Doublet Groups, Extended Lie Algebras, and Well Defined Gauge Theories for the Two Form Field.

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Abstract

We propose a symmetry law for a doublet of different form fields, which resembles gauge transformations for matter fields. This may be done for general Lie groups, resulting in an extension of Lie algebras and group manifolds. It is also shown that non-associative algebras naturally appear in this formalism, which are briefly discussed.

Afterwards, a general connection which includes a two-form field is settled-down, solving the problem of setting a gauge theory for the Kalb-Ramond field for general groups.

Topological Chern-Simons theories can also be defined in four dimensions, and this approach clarifies their relation to the so-called $B \wedge F$-theories. We also revise some standard aspects of Kalb-Ramond theories in view of these new perspectives.

1 Introduction

The (Abelian) Kalb-Ramond field [1, 2] (KR), $B_{\mu\nu}$, is a two-form field which appears in the low energy limit of String Theory [3], in Quantum Gravity [4] and in several other frameworks in Particle Physics [5]. In particular, most attempts to incorporate mass to gauge field models in four dimensions take into account this object added to a one form gauge field [6, 7, 8]. However, their actual underneath group structure is lacking. It is often implemented by hand in order to analyze the gauge invariance of certain $B \wedge F$ models.

The symmetry of the KR field is remarkably similar to that of a 1-form gauge field [8]:

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \Theta_{[\mu, \beta_{\nu}],}$$

where $\beta_{\nu}$ is a 1-form parameter. The question is: how can we associate the parameter $\beta_{\mu}$ to the manifold of some gauge group [9, 10]? This problem was rigorously analyzed in refs. [11, 13] where the representations were singlet tensor/spinor spaces with inner product, and the KR field was built in the connection [12, 13]. However, many difficulties arose involving Lorentz invariance of physical models in the non-Abelian case [12, 13, 14]. Also, it was not clear how the KR field could be built when spacetime would be non-flat. In this paper we propose a framework where these difficulties are solved.

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From the physical point of view, it is essential to ask if a genuine gauge theory may be formulated for this field, i.e., if the two-form gauge potential may be stated as a connection on some group manifold. This is important because, as it is known, this structure would be crucial for the identities which determine the finiteness or not of physical models. In particular, in ref. [15], it was proven that massive (non-Abelian) gauge models [6, 7] necessarily based on a gauge KR field, \( B_{\mu
u} \), and a usual one-form, \( A_\mu \), are ill defined in four spacetime dimensions. The first objective of this article is to establish clearly this group structure and to show that these theories may be formulated in a similar way as the Yang-Mills-Chern-Simons theories in 2 + 1 dimensions, which are known to be finite [16].

Another crucial question is how to define a minimal coupling of this field with matter fields, with the interaction with gauge fields appearing by replacing partial derivatives of the matter fields by covariant ones in the free Lagrangian. This is directly related to the charge conservation laws via Noether’s theorem. To do this, local gauge transformations for matter fields need to be defined. Up to now, this is unknown for transformations which involve a 1-form parameter. This is our second objective here. Recently, other perspectives have been presented for these questions [17], where its expected applications in gravitation with torsion and Kalb-Ramond cosmology are mentioned [17, 18, 19, 20].

To tackle all these problems, we explore here a new possibility in order to have a well defined two-form gauge field: relaxing the requirement of singlet tensorial representations\(^3\), imposed in preceding approaches [11, 12, 13]. We shall show that this allows us to construct well defined gauge models for KR fields, which may be minimally coupled with matter fields in a natural way. Once more, the simplest solution of the problem arises from considering doublets of tensors of different ranks as a representation for a Lie group. This kind of idea has been successfully used to solve other algebraic questions related to Hodge duality [21].

By considering a doublet field representation, we are able to include an 1-form parameter in an exponential-like symmetry/transformation law:

\[
\delta \left( \begin{array}{c} \phi \\ \phi_\mu \end{array} \right) = \left( \begin{array}{c} i\alpha \phi + i\beta^\mu \phi_\mu \\ i\alpha \phi_\mu + i\beta_\mu \phi \end{array} \right) = i \left( \begin{array}{cc} \alpha & \beta \\ \beta & \alpha \end{array} \right) \left( \begin{array}{c} \phi \\ \phi_\mu \end{array} \right)
\]

where the variation of the fields is proportional to themselves and to the group parameters. These simple expressions solve the problem of writing this kind of transformation law in a simple and satisfactory way, and are the key to define the group operations involving an 1-form parameter. Notice that, without a doublet representation (and a scalar parameter \( \alpha \)), individual fields (\( \phi \) and \( \phi_\mu \)) can never be combined with an 1-form \( \beta_\mu \) to give a tensor of the same type, and to define their variations. Furthermore, in ref. [13] (also in [14]) it has been shown that, if one insists in representing such groups with single fields, then \( \beta \) must be decomposed with respect to an orthogonal spacetime basis and the Lorentz symmetry is broken in gauge theories, except in the Abelian case, where relativistic invariance is restored in the gauge actions. In such context, it is also unclear how can it be generalized to non-flat base manifolds.

Clearly, (2) is the most general rule where both \( \beta \) and a minimal number of matter fields appear linearly and in a Lorentz-invariant way, such as was argued in refs. [13, 14]. This idea may sound technically trivial but it is meaningful, it has never been used before as the cornerstone for a gauge principle generating the two-form field.

This approach is more satisfactory than previous ones [12, 13], as everything can be expressed in a manifestly covariant form, i.e., we do not need to define the representation with respect to a spacetime coordinate basis, and the generalization to curved spacetime becomes rather immediate.

This work is organized as follows: in Section 2, we explicitly find out the Lie group corresponding to these transformations and the covariant derivative with the generic tensor field being part of the connection is defined, and in Section 3, gauge theories as Yang-Mills and Chern-Simons in four dimensions are discussed. Finally, in Section 4, we study the minimal interaction of this doublet connection with matter. Concluding remarks are collected in Section 5.

\(^3\)Namely, a group representation given by a single tensor or spinor (or tensor product of them).
2 Doublet Field Representations and Tensorial Symmetry Parameters.

Let \((M, g_{\mu\nu})\) be a four-dimensional oriented spacetime and \(G\) be a Lie group whose associated algebra is \(\mathcal{G}\); \(\tau^a\) are the matrices representing the generators of the group with \(a = 1, \ldots, \dim G\); \(\tau_{abc}\) are the structure constants.

As mentioned, consider the general transformations

\[
\delta \phi = i\alpha \phi + i\beta^\mu \phi_{\mu} \\
\delta \phi_{\mu} = i\alpha \phi_{\mu} + i\beta_{\mu} \phi
\]

where the doublet of parameters \((\alpha, \beta)\) consists of two Lie algebra valued 0- and 1-forms respectively \(^4\). Let us denote the doublet of fields by \(\Phi \equiv (\phi, \phi_{\mu})\). Thus, this transformation may be formally expressed as

\[
\delta_{\alpha, \beta} \Phi = (I\alpha + \sigma\beta)\Phi
\]

where \(I, \sigma\) are \(2 \times 2\) matrices, the identity and the first of Pauli's matrices (often denoted by \(\sigma_1\) respectively. The product of two elements of the algebra is well defined and is naturally given by the usual matrix product

\[
\delta_{\alpha', \beta'} \delta_{\alpha, \beta} \Phi = I(\alpha' \alpha + b\beta' \beta^\mu) + \sigma(\alpha \beta' + \alpha' \beta)\Phi,
\]

where we have introduced the real parameter \(b\) multiplying the metric \(g_{\mu\nu}\), a priori taken to be equal to one. However, one may explicitly verify that this algebra is non-associative, due precisely to the term which is quadratic in \(\beta\), since it involves scalar products. In structures like these (called quasi-algebras) the Jacobi identity must be replaced by a weaker expression \(^{27}\). This quasi-algebra generates a quasi-group, which has all properties of a group, except associativity. In the present case, associativity is satisfied for subsets of \(\beta\)-parameters which are all parallel between themselves. So, the set of Lie parameters may be thought to describe a collection of groups (parameterized by the set of orientations in spacetime). This coincides remarkably with the main result of ref. \(^{11}\), where singlet representations are analyzed.

Despite this, we can repeat most of the steps towards well defined theories with non-associative gauge symmetry. For instance, we could formally define a covariant derivative and a curvature tensor since, only the infinitesimal structure is required. We will return to this point in Sub-Section 2.1.

The whole structure may alternatively be expressed in terms of doublets, ordered pairs of 0- and 1-forms. The product of doublets reads as:

\[
(\alpha, \beta_{\mu})(\phi, \phi_{\mu}) = (\phi\alpha + b\phi_{\mu}\beta^\mu, \phi_{\mu}\alpha + \phi\beta_{\mu})
\]

thus

\[
(\delta \phi, \delta \phi_{\mu}) = i(\alpha, \beta_{\mu})(\phi, \phi_{\mu})
\]

Besides non-associativity, a second problem with this product is that, since it involves a spacetime metric, it cannot be a topological construction. However, this does not constitute a technical difficult in itself, if one is not interested in topological theories.

In order not to deal, in this paper, with the two problems mentioned before (which would conduct us to many interesting possibilities), we will concentrate in a \(b\)-product with \(b = 0\), which is associative and defines a Lie algebra. In this case, the symmetry transformation reads

\[
\left(\begin{array}{c}
\delta \phi \\
\delta \phi_{\mu}
\end{array}\right) = \left(\begin{array}{c}
i\phi\alpha \\
i\phi_{\mu}\alpha + i\phi\beta_{\mu}
\end{array}\right).
\]

For simplicity, we will consider a group structure \(G = G_\alpha \times G_\beta\), where \(\alpha = \alpha^a \tau^a \in G_\alpha\), \(\beta = \beta^a_{\mu} \tau^a_{\mu} \in G_\beta\) (\(\beta \in G_\beta \otimes \Lambda_1\)) are Lie algebra valued. Clearly, \([\alpha, \beta] = 0\). In this case, the calculation

\(^4\)We assume them in a matricial representation of the algebra.
of explicit form for the group elements simplifies considerably (see below), however, there is no technical obstacle in generalizing the procedure to any Lie algebra (namely, any $G$ where $[\alpha, \beta] \neq 0$).

The identity for this product is $(1, 0)$. The following formula may be easily shown by induction

$$(\alpha, \beta)^n = (\alpha^n, n\beta_\mu \alpha^{n-1}).$$

(8)

Defining $(\epsilon, \epsilon_\mu) = (\frac{\alpha}{n}, \frac{\beta}{n})$, an infinitesimal group transformation can be written

$$\Phi' = g(\epsilon, \epsilon_\mu)\Phi = \Phi + i(\epsilon, \epsilon_\mu)\Phi = ((1, 0) + i(\epsilon, \epsilon_\mu))\Phi = (1 + i\epsilon, i\epsilon_\mu)\Phi.$$  

(9)

We now compose this operation $n$-times, according to (8):

$$g(\epsilon, \epsilon_\mu)^n = (1 + i\epsilon, i\epsilon_\mu)^n \Phi = \Phi + i(n\epsilon, n\epsilon_\mu)\Phi = ((1 + i\epsilon)^n, in\epsilon_\mu (1 + i\epsilon)^{n-1})\Phi.$$

(10)

Taking the limit $n \to \infty$, when $n \sim n - 1$, we obtain:

$$\lim_{n \to \infty} g(\epsilon, \epsilon_\mu)^n = (\exp i\alpha, i\beta_\mu \exp i\alpha).$$

(11)

Thus, the final closed form for a generic group element is

$$g(\alpha, \beta_\mu) = (e^{i\alpha}, i\beta_\mu e^{i\alpha}),$$

(12)

which is one of our main results. The inverse element is $[g(\alpha, \beta_\mu)]^{-1} = g(-\alpha, -\beta_\mu)$.

In the Abelian case this has the properties of an exponential function since

$$g(\alpha, \beta_\mu)g(\alpha', \beta'_\mu) = g(\alpha + \alpha', \beta_\mu + \beta'_\mu)$$

(13)

and $g(0, 0) = 1d (= (1, 0))$. Notice that in this (separate) case, by virtue of the product defined above, all group elements may be factorized as:

$$g(\alpha, \beta_\mu) = g(\alpha, 0)g(0, \beta_\mu) = g(0, \beta_\mu)g(\alpha, 0).$$

(14)

These transformations are the crucial point in this paper; after that, the rest of the construction follows in a straightforward way.

Let us remark once more that here, for simplicity, we are going to construct gauge theories for these separable groups, whose elements are generically expressed by (12), since our main objective in this article is to show, in a concise way, some remarkable theoretical consequences of this formalism (say, minimal coupling from a gauge principle and the existence of a vector Noether charge), at least in the simplest situation. However, for completeness, in the next subsections we briefly discuss the other interesting generalizations, including the non-associative case.

### 2.1 Non-Associative Symmetry and Non-Separable Groups.

In the non-associative case (for instance, with $b = 1$), since all infinitesimal parameters $\epsilon_\mu$ are considered parallel (the product is clearly associative in this subset), $g(\alpha, \beta)$ can be found by a similar procedure:

$$g_{(b=1)}(\alpha, \beta_\mu \equiv e_\mu \beta) \equiv \left(g \left(\frac{\alpha}{n}, \frac{\beta_\mu}{n}\right)\right)^n = (\cos \beta e^{i\alpha}, i\epsilon_\mu \sin \beta e^{i\alpha}),$$

(15)

where $e_\mu$ is a unit one-form. From this, we can verify directly that, although the product of these objects (quasi-group elements) is indeed non-associative, a weaker associativity (quasi-associativity) of the form: $a(ba) = (ab)a$, is satisfied\(^5\).

\(^5\)This is actually stronger: $g_3(g_2g_1) = (g_3g_2)g_1$ if their corresponding $\epsilon_3, \epsilon_1$ coincide.
In fact, by composing a large number of infinitesimal, general transformations (3), one can represent the result as an exponential:

\[ g = \exp i(\alpha I + \beta \sigma_e) \]  

(16)

whose precise meaning is given according to the algebra:

\[ \sigma_e \sigma_e' = b(e,e') I, \]  

(17)

where \( e, e' \) are the respective unit directions of two arbitrary one-form parameters \(^6\). Thus, in the separate case, this can be expressed as a doublet, (15).

As it was previously commented, one can construct the same objects as those which appear in a standard associative gauge theory (namely, connection, curvature, actions) since this is a gauge symmetry in its own right, because the associative behavior is recovered to first order in the parameters.

The exponential notation is convenient even for the associative case: one can perform calculations and, at the end, take \( b \to 0 \) to recover a Lie structure. Equivalently, one can consider the leading order in \( b \), as we can see directly from expression (15). In this case, the algebra above becomes degenerate and all \( \sigma_e \) may be identified with a single \( \sigma \) (such that \( \sigma^2 = 0 \)). So, we can say that in general this is a quasi-group manifold, which locally (in a neighborhood of \( b = 0 \)), approaches a Lie group. In particular, this is the best way to represent an element of a non-separable group. By truncating (16) to first order in \( b \), in the doublet notation, we obtain the general form of a generic element of an extended Lie group:

\[ g(\alpha, \beta) = \left( e^{i\alpha}, \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[ \sum_{m=1}^{n} \alpha^{m-1} \beta^\mu \alpha^{n-m} \right] \right), \]  

(18)

where \( \alpha, \beta, \mu \) are valued in any given Lie algebra \( G \). If \( [\alpha, \beta] = 0 \), (18) yields (12), as expected. Notice that the expression (18) is not convenient for calculations. So, as mentioned, the exponential notation (up to \( o^2(\beta) \) or \( o(b) \) contributions) should be used instead of this.

### 2.2 Generalized Doublets and \((0, r)\)-Tensors as Gauge Fields.

Let us consider general representations of \( p \)-doublets of order \( r \), which consist in pairs \((\phi_p, \phi_{p+r}) \in \Pi_p \times \Pi_{p+r}\), where \( \Pi_p \) denotes the standard set of tensors of type \((0, p)\). So, the symmetry transformation can be built over doublets of order \( r \) using the same idea, in any of the spaces \( \Pi_p \times \Pi_{p+r}, \forall p \) \(^7\). We take \( \Phi = (\phi_p, \phi_{p+r}) \) and \((\alpha, \beta_r) \in \Pi_p \times \Pi_r\), and the \( r \)-generalized connection reads as \( A = (A_1, B_{r+1}) \in \Pi_1 \times \Pi_{r+1} \) and so on. In this case, in view of (2), the symmetry can be written as below:

\[ \delta \left( \begin{array}{c} \phi_p \\ \phi_{p+r} \end{array} \right) = i \left( \begin{array}{cc} \alpha & \beta_r \\ \beta_r & \alpha \end{array} \right) \left( \begin{array}{c} \phi_p \\ \phi_{p+r} \end{array} \right), \]  

(19)

with the \( b \)-product rule defined in Section 2, where, in the \( b \)-term \( \sim b \beta_r \phi_{p+r} \), we mean that \( r \) indices are contracted using the metric. Once more, this product leads to a (non-associative) quasi-group and such that a Lie group is recovered for \( b = 0 \) or, equivalently, to leading order in \( b \).

We introduce the partial derivative of a \((0, p)\) tensor \( T_p \) as a \((0, p+1)\) tensor given by

\[ T_p = T_{\mu_1...\mu_p} dx^{\mu_1} \otimes ... \otimes dx^{\mu_p}, \]

as

\[ \partial T_p := \partial_\mu T_{\mu_1...\mu_p} dx^\mu \otimes dx^{\mu_1} \otimes ... \otimes dx^{\mu_p}. \]

So, we can define the partial derivative of a doublet as the doublet consisting of the partial derivatives

\[ \partial(\phi_p, \phi_{p+r}) \equiv ((\partial \phi_p)_{p+1}, (\partial \phi_{p+r})_{p+r+1}). \]  

(20)

\(^6\)So, these objects may be represented as a \( 2 \times 2 \) matrix, \( \sigma_e = e \sigma \).

\(^7\)Which takes values in a representation of the Lie group.
It is easy to verify that this definition is consistent with the Leibnitz rule for the product of doublets.

Next, let us give some useful definitions for the associative case: the tensor product of two doublets of arbitrary orders and types, is the simple generalization of the rule \((A, B)(A', B') = (A \otimes A', A \otimes B' + B \otimes A')\). We also denote by \(\hat{X}\) the totally anti-symmetrized part of a \((0, p)\) tensor \(X\). When applied to doublets, we define it as \((\hat{x}, y) \equiv (\hat{x}, \hat{y})\). Furthermore, we can define the covariant derivative of a \(p\)-doublet (of \(r\)-order), \(\Phi_p = (\phi_p, \phi_{p+r+1})\), as a \((p + 1)\)-doublet:

\[
D\Phi_p = \partial\Phi_p - iA\Phi_p = (\partial\phi_p - iA\phi_p , \partial\phi_{p+r} - iA\phi_{p+r} - iB_{r+1}\phi_p) ,
\]

where the connection must be a 1-doublet \(\mathcal{A} \equiv (A_1, B_{r+1})\) of order \(r\).

Imposing that \(gD\Phi_p = D'\Phi'\), and using \(D' = \partial - iA'\) and \(\Phi'_p = g\Phi_p\), we obtain the transformation law for the connection:

\[
A' = g(\alpha, \beta)A\gamma(-\alpha, -\beta) - i(\partial g(\alpha, \beta))\gamma(-\alpha, -\beta),
\]

whose infinitesimal expression is \(\delta\mathcal{A} = \partial(\alpha, \beta) - i[A, (\alpha, \beta)] = D(\alpha, \beta)^8\); which reads, in terms of the doublet components:

\[
\delta A = \partial\alpha - i[A, \alpha] ,
\]

\[
\delta B = \partial\beta - i[B, \alpha] - i[A, \beta].
\]

### 3 Gauge Fields: BF/Chern-Simons Correspondence and Yang-Mills Models.

From now on, we will concentrate on separable (associative) groups and doublet representations of order 1, in order to point out some relevant differences with previous similar approaches in which \(B_{\mu\nu}\) is viewed as gauge fields, but without a precise description of the underlying symmetry.

So, in terms of tensor components, the curvature tensor \(\mathcal{F} = (F_2, H_3) \in \Pi_2 \times \Pi_3\) results as:

\[
F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + i[A_{\mu}, A_{\nu}] \tag{25}
\]

\[
H_{\mu\nu\rho} = 2\partial_{[\mu}B_{\nu\rho]} + 2i(B_{[\mu\rho}A_{\nu]} - A_{[\mu}B_{\nu\rho]}) \tag{26}
\]

where the symbol \(|\) before the \(\rho\) index means that \(\rho\) is not to be anti-symmetrized. Since we are considering \((0, p)\) tensors in our construction and not only \(p\)-forms, this curvature differs from the one considered in other approaches where the two-form field is considered, for us, \(H\) is not totally anti-symmetric but it contains more components. The Kalb-Ramond gauge field must then be identified with anti-symmetric part, \(\hat{B} \equiv B_{[\mu\nu]}\).

Next, we may define the topological Abelian Chern-Simons Action for the connection \(\mathcal{A} = (A, B)\) as:

\[
\mathcal{S}_{CS}[\mathcal{A}] = -\frac{k}{2} \int \mathcal{A} \wedge \hat{\mathcal{F}} = -\frac{k}{2} \int [A \wedge \hat{H} + \hat{B} \wedge \mathcal{F}] , \tag{27}
\]

where \(k\) denotes the inverse of the coupling constant. This is a well defined gauge invariant topological theory which generalizes to a non-Abelian group as:

\[
\mathcal{S}_{CS}[\mathcal{A}] = -k \int \text{tr} \left([A \wedge \partial + \hat{B} + \hat{B} \wedge \partial \wedge A] - i2[A \wedge A \wedge \hat{B}]\right) . \tag{28}
\]

It is indeed straightforward to check out that \(\mathcal{S}_{CS}\) is gauge invariant (up to a total derivative) as expected\(^9\). \(B \wedge F\) theories are similar to Chern-Simons in three dimensions and they are often formally identified, however, the actual connection between both never was clearly established \([21]\). In the present

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\(^8\)The canonical curvature tensor \(\mathcal{F} \in \Pi_2 \times \Pi_{2+r}\) transforms as \(\mathcal{F} = g(\alpha, \beta)\mathcal{F}(-\alpha, -\beta)\).

\(^9\)This may be verified by doing first order variations in \((\alpha, \beta)\) and by using the Bianchi identity (which is a consequence of Jacobi’s identity and the definition of \(\mathcal{F}\)), in the same way that it is usually done for a Chern-Simons theory in 3d.
framework, by defining \( \wedge \) for doublets as the totally anti-symmetrized tensor product (and taking the integral of the second component), this can be formulated as a **genuine** Chern-Simons theory for a doublet connection.

As a result, we observe that a self-interacting \( B \)-field can only be obtained for a non-associative gauge symmetry. In fact, as discussed in Section 2, even in the non associative case (where, for instance, \( b = 1 \)), one could define a gauge theory, since the infinitesimal algebra would also lead to gauge invariant models. In this case, one should add a term \(-i[B_{\mu \nu}, \beta^\nu]\) in expression (23), and \(i[B_{\mu \alpha}, B_{\nu \beta}]g^{\alpha \beta}\) in (25), however the Chern-Simons theory is not longer topological.

In the presence of a spacetime metric, \( g \), there exists a natural map \( A_p \mapsto A^p \) (\( \Pi_p \to \Pi_p^* \)) (rising the indices in ordered way), where \( \Pi_p^* \) is the dual space to \( \Pi_p \), defined as the set of linear maps, \(< A^p; ... >; \Pi_p \to \mathbb{R} \). This is linearly extended to doublets through the definition:

\[
< (A^p, B^{p+1}); (C_p, D_{p+1}) >= < A^p; C_p > + m_p < B^{p+1}; D_{p+1} > ,
\]

where \( m_p \) is a real constant which may depend of \( p \).10 In this way, the operation \(< ; >\) may be naturally extended to pairs of doublets, \((A_p, B_{p+1})\). Therefore, we may define the Yang-Mills Lagrangian for the generalized connection by:

\[
\mathcal{L}_{YM}[\{A_\mu, B_{\nu\sigma}\}] \equiv -\frac{1}{4\mu^2} \text{tr} < F; F > = -\frac{1}{4\mu^2} (\text{tr} F^2 + m_2 \text{tr} H^2),
\]

where \( \text{tr} \) is the trace in the Lie algebra and \( \mu \) is the coupling constant. This model is gauge invariant only in the special case \([A, \beta] = 0 \), since in this case the curvature transformations: \((F', H') = (\epsilon^{\alpha \beta} F e^{-i\alpha \beta}, \epsilon^{\alpha \beta} (H - i[F, \beta]) e^{-i\alpha \beta}) = (\epsilon^{\alpha \beta} F e^{-i\alpha \beta}, \epsilon^{\alpha \beta} H e^{-i\alpha \beta}) = \epsilon^{\alpha \beta} (F, H) e^{-i\alpha \beta} \). This is an remarkable result since, in this formalism, the standard Lagrangian (30) is gauge invariant in this special case (which contains the Abelian one) but, a Yang-Mills-type Lagrangian invariant for a general group symmetry, should require a functional of higher-order in \( F, H \).

It is remarkable that the theory \( S(A) := S_{CS} + S_{YM} \) coincides with the Cremmer-Scherk-Kalb-Ramond model (rigorously generalized here to non-Abelian groups), which is a gauge model with massive modes. A crucial “no go” result in this type of theories has been presented in ref. [15]. However, it is interesting to analyze this theory in view of the gauge group structure clarified here. Since this negative result is based on the impossibility of closing the BRS algebra, we expect that our group structure could be crucial in doing that and, thus, in proving the consistency of this topologically massive model (which can be an alternative to the Standard Model). Our work will continue along these lines and the results will be presented in a forthcoming work [25].

4 Coupling with Matter Fields via Minimal Substitution.

Let us consider the matter free Lagrangian for a doublet of complex fields \( \Phi = (\phi, \phi_\mu) \). Let us denote its complex conjugate by \( \bar{\Phi} = (\bar{\phi}, \bar{\phi}_\mu) \) which takes values in a representation of the Lie group \( G_{(\alpha)} \times G_{(\beta)} \). We can choose a number of Lagrangians for non-interacting matter 12,

\[
\mathcal{L}_G = \mathcal{L}[\bar{\Phi}, \Phi, \partial \bar{\Phi}, \partial \Phi],
\]

with global gauge invariance. Thus, if we consider the group parameters as local, this Lagrangian becomes locally gauge invariant if we perform “minimal substitution”, i.e., if we replace partial derivatives by covariant ones. Doing so, we obtain the full Lagrangian density which contains minimal interactions involving one and two-form gauge field:

\[
\mathcal{L}_L[\bar{\Phi}, \Phi, \partial \bar{\Phi}, \partial \Phi, A_\mu, B_{\nu\rho}] = \mathcal{L}[\bar{\Phi}, \Phi, \tilde{D} \Phi, D \Phi],
\]

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10 We may define arbitrarily an internal metric in each two dimensional doublet space.

11 In the separable case ( \( G \sim G_{(\alpha)} \times G_{(\beta)} \), \( [\alpha, \beta] = 0 \) that we are emphasizing here, the connection may be defined to satisfy this.

12 This means that it may to interact with itself but not with gauge fields.
where
\[ D\Phi = [\partial - iA]\Phi = (\partial_{\mu}\phi - iA_{\mu}\phi, \partial_{\mu}\phi_{\nu} - iA_{\mu}\phi_{\nu} - iB_{\mu\nu}\phi) , \] (33)
and
\[ \bar{D}\Phi = [\partial + iA]\Phi = (\partial_{\mu}\bar{\phi} + i\bar{\phi}A_{\mu}, \partial_{\mu}\bar{\phi}_{\nu} + i\bar{\phi}_{\nu}A_{\mu} + i\bar{\phi}B_{\mu\nu}) . \] (34)

The simplest globally invariant Lagrangian, which is second order and quadratic in the fields, involves only the first component (scalar) of the doublet:
\[ L_{\text{scalar}} = \frac{1}{2} \partial^{\mu}\partial_{\mu}\phi - \frac{M^{2}}{2} \bar{\phi}\phi. \] (35)
However, it is not very interesting because, when we replace the partial by the covariant derivatives, the only interaction that appears is with the 1-form gauge field \( A_{\mu} \), as expected for a complex scalar field. So, in order to get matter interacting with \( B_{\mu\nu} \), one must to consider bigger order Lagrangians (in the matter fields). The combinations
\[ \Psi_{0} = \Phi\Phi , \Psi_{1} = \Phi \partial_{\mu}\Phi , \Psi_{1} = (\partial_{\mu}\Phi)\Phi , \Psi_{2} = \partial_{\rho}\Phi\partial_{\mu}\Phi , \ldots \] (36)
and so on, are invariant doublets under global gauge transformation and we can form Lagrangian densities by constructing scalars with them. For instance, using the previous definition of the internal product, we may write down free Lagrangian densities in addition to (35):
\[ L_{\text{matter}} = \frac{1}{4} (c_{0} < \Psi_{0}; \Psi_{0} > + c_{1} < \Psi_{1}; \Psi_{1} > + c_{2} < \Psi_{2}; \Psi_{2} > + \ldots) , \] (37)
where \( c_{1,2,3,...} \) are constant coefficients.

Now, using the standard rule of the Noether’s theorem for the symmetry described here, we are able to find new conservation laws associated to the interaction with both, rank 1 and 2 gauge fields. The corresponding Noether’s currents are:
\[ J^{\mu} = i\bar{\phi} \frac{\partial L}{\partial \phi_{\mu} - i\bar{\phi}} \frac{\partial L}{\partial \phi_{\mu}} + i\phi_{\nu} \frac{\partial L}{\partial \phi_{\mu}} - i\phi_{\nu} \frac{\partial L}{\partial \phi_{\mu}}, \] (38)
and remarkably
\[ J^{\mu\nu} = i\bar{\phi} \frac{\partial L}{\partial \phi_{\mu}} - i\bar{\phi} \frac{\partial L}{\partial \phi_{\nu}}, \] (39)
which, by virtue of the (free) equations of motion, satisfy \( (\partial_{\mu}J^{\mu}; \partial_{\rho}J^{\mu\rho}) = 0. \)

As mentioned, we localize this symmetry by substituting \( \partial \) by \( D \) in the doublets \( \Psi_{1}, \Psi_{1}, \Psi_{2} \ldots \), which, for construction, become locally invariant objects; so finally, replacing them into (37), the proper locally gauge invariant Lagrangian interacting with the gauge fields (\( A, B \)) is canonically obtained.

As an example consider the simplest model, where an Abelian \( B \)-interaction arise from a gauge principle. Let us take \( c_{i \geq 2} \equiv 0 \) in eq. (37), which is quartic in the matter fields; so, we consider:
\[ L_{\text{matter}} = \frac{1}{4} (c_{0} < \Psi_{0}; \Psi_{0} > + c_{1} < \Psi_{1}; \Psi_{1} >) , \] (40)
where explicitly:
\[ < \Psi_{0}; \Psi_{0} > = \phi^{2}\bar{\phi}^{2} + (\phi\bar{\phi}_{\mu} + \bar{\phi}\phi_{\mu})(\phi\bar{\phi} + \bar{\phi}\phi) \] (41)
and
\[ < \Psi_{1}; \Psi_{1} > = \phi\bar{\phi}_{\mu}\phi\partial_{\mu}\bar{\phi} + (\phi\bar{\phi}_{\mu} + \bar{\phi}\phi_{\mu})(\phi\partial_{\mu}\phi_{\mu} + \bar{\phi}_{\mu}\partial_{\mu}\phi). \] (42)
This may be viewed as a Sigma model with a non-trivial metric on the fields manifold.

In components, the gauge transformations read as
\[ \phi' = e^{i\alpha} \phi \quad ; \quad \phi'_{\mu} = e^{i\alpha} (\phi_{\mu} + i\beta_{\mu}\phi) \]
\[ \bar{\phi}' = \phi e^{-i\alpha} \quad ; \quad \bar{\phi}'_{\mu} = (\bar{\phi}_{\mu} - i\beta_{\mu}\phi) e^{-i\alpha}, \] (43)
which are global symmetries of $L_{\text{Matter}}$ as can be easily verified. In order to preserve this symmetry when the parameters are considered functions of the spacetime point, one must to replace $\partial_\mu$ by covariant derivatives in expression (44), which according to our definitions reads:

$$<\bar{\Psi}_1:\Psi_1 \rangle_{\text{local}} = \phi \bar{\phi}(\partial_\mu - iA_\mu)\phi (\partial^\mu + iA^\mu)\bar{\phi} + (\phi(\partial_\nu + iA_\nu)\bar{\phi}) \phi_\nu(\partial^\nu - iA^\nu)\phi + \bar{\phi}(\partial_\mu - iA_\mu)\phi^\mu + \bar{\phi}^\mu(\partial^\mu - iA^\mu)\phi + \phi(\partial_\nu + iA_\nu)\bar{\phi}^\nu(\partial^\nu + iA^\nu)\phi + \bar{\phi}(\partial_\mu - iA_\mu)\phi^\mu + \bar{\phi}^\mu(\partial^\mu - iA^\mu)\phi)(\partial^\nu - iA^\nu)\phi + B_{\mu\nu}\phi\bar{\phi} + B_{\mu\nu}B^{\mu\nu}\phi^2\bar{\phi}^2. \quad (44)$$

This manifestly describes the minimal interaction of the $B$-field with the matter. So, from expressions (38), we have a standard current density

$$J^\nu = \frac{i}{4}c_1(\phi^2\bar{\phi}^\nu\bar{\phi} - \phi^3\bar{\phi}^\nu\phi) +$$

$$+ \frac{i}{4}c_1(\phi\bar{\phi}_\mu + \phi_\mu\bar{\phi})(\partial^\nu\bar{\phi}^\mu + \bar{\phi}^\mu\partial^\nu\phi - \bar{\phi}\partial_\mu\phi^\nu + \bar{\phi}^\mu\partial_\nu\phi), \quad (45)$$

where we have ignored the contribution of the additional Klein-Gordon Lagrangian (35). Furthermore, according to (38), we also have the tensorial current:

$$J^{\mu\nu} = \frac{i}{4}c_1\phi\bar{\phi}[\phi\partial^\mu\bar{\phi}^\nu - \phi^\mu\partial^\nu\phi - \bar{\phi}^\nu\partial_\mu\phi - \bar{\phi}^\nu\partial_\mu\phi]. \quad (46)$$

This result is new. It reveals the conservation of a charge that has a vector index and arises from a gauge symmetry, this is not however the momentum generator, as the symmetry is not a translation. Indeed, tensor-like charges may appear in higher dimensions whenever a supersymmetry algebra is settled with central charges [26].

5 Concluding Remarks.

In this paper we have found many answers to old questions related to the $B \wedge F$ field theories (where $B$ is a KR field). We cleared the group structure underlying these models and constructed the KR-field through a standard connection even for non-Abelian Lie groups, thus setting the formalism to decide definitively if well defined topologically massive models are possible or not in four space time dimensions. Finally, for the first time also, we built a theory where there is minimal interaction with a (gauge) tensor field and we found a conserved current associated with its gauge character (via Noether’s theorem). Many open possibilities on both, mathematics and physics, have been briefly pointed out in this article, which will be developed properly elsewhere.

Among several possible applications of this framework, we stress that perhaps it could be helpful in formulating gravitation as a genuine topological ($B \wedge F$) theory which nowadays is an strong research line; and in a mathematical context, it could provide new insights in order to find topological invariants in four or more dimensions and novel realizations of non-Associative algebras [27].

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