Statistics Of The Burst Model At Super-critical Phase

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Abstract

We investigate the statistics of a model of type-I X-ray burst [Phys. Rev. E, 51, 3045 (1995)] in its super-critical phase. The time evolution of the burnable clusters, places where fire can pass through, is studied using simple statistical arguments. We offer a simple picture for the time evolution of the percentage of space covered by burnable clusters. A relation between the time-average and the peak percentage of space covered by burnable clusters is also derived.

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I. INTRODUCTION

In recent years, spatially extended systems of many degrees of freedom have attracted considerable attention. A number of cellular automaton models, for example, Eden model, percolation, contact process, and various self-organized critical models have been investigated. The forest fire model (FFM) and some sandpile models with special geometries or toppling rules tell us that the existence of local particle conservation law is not a necessary condition for self-organized critical (SOC). Thus, some of the SOC models may be used to comprehend the underlying principles of dissipative systems.

As non-conserving systems are divers in many branches of study, we have proposed a simple cellular automaton model for diffusive and dissipative systems, whose rules are similar to that of the FFM. Our model is applicable to a number of phenomena including the type-I X-ray bursts, and the CO$_2$ gas outburst in some crater lakes. Because of this, we name our model the “X-ray burst model”, or simply the “burst model”. We studied the burst model by both numerical and semi-analytical means and we found that the model exhibits various phases when choosing different set of parameters. In spite of the similarity with the FFM, the burst model is not self-organized critical.

In this paper, we focus on the statistical behavior of the burst model at its super-critical phase. First, we briefly review the burst model in Section II. Then we discuss the relationship of the burst model at its super-critical phase with percolation in Section III. A relation between the time-average coverage fraction and the average coverage fraction just before burning (precise definitions will be given later in the text) is also derived using simple statistical arguments. Finally, we have a brief discussion in Section IV.

II. THE BURST MODEL

Consider a two-dimensional square lattice with periodic boundary conditions. (Rules for other lattice geometries are similar.) We denote the energy stored at each lattice site $r$ by $E(r) \geq 0$. At each timestep, the system is parallel updated according to the following rules:

1. Energy introduction: a unit of energy is added to a randomly chosen site $r_0$, i.e. $E(r_0) \rightarrow E(r_0) + 1$.

2. Fire triggering: if the energy in the chosen site is greater than or equals to a fix threshold $E_{c1}$, the energy will be dissipated (or burnt), i.e. if $E(r_0) \geq E_{c1}$, then $E(r_0) \rightarrow 0$.

3. Fire propagation: if the energy in the nearest neighbor of a burning site is greater than or equal to another fix threshold $E_{c2} \leq E_{c1}$, the energy in that neighboring site will be also dissipated. That is, if $E(r'_0) \geq E_{c2}$, then $E(r'_0) \rightarrow 0$, where $r'_0$ are the nearest neighbors of the burning site $r$. This process is repeated until the fire cannot propagate any longer. And we also assume that fire propagation is so fast that it takes only one timestep to set the relevant sites on fire.
4. Material diffusion: diffusion takes place at every site, i.e. \( E(r) \rightarrow E(r) + \Delta E(r) \) for all \( r \), where \( \Delta E \) depends on the geometry of the lattice and the diffusion constant \( D \).

The system is allowed to evolve for a sufficiently long time in order to attain a steady state before any statistics is taken.

The behavior of the burst model depends on four external parameters, namely, \( E_{c1} \), \( E_{c2} \), \( D \), and the size of each energy blob in comparison with the size of a lattice cell. Depending on various choice of these parameters, the occurrence probability \( P(S) \) of dissipation size \( S \) falls into one of the following three types of behaviors [7]: (a) \( P(S) \) shows a localized peak which is a signature of a super-critical state; (b) \( P(S) \) shows an early exponential cut-off which can be identified as a sub-critical state; and (c) power law is observed in \( P(S) \) at the transition point between the super-critical and sub-critical phase.

III. STATISTICS IN THE SUPER-CRITICAL PHASE

Let us introduce a few concepts before discussing the super-critical phase statistics and the relationship between the burst model at super-critical phase and percolation. A site is called burnable if its energy content is greater than or equals to \( E_{c2} \). Thus, a burnable site will catch fire when any one of its nearest neighbors burns. Then, we define a burnable cluster to be a collection of maximal burnable sites such that all of them catch fire in the next timestep provided that any one of them is on fire [7]. The size of a burnable cluster \( s \) is defined as the number of sites it contains.

During the evolution of the system, clusters of sites grow continuously while they burn suddenly to empty spaces (or voids). And it is interesting to note that the dissipation size \( S \) (i.e., the total energy released in the burning of a cluster) is generally proportional to the cluster size. The behavior of \( P(S) \) under different choice of parameters can be understood in terms of the size of the burnable clusters [7]. In particular, if there is only a few large burnable clusters in the system immediately before most of the burnings, \( P(S) \) will show a peak indicating that majority of the energy release in the system occurs in the “large events” which extend across the whole system. Therefore, the system is in the super-critical phase. Finally, we define the coverage fraction \( \rho \) to be the percentage of sites in the system that found to be burnable.

As we can see in Fig. [1], which is a snapshot of a typical system immediately after a large energy dissipation, a large void is observed with cluster islands of different sizes lying inside. So, in general, a burnable cluster for the system (in two-dimension) is not compact.

To investigate the relation between our model and percolation, we consider a region of space with linear size \( L \) inside the system. If the typical length of a burnable cluster \( \xi \) is much less than \( L \), we expect most of the dissipation in this area is initiated and confined or localized inside the region. On the other hand, if \( \xi \gtrsim L \), the coverage fraction in this area would gradually grow to a value \( \rho_b \) before a large dissipation sweeps through the whole

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1 Nevertheless, our study is focused on diffusionless system in this paper.

2 The collection of sites is maximal in the sense of inclusion.
region. Therefore, if the system is super-critical, we can take a sufficiently large $L$ and still have non-negligible probability of finding burnable clusters with size $\gtrsim L^2$. In order to burn down such a large cluster, the coverage fraction around in the large burnable cluster immediately before it burns must be greater than or equal to the (site) percolation threshold $\rho_c$ of the corresponding lattice. In general, the (time) mean coverage fraction $\bar{\rho}$ depends on the large as well as small dissipations. But at the super-critical phase, large dissipations dominate the behavior of the system and hence we can safely neglect the effect of small dissipations.

Using the same argument, we know that the coverage fraction $\rho$ attains its local minimum value just after a fire. Then as energy is gradually introduced into the system, $\rho$ increases. However, the rate of increase $\dot{\rho}$ will, in general, flatten out gradually with time due to the fact that adding energy into a site which is already burnable does not increase $\rho$. Eventually, $\rho$ reaches its local maximum value immediately before a fire. A typical time evolution of $\rho$ for a super-critical system is plotted in Fig. 2. In this respect, the time evolution of burnable clusters in the burst model can be viewed as a dynamical percolation problem where the “conductance probability” gradually increases with time. Any sudden drop in the conductance probability is due to a fire. Finally, we like to remark that both $\rho_b$ and $\bar{\rho}$ are functions of $E_{c1}$ and $E_{c2}$.

**A. The Case When $E_{c2} = 1$**

We first consider an easy case with $D = 0$, $E_{c2} = 1$ and $E_{c1} \gg E_{c2}$. With this set of parameters, the system is in the super-critical phase. Since $E_{c2} = 1$, an empty site turns burnable once an energy unit is dropped onto it. Moreover, burning is the only way to turn a burnable site back to an empty site because $D = 0$. Thus, a burnable site is formed out of an empty site with probability $(1 - \rho)/N$ at each timestep where $N$ is the total number of sites in the system. That is,

$$\frac{d\rho}{dt} = k(1 - \rho)$$

for some positive constant $k$. Integrating Eq. (1) gives us

$$\rho(t) = 1 - e^{-kt},$$

where $t$ is the time since the site is last burnt (and hence $\rho(0) \approx 0$). Since the system is in the super-critical phase and $E_{c2} = 1$, it is almost sure that immediately before each burning, almost the entire system is covered by a single large burnable cluster. So after each burning, almost all the sites becomes void. Everything starts over again almost independent of the history of the system. Thus, we may relate the mean coverage fraction, $\bar{\rho}$, to the average coverage fraction attained just before burning, $\rho_b$, by

$$\bar{\rho} = \left(1 + \frac{\rho'}{\ln(1 - \rho')}\right) \approx 1 + \frac{\rho_b}{\ln(1 - \rho_b)},$$

where the expectation is taken over all the values of coverage fractions, $\rho'$, just before burnings.
In the $E_{c1} \to \infty$ limit (while the lattice size remains fix), every grid point the system must be covered by a single burnable cluster right before each fire. So $\rho_b = \bar{\rho} = 1$, which agrees with the prediction of Eq. (3). As we lower the value of $E_{c1}$, some sites may not be contained in the burnable cluster before a fire. Therefore, we expect both $\rho_b$ and $\bar{\rho}$ to be less than 1. At this point, Eq. (3) still holds. As we further decrease the value of $E_{c1}$, the system eventually enters the critical, and then follows by the sub-critical phase. As there is negligible fire which extends system-wide, fluctuation of $\rho$ about its (time) mean value $\bar{\rho}$ is small. So for a infinitely large system, $\rho_b$ equals $\bar{\rho}$ and their relation given by Eq. (3) is no longer valid. Finally, as $E_{c1} \to E_{c2}$, both $\rho_b$ and $\bar{\rho}$ go to 0.

In order to study the local properties of the system near a burnable cluster, we define $(\rho_b)_{\text{Local}}$ to be the (time) average coverage fraction immediately before a fire at a sufficiently small region centered at the triggering site. Using the same argument in the derivation of Eq. (3), we find that

$$\bar{\rho} \approx 1 - \frac{(\rho_b)_{\text{Local}} - \rho(0)}{\log (1 - \rho(0)) - \log (1 - (\rho_b)_{\text{Local}})},$$

where $\rho(0)$ is determined by the amount of energy remains after the last fire in the nearby region, which should be close to 0. We expect $(\rho_b)_{\text{Local}}$ to be greater than its corresponding global value $\rho_b$. While $\rho_b$ does not fluctuate in the limit of large system size in the sub-critical phase, $(\rho_b)_{\text{Local}}$ near a burning cluster can deviate from $\rho_b$.

The above discussions are verified by numerical simulations\textsuperscript{3}. We fix $E_{c2} = 1$ and $D = 0$ on a $512 \times 512$ square lattice. $\bar{\rho}$ is obtained by calculating the time mean of $\rho$. On the other hand, $\rho_b$ and $(\rho_b)_{\text{Local}}$ are obtained by calculating the median of their corresponding values immediately before each fire. Their values obtain this way will not be seriously affected by the “pre-mature” burnings. As shown in Table I, Eq. (3) agrees very well with our numerical experiment in the super-critical phase. In addition, the relation between $\bar{\rho}$ and $(\rho_b)_{\text{Local}}$ in both the super-critical and sub-critical phase is also verified. As one can notice from Table I, the value of $\bar{\rho}$ predicted from $(\rho_b)_{\text{Local}}$ deviates systematically from the observed valued of $\bar{\rho}$ in the sub-critical phase. The deviation can be explained as follows: Eq. (3) tells us that the predicted value of $\bar{\rho}$ increases when we increase $\rho(0)$ but keeping $\rho_b$ fixed. Consequently, when $\bar{\rho}$ is fixed, $\rho_b$ decreases with increasing $\rho(0)$. Thus, in the sub-critical phase whose $\rho(0) > 0$, the prediction of $\bar{\rho}$ in Table I using the assumption that $\rho(0) = 0$ is systematically drifted to the lower side.

All the above observations are also verified for honeycomb, triangular, and simple cubic lattices (see Table I). In the supercritical phase, Eq. (3) holds very well in all these lattices.

Since we only use a modest set of assumptions in deriving these relations, we expect Eqs. (3)-(4) to hold for a number of models with similar growth mechanisms as long as there is a separation of timescale between the gradual growth and sudden dissipation. To test this hypothesis, we look into the version of FFM proposed by Drossel et al.\textsuperscript{5}. We have performed numerical experiment on a $1024 \times 1024$ square lattice with the probability of tree grow and lightning per site per timestep being $1 \times 10^{-9}$ and $1 \times 10^{-13}$ respectively.

\textsuperscript{3}All computer simulations in this paper are done in SUN SPARC 10 workstations in University of Hong Kong, and SGI Challenge in IAS
The average coverage fraction \( \bar{\rho} \) is found to be 0.408(1) which is consistent with the value 0.4081(7) found recently by Clar et al. [10]. We also obtain \( \rho_b = 0.423(2) \) and \( (\rho_b)_{\text{Local}} = 0.62(3) \). If we assume that \( \rho(0) = 0 \), \( \bar{\rho} \) deduced from the local coverage fraction equals 0.359(2) which is again drifted to the lower side. Nevertheless, within an error of 10\% or so, Eq. (4) gives a reasonably good estimate of \( \bar{\rho} \) based on \( (\rho_b)_{\text{Local}} \). In this respect, we believe that Eq. (4) is a good approximation for models with growth mechanisms similar to the burst model or the FFM.

**B. The Case When \( E_{c2} > 1 \)**

The situation is more complicated when the system is in super-critical phase with \( E_{c2} > 1 \). For example, if we take \( E_{c1} = 16, E_{c2} = 2 \) and \( D = 0 \) in the 512 \( \times \) 512 square lattice, Table I tells us that \( \rho_b = 0.874(3) \) and \( \bar{\rho} = 0.502(1) \). Although the system is in the super-critical phase, Eq. (3) does not hold. This discrepancy is due to the invalidity of Eq. (1).

Since the adding an unit of energy to a site with \( E < E_{c2} \) does not necessarily turn it into a burnable site, we expect \( d\rho/dt \) to be smaller than \( k(1 - \rho) \).

For simplicity, we consider a diffusionless system and write \( E_{c2} = n \). We also define \( \rho_j \) to be the coverage fraction of the system by sites with energy equals \( j \) for \( j = 0, 1, 2, \ldots, n-1 \). Moreover, \( \rho_n \) is the coverage fraction of the system by sites with energy greater than or equals to \( n \) (hence they are burnable and \( \rho_n \equiv \bar{\rho} \)). It is easy to see that

\[
\frac{d\rho_0}{dt} = -k\rho_0 , \quad (5a)
\]

\[
\frac{d\rho_j}{dt} = k\rho_{j-1} - k\rho_j \quad \text{for } j = 1, 2, \ldots, n-1 , \quad (5b)
\]

and

\[
\frac{d\rho_n}{dt} = k\rho_{n-1} , \quad (5c)
\]

where \( k \) is a positive constant. Note that the coverage fractions satisfy the conversation law

\[
\sum_{j=0}^{n} \rho_j = 1 . \quad (6)
\]

Clearly, Eq. (5c) reduces to Eq. (1) when \( n = 2 \).

Given the initial coverage fractions, we can integrate Eqs. (5) one at a time, which can eventually calculate the time evolution of \( \rho_j \) for all \( j < n \). Then \( \rho_n(t) \) can be computed using Eq. (3). After some simple but long computation, the solutions of Eqs. (5) are given by

\[
\rho_j(t) = \sum_{m=0}^{j} \frac{(kt)^m}{m!} \rho_{j-m}(0) e^{-kt} \quad \text{for } j = 0, 1, 2, \ldots, n-1 , \quad (7a)
\]

and
\[ \rho_n(t) = 1 - \sum_{j=0}^{n-1} \sum_{m=0}^{j} \frac{(kt)^m}{m!} \rho_{j-m}(0)e^{-kt}, \]  

(7b)

where \( t \) is the time since the last burning. Just after a large energy dissipation, \( \rho_0(0) \approx \rho_b \), \( \rho_j(0) \approx (1 - \rho_b)/(n-1) \) for \( j = 1, 2, \ldots, n-1 \), and \( \rho_n(0) = 0 \) are reasonable approximations. Thus, Eq. (7b) becomes

\[ \rho_n(t) \approx 1 - \rho_b \sum_{j=0}^{n-1} \frac{(kt)^j}{j!} e^{-kt} - \frac{1 - \rho_b}{n-1} \sum_{m=1}^{n-1} \frac{m!}{m!} (n-2-j)e^{-kt}. \]  

(8)

In particular, if \( E_{c1} \) is very large, \( \rho_b \approx 1 \), and Eq. (8) becomes

\[ \rho_n(t) \approx 1 - e^{-kt} \sum_{m=0}^{n-1} \frac{(kt)^m}{m!}. \]  

(9)

Unfortunately, there are no simple close forms for the time \( T \) when \( \rho(T) \equiv \rho_n(T) = \rho_b \) in general for both Eqs. (8) and (9). So we do not have a simple relationship between \( \bar{\rho} \) and \( \rho_b \) as in the case when \( E_{c2} \equiv n = 1 \).

To verify the validity of Eq. (8), we numerically solve for \( kT \) when \( \rho_n(T) = \rho_b \) and \( n = 2 \). Once \( kT \) is found, we can numerically integrate Eq. (8) to obtain \( \bar{\rho} \). The results are tabulated in Table II. As we can see, the prediction of Eq. (8) agrees quite well with our numerical experiments.

**IV. DISCUSSIONS**

In summary, we have investigated the statistics of the burst model at its super-critical phase. We have a simple picture for the time evolution of the coverage fraction \( \rho \). Using simple statistical argument, we also derive a relationship between the (time) mean coverage fraction \( \bar{\rho} \) and the average coverage fraction just before a fire \( \rho_b \). The relation agrees quite well with our numerical simulations within an error level of a few percent. Moreover, we find a local version of this relationship (see Eqs. (3)-(4)). Since these two relations are derived using a modest set of assumptions, we expect that they are valid in many other growth models with dissipations. Our preliminary analysis on the FFM tells us that this is indeed the case. And it will be very interesting to apply the same analysis to both the burst model in higher spatial dimensions and other growth and dissipation models.

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### TABLE I. Values of the average coverage fractions when $D = 0$ and $E_{c2} = 1$ in various underlying lattices with different values of $E_{c1}$.

| Lattice | $E_{c1}$ | $\bar{\rho}$ | $\rho_b$ | $(\rho_b)_{\text{Local}}$ | $(\bar{\rho})_{\text{proj}}^a$ | $(\bar{\rho})_{\text{proj}}^b$ | Phase |
|---------|--------|---------|--------|----------------|----------------|----------------|-------|
| 512 x 512 Square | 4 | 0.334(2) | 0.338(4) | 0.55(4) | 0.31(2) | | Sub-critical |
| | 6 | 0.381(1) | 0.382(2) | 0.63(3) | 0.36(2) | | |
| | 8 | 0.400(4) | 0.41(1) | 0.66(1) | 0.388(7) | | ↓ |
| | 10 | 0.426(5) | 0.61(1) | 0.71(1) | 0.35(1) | 0.426(6) | ↑ |
| | 11 | 0.509(5) | 0.79(1) | 0.80(1) | 0.49(1) | 0.503(4) | ↓ |
| | 12 | 0.590(4) | 0.92(2) | 0.82(2) | 0.51(1) | 0.590(6) | ↓ |
| | 13 | 0.633(3) | 0.920(3) | 0.920(4) | 0.63(4) | 0.636(4) | ↓ |
| 512 x 512 Honeycomb | 8 | 0.473(2) | 0.476(2) | 0.73(4) | 0.44(3) | | Sub-critical |
| | 10 | 0.495(4) | 0.52(1) | 0.76(3) | 0.47(3) | | ↑ |
| | 12 | 0.530(3) | 0.81(2) | 0.82(1) | 0.51(2) | 0.52(1) | ↓ |
| | 14 | 0.666(1) | 0.942(2) | 0.941(1) | 0.669(3) | 0.668(2) | ↓ |
| 512 x 512 Triangular | 6 | 0.323(4) | 0.326(4) | 0.52(4) | 0.29(3) | | Sub-critical |
| | 10 | 0.465(1) | 0.753(4) | 0.754(4) | 0.462(3) | 0.462(3) | ↑ |
| | 12 | 0.580(1) | 0.880(2) | 0.880(3) | 0.584(2) | 0.584(3) | ↑ |
| 64 x 64 x 64 Simple Cubic | 6 | 0.220(2) | 0.230(6) | 0.30(3) | 0.15(2) | | Sub-critical |
| | 8 | 0.294(4) | 0.488(4) | 0.510(5) | 0.27(1) | 0.285(4) | ↑ |
| | 10 | 0.464(2) | 0.756(2) | 0.756(4) | 0.464(2) | 0.464(3) | ↑ |

$a(\bar{\rho})_{\text{proj}}^a$ is the value of $\bar{\rho}$ obtained as if Eq. (3) is correct. The difference between this value and $\bar{\rho}$ we measure (in column 3) tells us how good Eq. (3) is. We only show this prediction when the system is in super-critical state.

$b(\bar{\rho})_{\text{proj}}^b$ is the value of $\bar{\rho}$ obtained by substituting $(\rho_b)_{\text{Local}}$ into Eq. (4) and assuming $\rho(0) = 0$. 

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TABLE II. Values of the average coverage fractions when $D = 0$ and $E_{c2} = 2$ in $512 \times 512$ square lattice using different values of $E_{c1}$.

| $E_{c1}$ | $\bar{\rho}$ | $\rho_b$ | $(\rho_b)_{\text{Local}}$ | $(\bar{\rho})_{\text{proj}^a}$ | $(\bar{\rho})_{\text{proj}^b}$ | Phase    |
|---------|--------------|----------|---------------------------|-------------------------------|-------------------------------|----------|
| 8       | 0.338(1)     | 0.338(1) | 0.60(5)                   | 0.32(4)                       |                               | sub-critical ↓↑ |
| 10      | 0.358(2)     | 0.361(3) | 0.64(4)                   | 0.34(4)                       |                               | ↓         |
| 12      | 0.368(2)     | 0.387(3) | 0.68(1)                   | 0.365(8)                      |                               | super-critical |
| 14      | 0.379(2)     | 0.557(3) | 0.685(6)                  | 0.297(3)                      | 0.368(5)                     |           |
| 16      | 0.502(1)     | 0.874(3) | 0.875(5)                  | 0.503(3)                      | 0.504(5)                     |           |

$^a(\bar{\rho})_{\text{proj}^1}$ is the value of $\bar{\rho}$ obtained as if Eq. (8) is correct. The difference between this value and $\bar{\rho}$ we measure (in column 2) tells us how good Eq. (8) is. We only show this prediction when the system is in super-critical state.

$^b(\bar{\rho})_{\text{proj}^2}$ is the value of $\bar{\rho}$ obtained using the local version of Eq. (8).
FIGURES

FIG. 1. Gray scale snapshot of a $512 \times 512$ system with $E_{c1} = 8$, $E_{c2} = 1$ and $D = 0$ just after a large dissipation showing a large void inside which there are some small burnable clusters.

FIG. 2. The time evolution of the coverage fraction $\rho$ (in solid line) in a typical super-critical system. Here, we use $E_{c1} = 13$, $E_{c2} = 1$, and $D = 0$ in a $512 \times 512$ square lattice. The values of $\bar{\rho}$ and $\rho_b$ are shown (in dash lines) for comparison.