Abstract

We consider the squared singular values of the product of $M$ standard complex Gaussian matrices. Since the squared singular values form a determinantal point process with a particular Meijer G-function kernel, the gap probabilities are given by a Fredholm determinant based on this kernel. It was shown by Strahov [1] that a hard edge scaling limit of the gap probabilities is described by Hamiltonian differential equations which can be formulated as an isomonodromic deformation system similar to the theory of the Kyoto school. We generalize this result to the case of finite matrices by first finding a representation of the finite kernel in integrable form. As a result we obtain the Hamiltonian structure for a finite size matrices and formulate it in terms of a $(M + 1) \times (M + 1)$ matrix Schlesinger system. The case $M = 1$ reproduces the Tracy and Widom theory which results in the Painlevé V equation for the $(0, s)$ gap probability. Some integrals of motion for $M = 2$ are identified, and a coupled system of differential equations in two unknowns is presented which uniquely determines the corresponding $(0, s)$ gap probability.
1 Introduction

Consider a point process on the line. The process is said to be determinantal if the $k$-point correlation functions $\rho(k)$ have the form

$$\rho(k)(x_1, \ldots, x_k) = \det[K(x_i, x_j)]_{i,j=1}^{k}, \quad (1.1)$$

for $K(x, y)$ — the so-called correlation kernel — independent of $k$. The eigenvalues of many ensembles of complex Hermitian matrices, and their various scaling limits are well known examples of determinantal point processes, as are the positions of nonintersecting random walkers on the line; see e.g. the monographs [2, Ch. 5] and [3].

For a one-dimensional point process, let $E(k; J)$ denote the probability that there are exactly $k$ eigenvalues in the interval $J$. With a slight abuse of notation, introduce the generating function

$$E(\lambda; J) = \sum_{k=0}^{\infty} (1 - \lambda)^k E(k; J). \quad (1.2)$$

A characterising feature of the determinant case is that (1.2) can be expressed as a Fredholm determinant

$$E(\lambda; J) = \det[I - \lambda\mathbb{K}_J]. \quad (1.3)$$

Here $\mathbb{K}_J$ denotes the integral operator on $J$ with kernel $K(x, y)$, as appears in (1.1).

Suppose furthermore that the correlation kernel has the additional structure

$$K(x, y) = \sum_{i=1}^{r} \frac{f_i(x)g_i(y)}{x - y}, \quad (1.4)$$

where $\sum_{i=1}^{r} f_i(x)g_i(x) = 0$. Kernels of the form (1.4) are termed integrable in [4]. They have the general property that the corresponding resolvent kernel is also an integrable kernel. The simplest case of (1.4) occurs when $r = 2$ and $f_2 = f_1, g_2 = -g_1$, giving

$$K(x, y) = \frac{f(x)g(y) - g(x)f(y)}{x - y}. \quad (1.5)$$

This is well known in random matrix theory. It results from unitary invariant ensembles, as a consequence of the Christoffel-Darboux summation formula (see e.g. [2, Prop. 5.1.3]). For example, with

$$f(x) = \frac{1}{\pi} \sin \pi x, \quad g(x) = \frac{1}{\pi} \cos \pi x, \quad (1.6)$$

(1.5) gives the sine kernel

$$K(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)}, \quad (1.7)$$

which is the correlation kernel for complex Hermitian random matrices with bulk scaling (see e.g. [2, Ch. 5]).

Note that (1.6) satisfies the first order matrix linear differential equation

$$\frac{1}{\pi} \frac{d}{dx} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}. \quad (1.8)$$
Tracy and Widom [5] shows that for kernels (1.5) with \((f,g)\) satisfying the first order matrix linear differential equation (1.8) corresponding to classical orthogonal polynomials or their scaling limits, quantities associated with Fredholm determinant (1.3) satisfy an integrable (Hamiltonian) system of non-linear differential equations. For certain intervals \(J\) depending on a single parameter, this system could be integrated to yield a characterisation of the logarithmic derivative of (1.3) as the solution of a Painlevé equation in sigma form (see e.g. [2, §8.1]). This work generalised, and in fact was inspired by, the work of the Kyoto school [9] in the case of the sine kernel (1.7), in which results of this type were first derived. See [2] Ch. 9 for a textbook treatment. At this point it is worth mentioning other remarkable appearances of Painlevé equations in the theory of integrable systems [10–12].

The first study in random matrix theory to give rise to a kernel of the form (1.4) with \(r = 3\) was that of the so-called Pearcey kernel [13–15]. It comes about as the critical scaling of the matrix sum \(tH + H_0\), where \(H\) is a member of the GUE (complex Hermitian random matrices) and \(H_0\) is a fixed matrix with half its eigenvalues at +1 and the other half at −1; \(t\) is a parameter. The \(f_i, g_i\) in (1.4) satisfy third order linear differential equations, and Brézin and Hikami [13] showed that the method of [5] could be adapted to this setting, obtaining a characterisation of the gap probability for \(J\) a symmetrical interval about the origin in terms of a pair of coupled nonlinear equations. For the parameter dependent extension (the Pearcey process), the kernel is again of the form (1.4) with \(r = 3\) and the \(f_i, g_i\) satisfying third order linear differential equations. PDEs for the corresponding gap probabilities have been derived in [6], and their numerical evaluation using the method of Bornemann [7] has been studied in [8].

More recently, the hard edge scaling (see Section 2.3) of the squared singular values of standard complex rectangular Gaussian random matrices has been shown to be of the form (1.4) with \(r = M + 1\) [16]. From this, Strahov [11] generalised the approach of Tracy and Widom to derive a system of nonlinear partial differential equations associated with (1.3) in the case that \(J\) is given by a disjoint union of positive intervals \((a_{2j-1}, a_{2j})\), \(j = 1, \ldots, m\),

\[
J = \bigcup_{j=1}^{m} (a_{2j-1}, a_{2j})
\]  

He also found the Hamiltonian of the associated dynamical system and derived its isomonodromic representation. For a single interval \(J = (0, s)\), Witte and Forrester [17] showed how these coupled equations could be integrated in the case \(M = 1\) to reclaim results obtained originally in [19] for the Bessel kernel (this reduction was also investigated in [11]). The same task was carried out in the case \(M = 2\), leading to the characterisation of the Fredholm determinant in terms of the solution of a certain fourth order nonlinear ordinary differential equation. The latter is lengthy; for a special choice of parameters a much simpler third order equation was found upon the basis of series expansions, but a proof has yet to be found. Notwithstanding its complex nature, as an application the 4th order equation was used to deduce the leading large \(s\) form of the gap probability. In a recent development, Claeys, Givotti and Stivigny [18] used a Riemann Hilbert analysis to extend this result to the first three orders, and also to general \(M\).

The Bessel kernel as is relevant to the case \(M = 1\) results as a hard edge scaling limit of the Laguerre kernel [20]. Tracy and Widom [4] applied their theory directly to the Laguerre kernel, and integrated the resulting system of coupled nonlinear equations in the case \(J = (0, s)\) to obtain a characterisation of (1.3) in terms of the solution of a \(\sigma\)-Painlevé V equation; see also [21–24]. Taking the hard edge scaling limit of the latter directly gives the \(\sigma\)-Painlevé III equation characterising the Bessel kernel.
This motivates us to embark on an analogous study of the finite matrix sizes kernel for the squared singular values of the product of $M$ rectangular complex Hermitian matrices. Specifically, in this work we show that the corresponding kernel can be written in integrable form (1.4), and that the analogue of Strahov’s equations can be derived. These equations can be written in Hamiltonian form, and as the isomonodromic deformation of a linear system. For $M = 1$ it is shown that they are equivalent to the system of equations for the gap probabilities associated with the Laguerre kernel, as isolated by Tracy and Widom [5]. For $M = 2$ several integrals of motion are deduced. Moreover, a coupled differential system in two unknowns is presented which uniquely determines the gap probability for no eigenvalues in $(0, s)$.

2 Singular values of products of complex Ginibre random matrices

2.1 The kernel

Complex Ginibre matrices are random matrices with independent standard complex Gaussian entries. Let $X_1, \ldots, X_M$, $M \geq 1$ be a sequence of such matrices with $X_m$ of size $N_m \times N_{m-1}$ ($1 \leq m \leq M$), and define the product

$$Y_M = X_M X_{M-1} \cdots X_1.$$  (2.1)

That the squared singular values of $Y_M$, or equivalently the eigenvalues of $Y_M^\dagger Y_M$, $\text{Spec}(Y_M^\dagger Y_M) = (x_1, \ldots, x_n)$ form a determinantal point process on $\mathbb{R} > 0$ was first established by Akemann, Ipsen and Kieburg [25], and further insights were given by Kuijlaars and Zhang [16]. The work [25] extended that of Akemann, Kieburg and Wei [26] in the case that each $X_m$ is square. A review of these recent developments is given in [27]. Here we record the explicit form of the correlation kernel, which is given in terms of Meijer G-functions (see the Appendix for the definition).

**Theorem 2.1.** Introduce the parameters

$$\nu_m = N_m - N_0, \quad \nu_m \geq 0 \quad m = 0, \ldots, M, \quad n = N_0$$ (2.2)

In terms of the Meijer G-function [28, 29] define

$$Q_n(x) = \frac{1}{n! \prod_{j=1}^{M} \Gamma(\nu_j + n + 1)} G_{1,M+1}^{M+1,0} \left( -n \Big| 0, \nu_1, \ldots, \nu_M \bigg| x \right)$$ (2.3)

(this is eq. (3.7) of [16] and eq. (47) of [25]) and

$$P_n(x) = -\prod_{j=0}^{M} \Gamma(\nu_j + n + 1) G_{1,M+1}^{1,0} \left( n+1 \bigg| -n, \nu_0, \ldots, -\nu_M \bigg| x \right)$$ (2.4)

(this is eq. (44) of [25] and eq. (3.11) of [25]). The correlation kernel for the determinantal point process specifying the statistical distribution of $\text{Spec}(Y_M^\dagger Y_M)$ is given by [25]

$$K_n^M(x,y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y)$$ (2.5)
or alternatively \cite[eq. (5.4)]{16}

\[
K^M_n(x, y) = -\prod_{j=0}^{M} (n + \nu_j) \int_0^1 P_{n-1}(ux)Q_n(uy)du.
\]  

(2.6)

We remark that \(P_n(x)\) is a polynomial of degree \(n\), and as revised in Appendix A, it can alternatively be written as a generalised hypergeometric function.

The singular values of products of complex Ginibre matrices is one of a number of random matrix ensembles which gives rise to a correlation kernel of the form (2.5), with \(P_n(x), Q_n(y)\) given in terms of Meijer G-functions. Others include the Cauchy two-matrix model \cite{30}, the closely related Bures ensemble of random density matrices \cite{31}, the singular values of products of complex Ginibre matrices and their inverses \cite{32}, and the singular values of products of truncated unitary matrices \cite{33}.

2.2 **Properties of biorthogonal functions \(P_n(x)\) and \(Q_n(x)\).**

Following \cite{16, 29}, for future use we make note of several properties of the biorthogonal functions \(P_n(x)\) and \(Q_n(x)\), following essentially from their definition as Meijer G-functions. We start with

**Proposition 2.2.** Let \(\delta_x = \frac{d}{dx}\). We have

\[
\prod_{i=0}^{M} (\delta_x + \nu_i)P_n(x) = x(\delta_x - n)P_n(x),
\]  

(2.7)

\[
\prod_{i=0}^{M} (\delta_x - \nu_i)Q_n(x) = (-1)^M x(\delta_x + n + 1)Q_n(x).
\]  

(2.8)

The proof follows from the differential equation for Meijer G-functions (A.4).

**Proposition 2.3.** Upon multiplication by \(x\), \(P_n(x)\) and \(Q_n(x)\) satisfy the recurrence relation

\[
xP_n(x) = P_{n+1}(x) + \sum_{k=0}^{M} P_{n-k}(x) a_{k,n},
\]  

(2.9)

\[
xQ_n(x) = Q_{n-1}(x) + \sum_{k=0}^{M} Q_{n+k}(x) a_{k,n+k},
\]  

(2.10)

where

\[
a_{k,n} = \prod_{j=0}^{M} (n - k + \nu_j + 1)_k \sum_{j=0}^{k+1} \frac{(-1)^j}{j!(k+1-j)!} \prod_{i=0}^{M} (n + 1 - j + \nu_i).
\]  

(2.11)

This proposition was proved in \cite[Section 4]{16}.

Next we note

\[\text{There is a misprint in (5.4) of } \cite{16} \text{ where the product should start from } j = 0 \text{ instead of } j = 1.\]
Proposition 2.4. Upon application of the operator $\delta_x$, $P_n(x)$ and $Q_n(x)$ satisfy the recurrence

\[
\prod_{i=0}^{M} (n + \nu_i) P_{n-1}(x) = (\delta_x - n) P_n(x),
\]
(2.12)

\[
\prod_{i=0}^{M} (n + \nu_i + 1) Q_{n+1}(x) = (\delta_x - n - 1) Q_n(x).
\]
(2.13)

Proof. From [16, Eq. (3.8)] we have

\[
P_n(x) = \frac{1}{2\pi i} \prod_{j=0}^{M} \Gamma(n + \nu_j + 1) \int_{\Sigma_n} \frac{\Gamma(t - n)}{\prod_{j=0}^{M} \Gamma(t + \nu_j + 1)} x^t dt,
\]
(2.14)

where $\Sigma_n$ is a closed contour encircling $0, \ldots, n$ in a positive direction.

Let us calculate the RHS of (2.12). We have the identity

\[
(\delta_x - n)[\Gamma(t - n)x^t] = \Gamma(t - (n - 1))x^t.
\]
(2.15)

Therefore, the pole of the integrand at $t = n$ disappears, the contour shrinks to $\Sigma_{n-1}$ and we come to the integral representation for $P_{n-1}(x)$. The extra factor in the LHS of (2.12) comes from the pre-factor in (2.14).

Similarly for $Q_n(x)$ we have from (3.6) in [16]

\[
Q_n(x) = \frac{1}{2\pi i} \prod_{j=0}^{M} \Gamma(n + \nu_j + 1) \int_{-i\infty}^{i\infty} \frac{\prod_{j=0}^{M} \Gamma(t + \nu_j)}{\Gamma(t - n)} x^{-t} dt.
\]
(2.16)

Using the identity

\[
(-\delta_x - n - 1) \left[ \frac{x^{-t}}{\Gamma(t - n)} \right] = \frac{x^{-t}}{\Gamma(t - (n + 1))}
\]
(2.17)

we immediately obtain (2.13).

A generalisation of Proposition 2.4 is

Proposition 2.5. We have

\[
\prod_{j=0}^{M} (n - m + \nu_j + 1) P_{n-m}(x) = (\delta_x - n)_m P_n(x),
\]
(2.18)

\[
\prod_{j=0}^{M} (n + \nu_j + 1) Q_{n+m}(x) = (-1)^m (\delta_x + n + 1)_m Q_n(x).
\]
(2.19)
Proof. Let us prove (2.18) by induction in \( m \). For \( m = 1 \) (2.18) coincides with (2.12). Consider the LHS of (2.18) with \( m \) replaced with \( m + 1 \). Using (2.12) with \( n \) replaced by \( n - m \) we obtain
\[
\prod_{j=0}^{M} (n - m + \nu_j)_{m+1} P_{n-m-1}(x) = \prod_{j=0}^{M} (n - m + \nu_j + 1)_m (\delta_x + m - n) P_{n-m}(x). 
\quad (2.20)
\]
Now applying (2.18) to the RHS of (2.20) we get
\[
(\delta_x + m - n) \prod_{j=0}^{M} (n - m + \nu_j + 1)_m P_{n-m}(x) = (\delta_x + m - n)(\delta_x - n)_m P_n(x) = (\delta_x - n)_{m+1} P_n(x),
\quad (2.21)
\]
which completes the proof. The proof of (2.19) is similar. \( \Box \)

An identity involving multiplication by \( x \) and the operator \( \delta_x \) is also of interest.

**Proposition 2.6.** We have
\[
P_n(x) - xP_{n-1}(x) + \sum_{k=0}^{M} \sum_{l=0}^{M-k} e_{M-k-l}(\nu) n^l \delta_x^k P_{n-l-1}(x) = 0,
\quad (2.22)
\]
\[
Q_{n-1}(x) - xQ_n(x) + \sum_{k=0}^{M} \sum_{l=0}^{M-k} e_{M-k-l}(\nu) n^l (-\delta_x^k) Q_{n-l}(x) = 0,
\quad (2.23)
\]
where \( e_k(\nu) \) is the \( k \)-th elementary symmetric function of \( M \) variables \( \nu = (\nu_1, \ldots, \nu_M) \).

Proof. Let us first write (2.22) in the form
\[
P_n(x) = \left( x - A_n(\delta_x) \right) P_{n-1}(x),
\quad (2.24)
\]
where \( A_n(\delta_x) \) is the differential operator given by a double sum in (2.22). Using (2.12) we can rewrite (2.24) in the form
\[
\left[ x(\delta_x - n) - A_n(\delta_x)(\delta_x - n) - \prod_{j=0}^{M} (n + \nu_j) \right] P_n(x) = 0.
\quad (2.25)
\]
Using (2.7), we obtain from (2.25)
\[
\left[ \prod_{j=0}^{M} (\delta_x + \nu_j) - \prod_{j=0}^{M} (n + \nu_j) - A_n(\delta_x)(\delta_x - n) \right] P_n(x) = 0.
\quad (2.26)
\]
Let us show that the differential operator in the LHS is identically equal to 0. We have
\[
A_n(\delta_x)(\delta_x - n) = \sum_{k=0}^{M} \sum_{l=0}^{M-k} e_{M-k-l}(\nu_1, \ldots, \nu_M)(n^l \delta_x^k - n^{l+1} \delta_x^k) =
\sum_{k=0}^{M+1} e_{M+1-k}(0, \nu_1, \ldots, \nu_M)(\delta_x^k - n^k) = \prod_{j=0}^{M} (\delta_x + \nu_j) - \prod_{j=0}^{M} (n + \nu_j).
\quad (2.27)
\]

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Here we used the fact that the double sum in (2.27) is telescopic and so only boundary one-dimensional sums survive. Thus, (2.26) is proved. The proof of (2.23) follows from (2.8) and (2.13) in a similar manner.

\[ \lim_{N_0 \to \infty} \frac{1}{N_0} K^{M}_{N_0} \left( \frac{x}{N_0} \! : \! \frac{y}{N_0} \right) = K^{M}(x, y), \]  

(2.28)

is well defined. Kuijlaars and Zhang \[16\] used (2.6) to deduce that

\[ K^{M}(x, y) = \int_{1}^{0} P(ux)Q(uy)du, \]  

(2.29)

with \( P(x) \) and \( Q(y) \) defined by

\[ P(x) = G^{1,0}_{0,M+1}(x \! : \! -\nu_0, -\nu_1, \ldots, -\nu_M), \quad Q(y) = G^{M,0}_{0,M+1}(\nu_1, \ldots, \nu_M, \nu_0 \! : \! y). \]  

(2.30)

Most significant for our present purposes is that these authors were able to deduce from (2.29) that \( K^{M} \) can be written as an integrable kernel.

**Theorem 2.7.** Let \( P(x) \) and \( Q(y) \) be given by (2.30). Let \( \mathcal{B}(.,.) \) be the bilinear operator defined by

\[ \mathcal{B}(f(x), g(y)) = (-1)^{M+1} \sum_{j=0}^{M} (-1)^{j} \delta x f(x) \sum_{i=0}^{M-j} \alpha_{i+j} \delta y g(y), \]  

(2.31)

with the constants \( \alpha_{i} \) given by

\[ \prod_{i=1}^{M} (x - \nu_{i}) = \sum_{i=0}^{M} \alpha_{i} x^{i}, \]  

(2.32)

or equivalently in terms of an elementary symmetric function

\[ \alpha_{i} = (-1)^{i} e_{M-i}(\nu_{1}, \ldots, \nu_{M}). \]  

(2.33)

We have

\[ K^{M}(x, y) = \frac{\mathcal{B}(P(x), Q(y))}{x - y}. \]  

(2.34)
3 The integrable form of the kernel $K_n^M(x, y)$

We would like to express the finite $n$ kernel (2.5) in integrable form. In light of the fact that the hard edge scaled kernel $K^M(x, y)$ was derived from the integral representation (2.29), it seems natural to start from the representation (2.6), and to use the differential equations for $P_k(x)$ and $Q_k(x)$ analogous to what was done in [16]; see also [35] in the closely related case of the hard edge scaled Muttalib–Borodin model [34, 36, 37]. However, the presence of the parameter $n$ makes it unclear as to how to implement this strategy.

We proceed instead by algebraic means. Our central result is

**Theorem 3.1.** The kernel $K_n^M(x, y)$ permits the integrable form

$$K_n^M(x, y) = \frac{D(P_n(x), Q_n(y))}{x - y},$$

(3.1)

valid for any $n \geq 1$, where the bilinear differential operator $D$ does not depend on $n$ and has the form

$$D(f(x), g(y)) = \sum_{j=0}^{M} \varphi_j(x)\psi_j(y)$$

(3.2)

with

$$\varphi_j(x) = (-1)^{j+1}\delta_{j} f(x), \quad \psi_j(x) = -\delta_{j,0} x g(x) + \sum_{i=0}^{M-j} \alpha_{i+j}\delta_{i} g(x),$$

(3.3)

and where the $\alpha_i$ are given by (2.33).

The simplicity of this result for any finite $n$ is striking. Comparing the bilinear differential operators $B$ from (2.31) and $D$ from (3.2) we see that they are almost identical except for the overall factor $(-1)^M$ and the extra term in $\psi_0(x)$ in (3.3).

To prove the above theorem we need some preparatory lemmas.

**Lemma 3.2.** For any $n \geq 0$ and $x, y \in \mathbb{C}$

$$\sum_{i=0}^{n} (-1)^i \frac{(x - i)^n}{i!(n - i)!(y + i)} = \frac{(x + y)^n}{(y)^{n+1}}$$

(3.4)

and

$$\sum_{i=0}^{n} (-1)^i \frac{(x - i)^{n+1}}{i!(n - i)!(y + i)} = \frac{(x + y)^{n+1}}{(y)^{n+1}} - 1.$$  

(3.5)

**Proof.** Consider first (3.4), and regard both sides as a function of the complex variable $y$. Both sides go to zero as $|y| \to \infty$ and have simple poles at $y = 0, -1, \ldots, -n$ with the same residues, and hence are identical functions of $y$. The same argument works for (3.5). \qed

**Lemma 3.3.** For $x, y, z \in \mathbb{C}$ and $l \geq 0$

$$\sum_{k=1}^{l} \sum_{j=0}^{k-1} \sum_{m=0}^{k+1} (-1)^{j+m} \frac{(x)(k-m)(y)_{m}}{j!(k+1-j)!} (z + m - j + 1)^{l+1} = \frac{z^{l+1}}{1 - y} - \frac{(x + z)^{l+1}}{1 - x - y} + \frac{x(1 - y + z)^{l+1}}{(1 - y)(1 - x - y)}$$

(3.6)
**Proof.** Let us change the order of summations in \( k \) and \( m \) and introduce a new variable \( s = k - m \). Then we can rewrite the LHS of (3.6) as

\[
\sum_{m=0}^{l-1} \sum_{j=0}^{l+m} (-1)^{j+m} \frac{(x)_m (y)_m}{j!(s + m + 1 - j)!} (z + m - j + 1)^{l+1}.
\]  

(3.7)

We extended the summation in \( j \) to \( l + 1 \) since it is truncated by the factor \((s + m + 1 - j)!\) in the denominator. If we interchange the summations in \( s \) and \( j \), we need to split the sum in \( j \) at the value \( j = m + 1 \) and write

\[
\sum_{s=0}^{l-m} \sum_{j=0}^{l-1} (-1)^{j+m} \frac{(x)_s (y)_m}{j!(s + m + 1 - j)!} (z + m - j + 1)^{l+1}.
\]

(3.8)

where we take into account the truncating condition \( s + m + 1 - j \geq 0 \) in (3.7).

Now the sum in \( s \) can be calculated. The simple identity

\[
\sum_{s=0}^{l-m} (x)_s s! = \frac{(x + 1)_k}{k!}
\]

(3.9)

shows

\[
\sum_{s=m+1}^{l-m} (x)_s s! = \frac{(x + 1)_k}{k!} - \frac{(x + 1)_m}{m!}, \quad k \geq m.
\]

(3.10)

Using (3.10) we can calculate the sums in \( s \) in both terms in (3.8). This allows (3.7) to be reduced to two double summations

\[
\sum_{m=0}^{l-1} \sum_{j=0}^{l+m} (-1)^{j+m} \frac{(x)_m (y)_m}{j!(1 + l - j)!} (1 + z - m + x - 1)^{l+1}
\]

(3.11)

\[
\sum_{m=0}^{l-1} \sum_{j=0}^{l+m} (-1)^{j+m+1} \frac{x (y)_m}{j!(1 + m - j)!} (1 + z - j + m)^{l+1}.
\]

(3.12)

Let us consider the first sum (3.11). The sum in \( j \) can be evaluated using Lemma 3.2

\[
\sum_{j=0}^{l+1} (-1)^j \frac{(1 + z - j + m)^{l+1}}{j!(1 + l - j)!} (1 + z - m + x - 1) = \frac{(x + z)^{l+1}}{(x - m - 1)!}.
\]

(3.13)

To calculate the sum over \( m \) in (3.11) we need the formula

\[
\sum_{m=0}^{n} \frac{(x)_m}{(y)_m} = \frac{y - 1}{1 + x - y} + \frac{(x)_{n+1}}{(1 + x - y)(y)_n}.
\]

(3.14)

which is easy to prove by induction. Using (3.14) we finally get the answer for the sum (3.11)

\[
\frac{x + z)^{l+1}}{1 - x - y} - (-1)^l \frac{(x)_{l}(y)_{l}}{1 - x - y}.
\]

(3.15)
Now let us turn to the second sum (3.12). Changing the summation variable \( j = m + 1 - r \) shows

\[
\sum_{m=0}^{l-1} \sum_{r=0}^{m+1} (-1)^r \frac{x(y)_m}{r!(1 + m - r)!}(x-r)^{l+1} = \sum_{m=0}^{l-1} \sum_{s=0}^{m} x(-1)^s y_m(z + 1 + s)^{l+1} \frac{(z + 1 + s)^{l+1}}{(l - s - 1)!(s + 1)!(y + s)(x - s - 1)}
\]

Let us separate the term at \( r = 0 \) and calculate it using (3.10). We obtain

\[
\sum_{m=0}^{l-1} \frac{(y)_m}{(1 + m)!} z^{l+1} = \frac{z^{l+1} + (y)_l}{(1-y)} + \frac{z^{l+1}(y)_l}{l!(y-1)}.
\]

In the remaining sum we substitute \( r = s + 1 \) and interchange summations in \( m \) and \( s \), giving

\[
\sum_{m=0}^{l-1} \sum_{s=0}^{m} x(-1)^{s+1} y_m(z + 1 + s)^{l+1} \frac{(z + 1 + s)^{l+1}}{(l - s - 1)!(s + 1)!(y + s)(x - s - 1)} = \sum_{s=0}^{l-1} \sum_{m=s}^{l-1} x(-1)^{s+1} y_m(z + 1 + s)^{l+1} \frac{(z + 1 + s)^{l+1}}{(l - s - 1)!(s + 1)!(y + s)(x - s - 1)}
\]

\[
= x(y)_l \sum_{s=0}^{l-1} (-1)^s \frac{(z + 1 + s)^{l+1}}{(l - s - 1)!(s + 1)!(y + s)(x - s - 1)}
\]

\[
= x(y)_l \sum_{t=0}^{l} (-1)^t \frac{(z + t)^{l+1}}{t!(l-t)!(y + t-1)(x - t)} - \frac{z^{l+1}(y)_l}{l!(y-1)}.
\]

where we used (3.9) to calculate the sum in \( m \), set \( s = t - 1 \) and subtracted the term with \( t = 0 \). Splitting the factors in the denominator as

\[
\frac{1}{(y + t - 1)(x - t)} = \frac{1}{(x + y - 1)(x - t)} + \frac{1}{(x + y - 1)(y + t - 1)},
\]

we can finally evaluate (3.18) using (3.5). The result reads

\[
\frac{x(y)_l(x + z)^{l+1}}{(1 - x - y)(-z)_l+1} + \frac{x(1 - y + z)^{l+1}}{(1 - y)(1 - x - y)} - \frac{z^{l+1}(y)_l}{l!(y-1)}.
\]

Combining (3.15), (3.17) and (3.20) we get the RHS of (3.6).}

Now we can give the proof of Theorem 3.1 based on the above identities, and the algebraic properties of \( P_n(x) \) and \( Q_n(x) \) from Section 2.2.

**Proof of Theorem 3.1.** From (2.9) and (2.10) we have

\[
xP_m(x)Q_m(y) = P_{m+1}(x)Q_m(y) + \sum_{k=0}^{M} a_{k,m}P_{m-k}(x)Q_m(y),
\]

\[
yP_m(x)Q_m(y) = P_m(x)Q_{m-1}(y) + \sum_{k=0}^{M} a_{k,m+k}P_m(x)Q_{m+k}(y).
\]

Subtracting these two relations and summing over \( m \) from 0 to \( n - 1 \), we get after simplifications

\[
(x - y)K_n^M(x, y) = P_n(x)Q_{n-1}(y) - \sum_{k=1}^{M} \sum_{m=0}^{k-1} a_{k,n+m}P_{n-k+m}(x)Q_{n+m}(y).
\]

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Here we used the fact that \( a_{k,n} = 0 \) for \( k > n \) as follows from (2.11) and \( Q_{-1}(x) = 0 \) which can be seen from the integral representation (2.10).

Using formulas (2.18,2.19,2.23) and the explicit form (2.11) of the coefficients \( a_{k,n} \) we can rewrite the RHS as a bilinear differential operator acting on the product \( P_n(x)Q_n(y) \). Expanding the result in the basis of symmetric functions \( e_{M-l}(\nu_1,\ldots,\nu_M) \), \( l = 0,\ldots,M \) and comparing with (3.2) we can reformulate the statement of the theorem as the identity

\[
\sum_{k=1}^{M} \sum_{m=0}^{k-1} \sum_{j=0}^{k+1} \frac{(-1)^j n}{j!(k+1-j)!} (\delta_x - n)_{k-m} (\delta_y + n + 1)_{m}(n + m - j + 1)^{l+1} = \sum_{i=0}^{l} \delta_x^{-i} (-\delta_y)^i - \sum_{i=0}^{l} n^{i-l} (-\delta_y)^i, \quad l = 0,\ldots,M. \tag{3.24}
\]

This identity contains three independent variables \( n, \delta_x \) and \( \delta_y \). First we notice that all contributions to the sum with \( k > l \) are equal to 0. This follows from the fact that for any fixed \( k \) and \( m = 0,\ldots,k-1 \), the sum over \( j \) is equal to 0 because of the identity

\[
\sum_{i=0}^{n} (-1)^i \frac{1}{i!(n-i)!} \frac{\delta_x^n}{\delta_x^n} = \sum_{i=0}^{n} (-1)^i \frac{\delta_x^n}{i!(n-i)!} \bigg|_{z=1} = \frac{1}{n!} \delta_x^n [(1-z)^n] \bigg|_{z=1} = 0,
\]

valid for \( 0 \leq p < n \). Restricting the summation in \( k \) to \( 1,\ldots,l \) we can use Lemma 3.3 to calculate the sum in the LHS of (3.24). We obtain

\[
-\frac{n^{l+1}}{n + \delta_y} + \frac{\delta_x^{l+1}}{\delta_x + \delta_y} + \frac{(\delta_x - n)(-\delta_y)^{l+1}}{(n + \delta_y)(\delta_x + \delta_y)}. \tag{3.25}
\]

Summing up geometric series in the RHS of (3.24) we get the same result, thus verifying (3.24) and establishing the Theorem. \( \square \)

4 Generalization of Tracy and Widom theory

The integrable form of the finite \( n \) kernel (3.1) enables the derivation of a set of partial differential equations for the gap probabilities. We closely follow Strahov’s generalization of the original approach of Tracy and Widom [5] in his derivation of analogous equations in relation to the hard edge kernel in integrable form (2.34), but with modifications due to the final value of \( N_0 = n \). We remark that the approach of Strahov has also been applied in [35] to study the gap probability in the hard edge scaled Muttalib–Borodin model.

Using the definitions (3.3) of the functions \( \varphi_j^{(n)} \), \( \psi_j^{(n)} \), and (2.33) of \( \alpha_i \) we can derive

\[
\delta_x \varphi_j(x) = -\varphi_{j+1}(x), \quad j = 0,\ldots,M - 1, \tag{4.1}
\]

\[
\delta_x \varphi_M(x) = \sum_{k=1}^{M} (-1)^{M-k+1} e_{M-k+1}(\nu) \varphi_k(x) + (-1)^{M-1} x(\varphi_1(x) + n\varphi_0(x)), \tag{4.2}
\]

\[
\delta_x \psi_0(x) = nx(-1)^M \psi_M(x), \tag{4.3}
\]

\[
\delta_x \psi_j(x) = \psi_{j-1}(x) + (-1)^{M-j} (e_{M-j+1}(\nu) - x\delta_{j,1}) \psi_M(x), \quad j = 1,\ldots,M. \tag{4.4}
\]
Denoting the characteristic function of the interval \( J \) by \( \chi_J(x) \), we introduce the operator \( K_{M,n} \) on \( L^2(0, \infty) \) with the kernel \( K_n^{M}(x,y)\chi_J(y) \) which we denote as
\[
K_{M,n}(x,y) = K_n^{M}(x,y)\chi_J(y).
\] (4.5)

Let us notice that to restore a dependence on the parameter \( \lambda \) entering (1.3) all we need is to make a substitution
\[
Q_n(x) \rightarrow \lambda Q_n(x)
\] (4.6)
in all formulas in subsequent sections. This will only effect initial conditions for primary variables satisfying equations (7.5-7.11) below. For simplicity we set \( \lambda = 1 \) and restore a dependence on \( \lambda \) in Sections 9 and 10 when we analyze cases \( M = 1, 2 \) in details.

Similarly define the operators \( K'_{M,n} \) and \( K_{M,n}^T \) with kernels
\[
K'_{M,n}(x,y) := K_n^{M}(y,x)\chi_J(y), \quad K_{M,n}^T(x,y) := K_n^{M}(y,x)\chi_J(x).
\] (4.7)

We also define operators
\[
\rho_{M,n} = (1 - K_{M,n})^{-1}, \quad R_{M,n} = (1 - K_{M,n})^{-1}K_{M,n},
\] (4.8)
\[
\rho'_{M,n} = (1 - K'_{M,n})^{-1}, \quad R'_{M,n} = (1 - K'_{M,n})^{-1}K'_{M,n},
\] (4.9)
\[
\rho_T^{M,n} = (1 - K_{M,n}^T)^{-1}, \quad R_T^{M,n} = (1 - K_{M,n}^T)^{-1}K_{M,n}^T,
\] (4.10)
as well as the functions
\[
Q_j^{(n)}(x; a) = (1 - K_{M,n})^{-1}\varphi_j^{(n)}(x),
\] (4.11)
\[
P_j^{(n)}(x; a) = (1 - K'_{M,n})^{-1}\psi_j^{(n)}(x), \quad a \equiv (a_1, \ldots, a_{2m}),
\] (4.12)
\[
V_{i,j}^{(n)}(a) = \int_J \varphi_i^{(n)}(x)P_j^{(n)}(x; a)dx = \int_0^\infty \varphi_i^{(n)}(x)P_j^{(n)}(x; a)\chi_J(x)dx
\] (4.13)
with \( 0 \leq i, j \leq M \). As a final preliminary, we note that by substituting \( P_{n-1}(x) \) from (2.12) in (2.6) gives
\[
K_n^{M}(x,y) = (n - \delta_x) \int_0^1 P_n(ux)Q_n(uy)du.
\] (4.14)

**Proposition 4.1.** For \( j = 0, \ldots, M - 1 \) the functions \( Q_j^{(n)}(x; a) \) satisfy the system of partial differential equations
\[
\delta_x Q_j^{(n)}(x; a) = (-1)^{M+1} \left[ nQ_0^{(n)}(x; a) + Q_1^{(n)}(x; a) \right] V_{j,M}^{(n)}(a) - Q_{j+1}^{(n)}(x; a)
- \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x; a_k)Q_j^{(n)}(a_k; a)
\] (4.15)
while for $j = M$

\[
\delta_x Q_{M}^{(n)}(x; a) = (-1)^{M+1} \left[ nQ_0^{(n)}(x; a) + Q_1^{(n)}(x; a) \right] (x + V_{M,M}(a)) + \\
+ (-1)^M \sum_{k=0}^{M} Q_k^{(n)}(x; a) \left[ nV_{0,k}^{(n)}(a) + V_{1,k}^{(n)}(a) - (-1)^k e_{M+1-k}(\nu) \right] \\
- \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x, a_k) Q_{M}^{(n)}(a_k; a),
\]

(4.16)

For $1 \leq k \leq 2m$, $0 \leq j \leq M$ we also have

\[
\frac{\partial}{\partial a_k} Q_j^{(n)}(x; a) = (-1)^k R_{M,n}(x, a_k) Q_j^{(n)}(a_k; a).
\]

(4.17)

**Proposition 4.2.** For $j = 2, \ldots, M$ the functions $P_j^{(n)}(x; a)$ satisfy the system of partial differential equations

\[
\delta_x P_j^{(n)}(x; a) = (-1)^{M+1} P_j^{(n)}(x; a) \left[ nV_{0,j}^{(n)}(a) + V_{1,j}^{(n)}(a) - (-1)^j e_{M-j+1}(\nu) \right] + \\
+ P_{j-1}^{(n)}(x; a) - (-1)^k a_k R_{M,n}(x, a_k) P_j^{(n)}(a_k; a),
\]

(4.18)

while for $j = 1$

\[
\delta_x P_1^{(n)}(x; a) = (-1)^{M+1} P_1^{(n)}(x; a) \left[ nV_{0,1}^{(n)}(a) + V_{1,1}^{(n)}(a) + e_{M}(\nu) - x \right] + \\
+ P_0^{(n)}(x; a) + (-1)^M \sum_{k=0}^{M} P_k^{(n)}(x; a)V_{k,M}^{(n)}(a) \\
- \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x, a_k) P_1^{(n)}(a_k; a),
\]

(4.19)

and for $j = 0$

\[
\delta_x P_0^{(n)}(x; a) = (-1)^{M+1} P_0^{(n)}(x; a) \left[ nV_{0,0}^{(n)}(a) + V_{1,0}^{(n)}(a) \right] \\
+ (-1)^M n x P_0^{(n)}(x; a) + (-1)^M n \sum_{k=0}^{M} P_k^{(n)}(x; a)V_{k,M}^{(n)}(a) - \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x, a_k) P_0^{(n)}(a_k; a).
\]

(4.20)

For $1 \leq k \leq 2m$, $0 \leq j \leq M$ we also have

\[
\frac{\partial}{\partial a_k} P_j^{(n)}(x; a) = (-1)^k R_{M,n}(x, a_k) P_j^{(n)}(a_k; a),
\]

(4.21)

and for $1 \leq l \leq 2m$, $0 \leq j \leq M$

\[
\frac{\partial}{\partial a_l} V_{i,j}^{(n)}(a) = (-1)^j Q_j^{(n)}(a_l; a) P_j^{(n)}(a_l; a).
\]

(4.22)

These two propositions generalise Propositions 4.1 and 4.2.
5 Proofs

Here we give proofs of Propositions 4.1, 4.2. To make it more structural we split the derivation of the partial differential equations for the functions (4.11–4.13) into several steps.

First, we notice that for any two operators $K$ and $L$ (see e.g. [2, Prop. 9.3.4])

\[ [L, (1 - K)^{-1}] = (1 - K)^{-1}[L, K](1 - K)^{-1}, \]

and

\[ \frac{d}{da}(1 - K)^{-1} = (1 - K)^{-1}\frac{dK}{da}(1 - K)^{-1}, \]

where we imply that the operator $K$ smoothly depends on a parameter $a$. And with $J$ given by (1.9) we have

\[
\frac{\partial}{\partial x}\chi_J(x) = 2m\sum_{k=1}^{2m}(-1)^{k-1}\delta(x - a_k), \quad \frac{\partial}{\partial a_k}\chi_J(x) = (-1)^k\delta(x - a_k).
\]

It is convenient to use the following notations

\[ D\varphi(x) = \frac{d}{dx}\varphi(x), \quad M\varphi(x) = x\varphi(x). \]

Obviously, the operator $\delta$ introduced earlier in (2.31) is equal to $\delta = MD$. We also notice that for any operator $L$ with a kernel $L(x, y)$

\[
\int_{\Omega} (\delta_x L(x, y) - L(x, y)\delta_y) f(y)dy = \int_{\Omega} (x \frac{d}{dx} L(x, y) - L(x, y) y \frac{d}{dy} f(y)dy =
\]

\[
-L(x, y)f(y)_{y\in\partial\Omega} + (\delta_x + 1) \int_{\Omega} L(x, y) f(y)dy + \int_{\Omega} f(y) \frac{\partial}{\partial y} L(x, y)dy.
\]

So if $L$ has a compact support, we have the identity for the kernel of the commutator $[\delta, L]$

\[ [\delta, L](x, y) = (\delta_x + \delta_y + 1)L(x, y). \]

Lemma 5.1. The kernel of the operator $[\delta, (1 - K_{M,n})^{-1}]$ is given by

\[
[\delta, (1 - K_{M,n})^{-1}] = (-1)^{M+1}\{nQ_0^{(n)}(x; a) + Q_1^{(n)}(x; a)\} P_{M}^{(n)}(y; a)\chi_J(y)
\]

\[
- \sum_{k=1}^{2m}(-1)^k a_k R_{M,n}(x, a_k) \rho_{M,n}(a_k, y).
\]

Similarly,

\[
[\delta, (1 - K'_{M,n})^{-1}] = (-1)^{M+1}\{nQ_0^{(n)}(y; a) + Q_1^{(n)}(y; a)\} P_{M}^{(n)}(x; a)\chi_J(y)
\]

\[
- \sum_{k=1}^{2m}(-1)^k a_k R'_{M,n}(x, a_k) \rho'_{M,n}(a_k, y).
\]
Proof. We start with the proof of (5.7). According to (5.6) we obtain for the operator $K_{M,n}$

$$[\delta, K_{M,n}] (x, y) = (\delta_x + \delta_y + 1)(n - \delta_x) \left\{ \chi_J(y) \int_0^1 P_n(ux)Q_n(uy)du \right\}$$

$$= K_n^M (x, y) \left\{ \chi_J(y) + y \frac{\partial}{\partial y} (\chi_J(y)) \right\} + \chi_J(y)(n - \delta_x) \int_0^1 u \frac{\partial}{\partial u} (P_n(ux)Q_n(uy))du =$$

$$= K_n^M (x, y) y \frac{\partial}{\partial y} (\chi_J(y)) + (n - \delta_x)P_n(x)Q_n(y)\chi_J(y) =$$

$$= (-1)^{M+1} \left\{ n\varphi_0^{(n)}(x) + \varphi_1^{(n)}(x) \right\} \psi_M^{(n)}(y)\chi_J(y) - \sum_{k=1}^{2m} (-1)^k a_k K_n^M(x, a_k)\delta(y - a_k), \tag{5.9}$$

where we used (5.3), (4.14) and (3.1) to express $P_n(x)$ and $Q_n(y)$ in terms of $\varphi$’s and $\psi$’s. Now

$$[\delta, (1 - K_{M,n})^{-1}] (x, y) = \int_0^\infty du \rho_{M,n}(x, u) \int_0^\infty dv \rho_{M,n}(v, y) [\delta, K_{M,n}](u, v) = \tag{5.10}$$

$$= \int_0^\infty du \rho_{M,n}(x, u) \left\{ n\varphi_0^{(n)}(u) + \varphi_1^{(n)}(u) \right\} \int_0^\infty dv \rho_{M,n}(v, y)\psi_M^{(n)}(v)\chi_J(v)$$

$$- \sum_{k=1}^{2m} (-1)^k a_k \int_0^\infty du \rho_{M,n}(x, u)K_n^M(u, a_k) \int_0^\infty dv \rho_{M,n}(v, y)\delta(v - a_k) =$$

$$= \int_0^\infty du \rho_{M,n}(x, u) [\delta, K_n^M](u, y) \int_0^\infty dv \rho_{M,n}(v, y)\chi_J(v)$$

$$- \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x, a_k) \rho_{M,n}(a_k, y)$$

where we used (5.3), (4.14), (4.15), (5.12) and

$$\rho_{M,n}(y, x)\chi_J(y) = \chi_J(x)\rho_{M,n}'(x, y). \tag{5.11}$$

For the kernel of $[\delta, K_M', n]$ we obtain in the same way

$$[\delta, K_M', n] (x, y) = (-1)^{M+1} \left\{ n\varphi_0^{(n)}(y) + \varphi_1^{(n)}(y) \right\} \psi_M^{(n)}(x)\chi_J(y)$$

$$- \sum_{k=1}^{2m} (-1)^k a_k K_n^M(a_k, x)\delta(y - a_k) \tag{5.12}$$

and (5.8) follows by calculation similar to (5.10).

\[ \square \]

5.1 Proof of Proposition 4.1

We have

$$\delta_x Q_j^{(n)}(x; a) = \delta_x \left\{ (1 - K_{M,n})^{-1} \varphi_j^{(n)}(x) \right\} =$$

$$[\delta, (1 - K_{M,n})^{-1}] \varphi_j^{(n)}(x) + (1 - K_{M,n})^{-1} \delta \varphi_j^{(n)}(x). \tag{5.13}$$

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For $j = 0, \ldots, M - 1$ we obtain from (3.3)
\[ \delta \varphi_j^{(n)}(x) = -\varphi_{j+1}^{(n)}(x) \] (5.14)
and (4.15) immediately follow by applying (5.7) from Lemma 5.1 to the RHS of (5.13).

Now
\[ \delta_x Q_M^{(n)}(x; a) = [\delta, (1 - K_{M,n})^{-1}] \varphi_M^{(n)}(x) + (1 - K_{M,n})^{-1} \delta \varphi_M^{(n)}(x). \] (5.15)
Using (5.7) we obtain for the first term in (5.15)
\[ [\delta, (1 - K_{M,n})^{-1}] \varphi_M^{(n)}(x) = (-1)^{M+1} \left[ nQ_0^{(n)}(x; a) + Q_1^{(n)}(x; a) \right] V_M^{(n)}(a) \]
\[ - \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x, a_k) Q_k^{(n)}(a_k; a). \] (5.16)

We can use (4.2) to calculate $\delta \varphi_M^{(n)}(x)$ in terms of $\delta \varphi_j^{(n)}(x)$, $j = 0, \ldots, M$
\[ (1 - K_{M,n})^{-1} \delta \varphi_M^{(n)}(x) = \sum_{k=1}^{M} (-1)^{M-k+1} e_{M-k+1}(\nu) Q_k^{(n)}(x; a) \]
\[ + (-1)^{M-1}(1 - K_{M,n})^{-1} M \left( \varphi_1(x) + n\varphi_0(x) \right). \] (5.17)

Next we require
\[ (1 - K_{M,n})^{-1} M \varphi_j(x) = [(1 - K_{M,n})^{-1}, M] \varphi_j(x) + xQ_j^{(n)}(x; a) \] (5.18)
for $j = 0, 1$. Using (5.13.2) we obtain
\[ [K_{M,n}, M](x, y) = - \sum_{k=0}^{M} \varphi_k(x) \psi_k(y) \chi_J(y) \] (5.19)
and
\[ [(1 - K_{M,n})^{-1}, M](x, y) = - \sum_{k=0}^{M} Q_k^{(n)}(x; a) F_k^{(n)}(y; a) \chi_J(y). \] (5.20)
As a result
\[ (1 - K_{M,n})^{-1} M \left\{ \varphi_1(x) + n\varphi_0(x) \right\} = x(nQ_0^{(n)}(x; a) + Q_1^{(n)}(x; a)) \]
\[ - \sum_{k=0}^{M} Q_k^{(n)}(x; a) \left( nV_{0,k}^{(n)}(a) + V_{1,k}^{(n)}(a) \right). \] (5.21)
Combining (5.15), (5.17) and (5.21) we obtain (4.16).

It remains to prove (4.17). From (5.3) we derive
\[ \frac{\partial}{\partial a_k} \left( K_M^{(n)}(x, y) \chi_J(y) \right) = (-1)^k K_n^{M}(x, a_k) \delta(y - a_k). \] (5.22)
we get similar to the calculation in \((5.10)\):

\[
\frac{\partial}{\partial a_k}(1 - K_{M,n})^{-1} = (1 - K_{M,n})^{-1} \frac{\partial}{\partial a_k} K_{M,n}(1 - K_{M,n})^{-1}.
\]  

(5.23)

we obtain

\[
\frac{\partial}{\partial a_k}(1 - K_{M,n})^{-1}(x,y) = (-1)^k R_{M,n}(x,a_k) p_{M,n}(a_k,y)
\]

and as a result

\[
\frac{\partial}{\partial a_k} Q_j^{(n)}(x; a) = \frac{\partial}{\partial a_k}(1 - K_{M,n})^{-1} \varphi_j(x) = (-1)^k R_{M,n}(x,a_k) Q_j^{(n)}(a_k; a).
\]

(5.25)

5.2 Proof of Proposition 4.2

We have

\[
\delta_x P_j^{(n)}(x; a) = [\delta, (1 - K'_{M,n})^{-1}] \psi_j(x) + (1 - K'_{M,n})^{-1} \delta \psi_j(x).
\]

(5.26)

We evaluate the first term on the RHS using \((5.8)\) from Lemma 5.1.

Noticing that

\[
\int J_{i,j}^{(n)}(x; y, a) \chi_j(x) dx = \int_{0}^{\infty} \rho_{M,n}(x, y) \varphi_i(y) dy \psi_i(x) \chi_j(x) dx = \int_{0}^{\infty} \chi_j(y) \varphi_i(y) dy \int_{0}^{\infty} \rho'_{M,n}(y, x) \psi_i(x) dx = \int_{j} \varphi_i(y) P_j^{(n)}(y; a) dy = V_{i,j}^{(n)}(a),
\]

(5.27)

we obtain

\[
[\delta, (1 - K'_{M,n})^{-1}] \psi_j(x) = (-1)^{M+1} P_M^{(n)}(x; a) \left[ n V_{0,j}^{(n)}(a) + V_{i,j}^{(n)}(a) \right] - \sum_{k=1}^{2m} (-1)^k a_k R'_{M,n}(x,a_k) P_j^{(n)}(a_k; a).
\]

(5.28)

For the second term in (5.26) we consider three cases. For \(2 \leq j \leq M\)

\[
(1 - K'_{M,n})^{-1} \delta \psi_j(x) = (1 - K'_{M,n})^{-1} \left\{ \psi_{j-1}(x) + (-1)^{M-j} e_{M-j+1}(\nu) \psi_{M}(x) \right\}
\]

\[
= P_{j-1}^{(n)}(x; a) + (-1)^{M-j} e_{M-j+1}(\nu) P_{M}^{(n)}(x; a)
\]

(5.29)

and this together with (5.28) gives (4.15). For \(j = 0, 1\) we need to calculate the commutator \([(1 - K'_{M,n})^{-1}, M]\). The calculation is similar to (5.19, 5.20) and gives

\[
[(1 - K'_{M,n})^{-1}, M](x, y) = \sum_{k=0}^{M} P_k^{(n)}(x; a) Q_k^{(n)}(y; a) \chi_j(y),
\]

(5.30)
implying
\[(1 - K'_{M,n})^{-1} M \psi_j(x) = x P_j^{(n)}(x; a) + \sum_{k=0}^{M} P_k^{(n)}(x; a) V_{k,j}^{(n)}(a). \tag{5.31}\]

Now for \(j = 1\) we obtain
\[
(1 - K'_{M,n})^{-1} \delta \psi_1(x) = 1 - K'_{M,n})^{-1} \{ \psi_0(x) + (-1)^M (x - e_M(\nu)) \psi_M(x) \}
\]
\[= P_0^{(n)}(x; a) + (-1)^M (x - e_M(\nu)) P_M^{(n)}(x; a) + (-1)^M \sum_{k=0}^{M} P_k^{(n)}(x; a) V_{k,M}^{(n)}(a). \tag{5.32}\]

Combining \((5.26 \, 5.28 \, 5.32)\) we come to \((4.19)\). Finally, for \(j = 0\) we use \((4.3)\)
\[
(1 - K'_{M,n})^{-1} \delta \psi_0(x) = (1 - K'_{M,n})^{-1} \{ (-1)^M n x \psi_M(x) \}
\]
\[= (-1)^M n x P_M^{(n)}(x; a) + (-1)^M n \sum_{k=0}^{M} P_k^{(n)}(x; a) V_{k,M}^{(n)}(a) \tag{5.33}\]
and we obtain \((4.20)\).

It remains to check \((4.21 \, 4.22)\). We have
\[
\frac{\partial}{\partial a_k} K'_{M,n}(x, y) = (-1)^k K(a_k, x) \delta(a_k - y), \tag{5.34}\]
\[
\frac{\partial}{\partial a_k} (1 - K'_{M,n})^{-1}(x, y) = (-1)^k \left\{ \int_0^\infty \rho'_{M,n}(x, u) K(a_k, u) \right\} \rho_{M,n}(a_k, y)
\]
\[= (-1)^k R'_{M,n}(x, a_k) \rho_{M,n}(a_k, y). \tag{5.35}\]

and \((4.21)\) follows. Similarly,
\[
\frac{\partial}{\partial a_l} V_{i,j}^{(n)}(a) = \int_0^\infty \varphi_i(x) \frac{\partial}{\partial a_l} \left\{ P_j^{(n)}(x; a) \chi_J(x) \right\}
\]
\[= (-1)^l \varphi_i(a_l) P_j^{(n)}(a_l; a) + (-1)^l \left\{ \int_0^\infty \varphi_i(x) \chi_J(x) R'_{M,n}(x, a_l) \right\} P_j^{(n)}(a_l; a)
\]
\[= (-1)^l \left\{ \int_0^\infty \varphi_i(x) \chi_J(x) \left( \delta(x - a_l) + R'_{M,n}(x, a_l) \right) \right\} P_j^{(n)}(a_l; a)
\]
\[= (-1)^l \left\{ \int_0^\infty \varphi_i(x) \chi_J(x) \rho_{M,n}(x, a_l) \right\} P_j^{(n)}(a_l; a) = (-1)^l Q_1^{(n)}(a_l; a) P_j^{(n)}(a_l; a). \tag{5.36}\]

\[\square\]

6 Kernels of resolvent operators

From \textbf{Theorem 3.1} the kernel \(K_{M,n}\) and its transpose \(K'_{M,n}\) in integrable form are
\[
K_{n}^{M}(x, y) \chi_J(y) = \sum_{j=0}^{M} \varphi_j(x) \psi_j(y) \quad \chi_J(y), \quad K_{n}^{M}(y, x) \chi_J(y) = - \sum_{j=0}^{M} \psi_j(x) \varphi_j(y) \quad \chi_J(y). \tag{6.1}\]

The following proposition is the direct analog of Proposition 4.4 from \cite{[1]}
Proposition 6.1. The kernels of the resolvent operators $R_{M,n}$ and $R'_{M,n}$ (4.8-4.9) are given by

$$R_{M,n}(x, y) = \sum_{j=0}^{M} Q_j^{(n)}(x; a) P_j^{(n)}(y; a) \frac{x - y}{x - y} \chi_J(y),$$  \hspace{1cm} (6.2)

$$R'_{M,n}(x, y) = -\sum_{j=0}^{M} P_j^{(n)}(x; a) Q_j^{(n)}(y; a) \frac{x - y}{x - y} \chi_J(y).$$  \hspace{1cm} (6.3)

Proof. Let us first calculate the kernel of the operator $R_{M,n}(x, y)$,

$$(x - y)R_{M,n}(x, y) = [M, R_{M,n}](x, y) = [M, -1 + (1 - K_{M,n})^{-1}](x, y) =$$

$$- [(1 - K_{M,n})^{-1}, M](x, y) = \sum_{j=0}^{M} Q_j^{(n)}(x; a) P_j^{(n)}(y; a) \chi_J(y),$$  \hspace{1cm} (6.4)

where we used the fact that $R_{M,n} = (1 - K_{M,n})^{-1} K_{M,n} = -1 + (1 - K_{M,n})^{-1}$ and (5.20).

Repeating these calculations for the operator $R'_{M,n}$ and using (5.30) we come to (6.3).

Another important property of the kernel $R_{M,n}$ generalises Proposition 4.6 of [1].

Proposition 6.2. The kernel $R_{M,n}(x, y)$ satisfies the partial differential equation

$$\left( \delta_x + \delta_y + \sum_{k=1}^{2m} \delta_{a_k} + 1 \right) R_{M,n}(x, y) = (-1)^{M+1} \{nQ_0^{(n)}(x; a) + Q_1^{(n)}(x; a)\} P_M^{(n)}(y; a) \chi_J(y).$$  \hspace{1cm} (6.5)

Proof. By (5.6) and Lemma 5.1 we have

$$(\delta_x + \delta_y + 1) R_{M,n}(x, y) = [\delta, R_{M,n}(x, y)] = [\delta, (1 - K_{M,n})^{-1}](x, y) =$$

$$(-1)^{M+1} \{nQ_0^{(n)}(x; a) + Q_1^{(n)}(x; a)\} P_M^{(n)}(y; a) \chi_J(y) - \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x, a_k) \rho_{M,n}(a_k, y).$$  \hspace{1cm} (6.6)

Finally, from (5.24) we have

$${\sum_{k=1}^{2m} a_k \frac{\partial}{\partial a_k} R_{M,n}(x, y)} = \sum_{k=1}^{2m} (-1)^k a_k R_{M,n}(x, a_k) \rho_{M,n}(a_k, y)$$  \hspace{1cm} (6.7)

and (6.5) follows.
7 Strahov’s equations for primary variables

We now define analogs of Strahov primary variables for finite \( n \) for \( j = 0, \ldots, M, k = 1, \ldots, 2m \)

\[
x_{j,k}^{(n)} = \epsilon_k (1 - K_{M,n})^{-1} \varphi_j^{(n)}(a_k), \quad y_{j,k}^{(n)} = \epsilon_k (1 - K_{M,n}')^{-1} \psi_j^{(n)}(a_k),
\]

where \( \epsilon_k \) are some constants. It is easy to see that they enter the equations in Theorems 4.1 4.2 only as \( \epsilon_k^2 \) and it is convenient to choose

\[
\epsilon_k^2 = (-1)^{k+1}.
\]

We realise this by following the choice of \( \Pi \),

\[
\epsilon_{2k} = i, \quad \epsilon_{2k-1} = 1, \quad k = 1, \ldots, m.
\]

Let also define variables for \( j = 0, \ldots, M \)

\[
\xi_j^{(n)} = (-1)^M \left( nV_{0,j}^{(n)}(a) + V_{1,j}^{(n)}(a) - (-1)^j e_{M+1-j}(\nu) \right), \quad \nu = (\nu_0, \ldots, \nu_M),
\]

\[
\eta_j^{(n)} = (-1)^M V_{J,j}^{(n)}(a).
\]

**Theorem 7.1.** The functions \( x_{j,k}^{(n)}, y_{j,k}^{(n)}, \xi_j^{(n)} \) and \( \eta_j^{(n)} \) satisfy systems of partial differential equations

1. For \( j = 0, \ldots, M-1, l = 1, \ldots, 2m \)

\[
\frac{\partial x_{j,l}^{(n)}}{\partial a_l} = -\eta_j^{(n)} (nx_{0,l}^{(n)} + x_{1,l}^{(n)}) - x_{j+1,l}^{(n)} + \sum_{k=1}^{2m} a_k x_{j,k}^{(n)} \sum_{i=0}^{M} x_{i,l}^{(n)} y_{i,k}^{(n)}
\]

2. For \( j = M, l = 1, \ldots, 2m \)

\[
\frac{\partial x_{M,l}^{(n)}}{\partial a_l} = -(\eta_M^{(n)} + (-1)^M a_l) (nx_{0,l}^{(n)} + x_{1,l}^{(n)}) + \sum_{i=0}^{M} x_{i,l}^{(n)} \left\{ \xi_i^{(n)} + \sum_{k=1}^{2m} a_k x_{j,k}^{(n)} \right\}.
\]

3. For \( j = 0, l = 1, \ldots, 2m \)

\[
a_l \frac{\partial y_{0,l}^{(n)}}{\partial a_l} = \left[ (-1)^M na_l - \xi_0^{(n)} \right] y_{M,l}^{(n)} + \sum_{i=0}^{M} y_{i,l}^{(n)} \left\{ \eta_i^{(n)} + \sum_{k=1}^{2m} a_k x_{i,k}^{(n)} \right\}.
\]

4. For \( j = 1, l = 1, \ldots, 2m \)

\[
a_l \frac{\partial y_{1,l}^{(n)}}{\partial a_l} = \left[ (-1)^M a_l - \xi_1^{(n)} \right] y_{M,l}^{(n)} + \sum_{i=0}^{M} y_{i,l}^{(n)} \left\{ \eta_i^{(n)} + \sum_{k=1}^{2m} a_k x_{i,k}^{(n)} \right\},
\]

and for \( j = 2, \ldots, M, l = 1, \ldots, 2m \)

\[
a_l \frac{\partial y_{j,l}^{(n)}}{\partial a_l} = \xi_j^{(n)} y_{M,l}^{(n)} + y_{j-1,l}^{(n)} + \sum_{k=1}^{2m} a_k y_{j,k}^{(n)} \sum_{i=0}^{M} x_{i,k}^{(n)} y_{i,l}^{(n)}.
\]
3. For \( j = 0, \ldots, M \) and \( k, l = 1, \ldots, 2m, k \neq l \)

\[
\frac{\partial x_{j,l}^{(n)}}{\partial a_k} = -\frac{x_{j,k}^{(n)}}{a_l - a_k} \sum_{i=0}^{M} x_{i,l}^{(n)} y_{i,k}^{(n)}, \quad \frac{\partial y_{j,l}^{(n)}}{\partial a_k} = -\frac{y_{j,k}^{(n)}}{a_l - a_k} \sum_{i=0}^{M} x_{i,k}^{(n)} y_{i,l}^{(n)}. \tag{7.10}
\]

4. For \( j = 0, \ldots, M, l = 1, \ldots, 2m \)

\[
\frac{\partial \xi_{j,l}^{(n)}}{\partial a_l} = (-1)^{M+1} (n x_{0,l}^{(n)} + x_{1,l}^{(n)}) y_{j,l}^{(n)}, \quad \frac{\partial \eta_{j,l}^{(n)}}{\partial a_l} = (-1)^{M+1} x_{l}^{(n)} y_{M,l}^{(n)}. \tag{7.11}
\]

The proof of this theorem is straightforward. We set \( x = a_l \) in Propositions \( 4.1-4.2 \) and in formulas for resolvent kernels \( \delta_{0,2}^{[0,3]} \).

Comparison with the corresponding result in \[1\] Prop. 3.3 one sees that the modification of Strahov equations to finite \( n \) is very simple. It looks even simpler at the level of symplectic structure. Consider a dynamical system with variables \((x_{j,k}^{(n)}, \xi_{j,l}^{(n)}, y_{j,k}^{(n)}, \eta_{j,l}^{(n)})\) and introduce the same Poisson brackets as in \[1\] eq. (3.24),

\[
\{x_{j,k}^{(n)}, y_{i,l}^{(n)}\} = \frac{1}{a_k} \delta_{k,l} \delta_{i,j}, \quad \{\xi_{j,l}^{(n)}, \eta_{i,l}^{(n)}\} = (-1)^{M} \delta_{i,j}. \tag{7.12}
\]

with all remaining Poisson brackets equal to 0.

**Theorem 7.2.** The system of equations from the Theorem 7.1 associated with the kernel \( K_{M,n} \) can be written in Hamiltonian form

\[
\frac{\partial x_{j,k}^{(n)}}{\partial a_l} = \left\{ x_{j,k}^{(n)}, H_{l}^{(n)} \right\}, \quad \frac{\partial y_{j,l}^{(n)}}{\partial a_l} = \left\{ y_{j,l}^{(n)}, H_{l}^{(n)} \right\}, \tag{7.13}
\]

\[
\frac{\partial \xi_{j,l}^{(n)}}{\partial a_l} = \left\{ \xi_{j,l}^{(n)}, H_{l}^{(n)} \right\}, \quad \frac{\partial \eta_{j,l}^{(n)}}{\partial a_l} = \left\{ \eta_{j,l}^{(n)}, H_{l}^{(n)} \right\}. \tag{7.14}
\]

for \( j = 0, \ldots, M, k, l = 1, \ldots, 2m \). The Hamiltonians are given by

\[
H_{l}^{(n)} = \left\{ (-1)^{M+1} a_l y_{M,l}^{(n)} - \sum_{i=0}^{M} \eta_{i,l}^{(n)} y_{i,l}^{(n)} \right\} (n x_{0,l}^{(n)} + x_{1,l}^{(n)}) -

- \sum_{j=0}^{M-1} x_{j+1,l}^{(n)} y_{j,l}^{(n)} + y_{M,l}^{(n)} \sum_{i=0}^{M} \xi_{i,l}^{(n)} x_{i,l}^{(n)} + \sum_{k=1}^{2m} \frac{a_k}{a_l - a_k} \sum_{i,j=0}^{M} x_{i,k}^{(n)} y_{i,l}^{(n)} y_{j,k}^{(n)}. \tag{7.15}
\]

The Hamiltonians \( H_{l}^{(n)} \) are in involution

\[
\{H_{l}^{(n)}, H_{r}^{(n)}\} = 0, \tag{7.16}
\]

where \( 1 \leq l, r \leq 2m \).
Using the Poisson brackets (7.12) it is easy to see that equations (7.5-7.11) follow from (7.13,7.14). The relation (7.16) is a tedious but straightforward calculation with the use of (7.12).

Comparing (7.15) with eq. (3.28) in [1] we see that the only modification of the Hamiltonians from the case \( n \to \infty \) to finite \( n \) reduces to the change
\[
x_{0,l} \to n x_{0,l}^{(n)} + x_{1,l}^{(n)}.
\]
in the first term of (7.15).

**Proposition 7.3.** The Hamiltonians \( H_l^{(n)} \) (7.15) can be written as
\[
H_l^{(n)} = a_l \frac{\partial}{\partial a_l} \log (\det(1 - K_{M,n})) .
\]

**Proof.** The proof is standard (see, for example, [2, Ex. 9.3 q.1]) and based on calculation of the trace of the resolvent operator \( R_{M,n} \). Using (4.8) and (5.22) we obtain
\[
\left( (1 - K_{M,n})^{-1} \frac{\partial K_{M,n}}{\partial a_l} \right)(x, y) = (-1)^l R_{M,n}(x, a_l) \delta(a_l - y)
\]
and
\[
\frac{\partial}{\partial a_l} \log (\det(1 - K_{M,n})) = -\text{Tr} \left( (1 - K_{M,n})^{-1} \frac{\partial K_{M,n}}{\partial a_l} \right) = (-1)^{l+1} R_{M,n}(a_l, a_l).
\]
We can calculate \( R_{M,n}(a_l, a_l) \) by taking the limit \( x \to y = a_l \) in (6.2). First notice that from continuity of the kernel \( R_{M,n}(x, y) \) at \( x = y \) we have
\[
\sum_{j=0}^M P_j^{(n)}(x; a_l) Q_j^{(n)}(x; a_l) = 0
\]
and as a result we obtain
\[
a_l R_{M,n}(a_l, a_l) = \sum_{j=0}^M P_j^{(n)}(a_l; a_l) x \frac{\partial}{\partial x} Q_j^{(n)}(x; a_l) \bigg|_{x=a_l} = (-1)^{l+1} H_l^{(n)},
\]
where we used (4.14, 4.19) to calculate the derivatives of \( Q_j^{(n)}(x; a) \) at \( x = a_l \) for \( j = 0, \ldots, m \) and the explicit expressions (7.15) for \( H_l^{(n)} \). Comparing (7.20) and (7.22) we obtain (7.18). □

We can now define a sequence of \( \tau \)-functions \( \tau_n, n = 1, 2, \ldots \)
\[
\tau_n(a) = \text{det}(1 - K_{M,n}),
\]
and the closed form
\[
w_n(a) = d \log \tau_n(a).
\]

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We have
\[ w_n(a) = \sum_{i=1}^{2m} H_i^{(n)} \frac{da_i}{a_i}. \quad (7.25) \]

For the simplest case \( m = 1, J = (0, s), a_1 = 0, a_2 = s \) the system (7.5-7.11) becomes the system of nonlinear differential equations in \( s \). The gap probability \( E_n^M(0; J) \) coincides with the tau-function \( \tau_n(s) \) which is given by
\[ \tau_n(s) = \exp \left\{ \int_0^s \frac{dt}{t} H_n(t) \right\}, \quad (7.26) \]
where \( H_n(s) = H_2^{(n)}(s) \). Using variables \( x_j^{(n)} = x_{j,2}^{(n)}, y_j^{(n)} = y_{j,2}^{(n)} \) we can rewrite (7.15) for \( l = 2 \) as
\[ H_n(s) = (-1)^{M+1} s(nx_0 + x_1)y_M - \sum_{i=0}^{M-1} x_{i+1}^{(n)} y_i^{(n)} + \sum_{i=0}^{M} \left\{ y_M s_i^{(n)} x_i^{(n)} - (nx_0 + x_1) \eta_i^{(n)} y_i^{(n)} \right\}. \quad (7.27) \]

Now **Proposition 6.2** gives
\[ (sR_{M,n}(s, s))' = (-1)^{M+1} \tilde{e}_{2}(nx_0 + x_1)y_M = (-1)^{M}(nx_0 + x_1)y_M. \quad (7.28) \]
Comparing this with the equations (7.11) we can integrate (7.28) and derive
\[ sR_{M,n}(s, s) = -(n\eta_0(s) + \eta_1(s)) + C_0, \quad (7.29) \]
where \( C_0 \) is the integration constant. Taking into account (4.13) and (7.4) we obtain in the limit \( s \to 0 \eta_j(0) = 0, j = 0, \ldots, M \) and as a result \( C_0 = 0 \). Comparing (7.29) with (7.22) we finally get an alternative expression for the Hamiltonian (7.27)
\[ H_n(s) = n\eta_0(s) + \eta_1(s) \quad (7.30) \]
and the expression for the tau-function in terms of primary variables
\[ \tau_n(s) = \exp \left\{ \int_0^s \frac{\eta_0(t) + \eta_1(t)}{t} dt \right\}. \quad (7.31) \]
The sequence of tau-functions \( \tau_n(s) \) should satisfy Toda-type recurrence relations, but we postpone investigation of this possibility to another occasion.

### 8 Isomonodromic deformation

Here we briefly discuss the isomonodromic deformation of the system given by the **Theorem 7.1**. Again this section is a straightforward generalization of results of Section 3.4 in [1] to finite \( n \). Introduce a set of \((M + 1) \times (M + 1)\) matrices
\[ E_n = (-1)^{M+1} \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ \eta & 1 & 0 & \ldots & 0 \end{pmatrix}, \\ C_n = \begin{pmatrix} -\eta_0 & -\eta_0 - 1 & 0 & \ldots & 0 \\ -\eta_1 & -\eta_1 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\eta_{M-1} & -\eta_{M-1} & 0 & \ldots & -1 \\ -\eta_M + \xi_0 & -\eta_M + \xi_1 & \xi_2 & \ldots & \xi_M \end{pmatrix} \quad (8.1) \]
and a set of residue matrices
\[
A_n^{(l)} = \begin{pmatrix}
    x_{0,l}^{(n)} \\
x_{1,l}^{(n)} \\
    \vdots \\
x_{M,l}^{(n)}
\end{pmatrix} \otimes (y_0^{(n)}, y_1^{(n)}, \ldots, y_M^{(n)}).
\tag{8.2}
\]

The following proposition is the analogue of Proposition 3.6 from [1]

**Proposition 8.1.** The differential equations (7.5-7.11) can be rewritten in the matrix form
\[
a_l \frac{\partial}{\partial a_l} A_n^{(l)} = [C_n + a_l E_n, A_n^{(l)}] + \sum_{k=1}^{2m} a_k \left[ A_n^{(k)}, A_n^{(l)} \right], \quad l = 1, \ldots, 2m,
\tag{8.3}
\]
\[
\frac{\partial}{\partial a_k} A_n^{(l)} = \frac{A_n^{(l)} A_n^{(k)}}{a_l - a_k}, \quad k \neq l = 1, \ldots, 2m,
\tag{8.4}
\]
\[
\frac{\partial}{\partial a_l} C_n = [E_n, A_n^{(l)}], \quad l = 1, \ldots, 2m.
\tag{8.5}
\]

The Hamiltonians \(H_i^{(n)}\) have the form
\[
H_i^{(n)} = \text{Tr}(C_n A_n^{(l)}) + a_l \text{Tr}(E_n A_n^{(l)}) + \sum_{k=1}^{2m} \frac{a_k}{a_l - a_k} \text{Tr}(A_n^{(k)} A_n^{(l)}), \quad l = 1, \ldots, 2m.
\tag{8.6}
\]

Now we can consider the linear system of ordinary differential equations for the function \(\Psi_n(z; a_1 \ldots, a_{2m})\)
\[
\frac{\partial \Psi_n}{\partial z} = \left\{ E_n + \frac{1}{z} \left( C_n - \sum_{j=1}^{2m} A_n^{(j)} \right) + \sum_{j=1}^{2m} \frac{A_n^{(j)}}{z - a_j} \right\} \Psi_n
\tag{8.7}
\]
and for \(1 \leq j \leq 2m\)
\[
\frac{\partial \Psi_n}{\partial a_j} = -\frac{A_n^{(j)}}{z - a_j} \Psi_n,
\tag{8.8}
\]
where the poles \(a_1, \ldots, a_{2m}\) play the role of deformation parameters.

The compatibility conditions for the system (8.7-8.8) lead to Schlesinger equations which exactly coincide with equations of motion (8.3-8.5). Therefore, we derive the isomonodromic deformation representation for finite \(n\) similar to Jimbo, Miwa, Mori, Sato theory [9].

Now let us consider the case \(m = 1\) in more details. We choose \(J = (0, s), a_1 = 0, a_2 = s\) and introduce variables
\[
x_i(s) = x_{i,2}^{(n)}(s), \quad y_i(s) = y_{i,2}^{(n)}(s), \quad \xi_i(s) = \xi_i^{(n)}(s), \quad \eta_i(s) = \eta_i^{(n)}(s).
\tag{8.9}
\]
We have only one nontrivial Hamiltonian $H_n(s)$ (7.21) and the equations (8.3-8.5) for $l = 2$ can be rewritten as

$$s \frac{\partial}{\partial s} A_n^{(2)} = [C_n + sE_n, A_n^{(2)}], \quad \frac{\partial}{\partial s} C_n = [E_n, A_n^{(2)}].$$  \hspace{1cm} (8.10)

The isomonodromic equations (8.7-8.8) for $\Psi_n(z,s)$ rewrite as

$$\frac{\partial \Psi_n}{\partial z} = \left\{ E_n + \frac{C_n - A_n^{(2)}}{z} + \frac{A_n^{(2)}}{z - s} \right\} \Psi_n, \quad \frac{\partial \Psi_n}{\partial s} = -A_n^{(2)} \frac{z}{z - s} \Psi_n.$$  \hspace{1cm} (8.11)

Using representation (8.10) it is not difficult to construct additional $M + 1$ conserved quantities.

**Proposition 8.2.** The eigenvalues $\text{Spec}(A_n^{(2)} - C_n)$ are integrals of the motion (8.10).

**Proof.** Let us denote $B_n(s) = A_n^{(2)}(s) - C_n(s)$ and introduce the characteristic polynomial

$$p(z,s) = \det (zI - B(s)).$$  \hspace{1cm} (8.12)

We have

$$\frac{\partial}{\partial s} p(z,s) = p(z,s) \frac{\partial}{\partial s} \text{Tr} \log (zI - B(s)) = -p(z,s) \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \text{Tr} \left( B_n^k(s) B_n'(s) \right),$$  \hspace{1cm} (8.13)

where the sum always converges for sufficiently large $z$.

Now

$$B_n'(s) = (A_n^{(2)}(s) - C_n(s))' = \frac{1}{s} [C_n, A_n^{(2)}] = \frac{1}{s} [C_n, B_n]$$  \hspace{1cm} (8.14)

by (8.10) and we have for any $k \geq 0$

$$\text{Tr} \left( B_n^k(s) B_n'(s) \right) = \frac{1}{s} \text{Tr} \left( B_n^k [C_n, B_n] \right) = 0.$$  \hspace{1cm} (8.15)

Therefore, the sum in (8.13) is equal to zero and the characteristic polynomial $p(z,s)$ does not depend on $s$. We conclude that the eigenvalues of the matrix $B_n(s)$ do not depend on $s$. \hfill \square

One can calculate the spectrum of the matrix $A_n^{(2)} - C_n$ using the small $s$ expansion. Let us assume that

$$\nu_{\text{min}} = \min(\nu_1, \ldots, \nu_M) > 0.$$  \hspace{1cm} (8.16)

We can use the integral representation (2.16) for $Q_n(x)$ and calculate the integral by closing the contour to the left. Using (7.1) we obtain

$$y_i(s) \sim s^{\nu_{\text{min}}(\text{const} + \text{possible log terms})} \to 0 \quad \text{at} \quad s \to 0.$$  \hspace{1cm} (8.17)

Similarly, using the representation for $P_n(x)$ in terms of hypergeometric function (A.6) we obtain

$$x_0(0) \sim \text{const}, \quad x_i(0) = 0, \quad i > 0.$$  \hspace{1cm} (8.18)

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Finally, $V_{i,j}(s) \to 0$ in (4.13) at $s \to 0$ and we obtain from (7.4)

$$\xi_i(0) = (-1)^{M+i+1}e_{M+1-i}(\nu), \quad \eta_i(0) = 0. \quad (8.19)$$

Therefore, the matrix $B_n(s)$ at $s = 0$ becomes

$$B_n(0) = -C_n(0) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & d_1 & d_2 & \ldots & d_M \end{pmatrix},$$

where $d_i = (-1)^{M+i}e_{M+1-i}(\nu)$.

The matrix (8.20) has the eigenvalue 0 with the eigenvector $(1, 0, \ldots, 0)$ and $M$ eigenvalues $\nu_i$ with the eigenvectors $(1, \nu_i, \nu_i^2, \ldots, \nu_i^M)$ which follows from the identity

$$\sum_{k=1}^{M}(-1)^{M+k}e_{M+1-k}(\nu)\nu_i^k = (-\nu_i)(-\nu_i^M + \prod_{j=1}^{M}(\nu_i - \nu_j)) = \nu_i^{M+1}. \quad (8.21)$$

Therefore, we conclude that

$$\text{Spec}(A_n^{(2)} - C_n) = (0, \nu_1, \ldots, \nu_M). \quad (8.22)$$

If $\nu_{\min} = 0$ in (8.16), then $y_i(s)$ can have constant or growing logarithmic asymptotics at $s \to 0$ and the calculation becomes more involved.

A topic for future study is a possible relationship of this isomonodromic deformation with the theory of so-called four dimensional Painlevé systems [38], as speculated in [17].

9 The case $M = 1$

We set $m = 1$, choose $J = (0, s)$ and use the variables (8.9). The system of partial differential equations (7.5-7.11) for $M = 1$ reads

$$sx_0' = -(nx_0 + x_1)\eta_0 - x_1, \quad (9.1)$$

$$sx_1' = -(nx_0 + x_1)(\eta_1 - s) + \xi_0x_0 + \xi_1x_1, \quad (9.2)$$

$$sy_1' = -(s + \xi_1)y_1 + y_0 + \eta_0y_0 + \eta_1y_1, \quad (9.3)$$

$$sy_0' = -(ns + \xi_0)y_1 + n(\eta_0y_0 + \eta_1y_1), \quad (9.4)$$

$$\xi_0' = (nx_0 + x_1)y_0, \quad (9.5)$$

$$\xi_1' = (nx_0 + x_1)y_1, \quad (9.6)$$

$$\eta_0' = x_0y_1, \quad (9.7)$$

$$\eta_1' = x_1y_1, \quad (9.8)$$

and the Hamiltonian (8.6) for $l = 2$ takes the form

$$H_2^{(n)} = (nx_0 + x_1)(sy_1 - \eta_0y_0 - \eta_1y_1) - x_1y_0 + y_1(\xi_0x_0 + \xi_1x_1). \quad (9.9)$$
Let us compare this with the results of [5] for the Laguerre kernel. From the Christoffel–Darboux formula (see e.g. [2, Prop. 5.1.3]) we define the Tracy-Widom kernel for the finite Laguerre ensemble of \( n \times n \) matrices by

\[
K_L(x, y) = \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y},
\]

(9.10)

with

\[
\varphi(x) = \sqrt{\lambda}(n(1 + \nu))^{1/4}\varphi_{n-1}(x), \quad \psi(x) = \sqrt{\lambda}(n(1 + \nu))^{1/4}\varphi_n(x),
\]

(9.11)

\[
\varphi_n(x) = \sqrt{\frac{n!}{\Gamma(n + \nu + 1)}}x^{\nu/2}e^{-\nu/2}L_n^{\nu}(x),
\]

(9.12)

as in the eqs. (1.2), (5.36) of [5]. \( L_n^{\nu}(x) \) are generalized Laguerre polynomials.

Taking into account the remark before (4.6) the functions \( P_n(x) \) and \( Q_n(x) \) in (2.3)-(2.4) for \( M = 1 \) take the form

\[
P_n(x) = (-1)^n n!L_n^{\nu}(x), \quad Q_n(x) = \lambda \frac{(1-n)x^\nu e^{-x}}{\Gamma(n + \nu + 1)}L_n^{\nu}(x),
\]

(9.13)

where we used the expression of Laguerre polynomials in terms of the hypergeometric function and (A.5) of the Appendix

\[
L_n^{\nu}(x) = \frac{(\nu + 1)_n}{n!} F_1 \left( \frac{-n}{1 + \nu} \right).
\]

(9.14)

Now we can rewrite the formula (3.1) for the kernel \( K_n^M(x, y) \) at \( M = 1 \) as

\[
K_n^1(x, y) = \frac{1}{x - y} \left( \delta_y - \delta_x + y - \nu \right) P_n(x)Q_n(y).
\]

(9.15)

Using the differentiation formula for the Laguerre polynomials

\[
dx \frac{d}{dx} L_n^{\nu}(x) = nL_n^{\nu}(x) - (n + \nu)L_n^{\nu-1}(x)
\]

(9.16)

and substituting (9.13) into (9.15) we find after straightforward calculations

\[
K_L(x, y) = h(x)K_n^1(x, y)h^{-1}(y), \quad h(x) = x^{\nu/2}e^{-x/2}.
\]

(9.17)

We thus see that the kernel \( K_n^1(x, y) \) is symmetrizable, and after the diagonal similarity transformation coincides with the symmetric kernel \( K_L(x, y) \).

The functions \( P_L(x) \) and \( Q_L(x) \) of [5 Eq. (1.5)] match with our \( Q_n^{(n)}(x; s) \) in (4.11) according to

\[
P_L(x) = (1 - K_L)^{-1}\varphi(x) = c_n \sqrt{\lambda}x^{\nu/2}e^{-x/2}Q_0^{(n)}(x; s),
\]

(9.18)

\[
Q_L(x) = (1 - K_L)^{-1}\varphi(x) = \frac{c_n}{\sqrt{n(n + \nu)}} \sqrt{\lambda}x^{\nu/2}e^{-x/2}(nQ_0^{(n)}(x; s) + Q_1^{(n)}(x; s)),
\]

(9.19)

\[
c_n = \frac{(-1)^{\nu+1}(n(n + \nu))^{1/4}}{\sqrt{n!\Gamma(n + \nu + 1)}}.
\]

(9.20)
Setting \( x = s \) we find from (9.18), a connection of Tracy and Widom’s variables \( q(s) \) and \( p(s) \) to our variables \( x_0(s) \) and \( x_1(s) \)

\[
q(s) = -i \frac{c_n}{\sqrt{n(n+\nu)}} \sqrt{\lambda s^{\nu/2}} e^{-s/2} (nx_0(s) + x_1(s)), \quad p(s) = -i c_n \sqrt{\lambda s^{\nu/2}} e^{-s/2} x_0(s). \tag{9.21}
\]

We also obtain from (9.13) and (9.17) that

\[
Q_n(x) = \frac{\lambda h^2(x)}{n!\Gamma(n+\nu+1)} P_n(x) \tag{9.22}
\]

and

\[
K_n^1(y, x) = h^2(x)K_n^1(x, y)h^{-2}(y). \tag{9.23}
\]

It immediately follows that for \( M = 1 \) we can express variables \( y_i(s) \) in terms of \( x_i(s), i = 0, 1 \)

\[
y_1(s) = \frac{\lambda s^{\nu} e^{-s}}{n!\Gamma(n+\nu+1)} x_0(s), \quad y_0(s) = -\frac{\lambda s^{\nu} e^{-s}}{n!\Gamma(n+\nu+1)} x_1(s). \tag{9.24}
\]

We can now calculate correct initial conditions for the variables (8.9) for small \( s \). Assuming that the parameter \( \nu > 0 \) is in generic position we obtain from (3.3, 4.11, 4.13, 7.1-7.4, 14.14) and explicit expressions (9.13, 9.14) for \( P_n(x) \), \( Q_n(x) \), \( x_0(s) \), \( x_1(s) \)

\[
x_0(s) = i(-1)^{n+1}(\nu + 1)\frac{1}{\Gamma(n+1)} \left( 1 + O(s) \right) - \frac{i\lambda(-1)^{n+1}(\nu + 1)^2}{(n+1)\Gamma(n)(\nu + 2)} \left( 1 + O(s) \right) + O(s^{2\nu+2}), \tag{9.25}
\]

\[
x_1(s) = i(-1)^{n+1}(\nu + 2)n\left( s + O(s^2) \right) - \frac{i\lambda(-1)^{n+1}(\nu + 1)^2}{(n+1)\Gamma(n)(\nu + 3)} \left( 1 + O(s) \right) + O(s^{2\nu+3}). \tag{9.26}
\]

\[
\eta_0(s) = -\frac{(\nu + 1)n}{\Gamma(n+1)\Gamma(\nu + 2)} \lambda s^{\nu+1} (1 + O(s)) + O(s^{2\nu+2}), \tag{9.27}
\]

\[
\eta_1(s) = -\frac{(\nu + 1)n}{\Gamma(n)\Gamma(\nu + 3)} \lambda s^{\nu+2} (1 + O(s)) + O(s^{2\nu+3}), \tag{9.28}
\]

\[
\xi_0(s) = -\frac{n(\nu + 1)n}{\Gamma(n)\Gamma(\nu + 2)} \lambda s^{\nu+2} (1 + O(s)) + O(s^{2\nu+3}), \tag{9.29}
\]

\[
\xi_1(s) = -\nu - \frac{\lambda s^{\nu+1}}{\Gamma(n)\Gamma(\nu + 2)} (1 + O(s)) + O(s^{2\nu+2}). \tag{9.30}
\]

So the recipe to restore a dependence on \( \lambda \) in initial conditions for primary variables is very simple — we replace each power \( s^\nu \) by \( \lambda s^\nu \).

With given initial conditions we can combine (9.6, 9.8) to obtain the first integral

\[
\xi_1(s) - n\eta_0(s) - \eta_1(s) + \nu = 0. \tag{9.31}
\]

The second integral was derived in (7.30)

\[
H_2^{(n)}(s) - n\eta_0(s) - \eta_1(s) = 0. \tag{9.32}
\]
Now let us see how integrals (8.22) are expressed in terms of basic variables. We have
\[ A_n^{(2)} - C_n = \begin{pmatrix} x_0 y_0 + n \eta_0 & x_0 y_1 + \eta_0 + 1 \\ x_1 y_0 + n \eta_1 - \xi_0 & x_1 y_1 + \eta_1 - \xi_1 \end{pmatrix} \quad (9.33) \]
The first integral \( \text{Tr}(A_n^{(2)} - C_n) = \nu \) is equivalent to (9.31) due to the orthogonality condition
\[ x_0(s) y_0(s) + x_1(s) y_1(s) = 0 \quad (9.34) \]
. The second integral \( \det(A_n^{(2)} - C_n) = 0 \) gives
\[ (x_0 y_0 + n \eta_0)(x_1 y_1 + \nu - n \eta_0) - (x_0 y_1 + \eta_0 + 1)(x_1 y_0 + n \eta_1 - \xi_0) = 0. \quad (9.35) \]
Taking the sum of (9.32) and (9.35) and using (9.1-9.8), we obtain the expression for \( \xi_0(s) \) in terms of \( \eta_0(s) \) and \( \eta_1(s) \)
\[ \xi_0(s) = \frac{s(n \eta_0'(s) + \eta_1'(s)) + (n \eta_0(s) - 1)(n \eta_0(s) + \eta_1(s)) + n(\eta_1(s) - \nu \eta_0(s))}{1 + \eta_0(s)}. \quad (9.36) \]
Using (9.36) to eliminate \( x_0, x_1, y_0, y_1 \) we can rewrite the integral (9.35) as
\[ n \eta_0(n \eta_0 + \eta_1 - \nu) + n \eta_1(1 + \eta_0') + (n + \nu - n \eta_0) \eta_1' - \xi_0(1 + \eta_0 + \eta_0') + (1 + \eta_0) \xi_0' = 0. \quad (9.37) \]
Finally, differentiating (9.7-9.8) and using (9.1-9.8), it is easy to obtain the relations
\[ sn_0'' + 2(1 + \eta_0) \eta_1' + (2n \eta_0 + s - \nu) \eta_0' = 0, \quad (9.38) \]
\[ sn_1'' - (1 + \eta_0)(n \eta_1' + \xi_0') + (n \eta_1 - ns - \xi_0) \eta_0' = 0. \quad (9.39) \]
Let us introduce the function
\[ \sigma(s) = -n \eta_0(s) - \eta_1(s). \quad (9.40) \]
Using (9.36) to eliminate \( \xi_0, \xi_0' \) from (9.37-9.39), we can express \( \sigma, \sigma' \) and \( \sigma'' \) in terms of \( \eta_0, \eta_0', \eta_0'' \) and obtain the 3rd order differential equation for \( \eta_0(s) \). The function \( \sigma(s) \) satisfies the \( \sigma \)-version of Painlevé V
\[ (s \sigma'')^2 = 4s(\sigma')^3 - 4s(\sigma')^2 + \sigma^2 + 2(\nu - s + 2n)\sigma \sigma' + ((\nu - s)^2 - 4sn)(\sigma')^2 \quad (9.41) \]
subject to the boundary condition
\[ \sigma(s) = \frac{(\nu + 1)n}{\Gamma(n) \Gamma(\nu + 2)} \lambda s^{\nu+1} \left( 1 - \frac{2n + \nu + 2}{\nu + 2}s + \frac{\nu(\nu + 1)^2 + 2n(\nu + \nu)(2\nu + 3)}{2(\nu + 1)^3}s^2 - \ldots \right) + O(s^{2\nu+2}). \]
Now let us find a correspondence with Tracy-Widom variables \( u(s), w(s) \) and \( R(s) \) given by (5.41-5.46) in [5]
\[ sq'(s) = \left( \frac{s - \nu}{2} - n \right) q(s) + \left( \sqrt{n(n + \nu) + u(s)} \right) p(s), \quad (9.42) \]
\[ sp'(s) = -\left( \frac{s - \nu}{2} - n \right) p(s) - \left( \sqrt{n(n + \nu) - w(s)} \right) q(s), \quad (9.43) \]
\[ sR(s) = (s - \nu + 2n)q(s)p(s) + \left( \sqrt{n(n + \nu) + u(s)} \right) p^2(s) \]
\[ + \left( \sqrt{n(n + \nu) - w(s)} \right) q^2(s), \quad (9.44) \]
\[ (sR(s))' = q(s)p(s). \quad (9.45) \]
and
\[ u'(s) = q^2(s), \quad w'(s) = p^2(s). \] (9.46)

After straightforward calculations we obtain that the equations (9.1-9.9) are consistent with (9.42-9.46) under the choice
\[ u(s) = \frac{n(\xi_1(s) + \nu) - \xi_0(s)}{\sqrt{n(n + \nu)}}, \] (9.47)
\[ w(s) = -\sqrt{n(n + \nu)\eta_0(s)}, \] (9.48)
\[ sR(s) = \sigma(s) = -\xi_2(s) = -n\eta_0(s) - \eta_1(s). \] (9.49)

10 The case \( M = 2 \)

Again we consider the case \( J = (0, s) \) with basic variables \( x, y, \eta, \) and \( \xi, j = 0, 1, 2 \) defined by (8.9). The system (7.5-7.11) is
\[ sx_0' = -(nx_0 + x_1)\eta_0 - x_1, \] (10.1)
\[ sx_1' = -(nx_0 + x_1)\eta_1 - x_2, \] (10.2)
\[ sx_2' = -(nx_0 + x_1)(\eta_2 + s) + (\xi_0x_0 + \xi_1x_1 + \xi_2x_2), \] (10.3)
\[ sy_0' = -(\xi_0 - ns)y_2 + n(\eta_0y_0 + \eta_1y_1 + \eta_2y_2), \] (10.4)
\[ sy_1' = -(\xi_1 - s)y_2 + y_0 + \eta_0y_0 + \eta_1y_1 + \eta_2y_2, \] (10.5)
\[ sy_2' = -\xi_2y_2 + y_1, \] (10.6)
\[ \xi_0' = -(nx_0 + x_1)y_0, \] (10.7)
\[ \xi_1' = -(nx_0 + x_1)y_1, \] (10.8)
\[ \xi_2' = -(nx_0 + x_1)y_2, \] (10.9)
\[ \eta_0' = -xy_2, \] (10.10)
\[ \eta_1' = -xy_2, \] (10.11)
\[ \eta_2' = -xy_2. \] (10.12)

For simplicity we assume that \( \nu_1, \nu_2 > 0 \) are generic, i.e. \( \nu_1, \nu_2, \nu_1 - \nu_2 \not\in \mathbb{Z} \). Although from the random matrix application the case of integer \( \nu_1, \nu_2 \) case is exactly the case of interest, these restrictions can be lifted in principle by use of a limiting procedure.

We start with the initial conditions for basic variables at \( s = 0 \). First, biorthogonal functions (2.3.2.4) can be expressed in terms of generalized hypergeometric functions
\[ P_n(x) = (-1)^n(n_1 + 1)(n_2 + 1)_{1F2} \left( \frac{-n}{1 + \nu_1, 1 + \nu_2} \mid x \right), \] (10.13)
\[ Q_n(x) = \frac{(-1)^n\lambda x^{\nu_1}\Gamma(\nu_2 - \nu_1)}{n!\Gamma(n_1 + 1)\Gamma(n_2 + 1)}_{1F2} \left( \frac{\nu_1 + n + 1}{1 + \nu_1, 1 + \nu_2} \mid x \right) + (\nu_1 \leftrightarrow \nu_2), \] (10.14)
where (10.13) follows from (A.6) and (10.14) is obtained by closing the contour in (2.16) to the left and summing over two series of poles. Similar to the previous section a dependence on \( \lambda \) is recovered by replacing \( s^{\nu_1} \rightarrow \lambda s^{\nu_1} \) and \( s^{\nu_2} \rightarrow \lambda s^{\nu_2} \).
After straightforward calculations we obtain

\[
x_0(s) = i(-1)^{n+1} \lambda s^{\nu_1 + 1} \frac{(\nu_1 + 2)^2}{\Gamma(n)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + i(-1)^{n+1}(\nu_1 + 1)\nu_2 + 1)(1 + O(s)) + O_{\nu_1+1, \nu_2+1},
\]

\[
x_1(s) = i(-1)^{n+1} \lambda s^{\nu_1 + 2} \frac{n(\nu_1 + 1)^2}{\Gamma(n)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + i(-1)^{n+1}n(\nu_1 + 2)\nu_2 + 1)(1 + O(s)) + O_{\nu_1+1, \nu_2+1},
\]

\[
x_2(s) = i(-1)^{n} \lambda s^{\nu_1 + 2} \frac{n(\nu_1 + 2)^2}{\Gamma(n)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + i(-1)^{n}n(\nu_1 + 2)\nu_2 + 1)(1 + O(s)) + O_{\nu_1+1, \nu_2+1},
\]

\[
y_0(s) = i\lambda s^{\nu_1 + 1} \frac{(-1)^n\Gamma(\nu_2 - \nu_1)}{\Gamma(n)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + n + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1+1, \nu_2+1},
\]

\[
y_1(s) = -i\lambda s^{\nu_1} \frac{(-1)^n\nu_2(\nu_2 - \nu_1)}{\Gamma(n + 1)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + n + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1, \nu_2},
\]

\[
y_2(s) = i\lambda s^{\nu_1} \frac{(-1)^n\Gamma(\nu_2 - \nu_1)}{\Gamma(n + 1)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + n + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1, \nu_2},
\]

\[
\eta_0(s) = -\lambda s^{\nu_1 + 1} \frac{(\nu_1 + 2)\Gamma(\nu_2 - \nu_1)}{\Gamma(n)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1+1, \nu_2+1},
\]

\[
\eta_1(s) = -\lambda s^{\nu_1 + 1} \frac{(\nu_1 + 1)\nu_2\Gamma(\nu_2 - \nu_1)}{\Gamma(n)\Gamma(\nu_1 + 3)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1+1, \nu_2+1},
\]

\[
\eta_2(s) = \lambda s^{\nu_1 + 1} \frac{(\nu_1 + 1)\nu_2\Gamma(\nu_2 - \nu_1)}{\Gamma(n)\Gamma(\nu_1 + 3)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1+1, \nu_2+1},
\]

\[
\xi_0(s) = -\lambda s^{\nu_1 + 1} \frac{(\nu_1 + 1)n\nu_2\Gamma(\nu_2 - \nu_1)}{\Gamma(n)\Gamma(\nu_1 + 3)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1+1, \nu_2+1},
\]

\[
\xi_1(s) = e_2 + \lambda s^{\nu_1 + 1} \frac{(\nu_1 + 1)n\nu_2\Gamma(\nu_2 - \nu_1)}{\Gamma(n)\Gamma(\nu_1 + 2)\Gamma(\nu_2)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1+1, \nu_2+1},
\]

\[
\xi_2(s) = -e_1 - \lambda s^{\nu_1 + 1} \frac{(\nu_1 + 1)n\nu_2\Gamma(\nu_2 - \nu_1)}{\Gamma(n)\Gamma(\nu_1 + 2)\Gamma(\nu_2 + 1)}(1 + O(s)) + (\nu_1 \leftrightarrow \nu_2) + O_{\nu_1+1, \nu_2+1},
\]

where we used a notation for higher order terms

\[
O_{\alpha, \beta} = O(s^{2\alpha}, s^{\alpha+\beta}, s^{2\beta})
\]

and as before \(e_1 = \nu_1 + \nu_2, e_2 = \nu_1\nu_2\).
For later convenience let us introduce new variables
\[ \chi_0 = n \eta_0 + \eta_1, \quad \chi_1 = n \eta_1 + \eta_2. \]  
(10.28)

The Hamiltonian
\[ H^{(n)}_2 = -(nx_0 + x_1)(sy_2 + \eta_0 y_0 + \eta_1 y_1 + \eta_2 y_2) - x_1 y_0 - x_2 y_1 + y_2(\xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2) \]  
(10.29)
leads to the first integral
\[ H^{(n)}_2 - \chi_0 = 0. \]  
(10.30)

The Proposition 8.2 together with the equation (8.22) gives three additional integrals. Similar to the case \( M = 1 \) we can combine them with (10.30) and the orthogonality condition
\[ x_0(s)y_0(s) + x_1(s)y_1(s) + x_2(s)y_2(s) = 0 \]  
(10.31)
to derive the expressions for \( \xi_i \)'s in terms of \( \eta_i \)'s. To do that we first express the variables \( y_0, y_1, x_0, x_1, x_2 \) from the equations (10.27) - (10.12) in terms of \( \xi_i \)'s and \( \eta_i \)'s and substitute into (10.30) and (8.22). The dependence on the variable \( y_2 \) drops out and after some algebra we obtain
\[ \xi_2 = \chi_0 - e_1, \]  
(10.32)
\[ \xi_1 = e_2 - (1 + e_1)\chi_0 + \chi_0^2 + s\chi_0' + \chi_1, \]  
(10.33)
\[ \xi_0 = \frac{1}{n(1 + \eta_0)} \left\{ n\chi_0(\chi_0 - 1 - \nu_1)(\chi_0 - 1 - \nu_2) + (\eta_2 - \chi_1)(\xi_1 + n(n + e_1 - \chi_0)) \right. \]
\[ + n\chi_1(\chi_0 + n - 2) - ns\chi_0'(1 + e_1 - 3\chi_0) + ns(s\chi_0'' + 2\chi_1') \}, \]  
(10.34)
where we also used the variables (10.28) to simplify final expressions.

Now the integral (10.30) will give a complicated differential equation for \( \eta_0, \eta_1 \) and \( \eta_2 \). Similar to the case \( M = 1 \) we would like to derive a closed differential equation for the \( \tau \)-function (10.31) which is expressed in terms of \( \chi_0(s) \). To do that we need another integral of the system (10.1) - (10.12). We were able to find such an additional integral and combining it with (10.30) and initial conditions (10.15), (10.20) to derive after tedious calculations a coupled system of differential equations for \( \chi_0(s) \) and \( \chi_1(s) \)
\[ \chi_1' \left[ 3\chi_1 + 3s\chi_0'' + 2\chi_0'(3\chi_0 - e_1) \right] + \chi_0'(s^2\chi_0'' + (1 - e_1)s\chi_0') + \chi_0'' + \chi_0'(3s\chi_0'' + \chi_0'(3\chi_0 - 2e_1 - 1) - 1) + (e_2 - s)\chi_0'' + 3s\chi_0''' = 0, \]  
(10.35)
\[ (n - 1)\chi_0'' \chi_0(\chi_0 - s\chi_0') + \chi_0''' \left[ (1 + e_1 + e_2 - s - (2 + e_1)\chi_0 + \chi_0^2)\chi_0 + s^2\chi_0'' \right] \]
\[ + \chi_0' \left[ 3s\chi_0'' - s(1 + e_1) \right] + 2\chi_1(1 - \chi_0')\chi_0'' + \chi_1'(\chi_1' + 3\chi_0\chi_0'' - e_1\chi_0) + s^2\chi_0''\chi_0\chi_0'' \]
\[ + \chi_1' \left[ \chi_0''(e_2 - s + 3\chi_0^2 + 3s\chi_0') - \chi_0'(1 + (1 + 2e_1)\chi_0') - s^2\chi_0'' \right] = 0. \]  
(10.36)

Currently we don’t know how to derive this additional integral algebraically in terms of isomonodromic formulation. The next natural step is to eliminate the function \( \chi_1 \) from the system (10.35), (10.36) and to obtain the differential equation for \( \chi_0 \) which coincides with the logarithmic
The gap probability \(7.26\) is given by we cannot give it here. In the hard edge scaling limit its simplified version was derived in \[17\].

Let us notice that for the case \(M = 1\) both functions \(\eta_0(s)\) and \(\eta_1(s)\) satisfy the third order differential equations but the equation for their linear combination \(\sigma(s)\) can be integrated once and gives the second order equation \[9.41\]. It is not clear whether the system \[10.35, 10.36\] can be integrated further to produce a simpler third order differential equation for some combination of \(\chi_0\) and \(\chi_1\). In the hard edge scaling limit with \(\nu_1 = -1/2, \nu_2 = 0\) such third order differential equation was found in \[17\], but it may be the case only at this special point.

With the given asymptotics at \(s \to 0\) which follow from \[10.21, 10.23\]

\[
\chi_0(s) = \alpha_0 s^{\nu_1+1} (1 + O(s)) + \beta_0 s^{\nu_2+1} (1 + O(s)) + O_{\nu_1+1, \nu_2+1},
\]

\[
\chi_1(s) = \alpha_1 s^{\nu_1+1} (s + O(s^2)) + \beta_1 s^{\nu_2+1} (s + O(s^2)) + O_{\nu_1+1, \nu_2+1},
\]

the system \[10.35, 10.36\] uniquely determines power series expansions for \(\chi_0, \chi_1\) in terms of two free parameters \(\alpha_0\) and \(\beta_0\). Parameters \(\alpha_1\) and \(\beta_1\) are fixed by

\[
\alpha_1 = \frac{(n-1)\alpha_0}{(\nu_1 + 2)(\nu_2 + 1)}, \quad \beta_1 = \frac{(n-1)\beta_0}{(\nu_1 + 1)(\nu_2 + 2)}.
\]

and the power series for \(\chi_0(s)\) and \(\chi_1(s)\) have the form

\[
\chi_0(s) = \alpha_0 s^{\nu_1+1} \left( 1 + \frac{2 + 2\nu_1 + \nu_1\nu_2 + n(2\nu_2 - \nu_1)}{(\nu_1 + 2)(\nu_2 + 1)(1 + \nu_1 - \nu_2)} s + O(s^2) \right)
\]

\[
+ \beta_0 s^{\nu_2+1} \left( 1 + \frac{2 + 2\nu_2 + \nu_1\nu_2 + n(2\nu_1 - \nu_2)}{(\nu_1 + 1)(\nu_2 + 2)(1 + \nu_2 - \nu_1)} s + O(s^2) \right)
\]

\[
- \alpha_0\beta_0 s^{\nu_1+\nu_2+2} \frac{\nu_1 + \nu_2 + 2}{(\nu_1 + 1)(\nu_2 + 1)} (1 + O(s)) + O_{\nu_1+1, \nu_2+1}. \tag{10.39}
\]

\[
\chi_1(s) = \alpha_0 (n-1)\nu_0\gamma_0^2 + \beta_0 \frac{(n-1)\gamma_0^2}{(\nu_1 + 1)(\nu_2 + 2)} (1 + O(s))
\]

\[
- \alpha_0\beta_0 s^{\nu_1+\nu_2+3} \frac{\nu_1 + \nu_2 + 4}{(\nu_1 + 1)(\nu_2 + 1)(\nu_2 + 2)} + O_{\nu_1+2, \nu_2+2}. \tag{10.40}
\]

The coefficients \(\alpha_0, \beta_0\) are fixed by the asymptotics of \(\eta_0(s)\) \[10.21\]

\[
\alpha_0 = \frac{\lambda (\nu_1 + 2)n^{-1}}{\Gamma(n)(\nu_1 + 1)} \Gamma(\nu_2 + 1), \quad \beta_0 = \frac{\lambda (\nu_2 + 2)n^{-1}}{\Gamma(n)(\nu_1 + 1)} \Gamma(\nu_2 + 1).
\]

The gap probability \[7.26\] is given by

\[
E_n^M(0; J) = \exp \left\{ \int_0^s \frac{\chi_0(t)}{t} dt \right\} \tag{10.42}
\]

and its expansion at \(s = 0\) has the form

\[
E_n^M(0; J) = 1 + \alpha_0 s^{\nu_1+1} \left( \frac{1}{\nu_1 + 1} + \frac{2 + 2\nu_1 + \nu_1\nu_2 + n(2\nu_2 - \nu_1)}{(\nu_1 + 2)(\nu_2 + 1)(1 + \nu_1 - \nu_2)} s + O(s^2) \right)
\]

\[
+ \beta_0 s^{\nu_2+1} \left( \frac{1}{\nu_2 + 1} + \frac{2 + 2\nu_2 + \nu_1\nu_2 + n(2\nu_1 - \nu_2)}{(\nu_1 + 1)(\nu_2 + 2)(1 + \nu_2 - \nu_1)} s + O(s^2) \right)
\]

\[
- \alpha_0\beta_0 s^{\nu_1+\nu_2+3} \frac{\nu_1 + \nu_2 + 2}{(\nu_1 + 1)^2(\nu_2 + 1)^2(\nu_2 + 2)^2(1 + O(s)) + O_{\nu_1+1, \nu_2+1}. \tag{10.43}
\]
11 Acknowledgments

We would like to thank V. Bazhanov and J.R. Ipsen for useful discussions and N. Witte for careful reading of the manuscript and his comments. We acknowledge support by the Australian Research Council through grant DP140102613 (PJF, VVM) and the ARC Centre of Excellence for Mathematical and Statistical Frontiers (PJF).

Appendix

In this appendix we give definitions for the generalized hypergeometric function and for the Meijer G-function and discuss some of their properties. We follow notations of [29].

The generalized hypergeometric function is defined by a power series

$$
pFq\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}.
$$

(A.1)

where the Pochhammer symbol is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{k=0}^{n-1} (a+k), \quad n \geq 0.
$$

(A.2)

We assume that $z$ is chosen in the region of convergence of (A.1). This region can be extended by a contour integral representation like for the Meijer G-function below.

The Meijer G-function is given by a contour integral

$$
G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{i=1}^{m} \Gamma(b_i - u) \prod_{i=1}^{n} \Gamma(1-a_i + u)}{\prod_{i=n+1}^{p} \Gamma(a_i - u) \prod_{i=m+1}^{q} \Gamma(1-b_i + u)} z^u du,
$$

(A.3)

where $m, n, p, q$ are integers such that $0 \leq m \leq q$, $0 \leq n \leq p$ and no pole of $\Gamma(b_j - u)$, $j = 1, \ldots, m$ coincides with any pole of $\Gamma(1 - a_k + u)$, $k = 1, \ldots, n$.

The contour $L$ runs from $-i\infty$ to $+i\infty$ separating the poles of $\Gamma(b_j - u)$, $j = 1, \ldots, m$ on the right and $\Gamma(1 - a_k + u)$, $k = 1, \ldots, n$ on the left. It can also be a loop starting and ending at $+\infty$ and encircling poles of $\Gamma(b_j - u)$ for $p < q$ or a loop starting and ending at $-\infty$ and encircling poles of $\Gamma(1 - a_k + u)$ for $p > q$.

The Meijer G-function satisfies the differential equation

$$
\left[-(1)^{p+m-n} z \prod_{j=1}^{p} (\delta_z - a_j + 1) - \prod_{j=1}^{q} (\delta_z - b_j)\right] G_{p,q}^{m,n} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| z \right) = 0.
$$

(A.4)
Let us set $m = 1$, $n = 0$, $b_1 = 0$ and assume that $b_j \notin \mathbb{Z}$, $j = 2, \ldots, q+1$ and $p \leq q+1$. Then we can evaluate the integral in (A.3) over the loop starting and ending at $\infty$ and encircling poles $\Gamma(-u)$. The sum of the residues gives the generalized hypergeometric function and we get the relation

$$pF_q \left( \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \left| z \right. \right) = \prod_{i=1}^{p} \Gamma(1 - a_i) \prod_{j=1}^{q} \Gamma(b_j) \mathcal{G}^{1,0}_{p,q+1} \left( \begin{matrix} 1 - a_1, \ldots, 1 - a_p \\ 0, 1 - b_1, \ldots, 1 - b_q \end{matrix} \left| (-1)^{p+1}z \right. \right) \quad \text{(A.5)}$$

If any of $a_i$, $i = 1, \ldots, p$ is equal to a negative integer $-n$, the hypergeometric series truncates and we get a polynomial of the degree $n$. In particular, setting $p = 1$, $q = M$ in (A.5) and comparing with [2.3] we obtain a representation of polynomials $P_n(x)$ in terms of the generalized hypergeometric function $1F_M$

$$P_n(x) = (-1)^n \prod_{j=1}^{M} (\nu_j + 1)_n \quad 1F_M \left( \begin{matrix} -n \\ 1 + \nu_1, \ldots, 1 + \nu_M \end{matrix} \left| x \right. \right) \quad \text{(A.6)}$$

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