Friedmann equations from nonequilibrium thermodynamics of the Universe: A unified formulation for modified gravity

David W. Tian

Faculty of Science, Memorial University, St. John’s, Newfoundland, Canada, A1C 5S7

Ivan Booth

Department of Mathematics and Statistics, Memorial University, St. John’s, Newfoundland, Canada, A1C 5S7

Inspired by the Wald-Kodama entropy $S = A/(4G_{\text{eff}})$ where $A$ is the horizon area and $G_{\text{eff}}$ is the effective gravitational coupling strength in modified gravity with field equation $R_{\mu\nu} - Rg_{\mu\nu}/2 = 8\pi G_{\text{eff}}T_{\mu\nu}^{\text{eff}}$, we develop a unified and compact formulation in which the Friedmann equations can be derived from thermodynamics of the Universe. The Hawking and Misner-Sharp masses are generalized by replacing Newton’s constant $G$ with $G_{\text{eff}}$, and the unified first law of equilibrium thermodynamics is supplemented by a nonequilibrium energy dissipation term $E$ which arises from the revised continuity equation of the perfect-fluid effective matter content and is related to the evolution of $G_{\text{eff}}$. By identifying the mass as the total internal energy, the unified first law for the interior and its smooth transit to the apparent horizon yield both Friedmann equations, while the nonequilibrium Clausius relation with entropy production for an isochoric process provides an alternative derivation on the horizon. We also analyze the equilibrium situation $G_{\text{eff}} = G = \text{constant}$, provide a viability test of the generalized geometric masses, and discuss the continuity/conservation equation. Finally, the general formulation is applied to the FRW cosmology of minimally coupled $f(R)$, generalized Brans-Dicke, scalar-tensor-chameleon, quadratic, $f(R, G)$ generalized Gauss-Bonnet and dynamical Chern-Simons gravity. In these theories we also analyze the $f(R)$-Brans-Dicke equivalence, find that the chameleon effect causes extra energy dissipation and entropy production, geometrically reconstruct the mass $\rho_mV$ for the physical matter content, and show the self-inconsistency of $f(R, G)$ gravity in problems involving $G_{\text{eff}}$.

PACS numbers: 04.20.Cv , 04.50.Kd , 98.80.Jk

I. INTRODUCTION

Ever since the discovery of black hole thermodynamics [1], physicists have been searching for more and deeper connections between relativistic gravity and fundamental laws of thermodynamics. One avenue of investigation by Gibbons and Hawking [2] found that the event horizon with radius $\ell$ for the de Sitter spacetime also produces Hawking radiation of temperature $1/(2\pi \ell)$. Jacobson [3] further showed within general relativity (GR) that on any local Rindler horizon, the entropy $S = A/4G$ and the Clausius relation $T dS = \delta Q$ could reproduce Einstein’s field equation, with $\delta Q$ and $T$ being the energy flux and the Unruh temperature [4].

Besides global and quasilocal black-hole horizons [2, 6] and the local Rindler horizon, another familiar class of horizons are the various cosmological horizons. Frolov and Kofman [7] showed that for the flat quasi-de Sitter inflationary universe, $dE = T dS$ yields the Friedmann equation for the rolling inflaton field, and with metric and entropy perturbations it reproduces the linearized Einstein equations. By studying the heat flow during an infinitesimal time interval on the apparent horizon of the FRW universe within GR, Cai and Kim [8] showed that the Clausius thermal relation $T dS = \delta Q = - A\psi$ yields the second Friedmann gravitational equation with any spatial curvature, from which the first Friedmann equation can be directly recovered via the continuity/conservation equation of the perfect-fluid matter content. This work soon attracted much interest, and cosmology in different dark-energy content and gravity theories came into attention.
In [9] it was found that extensions of this formulation from GR to $f(R)$ and scalar-tensor theories are quite nontrivial, and the entropy formulas $S = Af_R/4G$ and $S = Af(\phi)/4G$ for black-hole horizons prove inconsistent in recovering Friedmann equations. In the meantime, Eling et al. [10] studied nonequilibrium thermodynamics of spacetime and found that $f(R)$ gravity indeed corresponds to a nonequilibrium description and therefore needs an entropy production term to balance the energy supply; the nonequilibrium Clausius relation $\delta Q = T(ds + d\mu s)$ with $S = Af_R/(4G)$ then recovers the Friedmann equations. This nonequilibrium picture has been widely accepted, and relativistic gravity theories with nontrivial coefficient for $R_{\mu\nu}$ or equivalently $T_{\mu\nu}^{(m)}$ (hence nontrivial gravitational coupling strength $G_{\text{eff}}$) in their field equations always require a nonequilibrium description. Following [10], Friedmann equations are recovered from nonequilibrium thermodynamics within scalar-tensor gravity with horizon entropy $S = Af(\phi)/(4G)$ [11]. Besides the most typical $f(R)$ [9, 10] and scalar-tensor [9, 11] gravity, Friedmann equations from the Clausius relation are also studied in higher-dimensional gravity models like Lovelock gravity [8, 11] and Gauss-Bonnet gravity [8].

In the early investigations within modified and alternative theories of gravity, the standard definition of the Misner-Sharp mass [12] was used. However, the interesting fact that higher-order geometrical term or extra physical degrees of freedom beyond GR act like an effective matter content encourages the attempts to generalize such geometric definitions of mass in modified gravity. [13] generalized the Misner-Sharp mass in $f(R)$ gravity, and also for the FRW universe in the scalar-tensor gravity. In [14], a masslike function was employed in place of the standard Misner-Sharp mass, so that for $f(R)$ and scalar-tensor gravity the Friedmann equations on the apparent horizon could be recovered from the equilibrium Clausius relation $TdS = \delta Q$ without the nonequilibrium correction of [10]. Moreover, the opposite process of [8] to inversely rewrite the Friedmann equations into the thermodynamic relations has been investigated as well. For example, [15] studies such reverse process for GR, Lovelock and Gauss-Bonnet gravity, [16] for $f(R)$ gravity, [17] for the braneworld scenario, and [18] for generic $f(R, \phi, \nabla\phi, \nabla^2\phi)$ gravity. Also, the field equations of various modified gravity are recast into the form of the Clausius relation in [19]. One should carefully distinguish the problem of “thermodynamics to Friedmann equations” with “Friedmann equations to thermodynamics”, to avoid falling into the trap of cyclic logic.

Considering the discreteness of these works following [8] and the not-so-consistent setups of thermodynamic quantities therein, we are pursuing a simpler and more concordant mechanism behind them: the purpose of this paper is to develop a unified formulation which derives the Friedmann equations from the (non)equilibrium thermodynamics of the FRW universe within all relativistic gravity with field equation $R_{\mu\nu} - Rg_{\mu\nu}/2 = 8\pi G_{\text{eff}} T_{\mu\nu}^{(\text{eff})}$ with a possibly dynamical $G_{\text{eff}}$. These theories include fourth-order modified theories of gravity in the metric approach (as opposed to Palatini) (eg. [20, 21]) with Lagrangian densities like $\mathcal{L} = f(R) + 16\pi G_{\text{eff}} \mathcal{L}_m$ [22], $\mathcal{L} = f(R, G) + 16\pi G_{\text{eff}} \mathcal{L}_m$ [23] ($G$ denoting the Gauss-Bonnet invariant), $\mathcal{L} = f(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}R_{\rho\sigma}^{\mu\nu}) + 16\pi G_{\text{eff}} \mathcal{L}_m$ [24] and quadratic gravity [25]; alternative theories of gravity like Brans-Dicke [26] and scalar-tensor-chameleon [27] in the Jordan frame; typical dark-energy models $\mathcal{L} = R + f(\phi, \nabla\phi, \nabla^2\phi) + 16\pi G_{\text{eff}} \mathcal{L}_m$ [28], and even generic mixed models like $\mathcal{L} = f(R, \phi, \nabla\phi, \nabla^2\phi) + 16\pi G_{\text{eff}} \mathcal{L}_m$ (eg. [18]). All have minimal geometry-matter coupling with isolated matter Lagrangian density $\mathcal{L}_m$. The situation with nonminimal curvature-matter coupling terms [29, 30] like $R\mathcal{L}_m$ will not be considered in this paper, although the nonminimal chameleon coupling $\phi\mathcal{L}_m$ [27, 31] in scalar-tensor gravity is still analyzed.

This paper is organized as follows. Sec. II makes necessary preparations by locating the marginally inner trapped horizon as the apparent horizon of the FRW universe, revising the continuity equation for effective perfect fluids, and introducing the energy dissipation term $E$ for modified gravity with field equation $R_{\mu\nu} - Rg_{\mu\nu}/2 = 8\pi G_{\text{eff}} T_{\mu\nu}^{(\text{eff})}$. In Sec. III we generalize the geometric definitions of mass using $G_{\text{eff}}$, supplement the unified first law of thermodynamics into $dE = A\psi + WdV + E$ by $E$, and match the transverse gradient of the geometric mass with the change of total internal energy to directly obtain both Friedmann equations. We

\[1\] For brevity, we will use the terminology “modified gravity” to denote both modified and alternative theories of relativistic gravity without discrimination whenever appropriate.
continue to study the thermodynamics of the apparent horizon by taking the smooth limit from the interior to the horizon in Sec. [IV] and alternatively obtain the Friedmann equation from the nonequilibrium Clausius relation $T(dS + d_pS) = \delta Q = -(A\psi_t + \mathcal{E})$, where $d_pS$ represents entropy production which is generally nontrivial unless $G_{\text{eff}} = \text{constant}$. After developing the generic theories, Sec. [V] provides a viability test for the generalized geometric masses, discusses the continuity equation, and analyzes the equilibrium case of $G_{\text{eff}} = G = \text{constant}$ with vanishing dissipation $\mathcal{E} = 0$ and entropy production $d_pS = 0$. Finally in Sec. [VI] the theory is applied to $f(R)$, generalized Brans-Dicke, scalar-tensor-chameleon, quadratic, $f(R, \mathcal{G})$ generalized Gauss-Bonnet and dynamical Chern-Simons gravity, with comments on existing treatment in $f(R)$ and scalar-tensor theories. Throughout this paper, especially for Sec. VI we adopt the sign convention $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$, $R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} \cdots$ and $R_{\mu\nu} = R^\alpha_{\mu\nu\alpha}$ with the metric signature $(-, +, +, +)$.

II. PREPARATIONS AND SETUPS

II.1. FRW cosmology and location of the apparent horizon

The Friedman-Robertson-Walker (FRW) metric provides the most general solution describing a spatially homogeneous and isotropic Universe. It is not just a theoretical construct: it matches with observations. As such it must, a priori, be a solution of any aspiring modified or alternative theory of gravity [20]. In the comoving coordinates $(t, r, \theta, \phi)$ the line element reads (eg. [8])

$$ds^2 = -dt^2 + \frac{a(t)^2}{1 - kr^2} dr^2 + a(t)^2 r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

where the curvature index $k$ is normalized to one of $\{-1, 0, +1\}$ which correspond to closed, flat and open universes, respectively; the metric function $a(t)$ is the scale factor, which is an arbitrary function of the comoving time and is to be determined by the particular gravitational field equations. $h_{\alpha\beta} := \text{diag}\{-1, \frac{a(t)^2}{1 - kr^2}\}$ is the transverse two-metric spanned by $x^\alpha = (t, r)$, and $\Upsilon := a(t) r$ is the astrophysical circumference/areal radius. Although observations currently support a flat universe with $k = 0$, we will allow for all three situations $k = \{0, \pm 1\}$ of spatial homogeneity and isotropy throughout this paper.

This solution is spherically symmetric and so in studying its physical and geometric properties it is convenient to work with a null tetrad [3] adapted to this symmetry:

$$\ell^\mu = \left( 1, \frac{\sqrt{1 - kr^2}}{a}, 0, 0 \right), \quad n^\mu = \frac{1}{2} \left( 1, -\frac{\sqrt{1 - kr^2}}{a}, 0, 0 \right), \quad m^\mu = \frac{1}{\sqrt{2\Upsilon}} \left( 0, 0, 1, \frac{i}{\sin \theta} \right),$$

where the null vectors $\ell^\mu$ and $n^\mu$ have respectively been adapted to the outgoing and ingoing null directions. The tetrad obeys the cross normalization $\ell_\mu n^\mu = -1$ and $m_\mu \tilde{m}^\mu = 1$, and thus the inverse metric satisfies $g^{\mu\nu} = -\ell^\mu n^\nu - n^\mu \ell^\nu + m^\mu \tilde{m}^\nu + \tilde{m}^\mu m^\nu$. In this tetrad, the outward and inward expansions of radial null flow are found to be

$$\theta_\ell(t) = - (\rho_{\text{NP}} + \tilde{\rho}_{\text{NP}}) = \frac{2r\dot{a} + 2\sqrt{1 - kr^2}}{a r} = 2H + 2\Upsilon^{-1} \sqrt{1 - kr^2} \frac{a}{a^2}$$

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2 The null tetrad formalism and all Newman-Penrose quantities in use here are adapted to the metric signature $(-, +, +, +)$, which is the preferred convention for quasilocal black hole horizons (see eg. the Appendix B of [6]). Also, the tetrad can be rescaled by $\ell^\mu \mapsto e^\ell^\mu$ and $n^\mu \mapsto e^{-n^\mu}$ for an arbitrary function $f$, and consequently $\theta_\ell(t) \mapsto e^{\ell_\ell(t)}$ and $\theta_m(t) \mapsto e^{-m_\ell(t)}$. 
To that end note that the total derivative of the physical radius will be highlighted by the subscript so the FRW metric Eq.(1) can be rewritten into
\[
\frac{\partial}{\partial \Upsilon} = H - \Upsilon^{-1} \sqrt{1 - \frac{k\Upsilon^2}{a^2}},
\]
where \(\rho_{NP} := -m^\mu m^\nu \nabla_\nu \ell_\mu\) and \(\mu_{NP} := \bar{m}^\mu m^\nu \ell_\mu\) are two Newman-Penrose spin coefficients, and \(H\) is Hubble’s parameter
\[
H := \frac{\dot{a}}{a},
\]
with the overdot denoting the derivative with respect to the comoving time \(t\). In our universe in which \(\dot{a} > 0\) and \(H > 0\) the outward expansion \(\theta(t)\) is always positive while \(\theta(n)\) can easily be seen to vanish when
\[
r_A = \frac{1}{\sqrt{\dot{a}^2 + k}} \quad \Leftrightarrow \quad T_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}.\]

On this surface
\[
\theta(t) = 4H > 0,
\]
and thus \(\Upsilon = T_A\) is a marginally inner trapped horizon \(^5\) with \(\theta(n) < 0\) for \(\Upsilon < T_A\) and \(\theta(n) > 0\) for \(\Upsilon > T_A\). It is identified as the apparent horizon of the FRW universe\(^1\). Unlike the cosmological event horizon \(T_E := a \int_0^\infty a^{-1} \, dt\) \(^3\), which is the horizon of absolute causality and relies on the entire future history of the universe, the geometrically defined apparent horizon \(T_A\) is the horizon of relative causality and is observer-dependent: if we center our coordinate system on any observer comoving with the universe, then \(r_A\) is the coordinate location of the apparent horizon relative to that observer. \(T_A\) is practically more useful and realistic in observational cosmology as it can be identified by local observations in short duration. In fact, it has been found that \(^4\) for an accelerating universe driven by scalarial dark energy with a possibly varying equation of state, the first and second laws of thermodynamics hold on \(T_A\) but break down on \(T_E\). Moreover for black holes, Hajicek \(^5\) has argued that Hawking radiation happens on the apparent horizon rather than the event horizon. Hence in this paper we will focus on the cosmological apparent horizon \(T_A\). Note that in spherical symmetry \(T_A\) can equivalently be specified by setting \(g^\mu\nu \partial_\mu \Upsilon \partial_\nu \Upsilon = h^\alpha\beta \partial_\alpha \Upsilon \partial_\beta \Upsilon = 0\), which locates the hypersurface on which \(\partial_\alpha \Upsilon\) becomes a null vector. Hereafter, quantities related to or evaluated on the apparent horizon \(\Upsilon = T_A\) will be highlighted by the subscript \(A\).

In some calculations we will find it useful to work with the metric with radial coordinate \(\Upsilon\) rather than \(r\). To that end note that the total derivative of the physical radius \(\Upsilon = a(t) r\) yields
\[
ad r = d\Upsilon - H\Upsilon dt,\]
so the FRW metric Eq.(1) can be rewritten into
\[
ds^2 = \left(1 - \frac{k\Upsilon^2}{a^2}\right)^{-1} \left( - \left(1 - \frac{T^2}{T_A^2}\right) dt^2 - 2H\Upsilon dt d\Upsilon + d\Upsilon^2 \right) + T^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).\]

For Eqs.(1) and (2), the coordinate singularity \(\hat{r}^2 = 1/k\) or \(\Upsilon^2 = a^2/k\) can be removed in the isotropic radial coordinate \(\hat{r}\) with \(r := \hat{r} (1 + \frac{k\hat{r}^2}{a^2})^{-1}\). Following Eq.(9) and keeping in mind that \(t\) is not orthogonal to \(\Upsilon\) in the

\(^3\) By the original definition \(^3\) an apparent horizon is always marginally outer trapped with \(\theta(n) = 0\). However in this paper we follow the more general cosmological vernacular convention which defines an apparent horizon to be either a marginally outer trapped or marginally inner trapped surface. In a contracting universe with \(a < 0\) and \(H < 0\), however, we would have a more standard marginally outer trapped horizon with \(\theta(t) = 0\) and \(\theta(n) = 2H < 0\) at \(\Upsilon = T_A\).
(t, Υ, θ, φ) coordinates, the transverse component of the tetrad can be rebuilt as

\[ η^μ = \left(1, H \Upsilon + \sqrt{1 - \frac{k \Upsilon^2}{a^2}}, 0, 0 \right), \quad η^μ = \frac{1}{2} \left(1, H \Upsilon - \sqrt{1 - \frac{k \Upsilon^2}{a^2}}, 0, 0 \right), \]

with which we obtain the same expansion rates {θ(ℓ), θ(μ)} and the horizon location Υ as from the previous tetrad Eq.(2).

II.2. Modified gravity and energy dissipation

For modified theories of relativistic gravity such as \( f(R) \), \( f(R, G) \) and \( f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu
u\rho\sigma}R^{\mu
u\rho\sigma}) \) classes of fourth-order gravity, and alternative theories such as Brans-Dicke and generic scalar-tensor-chameleon gravity, the field equations can be recast into the following compact GR form,

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_{\text{eff}} T^{(\text{eff})}_{\mu\nu} \quad \text{with} \quad T^{(\text{eff})}_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(\text{MG})}_{\mu\nu}, \]

where the effective gravitational coupling strength \( G_{\text{eff}} \) relies on the specific gravity model and can be directly recognized from the stress-energy-momentum (SEM) density tensor \( T^{(m)}_{\mu\nu} \) for the physical matter content, which is defined from extremizing the matter action functional \( \delta I_m = -\frac{1}{2} \int d^4x \sqrt{-\hat{g}} \delta G^{\mu\nu}. \) For example, as will be extensively discussed later in Sec. VI, we have \( G_{\text{eff}} = G/\bar{f}_R \) for \( f(R) \) gravity, \( G_{\text{eff}} = G/\bar{f} \) for Brans-Dicke, \( G_{\text{eff}} = G/(1+2aR) \) for quadratic gravity, \( G_{\text{eff}} = G/(f_\pi + 2R f_\pi) \) for \( f(R, G) \) generalized Gauss-Bonnet gravity, and \( G_{\text{eff}} = G \) for dynamical Chern-Simons gravity. All terms beyond GR \( (G_{\mu\nu} = 8\pi G T^{(m)}_{\mu\nu}) \) have been packed into \( G_{\text{eff}} \) and \( T^{(\text{MG})}_{\mu\nu} \), which together with \( T^{(m)}_{\mu\nu} \) comprises the total effective SEM tensor \( T^{(\text{eff})}_{\mu\nu} \). Furthermore, we assume a perfect-fluid-type content, which in the metric-independent form is

\[ T^{(\text{eff})}_{\nu\nu} = \text{diag}[ -\rho_{\text{eff}}, P_{\text{eff}}, P_{\text{eff}}, P_{\text{eff}}] , \quad \rho_{\text{eff}} = \rho_m + \rho_{(\text{MG})}, \quad P_{\text{eff}} = P_m + P_{(\text{MG})}, \]

so that \( T^{(m)}_{\nu\nu} = \text{diag}[ -\rho_m, P_m, P_m, P_m] \) and \( T^{(\text{MG})}_{\nu\nu} = \text{diag}[ -\rho_{(\text{MG})}, P_{(\text{MG})}, P_{(\text{MG})}, P_{(\text{MG})}] \). Here \( \rho_m \) and \( P_m \) respectively collect the energy densities and pressures of all matter components in the universe, say \( \rho_m = \rho_m(\text{baryon dust}) + \rho_m(\text{radiation}) + \rho_m(\text{dark energy}) + \rho_m(\text{dark matter}) + \cdots \) and the same for \( P_m \), while the effects of modified gravity have been encoded into \( G_{\text{eff}}, \rho_{(\text{MG})}, \) and \( P_{(\text{MG})} \). For the spatially homogeneous and isotropic FRW universe of maximal spatial symmetry, the coupling strength \( G_{\text{eff}} \), the energy densities \( \{\rho_{\text{eff}}, P_{\text{eff}}, P_{(\text{MG})}\} \) and the pressures \( \{P_{\text{eff}}, P_m, P_{(\text{MG})}\} \), are all functions of the comoving time \( t \) only.

If we take the covariant derivative of the field equation (11), then it follows from the contracted Bianchi identities that the generalized stress-energy-momentum conservation \( \nabla_\mu G^{\mu\nu} = 0 = 8\pi \nabla_\mu (G_{\text{eff}} T^{(\text{eff})}_{\nu\nu}) \) holds for all modified gravity. With respect to the FRW metric Eq.(1), only the t-component of this conservation equation is nontrivial and leads to the universal relation

\[ \dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + P_{\text{eff}}) = \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}}, \]

which serves as the generalized continuity equation for the perfect fluid of Eq.(12). Compared with the continuity equation of a cosmological perfect fluid \( \dot{\rho}_m + 3H (\rho_m + P_m) = 0 \) within GR, the extra term \( -(\dot{G}_{\text{eff}}/G_{\text{eff}}) \rho_{\text{eff}} \) shows up in Eq.(13) to balance the energy flow. Since it has the same dimension as the effective density flow \( \dot{\rho}_{\text{eff}} \), we introduce the following differential energy by multiplying \( Vdt = \frac{4}{3}\pi r^3 dt \) to it,

\[ \mathcal{E} := -\frac{4}{3}\pi \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt. \]
and call it the term of nonequilibrium energy dissipation. Note that at this stage in Eq. (14) for $\mathcal{E}$, the $\frac{4\pi}{27} \rho \rho^{\text{eff}}$ should not be combined into some kind of physically defined mass $V \rho \rho^{\text{eff}} = M_{\text{eff}}$ as its meaning is not clear yet (this is just an issue for security to avoid cyclic logic).

$\mathcal{E}$ is related to the temporal evolution of $G_{\text{eff}}$ and its coupling to $\rho_{\text{eff}}$. Whether $\mathcal{E}$ drives the evolution of $G_{\text{eff}}$ or contrarily is produced by the evolution of $G_{\text{eff}}$ is however not yet certain. Also, as will be seen later, $\mathcal{E}$ plays an important role below in supplementing the unified first law of equilibrium thermodynamics and calculating the entropy production.

III. THERMODYNAMICS INSIDE THE APPARENT HORIZON

For the FRW universe as a solution to the generic field equation (11), we substitute the effective gravitational coupling strength $G_{\text{eff}}$ for Newton’s constant $G$ and thus generalize the Hawking mass $M_{\text{HK}}$ [39] for twist-free spacetimes into

$$M_{\text{HK}} := \frac{1}{4\pi G_{\text{eff}}} \left( \int \frac{dA}{4\pi} \right)^{\frac{1}{2}} \left( -\Psi_2 - \sigma_{\text{NP}} \lambda_{\text{NP}} + \Phi_{11} + \Lambda_{\text{NP}} \right) dA$$

$$= \frac{1}{4\pi G_{\text{eff}}} \left( \int \frac{dA}{4\pi} \right)^{\frac{1}{2}} \left( 2\pi - \int \rho_{\text{NP}} \mu_{\text{NP}} dA \right).$$

(15)

Since we are dealing with spherical symmetry, $M_{\text{HK}}$ can equivalently be written as

$$M_{\text{MS}} := \frac{\gamma}{2G_{\text{eff}}} \left( 1 - h^{a\beta} \partial_a \gamma \partial_\beta \right),$$

(16)

which similarly generalizes the Misner-Sharp mass $M_{\text{MS}}$ [12]. As will be shown later in Sec. V.1 the geometric definitions Eqs. (15) and (16) fully reflect the spirit of geometrodynamics that the effective matter content $\rho_{\text{eff}} = \rho_m + \rho_{\text{MG}}$ curves the space homogeneously and isotropically through the field equation (11) to form the FRW universe. Moreover, the Misner-Sharp mass of black holes in Brans-Dicke gravity with $G_{\text{eff}} = 1/\phi$ has been found to satisfy Eq. (16) [36], which also encourages us to make the extensions in Eqs. (15) and (16).

Note that the Hawking and Misner-Sharp masses restrict their attentions to the mass of the matter content and do not include the energy of gravitational field.

With $\Psi_2 = \sigma_{\text{NP}} = \lambda_{\text{NP}} = 0$, $\Phi_{11} = -(\dot{H} - \frac{k}{a^2})/4$, $\Lambda_{\text{NP}} = (\dot{H} + 2H^2 + \frac{k}{a^2})/4$ or $\rho_{\text{NP}} \mu_{\text{NP}} = -\theta_1 (\theta_{(a)})/4$ in the tetrad Eq. (2), and $h^{a\beta} = \text{diag}[-1, \beta_{-1-kl}, \beta_{-1-kl}]$ for the transverse two-metric in Eq. (1), either Eq. (15) and Eq. (16) yield that the mass enveloped by a standard sphere of physical radius $\gamma$ in the FRW universe is

$$M = \frac{\gamma^3}{2G_{\text{eff}}} \left( H^2 + \frac{k}{a^2} \right).$$

(17)

Immediately, the total derivative or the transverse gradient of $M = M(t, r)$ is

$$dM = \frac{\gamma^3 H}{2G_{\text{eff}}} \left( 2H + 3H^2 + \frac{k}{a^2} \right) dt + \frac{3\gamma^2}{2G_{\text{eff}}} \left( H^2 + \frac{k}{a^2} \right) dH - \frac{\gamma^3 \dot{G}_{\text{eff}}}{2G_{\text{eff}}^2} \left( H^2 + \frac{k}{a^2} \right) dt$$

$$= \frac{\gamma^3 H}{G_{\text{eff}}} \left( H - \frac{k}{a^2} \right) dt + \frac{3\gamma^2}{2G_{\text{eff}}} \left( H^2 + \frac{k}{a^2} \right) dH - \frac{\gamma^3 \dot{G}_{\text{eff}}}{2G_{\text{eff}}^2} \left( H^2 + \frac{k}{a^2} \right) dt,$$

(18)

(19)

where Eq. (8) has been used to reexpress Eq. (18) into Eq. (19) in terms of the $(t, \gamma)$ normal coordinates.

Hayward derived a unified first law of equilibrium thermodynamics [37, 38] for the differential element of energy change within GR under spherical symmetry, which however will be taken as a first principle in our
where the covector invariant $WdV = \psi$ is the energy/heat flux density, the scalar invariant $W$ is the work density, and $WdV = WAdT$. We formally inherit the original definitions of $\{\psi, W\}$ but make use of the total effective SEM tensor $T^{(\text{eff})}_{\mu\nu}$ rather than just $T^{(m)}_{\mu\nu}$ as in GR:

$$\psi_{\alpha} := T^{\beta}_{\alpha(\text{eff})} \partial_{\beta} T + W \partial_{\alpha} T \quad \text{with} \quad W := -\frac{1}{2} T^{\alpha\beta}_{\text{eff}} h_{\alpha\beta},$$

where $T^{(\text{eff})}_{\alpha\beta}$ denote the components of $T^{(\text{eff})}_{\mu\nu}$ along the transverse directions. Note that the definitions of $\psi$ and $W$ also guarantee that they are independent of the coordinate systems or observers and the choice of metric signature. Moreover, with the matter content of effective perfect fluid assumed in Eq. (12), $\psi$ and $W$ explicitly become

$$W = \frac{1}{2} \left( \rho_{\text{eff}} - P_{\text{eff}} \right) \quad \text{and}$$

$$\psi = -\frac{1}{2} \left( \rho_{\text{eff}} + P_{\text{eff}} \right) H' dt + \frac{1}{2} \left( \rho_{\text{eff}} + P_{\text{eff}} \right) a dr$$

$$= - \left( \rho_{\text{eff}} + P_{\text{eff}} \right) H' dt + \frac{1}{2} \left( \rho_{\text{eff}} + P_{\text{eff}} \right) dT,$$

where $W$ no longer preserves the generalized energy conditions as opposed to the situation of GR unless $G_{\text{eff}}$ is positive definite. Hence, the unified first law Eq. (20) leads to

$$dE = -A \psi H P_{\text{eff}} dt + A \rho_{\text{eff}} adr - \frac{4}{3} \pi G_{\text{eff}} \rho_{\text{eff}} dT$$

$$= -A \left( \rho_{\text{eff}} + P_{\text{eff}} \right) H' dt + A \rho_{\text{eff}} dT - \frac{4}{3} \pi G_{\text{eff}} \rho_{\text{eff}} dT.$$

Hence, by identifying the geometrically defined mass $M$ as the total internal energy, matching the coefficients of $dt$ and $dr$ in Eqs. (18) and (24) or the coefficients of $dt$ and $dT$ in Eqs. (19) and (25), we obtain

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_{\text{eff}}}{3} \rho_{\text{eff}}$$

$$H - \frac{k}{a^2} = -4\pi G_{\text{eff}} \left( \rho_{\text{eff}} + P_{\text{eff}} \right) \quad \text{or} \quad 2H + 3H^2 + \frac{k}{a^2} = -8\pi G_{\text{eff}} P_{\text{eff}},$$

where we have recognized the last term in Eqs. (18) and (19) for $dM$ equal to the dissipation $\mathcal{E}$ in $dE$ as they are both relevant to the evolution of $G_{\text{eff}}$.

In fact, by substituting the FRW metric Eq. (11) into the field equation (11), it can be verified that Eqs. (26) and (27) are exactly the first and the second Friedmann equations governing the dynamics of the scale factor

$^{4}$ For the field equation (11) along with $R = -8\pi G_{\text{eff}} T^{(\text{eff})}_{\mu
u}$ and $R_{\mu
u} = 8\pi G_{\text{eff}} \left( T^{(\text{eff})}_{\mu
u} - \frac{1}{2} g_{\mu\nu} T^{(\text{eff})}_{\mu\nu} \right)$, the Raychaudhuri equations (22) or the appendix of (21) imply the following null, weak and strong energy conditions (abbreviated into NEC, WEC and SEC respectively): $G_{\text{eff}} T^{(\text{eff})}_{\mu
u} \eta_{\mu} \eta_{\nu} \geq 0$ (NEC), $G_{\text{eff}} T^{(\text{eff})}_{\mu
u} \nu_{\mu} u_{\nu} \geq 0$ (WEC), $G_{\text{eff}} T^{(\text{eff})}_{\mu
u} \nu_{\mu} u_{\nu} \geq \frac{1}{2} G_{\text{eff}} T^{(\text{eff})}_{\mu
u} u_{\mu} u_{\nu}$ (SEC),

where $u_{\mu} u_{\nu} = -1$ in the SEC for the metric signature $(-, +, +, +)$ used in this paper. All energy conditions require $G_{\text{eff}} (\rho_{\text{eff}} - P_{\text{eff}}) \geq 0$ for the effective matter content Eq. (12).
For the FRW cosmology. Hence, the gravitational equations (26) and (27) have been derived from the unified first law of nonequilibrium thermodynamics $dE = A\psi + WdV + \dot{E}$ instead of the field equation (11), and this is not a result of cyclic logic as Eqs. (26) and (27) are preassumed as unknown. By the way, for the two versions of the second Friedmann equation in Eq. (27), the former is generally more preferred than the latter, because the former directly reflects the evolution of the Hubble parameter $H$ (especially for $k = 0$ of the observed universe), and in numerical simulations the values of $\dot{H}$ and $H^2$ can differ dramatically (e.g. [7] with $H^2 \gg \dot{H}$) and thus be problematic to work with when put together.

Once one of the Friedmann equations is known, the other one can be obtained using the continuity equation (13). For example, taking the time derivative of the first Friedmann equation $H^2 + k/\alpha^2 = 8\pi G_{\text{eff}} \rho_{\text{eff}} / 3$,

$$2H \left( H - \frac{k}{\alpha^2} \right) = \frac{8\pi}{3} \left( G_{\text{eff}} \rho_{\text{eff}} + G_{\text{eff}} \dot{\rho}_{\text{eff}} \right),$$

and applying the continuity equation

$$\dot{G}_{\text{eff}} \rho_{\text{eff}} + G_{\text{eff}} \dot{\rho}_{\text{eff}} + 3G_{\text{eff}} H \left( \rho_{\text{eff}} + P_{\text{eff}} \right) = 0,$$

one recovers the second Friedmann equation $\dot{H} - k/\alpha^2 = -4\pi G_{\text{eff}} (\rho_{\text{eff}} + P_{\text{eff}})$. Inversely, integration of the second Friedmann equation with the continuity equation leads to the first Friedmann equation by neglecting an integration constant or otherwise treat it as a cosmological constant [8] and incorporate it into $\rho_{\text{eff}}$.

IV. THERMODYNAMICS ON THE APPARENT HORIZON

Having derived the Friedmann equations from the thermodynamics of the FRW universe inside the apparent horizon $\Upsilon < \Upsilon_A$, we will continue to study this thermodynamics-gravity correspondence on the horizon $\Upsilon = \Upsilon_A$, and in the meantime require consistency between the interior and the horizon. In fact, existing papers about this problem almost exclusively focus on the horizon alone [8, 9, 11, 14], as a companion to the thermodynamics of black-hole and Rindler horizons. In this section, the apparent horizon $\Upsilon = \Upsilon_A$ will be studied via two methods: (1) Following Sec. III applying the nonequilibrium unified first law $dE = A\psi + WdV + \dot{E}$ and $dE = dM$ in the smooth limit $\Upsilon \to \Upsilon_A$; (2) Using the nonequilibrium Clausius relation $T(dS + d\rho S) = \delta Q = -(A\psi + \dot{E})$ with entropy production $d\rho S$ and the continuity equation (13).

IV.1. Method 1: Unified first law and $dE \equiv dM$

As shown by Eq. (6) in Sec. II, the cosmological apparent horizon, in this case a marginally inner trapped horizon of the expanding FRW universe locates at $\Upsilon_A = 1/\sqrt{H^2 + k/\alpha^2}$, and according to Eq. (17), the mass within the horizon is $M_A = \Upsilon_A/(2G_{\text{eff}})$. Following Sec. II.2 and taking the smooth limit $\Upsilon \to \Upsilon_A$ from the interior to the horizon, Eqs. (18) and (24) yield in the $(t, r)$ comoving transverse coordinates that

$$dM \equiv \frac{\Upsilon_A^3}{2G_{\text{eff}}} \left( 2H + 3H^2 + \frac{k}{\alpha^2} \right) dt + \frac{3a}{2G_{\text{eff}}} dr - \frac{\Upsilon_A G_{\text{eff}}}{2G_{\text{eff}}} \frac{dt}{dt},$$

$$dE \equiv -A_A \Upsilon_A H P_{\text{eff}} dt + A_A \rho_{\text{eff}} adr - \frac{4}{3} \Upsilon_A ^3 \frac{G_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt,$$
while Eqs. (19) and (25) in the \((t, \Upsilon)\) coordinates give rise to
\[
\frac{dM}{dt} \equiv \frac{\Upsilon_A^3 H^2}{G_{\text{eff}}} \left( \dot{H} - \frac{k}{a^2} \right) + \frac{3}{2 G_{\text{eff}}} d\Upsilon - \frac{\Upsilon_A \dot{G}_{\text{eff}}}{2 G_{\text{eff}}} dt
\]
\[
dE \equiv - A_A \left( \rho_{\text{eff}} + P_{\text{eff}} \right) H \Upsilon_A \, dt + A_A \rho_{\text{eff}} d\Upsilon - \frac{4}{3} \pi \Upsilon_A^3 \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt,
\]
where the symbol \(\equiv\) will be employed hereafter to denote “equality on the apparent horizon”, a standard denotation widely used for equality on quasilocal black-hole horizons (eg. [6]). Note that for the \(dr\) components in Eqs. (29) and (30) as well as the \(d\Upsilon\) components in Eqs. (31) and (32), one just needs to evaluate their coefficients in the limit \(\Upsilon \to \Upsilon_A\); although both horizon radii \(r_A = r_A(t)\) and \(\Upsilon_A = \Upsilon_A(t)\) are functions of \(t\) according to Eq. (6), the differentials \(dr\) and \(d\Upsilon\) should not be replaced by \(\dot{r}_A\) and \(\dot{\Upsilon}_A\) for \(\Upsilon \to \Upsilon_A\), because the horizon is not treated as a thermodynamical system alone by itself. As expected, in the limit \(\Upsilon \to \Upsilon_A\) the equality \(dM \equiv dE\) recovers the Friedmann equations again,
\[
H^2 + \frac{k}{a^2} \equiv \frac{8\pi G_{\text{eff}}}{a^2} \rho_{\text{eff}} \quad \text{and} \quad \dot{H} - \frac{k}{a^2} \equiv - 4\pi G_{\text{eff}} \left( \rho_{\text{eff}} + P_{\text{eff}} \right) \quad \text{or} \quad 2\dot{H} + 3H^2 + \frac{k}{a^2} \equiv -8\pi G_{\text{eff}} P_{\text{eff}} .
\]
Specifically note from Eqs. (31) and (32) that on the horizon the dissipation term satisfies
\[
\frac{4}{3} \pi \Upsilon_A^3 \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} \equiv \frac{1}{2} \Upsilon_A^3 \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \Upsilon_A^2 ,
\]
which, without being further simplified, will be used in the next subsection to reduce the expression of the on-horizon entropy production.

### IV.2. Method 2: Nonequilibrium Clausius relation

The modified theories of gravity under our consideration with the field equation (11) are all diffeomorphism invariant, and therefore we can obtain the Wald-Kodama dynamical entropy of the FRW apparent horizon by Wald’s Noether-charge method [38, 41, 42] as
\[
S := \int \frac{dA}{4G_{\text{eff}}} \equiv \frac{A_A}{4G_{\text{eff}}} \equiv \frac{\pi \Upsilon_A^2}{G_{\text{eff}}} ,
\]
with \(G_{\text{eff}} = G_{\text{eff}}(t)\). In fact, the field equations of modified and alternative gravity have been deliberately rearranged into the form of Eq. (11) with an effective gravitational coupling strength \(G_{\text{eff}}\) to facilitate the definition of the horizon entropy Eq. (34). Moreover, the absolute temperature of the horizon is assumed to be \([8]\)
\[
T \equiv \frac{1}{2\pi \Upsilon_A} ,
\]
which agrees with the temperature of the semiclassical thermal spectrum [40] for the matter tunneling into the region \(\Upsilon < \Upsilon_A\) from the exterior \(\Upsilon > \Upsilon_A\), as measured by a Kodama observer using the line element Eq. (9). In fact, if the dynamical surface gravity [43] for the FRW spacetime is defined as \(\kappa := -\frac{1}{2} \partial_r \Xi\) with \(\Xi := h^{\alpha \beta} \partial_\alpha \Upsilon \partial_\beta \Upsilon \equiv 1 - \Upsilon^2 (H^2 + \frac{k}{a^2}) = 1 - \Upsilon^2 / \Upsilon_A^2\), then \(\kappa = \Upsilon / \Upsilon_A^2 \equiv 1 / \Upsilon_A\) and the temperature ansatz Eq. (35) satisfies \(T = \kappa / (2\pi)\). This formally matches the Hawking temperature of (quasi-)stationary black holes in terms of the traditional definition of surface gravity [1] based on Killing vectors and Killing horizons. Hence
it follows from Eqs. (34) and (35) that

$$TdS \doteq \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} dt - \frac{1}{2} \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} G_{\text{eff}} dt \quad \text{with} \quad \dot{\Upsilon}_A = -H\Upsilon_A^3 \left( \frac{\dot{H}}{H} - \frac{k}{a^2} \right).$$

(36)

Assuming that at the moment $t = t_0$ the apparent horizon locates at $\Upsilon_{A0}$, then during the infinitesimal time interval $dt$ the horizon will move to $\Upsilon_{A0} + \dot{\Upsilon}_{A0} dt$. In the meantime, for the \textit{isochoric} process ($d\Omega = 0$) for the volume of constant radius $\Upsilon_{A0}$, the amount of energy across the horizon $\Upsilon = \Upsilon_{A0}$ during this $dt$ is just $dE \doteq A_A \dot{\Upsilon}_A + \mathcal{E}_A$ evaluated at $t = t_0$, as has been calculated in Eq. (32) with the $d\Omega$ component removed.

Compare $dE \doteq A_A \dot{\Upsilon}_A + \mathcal{E}_A$ with Eq. (36), and it turns out the Clausius relation $TdS \doteq \delta Q \doteq -dE$ for equilibrium thermodynamics does not hold. To balance the energy change, we have to introduce an extra entropy production term $d\rho S$ (subscript $p$ being short for “production”) so that

$$TdS + Td\rho S \doteq -dE \doteq -(A_A \dot{\Upsilon}_A + \mathcal{E}_A).$$

(37)

Hence, it follows from Eqs. (32) and (36) that

$$Td\rho S \doteq -TdS - A_A \dot{\Upsilon}_A - \mathcal{E}_A$$

$$\doteq \left( \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} dt + A_A \dot{\Upsilon}_A \right) + \frac{1}{2} \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} G_{\text{eff}} dt - \mathcal{E}_A$$

$$\doteq \left( \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} dt - A_A \left( \rho_{\text{eff}} + P_{\text{eff}} \right) \frac{H}{\Upsilon_A} \right) dt + \frac{1}{2} \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} G_{\text{eff}}^2 + \frac{4}{3} \pi \Upsilon_A^3 \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt.$$

(38)

We have combined the $\dot{\Upsilon}_A$ component of $TdS$ in Eq. (36) with $A_A \dot{\Upsilon}_A$, which reproduces the second Friedmann equation

$$\frac{\dot{\Upsilon}_A}{G_{\text{eff}}} dt - A_A \left( \rho_{\text{eff}} + P_{\text{eff}} \right) \frac{H}{\Upsilon_A} \doteq 0 \quad \Rightarrow \quad \frac{\dot{H}}{H} - \frac{k}{a^2} \doteq -4\pi G \left( \rho_{\text{eff}} + P_{\text{eff}} \right),$$

(39)

while the $\dot{G}_{\text{eff}}$ component of $TdS$ in Eq. (36) and the energy dissipation $\mathcal{E}_A$ add up together and give rise to the entropy production

$$Td\rho S \doteq \frac{1}{2} \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} G_{\text{eff}} dt + \frac{4}{3} \pi \Upsilon_A^3 \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt$$

and

$$d\rho S \doteq \frac{8}{3} \pi \Upsilon_A^3 \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt.$$ (40)

Hence, for the Wald-Kodama dynamical entropy Eq. (34), $TdS$ manifests its effects in two aspects: the $\dot{\Upsilon}_A$ bulk term is the equilibrium part related to the expansion of the universe and the apparent horizon, while the $\dot{G}_{\text{eff}}$ term is the nonequilibrium part associated to the evolution of the coupling strength. The former balances the energy flux $A_A \dot{\Upsilon}_A$ and leads to the Friedmann equation (39), while the latter, together with the generic energy dissipation $\mathcal{E}_A$ evaluated on the horizon, constitute the two sources shown up in Eq. (40) responsible for the entropy production.

As discussed before in Sec. III, the first Friedmann equation $H^2 + k/a^2 \doteq 8\pi G_{\text{eff}} \rho_{\text{eff}}/3$ can be obtained from Eq. (39) with the help of the continuity equation (13). For the consistency between the interior in the relation $dE \doteq dM \doteq -T(dS + d\rho S)$, we have adjusted the thermodynamic sign convention into $T(dS + d\rho S) \doteq \delta Q \doteq -dE \doteq -(A_A \dot{\Upsilon}_A + \mathcal{E}_A)$.

The second Friedmann equation (40) can be rewritten into the evolution equation for the apparent-horizon radius $\Upsilon_A$:

$$\dot{\Upsilon}_A = 4\pi H \Upsilon_A^3 G_{\text{eff}} \left( \rho_{\text{eff}} + P_{\text{eff}} \right).$$

(40)

which shows that for an expanding universe ($H > 0$), $\Upsilon_A$ can be either expanding, contracting or even static, depending on the values of $G_{\text{eff}}$ and the effective equation of state parameter $w_{\text{eff}} = P_{\text{eff}}/\rho_{\text{eff}}$. 

5 The second Friedmann equation (40) can be rewritten into the evolution equation for the apparent-horizon radius $\Upsilon_A$:
In this paper, following the spirit of [10], primarily we call the modified gravity an equilibrium or nonequilibrium theory from the thermodynamic point of view depending on whether the equilibrium Clausius relation \( TdS \equiv \delta Q \equiv -A_A \psi_t \) or its nonequilibrium extension with entropy production \( TdS + Td_p S \equiv -dE \equiv -(A_A \psi_t + E_A) \) works on the apparent horizon. Moreover, Eq. (40) clearly shows that both sources for the nonequilibrium entropy-production \( d_p S \) trace back to the dynamics/evolution of \( G_{\text{eff}} \). Hence, we further regard all those quantities containing \( \dot{G}_{\text{eff}} \) as nonequilibrium, such as the energy dissipation element introduced in Eq. (14). In the same sense, \( TdS \) itself in Eq. (36) is no longer a thermodynamical quasistationary expression, and we regard its \( \dot{\Upsilon}_A \) bulk component as equilibrium, while its \( \dot{\Upsilon}_A \) component as nonequilibrium. This way, the thermodynamic terminology “nonequilibrium” and “equilibrium” in our usage throughout this paper have been clarified.

Eq. (40) demonstrates that the entropy production effect is generally unavoidable in modified gravity unless \( G_{\text{eff}} = \text{constant} \). An increasing coupling strength \( G_{\text{eff}} \) leads to an entropy increment, while more interestingly, a decreasing \( G_{\text{eff}} \) would produce negative entropy for the universe. Yet Eq. (40) only reflects the entropy production \( d_p S \) on the horizon, and the total entropy change of the horizon as well as the entire universe needs further clarification within the generalized second law of thermodynamics within modified gravity. This problem is not tackled in this paper as we concentrate on the (unified) first law of thermodynamics. In addition, note that the dynamics of \( G_{\text{eff}} \) is different from the idea of varying gravitational constant in Dirac’s “large numbers hypothesis” [44], which means nonconstancy of Newton’s constant \( G \) over the cosmic time scale within GR.

If we take advantage of the on-horizon dissipation equation (33) in \( dM \equiv dE \), that is to say, with the assistance of the first method in Sec. IV .1, the entropy production equation (40) can be much simplified into

\[
Td_p S \equiv \dot{\Upsilon}_A \frac{G_{\text{eff}}}{G_{\text{eff}}^2} \frac{G_{\text{eff}}}{G_{\text{eff}}^2} dt \quad \text{and} \quad d_p S \equiv 2\pi \dot{\Upsilon}_A \frac{G_{\text{eff}}}{G_{\text{eff}}^2} \frac{G_{\text{eff}}}{G_{\text{eff}}^2} dt .
\]  

(41)

It can reduce the calculations in specifying the amount of entropy production, when we need not distinguish the two sources represented by the two terms in Eq. (40). This simplification also indicates the \( dM = dE \) method nicely complements the Clausius method.

V. FURTHER DISCUSSION ON THE UNIFIED FORMULATION

So far a unified formulation has been developed to derive the Friedmann equations from nonequilibrium thermodynamics within generic metric gravity \( R_{\mu\nu} - R g_{\mu\nu}/2 = 8\pi G_{\text{eff}} T_{\mu\nu}^{(\text{eff})} \), and the whole operation is:

1. Inside the apparent horizon \( \Upsilon < \Upsilon_A \), the total derivative \( dM \) of the geometric mass and the unified first law of nonequilibrium thermodynamics \( dE = A \psi + WdV + E \) yield Friedmann equations via \( dE = dM \). This method also applies to the horizon by taking the smooth limit \( \Upsilon \to \Upsilon_A \).

2. Alternatively, consider the change of total internal energy during the time interval \( dt \). When evaluated on the horizon \( \Upsilon = \Upsilon_A \), the extended nonequilibrium Clausius relation \( TdS + Td_p S \equiv \delta Q \) yields the second Friedmann equation, which can reproduce the first one with the continuity equation.

3. Derivations for the interior \( \Upsilon < \Upsilon_A \) and the horizon \( \Upsilon_A \) should be consistent, which sets up the thermodynamic sign convention \( T(dS + d_p S) \equiv \delta Q \equiv -dE \equiv -(A_A \psi_t + E_A) \).

In this section we will further investigate some problems involved in the unified formulation.
V.1. A viability test of the extended Hawking and Misner-Sharp masses

We have replaced $G$ with $G_{\text{eff}}$ to generalize the Hawking mass and the Misner-Sharp mass into Eqs. (15) and (16), respectively. Such geometric mass worked well in deriving the Friedmann equations in the unified formulation for the correctness of this extension. Here we provide another piece of evidence by demonstrating that equality between the physical effective mass $M = \rho_{\text{eff}} V$ and the generalized geometric masses automatically reproduces the Friedmann equations.

The total derivative of the physically defined effective mass $M = \rho_{\text{eff}} V = (\rho_m + \rho_{(MG)}) V$ reads

\[
dM = d(\rho_{\text{eff}} V) = \rho_{\text{eff}} dV + V \rho_{\text{eff}} dt = \rho_{\text{eff}} A d\Gamma - V \left(3H(\rho_{\text{eff}} + P_{\text{eff}}) + \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}}\right) dt = 4\pi r^2 \rho_{\text{eff}} d\Gamma - 4\pi r^3 H (\rho_{\text{eff}} + P_{\text{eff}}) - 4\pi \frac{1}{3} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt,
\]

where we have used the continuity equation (13) to replace $\dot{\rho}_{\text{eff}}$. Compare Eq. (42) with Eq. (19),

\[
dM = \frac{\dot{r}^2 H}{G_{\text{eff}}} \left(\dot{H} - \frac{k}{a^2}\right) dt + \frac{3\dot{r}^2}{2G_{\text{eff}}} \left(H^2 + \frac{k}{a^2}\right) d\Gamma - \frac{\dot{r}^2 G_{\text{eff}}}{2G_{\text{eff}}} \left(H^2 + \frac{k}{a^2}\right) dt,
\]

and straightforwardly, by assuming the physically defined effective mass $M = \rho_{\text{eff}} V$ equal to the geometric effective mass in Eq. (17), which comes from Eqs. (15) and (16) that are defined solely out of the spacetime metric, we will automatically recover the two Friedmann equations from $dM = dM$:

\[
\dot{H} + \frac{k}{a^2} = \frac{8\pi G_{\text{eff}}}{3} \rho_{\text{eff}}, \quad \dot{H} - \frac{k}{a^2} = -4\pi G_{\text{eff}} (\rho_{\text{eff}} + P_{\text{eff}}).
\]

In this sense we argue that the generalized definitions in Eqs. (15) and (16) for the Hawking and the Misner-Sharp masses are intuitive. Also, the equality to $M = \rho_{\text{eff}} V$ indicates that Eqs. (15) and (16) only refer to the effective matter content and do not include the energy of gravitational field.

Having obtained the first Friedmann equation (26), we can now combine Eqs. (17) and (26) to eventually see that

\[
M_{\text{MS}} = \frac{\dot{r}^3}{2G_{\text{eff}}} \left(H^2 + \frac{k}{a^2}\right) = \frac{\dot{r}^3}{2G_{\text{eff}}} \cdot \frac{8\pi G_{\text{eff}}}{3} \rho_{\text{eff}} = \frac{4}{3} \pi r^3 \rho_{\text{eff}} = V \rho_{\text{eff}} = M,
\]

so the geometric effective mass Eq. (17) is really equal to the physically defined mass $V \rho_{\text{eff}}$ with the effective density determined by Eqs. (11) and (12). Note that (13) has generalized the Misner-Sharp masses for the $f(R)$ gravity with $G_{\text{eff}} = G/f_R$ and the scalar-tensor gravity with $G_{\text{eff}} = G/f(\phi)$, and their results actually refer to the pure mass $V \rho_m$ of the physical matter content compared with our generalizations, as will be clearly shown in Sec. VI.1 and Sec. VI.4 later. Also, the following masslike function was assumed in [14]

\[
\text{Masslike} := \frac{\dot{r}}{2G_{\text{eff}}} \left(1 + h^{\alpha\beta} \partial_{\alpha} \dot{r} \partial_{\beta} \dot{r}\right) = \frac{\dot{r}}{2G_{\text{eff}}} \left(2 - \frac{\dot{r}^2}{\dot{r}_A^2}\right) \leq \frac{\dot{r}_A}{2G_{\text{eff}}},
\]

in an attempt to recover the Friedmann equations on the horizon itself from the equilibrium Clausius relation without the entropy-production correction $d_S$. However, it is not suitable in our more general formulation in Sec. III and Sec. IV, especially in the $dM = dE$ approach for the whole region $\Gamma < \Gamma_A$, and it does not pass the test just above as in Eq. (42).

On the other hand, recall that in recent studies on the interesting idea of “chemistry” of anti-de Sitter
black holes [45], the mass \( M \) has been treated as the enthalpy \( \mathcal{H} \) rather than total internal energy \( E \), i.e. \( M = \mathcal{H} = E + PV \) where the pressure \( P \) is proportional to the cosmological constant \( \Lambda \). Since \( \Lambda \) the the simplest modified-gravity term, similarly, is it possible to identify the mass \( M \) in a sphere of radius \( \gamma \leq \gamma_A \) in the FRW universe as the enthalpy \( \mathcal{H} = E + PV \) for some kind of pressure \( \tilde{P} \) (it can be \( P_{\text{eff}}, P_{(MG)} \), etc.)? We find that the answer seems to be negative. The equality between Eqs. (18)(19) for \( dM \) and Eqs. (24)(25) for \( dE \), as well as the consistency among Eqs. (19), (25) and (42) clearly shows that the mass \( M \) should be identified as the total internal energy \( E \). Moreover, if forcing the equality \( M = \mathcal{H} \), then \( dM = d\mathcal{H} = d(E + PV) \) implies that necessarily that \( \tilde{P} \equiv 0 \) and \( \tilde{P} \equiv 0 \) and thus we still have \( M \equiv E \).

V.2. The continuity/conservation equation

As emphasized before in Sec. [1], we are considering ordinary modified gravity under minimal geometry-matter coupling, \( \mathcal{L}_{\text{total}} = \mathcal{L}_{\text{gravity}} + 16\pi G \mathcal{L}_m \) with an isolated matter density \( \mathcal{L}_m \) in the total lagrangian density and thus no curvature-matter coupling terms like \( R \mathcal{L}_m \); or equivalently, the gravity/geometry part and the matter part in the total action are fully separable, \( I_{\text{total}} = I_{\text{gravity}} + I_m \). For the matter action \( I_m = \int d^4 x \sqrt{-g} \mathcal{L}_m \) itself, the SEM tensor \( T^{(m)}_{\mu\nu} \) is defined by the following stationary variation (eg. [21]),

\[
\delta I_m = \delta \int d^4 x \sqrt{-g} \mathcal{L}_m = -\frac{1}{2} \int d^4 x \sqrt{-g} T^{(m)}_{\mu\nu} \delta g^{\mu\nu} \quad \text{with} \quad T^{(m)}_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}} . \tag{45}
\]

On the other hand, since \( \mathcal{L}_m \) is a scalar invariant, Noether’s conservation law yields

\[
\nabla^\mu \left( \frac{1}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}} \right) = 0 . \tag{46}
\]

Comparison with Eq. (45) yields that Eq. (46) can be rewritten into \( -\frac{1}{2} \nabla^\mu T^{(m)}_{\mu\nu} = 0 \). Hence, the definition of the SEM tensor \( T^{(m)}_{\mu\nu} \) as in Eq. (45) is Noether-compatible, and the definition of \( T^{(m)}_{\mu\nu} \) by itself automatically guarantees stress-energy-momentum conservation

\[
\nabla^\mu T^{(m)}_{\mu\nu} = 0 . \tag{47}
\]

For a time-dependent perfect-fluid matter content \( T^{(m)}_{\mu\nu} = \text{diag} \{ -\rho_m(t), P_m(t), P_m(t), P_m(t) \} \) (say for the FRW universe), \( \nabla^\mu T^{(m)}_{\mu\nu} = 0 \) gives rise to the continuity equation

\[
\dot{\rho}_m + 3H (\rho_m + P_m) = 0 . \tag{48}
\]

Hence, the total continuity equation (13) can be reduced into

\[
\dot{\rho}_{(MG)} + 3H (\rho_{(MG)} + P_{(MG)}) = -\frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} (\rho_{(MG)} + \rho_{(MG)}) . \tag{49}
\]

Also, note that \( \rho_m \) collects the energy density of all possible physical material content,

\[
\rho_m \equiv \sum \rho_{m(i)} = \rho_m(\text{baryon dust}) + \rho_m(\text{radiation}) + \rho_m(\text{dark energy}) + \rho_m(\text{dark matter}) + \cdots , \tag{50}
\]
and for each type of component $\rho_{m(i)}$, by decomposing Eq. (48) we have individually

$$\dot{\rho}_{m(i)} + 3H (\rho_{m(i)} + P_{m(i)}) = Q_{m(i)} \quad \text{with} \quad \sum Q_{m(i)} = 0,$$

where $Q_{m(i)}$ denotes the energy exchange due to the possible self- and cross-interactions among different matter components.

These results are applicable to the situation of minimal geometry-matter couplings. The thermodynamics of nonminimally coupled theories like $\mathcal{L} = f(R) + 16\pi G \mathcal{L}_m [46]$ (where $T^{(m)} = g_{\mu\nu} T^{(m)}_{\mu\nu}$) and $\mathcal{L} = f(R, T^{(m)}, R_{\mu\nu} T^{(m)}_{\mu\nu}) + 16\pi G \mathcal{L}_m [47]$ have been attempted using the traditional formulation as in [9] for $f(R)$ gravity. However, more profound thermodynamic properties may hide in these theories, as there is direct energy exchange between spacetime geometry and the energy-matter content under nonminimal curvature-matter couplings [21, 29, 30]. For example, very recently Harko [48] has interpreted the generalized conservation equations in $\mathcal{L} = f(R, \mathcal{L}_m)$ and $\mathcal{L} = f(R, T^{(m)}) + 16\pi G \mathcal{L}_m$ gravity as a matter creation process with an irreversible energy flow from the gravitational field to the created matter in accordance with the second law of thermodynamics. The unusual thermodynamic effects in these theories go beyond the scope of this paper, but for the chameleon effect [27, 31] which is another type of nonminimal coupling in scalar-tensor alternative gravity, we manage to find the extra energy dissipation and entropy production caused by the chameleon field, as will be shown later in Sec. [VI.4]

V.3. “Negative temperature” on the horizon could remove the entropy production $d_p S$

In Sec. [IV.2] by studying the energy change during $dt$ across the horizon we have derived the second Friedmann equation from the nonequilibrium Clausius relation $T(dS + d_p S) = (A_\Lambda \psi_t + E_\Lambda)$ with a necessary entropy-production element $d_p S$. However, we also observe that if the geometric temperature of the horizon were to be defined by the following “negative temperature”

$$\mathcal{T} \equiv -\frac{1}{2\pi T_A} < 0,$$

which is the opposite to Eq. (35), then it is easily seen from Sec. [IV.2] that

$$\mathcal{T} dS - A_\Lambda \psi_t - E_\Lambda \approx \left( \frac{\dot{\rho}_A}{G_{\text{eff}}} dt - A_\Lambda \dot{\psi}_t \right) - \frac{1}{2} \frac{\dot{\rho}_A}{G_{\text{eff}}} \dot{G}_{\text{eff}} dt + E_\Lambda$$

$$\approx - \left( \frac{H \dot{Y}_A}{G_{\text{eff}}} \dot{H} - \frac{k}{d^2} + A_\Lambda (\rho_{\text{eff}} + P_{\text{eff}}) H \dot{Y}_A \right) dt - \frac{1}{2} \frac{\dot{\rho}_A}{G_{\text{eff}}} \dot{G}_{\text{eff}} dt - \frac{4}{3} \pi \frac{\dot{Y}_A^3}{G_{\text{eff}}} \rho_{\text{eff}} dt.$$

In the last row of Eq. (53), the vanishing of the former parentheses leads to the second Friedmann equation, while in the second parentheses, the $G_{\text{eff}}$ component of $\mathcal{T} dS$ and the overall energy dissipation term $E_\Lambda$ cancel out each other to yield the first Friedmann equation. Hence, with the negative horizon temperature Eq. (52), both Friedmann equations could be obtained from the standard equilibrium Clausius relation

$$\mathcal{T} dS \hat{=} dE \hat{=} A_\Lambda \psi_t + E_\Lambda$$

without employing an entropy-production term $d_p S$.

However, the negative temperature ansatz Eq. (52) is problematic in various aspects. For example, negative absolute temperature is forbidden by the third law of thermodynamics (as is well known, the so-called “negative temperature” state in atomic physics actually occurs at a unusual phase of very high temperature where the entropy decreases with increasing internal energy, $T^{-1} := \partial S / \partial E < 0$). Also, if tracing back to the past history
of the expanding Universe, one will find the horizon carrying a more and more negative temperature \( T \) while enclosing a more and more (positively) hot interior. From these perspectives, the observation from Eq. (\ref{52}) that \( T = -1/(2\pi \Upsilon A) \) could provide a most economical way to recover the Friedmann equations on the apparent horizon from equilibrium thermodynamics may just be an interesting coincidence.

V.4. Equilibrium situations with \( G_{\text{eff}} = G = \text{constant and thus} \ E = 0 \)

When the effective gravitational coupling strength \( G_{\text{eff}} \) reduces to become Newton’s constant \( G \), the field equation (\ref{11}) reduces to

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}^{(\text{eff})} = 8\pi G \left( T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\text{MG})} \right). \tag{55}
\]

For theories in this situation, the Lagrangian density generally takes the form

\[
\mathcal{L} = R + f(R_{\mu\nu} R^{\mu\nu}, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, R, \cdots) + \omega(\phi, \nabla_\mu \phi \nabla^\mu \phi) + 16\pi G \mathcal{L}_m, \tag{56}
\]

where \( R_i \) denotes an arbitrary algebraic or differential Riemannian invariant \( R_i = R_i(g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, \cdots) \), \( \nabla_\gamma \nabla_\gamma \cdots \nabla_\gamma R_{\mu\alpha\beta\gamma} \) which is beyond the Ricci scalar \( R \) and makes no contribution to the coefficient of \( R_{\mu\nu} \) in the field equation. \( \omega \) is a generic function of the scalar field \( \phi = \phi(x^\mu) \) and its kinetic term \( \nabla_\mu \phi \nabla^\mu \phi \). For example, the \( \mathcal{L} = R + f(R_{\mu\nu} R^{\mu\nu}, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}) + 16\pi G \mathcal{L}_m \) fourth-order gravity and typical scalarial dark-energy models \( \text{[28]} \) (like quintessence, phantom, k-essence) all belong to this class.

To apply the unified formulation developed in Sec. \( \text{[III]} \) and Sec. \( \text{[IV]} \) for this situation, we just need to replace \( G_{\text{eff}} \) by \( G \), set \( G_{\text{eff}} = 0 \), and remove the energy dissipation term \( E \). Hence, the Hawking or Misner-Sharp mass enclosed by a sphere of radius \( \Upsilon \) is \( M = (\Upsilon^3/2G)(H^2 + k/a^2) \). Compare the transverse gradient \( dM \) of the mass with the change of internal energy \( dE = A \psi + WdV \), and by matching the coefficients of

\[
dM = \frac{\Upsilon^3 H}{2G} \left( 2H + 3H^2 + \frac{k}{a^2} \right) dt + \frac{3\Upsilon^2}{2G} \left( H^2 + \frac{k}{a^2} \right) a dr \tag{57}
\]

\[
dE = -4\pi \Upsilon^3 H \rho_{\text{eff}} dt + 4\pi \Upsilon^2 \rho_{\text{eff}} a dr
\]

in the comoving coordinates \((t, r)\), or

\[
dM = \frac{\Upsilon^3 H}{G} \left( \dot{H} - \frac{k}{a^2} \right) dt + \frac{3\Upsilon^2}{2G} \left( H^2 + \frac{k}{a^2} \right) d\Upsilon \tag{58}
\]

\[
dE = -4\pi \Upsilon^3 H \left( \rho_{\text{eff}} + P_{\text{eff}} \right) dt + 4\pi \Upsilon^2 \rho_{\text{eff}} d\Upsilon \]

in the astrophysical areal coordinates \((t, \Upsilon)\), one obtains the Friedmann equations with \( G_{\text{eff}} = G \):

\[
H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_{\text{eff}} \quad \text{and} \quad \dot{H} - \frac{k}{a^2} = -4\pi G \left( \rho_{\text{eff}} + P_{\text{eff}} \right) \quad \text{or} \quad 2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi GP_{\text{eff}}. \tag{59}
\]

Moreover, in the smooth limit \( \Upsilon \to \Upsilon_A \) Eqs. (\ref{57}) and (\ref{58}) recover the complete set of Friedmann equations on the apparent horizon \( \Upsilon = \Upsilon_A \) by \( dM \equiv dE \). Alternatively, with the absolute temperature \( T \) and the entropy \( S \) of the horizon being

\[
T \equiv \frac{1}{2\pi \Upsilon_A} \quad \text{and} \quad S \equiv \frac{A_A}{4G} \equiv \frac{\pi \Upsilon_A}{G}, \tag{60}
\]
we have
\[ TdS = \frac{\dot{A}_A}{G} dt \quad \text{and} \quad A_A \psi_t = - A_A (\rho_{\text{eff}} + P_{\text{eff}}) H \dot{A}_A dt. \] (61)

Thus, the equilibrium Clausius relation \( TdS = \dot{Q} = - A_A \psi_t \) with Eq. (61) for an isochoric process leads to the second Friedmann equation \( \dot{H} - k/a^2 = -4\pi G (\rho_{\text{eff}} + P_{\text{eff}}) \). Taking into account the continuity equation with vanishing dissipation \( \dot{E} = 0 \):

\[ \dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + P_{\text{eff}}) = 0, \] (62)

integration of the second Friedmann equation leads to the first equation \( \dot{H} - \frac{k}{a^2} = -\frac{4}{3} \pi G \rho_{\text{eff}} \) where the integration constant has been neglected or absorbed into \( \rho_{\text{eff}} \). Moreover, the continuity/conservation equation (62) together with conservation of \( T_{\mu\nu}^{(m)} \) in Eq. (48) lead to

\[ \dot{\rho}_{\text{(MG)}} + 3H (\rho_{\text{(MG)}} + P_{\text{(MG)}}) = 0. \] (63)

For the componential contravariant Lagrangian density \( \sqrt{-g} f(R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\cdot \cdot} \cdot \cdot) \) in Eq. (56), this is actually the “generalized contracted Bianchi identities” \[21\] in perfect-fluid form under the FRW background.

VI. EXAMPLES

In this section, we will apply the unified formulation in Sec. III and Sec. IV to some concrete theories of modified gravity. Compatible with the FRW metric Eq. (1) in the signature \((-++,+++)\), we will adopt the geometric sign convention \( \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}, R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} \cdot \cdot \cdot \) and \( R_{\mu\nu} = R^a_{\mu\nu} \).

VI.1. \( f(R) \) gravity

The \( f(R) \) gravity \[22\] is the simplest class of fourth-order gravity, which straightforwardly generalizes the Hilbert-Einstein Lagrangian density \( \mathcal{L}_{\text{HE}} = R + 16\pi G \mathcal{L}_m \) into \( \mathcal{L} = f(R) + 16\pi G \mathcal{L}_m \) by replacing the Ricci scalar \( R \) with its arbitrary function \( f(R) \). The field equation in the form of Eq. (11) is

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G f^{(m)}_{\mu\nu} + \frac{1}{f_R} \left( \frac{1}{2} (f - f_R) g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f_R \right), \] (64)

where \( f_R := \partial f(R) / \partial R \) and \( \Box \equiv \nabla^\gamma \nabla_\gamma \) denotes the covariant d’Alembertian. From the coefficient of \( T_{\mu\nu}^{(m)} \) we learn that the effective gravitational coupling strength for \( f(R) \) gravity is

\[ G_{\text{eff}} = \frac{G}{f_R}, \] (65)

and thus the modified-gravity SEM tensor is

\[ T_{\mu\nu}^{(\text{MG})} = \frac{1}{8\pi G} \left( \frac{1}{2} (f - f_R) g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f_R \right), \] (66)

which has collected the contributions from nonlinear and fourth-order curvature terms. Substituting the FRW metric Eq. (26) into this \( T_{\nu\nu}^{(\text{MG})} \) and keeping in mind \( T_{\nu\nu}^{(\text{MG})} = \text{diag}[-\rho_{\text{(MG)}}, P_{\text{(MG)}}, P_{\text{(MG)}}, P_{\text{(MG)}}] \), the energy...
density and pressure from the $f(R)$ modified-gravity effect are found to be

$$
\rho_{\text{MG}} = \frac{1}{8\pi G} \left( \frac{1}{2} f_R R - \frac{1}{2} f - 3H \dot{f}_R \right) \quad \text{and} \quad P_{\text{MG}} = \frac{1}{8\pi G} \left( \frac{1}{2} f - \frac{1}{2} f_R R + \dot{f}_R + 2H \dot{f}_R \right).
$$

(67)

Given $G_{\text{eff}} = G/f_R$, the Hawking or Misner-Sharp mass in a sphere of radius $\mathcal{R}$ in the universe is

$$
M = \frac{f_R \mathcal{R}^3}{2G} \left( H^2 + \frac{k}{a^2} \right) \quad \text{with} \quad M_\Lambda \equiv \frac{f_R \mathcal{R}_\Lambda}{2G}.
$$

(68)

Also, the geometric nonequilibrium energy dissipation term associated with $G_{\text{eff}}$ and the geometric Wald-Kodama entropy of the horizon $\mathcal{Y}_\Lambda$

$$
\mathcal{E} = \frac{4}{3} \mathcal{Y} \frac{f_R}{f} \rho_{\text{eff}} dt \quad \text{and} \quad S = \frac{\mathcal{A}_\Lambda f_R}{4G}.
$$

(69)

Note that in $\mathcal{E}$ the term $\frac{4}{3} \mathcal{Y} \frac{f_R}{f} \rho_{\text{eff}}$ should not be combined into the mass $V \rho_{\text{eff}} = \mathcal{M}$ at this stage for the reason stressed after Eq.(14). Applying the unified formulation developed in Sec. III and Sec. IV to the FRW universe governed by $f(R)$ gravity, for the interior and the horizon $\mathcal{R} \leq \mathcal{R}_\Lambda$, the unified first law $dE = A\psi + WdV + \mathcal{E} = dM$ of nonequilibrium thermodynamics and the nonequilibrium Clausius relation $T(dS + dP S) \approx \delta Q = -(A\psi + \mathcal{E}_\Lambda)$ give rise to

$$
H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \frac{G}{f_R} \rho_m + \frac{1}{3} \frac{f_R}{f} \left( \frac{1}{2} f_R R - \frac{1}{2} f - 3H \dot{f}_R \right),
$$

(70)

$$
H - \frac{k}{a^2} = -4\pi \frac{G}{f_R} (\rho_m + P_m) - \frac{1}{2} \frac{f_R}{f} \left( \dot{f}_R - H \dot{f}_R \right).
$$

(71)

In the meantime, the nonequilibrium entropy production $d\rho S$ on the horizon turns out to be

$$
d\rho S \approx -2\pi \mathcal{Y}_\Lambda \frac{f_R}{G} dt.
$$

(72)

Substituting the FRW metric Eq.(11) into Eq.(64), we have verified that, Eqs.(70) and (71) are exactly the Friedmann equations of the FRW universe in $f(R)$ gravity. Such thermodynamics-gravity correspondence within $f(R)$ gravity has been investigated before in [9, 10] with different setups for the quantities $\{M, \rho_{\text{MG}}, P_{\text{MG}}, \cdots\}$ and thus $\{\psi, W, \cdots\}$; compared with these earlier works, we have revised the thermodynamic setups and improved the result of entropy production.

Also note that, compact notations have been used in Eqs.(70) and (71), and $f_R$ itself is treated as a function of the comoving time $t$. Otherwise, one can further write $\dot{f}_R$ into $f_{RR} \dot{R}$ and $\ddot{f}_R$ into $f_{RRR} \dot{R} + f_{RRR} \dot{R}^2$ as in [9, 15], and for the FRW spacetime with metric Eq.(11), we have already known the Ricci scalar that

$$
R = R(t) = 6 \left( \dot{H} + 2H^2 + \frac{k}{a^2} \right),
$$

(73)

which in turn indicates the third-derivative $\dddot{H}$ and thus fourth-derivative $\cdots$ get involved in Eqs.(70) and (71), and these terms are gone once we return to GR with $f_R = 1$.

In [13], Cai et al. have generalized the Misner-Sharp energy/mass to $f(R)$ gravity by the integration and the conserved-charge methods. Specifically for the FRW universe, they found that the energy/mass within a
sphere of radius $\Upsilon$ is

$$E_{\text{eff}} = \frac{\Upsilon}{2G} \left( (1 - h^{\alpha\beta} \partial_\alpha \partial_\beta \Upsilon) + \frac{1}{6} \Upsilon^2 (f - f_R R) - \Upsilon h^{\alpha\beta} \partial_\alpha f_R \partial_\beta \Upsilon \right)$$

$$= \frac{\Upsilon^3}{2G} \left( \frac{1}{\Upsilon^2} f_R + \frac{1}{6} (f - f_R R) + H f_R \right),$$

with $\Upsilon_\Lambda = 1/\sqrt{H^2 + k/a^2}$. What are the differences between this $E_{\text{eff}}$ and our extended Misner-Sharp mass in Eqs.\textcolor{red}{(16)} and \textcolor{red}{(17)} in this paper? In the first and second row of Eq.\textcolor{red}{(74)}, the first terms therein are respectively the definition Eq.\textcolor{red}{(16)} and the concrete mass Eq.\textcolor{red}{(17)} in our usage. To further understand the remaining terms in Eq.\textcolor{red}{(74)}, one can manipulate it into

$$E_{\text{eff}} = \frac{f_R \Upsilon^3}{2G} \left( H^2 + \frac{k}{a^2} \right) - \frac{\Upsilon^3}{2G} \left( \frac{1}{6} (f_R R - f) - H f_R \right)$$

$$= \frac{f_R \Upsilon^3}{2G} \left( H^2 + \frac{k}{a^2} \right) - \frac{4}{3} \Upsilon^2 \Upsilon^2 \left( \frac{1}{8\pi G} \left[ \frac{1}{2} f_R R - \frac{1}{2} f - 3H f_R \right] \right).$$

Recall that in Eq.\textcolor{red}{(43)}, we have already proved the geometric mass Eq.\textcolor{red}{(17)} with which we start our formulation is equal to the physically defined mass $\rho_{\text{eff}} V = (\rho_m + \rho_{(\text{MG})}) V$. Then from the density $\rho_{(\text{MG})}$ in Eq.\textcolor{red}{(67)} and the mass $M$ in Eq.\textcolor{red}{(68)} for $f(R)$ gravity in our unified formulation, it turns out that the $E_{\text{eff}}$ in Eq.\textcolor{red}{(75)} is actually

$$E_{\text{eff}} = M - \rho_{(\text{MG})} V = \left( \rho_m + \rho_{(\text{MG})} \right) V - \rho_{(\text{MG})} V = \rho_m V.$$

Hence, the “generalized Misner-Sharp energy $E_{\text{eff}}$” in \textcolor{red}{[13]} for the FRW universe within $f(R)$ gravity exactly match the pure mass of the physical matter content in our formulation of $f(R)$ cosmology.

VI.2. Generalized Brans-Dicke gravity with self-interaction potential

Now, consider a generalized Brans-Dicke gravity with self-interaction potential in the Jordan frame given by the following Lagrangian density,

$$\mathcal{L}_{\text{GBD}} = \phi R - \frac{\omega(\phi)}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16\pi G \mathcal{L}_m,$$

where, to facilitate the comparison with the proceeding case of $f(R)$ gravity, we have adopted the convention with an explicit $G$ in $16\pi G \mathcal{L}_m$, rather than just $16\pi \mathcal{L}_m$ which encodes $G$ into $\phi^{-1}$. The gravitational field equation $\delta(\sqrt{-g} \mathcal{L}_{\text{GBD}})/\delta g^{\mu\nu} = 0$ is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}^{(m)} + \frac{1}{\phi} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \phi + \frac{\omega(\phi)}{\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) - \frac{1}{2\phi} V g_{\mu\nu},$$

from which we directly read that the effective coupling strength and the modified-gravity SEM tensor are

$$G_{\text{eff}} = \frac{G}{\phi} \quad \text{and} \quad T_{\mu\nu}^{(\text{MG})} = \frac{1}{8\pi G} \left( (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \phi + \frac{\omega(\phi)}{\phi} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) - \frac{1}{2} V g_{\mu\nu} \right).$$
where $T_{\mu\nu}^{(MG)}$ encodes the gravitational effects of the scalar field $\phi$. Put the FRW metric Eq. (26) back to $T_{\mu\nu}^{(MG)}$ with $T_{\mu\nu}^{(MG)} = \text{diag}(-\rho^{(MG)}, P^{(MG)}, P^{(MG)}, P^{(MG)})$, and the energy density and pressure from $\phi$ are found to be
\[
\rho^{(MG)} = \frac{1}{8\pi G} \left( -3H\dot{\phi} + \frac{\omega}{2\phi} \dot{\phi}^2 + \frac{1}{2} V \right), \quad \text{with} \quad P^{(MG)} = \frac{1}{8\pi G} \left( \dot{\phi} + 2H\dot{\phi} + \frac{\omega}{2\phi} \dot{\phi}^2 - \frac{1}{2} V \right),
\]
since $G_{\text{eff}} = G/\phi$, the geometric mass enveloped in a sphere of radius $\Upsilon$ is
\[
M = \frac{\phi \Upsilon^3}{2G} \left( H^2 + \frac{k}{a^2} \right), \quad \text{with} \quad M_A \doteq \frac{\phi \Upsilon A}{2G},
\]
which in fact matches the Misner-Sharp mass of black holes in standard Brans-Dicke gravity in Eq. (36). Also the nonequilibrium energy dissipation term $E$ associated with the evolution of $G_{\text{eff}}$ and the Wald-Kodama entropy $S$ of the horizon are
\[
E = \frac{4}{3} \pi \Upsilon^3 \dot{\phi} \rho_{\text{eff}} dt \quad \text{and} \quad S \doteq \frac{A \dot{\phi}}{4G}.
\]
Following the unified formulation developed in Sec. III and Sec. IV to study $dM = dE = A \psi + WdV + E$ for the region $\Upsilon \leq \Upsilon_A$ and $T(dS + d\rho S) \doteq dQ \doteq -(A \psi + E)_{\Upsilon A}$ for the horizon itself, we find
\[
H^2 + \frac{k}{a^2} = \frac{8\pi G}{3\phi} \rho_m + \frac{1}{3\phi} \left( -3H\dot{\phi} + \frac{\omega}{2\phi} \dot{\phi}^2 + \frac{1}{2} V \right),
\]
\[
H - \frac{k}{a^2} = -4\pi G \frac{\phi}{\phi} (\rho_m + P_m) - \frac{1}{2\phi} \left( \ddot{\phi} - H\dot{\phi} + \frac{\omega}{\phi} \dot{\phi}^2 \right),
\]
where as we can see, the scalar kinetics $\frac{\omega(\phi)}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi$ and the potential $V(\phi)$ does not influence the evolution of the Hubble parameter $H$, and meanwhile the dynamics of $\phi$ and its nonminimal coupling to $R$ in Eq. (149) leads to the entropy production
\[
d\rho S \doteq -2\pi \Upsilon^2 \dot{\phi} \rho_{\text{eff}} dt
\]
for the horizon. We have already verified that Eqs. (84) and (85) are just the Friedmann equations of the FRW universe in the generalized Brans-Dicke gravity by directly applying the FRW metric Eq. (1) to the gravitational field equation (78). Specifically when $\omega(\phi) \equiv \omega_{BD}=$constant and $V(\phi) = 0$ (and erase $G$ as $G \rightarrow 1/\phi$ in standard Brans-Dicke), the thermodynamics-gravity correspondence just above reduces to the situation for the standard Brans-Dicke gravity [26] and its FRW cosmology. Moreover, our results improves the setups of $\{\rho^{(MG)}, P^{(MG)}, \psi, W \cdots \}$ and the entropy production in [9] and [11] for a similar scalar-tensor theory with $\mathcal{L} = \dot{f}(\phi)R/(16\pi G) - \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + \mathcal{L}_m$.

VI.3. Equivalence between $f(R)$ and modified Brans-Dicke without kinetic term

The two models analyzed just above have exhibited pretty similar behaviors. Next we consider a modified Brans-Dicke gravity $\mathcal{L} = \phi R - V(\phi) + 16\pi G \mathcal{L}_m$, which is just the Lagrangian density Eq. (149) in Sec. VI.2 without the kinetic term $-\frac{\omega(\phi)}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi$. Compare its field equation with that of the $\mathcal{L} = f(R) + 16\pi G \mathcal{L}_m$. 

gravity in Sec. VI.1:

\[\phi R_{\mu\nu} - \frac{1}{2}(\phi R - V(\phi)) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) \phi = 8\pi G T_{\mu\nu}^{\text{(m)}},\]  

\[f_R R_{\mu\nu} - \frac{1}{2}f(R) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu)f_R = 8\pi G T_{\mu\nu}^{\text{(m)}}.\]  

(87)

Clearly, these two field equations become identical with the following relations:

\[f_R = \phi \quad \text{and} \quad f(R) = \phi R - V(\phi) \implies f_R R - f(R) = V(\phi).\]  

(88)

That is to say, the \(f(R)\) fourth-order modified gravity in Sec. VI.1 and the generalized Brans-Dicke alternative gravity in Sec. VI.2 are not totally independent. Instead, the former can be regarded as a subclass of the latter with vanishing coefficient \(\omega(\phi) \equiv 0\) for the kinematic term \(\nabla_\alpha \phi \nabla^\alpha\), and the equivalence is built upon Eq. (88).

Applying the replacements \(f_R \mapsto \phi\) and \(f_R R - f(R) \mapsto V(\phi)\) to Sec. VI.1 we obtain the modified-gravity SEM tensor as

\[T_{\mu\nu}^{\text{(MG)}} = \frac{1}{8\pi G} \left( (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \phi - \frac{1}{2} V g_{\mu\nu} \right),\]  

(89)

the energy density and pressure in \(T_{\mu\nu}^{\text{(MG)}} = \text{diag}[-\rho^{\text{(MG)}}, P^{\text{(MG)}}, P^{\text{(MG)}}, P^{\text{(MG)}}]\) as

\[\rho^{\text{(MG)}} = \frac{1}{8\pi G} \left( -3H\dot{\phi} + \frac{1}{2} \dot{V} \right) \quad \text{and} \quad P^{\text{(MG)}} = \frac{1}{8\pi G} \left( \ddot{\phi} + 2H\dot{\phi} - \frac{1}{2} \dot{V} \right),\]  

(90)

as well as the geometric mass \(M\), nonequilibrium energy dissipation term \(E\), horizon entropy \(S\) and the nonequilibrium entropy production \(d_\rho S\) to be

\[M = \frac{\phi \Gamma^3}{2G} \left( H^2 + \frac{k}{a^2} \right) , \quad E = \frac{4}{3} \pi \frac{\Gamma^3 \dot{\phi}}{\dot{\phi}} \rho_{\text{eff}} \, dt , \quad S = \frac{\Lambda A^2 \phi}{4G} \quad \text{and} \quad d_\rho S = -2\pi \Gamma^2 \frac{\dot{\phi}}{G} dt.\]  

(91)

Finally the following equations are obtained from thermodynamics-gravity correspondence

\[H^2 + \frac{k}{a^2} = \frac{8\pi G}{3\phi} \rho_m + \frac{1}{3\dot{\phi}} \left( -3H\dot{\phi} + \frac{1}{2} \dot{V} \right) \quad \text{and} \quad \dot{H} - \frac{k}{a^2} = -4\pi \frac{G}{\phi} \left( \rho_m + P_m \right) - \frac{1}{2\dot{\phi}} \left( \dot{\phi} - H\dot{\phi} \right).\]  

(92)

It is easy to verify that, these thermodynamics quantities and equations precisely match the generalized Brans-Dicke in Sec. VI.2 with \(\omega(\phi) \equiv 0\).

Conversely, if start from these setups just above or those in Sec. VI.2 with \(\omega(\phi) \equiv 0\), the formulation in Sec. VI.1 can be recovered by applying the replacements \(\phi \mapsto f_R\) and \(V(\phi) \mapsto f_R R - f(R)\).

VI.4. Scalar-tensor-chameleon gravity

Consider the following Lagrangian density for the generic scalar-tensor-chameleon gravity [27] in the Jordan frame,

\[\mathcal{L}_{\text{STC}} = F(\phi) R - Z(\phi) \nabla_\alpha \phi \nabla^\alpha \phi - 2U(\phi) + 16\pi GE(\phi) \mathcal{L}_m.\]  

(93)

where \(\{F(\phi), Z(\phi), E(\phi)\}\) are arbitrary functions of the scalar field \(\phi\), and \(E(\phi)\) is the chameleon function describing the coupling between \(\phi\) and the matter Lagrangian density \(\mathcal{L}_m\). The name “chameleon” comes from the fact that in the presence of \(E(\phi)\), the wave equation \(\delta(\sqrt{-g} \mathcal{L}_{\text{STC}})/\delta \phi = 0\) of \(\phi\) becomes explicitly
dependent on the matter content of the universe (e.g., $\mathcal{L}_m$ or $T^{(m)} = g^{\mu \nu} T^{(m)}_{\mu \nu}$), which makes the wave equation change among different cosmic epoches as the dominant matter content varies [31]. The gravitational field equation $\delta(\sqrt{-g} \mathcal{L}_{STC})/\delta g^{\mu \nu} = 0$ is

$$ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8 \pi G \frac{E(\phi)}{F(\phi)} T^{(m)}_{\mu \nu} + \frac{1}{F(\phi)} \left( \nabla_\mu \nabla_\nu - g_{\mu \nu} \nabla^2 \right) F(\phi) + \frac{Z(\phi)}{F(\phi)} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) - \frac{U(\phi)}{F(\phi)} g_{\mu \nu}, $$

(94)

so from the coefficient of $T^{(m)}_{\mu \nu}$ we recognize

$$ G_{\text{eff}} = \frac{E(\phi)}{F(\phi)} G, $$

(95)

$$ T^{(MG)}_{\mu \nu} = \frac{1}{8 \pi G E(\phi)} \left( \nabla_\mu \nabla_\nu - g_{\mu \nu} \nabla^2 \right) F(\phi) + \frac{Z(\phi)}{F(\phi)} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) - \frac{U(\phi)}{F(\phi)} g_{\mu \nu}. $$

(96)

Note that [27] however adopted $G_{\text{eff}} = G/\phi$ to study the second law of thermodynamics for the flat FRW universe, the chameleon function $E(\phi)$ excluded from $G_{\text{eff}}$. Substituting the FRW metric Eq. (26) into $T^{(MG)}_{\mu \nu}$, the energy density and pressure for $T^{(MG)}_{\mu \nu} = \text{diag}[-\rho_{(MG)}, P_{(MG)}, P_{(MG)}, P_{(MG)}]$ are found to be

$$ \rho_{(MG)} = \frac{1}{8 \pi G E(\phi)} \left( -3 H \dot{F} + \frac{1}{2} Z(\phi) \dot{\phi}^2 + U \right) \quad \text{and} \quad P_{(MG)} = \frac{1}{8 \pi G E(\phi)} \left( \ddot{F} + 2 H \dot{F} + \frac{1}{2} Z(\phi) \dot{\phi}^2 - U \right), $$

(97)

where the compact notations $\ddot{F}$ and $\dot{F}$ can be replaced by $F_\phi \ddot{\phi}$ and $F_\phi \dot{\phi} + F_{\phi \phi} \dot{\phi}^2$, respectively. As $G_{\text{eff}} = G E(\phi)/F(\phi)$, the Hawking or Misner-Sharp geometric mass becomes

$$ M = \frac{F(\phi) \Gamma^3}{2 G E(\phi)} \left( H^2 + \frac{k}{a^2} \right) \quad \text{with} \quad M_A = \frac{F(\phi) \Gamma_A}{2 G E(\phi)}, $$

(98)

while the nonequilibrium energy dissipation $\mathcal{E}$ in the conservation equation and the Wald-Kodama entropy of the horizon $S$ are respectively

$$ \mathcal{E} = \frac{4}{3} \pi \Gamma^3 \frac{G}{F(\phi)^2} \left( E(\phi) \dddot{F} - F(\phi) \dddot{E} \right) \rho_{\text{eff}} dt \quad \text{and} \quad S = \frac{A_A F(\phi)}{4 G E(\phi)}, $$

(99)

where in $\mathcal{E}$ the compact notation $E(\phi) \dddot{F} - F(\phi) \dddot{E}$ can be expanded into $(E F_\phi - F E_\phi) \dot{\phi}$. Moreover, using the unified formulation developed in Sec. [III] and Sec. [IV] for the interior and the horizon we obtain

$$ H^2 + \frac{k}{a^2} = \frac{8 \pi G E(\phi)}{3 F(\phi)} \rho_m + \frac{1}{3 F(\phi)} \left( -3 H \dot{F} + \frac{1}{2} Z(\phi) \dot{\phi}^2 + U \right), $$

(100)

$$ H - \frac{k}{a^2} = -4 \pi G E(\phi) F(\phi) \left( \rho_m + P_m \right) - \frac{1}{2 F(\phi)} \left( \dddot{F} - H \dot{F} + Z(\phi) \dot{\phi}^2 \right). $$

(101)

With $\dot{F} = F_{\phi} \dot{\phi}$ and $\dddot{F} = F_{\phi \phi} \dddot{\phi} + F_{\phi \phi} \dot{\phi}^2$, they can be recast into

$$ H^2 + \frac{k}{a^2} = \frac{8 \pi G E(\phi)}{3 F(\phi)} \rho_m + \frac{1}{3 F(\phi)} \left( -3 H \dot{F} + \frac{1}{2} Z(\phi) \dot{\phi}^2 + U \right), $$

(102)
As a subclass of the generic scalar-tensor-chameleon gravity Eq.(93) with \[ E_{\phi} \phi Z(\phi) \dot{\phi}^2 - HF_\phi \phi + Z(\phi) \dot{\phi}^2 \]. (103)

At the same time, the nonequilibrium entropy production turns out to be
\[ d_\rho S = 2\pi \gamma^2 \frac{1}{G E(\phi)^2} (F E_\phi - E F_\phi) \dot{\phi} dt \]. (104)

We have verified by direct substitution of the FRW metric Eq.(1) into Eq.(94) that Eqs.(102) and (102) are indeed the Friedmann equations of the FRW universe in the scalar-tensor-chameleon gravity.

On the other hand, in the absence of the chameleon function, \( E(\phi) \equiv 1, E_\phi = 0 \), and with \( F(\phi) \mapsto \phi, F_\phi \mapsto 1, F_\phi Z(\phi) \mapsto \phi(\phi)/\phi, U \mapsto \frac{1}{2} V \), we recover the generalized Brans-Dicke in Sec. VI.2.

In [13], for the scalar-tensor gravity \( \mathcal{L} = F(\phi)R/(16\pi G) - \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + \mathcal{L}_m \), the generalized Misner-Sharp mass/energy in the FRW universe is found to be
\[ E_{\text{eff}} = \frac{\gamma^3}{2G} \left( F(\phi) \left( H^2 + \frac{k}{a^2} \right) + HF_\phi - \frac{4\pi}{3} \left( \frac{1}{2} \dot{\phi}^2 + V \right) \right) \]. (105)

(Note: A typo in Eq.(A8) of [13] is corrected here by either checking the derivation of Eq.(A8), or by referring to Eq.(74) with the correspondence \( f_R = \phi \) and \( f_R R - f(R) = V \) as in Eq.(88), despite the nonzero kinetic term \(-\frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi \). Compared with Eq.(93), [13] actually adopts a different scaling convention for the Lagrangian density; in accordance with Eq.(93), we rescale [13] by
\[ \mathcal{L} = F(\phi)R - \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) + 16\pi G \mathcal{L}_m \], (106)

and consequently
\[ E_{\text{eff}} = \frac{\gamma^3}{2G} \left( F(\phi) \left( H^2 + \frac{k}{a^2} \right) + HF_\phi - \frac{1}{6} \left( \frac{1}{2} \dot{\phi}^2 + V \right) \right) \], (107)

which can be expanded into
\[ E_{\text{eff}} = \frac{F(\phi) \gamma^3}{2G} \left( H^2 + \frac{k}{a^2} \right) - \frac{4\pi \gamma^3}{8\pi G} \left( -3HF_\phi + \frac{1}{4} \dot{\phi}^2 + \frac{1}{2} V \right) \]. (108)

As a subclass of the generic scalar-tensor-chameleon gravity Eq.(93) with \( E(\phi) \mapsto 1, Z(\phi) \mapsto \frac{1}{2} \) and \( U \mapsto \frac{1}{2} V \) for the Lagrangian density Eq.(106), the energy density \( \rho_{(\text{MG})} \) in Eq.(97) and the mass \( M \) in Eq.(98) reduce to become
\[ \rho_{(\text{MG})} = \frac{1}{8\pi G} \left( -3HF_\phi + \frac{1}{4} \dot{\phi}^2 + \frac{1}{2} V \right) \quad \text{and} \quad M = \frac{F(\phi) \gamma^3}{2G} \left( H^2 + \frac{k}{a^2} \right) \], (109)

which finally recast Eq.(110) into
\[ E_{\text{eff}} = M - \rho_{(\text{MG})} V = \left( \rho_m + \rho_{(\text{MG})} \right) V - \rho_{(\text{MG})} V = \rho_m V \]. (110)

Hence, the “generalized Misner-Sharp energy \( E_{\text{eff}} \)” for the FRW universe within the scalar-tensor gravity in [13] is in fact the pure Misner-Sharp mass of physical matter for the same gravity in our work, just like the case
VI.5. Reconstruction of the physical mass $\rho_m V$ in generic modified gravity

Before proceeding to analyze more examples, we would like to give some remarks on the problem of reconstructing physical mass. Recall that in GR the mass $\rho_m V$ of the physical matter (like baryon dust, radiation) can be geometrically recovered by the Hawking mass for twist-free spacetimes [39] and the Misner-Sharp mass for spherically symmetric spacetimes [12]. In modified gravity, the physical matter content determines the FRW spacetime geometry Eq.(1) through more generic field equations which usually contain nonlinear and higher-order curvature terms beyond GR. Thus, how to reconstruct the mass of the physical matter from the spacetime geometry?

In [13], Cai et al. generalized the Misner-Sharp mass of GR into higher-dimensional Gauss-Bonnet gravity and the $f(R)$ (plus the scalar-tensor FRW) gravity in four dimensions. As just shown in Sec.VI.1 and Sec. VI.4, for the FRW universe the results in [13] do match the physical material mass $\rho_m V$ in our unified formulation. In fact, for the FRW universe governed by generic modified gravity with the field equation $R_{\mu\nu} - Rg_{\mu\nu}/2 = 8\pi G_{\text{eff}} T_{\mu\nu}(\text{eff})$, the mass $M^{(m)} = \rho_m V$ of the physical matter content can be reconstructed from an geometric approach by

$$M^{(m)} = \frac{\Upsilon^3}{2G_{\text{eff}}} \left( H^2 + \frac{k}{a^2} \right) - \frac{4\pi \Upsilon^3}{3} \rho_{(\text{MG})},$$  

(111)

where $\rho_{(\text{MG})}$ is the density of modified-gravity effects collecting the nonlinear and higher-order geometric terms and joining $T^{\mu\nu}_{\text{eff}} = \text{diag}[-\rho_{(\text{MG})}, P_{(\text{MG})}, P_{(\text{MG})}, P_{(\text{MG})}]$, as concretely shown just before for $f(R)$, generalized Brans-Dicke and scalar-tensor-chameleon gravity. When going beyond the FRW geometry in modified gravity, however, the validity of

$$M^{(m)}_{\text{Hk}} = \frac{1}{4\pi G_{\text{eff}}} \left( \int \frac{dA}{4\pi} \right)^{1/2} \left( -\Psi_2 - \sigma_{NP\lambda\lambda\lambda} + \Phi_{11} + \Lambda_{\lambda\lambda\lambda} \right) dA - \frac{4\pi \Upsilon^3}{3} \rho_{(\text{MG})},$$  

(112)

to recover the physical mass $\rho_m V$ for an arbitrary twist-free spacetime based on the effective Hawking mass Eq.(15) in our unified formulation, and the feasibility of

$$M^{(m)}_{\text{MS}} = \frac{\Upsilon}{2G_{\text{eff}}} \left( 1 - \lambda_{\alpha\beta} \partial_{\alpha} T_{\beta\gamma} \right) - \frac{4\pi \Upsilon^3}{3} \rho_{(\text{MG})},$$  

(113)

for generic spherically symmetric spacetimes based on the effective Misner-Sharp mass Eq.(16), remain to be examined.

VI.6. Quadratic gravity

For quadratic gravity [25], the Lagrangian density is constructed by combining the Hilbert-Einstein density of GR with the linear superposition of some well-known quadratic (as opposed to cubic and quartic) algebraic curvature invariants such as $R^2$, $R_{\mu\nu}R^{\mu\nu}$, $S_{\mu\nu}S^{\mu\nu}$ (with $S_{\mu\nu} := R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}$), $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $C_{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ (Weyl tensor square), say $\mathcal{L} = R + a R^2 + b R_{\mu\nu}R^{\mu\nu} + c S_{\mu\nu}S^{\mu\nu} + d R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + e C_{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + 16\pi G_{\text{eff}} \mathcal{L}_m$ where $\{a, b, c, d, e\}$ are real-valued constants. However, these quadratic invariants are not totally independent of each
other, as \( S_{\mu\nu} S^{\mu\nu} = R_{\mu\nu} R^{\mu\nu} - \frac{1}{4} R^2 \), \( C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + R^2 / 3 \), and moreover \( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) can be absorbed into the Gauss-Bonnet invariant \( G := R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) which does not contribute to the field equation since \( \delta \int d^4 x \sqrt{-g} G / \delta g^{\mu\nu} \equiv 0 \) (eg. [21]). Hence, it is sufficient to consider the following Lagrangian density for quadratic gravity

\[
\mathcal{L}_{QG} = R + a R^2 + b R_{\mu\nu} R^{\mu\nu} + 16 \pi G \mathcal{L}_m ,
\]

and the field equation is [21]

\[
- \frac{1}{2} (R + a \cdot R^2 + b \cdot R_c^2) g_{\mu\nu} + (1 + 2 a R) R_{\mu\nu} + 2 a (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) R + b \cdot H^{QG}_{\mu\nu} = 8 \pi G T^{(m)}_{\mu\nu} ,
\]

where \( R_c^2 \) is the straightforward abbreviation for the Ricci tensor square \( R_{\mu\nu} R^{\mu\nu} \) to shorten some upcoming expressions below, and

\[
H^{QG}_{\mu\nu} = 2 R_{\mu\nu\rho\sigma} R^{\rho\sigma} + \left( \frac{1}{2} g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) R + \Box R_{\mu\nu} .
\]

It can be rewritten into

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8 \pi G \frac{G}{1 + 2 a R} \left( T^{(m)}_{\mu\nu} + T^{(MG)}_{\mu\nu} \right)
\]

where

\[
G_{\text{eff}} = \frac{G}{1 + 2 a R} \quad \text{and}
\]

\[
T^{(MG)}_{\mu\nu} = \frac{1}{8 \pi G} \left( \frac{1}{2} \left( b \cdot R_c^2 - a R^2 \right) g_{\mu\nu} + (2 a + b) \nabla_\mu \nabla_\nu R - (2 a + b) g_{\mu\nu} \Box R - 2 b \left( 2 R_{\mu\nu\rho\sigma} R^{\rho\sigma} + \Box R_{\mu\nu} \right) \right) .
\]

Substitute the FRW metric Eq.(1) into \( T^{(MG)}_{\mu\nu} \), and with \( T^{(MG)}_{\mu\nu} = \text{diag}\{-p_{(MG)}, p_{(MG)}, p_{(MG)}, p_{(MG)}\} \) we get

\[
p_{(MG)} = \frac{1}{8 \pi G} \left( \frac{a}{2} \left( 2 R_c^2 - a R^2 \right) g_{\mu\nu} + (2 a + b) \nabla_\mu \nabla_\nu R - \left( 4 a + b \right) H \dot{R} + 4 b R' \alpha_{\mu\nu} + 2 b \Box R_{\mu\nu} \right) ,
\]

\[
p_{(MG)} = \frac{1}{8 \pi G} \left( \frac{b}{2} \left( 2 R_c^2 - a R^2 \right) g_{\mu\nu} + (2 a + b) \nabla_\mu \nabla_\nu R - \left( 4 a + b \right) H \dot{R} - 4 b R' \alpha_{\mu\nu} R^{\alpha\beta} - 2 b \Box R_{\mu\nu} \right) .
\]

where we have used \( R' \alpha_{\mu\nu} = -R_{\alpha\mu\nu} \) and \( \Box R_{\mu\nu} = -\Box R_{\mu\nu} \) in \( p_{(MG)} \) under the FRW metric Eq.(1). Also, since \( G_{\text{eff}} = G / \phi \), the geometric mass enclosed in a sphere of radius \( \mathcal{I} \) is

\[
M = \frac{(1 + 2 a R) \gamma^3}{2 G} \left( H^2 + \frac{k}{a^2} \right) \quad \text{with} \quad M_\Lambda \equiv \frac{(1 + 2 a R) \gamma^3}{2 G} ,
\]

while the nonequilibrium energy dissipation \( \mathcal{E} \) associated with the evolution of \( G_{\text{eff}} \) and the Wald-Kodama entropy \( E \) of the horizon are respectively

\[
\mathcal{E} = \frac{4}{3} \pi \gamma^3 \frac{2 a R}{1 + 2 a R} \rho_{\text{eff}} d t \quad \text{and} \quad S = \frac{A_\Lambda (1 + 2 a R)}{4 G} .
\]
Following the unified formulation developed in Sec. III and Sec. IV to study \(dM = dE = A\psi + WdV + \mathcal{E}\) for the region \(\mathcal{T} \leq \mathcal{T}_A\) and \(T(dS + dpS) \doteq \delta Q \doteq -(A_\Delta \psi + \mathcal{E}_A)\) for the horizon itself, we find

\[
H^2 + \frac{k}{a^2} = \frac{8\pi G}{3(1 + 2aR)} \rho_m + \frac{1}{3(1 + 2aR)} \left[ \frac{a}{2} R^2 \right] - \frac{b}{2} R^2_c + \frac{b}{2} \dot{R} - \left( 4a + b \right) H \dot{R} + 4b R^t_{\alpha\beta} + 2b \Box R^t_i \right),
\]

(124)

\[
\dot{H} - \frac{k}{a^2} = -4\pi \frac{G}{1 + 2aR} (\rho_m + p_m) - \frac{1}{2(1 + 2aR)} \left[ (2a + b) \dot{R} - \frac{b}{2} H \dot{R} + 4b (R^t_{\alpha\beta} - R^t_{\alpha\beta}) R^\beta + 2b \Box (R^t_i - R^t_i) \right),
\]

(125)

while the nonequilibrium entropy production on the horizon is

\[
d_p S \doteq -4\pi \dot{\mathcal{T}}^2 \frac{a\dot{R}}{G} dt.
\]

(126)

We have verified that the thermodynamic relations Eqs. (124) and (125) are equivalent to the gravitational Friedmann equations by substituting the FRW metric Eq. (1) into the quadratic field equations (117) and (119).

Just like the treatment of \(f(R)\) gravity in Sec. VI.1 to keep the expressions of \(\rho_{(MG)}\), \(p_{(MG)}\) and the Friedmann equations (124) and (125) clear and readable, we continue using compact notations for \(R\), \(R^2\), \(\tilde{R}\), \(R^t\), \(R^t_{\alpha\beta}\), \(\Box R^t_{\alpha\beta}\), and \(\Box R^t_i\) and one should keep in mind that for the FRW metric Eq. (1), these quantities are already known and can be fully expanded into higher-derivative and nonlinear terms of \(H\) or \(a\).

**VI.7. \(f(R, \mathcal{G})\) generalized Gauss-Bonnet gravity**

The generalized Gauss-Bonnet gravity under discussion is given by the Lagrangian density \(\mathcal{L}_{GB} = f(R, \mathcal{G}) + 16\pi G \mathcal{L}_m\) where \(\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\) is the Gauss-Bonnet invariant. This is in fact a subclass of \(\mathcal{L} = f(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}) + 16\pi G \mathcal{L}_m\) gravity with explicit dependence on \(R^2\) and satisfying the “coherence condition” \(f_{R^2} = f_{R^2} = -f_R R^2 / 4\) \(\mathcal{L}_m\) \(R^2_m\) and \(R^2\) are the intuitive abbreviations for the Riemann tensor square \(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\) and the Ricci tensor square \(R_{\mu\nu} R^{\mu\nu}\), respectively. The field equation for \(f(R, \mathcal{G})\) gravity reads

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi \frac{G}{f_R + 2R f_{\mathcal{G}}} T^{(m)}_{\mu\nu} + (f_R + 2R f_{\mathcal{G}})^{-1} \left[ \frac{1}{2} \left( f - (f_R + 2R f_{\mathcal{G}}) R \right) g_{\mu\nu} \right.
\]

\[
+ (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f_R + 2R (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f_{\mathcal{G}} + 4R_{\mu\nu} \Box f_{\mathcal{G}} + H^{(GB)}_{\mu\nu} \right],
\]

(127)

where

\[
H^{(GB)}_{\mu\nu} \doteq 4f_{\mathcal{G}} R_{\mu\nu} - 4f_{\mathcal{G}} R_{\mu\nu\rho\sigma} R^{\rho\sigma} - 2f_{\mathcal{G}} R_{\mu\nu\rho\sigma} \nabla^{\rho\sigma} - 4R_{\mu\nu} \nabla^\alpha \nabla_\alpha f_{\mathcal{G}}
\]

\[
-4R_{\mu\nu} \nabla_\alpha \nabla_\beta f_{\mathcal{G}} - 4g_{\mu\nu} R_{\rho\sigma} \nabla^{\rho\sigma} - 4R_{\mu\nu} \nabla^\alpha \nabla_\alpha f_{\mathcal{G}},
\]

(128)

and \{\(f, f_R, f_{\mathcal{G}} = \partial f / \partial \mathcal{G}\)\} are all functions of \((R, \mathcal{G})\). Note that in \(H^{(GB)}_{\mu\nu}\) the second-order-derivative operators \(\Box, \nabla_\alpha \nabla_\nu, etc\) only act on the scalar functions \(f_{\mathcal{G}}\). Hence,

\[
G_{\text{eff}} = \frac{G}{f_R + 2R f_{\mathcal{G}}}
\]

(129)
The region following the unified formulation developed in Sec. III and Sec. IV to study entropy equations by substituting the FRW metric Eq. (1) into the generalized Gauss-Bonnet field equations (127) and we have verified that the thermodynamic relations Eqs. (135) and (136) are really the gravitational Friedmann relations. Substituting the FRW metric Eq. (1) into Notations we obtain

\[ T_{\mu\nu}^{(MG)} = \frac{1}{8\pi G} \left( \frac{1}{2} \left( f - (f_R + 2R f_G) R \right) g_{\mu\nu} + (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box) f_R + 2R (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box) f_G + 4R_{\mu\nu} \Box f_G + H_{\mu\nu}^{(GB)} \right). \] (130)

Substitute the FRW metric Eq. (1) into \( T_{\mu\nu}^{(MG)} \) with \( T_{\mu\nu}^{(MG)} = \text{diag}[\rho_{(MG)}, P_{(MG)}, P_{(MG)}, P_{(MG)}] \), and in compact notations we obtain

\[ \rho_{(MG)} = \frac{1}{8\pi G} \left( \frac{1}{2} (f_R + 2R f_G) R - \frac{1}{2} f - 3H f_R - 6RH f_G + 4R_{\mu}^\nu f_G + 3H f_G^\prime \right), \] (131)

\[ P_{(MG)} = \frac{1}{8\pi G} \left( \frac{1}{2} f - \frac{1}{2} (f_R + 2R f_G) R + \dot{f}_R + 2H (f_R + 2R f_G) - 4R_{\mu}^\nu f_G + 3H f_G^\prime + H_{r(GB)}^\prime \right), \] (132)

where we have used the properties \( R_{\mu}^\nu = -R_{\mu\nu} \) and \( H_{r(GB)}^\prime = -H_{r(GB)} \) in \( \rho_{(MG)} \) under the FRW metric Eq. (1).

Since \( G_{\text{eff}} = G/(f_R + 2R f_G) \), the geometric mass within a sphere of radius \( T \) is

\[ M = \frac{(f_R + 2R f_G) T^3}{2G} \left( H^2 + \frac{k}{a^2} \right) \text{ with } M_\Lambda \doteq \frac{(f_R + 2R f_G) T^3}{2G}, \] (133)

while the nonequilibrium energy dissipation \( \mathcal{E} \) associated with the evolution of \( G_{\text{eff}} \) and the Wald-Kodama entropy \( S \) of the horizon are respectively

\[ \mathcal{E} = \frac{4}{3} \pi \frac{1}{3} \frac{T^3}{f_R + 2R f_G} \rho_\text{eff} dt = \frac{A_\Lambda (f_R + 2R f_G)}{4G}. \] (134)

Following the unified formulation developed in Sec. III and Sec. IV to study \( dM = dE = A \psi + W dV + \mathcal{E} \) for the region \( T \leq T_\Lambda \) and \( T(dS + d\rho S) \doteq \delta Q \doteq (A_\Lambda \psi + S_\Lambda) \) for the horizon itself, we find

\[ H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \frac{G}{f_R + 2R f_G} \rho_\text{m} + \frac{1}{2} \frac{1}{f_R + 2R f_G} \left( \frac{1}{2} (f_R + 2R f_G) R - \frac{1}{2} f - 3H (f_R + 2R f_G) - 4R_{\mu}^\nu f_G + 3H f_G^\prime \right), \] (135)

\[ H - \frac{k}{a^2} = -4\pi \frac{G}{f_R + 2R f_G} (\rho_\text{m} + P_\text{m}) - \frac{1}{2} \frac{1}{f_R + 2R f_G} \left( \dot{f}_R - H f_R + 2R f_G + 2RH f_G + 4 (R_{\mu}^\nu R_{\nu}^\rho - R_{\mu}^\nu R_{\nu}^\rho) (f_G + 3H f_G) - H_{r(GB)}^\prime + H_{r(GB)}^\prime \right), \] (136)

while the nonequilibrium entropy production on the horizon is

\[ d_\rho S \doteq -2\pi T^2 \Lambda \frac{\dot{f}_R + 2R f_G + 2R f_G}{G} dt. \] (137)

We have verified that the thermodynamic relations Eqs. (135) and (136) are really the gravitational Friedmann equations by substituting the FRW metric Eq. (1) into the generalized Gauss-Bonnet field equations (127) and (128). Moreover, by setting \( f_G = 0 \) and thus \( f_G = f_G = 0 \), the situation of the \( f(R, G) \) generalized Gauss-Bonnet gravity reduces to become the case of \( f(R) \) gravity in Sec. VI.1.
VI.8. Self-inconsistency of \( f(R, G) \) gravity

The \( f(R, G) \) example just above is based on Eqs. (127) and (128), which together with their contravariant forms constitute the standard field equations of the \( f(R, G) \) gravity that are proposed in \cite{23} and adopted in existing papers related to generic dependence on \( G \). On the other hand, recall that in four dimensions the Gauss-Bonnet invariant \( G \) is proportional to the Euler-Poincaré topological density as

\[
G = \left( \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} R^{\gamma \delta} \right) \cdot \left( \frac{1}{2} \epsilon_{\eta \xi \mu \nu} R^{\eta \xi \mu \nu} \right) = \ast R^{\eta \xi} \ast R_{\eta \xi}^\alpha \beta ,
\]

where \( \epsilon_{\alpha \beta \gamma \delta} \) refers to the totally antisymmetric Levi-Civita (pseudo)tensor with \( \epsilon_{0123} = \sqrt{-g} \). The integral \( \int d^4 x \sqrt{-g} G \) is equal to the Euler characteristic number \( \chi \) (just a constant) of the spacetime, and thus

\[
\frac{\delta}{\delta g_{\mu \nu}} \int d^4 x \sqrt{-g} G \equiv 0.
\]

By explicitly carrying out this variational derivative, one could find the following Bach-Lanczos identity \cite{49}:

\[
2 R R_{\mu \nu} - 4 R^g_{\mu \alpha} R_{\alpha \nu} - 4 R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} + 2 R_{\mu \rho \gamma \delta} R_{\nu}^{\alpha \beta \gamma \delta} \equiv \frac{1}{2} G g_{\mu \nu},
\]

with which the standard field equations (127) and (128) of the \( f(R, G) \) gravity can be simplified into

\[
R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8 \pi \frac{G}{f_R} T^{(m)}_{\mu \nu} + \frac{1}{f_R} \left( \frac{1}{2} (f - f_g G - f_R R) g_{\mu \nu} + (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) f_R \right) + 2 R (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) f_R + 4 R_{\mu \rho \gamma \delta} f_{R} + 4 R_{\mu \nu} \Box f_R + \mathcal{H}^{(GB)}_{\mu \nu},
\]

where

\[
\mathcal{H}^{(GB)}_{\mu \nu} := -4 R^g_{\mu \alpha} \nabla_\alpha \nabla_\nu f_R - 4 R^g_{\alpha \beta} \nabla_\alpha \nabla_\beta f_R + 4 g_{\mu \nu} R_{\alpha \beta \gamma \delta} \nabla_\alpha \nabla_\beta f_{R} - 4 R_{\alpha \beta \gamma \delta} \nabla_\alpha \nabla_\beta f_{R}.
\]

This way, the effective gravitational coupling strength is recognized to be

\[
G_{\text{eff}} = \frac{G}{f_R},
\]

as opposed to the \( G_{\text{eff}} = G/(f_R + 2 R f_R) \) in Eq. (129); this is because the \( 2 f_R R_{\mu \nu} \) term directly joining Eq. (127) is now absorbed by the \( \frac{1}{2} f_R G g_{\mu \nu} \) term in Eq. (141) due to the Bach-Lanczos identity and thus no longer shows up in Eq. (141). The SEM tensor from modified-gravity effects becomes

\[
T^{(MG)}_{\mu \nu} = \frac{1}{8 \pi G} \left( \frac{1}{2} (f - f_g G - f_R R) g_{\mu \nu} + (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) f_R + 2 R (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) f_R + 4 R_{\mu \nu} \Box f_R + \mathcal{H}^{(GB)}_{\mu \nu} \right),
\]

which with the FRW metric Eq. (1) gives rise to

\[
\rho^{(MG)} = \frac{1}{8 \pi G} \left( \frac{1}{2} (f R + f_g G) + \frac{1}{2} f - 3 H f_R - 6 R H f_R + 4 R_{\nu} (\dot{f}_R + 3 H \dot{f}_R) - \mathcal{H}_{\nu} \right)
\]

and

\[
P^{(MG)} = \frac{1}{8 \pi G} \left( \frac{1}{2} f - \frac{1}{2} (f R + f_g G) + \dot{f}_R + 2 H \ddot{f}_R + 2 R (\dot{f}_R + 2 H \dot{f}_R) - 4 R_{\nu} (\ddot{f}_R + 3 H \dot{f}_R) + \mathcal{H}_{\nu} \right).
\]
Since the $G_{\text{eff}} = G/f_R$ coincides with that of $f(R)$ gravity, the Hawking or Misner-Sharp geometric mass $M$, the nonequilibrium energy dissipation $\dot{E}$, the horizon entropy $S$ and the entropy production element $d_pS$ are all the same with those of $f(R)$ gravity, as derived before in Eqs. (68), (69), and (72) in Sec. VI.1, respectively. Then the thermodynamical approach of Sec. III and Sec. IV yields

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3f_R}\rho_m + \frac{1}{3f_R}\left(\frac{1}{2}(f_R R + f_G G) - \frac{1}{2}f - 3H(\dot{f}_R + 2R\dot{f}_G) + 4R_t(\dot{f}_G + 3H\dot{f}_G) - \mathcal{H}_t^r\right)$$

(147)

and

$$H^2 - \frac{k}{a^2} = -4\pi Gf_R(\rho_m + P_m) - \frac{1}{2f_R}\left(\dot{f}_R - H\dot{f}_R + 2R \dot{f}_G - 2RH\dot{f}_G + 4(R_t^r - R_t^g)(\dot{f}_G + 3H\dot{f}_G) - \mathcal{H}_t^r + \mathcal{H}_t^{r,\prime}\right),$$

(148)

which match the Friedmann equations obtained from substituting the FRW metric Eq. (1) into the simplified $f(R, \mathcal{G})$ field equation (141).

However, these thermodynamical quantities and relations of $f(R, \mathcal{G})$ gravity differ dramatically with those in the previous Sec. VI.1. The contrast may be seen even more evidently in the $\mathcal{L} = R + f(\mathcal{G}) + 16\pi G \mathcal{L}_m$ modified Gauss-Bonnet gravity [50] which is a special subclass of the $f(R, \mathcal{G})$ theory. It follows from Sec. VI.1.7 that $G_{\text{eff}} = G/(1 + 2f_{G})$ for $f(R, \mathcal{G}) = R + f(\mathcal{G})$, and it is a nonequilibrium scenario with nonvanishing energy dissipation $\dot{E}$ and entropy production $d_pS$ on the apparent horizon. On the contrary, we have $G_{\text{eff}} = G$ in accordance with Eq. (141) as $f_R = 1$, which corresponds to an equilibrium gravitational thermodynamics with $\dot{E} = 0 = d_pS$.

Note that the existence of the two distinct formulations for the thermodynamics of $f(R, \mathcal{G})$ gravity does not indicate a failure of our unified formulation. Instead, it reveals a self-inconsistency feature of the $f(R, \mathcal{G})$ theory itself. Although the simplified field equations (141) and (142) are equivalent to Eqs. (127) and (128) in Sec. VI.1.7 via the identity Eq. (140), practically they will behave differently with each other in any problems relying on the input of the effective coupling strength $G_{\text{eff}}$. Moreover, we also expect this self-inconsistency of $f(R, \mathcal{G})$ gravity to arise in other problems such as the black-hole thermodynamics.

VI.9. Dynamical Chern-Simons gravity

So far we have applied our unified formulation to the $f(R)$, generalized Brans-Dicke, scalar-tensor-chameleon, quadratic and $f(R, \mathcal{G})$ gravity; they are all nonequilibrium theories with nontrivial $G_{\text{eff}}$ in the coefficient of $\mathcal{L}_m^{(m)}$. As a final example we will continue to consider the (dynamical) Chern-Simons modification of GR [51], which is a thermodynamically equilibrium theory with $G_{\text{eff}} = G$. Its Lagrangian density reads

$$\mathcal{L}_{\text{CS}} = R + \frac{a}{\sqrt{-g}}\frac{\alpha}{2}\mathring{\mathcal{R}}R - b\nabla_\mu \theta \nabla^\mu \theta - V(\theta) + 16\pi G \mathcal{L}_m,$$

(149)

where $\theta = \theta(x^\mu)$ is a scalar field, $\{a, b\}$ are constants, and $\mathring{\mathcal{R}}$ denotes the parity-violating Pontryagin invariant

$$\mathring{\mathcal{R}} = \mathring{R}_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} R_{\mu\nu\gamma\delta}.$$

(150)

$\mathring{\mathcal{R}}$ is proportional to the divergence of the Chern-Simons topological current $K^\mu$ [51]:

$$\mathring{\mathcal{R}} = -2 \partial_\mu K^\mu \quad \text{and} \quad K^\mu = 2\epsilon^{\mu\alpha\beta\gamma} \left(\frac{1}{2} \Gamma^\xi_{\alpha\tau} \partial_\beta \Gamma^\gamma_{\xi} + \frac{1}{3} \Gamma^\xi_{\alpha\tau} \Gamma^\tau_{\beta\rho} \Gamma^\rho_{\xi}\right).$$

(151)
with $e^{0123} = 1/\sqrt{-g}$, hence the name Chern-Simons gravity. Variational derivative of $\sqrt{-g}L_{CS}$ with respect to the inverse metric $g^{\mu\nu}$ yields the field equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8 \pi G T_{\mu\nu}^{(m)} - \frac{a}{\sqrt{-g}} C_{\mu\nu} + b \left( \nabla_{\mu} \theta \nabla_{\nu} \theta - \frac{1}{2} g_{\mu\nu} \nabla_{\alpha} \theta \nabla^{\alpha} \theta \right) - \frac{1}{2} V(\theta) g_{\mu\nu},$$

(152)

where

$$C_{\mu\nu} = \nabla^\alpha \theta \cdot \left( \epsilon_{\alpha\beta\gamma\mu} \nabla^\gamma R_{\nu}^{\beta} + \epsilon_{\alpha\beta\gamma\nu} \nabla^\gamma R_{\mu}^{\beta} \right) + \nabla^\alpha \nabla^\beta \theta \cdot \left( R_{\beta\mu\alpha} + R_{\beta\nu\alpha} \right).$$

(153)

Eq. (152) directly shows that the Chern-Simons gravitational coupling strength is just Newton’s constant, $G_{\text{eff}} = G$, and

$$T_{\mu\nu}^{(MG)} = - a C_{\mu\nu} + b \left( \nabla_{\mu} \theta \nabla_{\nu} \theta - \frac{1}{2} g_{\mu\nu} \nabla_{\alpha} \theta \nabla^{\alpha} \theta \right) - \frac{1}{2} V(\theta) g_{\mu\nu}.$$ 

(154)

With the FRW metric Eq. (26), this $T_{\mu\nu}^{(MG)}$ leads to

$$\rho_{\text{(MG)}} = \frac{1}{16 \pi G} \left( b \dot{\theta}^2 + V(\theta) \right) \quad \text{and} \quad P_{\text{(MG)}} = \frac{1}{16 \pi G} \left( b \dot{\theta}^2 - V(\theta) \right).$$

(155)

Since $G_{\text{eff}} = G = \text{constant}$, we can make use of the reduced formulation in Sec. V.4 for equilibrium situations. The geometric mass and the horizon entropy are respectively

$$M = \frac{\Upsilon^3}{2G} \left( H^2 + \frac{k}{a^2} \right) \quad \text{with} \quad M_A \equiv \frac{\Upsilon A}{2G},$$

(156)

and

$$S = \frac{A_A}{4G} = \frac{\pi \Upsilon^2 A}{G},$$

(157)

which are the same with those of GR [8]. Also, there are no energy dissipation $\mathcal{E}$ and the on-horizon entropy production $d_p S$,

$$\mathcal{E} = 0 \quad \text{and} \quad d_p S = 0.$$ 

(158)

Following the procedures in Sec. V.4 for the interior and the horizon we obtain from the thermodynamical approach that

$$H^2 + \frac{k}{a^2} = \frac{8 \pi G}{3} \rho_m + \frac{1}{6} \left( b \dot{\theta}^2 + V(\theta) \right)$$

(159)

and

$$\dot{H} - \frac{k}{a^2} = - \frac{4 \pi G}{3} \left( \rho_m + P_m \right) - \frac{b}{2} \dot{\theta}^2.$$ 

(160)

By substituting the FRW metric Eq. (11) into the field equation (152), we have confirmed that Eqs. (159) and (160) are really the Friedmann equations of the FRW universe governed by the Chern-Simons gravity.
VII. CONCLUSIONS

In this paper, we have developed a unified formulation to derive the Friedmann equations from (non)equilibrium thermodynamics within modified gravity with field equations of the form $R_{\mu\nu} - R g_{\mu\nu}/2 = 8\pi G_{\text{eff}} T_{\mu\nu}^{(\text{eff})}$. We firstly made the necessary preparations by locating the marginally inner trapped horizon $\mathcal{T}_A$ of the expanding FRW universe as the apparent horizon of relative causality, and then rewrote the continuity equation from $\nabla^{\mu}(G_{\text{eff}} T_{\mu\nu}^{(\text{eff})}) = 0$ to introduce the energy dissipation element $\mathcal{E}$ which is related with the evolution of $G_{\text{eff}}$.

With these preparations, we began to study the thermodynamics of the FRW universe. We have generalized the Hawking and Misner-Sharp geometric definitions of mass by replacing Newton’s constant $G$ with $G_{\text{eff}}$, and calculated the total derivative of $M$ in the comoving $(t, r)$ and the areal $(t, \mathcal{R})$ transverse coordinates. Also, we have supplemented Hayward’s unified first law of thermodynamics into $dE = A \psi + W dV + \mathcal{E}$ with the dissipation term $\mathcal{E}$, where the work density $W$ and the heat flux covector $\psi$ are computed using the effective matter content $T_{\mu\nu}^{(\text{eff})}$. By identifying the geometric mass $M$ enveloped by a sphere of radius $\mathcal{R} < \mathcal{T}_A$ as the total internal energy $E$, the Friedmann equations have been derived from the thermodynamic equality $dM = dE$.

On the horizon $\mathcal{R} = \mathcal{T}_A$, besides the smooth limit $\mathcal{R} \to \mathcal{T}_A$ of $dM = dE$ from the untrapped interior $\mathcal{R} < \mathcal{T}_A$ to the horizon, we have employed an alternative Clausius method. By considering the heat flow during the infinitesimal time interval $dt$ for an isochoric process using the unified first law $dE \equiv A \psi_{\mathcal{T}} + \mathcal{E}_{\mathcal{T}}$ and the generic nonequilibrium Clausius relation $T(dS + d_p S) \equiv \delta Q$ respectively, we have obtained the second Friedmann equation $H^2 - k/a^2 \equiv -4\pi G_{\text{eff}}(\rho_{\text{eff}} + P_{\text{eff}})$ from the thermodynamics equality $T(dS + d_p S) \equiv \delta Q \equiv -dE \equiv -(A_{\mathcal{T}} \psi_{\mathcal{T}} + \mathcal{E}_{\mathcal{T}})$, while the first Friedmann equation $H^2 + k/a^2 \equiv 8\pi G_{\text{eff}} P_{\text{eff}}/3$ can be recovered using the generalized continuity equation $\dot{G}_{\text{eff}} + G_{\text{eff}} \dot{\rho}_{\text{eff}} + 3G_{\text{eff}} H(\rho_{\text{eff}} + P_{\text{eff}}) = 0$. Here we have taken the temperature ansatz $T = 1/(2\pi \mathcal{T}_A)$ in [8] and the Wald-Kodama dynamical entropy $S \equiv A_{\mathcal{T}}/(4G_{\text{eff}})$ for the horizon, and the equality $T(dS + d_p S) \equiv -(A_{\mathcal{T}} \psi_{\mathcal{T}} + \mathcal{E}_{\mathcal{T}})$ has also determined the entropy production $d_p S$ which is generally nonzero unless $G_{\text{eff}} = \text{constant}$. In the meantime, we have adjusted the thermodynamic sign convention by the consistency between the thermodynamics of the horizon and the interior.

After developing the unified formulation for generic relativistic gravity, we have extensively discussed some important problems related to the formulation. A viability test of the generalized effective mass has been proposed, which shows that the equality between the physically defined effective mass $M = \rho_{\text{eff}} V = (\rho_m + \rho_{(MG)} V$ and the geometric effective mass automatically yields the Friedmann equations. Also, we have argued that for the modified-gravity theories under discussion with minimal geometry-matter coupling, the continuity equation can be further simplified due to the Noether-compatible definition of $T_{\mu\nu}^{(\text{eff})}$. Furthermore, we have discussed the reduced situation of the unified formulation for $G_{\text{eff}} = G = \text{constant}$ with vanishing dissipation $\mathcal{E} = 0$ and entropy production $d_p S = 0$, which is of particular importance for typical scalar dark-energy models and some fourth-order gravity.

Finally, we have applied our unified formulation to the $f(R)$, generalized Brans-Dicke, scalar-tensor-chameleon, quadratic, $f(R, \mathcal{G})$ generalized Gauss-Bonnet and dynamical Chern-Simons gravity, to derive the Friedmann equations from thermodynamics-gravity correspondence, where compact notations have been employed to simplify the thermodynamic quantities $(\rho_{(MG)}, P_{(MG)})$. In addition, we have verified that, the “generalized Misner-Sharp energy” for $f(R)$ and scalar-tensor gravity FRW cosmology in [13] matches the pure mass $\rho_m V$ of the physical matter content in our formulation, and then continued to reconstruct the physical mass $\rho_m V$ from the spacetime geometry for generic modified gravity. We also found the self-inconsistency of $f(R, \mathcal{G})$ gravity in such problems which require to specify the $G_{\text{eff}}$.

In our prospective studies, we will apply the unified formulation developed in this paper to the generalized second law of thermodynamics for the FRW universe, and extend our formulation to more generic theories of modified gravity which allow for nonminimal curvature-matter couplings. Moreover, we will try to loosen the restriction of spherical symmetry and look into the problem of thermodynamics-gravity correspondence in the Bianchi classes of cosmological solutions.
ACKNOWLEDGEMENT

The authors are grateful to Prof. Rong-Gen Cai (Beijing) for helpful discussion. This work was financially supported by the Natural Sciences and Engineering Research Council of Canada.

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