On the monotonicity property of the generalized eigenvalue for weakly-coupled cooperative elliptic systems

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Abstract. We consider general linear non-degenerate weakly-coupled cooperative elliptic systems and study certain monotonicity properties of the generalized principal eigenvalue in \( \mathbb{R}^d \) with respect to the potential. It is shown that monotonicity on the right is equivalent to the recurrence property of the twisted operator which is, in turn, equivalent to the minimal growth property at infinity of the principal eigenfunctions. The strict monotonicity property of the principal eigenvalue is shown to be equivalent with the exponential stability of the twisted operators. An equivalence between the monotonicity property on the right and the stochastic representation of the principal eigenfunction is also established.

1. Introduction

Regime switching diffusions are heavily used for modelling purposes in applied subjects like mathematical finance [1–4], wireless communications [5], production planning [6], predictive modelling [7,8]. See also the introduction of the book by Yin and Zhu [9] for further motivation in studying regime switching diffusions. Eigenvalue problems for weakly-coupled systems have also received a lot of attention. Most of the existing works in this direction are concerned with the maximum principle and Dirichlet principal eigenvalue problems in bounded domains, see for instance, Amann [10], Birindelli et al. [11], Cantrell and Schmitt [12], Cantrell [13], Hess [14], Sweers [15]. In this article we consider the eigenvalue problem in the whole space \( \mathbb{R}^d \) and study its monotonicity properties with respect to the potential \( c \), and provide some sharp characterizations. Our interest in these problems stems from its applications in risk-sensitive control problems [16,17]. Apart from this, eigenvalue problems are also important in understanding the large deviations behavior [18–20] and the Fisher-KPP type phenomenon [21]. In particular, given a potential function \( c \) we consider the exponential-to-integration (or risk-sensitive cost) function given by

\[
\mathcal{E}(x,i) := \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}_{x,i} \left[ e^{\int_0^T c(X_t,S_t) \, dt} \right],
\]

where \((X,S)\) represents the regime switching diffusion. Such functionals are the main object in the study of risk-sensitive controls and large deviations phenomena. It is often important to know under what circumstances we can have \( \mathcal{E}(x,i) = \lambda^* \) where \( \lambda^* \) is the principal eigenvalue of \( \mathcal{A} = \mathcal{L} + c \) in \( \mathbb{R}^d \) and \( \mathcal{L} \) is the extended generator of \((X,S)\). As shown in [16, Example 3.1] this equality does not hold in general and the concepts of monotonicity (see Definition 1.3 below) of the principal eigenvalues were introduced in [16] to provide a sufficient condition for this equality to hold. It turns out that the concept of monotonicity is also linked to the criticality of eigenfunctions used in potential theory (see [22] and references therein). In this article we extend the study to weakly-coupled cooperative

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2000 Mathematics Subject Classification. Primary 93E20, 60J60.

Key words and phrases. Principal eigenvalue, elliptic systems, regime switching diffusions, monotonicity of eigenvalues, Dirichlet eigenvalue problems.
systems. The results are closely related to the works of Ichihara [23, 24] where the eigenvalues of ergodic Hamilton-Jacobi equations are characterized through the recurrence/transience behavior of the diffusion process governed by optimal feedback control. An important aspect of this paper, is that most of the results are obtained by analytical methods, instead using probabilistic arguments. This is made possible by abstracting the notions of regularity, recurrence, and geometric ergodicity of a diffusion to analogous notions for an operator (see \text{Definition 1.2}). This article owes much to the work of Berestycki and Rossi [25] who recently study eigenvalue problems for scalar elliptic equations in unbounded domains and its relation to maximum principles. It is also possible to develop an analogous theory for systems but we do not pursue this direction in this article.

1.1. \textbf{The model and main results.} Let $\mathcal{S} := \{1, 2, \ldots, N\}$ be a discrete set. In this paper we consider the generalized eigenvalue problem for weakly-coupled elliptic systems $\mathcal{A}$ on $\mathbb{R}^d \times \mathcal{S}$ taking the form

$$ (\mathcal{A} f)_k(x) = \text{Tr} \left( a_k(x) \nabla^2 f_k(x) \right) + b_k(x) \cdot \nabla f_k(x) + \sum_{j \in \mathcal{S}} c_{kj}(x) f_j(x), \quad k \in \{1, 2, \ldots, N\}, $$

(1.1)

for $f = (f_i)_{i \in \mathcal{S}}$, with $f_i \in C^2(\mathbb{R}^d)$. Here, $\nabla^2$ denotes the Hessian, the coefficients $(a_k)_{k \in \mathcal{S}}$ are continuous, positive definite symmetric matrices. The coefficients $b_k : \mathbb{R}^d \to \mathbb{R}^d$, $k \in \mathcal{S}$ and $c_{ij} : \mathbb{R}^d \to \mathbb{R}$, $i, j \in \mathcal{S}$, are locally bounded and Borel measurable. We consider cooperative systems, that is, $\mathcal{A}$ as above with $c_{ij} \geq 0$ for $i \neq j$. This property is also known as quasi-monotonicity property. We also impose the following non-degeneracy condition throughout this article: for each $R > 0$, it holds that

$$ \sum_{i,j=1}^d a_{ij}^k(x) \zeta_i \zeta_j \geq C_R^{-1} |\zeta|^2 \quad \forall k \in \mathcal{S}, x \in B_R, $$

and for all $\zeta = (\zeta_1, \ldots, \zeta_d)^T \in \mathbb{R}^d$. We prefer to write (1.1) in a form that has a probabilistic interpretation. Let $c_k := \sum_{j \in \mathcal{S}} c_{kj}$, and the matrix $M = [m_{ij}]_{i,j \in \mathcal{S}}$ be defined by

$$ m_{ij} := c_{ij} \quad \text{for } i \neq j, \quad \text{and } m_{ii} := - \sum_{j \in \mathcal{S}, j \neq i} c_{ij}. $$

Then, for each $x \in \mathbb{R}^d$, $M(x)$ is a stochastic rate matrix of a finite state Markov process. We can write (1.1) in vector form as

$$ \mathcal{A} f(x) = \mathcal{L} f(x) + c f(x), \quad x \in \mathbb{R}^d, $$

with

$$ (\mathcal{L} f)_k(x) := \text{Tr} \left( a_k(x) \nabla^2 f_k(x) \right) + b_k(x) \cdot \nabla f_k(x) + \sum_{j \in \mathcal{S}} m_{kj}(x) f_j(x), \quad (x,k) \in \mathbb{R}^d \times \mathcal{S}, $$

(1.2)

and

$$ (c f)_k = c_k f_k. $$

The operator $\mathcal{L}$ in (1.2) is the extended generator of a regime switching diffusion in $\mathbb{R}^d$ (see description in \text{Section 1.1.1}). One could go one more step further in this representation. Define the collection $(L_k)_{k \in \mathcal{S}}$ of elliptic operators on $\mathbb{R}^d$ by

$$ L_k g(x) := \text{Tr} \left( a_k(x) \nabla^2 g(x) \right) + b_k(x) \cdot \nabla g(x), \quad (x,k) \in \mathbb{R}^d \times \mathcal{S}, \quad g \in C^2(\mathbb{R}^d). $$

(1.3)

Then, if we let $L := \text{diag}(L_1, \ldots, L_N)$, the operator $\mathcal{L}$ can be written in vector notation as

$$ \mathcal{L} f(x) = L f(x) + M(x) f(x), \quad x \in \mathbb{R}^d. $$
Throughout the paper, if $\mathcal{X}(\mathbb{R}^d)$ is a space of real-valued functions on $\mathbb{R}^d$ then we define the corresponding space $\mathcal{X}(\mathbb{R}^d \times S) := (\mathcal{X}(\mathbb{R}^d))^N$, and endow it with the product topology, if applicable. Thus, a function $f \in \mathcal{X}(\mathbb{R}^d \times S)$ is identified with the vector-valued function
\[
f := (f_1, \ldots, f_n) \in (\mathcal{X}(\mathbb{R}^d))^N, \quad \text{where} \quad f_k(\cdot) := f(\cdot, k), \quad k \in S.
\] (1.4)

With a slight abuse in notation we write $f \in \mathcal{X}(\mathbb{R}^d \times S)$. Naturally, inequalities such as $f \geq 0$ are meant to hold componentwise. Also, the product of two functions in $\mathcal{X}(\mathbb{R}^d \times S)$ should be understood componentwise.

We also identify an irreducibility property of the matrix $M$ which is used in many results.

**Definition 1.1.** The matrix $M$ is **irreducible** in a bounded domain $D$ if for any non-empty sets $S_1, S_2 \subset S$ satisfying $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$, there exists $i \in S_1$ and $j \in S_2$ satisfying
\[
|\{x \in D : m_{ij}(x) > 0\}| > 0.
\] (1.5)

We say that $M$ is irreducible in an unbounded domain if it is so in some bounded subdomain.

Throughout the rest of the paper, with the exception of Theorem 1.1 we assume that $M$ is irreducible on $\mathbb{R}^d$.

For a second order elliptic operator which is the extended generator of a diffusion process, the ergodic properties of the diffusion and the corresponding twisted diffusion play a crucial role in the study of the eigenvalue problem. This was thoroughly investigated in [16, 26]. In this paper, we wish to adopt an analytical approach and avoid, for the most part, probabilistic arguments. First, it avoids imposing unnecessary regularity hypotheses on the coefficients of the operator to ensure the existence of an associated stochastic process. Second by abstracting probabilistic properties into analytical ones, the results are not restricted to an operator of the form (1.2) but apply to a much larger class of elliptic operators which satisfies these properties. Third, by ‘translating’ the probabilistic arguments into analytical ones, it makes the arguments more accessible to the pde community, and, leads to a unified treatment of the problem.

**Definition 1.2** below abstracts the notions of regularity, recurrence, and geometric ergodicity of a diffusion to analogous notions for an operator. For nondegenerate diffusions on $\mathbb{R}^d$ the first two abstractions date back to the work of Hasminskii [27]. Here, we develop them further.

Before stating this definition, we comment on the Dirichlet problem for the operator $\mathcal{L}$. Let $S_1$ be a nonempty subset of $S$, and $D_i$, $i \in S_1$, bounded domains with smooth boundary. Let
\[
\Omega := \bigcup_{i \in S_1} (D_i \times \{i\}), \quad \text{and} \quad K := \left(\bigcup_{i \in S_1} D_i \times S\right) \setminus \Omega.
\] (1.6)

If $g : K \to \mathbb{R}$ is a continuous function and $f \in L^p(\Omega)$, then the Dirichlet problem $\mathcal{L}u = f$ on $\Omega$, and $f = g$ on $K$ has a unique solution in $u \in W^{2,p}_0(\Omega) \cap C(\Omega \cup K)$. This is shown in Lemma 2.1 in a slightly different form.

The definition below applies to a general operator $\mathcal{L}$ on $C^2(\mathbb{R}^d \times S)$, not necessarily of the form (1.2). For example, it applies to elliptic operators containing a nonlocal component.

**Definition 1.2.** Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and $S_1$ a nonempty subset of $S$. We say that $D \times S_1$ is **recurrent** (for $\mathcal{L}$) if the Dirichlet problem
\[
\mathcal{L}u = 0 \quad \text{in} \quad (\bar{D} \times S_1)^c, \quad u = g \quad \text{in} \quad \bar{D} \times S_1
\]
for any $g \in C(\bar{D} \times S_1)$ has a unique bounded solution $u \in W^{2,p}_0((\bar{D} \times S_1)^c) \cap C(\mathbb{R}^d \times S)$ for $p > d$.

An operator $\mathcal{L}$ on $C^2(\mathbb{R}^d \times S)$ is called

(i) **regular** if the equation $\mathcal{L}u = Cu$ has no bounded positive solution for any constant vector $C > 0$.

(ii) **recurrent** if every set of the form $B_r(x_0) \times S_1$, with nonempty $S_1 \subset S$, is recurrent.
(iii) **exponentially stable** if it is regular and there exists a \( \mathbf{V} \in \mathcal{W}_{loc}^{2,p}(\mathbb{R}^d \times S) \), \( p > d \), with \( \mathbf{V} \geq 1 \) and positive constants \( \kappa_0 \) and \( \kappa_1 \) such that

\[
\mathcal{L} \mathbf{V} \leq \kappa_0 \mathbf{1}_{K \times S} - \kappa_1 \mathbf{V} \quad \text{in} \quad \mathbb{R}^d \times S, \tag{1.7}
\]

for some compact set \( K \).

In this article we shall refer to \( \mathbf{V} \) as Lyapunov function. **Definition 1.2** (i) is also known as \( L^\infty \)-Liouville property and is a key property in the study of stochastic completeness of Riemannian manifolds [28, 29]. The above definitions are motivated from the associated probabilistic model as can be seen from the Remark 1.1 below. As mentioned earlier, we do not impose any regularity hypotheses on the coefficients to ensure existence of a stochastic process corresponding to the extended generator \( \mathcal{L} \).

We state two versions of the strong maximum principle for the operator \( \mathcal{L} \) which we use often. These do not require irreducibility of \( M \).

(P2a) Suppose that \( \mathbf{u} \in \mathcal{W}_{loc}^{2,d}(D \times S) \) satisfies \((\mathcal{L} \mathbf{u})_i - cu_i \leq 0 \) on a domain \( D \) for some constant \( c \geq 0 \) and \( i \in S \). Then \( (x, i) \mapsto u(x, i) \) cannot attain a negative minimum in \( D \times \{i\} \).

(P2b) Let \( \Omega \) and \( K \) be as in (1.6). Suppose that \( \mathbf{u} \in \mathcal{W}^{2,d}(\Omega) \cap C(\Omega \cup K) \) is nonnegative on \( K \) and satisfies \((\mathcal{L} \mathbf{u})_i - cu_i \leq 0 \) on \( \Omega \) for some constant \( c \geq 0 \) and \( i \in S_1 \). Then \( u_i \) cannot have a nonpositive local minimum in \( D_i \) unless it is equal to a constant.

**Remark 1.1.** Suppose that \((X, S)\) is a (local) regime switching diffusion corresponding to the generator \( \mathcal{L} \) where \( X = \{X_t\}_{t \geq 0} \) represents the diffusion component and \( S = \{S_t\}_{t \geq 0} \) represents the finite state Markov process taking values in \( S \). In particular, \((X, S)\) solves the associated local martingale problem. In fact, if \( a \) has at most quadratic growth, \( b \) has at most linear growth and \( M \) is bounded, then the corresponding martingale problem is well-posed (this follows combining [30], [31, Theorem 5.2] and [32, Theorem 5.2]). Let \( B_n \) denote the ball of radius \( n \) around 0, and \( \tau_n \) the first exit time from \( B_n \times S \), that is,

\[ \tau_n := \inf \{ t > 0 : X_t \notin B_n \} . \]

The diffusion \((X, S)\) is said to be regular [9, Definition 2.6] if for all \((x, i) \in \mathbb{R}^d \times S\) we have \( P_{x,i}(\lim_{n \to \infty} \tau_n < \infty) = 0 \). Using the Itô-Krylov formula [50, p. 122] it is easily seen that \( u_n(x, i) = E_{x,i}[e^{-C\tau_n}] \) solves the Dirichlet problem

\[ \mathcal{L} u_n = C u_n \quad \text{in} \quad B_n \times S, \quad \text{and} \quad u_n = 1 \quad \text{on} \quad \partial B_n \times S. \tag{1.8} \]

Suppose that \( v \) is a bounded positive solution of \( \mathcal{L} v = C v \). Without loss of generality, assume that \( v \leq 1 \). By the strong maximum principle, \( u_n \geq v \) in \( B_n \). Thus taking limits it follows that \( E_{x,i}[e^{-C\tau_n}] > 0 \), where \( \tau_n = \lim_{n \to \infty} \tau_n \), from which it is straightforward to deduce that \( \tau_n < \infty \) with positive probability. Thus the diffusion cannot be regular. Conversely, if the diffusion is not regular, then using standard elliptic pde estimates it can be easily shown that \( u = \lim_{n \to \infty} u_n \) is a solution of

\[ \mathcal{L} u = C u \quad \text{in} \quad \mathbb{R}^d \times S. \]

Since \( P(\tau_n < \infty) > 0 \) by hypothesis, it follows that \( u(x, i) = E_{x,i}[e^{-C\tau_n}] > 0 \), and hence this solution is bounded and positive. Thus the regularity of \((X, S)\) is equivalent to the regularity of \( \mathcal{L} \) in **Definition 1.2** (i). The above argument is due to Hasminskii [27, Lemma 4.1].

Next, we show that nonexistence of positive bounded solution to \( \mathcal{L} u = u \) characterizes regularity.

**Lemma 1.1.** Suppose that, for all large enough \( n \in \mathbb{N} \), the operator \( \mathcal{L} \) satisfies the strong maximum principle (P2a) on \( B_n \times S \) and the Dirichlet problem in (1.8) has a unique solution for every \( C > 0 \). Then \( \mathcal{L} \) is regular if and only if the equation \( \mathcal{L} u = u \) has no positive bounded solution.
Proof. Necessity follows from Definition 1.2 (i). To show sufficiency, we argue as follows. First, suppose that \( v < 1 \) is a bounded positive subsolution of \( \mathcal{L}v \geq \kappa v \) for some \( \kappa > 0 \). Let \( C \in (0, \kappa] \).

The solution \( u_n \) of (1.8) satisfies

\[
\mathcal{L}(u_n - v) - C(u_n - v) \leq (C - \kappa)v \leq 0 \quad \text{in } B_n,
\]

and thus we must have \( u_n > v \) on \( B_n \) by the strong maximum principle. Taking limits as \( n \to \infty \), we deduce that \( \mathcal{L}u = Cu \) must have a positive bounded solution for any \( C \in (0, \kappa] \). To finish the proof, it suffices to show that \( \mathcal{L}u \geq 2\kappa u \) has a positive bounded solution. We argue by contradiction. Suppose that this equation has no positive bounded solution. Consider the Dirichlet problems \( \mathcal{L}u_n = 2\kappa u_n \) in \( B_n \times S \) with \( u_n = \|v\|_{\infty} \) on \( \partial B_n \times S \). According to the hypothesis \( u_n \) must converge pointwise to 0 as \( n \to \infty \). Let \( u_{i,n} := (u_n)_i \) for \( i \in S \). Then, for all large enough \( n \), we must have that

\[
\inf_{B_n \in S} \min_{i \in S} (u_{i,n} - v_i) \leq -\frac{2}{3}\|v\|_{\infty}.
\]

Hence if \((x_n, i) \in B_n \times S\) is a point where \( \min_{i \in S} (u_{i,n} - v_i) \) attains its infimum in \( B_n \), it follows from (1.9) that

\[
2u_{i,n}(x_n) - v_i(x_n) \leq u_i(x_n) - \frac{4}{3}\|v\|_{\infty} < 0.
\]

Since \( (\mathcal{L}(u_n - v))_i \leq \kappa(2u_{i,n} - v_i) \) in \( B_n \), (1.10) shows that \( (\mathcal{L}(u_n - v))_i < 0 \) in some open neighborhood of \( x_n \) which contradicts (P2a).

We can also relate Definition 1.2 (ii) with the recurrence property of a process \((X, S)\) associated with \( \mathcal{L} \). Recall that a bounded domain \( D \times S_1 \) is said to be recurrent for the process \((X, S)\) if \( \mathbb{P}_{x,i}(\hat{\tau}(D \times S_1) < \infty) = 1 \) for all \((x, i) \in (D \times S_1)^c \) where \( \hat{\tau}(A) \) denotes the first hitting time of the set \( A \), that is,

\[
\hat{\tau}(A) := \inf\{t > 0 : (X_t, S_t) \in A\}.
\]

Also, a process \((X, S)\) is said to be recurrent if any such domain \( D \times S_1 \) is recurrent. With the help of Itô-Krylov formula it can be easily checked that \( u(x, i) = \mathbb{P}_{x,i}(\hat{\tau}(D \times S_1) < \infty) \) is a solution to

\[
\mathcal{L}u = 0 \quad \text{in } (\hat{D} \times S_1)^c, \quad u = 1 \quad \text{in } \hat{D} \times S_1.
\]

One can easily relate the recurrence of \((X, S)\) with Definition 1.2 (ii). This is also discussed in [27]. We want to note here that the recurrence properties depend on the irreducibility of \( M \) in Definition 1.1. If \( M \) is irreducible on \( \mathbb{R}^d \), then for the process \((X, S)\) to be recurrent it suffices that any bounded domain \( D \times S_1 \) is recurrent. Also, we have a dichotomy: the process is either recurrent of transient. However, if \( M \) is reducible on \( \mathbb{R}^d \), this dichotomy does not hold. For a counterexample see Example 1.1 below.

The following example shows that exponential stability as in Definition 1.2 (iii) does not immediately imply recurrence, unless \( M \) is irreducible on \( \mathbb{R}^d \).

**Example 1.1.** Let \( S = \{1, 2\} \), and dynamics given by

\[
\begin{align*}
\frac{dX_1(t)}{} &= \text{sign}(X_1(t)) + \sqrt{2}dW_1(t), \\
\frac{dX_2(t)}{} &= -X_2(t) + \sqrt{2}dW_2(t).
\end{align*}
\]

Here, \( X_1 \) and \( X_2 \) are 1-dimensional Itô processes. Suppose that \( M = 0 \) on \( B_2 \times S \), while on \( B_2^c \times S \) we have \( m_{11} = -\delta, m_{12} = \delta, m_{21} = -m_{22} = 0 \), for some \( \delta > 0 \). It can be seen, by using the Lyapunov function \( \mathcal{V}_2(x) = \frac{x^2}{1 + x^2} \) and \( \mathcal{V}_1(x) = 3 - \mathcal{V}_2x \), the switched diffusion is exponentially stable according to Definition 1.2 (iii). However, it is clear that the set \( B \times \{1\} \) cannot be recurrent. The lack of irreducibility of \( M \) is responsible for this. On the other hand, the set \( B \times S \) is recurrent.

If we modify \( M \) and let \( m_{11} = -m_{22} = \epsilon \) in \( B_2^c \times S \) for some \( \epsilon > 0 \), then the Lyapunov equation (1.7) still holds, and \( B \times \{1\} \) is now recurrent.
Theorem 1.1 below, concerns the relations among (i)–(iii) in Definition 1.1. These are well-known if \( \mathcal{L} \) is the extended generator of a stochastic process \( (X, S) \) and \( M \) is irreducible on \( \mathbb{R}^d \) (cf. [9, 33, 34]). Our objective though is to provide analytical proofs in a very general setting without using the probabilistic structure. The results apply to any elliptic operator \( \mathcal{L} \) satisfying the strong maximum principle in (P2a) or (P2b) and for which the Dirichlet problem on a bounded domain \( D \times S_1 \) has a solution.

**Theorem 1.1.** The following hold.

(a) A recurrent operator \( \mathcal{L} \) is regular.

(b) Provided that \( M \) is irreducible on \( \mathbb{R}^d \), an exponentially stable operator \( \mathcal{L} \) is recurrent.

(c) Irrespective of the irreducibility properties of \( M \), if \( \mathcal{L} \) is exponentially stable, then any bounded domain of the form \( D \times S \) is recurrent.

The proof of Theorem 1.1 is in Section 2.

As already mentioned, this paper is devoted to the study of the generalized principal eigenvalue \( \lambda^* \) in \( \mathbb{R}^d \) of \( \mathcal{A} \). Let

\[
\Psi^+(\lambda) := \{ f \in W^{2, d}_{loc}(\mathbb{R}^d \times S) : f > 0, \mathcal{A} f + \lambda f \leq 0 \text{ in } \mathbb{R}^d \}, \quad \lambda \in \mathbb{R}.
\]

The principal eigenvalue \( \lambda^* \) is defined as

\[
\lambda^* := \sup \{ \lambda \in \mathbb{R} : \Psi^+(\lambda) \neq \emptyset \}.
\]

We refer to the parameter \( c \) as the potential and, when needed, we indicate the dependence of \( \lambda^* \) on \( c \) explicitly in the notation by writing \( \lambda^*(c) \). Some early works on generalized principal eigenvalue for scalar elliptic equation appeared in Protter–Weinberger [35], Nussbaum [36] and Nussbaum–Pinchover [37]. Generalized eigenvalues and its relation to maximum principles in bounded domains are established in the seminal work of Berestycki–Nirenberg–Varadhan [38]. Later, this was extended to more general operators. Recently, Berestycki–Rossi [25] studied the principal eigenvalue problem for scalar elliptic operators in unbounded domains and established several interesting properties (see also [37]).

We say that a constant \( \lambda \in \mathbb{R} \) and a positive \( \Psi \in W^{2,p}_{loc}(\mathbb{R}^d \times S) \), \( p > d \), solve the eigenvalue equation for \( \mathcal{A} \), if \( \mathcal{A} \Psi = -\lambda \Psi \). In such a case we call \( \Psi \) the eigenfunction and \( (\Psi, \lambda) \) the *eigenpair*. An eigenfunction is always meant to be a positive function. The theorem which follows is also a generalization of [25, Theorem 1.4].

**Theorem 1.2.** For any \( \lambda \leq \lambda^* \) there exists \( \Psi \in W^{2,p}_{loc}(\mathbb{R}^d \times S) \), \( \Psi > 0 \), satisfying \( \mathcal{A} \Psi = -\lambda \Psi \) in \( \mathbb{R}^d \times S \).

In view of Theorem 1.2, the following question seems natural.

**Question.** Given an eigenpair \( (\Psi, \lambda) \), when can we identify it as a principal eigenpair?

A main goal of this article is to answer this question by exploiting the ‘stability’ properties of the twisted operators. Recall the operator \( \mathcal{L} \) in (1.2). Corresponding to an eigenpair \( (\Psi, \lambda) \), we let \( \psi := \log \Psi \) componentwise, and define the twisted operator \( \mathcal{L}^\psi \) as follows. We first define the operators \( \{ L_k^\psi \}_{k \in S} \), by

\[
L_k^\psi g(x) := \text{Tr}(a_k(x) \nabla^2 g(x)) + (b_k(x) + 2a_k(x) \nabla \psi_k(x)) \cdot \nabla g(x), \quad x \in \mathbb{R}^d, \quad g \in C^2(\mathbb{R}^d),
\]

and the matrix \( \mathcal{M} = [\tilde{m}_{ij}]_{i,j \in S} \) by

\[
\tilde{m}_{ij} := m_{ij} \frac{\Psi_j}{\Psi_i} \quad \text{for } i \neq j, \quad \text{and } \tilde{m}_{ii} := -\sum_{j \in S, j \neq i} \tilde{m}_{ij}.
\]
With these definitions, the twisted operator is given by
\[ \tilde{\mathcal{L}}^\psi f(x) := \mathcal{L}^\psi f(x) + \tilde{M}(x)f(x), \quad x \in \mathbb{R}^d. \] (1.12)

Let us first consider the case of bounded domains. Let \( D \) be a smooth bounded domain in \( \mathbb{R}^d \), and \((\Psi_D, \lambda_D)\) be the Dirichlet principal eigenpair of \( \mathcal{A} \) in \( D \), that is,
\[
\begin{align*}
\mathcal{A} \Psi_D &= \lambda_D \Psi_D \quad \text{in } D \times S, \\
\Psi_D &= 0 \quad \text{on } \partial D \times S, \\
\Psi_D &> 0 \quad \text{in } D \times S, \tag{1.13}
\end{align*}
\]
and \( \Psi_D \in C_0(\overline{D} \times S) \cap \mathcal{W}^{2,p}_\text{loc}(D \times S) \) for \( p > d \) (see Theorem A.1). We note that \( \lambda_D \) is the only eigenvalue with a positive eigenfunction. We let \( \tilde{\mathcal{L}}_D \) denote the twisted operator corresponding to the eigenpair \((\Psi_D, \lambda_D)\). Then we have the following result.

**Theorem 1.3.** There exists a inf-compact function \( \mathcal{V}_D : D \times S \to [1, \infty) \), \( \mathcal{V}_D \in \mathcal{W}^{2,p}_\text{loc}(D \times S) \), satisfying
\[ \tilde{\mathcal{L}}_D \mathcal{V}_D \leq -\delta_1 \mathcal{V}_D + \delta_2 \quad \text{in } D \times S \] (1.14)
for some constants \( \delta_1, \delta_2 > 0 \). Furthermore, \( \tilde{\mathcal{L}}_D \) is regular when restricted to \( D \) in the sense of Definition 1.2 (i).

The function \( \mathcal{V}_D \) in (1.14) is commonly known as Lyapunov function. It turns out that the existence of a Lyapunov function in a bounded domain follows from the monotonicity property (cf. Theorems A.2 and A.3) of the principal eigenvalue; this does not always hold in \( \mathbb{R}^d \).

**Remark 1.2.** The process associated with \( \tilde{\mathcal{L}}_D \) in Theorem 1.3 is confined in the domain \( D \), and is known in the literature as the \( Q \)-process. There is an extensive literature on the \( Q \)-process covering various classes of Markov processes. We cite here [39–41].

**Remark 1.3.** Lyapunov functions play a central role in the study of exponential ergodicity of regime switching diffusions. In fact, finding sufficient condition for the existence of Lyapunov function is an important issue. See, for instance, [9, 42–45] and references therein. In particular, by [44, Theorem 5.3], if \( \mathcal{V} \) is inf-compact, \( M \) is bounded and \( a \) is uniformly elliptic, we get \( \mathcal{V} \)-geometric ergodicity for the regime switching diffusion. In this paper, we only concentrate on the existence of Lyapunov functions and do not address the delicate issue of exponential ergodicity.

The existence of a Lyapunov function for \( \tilde{\mathcal{L}} \) in \( \mathbb{R}^d \) is related to a certain monotonicity property of the principal eigenvalue in \( \mathbb{R}^d \), which we state next. By \( \mathcal{B}_0^+(\mathbb{R}^d \times S) \) we denote the class of all nontrivial, nonnegative bounded measurable functions \( h : \mathbb{R}^d \times S \to [0, \infty) \) that vanish at infinity.

**Definition 1.3.** We say \( \lambda^* \) is **monotone at \( c \) on the right**, if \( \lambda^*(c) > \lambda^*(c+h) \) for all \( h \in \mathcal{B}_0^+(\mathbb{R}^d \times S) \), and we say \( \lambda^* \) is **strictly monotone at \( c \)**, if \( \lambda^*(c-h) > \lambda^*(c) \) for some \( h \in \mathcal{B}_0^+(\mathbb{R}^d \times S) \).

**Remark 1.4.** It is shown later in Theorem 1.6 that strict monotonicity implies \( \lambda^*(c-h) > \lambda^*(c) \) for all \( h \in \mathcal{B}_0^+(\mathbb{R}^d \times S) \). Furthermore, since \( c \mapsto \lambda^*(c) \) is concave by Lemma 2.3, strict monotonicity implies monotonicity on the right.

Throughout the rest of the paper, we always assume that the principal eigenvalue is finite:

**Hypothesis 1.1.** \( \lambda^* = \lambda^*(c) < \infty \).

The next result shows that twisted operators corresponding to the lower eigenvalues are not recurrent. This should be compared with [20, Proposition 3.3] and [16, Theorem 2.1].

**Theorem 1.4.** For any \( \lambda < \lambda^* \), the twisted operator \( \tilde{\mathcal{L}} \) corresponding to \((\Psi, \lambda)\) is not recurrent.
This of course, brings us to the question what happens to the twisted operator corresponding to the principal eigenpair. As the following example suggests, the twisted operator may be non-recurrent for the principal eigenpair even when $M$ is irreducible.

Example 1.2. Let $N = 2$ and $a = I$, $b = 0$, $c = 0$, and $m_{12} = m_{21} = 1$. Then the constant functions are the principal eigenfunctions and $\lambda^* = 0$. Note that $(\sin(n^{-1}\pi x), \sin(n^{-1}\pi x))$ are the eigenfunctions in $B_n \times S$ with eigenvalue $\frac{\pi^2}{n^2}$. Therefore, $\lambda^* = \lim_{n \to \infty} \frac{\pi^2}{n^2} = 0$. Since constants are principal eigenfunctions, the corresponding twisted operator is the same as $L$. Therefore, the twisted operator is not recurrent for $d \geq 3$. Indeed, setting $u_i(x) = 1 - |x|^{2-d}$ and $B = B_1$ we see that

$$Lu = 0 \text{ in } (B \times S)^c, \text{ and } u = 0 \text{ on } \partial B \times S.$$ 

It turns out that the recurrence of the twisted operator is equivalent to the monotonicity of $\lambda^*$ on the right.

Theorem 1.5. The following are equivalent.

(a) The twisted operator corresponding to $(\Psi^*, \lambda^*)$ is recurrent.
(b) $\lambda^*$ is monotone on the right at $c$.

In addition, under either (a) or (b), $\lambda^*$ is a simple eigenvalue.

The next result characterizes the strict monotonicity property.

Theorem 1.6. The following are equivalent.

(a) The twisted operator corresponding to $(\Psi^*, \lambda^*)$ is exponentially stable.
(b) $\lambda^*$ is strictly monotone at $c$.
(c) For any $h \in B_0^+(\mathbb{R}^d \times S)$, we have $\lambda^*(c - h) > \lambda^*(c)$.

It is interesting to observe from Theorems 1.5 and 1.6 that monotonicity of the principal eigenvalue implies simplicity of the eigenvalue. Another criterion that is often used to ensure simplicity of principal eigenvalue is Agmon’s minimal growth at infinity introduced by Agmon in [46] (see also [25, Definition 8.2] and [47, 48]).

Definition 1.4 (Minimal growth at infinity). An eigenpair $(\Psi, \lambda)$ is said to have minimal growth at infinity, if for any compact set $K \times S_1 \subset \mathbb{R}^d \times S$ and for any $\Phi \in W_{loc}^{2,p}((K \times S_1)^c)$, $p > d$, continuous and positive in $\mathbb{R}^d \times S$, satisfying

$$L\Phi + (c + \lambda)\Phi \leq 0 \text{ in } (K \times S_1)^c,$$

we have $\Phi \geq \kappa \Psi$ in $\mathbb{R}^d \times S$, for some $\kappa > 0$.

Then the following result is immediate.

Theorem 1.7. Suppose that $(\Psi, \lambda)$ has the minimal growth at infinity. Then $(\Psi, \lambda)$ is a principal eigenpair and $\lambda$ is simple.

Proof. Let $(\Psi^*, \lambda^*)$ be a principal eigenpair. Then it follows from Theorem 1.2 that $\lambda^* \geq \lambda$, and

$$L\Psi^* + (c + \lambda)\Psi^* \leq L\Psi^* + (c + \lambda^*)\Psi^* = 0 \text{ in } \mathbb{R}^d.$$ 

Thus, the minimal growth at infinity of $(\Psi, \lambda)$ implies that $\Psi^* > \kappa \Psi$ for some $\kappa > 0$. Let $\kappa_1 := \min_{k \in S} \inf_{\mathbb{R}^d} \frac{\Psi_k^*}{\Psi_k}$. We claim that $\Psi^* - \kappa_1 \Psi \geq 0$, and that at least one of the components must vanish in $\mathbb{R}^d$. If not, then we get $\Phi = \Psi^* - \kappa_1 \Psi > 0$ and

$$L\Phi + (c + \lambda)\Phi \leq 0, \text{ in } \mathbb{R}^d.$$
which implies that $\Phi > \kappa_2 \Psi$ for some $\kappa_2 > 0$. But this contradicts the definition of $\kappa_1$. Thus, one of the components of $\Phi$ must vanish in $\mathbb{R}^d$. The strong maximum principle then implies that $\Phi = 0$. Hence $\lambda = \lambda^*$ and $\Psi^* = \kappa_1 \Psi$ in $\mathbb{R}^d \times S$. This completes the proof. \hfill \Box

Continuing, we show that minimal growth at infinity is equivalent to monotonicity of the principal eigenvalue on the the right. For the scalar equation an analogous result is established in [22] using probabilistic methods. In contrast, the proof of Theorem 1.8 is analytical, and thus more general in nature.

**Theorem 1.8.** The following are equivalent.

(a) $\lambda^*$ is monotone on the right at $c$.

(b) $(\Psi^*, \lambda^*)$ has minimal growth at infinity.

Next we relate the monotonicity property of $\lambda^*$ on the right with the stochastic representation of the principal eigenfunction $\Psi^*$. To do so we need to impose mild restrictions on the coefficients of $\mathcal{L}$ to ensure the existence of a strong solution.

1.1.1. **Description of the probabilistic model.** We introduce the regime switching diffusion process. This is a process $(X_t, S_t)$ in $\mathbb{R}^d \times S$ governed by the following stochastic differential equations:

\[
\begin{align*}
\text{d}X_t &= b(X_t, S_t)\text{d}t + \sigma(X_t, S_t)\text{d}W_t, \\
\text{d}S_t &= \int_{\mathbb{R}} h(X_t, S_t, \rho, z)\varphi(\text{d}t, \text{d}z),
\end{align*}
\]

for $t \geq 0$. Here

(i) $S_0$ is a prescribed $S = \{1, 2, \ldots, N\}$ valued random variable;

(ii) $X_0$ is a prescribed $\mathbb{R}^d$ valued random variable;

(iii) $W$ is a $d$-dimensional standard Wiener process;

(iv) $\varphi(\text{d}t, \text{d}z)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\text{d}t \times \mu(\text{d}z)$, where $\mu$ is the Lebesgue measure on $\mathbb{R}$;

(v) $\varphi(\cdot, \cdot), W(\cdot), X_0,$ and $S_0$ are independent;

(vi) The function $h: \mathbb{R}^d \times S \times \mathbb{R} \to \mathbb{R}$ is defined by

\[
h(x, i, z) := \begin{cases} 
  j - i & \text{if } z \in \Delta_{ij}(x), \\
  0 & \text{otherwise,}
\end{cases}
\]

where for $i, j \in S$ and fixed $x$, $\Delta_{ij}(x)$ are left closed right open disjoint intervals of $\mathbb{R}$ having length $m_{ij}(x)$.

Note that $M(x)$ can be interpreted as the rate matrix of the Markov chain $S_t$ given that $X_t = x$. In other words,

\[
P(S_{t+h} = j \mid X_t, S_t) = \begin{cases} 
  m_{S_{ij}}(X_t)h + a(h) & \text{if } S_t \neq j, \\
  1 + m_{S_{ij}}(X_t)h + a(h) & \text{if } S_t = j,
\end{cases}
\]

and $X$ behaves like an ordinary diffusion process governed by (1.15) between two consecutive jumps of $S$. In addition to (1.5), we impose the following assumptions to guarantee existence of solution of (1.15).

(A1) **Local Lipschitz continuity:** The function $\sigma = \begin{bmatrix} \sigma^{ij} \end{bmatrix}: \mathbb{R}^d \times S \to \mathbb{R}^{d \times d}$ is continuous and locally Lipschitz in $x$ with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, with $\|\sigma\| := \sqrt{\text{Tr}(\sigma^* \sigma)}$, we have

\[
\|\sigma(x, k) - \sigma(y, k)\|^2 \leq C_R |x - y|^2 \quad \forall x, y \in B_R, \forall k \in S.
\]

The function $b: \mathbb{R}^d \times S \to \mathbb{R}^d$ is assumed to be Borel measurable and locally bounded.
(A2) Affine growth condition: $b(x, k)$ and $\sigma(x, k)$ satisfy a global growth condition of the form
\[
(b(x, k), x)^T + \|\sigma(x, k)\|^2 \leq C_0 (1 + |x|^2) \quad \forall (x, k) \in \mathbb{R}^d \times \mathcal{S},
\]
for some constant $C_0 > 0$.

(A3) Nondegeneracy: For each $R > 0$, it holds that
\[
\sum_{i,j=1}^d a^{ij}_k(x)\zeta_i\zeta_j \geq C_R^{-1} |\zeta|^2 \quad \forall (x, k) \in B_R \times \mathcal{S},
\]
and for all $\zeta = (\zeta_1, \ldots, \zeta_d)^T \in \mathbb{R}^d$, where, $a := \frac{1}{2} \sigma \sigma^T$.

It is well known that under hypotheses (A1)–(A3), (1.1) has a unique strong solution with $X \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^d)$, and $S_t \in \mathcal{D}(\mathbb{R}_+; \mathcal{S})$, where $\mathcal{D}(\mathbb{R}_+; \mathcal{S})$ denotes the space of all right continuous functions from $\mathbb{R}_+$ to $\mathcal{S}$ having left limit [49] (cf. [33, Remark 5.1.2]). Moreover, the solution $(X_t, S_t)$ is a Feller process (see [49, Theorem 2.1] and [33, Remark 5.1.6]) and therefore, a strong Markov process. Also, the ergodic behavior of $Y_t := (X_t, S_t)$ depends heavily on the coupling coefficients $\{m_{ij}\}$ (cf. [33], [9, Chapter 2]).

For a ball $\mathcal{B}$, centered at 0, we denote by $\check{\tau}$ the first hitting time to $\mathcal{B} \times \mathcal{S}$, that is,
\[
\check{\tau} := \inf\{t > 0 : X_t \in \mathcal{B}\}.
\]

**Theorem 1.9.** The following are equivalent.

(a) $\lambda^*$ is monotone on the right at $c$.

(b) For some ball $\mathcal{B}$ we have
\[
\Psi^*_k(x) = \mathbb{E}_{x,k} \left[ e^{(c(X_t, S_t) - \lambda^*)^+ d t} \Psi^*(X_t, S_t) \mathbb{1}_{\{\check{\tau} < \infty\}} \right], \quad (x, k) \in \mathcal{B}^c \times \mathcal{S}. \tag{1.16}
\]

Before we conclude this section, let us compare the contribution of this paper with the existing work. The notion of monotonicity was introduced in the [16], and results analogous to Theorems 1.4 to 1.6 were proved in for a scalar operator using probabilistic methods. In particular, $a$ was assumed to be locally Lipschitz, $b$ was assumed to satisfy (A2) and $c$ was assumed to be bounded from below. In this article we do not impose such restrictions. So the arguments in [16] do not work for us in this article. As can be seen, Theorem 1.9 is the only result that relies on the probabilistic model, but the proof does not use the results in [16]. This is because of the nonavailability of a suitable Girsanov transformation for a general regime switching diffusion. Instead, we study the parabolic system (see Lemma 2.4) to find a substitute for Girsanov’s transformation for this model. In this manner, we obtain an explicit form for the twisted operator in (1.12) for elliptic systems.

1.2. Notation. $F > \kappa$ would mean $F_k > \kappa$ for all $k \in \mathcal{S}$ and $F \geq 0$ means $F_k \geq 0$ for all $k \in \mathcal{S}$. $F \geq 0$ means $F_k \geq 0$ for all $k \in \mathcal{S}$ and $\sum_{k \in \mathcal{S}} F_k > 0$ on a set of positive Lebesgue measure in $\mathbb{R}^d$. By $B_r(x)$ we denote the ball of radius $r$ around $x$ and for $x = 0$ we simply denote it by $B_r$.

If $\mathcal{X}(Q)$ is a topological space of real-valued functions on a domain $Q \subset \mathbb{R}^d$, we denote by $\mathcal{X}(Q \times \mathcal{S})$ the space $(\mathcal{X}(Q))^\mathcal{S}$ endowed with the product topology inherited from $\mathcal{X}(Q)$. As already explained in (1.4), if $f$ is a real valued function on $Q \times \mathcal{S}$, we let $f_k(\cdot) := f(\cdot, k)$, and identify $f$ with $f := (f_1, \ldots, f_N)$, which is viewed as a vector-valued function on $Q$. If $\mathcal{X}(Q)$ is endowed with a norm $\| \cdot \|_{\mathcal{X}(Q)}$, we let $\| f \|_{\mathcal{X}(Q \times \mathcal{S})} := \sum_{k \in \mathcal{S}} \| f_k \|_{\mathcal{X}(Q)}$ for $f \in \mathcal{X}(Q \times \mathcal{S})$. 

2. Proofs of main results

In this section we present the proofs of the main results. The proof of Theorem 1.1 requires the following Liouville property.

**Proposition 2.1.** Suppose that \( \mathcal{L} \) is recurrent. Then any \( \mathbf{V} \in W_{\text{loc}}^{2,d}(\mathbb{R}^d \times \mathcal{S}) \) which is bounded from below in \( \mathbb{R}^d \times \mathcal{S} \) and satisfies \( \mathcal{L} \mathbf{V} \leq 0 \) in \( \mathbb{R}^d \times \mathcal{S} \) must be equal to a constant, that is, \( \mathbf{V} = (c, c, \ldots, c) \).

**Proof.** With no loss of generality we may assume that \( \mathbf{V} > 0 \). We pick some \( z \in \mathbb{R}^d \) and then we show that \( \inf_{j \in \mathcal{S}} \inf_{\mathbb{R}^d} V_j \geq \min_{j \in \mathcal{S}} V(z, j) \). Let \( i \in \mathcal{S} \) be such that \( \min_{j \in \mathcal{S}} V(z, j) = V(z, i) \). Given \( \delta \in (0, V(z, i)) \), fix \( \varepsilon > 0 \) small enough so that \( V(x, j) > V(z, i) - \delta \) for all \( x \in B_{\varepsilon}(z) \) and \( j \in \mathcal{S} \). Consider the sequence of solutions \( w_n \), satisfying

\[
\begin{align*}
\mathcal{L} w_n &= 0 \quad \text{in} \quad (B_n(z) \setminus B_{\varepsilon}(z)) \times \mathcal{S}, \\
w_n &= 0 \quad \text{on} \quad \partial B_n(z) \times \mathcal{S}, \\
w_n &= V(z, i) - \delta \quad \text{on} \quad \partial B(z, \varepsilon) \times \{i\}.
\end{align*}
\]

Applying the maximum principle [51, Theorem 3], it is easy to see that

\[0 \leq w_n(x, j) \leq \min \left\{ \max_{j \in B(z)} V, V(x, j) \right\} \text{ in } (B_n(z) \setminus B_{\varepsilon}(z)) \times \mathcal{S} \tag{2.1} \]

Letting \( n \to \infty \) and using standard elliptic estimates we find a bounded solution \( \mathbf{w} \) of

\[ \mathcal{L} \mathbf{w} = 0 \quad \text{in} \quad B_{\varepsilon}(z) \times \mathcal{S}, \quad \mathbf{w} = V(z, i) - \delta \quad \text{on} \quad \partial B(z, \varepsilon) \times \{i\}. \]

Using the recurrence of \( \mathcal{L} \) it is evident that \( \mathbf{w} = V(z, i) - \delta \) and then, using (2.1) we obtain \( V(z, i) - \delta \leq V(x, j) \) for all \( x \in B_{\varepsilon}(z) \) and \( j \in \mathcal{S} \). Now letting \( \delta, \varepsilon \to 0 \) gives us

\[ \inf_{j \in \mathcal{S}} \inf_{\mathbb{R}^d} V_j \geq V(z, i) = \min_{j \in \mathcal{S}} V(z, j). \]

This of course, implies that \( \mathbf{V} \) attains its minimum in \( \mathbb{R}^d \times \mathcal{S} \). Let \( \xi(x, j) = V(x, j) - V(z, i) \). Then \( \xi \geq 0 \) and

\[ \text{Tr}(a_{ii}(x)\nabla^2 \xi_i(x)) + b_i(x) \cdot \nabla \xi_i(x) + m_{ii}(x)\xi_i(x) \leq (\mathcal{L} \xi)_i \leq 0 \quad \text{in} \quad \mathbb{R}^d. \]

Since \( m_{ii} \leq 0 \), by the strong maximum principle, this implies \( \xi_i = 0 \) in \( \mathbb{R}^d \). This also implies that

\[ 0 = (\mathcal{L} \xi)_i = \sum_{k \neq j} m_{jk}(x)\xi_k(x) \quad \text{in} \quad \mathbb{R}^d. \tag{2.2} \]

Using (1.5) we find \( k \in \mathcal{S} \setminus \{i\} \) so that \( m_{ik}(y) > 0 \) for some \( y \in \mathbb{R}^d \). Hence from (2.2) we get \( \xi_k(y) = 0 \). Then repeating the above argument once again we have \( \xi_k = 0 \) in \( \mathbb{R}^d \). Now we can repeat the same argument with the help of (1.5) to arrive at \( \xi = 0 \) in \( \mathbb{R}^d \times \mathcal{S} \). This completes the proof. \( \square \)

We also need a maximum principle which is a mild extension of [51, Theorem 1]. Consider a collection of smooth bounded domains \( \{D_i\} \) with the property that \( D_i \subset D \) for \( 1 \leq i \leq N \). Let \( g_i : \overline{D} \setminus D_i \to \mathbb{R} \) be given continuous functions for \( 1 \leq i \leq N \), and

\[ G := \max_{D \setminus D_i} \max_i g_i^+. \]

**Lemma 2.1.** Let \( D_i \subset D \subset \mathbb{R}^d \), \( i \in \mathcal{S} \), be bounded domains, and \( c \leq 0 \). Suppose that \( u_i \in W_{\text{loc}}^{2,d}(D_i) \cap C(\overline{D}) \) satisfy

\[ (\mathcal{L} u)_i + c_i u_i \geq -f_i^+ \quad \text{in} \quad D_i, \quad u_i = g_i \quad \text{in} \quad D \setminus D_i, \quad \text{for all} \quad i \in \mathcal{S}, \]

then \( u_i \geq u \) in \( D \), where \( u \) is a solution of the following problem:

\[ (\mathcal{L} u) + \epsilon u \geq -\epsilon f_i^+ \quad \text{in} \quad D \setminus D_i, \quad u = g_i \quad \text{on} \quad \partial D \setminus \partial D_i, \quad \text{for all} \quad i \in \mathcal{S}. \]
with \( c \leq 0 \) Then for some constant \( C \), not dependent on \( u, f \) and \( g \), we have
\[
\max_i \sup_{D_i} u_i^+ \leq \left( G + C \sum_{i=1}^d \| f_i^+ \|_{L^d(D_i)} \right). \tag{2.3}
\]
Furthermore, if \( f_i \in L^d(D_i) \) for \( 1 \leq i \leq N \), then there exists a unique solution to
\[
(\mathcal{L} u)_i + c_i u_i = f_i \quad \text{in } D_i, \quad u_i = g_i \quad \text{in } D \setminus D_i, \quad \text{for all } i \in S. \tag{2.4}
\]

**Proof.** To establish (2.3) we follow the idea of [51]. Let \( j \) be such that \( \max_i \sup_{D_i} u_i^+ = \sup_{D_j} u_j^+ \). Replacing \( u_i \) by \( u_i - G \) we may assume that \( G = 0 \). Let \( c_{jj} := c_j + m_{jj} \). Since the equations are cooperative we have \( c_{jj} \leq -\sum_{k \neq j} m_{jk} \leq 0 \) in \( D_j \). Therefore, if \( c_{jj} = 0 \) in \( D_j \), then \( \sum_{k \neq j} m_{jk} = 0 \) in \( D_j \) which in turn, makes the \( j \) equation a scalar equation. Then we can apply the standard ABP estimate to obtain (2.3). Thus, we assume that \( c_{jj} \leq 0 \) in \( D_j \). Recall from (1.3) that
\[
L_j g = \text{Tr} \left( a_j(x) \nabla^2 g(x) \right) + b_j(x) \cdot \nabla g(x).
\]

Let \( v, w \in W^{2,d}_0(D_j) \cap C(\overline{D_j}) \) be such that
\[
L_j w = -f_j^+ \quad \text{in } D_j, \quad w = 0 \quad \text{on } \partial D_j,
\]
and
\[
L_j v + c_{jj} v = c_{jj} \quad \text{in } D_j, \quad v = 0 \quad \text{on } \partial D_j.
\]
Applying [51, Lemma 2.1] we find \( \delta > 0 \), dependent on \( D_j \) and the coefficients of \( \mathcal{L} \), satisfying
\[
0 \leq v \leq 1 - \delta \quad \text{in } D_j.
\]
Let \( M = \max_i \sup_{D_i} u_i^+ = \max_i \sup_{D_i} u_i^+ \) (otherwise, there is nothing to prove). We observe that
\[
L_j u_j + c_{jj} u_j \geq -f_j^+ - \sum_{k \neq j} m_{jk} u_k
\]
\[
\geq -f_j^+ - \sum_{k \neq j} m_{jk} u_k^+
\]
\[
\geq -f_j^+ + M c_{jj} \quad \text{in } D_j.
\]
Again, for \( h = w + M v \), we have \( L_j h + c_{jj} h \leq -f_j^+ + M c_{jj} \) in \( D_j \). Thus, by the strong maximum principle, we get \( u_j \leq h \) in \( D_j \) giving us
\[
M = \sup_{D_j} u_j \leq \sup_{D_j} h \leq C \| f_j^+ \|_{L^d(D_j)} + M(1 - \delta),
\]
where we used the ABP estimate for \( w \). This gives us (2.3).

Now that we have established the maximum principle, existence of a unique solution follows from a fixed point theorem. In particular, it is enough to prove existence of a solution assuming \( g_i \in C^2(\overline{D}) \) for all \( i \). For continuous \( g \) we can use a standard approximation argument. Now, replacing \( u_i \) by \( u_i - g_i \) we may assume that \( g_i = 0 \) for all \( i \in S \). Consider the set
\[
\mathcal{K} := \{ v \in C(\overline{D} \times S) : v_i = 0 \text{ in } \overline{D} \setminus D_i, \text{ for } i \in S \}.
\]
For \( v \in \mathcal{K} \) we define the map \( w = tv \in \mathcal{K} \) as follows: \( w_i \in W^{2,d}(D_i) \cap W^{1,d}_0(D_i) \) solves
\[
L_i w_i + c_{ii} w_i = -f_i - \sum_{k \neq i} m_{ki} v_k \quad \text{in } D_i, \quad \text{and } w_i = 0 \quad \text{on } \partial D_i.
\]
In view of [52, Theorem 9.15], \( T \) is well defined. Further more \( T \) is linear, continuous and compact. Now setting \( \mathcal{K} \subset \mathcal{K} \) as the collection of \( \beta \)-Hölder continuous functions for some small \( \beta \), we note from [52, Corollary 9.29] that \( T : \mathcal{K} \to \mathcal{K} \). Since \( \mathcal{K} \) is a compact, convex subset of \( \mathcal{K} \), using Schauder fixed point theorem we get a fixed point \( u \) of \( T \). It is easily seen that \( u \) is the solution of (2.4). \( \square \)
We are now ready to prove Theorem 1.1

**Proof of Theorem 1.1.** First we consider (a). Suppose, on the contrary, that there exists a bounded, positive \( u \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^d \times S) \) solving \( \mathcal{L}u = C \) for some constant vector \( C > 0 \). Since \( \mathcal{L}u \geq 0 \) in \( \mathbb{R}^d \times S \). From the proof of Proposition 2.1 it follows that \( u \) attains its maximum in \( \mathbb{R}^d \times S \), say in the component \( u_i \), and \( u_i \) is constant in \( \mathbb{R}^d \). Then

\[
0 < C_i u_i = (\mathcal{L}u)_i = \sum_{k \neq i} m_{ik} (u_j - u_i) \leq 0,
\]

which is a contradiction. This proves (a).

Next we consider (b). Fix a ball \( B \subset \mathbb{R}^d \) and \( S_1 \subset S \). Given a continuous function \( g : \bar{B} \times S_1 \rightarrow \mathbb{R} \), we first prove existence of solution to

\[
\mathcal{L}u = 0 \quad \text{in } (B \times S_1)^c, \quad u = g \quad \text{on } B \times S_1.
\]

Applying Lemma 2.1, we consider a sequence of solutions satisfying

\[
\begin{align*}
\mathcal{L}u_n &= 0 \quad \text{in } (B_n \times S) \setminus (B \times S_1), \\
\ u_n &= g \quad \text{on } B \times S_1, \ \\
\ u_n &= 0 \quad \text{on } \partial B_n \times S.
\end{align*}
\]

One more application of Lemma 2.1 gives

\[
-\max_i \max_B |g_i| \leq u_n(x,i) \leq \max_i \max_B |g_i|, \quad (x,i) \in (B_n \times S) \setminus (B \times S_1).
\]

Thus, using standard elliptic estimates we can pass to the limit in (2.6) to find a solution \( u \) of (2.5).

Next we divide the proof of uniqueness in three steps.

**Step 1.** Let \( K \subset B \) and \( S_1 = S \) where \( K \) is from (1.7). We claim that for a solution \( u \) of (2.5) we have

\[
\sup_j \sup_{B^c} u_j = \max_j \max_{\partial B} u_j.
\]

Replacing \( u_i \) by \( u_i - \max_j \max_{\partial B} u_j \) we may assume that the rhs of (2.7) is 0. Suppose, on the contrary, that the claim (2.7) is not true. Then we must have

\[
0 < \max_j \sup_{B^c} u_j < \infty.
\]

Since \( V \geq 1 \) is bounded below, without any loss of generality we may assume \( V > u \). Otherwise, multiply \( u \) with a suitable positive constant. Now we choose \( n \) large enough so that \( (u - \frac{1}{n} V) (x_0, j_0) > 0 \) for some \( (x_0, j_0) \in B^c \times S \). Again, we choose \( \kappa > 0 \) small enough so that for \( \phi = (u - \frac{1}{n} V) \) we have

\[
\mathcal{L} \phi = \frac{-1}{n} \mathcal{L} V \geq \frac{\kappa}{n} V > \kappa \phi \quad \text{in } B^c \times S,
\]

where the inequality follows from (1.7). Let \( w_n \) be the solution of

\[
\mathcal{L} w_n = \kappa w_n \quad \text{in } B_n \times S, \quad \text{and} \quad w_n = \| \phi \|_{L^\infty} \quad \text{on } \partial B_n \times S.
\]

Using the scalar maximum principle we have \( w_n > 0 \) in \( B_n \times S \). Indeed, if \( \min_j \min_{B_n} (w_n)_j = (w_n)_i(z) \leq 0 \), then using (1.3) we write

\[
L_i ((w_n)_i - (w_n)_i(z)) + (m_{ii} - \kappa) ((w_n)_i - (w_n)_i(z)) \leq \left( \mathcal{L} (w_n - (w_n)_i(z)) \right)_i - \kappa ((w_n)_i - (w_n)_i(z)) \leq 0,
\]

and therefore, by the strong maximum principle, we must have \((w_n)_i = (w_n)_i(z) \) in \( B_n \) which is not possible since \((w_n)_i > 0 \) on \( \partial B_n \). Since \( \mathcal{L} \phi - \kappa \phi \geq 0 \) in \((B_n \cap B^c) \times S \) and \( \phi \leq 0 \) in \( \partial B \times S \), using Lemma 2.1 it follows that \( w_n \geq \phi \) in \((B_n \cap B^c) \times S \). Furthermore, since \( w_n \) attains its
Lemma 2.1. Proposition 2.1). Likewise, we can also show that, we get, given a function, we find a solution \( v \) of \( \mathcal{L}v = \kappa v \) which is bounded and non-negative. Again, \( v(x_0, j_0) \geq \phi(x_0, j_0) \) implies that \( v \) is positive, due to the maximum principle. This of course, contradicts regularity of \( \mathcal{L} \). Hence, we must have

\[
\sup_{j} \sup_{B} u_j \leq 0,
\]

which establishes the claim (2.7). Now using (2.7) we can easily obtain uniqueness of solution (2.5) when \( B \ni K \) and \( S_1 = \bar{S} \).

**Step 2.** We show that if the exterior problem (2.5) with respect to a set \( B' \times S' \) has a unique bounded solution with boundary data 0, then the same is the case for the exterior problem with respect to any domain \( B'' \times S'' \) where \( B' \subset B'' \) and \( S' \subset S'' \subset S \). Suppose, on the contrary, that there exists \( B'' \times S'' \) such that (2.5) has a non-zero bounded solution \( v \) with boundary data 0 given in \( B'' \times S'' \). With no loss of generality, we may assume that \( v^+ \geq 0 \). Now consider the sequence of solutions \( v_n \) of

\[
\mathcal{L}v_n = 0 \quad \text{in} \quad (B_n \times S) \setminus (B' \times S'),
\]

\[
v_n = 0 \quad \text{on} \quad \bar{B}' \times S', \quad v_n = \|v^+\|_{L^\infty} \quad \text{on} \quad \partial B_n \times S.
\]

Furthermore, \( 0 \leq v_n \leq \|v^+\|_{L^\infty} \), by an argument similar to (2.8). Again, by the comparison principle in Lemma 2.1, we get \( v^+ \leq v_n \leq \|v^+\|_{L^\infty} \) in \( (B'' \times S'') \). Therefore, using standard elliptic pde estimates we can pass to the limit in (2.9), as \( n \to \infty \), to obtain a solution \( u \) satisfying

\[
\mathcal{L}u = 0 \quad \text{in} \quad (B' \times S')^c, \quad u = 0 \quad \text{on} \quad \bar{B}' \times S',
\]

and \( u \geq v^+ \). But this contradicts the uniqueness hypothesis with respect to the domain \( B' \times S' \).

**Step 3.** In view of Step 2, it is enough to prove uniqueness of (2.5) with respect to domains of the form \( B \times \{i\} \). Again, we may choose \( |B| \) small enough so that \( M \) is irreducible (see (1.5)) in \( \mathbb{R}^d \setminus B \). Now consider a solution \( u \) of the problem

\[
\mathcal{L}u = 0 \quad \text{in} \quad (B \times \{i\})^c, \quad u = 0 \quad \text{on} \quad \bar{B} \times \{i\}.
\]

We have to show that \( u = 0 \). Choose \( \bar{B} \ni K \cap B \). Then, from (2.7), we get

\[
\sup_{j} \sup_{\bar{B} \cap B} u_j = \max_{j} \max_{\partial B} u_j.
\]

This of course, implies that \( u \) attends its maximum in \( \bar{B} \times S \). Using irreducibility, it is now easy to show that \( u \leq 0 \) (see the argument in Proposition 2.1). Likewise, we can also show that \( u \geq 0 \).

Part (c) can be treated as a special case, using the arguments in part (b). This completes the proof.

Let us also include the following useful characterization of recurrence. A similar result can be found in [9, Theorem 3.12] in a more restrictive setting.

**Proposition 2.2.** Suppose that for some ball \( B_r(x_0) \) and \( S_1 \subset \bar{S} \) the exterior Dirichlet problem

\[
\mathcal{L}u = 0 \quad \text{in} \quad (\bar{B}_r(x_0) \times S_1)^c,
\]

with given continuous boundary values on \( \bar{B}_r(x_0) \times S_1 \) has a unique bounded solution. Then \( \mathcal{L} \) is recurrent.

**Proof.** Without any loss of generality, we may assume that \( B_r(x_0) \times S_1 = B_1(0) \times S_1 \). For some ball \( B \) consider the set \( B \times S_2 \). As shown in the proof of Theorem 1.1, given a function \( g : \bar{B} \times S_2 \to \mathbb{R} \), there exists a bounded solution to

\[
\mathcal{L}u = 0 \quad \text{in} \quad (\bar{B} \times S_2)^c, \quad u = g \quad \text{on} \quad \bar{B} \times S_2.
\]

Thus we only need to establish the uniqueness of (2.10), in other words, that \( g = 0 \) implies \( u = 0 \).
First, consider the case when $B_1(0) \times S_1 \subset B \times S_2$. Suppose, on the contrary, that $u \neq 0$. Then repeating the argument of step 2 in Theorem 1.1 we can construct a solution $v$ to

$$\mathcal{L}v = 0 \quad \text{in } (B_1(0) \times S_2)^c, \quad \mathcal{v} = 0 \quad \text{on } \bar{B}_1(0) \times S_2,$$

and $v \geq u^+$. This clearly, contradicts the hypothesis of the proposition. Thus $u = 0$.

Next, we examine the case where $B \times S_2 \subset B_1(0) \times S_1$. We claim that for any solution $v$ of

$$\mathcal{L}v = 0 \quad \text{in } (B \times S)^c,$$

we have

$$\max_j \max_B v_j = \max_j \sup_{B^c} v_j. \quad (2.11)$$

(2.11) follows from the uniqueness of solution, comparison principle in bounded domains and approximation of $v$ by a sequence of solution as done in (2.6). Now using (2.11) we can complete the proof of uniqueness repeating an argument similar to step 3 of Theorem 1.1.

The following observation is used in several places.

**Lemma 2.2.** Suppose that $(\Psi, c, \lambda)$ and $(\tilde{\Psi}, \tilde{c}, \tilde{\lambda})$ be two tuples satisfying

$$\mathcal{L}\Psi + c\Psi = -\lambda\Psi, \quad \text{and} \quad \mathcal{L}\tilde{\Psi} + \tilde{c}\tilde{\Psi} = -\tilde{\lambda}\tilde{\Psi}$$

on $\mathbb{R}^d$. Define $\Phi_k(x) := \frac{\tilde{\Psi}_k(x)}{\Psi_k(x)}$, and $\Phi := (\Phi_1, \ldots, \Phi_N)$. Then, the following identity holds:

$$\mathcal{L}_\Psi \Phi + (\tilde{c} - c - (\lambda - \tilde{\lambda}))\Phi = 0.$$

Moreover, for $\tilde{\Psi}_k = \Phi_k \Psi_k$ we have

$$(\mathcal{L}\Phi)_k = \Phi_k (\mathcal{L}\Psi)_k + \Psi_k (\mathcal{L}_\Psi \Phi)_k \quad \forall k \in \mathcal{S}.$$

**Proof.** Note that the first identity follows from the second one, which can be shown as follows:

$$(\mathcal{L}\Phi)_k = \text{Tr}(a_k(x)\nabla^2 \tilde{\Psi}_k(x)) + b_k(x) \cdot \nabla \tilde{\Psi}_k(x) + \sum_{j \neq k} m_{kj}(x) (\tilde{\Psi}_j(x) - \tilde{\Psi}_k(x))$$

$$= \Phi_k(x) \left[\text{Tr}(a_k(x)\nabla^2 \Psi_k(x)) + b_k(x) \cdot \nabla \Psi_k(x) + \sum_{j \neq k} m_{kj}(x) (\Psi_j(x) - \Psi_k(x))\right]$$

$$+ \Psi_k(x) \left[\text{Tr}(a_k(x)\nabla^2 \Psi_k(x)) + b_k(x) + 2a_k(x)\nabla \psi_k(x) \cdot \nabla \Psi_k(x)\right]$$

$$+ \sum_{j \neq k} m_{kj}(x)(\Phi_j(x) - \Phi_k(x))\Psi_j(x)$$

$$= \Phi_k(x) (\mathcal{L}\Psi)_k + \Psi_k(x)L_k \Phi_k + \Psi_k(x) \sum_{j \neq k} \tilde{m}_{kj}(x)(\Phi_j(x) - \Phi_k(x))$$

$$= \Phi_k(x) (\mathcal{L}\Psi)_k + \Psi_k(\mathcal{L}_\Psi \Phi)_k.$$

This completes the proof.

Next we prove Theorem 1.3.

**Proof of Theorem 1.3.** Let $(\Psi_D, \lambda_D)$ be the Dirichlet principal eigenpair solving (1.13). Consider a closed ball $B \subset D$ and let $K = B \times S$. Applying Theorem A.2 it follows that $\lambda_D(c - 1_K) > \lambda_D(c) = \lambda_D$. Again, using Theorems A.3 and A.4 we can find a smooth domain $D_1$ which contains $\bar{D}$ and $\lambda_1 := \lambda_{D_1}(c - 1_K) > \lambda_D$. 

Let $\Psi_{D_1}$ be the Dirichlet principal eigenfunction corresponding to the eigenvalue $\lambda_{D_1}(c - 1_K)$ in the domain $D_1$. That is, $\Psi_{D_1} \in C_0(\overline{D_1} \times S) \cap W^{2,p}_{\text{loc}}(D_1 \times S)$, $p > d$, and
\begin{align}
\mathcal{L}\Psi_{D_1} + (c - 1_K)\Psi_{D_1} = -\lambda_1\Psi_{D_1} & \quad \text{in } D_1 \times S, \\
\Psi_{D_1} = 0 & \quad \text{on } \partial D_1 \times S, \\
\Psi_{D_1} > 0 & \quad \text{in } D_1 \times S. \tag{2.12}
\end{align}

Now define
\[ (\mathcal{V}_D)_k := \frac{(\Psi_{D_1})_k}{(\Psi_D)_k} \quad \text{for } k \in S. \]

Since $\Psi_D = 0$ on $\partial D \times S$, it is evident that $\mathcal{V}_D$ is inf-compact and $\mathcal{V}_D \in W^{2,p}(D \times S)$ for any $p > d$. Using Lemma 2.2 and (2.12) we then obtain
\[ \mathcal{L}_D\mathcal{V}_D = (\lambda_D - \lambda_1 + 1_K)\mathcal{V}_D \leq -\delta_1\mathcal{V}_D + \delta_2 1_K, \tag{2.13} \]
where $\delta_1 = \lambda_1 - \lambda_D$ and $\delta_2 = \max_k \max_{\bar{S}} (\mathcal{V}_D)_k$. This proves (1.14).

Now we show that the twisted operator $\tilde{\mathcal{L}}_D$ is regular. Suppose, on the contrary, that it is not. Then we can find a positive vector $\mathbf{C}$ and a bounded, positive solution $u$ of
\[ \tilde{\mathcal{L}}_D u = Cu \quad \text{in } D \times S. \]

Define $\Phi_k := (\mathcal{V}_D)_k u_k$. Using Lemma 2.2, we then have
\[ \mathcal{L}\Phi = u\mathcal{L}\Psi_D + \Psi_D\mathcal{L}_D u = -\lambda_D\Phi + C\Phi \geq (-\lambda_D + C_0)\Phi \]
in $D \times S$, where $C_0 = \min\{C_1, \ldots, C_N\} > 0$. But this is not possible due to (A.4). Hence $\tilde{\mathcal{L}}_D$ is regular, completing the proof.

Next we consider the eigenvalue problem for $\mathcal{A}$ in $\mathbb{R}^d$. Using (1.5), we can find a ball $B_{n_0}$ such that the matrix $M$ is irreducible in $B_{n_0}$. Then, by Theorem A.1, there exists $n_0 \in \mathbb{N}$, and a unique pair $(\Psi_n, \lambda_n) \in C_0(B_n \times S) \cap W^{2,p}_{\text{loc}}(B_n \times S) \times \mathbb{R}$, $p > d$, satisfying
\begin{align}
\mathcal{A}\Psi_n = -\lambda_n \Psi_n & \quad \text{in } B_n \times S, \\
\Psi_n = 0 & \quad \text{on } \partial B_n \times S, \\
\Psi_n > 0 & \quad \text{in } B_n \times S, \tag{2.14}
\end{align}

for all $n \geq n_0$. The uniqueness of $\Psi_n$ holds up to a multiplicative constant. Furthermore, by Theorem A.3 and (A.1), we have $\lambda_n > \lambda_{n+1} \geq \lambda^*$. Hence, it suffices show that the Dirichlet principal eigenvalues $\{\lambda_n\}$ form a monotone sequence that tends to the principal value $\lambda^*$ as $n \to \infty$. But this is a simple generalization of [16, Lemma 2.2] to systems.

**Lemma 2.3.** Suppose that $\tilde{\lambda} = \lim_{n \to \infty} \lambda_n > -\infty$. Then the following hold:

(a) There exists a function $\Psi^* \in W^{2,p}_{\text{loc}}(\mathbb{R}^d \times S)$, $\Psi^* > 0$, satisfying
\[ \mathcal{A}\Psi^* = -\tilde{\lambda}\Psi^* \quad \text{in } \mathbb{R}^d \times S. \tag{2.15} \]

(b) It holds that $\tilde{\lambda} = \lambda^*$.

(c) $\lambda^*$ is concave in $c$.

**Proof.** By (2.14), for each $n \geq n_0$, the function $\Psi_n = (\Psi_{n,1}, \ldots, \Psi_{n,N})$ satisfies
\[ (\mathcal{A}\Psi_n)_k(x) = -\lambda_n(\Psi_{n})_k(x), \quad x \in B_n, \quad k \in S. \]

Let $\mathcal{K} \subset B_n$ be any compact set, and without loss of generality, assume that $0 \in \mathcal{K}$. Scale $\Psi_n$, so that $\min\{\Psi_{n,1}(0), \ldots, \Psi_{n,N}(0)\} = 1$. Applying Harnack’s inequality [51, Theorem 2] (see also [49,53]), we obtain
\[ \sup_{y \in \mathcal{K}} \max_{k \in S} \Psi_{n,k}(y) \leq C_H, \]
for some constant $C_H$ independent of $n$. Thus, by [52, Theorem 9.11], it follows that for any domain $Q \subset \mathcal{K}$ and any $p > d$, there exists a constant $\kappa_1$ such that

$$\|\Psi_n\|_{W^{2,p}(Q \times \mathcal{S})} \leq \kappa_1 \quad \forall n \geq n_0.$$  

Hence, by a standard diagonalization argument, we can extract a subsequence $\{\Psi_{n_k}\}$ such that

$$\Psi_{n_k} \to \Psi^* \quad \text{in} \ W^{2,p}_0(\mathbb{R}^d \times \mathcal{S}), \quad \text{and} \quad \Psi_{n_k} \to \Psi^* \quad \text{in} \ C^{1,\alpha}_{\text{loc}}(\mathbb{R}^d \times \mathcal{S})$$

for some $\Psi^* \in W^{2,p}_0(\mathbb{R}^d \times \mathcal{S})$. Moreover, we have

$$\mathcal{A}\Psi^* = -\tilde{\lambda}\Psi^* \quad \text{in} \ \mathbb{R}^d. \quad (2.16)$$

Since $\min_{c \in \mathcal{S}} \Psi^*_k(0) \geq 1$, another application of Harnack’s inequality shows that $\Psi^*> 0$ in $\mathbb{R}^d \times \mathcal{S}$. This gives us (2.16) and hence the proof of part (a) is complete.

We continue with part (b). It is clear from (2.16) and (1.11) that $\tilde{\lambda} \leq \lambda^*$. Again, from the definition in (A.1), we have $\lambda_n \geq \lambda^*$ for all $n \in \mathbb{N}$. This implies that $\tilde{\lambda} \geq \lambda^*$. Therefore, we obtain $\tilde{\lambda} = \lambda^*$. This proves part (b).

Part (c) follows from Lemma A.2 and (b). \qed

Remark 2.1. Since $\tilde{\lambda} = -\infty$ implies $\lambda^* = -\infty$, it follows from the proof of Lemma 2.3 that $\tilde{\lambda} = \lambda^* \in (-\infty, \infty]$.

Now we can prove Theorem 1.2

Proof of Theorem 1.2. In view of Lemma 2.3 we only need to consider the case $\lambda < \lambda^*$. It follows from Theorem A.3 that for any bounded domain $D$ we have $\lambda_D(\mathcal{A} + \lambda) > 0$ where $\lambda_D(\mathcal{A} + \lambda)$ is the Dirichlet principal eigenvalue of $\mathcal{A} + \lambda$ in $D$ (see Appendix A). Let $n \geq n_0$. Let $D_n \subseteq B_n \setminus B_{n-1}$ be a ball of radius $1/4$. Then using Lemma A.3 we can find a positive function $\tilde{\Psi}_n$ satisfying $(\mathcal{A} + \lambda)\tilde{\Psi}_n = -I_{D_n \times \mathcal{S}}$ in $B_n \times \mathcal{S}$. We scale $\tilde{\Psi}_n$ so that $\min \{\tilde{\Psi}_{n,1}(0), \ldots, \tilde{\Psi}_{n,N}(0)\} = 1$. Now using Harnack’s inequality [51], it is easy to show that the functions $\{\tilde{\Psi}_n\}$ are locally uniformly bounded, and therefore by standard elliptic estimates they are locally uniformly bounded in $W^{2,p}$. Thus we can extract a subsequence and find an eigenfunction $\Psi$ solving $\mathcal{A}\Psi + \lambda^*\Psi = 0$. Again, we have $\Psi > 0$ by the strong maximum principle. This completes the proof. \qed

At this point we recall that for the remaining part we need Hypothesis 1.1, which is enforced without any further mention. Next we produce a proof of Theorem 1.4

Proof of Theorem 1.4. Let $(\Psi, \lambda)$ be an eigenpair with $\lambda < \lambda^*$ and $\mathcal{L}$ be the corresponding twisted operator. Suppose, on the contrary, that $\mathcal{L}$ is recurrent. Let $\Phi_k = \frac{\Psi^*_k}{\Psi^*_k}$. Using Lemma 2.2 we then get

$$\mathcal{L}\Phi = (\lambda - \lambda^*)\Phi < 0 \quad \text{in} \ \mathbb{R}^d \times \mathcal{S}.$$  

Since $\mathcal{L}$ is recurrent, applying Proposition 2.1 we see that $\Phi$ is constant, i.e., for some $c > 0$ we have $\Phi = (c, c, \ldots, c)$. This also gives us $(\lambda - \lambda^*) = 0$ which is a contradiction. This completes the proof. \qed

Next we show that minimal growth at infinity is equivalent to the monotonicity property on the right.

Proof of Theorem 1.8. First we show that (b) $\Rightarrow$ (a). Consider a function $h \in \mathcal{B}_+^1(\mathbb{R}^d \times \mathcal{S})$. From the definition of $\lambda^*$ in (1.11) it follows that $\lambda^*(c + h) \leq \lambda^*(c)$. Now suppose, to the contrary, that $\lambda^*(c + h) = \lambda^*(c)$. Let $\tilde{\Psi}$ be a principal eigenfunction corresponding to $\lambda^*(c + h)$. Then

$$\mathcal{L}\tilde{\Psi} + (c + \lambda^*)\tilde{\Psi} \leq \mathcal{L}\tilde{\Psi} + (c + h + \lambda^*)\tilde{\Psi} = 0 \quad \text{in} \ \mathbb{R}^d \times \mathcal{S}. \quad (2.17)$$
Since $\Psi^*$ has minimal growth property at infinity, it follows from the proof of Theorem 1.7 that $\tilde{\Psi} = \kappa \Psi^*$ for some $\kappa > 0$. From (2.17) we then get $\hat{h} \tilde{\Psi} = 0$, which contradicts to the fact that $\hat{h} \neq 0$. Hence we must have $\lambda^*(c + h) < \lambda^*(c)$. Thus we get (a).

Next, we show that (a) $\Rightarrow$ (b). We construct a principal eigenfunction $\Psi^*$ with the minimal growth at infinity. Let $f = 1_{B_1 \times S}$ and $\Psi_n \in W^2_{\text{loc}}(B_n \times S) \cap C(B_n \times S)$ be the unique solution of

$$
\mathcal{L} \Psi_n + (c + \lambda^*) \Psi_n = -f \quad \text{in } B_n \times S,
$$

$$
\Psi_n > 0 \quad \text{in } B_n \times S,
$$

$$
\Psi_n = 0 \quad \text{on } \partial B_n \times S.
$$

Existence of $\Psi_n$ follows from Lemma A.3. Let

$$
\beta_n := \max_{j \in S} \max_{B_1} (\Psi_n)_j.
$$

We claim that $\beta_n \to \infty$ as $n \to \infty$. Arguing by contradiction, suppose that $\{\beta_{n_k}\}$ is bounded for some subsequence $\{n_k\}$. Let

$$
\kappa_n := \sup \{t : \Psi^* - t \Psi_n > 0 \text{ in } \bar{B}_1 \times S\} \wedge 1,
$$

where $(\Psi^*, \lambda^*)$ denotes the principal eigenpair. Note that, by this hypothesis, we have $\inf_{n_k} \kappa_{n_k} > 0$. Letting $v_n = \kappa_n \Psi_n$, we note that $v_{n_k} \leq \Psi^*$ in $\bar{B}_1 \times S$, and

$$
\mathcal{L} (\Psi^* - v_{n_k}) + (c + \lambda^*) (\Psi^* - v_{n_k}) = 0 \quad \text{in } (B_n \setminus \bar{B}_1) \times S.
$$

Since $\lambda_{B_n \cap B_1^c} > 0$, it follows from (A.4) that $v_{n_k} \leq \Psi^*$ in $B_n \times S$. Now using standard elliptic pde estimates we can find a subsequence of $v_{n_k}$ converging to some non-negative $v$, and using (2.18) we have

$$
\mathcal{L} v + (c + \lambda^*) v = -\kappa f \leq 0 \quad \text{in } \mathbb{R}^d \times S,
$$

where $\kappa > 0$ is obtained along some subsequential limit of $\{\kappa_{n_k}\}$. By the strong maximum principle, either $v = 0$ or $v > 0$ in $\mathbb{R}^d$. The former is not possible as $f \neq 0$. So we must have $v > 0$. Letting $h_k = \kappa \frac{f}{\kappa}$ we obtain from above that

$$
\mathcal{L} v + (c + h + \lambda^*) v = 0 \quad \text{in } \mathbb{R}^d \times S.
$$

From (1.11) we then have $\lambda^*(c + h) \geq \lambda^*$ which contradicts the hypothesis of monotonicity on the right. Therefore, we must have $\beta_n \to \infty$ as $n \to \infty$. In this case we have $\kappa_n \to 0$ as $n \to \infty$. Also, we note that for all large $n$ we have

$$
\min_{j} \min_{\bar{B}_1} (\Psi_j^* - (v_n)_j) = 0. \quad \text{(2.19)}
$$

As before, we can pass to the limit in (2.18) (after multiplying both sides by $\kappa_n$) to obtain a non-negative solution $v$ of

$$
\mathcal{L} v + (c + \lambda^*) v = 0 \quad \text{in } \mathbb{R}^d \times S.
$$

In view of (2.19), we must have

$$
\min_{j} \min_{\bar{B}_1} (\Psi_j^* - v_j) = 0,
$$

which means $v > 0$ in $\mathbb{R}^d \times S$.

To complete the proof, it is enough to show that $v$ has minimal growth at infinity. Consider a positive $\Phi \in W^2_{\text{loc}}((K \times S_1)^c)$, $p > d$, satisfying

$$
\mathcal{L} \Phi + (c + \lambda^*) \Phi \leq 0 \quad \text{in } (K \times S_1)^c.
$$

Let $B$ be a ball large enough so that $B \supseteq K \cup B_1$. Choose $g$ large enough to that

$$
\sup_{n} \sup_{j} \sup_{\bar{B}} ((v_n)_j - g \Phi_j) \leq 0.
$$
As earlier, applying the maximum principle we see that $v_n \leq \varphi \Phi$ in $B_n \times S$. Passing to limit, as $n \to \infty$, we obtain $v \leq \varphi \Phi$. This completes the proof. \hfill \square

We continue with the proof of Theorem 1.5.

**Proof of Theorem 1.5.** First we show that (a) $\Rightarrow$ (b). Take $h \in \mathbb{B}_0^+(\mathbb{R}^d \times S)$. We need to show that $\lambda^*(c + h) < \lambda^*(c)$. Suppose, on the contrary, that $\lambda^*(c + h) = \lambda^*(c)$. Let $\tilde{\Psi}$ be a principal eigenfunction corresponding to the eigenvalue $\lambda^*(c + h)$, that is,

$$\mathcal{L}_{\tilde{\Psi}} + (c + h + \lambda^*(c))\tilde{\Psi} = 0 \quad \text{in } \mathbb{R}^d \times S.$$ 

Let $\Phi_k := \tilde{\Psi}_{\frac{1}{k}}$. Then from Lemma 2.2 we obtain

$$\mathcal{L}_{\Phi_k} \Phi = -h \Phi \leq 0 \quad \text{in } \mathbb{R}^d \times S. \quad (2.20)$$

Since $\mathcal{L}_{\Phi_k}$ is recurrent, it follows from Proposition 2.1 that $\Phi$ is a constant. This implies from (2.20) that $h \Phi = 0$, which contradicts the fact that $h \neq 0$. Thus we must have $\lambda^*(c + h) < \lambda^*(c)$.

Next, we prove that (b) $\Rightarrow$ (a). From Theorem 1.8 we know that monotonicity property on the right is equivalent to the minimal growth at infinity of the principal eigenfunction. Therefore, we assume that the principal eigenpair $(\Psi^*, \lambda^*)$ has minimal growth property at infinity and show that (a) holds. Suppose, on the contrary, that $\mathcal{L}_{\Psi^*}$ is not recurrent. In view of step 2 of the proof of Theorem 1.1, we can find a non-zero bounded solution $u$ of

$$\mathcal{L}_{\Psi^*} u = 0 \quad \text{in } (B \times \{i\})^c, \quad u = 0 \quad \text{in } \bar{B} \times \{i\},$$

for some ball $B$ and some $i \in S$. We may also choose $B$ small enough so that $M$ is irreducible in $\mathbb{R}^d \setminus B$. Scale $u$ in such a fashion that $\sup_i \sup_{\mathbb{R}^d} u_j = 1$. Since $u$ is non-zero, this supremum value is not attained in $\mathbb{R}^d \times S$. Define $\Phi_k := \Psi^*_{\frac{1}{k}} (1 - u_k)$. Then $\Phi$ is continuous and positive in $\mathbb{R}^d \times S$. Furthermore, $\Phi \in \mathcal{W}^{2,p}_{\text{loc}}((B \times \{i\})^c)$ for $p > d$, and applying Lemma 2.2 we also have

$$\mathcal{L}_{\Psi^*} (1 - u)\Psi^* - \Psi^* \mathcal{L}_{\Psi^*} u = -(c + \lambda^*)\Phi \quad \text{in } (B \times \{i\})^c.$$ 

Since $\sup_i \sup_{\mathbb{R}^d} u_j = 1$ there is no constant $\kappa > 0$ satisfying $(1 - u)\Psi^* \geq \kappa\Psi^*$ in $\mathbb{R}^d \times S$. This contradicts the minimal growth hypothesis of $\Psi^*$. Hence $\mathcal{L}_{\Psi^*}$ must be recurrent, and this implies (a).

Finally, simplicity of $\lambda^*$ follows from Theorem 1.7. \hfill \square

We continue with the proof of Theorem 1.6.

**Proof of Theorem 1.6.** First we show that if (b) holds then there exists a principal eigenfunction $\Psi^*$ such that for some Lyapunov function $\mathcal{V}$ we have

$$\mathcal{L}_{\Psi^*} \mathcal{V} \leq \kappa_0 1_{K \times S} - \kappa_1 \Psi^* \quad \text{in } \mathbb{R}^d \times S, \quad (2.21)$$

for some constants $\kappa_0, \kappa_1$ and a compact set $K$. Let $h \in \mathbb{B}_0^+(\mathbb{R}^d \times S)$ be such that $\lambda^*(c - h) > \lambda^*(c)$, and $(\Psi_h, \lambda^*(c - h))$ be a principal eigenpair with respect to the potential $c - h$. Let $2\delta = \lambda^*(c - h) - \lambda^*(c)$ and $\mathcal{B}$ be such that $\sup_{\mathcal{B} \times \mathcal{S}} h(x, i) \leq \delta$.

Recall that $(\Psi_n, \lambda_n)$ is the Dirichlet principal eigenpair in the ball $B_n$, that is,

$$\mathcal{A}\Psi_n = -\lambda_n \Psi_n, \quad \text{in } B_n \times S, \quad \text{and } \Psi_n = 0 \quad \text{on } \partial B_n \times S.$$ 

Define

$$\kappa_n := \sup \{ \kappa > 0 : \Psi_h > \kappa \Psi_n \text{ in } B_n \}.$$ 

Thus $\kappa_n \Psi_n$ must touch $\Psi_h$ at some point from below. We claim that any such point must lie in $\mathcal{B} \times S$. Note that for $x \in \mathcal{B}^c$ we have

$$\left(\mathcal{L}(\Psi_h - \kappa_n \Psi_n)\right)_k + (c_k - \lambda_n)((\Psi_h)_k - \kappa_n(\Psi_n)_k) \leq 0,$$ 

This completes the proof.
for all $k$, for all $n$ large so that $\lambda'(c - h) > \lambda_n + \delta$. If $\kappa_n \Psi_n$ touches $\Psi_h$ in $B_n \setminus B$, then by strong maximum principle some component of $\Psi_h - \kappa_n \Psi_n$ must vanish in $B_n \setminus B$ which is not possible since $\Psi_h - \kappa_n \Psi_n > 0$ on $\partial B_n \times S$. This proves the claim. Therefore, using Harnack’s inequality \[51\], we take limits as $n \to \infty$, and see that $\kappa_n \Psi_n$ converges to the principal eigenfunction $\Psi^*$ which touches $\Psi_h$ from below in $B \times S$. In particular, we have $\Psi^* \leq \Psi_h$.

Now define $V_k(x) := \frac{\Psi_h(x,k)}{\Psi(x,k)}$. It is then evident that $V \geq 1$. Then applying Lemma 2.2 we deduce that
\[
\mathcal{L}^{\Psi^*} V = (h - \lambda'(c - h) + \lambda')V \leq \kappa_0 1_{B \times S} - \delta V \quad \text{in } \mathbb{R}^d \times S,
\]
for some constant $\kappa_0$. This proves (2.21).

We continue by showing that (a) $\Rightarrow$ (c). In fact, we show that the Lyapunov condition (2.21) implies (c). Suppose, on the contrary, that for some $h \in B_0^+(\mathbb{R}^d \times S)$ we have $\lambda'(c - h) = \lambda'(c)$. Without any loss of generality, we may assume $h$ is compactly supported. Let $(\xi_n, \beta_n)$ be the sequence of Dirichlet eigenpairs corresponding to the potential $c - h$ in $B_n$. That is
\[
\mathcal{L}\xi_n + (c - h + \beta_n)\xi_n = 0 \quad \text{in } B_n \times S,
\]
\[
\xi_n > 0 \quad \text{in } B_n \times S,
\]
\[
\xi_n = 0 \quad \text{on } \partial B_n \times S. \tag{2.22}
\]
Then we have $\beta_n \to \lambda^*$. Let $(u_k)_k := (\xi_n)_k$. Using Lemma 2.2 and (2.22) we then obtain that
\[
\mathcal{L}^{\Psi^*} u_k = (\lambda^* - \beta_n + h)u_k \quad \text{in } B_n \times S. \tag{2.23}
\]
Recall the compact set $K$ from (2.21), and scale $u_n$ to satisfy $\max_j \max_K(u_n)_j = 1$. Let $u$ be the limit of $u_n$, which exists due to Harnack inequality \[51\] and (2.23). We claim that $u \leq V^\alpha$ in $\mathbb{R}^d \times S$, for any $\alpha \in (0,1)$, where $V^\alpha = (V_{i_1}^\alpha, V_{i_2}^\alpha, \ldots, V_{i_d}^\alpha)$. To prove the claim, first note that in $(K \times S)^c$ we have
\[
\mathcal{L}^{\Psi^*} V^\alpha \leq -\alpha \kappa_1 V^\alpha. \tag{2.24}
\]
Indeed, for $\alpha \in (0,1)$, we have from Hölder’s inequality that
\[
V_k^\alpha = V_k^{\alpha}\beta^{\alpha(1-\alpha)} V_i^{\alpha(1-\alpha)} \leq \alpha V_k V_i^{\alpha(1-\alpha)} + (1 - \alpha) V_i^{\alpha} \quad \text{for all } i, k \in S.
\]
Therefore, in $(K \times S)^c$, we get
\[
\left(\mathcal{L}^{\Psi^*} V^\alpha\right)_i = \alpha V_i^{\alpha-1}(\mathcal{L}^{\Psi^*} V)_i - \alpha V_i^{\alpha-1} \sum_{k \neq i} m_{ik}(V_k - V_i) + \alpha(1-\alpha) V_i^{\alpha-2} \nabla V_i a_i \cdot \nabla V_i \\
+ \sum_{k \neq i} m_{ik}(V_k^{\alpha} - V_i^{\alpha}) \\
\leq -\alpha \kappa_1 V_i^{\alpha} + \sum_{k \neq i} m_{ik}(V_k^{\alpha} - \alpha V_i V_i^{\alpha-1}) - (1 - \alpha) \sum_{k \neq i} m_{ik} V_i^{\alpha} \\
\leq -\alpha \kappa_1 V_i^{\alpha}. \tag{2.24}
\]
Next, enlarge $K$ to contain support of $h$. Now suppose that $u_n > V^\alpha$ at some point in $B_n \times S$. Since $V \geq 1$ and $\max_j \max_K(u_n)_j = 1$, there should be a point in $K^c \times S$ where $u_n > V^\alpha$. Choose $\kappa \in (0,1)$ so that $\kappa u_n \leq V^\alpha$ in $B_0 \times S$ and $\kappa u_n(x_0, i_0) = V^\alpha(x_0, i_0)$ for some $x_0 \in B_n \cap K^c$ and some $i_0 \in S$. Using (2.23) and (2.24), we obtain
\[
\mathcal{L}^{\Psi^*} (V^\alpha - \kappa u_n) \leq -\alpha \kappa_1 V^\alpha - (\lambda^* - \beta_n + h)\kappa u_n \leq (-\alpha \kappa_1 + |\lambda^* - \beta_n|) \kappa u_n < 0
\]
for all large enough $n$. Thus, writing $\zeta = V^\alpha - \kappa u_n$, we see from above that
\[
\text{Tr}(a_{i_0} \nabla^2 \zeta_{i_0}) + (b_{i_0} + 2a_{i_0} \nabla \psi_{i_0}) \cdot \nabla \zeta_{i_0} + m_{i_0 i_0} \zeta_{i_0} \leq (\mathcal{L}^{\Psi^*} \zeta)_{i_0} \leq 0 \quad \text{in } B_n \cap K^c.
\]
Since \( \zeta_0(x_0) = 0 \), we must have \( \zeta_0 = 0 \) in \( B_n \cap K^c \) by the strong maximum principle. But this is not possible since \( \zeta_0 > 0 \) on \( \partial B_n \). Therefore, we must have \( u_n \leq V^\alpha \) in \( B_n \times \mathcal{S} \). Now letting \( n \to \infty \), we establish the claim of \( u \leq V^\alpha \) in \( \mathbb{R}^d \times \mathcal{S} \), for any \( \alpha \in (0, 1) \).

Letting \( \alpha \to 0 \) we get \( u \leq 1 \). This of course, implies that \( u \) attains its maximum in \( K \times \mathcal{S} \). Since

\[
\mathcal{L}^\psi u = hu \geq 0 \quad \text{in} \quad \mathbb{R}^d \times \mathcal{S},
\]

we must have \( u = (1, 1, \ldots, 1) \) by the strong maximum principle. But this implies \( hu = 0 \) which is not possible since \( h \neq 0 \). Therefore, the original hypothesis that \( \lambda^*(c - h) = \lambda^*(c) \) cannot be correct. This establishes the inequality \( \lambda^*(c - h) > \lambda^*(c) \), giving us (c).

It is obvious that (c) \( \Rightarrow \) (b). Next suppose (b) holds. Then as have shown in (2.21), a Lyapunov function exists for \( \mathcal{L}^\psi \). Applying the preceding argument we see that \( \lambda^*(c + h) > \lambda^*(c) \) for every \( h \in \mathbb{B}_0^+ \). Since \( \mathbb{R} \ni s \to \lambda^*(c + sh) \) is decreasing and concave (Lemma 2.3), we get that \( \lambda^*(c + h) < \lambda^*(c) \) for all \( h \in \mathbb{B}_0^+ \). From Theorem 1.5 we then see that \( \mathcal{L}^\psi \) is recurrent and hence regular. Combining this with (2.21) we get (a). This completes the proof.

The remaining part of this section is devoted to the proof of Theorem 1.9. Let us first introduce the twisted switching diffusion process. Given an eigenpair \((\Psi, \lambda)\) we recall the twisted operator from (1.12), which was defined as

\[
\mathcal{L}^\Psi f(x) := L^\Psi f(x) + M(x)f(x), \quad x \in \mathbb{R}^d.
\]

The corresponding twisted switching diffusion is defined by

\[
d\tilde{X}_t = b(\tilde{X}_t, \tilde{S}_t) dt + 2a(\tilde{X}_t, \tilde{S}_t) \nabla \psi(\tilde{X}_t, \tilde{S}_t) dt + \sigma(\tilde{X}_t, \tilde{S}_t) d\tilde{W}_t,
\]

\[
d\tilde{S}_t = \int_{\mathbb{R}} \tilde{h}(\tilde{X}_t, \tilde{S}_{t-}, z) \tilde{\nu}(dt, dz)
\]

for \( t \geq 0 \), where

(i) \( \tilde{W} \) is a \( d \)-dimensional standard Wiener process;
(ii) \( \tilde{\nu}(dt, dz) \) is a Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R} \) with intensity \( dt \times dz \);
(iii) \( \tilde{\nu}(\cdot, \cdot) \) and \( \tilde{W}(\cdot) \) are independent;
(iv) The function \( \tilde{h} : \mathbb{R}^d \times \mathcal{S} \times \mathbb{R} \to \mathbb{R} \) is defined by

\[
\tilde{h}(x, i, z) := \begin{cases} 
 j - i & \text{if } z \in \Delta_{ij}(x), \\
 0 & \text{otherwise},
\end{cases}
\]

where for \( i, j \in \mathcal{S} \) and fixed \( x \), \( \Delta_{ij}(x) \) are left closed right open disjoint intervals of \( \mathbb{R} \) having length \( \bar{m}_{ij}(x) = \frac{\Psi_j(x)}{\Psi_i(x)} m_{ij}(x) \) for \( j \neq i \), and

\[
\bar{m}_{ij}(x) = -\sum_{j \neq i} \bar{m}_{ij}(x).
\]

In what follows, we use the notation \( \tilde{\mathbb{P}} \) and \( \tilde{\mathbb{E}} \) to denote the law of the twisted switching diffusion and the expectation with respect to this probability measure, respectively. It is clear that the extended generator of \( (\tilde{X}, \tilde{S}) \) is given by (2.25). We write the generator simply as \( \mathcal{L} \) when the dependence on \( \psi \) is clear. Below we recall the probabilistic definition of recurrence and transience of a regime switching diffusion. For a process \( \{(X_t, S_t)\}_{t \geq 0} \) on \( \mathbb{R}^d \times \mathcal{S} \), denote by \( \tau(A) \) the first exit time from the set \( A \subset \mathbb{R}^d \times \mathcal{S} \), defined by

\[
\tau(A) := \inf \{ t > 0 : (X_t, S_t) \notin A \}.
\]

The open ball of radius \( r \) in \( \mathbb{R}^d \), centered at the origin, is denoted by \( B_r \), and we let \( \tau_r := \tau(B_r) \), \( \tau_{ir} := \tau(B_r)^c \), and \( \tau_{i} := \tau((B_r \times \{i\})^c) \) for \( i \in \mathcal{S} \).
**Definition 2.1.** The regime switching diffusion is said to be *recurrent* if for any ball \( B \subseteq \mathbb{R}^d \) and \( j \in S \) we have

\[
\mathbb{P}_{x,i}(\tau(B \times \{j\}) < \infty) = 1,
\]

for all \((x, i) \in \mathbb{R}^d \times S\). It is said to be *transient* if for all \((x, i) \in \mathbb{R}^d \times S\) we have

\[
\mathbb{P}_{x,i} \left( \lim_{t \to \infty} |X(t)| = \infty \right) = 1.
\]

Given a set \( A \) and \( j \in S \), we define the hitting time to \( A \times \{j\} \) by

\[
\tilde{\tau}(A, j) := \inf \left\{ t > 0 : (X_t, S_t) \in A \times \{j\} \right\}.
\]

It is well known that a regime switching diffusion is either transient or recurrent [33, Chapter 5], [9, Chapter 3] (the irreducibility of \( M \) is crucial for this statement). Furthermore, a regime switching diffusion is transient if and only if for every \( g \in C_c(\mathbb{R}^d \times S) \) we have

\[
\mathbb{E}_{x,i} \left[ \int_0^\infty g(X_t, S_t) dt \right] < \infty
\]

for all \((x, i)\) (cf. [9, Proposition 3.19]). Furthermore, as discussed in Remark 1.1, the recurrence of \((X, S)\) is equivalent to the recurrence of its extended generator \( \mathcal{L} \) in the sense of Definition 1.2.

The next lemma is the heart of the proof of Theorem 1.9. It is basically a change of measure type result. This should be compared with [16, Lemma 2.3]. In the case of scalar equation this change of measure is a straight-forward application of Girsanov’s transformation. But in the case of a regime switching diffusion such a transformation is not known for the general model given above. The Girsanov transformation for a simpler setting can be found in [54, Theorem 3.2].

**Lemma 2.4.** For any \( g \in C_c(\mathbb{R}^d \times S) \), and any eigenpair \((\Psi, \lambda)\), that is, a pair satisfying \( \mathcal{A} \Psi + \lambda \Psi = 0 \), we have

\[
\mathbb{E}_{x,k} \left[ e^{\int_0^T (c(X_s, S_s) + \lambda) ds} g(X_T, S_T) \Psi(X_T, S_T) \right] = \Psi(x, k) \mathbb{E}_{x,k}^\Psi \left[ g(\tilde{X}_T, \tilde{S}_T) 1_{\{T < \tau_\infty\}} \right],
\]

(2.27)

for all \((x, k) \in \mathbb{R}^d \times S\), where \( \mathbb{E}_{x,k}^\Psi \) is the expectation operator with respect to the law of the twisted process (2.26), and \( \tau_\infty := \lim_{n \to \infty} \tau_n \) is the explosion time.

**Proof.** Let \( g \in C_2^2(\mathbb{R}^d \times S) \) with support \( g \subset B \times S \) for some ball \( B \subset \mathbb{R}^d \). Select \( n_0 \in \mathbb{N} \) sufficiently large such that \( B \times S \subset B_{n_0} \times S \) for all \( n \geq n_0 \). Now fix any \( n \geq n_0 \). By [55, Theorem 1.1, p. 573], there exists \( \phi \in W^{1,1,2}( 0, T \times B_n \times S ) \) satisfying

\[
\frac{\partial \phi_k}{\partial t}(t, x) - (\mathcal{L}_k \phi_k)(x, t) = 0, \\
\phi_k(x, 0) = g(x, k), \\
\phi_k(x, t) = 0 \quad \text{on} \quad \partial B_n \times S \times [0, T],
\]

(2.28)

for all \( k \in S \), where

\[
(\mathcal{L}_k \phi)_k = (\mathcal{L}^\Psi \phi)_k = \text{Tr} \left( a_k \nabla^2 \phi_k \right) + \left< b_k, \nabla \phi_k \right> + 2 \left< \nabla \psi_k, a_k \nabla \phi_k \right> + \sum_{j \neq k} m_{kj} \left[ \frac{\Psi_j}{\Psi_k} \phi_j - \phi_k \right], \quad \forall k \in S.
\]

(2.29)

By the Gagliardo–Nirenberg–Sobolev inequality, we have \( \phi \in L^2((0, T) \times B_n \times S) \). Then considering (2.28) as an equation in \( \phi_k \), it is clear from [56, Theorem 9.2.5] and [57, Theorem 3.4, p. 89] that \( \phi_k \in W^{1,2,2}((0, T) \times B_n) \) for all \( k \in S \). Repeating the above argument it is easy to see that
\( \phi \in W^{1,2,p}((0,T) \times B_n \times \mathcal{S}) \cap C([0,T] \times \bar{B}_n \times \mathcal{S}), \ p \geq d. \) Now using (2.28), and applying the Itô–Krylov formula [50, p. 122] on \( \phi(\bar{X}_t, \bar{S}_t, T-t) \), it follows that
\[
\phi(x,k,T) = \mathbb{E}^x_k \left[ \phi(\bar{X}_{T \wedge \tau_n}, \bar{S}_{T \wedge \tau_n}, T - T \wedge \tau_n) \right].
\] (2.30)

Rewriting (2.29), we obtain
\[
\Psi_k \tilde{L}_k \phi = \Psi_k \left[ \text{Tr} \left( a_k \nabla^2 \phi_k \right) + \langle b_k, \nabla \phi_k \rangle + 2 \langle \nabla \psi_k, a_k \nabla \phi_k \rangle \right] + \sum_{j \neq k} m_{kj}(\Psi_j \phi_j - \Psi_j \phi_k)
\] (2.31)

for all \( k \in \mathcal{S} \). Let \( \hat{\phi}_k = \phi_k \Psi_k \). Then, using Lemma 2.1, we get
\[
(\mathcal{L} \hat{\phi})_k = (\lambda - c_k) \hat{\phi}_k \Psi_k + \Psi_k (\mathcal{L} \phi)_k.
\]

Therefore, from (2.28), we deduce that
\[
\frac{\partial \hat{\phi}_k}{\partial t}(x,t) = \Psi_k(x) (\mathcal{L} \hat{\phi})_k(x,t)
\] (2.32)

Thus, since we have \( \hat{\phi}(x,k,0) = g(x,k) \Psi_k(x) \), and \( \hat{\phi}(x,k,t) = 0 \) on \( \partial B_n \times \mathcal{S} \times [0,T] \) for all \( k \in \mathcal{S} \), it follows from (2.32) by an application of the Itô–Krylov formula, that
\[
\phi(x,k,T) \Psi_k(x) = \mathbb{E}^x_k \left[ e^{\int_0^{T \wedge \tau_n} \left( c(X_t,S_t) + \lambda \right) dt} \phi(X_{T \wedge \tau_n}, S_{T \wedge \tau_n}, T - T \wedge \tau_n) \Psi(X_{T \wedge \tau_n}, S_{T \wedge \tau_n}) \right]
\] (2.33)

Now combining (2.30), (2.31), and (2.33), we obtain
\[
\Psi_k(x) \tilde{L}_k \phi = \mathbb{E}^x_k \left[ e^{\int_0^{T \wedge \tau_n} \left( c(X_t,S_t) + \lambda \right) dt} \phi(X_t,S_t) \Psi(X_t,S_t) \mathbbm{1}_{\{T < \tau_n\}} \right].
\] (2.34)

Applying the monotone convergence theorem to take limits as \( n \to \infty \) in (2.34), we obtain (2.27) for \( g \in C^2_c(\mathbb{R}^d \times \mathcal{S}) \). A standard approximation argument shows that (2.27) also holds for \( g \in C_c(\mathbb{R}^d \times \mathcal{S}) \). This completes the proof. \( \square \)

Finally, we prove Theorem 1.9. For a ball \( \mathcal{B} \), centered at 0, we denote by \( \tau \) the first hitting time to \( \mathcal{B} \times \mathcal{S} \).

**Proof of Theorem 1.9.** First we show that (a) \( \Rightarrow \) (b). From Theorem 1.5 we know that \( (\bar{X}, \bar{S}) \) is recurrent. Thus for some compactly support \( g \in C^\infty_c(\mathbb{R}^d \times \mathcal{S}) \) we have
\[
\mathbb{E}^x_k \left[ \int_0^\infty \xi(\bar{X}_t, \bar{S}_t) dt \right] = \infty
\]
for some \( (x,k) \), where
\[
\xi_k(x) := \frac{g_k(x)}{\Psi_k^1(x)}, \quad k \in \mathcal{S}.
\]

With no loss of generality we assume that \( (x,k) = (0,1) \). For \( \alpha > 0 \) we define \( F_\alpha(x,i) := c(x,i) + \lambda^* - \alpha \) and
\[
\Gamma_\alpha := \mathbb{E}_0,1 \left[ \int_0^\infty e^{\int_0^t F_\alpha(x(s),S_s) ds} g(X_t,S_t) dt \right].
\]

Then as shown in [16, Lemma 2.7], we have \( \Gamma_\alpha < \infty \) for all \( \alpha > 0 \), and using Lemma 2.4 we get \( \Gamma_\alpha \to \infty \) as \( \alpha \to 0 \). Let \( \Phi^\alpha_n \in C_0(B_n \times \mathcal{S}) \cap W^{2,p}_{loc}(B_n \times \mathcal{S}) \) be the unique solution to
\[
\mathcal{L}_k \Phi^\alpha_n + F_\alpha \Phi^\alpha_{n,k} = -\Gamma_\alpha^{-1} g_k \quad \text{in } B_n,
\] (2.35)
for all \( k \). The existence follows from Lemma A.3. Again following the arguments in [16, Lemma 2.7] and the Harnack inequality in [51, Theorem 2], we can show that the family \( \{ \Phi_n \}_{n \geq n_0} \) is locally uniformly bounded in \( \mathcal{W}^{2,p} \)-norm and therefore, we can extract a subsequence converging to a positive \( \Phi^\alpha \in \mathcal{W}^{2,p}_{\text{loc}}(\mathbb{R}^d) \), \( p > d \), satisfying
\[
\mathcal{L}_k \Phi^\alpha + F_{a,k} \Phi^\alpha = -\Gamma_\alpha^{-1} g_k \quad \text{in } \mathbb{R}^d,
\] (2.36)
for all \( k \). In addition, we also have \( \Phi^\alpha(0,1) = 1 \) for all \( \alpha > 0 \). Let \( B \) be a ball satisfying support\((g) \in B \times S \). Then applying the Itô-Krylov formula to (2.35) we obtain
\[
\Phi_n(x,k) = E_{x,k} \left[ e^{\int_0^{T_h} F_o(x_t,S_t) dt} \Phi_0(X_{T \wedge \tau},S_{T \wedge \tau}) \mathbb{1}_{\{T \wedge \tau < \infty\}} \right], \quad \text{for } x \in B \setminus \overline{B}.
\]
As shown in [16, Lemma 2.7], we can let \( T \to \infty \) first and then \( n \to \infty \) to arrive at
\[
\Phi^\alpha(x,k) = E_{x,k} \left[ e^{\int_0^{T} F_o(x_t,S_t) dt} \Phi^\alpha(X_T,S_T) \mathbb{1}_{\{T < \infty\}} \right], \quad \text{for } x \in B^c.
\] (2.37)
Since \( \Phi^\alpha(0,1) = 1 \), using Harnack’s inequality and the Sobolev estimate we can extract a subsequence of \( \{ \Phi^\alpha \}_{\alpha \in (0,1)} \) converging to \( \Psi^\ast \) in \( \mathcal{W}^{2,p}_{\text{loc}}(\mathbb{R}^d) \) as \( \alpha \to 0 \). It is then evident from (2.36) that
\[
(L \Psi^\ast) + (c_k(x) + \lambda^\ast) \Psi^\ast_k = 0 \quad \text{in } \mathbb{R}^d, \quad \forall k \in S,
\]
and passing the limit in (2.37) with the help of monotone convergence theorem we obtain
\[
\Psi^\ast(x,k) = E_{x,k} \left[ e^{\int_0^{T} (c(X_t,S_t) + \lambda^\ast) dt} \Psi^\ast(X_T,S_T) \mathbb{1}_{\{T < \infty\}} \right], \quad \text{for } x \in B^c.
\] (2.38)
Thus we get (1.16). This gives us (b).

Next we show that (b) \( \Rightarrow \) (a). Let \( h \in \mathcal{B}^+ (\mathbb{R}^d \times S) \), and suppose, on the contrary, that \( \lambda^\ast (c+h) = \lambda^\ast (c) = \lambda^\ast \). Let \( \tilde{\Psi} > 0 \) be an principal eigenfunction with potential \( c + h \). Then
\[
(L \tilde{\Psi}) + (c_k + \lambda^\ast) \tilde{\Psi}_k \leq (L \hat{\Psi}) + (c_k + h_k + \lambda^\ast) \hat{\Psi}_k = 0, \quad \text{in } \mathbb{R}^d, \quad \forall k \in S.
\] (2.39)
Let \( B \) be the ball given in (b) and \( \tilde{\tau} \) be the first hitting time to \( B \times S \). Then using Itô-Krylov formula and Fatou’s lemma to (2.39) we obtain that
\[
\tilde{\Psi}(x,k) \geq E_{x,k} \left[ e^{\int_0^{	ilde{\tau}} (c(X_t,S_t) + \lambda^\ast) dt} \tilde{\Psi}(X_{\tilde{\tau}},S_{\tilde{\tau}}) \mathbb{1}_{\{\tilde{\tau} < \infty\}} \right], \quad \text{for } x \in B^c, \ k \in S.
\] (2.40)
Define
\[
k := \min_k \min_B \frac{\tilde{\Psi}_k(x)}{\Psi^\ast_k}.
\]
Then using (2.38) and (2.40) we have \( \tilde{\Psi} \geq k \Psi^\ast \) and \( \min_k \min_B (\tilde{\Psi}_k - k \Psi^\ast) = 0 \). Since \( L (\tilde{\Psi} - k \Psi^\ast) \leq 0 \), it follows from the strong maximum principle that \( \tilde{\Psi} = k \Psi^\ast \). From (2.39) this also gives us \( h \tilde{\Psi} = 0 \), which contradicts the fact that \( h \neq 0 \). Hence \( \lambda^\ast (c+h) < \lambda^\ast (c) \), establishing (a). \( \square \)

**Appendix A. The Dirichlet eigenvalue problem in bounded domains**

In this section we consider the principal eigenvalue problem in a smooth bounded domain \( D \subset \mathbb{R}^d \).

Some of the results obtained below can also be found in [53] (see Theorems 13.1 and 13.2 there). Without any loss of generality we may assume that \( 0 \in D \). For this section, the only hypotheses we require are summarized in the following assumption.

**Assumption A.1.** The following hold.

(i) \( a \in (C(D \times S))^{d \times d} \), and, for some \( \Lambda > 0 \), we have
\[
\Lambda^{-1} I \leq a_k(x) \leq \Lambda I \quad \forall x \in \overline{D}, \forall k \in S.
\]

(ii) \( b: D \times S \to \mathbb{R}^d \), \( m_{ij}: D \to \mathbb{R} \), \( i,j \in S \), and \( c: D \times S \to \mathbb{R} \) are bounded, Borel measurable functions.
(iii) $M$ is irreducible in $D$, that is, (15) holds.

For $\lambda \in \mathbb{R}$, consider the set

$$\Psi_D^+(\lambda) := \{ \Phi \in W^{2,p}_{\text{loc}}(D \times S) \cap C(D \times S) : \Phi > 0 \text{ in } D \times S, (A\Phi)_k(x) + \lambda \Phi_k(x) \leq 0 \text{ in } D \forall k \in S \}.$$  

We define the generalized Dirichlet principal eigenvalue $\lambda_D$ of $A$ in the domain $D$ by

$$\lambda_D := \sup \{ \lambda \in \mathbb{R} : \Psi_D^+(\lambda) \neq \emptyset \}. \tag{A.1}$$

**Theorem A.1.** There exists a unique pair $(\varphi, \rho) \in C_0(\overline{D} \times S) \cap W^{2,p}_{\text{loc}}(D \times S) \times \mathbb{R}, p > d$, satisfying

$$A\varphi = -\rho \varphi \quad \text{in } D \times S,$$

$$\varphi = 0 \quad \text{on } \partial D \times S,$$

$$\varphi > 0 \quad \text{in } D \times S. \tag{A.2}$$

In addition $\rho = \lambda_D$.

**Proof.** Since $c$ is bounded, using $(\|c\|_{L^\infty(D \times S)} - c)$ as the coefficient of the zeroth order term, the existence of unique solution $(\varphi_D, \rho) \in C_0(\overline{D} \times S) \cap W^{2,p}_{\text{loc}}(D \times S) \times \mathbb{R}, p > d$, to (A.2) follows from [15, Remark 1.3 and Corollary 2.1]. Uniqueness of $\varphi_D$ is of course only up to a multiplicative constant.

We claim that if $w \in C(\overline{D} \times S) \cap W^{2,p}_{\text{loc}}(D \times S)$, with $w > 0$ in $D$, satisfies

$$Aw \leq -\rho w \quad \text{in } D \times S,$$

then $w = t\varphi_D$ for some constant $t > 0$. This clearly implies that $\rho = \lambda_D$.

In order to prove the claim, we define $u_t := t\varphi_D - w$. Let $K \subset D$ be a compact set such that $|D \setminus K| < \varepsilon$, for some small number $\varepsilon > 0$. Then for some suitable choice of $t > 0$ (small enough), we have $u_t \leq 0$ in $K \times S$. Also, for all $k \in S$ we have

$$(\mathcal{L}u_t)_k(x) - (c_k(x) + \rho)^-(u_t)_k(x) \geq -(c_k(x) + \rho)^+(u_t)_k(x).$$

Now choosing $\varepsilon$ sufficiently small, and applying [51, Theorem 1] on the domain $D \setminus K$ we see that

$$\sup_{D \setminus K} \max_{k \in S} (u_t)_k \leq \theta_0 \sup_{D \setminus K} \max_{k \in S} (u_t)_k$$

for some $\theta_0 \in (0, 1)$. This is possible only if $u_t \leq 0$ in $D \setminus K$. Thus, $u_t \leq 0$ in $D \times S$, and it satisfies

$$\text{Tr}(a_k \nabla^2(u_t)_k(x) + b_k(x) \cdot \nabla(u_t)_k(x) - (c_k(x) + m_{k,k}(x) + \rho)^-(u_t)_k(x) \geq 0.$$ 

Therefore, we must either have $(u_t)_k = 0$ or $(u_t)_k < 0$ in $D$ by the strong maximum principle [52, Theorem 9.6]. If $(u_t)_k < 0$ for some $k \in S$, the irreducibility condition in **Assumption A.1 (iii)** implies that $(u_t)_j < 0$ for all $j \in S$. Thus we either have $u_t = 0$ or $u_t < 0$ in $D \times S$. Suppose that $u_t < 0$ in $D \times S$. Define

$$t := \sup \{ t > 0 : u_t < 0 \text{ in } D \times S \}.$$ 

By the above argument, $t > 0$, and by the strong maximum principle, we must have either $u_t = 0$ or $u_t < 0$. If $u_t < 0$, then for some $\delta > 0$ we have $u_{t+\delta} < 0$ in $K \times S$, and repeating the argument above, we obtain $u_{t+\delta} < 0$ in $D \times S$. This contradicts the definition of $t$. So the only possibility is $u_t = 0$. This indeed implies that $\rho = \lambda_D$ and completes the proof. \qed

**Theorems A.2 and A.3** which follow, concern the strict monotonicity of the principal eigenvalue with respect to the potential and the domain. We denote the eigenvalue as $\lambda_D(c)$ when we want to explicitly indicate its dependence on the potential $c$.

**Theorem A.2.** If two potentials satisfy $c \leq c'$, then $\lambda_D(c) > \lambda_D(c')$. 

Proof. Let \( \varphi_c \) and \( \varphi' \) denote the principal eigenfunctions corresponding to \( c \) and \( c' \), respectively. It is clear from (A.1) that \( \lambda_D(c) \geq \lambda_D(c') \). Suppose that \( \lambda_D(c) = \lambda_D(c') \). Then, we obtain
\[
\mathcal{A} \varphi'(x) \leq -\lambda_D(c) \varphi'(x), \quad \text{in } D \times \mathcal{S}.
\]
Now, it follows from the proof of Theorem A.1 that \( \varphi = t \varphi_c \) for some positive constant \( t \). But this contradicts the fact that \( c \leq c' \). Therefore, we have \( \lambda_D(c) > \lambda_D(c') \).

**Theorem A.3.** If \( D_1 \subseteq D_2 \), then \( \lambda_{D_1} > \lambda_{D_2} \).

Proof. Let \( \varphi_1 \) and \( \varphi_2 \) denote the principal eigenfunctions corresponding to \( \lambda_{D_1} \) and \( \lambda_{D_2} \), respectively. From the definition in (A.1), it follows that \( \lambda_{D_1} \geq \lambda_{D_2} \). If \( \lambda_{D_1} = \lambda_{D_2} \), then
\[
\mathcal{A} \varphi_2 \leq -\lambda_{D_1} \varphi_2 \quad \text{in } D \times \mathcal{S}.
\]
As in the proof of Theorem A.1, this implies \( \varphi_2 = t \varphi_1 \) on \( D_1 \times \mathcal{S} \) for some \( t > 0 \). This contradicts the fact that \( \varphi_2 > 0 \) in \( D_2 \times \mathcal{S} \), because \( D_1 \subseteq D_2 \) and \( \varphi_1 = 0 \) on \( \partial D_1 \times \mathcal{S} \). Thus, we must have \( \lambda_{D_1} > \lambda_{D_2} \).

Next, we address the continuity properties of the principal eigenvalue with respect to the domain \( D \). We say that a domain \( D \) has the exterior sphere property of radius \( r > 0 \), if every point of \( \partial D \) can be touched from outside of \( D \) with a ball of radius \( r \). We need the following boundary estimate. For a proof, see [58, Lemma 6.1].

**Lemma A.1.** Suppose that \( \|w\|_{L^\infty(D \times \mathcal{S})} \leq 1 \), and it satisfies
\[
\text{Tr}(a_k \nabla^2 w_k) + \delta |\nabla w_k| \geq L \quad \text{in } D, \quad \forall k \in \mathcal{S}, \quad \text{and } w = 0 \quad \text{on } \partial D \times \mathcal{S},
\]
where \( D \) has the exterior sphere property of radius \( r > 0 \). Then for \( s \in (0,1) \), there exist constants \( M \), and \( \varepsilon \), depending only on \( \delta \), \( L \), \( r \), and \( s \), such that
\[
\max_{k \in \mathcal{S}} w_k^+(x) \leq M \text{dist}(x, \partial D)^s, \quad \text{for all } x \text{ such that } \text{dist}(x, \partial D) < \varepsilon.
\]

**Theorem A.4.** Let \( \{D_n\}_{n \in \mathbb{N}} \) be a decreasing sequence of smooth domains whose intersection is denoted as \( D \), and which have the exterior sphere property of radius \( r \) uniformly in \( n \in \mathbb{N} \). Then \( \lambda_{D_n} \to \lambda_D \), as \( n \to \infty \).

Proof. From Theorem A.3 it is clear that \( \lambda_{D_n} \) is an increasing sequence which is bounded above by \( \lambda_D \). Thus, \( \lambda_{D_n} \) converges to some number \( \lambda = \lambda_D \). We normalize the eigenfunctions so that \( \|\varphi_{D_n}\|_{L^\infty(D_n \times \mathcal{S})} = 1 \). Now, using Lemma A.1 and the standard interior estimate, it can be easily seen that the family \( \{\varphi_{D_n}\} \) is equicontinuous and each limit point \( \phi \in C(\overline{D} \times \mathcal{S}) \cap W^{2,p}_{\text{loc}}(D \times \mathcal{S}) \) is a nonnegative solution to
\[
\mathcal{A} \phi = -\lambda \phi \quad \text{in } D \times \mathcal{S}.
\]
By the strong maximum principle, we must have \( \phi > 0 \) in \( D \times \mathcal{S} \). Thus, the equality \( \lambda = \lambda_D \) follows from the proof of Theorem A.1.

The next result shows that \( \lambda_D \) is convex with respect to the potential \( c \).

**Lemma A.2.** It holds that
\[
\lambda_D(\theta c_1 + (1 - \theta) c_2) \geq \theta \lambda_D(c_1) + (1 - \theta) \lambda_D(c_2) \quad \forall \theta \in [0,1].
\]

Proof. Let \( \varphi_i \) denote the principal eigenfunction with respect to the potential \( c_i \), \( i = 1,2 \). Define \( f_k := \frac{\varphi_i}{\varphi_j} \). Then, by Young’s inequality we have
\[
\sum_{j \neq k} m_{kj} \frac{f_j}{f_k} = \sum_{j \neq k} m_{kj} \frac{\varphi_i \varphi_j}{\varphi_1 k \varphi_2 k} \leq \sum_{j \neq k} m_{kj} \left( \frac{\varphi_1 j + (1 - \theta) \varphi_2 j}{\varphi_1 k \varphi_2 k} \right).
\]
Also, it is straightforward to show that
\[
\frac{1}{f_k} \text{Tr}(a_k \nabla^2 f_k) \leq \frac{\theta}{\varphi_{1,k}} \text{Tr}(a_k \nabla^2 \varphi_{1,k}) + \frac{(1-\theta)}{\varphi_{2,k}} \text{Tr}(a_k \nabla^2 \varphi_{2,k}) \quad \forall k \in \mathcal{S}.
\]
Thus, we obtain
\[
\frac{1}{f_k} (\mathcal{L}_k f + (\theta c_{1,k} + (1-\theta) c_{2,k}) f_k) \leq \frac{\theta}{\varphi_{1,k}} (\mathcal{L}_k \varphi_{1,k} + c_{1,k} \varphi_{1,k}) + \frac{(1-\theta)}{\varphi_{2,k}} (\mathcal{L}_k \varphi_{2,k} + c_{2,k} \varphi_{2,k})
\]
for all \( k \in \mathcal{S} \). Simplifying the above inequality, we obtain
\[
\mathcal{L}_k f + (\theta c_{1,k} + (1-\theta) c_{2,k}) f_k \leq -((\theta \lambda_D(c_1) + (1-\theta) \lambda_D(c_2)) f_k \quad \forall k \in \mathcal{S}.
\]
In view of (A.1), this implies (A.3).

We conclude the Appendix with the following result.

**Lemma A.3.** Suppose that \( \lambda_D > 0 \). Then for any \( f \leq 0 \) in \( D \times \mathcal{S} \), there exists a unique positive solution \( \varphi \) satisfying \( \mathcal{A} \varphi = f \) in \( D \times \mathcal{S} \), with \( \varphi = 0 \) on \( \partial D \times \mathcal{S} \).

**Proof.** The proof is quite standard and uses the refined maximum principle. The latter follows from the following characterization of \( \lambda_D \). Let \( F(D \times \mathcal{S}) \) denote the collection of all functions in \( C(\bar{D} \times \mathcal{S}) \cap W^{2,d}_{\text{loc}}(D \times \mathcal{S}) \) which have non-positive values on \( \partial D \times \mathcal{S} \) and are positive at some point in \( D \times \mathcal{S} \). Then
\[
\lambda_D = \inf \{ \lambda \in \mathbb{R} : \exists \Psi \in F(D \times \mathcal{S}) \text{ such that } \mathcal{A} \Psi + \lambda \Psi \geq 0 \text{ in } D \times \mathcal{S} \}.
\]
Let \( \lambda' \) denote the right hand side of (A.4). It follows from Theorem A.1 that \( \lambda' \leq \lambda_D \). Now suppose that for some \( \Psi \in F(D \times \mathcal{S}) \) we have
\[
\mathcal{A} \Psi \geq -\lambda \Psi
\]
in \( D \times \mathcal{S} \) with \( \lambda < \lambda_D \). Since \( \varphi_D \) in Theorem A.1 is a super-solution to \( \mathcal{A} + \lambda \), repeating a similar argument as in Theorem A.1, we see that \( \varphi_D = t \Psi \) for some \( t > 0 \) which contradicts the fact that \( \lambda < \lambda_D \). Thus \( \lambda' = \lambda_D \).

As a consequence of the above characterization we have a maximum principle which can be stated as follows: if \( u \) is a solution to \( \mathcal{A} u \geq 0 \) and \( u \leq 0 \) on \( \partial D \times \mathcal{S} \), then \( u \leq 0 \) in \( D \times \mathcal{S} \). Now it is standard to apply a monotone iteration to find a solution \( \varphi \) as stated in the lemma.

**Acknowledgement.** The authors would like to thank Anindya Goswami for the helpful discussions. The research of Ari Arapostathis was supported in part by the National Science Foundation through grant DMS-1715210, in part by the Army Research Office through grant W911NF-17-1-001, and in part by Office of Naval Research through grant N00014-16-1-2956 and was approved for public release under DCN# 43-7339-20. The research of Anup Biswas was supported in part by a SwarnaJayanti fellowship and DST-SERB grants EMR/2016/004810, MTR/2018/000028.

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