TRANSFINITE DIAMETER NOTIONS IN $\mathbb{C}^N$ AND INTEGRALS OF VANDERMONDE DETERMINANTS

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Abstract. We provide a general framework and indicate relations between the notions of transfinite diameter, homogeneous transfinite diameter, and weighted transfinite diameter for sets in $\mathbb{C}^N$. An ingredient is a formula of Rumely [19] which relates the Robin function and the transfinite diameter of a compact set. We also prove limiting formulas for integrals of generalized Vandermonde determinants with varying weights for a general class of compact sets and measures in $\mathbb{C}^N$. Our results extend to certain weights and measures defined on cones in $\mathbb{R}^N$.

1. Introduction.

Given a compact set $E$ in the complex plane $\mathbb{C}$, the transfinite diameter of $E$ is the number

$$d(E) := \lim_{n \to \infty} \max_{\zeta_1, \ldots, \zeta_n \in E} |VDM(\zeta_1, \ldots, \zeta_n)|^{1 / (\binom{n}{2})} := \max_{\zeta_1, \ldots, \zeta_n \in E} \prod_{i < j} |\zeta_i - \zeta_j|^{1 / \binom{2}{2}}.$$  

It is well-known that this quantity is equivalent to the Chebyshev constant of $E$:

$$T(E) := \lim_{n \to \infty} \left[ \inf \{ \|p_n\|_E : p_n(z) = z^n + \sum_{j=1}^{n-1} c_j z^j \} \right]^{1/n}$$  

(here, $\|p_n\|_E := \sup_{z \in E} |p(z)|$) and also to $e^{-\rho(E)}$ where

$$\rho(E) := \lim_{|z| \to \infty} \left[ g_E(z) - \log |z| \right]$$

is the Robin constant of $E$. The function $g_E$ is the Green function of logarithmic growth associated to $E$. Moreover, if $w$ is an admissible weight function on $E$, weighted versions of the above quantities can be defined. We refer the reader to the book of Saff-Totik [20] for the definitions and relationships.

For $E \subset \mathbb{C}^N$ with $N > 1$, multivariate notions of transfinite diameter, Chebyshev constant and Robin-type constants have been introduced and...
studied by several people. For an introduction to weighted versions of some of these quantities, see Appendix B by Bloom in [20]. In the first part of this paper (section 2), we discuss a general framework for the various types of transfinite diameters in the spirit of Zaharjuta [23]. In particular, we relate (Theorem 2.7) two weighted transfinite diameters, \( d^w(E) \) and \( \delta^w(E) \), of a compact set \( E \subset \mathbb{C}^N \) using a remarkable result of Rumely [19] which itself relates the (unweighted) transfinite diameter \( d(E) \) with a Robin-like integral formula. Very recently Berman-Boucksom [2] have established a generalization of Rumely’s formula which includes a weighted version of his result.

In the second part of the paper (section 3) we generalize to \( \mathbb{C}^N \) some results on strong asymptotics of Christoffel functions proved in [8] in one variable. For \( E \) a compact subset of \( \mathbb{C} \), \( w \) an admissible weight function on \( E \), and \( \mu \) a positive Borel measure on \( E \) such that the triple \( (E, w, \mu) \) satisfies a weighted Bernstein-Markov inequality (see (3.5)), we take, for each \( n = 1, 2, ..., \), a set of orthonormal polynomials \( q_{1}^{(n)}, ..., q_{n}^{(n)} \) with respect to the varying measures \( w(z)2^{n}d\mu(z) \) where \( \text{deg} q_{j}^{(n)} = j - 1 \) and form the sequence of Christoffel functions \( K_{n}(z) := \sum_{j=1}^{n} |q_{j}^{(n)}(z)|^{2} \). In [8] we showed that

\[
(1.1) \quad \frac{1}{n}K_{n}(z)w(z)^{2n}d\mu(z) \rightarrow d_{\mu_{\text{eq}}}(z)
\]

weak-* where \( \mu_{\text{eq}} \) is the potential-theoretic weighted equilibrium measure. The key ingredients to proving (1.1) are, firstly, the verification that

\[
(1.2) \quad \lim_{n \to \infty} Z_{n}^{1/n^{2}} = \delta^w(E)
\]

where

\[
(1.3) \quad Z_{n} = Z_{n}(E, w, \mu) := \int_{E^n} |VDM(\lambda_1, ..., \lambda_n)|^{2} w(\lambda_1)^{2n} \cdots w(\lambda_n)^{2n} d\mu(\lambda_1) \cdots d\mu(\lambda_n);
\]

and, secondly, a “large deviation” result in the spirit of Johansson [16]. We generalize these two results to \( \mathbb{C}^N, N > 1 \) (Theorems 3.1 and 3.2). The methods are similar to the corresponding one variable methods and were announced in [8], Remark 3.1. In particular, \( \delta^w(E) \) is interpreted as the transfinite diameter of a circled set in one higher dimension. We also discuss the case where \( E = \Gamma \) is an unbounded cone in \( \mathbb{R}^N \) for special weights and measures. Some of our results were proved independently by Berman-Boucksom [2].

We end the paper with a short section which includes questions related to these topics. We are grateful to Robert Berman for making reference [2]
available to us. The second author would also like to thank Sione Ma’u and Laura DeMarco for helpful conversations.

2. Transfinite diameter notions in $\mathbb{C}^N$.

We begin by considering a function $Y$ from the set of multiindices $\alpha \in \mathbb{N}^N$ to the nonnegative real numbers satisfying:

\[(2.1) \quad Y(\alpha + \beta) \leq Y(\alpha) \cdot Y(\beta) \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^N.\]

We call a function $Y$ satisfying (2.1) submultiplicative; we have three main examples below. Let $e_1(z), \ldots, e_j(z), \ldots$ be a listing of the monomials \(\{e_i(z) = z^{\alpha(i)} = z_1^{\alpha_1} \cdots z_N^{\alpha_N}\}\) in $\mathbb{C}^N$ indexed using a lexicographic ordering on the multiindices $\alpha = \alpha(i) = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, but with $\text{deg}e_i = |\alpha(i)|$ nondecreasing. We write $|\alpha| := \sum_{j=1}^N \alpha_j$.

We define the following integers:

1. \(m_d^{(N)} = m_d := \text{the number of monomials } e_i(z) \text{ of degree at most } d \text{ in } N \text{ variables};\)
2. \(h_d^{(N)} = h_d := \text{the number of monomials } e_i(z) \text{ of degree exactly } d \text{ in } N \text{ variables};\)
3. \(l_d^{(N)} = l_d := \text{the sum of the degrees of the } m_d \text{ monomials } e_i(z) \text{ of degree at most } d \text{ in } N \text{ variables}.\)

We have the following relations:

\[(2.2) \quad m_d^{(N)} = \binom{N+d}{d}; \quad h_d^{(N)} = m_d^{(N)} - m_{d-1}^{(N)} = \binom{N+1+d}{d};\]

and

\[(2.3) \quad h_d^{(N+1)} = \binom{N+d}{d} = m_d^{(N)}; \quad l_d^{(N)} = N \binom{N+d}{N+1} = \frac{N}{N+1} \cdot dm_d^{(N)}.\]

The elementary fact that the dimension of the space of homogeneous polynomials of degree $d$ in $N+1$ variables equals the dimension of the space of polynomials of degree at most $d$ in $N$ variables will be crucial in sections 4 and 5. Finally, we let

\[r_d^{(N)} = r_d := dh_d^{(N)} = d(m_d^{(N)} - m_{d-1}^{(N)})\]

which is the sum of the degrees of the $h_d$ monomials $e_i(z)$ of degree exactly $d$ in $N$ variables. We observe that

\[(2.4) \quad l_d^{(N)} = \sum_{k=1}^d r_k^{(N)} = \sum_{k=1}^N k h_k^{(N)}.\]
Let $K \subset \mathbb{C}^N$ be compact. Here are our three natural constructions associated to $K$:

1. **Chebyshev constants**: Define the class of polynomials
   \[ P_i = P(\alpha(i)) := \{e_\alpha(z) + \sum_{j<i} c_j e_j(z)\}; \]
   and the Chebyshev constants
   \[ Y_1(\alpha) := \inf \{||p||_K : p \in P_i\}. \]
   We write $t_{\alpha,K} := t_{\alpha(\cdot),K}$ for a Chebyshev polynomial; i.e., $t_{\alpha,K} \in P(\alpha(i))$ and $||t_{\alpha,K}||_K = Y_1(\alpha)$.

2. **Homogeneous Chebyshev constants**: Define the class of homogeneous polynomials
   \[ P_i^{(H)} = P^{(H)}(\alpha(i)) := \{e_\alpha(z) + \sum_{j<i, \deg(e_j) = \deg(e_i)} c_j e_j(z)\}; \]
   and the homogeneous Chebyshev constants
   \[ Y_2(\alpha) := \inf \{||p||_K : p \in P_i^{(H)}\}. \]
   We write $t_{\alpha,K}^{(H)} := t_{\alpha(\cdot),K}^{(H)}$ for a homogeneous Chebyshev polynomial; i.e., $t_{\alpha,K}^{(H)} \in P^{(H)}(\alpha(i))$ and $||t_{\alpha,K}^{(H)}||_K = Y_2(\alpha)$.

3. **Weighted Chebyshev constants**: Let $w$ be an admissible weight function on $K$ (see below) and let
   \[ Y_3(\alpha) := \inf \{||w|^{\alpha(i)} p||_K := \sup_{z \in K} \{w(z)|^{\alpha(i)}| p(z)\} : p \in P_i\} \]
   be the weighted Chebyshev constants. Note we use the polynomial classes $P_i$ as in (1). We write $t_{\alpha,K}^w$ for a weighted Chebyshev polynomial; i.e., $t_{\alpha,K}^w$ is of the form $w^{\alpha(i)} p$ with $p \in P(\alpha(i))$ and $||t_{\alpha,K}^w||_K = Y_3(\alpha)$.

Let $\Sigma$ denote the standard $(N-1)$–simplex in $\mathbb{R}^N$; i.e.,
\[ \Sigma = \{\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N : \sum_{j=1}^N \theta_j = 1, \ \theta_j \geq 0, \ j = 1, \ldots, N\}, \]
and let
\[ \Sigma^0 := \{\theta \in \Sigma : \theta_j > 0, \ j = 1, \ldots, N\}. \]
Given a submultiplicative function $Y(\alpha)$, define, as with the above examples, a new function
\[ \tau(\alpha) := Y(\alpha)^{1/|\alpha|}. \]
An examination of lemmas 1, 2, 3, 5, and 6 in [23] shows that (2.1) is the only property of the numbers \( Y(\alpha) \) needed to establish those lemmas. That is, we have the following results for \( Y : \mathbb{N}^N \rightarrow \mathbb{R}^+ \) satisfying (2.1) and the associated function \( \tau(\alpha) \) in (2.5):

**Lemma 2.1.** For all \( \theta \in \Sigma^0 \), the limit
\[
T(Y, \theta) := \lim_{\alpha/|\alpha| \to \theta} Y(\alpha)^{1/|\alpha|} = \lim_{\alpha/|\alpha| \to \theta} \tau(\alpha)
\]
exists.

**Lemma 2.2.** The function \( \theta \to T(Y, \theta) \) is log-convex on \( \Sigma^0 \) (and hence continuous).

**Lemma 2.3.** Given \( b \in \partial \Sigma \),
\[
\liminf_{\theta \to b, \theta \in \Sigma^0} T(Y, \theta) = \liminf_{i \to \infty, \alpha(i)/|\alpha(i)| \to b} \tau(\alpha(i)).
\]

**Lemma 2.4.** Let \( \theta(k) := \alpha(k)/|\alpha(k)| \) for \( k = 1, 2, \ldots \) and let \( Q \) be a compact subset of \( \Sigma^0 \). Then
\[
\limsup_{|\alpha| \to \infty} \{ \log \tau(\alpha(k)) - \log T(Y(\theta(k))) : |\alpha(k)| = \alpha, \theta(k) \in Q \} = 0.
\]

**Lemma 2.5.** Define
\[
\tau(Y) := \exp\left[ -\frac{1}{\text{meas}(\Sigma)} \int_{\Sigma} \log T(Y, \theta) d\theta \right]
\]
Then
\[
\lim_{d \to \infty} \frac{1}{h_d} \sum_{|\alpha| = d} \log \tau(\alpha) = \log \tau(Y);
\]
i.e., using (2.5),
\[
\lim_{d \to \infty} \left[ \prod_{|\alpha| = d} Y(\alpha) \right]^{1/dh_d} = \tau(Y).
\]

One can incorporate all of the \( Y(\alpha)'s \) for \( |\alpha| \leq d \); this is the content of the next result.

**Theorem 2.6.** We have
\[
\lim_{d \to \infty} \left[ \prod_{|\alpha| \leq d} Y(\alpha) \right]^{1/d} \text{ exists and equals } \tau(Y).
\]

**Proof.** Define the geometric means
\[
\tau_d^0 := \left( \prod_{|\alpha| = d} \tau(\alpha) \right)^{1/h_d}, \quad d = 1, 2, \ldots
\]
The sequence
\[ \log \tau_0^0, \log \tau_1^0, \ldots (r_1 \text{ times}), \ldots, \log \tau_d^0, \log \tau_d^0, \ldots (r_d \text{ times}), \ldots \]
converges to \( \log \tau(Y) \) by the previous lemma; hence the arithmetic mean of the first \( l_d = \sum_{k=1}^d r_k \) terms (see (2.4)) converges to \( \log \tau(Y) \) as well. Exponentiating this arithmetic mean gives
\[ (2.6) \quad \left( \prod_{k=1}^d (\tau_k^0)^{r_k} \right)^{1/l_d} = \left( \prod_{k=1}^d \tau(\alpha)^k \right)^{1/l_d} = \left( \prod_{|\alpha| \leq d} Y(\alpha) \right)^{1/l_d} \]
and the result follows. \( \square \)

Returning to our examples (1)-(3), example (1) was the original setting of Zaharjuta [23] which he utilized to prove the existence of the limit in the definition of the transfinite diameter of a compact set \( K \subset \mathbb{C}^N \). For \( \zeta_1, \ldots, \zeta_n \in \mathbb{C}^N \), let
\[ (2.7) \quad VDM(\zeta_1, \ldots, \zeta_n) = \det[e_i(\zeta_j)], i,j=1,\ldots,n \]
\[ = \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \cdots & e_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_n(\zeta_1) & e_n(\zeta_2) & \cdots & e_n(\zeta_n) \end{bmatrix} \]
and for a compact subset \( K \subset \mathbb{C}^N \) let
\[ V_n = V_n(K) := \max_{\zeta_1,\ldots,\zeta_n \in K} |VDM(\zeta_1, \ldots, \zeta_n)|. \]

Then
\[ (2.8) \quad d(K) = \lim_{d \to \infty} V_n^{1/l_d} \]
is the transfinite diameter of \( K \); Zaharjuta [23] showed that the limit exists by showing that one has
\[ (2.9) \quad d(K) = \exp\left[\frac{1}{\text{meas}(\Sigma)} \int_{\Sigma^0} \log \tau(K,\theta)d\theta\right] \]
where \( \tau(K,\theta) = T(Y_1,\theta) \) from (1); i.e., the right-hand-side of (2.9) is \( \tau(Y_1) \).

This follows from Theorem 2.6 for \( Y = Y_1 \) and the estimate
\[ \left( \prod_{k=1}^d (\tau_k^0)^{r_k} \right)^{1/l_d} \leq V_{m_d}^{1/l_d} \leq (m_d!)^{1/l_d} \left( \prod_{k=1}^d (\tau_k^0)^{r_k} \right)^{1/l_d} \]
in [23] (compare (2.6)).

For a compact circled set \( K \subset \mathbb{C}^N \); i.e., \( z \in K \) if and only if \( e^{i\phi}z \in K \), \( \phi \in [0,2\pi] \), one need only consider homogeneous polynomials in the definition of
the directional Chebyshev constants $\tau(K, \theta)$. In other words, in the notation of (1) and (2), $Y_1(\alpha) = Y_2(\alpha)$ for all $\alpha$ so that

$$T(Y_1, \theta) = T(Y_2, \theta)$$

for circled sets $K$.

This is because for such a set, if we write a polynomial $p$ of degree $d$ as $p = \sum_{j=0}^{d} H_j$ where $H_j$ is a homogeneous polynomial of degree $j$, then, from the Cauchy integral formula, $||H_j||_K \leq ||p||_K$, $j = 0, \ldots, d$. Moreover, a slight modification of Zaharjuta’s arguments proves the existence of the limit of appropriate roots of maximal homogeneous Vandermonde determinants; i.e., the homogeneous transfinite diameter $d^{(H)}(K)$ of a compact set (cf., [15]). From the above remarks, it follows that

(2.10) for circled sets $K$, $d(K) = d^{(H)}(K)$.

Since we will be using the homogeneous transfinite diameter, we amplify the discussion. We relabel the standard basis monomials $\{e_i^{(H,d)}(z) = z^{\alpha(i)}\}$ where $|\alpha(i)| = d$, $i = 1, \ldots, h_d$, we define the $d$–homogeneous Vandermonde determinant

(2.11) $VDMH_d(\zeta_1, \ldots, \zeta_{h_d}) := \det [e_i^{(H,d)}(\zeta_j)]_{i,j=1,\ldots,h_d}$

Then

(2.12) $d^{(H)}(K) = \lim_{d \to \infty} \left[ \max_{\zeta_1, \ldots, \zeta_{h_d} \in K} |VDMH_d(\zeta_1, \ldots, \zeta_{h_d})| \right]^{1/dh_d}$

is the homogeneous transfinite diameter of $K$; the limit exists and equals

$$\exp\left[ \frac{1}{\text{meas}(\Sigma)} \int_{\Sigma^0} \log T(Y_2, \theta) d\theta \right]$$

where $T(Y_2, \theta)$ comes from (2).

Finally, related to example (3), there are similar properties for the weighted version of directional Chebyshev constants and transfinite diameter. To define weighted notions, let $K \subset \mathbb{C}^N$ be closed and let $w$ be an admissible weight function on $K$; i.e., $w$ is a nonnegative, usc function with $\{z \in K : w(z) > 0\}$. Let $Q := -\log w$ and define the weighted pluricomplex Green function $V_{K,Q}^*(z) := \limsup_{\zeta \to z} V_{K,Q}(\zeta)$ where

$$V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}^N), u \leq Q \text{ on } K\}$$

Here, $L(\mathbb{C}^N)$ is the set of all plurisubharmonic functions $u$ on $\mathbb{C}^N$ with the property that $u(z) - \log |z| = o(1)$, $|z| \to \infty$. If $K$ is closed but not necessarily bounded, we require that $w$ satisfies the growth property

(2.13) $|z| w(z) \to 0$ as $|z| \to \infty$, $z \in K$,

so that $V_{K,Q}$ is well-defined and equals $V_{K \cap B_R,Q}$ for $R > 0$ sufficiently large where $B_R = \{z : |z| \leq R\}$ (Definition 2.1 and Lemma 2.2 of Appendix B in
The unweighted case is when \( w \equiv 1 \) (\( Q \equiv 0 \)); we then write \( V_K \) for the pluricomplex Green function. The set \( K \) is called regular if \( V_K = V^*_K \); i.e., \( V_K \) is continuous; and \( K \) is locally regular if for each \( z \in K \), the sets \( K \cap B(z,r) \) are regular for \( r > 0 \) where \( B(z,r) \) denotes the ball of radius \( r \) centered at \( z \). We define the weighted transfinite diameter
\[
d^w(K) := \exp\left[ \frac{1}{\text{meas}(\Sigma)} \int_{\Sigma^0} \log \tau^w(K,\theta) d\theta \right]
\]
as in [9] where \( \tau^w(K,\theta) = T(Y,3,\theta) \) from (3); i.e., the right-hand-side of this equation is the quantity \( \tau(Y,3) \).

We remark for future use that if \( \{K_j\} \) is a decreasing sequence of locally regular compacta with \( K_j \downarrow K \), and if \( w_j \) is a continuous admissible weight function on \( K_j \) with \( w_j \downarrow w \) on \( K \) where \( w \) is an admissible weight function on \( K \), then the argument in Proposition 7.5 of [9] shows that
\[
\lim_{j \to \infty} \tau^w_j(K_j,\theta) = \tau^w(K,\theta) \quad \text{for all} \quad \theta \in \Sigma^0
\]
and hence
\[
\lim_{j \to \infty} d^{w_j}(K_j) = d^w(K).
\]
In particular, (2.14) holds in the unweighted case (\( w \equiv 1 \)) for any decreasing sequence \( \{K_j\} \) of compacta with \( K_j \downarrow K \); i.e.,
\[
\lim_{j \to \infty} d(K_j) = d(K)
\]
(cf., [9] equation (1.13)).

Another natural definition of a weighted transfinite diameter uses weighted Vandermonde determinants. Let \( K \subset \mathbb{C}^N \) be compact and let \( w \) be an admissible weight function on \( K \). Given \( \zeta_1, \ldots, \zeta_n \in K \), let
\[
W(\zeta_1, \ldots, \zeta_n) := \det M(\zeta_1, \ldots, \zeta_n) w(\zeta_1)^{|\alpha(n)|} \cdots w(\zeta_n)^{|\alpha(n)|}
\]
be a weighted Vandermonde determinant. Let
\[
W_n := \max_{\zeta_1, \ldots, \zeta_n \in K} |W(\zeta_1, \ldots, \zeta_n)|
\]
and define an \( n \)-th weighted Fekete set for \( K \) and \( w \) to be a set of \( n \) points \( \zeta_1, \ldots, \zeta_n \in K \) with the property that
\[
|W(\zeta_1, \ldots, \zeta_n)| = \sup_{\xi_1, \ldots, \xi_n \in K} |W(\xi_1, \ldots, \xi_n)|.
\]
Also, define
\[
\delta^w(K) := \lim_{d \to \infty} \frac{W_n^{1/l_d}}{n^d}.
\]
We will show in Proposition 2.1 that \( \lim_{d \to \infty} W_{m_d}^{1/d} \) (the weighted analogue of (2.8)) exists. The question of the existence of this limit if \( N > 1 \) was raised in [9]. Moreover, using a recent result of Rumely, we show how \( \delta_w(K) \) is related to \( d_w(K) \):

\[
\delta_w(K) = \left[ \exp \left( -\int_K Q(dd^*V_{K,Q}^*)^N \right) \right]^{1/N} \cdot d_w(K)
\]

where \( (dd^*V_{K,Q}^*)^N \) is the complex Monge-Ampere operator applied to \( V_{K,Q}^* \).

We refer the reader to [17] or Appendix B of [20] for more on the complex Monge-Ampere operator.

We begin by proving the existence of the limit in the definition of \( \delta_w(E) \) in (2.1) for a set \( E \subset \mathbb{C}^N \) and an admissible weight \( w \) on \( E \) (see also [2]).

**Proposition 2.1.** Let \( E \subset \mathbb{C}^N \) be a compact set with an admissible weight function \( w \). The limit

\[
\delta_w(E) := \lim_{d \to \infty} \left[ \max_{\lambda \in E} \left| VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d^{(N)})}) \right| \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(m_d^{(N)})})^d \right]^{1/d(N)}
\]

exists.

**Proof.** Following [6], we define the circled set

\[
F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E, \ |t| = w(\lambda)\}.
\]

We first relate weighted Vandermonde determinants for \( E \) with homogeneous Vandermonde determinants for \( F \). To this end, for each positive integer \( d \), choose

\[
m_d^{(N)} = \binom{N + d}{d}
\]

(recall (2.2)) points \( \{(t_i, z^{(i)})\}_{i=1,\ldots,m_d^{(N)}} = \{(t_i, t_i\lambda^{(i)})\}_{i=1,\ldots,m_d^{(N)}} \) in \( F \) and form the \( d \)-homogeneous Vandermonde determinant

\[
V DMH_d((t_1, z^{(1)}), \ldots, (t_{m_d^{(N)}}, z^{(m_d^{(N)})})).
\]

We extend the lexicographical order of the monomials in \( \mathbb{C}^N \) to \( \mathbb{C}^{N+1} \) by letting \( t \) precede any of \( z_1, \ldots, z_N \). Writing the standard basis monomials of degree \( d \) in \( \mathbb{C}^{N+1} \) as

\[
\{t^{d-j}\epsilon_k^{(H_d)}(z) : j = 0, \ldots, d; \ k = 1, \ldots, h_j\};
\]

i.e., for each power \( d - j \) of \( t \), we multiply by the standard basis monomials of degree \( j \) in \( \mathbb{C}^N \), and dropping the superscript \( (N) \) in \( m_d^{(N)} \), we have the
Thus \( \lambda \) is a homogeneous Vandermonde matrix

\[
\begin{bmatrix}
    t^d_1 & t^d_2 & \cdots & t^d_{md} \\
    t^{d-1}_1 z^{(1)} & t^{d-1}_2 z^{(2)} & \cdots & t^{d-1}_{md} z^{(md)} \\
    \vdots & \vdots & \ddots & \vdots \\
    e_{md}(z^{(1)}) & e_{md}(z^{(2)}) & \cdots & e_{md}(z^{(md)})
\end{bmatrix}
\]

Factoring \( t_i^d \) out of the \( i \)-th column, we obtain

\[
VDMH_d((t_1, z^{(1)}), \ldots, (t_{md}, z^{(md)})) = t^d_1 \cdots t^d_{md} \cdot VDM(\lambda^{(1)}, \ldots, \lambda^{(md)});
\]

thus, writing \( |A| := |\det A| \) for a square matrix \( A \),

\[
(2.19)
\]

\[
= |t_1|^d \cdots |t_{md}|^d
\]

\[
\begin{bmatrix}
    1 & 1 & \cdots & 1 \\
    \lambda^{(1)} & \lambda^{(2)} & \cdots & \lambda^{(md)} \\
    \vdots & \vdots & \ddots & \vdots \\
    (\lambda^{(1)}_N)^d & (\lambda^{(2)}_N)^d & \cdots & (\lambda^{(md)}_N)^d
\end{bmatrix}
\]

where \( \lambda^{(j)}_k = z^{(j)}_k / t_j \) provided \( t_j \neq 0 \). By definition of \( F \), since \((t_i, z^{(i)}) = (t_i, t_i \lambda^{(i)}) \in F\), we have \( |t_i| = w(\lambda^{(i)}) \) so that from (2.19)

\[
VDMH_d((t_1, z^{(1)}), \ldots, (t_{md}, z^{(md)})) = VDM(\lambda^{(1)}, \ldots, \lambda^{(md)}) \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(md)})^d.
\]

Thus

\[
\max_{(t_i, z^{(i)}) \in F} |VDMH_d((t_1, z^{(1)}), \ldots, (t_{md}, z^{(md)}))| =
\]

\[
\max_{\lambda^{(i)} \in E} |VDM(\lambda^{(1)}, \ldots, \lambda^{(md)})| \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(md)})^d.
\]

Note that the maximum will occur when all \( t_j = w(\lambda^{(j)}) > 0 \). As mentioned in section [3] the limit

\[
\lim_{d \to \infty} \left[ \max_{(t_i, z^{(i)}) \in F} |VDMH_d((t_1, z^{(1)}), \ldots, (t_{md}, z^{(md)}))| \right]^{1/d_{\lambda}^{(N+1)}} = d^{(H)}(F)
\]
exists \cite{15}; thus the limit

$$\lim_{d \to \infty} \left[ \max_{\lambda \in E} |VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d)})| \cdot w(\lambda^{(1)})^{d} \cdots w(\lambda^{(m_d)})^{d} \right]^{1/l_d^{(N)}} := \delta^w(E)$$

exists.

\textbf{Corollary 2.1.} For $E \subset \mathbb{C}^N$ a nonpluripolar compact set with an admissible weight function $w$ and

$$F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E, |t| = w(\lambda)\},$$

(2.20)

$$\delta^w(E) = d^{H(F)} \left( F \right)^{\frac{N+1}{N}} = d(F)^{\frac{N+1}{N}}.$$

\textbf{Proof.} The first equality follows from the proof of Proposition 2.1 using the relation

$$l_d^{(N)} = \left( \frac{N}{N+1} \right) \cdot dh_d^{(N+1)}$$

(see (2.3)). The second equality is (2.10). \hfill \square

We next relate $\delta^w(E)$ and $d^w(E)$ but we first recall the remarkable formula of Rumely \cite{19}. For a plurisubharmonic function $u$ in $L(\mathbb{C}^N)$ we can define the \textit{Robin function} associated to $u$:

$$\rho_u(z) := \limsup_{|\lambda| \to \infty} \left[ u(\lambda z) - \log(|\lambda|) \right].$$

This function is plurisubharmonic (cf., \cite{5}, Proposition 2.1) and logarithmically homogeneous:

$$\rho_u(tz) = \rho_u(z) + \log |t| \text{ for } t \in \mathbb{C}.$$

For $u = V_{E,Q}^*$ ($V_E^*$) we write $\rho_u = \rho_{E,Q} (\rho_E)$. Rumely’s formula relates $\rho_E$ and $d(E)$:

$$\begin{align*}
- \log d(E) &= \frac{1}{N} \int_{\mathbb{C}^{N-1}} \rho_E(1, t_2, \ldots, t_N)(dd^c \rho_E(1, t_2, \ldots, t_N))^{N-1} \\
&\quad + \int_{\mathbb{C}^{N-2}} \rho_E(0, 1, t_3, \ldots, t_N)(dd^c \rho_E(0, 1, t_3, \ldots, t_N))^{N-2} \\
&\quad + \cdots + \int_{\mathbb{C}} \rho_E(0, \ldots, 0, 1, t_N)(dd^c \rho_E(0, \ldots, 0, 1, t_N) + \rho_E(0, \ldots, 0, 1)).
\end{align*}$$

(2.21)

Here we make the convention that $dd^c = \frac{1}{2\pi} (2i\partial \bar{\partial})$ so that in any dimension $d = 1, 2, \ldots$,

$$\int_{\mathbb{C}^d} (dd^c u)^d = 1$$

for any $u \in L^+(\mathbb{C}^d)$; i.e., for any plurisubharmonic function $u$ in $\mathbb{C}^d$ which satisfies

$$C_1 + \log(1 + |z|) \leq u(z) \leq C_2 + \log(1 + |z|)$$

for some constants $C_1, C_2$. \hfill \square
for some \( C_1, C_2 \).

We begin by rewriting (2.21) for regular circled sets \( E \) using an observation of Sione Ma'u. Note that for such sets, \( V_E^* = \rho_E^+ := \max(\rho_E, 0) \). If we intersect \( E \) with a hyperplane \( H \) through the origin, e.g., by rotating coordinates, we take \( H = \{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N : z_1 = 0 \} \), then \( E \cap H \) is a regular, compact, circled set in \( \mathbb{C}^{N-1} \) (which we identify with \( H \)). Moreover, we have

\[
\rho_{H \cap E}(z_2, \ldots, z_N) = \rho_E(0, z_2, \ldots, z_N)
\]

since each side is logarithmically homogeneous and vanishes for \((z_2, \ldots, z_N) \in \partial(H \cap E)\). Thus the terms

\[
\int_{\mathbb{C}^{N-2}} \rho_E(0, 1, t_3, \ldots, t_N)(dd^c \rho_E(0, 1, t_3, \ldots, t_N))^{N-2}
\]

\[
+ \cdots + \int_{\mathbb{C}} \rho_E(0, 0, 1, t_N)(dd^c \rho_E(0, 0, 1, t_N)) + \rho_E(0, 0, 1)
\]

in (2.21) are seen to equal

\[
(N - 1)d^{\mathbb{C}^{N-1}}(H \cap E)
\]

(where we temporarily write \( d^{\mathbb{C}^{N-1}} \) to denote the transfinite diameter in \( \mathbb{C}^{N-1} \) for emphasis) by applying (2.21) in \( \mathbb{C}^{N-1} \) to the set \( H \cap E \). Hence we have

\[
-\log d(E) = \frac{1}{N} \int_{\mathbb{C}^{N-1}} \rho_E(1, t_2, \ldots, t_N)(dd^c \rho_E(1, t_2, \ldots, t_N))^{N-1}
\]

\[
+ (N - 1)\left[ -\log d^{\mathbb{C}^{N-1}}(H \cap E) \right].
\]

**Theorem 2.7.** For \( E \subset \mathbb{C}^N \) a nonpluripolar compact set with an admissible weight function \( w \),

\[
\delta^w(E) = \left[ \exp \left( -\int_E Q(dd^c V_{E,Q}^*)^N \right) \right]^{1/N} \cdot w(E).
\]

**Proof.** We first assume that \( E \) is locally regular and \( Q \) is continuous. It is known in this case that \( V_{E,Q} = V_{E,Q}^* \) (cf., [21], Proposition 2.16). As before, we define the circled set

\[
F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E, \ |t| = w(\lambda) \}.
\]

We claim this is a regular set; i.e., \( V_F \) is continuous. First of all, \( V_F^*(t, z) = \max[\rho_F(t, z), 0] \) (cf., Proposition 2.2 of [6]) so that it suffices to verify that \( \rho_F(t, z) \) is continuous. From Theorem 2.1 and Corollary 2.1 of [6],

\[
V_{E,Q}(\lambda) = \rho_F(1, \lambda) \text{ on } \mathbb{C}^N.
\]
which implies, by the logarithmic homogeneity of \( \rho_F \), that \( \rho_F(t, z) \) is continuous on \( \mathbb{C}^{N+1} \setminus \{ t = 0 \} \). Corollary 2.1 and equation (2.8) in [6] give that
\[
\rho_F(0, \lambda) = \rho_{E,Q}(\lambda) \quad \text{for } \lambda \in \mathbb{C}^N
\]
and \( \rho_{E,Q} \) is continuous by Theorem 2.5 of [9]. Moreover, the limit exists in the definition of \( \rho_{E,Q} \):
\[
\rho_{E,Q}(\lambda) := \limsup_{|t| \to \infty} [V_{E,Q}(t\lambda) - \log |t|] = \lim_{|t| \to \infty} [V_{E,Q}(t\lambda) - \log |t|];
\]
and the limit is uniform in \( \lambda \) (cf., Corollary 4.4 of [11]) which implies, from (2.24) and (2.25), that \( \lim_{t \to 0} \rho_F(t, \lambda) = \rho_F(0, \lambda) \) so that \( \rho_F(t, z) \) is continuous. In particular,
\[
V_{E,Q}(\lambda) = Q(\lambda) = \rho_F(1, \lambda) \quad \text{on the support of } (dd^c V_{E,Q})^N
\]
so that
\[
\int_E Q(\lambda)(dd^c V_{E,Q}(\lambda))^N = \int_{\mathbb{C}^N} \rho_F(1, \lambda)(dd^c \rho_F(1, \lambda))^N.
\]
On the other hand, \( E^w_\rho := \{ \lambda \in \mathbb{C}^N : \rho_{E,Q}(\lambda) \leq 0 \} \) is a circled set, and, according to eqn. (3.14) in [9], \( d^w(E) = d(E^w_\rho) \). But
\[
\rho_{E,Q}(\lambda) = \limsup_{|t| \to \infty} [V_{E,Q}(t\lambda) - \log |t|]
\]
\[
= \limsup_{|t| \to \infty} [\rho_F(1, t\lambda) - \log |t|] = \limsup_{|t| \to \infty} \rho_F(1/t, \lambda) = \rho_F(0, \lambda).
\]
Thus
\[
E^w_\rho = \{ \lambda \in \mathbb{C}^N : \rho_F(0, \lambda) \leq 0 \} = F \cap \mathcal{H}
\]
where \( \mathcal{H} = \{ (t, z) \in \mathbb{C}^{N+1} : t = 0 \} \) and hence
\[
(2.27) \quad d^w(E) = d(E^w_\rho) = d(F \cap \mathcal{H}).
\]
From (2.22) applied to \( F \subset \mathbb{C}^{N+1} \),
\[
(2.28) \quad - \log d(F) = \frac{1}{N+1} \int_{\mathbb{C}^N} \rho_F(1, \lambda)(dd^c \rho_F(1, \lambda))^N
\]
\[
+ \left( \frac{N}{N+1} \right) [- \log d(F \cap \mathcal{H})].
\]
Finally, from (2.20),
\[
(2.29) \quad \delta^w(E) = d(F)^{\frac{N+1}{N}};
\]
putting together (2.26), (2.27), (2.28) and (2.29) gives the result if \( E \) is locally regular and \( Q \) is continuous.

The general case follows from approximation. Take a sequence of locally regular compacta \( \{ E_j \} \) decreasing to \( E \) and a sequence of weight functions
\{w_j\}$ with $w_j$ continuous and admissible on $E_j$ and $w_j \downarrow w$ on $E$ (cf., Lemma 2.3 of [6]). From (2.14) we have
\begin{equation}
\lim_{j \to \infty} d^{w_j}(E_j) = d^w(E).
\end{equation}
Also, by Corollary 2.1 we have
\begin{equation}
\delta^{w_j}(E_j) = d(F_j) \frac{N+1}{N},
\end{equation}
where
\begin{equation*}
F_j = F_j(E_j, w_j) = \{(t(1, \lambda) : \lambda \in E_j, |t| = w_j(\lambda)\}.
\end{equation*}
Since $E_{j+1} \subset E_j$ and $w_{j+1} \leq w_j$, the sets
\begin{equation*}
\tilde{F}_j = \tilde{F}_j(E_j, w_j) = \{(t(1, \lambda) : \lambda \in E_j, |t| \leq w_j(\lambda)\}
\end{equation*}
satisfy $\tilde{F}_{j+1} \subset \tilde{F}_j$ and hence
\begin{equation*}
d(\tilde{F}_{j+1}) = d(F_{j+1}) \leq d(\tilde{F}_j) = d(F_j).
\end{equation*}
Since $F_j \downarrow F$, we conclude from (2.15) and (2.31) that
\begin{equation}
\lim_{j \to \infty} \delta^{w_j}(E_j) = \delta^w(E).
\end{equation}
Applying (2.23) to $E_j, w_j, Q_j$ and using (2.30) and (2.32), we conclude that
\begin{equation*}
\int_{E_j} Q_j(dd^c V_{E_j, Q_j})^N \to \int_E Q(dd^c V_{E, Q})^N,
\end{equation*}
completing the proof of (2.23). \hfill \Box

3. Integrals of Vandermonde determinants.

In this section, we first state and prove the analogue of an “unweighted” generalization to $\mathbb{C}^N$ of Theorem 2.1 of [8] as it has a self-contained proof. We first recall some terminology. Given a compact set $E \subset \mathbb{C}^N$ and a measure $\nu$ on $E$, we say that $(E, \nu)$ satisfies the Bernstein-Markov inequality for holomorphic polynomials in $\mathbb{C}^N$ if, given $\epsilon > 0$, there exists a constant $M = M(\epsilon)$ such that for all such polynomials $Q_n$ of degree at most $n$
\begin{equation}
||Q_n||_E \leq M(1 + \epsilon)^n ||Q_n||_{L^2(\nu)}.
\end{equation}

**Theorem 3.1.** Let $(E, \mu)$ satisfy a Bernstein-Markov inequality. Then
\begin{equation}
\lim_{d \to \infty} Z_d^{1/2l(N)} = d(E)
\end{equation}
where
\begin{equation}
Z_d = Z_d(E, \mu) :=
\end{equation}
\[
\int_{E^m(N)} |VDM(\lambda(1), ..., \lambda^{m(N)}_d)|^2 d\mu(\lambda(1)) \cdots d\mu(\lambda^{m(N)}_d).
\]

Proof. Since \( VDM(\zeta_1, ..., \zeta_n) = \det[e_i(\zeta_j)]_{i,j=1,...,n} \) for any positive integer \( n \), if we apply the Gram-Schmidt procedure to the monomials \( e_1, ..., e^{m(N)}_d \) to obtain orthogonal polynomials \( q_1, ..., q^{m(N)}_d \) with respect to \( \mu \) where \( q_j \in P_j \) has minimal \( L^2(\mu) \)-norm among all such polynomials, we get, upon using elementary row operations on \( VDM(\zeta_1, ..., \zeta^{m(N)}_d) \) and expanding the determinant (cf., [14] Chapter 5 or section 2 of [8])

(3.3)

\[
\int_{E^m(N)} |VDM(\zeta_1, ..., \zeta^{m(N)}_d)|^2 d\mu(\zeta_1) \cdots d\mu(\zeta^{m(N)}_d) = m^{(N)}_d \prod_{j=1}^{m^{(N)}_d} ||q_j||_{L^2(\mu)}^2.
\]

Let \( t_{\alpha,E} \in P(\alpha) \) be a Chebyshev polynomial; i.e., \( ||t_{\alpha,E}||_E = Y_1(\alpha) \). Then Theorem 2.6 shows that

\[
\lim_{d \to \infty} \left( \prod_{|\alpha| \leq d} ||t_{\alpha,E}||_E \right)^{1/l_d} = \tau(Y_1)
\]

since

\[
\lim_{d \to \infty} \left( m^{(N)}_d \right)^{1/l_d} = 1.
\]

Zaharjuta’s theorem (2.9) shows that \( \tau(Y_1) = d(E) \) so we need show that

(3.4)

\[
\lim_{d \to \infty} \left( \prod_{|\alpha| \leq d} ||t_{\alpha,E}||_E \right)^{1/l_d} = \lim_{d \to \infty} \left( \prod_{|\alpha| \leq d} ||q_\alpha||_{L^2(\mu)} \right)^{1/l_d}.
\]

This follows from the Bernstein-Markov property. First note that

\[
||q_\alpha||_{L^2(\mu)} \leq ||t_{\alpha,E}||_{L^2(\mu)} \leq \mu(E) \cdot ||t_{\alpha,E}||_E
\]

from the \( L^2(\mu) \)-norm minimality of \( q_\alpha \); then, given \( \epsilon > 0 \), the Bernstein-Markov property and the sup-norm minimality of \( t_{\alpha,E} \) give

\[
||t_{\alpha,E}||_E \leq ||q_\alpha||_E \leq M(1 + \epsilon)^{|\alpha|} ||q_\alpha||_{L^2(\mu)}
\]

for some \( M = M(\epsilon) > 0 \). Taking products of these inequalities over \( |\alpha| \leq d \); taking \( l_d \)-th roots; and letting \( \epsilon \to 0 \) gives the result. This reasoning is adapted from the proof of Theorem 3.3 in [7]. \( \square \)

A weighted polynomial on \( E \) is a function of the form \( w(z)^n p_n(z) \) where \( p_n \) is a holomorphic polynomial of degree at most \( n \). Let \( \mu \) be a measure with support in \( E \) such that \( (E, w, \mu) \) satisfies a Bernstein-Markov inequality for weighted polynomials (referred to as a weighted B-M inequality in [6]):
given $\epsilon > 0$, there exists a constant $M = M(\epsilon)$ such that for all weighted polynomials $w^n p_n$

\[(3.5) \quad ||w^n p_n||_E \leq M(1 + \epsilon^n)||w^n p_n||_{L^2(\mu)}.
\]

Generalizing Theorem 3.1, we have the following result.

**Theorem 3.2.** Let $(E, w, \mu)$ satisfy a Bernstein-Markov inequality (3.5) for weighted polynomials. Then

\[
\lim_{d \to \infty} Z_d^{1/2d^{(N)}} = \delta^w(E)
\]

where

\[(3.6) \quad Z_d = Z_d(E, w, \mu) := \int_{E^{m_d^{(N)}}} |VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d^{(N)})})|^2 \times \]

\[w(\lambda^{(1)})^{2d} \ldots w(\lambda^{(m_d^{(N)})})^{2d} \mu(\lambda^{(1)}) \ldots \mu(\lambda^{(m_d^{(N)})}).\]

The proof of Theorem 3.2 follows along the lines of section 3 of [8]. Let $E \subset \mathbb{C}^N$ be a nonpluripolar compact set with an admissible weight function $w$ and let $\mu$ be a measure with support in $E$ such that $(E, w, \mu)$ satisfies a Bernstein-Markov inequality for weighted polynomials. The integrand

\[|VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d^{(N)})})|^2 \cdot w(\lambda^{(1)})^{2d} \ldots w(\lambda^{(m_d^{(N)})})^{2d}\]

in the definition of $Z_d$ in (3.6) has a maximal value on $E^{m_d^{(N)}}$ whose $1/2d^{(N)}$ root tends to $\delta^w(E)$. To show that the integrals themselves have the same property, we begin by constructing the circled set $F \subset \mathbb{C}^{N+1}$ defined as in section 4:

\[F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E, |t| = w(\lambda)\}.
\]

We construct a measure $\nu$ on $F$ associated to $\mu$ such that $(F, \nu)$ satisfies the Bernstein-Markov property for holomorphic polynomials in $\mathbb{C}^{N+1}$; i.e., (3.1) holds. Define

\[\nu := m_\lambda \otimes \mu, \lambda \in E\]

where $m_\lambda$ is normalized Lebesgue measure on the circle $|t| = w(\lambda)$ in the complex $t-$plane given by

\[C_\lambda := \{(t, t\lambda) \in \mathbb{C}^{N+1} : t \in \mathbb{C}\}.
\]

That is, if $\phi$ is continuous on $F$,

\[\int_F \phi(t, z)d\nu(t, z) = \int_E \left[ \int_{C_\lambda} \phi(t, t\lambda)dm_\lambda(t) \right]d\mu(\lambda).
\]

Equivalently, if $\pi : \mathbb{C}^{N+1} \to \mathbb{C}^N$ via $\pi(t, z) = z/t := \lambda$, then $\pi_*(\nu) = \mu$. The fact that $(F, \nu)$ satisfies the Bernstein-Markov property follows from
Theorem 3.1 of [6]. Moreover, if \( p_1(t, z) \) and \( p_2(t, z) \) are two homogeneous polynomials in \( \mathbb{C}^{N+1} \) of degree \( d \), say, and we write
\[
p_j(t, z) = p_j(t, t\lambda) = t^d p_j(1, \lambda) =: t^d G_j(\lambda), \quad j = 1, 2
\]
for univariate \( G_j \), then it is straightforward to see that
\[
\int_F p_1(t, z)p_2(t, z)dv(t, z) = \int_E G_1(\lambda)G_2(\lambda)w(\lambda)^{2d}d\mu(\lambda)
\]
(cf., [6], Lemma 3.1 and its proof). Note that if
\[
p(t, z) = t^d z^\alpha = t^d z_1^{\alpha_1} \cdots z_N^{\alpha_N}
\]
with \( |\alpha| = \alpha_1 + \cdots + \alpha_N = d - i \), then
\[
p(t, z) = t^d(z/t)^\alpha = t^d G(\lambda) = t^d \lambda_1^{\alpha_1} \cdots \lambda_N^{\alpha_N}
\]
where \( G(\lambda) = \lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_N^{\alpha_N} \).

**Proposition 3.1.** Let
\[
\tilde{Z}_d := \int_{F^{m_d(N)}} |VDM_H_d((t_1^{(1)}, \ldots, (t_{m_d(N)}, z(m_d(N))))|^2
d\nu(t_1^{(1)} \cdots d\nu(t_{m_d(N)}^{(1)}, z(m_d(N)))),
\]
Then \( \tilde{Z}_d = Z_d \) where \( m_d^{(N)} = \binom{N+d}{d} \) and (recall (3.6))
\[
Z_d = \int_{F^{m_d(N)}} |VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d(N))})|^2 \times
w(\lambda^{(1)})^{2d} \cdots w(\lambda^{(m_d(N))})^{2d} d\mu(\lambda^{(1)}) \cdots d\mu(\lambda^{(m_d(N))}).
\]

**Proof.** Recall from section 2 that the \( d \)--homogeneous Vandermonde determinant \( VDM_H_d((t_1^{(1)}, \ldots, (t_{m_d(N)}, z(m_d(N)))) \) equals
\[
\det \begin{bmatrix}
t_1^d & t_2^d & \cdots & t_{m_d(N)}^d \\
\vdots & \vdots & \ddots & \vdots \\
e_{m_d(N)}^{(1)}(z) & e_{m_d(N)}^{(2)}(z) & \cdots & e_{m_d(N)}^{(m_d(N)}(z)
\end{bmatrix}.
\]

Expanding this determinant in \( \tilde{Z}_d \) gives
\[
\tilde{Z}_d = \sum_{I, S} \sigma(I) \cdot \sigma(S) \left[ \int_{F_d^{d-deg(e_{s_1})}} t_1^{d-deg(e_{s_1})} e_{s_1}^{(1)}(z) d\nu(t_1, z(1)) \cdots \\
\cdots \int_{F_d^{m_d(N)-deg(e_{s_1})}} t_{m_d(N)}^{d-deg(e_{s_1})} e_{s_1}^{(m_d(N)}(z) d\nu(t_{m_d(N)}, z(m_d(N))) \right]
\]
where \( I = (i_1, \ldots, i_{m_d^{(N)}}) \) and \( S = (s_1, \ldots, s_{m_d^{(N)}}) \) are permutations of \( (1, \ldots, m_d^{(N)}) \) and \( \sigma(I) \) is the sign of \( I \) (+1 if \( I \) is even; −1 if \( I \) is odd). Expanding the ordinary Vandermonde determinant in \( Z \) and \( \sigma \)

\[
Z_d = \sum_{I, S} \sigma(I) \cdot \sigma(S) \left[ \int_E e_{i_1}(\lambda(1))e_{s_1}(\lambda(1))w(\lambda(1))^2d\mu(\lambda(1)) \cdots \right.
\]
\[
\cdots \left. \int_E e_{i_{m_d^{(N)}}}(\lambda(m_d^{(N)}))e_{s_{m_d^{(N)}}}(\lambda(m_d^{(N)}))w(\lambda(m_d^{(N)}))^2d\mu(\lambda(m_d^{(N)})) \right].
\]

Since \( |t_j| = w(\lambda^j) \), using (3.7) completes the proof. □

We need to work in \( \mathbb{C}^{N+1} \) with the \( \tilde{Z}_d \) integrals and verify the following.

**Proposition 3.2.** We have

\[
\lim_{n \to \infty} \tilde{Z}_d^{1/2m_d^{(N)}} = d^{(H)}(F).
\]

**Proof.** Fix \( d \) and consider the \( m_d^{(N)} \) monomials

\[
t^d, t^{d-1}z_1, \ldots, z_N^d,
\]

utilized in \( \tilde{VDMH}_d((t_1, z(1)), \ldots, (t_n, z(m_d^{(N)}))) \). Use Gram-Schmidt in \( L^2(\nu) \) to obtain orthogonal homogeneous polynomials

\[
q_1^{(H)}(t, z) = t^d, \ q_2^{(H)}(t, z) = t^{d-1}z_1 + \cdots, q_{m_d^{(N)}}^{(H)}(t, z) = z_N^d + \cdots.
\]

Then

\[
\tilde{VDMH}_d((t_1, z(1)), \ldots, (t_{m_d^{(N)}}, z(m_d^{(N)}))) = \det [q_i^{(H)}(t_j, z(j))]_{i,j=1,\ldots,m_d^{(N)}}.
\]

By orthogonality, as in (3.3), we obtain

\[
\tilde{Z}_d = m_d^{(N)}! ||q_1^{(H)}||_{L^2(\nu)}^2 \cdots ||q_{m_d^{(N)}}^{(H)}||_{L^2(\nu)}^2.
\]

Note that from (2.2) and (2.3) \( (m_d^{(N)})! \to 1 \) as \( d \to \infty \). Now from Lemma 2.5 we have

\[
\lim_{d \to \infty} \left( \prod_{|\alpha|=d} ||t_{\alpha,F}^{(H)}||_F \right)^{1/dm_d^{(N)}} = \tau(Y_2) = \tau(Y_1) = d^{(H)}(F).
\]

Thus we need to show that

\[
\lim_{d \to \infty} \left( \prod_{|\alpha|=d} ||t_{\alpha,F}^{(H)}||_F \right)^{1/dm_d^{(N)}} = \lim_{d \to \infty} \left( \prod_{i=1}^{m_d^{(N)}} ||q_i^{(H)}||_{L^2(\nu)} \right)^{1/dm_d^{(N)}}.
\]
This is analogous to (3.4) in the proof of Theorem 3.1 and it follows in the same manner from the Bernstein-Markov property for $(F, \nu)$ and the minimality properties of $t^{(H)}_{\alpha, F}$ and $q^{(H)}_i$.

Combining Propositions 3.1 and 3.2 with equation (2.20) and the second equation in (2.3) completes the proof of Theorem 3.2.

As a corollary, we get a “large deviation” result, which follows easily from Theorem 3.2. Define a probability measure $P_d$ on $E^{m(N)}_d$ via, for a Borel set $A \subset E^{m(N)}_d$,

$$P_d(A) := \frac{1}{Z_d} \int_A |VDM(z_1, ..., z_{m_d(N)})|^2 w(z_1)^{2d} \cdots w(z_{m_d(N)})^{2d} \, d\mu(z_1) \cdots d\mu(z_{m_d(N)}) .$$

**Proposition 3.3.** Given $\eta > 0$, define

$$A_{d, \eta} := \{(z_1, ..., z_{m_d(N)}) \in E^{m(N)}_d : |VDM(z_1, ..., z_{m_d(N)})|^2 w(z_1)^{2d} \cdots w(z_{n_d(N)})^{2d} \geq (\delta^w(E) - \eta)^{2d}\} .$$

Then there exists $d^* = d^*(\eta)$ such that for all $d > d^*$,

$$P_d(E^{m(N)}_d \setminus A_{d, \eta}) \leq (1 - \frac{\eta}{2\delta^w(E)})^{2d} .$$

**Proof.** From Theorem 3.2, given $\epsilon > 0$,

$$Z_d \geq [\delta^w(E) - \epsilon]^{2d}$$

for $d \geq d(\epsilon)$. Thus

$$P_d(E^{m(N)}_d \setminus A_{d, \eta}) =$$

$$\frac{1}{Z_d} \int_{E^{m(N)}_d \setminus A_{d, \eta}} |VDM(z_1, ..., z_{m_d(N)})|^2 w(z_1)^{2d} \cdots w(z_{n_d(N)})^{2d} \, d\mu(z_1) \cdots d\mu(z_{m_d(N)})$$

$$\leq \frac{[\delta^w(E) - \eta]^{2d}}{[\delta^w(E) - \epsilon]^{2d}}$$

if $d \geq d(\epsilon)$. Choosing $\epsilon < \eta/2$ and $d^* = d(\epsilon)$ gives the result.

Finally, we state a a version of (1.2) for $\Gamma$ an unbounded cone in $\mathbb{R}^N$ with $\Gamma = \text{int}\Gamma$. Precisely, our set-up is the following. Let $R(x) = R(x_1, ..., x_N)$ be a polynomial in $N$ (real) variables $x = (x_1, ..., x_N)$ and let

$$d\mu(x) := |R(x)| \, dx = |R(x_1, ..., x_N)| \, dx_1 \cdots dx_N .$$

Next, let $w(x) = \exp(-Q(x))$ where $Q(x)$ satisfies the inequality

$$Q(x) \geq c|x|^{\gamma}$$

for all $x \in \Gamma$ for some $c, \gamma > 0$. 

**Theorem 3.2**
Theorem 3.3. Let \( S_w := \text{supp}(dd^cV_{T,Q})^N \) where \( Q \) is defined as in (3.9). With \( \mu \) defined in (3.8),
\[
\lim_{d \to \infty} Z_d^{1/2d} = \delta^w(S_w)
\]
where
\[
Z_d = Z_d(\Gamma, w, \mu) := \int_{\Gamma_d(N)} |VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d(N))})|^2 \times w(\lambda^{(1)})^{2d} \ldots w(\lambda^{(m_d(N))})^{2d} d\mu(\lambda^{(1)}) \ldots d\mu(\lambda^{(m_d(N))}).
\]

Remark. The integrals considered in Theorem 3.3 may be considered as multivariate versions (i.e., with a multivariable Vandermonde determinant in the integrand rather than a one-variable Vandermonde determinant) of integrals of the form
\[
\int_{\mathbb{R}^d} |VDM(\lambda_1, \ldots, \lambda_d)|^2 e^{-dQ(\lambda_1)} \ldots e^{-dQ(\lambda_d)} d\lambda_1 \ldots d\lambda_d
\]
considered in [14], Chapter 6, arising in the joint probability distribution of eigenvalues of certain random matrix ensembles. They are also multivariate versions of Selberg integrals of Laguerre type (cf., [18], equation (17.6.5)) which, after rescaling by a factor of \( d \), are of the form, for \( \Gamma = [0, \infty) \subset \mathbb{R} \) and \( \alpha > 0 \),
\[
\int_{\Gamma_d} |VDM(\lambda_1, \ldots, \lambda_d)|^2 e^{-d\lambda_1} \ldots e^{-d\lambda_d} \prod_{j=1}^{d} \lambda_j^\alpha d\lambda_1 \ldots d\lambda_d.
\]

Proof. We begin by observing that
\[
VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d(N))})^2 \cdot w(\lambda^{(1)})^{2d} \ldots w(\lambda^{(m_d(N))})^{2d}
\]
(the integrand in (3.10) with the absolute value removed from the VDM) becomes, if all but one of the \( m_d(N) - 1 \) variables are fixed, a weighted polynomial in the remaining variable. Since \( w(x) \) is continuous, by Theorem 2.6 in Appendix B of [20], a weighted polynomial attains its maximum on \( S_w \subset \Gamma \). Hence the maximum value of (3.11) on \( \Gamma_d(N) \) is attained on \((S_w)^{m_d(N)}\). Since \( S_w \) has compact support (cf., Lemma 2.2 of Appendix B of [20]), we can take \( T > 0 \) sufficiently large with \( S_w \subset \Gamma \cap B_T \) where \( B_T := \{ x \in \mathbb{R}^N : |x| \leq T \} \) and
\[
\delta^w(S_w) = \delta^w(\Gamma \cap B_T).
\]
We need the following result.
Lemma 3.4. For all $T > 0$ sufficiently large, there exists $M = M(T) > 0$ with
\[ \|w^d p\|_{L^2(\Gamma, \mu)} \leq M \|w^d p\|_{L^2(\Gamma \cap B_T, \mu)} \]
if $p = p(x)$ is a polynomial of degree $d$.

Proof. By Theorem 2.6 (ii) in Appendix B of [20], we have
\[ |w(x)^d p(x)| \leq \|w^d p\|_{S_w} e^{d(V_T \cdot Q(x) - Q(x))} \]
for all $x \in \Gamma$. Since $V_{\Gamma, Q} \in L(\mathbb{C}^N)$ and $Q(x) \geq c|\gamma|$ for $x \in \Gamma$, there is a $c_0 > 0$ with
\[ |w(x)^d p(x)| \leq \|w^d p\|_{S_w} e^{-c_0 d|x|^\gamma} \]
for all $x \in \Gamma \cap B_T$ for $T$ sufficiently large. Hence
\[ \|w^d p\|_{L^2(\Gamma, \mu)} \leq \|w^d p\|_{L^2(\Gamma \cap B_T, \mu)} + \|w^d p\|_{S_w} \int_{\Gamma \cap \{|x| \geq T\}} e^{-c_0 d|x|^\gamma} |R(x)| \, dx. \]
Now $(\Gamma \cap B_T, \mu)$ satisfies the Bernstein-Markov property ([10], Theorem 2.1); thus by [6] Theorem 3.2, the triple $(\Gamma \cap B_T, w, \mu)$ satisfies the weighted Bernstein-Markov property. Thus, given $\epsilon > 0$, there is $M_1 = M_1(\epsilon) > 0$ with
\[ \|w^d p\|_{S_w} = \|w^d p\|_{\Gamma \cap B_T} \leq M_1 (1 + \epsilon)^d \|w^d p\|_{L^2(\Gamma \cap B_T, \mu)}. \]
A simple estimate shows that
\[ \int_{|x| \geq T} e^{-c_0 d|x|^\gamma} |R(x)| \, dx \leq e^{-c'd} \]
for some $c' > 0$. The result now follows by choosing $\epsilon$ sufficiently small. \( \square \)

We now expand the integrands in the formulas for $Z_d(\Gamma) := Z_d(\Gamma, w, \mu)$ and in $Z_d(\Gamma \cap B_T) := Z_d(\Gamma \cap B_T, w|\Gamma \cap B_T, \mu|\Gamma \cap B_T)$ as a product of $L^2-$norms of orthogonal polynomials as in [33], and then proceed as in the proof of Corollary 2.1 in section 5 of [3] to conclude that
\[ \lim_{d \to \infty} Z_d(\Gamma)^{1/2d^2} = \lim_{d \to \infty} Z_d(\Gamma \cap B_T)^{1/2d^2} = \delta^w(\Gamma \cap B_T) = \delta^w(S_w). \]
\( \square \)

4. Final remarks and questions.

In this section we discuss some further results in the literature and pose some questions. Recall from section 2 that a $d-$th weighted Fejér set for $E \subset \mathbb{C}^N$ and an admissible weight $w$ on $E$ is a set of $m_d$ points $\zeta_1^{(d)}, \ldots, \zeta_{m_d}^{(d)} \in E$ with the property that
\[ |W(\zeta_1, \ldots, \zeta_{m_d})| = \sup_{\xi_1, \ldots, \xi_{m_d} \in E} |W(\xi_1, \ldots, \xi_{m_d})| \]
where $W$ is defined in \[2.16\]. In \[9\] the authors asked if the sequence of probability measures

$$\mu_d := \frac{1}{m_d} \sum_{j=1}^{m_d} < \zeta_j^{(d)} >, \quad d = 1, 2, \ldots,$$

where $< z >$ denotes the point mass at $z$ and $\{\zeta_1^{(d)}, \ldots, \zeta_{m_d}^{(d)}\}$ is a $d-$th weighted Fejér set for $E$ and $w$, has a unique weak-* limit, and, if so, whether this limit is the Monge-Ampere measure, $\mu_{eq}^w := (dd^*V_{E,Q}^*E)^N$. From the proof of Proposition 2.1, a $d-$th weighted Fejér set for $E$ and $w$ corresponds to a $d-th$ homogeneous Fejér set for the circled set

$$F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E, \ |t| = w(\lambda)\};$$

i.e., a set of $m_d^{(N)} = h_d^{(N+1)}$ points in $F$ which maximize the corresponding homogeneous Vandermonde determinant \[2.11\] for $F$. From Theorem 2.2 of \[6\], to verify this conjecture for $E \subset \mathbb{C}^N$ and an admissible weight $w$ it suffices to verify it for homogeneous Fejér points associated to circled sets in $\mathbb{C}^{N+1}$.

Suppose now that $\mu$ is a measure on $E$ such that $(E, w, \mu)$ satisfies a Bernstein-Markov inequality for weighted polynomials. Define the probability measures

$$\mu_d(z) := \frac{1}{Z_d} R_1^{(d)}(z)w(z)^{2d} d\mu(z)$$

where $Z_d$ is defined in \[3.6\] and

\begin{equation}
R_1^{(d)}(z) := \int_{E^{m_d-1}} |VDM(\lambda^{(1)}, \ldots, \lambda^{(m_d-1)}, z)|^2 d\mu(\lambda^{(1)}) \ldots d\mu(\lambda^{(m_d-1)}).
\end{equation}

We observe that with the notation in \[(4.1)\] and \[(3.6)\]

\begin{equation}
\frac{R_1^{(d)}(z)}{Z_d} = \frac{1}{m_d} \sum_{j=1}^{m_d} |q_j^{(d)}(z)|^2
\end{equation}

where $q_1^{(d)}, \ldots, q_{m_d}^{(d)}$ are orthonormal polynomials with respect to the measure $w(z)^{2d} d\mu(z)$ forming a basis for the polynomials of degree at most $d$. To verify \[(4.2)\], we refer the reader to the argument in Remark 2.1 of \[8\]. Forming the sequence of Christoffel functions $K_d(z) := \sum_{j=1}^{m_d} |q_j^{(d)}(z)|^2$, in \[8\] Theorem 2.2 it was shown that if $N = 1$ then $\mu_d(z) \rightarrow \mu_{eq}^w(z)$ weak-*; i.e.,

\begin{equation}
\frac{1}{m_d} K_d(z) w(z)^{2d} d\mu(z) \rightarrow \mu_{eq}^w(z) \text{ weak-}^*.
\end{equation}
We conjecture that (4.3) should hold in $\mathbb{C}^N$ for $N > 1$. To this end, we remark that if $E = \overline{D}$ where $D$ is a smoothly bounded domain in $\mathbb{R}^N$, it follows from the proof of Theorem 1.3 in [1] that $\mu_{eq} := (dd^cV^*_E)^N = c(x)dx$ is absolutely continuous with respect to $\mathbb{R}^N$–Lebesgue measure $dx$ on $D$; and if $\mu(x) = f(x)dx$ is also absolutely continuous, then a conjectured version of (4.3) in the unweighted case $w \equiv 1$ is

$$\frac{1}{m_d}K_d(x)f(x) \to c(x) \text{ on } D.$$ 

Bos ([12], [13]) has verified this result for centrally symmetric functions $f(x)$ on the unit ball in $\mathbb{R}^N$ and Xu [22] proved this result for certain Jacobi-type functions $f(x)$ on the standard simplex in $\mathbb{R}^N$. For further results on subsets of $\mathbb{R}^1$, see references [10], [13], [15]-[17] in [8]. Berman ([8] and [4]) has shown that if $w = e^{-Q}$ is a smooth admissible weight function on $\mathbb{C}^N$ (recall [2,13]), then $\mu_{eq}^w := (dd^cV^*_C)^N = c(z)dz$ is absolutely continuous with respect to $\mathbb{C}^N$–Lebesgue measure $dz$ on the interior $I$ of the compact set $\{z \in \mathbb{C}^N : V_{\mathbb{C}^N,Q}(z) = Q(z)\}$ and

$$\frac{1}{m_d}K_d(z)Q(z)^d \to c(z) \text{ a.e. on } I.$$ 

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