Two-radii theorem for solutions of some mean value equations

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Abstract. A description of solutions of some integral equations has been obtained. A two-radii theorem is obtained as well.

1 Introduction

Characterization of solutions for differential equations in terms of various integral mean values has been studied by many authors (see [1] - [9] and references in these papers).

The classes of functions on subsets of the compact plane that satisfy the conditions of the next type is studied in this work

\[
\sum_{n=s}^{m-1} \frac{r^{2n+2}}{2(n-s)!(n+1)!} \left( \frac{\partial}{\partial z} \right)^{n-s} \left( \frac{\partial}{\partial \bar{z}} \right)^n f(z) = \frac{1}{2\pi} \int \int f(\zeta)(\zeta - z)^s d\xi d\eta, \quad (1)
\]

where \( m \in \mathbb{N} \) and \( s \in 0, \ldots, m - 1 \) are fixed. Also \( r \) is fixed or belongs to the set of two elements.

We point out that this equation holds for \( m \)-analytic functions (see [10]). Function from \( C^{2m-2-s} \) in some domain, that satisfies (1) with all possible \( z \) and \( r \) is of great interest.

The main results of this work are as follow.

1) The description of all smooth solutions for (1) in a disk with radius \( R > r \) with one fixed \( r \) is obtained (see Theorem 1 below);
2) The two-radii theorem is obtained. It turn out that this theorem characterizes class of solution for equation

\[
\left( \frac{\partial}{\partial z} \right)^{m-s} \left( \frac{\partial}{\partial \bar{z}} \right)^m f = 0
\]

in terms of equation (1) (see Theorem 2).

Note that the case \( s \geq m \) that corresponds to the zero integral mean value in the right hand side of (1), has been studied in the works of L.Zalcman and V.V.Volchkov (see [3], [11] - [12]). The first results that deal with the mean value theorem for polyanalytic functions, are contained in [13] - [14].

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2 Main results

Let $J_\nu$ be the Bessel function of the first kind with index $\nu$. For $\rho \geq 0, \lambda \in \mathbb{C}, k \in \mathbb{Z}$, let

$$\Phi_{\lambda, \eta, k}(\rho) = \left( \frac{d}{dz} \right)^n (J_k(z\rho)) |_{z=\lambda}.$$ 

Let also

$$g_r(z) = \frac{J_{s+1}(rz)}{(zr)^{s+1}} - \sum_{n=0}^{m-1} \frac{(zr)^{2(n-s)}(-1)^{n-s}}{(n+1)!(n-s)!2^{n-s+1}},$$

and $Z(g_r) = \{ z \in \mathbb{C} : g_r(z) = 0 \}$,

$Z_r = Z(g_r) \setminus \{ \{ z \in \mathbb{C} : \text{Re} \ z > 0 \} \cup \{ z \in \mathbb{C} : \text{Im} \ z \geq 0, \text{Re} \ z = 0 \} \}$. For $\lambda \in Z_r$ by the symbol $n_\lambda$ we denote the multiplicity of zero $\lambda$ of the entire function $g_r$.

Let $D_R = \{ z \in \mathbb{C} : |z| < R \}$. To any function $f \in C(D_R)$ there corresponds Fourier series

$$f(z) \sim \sum_{k=-\infty}^{\infty} f_k(\rho) e^{ik\rho},$$

where

$$f_k(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{it}) e^{-ikt} dt$$

and $0 \leq \rho < R$.

The next result gives a description for all solutions (1) in a class $C^\infty(D_R)$ with one fixed $r < R$.

**Theorem 1.** Let $r > 0, m \in \mathbb{N}$ and $s \in 0, ..., m - 1$ are fixed. Let also $R > r$ and a function $f$ belongs to $C^\infty(D_R)$. Then the next statements are equivalent.

1) With $|z| < R - r$ equality (1) holds.

2) For any $k \in \mathbb{Z}$ on $[0, R)$ the next equality holds

$$f_k(\rho) = \sum_{0 \leq p \leq s-1 \ p + k \geq 0} a_{k,p} \rho^{2p+k} + \sum_{p=0}^{m-s-1} b_{k,p} \rho^{2p+s+|k|+s} + \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda-1} c_{\lambda, \eta, k} \Phi_{\lambda, \eta, k}(\rho),$$

where $a_{k,p} \in \mathbb{C}, b_{k,p} \in \mathbb{C}, c_{\lambda, \eta, k} \in \mathbb{C}$ and

$$c_{\lambda, \eta, k} = O(|\lambda|^{-\alpha})$$

as $\lambda \to \infty$ for any fixed $\alpha > 0$.

Note that analogues of the Theorem 1 for other equations related to ball mean values, were obtained by V.V. Volchkov for the first time (see [5] - [6] and the references in these papers).

Then let $Z(r_1, r_2) = Z_{r_1} \cap Z_{r_2}$.

We formulate now the local two-radii theorem for equation (1).

**Theorem 2.** Let $r_1, r_2 > 0, m \in \mathbb{N}$ and $s \in 0, ..., m - 1$ are fixed. Then:

1) if $R > r_1 + r_2$, $Z(r_1, r_2) = \emptyset$, $f \in C^{2n-2-s}(D_R)$ and with $|z| < R - r$ holds (1), then $f \in C^\infty(D_R)$ and satisfies (2);  

2) if $\max\{r_1, r_2\} < R < r_1 + r_2$ and $Z(r_1, r_2) \neq \emptyset$, then there is $f \in C^\infty(D_R)$, that satisfies (1) with $|z| < R - r$ and does not satisfy (2).

As regards other two-radii theorems see papers [1] - [9] and references in these papers.
3 Auxiliary Statements

In this section we will obtain some auxiliary statements, that are necessary for the proof of main results. First of all, we note that the function \( g_r \) is an even entire function of exponential type, that grows as a polynomial on the real axis (see, for example, [15], § 29). This together with the Hadamard theorem implies that the set \( Z_r \) is infinite.

**Lemma 1.** Let \( \lambda \in Z_r \) and \( |\lambda| > 4/r \). Then

\[
|\text{Im}\lambda| \leq c_1 \ln(1 + |\lambda|),
\]

where constant \( c_1 \) is not depended on \( \lambda \).

Moreover, for all \( \lambda \) with sufficiently large absolute value

\[
|g'_r(\lambda)| > \frac{c_2}{|\lambda|},
\]

where \( c_2 \) is not depended on \( \lambda \). In addition, all zeros of the \( g_r \) with sufficiently large absolute value are simple.

**Proof.** From the condition \( g_r(\lambda) = 0 \) and asymptotic expansion for \( J_{s+1}(\lambda r) \) as \( \lambda \to \infty \) (see [15], § 29) we have

\[
\sqrt{\frac{2}{\pi \lambda r}} \left( \cos(\lambda r - \frac{\pi s}{2} - \frac{3\pi}{4}) - \frac{4(s^2 + 2s + 1) - 1}{8\lambda r} \sin(\lambda r - \frac{\pi s}{2} - \frac{3\pi}{4}) \right) + O \left( (\lambda r)^{-2} e^{\text{Im}(\lambda r)} \right) = (\lambda r)^{s+1} \sum_{n=s}^{m-1} \frac{(\lambda r)^{2n-2s}(-1)^{n-s-1}}{(2n+2)(n-s)!n!2^{2n-s}}.
\]

Hence, using \( \lambda \in Z_r \), we obtain

\[
\frac{e^{i(\lambda r - \frac{\pi s}{2} - \frac{\pi}{4})}}{2i} + O \left( \frac{e^{\text{Im}(\lambda r)}}{\lambda r} \right) = \sqrt{\frac{\pi \lambda r}{2}} \sum_{n=s}^{m-1} \frac{(\lambda r)^{2n-s+1}(-1)^{n-s-1}}{(2n+2)(n-s)!n!2^{2n-s}}.
\]

Denote by \( p_1(\lambda r) \) the polynomial from the right hand side of this equation. Then we have the following

\[
e^{i(\lambda r - \frac{\pi s}{2} - \frac{\pi}{4})} = 2ip_1(\lambda r) + O \left( \frac{2i e^{\text{Im}(\lambda r)}}{\lambda r} \right).
\]

Let us estimate

\[
|\text{Im}(\lambda r)| \leq |2ip_1(\lambda r)| + \left| \frac{2i e^{\text{Im}(\lambda r)}}{\lambda r} \right| \leq |2ip_1(\lambda r)| + \frac{|i e^{\text{Im}(\lambda r)}|}{2}.
\]

Now one has

\[
|\text{Im}(\lambda r)| \leq 4|p_1(\lambda r)|
\]

and inequality (7) is proved. Inequality (8) can be proved in a similar way, by using [15], formula (6.3). \( \square \)
Lemma 2. Let $\lambda \in \mathbb{C}$, $f(z) = e^{i\lambda(z \cos \alpha + y \sin \alpha)}$, $r > 0$. Then for $z \in \mathbb{C}$ we have

$$\int \int f(\zeta)(\zeta - z)^s d\zeta d\eta - \sum_{n=s}^{m-1} \frac{2\pi r^{2n+2}}{2(n-s)!(n+1)!} \left( \frac{\partial}{\partial z} \right)^{n-s} \left( \frac{\partial}{\partial \bar{z}} \right)^n f(z) =$$

$$= 2\pi g_r(\lambda)e^{ias} i^{s+2} T^{s+2} \frac{1}{\lambda} e^{i\lambda(z \cos \alpha + y \sin \alpha)}$$

Proof. We substitute the function $e^{i\lambda(z \cos \alpha + y \sin \alpha)}$ to the right hand side of equation (1). First, we have

$$\int \int f(w + z)w^s dudv = \int \int e^{i\lambda((x+u) \cos \alpha + (y+v) \sin \alpha)} w^s dudv =$$

$$= e^{i\lambda(z \cos \alpha + y \sin \alpha)} \pi \int_{-\pi}^{\pi} \int_{0}^{r} \rho e^{i\varphi} e^{i\lambda \rho \cos(\varphi - \alpha)} \rho d\rho d\varphi.$$

Let make the substitution $t = \varphi - \alpha$. Then

$$e^{i\lambda(z \cos \alpha + y \sin \alpha)} e^{ias} \pi \int_{-\pi}^{\pi} \int_{0}^{r} \rho^{s+1} e^{i\lambda \rho \cos t} dt d\rho =$$

$$= e^{i\lambda(z \cos \alpha + y \sin \alpha)} e^{ias} \times$$

$$\times \int_{0}^{r} \rho^{s+1} (-1) \int_{-\pi}^{\pi} e^{-i(t+\frac{\pi}{2})} e^{\frac{s}{2} \rho} e^{i\lambda \rho \sin \left(\frac{\pi}{2} + t\right)} d\left(\frac{\pi}{2} + t\right) d\rho.$$

Continuing consideration, we obtain

$$e^{i\lambda(z \cos \alpha + y \sin \alpha)} e^{ias} \pi \int_{-\pi}^{\pi} \int_{0}^{r} \rho^{s+1} e^{i\lambda \rho \cos t} dt d\rho =$$

$$= e^{i\lambda(z \cos \alpha + y \sin \alpha)} e^{ias} \frac{1}{i} 2\pi (-1) \int_{0}^{r} \rho^{s+1} J_s(\lambda \rho) d\rho.$$

Now from the properties of the Bessel function $J_s(z)$ we deduce the next

$$e^{i\lambda(z \cos \alpha + y \sin \alpha)} e^{ias} \frac{1}{i} 2\pi (-1) \int_{0}^{r} (\lambda \rho)^{s+1} J_s(\lambda \rho) d(\lambda \rho) =$$

$$= e^{i\lambda(z \cos \alpha + y \sin \alpha)} e^{ias} \left(\frac{-2\pi}{\lambda}\right)^{s+1} J_{s+1}(\lambda).$$

Then we substitute this function to the left hand side of our equation.

$$2\pi \sum_{n=s}^{m-1} \frac{r^{2n+2}}{(2n+2)(n-s)!n!} \left( \frac{\partial}{\partial z} \right)^{n-s} \left( \frac{\partial}{\partial \bar{z}} \right)^n \left( e^{i\lambda(z \cos \alpha + y \sin \alpha)} \right) =$$
Proof. The proof follows from the Lemma 2 and [5, formula (1.5.29)].

The statement was proved in [10]. In our case the proof is carried out by the same lines.

\[ z \text{ satisfies (1) for all } k \] \[ \text{and only if for all } k \]

The same statement is true for the function \( \Phi \).

\[ \text{Lemma 4. Let } f \in C^{2m-s}(D_R) \text{ satisfies (2) if and only if for all } k \in \mathbb{Z} \] \[ \text{and all } \rho \in [0, R) \text{ the next equality is true} \]

\[ f_k(\rho) = \sum_{0 \leq p \leq s-1} a_{k,p} \rho^{2p+k} + \sum_{p=0}^{m-s-1} b_{k,p} \rho^{2p+s+k}, \] \[ \text{(9)} \]

where \( a_{k,p} \in \mathbb{C} \) and \( b_{k,p} \in \mathbb{C} \).

\[ \text{Proof. In the case where } b_{k,p} = 0 \text{ and equality } (\partial f) = 0 \text{ is considered instead of a similar statement was proved in [10]. In our case the proof is carried out by the same lines.} \]

\[ \text{Lemma 4. Let } f \in C^{\infty}(D_R) \text{ satisfies (1) with fixed } r < R \text{ and all } z \in D_{R-r}. \]

\[ \text{Let } f = 0 \text{ in } D_r. \text{ Then } f \equiv 0. \]

\[ \text{Proof. The statement of Lemma 4 is a special case Theorem 1 from [16].} \]

4 PROOF OF THEOREM 1

Sufficiency. First, let \( f \in C^{\infty}(D_R) \) and equality (5) holds on \([0, R)\) for any \( k \in \mathbb{Z} \) with the coefficients, that satisfy (6). From Lemma 2 and Corollary 1 we see, that function \( f_k(\rho) e^{ik\phi} \) satisfies (1) with \( |z| < R - r \). Because of the arbitrariness of \( k \in \mathbb{Z} \) this together with (3), (4) implies (see, for example, [5, Section 1.5.2] that the function \( f \) also satisfies (1) with \( |z| < R - r \). Hence, implication 2) \( \rightarrow \) 1) is proved.

Now we prove the reverse statement.

Let \( E'_z(\mathbb{C}) \) denote the space of radial compactly supported distributions on \( \mathbb{C} \). Let \( f \in C^{\infty}(D_R) \) and assume that equality (1) holds for \( |z| < R - r \). From [5, statement 1.5.6] the functions \( F_k(z) = f_k(\rho) e^{ik\phi} \) satisfy this condition as well. Using the Paley-Wiener theorem for the spherical transform (see [5, Section 3.2.1 and Theorem 1.6.5]), we define the distribution \( T \in E'_z(\mathbb{C}) \) with support in \( \overline{D}_r \) by the following formula

\[ \tilde{T}(z) = g_r(z), z \in \mathbb{C}. \]
A calculation shows that equality (1) holds for the function $F_k$ with $|z| < R - r$. This is equivalent to the following convolution equation

$$F_k \ast \left( \frac{\partial}{\partial z} \right)^{m-s} \left( \frac{\partial}{\partial z} \right)^m T = 0$$

in $\mathbb{D}_{R-r}$.

We solve this equation by using Lemma 1 - 4. Then we have (see [5, Section 3.2.4]) statement 2).

Hence the theorem.

5 PROOF OF THEOREM 2

Let $R > r_1 + r_2$, $Z(r_1, r_2) = \emptyset$, $f \in C^{2m-2-s}(\mathbb{D}_R)$ and assume that equality (1) holds for $|z| < R - r$. Let us prove that $f$ satisfies (2) in $\mathbb{D}_R$.

Without loss of the generality, we can suppose that $f \in C^\infty(\mathbb{D}_R)$ (the general case can be reduced to this one by the standard smoothing, see [5, Section 1.3.3]).

By Theorem 1, for any $k \in \mathbb{Z}$ and $\rho \in [0, R)$ the next equality holds

$$f_k(\rho) e^{ik\varphi} = \sum_{0 \leq p \leq s-1} a_{k,p} \rho^{2p+k} e^{ik\varphi} + \sum_{p=0}^{m-s-1} b_{k,p} \rho^{2p+s+k+1} e^{ik\varphi} +$$

$$+ \sum_{\lambda \in Z_{r_1}} \sum_{\eta = 0}^{n-1} c_{\lambda,\eta,k} \Phi_{\lambda,\eta,k}(\rho) e^{ik\varphi},$$

(10)

where $a_{k,p}, b_{k,p} \in \mathbb{C}$ and the constants $c_{\lambda,\eta,k}$ satisfy (6).

From this condition it follows that the series in (10) converges in the space $C^\infty(\mathbb{D}_R)$ (see [5, Lemma 3.2.7]).

Let

$$F_k(z) = \left( \frac{\partial}{\partial z} \right)^{m-s} \left( \frac{\partial}{\partial z} \right)^m (f_k(\rho) e^{ik\varphi}) =$$

$$= \sum_{\lambda \in Z_{r_1}} \sum_{\eta = 0}^{n-1} c_{\lambda,\eta,k} \left( \frac{\partial}{\partial z} \right)^{m-s} \left( \frac{\partial}{\partial z} \right)^m \Phi_{\lambda,\eta,k}(\rho) e^{ik\varphi}.$$  

(11)

In view of (11) we see that $F_k \ast T_1 = 0$ in $\mathbb{D}_{R-r_1}$, where the distribution $T_1 \in \mathcal{E}_r'(\mathbb{C})$ with support in $\overline{\mathbb{D}_{r_1}}$ is determined by the equality $\widetilde{T_1}(z) = g_{r_1}(z)$ (see [5, Theorem 1.6.5]).

Similarly, using Theorem 1 for $r = r_2$, we conclude that $F_k \ast T_2 = 0$ in $\mathbb{D}_{R-r_2}$, where $T_2 \in \mathcal{E}_r'(\mathbb{C})$ with support in $\overline{\mathbb{D}_{r_2}}$ is determined by the equality $\widetilde{T_2}(z) = g_{r_2}(z)$.

If $Z(r_1, r_2) = \emptyset$ then from [5, Theorem 3.4.1] we conclude that $F_k = 0$.

Then it follows from (11) that the function $f_k(\rho) e^{ik\varphi}$ satisfies (2) for all $k \in \mathbb{Z}$. It means that (see [5, proof of the Lemma 2.1.4]) $f$ satisfies (2). Thus the first statement of Theorem 2 is proved.

We now establish the second statement.

If there is $\lambda \in Z(r_1, r_2)$ then the function $f(z) = \Phi_{\lambda,0,0}(|z|)$ does not satisfy (2). In addition, it satisfies (1) for all $z \in \mathbb{C}$ and $r = r_1, r_2$ (see Corollary 1). Then we henceforth assume that $Z(r_1, r_2) = \emptyset$.

Suppose that $T_1, T_2 \in \mathcal{E}_r'(\mathbb{C})$ are defined as above. If $R < r_1 + r_2$, in view of [5, Theorem 3.4.9] we conclude, that there is a nonzero radial function $f \in C^\infty(\mathbb{D}_R)$. It satisfies the conditions
Applying [5, Theorem 3.2.3] we infer that for \( r = r_1, r_2 \) the following equality holds

\[
f(z) = \sum_{\lambda \in \mathbb{Z}} \sum_{\eta = 0}^{n_\lambda - 1} c_{\lambda, \eta}(r) \Phi_{\lambda, \eta, 0}(|z|),
\]

where \( z \in \mathbb{D}_R \) and the constants \( c_{\lambda, \eta}(r) \) satisfy (5). Moreover, these constants are not all equal to zero.

From this equality and Corollary 1 one deduces that \( f \) satisfies (1) for \(|z| < R - r, r = r_1, r_2 \). Suppose now that \( f \) satisfies (2).

Then \( f(z) = \sum_{0 \leq p \leq s - 1} a_p |z|^{2p} + \sum_{p=0}^{m-s-1} b_p |z|^{2p+2s} \) in \( \mathbb{D}_R \) and the convolutions \( f \ast T_1 \) and \( f \ast T_2 \) are polynomials. This means that \( f \ast T_1 = f \ast T_2 = 0 \) in \( \mathbb{C} \).

Since \( Z(r_1, r_2) = \emptyset \), from [5, Theorem 3.4.1] we infer that \( f = 0 \). This contradicts by the definition of \( f \).

Therefore, the function \( f \) satisfies all the requirements of the second statement of Theorem 2.

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