THERMODYNAMICS OF THE BINARY SYMMETRIC CHANNEL

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Abstract. We study a hidden Markov process which is the result of a transmission of the binary symmetric Markov source over the memoryless binary symmetric channel. This process has been studied extensively in Information Theory and is often used as a benchmark case for the so-called denoising algorithms. Exploiting the link between this process and the 1D Random Field Ising Model (RFIM), we are able to identify the Gibbs potential of the resulting Hidden Markov process. Moreover, we obtain a stronger bound on the memory decay rate. We conclude with a discussion on implications of our results for the development of denoising algorithms.

1. Introduction

We study the binary symmetric Markov source over the memoryless binary symmetric channel. More specifically, let \( \{X_n\} \) be a stationary two-state Markov chain with values \( \{\pm 1\} \), and

\[ P(X_{n+1} \neq X_n) = p, \]

where \( 0 < p < 1 \). The binary symmetric channel will be modelled as a sequence of Bernoulli random variables \( \{Z_n\} \) with

\[ P(Z_n = -1) = \varepsilon, \quad P(Z_n = 1) = 1 - \varepsilon. \]

Finally, put

\[ Y_n = X_n \cdot Z_n \quad (1.1) \]

for all \( n \). The process \( \{Y_n\} \) is a hidden Markov process, because \( Y_n \in \{-1, 1\} \) is chosen independently for any \( n \) from an emission distribution \( \pi_{X_n} \) on \( \{-1, 1\} \): \( \pi_1 = (\varepsilon, 1 - \varepsilon) \) and \( \pi_{-1} = (1 - \varepsilon, \varepsilon) \).

The law \( \mathbb{Q} \) of the process \( \{Y_n\} \) is the push-forward of \( \mathbb{P} \times \mathbb{P}_Z \) under \( \psi : \{-1, 1\}^\mathbb{Z} \times \{-1, 1\}^\mathbb{Z} \mapsto \{-1, 1\}^\mathbb{Z} \), with \( \psi((x_n, z_n)) = x_n \cdot z_n \). We write \( \mathbb{Q} = (\mathbb{P} \times \mathbb{P}_Z) \circ \psi^{-1} \). For every \( m \leq n \), and \( y_m^n := (y_m, \ldots, y_n) \in \{-1, 1\}^{n-m+1} \), the measure of the corresponding cylindric
set is given by
\[ Q(y^m_n) := \mathbb{P}(x^m_n) \prod_{k=m}^n \mathbb{I}[y_k = x_k \cdot z_k] \]
(1.2)
\[ = \sum_{x^m_n \in \{-1,1\}^{n-m+1}} \frac{1}{2} \prod_{i=m}^{n-1} p_{x_i x_{i+1}} \cdot \#\{i \in [m,n]: x_i y_i = -1\} (1 - \varepsilon) \#\{i \in [m,n]: x_i y_i = 1\}. \]

2. Random Field Ising Model

It was observed in [12] that the probability \( Q(y^m_n) \) of a cylindric event \( \{Y_m = y_m, \ldots, Y_n = y_n\} \), \( m \leq n \), can be expressed via a partition function of a random field Ising model. We exploit this observation further. Assume \( p > 0 \) and \( \varepsilon > 0 \), and put
\[ J = \frac{1}{2} \log \frac{1 - p}{p}, \quad K = \frac{1}{2} \log \frac{1 - \varepsilon}{\varepsilon}. \]
Then for any \( (y^m_n) \in \{-1,1\}^{n-m+1} \), expression for the cylinder probability (1.2) can be rewritten as
\[ Q(y^m_n) = \frac{c_J}{\lambda_{J,K}^{n-m+1}} \sum_{x^m_n \in \{-1,1\}^{n-m+1}} \exp\left( J \sum_{i=m}^{n-1} x_i x_{i+1} + K \sum_{i=m}^n x_i y_i \right), \]
where
\[ c_J = \cosh(J), \quad \lambda_{J,K} = 2 (\cosh(J + K) + \cosh(J - K)) = 4 \cosh(J) \cosh(K). \]
The non-trivial part of the cylinder probability is the sum over all hidden configurations \( (x^m_n, \ldots, x^n_m) \):
\[ Z_{n,m}(y^m_n) := \sum_{x^m_n \in \{-1,1\}^{n-m+1}} \exp\left( J \sum_{i=m}^{n-1} x_i x_{i+1} + K \sum_{i=m}^n x_i y_i \right) \]
is in fact the partition function of the Ising model with the random field given by \( y \)'s. Applying the recursive method of [9], the partition function can be evaluated in the following fashion [1]. Consider the following functions
\[ A(w) = \frac{1}{2} \log \frac{\cosh(w + J)}{\cosh(w - J)}, \]
\[ B(w) = \frac{1}{2} \log \left[ 4 \cdot \cosh(w + J) \cosh(w - J) \right] = \frac{1}{2} \log \left[ e^{2w} + e^{-2w} + e^{2J} + e^{-2J} \right] \]
One readily checks that if \( s = \pm 1 \), then for all \( w \in \mathbb{R} \)
\[ \exp\left( sA(w) + B(w) \right) = 2 \cosh(w + sJ). \]
Now the partition function can be evaluated by summing the right-most spin. Namely, suppose \( m < n, \ y_m^n \in \{-1, 1\}^{n-m+1} \), then

\[
Z_{m,n}(y_m^n) = \sum_{x_m^{n-1} \in \{-1, 1\}^{m-n}} \exp\left( J \sum_{i=m}^{n-2} x_i x_{i+1} + K \sum_{i=m}^{n-1} x_i y_i \right) \sum_{x_n \in \{-1, 1\}} e^{x_n (J x_{n-1} + Ky_n)}
\]

\[
= \sum_{x_m^{n-1} \in \{-1, 1\}^{m-n}} \exp\left( J \sum_{i=m}^{n-2} x_i x_{i+1} + K \sum_{i=m}^{n-1} x_i y_i \right) \left\{ 2 \cosh(J x_{n-1} + Ky_n) \right\}
\]

\[
= \sum_{x_m^{n-1} \in \{-1, 1\}^{m-n}} \exp\left( J \sum_{i=m}^{n-2} x_i x_{i+1} + K \sum_{i=m}^{n-1} x_i y_i \right) \exp\left( x_{n-1} A(w^{(n)}_n) + B(w^{(n)}_n) \right)
\]

where

\[
w^{(n)}_m = Ky_n.
\]

Hence,

\[
Z_{m,n}(y_m^n) = \sum_{x_m^{n-1} \in \{-1, 1\}^{m-n}} \exp\left( J \sum_{i=m}^{n-2} x_i x_{i+1} + K \sum_{i=m}^{n-1} x_i y_i + x_{n-1} (Ky_{n-1} + A(w^{(n)}_n)) \right)
\]

\[
\times \exp\left( B(w^{(n)}_n) \right).
\]

and thus the new sum has exactly the same form, but instead of \( Ky_{n-1} \), we now have \( w^{(n)}_{n-1} = Ky_{n-1} + A(w^{(n)}_n) \). Continuing the summation over the remaining right-most \( x \)-spins, one gets

\[
Z_{m,n}(y_m^n) = 2 \cosh(w^{(n)}_m) \exp\left( \sum_{i=m+1}^{n} B(w^{(n)}_i) \right),
\]

where

\[
w^{(n)}_i = Ky_i + A(w^{(n)}_{i+1}) \quad \text{for every } i < n,
\]
equivalently, since \( A(0) = 0 \), we can define

\[
w^{(n)}_i = 0 \quad \forall i > n, \quad \text{and } w^{(n)}_i = Ky_i + A(w^{(n)}_{i+1}) \quad \forall i \leq n.
\]

Therefore, we obtain the following expressions for the cylinder and conditional probabilities

\[
Q(y_0^n) = \frac{c_J}{\lambda_{J,K}^{n+1}} \cosh(w^{(n)}_0) \exp\left( \sum_{i=1}^{n} B(w^{(n)}_i) \right),
\]

(2.2)

\[
Q(y_0^n | y_1^n) = \frac{1}{\lambda_{J,K}} \frac{\cosh(w^{(n)}_0) \exp\left( B(w^{(n)}_1) \right)}{\cosh(w^{(n)}_1)}.
\]
3. Thermodynamic formalism

Let $\Omega = A^{\mathbb{Z}_+}$, where $A$ is a finite alphabet, be the space of one-sided infinite sequences $\omega = (\omega_0, \omega_1, \ldots)$ in alphabet $A$ ($\omega_i \in A$ for all $i$). We equip $\Omega$ with the metric

$$d(\omega, \tilde{\omega}) = 2^{-k(\omega, \tilde{\omega})},$$

where $k(\omega, \tilde{\omega}) = 1$ if $\omega_0 \neq \tilde{\omega}_0$, and $k(\omega, \tilde{\omega}) = \max\{k \in \mathbb{N} : \omega_i = \tilde{\omega}_i \ \forall \ i = 0, \ldots, k - 1\}$, otherwise. Denote by $S : \Omega \to \Omega$ the left shift:

$$(S\omega)_i = \omega_{i+1} \text{ for all } i \in \mathbb{Z}_+.$$ 

Borel probability measure $\mathbb{P}$ is translation invariant if

$$\mathbb{P}(S^{-1}C) = \mathbb{P}(C)$$

for any Borel event $C \subseteq \Omega$.

Let us recall the following well-known definitions:

**Definition 3.1.** Suppose $\mathbb{P}$ is a fully supported translation invariant measure on $\Omega = A^{\mathbb{Z}_+}$, where $A$ is a finite alphabet.

(i) The measure $\mathbb{P}$ is called a $g$-measure, if for some positive continuous function $g : \Omega \to (0, 1)$ satisfying the normalization condition

$$\sum_{\omega_0 \in A} g(\omega_0, \omega_1, \omega_2, \ldots) = 1$$

for all $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$, one has

$$\mathbb{P}(\omega_0|\omega_1, \omega_2, \ldots) = g(\omega_0, \omega_1, \ldots)$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$.

(ii) The measure $\mathbb{P}$ is Bowen-Gibbs for a continuous potential $\phi : \Omega \to \mathbb{R}$, if there exist constants $P \in \mathbb{R}$ and $C \geq 1$ such that for all $\omega \in \Omega$ and every $n \in \mathbb{N}$

$$\frac{1}{C} \leq \frac{\mathbb{P}(\{\tilde{\omega} \in \Omega : \tilde{\omega}_0 = \omega_0, \ldots, \tilde{\omega}_{n-1} = \omega_{n-1}\})}{\exp((S_n\phi)(\omega) - nP)} \leq C,$$

where $(S_n\phi)(\omega) = \sum_{k=0}^{n-1} \phi(\sigma^k\omega)$.

(iii) The measure $\mathbb{P}$ is called an equilibrium state for continuous potential $\phi : \Omega \to \mathbb{R}$, if $\mathbb{P}$ attains maximum of the following functional

$$h(\mathbb{P}) + \int \phi \, d\mathbb{P} = \sup_{\mathbb{P} \in \mathcal{M}^*_1(\Omega)} \left[ h(\mathbb{P}) + \int \phi \, d\mathbb{P} \right],$$

where $h(\mathbb{P})$ is the Kolmogorov-Sinai entropy of $\mathbb{P}$ and the supremum is taken over the set $\mathcal{M}^*_1(\Omega)$ of all translation invariant Borel probability measures on $\Omega$.

It is known that every $g$-measure $\mathbb{P}$ is also an equilibrium state for $\log g$; and that every Bowen-Gibbs measure $\mathbb{P}$ for potential $\phi$ is an equilibrium state for $\phi$ as well.
Theorem 3.1. The measure $\mathcal{Q}$ on $\{-1, 1\}^{Z_+}$ (c.f., (2.2)) is a $g$-measure for some positive continuous function $g$ with an exponential decay of variation:

$$\text{var}_n(g) := \sup_{y, \tilde{y}: y_0^{n-1} = \tilde{y}_0^{n-1}} |g(y) - g(\tilde{y})| \leq C\rho^n,$$

where $C > 0$ and $\rho \in (0, 1)$. The measure $\mathcal{Q}$ is also a Bowen-Gibbs measure for a Hölder continuous potential $\phi : \{-1, 1\}^{Z_+} \to \mathbb{R}$.

The result of Theorem 3.1 is actually true in much greater generality: namely, for distributions of Hidden Markov Chains $\{Y_n\}$, where the underlying Markov chain $\{X_n\}$ has strictly positive transition probability matrix $P$, see [13] for review of several results of this nature. However, the present situation is rather exceptional since one is able to identify the $g$-function and the Gibbs potential $\phi$ explicitly. Another interesting question is the estimate of the decay rate $\rho$. In [13] a number of previously known estimates of the rate of exponential decay in (3.2) have been compared; the best known estimate for $\rho$

$$\rho \leq |1 - 2p|$$

is due to [7] and [6]. Quite surprisingly the estimate does not depend on $\varepsilon$, and in fact, it was conjectured in [13] that the estimate could be improved, e.g., by incorporating dependence on $\varepsilon$. The proof of Theorem 3.1 shows that this is indeed the case and one obtains a new estimate

$$\rho \leq \rho^*(p, \varepsilon) < |1 - 2p|.$$

We start with the following technical result.

Lemma 3.2. Fix $y = (y_0, y_1, \ldots) \in \{-1, 1\}^{Z_+}$. For every $n \in \mathbb{Z}_+$, define the sequence $w_i^{(n)} = w_i^{(n)}(y)$, $i \in \mathbb{Z}_+$, by letting $w_i^{(n)} = 0$ for every $i \geq n + 1$ and $w_i^{(n)} = Ky_i + A(w_{i+1}^{(n)})$ for $i \leq n$. Then for every $i \in \mathbb{Z}_+$

$$\lim_{n \to \infty} w_i^{(n)} =: w_i(y).$$

Moreover, there exist constants $q \in (0, 1)$ and $C > 0$, both independent of $y$, such that

$$|w_i^{(n)}(y) - w_i(y)| \leq Cq^n \quad \text{for all } n \geq i,$$

and therefore, $w_i : \{-1, 1\}^{Z_+} \to \mathbb{R}$ is Hölder continuous for every $i \in \mathbb{Z}_+$:

$$|w_i(y) - w_i(\tilde{y})| = |w_0(S^i y) - w_0(S^i \tilde{y})| \leq C' \left(d(S^i y, S^i \tilde{y})\right)\theta,$$

for some $C', \theta > 0$ and all $y, \tilde{y} \in \{-1, 1\}^{Z_+}$.

Proof. Suppose $i \leq n \leq m$. Then

$$|w_i^{(n)} - w_i^{(m)}| = |A(w_i^{(n)}(y) - A(w_i^{(m)}(y))| \leq |w_i^{(n)} - w_i^{(m)}| \cdot \sup_w \left|\frac{dA}{dw}\right|.$$
and hence

\[ (3.4) \quad \rho := \sup_w \left| \frac{dA}{dw} \right| = \left| \frac{\sinh(2J)}{\cosh(2J) + 1} \right| = |\tanh(J)| = |1 - 2p| < 1. \]

Combined with the fact that for all \( i \in \mathbb{Z}_+ \)

\[ |w_i^{(m)}| = |Ky_i + A(w_{i+1}^{(m)})| \leq |K| + |\arctanh(1 - 2p)| \leq |K| + |J| =: C_1. \]

Therefore for \( i \leq n \leq m \)

\[ |w_i^{(n)} - w_i^{(m)}| \leq \rho^{-i} |w_i^{(n)} - w_i^{(m+1)}| = \rho^{-i} |w_{i+1}^{(m)}| \leq C_1 \rho^{-i}. \]

Hence, \( \lim_{n \to \infty} w_i^{(n)} =: w_i \) exists and

\[ |w_i^{(n)} - w_i| \leq \sum_{m=n}^{\infty} |w_i^{(m)} - w_i^{(m+1)}| \leq C_1 \sum_{m=n}^{\infty} \rho^{-m} = C_1 \frac{1}{1 - \rho} \rho^{-i} =: C \rho^{-i}. \]

The estimate in (3.4) can be improved. Firstly, assume that \( \varepsilon < p \). In this case, \( |K| > |J| > 0 \), and if \( i \leq n \), then

\[ w_i^{(n)} \in [-|K| - |J|, -|K| + |J|] \cup [|K| - |J|, |K| + |J|], \]

i.e., \( |w_i^{(n)}| \) is bounded away from 0 (see Figure 1.(a)). Therefore, we can define \( \rho \) by

\[ \rho = \rho(J, K) = \rho(p, \varepsilon) = \sup_{w \in [K - |J|, K + |J|]} \left| \frac{dA}{dw} \right| = \left| \frac{\sinh(2J)}{\cosh(2J) + \cosh(2K - 2J)} \right| \]

\[ = \frac{\varepsilon(1 - \varepsilon)}{p^2 - 2p\varepsilon + \varepsilon} \leq \frac{\varepsilon(1 - \varepsilon)}{(p - \varepsilon)^2 + \varepsilon(1 - \varepsilon)} |1 - 2p| < |1 - 2p|. \]

Figure 1. Graphs of \( F_+(w) = K + A(w) \), \( F_-(w) = -K + A(w) \) for (a) \( \varepsilon < p \leq 0.5 \) and (b) \( p \leq \varepsilon \leq 0.5 \).

If \( \varepsilon > p \) (equivalently, \( K < J \)), then the maps \( F_+(w) = K + A(w) \) and \( F_-(w) = -K + A(w) \) have no longer disjoint images (c.f., Figure 1.(b)). Nevertheless, one can
consider second iterations:

\[ |w_i^{(n)} - w_i^{(m)}| = |A(w_i^{(n)} + 1) - A(w_i^{(m)})| = |A(Ky_{i+1} + A(w_i^{(n)}) - A(Ky_{i+1} + A(w_i^{(m)})| \]

\[ \leq (\sup_w |A(K + A(w))A'(w)|) =: \rho^{(2)}|w_i^{(n)} - w_i^{(m)}| \]

One can show that

\[ (3.5) \quad \rho^{(2)} = \sup_w |A(K + A(w))A'(w)| < (1 - 2p)^2. \]

Informally, it is evident that the maximal value of the derivative \(|A'|\), equal to \(|1 - 2p|\), is attained, if \(w = 0\) or if \(K + A(w) = 0\), but then \(K + A(w) \neq 0\) or \(w \neq 0\), respectively, and hence (3.5) holds. Similar argument generalises to all \(w\): firstly, note that

\[ (3.6) \quad |A'(K + A(w))A'(w)| = \frac{(1 - 2p)^2}{(\alpha + (1 - \alpha) \cosh(2K + 2A(w))) \cdot (\alpha + (1 - \alpha) \cosh(2w))}, \]

where \(\alpha = (1 - p)^2 + p^2, 1 - \alpha = 2p(1 - p)\). Let \(\Delta > 0\) be such that for all \(w \in [-\Delta, \Delta]\) one has \(|A(w)| < |K|/2\) and \(\cosh(2K + 2A(w)) > \cosh(K)\), and hence

\[ |A'(K + A(w))A'(w)| \leq \frac{(1 - 2p)^2}{(\alpha + (1 - \alpha) \cosh(K))} < (1 - 2p)^2. \]

For \(w \notin [-\Delta, \Delta]\), one has

\[ |A'(K + A(w))A'(w)| \leq \frac{(1 - 2p)^2}{1 \cdot (\alpha + (1 - \alpha) \cosh(\Delta))} < (1 - 2p)^2. \]

Hence,

\[ \rho^{(2)} = \min \left\{ \frac{(1 - 2p)^2}{(\alpha + (1 - \alpha) \cosh(K))}, \frac{(1 - 2p)^2}{(\alpha + (1 - \alpha) \cosh(\Delta))} \right\} < (1 - 2p)^2, \]

and hence \(g = \sqrt{\rho^{(2)}} < |1 - 2p|\). Sharper bounds can be achieved by studying minimum of the denominator in (3.6).

**Proof of Theorem 3.1.** To show that \(Q\) is a \(g\)-measure it is sufficient to show that conditional probabilities \(Q(y_0|y_1^n)\) converge uniformly as \(n \to \infty\). Given that

\[ (3.7) \quad Q(y_0|y_1^n) = \frac{1}{\mu_{J,K}} \frac{\cosh(w_0^{(n)}) \exp(B(w_1^n))}{\cosh(w_1^n)}, \]

and using the result of Lemma 3.2: \(w_i^n(y) \Rightarrow w_i(y)\) as \(n \to \infty\), we obtain uniform convergence of conditional probabilities, and hence, \(Q\) is a \(g\)-measure with \(g\) given by

\[ (3.8) \quad g(y) = \frac{1}{\mu_J} \frac{\cosh(w_0(y)) \exp(B(w_1(y)))}{\cosh(w_1(y))}. \]

Let us introduce the following functions: for \(y \in \{-1, 1\}^Z\), put

\[ \phi(y) = B(w_0(y)), \quad h(y) = \cosh(w_0(y)) \exp(-B(w_0(y))). \]
Taking into account that \( w_1(y) = w_0(Sy) \), one has
\[
g(y) = \frac{e^{\phi(y)} h(y)}{\lambda_{J,K} h(Sy)}.
\]
Since every \( g \)-measure is also an equilibrium state for \( \log g \), we conclude that \( \mathbb{Q} \) is an equilibrium state for
\[
\tilde{\phi}(y) = \phi(y) + \log h(y) - \log h(Sy) - \log \lambda_{J,K}.
\]
The difference \( \tilde{\phi}(y) - \phi(y) \) has a very special form: it is a sum of a so-called coboundary \((\log h(y) - \log h(Sy))\) and a constant \((-\log \lambda_{J,K})\). Two potentials whose difference is of a such form, have identical sets of equilibrium states. The reason is that for any translation invariant measure \( \mathbb{Q}' \) one has
\[
\int (\log h(y) - \log h(Sy) - \log \lambda_{J,K}) d\mathbb{Q}' = -\log \lambda_{J,K} = \text{const}.
\]
Therefore, if \( \mathbb{Q}' \) achieves maximum in the righthand side of (3.1) for \( \tilde{\phi} \), then \( \mathbb{Q}' \) achieves maximum for \( \phi \) as well. Thus \( \mathbb{Q} \) is also an equilibrium state for
\[
\phi(y) = B(w_0(y)) = \frac{1}{2} \log \left[ 4 \sinh^2(w_0(y)) + \frac{1}{p(1-p)} \right].
\]
Any equilibrium measure for a Hölder continuous potential \( \phi \) is also a Bowen-Gibbs measure [3]. In our particular case, direct proof of the Bowen-Gibbs property for \( \mathbb{Q} \) is straightforward. Indeed, using the result of (2.2) and the notation introduced above, for every \( y = (y_0, y_1, \ldots) \) one has
\[
\mathbb{Q}(y_0^n) = \frac{c_J}{\lambda_{J,K}^{n+1}} \exp \left( \sum_{i=1}^{n} B(w_i^{(n)}(y)) \right) \cosh(w_0^{(n)}(y))
= \frac{c_J \cdot \cosh(w_0^{(n)}(y))}{\exp(B(w_0(y)))} \exp \left( \sum_{i=1}^{n} [B(w_i^{(n)}(y)) - B(w_i(y))] \right)
\times \exp \left( \sum_{i=0}^{n} B(w_i(y)) - (n + 1) \log \lambda_{J,K} \right).
\]
Therefore, for \( P = \log \lambda_{J,K} \),
\[
\mathbb{Q}(y_0^n) \frac{\exp((S_{n+1}\phi)(y) - (n + 1)P)}{\exp(B(w_0(y)))} = \frac{c_J \cdot \cosh(w_0^{(n)}(y))}{\exp(B(w_0(y)))} \exp \left( \sum_{i=1}^{n} [B(w_i^{(n)}(y)) - B(w_i(y))] \right)
\]
It only remains to demonstrate that the right hand side is uniformly bounded (both in \( n \) and \( y = (y_0, y_1, \ldots) \)) from below and above by some positive constants \( C_-, C_+ \), respectively. Indeed, since \( p, \varepsilon > 0 \), \( I = [-|K| - |J|, |K| + |J|] \) is a finite interval, by the result of the previous Lemma, \( w_i^{(n)}(y) \in I \) for all \( i \) and \( n \). Using (3.3), one readily checks that the
following choice of constants suffices:

\[
C = c_j \sup_{w \in I} \cosh(w) \exp\left(\frac{C}{1 - \eta} \sup_{w \in I} \left| \frac{dB}{dw} \right| \right) < \infty,
\]

\[
C = c_j \inf_{w \in I} \cosh(w) \exp\left(-\frac{C}{1 - \eta} \sup_{w \in I} \left| \frac{dB}{dw} \right| \right) > 0.
\]

We complete this section with a curious continued fraction representation of the \(g\)-function (3.8).

**Proposition 3.3.** For every \(y = (y_0, y_1, \ldots) \in \{-1, 1\}^{\mathbb{Z}_+}\), one has

\[
2g(y) = a_1 - \frac{b_1}{a_2 - \frac{b_2}{a_3 - \frac{b_3}{a_4 - \ldots}}},
\]

where for \(i \geq 1\)

(3.9) \[q_i = (1 - 2p)y_{i-1}y_i, \quad a_i = 1 + q_i, \quad b_i = 4\varepsilon(1 - \varepsilon)q_i.\]

**Proof.** Using elementary transformations, one can show that for every \(y = (y_0, y_1, \ldots) \in \{-1, 1\}^{\mathbb{Z}_+}\) one has

\[
g(y) = \frac{1}{\lambda_{J,K}} \frac{\cosh(w_0(y))}{\cosh(w_1(y))} \exp\left( B(w_1(y)) \right)
\]

(3.10) \[
= \frac{1}{2} + \frac{1}{2}(1 - 2p)(1 - 2\varepsilon)y_0 \tanh(w_1(y))
\]

Since

\[\tanh(A(w)) = \tanh(J) \tanh(w) = (1 - 2p) \tanh(w) \quad \text{for all } w \in \mathbb{R},\]

for every \(i \in \mathbb{Z}_+\), one has

\[
\tanh(w_i) = \frac{\tanh(Ky_i) + \tanh(A(w_{i+1}))}{1 + \tanh(Ky_i) \cdot \tanh(A(w_{i+1}))}
= \frac{(1 - 2\varepsilon)y_i + (1 - 2p) \tanh(w_{i+1})}{1 + (1 - 2\varepsilon)(1 - 2p)y_i \tanh(w_{i+1})}
= y_i \frac{(1 - 2\varepsilon) + (1 - 2p)y_i \tanh(w_{i+1})}{1 + (1 - 2\varepsilon)(1 - 2p)y_i \tanh(w_{i+1})},
\]

Therefore, if we let \(z_i = (1 - 2p)(1 - 2\varepsilon)y_{i-1} \tanh(w_{i}), i \in \mathbb{N}\), then

\[
z_i = (1 - 2p)y_{i-1}y_i - \frac{4\varepsilon(1 - \varepsilon)(1 - 2p)y_{i-1}y_i}{1 + z_{i+1}} = q_i - \frac{b_i}{1 + z_{i+1}}.
\]

Since \(g(y) = \frac{1}{2} + \frac{1}{2}z_1\), we obtain the continued fraction expansion (3.3) \(\square\)
4. Two-sided conventional probabilities and denoising

In the previous section we established that $Q$ is a Bowen-Gibbs measure. The notion of a Gibbs measure originates in Statistical Mechanics, and is not equivalent to the Bowen-Gibbs definition. In Statistical Mechanics, one is interested in two-sided conditional probabilities

$$Q(y_0 | y_{-m}, y_1^n)$$

or

$$Q(y_0 | y_{<0}, y_{>0}) := Q(y_0 | y_{-\infty}, y_1^\infty).$$

The method of section 2 can be used to evaluate continual probabilities $Q(y_0 | y_{-m}, y_1^n)$, $m, n > 0$ for $y = (\ldots, y_{-1}, y_0, y_1, \ldots) \in \{-1, 1\}^Z$. Indeed,

$$Q(y_0 | y_{-m}, y_1^n) = \frac{Q(y_{-m}, y_0, y_1^n)}{Q(y_{-m}, y_0, y_1^n) + Q(y_{-m}, y_0, y_1^n)},$$

where $\bar{y}_0 = -y_0$. We can evaluate

$$Q(y_{-m}, \ldots, y_{-1}, y_0, y_1, \ldots, y_n) = \frac{c_j}{\lambda_{J, K}^{n+m+1}} \sum_{x_{-m}, \ldots, x_{0}} \exp \left( J \sum_{i=-m}^{n-1} x_i x_{i+1} + K \sum_{i=-m}^{n} x_i y_i \right)$$

by first summing over spins on the right: $x_n, \ldots, x_1$, and then summing over spins on the left: $x_{-m}, \ldots, x_{-1}$. One has

$$Z_{-m,n}(y_{-m}) = \sum_{x_{-m}, \ldots, x_{0}} \exp \left( J \sum_{i=-m}^{-1} x_i x_{i+1} + K \sum_{i=-m}^{0} x_i y_i + x_0 A(w_{i}^{(n)}) \right) \exp \left( \sum_{i=1}^{n} B(w_{i}^{(n)}) \right)$$

$$= \exp \left( \sum_{j=-m}^{-1} B(w_{j}^{(-m)}) \right) 2 \cosh(w_{0}^{(-m,n)}) \exp \left( \sum_{i=1}^{n} B(w_{i}^{(n)}) \right)$$

where now $w_{(-m)} = K y_{-m}$, $w_{j+1}^{(-m)} = K y_{j+1} + A(w_{j}^{(-m)})$, $j = -m, \ldots, -2$. and

$$w_{0}^{(-m,n)} = K y_0 + A(w_{-1}^{(-m)}) + A(w_{1}^{(n)}).$$

Therefore,

$$Q(y_0 | y_{-m}, y_1^n) = \frac{Z_{-m,n}(y_{-m}, y_0, y_1^n)}{Z_{-m,n}(y_{-m}, y_0, y_1^n) + Z_{-m,n}(y_{-m}, y_0, y_1^n)}$$

$$= \frac{\cosh(K y_0 + A(w_{-1}^{(-m)}) + A(w_{1}^{(n)}))}{\cosh(K y_0 + A(w_{-1}^{(-m)}) + A(w_{1}^{(n)}) + \cosh(-K y_0 + A(w_{-1}^{(-m)}) + A(w_{1}^{(n)})))}.$$
Thus the two sided conditional probabilities are also regular, c.f. Theorem 3.1.

4.1. Denoising. Reconstruction of signals corrupted by noise during the transmission is one of the classical problems in Information Theory. Suppose we observe a sequence \( \{y_n\} \), \( n = 1, \ldots, N \), given by (1.1), i.e.,

\[
y_n = x_n \cdot z_n,
\]

where \( \{x_n\} \) is some unknown realisation of the Markov chain, and \( \{z_n\} \) is unknown realisation of the Bernoulli sequence \( \{Z_n\} \). The natural question is, given the observed data \( y^N = (y_1, \ldots, y_N) \), what is the optimal choice of \( \hat{X}_n = \hat{X}_n(y^N) \) – the estimate of \( X_n \), such that the empirical zero-one loss (bit error rate)

\[
L_N = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}[\hat{X}_n \neq x_n].
\]

is minimal. The corresponding standard maximum a posteriori probability (MAP) estimator (denoiser) is given by

\[
\hat{X}^n = \hat{X}^n(y^N) = \arg \max_{x \in \{-1, 1\}} \mathbb{P}[X_n = x \mid Y^N = y^1], \quad n = 1, \ldots, N.
\]

In case, parameters of the Markov chain (i.e., \( P \)) and of the channel (i.e., \( \Pi \)) are known, conditional probabilities \( \mathbb{P}[X_n = x \mid Y^N = y^N] \) can be found using the \textbf{backward-forward algorithm}. Namely, one has

\[
\mathbb{P}[X_n = x \mid Y^N = y^N] = \frac{\alpha_n(x)\beta_n(x)}{\sum_{\tilde{x} \in A} \alpha_n(\tilde{x})\beta_n(\tilde{x})}
\]

where

\[
\alpha_n(x) = \mathbb{P}[Y_1^n = y_1^n, X_n = x], \quad \beta_n(x) = \mathbb{P}[Y_{n+1}^N = y_{n+1}^N \mid X_n = x]
\]

are the so-called forward and backward variables, satisfying simple recurrence relations:

\[
\alpha_{n+1}(x) = \sum_{\tilde{x} \in A} \alpha_n(\tilde{x}) \cdot P_{\tilde{x}, x} \cdot \Pi_{x, y_{n+1}}, \quad n = 1, \ldots, N - 1, \quad \text{with} \quad \alpha_1(x) = \mathbb{P}(X_1 = x)\Pi_{x, y_1},
\]

\[
\beta_{n}(x) = \sum_{\tilde{x} \in A} \beta_{n+1}(\tilde{x}) \cdot P_{x, \tilde{x}} \cdot \Pi_{\tilde{x}, y_{n+1}}, \quad n = 1, \ldots, N - 1, \quad \text{with} \quad \beta_N(x) = 1.
\]

The key observation of [10] is that the probability distribution \( \mathbb{P}[X_n = \cdot \mid Y^N = y^N] \), viewed as a column vector, can be expressed in terms of two-sided conditional probabilities \( \mathbb{Q}[Y_n = \cdot \mid Y^{N\setminus n} = y^{N\setminus n}] \), with \( N \setminus n = \{1, \ldots, N\} \setminus \{n\} \), as follows

\[
\mathbb{P}[X_n = \cdot \mid Y^N = y^N] = \frac{\pi_{y_n} \circ \Pi^{-1} \mathbb{Q}[Y_n = \cdot \mid Y^{N\setminus n} = y^{N\setminus n}]}{\langle \pi_{y_n} \circ \Pi^{-1} \mathbb{Q}[Y_n = \cdot \mid Y^{N\setminus n} = y^{N\setminus n}], 1 \rangle},
\]

where \( \Pi \) is the emission matrix, and \( \pi_{-1}, \pi_1 \) are the columns of \( \Pi \):

\[
\Pi = \begin{bmatrix} 1 - \epsilon & \epsilon \\ \epsilon & 1 - \epsilon \end{bmatrix}, \quad \pi_{-1} = \begin{bmatrix} 1 - \epsilon \\ \epsilon \end{bmatrix}, \quad \pi_1 = \begin{bmatrix} \epsilon \\ 1 - \epsilon \end{bmatrix}, \quad \Pi^{-1} = \frac{1}{1 - 2\epsilon} \begin{bmatrix} 1 - \epsilon & -\epsilon \\ -\epsilon & 1 - \epsilon \end{bmatrix}.
\]
and \( \odot \) is componentwise product of vectors of equal lengths,

\[ u \odot v = (u_1 \cdot v_1, \ldots, u_d \cdot v_d). \]

Expression (4.2) opens a possibility of constructing denoisers when parameters of the underlying Markov chains are unknown; we continue to assume that the channel remains known. Indeed, two-sided conditional probabilities \( \bar{Q}(Y_n = c | Y_{n}^{N\backslash n} = y^{N\backslash n}) \) could be estimated from the data. The Discrete Universal Denoiser (DUDE) [10] algorithm estimates conditional probabilities

\[
\bar{Q}(Y_n = c | Y_{n-k_N}^{n-1} = a_{-k_N}, Y_{n+1+k_N}^{n+1} = b_{1}^{k_N}) = \frac{m(a_{-k_N}, c, b_{1}^{k_N})}{\sum_c m(a_{-k_N}, c, b_{1}^{k_N})}
\]

where \( m(a_{-k_N}, c, b_{1}^{k_N}) \) is the number of occurrences of the word \( a_{-k_N}b_{1}^{k_N} \) in the observed sequence \( y^N = (y_1, \ldots, y_N) \); the length of right and left contexts is set to \( k_N = c \log N, \) \( c > 0 \). DUDE has shown excellent performance in a number of test cases. In particular, in case of the binary memoryless channel and the symmetric Markov chain, considered in this paper, performance in comparable to the one of the backward-forward algorithm (4.1), which requires full knowledge of the source distribution, while DUDE is completely oblivious in that respect. In our opinion, excellent performance of DUDE in this case is partially due to the fact that \( \bar{Q} \) is a Gibbs measure, admitting smooth two-sided conditional probabilities, which are well approximated by (4.3) and thus can be estimated from the data. It will be interesting to evaluate performance in cases when the output measure is not Gibbs.

Invention of DUDE sparked a great interest in two-sided approaches to information-theoretic problems. It turns out that despite the fact the efficient algorithms for estimation of one-sided models exist, the analogous two-sided problem is substantially more difficult. As alternatives to (4.3), other methods to estimate two-sided conditional probabilities have been suggested, e.g., [5, 8, 11]. For example, Yu and Verdú [11] proposed a **Backward-Forward Product (BFP)** model:

\[
\bar{Q}(y_0 | y_{<0}, y_{>0}) \propto \bar{Q}(y_0 | y_{<0})\bar{Q}(y_0 | y_{>0}),
\]

and the one-sided conditional probabilities \( \bar{Q}(y_0 | y_{<0}), \bar{Q}(y_0 | y_{>0}) \) can be estimated using standard one-sided algorithms. Note, that in our model,

\[
\frac{\bar{Q}(y_0 | y_{<0})\bar{Q}(y_0 | y_{>0})}{\bar{Q}(y_0 | y_{<0})\bar{Q}(y_0 | y_{>0}) + \bar{Q}(y_0 | y_{<0})\bar{Q}(y_0 | y_{>0})} = \frac{\cosh(Ky_0 + A(w_{-1})) \cosh(Ky_0 + A(w_1))}{\cosh(Ky_0 + A(w_{-1})) \cosh(Ky_0 + A(w_1)) + \cosh(-Ky_0 + A(w_{-1})) \cosh(-Ky_0 + A(w_1))}
\]

in general does not coincide with

\[
\frac{\cosh(Ky_0 + A(w_{-1}) + A(w_1))}{\cosh(Ky_0 + A(w_{-1}) + A(w_1)) + \cosh(-Ky_0 + A(w_{-1}) + A(w_1))} = \bar{Q}(y_0 | y_{<0}, y_{>0}).
\]

Nevertheless, the BFP model seems to perform extremely well [11].
Among other alternatives, let us mention the possibility to extend standard one-sided algorithms to produce algorithms for estimating two-sided conditional probabilities from data. This approach is investigated in [2], where the denoising performance of the resulting Gibbsian models is evaluated. Gibbsian algorithm performs better than DUDE: bit error rates are given in the table below for noise level $\epsilon = 0.2$ and various values of $p$ (smaller rates are better).

| $p$  | Gibbs  | DUDE   |
|------|--------|--------|
| 0.05 | 5.30%  | 5.58%  |
| 0.10 | 9.91%  | 10.48% |
| 0.15 | 13.20% | 13.77% |
| 0.20 | 18.34% | 18.77% |

One could also try to estimate the Gibbsian potential directly, e.g., using the estimation procedure proposed in [4]. This method showed promising performance in experiments on language classification and authorship attribution. In conclusion, let us also mention that the direct two-sided Gibbs modeling of stochastic processes opens possibilities for applying semi-parametric statistical procedures, as opposed to the universal (parameter free) approach of DUDE.

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