Robust portfolio selection has become a popular problem in recent years. In this study, we focus on the purpose of risk control and seek to minimize the probability of lifetime ruin. This study is motivated by the work of [3], except that we use a standardized penalty for ambiguity aversion. The advantage of taking a standardized penalty is that the closed-form solutions to both the robust investment policy and the value function can be obtained. More interestingly, we use the “Ambiguity Derived Ratio” to characterize the existence of model ambiguity which significantly affects the optimal investment policy. Finally, several numerical examples are given to illustrate our results.

1. Introduction. In recent decades, many models depicting individuals’ behavioral dynamics have been developed. Merton [12, 13] investigated the optimal consumption and portfolio policies in continuous-time models, and as a result, this researcher has become one of the pioneers in the theory of portfolio selection. Since these early publications, many researchers have devoted their careers to relaxing the assumptions of Merton’s model to generalize the model and increase its relevance to real-world applications. To name a few, Browne [4] extended the discussion in Merton [13] to a dynamic active portfolio management problem in which the objective is to analyze the performance of a portfolio compared with a given benchmark. Young [19] studied the optimal investment strategy of an individual who aims to minimize the probability of lifetime ruin. Zhang et al. [21] investigated a portfolio
selection problem in an enlarged Markovian regime-switching model. Meng et al. [11] considered optimal portfolio strategies in a continuous-time economy, in which the price dynamics of a risky asset are governed by a continuous-time self-exciting threshold model. Other extensions can be found in [14, 20, 2, 15, 18, 22, 23], etc.

In most of the previous literature on this topic, one fundamental assumption is that individuals know the true probability distribution of the financial assets’ returns. In fact, an agent formulates a reference model of the probability measure based on the data available from the financial market; however, this reference model is only an approximation to the true model and will result in some inevitable bias. Therefore, in recent years, many researchers considered optimal investment policies under the existence of model ambiguity. For example, Anderson et al. [1] used a statistical theory of detection and a robust control theory to handle the decision maker’s approximating model, which would otherwise be subject to mis-specification. Maenhout [10] presented a new approach to the dynamic portfolio and consumption problem of an investor who worries about model ambiguity and seeks for robust policies, along the lines of what Anderson et al. [1] previously presented. In particular, the penalty function in Maenhout [10] is standardized. Uppal and Wang [16] developed a framework that allows for ambiguity in the joint distribution of returns for all stocks being considered for the portfolio, as well as for different levels of ambiguity for the marginal distribution of returns for any subset of these stocks. In these previous works, the authors applied the relative entropy penalty to construct the framework of the robust control problem; by subjecting the model to expected utility and using the principle of dynamic programming, these authors obtained the robust optimal investment policies for the investigated scenarios. For further details of robust control, see Hansen and Sargent [6, 7, 8].

Instead of using a conventional value function such as the utility of terminal wealth maximization, Bayraktar and Zhang [3] studied the optimal robust investment strategy from the perspective of the probability of lifetime ruin. In their model, the relative entropy is taken as the penalty to build the value function. Applying the principle of dynamic programming, the authors characterized the value function as the unique classical solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation. Unfortunately, explicit solutions to the value function and the optimal policy are not obtained in the general case, so the authors only give the existence and uniqueness of the viscosity solution.

In this paper, we borrow the framework of [3] to consider the optimal investment policy for an individual who worries about model ambiguity and seeks to minimize the probability of lifetime ruin. In this paper, we present a new form of the value function in which a standardized penalty on model ambiguity is used to the probability of lifetime ruin. As a result, the closed-form solutions to both the robust investment policy and the value function can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function. The reason for taking a standardized penalty is that this approach can convert the penalty to units of the value function and gives meaning to the penalty in the definition of the value function. We can find a similar penalty but on the value function of expected utility in [10, 16, 17] and so on.

The rest of the paper is organized as follows. In Section 2, we provide the model formulation. In Section 3, we define the value function with a standardized penalty. To do so, we derive the HJB equation satisfied by the value function, and then
we give the explicit solutions of optimal investment policy and the value function. Finally, in Section 4, we provide several numerical examples to illustrate our results.

2. The basic problem. We start with a complete probability space \((\Omega_1, \mathcal{F}_1, P_1)\) with filtration \(\{\mathcal{F}_1(t)\}_{t \geq 0}\) that satisfies the usual condition. \(B = \{B(t)\}_{t \geq 0}\) is a standard Brownian motion in this probability space. In addition, \(N = \{N(t)\}_{t \geq 0}\) is a time-homogeneous Poisson process defined in another probability space \((\Omega_2, \mathcal{F}_2, P_2)\). Assume that \(\iota\) is the first time that the Poisson process jumps, which follows an exponential distribution with a constant hazard rate \(\lambda > 0\). Assume that the two processes \(B\) and \(N\) are independent, then \(\iota\) is independent of \(B(t)\). In this paper, \(\iota\) serves as the random death time of a person. We construct a new probability space as

\[
(\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2),
\]

where \(P(A \times B) = P_1(A)P_2(B)\), for \(A \in \mathcal{F}_1, B \in \mathcal{F}_2\).

2.1. The wealth process. Only one risk-free asset and one risky asset are assumed to be present in the financial market. The price process \(A(t)\) of the risk-free asset satisfies

\[
dA(t) = rA(t)dt,
\]

where \(r \geq 0\) is the risk-free interest rate force. Furthermore, the price process \(S(t)\) of the risky asset satisfies a geometric Brownian motion:

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t),
\]

where both \(\mu\) and \(\sigma\) are positive constants, and \(B(t)\) is a standard Brownian motion. To avoid triviality, we assume \(\mu > r\).

We assume that an individual, invests all of his wealth into the financial market during his lifetime. Short selling and borrowing at risk-free interest rates are allowed. We denote by \(\iota\) the death time of the investor. Naturally we can assume that \(\iota\) is independent of \(B(t)\). In addition, the total amount of money invested in the risky asset at time \(t\) is denoted as \(f_t\). Therefore, the wealth process of the individual can be written as

\[
dW(t) = [rW(t) + (\mu - r)f_t - c]dt + \sigma f_t dB(t), \quad W(0) = \omega,
\]

where \(c > 0\) means that the individual consumes at a constant rate. In fact, within this model Young [19] investigated the optimal investment policy to minimize the probability of lifetime ruin under a true probability measure \(P\). In [19], the objective function is

\[
V_1(\omega) = \inf_{\bar{f}} P[\tau_a < \iota \mid W(0) = \omega],
\]

where \(\tau_a = \inf\{t \geq 0 : W(t) \leq a\}\) represents the first time at which the wealth falls below a specified ruin level \(a\), \(\omega \geq a\), and \(\bar{f}\) is an admissible policy. A basic assumption behind this problem is that individuals precisely and completely understand the true probability measure \(P\). However, as many authors have argued, this assumption is too strong, and thus, here we assume that the individual worries about misspecification of the model in the financial market. Therefore, we must study the optimal investment policy under model ambiguity. In the following subsection, we present the model ambiguity in our optimal control problem.
2.2. Model ambiguity. We know that the probability measure $P$ is constructed from a limited amount of information collected from the financial market. We call this $P$ the reference probability or reference model. Obviously, individuals are not sure whether $P$ is correct or whether $P$ may be subject to misspecification error with regard to the financial market. Therefore, individuals would consider alternative models. In this paper, for simplicity, we only consider the ambiguity of the financial model and assume that the death time random variable $\iota$ is always exponentially distributed with a constant hazard rate $\lambda$. Because the death time is independent of the financial market, we can choose alternative models that will not change the distribution of the death time $\iota$. In addition, the considered alternative models should be similar to the reference model, so that we define the alternative models by a class of probability measures that are equivalent to $P$ and are not able to change $\iota$, i.e.,

$$ Q := \{ Q | Q \sim P; Q \text{ isn’t able to change the distribution of } \iota \}. $$

Because $Q \in Q$ is equivalent to $P$ and is not able to change $\iota$, we can denote $Q = Q_1 \times P_2$; then, by applying Girsanov’s theorem (Klebaner [9]), $Q$ and $Q_1$ should satisfy

$$ \frac{dQ}{dP}(B[0,t]) = \frac{dQ_1}{dP_1}(B[0,t]) = \Lambda(t), \quad t \geq 0; \quad (4) $$

where

$$ \Lambda(t) = \exp \left\{ \int_0^t m(s)dB(s) - \frac{1}{2} \int_0^t [m(s)]^2 ds \right\} $$

is a $P$-martingale with filtration $\{ F_t \}_{t \geq 0}$, and $m(t)$ is a regular adapted process satisfying Novikov’s condition, i.e.,

$$ E^P \left[ \exp \left( \frac{1}{2} \int_0^t (m(s))^2 ds \right) \right] < \infty, \quad t \geq 0. $$

Therefore, we can rewrite $Q$ as

$$ Q = \{ Q | Q = Q_1 \times P_2 \}, $$

where $Q_1$ satisfies (4). By Girsanov’s theorem, the standard Brownian motion $B(t)$ under probability measure $P$ can be represented as

$$ dB(t) = m(t)dt + dB^Q(t), \quad (5) $$

where $B^Q(t)$ is the standard Brownian motion under probability measure $Q$. Then, (1) can also be written as

$$ \frac{dS(t)}{S(t)} = [\mu + \sigma m(t)]dt + \sigma dB^Q(t), \quad (6) $$

where $m(t)$ can be served as a drift distortion. Inserting (5) into (2), we can rewrite the wealth process of the individual as

$$ dW(t) = [rW(t) + (\mu - r + \sigma m(t))f_t - c]dt + \sigma f_t dB^Q(t), \quad W(0) = \omega \geq a. \quad (7) $$

For the purpose of considering the alternative model $Q$, we must measure the discrepancy between each alternative model and reference model by using the relative entropy. The relative entropy is a well-established approach in measuring the discrepancy between probability distributions.
discrepancy between probability measures $Q$ and $P$. For example, see [10, 17]. The relative entropy between $Q$ and $P$ is defined by

$$H_{[0,t]}(Q \parallel P) = E_{[0,t]}^Q \left[ \ln \frac{dQ}{dP} \right] = E^Q \left\{ \int_0^t m(s)dB(s) - \frac{1}{2} \int_0^t [m(s)]^2 ds \right\} = E^Q \left\{ \int_0^t m(s)dB^Q(s) + \frac{1}{2} \int_0^t [m(s)]^2 ds \right\}.$$  

Because $B^Q(t)$ is the standard Brownian motion under probability measure $Q$, we have

$$H_{[0,t]}(Q \parallel P) = E^Q \left\{ \int_0^t \frac{1}{2} [m(s)]^2 ds \right\} := E^Q \left\{ \int_0^t Z(s) ds \right\}, \quad (8)$$

where $Z(s) = \frac{1}{2} [m(s)]^2$. Hence, $Z(t)$ measures $H_{[0,t]}(Q \parallel P)$. If the individual rejects the reference model $P$ and accepts the alternative model $Q$, a penalty will be incurred. Obviously, the larger that $H_{[0,t]}(Q \parallel P)$ is, then the larger the penalty should be. In the next section, we formulate the robust control problem in terms of the relative entropy.

3. **Minimize the probability of lifetime ruin with ambiguity.** Following [19], Bayraktar and Zhang [3] investigated the minimization of the probability of lifetime ruin under ambiguity aversion with the objective function defined as

$$V_2(\omega) = \inf_{f \in V} \sup_{Q \in Q} \left\{ Q_\omega(\tau_a < \iota) - \frac{1}{\epsilon} H_{[0,\iota]}(Q \parallel P) \right\}, \quad (9)$$

where the subscript $\omega$ represents the initial wealth $W(0) = \omega$ and the time of ruin is defined as $\tau_a = \inf \{ t \geq 0 : W(t) \leq a \}$. Clearly, the penalty function in (9) is $\frac{1}{\epsilon} H_{[0,\iota]}(Q \parallel P)$. Building off this approach, Bayraktar and Zhang [3] established the existence and uniqueness of the value function in terms of the viscosity solution. Unfortunately, the closed-form solution does not exist for the general case, such that the authors only obtained the closed-form solution when the hazard rate of $\iota$ was $\lambda = 0$, i.e., $\iota = \infty$. Distinct from the penalty used previously, in this paper, we incorporate a standardized penalty function; that is also used in [10, 16, 17], among others. The advantage of taking a standardized penalty is that we can obtain the closed-form solutions to both the optimal investment policy and the value function in a more general case. Instead, the objective function in our work is defined as

$$V(\omega) = \inf_{f \in V} \sup_{Q \in Q} \left\{ Q_\omega(\tau_a < \iota) - E^Q_\omega \left[ \int_0^{\iota \wedge \tau_a} \xi \phi(V(W(s))) Z(s) ds \right] \right\}, \quad (10)$$

where $Q_\omega(\cdot) = Q(\cdot | W(0) = \omega)$, $E^Q_\omega(\cdot) = E^Q[\cdot | W(0) = \omega]$, $\phi(\cdot) > 0$ is a normalization factor that converts the penalty to the same order of magnitude as the order of $V(\omega)$, and the constant $\xi > 0$ denotes the degree of the decision maker’s confidence in reference model $P$. The larger that $\xi$ is, the greater the decision maker’s confidence in $P$. In addition, $V$ is the set of admissible policies, detailed later in this work. The sup term reflects individuals’ aversion to ambiguity, which implies that the individual is conservative such that he will consider the most robust results with ambiguity. In the extreme case, $\xi \rightarrow \infty$ implies that the use of any alternative model will result in a heavy penalty, meaning that the individual is extremely confident in the reference model. In such a case, no model ambiguity exists, and the problem reduces to (3). However, $\xi \rightarrow 0$ means that the individual
has no more information regarding the reference model compared with other models. Hence, we assume $0 < \xi < \infty$ in the following study.

**Definition 3.1.** We say that $\mathbb{V} = \{f_t, t \geq 0\}$ is an admissible set, if $f := \{f_t, t \geq 0\}$ is a predictable process and satisfies $E^Q \left[\int_0^\infty f_s^2 ds\right] < \infty$, $t \geq 0$, $Q \in \mathcal{Q}$. $f$ is called an admissible investment policy if $f \in \mathbb{V}$. Note that both short selling and borrowing are allowed here.

According to the definition of the value function (10), it is obvious that $V(a) = 1$. Furthermore, we can see that when the individual’s initial wealth $\omega \geq \frac{c}{r}$, the individual can invest all of his money in bonds to earn risk-free interest and allowing his consumption to be completely sustained by the interest. In this case, ruin in the individual’s lifetime will not occur, so we can show that $V(\omega) = 0$ for all $\omega \geq \frac{c}{r}$.

For an admissible investment policy $f \in \mathbb{V}$ and any function $g(\cdot) \in C^{1,2}$, we define a generator corresponding to (7) as

$$A^f g(\omega) = \left[r\omega + (\mu - r + \sigma m)f - c\right]g'(\omega) + \frac{1}{2}\sigma^2 f^2 g''(\omega).$$

For simplicity, we assume that the death time of the individual $\iota$ is exponentially distributed with a constant hazard rate $\lambda > 0$. According to the technique of the principle of dynamic programming, if $V(\omega) \in C^{1,2}$, then we can show that the value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation (see [5])

$$\lambda V(\omega) = \inf_{f \in \mathbb{V}} \sup_m \left\{A^f V(\omega) - \frac{1}{2}\xi \phi(V(\omega))m^2\right\}$$

(11)

and the boundary conditions $V(a) = 1$ and $V(\xi) = 0$.

**Lemma 3.2.** (Verification Theorem) If $\nu(\omega) \in C^{1,2}$ is the solution to the HJB equation (11) with the boundary conditions $\nu(a) = 1$ and $\nu(\xi) = 0$, then $\nu(\omega) = V(\omega)$.

**Proof.** See Appendix A.

In the following, we provide closed-form solutions to the optimal problem (10). For analytical convenience, we take a special case

$$\phi(x) = x, \quad x \geq 0.$$  

(12)

As mentioned by [10], the reason for taking the special case is to ensure the homotheticity or scale invariance of the value function and, therefore, to ensure that the the optimal decision problem (10) has a natural economic justification. Because the relative entropy $\frac{1}{2}[m(s)]^2$ is actually unitless, a suitable form of $\phi(x) = x$ can convert the penalty to the same order of magnitude as the order of the value function.

Substituting (12) into (11), we have the HJB equation

$$\lambda V(\omega) = \inf_{f \in \mathbb{V}} \sup_m \left\{\left[r\omega + (\mu - r + \sigma m)f - c\right]V'(\omega)ight.\right.$$

$$\left.\left.\quad + \frac{1}{2}\sigma^2 f^2 V''(\omega) - \frac{1}{2}\xi \phi(V(\omega))m^2\right\}.$$  

(13)

for $a < \omega < \xi$. To solve the HJB equation (13), we firstly assume that $V(\omega) > 0$ for $a < \omega < \xi$. In accordance with the first-order conditions, we can obtain that
$m^*$ has the following form.

$$m^* = f \frac{\sigma V'(\omega)}{\xi V(\omega)}.$$  \hfill (14)

Combining (13) with (14) yields

$$\lambda V(\omega) = \inf_{f \in V} \left\{ \left[ r\omega + (\mu - r)f - c \right] V'(\omega) + \frac{1}{2} \sigma^2 f^2 \left[ V''(\omega) + \frac{[V'(\omega)]^2}{\xi V(\omega)} \right] \right\}.$$  \hfill (15)

Second, we assume that $V''(\omega) + \frac{[V'(\omega)]^2}{\xi V(\omega)} > 0$ for $a < \omega < \frac{c}{r}$. Using the first-order condition again, we have

$$f^* = -\frac{(\mu - r)}{\sigma^2} \frac{V'(\omega)}{V''(\omega) + \frac{[V'(\omega)]^2}{\xi V(\omega)}}.$$  \hfill (16)

Substituting (16) into (15), we have

$$\lambda V(\omega) = \left[ r\omega - c \right] V'(\omega) - \frac{1}{2} \frac{(\mu - r)^2 [V'(\omega)]^2}{\sigma^2 (V''(\omega) + \frac{[V'(\omega)]^2}{\xi V(\omega)})}.$$  \hfill (17)

To obtain the optimal policies, we should have an explicit solution to (17). With the boundary conditions $V(a) = 1$ and $V(\frac{c}{r}) = 0$, we speculate that the value function has the following form, for some $d > 0$,

$$V(\omega) = \left( \frac{c - r\omega}{c - ra} \right)^d, \quad a \leq \omega \leq \frac{c}{r}.$$  \hfill (18)

Therefore, we know that

$$V'(\omega) = -\frac{dr}{(c - rw)} V(w),$$  \hfill (19)

$$V''(\omega) = \frac{d(d - 1)r^2}{(c - rw)^2} V(w).$$  \hfill (20)

Substituting (18), (19) and (20) into (17) yields

$$\lambda = rd - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{d}{d - 1 + \frac{d}{\xi}}.$$  \hfill (21)

i.e., $d$ satisfies the following quadratic equation

$$r(1 + \frac{1}{\xi})d^2 - [r + D + \lambda(1 + \frac{1}{\xi})]d + \lambda = 0,$$  \hfill (21)

where $D = \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2$. Note that this equation has a positive discriminant,

$$\Delta : = [r + D + \lambda(1 + \frac{1}{\xi})]^2 - 4\lambda r(1 + \frac{1}{\xi})$$

$$= [\lambda(1 + \frac{1}{\xi}) + D - r]^2 + 4rD > 0.$$  \hfill (22)

Thus (21) determines two positive roots

$$d_1 = \frac{r + D + \lambda(1 + \frac{1}{\xi}) - \sqrt{\Delta}}{2r(1 + \frac{1}{\xi})}$$

and

$$d_2 = \frac{r + D + \lambda(1 + \frac{1}{\xi}) + \sqrt{\Delta}}{2r(1 + \frac{1}{\xi})}.$$
To guarantee that $V''(\omega) + \frac{|V'(\omega)|^2}{\xi V(\omega)} > 0$ follows, by (18), (19) and (20) we should have $d > \frac{1}{1+\xi}$. Because

$$d_1 < \frac{r + D + \lambda(1 + \frac{1}{\xi}) - |\lambda(1 + \frac{1}{\xi}) + D - r|}{2r(1 + \frac{1}{\xi})} \leq \frac{1}{1+\xi},$$

$$d_2 > \frac{r + D + \lambda(1 + \frac{1}{\xi}) + |\lambda(1 + \frac{1}{\xi}) + D - r|}{2r(1 + \frac{1}{\xi})} \geq \frac{1}{1+\xi},$$

it follows that

$$d = d_2 = \frac{r + D + \lambda(1 + \frac{1}{\xi}) + \sqrt{\Delta}}{2r(1 + \frac{1}{\xi})}. \tag{23}$$

Then the optimal investment policies $f^*$ can be represented as

$$f^*(w) = \frac{\mu - r}{\sigma^2} \frac{\frac{c}{r} - w}{d(1 + \frac{1}{\xi}) - 1}, \quad a \leq w \leq \frac{c}{r}. \tag{24}$$

Substituting (18), (19) and (24) into (14), we know that

$$m^* = -\frac{\mu - r}{\sigma} \frac{1}{\xi(1 - \frac{1}{d}) + 1},$$

and whether Novikov’s condition is satisfied can be readily verified.

Thus far, we have obtained a solution (18) to the HJB equation (13) and the optimal policies associated with the solution. According to the verification theorem, we have the following theorem:

**Theorem 3.3.** For problem (10), the optimal investment policy is

$$f_t^* = \begin{cases} \frac{(\mu - r)(\frac{c}{r} - W(t))}{d(1 + \frac{1}{\xi}) - 1}, & a \leq W(t) \leq \frac{c}{r}; \\ 0, & W(t) > \frac{c}{r}. \end{cases}$$

And the value function is

$$V(\omega) = \begin{cases} \left(\frac{c - r\omega}{c - ra}\right)^d, & a \leq \omega \leq \frac{c}{r}; \\ 0, & \omega > \frac{c}{r}; \end{cases}$$

where $d$ is defined by (23).

**Remark 1. (Ambiguity Derived Ratio)** When the individual has no ambiguity in the financial market, we know from Young [19] that the optimal investment policy is

$$\tilde{f}^* = \frac{\mu - r}{\sigma^2} \frac{(\frac{c}{r} - \omega)}{(p - 1)},$$

where

$$p = \frac{r + D + \lambda + \sqrt{(r + D + \lambda)^2 - 4r\lambda}}{2r} > 1.$$

When there exists model ambiguity in the financial market, we know from (24) that the optimal investment policy

$$f^* = \tilde{f}^* \frac{(p - 1)}{d(1 + \frac{1}{\xi}) - 1}.$$
Therefore, we can define the Ambiguity Derived Ratio (ADR) as

\[
ADR = \frac{(p - 1)}{d(1 + \frac{1}{\xi}) - 1}.
\]

Note that

\[
d(1 + \frac{1}{\xi}) - 1 - (p - 1) = \frac{\lambda^{\frac{1}{\xi}} + \sqrt{(r + D + \lambda(1 + \frac{1}{\xi}))^2 - 4r\lambda(1 + \frac{1}{\xi}) - \sqrt{(r + D + \lambda)^2 - 4r\lambda}}}{2r} \geq 0,
\]

so we have \(ADR \leq 1\).

As we can see from (24), what robustness \((ADR \leq 1)\) does is to make the individual less willing to invest in risky assets. The optimal investment amount in the risky asset is proportionally reduced due to the model ambiguity. And the optimal investment amount is increasing with respect to the parameter \(\xi\), which means that the more confidence the investor has in the market, the more money he/she is willing to invest in it.

In addition, we observe from (22)-(24) that the optimal investment policy with considering model ambiguity is exactly the same as the optimal policy without considering model ambiguity when the individual’s hazard rate increases from \(\lambda\) to \(\lambda(1 + \frac{1}{\xi})\). Therefore, model ambiguity on hazard rate is another interesting topic.

**Remark 2.** (The concavity and convexity of the value function) Note that the concavity and convexity of the value function \(V(w)\) depend on the sign of \(d - 1\).

In fact, we can see that the value function \(V(w)\) can be either convex or concave for \(a \leq w \leq c\).

(i) For \(\lambda \geq r\),

\[
d - 1 = \frac{r + D + \lambda(1 + \frac{1}{\xi}) + \sqrt{\Delta}}{2r(1 + \frac{1}{\xi})} - 1 \geq \frac{D + \lambda(1 + \frac{1}{\xi})}{r(1 + \frac{1}{\xi})} - 1 > 0.
\]

Hence, \(V''(\omega) > 0\), and \(V(\omega)\) is strictly convex.

(ii) For \(\lambda < r\),

\(\triangleright\) if \(\xi > \frac{r - \lambda}{D}\), then \(V(\omega)\) is strictly convex;

\(\triangleright\) if \(\xi = \frac{r - \lambda}{D}\), then \(V(\omega)\) is a linear function;

\(\triangleright\) if \(\xi < \frac{r - \lambda}{D}\), then \(V(\omega)\) is strictly concave.

The proof is given in Appendix B. The concavity and convexity of the value function can be understood as follows. As we can see, the optimization problem (10) is a max-min problem. When the parameter \(\xi\) is large, the individual is more confident in the reference model. So the dominate part in (10) is to find optimal investment policy to minimize the lifetime ruin probability, which leads to a convex value function. On the other hand, if the parameter \(\xi\) is small, the individual knows little about the reference model. Then the dominate part in (10) is to find a worst-scenario to maximize the lifetime ruin probability, which leads to a concave value function.

**Remark 3.** \((\iota = \infty)\) If the risk of death does not exist, i.e., \(\lambda = 0\), then we can see that the optimal investment policy reduces to

\[
f^*_t = \begin{cases} \frac{2(c - rW(t))}{\mu - r}, & a \leq W(t) \leq \frac{c}{r}; \\ 0, & W(t) > \frac{c}{r}; \end{cases}
\]
which is the same as the corresponding result of Bayraktar and Zhang [3].

4. **Numerical calculations.** In this section, we present several numerical examples to illustrate the impacts of model ambiguity and the various model parameters on the optimal policy and value function. Unless otherwise stated, we take the following values for the parameters: $a = 1$, $c = 1$, $\mu = 0.1$, $\sigma = 0.15$, $\lambda = 0.04$, and $r = 0.02$ or $0.06$. These parameters are applied as in Bayraktar and Zhang [3].

**Figure 1.** Optimal investment policies with respect to the wealth and the model ambiguity.

In Figure 1, we can see that the optimal investment policy decreases with respect to the wealth $\omega$, which means that the more wealth the individual owns, the less he is willing to invest in risky assets. Once his wealth is larger than the barrier $c/r$, the individual will no longer assume financial risk. From this point of view, we can say that ruin probability is a conservative value function. In addition, from Figures 1 and 5, we can see that the optimal investment policy is increasing with respect to $\xi$. The Ambiguity Derived Ratio (ADR) increases to 1 as $\xi$ increases to $\infty$, and is concave as a function of $\xi$. Thus, we empirically have

$$\frac{\partial f^*}{\partial \xi} \geq 0 \text{ and } \frac{\partial^2 f^*}{\partial \xi^2} \leq 0.$$

This implies that the less confident that the individual is in the financial market, the less he is willing to invest in risky assets. The effect of model ambiguity on the optimal amount of an investment in risky assets decreases with respect to $\xi$. All of these results are consistent with intuition.

**Figure 2** displays the effect of the individual’s lifetime on the optimal investment policy. We can see that the optimal amount of an investment in a risky asset is decreasing with respect to $\lambda$, which means that the lower the individual’s mortality rate is (i.e., the longer the individual is expected to live), the more money is invested in the risky asset. Because greater investment profits can be expected from investment in risky assets, an individual is more likely to invest in such assets to cover his future consumptions. Moreover, the effect of mortality rate $\lambda$ on the optimal investment policy is increasing. Thus, we empirically have

$$\frac{\partial f^*}{\partial \lambda} \leq 0 \text{ and } \frac{\partial^2 f^*}{\partial \lambda^2} \geq 0.$$
In this paper, we fixed the consumption rate of the individual to minimize the probability of lifetime ruin. It would also be interesting to study the optimal consumption rate under an alternative value function.

Figure 3 shows the effect of model ambiguity $\xi$ on the value function $V(\omega)$. We can see that $V(\omega)$ is decreasing with respect to $\xi$. As mentioned in Remark 2, the value function is strictly convex when the risk-free interest rate $r$ is smaller than the hazard rate $\lambda$; if the interest rate is larger than the hazard rate, the concavity or convexity of the value function depends on the level of $\xi$. The value function is strictly convex if $\xi > 0.5625$, strictly concave if $\xi < 0.5625$, and a linear function if $\xi = 0.5625$.

Due to the existence of model ambiguity in the financial market, the individual will adjust the rate of return of a risky asset, depending on how much he trusts the reference model $P$. Now, let’s see how the model ambiguity $\xi$ affects the expected return rate of the risky asset under the robust risk measure. To facilitate this analysis, we rewrite (6) as

$$\frac{dS(t)}{S(t)} = [\mu + \sigma m^*(t)]dt + \sigma dB^Q(t).$$

(25)
Figure 4. Return rate of the risky asset under robust risk measure ($\mu = 0.1$).

From Figure 4, we can see that the drift coefficient of (25) increases to $\mu = 0.1$ as $\xi$ increases to $\infty$. This illustrates that the more confidence the individual has in the financial market, the larger the rate of return the risky asset has, not to exceed the rate $\mu$ referenced in (1).

Figure 5. Ambiguity Derived Ratio with respect to model ambiguity.

Appendix A. Proof of Lemma 3.2.

Proof. This proof is similar to Theorem 6.1 in Bayraktar and Zhang [3]. It is obvious that $\{N(t)\}_{t\geq 0}$ is also a Poisson process with rate $\lambda$ under $Q$. Naturally, $t$ can be served as the first time that the Poisson process jumps. Let $\Xi$ be the “special state”, and $[a,\infty) \cup \Xi$ be the one point compactification of $[a,\infty)$, and $\nu(\Xi) = 0$. Let $\tilde{W}(t) = W(t)1_{\{t<\xi\}} + \Xi1_{\{t\geq \xi\}}$. It is obvious that $\tilde{W}(t)$ can be regarded as the enlargement by $\{N(t)\}_{t\geq 0}$, and $W(t)$ can be written as

$$d\tilde{W}(t) = [rW(t) + (\mu - r + \sigma m(t))f_t - c]dt + \sigma f_t dB^Q(t) + (\Xi - W(t-))dN(t), \quad W(t) = \omega.$$
i) Assuming that \( f^*_t \) satisfies the inf term of \((11)\). We denote that \( \tau^*_a := \inf\{ t : W(t) | f_t^* \leq a \} \) and \( \tau^* := \inf\{ t : W(t) | f_t^* \geq \xi \} \), where \( W(t) | f_t^* \) means that we take value \( f_t^* \) for \( f_t \) in \( W(t) \), and similarly below. Applying Itô’s lemma to \( \nu(W(t) | f_t^*) \), we have

\[
\nu(W(\tau^*_a \land \tau^*) | f_{\tau^*}) = \nu(\omega) + \int_{0}^{\tau^*_a \land \tau^*} \mathcal{A}^f \nu(W(s) | f_{\tau^*}) - \lambda \nu(W(s) | f_{\tau^*}) ds \\
+ \int_{0}^{\tau^*_a \land \tau^*} \nu'(W(s) | f_{\tau^*}) dB^Q(s) \\
- \int_{0}^{\tau^*_a \land \tau^*} \nu(W(s) | f_{\tau^*}) d(N(s) - \lambda s).
\]

We know that there is one \( m(t) \) corresponding to one \( Q \). Thus, taking \( E^Q_w \) on both sides of \((26)\), we have

\[
E^Q_w \left[ \nu(W(\tau^*_a \land \tau^*) | f_{\tau^*}) \right] \\
= \nu(\omega) + E^Q_w \left[ \int_{0}^{\tau^*_a \land \tau^*} \mathcal{A}^f \nu(W(s) | f_{\tau^*}) - \lambda \nu(W(s) | f_{\tau^*}) ds \right].
\]

Because \( f^*_t \) satisfies the inf term of \((11)\), we have

\[
0 = \sup_m \left\{ \mathcal{A}^f \nu(W(s) | f_{\tau^*}, m) - \frac{1}{2} \xi \phi(\nu(W(s) | f_{\tau^*}, m)) m^2(s) \right\} - \lambda \nu(W(s) | f_{\tau^*}, m) \\
\geq \mathcal{A}^f \nu(W(s) | f_{\tau^*}, m) - \frac{1}{2} \xi \phi(\nu(W(s) | f_{\tau^*}, m)) m^2(s) - \lambda \nu(W(s) | f_{\tau^*}, m)
\]

Thus, we have

\[
E^Q_w \left[ \nu(W(\tau^*_a \land \tau^*) | f_{\tau^*}, m) \right] \\
\leq \nu(\omega) + E^Q_w \left[ \int_{0}^{\tau^*_a \land \tau^*} \frac{1}{2} \xi \phi(\nu(W(s) | f_{\tau^*}, m)) m(s)^2 ds \right],
\]

which is

\[
\nu(\omega) \geq E^Q_w \left[ \nu(W(\tau^*_a \land \tau^*) | f_{\tau^*}, m) \right] - \int_{0}^{\tau^*_a \land \tau^*} \frac{1}{2} \xi \phi(\nu(W(s) | f_{\tau^*}, m)) m(s)^2 ds.
\]

We know that \( \nu(W(\tau^*_a \land \tau^*) | f_{\tau^*}, m) = 1_{(\tau^*_a \leq \tau^*)} = 1_{(\tau^*_a \leq \xi)} \), because if \( W(s) | f_{\tau^*}, m \) reaches \( \xi \) it will stay constant until death. In addition, it is obvious that \( \tau^*_a \land \tau^* \leq \tau^*_a \land \xi \). As a result, we have

\[
\nu(\omega) \geq E^Q_w \left[ 1_{(\tau^*_a \leq \xi)} - \int_{0}^{\tau^*_a \land \xi} \frac{1}{2} \xi \phi(\nu(W(s) | f_{\tau^*}, m)) m(s)^2 ds \right].
\]

Because this statement holds for all \( Q \), we have

\[
\nu(\omega) \geq \sup_{Q \in \mathcal{Q}} E^Q_w \left[ 1_{(\tau^*_a \leq \xi)} - \int_{0}^{\tau^*_a \land \xi} \frac{1}{2} \xi \phi(\nu(W(s) | f_{\tau^*})) m(s)^2 ds \right] \\
\geq \inf_{f \in \mathcal{F}} \sup_{Q \in \mathcal{Q}} E^Q_w \left[ 1_{(\tau^*_a \leq \xi)} - \int_{0}^{\tau^*_a \land \xi} \frac{1}{2} \xi \phi(\nu(W(s))) m(s)^2 ds \right] \\
= V(\omega).
\]
ii) Assuming that $m^*(t)$ satisfies the sup term of (11), and there exists a probability measure $Q^*$ corresponding to it, we can denote $\tau_{m^*}^* := \inf\{t : W(t) \mid_{m=m^*} \leq a\}$ and $\tau_{m^*}^{-} := \inf\{t : W(t) \mid_{m=m^*} \geq \frac{a}{2}\} \land t$. Applying Itô’s lemma to $\nu(W(t) \mid_{m=m^*})$, we have

$$
\nu(\widetilde{W}(\tau_{m^*}^* \land \tau_{m^*}^- \land t) \mid_{m=m^*})
= \nu(\omega) + \int_{0}^{\tau_{m^*}^* \land \tau_{m^*}^- \land t} A^{f,m^*} \nu(W(s) \mid_{m=m^*}) - \lambda \nu(W(s) \mid_{m=m^*}) ds
+ \int_{0}^{\tau_{m^*}^* \land \tau_{m^*}^- \land t} \nu'(W(s) \mid_{m=m^*}) dB^{Q^*}(s)
- \int_{0}^{\tau_{m^*}^* \land \tau_{m^*}^- \land t} \nu(W(s) \mid_{f=m^*}) d(N(s) - \lambda s).
$$

(27)

Taking $E_{\omega}^{Q^*}$ on both sides of (27), we have

$$
E_{\omega}^{Q^*} \left[ \nu(\widetilde{W}(\tau_{m^*}^* \land \tau_{m^*}^- \land t) \mid_{m=m^*}) \right]
= \nu(\omega) + E_{\omega}^{Q^*} \left[ \int_{0}^{\tau_{m^*}^* \land \tau_{m^*}^- \land t} A^{f,m^*} \nu(W(s) \mid_{m=m^*}) - \lambda \nu(W(s) \mid_{m=m^*}) ds \right].
$$

Because $m^*(t)$ satisfies the sup term of (11), we have

$$
0 = \inf_{f \in \mathcal{V}} \left\{ A^{f,m^*} \nu(W(s) \mid_{f,m=m^*}) - \frac{1}{2} \xi \phi(\nu(W(s) \mid_{f,m=m^*})) [m^*(s)]^2 \right\}
- \lambda \nu(W(s) \mid_{f,m=m^*})
\leq A^{f,m^*} \nu(W(s) \mid_{f,m=m^*}) - \frac{1}{2} \xi \phi(\nu(W(s) \mid_{f,m=m^*})) [m^*(s)]^2 - \lambda \nu(W(s) \mid_{f,m=m^*})
$$

Therefore, we obtain

$$
E_{\omega}^{Q^*} \left[ \nu(\widetilde{W}(\tau_{m^*}^* \land \tau_{m^*}^- \land t) \mid_{m=m^*}) \right]
\geq \nu(\omega) + E_{\omega}^{Q^*} \left[ \int_{0}^{\tau_{m^*}^* \land \tau_{m^*}^- \land t} \frac{1}{2} \xi \phi(\nu(W(s) \mid_{f,m=m^*})) [m^*(s)]^2 ds \right].
$$

Letting $t \to \infty$, we have

$$
\nu(\omega)
\leq E_{\omega}^{Q^*} \left[ \nu(\widetilde{W}(\tau_{m^*}^* \land \tau_{m^*}^-) \mid_{m=m^*}) - \int_{0}^{\tau_{m^*}^* \land \tau_{m^*}^-} \frac{1}{2} \xi \phi(\nu(W(s) \mid_{f,m=m^*})) [m^*(s)]^2 ds \right].
$$

Because $\nu(\widetilde{W}(\tau_{m^*}^* \land \tau_{m^*}^-) = 1_{\{\tau_{m^*}^* < \tau_{m^*}^-\}} = 1_{\{\tau_{m^*}^* < \iota\}}$, we have

$$
\nu(\omega) \leq E_{\omega}^{Q^*} \left[ 1_{\{\tau_{m^*}^* < \iota\}} - \int_{0}^{\tau_{m^*}^* \land \tau_{m^*}^-} \frac{1}{2} \xi \phi(\nu(W(s) \mid_{f,m=m^*})) [m^*(s)]^2 ds \right].
$$
We can see that if \( \tau^m \wedge \tau^* \leq s < \tau^m \wedge t \), then \( m^*(s) = 0 \), which is due to the fact that \( m(s) = 0 \) if \( W(t) \notin (a, c) \). Therefore,

\[
\nu(\omega) \leq E^Q_w \left[ 1_{(\tau^m \wedge s < \tau^*)} - \int_0^{\tau^m \wedge s} \frac{1}{2} \xi \phi(\nu(W(s) | f, m = m^*)) |m^*(s)|^2 ds \right]
\]

\[
\leq \sup_{Q \in \mathcal{Q}} E^Q_w \left[ 1_{(\tau < s)} - \int_0^{\tau} \frac{1}{2} \xi \phi(\nu(W(s))) |m(s)|^2 ds \right].
\]

This holds for all \( f \), so we have

\[
\nu(\omega) \leq \inf_{f \in V} \sup_{Q \in \mathcal{Q}} E^Q_w \left[ 1_{(\tau < s)} - \int_0^{\tau} \frac{1}{2} \xi \phi(\nu(W(s))) |m(s)|^2 ds \right]
\]

\[
= V(\omega).
\]

Then, we complete the proof. \( \square \)

**Appendix B.**

**Proof.** We just consider the case \( \lambda < r \). It is easy to see that

\[
d - 1 = \frac{r + D + \lambda(1 + \frac{1}{\xi}) + \sqrt{\Delta}}{2r(1 + \frac{1}{\xi})} - 1 > 0
\]

and

\[
\sqrt{\Delta} > 2r(1 + \frac{1}{\xi}) - \lambda(1 + \frac{1}{\xi}) - r - D
\]

are equivalent. By direct calculations, we have

\[
\Delta - [2r(1 + \frac{1}{\xi}) - \lambda(1 + \frac{1}{\xi}) - r - D]^2 = \Delta - [\lambda(1 + \frac{1}{\xi}) + D - r - 2r\frac{1}{\xi}]^2
\]

\[
= 4rD(1 + \frac{1}{\xi})\frac{1}{\xi} [\xi - \frac{r - \lambda}{D}],
\]

which implies that

- if \( \xi > \frac{r - \lambda}{D} \), then \( \Delta > [2r(1 + \frac{1}{\xi}) - \lambda(1 + \frac{1}{\xi}) - r - D]^2 \) and \( d - 1 > 0 \);
- if \( \xi = \frac{r - \lambda}{D} \), then \( \Delta = [2r(1 + \frac{1}{\xi}) - \lambda(1 + \frac{1}{\xi}) - r - D]^2 \) and \( d - 1 = 0 \).
- if \( 0 < \xi < \frac{r - \lambda}{D} \), then \( \Delta < [2r(1 + \frac{1}{\xi}) - \lambda(1 + \frac{1}{\xi}) - r - D]^2 \) and \( d - 1 < 0 \).

In conclusion, for \( \lambda < r \), we obtain that

- if \( \xi > \frac{r - \lambda}{D} \), then \( V(\omega) \) is strictly convex;
- if \( \xi = \frac{r - \lambda}{D} \), then \( V(\omega) \) is a linear function;
- if \( \xi < \frac{r - \lambda}{D} \), then \( V(\omega) \) is strictly concave. \( \square \)

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