Abstract
We study the decomposition of the Feynman kernel for a particle in a box with
1/\sin^2 \theta potential to find that the well-known phase factor $-1$, which is correct
for the case of the free particle, for reflection at boundaries should be generalized
depending on the parameter of the potential.

1. Introduction

The Feynman kernel for the free particle on a circle\[1, 2, 4]\ and for the
one in a one dimensional box\[3, 4]\ are well-known examples of solvable path
integrals as a sum over classical paths by taking the topology of the system into
account. The topology of a box appears as a phase factor $-1$ at each reflection
by boundaries while the winding number of the path is entering as phase for the
free particle on a circle. On the other hand, in Schrödinger picture of quantum
mechanics, there exists a simple and exactly solvable model\[5]\ with a potential
proportional to $1/ \sin^2 \theta$ ($\theta = \pi x/L$) on the interval $0 < x < L$. This model is
also known to be exactly solvable in Heisenberg picture thanks to the existence
of the sinusoidal coordinate\[6].

We aim in this note to make it clear whether the phase factor $-1$ above
can be universal and independent of the potential. To achieve this, we convert
the eigenfunction expansion of the Feynman kernel for the Hamiltonian with
1/\sin^2 \theta potential into the form of a sum over paths with difference in the
number of reflections.

2. Feynman kernel for the 1/\sin^2 \theta potential

Stationary Schrödinger equation
$$\mathcal{R} \left\{ -\frac{d^2}{d\theta^2} + \frac{\nu(\nu - 1)}{\sin^2 \theta} \right\} \psi(\theta) = E\psi(\theta), \quad \mathcal{R} \equiv \frac{1}{2m} \left( \frac{\pi \hbar}{L} \right)^2$$

\[2.1\]
can be solved by the eigenfunction
\[ \phi_n^{(\nu)}(\theta) = 2^{\nu} \Gamma(\nu) \sqrt{\frac{(n + \nu)n!}{2\pi \Gamma(n + 2\nu)}} \sin^\nu \theta C_n^{(\nu)}(\cos \theta) \] (2.2)
to yield the n-th eigenvalue \( E_n^{(\nu)} = (n + \nu)^2 \mathcal{R} \) for \( n = 0, 1, 2, \ldots \). Here the parameter \( \nu \) is assumed to be \( \nu \geq 1/2 \) and \( C_n^{(\nu)}(\cos \theta) \) describes the Gegenbauer polynomial of the n-th order.

The stationary Schrödinger equation defines a Hamiltonian
\[ H^{(\nu)} = \frac{p^2}{2m} + \frac{\nu(\nu - 1)\mathcal{R}}{\sin^2 \theta} \] (2.3)
where \( p^2 \) acts as \(-\hbar^2 d^2/dx^2\) on wavefunctions. Since we know the complete set of eigenfunctions of this Hamiltonian, we can immediately write out the eigenfunction expansion of the Feynman kernel as
\[ K^{(\nu)}(\theta_a, \theta_b; \beta) = \sum_{n=0}^{\infty} e^{-\beta E_n^{(\nu)}/\hbar} \phi_n^{(\nu)}(\theta_a) \phi_n^{(\nu)}(\theta_b) \] (2.4)
for the Euclidean time evolution operator \( e^{-\beta H^{(\nu)}/\hbar} \). Note that this kernel is normalized to be suitable for integration with respect to \( x_a \) or \( x_b \).

We can therefore rewrite the short time kernel as
\[ K^{(\nu)}(\theta, \theta'; \epsilon) \sim 2^{2\nu}\left\{ \Gamma(\nu) \right\}^2 \frac{I_{\nu+n}(\lambda)}{\sqrt{2\pi \lambda}} \Gamma(2\nu + n) I_{\nu+n}(\frac{1}{\lambda}) \exp \left\{ -\frac{1}{\lambda} - \frac{\lambda}{8} \right\} \] (2.5)
for infinitesimally small positive \( \lambda \), we observe
\[ e^{-\lambda(n+\nu)^2/2} \sim \sqrt{\frac{2\pi}{\lambda}} I_{\nu+n}(\frac{1}{\lambda}) \exp \left\{ -\frac{1}{\lambda} - \frac{\lambda}{8} \right\}. \] (2.6)

We can therefore rewrite the short time kernel as
\[ K^{(\nu)}(\theta, \theta'; \epsilon) \sim \frac{2^{2\nu}\{\Gamma(\nu)\}^2}{\sqrt{2\pi \lambda}} (\sin \theta \sin \theta')^\nu \exp \left\{ -\frac{1}{\lambda} - \frac{\lambda}{8} \right\} \] (2.7)
The sum in the right hand side above can be converted into a simplified form by the formula (see e.g. Ch. 11.5 of ref.[7] or Ch. 8.8 of ref.[4])

\[
2^{2\nu} \left\{ \Gamma(\nu) \right\}^2 \sqrt{2\pi \lambda} (\sin \theta \sin \theta')^{1/2} \sum_{n=0}^{\infty} \frac{n!(\nu + n)}{\Gamma(2\nu + n)} I_{\nu+n} \left( \frac{1}{\lambda} \right) C_n^\nu(\cos \theta) C_n^\nu(\cos \theta')
\]

\[
= \frac{(\sin \theta \sin \theta')^{1/2}}{\lambda} \exp \left( \frac{\cos \theta \cos \theta'}{\lambda} \right) I_{\nu-1/2} \left( \frac{\sin \theta \sin \theta'}{\lambda} \right)
\]

(2.8)

to result in

\[
K^{(\nu)}(\theta, \theta'; \epsilon) \sim \frac{(\sin \theta \sin \theta')^{1/2}}{\lambda} \exp \left( - \frac{1 - \cos \theta \cos \theta'}{\lambda} - \frac{\lambda}{8} \right) I_{\nu-1/2} \left( \frac{\sin \theta \sin \theta'}{\lambda} \right).
\]

(2.9)

We have thus obtained a closed expression for the short time kernel for arbitrary values of \( \nu \geq 1/2 \).

To check the validity of (2.9), let us first set \( \nu \) to be unity and consider the case of the free particle in a box. Since \( I_{1/2}(z) \) can be expressed as

\[
I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z,
\]

the modified Bessel function in (2.9) yields

\[
\frac{\lambda}{2\pi \sin \theta \sin \theta'} \left\{ \exp \left( \frac{\sin \theta \sin \theta'}{\lambda} \right) - \exp \left( - \frac{\sin \theta \sin \theta'}{\lambda} \right) \right\}
\]

(2.10)

for \( \nu = 1 \). Hence the right hand side of (2.9) can now be rewritten as

\[
\frac{e^{\lambda/8}}{\sqrt{2\pi \lambda}} \left[ \exp \left\{ - \frac{1 - \cos(\theta - \theta')}{\lambda} \right\} - \exp \left\{ - \frac{1 - \cos(\theta + \theta')}{\lambda} \right\} \right].
\]

(2.11)

This kernel possesses infinitely many saddle points to make us replace it with

\[
\frac{1}{\sqrt{2\pi \lambda}} \sum_{k=-\infty}^{\infty} \left[ \exp \left\{ - \frac{(\theta - \theta' - 2k\pi)^2}{2\lambda} \right\} - \exp \left\{ - \frac{(\theta + \theta' - 2k\pi)^2}{2\lambda} \right\} \right]
\]

(2.12)

for infinitesimally small \( \lambda \). Here use has been made of the method given by the present author in ref.[8] for converting complex kinetic term in the path integral into the stand one with additional potential terms. For the present case the additional potential is same for all saddle points and given by \(-\lambda/8\) in the exponent of the Feynman kernel to cancel the factor \(e^{\lambda/8}\) in (2.11). We now observe that (2.12) is nothing but the Feynman kernel, that is normalized to fit integration with respect to \(\theta(\theta')\) instead of \(x(x')\), for the free particle in a box. Therefore our derivation of the short time kernel (2.9) is correct for \(\nu = 1\). It should be emphasized here that, in the calculation above, the origin of the minus sign in front of the second term in the sum in (2.12) is the coefficient of \(e^{-z}\) in \(\sinh z\) appears in \(I_{1/2}(z) = \sqrt{2/(\pi z)} \sinh z\).
For \( \nu = 2 \), we make use of \( I_{3/2}(z) = \sqrt{2/(\pi z)} (\cosh z - z^{-1} \sinh z) \) to obtain

\[
(1 - \frac{\lambda}{\sin \theta \sin \theta'}) \exp \left( \frac{\sin \theta \sin \theta'}{\lambda} \right) + \left( 1 + \frac{\lambda}{\sin \theta \sin \theta'} \right) \exp \left( -\frac{\sin \theta \sin \theta'}{\lambda} \right)
\]

\[
= \exp \left( \frac{\sin \theta \sin \theta'}{\lambda} - \frac{\lambda}{\sin \theta \sin \theta'} \right) + \exp \left( -\frac{\sin \theta \sin \theta'}{\lambda} + \frac{\lambda}{\sin \theta \sin \theta'} \right)
\]

by discarding irrelevant terms, with the same pre-factor as in (2.10) for the modified Bessel function in (2.9). We then find that the short time kernel can be written as

\[
K^{(2)}(\theta, \theta'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \sum_{k = -\infty}^{\infty} \left[ \exp \left\{ -\frac{(\theta - \theta' - 2k\pi)^2}{2\lambda} - \frac{\lambda}{\sin \theta \sin \theta'} \right\} \right.
\]

\[
+ \exp \left\{ -\frac{(\theta + \theta' - 2k\pi)^2}{2\lambda} + \frac{\lambda}{\sin \theta \sin \theta'} \right\} \right].
\]

(2.13)

This is the decomposition of the short time kernel for \( \nu = 2 \) into the sum over paths with different number of reflections. The second term in the sum above expresses contributions from paths reflected odd times at boundaries. Surprisingly, the coefficient of this terms is now +1. For integral values of \( \nu \), we can repeat the similar process to find that the coefficient is -1 if \( \nu \) is odd integer and +1 for even integers.

We now proceed to consider the case of non-integral values for \( \nu \). For this case we may resort to making use of the asymptotic form of the modified Bessel function in (2.9). To take all possible contributions from reflected paths into account, we have to determine the \( \arg(\sin \theta \sin \theta') \). In the original domain, both \( \sin \theta \) and \( \sin \theta' \) are positive real. We thus define \( \arg(\sin \theta \sin \theta') = 0 \) there. It is then natural to define \( \arg(\sin \theta \sin \theta') = -\pi \) for the saddle point at \( \theta + \theta' = 0 \), \( \arg(\sin \theta \sin \theta') = -2\pi \) for \( \theta - \theta' = -2\pi, \ldots \) and \( \arg(\sin \theta \sin \theta') = \pi \) for the saddle point at \( \theta + \theta' = 2\pi \), \( \arg(\sin \theta \sin \theta') = 2\pi \) for \( \theta - \theta' = 2\pi, \ldots \). In this way, we obtain, by keeping only relevant terms, the asymptotic form of the modified Bessel function in (2.9) for the saddle point at \( \theta - \theta' = 2k\pi (k = 0, \pm 1, \pm 2, \ldots) \)

\[
e^{2k\nu\pi i} \exp \left\{ \frac{\sin \theta \sin \theta'}{\lambda} - \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right\}
\]

(2.15)

and

\[
e^{(2k-1)\nu\pi i} \exp \left\{ -\frac{\sin \theta \sin \theta'}{\lambda} + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right\}
\]

(2.16)

for saddle point at \( \theta + \theta' = 2k\pi \), with the same pre-factor that appears in (2.10).

We thus obtain

\[
K^{(\nu)}(\theta, \theta'; \epsilon) = \frac{1}{\sqrt{2\pi \lambda}} \sum_{k = -\infty}^{\infty} \left[ e^{2k\nu\pi i} \exp \left\{ -\frac{(\theta - \theta' - 2k\pi)^2}{2\lambda} - \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right\} \right.
\]

\[
+ e^{(2k-1)\nu\pi i} \exp \left\{ -\frac{(\theta + \theta' - 2k\pi)^2}{2\lambda} + \frac{\lambda}{2} \frac{\nu(\nu - 1)}{\sin \theta \sin \theta'} \right\} \right].
\]

(2.17)
as the decomposition of the Feynman kernel \((2.4)\) into the sum over paths with difference in the number of reflections for infinitesimally small \(\epsilon\). The phase factor in front of each component above may change if we choose another prescription to determine \(\arg(\sin \theta \sin \theta')\) outside the original domain. We may choose such that \(\arg(\sin \theta \sin \theta') = 0\) for \(\theta\) in \(2k\pi < \theta < (2k + 1)\pi\) and \(\arg(\sin \theta \sin \theta') = \pi\) for \((2k - 1)\pi < \theta < 2k\pi\) for example. For this choice all coefficients in the sum of contributions from paths reflected even times become unity and those in the sum of contributions from paths reflected odd times reduce to \(e^{\nu \pi i}\). We therefore observe here that the factor \(-1 = e^{\pi i}\) for the reflection of the free particle in a box is not the universal one; it rather depends on the parameter \(\nu\) that characterizes the potential.

3. Summary

We have studied the decomposition of the Feynman kernel for a particle in a box with \(1/\sin^2 \theta\) potential to find that the phase the kernel acquires at each reflection by boundaries depends on the parameter of the potential. The form of the decomposition possesses Lagrangian form of the Euclidean action and allows us to consider that the Feynman kernel can be expressed as a sum over paths if we treat the phase generated by reflection at boundaries carefully. The phase which appears in Euclidean path integral may have some geometric origin. It will be, therefore, interesting to find its meaning. Finally, we must add the following comment; although we have obtained our result starting from the eigenfunction expansion of the Feynman kernel, it will be desired to find a method to arrive the same result from the Hamiltonian path integral, as we usually do in obtaining the Lagrangian path integral for systems on the whole real line, by keeping good connection with the operator formalism so that we can deduce the eigenfunction expansion of the kernel solely by means of the path integral technique. Such a method will be reported elsewhere[9].

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