Omni-Lie algebroids *

Z. Chen and Z.-J. Liu
Department of Mathematics and LMAM
Peking University, Beijing 100871, China
email: chenzhuo@math.pku.edu.cn, liuzj@pku.edu.cn

Abstract

A generalized Courant algebroid structure is defined on the direct sum bundle $\mathcal{D}E \oplus \mathcal{J}E$, where $\mathcal{D}E$ and $\mathcal{J}E$ are the gauge Lie algebroid and the jet bundle of a vector bundle $E$ respectively. Such a structure is called an omni-Lie algebroid since it is reduced to the omni-Lie algebra introduced by A.Weinstein if the base manifold is a point. We prove that any Lie algebroid structure on $E$ is characterized by a Dirac structure as the graph of a bundle map from $\mathcal{J}E$ to $\mathcal{D}E$.

1 Introduction

The notion of omni-Lie algebras was introduced by Weinstein [21] by defining some kind of algebraic structures on $\mathfrak{gl}(V) \oplus V$ for a vector space $V$. Such an algebra is not a Lie algebra but all possible Lie algebra structures on $V$ can be characterized by its Dirac structures. This is why the term omni is used here. The omni-Lie algebra can be regarded as the linearization of the Courant algebroid structure on $TM \oplus T^*M$ at a point and are studied from several aspects recently ([2], [8], [19]).

Our purpose is to generalize the omni-Lie algebra from a vector space to a vector bundle $E$ in order to characterize all possible Lie algebroid structures on $E$. It will be seen that the omni-Lie algebroid is of the form $E = \mathcal{D}E \oplus \mathcal{J}E$, where $\mathcal{D}E$ and $\mathcal{J}E$ denote the gauge algebroid and the jet bundle of $E$ respectively.

An important fact to be discussed in Section 2 is that $\mathcal{D}E$ and $\mathcal{J}E$ can regarded as $E$-dual bundles for each other, i.e., there is a non-degenerate $E$-valued pairing between these two vector bundles. As a maximal isotropic and integrable subbundle of $\mathcal{E}$, a Dirac structure in the omni-Lie algebroid turns out to be a Lie algebroid with a representation on $E$. We prove that there is a one-to-one correspondence between a Dirac structure coming from a bundle map $\mathcal{J}E \to \mathcal{D}E$ and a Lie algebroid (local Lie algebra) structure on $E$ when rank($E$) $\geq 2$ ($E$ is a line bundle).

Let’s fix some notations firstly. In this paper, $M$ denotes a smooth manifold, $1_C$ the identity map for any set $C$ and $E \xrightarrow{\pi} M$ a vector bundle. Before going to construct an omni-Lie algebroid, let us review some related notions. Assume that the readers are familiar

---

0 Keywords: gauge Lie algebroid, jet bundle, omni-Lie algebroid, Dirac structure, local Lie algebra.

0 MSC: 17B66.

* Research partially supported by NSFC(19925105) and CPSF(20060400017).
with Lie algebroids, which unify the structures of a Lie algebra and the tangent bundle of a manifold (please see [14] for more details). The notion of Leibniz algebras was introduced by Loday [13] as follows:

**Definition 1.1.** A Leibniz algebra is a vector space $L$ with a bilinear operation (not necessarily skew-symmetric) $\{\cdot,\cdot\} : L \times L \to L$ such that the following identity holds:

$$\{X,\{Y,Z\}\} = \{\{X,Y\},Z\} + \{Y,\{X,Z\}\}, \quad \forall X,Y,Z \in L.$$  

• **Courant algebroids and Dirac structures.** The Courant bracket on the sections of $T = TM \oplus T^* M$ was introduced by Courant [3]:

$$[x_1 + \alpha_1, x_2 + \alpha_2] = [x_1, x_2] + \mathcal{L}_{x_2}\alpha_1 - \mathcal{L}_{x_1}\alpha_2 - \frac{1}{2} d(\langle x_1, \alpha_2 \rangle - \langle x_2, \alpha_1 \rangle).$$

For the inner product defined by $(x_1 + \alpha_1, x_2 + \alpha_2)_+ = \frac{1}{2}(\langle x_1, \alpha_2 \rangle + \langle x_2, \alpha_1 \rangle)$, a Dirac structure is a maximal isotropic subbundle $L \subset T$ whose sections are closed under the Courant bracket. The Dirac structures include not only Poisson and presymplectic structures, but also foliations on $M$. A Dirac structure is also a Lie algebroid on $M$, whose bracket and anchor are the restrictions of the Courant bracket and the projection on $TM$. The properties of Courant’s bracket are the basis for the definition of a Courant algebroid ([12], [18]). Recently, several applications of the Courant algebroid and the Dirac structure have been found in different fields, e.g., gerbes and generalized complex geometry (see [2], [6] for more details). By introducing a non skew-symmetric bracket,

$$\{x_1 + \alpha_1, x_2 + \alpha_2\} \triangleq [x_1, x_2] + \mathcal{L}_{x_2}\alpha_1 - \mathcal{L}_{x_1}\alpha_2 + d\langle x_2, \alpha_1 \rangle,$$

the pair $(\Gamma(T), \{\cdot,\cdot\})$ turns out to be a Leibniz algebra with the following nice properties.

$$\rho \{e_1, e_2\} = [\rho(e_1), \rho(e_2)],$$

$$\{e_1, fe_2\} = f \{e_1, e_2\} + \rho(e_1)(f)e_2,$$

$$\{e_1, e_1\} = d(e_1)_+,$$

$$\rho(e_1)(e_2, e_3)_+ = \{(e_1, e_2), e_3\}_+ + (e_2, \{e_1, e_3\})_+,$$

for all $e_i \in \Gamma(T)$, $f \in C^\infty(M)$, where $\rho : T \to TM$ is the projection. Thus a Courant algebroid is a Leibniz algebroid ([7]) and the twisted bracket ([10]) is known as the Dorfman bracket ([5]). This bracket is mentioned in [12] and the Leibniz rule is shown in [16].

• **Omni-Lie algebras.** Motivated by an integrability problem of the Courant bracket, A. Weinstein gives a linearization of the Courant bracket at a point [21]. Let $V$ be a vector space. Weinstein’s bracket is defined on the direct sum $\mathcal{E} = \mathfrak{gl}(V) \oplus V$:

$$[[\xi_1, v_1], \xi_2, v_2]] \triangleq ([\xi_1, \xi_2], \frac{1}{2}(\xi_1(v_2) - \xi_2(v_1))).$$

This bracket does not satisfy the Jacobi identity. He called $\mathcal{E} = \mathfrak{gl}(V) \oplus V$ with the bracket above an omni-Lie algebra because of the following property:

**Theorem 1.2.** [21] There is a one-to-one correspondence between a Lie algebra structure on $V$ and a Dirac structure in $\mathcal{E}$ coming from a linear map in $\text{Hom}(V, \mathfrak{gl}(V))$. 

2
Here a Dirac structure is a subspace of $\mathcal{E}$ closed under the bracket $[,]$ and maximal isotropic with respect to the $V$-valued nondegenerate symmetric bilinear form:

$$((\xi_1, v_1), (\xi_2, v_2))_V \triangleq \frac{1}{2}(\xi_1(v_2) + \xi_2(v_1)).$$

That is, every Lie algebra structure on $V$ can be characterized by a Dirac structure, which is similar to a Poisson structure on a manifold.

**Local Lie algebras and Jacobi manifolds.**

A Lie algebroid is a special case of local Lie algebras in the sense of Kirillov [9]. Recall that a local Lie algebra is a vector bundle $E$ whose section space $\Gamma(E)$ has a $\mathbb{R}$-Lie algebra structure $[\cdot, \cdot]_E$ with the local property, supp$[u, v] \subset$ supp$u \cap$ supp$v$, for all $u, v \in \Gamma(E)$, which is also called a Jacobi-line-bundle if rank$E = 1$. In particular, $M$ is called a Jacobi manifold if the trivial bundle $M \times \mathbb{R}$ is a local Lie algebra, which is equivalent to that there is a triple $(M, \Lambda, X)$, where $\Lambda$ is a bi-vector field and $X$ is a vector field on $M$ such that $[\Lambda, \Lambda] = 2X \wedge \Lambda$ and $[\Lambda, X] = 0$ ([11]).

In [20], the Courant bracket was extended to the direct sum of the vector bundle $TM \times \mathbb{R}$ with its dual bundle, the jet bundle $J^1 = T^*M \times \mathbb{R}$. From this way, it allows one to interpret many structures encountered in differential geometry in terms of Dirac structures such as homogeneous Poisson manifolds, Jacobi structures and Nambu manifolds.

**The jet bundle of a vector bundle.** For a vector bundle $E \xrightarrow{\nabla} M$, one can define its 1-jet vector bundle $J^1 E$ by taking an equivalence relation in $\Gamma(E)$:

$$u_1 \sim u_2 \iff u_1(m) = u_2(m) \quad \text{and} \quad d\langle u_1, \xi \rangle_m = d\langle u_2, \xi \rangle_m, \quad \forall \xi \in \Gamma(E^*).$$

$(J^1 E)_m$ is the collection of all equivalence classes. So any $u \in (J^1 E)_m$ has a representative $u \in \Gamma(E)$ such that $\mu = [u]_m$. There are several equivalent descriptions for jet bundles (see [1] and the references thereof). It is shown in [4] that for any Lie algebroid $E$, each $k$-order jet bundle $J^k E$ inherits a natural Lie algebroid structure. Let $\mathfrak{p}$ be the projection which sends $[u]_m$ to $u(m)$. It is known that $\text{Ker} \mathfrak{p} \cong \text{Hom}(TM, E)$ and there is an exact sequence, referred as the jet sequence of $E$:

$$0 \longrightarrow \text{Hom}(TM, E) \xrightarrow{e} J^1 E \xrightarrow{\partial} E \longrightarrow 0. \quad (2)$$

Moreover, $\Gamma(J^1 E)$ is isomorphic to $\Gamma(E) \oplus \Gamma(T^*M \otimes E)$ as a $\mathbb{R}$-vector space and any $u \in \Gamma(E)$ has a lift $[u] \in \Gamma(J^1 E)$ such that

$$[fu] = f[u] + df \otimes u, \quad \forall f \in C^\infty(M). \quad (3)$$

**The gauge algebroid of a vector bundle.** For a vector bundle $E \xrightarrow{\nabla} M$, its gauge Lie algebroid $\mathfrak{D}E$ is just the gauge Lie algebroid of the frame bundle $F(E)$, which is also called the covariant differential operator bundle of $E$ (see [14 Example 3.3.4] and [15]). Here we treat each element $\mathfrak{d}$ of $\mathfrak{D}E$ at $m \in M$ as a $\mathbb{R}$-linear operator $\Gamma(E) \rightarrow E_m$ together with some $x \in T_m M$ (which is uniquely determined by $\mathfrak{d}$ and called the anchor of $\mathfrak{d}$) such that

$$\mathfrak{d}(fu) = f(m)\mathfrak{d}(u) + x(f)u(m), \quad \forall f \in C^\infty(M), u \in \Gamma(E).$$

It is known that $\mathfrak{D}E$ is a transitive Lie algebroid over $M$ ([10]). The anchor of $\mathfrak{D}E$ is given by $\alpha(\mathfrak{d}) = x$ and the Lie bracket $[ , ]_{\mathfrak{D}}$ of $\Gamma(\mathfrak{D}E)$ is given by the usual commutator of two operators. The corresponding exact sequence,

$$0 \longrightarrow \mathfrak{g}(E) \xrightarrow{\mathfrak{i}} \mathfrak{D}E \xrightarrow{\alpha} TM \longrightarrow 0, \quad (4)$$

3
is usually called the Artiyah sequence. The embedding maps e and i in above two exact sequences will be ignored somewhere if there is no confusion.

2 The E-Duality Between $\mathcal{D}E$ and $\mathcal{J}^1E$

**Definition 2.1.** Let $A$ and $E$ be two vector bundles over $M$. A vector bundle $B \subset \text{Hom}(A, E)$ is called an $E$-dual bundle of $A$ if the $E$-valued pairing $\langle \cdot, \cdot \rangle_E : A \times_M B \to E$, $\langle a, b \rangle_E \triangleq b(a)$ (where $a \in A$, $b \in B$) is nondegenerate.

It is easy to see that $B$ is an $E$-dual bundle of $A$ if and only if $A$ is an $E$-dual bundle of $B$. In this section we show that the first jet bundle $\mathcal{J}^1E$ of a vector bundle $E \xrightarrow{q} M$ is an $E$-dual of $\mathcal{D}E$ with some nice properties. Let us now illustrate a procedure that will yield a new exact sequence from the Artiyah sequence [4]. First we consider the dual sequence

$$0 \to T^*M \xrightarrow{\alpha^*} (\mathcal{D}E)^* \xrightarrow{\beta^*} E \to E^* \to 0.$$  

Applying the functor $"- \otimes E"$, the right end becomes $E \otimes E^* \otimes E \cong E \otimes \text{gl}(E)$. Then using the decomposition $\text{gl}(E) \cong \text{sl}(E) \oplus \mathbb{R}1_E$, we are able to get a pull-back diagram:

$$0 \to T^*M \otimes E \xrightarrow{\epsilon} \mathcal{J}E \xrightarrow{\beta} E \to E \otimes E^* \to 0 \quad (5)$$

$$0 \to T^*M \otimes E \xrightarrow{\alpha^* \otimes 1_E} (\mathcal{D}E)^* \otimes E \xrightarrow{\beta^* \otimes 1_E} E \otimes \text{sl}(E) \oplus E \to 0.$$  

Here the right down arrow I is the canonical embedding of $E$ into $E \otimes \text{sl}(E) \oplus E$ and $\mathcal{J}E$ is the pull-back of $(\mathcal{D}E, 1, 1)$. In other words, for each $m \in M$,

$$\mathcal{J}E = \{ \nu \in \text{Hom}(\mathcal{D}E, E) | \nu(\Phi) = \Phi \circ \nu(1_E), \ \forall \Phi \in \text{gl}(E) \}.$$  

Moreover, the maps $\epsilon$ and $\beta$ in Diagram (5) are given respectively by:

$$\epsilon(\eta)(\mathcal{D}) = \eta \circ \alpha(\mathcal{D}), \ \forall \mathcal{D} \in \mathcal{D}E, \ \eta \in \text{Hom}(TM, E);$$

$$\beta(\nu) = \nu(1_E), \ \forall \nu \in \mathcal{J}E.$$  

It is easy to see that $\mathcal{J}E$ is $E$-dual to $\mathcal{D}E$ and it is called the standard $E$-dual bundle of $\mathcal{D}E$. Analogously, one can define the standard $E$-dual bundle of $\mathcal{J}E$, denoted by $\mathcal{J}E$, which is given in the following pull-back diagram.

$$0 \to E^* \otimes E \xrightarrow{i} \mathcal{J}E \xrightarrow{\alpha} TM \to TM \otimes TM \to 0 \quad (6)$$

$$0 \to E^* \otimes E \xrightarrow{\beta^* \otimes 1} (\mathcal{J}E)^* \otimes E \xrightarrow{\epsilon^* \otimes 1} TM \otimes \text{sl}(E) \oplus TM \to 0.$$  

In other words,

$$\mathcal{J}E \triangleq \{ \delta \in \text{Hom}(\mathcal{J}E, E) | \exists x \in TM, \ \delta(\eta) = \eta(x), \ \forall \eta \in \text{Hom}(TM, E) \}.$$
There is a canonical isomorphism as follows:

\[(\cdot) : \mathcal{D}E \cong \mathcal{D}E \quad \text{s.t.} \quad \nu'(\vartheta) = \nu(\vartheta), \quad \forall \vartheta \in \mathcal{D}E, \ \nu \in \mathcal{J}E. \quad (7)\]

Under this isomorphism, the Atiyah sequence \[8\] is isomorphic to the first row in Diagram \[9\]. Moreover, there is a canonical isomorphism between the jet bundle of \(E\) and the standard \(E\)-dual bundle of \(\mathcal{D}E\).

**Theorem 2.2.** \(\mathcal{J}^1E\) is canonically isomorphic to \(\mathcal{J}E\).

**Proof.** We should define a bijective linear map \(\sim : \mathcal{J}^1E \to \mathcal{J}E\) such that the following diagram commutes:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(TM, E) & \overset{e}{\longrightarrow} & \mathcal{J}^1E & \overset{\mathcal{D}}{\longrightarrow} & E & \longrightarrow 0 \\
& & \downarrow & & \downarrow & & 1_E & \\
0 & \longrightarrow & \text{Hom}(TM, E) & \overset{\epsilon}{\longrightarrow} & \mathcal{J}E & \overset{\beta}{\longrightarrow} & E & \longrightarrow 0.
\end{array}
\]

For each \(\mu \in (\mathcal{J}^1E)_m\), \(p(\mu) = e \in E_m\), if \(\mu = |u|_m\), for some \(u \in \Gamma(E)\), then we define \(\tilde{\mu} \in (\mathcal{J}E)_m\) by

\[
\tilde{\mu}(\vartheta) = \widetilde{|u|_m}(\vartheta) \triangleq \vartheta u, \quad \forall \vartheta \in \mathcal{D}E_m. \quad (9)
\]

To see that the RHS of \[8\] is well defined, we need the following two lemmas.

**Lemma 2.3.** As Lie algebroids over \(M\), \(\mathcal{D}E\) and \(\mathcal{D}E^\ast\) are isomorphic via \((\cdot)^\sim\) defined by

\[
\langle \vartheta^\sim \phi, u \rangle = \alpha(\vartheta) \langle \phi, u \rangle - \langle \phi, \vartheta u \rangle, \quad \forall \vartheta \in \Gamma(\mathcal{D}E), \ u \in \Gamma(E), \ \phi \in \Gamma(E^\ast). \quad (10)
\]

This fact comes from the isomorphism between the principal frame bundles \(F(E)\) and \(F(E^\ast)\) by sending a frame to its dual frame. For this reason, we can identify \(\mathcal{D}E^\ast\) with \(\mathcal{D}E\) such that both \(\vartheta \phi\) and \(\vartheta u\) make sense, where \(\vartheta\) depends on what is put after it. Notice that, by this convention, if one treats \(\Phi \in \mathfrak{gl}(E)\) as in \(\mathfrak{gl}(E^\ast)\), it should be \(-\Phi^\ast\).

**Lemma 2.4.** Let \(u \in \Gamma(E)\) and suppose that \(u(m) = 0\). Then for any \(\vartheta \in (\mathcal{D}E)_m\), one has

\[
\langle \vartheta u \rangle^\uparrow = u_* (\alpha(\vartheta)) - 0_* (\alpha(\vartheta)). \quad (11)
\]

Here by \(e^\uparrow = \frac{d}{dt}|_{t=0} e \in T_mE (e \in E_m)\) we denote the vertical tangent vector and by \(0_*\) we mean the canonical inclusion of \(TM\) into \(TE\).

**Proof.** The RHS of \[11\] is clearly a vertical tangent vector of \(T_mE\). Thus we need only to show that the results of the two hand sides acting on an arbitrary fiber-wise linear function, say \(l_\phi\), for some \(\phi \in \Gamma(E^\ast)\), are equal. We see that

\[
\langle \vartheta u \rangle^\uparrow (l_\phi) = \langle \vartheta u, \phi(m) \rangle = \alpha(\vartheta) \langle u, \phi \rangle - \langle u(m), \vartheta \phi \rangle \quad \text{(by \[11\])}
\]

\[
= \alpha(\vartheta) \langle u, \phi \rangle = (u_*(\alpha(\vartheta)))l_\phi \]

\[
= (u_* (\alpha(\vartheta)) - 0_* (\alpha(\vartheta)))l_\phi.
\]

This completes the proof. 

\[\square\]
Now we continue to prove Theorem 2.2. Suppose that \( \mu \in (\mathfrak{J}^1 E)_m \) has two representatives \( u^1, u^2 \in \Gamma(E) \), i.e., \( \mu = [u^1]_m = [u^2]_m \). This means that
\[
u1(m) = \nu2(m), \quad \nu1m(x) = \nu2m(x), \quad \forall x \in T_m M.
\]
To guarantee \( \bar{\mu} \) is well-defined, we need to show that \( \mathfrak{d}(\nu1) = \mathfrak{d}(\nu2) \) holds for all \( \mathfrak{d} \in (\mathcal{D}E)_m \). In fact, let \( v = u^1 - u^2 \in \Gamma(E) \), which satisfies: \( v(m) = 0 \) and \( v|_m = 0|_m \). Then the lemma above claims that \( (\mathfrak{d}v)^1 = 0 \), so that \( \bar{\mu} \) is well-defined. Moreover, by Definition (3), we have
\[
\bar{\mu}(\Phi) = \Phi(e) = \Phi(\mathfrak{J}(1_m)), \quad \forall \Phi \in \mathfrak{gl}(E_m),
\]
and hence \( \bar{\mu} \) is indeed an element of \( (\mathfrak{J}E)_m \). Clearly \( \mathfrak{J} \) is a morphism of vector bundles.

The next step is to prove that \( \mathfrak{J} \) is a commutative diagram. But we first need the meaning of the embedding map \( \mathfrak{e} : \text{Hom}(TM, E) \hookrightarrow \mathfrak{J}E \). Take a local trivialization \( E|_U \cong U \times E_m \) for some open neighborhood \( U \cong \mathbb{R}^k \) \( (k = \dim(M)) \) containing \( m = 0 \). Then for any \( \eta \in \text{Hom}(TM, E)_m \), define a local section \( u \in \Gamma(E|_U) \) by
\[
u(p) = (p, \mathfrak{e}(\eta))(p), \quad \forall p \in U,
\]
where \( \mathfrak{e}(\eta) \) denotes the tangent vector from point 0 to point \( p \). Then \( u \) is a representative of \( \mathfrak{e}(\eta) \). Following Lemma 2.4, we get
\[
\mathfrak{e}(\eta)(\mathfrak{d}) = \widetilde{\nu}(\mathfrak{d}) = \mathfrak{d}(u) = \eta(\mathfrak{d}) = \mathfrak{e}(\eta)(\mathfrak{d}), \quad \forall \mathfrak{d} \in \mathcal{D}E.
\]
This means that the left square is commutative. The right square is commutative from the fact that
\[
\beta(\bar{\mu}) = \bar{\mu}(1_m) = \mathfrak{p}(\mu), \quad \forall \mu \in \mathfrak{J}^1 E.
\]
Thus the proof of the theorem is finished. ■

By means of this theorem we identify \( \mathfrak{J}E \) with \( \mathfrak{J}E \) from now on. Therefore any element \( \mu \in (\mathfrak{J}^1 E)_m = (\mathfrak{J}E)_m \) can be considered as a linear map from \( (\mathcal{D}E)_m \) to \( E_m \) satisfying
\[
\mu(\Phi) = \Phi \circ \mu(1_m), \quad \forall \Phi \in \mathfrak{gl}(E_m).
\]
Consequently, the jet sequence (2) has a new interpretation such that the projection \( \mathfrak{p} \) and the embedding \( \mathfrak{e} \) of \( \text{Hom}(TM, E) \) into \( \mathfrak{J}E \) are given by
\[
\mathfrak{p}(\mu) = \mu(1_m), \quad \mathfrak{e}(\eta)(\mathfrak{d}) \equiv \eta \circ \alpha(\mathfrak{d}), \quad \forall \mathfrak{d} \in (\mathcal{D}E)_m,
\]
By the canonical isomorphism (7), we can regard \( \mathcal{D}E \) as a subbundle of \( \text{Hom}(\mathfrak{J}E, E) \) and as the standard \( E \)-dual bundle of \( \mathfrak{J}E \). Therefore, there is an \( E \)-pairing between \( \mathfrak{J}E \) and \( \mathcal{D}E \) by setting:
\[
\langle \mu, \mathfrak{d} \rangle_E \equiv \check{\mu}(\mathfrak{d}) = \mathfrak{d}(u), \quad \forall \mu \in \mathfrak{J}E, \ \mathfrak{d} \in \mathcal{D}E,
\]
where \( u \in \Gamma(E) \) satisfies \( \mu = [u]_m \). Particularly, one has
\[
\langle \mu, \Phi \rangle_E = \Phi \circ \mathfrak{p}(\mu), \quad \forall \Phi \in \mathfrak{gl}(E), \ \mu \in \mathfrak{J}E;
\]
\[
\langle \eta, \mathfrak{d} \rangle_E = \eta \circ \alpha(\mathfrak{d}), \quad \forall \eta \in \text{Hom}(TM, E), \ \mathfrak{d} \in \mathcal{D}E.
\]
Similarly, $DE^*$ and $JE^*$ are $E^*$-dual for each other. Meanwhile, there is also a $T^*M$-pairing between $JE$ and $JE^*$ given by

$$\langle \mu, \varsigma \rangle_{T^*M} \triangleq d \langle u, \phi \rangle, \quad \forall \mu \in (JE)_m, \varsigma \in (JE^*)_m,$$

where $u \in \Gamma(E)$, $\phi \in \Gamma(E^*)$ satisfy $\mu = [u]_m$, $\varsigma = [\phi]_m$ respectively. Combining with the isomorphism given in Lemma 2.3, we can describe the relations among these four vector bundles by the following diagram:

![Diagram](image)

The relations above are similar to the following dual relations, where $TE$ and $TE^*$ are usual dual as two vector bundles over $T M$.

![Diagram](image)

The following diagram is a typical double vector bundle, by which and the duality theory of double vector bundles (see [14]) one can explain clearly the relationship between Diagrams (16) and (17).

![Diagram](image)

3 Omni-Lie Algebroids and Dirac Structures

Since the gauge Lie algebroid $DE$ has a natural representation on $E$, there is the Lie algebroid cohomology coming from the complex $(\Gamma(\text{Hom}(\wedge \cdot DE, E)), d)$. In fact, one can check that

$$du = [u] \in \Gamma(JE) \subset \Gamma(\text{Hom}(DE, E)), \quad \forall u \in \Gamma(E).$$

Furthermore, we claim that $\Gamma(JE)$ is an invariant subspace of the Lie derivative $L_d$ for any $d \in \Gamma(\mathcal{D}E)$, which can be defined by the Leibnitz rule as follows:

$$\langle L_d \mu, \Phi \rangle_E \triangleq d \langle \mu, \Phi \rangle_E - \langle \mu, [d, \Phi]_E \rangle, \quad \forall \mu \in \Gamma(JE), \Phi \in \Gamma(\mathcal{D}E).$$

Actually, it is easy to check that

$$\langle L_d \mu, \Phi \rangle_E = \Phi \circ d \circ p(\mu), \quad \forall \Phi \in \Gamma(gl(E)), \quad \Rightarrow \quad p(L_d \mu) = d \circ p(\mu). \quad (19)$$

This implies that $L_d \mu \in \Gamma(JE)$ by (14).
Now let $E = D E \oplus J E$, which has a nondegenerate symmetric 2-form from the $E$-duality:

$$(\mathfrak{d} + \mu, r + \nu)_E \equiv \frac{1}{2} (\langle \mathfrak{d}, \nu \rangle_E + \langle \mathfrak{r}, \mu \rangle_E), \quad \forall \mathfrak{d}, r \in DE, \, \mu, \nu \in JE.$$

We define the Dorfman bracket on $\Gamma(E)$, similar to that one mentioned in Section 1,

$$\{\mathfrak{d} + \mu, r + \nu\} \triangleq [\mathfrak{d}, r]_D + \mathcal{L}_\mathfrak{d} \nu - \mathcal{L}_r \mu + \mathfrak{d} \langle \mu, r \rangle_E,$$

and call the quadruple $(E, \{\cdot, \cdot\}, (\cdot, \cdot)_E, \rho)$ an omni-Lie algebroid, where $\rho$ is the projection of $E$ onto $DE$. Comparing with the Courant algebroid, we can prove that an omni-Lie algebroid has the similar properties as follows:

**Theorem 3.1.** With the notation above, an omni-Lie algebroid satisfies the following properties, where $\alpha$ is the projection from $DE$ to $TM$ in [4]. For all $X, Y, Z \in \Gamma(E), f \in C^\infty(M)$,

1) $(\Gamma(E), \{\cdot, \cdot\})$ is a Leibniz algebra,

2) $\rho \{X, Y\} = [\rho(X), \rho(Y)]_D$,

3) $\{X, fY\} = f \{X, Y\} + (\alpha \circ \rho(X))(f)Y$,

4) $\{X, X\} = \mathfrak{d} \langle X, X \rangle_E$,

5) $\rho(X)(Y, Z)_E = (\{X, Y\}, Z)_E + (Y, \{X, Z\})_E$.

When $E = M \times \mathbb{R}, \, E \cong (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, the structure above is studied by Wade in [20]. The simplest case is that $E = V$, a vector space. Then $E \cong \mathfrak{gl}(V) \oplus V$ and the structure is isomorphic to Weinstein’s omni-Lie algebra. A similar algebraic structure was defined in [17] and named as a generalized Lie bialgebra.

**Definition 3.2.** A Dirac structure in the omni-Lie algebroid $E$ is a subbundle $L \subset E$ being maximal isotropic with respect to $(\cdot, \cdot)_E$ and its section space $\Gamma(L)$ is closed under the bracket operation $\{\cdot, \cdot\}$.

**Proposition 3.3.** A Dirac structure $L$ is a Lie algebroid with the restricted bracket and anchor map $\alpha \circ \rho$. Moreover, $\rho|_L : L \to DE$ gives a representation of $L$ on $E$.

This fact is easy to be checked by the theorem above. Next we are going to study some special Dirac structures and generalize Theorem 1.2 of Weinstein from a vector space to a vector bundle. As we shall see, this includes two special cases, namely the jet algebroid of a Lie algebroid and the 1-jet algebroid of a Jacobi manifold. First let us mention the following basic fact, for which the proof is merely some calculations and is ignored.

**Lemma 3.4.** Given a bundle map $\pi : J E \to DE$, then its graph

$$L_\pi = \{(\pi(\mu), \mu) | \mu \in JE\} \subset E$$

is a Dirac structure if and only if

1) $\pi$ is skew-symmetric, i.e., $\langle \pi(\mu), \nu \rangle_E = -\langle \pi(\nu), \mu \rangle_E$, $\forall \mu, \nu \in JE$;
2) the following equation holds for all $\mu, \nu \in \Gamma(\mathfrak{J}E)$.

$$\pi [\mu, \nu]_{\pi} = [\pi(\mu), \pi(\nu)]_{\mathfrak{D}},$$

where the bracket $[\cdot, \cdot]_{\pi}$ on $\Gamma(\mathfrak{J}E)$ is defined by:

$$[\mu, \nu]_{\pi} \triangleq \mathcal{L}_{\pi(\mu)}\nu - \mathcal{L}_{\pi(\nu)}\mu - d\langle \pi(\mu), \nu \rangle_E.$$ (21)

Moreover, such a Dirac structure induces a Lie algebroid $(\mathfrak{J}E, [, , ]_{\pi}, \alpha \circ \pi)$.

**Lemma 3.5.** For the Lie algebroid $\mathfrak{J}E$ induced from a Dirac structure $L_{\pi}$ given above, then the following statements are equivalent.

1) $\alpha \circ \pi \circ d : \Gamma(E) \to \mathcal{X}(M)$ induces a bundle map $E \to TM$.

2) $\alpha \circ \pi \circ e = 0$ (i.e., $\pi(\text{Im}\ e) \subset \text{Im}\ \alpha$).

3) $\text{Hom}(TM, E)$ is an ideal of $\mathfrak{J}E$.

4) The quotient Lie algebroid structure on $E \cong \mathfrak{J}E/\text{Im}(e)$ is given by

$$\rho_E = \alpha \circ \pi \circ d, \quad [u, v]_E = p[d\mu, d\nu]_\pi = \pi(d\mu)v, \quad \forall u, v \in \Gamma(E).$$ (22)

Here the bundle maps $e, \alpha, i$ and $p$ are given in exact sequences $\{\mathfrak{J}\}$ and $\{D\}$.

**Proof.**

1) $\Rightarrow$ 2) Recall Eqn.$\{\mathfrak{J}\}$ and observe that for all $f \in C^\infty(M), u \in \Gamma(E)$,

$$\alpha \circ \pi \circ d(fu) = f\alpha \circ \pi(d\mu) + \alpha \circ \pi \circ e(df \otimes u),$$

this implication is obvious.

2) $\Rightarrow$ 3) For any $\eta \in \Gamma(\text{Hom}(TM, E))$ and $\mu \in \Gamma(\mathfrak{J}E)$, we have

$$p[\eta, \mu]_\pi = p(\mathcal{L}_{\pi(\eta)}\mu - \mathcal{L}_{\pi(\mu)}\eta - d\langle \pi(\eta), \mu \rangle_E) = \langle \pi(\eta), d\mu \rangle_E - \langle \pi(\eta), \mu \rangle_E$$

using (19).

$$= \langle \pi(\eta), d\mu \rangle_E - \langle \pi(\eta), \mu \rangle_E = (d\mu \circ \alpha \circ \pi \circ e)(\eta),$$

since $d\mu \circ \alpha \circ \pi \circ e \in \Gamma(\text{Hom}(TM, E))$. So condition 2) implies that $\rho_E \circ \pi \circ d = 0$, as required.

3) $\Rightarrow$ 4) This implication is obvious.

4) $\Rightarrow$ 1) For all $u, v \in \Gamma(E), f \in C^\infty(M)$, we have

$$[u, fv]_E = \pi(d\mu)(fv) = (\alpha \circ \pi \circ d\mu)(f)v + f\pi(d\mu)v \in \langle \pi, \cdot \rangle_E,$$

This shows that the anchor of the Lie algebroid $E$ should be $\alpha \circ \pi \circ d$, which must be a bundle map. ■

Suppose that a Lie algebroid $(E, [, , ]_E, \rho_E)$ is reduced from a bundle map $\pi$ satisfying the conditions in Lemma 3.5, it is not difficult to see that the anchor $\rho_E : E \to TM$ can be lift to a Lie algebroid morphism by setting

$$\hat{\rho}_E : \mathfrak{J}E \to \mathfrak{D}(TM), \quad \hat{\rho}_E[u]_m = [\rho_E(u), \cdot ](m), \quad \forall u \in \Gamma(E).$$
Moreover, one has the following commutative diagram such that all the arrows are Lie algebroid morphisms.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathfrak{gl}(E) & \longrightarrow & \mathcal{D}E & \longrightarrow & TM & \longrightarrow & 0 \\
& & \downarrow{(\rho_E)^*} & & \downarrow{\pi} & & \downarrow{\rho_E} & & \\
0 & \longrightarrow & \text{Hom}(TM, E) & \longrightarrow & \mathcal{J}E & \longrightarrow & E & \longrightarrow & 0 \\
& & \downarrow{-1} & & \downarrow{\tilde{\rho}_E} & & \downarrow{\rho_E} & & \\
0 & \longrightarrow & \mathfrak{gl}(TM) & \longrightarrow & \mathcal{D}(TM) & \longrightarrow & TM & \longrightarrow & 0.
\end{array}
$$

Now we have two representations of $\mathcal{J}E$ on $T^*M \otimes E \cong \text{Hom}(TM, E)$: (1) the adjoint representation since $\text{Hom}(TM, E)$ is an ideal of $\mathcal{J}E$ by Lemma 3.5; (2) the tensor representation of $\pi$ and $\tilde{\rho}_E$ in the above diagram by identifying $\mathcal{D}(TM)$ with $\mathcal{D}(T^*M)$. After some straightforward computations, we have

**Corollary 3.6.** The above two representations of $\mathcal{J}E$ on $T^*M \otimes E$ are equivalent.

Conversely, one can get the above diagram from a Lie algebroid $E$ over $M$.

**Lemma 3.7.** From a Lie algebroid $(E, [\cdot, \cdot]_E, \rho_E)$ one can get Diagram (23) by constructing a Dirac structure $L_\pi$ in $\mathcal{E}$ such that

$$
\pi(du) = [u, \cdot]_E, \quad \forall \ u \in \Gamma(E).
$$

Here $[u, \cdot]_E$ denotes the corresponding derivation of $E$.

**Proof.** Since $\Gamma(\mathcal{J}E) \cong \Gamma(E) \oplus \Gamma(T^*M \otimes E)$, for a Lie algebroid $(E, [\cdot, \cdot]_E, \rho_E)$, one can define a map $\pi : \Gamma(\mathcal{J}E) \rightarrow \Gamma(\mathcal{D}E)$ such that (24) holds and for all $f \in C^\infty(M), u \in \Gamma(E)$, set

$$
\pi(df \otimes u) = \pi(df(u) - dfu) = [fu, \cdot]_E = f[u, \cdot]_E.
$$

Then it is easy to check that $\pi$ is $C^\infty(M)$-linear and hence it well defines a morphism of vector bundles $\mathcal{J}E \rightarrow \mathcal{D}E$. By fact that any section of $\mathcal{J}E$ can be written as a linear combination of the elements with form $fdu$ as well as the property of anchor:

$$
[u, fv]_E = f[u,v]_E + ((\rho_E u) f) v, \quad \forall u, v \in \Gamma(E), \ f \in C^\infty(M),
$$

we can check that the $\pi$-bracket $[\cdot, \cdot]_\pi$ on $\Gamma(\mathcal{J}E)$ defined by (21) satisfies the following properties:

1. $[du_1, du_2]_\pi = d[u_1, u_2]_E$;
2. $[du_1, \omega \otimes u_2]_\pi = \chi_{\rho_E(u_1)}(\omega \otimes u_2 + \omega \otimes [u_1, u_2]_E$;
3. $[\omega_1 \otimes u_1, \omega_2 \otimes u_2]_\pi = \langle \omega_2, \rho_E(u_1) \rangle (\omega_1 \otimes u_2) - \langle \omega_1, \rho_E(u_2) \rangle (\omega_2 \otimes u_1)$,

where $u_i \in \Gamma(E), \omega_i \in \Omega(M)$. It is easy to see that these relations imply that Eqt. (20) is valid and hence $L_\pi$ is a Dirac structure. Moreover, one can check that $\alpha \circ \pi \circ \phi = 0$. Thus, by Lemma 3.5 the proof is completed. ■
Remark 3.8. Actually, it is already known to construct the jet Lie algebroid and a representation on $E$ from a given Lie algebroid $E$ (see [4]). Our discoveries include that: (1) the Lie algebroid structure of $\mathfrak{J}E$ is written clearly in form [21] by means of $\pi$ and can be characterized by a Dirac structure; (2) we find another representation $\hat{\rho}_E$ related to $\pi$ as showing in diagram [23] and Corollary 3.6.

The follows are two special cases for $E = TM$ with the usual Lie algebroid structure and $E = T^*M$ with the Lie algebroid structure coming from a Poisson structure.

Corollary 3.9. There is a canonical Lie algebroid isomorphism $\hat{\iota}_TM : \mathfrak{J}(TM) \cong \mathcal{D}(TM)$ with the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & \mathfrak{gl}(TM)^{op} & \rightarrow & \mathfrak{J}(TM) & \rightarrow & TM & \rightarrow & 0 \\
& & \downarrow \phi(TM) & & \downarrow \hat{\iota}_TM & & \downarrow 1_{TM} & & \\
0 & \rightarrow & \mathfrak{gl}(TM) & \rightarrow & \mathcal{D}(TM) & \rightarrow & TM & \rightarrow & 0.
\end{array}
$$

Corollary 3.10. Let $(M, \Pi)$ be a Poisson manifold. Then there is a Lie algebroid morphism $\hat{\Pi} : \mathfrak{J}(T^*M) \rightarrow \mathcal{D}(TM)$ such that the following diagram commutes.

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}(TM, T^*M) & \rightarrow & \mathfrak{J}(T^*M) & \rightarrow & T^*M & \rightarrow & 0 \\
& & \downarrow \Pi & & \downarrow \hat{\Pi} & & \downarrow \Pi & & \\
0 & \rightarrow & \mathfrak{gl}(TM) & \rightarrow & \mathcal{D}(TM) & \rightarrow & TM & \rightarrow & 0.
\end{array}
$$

In particular, $\mathfrak{J}(T^*M) \cong \mathcal{D}(TM)$ if $M$ is a symplectic manifold.

Now we mention the main result of this paper as follows:

Theorem 3.11. If $\text{rank}(E) \geq 2$, then there is a one-to-one correspondence between Lie algebroid structures on $E$ and Dirac structures in $\mathcal{E}$ coming from bundle maps $\mathfrak{J}E \rightarrow \mathcal{D}E$.

Proof. One direction is true by Lemma 3.7. For the converse direction, we assume that $L_\pi$ is a Dirac structure coming from a skew-symmetric bundle map $\pi : \mathfrak{J}E \rightarrow \mathcal{D}E$ given in Lemma 3.4. We claim that if $\text{rank}(E) \geq 2$, then $\alpha \circ \pi \circ \iota = 0$. By equalities (14) and (15), it is seen that $\alpha \circ \pi \circ \iota = 0$ is equivalent to

$$
(\pi(\eta), \mathfrak{J})_E = 0, \quad \forall \eta, \mathfrak{J} \in \text{Hom}(TM, E).
$$

Taking $\eta = \omega_1 \otimes e_1$, $\mathfrak{J} = \omega_2 \otimes e_2$, for any $\omega_1, \omega_2 \in T^*M$, $e_1, e_2 \in E$, we have

$$
\langle \pi(\omega_1 \otimes e_1), \omega_2 \otimes e_2\rangle_E = \langle \alpha \circ \pi(\omega_1 \otimes e_1), \omega_2 \rangle e_2 = -\langle \alpha \circ \pi(\omega_2 \otimes e_2), \omega_1 \rangle e_1.
$$

Since $\text{rank}(E) \geq 2$, $e_1$ and $e_2$ can be independent so that the coefficients ahead of them must be zero. This means that (20) is true. Finally, by Lemma 3.5, we know that $E$ has an induced Lie algebroid structure.

Example 3.12. Given a Lie algebroid $(E, [\cdot, \cdot], \rho)$, $\text{rank}(E) \geq 2$, with the Dirac structure $L_\pi$ as shown above. For a bundle map $N : E \rightarrow E$, i.e., $N \in \Gamma(\mathfrak{gl}(E)) \subset \Gamma(\mathcal{D}E)$, the Nijenhuis torsion of $N$ is defined by, $\forall u, v \in \Gamma(E)$,

$$
T^N(u, v) \triangleq N[u, v]^N - [Nu, Nv], \quad \text{where} \quad [u, v]^N = [Nu, v] + [u, Nv] - N[u, v].
$$
We define a twisted bundle map \( \pi \circ \tilde{N} - \text{ad}_N \circ \pi : \mathfrak{J}E \to \mathcal{D}E \), where \( \tilde{N} : \mathfrak{J}E \to \mathfrak{J}E \), \([u] \mapsto [Nu]\), is the lift of \( N \). Then the following three statements are equivalent.

1) The graph of \( \pi \circ \tilde{N} - \text{ad}_N \circ \pi \) is a Dirac structure.

2) \((E, [,]_N, \rho \circ N)\) is a Lie algebroid.

3) \([T^N(u, v), w] + T^N([u, v], w) + \text{c.p.} = 0, \ \forall u, v, w \in \Gamma(E)\).

In particular, \( N \) is a Nijenhuis operator if and only if \( T^N = 0 \).

**Example 3.13.** Let \( E = M \times V \) be a trivial vector bundle, where \( \dim V \geq 2 \). In this case,

\[
\mathcal{D}E = (M \times \mathfrak{gl}(V)) \oplus TM, \quad \mathfrak{J}E = \text{Hom}(TM, M \times V) \oplus (M \times V).
\] (27)

One can check that any skew-symmetric bundle map \( \pi : \mathfrak{J}E \to \mathcal{D}E \) is determined by a pair of bundle maps \((\theta, \Omega)\), where \( \theta : M \times V \to TM \) and

\[
\Omega : M \times V \to M \times \mathfrak{gl}(V), \quad \Omega(v_1)(v_2) + \Omega(v_2)(v_1) = 0, \ \forall v_1, v_2 \in V,
\]

such that

\[
\pi(n, v) = (\Omega(v) - n \circ \theta, \theta(v)), \quad \forall (n, v) \in \text{Hom}(TM, M \times V) \oplus (M \times V).
\]

Moreover, the graph of \( \pi \) is a Dirac structure if and only if, \( \forall v_1, v_2, v_3 \in V \) (as constant sections of \( E \)),

\[
\theta \circ \Omega(v_1, v_2) = [\theta(v_1), \theta(v_2)], \quad \Omega(v_1, \Omega(v_2, v_3)) + L_{\theta(v_1)}\Omega(v_2, v_3) + \text{c.p.} = 0.
\] (28) (29)

The reduced Lie algebroid structure on \( E \) is given by the anchor \( \theta \) and

\[
[u, v]_E = \Omega(u, v) + L_{\theta(u)}v - L_{\theta(v)}u, \quad \forall u, v \in C^\infty(M, V).
\] (30)

In particular, \( \Omega \) is constant if and only if \( E \) is an action Lie algebroid coming from the action of Lie algebra \((V, \Omega)\) on \( M \) by \( \theta \).

From now on we consider the line bundle case. The next example shows that Equation

\[26\] is not always true for a skew-symmetric bundle map \( \pi : \mathfrak{J}E \to \mathcal{D}E \) if \( \text{rank}(E) = 1 \).

**Example 3.14.** Suppose that \( E \) is the trivial bundle \( M \times \mathbb{R} \). Then \( \mathfrak{J}E \cong T^*M \times \mathbb{R} \) and \( \mathcal{D}E \cong TM \times \mathbb{R} \). For a skew-symmetric bivector field \( \Lambda \in \Gamma(\wedge^2TM) \), define a map \( \pi : \mathfrak{J}E \to \mathcal{D}E \) : \((\xi, t) \mapsto (\Lambda^2(\xi), 0)\) by means of the map \( \Lambda^2 : T^*M \to TM \). It is easy to see that \( \pi \) is skew-symmetric but \( \alpha \circ \pi \circ e = \Lambda^2 \neq 0 \).

In fact, we can also construct a Dirac structure for a Jacobi-line-bundle as doing in Lemma \[3.7\] But it needs more calculations because there is no anchor in this case.

**Lemma 3.15.** From a Jacobi-line-bundle \((E, [\cdot, \cdot]_E)\), one can construct a Dirac structure \( L_\pi \) in \( \mathcal{E} \) such that \( \pi(du) = [u, \cdot]_E, \ \forall u \in \Gamma(E) \).
Proof. We still define \( \pi \) by Eq.(24) and show that \( \pi \) is really a bundle map and takes values in \( \Gamma(\mathfrak{D}E) \). Since all calculations are local, without losing the generality, one can assume that \( E = M \times \mathbb{R} \) and identify \( \Gamma(E) \) with \( C^\infty(M) \). By the result in [9], there exists a pair \( (\Lambda, X) \), where \( \Lambda \) is a smooth bivector field and \( X \) is a smooth vector field such that
\[
[f, g]_E = \Lambda(df, dg) + fX(g) - gX(f), \quad \forall \ f, g \in C^\infty(M).
\]
Thus we have two expressions of \( \pi \):
\[
\pi(\text{d}u)v = [u, v]_E = \Lambda(\text{d}u, \text{d}v) + uX(v) - vX(u);
\]
\[
\pi(\text{d}f \otimes u)(v) = [fu, v]_E - f[u, v]_E = u\Lambda(df, \text{d}v) - uvX(f).
\]
Therefor we have, for any \( h \in C^\infty(M) \),
\[
\pi(\text{d}u)(hv) = h\pi(\text{d}u)v + (\Lambda^1(\text{d}u) + uX)(h)v;
\]
\[
\pi(\text{d}f \otimes u)(hv) = h\pi(\text{d}f \otimes u)v + u\Lambda^1(\text{d}f)(h)v,
\]
which mean that both \( \pi(\text{d}u) \) and \( \pi(\text{d}f \otimes u) \in \Gamma(\mathfrak{D}E) \). Using these formulas, it is also easy to check that \( \pi \) is a bundle map. Next we prove that \( L_\pi \) is a Dirac structure. By some simple calculations, we get \( \text{d}[u, v]_E = \text{d}[u, v]_E, \quad \forall \ u, v \in \Gamma(E) \), which implies that
\[
\pi(\text{d}[u, v]_E)(w) = \pi(\text{d}[u, v]_E)(w)
\]
\[
= \pi([u, v]_E, w)_E = [[u, w]_E, v]_E + [u, [v, w]_E]_E
\]
\[
= \pi(\text{d}[u, v]_E)_E(w), \quad \forall \ w \in \Gamma(E).
\]
Since any local section of \( \mathfrak{J}E \) can be written as a linear combination of elements of the form \( f \text{d}u \), the above equality implies that Eq.(20) is valid. 

For a line bundle, we have the following theorem analogous to Theorem 3.11. The difference is that the quotient structure on \( E \) can not be claimed directly since \( \alpha \circ \pi \circ \varepsilon \) may be not zero in this case.

Theorem 3.16. For any line bundle \( E \), there is a one-to-one correspondence between local Lie algebra structures on \( E \) and Dirac structures in \( \mathcal{E} \) coming from bundle maps \( \mathfrak{J}E \to \mathfrak{D}E \). In particular, Dirac structure \( L_\pi \) corresponds to a Lie algebroid structure of \( E \) if and only if \( \alpha \circ \pi \circ \varepsilon = 0 \).

Proof. One implication is shown in Lemma 3.15. For the converse part, let us show that a Dirac structure \( L_\pi \) of the line bundle \( E \) determines a local Lie algebra structure \( (E, [\cdot, \cdot]_E) \) by setting
\[
[u, v]_E \triangleq \text{p}[\text{d}[u, v]_\pi] = \pi(\text{d}u)v, \quad \forall \ u, v \in \Gamma(E).
\]
It clearly satisfies the local condition. Moreover, we have
\[
\text{d}[u, v]_E = \text{d}\pi(\text{d}u)v = \mathfrak{L}_{\pi(\text{d}u)}\text{d}v = [\text{d}u, \text{d}v]_\pi.
\]
To see that \( [\cdot, \cdot]_E \) enjoys the Jacobi identity, we compute, for all \( u, v, w \in \Gamma(E) \),
\[
[[u, v]_E, w]_E = \pi(\text{d}[u, v]_E)w = \pi([\text{d}u, \text{d}v]_\pi)w
\]
\[
= [\pi(\text{d}u), \pi(\text{d}v)]_\mathfrak{D}(w) \quad \text{(since } L_\pi \text{ is a Dirac structure)}
\]
\[
= [u, [v, w]]_E - [v, [u, w]]_E.
\]
The last statement of the theorem is already implied by Lemma 3.5. 

Finally, notice that \( \alpha \circ \pi \circ \varepsilon : \Gamma(E) \to \Gamma(TM) \) is generally not a bundle map for a Jacobi-line bundle. But it plays a similar role as the anchor of a Lie algebroid as follows:
Corollary 3.17. For a Jacobi-line bundle and \((E, [, ]_E)\), one has
\[ [u, f v]_E = f[u, v]_E + ((\alpha \circ \pi \circ d \alpha) f) v, \quad \forall u, v \in \Gamma(E), \quad f \in C^\infty(M). \]

This equation follows directly from formula (31). In fact, the only obstruction for a Jacobi-line bundle to be a Lie algebroid is that \(\alpha \circ \pi \circ d \alpha\) is not a bundle map.

References

[1] R. Almeida and A. Kumpera, Structure produit dans la catégorie des algèbroids de Lie, An Acad. Bra. Ciênc. 53(1981), 247-250.

[2] H. Bursztyn, G. Cavalcanti, M. Gualtieri, Reduction of Courant algebroids and generalized complex structures, arXiv:math.DG/0509640.

[3] T. Courant, Dirac manifolds, Trans. Amer. Math. Soc. 319(1990), 631-661.

[4] M. Crainic and R. L. Fernandes, Secondary characteristic classes of Lie algebroids, Lect. Notes. Phys., 662(2005), 157-176.

[5] I. Ya. Dorfman, Dirac structures of integrable evolution equations, Phys. Lett. A 125 (1987), 240-246.

[6] M. Gualtieri, Generalized Complex Geometry, PhD thesis, St John’s College, University of Oxford, Nov. 2003.

[7] R. Ibáñez, M. de León, J.C. Marrero and E. Padrón, Leibniz algebroid asociated with a Nambu-Poisson structure, J. Phys. A 32(1999), 8129-8144.

[8] Kinyon, K. and A. Weinstein, Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces, Amer. J. math., 123(2001), 525-550.

[9] A. Kirillov, Local Lie algebras, Russian Math. Surveys, 31(1976), 55-76.

[10] Y. Kosmann-Schwarzbach and K. Mackenzie, Differential operators and actions of Lie algebroids, Contemp. Math., 315(2002), 213-233.

[11] A. Lichnerowicz, Les variétés de Jacobi et leurs algèbres de Lie associées, J. Math. Pures et Appl., 57 (1978), 453-488.

[12] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, J. Diff. Geom., 45(1997), 547-574.

[13] J. L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, Enseign. Math. J. 5(1998), 263-276.

[14] K. Mackenzie, General theories of Lie groupoids and Lie algebroids, Cambridge University Press, 2005.

[15] K. Mackenzie and P. Xu, Lie bialgebroids and Poisson groupoids, Duke Math. J., 73(2)(1994),415-452.
[16] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, PhD thesis, UC Berkeley, 1999, [arXiv:math.DG/9910078](http://arxiv.org/abs/math.DG/9910078).

[17] Y.-J. Tan and Z.-J. Liu, Generalized Lie bialgebras, *Comm. Alg.* 26(7) (1998), 2293-2319.

[18] K. Uchino, Remarks on the definition of a Courant algebroid, *Lett. Math. Phys.* 60(2002): 171-175.

[19] K. Uchino, Courant brackets on noncommutative algebras and omni-Lie algebras, [arXiv:math.SG/0604101](http://arxiv.org/abs/math.SG/0604101).

[20] A. Wade, Conformal Dirac structures, *Lett. Math. Phys.* 53(2000), 331-348.

[21] A. Weinstein, Omni-Lie algebras, Microlocal analysis of the Schrodinger equation and related topics (Kyoto, 1999). No. 1176(2000), 95-102.