Signatures of trans-Planckian dispersion in inflationary spectra

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The primordial spectra are calculated using dispersion relations which deviate from the relativistic one above a certain energy scale \( \Lambda \). We determine the properties of the leading modifications with respect to the standard spectra when \( \Lambda \gg H \), where \( H \) is the Hubble scale during inflation. To be generic, we parameterize the lowest order deviation from the relativistic law by \( \alpha \), the power of \( P/\Lambda \) where \( P \) is the proper momentum. When working in the asymptotic vacuum, the leading modification scales as \( (H/\Lambda)^\alpha \) for all \( \alpha \), except for a discrete set where the power is higher. Moreover, this modification is robust against introducing higher order terms in the dispersion relation. We then algebraically deduce the modifications of scalar and tensor power spectra in slow roll inflation from modifications calculated in de Sitter space. The modifications do not exhibit oscillations unless the dispersion relation induces some non-adiabaticity near a given scale. Finally, we explore the much less studied regime where \( H \) and \( \Lambda \) are comparable. Our results indicate that the project of reconstructing the inflaton potential cannot be pursued without making some hypothesis about the dispersion relation of the fluctuation modes.

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I. INTRODUCTION

In agreement with the observed temperature anisotropies in the Cosmic Microwave Background [1], inflation predicts an almost scale invariant spectrum of primordial density perturbations [2, 3]. These adiabatic perturbations arise from the amplification of vacuum fluctuations of linearized metric and inflaton perturbations. This mechanism relies on a semi-classical description, i.e., quantum fields propagating in a background spacetime. However in most models, the inflationary phase lasts so many e-foldings that the fluctuations we today observe stem from vacuum fluctuations characterized by wavelengths much shorter than the Planck length at the onset of inflation. In this regime the semiclassical description is no longer trustworthy. It is therefore of importance to find out to what extent the inflationary predictions depend on the physics that takes place above (or near) the Planck scale.

In the absence of a theory of quantum gravity, there is no obvious way to address this question. In fact several approaches have been adopted. The first one is directly inspired by what was done in black hole physics where a similar problem occurs [4, 5, 6]: to test the robustness of the predictions, one introduces some dispersion above a certain energy scale \( \Lambda \), and studies the sensitivity of the power spectrum to this modification [7, 8]. It was shown that the properties of the power spectrum are robust, i.e., the deviations with respect to the standard spectrum are small whenever the adiabaticity of the evolution is preserved and \( \Lambda \) taken well above the Hubble scale \( H \) during inflation [9]. However, the precise relationship between the modifications of the dispersion relation and the induced modifications of the spectrum was not obtained in the general case. To get a generic estimate of the modifications a simplified approach was proposed in Ref. [10]. Instead of introducing dispersion around the scale \( \Lambda \), the vacuum state was imposed when the momentum \( P = \Lambda \) rather than in the asymptotic regime \( P \rightarrow \infty \) as done when working in the Bunch-Davies vacuum. A third approach based on an effective action obtained by integrating out heavy degrees of freedom of characteristic mass \( \Lambda \), was also used in Ref. [11] and a different estimate of the modifications was obtained.

If all approaches agree on the fact that the inflationary predictions are robust when \( \Lambda/H \gg 1 \), there is indeed no agreement about the generic properties of the modifications of the spectra. In particular, there is no agreement concerning the power of \( H/\Lambda \) which characterizes their amplitude. In Ref. [10] it was argued that the deviations should generically be first order in \( H/\Lambda \), whereas in the effective lagrangian approach it was argued that the corrections should be (at least) second order and contain only even powers of \( H/\Lambda \). Moreover in a reformulation of the model of [10] where the state imposed when \( P = \Lambda \) is the (properly defined) instantaneous adiabatic vacuum [12], it was shown that the corrections are third order. As of the first method based on dispersion, as we said, no general result seems to have been obtained.

In addition, besides the question of the amplitude of the deviations, there is no agreement either on the generic character of the rapid oscillations which have been found in [12, 13, 14] (but not in [11]), confronted with observational data in [13, 15], and criticized in [17].

In the present work, we aim to settle the questions concerning the amplitude of the deviations and the presence of fast oscillations when using dispersion, and still work-
ing with the asymptotic (Bunch-Davies) vacuum. To get results which are not bound to a particular dispersion relation (or to a particular class thereof), we parameterize the first deviation with respect to the relativistic law by a scale $\Lambda$ and a power $\alpha$ in the following way

$$\Omega^2 = F^2(P^2) = P^2 \left(1 \pm (P/\Lambda)^\alpha + O\left((P/\Lambda)^{2\alpha}\right)\right). \quad (1)$$

The sign determines whether the propagation is super- (+) or subluminous (−). As in former works, the preferred frame which is used to define $\Omega$ and $P$ is taken to coincide with the cosmological frame: $\Omega$ is thus the proper frequency and $P$ the norm of the spatial momentum as measured by comoving observers. In these models the isotropy and the homogeneity of FLRW are preserved, but the local Lorentz invariance has been broken. (This is not the case in [11].) It should also be noticed that these models respect a modified (weak) Equivalence Principle [19] in that, when considering high momenta (in the preferred frame) and neglecting the gradients of the metric, the physics is the same as in Minkowski space (in the preferred rest frame).

To separate the modifications due to the dispersion above the scale $\Lambda$ from those governed by slow-roll parameters, we first compute the deviations of the spectrum of a scalar field propagating in de Sitter space and then show how these determine, by simple substitutions, those of scalar and tensor modes in slow-roll inflation.

In Section II, to start the analysis, we algebraically solve a particular case: a quartic superluminous dispersion ($\alpha = 2$ in eq. (1)), in de Sitter space. We compute the resulting spectrum for all values of $H/\Lambda$. This analytical treatment allows us to identify the nature of the signatures, and to prepare the numerical treatment.

In Section III, using numerical integration techniques, we consider the general case, with $\alpha$ ranging from 1 to 6. When $H/\Lambda \ll 1$, we first show that the dominant deviation of the power spectrum scales as $(H/\Lambda)^\alpha$. In other words the leading modification is linear in the lowest order deviation of the dispersion relation. This is true for both super- and subluminous dispersion, and for all values of $\alpha$ but for a discrete set of powers ($\alpha_c = 3 + 2i$) where an overall $\alpha$-dependent factor vanishes and where the leading modification scales with a power higher than $\alpha_c$. Secondly, we show that higher order terms in the dispersion relation are irrelevant in that they induce sub-dominant deviations which vanish faster than $(H/\Lambda)^\alpha$ in the limit $H/\Lambda \to 0$. The signatures are thus robust against modifying the dispersion relation by adding higher order terms. Thirdly, we show that the signatures of super- and subluminous dispersion have equal magnitude and opposite sign for all $\alpha \neq \alpha_c$, and all values of $H/\Lambda \ll 1$. Fourthly, when the dispersion relation is smooth and the asymptotic vacuum well-defined, the modifications of the spectrum do not display oscillations. On the contrary, (fast) oscillations appear when some non-adiabaticity is localized near a certain (UV) scale, in agreement with the conclusions of Ref. [17]. We also briefly study the other regime where $H/\Lambda$ is comparable or greater than 1. (For subluminous dispersion we chose the asymptotic behavior of $F$ so that the the adiabatic vacuum is well defined.) As one might have expected, the deviations are no longer governed by the lowest order modification of the dispersion relation. This does not mean however that this regime should be discarded because it is not known at which scale the semi-classical description loses its validity. (In some higher dimensional models (see [20] and references therein), this scale could be much smaller than the (4 dimensional) Planck mass, and therefore could be smaller than $H$.)

We then establish in Section IV that the modifications of the spectra in slow-roll inflation can be obtained from the above results. The main result is that the modifications depend on the wave vector $k$ only through the ratio $H_k/\Lambda$, $H_k$ being the value of the Hubble scale when the $k$-mode exits the Hubble radius.

In the Conclusions we briefly discuss the possibility of distinguishing between modifications of the spectra coming from dispersion and other modifications, like those stemming from a change in the inflaton potential. In particular, we point out that our results predict a violation of the consistency relation, even in the case when the scalar and tensor modes obey the same dispersion relation.

## II. MODIFIED POWER SPECTRUM

### A. The model

In this Section, we consider a minimally coupled massless scalar field $\Phi$ propagating in a FLRW background spacetime in comoving coordinates:

$$ds^2 = a(\eta)^2 (-d\eta^2 + d\xi^2). \quad (2)$$

The spectrum of scalar perturbations and gravitational waves can be deduced from the power spectrum of this field, see [8, 21] and Section IV. We then introduce a non-linear dispersion relation that we parameterize by $F$, see eq. (1). We assume that one recovers the standard relation in the IR, i.e. $F^2(P^2) \to P^2$. We call $\Lambda$ the UV scale which weighs the first non-linear term of $F^2$.

Decomposing the field in Fourier modes with fixed comoving momentum $k = aP$, the equation of the rescaled mode $\phi_k = a\Phi_k$ reads

$$\left(\frac{\partial^2}{\partial \eta^2} + a^2 F^2 \left(\frac{k^2}{a^2} \right) - \frac{\partial^2 a}{a}\right) \phi_k = 0. \quad (3)$$

Before considering slow-roll inflation, we first specialize to de Sitter space, where

$$\left(\frac{\partial^2}{\partial \eta^2} + \frac{1}{H^2 \eta^2} F^2 \left(k^2 H^2 \eta^2 - \frac{2}{\eta^2}\right) \phi_k = 0, \quad (4)$$

since $a = -1/(H\eta)$. All dependencies in $k$ drop out when
using the variable $x = -k\eta = P/H$:

$$
\left( \partial_x^2 + \frac{1}{H^2x^2}F^2(H^2x^2) - \frac{2}{x^4} \right) \phi = 0.
$$

(5)

Thus, the power spectrum remains scale invariant in de Sitter space when the dispersion relation is expressed in terms of proper frequency and momentum, i.e., when it respects the Equivalence Principle, and when the preferred frame coincides with the cosmological frame. [39]

B. Relativistic case

Let us briefly recall the calculation of the power spectrum for the standard relativistic case $F^2 = P^2$. The mode equation (5) reduces to:

$$
\left( \partial_x^2 + 1 - \frac{2}{x^4} \right) \phi = 0.
$$

(6)

To obtain the power spectrum one needs to identify the asymptotic positive frequency solution of the above equation. Indeed, in any successful model of inflation, the relevant modes we today observe obeying $P \gg H$ at the onset of inflation, and hence were all in their ground state [22]. The power spectrum $P$ of the relevant fluctuations is thus given by the following VEV

$$
P(\eta, k) = \frac{k^3}{2\pi^2} \int d^4 \xi \, e^{ik\cdot \xi} \langle 0|\hat{\Phi}(\eta, \xi) \hat{\Phi}(\eta, \vec{0})|0\rangle
$$

$$
= \frac{k^3}{2\pi^2} |\Phi^\text{in}_k(\eta)|^2,
$$

(7)

where $|0\rangle$ is the Bunch-Davies vacuum [23] and where $\Phi^\text{in}_k$ is the positive unit norm Fourier mode associated with this asymptotic state.

The observationally relevant quantity is the value of $P(k)$ at late times, long after horizon exit, $k/aH = x \to 0$. When $\Phi^\text{in}_k(\eta)$ is written in terms of

$$
u^\text{in}(x) = \frac{1}{\sqrt{x}} \left( 1 + \frac{i}{x} \right) e^{ix},
$$

the unit wronskian solution of eq. (6) that is purely positive frequency for $k\eta = -x \to -\infty$, one obtains

$$
2\pi^2 P_0 = H^2 \left( x^2 |\nu^\text{in}|^2 \right)_{x \to 0} = \frac{H^2}{2}.
$$

(9)

The index 0 stands for the unperturbed relativistic case.

Had we worked in slow-roll inflation rather than de Sitter space, $P_0(k)$ would have been given by the above with $H$ replaced by $H_k$, the value of $H$ when the $k$-mode exited the Hubble radius, i.e., $k/(a_kH_k) = 1$, see Section [IV] for details.

C. Quartic dispersion relation

In this subsection, we compute the power spectrum in the particular case

$$
F^2_{2+} = P^2 \left( 1 + \frac{P^2}{\Lambda^2} \right).
$$

(10)

The subscript 2 indicates that the first (and only) non-linearity in $F$ is quadratic in $P/\Lambda$ and the + sign indicates that the dispersion is superluminous. This dispersion relation [24] has been studied in a cosmological context in Ref. [25]. However, to our knowledge, the following calculation has never been made. We believe this is the only dispersive case where an exact calculation of the power spectrum is possible in terms of hypergeometric functions.

1. Analytical expression of the power spectrum

When $F$ is given by eq. (10), the wave equation becomes:

$$
\left( \partial_x^2 + 1 + \frac{x^2}{4\Lambda^2} - \frac{2}{x^4} \right) \phi_{2+} = 0.
$$

(11)

From the comparison of this equation with eq. (6), one can deduce that the modifications of observables with respect to the relativistic case will all be governed by the dimensionless parameter

$$
\lambda = \frac{\Lambda}{2H}.
$$

(12)

(The factor 1/2 has been introduced for convenience and will be retained throughout the paper.)

As in the relativistic case, the state of the field is chosen to be the Bunch-Davies vacuum, that is, the ground state associated with the asymptotic positive frequency solution of eq. (11). This solution, normalized to unit wronskian, is given by

$$
u^\text{in}_{2+}(x) = \sqrt{\Lambda} e^{-\frac{x^2}{4\Lambda}} W_{1/4} \left( -i\frac{x^2}{2\Lambda} \right).
$$

(13)

The definition of the Whittaker function $W_{\kappa, \mu}$ can be found in [26], p.505. That the function $\nu^\text{in}_{2+}$ has unit wronskian and is positive frequency at early times can be seen from its asymptotic form for large $x$ (see equation 13.5.2 in [26]):

$$
u^\text{in}_{2+}(x) \sim \sqrt{\frac{\Lambda}{x}} e^{i\left( \frac{x^2}{4\Lambda} + \frac{1}{4\Lambda} \ln \frac{x^2}{4\Lambda} \right)}.
$$

(14)

This asymptotic form is precisely the WKB solution of eq. (11) with positive frequency. One verifies that the corrections vanish as $1/x^2$ when $x \to \infty$. Therefore the Bunch-Davies vacuum is well defined.
The power spectrum is then straightforwardly obtained from the behavior of $W_{+\frac{3}{2}}^+$ for $x \to 0$ (equation 13.5.6 in [26]). This gives

$$u_{2+}^*(x) = \frac{e^{-\left(\frac{x\lambda}{\Lambda} - i\frac{\pi}{4}\right)}}{x} \sqrt{\frac{\pi}{8}} \frac{(2\lambda)^{3/4}}{\Gamma\left(\frac{3}{4} - i\frac{\pi}{4}\right)} \times (1 + O(x^2)) .$$

(15)

Using this expression we obtain

$$2\pi^2 P_{2+}(\lambda) = H^2 \left(x^2 |u_{2+}^*(x)|^2\right)_{x\to0}$$

$$= \frac{H^2}{2} \times \frac{\pi (2\lambda)^{3/2} e^{-\frac{\pi\lambda}{\Lambda}}}{4 |\Gamma\left(\frac{3}{4} - i\frac{\pi}{4}\right)|^2} .$$

(16)

This expression is exact and valid for all values of $H/\Lambda$. It gives the power spectrum when using the dispersion relation of eq. (10) and when working in the Bunch-Davies vacuum.

Using equation 6.1.45 of [26], one verifies that the factor of the relativistic spectrum $P_0 = H^2/2$ tends to 1 when $\lambda = \Lambda/2H \to \infty$. In this we recover that the spectrum is robust, that is, the standard value obtains when adiabaticity is preserved, as it is here the case when $H/\Lambda \ll 1$.

2. Signatures for small $H/\Lambda$

Since we have the exact expression of the power spectrum, we can properly extract the first corrections in $H/\Lambda = 1/2$. When $\lambda \gg 1$, using equation 6.1.47 of [26], the r.h.s. of (16) can be expanded into:

$$P_{2+}(\lambda) = P_0 \left(1 - \frac{5}{16} \lambda^{-2} + O(\lambda^{-4})\right) \left(1 + e^{-\pi\lambda}\right) .$$

(17)

This expression contains three factors: the relativistic spectrum and two factors which tend to one when $\lambda \to \infty$. The first one contains a polynomial starting with a quadratic correction in $1/\lambda$, whereas the correction term of the second is exponentially suppressed. In Appendix A, we show that an approximate calculation based on WKB waves correctly reproduces these features, namely that there exist two sources of corrections: polynomial corrections related to local modifications of the mode with respect to the relativistic case, and exponentially small corrections related to non-adiabatic transitions.

3. Figures

To visualize the signatures and to prepare the numerical analysis of the next Section, we have represented several figures. In Figure 1, we show $P_{2+}$ of eq. (16) divided by $P_0$ as a function of $\lambda^{-1}$ for $10^{-2} < \lambda^{-1} < 10^2$. For large $\lambda$, in the adiabatic regime, the modified spectrum asymptotes to the standard result. Instead, for small $\lambda$, the modified power spectrum tends to zero as

$$P_{2+}(\lambda) \sim P_0 \times \frac{4\pi}{\Gamma^2\left(\frac{3}{4}\right)} (2\lambda)^{3/2} .$$

(18)

In Figure 2, to display the signatures of the dispersion, we analyze the relative deviation with respect to the relativistic spectrum, $\Delta P_{2+}/P_0 = (P_{2+} - P_0)/P_0$.

In the upper plot we have represented $\log |\Delta P_{2+}/P_0|$ as a function of $\log \lambda^{-1}$. One clearly sees that when $1/\lambda = 2H/\Lambda < 0.1$, the polynomial correction term in (17) largely dominates all other contributions. Indeed the curve asymptotes to a straight line, with the expected slope, equal to 2, and the expected intercept $= 5/16 \simeq -0.505$. In the lower plot, in anticipation to the numerical analysis, we have represented the quantity

$$\Delta_F(\lambda) = \lambda^\alpha \frac{P_F - P_0}{P_0} ,$$

(19)

where $\alpha$ is the power of $P/\Lambda$ in the first non-linearity in $F$, $\alpha = 2$ here. The interest of this representation is that as soon as the deviation in $\lambda^{-\alpha}$ becomes the dominant one, the curve becomes horizontal.

III. NUMERICAL ANALYSIS

A. Parameterization of the dispersion relation

In the previous section, when studying the dispersion relation with a quadratic correction term in $P/\Lambda$, we found that the first deviation of the power spectrum was quadratic in $H/\Lambda$. Our first aim is thus to verify...
whether the power of $H/\Lambda$ characterizing the first deviation is always the same as that of $P/\Lambda$ in the leading non-linear term of the dispersion relation. Secondly, we want to show that when adding subleading correction terms to the dispersion relation, i.e., terms characterized by a power of $P/\Lambda$ higher than $\alpha$, the signatures are robust, i.e., they are insensitive to these additional terms. Third, we want to analyze in a symmetrical manner super- and subluminous dispersion. Therefore, we select subluminous dispersion relations which possess a well defined asymptotic regime $P \to \infty$ so that the adiabatic vacuum condition can be implemented for early times $x = -k\eta \to \infty$. (Remember that subluminous dispersion relations containing only one non-linear term do not possess such a well defined regime, because $\Omega^2$ becomes negative for $P/\Lambda > 1$.)

To these ends, we consider the following parameterization of the dispersion relation:

$$\Omega^2 = F^2(P, \Lambda; \alpha, \beta, N) = \left\{ \beta P + \gamma \Lambda \tanh^{\frac{3}{2}} \left[ \left( \frac{\zeta P}{\Lambda} \right)^{\frac{2}{1+2}} \right] e^{-(x/\lambda)^{2+2}} \right\}^2,$$

where the extra parameters $\zeta, \gamma$ verify the following equations:

$$\zeta = \left( \frac{3}{4\alpha(1-\beta)} \right)^{1/\alpha}, \quad \gamma = 1 - \frac{\beta}{\zeta}. \quad (21)$$

When these are verified, the Taylor expansion of $F^2$ for $P/\Lambda \ll 1$ is

$$F^2 = P^2 \left[ 1 + \text{sign}(\beta - 1) \left( \frac{P}{\Lambda} \right)^{2} + \frac{\beta}{\alpha} \left( \frac{P}{\Lambda} \right)^{2+2} \right]. \quad (22)$$

We obtain a superluminous relation when $\beta > 1$ and a subluminous one when $\beta < 1$. In addition, in each sector, at fixed $\alpha$, varying $\beta$ modifies only the coefficient of the subdominant deviations (with powers of $P/\Lambda$ equal to or greater than $2\alpha$).

Returning to the exact expression (20), we notice that $\partial_x \ln F \to 0$ in the limit $P/\Lambda \to \infty$. Therefore, in all cases we reach an adiabatic regime which allows to properly define the asymptotic vacuum, see below for more details. The exponential factor in eq. (20) might seem a priori useless since it does not affect any of the first three terms in eq. (22). It has been introduced to facilitate the numerical integration because it improves the validity of the WKB approximation for large $x$. The initial vacuum conditions can then be specified for smaller values of $x$ (typically $x_{in} \approx 500 \Lambda/\zeta$), thus avoiding a too large accumulation of numerical error coming from the (physically irrelevant) early evolution where $P/\Lambda \gg 1$. In the numerical integration we shall put $N = 100/\zeta = x_{in}/5\Lambda$, thereby switching on the tanh about two $e$-foldings after having started the integration. (We have checked the stability of the results when using higher values of $N$ and $x_{in}/\Lambda$.)

Figure 3 represents the dispersion relation (20) for $\alpha = 2$ and several choices of $\beta$. In the left plot the cases $\beta = 0.2$ and $\beta = 1.8$ illustrate the fact that the parameterization of eq. (20) encompasses both super- and subluminous dispersion relations. In the right plot, the subluminous dispersion relations with $\beta = 0.1$ and $\beta = 0.5$ are represented along with the quartic dispersion relation $F^2 = P^2(1 - P^2/\Lambda^2)$. Since the leading (quadratic) deviation is the same for all three cases, the curves coincide even after having left the linear regime, until $P \approx \Lambda/2$.

To study the adiabaticity, we write the corresponding wave equation in the variable $x = -k\eta$ as

$$\left[ \partial_x^2 + \omega_F^2 \right] \phi(x) = 0, \quad (23)$$

with the effective square frequency

$$\omega_F^2 = \frac{1}{H^2 x^2} F^2(H^2 x^2, \Lambda; \alpha, \beta, N) - \frac{2}{x^2}. \quad (24)$$

The evolution is adiabatic whenever $\sigma_F$ given by

$$\sigma_F(x) = \frac{\left| \partial_x \ln \omega_F^2 \right|}{\omega_F^2}, \quad (25)$$

Figure 2: Plot of the relative corrections $\Delta P_{2\lambda} / P_0$, obtained from the exact expression (16). In the log-log representation (upper plot), the dominance of the first signature is seen through the linearity of the curve: when $\lambda^{-1} < 0.1$ the curve is indistinguishable from the straight line with slope 2 and intercept $\log(5/16) \approx -0.505$ (dashed line) associated with the quadratic correction in eq. (17). In the lower plot, it is seen through the constancy of $\Delta_{2\lambda} = \lambda^2 \Delta P_{2\lambda} / P_0$ for $\lambda^{-1} < 0.1$, with a value equal to the coefficient of the first signature term in (17), $-5/16 = -0.3125$. 
remains much smaller than 1 [6, 9].

In the absence of dispersion, the non-adiabaticity only arises from the $-2/x^2$ term and $\sigma_0$ decreases for $x \gg 1$ as $1/x^3$. When adding dispersion, the non-linear behavior of $F$ brings another source of non-adiabaticity. However, when $x/\lambda \gg 1$ is also satisfied, the non-linearities introduced by the hyperbolic tangent in (20) are suppressed by the exponential factor. The remaining terms in $\omega_F$ are a constant term ($= \beta^2$) and the (small) negative term $-2/x^2$. Thus in this asymptotic regime, $\sigma_F$ decreases as $(\beta^2 x^3)^{-1}$ thereby guaranteeing that the adiabatic vacuum stays as well-defined as in the absence of dispersion. (Had we not introduced the exponential factor, the hyperbolic tangent would asymptote to a constant which would bring a contribution to $\sigma_F$ decreasing only as $1/x^2$, thereby imposing to use a larger value of $x_{in}$ to describe with a high precision the asymptotic vacuum.)

The behavior of $\sigma_F$ is represented in Figure 3 for several values of the parameter $\lambda$ and for subluminous (left plot) and superluminous (right plot) dispersion. The evolution is globally more adiabatic in the superluminous case, as expected since in this case $\omega_F$ is greater than in the subluminous case. For each value of $\lambda$, there is a local maximum around $x \simeq \lambda$, whose height diminishes when $\lambda$ increases. When scale separation is not realized ($\lambda = 1$), in the superluminous case $\alpha = 2$ and $\beta = 1.8$, this local maximum merges with the rapid increase of $\sigma_F$ when $x$ approaches 1. In the subluminous case $\alpha = 2$ and $\beta = 0.2$, $\sigma_F$ takes non-negligible values, of order $10^{-1}$ around $x \simeq 6$, significantly before horizon exit.

Following conventional techniques, the second order differential equation (20) is separated into a system of 4 first order equations, 2 for the real part of $\phi$ and its derivative, and 2 for the imaginary part of $\phi$ and its derivative. After setting the initial conditions, discussed below eq. (26), this system is integrated using the embedded 8th order Runge-Kutta-Prince-Dormand algorithm provided in the GNU Scientific Library, from $x_i$ to some $x_f$ long after the Hubble scale exit ($x = 1$), typically $x_f \simeq 10^{-5}$. In this algorithm, the error is estimated at each step and the stepsize adapted to keep the errors within fixed bounds. The use of this algorithm was necessary, because the relative deviation $\Delta P_F/P_0$ reaches very small values when $\lambda$ is large. For instance it is of order $10^{-8}$ for $\alpha = 4$ and $\lambda = 100$. Since the high values of $\lambda$ also require a larger number $n$ of integration points (because $x_i \simeq 500\lambda/\zeta$, see below, is further away), the absolute error for each step must be kept under $10^{-10}/n$ if we want to be able to distinguish values of $\Delta P_F/P_0$ as low as $10^{-10}$.

The initial conditions are fixed using the fact that the WKB solution of eq. (23), see eq. (A2), becomes exact when $x \to \infty$. For some finite $x_i \gg \lambda$, the difference between the WKB solution and the exact positive frequency solution is of the order of $\sigma_F$ [12]. Thus the WKB approximation becomes excellent whenever both conditions $x/\lambda \gg 1$ and $x \gg 1$ are satisfied, since in this case $\sigma_F \sim (\beta^2 x_i^3)^{-1}$. In practice, the initial conditions are fixed at $x_i = 500\lambda$ for $\lambda > 10$ and at $x_i = 5000$ for

![Figure 3: Plots of the dispersion relation eq. (20) for $\alpha = 2$ and various values of $\beta$. The linear relativistic dispersion relation is also represented in both plots, as well as the quartic subluminous dispersion relation in the right plot. In plot (b), $\Omega^2$ is shown as a function of $P^2$ so that the quartic dispersion relation be simply a parabola.](image-url)
λ < 10, so that σF < 10−10/β2. We thus safely impose

\[ \phi_F^{in}(x_i) = \frac{1}{\sqrt{2\omega_F(x_i)}} , \]

\[ \partial_x \phi_F^{in} \big|_{x_i} = \frac{i\omega_F}{\sqrt{2\omega_F}} \left( 1 + i \frac{1}{2} \frac{\partial_x \omega_F}{\omega_F} \right) \big|_{x_i} . \]  

(26)

(The arbitrary phase of the mode has been put to 0 at \( x_i \) without affecting the results.)

At the end of the integration of the wave equation, the relative deviation of the power spectrum wrt the relativistic spectrum is evaluated through:

\[ \frac{\Delta P_F(x_f)}{P_0} = \frac{P_F(x_f) - P_0}{P_0} = 2x_f^2 |\phi_F^{in}(x_f)|^2 - 1 , \]  

(27)

which follows from eq. (9). Since \( x_f \) is not exactly zero, the value obtained contains a finite contribution from the decaying mode. For \( \lambda < 1 \), this contribution is negligible wrt the modifications due to dispersion and we simply ignore it. However, since for \( \lambda \gg 1 \) it decreases as \( x^2 \) and since \( x_f \sim 10^{-3} \), the decaying mode contribution in (27) is of order 10⁻⁶. It largely dominates the small corrections we are after. One could consider smaller values of \( x_f \) but this leads to an increase of the number of integration points and of the numerical error. We thus use the fact that for large \( \lambda \) and \( x \ll 1 \), \( \Delta P_F(x) \) is of the form:

\[ \Delta P_F(x) = P_0(A_F + B_F x^2) , \]  

(28)

where \( A_F \) and \( B_F \) depend on the values of the various parameters. \( A_F \) is the contribution of the growing mode that we seek to extract. To do so, we compute the relative deviation (27) at 10\( x_f \) and \( x_f \). Using twice (28), we properly extract \( A_F \).

C. Results

1. Properties of the signatures when \( H/\Lambda \ll 1 \)

The expansion at large \( \lambda \) of the analytical result of Section II C 1 and the qualitative arguments in Appendix A led us to conjecture that \( \Delta P_F/P_0 \) is linearly related to the first deviation in the dispersion relation (22) and thus possesses the following asymptotic form:

\[ \frac{\Delta P_F}{P_0} \sim \frac{\delta^\pm(\alpha)}{\lambda^\alpha} , \quad \text{for } \lambda \to \infty . \]  

(29)

Since only the sign of \( \beta - 1 \) appears in the first deviation, \( \delta^\pm \) is expected to depend on the superluminous (+ exponent) or subluminous (− exponent) character of the dispersion, but not on the actual value of \( \beta \).

In Figure 5 \( \log \Delta P_F/P_0 \) is represented as a function of \( \log \lambda^{-1} \) for a series of subluminous dispersion relations with \( \beta = 0.2 \) and values of \( \alpha \) from 2 to 4. For high values of \( \lambda \) up to 10³, we obtain a set of straight lines, with a slope precisely equal to \( \alpha \), thereby confirming the validity of the asymptotic behavior given in eq. (29). Had we continued the curves toward \( \lambda = 1 \), we would have seen deviations from this linear behavior. This (non-adiabatic) regime around and beyond scale crossing will
be studied in Section III.D. For superluminous dispersion we obtain similar results which confirm the validity of eq. (29).

2. Robustness of the signatures for $H/\Lambda \ll 1$

To establish that higher order terms in $P/\Lambda$ in the dispersion relation are irrelevant in the limit $H/\Lambda \ll 1$, in Figure 5 we have plotted $\Delta_{F}(\lambda)$ of eq. (19) for $\alpha = 2$ and various values of $\beta$ which weigh the term scaling as $(P/\Lambda)^{2\alpha}$ in eq. (22). The value of $\beta$ ranges from 0.1, in which case the coefficient of this term is of order 1, to $\beta = 1 - 10^{-3}$ in which case the coefficient is $\approx 10^{3}$. Similarly for superluminous dispersion relations, $\beta$ ranges from $1 + 10^{-3}$ to 1.9. As expected, when $\lambda \gg 1$, all curves agree irrespectively of the value of $\beta$. We have numerically verified that the next order deviations (i.e. the departure from the asymptotic behavior of eq. (29) which are clearly seen in Figure 6) scale as the square of the first correction term in that equation.

Since the function $\Delta_{F}$ is asymptotically constant when $\Lambda/H \to \infty$, the asymptotic properties of the signatures of UV-dispersion are governed by the two functions $\delta^{\pm}(\alpha)$ of eq. (29). We have verified that similar results hold for all $\alpha$ ranging from 1 to 6, except for $\alpha = 3$ and $\alpha = 5$, which are particular cases as explained below.

3. Properties of $\delta(\alpha)$

To further investigate the asymptotic behavior of the signatures, we have represented in Figure 7 the function $\Delta_{F}(\lambda)$ of eq. (19) for various values of $\alpha$ and for both sub- ($\beta = 0.2$) and superluminous ($\beta = 1.8$) dispersion relations.

Besides confirming the fact that $\Delta_{F}$ is indeed constant when $\lambda \to \infty$, we learn from Figure 7 two rather unexpected results. First, when comparing super- and subluminous cases for each value of $\alpha$, we see that $\delta^{+}(\alpha) = -\delta^{-}(\alpha)$. More precisely, we have numerically checked that the sum $\Delta_{\alpha,+}(\lambda) + \Delta_{\alpha,-}(\lambda)$ scales as $\lambda^{-\alpha}$ at large $\lambda$, and thus vanishes when $\lambda \to \infty$. This result allows us to consider the unique function

$$\delta(\alpha) = \text{sign}(1 - \beta) \delta^{\pm}(\alpha)$$

which does not depend on $\beta$. We also point out that we have not been able to find any analytical explanation for this simple fact. See Appendix A for an attempt in this sense.

Second, we see that the sign of $\delta$ changes between 2.5 and 3.5. In fact it flips sign exactly for $\alpha = 3$ when the dimensionality of spatial sections is 3. To establish this second fact, we consider the generalization of eq. (24) for a de Sitter spacetime of spatial dimension $d$:

$$\omega_{F,d}^{2} = \frac{1}{H^{2}x^{2}}F^{2}(H^{2}x^{2}) - \frac{d^{2} - 1}{4x^{2}}.$$  (31)

To obtain $\delta(\alpha, d)$, we constructed a large number of curves $\Delta_{\alpha,d}(\lambda)$ for each $d$ and found the asymptotic constant value. We repeated this for different values of $\alpha$ (including unphysical non-integer values). The result is shown in Figure 8. (We verified that, for every $d$, $\delta^{+}(\alpha, d) = -\delta^{-}(\alpha, d)$, thereby extending the validity of this peculiar correspondence.)

We find that $\delta(\alpha, d)$ vanishes precisely when $\alpha$ is equal to $d$. We also find that after the first zero, it flips sign
Figure 7: Comparison of super- and subluminous signatures. The legend applies to both plots. In (a) the dispersion is superluminous with $\beta = 1.8$. In (b) the dispersion is subluminous with $\beta = 0.2$. In (a) we represented $-\Delta F$ as a function of $\log \lambda$ so that the asymptotic values reached in each plot can be compared at the center of the figure. We clearly see that $\delta^+(\alpha) = -\delta^-(\alpha)$.

4. Particular case $\alpha = 3$ when $d = 3$

In this subsection we return to 3 dimensions and consider the particular case $\alpha = 3$ for which $\delta$ vanishes. In Figure 8 we have represented $\log \Delta P_{F}/P_0$ as a function of $\log \lambda^{-1}$ for $\alpha = 3$ and several values of $\beta$.

From the linear behavior for small $\lambda^{-1}$, we see that the corrections are still given by a power law. However, the slope is now 6, thereby establishing that the signature is given by the square of $(H/\Lambda)^{\alpha}$. This dominant power of $\lambda^{-1}$, i.e., the slope, is stable against changes of $\beta$, but its coefficient, related to the intercept, is not. We verified that this coefficient depends linearly on $(1 - \beta)^{-1}$, like the coefficient $\tilde{N}$ of the second correction term in the dispersion relation (see eq. (22)), but that it is not simply proportional to $\tilde{N}$. Thus when $\delta$ vanishes, the dominant deviation of the spectrum comes both from the square of the first non-linear term in the dispersion relation and from a linear contribution of the second non-linear term.

5. Fast oscillations

In various works [10, 12, 13], it was found that the deviations from the standard spectrum display oscillations with a high frequency proportional to $\Lambda/H$. In these works the UV scale $\Lambda$ was introduced through the choice of the initial vacuum which was imposed when $P_{in} = k/a_{in} = \Lambda$ (i.e. $x_{in} = 2\lambda$ in our notations). Depending on the adiabaticity of the choice of the vacuum, the power in $H/\Lambda$ of the norm of the deviations runs from 1 [10] to 3 [12], but in all cases, the deviations oscillate.
with a high frequency. In Ref. [17], the origin of these oscillations was attributed to the instantaneous characterization of the initial state. It was also shown that the oscillations are suppressed when smearing the UV scale (completely or partially depending on the width of $\lambda$ being larger or smaller than $H$).

Given this, it is interesting to look for oscillations when the UV scale $\Lambda$ is introduced through a dispersion relation. In agreement with the conclusions of Ref. [17], we find that the deviations display no oscillations when the dispersion relation is smooth, and when the initial state is sufficiently close to the asymptotic adiabatic vacuum. When instead the dispersion relation possesses a kink or a bump near a given UV scale, fast oscillations appear in the deviations. [40] We have verified this in several examples.

From this we get two conclusions. On the one hand, fast oscillations of the type found in Refs. [10, 12, 13, 15] can also be obtained with dispersion even when working in the asymptotic adiabatic vacuum. On the other hand, however, to obtain them requires some odd feature in the dispersion relation well localized near a UV scale. Thus fast oscillations (with significant amplitude when compared to the deviations) cannot be considered as a generic property of the deviations of the spectra engendered by modifying the physics in the UV sector.

D. Power spectrum for $H/\Lambda \geq 1$

In our parameterization of the dispersion relation the asymptotic vacuum is well defined for all values of $H/\Lambda$. Hence we can investigate the properties of the power spectrum in the much less often studied regime where $H$ is comparable and greater than $\Lambda$. (To our knowledge, this was never done before with dispersion, see [20, 27] for an analysis in the presence of dissipative effects.) It is also of value to note that the simplest approach to the trans-Planckian question where there is no dispersion but where the quantum state of the field is fixed at some scale $\Lambda$ (see [10, 12, 14]), is not defined in the regime $H/\Lambda > 1$.

Figure 10 represents $\Delta P_E/P_0$ as a function of $\log \lambda^{-1}$ for one super- and three subluminous dispersion relations with $\alpha = 2$. The main plot shows that the different spectra, that merge into each other for $\alpha \ll 1$, depart from each other and depend strongly on $\lambda$ when $\lambda$ approaches 1. However, for $\lambda \ll 1$, all curves asymptote to a constant value that strongly depends on $\beta$. As already pointed out in Section III A, the asymptotic constant behavior can be understood from eq. [20]: when $\lambda \ll 1$ there is actually no dispersion before and at horizon exit, but the modified speed of the modes is $\beta \neq 1$. Thus, when dealing with eq. [20], the power spectrum does not depend on $\lambda$ and is equal to $P_0/\beta^3$ (for all $\beta$, lower or greater than 1), in agreement with what we see in Figure 10.

Thus, as expected, when the condition $\lambda \gg 1$ is not satisfied, the knowledge of the full dispersion relation is needed to make predictions. However, the power spectrum is still well defined as a VEV when the asymptotic vacuum makes sense. Therefore there is a priori no reason to exclude the possibility that the regime $H \geq \Lambda$ be relevant for observational cosmology. And were this the case, this would complicate the identification of the slow roll parameters, not to mention the reconstruction of the inflaton potential.

IV. SLOW-ROLL INFLATION

To show that the above results apply mutatis mutandis to slow-roll inflation, we need to say a few words about the kinematics of slow-roll and power law inflation. A large class of realistic quasi-de Sitter backgrounds can be described in the framework of the slow-roll approximation where the evolution of the Hubble scale is characterized by the smallness of the parameters

$$\epsilon_1 = -\frac{1}{H} \partial_\eta \ln H \ll 1,$$

$$\epsilon_2 = \frac{1}{2H} \partial_\eta \ln \epsilon_1 \ll 1.$$

The linear order slow-roll approximation consists in keeping only the terms that are linear in $\epsilon_1$ and $\epsilon_2$, as well as treating these parameters as constants. This requires that the condition $\partial_\eta \epsilon_2 = O(\epsilon_1^2, \epsilon_2^2, \epsilon_1 \epsilon_2)$ be fulfilled. In what follows, expressions in slow-roll inflation should be understood as given to linear order in $\epsilon_1$ and $\epsilon_2$. For the particular case of power-law inflation, $\epsilon_1$ is constant, $\epsilon_2 = 0$, and the scale factor is exactly given by:

$$a_{pl} = \left(\frac{1}{-H_0 \eta}\right)^{1/(1-\epsilon_1)}.$$

When $\epsilon_2 \neq 0$, to linear order in the slow-roll parameters,
\(a_{sl}\) only depends on \(\epsilon_1\):

\[
a_{sl} \simeq \left(\frac{1}{-H_0 \eta}\right)^{1+\epsilon_1},
\]

which agrees to (36) to first order in \(\epsilon_1\). This equation follows from

\[
\eta = \int \frac{da}{a^2 H} = -\frac{1+\epsilon_1}{a H} + O(\epsilon_1^2, \epsilon_2, \epsilon_1 \epsilon_2).
\]

In slow-roll inflation, the time-dependent term in the square frequency of the tensor modes is

\[-\frac{\partial_+^2 a}{a}, \quad \text{as in eq. (3),}\]

whereas that of scalar modes is

\[-\frac{\partial_+^2 f}{f} \quad \text{with} \quad f = \alpha \sqrt{\tau_i} \Rightarrow \text{[3].}\]

In power-law and in first order slow-roll inflation, these terms are always of the form \(-\mu \eta^2\) where \(\mu\) is a constant. In power-law inflation, the value of the constant is the same for scalar (S) and tensor (T) modes and is given by

\[\mu_{pl} = \frac{2-\epsilon_1}{(1-\epsilon_1)^2}.\]

For slow-roll inflation, the values differ and are respectively

\[\mu_T \simeq 2 + 3\epsilon_1,\]

\[\mu_S \simeq 2 + 3(\epsilon_1 + \epsilon_2).\]

Given the above discussion, we can deduce the properties of dispersion-induced modifications of the spectra for scalar or tensor modes in slow-roll inflation, by simply comparing the mode equation to that of the test field in a de Sitter background. Moreover, when considering the leading modification of the spectrum, since it is governed by the lowest order non-linear term of \(F\), we can restrict our comparison of the mode equations to the following. In de Sitter space, in \(d\) spatial dimensions, the relevant terms for a test field are

\[
\left(\partial_+^2 + \frac{x}{4\lambda} \phi + \frac{d^2 - 1}{4x^2} \mu \right) \phi = 0.
\]

In slow-roll inflation, in 3 dimensions, using again the variable \(x = -k \eta\), they are given by

\[
\left(\partial_+^2 + \frac{x}{2\lambda(\epsilon_1, k)} \phi + \frac{\mu}{x^2} \right) \phi_k = 0,
\]

where the constant \(\mu\) is \(\mu_{pl}, \mu_T\) or \(\mu_S\). In power-law inflation, the new parameters are

\[\alpha_{1,pl} = \frac{\alpha}{1-\epsilon_1},\]

\[2\lambda_{pl}(\epsilon_1, k) = \frac{1}{1-\epsilon_1} \left(\frac{\Lambda}{H_k}\right)^{1-\epsilon_1},\]

where \(H_k\) is

\[H_k = \frac{k}{a_{pl}(\eta_k)} = \left(\frac{H_0}{1-\epsilon}\right)^{1/(1-\epsilon)} k^\epsilon/(1-\epsilon).\]

In first order slow-roll inflation, \(\lambda_{sl}(\epsilon_1, k)\) is given by the linearization in \(\epsilon_1\) of eq. (41).

Comparing equation (12) with eq. (11), one sees that the results of the previous sections apply with \(\lambda\) replaced by \(\lambda(\epsilon_1, k)\), \(\alpha\) by \(\alpha_1\), and \(d\) given by

\[d_\epsilon = \sqrt{4\mu + 1}.
\]

The condensed notation \(d_\epsilon\) indicates a dependence on both \(\epsilon_1\) and \(\epsilon_2\), and \(\mu\) can be \(\mu_{pl}\), \(\mu_T\) or \(\mu_S\) according to the case considered. Thus, if we denote by \(P_{0,\epsilon}\) and \(P_{a,\epsilon}\) the relativistic and modified spectrum respectively, we have, in place of eq. (29),

\[
\frac{P_{a,\epsilon} - P_{0,\epsilon}}{P_{0,\epsilon}} \sim \frac{\delta^+ (\epsilon_1, d_\epsilon)}{\lambda(\epsilon_1, k)^{\alpha_1}},
\]

where \(\lambda(\epsilon_1, k) \to \infty\). Using eqs. (43) and (44), we get

\[
\frac{P_{a,\epsilon} - P_{0,\epsilon}}{P_{0,\epsilon}} \sim \frac{\delta^+ (\epsilon_1, d_\epsilon)}{(2 - 2\epsilon_1)^{-\alpha_1}} \times \left(\frac{H_k}{\Lambda}\right)^\alpha,
\]

where the power of \(H_k/\Lambda\) is \(\alpha\), independently of \(\epsilon_1, \epsilon_2\).

Eq. (43) is the main result of the paper. For power-law inflation, it gives the leading modification to the power spectra to all orders in \(\epsilon_1\). For slow-roll inflation, it gives the leading modification at linear order in the slow-roll parameters. It applies both to scalar and tensor perturbations, with different expressions for \(d_\epsilon\), see above. It establishes that the \(k\)-dependence of the modifications to the spectra only arises through \((H_k/\Lambda)^\alpha\).

Given the above substitutions, the overall coefficient of \((H_k/\Lambda)^\alpha\) vanishes for the discrete set of values

\[\alpha_{i,S} = d_\epsilon + 2i.\]

Using (48), (39) and (10), one finds

\[\alpha_{i,pl} = \sqrt{9 - 6\epsilon_1 + \epsilon_1^2} + 2i,\]

\[\alpha_{i,T,sl} = 3 - \epsilon_1 + 2i,\]

\[\alpha_{i,S,sl} = 3 - \epsilon_1 - 2i + 2i.\]

For these values, the leading modification scales with a power higher than \(\alpha_i\).

A few remarks are in order here. Firstly, as long as \(H_k/\Lambda \ll 1\), one can extend the correspondence between de Sitter space and slow-roll inflation to an arbitrary number of terms of a general dispersion relation. Indeed, writing the \(n^{\text{th}}\) term of the dispersion relation as \(c_n(P/\Lambda)^{\alpha_n}\), this yields \(c_n(x/2\lambda)^{2\alpha_n}\) in the wave equation in de Sitter space. Thus, as above, the corresponding term in slow-roll inflation will be given by \(\lambda \to \lambda(\epsilon_1, k)\), \(\alpha_n \to \alpha_n(1 - \epsilon_1)\) (or \(\alpha_n \to \alpha_n(1 + \epsilon_1)\) in slow roll inflation). This correspondence term by term can be used to predict the power of \(\lambda(\epsilon_1, k)\) for an arbitrary number of subleading modifications to the spectra.

Secondly, in the particular cases where the dispersion relation is of the type (20) when \(N \to \infty\), i.e. where
all powers of $P/\Lambda$ are multiples of $\alpha$, the complete mode equation for slow-roll inflation can be obtained by making the above replacements directly in the effective frequency of eq. (21). Thus in this case, the complete modification to the spectra in slow-roll inflation can be obtained from the de Sitter result, for arbitrary values of $H_k/\Lambda$.

V. CONCLUSIONS

In this paper, we have determined the properties of the modifications of the inflationary power spectra engendered by dispersion when the quantum state is the asymptotic vacuum. When the leading non-linear term in the dispersion relation is $(P/\Lambda)^\alpha$, see eq. (22), we have established that the following holds in the regime $H/\Lambda \ll 1$,

- The leading modification of the spectrum behaves as $(\delta(\alpha)(H/\Lambda)^\alpha$, for all values of $\alpha$, except for a discrete set of values where $\delta$ vanishes.
- This leading modification is robust in the sense that when adding to the dispersion relation terms containing higher powers of $P/\Lambda$ than $\alpha$, these give rise to subdominant modifications of the power spectrum, as clearly shown in Figure 6.
- For all values of the power $\alpha$, except those where $\delta$ vanishes, the leading modification of the spectrum has equal magnitude and opposite sign when comparing sub- and superluminous dispersion, see Figures 7 and 8.
- When the state is the asymptotic vacuum, the modifications of the spectra display no oscillations, unless the dispersion relation has some localized bump that would engender some non-adiabaticity at that scale.
- These results, which have been derived in de Sitter space, apply to slow-roll and power-law inflation, both for scalar and tensor modes, by replacing $H$ by $H_k = k/a_k$ and adjusting some constants which appear in the mode equation, see eq. (18).

In brief, when $H_k/\Lambda \ll 1$, when the evolution stays adiabatic, and when the state is the asymptotic vacuum, we see that it is not necessary to exactly know the dispersion relation, since only the first non-linear term matters. In particular, in slow-roll inflation, this implies that the dependence on $k$ of the modifications only arises through $(H_k/\Lambda)^\alpha$. We also learn that, when $\alpha$ is unknown, no general prediction concerning the properties of the signatures of high energy dispersion can be drawn even in the regime $H/\Lambda \ll 1$.

We have also analyzed the ‘non-adiabatic’ regime where $H \geq \Lambda$ that might be relevant in some theories with large extra-dimensions. The modifications of the spectrum are still well defined and governed by the dispersion (when working in the asymptotic vacuum) even though they are no longer governed by the lowest order non-linear term of the dispersion relation, see Figure 10.

To conclude we discuss to what extent the modifications to the power spectrum we obtain could be distinguished, given some observations, from some other change in the scenario, such as, for instance, a change in the inflaton potential.

The main point is that the absence of (rapid) oscillations in eq. (18) greatly complicates this identification, since smooth deformations of the spectra can be accounted for by various means. As a consequence, the project of reconstructing ‘the’ inflaton potential cannot be pursued without making some hypothesis concerning the dispersion relation of the fluctuation modes.

Second, when tensor and scalar fluctuations obey different dispersion relations, the consistency condition between their spectra is modified, as one might have expected. What is less trivial is that, in slow roll inflation ($\epsilon_2 \neq 0$), even if both types of fluctuations are subject to the same dispersion relation, the modifications of the spectra do not coincide, since the prefactor $\delta$ in eq. (18) depends on the value of the parameter $\mu$, see eqs. (39), (40), and (41). This remark can be important in scenarios where $H \geq \Lambda$.

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Appendix A: WKB EVALUATION OF THE FIRST CORRECTIONS

We show that WKB waves can be used to reproduce the main features of the analytical solution (10), to explain their physical origin, and to draw predictions for the corrections induced by an arbitrary dispersion relation. However the normalisation of the deviations of the spectrum cannot be obtained by this method.

The basis of the argument is that the exact solution of eq. (5) can always be rewritten as a combination of WKB modes:

$$u(x) = C(x) v(x) + D(x) (v(x))^*, \quad (A1)$$

where

$$v(x) = \frac{1}{\sqrt{2\omega(x)}} e^{i \int \omega(x') dx'}, \quad (A2)$$

$$\omega(x) = \sqrt{\frac{1}{H^2 x^2} F^2 (H^2 x^2) - \frac{2}{x^2}}.$$

Such a decomposition, however, introduces two unknown functions and is thus underconstrained. Following section
We also assume that there is a second interval later on where the contribution from the saddle point is approximately constant in it. We restrict attention to the dispersion relations such that this is realized for $x \to \infty$. In this case the $\text{in}$-mode is simply the asymptotic positive frequency WKB solution, the asymptotic values of $\mathcal{C}(x)$ and $\mathcal{D}(x)$ are $\mathcal{C} = 1$ and $\mathcal{D} = 0$. We also assume that there is a second interval later on where this condition is realized. Then $\mathcal{D}$ is constant in this interval with a non-zero value $\mathcal{D}_f$. This value represents the probability amplitude of non-adiabatic transitions and is computed to first order by setting $\mathcal{C} = 1$ in (A5):

$$\mathcal{D}_f \approx -\int_0^\infty dx \frac{\partial_x \omega}{2\omega} e^{+2i \int^x \omega'(x')}.$$  

This integral can be evaluated by a saddle point method when the non-adiabaticity is weak.

We perform this calculation in the particular case of the quartic dispersion relation of equation (10). To start the analysis we first drop the $-\frac{\omega^4}{2}$ term in $\omega(x)$, which allows us to integrate all the way from $x = +\infty$ to $x = 0$. The path of integration can be deformed continuously into $i \mathbb{R}_+$. The integral is dominated by the contribution from the saddle point $x^* = 2i \lambda$ where $\omega(x^*) \approx \sqrt{1 + \frac{x^*}{\lambda}} = 0$. Using

$$\int_0^x \sqrt{1 + \frac{x'^2}{4\lambda^2}} dx' = \text{arsinh} \left( \frac{x}{2\lambda} \right) + x \sqrt{1 + \frac{x^2}{4\lambda^2}},$$

this yields

$$|\mathcal{D}_f| = d e^{2i \text{Im} \int^x \omega(x') dx} = d e^{-\pi \lambda},$$

where the constant $d$ can be shown to be equal to 1 in the limit $\lambda \gg 1$. In this limit, $\mathcal{D}_f$ being exponentially suppressed, (A3) makes the approximation $\mathcal{C} \approx 1$, and the squared amplitude of the mode for $1 \ll |x| \ll \sqrt{\lambda}$ is given by:

$$|u^{\text{in}}(x)|^2 \approx |v(x)|^2 \left( 1 + 2 e^{-\pi \lambda} \cos 2\varphi(x) + e^{-2\pi \lambda} \right),$$

where $\varphi(x)$ is the phase of $v(x)$ accumulated from zero to $x$. The important point for us is that $\mathcal{D}_f$ is exponentially suppressed and not power law suppressed as $\lambda \to \infty$.

We can now qualitatively understand the form of the rhs in (17) by the following argument: neglecting the term $O(e^{-2\pi \lambda})$, before horizon exit, but before the WKB approximation completely ceases to be valid, the ratio of the instantaneous values of the power spectrum with and without dispersion is,

$$\frac{P^2}{P_0^2}(x) \approx \frac{|v^2(x)|^2}{|v_0^2(x)|^2} \left( 1 + 2 e^{-\pi \lambda} \cos 2\varphi(x) + O(1) \right) \approx \frac{\omega_0(x)}{\omega(x)} \left( 1 + 2 e^{-\pi \lambda} \cos 2\varphi(x) \right),$$

where $\omega_0(x)^2 = 1 - \frac{2}{x^2}$. To first order in $(F^2 - P^2)/P^2$ we have:

$$\frac{\omega_0(x)}{\omega(x)} \approx 1 - \frac{F^2 - P^2}{2P^2}.$$  

For the quartic case, one thus has

$$\frac{\omega_0(x)}{\omega(x)} \approx 1 - \frac{x^2}{8\lambda^2}.$$  

Thus, eqs. (A12) and (A10) allow us to explain the origin of the features observed in (17):

i) The exponentially small corrections come from the non adiabatic transitions engendered by the dispersion. They are thus the result of a global (cumulative) effect.

ii) The polynomial corrections in (17) can be viewed as an imprint of the different normalisation between the relativistic and dispersive modes, around horizon exit (42). These corrections are thus local in the sense that they involve only the late time behavior of the modes. The fact that these corrections scale as $\lambda^{-2}$ directly comes from the power of $\lambda$ in the non relativistic term in the modified dispersion relation. (In addition, since the normalisation affects equally the positive and negative frequency WKB modes, this explains why the polynomial corrections are in factor of both the adiabatic and non-adiabatic terms in (17).)

We see in (A12) that the normalisation of the superluminous dispersive mode is always smaller than the relativistic normalisation, since $\omega_2(x) > \omega_0(x)$ for all $x$. From this it is tempting to deduce that the deviation of the power spectrum is always negative for superlumino- nus dispersion relations, and positive for subluminous ones. However this conjecture is not correct: the numerical analysis shows that the sign of the deviations flips at $\alpha = 3$. Therefore the above treatment of the corrections to the WKB approximation is only indicative when there is mode amplification, that is, strong departure from adiabaticity. Thus it cannot be used to explain why the leading signatures of sub- and superluminous dispersion found in Figures 7 have exactly the opposite sign.

However this treatment does contain correct elements. To show this, we consider in the next appendix a toy model in which the UV evolution is identical to that considered here but in which there is no horizon exit, and
therefore no mode amplification. From the exact solutions, we shall see that only the global exponentially suppressed corrections remain, thereby demonstrating that the polynomial corrections arise near horizon exit and are linked with the mode amplification (the instability) occurring after horizon exit.

Appendix B: GLOBAL CORRECTIONS AND BACK-SCATTERING

In this appendix, we consider a toy model where the evolution is governed by the same mode equation as in \[ \text{(11)} \] but with the sign of the non-conformal term \(-\frac{\gamma}{x^2}\) reversed:

\[
\left( \partial_x^2 + 1 + \frac{\mu^2}{x^2} \right) \phi = 0 .
\]

(B1)

we have also slightly generalized the situation by introducing the extra parameter \(\mu\).

In the present case there is no mode amplification since the square frequency remains positive. We thus compare the pair creation probabilities with and without dispersion, and show that only non-adiabatic, exponentially suppressed corrections appear.

It is worth noticing the relationship between equation \(\text{(11)}\) and the wave equation for a massless scalar field in a black hole spacetime. When studying the radial wave function of s-waves in the momentum representation, after having factorized out the dependence on the Killing frequency (see for instance the appendix of \[29\] for details) one obtains in the near horizon region the same equation as that of eq. \(\text{(11)}\). Thus we do not exclude that the following exercise be relevant for the trans-Planckian signature in Hawking radiation.

1. Particle creation rate without dispersion

In the absence of dispersion, the wave equation \(\text{(11)}\) reads:

\[
\left( \partial_x^2 + 1 + \frac{2\mu^2}{x^2} \right) \phi = 0 .
\]

(B2)

where we restrain ourselves to \(\mu > 1/\sqrt{8}\) not to have mode amplification when \(x \to 0\).

The asymptotic \(\text{in}\)-mode with positive frequency mode with respect to the time \(\eta \propto -x\), for \(x \to \infty\) is:

\[
\phi^{\text{in}} = \sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{4} x} H^{(1)}_{i\gamma} (x) ,
\]

(B3)

where \(H^{(1)}_{i\gamma}\) is the Hankel function of the first kind, and \(\gamma = \frac{i}{4} \sqrt{8\mu^2 - 1}\). When \(x \to 0^+\), the positive frequency mode is:

\[
\phi^{\text{out}} = \sqrt{\frac{\pi}{2 \sinh \pi \gamma}} \sqrt{x} J_{i\gamma} (x) ,
\]

(B4)

where \(J_{i\gamma}\) is the Bessel function of the first kind.

We want to compute the Bogoliubov coefficients defined by:

\[
\phi^{\text{in}} = \alpha \phi^{\text{out}} + \beta \phi^{\text{out}}^* .
\]

(B5)

The squared modulus of \(\beta\) gives the particle creation probability. Using the identity:

\[
H^{(1)}_{i\gamma} = \frac{\cosh \frac{\pi x}{2}}{\sinh \frac{\pi x}{2}} J_{i\gamma} - \frac{1}{\sinh \frac{\pi x}{2}} J_{-i\gamma} ,
\]

(B6)

we get:

\[
\beta_0 = -\frac{1}{\sqrt{2 \pi \gamma - 1}} .
\]

(B7)

2. Particle creation rate in the presence of dispersion

We now consider the wave equation \(\text{(13)}\). The positive frequency \(\text{in}\)-mode is:

\[
\phi^{\text{in}} = \frac{\sqrt{\pi} e^{-\frac{\pi}{4} x}}{\sqrt{x}} W_{i\frac{1}{2}, i\frac{1}{2}} \left( -\frac{x^2}{2\lambda} \right) ,
\]

(B8)

where \(W_{i\frac{1}{2}, i\frac{1}{2}}\) is another Whittaker function defined in \[26\].

In this case as well, the Bogoliubov coefficients can be directly read from the identity (equation 13.1.34 in \[26\]):

\[
W_{i\frac{1}{2}, i\frac{1}{2}} = \frac{\Gamma(-i\gamma)}{\Gamma(\frac{1}{2} + i\frac{1}{2} - i\frac{1}{2})} M_{i\frac{1}{2}, i\frac{1}{2}} M_{i\frac{1}{2}, i\frac{1}{2}}^* \left( M_{i\frac{1}{2}, i\frac{1}{2}} \right)^* .
\]

(B10)

Hence we get

\[
\beta_{2+} = -i \sqrt{\pi} e^{-\frac{\pi}{4} x} \Gamma(i\gamma) \Gamma(\frac{1}{2} + i\frac{1}{2} - i\frac{1}{2}) .
\]

(B11)

The squared modulus simplifies

\[
|\beta_{2+}|^2 = \frac{1}{e^{2\pi \gamma - 1}} (1 + e^{-\pi \lambda e^{\pi \gamma}}) = |\beta_0|^2 (1 + e^{-\pi \lambda e^{\pi \gamma}}).
\]

(B12)
From the second equation, one immediately sees that in the limit $\lambda \to \infty$ the pair creation probability in the absence of dispersion is recovered. And, as announced, in the case when there is no mode amplification, only exponentially suppressed corrections are present, and these correspond to non-adiabatic effects. A simplified version of this result can be found in a black hole context in subsection 5.2 of [30].

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[39] There is *a priori* no reason for the local frame, which governs the violation of Lorentz Invariance in the UV, to coincide with the cosmological frame which is associated with the homogeneous inflationary patch. However, when taking into account the fact that the orientation of the local frame should be dynamically determined [31], we expect on general grounds that it will progressively align with the cosmological orientation as inflation proceeds [34]. However the coupling between the inflaton and a dynamical unit timelike vector field specifying the preferred frame could also be described by the parameterization given in eq. (37) of Ref. [17]. The modifications found in that reference followed from a different scheme where there is no dispersion but where the state of the modes is imposed when $k/(aH) = \Lambda$. We also notice that the parameter $B$ in eq. (37) is zero in our case since we find no oscillations.
[40] Another way to obtain fast oscillations related to a non-adiabatic evolution has been considered in Refs. [32, 33]. In these works, the non-adiabaticity results from the fact that the proper frequency $\Omega = F(P)$ becomes smaller than $H$ for some high momentum $P$.
[41] It is interesting to note that the modifications we find can be described by the parameterization given in eq. (37) of Ref. [17]. The modifications found in that reference followed from a different scheme where there is no dispersion but where the state of the modes is imposed when $k/(aH) = \Lambda$. We also notice that the parameter $B$ in eq. (37) is zero in our case since we find no oscillations.
[42] This statement can be made precise using the interesting techniques presented in [33]: one can factorize a singular part in the mode solution so that the WKB approximation be applicable to the remaining part after horizon exit. The power spectrum can then be computed in this approximation. However, we checked that the normalization of the leading deviation cannot be obtained in this way. In particular, this approximate calculation cannot explain why this coefficient vanishes for $\alpha = d$. 