Rate Optimal Estimation and Confidence Intervals for
High-dimensional Regression with Missing Covariates

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Abstract

We consider the problem of estimating and constructing component-wise confidence intervals of a sparse high-dimensional linear regression model when some covariates of the design matrix are missing completely at random. A variant of the Dantzig selector (Candes & Tao, 2007) is analyzed for estimating the regression model and a de-biasing argument is employed to construct component-wise confidence intervals under additional assumptions on the covariance of the design matrix. We also derive rates of convergence of the mean-square estimation error and the average confidence interval length, and show that the dependency over several model parameters (e.g., sparsity \(s\), portion of observed covariates \(\rho_\ast\), signal level \(\|\beta_0\|_2\)) are optimal in a minimax sense.

1 Introduction

High-dimensional regression has been an active topic of research in statistics and machine learning over the past 20 years (Tibshirani, 1996; Efron et al., 2004; Donoho, 2006; Candes et al., 2006; Fan & Li, 2001). Generally speaking, the high-dimensional estimation problem concerns the setting where the number of variables (or features) is on par with, or even far exceeds the number of observations (data points) available. To make the estimation problem well-defined, it is usually assumed that only a small portion of the variables are related to the response variable. A typical high-dimensional regression model is defined as

\[ y = X\beta_0 + \varepsilon; \quad X \in \mathbb{R}^{n \times p}, \quad \varepsilon | X \sim \mathcal{N}_n(0, \sigma^2 I) \]  

(1)

where \(\beta_0\) is a \(p\)-dimensional sparse linear model to be estimated and \(\varepsilon\) is i.i.d. Gaussian noise with zero mean and variance \(\sigma^2\). The number of observations \(n\) is assumed to be smaller than the dimension \(p\) of each data point, while it is assumed that \(s < n\) components of \(\beta_0\) is non-zero. Popular estimators for Eq. (1) include the Lasso (Tibshirani, 1996), the SCAD (Fan & Li, 2001) and the Dantzig selector (Candes & Tao, 2007), whose asymptotic rates of convergence and
model selection properties are well understood (Zhao & Yu, 2006; Bach, 2008; Bickel et al., 2009; Wainwright, 2009)

In many statistical applications, however, the full design (data) matrix $X$ is not fully observed and missing/corrupted entries are common. For example, in a data set that records characterization of $p = 5520$ genes for $n = 46$ patients with soft tissue tumors (Nielsen et al., 2002), a total of 6.7% entries are missing; in addition, 78.6% of the 5520 genes and all of the 46 patients have at least one missing covariate. Under such scenario, classical methods like list-deletion is no longer applicable; imputation based methods require additional assumptions on the data generative model and might lead to invalid confidence intervals because the noise of the imputed values is not taken into consideration.

In this paper, we consider the problem of estimating and building component-wise confidence intervals of $\beta_0$ without imputing an incomplete design $X$. Let $R_{ij} \in \{0, 1\}$ be indicator variables of whether $X_{ij}$ is missing and define the observable zero-filled $n \times p$ matrix $\tilde{X}$ as

$$
\tilde{X}_{ij} = R_{ij} X_{ij}, \quad 1 \leq i \leq n, 1 \leq j \leq p.
$$

Following Loh & Wainwright (2012a), we assume that each element $X_{ij}$ is missing independently and completely at random, and impose a random design model over $X$ where each row of $X$ is sampled i.i.d. from a sub-Gaussian distribution with zero mean and covariance $\Sigma_0$. The main contributions of this paper are as follows:

**Estimation with unknown $\Sigma_0$** We analyze the noisy Dantzig selector estimator and show that its average squared error of estimating $\beta_0$ (i.e., $E[\|\hat{\beta} - \beta_0\|^2_2]$) depends quadratically on $\rho$, the probability of observing each $X_{ij}$, and linearly on $\|\beta_0\|^2_2$. The dependency over $\rho$ is better than existing estimators (Loh & Wainwright, 2012a; Chen & Caramanis, 2013) under similar settings, which depend on the 4th power of $\rho$. It is also proved that the dependency over $\rho^2$ cannot be removed in the minimax sense. Our bounds are not directly comparable to (Rosenbaum & Tsybakov, 2010, 2013) that do not make random design assumptions, whose statistical rates depend on $\|\beta_0\|_1$ instead of $\|\beta_0\|_2$.

**Estimation with known $\Sigma_0$** We analyze a variant of the noisy Dantzig selector and show that its averaged square error dependes linearly on $\rho$ and $\|\beta_0\|^2_2$. This improves over existing estimators (Loh & Wainwright, 2012b) for the known $\Sigma_0$ setting, which depend quadratically on $\rho$. In addition, it is shown that under the identity covariance case $\Sigma_0 = I$ the dependency over both $\rho$ and $\|\beta_0\|_2$ is minimax optimal, as well as the dependency over conventional quantities $s, p$ and $n$.

**Component-wise Confidence Intervals with unknown $\Sigma_0$** Under the additional assumption that $\Sigma_0^{-1}$ is sparse, coordinate-wise confidence intervals of $\beta_0$ are constructed by de-biasing the noisy Dantzig selector. The constructed confidence intervals are conditioned on $X$, with randomness over both the missing pattern $\{R_{ij}\}$ and noise $\varepsilon$. Furthermore, under the identity covariance case $\Sigma_0 = I$ it is shown that the length of the confidence interval matches the minimax rate up to universal constants.

One important difference from existing de-biased sparse estimators (Javanmard & Montanari, 2014; Cai et al., 2014; Zhang & Zhang, 2014; van de Geer et al., 2014) is that when $\tilde{X}$ contains missing values, the de-biasing matrix $\Theta$ (defined in Eq. (7)) correlates with the sparse estimator $\hat{\beta}$ (cf. Eqs. (2,3)), and the limiting distribution of the de-biased estimator depends on unseen covariates.
in $X$. We use a variant of the CLIME estimator (Cai et al., 2011) to resolve the correlation issue and propose a data-driven estimator for the limiting variance of the de-biased estimator.

Notations For a vector $x$, we use $\|x\|_p = \left(\sum_j |x_j|^p\right)^{1/p}$ to denote the $p$-norm of $x$. For a matrix $A$, we use $\|A\|_{L_p}$ to denote the operator $p$-norm of $A$; that is, $\|A\|_{L_p} = \sup_{x \neq 0} \|Ax\|_p/\|x\|_p$. We also write $\|A\|_\infty$ for the max norm of a matrix: $\|A\|_\infty = \max_{j,k} |A_{jk}|$. For a positive semi-definite matrix $A$, let $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ be the largest and smallest eigenvalues of $A$. We use $\mathbb{B}_p(M) = \{x : \|x\|_p \leq M\}$ to denote the centered $\ell_p$ ball of radius $M$.

1.1 Related work

Rosenbaum & Tsybakov (2010) proposed the MU-selector for high-dimensional regression under an error-in-variable model, where the design matrix $X$ is observed with deterministic (adversarial) measurement error $W$ that is bounded in matrix max norm. The estimator was further generalized to handle missing data in (Rosenbaum & Tsybakov, 2013) for which de-biasing the covariance estimator leads to improved error bounds. Optimization algorithms and minimax rates when $W$ is Gaussian white noise are derived in (Belloni et al., 2016).

Loh & Wainwright (2012a) analyzed a gradient descent algorithm for optimizing a non-convex Lasso-type loss function and derived rates of convergence from both statistical and optimization perspectives. Their analysis shows a quadruple dependency over the observation/missing rate for the mean-square estimation error and requires an upper RE condition. Loh & Wainwright (2012b) derived lower bounds on the minimax rate, assuming identity covariance for the design points and bounded signal level $\|\beta_0\|_2$. The error lower bound depends linear in observation/missing rate (Loh & Wainwright, 2012b). A general analytical framework for non-convex optimization in high-dimensional regression problems is presented in (Loh & Wainwright, 2015). A similar rate of convergence was established in (Chen & Caramanis, 2013) for orthogonal matching pursuit (OMP) type estimators.

Datta & Zou (2015) proposed COCOCOASSO, a variant of Lasso for error-in-variable models where a covariance estimate $\hat{\Sigma}$ is projected onto a positive semi-definite cone so that the resulting optimization problem is convex. Both additive and multiplicative measurement error models were considered in (Datta & Zou, 2015) and corresponding rates of convergence were derived.

2 Rate-optimal estimation of $\beta_0$

2.1 Problem setup and assumptions

Throughout this paper we make the following assumptions:

(A1) Homogenous Gaussian noise: $\varepsilon \sim N_n(0, \sigma_\varepsilon^2 I)$ for some $\sigma_\varepsilon < \infty$.

(A2) Sub-Gaussian random design: each row of $X$ is sampled i.i.d. from some underlying sub-Gaussian distribution with covariance $\Sigma_0$ and sub-Gaussian parameter $\sigma_x < \infty$. Assume $0 < \lambda_{\text{min}}(\Sigma_0) \leq \lambda_{\text{max}}(\Sigma_0) < \infty$. For notational simplicity we drop $\Sigma_0$ and use $\lambda_{\text{min}}, \lambda_{\text{max}}$ instead in the rest of this paper.
(A3) *Missing completely at random*: $R_{ij}$ are independent Rademacher variables with $\Pr[R_{ij} = 1] = \rho_j$ for some $\rho_1, \cdots, \rho_p \in (0, 1)$. Also assume $\{R_{ij}\}_{i,j} \perp X, \varepsilon$ and $\rho_\ast = \min_{1 \leq j \leq p} \rho_j > 0$.

(A4) *Sparsity*: The support set $J_0 = \text{supp}(\beta_0) = \{j : |\beta_{0j}| \neq 0\}$ satisfies $|J_0| \leq s$ for some $s \ll n$.

(A1), (A3) and (A4) are standard assumptions for high-dimensional regression with missing data, and (A2) implies (with high probability) a deterministic Restricted Eigenvalue (RE) condition (Bickel et al., 2009) on the sample covariance of $X$, which leads to a $s \log p/n$ fast rate for estimating $\beta_0$.

### 2.2 The noisy Dantzig selector

Define $\tilde{X} \in \mathbb{R}^{n \times p}$ and $\tilde{\Sigma} \in \mathbb{R}^{p \times p}$ as

$$
\tilde{X}_{ij} = \frac{R_{ij}X_{ij}}{\rho_j}, \quad \tilde{\Sigma} = \frac{1}{n} \tilde{X}^\top \tilde{X} - D \text{diag} \left( \frac{1}{n} \tilde{X}^\top \tilde{X} \right),
$$

where $D = \text{diag}(1 - \rho_1, \cdots, 1 - \rho_p)$ is a known $p \times p$ diagonal matrix. It is a simple observation that, conditioned on $X$, $\mathbb{E} \tilde{X} = X$ and $\mathbb{E} \tilde{\Sigma} = \tilde{\Sigma} = \frac{1}{n} X^\top X$. Define the noisy Dantzig selector as

$$
\hat{\beta}_n = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|\beta\|_1 : \left\| \frac{1}{n} \tilde{X}^\top y - \tilde{\Sigma} \beta \right\|_\infty \leq \tilde{\lambda}_n \right\}, \quad (2)
$$

where $\tilde{\lambda}_n > 0$ are tuning parameters. Eq. (2) is a variant of the Dantzig selector Candes & Tao (2007) and is in principle similar to the MU-selector in Rosenbaum & Tsybakov (2010). Note that Eq. (2) is always a convex optimization problem (regardless of whether $\tilde{\Sigma}$ is positive semi-definite) and hence can be efficiently solved.

We also consider a variant of the noisy Dantzig selector under the idealized scenario where the population covariance $\Sigma_0$ for the design matrix is known. In particular, define $\hat{\beta}_n$ as the solution of

$$
\hat{\beta}_n = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|\beta\|_1 : \left\| \frac{1}{n} \tilde{X}^\top y - \Sigma_0 \beta \right\|_\infty \leq \tilde{\lambda}_n \right\}. \quad (3)
$$

Note that the covariance estimate $\tilde{\Sigma}$ is replaced with the known population covariance $\Sigma_0$ in Eq. (3). The estimator $\hat{\beta}_n$ is primarily for theoretical considerations, as $\Sigma_0$ is in general unknown in most applications. In following sections we show that $\hat{\beta}_n$ achieves exact minimax rates for recovering $\beta_0$ with missing covariates.

### 2.3 Rates of convergence and minimax lower bounds

Theorem 2.1 establishes upper bounds on the mean square estimation error of $\beta_0$. Eq. (4) corresponds to the setting where the population covariance $\Sigma_0$ is known and Eq. (5) holds when $\Sigma_0$ is unknown.
Theorem 2.1. Assume (A1) to (A4). If \( \frac{\log p}{\rho^2 n} \to 0 \) and \( \lambda_n \asymp (\sigma^2 \|\beta_0\|_2 + \sigma_x \sigma_\varepsilon) \sqrt{\frac{\log p}{\rho^* n}} \), then
\[
\|\hat{\beta}_n - \beta_0\|_2 \leq O_P \left\{ \frac{\sigma^2 \sqrt{s}}{\lambda_{\min}} \left( \|\beta_0\|_2 + \frac{\sigma_x}{\sigma_\varepsilon} \sqrt{\frac{\log p}{\rho^* n}} \right) \right\}.
\] (4)

In addition, if \( \max \left\{ \frac{\sigma^2 \log(\sigma^2 p/\rho^* s)}{p^2 \lambda_{\min}^2 n}, \frac{\log p}{\rho^* n} \right\} \to 0 \) and \( \lambda_n \asymp \sigma^2 \|\beta_0\|_2 + \sigma_\varepsilon \sqrt{\frac{\log p}{\rho^* n}} \), then
\[
\|\hat{\beta}_n - \beta_0\|_2 \leq O_P \left\{ \frac{\sigma^2 \sqrt{s}}{\lambda_{\min}} \left( \|\beta_0\|_2 + \frac{\sigma_x}{\sigma_\varepsilon} \sqrt{\frac{\log p}{\rho^* n}} \right) \right\}.
\] (5)

Finally, \( \|\hat{\beta}_n - \beta_0\|_1 \leq 2\sqrt{s} \|\hat{\beta}_n - \beta_0\|_2 \) and \( \|\hat{\beta}_n - \beta_0\|_1 \leq 2\sqrt{s} \|\hat{\beta}_n - \beta_0\|_2 \) with probability \( 1 - o(1) \).

Compared to Loh & Wainwright (2012a) our bounds are better by an \( O(1/\rho^*) \) factor for \( \hat{\beta}_n \) when \( \Sigma_0 \) is unknown and an \( O(1/\rho^*^{3/2}) \) factor better when \( \Sigma_0 \) is known. Our bounds are not directly comparable to Rosenbaum & Tsybakov (2010) that considers a fixed-design setting with no stochastic model assumed over \( X \). We however remark that error bounds in Rosenbaum & Tsybakov (2010) depend on \( \|\beta_0\|_1 \), which could be a factor of \( \sqrt{s} \) worse than \( \|\beta_0\|_2 \).

We next present minimax lower bounds for the \( L_2 \) estimation error \( \|\beta_n - \beta_0\|_2^2 \). We first consider the simple case where the population covariance \( \Sigma_0 \) is the identity.

Theorem 2.2. Suppose \( 4 \leq s < 4p/5 \), \( \frac{s \log(p/s)}{\rho^* n} \to 0 \) and \( \Sigma_0 = I \). Then
\[
\inf_{\hat{\beta}_n} \sup_{\beta_0 \in \mathbb{B}_2(M) \cap \mathbb{B}_0(s)} \mathbb{E} \|\hat{\beta}_n - \beta_0\|_2^2 \geq C_0 \cdot \min \left\{ \sigma^2_\varepsilon + \frac{1 - \rho^*}{1 + 2c} M^2, c^{0.5c^2(1-\rho^*)} \sigma^2_\varepsilon \right\} \cdot \min \left\{ \sqrt{\frac{s \log(p/s)}{(1 - \rho^*)^2 n}}, \frac{s \log(p/s)}{\rho^* n} \right\}.
\] (6)

Here \( C_0 > 0 \) is a universal constant and \( c > 0 \) is an arbitrary constant.

Remark 2.1. Under the additional assumption that \( \frac{(1-\rho^*)^2 s \log(p/s)}{\rho^2 n} \to 0 \), the right-hand side of Eq. (6) can be simplified to
\[
C_0 \cdot \min \left\{ \sigma^2_\varepsilon + \frac{1 - \rho^*}{1 + 2c} M^2, c^{0.5c^2(1-\rho^*)} \sigma^2_\varepsilon \right\} \frac{s \log(p/s)}{\rho^* n}.
\]

Suppose that \( s/p \to 0 \) and hence \( \log p \) and \( \log(p/s) \) are of the same order. If the missing rate \( (1 - \rho^*) \) is at least a constant and the sparsity level \( s \) or the noise level \( \sigma_\varepsilon \) is not too small, the term \( c^{0.5c^2(1-\rho^*)} \sigma^2_\varepsilon \) is negligible because it increases exponentially with \( s \). This term arises in the lower bound because on a subset of of the design points whose size decreases exponentially fast with \( s \), the covariates corresponding to the support of \( \beta_0 \) are fully observed. Apart from this term, the lower bound matches the estimation error rate of \( \hat{\beta}_n \) when \( \Sigma_0 = I \), corresponding to \( \lambda_{\min} = \sigma_\varepsilon = 1 \).

The setting where \( \Sigma_0 \) is unknown is more complicated because of the \( 1/\rho^2_\varepsilon \) dependency in the upper bound of \( \hat{\beta}_n \) (of squared \( \ell_2 \) estimation error). The following theorem shows that such dependency is unavoidable if the population covariance \( \Sigma_0 \) is unknown.
Theorem 2.3. Let \( \gamma_0 \in (0, 1/2) \) be an arbitrary small positive constant and suppose \( s \geq 4 \), \( \max \{ \sigma^2 M, \frac{1}{\gamma_0 \rho_2 n} \} \to 0 \). Define \( \Lambda(\gamma_0) = \{ \Sigma_0 \in S^p_+ : 1 - \gamma_0 \leq \lambda_{\min}(\Sigma_0) \leq \lambda_{\max}(\Sigma_0) \leq 1 + \gamma_0 \} \), where \( S^p_+ \) is the class of all positive definite \( p \times p \) matrices. Then for any fixed \( j \in \{1, \ldots, p\} \) it holds that

\[
\inf_{\beta_n} \sup_{\beta_0 \in \mathbb{B}_0} \mathbb{E} |\hat{\beta}_{nj} - \beta_{0j}|^2 \geq C_1 \cdot \max \left\{ \sigma^2 \rho_n, \min \left( \frac{1 - \rho^*}{1 + 2c M^2}, e^{0.5c^2(1 - \rho^*)} \sigma^2 \right) \right\},
\]

where \( C_1 > 0 \) is a universal constant that only depends on \( \gamma_0 \) and \( c > 0 \) is an arbitrary constant.

3 Confidence intervals of regression coefficients

We describe a method that builds confidence intervals over the estimated \( \hat{\beta}_n \) by de-biasing the noisy Dantzig selector. We need the following additional assumption to justify the proposed approach:

(A5) There exist \( b_0, b_1 < \infty \) such that each row (and column) of \( \Sigma_0^{-1} \) belongs to \( \mathbb{B}_0(b_0) \cap \mathbb{B}_1(b_1) \).

That is, each row of \( \Sigma_0^{-1} \) is \( b_0 \)-sparse and \( \| \Sigma_0^{-1} \|_1 \leq b_1 \).

Condition (A5) allows the usage of CLIME or node-wise Lasso to estimate an approximate “inverse” of \( \Sigma_0 \) that asymptotically de-biases an estimate \( \hat{\beta}_n \). Similar conditions for high-dimensional inference were studied in (van de Geer et al., 2014). We discuss potential settings where (A5) could be relaxed in Sec. 5.

3.1 The de-biased noisy Dantzig selector

Let \( \hat{\Theta} \) be a \( p \times p \) matrix obtained by solving the following optimization problem:

\[
\hat{\Theta} = \arg\min_{\Theta \in \mathbb{R}^{p \times p}} \left\{ \| \Theta \|_1 : \| \hat{\Sigma} \Theta - I_p \|_{L_1} \leq \tilde{\nu}_n \right\}, \quad (7)
\]

where \( \tilde{\nu}_n > 0 \) is some tuning parameter to be specified later. Eq. (7) is a missing data variant of the CLIME estimator proposed in Cai et al. (2011) for estimating precision matrices in high dimension. The following lemma formally establishes the performance of \( \hat{\Theta} \) for estimating \( \Sigma_0^{-1} \). Its proof is standard and for completeness we include it in the supplementary materials.

Lemma 3.1. Under (A1), (A3) and (A5), suppose \( \frac{\log p}{\rho_n^2} \to 0 \) and \( \tilde{\nu}_n \asymp \sigma^2 b_1 \sqrt{\frac{\log p}{\rho_n^2}} \). Then with probability \( 1 - o(1) \) it holds that \( \max\{\| \hat{\Theta} \|_{L_1}, \| \hat{\Theta} \|_{L_{\infty}} \} \leq b_1 \) and \( \max\{\| \hat{\Theta} - \Sigma_0^{-1} \|_{L_1}, \| \hat{\Theta} - \Sigma_0^{-1} \|_{L_{\infty}} \} \leq 2\tilde{\nu}_n b_0 b_1 \).

With an estimator \( \hat{\beta}_n \) of \( \beta_0 \) and \( \hat{\Theta} \) obtained from solving Eq. (7), the de-biased estimator \( \hat{\beta}_n^u \) is defined as

\[
\hat{\beta}_n^u = \hat{\beta}_n + \hat{\Theta} \left( \frac{1}{n} \hat{X}^T y - \hat{\Sigma} \hat{\beta}_n \right). \quad (8)
\]

We then have the following theorem that derives the limiting variance of \( \hat{\beta}_n^u \).
Theorem 3.1. Define $\hat{\Gamma} \in \mathbb{R}^{p \times p}$ as

$$\hat{\Gamma} = \frac{\sigma_x^2}{n}X^T X + \frac{\sigma_z^2}{n}\tilde{D}\text{diag}(X^T X) + \hat{\Upsilon},$$

where $\tilde{D} = \text{diag}(\frac{1}{\rho_1} - 1, \ldots, \frac{1}{\rho_p} - 1)$ and

$$\hat{\Upsilon}_{jk} = \begin{cases} \frac{1}{n}\sum_{i=1}^n \sum_{t \neq j} \frac{1}{\rho_i}X_{ij}^2X_{it}\beta_{0t}, & j = k; \\ \frac{1}{n}\sum_{i=1}^n \sum_{t \neq j, k} \frac{1}{\rho_i}X_{ij}X_{ik}\beta_{0t}, & j \neq k. \end{cases}$$

If the conclusion in Lemma 3.1 holds and

$$\sigma_x^2 b_0 b_1 \hat{v}_n \left( \frac{\sigma_x}{\sigma_x} \sqrt{\frac{\log p}{\rho_*}} + \|\beta_0\|_2 \sqrt{\frac{\log p}{\rho_*^2}} + \frac{\sqrt{n}||\hat{\beta}_n - \beta_0||_1}{\sigma_x^2 b_0 b_1} \right) \xrightarrow{p} 0, \quad (9)$$

then for any variable subset $S \subseteq [p]$ with constant size it holds that with probability $1 - o(1)$ over the random design $X$,

$$\sqrt{n} \left( \hat{\beta}_n^u - \beta_0 \right)_{SS} \overset{d}{\rightarrow} \mathcal{N}_{|S|} \left( 0, \left[ \Sigma_0^{-1}\hat{\Sigma}_0^{-1} \right]_{SS} \right) \quad \text{conditioned on } X.$$

Remark 3.1. When the noisy Dantzig selector Eq. (2) is used for the initial estimation $\hat{\beta}_n$ and both $\hat{v}_n$ and $\hat{\lambda}_n$ are chosen at the rates specified in Theorem 2.1 and Lemma 3.1, then a sufficient condition for Eq. (9) to hold is that

$$\sigma_x^4 b_0^2 b_1^2 \sqrt{\frac{\log p}{\rho_* n}} \left( \frac{\sigma_x \sqrt{\rho_*}}{\sigma_x} + \|\beta_0\|_2 \right) \left( 1 + \frac{s}{\lambda_{\min} b_0 b_1} \right) \rightarrow 0. \quad (10)$$

Remark 3.2. We derive the asymptotic variance of $\sqrt{n}(\hat{\beta}_{n_j}^u - \beta_{0j})$ for a specific coordinate $j$ under the identity covariance setting $\Sigma_0 = I$ and demonstrate its rate optimality. For simplicity we also assume uniform observation rates $\rho_1 = \rho_2 = \cdots = \rho_p = \rho_*$. Fix $j$ as a single coordinate and let

$$V_j = V_{jj}\sqrt{n}(\hat{\beta}_{n_j}^u - \beta_0).$$

By Theorem 3.1, when $n$ is sufficiently large

$$V_j \xrightarrow{p} \hat{\Gamma}_{jj} \xrightarrow{p} \frac{\sigma_x^2}{\rho_*} + \frac{1 - \rho_*}{\rho_*^2} \sum_{t \neq j} \beta_{0t}^2 \leq \frac{\sigma_x^2}{\rho_*} + \frac{1 - \rho_*}{\rho_*^2} \|\beta_0\|_2^2. \quad (11)$$

Comparing Eq. (11) with Theorem 2.3, the variance $V_j$ achieves the minimax rates of coordinate-wise estimation up to problem independent constants. Formally, under the additional assumption $\sigma_x^2 \geq e^{-0.5c^2(1-\rho_*)\|\beta_0\|_2^2}$ that $\sigma_x$ is not exponentially small, we have that

$$\limsup_{p,n \to \infty} \inf_{\beta_n \in B_2(\|\beta_0\|_2)} \sup_{\beta_0' \in B_2(\|\beta_0\|_2) \cap B_0(s), \Sigma_0' \in \Lambda(\gamma_0)} \mathbb{E}_0' \mathbb{E}_0' \left| \hat{\beta}_{n_j}^u - \beta_{0j} \right|^2 \leq 2C_1^{-1}(1 + 2c),$$

where $C_1 > 0$ is a universal constant in Theorem 2.3.
3.2 Data-driven approximation of the limiting covariance

The limiting variance \( \Sigma_0^{-1} \Gamma \Sigma_0^{-1} \) derived in Theorem 3.1 depends on the full design matrix \( X \) and the population precision matrix \( \Sigma_0^{-1} \), both of which are unavailable in practice when some covariates of \( X \) are missing. To overcome this difficulty, in this section a data-driven approximation of \( \Sigma_0^{-1} \Gamma \Sigma_0^{-1} \) is proposed, which only depends on the observed data \( \tilde{X} \) and \( y \).

Define \( \tilde{\Gamma} = \frac{\sigma_x^2}{n} \tilde{X}^\top \tilde{X} + \tilde{\Upsilon} \), where \( \tilde{\Upsilon}_{jk} = \frac{1}{n} \sum_{i=1}^{n} \sum_{t \neq j,k} (1 - \rho_t) \tilde{X}_{ij} \tilde{X}_{ik} \tilde{X}_{it}^2 \hat{\beta}_{nt} \), for \( j, k \in \{1, \cdots, p\} \). The following theorem shows that \( \hat{\Theta} \tilde{\Gamma} \hat{\Theta}^\top \) is a good approximation of \( \Sigma_0^{-1} \Gamma \Sigma_0^{-1} \) when \( n \) is sufficiently large:

**Theorem 3.2.** Suppose the conclusion in Lemma 3.1 holds, \( \frac{\log p}{\rho^*_n} \rightarrow 0 \) and \( \| \hat{\beta}_n - \beta_0 \|_2 \rightarrow 0 \). Then

\[
\left\| \hat{\Theta} \tilde{\Gamma} \hat{\Theta}^\top - \Sigma_0^{-1} \Gamma \Sigma_0^{-1} \right\|_\infty \leq O_P \left( \frac{\sigma_x^4 b^2 \log^2 p}{\rho^*_n} \left( \| \beta_0 \|_2^2 + \frac{\rho_x \sigma_x^2}{\sigma_x^2} \right) \left( b_0 \tilde{\nu}_n + \sqrt{\frac{\log p}{\rho^*_n}} + \| \beta_0 \|_2 \| \hat{\beta}_n - \beta_0 \|_1 \right) \right).
\]

Based on Theorems 3.1 and 3.2, an asymptotic \((1 - \alpha)\) confidence interval of \( \beta_{0j} \) can be computed as

\[
\text{CI}_j(\alpha) = \left[ \hat{\beta}_{uj} \frac{\Phi^{-1}(1 - \alpha/2) \sqrt{(\hat{\Theta} \tilde{\Gamma} \hat{\Theta})_{jj}}}{\sqrt{n}}, \hat{\beta}_{uj} \frac{\Phi^{-1}(1 - \alpha/2) \sqrt{(\hat{\Theta} \tilde{\Gamma} \hat{\Theta})_{jj}}}{\sqrt{n}} \right],
\]

where \( \Phi^{-1}(\cdot) \) is the inverse function of the CDF of the standard Gaussian distribution.

4 Simulation results

4.1 Synthetic data

We fix \( \sigma_x = 0.1 \) and set \( \Sigma_0 = \Omega^{-1} \) where \( \Omega \) is chosen to be the following banded matrix:

\[
\Omega_{ij} = \begin{cases} 
0.5|i-j| & \text{if } |i-j| \leq 5 \\
0 & \text{otherwise}
\end{cases}
\]

Assume uniform observation rate \( \rho_1 = \cdots = \rho_p = \rho \), which ranges from 0.5 to 0.9. The support set \( J_0 \subset \{p\} \) of \( \beta_0 \) is selected uniformly at random, with \( |J_0| = 10 \). \( \beta_0 \) is then generated as \( \beta_{0j} \overset{i.i.d.}{\sim} \text{Bernoulli}(+1, -1) \) for \( j \in J_0 \) and \( \beta_{0j} = 0 \) for \( j \notin J_0 \). Both the noisy Dantzig selector Eq. (2) and the noisy CLIME estimator Eq. (7) are solved using alternating direction methods of multipliers (ADMM).
4.1.1 Verification of asymptotic normality

We run 1000 independent realizations of \( \{ R, \bar{X}, y \} \) and study the distributions of \( \sqrt{n}(\hat{\beta}_n - \beta_0) \). We plot the empirical distribution of

\[
\hat{\delta}_j = \frac{\sqrt{n}(\hat{\beta}_{nj} - \beta_{0j})}{\sqrt{(\Theta \Gamma \Theta^\top)_{jj}}}
\]

against the standard Normal distribution. Figure 1 shows that the empirical distribution of \( \hat{\delta}_j \) agrees quite well with \( \mathcal{N}(0,1) \). In addition, more observations \( (n) \) are required to deliver asymptotic normality when observation rates are low (e.g., \( \rho = 0.5 \)).

4.1.2 Average CI coverage and length

We calculate the average coverage and length of the constructed confidence intervals from \( T \) independent realizations, defined as

\[
\text{Avgcov}(j) = \frac{1}{T} \sum_{i=1}^{T} I(\beta_{0j} \in \text{CI}_j^{(i)}(\alpha)), \quad \text{and} \quad \text{Avglen}(j) = \frac{1}{T} \sum_{i=1}^{T} \text{length}(	ext{CI}_j^{(i)}(\alpha)),
\]

where \( \text{CI}_j(\alpha) \) is defined in Eq. (12). We also report the average coverage and length of coordinate-wise confidence intervals across a coordinate subset \( J \subseteq [p] \), defined as

\[
\text{Avgcov}(J) = \frac{1}{|J|} \sum_{j \in J} \text{Avgcov}(j) \quad \text{and} \quad \text{Avglen}(J) = \frac{1}{|J|} \sum_{j \in J} \text{Avglen}(j).
\]

Tables 1 summarize the results for various \( (n, p, \rho) \) settings.

4.2 Real data

In this section we conduct experiments on two datasets: DNA and Madelon\(^1\), where the distribution of the design matrices are not necessarily sampled sub-Gaussian distributions. The DNA data contains 2000 instances and 180 covariates, while Madelon contains 2000 data points and 500 covariates. For these two datasets, we only uses their data matrix \( X \) and construct the response \( y \) according to the sparse linear regression model as specified in previous section. Following the simulation study, we randomly remove the observed covariates with probability \( 1 - \rho \), and then perform statistical inference based on the datasets with missing covariates. The performance of the constructed confidence intervals are reported in Table 3. We see that the proposed procedure can produced roughly normal estimates for the parameters of interest when \( \rho \) is not too small, this demonstrate that the method can be robust to violations of the statistical assumptions on the design matrix.

5 Discussion

Quadratic dependency over \( \rho_* \)  Theorem 2.3 shows that if the population covariance \( \Sigma_0 \) of the design matrix \( X \) is unknown, then mean square estimation error of a particular component in \( \beta_0 \)

\(^1\)Available from https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
\( n = 1500, p = 500, \rho = 0.9 \)

\( n = 5000, p = 500, \rho = 0.7 \)

\( n = 8000, p = 500, \rho = 0.5 \)

\( n = 12000, p = 500, \rho = 0.5 \)

Figure 1: Empirical distribution and density of \( \hat{\delta}_j = \frac{\sqrt{n}(\hat{\beta}_{n,j} - \beta_0)}{\sqrt{(\Theta \Gamma \Theta)^{-1})_{jj}}} \) of 1000 independent realizations. Top row in each subfigure: two coordinates randomly chosen from \( J_0 \); bottom row in each subfigure: two coordinates randomly chosen from \( J_0^c \). Red curve: density of the standard Normal distribution.
Table 1: 95% confidence intervals for high-dimensional regression with missing data when $\rho \in [0.7, 0.9]$.

| $(n, p, \rho)$      | $\text{Random } j \in J_0$ | $\text{Random } j \not\in J_0$ | $J_0$     | $J_0^c$    |
|---------------------|------------------------------|------------------------------|-----------|-----------|
|                     | Avgcov | Avglen | Avgcov     | Avglen | Avgcov | Avglen | Avgcov | Avglen | Avgcov | Avglen |
| (1000,200,0.9)      | 0.941  | 0.182  | 0.951      | 0.192  | 0.938  | 0.208  | 0.966  | 0.187  |
| (1000,200,0.8)      | 0.945  | 0.318  | 0.948      | 0.329  | 0.944  | 0.334  | 0.979  | 0.331  |
| (1000,200,0.7)      | 0.952  | 0.494  | 0.983      | 0.540  | 0.949  | 0.547  | 0.989  | 0.529  |
| (1500,500,0.9)      | 0.931  | 0.155  | 0.966      | 0.170  | 0.945  | 0.183  | 0.971  | 0.158  |
| (1500,500,0.8)      | 0.927  | 0.278  | 0.982      | 0.294  | 0.937  | 0.308  | 0.985  | 0.284  |
| (1500,500,0.7)      | 0.963  | 0.415  | 0.994      | 0.469  | 0.971  | 0.497  | 0.995  | 0.450  |
| (2000,1000,0.9)     | 0.947  | 0.144  | 0.974      | 0.144  | 0.949  | 0.160  | 0.975  | 0.139  |
| (2000,1000,0.8)     | 0.967  | 0.249  | 0.987      | 0.264  | 0.939  | 0.281  | 0.990  | 0.254  |
| (2000,1000,0.7)     | 0.952  | 0.378  | 0.995      | 0.422  | 0.930  | 0.451  | 0.997  | 0.409  |
| (3000,2000,0.9)     | 0.958  | 0.116  | 0.954      | 0.118  | 0.951  | 0.133  | 0.981  | 0.115  |
| (3000,2000,0.8)     | 0.919  | 0.202  | 0.979      | 0.220  | 0.948  | 0.236  | 0.993  | 0.212  |
| (3000,2000,0.7)     | 0.891  | 0.315  | 0.998      | 0.349  | 0.950  | 0.372  | 0.998  | 0.348  |

Table 2: 95% confidence intervals for regression with missing data when $\rho = 0.5$.

| $(n, p, \rho)$      | $\text{Random } j \in J_0$ | $\text{Random } j \not\in J_0$ | $J_0$     | $J_0^c$    |
|---------------------|------------------------------|------------------------------|-----------|-----------|
|                     | Avgcov | Avglen | Avgcov     | Avglen | Avgcov | Avglen | Avgcov | Avglen | Avgcov | Avglen |
| (1000,200,0.5)      | 0.928  | 1.051  | 0.998      | 1.223  | 0.942  | 1.384  | 0.999  | 1.194  |
| (2000,200,0.5)      | 0.971  | 0.715  | 0.997      | 0.849  | 0.971  | 0.799  | 0.995  | 0.813  |
| (3000,200,0.5)      | 0.956  | 0.574  | 0.976      | 0.644  | 0.961  | 0.668  | 0.989  | 0.640  |
| (4000,200,0.5)      | 0.936  | 0.468  | 0.984      | 0.541  | 0.943  | 0.527  | 0.986  | 0.534  |
| (1500,500,0.5)      | 0.986  | 0.795  | 0.978      | 0.911  | 0.756  | 0.954  | 1.000  | 0.896  |
| (3000,500,0.5)      | 0.849  | 0.510  | 0.899      | 0.575  | 0.479  | 0.634  | 0.998  | 0.572  |
| (8000,500,0.5)      | 0.972  | 0.352  | 0.978      | 0.408  | 0.908  | 0.417  | 0.988  | 0.403  |
| (12000,500,0.5)     | 0.941  | 0.272  | 0.965      | 0.315  | 0.936  | 0.328  | 0.976  | 0.309  |
Table 3: 95% confidence intervals for regression with missing data on real world datasets.

| (dataset, ρ) | Random $j \in J_0$ | Random $j \notin J_0$ | $J_0$ | $J'_0$ |
|--------------|---------------------|----------------------|-------|-------|
|              | Avgcov | Avglen | Avgcov | Avglen | Avgcov | Avglen | Avgcov | Avglen |
| (DNA,0.9)    | 0.924 | 0.120  | 0.956 | 0.128  | 0.937 | 0.128  | 0.957 | 0.129  |
| (DNA,0.8)    | 0.908 | 0.195  | 0.959 | 0.216  | 0.926 | 0.212  | 0.965 | 0.218  |
| (DNA,0.7)    | 0.888 | 0.286  | 0.967 | 0.318  | 0.925 | 0.314  | 0.973 | 0.317  |
| (DNA,0.5)    | 0.713 | 0.464  | 0.964 | 0.516  | 0.745 | 0.512  | 0.976 | 0.519  |
| (Madelon,0.9)| 0.943 | 0.095  | 0.963 | 0.101  | 0.949 | 0.098  | 0.945 | 0.105  |
| (Madelon,0.8)| 0.966 | 0.167  | 0.976 | 0.174  | 0.961 | 0.181  | 0.971 | 0.223  |
| (Madelon,0.7)| 0.962 | 0.229  | 0.977 | 0.236  | 0.956 | 0.253  | 0.977 | 0.261  |
| (Madelon,0.5)| 0.663 | 0.334  | 0.977 | 0.357  | 0.682 | 0.377  | 0.965 | 0.356  |

must depend quadratically on the observation ratio $ρ_∗$. We conjecture that such results also hold for the estimation error of the entire regression model $β_0$ as well. More specifically, we conjecture that under suitable finite-sample conditions,

$$\inf_{β_n} \sup_{β_0 \in \mathbb{B}_2(M) \cap \mathbb{B}_0(s)} \mathbb{E} \| \hat{β}_n - β_0 \|_2 \geq C'_1 \cdot \max \left\{ \frac{σ^2 s \log p}{ρ_∗ n}, \min \left( \frac{1 - ρ_∗}{1 + 2c M^2}, \frac{1}{\rho^2 n} \right) \right\}.$$ 

We are, however, unable to generalize our construction of difficult cases in the proof of Theorem 2.3 (cf. Sec. 6.4) to handle $\mathbb{E} \| \hat{β} - β_0 \|_2$. The current construction relies on a carefully designed set of covariance matrices that leak no information unless both $X_1$ and $X_j$ are observed. Extending such construction to multiple covariates requires new ideas.

**Finite-sample conditions** The finite-sample condition (i.e., relationship between $n$ and the other model parameters for the asymptotic error bound to hold) required for Eq. (5) in Theorem 2.1 is slightly more restrictive than the actual error bound suggests. The condition arises from the use of Bernstein-type concentration inequalities, where the variance of a concentrated empirical sum is much smaller than its high-order moments. We are not sure whether such finite-sample conditions are results of a fundamental information-theoretical limitation, or can be avoided by a more refined analysis.

Confidence intervals constructed in Sec 3 requires more stringent conditions to be asymptotically level $α$. Eq. (10) suggests that at least $n \gg s^2 ρ_∗^{-4} \log p$ needs to be satisfied. On the other hand, Cai & Guo (2015) shows that no adaptive confidence intervals exist under the regime of $n < s^2$.

**On Condition (A5)** (A5) requires the population precision matrix $Σ_0^{-1}$ to be sparse, which could be restrictive as the precision matrix is only a nuisance parameter and confidence intervals of $β_0$ do not necessarily require good estimation of $Σ_0^{-1}$. Javanmard & Montanari (2014) drops such sparse precision conditions at the cost of asymptotic efficiency of the average length of the resulting CI.
However, the techniques in (Javanmard & Montanari, 2014) cannot be easily adapted to the missing data case because both estimates $\hat{\Theta}$ and $\hat{\beta}_n$ depend on the randomness of the missing patterns $R$. In our proof we circumvent this issue by connecting to the deterministic population precision $\Sigma_0^{-1}$, which we do not know how to generalize to the case when $\Sigma_0^{-1}$ is not sparse.

6 Proofs

6.1 Additional notations on concentration bounds

Definition 6.1. Let $A, B$ be random or deterministic square matrices of the same size and $\varepsilon$ be a random vector of i.i.d. $N(0, \sigma^2_\varepsilon)$ components. Let $\varphi_{u,v}(A, B; \log N)$, $\varphi_{u,\infty}(A, B; \log N)$, $\varphi_{v,\infty}(A)$ be terms such that, with probability $1 - o(1)$, for all subset $S$ of vectors with $|S| \leq N$, the following hold for all $u, v \in S$:

$$|u^\top (A - B)v| \leq \varphi_{u,v}(A, B; \log N) \cdot \|u\|_2\|v\|_2;$$

$$\left\|A^\top \varepsilon\right\|_\infty \leq \varphi_{v,\infty}(A) \cdot \sigma_\varepsilon.$$

Note that $\varphi_{u,v}(\cdot, \cdot)$ is symmetric and satisfies the triangle inequality. Also, infinity norms like $\|A - B\|_\infty$ or $\|(A - B)u\|_\infty$ for a fixed $u$ can be upper bounded by $\varphi_{u,v}(A, B; O(\log \dim(A)))$, by considering the set of unit vectors $\{e_1, \cdots, e_{\dim(A)}\}$.

6.2 Proof of Theorem 2.1

We need the following two concentration lemmas, which are proved in the supplementary material.

Lemma 6.1. Denote random matrices $A^{(\ell)}$, $\ell \in \{0, 1, 2\}$ as $A^{(0)} = \hat{\Sigma}$, $A^{(1)} = \frac{1}{n} \hat{X}^\top X$ and $A^{(2)} = \hat{\Sigma}$, respectively. Then for $\ell \in \{0, 1, 2\}$:

$$\varphi_{u,v}(A^{(\ell)}; \log N) \leq O \left(\sigma_x^2 \max \left\{ \frac{\log N}{\rho_n^{1.5\ell}}, \sqrt{\frac{\log N}{\rho_n^2}} \right\}\right).$$

Lemma 6.2. If $\log p \rho_n \to 0$ then $\varphi_{\varepsilon,\infty}(\frac{1}{n} \hat{X}) \leq O(\sigma_x \sigma_\varepsilon \sqrt{\frac{\log p}{\rho_n^2}})$.

We present the following lemma commonly known as the basic inequality in high-dimensional inference literature. Its proof is given in the supplementary material.

Lemma 6.3 (Basic inequality). Suppose $\frac{\log p}{\rho_n^2} \to 0$ for $\hat{\beta}_n$ or $\frac{\log p}{\rho_n^2} \to 0$ for $\hat{\beta}_n$, and let $J_0 = \text{supp}(\beta_0)$ be the support of $\beta_0$. If $\tilde{\lambda}_n \geq \Omega\left\{\sigma_x \sqrt{\frac{\log p}{n}} (\frac{\sigma_x |J_0|}{\rho_n} + \frac{\sigma_\varepsilon}{\sqrt{\rho_n}})\right\}$ and $\tilde{\lambda}_n \geq \Omega\left\{\sigma_x \sqrt{\frac{\log p}{\rho_n}} (\sigma_x |J_0| + \sigma_\varepsilon)\right\}$, then with probability $1 - o(1)$ we have that $\|\hat{\beta}_n - \beta_0\|_1 \leq \|\hat{\beta}_n - \beta_0\|_1$ and $\|\hat{\beta}_n - \beta_0\|_1 \leq \|\hat{\beta}_n - \beta_0\|_1$. 

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**Definition 6.2** (Restricted eigenvalue condition). A \( p \times p \) matrix \( A \) is said to satisfy \( \text{RE}(s, \phi_{\min}) \) if for all \( J \subseteq [p], |J| \leq s \) the following holds:

\[
\inf_{h \neq 0, \|h_{j}\|_1 = 1} \frac{h^T Ah}{h^T h} \geq \phi_{\min}.
\]

The following lemma is proved in the supplementary material.

**Lemma 6.4.** Suppose \( \frac{s^2 \log(p) \log(p/\rho_s)}{\rho_s^2 \lambda_{\min}^2 n} \to 0 \). Then with probability \( 1 - o(1) \), \( \hat{\Sigma} \) satisfies \( \text{RE}(s, (1 - o(1)) \lambda_{\min}(\Sigma_0)) \).

We are now ready to prove Theorem 2.1 that establishes the rate of convergence of the noisy Dantzig selector estimators. We consider \( \hat{\beta}_n \) first. Define \( \hat{\lambda}_n \mu = \frac{1}{n} \hat{X}^T y - \hat{\Sigma} \hat{\beta}_n \). By \( y = X \beta_0 + \epsilon \), we have that

\[
\hat{\Sigma}(\hat{\beta}_n - \beta_0) = \left( \frac{1}{n} \hat{X}^T X - \Sigma_0 \right) \beta_0 + \left( \Sigma_0 - \hat{\Sigma} \right) \beta_0 - \hat{\lambda}_n \mu + \frac{1}{n} \hat{X}^T \epsilon.
\]

Multiply both sides by \( (\hat{\beta}_n - \beta_0) \) and apply Hölder’s inequality:

\[
(\hat{\beta}_n - \beta_0)^T \hat{\Sigma}(\hat{\beta}_n - \beta_0)
\]

\[
\leq \|\hat{\beta}_n - \beta_0\|_1 \left\{ \left\| \left( \frac{1}{n} \hat{X}^T X - \Sigma_0 \right) \beta_0 \right\|_\infty + \left\| \left( \Sigma_0 - \hat{\Sigma} \right) \beta_0 \right\|_\infty + \hat{\lambda}_n \|\mu\|_\infty + \left\| \frac{1}{n} \hat{X}^T \epsilon \right\|_\infty \right\}
\]

\[
\leq \|\hat{\beta}_n - \beta_0\|_1 \cdot O_P \left\{ \varphi_{u,v} \left( \frac{1}{n} \hat{X}^T X, \Sigma_0; \log p \right) \|\beta_0\|_2 + \varphi_{u,v} \left( \hat{\Sigma}, \Sigma_0; \log p \right) \|\beta_0\|_2 + \hat{\lambda}_n + \varphi_{\epsilon, \infty} \left( \frac{1}{n} \hat{X} \right) \sigma_\epsilon \right\}
\]

\[
\leq \|\hat{\beta}_n - \beta_0\|_1 \cdot O_P \left\{ \sigma_x^2 \|\beta_0\|_2 \sqrt{\frac{\log p}{\rho_s^2 n}} + \sigma_x \sigma_\epsilon \sqrt{\frac{\log p}{\rho_s^2 n}} + \hat{\lambda}_n \right\}.
\]

Here the last inequality is due to Lemmas 6.1 and 6.2. Suppose \( \frac{s^2 \log(p)\log(p/\rho_s)}{\rho_s^2 \lambda_{\min}^2 n} \to 0 \) and \( \hat{\lambda}_n \) is appropriately set as in Lemma 6.3. We then have

\[
\|\hat{\beta}_n - \beta_0\|_1 \leq 2\|(\hat{\beta}_n - \beta_0) J_0\|_1 \leq 2\sqrt{s} \|\hat{\beta}_n - \beta_0\|_2
\]

by Lemma 6.3 and

\[
(\hat{\beta}_n - \beta_0)^T \hat{\Sigma}(\hat{\beta}_n - \beta_0) \geq (1 - o(1)) \lambda_{\min} \|\hat{\beta}_n - \beta_0\|_2^2
\]

by Lemma 6.4. Chaining all inequalities we get

\[
\|\hat{\beta}_n - \beta_0\|_2 \leq O_P \left( \sqrt{\frac{s}{\lambda_{\min}}} \left\{ \sigma_x^2 \|\beta_0\|_2 \sqrt{\frac{\log p}{\rho_s^2 n}} + \sigma_x \sigma_\epsilon \sqrt{\frac{\log p}{\rho_s^2 n}} + \hat{\lambda}_n \right\} \right)
\]

\[
\leq O_P \left( \sqrt{\frac{s}{\lambda_{\min}}} \left\{ \sigma_x^2 \|\beta_0\|_2 \sqrt{\frac{\log p}{\rho_s^2 n}} + \sigma_x \sigma_\epsilon \sqrt{\frac{\log p}{\rho_s^2 n}} \right\} \right).
\]

The \( \ell_1 \) norm error bound \( \|\hat{\beta}_n - \beta_0\|_1 \) can be easily obtained by the fact that \( \|\hat{\beta}_n - \beta_0\|_1 \leq 2\sqrt{s} \|\hat{\beta}_n - \beta_0\|_2 \).
Finally, consider $\hat{\mu}_n$ and define $\hat{\lambda}_n \hat{\mu} = \frac{1}{n} \hat{X}^T y - \Sigma_0 \hat{\beta}_n$. Note that $\|\hat{\delta}\|_{\infty} \leq 1$ and

$$\Sigma_0 (\hat{\beta}_n - \beta_0) = \left( \frac{1}{n} \hat{X}^T X - \Sigma_0 \right) \beta_0 - \hat{\lambda}_n \hat{\mu} + \frac{1}{n} \hat{X}^T \varepsilon.$$  

Note in addition that $(\hat{\beta}_n - \beta_0)^T \Sigma_0 (\hat{\beta}_n - \beta_0) \geq \lambda_{\min} \|\hat{\beta}_n - \beta_0\|_2^2$ by Assumption (A3). Subsequently, the same line of argument as $\hat{\beta}_n$ yields

$$\|\hat{\beta}_n - \beta_0\|_2 \leq \frac{2\sqrt{s}}{\lambda_{\min}} \cdot O_\mathbb{P} \left\{ \varphi_{\varepsilon,\nu} \left( \frac{1}{n} X^T X, \Sigma_0; \log p \right) \|\beta_0\|_2 + \hat{\lambda}_n + \varphi_{\varepsilon,\infty} \left( \frac{1}{n} \hat{X} \right) \sigma_{\varepsilon} \right\}$$  

\[
\leq O_\mathbb{P} \left\{ \left( \sigma^2_s \|\beta_0\|_2 + \sigma_{\varepsilon} \right) \sqrt{\frac{s \log p}{\lambda_{\min}^2 p \varepsilon}} \right\}.
\]

### 6.3 Proof of Theorem 2.2

We consider the worst case with equal observation rates across covariates: $\rho_1 = \cdots = \rho_p = \rho_*$ and use Fano’s inequality (Lemma A.1) to establish the minimax lower bound in Theorem 2.2. Construct hypothesis $\beta$ as

$$\beta = (a, \ldots, a, 0, \pm \delta, 0, \ldots, \pm \delta, 0),$$  

where $\delta \to 0$ is some parameter to be chosen later and $a = \sqrt{\frac{2M^2}{s} - \delta^2}$ is carefully chosen so that $\|\beta\|_2 = M$. Clearly $\beta \in \mathbb{B}_2(M) \cap \mathbb{B}_0(s)$. Let $d_H(\beta, \beta') = \sum_{j=1}^p I[\beta_j \neq \beta'_j]$ be the Hamming distance between $\beta$ and $\beta'$. The following lemma shows that it is possible to construct a large hypothesis classes where any two models in the hypothesis class are far away under the Hamming distance:

**Lemma 6.5 (Raskutti et al. 2011), Lemma 4.** Define $\mathcal{H} = \{ z \in \{-1, 0, +1\}^p : \|z\|_0 = s \}$. For $p, s$ even and $s < 2p/3$, there exists a subset $\mathcal{H} = \mathcal{H} \subseteq \mathcal{H}$ with cardinality $|\mathcal{H}| \geq \exp\left( \frac{s}{2} \log \frac{e-s}{s/2} \right)$ such that $\rho_H(z, z') \geq s/2$ for all distinct $s, s' \in \mathcal{H}$.

Without loss of generality we shall restrain ourselves to even $p$ and $s/2$ scenarios. This does not affect the minimax lower bound to be proved. Using the above lemma and under the condition that $s \leq 4p/5$, one can construct $\Theta$ consisting of hypothesis of the form in Eq. (13) such that $\log |\Theta| \asymp s \log (p/s)$ and $\|\beta - \beta'\|_2$ for all distinct $\beta, \beta' \in \Theta$. It remains to evaluate the KL divergence between $P_\beta$ and $P_{\beta'}$.

Let $x_{\text{obs}}$ and $x_{\text{mis}}$ denote the observed and missing covariates of a particular data point and let $\beta_{\text{obs}}, \beta_{\text{mis}}$ be the corresponding partition of coordinates of $\beta$. The likelihood of $x_{\text{obs}}$ and $y$ can be obtained by integrating out $x_{\text{mis}}$ (assuming there are $q$ coordinates that are observed):

$$p(y, x_{\text{obs}}; \beta) = \rho_0^q (1 - \rho_*)^{p-q} \int N_p(x_{\text{obs}}, x_{\text{mis}}; 0, I) N(y - (x_{\text{obs}}^T \beta_{\text{obs}} - x_{\text{mis}}^T \beta_{\text{mis}}; 0, \sigma_{\varepsilon}^2) dx_{\text{mis}}$$  

$$= p(x_{\text{obs}}) \cdot \frac{1}{\sqrt{2\pi (\sigma_{\varepsilon}^2 + \|\beta_{\text{mis}}\|_2^2)}} \exp \left\{ \frac{(y - x_{\text{obs}}^T \beta_{\text{obs}})^2}{2(\sigma_{\varepsilon}^2 + \|\beta_{\text{mis}}\|_2^2)} \right\}.$$  

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Note that \( p(x_{\text{obs}}) \) does not depend on \( \beta \). Subsequently,

\[
\text{KL}(P_\beta \| P_{\beta'}) = \mathbb{E}_{\beta, \rho_*} \log \frac{p(y, x_{\text{obs}}; \beta')}{p(y, x_{\text{obs}}; \beta)}
\]

\[
= \mathbb{E}_{\beta, \rho_*} \left\{ \frac{1}{2} \log \frac{\sigma^2 + \|\beta'_{\text{mis}}\|^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} + \frac{1}{2} \left( \frac{(y - x_{\text{T}_{\text{obs}}})^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} - \frac{(y - x_{\text{T}_{\text{obs}}})^2}{\sigma^2 + \|\beta'_{\text{mis}}\|^2} \right) \right\}
\]

\[
= \mathbb{E}_{\rho_*} \left\{ \frac{1}{2} \log \frac{\sigma^2 + \|\beta'_{\text{mis}}\|^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} + \frac{1}{2} \left( \frac{\sigma^2 + \|\beta_{\text{mis}}\|^2}{\sigma^2 + \|\beta'_{\text{mis}}\|^2} - 1 \right) \right\}
\]

(\( a \)) \[
\leq \mathbb{E}_{\rho_*} \left\{ \frac{1}{2} \left( \frac{\sigma^2 + \|\beta'_{\text{mis}}\|^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} - 1 \right) \right\} + \frac{1}{2} \left( \frac{\|\beta_{\text{obs}} - \beta'_{\text{obs}}\|^2}{\|\beta'_{\text{obs}}\|^2} \right).
\]  

(14)

Here for (\( a \)) we apply the inequality that \( \log(1 + x) \leq x \) for all \( x > 0 \). For some constant \( c \in (0, 1/2) \), define \( \mathcal{E}(c) \) as the event that at least \( \frac{1 - \rho_*}{2} \) portion of the first \( s/2 \) coordinates in \( x \) are missing. By Chernoff bound, \(^2 \) \( \Pr[\mathcal{E}(c)] \geq 1 - e^{-c^2(1 - \rho_*)s} \). Note that under \( \mathcal{E}(c) \), \( \|\beta_{\text{mis}}\|^2, \|\beta'_{\text{mis}}\|^2 \geq \frac{(1 - \rho_*)s}{2(1 + 2c)}a^2 \) almost surely. Subsequently,

\[
\text{KL}(P_\beta \| P_{\beta'}) \leq \frac{1}{2} \mathbb{E}_{\rho_*}[\mathcal{E}(c)] \left( \frac{\|\beta'_{\text{mis}}\|^2 - \|\beta_{\text{mis}}\|^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} \right)^2 + \frac{1}{2} \mathbb{E}_{\rho_*}[\mathcal{E}(c)] \left( \frac{\|\beta_{\text{obs}} - \beta'_{\text{obs}}\|^2}{\sigma^2 + \|\beta'_{\text{mis}}\|^2} \right)
\]

\[
+ e^{-c^2(1 - \rho_*)s} \left( \frac{1}{2} \mathbb{E}_{\rho_*}[\mathcal{E}(c)] \left( \frac{\|\beta'_{\text{mis}}\|^2 - \|\beta_{\text{mis}}\|^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} \right)^2 + \frac{1}{2} \mathbb{E}_{\rho_*}[\mathcal{E}(c)] \left( \frac{\|\beta_{\text{obs}} - \beta'_{\text{obs}}\|^2}{\sigma^2 + \|\beta'_{\text{mis}}\|^2} \right) \right).
\]

Because \( \beta \) and \( \beta' \) are identical in the first \( s/2 \) coordinates, both \( \|\beta'_{\text{mis}}\|^2 - \|\beta_{\text{mis}}\|^2 \) and \( \|\beta_{\text{obs}} - \beta'_{\text{obs}}\|^2 \) are independent of \( \mathcal{E}(c) \). Therefore,

\[
\mathbb{E}_{\rho_*} \left( \frac{\|\beta'_{\text{mis}}\|^2 - \|\beta_{\text{mis}}\|^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} \right)^2 = \mathbb{E}_{\rho_*} \left( \frac{\|\beta'_{\text{mis}, > s/2}\|^2 - \|\beta_{\text{mis}, > s/2}\|^2}{\sigma^2 + \|\beta_{\text{mis}}\|^2} \right) \leq 4(1 - \rho_*)^2s^2 \delta^4;
\]

\[
\mathbb{E}_{\rho_*} \|\beta'_{\text{obs}} - \beta_{\text{obs}}\|^2 = \mathbb{E}_{\rho_*} \|\beta'_{\text{obs}, > s/2} - \beta_{\text{obs}, > s/2}\|^2 \leq 2\rho_*s^2 \delta^2.
\]

Here \( \beta_{\text{obs}, > s/2} \) denote the \( \beta \) vector without its first \( s/2 \) coordinates, and in both inequalities we note by construction that \( \|\beta_{\text{obs}, > s/2}\|^0, \|\beta'_{\text{obs}, > s/2}\|^0 \leq s/2 \). Because \( a^2 = \frac{2M^2}{s} - \delta^2 \), we have that

\[
\frac{(1 - \rho_*)s}{2(1 + 2c)}a^2 = \frac{1 - \rho_*}{1 + 2c}M^2 - \frac{(1 - \rho_*)s}{2(1 + 2c)} \delta^2.
\]

For now assume that \( \frac{1 - \rho_*}{1 + 2c}s \delta^2 \ll \sigma^2 + \frac{1 - \rho_*}{1 + 2c}M^2 \), then implies \( \sigma^2 + \frac{1 - \rho_*}{1 + 2c}a^2 \geq \frac{1}{2} \left( \sigma^2 + \frac{1 - \rho_*}{1 + 2c}M^2 \right) \). We will justify this assumption at the end of this proof. Combining all inequalities we have

\[
\text{KL}(P_\beta \| P_{\beta'}) \leq \frac{8(1 - \rho_*)^2s^2 \delta^4}{\sigma^2 + \frac{1 - \rho_*}{1 + 2c}M^2} + \frac{2\rho_*s^2 \delta^2}{\sigma^2 + \frac{1 - \rho_*}{1 + 2c}M^2} + e^{-c^2(1 - \rho_*)s} \left[ \frac{2(1 - \rho_*)^2s^2 \delta^4}{\sigma^2} + \rho_*s^2 \delta^2 \right].
\]

\(^2\)\text{If } X_1, \ldots, X_n \text{ are i.i.d. random variables taking values in } \{0, 1\} \text{ then } \Pr[\frac{1}{n} \sum_{i=1}^{n} X_i < (1 - \delta)\mu] \leq \exp\left(-\frac{\delta^2\mu}{2}\right) \text{ for } 0 < \delta < 1, \text{ where } \mu = \mathbb{E}X.
Let $P^n_\beta$ and $P^n_{\beta'}$ be the distribution of $n$ i.i.d. samples parameterized by $\beta$ and $\beta'$, respectively. Because the samples are i.i.d., we have that $\text{KL}(P^n_\beta || P^n_{\beta'}) = n \text{KL}(P_\beta || P_{\beta'})$. On the other hand, because $\log|\Theta| \asymp s \log(p/s)$, to ensure $1 - \frac{\text{KL}(P^n_\beta || P^n_{\beta'}) + \log 1/2}{\log|\Theta|} \geq \Omega(1)$ we only need to show $\text{KL}(P^n_\beta || P^n_{\beta'}) \asymp s \log(p/s)$, which is implied by

$$\frac{(1 - \rho_s)^2 s^2 \delta^4}{\sigma^2 \left( \frac{\rho_s s^2 \delta^2}{\sigma^2} + \frac{1 - \rho_s}{1 + 2c} M^2 \right)^2} \asymp \frac{s \log(p/s)}{n} \iff \delta^2 \asymp \left( \frac{\sigma^2}{\rho_s} + \frac{1 - \rho_s}{1 + 2c} M^2 \right) \sqrt{\frac{\log(p/s)}{(1 - \rho_s)^2 sn}};$$

$$\frac{\rho_s s^2 \delta^2}{\sigma^2} \asymp \frac{s \log(p/s)}{n} \iff \delta^2 \asymp \left( \frac{\sigma^2}{\rho_s} + \frac{1 - \rho_s}{1 + 2c} M^2 \right) \frac{\log(p/s)}{\rho_s n};$$

$$e^{-c^2(1 - \rho_s)s} \frac{(1 - \rho_s)^2 s^2 \delta^4}{\sigma^2} \asymp \frac{s \log(p/s)}{n} \iff \delta^2 \asymp e^{0.5c^2(1 - \rho_s)s} \frac{\log(p/s)}{\rho_s n}.$$

Combining all terms we have that

$$\delta^2 \asymp \min \left\{ \frac{\sigma^2}{\rho_s} + \frac{1 - \rho_s}{1 + 2c} M^2, e^{0.5c^2(1 - \rho_s)s} \sigma^2 \right\} \cdot \min \left\{ \sqrt{\frac{\log(p/s)}{(1 - \rho_s)^2 sn}}, \frac{s \log(p/s)}{\rho_s n} \right\}. \quad (15)$$

The bound for $||\beta - \beta'||^2_2$ can then be obtained by $||\beta - \beta'||^2_2 \geq \frac{\delta^2}{4}$. The final part of the proof is to justify the assumption that $\frac{1 - \rho_s}{1 + 2c} s \delta^2 \ll \sigma^2 + \frac{1 - \rho_s}{1 + 2c} M^2$. Invoking Eq. (15), the assumption is valid if $\frac{s \log(p/s)}{n} \max \left\{ \sqrt{\frac{\log(p/s)}{(1 - \rho_s)^2 sn}}, \frac{s \log(p/s)}{\rho_s n} \right\} \rightarrow 0$, which holds if $\frac{s \log(p/s)}{\rho_s n} \rightarrow 0$.

### 6.4 Proof of Theorem 2.3

We again take $\rho_1 = \cdots = \rho_p = \rho_s$. The first term $\frac{\sigma^2}{\rho_s n}$ in the minimax lower bound is trivial to establish: consider $\beta_0 = \delta e_j$ and $\beta_1 = -\delta e_j$ with $\Sigma_0 = \Sigma_1 = I$. By Eq. (14), we have that

$$\text{KL}(P^n_{\beta_0} || P^n_{\beta_1}) = n \cdot \text{KL}(P_{\beta_0} || P_{\beta_1}) \leq \frac{2\rho_s n \delta^2}{\sigma^2}.$$

Equating $\text{KL}(P^n_{\beta_0} || P^n_{\beta_1}) \asymp O(1)$ we have that $\delta^2 \asymp \frac{\sigma^2}{\rho_s n}$. Because $\frac{\sigma^2}{M^2 \rho_s n} \rightarrow 0$, we know that $\beta_0, \beta_1 \in \mathbb{B}_2(M) \cap \mathbb{B}_0(1)$ when $n$ is sufficiently large. Invoking Le Cam’s method (Lemma A.2) with $|\beta_0j - \beta_1j|^2 = 4\delta^2 \asymp \frac{\sigma^2}{\rho_s n}$, we prove the desired minimax lower bound of $\frac{\sigma^2}{\rho_s n}$.

We next focus on the second term in the minimax lower bound that involves $1/\rho_s^2 n$. Without loss of generality assume $j > s - 1$. Construct two hypothesis $(\beta_0, \Sigma_0)$ and $(\beta_1, \Sigma_1)$ as follows:

$$\beta_0 = \left( \frac{\tilde{a}}{\sqrt{s - 2}}, \cdots, \frac{\tilde{a}}{\sqrt{s - 2}} \right) \tilde{a}, 0, \cdots, 0, \tilde{a}^{\gamma}, 0, \cdots, 0), \quad \Sigma_0 = I_{p \times p} - \gamma (e_{s-1}e_j^\top + e_j e_{s-1}^\top);$$

$$\beta_1 = \left( \frac{\tilde{a}}{\sqrt{s - 2}}, \cdots, \frac{\tilde{a}}{\sqrt{s - 2}} \right) \tilde{a}^{\gamma}, 0, \cdots, 0, \tilde{a}, 0, \cdots, 0). \quad \Sigma_1 = I_{p \times p} - \gamma (e_j e_{s-1}^\top + e_{s-1} e_j^\top).$$
\[ \beta_1 = \left( \frac{\tilde{a}}{s-2}, \cdots, \frac{\tilde{a}}{s-2}, \tilde{a}, 0, \cdots, 0, -\tilde{a} \gamma, 0, \cdots, 0 \right), \quad \Sigma_1 = I_{p \times p} + \gamma (e_{s-1} e_j^T + e_j e_{s-1}^T). \]

Here \( \gamma \to 0 \) is some parameter to be determined later and \( \tilde{a} \) is set to \( \tilde{a} = \sqrt{\frac{M^2}{2 + \gamma}} \) to ensure that \( \| \beta_0 \|_2 = \| \beta_1 \|_2 = M \). It is immediate by definition that \( \beta_0, \beta_1 \in \mathbb{B}_2(M) \cap \mathbb{B}_0(s) \). In addition, by Gershgorin circle theorem all eigenvalues of \( \Sigma_0 \) and \( \Sigma_1 \) lie in \( [1 - \gamma, 1 + \gamma] \). As \( \gamma \to 0 \), it holds that \( \Sigma_0, \Sigma_1 \in \Lambda(\gamma_0) \) for any constant \( \gamma_0 \in (0, 1/2) \) when \( n \) is sufficiently large. A finite-sample statement of this fact is given at the end of the proof.

Unlike the identity covariance case, the likelihood \( p(y, x_{\text{obs}}; \beta, \Sigma) \) for incomplete observations are complicated when \( \Sigma \) has non-zero off-diagonal elements. The following lemma gives a general characterization of the likelihood when \( \beta \neq 0 \). Its proof is given in the supplementary material.

**Lemma 6.6.** Partition the covariance \( \Sigma \) as \( \Sigma = \left[ \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array} \right] \), where \( \Sigma_{11} \) corresponds to \( x_{\text{obs}} \) and \( \Sigma_{22} \) corresponds to \( x_{\text{mis}} \). Define \( \Sigma_{22:1} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \). Let \( q = \dim(\Sigma_{11}) \) be the number of observed covariates. Then

\[
p(y, x_{\text{obs}}; \beta, \Sigma) = \rho_s^p (1 - \rho_s)^{p-q} \cdot \frac{1}{\sqrt{2\pi q (\Sigma_{11})}} \exp \left\{ -\frac{1}{2} x_{\text{obs}}^T \Sigma_{11}^{-1} x_{\text{obs}} \right\} \cdot \frac{1}{\sqrt{2\pi (\sigma_e^2 + \beta_{\text{mis}}^T \Sigma_{22:1}^{-1} \beta_{\text{mis}})}} \exp \left\{ -\frac{(y - x_{\text{obs}}^T \beta_{\text{obs}} - \beta_{\text{mis}}^T \Sigma_{21} \Sigma_{11}^{-1} x_{\text{obs}})^2}{2(\sigma_e^2 + \beta_{\text{mis}}^T \Sigma_{22:1} \beta_{\text{mis}})} \right\}.
\]

We now present the following lemma, which is key to establish the \( 1/\rho_s^2 \) rate in the minimax lower bound. Its proof is given in the supplementary material.

**Lemma 6.7.** \( p(y, x_{\text{obs}}; \beta_0, \Sigma_0) = p(y, x_{\text{obs}}; \beta_1, \Sigma_1) \) unless both \( x_{s-1} \) and \( x_j \) are observed.

Let \( P_0 \) and \( P_1 \) denote the distributions parameterized by \( (\beta_0, \Sigma_0) \) and \( (\beta_1, \Sigma_1) \), respectively. Let \( \mathcal{A} \) denote the event that both \( x_{s-1} \) and \( x_j \) are observed. By Lemma 6.7, we have that

\[
\text{KL}(P_0||P_1) = \text{Pr}[\mathcal{A}] \mathbb{E}_0 \left[ \log \frac{p(y, x_{\text{obs}}; \beta_0, \Sigma_0)}{p(y, x_{\text{obs}}; \beta_1, \Sigma_1)} | \mathcal{A} \right] = \rho_s^2 \mathbb{E}_0 \left[ \log \frac{p(y, x_{\text{obs}}; \beta_0, \Sigma_0)}{p(y, x_{\text{obs}}; \beta_1, \Sigma_1)} | \mathcal{A} \right].
\]

Suppose \( \Sigma_0 = [\Sigma_{011}; \Sigma_{012}; \Sigma_{021}; \Sigma_{022}] \) and \( \Sigma_1 = [\Sigma_{111}; \Sigma_{112}; \Sigma_{121}, \Sigma_{122}] \) are partitioned in the same way as in Lemma 6.6. Conditioned on the event \( \mathcal{A} \), we have that \( \Sigma_{022} = \Sigma_{122} = I_{(p-q) \times (p-q)} \), \( \Sigma_{012} = \Sigma_{112} = \Sigma_{121} = 0_{q \times (p-q)} \), \( \Sigma_{011} = I_{q \times q} - \gamma (e_{s-1} e_j^T + e_j e_{s-1}^T) \), \( \Sigma_{111} = I_{q \times q} + \gamma (e_{s-1} e_j^T + e_j e_{s-1}^T) \), and by Lemma A.3, we have that \( \Sigma_{011}^{-1} = I + \frac{2}{1 - \gamma^2} (e_{s-1} e_j^T + e_j e_{s-1}^T) \) and \( \Sigma_{111}^{-1} = I + \frac{2}{1 - \gamma^2} (e_{s-1} e_j^T + e_j e_{s-1}^T) - \frac{2}{1 - \gamma^2} (e_{s-1} e_j^T + e_j e_{s-1}^T) \). In addition, \( \det(\Sigma_{011}) = \det(\Sigma_{111}) = 1 - \gamma^2 \). Note also that \( \Sigma_{022:1} = \Sigma_{122:1} = I_{(p-q) \times (p-q)} \) and hence \( \beta_{\text{mis}}^{\Sigma_{022:1}} = \beta_{\text{mis}}^{\Sigma_{122:1}} \) regardless of which covariates are missing. Define \( x_{\text{obs}, s} = \{x_j : x_j \text{ is observed}; j < s\} \) and \( \beta_{\text{obs}, s} = \{\beta_j : x_j \text{ is observed}; j < s\} \). Subsequently, invoking Lemma 6.6 we get

\[
\mathbb{E}_0[\mathcal{A}] \left[ \log \frac{P_0}{P_1} \right] = -\frac{2\gamma}{1 - \gamma^2} \mathbb{E}_0[x_{s-1} x_j] - \mathbb{E}_0[\mathcal{A}] \left\{ \frac{1}{2} \frac{(y - x_{\text{obs}}^T \beta_{\text{obs}})^2 - (y - x_{\text{obs}}^T \beta_{\text{obs}})^2}{\sigma_e^2 + \| \beta_{\text{mis}} \|^2_2} \right\}.
\]
we finish the proof of the minimax lower bound. Subsequently, here note that except for $k = j$, we justify the conditions $(a)$ due to $\beta_{0\text{obs},s} = \beta_{1\text{obs},s}$ and $\beta_{0,j}^2 = \beta_{1,j}^2$, and $(b)$ is because $\beta_{0,k} = 0$ for all $k \geq s$ except for $k = j$. Note also that under $A$, $x_j$ is observed and hence $\beta_{0,j}$ always belongs to $\beta_{0\text{obs}}$. For $(c)$, note that $x_{s-1}$ is observed under $A$ and $x_j$ is independent of $x_{<s-1}$ and $\varepsilon$ conditioned on $R$, thanks to the missing completely at random assumption (A3). For any constant $c \in (0,1/2)$ define $E'(c)$ as the event that at least $\frac{1 - \rho_s}{1 + 2c} \hat{a}^2$ portion of the first $(s - 2)$ coordinates in $x$ are missing. Note that $\|\beta_{0\text{mis}}\|_2^2 \geq \frac{1 - \rho_s}{1 + 2c} \hat{a}^2$ almost surely under $A \cap E'(C)$ and by Chernoff bound $\Pr[A] \geq 1 - e^{-c^2(1-\rho_s)(s-2)} \geq 1 - e^{-0.5c^2(1-\rho_s)s}$ for $s \geq 4$. Subsequently, by law of total expectation

$$\mathbb{E}_{R|A}\left\{ \frac{2\hat{a}^2\gamma^2}{\sigma^2 + \|\beta_{0\text{mis}}\|_2^2} \right\} \leq \frac{2\hat{a}^2\gamma^2}{\sigma^2 + \frac{1 - \rho_s}{1 + 2c} \hat{a}^2} + e^{-0.5c^2(1-\rho_s)s} \frac{2\hat{a}^2\gamma^2}{\sigma^2}.$$

Replace $\hat{a}^2 = \frac{M^2}{2 + \gamma^2}$. We then have that

$$\text{KL}(P^n_0 || P^n_1) \leq n\rho_s^2 \left[ \frac{2\gamma^2}{1 - \gamma^2} + \frac{2M^2\gamma^2}{(2 + \gamma^2)\sigma^2 + \frac{1 - \rho_s}{1 + 2c} M^2} + e^{-0.5c^2(1-\rho_s)s} \frac{2M^2\gamma^2}{(2 + \gamma^2)\sigma^2} \right].$$

Equating $\text{KL}(P^n_0 || P^n_1) \propto O(1)$ and applying the condition that $\gamma^2 \to 0$, we have that

$$\gamma^2 \sim \min \left\{ \frac{1 - \rho_s}{2(1 + 2c)} M^2, e^{0.5c^2(1-\rho_s)s} \frac{\sigma^2}{M^2} \right\} \frac{1}{\rho_s^2 n}. \quad (16)$$

Subsequently,

$$|\beta_{0j} - \beta_{1j}|^2 = 4\hat{a}^2\gamma^2 \sim \min \left\{ \frac{1 - \rho_s}{2(1 + 2c)} M^2, e^{0.5c^2(1-\rho_s)s} \frac{\sigma^2}{M^2} \right\} \frac{1}{\rho_s^2 n}.$$

Invoking Lemma A.2 we finish the proof of the minimax lower bound.

Finally, we justify the conditions $\gamma^2 \to 0$ and $\gamma < \gamma_0$ that are used in the proof. Eq. (16) yields $\gamma^2 \leq O(\frac{1}{\rho_s^2 n})$. So $\gamma^2 \to 0$ and $\gamma < \gamma_0$ is implied by $\frac{1}{\gamma_0 \rho_s^2 n} \to 0$. 

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6.5 Proof of Theorem 3.1

Using $y = X\beta_0 + \varepsilon$ we have that

$$\hat{\Sigma}(\hat{\beta}_n - \beta_0) + \left(\frac{1}{n} \bar{X}^T y - \hat{\Sigma}\right) = \left(\frac{1}{n} \bar{X}^T X - \hat{\Sigma}\right) \beta_0 + \frac{1}{n} \bar{X}^T \varepsilon. \tag{17}$$

Define $\Delta_n = \frac{1}{n} \bar{X}^T X - \hat{\Sigma}$, Recall that $\hat{\beta}_n = \hat{\beta} + \bar{\Theta} \left(\frac{1}{n} \bar{X}^T y - \hat{\Sigma}\right)$. Subsequently, multiplying both sides of Eq. (17) with $\sqrt{n} \bar{\Theta}$ and re-organizing terms we have

$$\sqrt{n}(\hat{\beta}_n - \beta_0) = \sqrt{n} \bar{\Theta} (\Delta_n \beta_0 + \frac{1}{n} \bar{X}^T \varepsilon) - \sqrt{n}(\bar{\Theta} \hat{\Sigma} - I)(\hat{\beta}_n - \beta_0)$$

$$= \sqrt{n} \Sigma_0^{-1} \left(\Delta_n \beta_0 + \frac{1}{n} \bar{X}^T \varepsilon\right) - \sqrt{n}(\bar{\Theta} \hat{\Sigma} - I)(\hat{\beta}_n - \beta_0) + \sqrt{n}(\bar{\Theta} - \Sigma_0^{-1})(\Delta_n \beta_0 + \frac{1}{n} \bar{X}^T \varepsilon).$$

Define $r_n = \sqrt{n}(\bar{\Theta} \hat{\Sigma} - I)(\hat{\beta}_n - \beta_0)$ and $\tilde{r}_n = \sqrt{n}(\bar{\Theta} - \Sigma_0^{-1})(\Delta_n \beta_0 + \frac{1}{n} \bar{X}^T \varepsilon)$

**Lemma 6.8.** Suppose $\frac{\log p}{\sqrt{n} \log n} \to 0$ and the conclusion in Lemma 3.1 holds. Then $\|r_n\|_{\infty} \leq O_P(\sqrt{n} \log n \|\hat{\beta}_n - \beta_0\|_1)$ and $\|\tilde{r}_n\|_{\infty} \leq O_P(\sigma_x \beta_0 b_1 \tilde{v}_n(\sigma_x \sqrt{\frac{\log p}{\sqrt{n}}}) + \sigma_x \|\beta_0\|_2 \sqrt{\log p}).$

Lemma 6.8 based on Hölder’s inequality and is proved in the supplementary materials. If the condition in Eq. (9) holds, Lemma 6.8 implies that $\max\{\|r_n\|_{\infty}, \|\tilde{r}_n\|_{\infty}\} \xrightarrow{p} 0$, which means both terms $r_n$ and $\tilde{r}_n$ are asymptotically negligible in the infinity norm sense. It then suffices to analyze the limiting distribution (conditioned on $X$) of $a_n = \sqrt{n} \Sigma_0^{-1} \left(\Delta_n \beta_0 + \frac{1}{n} \bar{X}^T \varepsilon\right)$. By Assumptions (A1) and (A3), $\mathbb{E}\Delta_n|X = 0$, $\mathbb{E}|X = 0$ and hence $\mathbb{E}a_n|X = 0$. We next analyze the conditional covariance $\mathbb{V}a_n|X$. Recall that $\Delta_n = \frac{1}{n} \bar{X}^T X - \hat{\Sigma}$. By definition, for any $j, k \in \{1, \cdots, p\}$

$$[\Delta_n]_{jk} = \left\{\begin{array}{ll}
\frac{1}{n} \sum_{i=1}^{n} \frac{R_{ij}}{\rho_j} \left(1 - \frac{R_{ik}}{\rho_k}\right) X_{ij} X_{ik}, & j \neq k; \\
0, & j = k.
\end{array}\right.$$  

Here $R_{ij} = 1$ if $X_{ij}$ is observed and $R_{ij} = 0$ otherwise. Subsequently, $a_n = \Sigma_0^{-1} a_n$ where

$$[\tilde{a}_n]_{j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{R_{ij} X_{ij}}{\rho_j} \varepsilon_i + \sum_{k \neq j} \frac{R_{ij}}{\rho_k} \left(1 - \frac{R_{ik}}{\rho_k}\right) X_{ij} X_{ik}/\beta_{0k}\right).$$

Because $R \perp X, \varepsilon$ and $\varepsilon \perp X$, we have that $\mathbb{E}\varepsilon T_{ij}|X = 0$. Therefore, for any $j \in \{1, \cdots, p\}$

$$\mathbb{V}T_{ij}|X = \mathbb{E} \left[|T_{ij}|^2|X \right] = \frac{\sigma_x^2 X_{ij}^2}{\rho_j} + \sum_{t \neq j} \frac{1 - \rho_{jt}}{\rho_j \rho_t} X_{ij}^2 X_{it}^2/\beta_{0t}^2.$$  

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and for $j \neq k$,
\[
\text{cov}(T_{ij}, T_{ik}|X) = \mathbb{E}[T_{ij}T_{ik}|X] = \sigma_x^2 X_{ij}X_{ik} + \sum_{t \neq j, k} \frac{1 - \rho_t}{\rho_t} X_{ij}X_{ik}X_{it}^2 \rho_{0t}^2.
\]

Because $\{T_{ij}\}_{i=1}^n$ are i.i.d. random variables, by central limiting theorem, for any subset $S \subseteq [p]$ with constant size
\[
[a_n]_{SS} \xrightarrow{d} \mathcal{N}|S| (0, \text{cov}_{SS}(a_n|X)) \xrightarrow{d} \mathcal{N}|S| \left(0, \left[\Sigma_0^{-1} \hat{\Gamma} \Sigma_0^{-1}\right]_{SS}\right),
\]
where all randomness is conditioned on $X$.

### 6.6 Proof of Theorem 3.2

By triangle inequality and Hölder’s inequality,
\[
\begin{align*}
\|\Sigma_0^{-1} \hat{\Gamma} \Sigma_0^{-1} - \hat{\Theta} \hat{\Theta}^\top\|_\infty &\leq \|\Sigma_0^{-1} \hat{\Theta} \Sigma_0^{-1}\|_\infty + \|\hat{\Theta} (\Sigma_0^{-1} - \hat{\Theta})\|_\infty + \|\hat{\Theta} (\hat{\Gamma} - \hat{\Gamma})\hat{\Theta}^\top\|_\infty \\
&\leq 2 \max \left\{\|\Sigma_0^{-1}\|_{L_1}, \|\hat{\Theta}\|_{L_1}, \|\hat{\Theta}\|_{L_\infty}\right\} \max \left\{\|\Sigma_0^{-1} - \hat{\Theta}\|_{L_1}, \|\Sigma_0^{-1} - \hat{\Theta}\|_{L_\infty}\right\} \|\hat{\Gamma}\|_\infty + \|\hat{\Theta}\|_{L_1}^2 \|\hat{\Gamma} - \hat{\Gamma}\|_\infty.
\end{align*}
\]

With Lemma 3.1, the bound can be simplified to (with probability $1 - o(1)$)
\[
\|\Sigma_0^{-1} \hat{\Gamma} \Sigma_0^{-1} - \hat{\Theta} \hat{\Theta}^\top\|_\infty \leq 4b_0 b_1^2 \bar{c}_n \|\hat{\Gamma}\|_\infty + b_1^2 \|\hat{\Gamma} - \hat{\Gamma}\|_\infty. \tag{18}
\]

Note that by standard concentration inequalities of supreme of sub-Gaussian random variables, $\|X\|_\infty \leq O_p(\sigma_x \sqrt{\log p})$. Also, by Hölder’s inequality $\|\hat{\Theta}\|_\infty \leq \rho_*^{-2} \|X\|_{2, \infty}^2 \rho_0^2 \|. \quad$ Subsequently,
\[
\|\hat{\Gamma}\|_\infty \leq \frac{\sigma_x^2}{\rho_*} \|X\|_{2, \infty}^2 + \|X\|_{2, \infty} \|\beta_0\|_2^2 \leq O_p \left\{ \sigma_x^4 \log^2 p \left( \frac{\sigma_x^2}{\sigma_x^2 \rho_*} + \frac{\|\beta_0\|_2^2}{\rho_*^2} \right) \right\}. \tag{19}
\]

It remains to upper bound $\|\hat{\Gamma} - \tilde{\Gamma}\|_\infty$. Decompose the difference as
\[
\|\hat{\Gamma} - \tilde{\Gamma}\|_\infty \leq \sigma_x \left\| \frac{1}{n} X^\top \tilde{X} - \frac{1}{n} X^\top X - \hat{D} \text{diag} \left( \frac{1}{n} X^\top X \right) \right\|_\infty + \|\hat{\Sigma} - \tilde{\Sigma}\|_\infty.
\]

We first focus on the first term. Recall that $\|\hat{D}\|_\infty \leq 1 - 1/\rho_*$ and $\frac{1}{n} X^\top \tilde{X} = \hat{\Sigma} + \hat{D} \text{diag} (\frac{1}{n} X^\top X)$.

Subsequently, the first infinity norm term is upper bounded by
\[
\|\hat{\Sigma} - \Sigma_0\|_\infty + \|\hat{D} \text{diag} \left( \frac{1}{n} X^\top \tilde{X} \right) - \hat{D} \text{diag} (\Sigma_0)\|_\infty + \frac{1}{\rho_*} \|\hat{\Sigma} - \Sigma_0\|_\infty.
\]

By Lemma 6.1, if $\frac{\log p}{\rho_*^2 n} \to 0$ then $\|\hat{\Sigma} - \Sigma_0\|_\infty \leq O_p \left( \sigma_x^2 \sqrt{\frac{\log p}{\rho_*^2 n}} \right)$ and $\|\hat{\Sigma} - \Sigma_0\|_\infty \leq O_p \left( \sigma_x^2 \sqrt{\frac{\log p}{\rho_*^2 n}} \right)$.

For the remaining term, we invoke the following lemma that is proved in the supplementary materials:

**Lemma 6.9**. If $\frac{\log p}{\rho_*^2 n} \to 0$ then $\|\hat{D} \text{diag} (\frac{1}{n} X^\top X) - \hat{D} \text{diag} (\Sigma_0)\|_\infty \leq O_p \left( \sigma_x^2 \sqrt{\frac{\log p}{\rho_*^2 n}} \right)$. 

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Consequently,
\[
\sigma_x^2 \left\| \frac{1}{n} \tilde{X}^T \tilde{X} - \frac{1}{n} X^T X - \tilde{D} \text{diag} \left( \frac{1}{n} X^T X \right) \right\|_\infty \leq O_P \left\{ \sigma_x^2 \sqrt{\frac{\log p}{\rho_x^2 n}} \right\}.
\] (20)

Finally, we derive the upper bound for \( \| \tilde{\Upsilon} - \bar{\Upsilon} \|_\infty \). We first construct a \( p \times p \) matrix \( \bar{\Upsilon} \) as an “intermediate” quantity defined as
\[
\tilde{\Upsilon}_{jk} = \frac{1}{n} \sum_{i=1}^{n} \sum_{t \neq j, k} (1 - \rho_t) \tilde{X}_{ij} \tilde{X}_{ik} \tilde{X}_{it}^2 \beta_0^2\]
for \( j, k \in \{1, \ldots, p\} \).

Note that \( \bar{\Upsilon} \) involves the missing design \( \tilde{X} \) and the true model \( \beta_0 \). Further define \( \tilde{\Upsilon}_{jkt} \) and \( \Upsilon_{jkt} \) for \( j, k, t \in \{1, \ldots, p\} \) as
\[
\tilde{\Upsilon}_{jkt} = \frac{1}{n} \sum_{i=1}^{n} (1 - \rho_t) \tilde{X}_{ij} \tilde{X}_{ik} \tilde{X}_{it}^2, \quad \Upsilon_{jkt} = E \tilde{\Upsilon}_{jkt} | X.
\]

We next state the following concentration results on \( \tilde{\Upsilon}_{jkt} \) and \( \Upsilon_{jkt} \), which will be proved in the supplementary material.

**Lemma 6.10.** Fix \( j, k \in [p] \) and suppose \( \frac{\log p}{\rho_x^2 n} \rightarrow 0 \). We then have that
\[
\max_{j, k \in [p]} \max_{t \neq j, k} | \tilde{\Upsilon}_{jkt} | \leq O_P \left( \frac{\sigma_x^4 \log^2 p}{\rho_x^2} \right)
\]
and
\[
\max_{j, k \in [p]} \max_{t \neq j, k} | \tilde{\Upsilon}_{jkt} - \Upsilon_{jkt} | \leq O_P \left( \sigma_x^4 \log^2 p \sqrt{\frac{\log p}{\rho_x^2 n}} \right).
\]

We then upper bound \( \| \tilde{\Upsilon} - \bar{\Upsilon} \|_\infty \) by bounding \( \| \tilde{\Upsilon} - \bar{\Upsilon} \|_\infty \) and \( \| \tilde{\Upsilon} - \bar{\Upsilon} \|_\infty \) separately.

**Upper bound for \( \| \tilde{\Upsilon} - \bar{\Upsilon} \|_\infty \)** By definition, \( \tilde{\Upsilon}_{jk} = \sum_{t \neq j, k} \tilde{\Upsilon}_{jkt} \beta_0^2 \) and \( \Upsilon_{jkt} = \sum_{t \neq j, k} \Upsilon_{jkt} \beta_0^2 \). Hölder’s inequality then yields
\[
\| \tilde{\Upsilon} - \bar{\Upsilon} \|_\infty \leq \max_{j, k \in [p]} \max_{t \neq j, k} | \tilde{\Upsilon}_{jkt} | \cdot \| \beta_0^2 - \bar{\beta}_n^2 \|_1.
\]

Under the condition that \( \frac{\log p}{\rho_x^2 n} \rightarrow 0 \), it holds that \( \max_{j, k} \max_{t \neq j, k} | \tilde{\Upsilon}_{jkt} | \leq O_P (1) \cdot \max_{j, k} \max_{t \neq j, k} | \Upsilon_{jkt} |.
\)

Furthermore, \( \| \beta_0^2 - \bar{\beta}_n^2 \|_1 \leq \| \beta_n + \beta_0 \|_\infty \| \beta_n - \beta_0 \|_1 \leq (\| \beta_0 \|_2 + \| \beta_n - \beta_0 \|_2) \| \beta_n - \beta_0 \|_1. \) Invoking Lemma 6.10 and the condition that \( \| \beta_n - \beta_0 \|_2 \rightarrow 0 \) we get
\[
\| \tilde{\Upsilon} - \bar{\Upsilon} \|_\infty \leq O_P \left\{ \sigma_x^4 \log^2 p \rho_x^2 \| \beta_0 \|_2 \| \beta_n - \beta_0 \|_1 \right\}.
\] (21)
Upper bound for $\|\hat{\Upsilon} - \bar{\Upsilon}\|_\infty$ Note that $\hat{\Upsilon}_{jk} = \sum_{t \neq j,k} \Upsilon_{jkt} \beta_{0t}^2$ and $\bar{\Upsilon}_{jk} = \sum_{t \neq j,k} \bar{\Upsilon}_{jkt} \beta_{0t}^2$. By Hölder’s inequality,
\[
\|\hat{\Upsilon} - \bar{\Upsilon}\|_\infty \leq \max_{j,k} \max_{t \neq j,k} |\hat{\Upsilon}_{jkt} - \Upsilon_{jkt}| \cdot \|\beta_{0t}\|_1.
\]

Invoking Lemma 6.10 we then have
\[
\|\hat{\Upsilon} - \bar{\Upsilon}\|_\infty \leq O_P \left\{ \sigma_x^4 \log^2 p \frac{\|\beta_0\|_2^2}{\rho^* n} \right\}. \tag{22}
\]

Finally, combining Eqs. (18, 19, 20, 21, 22) we complete the proof of Theorem 3.2.

Appendix A Technical Lemmas

Lemma A.1 (generalized Fano’s inequality, (Ibragimov & Has’minskii, 2013)). Let $\Theta$ be a parameter set and $d : \Theta \times \Theta \to \mathbb{R}_{\geq 0}$ be a semimetric. Let $P_\theta$ be the distribution induced by $\theta$ and $P_{\theta_0}^n$ be the distribution of $n$ i.i.d. observations from $P_\theta$. If $d(\theta, \theta') \geq \alpha$ and $\text{KL}(P_\theta \| P_{\theta'}) \leq \beta$ for all distinct $\theta, \theta' \in \Theta$, then
\[
\inf_{\hat{\theta}} \sup_{\theta \in \Theta} \mathbb{E}_{P_{\theta_0}^n} \left[ d(\hat{\theta}, \theta) \right] \geq \frac{\alpha}{2} \left( 1 - \frac{n\beta + \log 2}{\log |\Theta|} \right).
\]

Lemma A.2 (Le Cam’s method, (Le Cam, 2012)). Suppose $P_{\theta_0}$ and $P_{\theta_1}$ are distributions induced by $\theta_0$ and $\theta_1$. Let $P_{\theta_0}^n$ and $P_{\theta_1}^n$ be distributions of $n$ i.i.d. observations from $P_{\theta_0}$ and $P_{\theta_1}$, respectively. Then for any estimator $\hat{\theta}$ it holds that
\[
\frac{1}{2} \left[ \mathbb{P}_{P_{\theta_0}^n} (\hat{\theta} \neq \theta_0) + \mathbb{P}_{P_{\theta_1}^n} (\hat{\theta} \neq \theta_1) \right] \geq \frac{1}{2} - \frac{1}{2} \|P_{\theta_0}^n - P_{\theta_1}^n\|_{\text{TV}} \geq \frac{1}{2} - \frac{1}{2\sqrt{2}} n\text{KL}(P_{\theta_0} \| P_{\theta_1}).
\]

Lemma A.3 (Miller (1981), Eq. (13)). Suppose $H$ is a matrix of rank at most 2 and $(I + H)$ is invertible. Then
\[
(I + H)^{-1} = I - \frac{aH - H^2}{a + b},
\]
where $a = 1 + \text{tr}(H)$ and $2b = [\text{tr}(H)]^2 + \text{tr}(H^2)$.

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Supplementary Material for: Rate Optimal Estimation and Confidence Intervals for High-dimensional Regression with Missing Covariates

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This supplementary material provides detailed proofs for technical lemmas whose proofs are omitted in the main text.

A Proofs of concentration bounds

A.1 Proof of Lemma 6.1

Fix arbitrary \( u, v \in S \). For \( j, k \in [p] \) and \( \ell \in \{0, 1, 2\} \), define

\[
\xi_{jk}^{(0)}(R_i, \rho) = 1, \quad \xi_{jk}^{(1)}(R_i, \rho) = \frac{R_{ij}}{\rho_j}, \quad \xi_{jk}^{(2)}(R_i, \rho) = \begin{cases} \frac{R_{ij}}{\rho_j \rho_k}, & j = k; \\ \frac{R_{ij} R_{ik}}{\rho_j \rho_k}, & j \neq k. \end{cases}
\]

Also let \( T_i^{(\ell)} = \sum_{j,k=1}^p \xi_{jk}(R_i, \rho) X_{ij} X_{ik} u_j v_k. \) We then have that

\[
|u^\top (\hat{\Sigma} - \Sigma_0) v| = \left| \frac{1}{n} \sum_{i=1}^n T_i^{(0)} - \mathbb{E}T_i^{(0)} \right|, \quad \text{(S1)}
\]

\[
|u^\top (\frac{1}{n} \tilde{X}^\top X - \Sigma_0) v| = \left| \frac{1}{n} \sum_{i=1}^n T_i^{(1)} - \mathbb{E}T_i^{(1)} \right|, \quad \text{(S2)}
\]

\[
|u^\top (\hat{\Sigma} - \Sigma_0) v| = \left| \frac{1}{n} \sum_{i=1}^n T_i^{(2)} - \mathbb{E}T_i^{(2)} \right|, \quad \text{(S3)}
\]

The main idea is to use Berstein inequality with moment conditions (Lemma E.5) to establish concentration bounds and achieve optimal dependency over \( \rho \). Define \( V^{(\ell)} = \mathbb{E} \left[ |T_i^{(\ell)} - \mathbb{E}T_i^{(\ell)}|^2 \right] \).

We then have that

\[
V^{(\ell)} \leq \mathbb{E}|T_i^{(\ell)}|^2 = \sum_{j,k,j',k'=1}^p \mathbb{E} \left\{ \xi_{jk}^{(\ell)} \xi_{j'k'}^{(\ell)} \right\} \mathbb{E} \{ X_{ij} X_{ik} X_{ij'} X_{ik'} u_j v_k u_{j'} v_{k'} \}.
\]
It is then of essential importance to evaluate $E \left\{ \xi_j^{(l)} \xi_j^{(l)} \right\}$. For $\ell = 0$ the expectation trivially equals 1. For $\ell = 1$ and $\ell = 2$, we apply the following proposition, which is easily proved by definition.

**Proposition A.1.** $E \left\{ \xi_j^{(1)} \xi_j^{(1)} \right\} = 1 + I[j = j'] \left( \frac{1}{\rho_j} - 1 \right)$ and $E \left\{ \xi_j^{(2)} \xi_j^{(2)} \right\} = 1 + I[j = j'] \left( \frac{1}{\rho_j} - 1 \right) + I[k = k'] \left( \frac{1}{\rho_k} - 1 \right) + I[j = j' \land k = k'] \left( \frac{1}{\rho_j} - 1 \right)(\frac{1}{\rho_k} - 1)$. Here $I[\cdot]$ is the indicator function.

We are now ready to derive $E|T_i^{(l)}|^2$.

$$
E|T_i^{(0)}|^2 = E \left\{ |X_i^T u|^2 |X_i^T v|^2 \right\};
$$
$$
E|T_i^{(1)}|^2 = E \left\{ |X_i^T u|^2 |X_i^T v|^2 \right\} + \sum_{j=1}^{p} \left( \frac{1}{\rho_j} - 1 \right) u_j^2 \mathbb{E} \left\{ X_{ij}^2 |X_i^T v|^2 \right\};
$$
$$
E|T_i^{(2)}|^2 = E \left\{ |X_i^T u|^2 |X_i^T v|^2 \right\} + \sum_{j=1}^{p} \left( \frac{1}{\rho_j} - 1 \right) (u_j^2 + v_j^2) \mathbb{E} \left\{ X_{ij}^2 |X_i^T v|^2 \right\} + \sum_{j,k=1}^{p} \left( \frac{1}{\rho_j} - 1 \right) \left( \frac{1}{\rho_k} - 1 \right) u_j^2 v_k^2 \mathbb{E} \left\{ X_{ij}^2 X_{ik}^2 \right\};
$$

By Cauchy-Schwartz inequality and moment upper bounds of sub-Gaussian random variables (Lemma E.1), we have that

$$
E \left\{ |X_i^T a|^2 |X_i^T b|^2 \right\} \leq \sqrt{E|X_i^T a|^4} \sqrt{E|X_i^T b|^4} \leq 16\sigma_x^4 \|a\|_2^2 \|b\|_2^2.
$$

Consequently, there exists universal constant $c_2 > 0$ such that

$$
E|T_i^{(0)}|^2 \leq c_2\sigma_x^4 \|u\|_2^2 \|v\|_2^2, \quad E|T_i^{(1)}|^2 \leq \frac{c_2}{\rho_*} \sigma_x^4 \|u\|_2^2 \|v\|_2^2, \quad E|T_i^{(2)}|^2 \leq \frac{c_2}{\rho_*^2} \sigma_x^4 \|u\|_2^2 \|v\|_2^2.
$$

We next establish $L > 0$ so that the moment condition in Lemma E.5 is satisfied, namely $E|T_i^{(l)} - ET_i^{(l)}|^k \leq \frac{1}{2} V(\ell) L^{k-2} k!$ for all $k > 1$. Note that for all $\ell \in \{0, 1, 2\}$, there exist functions $\xi_j^{(\ell)}$ and $\tilde{\xi}_j^{(\ell)}$ only depending on $j$ such that $\xi_j^{(\ell)} = \xi_j^{(\ell)} \tilde{\xi}_j^{(\ell)} + I[j = k] \cdot \tilde{\xi}_j^{(\ell)}$ and furthermore $\max_j |\xi_j^{(\ell)}| \leq 1/\rho_*$, $\xi_j^{(0)} = \tilde{\xi}_j^{(1)} = 0$ and $\max_j |\tilde{\xi}_j^{(2)}| \leq 1/\rho_*^2$. Subsequently,

$$
E|T_i^{(\ell)} - ET_i^{(\ell)}|^k \leq \left| \sum_{j,k=1}^{p} \left( \xi_j^{(\ell)} \tilde{\xi}_k^{(\ell)} + I[j = k] \cdot \tilde{\xi}_j^{(\ell)} - 1 \right) X_{ij} X_{ik} u_j v_k \right|^k \leq 3^k \left( \sum_{j,k=1}^{p} \left| \sum_{j=1}^{p} \xi_j^{(\ell)} X_{ij} X_{ik} u_j v_k \right|^k + \sum_{j=1}^{p} \left| \sum_{j,k=1}^{p} \xi_j^{(\ell)} X_{ij}^2 u_j v_j \right|^k + \sum_{j,k=1}^{p} \left| \sum_{j=1}^{p} X_{ij} X_{ik} u_j v_k \right|^k \right).
$$
Here the second line is a consequence of the following inequality: for all $a, b, c \geq 0$ we have that 
$$(a + b + c)^k \leq (3 \max\{a, b, c\})^k \leq 3^k \max\{a^k, b^k, v^k\} \leq 3^k (a^k + b^k + c^k)$$.
Define $\bar{u}_j = u_j \xi_j^{(t)}$, $\bar{v}_k = v_k \xi_k^{(t)}$, $\bar{u}_j = u_j \sqrt{|\xi_j^{(t)}|}$ and $\bar{v}_j = v_j \sqrt{|\xi_j^{(t)}|}$. Apply Lemma E.6 with $|\sum_{j=1}^{p} \xi_j^{(t)} X_j^{2} u_j v_j| \leq X_i^T A X_i$, $A = \text{diag}([\bar{u}_1 \bar{v}_1', \ldots, \bar{u}_p \bar{v}_p'])$ and note that $\text{tr}(A) = \sqrt{\text{tr}(A^2)} = |\bar{u}|^T |\bar{v}| \leq \|\bar{u}\|_2 \|\bar{v}\|_2$ and $\|A\|_{op} = \max_{1 \leq j \leq p} |\bar{u}_j \bar{v}_j| \leq \|\bar{u}\|_2 \|\bar{v}\|_2$. Subsequently, for all $t > 0$

$$\Pr \left[ X_i^T A X_i > 3\sigma^2 \|\bar{u}\|_2 \|\bar{v}\|_2 (1 + t) \right] \leq e^{-t}. \tag{S4}$$

Let $F(x) = \Pr[X_i^T A X_i \leq x], x \geq 0$ be the CDF of $X_i^T A X_i$ and $G(x) = 1 - F(x)$. Using integration by parts, we have that

$$\mathbb{E}[X_i^T A X_i]^k = \int_0^\infty x^k dF(x) = - \int_0^\infty x^k dG(x) = \int_0^\infty kx^{k-1} G(x) dx.$$

Here in the last equality we use the fact that $\lim_{x \to \infty} x^k G(x) = 0$ for any fixed $k \in \mathbb{N}$, because $G(x) \leq \exp\{1 - \frac{1}{M}\}$ by Eq. (S4), where $M = 3\sigma_x^2 \|\bar{u}\|_2 \|\bar{v}\|_2$. Consequently,

$$\mathbb{E}[X_i^T A X_i]^k = \int_0^M kx^{k-1} G(x) dx + \int_M^\infty x^{k-1} G(x) dx$$

$$\leq M^k + k \int_0^\infty M^{k-1}(1 + z)^{k-1} e^{-z} \cdot M dz$$

$$= M^k + k M^k \int_0^\infty (1 + z)^{k-1} e^{-z} dz$$

$$\leq M^k + k M^k \cdot k! \leq (k + 1)! M^k.$$

Here in the second line we apply change-of-variable $x = M(1 + z)$ and the fact that $G(M(1 + z)) \leq e^{-z}$ in the integration term. Because $2^k \geq k + 1$ for all $k \geq 1$, we conclude that

$$\mathbb{E} \left| \sum_{j=1}^{p} \xi_j^{(t)} X_j^{2} u_j v_j \right|^k \leq 6^k \sigma_x^{2k} k! \mathbb{E} \|\bar{u}\|_2^k \|\bar{v}\|_2^k, \quad \forall k \geq 1.$$

Subsequently, applying Cauchy-Schwartz inequality together with moment bounds for sub-Gaussian random variables (Lemma E.1) we obtain

$$\mathbb{E}|T_i^{(t)} - \mathbb{E}T_i^{(t)}|^k$$

$$\leq 3^k \left( \sqrt{\mathbb{E}|X^T \bar{u}|^{2k}} \sqrt{\mathbb{E}|X^T \bar{v}|^{2k}} + 6^k \sigma_x^{2k} k! \mathbb{E}\|\bar{u}\|_2^k \|\bar{v}\|_2^k + \sqrt{\mathbb{E}|X^T u|^{2k}} \sqrt{\mathbb{E}|X^T v|^{2k}} \right)$$

$$\leq 3^k \cdot 2^k \cdot 6^k \Gamma(k) \sigma_x^{2k} \cdot \left( \sqrt{\mathbb{E}\|\bar{u}\|_2^k} \sqrt{\mathbb{E}\|\bar{v}\|_2^k} + \sqrt{\mathbb{E}\|\bar{u}\|_2^k} \sqrt{\mathbb{E}\|\bar{v}\|_2^k} + \mathbb{E}|u|_2^k \|v\|_2^k \right)$$

$$\leq \rho_{s}^{k/2} \left( \frac{C'}{\rho_{s}^4} \right)^{k!} \mathbb{E}|T_i^{(t)}|^k$$

where $C' < \infty$ is some absolute constant. Compare the bound of $\mathbb{E}|T_i^{(t)} - \mathbb{E}T_i^{(t)}|^k$ with the variance $\mathbb{E}|T_i^{(t)}|^2$ we obtained earlier, we have that $L = \sigma_x^2 \|u\|_2 \|v\|_2 \cdot C^d / \rho_{s}^{1.5} \rho_{s}^{1.5} \cdot \rho_{s}^{1.5}$ is sufficient to guarantee
\[ E|T_i^{(\ell)} - E T_i^{(\ell)}|^k \leq \frac{1}{2} V^{(\ell)} L^{k-2} k! \text{ for all } k > 1. \]

Applying Bernstein inequality with moment conditions (Lemma E.5) and union bound over all \( u, v \in S \), we have that

\[
\Pr \left[ \forall u, v \in S, \left| \frac{1}{n} \sum_{i=1}^{n} T_i^{(\ell)} - E T_i^{(\ell)} \right| > \|u\|_2 \|v\|_2 \epsilon \right] \leq 2N^2 \exp \left\{ - \frac{n \epsilon^2}{2(V^{(\ell)} + L \epsilon)} \right\}
\]

for all \( \epsilon > 0 \), where \( \hat{V}^{(\ell)} = \frac{V^{(\ell)}}{\|u\|_2 \|v\|_2} \) and \( \hat{L} = \frac{L}{\|u\|_2 \|v\|_2} \). Subsequently,

\[
\sup_{u, v \in S} \left| \frac{1}{n} \sum_{i=1}^{n} T_i^{(\ell)} - E T_i^{(\ell)} \right| \leq O_P \left( \|u\|_2 \|v\|_2 \max \left\{ \frac{\hat{L} \log N}{n}, \sqrt{\frac{\hat{V}^{(\ell)} \log N}{n}} \right\} \right)
\]

\[
\leq O_P \left( \sigma_x^2 \|u\|_2 \|v\|_2 \max \left\{ \frac{\log N}{\rho_1^{1.5} n}, \sqrt{\frac{\log N}{\rho_1^{1.5} n}} \right\} \right).
\]

**A.2 Proof of Lemma 6.2**

Define \( \delta_j = \frac{1}{n} \sum_{i=1}^{n} Z_{ij} \) where \( Z_{ij} = \bar{X}_{ij} \epsilon_i \). Because \( \mathbb{E}\epsilon_i X = 0 \), we have that \( \mathbb{E}Z_{ij} = 0 \). In addition,

\[
\mathbb{E}|Z_{ij}|^2 = \frac{\sigma_x^2 \sigma_x^2}{\rho_j} \leq \frac{\sigma_x^2 \sigma_x^2}{\rho_s} =: V
\]

and

\[
\mathbb{E}|Z_{ij}|^k = \rho_j \cdot \frac{1}{\rho_j^k} \cdot \mathbb{E}|X_{ij}|^k \leq \frac{k^2 (2\sigma_x \sigma_x)^k}{\rho_s^{k-1} k!} \leq \rho_s \left( \frac{4 \sigma_x \sigma_x}{\rho_s} \right)^k k!.
\]

By setting \( L = 64 \sigma_x \sigma_x / \rho_s \), we have that \( \mathbb{E}|Z_{ij}|^k \leq \frac{1}{2} V L^{k-2} k! \) for all \( k > 1 \). Subsequently, applying Bernstein inequality with moment conditions (Lemma E.5) and union bound over \( j = 1, \cdots, p \) we have that

\[
\Pr [ \|\delta\|_\infty > \epsilon ] \leq 2p \exp \left\{ - \frac{n \epsilon^2}{2(V^{(\ell)} + L \epsilon)} \right\}
\]

for any \( \epsilon > 0 \). Suppose \( \frac{\ell}{\epsilon} \to 0 \). We then have that

\[
\|\delta\|_\infty \leq O_P \left( \sigma_x \sigma_x \sqrt{\frac{\log p}{\rho_s n}} \right).
\]

The condition \( \frac{\ell}{\epsilon} \to 0 \) is satisfied with \( \frac{\log p}{\rho_s n} \to 0 \).

---

1The case of \( k = 2 \) is trivially true.
A.3 Proof of Lemma 6.9

Fix arbitrary \( j \in \{1, \cdots, p\} \) and consider

\[
T_{ij} = (1 - \rho_j) \tilde{X}_{ij}^2 = \frac{(1 - \rho_j)R_{ij}X_{ij}^2}{\rho_j^2}.
\]

It is easy to verify that \( |D\text{diag}(\frac{1}{n} \tilde{X}^\top \tilde{X})|_{jj} = \frac{1}{n} \sum_{i=1}^n T_{ij} \) and \( |\check{D}\text{diag}(\Sigma_0)|_{jj} = \frac{1}{n} \sum_{i=1}^n ET_{ij} = \frac{\rho_j}{n} \sum_{i=1}^n T_{ij} \). We use moment based Bernstein’s inequality (Lemma E.5) to bound the perturbation \( |\frac{1}{n} \sum_{i=1}^n T_{ij} - ET_{ij}| \). Define \( V_j = E(T_{ij} - ET_{ij})^2 \). We then have

\[
V_j \leq E|T_{ij}|^2 = \frac{(1 - \rho_j)^2 E X_{ij}^4}{\rho_j^2} \leq \frac{3\sigma_x^4}{\rho_j^2}
\]

and for all \( k \geq 3 \),

\[
E|T_{ij} - ET_{ij}|^k \leq 2^k \left( E|T_{ij}|^k + |ET_{ij}|^k \right) \leq \frac{4^{k+1}}{\rho_{k-1}} \sigma_x^{2k} k!.
\]

It can then be verified that \( E|T_{ij} - ET_{ij}|^k \leq \frac{4}{k} V_j L^{k-2} \) for all \( k \geq 2 \). By Lemma E.5 and a union bound over all \( j \in \{1, \cdots, p\} \), we have that

\[
\Pr \left[ \forall j, \left| \frac{1}{n} \sum_{i=1}^n T_{ij} - ET_{ij} \right| > \epsilon \right] \leq 2p \exp \left\{ -\frac{n\epsilon^2}{2(V + \epsilon)} \right\}
\]

for all \( \epsilon > 0 \), where \( V = \frac{3\sigma_x^4}{\rho_2^2} \) and \( L = \frac{512\sigma_x^2}{\rho_2^2} \). Under the assumption that \( \frac{\epsilon}{L} \to 0 \), we have that

\[
\left\| \check{D}\text{diag} \left( \frac{1}{n} \tilde{X}^\top \tilde{X} \right) - D\text{diag}(\Sigma_0) \right\|_\infty = \sup_{1 \leq j \leq p} \left| \frac{1}{n} \sum_{i=1}^n T_{ij} - ET_{ij} \right| \leq O_p \left( \frac{\sigma_x^2 \log p}{\rho_2^2 n} \right).
\]

The condition \( \frac{\epsilon}{L} \to 0 \) is then satisfied with \( \frac{\log p}{\rho_2 n} \to 0 \).

A.4 Proof of Lemma 6.10

By definition and the missing data model,

\[
\Upsilon_{jkt} = E \tilde{\Upsilon}_{jkt} | X = \begin{cases} \frac{1}{n} \sum_{i=1}^n \frac{1 - \rho_j}{\rho_{jk}} X_{ij}^2 X_{it}^2, & j = k; \\ \frac{1}{n} \sum_{i=1}^n \frac{1 - \rho_j}{\rho_{ik}} X_{ij} X_{ik} X_{it}^2, & j \neq k. \end{cases}
\]

Subsequently,

\[
\max_{j,k \in [p]} \max_{t \neq j,k} |\Upsilon_{jkt}| \leq \frac{\|X\|^4}{\rho_{\star}^4} \leq O_p \left( \frac{\sigma_x^4 \log p}{\rho_{\star}^2 n} \right).
\]

To prove the second part of this lemma, we first fix arbitrary \( j, k \in [p] \) and \( t \neq j, k \). Define

\[
T_{ijkt} = (\xi_{jkt}(R_i, \rho) - E\xi_{jkt}(R_i, \rho)) X_{ij} X_{ik} X_{it}^2,
\]

and consider
where $\xi_{jkt}(R_i, \rho) = \frac{(1-\rho_i)R_iR_kR_l}{\rho_i\rho_k\rho_l}$. It is easy to verify that $\tilde{Y}_{jkt} - Y_{jkt} = \frac{1}{n} \sum_{i=1}^{n} T_{ijkt}$ and $\mathbb{E}T_{ijkt}|X = 0$. We then use Bernstein inequality with support conditions (Lemma E.4) to bound the concentration of $\frac{1}{n} \sum_{i=1}^{n} T_{ijkt}$ towards zero. Define $A = \max_{i,j,k,t} |T_{ijkt}|$ and $V = \max_{i,j,k,t} \mathbb{E}|T_{ijkt}|^2$. By H"{o}lder’s inequality we have that

$$A \leq \frac{\|X\|_\infty^4}{\rho_s^4} \leq O_p \left( \frac{\sigma_x^4 \log^2 p}{\rho_s^4} \right).$$

Here in the $O_p(\cdot)$ notation the randomness is on the generating process of $X$ and is independent of the randomness of missing patterns $R$. In addition, note that

$$\mathbb{E} \left| (\xi_{jkt} - \mathbb{E}\xi_{jkt}) (\xi_{jkt'} - \mathbb{E}\xi_{jkt'}) \right| \leq \frac{1}{\rho_s^4}$$

for all $j, k, t, t' \in \{1, \ldots, p\}$ and $t, t' \neq j, k$. Subsequently,

$$V = \max_{i,j,k,t} \mathbb{E}|T_{ijkt}|^2 \leq \frac{1}{\rho_s^4} X_{ij}^2 X_{ik}^2 X_{it}^2 \leq \frac{\|X\|_\infty^8}{\rho_s^4} \leq O_p \left( \frac{\sigma_x^8 \log^4 p}{\rho_s^4} \right).$$

Applying Lemma E.4 conditioned on $\|X\|_\infty \leq O(\frac{\sigma_x^2 \log^2 p}{\rho_s^2})$, we have that with probability $1 - O(\delta)$ for some $\delta = o(1)$ the following holds:

$$\left| \frac{1}{n} \sum_{i=1}^{n} T_{ijkt} \right| \leq O \left( \frac{\sigma_x^4 \log^2 p \sqrt{\log(1/\delta)}}{\rho_s^4 n} \right) =: \epsilon,$$

provided that $\frac{\epsilon}{A} \to 0$. Applying union bound over all $j, k \in [p]$ and $t \in [p] \setminus \{j, k\}$ we get

$$\max_{j,k \in [p]} \max_{t \neq j,k} \left| \frac{1}{n} \sum_{i=1}^{n} T_{ijkt} \right| \leq O_p \left( \frac{\sigma_x^4 \log^2 p \sqrt{\log(p/n)}}{\rho_s^4 n} \right),$$

The condition $\frac{\epsilon}{A} \to 0$ is satisfied with $\frac{\log p}{\rho_s^4 n} \to 0$.

### B Proof of restricted eigenvalue conditions

**Lemma B.1.** Suppose $A, B$ are $p \times p$ random matrices with $\Pr[\|A - B\|_\infty \leq M] \geq 1 - o(1)$ for some $M < \infty$. If $A$ satisfies $\text{RE}(s, \phi_{\min})$ and $B$ satisfies $\text{RE}(s, \phi'_{\min})$, then with probability $1 - o(1)$ we have that

$$\phi'_{\min} \geq \phi_{\min} - \{O(1) \cdot \varphi_{u,v}(A, B; O(s \log(Mp))) + O(1/n)\}.$$

**Proof.** For any $h \in \mathbb{R}^p$ it holds that

$$\frac{h^\top B h}{h^\top h} \geq \frac{h^\top A h}{h^\top h} - \frac{h^\top (B - A) h}{h^\top h}.$$
With appropriate scalings, it suffices to bound
\[ \sup_{h : \|h_J\|_1 \leq \|h\|_1, \|h\|_2 \leq 1} |h^\top (B - A) h| \]
for all \( J \subseteq [p], |J| \leq s \) as the largest possible gap between \( \phi_{\min} \) and \( \phi'_{\min} \).

Define \( \mathbb{B}_p(r) = \{ x \in \mathbb{R}^p : \|x\|_p \leq r \} \) as the \( p \)-norm ball of radius \( r \). Because \( \|h_J\|_1 \leq \|h\|_1 \) implies \( \|h\|_1 \leq 2 \|h_J\|_1 \leq 2 \sqrt{s} \|h\|_2 \), we have that
\[ \sup_{h : \|h_J\|_1 \leq \|h\|_1, \|h\|_2 \leq 1} |h^\top (B - A) h| \leq \sup_{h \in \mathbb{B}_2(1) \cap \mathbb{B}_1(2 \sqrt{s})} |h^\top (B - A) h|. \]

By Lemma 11 in the supplementary material of Loh & Wainwright (2012a), we have that
\[ \mathbb{B}_2(1) \cap \mathbb{B}_1(2 \sqrt{s}) \subseteq 3 \text{conv} \{ \mathbb{B}_0(4s) \cap \mathbb{B}_2(1) \} \subseteq \text{conv} \{ \mathbb{B}_0(4s) \cap \mathbb{B}_2(3) \}. \]

Here \( \text{conv}(A) \) denotes the convex hull of set \( A \). Let \( K(4s) = \mathbb{B}_0(4s) \cap \mathbb{B}_2(3) \) and denote \( N_{\epsilon,\cdot,\cdot}(K(4s)) \) as the covering number of \( K(4s) \) with respect to the Euclidean norm \( \| \cdot \|_2 \). That is, \( N_{\epsilon,\cdot,\cdot}(K(4s)) \) is the size of the smallest covering set \( H \subseteq K(4s) \) such that \( \sup_{h \in K(4s)} \inf_{h' \in H} \|h - h'\|_2 \leq \epsilon \). By definition of the concentration bounds, we have that with probability \( 1 - o(1) \)
\[ \sup_{h \in H} |h^\top (A - B) h| \leq \varphi_{u,u}(A, B; \log |H|) \sup_{h \in H} \|h\|_2^2 \leq 9 \varphi_{u,u}(A, B; \log N_{\epsilon,\cdot,\cdot}(K(4s))). \]

Subsequently, for any \( \epsilon \in (0, 1) \) with probability \( 1 - o(1) \)
\[ \sup_{h \in \mathbb{B}_2(1) \cap \mathbb{B}_1(2 \sqrt{s})} |h^\top (B - A) h| \leq \sup_{h \in \text{conv}(K(4s))} |h^\top (A - B) h| \]
\[ \leq \sup_{\xi_1, \ldots, \xi_T \geq 0, \xi_1 + \cdots + \xi_T = 1, h_1, \ldots, h_T \in K(4s)} \sum_{i,j=1}^T \xi_i \xi_j |h_i^\top (A - B) h_j| \]
\[ \leq \sup_{h, h' \in K(4s)} |h^\top (A - B) h'| \]
\[ \leq \sup_{h, h' \in \mathcal{H}_{\epsilon,\cdot,\cdot}[K(4s)]} |h^\top (A - B) h'| + (6\epsilon + 3\epsilon^2) \|A - B\|_{L_2} \]
\[ \leq 36 \{ \varphi_{u,u}(A, B; \log N_{\epsilon,\cdot,\cdot}(K(4s))) + \epsilon p M \}. \]

Here the last inequality is implied by the condition that \( \|A - B\|_{\infty} \leq M \) with probability \( 1 - O(n^{-a}) \). Taking \( \epsilon = O(1/(p^2 M)) \) we have that \( \epsilon p M = O(1/p) = O(1/n) \).

The final part of the proof is to establish upper bounds for the covering number \( N_{\epsilon,\cdot,\cdot}(K(4s)) \).

First note that by definition
\[ K(4s) = \bigcup_{J \subseteq [p]: |J| \leq 4s} \{ h : \supp(h) = J \land \|h\|_2 \leq 3 \}. \]

The covering number of a union of subsets can be upper bounded by the following proposition:
Proposition B.1. Let \( K = K_1 \cup \cdots \cup K_m \). Then \( N_{\epsilon,\|\cdot\|_2}(K) \leq \sum_{i=1}^{m} N_{\epsilon,\|\cdot\|_2}(K_i) \).

Proof. Let \( H_i \subseteq K_i \) be covering sets of subset \( K_i \). Define \( H = H_1 \cup \cdots \cup H_m \). Clearly \( |H| \leq \sum_{i=1}^{m} |H_i| \leq \sum_{i=1}^{m} N_{\epsilon,\|\cdot\|_2}(K_i) \). It remains to prove that \( H \) is a valid \( \epsilon \)-covering set of \( K \). Take arbitrary \( h \in K \). By definition, there exists \( i \in [m] \) such that \( h \in K_i \). Subsequently, there exists \( h^* \in H_i \subseteq H \) such that \( \|h - h^*\|_2 \leq \epsilon \). Therefore, \( H \) is a valid \( \epsilon \)-covering set of \( K \).

Define \( K_J(r) = \{ h : \text{supp}(h) = J \land \|h\|_2 \leq r \} \). The covering number of \( K_J \) is established in the following proposition:

Proposition B.2. \( N_{\epsilon,\|\cdot\|_2}(K_J(r)) \leq \left( \frac{4\epsilon r}{\epsilon} \right)^{|J|} \).

Proof. \( K_J(r) \) is nothing but a centered \( |J| \)-dimensional ball of radius \( r \), locating at the coordinates indexed by \( J \). The covering number result of high-dimensional ball is due to Lemma 2.5 of van de Geer (2010).

Combining the three propositions, we obtain

\[
\log N_{\epsilon,\|\cdot\|_2}(K(4s)) \leq \log \left( \sum_{j=0}^{4s} \binom{p}{j} \right) + \log \left( \frac{12 + \epsilon/2}{\epsilon/2} \right)^{4s} \leq O(s \log(p/\epsilon)).
\]

With the configuration of \( \epsilon = O(1/(p^2 M)) \), we have that

\[
\log N_{\epsilon,\|\cdot\|_2}(K(4s)) \leq O(s \log(p M)).
\]

We are now ready to prove Lemma 6.4.

Proof of Lemma 6.4. Consider \( A = \hat{\Sigma} \) and \( B = \Sigma_0 \) in Lemma B.1. Lemma 6.1 yields

\[
\varphi_{u,v}(\hat{\Sigma}, \Sigma_0; O(s \log(M p))) \leq O \left( \sigma_x^2 \max \left\{ \frac{s \log(M p)}{\rho^2 n}, \sqrt{\frac{s \log(M p)}{\rho^2 n}} \right\} \right) =: \epsilon.
\]

By Lemma B.1, to prove this corollary it is sufficient to show that \( \frac{\epsilon}{\lambda_{\min}(\Sigma_0)} \to 0 \). Note also that \( M \geq \|\hat{\Sigma} - \Sigma_0\|_{\infty} \leq \frac{\|X\|_{\infty}}{\rho^2} \leq O \left( \frac{\sigma_x \sqrt{\log p \rho^2}}{\rho^2} \right) \) with probability \( 1 - o(1) \). The condition \( \frac{\epsilon}{\lambda_{\min}(\Sigma_0)} \to 0 \) can then be satisfied with \( \frac{\sigma_x^2 s \log(s \log p / \rho^2)}{\rho^2 \lambda_{\min}^2} \to 0 \).

C Proof of Lemma 3.1

Lemma C.1. Suppose \( \frac{\log p}{\rho^2 n} \to 0 \) and \( \tilde{v}_n \asymp \sigma_x^2 \sqrt{\log p} / \rho^2 n \). Then with probability \( 1 - o(1) \) the population precision matrix \( \Sigma_0^{-1} \) is a feasible solution to Eq. (7); that is, \( \max\{\|\hat{\Sigma} \Sigma_0^{-1} - I_p \times p\|_{\infty}, \|\Sigma_0^{-1} \hat{\Sigma} - I_p \times p\|_{\infty}\} \leq \nu_n \).
Here the last inequality is due to Eq. (S5) it is proved that the solution set of
\[ \hat{\Theta} = \{\hat{\omega}_i\}_{i=1}^p, \quad \hat{\omega}_i \in \arg\min_{\omega_i \in \mathbb{R}^p} \left\{ \|\hat{\omega}_i\|_1: \|\hat{\omega}_i\|_1 \leq e_i \right\}. \]
Because \( \Sigma_0^{-1} \) belongs to the feasible set of the above constrained optimization problem, we have that \( \|\hat{\omega}_i\|_1 \leq \|\Sigma_0^{-1}\|_1 \) for all \( i = 1, \ldots, p \) and hence \( \|\hat{\Theta}\|_1 \leq \|\Sigma_0^{-1}\|_1 \). The inequality \( \|\hat{\Theta}\|_\infty \leq \|\Sigma_0^{-1}\|_1 \) can be proved by applying the same argument to \( \hat{\Theta}^T \).

We next prove the infinity norm bound for the estimation error \( \hat{\Theta} - \Sigma_0^{-1} \). By triangle inequality,
\[ \|\Sigma_0(\hat{\Theta} - \Sigma_0^{-1})\|_\infty \leq \|\hat{\Theta}\|_\infty + \|\hat{\Theta} - \Sigma_0\|_\infty \leq \tilde{v}_n + \|\hat{\Theta} - \Sigma_0\|_\infty. \]
Using Hölder’s inequality, we have that
\[ \|\hat{\Sigma} - \Sigma_0\|_\infty \leq \|\hat{\Theta}\|_1 \|\hat{\Sigma} - \Sigma_0\|_\infty \leq \|\Sigma_0^{-1}\|_1 \|\hat{\Sigma} - \Sigma_0\|_\infty \leq \tilde{v}_n. \]
Here the last inequality is due to Eq. (S5). Subsequently, \( \|\Sigma_0(\hat{\Theta} - \Sigma_0^{-1})\|_\infty \leq 2\tilde{v}_n \). Applying Hölder’s inequality again we obtain
\[ \|\hat{\Theta} - \Sigma_0^{-1}\|_\infty \leq \|\Sigma_0^{-1}\|_1 \|\hat{\Theta} - \Sigma_0\|_\infty \leq 2\tilde{v}_n \|\Sigma_0^{-1}\|_1. \]

To translate the infinity-norm estimation error \( \Sigma_0^{-1} \) into an \( L_1 \)-norm bound that we desire, we need the following lemma that establishes basic inequality of the estimation error:

**Lemma C.3.** Suppose \( \Sigma_0^{-1} \) is a feasible solution to Eq. (7). Then under Assumption (A5) we have that
\[ \max\{|\hat{\Theta} - \Sigma_0^{-1}|_1, |\hat{\Theta} - \Sigma_0^{-1}|_\infty\} \leq 2b_0|\hat{\Theta} - \Sigma_0^{-1}|_\infty. \]
Proof. Let \( \hat{\omega}_i \) and \( \hat{\omega}_0 \) be the \( i \)th columns of \( \hat{\Theta} \) and \( \Sigma_0^{-1} \), respectively. Let \( J_i \) denote the support size of \( \hat{\omega}_0 \). Define \( \hat{h} = \hat{\omega}_i - \hat{\omega}_0 \). We then have that

\[
\| \hat{\omega}_i \|_1 = \| \hat{\omega}_0 + \hat{h} \|_1 + \| \hat{h} \|_1 \geq \| \hat{\omega}_0 \|_1 - \| \hat{h} \|_1 + \| \hat{h} \|_1.
\]

On the other hand, \( \| \hat{\omega}_i \|_1 \leq \| \hat{\omega}_0 \|_1 \) as shown in the proof of Lemma C.2. Subsequently, \( \| \hat{h} \|_1 \leq \| \hat{\omega}_0 \|_1 \) and hence

\[
\| \hat{\omega}_i - \hat{\omega}_0 \|_1 = 2\| \hat{h} \|_1 \leq 2\| \hat{h} \|_\infty \leq 2b_0 \| \hat{\omega}_i - \hat{\omega}_0 \|_\infty.
\]

Because the above inequality holds for all \( i = 1, \cdots, p \), we conclude that \( \| \hat{\Theta} - \Sigma_0^{-1} \|_{L_1} \leq 2b_0 \| \hat{\Theta} - \Sigma_0^{-1} \|_{L_\infty} \). The bound for \( \| \hat{\Theta} - \Sigma_0^{-1} \|_{L_\infty} \) can be proved by applying the same argument to \( \hat{\Theta}^\top \). \( \square \)

Combining all the above lemmas, we have that with probability \( 1-o(1) \)

\[
\max \{ \| \hat{\Theta} - \Sigma_0^{-1} \|_{L_1}, \| \hat{\Theta} - \Sigma_0^{-1} \|_{L_\infty} \} \leq 2\bar{c}_n b_0 \Sigma_0^{-1} \| L_1 \leq O \left\{ \sigma_x^2 b_0 b_1^2 \sqrt{\frac{\log p}{\rho_n^2}} \right\}.
\]

D  Proofs of the other technical lemmas

D.1  Proof of Lemma 6.3

We first show that under the conditions on \( n, \tilde{\lambda}_n \) and \( \tilde{\lambda}_n \) specified in the lemma, the true regression vector \( \beta_0 \) is feasible to both optimization problems with high probability; that is, \( \| \frac{1}{n} X^\top y - \hat{\Sigma} \beta_0 \|_\infty \leq \tilde{\lambda}_n \) and \( \| \frac{1}{n} X^\top y - \Sigma_0 \beta_0 \|_\infty \leq \tilde{\lambda}_n \) with probability \( 1-o(1) \).

Consider \( \beta_n \) first. Apply \( y = X \beta_0 + \varepsilon \) and Definition 6.1, we have that with probability \( 1-o(1) \)

\[
\left\| \frac{1}{n} X^\top y - \hat{\Sigma} \beta_0 \right\|_\infty \leq \left\| \frac{1}{n} X^\top X - \Sigma_0 \right\|_\infty \beta_0 + \left\| \hat{\Sigma} - \Sigma_0 \right\|_\infty + \left\| \frac{1}{n} X^\top \varepsilon \right\|_\infty \leq \left\{ \varphi_{u,v} \left( \frac{1}{n} X^\top X, \Sigma_0; \log p \right) + \varphi_{u,v} \left( \hat{\Sigma}, \Sigma_0; \log p \right) \right\} \| \beta_0 \|_2 + \sigma_x^2 \varphi_{\varepsilon,\infty} \left( \frac{1}{n} X \right).
\]

Now apply Lemmas 6.1 and 6.2: with probability \( 1-o(1) \)

\[
\left\| \frac{1}{n} X^\top y - \hat{\Sigma} \beta_0 \right\|_\infty \leq O \left\{ \sigma_x \sqrt{\frac{\log p}{n}} \left( \frac{\sigma_x \left\| \beta_0 \right\|_2}{\rho_n} + \frac{\sigma_x}{\sqrt{\rho_n}} \right) \right\} \leq \tilde{\lambda}_n,
\]

provided that \( \frac{\log p}{\rho_n^2} \to 0 \). The same line of argument applies to the second inequality by the following decomposition: under the condition that \( \frac{\log p}{\rho_n^2} \to 0 \), with probability \( 1-o(1) \)

\[
\left\| \frac{1}{n} X^\top y - \Sigma_0 \beta_0 \right\|_\infty \leq \left\| \frac{1}{n} X^\top X - \Sigma_0 \right\|_\infty \beta_0 + \left\| \frac{1}{n} X^\top \varepsilon \right\|_\infty \leq \varphi_{u,v} \left( \frac{1}{n} X^\top X, \Sigma_0; \log p \right) \| \beta_0 \|_2 + \sigma_x^2 \varphi_{\varepsilon,\infty} \left( \frac{1}{n} X \right)
\]

\[
\leq O \left\{ \sigma_x \sqrt{\frac{\log p}{\rho_n^2}} \left( \sigma_x \left\| \beta_0 \right\|_2 + \sigma_x \right) \right\} \leq \tilde{\lambda}_n.
\]
We are now ready to prove Lemma 6.3. We only prove the assertion involving \( \hat{\beta}_n \), because the same argument applies for \( \hat{\beta}_n \) as well. Let \( \hat{h} = \hat{\beta}_n - \beta_0 \). Because \( J_0 = \text{supp}(\beta_0) \), we have that
\[
\|\hat{\beta}_n\|_1 = \|\beta_0 + \hat{h}_J\|_1 + \|\hat{h}_{\bar{J}}\|_1 \geq \|\beta_0\|_1 - \|\hat{h}_J\|_1 + \|\hat{h}_{\bar{J}}\|_1.
\]
On the other hand, because both \( \hat{\beta}_n \) and \( \beta_0 \) are feasible, by definition of the optimization problem we have that \( \|\hat{\beta}_n\|_1 \leq \|\beta_0\|_1 \). Combining both chains of inequalities we arrive at \( \|\hat{h}_J\|_1 \leq \|\hat{h}_J\|_1 \), which is to be demonstrated.

D.2 Proof of Lemma 6.6

**Proposition D.1.** Suppose \( X \sim \mathcal{N}(\mu, \nu^2) \) for \( \mu \in \mathbb{R} \) and \( \nu > 0 \). Then for any \( b \in \mathbb{R} \) and \( a > 0 \), it holds that
\[
\mathbb{E} \left[ \frac{1}{\sqrt{2\pi a^2}} \exp \left\{ -\frac{(X - b)^2}{2a^2} \right\} \right] = \sqrt{\frac{\nu^2}{a^2 + \nu^2}} \exp \left\{ -\frac{(\mu - b)^2}{2(a^2 + \nu^2)} \right\}.
\]

**Proof.** Because \( X \sim \mathcal{N}(\mu, \nu^2) \),
\[
\sqrt{2\pi \nu^2} \mathbb{E} \left[ \exp \left\{ -\frac{(X - b)^2}{2a^2} \right\} \right] = \int \exp \left\{ -\frac{(x - \mu)^2}{2\nu^2} - \frac{(x - b)^2}{2a^2} \right\} dx = \int \exp \left\{ -\frac{(a^2 + \nu^2)x^2}{2a^2\nu^2} - \frac{2(a^2\mu + \nu^2b)x}{a^2 + \nu^2} - \frac{\nu^2b^2}{a^2 + \nu^2} \right\} dx
\]
\[
= \exp \left\{ -\frac{(\mu - b)^2}{2(a^2 + \nu^2)} \right\} \int \exp \left\{ -\frac{a^2 + \nu^2}{2a^2\nu^2} \left( x - \frac{a^2\mu + \nu^2b}{a^2 + \nu^2} \right)^2 \right\} dx = \exp \left\{ -\frac{(\mu - b)^2}{2(a^2 + \nu^2)} \right\} \sqrt{\frac{2\pi a^2\nu^2}{a^2 + \nu^2}}.
\]
The proposition is then proved by multiplying both sides by \( \sqrt{2\pi a^2/\nu^2} \). \( \square \)

We now consider the likelihood \( p(y, x_{\text{obs}}; \beta, \Sigma) \). Integrating out the missing parts \( x_{\text{mis}} \) we have
\[
p(y, x_{\text{obs}}; \beta, \Sigma) = p(x_{\text{obs}}) \int \frac{1}{\sqrt{2\pi \sigma^2 \Sigma}} \exp \left\{ -\frac{(y - x_{\text{obs}}^T \beta_{\text{obs}} - x_{\text{mis}}^T \beta_{\text{mis}})^2}{2\sigma^2 \Sigma} \right\} dP(x_{\text{mis}} | x_{\text{obs}})
\]
\[
= p(x_{\text{obs}}) \mathbb{E}_u \left[ \exp \left\{ -\frac{(y - x_{\text{obs}}^T \beta_{\text{obs}} - u)^2}{2\sigma^2} \right\} \right] | x_{\text{obs}}; \Sigma,
\]
where \( u = x_{\text{mis}}^T \beta_{\text{mis}} \) follows conditional distribution \( u | x_{\text{obs}} \sim \mathcal{N}(\mu, \nu^2) \) with \( \mu = x_{\text{obs}}^T \Sigma_{12} \Sigma_{22}^{-1} \beta_{\text{mis}} \) and \( \nu^2 = \beta_{\text{mis}} \Sigma_{22} \beta_{\text{mis}} \). Applying Proposition D.1 with \( a = \sigma \) and \( b = y - x_{\text{obs}}^T \beta_{\text{obs}} \), we have
Finally, $R \perp x, x_{\text{obs}} \sim \mathcal{N}_q(0, \Sigma_{11})$ and hence

$$p(x_{\text{obs}}) = \rho^q(1 - \rho)^{p-q} \cdot \frac{1}{\sqrt{2\pi} |\Sigma_{11}|} \exp\left\{-\frac{1}{2} x_{\text{obs}}^\top \Sigma_{11}^{-1} x_{\text{obs}}\right\}.$$  

### D.3 Proof of Lemma 6.7

We prove this lemma by discussing three cases separately when at least one covariate of $x_{s-1}$ and $x_j$ are missing. Assume in each case $\Sigma_0$ and $\Sigma_1$ are partitioned as in Lemma 6.6; that is, $\Sigma_0 = [\Sigma_{011}; \Sigma_{012}; \Sigma_{021}; \Sigma_{022}]$ and $\Sigma_1 = [\Sigma_{111}; \Sigma_{112}; \Sigma_{121}; \Sigma_{122}]$.

1. **Both $x_{s-1}$ and $x_j$ are missing.** In this case $\Sigma_{011} = \Sigma_{111} = I_{q \times q}$ and $\Sigma_{012} = \Sigma_{112} = \Sigma_{021} = \Sigma_{121} = 0_{q \times (p-q)}$. Therefore, $\Sigma_{011} = \Sigma_{111}$ and the first two terms in $p(y, x_{\text{obs}}; \beta_0, \Sigma_0)$ and $p(y, x_{\text{obs}}; \beta_1, \Sigma_1)$ are identical. In addition, $\Sigma_{022} = \Sigma_{122} = I - \gamma (e_{s-1} e_j^\top + e_j e_{s-1}^\top)$, $\Sigma_{122:1} = \Sigma_{122} = I + \gamma (e_{s-1} e_j^\top + e_j e_{s-1}^\top)$. Therefore, $\Sigma_{011} = \Sigma_{111}$ and hence the first two terms in the likelihood are identical. In addition, $\beta_{0\text{mis}}^{\text{mis}} \Sigma_{022:1} \beta_{0\text{mis}} = \beta_{0\text{mis}}^{\text{mis}} \Sigma_{022:1} \beta_{0\text{mis}} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j}$. Therefore, last term in both likelihoods are the same as well.

2. **$x_{s-1}$ is observed but $x_j$ is missing.** In this case, $\Sigma_{011} = \Sigma_{111} = I_{q \times q}$, $\Sigma_{022} = \Sigma_{122} = I_{(p-q) \times (p-q)}$, $\Sigma_{012} = \Sigma_{112} = -\gamma e_{s-1} e_j^\top$ and $\Sigma_{121} = -\gamma e_{s-1} e_j^\top$. Therefore, $\Sigma_{011} = \Sigma_{111}$ and hence the first two terms in the likelihood are identical. In addition, $\beta_{0\text{mis}}^{\text{mis}} \Sigma_{022:1} \beta_{0\text{mis}} = \beta_{0\text{mis}}^{\text{mis}} \Sigma_{022:1} \beta_{0\text{mis}} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j}$.

3. **$x_j$ is observed but $x_{s-1}$ is missing.** In this case, $\Sigma_{011} = \Sigma_{111} = I_{q \times q}$, $\Sigma_{022} = \Sigma_{122} = I_{(p-q) \times (p-q)}$, $\Sigma_{012} = \Sigma_{112} = -\gamma e_s e_j^\top$ and $\Sigma_{121} = -\gamma e_s e_j^\top$. Therefore, $\Sigma_{011} = \Sigma_{111}$ and hence the first two terms in the likelihood are identical. In addition, $\beta_{0\text{mis}}^{\text{mis}} \Sigma_{022:1} \beta_{0\text{mis}} = \beta_{0\text{mis}}^{\text{mis}} \Sigma_{022:1} \beta_{0\text{mis}} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j} = \|\beta_{0\text{mis}}^{\text{mis}}\|^2 - 2 \gamma \beta_{0,s-1} \beta_{0j}$.

Because $\beta_{0\text{obs}}^{\text{obs}} = \beta_{0\text{mis}}^{\text{mis}}$ when $x_j$ is missing and $\beta_{0\text{obs}}^{\text{obs}} = \beta_{0\text{mis}}^{\text{mis}}$ when $x_{s-1}$ is missing, we conclude that the last term of both likelihoods are the same.

### D.4 Proof of Lemma 6.8

We first prove the upper bound for $\|r_n\|_\infty$. By Hölder’s inequality,

$$\|r_n\|_\infty \leq \sqrt{n} \|\hat{\Theta} \hat{\Sigma} - I\|_\infty \|\hat{\beta}_n - \beta_0\|_1\leq \sqrt{n} \|\hat{\Theta} \hat{\Sigma} - I\|_\infty \|\hat{\beta}_n - \beta_0\|_1.$$
where the last inequality is due to Eq. (7).

We next focus on \( \| \tilde{r}_n \|_\infty \). Apply Hölder’s inequality and triangle inequality:

\[
\| \tilde{r}_n \|_\infty \leq \sqrt{n} \| \hat{\Theta} - \Sigma_0^{-1} \|_{L,\infty} \left( \| \Delta_n \beta_0 \|_\infty + \left\| \frac{1}{n} \tilde{X}^T \varepsilon \right\|_{\infty} \right) \\
\leq 2 \sqrt{n} b_0 b_1 \tilde{r}_n \left( \| \Delta_n \beta_0 \|_\infty + \left\| \frac{1}{n} \tilde{X}^T \varepsilon \right\|_{\infty} \right).
\]

Here in the second line we invoke the conclusion in Lemma 3.1. It then suffices to upper bound \( \| \Delta_n \beta_0 \|_\infty \) and \( \| \frac{1}{n} \tilde{X}^T \varepsilon \|_\infty \). With Definition 6.1, it holds with probability \( 1 - o(1) \) that

\[
\| \Delta_n \beta_0 \|_\infty \leq \left\| \left( \frac{1}{n} \tilde{X}^T X - \Sigma_0 \right) \beta_0 \right\|_\infty + \left\| \left( \hat{\Sigma} - \Sigma_0 \right) \beta_0 \right\|_\infty \\
\leq \left[ \varphi_{u,v} \left( \frac{1}{n} \tilde{X}^T X, \Sigma_0; \log p \right) + \varphi_{u,v} \left( \hat{\Sigma}, \Sigma_0; \log p \right) \right] \| \beta_0 \|_2
\]

and

\[
\left\| \frac{1}{n} \tilde{X}^T \varepsilon \right\|_{\infty} \leq \sigma_e \varphi_{e,\infty} \left( \frac{1}{n} \tilde{X} \right).
\]

By Lemmas 6.1 and 6.2, if \( \frac{\log n}{\rho^2 n} \to 0 \) then

\[
\| \Delta_n \beta_0 \|_\infty \leq O_p \left\{ \sigma_e^2 \| \beta_0 \|_2 \sqrt{\frac{\log p}{\rho^2 n}} \right\} \quad \text{and} \quad \left\| \frac{1}{n} \tilde{X}^T \varepsilon \right\|_{\infty} \leq O_p \left\{ \sigma_e \sqrt{\frac{\log p}{\rho^2 n}} \right\}.
\]

## E Tail inequalities

**Lemma E.1** (Sub-Gaussian concentration inequality). Suppose \( X \) is a univariate sub-Gaussian random variable with parameter \( \sigma > 0 \); that is, \( \mathbb{E}X = 0 \) and \( \mathbb{E}e^{tX} \leq e^{\sigma^2 t^2 / 2} \) for all \( t \in \mathbb{R} \). Then

\[
\Pr \left[ |X| \geq \epsilon \right] \leq 2 e^{-\frac{\epsilon^2}{2\sigma^2}}, \quad \forall \epsilon > 0;
\]

\[
\mathbb{E}|X|^r \leq r \cdot 2^{r/2} \cdot \sigma^r \cdot \Gamma \left( \frac{r}{2} \right), \quad \forall r = 1, 2, \ldots
\]

**Lemma E.2** (Sub-exponential concentration inequality). Suppose \( X_1, \ldots, X_n \) are i.i.d. univariate sub-exponential random variables with parameter \( \lambda > 0 \); that is, \( \mathbb{E}X_i = 0 \) and \( \mathbb{E}e^{tX_i} \leq e^{\lambda t^2 / 2} \) for all \( |t| \leq 1/\lambda \). Then

\[
\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^n X_i \right| \geq \epsilon \right] \leq 2 \exp \left\{ -\frac{n}{2} \min \left( \frac{\epsilon^2}{\lambda^2}, \frac{\epsilon}{\lambda} \right) \right\}.
\]

**Lemma E.3** (Hoeffding inequality). Suppose \( X_1, \ldots, X_n \) are independent univariate random variables with \( X_i \in [a_i, b_i] \) almost surely. Then for all \( t > 0 \), we have that

\[
\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}X_i \right| \geq t \right] \leq 2 \exp \left\{ -\frac{2n^2 t^2}{\sum_{i=1}^n (b_i - a_i)^2} \right\}.
\]
Lemma E.4 (Bernstein inequality, support condition). Suppose $X_1, \cdots, X_n$ are independent random variables with zero mean and finite variance. If $|X_i| \leq M < \infty$ almost surely for all $i = 1, \cdots, n$, then
\[
\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| > t \right] \leq 2 \exp \left\{ -\frac{\frac{1}{2}n^2 t^2}{\sum_{i=1}^{n} \mathbb{E}X_i^2 + \frac{1}{3}Mnt} \right\}, \quad \forall t > 0.
\]

Lemma E.5 (Bernstein inequality, moment condition). Suppose $X_1, \cdots, X_n$ are independent random variables with zero mean and $\mathbb{E}|X_i|^2 \leq \sigma^2 < \infty$. Assume in addition that there exists some positive number $L > 0$ such that
\[
\mathbb{E}|X_i|^k \leq \frac{1}{2}\sigma^2 L^{k-2}k!, \quad \forall k > 1.
\]
Then we have that
\[
\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| > t \right] \leq 2 \exp \left\{ -\frac{nt^2}{2(\sigma^2 + Lt)} \right\}, \quad \forall t > 0.
\]

Lemma E.6 (Hsu et al. (2012)). Suppose $X = (X_1, \cdots, X_p)$ is a $p$-dimensional zero-mean sub-Gaussian random vector; that is, there exists $\sigma > 0$ such that
\[
\mathbb{E}\exp \left\{ \alpha^\top X \right\} \leq \exp \left\{ \|\alpha\|_2^2 \frac{\sigma^2}{2} \right\}, \quad \forall \alpha \in \mathbb{R}^p.
\]
Let $A$ be a $p \times p$ positive semi-definite matrix. Then for all $t > 0$,
\[
\Pr \left[ X^\top AX > \sigma^2 \left( \text{tr}(A) + 2\sqrt{\text{tr}(A^2)t + 2\|A\|_{\text{op}}t} \right) \right] \leq e^{-t}.
\]

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