Universality of the Route to Chaos -Exact Analysis-

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(Dated: June 18, 2021)

The universality of the route to chaos is analytically proven for countably infinite number of maps by proposing the Super Generalized Boole (SGB) transformations. As one of the route to chaos, intermittency route was studied by Pomeau and Manneville numerically. They conjectured the universality in Type 1 intermittency, that the critical exponent of the Lyapunov exponent is $1/2$ in Type 1 intermittency. In order to prove their conjecture, we showed that for certain parameter ranges, the SGB transformations are exact and preserve the Cauchy distribution. Using the property of exactness, we proved that the critical exponent is $1/2$ for countably infinite number of maps where Type 1 intermittency occurs.

PACS numbers: 05.45.-a, 05.70.Jk
Universality in chaos  Route from stable states to chaotic (intermittent) states has caught much attention in broad fields in physics. This issue treats fundamental change of systems from stable state to unstable state and it is an essential theme to analyze the stability of physical systems. Route to chaos is also studied theoretically and experimentally such as Hamiltonian systems [1], map systems [2–7], coupled oscillators [8], Belousov-Zhabotinskii reaction [9], Rayleigh-Brnard convection [9], Couette Taylor flow [9], noise induced system [10], thermoacoustic system [11] and optomechanics [12–14]. There is the theoretical classification of routes to chaos such as intermittency route, period doubling route, frequency locking route, etc [9]. Frequently these researches have been motivated to discover the universality at the onset of chaos with respect to the critical exponent of the Lyapunov exponent, which is an indicator of chaos. The universality of the critical exponents in each route to chaos has been studied extensively by numerical simulations. For period doubling route, Huberman and Rudnick [5] estimated numerically the critical exponent $\nu$ as $\nu = \frac{\log 2}{\log 3}$ where $\delta$ represents the Feigenbaum constant. For intermittency route treated in this Letter, Pomeau and Manneville [3, 4] classified intermittency into three types and conjectured the universality of $\nu$ for each intermittent type. In particular, in Type 1 intermittency, they conjectured the universality that $\nu = \frac{1}{2}$, by the numerical simulations. After the work by Pomeau and Manneville, the critical exponent has been researched in various field with relations to intermittency such as Billiard system [15, 16], electronic circuit [17], plasma physics [18] and intermittent map [19]. Those studies by numerical simulations suggest that their conjecture $\nu = \frac{1}{2}$ would be right.

However, in these researches, the critical exponent $\nu$ is estimated by numerically or the analytical formulae of the Lyapunov exponent $\lambda$ was not obtained without any assumption.

On the other hand, the present authors [5] led the analytical formula of the Lyapunov exponent $\lambda$ as an explicit function in terms of the bifurcation parameter by showing the mixing property for the Generalized Boole (GB) transformation. They proved in the GB transformation that both Type 1 and Type 3 intermittency occur and that the conjecture by Pomeau and Manneville is correct.

In this Letter, it is analytically proved that for countably infinite number of maps, it holds that

$$\lambda \sim b |\alpha - \alpha_c|^\nu, \quad \nu = \frac{1}{2}, b > 0,$$

when Type 1 intermittency occurs where $\alpha$ and $\alpha_c$ represent a bifurcation parameter and the critical point, respectively. In order to prove this, we propose more generalized maps, the Super Generalized Boole (SGB) transformations and show that there are parameter ranges in which the SGB transformations are exact (stronger condition than ergodicity). That means one obtains countably infinite number of exact (ergodic) maps. Using this result, one can obtain explicitly the analytical formulae of the Lyapunov exponents and critical exponents.

We define two-parameterized one-dimensional maps, the Super Generalized Boole Transformations (SGB) $S_{K,\alpha} : \mathbb{R}\setminus B \to \mathbb{R}\setminus B$ as follows.

$$x_{n+1} = S_{K,\alpha}(x_n) \overset{\text{def}}{=} \alpha F_K(x_n),$$

where $\alpha > 0$, $K \in \mathbb{N}\setminus\{1\}$, the set $B$ is a set on $\mathbb{R}$ such that for any point $x$ on $B$, there is an integer $n$ where $S_{K,\alpha}^n x$ reaches a singular point, and the function $F_K$ corresponds to $K$-angle formula of cot function defined in Supplemental material. The GB transformation $T_{\alpha,\gamma}$ in [5] corresponds to the map $S_{2,\alpha}$. Figure 1 shows the return maps of $S_{3,\frac{1}{2}}, S_{4,\frac{1}{2}}$ and $S_{5,\frac{1}{4}}$.

Invariant density  In this paragraph, it is proven that the SGB transformations preserve the Cauchy distribution for certain condition. According to [1, 21], the map $S_{K,\alpha}$ is $K$ to one map as follows. $y = K\alpha \cot K\theta = K\alpha F_K(x_j), x_j = \cot(\theta + j\frac{\pi}{K}), j = 1, 2, \cdots, K$. If variables $\{x_j\}$ obey the Cauchy distribution $f_\gamma(x) = \frac{1}{\pi \sqrt{x^2 + \gamma^2}}$ whose scale parameter is $\gamma$, then according to [1], the variable $y$ obeys the density function $p(y) = \frac{1}{\pi \gamma^2 + \alpha^2 K^2 \gamma^2 G_K(\gamma)}$, where the function $G_K(x)$ corresponds to $K$-angle formula of cot function defined in Supplemental material. Then the scale parameter $\gamma$ is transformed in one iteration as $\gamma \mapsto \alpha K \gamma$. Now, for each $K$, let us obtain the fixed point $0 < \gamma_{K,\alpha} < \infty$ which satisfies the relation

$$\gamma_{K,\alpha} = \alpha K G_K(\gamma_{K,\alpha})$$

and clarify the condition of $\alpha$ that there exists a solution of (3). The Cauchy distribution whose scale parameter is a solution of (3), which corresponds to the invariant density. In order to approach this problem, we define the Condition A as follows.
Definition 1. Condition A is referred to as
\[
\begin{cases}
0 < \alpha < 1 & \text{in the case of } K = 2N, \\
\frac{1}{K^2} < \alpha < 1 & \text{in the case of } K = 2N + 1,
\end{cases}
\] (4)
where \( N \in \mathbb{N} \).

Then the following theorem holds.

Theorem A. When the Condition A is satisfied, the SGB transformations \( \{S_{K,\alpha}\} \) preserve the Cauchy distribution and the scale parameter can be chosen uniquely.

The proof of Theorem A is given in Supplemental material. Although it has been proven that the map \( S_{K,\alpha} \) preserves the Cauchy distribution and its scale parameter \( \gamma_{K,\alpha} \) can be determined uniquely when the Condition A is satisfied, it is not straightforward to obtain the explicit form of fixed point \( \gamma_{K,\alpha} \) for arbitrary \( K \), since we have to solve the \( K \)th-degree equations. From Theorem A, the condition that there exists only one solution of (3) which satisfied \( 0 < \gamma_{K,\alpha} < \infty \) is nothing but the Condition A.

Exactness According to [2–4], the exactness is defined as follows.

Definition 2 (Exactness). A dynamics \( T \) on a phase space \( X \) with transfer operator \( P_T \) and unique stationary density \( f_* \) is called to be exact if and only if
\[
\lim_{n \to \infty} \|P_T^n f - f_*\|_{L^1} = 0,
\] (5)
for every initial density \( f \in D \) where \( D \) denotes all densities on \( X \).

This definition is equivalent to as follows,
\[
\lim_{n \to \infty} \mu_*(T^n s) = 1, \quad \forall s \in \mathcal{B}, \quad \mu_*(s) > 0,
\] (6)
where \( \mathcal{B} \) denotes the \( \sigma \)-algebra and \( \mu_* \) denotes the invariant measure corresponding the invariant density \( f_* \).
In terms of exactness, we obtain the following theorem.

**Theorem B.** If the the Condition A is satisfied, the SGB transformations \( \{S_{K,\alpha}\} \) are exact.

The proof is given in Supplemental material. From Theorems A and B, when the Condition A is satisfied, the map \( S_{K,\alpha} \) preserves certain Cauchy distribution \( f_* \) and any initial density function \( f \) defined on \( \mathbb{R}\setminus B \) converges to \( f_* \) as

\[
\lim_{n \to \infty} \|P^n_{S_{K,\alpha}} f - f_*\|_{L^1} = 0.
\]

(7)

For example, \( S_{3,\alpha} \), \( S_{4,\alpha} \) and \( S_{5,\alpha} \) are exact for \( \frac{1}{9} < \alpha < 1 \), \( 0 < \alpha < 1 \) and \( \frac{1}{25} < \alpha < 1 \), respectively. According to [3], if the SGB transformations are exact, then the corresponding dynamical systems are mixing and ergodic. Therefore the following Corollary holds.

**Corollary 3.** Suppose that the Condition A is satisfied. Then the dynamical system \( (\mathbb{R}\setminus B, S_{K,\alpha}, \mu_*) \) has the mixing property and it is ergodic where \( \mu_* \) is the invariant measure corresponding to the invariant density \( f_* \).

Using the property of exactness, one can obtain the explicit formula of the Lyapunov exponent such that

\[
\lambda_{K,\alpha} = \frac{1}{\pi} \int_{-\infty}^{\infty} \log \left| \frac{dS_{K,\alpha}}{dx} \right| \frac{\gamma_{K,\alpha}}{x^2 + \gamma_{K,\alpha}^2} dx.
\]

(8)

From Pesin’s formula, one sees that the Kolmogorov-Sinai entropy is equivalent to the Lyapunov exponent since the SGB transformations are a one-dimensional map.

For \( \alpha > 1 \), changing variable as \( z_n = 1/x_n \), one obtains the map \( \tilde{S}_{K,\alpha} \) defined as

\[
\begin{align*}
\tilde{S}_{2N,\alpha}(z) & \equiv \frac{1}{\alpha K} \sum_{i=0}^{N-1} (-1)^i 2N C_{2N-2i-1} z^{2N-2i-1}, \\
\tilde{S}_{2N+1,\alpha}(z) & \equiv \frac{1}{\alpha K} \sum_{i=0}^{N} (-1)^i 2N+1 C_{2N-2i+1} z^{2N-2i+1}.
\end{align*}
\]

(9)

Then one has that \( \frac{d^3\tilde{S}_{K,\alpha}(0)}{dx^3} = \frac{1}{\alpha} < 1 \), so that for any \( K \), the orbits are attracted into the infinite point for \( \alpha > 1 \). Thus, the Lyapunov exponent \( \lambda_{K,\alpha} \) for \( \alpha > 1 \) is derived from the inclination at the infinite point as

\[
\lambda_{K,\alpha} = \log \alpha, \quad \text{for } \forall K \in \mathbb{N}\setminus\{1\}.
\]

(10)

**Scaling behavior** At the edges of the Condition A, one has that

\[
\begin{align*}
\gamma_{K,\alpha} &= \infty, \quad \text{for } \alpha = 1 \quad \forall K, \\
\gamma_{K,\alpha} &= 0, \quad \begin{cases} 
\text{for } \alpha = 0 & \text{in the case of } K = 2N, \\
\text{for } \alpha = \frac{1}{2N+1} & \text{in the case of } K = 2N+1.
\end{cases}
\end{align*}
\]

(11)

Then the Lyapunov exponent converges to zero at the edges of the Condition A. In order to discuss the critical phenomena, define critical points as \( \alpha_{c1} = 1 \), \( \alpha_{c2} = \frac{1}{2N+1} \) and \( \alpha_{c3} = 0 \) and define critical exponents \( \nu_1 \), \( \nu_2 \) and \( \nu_3 \) corresponding to \( \alpha_{c_i}, i = 1, 2, 3 \). In terms of the scaling behavior of the Lyapunov exponents \( \lambda \sim b |\alpha - \alpha_{c_i}|^{\nu_i}, b > 0, i = 1, 2, 3 \) the following theorem holds.

**Theorem C.** Suppose that the Condition A is satisfied.

- For any \( K \in \mathbb{N}\setminus\{1\} \), it holds that \( \nu_1 = \frac{1}{2} \) as \( \alpha \to 1 - 0 \).
- For any \( K \in \mathbb{N}\setminus\{1\} \), it holds that \( \nu_1 = 1 \) as \( \alpha \to 1 + 0 \).
- For \( K = 2N + 1 \), it holds that \( \nu_2 = \frac{1}{2} \) as \( \alpha \to \frac{1}{K^2} + 0 \).
The proof is given in Supplemental material.

Discuss the Floquet multipliers in the case of \((K, \alpha) = (\gamma K, \alpha_{c1})\), \((2N + 1, \alpha_{c2})\), and \((2N, \alpha_{c3})\) for \(N \in \mathbb{N}\). By changing variable as \(x = \cot \theta\), the derivative of the map \(S_{K, \alpha}\) is rewritten as \(\frac{dS_{K, \alpha}}{dx} = \alpha K^{3} \frac{\sin^{2} \theta}{\sin^{2} \theta} \). (i) In the case of \((\gamma K, \alpha_{c1})\), the derivatives at the infinite point are denoted as

\[
\lim_{x \to +\infty} \frac{dS_{K, \gamma}}{dx} = \lim_{\theta \to 0} \frac{(K \theta)^{3} \sin^{2} \theta}{\sin^{2} \theta} = 1. \tag{12}
\]

Thus, the Floquet multiplier \(\chi\) for \((K, \alpha) = (\gamma K, \alpha_{c1})\) is unity.
(ii) In the case of \((2N + 1, \alpha_{c2})\), applying scale transformation such that \(x = \sqrt[3]{\alpha} y\), one has \(y_{n+1} = \hat{S}_{K, \alpha}(y_{n})\) and \(y_{n+1} = \hat{S}_{K, \alpha}(y_{n}) = -\frac{1}{y_{n}}\). Then, it holds that \(\chi = -1\) at \((K, \alpha) = (2N, \alpha_{c3})\).

From (i), (ii) and (iii), one sees that in the case of \((K, \alpha) = (\gamma K, \alpha_{c1})\) and \((2N + 1, \alpha_{c2})\), only Type 1 intermittency occurs at \(\alpha = \alpha_{c1}\) and \(\alpha_{c2}\) and that in the case of \((K, \alpha) = (2N, \alpha_{c3})\), Type 1 intermittency occurs at \(\alpha = \alpha_{c1}\) and Type 3 intermittency occurs at \(\alpha = \alpha_{c3}\).

In the case of \(K = 3, 4\) and 5 In this paragraph, the examples corresponds to \(K = 3, 4\) and 5 are illustrated. The solutions of (3) which satisfies 0 < \(\gamma_{K, \alpha} < \infty\) are uniquely determined as follows.

\[
\begin{align*}
\gamma_{3, \alpha} &= \frac{9\alpha - 1}{3 - 3\alpha}, \\
\gamma_{4, \alpha} &= \frac{6\alpha - 1 + \sqrt{32\alpha^{2} - 8\alpha + 1}}{2(1 - \alpha)}, \\
\gamma_{5, \alpha} &= \frac{-5(1 - 5\alpha) + \sqrt{20(25\alpha^{2} - 6\alpha + 1)}}{5(1 - \alpha)}.
\end{align*} \tag{14}
\]

From the above discussion, one knows that in the case of \(K = 3, 5\) and only Type 1 intermittency occurs and in the case of \(K = 4\), both Type 1 and Type 3 intermittency occur.

The Lyapunov exponents in the case of \(K = 3, 4\) and 5 are given as follows.

\[
\begin{align*}
\lambda_{3, \alpha} &= \log \left[ \frac{1}{\alpha} \left( \frac{3(1 - \alpha)}{8} \right)^{2} \left[ 1 + \sqrt{\frac{9\alpha - 1}{3 - 3\alpha}^{4}} \right] \right], \\
\lambda_{4, \alpha} &= \log \left[ \frac{\alpha(1 + \gamma_{4, \alpha})^{4}}{\gamma_{4, \alpha}(1 + \gamma_{4, \alpha})^{2}} \right], \\
\lambda_{5, \alpha} &= \log \left[ \frac{(1 - \alpha)^{4}}{256 \alpha(1 + \gamma_{5, \alpha})^{4} |1 + \gamma_{5, \alpha}|^{8}} \right]. \tag{15}
\end{align*}
\]

Figures 2a, 2b and 2c show the Lyapunov exponents against \(\alpha\) in the case of \(K = 3, 4\) and 5, respectively. One sees that the numerical simulations are exactly consistent with the obtained analytical formulæ. Since it holds that at the critical points \(\partial \lambda_{K, \alpha}/\partial \alpha = \pm \infty\), one sees that the parameter dependence of the Lyapunov exponent diverges at the critical points. This means the computational difficulty in obtaining the true value of the Lyapunov exponent by numerical simulation. Figure 3 shows the scaling behavior of the Lyapunov exponent. One sees that \(\nu_2 = \frac{1}{2}\) and \(\nu_3 = \frac{1}{2}\) in the case of \(K = 3, 4\) and 5.

Conclusion This work is the first example in which the conjecture by Pomeau and Manneville expressed in (1) is analytically proven to be true for countably infinite number of maps (the proposed Super Generalized Boole transformations). This work shows the theoretical picture of stable-unstable transition for intermittent maps. In the course of proof, we have shown that the Super Generalized Boole (SGB) transformations preserve the unique Cauchy
distribution, together with proving the fact that SGB transformations are exact and that any initial density function converges to the invariant Cauchy distribution when the Condition A is satisfied.

Applying the property of exactness, one can obtain analytical formulae of the Lyapunov exponents for the SGB transformations. In the SGB transformations, the Lyapunov exponents $\lambda_{K,\alpha}$ are equivalent to the Kolmogorov-Sinai entropy applying to the Pesin’s theorem. Using the analytical formulae of the Lyapunov exponents, we have confirmed that for $K = 3, 4$ and $5$, the derivative $\partial \lambda_{K,\alpha}/\partial \alpha$ diverge at the critical points and we obtained $\nu_1 = \nu_2 = \nu_3 = \frac{1}{2}$. Thus, we have proven the universality of the route to chaos for a large class of the chaotic systems.

As future works, clarifying the scaling relation between the critical exponent $\nu$ and the other critical exponents, we can obtain a new perspective of chaos in physics.
FIG. 2: Relations between the Lyapunov exponents of the SGB transformations and $\alpha$ for $K = 3, 4$ and 5. Circles and triangles represent numerical results for $\alpha \leq 1$ and for $\alpha > 1$, respectively. The solid lines and broken lines represent the analytical results for $\alpha \leq 1$ and for $\alpha > 1$, respectively. The initial point is $x_0 = 5\sqrt{7}$. The iteration number is $1 \times 10^5$ for $\alpha \leq 1$ and 200 for $\alpha > 1$. A vertical line corresponds to $\alpha = \frac{1}{9}$, $\frac{1}{25}$, respectively.
FIG. 3: Scaling behavior of the Lyapunov exponents of the SGB transformation in the case of $K = 3$, $4$ and $5$. Circles represent numerical simulation and solid lines represent the order. The initial point is $x_0 = 5\sqrt{7}$. The iteration number is $2 \times 10^5$. 
Ken-ichi Okubo acknowledges the support of Grant-in-Aid for JSPS Research Fellow Grant Number JP17J07694.

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In this Supplemental material it is shown that

1. the Super Generalized Boole transformations preserve the Cauchy distribution for certain parameter ranges and the Cauchy distribution is determined uniquely,

2. the SGB transformations are exact for the parameter ranges, and

3. the critical exponent of Lyapunov exponent is \( \nu = \frac{1}{2} \) for all \( K \in \mathbb{N} \setminus \{1\} \) when Type 1 intermittency occurs.

The definitions of \( F_K, G_K, S_{K,\alpha} \) are written in the following.

**Definition 3.** The map \( F_K : \mathbb{R} \setminus A \rightarrow \mathbb{R} \setminus A \) is referred to as
\[
F_K(\cot \theta) \overset{\text{def}}{=} \cot K \theta, \tag{S1}
\]
where \( K \in \mathbb{N} \setminus \{1\} \) and the set \( A \) is a set on \( \mathbb{R} \) such that for any point \( x \in A \), \( F_K(x) \) reaches a singular point.

**Definition 4.** The map \( G_K : \mathbb{R} \rightarrow \mathbb{R} \) is referred to as
\[
G_K(\coth \theta) \overset{\text{def}}{=} \coth K \theta, \tag{S2}
\]
where \( K \in \mathbb{N} \setminus \{1\} \).

**Definition 5** (Super Generalized Boole Transformation). Super Generalized Boole Transformation \( S_{K,\alpha} : \mathbb{R} \setminus B \rightarrow \mathbb{R} \setminus B \) is referred to as
\[
x_{n+1} = S_{K,\alpha}(x_n) \overset{\text{def}}{=} \alpha K F_K(x_n), \tag{S3}
\]
where \( \alpha > 0, K \in \mathbb{N} \setminus \{1\} \) and the set \( B \) is a set on \( \mathbb{R} \) such that for any point \( x \) on \( B \), there is an integer \( n \) where \( S_{K,\alpha}^n x \) reaches a singular point.

For example, \( S_{3,\alpha}, S_{4,\alpha} \) and \( S_{5,\alpha} \) are as follows.
\[
S_{3,\alpha}(x_n) = 3\alpha \frac{x_n^3 - 3x_n}{3x_n^2 - 1}, \tag{S4}
\]
\[
S_{4,\alpha}(x_n) = 4\alpha \frac{x_n^4 - 6x_n^2 + 1}{4x_n^3 - 4x_n}, \tag{S5}
\]
\[
S_{5,\alpha}(x_n) = 5\alpha \frac{x_n^5 - 10x_n^3 + 5x_n}{5x_n^4 - 10x_n^2 + 1}. \tag{S6}
\]

The derivative of \( S_{K,\alpha} \) with respect to \( x \) is denoted as
\[
S'_{2N,\alpha}(x) = (2N)^2 \alpha (1 + x^2)^{2N-1} \left[ \sum_{r=0}^{N-1} (-1)^r 2N C_{2r+1} x^{2N-2r-1} \right]^2 > 0,
\]
\[
S'_{2N+1,\alpha}(x) = (2N+1)^2 \alpha (1 + x^2)^{2N} \left[ \sum_{r=0}^{N} (-1)^r 2N+1 C_{2r+1} x^{2(N-r)} \right]^2 > 0.
\]

**Existence of the invariant density function**

In this section, we prove that

- when \( K = 2N, N \in \mathbb{N} \) and \( 0 < \alpha < 1 \), \( \Rightarrow \) the map \( S_{2N,\alpha} \) preserves the Cauchy distribution and it can be determined uniquely, and
• when $K = 2N + 1, N \in \mathbb{N}$ and $\frac{1}{(2N+1)^2} < \alpha < 1$, $\implies$ the map $S_{2N+1,\alpha}$ preserves the Cauchy distribution and it can be determined uniquely.

According to \[1\], the map $S_{K,\alpha}$ is one to one map as follows.

$y = K\alpha \cot K\theta = K\alpha F_K(x_j), j = 1, 2, \cdots , K,$

$x_j = \cot \left( \theta + j \frac{\pi}{K} \right), j = 1, 2, \cdots , K.$

If variables $\{x_j\}$ obey the Cauchy distribution

$f(x) = \frac{1}{\pi x^2 + \gamma^2},$

then according to \[1\] it holds that

$p(y) = 1 \frac{\alpha KG_K(\gamma)}{\pi \alpha^2 K^2 G^2_K(\gamma) + y^2}.$ (S7)

Then the scale parameter $\gamma$ is transformed in one iteration as

$\gamma \mapsto \alpha KG_K(\gamma).$ (S8)

Now, for each $K$ let us obtain the fixed point $\gamma_{K,\alpha}$ which satisfies

$\gamma_{K,\alpha} = \alpha KG_K(\gamma_{K,\alpha}).$ (S9)

If we discover a real and positive solution of (S9), it is the evidence that the map $S_{K,\alpha}$ preserves the Cauchy distribution.

The map $G_K(x)$ is denoted as

$G_{2N}(x) = \sum_{k=0}^{n} 2N C_{2k} x^{2(N-k)},$ (S10)

$G_{2N+1}(x) = \sum_{k=0}^{N} 2N + 1 C_{2k} x^{2(N-k)+1}.$ (S11)

Since at $\alpha = \frac{1}{K}$, the SGB transformation $S_{K,\alpha}$ is equivalent to $K$-angle formula of cot function, it is obvious that the fixed point of scaling parameter $\gamma_{K,\alpha}$ is unity by simple calculation. In the following, discuss the case that $\alpha \neq \frac{1}{K}$. Such lemmas hold.

**Lemma 6.** For $\frac{1}{K} < \alpha < 1$, fix $\alpha$ and there is only one solution which satisfies (S9) in the range of $\gamma_{K,\alpha} > 1$ and for $0 < \alpha < \frac{1}{K}$ there is no solution in the range of $\gamma_{K,\alpha} > 1$.

**Proof.** In (S9), change the variable from $\gamma_{K,\alpha} > 1$ into $\coth y$. One has that

$\coth y = \alpha K \coth(Ky).$ (S12)

A function $f(y)$ is defined to be

$f(y) = \coth y - \alpha K \coth(Ky).$ (S13)

In the range where $y \geq 0$, since a function $\coth y$ decreases monotonically, it holds that

$\coth y > \coth(Ky).$ (S14)
Then, if the condition \( \frac{1}{K} < \alpha < 1 \) is satisfied, one has that
\[
f'(y) = (1 - \coth^2 y) - \alpha K^2 \{1 - \coth^2(Ky)\} < 0 \tag{S15}
\]
Thus, the function \( f(y) \) decreases monotonically. Let us discuss the value at \( y = 0 \). In the limit of \( y \to +0 \), it holds that from (S11),
\[
\lim_{y \to +0} \coth\{Ky\} = \begin{cases} 
\lim_{y \to +0} \frac{2N C_0 \coth^{2N} y}{2N C_1 \coth^{2N} y} = \frac{1}{2N}, K = 2N, \\
\lim_{y \to +0} \frac{2N+1 C_0 \coth^{2N} y}{2N+1 C_1 \coth^{2N} y} = \frac{1}{2N+1}, K = 2N + 1.
\end{cases}
\]
Thus, one has that
\[
\lim_{y \to +0} f(y) = \begin{cases} 
\lim_{y \to +0} \coth y \left(1 - \frac{\alpha K}{2N}\right) = +\infty, K = 2N, \\
\lim_{y \to +0} \coth y \left(1 - \frac{\alpha K}{2N+1}\right) = +\infty, K = 2N + 1
\end{cases}
\tag{S16}
\]
One also has that
\[
\lim_{y \to \infty} f(y) = 1 - \alpha K < 0. \tag{S17}
\]
Thus, from (S15), (S16) and (S17), it is proven that there is only one solution that satisfies \( f(y) = 0 \). Therefore, for \( \frac{1}{K} < \alpha < 1 \), there is a solution which satisfies (S9) in the range of \( \gamma_{K,\alpha} > 1 \).

In the case of \( 0 < \alpha < \frac{1}{K} \), it holds that for any \( y > 0 \)
\[
f(y) > 0. \tag{S18}
\]
Then, there is no solution which satisfies \( \gamma_{K,\alpha} > 1 \).

**Lemma 7.** In the case of \( K = 2N \) for \( 0 < \alpha < \frac{1}{K} \), fix \( \alpha \). There is only one solution which satisfies (S9) in the range of \( 0 < \gamma_{K,\alpha} < 1 \) and for \( \frac{1}{K} < \alpha \) there is no solution in the range of \( 0 < \gamma_{K,\alpha} < 1 \).

*Proof.* In (S9), change the variable from \( 0 < \gamma_{K,\alpha} < 1 \) into \( \tanh y \). One has that
\[
\tanh y = \frac{\alpha(2N)}{\tanh(2Ny)}. \tag{S19}
\]
A function \( h_{2N}(y) \) is defined to be
\[
h_{2N}(y) = \tanh y - \frac{\alpha(2N)}{\tanh(2Ny)}. \tag{S20}
\]
The derivative of \( h_{2N}(y) \) is denoted as
\[
h'_{2N}(y) = 1 - \tanh^2 y + \alpha(2N)^2 \frac{1 - \tanh^2(2Ny)}{\tanh^2(2Ny)} > 0. \tag{S21}
\]
One has that
\[
h_{2N}(0) = -\infty < 0, \quad h_{2N}(\infty) = 1 - \alpha K > 0, \text{ for } 0 < \alpha < \frac{1}{K}. \tag{S22}
\]
From (S21) and (S22), one sees that in the case of \( K = 2N \) for fixed \( \alpha \) which satisfied \( 0 < \alpha < \frac{1}{K} \), there is only one solution which satisfies (S9) in the range of \( 0 < \gamma_{K,\alpha} < 1 \).

In the case of \( \frac{1}{2N} < \alpha \), it holds that for all \( y > 0 \)
\[
h_{2N}(y) < 0. \tag{S23}
\]
Therefore, there is no solution which satisfies \( 0 < \gamma_{K,\alpha} < 1 \). \qed
Lemma 8. In the case of $K = 2N + 1$ for $\frac{1}{K} < \alpha < \frac{1}{K}$, fix $\alpha$. There is only one solution which satisfies (S9) in the range of $0 < \gamma_{K,\alpha} < 1$ and for $\frac{1}{K} < \alpha$, there is no solution in the range of $0 < \gamma_{K,\alpha} < 1$.

Proof. In (S9), change the variable from $0 < \gamma_{K,\alpha} < 1$ into $\tanh y$. One has that

$$\tanh y = \alpha(2N + 1)^2 \tanh \{(2N + 1)y\}. \tag{S24}$$

A function $h_{2N+1}(y)$ is defined to be

$$h_{2N+1}(y) = \tanh y - \alpha(2N + 1) \tanh \{(2N + 1)y\}. \tag{S25}$$

It holds that

$$h_{2N+1}(0) = 0, \quad h_{2N+1}(\infty) = 1 - \alpha(2N + 1) > 0, \text{ for } \frac{1}{(2N+1)^2} < \alpha < \frac{1}{2N+1}. \tag{S26}$$

The derivative of $h_{2N+1}(y)$ is

$$h'_{2N+1}(y) = 1 - \alpha(2N + 1)^2 + \alpha(2N + 1)^2 \tanh^2 \{(2N + 1)y\} - \tanh^2 y, \quad h'_{2N+1}(0) = 1 - \alpha(2N + 1)^2 < 0, \text{ for } \frac{1}{(2N+1)^2} < \alpha < \frac{1}{2N+1}. \tag{S27}$$

The derivative $h'_{2N+1}(y)$ is also expressed using sinh functions as follows

$$h'_{2N+1}(y) = \frac{\sinh^2 \{(2N + 1)y\} - n [\sinh \{(2N + 2)y\} - \sinh(2Ny)]}{\sinh^2 \{(2N + 1)y\}}. \tag{S28}$$

The function $J(y) = \frac{\sinh y}{\sinh \{(2N + 1)y\}}$ decreases monotonously since for the derivative

$$J'(y) = -\sinh(2Ny) - n [\sinh \{(2N + 2)y\} - \sinh(2Ny)], \tag{S29}$$

the numerator is denoted as

$$n \sinh \{(2N + 2)y\} \left[ \frac{n - 1}{n} \frac{\sinh(2Ny)}{\sinh(2N + 2)y} - 1 \right] < 0. \tag{S30}$$

Then $J'(y) < 0$. Considering the fact that

$$\lim_{y \to \infty} \frac{\sinh(y)}{\sinh(2N + 1)y} = 0, \tag{S31}$$

a part of $h'_{2N+1}(y), \left[ 1 - \alpha(2N + 1) \frac{\sinh^2 y}{\sinh^2 \{(2N + 1)y\}} \right]$ increases monotonously and there is a unique point $y_*$ at which the sign of $h'_{2N+1}(y)$ changes from minus to plus. Therefore, there is a unique point $0 < y_* < \infty$ at which it holds that $h_{2N+1}(y_*) = 0$.

From above discussion, one sees that in the case of $K = 2N + 1$ and $\frac{1}{K} < \alpha < \frac{1}{2N+1}$, there is only one solution which satisfies (S9) in the range of $0 < \gamma_{K,\alpha} < 1$.

In the case of $\frac{1}{K} < \alpha$, since it holds that for all $y > 0$,

$$h_{2N+1}(y) < 0, \tag{S32}$$

there is no solution in the range of $0 < \gamma_{K,\alpha} < 1$.

From the Lemmas 4, 5 and 6, such lemmas hold.

Lemma 9. Consider the case of $K = 2N$.

For $\frac{1}{K} \leq \alpha < 1$, there is a unique solution of (S9) and the solution $\gamma_{K,\alpha}$ is in the range of $\gamma_{K,\alpha} \geq 1$. For $0 < \alpha < \frac{1}{K}$, there is a unique solution of (S9) and the solution $\gamma_{K,\alpha}$ is in the range of $0 < \gamma_{K,\alpha} < 1$.

Lemma 10. Consider the case of $K = 2N + 1$.

For $\frac{1}{K} \leq \alpha < 1$, there is a unique solution of (S9) and the solution $\gamma_{K,\alpha}$ is in the range of $\gamma_{K,\alpha} \geq 1$. For $\frac{1}{K} < \alpha < \frac{1}{2N+1}$, there is a unique solution of (S9) and the solution $\gamma_{K,\alpha}$ is in the range of $0 < \gamma_{K,\alpha} < 1$. 

\hfill \Box
The Condition A is defined as follows

**Definition 11.** Condition A is referred to as

\[
\text{Condition A} : \begin{cases} 
\text{in the case of } K = 2N, \quad \alpha \text{ satisfies } 0 < \alpha < 1 \quad \text{and} \\
\text{in the case of } K = 2N + 1, \quad \alpha \text{ satisfies } \frac{1}{K^2} < \alpha < 1.
\end{cases}
\] (S33)

From Lemmas 9 and 10, such theorem holds.

**Theorem A.** When the Condition A is satisfied, the SGB transformations \(\{S_{K,\alpha}\}\) preserve the Cauchy distribution and the scale parameter can be chosen uniquely.

### Exactness

According to [2–4], the exactness is defined as follows.

**Definition 10 (Exactness).** A dynamics \(T\) on a phase space \(\mathcal{X}\) with transfer operator \(P_T\) and unique stationary density \(f_*\) is called to be exact if and only if

\[
\lim_{n \to \infty} \|P_n^T f - f_*\|_{L^1} = 0
\] (S34)

for every initial density \(f \in \mathcal{D}\) where \(\mathcal{D}\) denotes all densities on \(\mathcal{X}\).

This definition is equivalent to as follows,

\[
\lim_{n \to \infty} \mu_*(\mathcal{B}) = 1, \quad \forall s \in \mathcal{B}, \quad \mu_*(s) > 0.
\] (S35)

where \(\mathcal{B}\) denotes the \(\sigma\)-algebra and \(\mu_*\) denotes the invariant measure corresponding to the invariant density \(f_*\).

**Theorem 11.** If the the Condition A is satisfied, the SGB transformations \(\{S_{K,\alpha}\}\) are exact.

**Proof.** This proof is based on [5]. For the map \(S_{K,\alpha}\) defined by (S3), substituting \(\cot(\pi \theta_n)\) into \(x_n\), one has the map \(\bar{S}_{K,\alpha} : [0,1) \to [0,1)\) such that

\[
\cot(\pi \theta_{n+1}) = \alpha K \cot(\pi K \theta_n),
\theta_{n+1} = \bar{S}_{K,\alpha}(\theta_n) = \frac{1}{\pi} \cot^{-1} \{ \alpha K \cot(\pi K \theta_n) \}. \] (S36)

The derivative of \(\bar{S}_{K,\alpha}\) with respect to \(\theta\) is as follows.

\[
\bar{S}'_{K,\alpha}(\theta) = \frac{\alpha K^2 \{ 1 + \cot^2(\pi K \theta) \}}{\alpha^2 K^2 \cot^2(\pi K \theta) + 1} > 0 \quad \text{for } 0 < \alpha < 1.
\] (S37)

Then, \(\bar{S}_{K,\alpha}\) increases monotonously. Since it holds that

\[
\frac{1}{\pi} \cot^{-1} \{ \alpha K \cot(\pi K \theta_n) \} = \frac{1}{\pi} \cot^{-1} \left[ \alpha K \cot \left( \frac{\pi K \left( \theta_n + \frac{j}{K} \right)}{2} \right) \right], \quad j = 1, 2, \ldots, K - 1,
\] (S38)

form of \(\bar{S}_{K,\alpha}\) has the translational symmetry and it can be constructed by shifting the form on \([0, \frac{1}{K})\). That is, the map \(\bar{S}_{K,\alpha}\) is also \(K\) points to one points map and on any interval \(I_{j,1}\), the form of the map \(\bar{S}_{K,\alpha}\) is the same as that on the interval \(I_{0,1}\). Then by operating \(\tilde{\bar{S}}_{K,\alpha}^{-1}\), the measure on \([0,1)\) is divided into \(K\) equivalently. We obtain intervals \(\{I_{j,n}\}\) defined below by operating \(\bar{S}_{K,\alpha}^{-n}\) into \([0,1)\). The interval \(I_{j,n} \subset [0,1)\) is defined to be

\[
\begin{align*}
I_{j,n} & \quad \overset{\text{def}}{=} \left[ \eta_{j,n}, \eta_{j+1,n} \right), \quad \eta_{j,n} < \eta_{j+1,n}, \quad 0 \leq j \leq K^n - 1, \\
\eta_{0,n} & = 0 \quad \text{and} \quad \eta_{K^n,n} = 1, \\
\tilde{\bar{S}}_{K,\alpha}^{-n}(I_{j,n}) & = [0,1), \\
\mu(I_{j,n}) & = \frac{1}{K^n}.
\end{align*}
\] (S39)
For any non zero measure subset $C \subset [0, 1)$, the set $C$ includes cylinder sets $\bigcup_{j, n'} I_{j, n'}$. Then for an invariant density $f_*$ and associated measure $\mu_*$, it holds that

$$1 \geq \lim_{n \to \infty} \mu_*(\bar{S}_{K, \alpha}^n(C)) \geq \lim_{n \to \infty} \mu_* \left( \bar{S}_{K, \alpha}^n \left( \bigcup_{j, n'} I_{j, n'} \right) \right) = 1.$$  \tag{S40}

Therefore, the map $\bar{S}_{K, \alpha}$ on a phase space $[0, 1)$, is exact. Owing to the topological conjugacy, the map $S_{K, \alpha}$ is also exact. \hfill \Box

**Scaling behavior**

At first, discuss the case of $\frac{1}{K} < \alpha < 1 (\gamma_{K, \alpha} > 1)$.

**Lemma 12.** In the case of $K = 2N$, $\gamma_{2N, \alpha}$ behaves as

$$\frac{1}{\gamma_{2N, \alpha}} \sim O(\sqrt{1 - \alpha}).$$  \tag{S41}

in the limit of $\gamma_{2N, \alpha} \to \infty$.

**Proof.** In the case of $K = 2N$, (S9) is rewritten as

$$\alpha = \frac{\text{coth} y}{2N \text{coth}(2Ny)}.$$  \tag{S42}

Then, one has that

$$1 - \alpha = \frac{1}{2N} \sum_{k=0}^{n} 2N C_{2k} \gamma_{2N, \alpha}^{2N-2k} - \frac{1}{2N} \sum_{k=0}^{n-1} 2N C_{2k+1} \gamma_{2N, \alpha}^{2N-2k-2} \sum_{k=0}^{n-1} 2N C_{2k+1} \gamma_{2N, \alpha}^{2N-2k-1} \sum_{k=0}^{n} 2N C_{2k} \gamma_{2N, \alpha}^{2N-2k}$$

$$= \frac{1}{2N} \sum_{k=0}^{n} 2N C_{2k} \gamma_{2N, \alpha}^{-2k} - \frac{1}{2N} \sum_{k=0}^{n-1} 2N C_{2k+1} \gamma_{2N, \alpha}^{-2k-2} \sum_{k=0}^{n} 2N C_{2k} \gamma_{2N, \alpha}^{-2k}.$$  \tag{S43}

In the limit of $\gamma_{2N, \alpha} \to \infty$, it holds that

$$1 - \alpha \sim \frac{1}{2N} \left( 2N \cdot 2N C_{2N} \gamma_{2N, \alpha}^{-2} - 2N C_{3} \gamma_{2N, \alpha}^{-2} \right),$$  \tag{S44}

$$\therefore \frac{1}{\gamma_{2N, \alpha}} \sim \sqrt{1 - \alpha}.$$ \hfill \Box

**Lemma 13.** In the case of $K = 2N + 1$, $\gamma_{2N+1, \alpha}$ behaves as

$$\frac{1}{\gamma_{2N+1, \alpha}} \sim O(\sqrt{1 - \alpha}).$$  \tag{S45}

in the limit of $\gamma_{2N+1, \alpha} \to \infty$.

**Proof.** In the case of $K = 2N + 1$, (S9) is rewritten as

$$\alpha = \frac{\text{coth} y}{(2N + 1) \text{coth} \{(2N + 1)y\}}.$$  \tag{S46}
Then one has that
\[
1 - \alpha = \frac{1}{2N + 1} \left( \sum_{k=0}^{n} 2N+1C_{2k}\gamma^{-2k}_{2N+1,\alpha} - \sum_{k=0}^{n} 2N+1C_{2k+1}\gamma^{-2k}_{2N+1,\alpha} \right),
\]  
(S47)

In the limit of \( \gamma_{2N+1,\alpha} \to \infty \), it holds that
\[
1 - \alpha \sim \frac{1}{2N + 1} \left\{ (2N + 1) \cdot 2N+1C_{2}\gamma^{-2}_{2N+1,\alpha} - 2N+1C_{3}\gamma^{-2}_{2N+1,\alpha} \right\},
\]  
(S48)

\[
\therefore \quad \frac{1}{\gamma_{2N+1,\alpha}} \sim O(\sqrt{1 - \alpha}).
\]

\[\square\]

**Lemma 14.** In the case of \( K = 2N \), \( \gamma_{2N,\alpha} \) behaves as
\[
\gamma_{2N,\alpha} \sim O(\sqrt{\alpha}).
\]  
(S49)

in the limit of \( \gamma_{2N,\alpha} \to 0 \).

**Proof.** Discuss the case of \( K = 2N \) and \( 0 < \alpha < \frac{1}{K} \). (S9) is rewritten as
\[
\alpha = \frac{1}{2N} \tanh y \cdot \tanh(2Ny)
\]  
(S50)

Then one has that
\[
\alpha = \frac{\sum_{k=0}^{N-1} 2N+C_{2k+1}\gamma^{2k+2}_{2N,\alpha}}{\sum_{k=0}^{N} 2N+C_{2k}\gamma^{2k}_{2N,\alpha}}
\]  
(S51)

In the limit of \( \gamma_{2N,\alpha} \to 0 \), it holds that
\[
\alpha \sim \frac{1}{2N} \cdot 2N\gamma^{2}_{2N,\alpha} = \gamma^{2}_{2N,\alpha},
\]  
(S52)

\[\square\]

**Lemma 15.** In the case of \( K = 2N + 1 \), \( \gamma_{2N+1,\alpha} \) behaves as
\[
\gamma_{2N+1,\alpha} \sim O\left( \sqrt{\frac{1}{(2N + 1)^2}} \right)
\]  
(S53)

in the limit of \( \gamma_{2N+1,\alpha} \to 0 \).

**Proof.** Discuss the case of \( K = 2N + 1 \) and \( \frac{1}{K+1} < \alpha < \frac{1}{K} \). (S9) is rewritten as
\[
\alpha = \frac{\tanh y}{(2N + 1) \tanh \{(2N + 1)y\}}
\]  
(S54)
Then one has that,

\[
\alpha - \frac{1}{(2N+1)^2} = \frac{(2N+1)\tanh y \tanh \{2(N+1)y\}}{(2N+1)^2 \tanh \{2(N+1)y\}},
\]

\[
= \frac{1}{N} \sum_{k=0}^{N} \{(2N+1)_{2N+1} C_{2k} - 2N+1 C_{2k+1}\} \gamma_{2N+1, \alpha}^{2k},
\]

\[
= \frac{1}{N} \sum_{k=0}^{N} 2N+1 C_{2k+1} \gamma_{2N+1, \alpha}^{2k},
\]

(S55)

In the limit of \(\gamma_{2N+1, \alpha} \to 0\), it holds that

\[
\alpha - \frac{1}{(2N+1)^2} \sim \frac{\{(2N+1)_{2N+1} C_{2} - 2N+1 C_{3}\} \gamma_{2N+1, \alpha}^{2}}{2N+1},
\]

\[
\therefore \gamma_{2N, \alpha} \sim O \left( \sqrt{\alpha - \frac{1}{(2N+1)^2}} \right).
\]

(S56)

\[\Box\]

From Lemma 12, 13 and 14, one knows that there are relations between the parameter \(\alpha\) and the scaling parameter \(\gamma_{K, \alpha}\). For all \(\alpha\) which satisfied the Condition A, Lyapunov exponent is denoted as

\[
\lambda_{K, \alpha} = \frac{1}{\pi} \int_{R \setminus \{1\}} \log |S_{K, \alpha}(x)| \frac{\gamma_{K, \alpha}}{\gamma_{K, \alpha} + x^2} \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \log |S_{K, \alpha}(x)| \frac{\gamma_{K, \alpha}}{\gamma_{K, \alpha} + x^2} \, dx < \infty,
\]

(S57)

\[
= \begin{cases} 
\frac{\gamma_{K, \alpha}}{\pi} \int_{-\infty}^{\infty} \log |S_{K, \alpha}(x)| \frac{1}{\gamma_{K, \alpha} + x^2} \, dx, & \text{for } 0 < \gamma_{K, \alpha} < 1, \\
\frac{1}{\gamma_{K, \alpha}} \int_{-\infty}^{\infty} \log |S_{K, \alpha}(x)| \frac{1}{1 + (x/\gamma_{K, \alpha})^2} \, dx, & \text{for } 1 < \gamma_{K, \alpha}.
\end{cases}
\]

(S58)

Define a function \(f_1(\theta, \gamma_{K, \alpha})\) to be

\[
f_1(\theta, \gamma_{K, \alpha}) = \log \left| \frac{\alpha K^2 \sin^2 \theta}{\sin^2 K \theta} \right| \frac{\gamma_{K, \alpha}}{\gamma_{K, \alpha} \sin^2 \theta + \cos^2 \theta},
\]

(S59)

and also define a set of points \(\{a_n\}_{n=1}^{K-1}\) such that at \(x = a_n \in (0, \pi]\), the function \(\log \left| \frac{\alpha K^2 \sin^2 \theta}{\sin^2 K \theta} \right| \frac{\gamma_{K, \alpha}}{\gamma_{K, \alpha} \sin^2 \theta + \cos^2 \theta}\) is not continuous. By changing variable from \(x\) to \(\cot \theta\), Lyapunov exponent \(\lambda_{K, \alpha}\) is rewritten as

\[
\lambda_{K, \alpha} = \frac{1}{\pi} \int_{0}^{\pi} f_1(\theta, \gamma_{K, \alpha}) \, d\theta.
\]

(S60)

**Theorem C.** Suppose that the Condition A is satisfied.

- For any \(K \in \mathbb{N}\setminus\{1\}\), it holds that \(\nu_1 = \frac{1}{2}\) as \(\alpha \to 1 - 0\).
- For any \(K \in \mathbb{N}\setminus\{1\}\), it holds that \(\nu_1 = 1\) as \(\alpha \to 1 + 0\).
- For \(K = 2N + 1\), it holds that \(\nu_2 = \frac{1}{2}\) as \(\alpha \to \frac{1}{\pi} + 0\).

**Proof.** The integrand in (S60) are continuous in \((a_n, a_{n+1})\) for \(0 \leq n \leq K\) where \(a_0 = 0\) and \(a_K = \pi\). The derivative of \(f_1(\theta, \gamma_{K, \alpha})\) with respect to \(\gamma_{K, \alpha}\) is as follows.

\[
\frac{\partial f_1}{\partial \gamma_{K, \alpha}} = \frac{1}{\alpha} \frac{\partial \alpha}{\alpha} \frac{\gamma_{K, \alpha}}{\gamma_{K, \alpha} \sin^2 \theta + \cos^2 \theta} + \log \left| \frac{-\gamma_{K, \alpha}^2 \sin^2 \theta + \cos^2 \theta}{(\gamma_{K, \alpha} \sin^2 \theta + \cos^2 \theta)^2} \right| + \log \left( \frac{K^2 \sin^2 \theta}{\sin^2 K \theta} \right) \frac{\gamma_{K, \alpha}^2 \sin^2 \theta + \cos^2 \theta}{(\gamma_{K, \alpha} \sin^2 \theta + \cos^2 \theta)^2},
\]

(S61)

The derivative is continuous on each interval \((a_n, a_{n+1})\).
(i) In the limit of $\alpha \to 1 - 0$ ($\gamma_{K, \alpha} \to \infty$) for any $K$, change variable as $z = \frac{1}{\gamma_{K, \alpha}}$ and define a function $f_2(\theta, z)$ as

$$f_2(\theta, z) = \log \left| \frac{\alpha K^2 \sin^2 \theta}{\sin^2 K\theta} \right| \frac{z}{\sin^2 \theta + z^2 \cos^2 \theta}. \quad (S62)$$

It holds that

$$\frac{\partial f_2}{\partial z}(\theta, 0) \sim \log \left| \frac{K^2 \sin^2 \theta}{\sin^2 K\theta} \right| \frac{1}{\sin^2 \theta}, \quad (S63)$$

and it dose not depend on $z$. Thus, one has that in the limit of $z \to +0$ ($\gamma_{K, \alpha} \to \infty$, $\alpha \to 1 - 0$),

$$\lambda_{K, \alpha} = \lambda_K(z) \sim \lambda(0) + \left[ \frac{1}{\pi} \int_0^\pi \log \left| \frac{K^2 \sin^2 \theta}{\sin^2 K\theta} \right| \frac{1}{\sin^2 \theta} \, d\theta \right] z + O(z^2). \quad (S64)$$

where $\lambda_K(z = 0) = 0$. Therefore it holds that

$$\therefore \lambda_{K, \alpha} \sim O(z) \sim O \left( \sqrt{1 - \alpha} \right). \quad (S65)$$

(ii) In the case of $K = 2N + 1$, there is a relation as $\alpha \sim \gamma_{K, \alpha}^2 + \frac{1}{K^2}$ in the limit of $\gamma_{K, \alpha} \to +0$ ($\alpha \to \frac{1}{K^2} + 0$). Then it holds that

$$\frac{\partial f_1}{\partial \gamma_{K, \alpha}}(\theta, 0) \sim \log \left| \frac{\sin^2 \theta}{\sin^2 K\theta} \right| \frac{1}{\cos^2 \theta}, \quad K = 2N + 1 \quad (S66)$$

and it dose not depend on $\gamma_{K, \alpha}$. Thus by a Taylor expansion with respect to $\gamma_{K, \alpha}$, one has that in the limit of $\gamma_{K, \alpha} \to 0$,

$$\lambda_{K, \alpha} = \lambda_K(\gamma_{K, \alpha}) \sim \lambda_K(0) + \left[ \frac{1}{\pi} \int_0^\pi \left\{ \log \left| \frac{\sin^2 \theta}{\sin^2 K\theta} \right| \frac{1}{\cos^2 \theta} \right\} \, d\theta \right] \gamma_{K, \alpha} + O(\gamma_{K, \alpha}^2), \quad K = 2N + 1. \quad (S67)$$

where $\lambda_K(\gamma_{K, \alpha} = 0) = 0$. Therefore it holds that

$$\lambda_{K, \alpha} \sim O(\gamma_{K, \alpha}) \sim O \left( \sqrt{\alpha - \frac{1}{K^2}} \right), \quad K = 2N + 1. \quad (S68)$$

in the limit of $\alpha \to \frac{1}{K^2} + 0$.

(iii) In the limit of $\alpha \to 1 + 0$ for any $K$, Lyapunov exponent is denoted as for $\alpha > 1$,

$$\lambda_{K, \alpha} = \log \alpha = \log \{1 + (\alpha - 1)\}, \quad (S69)$$

$$= (\alpha - 1) - \frac{1}{2}(\alpha - 1)^2 + \frac{1}{3}(\alpha - 1)^3 - \cdots. \quad (S69)$$

Therefore, it holds that

$$\lambda_{K, \alpha} \sim O(\alpha - 1). \quad (S70)$$

From equations $(S68)$, $(S65)$ and $(S70)$, one sees that for any $K$, the critical exponent of Lyapunov exponent $\nu_1$ is $\frac{1}{2}$ in the limit of $\alpha \to 1 - 0$, that for $K = 2N + 1$, $\nu_2 = \frac{1}{2}$ in the limit of $\alpha \to \frac{1}{K^2} + 0$ and that for any $K$, $\nu_1 = 1$ in the limit of $\alpha \to 1 + 0$. \(\square\)

From above discussion, it is proven that the derivatives of Lyapunov exponent with respect to parameter $\alpha$ diverge at critical points, which means that the parameter dependence of Lyapunov exponent diverges at critical points. This result implies that difficulty of calculating Lyapunov exponent near the critical points.
Scaling behavior for $K = 3, 4$ and 5

The solutions of (S9) which satisfied $0 < \gamma_{K,\alpha} < \infty$ in the cases of $K = 3, 4$ and 5 are determined uniquely when the Condition A is satisfied as follows.

\begin{align*}
\gamma_{3,\alpha} &= \sqrt{\frac{9\alpha - 1}{3 - 3\alpha}}, \\
\gamma_{4,\alpha} &= \sqrt{\frac{6\alpha - 1 + \sqrt{32\alpha^2 - 8\alpha + 1}}{2(1 - \alpha)}}, \\
\gamma_{5,\alpha} &= \sqrt{\frac{-5(1 - 5\alpha) + \sqrt{20(25\alpha^2 - 6\alpha + 1)}}{5(1 - \alpha)}}.
\end{align*}

From (S57), one has analytical formulae of Lyapunov exponent as follows.

\begin{align*}
\lambda_{3,\alpha} &= \log \left| \frac{1}{\alpha} \left( \frac{3(1 - \alpha)}{8} \right)^2 \left[ 1 + \sqrt{\frac{9\alpha - 1}{3 - 3\alpha}} \right] \right|, \\
\lambda_{4,\alpha} &= \log \left| \frac{\alpha(1 + \gamma_4)^6}{\gamma_4^2(1 + \gamma_4)^2} \right|, \\
\lambda_{5,\alpha} &= \log \left| \frac{256(\sqrt{125\alpha^2 - 30\alpha + 5} + 11\alpha - 1)^2 |1 + \gamma_5|^8}{(1 - \alpha)^4} \right|,
\end{align*}

(S71)

In the case of $K = 3$, equation (S72) converges to zero in the limit of $\alpha \to \frac{1}{6} + 0$ and $\alpha \to 1 - 0$, and the derivative of Lyapunov exponent $\frac{\partial \lambda_{3,\alpha}}{\partial \alpha}$ diverges at $\alpha = \frac{1}{6}, 1$. When the parameter $\alpha$ is close to $\frac{1}{6}$, the Lyapunov exponent grows as follows.

\begin{align*}
\lambda_{3,\alpha} &= -\log \left| 1 + 9 \left( \alpha - \frac{1}{9} \right) \right| + 2\log \left| 1 - \frac{9}{8} \left( \alpha - \frac{1}{9} \right) \right| + 4\log \left| 1 + \sqrt{\frac{3(\alpha - \frac{1}{9})}{1 - \alpha}} \right|, \\
&\simeq -9 \left( \alpha - \frac{1}{9} \right) - \frac{9}{4} \left( \alpha - \frac{1}{9} \right) + 4\sqrt{\frac{3(\alpha - \frac{1}{9})}{1 - \alpha}}, \\
&\simeq 6\sqrt{\frac{3}{2}} \sqrt{\alpha - \frac{1}{9}}.
\end{align*}

(S72)

Then, the critical exponent $\nu_2$ of Lyapunov exponent at $\alpha = \frac{1}{6}$ is $\frac{1}{2}$. In the case of $\alpha \lesssim 1$, Lyapunov exponent $\lambda_{3,\alpha}$ behaves as follows.

\begin{align*}
\lambda_{3,\alpha} &= 2\log \left| 1 + \frac{9}{8} (\alpha - 1) \right| - \log |1 + (\alpha - 1)| + 4\log \left| 1 + \sqrt{\frac{1 - \alpha}{3(\alpha - \frac{1}{6})}} \right|, \\
&\simeq -\frac{9}{4}(\alpha - 1) + (\alpha - 1) + 4\sqrt{\frac{1 - \alpha}{3(\alpha - \frac{1}{6})}}, \\
&\simeq 2\sqrt{\frac{3}{2}} \sqrt{1 - \alpha}.
\end{align*}

(S73)

Then, at $\alpha = 1$ one has $\nu_1 = \frac{1}{2}$.

For $K = 3$ and for $\frac{1}{6} < \alpha < 1$, the fixed point is only

\begin{equation}
\alpha^* = 0.
\end{equation}

(S74)

Then Floquet multiplier $\chi$ at $\alpha = \frac{1}{6}$ and $\alpha = 1$ are denoted as follows.

\begin{align*}
\chi_{3,\frac{1}{6}} &= S_{3,\frac{1}{6}}(0) = 1, \\
\chi_{3,1} &= S_{3,1}(0) = 1.
\end{align*}

(S75)
From these results, we can say that only Type 1 intermittency occurs for $K = 3$. These results are new phenomena since for the Generalized Boole transformation, one can observe two different intermittent type, Type 1 and Type 3 [5].

In the case of $K = 4$, Lyapunov exponent (S72) is denoted as

$$
\lambda_{4,\alpha} = \log \left| \frac{\alpha}{\gamma_4^2} \right| + 6 \log |1 + \gamma_4| - 2 \log |1 + \gamma_4^2|. \tag{S77}
$$

Let us discuss the scaling behavior of $\lambda_{4,\alpha}$ at $\alpha = 0$. Now the first term of (S77) is rewritten as

$$
\log \left| \frac{\alpha}{\gamma_4^2} \right| = \log \left| \frac{2\alpha(1-\alpha)}{6\alpha - 1 + \sqrt{32\alpha^2 - 8\alpha + 1}} \right| = \log \left| \frac{(1-\alpha)\left\{\sqrt{32\alpha^2 - 8\alpha + 1} - 6\alpha + 1\right\}}{-2\alpha + 2} \right|, \tag{S78}
$$

$$
\therefore \lim_{\alpha \to 0} \log \left| \frac{\alpha}{\gamma_4^2} \right| = \frac{1 \times (\sqrt{1} - 0 + 1)}{2} = 1. \tag{S79}
$$

Then,

$$
\gamma_4 \simeq \sqrt{-1 + 1 + \frac{1}{2}(32\alpha^2 - 8\alpha) + 6\alpha} \simeq \sqrt{2}\alpha,
$$

$$
\Rightarrow \log |1 + \gamma_4| \simeq \sqrt{2}\alpha \text{ and } \log |1 + \gamma_4^2| \simeq 2\alpha,
$$

$$
\therefore \lambda_{4,\alpha} \simeq 2\sqrt{2} \alpha. \tag{S80}
$$

Therefore, the critical exponent of Lyapunov exponent for $K = 4, \alpha = 0$ is

$$
\nu_3 = \frac{1}{2}. \tag{S81}
$$

Next, consider the scaling behavior of $\lambda_{4,\alpha}$ at $\alpha = 1 - 0$. From (S78), it holds that near $\alpha = 1$,

$$
\log \left| \frac{\alpha}{\gamma_4^2} \right| = \log \left| \frac{(1-\alpha)\left\{\sqrt{32\alpha^2 - 8\alpha + 1} - 6\alpha + 1\right\}}{-2\alpha + 2} \right| = \log \left| \frac{5\sqrt{32(\alpha - 1)^2 + 56(\alpha - 1) + 25} - 6(\alpha - 1) - 5}{2} \right|, 

\simeq \log \left| \frac{\frac{16}{5}(\alpha - 1)^2 - \frac{2}{5}(\alpha - 1)}{2} \right|, 

\simeq \log \left| \frac{1 - \alpha}{5} \right| \simeq \alpha. \tag{S82}
$$

The second and third terms of (S77) are denoted as

$$
\log \left| \frac{(1 + \gamma_4^6)^6}{(1 + \gamma_4^2)^2} \right| = 6 \log \left| \sqrt{2}(1-\alpha) + \sqrt{6\alpha - 1 + \sqrt{32\alpha^2 - 8\alpha + 1}} \right| - \log 2(1-\alpha) - 2 \log \left| 1 + 4\alpha + \sqrt{32\alpha^2 - 8\alpha + 1} \right|, 

\simeq \log 5 + 6 \log \left( 1 + \frac{1 - \alpha}{5} \right) - \log (1 - \alpha), 

\therefore \lambda_{4,\alpha} \simeq \frac{6\sqrt{5}}{5} \sqrt{1 - \alpha}, \text{ for } \alpha \simeq 1. \tag{S83}
$$

Therefore, the critical exponent of Lyapunov exponent for $K = 4, \alpha = 1 - 0$ is

$$
\nu_1 = \frac{1}{2}. \tag{S84}
$$
In the case of $K = 4$, fixed points of $S_{4,\alpha}$ are as follows.

$$x_{4,*} = \begin{cases} 
\pm \sqrt{\frac{1 - 6\alpha + \sqrt{40\alpha^2 - 16\alpha + 1}}{1 - \alpha}}, & \alpha = 0, \quad 0 < \alpha < 1, \\
\pm \frac{1}{\sqrt{\alpha}}, & \alpha = 1.
\end{cases} \quad (S85)$$

In order to obtain the Floquet multiplier at $\alpha = 0$, apply scale transformation such that $x = \sqrt{\alpha}y$. Then, one obtains following equations as

$$y_{n+1} = \tilde{S}_{4,\alpha}(y_n) = \frac{\alpha^2 y_n^4 - 6\alpha y_n^2 + 1}{\alpha y_n^2 - y_n},$$

$$\tilde{S}_{4,0}(y_n) = -\frac{1}{y_n}. \quad (S86)$$

Then $y = \pm i$ are the fixed points for $\tilde{S}_{4,0}$ and one has that

$$\tilde{S}'_{4,0}(y_n) = \frac{1}{y_n^2}, \quad \tilde{S}'_{4,0}(\pm i) = -1. \quad (S87)$$

Thus, the Floquet multiplier at $\alpha = 0$ is -1.

The values of $S'_{4,\alpha}$ at the other fixed points are as follows.

$$\lim_{\alpha \to +0} S'_{4,\alpha} \left( \pm \sqrt{\frac{1 - 6\alpha + \sqrt{40\alpha^2 - 16\alpha + 1}}{1 - \alpha}} \right) = 0, \quad (S88)$$

$$S'_{4,1} \left( \pm \frac{1}{\sqrt{\alpha}} \right) = \frac{27}{2}, \quad S'_{4,1}(\pm \infty) = 1.$$  

For $K = 4$, from (S87) and (S88), one obtains the Floquet multiplier at $\alpha = 0$ and $\alpha = 1$ as

$$\chi_{4,0} = -1, \quad \chi_{4,1} = 1. \quad (S89)$$

This result indicates that *Type 3* intermittency occurs at $\alpha = 0$ and *Type 1* intermittency occurs at $\alpha = 1$. These results are consistent with that of Generalized Boole transformation [5].

In the case of $K = 5$, equation (S72) converges to zero in the limit of $\alpha \to \frac{1}{25} + 0$ and $\alpha \to 1 - 0$. In addition, it holds that $\frac{\partial \lambda_{5,\alpha}}{\partial \alpha} \left( \frac{1}{25} \right) = \frac{\partial \lambda_{5,\alpha}}{\partial \alpha}(0) = \infty$. Lyapunov exponent $\lambda_{5,\alpha}$ for $\frac{1}{25} < \alpha < 1$ is divided as

$$\lambda_{5,\alpha} = \log \frac{25}{256} - \log |\alpha| + 4 \log |1 - \alpha| - 2 \log \left| 1 - 11\alpha - \sqrt{125\alpha^2 - 30\alpha + 5} \right| + 8 \log |1 + \gamma_{5}^*|. \quad (S90)$$

The forth term of (S90) at $\alpha = \frac{1}{25}$ is denoted as

$$\log \left| -1 - \frac{11}{25} \right| > \log |1| = 0 \quad (S91)$$

and denoted at $\alpha = 1$ as

$$\log | - 20 | > \log |1| = 0. \quad (S92)$$

Then the forth term is not dominant near $\alpha = \frac{1}{25}$ and $\alpha = 1$. Thus in considering the scaling behavior, we do not have to care the forth term.
When the parameter $\alpha$ is close to $\frac{1}{25}$, the Lyapunov exponent grows as follows

$$\lambda_{5,\alpha} \simeq \gamma_5 = \frac{-5(1-5\alpha) + \sqrt{20(25\alpha^2 - 6\alpha + 1)}}{5(1-\alpha)},$$  \hspace{1cm} (S93)

$$= \frac{4 \left[ \sqrt{1 + 20 \left( 25 (\alpha - \frac{1}{25})^2 - 4 (\alpha - \frac{1}{25}) \right)} + \frac{25}{4} (\alpha - \frac{1}{25}) - 1 \right]}{5(1-\alpha)},$$

$$\simeq \frac{185}{4} \sqrt{\alpha - \frac{1}{25}}. \hspace{1cm} (S94)$$

Therefore, the critical exponent $\nu_2$ is denoted as

$$\nu_2 = \frac{1}{2}. \hspace{1cm} (S95)$$

For $\alpha \lesssim 1$, Lyapunov exponent grows as follows.

$$\lambda_{5,\alpha} \simeq 4 \log \left| (1 - \alpha)(1 + \gamma_5^*)^2 \right|,$$

$$\simeq 4 \log \left| 8 + \sqrt{1 - \alpha} \right|,$$

$$\simeq \sqrt{1 - \alpha}. \hspace{1cm} (S96)$$

Therefore, the critical exponent $\nu_1$ is denoted as

$$\nu_1 = \frac{1}{2}. \hspace{1cm} (S97)$$

In the case of $K = 5$, fixed points of $S_{5,\alpha}$ are as follows.

$$x_* = \begin{cases} 0, & 0 < \alpha \leq 1, \\ \pm \sqrt{\frac{5}{3}}, & \alpha = \frac{1}{25}, \\ \pm \sqrt{\frac{5(1 - 5\alpha) + 2\sqrt{5(25\alpha^2 - 6\alpha + 1)}}{5(1 - \alpha)}}, & \frac{1}{25} < \alpha < 1, \\ \pm \sqrt{\frac{3}{5}}, & \alpha = 1. \end{cases} \hspace{1cm} (S98)$$

At fixed points $x_* = 0, \pm \sqrt{\frac{5}{3}}$ and $\pm \sqrt{\frac{3}{5}}$, the derivatives $S'_{5,\alpha}(x) = \frac{25\alpha(1+x^2)^4}{(5x^4 - 10x^2 + 1)^2}$ are

$$S'_{5,\frac{1}{25}}(0) = 1,$$

$$S'_{5,1}(0) = 25,$$

$$S'_{5,\frac{1}{25}} \left( \pm \sqrt{\frac{5}{3}} \right) = 16,$$

$$S'_{5,1} \left( \pm \sqrt{\frac{3}{5}} \right) = 16,$$

$$S'_{5,1} (\pm \infty) = 1. \hspace{1cm} (S99)$$

From (S99), for $K = 5$ and at $\alpha = \frac{1}{25}$ and $\alpha = 1$, one obtains the Floquet multipliers as

$$\lambda_{5,\frac{1}{25}} = S'_{5,\frac{1}{25}}(0) = 1,$$

$$\lambda_{5,1} = S'_{5,1}(0) = 1. \hspace{1cm} (S100)$$

Therefore, similar to the case of $K = 3$, only Type 1 intermittency occurs, which is different from the case with Generalized Boole transformation and the case of $K = 4$.  

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