CONTRACTION PROPERTY OF CERTAIN CLASSES OF LOG-\(M\)--SUBHARMONIC FUNCTIONS IN THE UNIT BALL

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ABSTRACT. We prove a contraction property of certain classes of smooth functions, whose absolute values of elements are log-hyperharmonic functions in the unit ball, thus extending the results of Kulikov to higher-dimensional space (GAFA (2022)). Moreover, by applying those results we get some new results for harmonic mappings in the complex plane.

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1. INTRODUCTION

In this paper \(\mathbb{B} = \{x \in \mathbb{R}^n : |x| < 1\}\) is the unit ball. Here and in the sequel for \(x = (x_1, \ldots, x_n)\), \(|x| := \sqrt{\sum_{k=1}^{n} x_k^2}\).

Assume that \(\mathcal{M}\) is the group of Möbius transformations of the unit ball onto itself (see e.g. [1]). We introduce the Möbius invariant hyperbolic measure on the

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unit ball. For \( x \in \mathbb{B} \) we define it as
\[
d\tau(x) = \frac{2^n}{(1 - |x|^2)^n} \frac{dV(x)}{\omega_n},
\]
where \( \omega_n = V(B) \) is the volume of the unit ball.

A mapping \( u \in C^2(\mathbb{B}^n, \mathbb{C}) \) or more generally \( u \in C^2(\mathbb{B}^n, \mathbb{R}^k) \) is said to be hyperbolic harmonic or \( \mathcal{M} \)-harmonic if \( u \) satisfies the hyperbolic Laplace equation
\[
\Delta_h u(x) = (1 - |x|^2)^2 \Delta u(x) + 2(n - 2)(1 - |x|^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x) = 0,
\]
where \( \Delta \) denotes the usual Laplacian in \( \mathbb{R}^n \). We call \( \Delta_h \) the hyperbolic Laplace operator. See Rudin \[23\] and Stoll \[25\].

The Poisson kernel for \( \Delta_h \) is defined by
\[
P_h(x, \zeta) = \frac{(1 - |x|^2)^{n-1}}{|x - \zeta|^{2n-2}}, \quad (x, \zeta) \in \mathbb{B} \times \mathbb{S}.
\]

Then for fixed \( \zeta \), \( x \to P_h(x, \zeta) \) is \( M \)-harmonic function and for a mapping \( f \in L(S) \), the function
\[
u(x) = P_h[f](x) := \int_S P_h(x, \zeta)f(\zeta)d\sigma(\zeta)
\]
is the Poisson extension of \( f \) and it is \( \mathcal{M} \)-harmonic in \( \mathbb{B} \).

We say that a real function \( u \) is \( \mathcal{M} \)-subharmonic if \( \Delta_h u(x) \geq 0 \). The definition can be extended to the case of upper semicontinuous functions, by using the so-called invariant mean value property \[25\].

Now we define the Hardy type space.

(1) As in \[25\], Definition 7.0.1, for \( 0 < p \leq \infty \), we denote by \( S^p \) the Hardy-type space of non-negative \( \mathcal{M} \)-subharmonic functions \( f \) on \( \mathbb{B} \) such that
\[
\|f\|^p = \sup_{0 < r < 1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty.
\]

When \( p = \infty \) we put \( \|f\|_\infty = \sup_{x \in \mathbb{B}} f(x) \).

(2) For \( 0 < p < \infty \) we say that a Borel function \( f : \mathbb{B} \to \mathbb{C} \) belongs to the Hardy space \( h^p \) if \( |f| \in S^p \). Then we define \( \|f\|^p := \|f \|^p_p \). When \( p = \infty \) we put \( \|f\|_\infty = \sup_{x \in \mathbb{B}} |f(x)| \).

Further
\[
(\Delta_h u)(m(x)) = \Delta_h(u \circ m)(x),
\]
for every Möbius transformation \( m \in \mathcal{M} \) of the unit ball onto itself.

For \( n = 2 \) the \( \mathcal{M} \)-harmonic and \( \mathcal{M} \)-subharmonic functions are just harmonic and subharmonic functions.

If \( f \) is \( \mathcal{M} \)-subharmonic, then we have the following Riesz decomposition theorem of Stoll \[25\], Theorem 9.1.3:
\[
f(x) = F_f(x) - \int_B G_h(x, y)d\mu f(y),
\]
provided that $f \in \mathcal{S}^1$, where $F_f(x)$ is the least $\mathcal{M}$–harmonic majorant of $f$ and $\mu_f$ is the $\mathcal{M}$–Riesz measure of $f$, and $G_h(x,y)$ is the Green function of $\Delta_h$. If $f \in \mathcal{S}^p$, where $p > 1$, then $g(x) = F_f(x) = P_h[\hat{f}](x)$, where $\hat{f}$ is the boundary function of $f$ (Theorem 7.1.1).

From the formula (1.3), by putting $u = \text{Id}$, and $m \in \mathcal{M}$, we get

$$\Delta_h m = 2(n - 2)(1 - |m|^2)m.$$ 

So Möbius transformations are (considered as vectorial functions) hyperbolic harmonic only in the case $n = 2$.

By putting $u(x) = g(|x|)$ and inserting in (1.1) we arrive to the equation

$$\Delta_h u = (1 - r^2) \left( \frac{(2(-2 + n)r^2 + (-1 + n)(1 - r^2)}{r} g'(r) + (1 - r^2) g''(r) \right),$$

where $r = |x|$. The hypergeometric function $F$, which we use in this paper is defined by

$$F \left[ \begin{array}{c} a, b, c \\ u, v \end{array} ; t \right] := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{n!(u)_n(v)_n} t^n,$$

for $|t| < 1$,

and by the continuation elsewhere. Here $(a)_n$ denotes the shifted factorial, i.e., $(a)_n = a(a + 1) \cdots (a + n - 1)$ with any real number $a$.

We will also sometimes write $\Phi_n(x)$ instead of $\Phi_n(|x|)$. If $n = 2$, then $\Phi_n(|x|) = 1 - |x|^2$ and this coincides with the case treated in 

**Definition 1.1.** For $0 < p < \infty$ and $\alpha > 1$ we say that a complex smooth function $f$ in $\mathcal{B}$ belongs to the $\mathcal{M}$–Bergman space $\mathcal{B}_\alpha^p$ if

$$\|f\|_{\alpha,p}^p = c(\alpha) \int_\mathcal{B} |f(x)|^p |\Phi_n^\alpha(|x|)| d\tau(x) < \infty,$$

where

$$\frac{1}{c(\alpha)} = \int_\mathcal{B} \Phi_n^\alpha(x)(1 - |x|^2)^{-n} dV(x) / \omega_n,$$

and $\Phi_n$ is a function defined in (1.4) above.

Observe that in view of (6.2),

$$\frac{1}{c(\alpha)} \geq E_n^\alpha \int_0^1 r^n (1 - r^2)^{n-\alpha-1} dr.$$

This implies that

$$\lim_{\alpha \to 1^+} c(\alpha) = 0.$$
1.1. Admissible monoid. We define \( \mathcal{E} \) to be the set of complex continuous functions \( g \) in \( \mathbb{B} \) and real analytic in \( g^{-1}(\mathbb{C} \setminus \{0\}) \) such that \( f := \log |g| \) is \( \mathcal{M} \)-subharmonic. If \( g(a) = 0 \), then we put \( f(a) = -\infty \). Let us remind that in the definition of subharmonicity, based on the mean value property, continuity of the function is not assumed a priori, but only upper continuity and the value \(-\infty\) for the function is a legitimate value.

Let \( \mathcal{E}_+ = \{ f \in \mathcal{E} : f > 0 \} \). Observe that \( \mathcal{E} \) is a monoid where the operation is simply the multiplication of two functions. Observe that \( 1 = e^0 \), so \( 1 \in \mathcal{E} \). This monoid contains the Abelian group \( \mathcal{G} = \{ e^f : \Delta_h f = 0 \} \). Then for \( a, b \in \mathcal{E}_+, c, d \in \mathcal{E}, p \geq 0 \) and \( \alpha, \beta > 0 \) we have

1. \( a \cdot b \in \mathcal{E}_+ \), \( c \cdot d \in \mathcal{E} \),
2. \( a^p \in \mathcal{E}_+ \), \( e^p \in \mathcal{E} \),
3. \( \alpha a + \beta b \in \mathcal{E}_+ \).

In other words, \( \mathcal{E}_+ \) is a convex cone and at the same time a monoid.

Previous statements follow from the straightforward calculation of the invariant Laplacian. First of all, we have

\[
\alpha e^f = e^{\log \alpha + f}.
\]

So if \( a \in \mathcal{E}_+ \) and \( \alpha > 0 \), then so is \( \alpha a \). Further if \( a = e^f \) and \( b = e^g \), then \( c = a + b = e^{\log(e^f + e^g)} \). Thus

\[
\Delta_h \log c = \Delta_h \log(e^f + e^g) = \frac{e^f \Delta_h f + e^g \Delta_h g}{e^f + e^g} + \frac{e^{f+g}}{e^f + e^g} (1 - r^2)|\nabla (f - g)|^2,
\]

where

\[
A = \Delta_h g = f, \quad B = \Delta_h f = g.
\]

So if \( A, B \geq 0 \), then \( \Delta_h \log (e^f + e^g) \geq 0 \). We also refer to the paper \cite{6} for some similar properties of log-subharmonic functions.

Let us collect some additional features of \( \mathcal{M} \)-harmonic and \( \mathcal{M} \)-subharmonic functions.

1. If \( \log f \) is \( \mathcal{M} \)-subharmonic, then \( f = e^{\log f} \) is \( \mathcal{M} \)-subharmonic.
2. If \( \log f \) is \( \mathcal{M} \)-subharmonic, then \( p \log f \) is \( \mathcal{M} \)-subharmonic and so \( f^p = e^{p \log f} \) is \( \mathcal{M} \)-subharmonic.
3. If \( f \) is \( \mathcal{M} \)-harmonic, then \( |f| \) is \( \mathcal{M} \)-subharmonic.
4. If \( f \) is \( \mathcal{M} \)-harmonic, then \( f \circ m \) is \( \mathcal{M} \)-harmonic, for every Möbius transformation \( m \in \mathcal{M} \) of the unit ball onto itself.
5. If \( f \) is \( \mathcal{M} \)-harmonic, then \( |f|^p \) is \( \mathcal{M} \)-subharmonic for \( p \geq 1 \).

For the above facts, we refer to the monograph by Stoll \cite{25}. Let us illustrate the proof of one of the properties: If \( u \) is \( \mathcal{M} \)-subharmonic, then \( e^u \) is \( \mathcal{M} \)-subharmonic. Indeed

\[
\Delta_h e^u = (1 - |x|^2)^2 |\nabla u|^2 e^u
\]

\[
+ e^u ((1 - |x|^2)^2 \Delta u(x) + 2(n - 2)(1 - |x|^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x))
\]

\[
= (1 - |x|^2)^2 |\nabla u|^2 e^u + e^u \Delta_h u \geq 0.
\]
Remark 1.2. Unfortunately, the class of holomorphic functions in $B \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ for $n \geq 2$ is not contained in the monoid $E$. To see this, let $n = 2$ and take the holomorphic mapping $f(w, z) = g(z)$ which depends only on $z$. Here $g$ is a holomorphic non-vanishing function. Then straightforward computations give that

$$\Delta_h \log |f(w, z)| = 8(1 - |w|^2 - |z|^2)\Re\left(\frac{zg'(z)}{g(z)}\right), \quad |z| < 1.$$ 

Hence the function $f$ is $\mathcal{M}$-subharmonic if and only if $g$ is a starlike function, i.e. if $\Re\left(\frac{zg'(z)}{g(z)}\right) \geq 0$ for $|z| < 1$.

Definition 1.3. For $0 < p < \infty$ and $\alpha > 1$ we define

(1) the $\mathcal{M}$--Hardy monoid $h^p$ consisting of functions $f$ in $h^p \cap \mathcal{E}$ satisfying the additional condition that there is a sequence of bounded $\mathcal{M}$--log-subharmonic functions $f_m$ converging in $h^p$ norm and pointwisely to $f$.

(2) the $\mathcal{M}$--Bergman monoid $B^p_\alpha$ consisting of functions $f$ in $B^p_\alpha \cap \mathcal{E}$.

We also refer to [26] for the corresponding Bergman space of holomorphic mappings in the space which is different from our space.

Remark 1.4. Note that the additional condition in the previous definition is redundant for smooth log-subharmonic functions for $n = 2$. In this case the dilatations $f_m(z) = f(\rho z)$ are log-subharmonic functions, $\rho = \rho_m = m/m+1$ and converge in $h^p$ norm and pointwisely to $f$. See e.g. [8, Proposition 1.3]. We also believe it is redundant in higher-dimensional case.

Observe that for $f(x) \equiv 1$ we have $\|f\|_p = \|f\|_{\alpha,q} = 1$ for all $p, q > 0$ and $\alpha > 1$.

An important property of these spaces is that point evaluations are continuous functionals. For this fact see Proposition [6,2].

One of the interesting questions about those spaces is which space is the subset of the other space. We consider the case when the quotient $\frac{p}{\alpha}$ is held constant, in which case we have

$$B^p_\alpha \subset B^q_\beta, \quad \frac{p}{\alpha} = \frac{q}{\beta} = r, \quad p < q$$

and $h^r$ is contained in all these spaces. This follows from our results.

Recently for $n = 2$, it was asked whether these embeddings are contractions, that is whether the norm $\|f\|_{B^{p}_{\alpha}}$ is decreasing in $\alpha$. In the case of Bergman spaces, this question was asked by Lieb and Solovej [15]. They proved that such contractivity implies their Wehrl-type entropy conjecture for the $SU(1, 1)$ group. In the case of contractions from the Hardy spaces to the Bergman spaces, it was asked by Pavlović in [22] and by Brevig, Ortega-Cerdà, Seip, and Zhao [3] concerning the estimates for analytic functions. In a recent paper [13], Kulikov confirmed these conjectures, and he proved more general results where the function $t^r$ is replaced with a general convex or monotone function, respectively.

1.2. Statement of main results. In this paper, we extend all those results to the higher-dimensional space by proving the following theorems.
Theorem 1.5. Let \( p > 0 \). \( G : [0, \infty) \to \mathbb{R} \) be an increasing function. Then the maximum value of

\[
\int_{\mathbb{B}} G(|f(x)|^p \Phi_n(x)) d\tau(x)
\]

is attained for \( f(x) \equiv 1 \), subject to the condition that \( f \in h^p \) and \( \|f\|_p = 1 \).

Theorem 1.6. Let \( p > 0 \) and \( \alpha > 0 \). Let \( G : [0, \infty) \to \mathbb{R} \) be a convex function. Then the maximum value of

\[
\int_{\mathbb{B}} G(|f(x)|^p \Phi_n(x))^\alpha d\tau(x)
\]

is attained for \( f(x) \equiv 1 \), subject to the condition that \( f \in B_{\alpha}^p \) and \( \|f\|_{\alpha,p} = 1 \).

We will prove Theorem 1.6 in Section 4. We will first prove Theorem 1.5 for bounded log-subharmonic functions and use such a statement for proving Theorem 1.7 below. More precisely, Theorem 2.2 and Proposition 6.6 (6.11) below imply Theorem 3.1 for a bounded function \( f \). Further such a statement implies Theorem 1.5 for a bounded function \( f \). Furthermore, Theorem 1.6 and Theorem 1.5 for bounded functions imply Theorem 1.7. Then by using Theorem 1.7 we prove Proposition 6.7 which completes the missing part of the proof of Theorem 3.1 and this implies Theorem 1.5 in full generality. To prove Theorem 1.7 we apply the convex and increasing function \( G(t) = t^s, s > 1 \). Note that we get that all the embeddings above between Hardy and Bergman monoids are contractions.

Theorem 1.7. For all \( 0 < p < q < \infty \) and \( 1 < \alpha < \beta < \infty \) with \( \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{r} \) and for all \( f \in h^r \), we have

\[
\|f\|_{\beta,q} \leq \|f\|_{\alpha,p} \leq \|f\|_{h^r}
\]

with equality for \( f(z) \equiv c \), where \( c \in \mathbb{C} \), or for \( f \) belonging to the extremal set below.

Proof: First of all, by the definition, there is a sequence of bounded \( M \)-log-subharmonic functions \( f_m \) converging to \( f \). Then Lemma 3.2 (for bounded function \( f_m \)) imply the inequality

\[
\|f_m\|_{\alpha,p} \leq \|f_m\|_{h^r}.
\]

By letting \( m \to \infty \), using the poinwise convergence in the left-hand side and the \( h^r \) convergence of sequence in the right-hand side of inequality (1.11), in view of Fatou’s lemma we get

\[
\|f\|_{\alpha,p} \leq \|f\|_{h^r}.
\]

If \( f \in h^r \), then \( f \in B_{\alpha}^p \) and Theorem 1.6 implies that

\[
\|f\|_{\beta,q} \leq \|f\|_{\alpha,p}.
\]

\[\square\]
1.3. **Extremal set.** It is important to mention that the Möbius group acts not only on the measure $\tau$ but on the spaces $B^p_\alpha$ as well. More precisely, given a function $f \in B^p_\alpha$ and $m \in M$, the function

$$g(x) = f(m(x)) \frac{\Phi_n^{\alpha/p}(|m(x)|)}{\Phi_n^{\alpha/p}(|x|)}$$

also belongs to the space $B^p_\alpha$ and moreover it has the same norm as $f$ and the same distribution of the function $|f(x)| \Phi_n^{\alpha/p}(x)$ with respect to the measure $\tau$. We need to check that $\Delta_h \log g \geq 0$, if $\Delta_h \log f \geq 0$, and this follows from the formula (1.3) and straightforward calculations:

$$\Delta_h \log g(x) = \Delta_h \log (f(m(x))) + \Delta_h \log \frac{\Phi_n^{\alpha/p}(|m(x)|)}{\Phi_n^{\alpha/p}(|x|)}$$

$$= \Delta_h \log (f(y)|_{y=m(x)}) + \Delta_h \log \frac{\Phi_n^{\alpha/p}(|y|)|_{y=m(x)}}{\Phi_n^{\alpha/p}(|x|)} - \Delta_h \log \frac{\Phi_n^{\alpha/p}(|y|)|_{y=x}}{\Phi_n^{\alpha/p}(|y|)|_{y=x}}$$

$$\geq 0 + (4(n-1)^2 - 4(n-1)^2) \frac{\alpha}{p} = 0.$$

To prove the second statement, we only need to point out the well-known formula for the Jacobian of Möbius transformations of the unit ball onto itself

$$J_m(x) = \frac{(1 - |m(x)|^2)^n}{(1 - |x|^2)^n}.$$ 

See e.g. [25, p. vii].

In particular, when $f(x) \equiv 1$ we get $g(x) = \frac{\Phi_n^{\alpha/p}(|m(x)|)}{\Phi_n^{\alpha/p}(|x|)}$ and the function $g$ gives us the maximal value in (1.8) and (1.9) for every $m \in M$.

We believe that our results also can be formulated for the Hardy and Bergman spaces in the upper half-space, by using a conformal mapping from the unit ball or by directly translating our methods.

1.4. **Structure of the paper.** The paper contains 5 more sections. In Section 2 we prove a general monotonicity theorem for the hyperbolic measure of the superlevel sets of log-$M$-subharmonic functions, which is an adaptation of the beautiful method from [20, 13]. Then, in Sections 3 and 4 we deduce from it Theorems 1.5 and 1.6 respectively. Notice that the proof of Theorem 1.6 is even simpler than the proof of the analogous theorem in [13] for the planar case. Finally, in Section 5 we briefly discuss an application of Corollary 1.7 to coefficient estimates for harmonic functions and some important classes of log-subharmonic functions. In the Appendix below it is given a solution to the hyperbolic harmonic equation in the unit ball and there are proven some propositions that deal with Hardy and weight-Bergman spaces of $M$—subharmonic functions used in this paper.

2. **The proof of the main result**

We begin with
2.1. Isoperimetric inequality for the hyperbolic ball and the function $Υ$. For a Borel set $E \subset \mathbb{B}$ we recall the definition of the hyperbolic volume

$$|E|_h = V(E) = \int_E \left(\frac{2}{1 - |x|^2}\right)^n dx.$$ 

Moreover the hyperbolic perimeter is defined by

$$|\partial E|_h = P(E) = \int_{\partial E} \left(\frac{2}{1 - |x|^2}\right)^{n-1} dH^{n-1}(x).$$

Assume that $\mathbb{B}_s$ is the ball centered at the origin with the radius $\tanh \frac{s}{2}$. The isoperimetric property of hyperbolic ball was established by Schmidt [24] see also [2, 9]. He proved that for every Borel set $E \subset \mathbb{B}$ of finite perimeter $P(E)$, such that $V(E) = V(\mathbb{B}_s)$ and $s > 0$ we have

(2.1) $$P_s \leq P(E),$$

where $P_s$ is the perimeter of $\mathbb{B}_s$ defined by

$$P_s = n \omega_n \sinh^{n-1}(s) = P(\mathbb{B}(0, \tanh s/2)).$$

The volume of $\mathbb{B}_s$ is given by

$$V_s = v(s) := n \omega_n \int_0^s \sinh^{n-1}(t) dt = V(\mathbb{B}(0, \tanh s/2)).$$

Here $\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is the Euclidean volume of the unit ball. Since $s \rightarrow v = V_s$ is increasing, it has an inverse function $S(v) = s$. Then define the function $Υ$ by

(2.2) $$Υ(v) = \frac{v}{P^2_{S(v)}}$$

and thus by (2.1)

(2.3) $$P(E)^2/V(E) \geq 1/Υ(V(E)),$$

with an equality in (2.3) if and only if $E$ is a ball. In the following remark, we give a connection with the isoperimetric inequality in the Euclidean space and it is given a specific inequality for $n = 2$.

**Remark 2.1.** For $s > 0$ we have

$$\frac{V_s}{P_s^{n/(n-1)}} = \frac{n \omega_n \int_0^s \sinh^{n-1}(t) dt}{n^{n/(n-1)} \omega_n^{n/(n-1)} \sinh^n(s)} = \frac{\psi(s)}{\phi(s)} = \frac{\psi'(t)}{\phi'(t)} = \frac{c_0}{\cosh t} \leq c_0,$$

where

$$c_0 = n \frac{n - 1 + n \omega_n^{1/n}}{\omega_n^{1/n}}.$$ 

Now

$$P^2 \geq V/Υ(V) \geq cV^{2(n-1)/n}$$

where

$$c = n^2 \omega_n^{2/n}.$$ 

It can be proved that

(2.4) $$P^n - c^{n/2} V^{n-1} = P^n - n^n \omega_n V^{n-1} \geq (n-1)^n V^n.$$
If \( n = 2 \) then the estimate (2.4) is equivalent to the estimate (2.3) (for the hyperbolic plane of negative Gaussian curvature \(-1\) see [21]). In this case \( \Upsilon(V) = \frac{1}{\pi V} \). It seems unlikely that we can give an explicit expression for the function \( \Upsilon \) for a higher-dimensional case, but we don’t need it in the proofs of our results.

Let \( f \) be a real analytic complex valued function such that \( v = |f| \) is log-\( \mathcal{M} \)-log-subharmonic function in \( \mathbb{B} \) and such that \( u(x) = v(x)^{\alpha} \Phi_n^\alpha(x) \) is bounded and goes to 0 uniformly as \( |x| \to 1 \). Then the superlevel sets \( A_t = \{ x : u(x) > t \} \) for \( t > 0 \) are compactly embedded into \( \mathbb{B} \) and thus have finite hyperbolic measure \( \mu(t) = \tau(A_t) \).

In this section, we prove the following theorem which says that a certain function related to this measure is decreasing.

**Theorem 2.2.** Let \( \alpha \geq 1 \) and \( a \geq 0 \) and assume that \( f \) is a real analytic complex valued function such that \( v = |f| : \mathbb{B} \to [0, +\infty) \) is a log-\( \mathcal{M} \)-subharmonic function. Assume further that the function \( u(x) = |f(x)|^a \Phi_n^\alpha(x) \) is bounded and \( u(x) \) tends to 0 uniformly as \( |x| \to 1 \). Then the function

\[
g(t) = t \exp \left[ \int_0^{\mu(t)} \gamma \Upsilon(x)dx \right],
\]

is decreasing on the interval \( (0, t_0) \), where \( \gamma = \alpha(n - 1)^2 \), \( \Upsilon \) is defined in (2.2) and \( t_0 = \max_{x \in \mathbb{B}} u(x) \).

If \( f(x) \equiv 1 \), the function \( g \) turns out to be constant and this is an important property of \( g \).

The proof of this theorem is mostly based on the methods developed in [20], translated from the Euclidean to the hyperbolic setting, and in [13] translated from the planar case to the higher-dimensional case.

**Proof of Theorem 2.2.** We start with the coarea formula

\[
\mu(t) = \tau(A_t) = \int_{A_t} \frac{2^n}{(1 - |x|^2)^n} dx = \int_{t}^{\max_u} \int_{\{u(x) = \kappa\}} \frac{2^n |\nabla u|^{-1}}{(1 - |x|^2)^n} d\mathcal{H}^{n-1}(x) d\kappa.
\]

Then we get

\[
(2.5) \quad - \mu'(t) = \int_{u=t} |\nabla u|^{-1} \frac{2^n d\mathcal{H}^{n-1}(x)}{(1 - |x|^2)^n}
\]

along with the claim that \( \{ x : u(x) = t \} = \partial A_t \) and that this set is a smooth hypersurface for almost all \( t \in (0, t_0) \). Here \( dS = d\mathcal{H}^{n-1} \) is \( n - 1 \) dimensional Hausdorff measure. Observe that a similar formula has been proved in [13]. These assertions follow the proof of Lemma 3.2 from [20]. We point out that, since \( u \) is real analytic and non-constant, then it is a well-known fact from measure theory that the level set \( \{ x : u(x) = t \} \) has a zero measure (see [19]), and this is equivalent to the fact that the \( \mu \) is continuous.
Following the approach from [20][13], our next step is to apply the Cauchy–Schwarz inequality to the hyperbolic area of $\partial A_t$:

$$
|\partial A_t|^2_h = \left( \int_{\partial A_t} \frac{2^{n-1} dS}{(1 - |x|^2)^{n-1}} \right)^2
$$

(2.6)

$$
\leq \int_{\partial A_t} |\nabla u|^{-1} \frac{2^n dS}{(1 - |x|^2)^n} \int_{\partial A_t} |\nabla u|^{2n-2} dS
$$

Let $\nu = \nu(x)$ be the outward unit normal to $\partial A_t$ at a point $x$. Note that, $\nabla u$ is parallel to $\nu$, but directed in the opposite direction. Thus we have $|\nabla u| = -\langle \nabla u, \nu \rangle$. Also, we note that since for $x \in \partial A_t$ we have $u(x) = t$, we obtain for $x \in \partial A_t$ that

$$
\frac{|\nabla u(x)|}{t} = \frac{|\nabla u(x)|}{u} = \langle \nabla \log u(x), \nu \rangle.
$$

Now the second integral on the right-hand side of (2.6) can be evaluated by the Gauss’s divergence theorem:

$$
\int_{\partial A_t} |\nabla u| dS = -t \int_{A_t} \text{div} \left( \frac{\nabla \log u(x)}{(1 - |x|^2)^{n-2}} \right) dx
$$

$$
= -t \int_{A_t} \frac{1}{(1 - |x|^2)^n} \Delta_h \log u(x) dx.
$$

Now we plug $u = v(x)^n \Phi_n(x)$, where $v(x) = |f(x)|$, and calculate

$$
-t \Delta_h \log (v^n \Phi_n) = -(at \Delta_h \log v + t\alpha \Delta_h \log \Phi_n) \leq 0 + 4t \gamma,
$$

where $\gamma = \alpha(n - 1)^2$. By using (2.5) and (2.6) we obtain

$$
|\partial A_t|^2_h \leq (-\mu'(t)) \int_{\partial A_t} |\nabla u|^{2n-2} dS
$$

$$
\leq -2^{n-2} \cdot 4t \gamma \frac{\mu'(t) \mu(t)}{2^n}
$$

$$
= -t \gamma \frac{\mu'(t) \mu(t)}{2^n}.
$$

So by (2.3), we have

$$
(2.7)
$$

$$
\gamma \mu'(t) \mu(t) + \frac{\mu(t)}{\gamma(t)} \leq 0
$$

with equality in (2.7) if $v$ is a constant because in that case $A_t$ is a ball centered at the origin.

Thus

$$
G(t) = - \int_t^{t_0} \gamma \mu'(t) \gamma(t) dt - \int_t^{t_0} \frac{1}{t} dt = \int_0^{\mu(t)} \gamma \gamma(x) dx - \log \left( \frac{t}{t_0} \right)
$$

is non-increasing. Therefore

$$
g(t) := \exp(G(t)) = t \exp \left[ \int_0^{\mu(t)} \gamma \gamma(x) dx \right]
$$

is non-increasing.
Remark 2.3. Note that for the function $f(x) \equiv 1$ everywhere in the proof above we have equalities for all values of $a$ and $\alpha$.

3. Weak-type estimate for the $M$–Hardy class $h^p$ and the proof of Theorem 1.5

In this section, we are going to prove the following bound for the measure of the so-called superlevel sets of functions from the Hardy spaces. Theorem 1.5 is then an easy matter. In what follows we keep the same notation as in the previous section.

Theorem 3.1. Let $f \in h^p$ for $p > 0$ with $\|f\|_{h^p} = 1$ and put $u(x) = |f(x)|^p \Phi_n(x)$. Then for all $t \in (0, \infty)$ we have

\begin{equation}
\mu(t) \leq \mu_1(t),
\end{equation}

where

\[ \mu(t) = |\{x : u(x) \geq t\}|_h \quad \text{and} \quad \mu_1(t) = |\{x : \Phi_n(x) \geq t\}|_h. \]

Note that this theorem extends the corresponding result in [13], where [3, Conjecture 2] is verified. Indeed it is easy to check that

\[ |\{x : \Phi_n(x) \geq t\}|_h = 4\pi \max\{1/t - 1, 0\}, \]

for $n = 2$ which coincides with the corresponding result of Kulikov in [13] after normalization. First, we prove the following lemma.

Lemma 3.2. Theorem 3.1 and Theorem 1.5 are true provided that $f$ is bounded.

### 3.1. Proof of Theorem 3.1 for bounded functions.

Put $t_0 = \max_{x \in \mathbb{B}} u(x)$. This number is well-defined since $f$ is bounded. Assume without loss of generality that $p > 1$. Indeed, if $p > 0$ and if $f$ is a positive log- $M$–subharmonic function such that $f \in h^p$, then $g = f^{p/2}$ is positive log- $M$–subharmonic function such that $g \in h^2$. In particular, for $t \geq t_0$ the bound (3.1) holds trivially.

Assume that there exists some $0 < t_1 < t_0$ such that $\mu(t_1) > \mu_1(t_1)$. Then $\mu(t_1) = \mu_1(t_1/c)$ for some $c > 1$, because $\lim_{s \to 0} \mu_1(s) = +\infty$. We claim that in that case for all $0 < t < t_1$ we have $\mu(t) \geq \mu_1(t/c)$.

Indeed, by applying the pointwise bound together with $u(x) \to 0$ as $|x| \to 1$, we see that Theorem 2.2 can be applied to $f$ with $a = p, \alpha = 1$, and we get that

\[ g(t) = t \exp \left[ \int_0^{\mu(t)} \gamma \Upsilon(x) dx \right], \]

is decreasing. Since
\[ g(t_1) = t_1 \exp \left[ \int_0^{\mu(t_1)} \gamma \Upsilon(x) \, dx \right] \\
= t_1 \exp \left[ \int_0^{\mu_1(t/c)} \gamma \Upsilon(x) \, dx \right] \\
= c \cdot \frac{t}{c} \exp \left[ \int_0^{\mu(t/c)} \gamma \Upsilon(x) \, dx \right] \\
< g(t) = t \exp \left[ \int_0^{\mu(t)} \gamma \Upsilon(x) \, dx \right]. \]

From this we get \( \mu(t) \geq \mu_1(t/c) \).

Now we are going to use Proposition 6.6 which says that \( \| f \|_{p; pr} \to \| f \|_p = 1 \) as \( r \to 1^+ \). Note that we can express the \( B_{pr}^{pr} \) norms through the function \( \mu(t) \):

\[ (3.2) \quad \| f \|_{p; pr} = c_r \int_0^{t_1} \mu(t) t^{r-1} \, dt, \]

where \( c_r = c(r) \) is defined in (1.5), and it satisfies the relation \( c_r \int_0^1 \mu_1(r) t^{r-1} \, dt = 1 \). We now use the fact that \( c_r \to 0 \) as \( r \to 1 \) (see (1.6)). By the formula (3.2) and above bound we have

\[ (3.3) \quad \| f \|_{r; pr} \geq c_r \int_0^{t_1} (\mu_1(r/c)) t^{r-1} \, dt = c_r c \int_0^{t_1/c} (\mu_1(s)) s^{r-1} \, ds. \]

On the other hand

\[ 1 = c_r \int_0^1 \mu_1(t) t^{r-1} \, dt \\
= c_r \int_0^{t_1/c} \mu_1(t) t^{r-1} \, dt + c_r \int_{t_1/c}^1 \mu_1(t) t^{r-1} \, dt = P(r) + Q(r). \]

Since \( c_r \to 0 \) as \( r \to 1 \), we have that \( Q(r) \to 0 \) as \( r \to 1 \) because the function we are integrating is bounded. Therefore, \( P(r) \to 1 \) as \( r \to 1 \). On the other hand, the right-hand side of (3.3) is at least \( cP(r) \). Therefore \( 1 = \lim_{r \to 1} \| f \|_{p; pr} \geq c \lim_{r \to 1} P(r) = c \) which is a contradiction. Recall that \( c > 1 \).

3.2. **Proof of Theorem 1.5 for bounded functions.** As in [13], we can assume that \( \lim_{t \to 0^+} G(t) = 0 \). Then this integral can be expressed through the function \( \mu(t) \) as

\[ (3.4) \quad \int_0^\infty \mu(t) \, dG(t). \]

Note that here we used that the function \( \mu(t) \) is continuous, that is the sets \( \{ x \in \mathbb{B} : u(x) = t \} \), \( t > 0 \), have zero measure.
Since $G$ is increasing, measure $dG(t)$ is positive. Thus, by Lemma 3.2, (3.1) this integral is at most
\[ \int_0^\infty \mu_1(t)dG(t), \]
which is the value of (1.8) for $f(x) \equiv 1$.

**Proof of Theorem 3.1 and Theorem 1.5.** We repeat the proof of Lemma 3.2. We also have
\[ 1 = \lim_{r \to 1} \|f\|_{pr,r}, \]
because of Proposition 6.7 below. This yields Theorem 3.1. Now we repeat the proof of Theorem 1.5 for bounded functions, by using Theorem 3.1 instead of Lemma 3.2 and get Theorem 1.5 for unbounded function $f$. \qed

**4. Proof of Theorem 1.6**

As in [13], we restrict ourselves to the case \( \lim_{t \to 0^+} G(t) = 0 \). Let \( \mu(t) = \tau(\{x : u(x) > t\}) \) where \( u(x) = |f(x)|^p(\Phi_n(x))^{\alpha} \). Applying Theorem 2.2 to $f$ with \( a = p \), we get that the function
\[ g(t) = t \exp \left[ \int_0^\mu(t) \gamma \Upsilon(x) dx \right], \]
is decreasing on \((0, t_o)\) with \( t_o = \max_{x \in D} u(x) \). Proposition 6.3 ensures the existence of \( t_o \).

For \( f \equiv 1 \), $g$ is a constant function equal to 1. Let
\[ \Lambda(u) = \int_0^u \gamma \Upsilon(x) dx \]
and \( \Theta = \Lambda^{-1} \). Note that \( \Theta \) is increasing. Then
\[ \mu(t) = \Theta \left( \log \frac{g(t)}{t} \right). \]
We assume that \( \|f\|_{\alpha,p} = 1 \), that is
\[ I_1 = \int_0^{t_o} \mu(t)dt = \int_0^{t_o} \Theta \left( \log \frac{g(t)}{t} \right) dt = \frac{1}{c(\alpha)}. \]
Now the integral in (1.9) can be rewritten as
\[ I_2 = \int_0^{t_o} \Theta \left( \log \frac{g(t)}{t} \right) G'(t) dt. \]
Then by Lemma 4.1 below, by taking \( \Phi(s) = \Theta(\log(s)) \) and \( \Psi(t) = G'(t) \), the maximum of \( I_2 \) under \( I_1 = \frac{1}{c(\alpha)} \) is attained for $g \equiv 1$.

**Lemma 4.1.** Assume that $\Phi, \Psi$ are positive increasing functions and $g$ is a positive non-increasing function such that
\[ \int_0^{t_o} \Phi \left( \frac{g(t)}{t} \right) dt = \int_0^{t_o} \Phi \left( \frac{1}{t} \right) dt = c. \]
Then
\[ \int_0^{t_0} \Phi \left( g(t)/t \right) \Psi(t) dt \leq \int_0^{t_0} \Phi \left( 1/t \right) \Psi(t) dt. \]

Proof. Choose \( a \in [0, t_0] \) such that \( g(t) \geq 1 \) for \( t \leq a \) and \( g(t) \leq 1 \) for \( t \geq a \). Then
\[ \chi(t) := (\Phi(g(t)/t) - \Phi(1/t))(\Psi(t) - \Psi(a)) \leq 0 \]
for all \( t \in [0, t_0] \). By integrating \( \chi(t) \) for \( t \in (0, t_0) \) we get
\[ \int_0^{t_0} \Phi \left( 1/t \right) \Psi(a) - \Phi(g(t)/t) \Psi(a) - \Phi(1/t) \Psi(t) + \Phi(g(t)/t) \Psi(t) dt \]
\[ = \Psi(a) \int_0^{t_0} (\Phi(g(t)/t) - \Phi(1/t)) dt \]
\[ + \int_0^{t_0} \Psi(t) (\Phi(g(t)/t) - \Phi(1/t)) dt \leq 0. \]
Since
\[ \int_0^{t_0} (\Phi(g(t)/t) - \Phi(1/t)) dt = 0, \]
the result follows. \( \square \)

5. Some Applications for the Case \( n = 2 \)

Let \( B_{2/p}^2 \), \( 1 < p < 2 \) be the space of harmonic functions in the unit disk \( \mathbb{D} \) such that
\[ \|f\|_{2/p,2}^2 := \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{2/p-1} \frac{dxdy}{\pi} < \infty \]
and let \( h^p \) be the standard Hardy space with the norm
\[ \|f\|_{h^p}^2 := \| \sqrt{|a|^2 + |b|^2} \|^2. \]
Assume that \( f = a + \overline{b} \) is a harmonic function defined in the unit disk, where \( a \) and \( b \) are holomorphic functions. Then \( \log(|a|^2 + |b|^2) \) is subharmonic (see e.g. [10]).

Then we obtain a special case of contraction from the Hardy space to a Bergman space is \( (h^p, \| \cdot \|) \subset B_{2/p}^2 \) for \( 1 < p < 2 \), which extends a corresponding result in [13] and a classical result [4] and also the recent results [16]. Namely for
\[ f(z) = a + \overline{b} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \overline{b_n} z^n \in B_{2/p}^2, \]
where \( b_0 = 0 \), we can express its norm as follows
\[ \|f\|_{2/p,2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2 + |b_n|^2}{c_{2/p}(n)}, \quad c_{2/p}(n) = \binom{n + 2/p - 1}{n}. \]
Thus, for a function \( f \in h^p \), from Corollary [17] we have
\[ \sum_{n=0}^{\infty} \frac{|a_n|^2 + |b_n|^2}{c_{2/p}(n)} \leq \|f\|_{h^p}^2. \]
Now by [10, Theorem 2.1], for $f \in h^p$, $b_0 = 0$, for $1 < p \leq 2$, we obtain the following inequality

\[
\sum_{n=0}^{\infty} \frac{|a_n|^2 + |b_n|^2}{c_{2/p}(n)} \leq \frac{1}{1 - |\cos \pi/p|} \|f\|_p^2,
\]

where

\[
\|f\|_p^2 = \int_{\partial D} |f(\zeta)|^p |d\zeta|^{1/p}. \]

By using the result from [17], that the Jacobian of a harmonic diffeomorphism is log-superharmonic, and the fact that the Jacobian $J(f, z) = |a'(z)|^2 - |b'(z)|^2$ is real analytic, Corollary 1.7 implies

**Corollary 5.1.** Let $1 < p < 2$ and $\alpha > 1$. Assume that $f$ is a harmonic diffeomorphism of the unit disk $\mathbb{D}$ onto a two-dimensional domain $\Omega$. Assume further that $1/J_f(z) \in h^\alpha p/\alpha$. Then we have

\[
\left( (\alpha - 1) \int_{\mathbb{D}} \frac{1}{J_f^p(z)} (1 - |z|^2)^{\alpha-2} \frac{dxdy}{\pi} \right)^{1/p} \leq \|1/J_f(z)\|_{p/\alpha}.
\]

Since $\log(|a|^2 + |b|^2)$ is subharmonic, if $a$ and $b$ are analytic functions (see property 3 of the admissible monoid), because of Corollary 1.7, we obtain the following Corollaries

**Corollary 5.2.** Assume that $f = a + \bar{b} : \mathbb{D} \to \mathbb{C}$ is a harmonic mapping and that $\alpha > 1, p > 1$. Then

\[
\left( (\alpha - 1) \int_{\mathbb{D}} (|a|^2 + |b|^2)^{p/2} (1 - |z|^2)^{\alpha-2} \frac{dxdy}{\pi} \right)^{1/p} \leq \|(|a|^2 + |b|^2)^{1/2}\|_{p/\alpha}.
\]

**Corollary 5.3.** For $p > 1$ and a harmonic function $f \in h^p$ we have the following isoperimetric type inequality

\[
\|f\|_{B^{2p}} \leq C_p \|f\|_p,
\]

where

\[
C_p = \frac{\sqrt{2} \cos \frac{\pi}{4p}}{(1 - |\cos \frac{\pi}{p}|)^{1/2}}.
\]

Here $B^{2p}$ is the Bergman space of harmonic mappings belonging to the Lebesgue space $L^{2p}(\mathbb{D})$.

Observe that for $p \geq 2$,

\[
C_p = \frac{1}{2} \csc \frac{\pi}{4p}.
\]

The inequality (5.3) for $p > 1$ being an integer has been proved by the author in [10, Theorem 2.11]. We also refer to a recent improvement of (5.3) in [18] for
$p > 2$, which is based on (5.3) for $p = 2$ and Jensen’s inequality. For $p \in (1, 2]$ we have,

$$C_p = \cos \frac{\pi}{4p} \sec \frac{\pi}{2p}.$$  

The inequality (5.3) for such constant $C_p$ has been proved for real-valued harmonic functions in [11]. In this case we do not have such a restriction.

**Proof of Corollary 5.3.** Let $f(z) = a(z) + \bar{b}(z)$ and assume without loss of generality that $a(0) = 0$. By integrating \([10, \text{eq. 2.3}]\) in the interval $[0, 1]$ we get

$$\int_D |a + \bar{b}|^p \frac{dxdy}{\pi} \leq I := \sqrt{2} \cos \frac{\pi}{4p} \left( \int_D (|a|^2 + |b|^2)^p \frac{dxdy}{\pi} \right)^{\frac{1}{2p}}.$$  

Then by Corollary 5.2, by choosing $\alpha = 2$, we get

$$I \leq J := \sqrt{2} \cos \frac{\pi}{4p} \| |a|^2 + |b|^2 \|_{h_p}.$$  

Now \([10, \text{Theorem 2.1}]\) implies

$$J \leq \frac{\sqrt{2} \cos \frac{\pi}{4p}}{(1 - |\cos \frac{\pi}{p}|)^{1/2}} \|f\|_p.$$  

The result follows. \(\square\)

Assume now that $f : \mathbb{D} \to \Sigma \subset \mathbb{R}^n$ is a conformal parameterization of the minimal surface $\Sigma$. Since $\log |f_x(z)|^2 = \log (|p(z)|(1 + |q(z)|^2))$, where $p$ and $q$ are holomorphic functions in $z = x + iy$ (the so-called Enneper-Weierstrass parameters), we have the following corollary.

**Corollary 5.4.** Assume that $\alpha > 1, p > 1$ and $f : \mathbb{D} \to \mathbb{R}^n$ is the Enneper-Weierstrass parameterisation of a minimal surface in $\mathbb{R}^n$. Then

$$(\alpha - 1) \left( \int_D |f_x(z)|^p (1 - |z|^2)^{\alpha - 2} \frac{dxdy}{\pi} \right)^{\frac{1}{p}} \leq \|f_x\|_{p/\alpha}.$$  

For $p = \alpha = 2$, the above formula is simply the isoperimetric inequality for minimal surfaces.

6. **Appendix**

**Proposition 6.1.** Let $u(x) = g(|x|)$. Then a solution of $\Delta_h \log u = -4b$ which is differentiable at $x = 0$ is given by the formula

$$(6.1) \quad u_b(x) = \exp \left\{ \frac{b(2 - n)r^2}{(n - 1)n} \left( \begin{array}{c} 1, 1, 2 - \frac{n}{2} \\ 2, 1 + \frac{n}{2} \end{array} ; r^2 \right) \left( 1 - r^2 \right)^{\frac{b}{n-1}} \right\}.$$  

Then for $b = (n - 1)^2$ we define $\Phi_n(r) = u_b(x)$ and have

$$(6.2) \quad E_n (1 - r^2)^{n-1} \leq \Phi_n(r) \leq (1 - r^2)^{n-1}$$  

and inequality is strict for $n > 2$ and $r > 0$. Here

$$E_n = \exp \left\{ \frac{(n-1)(2-n)}{n} \left( \begin{array}{c} 1, 1, 2 - \frac{n}{2} \\ 2, 1 + \frac{n}{2} \end{array} ; 1 \right) \right\}.$$
Proof. The differential equation $\Delta h \log u = -4b$, reduces to

$$(1 - r^2) \left( \frac{2(-2 + n)r^2 + (-1 + n)(1 - r^2)}{r} h(r) + (1 - r^2) h'(r) \right) = -4b,$$

where $h(r) = g'(r)$, $g(r) = \log u(|x|)$ and $r = |x|$. Then the general solution of the last equation is given by

$$h(r) = \frac{r^{1-n} (1 - r^2)^{2+n} (nc - 4br^n F_0 \left[ \frac{n}{2}, n, 2 + n, r^2 \right])}{n}.$$

Here $F_0$ is the Gaussian hypergeometric function. The solution which is defined in $x = 0$ is given by

$$h(r) = -\frac{4b (1 - r^2)^{2+n} r F_0 \left[ \frac{n}{2}, n, 2 + n, r^2 \right]}{n} = \frac{-4br F_0 \left[ 1, 1 - \frac{n}{2}, 1 + \frac{n}{2}, r^2 \right]}{n(1 - r^2)}.$$

The last equality follows from Euler transformations.

Then after straightforward computations we obtain

$$g(r) = \int_0^r h(t)dt = -\frac{(-2 + n)r^2 F \left[ 1, 1, 2 - \frac{n}{2}, 2, 1 + \frac{n}{2} ; r^2 \right] - n \log \left[ 1 - r^2 \right]}{(-1 + n)n}.$$

and $u(x) = \exp(g(r))$ is given by (6.1).

To prove the (6.2) we need to prove the monotonicity of

$$\phi(r) := r^2 \frac{(n - 1)(2 - n)}{n} F \left[ 1, 1, 2 - \frac{n}{2}, 2, 1 + \frac{n}{2} ; r^2 \right].$$

First we have

$$\phi'(r) = 2 \frac{(n - 1)(2 - n)}{n} r F_0 \left[ 1, 2 - \frac{n}{2}, 2 + n, r^2 \right],$$

then observe that

$$F_0[a, b, c, z] = \frac{1}{B[b, c - b]} \int_0^1 x^{b-1}(1 - x)^{c-b-1}(1 - zx)^{-a}dx,$$

where $B$ is the beta function. This functions is clearly positive for $c = \frac{2+n}{2}$, $b = 2 - \frac{n}{2}$, $a = 1$ and $z = r^2 < 1$, and thus $\phi$ is decreasing. This implies (6.2). □

The point evaluations are continuous functionals in $h^p$ and $B^p_\alpha$. This is the content of the following proposition.

**Proposition 6.2.** Let $p > 1$ and $\alpha > 1$. There are two constants $C_1 = C_1(n)$ and $C_2 = C_2(n, \alpha)$ such that, if $f$ is $M-$subharmonic and belongs to the Hardy space $S^p$, then

$$|f(x)|^p \Phi_n(x) \leq C_1 \|f\|_p^p$$

and if $f \in B^p_\alpha$ then
\[ \text{Eq. 10.1.5}, \] which is formulated for the first of all relation (6.3) follows from (6.2) and [25, Lemma 7.2.1].

**Proof.** First of all relation (6.3) follows from (6.2) and [25, Lemma 7.2.1].

To prove (6.4) we proceed similarly. This time we make use of Lemma 7.2.1 and Eq. 10.1.5, which is formulated for \( \mathcal{M} \)-harmonic functions, but the same proof can be applied for the \( \mathcal{M} \)-subharmonic functions.

Prove now (6.5). Let \( f \) be a positive \( \mathcal{M} \)-subharmonic function on the unit ball belonging to the Hardy space \( S^p \).

Let \( F = \mathcal{P}_h[\hat{f}] \) be the least harmonic majorant of \( f \) (25, Theorem 7.1.1), which is in \( S^p \). Now we define the mapping \( F_r(x) = \mathcal{P}_h[\hat{f}_r(x)] \), where \( \hat{f}_r(x) = F(rx) \), \( x \in \mathbb{S} \) and \( 0 < r < 1 \). We claim that for all \( \epsilon > 0 \) there exists \( r < 1 \) such that such that

\[ (6.4) \quad |f(x)|^p \Phi_n^\alpha(x) \leq C_2 \| f \|_{\alpha,p}^p. \]

Moreover, if \( f \in S^p \) then

\[ (6.5) \quad \lim_{|x| \to 1} |f(x)|^p \Phi_n(x) = 0. \]

Proof. First of all relation (6.3) follows from (6.2) and [25, Lemma 7.2.1].

To prove (6.4) we proceed similarly. This time we make use of (6.2) and [25, Eq. 10.1.5], which is formulated for \( \mathcal{M} \)-harmonic functions, but the same proof can be applied for the \( \mathcal{M} \)-subharmonic functions.

Prove now (6.5). Let \( f \) be a positive \( \mathcal{M} \)-subharmonic function on the unit ball belonging to the Hardy space \( S^p \).

Let \( F = \mathcal{P}_h[\hat{f}] \) be the least harmonic majorant of \( f \) (25, Theorem 7.1.1), which is in \( S^p \). Now we define the mapping \( F_r(x) = \mathcal{P}_h[\hat{f}_r(x)] \), where \( \hat{f}_r(x) = F(rx) \), \( x \in \mathbb{S} \) and \( 0 < r < 1 \). We claim that for all \( \epsilon > 0 \) there exists \( r < 1 \) such that such that

\[ (6.6) \quad \| F_r - F \|_p \leq \epsilon. \]

Since \( \hat{f} \in L^p(\mathbb{S}) \), there exists a continuous function \( g \) in \( \mathbb{S} \), such that \( \| g - \hat{f} \|_p < \epsilon / 3 \). Let \( G(x) = \mathcal{P}_h[g](x) \) and \( G_r(x) = \mathcal{P}_h[g_r](x) \), where \( g_r(x) = G(rx) \), \( x \in \mathbb{S} \), \( 0 < r < 1 \). Then \( G_r - F_r \) is \( \mathcal{M} \)-harmonic and so \( |G_r - F_r| \) is \( \mathcal{M} \)-subharmonic for \( p \geq 1 \). The same hold for \( G - H \) and \( |G - H| \) for \( p \geq 1 \).

Then by [25, Theorem 5.4.2] we have

\[ \| G - F \|_p = \| \mathcal{P}_h[\hat{f} - g] \|_p \leq \| \hat{f} - g \|_p \leq \epsilon / 3 \]

and

\[ \| G - F \|_p = \| \mathcal{P}_h[\hat{f} - g] \|_p \leq \| \hat{f} - g \|_p \leq \epsilon / 3. \]

In the last inequality we used [25, Theorem 5.4.2], which states that \( \int_{\mathbb{S}} k(r) d\sigma(\eta) \) is an increasing function for \( r \in [0, 1) \), provided that \( k \) is \( \mathcal{M} \)-subharmonic.

Thus

\[ \| F - F_r \|_p \leq \| F_r - G_r \|_p + \| G_r - G \|_p + \| G - F \|_p \]

\[ \leq \| g - \hat{f} \|_p + \| g_r - g \|_p + \| g - \hat{f} \|_p \]

\[ \leq \frac{2\epsilon}{3} + \| g_r - g \|_p. \]

Now by [25, Corollary 5.3.4], because \( g \) is continuous on \( \mathbb{S} \), there is \( r < 1 \) such that \( \| g_r - g \|_p < \epsilon / 3 \). This implies (6.6).
Then we have
\[ |F(x)| \leq |F(x) - F_r(x)| + |F_r(x)|. \]
From (6.3) and (6.6), we get
\[ |F(x) - F_r(x)|^p \Phi_n(x) \leq C \epsilon. \]
On the other hand, because \( F_r \) is smooth up to the boundary we have
\[ \lim_{|x| \to 1} |F_r(x)|^p \Phi_n(x) = 0. \]
So
\[ \limsup_{|x| \to 1} |F(x)|^p \Phi_n(x) \leq C \epsilon. \]
Because \( F \) is the harmonic majorant of \( f \), by the previous relation we get
\[ \limsup_{|x| \to 1} |f(x)|^p \Phi_n(x) \leq \limsup_{|x| \to 1} |F(x)|^p \Phi_n(x) \leq C \epsilon. \]
Thus
\[ \limsup_{|x| \to 1} |f(x)|^p \Phi_n(x) = 0 \]
as was stated. \( \square \)

Now we prove also an analog to the vanishing condition (6.5) for the Bergman spaces.

**Proposition 6.3.** If \( f \) is \( \mathcal{M} \)-subharmonic and
\[ \int_\mathbb{B} |f(x)|^p \Phi_n^\alpha(x) d\tau(x) < \infty \]
then
\[ |f(x)|^p \Phi_n^\alpha(x) \]
is bounded and tends to 0 uniformly as \( |x| \to 1 \).

We start with a lemma

**Lemma 6.4.** If \( f \) is \( \mathcal{M} \)-subharmonic in \( \mathbb{B} \) then from [25, Theorem 4.3.5]:
\[ (6.7) \quad |f(0)| \leq C(n) \int_{|x|<1/2} |f(x)| dx. \]

By using the previous lemma and Hölder’s inequality we have

**Lemma 6.5.** If \( f \) is \( \mathcal{M} \)-subharmonic in \( \mathbb{B} \) and \( 1 < p < \infty \) then
\[ (6.8) \quad |f(0)|^p \leq C(n, p) \int_{|x|<1/2} |f(x)|^p dx. \]

**Proof of Proposition 6.3.** We already know that \( |f(0)|^p \) is bounded. Now, we will use the \( \mathcal{M} \)-invariance of our integral to translate this bound from 0 to any other point. To prove that it tends to 0, observe that by the standard measure theory, we have
\[ \lim_{r \to 1} \int_{r<|x|<1} |f(x)|^p \Phi_n^\alpha(x) d\tau(x) = 0. \]
Now, pick \( s > r \) so that for all \( x_0 \) with \( |x_0| > s \) when we consider the Möbius transformation \( \varphi = \varphi_{x_0} (\varphi^{-1} = \varphi) \) which sends 0 to \( x_0 \) then the ball of radius 1/2 around 0 is getting mapped into the subset of a set \( r < |y| < 1 \). We can take for example \( s = s(r) = \frac{r+1/2}{1+r/2} \), because

\[
|y| = |\varphi(x)| = \frac{|x-x_0|}{|x,x_0|} \geq \frac{|x_0|-|x|}{1 - |x_0||x|} > r
\]

if \( |x_0| > s \) and \( |x| < 1/2 \). Here

\[
[x,x_0]^2 := 1 + |x|^2|x_0|^2 - 2 \langle x, x_0 \rangle \geq (1 - |x||x_0|)^2.
\]

Then by the translated pointwise bound we get

\[
|f(x_0)|^p = |f(\varphi(0))|^p \leq C(n, p) \int_{|x| < 1/2} |f(\varphi(x))|^p dx = \int_{|\varphi(x)| < 1/2} |f(y)|^p J_\varphi(y) dy,
\]

where \( J_\varphi(y) \) is the Jacobian of \( \varphi \) at \( y \). Now we use the fact that \( \varphi(|x| < 1/2) \subset \{|y| : r < |y| < 1\} \) and

\[
J_\varphi(y) = \frac{(1 - |x_0|^2)^n}{|y,x_0|^{2n}}.
\]

Now because \( (1 - |x_0|) \leq |y,x_0| \), we have

\[
(1 - |x_0|^2)^{(n-1)\alpha} J_\varphi(y) = \frac{(1 - |x_0|^2)^{(n-1)\alpha+n}}{|y,x_0|^{2n}} = \frac{(1 - |x_0|^2)^{(n-1)\alpha+n}}{|y,x_0|^{(n-1)\alpha+n-(n-1)\alpha+n}} \leq C_n (1 - |y|^2)^{(n-1)\alpha-n}.
\]

Thus

\[
|f(x_0)|^p \Phi_n^\alpha(x_0) \leq C(p, n) \int_{\varphi(|x| < 1/2)} (1 - |x_0|^2)^{(n-1)\alpha}|f(y)|^p J(y, \varphi) dy \leq C_1(p, n) \int_{r<|y|<1} \Phi_n^\alpha(y)|f(y)|^p d\tau(y).
\]

The last quantity tends to zero as \( r \to 1 \) and this implies the claim.

\( \square \)

**Proposition 6.6.** Let \( p > 0 \) and \( \alpha > 1 \). Then

(6.9) \[ h^p \subset B^{op}_\alpha. \]

Furthermore assume that \( f \in h^p \). Then

(6.10) \[ \lim_{\alpha \to 1^+} \|f\|_{\alpha,p} = \|f\|_p. \]
Moreover, if \( f \) is bounded, then

\[
\lim_{\alpha \to 1^+} \|f\|_{\alpha,p} = \|f\|_p.
\]

**Proof.** We can assume that \( p > 1 \), otherwise we consider the function \( g = |f|^{p/2} \) and apply the following proof. To prove (6.9) assume that \( f \in h^p \) and let \( g \) be the last \( M \)-harmonic majorant of \( f \). We use the Hardy-Littlewood type inequality for \( M \)-harmonic functions in the unit ball by Stoll: [25, Theorem 10.6.3] (see also [5])

\[
\|g\|_{\alpha,\beta,p} \leq C(p,\alpha,n) \|g\|_p.
\]

Since \( \|g\|_p = \|f\|_p \) and \( \|f\|_{\alpha,p} \leq \|g\|_{\alpha,p} \), it follows that \( f \in B_\alpha^{op} \) and this concludes (6.9).

From (1.7) we know that \( \Delta_h f \geq 0 \), and by corresponding theorem [25, Theorem 5.4.2], we get

\[
\|f\|_p^p = \lim_{r \to 1} \int_\Sigma |f(r\zeta)|^p d\sigma(\zeta).
\]

Then from (1.2) and (6.13) we have

\[
\|f\|_{\alpha,p}^p = c(\alpha) \int_\Sigma |f(x)|^p \Phi_n^\alpha(|x|)(1 - |x|^2)^{-n} \frac{dV(x)}{\omega_n}
\]

\[
= c(\alpha) \int_0^1 \Psi(r) dr \int_\Sigma |f(r\zeta)|^p d\sigma(\zeta)
\]

\[
\leq \|f\|_p^p,
\]

here \( \alpha > 1 \), \( \Psi(r) = r^{n-1} \Phi_n^\alpha(r)(1 - r^2)^{-n} \) and

\[
c(\alpha) \int_0^1 \Psi(r) dr = 1.
\]

On the other hand (6.14) implies

\[
\limsup_{\alpha \to 1^+} \|f\|_{\alpha,p} \leq \|f\|_p.
\]

Furthermore, because

\[
\|f\|_p^p = \lim_{r \to 1} \int_\Sigma |f(r\zeta)|^p d\sigma(\zeta),
\]

for every \( \epsilon > 0 \) there exists \( r_1 < 1 \) such that for \( r \in (r_1, 1) \) we have

\[
\int_\Sigma |f(r\zeta)|^p d\sigma(\zeta) \geq \|f\|_p^p - \epsilon.
\]
Then
\[ \|f\|_{\alpha,p}^p = c(\alpha) \int_0^1 r^{n-1} \Phi^n_\alpha(r)(1 - r^2)^{-n} dr \int_S |f(r\zeta)|^p d\sigma(\zeta) \]
\[ \geq c(\alpha) \int_1^{r_1} r^{n-1} \Phi^n_\alpha(r)(1 - r^2)^{-n} dr \int_S |f(r\zeta)|^p d\sigma(\zeta) \]
\[ \geq c(\alpha) \int_1^{r_1} r^{n-1} \Phi^n_\alpha(r)(1 - r^2)^{-n} dr (\|f\|_p^p - \epsilon). \]

By letting \( \alpha \to 1^+ \), and noting that
(6.16) \[ 1 > c(\alpha) \int_1^{r_1} r^{n-1} \Phi^n_\alpha(r)(1 - r^2)^{-n} dr \to 1, \]
when \( \alpha \to 1^+ \), we get
\[ \lim \inf_{\alpha \to 1^+} \|f\|_{\alpha,p}^p \geq \|f\|_p^p - \epsilon \]
and this finishes the proof of (6.10).

By using the same method as in the first part of the proof precisely (6.14) with \( p\alpha \) instead of \( \alpha \), by using this time that \( f \) is bounded we get, we have
\[ \|f\|_{\alpha,p} \leq \|f\|_\alpha. \]

Then by using the Lesbegue dominated theorem we obtain
(6.17) \[ \lim \sup_{\alpha \to 1^+} \|f\|_{\alpha,p} \leq \|f\|_p. \]

The opposite inequality is much easier and follows from Jensen’s inequality. Indeed
\[ \int_S |f(r\zeta)|^{p\alpha} d\sigma(\zeta) \geq \left( \int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^\alpha. \]

Hence for \( r_1 \) satisfying (6.15) we have
\[ \|f\|^{p\alpha}_{\alpha,p} = c(\alpha) \int_0^{r_1} r^{n-1} \Phi^n_\alpha(r)(1 - r^2)^{-n} dr \int_S |f(r\zeta)|^{p\alpha} d\sigma(\zeta) \]
\[ \geq c(\alpha) \int_1^{r_1} r^{n-1} \Phi^n_\alpha(r)(1 - r^2)^{-n} dr \left( \int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^\alpha. \]

By (6.15) and (6.16) we obtain
\[ \lim \inf_{\alpha \to 1^+} \|f\|^{p\alpha}_{\alpha,p} \geq (\|f\|_p^p - \epsilon)^\alpha. \]

This finishes the proof of the proposition.

\[ \square \]

**Proposition 6.7.** Let \( p > 0 \) and let \( f \in h^p \). Then
(6.18) \[ \lim_{\alpha \to 1^+} \|f\|_{\alpha,p} = \|f\|_p. \]

**Proof of Proposition 6.18** We repeat the previous proof. To get (6.17), we use Theorem 1.7 more specifically (1.12). \[ \square \]
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