Detection of the number of principal components by maximizing the infimum of the log-likelihood function and generalized AIC-type method

JIANWEI HU AND JI ZHU
Central China Normal University & University of Michigan

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Abstract

Estimating the number of principal components is one of the fundamental problems in many scientific fields such as signal processing (or the spiked covariance model). In this paper, we first derive the asymptotic expansion of the log-likelihood function and its infimum. Then we select the number of signals $k$ by maximizing the order of the infimum of the log-likelihood function (MIL). We demonstrate that the MIL is consistent under the condition that $\text{SNR}/\sqrt{4\gamma(p-k/2+1/2)\log \log n/n}$ is uniformly bounded away from below by 1. By re-examining the BIC (or the MDL) for the spiked covariance model, we find that the BIC is not suitable for the spiked covariance model, unless $\text{SNR}/\sqrt{2(p-k/2+1/2)\log n/n}$ is uniformly bounded away from below by 1. Compared with the BIC, the MIL is consistent at much lower SNR. Moreover, we demonstrate that any penalty term of the form $k'(p-(k'-1)/2)C_n$ may lead to an asymptotically consistent estimator under the condition that $C_n \to \infty$ and $C_n/n \to 0$. Compared with the condition in Zhao, Krishnaiah and Bai (1986), i.e., $C_n/\log \log n \to \infty$ and $C_n/n \to 0$, this condition is significantly weakened. We also extend our results

*Jianwei Hu, Department of Statistics, Central China Normal University, Wuhan, P.R. China, 430079, Email: jwhu@mail.ccnu.edu.cn. Ji Zhu, Department of statistics, University of Michigan, Ann Arbor, Michigan, USA, Email: jizhu@umich.edu.
to the case $n, p \rightarrow \infty$, with $p/n \rightarrow c > 0$. At low SNR, since the AIC tends to underestimate the number of signals $k$, the AIC should be re-defined in this case. As a natural extension of the AIC for fixed $p$, we propose the generalized AIC (GAIC), i.e., the AIC-type method with tuning parameter $\gamma = \varphi(c) = 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c$, and demonstrate that the GAIC-type method, i.e., the AIC-type method with tuning parameter $\gamma > \varphi(c)$, can select the number of signals $k$ consistently. Moreover, we show that the GAIC-type method is essentially tuning-free and outperforms the well-known KN estimator proposed in Kritchman and Nadler (2008) and the BFC estimator proposed in Bai, Fujikoshi and Choi (2017). Numerical studies indicate that the proposed method works well.

**KEY WORDS:** Consistency; minimum description length; signal-to-noise ratio; spiked covariance model; Tracy-Widom distribution.

1 Introduction

Detection of the number of principal components from a noisy data is one of the fundamental problems in many scientific fields. It is often the starting point for the signal parameter estimation problem such as signal processing (Wax and Kailath, 1985), wireless communications (Nicoli, Simeone and Spagnolini, 2003), array processing (Bohme, 1991) and finance (Bai and Ng, 2002) to list a few and has primary importance (Anderson, 2003).

In the signal processing literature, the most common approach to solving this problem is by using information theoretic criteria, and in particular minimum description length (MDL) (Rissanen, 1978), Bayesian information criterion (BIC) and Akaike information criterion (AIC) (Wax and Kailath, 1985). Although Zhao, Krishnaiah and Bai (1986) proved that the BIC (or the MDL) is consistent, the BIC fails to detect signals at low signal-to-noise ratio (SNR), hence underestimating the number of signals at small sample size. The reason lies in that in the proof of consistency of the BIC in Zhao, Krishnaiah and Bai (1986), the SNR was implied to be uniformly bounded away from below by 0 and thus was not allowed
to converge to 0. As a result, Zhao, Krishnaiah and Bai (1986) did not give the accurate condition under which the BIC is not consistent. In contrast, while the AIC is able to detect low SNR signals, it has a non-negligible probability to overestimate the number of signals and thus is not consistent. To remedy the shortcoming of the AIC, Nadler (2010) proposed an modified AIC. The only difference between these two methods lies in that the penalty of the modified AIC is two times that of the AIC. In the presence of pure noise with no signals (i.e., \( k = 0 \)), Nadler (2010) showed that the modified AIC had a negligible overestimation probability for large \( n \). However, for \( k > 0 \), Nadler (2010) did not give an explanation why the modified AIC had a negligible overestimation probability. On the other hand, for small and medium \( n \), the modified AIC tends to underestimate the number of signals \( k \) in our simulations.

Kritchman and Nadler (2008) and Kritchman and Nadler (2009) proposed a very different method for estimating the number of signals, via a sequence of hypothesis tests, at each step testing the significance of the \( k \)th eigenvalue as arising from a signal. The main tools used in these two papers are recent results from random matrix theory regarding both the distribution of noise eigenvalues and of signal eigenvalues in the presence of noise. In the absence of signals, the matrix \( nS \) follows a Wishart distribution (Wishart, 1928) with parameter \( n, p \). In the joint limit \( n, p \to \infty \), with \( p/n \to c > 0 \), the distribution of the largest eigenvalue of \( S \) converges to a Tracy-Widom distribution (Johansson, 2000; Johnstone, 2001; El Karoui, 2006; Johnstone and Lu, 2009; Ma, 2012). For fixed \( p \), although Kritchman and Nadler (2009) proved the strong consistency of their estimator, they did not give an explicit condition that the SNR should satisfy. On the other hand, our simulations show that when the SNR is low, for fixed \( p \) and medium \( n \), the KN tends to underestimate the number of signals \( k \).

Before further proceeding, we mention that a similar, if not identical problem, also appears in other literatures, likelihood ratio test statistic (Muirhead, 2002), Kac-Rice test (Choi, Taylor and Tibshirani, 2017). For some other work in this topic, we refer to Paul
Throughout this paper, we assume that the number of signals $k$ is fixed, unless otherwise stated. By re-examining the BIC for the spiked covariance model, we find that the BIC (or the MDL) is not suitable for the spiked covariance model, unless $\frac{SNR}{\sqrt{2(p - k/2 + 1/2)}\log n/n}$ is uniformly bounded away from below by 1, i.e., there exists a sufficiently small $\epsilon > 0$ such that for any $n$, we have $\frac{SNR}{\sqrt{2(p - k/2 + 1/2)}\log n/n} > 1 + \epsilon$. This implies that when the SNR is low, the BIC tends to underestimate the number of signals $k$.

In this paper, we first derive the asymptotic expansion of the log-likelihood function. Then we select the number of signals $k$ by maximizing the order of the infimum of the log-likelihood function (MIL). We demonstrate that the MIL is consistent under the condition that $\frac{SNR}{\sqrt{4\gamma(p - k/2 + 1/2)\log\log n/n}}$ is uniformly bounded away from below by 1, where $\gamma > 0$ is a tuning parameter and we recommend using $\gamma = 1$ in simulations. Compared with the BIC, the MIL is consistent at much lower SNR. Moreover, we demonstrate that any penalty term of the form $k'(p - (k' - 1)/2)C_n$ may lead to an asymptotically consistent estimator under the condition that $C_n \to \infty$ and $C_n/n \to 0$. Compared with the condition in Zhao, Krishnaiah and Bai (1986), i.e., $C_n/\log\log n \to \infty$ and $C_n/n \to 0$, this condition is significantly weakened. We also extend our results to the case $n, p \to \infty$, with $p/n \to c > 0$. At low SNR, since the AIC tends to underestimate the number of signals $k$, the AIC should be re-defined in this case. As a natural extension of the AIC for fixed $p$, we propose the generalized AIC (GAIC), i.e., the AIC-type method with tuning parameter $\gamma = \varphi(c) = 1/2 + \sqrt{1/c - \log(1 + \sqrt{c})}/c$, and demonstrate that the GAIC-type method, i.e., the AIC-type method with tuning parameter $\gamma > \varphi(c)$, can select the number of signals $k$ consistently. Moreover, we show that the GAIC-type method is essentially tuning-free and outperforms the well-known KN estimator proposed in Kritchman and Nadler (2008) and the BFC estimator proposed in Bai, Fujikoshi and Choi (2017).
For the remainder of the paper, we proceed as follows. In Section 2, we derive the asymptotic expansion of the log-likelihood function and propose the MIL to determine the number of signals. In Section 3, we establish the consistency of the estimator for the number of signals. We extend our results to the case $n, p \to \infty$, with $p/n \to c > 0$ in Section 4. The numerical studies are given in Section 5. Some further discussions are made in Section 6. All proofs are given in Section 7.

2 Maximizing the infimum of the log-likelihood function

Consider the model

$$x(t) = As(t) + n(t),$$

(2.1)

where $A = (A(\Psi_1), \cdots, A(\Psi_k)$, $s(t) = (s_1(t), \cdots, s_k(t))^\prime$, $n(t) = (n_1(t), \cdots, n_p(t))^\prime$ and $k < p$. In (2.1), $n(t)$ is the noise vector distributed independent of $s(t)$ as multivariate normal with mean vector 0 and covariance matrix $\sigma^2 I_p$. $s(t)$ is distributed as multivariate normal with mean 0 and nonsingular matrix $\Omega$ and $A(\Psi_i) : p \times 1$ is a vector of functions of the elements of unknown vector $\Psi_i$ associated with $i$-th signal. Then, the covariance matrix $\Sigma$ of $x(t)$ is given by

$$\Sigma = A\Omega A^\prime + \sigma^2 I_p.$$

(2.2)

We assume that $x(t_1), \cdots, x(t_n)$ are independent observations on $x(t)$. Let $\lambda_1 \geq \cdots \lambda_k > \lambda_{k+1} = \cdots = \lambda_p = \lambda = \sigma^2$ denote the eigenvalues of $\Sigma$. Note that (2.2) is also called the spiked covariance model in Johnstone (2001).
Using the well-known spectral representation theorem from linear algebra, we can express \( \Sigma \) as
\[
\Sigma = \sum_{i=1}^{k} (\lambda_i - \lambda) \Gamma_i \Gamma_i' + \lambda I_p,
\]
where \( \Gamma_i \) is the eigenvector of \( \Sigma \) with eigenvalue \( \lambda_i \). Denoting by \( \theta \) the parameter vector of the model, it follows that
\[
\theta' = (\theta_1', \theta_2'),
\]
where \( \theta_1' = (\lambda_1, \cdots, \lambda_k, \lambda) \) and \( \theta_2' = (\Gamma_1', \cdots, \Gamma_k') \).

With this parameterization we now proceed to the derivation of the information theoretic criteria for the detection problem. Since the observation are regarded as statistically independent Gaussian random vectors with zero mean, their joint probability density is given by
\[
f(X \mid \theta) = \prod_{i=1}^{n} f(x(t_i) \mid \theta) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{p/2} | \Sigma |^{1/2}} \exp \left( -\frac{1}{2} x'(t_i) \Sigma^{-1} x(t_i) \right),
\]
where \( X' = (x'(t_1), \cdots, x'(t_n)) \). Taking the logarithm, the log-likelihood function is given by
\[
\log f(X \mid \theta) = \sum_{i=1}^{n} \log f(x(t_i) \mid \theta)
\]
\[
= \log \prod_{i=1}^{n} \frac{1}{(2\pi)^{p/2} | \Sigma |^{1/2}} \exp \left( -\frac{1}{2} x'(t_i) \Sigma^{-1} x(t_i) \right)
\]
\[
= -\frac{1}{2} n \log | \Sigma | - \frac{1}{2} n \text{tr}(\Sigma^{-1} S) - \frac{1}{2} np \log(2\pi)
\]
\[
= A - \frac{1}{2} np \log(2\pi),
\]
where \( S = \frac{1}{n} \sum_{i=1}^{n} x(t_i)x(t_i)' = \frac{1}{n} X'X \) is the sample covariance matrix.

Let \( \Delta = \text{diag}\{\lambda_1, \cdots, \lambda_p\} \) and \( \Sigma = \Gamma \Delta \Gamma' \), where \( \Gamma \) is unitary and its column \( \Gamma_i \) is the eigenvector of \( \Sigma \) with eigenvalue \( \lambda_i \). Also, denote by \( d_1 \geq \cdots \geq d_p \) the eigenvalues of \( S \).
Let \( D = \text{diag}\{d_1, \cdots, d_p\} \) and \( S = CDC' \), where \( C \) is unitary and its column \( C_j \) is the eigenvector of \( S \) with eigenvalues \( d_j \). Define \( P = \Gamma'C \) and \( P_{ij} = \Gamma'_i C_j \).

We first derive the asymptotic expansion of the log-likelihood function and its infimum. Then we select the number of signals \( k \) by maximizing the infimum of the log-likelihood
Lemma 1.

\[ A = B - \frac{1}{2} \rho n \sum_{i=1}^{k} \sum_{j>i}^{p} ((d_i - d_j)P_{ij}^2/\lambda_i), \]

where \( B = -\frac{1}{2} n (\sum_{i=1}^{k} \log \lambda_i + (p - k) \log \lambda) - \frac{1}{2} n \sum_{i=1}^{k} (d_i/\lambda_i) - \frac{1}{2} n \sum_{i=k+1}^{p} (d_i/\lambda) \) and \( \rho \geq 0. \)

Since \( P_{ij} \) is a function of \( \Gamma_i \), it can be regarded as a parameter. In this case, for \( i \neq j \), the maximum likelihood estimate of \( P_{ij} \) is 0 and the maximum likelihood estimate of \( P_{ii} \) is 1 (Anderson, 1963). On the other hand, since \( P_{ij} \) is a function of \( C_j \), it can also be regarded as a random variable. The following result gives the order of the inner product between the eigenvectors of \( \Sigma \) and \( S \).

Lemma 2.

\[ P_{ij} = \Gamma'_i C_j = \begin{cases} O_p(\sqrt{1/n}), & i \neq j, \\ 1 + O_p(\sqrt{1/n}), & i = j. \end{cases} \]

Corollary 1.

\begin{align*}
A &= -\frac{1}{2} n \sum_{i=1}^{k} E_i - \frac{1}{2} n E_{k+1} - \frac{1}{2} k (p - (k + 1)/2) O_p(1) \\
&= B - \frac{1}{2} k (p - (k + 1)/2) O_p(1),
\end{align*}

where \( E_i = \log \lambda_i + d_i/\lambda_i, \ i = 1, \cdots, k, \ E_{k+1} = (p - k) \log \lambda + \sum_{i=k+1}^{p} (d_i/\lambda) \) and \( B = -\frac{1}{2} n \sum_{i=1}^{k} E_i - \frac{1}{2} n E_{k+1}. \)

Define \( \hat{\lambda} = 1/(p - k) \sum_{i=k+1}^{p} d_i \). Note that for \( i = 1, 2, \cdots, k, \ d_i \) and \( \hat{\lambda} \) are the maximum likelihood estimates of \( \lambda_i \) and \( \lambda \), respectively.

Throughout this paper, we assume that \( \lambda \) is known. Without loss of generality, we assume that \( \lambda = 1 \). The signal-to-noise ratio is defined as

\[ SNR = \frac{\lambda_k - \lambda}{\lambda} = \frac{\lambda_k}{\lambda} - 1 = \lambda_k - 1. \]
The following result shows how close is the eigenvalue of $\Sigma$ to its maximum likelihood estimate.

**Lemma 3.** For $i = 1, 2, \cdots, p$,

$$(\lambda_i - d_i)^2 = O_p(1/n),$$

$$(p - k)(\lambda - \hat{\lambda})^2 = O_p(1/n).$$

**Theorem 1.**

$$A = \sup_{\theta_1, \cdots, \theta_k} B - \frac{1}{2} np \sum_{i=1}^k C_1 \nu_i^{-2} (\lambda_i - d_i)^2 - \frac{1}{2} n C_2 \nu^{-2} (p - k)(\lambda - \hat{\lambda})^2$$

$$- \frac{1}{2} k(p - (k + 1)/2) O_p(1)$$

$$= \sup_{\theta_1, \cdots, \theta_k} B - \frac{1}{2} (k(p - (k - 1)/2) + 1) O_p(1)$$

$$= -\frac{1}{2} np - \frac{1}{2} n (\sum_{i=1}^k \log d_i + (p - k) \log \hat{\lambda}) - \frac{1}{2} (k(p - (k - 1)/2) + 1) O_p(1),$$

where $\min \{\lambda_i, d_i\} \leq \nu_i \leq \max \{\lambda_i, d_i\}$, $\min \{\lambda, \hat{\lambda}\} \leq \nu \leq \max \{\lambda, \hat{\lambda}\}$, $C_1 = 2d_i/\nu_i - 1$ and $C_2 = 2/(p - k) \sum_{i=k+1}^p (d_i/\nu) - 1$.

Suppose

$$S - \Sigma = \Theta \Phi \Theta' = \sum_{i=1}^p \phi_i \theta_i \theta_i',$$

where $\phi_1, \cdots, \phi_p$ and $\theta_1, \cdots, \theta_p$ are the eigenvalues and eigenvectors of $S - \Sigma$, respectively.

By Lemmas 9 and 10 in Section 7, for $i = 1, 2, \cdots, p$,

$$| \lambda_i - d_i | \leq \max_{1 \leq i \leq p} | \phi_i | = ||S - \Sigma|| = O_p(\sqrt{1/n}).$$

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Suppose $\Sigma = (\sigma_{rs})_{p \times p}$. For $1 \leq r \leq p, 1 \leq s \leq p$, we have

$$\frac{1}{n} \sum_{i=1}^{n} (x_r(t_i)x_s(t_i) - \sigma_{rs}) = \sum_{i=1}^{p} \phi_i \theta_i \theta_i = O_p(\sqrt{1/n}).$$

That is,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (x_r(t_i)x_s(t_i) - \sigma_{rs}) = O_p(1).$$

By Lemma 8 in Section 7, $2 \log \log n$ is the supreme of $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (x_r(t_i)x_s(t_i) - \sigma_{rs})$ almost surely.

Noting that $2 \log \log n$ is the supreme of $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (x_r(t_i)x_s(t_i) - \sigma_{rs})$ almost surely, we have

$$A \geq \ell(k) = -\frac{1}{2} np - \frac{1}{2} n(\sum_{i=1}^{k} \log d_i + (p - k) \log \hat{\lambda}) - \gamma (k(p - (k - 1)/2) + 1) \log \log n,$$

where $\gamma > 0$ is a tuning parameter.

Then we estimate $k$ by maximizing $\ell(k')$:

$$\hat{k} = \arg \max_{k'} \ell(k'), \quad (2.3)$$

where

$$\ell(k') = -\frac{1}{2} n(\sum_{i=1}^{k'} \log d_i + (p - k') \log \hat{\lambda}_{k'}) - \gamma k'(p - (k' - 1)/2) \log \log n. \quad (2.4)$$

where $\hat{\lambda}_{k'} = 1/(p - k') \sum_{i=k'+1}^{p} d_i$ The MIL essentially maximizes the order of the infimum of the log-likelihood function.
3 Consistency of the MIL

In this section, we establish the consistency of the MIL in the sense that it chooses the correct $k$ with probability tending to one when $n$ goes to infinity.

Similar to the discussion in Zhao, Krishnaiah and Bai (1986), we consider an alternative method.

Lemma 4. $\ell(k)$ defined at (2.4) can be re-written as:

$$\tilde{\ell}(k) = -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n \sum_{i=k+1}^{p} (d_i - 1) - \gamma k (p - (k - 1)/2) \log \log n.$$ 

Then we estimate $k$ by maximizing $\tilde{\ell}(k')$:

$$\hat{k} = \arg \max_{k'} \tilde{\ell}(k'), \quad (3.1)$$

where

$$\tilde{\ell}(k') = \frac{1}{2} n \sum_{i=1}^{k'} \log d_i - \frac{1}{2} n \sum_{i=k'+1}^{p} (d_i - 1) - \gamma k' (p - (k' - 1)/2) \log \log n. \quad (3.2)$$

Simulations show that (2.3) and (3.1) behave almost the same. For simplicity and comparisons with other methods, we consider (2.3) in our simulations.

Note that $\lim_n \lambda_k > 1$ was implied in Zhao, Krishnaiah and Bai (1986). That is, the SNR was assumed to be uniformly bounded away from below by 0 and thus was not allowed to converge to 0. As a result, the proof of consistency in Zhao, Krishnaiah and Bai (1986) did not depend on the SNR. In this section, we assume that $\lim_n \lambda_k \geq 1$. Under this assumption, the SNR is allowed to converge to 0.

Theorem 2. Suppose that $\text{SNR}/\sqrt{4\gamma(p - k/2 + 1/2) \log \log n/n}$ is uniformly bounded away from below by 1. Let $\tilde{\ell}(k')$ be the penalized likelihood function defined at (3.2).
For \( k' < k \),
\[
P(\hat{\ell}(k) > \hat{\ell}(k')) \to 1.
\]

For \( k' > k \),
\[
P(\hat{\ell}(k) > \hat{\ell}(k')) \to 1.
\]

For simplicity, we may use \( \gamma = 1 \) in our simulations which gives good performance.

Consider the following general criterion:
\[
\hat{k} = \arg \max_{k'} \ell(k'),
\]
where
\[
\ell(k') = -\frac{1}{2} n \sum_{i=1}^{k'} \log d_i + (p - k') \log \hat{\lambda}_{k'} - k'(p - (k' - 1)/2)C_n,
\]
(3.3)

where \( C_n \to \infty \) and \( C_n/n \to 0 \).

Similar to the discussion in Zhao, Krishnaiah and Bai (1986), we consider an alternative method.

**Lemma 5.** \( \ell(k) \) defined at (3.3) can be re-written as
\[
\tilde{\ell}(k) = -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n \sum_{i=k+1}^{p} (d_i - 1) - k(p - (k - 1)/2)C_n.
\]

Then we estimate \( k \) by maximizing \( \tilde{\ell}(k') \):
\[
\hat{k} = \arg \max_{k'} \tilde{\ell}(k'),
\]
where
\[
\tilde{\ell}(k') = \frac{1}{2} n \sum_{i=1}^{k'} \log d_i - \frac{1}{2} n \sum_{i=k'+1}^{p} (d_i - 1) - k'(p - (k' - 1)/2)C_n.
\]
(3.4)
Corollary 2. Suppose that $\text{SNR}/\sqrt{4(p - k/2 + 1/2)C_n/n}$ is uniformly bounded away from below by 1. Let $\tilde{\ell}(k')$ be the penalized likelihood function defined at (3.4).

For $k' < k$,

$$P(\tilde{\ell}(k) > \tilde{\ell}(k')) \to 1.$$  

For $k' > k$,

$$P(\tilde{\ell}(k) > \tilde{\ell}(k')) \to 1.$$

Remark 1. Zhao, Krishnaiah and Bai (1986) demonstrated that any penalty term of the form $k'(p - (k' - 1)/2)C_n$ leads to an asymptotically consistent estimator under the condition that $C_n/\log\log n \to \infty$ and $C_n/n \to 0$. Corollaries 2 shows that this condition can be significantly weakened as $C_n \to \infty$ and $C_n/n \to 0$.

Remark 2. By Corollary 2, the BIC (or the MDL) is not suitable for the spiked covariance model, unless $\text{SNR}/\sqrt{2(p - k/2 + 1/2)\log n/n}$ is uniformly bounded away from below by 1. This implies that when the SNR is low, to get a consistent estimate, the BIC needs a very large $n$.

4 Extension to high dimension case

In this section, we extend our results to the case $n, p \to \infty$, with $p/n \to c > 0$. In this case, the MIL is not consistent and we consider the following AIC-type method.

We estimate $k$ by maximizing $\ell(k')$:

$$\hat{k} = \arg\max_{k'} \ell(k'),$$  \hspace{1cm} (4.1)
where
\[
\ell(k') = -\frac{1}{2} n \left( \sum_{i=1}^{k'} \log d_i + (p - k') \log \hat{\lambda}_{k'} - \gamma k'(p - (k' - 1)/2). \right. \tag{4.2}
\]

In the case \(n, p \to \infty\), with \(p/n \to c > 0\), simulations show that, at low SNR, the AIC (i.e., \(\gamma = 1\)) tends to underestimate the number of signals \(k\) (see Section 5). This implies that, in the case \(n, p \to \infty\), with \(p/n \to c > 0\), the AIC does not work and should be re-defined.

In the following Remark 3, we will show that the AIC-type method (4.1) with \(\gamma = 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c\) is a natural extension of the AIC for fixed \(p\).

Note that the AIC-type method (4.1) is different from that of Bai, Fujikoshi and Choi (2017) which proposes the following BFC for estimating the number of principal components \(k\).

In the case \(0 < c < 1\), the BFC defined in Bai, Fujikoshi and Choi (2017) estimates \(k\) by minimizing \(\ell^1(k')\):
\[
\hat{k} = \arg\min_{k'} \ell^1(k'), \tag{4.3}
\]
where \(\ell^1(k') = (p - k') \log \bar{d}_{k'} - \sum_{i=k'+1}^{p} \log d_i - (p - k' - 1)(p - k' + 2)/n, \bar{d}_{k'} = \sum_{i=k'+1}^{p} d_i/(p - k')\).

Note that, (4.3) is essentially equivalent to the AIC (i.e., \(\gamma = 1\)). In the following Remark 4, we will show that, at low SNR, the BFC defined in (4.3) tends to underestimate the number of signals \(k\).

In the case \(c > 1\), the BFC defined in Bai, Fujikoshi and Choi (2017) estimates \(k\) by minimizing \(\ell^2(k')\):
\[
\hat{k} = \arg\min_{k'} \ell^2(k'), \tag{4.4}
\]
where \(\ell^2(k') = (n - 1 - k') \log \bar{d}_{k'} - \sum_{i=k'+1}^{n-1} \log d_i - (n - k' - 2)(n - k' + 1)/p, \bar{d}_{k'} = \sum_{i=k'+1}^{n-1} d_i/(n - 1 - k')\).

Simulations show that, the BFC defined in (4.4) is better than the AIC. However, at low
SNR, the BFC defined in (4.4) still tends to underestimate the number of signals $k$. In the following Remark 4, we will give some reasons.

Similar to the discussion in Zhao, Krishnaiah and Bai (1986), we first consider an alternative method.

Similar to Lemmas 3 and 4, we have the following results.

**Lemma 6.** In the case $n, p \to \infty$ with $p/n \to c > 0$, for $i = 1, 2, \ldots, p$,

$$(d_i - \lambda_i)^2 = O_p(p/n),$$

$$(p - k)(\lambda - \hat{\lambda})^2 = O_p(1/n).$$

**Lemma 7.** In the case $n, p \to \infty$ with $p/n \to c > 0$, $\ell(k)$ defined at (4.2) can be re-written as:

$$\tilde{\ell}(k) = -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n \sum_{i=k+1}^{p} (d_i - 1) - \gamma k(p - (k - 1)/2).$$

Then we estimate $k$ by maximizing $\tilde{\ell}(k')$:

$$\hat{k} = \arg \max_{k'} \tilde{\ell}(k'),$$

where

$$\tilde{\ell}(k') = -\frac{1}{2} n \sum_{i=1}^{k'} \log d_i - \frac{1}{2} n \sum_{i=k'+1}^{p} (d_i - 1) - \gamma k'(p - (k' - 1)/2).$$

To achieve the consistency of (4.5), we need two additional conditions,

$$\psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2\gamma c,$$  \hspace{1cm} (4.7)

$$\lambda_k > 1 + \sqrt{c},$$  \hspace{1cm} (4.8)
where \( \psi(\lambda_k) = \lambda_k + \frac{c \lambda_k}{\lambda_k - 1} \).

Since there is a tuning parameter \( \gamma \) in (4.7), \( \lambda_k > 1 + \sqrt{c} \) does not imply \( \psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2\gamma c \).

**Theorem 3.** In the case \( n, p \to \infty \) with \( p/n \to c > 0 \), let \( \tilde{\ell}(k') \) be the penalized likelihood function defined at (4.6). Suppose that \( \lambda_1 \) is bounded and the number of candidate models, \( q \), satisfies \( q = o(p) \).

For \( k' < k \), if (4.7) and (4.8) hold,

\[
P(\tilde{\ell}(k) > \tilde{\ell}(k')) \to 1.
\]

For \( k' > k \), if \( \gamma > \varphi(c) \),

\[
P(\tilde{\ell}(k) > \tilde{\ell}(k')) \to 1.
\]

where \( \varphi(c) = 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c \).

By using the techniques developed in Bai, Fujikoshi and Choi (2017), we can also establish the consistency of (4.1).

**Theorem 4.** In the case \( n, p \to \infty \) with \( p/n \to c > 0 \), let \( \ell(k') \) be the penalized likelihood function defined at (4.2). Suppose that \( \lambda_1 \) is bounded and the number of candidate models, \( q \), satisfies \( q = o(p) \).

For \( k' < k \), if (4.7) and (4.8) hold,

\[
P(\ell(k) > \ell(k')) \to 1.
\]

For \( k' > k \), if \( \gamma > \varphi(c) \),

\[
P(\ell(k) > \ell(k')) \to 1.
\]

where \( \varphi(c) = 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c \).
Simulations show that (4.1) and (4.5) behave almost the same. For simplicity and comparisons with other methods, we consider (4.1) in our simulations.

In the following two cases,

(1) in the case $p$ fixed, for large $n$,

(2) in the case $n, p \to \infty$ with $p/n \to 0$,

c is defined as $c = p/n$. Noting that $c$ is very small, by Taylor’s expansion, we get

$$\varphi(c) = \frac{1}{2} + \sqrt{\frac{1}{c}} - \log(1 + \sqrt{c})/c$$

$$= \frac{1}{2} + \sqrt{\frac{1}{c}} - \left(\sqrt{c} - \frac{c}{2}(1 + o(1))\right)/c$$

$$= 1 + o(1)$$

$$\to 1.$$

**Remark 3.** Note that for fixed $p$ and large $n$, $\gamma > \varphi(c)$ becomes $\gamma > 1$. It is well-known that, for fixed $p$, the AIC (i.e., $\gamma = 1$) tends to overestimate the number of signals $k$. To achieve the consistency of the AIC-type method, the tuning parameter $\gamma$ should be larger than one. That is, in the case $n, p \to \infty$ with $p/n \to c > 0$, the AIC-type method with $\gamma = \varphi(c)$ can be seen as a natural extension of the AIC for fixed $p$. There is a similar discussion in the case $n, p \to \infty$ with $p/n \to 0$. As a result, in the following three cases,

(1) in the case $p$ fixed, for large $n$,

(2) in the case $n, p \to \infty$ with $p/n \to 0$,

(3) in the case $n, p \to \infty$ with $p/n \to 0 < c < \infty$,

approximately, the AIC may be uniformly written as the AIC-type method (4.1) with $\gamma = 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c$. Thus, throughout this paper, we call the AIC-type method
with $\gamma = \varphi(c)$ the generalized AIC (GAIC). Similarly, we call the AIC-type method with $\gamma > \varphi(c)$ the GAIC-type method.

On the other hand, in the following two cases,

(1) in the case $p$ fixed, for large $n$,

(2) in the case $n, p \to \infty$ with $p/n \to c = 0$,

if the AIC is defined as the degeneration of the AIC-type method with $\gamma = \varphi(c)$ in the case $n, p \to \infty$ with $p/n \to c > 0$, i.e., $\gamma = \lim_{c \to 0^+} \varphi(c) = 1$, then we have essentially demonstrated that, to achieve the consistency of the AIC-type method in the above two cases, $\gamma > 1$ is required.

**Remark 4.** We note that $\gamma > \varphi(c)$ is required in our theoretic analysis, to avoid overestimating the number of principal components $k$. On the other hand, (4.7) requires that $\gamma$ cannot be too large, to avoid underestimating the number of principal components $k$. Note that, in the case $n, p \to \infty$ with $p/n \to c > 0$, we always have $\varphi(c) < 1$. Thus, it is not surprising that the AIC tends to underestimate the number of signals $k$. Since $\gamma > \varphi(c)$ is required in our theoretic analysis, we set $\gamma = 1.1\varphi(c)$ for simulation studies which gives good performance. In this sense, the GAIC-type method is essentially tuning-free.

To prove the consistency of the BFC, Bai, Fujikoshi and Choi (2017) gives the following two conditions,

(1) in the case $0 < c < 1$,

$$
\psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2c \quad \text{and} \\
(4.9)
$$

(2) in the case $c > 1$,

$$
\psi(\lambda_k)/c - 1 - \log(\psi(\lambda_k)/c) > 2/c \\
(4.10)
$$
Compared with (4.9) and (4.10), (4.7) contains a tuning parameter $\gamma$. By simulations, we found that when $c \geq 0.12$, we have $\gamma = 1 \varphi(c) < 1$. That is, when $0.12 \leq c \leq 1$, (4.7) is weaker than (4.9). This implies that, when $0.12 \leq c \leq 1$, in the case where (4.7) holds while (4.9) does not hold, the GAIC-type method is better than the BFC. When $c > 1$, although it is difficult to compare (4.7) with (4.10), by simulations, we found that there do exist cases where (4.7) holds while (4.10) does not hold (see Section 5). This implies that, in the case where (4.7) holds while (4.10) does not hold, the GAIC-type method is still better than the BFC. We also note that in the case $c > 1$, if $c$ is close to one, (4.10) and (4.9) are almost the same. Thus, in the case where $c$ is close to one, the GAIC-type method outperforms the BFC. The reason lies in that, when $c = p/n$ is close to one, (4.7) is weaker than (4.9) and (4.10). As a result, the GAIC-type method outperforms the BFC at least in the following two cases,

(1) in the case $0.12 \leq c \leq 1$,

(2) in the case $c > 1$, and $c$ is close to one.

Moreover, note that (4.3) is essentially equivalent to the AIC (i.e., $\gamma = 1$). Since when $c \geq 0.12$, we have $\gamma = 1 \varphi(c) < 1$. That is, when (1) $0.12 \leq c \leq 1$; (2) $c$ is larger than one and is close to one, the BFC tends to underestimate the number of signals $k$.

On the other hand, in the case where $c$ is significantly larger than one, simulations show that the GAIC-type method is still comparable to the BFC (see Section 5).

**Remark 5.** Although the BFC defined in (4.3) and (4.4) is tuning-free and simulation results are encouraging, Bai, Fujikoshi and Choi (2017) essentially define two different criteria in the case $0 < c < 1$ and $c > 1$, respectively. As a result, to achieve the consistency of the BFC, two different consistency conditions are required (i.e., (4.9) and (4.10)). Compared with (4.3) and (4.4), (4.1) with $\gamma > \varphi(c)$ is a natural extension
of the AIC for fixed $p$. We also note that, both the formula (4.1) and the consistency
condition (4.7) are more simple.

Table 1: performance of MIL: $p = 12$, $k = 3$, $\gamma = 1$, $\lambda = 1$, $SNR = \delta \sqrt{4(p-k/2+1/2) \log \log n/n}$

| $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean |
|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|
| 1        | 100 | 0.45 | 2.45 | 1.25     | 100 | 0.63 | 2.59 | 1.5       | 100 | 0.76 | 2.78 | 1.75     | 100 | 0.93 | 2.95 | 2         | 100 | 0.96 | 2.98 |
| 1        | 200 | 0.50 | 2.45 | 1.25     | 200 | 0.84 | 2.84 | 1.5       | 200 | 0.97 | 2.97 | 1.75     | 200 | 1.00 | 3.00 | 2         | 200 | 1.00 | 3.00 |
| 1        | 500 | 0.54 | 2.54 | 1.25     | 500 | 0.85 | 2.85 | 1.5       | 500 | 0.96 | 2.96 | 1.75     | 500 | 1.00 | 3.00 | 2         | 500 | 1.00 | 3.00 |
| 1        | 800 | 0.51 | 2.51 | 1.25     | 800 | 0.92 | 2.92 | 1.5       | 800 | 0.99 | 2.99 | 1.75     | 800 | 1.00 | 3.00 | 2         | 800 | 1.00 | 3.00 |
| 1        | 1000| 0.51 | 2.51 | 1.25     | 1000| 0.97 | 3.01 | 1.5       | 1000| 1.00 | 3.00 | 1.75     | 1000| 1.00 | 3.00 | 2         | 1000| 1.00 | 3.00 |

Table 2: performance of BIC: $p = 12$, $k = 3$, $\lambda = 1$, $SNR = \delta \sqrt{4(p-k/2+1/2) \log \log n/n}$

| $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean |
|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|
| 1        | 100 | 0.04 | 2.03 | 1.25     | 100 | 0.19 | 2.17 | 1.5       | 100 | 0.39 | 2.38 | 1.75     | 100 | 0.56 | 2.55 | 2         | 100 | 0.72 | 2.22 |
| 1        | 200 | 0.07 | 2.07 | 1.25     | 200 | 0.21 | 2.21 | 1.5       | 200 | 0.47 | 2.47 | 1.75     | 200 | 0.76 | 2.76 | 2         | 200 | 0.90 | 2.90 |
| 1        | 500 | 0.02 | 2.02 | 1.25     | 500 | 0.15 | 2.15 | 1.5       | 500 | 0.48 | 2.48 | 1.75     | 500 | 0.80 | 2.80 | 2         | 500 | 0.94 | 2.94 |
| 1        | 800 | 0.01 | 2.01 | 1.25     | 800 | 0.14 | 2.14 | 1.5       | 800 | 0.45 | 2.45 | 1.75     | 800 | 0.79 | 2.79 | 2         | 800 | 0.95 | 2.95 |
| 1        | 1000| 0.03 | 2.03 | 1.25     | 1000| 0.17 | 2.17 | 1.5       | 1000| 0.45 | 2.45 | 1.75     | 1000| 0.84 | 2.84 | 2         | 1000| 0.97 | 2.97 |

Table 3: performance of AIC: $p = 12$, $k = 3$, $\lambda = 1$, $SNR = \delta \sqrt{4(p-k/2+1/2) \log \log n/n}$

| $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean | $\delta$ | $n$ | Prob | Mean |
|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|----------|-----|------|------|
| 1        | 100 | 0.72 | 2.93 | 1.25     | 100 | 0.84 | 3.07 | 1.5       | 100 | 0.86 | 3.12 | 1.75     | 100 | 0.86 | 3.12 | 2         | 100 | 0.87 | 3.15 |
| 1        | 200 | 0.88 | 3.06 | 1.25     | 200 | 0.91 | 3.10 | 1.5       | 200 | 0.91 | 3.10 | 1.75     | 200 | 0.90 | 3.12 | 2         | 200 | 0.91 | 3.10 |
| 1        | 500 | 0.80 | 3.10 | 1.25     | 500 | 0.85 | 3.15 | 1.5       | 500 | 0.86 | 3.15 | 1.75     | 500 | 0.86 | 3.16 | 2         | 500 | 0.85 | 3.16 |
| 1        | 800 | 0.83 | 3.18 | 1.25     | 800 | 0.84 | 3.17 | 1.5       | 800 | 0.84 | 3.18 | 1.75     | 800 | 0.84 | 3.18 | 2         | 800 | 0.84 | 3.18 |
| 1        | 1000| 0.78 | 3.23 | 1.25     | 1000| 0.76 | 3.28 | 1.5       | 1000| 0.80 | 3.25 | 1.75     | 1000| 0.76 | 3.28 | 2         | 1000| 0.75 | 3.29 |
Table 4: performance of modified AIC: $p = 12$, $k = 3$, $\lambda = 1$, $SNR = \delta \sqrt{4(p - k/2 + 1/2) \log \log n/n}$

| $\delta$ | $n = 100$ | $n = 200$ | $n = 500$ | $n = 800$ | $n = 1000$ |
|----------|-----------|-----------|-----------|-----------|-----------|
| 1        | 0.12 2.12 | 0.22 2.22 | 0.32 2.32 | 0.40 2.40 | 0.39 2.39 |
| 1.25     | 0.36 2.36 | 0.54 2.54 | 0.67 2.67 | 0.75 2.75 | 0.82 2.82 |
| 1.5      | 0.54 2.54 | 0.80 2.80 | 0.92 2.92 | 0.95 2.95 | 0.97 2.97 |
| 1.75     | 0.67 2.67 | 0.94 2.94 | 0.98 2.98 | 1.00 3.00 | 1.00 3.00 |
| 2        | 0.84 2.84 | 1.00 3.00 | 1.00 3.00 | 1.00 3.00 | 1.00 3.00 |

Table 5: performance of KN: $p = 12$, $k = 3$, $\lambda = 1$, $SNR = \delta \sqrt{4(p - k/2 + 1/2) \log \log n/n}$

| $\delta$ | $n = 100$ | $n = 200$ | $n = 500$ | $n = 800$ | $n = 1000$ |
|----------|-----------|-----------|-----------|-----------|-----------|
| 1        | 0.10 2.09 | 0.09 2.09 | 0.19 2.19 | 0.24 2.24 | 0.23 2.23 |
| 1.25     | 0.22 2.22 | 0.29 2.29 | 0.48 2.48 | 0.58 2.58 | 0.64 2.64 |
| 1.5      | 0.41 2.41 | 0.59 2.59 | 0.81 2.81 | 0.84 2.84 | 0.94 2.94 |
| 1.75     | 0.60 2.59 | 0.84 2.84 | 0.95 2.95 | 1.00 3.00 | 1.00 3.00 |
| 2        | 0.73 2.73 | 0.96 2.96 | 0.99 2.99 | 1.00 3.00 | 1.00 3.00 |

5 Experiments

We compare the performance of the MIL with the BIC, the AIC and the KN in a series of simulations with $\lambda = \sigma^2 = 1$. As suggested in Nadler (2010), the confidence level $\alpha = 10^{-4}$ was used in the KN. Our performance measure is the probability of the successful recovery of the number of signals,

$$Pr(\hat{k} = k).$$

We restrict our attention to candidate values for the true number of signals in the range $k' \in \{0, 1, \ldots, \min\{p - 1, 15\}\}$ in simulations. Each simulation in this section is repeated 200 times.
Table 6: performance of MIL, AIC, Modified AIC, GAIC-type, BFC, KN: $n = 500$, $p = 200$, $k = 10$, $\lambda = 1$, $SNR = \delta$

|       | $\delta = 0.5$ |       | $\delta = 1$ |       | $\delta = 1.5$ |       | $\delta = 2$ |       | $\delta = 2.5$ |
|-------|---------------|-------|---------------|-------|---------------|-------|---------------|-------|---------------|
| Prob  | Mean          | Prob  | Mean          | Prob  | Mean          | Prob  | Mean          | Prob  | Mean          |
| MIL ($\gamma = 1.00$) | 0.00   | 0.00 | 0.00           | 1.51  | 0.00           | 8.39  | 0.10           | 9.10  | 0.85           | 9.85  |
| AIC   | 0.00           | 3.07  | 0.24           | 9.23  | 1.00           | 10.0  | 1.00           | 10.0  | 1.00           | 10.0  |
| Modified AIC | 0.00 | 0.00 | 0.00           | 5.26  | 0.00           | 7.44  | 0.02           | 9.02  | 0.67           | 9.67  |
| GAIC-type | 0.00 | 4.40 | 0.52           | 9.52  | 1.00           | 10.0  | 1.00           | 10.0  | 1.00           | 10.0  |
| BFC   | 0.00           | 3.07  | 0.24           | 9.23  | 1.00           | 10.0  | 1.00           | 10.0  | 1.00           | 10.0  |
| KN    | 0.00           | 1.81  | 0.02           | 8.94  | 0.89           | 9.89  | 1.00           | 10.0  | 1.00           | 10.0  |

Table 7: performance of MIL, AIC, Modified AIC, GAIC-type, BFC, KN: $n = 200$, $p = 500$, $k = 10$, $\lambda = 1$, $SNR = \delta$

|       | $\delta = 1.5$ |       | $\delta = 2.5$ |       | $\delta = 2.68$ |       | $\delta = 3.5$ |       | $\delta = 4.5$ |
|-------|---------------|-------|---------------|-------|---------------|-------|---------------|-------|---------------|
| Prob  | Mean          | Prob  | Mean          | Prob  | Mean          | Prob  | Mean          | Prob  | Mean          |
| MIL ($\gamma = 1.00$) | 0.00   | 0.00 | 0.00           | 0.00  | 1.00           | 0.00  | 1.34           | 0.00  | 5.34          |
| AIC   | 0.00           | 1.16  | 0.00           | 7.05  | 0.00           | 8.00  | 0.12           | 9.10  | 0.70           | 9.70  |
| Modified AIC | 0.00 | 0.00 | 0.00           | 0.00  | 0.00           | 0.00  | 0.00           | 0.00  | 0.00           | 1.42  |
| GAIC-type | 0.00 | 5.60 | 0.31           | 9.31  | 0.43           | 9.43  | 0.92           | 9.92  | 1.00           | 10.0  |
| BFC   | 0.00           | 3.75  | 0.11           | 9.04  | 0.25           | 9.22  | 0.84           | 9.84  | 1.00           | 10.0  |
| KN    | 0.00           | 3.62  | 0.01           | 8.43  | 0.07           | 9.06  | 0.45           | 9.45  | 0.94           | 9.94  |

5.1 Fixed $p$

Simulation 1. In this simulation, we investigate how the accuracy of the MIL changes as the SNR varies. We set $p = 12$, $k = 3$, $\gamma = 1$, $\lambda_1 = \cdots = \lambda_{k-1} = 1 + 2SNR$, $\lambda_k = 1 + SNR$ and $\lambda = 1$. We let $\delta$ increase from 1 to 2. It can be seen from Table 1 that, when $SNR/\sqrt{4(p-k/2+1/2)\log\log n/n} = 1$, the rate of the successful recovery of the number of signals is low even for $n = 1000$. On the other hand, when $SNR/\sqrt{4(p-k/2+1/2)\log\log n/n}$ is uniformly bounded away from below by 1, the success rate is high. This simulation verifies that Theorem 2 is correct.

Simulation 2. We compare the MIL with the BIC, the AIC, the modified AIC and the KN. We set $p = 12$, $k = 3$, $SNR = \delta\sqrt{4(p-k/2+1/2)\log\log n/n}$. Compared with Table 1, from Tables 2-5, we can see that, when $\delta \geq 1.25$, the MIL outperforms other methods.
Table 8: performance of MIL, AIC, Modified AIC, GAIC-type, BFC, KN: \( n = 200, p = 200, k = 10, \lambda = 1, SNR = \delta \)

| \( \delta \) | MIL (\( \gamma = 1.00 \)) | AIC | Modified AIC | GAIC-type | BFC | KN |
|-----|----------------|-----|--------------|----------|-----|----|
| 1   | Prob | Mean | Prob | Mean | Prob | Mean | Prob | Mean | Prob | Mean |
| 1.5 | 0.00 | 0.00 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 |
| 2   | 0.00 | 0.00 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 |
| 2.5 | 0.00 | 0.00 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 |
| 3   | 0.00 | 0.00 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 | 0.00 | 3.22 |

Table 9: performance of GAIC-type: \( \gamma = 1.1 \varphi(c), k = 10, \lambda = 1, SNR = 2\sqrt{p/n} \)

| \( n \) | \( p \) | Prob | Mean | Prob | Mean | Prob | Mean | Prob | Mean |
|-------|-------|------|------|------|------|------|------|------|------|
| 100   | 100   | 0.55 | 9.76 | 0.82 | 9.90 | 0.85 | 9.89 | 0.93 | 9.97 |
| 200   | 100   | 0.65 | 9.89 | 0.85 | 9.89 | 0.85 | 9.89 | 0.93 | 9.98 |
| 300   | 100   | 0.66 | 9.96 | 0.85 | 9.89 | 0.85 | 9.89 | 0.93 | 9.96 |
| 400   | 100   | 0.66 | 9.96 | 0.85 | 9.89 | 0.85 | 9.89 | 0.93 | 9.96 |
| 500   | 100   | 0.66 | 9.96 | 0.85 | 9.89 | 0.85 | 9.89 | 0.93 | 9.96 |

5.2 Infinite \( p \)

Note that the AIC-type method with \( \gamma > \varphi(c) \) is called the GAIC-type method in this paper. In simulations, we set \( \gamma = 1.1 \varphi(c) \).

Simulation 3. In this simulation, in the case \( n,p \to \infty \) with \( p < n \), we investigate how the accuracy of the GAIC-type method changes as the SNR varies. We set \( n = 500, p = 200, k = 10, \lambda_1 = \cdots = \lambda_{k-1} = 1 + 2SNR, \lambda_k = 1 + SNR \) and \( \lambda = 1 \). Note that when \( c = p/n = 0.4, \gamma = 1.1 \varphi(c) = 0.94 \) and \( \sqrt{c} = 0.632 \). We let \( \delta \) increase from 0.5 to 2.5. We also note that when \( \delta = 0.5, (4.7) \) does not hold, while when \( 1 \leq \delta \leq 2.5, (4.7) \) holds. It can be seen from Table 6 that, when \( \delta = 0.5 \), the success rate of the GAIC-type method is low. On the other hand, when \( \delta \geq 1 \), the success rate is high. This simulation verifies that Theorem 3 is correct, i.e., if (4.7) and (4.8) hold, the GAIC-type method is consistent. It can also be seen from Table 6 that, in the case \( n,p \to \infty \) with \( p < n \), the GAIC-type method
Table 10: performance of BFC : $k = 10, \lambda = 1, SNR = 2\sqrt{p/n}$

| $n$ | Prob | Mean |
|-----|------|------|
| 100 | 0.30 | 9.13 |
| 200 | 0.63 | 9.63 |
| 300 | 0.91 | 9.91 |
| 400 | 0.89 | 9.93 |
| 500 | 0.89 | 9.93 |

is better than the KN and the BFC, especially for $\delta \leq 1$.

Note that when $\delta = 1$, (4.7) holds while (4.9) does not hold. Thus, it not surprising that in this case, the GAIC-type method is better than the BFC. On the other hand, although (4.7) holds, $\psi(\lambda_k) - 1 - \log \psi(\lambda_k) - 2\gamma c = 0.017$ is close to zero. As a result, when $\delta = 1$, the success rate of the GAIC-type method is not high.

**Simulation 4.** In this simulation, in the case $n, p \to \infty$ with $p > n$, we investigate how the accuracy of the GAIC-type method changes as the SNR varies. We set $n = 200$, $p = 500$, $k = 10$, $\lambda_1 = \cdots = \lambda_{k-1} = 1 + 2SNR$, $\lambda_k = 1 + SNR$ and $\lambda = 1$. Note that when $c = p/n = 2.5$, $\gamma = 1.1\varphi(c) = 0.83$ and $\sqrt{c} = 1.581$. We let $\delta$ increase from 1.5 to 4.5. We also note that when $\delta = 1.5, 2.5$, (4.7) does not hold, while when $2.68 \leq \delta \leq 4.5$, (4.7) holds. It can be seen from Table 7 that, when $\delta = 1.5, 2.5$, the success rate of the GAIC-type method is low. On the other hand, when $\delta \geq 3.5$, the success rate is high. This simulation verifies that Theorem 3 is correct, i.e., if (4.7) and (4.8) hold, the GAIC-type method is consistent. It can also be seen from Table 7 that, in the case $n, p \to \infty$ with $p > n$, the GAIC-type method is better than the KN and the BFC, especially for $\delta \leq 3.5$.

Note that when $\delta = 2.68$, (4.7) holds while (4.10) does not hold. Thus, it not surprising that in this case, the GAIC-type method is better than the BFC. On the other hand, although (4.7) holds, $\psi(\lambda_k) - 1 - \log \psi(\lambda_k) - 2\gamma c = 0.0085$ is close to zero. As a result, when $\delta = 2.68$, the success rate of the GAIC-type method is not high.

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Simulation 5. In this simulation, in the case $n, p \to \infty$ with $p = n$, we investigate how the accuracy of the GAIC-type method changes as the SNR varies. We set $n = 200$, $p = 200$, $k = 10$, $\lambda_1 = \cdots = \lambda_{k-1} = 1 + 2SNR$, $\lambda_k = 1 + SNR$ and $\lambda = 1$. Note that when $c = p/n = 1$, $\gamma = 1.1\varphi(c) = 0.89$ and $\sqrt{c} = 1$. We let $\delta$ increase from 1 to 3. We also note that when $\delta = 1, 1.5$, (4.7) does not hold, while when $2 \leq \delta \leq 3$, (4.7) holds. It can be seen from Table 8 that, when $\delta = 1, 1.5$, the success rate of the GAIC-type method is low. On the other hand, when $\delta \geq 2$, the success rate is high. This simulation verifies that Theorem 3 is correct, i.e., if (4.7) and (4.8) hold, the GAIC-type method is consistent. It can also be seen from Table 8 that, in the case $n, p \to \infty$ with $p = n$, the GAIC-type method is better than the KN and the BFC, especially for $\delta \leq 2.5$.

Note that when $\delta = 2$, (4.7) holds while (4.9) does not hold. Thus, it not surprising that in this case, the GAIC-type method significantly outperforms the BFC.

Simulation 6. In this simulation, in the case $n, p \to \infty$, for low SNR, i.e., $SNR = 2\sqrt{p/n}$, we further compare the GAIC-type method with the BFC. We set $k = 10$, $\lambda_1 = \cdots = \lambda_{k-1} = 1 + 2SNR$, $\lambda_k = 1 + SNR$ and $\lambda = 1$. It can be seen from Tables 9 and 10 that, when $c = p/n$ is less than or close to one, the GAIC-type method outperforms the BFC. By remarks 4, the reasons are two folds. First, when (1) $0.12 \leq c \leq 1$; (2) $c$ is larger than one and is close to one, (4.7) is weaker than (4.9) and (4.10). Second, note that (4.3) is essentially equivalent to the AIC, i.e., (4.1) with tuning parameter $\gamma = 1$. Since when $c \geq 0.12$, we have $\gamma = 1.1\varphi(c) < 1$. That is, when (1) $0.12 \leq c \leq 1$; (2) $c$ is larger than one and is close to one, the BFC tends to underestimate the number of signals $k$.

On the other hand, when $c = p/n$ is significantly larger than one, the GAIC-type method is still comparable to the BFC.
6 Discussion

In this paper, we have derived the asymptotic expansion of the log-likelihood function and the order of its infimum. Then we select the number of signals $k$ by maximizing the order of the infimum of the log-likelihood function. We have demonstrated that the MIL is consistent under the condition that $SNR/\sqrt{4\gamma(p-k/2+1/2)\log \log n/n}$ is uniformly bounded away from below by 1, where $\gamma > 0$ is a tuning parameter and we recommend using $\gamma = 1$ in simulations. Compared with the BIC which is consistent under the condition that $SNR/\sqrt{2(p-k/2+1/2)\log n/n}$ is uniformly bounded away from below by 1, the MIL is consistent at much lower SNR. Moreover, we have demonstrated that any penalty term of the form $k'(p-(k'-1)/2)C_n$ may lead to an asymptotically consistent estimator under the condition that $C_n \rightarrow \infty$ and $C_n/n \rightarrow 0$. Compared with the condition in Zhao, Krishnaiah and Bai (1986), i.e., $C_n/\log \log n \rightarrow \infty$ and $C_n/n \rightarrow 0$, this condition has been significantly weakened. We have also extended our results to the case $n,p \rightarrow \infty$, with $p/n \rightarrow c > 0$. At low SNR, since the AIC tends to underestimate the number of signals $k$, the AIC should be re-defined in this case. As a natural extension of the AIC for fixed $p$, we have proposed the generalized AIC (GAIC), i.e., the AIC-type method with tuning parameter $\gamma = \varphi(c) = 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c$, and demonstrated that the GAIC-type method, i.e., the AIC-type method with tuning parameter $\gamma > \varphi(c)$, can select the number of signals $k$ consistently. In simulations, we set $\gamma = 1.1\varphi(c)$ which gives good performance and outperforms the KN proposed in Kritchman and Nadler (2008) and the BFC proposed in Bai, Fujikoshi and Choi (2017). That is, the GAIC-type method is essentially tuning-free.

We have noted that in the case $n,p \rightarrow \infty$ with $p/n \rightarrow c > 0$, if (4.7) does not hold, $\gamma = 1.1\varphi(c)$ tends to underestimate the number of of principal components $k$. As a result, $0 < \gamma < 1.1\varphi(c)$ may be a better choice. In this case, we may use the cross-validation method to choose the tuning parameter $\gamma$, which will be explored in future work.
7 Appendix

**Lemma 8.** Suppose \(\{x_i, i \geq 1\}\) is a stationary real \(\phi\)-mixing sequence with \(E(x_1) = 0\) and \(E(|x_1|^2) < \infty\). Also, \(\phi\) is decreasing with \(\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty\). Then

\[
P(\lim \sup \frac{\sum_{i=1}^{n} x_i}{(2\nu^2 \log \log(n\nu^2))^{1/2}} = 1) = 1,
\]

where \(\nu^2 = E(x_1^2) + 2 \sum_{i=1}^{\infty} E(x_1x_{1+i}) \neq 0\).

We note that \(\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty\) implies that \(\delta^2 < \infty\). A proof of the above lemma is given in Hall and Heyde (1980). For some earlier work on this topic, the reader is referred to Reznik (1968) and Stout (1974).

**Lemma 9.** Suppose \(Y = (Y_1, \cdots, Y_p)'\) is sub-Gaussian with constant \(\rho > 0\) and with mean 0 and covariance matrix \(\Sigma\). Let \(Y^{(1)}, \cdots, Y^{(n)}\) be \(n\) independent copies of \(Y\). Then there exist some universal constant \(C > 0\) and some constant \(\rho_1\) depending \(\rho\), such that the sample covariance matrix of \(Y^{(1)}, \cdots, Y^{(n)}, S\), satisfies

\[
P(||S - \Sigma|| > t) \leq 2 \exp(-nt^2 \rho_1 + C\rho),
\]

for all \(0 < t < \rho_1\). Here \(|| \cdot ||\) is the spectral norm.

The above lemma is given in Cai, Ren and Zhou (2016). For more work on this topic, the reader is referred to Davidson and Szarek (2001), Baik, Ben Arous and Peche (2005), Baik and Silverstein (2006), Paul (2007) and Bai and Yao (2008).

We also need the following Weyl’s (1912) inequality.

**Lemma 10.** Let \(A\) and \(B\) be \(p \times p\) Hermitian matrices with eigenvalues ordered as \(\lambda_1 \geq \cdots \geq \lambda_p\).
\[
\sum_{j=1}^{p} | \mu_j - \lambda_j | \leq || A - B ||,
\]
where \( || \cdot || \) is the spectral norm.

According to Dumitriu and Edelman (2006), we have the following result.

**Lemma 11.** In the case \( n, p \to \infty \) with \( p/n \to c > 0 \), when \( k = 0 \),

\[
\sum_{i=1}^{p} (d_i - \lambda) \xrightarrow{d} N(0, 2\lambda^2 c).
\]

For more work on this topic, we refer to Johansson (1998), Jonsson (1982), Bai and Silverstein (2004) and Nadakuditi and Edelman (2008).

According to Wang and Yao (2013), we have the following result.

**Lemma 12.** In the case \( n, p \to \infty \) with \( p/n \to c > 0 \), when \( k > 0 \),

\[
\sum_{i=k+1}^{p} (d_i - \lambda) + \sum_{i=1}^{k} (d_i - \lambda_i) \xrightarrow{d} N(0, 2\lambda^2 c).
\]

For more work on this topic, we refer to Onatski, Moreira and Hallin (2013), Wang, Silverstein and Yao (2014), Passemier, Mckay and Chen (2015).

The \( i \)-th largest eigenvalue, \( \lambda_i \), is said to be a distant spiked eigenvalue if \( \psi'(\lambda_i) > 0 \) where \( \psi(x) = x + \frac{\sqrt{x}}{x-1} \). Equivalently, \( \lambda_i \) is a distant spiked eigenvalue if \( \lambda_i > 1 + \sqrt{c} \).

According to Bai, Fujikoshi and Choi (2017), we have the following result.

**Lemma 13.** In the case \( n, p \to \infty \) with \( p/n \to c > 0 \), suppose that \( \lambda_1 \) is bounded.
(1) If $\lambda_i$ is a distant spiked eigenvalue, then

$$d_i \overset{a.s.}{\longrightarrow} \psi(\lambda_i) = \lambda_i + \frac{c_\lambda_i}{\lambda_i - 1}.$$ 

(2) If $\lambda_i$ is not a distant spiked eigenvalue and $i/p \to \alpha$, then

$$d_i \overset{a.s.}{\longrightarrow} \mu_{1-\alpha},$$

where $\mu_{\alpha}$ is the $\alpha$-th quantile of the Marčenko-Pastur distribution and the convergence is uniform in $0 \leq \alpha \leq 1$.

The above definition and result are a special case of a more general definition and result in Bai and Yao (2012).

**Proof of Lemma 1**

Since

$$\Sigma = \Gamma \Delta \Gamma'$$

$$= (V_1, V_2) \begin{pmatrix} \Delta_1 & 0 \\ 0 & \lambda I_{p-k} \end{pmatrix} \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix}$$

$$= (V_1, V_2) \begin{pmatrix} \Delta_1 - \lambda I_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix} + \lambda I_p,$$
we have
\[
\Sigma^{-1} = (V_1, V_2) \begin{pmatrix}
\Delta^{-1}_1 & 0 \\
0 & \lambda^{-1} I_{p-k}
\end{pmatrix} \begin{pmatrix}
V_1' \\
V_2'
\end{pmatrix} + \lambda^{-1} I_p
\]
\[
= (V_1, V_2) \begin{pmatrix}
\Delta^{-1}_1 - \lambda^{-1} I_k & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
V_1' \\
V_2'
\end{pmatrix} + \lambda^{-1} I_p,
\]
where \( \Gamma = (V_1, V_2), V_1 = (\Gamma_1, \cdots, \Gamma_k), V_2 = (\Gamma_{k+1}, \cdots, \Gamma_p) \) and \( \Delta_1 = \text{diag}\{\lambda_1, \cdots, \lambda_k\} \).

Hence,
\[
A = -\frac{1}{2} n \log |\Sigma| - \frac{1}{2} \text{tr}(\Sigma^{-1} S)
\]
\[
= -\frac{1}{2} n \log |\Gamma \Delta \Gamma'|- \frac{1}{2} \text{tr}(((V_1, V_2) \begin{pmatrix}
\Delta^{-1}_1 - \lambda^{-1} I_k & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
V_1' \\
V_2'
\end{pmatrix} + \lambda^{-1} I_p) S)
\]
\[
= -\frac{1}{2} n \log |\Gamma \Delta \Gamma'|- \frac{1}{2} \text{tr}(((V_1, V_2) \begin{pmatrix}
\Delta^{-1}_1 - \lambda^{-1} I_k & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
V_1' \\
V_2'
\end{pmatrix} S) - \frac{1}{2} n \sum_{i=1}^p (d_i/\lambda)
\]
\[
= -\frac{1}{2} n \log |\Gamma \Delta \Gamma'|- \frac{1}{2} \text{tr}(((V_1, V_2) \begin{pmatrix}
\Delta^{-1}_1 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
V_1' \\
V_2'
\end{pmatrix} C \Delta C' - \frac{1}{2} n \sum_{i=k+1}^p (d_i/\lambda)
\]
\[
= -\frac{1}{2} n \log |\Delta| - \frac{1}{2} \text{tr}(\begin{pmatrix}
\Delta^{-1}_1 & 0 \\
0 & 0
\end{pmatrix} PDP' - \frac{1}{2} n \sum_{i=k+1}^p (d_i/\lambda).
\]

where \( P = \Gamma' C \).

That is,
\[
A = -\frac{1}{2} n \log |\Sigma| - \frac{1}{2} \text{tr}(\Sigma^{-1} S)
\]
\[
= -\frac{1}{2} n(\sum_{i=1}^k \log \lambda_i + (p-k) \log \lambda) - \frac{1}{2} n \sum_{i=1}^k \sum_{j=1}^p (d_j P_{ij}^2 / \lambda_i) - \frac{1}{2} n \sum_{i=k+1}^p (d_i/\lambda)
\]
\[
= -\frac{1}{2} n \sum_{i=1}^k \log \lambda_i + (p-k) \log \lambda - \frac{1}{2} n \sum_{i=1}^k (d_i/\lambda_i) - \frac{1}{2} n \sum_{i=k+1}^p (d_i/\lambda) +
\]
(\(-\frac{1}{2} n \sum_{i=1}^k \sum_{j=1}^p (d_j P_{ij}^2 / \lambda_i) - (-\frac{1}{2} n \sum_{i=1}^k (d_i/\lambda_i)))
\]
\[
= B + (-\frac{1}{2} n \sum_{i=1}^k \sum_{j=1}^p (d_j P_{ij}^2 / \lambda_i) - (-\frac{1}{2} n \sum_{i=1}^k (d_i/\lambda_i))
\]

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where \( B = -\frac{1}{2} n (\sum_{i=1}^{k} \log \lambda_i + (p-k) \log \lambda) - \frac{1}{2} n \sum_{i=1}^{k} (d_i / \lambda_i) - \frac{1}{2} n \sum_{i=k+1}^{p} (d_i / \lambda) \).

Note that

\[
- \frac{1}{2} n \sum_{i=1}^{k} \sum_{j=1}^{p} (d_j P_{ij}^2 / \lambda_i) - \left( - \frac{1}{2} n \sum_{i=1}^{k} (d_i / \lambda_i) \right) \\
= - \frac{1}{2} n \sum_{i=1}^{k} \sum_{j \neq i}^{p} (d_j P_{ij}^2 / \lambda_i) - \frac{1}{2} n \sum_{i=1}^{k} (d_i / \lambda_i) - \left( - \frac{1}{2} n \sum_{i=1}^{k} (d_i / \lambda_i) \right) \\
= - \frac{1}{2} n \sum_{i=1}^{k} \sum_{j \neq i}^{p} (d_j P_{ij}^2 / \lambda_i) - \frac{1}{2} n \sum_{i=1}^{k} (1 - \sum_{j \neq i}^{p} P_{ij}^2 / \lambda_i) - \left( - \frac{1}{2} n \sum_{i=1}^{k} (d_i / \lambda_i) \right) \\
= \frac{1}{2} n \sum_{i=1}^{k} \sum_{j \neq i}^{p} ((d_i - d_j) P_{ij}^2 / \lambda_i) \\
= \frac{1}{2} n \sum_{i=1}^{k} \sum_{j > i}^{p} ((d_i - d_j) P_{ij}^2 / \lambda_i) + \frac{1}{2} n \sum_{i=1}^{k} \sum_{j < i} (d_i - d_j) P_{ij}^2 / \lambda_i.
\]

Since \(- \frac{1}{2} n \sum_{i=1}^{k} \sum_{j=1}^{p} (d_j P_{ij}^2 / \lambda_i) - \left( - \frac{1}{2} n \sum_{i=1}^{k} (d_i / \lambda_i) \right) \leq 0\), we have

\[
0 \leq \frac{1}{2} n \sum_{i=1}^{k} \sum_{j > i}^{p} ((d_i - d_j) P_{ij}^2 / \lambda_i) \leq - \frac{1}{2} n \sum_{i=1}^{k} \sum_{j < i} (d_i - d_j) P_{ij}^2 / \lambda_i.
\]

That is,

\[
- \frac{1}{2} n \sum_{i=1}^{k} \sum_{j=1}^{p} (d_j P_{ij}^2 / \lambda_i) - \left( - \frac{1}{2} n \sum_{i=1}^{k} (d_i / \lambda_i) \right) = - \frac{1}{2} n \sum_{i=1}^{k} \sum_{j > i} (d_i - d_j) P_{ij}^2 / \lambda_i),
\]

where \( \rho \geq 0 \).

Hence,

\[
A = B - \frac{1}{2} \rho n \sum_{i=1}^{k} \sum_{j > i} ((d_i - d_j) P_{ij}^2 / \lambda_i).
\]

Proof of Lemma 2

Let

\[
C = \Gamma + O_p(\sqrt{\beta_n / n}).
\]
Noting that $C D C' = S$, we obtain

$$(\Gamma + O_p(\sqrt{\beta_n/n}))D(\Gamma' + O_p(\sqrt{\beta_n/n})) = S.$$ 

That is,

$$\Gamma D \Gamma' + O_p(\sqrt{\beta_n/n})(\Gamma D + D \Gamma') + O_p(\beta_n/n)D = S.$$

Then,

$$\Gamma(D - \Delta)\Gamma' + \Gamma \Delta \Gamma' + O_p(\sqrt{\beta_n/n})(\Gamma D + D \Gamma') + O_p(\beta_n/n)D = \Sigma + O_p(\sqrt{1/n}).$$

Hence,

$$\Gamma(D - \Delta)\Gamma' + O_p(\sqrt{\beta_n/n})(\Gamma D + D \Gamma') + O_p(\beta_n/n)D = O_p(\sqrt{1/n}).$$

By Lemmas 9 and 10, for $i = 1, 2, \cdots, p$,

$$|\lambda_i - d_i| \leq \max_{1 \leq i \leq p} |\phi_i| = ||S - \Sigma|| = O_p(\sqrt{1/n}).$$

That is,

$$\Delta = D + O_p(\sqrt{1/n}).$$

Then,

$$O_p(\sqrt{1/n}) + O_p(\sqrt{\beta_n/n})(\Gamma D + D \Gamma') + O_p(\beta_n/n)D = O_p(\sqrt{1/n}).$$
We have
\[ O_p(\sqrt{\beta_n/n})(\Gamma D + D\Gamma') + O_p(\beta_n/n)D = O_p(\sqrt{1/n}). \]

Hence,
\[ \sqrt{\beta_n/n} = O_p(\sqrt{1/n}). \]

Thus,
\[ C = \Gamma + O_p(\sqrt{1/n}), \]

For \( i \neq j \),
\[ P_{ij} = \Gamma'_i C_j = \Gamma'_i (\Gamma_j + O_p(\sqrt{1/n})) = O_p(\sqrt{1/n}). \]

Also,
\[ P_{ii} = \Gamma'_i C_i = \Gamma'_i (\Gamma_i + O_p(\sqrt{1/n})) = 1 + O_p(\sqrt{1/n}). \]

\[ \Box \]

**Proof of Corollary 1**

\[
A = B - \frac{1}{2} \rho n \sum_{i=1}^{k} \sum_{j>i}^{p} ((d_i - d_j)P_{ij}^2 / \lambda_i) \\
= -\frac{1}{2} n (\sum_{i=1}^{k} \log \lambda_i + (p - k) \log \lambda) - \frac{1}{2} n (\sum_{i=1}^{k} (d_i / \lambda_i) + \sum_{i=k+1}^{p} (d_i / \lambda)) \\
-\frac{1}{2} \rho n \sum_{i=1}^{k} \sum_{j>i}^{p} ((d_i - d_j)P_{ij}^2 / \lambda_i) \\
= -\frac{1}{2} n \sum_{i=1}^{k} E_i - \frac{1}{2} n E_{k+1} - \frac{1}{2} k (p - (k + 1)/2)O_p(1) \\
= B - \frac{1}{2} k (p - (k + 1)/2)O_p(1),
\]

where \( E_i = \log \lambda_i + d_i / \lambda_i, \ i = 1, \cdots, k, \ E_{k+1} = (p - k) \log \lambda + \sum_{i=k+1}^{p} (d_i / \lambda) \) and \( B = -\frac{1}{2} n \sum_{i=1}^{k} E_i - \frac{1}{2} n E_{k+1} \). 

\[ \Box \]
Proof of Lemma 3

By Lemmas 9 and 10, for \( i = 1, 2, \cdots, p \),

\[
| \lambda_i - d_i | \leq \max_{1 \leq i \leq p} | \phi_i | = ||S - \Sigma|| = O_p(\sqrt{1/n}).
\]

Thus, for \( i = 1, 2, \cdots, p \), we have

\[
(\lambda_i - d_i)^2 = O_p(1/n).
\]

For fixed \( p \), by Lemmas 9 and 10,

\[
(p - k)(\lambda - \hat{\lambda})^2 = (p - k)(\lambda - \frac{1}{p-k} \sum_{i=k+1}^{p} d_i)^2
= (\sum_{i=k+1}^{p} (\lambda - d_i))^2/(p - k)
\leq p \max_{1 \leq i \leq p} (d_i - \lambda_i)^2
\leq p \max_{1 \leq i \leq p} | \phi_i |^2
= p ||S - \Sigma||^2
= O_p(1/n)
\]

\[\square\]

Proof of Theorem 1

Note that \( E_i \) \((i = 1, \cdots, k)\) attains its minimum at \( \hat{\lambda}_i = d_i \) and \( E_{k+1} \) attains its minimum at \( \hat{\lambda} = 1/(p - k) \sum_{i=k+1}^{p} d_i \). By Taylor’s expansion, \( E_i - \hat{E}_i = \nu_i^{-2} (2d_i/\nu_i - 1)(\lambda_i - \hat{\lambda}_i)^2 \geq 0 \) and \( E_{k+1} - \hat{E}_{k+1} = \nu^{-2} (p - k) (2/(p - k) \sum_{i=k+1}^{p} (d_i/\nu) - 1)(\lambda - \hat{\lambda})^2 \geq 0 \), where \( \min \{ \lambda_i, \hat{\lambda}_i \} \leq \)
\[ \nu_i \leq \max\{\lambda_i, \hat{\lambda}_i\} \text{ and } \min\{\lambda, \hat{\lambda}\} \leq \nu \leq \max\{\lambda, \hat{\lambda}\}. \]

Hence,

\[ \sum_{i=1}^{k} (E_i - \hat{E}_i) = \sum_{i=1}^{k} C_{1i} \gamma_i^{-2} (\lambda_i - \hat{\lambda}_i)^2, \]

where \( C_{1i} = 2d_i / \nu_i - 1 \).

That is,

\[ \sum_{i=1}^{k} (E_i - \hat{E}_i) + (E_{k+1} - \hat{E}_{k+1}) = \sum_{i=1}^{k} C_{1i} \nu_i^{-2} (\lambda_i - \hat{\lambda}_i)^2 + C_2 \nu^{-2} (p - k)(\lambda - \hat{\lambda})^2, \]

where \( C_2 = 2 / (p - k) \sum_{i=k+1}^{p} (d_i / \nu) - 1 \).

Thus,

\[ \sum_{i=1}^{k} (E_i - \hat{E}_i) + (E_{k+1} - \hat{E}_{k+1}) = \sum_{i=1}^{k} C_{1i} \nu_i^{-2} (\lambda_i - \hat{\lambda}_i)^2 + C_2 \nu^{-2} (p - k)(\lambda - \hat{\lambda})^2. \]

That is,

\[ \sum_{i=1}^{k} E_i + E_{k+1} = \sum_{i=1}^{k} \hat{E}_i + \hat{E}_{k+1} + \sum_{i=1}^{k} C_{1i} \nu_i^{-2} (\lambda_i - \hat{\lambda}_i)^2 + C_2 \nu^{-2} (p - k)(\lambda - \hat{\lambda})^2. \]

Noting that \( \sup_{\theta_i \in \Theta_{k+1}} B = -\frac{1}{2} n \sum_{i=1}^{k} \hat{E}_i - \frac{1}{2} n \hat{E}_{k+1} \), we have

\[ B = \sup_{\theta_i \in \Theta_{k+1}} B - \frac{1}{2} n \sum_{i=1}^{k} C_{1i} \nu_i^{-2} (\lambda_i - \hat{\lambda}_i)^2 - \frac{1}{2} n C_2 \nu^{-2} (p - k)(\lambda - \hat{\lambda})^2. \]
By Lemma 3,

\[ A = \sup_{\theta_1 \in \Theta_{k+1}} \left( B - \frac{1}{2} n \sum_{i=1}^{k} C_{1i} \nu_i^{-2} (\lambda_i - \hat{\lambda}_i)^2 - \frac{1}{2} n C_2 \nu_i^{-2} (p - k) (\lambda_i - \hat{\lambda}_i)^2 \right. \\
- \frac{1}{2} k (p - (k + 1)/2) O_p(1) \\
\]

\[ = \sup_{\theta_1 \in \Theta_{k+1}} \left( B - \frac{1}{2} n \sum_{i=1}^{k} C_{1i} \nu_i^{-2} (\lambda_i - d_i)^2 - \frac{1}{2} n C_2 \nu_i^{-2} (p - k) (\lambda_i - \hat{\lambda}_i)^2 \right. \\
- \frac{1}{2} k (p - (k + 1)/2) O_p(1) \\
\]

\[ = \sup_{\theta_1 \in \Theta_{k+1}} \left( B - \frac{1}{2} k (p - (k - 1)/2) O_p(1) - \frac{1}{2} O_p(1) \right. \\
\]

\[ = \sup_{\theta_1 \in \Theta_{k+1}} B - \frac{1}{2} (k (p - (k - 1)/2) + 1) O_p(1). \]

\[ \square \]

**Proof of Lemma 4**

Noting that \((p - k)(\lambda - \hat{\lambda})^2 = O_p(1/n)\), by Taylor’s expansion, we get

\[ \log L_k = -\frac{1}{2} n \left( \sum_{i=1}^{k} \log d_i + (p - k) \log \hat{\lambda} \right) \\
= -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n (p - k) \log \left( \frac{1}{p - k} \sum_{i=k+1}^{p} d_i - \lambda \right) + \lambda \\
= -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n (p - k) \log \left( \frac{1}{p - k} \sum_{i=k+1}^{p} d_i - 1 \right) + 1 \\
= -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n (p - k) \left( \frac{1}{p - k} \sum_{i=k+1}^{p} d_i - 1 \right) \\
\quad - \frac{1}{2} \left( \frac{1}{p - k} \sum_{i=k+1}^{p} d_i - 1 \right)^2 \left( 1 + o(1) \right) \\
= -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n \sum_{i=k+1}^{p} d_i + \frac{1}{2} n (p - k) + O_p(1) \\
= -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n \sum_{i=k+1}^{p} d_i - 1 + O_p(1). \]

Noting that \(\log \log n \to \infty\), (3.2) holds.

\[ \square \]

**Proof of Theorem 2**

Let \( \log \tilde{L}_k = -\frac{1}{2} n \sum_{i=1}^{k} \log d_i - \frac{1}{2} n \sum_{i=k+1}^{p} d_i - 1 \).
Suppose \( k' < k \). We have

\[
\tilde{\ell}(k) - \tilde{\ell}(k') = \log \tilde{L}_k - \log \tilde{L}_{k'} - \gamma(k - k')(p - (k + k')/2 + 1/2).
\]

By Lemmas 9 and 10, for \( i = 1, 2, \cdots, p \),

\[
|\lambda_i - d_i| \leq \max_{1 \leq i \leq p} |\phi_i| = ||S - \Sigma|| = O_p(\sqrt{1/n}).
\]

In the case \( \lim_n \lambda_k = 1 \), by Taylor’s expansion, we get

\[
\begin{align*}
\log \tilde{L}_k - \log \tilde{L}_{k'} &= \frac{1}{2} n \sum_{i=k'+1}^{k}(d_i - (1 - \log d_i)) \\
&= \frac{1}{2} n \sum_{i=k'+1}^{k}(\lambda_i - 1 - \log \lambda_i) + \frac{1}{2} n (d_i - \lambda_i - \log(d_i/\lambda_i)) \\
&= \frac{1}{2} n \sum_{i=k'+1}^{k}(\lambda_i - 1 - \log(1 + (\lambda_i - 1))) \\
&+ \frac{1}{2} n \sum_{i=k'+1}^{k}(d_i - \lambda_i - \log((\lambda_i + (d_i - \lambda_i))/\lambda_i)) \\
&= \frac{1}{4} n (k' - k)(\lambda_k - 1)^2(1 - o(1)) + \frac{1}{2} n \sum_{i=k'+1}^{k}((d_i - \lambda_i)/\lambda_i)^2 \left( 1 + O_p(1) \right) \\
&\geq \frac{1}{4} n (k' - k)(\lambda_k - 1)^2(1 - o(1)) + O_p(1).
\end{align*}
\]

Thus,

\[
P\{\tilde{\ell}(k) > \tilde{\ell}(k')\} = P\{\log \tilde{L}_k - \log \tilde{L}_{k'} > \gamma(k - k')(p - (k + k')/2 + 1/2) \log \log n\} \\
\geq P\{\frac{1}{4} n (\lambda_k - 1)^2(1 - o(1)) + O_p(1) > \gamma(p - k/2 + 1/2) \log \log n\} \\
\rightarrow 1.
\]

In the case \( \lim_n \lambda_k > 1 \), for large \( n \),

\[
\lambda_k - 1 - \log \lambda_k > \epsilon,
\]

where \( \epsilon > 0 \) is a sufficiently small constant.
Since

\[(d_i - 1 - \log d_i) - (\lambda_i - 1 - \log \lambda_i) \xrightarrow{P} 0,\]

for large \(n\), we have

\[
P\{\tilde{\ell}(k) > \tilde{\ell}(k')\} = P\{\log \tilde{L}_k - \log \tilde{L}_{k'} > \gamma(k - k')(p - (k + k')/2 + 1/2) \log \log n\}
\]

\[
= P\{\frac{1}{2} n \sum_{i=k'+1}^{k} (d_i - 1 - \log d_i) > \gamma(k - k')(p - (k + k')/2 + 1/2) \log \log n\}
\]

\[
\geq P\{\frac{1}{2} n (k - k')(\lambda_k - 1 - \log \lambda_k) + o_p(n) > \gamma(k - k')(p - (k + k')/2 + 1/2) \log \log n\}
\]

\[
\geq P\{\frac{1}{2} n (\lambda_k - 1 - \log \lambda_k) + o_p(n) > \gamma(p - k/2 + 1/2) \log \log n\}
\]

\[
\rightarrow 1.
\]

Suppose \(k' > k\). Since \(d_i - 1 = O_p(\sqrt{1/n})\) for \(i > k\), by Taylor’s expansion, we get

\[
\log \tilde{L}_k - \log \tilde{L}_{k'} = \frac{1}{2} n \sum_{i=k+1}^{k'} (\log d_i + 1 - d_i)
\]

\[
= \frac{1}{2} n \sum_{i=k+1}^{k'} (\log(1 + (d_i - 1)) + 1 - d_i)
\]

\[
= -\frac{1}{4} n \sum_{i=k+1}^{k'} (d_i - 1)^2 (1 + o(1))
\]

\[
= -O_p(1).
\]

Thus,

\[
P\{\tilde{\ell}(k) > \tilde{\ell}(k')\} = P\{\log \tilde{L}_k - \log \tilde{L}_{k'} > \gamma(k - k')(p - (k + k')/2 + 1/2) \log \log n\}
\]

\[
> P\{-O_p(1) > \gamma(k - k')(p - (k + k')/2 + 1/2) \log \log n\}
\]

\[
\rightarrow 1.
\]

\[\square\]

**Proof of Lemma 5**

The proof is similar to that of Lemma 4.

\[\square\]
Proof of Corollary 2

The proof is similar to that of Theorem 2.

Proof of Lemma 6

By Lemmas 9 and 10, for \( i = 1, 2, \cdots, p \),

\[
| \lambda_i - d_i | \leq \max_{1 \leq i \leq p} | \phi_i | = ||S - \Sigma|| = O_p(\sqrt{p/n}).
\]

Thus, for \( i = 1, 2, \cdots, p \), we have

\[
(\lambda_i - d_i)^2 = O_p(p/n).
\]

In the case \( n, p \to \infty \) with \( p/n \to c > 0 \), when \( k = 0 \), by Lemma 11, we have

\[
\sum_{i=1}^{p} (d_i - \lambda) \overset{d}{\longrightarrow} N(0, 2\lambda^2 c). \tag{7.1}
\]

In the case \( n, p \to \infty \) with \( p/n \to c > 0 \), when \( k > 0 \), by Lemma 12, we have

\[
\sum_{i=k+1}^{p} (d_i - \lambda) + \sum_{i=1}^{k} (d_i - \lambda_i) \overset{d}{\longrightarrow} N(0, 2\lambda^2 c). \tag{7.2}
\]

By (7.2), for fixed \( k \),

\[
\sum_{i=k+1}^{p} (d_i - \lambda) + O_p(p/n) \overset{d}{\longrightarrow} N(0, 2\lambda^2 c). \tag{7.3}
\]
By (7.1) and (7.3), in the case $n, p \to \infty$ with $p/n \to c > 0$, when $k \geq 0$,

$$(p - k)(\lambda - \hat{\lambda}) = O_p(1).$$

Thus,

$$(p - k)(\lambda - \hat{\lambda})^2 = O_p(1/p) = O_p(1/n).$$

Proof of Lemma 7

The proof is similar to that of Lemma 4.

Proof of Theorem 3

Let $\log \tilde{L}_{k'} = -\frac{1}{2} n \sum_{i=1}^{k'} \log d_i - \frac{1}{2} n \sum_{i=k'+1}^{p} (d_i - 1)$.

Suppose $k' < k$. We have

$$\tilde{\ell}(k) - \tilde{\ell}(k') = \log \tilde{L}_k - \log \tilde{L}_{k'} - \gamma(k - k')(p - (k + k')/2 + 1/2).$$

By Lemma 13, for $k' < i \leq k$,

$$d_i \xrightarrow{a.s.} \psi(\lambda_i) = \lambda_i + \frac{c\lambda_i}{\lambda_i - 1}.$$
Since $\psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2\gamma c$, we have

\[
P\{\ell(k) > \ell(k')\} = P\{\log \tilde{L}_k - \log \tilde{L}_{k'} > \gamma(k - k')(p - (k + k')/2 + 1/2)\}
\]
\[
= P\left\{\frac{1}{2} n \sum_{i=k+1}^{k'} (d_i - 1 - \log d_i) > \gamma(k - k')(p - (k + k')/2 + 1/2)\right\}
\]
\[
= P\left\{\frac{1}{2} n \sum_{i=k+1}^{k'} (\psi(\lambda_i) - 1 - \log \psi(\lambda_i)) > \gamma(k - k')(p - (k + k')/2 + 1/2)\right\}
\]
\[
\geq P\{\psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2\gamma(p - k/2 + 1/2)/n\}
\]
\[
\rightarrow 1.
\]

Suppose $k' > k$. By Lemma 13, for $k < i \leq k' = o(p)$,

\[
d_i \overset{a.s.}{\rightarrow} \mu_1.
\]

According to Bai, Fujikoshi and Choi (2017), $\mu_1 = (1 + \sqrt{c})^2$.

Since $\gamma > 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c$, we get

\[
P\{\ell(k) > \ell(k')\} = P\{\log \tilde{L}_k - \log \tilde{L}_{k'} > -\gamma(k' - k)(p - (k + k')/2 + 1/2)\}
\]
\[
= P\left\{-\frac{1}{2} n \sum_{i=k+1}^{k'} (d_i - 1 - \log d_i) > -\gamma(k' - k)(p - (k + k')/2 + 1/2)\right\}
\]
\[
= P\left\{-\frac{1}{2} n ((1 + \sqrt{c})^2 - 1 - 2 \log(1 + \sqrt{c})) > -\gamma(p - (k + k')/2 + 1/2)\right\}
\]
\[
= P\left\{\gamma > \frac{p}{\gamma(p - (k + k')/2 + 1/2)} (\frac{c}{2} + \sqrt{c} - \log(1 + \sqrt{c}))\right\}
\]
\[
\rightarrow 1.
\]

\[
= P\left\{\gamma > \frac{p}{\gamma(p - (k + k')/2 + 1/2)} \frac{p}{2} (\frac{c}{2} + \sqrt{c} - \log(1 + \sqrt{c}))\right\}
\]
\[
\rightarrow 1.
\]

\[
\text{Proof of Theorem 4}
\]

Let $\log L_{k'} = -\frac{1}{2} n \sum_{i=1}^{k'} \log d_i - \frac{1}{2} n (p - k') \log \hat{\lambda}_{k'}$. According to Bai, Fujikoshi and Choi (2017),

\[
\hat{\lambda}_{k'} = 1/(p - k') \sum_{i=k'+1}^{p} d_i \overset{a.s.}{\rightarrow} 1.
\]
Suppose $k' < k$. We have

$$
\ell(k) - \ell(k') = \log L_k - \log L_{k'} - \gamma(k - k')(p - (k + k')/2 + 1/2).
$$

By Lemma 13, for $k' < i \leq k$,

$$
d_i \xrightarrow{a.s.} \psi(\lambda_i) = \lambda_i + c\lambda_i / \lambda_i - 1.
$$

Since $\psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2\gamma c$, by Taylor’s expansion, we get

$$
P\{\ell(k) > \ell(k')\} = P\{\log L_k - \log L_{k'} > \gamma(k - k')(p - (k + k')/2 + 1/2)\}
= P\{\frac{1}{2}n[(p - k') \log((1 - (k - k')/(p - k'))(1 + \sum_{i=k'}^{k+1} d_i/(p - i)) - \sum_{i=k'}^{k+1} \log d_i] > \gamma(k - k')(p - (k + k')/2 + 1/2)\}
= P\{\frac{1}{2}n \sum_{i=k'}^{k+1} (\psi(\lambda_i) - 1 - \log \psi(\lambda_i)) + O(1) > \gamma(k - k')(p - (k + k')/2 + 1/2)\}
\geq P\{\psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2\gamma(p - (k + k')/2 + 1/2)/n + O(1/n)\}
\geq P\{\psi(\lambda_k) - 1 - \log \psi(\lambda_k) > 2\gamma p/n + O(1/n)\}
\rightarrow 1.
$$

Suppose $k' > k$. By Lemma 13, for $k < i \leq k' = o(p)$,

$$
d_i \xrightarrow{a.s.} \mu_1.
$$

According to Bai, Fujikoshi and Choi (2017), $\mu_1 = (1 + \sqrt{c})^2$. 

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Since $\gamma > 1/2 + \sqrt{1/c} - \log(1 + \sqrt{c})/c$, by Taylor’s expansion, we get

$$P\{\ell(k) > \ell(k')\} = P\{\log L_k - \log L_{k'} > -\gamma(k' - k)(p - (k + k')/2 + 1/2)\}$$

$$= P\{\frac{1}{2}n[(p - k)\log((1 + (k' - k)/(p - k))(1 - \sum_{i=k+1}^{k'} d_i/((p - i)\hat{\lambda})) - \sum_{i=k+1}^{k'} \log \hat{\lambda}_i + \sum_{i=k+1}^{k'} \log d_i] > -\gamma(k' - k)(p - (k + k')/2 + 1/2)\}$$

$$= P\{-\frac{1}{2}n \sum_{i=k+1}^{k'} (d_i - 1 - \log d_i) + O(1) > -\gamma(k' - k)(p - (k + k')/2 + 1/2)\}$$

$$= P\{-\frac{1}{2}n((1 + \sqrt{c})^2 - 1 - 2\log(1 + \sqrt{c})) + O(1) > -\gamma(p - (k + k')/2 + 1/2)\}$$

$$= P\{\gamma > \frac{n}{(p - (k + k')/2 + 1/2)}(\frac{c}{2} + \sqrt{c} - \log(1 + \sqrt{c}) + O(1/n))\}$$

$$= P\{\gamma > \frac{p}{(p - (k + k')/2 + 1/2)}(\frac{c}{2} + \sqrt{c} - \log(1 + \sqrt{c}) + O(1/n))\}$$

$$\to 1.$$
Bai, Z. D. and Yao, J. F. (2012). Central limit theorems for eigenvalues in a spiked population model. *Journal of Multivariate Analysis, 106*, 167–177.

Baik, J. and Silverstein, J. W. (2006). Eigenvalues of large sample covariance matrices of spiked population models. *Journal of Multivariate Analysis, 97*, 1382–1408.

Baik, J., Ben Arous, G. and Peche, S. (2005). Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *The Annals of Probability, 33*, 1643–1697.

Bao, Z., Pan, G. and Zhou, W. (2015). Universality for the largest eigenvalue of sample covariance matrices with general population. *The Annals of Statistics, 43*, 382–421.

Birnbaum, A., Johnstone, I. M., Nadler, B. and Paul, D. (2013). Minimax bounds for sparse PCA with noisy high-dimensional data. *The Annals of Statistics, 41*, 1055–1084.

Bohme, J. F. (1991). Array processing. *in advance in Spectrum Analysis and Array Processing*, S. Haykin, Ed. Englewood Cliffs, NJ: Pretice-Hall, 1-63.

Cai, T. T., Ma, Z. and Wu, Y. (2015). Optimal estimation and rank detection for sparse spiked covariance matrices. *Probability Theory and Related Fields, 161*, 781–815.

Cai, T. T., Ren, Z. and Zhou, H. H. (2016). Estimating structured high-dimensional covariance and precision matrices: Optimal rates and adaptive estimation (with discussion). *Electronic Journal of Statistics, 10*, 1–89.

Choi, Y., Taylor, J. and Tibshirani, R. (2017). Selecting the number of pricipal components: Estimation of the true rank of a noisy matrix. *The Annals of Statistics*, to appear.

Davidson, K.R. and Szarek, S.J. (2001). Local operator theory, random matrices and banach spaces. *Handbook of the geometry of Banach spaces, 1*, 317–366.

Dumitriu, I. and Edelman, A. (2006). Global spectrum fluctuations for the $\beta$-Hermite and $\gamma$-Laguerre ensembles via matrix models *J. Math. Phys., 47*, no. 6, 063302.

El Karoui, N. (2006). A rate of convergence result for the largest eigenvalue of complex white Wishart matrices. *The Annals of Probability, 36*, 2077–2117.

Hall, P. and Heyde, C. C. (1980). *Martingale limit theory and its application*. Academic Press, New York.
Johansson, K. (1998). On fluctuations of random Hermitian matrices. *Duke. Math. J.*, 91, 151–203.

Johansson, K. (2000). Shape fluctuations and random matrices. *Commn. Math. Phys.*, 12, 437–474.

Johnstone, I. M. (2001). On the distribution of the largest eigenvalue in principal component analysis. *The Annals of Statistics*, 29, 295–327.

Johnstone, I. M. and Lu, A. (2009). On consistency and sparsity for principal component analysis in high dimensions. *Journal of the American Statistical Association*, 104, 682–693.

Jonsson, D. (1982). Some limit theorem for the eigenvalues of a sample covariance matrix. *Journal of Multivariate Analysis*, 12, 1–28.

Kritchman, S. and Nadler, B. (2008). Determining the number of components in a factor model from limited noise data. *Chem. Intell. Lab. Syst.*, 94, 19–32.

Kritchman, S. and Nadler, B. (2009). Non-parametric detection of the number of signals, hypothesis tests and random matrix theory. *IEEE Trans. Signal Process.*, 57, 3930–3941.

Ma, Z. (2012). Accuracy of the Tracy-Widom limit for the extreme eigenvalues in the white Wishart matrices. *Bernoulli*, 18, 322-359.

Nadakuditi, R. R. and Edelman, A. (2008). Sample Eigenvalues based detection of high-dimensional signals in white noise using relatively few samples. *IEEE Trans. Signal Process.*, 56, 2625–2638.

Nadler, B. (2008). Finite sample approximation results for principal component analysis: A matrix perturbation approach. *The Annals of Statistics*, 36, 2791–2817.

Nadler, B. (2010). Nonparametric detection of signals by information theoretic criteria: Performance analysis and an improved estimator. *IEEE Trans. Signal Process.*, 58, 2746–2756.

Nicol, M., Simeone, O. and Spagnolini, U. (2003). Multislot estimation of frequency-selective fast-varying channels. *IEEE Trans. Commun.*, 51, 1337–1347.

Muirhead, R. (2002). *Aspects of multivariate statistical theory*, John Wiley and Sons, Inc., New York Wiley Series in Probability and Mathematical Statistics.
Onatski, A., Moreira, M. J. and Hallin, M. (2013) Asymptotic power of sphericity test for high-dimensional data. *The Annals of Statistics, 43*, 1204–1231.

Passemier, D. and Yao, J. (2014). Estimation of the number of spikes, possibly equal, in the high-dimensional case. *Journal of Multivariate Analysis, 127*, 173–183.

Passemier, D., Matthew and Chen, Y. (2015). Asymptotic linear spectral statistics for spiked Hermitian random matrices. *Journal of Statistical Physics, 160*, 120–150.

Paul, D. (2007). Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statistica Sinica, 17*, 1617–1642.

Reznik, M. K. (1968). The law of the iterated logarithm for some classes of stationary process. *Theory of Probability and Its Applications, 8*, 606–621.

Rissanen, J. (1968). Modeling by shortest data description. *Automatica, 14*, 465–471.

Schwarz, G. (1978). Estimating the dimension of a model. *The Annals of Statistics, 6*, 461–464.

Stout, W. F. (1974). *Almost sure convergence*. Academic Press, New York.

Wax, M. and Kailath, T (1985). Detection of signals by information theoretic criteria. *IEEE Trans. Acoust., Speech, Signal Process., 33*, 387–392.

Wang, Q., Silverstein, J. and Yao, J. (2014). A note on the CLT of the LSS for sample covariance matrix from a spiked population model. *Journal of Multivariate Analysis, 130*, 194–207.

Wang, Q. and Yao, J. (2013). On the sphericity test with large-dimensional observations. *Electronic Journal of Statistics, 7*, 2164–2192.

Weyl, H. (1912). Der asymptotische Verteilungs gesetz der Eigenwerte linearer partieller Differentialgleichungen. *Mathematische Annalen, 71*, 441–479.

Wishart, J. (1928). The generalized product moment distribution in samples from a normal multivariate population. *Biometrika, 20*, 32–52.

Zhao, L. C., Krishnaiah, P. R. and Bai, Z. D. (1986). On detection of the number of signals in presence of white noise. *Journal of Multivariate Analysis, 20*, 1–25.