A Bound for Orders in Differential Nullstellensatz

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Abstract

We give the first known bound for orders of differentiations in differential Nullstellensatz for both partial and ordinary algebraic differential equations. This problem was previously addressed in [1] but no complete solution was given. Our result is a complement to the corresponding result in algebraic geometry, which gives a bound on degrees of polynomial coefficients in effective Nullstellensatz [2,3,4,5,6,7,8,9].

This paper is dedicated to the memory of Eugeny Pankratiev, who was the advisor of the first three authors at Moscow State University.

Key words: differential algebra, characteristic sets, radical differential ideals, differential Nullstellensatz

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1 Introduction

Given a system of algebraic partial differential equations \( F = 0 \), where \( F = f_1, \ldots, f_k \), and a differential equation \( f = 0 \), one can effectively test if \( f \) is a differential algebraic consequence of \( F \). In this paper we develop a method that leads to an effective procedure which finds an algebraic expression of some power of \( f \) in terms of the elements of \( F \) and their derivatives (or shows that such an expression does not exist). This procedure is called \textit{effective differential Nullstellensatz}. A brute-force algorithm solving this problem consists of two steps:

1. find an upper bound \( h \) on the number of differentiations one needs to apply to \( F \) and
2. find an upper bound on the degrees of polynomial coefficients \( g_i \) and a positive integer \( k \)

such that \( f^k \) is a combination of the elements of \( F \) together with the derivatives up to the order \( h \) and the coefficients \( g_i \). We solve the first problem in the paper. The second problem was addressed and solved in [2] and further analyzed and improved in [3,4]. A purely algebraic solution was given in [5]. Most of the references on the subject can be found in [6,7,8,9].

More precisely, our problem is as follows. We are given a finite set \( F \) of differential polynomials such that a differential polynomial \( f \) belongs to the radical differential ideal generated by \( F \) in the ring of differential polynomials. Knowing \textbf{only} the orders and degrees of the elements of \( F \) and the order of \( f \), we find a non-negative integer \( h \) such that \( f \) belongs to the radical of the algebraic ideal generated by \( F \) and its derivatives up to the order \( h \).

We give a complete solution to this problem using differential elimination. The problem is non-trivial: the first (unsuccessful) attempt was made by Seidenberg [1], where it was conjectured that most likely such a bound would not be found. Here is where the main difficulty is coming from. In order to get the bound using a differential elimination algorithm we need to estimate how many differentiation steps this algorithm makes. Originally, termination proofs for such algorithms were based on the Ritt-Noetherianity of the ring of

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differential polynomials, that is: every increasing chain of radical differential ideals terminates. And this result does not say when the sequence terminates. We overcome this problem in our paper.

The article is organized as follows. We introduce basic notions of differential algebra in Section 2. Then we formulate the main result, Theorem 1, in Section 3. In order to achieve this, we bound the length of increasing sequences of radical differential ideals appearing in our differential elimination in Section 5 (see Proposition 9). For that, in Section 4 we first bound the length of dicksonian sequences of tuples of natural numbers with restricted growth of the maximal element in these tuples (Lemma 8). Finally, we apply this to obtain the bound for the differential Nullstellensatz in Theorem 15, from which Theorem 1 follows. We conclude by giving in Section 6 an alternative non-constructive proof of existence of the bound, based on model theory.

There is some previous work on bounding orders in differential elimination algorithms. In the ordinary case, we can bound the orders of derivatives of the output and all intermediate steps of differential elimination [10], and this bound holds for any ranking. Also in the ordinary case, one can give bounds for quantifier elimination [11] and for the orders and degrees of resolvents of prime differential ideals of a certain type [12]. A related bound for involutive prolongation, based on the analysis of stability of Spencer sequences, is obtained in [13].

Note that, unlike the bounds for differential elimination mentioned above, the bound for the differential Nullstellensatz proposed in this paper holds for the PDE case. Our bound is also based on the analysis of differential elimination. But, due to the ranking-independent nature of the differential Nullstellensatz, we could restrict our analysis to orderly rankings, which allowed us to treat not only the ordinary case, but the PDE case as well.

2 Basic differential algebra

One can find recent tutorials on the constructive theory of differential ideals in [14,15,16]. One also refers to [1,17,18,19,20,21,22,23,24,25,26] for differential elimination theory. A differential ring is a commutative ring with unity endowed with a set of derivations \( \Delta = \{ \partial_1, \ldots, \partial_m \} \), which commute pairwise. The case of \( \Delta = \{ \delta \} \) is called ordinary. Construct the multiplicative monoid \( \Theta = \{ \partial_1^{k_1} \partial_2^{k_2} \cdots \partial_m^{k_m} \mid k_i \geq 0 \} \) of derivative operators. Let \( Y = \{ y_1, \ldots, y_n \} \) be a set whose elements are called differential indeterminates. The elements of the set \( \Theta Y = \{ \theta y \mid \theta \in \Theta, y \in Y \} \) are called derivatives. Derivative operators from \( \Theta \) act on derivatives as \( \theta_1(\theta_2 y_i) = (\theta_1 \theta_2) y_i \) for all \( \theta_1, \theta_2 \in \Theta \) and \( 1 \leq i \leq n \).
The ring of differential polynomials in differential indeterminates $Y$ over a differential field $k$ is a ring of commutative polynomials with coefficients in $k$ in the infinite set of variables $\Theta Y$. This ring is denoted by $k\{y_1, \ldots, y_n\}$. We consider the case of $\text{char} k = 0$ only. An ideal $I$ in $k\{y_1, \ldots, y_n\}$ is called differential, if for all $f \in I$ and $\delta \in \Delta$, $\delta f \in I$. Let $F \subset k\{y_1, \ldots, y_n\}$ be a set of differential polynomials. For the differential and radical differential ideal generated by $F$ in $k\{y_1, \ldots, y_n\}$, we use notations $[F]$ and $\{F\}$, respectively.

A ranking is a total order $>$ on the set $\Theta Y$ satisfying the following conditions for all $\theta \in \Theta$ and $u, v \in \Theta Y$:

1. $\theta u \geq u$,
2. $u \geq v \Rightarrow \theta u \geq \theta v$.

Let $u$ be a derivative, that is, $u = \theta y_j$ for $\theta = \partial_1^{k_1} \partial_2^{k_2} \cdots \partial_m^{k_m} \in \Theta$ and $1 \leq j \leq n$. The order of $u$ is defined as

$$\text{ord } u = \text{ord } \theta = k_1 + \ldots + k_m.$$ 

If $f$ is a differential polynomial, $f \not\in k$, then ord $f$ denotes the maximal order of derivatives appearing effectively in $f$.

A ranking $>$ is called orderly if ord $u >$ ord $v$ implies $u > v$ for all derivatives $u$ and $v$. Let a ranking $>$ be fixed. The derivative $\theta y_j$ of the highest rank appearing in a differential polynomial $f \in k\{y_1, \ldots, y_n\} \setminus k$ is called the leader of $f$. We denote the leader by $\text{ld } f$ or $u_f$. Represent $f$ as a univariate polynomial in $u_f$:

$$f = 1_f u_f + a_1 u_f^{d_1} + \ldots + a_d.$$ 

The monomial $u_f^d$ is called the rank of $f$ and is denoted by $\text{rk } f$. Extend the ranking relation on derivatives to ranks: $u_1^{d_1} > u_2^{d_2}$ if either $u_1 > u_2$ or $u_1 = u_2$ and $d_1 > d_2$. The polynomial $1_f$ is called the initial of $f$. Apply any derivation $\delta \in \Delta$ to $f$:

$$\delta f = \frac{\partial f}{\partial u_f} \delta u_f + \delta_1 u_1 + \delta a_1 u_1^{d_1} + \ldots + \delta a_d.$$ 

The leader of $\delta f$ is $\delta u_f$ and the initial of $\delta f$ is called the separant of $f$, denoted $s_f$. If $\theta \in \Theta \setminus \{1\}$, then $\theta f$ is called a proper derivative of $f$. Note that the initial of any proper derivative of $f$ is equal to $s_f$.

We say that a differential polynomial $f$ is partially reduced w.r.t. $g$ if no proper derivative of $u_g$ appears in $f$. A differential polynomial $f$ is algebraically reduced w.r.t. $g$ if $\deg_{u_g} f < \deg_{u_g} g$. A differential polynomial $f$ is reduced w.r.t. a differential polynomial $g$ if $f$ is partially and algebraically reduced w.r.t. $g$. Consider any subset $\mathcal{A} \subset k\{y_1, \ldots, y_n\} \setminus k$. We say that $\mathcal{A}$ is autoreduced (respectively, algebraically autoreduced) if each element of $\mathcal{A}$ is reduced (respectively, algebraically reduced) w.r.t. all the others.
Every autoreduced set is finite [18, Chapter I, Section 9] (but an algebraically autoreduced set in a ring of differential polynomials may be infinite). For autoreduced sets we use capital calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ and notation $\mathcal{A} = A_1, \ldots, A_p$ to specify the list of the elements of $\mathcal{A}$ arranged in order of increasing rank. We denote the sets of initials and separants of elements of $\mathcal{A}$ by $i_\mathcal{A}$ and $s_\mathcal{A}$, respectively. Let $H_\mathcal{A} = i_\mathcal{A} \cup s_\mathcal{A}$. For a finite set $S$ of differential polynomials denote by $S^\infty$ the multiplicative set containing 1 and generated by $S$. Let $I$ be an ideal in a commutative ring $R$. The saturated ideal $I : S^\infty$ is defined as $\{ a \in R \mid \exists s \in S^\infty : sa \in I \}$. If $I$ is a differential ideal then $I : S^\infty$ is also a differential ideal (see [18]).

Let $\mathcal{A} = A_1, \ldots, A_r$ and $\mathcal{B} = B_1, \ldots, B_s$ be (algebraically) autoreduced sets. We say that $\mathcal{A}$ has lower rank than $\mathcal{B}$ if

- there exists $k \leq \min(r, s)$ such that $\text{rk} A_i = \text{rk} B_i$ for $1 \leq i < k$, and $\text{rk} A_k < \text{rk} B_k$,
- or if $r > s$ and $\text{rk} A_i = \text{rk} B_i$ for $1 \leq i \leq s$.

We say that $\text{rk} \mathcal{A} = \text{rk} \mathcal{B}$ if $r = s$ and $\text{rk} A_i = \text{rk} B_i$ for $1 \leq i \leq r$. Let $v$ be a derivative in $k\{y_1, \ldots, y_n\}$. Denote by $\mathcal{A}_v$ the set of the elements of $\mathcal{A}$ and their derivatives that have a leader ranking strictly lower than $v$. A set $\mathcal{A}$ is called coherent if whenever $A, B \in \mathcal{A}$ are such that $u_A$ and $u_B$ have a common derivative: $v = \psi u_A = \phi u_B$, then $s_B \psi A - s_A \phi B \in (\mathcal{A}_v) : H_\mathcal{A}^\infty$.

### 3 Main result

For a finite set of differential polynomials $F \subset k\{y_1, \ldots, y_n\}$ let $D(F)$ be the maximal total degree of a polynomial in $F$. For each $i, 1 \leq i \leq n$, let

$$h_i(F) = \text{ord}_{y_i}(F), \quad H(F) = \max_{1 \leq i \leq n} h_i(F).$$

For $h \in \mathbb{Z}_{\geq 0}$ let $F^{(\leq h)}$ denote the set of derivatives of the elements of $F$ of order less than or equal to $h$. The Ackermann function appearing in our main result is defined as follows [27, Section 2.5.5]:

$$A(0, n) = n + 1$$
$$A(m + 1, 0) = A(m, 1)$$
$$A(m + 1, n + 1) = A(m, A(m + 1, n)).$$

**Theorem 1** Let $F \subset k\{y_1, \ldots, y_n\}$ be a finite set, $0 \neq f \in \{F\}$ and let
PROOF. This result will be proved step-by-step in the following sections as described in the introduction and finally established in Theorem 15.

Remark 2 It is our own choice here to bound \( t(F, f) \) using solely the maximal orders and degrees of \( F \) and \( f \). One might come up with another bound using more information of \( F \) and \( f \). But we emphasise that the bound on orders must depend on the degrees, number of differential indeterminates, and number of basic differentiations as the following examples show.

Example 3 Let \( f = 1 \) and \( F = \{y' - 1, y^k\} \) in \( k\{y\} \), the ordinary case. In order to express 1 in terms of the elements of \( F \), one has to differentiate \( y^k \) \( k \) times.

In the linear and non-linear cases, consider the following examples showing that the bound must depend on the number of variables and derivations.

Example 4 Let \( f = 1 \) and \( F = \{y_1', y_2' - y_2', \ldots, y_{n-1}' - y_{n-1}', y_n - a\} \) in the ordinary differential ring \( k\{y_1, \ldots, y_n\} \), where \( a \in k \) is such that \( a^{(n)} = 1 \). We have to differentiate the first \( n - 1 \) generators \( n \) times to get \( y_n^{(n)} \) into the corresponding algebraic ideal. Hence, \( t(F, f) = n \).

Example 5 Let \( f = 1 \) and \( F = \{y_1^2, y_1 - y_2^2, \ldots, y_{n-1}^2 - y_n^2, 1 - y_n'\} \subset k\{y_1, \ldots, y_n\} \), again in the ordinary case. One can show that

\[
F \subseteq (y_1, y_2, \ldots, y_n, 1 - y_n') = I_0,
F^{(\leq 1)} \subseteq \left( I_0, y_1', y_2', \ldots, y_{n-1}', y_n'' \right) = I_1,
F^{(\leq 2)} \subseteq \left( I_1, y_2'', \ldots, y_{n-2}'', y_{n-1}'', y_n^{(2)} \right) = I_2,
F^{(\leq 3)} \subseteq \left( I_2, y_3'', \ldots, y_{n-2}'', y_{n-1}'', y_n^{(4)} \right) = I_3,
F^{(\leq 4)} \subseteq \left( I_3, y_1^{(4)}, \ldots, y_{n-2}^{(4)} - 2^2 \binom{4}{2}, y_n^{(4)} \right) = I_4,
\]

\[
F^{(\leq (2n-1))} \subseteq \left( I_{2^{n-1}-1}, y_1^{(2n-1)}, \prod_{k=1}^{n-1} \binom{2k}{2k-1}^{2n-k-1} y_2^{(2n-1)}, \ldots, y_n^{(2n-1)+1} \right) = I_{2^{n-1}},
\]

\[
F^{(\leq (2n-1))} \subseteq \left( I_{2^{n-2}}, y_1^{(2n-1)}, y_2^{(2n-1)}, \ldots, y_n^{(2n)} \right).
\]
Therefore, \(1 \notin (F^\leq (2^n - 1))\). Thus, \(t(F, f) = 2^n\), because modulo \(1 - y'_n\) we have
\[
(y_n^{2n})^{(2^n)} = 2^n \left((y_n^{2n-1}) y'_n\right)^{(2^n-1)} \equiv 2^n \left((y_n^{2n-1}) (2^n-1)\right) \equiv \ldots \equiv (2^n)! (y_n y'_n)' \equiv 2^n y_n^2 \equiv 1.
\]

**Example 6** If we replace \(F\) in the previous example by
\[
G = \left\{ u_{x_1}^2, u_{x_1} - u_{x_2}^2, \ldots, u_{x_{m-1}} - u_{x_m}^2, 1 - u_{x_m} \right\} \subset k\{u\}
\]
with partial derivatives \(\partial_{x_1}, \ldots, \partial_{x_m}\), we obtain an example which shows that the bound on orders must depend on the number \(m\) of derivations. Again, the generators will have to be differentiated \(2^m\) times to express 1.

### 4 Bounds on lengths of sequences

The results of this section with be further used in Section 5 to bound lengths of decreasing sequences of autoreduced sets appearing in the differential elimination algorithm that we use. In this section the letters \(m\) and \(n\) will not mean the number of derivations and differential indeterminates, respectively.

We begin by bounding the length of certain sequences of non-negative \(n\)-tuples. Call a sequence \(t_1, t_2, \ldots, t_k\) of \(n\)-tuples *dicksonian*, if for all \(1 \leq i < j \leq k\), there does not exist a non-negative \(n\)-tuple \(t\) such that \(t_i + t = t_j\). For example, any lexicographically decreasing sequence is dicksonian. By Dickson’s Lemma, every dicksonian sequence is finite. Our goal is to obtain an explicit upper bound for the length of a dicksonian sequence, whose elements do not grow faster than a given function, in terms of this function, the first element, and the size \(n\) of the tuples. Let
\[
(a_1^1, \ldots, a_n^1), (a_1^2, \ldots, a_n^2), \ldots, (a_1^k, \ldots, a_n^k)
\]
be a dicksonian sequence of \(n\)-tuples of non-negative integers such that
\[
\max\left(a_1^j, \ldots, a_n^j\right) \leq f(j)
\]
for all \(j, 1 \leq j \leq k\), where
\[
f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}
\]
is a fixed function. We say that the *growth of this sequence is bounded by the function* \(f\).

The following proposition closely resembles a particular case of our problem, namely that of \(f(i) = m + i - 1\). However, in Proposition 7 the maximal coordinate must increase by 1 at each step, whereas in our case it is allowed to decrease or remain the same. We will reduce the case of a dicksonian sequence
with the growth bounded by a function $f$ from a certain large class of functions that “do not grow too fast”, to the one treated in Proposition 7.

**Proposition 7** [28, Proposition 1] Let $t_1, t_2, \ldots, t_k$ be a dicksonian sequence of $n$-tuples, such that the maximal coordinate of $t_i$ equals $m + i - 1$, for all $1 \leq i \leq k$. Then the maximal coordinate in the last tuple, $t_k$, does not exceed $A(n, m - 1) - 1$, and there exists such a dicksonian sequence for which this bound is reached.

Note that in Proposition 7 we have: $m$ is the maximal coordinate of $t_1$ and the length $k$ of the sequence is bounded by $A(n, m - 1) - m$. The general case (of any function $f$, not necessarily from our class) has also been studied in [29] using a different approach. It is shown that the maximal possible length is primitive recursive in $f$ and recursive, but not primitive recursive (if $f$ increases at least linearly), in $n$. Sequences yielding the maximal possible length are constructed. Moreover, if $f$ is linear, an explicit expression for the maximal length is given in terms of a generalized Ackermann function. Our statement was motivated by the need to obtain an explicit expression for the bound for a wider class of growth functions.

Let $L_{f,n}$ denote the maximal length of a dicksonian sequence of $n$-tuples, whose growth is bounded by $f$. For an increasing function $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$, let $\lceil f^{-1}(x) \rceil$ be the least number $k$ such that $f(k) \geq x$.

**Lemma 8** Let $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ be an increasing function, and $d \in \mathbb{Z}_{\geq 0}$ be a number such that $f(i + 1) - f(i) \leq A(d, f(i) - 1)$ for all $i > 0$. Then

$$L_{f,n} < \lceil f^{-1}(A(n + d, f(1) - 1)) \rceil$$

and the maximal entry of the last $n$-tuple does not exceed $A(n + d, f(1) - 1)$.

**PROOF.** Consider a disckonian sequence

$$(a_1^1, \ldots, a_n^1), (a_1^2, \ldots, a_n^2), \ldots, (a_1^k, \ldots, a_n^k),$$

whose growth is bounded by $f$. Construct from (3) a new sequence satisfying the conditions of Proposition 7. Append to the first tuple $d$ new coordinates, each equal to $f(1)$, obtaining the following $(n + d)$-tuple:

$$(a_1^1, \ldots, a_n^1, f(1), \ldots, f(1)).$$

Then add $f(2) - f(1) - 1$ new $(n + d)$-tuples. The first $n$ coordinates of these tuples are $(a_1^1, \ldots, a_n^1)$. The last $d$ coordinates form a dicksonian sequence of $d$-tuples, starting with $(f(1), \ldots, f(1))$, with the maximum coordinate growing exactly by 1 at each step. From Proposition 7 and condition $f(2) - f(1) \leq$
$A(d, f(1) - 1)$, such sequence exists. The last tuple will have the maximum coordinate equal to $f(2) - 1$. Next, add the tuple

$$(a_1^2, \ldots, a_n^2, f(2), \ldots, f(2)).$$

Since the growth of (3) is bounded by $f$, the maximal coordinate in this tuple equals $f(2)$. Continue by adding $f(3) - f(2) - 1$ new $(n + d)$-tuples, whose first $n$ coordinates are $(a_1^2, \ldots, a_n^2)$ and last $d$ coordinates form a dicksonian sequence growing by 1 at each step. Finally, when the tuple

$$(a_k^k, \ldots, a_n^k, f(k), \ldots, f(k))$$

is reached, stop. We obtain a sequence of $(n + d)$-tuples in which the maximal coordinate grows by 1 at each step. We will show that this sequence is dicksonian. Suppose that it is not. Let $t_j, t_l, j < l$, be two $(n + d)$-tuples from this sequence, for which there exists a tuple $t$ such that $t_l = t_j + t$. Let $t^I$, $t^{II}$ denote the first $n$ coordinates and the last $d$ coordinates of an $(n + d)$-tuple $t$, respectively. Then we have $t^I_j = t^I_l + t^I$ and $t^{II}_l = t^{II}_j + t^{II}$. If $t_j$ and $t_l$ have been added after the same tuple of the form

$$p_i = (a_1^i, \ldots, a_n^i, f(i), \ldots, f(i)),$$

or if $t_j$ coincides with such a tuple $p_i$ and $t_l$ has been added after $p_i$, the equality $t^{II}_l = t^{II}_j + t^{II}$ contradicts the fact that the last $d$ coordinates of the tuples between $p_i$ and $p_{i+1}$, including $p_i$ and excluding $p_{i+1}$, form a dicksonian sequence. If $t_j$ and $t_l$ have been added after different tuples $p_i$ and $p_{i'}$, the equality $t^I_j = t^I_l + t^I$ contradicts the fact that sequence (3) is dicksonian. Therefore, our assumption was false and the constructed sequence is dicksonian.

By Proposition 7, the maximum coordinate of its last element does not exceed $A(n + d, m - 1) - 1$. Since the maximum coordinate in the first element is $f(1)$ and grows by 1 at each step, the number of elements in the constructed sequence does not exceed $A(n + d, f(1) - 1) - f(1)$. On the other hand, the number of elements in the constructed sequence is:

$$f(2) - f(1) + f(3) - f(2) + \ldots + f(k) - f(k - 1) + 1 = f(k) - f(1) + 1.$$

Therefore,

$$f(k) - f(1) + 1 \leq A(n + d, f(1) - 1) - f(1),$$

that is,

$$f(k) < A(n + d, f(1) - 1),$$

and

$$k < \lceil f^{-1}(A(n + d, f(1) - 1)) \rceil.$$
Using the result of the previous section, we obtain an upper bound for the length of sequences of autoreduced sets of decreasing rank produced by a differential elimination algorithm. The idea is to put in correspondence with such a sequence a dicksonian sequence of tuples, whose growth is bounded by a function derived from the algorithm.

We fix an **orderly** ranking. Algorithm \( \text{RGBound} \) computes a characteristic decomposition of a radical differential ideal given by a set of generators. It is designed in such a way that allows us to control the orders and degrees of differential polynomials occurring in all intermediate steps together with a **bound** on the number of iterations of this algorithm. Also, in the algorithm the procedure \( \text{algrem} \) computes an algebraic pseudo-remainder of a polynomial with respect to an algebraic triangular set (a set is called **triangular** if the leaders of its elements are distinct). Algorithm \( \text{MinimalTriangularSubset} \) inputs a finite set of differential polynomials and outputs one of its least rank triangular subsets. Algorithm \( \text{CharSet} \) inputs a finite set of differential polynomials and outputs one of its characteristic sets, that is, an autoreduced subset of the least rank. Denote by \( \Delta(C) \) the set of “differential S-polynomials” of \( C \) defined in [14, Definition 4.2]. From now on, \( m \) and \( n \) again denote the numbers of derivations and differential indeterminates, respectively.

**Algorithm 1** \( \text{RGBound}(F_1) \)
INPUT: a set $F_1 \subset k\{y_1, \ldots, y_n\}$ with derivations $\{\partial_1, \ldots, \partial_m\}$

OUTPUT: A finite set $T$ of triangular sets such that 

$\{F_1\} = \bigcap_{C \in \mathcal{T}} [C] : H_C^\infty$;

if $1 \notin [C] : H_C^\infty$ then $C$ is coherent and autoreduced

otherwise $1 \in (C) : H_C^\infty$.

$T := \{\emptyset\}; \quad U := \{(F_1, \emptyset)\}$

while $U \neq \emptyset$ do

Take and remove any $(F, C) \in U$

$f :=$ an element of $F$ reduced w.r.t. $C$ of the least rank

if $s_f \notin k$ then $U := U \cup (F \cup s_f, C)$ end if

if $1_f \notin k$ then $U := U \cup (F \cup 1_f, C)$ end if

$D := \{C \in \mathcal{C} \mid \text{ld } C = \theta \text{ld } f \text{ for some } \theta \in \Theta\}$

$\bar{C} := C \setminus D \cup \{f\}; \quad G := F \cup \Delta(\bar{C}) \cup D \setminus \{f\}$

$b := \max_{g \in \mathcal{G}} \text{ord } g$

$\mathcal{B} := \text{MinimalTriangularSubset}(\{\theta C \mid C \in \bar{C}, \text{ord } \theta C \leq b\})$

$\bar{\mathcal{B}} := \{\text{algrem}(h, \mathcal{B} \setminus \{h\}) \mid h \in \mathcal{B}\}$

if $\text{rk } \bar{\mathcal{B}} \neq \text{rk } \mathcal{B}$ then $T := T \cup \{\mathcal{B}\}; \quad \text{continue}; \quad$ end if

$R := \{\text{algrem}(g, \mathcal{B}) \mid g \in \mathcal{G} \setminus \{0\}\}; \quad \mathcal{C} := \text{CharSet}(\bar{\mathcal{B}})$

if $R = \emptyset$ then $T := T \cup \{\mathcal{C}\}$ else $U := U \cup (R \cup F, C)$ end if

end while

return $T$

We get the following bounds for the growth of the maximal degrees of the polynomials computed at the $i$-th step of Algorithm 1.

**Proposition 9** Fix $(F_i, \mathcal{C}_i) \neq \emptyset \in U$ and let $(F_{i+1}, \mathcal{C}_{i+1}) \neq \emptyset$ be any of the elements obtained from $(F_i, \mathcal{C}_i)$ after one iteration of the while-loop. We then have

$$D(F_{i+1} \cup \mathcal{C}_{i+1}) \leq (4D(F_i \cup \mathcal{C}_i))^{(2H_{m-1})+m+1}.$$
mials may increase are calls to \texttt{algrem}(g, B) for g \in G or \texttt{algrem}(h, B \setminus \{h\}) for h \in B. In both cases it is a sequence of at most |B| algebraic pseudodivision. We will prove the bound for the reduction of a fixed g \in G modulo B, and the other case follows similarly.

Assume that |B| = N, and let B = \{B_1, \ldots, B_N\} be ordered such that

\[ \text{ld}(B_1) > \text{ld}(B_2) > \cdots > \text{ld}(B_N). \]

Let \(g^{(0)} := g\) and \(g^{(t)} := \texttt{algrem}(g, \{B_1, \ldots, B_t\})\) for \(t > 0\). Denote the maximal total degree of \(g^{(t)} \cup B\) by \(\delta(t)\), \(t \geq 0\). Note that \(\delta(0) \leq 2D(F_i \cup C_i)\). Then \(g^{(t+1)}\) is obtained by the pseudo-division of \(g^{(t)}\) with respect to the polynomial \(B_{t+1}\). Thus,

\[ g^{(t+1)} = \epsilon_{B_{t+1}} g^{(t)} - qB_{t+1}, \]

where \(\epsilon\) is a sufficiently large exponent specified below, \(q\) is the pseudo-quotient, and the degree of \(g^{(t+1)}\) in \(\text{ld}(B_{t+1})\) is smaller that the same degree of \(B_{t+1}\). The exponent \(\epsilon\) is bounded by

\[ \deg_{\text{ld}(B_{t+1})} (g^{(t)}) - \deg_{\text{ld}(B_{t+1})}(B_{t+1}) + 1 \leq \delta(t). \]

Therefore, the total degree of \(g^{(t+1)}\) is bounded by

\[ \delta(t + 1) \leq \delta(t)\delta(0) + \delta(t) + \delta(0) \leq \begin{cases} 2\delta(0)\delta(t), & \text{if } t \geq 1 \text{ or } \delta(0) \geq 2; \\ 3, & \text{if } t = 0 \text{ and } \delta(0) = 1. \end{cases} \]

This implies that \(\delta(N) \leq (2\delta(0))^{N+1}\). Using that \(\delta(0) \leq 2D(F_i \cup C_i), N \leq \binom{b_i + m}{b_i}\), and \(b_i \leq 2H(F_i \cup C_i)\), we get the claim (the numbers \(b_i\) are defined in Algorithm 1).

### 5.1 Differential bounds for splitting

Consider now the splitting part of differential elimination. Algorithm 1 removes an element \((F, C)\) from \(U\) and within one iteration of the while-loop it converts this element into one, two, or three elements. Moreover, if the set \(C\) does not change, the orders and degrees of the elements of \(F\) do not increase after the conversion. We call such an iteration incomplete. Denote the maximal number of all (complete and incomplete) iterations of the while-loop of Algorithm 1 by \(L(F)\).

**Proposition 10** Algorithm 1 is correct and terminates. Moreover,

\[ L(F) \leq \log_2(A(m + 7, Q(F) - 1)), \]
where

\[ Q(F) = \max \left( 9, n, 2^{9H(F)}, D(F) \right). \]  \hspace{1cm} (4)

**PROOF.** To demonstrate **correctness** we will show that the **while**-loop of Algorithm 1 has the following invariant

\[ \{ F_1 \} = \bigcap_{(F, C) \in U} \{ F \cup C \} : H_c^\infty \bigcap_{A \in T} [A] : H_A^\infty. \]  \hspace{1cm} (5)

Indeed, since \( F_1 \subset F \) for all \((F, C) \in U\) and for any \( A \in T \) every element of \( F_1 \) is reducible to zero with respect to \( A \), we have the inclusion “\( \subset \)”. We will show the opposite inclusion by induction on the number of iterations of the loop (not assuming that it is finite). Invariant (5) holds at the beginning of the first iteration \((T = \emptyset, U = (F_1, \emptyset))\). Let a finite number of iterations of the loop be executed preserving the invariant. Suppose that at the next step we remove an element \((F, C)\) from \( U \). Let \( \bar{C} \) be either the set added to \( T \) (in this case we let \( \bar{F} = \emptyset \)), or the second element of the pair \((\bar{F}, \bar{C})\) returned back to \( U \). Note that in both cases we have

\[ \{ \bar{F}, \bar{C} \} \subset \{ F, C \} \]  \hspace{1cm} (6)

by construction. We will show the inclusion

\[ \{ \bar{F}, \bar{C} \} : H_{\bar{C}}^\infty \cap \{ F, 1_f, C \} : H_{\bar{C}}^\infty \cap \{ F, s_f, C \} : H_{\bar{C}}^\infty \subset \{ F, C \} : H_{\bar{C}}^\infty, \]

that together with the inductive hypothesis shows the result. Indeed, applying [14, Proposition 6.6], since the polynomial \( f \) chosen in the loop is reduced with respect to \( C \), we have

\[ \{ F, C \} : (H_C \cup H_f)^\infty \cap \{ F \cup 1_f, C \} : H_{\bar{C}}^\infty \cap \{ F \cup s_f, C \} : H_{\bar{C}}^\infty = \{ F, C \} : H_{\bar{C}}^\infty. \]

It remains to note that

\[ \{ \bar{F}, \bar{C} \} : H_{\bar{C}}^\infty \subset \{ F, C \} : (H_C \cup H_f)^\infty. \]

Indeed, according to Algorithm 1 every \( g \in \bar{C} \) comes from some \( C \in C \) as a remainder with respect to the triangular set \( B \subset [C] \). Applying (6) together with [10, Lemma 5] and [14, Lemma 6.9] to \( K = s_g, H = H_C \), we obtain that

\[ \{ \bar{F}, \bar{C} \} : H_{\bar{C}}^\infty \subset \{ \bar{F}, \bar{C} \} : (H_{\bar{C}} \cup H_C)^\infty \subset \{ F, C \} : (H_C \cup H_f)^\infty = \{ F, C \} : (H_C \cup H_f)^\infty. \]

It remains to note that in the case of \( \text{rk} \bar{B} \neq \text{rk} B \) we have \( 1 \in (B) : H_{\bar{B}}^\infty \) by [10, Lemma 5].
Termination of the algorithm and the bound for \(L(F)\) will be proved as follows. To each element of \(U\) we associate an \((m+4)\)-tuple in such a way that the sequence of these vectors for each element of \(U\) is dicksonian. Note that by our definition \(L(F)\) is the maximal length of such dicksonian sequence. At the beginning, the set \(U\) consists of one element only. We associate the vector 

\[ \tau_0 = (0, \ldots, 0, n, n, D(F)) \]

to it. Let an element \((F,C)\) with vector \(\tau\) be under processing of the loop and let \(f\) be an element of \(F\) reduced with respect to \(C\) and of minimal rank with this property.

If an incomplete iteration occurs, we add an element of the form \((F \cup \{g\}, C)\) to \(U\). The degree of \(g\) is less than the one of \(f\) (this is either the initial or separant of \(f\)). We then transform the vector \(\tau\) by replacing the last its component by \(D(g)\). If the iteration is complete then \(U\) is concatenated with a converted element \((\bar{F}, \bar{C})\). In this case we change the components of \(\tau\) in the following way. Let \(r_k f = (\theta y_j)^d\) be the rank of \(f\), where \(\theta = \partial_1 f \ldots \partial_m f\). Consider the \((m+4)\)-tuple 

\[ \bar{\tau} = (i_1, \ldots, i_m, d, j, n-j, \deg(g)), \]

where \(g\) is the element of \(\bar{F}\) reduced with respect to \(\bar{C}\) and of minimal rank.

Proceeding in the described way from \(\tau\) to \(\bar{\tau}\), let \(\tau_k\) denote the \((m+4)\)-tuple corresponding to the \(k\)-th iteration of the while-loop. Since \(f\) is reduced w.r.t. \(C\), the sequence \(\tau_0, \tau_1, \tau_2, \ldots\) is dicksonian. Therefore, the number of iterations for each element is bounded. This proves termination of Algorithm 1. Note that if we remove \(n-j\) from \(\bar{\tau}\), the sequence might not be dicksonian. Indeed, let \(F = \{y_1, y_2\} \subset k\{y_1, y_2, y_3\}\) with \(y_1 < y_2 < y_3\). We then would have \(\tau_1 = (0, 1, 1, 1)\) and \(\tau_2 = (0, 1, 2, 1)\).

Let \(H_1 = \max(H(F_1), m)\), \(H_{k+1} = 2H_k\), \(D_1 = D(F_1)\), and 

\[ D_{k+1} = (4D_k)^{(2H_k^m)^{+}}. \]

By Proposition 9, the maximal coordinate of the \((m+4)\)-tuple \(\tau_k\) does not exceed \(\max(H_k, D_k, n)\). Let 

\[ u_1 = \max \left( n, 9, 2^{2H(F)}, D(F) \right), \quad u_{k+1} = 2^{\frac{3}{\theta} u_k (2 + \log_2 u_k)}. \]

Then the maximal coordinate of \(\tau_k\) does not exceed \(u_k\) for all \(k \geq 1\). Indeed, we will prove by induction that \(H_k \leq \frac{1}{9} \log_2 u_k\) and \(D_k \leq u_k\). For \(k = 1\) these inequalities hold by definition of \(u_1\). Assuming that they hold for \(H_k, D_k,\) and \(u_k\), prove them for \(H_{k+1}, D_{k+1}, u_{k+1}\). Since \(H_{k+1} = 2H_k\), we have 

\[ 2^{0H_{k+1}} = 2^{0H_k} = \left(2^{0H_k}\right)^2 \leq u_k^2 = 2^{2\log_2 u_k} \leq 2^{\frac{3}{\theta} u_k \log_2 u_k} < u_{k+1}, \]
because $u_k \geq 9$. Next,

$$
\log_2 D_{k+1} = \left( \left( \frac{2H_k + m}{m} \right) + 1 \right) (2 + \log_2 D_k) \leq 2^{2H_k + m}(2 + \log_2 u_k) \leq 2^{3H_k}(2 + \log_2 u_k) \leq \sqrt[3]{u_k}(2 + \log_2 u_k) = \log_2 u_{k+1}.
$$

Here we used the fact that $H_k \geq m$, as well as the inequality $H_k \leq \frac{1}{3} \log_2 u_k$ proven above. Now observe that for $x \geq 9$, we have

$$
2^{\sqrt[3]{x} + \log_2 x} \leq 2^{x+2} - 3 = A(3, x - 1).
$$

Therefore, the sequence of $(m + 4)$-tuples $\tau_0, \tau_1, \tau_2, \ldots$ satisfies the conditions of Lemma 8 with $d = 3$. And, according to this lemma, the length of this sequence does not exceed

$$
\left\lceil f^{-1}(A(m + 7, f(1) - 1)) \right\rceil,
$$

where $f(k) = u_k \geq 2^k$. We can now plug in $f(1) = u_1$, and replace $f^{-1}$ with $\log_2$.

**Corollary 11** The maximal orders and degrees of polynomials computed by Algorithm 1 do not exceed

$$
A(m + 7, Q(F) - 1).
$$

**PROOF.** Follows directly from Lemma 8 and Proposition 10.

5.2 Lifting the final bound from splitting

Note that for $f \in \mathbf{k}\{y_1, \ldots, y_n\}$ and $F \subset \mathbf{k}\{y_1, \ldots, y_n\}$ we have

$$
1 \in \{F\} : f \iff f \in \{F\}.
$$

**Lemma 12** Let $A_1, \ldots, A_p = \mathcal{A} \subset [F]$ be a coherent autoreduced set that reduces all elements of $F$ to zero. Then for all $f \in \{F\}$

$$
1 \in \left( \mathcal{A}^{(\leq q)} : (H^\infty_{\mathcal{A}} \cup f) \right),
$$

where $q := \max \left( 0, \ord f - \min_{g \in \mathcal{A}} \ord g \right)$. 

PROOF. By our assumption, $1 \in \{F\} : f$. Moreover, since $[F] \subset [\mathcal{A}] : H_{\mathcal{A}}^\infty$, it follows from [19, Section 5.2] that

$$\{F\} : f = [\mathcal{A}] : (H_{\mathcal{A}}^\infty \cup \{F\}) \cap \bigwedge_{i=1}^{p} \{F, s_i\} : f,$$

where $i$ and $s_i$ are the initial and separant of $A_i$, respectively. Hence, $1 \in [\mathcal{A}] : (H_{\mathcal{A}}^\infty \cup \{F\})$, that is, $f \in [\mathcal{A}] : H_{\mathcal{A}}^\infty$. Let $g$ be a partial pseudo-remainder of $f$ with respect to $\mathcal{A}$. Then by the Rosenfeld lemma [18, Lemma 5, III.8], we have $g \in (\mathcal{A}) : H_{\mathcal{A}}^\infty$. Since the ranking is orderly, there exists $h \in H_{\mathcal{A}}^\infty$ such that $h \cdot f - g \in (\mathcal{A}^{(\infty)})$. Indeed, at each step of the partial pseudo-division the order of the resulting differential polynomial is less than or equal to the one of the previous polynomial. And $q$ represents the maximal number of times one possibly needs to differentiate elements of $\mathcal{A}$ to perform one step of the partial pseudo-reduction.

Lemma 13 Let $F \subset \mathbb{k}\{y_1, \ldots, y_n\}$ and $f, f_1, \ldots, f_k \in \mathbb{k}\{y_1, \ldots, y_n\}$. Suppose that for some $d \in \mathbb{Z}_{\geq 1}$ we have $(f_1 \cdot \ldots \cdot f_k)^d \in (F) : f$. Then

$$\theta_1 f_1 \cdot \ldots \cdot \theta_k f_k \in \sqrt{\left(F^{(\leq 4(k+1)H+1)d}\right)} : f,$$

where $\theta_i \in \Theta$ with $\text{ord} \theta_i \leq H$ for all $i$, $1 \leq i \leq k$.

PROOF. It follows from the proof of [30, Lemma 1.7] that if $a^d \in (F)$ then $(\partial_i a)^{2d-1} \in (\mathcal{F}^{(\leq d)})$ for any $a \in \mathbb{k}\{y_1, \ldots, y_n\}$ and $\partial_i \in \Delta$. Therefore,

$$((\partial_i f_1) \cdot f_2 \cdot \ldots \cdot f_k + \ldots + f_1 \cdot \ldots \cdot f_{k-1} \cdot (\partial_i f_k))^{2d-1} \in \left(F^{(\leq d)}\right) : f^\infty.$$

Multiplying by $((\partial_i f_1) \cdot f_2 \cdot \ldots \cdot f_k)^{2d-1}$, we obtain

$$((\partial_i f_1) \cdot f_2 \cdot \ldots \cdot f_k)^{2(2d-1)} \in \left(F^{(\leq d)}\right) : f^\infty.$$

To make the computation simpler, we have

$$((\partial_i f_1) \cdot f_2 \cdot \ldots \cdot f_k)^{4d} \in \left(F^{(\leq d)}\right) : f^\infty.$$

By induction, we conclude that

$$((\theta_1 f_1) \cdot f_2 \cdot \ldots \cdot f_k)^{4Hd} \in \left(F^{(\leq d(1+4\ldots+4H))}\right) : f^\infty \subset \left(F^{(\leq 4H+1)d}\right) : f^\infty.$$

Similarly, we obtain

$$((\theta_1 f_1) \cdot (\theta_2 f_2) \cdot \ldots \cdot f_k)^{4Hd} \in \left(F^{(\leq 4H+1d+4H+14Hd)}\right) : f^\infty = \left(F^{(\leq 4H+1d(1+4H))}\right) : f^\infty.$$
Finally, by induction we get
\[
((\theta_1 f_1) \cdot (\theta_2 f_2) \cdot \ldots \cdot (\theta_k f_k))^{4^k H d} \in \left( F(\leq 4^{H+1}d(1+4^H+\ldots+4^{(k-1)H})) \right) : f^\infty \subset \left( F(\leq 4^{(k+1)H+1}d) \right) : f^\infty ,
\]
which finishes the proof.

**Lemma 14** For a finite subset \( F \subset k\{y_1, \ldots, y_n\} \) we have:
\[
t(F, f) \leq \text{ord } f + H(F) \cdot 2^{L(F)} + 4n2^{H(F) - L(F) + 1}\cdot t(G, f) + 1 d, \quad (9)
\]
where
\[
d := \max (D(f), A(m + 7, Q(F) - 1)) n2^{H(F) - L(F) + m + \text{ord } f} , \quad (10)
\]
\( G \subset k\{y_1, \ldots, y_n\} \) is such that \( L(G) \leq L(F) - 1 \) and \( H(G) \leq H(F) \cdot 2^{L(F)} \) and \( Q(F) \) is defined in formula \( (4) \).

**PROOF.** Let \( \mathcal{A} \) be the first component computed by Algorithm 1 and \( p \) be the number of elements of \( \mathcal{A} \). Note that
\[
p \leq n \cdot 2^{H(\mathcal{A}) + m} . \quad (11)
\]
We then have
\[
\{F\} = [\mathcal{A}] : H_{\mathcal{A}}^\infty \cap \bigcap_{i=1}^p \{F, i_1\} \cap \bigcap_{i=1}^p \{F, s_i\} .
\]
If \( 1 \in (\mathcal{A}) : H_{\mathcal{A}}^\infty \), we have
\[
s_1 \ldots s_p \cdot 1_1 \ldots 1_p \in \sqrt{(\mathcal{A})} .
\]
Therefore,
\[
f \cdot s_1 \ldots s_p \cdot 1_1 \ldots 1_p \in \sqrt{(\mathcal{A})} .
\]
Consider now the case when \( \mathcal{A} \) is as in Lemma 12 that gives us
\[
f \cdot s_1^{j_1} \ldots s_p^{j_p} \cdot 1_1^{k_1} \ldots 1_p^{k_p} \in (\mathcal{A}(\leq \text{ord } f))
\]
for some non-negative integers \( j_1, \ldots, j_p \) and \( k_1, \ldots, k_p \). Therefore,
\[
f \cdot s_1 \ldots s_p \cdot 1_1 \ldots 1_p \in \sqrt{(\mathcal{A}(\leq \text{ord } f))} .
\]
Hence, by \( [5, \text{Corollary 1.7}] \), which gives an upper bound for degrees in the algebraic Nullstellensatz when the degrees of the generating polynomials are
not equal to two, after squaring all elements of $A^{(\leq \text{ord} f)}$ of degree two, we obtain that

$$(f \cdot s_1 \cdots \cdot s_p \cdot 1_1 \cdots \cdot 1_p)^d \in \left(A^{(\leq \text{ord} f)}\right),$$

(12)

where $d$ is defined by (10). Indeed, $n \cdot 2^{H(A)+m+\text{ord} f}$ bounds the number of algebraic indeterminates in $A^{(\leq \text{ord} f)}$ and $f$, $H(A) \leq H(f) \cdot 2^{L(F)}$ (because at each step of Algorithm 1 the maximal order doubles at most), and by Corollary 11 we have

$$\max \left(4, D\left(A^{(\leq \text{ord} f)}\right)\right) = \max(4, D(A)) \leq A(m + 7, Q(F) - 1).$$

We also have

$$1 \in \{F, s_i\} : f \text{ and } 1 \in \{F, i_i\} : f$$

for all $i, 1 \leq i \leq p$. Let $G$ be $F, i_i$ or $F, s_i$ with the maximal $t(G, f)$. It then follows that

$$f^{k_j} = f_j + h_j,$$

where $f_j \in \sqrt{(F^{(\leq t(G, f))})}$, $h_j \in \sqrt{\left(i_i^{(\leq t(G, f))}\right)}$ or $\sqrt{\left(s_i^{(\leq t(G, f))}\right)}$, and $k_j$ is a natural number. Multiplying the above expressions, we obtain that

$$f \sum^{k_j} = g + h,$$

(13)

where $g \in \left(F^{(\leq t(G, f))}\right)$ and

$$h \in \left(\left(\theta_1 s_1\right) \cdots \left(\theta_p s_p\right) \cdot \left(\theta'_1 i_1\right) \cdots \left(\theta'_p i_p\right)\right) \cdot \text{ord}(\theta_k), \text{ord}(\theta'_l) \leq t(G, f)\right).$$

Moreover, again since at each step of Algorithm 1 the maximal order doubles at most, we have

$$\left(A^{(\leq q)}\right) \subset \left(F^{(\leq q + H(F) \cdot 2^{L(F)})}\right)$$

for any $q \in \mathbb{Z}_{\geq 0}$. By Lemma 13 and inclusion (12) we have

$$\left(\left(\left(\theta_1 s_1\right) \cdots \left(\theta_p s_p\right) \cdot \left(\theta'_1 i_1\right) \cdots \left(\theta'_p i_p\right)\right) \cdot \text{ord}(\theta_k), \text{ord}(\theta'_l) \leq t(G, f)\right) : f,$$

where $d$ is defined in (10). Hence,

$$\left(\left(\left(\theta_1 s_1\right) \cdots \left(\theta_p s_p\right) \cdot \left(\theta'_1 i_1\right) \cdots \left(\theta'_p i_p\right)\right) \cdot \text{ord}(\theta_k), \text{ord}(\theta'_l) \leq t(G, f)\right) : f,$$

(14)
for all $\theta_k$ and $\theta'_k$ with $\text{ord}(\theta_k), \text{ord}(\theta'_k) \leq t(G, f)$, $1 \leq k \leq p$. Thus, from inequality (11) and inclusion (14) it follows that

$$h \in \left( F^{\leq \text{ord} f + H(F) 2^L(F) + 4^{(2n 2^H(A) + 1) t(G, f) + 1} d} \right) : f,$$

which finishes the proof because we have representation (13) and $g \in \left( F^{(\leq t(G, f))} \right)$.

**Theorem 15** We have

$$t(F, f) \leq A(m + 8, \max(n, H(F \cup f), D(F \cup f))).$$

(15)

**PROOF.** We begin the proof by recalling Corollary [11] which states that at any stage of Algorithm 1 the orders and degrees of differential polynomials computed by this algorithm do not exceed the bound

$$E := A(m + 7, Q(F \cup f) - 1),$$

where the function $Q$ is defined in formula (11) and $Q(F \cup f) \geq Q(F)$. In particular,

$$\text{ord} f \leq E, \quad H(F) \leq E,$$

and by Proposition [10]

$$L(F) \leq \log_2 E.$$

It follows directly from the definition of $Q(F)$ that

$$n \leq E, \quad m \leq E, \quad 100 \leq E.$$

Using these inequalities, we can bound the quantity $d$ defined in (10) as

$$d \leq E^{E 2^{E} 2^{E} 2^{E}} \leq E^{E^{E^{E}}},$$

whence

$$d \cdot 4^d \leq 4^{2d} \leq E^{E^{E^{E}}}. $$

To simplify the latter formula, we note that

$$k^k \ldots \leq 2^{2^2} \ldots,$$

for all natural numbers $k$ and $p$, which can be easily derived by induction, using the inequality $ab \leq a^b$, which holds for all natural numbers $a, b \geq 2$. Thus we get:

$$d \cdot 4^d \leq 4^{2d} \leq 2^{2^{E \times E \times E}}.$$
This allows us to obtain the following inequality from [9]:

\[
t(F, f) + 1 \leq E + 1 + E^2 + \left( \frac{2^{2^2 \cdot \cdot \cdot}}{5E \text{ times}} \right)^{t(G,f)+1} \leq \left( 3 \cdot \frac{2^{2^2 \cdot \cdot \cdot}}{5E \text{ times}} \right)^{t(G,f)+1} \leq \left( \frac{2^{2^2 \cdot \cdot \cdot}}{6E \text{ times}} \right)^{t(G,f)+1}.
\]

Now we use the fact that

\[A(4, k) = \frac{2^{2^2 \cdot \cdot \cdot}}{k+3 \text{ times}} -3\]

and simplify the above formula as

\[
t(F, f) + 1 \leq A(4, 6E)^{t(G,f)+1},
\]

and noting that \(A(4, 6E) \leq A(4, A(5, E - 1)) = A(5, E)\) yields

\[
t(F, f) + 1 \leq A(5, E)^{t(G,f)+1}.
\]

Now recall that the set \(G\), defined in the proof of Lemma 14, is the set obtained from \(F\) by Algorithm 1 by adding to it an initial or separant. If now we take

\[E_G := A(m + 7, Q(G) - 1),\]

and let \(G_2\) be the set obtained from \(G\) at the next iteration of the Algorithm 1 by adding an initial or separant, we can similarly write

\[
t(G, f) + 1 \leq A(5, E_G)^{t(G_2,f)+1}.
\]

We continue recursively writing similar inequalities for \(G_2, G_3, \ldots\), noting that the length of this chain does not exceed the number of iterations in Algorithm 1 that is, \(L(F) \leq \log_2 E\). We note also that all quantities \(E, E_G, E_{G_2}, \ldots\) arising in these inequalities can be uniformly bounded by

\[
\bar{E} := A \left( m + 7, \max \left( 9, n, 2^9E, E \right) \right) = A \left( m + 7, 2^9E \right) \leq A(m + 7, A(4, E)),
\]

since the orders and degrees are uniformly bounded by \(E\). Thus, we have

\[
t(F, f) + 1 \leq \underbrace{W \cdots W}_{E \text{ times}},
\]

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where $W = A(5, E)$, which implies that

$$t(F, f) + 1 \leq 2^{2^{2^{\ldots}}} \leq A(4, WE) \leq A(4, W^2) \leq A(4, A(5, W - 1)) =$$

$$= A(5, W) \leq A(5, A(6, E - 1)) = A(6, E) =$$

$$= A(6, A(m + 7, A(4, E))) \leq A(m + 6, A(m + 7, A(4, E))) =$$

$$= A(m + 7, A(4, E) + 1)) =$$

$$= A(m + 7, A(4, A(m + 7, Q(F \cup f) - 1)) + 1) \leq$$

$$\leq A(m + 7, A(m + 6, A(m + 7, Q(F \cup f) - 1))) + 1) =$$

$$= A(m + 7, A(m + 7, Q(F \cup f)) + 1).$$

Note that

$$Q(F \cup f) \leq A(4, B) \leq A(m + 8, B - 1) - 1,$$

where $B := \max(n, H(F \cup f), D(F \cup f)) \geq 1$, since $n \geq 1$. Therefore,

$$A(m + 7, A(m + 7, Q(F \cup f)) + 1) \leq$$

$$\leq A(m + 7, A(m + 7, A(m + 8, B - 1) - 1) + 1) \leq$$

$$\leq A(m + 7, A(m + 7, A(m + 8, B - 1)) - 1) =$$

$$= A(m + 7, A(m + 8, B) - 1) =$$

$$= A(m + 8, B).$$

### 6 Model-theoretic proof of existence of the bound

The following argument was shown to the authors by Michael Singer. In this section we refer the reader for ultrafilters and construction of ultraproducts to books in model theory, for instance [31,32]. Let $\hat{k}$ be the differential closure of $k$ (see [33, Definition 3.2] and the references given there) and $q \in \mathbb{Z}_{\geq 0}$. We would like to emphasise that in the statement below we had to fix in advance the number $r$ of differential polynomials in $F$ to be able to use ultraproducts. So, in Theorem 16 the variable $r$ is quantified before the bounding function $\beta$. However, the constructive bound that we obtained in Theorem 15 does not have such a restriction, because it depends solely on the orders and degrees in $F$ and $f$, but not on the number of elements in $F$.

**Theorem 16** For every $r \in \mathbb{Z}_{\geq 0}$ there exists a function $\beta : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ such that for any $q \in \mathbb{Z}_{\geq 0}$ and $F \subset \hat{k}\{y_1, \ldots, y_n\}$ with

$$|F| = r, \quad \max(H(F), D(F)) \leq q, \quad \text{and} \quad 1 \in [F]$$

we have

$$1 \in \left(F^{(\leq \beta(q,r,n))}\right).$$
PROOF. Assume that the statement is wrong, that is, there exist $r, q \in \mathbb{Z}_{\geq 0}$ such that for any $\alpha \in \mathbb{Z}_{\geq 0}$ there exist $p_{1,\alpha}, \ldots, p_{r,\alpha} \in \mathbf{k}\{y_1, \ldots, y_n\}$ with

$$\max(H(p_{ij}), D(p_{ij})) \leq q$$

such that

$$1 \in [p_{1,\alpha}, \ldots, p_{r,\alpha}]$$

and

$$1 \neq \sum_{i=1}^{r} \sum_{j=0}^{\alpha} q_{i,j} p_{i,\alpha}^{(j)}$$

(16)

for all $q_{i,j} \in \mathbf{k}\{y_1, \ldots, y_n\}$ of order less than or equal to $q + \alpha$. Again, it is essential here that $r$ does not depend on $\alpha$. For a maximal differential ideal $M$ in the differential ring $\prod_{i \in \mathbb{Z}_{\geq 0}} \bar{k}$ denote the differential ring $(\prod_{i \in \mathbb{Z}_{\geq 0}} \bar{k})/M$ by $K_M$. There is a natural differential ring homomorphism

$$\left( \prod_{i \in \mathbb{Z}_{\geq 0}} \bar{k} \right) \{y_1, \ldots, y_n\} \to K_M \{y_1, \ldots, y_n\} =: R.$$ 

We shall now make a special choice of the maximal differential ideal $M$. Let $\mathcal{F}$ be the filter consisting of all cofinite subsets of $\mathbb{Z}_{\geq 0}$. Then, there exists an ultrafilter $\mathcal{U}$ containing $\mathcal{F}$. Since the field $\bar{k}$ is differentially closed, by Loš’ theorem [31, Theorem 8.5.3] the ultraproduct

$$K := \prod \bar{k}/\mathcal{U}$$

is a differentially closed field with the following property.

Let $\bar{a} = (a_0, a_1, a_2, \ldots)$ and $\bar{b} = (b_0, b_1, b_2, \ldots) \in K$. Then, we have: if $\bar{a} = \bar{b}$ then $a_i = b_i$ for infinitely many indices $i$. We now take $M$ to be the kernel of the differential ring homomorphism

$$\prod_{i \in \mathbb{Z}_{\geq 0}} \bar{k} \to K.$$ 

Let $\bar{p}_i$ be the image of $(p_{i,1}, p_{i,2}, p_{i,3}, \ldots)$ in $R$. This is defined correctly as all $p_{i,j}$ have orders and degrees bounded. Assume that $(z_1, \ldots, z_n) \in (K_M)^n$ is a zero of $\bar{p}_i$ for all $i$. Then, for each $i$, $1 \leq i \leq r$, there exists $V_i \subset \mathbb{Z}_{\geq 0}$, $V_i \in \mathcal{U}$, such that

$$p_{i,j}(z_{1,j}, \ldots, z_{n,j}) = 0$$

for all $j \in V_i$, where $(z_{1,1}, z_{2,1}, z_{3,1}, \ldots)$ is mapped to $z_i$ under the mentioned differential ring homomorphism for each $t$, $1 \leq t \leq n$. Since $V_1 \cap \ldots \cap V_r \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$, there is an index

$$j \in V_1 \cap \ldots \cap V_r.$$
Therefore,
\[ p_{1,j}(z_{1,j}, \ldots, z_{n,j}) = \ldots = p_{r,j}(z_{1,j}, \ldots, z_{n,j}) = 0. \]

Since \( \bar{k} \) is differentially closed, this contradicts to \( 1 \in [p_{1,j}, \ldots, p_{r,j}] \) in the differential ring \( \bar{k}\{y_1, \ldots, y_n\} \). Therefore, since the field \( K_M \) is differentially closed, we have \( 1 \in [\bar{p}_1, \ldots, \bar{p}_r] \). Hence, there exist \( \gamma \in \mathbb{Z}_{\geq 0} \) and differential polynomials \( \bar{q}_{ij} \in K_M\{y_1, \ldots, y_n\} \) with \( \text{ord} \bar{q}_{ij} < \gamma + \alpha \) so that
\[
1 = \sum_{i=1}^{r} \sum_{j=0}^{\gamma} q_{ij} \bar{p}_i^{(j)}. \]

Again, due to our choice of \( \alpha \) (that is, due to the fact that \( \mathcal{U} \) is an ultrafilter), there exists \( \alpha \in \mathbb{Z}_{\geq 0} \) with \( \alpha > \gamma \) such that
\[
1 = \sum_{i=1}^{r} \sum_{j=0}^{\gamma} q_{ij} \bar{p}_{i,\alpha}^{(j)}, \]

where \( q_{i,j} \in \bar{k}\{y_1, \ldots, y_n\} \) of order less than \( \alpha + \gamma \). Since \( p_{i,\alpha} \in k\{y_1, \ldots, y_n\} \) for all \( i, 1 \leq i \leq r \), by taking a basis of \( \bar{k} \) over \( k \) we may assume that in fact \( q_{i,j} \in k\{y_1, \ldots, y_n\} \) for all \( i \) and \( j, 1 \leq i \leq r, 0 \leq j \leq \gamma \). This contradicts to (16). Thus, our initial assumption was wrong.

7 Conclusions

We have obtained the first bound on orders for the differential Nullstellensatz. Surely, one can improve the bound and find many applications of it. A general programme which is being realized here is as follows. The differential elimination algorithms would be very useful for applications if there were faster versions of them. Our work on bounding orders could lead to:

1. understanding complexity estimates for the differential elimination,
2. developing combined and separated differential and high performance algebraic algorithms.

One of the ideas is, instead of using the usual differential elimination, perform all differentiations at the beginning of the process and then use only fast algebraic methods. We hope that our bounds will contribute to this programme.

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