Semiclassical Collapse of a Sphere of Dust and Hawking Radiation

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1. The semiclassical collapse

The total classical action for a self gravitating sphere of homogeneous dust immersed in the vacuum (Schwarzschild) space–time is taken to be a generalization of the Oppenheimer–Snyder model (c = \( K_{\text{Boltzmann}} = 1 \), \( \kappa \equiv 8 \pi G = \ell_p^2/h \)):

\[
S = -\int \frac{d\tau}{2\kappa} \left( K_e K_e^2 + \epsilon K_e \right) + \int \frac{d\tau}{2} K_e^3 \left( \phi^2 - \mu^2 \phi^2 \right),
\]

where the dust is described by the scalar field \( \phi = \phi(\tau) \) (to allow for change in the number of particles in the second quantized development), \( \mu = m/h \) is the inverse of its Compton wave length and \( K_e = K_0 \partial_n h_e(\eta) \), with

\[
h_e(\eta) = \begin{cases} 
\eta - \sin \eta & \epsilon = +1 \\
\sinh \eta - \eta & \epsilon = -1 \\
\eta^3/6 & \epsilon = 0,
\end{cases}
\]

is the conformal factor of the interior (Robertson–Walker) metric with positive, negative and zero curvature respectively,

\[
ds^2 = -d\tau^2 + K_e^2 \left[ \frac{d\rho^2}{1 - \epsilon \rho^2} + \rho^2 d\Omega^2 \right]
\]

\( 0 \leq \rho \leq \rho_o \).

The constant \( K_0 \) is related to the mass parameter \( M \) of the external Schwarzschild metric by the junction condition \( M = \rho_o^3 K_0 \) at the surface of the sphere \( r_o \equiv \rho_o K_0 \).

Canonical quantisation leads to the Wheeler–DeWitt equation for the total wave function \( \Psi(K, \phi) \)

\[
\left[ \frac{\ell_p^2}{\kappa} \frac{\partial^2}{\partial K^2} - \frac{\epsilon}{\kappa} K - \frac{\hbar^2}{K^3} \frac{\partial^2}{\partial \phi^2} + \mu^2 \phi^2 K^3 \right] \Psi = 0.
\]

One may now choose a suitable operator ordering in the gravitational kinetic term and, following a previously employed procedure (analogous to the Born–Oppenheimer approximation used in molecular dynamics), express \( \Psi \) in the factorized form

\[
\Psi(K, \phi) = K \tilde{\psi}(K) \tilde{\chi}(\phi, K),
\]

where \( \tilde{\psi} \) and \( \tilde{\chi} \) satisfy the following equations:

\[
\left[ \hat{H}_M + K \langle \hat{H}_M \rangle \right] \tilde{\psi} = -\frac{\kappa \hbar^2}{2} \left( \partial^2 K \right) \tilde{\psi},
\]

\[
\left[ \hat{H}_M - \langle \hat{H}_M \rangle \right] \tilde{\chi} + \kappa \hbar^2 \left( \frac{\partial \tilde{\psi}}{\partial K} \right) \frac{\partial \tilde{\chi}}{\partial K}
\]

\[
= \frac{\kappa \hbar^2}{2} \tilde{\psi} \left( \frac{\partial^2}{\partial K^2} - \partial^2 K \right) \tilde{\chi},
\]

and \( \langle \hat{A} \rangle \equiv \int d\phi \tilde{\chi}^*(K, \phi) \hat{A} \tilde{\chi}(K, \phi) \) for any operator \( \hat{A} \).

1.1. The dust wave function

If we consider the semiclassical (WKB) approximation to the wave function \( \tilde{\psi} \), one has

\[
\frac{\partial \ln \tilde{\psi}}{\partial K} \simeq -\frac{i}{\hbar} \frac{\partial S_{\text{eff}}}{\partial K} = -\frac{i}{\hbar} \pi_K,
\]

\[\tag{8}\]

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where $S_{eff}$ is the effective action satisfying the Hamilton–Jacobi equation associated with the L.H.S. of Eq. (3). In such a semiclassical limit $\hat{\psi}$ will be peaked at the classical trajectory $K_c$. One may then define a (conformal) time variable
\[
\frac{\partial}{\partial \eta} \equiv -i \hbar \kappa \frac{\partial \ln \hat{\psi}}{\partial K} \frac{\partial}{\partial K} .
\]
Further if the R.H.S. of Eq. (3) is small one gets from Eq. (3) the Schrödinger equation for a harmonic oscillator
\[
K_c \hat{H}_M \chi_s = i \hbar \frac{\partial \chi_s}{\partial \eta} ,
\]
where we have scaled the dynamical phase, $\chi_s \equiv \hat{\chi} \exp \left\{ -\frac{i}{\hbar} \int^\eta \langle \hat{H}_M (\eta') \rangle K_c d\eta' \right\}$. In the adiabatic approximation for $K_c$ the solutions are [3]:
\[
\tilde{\chi}_N = N_{\nu} H_N (\gamma \phi) e^{-\gamma^2 \phi^2/2} ,
\]
where $H_N$ is the Hermite polynomial of degree $N$, $N_{\nu}$ a normalization factor and $\gamma^2 \equiv \mu K_c^3 / \hbar$. The expectation value of the matter Hamiltonian on the solutions is given by
\[
K_c \langle \hat{H}_M \rangle_N = (N + 1/2) \hbar \mu K_c .
\]

1.2. The gravitational wave function

It is straightforward to derive an expression for $\hat{\psi}$ which satisfies Eq. (3) in the semiclassical limit with vanishing R.H.S. (fluctuations). Suppose that at time $\eta_0$ for which $\partial_{\eta} K_c = 0$ the gravitational wave function is a gaussian packet centred at the value $K_c(\eta_0)$ with width $b$. At a succeeding time $\eta$, $\hat{\psi}$ will be given by
\[
\hat{\psi}(K, \eta) = \int dK' G_x(K, \eta; K', \eta_0) \hat{\psi}(K'; \eta_0) ,
\]
where $G_x$ is the Green’s function for the (inverted) harmonic oscillator for $\epsilon = +1 (-1)$ or for a particle moving in a linear potential for $\epsilon = 0$. One then obtains [3]
\[
|\hat{\psi}(K, \eta)|^2 = \exp \left\{ -\alpha^2 \left( K - K_c(\eta) \right)^2 \right\} ,
\]
where
\[
\alpha = \frac{b}{\left[ \kappa^2 \hbar^2 (\partial^2_{\eta} h_e) + b^4 \left( 1 - \epsilon (\partial^2_{\eta} h_e)^2 \right) \right]^{1/2} .
\]
We further note that the total gravitational wave function $\hat{\psi}(K)$ is 0 for $K = 0$, thus giving no finite probability to fall exactly into the point–like singularity $K_c = 0$.

If one now considers the limit $\hbar \to 0$ followed by $b \to 0$ (classical point–like limit) the classical trajectory is obtained:
\[
\alpha \to \infty \quad \Rightarrow \quad \frac{K^2 |\hat{\psi}|^2}{\langle \hat{\psi} | \hat{K}^2 | \hat{\psi} \rangle} \to \delta(K - K_c) ,
\]
where $\langle \hat{\psi} | \hat{K}^2 | \hat{\psi} \rangle$ is the norm of the complete wave function $\hat{\phi}$. With the above limits interchanged, that is $b \to 0$ with $\hbar$ finite, one would obtain
\[
\alpha \to 0 \quad \Rightarrow \quad \frac{K^2 |\hat{\psi}|^2}{\langle \hat{\psi} | \hat{K}^2 | \hat{\psi} \rangle} \to 0 .
\]
However, we expect that when $b$ becomes smaller than the Planck length $\ell_p$ quantum gravitational effects (fluctuations) become significant. Hence it is more sensible to consider the limit for which one has $b \sim \ell_p$ (corresponding to a minimum size wave packet of the order of the Planck length) and then consider $\ell_p \to 0$, which again leads to Eq. (16).

From Eq. (12) and taking into account the junction condition at the surface $\rho_0$, one may conclude that the total mass $M_G \equiv M / \kappa$ of the collapsing sphere is proportional to the number of dust quanta $N$,
\[
M_G = \rho_0^3 N \hbar \mu .
\]
The above apparently differs from the result obtained previously, $M_G \sim m_p \sqrt{n}$ [3], but we observe that the $N$ appearing in Eq. (15) is an energy quantum number of the dust, while the $n$ in Bekenstein’s formula can be obtained in the present context from quantising the gravitational Hamiltonian with purely classical matter source in the $\epsilon = +1$ case only [3].

1.3. Consistency conditions

Even if $\hat{\psi}(K, \eta)$ is peaked on the classical value $K_c$, there are fluctuations around it and one must...
require Ehrenfest’s theorem to hold in order to allow for the junction condition. In our case this means that one must have

$$\Delta \equiv |(\hat{\mathcal{K}}^2) - (\hat{K})^2 |/\hat{K}^2 \ll 1 .$$

(19)

For the wave function $\tilde{\psi}$ in Eq. (14) one finds $\Delta \sim 1/\alpha^2 \to 0$ in the classical point–like limit, Eq. (16). However if one considers the limiting procedure used in Eq. (13) one has $\Delta \approx 0.18$.

It is particularly interesting to consider small black holes. Let us then take $K_c$ small at fixed $\hbar$ and $b \sim \ell_p$, one then has $\Delta \sim \ell_p/K_c$, which implies that one must have $r_o(\eta) = \rho_o \hat{K}_c(\eta) \gg \ell_p$, or, in the limit for which $r_o$ approaches $r_H \equiv 2 M$ from outside,

$$M \gg \ell_p \iff M_c \gg m_p .$$

(20)

This coincides with what one would expect from a naive quantum mechanical argument: if $\kappa \hbar/M$ is the Compton wave length of the black hole, it should be much less than its Schwarzschild radius $2 M$ for the semiclassical approximation to work.

The remaining consistency conditions are that the R.H.S.s of Eqs. (6) and (7) be much smaller than $\langle \hat{H}_M \rangle$ (for fluctuations in the number of dust particles to be negligible) and that in Eq. (9), the second term on the L.H.S. be larger than the third (for the adiabatic approximation to hold). For $N \gg 1$, the above three conditions are essentially the same and, on using Eq. (8), one obtains the following result

$$M \gg 1/\mu \iff M_c \gg m_p^2/m ,$$

(21)

which means that the Schwarzschild radius of the dust must be much larger than the Compton wave length of the dust particles. Since $m \ll m_p$ for all known elementary particles, the latter expression is a stronger condition than Eq. (20) so that, when it holds, one can safely match the interior with an external classical space–time. The fact that the adiabatic approximation is not valid if Eq. (21) is not satisfied is not surprising. Indeed in such a case fluctuations, corresponding to the creation of matter particles, are large (and the evaporation time small) (3).

2. Hawking radiation

Let us consider the outer dust shell of the sphere situated at $\rho_o$. From the point of view of a static distant observer the position of the shell is not a classical variable but is a quantum observable determined by the gravitational wave function describing the semiclassical collapse obtained in the previous section, Eq. (14).

We consider an isotropic massless scalar field $\varphi = \varphi(\rho, \eta)$ conformally coupled to gravity (10) and to a static emitter. Its Lagrangian density will be given by

$$\mathcal{L}_\varphi = -\frac{1}{2} \left[ \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{6} R \varphi^2 \right] + \int d\tau \int d\rho \int_0^\infty dK \frac{|\tilde{\psi}(K, \tau)|^2}{\langle \hat{\psi} | \hat{K}^2 | \hat{\psi} \rangle} \times \delta(\rho - \rho_o) \mathcal{Q}(\rho, \tau) \varphi(\rho, \tau) ,$$

(22)

where $R$ is the curvature scalar, $\mathcal{Q}(\rho, \tau)$ describes a particle (monopole) emitter and the factor $\delta(\rho - \rho_o) |\tilde{\psi}|^2/(\langle \hat{\psi} | \hat{K}^2 | \hat{\psi} \rangle)$ forces the interaction to be localized on the outer shell.

Let us assume the emitter has a discrete set of internal energy eigenstates described by $| E \rangle$ where $E$ is the energy observed by our distant observer. We suppose the emitter is initially in an excited state $E'_o$, decays by emitting quanta of the scalar field $\varphi$ to a state $E_o - \hbar \omega$. One may then estimate, using first order perturbation theory, the total probability amplitude $P(\omega, \bar{\eta})$ for the emitter to decay in a finite conformal time. It will be given by

$$P(\omega, \bar{\eta}) = \frac{Q_\omega}{h^2} \int_{\tau'(\eta_o)}^{\tau(\eta_o)} d\tau'' \int_{\tau'(\bar{\eta})}^{\tau(\bar{\eta})} d\tau' \times e^{i \omega (\tau'' - \tau')} D^*_\tau(\eta''; \eta'; \rho) ,$$

(23)

where $Q_\omega \equiv |\langle E_0 | \mathcal{Q}(\rho_o, 0) | E_0 - \hbar \omega \rangle|$ is the absolute value of the matrix element of the monopole between the states with energy $E_0$ and $E_0 - \hbar \omega$, $t \simeq -2 M \ln(\eta - \eta_H)$ is the Schwarzschild time measured by the static distant observer expressed in term of the conformal time, $r(\eta_H) = r_H$ and $\bar{\eta}$ is an upper cut off such that $\eta_H < \bar{\eta} < \eta_o$. Further during the interval $(\eta_o, \bar{\eta})$ the static monopole emitter near the horizon is immersed in
the gravitational wave packet associated with the last dust shell and $D^+_{\rho'}$ is the Wightman function for the isotropic conformal scalar field in the Robertson–Walker metric evaluated at the same spatial point $\rho'' = \rho' = \rho$, but with the $K_c$ dependence “smeared” by the wave function $K_\tilde{\psi}$.

One obtains a probability amplitude per unit (Schwarzschild) time equal to

$$\lim_{\tau \to +\infty} \frac{P(\omega, T)}{T} \simeq \frac{Q \omega^2 R^2(\infty)}{2 \pi \hbar} \frac{\omega}{1 - e^{-\beta \hbar \omega}}, \quad (24)$$

which is a Planck distribution with the usual Hawking temperature $T_H = 1/\beta = \hbar/(8 \pi M)$ and $R(\infty)$ is a non zero constant given by $R(\eta_H)$ with

$$R(\eta) \equiv K_c(\eta) \frac{\langle \tilde{\psi}(\eta) | \hat{K} \tilde{\psi}(\eta) \rangle}{\langle \tilde{\psi}(\eta) | K^2 \tilde{\psi}(\eta) \rangle}, \quad (25)$$

(see Ref. [2] for explicit expressions).

In the classical point–like limit, Eq. (16), $R \to 1$ and one recovers the usual field theory in the fixed Robertson–Walker–like background. However, in alternative limit, Eq. (17), one finds that the emitter decouples from the conformal field, since $R \sim \alpha \to 0$, and there is no emission within any finite time $T$,

$$P(\omega, T) \to 0. \quad (26)$$

One may speculate that this effect can be used to eliminate ultra–planckian effects. In fact, it is known that, if $\omega$ is the frequency of the emitted quanta as is measured by a distant observer, a fixed observer located near the point of emission at $r = r_o$ will measure instead a blue–shifted frequency $\omega^* = (1 - 2 M/r_o)^{-1/2} \omega$ which gives $\omega^* > m_p/\hbar = \ell_p^{-1}$ for $r_o$ sufficiently close to $r_H$. In order to create these modes, one must use an emitter localized in a region smaller than $\omega^{-1} \sim \ell_p$. So one expects that $b$ in the wave function Eq. (14) should be less than $\ell_p$ for our collapsing shell to couple with conformal quanta of ultra–planckian energies and this would correspond to the limit in Eq. (17) which in turn implies Eq. (26).

A further point worth noting is that $R(\infty)$ actually depends on $\epsilon$ through $\alpha$ (see Eq. (13)). This of course implies that the probability amplitude in Eq. (24) depends on the internal dust sphere metric and one has different emission rates depending on the internal geometry.

Let us end with a speculation: we considered the matching condition in the classical limit; if one wished to consider quantum mechanical corrections one should replace it by the continuity of the gravitational wave function inside the dust (Robertson–Walker) with that outside (Schwarzschild). This suggests the results in Eq. (24) should not change dramatically immediately outside the dust, implying a form of quantum hair leading to information on the geometry in the black hole through the intensity of the radiation.

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