Long-lived quantum coherence of two-level spontaneous emission models within structured environments

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We investigate the long-lived quantum coherence of two-level spontaneous emission models within structured environments. The population of the system under the asymptotic non-Markovian dynamics is linked to the spectral density of the reservoir through a general functional relation between them. We figure out explicitly the preservation of quantum coherence, via notions of entanglement and quantum discord, in connection with the spectral parameters of Ohmic class reservoirs and then show how to achieve them optimally. We expect these results to contribute towards reservoir engineering with the aim of enhancing stationary quantum coherence in noisy environments.

Non-Markovian dynamics of open quantum systems attracts intensive attention recently.1 Differing from the conventional Markovian dissipative process, the memory effects of the non-Markovian environment will lead to non-exponential decay of the system and even result in dissipationless behavior. In general, these peculiarities become evident in cases of low-temperature environments and strong system-reservoir couplings. The corresponding dynamics of the system will depend dramatically on the spectrum structure of the reservoir. Besides the fundamental interest to the statistical physics itself, the study on this subject is stimulated by the progress of quantum information science,2 since accurate coherence control of quantum systems under noisy environments requires that the memory effect of the environment should be taken into account.

Basically, the memory effect of the non-Markovian environment will prolong quantum coherence, e.g., it may lead to entanglement revival and protect a composite system against sudden death of entanglement.3 Modifying the property of the reservoir to reach the non-Markovian regime was shown to be practicable in some physical systems, including the structured environment of photonic crystal materials4–6 and the optically confined Bose-Einstein condensate reservoir.7 For the typical spontaneous emission model, it is known for a long time that incomplete decay of atomic excitation can occur in the medium of photonic crystals8–9. The associated phenomenon of entanglement trapping was unveiled recently10,12. To characterize the connection between the long-lived quantum coherence and the reservoir spectrum hence is not only an issue of the non-Markovian dynamics itself, but also a task for reservoir engineering to enhance stationary quantum correlations in noisy environments.

Here we focus on the two-level spontaneous emission model and characterize its longtime behavior in connection with the reservoir spectra. Interaction of the model in the rotating-wave approximation is described by

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k [\omega_k a_k^\dagger a_k + (g_k a_k \sigma_+ + g_k^* a_k^\dagger \sigma_-)],$$ (1)

where $\omega_0$ is the fixed frequency of the system, $\sigma_{\pm}$ are shift operators acting on levels $|\pm\rangle$, and $a_k^\dagger$ ($a_k$) are creation (annihilation) operators of bosonic field modes with frequencies $\omega_k$. In the case that the environment is initially in the vacuum state $|0\rangle_E$, evolution of the total system is described by $|\Psi(t)\rangle = c(t)|0\rangle_E + \sum_k c_k(t)|k\rangle$, where $|k\rangle \equiv a_k^\dagger |0\rangle_E$ denotes the single excitation of the $k$th field mode. The corresponding Schrödinger equation is amenable to an exact resolution as the spectral function of the reservoir is given, which indeed has ever been explored intensively, e.g., via numerical calculations13 and analytical approaches8,9,14. However, since most studies were focused on the time evolution of the system, explicit revelation upon the connection between the population and the reservoir spectrum has only been obtained for very few cases with particular forms of the spectral density8,12.

In this work we expose explicitly the link between the long-lived coherent population of the system and the spectral density of the reservoir for the spontaneous emission model. By exploiting the functional relationship between the asymptotic population and the spectral density, we carry on detailed analyses upon how to engineer the parameters for Ohmic class spectra to achieve optimally the stationary coherent population of the asymptotic process. This enables us to characterize further the trapping phenomenon of quantum correlations—via notions of entanglement and quantum discord—in connection with spectral parameters for a two-qubit system undergoing local dissipative channels.

Let us start by considering the eigenvalue equation of the Hamiltonian11 in the single-excitation sector. By substituting $|\Psi_{BS}\rangle = |b+, (0_k)\rangle + \sum_k b_k |1_k\rangle$ into $H|\Phi_{BS}\rangle = E|\Phi_{BS}\rangle$, one obtains the secular equation

$$\omega_0 - \int_0^\infty \frac{J(\omega)}{\omega - E} d\omega = E,$$ (2)

where $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$ is the spectral density of the mode continuum of the reservoir. The corresponding
coefficients of the eigenstate $|\Phi_{BS}\rangle$ are expressed as $b_k = g_k b/(E - \omega_k)$ and
\[
b = \left[1 + \int_0^\infty \frac{J(\omega)}{(\omega - E)^2} d\omega\right]^{-1/2}.
\] (3)

Since we are considering the coupling between the two-level system and a continuous spectrum, the solution of Eq. (2) highly depends on the explicit form of $J(\omega)$. Note that this kind of eigenvalue problem has ever been investigated in a different physical context [16, 17] and one can even retrospect it to the work by von Neumann and Wigner [18]. For the spontaneous emission of an atom within photonic band gap mediums, existence of the eigen-solution of the above equation, so called as atom-photon bound states, was known well and the associated phenomenon of partial inhibition of radiative decay of the atomic excited state has been intensively studied in early works [8, 9, 13, 14].

Here, we suppose that Eq. (2) possesses only a single real root, which is possible if the spectral function fulfills $J(\omega) > 0$ for $\omega > 0$ [19, 20]. Note that due to the possible divergence of the integral contained in Eq. (2), a positive real root $E > 0$ must not exist under this condition. So the real root of Eq. (2), if do exist, should be unique in view that the left hand of Eq. (2) decreases monotonically with $E$ in the range $E \in (-\infty, 0)$. As a result, for a system initially in the excited state $|\uparrow\rangle$ with the amplitude $c(0) = 1$, the residual population after evolution in the longtime limit is given by
\[
P(t) \equiv |c(t)|^2 \lim_{t \to \infty} P_\infty = b^4.
\] (4)

To make this result clear, we recall that according to the Schrödinger equation $i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$, the time evolution of the amplitude $c(t)$ satisfies the integro-differential equation
\[
\dot{c}(t) + i\omega_0 c(t) + \int_0^t c(\tau) f(t - \tau) d\tau = 0,
\] (5)
with $f(t - \tau) = \int_0^\infty J(\omega) e^{-i\omega(t-\tau)} d\omega$. In the longtime limit $t \to \infty$, there will be no ingredient of the excited state $|\uparrow\rangle$ retained unless Eq. (2) allows the solution of bound states. Particularly, for the case that Eq. (2) allows only a single bound state $|\Phi_{BS}\rangle$, one can express the excited state as $|\uparrow, \{0_k\}\rangle = b|\Phi_{BS}\rangle + b|\Phi_D(0)\rangle$, where $|\Phi_D(0)\rangle$ represents the projective state of $|\uparrow, \{0_k\}\rangle$ over the complementary subspace orthogonal to $|\Phi_{BS}\rangle$ and the coefficient $b = (1 - b^2)^{1/2}$. The generated evolution by the Hamiltonian (1), $|\Phi_D(t)\rangle = e^{-iHt}|\Phi_D(0)\rangle$, will decay entirely to the ground state as $t \to \infty$. So the amplitude of the excited state in the longtime limit is contributed solely by the ingredient of it in $|\Phi_{BS}\rangle$, which leads promptly to that $|c_\infty| = b^2$. Note that this fact has ever been displayed in the literature [1] to describe the incomplete decay of an atom in photonic band gap mediums.

Combination of the above Eqs. (2)-(4) suggests a conclusive functional relationship between the long-lived population and the spectral function: $P_\infty = b^4[\omega_0, J(\omega)]$. Although each of these expressions, Eqs. (2), (3) and (4), has ever been obtained previously, say, in literatures [8, 9], the importance of this functional relation to explore the long-lived quantum coherence was not revealed so far: it renders indeed a peculiar perspective to expose the asymptotically dissipationless behavior of the system in connection with details of the spectral structure of the reservoir. This offers straightly the information towards reservoir engineering, e.g., to the aim of enhancing stationary quantum coherence in noisy environments.

To proceed, we mention the fact that $P_\infty$ in general is not a monotonically increasing quantity of the dissipation strength. This is somewhat counter-intuitive since a strong dissipation strength is regarded as a necessary condition for the existence of the bound state. To make it clear, we record $J(\omega) = \eta J(\omega)$ and substitute it into Eq. (2). One finds that $E$ will descend as the dissipation strength $\eta$ increases. The variation of $b$ in Eq. (3), relying on its contained integrand, is determined by the competition between the numerator and the denominator, both terms increasing with the strength $\eta$. As a consequence, the monotonicity of $P_\infty$ as a function of $\eta$ is conditioned to the concrete form of $j(\omega)$.

Let us go ahead by considering the widely used Ohmic class reservoir with $J(\omega) = \eta \omega^{s-\omega_0} e^{-\omega/\omega_c}$, where $\omega_c$ is the cutoff frequency and $s$ is a parameter whose scope, $s < 1$, $s = 1$, $s > 1$, corresponds to sub-Ohmic reservoirs, Ohmic and super-Ohmic reservoirs, respectively. It turns out that in this case the solution of Eq. (2) could exist as long as the parameters satisfy $\eta^{-1} \leq (\omega_c/\omega_0)\Gamma(s)$, where $\Gamma(s)$ is the gamma function. The equality contained here is obtained by substituting directly $E = 0$ into the secular equation (2). This critical condition characterizes actually the occurrence of a quantum phase transition of the model with or without a bound state (ground state) [21]. As the solution of the bound state is unique, the functional relation of Eqs. (2), (3) yields
\[
P_\infty(\eta, s, \omega_0/\omega_c) = \left[1 + \int_0^\infty \frac{\eta \omega^s e^{-x}}{(x - \kappa)^2} dx\right]^{-2},
\] (6)
where $\kappa \equiv E/\omega_c$ is determined by
\[
\omega_0/\omega_c - \int_0^\infty \frac{\eta \omega^s e^{-x}}{x - \kappa} dx = \kappa.
\] (7)
Numerical calculations to the latter transcendental equation are required to obtain $\kappa$ for specified parameters $(\eta, s, \omega_0/\omega_c)$, with which the exact population $P_\infty$ can be achieved from Eq. (6).

The dependence of $P_\infty$ on spectral parameters of the reservoir determined by the implicit function of Eq. (6) is quite sophisticated. We present below a detailed analysis on how to achieve $P_\infty$ optimally with respect to different zones of the spectral parameters $(\eta, s, \omega_0/\omega_c)$. In the case of low $\omega_c/\omega_0$, a high value of $\eta \Gamma(s)$ indicates...
that the scope of the parameters \((\eta, s)\) is relatively narrow to achieve a nonvanishing \(P_\infty\). Note that there is a physical constraint of the coupling strength in order to validate the rotating-wave approximation for the model Hamiltonian \(\mathcal{H}\) (reasonably \(\eta \lesssim 0.1\) owing to \(|g| \ll \omega_0|\)). The dependence of \(P_\infty\) on \(\eta\) is shown in Fig. 1(a) with \(\omega_c/\omega_0 = 0.3\). Our calculation displays that high Ohmicity about \(s \gtrsim 5.25\) is required, which may challenge the technology of the reservoir engineering. For the situation with high cutoff frequencies, the dependence of the population \(P_\infty\) on the parameter \(s\) is shown in Fig. 1(b). Note that in the limit of \(\omega_0/\omega_c \to 0\), the transcendental equation defines an implicit function \(\kappa = \kappa(\eta, s)\), hence \(P_\infty = P_\infty(\eta, s)\) according to Eq. \(\mathcal{H}\). For \(\eta = 0.08\), the maximum of the asymptotic population \(P_\infty \simeq 0.9\) is achieved at \(s \simeq 2.34\).

\[
\Gamma_1(t) = \begin{bmatrix} 0 & 0 \\ \bar{c}(t) & 0 \end{bmatrix}, \quad \Gamma_2(t) = \begin{bmatrix} \bar{c}(t) & 0 \\ 0 & 1 \end{bmatrix}
\]

in which \(\bar{c}(t) \equiv [1 - |\bar{c}(t)|^2]^{1/2}\). Therefore one has

\[
\rho_{AB}(t) = \sum_{i=1}^{2} \Gamma_i(t) \otimes I_B \rho_{AB}(0) \Gamma_i(t) \otimes I_B.
\]

We characterize below quantum correlations for the case of pure input states: \(|\psi_{AB}(0)\rangle = \alpha|+\rangle + \beta|-\rangle + \gamma|0\rangle\) with \(|\alpha|^2 + |\beta|^2 = 1\). The concurrence of the corresponding \(\rho_{AB}(t)\) is obtained readily as \(C_{AB}(t) = 2|\alpha\beta c(t)|\). Also the discord can be worked out via the method in [22], expressed analytically as: \(Q_{AB}(t) = h(\lambda) + h(\lambda_A) - h(\lambda_{AB})\), where \(h(x) = -x \log_2 x - (1-x) \log_2 (1-x)\) is the binary entropy function and the parameters \(\lambda_A = |\alpha(c(t))|^2\), \(\lambda_{AB} = |\alpha(c(t))|^2\), and \(\lambda = 1/2[1 + |1 - 4\alpha(\beta c(t))|^2]^{1/2}\). In the limit \(t \to \infty\), a steady population \(|c_\infty|^2 = b^2[J(\omega)]\) is yielded, hence there are \(Q_{AB}^\infty = Q_{AB}^\infty[J(\omega)]\) and \(C_{AB}^\infty = C_{AB}^\infty[J(\omega)]\). We depict in Fig. 2 the two quantities varying with the parameter \(s\) of the Ohmic class spectra with \(\eta = 0.08\) and \(\omega_c \gg \omega_0\).

Since both the two quantities of the output state are monotonic functions of the amplitude \(|c_\infty|\), the maximal values of \(C_{AB}^\infty\) and \(Q_{AB}^\infty\) are obtained at the same point with \(s = 2.34\).

As an application of the above results, we investigate the long-lived behavior of quantum correlations of an initially entangled two-qubit state \(\rho_{AB}\), in which the subsystem \(A\) is subject to a local channel of spontaneous emission. Referring to different notions, quantum correlation of the system is often depicted either by entanglement or quantum discord. The amount of entanglement of the two-qubit state \(\rho_{AB}\) can be expressed explicitly via the measure of concurrence [22]:

\[
C_{AB} = \max\{0, \lambda_{1/2} - \sum_{i=2}^{\infty} \lambda_i^{1/2}\}, \quad \text{where} \ \lambda_i \text{ are eigenvalues of the matrix } \rho_{AB} \otimes I \rho_{AB} \psi \otimes \psi \text{ in descending order.}
\]

The discord of \(\rho_{AB}\), according to the definition of \(\mathcal{H}\), is given by \(Q_{AB} = S(\rho_A) - S(\rho_{AB}) + \min_{\{A\}} S(\rho_B[\{A\}])\), where \(S(\rho) = -\text{tr}(\rho \log_2 \rho)\) is the von Neumann entropy. To describe the evolution of \(\rho_{AB}\), we note that the damping channel of the qubit \(A\) undergoing the spontaneous emission can be depicted by the Kraus representation \(\rho_A(t) = \sum_{i=1}^{2} \Gamma_i(t) \rho_A(0) \Gamma_i(t)^\dagger\) of

\[
\begin{aligned}
\text{FIG. 1: Long-lived population } P_\infty = b^4 \text{ in relation to the parameters of Ohmic class spectra. (a) } P_\infty \text{ as a function of } \eta \text{ with } \omega_c/\omega_0 = 0.3, 5, 5.25, \text{ and } 5.5, \text{ respectively. The green-dotted line figures out the boundary determined by the critical condition } \eta \Gamma(s) = \omega_0/\omega_c. \text{ For } s = 5.5, \text{ the maximal } P_\infty = 0.33 \text{ is achieved at } \eta = 0.08. \text{ (b) } P_\infty \text{ as a function of } s \text{ with } \eta = 0.08, \omega_c/\omega_0 = 15, 50 \text{ and } \infty. \text{ In the limit } \omega_c/\omega_0 \to \infty, \text{ the maximal } P_\infty = 0.90 \text{ is achieved at } s = 2.34.
\end{aligned}
\]

\[
\begin{aligned}
\text{FIG. 2: Long-lived entanglement and quantum discord as a function of } s \text{ for the Ohmic class reservoirs with } \eta = 0.08 \text{ and } \omega_c \gg \omega_0. \text{ The two-qubit system is initially in a pure state } |\psi_{AB}(0)\rangle = \alpha|+\rangle + \beta|-\rangle + \gamma|0\rangle. \text{ The maximal values } C_{AB}^\infty \simeq 0.95 \text{ and } Q_{AB}^\infty \simeq 0.88 \text{ are obtained at } s \simeq 2.34 \text{ for the maximally entangled input state.}
\end{aligned}
\]

To summarize, we have studied the asymptotic behavior of two-level spontaneous emission models within structured environments. As the occurrence of long-lived quantum coherence is clearly a consequence of the memory effect of the non-Markovian dynamics, our calculations reveal that it can occur for a wide range of Ohmic class reservoirs. We expect that our derived results, figuring out explicitly the connection between the system behavior and the reservoir spectra, could contribute useful
information towards reservoir engineering to enhance stationary quantum correlations under noisy environments. Finally, we mention that we have assumed a model with a bosonic reservoir in the vacuum state initially. Further studies to explore the influence of the existence of the bound state upon the coherence of the system dynamics for a realistic environment with nonzero temperature should be a subject of future researches.

We acknowledge Paolo Zanardi and Jiushu Shao for helpful discussions. This work was supported by the Natural Science Foundation of China.

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