Quantum Geometry of a Configuration Space in a Covariant Dynamical Theory

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A quantum version of the action principle in a simple covariant dynamical theory of two relativistic particles is formulated. The central object of this new formulation of quantum theory is a stationary eigenvalue of the quantum action. This quantity defines a quantum geometry in a configuration space. In the presence of "probe" fields it plays the role of a generation function of observables.

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I. INTRODUCTION

In the work \cite{1} the formulation of quantum mechanics in terms of a quantum version of the action principle was proposed. In this formulation the action functional $I$ is replaced by an operator $\hat{I}$ in a space of wave functionals. A wave functional $\Psi[q(t)]$ is defined on trajectories $q(t)$ with fixed end points in a configuration space of a dynamical system. The eigenvalue problem of the action operator $\hat{I}$ is formulated as follows:

$$\hat{I}\Psi = \Lambda \Psi. \quad (1)$$

Quantum dynamics of a system is described by an extremal eigenvalue $\Lambda_0$ and a corresponding eigenfunctional $\Psi_0$ (the quantum action principle). In the work \cite{2}, attention was paid to the fact that the new formulation is the most appropriate for covariant systems. As an example, the dynamics of a relativistic particle in the Minkowsky space was considered. In contrast to the ordinary formulation of relativistic quantum mechanics based on the Klein-Gordon equation, where the problem of probabilistic interpretation of a wave functional exists \cite{3}, the probabilistic interpretation of a wave functional $\Psi[q(t)]$ is natural. Incidentally, all symmetries of the original classical theory - the relativistic invariance and the invariance with respect to reparametrizations of a world line of a particle are preserved.

A special feature of the covariant theory is that the action, being invariant with respect to reparametrizations of a trajectory, defines a geometry which in general belongs to the class of the Finsler geometry \cite{4} in a configuration space of a system. In the case of a relativistic particle it is the Minkowsky geometry. As the consequence, quantization of the action introduces a quantum geometry of the configuration space. This geometry is defined by the extremal eigenvalue $\Lambda_0$ of the action which is a function of end points of a trajectory and, therefore, may be taken as a measure of distance. Quantum geometry of the configuration space defined in this way has direct physical meaning: it plays the role of a generation function for mean values of currents and correlators of currents, if the dynamics of system is "probed" by an external field. These quantities form the set of physical observables.

In the present work a quantum geometry of a configuration space of a system, that may be symbolically called the system of two particles, is considered. According to the structure of the algebra of constraints this dynamical system may serve as a simple model of General Relativity and string theories.

II. COVARIANT SYSTEM OF TWO PARTICLES

Classical dynamics of a system is described by a trajectory $(u^\mu(\tau), v^\mu(\tau))$ in the configuration space $M_4 \times M_4$, where $M_4$ is the four-dimensional Minkowsky space, and $\tau \in [0,1]$ is an arbitrary parameter. The dynamics is determined by the action:

$$I = \int_0^1 \left[ \frac{1}{4N_1} (u - \lambda u)^2 + \frac{1}{4N_2} (v + \lambda v)^2 + N_1 \dot{u}^2 + N_2 \dot{v}^2 + A_1^\lambda (u) \dot{u}^\mu + A_2^\lambda (v) \dot{v}^\mu \right] d\tau \quad (2)$$

where the dot denotes the derivative on the parameter $\tau$. Short notations for scalar products and squares of vectors in the Minkowsky space are used, for example, $u^2 \equiv \eta_{\mu\nu} u^\mu u^\nu$, where

$$\eta_{\mu\nu} \equiv diag(+1, -1, -1, -1) \quad (3)$$

is the metric of the Minkowsky space. Apart from the basic dynamical variables $(u^\mu, v^\mu)$, the action (2) depends on $N_{1,2}$ and $\lambda$ which are analogous to lapse and shift functions in the Arnowitt, Deser and Misner framework of General Relativity \cite{5}. Transformation properties of these variables ensure the invariance of the action with respect to reparametrizations of a trajectory. External "probe" fields $A_1^\lambda (u)$ and $A_2^\lambda (v)$ are included for the subsequent determination of observables (the charges of particles are equal to unity).
We obtain the classical geometry of the configuration space, induced by the action (2), by turning, first of all, to the geometrical form of the action. For this purpose let us exclude lapse and shift functions from the action (2), solving corresponding Euler-Lagrange (EL) equations:

\[- \frac{1}{4N_1^2} \left( \dot{u} - \lambda u \right)^2 + v^2 = 0, \]
\[- \frac{1}{4N_2^2} \left( \dot{v} + \lambda v \right)^2 + u^2 = 0, \]
\[- \frac{u (\dddot{u} - \lambda u)}{4N_1} + \frac{v (\dddot{v} + \lambda v)}{4N_2} = 0. \]

(4)

After substitution of a solution of the set (4) with respect to \(N_{1,2}, \lambda \) into (2), the action takes the geometrical form with the integrand which is homogeneous function of the first order in the velocities:

\[
I = \int_0^1 \left[ \sqrt{v^2 + (\dot{u} - \lambda u)^2} + \sqrt{u^2 + (\dot{v} + \lambda v)^2} + A^1 \dot{u} + A^2 \dot{v} \right] \, d\tau
\]

(5)

where

\[
\lambda = \frac{-b + \sqrt{b^2 - 4ac}}{2a},
\]
\[
a = (v^2)^2 \left( (\dot{u}u - u \ddot{u}^2) - (u^2)^2 \left( (\dot{v}v - v \ddot{v}^2) \right) \right),
\]
\[
b = 2 \left[ (\dot{v}v) v^2 \left( (\dot{u}u)^2 - u \ddot{u}^2 \right) \right. \]
\[
+ \left. (\dot{u}u) u^2 \left( (\dot{v}v)^2 - v \ddot{v}^2 \right) \right],
\]
\[
c = (\dot{v}v)^2 v^2 - (\dot{u}u)^2 u^2.
\]

Considering the action (5) as a measure of length of a trajectory \((u^\mu (\tau), v^\mu (\tau)) , \tau \in [0,1]\), let us define a "shortness" trajectory from the condition that the action (5) is stationary with respect to the basic variables, i.e., from the equations of motion of particles:

\[
\delta_u I = \delta_v I = 0.
\]

(7)

The equations (7) are differential equations of the second order in the parameter \(\tau\) with respect to functions \(u^\mu (\tau), v^\mu (\tau)\). Substitution of a solution of these equations at fixed end points of a trajectory, \((u_0, v_0)\) and \((u_1, v_1)\), into the action (5) determines a distance \(I_{01}\) between end points in a classical geometry of a configuration space.

Taking into account that the quantity \(I_{01}\) is a functional of "probe" fields, let us define current density vectors of particles:

\[
\frac{\delta I_{01}}{\delta A^\mu_1 (u)} \bigg|_{A=0} \equiv j^\mu_1 (u) = \int_0^1 d\tau \delta^4 \left( (u - u(\tau)) u^\mu (\tau) \right),
\]
\[
\frac{\delta I_{01}}{\delta A^\mu_2 (v)} \bigg|_{A=0} \equiv j^\mu_2 (v) = \int_0^1 d\tau \delta^4 \left( (v - v(\tau)) v^\mu (\tau) \right).
\]

(8)

where integrals are taken on classical world lines of particles. Higher order variational derivatives describe dependence of currents on external fields. These quantities contain all details of internal interactions in a system. In analogy with electrodynamics we call them correlators of currents. These quantities form the complete set of observables of the theory.

III. CANONICAL FORM OF THE ACTION

As usually, we begin the transition to a canonical form of the action, defining canonical momenta of particles:

\[
p_\mu = \frac{\partial L}{\partial u^\mu} = \frac{1}{2N_1} (u^\mu - \lambda u^\mu) + A^1_\mu, \]
\[
\pi_\mu = \frac{\partial L}{\partial v^\mu} = \frac{1}{2N_2} (v^\mu + \lambda v^\mu) + A^2_\mu.
\]

(9)

Then we obtain the Hamiltonian:

\[
H = \left( p^\mu + \pi^\mu - L \right) \bigg|_{uw} = N_1 H_1 + N_2 H_2 + \lambda D,
\]

(10)

where velocities are supposed to be excluded by use of Eq. (9), and

\[
H_1 \equiv (p - A^1)^2 - v^2 \approx 0,
\]
\[
H_2 \equiv (\pi - A^2)^2 - u^2 \approx 0,
\]
\[
D \equiv u (p - A^1) - v (\pi - A^2) \approx 0,
\]

(11)

are constraints of the first class. The wavy equalities mean that the equations (11), being the EL equations for \(N_{1,2}\) and \(\lambda\), have to be solved, but only after calculation of all necessary Poisson brackets. Among the latter are commutators of a Lee algebra of the constraints:

\[
\{H_1, H_2\} = 4D, \{D, H_1\} = 2H_1, \{D, H_2\} = -2H_2.
\]

(12)

In that case, the group of covariance is three-parametric. It includes independent reparametrizations of world lines of particles.

Canonical quantum theory of a dynamical system in ordinary formulation was considered in the work [6], where solutions of a set of wave equations:
for a wave function $\psi(u,v)$ were investigated. Operators of the constraints in the Schrödinger representation are obtained by a replacement of the canonical momenta by differential operators acting on a wave function:

$$\hat{\mathcal{H}}_1 \psi = \hat{\mathcal{H}}_2 \psi = \hat{D} \psi = 0, \quad (13)$$

However, we are forced to note, once again, that any probabilistic interpretation of solutions of the set (13) is absent. Proper probabilistic interpretation will be found in the new form of quantum theory based on the quantum action principle.

IV. QUANTUM ACTION PRINCIPLE

The central point in the modified procedure of the canonical quantization, proposed in [1], is the replacement of partial derivatives (14) acting on a wave function $\psi(u,v)$ by variational derivatives acting on a wave functional $\Psi(u(\tau),v(\tau))$. This functional also depends on the lapse functions $N_{1,2}(\tau)$, but at present the latter are assumed to be fixed. More precisely, the functional realization of basic canonical variables is as follows:

$$\hat{u}^\mu(\tau) \Psi = \partial^\mu(\tau) \Psi, \quad \hat{v}^\mu(\tau) \Psi = \partial^\mu(\tau) \Psi, \quad (15)$$

$$\hat{p}_\mu(\tau) \Psi = \frac{\hbar}{i} \frac{\delta \Psi}{\delta u^\mu(\tau)}, \quad \hat{\pi}_\mu(\tau) \Psi = \frac{\hbar}{i} \frac{\delta \Psi}{\delta v^\mu(\tau)}, \quad (16)$$

where the variational derivatives are defined as follows:

$$\delta \Psi = \int_0^1 \left[ \frac{\delta \Psi}{\delta u^\mu(\tau)} \delta u^\mu(\tau) N_1(\tau) \right. \right.$$

$$\left. + \frac{\delta \Psi}{\delta v^\mu(\tau)} \delta v^\mu(\tau) N_2(\tau) \right] d\tau. \quad (17)$$

The lapse functions $N_{1,2}$ are included into the measure of integration to ensure the covariance of the new quantum theory. The constant $\hbar$ differs from the "ordinary" Plank constant $\hbar$, its physical dimensionality is $[\hbar] = \text{Joule} \cdot \text{s}^2$.

A relationship between two constants may be established after a determination of observables and comparison between the theory and the experiment. Operators defined in this way obey the following permutation relations:

$$\left[ \hat{u}^\mu(\tau), \hat{p}_\nu(\tau') \right] = i\hbar \delta^\mu_\nu \frac{1}{N_1}(\tau - \tau'),$$

$$\left[ \hat{v}^\mu(\tau), \hat{\pi}_\nu(\tau') \right] = i\hbar \delta^\mu_\nu \frac{1}{N_2}(\tau - \tau'). \quad (18)$$

They are formally Hermitian with respect to the scalar product in the space of wave functionals:

$$\langle \Psi_1, \Psi_2 \rangle = \int_0^1 d^4 u(\tau) d^4 v(\tau) \overline{\Psi}_1[u(\tau),v(\tau)] \Psi_2[u(\tau),v(\tau)]. \quad (19)$$

The new realization of the basic dynamical variables in a space of wave functionals permits us also to determine an action operator in this space:

$$\hat{I} = \int_0^1 \left[ - \frac{\hbar}{i} \frac{\delta \Psi}{\delta u^\mu(\tau)} + \frac{\hbar}{i} \frac{\delta \Psi}{\delta v^\mu(\tau)} \right.$$

$$\left. - \left( N_1 \hat{H}_1 + N_2 \hat{H}_2 + \lambda \hat{D} \right) \right] d\tau, \quad (20)$$

where operators of the constraints are now obtained by substitution of Eqs. (15) and (16) into Eq. (11). The operator $\hat{I}$ is also formally Hermitian with respect to the scalar product (19).

Let us begin formulation of the quantum action principle. It is useful to re-formulate the eigenvalue problem (11), introducing for any wave functional $\Psi$ a functional:

$$\Lambda[u,v] = \frac{\hat{I} \Psi[u,v]}{\Psi[u,v]}. \quad (21)$$

For the wave functional the exponential representation:

$$\Psi[u,v] = \exp \left( \frac{i}{\hbar} S[u,v] + R[u,v] \right) \quad (22)$$

will be useful, in particular, to obtain the classical limit $\hbar \to \infty$. We suppose that the functionals $S[u,v], R[u,v]$ are real and analytical, i.e., they are represented by functional series, for example,

$$S[u,v] = \int_0^1 \left[ S_{1\mu}(\tau) u^\mu(\tau) N_1(\tau) + T_{1\mu}(\tau) v^\mu(\tau) N_2(\tau) \right] d\tau \right.$$

$$\left. + \frac{1}{2} \int_0^1 \int_0^1 d\tau' d\tau' \left[ S_{2\mu\nu}(\tau,\tau') u^\mu(\tau) v^\nu(\tau') N_1(\tau) N_1(\tau') \right. \right.$$

$$\left. + T_{2\mu\nu}(\tau,\tau') v^\mu(\tau) v^\nu(\tau') N_2(\tau) N_2(\tau') \right.$$ \n
$$+ V_{2\mu\nu}(\tau,\tau') u^\mu(\tau) v^\nu(\tau') N_1(\tau) N_2(\tau') + \ldots, \quad (23)$$

plus a similar representation for $R[u,v]$. The coefficients $S_{1\mu}, T_{1\mu}, S_{2\mu\nu}, T_{2\mu\nu}, V_{2\mu\nu}, \ldots$ are real functions of the parameter $\tau$, there transformation properties ensure the invariance of Eq. (23) with respect to Lorentz rotations.
Let us formulate conditions of equality of the functional \( 21 \) to an eigenvalue of the action operator. These conditions must be applied to coefficients of the series \( 23 \). First of all, an eigenvalue must be independent on internal points \((u(\tau), v(\tau))\) of a trajectory. It can only depend on the end points \((u_0, v_0)\) and \((u_1, v_1)\) which are fixed. Then, we assume that eigenvalues of the action operator are real. This gives additional conditions on the coefficients of the series. As we shall see, all these conditions give a set of first order differential equations in the parameter \( \tau \in [0, 1] \) for the coefficients \( S_{1\mu}, T_{1\mu}, S_{2\mu}, T_{2\mu}, V_{2\mu}, \ldots \). This set is an analog of the Schrödinger equation in ordinary quantum mechanics. In the framework of Cauchy problem, a solution of this set depends on the initial data at the moment \( \tau = 0 \). This solution also depends functionally on lapse and shift functions. Substituting this solution into Eq. \( 21 \), we obtain an eigenvalue of the action operator as a function (functional) of enumerated parameters. Therefore, the eigenvalue problem \( 1 \) is solved. All enumerated parameters are free and have to be fixed by an additional condition. The demand of stationarity of the eigenvalue function (functional) with respect to all free parameters gives a missing condition. Precisely this condition forms the content of the quantum action principle. We want to note, that not all of free parameters will be fixed by this stationarity condition. In this case the stationary eigenvalue \( \Lambda_0 \) of the action operator is degenerate \( 2 \). This remark also refers to lapse and shift functions.

The stationary eigenvalue \( \Lambda_0 \) depends on the end points \((u_0, v_0)\) and \((u_1, v_1)\). Therefore, it can be chosen as a measure of a distance in a configuration space. So, we arrive at the notion of quantum geometry. The corresponding eigenfunctional \( \Psi_0 [u, v] \) describes the quantum dynamics of two particles and has the natural probabilistic interpretation: \(|\Psi_0 [u, v]|^2 \) is a probability density of that a system moves along a trajectory in neighbourhood of a given trajectory \((u(\tau), v(\tau))\) \( 3 \). However, the physical interpretation of the theory can be obtained independently by use of the stationary eigenvalue \( \Lambda_0 \). Let us take into account a functional dependence of \( \Lambda_0 \) on "probe" fields \( A^{1,2} \). We shall consider the quantity \( \Lambda_0 \) as a generating function of mean values of current densities of particles and their mean correlators. In particular, quantum analogs of \( 8 \) are:

\[
\langle j_1^\mu (u) \rangle \equiv \frac{\delta \Lambda_0}{\delta A_\mu^1 (u)} \bigg|_{A=0}, \quad \langle j_2^\mu (v) \rangle \equiv \frac{\delta \Lambda_0}{\delta A_\mu^2 (v)} \bigg|_{A=0}.
\]

In the next section all parameters related the quantum action principle will be considered in the classical limit \( \hbar \to 0 \).

V. CLASSICAL LIMIT OF QUANTUM GEOMETRY

Taking into account the exponential representation of the wave functional \( 22 \), one can write the functional \( 21 \) in the classical limit as follows:

\[
\Lambda [u, v] = \int_0^1 d\tau \left\{ \delta S - u \frac{\delta S}{\delta u} + v \frac{\delta S}{\delta v} \right\} \equiv \frac{1}{N_1(\tau)} \delta (\tau - \tau') ,
\]

\[
S_{2\mu} (\tau, \tau') \equiv S_2 (\tau) \eta_{\mu
u} \frac{1}{N_2(\tau)} \delta (\tau - \tau') ,
\]

\[
T_{2\mu} (\tau, \tau') \equiv T_2 (\tau) \eta_{\mu
u} \frac{1}{N_2(\tau)} \delta (\tau - \tau') ,
\]

and \( V_{2\mu} (\tau, \tau') \equiv 0 \). Then

\[
\Lambda [u, v] = \int_0^1 \left\{ \left( u^\mu - \lambda u^\mu \right) (S_{1\mu} - S_{2\mu} u_\mu) + \left( v^\mu + \lambda v^\mu \right) (T_{1\mu} + T_{2\mu} v_\mu) \right\} d\tau.
\]

Integrating by parts the first two terms under the integral, we eliminate velocities \( u, v \). In the remaining integral, we assume that coefficients in front of the first and second orders in \( u^\mu (\tau) \) and \( v^\mu (\tau) \) are equal to zero. The latter condition gives us a set of differential equations:
\[ S_{1\mu} - \lambda S_{1\mu} + 2N_1 S_2 S_{1\mu} = 0, \]
\[ S_2 - \lambda S_2 + 2N_1 S_2^2 - 2N_2 = 0, \]
\[ T_{1\mu} - \lambda T_{1\mu} + 2N_2 T_2 T_{1\mu} = 0, \]
\[ T_2 - \lambda T_2 + 2N_2 T_2^2 - 2N_1 = 0, \]

and the remaining nonzero part of Eq. (27) becomes equal to an eigenvalue of the action operator:

\[
A = u\mu \left( S_{1\mu} + \frac{1}{2} S_{2u\mu} \right) \bigg|_0^1 + v\mu \left( T_{1\mu} + \frac{1}{2} T_{2u\mu} \right) \bigg|_0^1 - \frac{1}{2} (S_1^2 N_1 + T_2^2 N_2) \, d\tau. \tag{29}
\]

Formulation of the quantum action principle is now extremely transparent. The solution of the Cauchy problem for the system (28), being a function of initial parameters \( S_1^{(0)}, S_2^{(0)}, T_1^{(0)}, T_2^{(0)} \), and a functional of the lapse \( N_{1,2}(\tau) \) and shift \( \lambda(\tau) \) functions, must be substituted in Eq. (29). As the result, one obtains an eigenvalue of the action operator as a function (functional) of all enumerated parameters. At the final step we have to take the extremum of that function (functional) with respect to all enumerated free parameters. The stationary value \( \Lambda_0 \) of that eigenvalue function (functional) is a function of only the end points \((u_0, v_0)\) and \((u_1, v_1)\) of a trajectory. It is this function which is associated with a measure of distance in a quantum geometry of configuration space. Let us remember, that here we consider only the classical limit \( \hbar \to 0 \). The following question naturally arises: is the classical limit of the quantum geometry \( \Lambda_0 \) equal to the classical distance \( J_{01} \) defined in the first section? Within the local approximation (25) the answer is the following. The classical limit of the quantum geometry corresponds to an approximation of the classical Finsler geometry (5) for a special choice: \( \lambda = 0 \). In general case we have no answer on this question. This will be a subject of a next work. However, an evidence for this equality is that in the case of one particle the equality of two measures of distance takes place (2).

Let us return to the problem of determination of observables. "Switching on" "probe" fields modifies the previous consideration as follows. Let "probe" fields are real-analytical functions, i.e., they are represented by series:

\[
A^1_{\mu}(u) = \alpha_{0\mu}^1 + \alpha_{1\mu\nu}^1 u^{\nu} + \frac{1}{2} \alpha_{2\mu\nu\gamma}^1 u^{\nu} u^{\gamma} + \ldots,
\]
\[
A^2_{\mu}(v) = \alpha_{0\mu}^2 + \alpha_{1\mu\nu}^2 v^{\nu} + \frac{1}{2} \alpha_{2\mu\nu\gamma}^2 v^{\nu} v^{\gamma} + \ldots \tag{30}
\]

In this case, the "harmonic" approximation for functional is a function \( \Lambda_0 \). In general case the corresponding eigenvalue functional has a unique stationary value \( \Lambda_0 \) with respect to all free parameters. This quantity plays the role of a generating function of physical observables. In particular, for mean value of current density of the first particle one obtains the infinite set of equations:

\[
\frac{\delta \Lambda_0}{\delta \alpha^0_{\nu\mu}} \bigg|_{\alpha=0} = \int d^4u \left\langle j^\mu_1(u) \right\rangle ,
\]
\[
\frac{\delta \Lambda_0}{\delta \alpha^{1\mu\nu}} \bigg|_{\alpha=0} = \int d^4u \left\langle j^{1\mu}_1(u) \right\rangle u^{\nu} ,
\]
\[
\frac{\delta \Lambda_0}{\delta \alpha^{0\mu\nu\gamma}} \bigg|_{\alpha=0} = \int d^4u \left\langle j^{1\mu}_1(u) \right\rangle u^{\nu} u^{\gamma} , \ldots \tag{31}
\]

where all integrals are over \( M_4 \).

VI. CONCLUSIONS

In conclusion, the quantum action principle gives us a possibility to derive a correct quantum version of a covariant dynamical theory with a proper probabilistic interpretation. It also gives us an alternative tool for determination of observables. In the presence of "probe" fields, it is the stationary eigenvalue of the quantum action that plays the role of a generation function of currents and their correlators.

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[1] N.N. Gorobey, and A.S. Lukyanenko, arXiv: 0807.3508 (July 2008).
[2] Natalya Gorobey, and Alexander Lukyanenko, arXiv: 0812.1336 (December 2008).
[3] James D. Bjorken, Sidney D. Drell, Relativistic Quantum Mechanics (McGraw-Hill Book Company 1976).
[4] Hanno Rund, The Differential Geometry of Finsler Spaces (Springer-Verlag, Berlin-Göttingen-Heidelberg 1959).
[5] Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).

[6] Mersed Montesinos, Carlo Rovelli, and Thomas Thiemann, gr-qc/9901073 (January 1999).