ONE PROPERTY OF A PLANAR CURVE WHOSE CONVEX HULL COVERS A GIVEN CONVEX FIGURE

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Abstract. In this note, we prove the following conjecture by A. Akopyan and V. Vysotsky: If the convex hull of a planar curve \( \gamma \) covers a planar convex figure \( K \), then
\[
\text{length}(\gamma) \geq \text{per}(K) - \text{diam}(K).
\]
In addition, all cases of equality in this inequality are studied.

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1. Introduction

The authors of [1] (along with obtained interesting results) posed the following

Conjecture 1 (A. Akopyan and V. Vysotsky, [1]). Let \( \gamma \) be a curve such that its convex hull covers a planar convex figure \( K \). Then
\[
\text{length}(\gamma) \geq \text{per}(K) - \text{diam}(K).
\]

It should be noted that this hypothesis is confirmed in the case when \( \gamma \) is passing through all extreme points of \( K \) (see Theorem 7 in [1]). This note is devoted to the proof of the above conjecture in the general case.

We identify the Euclidean plane with \( \mathbb{R}^2 \) supplied with the standard Euclidean metric \( d \), where \( d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \). For any subset \( A \subset \mathbb{R}^2 \), co\( (A) \) means the convex hull of \( A \). For every points \( B, C \in \mathbb{R}^2 \), \([B, C]\) denotes the line segment between these points.

A convex (planar) figure is any compact convex subset of \( \mathbb{R}^2 \). We shall denote by per\( (K) \), bd\( (K) \) and int\( (K) \) respectively the perimeter, the boundary, and the interior of a convex figure \( K \). Note that the perimeter of any line segment (i.e. a degenerate convex figure) is assumed to be equal to its double length. Note also that the diameter \( \text{diam}(K) := \max \{d(x, y) \mid x, y \in K\} \) of a convex figure \( K \) coincides with the maximal distance between two parallel support lines of \( K \). Recall that an extreme point of \( K \) is a point in \( K \) which does not lie in any open line segment joining two points of \( K \). The set of extreme points of \( K \) will be denoted by ext\( (K) \). It is well-known that ext\( (K) \) is closed and \( K = \text{co}(\text{ext}(K)) \) for any convex figure \( K \subset \mathbb{R}^2 \).

A planar curve \( \gamma \) is the image of a continuous mapping \( \varphi : [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^2 \). From now on we will call planar curves simply curves for brevity, since no other curves are considered in this note. As usually, the length of \( \gamma \) is defined as
\[
\text{length}(\gamma) := \sup \{\sum_{i=1}^m d(\varphi(t_{i-1}), \varphi(t_i))\},
\]
where the supremum is taken over all finite increasing sequences \( a = i_0 < i_1 < \cdots < i_{m-1} < i_m = b \) that lie in the interval \([a, b]\). A curve \( \gamma \) is called rectifiable if \( \text{length}(\gamma) < \infty \).

We call a curve \( \gamma \subset \mathbb{R}^2 \) convex (closed convex) if it is a closed connected subset of the boundary (respectively, the whole boundary) of the convex hull co\( (\gamma) \) of \( \gamma \).
Let us consider the following

**Example 1.** Suppose that the boundary \( \text{bd}(K) \) of a convex figure \( K \) is the union of a line segment \([A, B]\) and a convex curve \( \gamma \) with the endpoints \( A \) and \( B \). Then \( K \subset \text{co}(\gamma) \) and \( \text{length}(\gamma) = \text{per}(K) - d(A, B) \). Moreover, \( \text{length}(\gamma) = \text{per}(K) - \text{diam}(K) \) if and only if \( d(A, B) = \text{diam}(K) \).

The main result of this note is the following

**Theorem 1.** For a given convex figure \( K \) and for any planar curve \( \gamma \) with the property \( K \subset \text{co}(\gamma) \), the inequality

\[
\text{length}(\gamma) \geq \text{per}(K) - \text{diam}(K)
\]

(1)

holds. Moreover, this inequality becomes an equality if and only if \( \gamma \) is a convex curve, \( \text{bd}(K) = \gamma \cup [A, B] \), and \( \text{diam}(K) = d(A, B) \), where \( A \) and \( B \) are the endpoints of \( \gamma \).

**Remark 1.** Since obviously \( \text{per}(K) \geq 2 \text{diam}(K) \), the inequality (1) immediately implies the following widely known inequality: \( \text{length}(\gamma) \geq \frac{1}{2} \text{per}(K) \), see e.g. [4].

The strategy of our proof is as follows. We fix a convex figure \( K \subset \mathbb{R}^2 \). Then we prove the existence of a curve \( \gamma_0 \) of minimal length among all curves \( \gamma \) satisfying the condition \( K \subset \text{co}(\gamma) \) (Section 2). After that we prove the inequality \( \text{length}(\gamma_0) \geq \text{per}(K) - \text{diam}(K) \) and study all possible cases of the equality \( \text{length}(\gamma_0) = \text{per}(K) - \text{diam}(K) \), where \( \gamma_0 \) is an arbitrary curve of minimal length among all curves \( \gamma \) satisfying the condition \( K \subset \text{co}(\gamma) \) (Section 3). This allow us to get the proof of Theorem 1 in Section 4.

### 2. Some auxiliary results

To prove the desired results, we first recall some important properties of curves and convex figures.

Let us recall the following useful definition. A sequence of curves \( \{\gamma_i\}_{i \in \mathbb{N}} \) converges uniformly to a curve \( \gamma \) if the curves \( \gamma_i \) admits parameterizations with the same domain that uniformly converges to some parametrization of \( \gamma \). We will need the following result (see e.g. Theorem 2.5.14 in [3]):

**Proposition 1** (Arzela – Ascoli theorem for curves). *Given a compact metric space. Any sequence of curves which have uniformly bounded lengths has an uniformly converging subsequence.*

We also note one important property (the lower semi-continuity of length) of the limit curve in the above assertion (see e.g. Proposition 2.3.4 in [3]).

**Proposition 2.** *Given a sequence of rectifiable curves \( \{\gamma_i\}_{i \in \mathbb{N}} \) which converges point-wise to \( \gamma \) (with respect to parameterizations with the same domain). Then the inequality \( \liminf_{i \to \infty} \text{length}(\gamma_i) \geq \text{length}(\gamma) \) holds.*

The following property (of the monotonicity of perimeter) of convex figures is well-known (see e.g. [2, §7]).

**Proposition 3.** *If convex figures \( K_1 \) and \( K_2 \) in the Euclidean plane are such that \( K_1 \subset K_2 \), then \( \text{per}(K_1) \leq \text{per}(K_2) \), and the equality holds if and only if \( K_1 = K_2 \).*
We need also the following well-known result (it could be proved using the Crofton formula, see e.g. [1, pp. 594–595]):

**Proposition 4.** Let \( \varphi : [c,d] \to \mathbb{R} \) be a parametric continuous curve with \( \varphi(c) = \varphi(d) \). Then the length of the curve \( \gamma = \{ \varphi(t) \mid t \in [c,d] \} \) is greater or equal to \( \operatorname{per}(\operatorname{co}(\gamma)) \). Moreover, the equality holds if and only if \( \gamma \) is closed convex curve.

Now, we are going to prove the following

**Proposition 5.** For a given convex figure \( K \subset \mathbb{R}^2 \), there is a curve \( \gamma_0 \) of minimal length among all curves \( \gamma \) satisfying the condition \( K \subset \operatorname{co}(\gamma) \).

**Proof.** If \( \operatorname{int}(K) = \emptyset \), then the proposition is trivial. In what follows, we assume \( \operatorname{int}(K) \neq \emptyset \). Denote by \( \Delta(K) \) the set of all planar curve \( \gamma \) such that \( K \subset \operatorname{co}(\gamma) \). Let us consider \( \inf \{ \text{length}(\gamma) \mid \gamma \in \Delta(K) \} =: M \). It is clear that \( M \leq \operatorname{per}(K) \), since \( \operatorname{bd}(K) \) could be considered as a curve \( \gamma \). Now, we consider the sequence of curves \( \{ \gamma_i \}_{i \in \mathbb{N}} \) from \( \Delta(K) \) such that \( \text{length}(\gamma_i) \to M \) as \( i \to \infty \). Without loss of generality we may assume that \( \text{length}(\gamma_i) \leq M + 1 \) for all \( i = 1, 2, 3, \ldots \).

Let us take a point \( O \in \operatorname{int}(K) \). There is \( r > 0 \) such that the ball with center \( O \) and radius \( r \) is inside \( K \). For a fixed \( i \in \mathbb{N} \), we consider the point \( C_i \in \gamma_i \) such that \( d(C_i, O) = \max \{ d(x, O) \mid x \in \gamma_i \} \) and the straight line \( l_i \) passing through \( O \) perpendicular to the straight line \( OC_i \). Since \( K \subset \operatorname{co}(\gamma_i) \), there is a point \( D_i \in \gamma_i \) such that the line segment \( [C_i, D_i] \) intersects \( l_i \). This means that \( M + 1 \geq \text{length}(\gamma_i) \geq d(C_i, D_i) \geq d(C_i, O) \geq r > 0 \). It implies that \( M \geq r > 0 \) and

\[
\gamma_i \subset B(O, M + 1) := \{ x \in \mathbb{R}^2 \mid d(x, O) \leq M + 1 \}.
\]

Since the ball \( B(O, M + 1) \) is compact and the lengths of the curves \( \gamma_i, i = 1, 2, 3, \ldots \), are uniformly bounded, then the sequence \( \{ \gamma_i \} \) has an uniformly converging subsequence by Proposition 1. Passing to a subsequence if necessary, we can assume that the sequence \( \{ \gamma_i \}_{i \in \mathbb{N}} \) converges uniformly to some curve \( \gamma_0 \). Since \( K \subset \operatorname{co}(\gamma_i) \) for \( i = 1, 2, 3, \ldots \), then \( K \subset \operatorname{co}(\gamma_0) \) too. The lower semi-continuity of length (see Proposition 2) implies \( M = \lim \text{length}(\gamma_i) \geq \text{length}(\gamma_0) \), therefore, \( \text{length}(\gamma_0) = M \). This proves the proposition. \( \blacksquare \)

**Remark 2.** Note that the curve \( \gamma_0 \) in Proposition 5 may not be unique. For instance, if \( K \) is an equilateral triangle, then the union of any two of its sides is such a curve.
Remark 3. Note also that the curve $\gamma_0$ in Proposition 5 could be non-convex. For instance, let $K$ be the parallelogram $ABCD \subset \mathbb{R}^2$ with $A = (0, 0)$, $B = (1, 1)$, $C = (t+1, 1)$, and $D = (t, 0)$, where $t \geq 1$. It is easy to see that the broken line $ABCE$ with $E = (t+1, 0)$ is one of the shortest convex curves, whose convex hull covers $K$, and its length is $1 + \sqrt{2} + t$, see Fig. 1. On the other hand, the length of the broken line $ABDC$ (which convex hull is $K$) is equal to $2\sqrt{2} + \sqrt{2 - 2t + t^2}$. It is easy to check that $2\sqrt{2} + \sqrt{2 - 2t + t^2} < 1 + \sqrt{2} + t$ for $t > (3\sqrt{2} + 2)/4 \approx 1.56066$.

The above discussion leads to the following natural problem.

**Problem 1.** Give a comprehensive description of the class of planar curves $\gamma$ with the following property: there is a compact convex figure $K \subset \mathbb{R}^2$ such that $\gamma$ is the shortest curve among all curves, whose convex hulls cover $K$.

In the next section, we consider more detail information about any curve of shortest length among all curves $\gamma$ satisfying the condition $K \subset \text{co}(\gamma)$ for a given $K$.

3. **Some properties of shortest curves $\gamma$ with $K \subset \text{co}(\gamma)$**

Let $U \subset \mathbb{R}^2$ be a convex figure. We say that a straight line $l \subset \mathbb{R}^2$ divides $U$ into $U_1$ and $U_2$, if $U_1$ and $U_2$ are convex figures lying in different half-planes relatively $l$, such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = U \cap l$.

We need the following two simple results.

**Lemma 1.** Let $U \subset \mathbb{R}^2$ be a convex figure and let us consider some points $E, F \in \text{ext}(U)$. Then the straight line $l = EF$ divides $U$ into convex figures $U_1$ and $U_2$ such that $U_i = \text{co}(\text{ext}(U) \cap U_i)$, $i = 1, 2$.

**Proof.** It is clear that $\text{co}(\text{ext}(U) \cap U_i) \subset U_i$. Let us suppose that $\text{co}(\text{ext}(U) \cap U_i) \neq U_i$. Then there is a point $C \in \text{ext}(U_i)$ such that $C \notin \text{co}(\text{ext}(U) \cap U_i)$. On the other hand, $\text{ext}(U_i) \subset \text{ext}(U)$ and we obtain the contradiction. □

**Lemma 2.** Let $U \subset \mathbb{R}^2$ be a convex figure. Let us suppose that a point $A \notin U$ and points $A_1, A_2 \in U$ are such that the straight lines $AA_1$ and $AA_2$ are support lines for $U$ and $AA_1 \perp AA_2$. Then $d(A, A_1) + \text{per}(U) > \text{per}(\text{co}(U \cup \{A\}))$. 

**Figure 2.** Illustration to Lemma 2: the convex figure $U$ and $\triangle A_1A_2A$. 

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Proof. Let us consider the triangle $A_1A_2$ and let $\gamma^*$ be a part of $\text{bd}(U)$ between the points $A_1$ and $A_2$ such that $U \subset \text{co}(\gamma^* \cup \{A\})$ (see Fig. 2). It is clear that $\text{bd}(\text{co}(U \cup \{A\})) = \gamma^* \cup [A, A_1] \cup [A, A_2]$. It is clear also that $\text{per}(U) - \text{length}(\gamma^*)$ is the length of the complementary to $\gamma^*$ part of $\text{bd}(U)$ between the points $A_1$ and $A_2$, hence $\text{per}(U) - \text{length}(\gamma^*) \geq d(A_1, A_2) > d(A, A_2)$ and we get
\[
d(A, A_1) + \text{per}(U) > d(A, A_1) + \text{length}(\gamma^*) + d(A, A_2) = \text{per}(\text{co}(U \cup \{A\}))
\]
that proves the lemma. ■

Let us fix a curve $\gamma$ with an arc length parametrization $\varphi(t), t \in [a, b]$, such that $K \subset \text{co}(\gamma)$ and has minimal possible length among all curve with this property. We put $A := \varphi(a)$, $B := \varphi(b)$, and $\tilde{K} := \text{co}(\gamma).

Lemma 3. In the above notations, we have $A, B \in \text{ext}(\tilde{K})$ and $A \neq B$. Moreover, $K \cap [A, B] \neq \emptyset$.

Proof. Let us suppose that $A \notin \text{ext}(\tilde{K})$, then there a sufficiently small $\varepsilon > 0$ such that $\varphi([a, a + \varepsilon]) \cap \text{ext}(\tilde{K}) = \emptyset$ (recall that $\text{ext}(\tilde{K})$ is a closed set in $\mathbb{R}^2$). Hence, if we modify $\gamma$ up to $\gamma_1 := \{\varphi(t) | t \in [a + \varepsilon, b]\}$ then we get a shorter curve with the same convex hull. This contradictions shows that $A = \varphi(a) \in \text{ext}(\tilde{K})$. Similar arguments imply $B = \varphi(\beta) \in \text{ext}(\tilde{K})$.

Suppose that $B = A$. Let us consider a support line $l_1$ for $\tilde{K}$ through the point $B$. Since $B \in \text{ext}(\tilde{K})$, we may take a point $C \in l_1$ and a support line $l_2$ for $\tilde{K}$ through $C$, such that $C \notin \tilde{K}$ and $l_2$ is perpendicular to $l_1$. Now, take a point $D \in \tilde{K} \cap l_2$. Let $\gamma^*$ be a part of $\text{bd}(\tilde{K})$ between the points $B$ and $D$ such that $\tilde{K} \subset \text{co}(\gamma^* \cup [C, D])$. Lemma 2 and Proposition 4 imply $d(C, D) + \text{length}(\gamma^*) < \text{per}(\tilde{K}) \leq \text{length}(\gamma)$. Hence, the curve $\gamma^* \cup [C, D]$ is shorter than $\gamma$, and we get a contradiction due to $\tilde{K} \subset \text{co}(\gamma^* \cup [C, D])$. Therefore, $B \neq A$.

Let us suppose that $K \cap [A, B] = \emptyset$. Then the distance $\min\{d(x, y) | x \in K, y \in l\}$ between $K$ and the straight line $AB =: l$ is positive (recall that $K \subset \tilde{K}$ and $A, B$ are extreme points of $\tilde{K}$). Therefore, $K \subset \text{co}\{\tilde{\psi}(t) | t \in [a + \varepsilon, b - \varepsilon]\} \subset \text{co}(\gamma)$ for sufficiently small $\varepsilon > 0$. Since the curve $\gamma_2 := \{\tilde{\psi}(t) | t \in [a + \varepsilon, b - \varepsilon]\}$ is shorter than $\gamma$, we get a contradiction. This proves that $K \cap [A, B] \neq \emptyset$. ■

Proposition 6. Let us consider $\alpha, \beta \in [a, b]$ such that $\varphi(\alpha), \varphi(\beta) \in \text{ext}(\tilde{K})$. Then one of the following assertions holds:

1) $[\varphi(\alpha), \varphi(\beta)] \subset \text{bd}(\tilde{K})$;

2) the straight line $l$ through the points $\varphi(\alpha)$ and $\varphi(\beta)$ divided $\tilde{K}$ into $\tilde{K}_1$ and $\tilde{K}_2$ such that $\left(\tilde{K}_i \setminus [\varphi(\alpha), \varphi(\beta)]\right) \cap K \neq \emptyset, i = 1, 2$.

Proof. Let us suppose that $[\varphi(\alpha), \varphi(\beta)] \notin \text{bd}(\tilde{K})$, then every $\tilde{K}_i, i = 1, 2$, has a point $C_i$ from $\text{ext}(\tilde{K}) \setminus \{\varphi(\alpha), \varphi(\beta)\}$. It is clear that $C_i = \varphi(t_0)$ for some $t_0 \in [a, b]$.

If $\left(\tilde{K}_i \setminus [\varphi(\alpha), \varphi(\beta)]\right) \cap K = \emptyset$, then $K \subset \text{co}(\text{ext}(\tilde{K}_j))$, $j \in \{1, 2\} \setminus \{i\}$, by Lemma 3. Now, we will show how one can modify $\gamma$ to a curve $\gamma_1$ which is shorter than $\gamma$, but $K \subset \text{co}(\gamma_1)$.
If \( t_0 = a \) (\( t_0 = b \)), then we can take a sufficiently small \( \varepsilon > 0 \) such that \( \varphi([a, a + \varepsilon]) \cap l = \emptyset \) (respectively \( \varphi([b - \varepsilon, b]) \cap l = \emptyset \)). Then we see that \( K \subset \text{co}(\text{ext}(\tilde{K})) \subset \text{co}(\gamma_1) \),
where \( \gamma_1 = \{ \varphi(t) \mid t \in [a + \varepsilon, b] \} \) (respectively, \( \gamma_1 = \{ \varphi(t) \mid t \in [a, b - \varepsilon] \} \)). Hence, we have found a curve that is shorter than \( \gamma \) and which convex hull contains \( K \), that is impossible.

If \( t_0 \in (a, b) \), then we can take \( t_1, t_2 \in [a, b] \), \( t_1 < t_2 \), such that \( t_0 \in (t_1, t_2) \) and \( \varphi([t_1, t_2]) \cap l = \emptyset \). Since \( \varphi(t_0) \in \text{ext}(\tilde{K}) \), then \( \varphi([t_1, t_0]) \neq [\varphi(t_1), \varphi(t_2)] \). Now we consider a curve \( \gamma_2 = (\gamma \setminus \varphi([t_1, t_0])) \cup [\varphi(t_1), \varphi(t_2)] \). Obviously, \( \gamma_2 \) is shorter than \( \gamma \), but its convex hull still contains \( K \). This contradiction proves the proposition. 

**Corollary 1.** Suppose that \( \varphi(t_0) \) is an extreme point of \( \tilde{K} \) and it is not isolated in the set \( \text{ext}(\tilde{K}) \). Then \( \varphi(t_0) \in K \).

**Proof.** Let us take a sequence \( \{t_n\}_{n \in \mathbb{N}}, t_n \in [a, b] \), such that all points \( \varphi(t_n) \) are extreme for \( \tilde{K} \), \( \varphi(t_n) \neq \varphi(t_0) \), \( [\varphi(t_0), \varphi(t_n)] \subset \text{bd}(\tilde{K}) \), and \( \varphi(t_n) \to \varphi(t_0) \) as \( n \to \infty \). By Proposition 6, the straight line \( l_n \) through the points \( \varphi(t_n) \) and \( \varphi(t_0) \) divides \( \tilde{K} \) into two convex figures, each of them contains some point of \( K \). Let \( \tilde{K}_n \) be a one of these two figures, which has a smaller diameter. It is clear that \( \text{diam}(\tilde{K}_n) \to 0 \) as \( n \to \infty \).

If \( C_n \in \tilde{K}_n \cap K \), then \( C_n \to \varphi(t_0) \) as \( n \to \infty \). Since \( K \) is closed, we get \( \varphi(t_0) \in K \).

By Lemma 3, the points \( A \) and \( B \) are extreme points of \( \tilde{K} \). If \( A \) (respectively, \( B \)) is not an isolated point in the set \( \text{ext}(\tilde{K}) \), then \( A \in \tilde{K} \) (respectively, \( B \in \tilde{K} \)). The following proposition deals with the case, when \( A \) (or \( B \)) is isolated in \( \text{ext}(\tilde{K}) \).

**Proposition 7.** If \( A = \varphi(a) \) is isolated in \( \text{ext}(\tilde{K}) \), then there are \( \tau_1, \tau_2 \in (a, b) \), \( \tau_1 < \tau_2 \), such that the following assertions holds:

1) \([A, \varphi(\tau_1)] \cup [A, \varphi(\tau_2)] \) covers some neighborhood of \( A \) in \( \text{bd}(\tilde{K}) \);
2) \( \varphi([a, \tau_1]) = [A, \varphi(\tau_1)] \),
3) \( \varphi([a, \tau_2]) \cup [A, \varphi(\tau_2)] \) is a closed convex curve;
4) \([A, \varphi(\tau_1)] \cap K \neq \emptyset \);
5) The angle between the line segments \([A, \varphi(\tau_1)]\) and \([A, \varphi(\tau_2)]\) is equal to \( \pi/2 \).

Similar results hold for \( B = \varphi(b) \), if it is isolated in \( \text{ext}(\tilde{K}) \).

**Proof.** Since the point \( A \) is extreme in \( \tilde{K} \) and isolated in \( \text{ext}(\tilde{K}) \), then there are points \( A_1, A_2 \in \text{ext}(\tilde{K}) \subset \text{bd}(\tilde{K}) \) such that \( [A_1, A_2] \cap [A, A_2] \subset \text{bd}(\tilde{K}) \) and \([A, A_1] \cup [A, A_2]\) covers some neighborhood of \( A \) in \( \text{bd}(\tilde{K}) \) (roughly speaking, \( A_1 \) and \( A_2 \) are closest extreme points to \( A \) with respect to different directions on \( \text{bd}(\tilde{K}) \)). It is clear that \( A_1 = \varphi(\tau_1) \) and \( A_2 = \varphi(\tau_2) \) for some \( \tau_1, \tau_2 \in (a, b) \). Without loss of generality we may suppose that \( 0 < \tau_1 < \tau_2 \).

Let us consider the following closed curves:

\[ \gamma_1 = \varphi([a, \tau_1]) \cup [A, \varphi(\tau_1)], \quad \gamma_2 = \varphi([a, \tau_2]) \cup [A, \varphi(\tau_2)]. \]

By Proposition 4, we get that \( \text{length}(\gamma_1) \geq \text{per}(\text{co}(\gamma_1)) \) and \( \text{length}(\gamma_2) \geq \text{per}(\text{co}(\gamma_2)) \). Since \([A, A_1]\) and \([A, A_2]\) cover \( \text{bd}(\tilde{K}) \), then \([A, A_1] \subset \text{bd}(\text{co}(\gamma_1)) \) and \([A, A_2] \subset \text{bd}(\text{co}(\gamma_2)) \).

Due to the inclusion, \( \gamma_i \subset \text{co}(\gamma_i), i = 1, 2 \), we may replace the curve \( \gamma \) with the curve
\[ \gamma_i := \varphi([\tau_i, b]) \cup (\text{bd}(\text{co}(\gamma_i)) \setminus [A, A_i]) \]
with the same convex hull $\tilde{K}$. Since $\gamma$ has minimal length among all curves which convex hull covers $K$, we get $\text{length}(\gamma) = \text{per}(\text{co}(\gamma))$ by Proposition 4. It means that $\gamma_1$ and $\gamma_2$ are closed convex curves by Proposition 4 (see Fig. 3).

Since $A, A_1 \in \text{co}(\gamma_2)$, then $[A, A_1] \subset \text{co}(\gamma_2)$. On the other hand, $[A, A_1] \subset \text{bd}(\tilde{K})$. Since $\text{co}(\gamma_2) \subset \tilde{K}$, we get $[A, A_1] \subset \text{bd}(\text{co}(\gamma_2))$. It implies that $[A, A_1] = \varphi([a, \tau_1])$ and $[A, A_2] \neq \varphi([a, \tau_2])$. Therefore, assertions 1)–3) are proved.

Let us prove 4). If $[A, \varphi(\tau_1)] \cap K = \emptyset$, then there is $\varepsilon > 0$ such that $\text{co}(\varphi([a, \tau_1 + \varepsilon]))$ and $K$ are situated in different half-planes with respect to some straight line. Therefore, $K \subset \text{co}(\gamma_3)$, where $\gamma_3 := \varphi([\tau_1 + \varepsilon, b]) \cup [A, \varphi(\tau_1 + \varepsilon)]$. On the other hand, $\gamma_3$ is shorter than $\gamma$ (recall that $\varphi(\tau_1)$ is extreme in $\tilde{K}$, hence $\varphi([a, \tau_1 + \varepsilon]) \neq [A, \varphi(\tau_1 + \varepsilon)])$. This contradiction implies $[A, \varphi(\tau_1)] \cap K \neq \emptyset$.

Finally, let us prove 5). If $\angle A_1AA_2 \neq \pi/2$, then we can take $A' \in [A, A_1]$ such that $A' \neq A$ and $d(A, A')$ is less than distance from $A$ to $\tilde{K}$. Then $A' = \varphi(\tau')$ for some $\tau' \in (a, \tau_1)$. Now, take a point $A'' \in [A, A_2]$ such that $[A', A'']$ is orthogonal to $[A, A_2]$. If we consider $\gamma_4 := \varphi([\tau', b]) \cup [A', A'']$, then $K \subset \text{co}(\gamma_4)$ and $\text{length}(\gamma_4) < \text{length}(\gamma)$ (since the leg is shorter than the hypotenuse in any right triangle). This contradiction shows that $\angle A_1AA_2 = \pi/2$.

Similar results for the point $B$ we get automatically, reversing the parameterization of the curve $\gamma$. The proposition is completely proved.

Proposition 8. In the above notations, let $\eta_1$ be the smallest number in $T$ and let $\eta_2$ be the largest number in $T$, where $T = \{t \in [a, b] \mid \varphi(t) \in K\}$. Then the following inequality holds:

$$\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \text{per}(\tilde{K}) \geq \text{per}(K).$$

Proof. Since $K \subset \tilde{K}$, then the inequality $\text{per}(\tilde{K}) \geq \text{per}(K)$ follows directly from Proposition 3. Therefore, it suffices to prove the inequality

$$\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \text{per}(\tilde{K}). \quad (2)$$

We have $\varphi([a, \eta_1]) = [A, \varphi(\eta_1)] \subset \text{bd}(\tilde{K})$ and $\varphi([\eta_2, b]) = [\varphi(\eta_2), B] \subset \text{bd}(\tilde{K})$ by Proposition 7. Proposition 7 also implies that there is $\theta_1 \in (a, b)$ such that $[A, \varphi(\eta_1)] \cup [A, \varphi(\theta_1)]$ covers a neighborhood of $A$ in $\text{bd}(\tilde{K})$ if $A \notin K$ and there is $\theta_2 \in (a, b)$ such...
that $[B, \varphi(\eta_2)] \cup [B, \varphi(\theta_2)]$ covers a neighborhood of $B$ in $\text{bd}(\tilde{K})$ if $B \not\in K$ (note that $\theta_1 = b$ if and only if $\theta_2 = a$).

Let us consider $\tilde{\gamma} = \varphi([\eta_1, \eta_2])$ and $\tilde{K} = \text{co}(\tilde{\gamma})$. Note that $\tilde{K} \subset \tilde{K}$ and $\tilde{K}$ contains all extreme points of $K$ with the possible exception of points $A$ and $B$ (the latter is possible only if $A$ or $B$ is not in $K$). Therefore, $\tilde{K} = \text{co}(\tilde{K} \cup \{A, B\})$.

Since $\tilde{\gamma} \cup [\varphi(\eta_1), \varphi(\eta_2)]$ is a closed curve, Proposition 4 implies the inequality

$$\text{length}(\tilde{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \text{per}(\tilde{K}).$$  \hspace{1cm} (3)

Let us consider the following four cases: 1) $A, B \in K$, 2) exactly one of the points $A$ and $B$ is in $K$, 3) $A, B \not\in K$ and $\theta_1 < b$, 4) $A, B \not\in K$ and $\theta_1 = b$.

In Case 1 we have $\gamma = \tilde{\gamma}$ and $\tilde{K} = \tilde{K}$, hence (3) implies $\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \text{per}(\tilde{K})$ and we get what we need.

Let us consider Case 2. Without loss of generality we may assume that $B \in K$ (hence, $\eta_2 = b$ and $\varphi(\eta_2) = B$) and $A \not\in K$. Hence, $\tilde{K} = \text{co}(\tilde{K} \cup \{A\})$. Let us consider the triangle $A_1A_2$, where $A_1 = \varphi(\eta_1)$ and $A_2 = \varphi(\theta_1)$. By Proposition 7 we have $\angle A_1A_2 = \pi/2$. Since $A_1, A_2, \in \text{bd}(\tilde{K}) \cap \tilde{\gamma}$, we get that $A_1, A_2 \in \text{bd}(\tilde{K})$. Then (3) and Lemma 2 imply

$$\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) = d(A, A_1) + \text{length}(\tilde{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \\
\geq d(A, A_1) + \text{per}(\tilde{K}) > \text{per}(\tilde{K}),$$

that proves (2).

To deal with Case 3, let us consider the triangles $A_1A_2$ and $B_1B_2$, where $A_1 = \varphi(\eta_1)$, $A_2 = \varphi(\theta_1)$, $B_1 = \varphi(\eta_2)$, and $B_2 = \varphi(\theta_2)$. By Proposition 7 we have $\angle A_1A_2 = \angle B_1B_2 = \pi/2$. Note that $\theta_1 < \eta_2$ and $\eta_1 < \theta_2$. Since $A_1, A_2, B_1, B_2 \in \text{bd}(\tilde{K}) \cap \tilde{\gamma}$, we get that $A_1, A_2, B_1, B_2 \in \text{bd}(\tilde{K})$. Then (3) and Lemma 2 imply

$$\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) = d(A, A_1) + d(\tilde{B}, B_1) + \text{length}(\tilde{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \\
\geq d(A, A_1) + d(\tilde{B}, B_1) + \text{per}(\tilde{K}) > d(A, A_1) + \text{per}(\text{co}(\tilde{K} \cup \{B\})) \\
> \text{per}(\text{co}(\tilde{K} \cup \{B\} \cup \{A\})) = \text{per}(\text{co}(\tilde{K} \cup \{A, B\})) = \text{per}(\tilde{K}),$$

that proves (2).

Finally, let us consider Case 4. In this case we have $[A, B] \subset \text{bd}(\tilde{K})$, $A_2 = B$, and $A = B_2$. Let as consider the quadrangle $A_1A_2B_1B_2$, where $A_1 = \varphi(\eta_1)$ and $B_1 = \varphi(\eta_2)$. By Proposition 7 we have $\angle A_1A_2 = \angle B_1B_2 = \pi/2$. Since $A_1, B_1 \in \text{bd}(\tilde{K}) \cap \tilde{\gamma}$, we get that $A_1, B_1 \in \text{bd}(\tilde{K})$.

We denote by $\gamma_3$ a part of $\text{bd}(\tilde{K})$ between $A_1$ and $A_2$ such that $\tilde{K} \subset \text{co}(\gamma_3 \cup \{A, B\})$ (see Fig. 4). It is clear that $\text{bd}(\tilde{K}) = \gamma_3 \cup [A, A_1] \cup [B, B_1] \cup [A, B]$. Note that $\text{per}(\tilde{K}) - \text{length}(\gamma_3)$ is the length of the curve $(\text{bd}(\tilde{K}) \setminus \gamma_3) \cup \{A_1, B_1\}$, connecting the points $A_1$ and $B_1$. Hence, $\text{per}(\tilde{K}) - \text{length}(\gamma_3) \geq d(A_1, B_1) \geq d(A, B)$ and we get

$$\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) = d(A, A_1) + d(\tilde{B}, B_1) + \text{length}(\tilde{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \\
\geq \text{per}(\tilde{K}) + d(A, A_1) + d(B, B_1) \\
\geq \text{length}(\gamma_3) + d(A, B) + d(A, A_1) + d(B, B_1) = \text{per}(\tilde{K}).$$
Hence, we have proved (2) for all possible cases. The proposition is proved.

Remark 4. We see from the above proof that the equality

$$\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) = \text{per}(\tilde{K})$$

is fulfilled if and only if $\varphi([a, b]) \cup [A, B]$ is a convex curve (that coincides with $\text{bd}(\tilde{K})$) and the quadrangle $AA_1B_1B$, where $A_1 = \varphi(\eta_1)$ and $A_2 = \varphi(\theta_1)$, is a rectangle (in particular, $A_1 = A$ and $B_1 = B$). Consequently, since $\text{per}(\tilde{K}) = \text{per}(K)$ implies $\tilde{K} = K$, the equality

$$\text{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) = \text{per}(K)$$

is fulfilled if and only if $\varphi([a, b]) \cup [A, B] = \text{bd}(K)$.

Since $\text{diam}(K) \geq d(\varphi(\eta_1), \varphi(\eta_2))$, then Proposition 8 and Remark 4 imply the following

Corollary 2. If a curve $\gamma$ has shortest length among all curves whose convex hulls cover a given compact convex figure $K$, then the following inequality holds:

$$\text{length}(\gamma) + \text{diam}(K) \geq \text{per}(K).$$

Moreover, the equality in this inequality is fulfilled if and only if $\gamma$ is convex, $\text{bd}(K) = \gamma \cup [A, B]$, and $\text{diam}(K) = d(A, B)$, where $A$ and $B$ are the endpoints of the curve $\gamma$.

4. Proof of Theorem 1

Let us fix a convex figure $K \subset \mathbb{R}^2$. By Proposition 5, there is a curve $\gamma_0$ of minimal length among all curves $\gamma$ satisfying the condition $K \subset \text{co}(\gamma)$. By Corollary 2, we get

$$\text{length}(\gamma) + \text{diam}(K) \geq \text{length}(\gamma_0) + \text{diam}(K) \geq \text{per}(K)$$

for any curve $\gamma$ such that $K \subset \text{co}(\gamma)$, that proves (1). We have the equality in (1) if and only if $\text{length}(\gamma) = \text{length}(\gamma_0)$ (hence, we may assume that $\gamma = \gamma_0$ without loss of generality), $\gamma$ is convex, $\gamma \cup [A, B] = \text{bd}(K)$, and $\text{diam}(K) = d(A, B)$, where $A$ and $B$ are the endpoints of the curve $\gamma$. Therefore, we obtain just convex figures $K$ and corresponding curves $\gamma$ exactly as in Example 1.
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