A Nonlinear Analysis of the Averaged Euler Equations

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Dedicated to Vladimir Arnold on the occasion of his 60th birthday

Abstract

This paper develops the geometry and analysis of the averaged Euler equations for ideal incompressible flow in domains in Euclidean space and on Riemannian manifolds, possibly with boundary. The averaged Euler equations involve a parameter $\alpha$; one interpretation is that they are obtained by ensemble averaging the Euler equations in Lagrangian representation over rapid fluctuations whose amplitudes are of order $\alpha$. The particle flows associated with these equations are shown to be geodesics on a suitable group of volume preserving diffeomorphisms, just as with the Euler equations themselves (according to Arnold’s theorem), but with respect to a right invariant $H^1$ metric instead of the $L^2$ metric. The equations are also equivalent to those for a certain second grade fluid. Additional properties of the Euler equations, such as smoothness of the geodesic spray (the Ebin-Marsden theorem) are also shown to hold. Using this nonlinear analysis framework, the limit of zero viscosity for the corresponding viscous equations is shown to be a regular limit, even in the presence of boundaries.
1 Introduction

More than twenty-five years ago, using a setting introduced by Arnold [1966], Ebin and Marsden [1970] proved that on manifolds with no boundary (such as spatially periodic flow in Euclidean three space $\mathbb{R}^3$), the solutions of the Navier-Stokes equation converge to those of the Euler equations as the viscosity tends to zero. Marsden, Ebin, and Fischer [1972] conjectured that although in a region with boundary, solutions of the Navier-Stokes equations would not in general converge to the solutions of the Euler equations, a certain averaged quantity of the flow may converge. Kato [1984] showed that the problem in the context of weak solutions of the Navier-Stokes equations has fundamental difficulties.

Recently, Barenblatt and Chorin [1998a,b] also speculated that certain average properties of the flow possess well-defined limits as the viscosity tends to zero. One of the main purposes of this paper is to develop extensions of the tools of Arnold, Ebin and Marsden and to use them to prove that, in a sense that will be made precise, viscous flow, with an appropriate ensemble averaging over rapid fluctuations or spatial averaging over small scales, does indeed converge to solutions of the corresponding inviscid limit equations, even in the presence of boundaries.

Some History. The geometric approach to fluid mechanics has a long and complex history, going back at least to the basic work of Poincaré (see the bibliographical references, but especially [1901b, 1910]) and the thesis of Ehrenfest (see Klein [1970]). In more recent times, it was the work of Arnold [1966] that was a critical contribution. Arnold showed, amongst other things, that if $u(x, t)$ is a time dependent divergence free vector field on a compact Riemannian $n$-manifold $M$, possibly with boundary, if $u$ is parallel to the boundary and if $\eta(x, t)$ is its volume preserving flow, then $u$ satisfies the Euler equations

$$\frac{\partial u}{\partial t} + \nabla_u u = -\text{grad } p$$

if and only if the curve $t \mapsto \eta(\cdot, t)$ is an $L^2$ geodesic in $\mathcal{D}_\mu(M)$, the group of $C^\infty$ volume preserving diffeomorphisms of $M$. Of course, the Euler equations in Eu-
clidean space in coordinates are given as follows (using the summation convention for repeated indices):

\[ \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} = -\frac{\partial p}{\partial x^i}, \]

and on a Riemannian manifold, the Euler equations take the following form in coordinates:

\[ \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \Gamma^i_{jk} u^j u^k = -g^{ij} \frac{\partial p}{\partial x^j}, \]

where \( g_{ij} \) is the Riemannian structure, \( g^{ij} \) is the inverse metric tensor and \( \Gamma^i_{jk} \) are the associated Christoffel symbols.

Ebin and Marsden [1970] proved a remarkable result: the geodesic spray of the \( L^2 \) right invariant metric on \( D^s(M) \), the group of volume preserving Sobolev \( H^s \) diffeomorphisms \( (s > (n/2) + 1) \) is \( C^\infty \) (recall that \( n \) is the dimension of the underlying manifold \( M \)). They derived a number of interesting consequences from this result, including theorems on the convergence of solutions of the Navier-Stokes equations to solutions of the Euler equations as the viscosity goes to zero when \( M \) has no boundary. Since that time, several papers (which we shall not review here) reproved and in some cases, rediscovered, these results using more traditional PDE methods in Eulerian representation. However, the depth and beauty of the direct approach in Lagrangian representation remains compelling.

There have been, of course, many other developments since that time, such as developments in the theory of hydrodynamic stability, bifurcations, and other aspects of the geometric approach. We refer to Arnold and Khesin [1998], Holm, Marsden, Ratiu and Weinstein [1985], Marsden [1992] and Marsden and Ratiu [1999] and references therein for stability studies and to Misiolek [1993, 1996] and Shkoller [1998] and references therein for the Riemannian geometry developments of the group of diffeomorphisms.

**\( H^1 \)-Geodesics.** The \((\kappa = 0)\) Camassa-Holm equation in one spatial dimension is given by

\[ u_t - u_{xxx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \]  

or equivalently,

\[ \frac{\partial v}{\partial t} + uv_x + 2vu_x = 0 \]

where \( v = u - u_{xx} \). This equation has attracted much attention; amongst many other features, it is a completely integrable bi-Hamiltonian system with non-smooth solitons that have very interesting geometry. See, for example, Camassa and Holm [1993], Camassa, Holm and Hyman [1994], and Alber, Camassa, Holm and Marsden [1994]. This one dimensional equation was given an Arnold-type interpretation in terms of \( H^1 \) geodesics on an infinite dimensional group by Misiolek [1998] (who also
considered the case \( \kappa \neq 0 \) and used the Bott-Virasoro group) and Kouranbaeva [1999] (who explored the geometry of the \( \kappa = 0 \) equations using the diffeomorphism group) and Shkoller [1998] for the well posedness in \( H^{3/2+\epsilon}(S^1) \) (See also Theorem 3.3 of the present paper).

The Averaged Euler Equations. Using recent developments in Lagrangian reduction and the Euler–Poincaré equations, Holm, Marsden and Ratiu [1998a,b,c], developed the “Arnold view” of fluid mechanics further by applying it to many other types of fluid equations, such as those in geophysics. One of the equations they studied is quite remarkable and is the subject of this work.

The equations of concern to us may be described in two mathematically equivalent ways. This equivalence is the \( H^1 \)-analogue of the theorem of Arnold mentioned above:

1. First of all, of course, there are the explicit **averaged Euler equations** (these are written in Euclidean coordinates on Euclidean \( n \)-space; we shall write the equations on manifolds in the main text):

\[
\frac{\partial v^i}{\partial t} + u^j \frac{\partial v^i}{\partial x^j} - \alpha^2 \left[ \frac{\partial u^j}{\partial x^i} \right] \Delta u_j = - \frac{\partial p}{\partial x^i},
\]

where \( \alpha \) is a positive constant, \( v = u - \alpha^2 \Delta u \) and \( \Delta \) denotes the componentwise Laplacian, and there is a summation over repeated indices (and in Euclidean coordinates, there is no difference between indices up or down).

As with the usual Euler equations, the function \( p \) is determined from the condition of incompressibility: \( \text{div} \, u = 0 \); the pressure satisfies a Poisson equation that is determined by taking the divergence of the equation.

The no slip boundary conditions \( u = 0 \) on the boundary will be assumed for equation (1.2). A second possible choice will be briefly discussed later.

2. The second description is to say that the flow \( \eta_t(\cdot) := \eta(\cdot, t) \) of the time dependent vector field \( u \) is a geodesic in a subgroup of \( \mathcal{D}^*_\mu(M) \) with respect to the right invariant \( H^1 \) metric.

There are two interesting and distinct ways of deriving the equations from the point of view of mechanics:

1. The first derivation is by means of averaging the Euler equations over rapid fluctuations whose amplitude is of order \( \alpha \) (see Holm, Marsden and Ratiu [1998b] for the case of Euclidean space). This method is explored in Marsden, Ratiu and Shkoller [1999b].

2. The second derivation uses the notion of second grade fluids, as will be discussed in the last section of the paper.
The Main Results of the Paper. In this paper we show that the geodesic spray of the right invariant $H^1$ metric on compact manifolds with boundary is smooth and has a unique flow which corresponds to the solution of the averaged Euler equations \( (1.2) \), the correspondence given by right translation. This has consequences that are similar to those for the usual Euler equations, namely, local existence and uniqueness, smooth dependence on initial condition and time, convergence of Chorin-Marsden type product formulas, etc.

Using this set up, we establish one of the main corollaries of our results; namely, for the viscous analog of the averaged Euler equations \( (1.2) \),

\[
\frac{\partial v^i}{\partial t} - \nu \triangle v^i + u^j \frac{\partial v^i}{\partial x^j} - \alpha^2 \left[ \frac{\partial u^j}{\partial x^i} \right] \triangle u^j = -\frac{\partial p}{\partial x^i},
\]

which we call the averaged Navier-Stokes equations, or the Navier-Stokes-$\alpha$ equations, we prove that

*The solutions for the corresponding viscous problem converge to those for the ideal problem, as the viscosity goes to zero, even in the presence of boundaries on uniform time intervals \([0, T]\), for \( T > 0 \), independent of the viscosity. The size of the interval \([0, T]\) is governed by the time of existence for the averaged Euler equations with the initial data fixed.*

The inclusion of boundaries is a major difference from the situation with the usual Navier-Stokes equations and the Euler equations, for which such convergence is believed to not hold near boundaries because of the generation of vorticity at the boundary. (See, for example, Marsden, Ebin and Fischer [1972] and Chorin, Hughes, Marsden, and McCracken [1978] for discussions).

Product Formulas. Once one has the above geometric setting and the smoothness of the spray is established, one can “read off” a number of interesting consequences. One of these is a product formula stated in the following equation. This is one of several possible product formulas, some of which are useful in computational settings. The product formula we have in mind is the mean motion version of a formula for the Euler equations having its origins in Chorin [1969] and stated, along with a very simple proof (based on the smoothness of the Lagrangian representation of the equations) in Marsden, Ebin and Fischer [1972]. This formula is

\[
E_t = \lim_{n \to \infty} \left( P^\alpha \circ G_t/n \right)^n,
\]

where \( E_t \) is the flow on the space of divergence free velocity fields \( u \) of the averaged Euler equations (with, say, zero boundary conditions), \( P^\alpha \) is the \( H^2 \)-orthogonal projection onto the divergence free vector fields zero on the boundary (defined below), and where \( G_t \) is the unconstrained \( H^1 \) spray—that is, the problem with the incompressibility condition dropped.

If there is no boundary, then \( P^\alpha \) is the same as \( P \), the usual Hodge \( L^2 \) orthogonal projection. For the case of zero slip boundary conditions, we have

\[
P^\alpha = (1 - \alpha^2 \triangle)^{-1} P (1 - \alpha^2 \triangle)
\]
where the domain of \((1 - \alpha^2 \Delta)\) is \(H^2(TM) \cap H_0^1(TM)\) (the Sobolev space of \(H^2\) vector fields vanishing on the boundary). Notice that \(P^\alpha\) is idempotent, so that it is a projection in \(H^2\) since

\[
\langle (1 - \alpha^2 \Delta)P^\alpha u, (1 - \alpha^2 \Delta)u \rangle_0 \geq 0,
\]

where \(\langle \cdot, \cdot \rangle_0\) is the \(L^2\) inner product. Note, however, that the \(H^1\) inner product of \((1 - \alpha^2 \Delta)^{-1}\)\(\text{grad} \; p\) with divergence free vectors \(U\) in \(H^s \cap H_0^1\) vanishes in the sense that

\[
\langle (1 - \alpha^2 \Delta)^{-1}\text{grad} \; p, U \rangle_1 := \langle (1 - \alpha^2 \Delta)(1 - \alpha^2 \Delta)^{-1}\text{grad} \; p, U \rangle_0 = \langle \text{grad} \; p, U \rangle_0 = 0.
\]

In the case of the viscous version of the equations, it is very interesting that now one does not require the vorticity creation operator to correct the boundary term (see also Marsden [1973, 1974], Chorin, Hughes, Marsden and McCracken [1978] and Benfatto and Pulvirenti [1986]). The form of this product formula is

\[
F_t = \lim_{n \to \infty} \left( S_{t/n} \circ E_{t/n} \right)^n = \lim_{n \to \infty} \left( S_{t/n} \circ P^\alpha \circ G_{t/n} \right)^n
\]

where \(S_t\) is the Stokes-\(\alpha\) flow.

**Smoothness of the Spray.** A crucial central result for this paper is the smoothness of the spray of the averaged Euler equations. Concretely, when we refer to the spray, we mean the infinite dimensional vector field governing the dynamics written in Lagrangian representation. It is this vector field, which is shown to be smooth in the standard sense of smooth vector fields on infinite dimensional manifolds.

Smoothness of the spray has the direct consequence that the averaged Euler equations are well-posed using the dynamical variable \(u\), for short time in the \(H^s\) topology for \(s > n/2 + 1\). Note that the vector field \(v = (1 - \alpha^2 \Delta)u\) is only in class \(H^{s-2}\) and need not be \(C^0\). Smoothness also plays a fundamental role for the derivation of the consequences already mentioned, such as the limit of zero viscosity and the product formulas.

Some additional results which are a consequence of the smoothness of the spray are also worth stating. For example, if \(\eta\) is a volume preserving diffeomorphism close to the identity, then there is a unique initial velocity field whose averaged Euler flow reaches \(\eta\) in time one (and stays in a neighborhood of the identity diffeomorphism). In addition, even if the initial condition is only \(H^s\), the flow is infinitely differentiable in time. This latter fact is surprising even for the usual Euler equations.\(^1\)

We conclude by outlining the general strategy for the proof of smoothness of this spray as well as stating a number of related results that follow from our method of proof.

\(^1\)As was pointed out by T. Kato (in an unpublished manuscript), this fact for the Euler equations was rediscovered by several authors who were evidently unaware that it is an immediate consequence of Ebin and Marsden [1970].
1. The spray of the Camassa-Holm equation on the circle is smooth on the tangent bundle of the $H^s$ diffeomorphism group of the circle, for $s > 3/2$. This follows from the fact that the spray has no loss of derivatives (see Remark 3.5 of Shkoller [1998] and Kouranbaeva [1999]) and the method of proof in the present paper. We also note that the noncompact case (replace the circle with the real line) should follow using weighted Sobolev spaces, as in Cantor [1975].

2. More generally, the right invariant $H^1$ metric on the full $H^s$ diffeomorphism group for $s > n/2 + 2$ of a compact Riemannian $n$-manifold possibly with boundary and with pointwise fixed boundary conditions, has a smooth spray. Using the Moser [1966] estimates on products, we conjecture that the condition on $s$ can be weakened to $s > n/2 + 1$.

3. The right translated projection $P^\alpha$ as a map from the tangent bundle of the full diffeomorphism group (restricted to the volume preserving ones) to the tangent bundle of the volume preserving diffeomorphism group is smooth.

4. The sprays on the group of volume preserving diffeomorphisms and that on the full group of diffeomorphism are related by the derivative of the right translated projection $P^\alpha$.

5. Combining the last two facts, one arrives at smoothness of the spray of the $H^1$ metric on the volume preserving diffeomorphism group in the $H^s$ topology, and a careful examination of the proof shows that $s > n/2 + 1$ is sufficient.

2 Diffeomorphism Groups

In this section we set up the relevant groups of diffeomorphisms that we shall need to study the averaged Euler equations in Lagrangian representation.

**Sobolev Spaces of Mappings.** Let $(M, \langle \cdot, \cdot \rangle)$ be a compact oriented $C^\infty$ $n$-dimensional Riemannian manifold possibly with boundary, and let $(Q, \langle \cdot, \cdot \rangle^Q)$ be a $p$-dimensional compact Riemannian manifold without boundary. By Sobolev’s embedding theorem, when $s > n/2 + k$, the set of Sobolev mappings $H^s(M, Q)$ is a subset of $C^k(M, Q)$ with continuous inclusion, and so for $s > n/2$, an $H^s$-map of $M$ into $Q$ is pointwise well-defined. Mappings in the space $H^s(M, Q)$ are those whose first $s$ distributional derivatives are square integrable in any system of charts covering the two manifolds.

For $s > n/2$, it is known that the space $H^s(M, Q)$ is a $C^\infty$ differentiable Hilbert manifold (see Palais [1968], Ebin and Marsden [1970] and references therein). Let $\exp: TQ \to Q$ be the exponential mapping associated with $\langle \cdot, \cdot \rangle^Q$. Then for each $\phi \in H^s(M, Q)$, the map $\omega_{\exp} : T_\phi H^s(M, Q) \to H^s(M, Q)$ is used to provide a differentiable structure which is independent of the chosen metric, where $\omega_{\exp}(v) = \exp \circ v$. 
The set of $H^s$ mappings from $M$ to itself is not a smooth manifold; however, if we embed $M$ in its double $\tilde{M}$, then the set $H^s(M, \tilde{M})$ is a $C^\infty$ Hilbert manifold, and for $s > n/2 + 1$, we may form the set $D^s(M)$ consisting of $H^s$ maps $\eta$ mapping $M$ to $M$ with $H^s$ inverses. This space is a smooth manifold. It is a well-known fact that the diffeomorphism group $D^s(M)$ is a $C^\infty$ topological group for which the left translation operators are continuous and the right translation operators are smooth (Ebin and Marsden [1970] and references therein). One also knows that $\eta : M \to M$ has an extension to an element of (the connected component of the identity of) $D^s(\tilde{M})$ if and only if $\eta$ lies in (the connected component of the identity of) $D^s(M)$.

Let $\mu$ be the Riemannian volume form on $M$, and denote by $D^s_\mu(M) := \{ \eta \in D^s(M) \mid \eta^*(\mu) = \mu \}$ the subgroup of $D^s(M)$ consisting of all volume preserving diffeomorphisms of class $H^s$. For each $\eta \in D^s_\mu(M)$, we may use the $L^2$ Hodge decomposition to define the projection $P_\eta : T_\eta D^s(M) \to T_\eta D^s_\mu(M)$ given by

$$P_\eta(X) = (P_e(X \circ \eta^{-1})) \circ \eta,$$

where $X \in T_\eta D^s_\mu(M)$, and $P_e$ is the $L^2$ orthogonal projection onto the divergence-free vector fields on $M$. Recall that this projection is given by

$$P_e(v) = v - \text{grad } p(v) - \text{grad } b(v),$$

where $p$ is the solution of the boundary value problem

$$\begin{align*}
\Delta p(v) &= \text{div } v \quad \text{in } M \\
p(v) &= 0 \quad \text{on } \partial M,
\end{align*}$$

and $b$ solves

$$\begin{align*}
\Delta b(v) &= 0 \quad \text{in } M \\
\langle \text{grad } b(v), n \rangle &= \langle v - \text{grad } p, n \rangle \quad \text{on } \partial M,
\end{align*}$$

where $n$ is the orientation preserving normal vector field on $\partial M$. The function $p$ is the pressure associated with $v$, while the function $b$ is a smooth extension of the normal component of $v$ along $\partial M$ to the interior of $M$. Subtraction of $\text{grad } b(v)$ is necessary as volume preserving diffeomorphisms of a manifold with boundary leave the boundary invariant.

**Diffeomorphisms Leaving the Boundary Pointwise Fixed.** We define the set

$$D^s_{\mu, \text{fix}}(M) = \{ \eta \in D^s_\mu(M) \mid \eta(x) = x \text{ for all } x \in \partial M \},$$

that is, the volume preserving diffeomorphisms that leave the boundary pointwise fixed.
Theorem 2.1 (Ebin and Marsden (1970)): \( D_{\mu, \text{fix}}^s(M) \) is a smooth subgroup of \( D^s(M) \) with Lie algebra \( T_e D_{\mu, \text{fix}}^s(M) \) consisting of the space of divergence free \( H^s \) vector fields that vanish on \( \partial M \).

In other words, this group corresponds to the choice of the no slip boundary conditions \( u = 0 \) for the averaged Euler equations.

We can now define \( P_\alpha^\eta : T_\eta D^s(M) \to T_\eta D_{\mu, \text{fix}}^s(M) \) by

\[
P_\alpha^\eta(X) = [(1 - \alpha^2 \triangle)^{-1} P_e (1 - \alpha^2 \triangle)(X \circ \eta^{-1})] \circ \eta.
\] (2.1)

Alternative Boundary Conditions. As is described in Holm, Marsden and Ratiu [1998a], there is an alternative choice of boundary conditions for the averaged Euler equations that are more like normal boundary conditions than no slip conditions. We will describe these conditions briefly and the geometry and analysis of this situation is explored in detail in Marsden, Ratiu and Shkoller [1999a].

Let \( n \) denote the outward unit normal vector field along the boundary of \( M \) and let \( N \) denote the corresponding normal bundle. We set

\[
D_{\mu, \text{normal}}^s(M) = \{ \eta \in D^s(M) | T_\eta|_{\partial M} : N \to N \}.
\]

This group \( D_{\mu, \text{normal}}^s(M) \) corresponds to an alternative choice of boundary conditions, namely those divergence free vector fields \( u \) that are tangent to \( \partial M \) and satisfy the boundary condition

\[
\langle \nabla_n u, v \rangle = H_n(u, v) = \langle S_n(u), v \rangle,
\] (2.2)

at points of \( \partial M \), for all divergence free vector fields \( v \) that are tangent to the boundary, where \( H_n \) (or equivalently \( S_n \)) denotes the second fundamental form of the boundary.

3 Mean Hydrodynamics on the Subgroup \( D_{\mu, \text{fix}}^s(M) \)

\( H^1 \) Metric on \( D_{\mu, \text{fix}}^s(M) \). In this section, we shall consider geodesic motion of the weak \( H^1 \) right invariant metric on the group \( D_{\mu, \text{fix}}^s(M) \) which is defined as follows. For \( X, Y \in T_e D^s_{\mu, \text{fix}}(M) \), we set

\[
\langle X, Y \rangle_1 = \int_M \left( \langle X(x), Y(x) \rangle + \alpha^2 \langle \nabla X(x), \nabla Y(x) \rangle \right) \mu(x),
\] (3.1)

and extend \( \langle \cdot, \cdot \rangle \) to \( D_{\mu, \text{fix}}^s(M) \) by right invariance.

Submanifold Geometry. Theorem 2.1 ensures that \( D_{\mu, \text{fix}}^s(M) \) is submanifold of \( D^s(M) \) in the strong \( H^s \) topology. With the induced metric (3.1), the submanifold \( D_{\mu, \text{fix}}^s(M) \) also inherits weak \( H^1 \) structure. Our definition of the projection \( P^\alpha \) in (2.1) from tangent spaces of \( D^s(M) \) to tangent spaces of \( D_{\mu, \text{fix}}^s(M) \) is orthogonal with respect to this weak \( H^2 \) structure, but \( P^\alpha \) is not a Hodge projection. In order
to obtain a Hodge projection which is $H^1$ orthogonal, one redefines the weak $H^1$ metric in terms of the de-Rham Laplacian which is related to the rough Laplacian $\Delta = \text{Tr} \nabla \nabla$ by an additional term involving the Ricci tensor on $M$ (see, for example, Section 2.3 of Shkoller [1998]).

Using the de Rham Laplacian, Shkoller [1998] has obtained expressions for the unique $H^1$ Riemannian connection on $D^s_{\mu}(M)$ when $M$ is a compact boundaryless manifold. Using this formula, together with submanifold geometry, he has proven that the weak $H^1$ curvature operator on $D^s_{\mu}(M)$ is a continuous trilinear map in the strong $H^s$ topology. This immediately implies the existence of unique solutions to the Jacobi equations along the geodesics of the $H^1$ Riemannian connection, and hence establishes the foundations for a Lagrangian stability analysis. All of this analysis extends trivially to manifolds with boundary on the subgroup $D^s_{\mu,\text{fix}}(M)$ if one replaces the $H^1$ Hodge projection with our projection $P^\alpha$.

**Euler-Poincaré equations on $T_e D^s_{\mu,\text{fix}}(M)$**. The Euler-Poincaré theorem (for background, see, for example, Marsden and Ratiu [1999] and Holm, Marsden and Ratiu [1998a]) uses an additional fact coming from the group structure of the problem, namely that the right translation maps are smooth on $D^s_{\mu,\text{fix}}(M)$ which can be used to translate geodesic motion over the entire topological group into motion in the “Lie algebra” $T_e D^s_{\mu,\text{fix}}(M)$. We shall state this theorem in this context.

**Theorem 3.1 (Euler-Poincaré)** Equip $D^s_{\mu,\text{fix}}(M)$ with the right invariant metric $\langle \cdot, \cdot \rangle_1$. Then, a curve $\eta(t)$ in $D^s_{\mu,\text{fix}}(M)$ is a geodesic of this metric if and only if $u(t) = T_{\eta(t)} R_{\eta(t)^{-1}} \dot{\eta}(t) = \dot{\eta}(t) \circ \eta(t)^{-1}$ satisfies

$$\frac{d}{dt} u(t) = -P^\alpha \circ \text{ad}^*_{u(t)} u(t) \quad (3.2)$$

where $\text{ad}^*_u$ is the formal adjoint of $\text{ad}_u$ with respect to the metric $\langle \cdot, \cdot \rangle_1$ at the identity, i.e.,

$$\langle \text{ad}^*_u v, w \rangle_1 = \langle v, [u, w] \rangle_1$$

for all $u, v, w \in T_e D^s_{\mu,\text{fix}}(M)$.

Next, we shall prove that a unique continuously differentiable geodesic spray exists for the right invariant $H^1$ metric obtained by right translating $\langle \cdot, \cdot \rangle_1$ over the entire group, but we note that even if a given metric does not have a $C^1$ spray from $H^s$ into $H^1$, but there is still an existence theorem for geodesics, then the theorem is still true, when appropriately interpreted.

We note here that use of the right invariant $H^1$ metric (as opposed to the usual, or naive, $H^1$ metric) is essential. For example, denote by $\mathcal{M}^1$ ($\mathcal{M}^0$) the manifold of $H^s$ maps from $\mathbb{T}^n$ into $\mathbb{T}^n$ with the weak $H^1$ ($L^2$) topology induced by the usual $H^1$ ($L^2$) metric. On the orthogonal complement of Ker$(1 - \Delta)$, $\mathcal{M}^1$ is totally geodesic in $\mathcal{M}^0$, and so the geodesic equations for both metrics are the same.

Now, in the context of the Euler-Poincaré theorem we have just stated, a straightforward computation of the geodesic spray of the right invariant metric restricted
to right invariant vector fields on $\mathcal{D}_{\mu,\text{fix}}^s(M)$ shows that the spray coincides with the equations expressed in terms of the coadjoint action (of the group $\mathcal{D}_{\mu,\text{fix}}^s(M)$ on the dual of its Lie algebra). Equivalently, there are two natural connections given by the Riemannian structure and right translation, and the Euler-Poincaré equations can be obtained from subtraction of the latter from the former. For the general Euler-Poincaré theorem for arbitrary Lagrangians, see Theorem 5.1 and the Appendix of Bloch, Krishnaprasad, Marsden, and Ratiu [1996] and Holm, Marsden and Ratiu [1998a] for the semidirect product theory.

The form (3.2) is obtained by expressing the variations $\delta u(t)$ of curves $u(t) \in T_e \mathcal{D}_{\mu,\text{fix}}^s(M)$ in terms of the variations $\delta \eta(t)$ of curves $\eta(t)$ in $\mathcal{D}_{\mu,\text{fix}}^s(M)$. In particular let $\eta : U \subset \mathbb{R}^2 \to \mathcal{D}_{\mu,\text{fix}}^s(M)$ be a smooth map and let

$$u(t,s) = T_{\eta(t,s)} R_{\eta(t,s)}^{-1} \left[ \frac{\partial \eta(t,s)}{\partial t} \right]$$

and

$$\xi(t,s) = T_{\eta(t,s)} R_{\eta(t,s)}^{-1} \left[ \frac{\partial \eta(t,s)}{\partial s} \right].$$

Then $(\partial u/\partial s) - (\partial \xi/\partial t) = [u, \xi]$ and we obtain

$$\delta u(t) = \dot{\xi} - \text{ad}_u \xi.$$

In the case of the right invariant $H^1$ metric on $\mathcal{D}_{\mu,\text{fix}}^s(M)$, a straightforward computation (as in Holm, Marsden and Ratiu [1998a]) shows that

$$\text{ad}^*_u u = (1 - \alpha^2 \Delta)^{-1} \left[ \nabla u(t) v(t) - \alpha^2 [\nabla u(t)]^t \cdot \Delta u(t) \right],$$

where

$$v = (1 - \alpha^2 \Delta) u.$$

Thus, the Euler–Poincaré equation $\dot{u} = -P^\alpha \text{ad}^*_u u$ takes the form

$$\dot{v}(t) + \nabla u(t) v(t) - \alpha^2 [\nabla u(t)]^t \cdot \Delta u(t) = -\text{grad} p(t)$$

(3.3)

together with the divergence constraint $\text{div} u = 0$, the no slip boundary conditions $u = 0$ on the boundary $\partial M$, and the initial conditions $u(0) = u_0$. We call this equation the **averaged Euler equations** or the **Euler-\( \alpha \) equations**.

In components, and with the standard summation conventions and choice of Levi-Civita connection and rough Laplacian, these equations read as follows

$$\frac{\partial u^i}{\partial t} + \frac{\partial u^i}{\partial x^j} u^j + \Gamma^i_{kj} v^k u^j - \alpha^2 \left[ g^{ij} \left( \frac{\partial u^j}{\partial x^l} + \Gamma^j_{kl} u^k \right) \right] (\Delta u) = -g^{ij} \frac{\partial p}{\partial x^j},$$

where

$$(\Delta u)_j = \left( \delta^i_j \frac{\partial}{\partial x^j} + \Gamma^i_{ij} \right) \left( \frac{\partial u^i}{\partial x^l} + \Gamma^i_{ml} u^m \right).$$

Being Euler–Poincaré equations, of course these equations share all the properties given by the general theory, such as a Kelvin-Noether theorem, a Lie–Poisson Hamiltonian structure and so on.
The geodesic spray on $D_{\mu,\text{fix}}^s(M)$. Now we are ready to state and prove the first main result of the paper.

**Theorem 3.2** For $s > n/2 + 1$, there exists a unique continuously differentiable geodesic spray of the metric $\langle \cdot, \cdot \rangle_1$ on the group $D_{\mu,\text{fix}}^s(M)$.

**Proof.** We compute the first variation of the action function

$$E(\eta) = \frac{1}{2} \int_{\mathbb{R}} \langle \dot{\eta}(t), \dot{\eta}(t) \rangle_1 \, dt,$$

which we decompose as

$$E^0(\eta) = \frac{1}{2} \int_{\mathbb{R}} \langle \dot{\eta}, \dot{\eta} \rangle_0 \, dt$$

and

$$E^1(\eta) = \frac{\alpha^2}{2} \int_{\mathbb{R}} \langle \nabla (\eta \circ \eta^{-1}), \nabla (\eta \circ \eta^{-1}) \rangle_0 \, dt.$$

We have

$$E^1(\eta) = \frac{\alpha^2}{2} \int_{\mathbb{R}} \int_{M} \langle \nabla (\eta \circ \eta^{-1})(y), \nabla (\eta \circ \eta^{-1})(y) \rangle_y \, dy \, dt$$

$$= \frac{\alpha^2}{2} \int_{\mathbb{R}} \int_{M} \langle \nabla \eta(x) \cdot [T\eta(x)]^{-1}, \nabla \eta(x) \cdot [T\eta(x)]^{-1} \rangle_{\eta(x)} \, dx \, dt.$$

Let $\epsilon \mapsto \eta^\epsilon$ be a smooth curve in $D_{\mu,\text{fix}}^s(M)$ such that $\eta^0 = \eta$ and $(d/d\epsilon)_{\epsilon=0} \eta^\epsilon = \delta \eta$. Then

$$dE^1(\eta) \cdot \delta \eta = \alpha^2 \int_{\mathbb{R}} \int_{M} \left( \frac{D}{d\epsilon} \right)_0 \left\{ \nabla \eta^\epsilon \cdot [T\eta^\epsilon]^{-1}, \nabla \eta \cdot [T\eta]^{-1} \right\}_{\eta(x)} \, dx \, dt$$

$$= \alpha^2 \int_{\mathbb{R}} \int_{M} \left[ \left( \frac{D}{d\epsilon} \right)_0 \left\{ \nabla [T\eta]^{-1}(\cdot) \eta^\epsilon, \nabla \eta \cdot [T\eta]^{-1} \right\}_{\eta(x)} 
- \langle \nabla \delta \eta \cdot [T\eta]^{-1}, (\nabla \eta \cdot [T\eta]^{-1})' (\nabla \eta \cdot [T\eta]^{-1}) \right\} \right] \, dx \, dt.$$

Now fix $x \in M$, and let $s \mapsto \sigma_s$ be a smooth curve in $M$ such that $\sigma_0 = x$, and $(d/ds)_{s=0} \sigma_s = [T\eta(x)]^{-1} \cdot a$ for some $a \in T_{\eta(x)}M$. Define the parameterized surface $(s, \epsilon) \mapsto f(s, \epsilon) := \eta^\epsilon \circ \sigma_s$. Denote $(d/d\epsilon)(\cdot)$ by $(\cdot)'$ and $(d/ds)(\cdot)$ by $(\cdot)^\prime$. Then,

$$f(0,0) = \eta(x), \quad f(0,\epsilon) = \eta^\epsilon(x), \quad f(s,0) = \eta(\sigma_s),$$

and

$$f'((0,\epsilon) = T\eta^\epsilon \cdot \sigma_s', \quad \hat{f}(s,\epsilon) = (d/d\epsilon) \cdot \eta^\epsilon(\sigma_s),$$

so that

$$f'(0,0) = a, \text{ and } \hat{f}(0,0) = \delta \eta(x).$$
Hence, for any \( a \in T_{\eta(x)}M \), and using \( L(a) \) to denote linear operation on \( a \) by the linear operator \( L \),

\[
\frac{D}{de}\bigg|_0 \nabla_{[\eta(x)]^{-1}(a)} = \frac{D}{de}\bigg|_0 \nabla \sigma'(0) = \frac{D}{de}\bigg|_0 \frac{D}{ds}\bigg|_0,
\]

and since

\[
\frac{D}{de}\bigg|_0 \frac{D}{ds}\bigg|_0 \eta^e = \frac{D}{ds}\bigg|_0 \frac{D}{de}\bigg|_0 \eta^e + R(\dot{f}, f')|_{0,0}\eta^e,
\]

where \( R \) is the curvature tensor on \( M \), we get

\[
\frac{D}{de}\bigg|_0 \nabla_{[\eta(x)]^{-1}(a)} \dot{\eta}^e = \nabla_{[\eta(x)]^{-1}(a)} \frac{D}{de}\bigg|_0 \dot{\eta}^e + R(\dot{f}, f')|_{0,0}\eta.
\]

Thus,

\[
d^\mathfrak{E}^1(\eta) \cdot \delta \eta = \alpha^2 \int_{\mathbb{R}} \int_M \left[ \langle \nabla (\nabla / dt) \delta \eta \cdot [\eta]^{-1} + R(\delta \eta, \cdot) \eta, \nabla \cdot [\eta]^{-1} \rangle \right. \\
\left. \langle \nabla \delta \eta, (\nabla \dot{\eta} \cdot [\eta]^{-1})^t \cdot (\nabla \dot{\eta} \cdot [\eta]^{-1}) \cdot [\eta]^{-1t} \rangle \right] dxdt.
\]

Now, using the same argument as above, we find that

\[
\left\langle \nabla \left( \frac{D}{dt} \delta \eta \cdot [\eta]^{-1}, \nabla \dot{\eta} \cdot [\eta]^{-1} \right) \right\rangle \\
= \left\langle \frac{D}{dt} \nabla \delta \eta + R(\eta, \dot{\eta} \eta) \delta \eta, \nabla \dot{\eta} \cdot [\eta]^{-1} \cdot [\eta]^{-1t} \right\rangle \\
= - \left\langle \nabla \delta \eta, \frac{D}{dt} \left\{ \nabla \dot{\eta} \cdot [\eta]^{-1} \cdot [\eta]^{-1t} \right\} \right\rangle + \left\langle R(\dot{\eta}, \cdot) \nabla (\dot{\eta} \circ \eta^{-1}), \delta \eta \right\rangle.
\]

Finally,

\[
\frac{D}{dt} \left\{ \nabla \dot{\eta} \cdot [\eta]^{-1} \cdot [\eta]^{-1t} \right\} = \nabla \frac{D}{dt} \dot{\eta} \cdot [\eta]^{-1} \cdot [\eta]^{-1t} + R(\dot{\eta}, T\eta) \dot{\eta} \cdot [\eta]^{-1} \cdot [\eta]^{-1t} \\
- (\nabla \dot{\eta} \cdot [\eta]^{-1}) \cdot (\nabla \dot{\eta} \cdot [\eta]^{-1}) \cdot [\eta]^{-1t} \\
- (\nabla \dot{\eta} \cdot [\eta]^{-1}) \cdot (\nabla \dot{\eta} \cdot [\eta]^{-1})^t \cdot [\eta]^{-1t}.
\]

Integrating by parts, noting that the boundary terms vanish by virtue of the subgroup \( D^\mu_{\text{fix}}(M) \), and denoting by \( \nabla^* \) the L2 formal adjoint of \( \nabla \), we have that

\[
d^\mathfrak{E}^1(\eta) \cdot \delta \eta = \alpha^2 \int_{\mathbb{R}} \int_M - \left\langle \delta \eta, \nabla^* \left\{ \nabla (\nabla / dt) \eta \cdot [\eta]^{-1} \cdot [\eta]^{-1t} \right\} \right\rangle \\
+ \nabla^* \left[ (\nabla \dot{\eta} \cdot [\eta]^{-1})^t \cdot (\nabla \dot{\eta} \cdot [\eta]^{-1}) \cdot [\eta]^{-1t} \\
- (\nabla \dot{\eta} \cdot [\eta]^{-1}) \cdot (\nabla \dot{\eta} \cdot [\eta]^{-1}) \cdot [\eta]^{-1t} \right]
\]
\begin{equation}
+ \langle \text{Tr} R(\nabla \dot{\eta} \cdot [T\eta]^{-1}, \dot{\eta}) \rangle + \text{Tr} R(\dot{\eta}, \cdot) \nabla \dot{\eta} \cdot [T\eta]^{-1}, \delta \eta)
- \langle \nabla^s (R(\dot{\eta}, \cdot) \dot{\eta} \cdot [T\eta]^{-1})^t, \delta \eta) \rangle.
\end{equation}

Computing the first variation of $\mathcal{E}^0$, we obtain
\begin{equation}
d \mathcal{E}^0(\eta) \cdot \delta \eta = \int_{\mathbb{R}} \int_{M} \left( \frac{D}{dt} \right) \left. \delta \eta \right|_0 \cdot \delta \eta \, dx dt = \int_{\mathbb{R}} \int_{M} \left( \frac{D}{dt} \delta \eta, \dot{\eta} \right) \, dx dt
- \int_{\mathbb{R}} \int_{M} \left( - \frac{D}{dt} \dot{\eta}, \delta \eta \right) \, dx dt.
\end{equation}

Setting $d \mathcal{E} \cdot \delta \eta = 0$, and using the projector $P^\alpha$ given by (3.1) gives
\begin{equation}
P^\alpha_{\eta} \circ \nabla \dot{\eta} \dot{\eta} = P^\alpha_{\eta} \circ (1 - \alpha^2 \Delta) \left[ \nabla \dot{\eta} \big( - \left( \nabla \dot{\eta} \cdot [T\eta]^{-1} \right)^t \left( \nabla \dot{\eta} \cdot [T\eta]^{-1} \right) \right]
+ \nabla \dot{\eta} \cdot [T\eta]^{-1} \nabla \dot{\eta} \cdot [T\eta]^{-1} + \left( \nabla \dot{\eta} \cdot [T\eta]^{-1} \right)^t \left( \nabla \dot{\eta} \cdot [T\eta]^{-1} \right)^t \right]
- \left[ \text{Tr} R(\nabla \dot{\eta} \cdot [T\eta]^{-1}, \dot{\eta}) \right] + \text{Tr} (R(\nabla \dot{\eta} \cdot [T\eta]^{-1}, \dot{\eta}) \cdot + R(\dot{\eta}, \cdot) \nabla \dot{\eta} \cdot [T\eta]^{-1} \cdot \dot{\eta})
+ \nabla^s \left( R(\dot{\eta}, [T\eta]^{-1}) \right),
\end{equation}

where
\begin{equation}
\Delta \eta = - \nabla^s \left( \nabla \cdot [T\eta]^{-1} \right) \cdot [T\eta]^{-1},
\end{equation}

and where again $T\eta^{-1}$ defines linear operation on a vector; explicitly, operating on a vector $V(x)$, this is just shorthand notation for $[T\eta(x)]^{-1} (V(x)) = [T\eta(x)]^{-1} V(x)$. Denote the quadratic form on the right-hand-side of (3.4) by $P^\alpha_{\eta} \circ F^\alpha_{\eta}$. Notice that $F^\alpha_{\eta}$ has no derivative loss and by standard arguments is continuous from $H^s$ into $H^s$. This is analogous to the equation (3.7) of Theorem 3.3 in Shkoller [1998], so we shall follow the proof of this theorem. The following is a combination of Lemmas 3.1 and 3.2 in Shkoller [1998], where the detailed proofs can be found.

**Lemma 3.3** Let
\[
\Delta: \cup_{\eta} \in D_{\mu}(M) H^s_{\eta} (TM) \rightarrow \cup_{\eta} \in D_{\mu}(M) H^s_{\mu} (TM) \rightarrow \cup_{\eta} \in D_{\mu}(M) H^s_{\eta} (TM) \rightarrow \cup_{\eta} \in D_{\mu}(M) H^s_{\mu} (TM)
\]
be defined by equation (3.5) and be the identity on $D^s_{\mu}(M)$. Then $\Delta$ is a $C^1$ bundle map. Also, the operator $(1 - \Delta) \cdot [T\eta]^{-1}$ is a $C^1$ bundle map as well.

We also have the following lemma which is Lemma 4 of Appendix A in Ebin and Marsden [1970].

**Lemma 3.4** The $L^2$ Hodge projection $\eta \rightarrow P_{\eta}$ is smooth as a function of $\eta$.

Choosing a local chart $(\mathcal{U}, x^i)$ on $M$, we may express the geodesic spray $S : T'D_{\mu}(M) \rightarrow T'T'D_{\mu}(M)$ in this chart by
\[
S_{\eta}(\dot{\eta}) = \frac{d}{dt} (\eta, \dot{\eta}) = (\dot{\eta}, (I - P_{\eta}) \circ [\nabla_{\eta, \eta^{-1}} (\dot{\eta} \circ \eta^{-1})])
\]
\[-P_\eta^\alpha \left[ \Gamma_\eta(\dot{\eta}, \dot{\eta}) \right] + P_\eta^\alpha \circ F_\eta \right), \tag{3.6}\]

where

\[\Gamma_\eta(\dot{\eta}, \ddot{\eta}) = \left\{ \Gamma_j^i \left( \dot{\eta}^i \circ \eta^{-1} \right) \left( \ddot{\eta}^g \circ \eta^{-1} \right) \frac{\partial}{\partial x^i} \right\} \circ \eta.\]

The fact that the projection \(P_\eta^\alpha\) is \(C^1\) in \(\eta\) follows from Lemma 3.3 and Lemma 3.4 since we may write \(P_\eta^\alpha(X) = (1 - \alpha^2 \Delta_\eta)^{-1} P_\eta(1 - \alpha^2 \Delta_\eta) X\).

Lemma 3.3 also shows that \(F_\eta^\alpha\) is \(C^1\) as well. Hence, the term \(P_\eta^\alpha \circ (F_\eta - \Gamma_\eta(\dot{\eta}, \dot{\eta}))\) is \(C^1\) in \(\eta\) from \(H^s\) into \(H^s\).

As for the remaining term,

\[(\text{Id} - P_\eta^\alpha) \circ \nabla_{\dot{\eta} \circ \eta^{-1}}(\dot{\eta} \circ \eta^{-1}) = (1 - \alpha^2 \Delta)^{-1} \text{grad} \Delta^{-1} (1 - \alpha^2 \Delta) \text{div} \nabla_{\dot{\eta} \circ \eta^{-1}}(\dot{\eta} \circ \eta^{-1}) + (1 - \alpha^2 \Delta)^{-1} \text{grad} \Delta^{-1} \text{div} (1 - \alpha^2 \Delta) \nabla_{\dot{\eta} \circ \eta^{-1}}(\dot{\eta} \circ \eta^{-1}).\]

The fact that the first term on the right-hand-side is a \(C^1\) map from \(H^s\) into \(H^s\) follows identically the argument used by Ebin and Marsden [1970] to prove smoothness of the spray of the right invariant \(L^2\) metric on \(D_\mu^s(M)\). The second term is \(C^1\) by similar reasoning as \([\text{div}, (1 - \alpha^2 \Delta)]\) is a differential operator of order 2. This proves that \(S\) is \(C^1\).

The standard theorem for existence and uniqueness of ordinary differential equations in a Banach space shows that unique solutions locally exist and depend smoothly on initial conditions. Q.E.D.

We remark that although the statement of Lemma 3.3 is \(C^1\), it should be clear that the \(C^k\) result could be obtained for any positive integer \(k\) by (a messy) induction; this type of reasoning led to the \(C^\infty\) regularity of the \(L^2\) Hodge projection \(P\) in Lemma 1.4.

As we remarked earlier, the method of proof we have given for the smoothness of the spray of the right invariant \(H^1\) metric on \(D_\mu^{s, \text{fix}}(M)\) also provides well-posedness for the one dimensional Camassa-Holm equation on \(S^1\), since the spray in this case does not have a loss of derivative (c.f. Remark 3.5 in Shkoller [1998]). The situation of the present paper is, however, complicated in a nontrivial way by the presence of boundaries. We state the situation for the CH equation as the following theorem.

**Theorem 3.5** The Cauchy problem for the 1D CH equation (1.4), given by

\[
\ddot{\eta} = - \left[(1 - \partial_y^2)^{-1} \partial_y \left( (\dot{\eta} \circ \eta^{-1})^2 + \frac{1}{2} (\dot{\eta} \circ \eta^{-1})_y \right) \right] \circ \eta \tag{3.7}
\]

with initial conditions

\[
\eta(0) = \epsilon, \quad \dot{\eta}(0) = u_0,
\]

has a unique solution \((\eta, \dot{\eta})\) in \(D^s(S^1) \times H^s(S^1)\) for \(s > \frac{3}{2}\) on a finite time interval where the solution has \(C^1\) dependence on time and smooth dependence on initial data.

Theorem 3.2 implies many other results too, such as finite smoothness with respect to initial conditions in Eulerian representation if one is willing to map between different Sobolev spaces, smoothness with respect to the time variable, etc.
Remark. The subgroup $D^s_{\mu,\text{fix}}(M)$ ensures that the boundary terms vanish in the above variational principle, as the variation $\delta\eta(x) = 0$ for all $x \in \partial M$. For variations which are not required to vanish on the boundary, one obtains natural boundary conditions on $\partial M$, which are discussed in Marsden, Ratiu and Shkoller [1999a].

4 The limit of zero viscosity on $D^s_{\mu,\text{fix}}(M)$

The Navier-Stokes $\alpha$-model is obtained by adding viscous diffusion to the Euler-$\alpha$ model. The equations are given by

$$\partial_t u - \nu \Delta u + (1 - \alpha^2 \Delta)^{-1} \left[ \nabla u (1 - \alpha^2 \Delta) u - \alpha \nabla u^t \cdot \Delta u \right] = -(1 - \alpha^2 \Delta)^{-1} \text{grad } p.$$  

(4.1)

Foias, Holm, and Titi [1999] have proven global well-posedness of (4.1) and have obtained estimates on the dimension of the global attractor. Chen et al. [1999] have proven existence of the limit of zero viscosity of (4.1). In the case that $\alpha = 0$, this limiting procedure is believed to be valid only for compact manifolds without boundary (e.g., for flows with periodic boundary conditions), as the Navier-Stokes equations and the Euler equations do not share the same boundary conditions on manifolds with boundary. When, $\alpha \neq 0$, however, as we shall discuss in the last section, a certain type of elasticity is added into the Euler-$\alpha$ model, and the mean motion of the fluid exhibits normal stress effects. Because of this, we may prescribe zero velocity boundary conditions even in the inviscid limit, and thus extend the limit of zero viscosity theorems for the averaged Euler equations to manifolds with boundary.

The following is the Euler-$\alpha$ version of Theorem 13.1 of Ebin and Marsden [1970].

**Theorem 4.1** Let $S : T^D^s_{\mu,\text{fix}}(M) \to TT^D^s_{\mu,\text{fix}}(M)$ be the Euler-$\alpha$ vector field given by (3.4). Let $T : T_e D^s_{\mu,\text{fix}}(M) \to T_e D^{s-\sigma}(M)$ be a given map, where $\sigma \geq 2$. Let $T$ be a bounded linear map which generates a strongly-continuous semi-group $F_t : T_e D^s_{\mu,\text{fix}}(M) \to T_e D^s_{\mu}(M)$, $t \geq 0$, and satisfies $\|F_t\|_s \leq e^{\beta t}$ for some $\beta > 0$ and some $s$. Extend $F_t$ to $TT^D^s_{\mu,\text{fix}}(M)$ by

$$\tilde{F}_t(X_\eta) = TR_\eta \cdot F_t \cdot TR_\eta^{-1}(X_\eta)$$

for $X_\eta \in T_\eta D^s_{\mu,\text{fix}}(M)$, and let $\tilde{T}$ be the vector field $\tilde{T} : T^D^s_{\mu,\text{fix}}(M) \to TT^D^{s-\sigma}_{\mu}(M)$ associated to the flow $\tilde{F}_t$.

Then $S + \nu \tilde{T}$ generates a unique local uniformly Lipschitz flow on $T^D^s_{\mu,\text{fix}}(M)$ for $\nu \geq 0$, and the integral curves $c^\nu(t)$ with $c^\nu(0) = x$ extend for a fixed time $\tau > 0$ independent of $\nu$ and are unique. Further,

$$\lim_{\nu \to 0} c^\nu(t) = c^0(t)$$

for each $t$, $0 \leq t < \tau$, the limit being in the $H^s$ topology, $s > (n/2) + 1 + 2\sigma$. 

16
See Lemmas 2-5 in Appendix B of Ebin and Marsden [1970] for an explanation of $2\sigma$ in the hypothesis of Theorem 4.1. Since Theorem 3.2 proves that the geodesic spray is smooth on $D_{\mu,\text{fix}}^s(M)$, we must verify the hypothesis of Theorem 4.1 for $T = -\Delta$.

**Lemma 4.2** The operator $\Delta$ generates a strongly continuous contraction semigroup on the space $T_e D_{\mu,\text{fix}}^s(M)$, for $s > (n/2) + 1$.

**Proof.** Standard energy estimates give the result. See, for example, Theorem 3.2 of Temam [1988], page 70, and Theorem 13.2 of Ebin and Marsden [1970]. Q.E.D.

**Remark.** With initial data $u_0 \in T_e D_{\mu,\text{fix}}^{\infty}(M)$, the solution $u(t)$ is also $C^\infty$ as a consequence of the regularization of parabolic flows.

The way one proves the product formulas mentioned in the introduction is exactly the same as is outlined in Ebin and Marsden [1970] and Marsden, Ebin and Fischer [1972]; one uses the general theorems on product formulas (formulas of Lie-Trotter type) for smooth vector fields on infinite dimensional manifolds, as discussed in, for example, Chorin, Hughes, Marsden and McCracken [1978] and in Abraham, Marsden and Ratiu [1988] using the smoothness of the spray and the projection operator. For product formulas involving the Stokes operator, one couples the smoothness of the spray with the analytic property of the generated semigroup, as explained in Ebin and Marsden [1970]. Of course this technique is closely related to Theorem 4.1 discussed above.

## 5 Second-grade Fluids

As noted in Chen et al. [1999], non-Newtonian fluids of second grade satisfy the same type of equations as the mean motion model equations for incompressible ideal fluids. In fact, we shall show that in the incompressible case, the equations are identical in the limit of zero viscosity. For simplicity we shall make the presentation in Euclidean space. (See Marsden and Hughes [1983] for how to put this discussion on manifolds).

The constitutive equation of fluids for a fluid of grade two is given by (see, for example, Noll and Truesdell [1965])

$$ \mathbf{T} = -\bar{p}\mathbf{1} + \nu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \quad (5.1) $$

where $\mathbf{T}$ is the stress tensor, $\bar{p}$ is the hydrostatic pressure, $\mathbf{1}$ is a unit tensor, $\nu$ is viscosity, $\alpha_1$ and $\alpha_2$ are normal stress moduli. Here, $\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivlin-Ericksen tensors (Rivlin and Ericksen [1955]) defined by

$$ \mathbf{A}_1 = \nabla u + (\nabla u)^t, $$
$$ \mathbf{A}_2 = \frac{D\mathbf{A}_1}{dt} + (\nabla u)^t \mathbf{A}_1 + \mathbf{A}_1 \nabla u. $$
Here, the tensors are of type \((1,1)\) and \(D/dt = (\partial/\partial t) + u \cdot \nabla\) is the material derivative.

To satisfy the Clausius-Duhem inequality, the first material constant has to satisfy \(\nu \geq 0\) and the constants \(\alpha_1\) and \(\alpha_2\) satisfy \(\alpha_1 + \alpha_2 = 0\). If in addition one asks that the specific Helmholtz free energy is a minimum when the fluid is in equilibrium, then

\[\alpha_1 \geq 0.\]

The results expressed by the last inequalities are derived in Dunn and Fosdick [1974].

We now turn to the incompressible case, i.e., \(\text{div } u = 0\). Under this assumption, we make the following calculations on the stress tensor:

\[
\begin{align*}
\text{div } T &= -\text{div } p \mathbf{1} + \nu \text{div } \mathbf{A}_1 + \alpha_1 (\text{div } \mathbf{A}_2 - \text{div } \mathbf{A}_1^2), \\
\text{div } \mathbf{A}_1 &= \text{div}(\nabla u)^t = \Delta u, \\
\text{div } \mathbf{A}_2 &= \frac{\partial}{\partial t} \text{div } \mathbf{A}_1 + \text{div}((u \cdot \nabla) \mathbf{A}_1) \\
&\quad + \text{div}(\mathbf{A}_1 \nabla u) + \text{div}((\nabla u)^t \mathbf{A}_1^t), \\
\text{div}((u \cdot \nabla) \mathbf{A}_1) &= \nabla u \cdot \nabla \mathbf{A}_1 + u \cdot \nabla(\Delta u), \\
\text{div}(\mathbf{A}_1 \nabla u) &= \sum_k (\Delta u)^k \nabla u^k + \frac{1}{2} \nabla(\text{Tr}(\nabla u)^2) + \frac{1}{2} \nabla(\text{Tr}(\nabla u(\nabla u)^t)), \\
\text{div}((\nabla u)^t \mathbf{A}_1) &= \Delta u \cdot \mathbf{A}_1 + (\nabla u)^t \cdot \nabla \mathbf{A}_1, \\
\text{div}(\mathbf{A}_1^2) &= \Delta u \cdot \mathbf{A}_1 + (\nabla u)^t \cdot \nabla \mathbf{A}_1 + \nabla u \cdot \nabla \mathbf{A}_1, \\
\text{div } \mathbf{A}_2 - \text{div } (\mathbf{A}_1^2) &= \frac{\partial}{\partial t} (\Delta u) + u \cdot \nabla(\Delta u) \\
&\quad + (\nabla u)^t \cdot \Delta u + \frac{1}{2} \nabla(\text{Tr}(\nabla u)^2) + \frac{1}{2} \nabla(\text{Tr}(\nabla u(\nabla u)^t)).
\end{align*}
\]

Substituting \(\text{div } T\) into the balance of momentum with no body force acting on the fluid,

\[
\rho \frac{Du}{dt} = \text{div } T,
\]

and making use of the calculations above, we obtain the following equation for a second-grade fluid

\[
\frac{D}{dt}(1 - \tilde{\alpha}_1 \triangle)u = \tilde{\nu} \triangle u + \tilde{\alpha}_1 (\nabla u)^t \cdot \Delta u + \nabla \left( \tilde{\alpha}_1 \text{D}: \text{D} - \frac{\bar{p}}{\rho} \right), \tag{5.2}
\]

where \(\text{D}: \text{D} = \text{Tr}(\text{DD}^t)\) and

\[
\tilde{\alpha}_1 = \frac{\alpha_1}{\rho}, \quad \tilde{\nu} = \frac{\nu}{\rho},
\]

and the stretch tensor is given by

\[
\text{D} = \frac{1}{2} \mathbf{A}_1.
\]
The theory of second-grade fluids has been developed over the last 40 years by a number of people, such as Noll and Truesdell [1965], Coleman and Markovitz [1964], Markovitz and Coleman [1964], and Coleman, Markovitz, and Noll [1966]. These authors have shown that second grade fluids can be regarded (like the Navier-Stokes theory) as an approximation to the general theory of simple fluids with fading memory, valid in sufficiently slow flows.

**Inviscid Second Grade Fluids.** Letting the viscosity $\nu = 0$ in (5.2) and defining $\alpha^2 = \tilde{\alpha}_1$, $v = u - \alpha^2 \Delta u$ gives the following equation

$$\frac{Dv}{dt} - \alpha^2 (\nabla u)^t \cdot \Delta u = -\text{grad} P, \quad \text{div} u = \text{div} v = 0,$$

(5.3)

where the second-grade fluid pressure function is given by

$$P = \frac{\bar{p}}{\rho} - \frac{\alpha^2}{2} \text{Tr}(\nabla u \cdot (\nabla u)^t) - \frac{\alpha^2}{2} \text{Tr}(\nabla u)^2,$$

and where $P$ is the generalized pressure.

We observe that the equations (5.3) are identical to the averaged Euler equations (3.3). In either case, one can eliminate the pressure terms in the standard fashion using the projection to the divergence free part or, if one prefers, one can view the pressure as being determined implicitly from the incompressibility constraint. While these equations are the same, the physical interpretation and the basic mechanics are different.

**Comments on the Limit of Zero Viscosity.** The equations for a second grade fluid with viscosity (5.2) are not the same as the Navier-Stokes–$\alpha$ equations (1.3). The difference is simply that in the case of second grade fluids, one uses the Laplacian of the velocity vector $u$ while in the Navier-Stokes–$\alpha$ equations one uses the Laplacian of the momentum $v$.

The theory for second grade fluids is, in a sense, much easier than that for the averaged Navier-Stokes equations. This is because when the equations are written with the dynamical variable $u$, there is no derivative loss in the dissipative term and so the vector field in Lagrangian coordinates is already smooth. The result on the limit of zero viscosity for second grade fluids corresponding to that for the Navier-Stokes–$\alpha$ equations is elementary, following simply from the smoothness of the flow of a smooth vector field as a function of parameters.

In the literature on second-grade fluids it is generally assumed that the viscosity does not vanish. However following the approximation procedure of Coleman and Markovitz [1964],! and Noll and Truesdell [1965] one can show that the case of zero viscosity does not contradict their theory. Hence for a simple fluid with fading memory, a second-grade fluid with zero viscosity furnishes a complete second-order correction, for normal stress effects, to the theory of perfect (in the terminology of Noll and Truesdell) elastic fluids, in the limit of slow motion. Notice that $\alpha_1$ is a material constant in second-grade fluid theory, whereas $\alpha$ is the fluctuation amplitude scale parameter (i.e., a flow regime parameter) in the average Euler equations.
Another aspect to be mentioned here is that the theory of simple fluids obeys the basic mechanical principles: the principle of material frame-indifference, which assert that the response of a material is the same for all observers, and the principles of determinism and local action, which asserts that the present stress at a particle is determined by the history of an arbitrary small neighborhood of that particle. The same principles are true of the averaged Euler equations and the Navier-Stokes—\( \alpha \) equations.

**Existence and Uniqueness.** In the viscous case, existence and uniqueness has already been shown by Cioranescu and Gorait [1997] and Galdi [1995], although as we have mentioned, many results, such as short time existence and uniqueness and the limit of zero viscosity results follow from our work too.

The question of the existence and dimension of attractors for the Navier-Stokes—\( \alpha \) equations is addressed in Foias, Holm and Titi [1999]. We note that these techniques do not allow one to take the limit of zero viscosity, whereas ours do allow such a limit.

**Specific Solutions.** Several formal solutions (ignoring boundary conditions) have been given for planar homogeneous incompressible flows of second grade fluids. The stream function formulation of the dynamics of such flows in two dimensions, with stream function \( \psi \) and velocity \( u \) related by

\[
  u = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right),
\]

is given by

\[
  q_t + \nabla_u q = \nu \Delta \omega, \quad (5.4)
\]

where

\[
  q = -(1 - \alpha^2 \Delta) \Delta \psi \quad \text{and} \quad \omega = -\Delta \psi.
\]

For example, Ting [1963] provided a set of exact solutions for planar startup flows. Further, Rajagopal and Gupta [1981] showed that equation (5.4) has separable solutions that may be interpreted as, e.g., a damped lattice of vortices, two impinging rotational flows, and a viscoelastic analog to Kelvin’s “Cat’s eyes” vortices. These formal separable solutions impose the condition

\[
  \Delta \psi = c \psi, \quad (5.5)
\]

where \( c \) is a nonzero constant. Of course, this condition implies the linear relation

\[
  q = -(1 - \alpha^2 c) c \psi.
\]

Hence, the transport term \( \nabla_u q = J(q, \psi) \) vanishes in (5.4) and the remaining terms impose

\[
  q_t = \frac{\nu q}{1 - \alpha^2 c}.
\]
which implies exponential time dependence for \( q \) and, thus, for \( \psi \). For example, the stream function for the viscoelastic analog to Kelvin’s “Cat’s eye” vortices found in Rajagopal and Gupta [1981] is given in separated form by

\[
\psi = A \cosh(ax) \cos(by) \exp(\lambda t). 
\] (5.6)

This stream function is a solution of (5.4) under condition (5.5) for arbitrary real \( A, a, b, \) with \( c = a^2 - b^2 \) and \( \lambda = \nu/(1 - \alpha^2 c) \). Relation (5.6) with \( \lambda = 0 \) and \( A, a, b \) arbitrary is also a \textit{stationary} solution of equation (5.4) in the limiting case that \( \nu = 0 \).

Two-dimensional incompressible Navier-Stokes-\( \alpha \) dynamics may be expressed in a form similar to equation (5.4), namely,

\[
q_t + \nabla u q = \nu \Delta q, \quad (5.7)
\]

where

\[
q \equiv \hat{z} \cdot \text{curl} v = -(1 - \alpha^2 \Delta) \Delta \psi
\]

is potential vorticity. Thus, under condition (5.5) (and still ignoring boundary conditions), separable solutions for second-grade fluids satisfying (5.4) become separable solutions for the two-dimensional Navier-Stokes-\( \alpha \) model satisfying (5.7) under the change \( \nu \to (1 - \alpha^2 c) \nu \). In the case \( \nu = 0 \), formal solutions satisfying the linear relation (5.5) are common to both theories. Another example of a steady flow that is common to both theories in the case \( \nu = 0 \) is the Gaussian jet, for which \( \psi = \exp(-y^2) \).

For the ideal limiting case in which \( \nu = 0 \) it should be clear that steady incompressible flows of \textit{both} second grade fluids \textit{and} the two-dimensional Euler-\( \alpha \) model occur whenever \( q \) and \( \psi \) are functionally (not just linearly) related. For example, for periodic boundary conditions, critical points of the functional

\[
H_{\Phi} = H + C_{\Phi} = \frac{1}{2} \int [q \psi + \Phi(q)] dxdy,
\]

(the sum of the energy and the domain integral of an arbitrary function \( \Phi \) of the potential vorticity) are steady flows of both models. This observation leads immediately to a unified viewpoint of nonlinear stability criteria for steady periodic flows of either model, following the energy-Casimir method, as in Holm, Marsden, Ratiu and Weinstein [1985].

Exact solutions of the three-dimensional Navier-Stokes-\( \alpha \) model may also be sought by adapting the classical solutions for the two-dimensional Navier-Stokes equations, e.g., for flows between two rotating disks, Beltrami flows, etc., as reviewed, e.g., in Wang [1991].

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