On Deciding Feature Membership in Explanations of SDD & Related Classifiers

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Abstract

When reasoning about explanations of machine Learning (ML) classifiers, a pertinent query is to decide whether some sensitive features can serve for explaining a given prediction. Recent work showed that the feature membership problem (FMP) is hard for $\Sigma^p_2$ for a broad class of classifiers. In contrast, this paper shows that for a number of families of classifiers, FMP is in NP. Concretely, the paper proves that any classifier for which an explanation can be computed in polynomial time, then deciding feature membership in an explanation can be decided with one NP oracle call. The paper then proposes propositional encodings for classifiers represented with Sentential Decision Diagrams (SDDs) and for other related propositional languages. The experimental results confirm the practical efficiency of the proposed approach.

1 Introduction

There is a growing interest in eXplainable Artificial Intelligence (XAI) (Guidotti et al. 2019; Xu et al. 2019). This interest is explained in part by the ongoing advances in Machine Learning (ML) and the resulting uses of ML in settings that impact humans, including high-risk and safety-critical applications (EU 2021). However, XAI finds other important uses (Weld and Bansal 2019). XAI can serve for diagnosing systems that exploit ML. XAI can be used to train human operators so that they learn from ML-enabled systems. Most importantly, XAI offers a general instrument for building trust in the use of systems of ML.

Most of past work on XAI involves so-called model-agnostic approaches. Model-agnostic XAI offers a practical solution for explaining complex ML models, and has been deployed in a number of relevant applications\textsuperscript{1}. However, model-agnostic XAI offers no guarantees of rigor, and can even (and often) produce unsound explanations (Ignatiev 2020). Thus, the use of model-agnostic XAI solutions in high-risk and safety-critical applications is ill-advised, as the lack of rigor could induce human decision makers in error. Recent years have seen the inception of formal approaches to XAI (FXAI) (Shih, Choi, and Darwiche 2018; Ignatiev, Narodytska, and Marques-Silva 2019; Darwiche and Hirth 2020). FXAI offers the strongest guarantees of rigor, since reasoning is in most cases model-precise, i.e. the actual ML model is accounted for when reasoning about explanations, and so explanations are rigorous with respect to the (logic) representation of the ML model.

Besides the computation of formal explanations, FXAI can answer a number of additional queries (Audemard, Koriche, and Marquis 2020; Huang et al. 2021b). Concretely, this paper studies the problem of deciding whether a feature can occur in some explanation of a given prediction for an ML classifier. In some practical uses of an ML classifier, it may be critical to decide whether a sensitive feature can be used in some explanation. For example, for a bank loan application, it would be troubling if a feature like gender, age, or ethnic origin might serve to explain a decision on a bank loan. Recent work (Huang et al. 2021b) proved that, for classifiers represented as DNF (disjunctive normal form) formulas, feature membership is hard for $\Sigma^p_2$. Thus, deciding feature membership should in general be at least as hard as solving a quantified boolean formula with two levels of quantifiers. However, it was also shown (Huang et al. 2021b) that FMP can be decided in polynomial time in the case of decision trees (DTs), and that the problem is in NP for the case of classifiers that can be represented with explanation graphs (XpG’s).

The gap in the computational complexity of FMP between DNF formulas and DTs (and also XpG’s) suggests that, for classifiers represented with specific propositional languages, the complexity of FMP could be simpler than that of DNF formulas. This paper proves that this is indeed the case. The paper starts by proving a more general result, namely that for any classifier for which one explanation can be computed in polynomial time, then FMP is in NP (and so FMP can be decided with an oracle for NP). The proof of this result offers a general approach for solving FMP, which entails devising propositional encodings for the target classifiers. However, the general approach can require large propositional encodings, which Boolean satisfiability (SAT) reasoners may be unable to solve efficiently. As a result, the paper refines the general result, proposing an alternative simpler approach for deciding FMP. As demonstrated by the experiments, the proposed refined approach yields much more compact encodings, which in turn enables SAT solvers to efficiently decide FMP for different families of classifiers. Furthermore, the paper details how the proposed approach can be instantiated for two concrete families of classifiers, namely those represented with Sentential Decision Diagrams (SDDs) (Darwiche

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\textsuperscript{1}E.g. https://cloud.google.com/explainable-ai.
2 Preliminaries

Classification problems. This paper considers classification problems, which are defined on a set of features (or attributes) \( \mathcal{F} = \{1, \ldots, m\} \) and a set of classes \( \mathcal{K} = \{c_1, c_2, \ldots, c_K\} \). Each feature \( i \in \mathcal{F} \) takes values from a domain \( \mathcal{D}_i \). In this paper, and unless otherwise indicated, the domains and classes are assumed to be boolean, i.e. \( \mathcal{B} = \{0, 1\} \) and \( \mathcal{K} = \{0, 1\} \). (It will also be convenient to allow \( \mathcal{K} = \{\bot, \top\} \) for propositional languages.) Feature space is defined as \( \mathcal{F} = \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_m = \mathcal{B}^m \); The notation \( x = (x_1, \ldots, x_m) \) denotes an arbitrary point in feature space, where each \( x_i \) is a variable taking values from \( \mathcal{D}_i \). The set of variables associated with features is \( \mathcal{X} = \{x_1, \ldots, x_m\} \). Moreover, the notation \( v = (v_1, \ldots, v_m) \) represents a specific point in feature space, where each \( v_i \) is a constant representing one concrete value from \( \mathcal{D}_i \). An ML classifier \( \mathcal{C} \) is characterized by a (non-constant) classification function \( \kappa : \mathcal{F} \to \mathcal{K} \). An instance \( \mathbf{x} \) denotes a pair \( (v, c) \), where \( v \in \mathcal{F} \) and \( c \in \mathcal{K} \), with \( c = \kappa(v) \).

Formal explanations. In contrast with well-known model-agnostic explanation approaches (Ribeiro, Singh, and Guestrin 2016; Lundberg and Lee 2017; Ribeiro, Singh, and Guestrin 2018; Guidotti et al. 2019), formal explanations are rigorously defined in terms of the function computed by the classifier. Prime implicant (PI) explanations (Shih, Choi, and Darwiche 2018) denote a minimal set of literals (relating a feature value \( x_i \) and a constant \( v_i \in \mathcal{D}_i \)) that are sufficient for the prediction. PI-explanations are related with abduction, and so are also referred to as abductive explanations (AXp) (Ignatiev, Narodytska, and Marques-Silva 2019). More recently, PI-explanations have been studied in terms of their computational complexity (Barceló et al. 2020; Audemard et al. 2021). Additional examples of recent work on formal explanation include (Wildchen et al. 2021; Malfa et al. 2021; Boumazouza et al. 2021; Blanc, Lange, and Tun 2021).

Formally, given \( v = (v_1, \ldots, v_m) \in \mathcal{F} \), with \( \kappa(v) = c \), an AXp is any minimal subset \( \mathcal{X} \subseteq \mathcal{F} \) such that
\[
\forall (\mathbf{x} \in \mathcal{F}, \left[ \bigwedge_{i \in \mathcal{X}} (x_i = v_i) \right] \rightarrow (\kappa(\mathbf{x}) = c) \tag{1}
\]
i.e. if the features in \( \mathcal{X} \) are sufficient for the predictions when these take the values dictated by \( v \), \( \mathcal{X} \) is irreducible. AXp’s can be viewed as answering a “Why?” question, i.e. why is some prediction made given some point in feature space. Besides, any subset \( \mathcal{X} \subseteq \mathcal{F} \) satisfying (1) is called a weak AXp (WeakAXp). In other words, an AXp is a subset-minimal or irreducible WeakAXp. Given a set \( \mathcal{X} \subseteq \mathcal{F} \), the predicate WeakAXp(\( \mathcal{X} \)) is true iff \( \mathcal{X} \) is a weak AXp. Similarly, AXp(\( \mathcal{X} \)) is true iff \( \mathcal{X} \) is a subset-minimal WeakAXp. A different view of explanations is a contrastive explanation (Miller 2019), which answers a “Why Not?” question, i.e. which features can be changed to change the prediction. A formal definition of contrastive explanation is proposed in recent work (Ignatiev et al. 2020). Given \( v = (v_1, \ldots, v_m) \in \mathcal{F} \) with \( \kappa(v) = c \), a contrastive explanation (CXp) is any minimal set \( \mathcal{Y} \subseteq \mathcal{F} \) such that
\[
\exists (\mathbf{x} \in \mathcal{F}, \bigwedge_{j \in \mathcal{Y}} (x_j = v_j) \land (\kappa(\mathbf{x}) \neq c)) \tag{2}
\]
Likewise, any \( \mathcal{Y} \subseteq \mathcal{F} \) satisfying (2) is called weak CXp (WeakCXp). Given a set \( \mathcal{Y} \subseteq \mathcal{F} \), the predicate WeakCXp(\( \mathcal{Y} \)) is true iff \( \mathcal{Y} \) is a weak CXp. Similarly, CXp(\( \mathcal{Y} \)) is true iff \( \mathcal{Y} \) is a subset-minimal WeakCXp. A consequence of the definition of WeakAXp(\( \mathcal{X} \)) and WeakCXp(\( \mathcal{Y} \)) is that these predicates are monotone:

**Proposition 1.** If WeakAXp(\( \mathcal{X} \)) (resp. WeakCXp(\( \mathcal{Y} \))) holds for \( \mathcal{X} \subseteq \mathcal{F} \) (resp. \( \mathcal{Y} \subseteq \mathcal{F} \)), then WeakAXp(\( \mathcal{X}' \)) (resp. WeakCXp(\( \mathcal{Y}' \))) also holds for any \( \mathcal{X}' \subseteq \mathcal{X} \subseteq \mathcal{F} \) (resp. \( \mathcal{Y}' \subseteq \mathcal{Y} \subseteq \mathcal{F} \)).

Building on the results of R. Reiter in model-based diagnosis (Reiter 1987), (Ignatiev et al. 2020) proves a minimal hitting set (MHS) duality relation between AXps and CXps, i.e. AXps are MHSes of CXps and vice-versa.

**SDD classifiers.** SDDs represent a well-known propositional language (Darwiche 2011; den Broeck and Darwiche 2015) that support efficient operations for building and manipulating Boolean functions. Similar to other circuit-based representations, e.g. binary decision diagrams (BDDs) or decision graphs (Huang et al. 2021b), SDDs can be used as binary classifiers (Huang et al. 2022; Huang et al. 2021a). SDDs are based on a decomposition type (Darwiche 2011) called partitions which can decompose a Boolean function as \( (p_1 \land q_1) \lor \cdots \lor (p_n \land q_n) \), where each \( p_i \) is called a prime and each \( q_i \) is called a sub. The primes are mutually exclusive, exhaustive and non-false. What’s more, the process of decomposition is governed by a variable tree (vtree) (Darwiche 2011).

As depicted in Figure 1a, an SDD is a directed acyclic graph (DAG) defined on \( \mathcal{B}^m \). Each circled node with outgoing edges is a decision node and represents the disjunction of its children. Each paired-box node is an element and represents the conjunction of the two boxes. The left (resp. right) box represents the prime (resp. sub). A box either contains a terminal SDD (i.e. \( \top, \bot \) or a literal) or a link to a decision node. Figure 1b shows a balanced vtree, where each leaf is a feature/variable.

**Queries and transformations.** In this paper, we only consider a number of queries and transformations that are supported by SDDs; these are the query CO (polynomial consistency check), and the transformations CD (polynomial con-
conditioning) and \( \neg C \) (polynomial negation). Let \( L \) denotes a propositional language and \( \rho \) denotes a term (i.e. conjunction of literals), we have the following standard definitions.

**Definition 1** (Conditioning (Darwiche and Marquis 2002)). Let \( \Phi \) represent a propositional formula and let \( \rho \) denote a consistent term. The conditioning of \( \Phi \) on \( \rho \), i.e. \( \Phi|_\rho \), is the formula obtained by replacing each variable \( x_i \) by \( \top \) (resp. \( \bot \)) if \( x_i \) (resp. \( \neg x_i \)) is a positive (resp. negative) literal of \( \rho \).

**Definition 2** (Queries & transformations (Darwiche and Marquis 2002)). The following queries and transformations are used throughout with respect to a propositional language \( L \):

- **CO** satisﬁes if there exists a polynomial algorithm that maps every formula \( \Phi \) from \( L \) to \( 1 \) if \( \Phi \) is consistent, and to \( 0 \) otherwise.
- **CD** satisﬁes if there exists a polynomial algorithm that maps every formula \( \Phi \) from \( L \) and every consistent term \( \rho \) into a formula from \( L \) that is logically equivalent to \( \Phi|_\rho \).
- **C** satisﬁes \( \neg C \) if there exists a polynomial algorithm that maps every formula \( \Phi \) from \( L \) to a formula of \( L \) that is logically equivalent to \( \neg \Phi \).

**Related classifiers & XpGs.** Apart from SDDs, we also consider other graph-based classifiers, for which the computation of one explanation can be represented with explanation graphs (XpG’s) (Huang et al. 2021b) (and references therein). Concrete examples include Decision Trees (DTs) (Quinlan 1986), Ordered Binary Decision Diagrams (OBDDs) (Bryant 1986), Ordered Multi-Valued Decision Diagrams (OMDDs) (Kam and Brayton 1990) and Decision Graphs (DGs) (Oliver 1992). (For DTs, DGs and OMDDs both the domains of features and the set of classes may not be boolean.) We include below a brief overview XpG’s (Huang et al. 2021b).

**Definition 3** (Explanation Graph (XpG)). An XpG is a 5-tuple \( \mathcal{D} = (G_D, S, v, \alpha_V, \alpha_E) \), where:

1. \( G_D = (V_D, E_D) \) is a labeled DAG, such that:
   - \( V_D = T_D \cup N_D \) is the set of nodes, partitioned into the terminal nodes \( T_D \) (with \( \deg^+(q) = 0, q \in T_D \)) and the non-terminal nodes \( N_D \) (with \( \deg^+(p) > 0, p \in N_D \));
   - \( E_D \subseteq V_D \times V_D \) is the set of (directed) edges.
   - \( G_D \) is such that there is a single node with indegree equal to \( 0 \), i.e. the root (or source) node.
2. \( S = \{s_1, \ldots, s_m\} \) is a set of variables;
3. \( v : N_D \rightarrow S \) is a total function mapping each non-terminal node to one variable in \( S \).
4. \( \alpha_V : V_D \rightarrow \{0, 1\} \) labels nodes with one of two values.
   - \( \alpha_V \) is required to be deﬁned only for terminal nodes,
5. \( \alpha_E : E_D \rightarrow \{0, 1\} \) labels edges with one of two values.
   - In addition, an XpG \( \mathcal{D} \) must respect the following properties:
   i. For each non-terminal node, there is at most one outgoing edge labeled 1; all other outgoing edges are labeled 0.
   ii. There is exactly one terminal node \( t \in T \) labeled 1 that can be reached from the root node with (at least) one path of edges labeled 1.

We refer to a tree XpG when the DAG associated with the XpG is a tree. Given a DAG \( \mathcal{G} \) representing a classifier \( C \in \{DTs, OBDDs, OMDDs, DGs\} \), and an instance \((v, c)\), the (unique) mapping to an XpG is obtained as follows:
1. The same DAG is used.
2. Terminal nodes labeled \( c \) in \( \mathcal{G} \) are labeled 1 in \( \mathcal{D} \). Terminal nodes labeled \( \neq c \) in \( \mathcal{G} \) are labeled 0 in \( \mathcal{D} \).
3. A non-terminal node associated with feature \( i \) in \( \mathcal{G} \) is associated with \( s_i \) in \( \mathcal{D} \).
4. Any edge labeled with a literal that is consistent with \( v \) in \( \mathcal{G} \) is labeled 1 in \( \mathcal{D} \). Any edge labeled with a literal that is not consistent with \( v \) in \( \mathcal{G} \) is labeled 0 in \( \mathcal{D} \).

Figure 2b shows an XpG mapped from an OBDD classifier (Figure 2a) and an instance.

**Evaluation of XpG’s.** Given an XpG \( \mathcal{D} \), let \( S = \mathbb{B}^m \), i.e. the set of possible assignments to the variables in \( S \). The evaluation function of the XpG, \( \sigma_D : S \rightarrow \{0, 1\} \), is based on the auxiliary activation function \( \varepsilon : S \times V_D \rightarrow \{0, 1\} \). Moreover, for a point \( s \in S \), \( \sigma_D \) and \( \varepsilon \) are deﬁned as follows:
1. If \( j \) is the root node of \( G_D \), then \( \varepsilon(s, j) = 1 \).
2. Let \( p \in \text{parent}(j) \) (i.e. a node can have multiple parents) and let \( s_i = v(p) \), \( \varepsilon(s,j)=1\) iff \( \varepsilon(s,p)=1 \) and either \( \alpha_E(p,j)=1 \) or \( s_i = 0 \), i.e.

\[
\varepsilon(s,j) \leftrightarrow \bigvee_{p \in \text{parent}(j) \land \neg \alpha_E(p,j)} \varepsilon(s,p) \bigvee_{p \in \text{parent}(j) \land \alpha_E(p,j)} \varepsilon(s,p) \tag{3}
\]

3. \( \sigma_D(s) = 1 \) iff for every terminal node \( j \in T_D \), with \( \alpha_V(j) = 0 \), it is also the case that \( \varepsilon(s,j) = 0 \), i.e.

\[
\sigma_D(s) \leftrightarrow \bigwedge_{j \in T_D \land \neg \alpha_V(j)} \neg \varepsilon(s,j) \tag{4}
\]

Terminal nodes labeled 1 are irrelevant for deﬁning \( \sigma_D \). Their existence is implicit (i.e. at least one terminal node with label 1 must exist and be reachable from the root when all the \( s_i \) variables take value 1), but the evaluation of \( \sigma_D \) is oblivious to their existence. Furthermore, and as noted above, we must have \( \sigma_D(1, \ldots, 1) = 1 \). If the graph has some terminal node labeled 0, then \( \sigma_D(0, \ldots, 0) = 0 \). This implies that \( \sigma_D = 1 \) if the prediction of the original classifier remain unchanged, and \( \sigma_D = 0 \) if the prediction of the original classifier changed.
Feature membership. Let $C$ be a classifier defined on a set of features $F$, a set of classes $K$, with feature space $E$, and computing function $\kappa$. The feature membership considered in this paper is adapted from earlier work (Huang et al. 2021b):

**Definition 4.** Given a classifier $C$, an instance $(v, c)$ and a feature $r \in F$, the feature membership problem (FMP) is to decide whether target feature $t$ is included in some explanation of instance $(v, c)$.

Previous work (Huang et al. 2021b) established that for a DNF classifier, FMP is $\Sigma^p_2$-hard, but that for DTs, FMP is in P. Moreover, (Huang et al. 2021b) proved that a target feature $t$ is included in some of the AXps iff it is included in some of the CXps. As a result, in this paper, we will focus mainly on deciding FMP on some AXps. One additional result in (Huang et al. 2021b) is a proof that FMP for XpG’s is in NP.

**Example 1.** Throughout the paper, we consider a staff recruitment scenario as our running example. In this scenario, we have a binary classification function $\kappa(P, Y, M, W) = (Y \land P) \lor (P \land W) \lor (W \land M)$. Its input are four features:
1) Young is true if the age of an applicant is less than 24; 2) Top is true if the applicant graduated from a top university. 3) Male is true if the applicant is male. 4) Work is true if the applicant has work experience. Its output is either $\top$ (accept) or $\bot$ (reject). Applicant Ella = $\{Y, \neg P, W, \neg M\}$ get $\bot$. Figure 1a shows the SDD representation of this classification function. Figure 2a shows the OBDD representation of this classification function, and Figure 2b shows the XpG representation of this OBDD and this applicant Ella. To test if this classifier is biased on feature Male, we solve the query: is there an AXp containing feature Male.

3 Classifiers with FMP in NP

This section proves results that are used throughout. First, we prove that finding an AXp/CXp of an SDD classifier runs in polynomial time. Second, we prove that for any classifier for which computing one AXp/CXp runs in polynomial time, then deciding FMP is in NP.

### 3.1 Finding one AXp and CXp for SDD Classifiers

We assume that the target binary classification functions are completely specified. This means that for any point in feature space, the classifier either predicts $\top$ or $\bot$.

**Proposition 2.** Finding one AXp of a decision taken by a SDD $C$ is polynomial-time.

**Proof.** Let $(v, c)$ be such that $c = \bot$. Our goal is then to find a $X$ such that $\kappa[\bigwedge_{i \in X}(x_i=v_i)]$ is inconsistent with the features in $X$ fixed, but becomes consistent if any feature $i$ is removed from $X$. Since SDD satisfies CD and CO, then fixing feature $i \in X$ to the $v_i$ (i.e. coordinate $i$ of $v$) can be done in polynomial time, and checking the consistency of the $\kappa[\bigwedge_{i \in X}(x_i=v_i)]$ can also be done in polynomial time.

In the case of $c = \top$. Since SDD satisfies $\neg C$, then we can construct a new SDD classifier $C'$ in polynomial time by using the negation operation. Then any instance classified as $\top$ in the original classifier $C$ is classified as $\bot$ in the new classifier $C'$. This means finding an AXp $X$ of an instance with prediction $\top$ in the original classifier $C$ can be done in polynomial time in the new classifier $C'$.

**Proposition 3.** Finding one CXp of a decision taken by a SDD $C$ is polynomial-time.

Proposition 3 can be proved with the similar argument described in the proof of Proposition 2. But the difference is to find a $Y$ such that $\kappa[\bigwedge_{i \in Y}(x_i=v_i)]$ is consistent with the features in $F \setminus Y$ fixed, but becomes inconsistent if any feature $i \in Y$ is added to $F \setminus Y$.

### 3.2 Classifiers with Polynomial-Time Explanations

This section proves that, for several families of classifiers, FMP is in NP, and can be decided with an NP oracle call. (In contrast with earlier work (Huang et al. 2021b), that includes a similar proof for XpG’s, our proof is independent of a concrete classifier, depending only on the fact that one explanation is computed in polynomial time.) Concretely, we prove that, if given $X \subseteq F$, deciding (1) (or (2)) is in P, then deciding FMP is in NP.

**Proposition 4.** Given a classifier for which (1) (or (2)) can be decided in polynomial time, then FMP is in NP.

**Proof.** We reason in terms of (1), but a similar argument could be used in the case of (2).

To prove that a set $X$ is an AXp, it suffices to prove that:
1. $\text{WeakAXp}(X) = \top$.
2. $\forall i \in X.\text{WeakAXp}(X \setminus \{i\}) = \bot$, that is, $X$ is subset-minimal.

Now, since by hypothesis, we can decide (1) in polynomial time, then we can decide whether any guessed set $X$ containing feature $t$ is an AXp in polynomial-time, as follows. For step 1., check that $X$ is a WeakAXp. For step 2., iteratively check, for each feature $i \in X$, $X \setminus \{i\}$ is not a WeakAXp. Clearly, given $X$, this procedure runs in polynomial time. Thus FMP is in NP.
Given Proposition 4 (which offers an alternative proof to the result in (Huang et al. 2021b) for XpG’s), we need now to devise ways to exploit NP oracles for solving FMP. This is the topic of the next sections.

3.3 Deciding Membership Without Witnesses

As argued in the previous section, the proof of Proposition 4 offers a solution for solving FMP in the case computing AXp’s or CXP’s is in P. As shown later, for classifiers for which there exists a propositional encoding for deciding whether a set of features is a WeakAXp, one can use Proposition 4 to devise a propositional encoding for deciding FMP. However, a straightforward encoding of the approach outlined in Proposition 4 often requires large propositional formulas. These formulas must encode one copy of the classifier to decide whether a pick \( X \) of the features is a WeakAXp, and then \( m \) copies (one for each feature) of the classifier to decide whether \( X \) is indeed subset-minimal. Observe that, since the size of \( X \) must be guessed, one must be prepared to check \( m \) features in the worst-case, and so the encoding must indeed account for \( m + 1 \) copies of the classifier.

In this section, we propose an approach that leads to drastically tighter encodings, premised on a simplification to the conditions proposed in the proof of Proposition 4. (The conditions of Proposition 4 were also considered in earlier work (Huang et al. 2021b) for a concrete family of classifiers.) Furthermore, one apparent downside of this alternative approach is that the picked set of features \( X \) may not represent a witness AXp. However, we also show how a witness AXp can still be computed from \( X \) in polynomial time.

The approach proposed in this section hinges on the following result:

**Proposition 5.** Let \( X \subseteq F \) represent a pick of the features, such that WeakAXp(\( X \)) holds and WeakAXp(\( X \setminus \{ t \} \)) does not hold. Then, for any AXp \( Z \subseteq X \subseteq F \), it must be the case that \( t \in Z \).

**Proof.** Let \( Z \subseteq F \) by any AXp such that \( Z \subseteq X \). Clearly, by definition WeakAXp(\( Z \)) must hold. Moreover, from Proposition 1, it is also the case that WeakAXp(\( Z' \)) must hold, with \( Z' = Z \cup (\{ t \} \cup (Z \setminus \{ t \})) \), since \( Z \subseteq Z' \subseteq F \). However, by hypothesis, WeakAXp(\( X \setminus \{ t \} \)) does not hold; a contradiction.

When compared with Proposition 4, Proposition 5 offers a simpler test to decide whether \( t \) is included in AXp, in that it suffices to guess a set \( X \) which is a WeakAXp, and such that removing \( t \) will cause \( X \setminus \{ t \} \) not to be a WeakAXp. An apparent drawback of this simpler test to decide AXp membership is that the guessed set \( X \) need not represent an AXp.

Nevertheless, we can use Proposition 5 to devise an efficient algorithm for producing a witness of \( t \) being included in some AXp. Let \( X \subseteq F \) be some guessed set which satisfies the conditions of Proposition 5. Because the working assumption is that the classifier is such that an AXp can be computed in polynomial-time, and since any AXp contained in \( X \) must include \( t \), then we can simply extract any AXp (in polynomial time) starting from set \( X \) (which can be viewed as a seed in algorithms proposed in earlier work (Huang et al. 2021b; Huang et al. 2022)).

Since the witness AXp is computed in a second step, this approach is referred to as the two-step method, in contrast with the approach detailed in the proof of Proposition 4, which we refer to as the one-step method. As shown in Section 5, very significant performance gains can be obtained by using the two-step method.

4 SAT encodings of FMP for SDDs and XpGs

This section proposes solutions for deciding FMP in the case of SDDs and also in the case of XpG’s. The proposed propositional encoding follows the approach described in the proofs of Proposition 4 and Proposition 5.

**One-step method.** This approach is based on the proof of Proposition 4. The whole problem is encoded into \( m + 1 \) replicas (where \( m = |F| \)), such that the 0-th replica asserts that there is a WeakAXp \( X' \), and each \( k \)-th replica asserts that if feature \( k \) is included in the candidate \( X' \), then \( k \) cannot be removed from \( X' \). Apparently, as \( m + 1 \) replicas are required, this encoding is polynomial on the number of features and the size of the classifier’s representation. What’s more, it can be expected that for SDD/XpG with a large number of features and/or number of nodes, the size of resulting propositional encoding can be unmanageable, reaching the limits of the scalability of SAT solvers.

**Two-step method.** In this approach, we seek to identify a set of features \( X' \) that is a WeakAXp and that contains the target feature \( t \). More importantly, and given Proposition 5, it is also the case that such a set \( X' \) ensures that \( t \) must be included in any AXp that is contained in \( X' \). Clearly, this can be achieved with only 0-th replica and \( t \)-th replica. The encoding is polynomial on the size of classifier’s representation, and in practice it scales better than the one-step method. After deciding whether there exists a WeakAXp \( X' \) containing \( t \), we can use any existing algorithm (Huang et al. 2021b; Huang et al. 2021a) for extracting one AXp starting from \( X' \).

4.1 Feature Membership for SDD’s

This section details, in the case of SDDs, the propositional encoding for deciding whether a subset \( X \subseteq F \) is a WeakAXp. Note that this encoding is not applicable to instances predicted to \( \top \). To present the constraints included in this encoding, we need to introduce some auxiliary boolean variables and predicates.

1. \( s_i, 1 \leq i \leq m \), \( s_i \) is a selector such that \( s_i = 1 \) iff feature \( i \) is included in \( X \). Moreover, in the context of finding one AXp, \( s_i = 1 \) also means that feature \( i \) must be fixed to its given value \( v_i \), while \( s_i = 0 \) means that feature \( i \) can take any value from its domain.
2. \( n^k_j, 1 \leq j \leq |C| \) and \( 0 \leq k \leq m \), \( n^k_j \) is the indicator of a node \( j \) of SDD C for replica \( k \). The indicator for the root node of \( k \)-th replica is \( n^k_j \). Moreover, the semantics of \( n^k_j \) is \( n^k_j = 1 \) iff the sub-SDD rooted at node \( j \) in \( k \)-th replica is consistent, otherwise inconsistent.
3. \( \text{Terminal}(j) = 1 \) if the node \( j \) is a terminal node.
4. \( \text{Element}(j) = 1 \) if the node \( j \) is an element.
5. \( \text{Decision}(j) = 1 \) if the node \( j \) is a decision node.
Table 1: Encoding for SDD to decide whether there exists an AXp that includes feature \( t \)

| General Conditions on Indexes | Specific Conditions | Constraints | Fml # |
|------------------------------|--------------------|-------------|-------|
| \( 0 \leq k \leq m, 1 \leq i \leq m, 1 \leq j \leq |C| \) | **Terminal(\( j \)), Feat(\( j, i \)), \text{Sat}(\text{Lit}(\( j \)), v_1)\) | \( n^k_j \) | (1.1) |
| | **Terminal(\( j \)), Feat(\( j, i \)), \neg\text{Sat}(\text{Lit}(\( j \)), v_1), i = k\) | \( n^k_j \) | (1.2) |
| | **Terminal(\( j \)), Feat(\( j, i \)), \neg\text{Sat}(\text{Lit}(\( j \)), v_1), i \neq k\) | \( n^k_j \leftrightarrow \neg s_i \) | (1.3) |
| | **Decision(\( j \))** | \( n^k_j \leftrightarrow \bigvee_{l \in \text{children}(j)} n^k_l \) | (1.4) |
| | **Element(\( j \))** | \( n^k_j \leftrightarrow \bigwedge_{l \in \text{children}(j)} n^k_l \) | (1.5) |
| | \( \kappa(v) = \bot \) | \( \neg n^k_j \) | (1.6) |
| | \( \kappa(v) = \top \) | \( s_i \leftrightarrow n^1_j \) | (1.7) |
| | | \( s_t \) | (1.8) |

Table 2: Encoding for XpG to decide whether there exists an AXp that includes feature \( t \)

| General Conditions on Indexes | Specific Conditions | Constraints | Fml # |
|------------------------------|--------------------|-------------|-------|
| \( 0 \leq k \leq m, 1 \leq i \leq m, 1 \leq j \leq |D| \) | \( \neg\text{Terminal}(\( j \)), k = 0 \) | \( n^0_j \leftrightarrow \bigvee_{p \in \text{parent}(r)} \left( n^0_p \land \neg s_i \right) \bigvee_{p \in \text{parent}(r)} n^0_p \) | (2.1) |
| | \( \neg\text{Terminal}(\( j \)), k > 0 \) | \( n^k_j \leftrightarrow \bigvee_{p \in \text{parent}(r)} \left( n^k_p \land \neg s_i \right) \bigvee_{p \in \text{parent}(r)} n^k_p \) | (2.2) |
| | | \( \sigma^D_p \leftrightarrow \bigwedge_{j \in \text{To}s, \kappa(s_i) = \bot} \neg n^k_j \) | (2.3) |
| | | \( \sigma^D_p \) | (2.4) |
| | | \( s_i \leftrightarrow \neg \sigma^D_p \) | (2.5) |
| | | \( s_t \) | (2.7) |

6. \( \text{Feat}(\( j, i \)) = 1 \) if the terminal node \( j \) labeled with feature \( i \).

7. \( \text{Sat}(\text{Lit}(\( j \)), v_1) = 1 \) if for terminal node \( j \), its the literal on feature \( i \) is satisfied by the value \( v_i \).

The encoding is summarized in Table 1. As literals are terminal SDDs, the values of the selector variables only affect the values of the indicator variables of terminal nodes. Constraint (1.1) states that for any terminal node \( j \) whose literal is consistent with the given instance, its indicator \( n^k_j \) is always consistent regardless the value of \( s_i \). On the contrary, constraint (1.3) states that for any terminal node \( j \) whose literal is inconsistent with the given instance, its indicator \( n^k_j \) is consistent iff feature \( i \) is not picked, in other words, feature \( i \) can take any value. Because replica \( k \) (\( k > 0 \)) is used to check the necessity of including feature \( k \) in \( \chi \), we assume the value of the local copy of selector \( s_k \) is 0 in replica \( k \). In this case, as defined in constraint (1.2), even though terminal node \( j \) labeled feature \( k \) has a literal that is inconsistent with the given instance, its indicator \( n^k_j \) is consistent. Constraint (1.4) defines the indicator for an arbitrary decision node \( j \). Constraint (1.5) defines the indicator for an arbitrary element node \( j \) (this constraint will be simplified when the sub is \( \top \) or \( \bot \)). Together, these constraints declare how the consistency is propagated through the entire SDD. Constraint (1.6) states that the prediction of the SDD classifier \( \mathcal{C} \) remains \( \bot \) since the selected features form a WeakAXp. Constraint (1.7) states that if feature \( i \) is selected, then removing it will change the prediction of \( \mathcal{C} \). Finally, constraint (1.8) indicates that feature \( t \) must be included in \( \chi \).

**Example 2.** For the SDD in Figure 1, we summarize the propositional encoding for deciding whether there is an AXp containing feature \( \text{Male} \). We have selectors \( s = \{ s_p, s_y, s_M, s_W \} \). If one-step method is adopted, then the encoding is as follows (otherwise if two-step method is adopted, then formerulas 0. and 3. are enough to check the existence of a WeakAXp):

0. \( (n^0_1 \leftrightarrow n^0_2 \lor n^0_3 \lor n^0_4) \land (n^0_5 \leftrightarrow n^0_6) \land (n^0_7 \leftrightarrow n^0_8) \land (n^0_9 \leftrightarrow n^0_{10}) \land (n^0_{11} \leftrightarrow -s_p) \land (n^0_{12} \leftrightarrow -s_p \land -s_y) \land (n^0_{13} \leftrightarrow -s_M) \land (s_M) \)

1. \( (n^1_1 \leftrightarrow n^1_2 \lor n^1_3 \lor n^1_4) \land (n^1_5 \leftrightarrow n^1_6) \land (n^1_7 \leftrightarrow n^1_8) \land (n^1_9 \leftrightarrow n^1_{10}) \land (n^1_{11} \leftrightarrow -s_p) \land (n^1_{12} \leftrightarrow -s_p \land -s_y) \land (n^1_{13} \leftrightarrow -s_M) \land (s_M) \)

2. \( (n^2_1 \leftrightarrow n^2_2 \lor n^2_3 \lor n^2_4) \land (n^2_5 \leftrightarrow n^2_6) \land (n^2_7 \leftrightarrow n^2_8) \land (n^2_9 \leftrightarrow n^2_{10}) \land (n^2_{11} \leftrightarrow -s_p) \land (n^2_{12} \leftrightarrow -s_p \land -s_y) \land (n^2_{13} \leftrightarrow -s_M) \land (s_M \leftrightarrow n^3_1) \)

3. \( (n^3_1 \leftrightarrow n^3_2 \lor n^3_3 \lor n^3_4) \land (n^3_5 \leftrightarrow n^3_6) \land (n^3_7 \leftrightarrow n^3_8) \land (n^3_9 \leftrightarrow n^3_{10}) \land (n^3_{11} \leftrightarrow -s_p) \land (n^3_{12} \leftrightarrow -s_p \land -s_y) \land (n^3_{13} \leftrightarrow -s_M) \land (s_M \leftrightarrow n^4_1) \)
4. \((n_1 \leftrightarrow n_2) \lor (n_3 \leftrightarrow n_4) \land (n_5 \leftrightarrow n_6) \land (n_7 \leftrightarrow n_8) \land (n_9 \leftrightarrow n_{10}) \land (n_11 \leftrightarrow n_{12}) \land (n_{13} \leftrightarrow -s_P) \land (n_{14} \leftrightarrow -s_P \land -s_Y) \land (n_{15} \leftrightarrow -s_M) \land (s_W \leftrightarrow n_{16})\)

Solving these formulas, we find that for applicant Ella, there is an AXp \(\{\neg P, \neg M\}\) containing feature Male, so the classifier is biased.

4.2 Feature Membership for XpG’s

Similarly to the previous section, this section details the propositional encoding for deciding whether a subset \(X \subseteq F\) is a WeakAXp, but considers instead the case of XpG’s. The encoding is based on the evaluation function \(f_D\). The boolean variables \(s_i\) of XpG’s also play the role of selectors, namely, \(s_i = 1\) if feature \(i\) is included in \(X\) (meanwhile, \(s_i = 1\) also means that feature \(i\) must be fixed to its given value \(v_i\)).

All the constraints are summarized in Table 2. Moreover, to simplify the encoding, for an arbitrary node \(k\), we replace the notation of its auxiliary activation function \(\varepsilon(s_i, j)\) by \(n_j\) (i.e. \(\varepsilon(s_i, j) \leftrightarrow n_j\)) and omit the assignment \(s_i\) to \(S\). Constraints (2.1), (2.3) and (2.4) together form the encoding of an evaluation function \(f_D\). Replica \(k (k > 0)\) is used to check feature \(k\). Thus for a non-terminal node \(j\) of this replica \(k\), its auxiliary activation function is defined as constraint (2.2). Similar to the encoding for SDDs, constraint (2.5) states that the prediction of the original classifier \(C\) remains unchanged. Constraint (2.6) states that if feature \(i\) is selected, then removing it will change the prediction of \(C\). Finally, constraint (2.7) indicates that feature \(t\) must be included in \(X\).

Example 3. For the XpG in Figure 2, we summarize the propositional encoding for deciding whether there is an AXp containing feature Male. We have selectors \(s = \{s_P, s_Y, s_M, s_W\}\). If one-step method is adopted, then the encoding is as follows:

0. \([n_1] \land [n_2 \leftrightarrow n_1] \land [n_3 \leftrightarrow n_1] \land [n_4 \leftrightarrow n_1 \land -s_P] \land [n_4 \leftrightarrow n_1 \land -s_M] \lor [n_3 \land -s_Y] \land [n_5 \land -s_Y] \land [n_6 \land -s_Y] \land [n_7 \land -s_Y] \land [n_8 \land -s_Y] \land [n_9 \land -s_Y] \land [n_10 \land -s_Y] \land [n_11 \land -s_Y] \land [n_12 \land -s_Y] \land [s_P \land -s_P] \land [s_M \land -s_M]

1. \([n_1] \land [n_2 \leftrightarrow n_1] \land [n_3 \leftrightarrow n_1] \land [n_4 \leftrightarrow n_1 \land -s_P] \land [n_4 \leftrightarrow n_1 \land -s_M] \lor [n_3 \land -s_Y] \land [n_5 \land -s_Y] \land [n_6 \land -s_Y] \land [n_7 \land -s_Y] \land [n_8 \land -s_Y] \land [n_9 \land -s_Y] \land [n_10 \land -s_Y] \land [n_11 \land -s_Y] \land [n_12 \land -s_Y] \land [s_P \land -s_P] \land [s_M \land -s_M]

2. \([n_1] \land [n_2 \leftrightarrow n_1] \land [n_3 \leftrightarrow n_1] \land [n_4 \leftrightarrow n_1 \land -s_P] \land [n_4 \leftrightarrow n_1 \land -s_M] \lor [n_3 \land -s_Y] \land [n_5 \land -s_Y] \land [n_6 \land -s_Y] \land [n_7 \land -s_Y] \land [n_8 \land -s_Y] \land [n_9 \land -s_Y] \land [n_10 \land -s_Y] \land [n_11 \land -s_Y] \land [n_12 \land -s_Y] \land [s_P \land -s_P] \land [s_M \land -s_M]

3. \([n_1] \land [n_2 \leftrightarrow n_1] \land [n_3 \leftrightarrow n_1] \land [n_4 \leftrightarrow n_1 \land -s_P] \land [n_4 \leftrightarrow n_1 \land -s_M] \lor [n_3 \land -s_Y] \land [n_5 \land -s_Y] \land [n_6 \land -s_Y] \land [n_7 \land -s_Y] \land [n_8 \land -s_Y] \land [n_9 \land -s_Y] \land [n_10 \land -s_Y] \land [n_11 \land -s_Y] \land [n_12 \land -s_Y] \land [s_P \land -s_P] \land [s_M \land -s_M]

4. \([n_1] \land [n_2 \leftrightarrow n_1] \land [n_3 \leftrightarrow n_1] \land [n_4 \leftrightarrow n_1 \land -s_P] \land [n_4 \leftrightarrow n_1 \land -s_M] \lor [n_3 \land -s_Y] \land [n_5 \land -s_Y] \land [n_6 \land -s_Y] \land [n_7 \land -s_Y] \land [n_8 \land -s_Y] \land [n_9 \land -s_Y] \land [n_10 \land -s_Y] \land [n_11 \land -s_Y] \land [n_12 \land -s_Y] \land [s_P \land -s_P] \land [s_M \land -s_M]

Likewise, solving these formulas will return us an AXp \(\{\neg P, \neg M\}\) containing feature Male.

5 Preliminary Experimental Results

This section presents preliminary experimental results on assessing the practical efficiency of the proposed methods. The experiments were performed on a MacBook Pro with a 6-Core Intel Core i7 2.6 GHz processor with 16 GByte RAM, running macOS Monterey.

Classifiers and Benchmarks. We consider SDD, DT, and OBDD classifiers (DTs and OBDDs were then mapped into XpGs). For SDDs, we selected 16 circuits from ISCAS89 suite, 6 circuits from ISCAS93 suite \(^4\), and 11 datasets from Density Estimation Benchmark Datasets\(^5\). (Lowd and Davis 2010; Haaren and Davis 2012; Larochelle and Murray 2011). 22 circuits were compiled into SDDs by using the well-known SDD package\(^6\). 11 datasets were used to learn SDD via using LearnSDD\(^7\) (Bekker et al. 2015) (with parameter maxEdges=20000). The obtained SDDs were used as binary classifiers (albeit the selected circuits/datasets might not originally target classification tasks.) For XpG, we selected 8 classification datasets from the Penn Machine Learning Benchmarks (Olson et al. 2017) and 14 test cases from a graph colouring problems benchmark flat-30-60 \(^8\) (the rest test cases are filtered out since their size are below 7500 nodes). 8 datasets were used to learn DTs by using Orange3 (Demšar et al. 2013). 14 test cases were compiled into OBDDs by using cd\(^9\) package which integrated well-known CUDD\(^10\) (Somnenzi 2012) package.

Prototype implementation. A prototype implementation of the proposed approach was implemented in Python\(^11\). The PySAT toolkit (Ignatiev, Morgado, and Marques-Silva 2018) was employed to perform feature membership encoding, and called Glucose 4 (Audemard and Simon 2018) SAT solver. SDD/XpG models were loaded by using PySDD\(^12\)/xpg\(^13\) package.

Experimental procedure. To assess the efficiency of deciding feature membership, and for each classifier, 100 test instances were randomly generated/selected. For SDDs, all tested instances have prediction \(\bot\). (We didn’t pick instances predicted to class \(\top\) as this requires the compilation of a new classifier which may have different size). Besides, for each instance, we randomly picked a feature appearing in the model. Hence for each SDD/XpG, we solved 100 queries. The time for deciding FMP was limited to 1800 seconds. And the time for finishing 100 queries was limited to 10 hours, this means the average time for deciding FMP cannot exceed 6 minutes. Note that for SDDs learned from LearnSDD, the reported number of features includes both original features and generated features (e.g. for Audio the original number of features is 100). Also note that PySDD offers canonical SDDs whose conditioning may take exponential time in the worst-case. Nevertheless, this worst-case behaviour was not observed in the experiments.

Results. Table 3 summarizes the obtained results of deciding FMP on SDDs with two methods. In this experiment, it

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\(^4\)http://www.cril.univ-artois.fr/KC/benchmarks.html
\(^5\)https://github.com/UCLA-StarAI/Density-Estimation-Datasets
\(^6\)http://reasoning.cs.ucla.edu/sdd/
\(^7\)https://github.com/ML-KULeuven/LearnSDD
\(^8\)https://www.cs.ubc.ca/~hoos/SATLIB/benchm.html
\(^9\)https://github.com/tulip-control/dd
\(^10\)https://github.com/vmari/cudd
\(^11\)https://github.com/XuanxiangHuang/fmp-experiments
\(^12\)https://github.com/wannesm/PySDD
\(^13\)https://github.com/yizha91/xpg
Table 3: Solving FMP for SDDs with two methods. First column reports the name of each test case. Column #TI reports the number of tested instances. Sub-column #F reports the number of features that appear in the SDD. Sub-column #N reports the number of nodes of a SDD. Column Succ (Test) shows the number of solved queries, inside the parentheses is the number of tested queries. Column Yes% shows the percentage of answering ‘Yes’ to the queries. Sub-Colums Avg. #var and Avg. #cls show, respectively, the average number of variables and clauses in a CNF encoding. Sub-columns Max and Avg. reports, respectively, maximal and average time in seconds for answering a query.

| Name     | #TI | SDD | Succ (Test) | Yes% | CNF Runtime (s) | two-step method |
|----------|-----|-----|-------------|------|-----------------|-----------------|
|          | #F  | #N  |                  |      |                 |                 |
| s1196    | 100 | 560 | 2230         | 100  | 63              | 8.3             | 63  |
| s1423    | 100 | 748 | 3493         | 100  | 60              | 200             | 30.6|
| s1448    | 100 | 667 | 3248         | 100  | 65              | 17.0            | 15.5|
| s1494    | 100 | 661 | 2644         | 100  | 54              | 10.1            | 8.9 |
| s400     | 100 | 189 | 2150         | 100  | 52              | 84.6            | 24.6|
| s420.1   | 100 | 252 | 2525         | 100  | 100             | 48.9            | 22.6|
| s444     | 100 | 205 | 2586         | 100  | 96              | 170.2           | 48.3|
| s510     | 100 | 236 | 4180         | 100  | 100             | 108.9           | 39.4|
| s526     | 100 | 217 | 3451         | 100  | 99              | 1800            | 338.6|
| s526n    | 100 | 554 | 100          | 99   | 100             | 2419046         | 7092974 |
| s641     | 100 | 433 | 2044         | 100  | 58              | 50436           | 9576   |
| s713     | 100 | 447 | 2050         | 100  | 56              | 51737           | 9637   |
| s820     | 100 | 312 | 1409         | 100  | 60              | 25723           | 9637   |
| s832     | 100 | 310 | 1420         | 100  | 51              | 21259           | 9637   |
| s838.1   | 100 | 512 | 5341         | 100  | 100             | 7526738         | 1390.6|
| s953     | 100 | 417 | 1692         | 100  | 39              | 210860          | 4.2   |
| s344     | 100 | 184 | 2581         | 100  | 80              | 2292067         | 322.9 |
| s499     | 100 | 175 | 2282         | 100  | 56              | 1380722         | 25.7  |
| s635     | 100 | 320 | 2972         | 100  | 1217532         | 183.5           | 45.8 |
| s938     | 100 | 512 | 5615         | 100  | 99              | 7862753         | 443.3 |
| s967     | 100 | 416 | 2292         | 100  | 72              | 839976          | 10.9  |
| s991     | 100 | 603 | 2799         | 100  | 74              | 1511707         | 27.4  |
| Accidents| 100 | 415 | 8863         | 23   | 23              | 5428799         | 16280994|
| Audio    | 100 | 272 | 7224         | 23   | 23              | 4214846         | 322.9 |
| DNA      | 100 | 513 | 8570         | 5    | 5               | 23460504        | 25.7  |
| Jester   | 100 | 254 | 7857         | 19   | 19              | 15492017        | 85   |
| KDD      | 100 | 306 | 8109         | 31   | 31              | 12813875        | 95   |
| Mushrooms| 100 | 248 | 7096         | 53   | 53              | 2941685         | 10.9  |
| Netflix  | 100 | 292 | 7039         | 34   | 34              | 3696194         | 97   |
| NLTCS    | 100 | 183 | 6661         | 34   | 34              | 1022667         | 91   |
| Plants   | 100 | 244 | 6724         | 34   | 34              | 1022667         | 91   |
| RCV-1    | 100 | 410 | 9472         | 10   | 10              | 21063341        | 96   |
| Retail   | 100 | 341 | 3704         | 87   | 87              | 1754801         | 90.9  |

Figure 3: Running times of Audio, Jester, Mushrooms and Plants.
can be observed that the number of nodes of the tested SDD is in the range of 1409 and 9472, and the number of features of tested SDD is in the range of 175 and 748. The one-step method requires \( m + 1 \) replicas, often leading to large CNF encodings. The increase on both the number of features and the number of nodes, can results in timeouts being observed. One observation is that the performance correlates inversely with propositional formula size. For the one-step method this is noticeable when the number of clauses in the CNF formulas exceeds 7,000,000. For \( s526n \) formulas exceeds 7,000,000. For \( s526n \), the one-step method can solve a small number of queries (e.g. for DNA, only 18 queries are tested, and only 5 queries are solved, 13 queries out of 18 cannot be solved in 1800 seconds time limit, and the rest 82 queries were not tested due to the 10 hours time limit.)

In contrast, the two-step method is much more efficient as the CNF encoding of two-step method is much smaller (the average number of CNF clauses does not exceed 130,000). For the SDDs compiled from 16 circuits, the two-step method successfully solve all the queries. For any of the examples considered, the two-step method never requires more than a few seconds to answer a query, and the average running time is at least one order of magnitude smaller than that of the one-step method. For the remaining SDDs, the average running time for two-step method to solve a query is less than 25 seconds; this highlights the scalability of the two-step method. However, notice that for SDDs representing Audio, Jester, Mushrooms and Plants, the largest running time for deciding FMP with the two-step method can exceed 3 minutes. As a result, we analyzed these results in greater detail. Figure 3 depicts a cactus plot showing the running time (in seconds) of deciding FMP for these 4 datasets (note that the runtime axis is scaled logarithmically, and the instances axis starts from 60). As can be observed, for each dataset, around 85 queries can be solved in a few seconds. This means that the running times of the two-step method only exceed a few seconds for a few concrete examples, and for a few of the datasets considered.

Table 4 summarizes the obtained results of deciding FMP on XpGs with two methods. No timeout occurs in this experiment. For XpGs reduced from DTs, the running time for deciding FMP is negligible regardless the method we adopt, this is due to the number of nodes of each tree XpG is small. For XpGs reduced from OBDDs, despite the size of each XpG is not small, using two-step method only takes maximal few seconds to solve a query. Furthermore, even though the average running time of the one-step method is not prohibitive, the two-step method still outperforms the one-step method by at least one order of magnitude.

### 6 Conclusions

This paper proves that, for classifiers for which one explanation can be computed in polynomial time, then the feature membership problem is in NP. Furthermore, for SDDs and also classifiers that can be mapped to explanation graphs (XpG’s), this paper details two propositional encodings to decide the existence of one explanation containing desired feature. The experiments confirm the practical efficiency and scalability of one of the proposed encodings, both for SDDs and XpGs.

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| Name       | #TI | #F | #N | XpG | Yes% | Avg. #var | Avg. #cls | Runtime (s) | Max | Avg. |
|------------|-----|----|----|-----|------|-----------|-----------|-------------|-----|------|
| one-step method |     |    |    |     |      |           |           |             |     |      |
| two-step method |     |    |    |     |      |           |           |             |     |      |

Table 4: Solving FMP for XpGs with two methods. The columns hold the same meaning as described in the caption of Table 3.
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