Hall-Littlewood polynomials and vector bundles on the Hilbert scheme

Erik Carlsson, Simons center for geometry and physics
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Abstract

Let $E$ be the bundle defined by applying a polynomial representation of $GL_n$ to the tautological bundle on the Hilbert scheme of $n$ points in the complex plane. By a result of Haiman [11], the Cech cohomology groups $H^i(E)$ vanish for all $i > 0$. It follows that the equivariant Euler characteristic with respect to the standard two-dimensional torus action has nonnegative coefficients in the torus variables $z_1, z_2$, because they count the dimensions of the weight spaces of $H^0(E)$. We derive a very explicit asymmetric formula for this Euler characteristic which has this property, by expanding known contour integral formulas for the Euler characteristic stemming from the quiver description [18, 19] in $z_2$, and calculating the coefficients using Jing’s Hall-Littlewood vertex operator with parameter $z_1$ [12].

1 Introduction

Let $\text{Hilb}_n \mathbb{C}^2$ denote the Hilbert scheme of $n$ points in the complex plane, and consider the standard two-dimensional torus action induced from

$$T = (\mathbb{C}^*)^2 \circ \mathbb{C}^2, \quad (z_1, z_2) \cdot (x, y) = (z_1^{-1}x, z_2^{-1}y)$$

by pullback of ideals. We also have an $n$-dimensional tautological bundle $\mathcal{U}$ on the Hilbert scheme, whose fiber over a subscheme $[Z] \in \text{Hilb}_n \mathbb{C}^2$ is simply the space of sections of $\mathcal{O}_Z$, and which inherits an action of $T$. See [15] for details.
Given a representation $\rho$ of $GL_n$, we obtain a new equivariant bundle $E = \rho(U)$, and we may consider its Cech cohomology groups $H^i(E)$, as well as its equivariant Euler characteristic

$$
\chi_n(E) = \sum_i (-1)^i \text{ch} H^i_{\text{Hilb}_n}(E) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}],
$$

where $\text{ch}$ denotes the character of $H^i(E)$ as a representation of $T$. If $\Lambda$ is the ring of symmetric polynomials in infinitely many variables, the polynomial representations are in the image of the map

$$
\Lambda \rightarrow K_T(\text{Hilb}_n \mathbb{C}^2), \quad s_\mu \mapsto S_\mu(U),
$$

where $s_\mu \in \Lambda$ is the Schur polynomial, $S_\mu$ is the corresponding representation of $GL_n$ (the Schur functor), and $\mu$ is a partition. Since the Euler characteristic is defined at the level of $K$-theory, we have a well defined Euler characteristic $\chi_n(f(U))$, for any symmetric function $f \in \Lambda$.

The main result of this paper is the following formula for the Euler characteristic,

**Theorem A.** The Euler characteristic is given by

$$
\chi_n(f(U)) = \sum_{\mu, \nu} z_2^{|\mu|} z_1^{-|\mu|} + k_{\mu\nu} b_{\nu,n}(z_1) f_{\nu\mu}(z_1).
$$

Here $k_{\mu\nu}$ is an integer, $b_{\nu,n}(z)$ is the norm squared of the Hall-Littlewood polynomial $P_\nu(X; z)$ in $n$ variables, and $f_{\nu\mu}(z)$ is the matrix element of the operator of multiplication by $f$ in the Hall-Littlewood basis. The significance of this formula is that the coefficients of its power series about the origin are nonnegative integers whenever $f$ is an honest representation, i.e. a nonnegative integral linear combination of Schur polynomials. This nonnegativity follows from a result of Haiman which says that the Cech cohomology groups $H^i(\mathcal{U}^{\otimes l} \otimes P)$ vanish for $i > 0$, where $P$ is the Procesi bundle. Since the trivial bundle is a summand of the Procesi bundle, and every $E = f(U)$ appears as a summand of $\mathcal{U}^{\otimes l}$, it follows that the desired Euler characteristic is the character of the honest representation $H^0(E)$.

It is not clear how this formula relates to the special case of Haiman’s formulas, corresponding to the trivial component of the Procesi bundle. Haiman’s answers are expressed in terms of the Macdonald polynomials, via the isomorphism of Bridgeland, King, and Reid.
Both sets of formulas are expressed in terms of symmetric functions, but in our formula, the rank of the Hilbert scheme corresponds to the number of variables, whereas in Haiman’s formulas, the number of variables is infinite, and \( n \) corresponds to the degree. The combinatorics of Macdonald polynomials are of course more difficult, but we can check the agreement when the bundle is trivial, explained in corollary 1.

It is interesting to note that the Euler characteristic is symmetric in \( z_1, z_2 \), which is not obvious from theorem A. This is also the case in the well-known \( q,t \)-Catalan number formulas studied by Garsia, Haglund, and Haiman [5, 6], which the author first learned about from Gorsky, Mazin, and Shende. In fact, Hall-Littlewood polynomials have been used to study this topic in a paper by Garsia, Xin, and Zabrocki [7]. It would also be of interest to relate them to the results and conjectures of Gorsky, Oblomkov, Rasmussen and Shende [8, 21], and any connections with the Hall-Littlewood formulas of Mironov, Morozov, Shakirov, and Sleptsov [14].

Our proof is based on a contour integral formula for the Euler characteristic (1) coming from the quiver description on the Hilbert scheme, which can be found in Negut [18], and is a \( K \)-theoretic version of a cohomological formula by Nekrasov [19]. We expand this formula in the \( z_2 \) variable, and calculate the coefficients in terms of the Hall-Littlewood inner product in \( n \) variables, with parameter \( z_1 \). An essential role is played by a vertex operator due to Jing [12], which extends Bernstein’s vertex operator [22] from Schur to Hall-Littlewood polynomials.

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2 Contour integrals

The Atiyah-Bott-Lefschetz localization formula gives an explicit formula for the Euler characteristic defined in the introduction,

\[
\chi_n(f(U)) = \sum_{|\mu| = n} U_\mu \text{ch}(T^*_\mu) \in \mathbb{C}(z_1, z_2).
\]

Here the fixed points of \( \text{Hilb}_n \mathbb{C}^2 \) are indexed by partitions \( \mu \) of \( n \), \( U_\mu, T^*_\mu \) denote the fibers of \( U \), and the cotangent bundle respectively.
\[
\Omega(V) = \left( \text{ch} \sum_i (-1)^i \Lambda^i V \right)^{-1} \in \mathbb{C}(z_1, z_2),
\]
for a torus representation \( V \). More generally, \( \Omega \) may be extended to the whole representation ring \( \mathbb{Z}(T) \) by
\[
\Omega(A + B) = \Omega(A)\Omega(B), \quad \Omega(x) = (1 - x)^{-1}, \quad (3)
\]
for any monomial \( x \). See [10, 15] for combinatorial formulas for the summands.

Strictly speaking, the localization formula does not apply in this situation because the Hilbert scheme is not compact. However, by a result of Nakajima [16], the weight spaces of the Cech cohomology group are finite-dimensional, and the Euler characteristic lives in \( \mathbb{Z}((z_1, z_2)) \), which represents the expansion of (2) about the origin. If \( \rho \) is a polynomial representation, as it is in this paper, the Euler characteristic turns out to be holomorphic at the origin. As usual, we cannot extract the signed dimensions of the weight spaces from the localization formula without simplifying the expression. In fact most of the terms have a singularity along a one-dimensional curve through the origin in the \( z_1, z_2 \) plane, and their expansions change depending on which \( z_a \) we expand about first.

This issue can be resolved using the following contour integral formula,
\[
\chi_n(f(\mathcal{U})) = \frac{1}{n!} \Omega(1 - M)^n \oint_{|x_1|=r} \frac{dx_1}{x_1} \cdots \oint_{|x_n|=r} \frac{dx_n}{x_n}
\]
\[
f(X)\Omega(X)\Omega(z_1z_2X)\Omega(-M\Delta), \quad (4)
\]
where
\[
M = (1 - z_1)(1 - z_2), \quad X = x_1 + \cdots + x_n,
\]
\[
\Delta = \sum_{i \neq j} x_i x_j^{-1} = XX - n, \quad f(X) = f(x_1, ..., x_n), \quad \overline{x_i} = x_i^{-1}.
\]
We refer to [18] for an explanation of this formula, or [19] for the original cohomological version. These formulas come from the description of the Hilbert scheme as a quiver variety, and apply to the more general moduli space of higher rank sheaves on \( \mathbb{P}^2 \), see [15]. They are shown to agree with (2) directly by applying the Cauchy residue
formula, one variable at a time. See also \cite{2}, which produces similar formulas, by considering the Hilbert scheme as a subvariety of an infinite-dimensional Grassmannian.

Under formula (11), we find that $\chi_n(f(U))$ is manifestly holomorphic at the origin, simply because the expansion of the integrand in $z_n$ is valid in the interior of the contour. Furthermore, we may count the signed dimension of the weight spaces by applying the contour integral to each coefficient. Each such integral may be expressed in terms of the standard Hall inner product on Symmetric functions in $n$ variables, establishing that it is an integer.

3 Hall-Littlewood polynomials

Let us recall briefly some notation about Hall-Littlewood polynomials and the plethystic notation, which we standardize with chapter 3 of Macdonald’s book \cite{13}, and Haiman \cite{9}.

Let $\Lambda$ denote the ring of symmetric functions, and consider the Hall-Littlewood inner product in finitely many variables,

$$ (f,g)_{z,n} = \frac{1}{n!} [X]_1 f(X) g(\overline{X}) \Omega(- (1 - z) \Delta_n), \quad (5) $$

where

$$ X = x_1 + \cdots + x_n, \quad \Delta_n = \sum_{i \neq j} x_i x_j^{-1}, $$

as in the introduction. The constant term $[X]_1$ may be defined either as a contour integral for any fixed value of $z$, or by expanding the integrand in $z$, and simply extracting the constant term of each coefficient, which is a Laurent polynomial in $x_i$. We also have its limit as the number of variables tends to infinity, normalized so that the norm of 1 $\in \Lambda$ is one, defined by

$$ (p_\mu, p_\nu)_z = \delta_{\mu\nu} 3(\mu) \prod_i (1 - z^{\mu_i})^{-1}, \quad (6) $$

where $p_\mu$ are the symmetric power sums

$$ p_\mu = \prod_k p_{\mu_k}, \quad p_k = x_1^k + x_2^k + \cdots. $$

The Hall-Littlewood polynomials $P_\mu(X; z)$ for partitions of length $\ell(\mu) \leq n$ constitute an orthogonal basis for (5), and satisfy

$$ (P_\mu, P_\nu)_{z,n} = \delta_{\mu\nu} (1 - z)^n b_{\mu,n}(z)^{-1}, \quad (7) $$

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where
\[ b_{\mu,n}(z) = \prod_{i \geq 0} [m_i(\mu)]_z, \quad [k]_z = \prod_{1 \leq j \leq k} (1 - z^j), \]
and \( m_i(\mu) \) is the number of times that \( i \) appears in \( \mu \), with the multiplicity of zero defined as \( n - \ell(\mu) \). In the limit as \( n \) tends to infinity, we get
\[ (P_\mu, P_\nu)_z = \delta_{\mu \nu} b_\mu(z)^{-1}, \]
\[ b_\mu(z) = \lim_{n \to \infty} b_{\mu,n}(z) b_{\emptyset,n}(z)^{-1} = \prod_{i \geq 1} [m_i(\mu)]_z. \]

Given a rational function \( A \) in some set of variables, let \( A_k \) denote the evaluation at \( z = z^k \), for each indeterminant \( z \) that appears in \( A \). We will make heavy use of the following multiplication operator
\[ \Gamma_-(A) : \Lambda \to \Lambda, \quad \Gamma_-(A)g = \exp \left( \sum_{k \geq 1} \frac{1}{k} f_k p_k \right) g, \]
which is technically only defined as a power series in all variables present in \( A \) with values in \( \text{End}(\Lambda) \). This is a harmless issue for our purposes, but see Frenkel and Ben-Zvi [4] for a full exposition. For instance, we have
\[ \Gamma_-(x) \cdot 1 = \sum_{k \geq 0} x^k h_k. \]
Its dual under the standard Hall inner product is a ring homomorphism, defined on generators by
\[ \Gamma_+(A)p_k = p_k + A_k. \]
The following relations are easily verified,
\[ (\Gamma_-(A)f, g)_z = (f, \Gamma_+(A(1 - z)^{-1}))_z, \]
\[ \Gamma_\pm(A + B) = \Gamma_\pm(A) \Gamma_\pm(B), \quad \Gamma^m_\pm(A) = \Gamma_\pm(mA), \]
\[ \Gamma_\pm(A)x^d = x^d \Gamma_\pm(x^{\pm 1}A), \quad x^d \cdot p_\mu = x^{|\mu|} p_\mu, \]
\[ \Gamma_+(A)\Gamma_-(B) = \Gamma_-(B)\Gamma_+(A)\Omega(AB), \] (8)
where
\[ \Omega(A) = \exp \left( \sum_{k \geq 1} \frac{1}{k} A_k \right), \]
and the convergence of $\Omega(AB)$ puts restrictions on $A, B$. Notice that this definition of $\Omega(A)$ is consistent with (3).

Next, we recall Jing’s vertex operator [12], which generates the Hall-Littlewood polynomials by successive applications to $1 \in \Lambda$. In this notation, it is defined by

$$J_k = [x^k]\Gamma_-(x(1 - z))\Gamma_+^{-1}(x^{-1}),$$

and has the property that

$$Q_{\mu} = J_{\mu_1} \cdots J_{\mu_n} \cdot 1,$$

where

$$Q_{\mu}(X; z) = b_{\mu}(z)P_\mu(X; z),$$

is the dual basis to $P_\mu$ under (6), as in MacDonald’s book. Upon setting $z = 0$, it becomes the vertex operator defined by Bernstein [22], which acts on the Schur polynomials.

Given any operator $\varphi : \Lambda \to \Lambda$, let us label its matrix elements in the Hall-Littlewood basis by

$$\varphi \cdot P_\nu(X; z) = \sum_{\mu} \varphi_{\mu\nu}(z)P_\mu(X; z).$$

If $f \in \Lambda$ is a polynomial, then we define $f_{\mu\nu}(z)$ to be the matrix elements of multiplication by $f$. We will also set

$$\psi_{\mu\nu}(z) = \Gamma_-(1)_{\mu\nu}(z),$$

which is the same thing as multiplication by the complete symmetric polynomial $h_k$, for $k = |\nu| - |\mu|$. The Pieri rules for Hall-Littlewood polynomials provide a combinatorial description of these coefficients, which we will not need.

4 Proof of the theorem

We may now state and prove our main result:

**Theorem 1.** If $f \in \Lambda$ is a symmetric function, and $U$ is the tautological $n$-dimensional bundle on $\text{Hilb}_n \mathbb{P}^2$ with the torus action (1), then we have

$$\chi_n(f(U)) = \sum_{\mu, \nu} z_2^{\mu_1}\bar{z}_1^{\mu_2} b_{\nu,n}(z_1)z_{1}^{-1}f_{\nu\mu}(z_1),$$

(10)
where
\[ k_{\mu\nu} = \sum_{i} \left( \frac{\mu'_i}{2} \right) + \left( \frac{\nu'_i}{2} \right) - \mu'_i \nu'_i, \]
and \( \mu' \) is the conjugate partition to \( \mu \).

If \( f \) is a linear combination of the Schur polynomials with nonnegative integer coefficients, then \( f_{\mu\nu}(z) \) is polynomial in \( z \) with nonnegative integer coefficients. The coefficients of the power series expansion of (10) are therefore nonnegative integers, representing the result of Haiman explained the introduction that the higher Čech cohomology groups vanish.

An immediate corollary is the well-known formula for the space of sections of \( O \).

**Corollary 1.** The space of sections of the trivial bundle is given by
\[
\sum_{n \geq 0} q^n \chi_n(O) = \Omega(qM^{-1}) = \prod_{i,j \geq 0} (1 - z_1^i z_2^j q)^{-1}. \quad (11)
\]

**Proof.** The partition function corresponds to \( f = 1 \), hence \( f_{\mu\nu} = \delta_{\mu\nu} \). We may easily check that \( k_{\mu\mu} = -|\mu| \), so that formula (10) becomes
\[
\chi_n(O) = \sum_{\mu} z^{|\mu|} b_{\mu,n}(z_1)^{-1}.
\]

Fixing \( n \), we may associate to \( \mu \) another partition \( \tilde{\mu} \), whose terms are the multiset of positive integers \( m_i(\mu) \), including the multiplicity of zero, as defined above. We can rewrite the above expression as
\[
\sum_{|\lambda|=n} \sum_{\tilde{\mu} = \lambda} z^{|\mu|} \prod_{k} [\lambda_k]_{z_1}^{-1}.
\]

One may easily check that
\[
\sum_{\tilde{\mu} = \lambda} z^{|\mu|} = m_\lambda(1, z_2, z_2^2, ...), \quad [\lambda_k]_{z_1}^{-1} = h_{\lambda_k}(1, z_1, z_1^2, ...),
\]
where \( m_\mu, h_\mu \) are the monomial and complete symmetric polynomials. Then
\[
\chi_n(O) = \sum_{|\lambda|=n} m_\lambda(1, z_2, z_2^2, ...) h_\lambda(1, z_1, z_1^2, ...).
\]

Converting this to (11) is precisely chapter I, formula (4.2) of [13]. \( \square \)
Before proving the theorem, we need a technical lemma:

**Lemma 1.** We have

\[
\sum_{\lambda} z^{-|\lambda|} b_{\lambda}(z) \psi_{\mu\lambda}(z) \psi_{\nu\lambda}(z) = z^{k_{\mu\nu}}. \tag{12}
\]

**Proof.** The exponent \( k_{\mu\nu} \) satisfies

\[
k_{\emptyset\emptyset} = 0, \quad k_{\mu\nu} = k_{\nu\mu}, \quad k_{[a,\mu]\nu} - k_{\mu\nu} = |\mu| - |\nu|,
\]

\[
a \geq \mu_1, \nu_1, \quad [a, \mu] = [a, \mu_1, ..., \mu_n]. \tag{13}
\]

It is uniquely determined by these properties by successively adding terms to \( \mu, \nu \), in increasing order.

Now, we may rewrite the left hand side of (12) as

\[
\sum_{\lambda} z^{-|\lambda|} (\Gamma_-(1) P_{\lambda}, Q_\mu) z (\Gamma_-(1) P_{\lambda}, Q_\nu) z (P_{\lambda}, P_{\lambda})^{-1} = (Q_\mu, Q_\nu)',
\]

where

\[
(f, g)' = \left( \Gamma_+(A^{-1}) f, z^{-d} \Gamma_+(A^{-1}) g \right)_z, \quad A = 1 - z,
\]

and \( d \) is the operator of multiplication by the norm on \( \Lambda \). It suffices to prove that this inner product satisfies

\[
(J_a f, Q_\nu)' = (z^d f, z^{-d} Q_\nu)
\]

whenever \( a \geq \nu_1 \), which would establish the last property in (13).

Inserting the definition (9), we get the coefficient of \( x^a \) in

\[
\left( \Gamma_+(A^{-1}) \Gamma_-(xA) \Gamma_+^{-1}(x^{-1}) f, z^{-d} \Gamma_+(A^{-1}) Q_\nu \right)_z =
\]

\[
(1 - x)^{-1} \left( \Gamma_+(A^{-1} - x^{-1}) f, \Gamma_+(x) z^{-d} \Gamma_+(A^{-1}) Q_\nu \right)_z =
\]

\[
(1 - x)^{-1} \left( \Gamma_+(A^{-1} - x^{-1}) f, z^{-d} \Gamma_+(A^{-1} + xz^{-1}) Q_\nu \right)_z.
\]

using the vertex operator relations (8).

The final expression may be written as \((1-x)^{-1} F(x)\), where \( F(x) \) is a Laurent polynomial in \( x \) with coefficients in \( \mathbb{C}(z) \). Since \( Q_\nu \) is lower-triangular with respect to the monomial basis \( m_\mu \), we may bound the degree of \( F(x) \) by

\[
\operatorname{deg}_x F(x) \leq \max_{\mu \leq \nu} \operatorname{deg}_x \Gamma_+(x) m_\mu,
\]

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where $\mu \leq \nu$ refers to the dominance ordering. Since

$$(f, \Gamma_+(x)m_\mu) = (\Gamma_-(x)f, m_\mu) = \sum_{i \geq 0} x^k (fh_k, m_\mu),$$

and $h_\mu, m_\mu$ are dual bases, we find that the degree of $F(x)$ is at most $\nu_1$.

Then for $a \geq \nu_1 \geq \deg_x F(x)$, we have

$$[x^a](1 - x)^{-1}F(x) = F(1) = \left(\Gamma_+ (zA^{-1}) f, z^{-d} \Gamma_+ (z^{-1}A^{-1}) Q_\nu \right) = (z^d f, z^{-d} Q_\nu)' .$$

We can now prove the main result.

**Proof.** We may rewrite the contour integral formula from the introduction as

$$\chi_n(f(\mathcal{U})) = \frac{1}{n!} \Omega(1 - M)^n [X_1 f(X) \Omega(X)\Omega(z_1 z_2 X)\Omega(-M\Delta), \quad (14)$$

where the constant term is taken from each term in the expansion of the integrand in $z_1, z_2$. Let us expand the rightmost term in the $z_2$ variable,

$$\Omega(-M\Delta) = \Omega(-(1 - z_1)\Delta)\Omega(z_2(1 - z_1)\Delta) = \Omega(-(1 - z_1)\Delta)\Omega(z_2(1 - z_1))^{-n} \sum_\lambda z_2^{||\lambda||} b_\lambda(z_1) P_\lambda(X; z_1) P_\lambda(\overline{X}; z_1),$$

by the expansion

$$\Omega((1 - z)XY) = \sum_\lambda x^{||\lambda||} b_\lambda(z) P_\lambda(X; z) P_\lambda(Y; z),$$

which can be found in [13], chapter III, equation (4.4). Inserting this into equation (14), and using (5), we get

$$(1 - z_1)^{-n} \sum_\lambda z_2^{||\lambda||} b_\lambda(z_1) (\Gamma_- (z_1 z_2) P_\lambda f, \Gamma_- (1) P_\lambda)_{z_1, n} = \sum_{\lambda, \mu, \nu} z_2^{||\mu|| - ||\lambda||} b_\lambda(z_1) b_{\nu, n}(z_1)^{-1} f_{\nu\mu}(z_1) \psi_{\mu\lambda}(z_1) \psi_{\nu\lambda}(z_1).$$

The result now follows by summing over $\lambda$, and applying lemma [1] \qed
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