On How the Introducing of a New $\theta$ Function Symbol Into Arithmetic’s Formalism Is Germaine to Devising Axiom Systems that Can Appreciate Fragments of Their Own Hilbert Consistency

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Abstract

A new $\theta$ function primitive is proposed that almost achieves the combined efficiency of the addition, multiplication and successor growth operations. This $\theta$ function symbol enables the constructing of an “IQFS(PA+)” axiom system that can corroborate a fragmentary definition of its own Hilbert consistency, while it will simultaneously verify isomorphic counterparts of all Peano Arithmetic’s $\Pi_1$ theorems. Many propositions and intermediate results are also established. Only one intermediate result, which most readers will intuit should be true, does remain formally unproven.

Keywords and Phrases: Gödel’s Second Incompleteness Theorem, Consistency, Hilbert’s Second Open Question, Hilbert-styled Deduction (and its Frege-like analogs).

Mathematics Subject Classification: 03B52; 03F25; 03F45; 03H13

Comment: All the theorems and propositions are the same in this Version 5 as in Version 4. The difference is that the writing style is now significantly more polished.

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1 Introduction

Two historic results were established by Gödel’s millennial paper [20]. The First Incompleteness
Theorem showed no decision procedure exists for identifying all the true statements of Arithmetic.
The Theorem XI in [20], later known as the “Second Incompleteness Theorem”, demonstrated no
extension of the Russell-Whitehead Principia Mathematica formalism can verify its own consist-
tency. Gödel’s two observations were historic because they demonstrated, unequivocally, that the
initial objectives of Hilbert’s Consistency Program were much too far-reaching. Thus at best, only
a sharply curtailed form of Hilbert’s goals would be plausible.

These observations were reinforced by a new version of the Second Incompleteness Theorem,
due to the combined work of Pudlák and Solovay [39, 44], enhanced by some added techniques of
Nelson and Wilkie-Paris [35, 49]. It established the prohibition (stated in §3’s Theorem ++) that
no conventional arithmetic formalism can verify its own “Hilbert-style” consistency, when it merely
recognizes Successor as a formally total functional operation.

Within such curtailed limits, we have published since 1993 a series of articles [50]-[59], outlining
generalizations of the Second Incompleteness Theorem and its sometimes-feasible boundary-case
exceptions. Pavel Pudlák examined an early preprint of our article [55] and suggested [41] we
consider attempting to hybridize its formalism with some of Ajtai’s observations about the Pigeon
Hole effects [3]. On a more informal basis during a lunch at a 1997 conference, Sam Buss [10]
conveyed to us an approximately similar suggestion. The Section 6 of [55] did subsequently formalize
one type of response to these insightful observations. Our new results in this current paper will
examine a much more ambitious approach to [55]’s particular topic, that is germane to an arithmetic
appreciating some very delicatedly defined fragments of its Hilbert-styled formalizations of its self
consistency.

During our discussion, Add(x, y, z) and Mult(x, y, z) will denote two 3-way predicate symbols
specifying that $x + y = z$ and $x \cdot y = z$, which are known by our formalism to satisfy $\Pi_1$ encodings
for axioms formalizing the associative, commutative, identity and distributive principles. Let us
say an axiom system $\alpha$ recognizes successor, addition and multiplication as Total Functions iff
it can additionally prove (1) - (3) as theorems.

\[
\forall x \exists z \quad \text{Add}(x, 1, z) \quad (1)
\]
\[
\forall x \forall y \exists z \quad \text{Add}(x, y, z) \quad (2)
\]
\[
\forall x \forall y \exists z \quad \text{Mult}(x, y, z) \quad (3)
\]

Our studied axiomatizations for integer arithmetics will differ from their conventional counterparts
by neither possessing an ability to prove any of the totality statements (1) - (3) above, nor containing
function symbols for formalizing the traditional addition, multiplication and successor function
operations. Instead, we will rely upon an entirely new different type of primitive function, called
the “θ” operator, to construct the endless sequence of integers 3, 4, 5, ... from merely the three initial starting constants of 0, 1 and 2. (The exact number of logical symbols for encoding a term $T_n$, that represents an integer $n \geq 3$, will satisfy either an $O(\log^3 n)$ or $O(\log n)$ upper bound under our Propositions 4.3 and 7.1 depending on whether a tree, as opposed to a Dag-oriented methodology, is used to encode $T_n$.)

Assuming only the very modest assumptions of our Conjecture 6.6 are true\(^1\), Theorem 6.7 will imply any consistent arithmetic axiom system $A$ (including Peano Arithmetic itself) can be mapped onto a consistent axiom system “IQFS($A$)”, where IQFS($A$) can simultaneously:

1. corroborate its own Hilbert consistency,

2. confirm the validity of all $A$’s $\Pi_1$ theorems in a slightly revised language that uses the preceding $Add(x, y, z)$ and $Mult(x, y, z)$ 3-way predicate symbols, and

3. use only the three starting constants of 0, 1 and 2, to construct any term $T_n$ for representing any natural number $n \geq 3$, while satisfying Propositions 4.3 and 7.1’s specified $O(\log^3 n)$ and $O(\log n)$ bounds for the length of $T_n$’s construction.

Items 1-3 are potentially significant because [55]’s earlier ISCE formalism could achieve the first two objectives, only when it had used an unfortunately infinite number of constant symbols.

Theorem 6.7’s result should be viewed, cautiously, as only a boundary-case evasion of the Second Incompleteness Theorem. This is because its “IQFS($A$)” formalism will be too weak to treat even Successor as a total function. Yet assuming our Conjecture 6.6 is correct (as the Appendix will demonstrate it is nearly 100% certain to be), the semantics of IQFS shall partially answer an open question, that was raised by Harvey Friedman, at the end of the fourth lecture within his 5-part year-2014 YouTube series [18].

It concerned whether some exotic boundary case exceptions to the Second Incompleteness Theorem might exist. The intuition behind IQFS($A$)’s improved behavior will be that Proposition 4.3’s $\theta$ primitive will capture much of the growth properties of the addition, multiplication and successor operations, without employing those of their special functionalities, that are known to particularly support [39, 44]’s broad-scale generalizations of the Second Incompleteness Theorem.

The present article, fortunately, can be read without examining any of our previous papers. If a reader does wish to skim one of our earlier articles, we recommend Sections 3 & 4 of [55]. (The reading of [55] is optional because a later chapter will summarize its content in more than adequate detail.)

\(^1\)Conjecture 6.6 is the result, mentioned in the abstract, which shall be demonstrated to be almost certainly true by the appendix attached to 40
2 Returning to the 1931-1939 Period

Gödel’s Second Incompleteness Theorem was published in two stages during the 1931-1939 period. Its initial 1931 announcement, formalized by Theorem XI in Gödel’s millenial paper [20], demonstrated that no extension of the Russell-Whitehead Principia Mathematicae formalism could corroborate its own consistency. The widely quoted more general result, that every consistent r.e. extension of Peano Arithmetic must be unable to corroborate its own consistency, was, technically, first published in the 1939 edition of the Hilbert-Bernays textbook [27]. The latter has been considered to be the first definitive demonstration of the broad reach of the Second Incompleteness Effect. Its formalism established, beyond any reasonable doubt, that any kind of axiom system corroborating its own consistency must rely upon a foundational structure different from Peano Arithmetic. (This is because the Hilbert-Bernays textbook formalized the forerunner of what is now known as the Hilbert-Bernays Derivability Conditions [27, 25, 33, 34], as a mechanism for envisioning the broad generality of the Second Incompleteness Effect.)

It is, thus, fascinating that Hilbert, as a co-author of an important generalization of the Incompleteness Theorem, never withdrew the justification [26] for his consistency program:

∗ “Let us admit that the situation in which we presently find ourselves with respect to paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?”

Indeed, Hilbert always insisted that some special new formalism would at least partially vindicate the prior goals of his consistency program. He thus arranged to have its often-quoted motto ("Wir müssen wissen—Wir werden wissen") engraved on his tombstone.

It is known [12, 22, 61] that Gödel was also doubtful about the generality of the Second Incompleteness Theorem for at least two years after its publication. He thus inserted the following historically noteworthy caveat into his famous 1931 millenial paper [20]:

** “It must be expressly noted that Theorem XI” (e.g. the Second Incompleteness Theorem) “represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and there might conceivably be finite proofs which cannot be stated in P or in ...”

The statement ** has had numerous different interpretations. All Gödel’s biographers [12, 22, 61] have noted his intention (prior to 1930) was to establish Hilbert’s proposed objectives, before

2 Boolos states there has been some debate among scholars whether Peano Arithmetic’s precise generalization of the Second Incompleteness Theorem should be fully credited to Gödel’s seminal paper [20], as opposed to having been implicit from it. In any case, the Hilbert-Bernays textbook [27] vented this generalization in its 1939 second edition, and von Neumann (unpublished) is also known [12, 22, 61] to have made similar observations.

3 English translation: “We must know, We will know.”

4 Some scholars have interpreted ** as anticipating attempts to confirm Peano Arithmetic’s consistency, via either Gentzen’s formalism or Gödel’s Dialectica interpretation. Other scholars have viewed ** as having more ambitious goals in 1931 (that is, goals which Gödel essentially formally withdrew later).
he proved a result leading in the opposite direction. Yourgrau [61] records how von Neumann surprisingly “argued against Gödel himself” in the early 1930’s, about the definitive termination of Hilbert’s consistency program, which “for several years” after [20]’s publication, Gödel “was cautious not to prejudge”. It is known Gödel began to more fully endorse the Second Incompleteness Theorem during a 1933 lecture [21], and he completely embraced it after learning about Turing’s work [47].

Our research in [50]-[59] is related to issues similar to those raised by Hilbert and Gödel in statements * and **. This is because it is awkward to explain how human beings can maintain the psychological drive and needed energy-desire to cogitate, without being stimulated by an instinctive faith in their own consistency (under a definition of such that is suitably gentle and soft to be consistent with the Incompleteness Effect).

Accordingly, our research in [50]-[59] has explored both generalizations and boundary-case exceptions for the Incompleteness Effect, so as to determine exactly which boundary-case evasions are permissible. Our prior research in [50]-[59] used mostly cut-free forms of deduction to evade the restrictions imposed by the Second Incompleteness Effect. The current article will instead focus on the more pristine Hilbert-Frege methods of deduction. Assuming the tempting assumptions of our Conjecture 6.6 are correct (as we do), the present article will show how a new type of \( \theta \) function symbol, employing an “indeterminate-styled” definition of growth (formalized in \( \S 4 \)), will usher in a surprising evasion of the Second Incompleteness Effect.

More details about \( \theta \) will be explained in the next two sections. Essentially, \( \theta \) is needed because a version of the Second Incompleteness Theorem, due to the combined work of Pudlák, Solovay, Nelson and Wilkie-Paris [35, 39, 44, 49], demonstrates that if an axiom system \( \alpha \) proves merely any one of (1) - (3)’s totality statements, then it will be incapable of confirming its own consistency under a Hilbert-style deductive method.

Our Theorem 6.7 will suggest it is possible to obtain a partially positive interpretation for what Hilbert and Gödel were seeking in their statements * and **, in a context where it is known the Second Incompleteness Effect, clearly, precludes a full achievement of their earlier objectives.

3 Motivation for Research and its Broad Perspective

It is helpful to employ a flexible vocabulary so our results will apply to any of the textbook formalisms of say Enderton, Fitting, Hájek-Pudlák, or Mendelson [13, 15, 25, 34]. Let us call an ordered pair \((\alpha, d)\) a “Generalized Arithmetic” iff its two components are defined as follows:

1. The Axiom Basis “\( \alpha \)” of an arithmetic is defined as its set of proper axioms it employs.

2. The Deductive Apparatus “\( d \)” of an arithmetic is defined as the combination of its formal rules for inference and the built-in logical axioms “\( L_d \)” that are used by these rules.
Example 3.1 This notation allows one to conveniently separate the logical axioms $L_d$, associated with $(\alpha, d)$, from the “basis axioms” $\alpha$. It also allows one to isolate and compare various apparatus techniques, including the $d_E$, $d_M$, $d_H$, and $d_F$ methods defined below:

i. The $d_E$ apparatus, introduced in §2.4 of Enderton’s textbook, uses only modus ponens as a rule of inference. The latter will be accompanied by a 4-part system of logical axioms, called $L_{d_E}$, to endow $d_E$ with an ability to support Gödel’s Completeness Theorem.

ii. The $d_M$ apparatus in §2.3 of Mendelson’s textbook and the $d_H$ apparatus in §0.10 of the Hájek-Pudlák’s textbook employ a more compressed set of logical axioms than $d_E$, in a context where they use two rules of inference (modus ponens and generalization). In the end, $d_M$ and $d_H$ prove the same set of theorems as $d_E$ with only minor and unimportant changes in proof length.

iii. The “semantic tableaux” $d_F$ apparatus in Fitting’s and Smullyan’s textbooks [15, 43] was the main focus of our investigations in [50, 51, 53, 54, 56, 59]. It will be rarely used in the current article, however. Unlike $d_E$, $d_M$ and $d_H$, it employs no logical axioms. It instead uses more complicated rules of inference. It has many applications in automated deduction, but is less efficient than $d_E$, $d_M$ and $d_H$ in worst-case environments.

Definition 3.2 The term “Hilbert-style” deductive method shall refer to any deductive apparatus $d$ which employs a modus ponens rule and also satisfies Gödel’s Completeness Theorem. (Thus, each of the $d_E$, $d_M$ and $d_H$ are examples of Hilbert-style deductive methodologies.)

Example 3.3 Some added notation is needed to explain why a Hilbert style deductive apparatus, such as $d_E$, $d_H$ or $d_M$, should be distinguished from $d_F$’s “tableaux” apparatus. Let $Add(x, y, z)$ and $Mult(x, y, z)$ again denote two 3-way predicate symbols specifying that $x + y = z$ and $x \times y = z$. Also, let us recall an axiom basis “$\alpha$” is said to recognize successor, addition and multiplication as “Total Functions” iff (4)-(6) are among its derived theorems.

\[
\forall x \exists z \quad Add(x, 1, z) \quad \text{(4)}
\]

\[
\forall x \forall y \exists z \quad Add(x, y, z) \quad \text{(5)}
\]

\[
\forall x \forall y \exists z \quad Mult(x, y, z) \quad \text{(6)}
\]

An “axiom basis” $\alpha$ is called Type-M iff it includes (4)-(6) as theorems, Type-A if it includes (4) and (5) as theorems, and Type-S if it contains only (4) as a theorem. Also, $\alpha$ is called Type-NS iff it can prove none of these theorems. A summary of the prior literature, using this particular notation, is provided below:

a. The combined research of Pudlák, Solovay, Nelson and Wilkie-Paris [35, 39, 44, 49], as is formalized by statement ++ , implies no natural Type-S generalized arithmetic $(\alpha, d)$ can
recognize its own consistency when \( d \) is one of Example 3.1’s three Hilbert-style deduction operators of \( d_E, d_H \) or \( d_M \):

\[ ++ \quad \text{(Solovay’s modification} [44]\text{ of Pudlák} [39]\text{'s formalism with Nelson and Wilkie-Paris} [32, 49]\text{'s methods): Let} (\alpha, d) \text{ denote a generalized arithmetic supporting the Line (4)’s Type-S statement and assuring the successor operation will satisfy both} x' \neq 0 \text{ and} x' = y' \iff x = y. \text{ Then} (\alpha, d) \text{ cannot verify its own consistency whenever simultaneously} d \text{ is a Hilbert-style deductive apparatus and} \alpha \text{ treats addition and multiplication as 3-way relations, satisfying their usual associative, commutative distributive and identity axioms.}

Essentially, Solovay [44] privately communicated to us in 1994 an analog of theorem ++. Many authors have noted Solovay has been reluctant to publish his nice privately communicated results on many occasions [11, 25, 35, 37, 39, 49]. Thus, approximate analogs of ++ were explored subsequently by Buss-Ignjatovic, Hájek and Švejdar in [11, 23, 45], as well as in Appendix A of our paper [51]. Also, Pudlák’s initial 1985 article [39] captured the majority of ++’s essence, chronologically before Solovay’s observations. Also, Friedman did related work in [16].

b. Part of what makes ++ interesting is that [51, 54, 55] developed two methods for generalized arithmetics to confirm their own consistency, whose natural hybridizations are precluded by ++. Specifically, these results involve either a Type-NS system [51, 55] verifying its own consistency under any of our three discussed variants of Hilbert-style methods, or a Type-A system [50, 51, 54, 56, 59] verifying its self-consistency under \( d_E \)’s tableaux apparatus. Also, Willard [52, 57] observed how one could refine ++ with Adamowicz-Zbierski’s methodology [1, 2] to show Type-M systems cannot recognize their semantic tableaux consistency.

The roles of Items (a) and (b) in our research will become more evident as this article progresses. Essentially, our prior research had focused mostly on Type-A arithmetics that could verify their consistency under either semantic tableaux deduction or some near-cousin of this concept (as was explained in [50]’s short 16-page summary of [50]-[58]’s results). The main challenge, faced by our prior research, was that ++’s generalization of the Second Incompleteness Theorem for Type-S arithmetics caused us to rely on analogs of [55]’s “ISCE” Type-NS framework, when a logic recognizes its own Hilbert consistency. (The latter, unfortunately, dropped the assumption that successor is a total function. It instead constructed the infinite set of integers by employing an infinite set of distinct constant symbols \( C_1, C_2, C_3, \ldots \) where \( C_j = 2^{j-1} \).) In this article, we will improve upon ISCE by showing how one needs only three built-in constant symbols, corresponding to the integers of 0, 1 and 2, to construct an “IQFS” axiom system that can verify its own Hilbert consistency \(- - -\) where here IQFS has access to a new type of “\( \theta \)” function primitive (defined later) for constructing the infinite range of integers from these three starting constants.
**Definition 3.4** Let $\alpha$ again denote an axiom basis, and $d$ designate a deduction apparatus. Then the ordered pair $(\alpha, d)$ will be called a **Self Justifying** configuration when:

i. one of $(\alpha, d)$’s theorems (or possibly one of $\alpha$’s axioms) do state that the deduction method $d$, applied to the basis system $\alpha$, produces a consistent set of theorems, and

ii. the axiom system $\alpha$ is in fact consistent.

**Example 3.5** Using Definition 3.4’s notation, our prior research in [50, 51, 55, 58, 59] developed arithmetics $(\alpha, d)$ that were “Self Justifying”. It also proved the Second Incompleteness Theorem implies specific limits beyond which self-justifying formalisms cannot transgress. For any $(\alpha, d)$, it is almost trivial to construct a system $\alpha^d \supseteq \alpha$ that satisfies the Part-i condition (in an isolated context where the Part-ii condition is not also satisfied). For instance, $\alpha^d$ could consist of all of $\alpha$’s axioms plus an added “SelfRef$(\alpha, d)$” sentence, defined as stating:

$\oplus$ There is no proof (using $d$’s deduction method) of $0 = 1$ from the union of the axiom system $\alpha$ with this sentence “SelfRef$(\alpha, d)$” (looking at itself).

Kleene discussed in [29] how to encode rough analogs of the above “I Am Consistent” axiom statement. Each of Kleene, Rogers and Jeroslow [29, 42, 28], however, emphasized that $\alpha^d$ may be inconsistent (e.g. violating Part-ii of self-justification’s definition), despite SelfRef$(\alpha, d)$’s formalized assertion. This is because if the pair $(\alpha, d)$ is too strong then a classic Gödel-style diagonalization argument can be applied to the axiom system $\alpha^d = \alpha + \text{SelfRef}(\alpha, d)$, where the added presence of the statement SelfRef$(\alpha, d)$ will cause this extended version of $\alpha$, ironically, to become automatically inconsistent. Thus, the encoding of “SelfRef$(\alpha, d)$” is relatively easy, via an application of the Fixed Point Theorem, but this sentence is ironically **typically useless**!

Unlike our earlier work, which focused mostly on a semantic tableau apparatus for deduction, the current paper will explore Definition 3.2’s more pristine Hilbert-style methodologies. There are, of course, many types of generalizations of the Second Incompleteness Theorem known to arise in Hilbert-like settings [4, 5, 6, 7, 9, 11, 14, 16, 20, 23, 24, 25, 27, 33, 30, 31, 36, 38, 39, 40, 44, 45, 48, 49, 51, 55]. Each such generalization formalizes a paradigm where self-justification is infeasible under a Hilbert-style apparatus.

Our main prior research about Hilbert consistency appeared in [55]. Its ISCE($\beta$) formalism could recognize its own Hilbert consistency and prove analogs of any r.e. extension $\beta$ of Peano Arithmetic’s $\Pi_1$ theorems. It unfortunately required the use of an infinite number of constant symbols, with ISCE($\beta$) using one built-in constant symbol $C_i$, for each power of 2.

Prior to [55]’s publication, Pavel Pudlák [41] examined our article and asked whether one could improve upon ISCE’s properties by using Ajtai’s observations [3] about the Pigeon Hole principle.
In a briefer and more informal respect, Sam Buss asked us a similar question during a lunch at a 1997 logic conference [10]. Our prior partial answer to these questions was offered in Sections 6 of [55]. A different and substantially more interesting reply to these questions will be offered in the current paper.

**Definition 3.6** Relative to any fixed given axiom system \( \gamma \), a formal function symbol \( F \) will be called a Q-Function iff \( \gamma \) is sufficiently ambiguous for there to exist an UNCOUNTABLY infinite number of different plausible sequences of formally enumerated ordered pairs \((i, a_i)\) in expression (7) where \( F(i) = a_i \) is allowed as a permissible representation for \( F \) under \( \gamma \)'s axioms.

\[
(0, a_0), (1, a_1), (2, a_2), (3, a_3), (4, a_4) \ldots
\]  

Most Q-Function symbols are unsuitable for analyzing Hilbert's Second Open Question or most issues in mathematics. This is because the presence of an uncountably infinite number of different plausible sequences, using Line (7) to specify \( F(i) = a_i \), is more of a burden than a benefit (in most cases). An exception to this general rule of thumb will be provided by the next section’s \( \theta \) operator. It will be germane to ++’s generalization of the Second Incompleteness and suggest a mechanism whereby an efficient form of “Type-NS” self-justifying arithmetic can recognize its own Hilbert consistency in a quite substantial sense.

**4 Starting Notation Conventions**

A function \( H \) will be called Non-Growth whenever \( H(a_1, a_2, \ldots, a_j) \leq \text{Maximum}(a_1, a_2, \ldots, a_j) \) for all \( a_1, a_2, \ldots, a_j \). Six examples of non-growth functions are:

1. Integer Subtraction where “ \( x - y \) ” is defined to equal zero when \( x \leq y \),
2. Integer Division where “ \( x \div y \)” equals 0 when \( y = 0 \) and it equals \( \lfloor x/y \rfloor \) otherwise,
3. Maximum\((x,y)\),
4. Root\((x,y)\) which equals \( \lfloor x^{1/y} \rfloor \) when \( y \geq 1 \) (and zero otherwise),
5. Logarithm\((x) = \lfloor \log_2(x) \rfloor \) and
6. Count\((x,j) = \) the number of “1” bits among \( x \)'s rightmost \( j \) bits.

These operations were called either the “Grounding” or “Ground-Level” functions in our articles [51, 53, 54, 59]. We will use the latter nomenclature in the current article because the notion of a “Ground-Level” function should not be confused with the very different notion of a “Grounded Term” employed by Definition 4.5.

Our starting language \( L^G \) will also contain the two atomic symbols of “ \( = \) ” and “ \( \leq \) ” and three built in constants symbols, \( C_0 \), \( C_1 \) and \( C_2 \), for representing the values of 0, 1 and 2. In this context, expressions (8) and (9) formalize exactly how addition and multiplication can be encoded as two 3-way predicates, denoted as Add\((x, y, z)\) and Mult\((x, y, z)\).

\[
z - x = y \land z \geq x
\]  

(8)
\[
(x = 0 \lor y = 0) \Rightarrow z = 0 \quad \land 
(x \neq 0 \land y \neq 0) \Rightarrow \left( \frac{z}{x} = y \quad \land \quad \frac{z-1}{x} < y \right)
\] (9)

Also, our constant symbols, \(C_1\) and \(C_2\), for representing the quantities 1 and 2, allow us to formalize the following further non-growth operations:

- \(\text{Pred}(x) = x - 1\) (in a context where the prior paragraph’s definition for Subtraction implies \(\text{Pred}(0) = 0\).)
- \(\text{Half}(x) = x \div 2\) (in a context where “\(x \div 2\)” equals technically \(\lfloor \frac{x}{2} \rfloor\) under the prior paragraph’s notation).
- \(\text{Pred}^n(x)\) defined to be \(n\) iterations of the Predecessor operation
- \(\text{Half}^n(x)\) defined to be \(n\) iterations of the halving operation.

Let us say that a function symbol \(H(x_1, x_2, \ldots, x_j)\) is **Growth Permitting** iff for each integer \(k \geq 2\) there exists a “growth-tuple” \((a_1, a_2, \ldots, a_j)\) satisfying \(H(a_1, a_2, \ldots, a_j) > k\) and also having each \(a_i \leq k\). It will be necessary to employ either an infinite number of constant symbols or some Growth-Permitting function, so that an extension of our language \(L^G\) can construct the naturally extended broader set of integers \(3, 4, 5, 6, \ldots\) from its starting objects of 0, 1 and 2.

One method for resolving this problem has been presented in [55]. Its ISCE(\(\beta\)) axiom basis did employ an infinite number of further constant symbols. It was compatible with self-justification, but deviated from Hilbert’s intended goals because it employed a *highly awkward* infinite number of distinct constant symbols.

The challenge posed by ++ is, thus, formidable. Our goal in this article will be to suggest how a Q-function primitive \(F(x)\), constrained by deliberately ambiguous axioms, can help overcome the constraint that ++ imposes. Such an unusual primitive \(F\) will have an uncountable number of vectors, analogous to Line (7), as permissible solutions for \(F\)’s definition. Our basic goal will be to outline how this unusual concept is germane to the self-justifying aspirations that Hilbert and Gödel had expressed in \(\ast\) and \(\ast \ast\).

**Definition 4.1** Let us say a formal function symbol \(F(x_1, x_2, \ldots, x_j)\) is **“1-Definitive”** iff it has only one permissible solution under an axiom system \(\gamma\)’s definition of it. Let us call the function \(F\) **“Indeterminate”** otherwise.

Mathematicians obviously typically avoid using axiom systems that entail using ambiguous function definitions (e.g. they certainly usually avoid analogs of Line (7)’s dizzying quantity of possibly \(\aleph_1\) distinct solutions for its defined function \(F\)). Our effort to overcome ++’s broad-scale generalization of the Second Incompleteness Theorem will be unconventional because we will use an indeterminate function, called the \(\theta\) operator, to overcome ++’s challenge. It will turn out that the \(\theta\) operator will almost duplicate the logarithmic efficiency of the traditional addition and multiplication function symbols for constructing the natural numbers, while simultaneously dovetail around ++’s generalization of the Second Incompleteness Theorem.
The indeterminate nature of $\theta$ will be crucial to our evasion of the Second Incompleteness Effect. Our research into this topic was greatly influenced by an email we received from Pavel Pudlák [41], shortly after he received an advanced copy of our article [55]. He appreciated the nature of the challenge the ISCE axiom framework faced, when it used an awkwardly infinite number of constant symbols. (ISCE relied on this infinite magnitude to evade $++$’s requirement that no evasion of the Second Incompleteness Theorem can take place when an arithmetic recognizes Successor as a total function.) Pudlák’s emailed communications [41], thereby suggested we look at Ajtai’s work [3] about a Pigeon-Hole function $\zeta(x)$, defined by the identities (10) and (11).

$$\forall x \quad \zeta(x) \neq 0$$

$$\forall x \; \forall y \; x \neq y \; \Rightarrow \; \zeta(x) \neq \zeta(y)$$

The relevance of $\zeta$ can be best appreciated when $\zeta^n(x)$ denotes a term $\zeta(\zeta(\ldots(\zeta(x))))$ consisting of $n$ iterations of the $\zeta$ operator. Line (12)’s composite term $S_n$ will then satisfy the lower bound of $S_n > n$.

$$S_n = \text{Max}[\zeta(0), \zeta^2(0), \zeta^3(0), \ldots, \zeta^n(0)]$$

Pudlák observed $\zeta(x)$ will grow too slowly (in the worst case) for one to be able to deduce successor is a total function from its properties [5]. His email [41] hinted the inequality $S_n > n$ might enable a formalism, utilizing the $\zeta$ operative, to improve upon [55]’s results.

It was initially unclear whether a positive answer to Pudlák’s interesting question would resolve ISCE’s main difficulties. This is because (12)’s term $S_n$ requires $O(n^2)$ logic symbols to encode an integer quantity greater than $n$ (since its term $\zeta^j(0)$ uses $O(j)$ logic symbols). Thus, the integer $2^{100}$, whose binary encoding uses 100 bits, would actually require in excess of $2^{200}$ bits to encode. Such quantities, exceeding the number of atoms in the universe, were troubling because our general goal has been to construct self-justifying arithmetics that possessed, at least, some partial facets of pragmatic value.

The remainder of this section will outline how a different type of Q-Function operator will be much better than $\zeta$ for meeting our needs. During our discussion, Power($x$) will denote a primitive specifying $x$ is a power of 2. It is encoded by (13) because our Grounding language has “Logarithm($x$)” $= \lfloor \log_2(x) \rfloor$.

$$x = 1 \lor \text{Logarithm}(x) \neq \text{Logarithm}(x - 1)$$

The operation $\zeta(x)$ will grow at a slower rate than Successor, if it equals $x + 1$ for all standard numbers $x$ and if $\zeta(x) = x - 1$ when $x$ is a non-standard integer. This seemingly minute detail implies one cannot infer Successor is a total function from $\zeta$’s behavior, since the latter is contradicted by a model where all non-standard numbers have sizes bounded by some fixed number B. (This subtle detail, raised by Pudlák’s email [41], was fascinating because it raised the question about whether a partial exception to Example 3.3-a’s invariant $++$ might plausibly exist.)
In this context, \( \theta(x) \) will denote our new analog of the \( \zeta(x) \) function that walks among the powers of 2 in a manner similar to \( \zeta(x) \)’s walk through conventional integers. It is defined by (14) -(17).

\[
\forall x \quad \text{Power}(x) \Rightarrow \text{Power}(\, \theta(x) \,)
\]

\[
\forall x \quad \theta(x) \neq 1
\]

\[
\forall x \forall y \quad [x \neq y \land \text{Power}(x)] \Rightarrow \theta(x) \neq \theta(y)
\]

\[
\forall x \quad \neg \text{Power}(x) \Rightarrow \theta(x) = 0
\]

It needs to be emphasized that (14) – (17) will be the only vehicle our self-justifying axioms have available to construct integers \( \geq 3 \). These axioms will be called the Up-Walking axioms. (The axiom (17) is, technically, unnecessary to construct integers \( \geq 1 \), but it is helpful because it formalizes how our methodology will treat integers which are not powers of 2.)

Both the Q-functions \( \zeta \) and \( \theta \) are challenging because there are a dizzying \( \aleph_1 \) distinct vectors, analogous to Line (7), that are representations of these functions. We will soon see, however, that there is a sharp distinction between these concepts from a computational complexity perspective.

**Definition 4.2** Let \( L^Q \) and \( L^Q^* \) denote the extensions of \( L^G \)’s Grounding language that contain the respective additional function symbols of \( \theta \) and \( \zeta \). Then \( L^Q \) shall be called the Q-Grounding language, and \( L^Q^* \) will be called the Q* Grounding language.

**Proposition 4.3** In contrast to the Q* Grounding language that requires \( O(\, n^2 \, ) \) function symbols for defining a term \( T^*_n \) for representing the integer \( n \), the Q-Grounding language needs only \( O(\, \log^3 n \, ) \) symbols to encode a term \( T_n \) representing \( n \).

Proposition 4.3’s proof will rely upon the following notation convention:

**Definition 4.4** Let \( \theta^j(x) \) denote the term \( \theta(\, \theta(\ldots\theta(x))\,) \) where there are \( j \) iterations of the \( \theta \) operation. Let \( E_j \) denote the quantity produced by (18)’s division operation:

\[
\text{Max} \left\{ \theta^j(1), \, \theta^{j-1}(1), \, \ldots, \theta(1) \right\}
\]

\[
\text{Half}^j \left\{ \text{Max} \left\{ \theta^j(1), \, \theta^{j-1}(1), \, \ldots, \theta(1) \right\} \right\}
\]

(18)

It is easy to establish that \( E_j = 2^j \) for every \( j \geq 1 \). This is because (18)’s twice-repeating term of Max \( \left[ \theta^j(1), \, \theta^{j-1}(1), \, \ldots, \theta(1) \right] \) is a power of 2 exceeding \( 2^j \). For the additional case where \( j = 0 \), we shall define \( E_0 = 1 \) (by having the formal constant symbol of \( C_1 \) stand for “1” ).

Proof of Proposition 4.3: Easy consequence of Definition 4.4’s machinery. Thus if \( n \) is a power of 2 of the form \( 2^j \) then expression \( E_j \) is a term representing \( n \)’s value that employs \( O(\, \log^2 n \, ) \) logical symbols. On the other hand, if \( n \) is not a power of 2 then it can be defined with \( O(\, \log^3 n \, ) \) symbols by setting \( E_j \) equal to the least power of 2 greater than \( n \) and subtracting from \( E_j \) those powers of 2 that are needed to produce \( n \)’s value. For example since 76 = 128 – 32 – 16 – 4, it can be formalized as a term \( T_{76} \) defined by \( E_7 - E_5 - E_4 - E_2 \).
Definition 4.5 A “term” in mathematical logic is defined to be a syntactic object, built out of solely symbols for representing functions, constants and variables. Such an object is called a “Ground Term” when it is built out of solely function and constant symbols. For example in our Q-Grounding language (which uses $C_0$, $C_1$ and $C_2$ as built-in constants), the expression “$C_2 - C_1$” is a ground term. Two more complex examples of ground terms are “Max($C_2, C_1 - C_0$)” and “Max( $\theta(C_1), C_2$ )”. Also, expression $E_j$ in Line (18) should be viewed as a ground term (when one views its use of the symbol “1” as an abbreviation for the constant “$C_1$”).

Remark 4.6. We will distinguish in Proposition 7.1 and in other parts of §7 between two kinds of ground terms, that are called the “Tree-Oriented” and “Dag-Oriented” formats. The latter will differ from a more conventional tree structure by having a Directed Acyclic Graph structure replace a term’s usual tree format for defining its quantitative values. Our discussion in the next several sections will be simplified if we use the shorter phrase of “Ground Term” to refer to what §7 will more accurately called a “Tree-Oriented Ground Term”. (It will turn out Proposition 7.1 will later explain how Dag-oriented ground terms differ from their tree-oriented counterparts by reducing the $O(\log^3 n)$ length of a tree-oriented term to a more compact $O(\log n)$ size.)

Definition 4.7 A ground term $T$ will be called an “Observable” object iff there is an unique interpretation of its quantified value in the Q-Grounding language. It is called an “Unobservable” iff it has multiple such interpretations due to $\theta$’s “indeterminate” definition (e.g. see Definition 4.1).

Example 4.8 The previously mentioned ground term Max( $\theta(C_1), C_2$ ) is an “unobservable” because it can assume any of the plausible integer values of 2, 4, 8, 16 … . On the other hand, (19) is an “observable” that represents the integer value of “3”. (This is because its twice-repeating term “Max[ $C_2, \theta(C_2), \theta^2(C_2)$ ]” is bounded below by 4, causing the left and right sides of its subtraction operation to differ by exactly 3.)

$$\text{Max}[ C_2, \theta(C_2), \theta^2(C_2) ] - \text{Pred}^3 \{ \text{Max}[ C_2, \theta(C_2), \theta^2(C_2) ] \}$$

(19)

Our notation also implies that Line (18)’s expression $E_j$ is an “observable”. This implies, in turn, that Proposition 4.3’s term $T_n$ is an “observable” employing no more than $O(\log^3 n)$ logical symbols. For example since $76 = 128 - 32 - 16 - 4$, it follows that $T_{76}$ corresponds to the observable term $E_7 - E_5 - E_4 - E_2$, where each $E_j$ employs only $O(\log^2 j)$ symbols.

Thus, Definition 4.7 and Example 4.8 have illustrated how the realm of “observable” objects is a broad and accessible world, of non-trivial significance. It allows every integer $n$ to be represented by a term $T_n$ with a tight $O(\log^3 n)$ length (in a context where Proposition 7.1’s more elaborate formalism will allow us to reduce this length to a yet more attractive $O(\log n)$ size).

The distinction between “Observables” and “Unobservables” will offer a new perspective on the aspirations which Hilbert and G"odel stated in their statements * and **, under the new IQFS
axiomatic framework that is proposed later in this article. It will suggest how the Second Incompleteness Theorem can be seen as a fascinating majestic result from a purist perspective, while a well-defined fragment of what Hilbert and Gödel sought in * and ** can be satisfied, in at least a diluted sense.

Remark 4.9 (explaining the goals of this paper): Let us say a basis axiom system $\alpha$ owns a “Finitized Perspective” of the Natural Numbers if it requires only a finite number of proper axioms to construct the full set of integers $0, 1, 2, 3, \ldots$. All conventional arithmetics have this property. Such logics fall into two categories, called Single and Double-Formatted systems. They are defined below:

a. Single-Formatted Arithmetics consist of axiomatic basis systems whose ground terms are all Observables. (Most conventional arithmetics lie in this category because they typically employ the Successor operation in a traditional manner.)

b. Double-Formatted Arithmetics represent systems whose ground terms may be either Observables or Unobservables. (Axiomizations for Q-Grounded logics are “Double-Formatted” because they allow $\theta$’s analog of Line 7’s function symbol $F$ to have an uncountable number of different allowed representations).

The distinction between categories (a) and (b) is significant because Example 3.3-a explained how statement ++’s generalization of the Second Incompleteness Theorem applies to any formalism recognizing Successor as a total function. Thus, Double-Format logics are useful, if one wishes to consider alternatives where successor is not seen as a total function. In this context, Hilbert’s Second Open Problem can be viewed as a 2-part question, composed of sub-queries Q-1 and Q-2:

**Question Q-1** Are any axiom systems able to prove theorems verifying their own consistency in a robust sense? The answer to Q-1 is clearly “No” because the combination Gödel’s initial 1931 result [20] with Hilbert-Bernays’s result [27] and the Pudlák-Solovay invariant ++ (from Example 3.3-a) imply arithmetics of ordinary strength cannot prove their own consistency in a robust sense.

**Question Q-2** Can logic systems “appreciate” their own consistency in some reduced sense, that is diluted but not fully immaterial? The answer to Q-2 is complex because some logics, such as our proposal in [50, 51, 54, 55, 58, 59]’s weaker and earlier paradigms, can formalize their own consistency, using Example 3.5’s Fixed-Point “I am consistent” axiom.

A theme of this article will be that the distinguishing between questions Q-1 and Q-2 and between Single and Double-Formatted Logics is related to the mystery enshrouding the Second Incompleteness Theorem. This is because there can be no doubt that the Second Incompleteness Theorem is fully robust from a purist mathematical perspective. Yet, it is still problematic to fully
dismiss Hilbert’s 1926 suggestion that some specialized forms of logics should possess a type of well-defined knowledge about their own internal consistency. (This is because it is very awkward to explain how human beings are able to motivate their own cognitive process, otherwise.)

Thus the distinguishing between the questions Q-1 and Q-2, combined with Example 3.5’s “SelfRef” formalism and Proposition 4.3’s “θ primitive”, will usher in our new approach.

Before discussing our new “IQFS” methodology, it should be mentioned that other unusual interpretations of the Second Incompleteness Theorem have followed from Gentzen’s perspectives about transfinite induction under his $\varepsilon_0$ ordinal [19, 46], the Kreisel-Takeuti’s “CFA” system [32] and also the interpretational frameworks of Friedman, Nelson, Pudlák and Visser [17, 35, 39, 48]. These systems are unrelated to our methods. They do not use Kleene-like “I am consistent” axioms. Also, they employ “cut-free” logics (rather than a preferable Hilbert-style deductive apparatus). Their fascinating perspective should be examined by researchers interested in the Second Incompleteness Theorem, although it is unrelated to our approach in the next section.

5 The ISCE and IQFS Axiomatic Formalisms

The only aspect of our prior research that will be directly related to our new IQFS axiomatic framework is [55]’s ISCE formalism. The next several paragraphs will review [55]’s definitions of ISCE, given in its Sections 3 & 4. After reviewing ISCE’s properties, the remainder of this section will explain, intuitively, how our new IQFS framework can be incrementally defined, by suitably revising ISCE’s prior definition. (A subsequent formal proof, affirming IQFS’s exact self-justification properties, will appear in §6.)

During our discussion, $L^G$ will again denote our Grounding-level language built out of our six basic non-growth functions (e.g. the Subtraction, Division, Maximum, Logarithm, Root and Count operations). Also, $C_0$, $C_1$ and $C_2$ will again denote three constant symbols designating the integers values of “0”, “1” and “2”. In a context where $\text{Pred}(x)$ is an abbreviation for “$x - 1$” (or more precisely “$x - C_1$”), the ISCE axiom system from [55] had used (20)’s axiom statement to define $C_0$, $C_1$ and $C_2$:

$$\text{Pred}(C_0) = C_0 \land C_1 \neq C_0 \land \text{Pred}(C_1) = C_0 \land \text{Pred}(C_2) = C_1 \land \forall x \neg (C_0 < x < C_1)$$ (20)

The challenge [55] faced was its formalism could not use any of the function-operations of successor, addition or multiplication to infer the existence of larger integers from the initial constants of $C_0$, $C_1$ and $C_2$. This was because the Pudlák-Solovay result ++ indicated the presumption successor is a total function precludes most systems from recognizing their own Hilbert consistency.

Our article [55] considered two alternatives for overcoming these difficulties, called the Additive and Multiplicative Naming conventions. They defined some further constant symbols
where \( C_j = 2^{j - 1} \) and \( C^*_j = 2^{2^{2j - 2}} \) under these two conventions.

The definition of these constants is easy under \( L^G \)’s language. This is because Lines (8) and (9) specify how two 3-way predicates, called Add\((x, y, z)\) and Mult\((x, y, z)\), encode the identities of \( x = y + z \) and \( x \times y = z \). Our additive and multiplicative conventions can, then, define \( C_3, C_4, C_5, \ldots \) and \( C^*_3, C^*_4, C^*_5, \ldots \) via an infinite number of instances of (21) and (22)’s particular axiom schemas, respectively:

\[
\begin{align*}
\text{Add}(C_{j-1}, C_{j-1}, C_j) \quad &\quad (21) \\
\text{Mult}(C^*_{j-1}, C^*_{j-1}, C^*_j) \quad &\quad (22)
\end{align*}
\]

The methodology in \([55]\) presumed the “names” for its constants \( C_j \) and \( C^*_j \) had nice compact encodings using \( O(\log(j)) \) bits. Its formalism calculated the values of “unnamed” integers from named entities via the non-growth Subtraction and Division primitives. For instance since \( 20 = 32 - 8 - 4 \), the quantity 20 can be encoded as \( C_6 - C_4 - C_3 \).

The challenge \([55]\) faced was to determine whether self-justification was possible under either (21)’s or (22)’s schema. It found (22)’s multiplicative convention was incompatible with self-justification (due to its speedy growth rate), but (21)’s additive schema did, conveniently, permit self-justification.

Our new proposed Double-Formatted form of a self-justifying axiom system is easiest to describe, if we first review \([55]\)’s Single-Formatted formalism and then refine it.

The extension of our base-language \( L^G \) that includes the Additive Naming Convention (ANC)’s additional constants \( C_3, C_4, C_5, \ldots \) will be called an ANC-Based Language and be denoted as \( L^{ANC} \). Also if \( t \) denotes any term in \( L^{ANC} \)’s language, then the quantifiers in the two wffs of \( \forall v \leq t \ \Psi(v) \) and \( \exists v \leq t \ \Psi(v) \) will be called \( L^{ANC} \)’s “Bounded Quantifiers”.

**Definition 5.1** The analogs of a conventional arithmetic’s \( \Delta_0, \Pi_n, \Sigma_n \) formulae in the language \( L^{ANC} \) will be denoted as \( \Delta_0^{ANC}, \Pi_n^{ANC}, \Sigma_n^{ANC} \). Thus, a formula is defined to be \( \Delta_0^{ANC} \) iff all its quantifiers satisfy the prior paragraph’s bounding condition. The definitions of \( \Pi_n^{ANC} \) and \( \Sigma_n^{ANC} \) formulae, in Items 1-3, are also quite conventional:

1. Every \( \Delta_0^{ANC} \) formula is considered to be also a \( \Pi_0^{ANC} \) and an \( \Sigma_0^{ANC} \) expression.

2. A formula is called \( \Pi_n^{ANC} \) when it can be encoded as \( \forall v_1 \ldots \forall v_k \Phi \) where \( \Phi \) is \( \Sigma_{n-1}^{ANC} \)

3. A formula is called \( \Sigma_n^{ANC} \) when it can be encoded as \( \exists v_1 \ldots \exists v_k \Phi \), where \( \Phi \) is \( \Pi_{n-1}^{ANC} \).
Given an initial axiom system $\beta$, the Theorem 3 of [55] did formalize a self-justifying logic, called ISCE($\beta$), that could prove all $\beta$’s $\Pi^1_{ANC}$ theorems and which could also verify its own consistency under any Hilbert-style deductive apparatus (including the $d_E$, $d_M$ and $d_H$ deductive methodologies) The axiom basis for ISCE($\beta$) was comprised, formally, of the following four distinct groups of axioms:

**GROUP-ZERO:** This schema will use Line (20)’s axiom to define the constants of $C_0$, $C_1$ and $C_2$ and Line (21)’s Additive Naming schema to define the further constants $C_3, C_4, C_5, \ldots$

**GROUP-1:** It is convenient to define ISCE’s Group-1 schema using a notation that will support [55]’s Theorem 3 in a more general sense than had appeared in [55], so that our new “IQFS” formalism (appearing later in this section) will be easier to describe. Let us therefore say a $\Pi^1_{ANC}$ sentence is **Simple** iff the only built-in constants it employs are $C_0$, $C_1$ and $C_2$. Then ISCE’s Group-1 scheme will be allowed to be any finite set of simple $\Pi^1_{ANC}$ axioms, called $S$, that is consistent with the Group-zero schema and which has the following properties:

1. The union of $S$ with ISCE’s Group-Zero axioms will be capable of proving all $\Delta^0_{ANC}$ statements which are true.
2. The union of $S$ with ISCE’s Group-Zero scheme will also be capable of proving that the “$=$” and “$\leq$” predicates support their conventional transitivity, reflexivity, symmetry and total ordering properties.

Any finite set of $\Pi^1_{ANC}$ axioms with the above properties can be used to define $S$ and support an analog of [55]’s Theorem 3, by an easy generalization of the methodologies from Sections 3 and 4 of [55]. (Thus, any such finite set $S$ supporting Conditions (1) and (2) can be employed to define ISCE’s Group-1 axiom schema.)

**GROUP-2:** Let $\Phi^\gamma$ denote $\Phi$’s Gödel number, and $\text{HilbPrf}_\beta(x,y)$ denote a $\Delta^0_{ANC}$ formula indicating $y$ is a Hilbert-styled proof from axiom system $\beta$ of the theorem $x$. For each $\Pi^1_{ANC}$ sentence $\Phi$, the Group-2 schema for ISCE($\beta$) will contain a $\Pi^1_{ANC}$ axiom of the form:

\[ \forall y \{ \text{HilbPrf}_\beta (\Phi^\gamma, y) \Rightarrow \Phi \} \quad (23) \]

**GROUP-3:** This last part of [55]’s ISCE($\beta$) formalism is a single self-referencing $\Pi^1_{ANC}$ sentence stating:

\[ \otimes \otimes \quad \text{“There exists no Hilbert-style proof of } 0=1 \text{ from the union of the Group-0, 1 and 2 axioms with THIS SENTENCE (referring to itself)”} \]

**Clarifying $\otimes \otimes$’s Meaning:** Several of our articles [51 54 55 58] employed self-referential $\Pi^1_{ANC}$ constructions, similar to $\otimes \otimes$, as the Example [55] did mention. A reader can, thus, find several illustrations about how $\otimes \otimes$ is encoded in these articles.

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6A proof of this specific generalization of [55]’s results is entirely routine. For the sake of brevity, it is omitted.
**Definition 5.2** Let $I(\bullet)$ denote an operation that maps an initial axiom basis $\beta$ onto an alternate system $I(\beta)$. (One example of such an operation is the ISCE(\bullet) framework, that maps an initial axiom basis of $\beta$ onto the alternate formalism of ISCE(\beta).) Such an operation $I(\bullet)$ is called **Consistency Preserving** iff $I(\beta)$ is consistent whenever the union of $\beta$ with the Groups 0 and 1 axiom schemas is consistent.

**Example 5.3** Several of our research projects had employed Definition 5.2’s framework. For instance, [55] demonstrated the ISCE(\bullet) mapping was consistency preserving. Thus if PA+ denotes the extension of Peano Arithmetic that includes PA’s traditional Addition and Multiplication functions plus $L^G$’s six added Grounding-level function primitives, then ISCE(PA+) must automatically be consistent (because PA+ was consistent). Hence while Peano Arithmetic is unable to verify its own consistency (on account of Gödel’s seminal 1931 discovery), it is known to be sufficiently agile to prove the following relative-consistency statement:

# If PA is consistent then ISCE(PA+) is self-justifying.

The preceding relative-consistency statement provides a partial positive answer to the Q-2 version of Hilbert’s Second Question. It captures one respect in which ISCE(PA+) can appreciate its own consistency. This respect is, obviously, of a sharply limited nature because ‘‘’s generalization of the Incompleteness Theorem indicates no Type-S arithmetic can recognize its Hilbert consistency and view successor as a total function. It does, however, raise an enticing question:

# # Can the infinite number of distinct constant symbols, employed by ISCE’s Group-Zero schema, be reduced to a finite size by a Type-NS Self-Justifying Logic in some type of formal manner?

The next two sections of this paper will outline how a quite encouraging answer to # #’s query shall arrive, when one modifies ISCE’s formalism with the Q-function operative of $\theta$.

**Definition 5.4** Let $L^Q$ once again denote the special extension of $L^G$’s Grounding language that includes the added new Q-function symbol of $\theta$. Then $\Delta^Q_0$, $\Pi^Q_n$ and $\Sigma^Q_n$ will denote the analogs of Definition 5.1’s $\Delta^{ANC}_0$, $\Pi^{ANC}_n$ and $\Sigma^{ANC}_n$’s formulae under $L^G$’s modified language. In particular, if $\Phi$ is one of an $\Delta^{ANC}_0$, $\Pi^{ANC}_n$ or $\Sigma^{ANC}_n$ formula, then $\Phi^Q$ will be called $\Delta^Q_0$, $\Pi^Q_n$ or $\Sigma^Q_n$ when it differs from $\Phi$ by exactly replacing each constant $C_j$ from the set $C_3, C_4, C_5...$ with Line 18’s term $E_{j-1}$.

**Example 5.5** Suppose $\Phi$ is one of a $\Delta^{ANC}_0$, $\Pi^{ANC}_n$ or $\Sigma^{ANC}_n$ sentence that employs the three constant symbols of $C_4$, $C_6$ and $C_{10}$ for representing the three numbers of 8, 32 and 512. Let us recall that $E_3$, $E_5$ and $E_9$ formulate these three quantities under Line 18’s notation. Then $\Phi^Q$ will have an identical definition as $\Phi$ except each $C_j$ is replaced by $E_{j-1}$.
A formula is, moreover, defined to lie in one of the $\Delta^0_n$, $\Pi^0_n$ or $\Sigma^0_n$ classes if and only if it is constructed in exactly such a manner. This fact ensures that all the main primary terms employed in our three major classes of sentences are “Observable” terms. Hence “Unobservable” ground terms are allowed in $L^Q$’s language, but they are excluded from occurring as the “end-product primary” terms in the $\Delta^0_n$, $\Pi^0_n$ or $\Sigma^0_n$ theorems that do encapsulate the intended use of our formalism. We will need one more preliminary definition before “IQFS” can be defined.

**Definition 5.6** Let us recall that Example 3.1 defined three examples of Hilbert-style deductive methodologies. The symbol $d_{ER}$ will denote the particular version of Hilbert-style deduction that we will use in this article. It will be a minor revision of Enderton’s $d_E$ apparatus (that was mentioned in Example 3.1). Unlike Enderton’s schema, it will not take all tautologies as axioms. Instead, it will use the following three axiom schemas, from Mendelson’s textbook [34], to prove all tautologies as theorems:

I. $B \Rightarrow (C \Rightarrow B)$

II. $[B \Rightarrow (C \Rightarrow D)] \Rightarrow [(B \Rightarrow C) \Rightarrow (B \Rightarrow D)]$

III. $[(\neg C) \Rightarrow (\neg B)] \Rightarrow \{[(\neg C) \Rightarrow B] \Rightarrow C\}$

This approach is preferable because the problem of identifying tautologies is NP-hard (e.g. it is preferable to avoid viewing tautologies as axioms and to instead employ the axiom schemas I-III to prove tautologies as theorems). Thus, the logical axioms of $d_{ER}$ shall consist of the axiom schemas I-III plus the below schemas 2-4 from §2.4 of Enderton’s textbook. The deductive apparatus $d_{ER}$ will also follow [13]’s example by treating $\forall x \Psi$ automatically as a logical axiom whenever $\Psi$ is a logical axiom.

2. $\forall x \phi(x) \Rightarrow \phi(t)$ when $t$ is substitutable for $x$ in $\phi$.

3. $\forall x (\phi \Rightarrow \psi) \Rightarrow \{ (\forall x \phi) \Rightarrow (\forall x \psi) \}$

4. $\psi \Rightarrow \forall x \psi$ when $x$ does not occur free in $\psi$.

The symbol $L_{d_{ER}}$ will henceforth denote this set of six basic logical axiom schemas (e.g. items I-III combined with 2-4). Our $d_{ER}$ deduction methodology will follow Enderton’s example by using modus ponens as its only employed rule of inference.\footnote{Unlike several other textbooks, Enderton [13] does not view generalization as a formalized rule of inference because he takes Items (3) and (4) as logical axioms.}

The remainder of this section will combine the formalisms of Definitions 5.4 and 5.6 to define our new axiom system, IQFS($\beta$), that can prove all $\beta$’s $\Pi^0_1$ theorems as well as recognize its own consistency under $d_{ER}$’s form of Hilbert-style deduction. In essence, IQFS($\beta$) will be the direct
analog of [55]’s ISCE(\(\beta\)) axiom system (when the logic’s base language is changed from \(L^{ANC}\) to \(L^{Q}\) and our focus changes from \(\Pi_{1}^{ANC}\) to \(\Pi_{1}^{Q}\) theorems).

**Definition 5.7** The acronym “IQFS” stands for “Introspective Q-Function Semantics”. In a context where \(\beta\) is an initial axiom system that proves theorems in the language \(L^{Q}\), the system, IQFS(\(\beta\)) will be designed to be a 4-part formalism, analogous to ISCE(\(\beta\)), that can prove all \(\beta\)’s \(\Pi_{1}^{Q}\) theorems and recognize its own consistency. In essence, IQFS(\(\beta\))’s definition will be identical to ISCE(\(\beta\)), except for the following four changes (three of which are trivial):

a. The **GROUP-ZERO** schema of IQFS will differ from ISCE’s analog by replacing Line (21)’s “Additive Naming” schema with the Up-Walking axioms, from Lines (14)–(17). (This is because the language \(L^{Q}\) differs from \(L^{ANC}\) by having the Q-function operator of \(\theta\) define the formal quantities that are represented by the constant symbols of \(C_{3}, C_{4}, C_{5}, \ldots\) under \(L^{ANC}\).) In order to assure that the four Up-Walking axioms are adequate to determine \(E_{j}\) is numerically equivalent to \(C_{j+1}\), our new Group-Zero schema will include Line (20)’s axiom (for defining the starting constants of \(C_{0}\), \(C_{1}\), and \(C_{2}\)) as well as (24)’s added axiom:

\[
\forall x \left\{ \left[ \text{Power}(x) \land x \geq 2 \right] \Rightarrow \frac{x}{\text{Half}(x)} = 2 \right\}
\]

(24)

b. The **GROUP-1** scheme of IQFS will be identical to ISCE’s counterpart, except it will reflect Item a’s modification of the Group-Zero scheme for \(L^{Q}\)’s particular revised language (e.g. the footnote \(^8\) describes this easily straightforward revision of ISCE’s Group-1 scheme.)

c. All the \(\Pi_{1}^{Q}\) axioms lying in IQFS’s **GROUP-2** scheme will be identical to their counterparts under ISCE, except they will employ Definition [55]’s machinery for translating \(\Pi_{1}^{ANC}\) sentences into their equivalent \(\Pi_{1}^{Q}\) counterparts. In particular, let us assume the Gödel number “\(\neg \Phi\)” is defined via the “Byte-Style” encoding method (defined in the next section) and \(\text{HilbPrf}_{\beta}(x, y)\) denotes a \(\Delta_{0}^{Q}\) formula indicating \(y\) is a Hilbert-styled proof of the theorem \(x\) from axiom system \(\beta\). Then for each \(\Pi_{1}^{Q}\) sentence \(\Phi\), IQFS(\(\beta\))’s Group-2 schema will contain a \(\Pi_{1}^{Q}\) axiom of the form:

\[
\forall y \left\{ \text{HilbPrf}_{\beta} \left( \neg \Phi, y \right) \Rightarrow \Phi \right\}
\]

(25)

d. The **GROUP-3** axiom of IQFS will be essentially similar to ISCE’s Group-3 “I am consistent” axiom, except the latter’s notion of “I” will trivially reflect the above changes in the Groups 0, 1 and 2 schemes. It will, thus, be a \(\Pi_{1}^{Q}\) sentence declaring that:

\(^8\) In a context where a \(\Pi_{1}^{Q}\) sentence is called “Simple” when it contains no \(E_{j}\) term with \(j \geq 2\), the Group-1 scheme of IQFS will be analogous to ISCE’s counterpart by consisting of any finite set of simple \(\Pi_{1}^{Q}\) axioms, called \(S^{*}\), that is consistent with Group-zero schema and which has the following properties:

1. The union of \(S^{*}\) with IQFS’s Group-Zero axioms will be capable of proving all \(\Delta_{0}^{Q}\) statements which are true.
2. The union of \(S^{*}\) with IQFS’s Group-Zero scheme will also be capable of proving that the “=” and “≤” predicates support their conventional transitivity, reflexivity, symmetry and total ordering properties.

The above two properties are the 1-to-1 analogs of their counterparts used by ISCE’s Group-1 scheme. As was the case with ISCE’s formalism, any finite set of simple \(\Pi_{1}^{Q}\) axioms with the above properties can be used to define \(S^{*}\). Once again, it is unimportant which particular finite-sized definition for \(S^{*}\) is used.
“Under Definition 5.6’s Hilbert-style deductive methodology, there will exist no proof of 0=1 from the union of the preceding axioms with THIS SENTENCE (looking at itself”).

Readers familiar with our previous papers [50, 51, 54, 55] about “I am consistent” axioms should be able to appreciate, intuitively, it is feasible to formulate a $\Pi^Q_1$ encoding for $\oplus \oplus \oplus$. This fourth part of our definition of the IQFS’s axiom system should, however, be articulated in greater detail because it is more complicated than IQFS(\beta)’s other three parts. Therefore, the next chapter’s Lemma 6.3 will formalize precisely how $\oplus \oplus \oplus$ is endowed with a $\Pi^Q_1$ encoding.

Our main goal will be to show that IQFS satisfies Definition 5.2’s Consistency-Preserving property, analogous to [55]’s ISCE methodology. The particular virtue of IQFS is that it needs only three starting constants, $C_0$, $C_1$, and $C_2$, to define the full infinity of natural numbers, unlike ISCE. (Moreover, our Proposition 4.3 will imply IQFS can encode every integer $n$ efficiently with a term $T_n$ with an $O\{ [\log(n)]^3 \}$ length, and Proposition 7.1 will show this quantity can be reduced to a yet-more-efficient logarithmic size, when ground terms are encoded as directed acyclic graphs.)

6 Main Results about IQFS

Our main results about the IQFS formalism will be proven in this chapter. Its discussion will be divided into two parts where:

1. Section 6.1 will formalize the encoding of $\oplus \oplus \oplus$’s Group-3 axiom (e.g. the Item (d) from Definition 5.7).

2. Section 6.2 will display our main theorems about IQFS’s consistency-preservation property.

6.1 The $\Pi^Q_1$ Encoding for IQFS’s Group- 3 Axiom

Before discussing how to endow $\oplus \oplus \oplus$ with a $\Pi^Q_1$ encoding, it is necessary to first outline our methodology for generating a proof’s Gödel number. It will be analogous to the natural B-adic encoding methods used by Buss [9], Hájek-Pudlák [25] and Wilkie-Paris [49] — insofar as the number of utilized bits to encode a semantic object will be approximately proportional to the length of such an expression written by hand. We will say a byte is a string of 6 bits, and encode each proof as an integer, written in base 64, comprising a sequence of bytes. Our encoding will use less than 32 logical symbols; thus, each symbol will be encoded by an unique integer between 32 and 63. These symbols shall include:

1. The standard connective symbols of $\land$, $\lor$, $\neg$, $\Rightarrow$, $\forall$ and $\exists$.
2. The left and right parenthesis symbols, and also a comma symbol (to separate terms) and a period symbol (to specify the end of a sentence).
3. Seven function symbols for representing our six grounding functions and the new \( \theta \) primitive.

4. The relation symbols of “=” and “\( \leq \).”

5. The symbol \( \hat{V} \) for designating the presence of a basic variable symbol.

6. Three constant symbols to represent the built in constants of 0, 1 and 2.

The \( \hat{V} \) symbol is somewhat different from our other logical symbols because there are an infinite number of different variable names that may occur in a proof. The \( j \)-th variable will be represented by a sequence of \( 1 + \lceil \log_{32} (j + 1) \rceil \) bytes, where the first byte encodes the \( \hat{V} \) symbol and the remaining bytes encode \( j \) as a Base-32 number.

**Definition 6.1** Let \( \alpha \) denote a set of proper axioms. Then the symbol \( \text{AxiomCheck}_\alpha(x) \) will denote a predicate formula that yields a value of True when \( x \) is the Gödel number of one of \( \alpha \)'s axioms. Also, \( \text{ProofCheck}_\alpha(s) \) will denote a formula that is True when \( s \) represents a byte string for a proof, whose proper axioms come from \( \alpha \) and whose logical axioms and deductive apparatus were formalized by Definition 5.6’s “\( d_{\text{ER}} \)” formalism. The relationship between these two concepts is formalized by Lemma 6.2.

**Lemma 6.2** The predicate \( \text{ProofCheck}_\alpha(s) \) is \( \Delta^0_0 \) encodable whenever \( \text{AxiomCheck}_\alpha(x) \) is \( \Delta^0_0 \) encodable. (The latter condition does hold for the majority of r.e. axiom systems \( \alpha \).)

**Proof Summary:** It is well known that Lemma 6.2 holds for classic arithmetics that use \( \Delta_0 \) encodings, instead of \( \Delta^0_0 \) encodabilities. (There are many proofs of this fact. One can use the fact that Wrathall’s notion of LinH functions are known [25, 31, 60] to have the property that a formula has a \( \Delta_0 \) encoding if and only if it can be corroborated by a LinH decision procedure.) A similar paradigm trivially generalizes for \( \Delta^0_0 \) encodings. Other routine details about Lemma 6.2’s proof are omitted for the sake of brevity. \( \square \)

Lemma 6.2 is obviously trivial, but it was worth mentioning because we will use it during one of the footnotes in Lemma 6.3’s proof.

**Lemma 6.3** The Group-3 axiom of IQFS(\( \beta \)), introduced informally by Definition 5.7’s statement \( \oplus \oplus \oplus \), can be formally encoded as a \( \Pi^1_Q \) axiom sentence.

**Proof.** Let \( \text{UNION}(\beta) \) denote the union of IQFS(\( \beta \))’s Group-Zero, Group-1 and Group-2 axioms (given in Items a-c of Definition 5.7 ). We will also use the notation conventions below:

i. \( \text{Prf}_{\text{UNION}(\beta)}(t, p) \) will denote a formula designating that \( p \) is a proof of the theorem \( t \) from the axiom system \( \text{UNION}(\beta) \), using Definition 5.6’s deduction method \( d_{\text{ER}} \).

ii. \( \text{ExPrf}_{\text{UNION}(\beta)}(h, t, p) \) will be a formula stating that \( p \) is a proof (using deduction method \( d_{\text{ER}} \)) of a theorem \( t \) from the union of the axiom system \( \text{UNION}(\beta) \) with an added axiom whose Gödel number equals \( h \).
iii. Subst \((g, h)\) will denote Gödel’s classic substitution formula — which yields TRUE when \(g\) is an encoding of a formula and \(h\) is an encoding of a sentence that replaces all occurrence of free variables in \(g\) with the Gödel-encoded term \(\bar{g}\).

iv. \(\text{SubstPrf}_{\text{UNION}(\beta)} (g, t, p)\) will denote the natural hybridizations of the constructs from Items (ii) and (iii). It will yield a Boolean value of TRUE exactly when there exists some integer \(h\) simultaneously satisfying both the conditions \(\text{Subst} (g, h)\) and \(\text{ExPrf}_{\text{UNION}(\beta)} (h, t, p)\).

Each of (i)–(iii) can be easily encoded as \(\Delta^0_0\) formulae. In such a context, Line (26) illustrates one possible \(\Delta^0_0\) encoding for \(\text{SubstPrf}_{\text{UNION}(\beta)} (g, t, p)\)’s graph. (It is equivalent to “\(\exists h \ [ \text{Subst}(g,h) \land \text{ExPrf}_{\text{UNION}(\beta)}(h, t, p) \]”, but Line (26) is a \(\Delta^0_0\) formula — unlike the quoted expression.)

\[
\text{Prf}_{\text{UNION}(\beta)} (t, p) \lor \exists h \leq p \ [ \text{Subst} (g, h) \land \text{ExPrf}_{\text{UNION}(\beta)} (h, t, p) ]
\]  

(26)

Utilizing (26)’s particular \(\Delta^0_0\) encoding for \(\text{SubstPrf}_{\text{UNION}(\beta)}(g,t,p)\), it is easy to encode IQFS(\(\beta\))’s Group-3 axiom, specified by Definition 5.7’s statement \(\oplus \oplus \oplus\), as a \(\Pi^Q_1\) sentence. Thus, let \(\Gamma(g)\) denote Line (27)’s formula, and let \(n\) denote \(\Gamma(g)\)’s Gödel number. Then “\(\Gamma(\bar{n})\)” is a \(\Pi^Q_1\) sentence encoding IQFS(\(\beta\))’s Group-3 axiom.

\[
\forall p \neg \text{SubstPrf}_{\text{UNION}(\beta)} (g, \bar{\top}, 0 = 1^\top, p)
\]  

(27)

In other words, \(\oplus \oplus \oplus\) is encoded as: “\(\forall p \neg \text{SubstPrf}_{\text{UNION}(\beta)} (\bar{n}, \top, 0 = 1^\top, p)\)”.

\(\square\)

**Remark 6.4** We shall not provide IQFS(\(\beta\))’s Groups 1 and 2 axioms with \(\Pi^Q_1\) encodings, similar to Lemma 6.2’s construction. Such is unnecessary because these Group-1 and Group-2 axioms have obvious encodings, unlike the preceding encoding of IQFS(\(\beta\))’s Group-3 axiom, described above.

### 6.2 Main Conjecture and Main Theorem

This section will introduce our main conjecture and accompanying theorem. The attached appendix will provide evidence, very strongly suggesting that Conjecture 6.6 is correct. This evidence shall almost reach the critical-mass level of amounting to a proof. We will begin our discussion by introducing one preliminary definition, that will help facilitate the relationship between Conjecture 6.6 and Theorem 6.7.

**Definition 6.5** A \(\Pi^Q_1\) sentence \(\forall x \phi(x)\) will be said to be a **Size K Breaking Pont** iff \(\phi(K)\) is false and \(\phi(x)\) is true for all \(x < K\).

---

9. In particular, the \(\Delta^0_0\) encodings for Items (i) and (ii) are immediate consequences of Lemma 6.2.
Conjecture 6.6 Suppose $\gamma$ is an axiom system that includes:

A. All IQFS’s Group-zero and Group-1 axioms (from Definition 5.7),

B. A set of additional $\Pi_1^Q$ axiom statements, all of which hold true in the Standard Model, except for one special $\Pi_1^Q$ sentence, called $\Psi$, that constitutes a Size $K$ Breaking Pont (with $K > 2$).

Suppose $P$ is a proof from $\gamma$ of $0 = 1$ (using again Definition 5.6’s $d_{ER}$ deductive method). Then $P$, viewed as a Gödel number, will impose the following constraint on $K$’s magnitude

$$\log_2 K < \frac{1}{6} \log_2 P$$

(28)

We are essentially 100% confident that Conjecture 6.6 is true. Indeed, Line (28)’s estimate of an upper bound on $K$’s length is actually an excessively conservative over-estimate. An approximate intuitive justification of this inequality is provided in the attached appendix. (It actually falls only one tiny iota short of being a formal proof.) Our immediate goal is to discuss and prove the Theorem 6.7. We recommend that the reader postpone examining the Appendix’s justification of Conjecture 6.6 until after Theorem 6.7 and its proof have been both first read.

Theorem 6.7 The Conjecture 6.6 does imply that IQFS’s axiomatic framework satisfies Definition 5.2’s Consistency Preservation property. (E.g. that if $\beta$ is an axiom system consistent with IQFS’s Group-0 and 1 axioms then IQFS($\beta$) is also consistent.)

Proof. Similar to the Consistency Preservation analysis in our earlier papers, the proof of Theorem 6.7 will be a proof by contradiction.

Let UNION($\beta$) once again denote the union of IQFS($\beta$)’s Group-Zero, Group-1 and Group-2 axioms. If Theorem 6.7 was false then it easily follows that all the axioms in UNION($\beta$) will hold true in the standard model, but IQFS($\beta$) will be contradictory. We will show this is impossible.

In particular, if IQFS($\beta$) was contradictory, then there must exist a minimal sized proof of $0 = 1$ from IQFS($\beta$), called say $P$. We call the reader’s attention to the fact that UNION($\beta$) consists of all of IQFS($\beta$)’s axioms, except for its Group-3 axiom. Lemma 6.3 showed that IQFS($\beta$)’s Group-3 axiom can be encoded as a $\Pi_1^Q$ sentence of the form:

$$\forall x \psi(x)$$

(29)

Then since $P$ denotes the minimal proof of $0 = 1$ and since IQFS($\beta$)’s Group-3 axiom states “There exists no proof of $0 = 1$ from IQFS($\beta$)”, it follows that this italicized sentence is false. Thus, Line (29) must own a Size-$K$ Breaking Point, for some $K$.

The Conjecture 6.6 indicates $K < P$ (because of (28)’s inequality). This is impossible because $P$ was defined to be the minimal proof of $0 = 1$ , and $K$ represents a smaller such proof of $0 = 1$. Hence, the IQFS( • ) framework must be consistency-preserving to avoid this contradiction. □
Remark 6.8 Our earlier Consistency Preservation results in [50, 51, 54, 58] also relied upon proofs-by-contradiction, mostly analogous to Theorem 6.7’s verification. One important distinction is, however, that Theorem 6.7’s proof employed Conjecture 6.6’s paradigm as an intermediate step. There are good reasons for believing this conjecture is correct. In particular, the attached appendix provides strong evidence confirming this conjecture, in a context where its informal justification falls only one small iota short of being a fully formalized proof. At this juncture, we encourage the reader to take a quick glance at the Appendix’s justification for Conjecture 6.6.

7 Further Results and Useful Added Perspectives

Both the virtues and drawbacks of the IQFS formalism are consequences of Proposition 4.3’s characterization of the $O\{\lceil \log(n) \rceil^3\}$ quantity of logical symbols used for encoding an integer $n$ as a grounded term $T_n$. Thus, this quantity is clearly significantly better than the alternate $O(n^2)$ length that arises when the $\theta$ function symbol is replaced by the less efficient primitive $\zeta$, defined by Lines (10) and (11). On the other hand, one would ideally prefer our ground terms to resemble conventional encodings of a binary number that use $O\{\log(n)\}$ logical symbols to encode an arbitrary number $n$.

It turns out it is possible to improve Proposition 4.3’s encodings to such a compressed $O\{\log(n)\}$ size, if one adds only a minor wiggle to the logic’s notation convention. This distinction arises because most traditional logic languages formalize Definition 4.5’s “Ground terms” as tree-like structures. An easy alternative mechanism will allow these terms to own the more generalized structure of a Directed Acyclic Graph (Dag).

Traditionally, this distinction has been viewed as an unimportant wrinkle because it can be easily proven that almost every Dag-oriented term can be converted into its Tree-oriented counterpart with usually only a Polynomial increase in length. The reason for our special interest in Dag-formulated ground terms is that the IQFS formalism possesses access only to the specialized $\theta$ primitive for generating growth among integers. Thus, Proposition 7.1 will indicate that Proposition 4.3’s earlier ground terms can have their lengths nicely compressed from an $O\{\lceil \log(n) \rceil^3\}$ size into a more enticing $O\{\log(n)\}$ magnitude, when a Dag notation is employed.

Proposition 7.1 Let us consider a Dag-analog of §4’s formalism where once again:

1. $\theta$ is the only available growth permitting function symbol,

2. the only built-in constant symbols are, again, $C_0, C_1$ and $C_2$, for representing 0, 1 and 2,

3. and the three function symbols for representing integer-subtraction, integer-division and the maximum operation are, once again, available.

In this context, any integer $n$ can be encoded by a Dag-oriented Ground term $G_n$ using only $O\{\log(n)\}$ logical symbols. (As the pointers needed to separate these $O\{\log(n)\}$ logical objects
use $O\{ \log \log(n) \} \text{ bits, the total amount of memory required for encoding a Dag-oriented Ground term will employ at most } O\{ \log(n) \cdot \log \log(n) \} \text{ bits.}$

The proof for Proposition 7.1 rests essentially on a more elaborate version of §4’s justification of Proposition 4.3. It uses the fact that Proposition 4.3’s ground term $T_n$ do have many repeating subterms that can be compressed into single objects under a Dag-style notation.

**Detailed Proof:** It is easy to convert the preceding paragraph’s summary of the intuition behind Proposition 7.1 into a formal proof. Let $G_n$ denote our Directed Acyclic Graph (Dag) for representing an integer $n$, and let $M$ be an abbreviation for the quantity $\lceil 1 + \log_2(n) \rceil$. Our directed graph $G_n$ will consist of approximately $5 \cdot \log(n)$ nodes. The first five of its six groups of nodes in $G_n$’s graph are defined below in bottom-up order:

1. The bottom-most nodes in $G_n$’s graph will correspond to the three built-in constant symbols of $C_0$, $C_1$ and $C_2$, that represent the values of 0, 1 and 2.

2. Let $\theta^j(x)$ denote the term $\theta(\theta(\theta(\ldots \theta(x))))$ where there are $j$ iterations of the $\theta$ operation. For each $j \leq M$, the next $j$ levels of $G_n$’s directed graph will define nodes $A_j$ that formalize the quantity $\theta^j(1)$. In a context where $A_0$ is an abbreviation for $C_1$, the remaining $A_j$ are defined by:

   $$A_j = \theta(A_{j-1}) \quad (30)$$

3. For each $j \leq M$, let $B_j$ denote the value of $\max(A_0, A_1, A_2, \ldots, A_j)$. In a context where $B_0$ is an abbreviation for the entity $C_1$, the remaining $B_j$ are defined chronologically in $G_n$’s directed graph by:

   $$B_j = \max(A_j, B_{j-1}) \quad (31)$$

4. For each $j \leq M$, let $D_j$ denote the value of $2^{-j} \cdot B_M$. In a context where $D_0$ is equivalent to the prior entry $B_M$ in our directed graph, the remaining $D_j$ nodes in our graph will be defined via (32)’s Division operation:

   $$D_j = \frac{D_{j-1}}{2} \quad \text{e.g.} \quad D_j = \frac{D_{j-1}}{C_2} \quad (32)$$

5. For each $j \leq M$, the node $E_j$ will represent the quantity $2^j$. These nodes in $G_n$’s graph will be defined by (33)’s Division operation:

   $$E_j = \frac{D_M}{D_{M-j}} \quad (33)$$

Some added notation is needed to describe the last part of $G_n$’s graph for formalizing $n$’s representation as a Dag-oriented ground term employing $O\{ \log(n) \} \text{ logic symbols. Let } T_n \text{ denote Proposition 4.3’s formulation of } n \text{ as a Tree-oriented ground term, and } G_n \text{ denote its Dag counterpart (using the five intermediate steps itemized above). Our prior proof of Proposition 4.3 noted } T_n \text{ at }$
was constructed by setting \( E_M \) equal to the least power of 2 greater than \( n \) and then subtracting from it those powers of 2 which are needed to produce the quantity \( n \).

The exact same methodology will now be used to construct our \( G_n \) representation of \( n \), except we will now obviously use the methodologies from Items 1-5 to assure no more than \( O\{ \log(n) \} \) graph nodes are used to construct all Line (33)’s \( E_j \) terms. (For example since \( 86 = 128 - 32 - 8 - 2 \) which in turn equals “ \( E_7 - E_5 - E_3 - E_1 \)”, the final stage of our construction of \( G_{86} \) will first set node \( F_1 \) equal to “ \( E_7 - E_5 \)”, then set node \( F_2 \) equal to “ \( F_1 - E_3 \)” and lastly have the output node \( F_3 \) represent the final answer as the quantity of “ \( F_2 - E_1 \)”.

It is easy to see that this methodology will never use more than \( O\{ \log(n) \} \) logical symbols to encode \( G_n \) as a Dag-oriented ground term. Moreover, the needed pointers in the Dag graph \( G_n \) will require no more than \( \log \log(n) \) bits to distinguish between these \( O\{ \log(n) \} \) separate objects. Hence if one selects to use a pointer methodology to formulate \( G_n \)’s graph, then no more than \( O\{ \log(n) \cdot \log \log(n) \} \) bits will be needed to encode all these pointers (as the last sentence of Proposition 7.1 had claimed). □

**Remark 7.2.** Let IQFS* denote the analog of our IQFS framework that has Proposition 7.1’s Dag-oriented Ground Terms replace §4’s earlier Tree-oriented Ground Terms in an accordingly revised language. A construction exactly analogous to Theorem 6.7 will show IQFS* also satisfies the Consistency Preserving paradigm, assuming (as we do again) that Conjecture 6.6 is correct. Thus, IQFS* is a better formalism, although somewhat more complicated to describe.

**Remark 7.3.** It should be mentioned that the infinite number of axiom sentences, appearing in the Group-2 schemas for ISCE(\( \beta \)), IQFS(\( \beta \)) and IQFS*(\( \beta \)), can be nicely reduced to a purely finite sizes, with almost no loss in information. This was done in [59] for the Group-2 scheme of its ISD(\( \beta \)) formalism, with the latter producing isomorphic counterparts of all of \( \beta \)’s full set of \( \Pi_1 \) theorems (e.g. see Sections 5 and 6 of [59]). The same methods will routinely generalize for the ISCE, IQFS and IQFS* frameworks. It will imply that easy modifications of IQFS(PA+) and IQFS*(PA+), owning a strictly finite number of axiom sentences, are able to prove isomorphic counterparts of the full infinity of \( \Pi_1 \) theorems, produced by Peano Arithmetic

**Remark 7.4** There is one other amendment to the IQFS and IQFS* formalisms that should be mentioned because of its pragmatic significance. Let \( S_j \) denote §4’s sentence. Let IQFS\(_R \) and IQFS*\(_R \) denote the minor modifications of IQFS and IQFS* that include \( S_1 \), \( S_2 \), \( S_3 \), ... as axioms.

\[
E_j - E_{j-1} = E_{j-1}
\]

(34)

The addition of these \( S_j \) as axioms cannot possibly affect IQFS’s consistency because each \( S_j \) is provable from the Group-Zero and Group-1 axioms as a theorem. The advantage of treating such \( S_j \) as such built-in axioms (rather than as theorems) is that many of IQFS(\( \beta \)’s other theorems can have their proofs conveniently shortened in length in this case.

**Remark 7.5** Our proposed \( \theta \) primitive is likely to be of philosophical interest, quite apart from its implications for Hilbert’s Second Open Question. This is because:
1. It is interesting that the $\theta$ primitive, combined with the non-growth subtraction, division and maximum operands, can represent any integer $n$ with a complexity comparable to the $O(\log(n))$ sized representations of an integer under its conventional binary encoding. Does this suggest the foundational reasoning capacities of either a child or of the anthropological predecessors of modern man possess access to a technique for reasoning that constructs the natural numbers without the traditional uses of the addition and multiplication functions? There is likely to be no easy yes-or-no answer to this question. (Our suspicion is that an infant-child’s primitive thoughts rely upon an analog of $\theta$’s basic indeterminate functionality.)

2. Also, it is philosophically curious whether or not the Group-2 axioms of Definition 5.7’s IQFS formalism capture the majority of the engineering implications of Peano Arithmetic? This is because one may wonder whether the purist engineering implications of Peano Arithmetic consist of its particular theorems, that can be encoded in such “$\Pi^0_1$ formats”?

There is no space available in this article to adequately address these philosophical issues. Our point is, however, that IQFS raises other issues besides besides this article’s main theme, that some fragments of the historic aspirations of Hilbert and Gödel in * and ** should be revisited.

8 Disentangling Some Confusing Interpretations of Theorem 6.7 and Its Corollaries that Could Otherwise Plausibly Arise

It was during one of the final stages of this paper, as its draft was nearing completion, that we elected to add a very short further chapter into this article, “disentangling” some confusions that might “otherwise plausibly arise”. It was desirable to insert a chapter into this article, whose title contained the preceding italicized words, so that the exact meaning of Theorem 6.7’s central result could not be confused.

Theorem 6.7’s meaning is complicated by the fact that IQFS($PA^+$), as well as its several predecessors in [50]-[59], were able to prove theorems, asserting their own consistency, only by employing Example 3.5’s built-in “I am consistent” statements, as invoked axiomatic declarations. This approach, which was centered around Item $\oplus$ from Example 3.5 will naturally cause many readers to raise the following skeptical question:

# # # Is it not almost cheating when an axiom system verifies its own consistency by using $\oplus$’s formalized “I am consistent” axiom as an intermediate step, to verify its own consistency? After all, such a technique can verify its own consistency only in a technically purely legalistic sense. (It is certainly, however, not meaningful in the much broader almost philosophical respect, that Hilbert was grasping for in his famous year-1900 Second Open Question.)

It will be necessary for us to introduce the formalisms of Definition 8.1 and Corollary 8.2 before replying to # # #’s query.

---

10 $PA^+$ was defined in our articles to be the trivial extension of Peano Arithmetic that uses a language that includes the six Grounding level functions, in addition to the usual addition and multiplication primitives.
Definition 8.1 Let $\beta$ again denote an axiom system that uses $L^Q$’s language (whose collection of function symbols include the six usual Grounding-level primitives plus $\theta$’s special indeterminately-defined function operation). Also, let IQFS($\beta$) once again denote Definition 5.7’s self-referencing formalism, that proves all of $\beta$’s $\Pi^Q_1$ theorems and which additionally recognizes its own formalized Hilbert consistency, via its use of an invoked “I am consistent” axiom. Then $\beta$ will be called Platonically Stable iff IQFS($\beta$) is a consistent axiom system. (Likewise, if IQFS($\beta$) is inconsistent then $\beta$ will be called “Platonically Unstable”.)

Corollary 8.2 Let us assume Conjecture 6.6 is correct (as we are almost 100% certain it is). Suppose $\beta$ is an axiom system which is consistent with IQFS’s Group-0 and 1 axioms. Then $\beta$ is automatically “Platonically Stable” (e.g. the formalism IQFS($\beta$) is consistent).

The formal statements of Theorem 6.7 and Corollary 8.2 are closely related. This is because Corollary 8.2 applies Definition 8.1’s notion of “Platonic Stability” in essentially every place where Theorem 6.7 had used Definition 5.2’s “Consistency Preserving” paradigm. Thus, there is no need to formally prove Corollary 8.2 because it is almost a direct consequence of Theorem 6.7.

The nice aspect of Corollary 8.2 is its vocabulary will allow us to better frame a reply to # # #’s query. In particular, it would certainly be preferable (and more Utopian) if human beings could muster more sophisticated justifications for their thought processes than had appeared in Example 3.5’s short 1-sentence axiom statement $\oplus$ (which simply declared “I am consistent”). However, such a reply to # # #’s query is unnecessary for the planet Earth’s high-IQ primates to gain adequate confidence to justify their thought processes. All that is strictly necessary, from a purist perspective, is for an advanced Thinking Being to find a robust formalism that satisfies Definition 8.1’s criteria of “Platonic Stability”. The significant aspect of our Corollary 8.2 is it shows adequate forms of Platonic Stability are available to allow an introspective thinker to simultaneously:

1. presume its own consistency as a built-in assumption, and
2. rest assured this assumption will not spin its IQFS($\beta$) formalism into a cycle of inconsistency.

Corollary 8.2’s notion of Platonic Stability, thus, partially reinforces Gödel’s philosophy of Mathematical Platonism, and it also helps make comprehensible the aspirations that motivated Hilbert’s and Gödel’s famous statements $\ast$ and $\ast \ast$. This perspective is useful when we remind ourselves that the invariant $++$, which Example 3.3 had attributed to the joint work of Pudlák, Solovay, Nelson and Wilkie-Paris, 35 39 44 49, established that self-justifying formalisms, much stronger than IQFS($\beta$), are simply impossible.

Remark 8.3 (Snapshot Perspective): Summarizing our last 25 years of research into one short paragraph, it is certainly true that any proof that relies upon Example 3.5’s “I am consistent” axiom is, in some respects, a quite skinny form of proof, that one is almost first tempted to ignore. On the other hand, this perspective is actually quite useful because Corollary 8.2 does reminds us that we do live in what Definition 8.1 had called a “Platonically Stable” world.
9 Concluding Remarks

All our published articles about self-justifying arithmetics have emphasized that evasions of the Second Incompleteness Effect rested on using arithmetics that were weaker than traditional arithmetics in, at least, some respects. (The Second Incompleteness Theorem’s significance in refuting the original objectives of Hilbert’s Consistency Program is thus, simply, undeniable.) It would, nevertheless, be of interest if some fragments of Hilbert’s and Gödel’s objectives, as announced in their formal statements * and **, were achieved.

It is in this context and also with knowledge about how the scope of the Second Incompleteness Theorem has been enhanced in many respects during the last 35 years, that we hope our boundary-case exceptions to the Second Incompleteness paradigm shall be found to be useful.

This is not merely because it would be pleasantly reassuring to see achieved some select portions of what early 20-th century articles had advocated. (For instance, Gödel’s preference [12, 22] for a “Platonist” philosophy is partially reinforced by Corollary 8.2’s “Platonic Stability”.) Our results are, also of interest because future computers will, perhaps, have their capacities enhanced, if they can simulate a human’s gut instinctive faith in his own consistency.

Thus, it is in such respects that Theorem 6.7 and the related Propositions 4.3 and 7.1 will help us better appreciate the aspirations that Hilbert and Gödel were pursuing in their formal statements * and **, as well as to formalize what limited aspects of their stated goals in * and ** can actually be achieved.

It also should be mentioned that some special philosophical issues were raised by Remark 7.5 and in Remark 8.3’s observation that symbolic logic has a “Platonically Stable” structure.

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Appendix Justifying Conjecture 6.6

This appendix should be read only after Section 6 has been finished. It will have two purposes. They will be to:

1. Outline an approximate intuitive justification for Conjecture 6.6’s validity.

2. Explore this conjecture’s broader significance.

A.1 An Approximate Intuitive Sketch Justifying Conjecture 6.6

The underlying intuition supporting Conjecture 6.6 is easy to summarize. It stems from the fact that \( \theta \) is the only growth-permitting function (or primitive) in \( \mathcal{L}_Q \)’s language. Moreover, the definitions of \( \Pi_1^Q \) and \( \Sigma_1^Q \) sentences (in Item 5.4) forbid \( \theta \)'s appearance anywhere in these sentences, outside of the particular places where Definition 4.4’s \( E_j \) terms use the \( \theta \) primitive as an intermediate device for defining the quantity \( 2^j \).

These facts seem to obviously imply that a proof can construct integers \( n > 2 \) only by applying the Up-Walking axioms, from Lines (14)–(17), in the directly canonical and cumbersome manner. That is, a proof \( P \) appears to be able to verify the existence of an integer \( n \geq 2^d \) only after it has isolated \( d + 1 \) distinct terms, which correspond to \( d + 1 \) distinctly different powers of 2.

Let \( M_d \) denote a finite-sized model for the subset of Natural Numbers, all of which satisfy the exact inequality of \( x < 2^d \). Also, let \( S \) denote the set of \( \Pi_1^Q \) axiom sentences that were defined by Part (B) of Conjecture 6.6. If \( d = \lfloor \log_2 K \rfloor \) then Part (B) implies that all of the axioms in \( S \) hold true within the model \( M_d \). However, its particular “Breaking Point” sentence \( \Psi \) does become false in the model \( M_{d+1} \), while all the other axioms of \( S \) continue to hold true within each of the models of \( M_{d+1}, M_{d+2}, M_{d+3}, \ldots \).

At an intuitive (somewhat informal) level, these facts suggest that the proof \( P \), alluded to in Line (28), can establish “0 = 1” only after it has first constructed more than \( 1 + \lfloor \log_2 K \rfloor \) distinct powers of 2. (This is because the sentence \( \Psi \) ultimately forces a contradiction, in such a context, because of its “Size-\( K \) Breaking Point” property.)
Under our rules for encoding $P$ (which requires six bits to encode each of $P$’s 23 logical symbols), such a lengthy sequence of $d+1$ distinctly different terms will cause $P$ to be so large that $\log_2(P)$ exceeds by at least a factor of 6 the size of $\log_2(K)$. Indeed, the quantity $\frac{1}{6}$ in Line (28)’s upper bound is actually a conservative overestimate of the actual bounding quantity. But we will not delve further into such a discussion, here, because this section was intended to provide only an intuitive sketch justifying Conjecture 6.6.

A 1-sentence summary of the intuition behind Conjecture 6.6 is that its axiom system $\gamma$ is so weak that it cannot construct a required integer, larger than $K$, without engaging in a proof that directly requires more than $6 \cdot \log_2 K$ bits.

A.2 More Details about Conjecture 6.6

Our IQFS($\beta$) axiom system does not need an actual formal proof of Conjecture 6.6 for it to become mathematically very significant. All it needs, from a strictly minimalistic perspective, is for the Conjecture 6.6 to hold true under the Standard Model (even if its formal proof is independent from the exact axioms of say Peano Arithmetic and/or ZF Set Theory).

A rigorous proof of Conjecture 6.6 is, obviously, still desirable. This proof is likely to be tedious and lengthy because the analog of Conjecture 6.6’s inequality (28) is, simply, not true when IQFS’s Group-zero axiom replaces our new $\theta$ function operation with the Successor primitive. In particular, the Pudlák-Solovay Assertion ++, from Example 3.3-a, was able to generalize the Second Incompleteness Theorem, only by using the fact that analogs of Conjecture 6.6 are false when Successor replaces $\theta$.

The intuitive reason that $\theta$ operates so differently from Successor is that these two operations differ in the following three significant respects:

1. Integers grow at a monotonically uniform rate under the successor operation, while any finite sequence $\theta(2), \theta(\theta(2)), \theta(\theta(\theta(2))) \ldots \theta^k(2)$ is allowed to be monotonically decreasing under the $\theta$ primitive.

2. The $\theta$ primitive satisfies Definition 3.6’s property of being a “Q-function” with $\aleph_1$ different possible solution vectors, while there is only one unique representation of the successor function.

3. The $\Pi_1$ encodings for axioms about the 3-way arithmetic predicates $Add(x, y, z)$ and $Mult(x, y, z)$, such as the associative rules for addition and multiplication, help one characterize the implications of the Successor function. But these rules are fully irrelevant to $\theta$’s behavior.

Thus, Items 1-3 make us essentially 100% confident that the $\theta$ and Successor primitives shall differ sufficiently for Conjecture 6.6 to apply to the former primitive, although certainly not also to its counterpart under the Successor operation.