A RELATION BETWEEN HIGHER-RANK PT STABLE OBJECTS AND QUOTIENTS OF COHERENT SHEAVES

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ABSTRACT. On a smooth projective threefold, we construct an essentially surjective functor \( F \) from a category of two-term complexes to a category of quotients of coherent sheaves, and describe the fibers of this functor. Under a coprime assumption on rank and degree, the domain of \( F \) coincides with the category of higher-rank PT stable objects, which appear on one side of Toda’s higher-rank DT/PT correspondence formula. The codomain of \( F \) is the category of objects that appear on one side of another correspondence formula by Gholampour-Kool, between the generating series of topological Euler characteristics of two types of quot schemes.

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1. INTRODUCTION

On a smooth projective threefold \( X \), Gholampour-Kool computed the generating series of some moduli spaces of slope stable sheaves of homological dimension at most one \([5]\). An integral part of their argument was the following counting formula, where \( \text{Quot}_X(\cdot, n) \) denotes the quot scheme of length-\( n \) quotients of a coherent sheaf, \( e(\cdot) \) denotes the topological Euler characteristic, and \( M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \) is the MacMahon function:

**Theorem 1.1.** \([5\text{, Theorem 1.1}]\) For any rank \( r \) torsion-free sheaf \( F \) of homological dimension at most 1 on a smooth projective threefold \( X \), we have

\[
\sum_{n=0}^{\infty} e(\text{Quot}_X(F, n))q^n = M(q)^{\text{re}(X)} \sum_{n=0}^{\infty} e(\text{Quot}_X(\text{Ext}^1(F, O_X), n))q^n.
\]

On the other hand, on a smooth projective Calabi-Yau threefold \( X \), Toda proved a correspondence formula between higher-rank Donaldson-Thomas (DT) and Pandharipande-Thomas (PT) invariants. While DT invariants virtually count slope stable sheaves on \( X \), PT invariants count PT stable objects in the derived category of coherent sheaves \( D^b(X) = D^b(\text{Coh}(X)) \) on \( X \). PT stability is a type

\[\text{(1.1.1)}\]

2010 Mathematics Subject Classification. Primary 14J30; Secondary: 14D23.

Key words and phrases. stable pair, PT stable object, quot scheme.
of polynomial stability on $D^b(X)$ in the sense of Bayer \cite{11}; a rank-one PT stable object of trivial determinant is exactly a stable pair

$$ (1.1.2) \quad \mathcal{O}_X \xrightarrow{\sim} F $$

in the sense of Pandharipande-Thomas \cite{13}, which we call a PT stable pair, where $F$ is a pure 1-dimensional sheaf and the cokernel of the morphism of sheaves $s$ is 0-dimensional. The properties of PT stable objects were studied and their moduli spaces constructed by the author in \cite{9, 11, 10}. For any ample divisor $\omega$ on $X$ and any $(r, D, -\beta, -n) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X)$ where $r \geq 1$ with $r, D, -\beta$ coprime, let us write $\text{DT}(r, D, -\beta, -n)$ to denote the DT invariant virtually counting $\mu_\omega$-stable sheaves of Chern character $(r, D, -\beta, -n)$, and $\text{PT}(r, D, -\beta, -n)$ to denote the PT invariant virtually counting PT stable objects of that Chern character. Then Toda’s correspondence formula reads:

**Theorem 1.2.** \cite{17} Theorem 1.2 For a fixed $(r, D, \beta)$, we have

$$ (1.2.1) \quad \sum_{6n \in \mathbb{Z}} \text{DT}(r, D, -\beta, -n)q^n = M(((-1)^r) r^{\text{ch}(X)} \sum_{6n \in \mathbb{Z}} \text{PT}(r, D, -\beta, -n)q^n. $$

The case $(r, D) = (1, 0)$ of the formula (1.2.1), i.e. rank-one DT/PT correspondence, was first conjectured in \cite{13} and first proved by Bridgeland \cite{2}. Toda also gave a proof under an assumption on the local structure of the moduli stacks involved \cite{16}; the assumption was later removed in \cite{17}.

In this article, we describe a relation between the objects that appears on the right-hand side of Gholampour-Kool’s formula (1.1.1) and the objects that appears on the right-hand side of Toda’s formula (1.2.1). More precisely, on a smooth projective threefold $X$, we define a category $\mathcal{E}_0$ of 2-term complexes in $D^b(X)$ with cohomology at degrees $-1, 0$, that contains all the PT semistable objects in $D^b(X)$. The category $\mathcal{E}_0$ also contains all the ‘frozen triples’ in the sense of Sheshmani \cite{15}, which gives an alternative approach for generalising Pandharipande-Thomas’ stable pairs (1.1.2) to higher ranks. We write $\text{Mor}(\text{Coh}(X))$ to denote the category where the objects are morphisms of coherent sheaves on $X$, and morphisms are given by commutative squares in $\text{Coh}(X)$. For any coherent sheaf $A$ on $X$, let us write $S(A)$ to denote the full subcategory of $\text{Mor}(\text{Coh}(X))$ consisting of objects of the form $A \to Q$ where $Q$ is a 0-dimensional sheaf, and $q$ is a surjection of sheaves. We construct a (contravariant) functor

$$ F : \mathcal{E}_0 \to \prod_{F \in \text{Coh}(X), \text{hd}(F) \leq 1}\text{S}(\mathcal{Ext}^1(F, \mathcal{O}_X))^{\text{op}} $$

(see Definition 5.4) and prove our main result:

**Theorem 1.3.** (Theorem 5.9) The functor $F$ is essentially surjective. If we fix an ample class on $X$, fix $r \in H^0(X), D \in H^2(X)$ such that $r, D, -\beta$ are coprime, restrict the domain of $F$ to PT stable objects $E$ with $\text{ch}_0(E) = -r, \text{ch}_1(E) = -D$ and restrict the codomain by requiring $F$ above to be $\mu_\omega$-stable with $\text{ch}_0(F) = r, \text{ch}_1(F) = D$, then the restriction of $F$ is also essentially surjective.

In Section 6, we analyse the fibers of the functor $F$. We describe how to enumerate all the objects in a given fiber of $F$ in Lemma 6.2. For two objects $E, \tilde{E}$ of $\mathcal{E}_0$, we pin down the difference between $E, \tilde{E}$ using $\mathcal{E}_0$ being isomorphic in $D^b(X)$ and $F(E), F(\tilde{E})$ being isomorphic in $\text{Mor}(\text{Coh}(X))$. Finally, we recall a construction mentioned in Gholampour-Kool’s work in which a sheaf quotient $\mathcal{E}_{\text{Ext}}^1(I_C, \mathcal{O}_X) \rightarrow Q$, where $I_C$ is the ideal sheaf of a Cohen-Macaulay curve $C$ on $X$ and $Q$ is a 0-dimensional sheaf, can be used to construct a PT stable pair (i.e. a rank-one PT stable object). We generalise this construction to higher ranks in 6.8, so that given a higher-rank sheaf quotient, we produce a higher-rank PT stable object. We end the article with Lemma 6.9 which compares this higher-rank construction and the functor $F$ constructed in Section 5.

1.1. Acknowledgements. The author would like to thank Yunfeng Jiang, Martijn Kool and Zhenbo Qin for answering his various questions, and Jun Li and Ziyu Zhang for helpful discussions.
2. Preliminaries

2.1. Notation. Unless otherwise stated, we will write $X$ for a smooth projective threefold in this article, $\text{Coh}(X)$ for the category of coherent sheaves on $X$, and $D^b(X) = D^b(\text{Coh}(X))$ for the bounded derived category of coherent sheaves on $X$.

2.1. For any category $\mathcal{C}$, we will write $\text{Mor}(\mathcal{C})$ to denote the category of morphisms in $\mathcal{C}$. That is, the objects of $\text{Mor}(\mathcal{C})$ are morphisms $f : A \to B$ in $\mathcal{C}$, and a morphism between two objects $f : A \to B, f' : A' \to B'$ of $\text{Mor}(\mathcal{C})$ is a commutative diagram in $\mathcal{C}$

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \quad \downarrow \\
A' \xrightarrow{f'} B'
\end{array}
\]

2.2. For any object $E \in D^b(X)$ and any subcategory $\mathcal{C}$ of $D^b(X)$, we will write $\text{Hom}(\mathcal{C}, E) = 0$ to mean $\text{Hom}_{D^b(X)}(C, E) = 0$ for all $C \in \mathcal{C}$, and similarly for $\text{Hom}(E, \mathcal{C}) = 0$.

2.3. For any integer $d$, we will write $\text{Coh}^{\leq d}(X)$ to denote the Serre subcategory of $\text{Coh}(X)$ consisting of sheaves $E$ supported in dimension at most $d$. We will also write $\text{Coh}^{= d}(X)$ to denote the full subcategory of $\text{Coh}(X)$ consisting of sheaves $E$ such that $\text{Hom}(\text{Coh}^{\leq d-1}(X), E) = 0$, i.e. sheaves $E$ that have no subsheaves supported in dimension $d-1$ or lower. Then we set $\text{Coh}^{= d}(X) = \text{Coh}^{\leq d}(X) \cap \text{Coh}^{\leq d}(X)$, which is the category of pure $d$-dimensional sheaves on $X$.

2.4. Given any object $E \in D^b(X)$ and any integer $i$, we will write $H^i(E)$ to denote the degree-$i$ cohomology of $E$ with respect to the standard $t$-structure on $D^b(X)$. We then define

\[D^{\leq i}_{\text{Coh}(X)} = \{ E \in D^b(X) : H^k(E) = 0 \text{ for all } k < i \}\]

and similarly $D^{\leq i}_{\text{Coh}(X)}$. For any integers $i \leq j$, we set

\[D^{[i,j]}_{\text{Coh}(X)} = D^{\leq i}_{\text{Coh}(X)} \cap D^{\leq j}_{\text{Coh}(X)}\]

2.5. Given a coherent sheaf $F$ on $X$, we will refer to the dimension (resp. codimension) of $\text{supp}(F)$ simply as the dimension (resp. codimension) of $F$, and denote it as $\dim F$ (resp. $\text{codim} F$).

2.6. For any $F \in D^b(X)$, we will write $F^\vee$ to denote the derived dual $R\mathcal{H}om(F, \mathcal{O}_X)$ of $F$. When $F$ is a coherent sheaf of codimension $c$, we will write $F^*$ to denote the usual sheaf dual of $F$, i.e. $\mathcal{E}xt^c(F, \mathcal{O}_X)$; note that $H^c(F^\vee) = F^*$. Given a pure codimension-$c$ coherent sheaf $F$ on $X$, we will say $F$ is reflexive if the natural injection $F \hookrightarrow F^{**}$ is an isomorphism.

2.7. Recall that the homological dimension of a coherent sheaf $F$ on a smooth projective variety $X$ is defined to be the minimal length of a locally free resolution of $F$, and that a coherent sheaf of homological dimension $n$ satisfies $\mathcal{E}xt^i(F, \mathcal{O}_X) = 0$ for all $i > n$ and hence $F^{\vee} \in D^{[0,n]}(\text{Coh}(X))$. We will write $\text{hd}(F)$ to denote the homological dimension of a coherent sheaf $F$ on $X$.

2.8. Stable pairs. On a smooth projective threefold $X$, a stable pair in the sense of Pandharipande-Thomas [13] is a pure 1-dimensional sheaf $F$ together with a section $\mathcal{O}_X \rightarrow F$ such that coker $(s)$ is 0-dimensional. The purity of $F$ implies that the support of $F$ is a Cohen-Macaulay curve. We often think of a stable pair $\mathcal{O}_X \rightarrow F$ as a 2-term complex representing an object in $D^b(X)$, with $F$ sitting at degree 0. We will refer to a stable pair in the sense of [13] as a PT stable pair, or simply a stable pair.

2.9. PT stable objects. Bayer characterised PT stable pairs using the notion of polynomial stability in [11]. There is a particular polynomial stability $\sigma_{PT}$ on $D^b(X)$, referred to as PT stability by Bayer,
such that the $\sigma_{PT}$-stable objects objects $E$ in the heart

$$A^0 := \langle \text{Coh}^{\geq 2}(X)[1], \text{Coh}^{\leq 1}(X) \rangle$$

with $\text{ch}_0(E) = -1, \text{ch}_1(E) = 0$ and $\det E = O_X$ are precisely the PT stable pairs in [2.8]. We will refer to $\sigma_{PT}$-(semi)stable objects in $A^0$ of any Chern character as PT (semi)stable objects. The properties of higher-rank PT stable objects and their moduli spaces were studied in [9, 11].

2.9.1. Suppose $X$ is a smooth projective threefold, and $\omega$ is a fixed ample class on $X$ that appears in the definition of PT stability. Then every PT semistable object $E$ with nonzero $\text{ch}_0$ satisfies the following properties:

(i) $H^{-1}(E)$ is torsion-free and $\mu_\omega$-semistable,

(ii) $H^0(E)$ is 0-dimensional,

(iii) $\text{Hom}_{D^b(X)}(\text{Coh}^{\leq 0}(X), E) = 0$;

moreover, when $\text{ch}_0(E)$ and $\omega^2 \text{ch}_1(E)$ are coprime, every object in $A^0$ satisfying (i) through (iii) is a PT stable object, and PT stability coincides with PT semistability [11, Proposition 2.24]. Also, properties (i) and (ii) implies that, if $E$ is a PT-semistable object, then $\text{ch}_0(E) = -n$ for some nonnegative integer $n$; we will sometimes refer to such an $E$ as a rank $n$ PT semistable object by abuse of notation.

2.9.2. Under derived dual and up to a shift, PT stability corresponds to another polynomial stability $\sigma_{PT}$, meaning $\sigma_{PT}$-stable objects and $\sigma_{PT}$-stable objects correspond to each other via derived dual. We will refer to the $\sigma_{PT}$-(semi)stable objects as dual-PT (semi)stable objects; their properties and moduli spaces were studied in [10].

2.9.3. Suppose $X$ is a smooth projective threefold, and $\omega$ is a fixed ample class on $X$ that appears in the definition of dual-PT stability. Then a standard argument shows that every dual-PT semistable object $E$ with nonzero $\text{ch}_0$ satisfies the following properties besides lying in $A^0$:

(i) $H^{-1}(E)$ is torsion-free and $\mu_\omega$-semistable.

(ii) $\text{Hom}_{D^b(X)}(\text{Coh}^{\leq 1}(X), E) = 0$.

Property (ii) implies that $H^{-1}(E)$ is a reflexive sheaf. Also, when $\text{ch}_0(E)$ and $\omega^2 \text{ch}_1(E)$ are coprime, every object in $A^0$ satisfying (i) and (ii) is a dual-PT stable object, and dual-PT stability coincides with dual-PT semistability [10] Lemma 3.5).

Remark 2.10. In Toda’s work [17], he directly defines PT semistable objects to be the objects in $D^b(X)$ satisfying properties (i) through (iii). All the computations in [17], however, are performed under the assumption that $\text{ch}_0, \omega^2 \text{ch}_1$ are coprime; under this assumption, the PT semistable objects Toda studies coincide with the PT semistable objects defined using Bayer’s polynomial stability (as in 2.9).

3. The dualising functor

In this section, we study the behaviour of a class of 2-term complexes under the derived dual functor $^\vee$. These 2-term complexes can be taken to be various stable objects (see Section 5) and, in particular, PT stable objects.

Lemma 3.1. Let $E$ be an object of $D^b(X)$ satisfying $E^\vee \in D^C_{\text{Coh}(X)}$. Then

$$\text{Hom}_{D^b(X)}(\text{Coh}^{\leq 0}(X), E) = 0 \text{ if and only if } H^3(E^\vee) = 0.$$ 

Proof. For any $E \in D^b(X)$ and $T \in \text{Coh}^{\leq 0}(X)$ we have

$$\text{Hom}(T, E) \cong \text{Hom}(E^\vee, T^*[−3]).$$
Therefore, when $E$ satisfies $E^\vee \in D^{\leq 3}_{\text{Coh}(X)}$, we have $H^3(E^\vee) = 0$ if and only if $\text{Hom}(T, E) = 0$ for all $T \in \text{Coh}^{\leq 0}(X)$, i.e. $\text{Hom}(\text{Coh}^{\leq 0}(X), E) = 0$.

Example 3.2. For any $E \in \langle \text{Coh}^{\leq 1}(X)[1], \text{Coh}^{\leq 1}(X) \rangle$, in the associated exact triangle

$$H^0(E)^\vee \to E^\vee \to H^{-1}(E)^\vee[-1] \to H^0(E)^\vee[1]$$

we have $H^0(E)^\vee \in D^{[2,3]}_{\text{Coh}(X)}$ and $H^{-1}(E)^\vee \in D^{[0,2]}_{\text{Coh}(X)}$ [8 Proposition 1.1.6] and hence $E^\vee \in D^{\leq 3}_{\text{Coh}(X)}$.

Thus

$$\text{Hom}(\text{Coh}^{\leq 0}(X), E) = 0 \text{ if and only if } H^3(E^\vee) = 0$$

for such $E$ by Lemma 3.1.

Lemma 3.3. Suppose $X$ is a smooth projective threefold and $F \in \text{Coh}^{\geq 2}(X)$. Then

(i) $F$ has homological dimension at most 1 if and only if

(3.3.1) $\text{Hom}(\text{Coh}^{\leq 0}(X), F[1]) = 0$.

(ii) If $F$ is torsion-free, then $F$ is reflexive if and only if

(3.3.2) $\text{Hom}(\text{Coh}^{\leq 1}(X), F[1]) = 0$.

Proof. (i) Taking derived dual, we observe that

(3.3.3) $\text{Hom}(\text{Coh}^{\leq 0}(X), F[1]) = 0 \text{ if and only if } \text{Hom}(F^\vee, \text{Coh}^{\leq 0}(X)[-2]) = 0$

for any $F \in \text{D}^b(X)$. For any $F \in \text{Coh}^{\geq 2}(X)$, we know $F^\vee \in D^{[0,2]}_{\text{Coh}(X)}$ from [8 Proposition 1.1.6], and so $F$ satisfies the equivalent conditions in (3.3.3) if and only if $F^\vee \in D^{[0,1]}_{\text{Coh}(X)}$, which in turn is equivalent to $F$ having homological dimension at most 1 [6 III.6].

(ii) This is a special case of [8 Lemma 4.20].

The results in the remainder of this provide a common ground across the constructions in this article, Toda’s work [17], Gholampour-Kool’s work [5], and the author’s previous work [10].

Lemma 3.4. The category

(3.4.1) $\{ E \in \langle \text{Coh}^{\geq 3}(X)[1], \text{Coh}^{\leq 1}(X) \rangle : \text{Hom}(\text{Coh}^{\leq 0}(X), E) = 0 \}$

is invariant under the functor $(-^\vee)[2]$. Moreover, $H^{-1}(E)$ is slope (semi-)stable if and only if $H^{-1}(E^\vee[2])$ is so.

Proof. Take any object $E$ in the category (3.4.1). If $H^{-1}(E) = 0$, then $E = H^0(E)$ is a pure 1-dimensional sheaf, in which case $E^\vee[2] \cong \text{Ext}^2(E, \mathcal{O}_X)$ is also a pure 1-dimensional sheaf and hence again lies in the category (3.4.1). So let us suppose $H^{-1}(E) \neq 0$ from now on. The exact triangle

$$H^0(E)^\vee \to E^\vee \to H^{-1}(E)^\vee[-1] \to H^0(E)^\vee[1]$$
Lemma 3.6. The functor \( H_E \) where the first isomorphism follows from taking derived dual, and the last isomorphism uses the proof.

As a result, Lemma 3.6 can be considered as a purely homological version of the statement that ‘PT stable objects and dual-PT stable objects correspond to each other under derived dual’. We also note that this Lemma had essentially appeared in Piyaratne-Toda’s work [14, Lemma 4.16] in their study of the moduli spaces of Bridgeland semistable objects on threefolds.

Also, by Lemma 3.3 all the torsion-free coherent sheaves of homological dimension at most 1 (resp. torsion-free reflexive sheaves) sitting at degree -1 lie in the left-hand side (resp. right-hand side) of (3.6.1).

Proof. Suppose \( E \) is an object in the left-hand side of (3.6.1). By Lemma 3.4, it suffices to show that \( \text{Hom}(\text{Coh}^{=1}(X), E^\vee[2]) = 0 \). For any \( T \in \text{Coh}^{=1}(X) \), we have \( \text{Hom}(T, E^\vee[2]) \cong \text{Hom}(H^0(E), T^*) \)
by (3.5.1); the latter Hom vanishes since \( H^0(E) \in \text{Coh}^{\leq 0}(X) \) by assumption while \( T^* \) is pure 1-dimensional.

Conversely, suppose \( E \) is an object in the right-hand side of (3.6.1). Again, by Lemma 3.4 it suffices to show that \( H^0(E^\vee[2]) \in \text{Coh}^{\leq 0}(X) \). Since Lemma 3.4 already gives \( H^0(E^\vee[2]) \in \text{Coh}^{\leq 1}(X) \), it suffices to show \( \text{Hom}(H^0(E^\vee[2]), \text{Coh}^{-1}(X)) = 0 \). To this end, take any \( T \in \text{Coh}^{-1}(X) \); note that \( T \) is reflexive. Then (3.5.1) gives \( \text{Hom}(T^*, E) \cong \text{Hom}(H^0(E^\vee[2]), T) \), in which the former Hom vanishes because \( \text{Hom}(\text{Coh}^{\leq 1}(X), E) = 0 \) by assumption.

\[ \text{Corollary 3.7.} \quad \text{The category} \]

\[ \{ E \in \langle \text{Coh}^{=3}(X)[1], \text{Coh}^{\leq 0}(X) \rangle : \text{Hom}(\text{Coh}^{\leq 1}(X), E) = 0 \} \]

is invariant under the functor \( -^\vee[2] \).

Note that the category (3.7.1) contains all the reflexive sheaves (shifted by 1); in fact, it contains all the objects that are PT semistable and dual-PT semistable at the same time, the moduli space of which was studied in [10, Theorem 1.2] with a coprime assumption on the rank and degree of the objects.

### 4. A Functor Taking Objects to Morphisms

In this section, we construct a functor \( \tilde{F} \) that takes a subcategory \( \mathcal{E} \) of \( E \in D_{[1]}^{-1,0} \) into the category \( \text{Mor}(D^b(X)) \) (Proposition 4.4). Composing with the cohomology functor \( H^0 \), we obtain a functor from \( \mathcal{E} \) into \( \text{Mor}(\text{Coh}(X)) \) (see 4.4.5).

#### 4.1. We define the full subcategory of \( D^b(X) \)

\[ \mathcal{E} = \{ E \in D_{\text{Coh}(X)}^{-1,0} : \text{hd}(H^{-1}(E)) \leq 1, H^0(E) \in \text{Coh}^{\leq 0}(X) \} \]

and the full subcategory of \( \text{Mor}(D^b(X)) \)

\[ \mathcal{L} = \{ A^\vee[1] \rightarrow B : A \in \text{Coh}(X), \text{hd}(A) \leq 1, B \in \text{Coh}^{\leq 0}(X) \} \].

#### 4.1.1. Note that for any coherent sheaf \( F \) on \( X \), the condition \( \text{hd}(F) \leq 1 \) implies \( \text{Coh}^{\geq 2}(X) \) by [8, Proposition 1.1.6]. Hence \( H^{-1}(E) \in \text{Coh}^{\geq 2}(X) \) for any \( E \in \mathcal{E} \).

#### 4.2. For any \( E \in \mathcal{E} \), truncation functors with respect to the standard t-structure on \( D^b(X) \) give the canonical exact triangle

\[ (4.2.1) \quad H^{-1}(E)[1] \rightarrow E \rightarrow H^0(E) \rightarrow H^{-1}(E)[2] \]

Applying the derived dual functor to (4.2.1) followed by the shift functor [2] gives the exact triangle

\[ (4.2.2) \quad H^0(E)^\vee[2] \rightarrow E^\vee[2] \rightarrow H^{-1}(E)^\vee[1] \rightarrow H^0(E)^\vee[3] \]

where \( H^{-1}(E)^\vee[1] \in D_{\text{Coh}(X)}^{-1,0} \) by the assumption \( \text{hd}(H^{-1}(E)) \leq 1 \), and \( H^0(E)^\vee[3] \in \text{Coh}^{\leq 0}(X) \). That is, the morphism \( w^\vee[3] \) is an object of \( \mathcal{L} \).

\[ \text{Definition 4.3.} \quad \text{For any} \ E \in \mathcal{E}, \text{we define} \ \tilde{F}(E) \text{ to be the object} \ w^\vee[3] \text{ of} \ \mathcal{L} \text{ in the notation of 4.2} \]

\[ \text{Proposition 4.4.} \quad \tilde{F} \text{ is an essentially surjective (contravariant) functor from} \ \mathcal{E} \ \text{to} \ \mathcal{L}, \text{and induces a bijection between the isomorphism classes in} \ \mathcal{E} \ \text{and} \ \mathcal{L}. \]
Proof. Step 1. Given any morphism \( E_1 \xrightarrow{f} E_2 \) in \( \mathcal{E} \), the truncation functors give a morphism of exact triangles in \( D^b(X) \)

\[
\begin{array}{c}
H^{-1}(E_1)[1] & \xrightarrow{f} & E_1 & \xrightarrow{H^0(E_1)} & H^{-1}(E_1)[2] \\
\downarrow & & \downarrow & & \downarrow \\
H^{-1}(E_2)[1] & \xrightarrow{E} & E_2 & \xrightarrow{H^0(E_2)} & H^{-1}(E_2)[2]
\end{array}
\]

Applying the derived dual functor followed by the shift functor \([2]\) gives

\[
\begin{array}{c}
H^0(E_2)[2] & \xrightarrow{f^\vee[2]} & E_2 & \xrightarrow{H^0(E_2)[2]} & H^{-1}(E_2)[2] \\
\downarrow & & \downarrow & & \downarrow \\
H^0(E_1)[2] & \xrightarrow{\omega^0[2]} & E_1 & \xrightarrow{H^0(E_1)[2]} & H^{-1}(E_1)[2]
\end{array}
\]

The right-most square now gives a morphism from \( \omega^1[3] \) to \( \omega^2[3] \) in \( \mathcal{L} \). It is clear that \( \mathcal{F} \) respects composition of morphisms in \( \mathcal{E} \), and so \( \mathcal{F} \) is a functor from \( \mathcal{E} \) to \( \mathcal{L} \).

Step 2. To show the essential surjectivity of \( \mathcal{F} \), let us take an arbitrary element of \( \mathcal{L} \), say the diagram in \( D^b(X) \)

\[
A^\vee[1] @>>> B
\]

where \( A \) is a sheaf of homological dimension at most 1, and \( B \) is a sheaf supported in dimension 0. We first complete \( m \) to an exact triangle in \( D^b(X) \)

\[
(4.4.1) \quad A^\vee[1] @>>> B \rightarrow C \rightarrow A^\vee[2].
\]

Applying \([-3]^\vee\) now gives the exact triangle

\[
A[1] @>>> G \rightarrow B^\vee[3] @>>> A[2]
\]

where \( G := C^\vee[3] \) is an object of \( \mathcal{E} \). Since \( A[1] \in D^b_{\text{Coh}(X)} \) and \( B^\vee[3] \in D^b_{\text{Coh}(X)} \), there is a canonical isomorphism of exact triangles in \( D^b(X) \) [4 Lemma 5, IV.4]

\[
(4.4.2) \quad A[1] @>>> G @>>> B^\vee[3] @>>> A[2] \quad .
\]

Applying the functor \( \cdot^\vee[2] \) gives the isomorphism of exact triangles

\[
(4.4.3) \quad H^0(G)^\vee[2] @>>> G^\vee[2] @>>> H^{-1}(G)^\vee[1] @>>> H^0(G)^\vee[3] \quad .
\]

Since \( \omega^\vee[3] \) is precisely \( \mathcal{F}(G) \), we have shown the essential surjectivity of \( \mathcal{F} \).

Step 3. To show that \( \mathcal{F} \) induces a bijection between the isomorphism classes in \( \mathcal{E} \) and \( \mathcal{L} \), let us take two objects \( E_1, E_2 \) in \( \mathcal{E} \) and suppose there is an isomorphism from \( \mathcal{F}(E_2) \) to \( \mathcal{F}(E_1) \) in \( \mathcal{L} \), say given by
the diagram in $D^b(X)$

\[
\begin{array}{ccc}
H^{-1}(E_2)^\vee[1] & \longrightarrow & H^0(E_2)^\vee[3] \\
\downarrow h & & \downarrow i \\
H^{-1}(E_1)^\vee[1] & \longrightarrow & H^0(E_1)^\vee[3]
\end{array}
\]

where $h, i$ are isomorphisms. We can complete the rows of this square to exact triangles of the form

\[
\begin{array}{ccc}
H^0(E_2)^\vee[2] & \longrightarrow & E_2^\vee[2] \longrightarrow H^{-1}(E_2)^\vee[1] \longrightarrow H^0(E_2)^\vee[3]; \\
\downarrow i[-1] & & \downarrow g \\
H^0(E_1)^\vee[2] & \longrightarrow & E_1^\vee[2] \longrightarrow H^{-1}(E_1)^\vee[1] \longrightarrow H^0(E_1)^\vee[3]
\end{array}
\]

and then $h, i$ can be completed with an isomorphism $g : E_2^\vee[2] \rightarrow E_1^\vee[2]$ to an isomorphism of exact triangles [4, Corollary 4a, IV.1]. Hence $E_1$ and $E_2$ are isomorphic in $D^b(X)$. This completes the proof of the proposition.

4.4.4. In the proof of essential surjectivity in Proposition [4.4] (i.e. Step 2 of the proof), it is not clear that the construction taking the object $m$ in $\mathcal{L}$ to the object $G$ in $\mathcal{E}$ is a functor, since the object $C$ is defined up to an isomorphism that is not necessarily canonical.

4.4.5. The degree-zero cohomology functor with respect to the standard t-structure $H^0 : D^b(X) \rightarrow \text{Coh}(X)$ induces a functor $\text{Mor}(D^b(X)) \rightarrow \text{Mor}(\text{Coh}(X))$ which we will also denote by $H^0$. For any object in $\mathcal{L}$ of the form

\[
A^\vee[1] \xrightarrow{m} B,
\]

its image under $H^0$ is

\[
H^0(A^\vee[1]) = \text{Ext}^1(A, \mathcal{O}_X) \xrightarrow{H^0(m)} B.
\]

Let us use the notation in the proof of Proposition [4.4] and take $C, w$ as in (4.4.1), (4.4.2), respectively. Then from (4.4.3), the morphism of sheaves $H^0(m)$ is surjective if and only if the morphism of sheaves $H^0(w^\vee[3])$ is surjective.

5. Essential Surjectivity of the Functor $\mathcal{F}$

In this section, we will modify the functor $\mathcal{F}$ to a new functor $\mathcal{F}$, and show that each object $E \in D^{[-1,0]}_{\text{Coh}(X)}$ of the following types is taken by $\mathcal{F}$ to a surjective morphism of sheaves $\text{Ext}^1(H^{-1}(E), \mathcal{O}_X) \rightarrow H^0(E)^*$: PT-semistable objects, dual-PT semistable objects, objects giving rise to $L$-invariants in the sense of Toda [17], and stable frozen triples in the sense of Sheshmani [15]. In particular, under a coprime assumption on rank and degree, we prove in Theorem 5.9 that $\mathcal{F}$ restricts to an essentially surjective functor from the category of PT stable objects to a category of surjective morphisms of coherent sheaves.

Lemma 5.1. Suppose $E$ is an object in $(\text{Coh}^{\geq 2}(X))[1], \text{Coh}^{\leq 1}(X))$ and $H^{-1}(E)$ has homological dimension at most 1. Suppose

\[
H^{-1}(E)[1] \rightarrow E \rightarrow H^0(E) \xrightarrow{\alpha} H^{-1}(E)[2]
\]

is the associated canonical exact triangle in $D^b(X)$. Then

\[
H^0(\mathcal{F}(E)) = H^0(w^\vee[3]) : \text{Ext}^1(H^{-1}(E), \mathcal{O}_X) \rightarrow H^0(E)^*
\]

is a surjection in $\text{Coh}(X)$ if and only if $\text{Hom}_{D^b(X)}(\text{Coh}^{\leq 0}(X), E) = 0$. 


Proof. Applying $\vee[2]$ to (4.2.1) gives us the exact triangle
\[
H^0(E)^\vee[2] \to E^\vee[2] \to H^{-1}(E)^\vee[1] \xrightarrow{w^\vee[3]} H^0(E)^\vee[3],
\]
which has long exact sequence of cohomology
\[
\cdots \to \mathcal{E}xt^1(H^{-1}(E), \mathcal{O}_X) \xrightarrow{H^0(w^\vee[3])} H^0(E)^* \to H^1(E^\vee[2]) \to \mathcal{E}xt^2(H^{-1}(E), \mathcal{O}_X)
\]
where the last term $\mathcal{E}xt^2(H^{-1}(E), \mathcal{O}_X)$ vanishes since $H^{-1}(E)$ has homological dimension at most 1. Therefore, the morphism $H^0(\mathcal{F})(E) = H^0(w^\vee[3])$ is surjective if and only if $H^3(E^\vee) = H^1(E^\vee[2]) = 0$. On the other hand, $H^3(E^\vee) = 0$ if and only if $\text{Hom}(\text{Coh} \leq 0(X), E) = 0$ by Example 5.2 and so we are done.

\section*{Example 5.2.} Suppose $E$ is an object in $\langle \text{Coh} \leq 3(X)[1], \text{Coh} \leq 1(X) \rangle$ that satisfies the vanishing
\[
\text{Hom}(\text{Coh} \leq 0(X), E) = 0.
\]
Note that the category $\mathcal{A} := \langle \text{Coh} \geq 2(X)[1], \text{Coh} \leq 1(X) \rangle$ is the heart of a t-structure on $D^b(X)$ and hence an abelian category [1, Section 3]. Also, the subcategory $\text{Coh} \leq 0(X)$ is closed under quotients in $\mathcal{A}$. Hence the vanishing $\text{Hom}(\text{Coh} \leq 0(X), E) = 0$ implies $\text{Hom}(\text{Coh} \leq 0(X), H^{-1}(E)[1]) = 0$, which in turn implies $H^{-1}(E)$ has homological dimension at most 1 by Lemma 3.3(i). That is, $E$ satisfies the hypotheses of Lemma 5.1. As a result, all of the following objects satisfy the hypotheses of Lemma 5.1 in addition to the vanishing $\text{Hom}(\text{Coh} \leq 0(X), -) = 0$:

(a) Objects in the left-hand side of the equivalence (3.6.1), i.e., in the category
\[
\{ E \in \langle \text{Coh} \leq 3(X)[1], \text{Coh} \leq 0(X) \rangle : \text{Hom}(\text{Coh} \leq 0(X), E) = 0 \}.
\]
These include all the PT-semistable objects (see 2.9.1).

(b) Objects in the right-hand side of the equivalence (3.6.1), i.e., in the category
\[
\{ E \in \langle \text{Coh} \leq 3(X)[1], \text{Coh} \leq 1(X) \rangle : \text{Hom}(\text{Coh} \leq 1(X), E) = 0 \}.
\]
These include all the dual-PT semistable objects (see 2.9.3).

(c) Objects in
\[
\{ E \in \langle \text{Coh}_{\mu}(X), \mathcal{C}_{[0, \infty]} \rangle : \text{Hom}(\mathcal{C}_{[0, \infty]}, E) = 0 \},
\]
which are the objects giving rise to the $L$-invariants defined by Toda in proving a higher-rank DT/PT correspondence in [17]. Here, we have some fixed ample class $\omega$ on $X$, and $\text{Coh}_{\mu}(X)$ is the category of all $\mu_{\omega}$-semistable coherent sheaves $E$ with $\mu_{\omega}(E) := \omega^2 \text{ch}_1(E)/\text{ch}_0(E) = \mu$.

On the other hand, the category $\mathcal{C}_{[0, \infty]}$ consists of coherent sheaves $F$ supported in dimension at most 1, such that all its Harder-Narasimhan factors with respect to the slope function $\text{ch}_3(-)/\omega \text{ch}_2(-)$ have slopes lying in the interval $[0, \infty]$.

5.3. Lemma 5.1 motivates us to define the full subcategory of $\mathcal{E}$
\[
\mathcal{E}_0 = \{ E \in \mathcal{E} : \text{Hom}_{D^b(X)}(\text{Coh} \leq 0(X), E) = 0 \}.
\]
For any coherent sheaf $F$ on $X$, we will also define the full subcategory of $\text{Mor}(\text{Coh}(X))$
\[
\mathcal{S}(F) = \{ F \xrightarrow{q} Q : q \text{ is a surjection in } \text{Coh}(X), Q \in \text{Coh} \leq 0(X) \}.
\]
For any $E \in \mathcal{E}_0$, we have $H^{-1}(E) \in \text{Coh} \leq 2(X)$ by 4.1.1 and so $E$ satisfies the hypotheses of Lemma 5.1. Since $H^0(E)$ is 0-dimensional, we obtain that $H^0(\mathcal{F})(E)$ lies in the category $\mathcal{S}(\mathcal{E}xt^1(H^{-1}(E), \mathcal{O}_X))$. Note that $H^{-1}(E)$ has homological dimension at most 1 from the definition of $\mathcal{E}$. This allows us to make the following definition:

\section*{Definition 5.4.} We write $\mathcal{F}$ to denote the restriction of $\mathcal{F}$
\[
(5.4.1) \quad \mathcal{F} = (H^0 \circ \mathcal{F})|_{\mathcal{E}_0} : \mathcal{E}_0 \to \coprod_{F \in \text{Coh}(X), \text{hd}(F) \leq 1} \mathcal{S}(\mathcal{E}xt^1(F, \mathcal{O}_X)).
\]
Example 5.5. (PT semistable objects) Every PT semistable object $E$ of nonzero $c_h$ lies in $\mathcal{E}_o$. To see this, note that the canonical exact triangle $H^{-1}(E)[1] \to E \to H^0(E) \to H^{-1}(E)[2]$ gives a short exact sequence $0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0$ in $\mathcal{A}_p$. Since $\text{Coh}^{\leq 0}(X)$ is closed under quotient in $\mathcal{A}_p$, the vanishing $\text{Hom}(\text{Coh}^{\leq 0}(X), E) = 0$ implies the vanishing $\text{Hom}(\text{Coh}^{\leq 0}(X), H^{-1}(E)[1]) = 0$, and so $\text{ld}(H^{-1}(E)) \leq 1$ by Lemma 3.3(i). The claim then follows from 2.9.1.

Example 5.6. (Sheshmani’s frozen triples) Every stable frozen triple on a smooth projective threefold $X$ in the sense of Sheshmani [15] represents an object in $\mathcal{E}_o$. A frozen triple $(G,F,\varphi)$ consists of a locally free sheaf $G \cong O_X^{\oplus r}$ on $X$ (for some positive integer $r$) together with a morphism of coherent sheaves $\varphi : G \to F$ where $F$ is pure 1-dimensional. Such a frozen triple is called stable (or $r'$-stable in [15]) if the cokernel of $\varphi$ is 0-dimensional. Let us write $E$ to represent the 2-term complex $[G \xrightarrow{\varphi} F]$ in $D^b(X)$ with $F$ sitting at degree 0. Then clearly $E \in \mathcal{A}_p$. For any $T \in \text{Coh}^{\leq 0}(X)$, applying $\text{Hom}(T,-)$ to the exact triangle in $D^b(X)$

$$G \xrightarrow{\varphi} F \to E \to G[1]$$

gives $\text{Hom}(T,E) = 0$ and so $\text{Hom}(\text{Coh}^{\leq 0}(X), E) = 0$. The same argument as in Example 5.5 then shows $\text{ld}(H^{-1}(E)) \leq 1$, and so $E \in \mathcal{E}_o$. Note that this example is already implicitly stated in [17, Example 3.2].

Lemma 5.7. Given any $A \in \text{Coh}(X)$ with $\text{ld}(A) \leq 1$ and a morphism of sheaves $r : \mathcal{E}.x.t^1(A,O_X) \to Q$ where $Q \in \text{Coh}^{\leq 0}(X)$, there exists an object $G \in \mathcal{E}$ such that $H^0(\tilde{F}(G))$ is isomorphic to $r$ in $\text{Mor}(\text{Coh}(X))$. Moreover, if $r$ is a surjection in $\text{Coh}(X)$, then we can take $G$ to be in $\mathcal{E}_o$.

Proof. Consider the composition of morphisms in $D^b(X)$

$$A^r[1] \xrightarrow{\varphi} H^0(A^r[1]) = \mathcal{E}.x.t^1(A,O_X) \xrightarrow{\varphi} Q$$

where $c$ is canonical. Since $rc \in \mathcal{L}$, the essential surjectivity of $\tilde{F}$ from Proposition 4.4 implies that there exists some $G \in \mathcal{E}$ such that $\tilde{F}(G) \cong rc$ in $\mathcal{L}$, which in turn implies $r \cong H^0(\tilde{F}(G))$ in $\text{Mor}(\text{Coh}(X))$.

In fact, from the construction of the object $G$ (see Step 2 of the proof of Proposition 4.4), we have the exact triangle in $D^b(X)$

$$A[1] \to G \to Q^r[3] \to A[2]$$

which gives $H^{-1}(G) \cong A$ and $H^0(G) \cong Q^r$, i.e. $G$ satisfies the hypotheses of Lemma 5.1. Therefore, if $r$ is a surjection in $\text{Coh}(X)$, the morphism of sheaves $H^0(\tilde{F}(G))$ is also a surjection in $\text{Coh}(X)$, and Lemma 5.1 implies that $\text{Hom}(\text{Coh}^{\leq 0}(X), G) = 0$, meaning $G \in \mathcal{E}_o$. 

5.8. We have the following restriction of $\mathcal{F}$:

$$\mathcal{F}_{\text{tf}} : \{ E \in \mathcal{E}_o : H^{-1}(E) \text{ is torsion free} \} \to \bigcup_{A \in \text{Coh}^{\leq 3}(X), \text{ld}(A) \leq 1} \text{S}(\mathcal{E}.x.t^1(A,O_X))$$

$$E \mapsto \mathcal{F}(E).$$

If we fix an ample divisor $\omega$ on $X$ and $r \in H^0(X), D \in H^2(X)$ such that $r, \omega^2 D$ are coprime, then from [2.9.1] we know that an object $E$ with $c_{h_0}(E) = -r$ and $c_{h_1}(E) = -D$ lies in $\mathcal{E}_o$ if and only if it is a PT stable object. Let us write

$$\mathcal{H} = \{ A \in \text{Coh}^{= 3}(X) : \text{ld}(A) \leq 1, A \text{ is } \mu_{\omega}-\text{stable} \}. $$
Then we can further restrict \( \mathcal{F} \) to:
\[
\mathcal{F}_{PT,r,D} : \{ E \text{ is a PT stable object} : \text{ch}_0(E) = -r, \text{ch}_1(E) = -D \} \to \bigoplus_{A \in H \text{ ch}_0(A) = r, \text{ch}_1(A) = D} \mathcal{S}(\mathcal{E}xt^1(A, \mathcal{O}_X))
\]
\[
E \mapsto \mathcal{F}(E)
\]

**Theorem 5.9.** The functors \( \mathcal{F}, \mathcal{F}_{\text{tt}} \) are essentially surjective. If \( \omega \) is an ample divisor on \( X \) and \( r \in H^0(\mathcal{X}), D \in H^2(\mathcal{X}) \) are such that \( r, \omega^2 D \) are coprime, then \( \mathcal{F}_{PT,r,D} \) is also essentially surjective.

**Proof.** The essential surjectivity of \( \mathcal{F} \) follows from Lemma 5.7, while that of \( \mathcal{F}_{\text{tt}}, \mathcal{F}_{PT,r,D} \) follows from the essential surjectivity of \( \mathcal{F} \) itself and the discussion in 5.8.

### 6. Fibers of the functor \( \mathcal{F} \)

Given an object \( E \) of \( \mathcal{E}_0 \), the functor \( \mathcal{F} \) constructed in Section 5 takes \( E \) to a surjective morphism of coherent sheaves
\[
\mathcal{F}(E) : \mathcal{E}xt^1(H^{-1}(E), \mathcal{O}_X) \to H^0(E)^*.
\]

In this section, we answer the following questions:

1. Given an object \( E \in \mathcal{E}_0 \), how do we enumerate all the objects \( \overline{E} \in \mathcal{E}_0 \) such that \( \mathcal{F}(E) \) and \( \mathcal{F}(\overline{E}) \) are isomorphic in \( \text{Mor}(\text{Coh}(X)) \)?
2. Given two objects \( E, \overline{E} \) of \( \mathcal{E}_0 \) such that \( \mathcal{F}(E), \mathcal{F}(\overline{E}) \) are isomorphic in \( \text{Mor}(\text{Coh}(X)) \), precisely when are \( E, \overline{E} \) isomorphic in \( \mathcal{E}_0 \)?

These questions are answered in Lemma 6.2 and Lemma 6.3, respectively.

For the purpose of computing invariants, however, it may help to think of \( \mathcal{F}(\overline{E}) \) as a point of a quot scheme.

**Definition 6.1.** For any coherent sheaf \( F \) on \( X \), we will write \( Q(F) \) to denote the subcategory of \( \text{Mor}(\text{Coh}(X)) \) where the set of objects is
\[
Q(F) = \{ F \xrightarrow{q} Q : q \text{ is a surjection in } \text{Coh}(X), Q \in \text{Coh}^{\le 0}(X) \},
\]

and where the morphisms from an object \( F \xrightarrow{q_2} Q_2 \) to another \( F \xrightarrow{q_1} Q_1 \) are commutative diagrams in \( \text{Coh}(X) \) of the form
\[
\begin{array}{ccc}
F & \xrightarrow{q_1} & Q_1 \\
\downarrow 1_F & & \downarrow f \\
F & \xrightarrow{q_2} & Q_2
\end{array}
\]

A morphism in \( Q(F) \) as above is an isomorphism if and only if \( f \) is an isomorphism in \( \text{Coh}(X) \). Note that \( Q(F) \) is not a full subcategory of \( \text{Mor}(\text{Coh}(X)) \), i.e. \( Q(F) \) has ‘fewer’ arrows than \( \text{Mor}(\text{Coh}(X)) \).

If we have an isomorphism \( E \xrightarrow{\phi} \overline{E} \) in \( \mathcal{E}_0 \), then this isomorphism induces an isomorphism \( \mathcal{F}(E) \xrightarrow{\phi} \mathcal{F}(\overline{E}) \) in \( \text{Mor}(\text{Coh}(X)) \) by virtue of \( \mathcal{F} \) being a functor. When \( H^{-1}(E) = H^{-1}(\overline{E}) \), however, it is not necessarily the case that an isomorphism \( E \xrightarrow{\phi} \overline{E} \) in \( \mathcal{E}_0 \) induces an isomorphism in the category \( Q(\mathcal{E}xt^1(H^{-1}(E), \mathcal{O}_X)) \). If \( E, \overline{E} \in \mathcal{E}_0 \) satisfy \( H^{-1}(E) = H^{-1}(\overline{E}) \) and \( \mathcal{F}(E), \mathcal{F}(\overline{E}) \) are isomorphic as objects of \( Q(\mathcal{E}xt^1(H^{-1}(E), \mathcal{O}_X)) \), however, it is indeed true that \( E \) and \( \overline{E} \) are isomorphic in \( D^b(X) \) by 6.3.3. We phrase this formally in Lemma 6.5.

In the last part of this section, we revisit a construction mentioned in [5, p.3] in which a surjection of sheaves \( \mathcal{E}xt^1(I_C, \mathcal{O}_X) \to Q \), where \( I_C \) is the ideal sheaf of a Cohen-Macaulay curve \( C \) on \( X \) and \( Q \) is a 0-dimensional sheaf, gives rise to a PT stable pair (i.e. a rank-one PT stable object). We generalise
Lemma 6.2. Fix an element $\mathcal{E} \in \mathcal{E}_0$. The functor

\[ G : \{ E \in \mathcal{E}_0 : F(E) \cong F(\mathcal{E}) \} \to \{ A \in \text{Coh}(X) : \text{hd}(A) \leq 1, \mathcal{E}xt^1(A, \mathcal{O}_X) \cong \mathcal{E}xt^1(H^{-1}(\mathcal{E}), \mathcal{O}_X) \} \]

\[ E \mapsto H^{-1}(E) \]

is essentially surjective.

Proof. That $G$ is a functor from the stated domain to the stated codomain follows from the definitions of $\mathcal{E}$ and the construction of the functor $F$. To see the essential surjectivity of $G$, take any $A$ in the codomain of $G$ and fix an isomorphism of sheaves $\alpha : \mathcal{E}xt^1(A, \mathcal{O}_X) \to \mathcal{E}xt^1(H^{-1}(\mathcal{E}), \mathcal{O}_X)$. Let $c$ denote the canonical map $A^\vee[1] \to H^0(A^\vee[1])$. The composite map

\[ (6.2.1) \]

\[ A^\vee[1] \xrightarrow{\alpha^\vee} H^0(A^\vee[1]) \xrightarrow{\mathcal{F}(\mathcal{E})} H^0(H^{-1}(\mathcal{E})^\vee[1]) \xrightarrow{\mathcal{F}(\mathcal{E})} H^0(\mathcal{E})^\vee[3] \]

can be completed to an exact triangle in $D^b(X)$

\[ A^\vee[1] \xrightarrow{\mathcal{F}(\mathcal{E}) \circ \alpha^\vee} H^0(\mathcal{E})^\vee[3] \to G^\vee[3] \to A^\vee[2] \]

for some object $G$ (which is unique up to a non-canonical isomorphism [7, TR3, 1.2]). Applying $-^\vee[3]$ gives us the second row of the following diagram; the first row is constructed using truncation functors, while the unmarked vertical maps are canonical and induced by the identity map on $G$ [4 IV.4 Lemma 5b]:

\[
\begin{array}{cccccc}
H^{-1}(G)[1] & \xrightarrow{1} & G & \xrightarrow{1} & H^0(G) & \xrightarrow{1} & H^{-1}(G)[2] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A[1] & \xrightarrow{1} & G & \xrightarrow{1} & H^0(\mathcal{E}) & \xrightarrow{1} & A[2]
\end{array}
\]

Applying $-^\vee[2]$ to the entire diagram above now gives the isomorphism of exact triangles

\[
\begin{array}{cccccc}
H^0(\mathcal{E})^\vee[2] & \xrightarrow{1} & G^\vee[2] & \xrightarrow{1} & A^\vee[1] & \xrightarrow{\mathcal{F}(\mathcal{E}) \circ \alpha^\vee} H^0(\mathcal{E})^\vee[3] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(G)^\vee[2] & \xrightarrow{1} & G^\vee[2] & \xrightarrow{1} & H^{-1}(G)^\vee[1] & \xrightarrow{1} & H^0(G)^\vee[3]
\end{array}
\]

in which the right-most square factorises as

\[ (6.2.2) \]

\[ A^\vee[1] \xrightarrow{c} H^0(A^\vee[1]) \xrightarrow{\mathcal{F}(\mathcal{E}) \circ \alpha} H^0(\mathcal{E})^\vee[3] \]

\[ H^{-1}(G)^\vee[1] \xrightarrow{c'} H^0(H^{-1}(G)^\vee[1]) \xrightarrow{(H^0(\mathcal{F}))(G)} H^0(G)^\vee[3] \]

through canonical maps $c, c'$.

Since $\mathcal{E} \in \mathcal{E}_0$ by assumption, the morphism of sheaves $F(\mathcal{E})$, and hence $F(\mathcal{E}) \circ \alpha$, is surjective. It follows that $(H^0(\mathcal{F}))(G)$ is also surjective by the commutativity of the right-hand square of (6.2.2), and so $G \in \mathcal{E}_0$ by Lemma 5.1. Hence we can write $(H^0(\mathcal{F}))(G)$ as $F(G)$.

Note that in (6.2.2), the map $c'$ is canonical and all the vertical maps are isomorphisms. Taking inverses of the vertical maps in the right-hand square in (6.2.2), the following concatenation gives an
isomorphism $\mathcal{F}(G) \to \mathcal{F}(\bar{G})$:

$$
\begin{array}{c}
\begin{array}{c}
H^0(H^{-1}(\bar{E})^\vee[1]) \\
\downarrow \alpha
\end{array}
\begin{array}{c}
\begin{array}{c}
H^0(A^\vee[1]) \\
\downarrow \mathcal{F}(\bar{E}) \circ \alpha
\end{array}
\begin{array}{c}
\begin{array}{c}
H^0(H^{-1}(E)^\vee[1]) \\
\downarrow \mathcal{F}(G)
\end{array}
\begin{array}{c}
\begin{array}{c}
H^0(\bar{E})^\vee[3] \\
\downarrow H^0(E)^\vee[3]
\end{array}
\begin{array}{c}
\begin{array}{c}
H^0(G)^\vee[3]
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

That is, $G$ is an object in the domain of the functor $\mathcal{G}$ such that $\mathcal{G}(G) = H^{-1}(G) \cong A$ (this isomorphism follows from the left-most vertical map in (6.2.2)), proving the essential surjectivity of $\mathcal{G}$. \hfill \blacksquare

6.2.3. Given a fixed object $\bar{E} \in \mathcal{E}_0$, the proof of Lemma [6.2] says we can construct all the objects in $\{E \in \mathcal{E}_0 : \mathcal{F}(E) \cong \mathcal{F}(\bar{E})\}$ by first going through all the coherent sheaves $A$ of homological dimension at most 1 for which there exists a sheaf isomorphism $\alpha : \mathcal{E}xt^1(A, \mathcal{O}_X) \to \mathcal{E}xt^1(H^{-1}(\bar{E}), \mathcal{O}_X)$, and then completing composite maps (6.2.1) to exact triangles.

For objects $E, \bar{E} \in \mathcal{E}_0$, the following lemma gives a comparison between the condition of $E, \bar{E}$ being isomorphic in $D^b(X)$ and the condition of $\mathcal{F}(E), \mathcal{F}(\bar{E})$ being isomorphic in $\text{Mor}(\text{Coh}(X))$.

**Lemma 6.3.** Given $E, \bar{E} \in \mathcal{E}_0$, the following are equivalent:

(i) $E, \bar{E}$ are isomorphic in $D^b(X)$.

(ii) There exists an isomorphism $\mathcal{F}(E) \to \mathcal{F}(\bar{E})$ in $\text{Mor}(\text{Coh}(X))$

(6.3.1) \[ H^0(H^{-1}(\bar{E})^\vee[1]) \xrightarrow{\pi} H^0(\bar{E})^\vee[3] \]

\[ \downarrow i \quad \downarrow k \]

\[ H^0(H^{-1}(E)^\vee[1]) \xrightarrow{\alpha} H^0(E)^\vee[3] \]

and an isomorphism $H^{-1}(\bar{E})^\vee[1] \to H^{-1}(E)^\vee[1]$ in $D^b(X)$ such that $H^0(i) = j$.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows from the construction of the functor $\mathcal{F}$.

Let us now assume (ii) holds. We can concatenate (6.3.1) with the commutative square induced by $i$ to form

\[ H^{-1}(\bar{E})^\vee[1] \xrightarrow{\alpha} H^0(H^{-1}(\bar{E})^\vee[1]) \xrightarrow{\pi} H^0(\bar{E})^\vee[3] \]

\[ \downarrow i \quad \downarrow k \]

\[ H^{-1}(E)^\vee[1] \xrightarrow{\alpha} H^0(H^{-1}(E)^\vee[1]) \xrightarrow{\alpha} H^0(E)^\vee[3] \]

where $\pi, \alpha$ are canonical maps. Since $\mathcal{F}(E) = u$ and $\mathcal{F}(\bar{E}) = \pi$ in our notation, from the construction of $\mathcal{F}$ we know the composite maps $\bar{u} \circ \alpha, uc$ can be completed to exact triangles with $\bar{E}^\vee[3], E^\vee[3]$:

(6.3.2) \[ H^{-1}(\bar{E})^\vee[1] \xrightarrow{\bar{u} \circ \alpha} H^0(\bar{E})^\vee[3] \xrightarrow{E^\vee[3]} H^{-1}(\bar{E})^\vee[2] \]

\[ \downarrow \quad \downarrow i \quad \downarrow k \]

\[ H^{-1}(E)^\vee[1] \xrightarrow{uc} H^0(E)^\vee[3] \xrightarrow{E^\vee[3]} H^{-1}(E)^\vee[2] \]

and then $i, k$ induce an isomorphism $l$ in $D^b(X)$, and so (i) holds. \hfill \blacksquare
6.3. Suppose $E, \overline{E} \in \mathcal{E}_0$ satisfy $H^{-1}(E) = H^{-1}(\overline{E})$ and $F(E), F(\overline{E})$ are isomorphic in $\mathcal{Q}(H^{-1}(E))$. Then by taking $i = 1_{H^{-1}(E)[1]}$ in the proof of Lemma 6.3, we see that $E, \overline{E}$ are isomorphic in $D^b(X)$. Let us phrase this in a slightly more formal framework in Lemma 6.5 below.

6.4. Let us define a subcategory $\mathcal{E}_0'$ of $\mathcal{E}_0$ where the objects of $\mathcal{E}_0'$ are the same as those of $\mathcal{E}_0$, but where a morphism $E \to \overline{E}$ in $\mathcal{E}_0'$ is a morphism in $\mathcal{E}_0$ with the extra requirement that, in the induced morphism of exact triangles

$$
\begin{array}{cccc}
H^{-1}(E)[1] & \longrightarrow & E & \longrightarrow & H^0(E) & \longrightarrow & H^{-1}(E)[2] \\
\downarrow H^{-1}(f)[1] & & \downarrow f & & \downarrow & & \downarrow \\
H^{-1}(\overline{E})[1] & \longrightarrow & \overline{E} & \longrightarrow & H^0(\overline{E}) & \longrightarrow & H^{-1}(\overline{E})[2]
\end{array}
$$

we must have

$$
H^{-1}(f) = 1_{H^{-1}(E)}.
$$

As a result, if $E, \overline{E}$ are two objects of $\mathcal{E}_0$ such that $H^{-1}(E) \neq H^{-1}(\overline{E})$, then $\text{Hom}_{\mathcal{E}_0'}(E, \overline{E}) = \emptyset$, meaning $\mathcal{E}_0'$ is a non-full subcategory of $\mathcal{E}_0$. It is easy to see that $\mathcal{F}$ restricts to a functor

$$
\mathcal{F}' : \mathcal{E}_0' \to \prod_{F \in \text{Coh}(X), \text{hd}(F) \leq 1} \mathcal{Q}(\mathcal{E}xt^1(F, \mathcal{O}_X))
$$

For a fixed object $\overline{E}$ of $\mathcal{E}_0'$, we also define the full subcategory of $\mathcal{E}_0'$

$$
\mathcal{E}_0'(\overline{E}) := \{ E \in \mathcal{E}_0' : H^{-1}(E) = H^{-1}(\overline{E}), \mathcal{F}'(E) \cong \mathcal{F}'(\overline{E}) \text{ in } \mathcal{Q}(\mathcal{E}xt^1(H^{-1}(\overline{E}), \mathcal{O}_X)) \}.
$$

Lemma 6.5. Let $\overline{E}$ be a fixed object of $\mathcal{E}_0'$. Then $\mathcal{F}'$ further restricts to a functor

$$
\mathcal{E}_0'(\overline{E}) \to \mathcal{Q}(\mathcal{E}xt^1(H^{-1}(\overline{E}), \mathcal{O}_X))
$$

which induces an injection from the set of isomorphism classes in the domain to the set of isomorphism classes in the codomain.

Proof: Suppose $E, \overline{E}$ are two objects of $\mathcal{E}_0'$ such that $H^{-1}(E) = H^{-1}(\overline{E})$ and $\mathcal{F}'(E) \cong \mathcal{F}'(\overline{E})$ in the category $\mathcal{Q}(\mathcal{E}xt^1(H^{-1}(\overline{E}), \mathcal{O}_X))$. Then we have a commutative diagram in $D^b(X)$

$$
\begin{array}{cccc}
H^{-1}(E)^{\vee}[1] & \overset{c}{\longrightarrow} & \mathcal{E}xt^1(H^{-1}(E), \mathcal{O}_X) & \overset{q}{\longrightarrow} & H^0(E)^{\vee}[3] \\
\downarrow & & \downarrow \phi & & \downarrow \\
H^{-1}(\overline{E})^{\vee}[1] & \overset{c}{\longrightarrow} & \mathcal{E}xt^1(H^{-1}(\overline{E}), \mathcal{O}_X) & \overset{q}{\longrightarrow} & H^0(\overline{E})^{\vee}[3]
\end{array}
$$

where $c$ is the canonical map, the right-hand vertical arrow is an isomorphism, and $q = \mathcal{F}'(E), \overline{q} = \mathcal{F}'(\overline{E})$. From the construction of the functor $\mathcal{F}$, we can complete $qc, \overline{q}c$ to exact triangles with $E^{\vee}[2], \overline{E}^{\vee}[2]$ and obtain an isomorphism of exact triangles

$$
\begin{array}{cccc}
E^{\vee}[2] & \longrightarrow & H^{-1}(E)^{\vee}[1] & \overset{q_c}{\longrightarrow} & H^0(E)^{\vee}[3] & \longrightarrow & E^{\vee}[3] \\
\downarrow g^{\vee}[2] & & \downarrow 1 & & \downarrow & & \downarrow \\
\overline{E}^{\vee}[2] & \longrightarrow & H^{-1}(\overline{E})^{\vee}[1] & \overset{\overline{q}_c}{\longrightarrow} & H^0(\overline{E})^{\vee}[3] & \longrightarrow & E^{\vee}[3]
\end{array}
$$

for some isomorphism $\overline{E} \overset{q}{\to} E$ in $D^b(X)$. Dualising, we obtain the isomorphism of triangles

$$
\begin{array}{cccc}
H^{-1}(E)[1] & \longrightarrow & \overline{E} & \longrightarrow & H^0(E) & \longrightarrow & H^{-1}(E)[2] \\
\downarrow & & \downarrow g & & \downarrow & & \downarrow \\
H^{-1}(E)[1] & \longrightarrow & E & \longrightarrow & H^0(E) & \longrightarrow & H^{-1}(E)[2]
\end{array}
$$
from which we see \( H^{-1}(g) = 1_{H^{-1}(\mathcal{E})} \) by \([4\ IV.4 \text{ Lemma 5b}]\). That is, the morphism \( g \) is a morphism in the category \( \mathcal{E}'_0 \).

\[ \]

**Lemma 6.6.** We have

\[ \{ E \in \mathcal{E}_0 : \operatorname{ch}_0(E) = -1, \operatorname{ch}_1(E) = 0, H^{-1}(E) \text{ is torsion-free} \} = \{ \text{rank-one PT stable objects} \}. \]

**Proof.** The inclusion from left to right follows from \([2.9.1]\). To see the other inclusion, take any rank-one PT stable object \( E \). Then \( E \in \mathcal{A}^p \), and the canonical exact triangle \([4.2.1]\) makes \( H^{-1}(E)[1] \) an \( \mathcal{A}^p \)-subobject of \( E \). Since \( \operatorname{Coh}^{\leq 0}(X) \) is closed under quotient in \( \mathcal{A}^p \) and we have the vanishing \( \operatorname{Hom}(\operatorname{Coh}^{\leq 0}(X), E) = 0 \) from the PT stability of \( E \), it follows that \( \operatorname{Hom}(\operatorname{Coh}^{\leq 0}(X), H^{-1}(E)[1]) = 0 \). Lemma \([3.3]\) then implies \( \operatorname{ld}(H^{-1}(E)) \leq 1 \), and so \( E \) lies in the category on the left-hand side. □

6.7. For a fixed Cohen-Macaulay curve \( C \) on a smooth projective threefold \( X \), Gholampour-Kool describes a construction that takes an element of \( Q(\operatorname{Ext}^1(I_C, \mathcal{O}_X)) \) to a stable pair on \( X \) \([5 \ p.3]\). This construction is as follows: given a surjection \( q : \operatorname{Ext}^1(I_C, \mathcal{O}_X) \to Q \) in \( \operatorname{Coh}(X) \) where \( Q \) is 0-dimensional, let \( K = \ker(q) \) so that we have a short exact sequence of sheaves

\[
0 \to K \to \operatorname{Ext}^1(I_C, \mathcal{O}_X) \xrightarrow{q} Q \to 0.
\]

Taking derived dual and then taking cohomology, and noting that \( \operatorname{Ext}^1(I_C, \mathcal{O}_X) \cong \mathcal{O}_C^* \) where \( \mathcal{O}_C \) is reflexive, we obtain the short exact sequence

\[
0 \to \mathcal{O}_C \to K^* \to Q^* \to 0.
\]

Taking the Yoneda product of the last exact sequence with the structural exact sequence

\[(6.7.1) \quad 0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0\]

then gives the four-term exact sequence

\[
0 \to I_C \to \mathcal{O}_X \xrightarrow{\delta} K^* \to Q^* \to 0
\]

and hence a stable pair

\[(6.7.2) \quad \mathcal{O}_X \xrightarrow{\delta} F
\]

where \( F := K^* \).

6.8. The construction in \([6.7]\) can be generalised to higher ranks. This is already mentioned in \([17 \ Lemma \ 3.3]\), but we describe the details here so that we can compare the construction with our functor \( F \): Suppose \( A = H^{-1}(E) \) for some \( E \in \mathcal{E}_0 \), and that \( A \) is torsion-free with \( A^{**} \) locally free. For any surjection \( q : \operatorname{Ext}^1(A, \mathcal{O}_X) \to Q \) in \( \operatorname{Coh}(X) \) where \( Q \) is 0-dimensional, let \( K = \ker(q) \). We have a natural short exact sequence of sheaves

\[(6.8.1) \quad 0 \to A \xrightarrow{\beta} A^{**} \xrightarrow{\gamma} T \to 0\]

where \( T \in \operatorname{Coh}^{\leq 1}(X) \). Taking derived dual and noting that \( A^{**} \) is locally free, we obtain the isomorphism

\[
\operatorname{Ext}^1(A, \mathcal{O}_X) \to \operatorname{Ext}^2(T, \mathcal{O}_X)
\]

where \( \operatorname{Ext}^2(T, \mathcal{O}_X) \) is a pure sheaf in \( \operatorname{Coh}^{\leq 1}(X) \) by \([8 \ Proposition \ 1.1.6]\). Dualising brings the short exact sequence

\[
0 \to K \to \operatorname{Ext}^1(A, \mathcal{O}_X) \xrightarrow{\delta} Q \to 0
\]

to the short exact sequence

\[
0 \to \operatorname{Ext}^2(\operatorname{Ext}^1(A, \mathcal{O}_X), \mathcal{O}_X) \xrightarrow{\delta} \operatorname{Ext}^2(K, \mathcal{O}_X) \xrightarrow{\gamma} Q^* \to 0
\]
where we have an isomorphism $\mathcal{E}xt^2(\mathcal{E}xt^2(T, O_X), O_X) \xrightarrow{\delta} \mathcal{E}xt^2(\mathcal{E}xt^1(A, O_X), O_X)$. Note that if $T$ is nonzero, then it must be pure 1-dimensional. To see this, notice that the short exact sequence of sheaves (6.8.1) induces the short exact sequence in $\mathcal{A}^p$

$$0 \to T \to A[1] \to A^{**}[1] \to 0.$$ 

Since $\text{Coh}^{\leq 0}(X)$ is a Serre subcategory of $\mathcal{A}^p$, any nonzero 0-dimensional subsheaf $T'$ of $T$ would be an $\mathcal{A}^p$-subobject of $A[1]$, hence of $E$, contradicting the vanishing $\text{Hom}(\text{Coh}^{\leq 0}(X), E) = 0$ from the definition of $E_0$. Hence when $T$ is nonzero, it is pure 1-dimensional, hence reflexive [8 Proposition 1.1.10], and so the double dual map $T \xrightarrow{\alpha_2} \mathcal{E}xt^2(\mathcal{E}xt^2(T, O_X), O_X) = T^{**}$ is an isomorphism. Putting everything together, the composition of morphisms in $\text{Coh}(X)$

$$A^{**} \xrightarrow{\gamma} T \xrightarrow{\alpha_2} T^{**} = \mathcal{E}xt^2(\mathcal{E}xt^2(T, O_X), O_X) \xrightarrow{i} \mathcal{E}xt^2(\mathcal{E}xt^1(A, O_X), O_X) \xrightarrow{\delta} \mathcal{E}xt^2(K, O_X)$$

has $A$ as the kernel and a 0-dimensional sheaf as the cokernel. Overall, we have constructed a 2-term complex

(6.8.2) $$A^{**} = H^{-1}(E)^{**} \xrightarrow{\gamma} \mathcal{E}xt^2(K, O_X)$$

where $s = \delta \circ \alpha_2$. When $A = I_C$ is the ideal sheaf of a Cohen-Macaulay curve $C$ on $X$, we have $T = O_C$, and the above construction reduces to the construction in [6.7] while the 2-term complex (6.8.2) coincides with the stable pair (6.7.2).

Recall that the proof of Lemma [5.7] describes a construction taking any element of $Q(\mathcal{E}xt^1(I_C, O_X))$ to a rank-one PT stable object. The following lemma describes the precise relation between this construction in the proof of Lemma [5.7] and the construction in [6.8].

**Lemma 6.9.** Let $E$ be an object of $E_0$ such that $H^{-1}(E)$ is torsion-free and $H^{-1}(E)^{**}$ is locally free. Suppose $q : \mathcal{E}xt^1(H^{-1}(E), O_X) \to Q$ is a surjection in $\text{Coh}(X)$ where $Q$ is 0-dimensional. Let $G$ be the object in $E_0$ satisfying $F(G) \equiv q$ as constructed in the proof of Lemma [5.7] and let $[H^{-1}(E)^{**} \xrightarrow{\alpha_2} \mathcal{E}xt^2(K, O_X)]$ be the 2-term complex (6.8.2) constructed from $q$ as in [6.8] Then $G$ fits in an exact triangle

$$(H^{-1}(E)^{**}) \xrightarrow{\delta} K^{**}[2] \to G \to (H^{-1}(E)^{**})^{\vee}[1],$$

and $s$ and $G$ are related by $H^0(\phi) = s$.

**Proof.** Let us use the notation in [6.8] and write $A = H^{-1}(E)$, $K = \ker(q)$. Let us also write $c$ to denote the canonical map $A^{**}[1] \to \mathcal{E}xt^1(A, O_X)$ as in the proof of Lemma [5.7] Let $C$ be an object that completes $q_c$ to an exact triangle

$$A^{**}[1] \xrightarrow{q_c} Q \to C \to A^{**}[2].$$

Applying the octahedral axiom to the composition $q_c$ then gives us the diagram
in which every straight line is an exact triangle. Taking \(-^\mathbb{V}[3]\) then yields

\[
\begin{array}{c}
(A^*)^\mathbb{V}[1] \\
\end{array}
\]

where we write \(G = C^\mathbb{V}[3]\). The vertical line in the last diagram gives an exact triangle

\[
(A^*)^\mathbb{V} \xrightarrow{\phi} K^\mathbb{V}[2] \rightarrow G \rightarrow (A^*)^\mathbb{V}[1].
\]

Now we apply the formulation of the octahedral axiom in [12, Lemma 1.4.6] to the composition \(K^\mathbb{V}[2] \rightarrow G \rightarrow Q^\mathbb{V}[3]\), which yields the diagram

\[
\begin{array}{c}
\end{array}
\]

in which every row and every column is an exact triangle. Applying the cohomology functor \(H^{-1}\) to the top-right square gives

\[
\begin{array}{c}
\end{array}
\]

where \(\delta\) is as in 6.8 and \(H^0(\alpha^\mathbb{V}[2])\) is precisely the surjection \(i\alpha\gamma\) in 6.8. Thus \(H^0(\phi)\) coincides with \(s\), which was constructed as \(\delta i\alpha\gamma\).

**Example 6.10.** Suppose \(E\) is an object satisfying the hypotheses of Lemma 6.9, such that \(H^{-1}(E)\) is not locally free while its dual \(H^{-1}(E)^*\) is locally free. Using the notation in 6.8 and Lemma 6.9 we have that \(T\) is nonzero and hence pure 1-dimensional by 6.8. It follows that \(\mathcal{E}xt^1(A, \mathcal{O}_X)\) and \(K\) are both nonzero and pure 1-dimensional, as is \(\mathcal{E}xt^2(K, \mathcal{O}_X) = K^*\). On the other hand, we have \((H^{-1}(E)^*)^\mathbb{V} = H^{-1}(E)^{**}\). Hence \(\phi = H^0(\phi)\) and the morphism \(\phi\) coincides with the morphism \(s\) in Lemma 6.9. Examples of such \(E\) include:
(1) $E$ is a rank-one PT stable object. In this case, we have $H^{-1}(E) = L \otimes I_C$ where $L$ is some line bundle and $I_C$ is the ideal sheaf of some Cohen-Macaulay curve $C$ on $X$, so that $\phi = s : L \to \mathcal{E}xt^2(K, O_X)$. When $L = O_X$, this morphism is a PT stable pair.

(2) $E$ is a 2-term complex of coherent sheaves $[G \xrightarrow{\phi} F]$ in $D^b(X)$ with $F$ sitting at degree 0, and where $(G, F, \phi)$ is a stable frozen triple (see Example 5.6). To see why $H^{-1}(E)$ satisfies the requirements of Lemma 6.9, consider the short exact sequence of sheaves

$$0 \to H^{-1}(E) \to G \to \text{im} (\phi) \to 0$$

where $\text{im} \phi$ is a pure 1-dimensional sheaf. This gives the isomorphism $\mathcal{E}xt^1(H^{-1}(E), O_X) \cong \mathcal{E}xt^2(\text{im} (\phi), O_X) \neq 0$, and so $H^{-1}(E)$ cannot be locally free. We also have $H^{-1}(E)^* = \mathcal{E}xt^0(H^{-1}(E), O_X) \cong \mathcal{E}xt^0(G, O_X)$, which is locally free. Hence $E$ satisfies all the requirements in Lemma 6.9.

REFERENCES

[1] A. Bayer. Polynomial Bridgeland stability conditions and the large volume limit. Geom. Topol., 13:2389–2425, 2009.
[2] T. Bridgeland. Hall algebras and curve-counting invariants. J. Amer. Math. Soc., 24(4):969–998, 2011.
[3] W.-Y. Chuang and J. Lo. Stability and Fourier-Mukai transforms on higher dimensional elliptic fibrations. Comm. Anal. Geom., 24(5):1047–1084, 2016.
[4] S.I. Gelfand and Y.I. Manin. Methods of Homological Algebra. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2002.
[5] A. Gholampour and M. Kool. Higher rank sheaves on threefolds and functional equations. Preprint. arXiv:1706.05246 [math.AG], 2017.
[6] R. Hartshorne. Algebraic geometry. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
[7] D. Huybrechts. Fourier-Mukai Transforms in Algebraic Geometry. Oxford Mathematical Monographs. Clarendon Press, 2006.
[8] D. Huybrechts and M. Lehn. The Geometry of Moduli Spaces of Sheaves, volume 31 of Aspects of Mathematics. Vieweg, Braunschweig, 1997.
[9] J. Lo. Moduli of PT-semistable objects I. J. Algebra, 339(1):203–222, 2011.
[10] J. Lo. Polynomial Bridgeland stable objects and reflexive sheaves. Math. Res. Lett., 19(4):873–885, 2012.
[11] J. Lo. Moduli of PT-semistable objects II. Trans. Amer. Math. Soc., 365(9):4539–4573, 2013.
[12] A. Neeman. Triangulated Categories. (AM-148). Annals of Mathematics Studies. Princeton University Press, 2014.
[13] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. Invent. Math., 178:407–447, 2009.
[14] D. Piyaratne and Y. Toda. Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants. J. Reine Angew. Math., 2015.
[15] A. Sheshmani. Higher rank stable pairs and virtual localization. Comm. in Anal. and Geom., 24(1):139–193, 2016.
[16] Y. Toda. Curve counting theories via stable objects I. DT/PT correspondence. J. Amer. Math. Soc., 23(4):1119–1157, 2010.
[17] Y. Toda. Hall algebras in the derived category and higher rank DT invariants. Preprint. arXiv:1601.07519 [math.AG], 2016.

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