QED(1+1) at Finite Temperature –
a Study with Light-Cone Quantisation.

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Abstract

We explore quantum electrodynamics in (1+1) dimensions at finite temperature using the method of Discretized Light-Cone Quantisation. The partition function, energy and specific heat are computed in the canonical ensemble using the spectrum of invariant masses computed with a standard DLCQ numerical routine. In particular, the specific heat exhibits a peak which grows as the continuum limit is numerically approached. A critical exponent is tentatively extracted. The surprising result is that the density of states contains significant finite size artifacts even for a relatively high harmonic resolution. These and the other outstanding problems in the present calculation are discussed.

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1 Introduction

Developing nonperturbative methods of performing computations at finite temperature for hadronic systems is soon to become in demand with the advent of experiments such as the RHIC to explore the phase transition to and properties of the putative Quark-Gluon Plasma. For zero temperature, there is now much activity in also developing Dirac’s front form \cite{1} or Hamiltonian light-cone field theory for studying nonperturbative field theories \cite{2} and extracting, for example, the relativistic bound state spectra. Reviews can be found in \cite{3}. Since the purpose of this method is to write down a Hamiltonian, diagonalise it and thus obtain a spectrum there would appear to be nothing conceptual in stepping from there to the computation of thermodynamic quantities: the spectrum, as the basis for developing the partition function, has – at least in principle – all the information one could require.

For this reason we tackle in the present work the finite temperature properties of quantum electrodynamics in (1+1) dimensions. Over several years and in a number of separate works \cite{4, 5, 6}, the Hamiltonian and spectrum of the theory have been studied with Discretized Light-Cone Quantisation (DLCQ). Comparison with what lattice data is available \cite{7} has been encouraging. It is also known that at least some – though not all – of the physics of the fermion condensate in the small fermion mass limit \cite{8} are contained in the spectrum even with the trivial vacuum \cite{9}. QED(1+1) is thus the ideal place to test the light-cone Hamiltonian method for finite temperature and even to make some contribution to what little is known about the ‘massive Schwinger model’ at finite temperature.

The main subtlety in the DLCQ program is the continuum limit. One works at any stage in the numerical work with a finite length interval manifested in momentum space by a finite total momentum $K$, or harmonic resolution, available for distribution amongst the ‘partons’ of the theory. Numerically, one computes at various $K$-values and, when feasible, extrapolates to extract the continuum limit. For any finite $K$ the resulting spectrum is finite and this of course has an impact on thermodynamic quantities in the sum over all eigenstates of the Hamiltonian. We study how this works its way into the various thermodynamic quantities such as partition function, energy and specific heat. The partition function for various $K$-values varies enormously – however this is not directly a physical quantity. We observe that in the energy for low $T$ the results are roughly $K$-independent while above a certain temperature regime the energy becomes linear in the temperature with the slope being independent of $K$. This leads, in the specific heat, to a $K$-independent result except in the neighborhood of $T = 0.5 - 1.0$, in units of $g/\sqrt{\pi}$, where a peak with increasing height for increasing $K$ appears. The growing peak is suggestive of a property of the system as we approach an infinite number of degrees of freedom. We tentatively accept this to indicate a phase transition, though it is not associated with the chiral condensate as order parameter (which seems to be nonzero for all finite temperatures anyway \cite{10}). Rather, it appears to be related to the change in the spectrum from a discrete one of bound mesonic states to a continuous spectrum of scattering states.

First, we briefly review the light-cone formalism as applied to QED(1+1), discuss the basic spectrum and give the corresponding density of states as determined in DLCQ. We work in the canonical ensemble and present results for the partition function, energy and specific heat for various values of harmonic resolution. The letter concludes with a discussion of the results and the outstanding problems.
2 DLCQ for QED(1+1)

The literature is already very extensive in references to the basic method of DLCQ [2, 3]. We can therefore afford to be brief. Our light-cone conventions are those of [11]. Light-cone quantization involves initialising independent quantum fields at equal light-cone time, say $x^+ = (x^0 + x^3)/\sqrt{2} = 0$. The orthogonal $x^- = (x^0 - x^3)/\sqrt{2}$ is the longitudinal or light-cone space. The appropriate time evolution generator is $P^-$, itself a constant of the motion under evolution in $x^+$, together with the momentum or $x^-$-translation generator $P^+$. Thus these operators can be built via the energy-momentum tensor from the bare Lagrangian fields once all redundant variables have been eliminated. In order to regularise the infrared, a finite interval is employed: $x^- \in [-L, +L]$. Bosons are assigned periodic and fermions antiperiodic boundary conditions. In the following we ignore zero modes\(^1\). Then the light-cone gauge $A^+ = 0$ is permissible in which the remaining $A^-$ gauge potentials are constrained and can be eliminated by solving the Gauss law constraints. Thus, apart from the resulting linear Coulomb potential, there is no trace left of the photons in two dimensional QED. Similarly, only half the fermion degrees of freedom are independent, the rest being eliminated by the Dirac equation – a strictly light-cone peculiarity. One is thus able to completely specify the Poincaré generators in terms of the so-called ‘right-mover’ fermion field which is quantised canonically. The vacuum is just the perturbative Fock vacuum in this framework [12]. Hence states can be built up by application of creation operators in the right-movers after an expansion in plane wave modes.

The relativistic bound-state problem can be formulated as the eigenvalue equation

$$2P^+P^-|\Psi_i\rangle = m_i^2|\Psi_i\rangle.$$  \hspace{1cm} (1)

Here, $P^\pm$ is expressed in terms of the Fock operators into which the fields are expanded, and $|\Psi_i\rangle$ is similarly built from Fock operators on the trivial vacuum. Inserting a complete set of Fock states results in a finite matrix equation

$$\langle i | 2P^+P^- : j \rangle \langle j | |\Psi\rangle = m^2(i|\Psi),$$  \hspace{1cm} (2)

which has been diagonalized numerically [4, 5]. The reader can find the explicit form of the Hamiltonian in [4, 5]. We give here just the conventions for scaling of the coupling and invariant mass. With $g$ the gauge coupling and $m_F$ the fermion mass in the QED(1+1) Lagrangian, we define a coupling

$$\lambda \equiv \left[1 + \frac{\pi m_F^2}{g^2}\right]^{-1}$$  \hspace{1cm} (3)

and a mass scale

$$m_E^2 \equiv m_F^2 + \frac{g^2}{\pi}.$$  \hspace{1cm} (4)

The coupling $\lambda$ allows us to explore the weak ($\lambda \to 0$) to strong ($\lambda \to 1$) coupling regimes within a finite range plot. In the following, all energies shall be in units of $m_E$ as shall be the units of temperature as well.

On the computer it is more convenient to express the continuum limit in terms of the ‘harmonic resolution’ $K = LP^+ / \pi$ which is related to the total (integer) momentum available to be distributed to the partons of the theory. Evidently, in order to take $L \to \infty$

\(^1\)Their complete incorporation into the analogue of these calculations is still under investigation.
with fixed total momentum one must then take $K \to \infty$. For some physical quantities it is feasible to compute for a range of values of $K$ and extrapolate, however in the present work we are restricted to studying thermodynamic quantities for finite but increasing $K$. Herein lies one problem to which we will have cause to refer frequently: computing with $K \neq \infty$ generates an error in the numerical results, however as yet this error lacks a physical interpretation. Nonetheless, the maximum value of $K$ chosen enables reproduction of results for the low energy mass spectrum consistent with lattice gauge theory but still permits computation on a workstation. The programs are described in some detail in [6].

3 Spectrum and Density of States

Before diving into the thermodynamics, we prepare ourselves by reviewing the basic properties of the spectrum of QED(1+1). A typical spectrum can be found in [4]. For completeness we reproduce one used to calculate the quantities of this letter; applying slightly different boundary conditions as discussed in [5].

There are a number of basic features common to all values of coupling constant. The lowest bound state is essentially a fermion-antifermion pair which becomes the ‘Schwinger boson’ of mass $g/\sqrt{\pi}$ as fermion mass $m_F$ vanishes [13]. For non-zero fermion mass, this mesonic state has a ‘size’ and so we refer to it as an ‘extended’ Schwinger boson. Above this state lie several bound ‘molecule’ states of the extended Schwinger boson state. The number of these depend on the strength of the coupling and the value of the $\theta$ parameter which is essentially a background electric field [14]. Finally there is a continuous spectrum consisting of extended Schwinger boson states in relative motion. Embedded within this
continuum are discrete bound states of larger ‘molecules’ of extended Schwinger bosons, for example one of three bosons, namely predominately a six-particle state \[15\].

Numerically, for any finite \(K\) we always obtain a discrete spectrum which in the limit should become a continuum. Put another way, for any \(K \neq \infty\) there are no scattering states, rather states of progressively weaker binding as \(K\) increases. In particular, the spectrum on its own is rather misleading in illustrating how close one is to a continuous spectrum. We give the density of states for a particular value of coupling for various values of \(K\) in Fig.2.

For other couplings the results are qualitatively similar. One observes that at some point in the spectrum the density of states begins to decrease and this is purely an artifact. The point at which the density begins decreasing does not appear to vary strongly with \(K\) within the range of values explored here. It is difficult to draw any conclusions on this aspect in the \(K = \infty\) case. We do learn here that we should not trust the thermodynamics for temperatures commensurate with energies above the point where the density of states decreases. This will be reflected in the results to be given in the next section, where we take a rather conservative range of temperatures.

These features aside, we conclude this section by observing the basic structure of the density of states: it is basically a rapidly rising distribution breaking down at a certain temperature, thus roughly resembling a Gaussian with height and width which increase with increasing \(K\). This will be useful at the end in order to understand certain properties of the thermodynamics to which we turn now.

### 4 Canonical Ensemble

As mentioned, the DLCQ method involves a calculation of the spectrum for a fixed ‘size’ of system, which is subsequently varied in order to approach the continuum. One should note that this size is not related to that measured in any conventional frame accessible by proper Lorentz transformations, and so it is not intuitive to picture this as being in
a certain frame of reference. Rather, the method tends to a system of infinite length in a standard reference frame. Put another way, the results of DLCQ for, say, a spectrum acquire relevance to the ‘real world’ with no boundary walls only when there is some converging behaviour as $K \to \infty$.

Now, as we attempt to describe the thermodynamics of such a system we are lead to a canonical ensemble of such finite size systems, there being no external source of particles in this problem. Again, these canonical ensembles bear no resemblance to any finite size system in a meaningful rest frame. Rather, by taking $K \to \infty$ we (hope to) approach the same thermodynamics of a set of canonical ensembles of progressively larger volumes as determined in the usual sense. Thus, it is wrong to interpret the results for any finite $K$. Rather we must always (where possible) extrapolate at least qualitatively to the continuum.

We now compute the partition function in the canonical ensemble using the basic relation $Z = \sum_i \exp(-\beta E_i)$ where $E_i$ is the energy measured in some well-defined rest-frame. We consider the system in some given frame attainable by proper Lorentz transformations, and now relate the energy $E_i$ to the relativistic-invariant mass squared of the particles in the system. Here we make our first genuine approximation: we shall treat all the states over which the partition is sum is taken as free discrete bound states, enabling us to take $M^2$ independent of $P^+$ and to use the continuum energy-momentum relation:

$$Z = \sum_{\text{states}} \sum_{\text{momenta}} e^{-\beta \sqrt{M^2_i + p^2_i}}. \quad (5)$$

The momenta $p_i$ are discrete given that the system is taken to be of finite size $d \neq L$. They are the momenta of bound states measured in the given frame for which the size is $d$. Eq.(5) is valid for the low energy spectrum which is indeed that of free bound states, either of a single extended Schwinger boson, or the molecule states. We now use for $M^2_i$ the values obtained from a DLCQ calculation for a given coupling $\lambda$ and harmonic resolution $K$. Insofar as for any finite $K$ all the DLCQ states are discrete, Eq.(5) would appear to be correct. Where the error lies is in concluding from finite $K$ results the thermodynamic properties of the finite size system in some Lorentz frame. Unfortunately, at this point we cannot determine the error being made following this route.

From Eq.(5) the procedure is now straightforward. A useful trick is to approximate the momentum sum by an integral,

$$Z = \frac{d}{2\pi} \sum_{\text{states}} \sum_{\text{momenta}} 2 \int_0^\infty dp_i e^{-\beta \sqrt{M^2_i + p^2_i}}$$

$$= \frac{d}{\pi} \sum_i M_i K_1(\beta M_i) \quad (6)$$

with $K_1$ the first Bessel function in the notation of [14], where we add that integrating over $P^+$ in the same approximation results of course in the same expression. We take Boltzmann’s constant to be one so that $T = \beta^{-1}$. The energy $e$ and specific heat $c$ are calculated via the relations

$$e = \frac{\partial \ln Z}{\partial T} \cdot T^2 \quad \text{and}$$

$$c = \frac{\partial e}{\partial T}. \quad (7)$$
Figure 3: Partition function, Energy and Specific Heat. Results of a DLCQ calculation with antiperiodic boundary conditions and coupling $\lambda = 0.30$ for the following quantities: a) Partition function, b) Energy, c) Specific heat.

Figure 4: Specific Heat for Three Values of Coupling. The specific heat for couplings a) $\lambda = 0.05$, b) $\lambda = 0.30$, c) $\lambda = 0.85$.

Now, given the spectrum for a given $K$ and $\lambda$ the above quantities are straightforwardly calculated. We first present the results for an intermediate value of coupling and various values of $K$ in Fig.3.

The most significant feature we observe is the rising peak in the specific heat as $K$ increases. This region aside, for other temperatures the different $K$ values coincide. Secondly, we see a rise in the energy above this peak temperature with increasing $K$.

For different couplings, the results qualitatively do not change. Quantitatively, the changes are most evident by looking just at the specific heat. In Fig.4 we compare this for three different couplings. The only slight change is in the position and height of the peak as the coupling increases. More precisely, inspection shows the temperature at which the peak occurs appears to approach $T = 1$ as $\lambda \to 1$, which coincides with the strong
coupling or alternately massless fermion case. We mention that in our units the lowest state in this regime, the Schwinger boson, has mass $M = 1$.

## 5 Discussion

Before attempting to understand these results, we first note that we have given values in temperature which do not correspond to systems in the ensemble with states occupying deeply into the putative continuum spectrum. Insofar as, with increasing $K$, we trust the low energy derived spectrum the only error we could be making is in treating even the low part of the scattering spectrum on the same footing as the discrete one. As mentioned, we are unable at this point to quantify how significant this error is.

Can we nonetheless understand the origin of the peak? Based on our earlier observation that the density of states roughly resembles a Gaussian distribution, we can try simple analytic calculations using

$$\rho(E) = \exp\left(-\frac{(E - E_0)^2}{A}\right).$$

(8)

This indeed gives a qualitatively similarly increasing peak in the specific heat with decreasing $A$. In other words, the height of the peak comes from the steepness of the rise in the density of states as $K$ increases.

The question now is: can these results be extrapolated to the continuum to draw physically relevant information? This is impeded by several features. The first is a conceptual problem: relating $K$ to a physical length scale for a finite size scaling analysis. The second impediment is entirely a practical one: going beyond the present values of $K$ while maintaining quick CPU time. The question is whether there is some convergence to a finite value for the specific heat at the turning point, or whether indeed it diverges to infinity. The associated question is whether the profile of the rise in the density of states actually converges to some fixed form in the continuum limit. Either way, increased computing power is necessary which could be complemented by algorithms such as the Lanczos method [17]. Only via such computations could we say with any confidence whether the peak is indicative of a phase transition.

Let us, nonetheless, assume scaling behaviour, and use the values of the peak in the specific heat to attempt a numerical extrapolation over increasing values of $K$. In this way we have estimated the critical exponent $\alpha$ (see, for example, [18]) associated with the suggested second-order phase transition. Direct fits ranging from $K = 10$ to 25 resulted in increasing exponents from $\alpha \approx 0.38$ to 0.61. At best, if this is critical behaviour we estimate a lower bound for the exponent at $\alpha > 0.7$. A scaling analysis using $K$ as our relevant length scale (in the absence of any other choice) resulted in a determination of $\alpha$ and $T_c$ for $\lambda = 0.05$ to $\alpha = 0.89 \pm 0.04$ and $T_c = 0.54 \pm 0.04$ consistent with the above bounds. Of course, all this is severely limited by the problems discussed above but we put these estimates forward as the basis for further discussion.

At any rate, for low to moderate temperatures the method of calculation would appear to be consistent and yields physically sensible results. An interesting calculation which could be pursued with the present method is that of the temperature dependence of the chiral condensate. There exist analytic results in the literature which allow some comparison [14].

In summary, we have presented a method for extracting thermodynamic quantities from the spectrum of a given field theory as computed using Discretised Light-Cone Quantisation. The particular example we chose was QED in (1+1) dimensions. Were the DLCQ
program for QCD to be achieved, namely the generation of a hadron spectrum, then the method presented here could be generalized with little effort. Of course, computing the spectrum of QCD from a ‘first principles’ calculation remains the difficult challenging task, but nonetheless the extension of the method proposed in the present work will be essential for such computations.

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