(i/n)} + \frac{\log \frac{x}{y}}{2 \sqrt{m(i/n)}} \right) \left( 1 - \frac{\log \frac{x}{y} + 2}{2m(i/n)} + o \left( \frac{1}{m(i/n)} \right) \right),
\]

\[
I_1(x, y; m(i/n)) = 2x \phi \left( \sqrt{m(i/n)} + \frac{\log \frac{x}{y}}{2 \sqrt{m(i/n)}} \right) (\log(-y) - 2 + o(1)),
\]

\[
I_2(x, y; m(i/n)) = -2x \phi \left( \sqrt{m(i/n)} + \frac{\log \frac{x}{y}}{2 \sqrt{m(i/n)}} \right) (\log(-y))^2 - 4 \log(-y) + 8 + o(1),
\]

\[
I_3(x, y; m(i/n)) = 2x \phi \left( \sqrt{m(i/n)} + \frac{\log \frac{x}{y}}{2 \sqrt{m(i/n)}} \right) ((\log(-y))^3 - 6(\log(-y))^2 + 24 \log(-y) - 48 + o(1)),
\]
which implies that

\[
\int_y^{\log n} t d\Phi \left( \frac{\Phi^{-1}(1 + \frac{x}{n}) - \rho_n \Phi^{-1}(1 + \frac{y}{n})}{\sqrt{1 - \rho^2_n}} \right) = \int_y^{\log n} \frac{(-t)\rho_n}{\sqrt{1 - \rho^2_n}} \phi(\Phi^{-1}(1 + \frac{y}{n})) dt
\]

\[
= \frac{1}{2\sqrt{m(i/n)}} \left( 1 - \frac{3m(i/n)}{4\log n} (1 + o(1)) \right)
\]

\[
\times \left[ \left( 1 + \frac{(1 + m(i/n)) \log \log n}{4 \log n} - \frac{(m(i/n))^2}{4 \log n} + o \left( \frac{(m(i/n))^2}{\log n} \right) \right) I_0(x, y; m(i/n))
\]

\[
+ \left( \frac{1}{16 \log n} - \frac{\log \log n + \log 4\pi - 2}{16 m(i/n) \log n} \right) (I_2(x, y; m(i/n)) + 2 \log(-x) I_1(x, y; m(i/n))
\]

\[
+ (\log(-x))^2 I_0(x, y; m(i/n))) + \frac{1}{16 m(i/n) \log n} (I_3(x, y; m(i/n)) + \log(-x) I_2(x, y; m(i/n))
\]

\[
- (\log(-x))^2 I_1(x, y; m(i/n)) - (\log(-x))^3 I_0(x, y; m(i/n)))
\]

\[
- \left( \frac{m(i/n)}{4 \log n} + o \left( \frac{m(i/n)}{4 \log n} \right) \right) I_1(x, y; m(i/n)) \right]
\]

\[
= \frac{(xy)^{\frac{1}{2}}}{\sqrt{2\pi} \sqrt{m(i/n)}} \exp \left( -\frac{m(i/n)}{2} \right) \left( 1 - \left( \frac{\log \frac{2}{y}}{2} \right)^2 + 4 \frac{\log \frac{2}{y}}{8m(i/n)} + o \left( \frac{1}{m(i/n)} \right) \right) .
\]

Hence, by using (3.28)-(3.30) we have

\[
\int_y^{\log n} t d\Phi \left( \frac{\Phi^{-1}(1 + \frac{x}{n}) - \rho_n \Phi^{-1}(1 + \frac{y}{n})}{\sqrt{1 - \rho^2_n}} \right) dt
\]

\[
= \frac{1}{\log n} \left( 1 - \Phi \left( \frac{\Phi^{-1}(1 + \frac{x}{n}) - \rho_n \Phi^{-1}(1 + \frac{y}{n})}{\sqrt{1 - \rho^2_n}} \right) \right)
\]

\[
- y \left( 1 - \Phi \left( \frac{\Phi^{-1}(1 + \frac{x}{n}) - \rho_n \Phi^{-1}(1 + \frac{y}{n})}{\sqrt{1 - \rho^2_n}} \right) \right)
\]

\[
+ \int_y^{\log n} t d\Phi \left( \frac{\Phi^{-1}(1 + \frac{x}{n}) - \rho_n \Phi^{-1}(1 + \frac{y}{n})}{\sqrt{1 - \rho^2_n}} \right)
\]

\[
= \frac{1}{\pi} \frac{(xy)^{\frac{1}{2}}}{\sqrt{m(i/n)}} \exp \left( -\frac{m(i/n)}{2} \right) (1 + o(1)) .
\]

(3.31)

as \( n \to \infty \).
It follows from (3.23) and (3.31) that

\[
P(n(\max_{1 \leq i \leq n} F_1(X_i) - 1) \leq x, n(\max_{1 \leq i \leq n} F_2(Y_i) - 1) \leq y) - e^{x+y}
\]

\[
= e^{x+y} \left( \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{\frac{2(xy)^{\frac{1}{2}}}{\sqrt{2\pi m(i/n)}}} \exp \left( -\frac{m(i/n)}{2} \right) \right) (1 + o(1))
\]

\[
= e^{x+y} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{2(xy)^{\frac{1}{2}}}{\sqrt{2\pi m(i/n)}} \exp \left( -\frac{m(i/n)}{2} \right) \right) (1 + o(1)).
\]

The proof is complete. \(\square\)

**Proof of Theorem 2.3.** Here we only prove the case of \(x \neq y\) since the proof of case \(x = y\) is similar. For \(\max(x, y) \leq t \leq -\frac{1}{\log n}, x < 0, y < 0\), we have

\[
\Phi^{-1} \left( 1 + \frac{\min(x, y)}{n} \right) - \rho_n \Phi^{-1} \left( 1 + \frac{1}{n} \right)
\]

\[
\sqrt{1 - \rho_n^2}
\]

\[
= \sqrt{m(i/n)} + \log \frac{t}{\min(x, y)} - \frac{\log t}{\min(x, y)} + \frac{(\log 4\pi + \log \log n) \log \frac{t}{\min(x, y)}}{8 \sqrt{m(i/n) \log n}}
\]

\[
+ \frac{(\log(-t))^2 - (\log(-\min(x, y)))^2}{8 \sqrt{m(i/n) \log n}} - \frac{\log \log n}{4 \log n} \sqrt{m(i/n)} - \frac{\log(-t)}{2 \log n} \sqrt{m(i/n)}
\]

\[
+ \frac{\sqrt{m(i/n) \log \frac{t}{\min(x, y)}}}{8 \log n} + o \left( \frac{\log \log n \sqrt{m(i/n)}}{\log n} \right),
\] (3.32)

due to (3.16), \(\lim_{n \to \infty} \max_{1 \leq i \leq n} m(i/n) = 0\) and \(\lim_{n \to \infty} (\log \log n) \min_{1 \leq i \leq n} m(i/n) = \infty\). Noting that

\[
\Phi^{-1} \left( 1 + \frac{\min(x, y)}{n} \right) - \rho_n \Phi^{-1} \left( 1 + \frac{1}{n} \right) \to -\infty \text{ for } t \in [\max(x, y), -\frac{1}{\log n}],
\]

we have

\[
\Phi \left( \frac{\Phi^{-1} \left( 1 + \frac{\min(x, y)}{n} \right) - \rho_n \Phi^{-1} \left( 1 - \frac{1}{n \log n} \right)}{\sqrt{1 - \rho_n^2}} \right)
\]

\[
= 1 - \Phi \left( \frac{\Phi^{-1} \left( 1 + \frac{\min(x, y)}{n} \right) - \rho_n \Phi^{-1} \left( 1 - \frac{1}{n \log n} \right)}{\sqrt{1 - \rho_n^2}} \right)
\]

\[
= O \left( \sqrt{m(i/n) \log n} \frac{\log n}{\log \log n} \exp \left( -\frac{(\log(-\min(x, y) \log n))^2}{8 \log n \sqrt{m(i/n)}} \right) \right),
\] (3.33)
\[
\Phi\left( \frac{\Phi^\left(1 + \frac{\min(x,y)}{n}\right) - \rho_n \Phi^\left(1 + \frac{\max(x,y)}{n}\right)}{\sqrt{1 - \rho_n^2}} \frac{1}{\sqrt{1 - \rho_n^2}} \right)
\]

\[
= 1 - \Phi\left( - \frac{\Phi^\left(1 + \frac{\min(x,y)}{n}\right) - \rho_n \Phi^\left(1 + \frac{\max(x,y)}{n}\right)}{\sqrt{1 - \rho_n^2}} \right)
\]

\[
= \frac{\sqrt{\frac{2\pi}{m(i/n)}} \left( \frac{\min(x,y)}{\max(x,y)} \right)}{\left( \log \frac{\min(x,y)}{\max(x,y)} \right)} \exp \left( - \frac{\left( \log \frac{\max(x,y)}{\min(x,y)} \right)^2}{8m(i/n)} \right)
\]

\[
\times \left( 1 - \frac{1}{2} m(i/n) + \frac{2m(i/n)}{\log \frac{\min(x,y)}{\max(x,y)}} - \frac{4m(i/n)}{\left( \log \frac{\max(x,y)}{\min(x,y)} \right)^2} + o(m(i/n)) \right).
\]

From (3.12)-(3.15), \( \lim_{n \to \infty} \max_{1 \leq i \leq n} m(i/n) = 0 \) and \( \lim_{n \to \infty} (\log \log n) \min_{1 \leq i \leq n} m(i/n) = \infty \), it follows that

\[
I_0(\min(x, y), \max(x, y); m(i/n))
\]

\[
= - \frac{4 \min(x, y) m(i/n) \phi \left( \sqrt{m(i/n)} + \frac{\log \frac{\min(x,y)}{\max(x,y)}}{2\sqrt{m(i/n)}} \right)}{\left( \log \frac{\min(x,y)}{\max(x,y)} \right)} \left( 1 - \frac{4m(i/n)}{\left( \log \frac{\min(x,y)}{\max(x,y)} \right)^2} (1 + o(1)) \right)
\]

\[
\times \left( \log \log n \right)^\frac{1}{2} \log \log n
\]

\[
I_1(\min(x, y), \max(x, y); m(i/n))
\]

\[
= \frac{4 \min(x, y) m(i/n) \phi \left( \sqrt{m(i/n)} + \frac{\log \frac{\min(x,y)}{\max(x,y)}}{2\sqrt{m(i/n)}} \right)}{\left( \log \frac{\min(x,y)}{\max(x,y)} \right)} \left( 1 - \frac{4m(i/n)}{\left( \log \frac{\min(x,y)}{\max(x,y)} \right)^2} (1 + o(1)) \right)
\]

\[
\times \left( \log (- \max(x, y)) - \frac{4m(i/n)}{\log \frac{\min(x,y)}{\max(x,y)}} + o(m(i/n)) \right),
\]

\[
I_2(\min(x, y), \max(x, y); m(i/n))
\]
\[
\begin{align*}
&= -\frac{4 \min(x, y) m(i/n) \phi \left( \sqrt{m(i/n)} + \frac{\log \min(x, y)}{2 \sqrt{m(i/n)}} \right)}{(\log \frac{\min(x, y)}{\max(x, y)} \left( 1 + \frac{2m(i/n)}{\log \frac{\min(x, y)}{\max(x, y)}} \right) \left( 1 - \frac{4m(i/n)}{(\log \frac{\min(x, y)}{\max(x, y)})^2 (1 + o(1))} \right)} \\
&\times \left( \log(- \max(x, y))^2 - \frac{8(\log(- \max(x, y))) m(i/n)}{\log \frac{\min(x, y)}{\max(x, y)}} + o(m(i/n)) \right)
\end{align*}
\]

and

\[
I_3(\min(x, y), \max(x, y); m(i/n))
\]

\[
\begin{align*}
&= \frac{4 \min(x, y) m(i/n) \phi \left( \sqrt{m(i/n)} + \frac{\log \min(x, y)}{2 \sqrt{m(i/n)}} \right)}{(\log \frac{\min(x, y)}{\max(x, y)} \left( 1 + \frac{2m(i/n)}{\log \frac{\min(x, y)}{\max(x, y)}} \right) \left( 1 - \frac{4m(i/n)}{(\log \frac{\min(x, y)}{\max(x, y)})^2 (1 + o(1))} \right)} \\
&\times \left( \log(- \max(x, y))^3 - \frac{12(\log(- \max(x, y)))^2 m(i/n)}{\log \frac{\min(x, y)}{\max(x, y)}} + o(m(i/n)) \right),
\end{align*}
\]

which implies that

\[
\int_{\max(x,y)} \frac{1}{\log n} t d\Phi \left( \phi \left( \Phi^{-1} \left( \frac{1 + \min(x, y)}{n} \right) - \rho_{ni} \Phi^{-1} \left( 1 + \frac{1}{n} \right) \right) \right) \]

\[
= \int_{\max(x,y)} \frac{1}{\log n} (-t) \rho_{ni} \phi \left( \Phi^{-1} \left( \frac{1 + \frac{1}{n}}{n} \right) - \rho_{ni} \Phi^{-1} \left( 1 + \frac{1}{n} \right) \right) dt \\
= 1 - \frac{3m(i/n)}{4 \log n} (1 + o(1)) \\
\times \left[ \left( 1 + \frac{(1 + m(i/n)) \log log n}{4 \log n} + o \left( \frac{\log log n}{\log n} m(i/n) \right) \right) I_0(\min(x, y), \max(x, y); m(i/n)) \\
+ \left( \frac{2 - \log 4\pi - \log log n}{16m(i/n) \log n} + \frac{1}{16 \log n} \right) \left( I_2(\min(x, y), \max(x, y); m(i/n)) + 2(\log(- \min(x, y))) I_1(\min(x, y), \max(x, y); m(i/n)) + (\log(- \min(x, y)))^2 I_0(\min(x, y), \max(x, y); m(i/n)) \right) \right]
\]

\[
+ \frac{1}{16m(i/n) \log n} \left( I_3(\min(x, y), \max(x, y); m(i/n)) \right) \\
- (\log(- \min(x, y)))^2 I_1(\min(x, y), \max(x, y); m(i/n)) \\
+ (\log(- \min(x, y))) I_2(\min(x, y), \max(x, y); m(i/n))
\]

\[
= 20
\]
Hence, by using (3.36) and (3.23), we have

\[-(\log(-\min(x,y)))^3I_0(\min(x,y), \max(x,y); m(i/n)) + o\left(\frac{\log \log n}{\log n}\right) I_1(\min(x,y), \max(x,y); m(i/n))\]

\[= \sqrt{\frac{2}{\pi}} \sqrt{m(i/n)(xy)^2} \exp\left(-\frac{\left(\log \frac{\min(x,y)}{\max(x,y)}\right)^2}{8m(i/n)}\right)\]

\[\times \left(1 - \frac{1}{2} m(i/n) - \frac{2m(i/n)}{\log \frac{\min(x,y)}{\max(x,y)}} - \frac{4m(i/n)}{\left(\log \frac{\min(x,y)}{\max(x,y)}\right)^2} + o(m(i/n))\right).\] (3.35)

Hence, it follows from (3.33)-(3.35) that

\[\int_{\max(x,y)}^{1} \Phi\left(\frac{\Phi^{-1}\left(1 + \frac{\min(x,y)}{n}\right) - \rho_n \Phi^{-1}\left(1 + \frac{1}{n}\right)}{\sqrt{1 - \rho_n^2}}\right) dt = -\frac{1}{\log n} \Phi\left(\frac{\Phi^{-1}\left(1 + \frac{\min(x,y)}{n}\right) - \rho_n \Phi^{-1}\left(1 + \frac{1}{n}\right)}{\sqrt{1 - \rho_n^2}}\right) - \max(x,y) \Phi\left(\frac{\Phi^{-1}\left(1 + \frac{\min(x,y)}{n}\right) - \rho_n \Phi^{-1}\left(1 + \frac{1}{n}\right)}{\sqrt{1 - \rho_n^2}}\right)\]

\[-\int_{\max(x,y)}^{1} t d\Phi\left(\frac{\Phi^{-1}\left(1 + \frac{\min(x,y)}{n}\right) - \rho_n \Phi^{-1}\left(1 + \frac{1}{n}\right)}{\sqrt{1 - \rho_n^2}}\right)\]

\[= \frac{8m(i/n)^{\frac{3}{2}}(xy)^{\frac{1}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{\left(\log \frac{\min(x,y)}{\max(x,y)}\right)^2}{8m(i/n)}\right) (1 + o(1)).\] (3.36)

Hence, by using 3.36 and 3.23, we have

\[\mathbb{P}\left(n\left(\max_{1\leq i\leq n} F_1(X_i) - 1 \leq x, n\left(\min_{1\leq i\leq n} F_2(Y_i) - 1 \leq y\right) - e^{\min(x,y)}\right)\]

\[= e^{\min(x,y)} \left(-\frac{1}{n} \sum_{i=1}^{n} \int_{\max(x,y)}^{1} \Phi\left(\frac{\Phi^{-1}\left(1 + \frac{\min(x,y)}{n}\right) - \rho_n \Phi^{-1}\left(1 + \frac{1}{n}\right)}{\sqrt{1 - \rho_n^2}}\right) dt + O\left(\frac{1}{\log n}\right)\right) (1 + o(1))\]

\[= -e^{\min(x,y)} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{8(m(i/n))^{\frac{3}{2}}(xy)^{\frac{1}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{\left(\log \frac{\min(x,y)}{\max(x,y)}\right)^2}{8m(i/n)}\right) (1 + o(1)),\right]

which is the desired result. The proof is complete. \[\square\]
Acknowledgements  This work was supported by the National Natural Science Foundation of China (grant No. 11601330), and the Funding Program for Junior Faculties of College and Universities of Shanghai Education Committee (grant No. ZZslg16020).

References

[1] Castro, L.C.E. (1987) Uniform rate of convergence in extreme-value theory: Normal and Gamma models. Annales Scientifiques de l’Université de Clermont-Ferrand 2, série Probabilités et applications, 6(6), 25-41.

[2] Chu, X. (2015) Modelling impact of monetary policy on stock market liquidity: a dynamic copula approach. Applied Economics Letters, 22(10), 820-824.

[3] Embrechts, P., McNeil, A. and Straumann, D. (2002) Correlation and dependence in risk management: properties and pitfalls. In Dempster, M. H. A. (editor), Risk management: Value at Risk and Beyond, pages 176-233. Cambridge University Press, Cambridge.

[4] Frick, M. and Reiss, R.D. (2013) Expansions and penultimate distributions of maxima of bivariate normal random vectors. Statistics and Probability Letters, 83(11), 2563-2568.

[5] Hashorva, E., Peng, Z. and Weng, Z. (2016) Higher-order expansions of distributions of maxima in a Hüsler-Reiss model. Methodology and Computing in Applied Probability, 18(1), 181-196.

[6] Hüsler, J. and Reiss, R.D. (1989) Maxima of normal random vectors: between independence and complete dependence. Statistics and Probability Letters, 7, 283-286.

[7] Liao, X. and Peng, Z. (2014) Convergence rate of maxima of bivariate Gaussian arrays to the Hüsler-Reiss distribution. Statistics and Its Interface, 7(3), 351-362.

[8] Liao, X. and Peng, Z. (2015) Asymptotics for the maxima and minima of Hüsler-Reiss bivariate Gaussian arrays. Extremes, 18(1), 1-14.

[9] Liao, X. and Peng, Z. (2017) Asymptotics and statistical inferences on independent and non-identically distributed bivariate Gaussian triangular arrays. Acta Mathematica Sinica, Chinese Series, 2, 297-314.

[10] Liao, X., Peng, L., Peng, Z. and Zheng, Y. (2016) Dynamic bivariate normal copula. Science China Mathematics, 59(5), 955-976.
[11] Lu, Y. and Peng, Z. (2017) Maxima and minima of independent and non-identically distributed bivariate Gaussian triangular arrays. Extremes, 20, 187-198.

[12] Naldi, M. and D’Acquisto, G. (2008) A normal copula model for the economic risk analysis of correlated failures in communications networks. Journal of Universal Computer Science, 14(5), 786-799.

[13] Salvatierra, I.D.L. and Patton, A.J. (2015) Dynamic copula models and high frequency data. Journal of Empirical Finance, 30, 120-135.

[14] Sibuya, M. (1960) Bivariate extreme statistics. Annals of the Institute of Statistical Mathematics, 11, 195-210.

[15] Taylor, J.M.G., Conlon, A.S.C. and Elliott, M.R. (2015) Surrogacy assessment using principal stratification with multivariate normal and Gaussian copula models. Clinical Trials, 12(4), 317-322.

[16] Wang, C.W., Yang, S.S. and Huang, H.C. (2015) Modeling multi-country mortality dependence and its application in pricing survivor index swaps- A dynamic copula approach. Insurance Mathematics and Economics, 63, 30-39.