Asymptotically solvable model for a solitonic vortex in a compressible superfluid

I. A. Toikka and J. Brand

Dodd-Walls Centre for Photonic and Quantum Technologies, Centre for Theoretical Chemistry and Physics, and New Zealand Institute for Advanced Study, Massey University, Private Bag 102904 NSMC, Auckland 0745, New Zealand

1. Introduction

The quantisation of vortices in superfluids is one of the most prominent quantum phenomena of condensed matter physics. Yet the dynamical properties of vortices are poorly understood, with the mass [1–3] and applicable forces [4–6] still the subject of controversies after decades of research [7–12]. Recent progress in ultracold atomic gases has demonstrated that the dynamics of individual vortices can be observed in defect-free superfluids near zero temperature [13–17]. In particular the determination of the inertial to physical mass ratio from the observed oscillation frequency of a solitonic vortex in a trapped unitary Fermi gas in [15] has demonstrated that, in principle, high precision measurements of the dynamical properties are feasible. The ability to derive a deeper understanding of vortex motion from such experiments is currently impeded by the limited precision of available theoretical predictions. For harmonically trapped quantum gases [16, 18–20] these are typically performed at logarithmic accuracy, i.e. up to an undetermined factor that varies only slowly (logarithmically) with system parameters. In addition to the long-range hydrodynamic laws governing the vortex motion, one can also expect the short-range, mesoscopic structure of the vortex core and its interaction with the fluid to influence the precise value of the vortex mass and other dynamical properties. This is particularly interesting for strongly correlated quantum fluids such as superfluid helium [21] where no quantitative microscopic theory exists. In Fermi superfluids such as neutron matter [22] and the unitary Fermi gas [23] one may further expect the Kopnin [6] and Iordanskii forces [4] to become relevant even though their effects may easily be dwarfed by dominant contributions from long-range hydrodynamics including the Magnus force [24]. Hence it is important to have precise control and understanding of the long-range hydrodynamic
effects on the vortex motion. In this work, we propose a simple slab geometry that provides this precise control and solve the hydrodynamic Euler equations for the vortex motion by a series expansion, where each term can be computed exactly. Thouless and Anglin concluded from a theoretical study that the vortex mass is poorly defined as its value depends on the process of measurement [7]. This ambiguity originates in the long-range nature of the velocity field associated with the vortex. In this work, we show how the ambiguity can be resolved and the vortex mass be properly defined by specifying a suitable geometry. Even though the value of the vortex mass now becomes specific to the chosen geometry, this enables us to control the universal long-range hydrodynamics in order to gain access to the short-range physics of the vortex core, which provides an important window into the quantum-many-body physics of the superfluid.

We consider the geometry of a long but transversely confined channel, or slab. A vortex line perpendicular to the channel’s long axis and traversing the channel’s narrow diameter is known as a solitonic vortex [25–27]. The concept was introduced in [25] in order to describe localised nonlinear wave excitations with vortex structure that travel at sub-sonic velocities along the channel [27]. Solitonic vortices have recently been observed as long-lived products from the decay of dark solitons [16, 28, 29] and as spontaneously formed defects during the Bose–Einstein condensation transition [30, 31]. The solitonic vortex appears as a decay product of other families of nonlinear localised excitations and is, in fact, the only known stable and slow-moving solitary wave excitation in channels that are sufficiently wide to host a vortex [32].

By virtue of the transverse confinement, the vortex becomes a localised object. Indeed, the exact solutions obtained in this work and visualised in figure 1 show that the associated velocity field and the corresponding excitation energy density are exponentially localised in the longitudinal direction on the length scale of the transverse dimension. This is a remarkable property given the well-known fact that the energy density of isolated vortices and vortex–antivortex pairs in two-dimensions or vortex rings in three-dimensions decay only algebraically in the absence of fluid boundaries [24, 33]. Mathematically speaking, the exponential localisation results from the infinite and periodic structure of the image vortices generated by the channel walls and the fact that the phase \( S_0 \) of the incompressible fluid satisfies Laplace’s equation (see figure 1(a) and section 2.1). A solution to Laplace’s equation that is periodic and infinite in extent in two-dimensions \((y, z)\) must be exponentially localised in the third-dimension \((x)\), which can be shown by using a separable ansatz to Laplace’s equation [34]. The incompressible fluid is the starting point of this work and physically represents the limit of a vanishing healing length scale. Accounting for a finite size of the healing length, or for non-abrupt trapping potentials that maintain the general shape of a long and transversely confined channel will smoothly deform the velocity fields but not change the property of exponential localisation. Thus the vortex loses its long-range nature beyond the transverse length scale of the channel, which justifies our use of the terminology ‘solitonic vortex’.

Figure 1. Solitonic vortex in a compressible superfluid. (a) Method of images: placement of two infinite families of images of alternating sign, of which the two first images are shown. Pluses (circles) correspond to images with the same (opposite) charge as the real vortex \((+)\). (b) The first-order compressible density profile \( n_1 = -q^2 n_0 D \nabla S_0 \) relative to the leading-order incompressible constant background \( n_0 \) together with \( S_0 \), the incompressible phase profile. Here \( h = D/2 \). (c) Numerical evaluation of the renormalised first-order compressible phase correction \( S_1 \) with \( h/D = 0.7 \). (d) Same as (b), but with the second-order density correction \( n_2 \), which depends on \( S_0 \) and \( S_1 \). For \( n_1 + n_2 \), we have chosen \( q = 1 \), \( r/2D = 0.05 \), and \( \gamma = 3/2 \), representing the unitary Fermi gas (UFG). Note that \( n_1 \) is independent of the equation-of-state exponent \( \gamma \).
The exponential localisation further makes it possible to consider the solitonic vortex as a quasiparticle with a well-defined inertial mass $M^*$ within the framework of Landau quasiparticle dynamics of solitary waves [35]. It allows us to derive a Newtonian equation of motion under the condition that the localised quasiparticle moves sufficiently slowly between regions of different background density, to suppress radiation and to conserve the quasiparticle’s energy. The effects of a trapping potential are hereby encapsulated in the chemical potential of local equilibrium $\mu (X) = \mu_0 - V_{\text{trap}} (X)$, which is assumed to vary by negligible amounts over the length scale of the solitonic vortex. If $E (\mu, V)$ denotes the excitation energy associated with a solitonic vortex at chemical potential $\mu$ moving with velocity $V = \dot{X}$, then requiring conservation of energy, $dE (\mu (X), \dot{X}) / dt = 0$, leads to

$$F = M^* \ddot{X},$$

where

$$M^* = \frac{1}{V} \frac{\partial E}{\partial V} \bigg|_{\mu}$$

is the inertial or effective mass. Since the vortex depletes particles from its core, we also have $F = M_{\text{ph}} g$, a buoyancy-type force with the physical mass

$$M_{\text{ph}} = - \frac{\partial E}{\partial \mu} \bigg|_V,$$

where $g = m^{-1} d\mu / dX$ is a buoyancy parameter with dimension of acceleration, and $m$ the mass of the elementary bosons. While $g$ characterises the environment of the solitonic vortex, the mass parameters $M^*$ and $M_{\text{ph}}$ are measurable and well-defined intrinsic properties that depend both on the microscopic details of the many-body physics, e.g., the vortex core structure, as well as the transverse confinement. In this way the Landau quasiparticle picture of the solitonic vortex provides a framework for defining vortex properties, where additional effects like non-conservative and fluctuating forces can be added later, or be read off from experimental observations where they deviate from the universal predictions of the hydrodynamic theory.

Here, we will calculate the mass parameters $M^*$ and $M_{\text{ph}}$ that arise as a consequence of the superfluid hydrodynamics given by the Euler and continuity equations. These equations govern the fluid dynamics at length scales large compared to the healing length $\xi$ and other microscopic length scales. The ratio $\xi / D$ of the healing length to the channel width $D$ thus naturally emerges as a small parameter of the theory. The key to finding exact solutions is to choose a geometry where simple analytic solutions are available for the simplified model of an incompressible fluid. From this starting point, compressible corrections are calculated as a perturbation series that is closely related to the Janzen–Rayleigh expansion [36–38]. The convergence properties of the series expansion are found to be well behaved and are discussed in detail in section 2.2. We obtain universal expressions for the mass parameters $M^*$ and $M_{\text{ph}}$ as well as the oscillation period for large-amplitude motion as series expansions in the small parameter $\xi / D$. Comparing these expressions with experimental data will permit the determination of the characteristic power of the equation of state and deviations from the universal behaviour indicate interesting quantum-many-body effects. In particular we will discuss the influence of quantum pressure, which is only well understood for weakly interacting Bose–Einstein condensates (BECs).

The paper is organised as follows: the main hydrodynamic theory is laid out in section 2, starting with the definition of the model and the analytic solution for the solitonic vortex in an incompressible fluid followed by the perturbation theory for the effects of compressibility in the Euler equations. Section 3 then reports on a number of results for the properties of the solitonic vortex at low orders of perturbation theory. The consequences of these results for the vortex dynamics under the influence of weak harmonic trapping in the quasiparticle dynamics framework are reported in section 4 and the relevance of the results for further experimental study are discussed in section 5. Two appendices discuss the derivation of the Euler equation in the form used in this work and an analytic approximation for the first order correction to the superfluid phase, respectively.

2. Hydrodynamic theory

2.1. Solitonic vortex in an incompressible fluid

We consider a three-dimensional uniform slab geometry as sketched in figure 1(a). Hard-wall boundaries confine the superfluid to a rectangular region in the yz-plane where the potential is uniform. Such a geometry can be realised experimentally with a flat-bottom trap for atomic gases [39, 40]. For simplicity, we assume an infinite extent along the longitudinal x-direction, a width $D$ in the y-direction and additional confinement $< D$
along the third direction, with which the vortex is aligned. Since the third-dimension is irrelevant for the further discussion we will work in the projected two-dimensional xy-plane. For the geometry of a two-dimensional channel the velocity field $v_0(x, y)$ of a vortex in an incompressible inviscid fluid can be determined with the method of images [24].

The velocity field is related by $v_0 = \frac{\lambda}{m} \nabla S_0$ to the superfluid phase $S_0(x, y)$ shown in figure 1(b). The phase $S_0$ of the incompressible superfluid with a vortex at position $\rho$ and hard wall boundaries can be constructed by superimposing the known solution for a vortex without boundaries with those of image vortices, such as to satisfy the condition of no flow normal to the boundary in the resulting velocity field. While a single straight wall requires a single image vortex, the two parallel walls of the channel generate a (doubly) infinite array of image vortices. The problem can be solved in an elegant way on the complex plane by introducing the meromorphic velocity potential $w(z)$ with poles as the vortex locations, where $S_0(x, y) = \Re \{ w(x + iy) \}$. The solution for the channel can then be found with the help of a conformal transformation between the channel and the half plane, where a single image vortex suffices [24, 41]:

$$w = -iq \ln \left[ \frac{\sinh \left( \frac{x}{2D}(z - ih) \right)}{\sinh \left( \frac{x}{2D}(z - i(2D - h)) \right)} \right].$$

(4)

Here, $q \in \mathbb{Z}$ is the quantised charge with $q = 1$ for a right-handed singly charged vortex, the location of the vortex is $\rho = (0, h)$, and the two-dimensional coordinate space is now represented by the complex plane $z = x + iy$. The $x$-axis is taken parallel to the walls of the channel, the origin is at the bottom wall, and the $y$-axis passes through the vortex. $S_0$ is given by the real part of $w$ and can be written as

$$\tan \left( \frac{S_0(x)}{q} \right) = \frac{\sin \left( \frac{2\pi x}{D} \right)}{\cos \left( \frac{2\pi x}{D} \right)} \sinh \left( \frac{\pi x}{D} \right) - \cos \left( \frac{2\pi x}{D} \right) \cosh \left( \frac{\pi x}{D} \right).$$

(5)

For practical purposes, it is useful to introduce the stream function $\chi(x, y) = \Im \{ w(x + iy) \}$, which has the important property of being single-valued, as opposed to the phase $S_0$. By virtue of the Cauchy–Riemann equations $\partial_x S_0 = \partial_y \chi$ and $\partial_y S_0 = -\partial_x \chi$ any expression involving derivatives of $S_0$ are easily rewritten in terms of $\chi$.

Even though the phase singularity at the position of the vortex results in a locally divergent velocity field, meaning that for example the energy $E_i$ (figure 2(a)) obtained as an integral over the slab of the kinetic energy density is formally divergent, it turns out that our model can be renormalised. Importantly, the compressible corrections to the density and phase fields that we calculate here are renormalisable. This means that we can meaningfully assign several interesting properties of the solitonic vortex; in particular, we find that the vortex moves along the channel in the $x$-direction with a velocity $V_0 = \frac{\pi h}{2Dm_0 \cos(h\pi/D)}$ (figure 2(b)), where $q$ is the vortex charge and $h$ the distance from the left channel wall as in figure 1(a). With the non-divergent canonical momentum $P_0 = 2\pi i \hbar q m_0 h$ (figure 2(c)), the inertial mass can be calculated from $M_i^* = dP_0/dV_0$ as

$$M_i^* = -4 \frac{m_0 D^2}{\pi} \sin^2 \left( \frac{\pi h}{D} \right).$$

(6)

where $n_0$ is the two-dimensional density of the incompressible fluid. When the vortex is located at the centre of the channel ($h = D/2$), it is at rest ($V_0 = 0$), and the inertial mass $M_i^* = -4 \frac{m_0 D^2}{\pi}$ is simply proportional to the mass of the fluid contained in the square of the channel width $D$. While the inertia is thus determined by the fluid inside the localisation volume, the negative sign indicates that the solitonic vortex accelerates in the direction opposite to an applied force according to equation (1).

The incompressible fluid model is directly relevant to real superfluids where it corresponds to the Thomas–Fermi approximation (density follows the local chemical potential) in a flat-bottom trap. Significantly, equation (6) for the inertial mass presents an exact result for strongly and weakly correlated superfluids in the Thomas–Fermi limit $\xi/D \to 0$, in contrast to previous results obtained for harmonically trapped superfluids [16, 18–20]. A (finite) healing length $\xi$, then, defined by $d\mu/dn = h^2/(2\pi m^2 \xi^2)$, is associated with the compressibility of the fluid. All physical fluids are always compressible.

2.2. Perturbation expansion for compressible corrections

Limitations of the incompressible model appear when we try to evaluate the force term of the equation of motion (1), wherein the physical mass formally vanishes. A suitable model that overcomes this limitation is that of inviscid isentropic flow described by the Euler equation, which in the co-moving reference frame of the vortex becomes (appendix A)
Here $\mu_0$ is the bulk chemical potential, and $\mu [n]$ is the chemical potential in a LDA evaluated at $n(r)$. For definiteness, we assume a power-law equation of state $n \sim \rho^{\gamma}$; for example, for a weakly interacting BEC $\gamma = 1$, while for the unitary Fermi gas $\gamma = 3/2$ [42]. The dimensionless factor $\epsilon$ is used for collecting orders in a perturbation expansion, and will be set to 1 at the end.

We note at this point that the Gross–Pitaevskii equation for BECs can be rewritten as an Euler equation with an additional term known as quantum pressure [43]

$$\frac{\hbar^2}{2m} \nabla S^2 + \rho \mu(n(r)) - \mu_0 = 0,$$

(7)

where we have defined the quantum pressure term (first term on the left) to be of order $\epsilon$. Together with the usual continuity equation (see below), equation (8) is fully equivalent to Gross–Pitaevskii theory and provides a quantitative description of dilute-gas BECs. The quantum pressure term governs the detailed structure of solitons and vortex cores. Whether a similar term exists for the crossover superfluid Fermi gas, and the unitary Fermi gas in particular, is not known. Comparing our predictions with experimental results could shed light on this issue, as one could use the experiment to help inform the correct modelling of the Fermi system at the level of the Euler equation.

Complemented with the continuity equation

$$\nabla \cdot (n\vec{v}) = 0,$$

(9)

the Euler equation (7) describes a compressible superfluid at zero temperature. Setting $\epsilon = 0$ recovers the incompressible problem already solved with the method of images. The more general problem can be solved with a perturbation expansion for the density $n = n_0 + \epsilon n_1 + \epsilon^2 n_2 + \cdots$ and phase $S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots$. Requiring that $n$ and $S$ solve equations (7) and (9), sorting orders of $\epsilon$ reveals that the density $n_k$ at order $k$ can be expressed through derivatives of the lower-order phase fields, while the phase $S_k$ at...
order $k$ can be determined from a Poisson equation with a source term containing derivatives of $n_k$ and lower-order phase fields. Through the perturbation expansion, we have essentially linearised the problem by writing the nonlinear Euler and continuity equations in a different form as an infinite set of coupled, but linear Poisson equations. By integrating the Poisson equations it is thus possible to determine the phase and density fields to arbitrary order.

Specifically, the density can be obtained using the Euler equation (7) and inverting the equation of state $\mu [\nu]$. For definiteness we assume a power-law relation $\mu_{3D} = \alpha \mu^2$ for the three-dimensional density. Further assuming uniform confinement perpendicular to the $xy$ plane with a length scale $B < D$, we obtain for the two-dimensional density

$$n(r) = B \alpha \left[ \mu_0 - \epsilon \frac{1}{2} mv^2 (r) \right].$$

(10)

The power-law equation of state covers in particular the case of a weakly interacting BEC, where $\gamma = 1$ and $\mu = g n_{3D}$, where $g$ is the Gross–Pitaevskii coupling so that $\alpha = 1/g$. For the unitary Fermi gas we have $\gamma = 3/2$, and $\alpha = \left(2m/[\hbar^2 \xi_{\text{sub}} (3\pi^2)]\right)^2$, where $\xi_{\text{sub}} \approx 0.37$ is known as the Bertsch parameter. An important length scale related to the equation of state is the healing length $\xi$ with $\mu_0 = \gamma \hbar^2/(2m \xi^2)$.

We expand $n = n_0 + \epsilon n_1 + \epsilon^2 n_2 + \cdots$, $S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots$, where $n_0$ and $S_0$ are 4th order terms in $\epsilon$ in the perturbation expansion. Comparing coefficients of $\epsilon$ in equation (10) gives the density corrections

$$n_0 = B \alpha \mu_0^4,$$

$$n_1 = - \frac{\hbar^2}{2m} \mu_0 \gamma |\nabla S_0|^2,$$

$$n_2 = \frac{\hbar^2}{8m} \frac{n_0}{\mu_0^2} \left[ \frac{\gamma}{\gamma - 2} \frac{\hbar^2}{(\gamma - 1)!} |\nabla S_0|^4 - 8 \frac{\gamma}{(\gamma - 1)!} \mu_0 \nabla S_0 \cdot \nabla S_1 \right],$$

$$n_3 = \frac{\hbar^2}{48m} \frac{n_0}{\mu_0^2} \left[ - \frac{\gamma}{\gamma - 3!} \frac{\hbar^4}{m} |\nabla S_0|^6 + 24 \frac{\gamma}{(\gamma - 2)!} \mu_0 \frac{\hbar^2}{m} |\nabla S_0|^2 \nabla S_0 \cdot \nabla S_1 \right]$$

$$- 24 \frac{\gamma}{(\gamma - 1)!} \mu_0^2 \left( 2 \nabla S_0 \cdot \nabla S_2 + |\nabla S_2|^2 \right) \right].$$

(11)

Expressions at arbitrary order can be generated by writing $n$ as a generalised Cauchy product:

$$n = n_0 \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} \ldots \sum_{k_{\ell}=-\infty}^{\infty} \left( \frac{\gamma}{\ell} \right) \left( \frac{-\hbar^2}{2m \mu_0} \right)^\ell \epsilon^\ell \epsilon^{k_{\ell}} \epsilon^{k_{\ell} - k_{\ell}} \nabla S_{k_{\ell}} \cdot \nabla S_{l_{\ell}} \cdot \nabla S_{k_{\ell} - k_{\ell}} \ldots \epsilon^{k_{\ell} - k_{\ell} - k_{\ell}} \nabla S_{k_{\ell} - k_{\ell}} \cdot \nabla S_{k_{\ell} - k_{\ell}}.$$  

(12)

Equation (10) is of the form of a product of $\gamma$ infinite series, where $\gamma$ can be generalised to fractional values through the binomial distribution that appears in equation (12); equation (12) is the Cauchy product of $\gamma$ individual series. Mertens’s theorem states that if the series $\sum_{n=0}^{\infty} a_n$ converges to $A$ and the series $\sum_{n=0}^{\infty} b_n$ converges to $B$, and at least one of them converges absolutely, where $(a_n)$ and $(b_n)$ ($\nu \geq 0$) are real or complex sequences, then their Cauchy product converges to $AB$. Thus, for equation (12) to converge, we require that the velocity $v = v_0 + \epsilon v_1 + \cdots = \frac{\partial}{\partial t} \left( \nabla S_0 + \epsilon \nabla S_1 + \cdots \right)$ in the chemical potential $\mu (r) = \mu_0 - \frac{\hbar^2}{2m} \mu_0 \nabla^2 (r)$ converges absolutely. With the vortex, all the individual density terms $n_k$ and any finite approximation to the series expansion of the density $n$ diverge near the vortex position. Similarly for the individual velocity terms $v_k$ and the velocity $v$. We provide a renormalisation scheme for the density in section 3.4, which while not a rigorous proof for convergence suggests that if we never truncate the infinite series for $n$, we can assign a well-defined value for the density everywhere except for a set of measure zero: the vortex cores. The density approaches zero in the neighbourhood of these points.

Similarly, the phase corrections can be deduced using the continuity equation under the condition of stationary flow: $\nabla \cdot (\nu v) = - \frac{dn}{dt} = 0$. We get

$$[\nabla n_0 + \epsilon \nabla n_1 + \mathcal{O}(\epsilon^2)] \cdot [\nabla S_0 + \epsilon \nabla S_1 + \mathcal{O}(\epsilon^2)]$$

$$= - [n_0 + \epsilon n_1 + \mathcal{O}(\epsilon^2)] [\nabla^2 S_0 + \epsilon \nabla^2 S_1 + \mathcal{O}(\epsilon^2)].$$  

(13)

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4 Our definition of the exponent $\gamma$ differs from the literature by that $\gamma$ is the inverse of the exponent used in [42].
Clearly $\nabla n_0 = 0$, $\nabla^2 S_0 = 0$, and so

$$\nabla^2 S_1 = -\frac{\nabla n_1 \cdot \nabla S_0}{n_0},$$

$$\nabla^2 S_2 = -\frac{1}{n_0}\left(\nabla n_1 \cdot \nabla S_1 + \nabla n_2 \cdot \nabla S_0 - \frac{n_2}{n_0}\nabla n_1 \cdot \nabla S_0\right),$$

$$\nabla^2 S_3 = \ldots.$$  

(14)

Equations (11)–(14) were derived from the classical Euler equation (7). If we include the quantum pressure term for BECs from equation (8), the results are modified by an additive correction to the density expressions of equation (11) at each order in $\epsilon$ with $n_1 \to n_1 + n_1^{\text{corr}}$ but otherwise unchanged (Note that $\gamma = 1$ for BECs.) The quantum pressure corrections can be calculated order by order and read

$$n_0^{(\text{qp})} = 0,$$

$$n_1^{(\text{qp})} = 0,$$

$$n_2^{(\text{qp})} = -\frac{n_0}{4}\xi^4 \nabla^4 S_0^2,$$

$$n_3^{(\text{qp})} = \ldots.$$  

(15)

In figure 1, we show the results for the phase up to first order and the density up to second order. After renormalisation, every order of the phase and all compressible corrections to the density formally diverge only at the vortex position (appendix B). We note that only $S_0$ has a logarithmic branch point. Even though our counting device $\epsilon$ is not a small parameter itself, it turns out that successive orders in $\epsilon$ also accumulate factors of $\xi^2$, which is small compared to the relevant length scale $D^2$. The addition of a quantum pressure term to the Euler equation (7), which would formally make it equivalent to the Gross–Pitaevskii equation (with $\gamma = 1$), will affect the density and phase correction terms at order $\mathcal{O}(\xi^4/D^4)$.

3. Solitonic vortex in a compressible superfluid

In this section, we obtain several physical properties of the vortex from the compressible hydrodynamics developed in the previous section.

3.1. Velocity and phase step

The vortex speed is given by the superfluid flow field at the vortex location after subtracting the free vortex $q/\tilde{r}$ divergence (and any divergences of the velocity corrections), where $\tilde{r} = |r - \rho|$ is the distance from the vortex. The vortex velocity is parallel to the channel walls:

$$V = \frac{\hbar}{m} \lim_{\bar{r} \to \rho} \left( \nabla S - \frac{q}{\bar{r}} \left( -\sin(\theta) \cos(\theta) \right) \right) = V \hat{x},$$  

(16)

where $\theta$ is an angle measured at the vortex and a series expansion of $V = V_0 + \epsilon V_1 + \cdots$ is defined through the corresponding expansion of the phase $S$. From the incompressible phase $S_0$ of equation (5) we easily obtain

$$V_0 = \frac{\pi a h}{2Dm} \cot \left( \frac{h \pi}{D} \right).$$  

(17)

It turns out that $\nabla S_1$ diverges at the vortex. Expanding the analytical approximation around the vortex,

$$\nabla S_1^{\text{approx}} = -\frac{\pi a^2 Dm}{4m} \xi^2 \cot \left( \frac{\pi h}{D} \right) \left( \frac{\cos(2\theta)}{\bar{r}} - \frac{\pi^2 \cos^2(\theta)}{12D^3} \right) + \mathcal{O}(\bar{r}),$$  

(18)

where we have taken only the leading term for $S_1^{\text{approx}}$. We can see that once the diverging $\bar{r}^{-2}$ term is removed, taking the limit $\bar{r} \to 0$ gives the following first-order correction to the vortex velocity (in the $x$-direction):

$$V_0 + V_1^{\text{approx}} = \frac{\pi a h}{2Dm} \cot \left( \frac{h \pi}{D} \right) \left( 1 + \frac{\pi^2 q^2 \xi^2}{24 \sin^2 \left( \frac{h \pi}{D} \right)} + 1 \right).$$  

(19)

We show how to obtain the full $V_1$ (figure 2(b)) numerically below in section 3.3.

The phase step across the solitonic vortex can be obtained by evaluating the difference of the limits $(x \to \infty) - (x \to -\infty)$ of the phase field $S_0 + S_1$. Using the analytical approximation:
\[ \Delta (S_0 + S^{\text{approx}}) = 2\pi q \frac{D - h}{D} + \frac{\pi^2 q^2 \xi^2}{2D^2} \cot \left( \frac{h\pi}{D} \right). \]  

(20)

A direct numerical evaluation of equation (83) yields an additional term of order \( \xi^2/D^2 \) with the same sign.

### 3.2. Canonical momentum

Similar to the case of a dark soliton \([45]\) or a vortex ring in a cylindrical waveguide \([20]\), we need to distinguish between the physical momentum \( P_{ph} \) and the canonical momentum \( P = P_{ph} + \hbar n_0 D (2\pi - \Delta S) \). The physical momentum is the integral of the momentum density of the solitonic vortex solution with open boundary conditions for \( x \to \pm \infty \) and a phase step \( \Delta S \). The canonical momentum is relevant in the context of Hamiltonian dynamics and can be defined by requiring \( V = dE_\phi / dP \). This property is automatically fulfilled for the physical momentum when periodic boundary conditions are imposed but in an infinite cylinder with open boundary conditions the momentum of a back flow current originating from the phase step has to be added \([20, 27]\). We thus initially express the canonical momentum through a fictitious velocity field that consists of the physical velocity field with an additional constant velocity field superimposed such as to satisfy periodic boundary conditions for the phase:

\[
P = m \int_{\Omega} m v_{\text{ph}}^2 \, d\mathbf{r}
\]

\[
= \hbar \int_{\Omega} n_0 \partial_\mathbf{r} S_{\text{ph}} \, dx + \hbar \int_{\Omega} n_1 \partial_\mathbf{r} S_0 \, dx + \mathcal{O}(\epsilon^2)
\]

\[
P = \hbar + \epsilon P_1 + \mathcal{O}(\epsilon^2),
\]

(21)

where \( S_{\text{ph}} = S + (2\pi - \Delta S) x / L \). The first term of the right-hand side of equation (21) is easy to evaluate since \( \int \partial_\mathbf{r} S_{\text{ph}} \, dx \) has the result \( 2\pi \) for \( y < h \) and 0 for \( y > h \). The result is (at all orders of \( \epsilon \) when expanding \( S \))

\[
P_0 = q\hbar n_0 \hbar.
\]

(22)

There are no branch cuts or logarithmic branch points in \( S_1 \). In this sense, the vortex is already contained within \( S_0 \). Hence, the integrations become straightforward as we do not have to worry about crossing any branch cuts. At first order in \( \epsilon \) we have

\[
P_1 = \hbar \int_{\Omega \setminus B_p^\text{ph}} n_1 \partial_\mathbf{r} S_0 \, d\mathbf{r}
\]

\[
= - n_0 D^2 \xi^2 \int_{\Omega \setminus B_p^\text{ph}} \left| \nabla S_0 \right|^2 \partial_\mathbf{r} S_0 \, d\mathbf{r} + \frac{\hbar (2\pi - \Delta S) \hbar}{L} \int_{\Omega \setminus B_p^\text{ph}} n_1 \, d\mathbf{r},
\]

(23)

where the second term on the right-hand side vanishes in the limit \( L \to \infty \) and thus can be discarded. In the region \( \Omega \setminus B_p^\text{ph} \), we can write the integrand of the first term as a total derivative:

\[
\left| \nabla S_0 \right|^2 \partial_\mathbf{r} S_0 = \nabla \cdot \left[ (\partial_\mathbf{r} S_0) \nabla \left( \frac{S_0^2}{2} \right) - \frac{S_0^2}{2} \nabla (\partial_\mathbf{r} S_0) \right]
\]

\[
= \nabla \cdot \left[ \chi (\partial_\gamma \chi) \nabla \chi - \frac{\chi^2}{2} \nabla (\partial_\gamma \chi) \right].
\]

(24)

The integral is solved by means of the divergence theorem and gives

\[
P_1 = \hbar n_0 D^2 \xi^2 \int_{\partial B_p^\text{ph}} \left[ \chi (\partial_\gamma \chi) \partial_\mathbf{r} \chi - \frac{\chi^2}{2} \partial_\mathbf{r} (\partial_\gamma \chi) \right] \, ds
\]

\[
= \frac{1}{2} \hbar n_0 D^2 \pi^2 q^2 \xi^2 \left( \frac{\hbar}{D} \right)^2 \cot \left( \frac{\hbar \pi}{D} \right) \left[ 1 - 4 \ln \left( \frac{2D}{\pi \hbar} \sin \left( \frac{\hbar \pi}{D} \right) \right) \right].
\]

(25)

Finally taking \( R = \xi \) we obtain for the momentum

\[
P = 2q \hbar n_0 \hbar + \frac{1}{2} \hbar n_0 D^2 \pi^2 q^2 \xi^2 \left( \frac{\hbar}{D} \right)^2 \cot \left( \frac{\hbar \pi}{D} \right) \left[ 1 - 4 \ln \left( \frac{2D}{\pi \xi} \sin \left( \frac{\hbar \pi}{D} \right) \right) \right] + \mathcal{O} \left( \frac{\xi^4}{D^2} \right).
\]

(26)

### 3.3. Effective mass

The effective mass \( M^* \) of the vortex can be calculated as

\[
M^* = \frac{dP}{dV} = \frac{dP}{dh} \frac{1}{dV} \left( \frac{dV}{dh} \right)^{-1},
\]

(27)

where the canonical momentum \( P \) is given in equation (26). We write the velocity derivative in terms of an expansion \( \beta(h) \equiv dV / dh = \beta_0 + \beta_1 + \cdots \), where \( \beta_i \) corresponds to a term of order \( (\xi^2/D^2)^i \). The first term
$\beta_0$ is easily found from equation (17) and leads to the inertial mass (6) of the solitonic vortex for the incompressible fluid. Higher order terms can be obtained through equation (16) by integration. To order $\xi^2/D^2$, we obtain $\beta_1 = \beta_0 q^2 \xi^2 h/D^4 m$, which is a dimensionless function shown in figure 3. At $h = D/2$, we obtain numerically $\beta_1 (\frac{1}{2}) = -17.1109 \approx -\left(\frac{\pi^4}{8} + \frac{\pi^2}{2}\right)$.

Evaluating equation (27), we obtain

$$M^* = -\frac{4}{\pi} D^2 \text{mmn}_0 \sin^2 \left(\frac{\pi h}{D}\right) + m_{\text{mnp}} q^2 \xi^2 \left[-4 \ln \left(\frac{2D}{\pi \xi}\right) + 2 \cos \left(\frac{2\pi h}{D}\right) + 3 - \frac{8\beta_1}{\pi^4} \sin^4 \left(\frac{\pi h}{D}\right) \right] + O\left(\frac{\xi^4}{D^2}\right)$$

For the effective mass of a stationary solitonic vortex at $h = D/2$, this yields

$$M^* h = \frac{\xi^2}{4} - \frac{4}{\pi} - \frac{1}{\pi^4} \left(4 \pi \ln \left(\frac{2D}{\pi \xi}\right) - 2\pi - \frac{4}{\pi} + \frac{\beta_1}{\pi^4} + \frac{\beta_1}{\pi^4} \right) + O\left(\frac{\xi^4}{D^2}\right),$$

where $|\Xi| \lesssim 10^{-4}$, as determined by our numerical accuracy. We note that the first-order correction to $M^* h = \frac{\xi^2}{4}$ has the same sign as the leading-order term provided that $\frac{\xi^2}{D^2} < \frac{1}{2} e^{-\frac{\pi^4}{8\pi^2}} \approx 0.349$, making the effective mass even more negative.

By definition, $\beta (h) = dV/dh$. For $\beta_0$, we need $V_0 = (h/m) \nabla S_0$ at the vortex with the divergent ‘free vortex’ terms removed as in appendix B and section 3.1. Using equation (84), we start by writing

$$\beta_1 = \frac{h}{m} \frac{d}{dh} \left(\frac{\partial_0 S_0}{\partial h}\right)$$

As before for $S_1$, we renormalise $\beta_1$ by expanding the integrand about the vortex position as a power series in the distance $r$ from the vortex. The diverging terms are analytic expressions that scale with inverse powers of the distance $r$, from $r^{-4}$ to $r^{-1}$. We then renormalise the original integrand by adding four counter terms. We integrate the resulting finite integrand numerically to obtain $V_1$, which we then differentiate with respect to $h$ to get $\beta_1$. The numerically thus evaluated $V_1$ (whose only component is along $x$) is shown in figure 2(b) as the red solid line.

### 3.4. Number of missing particles and physical mass

Vortex lines are usually associated with a core region of depleted particle number density due to large kinetic energy densities near the vortex filament. This depletion is quantified by the number of missing particles

$$N_0 = \int_{\Omega} (n - n_0) \, d^3 r,$$

which usually takes negative values. Here $n_0$ is taken to be local background density in the vicinity of the vortex and is usually fixed by the (local) chemical potential. The missing particle number $N_0$ is closely related to the physical mass defined by equation (3). In fact, for a zero-velocity solitonic vortex [46–48]

$$M_{ph} = m N_0,$$

where the zero-velocity condition is equivalent to setting $h = D/2$. In the following we outline the procedure for calculating $N_0$.
Since the individual terms \( n_i \) and any finite approximation to the series expansion of the density diverge near the vortex position, the integral (31) has to be renormalized. As a consequence, the approximate density may become negative in a small region near the vortex position. A renormalisation procedure can then be defined by excluding the area \( B \) from the integral where the approximate density has negative values, defined by \( n(r) = 0 \) at \( r \in \partial B \). With \( n = n_0 + n_1 \) and \( \xi/D \to 0 \) this condition yields a disk \( B_{\xi}^0 \) with a radius \( R = \xi \). Numerical evaluation shows that the excision, while not a disk anymore, becomes smaller for \( n = n_0 + n_1 + n_2 \) (figure 1(d)). Assuming convergence of the infinite series expansion for the density \( n \), the excision radius \( R \) will shrink to zero, as the density in the hydrodynamic description of a compressible superfluid vanishes only at the position of the vortex.

We start by formally expanding \( N_0 \) in powers of \( \epsilon \):

\[
N_0 = \int_{\Omega \setminus B_\xi^0} (n - n_0) \, d^2r
= \int_{\Omega \setminus B_\xi^0} \left[ n_1 + \epsilon^2 n_2 + \mathcal{O}(\epsilon^3) \right] \, d^2r
= N_0^{(\epsilon)} + N_0^{(\epsilon^2)} + \mathcal{O}(\epsilon^3).
\]

From equation (11), considering only \( n_1 \), we get

\[
N_0^{(\epsilon)} = -n_0 D^2 \frac{\xi^2}{D^2} \int_{\Omega \setminus B_\xi^0} |\nabla S_0|^2 \, d^2r
= -2\pi q^2 n_0 \xi^2 \ln \left( \frac{D}{R} \sin \left( \frac{\pi h}{D} \right) \right)
\]

(34)

Considering \( n_2 \), we can write

\[
N_0^{(\epsilon^2)} = -2n_0 \xi^2 \int_{\Omega \setminus B_\xi^0} \nabla S_0 \cdot \nabla S_1 \, d^2r + \frac{\gamma - 1}{\gamma} \frac{n_0 q^4}{2} \int_{\Omega \setminus B_\xi^0} |\nabla S_0|^4 \, d^2r,
\]

(35)

but this integral is more difficult to evaluate in closed form, and essentially would give \( S_1 \) as well.

The procedure of excising a disk of radius \( R \), which we then identify with the healing length \( \xi \), introduces complications with terms of order \( \epsilon^2 \) and higher. In particular, at \( h = D/2 \), we find both analytically and numerically the following dependence on \( R \):

\[
\int_{\Omega \setminus B_\xi^0} |\nabla S_0|^4 \, d^2r = \frac{\pi q^4}{R^2} - \frac{\pi^5 q^4}{9D^4R^2} + \mathcal{O}\left( \frac{R^4}{D^6} \right),
\]

\[
\int_{\Omega \setminus B_\xi^0} |\nabla S_0|^6 \, d^2r = \frac{\pi q^6}{2R^4} - \frac{\pi^5 q^6}{2D^4 \ln \left( \frac{R}{D} \right)} + \mathcal{O}\left( \frac{R^4}{D^6} \right),
\]

\[
\int_{\Omega \setminus B_\xi^0} |\nabla S_0|^n \, d^2r = \frac{2\pi q^n}{(n - 2)R^{n-2}} + \ldots, \quad n = 4, 6, \ldots
\]

(36)

The leading order terms do not depend on \( h \). After the identification \( R = \xi \), this means that we need all the terms in the perturbation series just to get the leading-order \( \mathcal{O}(\xi^2) \) term for \( N_0 \). Fortunately, it is possible to evaluate this contribution from all the terms of order \( \epsilon^2 \) and higher using the general expression (12). Summing over all the higher-order contributions of integrals of the form \( \int_{\Omega \setminus B_\xi^0} |\nabla S_0|^n \, d^2r \), we obtain

\[
N_0 = -2n_0 \xi^2 \int_{\Omega \setminus B_\xi^0} \nabla S_0 \cdot \nabla S_1 \, d^2r + \mathcal{O}\left( \frac{\xi^2}{D^2} \right)
- 2n_0 \xi^2 \int_{\Omega \setminus B_\xi^0} \nabla S_0 \cdot \nabla S_1 \, d^2r + \mathcal{O}\left( \frac{\xi^2}{D^2} \right)
\]

(37)

where \( _pF_q \) is the generalised hypergeometric function. With the unitary Fermi gas (\( \gamma = 3/2 \)), we have \( _pF_q \left( 1, 1; \frac{1}{2}; 2, 3; \frac{3}{2} \right) \approx 1.06829 \). For the case of a BEC (\( \gamma = 1 \)), the second term in equation (37) drops out, and the quantum pressure terms only contribute at order \( \mathcal{O}(\xi^2/D^2) \). At \( h = D/2 \), numerical evaluation shows that

\[
\int_{\Omega \setminus B_\xi^0} \nabla S_0 \cdot \nabla S_1 \, d^2r = -2\pi |\Xi_2| q^2 \frac{\xi^2}{D^2} \ln \left( \frac{D}{R} \right) + \ldots,
\]

(38)

where \( |\Xi_2 - 5| \ll 10^{-2} \), as determined by our numerical accuracy.

The quantum pressure corrections contribute to \( n_2 \), and so have an effect on the missing particle number at order \( \epsilon^2 \) and higher. As is the case without the quantum pressure, the excision procedure introduces problems when evaluating \( N_0 \). For example, we can analytically evaluate the first quantum pressure correction
\[ \int_{\Omega \setminus \partial \Omega} n_2^{(q)} d^2r = -\frac{n_0}{4} \xi^4 \int_{\Omega \setminus \partial \Omega} \nabla^2 S_0^2 d^2r \]
\[ R = \xi - 2\pi n_0 \xi^2 + O\left(\frac{\xi^6}{D^4}\right). \]

which, as before, means that we must evaluate higher-order corrections to obtain the full contribution to \( N_s^{(c)} \).

While this is possible in principle, we have not performed a fully detailed calculation to obtain this contribution.

3.5. Boundary effects from confining potentials

So far we have considered boundary conditions only for the velocity field (phase) as appropriate for hard wall boundaries and an incompressible fluid. A compressible fluid will additionally require a boundary condition for the density, which has to vanish at the boundary. For the more realistic situation of a flat-bottom potential induced by laser fields \([39, 40]\) the confinement of the superfluid is due to an external potential which rises sharply but still continuously to a large value. The effect of a hard-wall boundary requiring vanishing density on the motion of a vortex was treated in \([49]\) for the Gross–Pitaevskii equation with the conclusion that the method of images is still appropriate for this situation but to leading order the distance of the vortex to the hard wall is effectively reduced by an amount of \( \sqrt{\xi} \). It can thus be expected that the results of the previous sections are still applicable for hard walls that require the density to vanish if the length scales \( D \) and \( h \) are appropriately shortened.

The results of \([49]\), however, were derived with a variational approach and it would be more satisfying to treat this problem with the same approach that we are using to treat the finite vortex core size. Since the boundary effects are associated with the length scale \( \xi \) (assuming that even the realistic potential varies roughly on that length scale) they can be treated within the formalism developed in section 2.2 by treating the potential terms as a perturbation and introducing them with an \( \epsilon \) prefactor into the Euler equation (7) in a similar fashion as it is done with the kinetic energy density. We do not carry out the calculations here, which would lead to similar terms as we have found for the compressible corrections originating from the vortices, but leave this as an exercise for future work.

4. Equation of motion under weak harmonic trapping

In the previous sections 2 and 3 we have considered a homogeneous slab extending infinitely in the x-direction, which led to the vortex moving at constant velocity parallel to the x-axis. In this section we consider the effects of a weak harmonic trapping potential \( V_{\text{trap}}(x) = \frac{1}{2} m_p \omega_x^2 x^2 \), where \( m_p \) is the mass of the elementary particles confined by the trapping potential\(^5\). An important property of the solitonic vortex is that it is localised along the x-axis. Indeed it can be seen from equation (5) that the incompressible phase field \( S_0(\mathbf{r}) \) approaches a constant value (a vacuum) exponentially with a characteristic length scale \( D/h \) on either side of the vortex core and the exponential localisation remains valid at all orders of the perturbation expansions (11) and (14) for the density and phase of the compressible superfluid solutions. For this reason, in the presence of a trapping potential, the dynamics of the solitonic vortex can be treated in the framework of Landau quasiparticle dynamics, as outlined in section 1 and previously considered in [16].

4.1. Period of small amplitude oscillations

In the harmonic trapping potential the buoyancy-like restoring force \( F \) takes the form
\[ F = -\frac{m}{m_p} M_{ph} \omega_x^2 x, \]
and the solutions of Newton’s equation (1) are oscillations, as shown in figure 4. In the regime of small-amplitude oscillations, the inertial and physical masses \( M^* \) and \( M_{ph} \) can be replaced by their respective values in the centre of the channel. Equation (1) now becomes that of a harmonic oscillator with an oscillation period \( T_0 \) given by
\[ \left( \frac{T_0}{T_{\text{trap}}} \right)^2 = \frac{M_{ph}}{M^*} \left[ \frac{x}{\xi} \right] \]
\[ = a_2 \frac{\pi^2 q^2 \xi^2}{2 D^2} + a_4 \frac{\pi^4 q^4 \xi^4}{2 D^4} + O\left(\frac{\xi^6}{D^6}\right), \]

\(^5\)In the case of fermionic superfluids \( 2m_p = m \), where \( m \) is the mass of a Cooper pair. For bosons \( m_p = m \).
The coefficients $a_2$ and $a_4$ are at most logarithmic functions of $\xi^2/D^2$ with explicit expressions as follows:

$$a_2 = \ln \left( \frac{D}{\xi} \right) - F,$$

$$a_4 = - \left[ F + 9 \ln \left( \frac{D}{\xi} \right) \right] + \left[ \ln \left( \frac{D}{\xi} \right) - F \right] \left[ \pi^2 \ln \left( \frac{\pi \xi}{2D} \right) + \frac{\pi^2}{2} \right],$$

where $F \equiv \mu_0^2 \left( 1, 2 - \gamma_1, 2, 3; \frac{1}{\gamma_1} \right)^{(\gamma_1 - 1)}$. For the unitary Fermi gas $\gamma = 2^2$ and $F \approx 0.08902$. In the BEC case ($\gamma = 1$) $F = 0$ and quantum pressure terms are required to obtain the full coefficients. In principle, the coefficients can be evaluated to any order.

### 4.2. Trajectories as energy contours

Beyond the limit of small-amplitude oscillations, the oscillation period of the solitonic vortex depends on the amplitude. The trajectories of the solitonic vortex and its oscillation period can be calculated from the free energy $E_s(\mu, V)$ in LDA where $\mu$ is replaced by the local chemical potential at the position of the vortex $\mu(X) = \mu_0 - V_{\text{trap}}(X)$. Evaluating the energy expression in the homogeneous slab

$$E_s = \frac{\hbar^2}{2m} \int_{D^2} n S_{ij} \partial_i S_{jk} \partial_j S_{ik} \, d^2r,$$

leads to leading order yields

$$E_s^{(0)} = \frac{\hbar^2 n_0}{2m} \int_{D^2} n_{\text{LDA}} \partial_i S_{ij} \partial_j S_{ij} \, d^2r,$$

where we have regularised the integral by excising a disk with radius $R$ as in appendix B. In the presence of the harmonic trapping potential by virtue of the LDA the background density $n_0$ is replaced by

$$n_{\text{LDA}}(x) = n_0 \left( 1 - \frac{x^2}{R_{\text{TF}}^2} \right)^\gamma,$$

where we have made use of the power law equation of state, and $R_{\text{TF}} = \left( \frac{2\mu_0}{\sqrt{m \omega_z^2}} \right)$ is the Thomas–Fermi radius. Applying the LDA to the energy expression of equation (44) and identifying the cutoff radius $R$ with the healing length $\xi$, we obtain

$$E_s^{(0)}(x, y) = \frac{\hbar^2 q^2}{m} \pi n_0 \left( 1 - \frac{x^2}{R_{\text{TF}}^2} \right)^\gamma \ln \left[ \frac{D}{\xi} \sin \left( \frac{\pi y}{D} \right) \right].$$
where \( x \) and \( y \equiv h \) are the coordinates of the vortex position in the slab. Vortex trajectories are given by \( E_{x, \text{LDA}}^{0} = f_{0} \), where \( f_{0} \) is a positive constant. The higher the energy, the slower the vortex as its effective mass is negative, and the smaller the amplitude of the trajectory.

Instead of the constant \( f_{0} \), we can use the turning points to have \( x_{\text{max}} \), the amplitude of the oscillations, as our control parameter. Eliminating \( f_{0} \) using equation (46), we obtain an implicit function for \( y \) and \( x \) in terms of \( x_{\text{max}} \) that gives the same trajectories:

\[
\sin \left( \frac{\pi y}{D} \right) = \frac{x}{D} \exp \left[ \ln \left( \frac{D}{\xi} \right) \left( \frac{x_{\text{max}}}{1 - x_{\text{max}}/R_{\text{TF}}} \right)^{\gamma} \right] \quad (47)
\]

### 4.3. Oscillation period

The oscillation period can be calculated by integration over the trajectory as

\[
T_{s} = 4 \int_{0}^{x_{\text{max}}} \frac{dx}{V} \quad (48)
\]

Figure 4(b) shows the resulting dependence of the oscillation period on the amplitude \( x_{\text{max}} \) obtained from numerical integration, where the velocity \( V = V_{0} + V_{1} \) includes numerically obtained compressible corrections up to \( O(\xi^{2}/D^{2}) \).

An analytic approximation for the oscillation period can be obtained by approximating the velocity by \( V = V_{0} \) as given in equation (17). Integrating over the energy contour (47) and taking the limit \( x_{\text{max}} \to 0 \) reproduces \( T_{s} \) of equation (41) to leading order in \( \xi^{2}/D^{2} \). Expanding around \( x_{\text{max}} = 0 \), we obtain

\[
\frac{T_{s}}{T_{0}} = 1 + \frac{\gamma - 2\gamma \ln \left( \frac{D}{\xi} \right) - 3 \left( \frac{x_{\text{max}}}{R_{\text{TF}}} \right)^{2}}{8} - \frac{\gamma (\gamma + 10) + 4\gamma \ln \left( \frac{D}{\xi} \right) \left( 3\gamma + \gamma \ln \left( \frac{D}{\xi} \right) - 5 \right) - 23 \left( \frac{x_{\text{max}}}{R_{\text{TF}}} \right)^{4}}{256} + \cdots \quad (49)
\]

The second order expansion of the oscillation period is shown in figure 4(b) as dashed lines.

### 5. Discussion

The mass ratio \( M_{ph}/M^{*} \) can become small for a solitonic vortex compared to a dark soliton, which is consistent with the interpretation of recent experiments [16, 17] performed in cylindrical, all-harmonic trapping potentials. The physical reasons become clear from the derivations performed in this work. While the dark soliton has a fixed mass ratio of \( 1/3 \) [35, 50], the two mass parameters of the solitonic vortex depend differently on the available length scales. While the inertial mass (equations (6) and (29)) essentially involves all the fluid mass contained in a volume defined by the channel’s diameter \( D \), the physical mass (32) is closely related to the density depletion of the vortex and thus primarily determined by the vortex core length scale \( \xi \) with only a logarithmic factor depending on the confinement length scale \( D \) (see equation (37)). The mass ratio and oscillation frequency (41) thus strongly depend on this length scale ratio. Since the length scale ratio \( \xi/D \) can be tuned continuously in experiment, e.g. by moving the confining potentials or adjusting the particle number density, the individual terms in the series expansion can be compared with the predictions of equations (41) and (49). In particular this would allow the parameter \( \gamma \) to be read off, which is not generally known for strongly correlated superfluids.

Future experiments in a rectangular slab geometry with hard-wall potentials could provide quantitative comparison of the oscillation period with the results of this work—potentially at very high accuracy. This could lead to identifying effects that are not yet included in the theory including the Kopnin mass, a mass contribution to the vortex expected to arise from fermionic bound states in the vortex core of a fermionic superfluid [12]. Also the role of quantum pressure contributions to the hydrodynamic equations of Fermi gases in the BEC to BCS crossover is currently unclear and experiments or Monte-Carlo simulations could clarify the situation by comparison to the theory presented here.

It was already discussed in section 3.5 that steep-wall trapping potentials can be treated within the perturbative framework if their spatial variation is of the order of the healing length or smaller. If the potentials are slowly varying compared to the channel width \( D \) they can be treated in the adiabatic approximation as discussed in section 4. Previous experiments [15–17] have, however, used harmonic trapping potentials for the transverse confinement to realise cigar-shaped traps rather than slabs, which introduces a new length scale. While many of the conceptual and qualitative results carry over to the harmonic confinement, the use of image vortices becomes more problematic and the asymptotic solutions of the current work no longer apply. An
elegant theoretical analysis of the dynamics of solitonic vortices in this situation was performed in [16], albeit limited to logarithmic accuracy. Solutions of the hydrodynamic equations beyond logarithmic accuracy seem feasible by numerical means but are currently not available in analytical form, although they would be highly desirable.

Our results for the inertial vortex mass may be compared with the recent discussion of vortex mass contributions by Sonin [12]. Based on physical arguments but without recourse to a specific confining geometry, Sonin introduces the ‘core’ mass and the ‘compressibility’ mass. While no mass contribution that scales as the leading contribution to the inertial mass (29) appears in [12], the ‘core’ and ‘compressibility’ masses resemble the next to leading order contributions $\propto m q n \xi^2$ to the inertial mass of equation (29). The ‘core’ mass is related to the fluid displacement by the vortex core and is consistent with the non-logarithmic contribution to the $\xi^2$ term in equation (29). It is not possible to compare the terms quantitatively as any constant can be absorbed in the logarithmic term that depends on the geometry. Sonin’s ‘compressibility’ mass has the same scaling as but a different sign than the $\xi^2 \ln(2D/\pi \xi)$ term in equation (29), which does depend on the confining geometry. We find that the overall sign of this term becomes negative when $\xi/D \lesssim 0.35$ whereas both contributions discussed in [12] as being relevant for bosonic superfluids are positive. It should be noted that the contributions at this level are geometry-dependent and thus it is essential to define the geometry as done with the slab geometry in this paper in order to make quantitative predictions.

The series expansion for the physical and inertial vortex masses established here provides the basis to further improving the theoretical understanding of strongly correlated quantum liquids like the superfluid Fermi gas beyond the currently known hydrodynamic model (7). By measuring the oscillation period $T_{\text{osc}}$ future high-precision experiments in the slab geometry can determine the expansion coefficients order-by-order and inform the modelling of quantum pressure, transverse forces [4–6], and vortex core filling [51]. Moreover, our results can be further developed to make experimental measurements of certain microscopic features such as the level spacing of the Andreev bound states of the vortex core [51].

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Appendix A. Derivation of the Euler equation

Here we outline the derivation of equation (7) in the main text. At zero temperature, changes in the chemical potential $\mu$ for a bulk system are related to changes in the pressure $p$ by the Gibbs–Duhem relation $dp = n d\mu/n$, where $n$ is the number density. The third law of thermodynamics states $s = 0$ for a perfect crystal at $T = 0$. On the other hand, for an isentropic process ($ds = 0$, where $s$ is the entropy), we have $dp = n d\bar{w}$ [52], where $\bar{w}$ is the enthalpy. Let us therefore identify $\mu = m \bar{w}$ at zero temperature for isentropic processes. The Nernst–Simon formulation of the third law of thermodynamics states that $ds \to 0$ for any reversible isothermal (dT = 0) process as $T \to 0$, i.e. any reversible isothermal process at $T = 0$ is isentropic.

The Euler equation in equation (2.9) of [52] is given as:

$$\frac{\partial v}{\partial t} + \nabla \bar{w} + (v \cdot \nabla)v = 0. \tag{A1}$$

Applying a Galilean boost $r = r' + v_b t$, $t = t'$, where $v_b$ is a constant Galilean boost velocity, i.e. $v = v' + v_b$, we get $\frac{\partial}{\partial t'} = v_b \cdot \nabla + \frac{\partial}{\partial \tau'}$, and $\nabla = \nabla'$, and so

$$\frac{\partial v'}{\partial \tau'} - (v_b \cdot \nabla')v' + \nabla' \bar{w}' + [(v' + v_b) \cdot \nabla']v' = 0, \tag{A2}$$

where we have used $v_b = 0$ and that its gradients also vanish. Importantly, in the boosted frame that travels alongside the vortex, the velocity does not change, i.e. $\frac{\partial v'}{\partial \tau'} = 0$. In the main text we calculate $v'$. We obtain (dropping the primes)

$$\nabla \bar{w} + (v \cdot \nabla)v = 0. \tag{A3}$$

Using $\frac{1}{2} \nabla (v \cdot v) = (v \cdot \nabla)v + v \times (\nabla \times v) = (v \cdot \nabla)v$ where $\nabla \times v = \nabla \times (\nabla S) = 0$, we can rewrite the above equation as follows:

$$\nabla \bar{w} + \frac{1}{2} \nabla v^2 = 0. \tag{A4}$$
This is nothing more but the Bernoulli equation for a compressible fluid, equation (5.3) in [52]:

$$\frac{1}{2} v^2 + \tilde{\nu} = \text{const.} \quad \text{(A5)}$$

Equation (7) of the main text now follows after multiplying with $m$ by replacing $m\tilde{\nu}$ with the chemical potential at the local density and denoting the constant with $\mu_0$.

**Appendix B. First-order compressible phase correction $S_1$: regularisation and renormalisation**

**B.1. Poisson equation for $S_1$**

The first-order velocity correction can be obtained from equation (14), which amounts to solving Poisson’s equation in the domain $\Omega$ with zero Neumann boundary conditions (where $\Omega$ represents the channel as per figure 1(a) of the main text):

$$\nabla^2 S_1 = -\frac{1}{n_0} \nabla n_1 \cdot \nabla S_0$$

$$= \frac{\hbar^2 \gamma}{2m \mu_0} \nabla \cdot (|\nabla S_0|^2 \nabla S_0) + \frac{\hbar^2 \gamma}{2m \mu_0} |\nabla S_0|^2 \nabla^2 S_0,$$  \quad \text{(B1)}

where $S_1$ is the first order phase field, and $n_1 = -\frac{\hbar^2 n_0}{2m \mu_0} |\nabla S_0|^2$ is the first-order density correction. Note that the right-hand side would be zero for a free vortex since there the velocity field lines $\nabla S_0$ are orthogonal to the density gradient $\nabla n_1$. For the solitonic vortex this is not the case and thus the right-hand side is non-zero.

The Poisson equation (B1) can be solved with the help of a Green’s function that obeys the Neumann boundary conditions of the channel and satisfies $\nabla^2 G(\mathbf{r}, \mathbf{r}') = +\delta^{(2)}(\mathbf{r} - \mathbf{r}')$. The Neumann Green’s function for the channel satisfying $\partial_\nu G = 0$ on the channel walls along $y = 0$ and $y = D$, and $\partial_\nu G = 1/(2D)$ on the walls at $|x| \to \infty$, where $\partial_\nu$ is the unit outward normal derivative, is [53]

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \ln \left[ \left( \sinh^2 \left( \frac{\pi (x - x')}{2D} \right) + \sin^2 \left( \frac{\pi (y - y')}{2D} \right) \right) \times \left( \sinh^2 \left( \frac{\pi (x - x')}{2D} \right) + \sin^2 \left( \frac{\pi (y + y')}{2D} \right) \right) \right],$$  \quad \text{(B2)}

The solution is then found by integrating the Green’s function with the source term

$$S_1(\mathbf{r}) = \int_{\Omega \setminus \bar{B}_R^n} \frac{1}{n_0} \nabla \cdot (n_1(\mathbf{r}') \nabla S_0(\mathbf{r}')) \, G(\mathbf{r}, \mathbf{r}') \, d^3r',$$

where we have regularised the otherwise divergent integral by excising a disk $B_R^n$ with radius $R$. This integral can be solved numerically, and a renormalisation procedure that allows $R$ to be taken to zero is outlined below. Further analytical progress is made noting that $\nabla^2 S_0 = 0$ in the integration domain and writing

$$G \nabla n_1 \cdot \nabla S_0 = G \nabla \cdot [G n_1 \nabla S_0] = \nabla \cdot [G n_1 \nabla S_0] - n_1 \nabla S_0 \cdot \nabla G:

S_1(\mathbf{r}) = \int_{\Omega \setminus \bar{B}_R^n} \frac{1}{n_0} \nabla \cdot [G n_1 \nabla S_0] - n_1 \nabla S_0 \cdot \nabla G \, d^3r'$$. \quad \text{(B4)}

The first term on the right-hand side can be analytically evaluated as a boundary term on $\partial B_R^n$, which is the circle where $|\mathbf{r}' - \rho| = R$. Setting $\rho = (0, h)$, the position of the vortex, we have $(x')^2 + (y' - h)^2 = R^2$, so that $x' = R \cos(\theta)$ and $y' = h + R \sin(\theta)$, where $\theta$ is measured at the vortex. We have to integrate over all $\theta$, and obtain as an approximation for $S_1$ that ignores the second term on the right-hand side of equation (B4)

$$S_1^{\text{approx}}(\mathbf{r}) = -\frac{\pi^2 q^3}{4} \frac{\xi_1}{D^2} \left[ \cot \left( \frac{\pi x}{D} \right) \sinh \left( \frac{\pi y}{D} \right) \cosh \left( \frac{\pi h}{D} \right) - \cos \left( \frac{\pi h}{D} \right) \cos \left( \frac{\pi (y + h)}{D} \right) \right] + O\left( \frac{R^2}{D^2} \right),$$  \quad \text{(B5)}

where $\mathbf{r} = (x, y)$. The $(R/D)^2$ series corresponds to a ‘multipole-like’ expansion (figure B1). We take the limit $R \to 0$ so that in effect we are excising a point.

**B.2. Renormalisation**

At first sight, the integral (B3) for $S_1$ is divergent. However, as we outline here, our model is renormalisable, and the physical properties of the solitonic vortex can be meaningfully assigned.

First, we regularise the divergence by excising the disk $B_R^n$. The analytically solvable term $S_1^{\text{approx}}$ equates to a multipole-like expansion in the cut-off $R$ (figures B1(a)–(c)), while the term that was ignored in equation (B4) is...
divergent as $R \to 0$. More specifically, it integrates to a divergent dipole term (figure B1(d)), and an $R$-independent quadrupole term (figure B1(f)). The dipole terms from both the analytically solvable term and the numerically evaluated term vanish at $h = D/2$, and $S_1$ takes the form of a quadrupole field there (figure B1(f)). Since the quadrupole field has no $R$-dependence in the full $S_1$ (it has $R^2$ dependence only in the analytically solvable term $S_1^{\text{approx}}$), the integral for $S_1$ is already renormalised and indeed gives an identical result as the full renormalisation procedure.

To renormalise the divergences, we expand the integrand in equation (B4) around the vortex position, and add counter-terms to cancel the terms in the expansion with negative powers of $\tilde{r}$, where $\tilde{r}$ is the distance from the vortex core. We have

$$-\frac{1}{n_0} \nabla \cdot (n_1(r') \nabla S_0(r')) G(r, r') = \frac{a_3}{\tilde{r}^3} + \frac{a_2}{\tilde{r}^2} + \frac{a_1}{\tilde{r}} + \ldots,$$

and so we need three counter terms. Here $a_{1-3}$ are analytic expressions that depend on $r$ and $r'$. Then, we take $R \to 0$ and obtain finite values for the remaining integrals. By doing this we find that $S_1$ is in general a combination of a dipole term and a quadrupole term (figure B1(e)). As discussed, only the finite quadrupole term survives at $h = D/2$. Although the integral (B3) for $S_1$ formally diverges when $h = D/2$, the underlying physical model is thus renormalisable with only a finite number of counter terms required.

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