A handlebody calculus for topology change

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5 November 1997

Abstract

We consider certain interesting processes in quantum gravity which involve a change of spatial topology. We use Morse theory and the machinery of handlebodies to characterise topology changes as suggested by Sorkin. Our results support the view that the pair production of Kaluza-Klein monopoles and the nucleation of various higher dimensional objects are allowed transitions with non-zero amplitude.

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1 Introduction

The question of whether the topology of space can change is a basic one in the search for a theory of quantum gravity. The theorems of Geroch [1] and Tipler [2] are widely understood to show that there is no topology change in classical general relativity, so that we should look to the quantum theory to see it, if it occurs. Though the definitive statement about the occurrence of topology change may well have to wait until we have a fully developed theory of quantum gravity it is nevertheless generally believed that topology change does happen. A general calculus for topology change within the Sum Over Histories (SOH) approach, based on Morse theory, has been suggested by Sorkin [3]. In this paper we will review this picture, and use it to investigate certain interesting topology changing processes. In the rest of the introduction we set up some terminology and outline our aims and results. Let an \( n \)-geometry \((M,g)\) consist of an \( n \)-dimensional manifold \(M\) and a metric \(g\) on \(M\) –strictly, a geometry is an equivalence class of such pairs under diffeomorphism. A topology change in \(n\) spacetime dimensions is a transition from a Riemannian \((n-1)\)-geometry \((W_0,h_0)\) to another Riemannian \((n-1)\)-geometry \((W_1,h_1)\) in which \(W_0\) and \(W_1\) are non-diffeomorphic. We call \((M,V_0,V_1)\) a smooth manifold triad if \(M\) is a compact smooth \(n\)-manifold whose boundary is the disjoint union of the two closed submanifolds \(V_0\) and \(V_1\), \(\partial M = V_0 \cup V_1\). Given two closed smooth \((n-1)\)-manifolds, \(W_0\) and \(W_1\), a topological cobordism from \(W_0\) to \(W_1\) is a 5-tuple \((M,V_0,V_1,d_0,d_1)\) where \((M,V_0,V_1)\) is a smooth manifold triad and \(d_i : V_i \rightarrow W_i\) is a diffeomorphism, \(i = 0,1\). Cobordism gives rise to an equivalence relation on the set of \((n-1)\) manifolds. We say that \(W_0\) and \(W_1\) are in the same cobordism class if a topological cobordism exists between them. A Lorentzian cobordism from geometry \((W_0,h_0)\) to \((W_1,h_1)\) is a 6-tuple \((M,V_0,V_1,d_0,d_1,g)\) where \((M,V_0,V_1,d_0,d_1)\) is a topological cobordism and \(g\) is a Lorentzian metric on \(M\) such that \((d_i^{-1})^*(g_{V_i}) = h_i, i = 0,1\) and such that \(V_0\) is a past spacelike boundary and \(V_1\) is a future spacelike boundary. We will often drop the explicit mention of the diffeomorphisms in what follows and unless otherwise stated all cobordisms \(M\) will be compact.

A necessary and sufficient condition for a topological cobordism to exist between a given pair of manifolds is that their Steifel-Whitney and Pontrjagin numbers coincide when both are oriented or just their Steifel-Whitney
numbers in the non-oriented case [4, 5]. Hence the number of cobordism
classes equals that of distinct combinations of Stiefel-Whitney and Pontrjagin
numbers, which has a finite value, depending on the dimension. As it
happens, all 3-manifolds are cobordant, while 4-manifolds divide into four
cobordism classes.

Now that we have explained what we mean by topology changing transi-
tions between two spacelike hypersurfaces, we must decide how to investigate
them. Among the different approaches to quantum gravity, the Sum Over
Histories affords the most natural expression for topology changing transition
amplitudes:

$$\langle W_1, h_1; W_0, h_0 \rangle = \sum_{(M, V_0, V_1, d_0, d_1)} \omega(M, d_0, d_1) \int_C Dg e^{iS[g]}$$

where the sum is over topological cobordisms and $C$ is a class of metrics, $g$,
on $M$ such that $(d^{-1}_i)^*(g|_{V_i}) = h_i$, $i = 0, 1$. The weight $\omega(M, d_0, d_1)$ will not
concern us here but is discussed in [6]. Although this formal expression is
far from being defined, and indeed may never be so without recourse to a
possibly discrete underlying theory, we can already draw some conclusions
from its general form. For example, if $W_0$ and $W_1$ are not cobordant then
the amplitude for the topology change is zero.

There are various proposals for the type of metrics over which the func-
tional integral runs for each topological cobordism $M$. Following Sorkin [7]
we start with the view that the integral should be over all Lorentzian met-
rics but this immediately raises a problem. In the event of topology change,
the geometry $(M, g)$ cannot be both Lorentzian and causally ordered. This
follows from the following theorem of Geroch [1]:

**Theorem 1 (Geroch,1967)** If a smooth triad $(M, V_0, V_1)$, with $V_0$ and $V_1$
closed, admits a time-orientable Lorentzian metric $g$ without closed timelike
curves and such that $V_0$ and $V_1$ are spacelike with respect to $g$, then $V_0 \cong V_1$
and $M \cong V_0 \times I$ where $I$ is the unit interval, i.e., there is no topology change.

So which do we choose to keep: causal order or the equivalence principle?
Following Sorkin [8] we plump for casual order. For one thing, if we were
instead to insist on globally time-orientable Lorentzian metrics this would rule out the production of Kaluza-Klein monopole-antimonopole pairs since there does not exist such a metric on any topological cobordism for this process [8, 9]. Also, if causal sets are the correct description of the discrete substructure of spacetime then causal order is more fundamental than metric [7]. Pursuing this route, however, means we must allow singularities of some sort in the geometries \((M, g)\) that contribute to the amplitude for a topology changing process. So what singularities are allowed? Sorkin has suggested that Morse theory (see e.g. [10]) furnishes the appropriate metrics that are Lorentzian almost everywhere and exist on all topological cobordisms.

A Morse function on a manifold \(M\) is a smooth function \(f : M \to \mathbb{R}\) such that \(\partial_\mu f\) vanishes only at a finite number of points \(p_k\) where the Hessian \(\partial_\mu \partial_\nu f|_{p_k}\) is a non degenerate matrix. The Morse index \(\lambda_k\) of each critical point \(p_k\) is the number of negative eigenvalues of the Hessian matrix evaluated at \(p_k\). The critical values of \(f\) are the values it takes at the critical points; we will often denote them \(c_k = f(p_k)\). The abundance of Morse functions on a manifold is enough to ensure the following [11, 12]

\[\text{Theorem 2} \quad \text{For any smooth triad, } (M, V_0, V_1), \text{ there exists a Morse function } f : M \to [0, 1] \text{ such that:}\]

1. \(f^{-1}(0) = V_0\) and \(f^{-1}(1) = V_1\)
2. \(f\) has no critical points on \(\partial M = V_0 \cup V_1\)

Then given any Riemannian metric \(G\) on \(M\) and a real number \(\zeta > 1\), we can construct an almost Lorentzian metric \(g\) associated with \(f\) as follows:

\[
g_{\mu\nu} = \partial_\mu f \partial_\sigma f G^{\rho\sigma} G_{\mu\rho} - \zeta \partial_\mu f \partial_\nu f \tag{2}\]

and we call this a Morse metric. It is Lorentzian everywhere except for the Morse points and \(G^{\mu\nu} \partial_\nu f\) defines a timelike direction. If moreover Riemannian metrics are given on \(V_0\) and \(V_1\), we can demand that \(g\) has the correct restrictions by choosing \(G\) appropriately. We summarise these statements as

\[\text{Lemma 1} \quad \text{Let } (M, V_0, V_1, d_0, d_1) \text{ be a topological cobordism between } W_0 \text{ and } W_1 \text{ and let } h_0 \text{ and } h_1 \text{ be Riemannian metrics on } W_0 \text{ and } W_1. \text{ Then there exists a Morse metric } g \text{ on } M \text{ such that } (d_i^{-1})^* (g|_{V_i}) = h_i, i = 0, 1.\]
The proof is given in Appendix B. Any topological cobordism has an infinite number of Morse metrics associated with it. We call each such geometry \((M, g)\) an Almost Lorentzian (AL) cobordism and restrict the functional integral to be over such cobordisms. Since these geometries are singular, it will be necessary to extend the definition of the action \(S\) to these cases [13]. Note that in this view the critical points are not to be sent to infinity as in [14, 15] but rather remain part of the spacetime and indeed the causal order is well defined with the Morse points present.

Now, Sorkin suggested that it might be necessary to impose a stronger condition on the set of contributing metrics. This observation is motivated by a very simple example in \((1 + 1)\) dimensions: quantum field theory on the trousers cobordism. The \((1 + 1)\)-dimensional trousers admits an everywhere flat AL metric with a single index one Morse point at the crotch which singularity is the source for an infinite burst of energy of a scalar quantum field propagating on the trousers [16, 17]. Anderson and DeWitt have argued that this provides evidence against topology change. But the regular propagation of a quantum field on the \((1 + 1)\) “yarmulke” topology, a hemisphere mediating the transition \(\emptyset \to S^1\), suggests that it might be a particular feature of the trousers topology, and not a general flaw of all nontrivial cobordisms, that causes the unphysical energy burst [18]. A crucial difference between the trousers and the yarmulke topologies is that the former has a causal discontinuity whereas the latter does not (roughly speaking a causal discontinuity is a discontinuous change in the volume of the causal past or future of a continuously varied point[19]). Generalising this idea Sorkin conceived the following conjectures:

**Conjecture 1** A quantum field propagating on an AL cobordism \((M, g)\) has an unphysically singular behaviour if and only if \((M, g)\) is causally discontinuous.

**Conjecture 2** An \(n\)-dimensional AL cobordism \((M, g)\) is causally discontinuous if and only if the Morse function from which \(g\) is constructed has either an index 1 or index \((n - 1)\) critical point [20].

Plausibility arguments for the second conjecture are given in [20] [21], but rigorous demonstration of both is work in progress. Now in addition
we take the singular behaviour of the quantum perturbations around these causally discontinuous backgrounds to be an indication of their contribution to the SOH being infinitely suppressed. Evidence for this is presented in [13], though the issue is clouded by the fact that the backgrounds considered there are not stationary points of the action. We can now eliminate from the SOH those topological cobordisms which do not admit Morse functions without index 1 or \((n - 1)\) points. The remaining AL cobordisms, the causally continuous ones, are our candidates to contribute to the SOH transition amplitude. We will denote them Causally Continuous AL (CCAL) cobordisms. The remainder of the article is based on this criterion; thus we work under the assumption that the conjectures hold.

In section 2 we describe how to use handlebody decompositions as a method to identify CCAL cobordisms;\(^1\) in particular we prove an identity that will allow us to deduce handlebody decompositions in high dimensions from lower dimensional ones. In section 3 we apply this technique to analyse various interesting topology changing processes of semi-classical decay in quantum gravity. We find a favourable handlebody decomposition for their respective instanton cobordisms, and hence verify that these processes can occur. As a preliminary we demonstrate that if the topological non-trivialities of a non-compact cobordism \(L\) are confined to some compact region \(M \subset L\) then the Morse metrics on \(M\) give rise to Morse metrics on \(L\) with the same critical structure. In the last section we summarise our results and list some open problems.

\(^1\)We thank Sumati Surya for suggesting handlebodies as a technique.
2 Handlebody decompositions

Define the Morse structure of a Morse function \( f \) on \( M \) to be a complete ordered list \( \{ (p_k, \lambda_k) : k = 1, \ldots, r \} \) of its Morse points and corresponding Morse indices. As we shall see, a handlebody decomposition of a manifold, \( M \), implies the existence of Morse functions on \( M \) with totally determined Morse structure. The following definitions follow very closely the first pages of Kirby’s book [22]. They make extensive use of the concepts of closed or open \( n \)-balls, \( n \)-spheres and their respective boundaries, which are listed here:

\[
  B^n = \{ x \in \mathbb{R}^n : |x|^2 \leq 1 \} \quad S^n = \{ x \in \mathbb{R}^{n+1} : |x|^2 = 1 \}
\]

\[
  \partial B^n = S^{n-1} \quad \partial S^n = \emptyset
\]

\[
  \dot{B}^n = \{ x \in \mathbb{R}^n : |x|^2 < 1 \}
\]

By \( \dot{A} \) we mean the interior of the set \( A \), i.e., the largest open set contained in \( A \). Note for future reference that when \( A \) is a subset of the manifold with boundary \( M \) and \( \partial M \) is not empty \( \dot{A} \) may contain part of it.

A handlebody decomposition of an \( n \)-dimensional compact manifold \( M \) is a nested sequence of manifolds \( \emptyset = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_r = M \) where \( M_k \) is obtained by adjoining a \( \lambda_k \) handle to \( M_{k-1} \), i.e., \( M_k = M_{k-1} + h_k B^{\lambda_k} \times B^{n-\lambda_k} \) via an embedding, \( h_k : \partial B^{\lambda_k} \times B^{n-\lambda_k} \hookrightarrow \partial M_{k-1} \) of the boundary of the \( \lambda_k \) handle into the boundary of \( M_{k-1} \). Note that \( M_0 = B^n \) in any such handlebody sequence. This definition involves two operations whereby a pair of manifolds with boundary can be combined: adjunction (+) and product (\( \times \)). In appendix A we describe how endowing the adjunction or the product of two manifolds with a differentiable structure raises the issue of smoothing corners. We here simply intone the slogan “corners can be smoothed” (e.g. \( B^n \) and \( (B^1)^n \) are equivalent as far as we are concerned).

Associated with any smooth handlebody decomposition is a Morse function \( f : M \to [0, 1] \) with as many critical points as handles being attached. For the \( r+1 \)-handled-body in the definition, the function \( f \) would have \( r+1 \) non-degenerate critical points, \( \{ p_k \} \), \( k = 0, 1, \ldots, r \), which can be taken to lie in different level surfaces, i.e., \( f(p_0) < f(p_1) < \cdots < f(p_r) \). Each critical point may be located at the centre \( (0,0) \) of \( B^{\lambda_k} \times B^{n-\lambda_k} \); then \( B^{\lambda_k} \times \{ \bar{0} \} \)
is the descending manifold and \( \{0\} \times B^{n-\lambda_k} \) the ascending manifold. By this we mean that around \( p_k \) the function \( f \) admits an expansion (Morse lemma [10]):

\[
f(q) = f(p_k) - x_1^2 - x_2^2 - x_{\lambda_k}^2 + x_{\lambda_k+1}^2 + \cdots + x_n^2
\]

The first \( \lambda_k \) local coordinates parametrise \( B^{\lambda_k} \), the last \( n - \lambda_k \) local coordinates parametrise \( B^{n-\lambda_k} \) and \( p_k \) is identified as a Morse point of index \( \lambda_k \). In other words, we can define \( f \) following the sequence of manifolds. It is zero at some point of \( M_0 \), which is the index 0 critical point \( p_0 \); it then increases in a regular way except for the critical point associated with each handle attachment.

The Morse function, \( f \), associated with the handlebody decomposition given above is 1 on the boundary of \( M \). As such, it is appropriate for the case of the topology change from the empty set to \( \partial M \). We are interested in the more general case of topology change from \( V_0 \) to \( V_1 \). In that case we have a manifold, \( M \), whose boundary is the disjoint union of \( V_0 \) and \( V_1 \). A generalised handlebody decomposition of \( M \) is a nested sequence \( V_0 \times B^1 = M_0 \subset M_1 \subset \ldots M_r = M \) where \( M_k \) is obtained by attaching a \( \lambda_k \) handle to \( M_{k-1} \). But now there is a restriction on each embedding \( h_k \): its image must not intersect the initial \( V_0 \) component of the boundary of \( M_k \). In this handlebody calculus, the addition of each handle can be thought of as an elementary topological transition from \( \partial M_k \) to \( \partial M_{k+1} \).

A useful property of handlebody decompositions is “right distributivity” of a \( B^m \). It allows us to deduce from a handlebody decomposition for a manifold \( M \) a whole series of higher dimensional handlebodies for the manifolds \( M \times B^m \). The crucial point is that the new \( B^m \) factor does not actively partake in the induced imbedding \( \partial B^\lambda \times B^{n+m-\lambda} \hookrightarrow \partial(M \times B^m) \). More explicitly we have the following\(^2\):

**Lemma 2** Let \( M \) be an \( n \)-dimensional manifold; then if

\[
M = B^n + \sum_{k=1}^{r} B^{\lambda_k} \times B^{n-\lambda_k}
\]

\(^2\)We omit mention of the particular embeddings; one is understood with each handle in the sum (a total of \( r \) handles are attached in succession to the initial ball.)
it follows that

\[ M \times B^m = B^{n+m} + \sum_{k=1}^r B^\lambda_k \times B^{n+m-\lambda_k} \]

**Proof.** First we claim that if \( L \) is an \( n \)-dimensional manifold such that

\[ M = L + B^\lambda \times B^{n-\lambda} \]

then

\[ M \times B^1 = L \times B^1 + B^\lambda \times B^{n+1-\lambda} \]

Provided that the claim holds one can proceed by induction in \( m \), the dimension of the right-factored ball, to infer that

\[ M \times B^m = L \times B^m + B^\lambda \times B^{n+m-\lambda} \quad (4) \]

The only subtle point about this induction process is the diffeomorphism \( B^l \times B^1 \cong B^{l+1} \), but this is again the problem of smoothing corners. Then by applying eq.\( (4) \) at each step of a handlebody decomposition one obtains:

\[
M \times B^m = M_{r-1} \times B^m + B^{\lambda_r} \times B^{n+m-\lambda_r} = M_{r-2} \times B^m + B^{\lambda_{r-1}} \times B^{n+m-\lambda_r} + B^{\lambda_r} \times B^{n+m-\lambda_r} = \ldots = M_j \times B^m + \sum_{k=j+1}^r B^{\lambda_k} \times B^{n+m-\lambda_k} = \ldots = B^{n+m} + \sum_{k=1}^r B^{\lambda_k} \times B^{n+m-\lambda_k}
\]

Finally we prove the initial claim. Given that \( M \cong L + hL^\lambda \times B^{n-\lambda} \) through the embedding \( h: \partial B^\lambda \times B^{n-\lambda} \rightarrow \partial L \), we define

\[ \tilde{h}: \partial B^\lambda \times (B^{n-\lambda} \times B^1) \rightarrow \partial L \times B^1 \subset \partial(L \times B^1) \]

by

\[ \tilde{h}(x,t) = (h(x),t) \]

where \( x \in \partial B^\lambda \times B^{n-\lambda} \) and \( t \in B^1 \). This new embedding induces a map \( f: (L+hL^\lambda \times B^{n-\lambda}) \times B^1 \rightarrow L \times B^1 + \tilde{h}B^\lambda \times B^{n+1-\lambda} \) given, in the terminology of appendix A, by:

\[ f(([z],t)) = ([z],t) \tilde{h} \quad (5) \]
The map $f$ is well defined – independent of class representative – and since the same holds for its obvious inverse, $f$ is a bijection. It is also a homeomorphism of topological spaces. It can be shown that $f$ has differentiable local representatives even at the smoothed corner set \cite{23} once it has been composed with the relevant smoothing maps. Thus $f$ is a diffeomorphism; it expresses distributivity between adjunction and product of manifolds with boundary. Fig 2 illustrates a simple case of $B^1$ distributivity: the handlebody for the annulus $S^1 \times B^1 \cong B^2 + B^1 \times B^1$ gives rise to the handlebody $S^1 \times B^2 \cong B^3 + B^1 \times B^2$.

\begin{equation}
B^3 = B^3 + B^1 \times B^2 + B^2 \times B^1
\end{equation}

\begin{equation}
B^3 = B^3
\end{equation}

Figure 2: Right $B^1$ distributivity lifting a 2-dim handlebody, the hollow cylinder (bottom-right corner), to a 2-dim handlebody, the solid torus (top-right corner).

With the machinery of Morse theory and handlebodies in hand we can investigate topology changing processes. First of all the content of the conjectures translates into the following statements. If a smooth triad $(M, V_0, V_1)$ has a handlebody decomposition which does not include a $B^1 \times B^{n-1}$ nor a $B^{n-1} \times B^1$ handle, then it admits a CCAL metric and according to the premises of this paper $M$ is to be included in the SOH for the process. On the other hand, if a smooth triad has a handlebody decomposition which does contain a 1-handle or an $(n-1)$-handle then we cannot draw the contrary conclusion. For example consider two decompositions of $B^3$ (Fig 3):

\begin{equation}
B^3 = B^3 + B^1 \times B^2 + B^2 \times B^1
\end{equation}

\begin{equation}
B^3 = B^3
\end{equation}
In view of (6) alone we would be wrong to conclude that $B^3$ supports no CCAL cobordisms for the creation of $S^2$ since (7) shows that $B^3$ does support causally continuous cobordisms.

However, the Morse inequalities [10] do furnish a sufficient—but not necessary—criterion for automatically discarding certain cobordisms. Consider the triad $(M, V_0, V_1)$. Let $\beta_\lambda(M, V_0)$ be the $\lambda$th Betti number of $M$ relative to $V_0$ and let $\mu_\lambda$ denote the number of critical points of index $\lambda$ of a Morse function $f : M \to [0, 1]$ with $f^{-1}(0) = V_0$ and $f^{-1}(1) = V_1$. Then a weak version of the Morse inequalities establishes that:

$$\mu_\lambda \geq \beta_\lambda(M, V_0) \quad (8)$$

So if the first or $(n-1)^{th}$ homology of $M$ relative to $V_0$ has non-trivial torsion free part, any Morse function on $M$ must have index 1 or index $(n-1)$ points.

As an example consider the cobordism $B^4 \times S^1$ for creation of an $S^3 \times S^1$ ($V_0$ is empty here). We can compute its homology using the Kunneth formula [24] for the homology groups of the product of two spaces when both have torsion-free homologies, namely $H_q(X \times Y) = \sum_{p=0}^q H_p(X) \otimes H_{q-p}(Y)$.

Applying this to $B^4 \times S^1$ gives:

$$H_1(B^4 \times S^1) = H_0(B^4) \otimes H_1(S^1) + H_1(B^4) \otimes H_0(S^1)$$

$$= Z \otimes Z + 0 \otimes Z = Z$$

Figure 3: “Redundant” decomposition of the 3-ball.
The same, applied to $B^2 \times S^3$ gives:

$$H_1(B^2 \times S^3) = H_0(B^2) \otimes H_1(S^3) + H_1(B^2) \otimes H_0(S^3) = Z \otimes 0 + 0 \otimes Z = 0$$

Thus $\beta_1(B^4 \times S^1) = 1$, while $\beta_1(B^2 \times S^3) = 0$. In conjunction with eq.(8) $\beta_1(B^4 \times S^1) = 1$ tells us that there are no Morse functions on $B^4 \times S^1$ without index 1 critical points and so this cobordism does not admit CCAL metrics. But for general $M$ a vanishing $\beta_1$ does not guarantee that there is an allowed Morse function on $M$, since there is no reason why Morse functions should exist that saturate the inequalities [21]. In particular, from $\beta_1(B^2 \times S^3) = 0$ alone we could not infer that $B^2 \times S^3$ admits CCAL metrics. It is only in view of the handlebody decomposition given earlier that we can so conclude.

3 Handlebodies and instantons for semi-classical decay in quantum gravity

Instantons in quantum gravity are the analogues of tunnelling solutions in quantum mechanics. When we consider tunnelling of a point particle from an unstable minimum $x_\infty$ to a position $x_0$ of zero momentum, we calculate the transition amplitude $\langle x_0, 0 | x_\infty, -\infty \rangle$. One can show that the SOH is well approximated by $A e^{-S}$ where $A$ is a prefactor and $S$ is the action of the classical Euclidean solution. By analogy, an instanton in gravity is a solution of the Euclidean Einstein equations that interpolates between an initial unstable state $U_0$ –approached asymptotically– and a zero-momentum hypersurface $U_1$ which is initial data for the post-decay Lorentzian evolution. The existence of such an instanton is usually taken as strong evidence that the transition takes place and the amplitude is approximately given by $A e^{-S}$ where $S$ is the action of the instanton. We are investigating the suggestion that the SOH in quantum gravity be defined fundamentally as a sum over CCAL cobordisms –or over AL cobordisms with causal continuity enforced dynamically. That means first of all that there must be some CCAL cobordisms for the transition under consideration. Secondly, it seems reasonable that there would only be an instanton approximation if the instanton had a background topology that was included in the sum over manifolds in equation (1), i.e., one which admits CCAL metrics. Thus we want to check that when instantons are invoked as evidence that topology changing processes occur, the instanton manifolds admit CCAL metrics.
3.1 Localised topology change

Before turning to our specific examples, we first prove some results necessary because the processes to be considered are embedded in an ambient asymptotically flat region. We could think of this as the topology change taking place within a lab with fixed walls say. Clearly our Morse and handlebody technology will have to be adapted to apply to these non-compact manifolds. This will not be difficult because, with the assumption that the topology change is localised in space, we can reduce the questions to the closed case by, roughly speaking, closing off space. Once we demonstrate the existence of CCAL metrics in the compact cobordism, we open back to the physical manifolds. That this can be done without disrupting the Morse structure of the metric is the content of the “decompactifying” lemmas stated below. Their proof is deferred to appendix B.

In the statement of lemmas 3 and 4, we use the concept of a gradient-like vector field for a Morse function \( f \) on a manifold \( M \). Defining such a vector field amounts to covering \( M \) with a congruence of curves, along which \( f \) increases, without reference to any particular Riemannian metric on \( M \). We borrow the definition from Milnor (Lemma 3.2., [12]), while our construction of a concrete vector field is a simple generalisation to non-elementary cobordisms of the one given therein. Let \( f \) be a Morse function on the \( n \)-dimensional manifold \( M \) with a set of \( r \) Morse points \( P = \{p_k\} \).

For simplicity we assume that each Morse point occurs on a distinct level surface of \( f \) though this assumption can easily be dropped.

A gradient-like vector field \( \xi \) for \( f \) is a smooth vector field on \( M \) with properties:

(i) \( \xi(f) > 0 \quad \forall q \notin P \)

(ii) \( \xi \) has coordinates \((-2x_1, \cdots, -2x_{\lambda_k}, 2x_{\lambda_k+1}, \cdots, 2x_n)\) in a neighborhood of \( p_k \) where \( f \) admits expansion \( f(q) = f(p_k) - \sum_{1 \leq i \leq \lambda_k} x_i^2 + \sum_{\lambda_k < j \leq n} x_j^2 \).

A vector field satisfying these two conditions can always be found in \( M \). Indeed, pick an atlas \( \mathcal{A} = (U_\alpha, \phi_\alpha) \quad \alpha = 1, \cdots, N \) so that a single chart \( U_k \) contains the critical point \( p_k \) and so that, dividing the range \( \{\alpha\} \) as \( \{k, a\} \quad k = 1, \cdots, r \quad a = r + 1, \cdots, N \), the following hold:
1. For each $k$, there is a smaller neighbourhood $U'_k \subset U_k$ satisfying

$$U'_k \cap U_\alpha = \begin{cases} U'_k & \text{if } k = \alpha \\ \emptyset & \text{otherwise} \end{cases}$$

and $\phi_k(U'_k) = \{ \vec{x} \in \mathbb{R}^n : |\vec{x}|^2 < \varepsilon \}$ for some small $\varepsilon$

where $\bar{U}'_k$ means the closure of $U'_k$.

2. In $U_k$ $f$ has local representative $f_k(\vec{x}) \equiv f \circ \phi_k^{-1}(\vec{x}) = c_k - \sum_{i=1}^{\lambda_k} x_i^2 + \sum_{i=\lambda_k+1}^{n} x_i^2$

3. In $U_a$ $f$ has local representative $f_a(\vec{x}) \equiv f \circ \phi_a^{-1}(\vec{x}) = \text{const} + x_1^{(a)}$

We now define $\xi$ chart by chart. In $U_k$ we give it the components $\xi^{(k)} = (-2x_1, \cdots, -2x_{\lambda_k}, 2x_{\lambda_k+1}, \cdots, 2x_n)$ and in $U_a$ $\xi^{(a)} = (1, 0, \cdots, 0)$. Then we combine the local representatives $\xi^{(a)}$, through a partition of unity $\{\theta_\alpha\}$ for $A$ to obtain a vector field, $\xi = \sum_\alpha \theta_\alpha \xi^{(a)}$, which clearly satisfies condition (i) and condition (ii) in the neighbourhood $U'_k$ of $p_k$.

Covering the case of ordinary asymptotic flatness we have

**Lemma 3** Consider two non-compact asymptotically flat $(n-1)$-geometries $(U_0, h_0)$ and $(U_1, h_1)$. Suppose that the closed manifolds $V_0$ and $V_1$ are one-point compactifications of $U_0$ and $U_1$, in the sense that there are points $\bar{q}_i \in V_i$ and diffeomorphisms $\tilde{d}_i : V_i - \bar{q}_i \to U_i$, $i = 0, 1$. Further suppose that there is a triad $(M, V_0, V_1)$ with a Morse function $f : M \to [0, 1]$ with no index $n$ critical points.

Then,

(i) There is an integral curve $C$ of a gradient-like vector field for $f$ which traverses $M$, from $V_0$ to $V_1$ without intercepting any critical point.

(ii) The manifold $L \equiv M - C$ is a cobordism between $U_0$ and $U_1$ and there is an AL metric on $L$ which has the same Morse structure as $f$, is asymptotically flat and has the correct restrictions, the pull-backs of $h_0$ and $h_1$, on the boundary $\partial L = (V_0 - q_0) \cup (V_1 - q_1)$ where $q_i = V_i \cap C$. 

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For asymptotic Kaluza-Klein boundary conditions consider compactifying $\mathbb{R}^3 \times S^1$, the topology of a spatial section in the 5-dimensional Kaluza-Klein vacuum: we add a whole circle, one point at infinity of $\mathbb{R}^3$ for each point of $S^1$. In the reverse process an $S^1$ must be removed to recover the physical boundaries from the closed manifold. While all points in a manifold are equivalent, in general not all embedded circles are: given a manifold $V$, the manifolds $V - C$ and $V - \tilde{C}$ may not be diffeomorphic if $C$ and $\tilde{C}$ are different embedded circles. In order to decompactify to Kaluza-Klein boundary conditions along the lines of 3, we enlarge our list of hypotheses with a further condition which guarantees the equivalence of all subtracted circles in the closed boundary $V_1$.

**Lemma 4** Consider two asymptotically Kaluza-Klein flat $(n-1)$-geometries $(U_0, h_0)$ and $(U_1, h_1)$. Suppose that there exist closed manifolds $V_0$ and $V_1$, with $V_1$ connected and simply connected, and diffeomorphisms $d_i : V_i - \tilde{C}_i \to U_i$ with $\tilde{C}_i \subset V_i$ diffeomorphic to $S^1$, $i = 0, 1$. Further suppose that there is a triad $(M, V_0, V_1)$ with a Morse function $f : M \to [0, 1]$ with no index $n$ or $(n - 1)$ critical points.

Then,

(i) There is an “integral annulus” $A$ for the gradient-like vector field $\xi$ — by this we mean an $S^1$ worth of integral curves of $\xi$, i.e., an embedding $i : B^1 \times S^1 \hookrightarrow M$ such that for each point in the circle $\psi \in [0, 2\pi)$ the segment $i(B^1 \times \{\psi\})$ is an integral curve of $\xi$ — which traverses $M$, from $V_0$ to $V_1$ without intercepting any critical point.

(ii) The manifold $L \equiv M - A$ is a cobordism between $U_0$ and $U_1$ and there is an AL metric on $L$ which has the same Morse structure as $f$, is asymptotically flat and has correct restrictions, the pull-backs of $h_0$ and $h_1$, on the boundary $\partial L = (V_0 - C_0) \cup (V_1 - C_1)$ where $C_i = V_i \cap A$, $i=0,1$.

### 3.2 Pair production of black holes

Due to the positive energy theorems, the Minkowski vacuum $M^4$ is stable with respect to semi-classical decay [25]. However a cylindrically symmetric magnetic field described by the Melvin solution can decay into a pair of
oppositely charged black holes thanks to the extra energy contained in the field [26, 27]. The instanton that governs the decay is the Euclideanised Ernst solution. To see the topologies associated with the metrics involved, the reader is encouraged to consult [26]. We take them as the starting point for analysing the cobordism. They are a spacelike hypersurface of Melvin, $\mathbb{R}^3$, a post-tunnelling spacelike hypersurface containing a pair of black holes, $S^2 \times S^1 - \{\text{point}\}$ and the doubled instanton, or “bounce” topology, $S^2 \times S^2 - \{\text{point}\}$. Removing a point from a 4-dimensional closed manifold is equivalent to removing a closed ball $B^4$: it gives a non-compact manifold. We compactify by adding the point back in and cut the bounce in half to obtain $\overline{M} \cong S^2 \times B^2$. We then delete an open four-ball to create the initial boundary. The manifold $\overline{M} - B^4$ is $M$ in Lemma 1, $V_0 \cong S^3$ is the initial boundary and $V_1 \cong S^2 \times S^1$ is the final boundary.

Combining Fig 4 with right-distributivity gives the following handlebody decomposition for $\overline{M}$:

$$\overline{M} = S^2 \times B^2 \cong (S^2 \times B^1) \times B^1 \cong (B^3 + B^2 \times B^1) \times B^1 = B^4 + B^2 \times B^2$$

![Figure 4: Handlebody for $S^2 \times B^1$: a ball with a hole at the centre.](image)

Thus there exists a Morse function on $M$ which contains only a Morse point of index 2. Dowker and Surya gave an earlier proof by explicitly constructing an allowed Morse function [21] on $\overline{M}$ that can, in fact, be regarded as associated with the handlebody decomposition given above. Lemma 3 then shows that there is an asymptotically flat CCAL metric on the non-compact cobordism $L = M - C \cong S^2 \times B^2 - \dot{B}^4 - B^1$ where $C$ is an integral curve of a gradient-like vector field of the Morse function. $L$ is diffeomorphic to the original cobordism –half of $(S^2 \times S^2 - \{\text{point}\})–$ up to the observation that the initial boundary in $L$ is at some finite time in the past whereas in the original cobordism it is in the infinite past.

We can illustrate the location of the critical points and the critical levels in a lower dimension, $n = 3$. The equivalent process would be $S^2 \to S^1 \times S^1$,.
Figure 5: Levels of a Morse function $f$ in the cobordism $S^2 \rightarrow S^1 \times S^1$. On the left, an index 0 point accounts for the creation of $S^2$; on the right an index 1 point marks the transition to $S^1 \times S^1$. The function $f$ increases in the direction of the expanding spheres and then in the direction of the expanding tori. Critical character of the Morse points is reflected in a same behaviour of $f$ along a Cartesian direction and its opposite.

which is mediated by part of the handlebody $B^2 \times S^1 = B^3 + B^1 \times B^2$, and hence contains an unwanted index 1 point. The two critical points lie in the interior of the solid torus $B^2 \times S^1$ as depicted in Fig 5, a two dimensional section.

This construction generalises to higher dimensions, so that black hole pair creation is feasible whenever $n \geq 4$ as shown in [21]. Indeed, applying right distributivity of the $B^2$ ball to $S^{n-2} = B^{n-2} + B^{n-2}$ we obtain:

$$S^{n-2} \times B^2 = B^n + B^{n-2} \times B^2$$  \hfill (9)

Thus the cobordism $S^{n-1} \rightarrow S^{n-2} \times S^1$ contains only an index $(n-2)$ critical point, which respects causal continuity whenever $n \geq 4$. 


3.3 Decay of Kaluza-Klein vacuum and magnetic field

In five-dimensional Kaluza-Klein gravity a fifth compact dimension is added to ordinary 4-dimensional spacetime. The corresponding metric has fifteen degrees of freedom, which can be interpreted as one dilaton scalar, four components of the electromagnetic field and ten components of the spacetime 4-metric. The 4-dimensional spacetime associated with a given 5-geometry is obtained by reduction along a Killing vector field of closed orbits. Both the Kaluza-Klein vacuum and the Kaluza-Klein version of the Melvin solution have a background topology $\mathbb{R}^4 \times S^1$ and are semi-classically unstable [25, 28, 29].

Now the relevant instanton is the euclideanisation of a five-dimensional black hole: 5-d Schwarzschild for the decay of the vacuum and a rotating 5-d Kerr solution for the decay of a magnetic field. The latter is interpreted, upon four dimensional reduction, as pair production of Kaluza-Klein monopoles.

The topology change is the same in both cases, from the unstable spacelike hypersurface $\mathbb{R}^3 \times S^1$ to the starting hypersurface for post-decay $\mathbb{R}^2 \times S^2 \cong S^4 - S^1$. The double instanton has topology $\mathbb{R}^2 \times S^3 \cong S^5 - S^1$. Once more we compactify by replacing the circle and then halve the closed $S^5$ bounce to get $M \cong B^5$, with $\partial B^5 = S^4$. Finally we delete an open thickened circle $S^1 \times \tilde{B}^4$ from $M$ to create the initial boundary. This yields $M$ with $\partial M = S^1 \times S^3 \cup S^4$. That is, with the notation of Lemma 2, we have the triple $(M, V_0, V_1) = (B^5 - S^1 \times \tilde{B}^4, S^3 \times S^1, S^4)$. We seek a handlebody decomposition for $B^5$ which truncates into a cobordism from $S^3 \times S^1$ to $S^4$. The “redundant” $B^3$ decomposition eq.(6) and right-distributivity imply the identity:

$$B^5 = B^3 \times B^2 = \frac{(B^3 + B^1 \times B^2 + B^2 \times B^1) \times B^2}{B^2 \times S^1} = \frac{B^5 + B^1 \times B^4 + B^2 \times B^3}{B^4 \times S^1}$$

The first term, $B^5$, corresponds to the creation of $S^4$ from $\emptyset$ and the first handle addition corresponds to the transition from $S^4$ to $\partial (B^4 \times S^1) = S^3 \times S^1$, i.e., the (closed) KK vacuum space. The second handle addition is therefore the one that corresponds to the process we are investigating, $S^3 \times S^1 \to S^4$. This means that in the cobordism between $S^1 \times S^3$ and
$S^4$, which involves only the handle $B^2 \times B^3$, there is a Morse function with exactly one critical point of index 2, i.e., no index 1 or 4 points.

We apply Lemma 4 to $M$ and conclude that since the original instanton manifold (half of $S^5 - S^1$) is diffeomorphic to $L = M - C$ there exist asymptotically CCAL metrics on it.

The next figure represents a section of the 3-dimensional analogue of the cobordism $M$. The whole cobordism between $S^1 \times S^1$ and $S^2$ is generated by revolution around the z-axis. The reader can try and picture a cylinder between the inner boundary and the outer boundary that is orthogonal to the contours and does not touch the critical point at the centre. That would be the integral annulus $A$ of Lemma 4.

![Figure 6: Levels of the Morse function that represents the change $S^1 \times S^1 \rightarrow S^2$. Notice that this is in fact Figure 5 turned inside out: the time reversed cobordism if time progresses from inner to outer surfaces. Thus the central point, which was there an index 1 point is here an index 2 point: the Morse function increases in the z-direction and decreases in the other two.](image)

This result also generalises to a countable family of higher-dimensional cobordisms that mediate the nucleation of various p-branes [30]. In n-dimensional Kaluza-Klein theory the Kerr instanton manifold $\mathbb{R}^2 \times S^{n-2} \cong S^n - S^1$ is the double of the cobordism that mediates the transition $\mathbb{R}^{n-2} \times S^1 \rightarrow \mathbb{R}^2 \times S^{n-3}$, which in the closed case reads $S^{n-2} \times S^1 \rightarrow S^{n-1}$. These
are respectively the boundaries of $B^{n-1} \times S^1$ and $B^n$; since

$$B^n = B^3 \times B^{n-3} =$$

$$= B^n + B^1 \times B^{n-1} + B^2 \times B^{n-2}$$

(10)

Again, it is the second handle addition that corresponds to the process of interest and we see that there exists a Morse function with only one critical point of index 2, which respects causal continuity when $n \geq 4$.

### 4 Discussion

We have seen that several instantons for interesting topology changing processes have topologies that support Morse metrics without index 1 or $n-1$ points. These include the pair production of black holes in any dimension, the decay of the Kaluza-Klein vacuum and pair production of Kaluza-Klein monopoles. We have called such cobordisms CCAL based on the conjecture that causal discontinuity of Morse metrics is associated with and only with indices 1 and $n-1$. This conjecture remains to be proved.

Further work that would place our results on a more physical footing would include the study of quantum fields propagating on Morse metric backgrounds with different indices. Can a quantum field be non-singular on a spacetime with an index 2 point in 4 spacetime dimensions for example?

Another question is whether the equivalence relation of “cobordant via a CCAL cobordism” results in a finer subdivision of $(n-1)$-manifolds than simple cobordism. What we know is that for $n-1 = 3$ this doesn’t happen. Any two 3-manifolds are cobordant and also cobordant via a CCAL cobordism [21]. We do not know what the situation is in higher dimensions.

The higher dimensional question is interesting for the following possibility. As far as simple cobordism goes any topology of the form $S^k \times A$ where $A$ is any closed manifold is possible for space since it is cobordant to the empty set via the cobordism $B^{k+1} \times A$. If it is right to restrict to CCAL cobordisms in the SOH then this might constrain the possible topologies of the universe because some may not be CCAL cobordant to $\emptyset$. $A$ might be some Calabi-Yau manifold for example or a torus $T^{n-k}$ so CCAL cobordism
might restrict some of the possible string theory compactifications if the universe was created from nothing as is usually conceived.

We end on a cautionary note. The idea that only CCAL cobordisms be included in the SOH for quantum gravity must be scrutinised in the light of the result that the canonical “U-tube” cobordism for pair production of topological geons is not CCAL [21]. This would leave in serious trouble the proposal that a spin statistics correlation for quantum geons can be established by a “topological” argument similar to that for skyrmions [31][32]. On the other hand if it is true that there’s an underlying discrete substructure to spacetime, it will likely regulate infinities such as the singular quantum field behaviour. In that case the suppression of non-CCAL cobordisms will be only finite and they would need to be included after all.

Acknowledgements We would like to thank A. Chamblin, G. Gibbons, C. Isham, R. Penrose, R. Sorkin, S. Surya, P. Tod and N. Woodhouse for help and interesting discussions. H.F. Dowker is supported in part by an EPSRC Advanced Fellowship.
A Combining manifolds with boundary.

We indicate the difficulties in defining a differentiable structure in the product or adjunction of two manifolds with boundary –both operations are essential in the construction of handlebodies. What makes this problematic is the requirement that coordinate charts in a \( \partial \)-manifold \( M \) be homeomorphisms from a neighbourhood of a point \( p \in M \) onto an open set of the “hemiplane” \( H^n \equiv \{ \vec{x} \in \mathbb{R}^n : x^n \geq 0 \} \). The irrelevance of the order in which atlases are defined makes the map \( f \) in eq.(5) a diffeomorphism.

Product of two \( \partial \)-manifolds. An atlas in the product of two closed manifolds –or in the product of a closed manifold and a \( \partial \)-manifold– \( M \) and \( N \), of respective dimensions \( m \) and \( n \), is naturally induced by the individual coordinate charts of the two manifolds. In the usual notation, given a chart \( (U_\alpha,\phi_\alpha) \) around the point \( p \in M \) and a chart \( (V_\beta,\psi_\beta) \) around \( q \in N \), the chart \( (U_\alpha \times V_\beta,\phi_\alpha \times \psi_\beta) \) maps a neighbourhood of \( (p,q) \in M \times N \) homeomorphically onto an open subset of \( \mathbb{R}^{m+n} \) and differentiability of the transition functions between charts automatically follows from that in \( M \) and \( N \). However, for the product of two \( \partial \)-manifolds the set \( \phi_\alpha \times \psi_\beta (U_\alpha \times V_\beta) \) is not open in \( H^{m+n} \) when \( U_\alpha \times V_\beta \) contains points in \( \partial M \times \partial N \subset \partial(M \times N) \). The troublesome region \( \partial M \times \partial N \) is called the corner set of \( M \times N \).

Standard \( \partial \)-manifold charts can also be defined in this region by fixing a homeomorphism \( H^1 \times H^1 \rightarrow H^2 \), see [33]. Then using the collaring theorem the \( (\partial M \times H^1) \times (\partial N \times H^1) \) is identified with \( \partial M \times \partial N \times H^2 \), which is embedded in \( M \times N \) as a usual collar and thus a genuine part of the boundary. It is this “deformed” structure, where an atlas can be properly defined, what we mean by the product of two \( \partial \)-manifolds.

Attachment of \( \partial \)-manifolds through a boundary identification. Down in the category of topological spaces, handle attachment is a special case of adjunction of two spaces. The adjunction of spaces \( X \) and \( Y \) through a map \( h : A \subset X \rightarrow Y \) is the quotient space of the disjoint union \( X \uplus Y \) by the equivalence relation \( R_h \). Given two points \( z,z' \in X \uplus Y \),

\[
z \cong_{R_h} z' \text{ iff } \begin{cases} 
z = z' \\
z = h(z') \\
z' = h(z) \\
h(z) = h(z')
\end{cases}
\]

Throughout this appendix \( \partial \)-manifold means compact manifold with boundary.
The adjunction space, denoted \( X \sqcup Y \) or simply \( X +_h Y \), is endowed with a topology \([34]\) through the projection \( \pi : X \sqcup Y \to X +_h Y \). Points in \( X +_h Y \) are the equivalence classes \([z]\) of \( \mathcal{R}_h \).

We attach a handle \( H \) to a manifold \( M \) through an injection \( h \) from a closed subset \( A \subset \partial H \) onto a region of \( \partial M \) diffeomorphic\(^4\) to \( A \). A \( \partial \)-manifold structure is induced in \( H +_h M \) with an atlas \( \mathcal{A} \) such that \( \partial A \cong \partial(h(A)) \) remains in the boundary but \( \tilde{A} \) becomes part of the interior of \( H +_h M \). On defining the charts in \( \mathcal{A} \) from those in the atlases of \( H \) and \( M \), the region \( \partial A \) behaves as a corner set and smoothing is required again. Such smoothing is implicit when we talk of the manifold \( M +_h H \). A justification of the glueing process, that uses the collaring theorem can be found in \([23, 33]\).

The map \( f : (L +_h B^\lambda \times B^{n-\lambda}) \times B^1 \to L \times B^1 +_h B^\lambda \times B^{n+1-\lambda} \) of eq.(5) essentially reverses the order in which product and adjunction are performed. Continuity of \( f \) and its inverse is easily checked with respect to the topologies on either side. Finally, to establish differentiability of \( f \) as a map between the two smoothed manifolds we would have to display the modified local charts and combine \( f \) with the smoothing operations, an exercise in differential topology that would take us too far afield.

**B  A.L. Metrics with correct boundary conditions.**

**Proof of lemma 1.** Theorem 2 tells us that there is a Morse function \( f : M \to [0, 1] \) with:

1. \( V_0 = f^{-1}(0) \) and \( V_1 = f^{-1}(1) \)
2. \( f \) has no critical points on \( \partial M = V_0 \sqcup V_1 \)

We construct a Morse metric \( g \), associated with \( f \), such that \( g|_{V_i} = \tilde{h}_i \equiv d_i^*(h_i) \) \( i = 0, 1 \). Our proof is divided in two steps: first we find conditions on a Riemannian metric \( G \) that are sufficient for \( g \) defined by eq.(2) to have

\(^4\)In this case the fourth possibility for equivalence can be omitted; it implies the first by \( h \) injectivity.
the correct restrictions; then we show that a Riemannian metric exists on
$M$ satisfying those conditions.

When a spacelike hypersurface $V \subset M$ is defined as a level surface of a
function $f : M \to \mathbb{R}$, the restriction of a metric $g$ on $M$ to $V$ is given by:

$$ h^{(g)}_{\mu\nu} = g_{\mu\nu} \pm \frac{\partial_\mu f \partial_\nu f}{|g^{\rho\sigma} \partial_\rho f \partial_\sigma f|} \quad (11) $$

where the − or + sign applies respectively to the cases of Riemannian or
Lorentzian $g$. So if we wish to compute the restriction of the Morse metric
$g_{\alpha\beta}$ to the boundary manifolds $V_0 \equiv \{ x \in M : f(x) = 0 \}$ and $V_1 \equiv \{ x \in M : f(x) = 1 \}$ we need its inverse $g^{\alpha\beta}$. From

$$ g_{\mu
u} = \partial_\mu f \partial_\nu f \ G^{\rho\sigma} \partial_\rho f \partial_\sigma f $$

one easily finds that:

$$ g^{\mu\nu} = \frac{1}{(\partial f)^2} G^{\mu\nu} + \frac{\zeta}{(\partial f)^4 (1 - \zeta)} G^{\rho\sigma} \partial_\rho f \partial_\sigma f $$

where $(\partial f)^2$ denotes $G^{\rho\sigma} \partial_\rho f \partial_\sigma f$. The $g$-induced metric on $V_i$, $i = 0, 1$, is

$$ h^{(g)}_{\mu\nu} = g_{\mu\nu} + \frac{\partial_\mu f \partial_\nu f}{|g^{\rho\sigma} \partial_\rho f \partial_\sigma f|} \quad (12) $$

So evaluate

$$ g^{\rho\sigma} \partial_\rho f \partial_\sigma f = \left( \frac{1}{(\partial f)^2} G^{\rho\sigma} + \frac{\zeta}{(\partial f)^4 (1 - \zeta)} G^{\rho\sigma} \partial_\alpha f \partial_\beta f \right) \partial_\rho f \partial_\sigma f $$

$$ = \left( 1 + \frac{\zeta}{(\partial f)^4 (1 - \zeta)} (\partial f)^4 \right) = 1 + \frac{\zeta}{1 - \zeta} = \frac{1}{1 - \zeta} $$

and insert this in eq.(12) to obtain

$$ h^{(g)}_{\mu\nu} = (\partial f)^2 G_{\mu\nu} - \zeta \partial_\mu f \partial_\nu f + \left| 1 - \zeta \right| \partial_\mu f \partial_\nu f $$

$$ = (\partial f)^2 G_{\mu\nu} - \partial_\mu f \partial_\nu f = (\partial f)^2 \left( G_{\mu\nu} - \frac{\partial_\mu f \partial_\nu f}{(\partial f)^2} \right) $$

$$ = (\partial f)^2 h^{(G)}_{\mu\nu} $$
From this equation we immediately infer sufficient conditions on $G$. If $G$ restricts to $h_i^{(G)} = \tilde{h}_i$ and is such that $(\partial f)^2 = 1$ on $V_i$, then:

$$h_i^{(g)} = h_i^{(G)} = \tilde{h}_i$$

and we will be done.

These two conditions can be fulfilled by adapting $G$ to the level surfaces of $f$, in the sense of Theorem 3.1. in [10]. More explicitly, by compactness and isolation of the Morse points $\{p_k\}$, $k = 1, \ldots r$, the collaring theorem [5, 12] permits the following factorisation of an open neighbourhood $M_0 \subset M$ of $V_0$:

$$M_0 \equiv \{ p \in M : 0 \leq f(p) < \epsilon_0 \} \cong V_0 \times [0, \epsilon_0)$$

where $\epsilon_0 < c_1$ and $f(\Phi_0^{-1}(x, t)) = t$ for all $(x, t)$ in $V_0 \times [0, \epsilon_0)$. This foliation of $M_0$ is adapted to $f$ in the sense that $\partial_t f = 1$ and $\partial_x f = 0$ where $x^i$, $i = 1, \ldots n - 1$, are some local coordinates on $V_0$ and thence on $M_0$.

Similarly let

$$M_1 \equiv \{ p \in M : 1 - \epsilon_1 < f(p) \leq 1 \} \cong V_1 \times (1 - \epsilon_1, 1]$$

where $c_r < 1 - \epsilon_1$. Let $M_2 = f^{-1}((\delta, 1 - \delta))$ where $\delta = \frac{1}{2} \min(\epsilon_0, \epsilon_1)$. We can then express $M$ as the union of open subsets: $M = M_0 \cup M_2 \cup M_1$. Next we define a Riemannian metric $G_0$ on $M_0$ as the $\Phi_0$ pull-back of the metric on $V_0 \times [0, \epsilon_0)$ with interval $ds^2 = dt^2 + \tilde{h}_0(x)_{ij} \, dx^i \, dx^j$ and similarly we define $G_1$ on $M_1 \cong V_1 \times (1 - \epsilon_1, 1]$. Take an arbitrary Riemannian metric $G_2$ on $M_2$. One certainly exists by paracompactness of $M_2$ as a submanifold of the compact manifold $M$.

Now let $(U_\alpha, \phi_\alpha)$ be a finite atlas for $M$, $\alpha = 1, \ldots, N$ say, then the sets $W_{\alpha 0} \equiv U_\alpha \cap M_0$, $W_{\alpha 1} \equiv U_\alpha \cap M_1$ and $W_{\alpha 2} \equiv U_\alpha \cap M_2$ give another finite cover of $M$: a refinement of the atlas $(U_\alpha, \phi_\alpha)$. Using an associated partition of unity $\{\theta_\alpha\}$ we can construct a metric on $M$ by patching together local metrics [11]; so define

$$G_{\mu\nu}(p) = \sum_{\alpha \iota} \theta_{\alpha \iota}(p) \, G_{\mu\nu}^{\alpha \iota}(p)$$

with

$$G_{\mu\nu}^{\alpha \iota} = (G_{i|W_\alpha})_{\mu\nu}$$

Throughout $M_0 - M_2 \cap M_0$, a neighbourhood of $V_0$, we have:

$$G^{\mu\nu} \partial_\mu f \partial_\nu f = \partial_0 f \partial_0 f + \tilde{h}_0^{ij} \partial_i f \partial_j f = 1_{=1}$$

$$= 0$$

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and similarly in a neighbourhood of $V_1$. In particular, the unit-norm condition is satisfied on $V_0 \cup V_1$ and substituting in the expression for the induced metric $h^{(G)}_{\mu\nu} = G_{\mu\nu} - \frac{\partial_{\mu} f \partial_{\nu} f}{\partial_{\sigma} f \partial_{\sigma} f}$ gives:

\[
\begin{align*}
    h^{(G)}_{00} &= G_{00} - \partial_0 f \partial_0 f = 0 \\
    h^{(G)}_{0i} &= G_{0i} - \partial_0 f \partial_i f = 0 \\
    h^{(G)}_{ij} &= G_{ij} - \partial_i f \partial_j f = \tilde{h}_{0ij}
\end{align*}
\]

so that $G|_{V_0} = \tilde{h}_0$. Similarly $G|_{V_1} = \tilde{h}_1$. Hence the result. □

**Proof of Lemma 3.** We will find a solid cylinder $C$ of such integral curves and take $C$ to be the central curve. It is convenient to start by defining the past and future “shadows” of each Morse point $p_k$ in $M$. To do so we work with the covering of $M$ used in the construction of the vector field $\xi$. Relabel the coordinates in the critical chart $U_k$ by setting $y_i = x_i$ for $1 \leq i \leq \lambda_k$ and $z_j = x_{\lambda_k+j}$ for $1 \leq j \leq n - \lambda_k$. One can then verify [12, 35] that the integral curve of $\xi$ through $(\vec{y}, \vec{z}) \in \phi_k(U'_k)$ is

1. A straight line ending at the origin, $\phi_k(p_k) = \vec{0}$, if $\vec{z} = \vec{0}$.

2. A straight line beginning at the origin if $\vec{y} = \vec{0}$.

3. A hyperbola which does not pass through the origin if neither $\vec{y}$ nor $\vec{z}$ vanish; this hyperbola intersects the $S^{n-1}$ boundary of $\phi_k(U'_k)$ in two points, one in $-|\vec{y}|^2 + |\vec{z}|^2 = -\varepsilon$ and another in $-|\vec{y}|^2 + |\vec{z}|^2 = \varepsilon$.

The past and future shadows of $p_k$ are defined to be the regions:

\[
\begin{align*}
    S^{(k)}_P &\equiv \{ q \in M : \exists s > 0 \text{ with } \sigma^x_q(s) = p_k \} \\
    S^{(k)}_F &\equiv \{ q \in M : \exists s < 0 \text{ with } \sigma^x_q(s) = p_k \}
\end{align*}
\]

where $\sigma^x_q$ is the integral curve of $\xi$ which passes through $q$ at $s = 0$, i.e., $\sigma^x_q(0) = q$ and the parameter $s$ is fixed by $\xi(f)(q) = \frac{df \sigma^x_q}{ds}|_{s=0}$. Then choose for each $k$ a pair of real numbers $a_k$, $b_k$ close enough to the critical value, $0 < c_k - a_k < \varepsilon$ and $0 < b_k - c_k < \varepsilon$, so that the intersection of the past and
future “shadows” of \( p_k \) with respectively \( V_{a_k} \equiv f^{-1}(a_k) \) and \( V_{b_k} \equiv f^{-1}(b_k) \) are complete spheres. Specifically:

\[
S_L^{(k)} = V_{a_k} \cap S_P^{(k)} = \phi_k^{-1}\{((\bar{y}, \bar{z}) : \bar{z} = 0 \text{ and } |\bar{y}|^2 = c_k - a_k\} \cong S^{\lambda_k - 1}
\]

\[
S_R^{(k)} = V_{b_k} \cap S_F^{(k)} = \phi_k^{-1}\{((\bar{y}, \bar{z}) : \bar{y} = 0 \text{ and } |\bar{z}|^2 = b_k - c_k\} \cong S^{n-\lambda_k - 1}
\]

The left sphere \( S_L^{(k)} \) includes all the points of \( V_{a_k} \) in curves that end in \( p_k \), while the right sphere \( S_R^{(k)} \) includes all the points of \( V_{b_k} \) which begin at \( p_k \). By considering just the left spheres, we can now show that if the index of each critical point \( p_k \) satisfies \( 0 \leq \lambda_k < n \), then there is an embedded solid cylinder \( C \) in \( M \) diffeomorphic to \( B^{(n-1)} \times B^1 \) which contains no critical points and such that the \( B^1 \) coordinate of a point \( p \in C \) is the value \( f(p) \).

We proceed by induction. It is convenient to define \( a_{r+1} = 1 \) so that \( V_{r+1} \equiv V_1 \). Suppose that we have found an integral cylinder \( C_k \) from \( V_0 \) to \( V_{a_k} \) with no critical points. It intersects the \((n-1)\)-dimensional manifold \( V_{a_k} \) in a disk \( D_k \cong B^{n-1} \). We extend this cylinder forward to an integral cylinder from \( V_0 \) to \( V_{a_{k+1}} \).

Induction starts, because \( V_{a_1} \cong V_0 \) and we can choose any disk \( D_0 \in V_0 \) and project it forwards along the \( \xi \)-lines until another \( D_1 \cong D_0 \) is reached in \( V_{a_1} \). The whole set of \( \xi \)-lines from \( D_0 \) to \( D_1 \) is the integral cylinder \( C_1 \).

To demonstrate how a typical inductive step works, we use the fact that the \( \xi \)-curves ending at \( p_k \) intersect \( V_{a_k} \) in a closed embedded sphere \( S_L^{(k)} \cong S^{\lambda_k - 1} \). Since \( \text{dim}(S_L^{(k)}) = \lambda_k - 1 < n - 1 \), the open subset \( \hat{D}_k \cap (V_{a_k} - S_L^{(k)}) \) is not empty in \( V_{a_k} \). 5 Hence we can find a closed disk \( D'_k \subset \hat{D}_k \cap (V_{a_k} - S_L^{(k)}) \). Projecting \( D'_k \) backwards along the integral curves of \( \xi \) gives a cylinder \( C'_k \subset C_k \); projecting it forwards generates a cylinder \( C''_k \) from \( V_{a_k} \) to \( V_{a_{k+1}} \) that ends on what we define to be \( D_{k+1} \subset V_{a_{k+1}} \) and, by construction, does not intercept \( p_k \). The union \( C'_k \cup C''_k \) is the cylinder \( C_{k+1} \) between \( V_0 \) and \( V_{a_{k+1}} \) that contains no critical points, see Fig 7.

This procedure is repeated for each elementary cobordism \( f^{-1}[a_k, a_{k+1}] \)

---

5 The absence of index-n critical points is a necessary condition. Consider an N-shaped hollow cylinder. There is no curve along which the height function increases monotonically connecting the initial and final circles: it contains \( \lambda = 2 \) critical points.
until \( V_{a+1} = V_1 \) is reached. The final disk \( D_{r+1} \) has finite size for the number of inductive steps is finite. The cylinder \( C_{r+1} \) is the desired \( C \). We select the integral curve \( \mathbf{C} = \{ p \in C : \Psi(p) = (\vec{0},t) \quad t \in [0,1] \} \), where the diffeomorphism \( \Psi : C \to B^{n-1} \times B^1 \) is a parametrisation of the cylinder adapted to \( \xi \) in the sense that \( t(p) = f(p) \).

That \( L = M - C \) is a cobordism between \( U_0 \) and \( U_1 \) follows from the equivalence under diffeomorphism of all embedded \( B^k \) balls in any connected manifold of dimension \( n \geq k \) [36]; so in particular there is a diffeomorphism of \( V_i \) that maps \( q_i \) to \( \tilde{q}_i \), \( i = 0,1 \). This implies that \( V_i - q_i \cong V_i - \tilde{q}_i \) and so there exists a diffeo \( d_i : V_i - q_i \to U_i \). To define a Morse metric in the manifold \( L = M - C \) we can combine local metrics as was done in the proof of lemma 1, even though \( L \) is no longer compact. This is because partitions of unity exist for the more general class of paracompact manifolds, and by construction, \( L \) is certainly one of these [11]. But there are now two conditions to be simultaneously fulfilled: correct restrictions and asymptotic flatness. The former requires a sectioning of \( L \) as in 3; while the latter is achieved through a sectioning by cylinders concentric on asymptotic infinity. Note that, since \( L \) is a submanifold of \( M \), the restrictions \( f|L \) and \( \xi|L \) are a Morse function and a gradient-like vector field in \( L \). So using again small numbers \( \delta < \epsilon \) with \( \epsilon < c_1 \) and \( c_r < 1 - \epsilon \) we cover \( L \) with three open regions: the collarings \( L_0, L_1 \) of the initial and final boundaries and the inner submanifold \( L_2 \).

\[
\begin{align*}
L_0 &\equiv f^{-1}([0,\epsilon)) \cap L \cong (V_0 - q_0) \times [0,\epsilon) \\
L_1 &\equiv f^{-1}((1-\epsilon,1]) \cap L \cong (V_1 - q_1) \times (1-\epsilon,1]
\end{align*}
\]
\[ L_2 \equiv f^{-1}(\delta, 1 - \delta)) \cap L \]

Then we carry out the second sectioning: take two solid cylinders \( C' \) and \( C'' \) concentric to \( C \) in \( M \), namely \( C \subset C'' \subset C' \subset C \). Define the core \( L_1 \equiv L - C'' \) and the asymptotic region \( L_{II} \equiv C' - C \) so that \( L = L_1 \cup L_{II} \). Choose a locally finite atlas \( \mathcal{B} = (W_\alpha, \varphi_\alpha) \) of \( L \) and define a refinement of \( \mathcal{B} \) adapted to the double sectioning. We obtain new charts \( W_{\alpha i a} \equiv W_\alpha \cap L_i \cap L_a \quad i = 0, 1, 2 \quad a = I, II. \)

Endow \( V_0 - q_0 \) with the induced metric \( \tilde{h}_0 \equiv d_0^* (h_0) \) and let \( H_0 \) be the product Riemannian metric \( ds^2 = dt^2 + \tilde{h}_{0, ab} dx^a dx^b \) on \( (V_0 - q_0) \times [0, \epsilon) \). Define the metric \( G_0 \equiv \Phi_0^* H_0 \) on \( L_0 \). Similarly define the metric \( G_1 \) on \( L_1 \). By asymptotic flatness of \( \tilde{h}_0 \) and \( \tilde{h}_1 \), \( G_0 \) and \( G_1 \) are on the right track for overall asymptotic flatness on \( L \). The same asymptotic behaviour must hold throughout \( L_{II} \), i.e., the metric must become flat in the surroundings of the removed curve. This is ensured by taking \( G_{II} \) to be the pullback of \( ds^2 = dt^2 + dr^2 + r^2 d\Omega_{n-2}^2 \) on \( B^1 \times (1, \infty) \times S^{n-2} \cong L_{II} \) where the \( t \) coordinate is identified with \( f \) via the diffeo. The local metrics on the charts of the refinement are:

\[
G_{\mu \nu} = \begin{cases} 
(G_i |_{W_{\alpha i a}})_{\mu \nu} & \text{if } i = 0, 1 \\
(G_{II} |_{W_{\alpha 2I}})_{\mu \nu} & \forall \alpha
\end{cases}
\]

and any Riemannian metric \( G_{\alpha 2I} \) on the remaining charts \( W_{\alpha 2I} \). Finally, combine all these local metrics through a partition of unity \( \{ \vartheta_{\alpha i a} \} \) associated with the atlas \( (W_{\alpha i a}, \varphi_\alpha |_{W_{\alpha i a}}) \) to define a Riemannian metric

\[ G(p) = \sum_{\alpha, i, a} \vartheta_{\alpha i a}(p) G^{\alpha i a}(p) \]

which: (i) is asymptotically flat, (ii) has the correct restrictions \( \tilde{h}_0 \) and \( \tilde{h}_1 \) and (iii) ensures \( G^{\mu \nu} \partial_\mu f \partial_\nu f = 1 \) throughout \( L_0 - L_0 \cap L_2 \) and \( L_1 - L_1 \cap L_2 \). The Lorentzian metric formed from \( G \) and \( f \) as in equation (2) inherits from \( G \) properties (i) and (ii) and has the Morse structure of \( f \). Thus \( g \) is the metric that we are seeking. \( \square \)

**Proof of lemma 4.** (i) Along the lines of the proof just given, we will find an annular tube \( A \cong B^1 \times S^1 \times B^{n-2} \) with central \( A \cong B^1 \times S^1 \times \{ \tilde{0} \} \) such that \( V_i - \tilde{C}_i \cong V_i - C_i, i = 0, 1, \) where \( C_i = V_i \cap A \).
With the same notation as in the proof of lemma 3, induction starts: thicken the initial circle \( C_0 \) to an embedded “torus”\(^6\), \( i_0 : S^1 \times B^{n-2} \to T_0 \subset V_0 \) and project it along the \( \xi \)-lines to \( V_{a_1} \cong V_0 \). This gives a first annular tube \( A_1 \). Now suppose that an \( A_k \) has been found from \( V_0 \) to \( V_{a_k} \) that satisfies:

1. \( A_k \cap V_0 = i_0(S^1 \times B_{x_k}), \) where \( B_{x_k} \subset \dot{B}^{n-2} \) is a ball centred at a point \( x_k \in B^{n-2} \) which is not necessarily the origin.

2. \( A_k \cap V_{a_k} \equiv T_k = i_k(S^1 \times B_{x_k}) \) with \( i_k(\psi, x) \equiv \sigma^\xi_{i_0(\psi, x)} \cap V_{a_k} \) for each \( \psi \in [0, 2\pi], x \in B_{x_k} \). Here \( \sigma^\xi_\psi \) denotes the image in \( M \) of the integral curve starting at \( \phi \in V_0 \).

We find an annulus \( A_{k+1} \) of integral curves of \( \xi \) in \( M^{a_{k+1}} = f^{-1}([0, a_{k+1}]) \) whose intersection with \( V_0 \) is still a thickened circle of the form \( i_0(S^1 \times B_x) \). Recall that between \( V_{a_k} \) and \( V_{a_{k+1}} \) there is a Morse point \( p_k \) whose index is now constrained by \( \lambda_k \leq n - 2 \). So \( \dot{T}_k \cap (V_{a_k} - S_L^{(k)}) \) is not empty and since \( S_L^{(k)} \) has dimension less or equal to \( n - 3 \) we can in fact choose another \( T_k' \subset \dot{T}_k \) such that \( T_k' = i_k(S^1 \times B_{x_{k+1}}) \) for some \( (n-2) \)-ball \( B_{x_{k+1}} \subset B_{x_k} \) and that \( T_k' \cap S_L^{(k)} = \emptyset \). Projecting \( T_k' \) backwards along the integral curves of \( \xi \) gives an annular tube \( A_k' \subset A_k \); projecting it forwards generates another tube \( A_k'' \) from \( V_{a_k} \) to \( V_{a_{k+1}} \) that ends on what we denote \( T_{k+1} \) and does not intercept \( p_k \). The union \( A_k' \cup A_k'' \) is the desired thickened cylinder \( A_{k+1} \) between \( V_0 \) and \( V_{a_{k+1}} \) that contains no critical points.

Again, apply the inductive step to each elementary cobordism until \( V_{a_{r+1}} \equiv V_1 \) is reached. Like before, the thickened circle \( T_{r+1} \) has finite size. Now \( A_{r+1} \) is the annular tube \( A \) between \( V_0 \) and \( V_1 \) which can be parametrised as the product \( B^1 \times B^{n-2} \times S^1 \): for \( p \in A \) the coordinates are \( (f(p), i_0(q)), \) where \( q \) is the point of \( T_0 \) at which starts the \( \xi \)-curve through \( p \) and the second coordinate naturally splits in an \( S^1 \) and a \( B^{n-2} \) parts. We select the integral annulus \( A \equiv \{ p \in C : q = i_0(\psi, x_{r+1}) 0 \leq \psi < 2\pi \} = \{ \sigma_{i_0(\psi, x_{r+1})}^{(r+1)}(t) : 0 \leq \psi < 2\pi, 0 \leq t \leq 1 \} \). The subsets \( C_0 \equiv A \cap V_0 \) and \( C_1 \equiv A \cap V_1 \) are embedded circles in the initial and final boundaries.

To demonstrate that \( L \equiv M - A \) is a cobordism between \( U_0 \) and \( U_1 \) we

\(^6\)The existence of tubular neighbourhoods of submanifolds is a well established fact in differential topology.
need to show that $V_0 - C_0 \cong V_0 - \tilde{C}_0$ and $V_1 - C_1 \cong V_1 - \tilde{C}_1$. For the former, consider that $\tilde{C}_0$ is the image under $i_0$ of $\{ (\psi, \vec{0}) : \psi \in S^1 \}$ in $V_0$ and $C_0$ is the image under $i_0$ of $\{ (\psi, \vec{x}_{r+1}) : \psi \in S^1 \}$. The result by Palais shows that there is a diffeo of $S^1 \times B^{n-2}$ that maps $(\psi, \vec{0})$ to $(\psi, \vec{x}_{r+1})$ and is the identity on the boundary. Then $i_0$ turns this into a diffeo of $V_0$ that maps $\tilde{C}_0$ onto $C_0$ and hence the result. For the latter, recall that $V_1$ is simply connected and connected by assumption and dim($V_1$) $\geq 4$. These conditions guarantee the absence of knots and also imply that every embedded circle bounds a disk; in particular $\tilde{C}_1$ and $C_1$ do. There exists a diffeo of $V_1$ that maps one of these disks onto the other and so maps $\tilde{C}_1$ onto $C_1$. Hence the result.

Finally the asymptotically flat AL metric in $L$ is defined as in the proof of 3; the only difference resides in the sectioning of manifold $L$: to perform the transition between an arbitrary Riemannian metric in the core of $L$ and the flat metric around the annular cylinder at infinity $A$, we use concentric annular tubes instead of concentric cylinders. □

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