Functional representation of the Ablowitz-Ladik hierarchy.

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Abstract

The Ablowitz-Ladik hierarchy (ALH) is considered in the framework of the inverse scattering approach. After establishing the structure of solutions of the auxiliary linear problems, the ALH, which has been originally introduced as an infinite system of difference-differential equations is presented as a finite system of difference-functional equations. The representation obtained, when rewritten in terms of Hirota’s bilinear formalism, is used to demonstrate relations between the ALH and some other integrable systems, the Kadomtsev-Petviashvili hierarchy in particular.

1 Introduction.

Among various concepts of the theory of integrable nonlinear systems one of the most fruitful is a viewpoint when each integrable equation is considered as member of an infinite number of related equations — hierarchy \([1]\). So, e.g., the famous nonlinear Schrodinger equation is the simplest equation of the AKNS hierarchy, its discrete integrable analog is a member of the Ablowitz-Ladik hierarchy (ALH), etc. A distinguishing feature of integrable hierarchies is that corresponding flows commute, i.e., all its equations are compatible. This enables to consider a hierarchy as one system of equations, i.e., to consider an infinite number of, say, (1+1)-dimensional partial differential equations (PDE) (as, for example, in the case of the AKNS hierarchy) or difference-differential equations (DDE) (in the case of the ALH) as a (1+∞)-dimensional problem for functions depending on an infinite number of variables. Such an approach has been intensively studied for almost all ‘classical’ integrable equations and has been shown to be a rather powerful tool for tackling integrable nonlinear problems. A logical continuation of this method is to ‘convert’ this infinite number of PDE’s or DDE’s into one (or several) functional equation which can be viewed as a generating function for equations of the hierarchy: expanding this functional relation in power series in some auxiliary parameter one can obtain all equations of the hierarchy. Such functional equations naturally appear in the Sato theory of soliton equations \([2]\). Also, this question, especially in the case of the Kadomtsev-Petviashvili (KP) hierarchy, has been discussed in connection with the problem of characterization of the Jacobi varieties (see, e.g., \([3, 4]\) and references therein). Some recent examples of the functional representation of integrable systems one can find, for example, in \([5, 6, 7, 8]\) (KP and dispersionless KP hierarchies) and \([9, 10]\) (Toda hierarchy).

In the present paper I will consider the case of the ALH. After outlining some basic facts related to the inverse scattering transform (IST) (section \([1]\)), and discussing more comprehensively corresponding linear problems (section \([2]\)) I will obtain the functional representation of the ALH (section \([3]\)). In the section \([4]\) I will rewrite the results obtained in terms of Hirota’s bilinear operators. This will expose some relations between the ALH and other integrable hierarchies, KP hierarchy in particular.
2 Zero curvature representation of the ALH.

The ALH is an infinite set of ordinal differential-difference equations, that has been introduced by Ablowitz and Ladik in 1975 [1]. The most well-known of these equations is the discrete nonlinear Schrodinger equation (DNLSE)

\[ i\dot{q}_n = q_{n+1} - 2q_n + q_{n-1} - q_nr_n (q_{n+1} + q_{n-1}) \]  
(1)

and the discrete modified KdV equation (DMKdV),

\[ \dot{q}_n = p_n (q_{n+1} - q_{n-1}) \]  
(2)

where

\[ p_n = 1 - q_n r_n, \quad r_n = -\kappa \bar{q}_n, \quad \kappa = \pm 1 \]  
(3)

(see, e.g., [12]). All equations of the ALH can be presented as the compatibility condition for the linear system

\[ \Psi_{n+1} = U_n \Psi_n \]  
(4)

\[ \partial_t \Psi_n = V_n \Psi_n \]  
(5)

where \( \partial_t \) stands for \( \partial/\partial t \), which leads to their zero-curvature representation (ZCR):

\[ \partial_t U_n = V_{n+1} U_n - U_n V_n \]  
(6)

In the standard IST approach developed in [1] the matrix \( U_n \) for the ALH is given by

\[ U_n = \begin{pmatrix} \lambda & r_n \\ \bar{q}_n & \lambda^{-1} \end{pmatrix} \]  
(7)

where \( \lambda \) is auxiliary constant parameter. For the elements of the matrix \( V_n \),

\[ V_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \]  
(8)

one can then obtain from (6), the system of equations

\[ \lambda (a_{n+1} - a_n) = -q_n b_{n+1} + r_n c_n \]  
(9)

\[ \lambda^{-1} (d_{n+1} - d_n) = q_n b_n - r_n c_{n+1} \]  
(10)

\[ \partial_t q_n = q_n (d_{n+1} - a_n) + \lambda c_{n+1} - \lambda^{-1} c_n \]  
(11)

\[ \partial_t r_n = r_n (a_{n+1} - d_n) - \lambda b_n + \lambda^{-1} b_{n+1} \]  
(12)

According to [1], they can be chosen as Laurent polynomials in \( \lambda \) in such a way that (9)-(12) hold automatically for all \( \lambda \)'s provided \( q_n \)'s, \( r_n \)'s satisfy some differential relations. It should be noted that one can obtain an infinite number of the matrices \( V_n \) (which are Laurent polynomials of different order) which leads to the infinite number of differential equations \( \partial q_n / \partial t = F_n^l \), \( l = 1, 2, \ldots \). According to the widely accepted now viewpoint, as was mentioned in the Introduction, one can consider \( q_n \)'s and \( r_n \)'s as depending on the infinite number of 'times', \( q_n = q_n(t_1, t_2, \ldots) \) and consider the \( l \)th equation of the ALH as describing the flow with respect to the \( l \)th variable, \( \partial q_n/\partial t_l = F_n^l \). I will also adhere this conception of \( q_n \)'s being functions of the infinite number of variables, but my approach will slightly differ from the classical one in the following aspect. Traditionally it is implied that all 'times' \( t_l \) are real, which is grounded from the standpoint of physical applications, and also is convenient in the framework of the inverse scattering technique. I will use instead of real 'times' \( t_l \) some complex variables \( z_j, \bar{z}_j \) \( (j = 1, 2, \ldots) \), which, as will shown below, exhibit in a more transparent way some intrinsic properties of the ALH. A simple analysis yields that the family of possible solutions of the system (9)-(12) (and hence the equations of the hierarchy) can be divided in two subsystems. One of them consist of \( V \)-matrices which are polynomials in \( \lambda^{-1} \) (I will term the corresponding equations as a 'positive' part of hierarchy), and the other consist of matrices which are polynomials in \( \lambda \) ('negative' subhierarchy), while in the
standard, ‘real-time’ approach all the $V$-matrices contain terms proportional to $\lambda^m$ together with the terms proportional to $\lambda^{-m}$ ($m \geq 0$). Let us consider first the ‘positive’ case. An infinite number of polynomial in $1/\lambda$ solutions $V_n^j$ ($j = 1, 2, ...$) possesses the following structure:

$$V_n^j = \lambda^{-2}V_n^{j-1} + \left( \begin{array}{ccc}
\lambda^{-2} \alpha_n^j & \lambda^{-1} \beta_n^j \\
n^{-1} & \lambda^{-1} \gamma_n^j \\
\delta_n^j & \lambda^{-1} \epsilon_n^j & \lambda^{-1} \zeta_n^j 
\end{array} \right)$$

(13)

where the elements $\alpha_n^j, ..., \delta_n^j$ satisfy the equations

$$\alpha_n^{j+1} - \alpha_n^j = -q_n\beta_n^{j+1} + r_n\gamma_n^j$$

(14)

$$\beta_n^{j+1} = \beta_n^{j+1} + r_n\gamma_n^j$$

(15)

$$\gamma_n^{j+1} = q_n\delta_n^{j+1} + \gamma_n^{j+1}$$

(16)

$$\delta_n^{j+1} - \delta_n^j = -q_n\beta_n^{j+1} - \delta_n^{j+1}$$

(17)

with $\partial_j = \partial/\partial z_j$. Rewriting this system as

$$\alpha_n^j = -\beta_n^{j-1}$$

(18)

$$\beta_n^j = \beta_n^{j-1} + r_n\gamma_n^{j-1}$$

(19)

$$\gamma_n^j = \gamma_n^{j-1} + q_n\delta_n^{j-1}$$

(20)

$$\delta_n^j = \delta_n^{j+1} - q_n\beta_n^{j+1} + r_n\gamma_n^{j+1}$$

(21)

and choosing

$$a_n^0 = 0 \quad b_n^0 = 0 \quad c_n^0 = 0 \quad d_n^0 = -i$$

(22)

we can obtain consequently

$$a_n^1 = 0 \quad a_n^2 = -ir_n-1q_n$$

$$b_n^1 = -ir_n-1$$

$$b_n^2 = -ir_n-2p_n-1 + ir_n-1q_n$$

$$c_n^1 = -iq_n$$

$$c_n^2 = -ip_nq_n+1 + ir_n-1q_n$$

$$d_n^1 = ir_n-1q_n$$

$$d_n^2 = ir_n-1p_nq_n+1 + ir_n-1p_nq_n+1 - ir_n-1q_n$$

(23)

and, in principle, all other matrices $V_n^j$. This leads to the infinite system of equations for $q_n$, $r_n$ some first of which are

$$\partial_t q_n = -ip_nq_n+1$$

(24)

$$\partial_t r_n = ir_n-1p_n$$

(25)

$$\partial_2 q_n = ir_n-1p_nq_n+1 + ip_nq_n+1q_n+1 - ip_nq_n+1q_n+2$$

(26)

$$\partial_2 r_n = ir_n-2p_n-1p_n - ir_n-1p_nq_n - ir_n-1p_nq_n+1$$

(27)

Analogously, looking for the $V$-matrices of the form

$$V_n^{-j} = \lambda^2 V_n^{-j+1} + \left( \begin{array}{ccc}
n^{-j} & \lambda^{-1} \beta_n^{-j} \\
n^{-j+1} & \lambda^{-1} \gamma_n^{-j} \\
\delta_n^{-j} & \lambda^{-1} \epsilon_n^{-j} & \lambda^{-1} \zeta_n^{-j} 
\end{array} \right)$$

(28)

and repeating the procedure described above one can obtain the ‘negative’ part of the ALH. Some first of its equations are

$$\partial_{-1} q_n = -iq_{n-1}p_n$$

(29)

$$\partial_{-1} r_n = ip_nr_{n+1}$$

(30)
Indeed, expressing from (24) and (25)

\[ q \]

for the equations (38), (39):

like to mention only one remarkable fact. By simple calculations one can obtain an alternative representation

\[ D \]

(hereafter I will write \( q \) and \( q \) of (1+1)-dimensional evolution equations for \( q \) manifest of the fact that both 'positive' and 'negative' subhierarchies can be transformed into hierarchies

One can rewrite analogously also the equations (40), (41) as well as all other equations of the ALH. It is a

other words we have presented the differential-difference equa
tions (38), (39) as a partial differential equation.

indicate derivatives with respect to \( z \) and the overbar denotes the complex conjugation.

Before proceeding further I would like to note that the simplest equations of the ALH, (24) and (29), when rewritten in terms of the real variables \( x = \text{Re} z \) and \( y = \text{Im} z \) become exactly the DNLSE (1) modified by the substitution \( q_n \to q_n \exp(2ix) \) and the DMKdV (3)

All equations of the ALH, as well as all equations of other integrable hierarchies, can be presented in the bilinear form using Hirota's operators

\[
D^a_x \ldots D^b_y u \cdot v = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^a \ldots \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^b u(x, y, \ldots)v(x', y', \ldots) \]  \tag{33}

To this end consider the functions \( \tau_n, \sigma_n \) and \( \rho_n \) defined by

\[
P_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}, \quad q_n = \sigma_n^2 \tau_n^{-1} \quad r_n = \rho_n^2 \tau_n\]  \tag{34}

The first equations of the ALH, (24) and (29), can be rewritten then, using the designation

\[
D_j = D_{\partial z_j}, \quad \bar{D}_j = D_{\partial \bar{z}_j} \]  \tag{35}

as

\[
D_1 \sigma_n \cdot \tau_n = -i \sigma_{n+1} \tau_{n-1} \]  \tag{36}

\[
\bar{D}_1 \sigma_n \cdot \tau_n = -i \sigma_{n-1} \tau_{n+1} \]  \tag{37}

(the corresponding equations for the functions \( \rho_n \) one can obtain from these equations using the involution \( \sigma_n = -\kappa \rho_n \)). The next pair of equations of the ALH, (24) and (30), can be presented as

\[
D_2 \sigma_n \cdot \tau_n = D_1 \sigma_{n+1} \cdot \tau_{n-1} \]  \tag{38}

\[
\bar{D}_1 \sigma_{n+1} \cdot \tau_n = i \sigma_{n+1} \rho_n \]  \tag{39}

and

\[
\bar{D}_2 \sigma_n \cdot \tau_n = \bar{D}_1 \sigma_{n-1} \cdot \tau_{n+1} \]  \tag{40}

\[
\bar{D}_1 \sigma_{n+1} \cdot \tau_n = -i \sigma_n \rho_{n+1} \]  \tag{41}

The bilinear representation of the higher equations of the hierarchy will be discussed below, and here I would like to mention only one remarkable fact. By simple calculations one can obtain an alternative representation for the equations (38), (39):

\[
(i D_2 + D_{11}) \sigma_n \cdot \tau_n = 0 \]  \tag{42}

(hereafter I will write \( D_{x \ldots y} \) instead of \( D_{x \ldots y} \)) which involves functions for only one value of the index \( n \). In other words we have presented the differential-difference equations (38), (39) as a partial differential equation. One can rewrite analogously also the equations (40), (41) as well as all other equations of the ALH. It is a manifestation of the fact that both 'positive' and 'negative' subhierarchies can be transformed into hierarchies of (1+1)-dimensional evolution equations for \( q = q_n \) and \( r = r_n \) as functions of \( z = z_1 \) and \( z_j, j = 2, 3, \ldots \). Indeed, expressing from (24) and (27) \( q_{n+1}, q_{n+2} \) and \( r_{n-1} \) in terms of \( q, q_z, q_{zz}, r, r_z \) (here the subscripts indicate derivatives with respect to \( z \)) one can rewrite the equations (28), (27) as

\[
i \partial_z q + q_{zz} + \frac{2q q_z r_z}{1 - qr} = 0 \]  \tag{43}

\[
-i \partial_z r + r_{zz} + \frac{2r r_z q_z}{1 - qr} = 0 \]  \tag{44}
and all higher equations of the ‘positive’ hierarchy in a similar way. Analogous procedure can surely be performed also for the ‘negative’ part of the hierarchy. Some general formulae for such a representation of the ALH will be obtained in the section 5.

3 Solutions of the auxiliary problems.

Now I am going to construct some solutions of the linear problems (4) and (5) which I rewrite now as

$$\partial_j \Psi_n = V^j_n \Psi_n$$ (45)

The question of solving (4) and (45) is not the main one of the present paper, and I discuss it for an illustrative purpose, to show a way how one can ‘deduce’ the functional representation of the ALH which we are looking for, and which can be then proved independently, without invoking results of this section. That is why I will write down some results (namely formulae (62) and (64)) without presenting their rigorous proof.

In what follows I will restrict myself to the simplest case

$$\lim_{n \to \infty} q_n, r_n = 0$$ (46)

or, in the \{τ, σ, ρ\}-representation,

$$\lim_{n \to \infty} σ_n, ρ_n = 0, \quad \lim_{n \to \infty} τ_n = \text{constant}$$ (47)

Presenting the elements of the first column of the matrix $Ψ_n$ as

$$Ψ^{(1)}_n = \lambda^n \frac{τ_n}{τ_{n-1}} \begin{pmatrix} \varphi_n \\ -λψ_n \end{pmatrix}$$ (48)

one can obtain from (4) the following equations for the quantities $ϕ_n, ψ_n$:

$$p_n ϕ_{n+1} = ϕ_n - r_n ψ_n$$

$$-λ^2 p_n ψ_{n+1} = q_n \varphi_n - ψ_n$$

which I will solve under the boundary conditions

$$\lim_{n \to \infty} ϕ_n = 1, \quad \lim_{n \to \infty} ψ_n = 0$$ (51)

This problem admit solution that can be presented as power series in $λ^2$:

$$\binom{ϕ_n}{ψ_n} = \sum_{m=0}^{∞} λ^{2m} \binom{ϕ^m_n}{ψ^m_n}$$ (52)

Substituting these series in (49), (50) one can derive a system of equations for the quantities $ϕ^m_n$ and $ψ^m_n$, which can be written as follows:

$$ϕ^m_n - ϕ^m_{n-1} = -r_n ψ^{m-1}_n$$

$$ψ^m_n = q_n ϕ^m_n + p_n ψ^{m-1}_{n+1}$$

From this system and (51) one can obtain

$$ϕ^0_n = 1, \quad ψ^0_n = q_n$$ (55)

Using the identity

$$\partial_1 \ln \frac{τ_n}{τ_{n-1}} = i r_n q_n$$ (56)
which follows from (39) one can perform next iteration:

\[ \varphi_n^1 = \frac{i}{\tau_n} \partial_1 \tau_n, \quad \psi_n^1 = \frac{i}{\tau_n} \partial_1 \sigma_n \]  
\[ (57) \]

Further, using

\[ \partial_2 \ln \frac{\tau_n}{\tau_{n-1}} = \frac{ir_{n-2}}{\tau_{n-1}} p_{n-1} q_n + \frac{ir_{n-1}}{\tau_n} q_{n+1} - \frac{ir_{n-1}^2}{\tau_{n-1}} q_n^2 \]
\[ (58) \]

one can obtain

\[ \varphi_n^2 = \frac{1}{2\tau_n} (i\partial_2 - \partial_{11}) \tau_n, \quad \psi_n^2 = \frac{1}{2\tau_n} (i\partial_2 - \partial_{11}) \sigma_n \]
\[ (59) \]

Iterating further the system (53), (54) one can conclude that the quantities \( \tau_n \varphi_n^m \) and \( \tau_n \psi_n^m \) are the coefficients of the Taylor expansion for the functions \( \tau_n(z_1 + i\lambda^2, z_2 + i\lambda^4/2, \ldots) \) and \( \sigma_n(z_1 + i\lambda^2, z_2 + i\lambda^4/2, \ldots) \). Moreover, it can be shown that the column (48) with

\[ \varphi_n = \frac{\tau_n(z_k + i\lambda^2 k / \bar{z}_k)}{\tau_n(z_k, \bar{z}_k)}, \quad \psi_n = \frac{\sigma_n(z_k + i\lambda^2 k / \bar{z}_k)}{\tau_n(z_k, \bar{z}_k)} \]
\[ (60) \]

where \( \tau_n \) and \( \sigma_n \) are solutions of the 'positive' subhierarchy, solve the linear problems (15) for \( j = 1, 2, \ldots \). Here the designation

\[ f(z_k, \bar{z}_k) \equiv f(z_1, z_2, \ldots z_1, \bar{z}_2, \ldots) \]
\[ (61) \]

is used.

Considering in a similar way the second column of the matrix \( \Psi_n \) one can obtain the following matrix solution for the linear problems of the 'positive' subhierarchy (i.e the problems (4) and (15) for \( j = 1, 2, \ldots \)):

\[ \Psi_n^+ = \frac{1}{\tau_{n-1}(z_k, \bar{z}_k)} \begin{pmatrix} \lambda^n \tau_n(z_k + i\lambda^2 k / \bar{z}_k) & \lambda^{-n+1} \exp(-i\phi) \rho_{n-1}(z_k - i\lambda^2 k / \bar{z}_k) \\ -\lambda^{n+1} \rho_n(z_k + i\lambda^2 k / \bar{z}_k) & \lambda^{-n} \exp(-i\phi) \tau_{n-1}(z_k - i\lambda^2 k / \bar{z}_k) \end{pmatrix} \]
\[ (62) \]

where

\[ \phi = \sum_{k=1}^{\infty} \lambda^{-2k} z_k \]
\[ (63) \]

Analogously, for the linear problems of the 'negative' subhierarchy, (4) and (15) for \( j = -1, -2, \ldots \), one can obtain solution

\[ \Psi_n^- = \frac{1}{\tau_{n-1}(z_k, \bar{z}_k)} \begin{pmatrix} \lambda^n \exp(i\phi) \tau_{n-1}(z_k, \bar{z}_k + i\lambda^{-2k}) & -\lambda^{-n-1} \rho_n(z_k, \bar{z}_k - i\lambda^{-2k}) \\ \lambda^{n-1} \exp(i\phi) \sigma_{n-1}(z_k, \bar{z}_k + i\lambda^{-2k}) & \lambda^{-n} \tau_n(z_k, \bar{z}_k - i\lambda^{-2k}) \end{pmatrix} \]
\[ (64) \]

where

\[ \bar{\phi} = \sum_{k=1}^{\infty} \lambda^{2k} \bar{z}_k \]
\[ (65) \]

The obtained formulae (62), (64) are one more illustration for the fact that an integrable hierarchy is more than a collection of solvable equations, and considering an hierarchy one can sometimes obtain more 'transparent' results than if dealing with one particular equation. Such approach (and such results) is not entirely new; it had been applied earlier to other hierarchies, say AKNS [4], though, to my knowledge, for the ALH this has been done for the first time in this work.
4 The main result.

In the previous section we constructed the matrices $\Psi_{n+1}^+$ ($\Psi^-_n$) which are solutions of the discrete auxiliary problem (49) and the 'positive' ('negative') set of evolution linear problems (55). And though validity of these results should be discussed more accurately, they give us sufficient hint to obtain the main result of this work, namely the functional representation of the ALH. The matrix equation $\Psi_{n+1}^+ = U_n \Psi_n^+$ after some transformations can be rewritten in the following way:

$$\sigma_n(z_k + i\lambda^{2k}/k, \bar{z}_k)\tau_n(z_k, \bar{z}_k) - \sigma_n(z_k, \bar{z}_k)\tau_n(z_k + i\lambda^{2k}/k, \bar{z}_k) = \lambda^2 \tau_{n-1}(z_k, \bar{z}_k)\sigma_{n+1}(z_k + i\lambda^{2k}/k, \bar{z}_k)$$

(66)

$$\rho_n(z_k, \bar{z}_k)\tau_n(z_k + i\lambda^{2k}/k, \bar{z}_k) - \rho_n(z_k + i\lambda^{2k}/k, \bar{z}_k)\tau_n(z_k, \bar{z}_k) = \lambda^2 \rho_{n-1}(z_k, \bar{z}_k)\tau_{n+1}(z_k + i\lambda^{2k}/k, \bar{z}_k)$$

(67)

$$\tau_n(z_k + i\lambda^{2k}/k, \bar{z}_k)\tau_n(z_k, \bar{z}_k) - \tau_{n-1}(z_k, \bar{z}_k)\tau_{n+1}(z_k + i\lambda^{2k}/k, \bar{z}_k) = \sigma_n(z_k + i\lambda^{2k}/k, \bar{z}_k)\rho_n(z_k, \bar{z}_k)$$

(68)

Now we can forget about were these equations originate from and consider them as a system of three difference-functional equations for unknown functions $\tau_n$, $\sigma_n$ and $\rho_n$. This system is compatible with all 'positive' flows of the ALH: if $\tau_n$, $\sigma_n$ and $\rho_n$ solve (66) - (68) then $\Psi_n^+$ satisfy all equations (49) for $j = 1, 2, ...$. To prove this consider the quantities

$$X_n^j = \partial_j \varphi_n - a_n^j \varphi_n + \lambda b_n^j \psi_n$$

(69)

$$Y_n^j = \partial_j \psi_n + \lambda^{-1} c_n^j \varphi_n - d_n^j \psi_n$$

(70)

where $\varphi_n$, $\psi_n$ are defined by

$$\varphi_n = \frac{\tau_n(z_k + i\lambda^{2k}/k, \bar{z}_k)}{\tau_{n-1}(z_k, \bar{z}_k)}$$

and

$$\psi_n = \frac{\sigma_n(z_k + i\lambda^{2k}/k, \bar{z}_k)}{\tau_{n-1}(z_k, \bar{z}_k)}$$

(71)

(note that these functions $\varphi_n$, $\psi_n$ differ from ones defined by (50) in the factor $\tau_n/\tau_{n-1}$) and $a_n^j$, ..., $d_n^j$ are elements of the matrix $V_n^j$ (see (5)). Using the identities

$$\varphi_{n+1} - \varphi_n + r_n \psi_n = 0$$

(72)

$$\lambda^2 \psi_{n+1} - \psi_n + q_n \varphi_n = 0$$

(73)

which follow from (66), (68) one can straightforwardly verify the fact that the combination $X_{n+1} - X_n + r_n Y_n$ can be presented as

$$X_{n+1} - X_n + r_n Y_n = \partial_j \left[ \varphi_{n+1} - \varphi_n + r_n \psi_n \right] +$$

$$+ \left[ a_n^j - a_{n+1}^j - \lambda^{-1} q_n b_n^j + \lambda^{-1} r_n c_n^j \right] \varphi_n$$

$$+ \left[ -\partial_j r_n + \left( a_{n+1}^j - d_n^j \right) - \lambda b_n^j + \lambda^{-1} b_{n+1}^j \right] \psi_n$$

(74)

It can be easily seen from (72) together with (49) and (52) that all expressions in square brackets are equal to zero, i.e.

$$X_{n+1} - X_n + r_n Y_n = 0$$

(75)

Analogously, calculating in a similar way $\lambda^2 Y_{n+1} - Y_n + q_n X_n$ one can obtain

$$\lambda^2 Y_{n+1} - Y_n + q_n X_n = 0$$

(76)
Presenting $X^j_n$ and $Y^j_n$ as

\[ X^j_n = \frac{\tau_n}{\tau_{n-1}} \sum_{m=0}^{\infty} \lambda^{2m} X^{j,m}_{n,m} \]
\[ Y^j_n = \frac{\tau_n}{\tau_{n-1}} \sum_{m=0}^{\infty} \lambda^{2m} Y^{j,m}_{n,m} \]  

one can derive the following recurrence for the coefficients $X^{j,m}_{n,m}$, $Y^{j,m}_{n,m}$:

\[ X^{j,m}_{n,m} - X^{j,m-1}_{n-1,m} = -\rho_{n-1} Y^{j,m-1}_{n,m-1} \]  
\[ Y^{j,m}_{n,m} = \rho_n Y^{j,m}_{n+1,m-1} + q_n X^{j,m}_{n,m} \]

It can be shown that $X_n(\lambda = 0) = Y_n(\lambda = 0) = 0$, i.e. $X^{j,0}_n = Y^{j,0}_n = 0$. Then, (78) yields $X^{j,1}_n = \text{constant}$, this constant, as follows from the boundary conditions for $\varphi_n$ and $\psi_n$ is zero, $X^{j,1}_n = 0$.

Analogously, the 'positive' part of the ALH can be written as following functional equations:

\[ \sigma_n(z_k + i\delta^k/2k)\tau_n(z_k - i\delta^k/2k) - \sigma_n(z_k - i\delta^k/2k)\tau_n(z_k + i\delta^k/2k) = \]
\[ = \delta \tau_{n-1}(z_k - i\delta^k/2k)\sigma_{n+1}(z_k + i\delta^k/2k) \]  
\[ \rho_n(z_k - i\delta^k/2k)\tau_n(z_k + i\delta^k/2k) - \rho_n(z_k + i\delta^k/2k)\tau_n(z_k - i\delta^k/2k) = \]
\[ = \delta \rho_{n-1}(z_k - i\delta^k/2k)\tau_{n+1}(z_k + i\delta^k/2k) \]  
\[ \tau_n(z_k + i\delta^k/2k)\tau_n(z_k - i\delta^k/2k) - \tau_{n-1}(z_k - i\delta^k/2k)\tau_{n+1}(z_k + i\delta^k/2k) = \]
\[ = \sigma_n(z_k + i\delta^k/2k)\rho_n(z_k - i\delta^k/2k) \]

(Here $\delta$ is used instead of $\lambda$ and dependence on the conjugated coordinates, $z_\bar{k}$, is temporarily omitted) are indeed compatible with the 'positive' flows (84) for $j = 1, 2, ..., \tau_n$ one can consider them as being equivalent to the 'positive' part of the ALH. Expanding (81)-(83) in powers of $\delta$ one will obtain an infinite number of differential-difference equation that can be transformed to the ones from the ALH. So, e.g., the equations which correspond to the first power of $\delta$

\[ (\partial_1 \sigma_n) \tau_n - \sigma_n (\partial_1 \tau_n) = -i\tau_{n-1} \sigma_{n+1} \]  
\[ (\partial_1 \rho_n) \tau_n - \rho_n (\partial_1 \tau_n) = i\rho_{n-1} \tau_{n+1} \]

are obviously the equations (24), (25) rewritten in the $\{\tau_n, \sigma_n, \rho_n\}$-representation. The equations which correspond to the second power of $\delta$ are equivalent to the second pair of equations of the ALH, (26) and (27), etc.

Analogically, the 'negative' part of the ALH can be written as following functional equations:

\[ \sigma_n(\tilde{z}_k + i\tilde{\delta}^k/2k)\tau_n(\tilde{z}_k - i\tilde{\delta}^k/2k) - \sigma_n(\tilde{z}_k - i\tilde{\delta}^k/2k)\tau_n(\tilde{z}_k + i\tilde{\delta}^k/2k) = \]
\[ = \tilde{\delta} \sigma_{n-1}(\tilde{z}_k + i\tilde{\delta}^k/2k)\tau_{n+1}(\tilde{z}_k - i\tilde{\delta}^k/2k) \]  
\[ \rho_n(\tilde{z}_k - i\tilde{\delta}^k/2k)\tau_n(\tilde{z}_k + i\tilde{\delta}^k/2k) - \rho_n(\tilde{z}_k + i\tilde{\delta}^k/2k)\tau_n(\tilde{z}_k - i\tilde{\delta}^k/2k) = \]
\[ = \tilde{\delta} \tau_{n-1}(\tilde{z}_k + i\tilde{\delta}^k/2k)\rho_{n+1}(\tilde{z}_k - i\tilde{\delta}^k/2k) \]
\[ \begin{align*}
\tau_n(\bar{z}_k + i\tilde{\delta}_k/2k)\tau_n(\bar{z}_k - i\tilde{\delta}_k/2k) - \tau_{n-1}(\bar{z}_k + i\tilde{\delta}_k/2k)\tau_{n+1}(\bar{z}_k - i\tilde{\delta}_k/2k) &= \sigma_n(\bar{z}_k + i\tilde{\delta}_k/2k)\rho_n(\bar{z}_k - i\tilde{\delta}_k/2k)
\end{align*} \] (88)

where \( \tilde{\delta} \) is used instead of \( \lambda^{-2} \) and dependence on \( z_k \)'s is omitted.

Equations (81) - (83) and (86) - (88) are the main result of this paper. They present an infinite number of the difference-differential equations of the ALH under the zero boundary conditions (46) in the form of six difference-functional equations. Thus we have derived the 'functional' representation of the ALH. Analogous results can be obtained for some other classes of boundary conditions, say, for the so-called finite-density (\( q_n \rightarrow \text{constant as } n \rightarrow \pm\infty \)) or quasiperiodical ones. Before proceeding further I would like to say few words on the following question. We have split the ALH into two subhierarchies (the 'positive' and 'negative' ones) which seems to be rather natural: one of the subhierarchies can be obtained from the other using the complex conjugation. Nevertheless, it would be interesting to obtain, instead of two sets of functional equations ((81) - (83) for the 'positive' hierarchy and (86) - (88) for the 'negative' one) one set of equations which takes into account both 'positive' and 'negative' flows. I cannot do this at present, and it will be a subject of following studies.

5 Hirota’s representation of the ALH.

It is already known fact that the \( D \)-operators calculus that has been invented by Hirota is not only an ingenious tool for deriving some families of solutions for integrable equations. It is a convenient way of operating with integrable hierarchies, which enables to reveal some regularities in their structure. Now I am going to rewrite the main results in Hirota’s formalism, which, in addition, will demonstrate some interesting features of the ALH. In what follows I will deal only with 'positive' subhierarchy, because for the 'negative' one all results can be obtained using the complex conjugation (\( \sigma_n = -\kappa\rho_n, \) etc). Using the following property of the Hirota’s operators

\[ \exp(aD_z) f(z) \cdot g(z) = f(z + a)g(z - a) \] (89)

and introducing

\[ D(\delta) = \sum_{k=1}^{\infty} \delta_k D_k \] (90)

one can rewrite (81)-(83) as

\[ \exp \left( \frac{i}{2} D(\delta) \right) \left( \sigma_n \cdot \tau_n - \tau_n \cdot \sigma_n - \delta \cdot \sigma_{n+1} \cdot \tau_{n-1} \right) = 0 \] (91)
\[ \exp \left( \frac{i}{2} D(\delta) \right) \left( \rho_n \cdot \tau_n - \tau_n \cdot \rho_n + \delta \cdot \tau_{n+1} \cdot \rho_{n-1} \right) = 0 \] (92)
\[ \exp \left( \frac{i}{2} D(\delta) \right) \left( \tau_{n+1} \cdot \tau_{n-1} - \tau_n \cdot \tau_n + \sigma_n \cdot \rho_n \right) = 0 \] (93)

One of the advantages of this viewpoint is that one can obtain explicit form of the \( j \)th equation of the ALH, which hardly can be done in the framework of the standard IST technique discussed in the section 2. This can be done in terms of the Schur’s polynomials

\[ \exp \left( \sum_{m=1}^{\infty} x^m f_m \right) = \sum_{m=0}^{\infty} x^m \chi_m (f_1, f_2, ...) \] (94)

(I will use below the designation \( \chi_m (f_k) \equiv \chi_m (f_1, f_2, ...) \)). By simple calculations equations (91), (92), which can be rewritten as
where operators

\[ \sigma_n \cdot \tau_n = \delta \exp \left( \frac{i}{2} D(\delta) \right) \sigma_{n+1} \cdot \tau_{n-1} \]

(95)

\[ 2i \sin \left( \frac{1}{2} D(\delta) \right) \rho_n \cdot \tau_n = -\delta \exp \left( \frac{i}{2} D(\delta) \right) \tau_{n+1} \cdot \rho_{n-1} \]

(96)

and equation (103) can be presented as

\[
\begin{align*}
\chi_j \left( \frac{iD_k}{2k} \right) - \chi_j \left( -\frac{iD_k}{2k} \right) \sigma_n \cdot \tau_n &= \chi_{j-1} \left( \frac{iD_k}{2k} \right) \sigma_{n+1} \cdot \tau_{n-1} \\
\chi_j \left( \frac{iD_k}{2k} \right) - \chi_j \left( -\frac{iD_k}{2k} \right) \rho_n \cdot \tau_n &= -\chi_{j-1} \left( \frac{iD_k}{2k} \right) \tau_{n+1} \cdot \rho_{n-1} \\
\chi_j \left( \frac{iD_k}{2k} \right) (\tau_{n+1} \cdot \tau_{n-1} - \tau_n \cdot \tau_n + \sigma_n \cdot \rho_n) &= 0
\end{align*}
\]

(97)-(99)

for \( j = 1, 2, \ldots \)

It has been noticed in section 3 that 'positive' subhierarchy of the ALH (as well as the 'negative' one) can be presented as a hierarchy of partial differential equations, which can be easily derived from (81)-(83). Using the identities

\[
\begin{align*}
D_1 \sigma_n (z_k + i\delta k/2k) \cdot \tau_n (z_k - i\delta k/2k) &= -i\sigma_{n+1} (z_k + i\delta k/2k) \tau_{n-1} (z_k - i\delta k/2k) \\
D_1 \tau_n (z_k + i\delta k/2k) \cdot \rho_n (z_k - i\delta k/2k) &= -i\tau_{n+1} (z_k + i\delta k/2k) \rho_{n-1} (z_k - i\delta k/2k) \\
D_1 \tau_n (z_k + i\delta k/2k) \cdot \tau_n (z_k - i\delta k/2k) &= i\delta \sigma_n+1 (z_k + i\delta k/2k) \rho_{n-1} (z_k - i\delta k/2k)
\end{align*}
\]

(100)-(102)

which follow from (71) and (80) for \( j = 1 \), equations (81) - (83) can be rewritten as

\[
\hat{G}(\delta) \begin{pmatrix} \sigma \cdot \tau \\ \tau \cdot \rho \\ \sigma \cdot \rho + \tau \cdot \tau \end{pmatrix} = 0
\]

(103)

where \( \sigma, \rho \) and \( \tau \) stand for \( \sigma_n, \rho_n \) and \( \tau_n \) with \( n \) being fixed and the operator \( \hat{G}(\delta) \) is defined by

\[
\hat{G}(\delta) = 2i \sin \left( \frac{1}{2} D(\delta) \right) - i\delta D_1 \exp \left[ \frac{i}{2} D(\delta) \right]
\]

(104)

Expanding (103) in power series in \( \delta \) one can obtain a hierarchy of partial differential equations

\[
\hat{G}_j \begin{pmatrix} \sigma \cdot \tau \\ \tau \cdot \rho \\ \sigma \cdot \rho + \tau \cdot \tau \end{pmatrix} = 0 \quad j = 2, 3, \ldots
\]

(105)

where operators \( \hat{G}_j \) are defined by

\[
\hat{G}(\delta) = \sum_{j=2}^{\infty} \frac{\delta^j}{j!} \hat{G}_j
\]

(106)

Some first equations of this hierarchy are ones given by (107) with

\[
\begin{align*}
\hat{G}_2 &= iD_2 + D_{11} \\
\hat{G}_3 &= iD_3 + \frac{3}{4}D_{21} + \frac{i}{4}D_{111} \\
\hat{G}_4 &= iD_4 + \frac{2}{3}D_{31} + \frac{i}{4}D_{211} - \frac{1}{12}D_{1111}
\end{align*}
\]

(107)-(109)
A rather interesting consequence of (81) - (83) can be obtained by excluding \( \sigma_n \) and \( \rho_n \). It can be straightforwardly shown using (71) and (80) for \( j = 1, 2 \) that

\[
[2D_1 - \delta(D_2 + iD_{11})] \tau_n(z_k + i\delta^k/2k) \cdot \tau_n(z_k - i\delta^k/2k) = 0
\]

or, using again the \( [iD(\delta)/2] \) operator,

\[
[2D_1 - \delta(D_2 + iD_{11})] \exp \left[ \frac{i}{2} D(\delta) \right] \tau \cdot \tau = 0
\]

where \( \tau \equiv \tau_n \) (for any \( n \)). Expanding this equation in powers of \( \delta \) one can obtain again an infinite number of equations, this time for one function, \( \tau \). Few first of them are

\[
(4D_{31} - 3D_{22} + D_{1111}) \tau \cdot \tau = 0
\]

\[
(3D_{41} - 2D_{32} + D_{2111}) \tau \cdot \tau = 0
\]

\[
(96D_{51} - 60D_{42} + 20D_{3111} + 15D_{2211} - D_{111111}) \tau \cdot \tau = 0
\]

It seems to be interesting that equation (112) is nothing other than the Kadomtsev-Petviashvili equation. Indeed, it can be verified by straightforward (though rather cumbersome) calculations that the quantity

\[
u = r_{n-1}p_nq_{n+1}
\]

for any \( n \) solves the equation

\[
\partial_1 (4\partial_3 u + \partial_{111} u + 12u \partial_1 u) = 3\partial_{22} u
\]

So, we have obtained the remarkable result: the Kadomtsev-Petviashvili equation turns out to be 'embedded' in the ALH!

6 Conclusion.

In the present work it has been obtained a representation of the ALH in the form of difference-functional equations. This result seems to be interesting from several viewpoints. First, it clearly demonstrates common origin of all equations of the hierarchy. Second, such approach can be useful as an easy tool for generating of big number of solutions for the ALH, such as, first of all, multisoliton, 'Wronskian' and some other ones. An interesting transformation of the results obtained to the already known ones arises when one considers the problem of quasiperiodical solutions. The functional relations (81) - (83) and (86) - (88) become the Fay’s trisecant formulae for the \( \theta \)-functions of Riemann surfaces. Also, bilinear functional representation of the ALH can provide answers to some more general questions of the theory of integrability, such as, e.g., classification of the Hirota’s polynomials: note that (97) can be viewed as explicit expression for the Hirota’s polynomials of arbitrary order.

The last moment I would like to discuss here, and which seems to be one of the most interesting, is the question of, so to say, 'universality' of the ALH. It is a known fact that some equations can be 'embedded' into the ALH. It has been shown that the ALH 'contains' the 2D Toda lattice [8] (see also [13]), O(3,1) \( \sigma \)-model [4], the Davey-Stewartson (DS) equation and the Ishimori model [6]. In the last paper it has been shown that the derivative nonlinear Schrödinger equation can also be 'embedded' into the ALH, which implies that the same can be done for the AKNS as well. In the present paper I have shown that the Kadomtsev-Petviashvili equation (hierarchy) can also be composed of the ALH-flows. The results of the works [4, 6, 8, 8] (her I would like to mention also the papers [4, 6]) are in some sense 'empirical' facts: one can easily verify them by simple calculations, but one can hardly find there an answer on the question why do such apparently different models turn out to be interrelated. At the same time the approach described above, and especially the results presented in the section 5 provide some insight into this problem. An impression arises that the ALH is, in some sense, the most general hierarchy, at least among the 'classical' ones. Surely, such statements need much more careful grounding, but the fact that the AKNS, DS, KP hierarchies may can be obtained from the ALH via reductions \( \{\tau, \sigma, \rho\} \to \{\tau\} \), to my mind, indicate that the hypothesis of 'universality' of the ALH is not senseless and worth further studies.
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