Discovery of Quantum Hidden Variable

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Preface

The first clue, in the theory of relativity, the 4-vector force acting on a particle is orthogonal to the 4-vector velocity of the particle, this orthogonality means that there is some difference between the orthogonality and the usual statement: the Coulomb’s force (or gravitational force) acts along the line joining a couple of particles (in usual 3D space), so the direction of 4-vector Coulomb’s force is carefully investigated, it is found that Maxwell’s equations can be derived from classical Coulomb’s force and the orthogonality.

The second clue, a 4-vector force has 4 components, because of the orthogonality of 4-vector force and 4-vector velocity, the number of independent components of the 4-vector force reduces to 3, however we prove that 4-vector Coulomb’s force can merely provide 2 independent components, this situation means that there is an undefined component accompanying the 4-vector Coulomb’s force, hinting that this missing undefined component is a hidden variable. The third clue, the best way to study the hidden variable is to establish a new concept: Z-space, in which the undefined component of 4-vector Coulomb’s force can be clearly defined as the hidden variable for the quantum mechanics.

At the last, the undefined component is regarded as a fluctuating source that contributes to Lorentz force, so that the quantum wave equation can be derived out in the ensemble space of particle motion from the relativistic Newton’s second law.

Hidden variable Einstein advocated plays an important role in the quantum mechanics, it implies that indeterminacy is not a fact of nature, but a reflection of our ignorance. Finding the hidden variable inevitably concerns with many aspects of the quantum mechanics, must explicitly provide
some additional information to the usual quantum theory so that we can get a complete description of particle motion. For this purpose, the finding process is divided into 9 chapters, each chapter focuses on a related theme, then finally the hidden variable can correctly serve the quantum mechanics and develop the quantum theory.

The first chapter points out that the 4-vector force exerted on a particle by another particle is always in the direction orthogonal to the 4-vector velocity of the exerted particle in the 4-dimensional space time, rather than along the line joining the couple of the particles. This inference is obviously supported from the fact that the magnitude of the 4-vector velocity is always a constant. This orthogonality brings out many new aspects for force concept. In this chapter it is found that Maxwell’s equations can be derived from classical Coulomb’s force and the orthogonality.

In the second chapter, we prove that the 4-vector electromagnetic force of Maxwell’s theory has two independent components in the 4-dimensional space time. A 4-vector force has 4 components, because of the orthogonality of 4-vector force and 4-vector velocity, the number of independent components of the force reduces to 3, however the 4-vector electromagnetic force can merely provide 2 independent components, this situation means that there is an undefined component accompanying the 4-vector electromagnetic force. We also briefly and confirmedly discuss the possibility that the undefined component of the 4-vector electromagnetic force serves as a hidden variable for the quantum mechanics. The undefined component is regarded as a fluctuating source that contributes to Lorentz force, so that the quantum wave equation can be derived out in the ensemble space of particle motion from the relativistic Newton’s second law.

The third chapter, using the momentum-wavefunction relation derived out in the preceding chapter, the fine structure of hydrogen atom energy is calculated, as the results, the energy levels are completely the same as that in the calculation of Dirac wave equation for hydrogen atom, while the wavefunction is quite different from Dirac’s wavefunction. Besides, the present calculation brings out spin nature in a new way, indicating that electronic spin is a kind of orbital motion. The present calculation is characterized by using the momentum-wavefunction relation directly, it
provides an insight into the foundations of the quantum mechanics.

The fourth chapter, a momentum path integral method for calculating the quantum states of particle is developed, differing from Feynman’s path integral. It is characterized by using the momentum-wavefunction relation. The momentum path integral method is directly applied to hydrogen atom, the energy levels are calculated out with the same fine structure and spin effect as Dirac wave equation. The momentum path integral method is much simple in solving problems than Dirac’s theory, because it only concerns with integrations rather than differential equations. The successful application of the momentum path integral method to hydrogen atom provides an insight into the foundations of the quantum mechanics.

The fifth chapter, the momentum path integral method is successful extended to the quantum mechanics through some instances: two slit experiment, Aharonov-Bohm effect, Bohr-Somerfeld hydrogen atom, the motion of particle in a potential well, superconductivity. Specially, superconducting state is discussed in terms of relativistic quantum theory, some significant results are obtained, including quantized magnetic flux, London equation, Meissner effect and Josephson effect.

In sixth chapter, it is found that the gravitational effects can be explained in terms of the orthogonality between 4-vector gravitational force and 4-vector velocity, the results are the same as the theory of general relativity.

The seventh chapter, we regard Pythagoras theorem as an axiom, from this axiom we derive out four important consequences of physics, they are Newton’s three laws of motion, the formulae of relativistic dynamics. These results indicate that at least the mechanics is possible to be geometrized, so that it is possible to string together mechanics theorems with Pythagoras theorem. Knowing the internal relationship between them, which have never been clearly revealed by other author, will benefit our physics teaching, and benefit our theme of this book.

The eighth chapter, using Pythagoras theorem we establish a Z-space to study the acceleration of particle, we define a new term "torna” to represents the change rate of the force acting on the particle. It is found that every particle 4-vector acceleration has the same magnitude in the Z-space,
then we must fairly look at every particle in the Z-space, every particle must fairly share the same physical laws in the Z-space. It is found that any 4-vector torna acting on a particle can never change the magnitude of the 4-vector acceleration of the particle but can change its direction in the Z-space. The torna corresponding to Coulomb’s force and the electromagnetism based on the torna are investigated, it is found that the force calculated from the torna is different from the usual electromagnetic force. The hidden variable in quantum mechanics is defined as the difference between the torna-yielded force and the usual electromagnetic force, the hidden variable is just the undefined component of Maxwell’s electromagnetic force discussed in the preceding chapters. In the present work, we regard the undefined component as a fluctuating source, and derive out the quantum wave equation in the ensemble space relating to the fluctuating source.

The ninth chapter, A critical review is presented, and we also discuss the relationship between the three main quantum wave equations: Dirac wave equation, Klein-Gordon wave equation and Schrödinger wave equation, we show that they can be derived from the momentum wave relation in a rigorous manner.
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Chapter 1

Direction of 4-vector force

1.1 Introduction

In the theory of relativity, consider a particle moving in an inertial Cartesian coordinate system $S : (x_1, x_2, x_3, x_4 = i ct)$, as shown in Figure 1.1 its 4-vector velocity $u$ is given by

$$ u \equiv \begin{bmatrix} \frac{v}{\sqrt{1 - v^2/c^2}} & \frac{ic}{\sqrt{1 - v^2/c^2}} \end{bmatrix} $$

(1.1)

$$ u_\mu = \frac{dx_\mu/dt}{\sqrt{1 - v^2/c^2}} = \frac{v_\mu}{\sqrt{1 - v^2/c^2}} $$

(1.2)

The magnitude of the 4-vector velocity $u$ is given by

$$ u_\mu u_\mu = -c^2 $$

(1.3)

$$ |u| = \sqrt{u_\mu u_\mu} = \sqrt{-c^2} = ic $$

(1.4)

We use the Einstein summation convention throughout this book: any repeated index is understood to be summed over all number concerned, where since the Cartesian coordinate system is a frame of reference whose axes are orthogonal to one another, there is no distinction between covariant
CHAPTER 1. DIRECTION OF 4-VECTOR FORCE

The magnitude of the 4-vector velocity $u$ keeps a constant, this result is very interesting. Comparing to a planet rotating about the sun in uniform circular motion, as shown in Figure 1.2, where the centripetal force acting on the planet is orthogonal to the direction of the planet velocity as the motion progresses, while the speed of the planet keeps a constant, therefore, since the magnitude of the 4-vector velocity of the particle (in Figure 1.2) keeps a constant, then the 4-vector force acting on the particle must be orthogonal to the 4-vector velocity. We can prove this in a simple way: any 4-vector force satisfies the following orthogonality.
The sun is rotating about the sun in uniform circular motion, where the centripetal force acting on the planet is orthogonal to the direction of the planet velocity as the motion progresses.

\[ u_\mu f_\mu = u_\mu m \frac{d u_\mu}{d \tau} = \frac{m}{2} \frac{d(u_\mu u_\mu)}{d \tau} = \frac{m}{2} \frac{d(-c^2)}{d \tau} = 0 \quad (1.5) \]

Where \( d\tau \) is the proper time interval corresponding to the time interval \( dt \).

The orthogonality of 4-vector velocity and 4-vector force seems against some statements in physics, for example, ”Coulomb’s force (or gravitational force) acts along the line joining a couple of particles”, frequently taught in physics classrooms. So the orthogonality is deserved to be investigated both for the consistency of physics about Coulomb’s force (or gravitational force) and for finding new aspects of Coulomb’s force (or gravitational force). In this book, we will focus on the new aspects of Coulomb’s force (or gravitational force), that come from the orthogonality, because they provide an insight into the hidden variable of the quantum mechanics, they are a ”gold mine”.

Although the orthogonality has occasionally appeared in some textbooks[2],
few authors pay attention to the orthogonality. In this book, Eq. (1.5) has
been elevated to an essential requirement for any definition of any force,
which brings out many new aspects for Coulomb’s force and gravitational
force.

1.2 Coulomb’s Force and Maxwell’s Equations

1.2.1 Coulomb’s force and Lorentz force

Suppose there are two charged particles \( q \) and \( q' \) locating at the positions
\( x \) and \( x' \) respectively in the Cartesian coordinate system \( S \), and moving at
the 4-vector velocities \( u \) and \( u' \) respectively, as shown in Figure 1.3 where
we have used \( X \) to denote \( x - x' \). The 4-vector Coulomb’s force \( f \) acting on
the particle \( q \) is orthogonal to the velocity direction of \( q \), as illustrated in
Figure 1.3 using Euclidian geometry to represent the complex space-time,
like a centripetal force, the force \( f \) should make an attempt to rotate itself
about the particle path center, the center should locate at a point near the
particle \( q' \), (fore-and-aft, near the particle \( q' \)), so we assume that the force
\( f \) lies in the plane of \( u' \) and \( X \), then

\[
f = Au' + BX
\]

(1.6)

Where \( A \) and \( B \) are unknown coefficients. We will discuss the completion of
the expansion in the next chapter, but for the moment, the expansion would
be used to clarify a relationship between the orthogonality and Maxwell’s
electromagnetism. Using the orthogonality \( u \cdot f = 0 \), we get

\[
u \cdot f = A(u \cdot u') + B(u \cdot X) = 0
\]

(1.7)

Thus the coefficient \( B \) relates to the coefficient \( A \) as

\[
B = -\frac{(u \cdot u')}{u \cdot X}A
\]

(1.8)

By substituting the coefficient \( B \) into Eq. (1.6), we rewrite Eq. (1.6) as
CHAPTER 1. DIRECTION OF 4-VECTOR FORCE

Figure 1.3: The Coulomb’s force acting on $q$ is orthogonal to the 4-vector velocity $u$ of $q$, and lies in the plane of $u'$ and $X$ with a retardation with respect to $q'$, here Euclidian geometry is used to illustrate the complex 4D space.

$$f = \frac{A}{u \cdot X} [(u \cdot X)u' - (u \cdot u')X] \quad (1.9)$$

It follows from the direction of Eq. (1.9) that the unit vector of the Coulomb’s force direction is given by

$$\hat{f} = \frac{1}{c^2 r} [(u \cdot X)u' - (u \cdot u')X] \quad (1.10)$$

because we can check
\[ \hat{f} = \frac{1}{c^2 r}[(u \cdot X)u' - (u \cdot u')X] \]
\[ = \frac{1}{c^2 r}[(u \cdot R)u' - (u \cdot u')R] \]
\[ = -[(\hat{u} \cdot \hat{R})\hat{u}' - (\hat{u} \cdot \hat{u}')\hat{R}] \]
\[ = -\hat{u}' \cosh \alpha + \hat{R} \sinh \alpha \] (1.11)

\[ |\hat{f}| = 1 \] (1.12)

Where \( \alpha \) refers to the angle between \( u \) and \( R \), \( R \perp u', r = |R|, \hat{u} = u/ic, \hat{u}' = u'/ic, \hat{R} = R/r \), as shown in Figure 1.3. Suppose that the magnitude of the force \( f \) has the classical form

\[ |f| = k' \frac{qq'}{r^2} \] (1.13)

Combination of Eq. (1.13) with (1.10), we obtain a modified Coulomb’s force

\[ f = \frac{kq'v}{c^2 r^3}[(u \cdot X)u' - (u \cdot u')X] \]
\[ = \frac{kq'v}{c^2 r^3}[(u \cdot R)u' - (u \cdot u')R] \]
\[ = q[(u \cdot \left( \frac{kq'}{c^2 r^3} R \right))u' - (u \cdot u') \left( \frac{kq'}{c^2 r^3} R \right)] \] (1.14)

By using the relation

\[ \partial_{\mu} \left( \frac{1}{r} \right) = -\frac{R_{\mu}}{r^3} \] (1.15)

we obtain

\[ f_{\mu} = q[-(u_{\nu} \partial_{\nu} \left( \frac{kq'}{c^2 r^3} \right))u'_{\mu} + (u_{\nu} u'_{\nu}) \partial_{\mu} \left( \frac{kq'}{c^2 r} \right)] \]
\[ = q[-(u_{\nu} \partial_{\nu} \left( \frac{kq' u'_{\mu}}{c^2 r} \right)) + (u_{\nu}) \partial_{\mu} \left( \frac{kq' u'_{\nu}}{c^2 r} \right)] \] (1.16)
CHAPTER 1. DIRECTION OF 4-VECTOR FORCE

The force can be rewritten in terms of 4-vector components as

\[ f_\mu = q F_{\mu\nu} u_\nu \]  
(1.17)

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]  
(1.18)

\[ A_\mu = \frac{k q' u'_\mu}{c^2 r} \]  
(1.19)

Thus \( A_\mu \) expresses the 4-vector potential of the particle \( q' \). \( F_{\mu\nu} \) expresses the field tensor in terms of electric field \( E \) and magnetic field \( B \)

\[ F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix} \]  
(1.20)

It is easy to find that Eq.(1.17) contains the Lorentz force.

\[ f = qE + qv \times B \]  
(1.21)

1.2.2 Lorentz gauge condition

From Eq.(1.19), because of \( u' \perp R \), i.e. \( u'_\mu R_\mu = 0 \), we have

\[ \partial_\mu A_\mu = \frac{k q' u'_\mu}{c^2} \partial_\mu \left( \frac{1}{r} \right) = -\frac{k q' u'_\mu}{c^2} \left( \frac{R_\mu}{r^3} \right) = 0 \]  
(1.22)

It is known as the Lorentz gauge condition.

1.2.3 Maxwell’s equations

To note that \( R \) has three degrees of freedom under the condition \( R \perp u' \), so we have

\[ \partial_\mu R_\mu = 3 \]  
(1.23)
\( \partial_\mu \partial_\mu \left( \frac{1}{r} \right) = -4\pi \delta(R) \)  \hspace{1cm} (1.24)

They are the formulae in common use in the classical electrodynamics textbooks. From Eq.(1.18), we have

\[
\begin{align*}
\partial_\nu F_{\mu\nu} & = \partial_\nu \partial_\mu A_\nu - \partial_\nu \partial_\nu A_\mu = \partial_\mu (\partial_\nu A_\nu) - \partial_\nu \partial_\nu A_\mu \\
& = -\partial_\nu \partial_\nu A_\mu = -\frac{kq'u'_\mu}{c^2} \partial_\nu \partial_\nu \left( \frac{1}{r} \right) \\
& = \frac{kq'u'_\mu}{c^2} 4\pi \delta(R) = \mu_0 J'_\mu
\end{align*}
\]

Where we define \( J'_\mu = q'u'_\mu \delta(R) \) as the current density of the source \( q' \), \( \mu_0 = 4\pi k/c^2 \). From Eq.(1.18), by exchanging the indices and taking the summation of them, we have

\[
\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \hspace{1cm} (1.26)
\]

The Eq.(1.25) and (1.26) are known as the Maxwell’s equations. For continuous media, they are valid as well.

### 1.2.4 Lienard-Wiechert potential

From Maxwell’s equations, we know there is a retardation time for the electromagnetic action to propagate between the two particles, as illustrated in Figure 1.3, the retardation time \( \Delta t \) is measured by

\[
r = c\Delta t = c \frac{|PO|}{ic} = c \frac{\hat{u} \cdot X}{ic} = \frac{u'_\nu (x'_\nu - x_\nu)}{c} \hspace{1cm} (1.27)
\]

Then

\[
A_\mu = \frac{kq'u'_\mu}{c^2} \frac{1}{r} = \frac{kq'}{c} \frac{u'_\mu}{u'_\nu (x'_\nu - x_\nu)} \hspace{1cm} (1.28)
\]
Obviously, Eq.(1.28) is known as the Lienard-Wiechert potential for a moving particle.

The above formalism clearly shows that the Maxwell’s equations can be derived from the classical Coulomb’s force and the orthogonality of 4-vector force and 4-vector velocity. In other words, the orthogonality is hidden in the Maxwell’s equations. Specially, Eq.(1.9) directly accounts for the geometrical meanings of the curl of vector potential, the curl contains the orthogonality. The orthogonality of 4-vector force and 4-vector velocity is one of consequences from the relativistic Newton’s second law.

1.3 Discussion

The above formalism has important significance on guiding how to develop the theory of gravity. In analogy with the modified Coulomb’s force of Eq.(1.14), we can directly suggest a modified universal gravitational force. We will discuss this problem in the following chapters.

To note that the orthogonality of 4-vector force and 4-vector velocity is valid for any force: strong, electromagnetic, weak and gravitational interactions, therefore there are many new aspects of mechanics remaining for physics to explore.

1.4 Conclusion

This chapter points out that the 4-vector force exerted on a particle by another particle is always in the direction orthogonal to the 4-vector velocity of the exerted particle in the 4-dimensional space-time, rather than along the line joining the couple of the particles. This inference is obviously supported from the fact that the magnitude of the 4-vector velocity is always a constant. This orthogonality brings out many new aspects for force concept. In this chapter it is found that Maxwell’s equations can be derived from classical Coulomb’s force and the orthogonality.
Chapter 2

Hidden variable

2.1 Introduction

In the preceding chapter we clearly show that Maxwell’s equations can be derived from classical Coulomb’s force and the orthogonality of 4-vector force and 4-vector velocity. In other words, the orthogonality is hidden in the Maxwell’s equations.

But, in this chapter we will find that the electromagnetic force of the Maxwell’s theory can not completely satisfy this orthogonality, and this incompletion leads us to find that the 4-vector electromagnetic force has two independent components in the 4 dimensional space time. We know that a 4-vector force has 4 components, because of the orthogonality of 4-vector force and 4-vector velocity, the number of independent components of the force reduces to 3, however the 4-vector electromagnetic force can merely provide 2 independent components, this situation means that there is an undefined component accompanying the 4-vector electromagnetic force — does this missing undefined component is a hidden variable? — quantum mechanics is eager to find it for a long time.

The primary purpose of this chapter is to strictly prove that the 4-vector electromagnetic force has two independent components in the 4 dimensional space time, and there is an undefined component within the 4-vector force.
We also discuss the possibility that the undefined component of the 4-vector electromagnetic force serves as a hidden variable for the quantum mechanics.

### 2.2 A problem for Maxwell’s equations

#### 2.2.1 hidden Pythagoras theorem

There is some difference between the two statements: the first statement, the Coulomb’s force (or gravitational force) acts along the line joining a couple of particles (in usual 3D space); the second statement, the 4-vector force $f$ acting on a particle must be orthogonal to the 4-vector velocity $u$ of the particle, because we find $|u| = ic$ or

$$u_\mu f_\mu = u_\mu m \frac{d u_\mu}{d\tau} = \frac{m}{2} \frac{d(u_\mu u_\mu)}{d\tau} = \frac{m}{2} \frac{d(-c^2)}{d\tau} = 0 \quad (2.1)$$

Which one among the two statements seems robust? Consider a particle of rest mass $m$ in our frame of reference $S(x_1, x_2, x_3, t)$, the particle moves a distance $\Delta l$ during the infinitesimal time interval $\Delta t$ with the speed $v$, as shown in Figure 2.1 according to Pythagoras theorem, we have

$$\Delta l^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2$$
$$= v^2 \Delta t^2$$
$$= v^2 \Delta t^2 - c^2 \Delta t^2 + c^2 \Delta t^2$$
$$= -c^2 \Delta t^2 (1 - v^2/c^2) + c^2 \Delta t^2 \quad (2.2)$$

Where $c$ is the speed of light. Dividing by $\Delta t^2(1 - v^2/c^2)$, this pythagoras theorem reads

$$\frac{\Delta l^2/\Delta t^2}{(1 - v^2/c^2)} = -c^2 + \frac{c^2}{(1 - v^2/c^2)} \quad (2.3)$$

Defining the modified velocity
Figure 2.1: A particle moves in the frame of reference.

\[ u_1 = \frac{v_1}{\sqrt{1 - v^2/c^2}} \quad u_2 = \frac{v_2}{\sqrt{1 - v^2/c^2}} \quad (2.4) \]
\[ u_3 = \frac{v_3}{\sqrt{1 - v^2/c^2}} \quad u_4 = \frac{ic}{\sqrt{1 - v^2/c^2}} \quad (2.5) \]

Where \( v^2 = v_1^2 + v_2^2 + v_3^2 \). From Eq.(2.3), we obtain

\[ u_1^2 + u_2^2 + u_3^2 + u_4^2 = -c^2 \quad (2.6) \]
\[ u_\mu u_\mu = -c^2 \quad (2.7) \]

The 4-vector velocity \( u = \{u_\mu\} \) is known as the relativistic velocity. It is convenient to define the proper time interval \( d\tau = dt\sqrt{1 - v^2/c^2} \), thus the relativistic velocity is given by
\[ u_\mu = \frac{dx_\mu}{d\tau} \] (2.8)

Where \( x_4 = ict \). Eq. (2.7) is the magnitude formula of relativistic 4-vector velocity of particle in the Minkowsky space \( (x_1, x_2, x_3, x_4 = ict) \) in its squared form.

To note that we have introduced the speed of light, but have never introduced any physical assumption, our work is a pure mathematical derivation. Because the magnitude of 4-vector velocity of particle keeps a constant, the 4-vector force \( f \) acting on a particle must be orthogonal to the 4-vector velocity \( u \) of the particle. Thus, the second statement is robust, it get support from Pythagoras theorem— a sound foundation.

2.2.2 a brief review about 4-vector Coulomb’s force and Maxwell’s equations

Consider two charged particle \( q \) and \( q' \) locating at the positions \( x \) and \( x' \) respectively in a Cartesian coordinate system \( S \), and moving at the 4-vector velocities \( u \) and \( u' \) respectively, as shown in Figure 2.2, where we have used \( X \) to denote \( x - x' \).

According to the second statement, we assume that the force \( f \) lies in the plane of \( u' \) and \( X \), then we make an expansion for the force \( f \) using \( u' \) and \( X \) as the basis vectors

\[ f = Au' + BX \] (2.9)

Where \( A \) and \( B \) are unknown coefficients. Let \( A \) and \( B \) satisfy the orthogonality \( u \cdot f = 0 \), we find that the 4-vector Coulomb’s force \( f \) acting on a particle \( q \) by another particle \( q' \) must be in the direction whose unit vector \( \hat{f} \) reads

\[ \hat{f} = \frac{1}{c^2r} [(u \cdot X)u' - (u \cdot u')X] \] (2.10)

Suppose that the magnitude of the force \( f \) has the classical form

\[ |f| = k \frac{qq'}{r^2} \] (2.11)
then the modified Coulomb’s force is given by

\[ f = \frac{kqq'}{c^2r^3}[(u \cdot X)u' - (u \cdot u')X] \]
\[ = \frac{kqq'}{c^2r^3}[(u \cdot R)u' - (u \cdot u')R] \]
\[ = q[(u \cdot \frac{kq'}{c^2r^3} R))u' - (u \cdot u')\frac{kq'}{c^2r^3} R] \]  \tag{2.12}  

The 4-vector force can be rewritten in terms of 4-vector components as

\[ A_\mu = \frac{kq'u'_\mu}{c^2r} \]  \tag{2.13}
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.14) \]
\[ f_\mu = qF_{\mu\nu}u_\nu \quad (2.15) \]

Thus \( A_\mu \) expresses the 4-vector potential of the particle \( q' \). It is easy to prove that the following equations hold for the 4-vector Coulomb’s force:

\[ \partial_\nu F_{\mu\nu} = \frac{kq' u'_\mu}{c^2} 4\pi \delta(R) = \mu_0 j'_\mu \quad (2.16) \]
\[ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (2.17) \]

Where we define \( j'_\mu = q' u'_\mu \delta(R) \) as the current density of the source \( q' \). The Eq. (2.16) and (2.17) are known as the Maxwell’s equations. For continuous media, they are valid as well [3].

### 2.2.3 the completion of the electromagnetic force’s basis vectors

Our question: whether the expansion we have used in Eq. (2.9) is complete? Obviously, we choose two basis vector \( u' \) and \( X \) to construct the Maxwell’s electromagnetism successfully, this choice means that the electromagnetic force has two independent components in its Hilbert space based on the two basis vector \( u' \) and \( X \). We readily recognize that the expansion is incomplete, because, we know that a 4-vector force has 4 components, because of the orthogonality of 4-vector force and 4-vector velocity, the number of independent components of the force reduces to 3, thus a complete expansion of a 4-vector force needs 3 basis vectors.

Our argument also arises from that if the electromagnetic force of Maxwell’s theory has two independent components, then there is an undefined component accompanying the Maxwell’s electromagnetic force. This missing undefined component may serve a hidden variable for the quantum mechanics. In the next sections, we will strictly and comprehensively prove that the electromagnetic force of Maxwell’s theory does have two independent components, does have an undefined component in the physics. In order to clarify its aspects as much as possible, we present four methods to prove the theme.
2.3 The proofs that the electromagnetic force of Maxwell’s theory has two independent components

2.3.1 the first proof — by an intuitive method

In usual 3D space, the vector product of two vector \( \mathbf{a} \) and \( \mathbf{b} \) is defined as another vector \( \mathbf{c} \) given by

\[
\mathbf{c} = \mathbf{a} \times \mathbf{b}, \quad \text{or} \quad c_k = \varepsilon_{kij} a_i b_j,
\]

where the indexes \( i, j \) and \( k \) take over \( 1, 2, 3 \) in the 3D space, \( \varepsilon_{kij} \) is the Levi-Civita symbol defined in usual textbooks; as we know, the vector \( \mathbf{c} \) is orthogonal to both the vectors \( \mathbf{a} \) and \( \mathbf{b} \), i.e. \( \mathbf{c} \perp \mathbf{a} \) and \( \mathbf{c} \perp \mathbf{b} \), as shown in Figure 2.3.

![Figure 2.3: Three vectors are perpendicular to one another.](image)

In the Minkowsky space \( S : (x_1, x_2, x_3, x_4 = ict) \), we also introduce a definition for vector product: as shown in Figure 2.2 imaging that there is
a 4-vector $\Gamma$ which is orthogonal to the plain of $u'$ and $X$, then $\Gamma$ is defined by the vector product as

$$
\Gamma = u' \times X \quad \Gamma_\lambda = \varepsilon_{\lambda\mu\nu} u'_\mu X_\nu \quad (2.18)
$$

Where the indexes take 1,2,3,4, as shown in Figure 2.4.

![Figure 2.4: Three 4-vectors are orthogonal to one another.](image)

It is easy to obtain their orthogonalities $\Gamma \perp u'$ and $\Gamma \perp X$, because

$$
\Gamma \cdot X = \Gamma_\lambda X_\lambda = \varepsilon_{\lambda\mu\nu} u'_\mu X_\nu X_\lambda \\
= \varepsilon_{\lambda\mu\nu} u'_\mu X_\nu X_\lambda |_{\nu<\lambda} + \varepsilon_{\lambda\mu\nu} u'_\mu X_\nu X_\lambda |_{\nu>\lambda} \\
= \varepsilon_{\lambda\mu\nu} u'_\mu X_\nu X_\lambda |_{\nu<\lambda} + \varepsilon_{\nu\mu\lambda} u'_\mu X_\lambda X_\nu |_{\nu<\lambda} \\
= (\varepsilon_{\lambda\mu\nu} + \varepsilon_{\nu\mu\lambda}) u'_\mu X_\nu X_\lambda |_{\nu<\lambda} = 0 \quad (2.19)
$$

$$
\Gamma \cdot u' = 0 \quad (2.20)
$$
The vector \( \Gamma \) can also be easily imaged out intuitively in Figure 2.2.

The electromagnetic 4-vector force has 4 components, it subjects to the first constraint given by

\[
f \cdot u = 0 \tag{2.21}
\]

the second constraint on the 4-vector force is given by

\[
f \cdot \Gamma = (Au' + BX) \cdot \Gamma = 0 \tag{2.22}
\]

The first constraint is just the orthogonality of 4-vector force and 4-vector velocity, the second constraint reflects our choice that the electromagnetic force lies in the plan of \( u' \) and \( X \) to which the vector \( \Gamma \) is orthogonal. Subjecting to these two constraints, the number of independent components of the electromagnetic force reduces to 2. The two independent components allow the electromagnetic force to change in the plan of \( u' \) and \( X \). The second constraint is the unusual reason that leads the independent component number of the electromagnetic force to be 2.

### 2.3.2 the second proof — in its Hilbert space

The moving particle \( q' \) exerts an electromagnetic force \( f \) on another moving particle \( q \), the action has been sophisticatedly expressed as

\[
f_{\mu} = qF_{\mu\nu}u_{\nu} = qu_{\nu}\partial_{\mu}A_{\nu} - qu_{\nu}\partial_{\nu}A_{\mu} \tag{2.23}
\]

To note that in the present stage of human’s knowledge, the above each term can not be resolved into more details, therefore, we conclude that the electromagnetic force contains two terms, each term \( (qu_{\nu}\partial_{\mu}A_{\nu} \) and \( -qu_{\nu}\partial_{\nu}A_{\mu} \) represents a basis vector of its Hilbert space, as shown in Figure 2.5, the number of independent components of the electromagnetic force is 2.

This provides an insight into the electromagnetism that a full electromagnetic force should contain at least tree terms in its formalism.

### 2.3.3 the third proof — hydrogen atom as an instance

The nucleus of a hydrogen atom provides a spherical Coulomb’s electric potential for its outer electron, this 4-vector force subjects to the first and
second constraints, given by

\begin{align*}
  f \cdot u &= 0 \\
  f \times r &= 0
\end{align*}  \hspace{1cm} (2.24)  \hspace{1cm} (2.25)

The second constraint expresses that the torque of the centripetal force is zero, and it arises from the above mentioned \( f \cdot \Gamma = 0 \) for this instance. Because, as shown in Figure 2.6, \( f \cdot \Gamma = f \cdot (u' \times X) = f \cdot (u' \times R) = 0 \) permits the expansion of \( f = Au' + BR \), whereas the nucleus is at rest, \( u' = (0, 0, 0, ic) \), \( R = (r, 0) \), then \( f = (Br, Aic) \) i.e., \( f \parallel r \) or \( f \times r = 0 \). Inversely, if \( f \parallel r \), then we can construct a vector \( \Gamma \) so that \( f \cdot \Gamma = 0 \). You can see, if we consider the two constraints, then the 4-force in hydrogen atom has only two independent components.

In Bohr’s hydrogen atom, the electron of hydrogen moves in a circular
orbit, the attractive force subjects to the spherical symmetry as its third constraint, in that case, the number of independent components of the 4-vector force reduces to 1, in consistency with its circular motion.

2.3.4 the fourth proof — outside the electromagnetism

A 4-vector force has 4 components, because of the orthogonality of 4-vector force and 4-vector velocity, the number of independent components reduces to 3, while the electromagnetic force can merely provide 2 independent components, this situation means that there is an undefined component accompanying the electromagnetic force. what is the effect of the Undefined Component (UC) in the physics? Outside the electromagnetism, a variety of problems may concern with the UC, for example, quantum mechanics is eager to find a hidden variable for a long time for which the UC may serve.

Figure 2.6: The hydrogen atom model.
Since its complex, we put the fourth proof in the next section, where we choose and briefly discuss three basic questions: (1) the properties of the UC; (2) wavefunction and 4-vector velocity field in the UC-related ensemble space; (3) wavefunction momentum relation.

2.4 The proof outside the electromagnetism

2.4.1 the properties of undefined component (UC)

The properties of UC include: (1) UC is a small quantity which is hardly noted by physicists, if it serves as a hidden variable for the quantum mechanics, then it should be responsible to the Planck’s constant; (2) UC is a fluctuating quantity in the ensemble space of enormous identical experiments, its average value at a given point must be zero, like \(< f_{UC} > = 0\); (3) as a nature of independency, UC must be independent from our usual 3D electromagnetic field, for example, in a vacuum when any our electromagnetic field has vanished off, however the UC can still alive in the vacuum and cause the indeterminacy of the motion of the particles moving in the vacuum; (4) UC also correlates with our 4-vector electromagnetic field through the orthogonality \(f \cdot u = 0\); (5) The effect of UC on the physics must be comprehensive, it is not concern with individual cases, no body can really escape from UC fluctuation if the UC still remains at undefined status.

According to the above mentioned properties of UC, the proof outside the electromagnetism will carry out on ”the path” that leads us from the UC to the two key concepts of the quantum mechanics: wave function and hydrogen atom.

2.4.2 wave function and 4-vector velocity field in ensemble space

Consider a particle diffraction experiment in an electromagnetic field where the UC is active and cause the indeterminacy of quantum mechanics, Each particle in the field subjects to both the usual electromagnetic force \(f\) and UC \(f_{UC}\), then it satisfies the dynamic equation (the relativistic Newton’s
second law):

\[
\frac{m}{d\tau} \frac{du_\mu}{d\tau} = f_\mu + f_{UC\mu} = qF_{\mu\nu}u_\nu + f_{UC\mu} \tag{2.26}
\]

\[
u_\mu u_\mu = -c^2 \tag{2.27}
\]

In order to eliminate the UC term by the means of statistics, i.e. using \( <f_{UC}> = 0 \), we turn to study this dynamic equation in the ensemble space which consists of enormous particle paths recorded in enormous identical experiments, in the ensemble space we find their averages

\[
\frac{m}{d\tau} \frac{d<u_\mu>}{d\tau} = qF_{\mu\nu} < u_\nu > \tag{2.28}
\]

\[
<u_\mu><u_\mu> = -c^2 \tag{2.29}
\]

Strictly speaking, Eq.(2.29) is not derived from Eq.(2.27), it stands for the property that the magnitude of any 4-vector velocity keeps constant, otherwise this 4-vector velocity concept collapses. If we regard the enormous recorded particle paths in the ensemble space as a flow, as shown in Figure 2.7, it is easy to find that there is a 4-vector velocity field for the flow. We clearly emphasize two points: (1) At every point of the ensemble space, the mean 4-vector velocity \(<u>\) satisfies the above mean dynamic equation; (2) the mean 4-velocity \(<u>\) represents a 4-vector velocity field of the flow, \(<u>\) is a function of the position of the ensemble space.

For our convenience, we drop the mean sign \(<>\) for the 4-vectors in the followings, simply use \(u\) in place of \(<u>\). Thus, Eq.(2.28) and Eq.(2.29) become

\[
\frac{m}{d\tau} \frac{du_\mu}{d\tau} = qF_{\mu\nu}u_\nu \tag{2.30}
\]

\[
u_\mu u_\mu = -c^2 \tag{2.31}
\]

As mentioned above, the 4-vector velocity \(u\) is regarded as a 4-vector velocity field in the ensemble space, then

\[
\frac{du_\mu}{d\tau} = \frac{\partial u_\mu}{\partial x_\nu} \frac{dx_\nu}{d\tau} = \frac{dx_\nu}{d\tau} \frac{\partial u_\mu}{\partial x_\nu} = u_\nu \partial_\nu u_\mu \tag{2.32}
\]

\[
qF_{\mu\nu}u_\nu = qu_\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{2.33}
\]
Beware: Eq. (2.32) is the most important step in the present work after introducing the concept of 4-vector velocity field in the ensemble space. Substituting them back into the dynamic equation, and re-arranging these terms, we obtain

\[
\begin{align*}
    u_\nu \partial_\nu (mu_\mu + qA_\mu) &= u_\nu \partial_\mu (qA_\nu) \\
    &= u_\nu \partial_\mu (mu_\nu + qA_\nu) - u_\nu \partial_\mu (mu_\nu) \\
    &= u_\nu \partial_\mu (mu_\nu + qA_\nu) - \frac{1}{2} \partial_\mu (mu_\nu u_\nu) \\
    &= u_\nu \partial_\mu (mu_\nu + qA_\nu) - \frac{1}{2} \partial_\mu (-mc^2) \\
    &= u_\nu \partial_\mu (mu_\nu + qA_\nu) \\
\end{align*}
\]

(2.34)
CHAPTER 2. HIDDEN VARIABLE

Using the notation
\[ K_{\mu\nu} = \partial_\mu (m u_\nu + q A_\nu) - \partial_\nu (m u_\mu + q A_\mu) \]  
(2.35)

Eq.(2.34) is given by
\[ u_\nu K_{\mu\nu} = 0 \]  
(2.36)

Because \( K_{\mu\nu} \) contains the variables \( \partial_\mu u_\nu, \partial_\mu A_\nu, \partial_\nu u_\mu \) and \( \partial_\nu A_\mu \), they are independent from \( u_\nu \), then a solution satisfying Eq.(2.36) is actually
\[ K_{\mu\nu} = 0 \]  
(2.37)
\[ \partial_\mu (m u_\nu + q A_\nu) = \partial_\nu (m u_\mu + q A_\mu) \]  
(2.38)

The above equation allows us to introduce a potential function \( \Phi \) in mathematics, further set \( \Phi = -\bar{\hbar} \ln \psi \), we obtain a very important equation
\[ (m u_\mu + q A_\mu) = \partial_\mu \Phi \]  
(2.39)
\[ (m u_\mu + q A_\mu) \psi = -i \bar{\hbar} \partial_\mu \psi \]  
(2.40)

Where \( \psi \) may be a complex mathematical function, its physical meanings will be determined from experiments after the introduction of the Planck's constant \( \bar{\hbar} \), as we have know, it is wave function.

Substituting Eq.(2.40) into \( u_\mu u_\mu = -c^2 \), we obtain a wave equation
\[ (-i \bar{\hbar} \partial_\mu \psi - q A_\mu \psi)(-i \bar{\hbar} \partial_\mu \psi - q A_\mu \psi) = -m^2 c^2 \psi^2 \]  
(2.41)

It is a new quantum wave equation[4][5]. Where the left side corresponds to the product of momentum and momentum itself, does not correspond to the product of momentum operator and momentum operator.

In this subsection, by using the \( \langle f_{UC} \rangle = 0 \) statistics and 4-vector velocity field in the ensemble space, we construct a relation between wave function and particle momentum, as we know, the relation has been widely used in the quantum mechanics.

After we introduce the UC into the quantum mechanics, the quantum mechanics becomes reasonable at its key points, we think that we have indirectly proved the existence of UC outside the electromagnetism, and the 4-vector electromagnetic force of Maxwell's theory has two independent components in the 4 dimensional space-time.
2.5 Definition of UC

Although UC shows us that it is a key role for solving a series problems such as hidden variable of the quantum mechanics, non-radiation of stationary hydrogen atom, originating Planck constant, etc., directly giving out a definition of UC seems to be very difficult, because before we can make a definition of UC, many fundamental works must be done, this is an arduous work. Actually, we have done, we will define the UC as a matter in terms of Z-electromagnetism in the behind chapters.

2.6 Conclusions

In this chapter, we prove that the 4-vector electromagnetic force of Maxwell’s theory has two independent components in the 4 dimensional space time. A 4-vector force has 4 components, because of the orthogonality of 4-vector force and 4-vector velocity, the number of independent components of the force reduces to 3, however the 4-vector electromagnetic force can merely provide 2 independent components, this situation means that there is an undefined component accompanying the 4-vector electromagnetic force. We also briefly and confirmedly discuss the possibility that the undefined component of the 4-vector electromagnetic force serves as a hidden variable for the quantum mechanics. The undefined component is regarded as a fluctuating source that contributes to Lorentz force, so that the quantum wave equation can be derived out in the ensemble space of particle motion from the relativistic Newton’s second law.
Chapter 3

Fine structure of hydrogen atom

3.1 Introduction

Hydrogen atom is one of few systems that can be exactly solved using quantum wave equation, it has become a test model for any physical theory. In this chapter, we will use the momentum wavefunction relation derived in the preceding chapter to calculate the fine structure and electronic spin of hydrogen atom. This approach provides an insight into the foundation of the quantum mechanics and brings out a new aspect of the quantum mechanics. The purpose of the calculation is to test our formalism based on the momentum wavefunction relation, to understand the new aspect of the formalism for defining the hidden variable.

Consider a particle of rest mass $m$ and charge $q$ moving in an inertial frame of reference with the relativistic 4-vector velocity $u_{\mu}$, it satisfies\(^\text{[1]}\)

\[ u_{\mu}u_{\mu} = -c^2 \quad (3.1) \]

Where there is not distinction between covariant and contravariant components in the Cartesian coordinate system. Eq.\((3.1)\) is just the relativistic energy-momentum relation when multiplying it by squared mass. Let $A_{\mu}$
CHAPTER 3. FINE STRUCTURE OF HYDROGEN ATOM

denote the vector potential of the electromagnetic field, substituting the momentum wavefunction relation\[5\][6]

\[ mu_\mu = \frac{1}{\psi}( -i\hbar \partial_\mu - qA_\mu)\psi \] (3.2)

into Eq.(3.1), we obtain a new quantum wave equation\[5\] with single component wavefunction

\[ [(-i\hbar \partial_\mu - qA_\mu)\psi][(-i\hbar \partial_\mu - qA_\mu)\psi] = -m^2c^2\psi^2 \] (3.3)

Note, taking the right side of Eq.(3.2) as momentum operator to replace momentum in a physical equation is a traditional usage, but in this chapter we directly use the right side of Eq.(3.2) as momentum, as we have done in Eq.(3.3). Please note that Eq.(3.3) is not the Klein-Gordon wave equation.

In the recent years, H. Y. Cui\[4, 7, 13\] has reported some significant results of Eq.(3.3) for certain physical systems. This chapter points out that the fine structure of hydrogen atom can also be calculated by using Eq.(3.3), however its wavefunction has only one single component in contrast with Dirac’s wavefunction (with four components), the spin effect of electron in magnetic field can also be revealed by Eq.(3.3). The present calculation totally does not need multi-component wavefunction for specifying electronic spin, and indicates that we have an alternative approach to recognize spin concept.

### 3.2 The fine structure of hydrogen atom energy

In the following, we use Gaussian units, and use \( m_e \) to denote the rest mass of electron.

In a spherical polar coordinate system \((r, \theta, \varphi, ic\tau)\), the nucleus of a hydrogen atom provides a spherically symmetric potential \( V(r) = e/r \) for the electron motion, as shown in Figure 3.1.

The wave equation (3.3) for the hydrogen atom in the energy eigenstate \( \psi(r, \theta, \varphi)e^{-iEt/\hbar} \) may be written in the spherical coordinates:
By substituting $\psi = R(r)X(\theta)\phi(\varphi)$, we separate the above equation into

$$\frac{m_e^2c^2}{\hbar^2}\psi^2 = \left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)^2 + \left(\frac{1}{r \sin \theta}\frac{\partial \psi}{\partial \varphi}\right)^2 + \frac{1}{\hbar^2 c^2} \left(E + \frac{e^2}{r}\right)^2 \psi^2 \quad (3.4)$$

By substituting $\psi = R(r)X(\theta)\phi(\varphi)$, we separate the above equation into

$$\left(\frac{\partial \phi}{\partial \varphi}\right)^2 + \kappa \phi^2 = 0 \quad (3.5)$$

$$\left(\frac{\partial X}{\partial \theta}\right)^2 + \left[\lambda - \frac{\kappa}{\sin^2 \theta}\right] X^2 = 0 \quad (3.6)$$

$$\left(\frac{\partial R}{\partial r}\right)^2 + \left[\frac{1}{\hbar^2 c^2} \left(E + \frac{e^2}{r}\right)^2 - \frac{m_e^2 c^2}{\hbar^2} - \frac{\lambda}{r^2}\right] R^2 = 0 \quad (3.7)$$

Where $\kappa$ and $\lambda$ are constants introduced for the separation. Eq.(3.5) can
be solved immediately, with the requirement that $\phi(\varphi)$ must be a periodic function, we find its solution given by

$$\phi = C_1 e^{\pm i \sqrt{\kappa} \varphi} = C_1 e^{im \varphi} \quad m = \pm \sqrt{\kappa} = 0, \pm 1, ... \quad (3.8)$$

Where $C_1$ is an integral constant.

It is easy to find the solution of Eq. (3.6), it is given by

$$X(\theta) = C_2 e^{\pm i \int \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} d\theta} \quad (3.9)$$

Where $C_2$ is an integral constant. The requirement of periodic function for $X$ demands

$$\int_0^{2\pi} \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} d\theta = 2\pi k \quad k = 0, 1, 2, ... \quad (3.10)$$

This complex integration is evaluated (see the appendix for the details), we get

$$\int_0^{2\pi} \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} d\theta = 2\pi (\sqrt{\lambda} - |m|) \quad (3.11)$$
	hus, we obtain

$$\sqrt{\lambda} = k + |m| \quad (3.12)$$

We rename the integer $\lambda$ as $j^2$ for a convenience in the following, i.e. $\lambda = j^2$.

The solution of Eq. (3.7) is given by

$$R(r) = C_3 e^{\pm \frac{\lambda}{2m_e} \int \sqrt{(E + \frac{e^2}{r})^2 - m_e^2 c^4 - \frac{\lambda h^2 c^2}{r^2}} dr} \quad (3.13)$$

Where $C_3$ is an integral constant. The radical wavefunction forms a "standing wave" in the range from $r = 0$ to $r = \infty$, it demands
This complex integration is evaluated in the appendix, we get
\[
\frac{1}{\hbar c} \int_0^\infty \sqrt{(E + \frac{e^2}{r})^2 - m_e^2 c^4 - \frac{\lambda \hbar^2 c^2}{r^2}} dr = \pi s
\]
\[s = 0, 1, 2, ... \tag{3.14}\]

Where \( \alpha = \frac{e^2}{\hbar c} \) is known as the fine structure constant.

From the last Eq. (3.14) and Eq. (3.15), we obtain the energy levels given by
\[
E = m_e c^2 \left[ 1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2} + s)^2} \right]^{-\frac{1}{2}} \tag{3.16}\]

Where \( j = \sqrt{\lambda} = k + |m| \). Because \( j \neq 0 \) in Eq. (3.16), we find \( j = 1, 2, 3, ... \).

The result, Eq. (3.16), is completely the same as that in the calculation of the Dirac wave equation [9] for the hydrogen atom, it is just the fine structure of hydrogen atom energy.

### 3.3 The electronic spin

If we put the hydrogen atom into an external uniform magnetic field \( B \) which is along the \( x_3 \) axis with the vector potential \( (A_r, A_\theta, A_\varphi) = (0, 0, 1\frac{r}{2} \sin \theta B) \), then according to Eq. (3.14), as shown in Figure 3.2.

The wave equation is given by
\[
\frac{m_e^2 c^2}{\hbar^2} \psi^2 = \left( \frac{\partial \psi}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{1}{\hbar^2 c^2} (E + \frac{e^2}{r})^2 \psi^2
\]
\[+ \left( \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \varphi} - \frac{e r \sin \theta B}{i 2 \hbar} \right)^2 \tag{3.17}\]
By substituting $\psi = R(r)X(\theta)\phi(\varphi)$, we separate the above equation into

$$\frac{\partial \phi}{\partial \varphi} + \kappa \phi = 0 \quad (3.18)$$

$$\left( \frac{\partial X}{\partial \theta} \right)^2 + \left[ \left( \frac{-\kappa}{\sin \theta} - \frac{e\sin \theta r^2 B}{i2\hbar} \right)^2 + \xi(r) \right] X^2 = 0 \quad (3.19)$$

$$\left( \frac{\partial R}{\partial r} \right)^2 + \left[ \frac{1}{\hbar^2 c^2} \left( E + \frac{e^2}{r} \right)^2 - \frac{m^2 c^2}{\hbar^2} - \frac{\xi(r)}{r^2} \right] R^2 = 0 \quad (3.20)$$

Where we have used the unknown constant $\kappa$ and function $\xi(r)$ to connect these separated equations. Eq. (3.18) has the solution

$$\phi = C_1 e^{-im\varphi}, \quad \kappa = im, \quad m = 0, \pm 1, \pm 2, \ldots \quad (3.21)$$
Expanding Eq. (3.19) and neglecting the term $O(B^2)$, we find a constant term $-\frac{mer^2B}{ch}$ in it, by moving this term into Eq. (3.20) through $\xi(r) = \lambda + \frac{mer^2B}{ch}$, we obtain

$$
\left(\frac{\partial X}{\partial \theta}\right)^2 + [\lambda - \frac{m^2}{\sin^2\theta}]X^2 = 0
$$

(3.22)

$$
\left(\frac{\partial R}{\partial r}\right)^2 + \left[\frac{1}{\hbar^2c^2}(E + \frac{e^2}{r})^2 - \frac{m^2e^2}{\hbar^2} - \frac{meB}{ch} - \frac{\lambda}{r^2}\right]R^2 = 0
$$

(3.23)

The above two equations are the same as Eq. (3.6) and Eq. (3.7), except for the additional constant term $-meB/ch$. After the similar calculation as the preceding section, we obtain the energy levels of hydrogen atom in the magnetic field, given by

$$
E = \sqrt{m^2c^4 + mechB \left[1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2 + s})^2}\right]}^{-\frac{1}{2}}
$$

(3.24)

In the usual spectroscopic notation of quantum mechanics, four quantum numbers: $n$, $l$, $m_l$ and $m_s$ are used to specify the state of an electron in an atom. After the comparison, we get the relations between the usual notation and our notation.

$$
n = j + s, \quad s = 0, 1, \ldots; j = 1, 2, \ldots \quad (3.25)
$$

$$
l = j - 1 \quad (3.26)
$$

$$
\text{max}(m_l) = \text{max}(m) - 1 \quad (3.27)
$$

We find that $j$ takes over $1, 2, \ldots, n$; for a fixed $j$ (or $l$), $m$ takes over $-(l + 1), -l, \ldots, 0, \ldots, l, l + 1$. In the present work, spin quantum number is absent.

According to Eq. (3.24), for a fixed $(n, l)$, equivalent to $(n, j = l + 1)$, the energy level of hydrogen atom will split into $2l + 3$ energy levels in the magnetic field, given by
\[
E = (m_e c^2 + \frac{me \hbar B}{2m_e c}) \left[ 1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2 + s}^2) \right]^{-\frac{1}{2}} + O(B^2) \quad (3.28)
\]

Considering \(m = -(l+1), -l, ..., 0, ..., l, l+1\), this effect is equivalent to the usual Zeeman splitting in the usual quantum mechanics, given by

\[
E = E_{nl} + \frac{(m_l \pm 1)e \hbar B}{2m_e c} \quad (3.29)
\]

But our work works on it without spin concept, the so-called spin effect has been revealed by Eq. (3.28) without spin concept, this result indicates that electronic spin is a kind of orbital motion.

In Stern-Gerlach experiment, the angular momentum of ground state of hydrogen atom is presumed to be zero according to the usual quantum mechanics, thus ones need make use of the spin. But in the present calculation, the so-called spin has been merged with the orbital motion of the electron.

Recalling that the spin is a mysterious concept even for very learned persons, we have come to this point: we can understand the quantum mechanics without the well-established spin concept, can’t we? without multi-component wavefunction, can’t we? Bear in mind that simplicity is always a merit for the physics.

### 3.4 Conclusion

Using equations

\[
(mu_\mu + qA_\mu)\psi = -i\hbar \partial_\mu \psi \quad (3.30)
\]

\[
u_\mu u_\mu = -c^2 \quad (3.31)
\]

by eliminating \(u_\mu\), the above equations can be written as
\[ -m^2c^2\psi^2 = \left[ (-i\hbar \partial_\mu - qA_\mu)\psi \right]\left[ (-i\hbar \partial_\mu - qA_\mu)\psi \right] \quad (3.32) \]

Using this quantum wave equation, the fine structure of hydrogen atom energy is calculated, as the results, the energy levels are given by

\[
E = m_e c^2 \left[ 1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2} + n)^2} \right]^{-\frac{1}{2}} \quad (3.33)
\]

\[
j = 1, 2, \ldots; n = 0, 1, \ldots
\]

the result is completely the same as that in the calculation of the Dirac wave equation for the hydrogen atom, while the wavefunction is quite different from Dirac’s wavefunction. Besides, the present calculation brings out spin nature in a new way, indicating that electronic spin is a kind of orbital motion. The present calculation is characterized by using the usual momentum-wavefunction relation directly, it provides an insight into the foundations of the quantum mechanics.

### 3.5 Appendix: Evaluations of integrations

In this appendix we give out the evaluations of the integrations appeared in the preceding sections, i.e.\text{Eq.}(3.11) and \text{Eq.}(3.15).

#### 3.5.1 wave-attenuating boundary condition

Consider the integrand in \text{Eq.}(3.11), it is a multiple-valued function, may be written as

\[
\sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} = \sqrt{f(\theta)} \quad f(\theta) = \lambda - \frac{m^2}{\sin^2 \theta} \quad (3.34)
\]

Suppose that \( \lambda \) is a real positive number, the function \( f(\theta) \) may be divided into the three regions: \((0,a), (a,b), \) and \((b,\pi)\), where \( a \) and \( b \) are...
the turning points at where the function \( f(\theta) \) changes its sign, as shown in Figure 3.3. Like \( \sqrt{-5} = \pm i\sqrt{5} \) or \((\pm i\sqrt{5})^2 = -5\), we find

\[
\int_0^a \sqrt{f(\theta)}d\theta = \int_0^a \sqrt{-|f(\theta)|}d\theta = \pm i \int_0^a \sqrt{|f(\theta)|}d\theta = \pm iA
\]

\[
(3.35)
\]

\[
\int_a^b \sqrt{f(\theta)}d\theta = \int_a^b \sqrt{|f(\theta)|}d\theta = B
\]

\[
(3.36)
\]

\[
\int_0^\pi \sqrt{f(\theta)}d\theta = \int_0^\pi \sqrt{-|f(\theta)|}d\theta = \pm iA
\]

\[
(3.37)
\]

Where \( A \) and \( B \) are two real numbers, then the integration of Eq. (3.9) has three possible solutions given by

\[
\int_0^{2\pi} \sqrt{\lambda - \frac{m^2}{\sin^2\theta}}d\theta = \begin{cases} 
2(B + 2iA) \\
2B \\
2(B - 2iA)
\end{cases}
\]

\[
(3.38)
\]

The second branch of this result is reasonable, because only it can fulfil the requirement that the wavefunction is a periodic function of \( \theta \). The multiple-valued result arises from \( \sqrt{-|f(\theta)|} = \pm i\sqrt{|f(\theta)|} \).

How to determine the sign of the multiple-valued function reasonably? Let us turn to our experience that we have had in the usual quantum mechanics. Consider the motion of a particle in a finitely deep potential well as shown in Figure 3.4, there are also two turning points \( a \) and \( b \). If the particle moves over the turning point \( a \) or \( b \) for \( E < V_0 \) (bound states), its momentum will become imaginary \( \pm i|p| \) with uncertain sign. As we know in the usual quantum mechanics the wavefunction is given by

\[
\psi(x) = \begin{cases} 
De^{-i\frac{|p|}{\hbar}x} & x < a \\
G \sin(\frac{|p_0|}{\hbar}x + \delta) & a < x < b \\
De^{-i\frac{|p|}{\hbar}x} & x > b
\end{cases}
\]

\[
(3.39)
\]
This has involved our experience in which we have taken plus sign for the imaginary momentum in $x < a$ and minus sign in $x > b$, to satisfy the so-called wave-attenuating boundary condition for the regions over the turning points.

Figure 3.3: The function has sign-changed points $a$ and $b$.

Figure 3.4: A finitely deep potential well that has two turning points $a$ and $b$.

In the followings, we use this wave-attenuating boundary condition to determine the sign of double-valued imaginary momentum: take plus sign in the region over the left turning point, whereas take minus sign in the region over the right turning point.
3.5.2 integration 1

To apply the wave-attenuating boundary condition to the following wave-function

\[ X(\theta) = C_2 e^{-i \int \frac{\lambda - m^2}{\sin^2 \theta} d\theta} \]  

(3.40)

due to wave-attenuating for the turning points, the integrand must choose the signs as

\[ \int_a^0 \sqrt{\lambda - m^2 \sin^2 \theta} d\theta = +i \int_a^0 \sqrt{|f(\theta)|} d\theta = iA \]  

(3.41)

\[ \int_a^b \sqrt{\lambda - m^2 \sin^2 \theta} d\theta = \int_a^b \sqrt{f(\theta)} d\theta = B \]  

(3.42)

\[ \int_b^\pi \sqrt{\lambda - m^2 \sin^2 \theta} d\theta = -i \int_b^\pi \sqrt{|f(\theta)|} d\theta = -iA \]  

(3.43)

thus the integration may have a real solution, actually it may be written as

\[ \int_0^{2\pi} \sqrt{\lambda - m^2 \sin^2 \theta} d\theta = 2 \int_a^b \sqrt{\lambda - m^2 \sin^2 \theta} d\theta = 2B \]  

(3.44)

In order to evaluate the definite integral of Eq. (3.44), we make use of contour integral in complex plane [10]. Consider a contour \( C_\delta \) which is a unit circle around zero, as shown in Figure 3.5, using \( z = e^{i\theta} \), we have

\[ I_1 = \int_0^{2\pi} \sqrt{\lambda - m^2 \sin^2 \theta} d\theta = \int_{C_\delta} \sqrt{\lambda + \frac{4m^2 z^2}{(z^2 - 1)^2}} \frac{dz}{iz} \]

(3.45)

As we have known that \( \sqrt{f(\theta)}|_{\theta=\pi/2, or \theta=3\pi/2} = \sqrt{\lambda - m^2} \), substituting \( z = i \) or \( z = -i \) into the above integrand, we find the integrand must take the minus sign. Thus choose the minus sign for it.
Figure 3.5: Unit circle contour for evaluating the integral

\[
I_1 = \int_{C_\delta} \frac{\sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2} \,dz}{iz} \tag{3.46}
\]

For scrutinizing the sign of the integrand over the turning points, we have

\[
\sqrt{f(\theta)} = \frac{\sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2}}{-(z^2 - 1)}
= \frac{\sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2}}{-(2iz)(z^2 - 1)/(2iz)}
= \frac{\sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2}}{-(2iz)\sin \theta}
= i\frac{\sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2}}{2z \sin \theta} \tag{3.47}
\]

we find that the integrand takes plus sign over the left turning point \((\theta \rightarrow 0^+, z \rightarrow 1)\) and minus sign over the right turning point \((\theta \rightarrow \pi^-, z \rightarrow -1)\), in accordance with the sign requirement of Eq.\((3.41)\) and \((3.43)\).

Continue our calculation, we have

\[
I_1 = \int_{C_\delta} \frac{\sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2} \,dz}{iz}
\]
\[ \int_{C_\delta} \frac{1}{z - 1} - \frac{1/2}{z + 1} \sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2} \, dz \]

\[ (3.48) \]

Now we find that the integrand has the poles at \( z = 0 \) and \( z = \pm 1 \). We let the counter \( C_\delta \) pass by the pole \( z = +1 \) through the interior of the unite circle, as indicated by the dash line in Figure 3.5, likewise, let the counter \( C_\delta \) pass by the pole \( z = -1 \) through the exterior of the unite circle. The deformation made for \( C_\delta \) has no influence on the integration value because the left deformation cancels the right deformation in the integration due to the opposite signs of the integrand near the left and right poles. Let \( C'_\delta \) denote the deformed counter, we continue the calculation by using the residue theorem.

\[ I_1 = \int_{C'_\delta} \frac{1}{z - 1} - \frac{1/2}{z + 1} \sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2} \, dz \]

\[ = \int_{C'_\delta} \frac{1}{z} \sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2} \, dz \]

\[ - \int_{C'_\delta} \frac{1/2}{z - 1} \sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2} \, dz \]

\[ - \int_{C'_\delta} \frac{1/2}{z + 1} \sqrt{\lambda(z^2 - 1)^2 + 4m^2z^2} \, dz \]

\[ = \int_{C'_\delta} \sqrt{\lambda} + O(z^2) \, dz \]

\[ - \int_{C'_\delta} |m| + O(z^2 - 1) \, dz \]

\[ = 2\pi(\sqrt{\lambda} - |m|) \]  

\[ (3.49) \]

### 3.5.3 Integration 2

To apply the wave-attenuating boundary condition to the following wave-function

\[ R(r) = C_3 e^{-\frac{\pi}{\hbar} \int \sqrt{(E + \frac{e^2}{r})^2 - m^2c^4 - \frac{4\hbar^2e^2}{r^2}} \, dr} \]  

\[ (3.50) \]
Where it has also two turning points \( r_1 \) and \( r_2 \) from \( r = 0 \) to \( r = \infty \) when \( E^2 < m_e^2 c^4 \) (bound states), as shown in Figure 3.6 where

\[
\sqrt{g(r)}|_{r \to 0} = \sqrt{e^4 - j^2 \hbar^2 c^2 / r} = i \sqrt{j^2 \hbar^2 c^2 - e^4 / r}
\]

(3.52)

\[
\sqrt{g(r)}|_{r \to \infty} = \sqrt{E^2 - m_e^2 c^4} = -i \sqrt{m_e^2 c^4 - E^2}
\]

(3.53)

In order to evaluate the definite integral of Eq. (3.15), consider a contour \( C \) consisting of \( C_\gamma \), \( L_- \), \( C_\delta \) and \( L \) around zero in the plane as shown in Figure 3.7, the radius of circle \( C_\gamma \) is large enough and the radius of circle \( C_\delta \) is small enough. The integrand of the following equation has no pole inside the contour \( C \), so that we have

\[
\int_C \sqrt{(E + \frac{e^2}{z})^2 - m_e^2 c^4 - \frac{j^2 \hbar^2 c^2}{z^2}} dz =
\]
Now we evaluate the integration on each contour with our sign choice for the double-valued function.

\[
\begin{align*}
\int_{C_{\gamma}} &= \int_{C_{\gamma}} i \sqrt{m_e^2 c^4 + \frac{j^2 \hbar^2 c^2}{z^2} - (E + \frac{e^2}{z})^2} \, dz \\
&= -i \int_{C_{\gamma}} \frac{2\pi i E e^2}{\sqrt{m_e^2 c^4 - E^2}} = -\frac{2\pi E e^2}{\sqrt{m_e^2 c^4 - E^2}} \tag{3.55}
\end{align*}
\]

\[
\int_{C_{\delta}} = \int_{C_{\delta}} i \sqrt{m_e^2 c^4 + \frac{j^2 \hbar^2 c^2}{z^2} - (E + \frac{e^2}{z})^2} \, dz \\
&= i \int_{C_{\delta}} \frac{\sqrt{m_e^2 c^4 z^2 + j^2 \hbar^2 c^2 - (E z + e^2)^2}}{z} \, dz
\]
= \int_{C_\delta} i \frac{\sqrt{j^2 h^2 c^2 - e^4 + O(z)}}{z} dz
= i(-2\pi i)\sqrt{j^2 h^2 c^2 - e^4}
= 2\pi \sqrt{j^2 h^2 c^2 - e^4} \quad (3.56)

Because the integrand is a multiple-valued function, when the integral takes over the path \( L_\gamma \) we have \( z = e^{i2\pi r}e^{0i} \), thus

\[
\int_{L_\gamma} = \int_{L_\gamma} \sqrt{e^{-i2\pi(\ldots)}} = -\int_{L_\delta} = \int_{L_\delta} = \int_{L} \quad (3.57)
\]

For a further manifestation, to define \( z - H = w = \rho e^{i\beta} \), where

\[
H = \frac{m^2 c^4 r^2 + j^2 h^2 c^2 - E^2 r^2 - e^4}{2Ee^2} \quad (3.58)
\]

we have

\[
\int_{L_\gamma} = \int_{L_\gamma} \frac{\sqrt{2Ee^2 \sqrt{z - H}}}{z} dz = \int_{L_\delta} \frac{\sqrt{2Ee^2 \sqrt{w}}}{z} dz
= \int_{L_\gamma} \frac{\sqrt{2Ee^2 e^{i\beta/2}}}{z} dz
= \int_{L(\gamma \rightarrow \delta)} \frac{\sqrt{2Ee^2 e^{i(\beta+2\pi)/2}}}{ze^{i2\pi}} d(ze^{i2\pi})
= -\int_{L(\gamma \rightarrow \delta)} \frac{\sqrt{2Ee^2 e^{i\beta/2}}}{z} dz
= \int_{L(\delta \rightarrow \gamma)} \frac{\sqrt{2Ee^2 e^{i\beta/2}}}{z} dz = \int_{L} \quad (3.59)
\]

Where we have use the relation of \( z \) and \( w \) in the fourth step of the above equation, as shown in Figure 3.8, to note that \( w \) rotates around zero with \( z \). Thus we have
Thus Eq. (3.15) becomes

\[
\frac{1}{\hbar c} \int_0^\infty \sqrt{(E + \frac{e^2}{r})^2 - m_e^2 c^4 - \frac{j^2 \hbar^2 c^2}{r^2}} \, dr = \frac{\pi E \alpha}{\sqrt{m_e^2 c^4 - E^2}} - \pi \sqrt{j^2 - \alpha^2}
\]  

(3.61)

Where \( \alpha = e^2/\hbar c \) is known as the fine structure constant.

### 3.5.4 discussion: the motion over turning points

Following the sign change for imaginary momentum over turning points, discussed in the preceding sections, we find that the periodic conditions in hydrogen atom may written as
\[
2 \int_a^b \sqrt{\lambda - \frac{m^2}{\sin^2 \theta}} d\theta = 2\pi k \tag{3.62}
\]

\[
\frac{1}{\hbar c} \int_{r_1}^{r_2} \sqrt{(E + \frac{e^2}{r})^2 - m_e^2 c^4 - \frac{\lambda \hbar^2 c^2}{r^2}} dr = \pi s \tag{3.63}
\]

because the integrations in the regions over the turning points are eliminated automatically. What are their physical meanings? A direct explanation is that it is not necessary for the electron to enter the regions over the turning points, in compliance with the classical physics.

In addition, the residue theorem we used in the paper gives out accurate results for our integrations, not approximate ones.
Chapter 4

Momentum path integral

4.1 Introduction

In the quantum mechanics, the momentum wavefunction relation is expressed as

$$m u_\mu = \frac{1}{\psi} (-i\hbar \partial_\mu - qA_\mu) \psi$$  \hspace{1cm} (4.1)

Where we consider a particle of rest mass $m$ and charge $q$ moving at the relativistic 4-vector velocity $u_\mu$. The author is always puzzled with this question: why the physics pays a little attention to Eq.\((4.1)\), compared with Schrodinger wave equation or Dirac wave equation. Since the author never find any real fault of Eq.\((4.1)\) in the early works of the quantum mechanics, the answer seems to be: this equation has not yet shown us its apparent usefulness for solving quantum systems.

Its integral form is given by

$$\psi = e^{i\frac{\hbar}{\psi} \int_{x_0}^{x} (p_\mu + qA_\mu) dx_\mu}$$  \hspace{1cm} (4.2)

the momentum of the particle is $p_\mu = mu_\mu$, the integral takes over an arbitrary path $l$ from the initial point $x_0$ to the final point $x$, it is of path
independence. In the Cartesian coordinate system \((x_1, x_2, x_3, x_4 = ict)\), the 4-vector velocity and 4-vector momentum satisfy

\[
\begin{align*}
    u_\mu u_\mu &= -c^2 \\ 
    p_1^2 + p_2^2 + p_3^2 + p_4^2 &= -m^2 c^2
\end{align*}
\]  

(4.3)  

(4.4)

In the present chapter, Eq.(4.2) is called the momentum path integral, obviously it is different from Feynman’s path integral. In order to investigate the validity and usefulness of Eq.(4.2), we apply the momentum path integral to hydrogen atom. As the result, the energy levels are calculated out with the same fine structure and spin effect as Dirac wave equation. Thus, this calculation shows that the Eq.(4.1) is useful for solving quantum problems through the instance of the hydrogen atom.

Developing the momentum path integral method has very important significance on the framework of quantum mechanics: (1) the momentum path integral uses single component wave function while Dirac’s wave equation uses four-component wave function. Spin effect is accessible by the single component wave function of the momentum path integral, it indicates that the method give us a useful tool for dealing with elementary particle properties; (2) the momentum path integral method is much simple in solving problems than the Dirac’s theory, because it only concerns with integrations rather than differential equations [1]

4.2 Application of momentum path integral to hydrogen atom’s fine structure

4.2.1 fine structure of hydrogen atom

In the followings, we use Gaussian units, and use \(m_e\) to denote the rest mass of electron. In a spherical polar coordinate system \((r, \theta, \varphi, ict)\), as shown in Figure 4.1.

The nucleus of hydrogen atom provides a spherically symmetric potential \(V(r) = e/r\) for the electron \((q = -e)\), the displacement elements and
vector potential are given by
\[ dx_r = dr \quad (4.5) \]
\[ dx_\theta = rd\theta \quad (4.6) \]
\[ dx_\varphi = r \sin \theta d\varphi \quad (4.7) \]
\[ A_r = A_\theta = A_\varphi = 0 \quad (4.8) \]
\[ A_4 = iV = ie/r \quad (4.9) \]

Then, the wave function is given by
\[ \psi = e^{i \frac{\hbar}{\kappa} \int p_r dx_r} e^{i \frac{\hbar}{\kappa} \int p_\theta dx_\theta} e^{i \frac{\hbar}{\kappa} \int p_\varphi dx_\varphi} e^{i \frac{\hbar}{\kappa} \int (p_4 + qA_4/c) dx_4} \quad (4.10) \]

Here and below we drop the marks such as integral path \( l \), the initial point \( x_0 \) and the final point \( x \), if possible. For separating the variables, we use
\[\psi = R(r)X(\theta)\phi(\varphi)e^{-iEt/\hbar}\] for energy eigenstates, we expect

\[\phi(\varphi) = e^{\frac{i}{\hbar}\int p_\varphi dx_\varphi}\] (4.11)
\[X(\theta) = e^{\frac{i}{\hbar}\int p_\theta dx_\theta}\] (4.12)
\[R(r) = e^{\frac{i}{\hbar}\int p_r dx_r}\] (4.13)
\[e^{-iEt/\hbar} = e^{\frac{i}{\hbar}\int_0^t(p_4 + qA_4/c)dx_4}\] (4.14)

The angular momentum magnitude and its z-axis component magnitude are denoted by \(J\) and \(J_z\) respectively, we have

\[p_\varphi r \sin \theta = J_z \quad (const.)\] (4.15)
\[\left(\sqrt{p_\theta^2 + p_\varphi^2}\right) r = J \quad (const.)\] (4.16)

From Eq. (4.14), we have

\[p_4 = \frac{-E - icqA_4/c}{ic} = \frac{-E - e^2/r}{ic}\] (4.17)

and we have

\[p_r = \pm \sqrt{-m^2c^2 - p_\theta^2 - p_\varphi^2 - p_4^2}\] (4.18)
\[= \pm \sqrt{-m^2c^2 - \frac{J^2}{r^2} + \frac{1}{c^2}(E + \frac{e^2}{r})^2}\] (4.19)

thus we have

\[\phi(\varphi) = e^{\frac{i}{\hbar}\int p_\varphi dx_\varphi} = C_1 e^{\frac{i}{\hbar}J_z \varphi}\] (4.20)
\[X(\theta) = e^{\frac{i}{\hbar}\int p_\theta dx_\theta} = C_2 e^{\pm \frac{i}{\hbar}\int_0^\theta \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} d\theta}\] (4.21)
\[R(r) = C_3 e^{\pm \frac{i}{\hbar}\int_0^r \sqrt{-m^2c^2 - \frac{J^2}{r^2} + \frac{1}{c^2}(E + \frac{e^2}{r})^2} dr}\] (4.22)
Where $C_1$, $C_2$ and $C_3$ are integral constants. Since $\phi(\varphi)$ and $X(\theta)$ must be periodic functions, and the radical wave function $R(r)$ spans the range from $r = 0$ to $r = \infty$, these requirements demand

\[
\frac{1}{\hbar} \int_0^{2\pi} J_z d\varphi = 2\pi m \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{4.23}
\]

\[
\frac{1}{\hbar} \int_0^{2\pi} \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} d\theta = 2\pi k \quad (k = 0, 1, 2, \ldots) \tag{4.24}
\]

\[
\frac{1}{\hbar} \int_0^{\infty} \sqrt{-m_c^2 c^2 - \frac{J_z^2}{r^2} + \frac{1}{c^2} (E + \frac{e^2}{r})^2} dr = \pi s \quad (s = 0, 1, 2, \ldots) \tag{4.25}
\]

To note that the last two integrands may become multiple-valued functions when over their “turning points”. These definite integrals have been evaluated in the Appendix using the residue theorem and contour integrations in complex space, the results are given by

\[
J_z = m\hbar \tag{4.26}
\]

\[
\frac{1}{\hbar} \int_0^{2\pi} \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} d\theta = \frac{2\pi}{\hbar} (J - |J_z|) \tag{4.27}
\]

\[
J = (k + |m|)\hbar = j\hbar \tag{4.28}
\]

\[
\frac{1}{\hbar} \int_0^{\infty} \sqrt{-m_c^2 c^2 - \frac{J_z^2}{r^2} + \frac{1}{c^2} (E + \frac{e^2}{r})^2} dr = \frac{\pi E\alpha}{\sqrt{m_c^2 c^4 - E^2}} - \pi \sqrt{j^2 - \alpha^2} = \pi s \tag{4.29}
\]

Where $\alpha = e^2/\hbar c$ is known as the fine structure constant.

From the last Eq. (4.29), we obtain the energy levels given by
\[ E = m_e c^2 \left[ 1 + \frac{\alpha^2}{(\sqrt{j^2 - \alpha^2} + s)^2} \right]^{-\frac{1}{2}} \]  (4.30)

Where \( j = k + |m| \), because of the restriction of \( j \neq 0 \) in Eq.(4.30), we find \( j = 1, 2, 3, \ldots \).

The result, Eq.(4.30), is completely the same as the calculation of the Dirac wave equation[9] for the hydrogen atom, it is just the fine structure of hydrogen energy.

4.2.2 electronic spin

If we put the hydrogen atom into an external uniform magnetic field \( B \) which is along the \( x_3 \) axis with the vector potential \( (A_r, A_\theta, A_\phi) = (0, 0, \frac{1}{2} r \sin \theta B) \), i.e. \( B = B e_z \), where \( e_z \) is the unit vector along the \( z \) axis, as shown in Figure 4.2.

According to Eq.(4.32), the energy eigenstate of the hydrogen atom is described by

\[
\begin{align*}
\psi &= R(r)X(\theta)\phi(\phi)e^{-iEt/\hbar} & (4.31) \\
\phi(\phi) &= e^{\frac{i}{\hbar} \int_0^\phi (p_\phi + q A_\phi/c) \, dx_\phi} & (4.32) \\
X(\theta) &= e^{\frac{i}{\hbar} \int_0^\theta p_\theta \, dx_\theta} & (4.33) \\
R(r) &= e^{\frac{i}{\hbar} \int_0^r p_r \, dx_r} & (4.34) \\
e^{-iEt/\hbar} &= e^{\frac{i}{\hbar} \int_0^t (p_4 + q A_4/c) \, dx_4} & (4.35)
\end{align*}
\]

The magnitude of the angular momentum is denoted by \( J \) and its component along the \( z \)-axis by \( J_z \), then

\[
\begin{align*}
p_\phi r \sin \theta &= J_z \quad (\text{const.}) \quad (4.36) \\
(\sqrt{p_\theta^2 + p_\phi^2}) r &= \sqrt{J} \quad (\text{const.}) \quad (4.37)
\end{align*}
\]
we also have the same expressions as

\[ p_4 = -\frac{E - i q A_4}{ic} = -\frac{E - e^2}{r} \]  \hspace{1cm} (4.38)

\[ p_r = \pm \sqrt{-m_e^2 c^2 - p_\theta^2 - p_\varphi^2 - p_4^2} \]
\[ = \pm \sqrt{-m_e^2 c^2 - \frac{J^2}{r^2} + \frac{1}{c^2} (E + \frac{e^2}{r})^2} \]  \hspace{1cm} (4.39)

thus we have

\[ \phi(\varphi) = e^{i \frac{\hbar}{\pi} \int (p_{\varphi} + q A_{\varphi}/c) dx_{\varphi}} \]
\[ = e^{i \frac{\hbar}{\pi} \int (p_{\varphi} + q \frac{1}{2e} r \sin \theta B) r \sin \theta d\varphi} \]
\[ = C_1 e^{i \frac{\hbar}{\pi} (J_z - \frac{1}{2e} r^2 \sin^2 \theta B) \varphi} = C_1 e^{im \varphi} \]  \hspace{1cm} (4.41)
CHAPTER 4. MOMENTUM PATH INTEGRAL

\[ X(\theta) = e^{\frac{i}{\hbar} \int_{p_{0}dx_{0}}^{\theta}} = C_{2}e^{\pm \frac{i}{\hbar} \int_{0}^{\theta} \sqrt{J^{2} - \frac{m^{2}\hbar^{2}}{2c} r^{2} \sin^{2} \theta B^{2}/\sin^{2} \theta} d\theta} \]

\[ \approx C_{2}e^{\pm \frac{i}{\hbar} \int_{0}^{\theta} \sqrt{J^{2} - \frac{m^{2}\hbar^{2}}{2c} \sin^{2} \theta} d\theta} \quad (4.42) \]

\[ R(r) = C_{3}e^{\pm \frac{i}{\hbar} \int_{0}^{r} \sqrt{-m^{2}c^{2} - \frac{J^{2}}{r^{2}} + \frac{1}{c^{2}}(E + \frac{e^{2}}{r})^{2}} dr} \quad (4.43) \]

Where \( C_{1}, C_{2} \) and \( C_{3} \) are integral constants, we have neglected \( O(B^{2}) \) terms. Since \( \phi(\varphi) \) and \( X(\theta) \) must be periodic functions, and the radical wave function \( R(r) \) spans the range from \( r = 0 \) to \( r = \infty \), these requirements demand

\[ \frac{1}{\hbar} \int_{0}^{2\pi} (J_{z} - e^{\frac{1}{2}r^{2}\sin^{2} \theta B})d\varphi = 2\pi m \quad (m = 0, \pm 1, \pm 2...) \quad (4.44) \]

\[ \frac{1}{\hbar} \int_{0}^{2\pi} \sqrt{J^{2} - \frac{m^{2}\hbar^{2}}{2c} \sin^{2} \theta} - \frac{m\hbar r^{2}B}{c} d\theta = 2\pi k \quad (k = 0, 1, 2...) \quad (4.45) \]

\[ \frac{1}{\hbar} \int_{0}^{\infty} \sqrt{-m^{2}c^{2} - \frac{J^{2}}{r^{2}} + \frac{1}{c^{2}}(E + \frac{e^{2}}{r})^{2}} dr = \pi s \quad (s = 0, 1, 2...) \quad (4.46) \]

These definite integrals have been evaluated in the Appendix, given by

\[ J_{z} - e^{\frac{1}{2}r^{2}\sin^{2} \theta B} = mh \quad (4.47) \]

\[ \frac{1}{\hbar} \int_{0}^{2\pi} \sqrt{J^{2} - \frac{m^{2}\hbar^{2}}{2c} - \frac{m\hbar r^{2}B}{c}} d\theta = 2\pi (\frac{1}{\hbar} \sqrt{J^{2} - \frac{m\hbar r^{2}B}{c}} - |m|) \quad (4.48) \]

we get

\[ J^{2} - \frac{m\hbar r^{2}B}{c} = (k + |m|)^{2}\hbar^{2} = j^{2}\hbar^{2} \quad (4.49) \]
we have
\[
\frac{1}{\hbar} \int_{0}^{\infty} \sqrt{-\frac{m_e^2 c^2}{r^2} - \frac{J^2}{r^2} + \frac{1}{c^2} (E + \frac{e^2}{r})^2} \, dr
\]
\[
= \frac{1}{\hbar} \int_{0}^{\infty} \sqrt{-\frac{m_e^2 c^2}{r^2} - \frac{1}{r^2} \left( j^2 \hbar^2 + \frac{m_e \hbar^2 B}{c} \right) + \frac{1}{c^2} (E + \frac{e^2}{r})^2} \, dr
\]
\[
= \frac{\pi E \alpha}{\sqrt{m_e^2 c^4 + m_e \hbar c B - E^2}} - \pi \sqrt{j^2 - \alpha^2} = \pi s
\] (4.50)
we obtain the energy levels of the hydrogen atom in the magnetic field, given by
\[
E = \sqrt{m_e^2 c^4 + m_e \hbar c B} \left[ 1 + \frac{\alpha^2}{\left( \sqrt{j^2 - \alpha^2} + s \right)^2} \right]^{-\frac{1}{2}}
\] (4.51)
In the usual spectroscopic notation of quantum mechanics, four quantum numbers: \( n, l, m_l \) and \( m_s \) are used to specify the state of an electron in an atom. After the comparison, we get the relations between the usual notation and our notation.

\[
n = j + s, \quad s = 0, 1, \ldots; j = 1, 2, \ldots \quad (4.52)
\]
\[
l = j - 1, \quad (4.53)
\]
\[
\text{max}(m_l) = \text{max}(m) - 1 \quad (4.54)
\]
We find that \( j \) takes over \( 1, 2, \ldots, n \); for a given \( j \) (or \( l \)), \( m \) takes over \( -(l + 1), -(l - 1), \ldots, 0, \ldots, l, l + 1 \). In the present work, spin quantum number is absent.

According to Eq. (4.51), for a given \( (n, l) \), equivalent to \( (n, j = l + 1) \), the energy level of hydrogen atom will split into \( 2l + 3 \) energy levels in the magnetic field, given by
\[
E = \left( m_e c^2 + \frac{m_e \hbar B}{2m_e c} \right) \left[ 1 + \frac{\alpha^2}{\left( \sqrt{j^2 - \alpha^2} + s \right)^2} \right]^{-\frac{1}{2}} + O(B^2) \quad (4.55)
\]
Considering $m = -(l + 1), -l, ..., 0, ..., l, l + 1$, this effect is equivalent to the usual Zeeman splitting in the usual quantum mechanics, given by

$$E = E_{nl} + \frac{(m \pm 1)\hbar B}{2mc}$$

(4.56)

But our work works on it without spin concept, the so-called spin effect has been revealed by Eq. (4.55) without spin concept, this result indicates that electronic spin is a kind of orbital motion. In Stern-Gerlach experiment, the angular momentum of ground state of hydrogen atom is presumed to be zero according to the usual quantum mechanics, thus ones need make use of the spin. But in the present calculation, the so-called spin has been merged with the orbital motion of the electron.

Bear in mind that simplicity is always a merit for the physics.

4.3 Discussion

Since the momentum path integral method for quantum mechanics does not need to evaluate quantum wave equation such as the Dirac wave equation (with four-component wavefunction), in addition, the wave function of the path integral is one single component wave function, the momentum path integral method definitely is a rapid quantum computation method. This path integral method provides a great prospects for computer computation in some research fields such as $X\alpha$, ab-initio, LMTO, DV, etc.

The wave function $\psi$ we employed in the calculation for hydrogen atom differs from the wave function in the usual quantum mechanics, because it is found that the wave function $\psi$ keeps $|\psi| = 1$ everywhere in the hydrogen atom. But this kind of wave functions can interfere with each other, for the details, consult the paper [12].

4.4 Conclusion

A momentum path integral method for calculating the quantum states of particle is developed, differing from Feynman’s path integral. It is charac-
CHAPTER 4. MOMENTUM PATH INTEGRAL

55

terized by using the usual momentum wavefunction relation. The momentum path integral method is directly applied to hydrogen atom, the energy levels are calculated out with the same fine structure and spin effect as Dirac wave equation. The momentum path integral method is much simpler in solving problems than the Dirac’s theory, because it only concerns with integrations rather than differential equations. The successful application of the momentum path integral method to hydrogen atom provides an insight into the foundations of the quantum mechanics.

4.5 Appendix: Evaluations of integrations

In this appendix we give out the evaluations of the integrations appeared in the preceding sections, i.e. Eq. (4.24) and Eq. (4.25).

4.5.1 wave-attenuating boundary condition

Consider the integrand in Eq. (4.24), it is a multiple-valued function, may be written as

\[
\sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} = \sqrt{f(\theta)} \quad f(\theta) = J^2 - \frac{J_z^2}{\sin^2 \theta} \quad (4.57)
\]

The function \( f(\theta) \) may be divided into the three regions: \((0, a), (a, b), \) and \((b, \pi)\), where \( a \) and \( b \) are the turning points at which the function \( f(\theta) \) changes its sign, as shown in Figure 4.3, like \( \sqrt{-5} = \pm i \sqrt{5} \) or \((\pm i \sqrt{5})^2 = -5\), so \( \sqrt{-|f(\theta)|} = \pm i \sqrt{|f(\theta)|} \) (corresponding to plus ”angular momentum” and minus ”angular momentum”), we find

\[
\int_0^a \sqrt{|f(\theta)|} d\theta = \int_0^a \sqrt{|f(\theta)|} d\theta = \pm i \int_0^a \sqrt{|f(\theta)|} d\theta = \pm A \quad (4.58)
\]

\[
\int_a^b \sqrt{|f(\theta)|} d\theta = \int_a^b \sqrt{|f(\theta)|} d\theta = B \quad (4.59)
\]

\[
\int_b^\pi \sqrt{|f(\theta)|} d\theta = \int_b^\pi \sqrt{|f(\theta)|} d\theta = \pm i A \quad (4.60)
\]
Where $A$ and $B$ are two real numbers, then the integration of Eq. (4.27) has three possible solutions given by

$$
\int_0^{2\pi} \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} \, d\theta = \begin{cases} 
2(B + 2iA), \\
2B, \\
2(B - 2iA)
\end{cases}
$$

(4.61)

The second branch of this result is reasonable, because only it can fulfill the requirement that the wave function is a periodic function of the variable $\theta$.

How to determine the sign of the multiple-valued function reasonably? Let us turn to our experience that we have had in the usual quantum mechanics. Consider the motion of a particle in a finitely deep potential well as shown in Figure 4.4, there are also two turning points $a$ and $b$. If the particle moves over the turning point $a$ or $b$ for $E < V_0$ (bound states), its momentum will become imaginary $\pm i|p|$ with uncertain sign. As we know in the usual quantum mechanics the wave function is given by

$$
\psi(x) = \begin{cases} 
D e^{-i\frac{|p|}{\hbar}x}, & x < a \\
G \sin\left(\frac{|p|}{\hbar}x + \delta\right), & a < x < b \\
D e^{-i\frac{|p|}{\hbar}x}, & x > b
\end{cases}
$$

(4.62)

In which we have taken the plus sign for the imaginary momentum in $x < a$.
and minus sign in $x > b$, to satisfy the wave-attenuating boundary condition in the regions over the turning points.

In the followings, we use this wave-attenuating boundary condition to determine the sign of double-valued imaginary momentum: take plus sign in the region over the left turning point, whereas take minus sign in the region over the right turning point.

### 4.5.2 integration 1

To apply the wave-attenuating boundary condition to the following wave function

$$X(\theta) = C_2 e^{-\frac{i}{\hbar} \int \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} d\theta}$$ (4.63)

the integrand has to choose the signs as

\[
\begin{align*}
\int_0^a \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} d\theta &= +i \int_0^a \sqrt{|f(\theta)|} d\theta = iA \quad (4.64) \\
\int_a^b \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} d\theta &= \int_a^b \sqrt{f(\theta)} d\theta = B
\end{align*}
\]
\[
\int_{b}^{\pi} \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} \, d\theta = -i \int_{b}^{\pi} \sqrt{|f(\theta)|} \, d\theta = -iA \quad (4.66)
\]

thus the integration may have a real solution, actually it may be written as

\[
\int_{0}^{2\pi} \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} \, d\theta = 2 \int_{0}^{b} \sqrt{J^2 - \frac{J_z^2}{\sin^2 \theta}} \, d\theta = 2B \quad (4.67)
\]

In order to evaluate the definite integral of Eq. (4.67), we make use of contour integral method in complex plane. Consider a contour \( C_\delta \) which is a unit circle around zero, as shown in Figure 4.5, using \( z = e^{i\theta} \), we have

\[
I_1 = \int_{C_\delta} \sqrt{J^2 + \frac{4J_z^2}{z^2 - 1}} \, \frac{dz}{iz} \quad (4.68)
\]

As we have known that \( \sqrt{f(\theta)} \big|_{\theta = \pi/2, \text{or} \theta = 3\pi/2} = \sqrt{J^2 - J_z^2} \), substituting \( z = i \) or \( z = -i \) into the above integrand, we find the integrand has to take the minus sign. Thus we choose the minus sign for it.
\[ I_1 = \int_{C_\delta} \frac{\sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2} \, dz}{iz} \quad (4.69) \]

For scrutinizing the sign of the integrand over the turning points, we have

\[
\sqrt{f(\theta)} = \frac{\sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2}}{-(z^2 - 1)} = \frac{\sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2}}{-(2iz)(z^2 - 1)/(2iz)}
\]

\[
= \frac{\sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2}}{-(2iz) \sin \theta} = i\frac{\sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2}}{2z \sin \theta} \quad (4.70)
\]

we find that the integrand takes plus sign over the left turning point \((\theta \rightarrow 0^+, z \rightarrow 1)\) and minus sign over the right turning point \((\theta \rightarrow \pi^-, z \rightarrow -1)\), in accordance with the sign requirement of Eq. \((4.64)\) and \((4.66)\).

Continue our calculation, we have

\[
I_1 = \int_{C_\delta} \frac{\sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2} \, dz}{iz}
\]

\[
= \int_{C_\delta} \left( \frac{1}{z} - \frac{1/2}{z - 1} - \frac{1/2}{z + 1} \right) \sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2} \, dz \quad (4.71)
\]

Now we find that the integrand has the three poles at \(z = 0\) and \(z = \pm 1\). We let the contour \(C_\delta\) pass by the pole \(z = +1\) through the interior of the unite circle, as indicated by the dash line in Figure 4.5, likewise, let the contour \(C_\delta\) pass by the pole \(z = -1\) through the exterior of the unite circle. The deformation made for \(C_\delta\) has no influence on the integration value because the left deformation cancels the right deformation in the integration due to the opposite signs of the integrand near the left and right poles. Let \(C'_\delta\) denote the deformed contour, we continue the calculation by using Laurent’s series expansion and the residue theorem.

\[
I_1 = \int_{C'_\delta} \left( \frac{1}{z} - \frac{1/2}{z - 1} - \frac{1/2}{z + 1} \right) \sqrt{J^2(z^2 - 1)^2 + 4J_z^2 z^2} \, dz
\]
\[ \int_{C_{\delta}} \frac{1}{z} \sqrt{J^2 (z^2 - 1)^2 + 4J^2 z^2} \, dz \]

\[ - \int_{C_{\delta}} \frac{1}{z-1} \sqrt{J^2 (z^2 - 1)^2 + 4J^2 z^2} \, dz \]

\[ - \int_{C_{\delta}} \frac{1}{z+1} \sqrt{J^2 (z^2 - 1)^2 + 4J^2 z^2} \, dz \]

\[ = \int_{C_{\delta}} \frac{J + O(z^2)}{z} \, dz - \int_{C_{\delta}} \frac{|J_z| + O(z^2 - 1)}{z + 1} \, dz \]

\[ = 2\pi (J - |J_z|) \quad (4.72) \]

### 4.5.3 Integration 2

To apply the wave-attenuating boundary condition to the following wave function

\[ R(r) = C_3 e^{ \frac{-i}{\hbar} \int \sqrt{(E + \frac{c^2}{r})^2 - m_e^2 c^4 - \frac{j^2 h^2 c^2}{r^2}} \, dr } \quad (4.73) \]

Where it has also two turning points \( r_1 \) and \( r_2 \) from \( r = 0 \) to \( r = \infty \) when \( E^2 < m_e^2 c^4 \) (bound states), as shown in Figure 4.6 where

\[ g(r) = (E + \frac{c^2}{r})^2 - m_e^2 c^4 - \frac{j^2 h^2 c^2}{r^2} \quad (4.74) \]

we take the following signs for its asymptotic behavior, i.e.

\[ \sqrt{g(r)} \bigg|_{r \to 0} = \sqrt{e^4 - j^2 h^2 c^2/r} = i \sqrt{j^2 h^2 c^2 - e^4/r} \quad (4.75) \]

\[ \sqrt{g(r)} \bigg|_{r \to \infty} = \sqrt{E^2 - m_e^2 c^4} = -i \sqrt{m_e^2 c^4 - E^2} \quad (4.76) \]

In order to evaluate the definite integral of Eq. (4.25), consider a contour \( C \) consisting of \( C_{\gamma}, L_-, C_{\delta} \) and \( L \) around zero in the plane as shown in Figure 4.7 the radius of circle \( C_{\gamma} \) is large enough and the radius of circle \( C_{\delta} \) is small enough. The integrand of the following equation has no pole inside the contour \( C \), so that we have
Figure 4.6: The function has two sign-changed points \( r_1 \) and \( r_2 \).

\[
\int_C \sqrt{\left( E + \frac{e^2}{z} \right)^2 - m_c^2 c^4 - \frac{j^2 \hbar^2 c^2}{z^2}} \, dz = \int_{C_\gamma} + \int_{L_-} + \int_{C_\delta} + \int_L = 0 \quad (4.77)
\]

Now we evaluate the integration on each contour with our sign choice for the double-valued function, using Laurent’s series expansion and the residue theorem.

\[
\int_{C_\gamma} = \int_{C_\gamma} \sqrt{\left( E + \frac{e^2}{z} \right)^2 - m_c^2 c^4 - \frac{j^2 \hbar^2 c^2}{z^2}} \, dz \\
= -i \int_{C_\gamma} \left[ \sqrt{m_c^2 c^4 - E^2} - \frac{E e^2}{\sqrt{m_c^2 c^4 - E^2} z^2} \right] \, dz + O(\frac{1}{z^2}) dz \\
= -i \frac{2\pi i E e^2}{\sqrt{m_c^2 c^4 - E^2}} = -\frac{2\pi E e^2}{\sqrt{m_c^2 c^4 - E^2}} \quad (4.78)
\]
Figure 4.7: A unit circle contour for evaluating the integral

\[
\int_{\gamma} = \int_{\gamma} \sqrt{m^2c^4 + j^2\hbar^2c^2 - (E + \frac{e^2}{z})^2} dz
\]

\[
= i \int_{\gamma} \sqrt{\frac{m^2c^4z^2 + j^2\hbar^2c^2 - (Ez + e^2)^2}{z}} dz
\]

\[
= i \int_{\gamma} \sqrt{\frac{j^2\hbar^2c^2 - e^4 + O(z)}{z}} dz
\]

\[
= i(-2\pi i)\sqrt{j^2\hbar^2c^2 - e^4} = 2\pi \sqrt{j^2\hbar^2c^2 - e^4} \quad (4.79)
\]

Because the integrand is a multiple-valued function, when the integral takes on the path \(L_-\) we have \(z = e^{i2\pi r}e^{0i}\), thus

\[
\int_{L_-} = \int_{\gamma} e^{-i2\pi(...)} = -\int_{\gamma} = \int_{\delta} = \int_{L} \quad (4.80)
\]

For a further clarification, to define \(z - H = w = \rho e^{i\beta}\), where

\[
H = \frac{m^2c^4r^2 + j^2\hbar^2c^2 - E^2r^2 - e^4}{2Ee^2} \quad (4.81)
\]
we have

\[
\int_{L^-} = \int_{L^-} \frac{\sqrt{2Ee^2} \sqrt{z-H}}{z} \, dz = \int_{L^-} \frac{\sqrt{2Ee^2} \sqrt{w}}{z} \, dz \\
= \int_{L^-} \frac{\sqrt{2Ee^2} \rho e^{i\beta/2}}{z} \, dz \\
= \int_{L(\gamma \to \delta)} \frac{\sqrt{2Ee^2} \rho e^{i(\beta+2\pi)/2}}{ze^{i2\pi}} \, d(ze^{i2\pi}) \\
= -\int_{L(\gamma \to \delta)} \frac{\sqrt{2Ee^2} \rho e^{i\beta/2}}{z} \, dz \\
= \int_{L(\delta \to \gamma)} \frac{\sqrt{2Ee^2} \rho e^{i\beta/2}}{z} \, dz = \int_{L}
\]

(4.82)

Where we have use the relation of \(z\) and \(w\) in the fourth step of the above equation, as shown in Figure 4.8 to note that \(w\) rotates around zero with \(z\). Thus we have

\[
\int_{L} = \frac{1}{2}(\int_{L} + \int_{L^-}) = -\frac{1}{2}(\int_{C_\gamma} + \int_{C_\delta})
\]

Figure 4.8: A contour for evaluating the integral
Thus Eq. (4.25) becomes

\[
\frac{1}{\hbar c} \int_0^\infty \sqrt{(E + \frac{e^2}{r})^2 - m_e^2 c^4} \frac{j^2 h^2 c^2}{r^2} dr = \pi E\alpha \sqrt{m_e^2 c^4 - E^2} - \pi \sqrt{j^2 - \alpha^2} \tag{4.84}
\]

Where \( \alpha = e^2 / \hbar c \) is known as the fine structure constant.

4.5.4 discussion: the motion over turning points

Following the sign change for imaginary momentum over turning points, discussed in the preceding sections, we find that the periodic conditions in hydrogen atom may be written as

\[
2 \int_a^b \sqrt{J^2 - \frac{m^2}{\sin^2 \theta}} d\theta = 2\pi k \tag{4.85}
\]

\[
\frac{1}{\hbar c} \int_{r_1}^{r_2} \sqrt{(E + \frac{e^2}{r})^2 - m_e^2 c^4} - \frac{j^2 h^2 c^2}{r^2} dr = \pi s \tag{4.86}
\]

because the contributions of the integrands in the regions over the turning points are eliminated automatically. What are their physical meanings? A direct explanation is that it is not necessary for the electron to enter the regions over the turning points, in compliance with the classical physics.

In addition, the residue theorem we used in the paper gives out accurate results for our integrations, not approximate ones.
Chapter 5

Applications of momentum path integral

5.1 Introduction
In the preceding chapters, we put forward the momentum path integral method, we apply the momentum path integral method to hydrogen atom. As the results, the energy levels are calculated out with the same fine structure and spin effect as Dirac wave equation. Thus, we shows that the momentum path integral is a useful method for solving quantum problems through the instance of hydrogen atom. In this chapter we continue to discuss some typical applications of the momentum path integral method.

Consider a particle of mass $m$ and charge $q$ moving in an electromagnetic field in a Minkowsky’s space time $(x_1, x_2, x_3, x_4 = i\epsilon t)$, the 4-vector velocity of the particle is denoted by $u$, the 4-vector potential of the electromagnetic field is denoted by $A$, where and below we use Greek letters for subscripts that range from 1 to 4. As we know, the momentum wavefunction relation is given by

$$mu_\mu = \frac{1}{\psi}(-i\hbar\partial_\mu - qA_\mu)\psi$$  \hspace{1cm} (5.1)
Its integral form is given by

\[ \psi = e^{i\frac{\hbar}{\mu} \int_{x_0}^{x} (p_\mu + qA_\mu) dx_\mu + i\theta} \] (5.2)

Where \( \theta \) is an integral constant, the momentum of the particle is \( p_\mu = \mu u_\mu \), the integral takes over an arbitrary path \( l \) from the initial point \( x_0 \) to the final point \( x \), it is of path independence.

There are three mathematical properties of \( \psi \) that need to be mentioned here. First, if there is a path \( l_i \) joining initial point \( x_0 \) to final point \( x \), then

\[ \psi_i = e^{i\frac{\hbar}{\mu} \int_{x_0}^{x} (\mu u_\mu + qA_\mu) dx_\mu + i\theta} \] (5.3)

Second, the integral of Eq.(5.3) is of path independence. Third, the superposition principle is valid for \( \psi_i \), i.e., if there are \( N \) paths from \( x_0 \) to \( x \), then

\[ \psi = \sum_i^{N} \psi_i \] (5.4)

\[ \mu u_\mu = \sum_i^{N} \mu u_\mu \psi_i / \sum_i^{N} \psi_i \] (5.5)

\[ (\mu u_\mu + qA_\mu) \psi = -i \hbar \partial_\mu \psi \] (5.6)

Where \( \mu u_\mu \) is the average momentum, because the wave function is defined in the ensemble space (see the preceding chapters).

To gain further insight into physical meanings of this momentum path integral method, we shall discuss its several applications[12][13] in the following sections.

### 5.2 Two slit experiment

As shown in Figure [5.1], suppose that the electron gun emits a burst of electrons at \( x_0 \) at time \( t = 0 \), the electrons arrive at the point \( x \) on the screen.
Figure 5.1: A diffraction experiment in which electron beam from the gun through the two slits to form a diffraction pattern at the screen.

at time $t$. There are two paths for the electron to go to the destination, according to our above statement, $\psi$ is given by

$$
\psi = e^{i\frac{\hbar}{\hbar} \int_{x_0(l_1)}^x (mu_\mu)dx_\mu} + e^{i\frac{\hbar}{\hbar} \int_{x_0(l_2)}^x (mu_\mu)dx_\mu} \tag{5.7}
$$

Where we use $l_1$ and $l_2$ to denote the paths $a + b$ and $c + d$ respectively. Multiplying Eq. (5.7) by its complex conjugate gives

$$
W = \psi(x)\psi^*(x) = 2 + e^{i\frac{\hbar}{\hbar} \int_{x_0(l_1)}^x (mu_\mu)dx_\mu} - e^{i\frac{\hbar}{\hbar} \int_{x_0(l_2)}^x (mu_\mu)dx_\mu} + e^{i\frac{\hbar}{\hbar} \int_{x_0(l_2)}^x (mu_\mu)dx_\mu} - e^{i\frac{\hbar}{\hbar} \int_{x_0(l_1)}^x (mu_\mu)dx_\mu} = 2 + 2 \cos \left[ \frac{1}{\hbar} \int_{x_0(l_1)}^x (mu_\mu)dx_\mu - \frac{1}{\hbar} \int_{x_0(l_2)}^x (mu_\mu)d_\mu \right]
$$
CHAPTER 5. APPLICATIONS OF MOMENTUM PATH INTEGRAL

\[ 2 + 2 \cos \left( \frac{p \hbar}{\hbar} (l_1 - l_2) \right) \]  

(5.8)

Where \( p \) is the momentum of the electron. We find a typical interference pattern with constructive interference when \( l_1 - l_2 \) is an integral multiple of \( \hbar/p \), and destructive interference when it is a half integral multiple. This kind of experiment has been done a long age, no matter what kind of particle, the comparison of the experiment to Eq.(5.8) leads to two consequences: (1) the complex function \( \psi \) is found to be probability amplitude, i.e., \( \psi(x)\psi^*(x) \) expresses the probability of finding a particle at location \( x \) in the Minkowsky space time. (2) \( \hbar \) is the Planck’s constant.

5.3 Aharonov-Bohm effect

Let us consider the modification of the two slit experiment, as shown in Figure 5.2. Between the two slits there is located a tiny solenoid S, designed so that a magnetic field perpendicular to the plane of the figure can be produced in its interior. No magnetic field is allowed outside the solenoid, and the walls of the solenoid are such that no electron can penetrate to the interior. Like Eq.(5.7), the amplitude \( \psi \) is given by

\[ \psi = e^{\frac{i}{\hbar} \int_{x_0(l_1)}^{x} (mu_\mu + qA_\mu) dx_\mu} + e^{\frac{i}{\hbar} \int_{x_0(l_2)}^{x} (mu_\mu + qA_\mu) dx_\mu} \]  

(5.9)

and the probability is given by

\[ W = \psi(x)\psi^*(x) \]

\[ = 2 + 2 \cos \left( \frac{p \hbar}{\hbar} (l_1 - l_2) \right) + \frac{1}{\hbar} \int_{x_0(l_1)}^{x} qA_\mu dx_\mu \]

\[ - \frac{1}{\hbar} \int_{x_0(l_2)}^{x} qA_\mu dx_\mu \]
\[= 2 + 2 \cos \left( \frac{p}{\hbar} (l_1 - l_2) + \frac{1}{\hbar} \oint_{(l_1 + l_2)} qA_{\mu} dx_{\mu} \right)\]
\[= 2 + 2 \cos \left( \frac{p}{\hbar} (l_1 - l_2) + \frac{q\phi}{\hbar} \right) \quad (5.10)\]

Where \(l_2\) denotes the inverse path with respect to the path \(l_2\), \(\phi\) is the magnetic flux that passes through the surface between the paths \(l_1\) and \(l_2\), and it is just the flux inside the solenoid.

![Figure 5.2: A diffraction experiment with adding a solenoid.](image)

Now, constructive (or destructive) interference occurs when

\[\frac{p}{\hbar}(l_1 - l_2) + \frac{q\phi}{\hbar} = 2\pi n \quad (\text{or} \quad n + \frac{1}{2}) \quad (5.11)\]

Where \(n\) is an integer. \(\hbar\) takes the value of the Planck’s constant, we know that this effect is just the Aharonov-Bohm effect which was shown experimentally in 1960.
5.4 The hydrogen atom

The hydrogen atom is one of the few physically significant quantum-mechanical systems for which an exact solution can be found and the theoretical predictions compared with experiment.

Rutherford’s model of a hydrogen atom consists of a nucleus made up of a single proton and of a single electron outside the nucleus, the electron moves in an orbit about the nucleus. Here we consider two points denoted by \( x_0 \) and \( x \) in the orbit, and two paths \( l_1 \) and \( l_2 \) from \( x_0 \) to \( x \) along different directions, as shown in Figure 5.3. Then, according to our above statement, the probability amplitude \( \psi \) is given by

\[
\psi = e^{i \frac{\bar{\hbar}}{2} \int_{x_0(l_1)} (m u_\mu + q A_\mu) dx_\mu} + e^{i \frac{\bar{\hbar}}{2} \int_{x_0(l_2)} (m u_\mu + q A_\mu) dx_\mu}
\]  

(5.12)

and the probability is given by

\[
W = \psi(x) \psi^*(x)
\]
\[ W = \psi(x)\psi^*(x) \]
\[ = 2 + 2\cos\left[\frac{1}{\hbar} \oint_{(l_1+l_2)} (mu_\mu + qA_\mu)dx_\mu\right] \]
\[ = 2 + 2\cos\left[\frac{1}{\hbar} \oint_{(l_1+l_2)} (mu_k)dx_k\right] \]
\[ = 2 + 2\cos\left[\frac{1}{\hbar} \oint_{(l_1+l_2)} p_kdx_k\right] \] (5.13)

Where \( k = 1, 2, 3, \) \( p_k = mu_k. \) For the stationary states, the integral about time will be automatically eliminated because the probability should be stable. The probability of the electron at every point in the orbit should be the same because these points in the orbit are equivalent, this leads to

\[ \oint_{(orbit)} p_kdx_k = 2\pi\hbar n \] (5.14)

Eq. (5.14) is just the Bohr-Somerfeld quantization rule for the hydrogen atom.

The probability of the electron outside the orbit should vanish, in where the momentum of the electron should become imaginary.

5.5 The motion of particle in a potential well

Let us now restrict ourselves to one dimensional well. We choose point \( x_0 \) to locate at the left turning point and \( x \) at arbitrary point in the well, as shown in Figure 5.4, likewise, there are two paths \( l_1 \) and \( l_2 \) from \( x_0 \) to \( x \) to correspond to ”coming” \((l_1)\) and ”back” \((l_2)\) for the particle motion, like Eq. (5.12) and (5.13), we obtain the probability as

\[ W = \psi(x)\psi^*(x) \]
\[ = 2 + 2\cos\left[\frac{1}{\hbar} \oint_{(l_1+l_2)} (mu_\mu + qA_\mu)dx_\mu\right] \]
\[ = 2 + 2\cos\left[\frac{1}{\hbar} \oint_{(l_1+l_2)} pdx\right] \] (5.15)
Figure 5.4: The motion of particle in a potential well. The left and right turning points are indicated on the potential well.

The integral about time vanishes for the stationary state. The probability has a distribution in the well, but it will vanish at the right turning point for satisfying boundary condition, this leads to

$$\int p \, dx = 2\pi \hbar n$$

(5.16)

Where the integral is evaluated over one whole period of classical motion, from the left turning point to the right and back. We again meet the Bohr-Sommerfeld quantization rule for the old quantum theory, although it was originally written in the form of Eq. (5.14) in 1915 due to A. Sommerfeld and W. Wilson.
5.6 Superconductivity

Superconductor ring has the same topological structure as hydrogen atom, as shown in Figure 5.5, the electron(s) has to rotate about the nucleus in hydrogen atom or about multi-nuclei ring in superconductor ring. The electron of hydrogen atom in stationary state moves in circular motion without electromagnetic radiation, the electron of superconductor ring in superconducting state also moves without electromagnetic radiation. In this sense, the motion of electron in stationary atom is just the motion in superconducting state. As we know, it is easy to carry out experiments on macron superconductor ring, comparing to micron atom, such as electromagnetic radiation, phase transformation and mechanism etc. So, superconductor study provides a clear insight into both theoretical mechanisms and experimental measurements for atom study.

Figure 5.5: The compare between hydrogen atom and superconductor ring.
In BCS theory, it is believed that the mechanism responsible for the transition to superconductivity is a coupling between electrons via the positive ions of metallic lattice. The electron-lattice-electron interaction provides an attraction between electrons which can lead to a ground state separated from excited states by an energy gap. Whereas, in recent years the discovery of high $T_c$ superconductivity has offered a challenge for BCS theory, the first great difficult in the extension of BCS theory is to discover a nature of interaction responsible both for the traditional and the high $T_c$ superconductivity.

In the preceding chapter, the 4-vector Coulomb’s force between two particles $q$ and $q'$ is given by

$$f = \frac{kqq'}{c^2r^3}[(\mathbf{u} \cdot \mathbf{X})u' - (\mathbf{u} \cdot \mathbf{u'})X]$$

$$= \frac{kqq'}{c^2r^3}[(\mathbf{u} \cdot \mathbf{R})u' - (\mathbf{u} \cdot \mathbf{u'})R] \quad (5.17)$$

Where the symbols were defined in the preceding chapter[3]. Obviously, for an electron pair, the first term in the right side of Eq.(5.17) can give an attraction between the two electrons in some situations, the second term represents a repulsion which contributes to the classical Coulomb’s force. To note that this attraction of electron pair requires no phonon exchange, the attraction is definitely distinguishable from that in Cooper pair of BCS theory. How the attraction of electron pair takes effect on the superconductivity remains to explore. In this section, we show that the momentum path integral method can successfully applied to many aspects of superconductivity[13].

5.6.1 vortex structure

Let us consider the integration of Eq.(5.2) over a closed path $L$ in the space, at any instant $t(dx_4 = 0)$, by the two pathes $l_1$ and $l_2$ respectively, as shown in Figure 5.6.

$$\psi_1 = e^{\frac{i}{\hbar} \int_{x_0}^{x} (mu_\mu + qA_\mu)dx_\mu + i\theta} \quad (5.18)$$
Figure 5.6: The integration over a closed path \( L \) in the space.

\[
\psi_2 = e^{\frac{i}{\hbar} \int_{x_0(l_2)}^{x}(mu_\mu + qA_\mu)dx_\mu + i\theta}
\]  

(5.19)

Eq. (5.2) must be a single-valued wave function, this requires \( \psi_1 = \psi_2 \), thus

\[
\frac{1}{\hbar} \oint_L (mu + qA) \cdot dl = 2\pi b \quad b = 0, \pm 1, \pm 2, ... 
\]  

(5.20)

Where \( dl \) is an displacement element of integral path. It expresses the vortex structure in a superconducting state.
5.6.2 quantized magnetic flux in a superconducting ring

Let us take the integral path $L$ in a superconducting ring, as shown in Figure 5.7.

Figure 5.7: The integration over a closed path $L$ in the space.

Because the electronic current is zero in the interior, i.e. $u = 0$, thus the quantized magnetic flux through $L$ is given from Eq. (5.20) by

$$\phi = \oint_L B \cdot d\sigma = \oint_L A \cdot dl = \frac{2\pi \hbar}{q}$$  \hspace{1cm} (5.21)

Where $d\sigma$ is an element of area on the surface bounded by the integral path $L$. By experiment, $q = -2e$, the charge of an electron pair, thus the flux through the ring is quantized in integral multiples of $\pi \hbar/e$. 
5.6.3 London equation

Let $n$ denote the density of electrons in superconductor, as shown in Figure 5.8.

![Diagram of an electron current over a closed path](image)

Figure 5.8: The electronic current over a closed path $L$ in the space.

The electronic current is given by

$$\mathbf{j} = -ne\mathbf{u}$$  \hspace{1cm} (5.22)

Then Eq. (5.20) can be rewritten as

$$\frac{-m}{neh} \oint_L \left( (\mathbf{j} + \frac{ne^2}{m}\mathbf{A}) \cdot d\mathbf{l} \right) = \frac{-m}{neh} \oint_L \left( \nabla \times \mathbf{j} + \frac{ne^2}{m}\mathbf{B} \right) \cdot d\sigma = 2\pi b$$  \hspace{1cm} (5.23)
CHAPTER 5. APPLICATIONS OF MOMENTUM PATH INTEGRAL

If the closed integral path contains no vortex, at where we obtain

\[ \nabla \times \mathbf{j} + \frac{ne^2}{m} \mathbf{B} = 0 \quad (5.24) \]

It is the well known the London equation.

### 5.6.4 Meissner effect

As shown in Figure 5.9.

![Figure 5.9: The penetration depth.](image-url)

According to Maxwell’s equations, we have

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (5.25) \]

From London equation, we obtain
\[ \nabla^2 B = B/\lambda_L^2 \]  
(5.26)

Where \( \lambda_L^2 = m/(\mu_0 ne^2) \) is a constant called as the London penetration depth. Near the surface of superconductor, the magnetic field along the depth direction \( z \) is given by

\[ B(z) = B(0) \exp(-z/\lambda_L) \]  
(5.27)

Thus for superconductor, the London equation leads to the Meissner effect.

### 5.6.5 Josephson superconductor tunnelling

Consider an insulator film occupying the space of rang \((0, \delta)\) in \(x\)-axis with an applied voltage \(V\), according to Eq.(5.2), set the origin at \(x = 0\), the wave function of electron pair in the region \(x > \delta\) can be calculated by taking the integral path \(l_1\) across the insulator, as shown in Figure 5.10.

It gives

\[ \psi_1 = e^{i \frac{k}{\hbar} \int_0^\delta p\,dx + \frac{i}{\hbar} \int_\delta^x p\,dx + \frac{i}{\hbar} \int_0^t qA_4dx_4 + i\theta_1} \]  
(5.28)

the wave function of electron pair in the region \(x > \delta\) can also be calculated by taking another integral path \(l_2\) around the circuit without crossing the insulator (i.e. \(l_1\) and \(l_2\) forms a cycle, \(l_1\) crosses the insulator while \(l_2\) does not cross the insulator), it gives

\[ \psi_2 = e^{i \frac{k}{\hbar} \int_0^\delta p\,dx + \frac{i}{\hbar} \int_\delta^x p\,dx + i\theta} \]

\[ = e^{i \frac{k}{\hbar} \int_\delta^x p\,dx + i\theta_2} \]  
(5.29)

To note that the electrons have the same momentum \(p\) in the superconductor whereas have an imaginary momentum in the insulator, to denote \(\int_0^\delta p\,dx = i\hbar k\delta\), we obtain the wave function

\[ \psi = \psi_1 + \psi_2 \]

\[ = e^{-k\delta + \frac{i}{\hbar} \int_\delta^x p\,dx + \frac{i2eVt}{\hbar} + i\theta_1} + e^{i \frac{k}{\hbar} \int_\delta^x p\,dx + i\theta_2} \]  
(5.30)
According to Eq.(5.22), the superconducting current in $x > \delta$ region is given by

$$j = nqu = \frac{nq}{m}p = -i\frac{q\hbar}{2m}(\psi^* \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial \psi^*}{\partial x})$$

$$= \frac{2e}{m}pe^{-k\delta} \sin[\frac{2eVt}{\hbar} + i(\theta_1 - \theta_2)]$$

The result is just the Josephson tunnelling effect, including DC Josephson effect and AC Josephson effect.
5.7 Discussion

The above formulation based on the momentum path integral method is successfully applied to the quantum mechanics, but we emphasize that Eq. (5.2) is essentially different from Schrodinger wave equation. In the author’s previous paper [7] we have proved that we can derive the Schrodinger wave equation from our Eq. (5.2), inversely we can not obtain Eq. (5.2) from the Schrodinger wave equation.

We always assume that the path integral about time vanishes for stationary state, because we always investigate stable experimental phenomena. If we can be equipped to investigate dynamic processes, the path integral about time will display its effects.

5.8 Conclusion

The momentum path integral method is successfully applied to the quantum mechanics through some instances: two slit experiment, Aharonov-Bohm effect, Bohr-Sommerfeld hydrogen atom, the motion of particle in a potential well, superconductivity.

Specially, superconducting state is discussed in terms of relativistic quantum theory, some significant results are obtained, including quantized magnetic flux, London equation, Meissner effect and Josephson effect.
Chapter 6

A missing factor

6.1 Introduction

In the preceding chapter, we point out that the 4-vector Coulomb’s force acting on a particle $q$ by another particle $q'$ is given by

$$f = \frac{kqq'}{c^2 r^3}[(u \cdot X)u' - (u \cdot u')X]$$

$$= \frac{kqq'}{c^2 r^3}[(u \cdot R)u' - (u \cdot u')R] \quad (6.1)$$

Where all symbols have been shown in Figure 6.1, the relativistic dynamics of the particle $q$ is governed by

$$m \frac{du}{d\tau} = \frac{kqq'}{c^2 r^3}[(u \cdot X)u' - (u \cdot u')X] \quad (6.2)$$

Before we continue to go on, there is a little trouble in our minds when counting the factor $\sqrt{1 - v^2/c^2}$ in the above equation, or looking at the following explicit equation

$$m \frac{d}{\sqrt{1 - v^2/c^2}dt}\left(\frac{du}{\sqrt{1 - v^2/c^2}dt}\right) = \frac{kqq'}{c^2 r^3}[(\frac{dx}{\sqrt{1 - v^2/c^2}dt}\cdot X)u' - (\frac{dx}{\sqrt{1 - v^2/c^2}dt}\cdot u')X] \quad (6.3)$$
Figure 6.1: The Coulomb’s force acting on \( q \) is orthogonal to the 4-vector velocity \( u \) of \( q \), and lies in the plane of \( u' \) and \( X \) with a retardation with respect to \( q' \), here Euclidian geometry is used to illustrate the complex 4D space.

It is found that the left hand side contains factor \( \sqrt{1 - v^2/c^2} \) twice while the right hand side contains one single factor \( \sqrt{1 - v^2/c^2} \).

Since the unit vector \( \hat{f} \) of the 4-vector Coulomb’s force keeps unit one

\[
\hat{f} = \frac{1}{c^2 r} [(u \cdot X)u' - (u \cdot u')X]
\]
\[
= \frac{1}{c^2 r} [(u \cdot R)u' - (u \cdot u')R] \quad (6.4)
\]
\[
|\hat{f}| = 1 \quad (6.5)
\]

we think that the magnitude \( |f| \) of the 4-vector Coulomb’s force must be
modified as

\[ |f| = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{kqq'}{r^2} \]  

(6.6)

Thus the modified 4-vector Coulomb’s force reads

\[ f = |f| \hat{f} = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{kqq'}{c^2r^3} [(u \cdot X)u' - (u \cdot u')X] \]  

(6.7)

According to the experience of the preceding chapter, this newly added factor \(1/\sqrt{1 - v^2/c^2}\) in the 4-vector Coulomb’s force will allow us to further improve Maxwell’s electromagnetism, some new aspects of electromagnetism will be unveiled.

At the present stage, we focus on some theoretical problems that have had some experimental supports, for example, gravitational force field problems [14].

### 6.2 Gravitational Force

The above formalism has an important significance on guiding how to develop the theory of gravity. In analogy with the modified Coulomb’s force in Eq. (6.7), we directly suggest a modified universal gravitational force as

\[ f = -\frac{1}{\sqrt{1 - v^2/c^2}} \frac{Gmm'}{c^2r^3} [(u \cdot X)u' - (u \cdot u')X] \]

\[ = -\frac{1}{\sqrt{1 - v^2/c^2}} \frac{Gmm'}{c^2r^3} [(u \cdot R)u' - (u \cdot u')R] \]  

(6.8)

for a couple of particles with masses \(m\) and \(m'\) respectively, the gravitational force must satisfy the orthogonality of 4-vector force and 4-vector velocity.

We emphasize that the orthogonal relation of force and velocity must be hold for gravitational force if it can be defined as a kind of force. It follows from Eq. (6.8) that we can predict that there exist gravitational radiation and magnetism-like components for the gravitational force. Particularly,
the magnetism-like components will act as an important role in geophysics and atmosphere physics.

We rewrite the gravitational force again as

\[
m\frac{du}{d\tau} = -\frac{1}{\sqrt{1-v^2/c^2}} \frac{G mm'}{c^2 r^3} [(u \cdot X)u' - (u \cdot u')X] = -\frac{u_4 GM}{ic c^2 r^3} [(u \cdot X)u' - (u \cdot u')X]
\]

(6.9)

We will show that Eq.(6.9) can gives out the same results as the general theory of relativity for gravitational problems.

Consider a planet \( m \) moving about the sun \( M \) with the 4-vector velocity \( u = (u, u_4) \), \( u = u_r + u_\varphi \) in the polar coordinate system \( S : (r, \varphi, ic) \). We assume the sun is at rest at the origin, \( u' = (0, 0, 0, ic) \), as shown in Figure 6.2.

Then from Eq.(6.9) the motion of the planet is governed by

\[
\frac{du_4}{d\tau} = -\frac{1}{\sqrt{1-v^2/c^2}} \frac{GM}{c^2 r^3} (ru_r)ic = -\frac{u_4 GM}{ic c^2 r^3} (ru_r)ic
\]

(6.10)

\[
\frac{du}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}} \frac{GM}{c^2 r^3} (icu_4)r = \frac{u_4 GM}{ic c^2 r^3} (icu_4)r
\]

(6.11)

From the above equations we obtain their solutions

\[
u_4 = Ke^{s/(2r)}
\]

(6.12)

\[
u_r^2 + \nu_\varphi^2 + u_4^2 = -c^2
\]

(6.13)

\[ru_\varphi = h
\]

(6.14)

Where \( K \) and \( h \) are two integral constants, \( s = 2GM/c^2 \) is called the Schwarzschild radius, \( s/r \) is a small term. Eq.(6.13) agrees with the relation \( u_4^2 = -c^2 \).
Eq. (6.12) tells us the relation of the time interval $dt$ and the proper time interval $d\tau$, i.e.

$$u_4 = \frac{icdt}{d\tau} = Ke^{s/(2r)} \quad (6.15)$$

From this relation, we know that the proper time interval $d\tau$ at position $r$ and the proper time interval $d\tau_\infty$ at $r = \infty$ has the relation as

$$\frac{d\tau}{d\tau_\infty} = \frac{K}{Ke^{s/(2r)}} = e^{-s/(2r)} \simeq 1 - \frac{s}{2r} \quad (6.16)$$

For planet, because $v/c << 1$, from Eq. (6.13) we know that $K \simeq ic$ and $dt \simeq d\tau_\infty$. 

---

**Figure 6.2**: A planet moving about the sun.
6.2.1 gravitational red shift

According to Eq. (6.16), the period of oscillation of an atom at $r$ is related to the period of oscillation of another atom at $r = \infty$ by

$$\frac{T_r}{T_\infty} \simeq 1 - \frac{s}{2r}$$  \hspace{1cm} (6.17)

This gives the spectral shift of a photon, this effect is called the gravitational red shift \cite{Note1}, in agreement with experimental observations \cite{Note15}. As shown in Figure 6.3.

![Diagram](image)

Figure 6.3: The gravitational red shift.
6.2.2 the Perihelion advance of a planet

Consider the planet again, Substituting Eq. (6.16) into Eq. (6.13) and Eq. (6.14), we obtain

\[
\left( e^{s/(2r)} \frac{dr}{d\tau_\infty} \right)^2 + \left( e^{s/(2r)} \frac{dx_\varphi}{d\tau_\infty} \right)^2 + \left( Ke^{s/(2r)} \right)^2 = -c^2
\]

\[r(e^{s/(2r)} \frac{dx_\varphi}{d\tau_\infty}) = h\]

They also can be rewritten as

\[
\left( \frac{dr}{d\tau_\infty} \right)^2 + \left( \frac{h^2}{r^2} e^{-s/r} + K^2 \right) = -c^2 e^{-s/r}
\]

\[r(e^{s/(2r)} \frac{dx_\varphi}{d\tau_\infty}) = h\]  \hspace{1cm} (6.18)

\[
\left( \frac{dr}{d\tau_\infty} \right)^2 + \left( \frac{h^2}{r^2} e^{-s/r} + K^2 \right) = -c^2 e^{-s/r}
\]

\[r^2 \frac{d\varphi_\infty}{d\tau_\infty} = h\]  \hspace{1cm} (6.19)

In order to pursue the covariance of Eq. (6.19), we define the angular displacement \( d\varphi_\infty = e^{s/(2r)} d\varphi \), then we have

\[
\left( \frac{dr}{d\varphi_\infty} \right)^2 + \left( \frac{h^2}{r^2} e^{-s/r} + K^2 \right) = -c^2 e^{-s/r}
\]

\[r^2 \frac{d\varphi_\infty}{d\tau_\infty} = h\]  \hspace{1cm} (6.20)

Eliminating \( d\tau_\infty \) by dividing Eq. (6.20) by Eq. (6.21), we obtain

\[
\left( \frac{dr}{d\varphi_\infty} \right)^2 + r^2 e^{-s/r} + K^2 \frac{r^4}{h^2} = -c^2 \frac{r^4}{h^2} e^{-s/r}
\]

\[
\left( \frac{dU}{d\varphi_\infty} \right)^2 + U^2 e^{-sU} + \frac{K^2}{h^2} = -c^2 \frac{r^4}{h^2} e^{-sU}
\]  \hspace{1cm} (6.22)

Now making the change of variable \( U = 1/r \) gives

\[
\left( \frac{dU}{d\varphi_\infty} \right)^2 + U^2 e^{-sU} + \frac{K^2}{h^2} = -c^2 \frac{r^4}{h^2} e^{-sU}
\]  \hspace{1cm} (6.23)
Chapter 6. A Missing Factor

Differentiating with respect to $\phi_{\infty}$ gives

$$\frac{d^2 U}{d\phi_{\infty}^2} + U e^{-sU} - \frac{1}{2} s U^2 e^{-sU} = \frac{c^2 s}{2h^2} e^{-sU}$$ \hfill (6.24)

Since $s/r$ (or $s/(2h^2/c^2)$) is a very small quantity, we neglect all but the first-order terms, we obtain

$$\frac{d^2 U}{d\phi_{\infty}^2} + U - \frac{3}{2} s U^2 = \frac{c^2 s}{2h^2}$$ \hfill (6.25)

The last term on the left side of the equation is the relativistic correction, in the absence of this term the solution is

$$U = \frac{c^2 s}{2h^2} \left[ 1 + e \cos(\phi_{\infty} - \phi_0) \right] = U_0 \left[ 1 + e \cos(\phi_{\infty} - \phi_0) \right]$$ \hfill (6.26)

Where $e$ and $\phi_0$ are constants of integration. Writing $U = U_0 + U_1$, we find that Eq. (6.25) becomes

$$\frac{d^2 U_1}{d\phi_{\infty}^2} + (1 - 3sU_0)U_1 = \frac{3}{2} s (U_0^2 + U_1^2)$$ \hfill (6.27)

By inspection we see that $U_1$ is on oscillatory function of $\phi_{\infty}$ with the frequency

$$\omega = \sqrt{1 - 3sU_0}$$ \hfill (6.28)

From $\omega \phi = 2\pi$ for a cycle, we know that

$$\phi \approx 2\pi (1 + \frac{3}{2} sU_0)$$ \hfill (6.29)

Thus the perihelion advance of the planet is given by

$$\Delta \phi = \phi - 2\pi = 3\pi sU_0$$ \hfill (6.30)

This result is the same as that of the general theory of relativity. As shown in Figure 6.4.
6.2.3 The bending of light rays

Since the above solutions are independent from the planet mass $m$, if the planet is a photon without mass, it also satisfies the above motion like a planet. For a photon, $d\tau_{\infty} \to 0$, Eq.(6.21) becomes

$$h = r^2 \frac{d\varphi_{\infty}}{d\tau_{\infty}}|_{r=\infty} = \infty$$  \hspace{1cm} (6.31)

Then Eq.(6.25) becomes

$$\frac{d^2U}{d\varphi_{\infty}^2} + U - \frac{3}{2}sU^2 = 0$$  \hspace{1cm} (6.32)

The solution of this equation gives the path of the photon, it is the same as that of the general theory of relativity. As shown in Figure 6.5.
In its flight past the massive object $M$ of radius $R$ the photon is deflected through an angle $\delta = \frac{2s}{R}$ in agreement with experimental observations\cite{15}.

\begin{equation}
\delta = \frac{2s}{R}
\end{equation}

in agreement with experimental observations\cite{15}.

6.2.4 The bending of space

If there is not the nonlinear effect of photon flying, any straight line in the space-time is of the path of photon. Since the light is bent in the 4-dimensional space-time for an infinite distance observer, we consequently conclude that the space will bend due to the gravitational effect mentioned above near a massive object.
6.3 Lorentz transformation

When a particle is accelerated from a rest state at the $r$ position to a speed $v$ state at the $r'$ position by a gravitational field $\Phi$, the 4th axis of the frame $S$ fixed at the particle will rotates an angle $\theta$ (towards the direction of $v$) in Minkowsky space time, the frame $S$ will become a new frame $S'$ in the view on the ground, the angle is given by $u_4 = ic\cosh \theta$. As shown in Figure 6.6.

![Figure 6.6: The rotation of a reference frame, in Euclidian geometry representation.](image)

From Eq. (6.12), we have

$$u_4|_r = ic = Ke^{s/(2r)}$$  \hspace{1cm} (6.34)
$$u_4|_{r'} = ic\cosh \theta = Ke^{s/(2r')}$$  \hspace{1cm} (6.35)
cosh \theta = \frac{u_4|_{r'}}{u_4|_r} = e^{s/(2r') - s/(2r)} = e^{\Phi/c^2} \quad (6.36)

Where \( \Phi = c^2 s/(2r') - c^2 s/(2r) \) is the potential difference between \( S \) and \( S' \). Since the particle is accelerated by \( \Phi \), then we have

\[
\frac{1}{2}v^2 = \Phi \quad (6.37)
\]

\[
cosh \theta = e^{\Phi/c^2} = \sqrt{\frac{1}{e^{2\Phi/c^2}}} \approx \frac{1}{\sqrt{1 - v^2/c^2}} \quad (6.38)
\]

\[
sinh \theta = \sqrt{1 - \cosh^2 \theta} = \frac{iv/c}{\sqrt{1 - v^2/c^2}} \quad (6.39)
\]

Because the origins of \( S \) and \( S' \) coincide at the time \( t = 0 \), the coordinate transformations between them are

\[
dr = dr' \cosh \theta + dx'_4 \sinh \theta = \frac{1}{\sqrt{1 - v^2/c^2}}dr' + \frac{iv/c}{\sqrt{1 - v^2/c^2}}dx'_4 \quad (6.40)
\]

\[
dx_4 = -dr' \sinh \theta + dx'_4 \cosh \theta = \frac{-iv/c}{\sqrt{1 - v^2/c^2}}dr' + \frac{1}{\sqrt{1 - v^2/c^2}}dx'_4 \quad (6.41)
\]

It is known as the Lorentz transformations.

### 6.4 Discussion

To note that the orthogonality of 4-vector force and 4-vector velocity is valid for any force: strong, electromagnetic, weak and gravitational interactions, therefore there are many new aspects of mechanics remaining for physics to explore.
6.5 Conclusion

In this chapter it is found that the gravitational effects can be explained in terms of the orthogonality between 4-vector gravitational force and 4-vector velocity, the results are the same as the theory of general relativity.
Chapter 7

Pythagoras theorem and fair-gene

7.1 Introduction

Before we continue to study the hidden variable in the quantum mechanics, we begin to worry about some conceptual problems arising from the foundations of the quantum mechanics. Indeed, we need a sound foundation which not only supports the modern physics framework but also brings out new aspects for our theme. And we feel it is necessary to check the foundations for the coming works. In this chapter we begin with the famous Pythagoras theorem.

In the chapter 2, we have shown that Pythagoras theorem hides in the theory of relativity, now we feel that the quantum hidden variable must relate to the Pythagoras theorem in some way.

If there is one mathematical theorem that is familiar to every university student, it is surely the theorem of Pythagoras. The theorem is embedded in physics even at the initiation of physics, because the physics begins with describing the motion of a body in a frame of reference by mathematical language—inevitably including the Pythagoras theorem\[16\][17].

Consider a particle of rest mass $m$ in the frame of reference $S(x_1, x_2, x_3, t)$,
the particle moves a distance $\Delta l$ during the infinitesimal time interval $\Delta t$ with the speed $v$, according to Pythagoras theorem, we have

$$\Delta l^2 = v^2 \Delta t^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2$$  \hspace{1cm} (7.1)

The above equation directly forms the velocity formula given by

$$v^2 = v_1^2 + v_2^2 + v_3^2$$  \hspace{1cm} (7.2)

Multiplying the above equation by the rest mass $m$, and differentiating it with respect to time, we have

$$\frac{d(mv^2)}{dt} = 2v_1 \frac{d(mv_1)}{dt} + 2v_2 \frac{d(mv_2)}{dt} + 2v_3 \frac{d(mv_3)}{dt}$$  \hspace{1cm} (7.3)

Defining a symbol $f$ as

$$f = \frac{d(mv)}{dt}$$  \hspace{1cm} (7.4)

we obtain

$$\frac{d(\frac{1}{2}mv^2)}{dt} = f \cdot v$$  \hspace{1cm} (7.5)

From the last two equations, we have seen what we expect: when $f$ represents the force exerting on the particle, Eq.(7.4) is the Newton’s second law of motion, and Eq.(7.5) is the kinetic energy theorem.

### 7.2 Newton’s mechanics

We may rewrite the above section in an axiomatical way.

**Axiom 1:** Pythagoras theorem is valid only in inertial frame of reference.

From this Axiom we can derive out some useful consequences.

**Consequence 1:** The Newton’s second law of motion can be derived from the axiom 1, accompanied by deriving out the kinetic energy theorem for a particle.
Consequence 2: The Newton’s first law of motion can also be derived from the axiom 1, because of Eq. (7.4).

Now we consider a composite system which contains two particles Bob and Alice, whereas both Bob and Alice are also composed of many identical constituent particles, the number of the constituents in Bob is \( N_b \), the number in Alice is \( N_a \). As shown in Figure 7.1.

\[ v_c = \frac{N_b v_b + N_a v_a}{N_b + N_a} \quad (7.6) \]

Figure 7.1: The composite system.

The composite system of Alice and Bob may be regraded as a single particle whose geometric center has a velocity \( v_c \) given by

Without the lost of generality, we have supposed that the \( N_b \) constituents of Bob have the same velocity \( v_b \), likewise for Alice with \( v_a \). Since the
composite system of Bob and Alice can be regraded as a single particle, it obeys the Newton’s second law of motion as

\[ \frac{d[(N_b + N_a)m\mathbf{v}_c]}{dt} = \mathbf{F}_{\text{ext}}. \]  \hfill (7.7)

Where \( m \) is the mass of constituent, \( \mathbf{F}_{\text{ext}} \) represents the external force exerting on the composite system (Bob and Alice). If the external force vanishes, then the system becomes

\[ \frac{d(N_b m\mathbf{v}_b)}{dt} + \frac{d(N_a m\mathbf{v}_a)}{dt} = N_b \mathbf{F}_b + N_a \mathbf{F}_a = 0 \]  \hfill (7.8)

Where \( \mathbf{F}_b \) represents the force exerting on each constituent of Bob, likewise \( \mathbf{F}_a \) on each constituent of Alice. Eq. (7.8) has expressed the law of action and reaction. So we get

Consequence 3: The Newton’s third law of motion can also be derived from the axiom 1 for composite system.

Even if composite particle is not composed of identical constituent particles, the Consequence 3 is also valid. Because it is always possible to divide composite particle into many identical units, each unit is so small enough that the each unit has the same mass, theoretically it is not necessary for the unit to be a real elementary particle.

### 7.3 Relativistic mechanics

Again consider a particle of rest mass \( m \) in our frame of reference \( S(x_1, x_2, x_3, t) \), the particle moves a distance \( \Delta l \) during the infinitesimal time interval \( \Delta t \) with the speed \( v \), according to Pythagoras theorem, we have

\[ \Delta l^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 = v^2 \Delta t^2 \]
\[ = v^2 \Delta t^2 - c^2 \Delta t^2 + c^2 \Delta t^2 \]
\[ = -c^2 \Delta t^2 (1 - v^2/c^2) + c^2 \Delta t^2 \]  \hfill (7.9)

Where \( c \) is the speed of light. Defining the modified velocity
\[ u_1 = \frac{v_1}{\sqrt{1 - v^2/c^2}} \quad u_2 = \frac{v_2}{\sqrt{1 - v^2/c^2}} \quad (7.10) \]

\[ u_3 = \frac{v_3}{\sqrt{1 - v^2/c^2}} \quad u_4 = \frac{ic}{\sqrt{1 - v^2/c^2}} \quad (7.11) \]

Where \( v^2 = v_1^2 + v_2^2 + v_3^2 \), from Eq.(7.9), we obtain

\[ u_1^2 + u_2^2 + u_3^2 + u_4^2 = -c^2 \quad (7.12) \]

\[ u_\mu u_\mu = -c^2 \quad (7.13) \]

Where the repeated Greek indices take summation over values 1, 2, 3 and 4. The 4-vector velocity \( u = \{u_\mu\} \) is known as the relativistic velocity[1].

It is convenient to define the proper time interval \( d\tau = dt \sqrt{1 - v^2/c^2} \), thus the relativistic velocity is given by

\[ u_\mu = dx_\mu/d\tau \quad (7.14) \]

Where \( x_4 = ict \), Eq.(7.12) is the magnitude formula of relativistic 4-vector velocity of particle in the Minkowsky space time \((x_1, x_2, x_3, x_4 = ict)\) in its squared form.

To note that we have introduced the speed of light, but have never introduced any physical assumption, our work is pure mathematical derivation.

(1) The motion of the particle satisfies Eq.(7.12), we have

\[ mu_\mu mu_\mu = -m^2 c^2 \quad (7.15) \]

By defining the momentum \( P_\mu = mu_\mu, P = (P_1, P_2, P_3) \), we have

\[ P_\mu P_\mu = P \cdot P + P_4^2 = -m^2 c^2 \quad (7.16) \]

Differentiating Eq.(7.16) with respect to the proper time interval \( d\tau \), we get

\[ P \cdot \left( \frac{dP}{d\tau} \right) + P_4 \frac{dP_4}{d\tau} = 0 \quad (7.17) \]
By using $P_4 = mu_4 = micdt/d\tau = mic/\sqrt{1 - v^2/c^2}$, and introducing a 3-dimensional vector symbol $\mathbf{f}$, the above equation becomes

$$\mathbf{f} = \frac{d\mathbf{P}}{d\tau}$$  \hspace{1cm} (7.18)

$$-ic\frac{d(P_4)}{d\tau} = \mathbf{f} \cdot \mathbf{v}$$  \hspace{1cm} (7.19)

Eq. (7.19) can be understood as: the right side represents the power of the force $\mathbf{f}$ which exerts on the particle with the usual 3-dimensional velocity $\mathbf{v}$; the left side represents the change rate of the energy $E$ of the particle, i.e.

$$E = -icP_4 = mc^2 \frac{dt}{d\tau} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$  \hspace{1cm} (7.20)

Obviously, Eq. (7.20) is just the relativistic energy of the particle in relativistic mechanics, Eq. (7.18) and Eq. (7.19) are just the relativistic Newton’s second law[6]. Furthermore,

$$m_r = \frac{m}{\sqrt{1 - v^2/c^2}}$$  \hspace{1cm} (7.21)

$m_r$ is called the relativistic mass of the particle. This leads to $E = m_r c^2$[18]. By rewriting, Eq. (7.16) is given by

$$E^2 = |\mathbf{P}|^2 c^2 + m^2 c^4$$  \hspace{1cm} (7.22)

Eq. (7.22) is just the Einstein’s relationship of energy and momentum.

(2) In the Minkowsky space time, according to Eq. (7.18) and Eq. (7.19), we may define a 4-vector force as

$$\mathbf{f} \equiv (\mathbf{f}, f_4)$$  \hspace{1cm} (7.23)

$$f_4 = -\frac{\mathbf{f} \cdot \mathbf{u}}{u_4}$$  \hspace{1cm} (7.24)
then Eq. (7.18) and Eq. (7.19) may be rewritten as
\[ f = \frac{d(mu)}{d\tau} \quad \text{or} \quad f_{\mu} = \frac{d(mu_{\mu})}{d\tau} \] (7.25)

Obviously, the 4-vector force \( f \) and 4-vector velocity \( u \) satisfies the orthogonal relation \[2\]

\[ u \cdot f = u_{\mu}f_{\mu} = u_{\mu} \frac{d(mu_{\mu})}{d\tau} = \frac{md(u_{\mu}u_{\mu})}{2d\tau} = \frac{md(c^2)}{2d\tau} = 0 \] (7.26)

This result is easy to be understood when we note that the magnitude of the 4-vector velocity keeps constant, i.e. \(|u| = \sqrt{u_{\mu}u_{\mu}} = i\), any 4-vector force can never change the magnitude of the 4-vector velocity but can change its direction in the Minkowsky space-time.

The orthogonality of 4-vector force and 4-vector velocity is a very important feature for basic interactions \[3\].

(3) If we define symbols

\[ \cosh \alpha_{\mu} \equiv \frac{u_{\mu}}{ic} = \frac{dx_{\mu}}{icd\tau} \] (7.27)

Eq. (7.12) can be rewritten as

\[ \cosh^2 \alpha_1 + \cosh^2 \alpha_2 + \cosh^2 \alpha_3 + \cosh^2 \alpha_4 = 1 \]
\[ \text{or} \quad \cosh \alpha_{\mu} \cosh \alpha_{\mu} = 1 \] (7.28)

The above equation indicates that there is a definite like-Euclidean trigonometry in the Minkowsky space-time, despite all that the fourth axis is imaginary, the quantity \( \alpha_{\mu} \) may be understood as the angle between \( dx_{\mu} \) and \( icd\tau \).

As the most simple case, only consider the motion taking place in the plane \((x_1, x_4)\), (one dimensional straight line motion), the length \( icd\tau \) makes
angles $\alpha_1$ and $\alpha_4$ with respect to the lengths $dx_1$ and $dx_4$ respectively, we have

$$\cosh^2 \alpha_1 + \cosh^2 \alpha_4 = 1 \quad (7.29)$$

The above equation indicates that the two angles $\alpha_1$ and $\alpha_4$ have formed a "right-angled triangle" in the plane $(x_1, x_4)$. As shown in Figure 7.2.

Figure 7.2: The rotation of a reference frame, in Euclidian geometry representation.

By defining symbol

$$\sinh \alpha_4 = \cosh \alpha_1 \quad (7.30)$$

we get

$$\sinh^2 \alpha_4 + \cosh^2 \alpha_4 = 1 \quad (7.31)$$
From Eq. (7.27), we know

\[
\cosh \alpha_4 = \frac{u_4}{ic} = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (7.32)
\]

\[
\sinh \alpha_4 = \cosh \alpha_1 = \frac{u_1}{ic} = \frac{v}{ic\sqrt{1 - v^2/c^2}} \quad (7.33)
\]

Where \( v = v_1 = dx_1/dt \) for this simple case (one dimensional straight line motion).

(4) In the Minkowsky space time, if the coordinate system \( S(x_1, x_2, x_3, x_4) \) "rotates" through an angle \( \alpha \) in the plane \((x_1, x_4)\), and becomes another new coordinate system \( S'(x'_1, x'_2, x'_3, x'_4) \). According to Eq. (7.31), the transformation of the two systems \( S \) and \( S' \) will be given by

\[
x'_1 = x_1 \cosh \alpha - x_4 \sinh \alpha \\
x'_2 = x_2 \\
x'_3 = x_3 \\
x'_4 = x_1 \sinh \alpha + x_4 \cosh \alpha 
\]  

(7.34)

(7.35)

By comparison, we know that this \( \alpha \) is just that \( \alpha_4 \) in Eq. (7.32). Substituting Eq. (7.32) and Eq. (7.33) into the above equations, and using \( x_4 = i ct \) and \( x'_4 = i ct' \), we get

\[
x'_1 = \frac{x_1}{\sqrt{1 - v^2/c^2}} - \frac{vt}{\sqrt{1 - v^2/c^2}} \\
x'_2 = x_2 \\
x'_3 = x_3 \\
t' = -\frac{x_1v/c^2}{\sqrt{1 - v^2/c^2}} + \frac{t}{\sqrt{1 - v^2/c^2}}
\]  

(7.36)

(7.37)

(5) If Bob is at rest at the origin of the system \( S' \), i.e. \( x'_1 = 0 \), from Eq. (7.36), we know that Bob will be moving at speed \( v = dx_1/dt \) in the
system $S$. In other words, the system $S'$ is fixed at Bob, while Bob is moving at the speed $v$ along the axis $x_1$ of the system $S$. Coupling with this explanation, Eq.(7.36) and Eq.(7.37) are just the Lorentz transformation.

The constancy of the speed of light, length contraction and time dilation can evidently be derived from the Lorentz transformations.

Therefore, we get

Consequence 4: The formula of relativistic dynamics can be derived from the axiom 1.

7.4 Fair-gene

To note that the relativistic velocity of particle in the Minkowsky space time $(x_1, x_2, x_3, x_4 = i ct)$ has the same magnitude

$$|u| = i c$$

This result is very important for us because from it we can extract out an idea: the same magnitude of 4-vector velocity is called as the velocity fair-gene in this book. The velocity fair-gene has the following aspects that are deserved to discuss:

(1) every particle velocity has the same magnitude in the Minkowsky space time, then we must fairly look at every particle in the Minkowsky space time, every particle must fairly share the same physical laws in the Minkowsky space time, for example, every particle satisfies Eq.(7.25) which is derived from the Pythagoras theorem. This is equal to the Einstein’s Postulate of relativity: the laws of physics are the same in all inertial frames.

(2) every particle velocity has the same magnitude $i c$ in the Minkowsky space time, photons have the same speed of light $i c$ in the Minkowsky space time and (for example) take direction in the $x_1$ axis (or $x_2$ axis, or $x_3$ axis), thus photons have the same speed of light $c$ in the Minkowsky space time. This is equal to Einstein’s Postulate of the absolute speed of light: the speed of light in vacuum is the same in all inertial frames. Moreover, the
velocity fair-gene promises that any particle (electron, proton, planet, etc, only photons) have the same magnitude $ic$ for its 4-vector velocity in the Minkowsky space time.

(3) Suppose that an inertial frame $S'$ is moving with constant velocity $v$ relative to another frame of reference $S$. In order to keep the same magnitude $ic$ of the 4-vector velocity of the particle in both the frame $S$ and frame $S'$, the transformation from coordinates $x_1, x_2, x_3, x_4$ of $S$ and $x'_1, x'_2, x'_3, x'_4$ of $S'$ was proved to be the Lorentz transformation (in the preceding section). This result is easy to be understood when we imagine that the Minkowsky space time of the frame $S$ "rotates onto" the Minkowsky space time of the frame $S'$ and the coordinate transformation between $S$ and $S'$ corresponding to the "rotation" is just the Lorentz transformation, while the "rotation" keeps the same magnitude $ic$ of the 4-vector velocity of the particle in both the frame $S$ and frame $S'$.

(4) the velocity fair-gene provides us a thinking way for establishing a sound framework for the physics.

7.5 Y-space

In modern physics, complex number or imagine number have been widely used for describing physical quantities. For example, the fourth component of Minkowsky space time $(x_1, x_2, x_3, x_4 = i ct)$ is an imagine number, the relativistic mechanics based on the Minkowsky space time inevitably contains imagine numbers in its every formula. Moreover, for example, in the quantum mechanics, almost all physical quantities have (less or more) involved imagine number $i$, for example, energy and momentum are related with energy operator and momentum operator like $-i\hbar \partial_t \psi$ and $-i\hbar \partial_x \psi$ respectively. Complex number or imagine number makes physics to be ugly and dubious for the public, as we know, most students and public dislike complex number or imagine number in physics.

What physics pursues is simplicity, beauty and reality. In addition, it also is the most important that the values and models of human kind must be close to the nature.
Is it possible to build physics theory without complex number? Yes, it is possible right now, in this chapter we would use Pythagoras theorem and the velocity fair-gene to establish a new framework for real physics without complex number or imagine number, physics will become a beautiful swan from a ugly little duck for the public.

Again consider a particle of rest mass $m$ in our frame of reference $S(x_1, x_2, x_3, t)$, the particle moves a distance $\Delta l$ during the infinitesimal time interval $\Delta t$ with the speed $v$, according to Pythagoras theorem, we have

$$\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 = v^2 \Delta t^2$$
$$v_1^2 + v_2^2 + v_3^2 = v^2$$
$$v_1^2 + v_2^2 + v_3^2 + c^2 = v^2 + c^2$$
$$v_1^2 + v_2^2 + v_3^2 + c^2 = c^2(1 + v^2/c^2) \quad (7.39)$$

$$\frac{v_1^2}{1 + v^2/c^2} + \frac{v_2^2}{1 + v^2/c^2} + \frac{v_3^2}{1 + v^2/c^2} + \frac{c^2}{1 + v^2/c^2} = c^2 \quad (7.40)$$

Where $c$ is the speed of light. Defining the modified 4 vector velocity

$$u_1 = \frac{v_1}{\sqrt{1 + v^2/c^2}} \quad u_2 = \frac{v_2}{\sqrt{1 + v^2/c^2}} \quad (7.41)$$
$$u_3 = \frac{v_3}{\sqrt{1 + v^2/c^2}} \quad u_4 = \frac{c}{\sqrt{1 + v^2/c^2}} \quad (7.42)$$

Where $v^2 = v_1^2 + v_2^2 + v_3^2$, from Eq. (7.40), we obtain

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = c^2 \quad (7.43)$$
$$u_\mu u_\mu = c^2 \quad (7.44)$$
Where the repeated Greek indices take summation over values 1, 2, 3 and 4. In order to distinguish from the Minkowsky space time \((x_1, x_2, x_3, x_4 =ict)\), we define a new coordinate system \(Y: (y_1, y_2, y_3, y_4)\) as

\[
\begin{align*}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= x_4 \\
y_4 &= ct
\end{align*}
\]

then, the coordinate system \(Y: (y_1, y_2, y_3, y_4)\) is called as the Y-space in this book, in which all coordinates are real number!

The 4-vector velocity \(u = \{u_\mu\}\) is known as the relativistic velocity in the Y-space. It is convenient to define the proper time interval \(d\eta = dt\sqrt{1 + v^2/c^2}\), thus the relativistic velocity is given by

\[
u_\mu = \frac{dx_\mu}{d\eta}
\]

Where \(y_4 = ct\), Eq.(7.43) is the magnitude formula of relativistic 4-vector velocity of particle in the Y-space \((y_1, y_2, y_3, y_4 = ct)\) in its squared form.

(1) The motion of the particle satisfies Eq.(7.43), we have

\[
mu_\mu mu_\mu = m^2 c^2
\]

By defining the momentum \(P_\mu = mu_\mu, \, P = (P_1, P_2, P_3)\), we have

\[
P_\mu P_\mu = P \cdot P + P_4^2 = m^2 c^2
\]

Differentiating Eq.(7.47) with respect to the proper time interval \(d\eta\), we get

\[
P \cdot \left(\frac{dP}{d\eta}\right) + P_4 \frac{dP_4}{d\eta} = 0
\]

By using \(P_4 = mu_4 = mcdt/d\eta = mc/\sqrt{1 + v^2/c^2}\), and introducing a 3-dimensional vector symbol \(\mathbf{f}\), the above equation becomes
\[ f = \frac{dP}{d\eta} \quad (7.49) \]
\[ -c \frac{d(P_4)}{d\eta} = f \cdot v \quad (7.50) \]

Eq.(7.50) can be understood as: the right side represents the power of the force \( f \) which exerts on the particle with the usual 3-dimensional velocity \( v \); the left side represents the change rate of the energy \( E \) of the particle, i.e.

\[
E = -cP_4 = -mc^2 \frac{dt}{d\eta} = \frac{-mc^2}{\sqrt{1 + v^2/c^2}} \\
= -mc^2 + \frac{1}{2}mv^2 + ... \quad (7.51)
\]

Obviously, Eq.(7.51) is just the relativistic energy of the particle in the \( Y \)-space, Eq.(7.49) and Eq.(7.50) are just the relativistic Newton’s second law in the \( Y \)-space. Further more,

\[
m_r = \frac{m}{\sqrt{1 + v^2/c^2}} \quad (7.52)
\]

\( m_r \) is called the relativistic mass of the particle in the \( Y \)-space. This leads to \( E = -m_r c^2 \). By rewriting, Eq.(7.47) is given by

\[
E^2 = |P|^2 c^2 - m^2 c^4 \quad (7.53)
\]

Eq.(7.53) is just the Einstein’s relationship of energy and momentum in the \( Y \)-space.

(2) In the \( Y \)-space, according to Eq.(7.49) and Eq.(7.50), we may define a 4-vector force as

\[
f \equiv (f, f_4) \quad (7.54)
\]
\[
f_4 = -\frac{f \cdot u}{u_4} \quad (7.55)
\]
then Eq. (7.49) and Eq. (7.50) may be rewritten as

\[ f = \frac{d(mu)}{d\eta} \quad (7.56) \]
\[ f_\mu = \frac{d(mu_\mu)}{d\eta} \quad (7.57) \]

Obviously, the 4-vector force \( f \) and 4-vector velocity \( u \) satisfies the orthogonal relation

\[ u \cdot f = u_\mu f_\mu = u_\mu \frac{d(mu_\mu)}{d\eta} = \frac{md(u_\mu u_\mu)}{2d\eta} = \frac{md(c^2)}{2d\eta} = 0 \quad (7.58) \]

This result is easy to be understood when we note that the magnitude of the 4-vector velocity keeps constant, i.e. \( |u| = \sqrt{u_\mu u_\mu} = c \), any 4-vector force can never change the magnitude of the 4-vector velocity but can change its direction in the Y-space.

The orthogonality of 4-vector force and 4-vector velocity is a very important feature for basic interactions.

(3) If we define symbols

\[ \cos \alpha_\mu \equiv \frac{u_\mu}{c} = \frac{dx_\mu}{cd\eta} \quad (7.59) \]

Eq. (7.43) can be rewritten as

\[ \cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 + \cos^2 \alpha_4 = 1 \]
\[ \cos \alpha_\mu \cos \alpha_\mu = 1 \quad (7.60) \]

The above equation indicates that there is a definite like-Euclidean trigonometry in the Y-space, the quantity \( \alpha_\mu \) may be understood as the angle between \( dx_\mu \) and \( cd\eta \).
As the most simple case, only consider the motion taking place in the plane \((y_1, y_4)\), the length \(cd\eta\) makes angles \(\alpha_1\) and \(\alpha_4\) with respect to the lengths \(dy_1\) and \(dy_4\) respectively, we have
\[
\cos^2 \alpha_1 + \cos^2 \alpha_4 = 1 \tag{7.61}
\]
The above equation indicates that the two angles \(\alpha_1\) and \(\alpha_4\) have formed a "right-angled triangle" in the plane \((y_1, y_4)\). As shown in Figure 7.3.

By defining symbol
\[
\sin \alpha_4 = \cos \alpha_1 \tag{7.62}
\]
we get
\[ \sin^2 \alpha_4 + \cos^2 \alpha_4 = 1 \quad (7.63) \]

From Eq. (7.59), we know

\[
\cos \alpha_4 = \frac{u_4}{c} = \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \quad (7.64)
\]
\[
\sin \alpha_4 = \cos \alpha_1 = \frac{u_1}{c} = \frac{v}{c\sqrt{1 + \frac{v^2}{c^2}}} \quad (7.65)
\]

Where \( v = v_1 = dx_1/dt \) for this simple case.

(4) In the Y-space, if the coordinate system \( Y(y_1, y_2, y_3, y_4) \) "rotates" through an angle \( \alpha \) in the plane \( (y_1, y_4) \), and becomes another new coordinate system \( Y'(y'_1, y'_2, y'_3, y'_4) \). According to Eq. (7.63), the transformation of the two systems \( Y \) and \( Y' \) will be given by

\[
y'_1 = y_1 \cos \alpha - y_4 \sin \alpha \quad (7.66)
\]
\[
y'_2 = y_2
\]
\[
y'_3 = y_3
\]
\[
y'_4 = y_1 \sin \alpha + y_4 \cos \alpha \quad (7.67)
\]

By comparison, we know that this \( \alpha \) is just that \( \alpha_4 \) in Eq. (7.64). Substituting Eq. (7.64) and Eq. (7.65) into the above equations, and using \( y_4 = ct \) and \( y'_4 = ct' \), we get

\[
y'_1 = \frac{y_1}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{vt}{\sqrt{1 + \frac{v^2}{c^2}}} \quad (7.68)
\]
\[
y'_2 = y_2
\]
\[
y'_3 = y_3
\]
\[
t' = \frac{y_1 v/c^2}{\sqrt{1 + \frac{v^2}{c^2}}} + \frac{t}{\sqrt{1 + \frac{v^2}{c^2}}} \quad (7.69)
\]
(5) If Bob is at rest at the origin of the system $Y'$, i.e. $y_1' = 0$, from Eq. (7.68), we know that Bob will be moving at speed $v = dy_1'/dt$ in the system $Y$. In other words, the system $Y'$ is fixed at Bob, while Bob is moving at the speed $v$ along the axis $y_1$ of the system $Y$. Coupling with this explanation, Eq. (7.68) and Eq. (7.69) are just Lorentz transformations in the $Y$-space.

(6) The constancy of the speed of light, length contraction and time dilation can evidently be derived from the Lorentz transformation equations in the $Y$-space. But, our explanation of length contraction and time dilation in the $Y$-space counters to that in the Minkowsky space time. For example, according to Eq. (7.69), the time interval $\Delta t$ between two events at $y_1 = 0$ in the frame $Y$ relates to the interval time $\Delta t'$ in the frame $Y'$ by

$$\Delta t' = \frac{\Delta t}{\sqrt{1 + v^2/c^2}}$$

Note that $\Delta t > \Delta t'$, the time interval $\Delta t$ observed in $Y$ is longer than $\Delta t'$, this is the time dilation effect.

To note that in the $Y$-space the relativistic velocity of particle has the same magnitude

$$|u| = c$$

(7.70)

This result is very important for us because from it we can extract out the velocity fair-gene just as we have done in the Minkowsky space time. About the $Y$-space, the following aspects are deserved to discuss:

(1) we have a new dynamics in the $Y$-space in which all physical quantities are real numbers, but the $Y$-space dynamics is equivalent to the usual relativistic dynamics in the Minkowsky space time, the reader can check it. The advantage of the $Y$-space dynamics seems to be that it is easy for student and public to understand the dynamics and electromagnetism, but I think it has deep significance on the development of physics.

(2) every particle velocity has the same magnitude in the $Y$-space, then we must fairly look at every particle in the $Y$-space, every particle must fairly share the same physical laws in the $Y$-space, for example, every particle satisfies Eq. (7.56) which is derived from Pythagoras theorem.
(3) the velocity fair-gene provides an insight into the geometrization of physics.

7.6 Discussion

(1) The four consequences are derived from Pythagoras theorem in the preceding sections for the fundamentals of mechanics, from them other useful theorems, such as Lagrange’s equation and Hamilton’s principle, etc., can also be derived out. With these results, we recognize that the Pythagoras theorem is the origin of the whole mechanics. It becomes possible to establish an axiomatical system for the subject, both for teaching and research, especially, the experiments addressing to test the mechanics may be cancelled from our classrooms because an axiomatical system couldn’t tolerate the existence of many initiations in the subject.

(2) What is the inertial frame of reference? Our answer is that a frame in which Pythagoras theorem is valid is just the inertial frame of reference. This answer has a little different from that in traditional textbooks.

(3) Both Newton’s mechanics and relativistic mechanics can be derived from the same Pythagoras theorem in rigorous manner, but modern physics have clearly show that it prefers to the relativistic mechanics, it was said that the Newton’s mechanics is an excellent approximation when the motion is slow with respect to the speed of light.

If we use the speed of sound to replace the speed of light in Eq. (7.40), we would obtain a dynamics that resembles to the relativistic mechanics, in which the speed of sound takes place the role of the speed of light, how do physics accept it? This is a problem.

Generally speaking, we have absolute confidence in Pythagoras theorem, we have no doubts on the preceding results which have realized the geometrization of physics. The author is to pursue the faith coming from the Pythagoras theorem, the Pythagoras theorem is also an well established law which is at least 2500 years old.

Why the nature seems to prefer to the relativistic mechanics? We here gives a trial explanation. In one hand, to note that all measurements about
distance and time must use facilities whose principles are directly or indirectly based on the light, therefore, all phenomena in our optical eyes or instruments have had to involve the speed of light, so that the action we perceive is the force $f$ of Eq. (7.49) rather than the force $f$ of Eq. (7.4), hence the nature we perceive is the world obeying the relativistic mechanics. In another hand, if our measurements are accomplished by virtue of infinite communicating speed (if it exists), the nature we perceive will be the world obeying the Newton’s mechanics. This explanation is completely compatible with the standard contents of modern physics, because we knew in textbooks [19][20] that relativistic mechanics reduces to Newton’s mechanics when $c$ becomes infinite. Our efforts are to provide a new insight into the subject.

(4) the Pythagoras theorem has a long history, after Greek philosopher Pythagoras (500 B.C.). Other ancient civilizations also have the same theorem, such as China (1100 B.C.—200 B.C.) [21][22], India (500 B.C.—200 B.C.) [23]. In China, the Pythagoras theorem is called the GoGood theorem (on phonetic translation) or Chenzi’s theorem (after Chenzi, B.C.700). Both western and eastern share the common wisdom of their ancient civilizations.

7.7 Conclusion

In this chapter, we regard the Pythagoras theorem as an axiom, from this axiom we derive out four important consequences of physics, they are Newton’s three laws of motion, the formulae of relativistic dynamics. The results indicate that at least the mechanics is possible to be geometrized, so that it is possible to string together mechanics theorems with the Pythagoras theorem. Knowing the internal relationships between them, which have never been clearly revealed by other author, will benefit our physics teaching, and benefit our theme of this book.
Chapter 8

Acceleration and electromagnetism

8.1 Introduction

In the preceding chapter, we have use Pythagoras theorem and velocity fair-gene to successfully establish the framework of relativistic dynamics, we have established a Y-space. In this chapter, based on Minkowsky’s space and the Y-space, we will use the same method (Pythagoras theorem and fair-gene) to establish a new Z-space, and develop electromagnetism in the Z-space. The UC as the hidden variable of quantum mechanics will be defined in terms of Z-electromagnetism in the Z-space.

Attention: the readers are required to read the preceding chapter before reading this chapter, because this chapter uses the same manner and terms as the preceding chapter.

8.1.1 Z-space

Consider a particle moving at the coordinates $x_1, x_2, x_3$ at the time $t$ in an inertial frame $S$. The particle has the 4-vector velocity $u$, actually, only the first three components of $u$ is measurable in the usual 3D space because its
fourth component is imaginary number. Let \( a \) denote the magnitude of the first three components of the 4-vector acceleration, according to Pythagoras theorem, then

\[
\left( \frac{du_1}{d\tau} \right)^2 + \left( \frac{du_2}{d\tau} \right)^2 + \left( \frac{du_3}{d\tau} \right)^2 = a^2 \tag{8.1}
\]

Now we make a mathematical transformation about it, by introducing a constant \( w \) (whose physical meanings will be discussed at a right time)

\[
\left( \frac{du_1}{d\tau} \right)^2 + \left( \frac{du_2}{d\tau} \right)^2 + \left( \frac{du_3}{d\tau} \right)^2 + w^2 = w^2 + a^2 \tag{8.2}
\]

\[
\left( \frac{du_1}{d\tau} \right)^2 + \left( \frac{du_2}{d\tau} \right)^2 + \left( \frac{du_3}{d\tau} \right)^2 + w^2 = w^2(1 + a^2/w^2) \tag{8.3}
\]

Thus we can define a new 4-vector acceleration \( b \) of the particle as

\[
b_1 = \frac{du_1/d\tau}{\sqrt{1 + a^2/w^2}} = \frac{1}{\sqrt{1 + a^2/w^2}} \left( \frac{d^2x_1}{d\tau^2} \right) \tag{8.5}
\]

\[
b_2 = \frac{du_2/d\tau}{\sqrt{1 + a^2/w^2}} = \frac{1}{\sqrt{1 + a^2/w^2}} \left( \frac{d^2x_2}{d\tau^2} \right) \tag{8.6}
\]

\[
b_3 = \frac{du_3/d\tau}{\sqrt{1 + a^2/w^2}} = \frac{1}{\sqrt{1 + a^2/w^2}} \left( \frac{d^2x_3}{d\tau^2} \right) \tag{8.7}
\]

\[
b_4 = \frac{w}{\sqrt{1 + a^2/w^2}} = \frac{1}{\sqrt{1 + a^2/w^2}} \left( \frac{d^2(w\tau^2/2)}{d\tau^2} \right) \tag{8.8}
\]

Therefore the 4-vector acceleration \( b \) of the particle satisfies

\[
b_1^2 + b_2^2 + b_3^2 + b_4^2 = w^2 \tag{8.9}
\]

Without any postulate, we obtain a new formalism from Pythagoras theorem like in the relativistic mechanics. We define a new coordinates \( z_1 = \)
$x_1, z_2 = x_2, z_3 = x_3, z_4 = w\tau^2/2$. It is convenient to define the proper time interval $d\xi = d\tau\sqrt{1 + a^2/w^2}$, the particle now is moving at the coordinates $z_1, z_2, z_3, z_4$ with the 4-vector acceleration

$$\begin{align*}
b_1 &= \frac{1}{\sqrt{1 + a^2/w^2}}(\frac{d^2x_1}{d\tau^2}) = \frac{d^2z_1}{d\xi d\tau} \quad (8.10) \\
b_2 &= \frac{1}{\sqrt{1 + a^2/w^2}}(\frac{d^2x_2}{d\tau^2}) = \frac{d^2z_2}{d\xi d\tau} \quad (8.11) \\
b_3 &= \frac{1}{\sqrt{1 + a^2/w^2}}(\frac{d^2x_3}{d\tau^2}) = \frac{d^2z_3}{d\xi d\tau} \quad (8.12) \\
b_4 &= \frac{1}{\sqrt{1 + a^2/w^2}}(\frac{d^2(w\tau^2/2)}{d\tau^2}) = \frac{d^2z_4}{d\xi d\tau} \quad (8.13) \\
b_\mu b_\mu &= w^2 \quad (8.14)
\end{align*}$$

Where the repeated Greek indexes take summation over values $1, 2, 3, 4$.

Definition: $Z$-space is defined as the coordinates system $(z_1 = x_1, z_2 = x_2, z_3 = x_3, z_4 = wt^2/2)$ in which the 4-vector acceleration of the particle is defined as $b_\mu = \frac{d^2z_\mu}{d\xi d\tau}$, where the proper time interval is defined as $d\xi = d\tau\sqrt{1 + a^2/w^2}$.

The purpose we establish the $Z$-space is to study the dynamics about the acceleration of particle in accordance with the manner used in the preceding chapter.

If the particle has the mass $m$, then the motion of the particle satisfies

$$mb_\mu b_\mu = m^2w^2 \quad (8.15)$$

Differentiating the above equation with respect to the time $t$, we get

$$mb_\mu \frac{d(mb_\mu)}{dt} = 0 \quad (8.16)$$

Here we define a new concept: torna, denoted by $T$, defined as

$$T = \frac{d(mb)}{dt} \quad (8.17)$$

$$T_\mu = \frac{d(mb_\mu)}{dt} \quad (8.18)$$
Torna, after word “tornado”, means the strong power in a tornado, we may think tornado=torna+do. Torna represents the change rate of the acceleration of the particle multiplied by the particle mass, or the change rate of the force acting on the particle \( f = ma \simeq mb \). So

\[
mb_{\mu} = \int_{0}^{t} T_{\mu} dt + C \tag{8.19}
\]

Where \( C \) is an integral constant. According to Eq. \((8.16)\), the 4-vector torna \( T \) is orthogonal to the 4-vector acceleration \( b \). This result is easy to be understood when we note that the magnitude of the 4-vector acceleration keeps constant, i.e., \( |b| = w \), and any 4-vector torna acting on the particle can never change the magnitude of the 4-vector acceleration but can change its direction in the Z-space.

Obviously, the 4-vector torna \( T \) and 4-vector acceleration \( b \) satisfies the orthogonality, so

\[
b \cdot T = b_{\mu} T_{\mu} = b_{\mu} \frac{d(mb_{\mu})}{dt} = \frac{m}{2} \frac{d(b_{\mu} b_{\mu})}{d\tau}
\]

\[
= \frac{m}{2} \frac{d(w^2)}{d\tau} = 0 \tag{8.20}
\]

About the Z-space, the following aspects are deserved to discuss:

1) The first three components of the 4-vector force can be calculated by

\[
f_{\mu} = m \frac{du_{\mu}}{d\tau} = m \left( \frac{d\xi}{d\tau} \right) \left( \frac{d\xi}{d\tau} \right)
\]

\[
= \sqrt{1 + a^2/w^2} \frac{du_{\mu}}{d\xi}
\]

\[
= \sqrt{1 + a^2/w^2} mb_{\mu}
\]

\[
= \sqrt{1 + a^2/w^2} \left( \int_{0}^{t} T_{\mu} dt + C \right) \tag{8.21}
\]

\[
\mu = 1, 2, 3 \tag{8.22}
\]
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The 4-vector torna $T$ have an influence on the change rate of the 4-vector force $f$ (as we know, $f$ is almost proportion to $b$). Mathematically, the 4-vector force $f$ is (almost) equal to the integration of $T$ over a period time. So the orthogonality of 4-vector acceleration and 4-vector torna may provide a new approach to calculate the 4-vector force $f$, the new approach may bring out "hidden variable" comparing with the usual relativistic 4-vector $f$, the "hidden variable" may allow us to successfully explain the quantum behavior of particle. This is just our motivation to develop this formalism.

(2) Every particle 4-vector acceleration has the same magnitude in the Z-space, $|b| = w$, the same magnitude is called the acceleration fair-gene of particle in the Z-space, then we must fairly look at every particle in the Z-space, every particle must fairly share the same physical laws in the Z-space, for example, every particle satisfies Eq. (8.17) which is derived from Pythagoras theorem. Every particle has the same gene in the Z-space, so we must fairly treat them, this is the spirits of the acceleration fair-gene concept.

(3) The acceleration fair-gene $w$ has a deep significance on finding the hidden variable in the quantum mechanics. In Z-space, to note that $b \simeq f/m$, "every particle 4-vector acceleration has the same magnitude", this sentence means that every particle would be exerted by a force which never vanishes, because of $|b| = w$. The unvanished force is a new aspect of the dynamics in the Z-space. Even in a vacuum, an electron without any action on in the usual 3D space, but the electron still has an acceleration $b$ in its Z-space or has been exerted by a force in its Z-space, this unvanished acceleration or force in its Z-space explains why the electron would experience a indeterminate quantum behavior even in vacuum. The vacuum has unvanished action in its Z-space, the unvanished action in vacuum is just the hidden variable of quantum mechanics! (it almost take the author’s many years to realize this point, only the Z-space can explain why quantum behavior can happen in vacuum.), The next work is to cut thing fine.

(4) Suppose that a frame $S'$ is moving with constant acceleration $a$ relative to another frame of reference $S$. In order to keep the same magnitude $w$ of the 4-vector acceleration of the particle in both the frame $S$ and frame
$S'$, the transformation from coordinates $z_1, z_2, z_3, z_4$ of $S$ to $z'_1, z'_2, z'_3, z'_4$ of $S'$ must be a "rotation transformation": the Z-space of the frame $S$ "rotates onto" the Z-space of the frame $S'$ and the coordinate transformation between $S$ and $S'$ corresponding to the "rotation" is just a Lorentz-like transformation, the "rotation" keeps the same magnitude $w$ of the 4-vector acceleration of the particle in both the frame $S$ and frame $S'$.

(5) According to the acceleration fair-gene, if Coulomb law is valid for a point charge at rest in the Z-space of the frame $S$, then the Coulomb law must also be valid for a point charge at rest in the Z-space of the frame $S'$, because the the frame $S$ and frame $S'$ must share the same physical laws, we can not distinguish out which one is superior than another one.

In the Z-space of the frame $S$, consider two particle $q$ and $q'$ separated by a distance $x_1$. If the particle acceleration is small enough: $a_1/w \ll 1$, then $d\xi = dt$ for the particle $q$. Then the torna $T$ acting on the particle $q$ by another particle $q'$ is given by

$$
T_1 = \frac{d(mb_1)}{dt} = \frac{d}{dt}(m\frac{d^2z_1}{dt^2}) = \frac{d}{dt}(m\frac{d^2x_1}{dt^2}) = \frac{d}{dt}(mf) = \frac{d}{dt}(k\frac{qq'}{x_1^2})
$$

$$
= -2k\frac{qq'}{x_1^3}\frac{dx_1}{dt} = -2k\frac{qq'v_1}{x_1^3}
$$

(8.23)

From this special case, we know that the magnitude of the torna acting on the particle $q$ by another particle $q'$ may be generalized as

$$
|T| = -2k\frac{qq'v_r}{r^3}
$$

(8.24)

Where $r$ is the distance from $q$ to $q'$, $v_r$ is the usual speed of the particle $q$ along the radical direction $r$. The direction of the torna in the Z-space will discussed in details in the next section. This result can be regarded as the Coulomb’s law in the Z-space.

(6) We can calculate any force $f$ from its torna $T$, it follows Eq. (8.21) that
\[ mb_\mu = \int_0^t T_\mu dt + C \quad (8.25) \]
\[ f_\mu = \sqrt{1 + a^2/w^2} mb_\mu = \sqrt{1 + a^2/w^2}(\int_0^t T_\mu dt + C) \quad (8.26) \]

Obviously, we must face the new element \( w \), there is some “fine” difference between the torna-yielded force \( f_\mu \) and the usual Newtonian mechanical force (or relativistic mechanical force) for same subject (for example, Coulomb’s force or gravitational force). We hope to be able at once to speak out that the difference is just the hidden variable that responds to the quantum behavior of particle. Now we have arrived at a point very close to our aim: defining the hidden variable in quantum mechanics.

### 8.1.2 Z-electric wave

Remembering in a preceding chapter we have successfully established Maxwell’s electromagnetism using Coulomb law and the orthogonality of 4-vector force and 4-vector velocity, the same manner can be safely applied to establish a torna-based-electromagnetism, because Eq. (8.20) clearly shows that the 4-vector torna \( T \) and 4-vector acceleration \( b \) are orthogonal to one another. In this section we investigate the torna-based-electromagnetism. We begin the investigation with the direction of the torna acting on a particle.

Suppose there are two charged particle \( q \) and \( q' \) locating at the positions \( x \) and \( x' \) respectively in a Cartesian coordinate system \( S \), and moving with the 4-vector velocity \( u \) and \( u' \) respectively, and with the 4-vector accelerations \( b \) and \( b' \) respectively, as shown in Figure 8.1, where we have used \( X \) to denote \( x - x' \). The Coulomb’s torna \( T \) acting on the particle \( q \) is orthogonal to the direction of the 4-vector acceleration \( b \) of \( q \), as illustrated in Figure 8.1 using the Euclidian geometry to represent the 4 dimensional Z-space, according to the experiences we have done in the preceding section about Maxwell’s electromagnetism, we know that the torna \( T \) should be in the plane of \( b' \) and \( X \), so that we make an expansion about \( T \) as

\[ T = Ab' + BX \quad (8.27) \]
Figure 8.1: The Coulomb’s torna acting on \( q \) is orthogonal to the 4-vector acceleration \( b \) of \( q \), and lies in the plane of \( b' \) and \( X \), here the Euclidian geometry is used to illustrate the Z-space.

Where \( A \) and \( B \) are unknown coefficients, \( b' \) and \( X \) are chosen as two independent basis vectors. The expansion would be used to clarify a relationship between the orthogonality and the torna-based-electromagnetism.

Using the orthogonality \( b \perp T \), we get

\[
b \cdot T = A(b \cdot b') + B(b \cdot X) = 0 \tag{8.28}
\]

By eliminating the coefficient \( B \), we rewrite Eq.(8.27) as

\[
T = \frac{A}{b \cdot X}[(b \cdot X)b' - (b \cdot b')X] \tag{8.29}
\]

It follows from the direction of Eq.(8.29) that the unit vector \( \hat{T} \) of the
Coulomb’s Torna is given by

\[ \hat{T} = \frac{1}{w^2 r^2} [(b \cdot X)b' - (b \cdot b')X] \]  

(8.30)

because of \(|\hat{T}| = 1\), where \(r = |R|\), \(R \perp u'\) as illustrated in Figure 8.1.

It follows from Eq. (8.24) that the magnitude of the torna acting on the particle \(q\) is

\[ |T| = -2k\frac{qq'v_r}{r^3} \]  

(8.31)

Where the \(r\) denotes the distance from \(q\) to \(q'\) in the frame \(S\) along the 4-vector \(R\). as shown in the figure 8.1. Because the magnitude of the torna is invariance during the coordinate transformation (corresponding to a "rotation" in the Z-space), therefore, the magnitude of the torna \(T\) in the frame \(S\) takes the same value as that in the frame \(S'\) fixed at the particle \(q'\).

Combination of Eq. (8.30) with Eq. (8.31), we obtain a torna formula base on the Coulomb law and the orthogonality of 4-vector torna and 4-vector acceleration:

\[ T = -2k\frac{qq'v_r}{w^2 r^4} [(b \cdot X)b' - (b \cdot b')X] \]

\[ = -2k\frac{qq'v_r}{w^2 r^4} [(b \cdot R)b' - (b \cdot b')R] \]

\[ = qv_r[(b \cdot (-2 R'^2))b' - (b \cdot b')(-2 R'^2)] \]  

(8.32)

Using the relation

\[ \partial_\mu \left( \frac{1}{r^2} \right) = -\frac{2R_\mu}{r^4} \]  

(8.33)

we obtain

\[ T_\mu = qv_r[-(b_\nu \partial_\nu (\frac{kk'}{w^2 r^2}))b'_\mu + (b_\nu b'_\nu)\partial_\mu (\frac{kk'}{w^2 r^2})] \]

\[ = qv_r[-(b_\nu \partial_\nu (\frac{kk'}{w^2 r^2})) + (b_\nu)\partial_\mu (\frac{kk'}{w^2 r^2})] \]  

(8.34)
The torna can be rewritten in terms of 4-vector components as

\[ U_\mu = \frac{kq'b'_\mu}{w^2r^2} \]  
\[ G_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu \]  
\[ T_\mu = qv_r G_{\mu\nu} b_\nu \]  

Thus \( U_\mu \) expresses the 4-vector acceleration-potential of the particle \( q' \).

The gauge condition in the Z-space: From Eq. (8.20), because of \( b'_\perp R \), i.e. \( b'_\mu R_\mu = 0 \), we have

\[ \partial_\mu U_\mu = \frac{kq'b'_\mu}{w^2} \partial_\mu \left( \frac{1}{r^2} \right) = \frac{kq'b'_\mu}{w^2} \left( -\frac{2R_\mu}{r^4} \right) = 0 \]  

It is called as the gauge condition in the Z-space.

To note that \( R \) has three degrees of freedom under the condition \( R \perp u' \), so we have

\[ \partial_\mu R_\mu = 3 \]
\[ \partial_\mu \left( \frac{1}{r} \right) = -4\pi \delta(R) \]
\[ \partial_\mu \partial_\mu \left( \frac{1}{r^2} \right) = \partial_\mu \left[ \frac{2}{r} \partial_\mu \left( \frac{1}{r} \right) \right] = \frac{2}{r} \partial_\mu \partial_\mu \left( \frac{1}{r} \right) + 2 \partial_\mu \left( \frac{1}{r} \right) \partial_\mu \left( \frac{1}{r} \right) = \frac{8\pi \delta(R)}{r} + 2 \left( -\frac{2R_\mu}{r^3} \right) \left( -\frac{2R_\mu}{r^3} \right) = \frac{8\pi \delta(R)}{r} + \frac{2}{r^4} \]  

From Eq. (8.36), we have

\[ \partial_\nu G_{\mu\nu} = \partial_\nu \partial_\mu G_\nu - \partial_\nu \partial_\nu G_\mu = \partial_\mu (\partial_\nu G_\nu) - \partial_\nu (\partial_\nu G_\mu) = -\partial_\nu \partial_\nu G_A_\mu = \partial_\mu \partial_\nu \left( \frac{1}{r^2} \right) = \frac{kq'b'_\mu}{w^2} \partial_\nu \partial_\nu \left( \frac{1}{r^2} \right) = \frac{kq'b'_\mu}{w^2r^2} \frac{8\pi \delta(R)}{r} + \frac{2kq'b'_\mu}{w^2r^4} \]
Where we define \( J'_u \) as the acceleration-current density of the source \( q' \).

From Eq. (8.36), by exchanging the indices and taking the summation of them, we have

\[
\partial_\lambda G_{\mu\nu} + \partial_\mu G_{\nu\lambda} + \partial_\nu G_{\lambda\mu} = 0 \quad (8.44)
\]

The Eq. (8.42) and (8.44) are recognized as Maxwell-like equations. For continuous media, they are valid as well.

The quantity \( G_{\mu\nu} \) is an antisymmetric tensor, \( G_{\mu\nu} = -G_{\nu\mu} \), which we call it as the 2-electric field tensor. Like the Maxwell’s electromagnetism, written as a 4X4 matrix, the field tensor is

\[
G_{\mu\nu} = \begin{bmatrix}
0 & H_3 & -H_2 & -D_1 \\
-H_3 & 0 & H_1 & -D_2 \\
H_2 & -H_1 & 0 & -D_3 \\
D_1 & D_2 & D_3 & 0 \\
\end{bmatrix} \quad (8.45)
\]

Where the quantity \( D \) is a electric-field-like physical quantity, the quantity \( H \) is a magnetic-field-like physical quantity.

The fourth component (\( \mu = 4 \)) of Eq. (8.42) is Gauss-like law,

\[
\partial_\nu G_{4\nu} = \frac{k}{w^2} J'_4 \quad (8.46)
\]

which is

\[
\nabla \cdot \mathbf{D} = \frac{k}{w^2} \left[ \frac{q'b'_4}{r} 8\pi \delta(R) + \frac{2q'b'_4}{r^4} \right] \\
= \frac{k}{w^2} \left[ \frac{q'}{r} \frac{w}{\sqrt{1 + a^2/w^2}} 8\pi \delta(R) + \frac{2q'}{r^4} \frac{w}{\sqrt{1 + a^2/w^2}} \right] \\
= \frac{kq'}{w\sqrt{1 + a^2/w^2}} \left[ \frac{8\pi \delta(R)}{r} + \frac{2}{r^4} \right] \quad (8.47)
\]
This equation cannot guarantee vector field $\mathbf{D}$ to be a converging field (like point charge electric field). The first three components ($\mu = 1, 2, 3$) of Eq. (8.42) is the spatial components like Ampere law.

Note that $\partial_\nu G_{\mu\nu}$ is a sum of $\mu$ terms. The term with $\nu = 4$ is

$$\frac{\partial D_i}{\partial z_4} = \frac{2}{w^2} \frac{\partial D_i}{\partial (t^2)}$$  \hspace{1cm} (8.48)

and the sum over $\nu = 1, 2, 3$ is the $i$ component of $\nabla \times \mathbf{H}$. Hence the spatial components of Eq. (8.42) are equivalent to

$$\nabla \times \mathbf{H} - \frac{2}{w^2} \frac{\partial \mathbf{D}}{\partial (t^2)} = \frac{k}{w^2} \mathbf{J}$$  \hspace{1cm} (8.49)

It is straightforward to verify that the fourth component of Eq. (8.44) is Gauss-like law

$$\nabla \cdot \mathbf{H} = 0$$

and the spatial components make up Faraday-like law

$$\nabla \times \mathbf{D} - \frac{2}{w^2} \frac{\partial \mathbf{H}}{\partial (t^2)} = 0$$  \hspace{1cm} (8.50)

For the Maxwell-like equations there are plenty of physical meanings deserved to exploit. In the present stage the first thing we care about is the wave concerning to $\mathbf{D}$ and $\mathbf{H}$ in the Z-space, which is called as the Z-electric wave.

The Maxwell’s electromagnetic wave can relate with the Z-electric wave through Eq. (8.26), but they are also apparently different.

1. When the particle $q'$ is moving with a constant velocity, according to the Maxwell’s theory, there are electric field and magnetic field around the particle $q'$, without electromagnetic wave radiation. For the constant velocity, according to Eq. (8.49) for $\mathbf{J}' = 0$, there is not Z-electric wave around the particle $q'$.

2. When the particle $q'$ is moving with a constant acceleration, according to the Maxwell’s theory, there are electromagnetic wave radiation around the particle $q'$. For the constant acceleration, according to Eq. (8.49)
for $\mathbf{J}' = \text{const.}$, there are static Z-electric field $\mathbf{D}$ and static Z-magnetic field $\mathbf{H}$ around the particle $q'$, without Z-electric wave radiation.

(3) When the particle $q'$ is moving with an alternating acceleration (for example, electric dipole), according to the Maxwell’s theory, there are electromagnetic wave radiation around the particle $q'$. For the alternating acceleration, according to Eq. (8.49), there are alternating Z-electric field $\mathbf{D}$ and Z-magnetic field $\mathbf{H}$ around the particle $q'$, there is the Z-electric wave radiation around the particle $q'$. Because the source is inversely proportional to the radial distance, the Z-electric wave drops rapidly comparing to the Maxwell’s electromagnetic wave. Thus, this situation can divided into to cases. The first case is macroscopic source like Hertzian dipole, the usual Maxwell electromagnetic wave radiation is dominant while the Z-electric wave radiation weakens off. The second case is hydrogen atom in a smaller size comparing to the Hertzian diploe, the Z-electric wave radiation becomes the dominant one and depresses the usual Maxwell electromagnetic wave radiation. Remembering the Maxwell’s theory is incomplete we have discussed in the earlier sections, the Z-electric wave can destroy the framework of the Maxwell’s theory in the hydrogen atom, thus destroy the mechanism of the usual electromagnetic wave radiation. This idea partially explains the famous problem: why the hydrogen atom in stationary states emits no radiation.

(4) The Z-electric wave may play a stronger role in interaction of nucleus than what we currently know, even more than what the quantum mechanics can tell us.

(5) The expansion of Eq. (8.27) is also incomplete for torna in the view of the chapter 2, but we believe that the expansion has a good enough approximation.

### 8.2 Residual force

As we know, magnetization curves for ferromagnetic materials are not retraced as we increase and then decrease the external magnetic field. When the applied magnetic field is increased and then decreased back to its initial
value, there is residual magnetism in ferromagnetic materials.

Analogy with residual magnetism, when the applied torna is increased and then decreased back to its initial value, there may be residual force in vacuum. Theoretically, when torna vanishes off, the force after the integration on the torna may not return its initial value (zero).

\[ f_\mu = \sqrt{1 + \frac{a^2}{w^2}mb_\mu} = \sqrt{1 + \frac{a^2}{w^2}} \left( \int_0^t T_\mu dt + C \right) \]  

(8.51)

Where the integral constant \(C\) may be zero or not zero. This is a new aspect of the Z-space. This residual force can be used to explain why quantum behavior can happen in vacuum.

In electronic diffraction experiment, the first electron passes the space considered, where the torna increases and then decreases, it may leave residual force in its trace, thus influences the next coming electrons.

### 8.3 Hidden variable in quantum mechanics

(1) We have two reasons to modify the relativistic dynamics, the first reason is clear when we note that the relativistic mechanics never contains the influence of the acceleration of particle, Lorentz force formula is given by

\[ \mathbf{f} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \]  

(8.52)

It contains the velocity of the particle considered but fails to contain the acceleration of the particle. According to the acceleration fair-gene in the Z-space, the acceleration has an essential influence on the the motion of the particle. When the Z-electromagnetism in some microscopic cases puts an addition force to the Eq.(8.52), it is not surprising to recognize that the addition force is just the hidden variable that has been discussed for a long time in the quantum mechanics history. The second reason arises from the incompleteness of the Maxwell’s theory we have discussed in the earlier chapters. When the interaction between microscopic particles appears beyond the electric force or magnetic force, it is also not surprising to recognize that the addition force is just the hidden variable.
The hidden variable $H$ in the quantum mechanics is defined as the difference between the torna-yielded force $f_{\text{torna}}$ and the relativistic electromagnetic force $f_{\text{relativistic}}$.

\[ H_\mu = f_{\text{torna}} - f_{\text{relativistic}} \]  
\[ = \sqrt{1 + \frac{a^2}{w^2}} \left( \int_0^t T_\mu dt + C' \right) - qF_{\mu\nu}u_\nu \]  

(2) The most simple modification on the Lorentz force formula seems to add an addition term to represent all effects that are not contained in Maxwell’s theory. In Minkowsky space time, the modified dynamics equation is given by

\[ m \frac{du_\mu}{d\tau} = f_\mu + f_{UC\mu} = qF_{\mu\nu}u_\nu + f_{UC\mu} \]  

Where $f$ denotes Lorentz force, $f_{UC}$ denotes the undefined component we have discussed in the preceding chapter. In the earlier chapter, we have successfully derived out the quantum wave equation for microscopic particle.

(3) In the present work, we are not to derive or analyse the undefined component $f_{UC}$, we regard $f_{UC}$ as a fluctuating source that contributes to the Lorentz force, we want to derive out the quantum wave equation in the ensemble space relating to the fluctuating source. The statistics will tell us the main influence of the undefined component $f_{UC}$ on the motion of particle while avoids the tedious mathematic complexity about UC.

(4) Why we say that the undefined component $f_{UC}$ is a fluctuating source? Consider an electron in a free motion, in its Z-space the acceleration fair-gene $|b| = w$ would applied unvanished force (acceleration) to the electron, while in its Y-space (or Minkowsky space time) the velocity fair-gene $|u| = c$ would have to allow to change the direction of the electron so that a 3D acceleration of the electron has to happen, then in its Z-space the acceleration fair-gene $|b| = w$ would adjust its acceleration direction and takes a further effect on its velocity, thus the electron would have no way go on its initial straight line, the interaction between the velocity fair-gene
\(|u| = c|b| = w| will be alternating (otherwise the system will go on to an explosion), the motion of the electron will go on at an "alternating" velocity to respond to the "alternating" acceleration, so the electron will move in the way of particle-wave duality, this is just the quantum behavior. Because the interaction between the velocity \(|u| = c\) and the acceleration \(|b| = w\) are nonlinear, maybe chaotic, we simply regard the undefined component \(f_{UC}\) as a fluctuating source.

(5) In a preceding chapter, we have derived out the quantum wave equation in the ensemble space relating to the UC fluctuating source. Reviewing the derivation will allow us more clearly to understand the relationship between the undefined component \(f_{UC}\) and the particle-wave duality, after confidence grows.

Consider a particle diffraction experiment in an electromagnetic field where the UC is active and cause the indeterminacy of quantum mechanics, Each particle in the field subjects to both the electromagnetic force \(f\) and UC \(f_{UC}\), then it satisfies the relativistic dynamic equation:

\[
m \frac{du_\mu}{d\tau} = f_\mu + f_{UC\mu} = qF_{\mu\nu}u_\nu + f_{UC\mu} \tag{8.56}
\]

\[
u_\mu u_\mu = -c^2 \tag{8.57}
\]

In order to eliminate the UC term by the means of statistics, i.e. using \(<f_{UC}> = 0\), we turn to study this dynamic equation in the ensemble space which consists of enormous particle paths recorded in enormous identical experiments, in the ensemble space we find their averages

\[
m \frac{d<u_\mu>}{d\tau} = qF_{\mu\nu} <u_\nu> \tag{8.58}
\]

\[
<u_\mu><u_\mu> = -c^2 \tag{8.59}
\]

Strictly speaking, Eq.(8.59) is not derived from Eq.(8.57), it stands for the property that the magnitude of any 4-vector velocity keeps constant, otherwise this 4-vector velocity concept collapses. If we regard the enormous recorded particle paths in the ensemble space as a flow, it is easy to find that there is a 4-vector velocity field for the flow, as shown in Figure 8.2.
We clearly emphasize two points: (1) At every point of the ensemble space, the mean 4-vector velocity \( <u> \) satisfies the above mean dynamic equation; (2) the mean 4-velocity \( <u> \) represents a 4-vector velocity field of the flow, \( <u> \) is a function of the position of the ensemble space.

For our convenience, we drop the mean sign \( <> \) for the 4-vectors in the followings, simply use \( u \) in place of \( <u> \). As mentioned above, the 4-vector velocity \( u \) is regarded as a 4-vector velocity field in the ensemble space, then

\[
\frac{du_\mu}{d\tau} = \frac{\partial u_\mu}{\partial x_\nu} \frac{dx_\nu}{d\tau} = \frac{dx_\nu}{d\tau} \frac{\partial u_\mu}{\partial x_\nu} = \partial_\nu u_\mu \quad \text{(8.60)}
\]

\[
qu F_{\mu\nu}u_\nu = q u_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \quad \text{(8.61)}
\]

Beware: Eq. (8.60) is the most important step in the present work after introducing the concept of 4-vector velocity field in the ensemble space.
Substituting them back into the dynamic equation, and re-arranging these terms, we obtain

\[ u_\nu \partial_\nu (mu_\mu + qA_\mu) = u_\nu \partial_\mu (qA_\nu) \]
\[ = u_\nu \partial_\mu (mu_\nu + qA_\nu) - u_\nu \partial_\mu (mu_\nu) \]
\[ = u_\nu \partial_\mu (mu_\nu + qA_\nu) - \frac{1}{2} \partial_\mu (mu_\nu u_\nu) \]
\[ = u_\nu \partial_\mu (mu_\nu + qA_\nu) - \frac{1}{2} \partial_\mu (-mc^2) \]
\[ = u_\nu \partial_\mu (mu_\nu + qA_\nu) \]  

(8.62)

Using the notation

\[ K_{\mu\nu} = \partial_\mu (mu_\nu + qA_\nu) - \partial_\nu (mu_\mu + qA_\mu) \]  

(8.63)

Eq. (8.62) is given by

\[ u_\nu K_{\mu\nu} = 0 \]  

(8.64)

Because \( K_{\mu\nu} \) contains the variables \( \partial_\mu u_\nu, \partial_\mu A_\nu, \partial_\nu u_\mu \) and \( \partial_\nu A_\mu \), they are independent from \( u_\nu \), then a solution satisfying Eq. (8.64) is actually

\[ K_{\mu\nu} = 0 \]  

(8.65)

\[ \partial_\mu (mu_\nu + qA_\nu) = \partial_\nu (mu_\mu + qA_\mu) \]  

(8.66)

The above equation allows us to introduce a potential function \( \Phi \) in mathematics, further set \( \Phi = -i\hbar \ln \psi \), we obtain a very important equation

\[ (mu_\mu + qA_\mu) = \partial_\mu \Phi \]  

(8.67)

\[ (mu_\mu + qA_\mu)\psi = -i\hbar \partial_\mu \psi \]  

(8.68)

Where \( \psi \) may be a complex mathematical function, its physical meanings will be determined from experiments after the introduction of the Planck’s constant \( \hbar \), as we have know, it is wave function.

Substituting Eq. (8.67) into \( u_\mu u_\mu = -c^2 \), we obtain a wave equation

\[ (-i\hbar \partial_\mu \psi - qA_\mu \psi)(-i\hbar \partial_\mu \psi - qA_\mu \psi) = -m^2 c^2 \psi^2 \]  

(8.69)
It is a new quantum wave equation\cite{4}\cite{7}. Where the left side corresponds to the product of momentum and momentum itself, does not correspond to the product of momentum operator and momentum operator!

8.4 Conclusions

In this chapter, using Pythagoras theorem we establish a Z-space to study the acceleration of particle, we define a new term ”torna” to represent the change rate of the force acting on the particle. It is found that every particle 4-vector acceleration has the same magnitude in the Z-space, then we must fairly look at every particle in the Z-space, every particle must fairly share the same physical laws in the Z-space. It is found that any 4-vector torna acting on a particle can never change the magnitude of the 4-vector acceleration of the particle but can change its direction in the Z-space. The torna corresponding to Coulomb’s force and the torna-based-electromagnetism are investigated, it is found that the force calculated from the torna is different from the usual electromagnetic force. The hidden variable in the quantum mechanics is defined as the difference between the torna-yielded force and the usual electromagnetic force, the hidden variable is just the undefined component of Maxwell's electromagnetic force discussed in the preceding chapter. In the present work, we regard the undefined component as a fluctuating source, and derive out the quantum wave equation in the ensemble space relating to the fluctuating source.
Chapter 9

Discussion

9.1 A critical review

In recent years, the awareness of physics knowledge and famous physicists in the public has dropped dramatically, more and more students majoring in physics feel harder to find a good jobs for their professional careers in physics related sectors. In this discussion section, we would like to make a comparison between physics and sport so that we can absorb some healthy spirits from other prospective sciences and societies.

(1) Sport is a powerful vehicle for peace by forging closer relations, mutual respect and understanding between peoples. Sport is also an important learning tool for young people as it is often during the playing of sport that children learn important values and models of good conduct that last a lifetime. In sport, misuse of substances or methods has led to cardiovascular disorders, liver and kidney disease, psychological or physical dependence, even death. Blood doping and gene manipulation to enhance sporting performance may raise further harm that affects society as a whole. Therefore, adopting is prohibited for a long time in modern sport.

In the history of physics in the 20 century, physicists had developed some special approaches that quickly solved some hard physical problems but also cause irreversible damage to the reasonable framework of classical physics.
For example, in the quantum mechanics, the energy and momentum are related with differential operators that act on the wave function $\psi$

$$E \psi = i \hbar \frac{\partial \psi}{\partial t} \quad (9.1)$$

$$P_x \psi = i \hbar \frac{\partial \psi}{\partial x} \quad (9.2)$$

Every thing goes very well, but when we see that the energy momentum relation for a free particle $m$ was replace by the energy operator and momentum operator in this way:

$$E = \frac{P^2}{2m} \quad (9.3)$$

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \quad (9.4)$$

we feel astonishing, the idea that squared momentum are replaced by squared momentum operator is quite beyond our ken. The key point is that Eq. (9.4) (Schrodinger wave equation) was so successful and was regarded as one of the most important achievement in the 20 century physics. What we can do is to keep silent, obey instructions with wounded logic. The wounded logic must recover, no matter how long it will take, 100 years or 200 years. We keep our faith: momentum is momentum, not momentum operator. Actually, we have made a great effort to prove in the earlier chapters that momentum is just momentum.

(2) Anabolic steroids are derived from the male hormone testosterone. They build muscles by altering chemical balances in the body. They are a dangerous drug for sport competitions with health consequences: hardening of the arteries, high cholesterol, high blood pressure, heart disease, liver damage, kidney damage, depression, psychic effects.

In the quantum mechanics, almost all physical quantities have (less or more) involved imagine number $i$, for example, energy and momentum are related with energy operator and momentum operator like in Eq. (9.1) and (9.2). Is it possible to build physics theory without complex number? Yes,
it is possible, if we use the Y-space to replace Minkowski space time as a real framework. What physics pursues is simplicity, beauty and reality, in addition, it also is the most important that the values and models of human kind must be close to the nature.

(3) Competitive athletes have always looked for ways to improve performance. Student athletes face pressure to win, from teammates, peers, parents and coaches. But some coaches and athletic directors have used their own story to urge young athletes to avoid the use of drug. Officials from the Olympic sports believes most Olympic athletes are drug-free, more and more coaches and sports officials believe education of young athletes and systematic testing will curb drug use. Pure Performance: the most important sporting record is a clean one.

Our blood pressure inevitably will rise to a higher level when we use the Feynman’s path integration method which is based on sum over all possible paths in the spacetime plane with the end point fixed, by no ways we can let our logic become clean. The Feynman path integration method, motivated by Dirac’s mysterious remark, is not too convenient for attacking practical problems in nonrelativistic quantum mechanics. Even for the simple harmonic oscillator it is rather cumbersome to evaluate explicitly the relevant path integral. But, Feynman’s path integration method have been found to be very powerful in all branches of modern physics, such as quantum field theory and statistical mechanics. So, we have always looked for ways to understand the path integration method, not to follow Feynman’s practicism.

9.2 Three quantum wave equations

In the past century, many attempts were made to address to understand the quantum wave nature from the classical mechanics, however, much of the connection with the classical physics is rather indirect. In this section, we show that Dirac wave equation, Klein-Gordon wave equation and Schrodinger wave equation can be derived from the momentum wave-function relation in a rigorous manner.
In the preceding chapter, we have derived out the momentum wavefunction relation from the relativistic Newton’s second law in its ensemble space\(^7\).

\[
(mu_\mu + qA_\mu)\psi = -i\hbar \partial_\mu \psi \tag{9.5}
\]

Where \(\psi\) is wave function, \(\hbar\) is the Planck’s constant. Multiplying the two sides of the following equation by \(\psi\)

\[
-m^2c^2 = m^2u_\mu u_\mu \tag{9.6}
\]

Using Eq.(9.5), we obtain

\[
-m^2c^2\psi = (mu_\mu)(-i\hbar \partial_\mu - qA_\mu)\psi
\]

\[
= (-i\hbar \partial_\mu - qA_\mu)(mu_\mu \psi)[-[i\hbar \psi \partial_\mu (mu_\mu)]
\]

\[
= (-i\hbar \partial_\mu - qA_\mu)(-i\hbar \partial_\mu - qA_\mu)\psi
\]

\[
-[-i\hbar \psi \partial_\mu (mu_\mu)] \tag{9.7}
\]

According to the continuity condition for particle motion

\[
\partial_\mu (mu_\mu) = 0 \tag{9.8}
\]

we have

\[
-m^2c^2\psi = (-i\hbar \partial_\mu - qA_\mu)(-i\hbar \partial_\mu - qA_\mu)\psi \tag{9.9}
\]

It is known as the Klein-Gordon equation.

On the condition of non-relativity, the Schrodinger wave equation can be derived from the Klein-Gordon wave equation \(^9\).

However, we must admit that we are careless when we use the continuity condition of Eq.(9.8), because from Eq.(9.5) we obtain

\[
\partial_\mu (mu_\mu) = \partial_\mu(-i\hbar \partial_\mu \ln \psi - qA_\mu) = -i\hbar \partial_\mu \partial_\mu \ln \psi \tag{9.10}
\]

Where we have used Lorentz gauge condition \(\partial_\mu A_\mu = 0\). Thus from Eq.(9.6) to Eq.(9.7) we obtain
\[- m^2 c^2 \psi = (\psi (-i \hbar \partial_\mu - q A_\mu)(-i \hbar \partial_\mu - q A_\mu)\psi \\
+ \hbar^2 \psi \partial_\mu \partial_\mu \ln \psi \]  
(9.11)

This is a complete wave equation for describing accurately the motion of the particle. The Klein-Gordon wave equation is a linear equation so that the principle of superposition remains valid, however with the addition of the last term in Eq.(9.11), Eq.(9.11) turns out to display nonlinear effect which can be understood as a sort of chaos[26].

In the following we shall show Dirac wave equation from Eq.(9.5) and Eq.(9.6). From Eq.(9.5), the wave function can be given in integral form by

\[ \Phi = -i \hbar \ln \psi = \int_{x_0}^{x} (mu_\mu + q A_\mu)dx_\mu + \theta \]  
(9.12)

Where \( \theta \) is an integral constant, \( x_0 \) and \( x \) are the initial and final points of the integral with an arbitrary integral path. Since Maxwell’s equations are gauge invariant, Eq.(9.5) should preserve invariant form under a gauge transformation, specified by

\[ A_\mu' = A_\mu + \partial_\mu \chi, \quad \psi' \leftarrow \psi \]  
(9.13)

Where \( \chi \) is an arbitrary function. Then Eq.(9.12) under the gauge transformation is given by

\[ \psi' = \exp \left\{ \frac{i}{\hbar} \int_{x_0}^{x} (mu_\mu + q A_\mu)dx_\mu + \frac{i}{\hbar} \theta \right\} \exp \left\{ \frac{i}{\hbar} q \chi \right\} \]

\[ = \psi \exp \left\{ \frac{i}{\hbar} q \chi \right\} \]  
(9.14)

Then \( \psi' \) and \( \psi \) share the same velocity field. The situation in which a wave function can be changed in a certain way without leading to any observable effects is precisely what is entailed by a symmetry or invariant principle.
in quantum mechanics. Here we emphasize that the invariance of velocity field is held for the gauge transformation.

Suppose there is a family of wave functions $\psi^{(j)}$, $j = 1, 2, 3, ..., N$, which correspond to the same velocity field denoted by $P_\mu = m u_\mu$, they are distinguishable from their different phase angles $\theta$ as in Eq.(9.12). Then Eq.(9.16) can be given by

$$0 = P_\mu P_\mu \psi^{(j)} \psi^{(j)} + m^2 c^2 \psi^{(j)} \psi^{(j)}$$  \hspace{1cm} (9.15)

Suppose there are matrices $a_\mu$ which satisfy

$$a_\nu j a_\mu j k + a_\mu j l a_\nu j k = 2 \delta_\mu \nu \delta_l k$$  \hspace{1cm} (9.16)

then Eq.(9.15) can be rewritten as

$$0 = a_\mu k j a_\mu j k P_\mu \psi^{(k)} P_\mu \psi^{(k)} + (a_\nu j a_\mu j k + a_\mu j l a_\nu j k) P_\nu \psi^{(l)} P_\mu \psi^{(k)} |_{\nu \geq \mu, \text{when} \nu = \mu, l \neq k} + m c \psi^{(j)} m c \psi^{(j)}$$

$$= [a_\nu j P_\nu \psi^{(l)} + i \delta_\nu j m c \psi^{(l)}] [a_\mu j k P_\mu \psi^{(k)} - i \delta_j k m c \psi^{(k)}]$$  \hspace{1cm} (9.17)

Where $\delta_{jk}$ is Kronecker delta function, $j, k, l = 1, 2, ..., N$. To note that we have added many zero terms in the above equation. For the above equation there is a special solution given by

$$[a_\mu j k P_\mu - i \delta_{jk} m c] \psi^{(k)} = 0$$  \hspace{1cm} (9.18)

There are many solutions for $a_\mu$ which satisfy Eq.(9.16), we select a set of $a_\mu$ as

$$N = 4, \quad a_\mu = \gamma_\mu \quad (\mu = 1, 2, 3, 4)$$  \hspace{1cm} (9.19)

$$\gamma_n = -i \beta \alpha_n \quad (n = 1, 2, 3), \quad \gamma_4 = \beta$$  \hspace{1cm} (9.20)

Where $\gamma_\mu, \alpha$ and $\beta$ are the matrices defined in Dirac algebra[27]. Substituting them into Eq.(9.18), we obtain
\[ [i c(-i\hbar\partial_4 - q A_4) + c\alpha_n(-i\hbar\partial_n - q A_n) + \beta mc^2]\psi = 0 \]  

(9.21)

Where \(\psi\) is an one-column matrix to represent \(\psi^{(k)}\).

Let index \(s\) denote a velocity field, then the four component functions of \(\tilde{\psi}_s(x)\) correspond to the same velocity field \(s\), the wave function \(\psi_s(x)\) may be regarded as the eigenfunction of the velocity field \(s\), it may be different from the eigenfunction of energy. Because velocity field is an observable in a physical system, in quantum mechanics we know that \(\psi_s(x)\) constitute a complete basis in which arbitrary function \(\phi(x)\) can be expanded in terms of them

\[ \phi(x) = \int C(s)\tilde{\psi}_s(x)ds \]  

(9.22)

Obviously, \(\phi(x)\) also satisfies Eq.(9.21). Therefore, Eq.(9.21) is just the Dirac wave equation.

Alternatively, another method to show the Dirac equation is more traditional: At the first, we show the Dirac equation of free particle by employing plane waves, we easily obtain Eq.(9.21) on the condition of \(A_\mu = 0\); Next, adding electromagnetic field, plane waves are valid in any finite small volume with the momentum of Eq.(9.5) when we regard the field to be uniform in the volume, so the Dirac equation Eq.(9.21) is valid in the volume even if \(A_\mu \neq 0\), plane waves constitute a complete basis in the volume; Third, the finite small volume can be chosen to locate at anywhere, then anywhere have the same complete basis, therefore the Dirac wave equation Eq.(9.21) is valid at anywhere.

Of course, on the condition of non-relativity, the Schrodinger wave equation can be derived from the Dirac wave equation [28].

By further calculation, the Dirac wave equation can arrive at the Klein-Gordon wave equation with an additional term which represents the effect of spin, this term is just the last term in Eq.(9.11) in a sense.

But, do not forget that the Dirac wave equation is a special solution of Eq.(9.17), therefore we believe there are some quantum effects beyond the Dirac wave equation.
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Afterword

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