Unique maximal Betti diagrams for Artinian Gorenstein $k$-algebras with the weak Lefschetz property

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ABSTRACT
We give an alternate proof for a theorem of Migliore and Nagel. In particular, we show that if $H$ is an SI-sequence, then the collection of Betti diagrams for all Artinian Gorenstein $k$-algebras with the weak Lefschetz property and Hilbert function $H$ has a unique largest element.

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1. Introduction and background

The graded Betti numbers of a module have received quite a bit of attention, especially since the advent of computer algebra systems allowing for the computation of examples in bulk. In this paper we explore the relationship between graded Betti numbers and the weaker Hilbert function invariant. In particular, we will take up a piece of the Gorenstein version of the question, given a Hilbert function $H$, what graded Betti numbers can occur?

To fix notation, we let $k$ be an infinite field of any characteristic. Then given a polynomial ring $R$ and a homogeneous ideal $I \subseteq R$, we write $H(R/I) : \mathbb{N} \to \mathbb{N}$ to be the Hilbert function of $R/I$, so $H(R/I, d) = \dim_k(R/I)_d$, and $\beta_{i,j}^I = \dim_k(\Tor_i(R/I, k))_j$ to be the $(i,j)$th graded Betti number of $R/I$.

We will write $\beta^I$ to refer to the Betti diagram of $R/I$ (which, following the notation of the computer algebra system Macaulay2, is a table whose $(i,j)$th entry is $\beta_{i+j}^I$) as a convenient way to refer to all the graded Betti numbers of a module at once. Given an $O$-sequence $H$ (that is, given any valid Hilbert function) with $H(0) = 1$ we write $B_H$ to be the set of all $\beta^I$ such that $I \subseteq R$ is homogeneous with $H(R/I) = H$.

To understand $B_H$ for fixed $H$, the first step is to show that this set has a sharp upper bound under the obvious component-wise partial order. This was accomplished independently by Bigatti [2] and Hulett [7] in characteristic zero, and later by Pardue [10] in characteristic $p$. These authors showed that if $L$ is the lex ideal (guaranteed by Macaulay to exist) such that $H(R/L) = H$, then $\beta^L$ is the unique largest element of $B_H$. Various other authors have asked if important subsets of $B_H$ also have unique upper bounds. For example, the Lex Plus Powers Conjecture of Evans and Charalambous [8] predicts that restricting $B_H$ to the Betti diagrams of quotients of ideals containing regular sequences in prescribed degrees should have a unique max. Aramova, Hibi, and Herzog [1] proved that restricting $B_H$ to the Betti diagrams of quotients of squarefree monomial ideals does in fact have a unique largest element.
One obvious subset of $\mathbb{B}_H$ to consider consists of the Betti diagrams of graded Artinian Gorenstein $k$-algebras. In general we cannot predict when this subset is nonempty since there is currently no analog to Macaulay's theorem for Gorensteins, but there is one well understood class of Hilbert functions, the so-called Stanley-irrobino sequences (hereafter SI-sequences), and we turn our attention there.

**Definition 1.1.** Given a $O$-sequence $H$, let $\Delta H$ be the sequence with $\Delta H(d) = H(d) - H(d - 1)$, and, for $i > 1$, let $\Delta^i H = \Delta \Delta^{i-1} H$.

When $H = \{0\}$ we say $\ell(H) = -1$, and for $H \neq \{0\}$, we write
$$\ell(H) = \max \{ d \in \mathbb{N} \mid H(d) \neq 0 \}$$
if such a $d$ exists and $\ell(H) = \infty$ otherwise. Finally, given an $O$-sequence $H$ with $0 < \ell(H) < \infty$, we say that $H$ is an SI-sequence if $H(d) = H(\ell(H) - d)$ for all $0 \leq d \leq \ell(H)$ (that is, $H$ is symmetric), and $\Delta H$ is an $O$-sequence for $d \leq \lfloor \frac{\ell(H)}{2} \rfloor$ (that is, the first half of $H$ is a differentiable $O$-sequence).

It is a result of Harima [4] that given an SI-sequence $H$, there is a Artinian Gorenstein $k$-algebra with Hilbert function $H$. In fact, more is true.

**Definition 1.2.** Let $M$ be a graded $R$-module. Then $M$ is said to have the weak Lefschetz property if there is $r \in R_1$ such that $M_i \rightarrow M_{i+1}$ given by multiplication by $r$ is injective or surjective for all $i$.

Harima actually proved that $H$ is an SI-sequence if and only if there is a Artinian Gorenstein $k$-algebra with the weak Lefschetz property and Hilbert function $H$. Harima’s result calls for $H$ to be unimodal, that is, $\Delta H(i) \geq 0$ for all $i \leq \lfloor \frac{\ell(H)}{2} \rfloor$, but this was shown to hold for all Artinian $k$-algebras with the Weak Lefschetz property in Remark 3.3 in [5].

**Remark 1.1.** Given an SI-sequence $H$, we will abuse notation slightly by saying that an $R$-module $M$ has Hilbert function $\Delta H$ when we really mean that its Hilbert function equals $\Delta H$ for $d \leq \lfloor \frac{\ell(H)}{2} \rfloor$, and is zero otherwise.

The Gorenstein question was first considered in this context by Geramita, Harima, and Shin [3] in 2000. They demonstrated how to embed a standard $k$-configuration $X$ (an iteratively defined sets of points in $\mathbb{P}^n$) in a so-called basic configuration $Z$ (an intersection of unions of hyperplanes), and then showed that the quotient of the sum of the defining ideals of $X$ and $Z - X$ is Gorenstein with the weak Lefschetz property and that its Betti diagram is larger than that of any other Artinian Gorenstein $k$-algebra with the weak Lefschetz property and the same Hilbert function. Of course, this only settled the question (granting the weak Lefschetz condition) for Hilbert functions which can be obtained via the weak Lefschetz property and the same Hilbert function. Of course, this only settled the question (granting the weak Lefschetz condition) for Hilbert functions which can be obtained via the construction, and one can show that this does not include every possible SI-sequence.

In 2003, Migliore and Nagel [9] extended Geramita, Harima, and Shin’s result by showing that for any SI-sequence $H$, the collection of Betti diagrams for Artinian Gorenstein $k$-algebras with the weak Lefschetz property and Hilbert function $H$ has a unique largest element. This is done in two steps—giving an upper bound for the Betti diagrams in question, and showing that this upper bound is sharp. Establishing the upper bound, which we record here for use later, turns out to be the easier step.

**Theorem 1.1** (Theorem 8.13 in [9]). Let $S = k[\mu_1, \ldots, \mu_c]$ for $k$ a field, $H$ be an SI-sequence with $H(1) \geq 1$, $c = H(1)$, $t = \ell(\Delta H)$, $L$ be the lex ideal in $S/(x_0)$ with Hilbert function $\Delta H$, and $I \subseteq S$ be a homogeneous ideal such that $S/I$ is a Artinian Gorenstein $k$-algebra with the weak Lefschetz property and $H(S/I) = H$. Then
$$\beta^L_{i,i+j} \leq \begin{cases} \beta^L_{i,i+j} & \text{if } j \leq \ell(H) - t - 1 \\ \beta^L_{i,i+j} + \beta^L_{c-i,c-i+j} & \text{if } \ell(H) - t \leq j \leq t \\ \beta^L_{c-i,c-i+j} & \text{if } j \geq t + 1. \end{cases}$$
Giving the result with respect to \((i, i + j)\) makes it straight forward to interpret the bound in terms of Betti diagrams. In particular, Migliore and Nagel’s theorem says that taking two copies of the Betti diagram of the lex ideal with Hilbert function \(\Delta \mathcal{H}\), rotating the second by 180 degrees and degree shifting appropriately, then super-imposing these tables and adding entries gives an upper bound for the Betti diagram of any Artinian Gorenstein \(k\)-algebra with the weak Lefschetz property and Hilbert function \(\mathcal{H}\).

Demonstrating that this bound is sharp turned out to be more difficult. Beginning with an arbitrary SI-sequence \(\mathcal{H}\), the authors define a special generalized stick figure (that is, a union of linear subvarieties of \(\mathbb{P}^n\) of the same dimension \(d\) such that the intersection of any three components has dimension \(\leq d - 2\)) whose Hilbert function is \(\Delta \mathcal{H}\), and embed this set in a second generalized stick figure with a Gorenstein coordinate ring and whose Hilbert function has a certain maximal property. The sum of the defining ideal of the original space with its link in the manufactured Gorenstein ideal gives a Gorenstein quotient with the weak Lefschetz property, the correct Hilbert function, and maximal graded Betti numbers.

Migliore and Nagel note that in an ambient space of large enough dimension, the linear forms used as building blocks in their construction can be taken as variables. We pursue this idea and by remaining monomial until the last possible moment are able to streamline their construction. The doubly iterative description we employ allows for proofs that are more naive (in the sense that they are mostly double induction) and, moreover, makes it easy to actually compute the ideals in question, for instance on the computer algebra system Macaulay 2, even for “large” \(\mathcal{H}\). In fact, it was mostly an attempt to compute examples of Migliore and Nagel’s ideals which led to this alternative approach. Admittedly, the economy and computability obtained by our approach comes at a steep cost, because the important geometric intuition and insight inherent in the generalized stick figures used in Migliore and Nagel’s work is lost in our machinery.

By section, our procedure will be as follows. In Section 2, we introduce the main building block of our construction—given a Hilbert function \(\mathcal{H}\) with \(\ell(\mathcal{H}) < \infty\), \(c \geq \mathcal{H}(1)\), and \(t \geq \ell(\mathcal{H})\), we iteratively define a squarefree monomial ideal \(I_{c,t}(\mathcal{H})\) via decomposition of \(\mathcal{H}\) into two \(O\)-sequences. In Section 3, we show that the quotient of \(I_{c,t}(\mathcal{H})\) (in the appropriate polynomial ring) is Cohen-Macaulay, and compute its dimension, Hilbert function, and graded Betti numbers. In Section 4, we consider a special case of our procedure for manufacturing the \(I_{c,t}(\mathcal{H})\), and thereby construct a family of Gorenstein ideals \(G_{c,t,s} \subseteq I_{c,t}(\mathcal{H})\) for \(s \geq 0\). Finally in Section 5, given an SI-sequence \(\mathcal{H}\) we form a Gorenstein ideal \(J_c(\mathcal{H})\) that has an Artinian reduction with the weak Lefschetz property, Hilbert function \(\mathcal{H}\), and extremal graded Betti numbers. For \(\mathcal{H}(1) \geq 2\), the procedure is to let \(c = \mathcal{H}(1)\), \(t = \ell(\Delta \mathcal{H})\), \(s = \ell(\mathcal{H}) - 2t + 1\), and then sum \(I_{c-1,t}(\Delta \mathcal{H})\) and its link with respect to \(G_{c-1,t,s}\).

2. The main building block

At the heart of our construction is the decomposition of a Hilbert function into two \(O\)-sequences as follows.

Definition 2.1. Given an \(O\)-sequence \(\mathcal{H}\), let \(c \geq \max\{1, \mathcal{H}(1)\}\), \(T = k[\mu_1, \ldots, \mu_c]\) for \(k\) a field, and \(L \subseteq T\) be the lex ideal attaining \(\mathcal{H}\). We define \(\mathcal{H}_c\) to be the \(O\)-sequence \(\mathcal{H}_c = H(T/(L : \mu_1))\), and \(\mathcal{H}^c\) to be the \(O\)-sequence \(\mathcal{H}^c = H(T/(L : \mu_1))\).

Given two \(O\)-sequences \(\mathcal{H}'\) and \(\mathcal{H}\), if we write \(\mathcal{H}' \leq \mathcal{H}\) to indicate that \(\mathcal{H}'(d) \leq \mathcal{H}(d)\) for all \(d \in \mathbb{N}\), then it is easy to see that \(\mathcal{H}' \leq \mathcal{H}\) implies \(\mathcal{H}'_c \leq \mathcal{H}_c\) and \(\mathcal{H}^c \leq \mathcal{H}^c\). A simple definition chase demonstrates that \((\mathcal{H}^c)_c \leq (\mathcal{H}_c)^c - 1\) (where the latter term makes sense since \(\mathcal{H}_c(1) \leq c - 1\)).

Lemma 2.1. Suppose that \(c \geq 2\) and \(\mathcal{H}\) is an \(O\)-sequence with \(H(1) \leq c\). Then \((\mathcal{H}^c)_c \leq (\mathcal{H}_c)^c - 1\).

Proof. Let \(L\) be the lex ideal in \(T = k[\mu_1, \ldots, \mu_c]\) attaining \(\mathcal{H}\). It is easy to see that \((L : \mu_1)\) is lex, so that

\[(\mathcal{H}^c)_c = (H(T/(L : \mu_1)))_c = H(T/((L : \mu_1) + \mu_1)).\]
On the other hand, \( T'/(L + \mu_1) \cong T'/L' \) for \( T' = k[\lambda_1, \ldots, \lambda_{c-1}] \) and \( L' \) the (obviously lex) preimage of \( L \cap k[\mu_2, \ldots, \mu_c] \) under the map \( \phi : T' \to T \) by \( \lambda_i \to \mu_{i+1} \). So

\[
(H_c)^{c-1} = (H(T/L + \mu_1))^{c-1} = H(T'/L' : \lambda_1)
\]

and it is enough to show that there is a surjective homomorphism \( T'/L' : \lambda_1 \to T/(L : \mu_1 + \mu_1) \). Obviously \( \phi \) induces a surjection from \( T' \) to \( T/(L : \mu_1 + \mu_1) \), so it is enough to show that if \( m \) is a monomial in \( (L' : \lambda_1) \), then \( \phi(m) \in (L : \mu_1) \). But \( m\lambda_1 \in L' \) implies that \( \phi(m\lambda_1) = \phi(m)\mu_2 \in L \cap k[\mu_2, \ldots, \mu_c] \subseteq L \), so \( \phi(m)\mu_1 \in L \) (since \( L \) is lex) and hence \( \phi(m) \in (L : \mu_1) \).

From the short exact sequence

\[
0 \to \frac{T}{L : \mu_1} \to \frac{T}{L} \to \frac{T}{L + \mu_1} \to 0
\]

it is evident that \( \mathcal{H}(d) = \mathcal{H}_c(d) + \mathcal{H}'(d - 1) \) for all \( d \in \mathbb{N} \), and we have already observed that \( \mathcal{H}_c(1) \leq c - 1 \). These observations provide the minor justification required to define the central object in our construction.

**Definition 2.2.** For \( c, s \in \mathbb{N}_{\geq 0} \) and \( t \in \mathbb{Z}_{\geq 1} \), let

\[
R_{c,t,s} = k[x_0, \ldots, x_{t+\lfloor \frac{s-1}{2} \rfloor}, y_0, \ldots, y_{t+\lfloor \frac{s-1}{2} \rfloor}, z_0, \ldots, z_{s-1}].
\]

Then given an \( O \)-sequence \( \mathcal{H} \) with \( \ell(\mathcal{H}) < \infty \), \( c \geq \mathcal{H}(1) \), and \( t \geq \ell(\mathcal{H}) \) we define the ideal \( I_{c,t}(\mathcal{H}) \subseteq R_{c,t,0} \) iteratively as follows:

- for \( c = 0 \), let \( I_{0,t}(\mathcal{H}) \subseteq R_{0,t,0} \) be the ideal

\[
I_{0,t}(\mathcal{H}) = \begin{cases} R_{0,t,0} & \text{if } \mathcal{H} = \{0\}, \\ 0 & \text{if } \mathcal{H} = \{1\}; \end{cases}
\]

- for \( t = -1 \) let

\[
I_{c,-1}(\mathcal{H}) = I_{c,-1}(\{0\}) = R_{c,-1,0};
\]

- for \( c > 0 \), and \( t > -1 \), let \( I_{c,t}(\mathcal{H}) \subseteq R_{c,t,0} \) be the ideal

\[
I_{c,t}(\mathcal{H}) = \begin{cases} R_{c,t,0} & \text{if } \mathcal{H} = \{0\} \\ I_{c-1,t}(\mathcal{H}_c)R_{c,t,0} + \omega_{c,t}I_{c,t-1}(\mathcal{H}_c)R_{c,t,0} & \text{if } \mathcal{H} \neq \{0\} \end{cases}
\]

where \( \omega_{c,t} = x_{t+\lfloor \frac{s-1}{2} \rfloor} \) if \( c \) is odd and \( y_{t+\lfloor \frac{s-1}{2} \rfloor} \) if \( c \) is even.

**Remark 2.1.** Note that depending on the values of \( c, t, \) and \( s \), one or more of \( \{x_1, \ldots, x_{\lfloor \frac{s-1}{2} \rfloor}\} \), \( \{y_1, \ldots, y_{\lfloor \frac{s-1}{2} \rfloor}\} \), and \( \{z_1, \ldots, z_{s-1}\} \) might be empty. Thus, \( R_{0,-1,0} = R_{1,-1,0} = R_{2,-1,0} = R_{0,0,0} = k \) and \( R_{3,-1,0} = R_{1,0,0} = k[x_0] \). We will suppress the obvious natural inclusions

\[
R_{c-1,t,s}, R_{c,t-1,s}, R_{c,t,s-1} \subseteq R_{c,t,s},
\]

and hence will write

\[
I_{c,t}(\mathcal{H}) = I_{c-1,t}(\mathcal{H}_c) + \omega_{c,t}I_{c,t-1}(\mathcal{H}_c)
\]

in cases for which there can be no confusion. For much of the initial construction \( s = 0 \) and hence we will typically write \( R_{c,t} \) for \( R_{c,t,0} \). In fact, whenever possible we will write \( R \) to denote \( R_{c,t,s} \), for example \( R/I_{c,t}(\mathcal{H}) \) for \( R_{c,t,s}/I_{c,t}(\mathcal{H}) \).
Remark 2.2. The ideal $I_{c,t}(\mathcal{H})$ should be compared with $I_{c,t}(\mathfrak{h})$ in Migliore and Nagel’s construction, and so Theorem 3.2 and Corollary 3.2 in this work with Theorem 5.8 and Corollary 5.10 in [9].

Example 2.1. Consider the Hilbert function $\mathcal{H} = \{1, 2\}$. Chasing the iteration, we have

\[
I_{3,2}([1, 2]) = I_{2,2}([1, 2]^{3}) + \omega_{3,2}I_{3,1}([1, 2]_{3}) = I_{2,2}([1, 2]) + x_{3}I_{3,1}([0])
\]

\[
I_{2,2}([1, 2]) = I_{1,2}([1, 2]^{2}) + \omega_{2,2}I_{2,1}([1, 2]_{2}) = I_{1,2}([1, 1]) + y_{2}I_{2,1}([1])
\]

\[
I_{3,1}([0]) = R_{3,1}
\]

\[
I_{1,2}([1, 1]) = I_{0,2}([1, 1]^{1}) + \omega_{1,2}I_{1,1}([1, 1]_{1}) = I_{0,2}([1]) + x_{2}I_{1,1}([1])
\]

\[
I_{2,1}([1]) = I_{1,1}([1]^{2}) + \omega_{2,1}I_{2,0}([1]_{2}) = I_{1,1}([1]) + y_{1}I_{2,0}([0])
\]

\[
I_{0,2}([1]) = 0
\]

\[
I_{1,1}([1]) = I_{0,1}([1]^{1}) + \omega_{1,1}I_{1,0}([1]_{1}) = I_{0,1}([1]) + x_{1}I_{1,0}([0])
\]

\[
I_{2,0}([0]) = R_{2,0}
\]

\[
I_{0,1}([1]) = 0
\]

\[
I_{1,0}([0]) = R_{1,0}
\]

and then back substituting

\[
I_{1,1}([1]) = I_{0,1}([1]) + x_{1}I_{1,0}([0]) = 0 + x_{1}R_{1,0} = (x_{1})
\]

\[
I_{1,2}([1, 1]) = I_{0,2}([1]) + x_{2}I_{1,1}([1]) = 0 + x_{2}(x_{1}) = (x_{1}x_{2})
\]

\[
I_{2,1}([1]) = I_{1,1}([1]) + y_{1}I_{2,0}([0]) = (x_{1}) + y_{1}R_{2,0} = (x_{1}, y_{1})
\]

\[
I_{2,2}([1, 2]) = I_{1,2}([1, 1]) + y_{2}I_{2,1}([1]) = (x_{1}x_{2}) + y_{2}(x_{1}, y_{1})
\]

\[
= (x_{1}x_{2}, x_{1}y_{2}, y_{1}y_{2})
\]

\[
I_{3,2}([1, 2]) = I_{2,2}([1, 2]) + x_{3}I_{3,1}([0]) = (x_{1}x_{2}, x_{1}y_{2}, y_{1}y_{2}) + x_{3}R_{3,1}
\]

\[
= (x_{1}x_{2}, x_{1}y_{2}, y_{1}y_{2}, x_{3}).
\]

One can easily show that for $\mathcal{H} \neq 0$ with $\mathcal{H}(1) \leq 1$ and $t \geq \ell(\mathcal{H})$, $I_{1,t}(\mathcal{H}) = (x_{0} \cdots x_{t})$. Note that using the convention that the empty product equals 1, $I_{1,-1}(\mathcal{H}) = (x_{0} \cdots x_{-1}) = (1) = R_{1,-1,0}$ is consistent with definition 2.2.

The $I_{c,t}(\mathcal{H})$ are relatively easy to manipulate inductively. To illustrate this (and for use later), we record here the following observations.

Proposition 2.1. Let $\mathcal{H}$ be an $O$-sequence with $\ell(\mathcal{H}) < \infty$, $c \geq \mathcal{H}(1)$, and $t \geq \ell(\mathcal{H})$. Then $I_{c,t}(\mathcal{H})$ is a squarefree monomial ideal.

Proof. The result is obvious if $\mathcal{H} = 0$, $c = 0$, or $t = -1$, so suppose that $\mathcal{H} \neq 0$ and $c > 0$, $t > -1$ and let $m \in I_{c,t}(\mathcal{H})$ be a minimal generator. But $m \in I_{c-1,t}(\mathcal{H}_{c}) + \omega_{c,t}I_{c-1,t}(\mathcal{H}_{c})$ and each of $I_{c-1,t}(\mathcal{H}_{c})$ and
Let \( I_{c,t-1}(\mathcal{H}^c) \) are squarefree by the induction hypothesis. The result then follows because \( I_{c,t-1} \subseteq R_{c,t-1} \) but \( \omega_{c,t} \not\in R_{c,t-1} \) by construction.

**Proposition 2.2.** Let \( \mathcal{H} \) be an \( \mathcal{O} \)-sequence with \(-1 < \ell(\mathcal{H}) < \infty, c \geq \mathcal{H}(1) \), and \( t \geq \ell(\mathcal{H}) \). Then \( I_{c,t}(\mathcal{H}) \) contains exactly \( c - \mathcal{H}(1) \) minimal linear generators.

**Proof.** This is obvious if \( c = 0 \). If \( c > 1 \), then there are two possibilities. If \( \mathcal{H}^c \neq 0 \), then (using the notation from **Definition 2.1**) \( \mu_1 \neq L \subseteq k[\mu_1, \ldots, \mu_t] \) so \( \mathcal{H}_c(1) = \mathcal{H}(1) - 1 \). By induction \( I_{c-1,t}(\mathcal{H}_c) \) contains exactly \( c - 1 - \mathcal{H}(1) = c - \mathcal{H}(1) \) linear generators and \( \omega_{c,t}I_{c,t-1}(\mathcal{H}^c) \) is generated in degrees \( \geq 2 \) (the latter point follows from the definition because \( I_{c,t-1}(\mathcal{H}^c) = R_{c,t-1} \) if and only if \( \mathcal{H}^c = 0 \)). Since \( I_{c,t}(\mathcal{H}) = I_{c-1,t}(\mathcal{H}_c) + \omega_{c,t}I_{c,t-1}(\mathcal{H}^c) \), we are finished. If \( \mathcal{H}^c = 0 \), then \( \mu_1 \in L \subseteq k[\mu_1, \ldots, \mu_t] \), so \( \mathcal{H}_c(1) = \mathcal{H}(1) \). By induction \( I_{c-1,t}(\mathcal{H}_c) \) contains exactly \( c - 1 - \mathcal{H}(1) = c - \mathcal{H}(1) \) linear generators, so \( I_{c,t}(\mathcal{H}) = I_{c-1,t}(\mathcal{H}_c) + \omega_{c,t}R_{c,t} \) contains exactly \( c - \mathcal{H}(1) \) linear generators as required.

**Proposition 2.3.** Let \( \mathcal{H} \) be \( \mathcal{O} \)-sequences with \( -1 < \ell(\mathcal{H}) < \infty, c \geq \mathcal{H}(1) \), and \( t \geq \ell(\mathcal{H}) \). Then \( I_{c,t}(\mathcal{H}) \supseteq I_{c,t}(\mathcal{H}') \).

**Proof.** We do induction on \( c \) and \( t \). The result is obvious if \( t = -1 \), \( \mathcal{H} = 0 \), or \( c = 0 \), so we suppose \( c > 0, t > -1, \) and \( \mathcal{H} \neq 0 \). Then

\[
I_{c,t}(\mathcal{H}') = I_{c-1,t}(\mathcal{H}_c') + \omega_{c,t}I_{c,t-1}(\mathcal{H}^c)
\supseteq I_{c-1,t}(\mathcal{H}_c) + \omega_{c,t}I_{c,t-1}(\mathcal{H}^c) = I_{c,t}(\mathcal{H})
\]

applying the induction hypothesis as well as the observations preceding **Lemma 2.1**.

It follows directly from the definition that \( I_{c-1,t}(\mathcal{H}_c) \subseteq I_{c,t}(\mathcal{H}) \). In fact, more is true.

**Proposition 2.4.** Let \( \mathcal{H} \) be an \( \mathcal{O} \)-sequence, \( c \geq \max\{1, \mathcal{H}(1)\} \), and \( t \geq \max\{0, \ell(\mathcal{H})\} \). Then \( I_{c-1,t}(\mathcal{H}_c)R_{c,t} \subseteq I_{c,t}(\mathcal{H}^c)R_{c,t} \).

**Proof.** The result is obvious if \( \mathcal{H} = 0 \), so presume not and proceed by induction on \( c \). If \( c = 1 \) then \( \mathcal{H}_1 \neq 0 \) and \( I_{0,1}(\mathcal{H}_1)R_{1,t} = (0) \subseteq I_{1,t-1}(\mathcal{H}_1^1)R_{1,t} \) trivially.

Now we suppose that \( c > 1 \). So

\[
I_{c-1,t}(\mathcal{H}_c)R_{c,t} = I_{c-2,t}(\mathcal{H}_c)R_{c,t} + \omega_{c-1,t}I_{c-1,t-1}(\mathcal{H}_c^{c-1})R_{c,t}
\supseteq I_{c-1,t-1}(\mathcal{H}_c^{c-1})R_{c,t} + \omega_{c-1,t}I_{c-1,t-1}(\mathcal{H}_c^{c-1})R_{c,t}
\supseteq I_{c-1,t-1}(\mathcal{H}_c^{c-1})R_{c,t} \subseteq I_{c-1,t-1}(\mathcal{H}_c^{c-1})R_{c,t}
\subseteq I_{c,t-1}(\mathcal{H}^c)R_{c,t},
\]

as required where the first containment is by the induction hypothesis and the third is by **Lemma 2.1** and **Proposition 2.3**.

The final result of this section gives some indication how we will chase information about Hilbert functions and graded Betti numbers through our iterative definition.

**Proposition 2.5.** Let \( \mathcal{H} \) be an \( \mathcal{O} \)-sequence with \( \mathcal{H}(1) \geq 1 \) and \( \ell(\mathcal{H}) < \infty, c \geq \mathcal{H}(1) \), and \( t \geq \min\{d \mid \mathcal{H}(d) \geq \mathcal{H}(d+1)\} \). Then we have a short exact sequence

\[
0 \xrightarrow{R} \frac{R}{\omega_{c,t}I_{c-1,t}(\mathcal{H}_c)} \xrightarrow{R} \frac{R}{I_{c-1,t}(\mathcal{H}_c)} \oplus \frac{R}{\omega_{c,t}I_{c,t-1}(\mathcal{H}^c)} \xrightarrow{R} \frac{R}{I_{c,t}(\mathcal{H})} \xrightarrow{0}.
\]
Proof. That $I_{c-1,t}^c(H_c) \cap \omega_{c,t} I_{c,t-1}^c(H_c^c) \supseteq \omega_{c,t} I_{c-1,t}^c(H_c^c)$ follows immediately from Proposition 2.4, so we suppose that $m$ is a monomial in

\[ I_{c-1,t}^c(H_c) \cap \omega_{c,t} I_{c,t-1}^c(H_c^c). \]

Then $m \in I_{c-1,t}^c(H_c)$ and $\omega_{c,t} | m$. But $\omega_{c,t}$ is a non-zero-divisor on $R/I_{c-1,t}^c(H_c)$ because $\omega_{c,t} \notin R_{c-1,t}$, so $m \in \omega_{c,t} I_{c-1,t}^c(H_c)$ as required.

\[ \square \]

3. Properties of the $I_{c,t}(H)$

We now show that $R/I_{c,t}(H)$ is Cohen-Macaulay and compute its dimension, graded Betti numbers, and Hilbert function.

Proposition 3.1. Let $H$ be an $O$-sequence with $-1 < \ell(H) < \infty$, $c \geq H(1)$, and $t \geq \ell(H)$. Then any minimal prime of $I_{c,t}(H)$ is generated by $c$ variables, exactly $\lfloor \frac{c-1}{2} \rfloor$ of which are from $\{y_0, \ldots, y_{t+\lfloor \frac{c}{2} \rfloor}\}$.

Proof. We do induction on $c$ and $t$. If $c = 0$ then $I_{0,t}(H) = (0)$ and the result is obvious. If $t = 0$, then $I_{c,0}(H) = (x_0, \ldots, x_{t+\lfloor \frac{c}{2} \rfloor}, y_0, \ldots, y_{\lfloor \frac{c}{2} \rfloor})$ and again the result is clear.

So suppose $c \geq 1$, $t \geq 1$, and $P$ is a minimal prime of

\[ I_{c,t}(H) = I_{c-1,t}(H_c) + \omega_{c,t} I_{c,t-1}(H_c^c). \]

If $\omega_{c,t} \in P$, then the prime obtained by removing $\omega_{c,t}$ from $P$ is minimal over $I_{c-1,t}(H_c)$ and hence consists of $c-1$ variables, $\lfloor \frac{c-1}{2} \rfloor$ from $\{y_0, \ldots, y_{t+\lfloor \frac{c}{2} \rfloor}\}$. If $c$ is even then $\omega_{c,t} = y_{t+\lfloor \frac{c}{2} \rfloor}$, $\lfloor \frac{c-1}{2} \rfloor = \lfloor \frac{c}{2} \rfloor - 1$, and the result follows. If $c$ is odd, then $\omega_{c,t} = x_{t+\lfloor \frac{c}{2} \rfloor}$ and $\lfloor \frac{c-1}{2} \rfloor = \lfloor \frac{c}{2} \rfloor$ as required.

If $\omega_{c,t} \notin P$ then $\omega_{c,t} \notin I_{c-1,t}(H_c)$, so $I_{c,t-1}(H_c^c) \not\supseteq R_{c,t-1}$ and thus $H_c^c \not\neq 0$. Moreover, $I_{c,t-1}(H_c^c) \subseteq P$, so there is a prime $Q \subseteq P \cap R_{c,t-1}$ minimal over $I_{c,t-1}(H_c^c)$. By induction on $t$, $Q$ has $c$ generators of which $\lfloor \frac{c}{2} \rfloor$ are in $\{y_0, \ldots, y_{t-1+\lfloor \frac{c}{2} \rfloor}\}$, and thus it is enough to show that $QR_{c,t} = P$. But this is immediate since $I_{c-1,t}(H_c) \subseteq I_{c,t-1}(H_c^c)R_{c,t}$ by Proposition 2.4.

\[ \square \]

Corollary 3.1. Let $H$ be an $O$-sequence with $-1 < \ell(H) < \infty$, $c \geq H(1)$, and $t \geq \ell(H)$, and let $s \geq 0$. Then $\dim R/I_{c,s}(H) = 2t + s$.

To show that $R/I_{c,t}(H)$ is Cohen-Macaulay requires a few ring theoretic observations. First recall the well known fact that for any graded ring $R$ and $I \subseteq R$ a homogeneous ideal, if $R/I$ is Cohen-Macaulay, then $R[x]/IR[x]$ is Cohen-Macaulay with $\dim (R[x]/IR[x]) = \dim (R/I) + 1$. Moreover by tensoring a minimal free resolution of $R/I$ by $R[x]$, we observe that $\text{pd}(R[x]/IR[x]) = \text{pd}(R/I)$, and the graded Betti numbers do not change.

A less standard but equally easy fact is as follows.

Lemma 3.1. Let $R$ be a graded ring and $I \subseteq R$ be a homogeneous ideal such that $R/I$ is Cohen Macaulay. If $f$ is a $d$-form of $R$ which is a nonzero-divisor on $R/I$, then $\text{depth}(R/I) = \text{depth}(R/I)$, $\text{pd}(R/I) = \text{pd}(R/I)$, and $\beta_{ij}^1 = \beta_{ij+d}^1$ for all $i, j \in \mathbb{N}$.

Proof. Let $(f_1, \ldots, f_r)$ be a minimal generating set for $I$ with $d_i = \deg f_i$ for $i = 1, \ldots, r$. Then $(f_1, \ldots, f_r)$ is a minimal generating set for $fI$ and setting $F_\bullet$ and $G_\bullet$ to be minimal free resolutions of $R/I$ and $R/I$ with respective differentials $\delta_i$ and $\partial_i$, we may take

\[ \delta_1 = \begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} \quad \text{and} \quad \partial_1 = \begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}. \]
Note that ker $\delta_1$ is a submodule of $R(-d_1) \oplus \cdots \oplus R(-d_r)$ while ker $\delta_0$ is a submodule of $R(-d_1 - d) \oplus \cdots \oplus R(-d_r - d)$. But $[g_1, \ldots, g_r] \in$ ker $\delta_1$ if and only if it is in the kernel of $\delta_0$, that is $\sum_{i=0}^{r} f_ig_i = 0$ if and only if $\sum_{i=0}^{r} f_ig_i = f$, and thus ker $\delta_1(-d) \cong$ ker $\delta_2$ in the obvious way. Thus the assertions concerning resolutions follows immediately and depth$(R/I) = \text{depth}(R/I')$ by the Auslander-Buchsbaum formula.

We can now show that $R/I_{c,t}(H)$ is Cohen-Macaulay.

**Theorem 3.1.** Let $H$ be an $O$-sequence with $-1 < \ell(H), c \geq H(1)$, and $t \geq \ell(H)$. Then $R_{c,t,s}/I_{c,t}(H)$ is a dimension $2t + s$ Cohen-Macaulay algebra with projective dimension $c$.

**Proof.** It suffices, by Corollary 3.1 and the Auslander-Buchsbaum formula, to show that $R_{c,t,s}/I_{c,t}(H)$ has projective dimension $c$.

We proceed by induction on $c$ and $t$. When $c = 0$, $I_{0,t}(H) = 0$ so the result is obvious. If $t = 0$ then $I_{c,0}(H) = \langle x_0, \ldots, x_{\frac{c-1}{2}}, y_0, \ldots, y_{\frac{c-1}{2}} \rangle$, and the Koszul complex gives the result.

Now suppose $c, t > 0$ so that $I_{c,t}(H) = I_{c-1,t}(H) + \omega_{c,t}I_{c-1,t}(H)$.

If $H^c = 0$, then $I_{c,t}(H) = I_{c-1,t}(H_c)$, and the mapping cone resolution on the short exact sequence

$$0 \to \frac{R}{I_{c-1,t}(H_c)} \xrightarrow{\omega_{c,t}} \frac{R}{I_{c-1,t}(H_c)} \to \frac{R}{I_{c,t}(H)} \to 0$$

is minimal (because $\omega_{c,t}$ is regular modulo $I_{c-1,t}(H_c)$), so the result follows immediately by induction.

If $H^c \neq 0$, then consider the sequence, exact by Proposition 2.5,

$$0 \to \frac{R}{\omega_{c,t}I_{c-1,t}(H_c)} \to \frac{R}{I_{c-1,t}(H_c)} \oplus \frac{R}{\omega_{c,t}I_{c-1,t}(H^c)} \to \frac{R}{I_{c,t}(H)} \to 0.$$ 

Let $F_\bullet, G_\bullet$, and $H_\bullet$ be minimal free resolutions of, respectively, $R/\omega_{c,t}I_{c-1,t}(H_c)$, $R/I_{c-1,t}(H_c)$, and $R/\omega_{c,t}I_{c-1,t}(H^c)$, and let $T_\bullet$ be the minimal free resolution of $R/I_{c,t}(H)$ living inside the mapping cone resolution. Thus $T_r \subseteq F_{r-1} \oplus G_r \oplus H_r = 0$ for $r > t$ since $\text{pd}(R/\omega_{c,t}I_{c-1,t}(H_c)) = \text{pd}(R/I_{c-1,t}(H_c)) = c - 1$, and $\text{pd}(R/\omega_{c,t}I_{c-1,t}(H^c)) = c$ (by induction and Lemma 3.1). Furthermore, $0 \neq H_r \subseteq T_r$ because $F_0 = 0$ and hence any non-minimality in the mapping cone resolution cannot involve $H_r$. Thus the projective dimension of $R/I_{c,t}(H)$ is $c$ as required.

Now we show that the Betti diagrams of $I_{c,t}(H)$ and the lex ideal attaining $H$ coincide. The first step is to demonstrate that the graded Betti numbers of the latter can be decomposed via $H_c$ and $H^c$.

**Lemma 3.2.** Let $H$ be an $O$-sequence with $-1 < \ell(H) < \infty$, $c \geq \max\{1, H(1)\}$, $L$ be the lex ideal in $T = k[\mu_1, \ldots, \mu_c]$ with Hilbert function $H$, and $L'$ be the lex ideal in $T' = k[\lambda_1, \ldots, \lambda_c-1]$ with Hilbert function $H(T/(L + \mu_1))$. Then for $(0,1) \neq (i,j) \neq (1,1)$, $\beta_{ij}^{L+\mu_1} = \beta_{ij}^{L+\mu_1} = \beta_{ij}^{L+\mu_1} = \beta_{ij}^{L+\mu_1}$ as required for the second equality.

**Proof.** Let $\phi : T' \to T$ by $\phi(\lambda_i) = \mu_{i+1}$. Then $L' = \phi^{-1}(L \cap k[\mu_2, \ldots, \mu_c])$ so $T/(L + \mu_1) \cong T/(\phi(L') + \mu_1)$ as $T$-modules. By the mapping cone (which is minimal since $\mu_1$ is regular modulo $\phi(L')$, $\beta_{ij}^{L+\mu_1} = \beta_{ij}^{L+\mu_1} = \beta_{ij}^{L+\mu_1} = \beta_{ij}^{L+\mu_1}$ as required for the second equality.

For the first equality, note that the result is obvious if $j = 0, 1$, so we take $j > 1$. Given $I \subseteq T$, let $G(I)$ be the degree $j$ minimal generators of $I$. Then it is easy to see that $G(I) = G(L + \mu_1) \cup \mu_1G(L : L_{\mu_1})$ (since $j > 1$) so the theorem holds if $i = 1$. The result is also obvious if $i = 0$ so we suppose $i > 1$.

Now write $G(I)_{j,q}$ to be set of all degree $j$ minimal monomial generators $m \in I$ with $\max\{p \mid p \in \text{supp}(m)\} = q$. Since $L$, $(L : \mu_1)$, and $L + \mu_1$ are all lex, the Eliahou-Kervaire formula (see [6], equation...
Theorem 3.2.

Consider first the chain map induced by \( G(1, \mu_1) \), \( \beta^{L+\mu_1}_{ij} = \sum_{q=1}^{c} \binom{q-1}{i-1} |G(L + \mu_1)_{j-i, q}| \).

Since \( m \neq \mu_1 \) (because \( q > 1 \)), \( q \in \text{supp}(m/\mu_1) \) and obviously \( m/\mu_1 \) is a minimal generator of \( L : \mu_1 \), so \( m \in \mu_1 G(L : \mu_1)_{j-i, q} \) as required.

Now computing the Betti diagram of \( R/I_{c,t} (\mathcal{H}) \) is simply a matter of unraveling the induction.

Theorem 3.2. Let \( \mathcal{H} \) be an \( \mathcal{O} \)-sequence with \( \ell(\mathcal{H}) < \infty \), \( c \geq \mathcal{H}(1) \), \( t \geq \ell(\mathcal{H}) \), \( T = k[\mu_1, \ldots, \mu_c] \), and \( L \subseteq T \) be the lex ideal attaining \( \mathcal{H} \). Then \( \beta^{L+\mu_1}_{ij}(\mathcal{H}) = \beta^{L}_{ij} \).

Proof. If \( \mathcal{H} = 0 \), then \( I_{c,t}(\mathcal{H}) = R_{c,t,0} \) and \( T = L \) and the result follows, so we assume \( \mathcal{H} \neq 0 \) and hence \( t > -1 \).

If \( c = 0 \), then \( I_{c,t}(\mathcal{H}) = 0 \) and \( \mathcal{H} = 1 \). It follows that \( L = (0) \subseteq T = k \) and we are done.

If \( c = 1 \), then \( I_{1,t}(\mathcal{H}) = (x_0 \cdots x_t) \) (see Example 2.1) and \( L = (\mu_1^{\ell(\mathcal{H})+1}) \) so the result is obvious.

Now suppose that \( c > 0 \), \( t > -1 \), and \( \mathcal{H} \neq 0 \). Since \( H(T/L : \mu_1) = \mathcal{H}_c \), by induction we have \( \beta^{L_{c-1},1}(\mathcal{H}_c) = \beta^{L_{c-1}}_{ij} \), and thus \( \beta^{L+\mu_1}_{ij} = \beta^{L_{c-1},1}(\mathcal{H}_c) \) (by Lemma 3.1). Similarly, if we let \( L' \) be the lex ideal in the polynomial ring with \( c - 1 \) variables and Hilbert function \( \mathcal{H}_c = H(T/L + \mu_1) \), then \( \beta^{L'}_{ij} = \beta^{L_{c-1},1}_j = \beta^{\omega_{c-1},1}_j(\mathcal{H}_c) \).

Obviously \( \beta^L_{0,1} = 0 = \beta^L_{0,1} \) and \( \beta^L_{1,1} = c - \mathcal{H}(1) = \beta^L_{1,1} \) (by Proposition 2.2 and because \( H(T/L) = \mathcal{H} \).

By Proposition 2.5, the sequence

\[
0 \rightarrow \mathcal{R} / \omega_{c,t}I_{c-1,t}(\mathcal{H}_c) \rightarrow \mathcal{R} / I_{c-1,t}(\mathcal{H}_c) \oplus \mathcal{R} / \omega_{c,t}I_{c,t}(\mathcal{H}_c) \rightarrow \mathcal{R} / I_{c,t}(\mathcal{H}_c) \rightarrow 0
\]

is exact, and we claim that the mapping cone resolution of \( R/I_{c,t}(\mathcal{H}) \) is minimal except in degree 1 between the zeroth and first step. This completes the proof since then, for \( (0, 1) \neq (i, j) \neq (1, 1) \), \( \beta^{L_{c-1},1}(\mathcal{H}) = \beta^{L_t}_{i,j-1} + \beta^{L_{c-1}}_{i} + \beta^{L_{c,1}}_{j} = \beta^{L_t}_{i,j} \), as required (the last equality from Lemma 3.2).

Consider first the chain map induced by \( R/\omega_{c,t}I_{c-1,t}(\mathcal{H}_c) \rightarrow R/I_{c-1,t}(\mathcal{H}_c) \). Note that every generator of \( \omega_{c,t}I_{c-1,t}(\mathcal{H}_c) \) is divisible by \( \omega_{c,t} \). Thus, the multidegree of each minimal generator at each step of a minimal free resolution of \( R/\omega_{c,t}I_{c-1,t}(\mathcal{H}_c) \) must be positive with respect to \( \omega_{c,t} \). Since \( I_{c-1,t}(\mathcal{H}_c) \subseteq \omega_{c,t}I_{c-1,t}(\mathcal{H}_c) \)
$R_{c-1,t}$, however, the multidegree of each minimal generator at each step of a minimal free resolution of $R/I_{c-1,t}(\mathcal{H}_c)$ must be zero with respect to $\omega_{c,t}$. It follows that the chain map induced by tensoring by $k$ must be zero.

Now consider the chain map induced by $R/\omega_{c,t}I_{c-1,t}(\mathcal{H}_c) \xrightarrow{\delta} R/\omega_{c,t}I_{c-1,1}(\mathcal{H}_c)$. Obviously if $\mathcal{T}$ is the generator of $R/\omega_{c,t}I_{c-1,t}(\mathcal{H}_c)$, then $\delta_0(\mathcal{T})$ is the generator of $R/\omega_{c,t}I_{c-1,1}(\mathcal{H}_c)$, so the mapping cone resolution is not minimal (between the zeroth and first step in degree 1). For $j > 1$, however, no cancelation can occur. Indeed, let $\alpha$ be the minimal exponent such that $\mu_{c}^{\alpha} \in L$ (we use here that $c > 1$). Then $\beta_{i,j}^{L'} = 0$ for $1 < j < \alpha + i - 1$, so $\beta_{i,j}^{\omega_{c,t}I_{c-1,t}(\mathcal{H}_c)} = 0$ for $2 < j < \alpha + i$ and it is enough to show that $\beta_{i,j}^{\omega_{c,t}I_{c-1,t}(\mathcal{H}_c)} = 0$ for $j \geq \alpha + i$, i.e., that $\beta_{i,j}^{L_t} = 0$ for $j \geq \alpha + i - 1$. So we need to show that the regularity of $T/(L : \mu_1)$ is $< \alpha - 1$. Now $\mu_2^c \in L$, so $\mu_1 \mu_2^{\alpha - 1} \in L$, and hence $\mu_2^{\alpha - 1} \in L : \mu_1$. Since it is well known that the regularity of a dimension zero quotient of a lex ideal is equal to one less than the minimal power of $\mu_2$ the ideal contains, we have that the regularity of the quotient of $L : \mu_1$ is $\leq \alpha - 2$ as required.

Thus for $j > 1$ the degrees of the generators at each step in minimal resolutions of $R/\omega_{c,t}I_{c-1,t}(\mathcal{H}_c)$ and $R/\omega_{c,t}I_{c-1,1}(\mathcal{H}_c)$ never coincide, hence we conclude that no cancelation can occur (for $j > 1$), and thus the mapping cone is minimal except in degree 1 between the zeroth and first step as required.

**Corollary 3.2.** Let $\mathcal{H}$ be an $O$-sequence with $\ell(\mathcal{H}) < \infty$, $c \geq \mathcal{H}(1)$, and $t \geq \ell(\mathcal{H})$. Then $\Delta^{2t+1}H(R_{c,t}/I_{c,t}(\mathcal{H})) = H$. 

**Proof.** This is trivial if $\mathcal{H} = 0$. If $\mathcal{H} \neq 0$, then use Theorem 3.1. Since $k$ is infinite an Artinian reduction of $R/I_{c,t}(\mathcal{H})$ and $T/L$ have the same graded Betti numbers and hence the same Hilbert function. □

### 4. Sub-ideals of $I_{c,t}(\mathcal{H})$

The goal of this section is to construct a Gorenstein ideal inside of $I_{c,t}(\mathcal{H})$ with which to form the link.

The first step is to identify two sub-ideals of $I_{c,t}(\mathcal{H})$ and determining how they relate to one another. We also record a few facts about these new families which will prove useful when we consider the weak Lefschetz property.

**Definition 4.1.** Let $c \geq 0$, $t \geq -1$, and $m$ be the unique homogeneous maximal ideal in $T = k[\mu_1, \ldots, \mu_c]$. Then we write $\mathbb{H}_{c,t} = H(T/m^{t+1})$.

**Remark 4.1.** The introduction of a doubly subscripted Hilbert function could turn following our iterative definition into an unmitigated disaster. We avoid this difficulty by introducing special notation for $I_{c,t}(\mathbb{H}_{c,t})$ which turns out to respect our inductive construction.

**Definition 4.2.** Given $c \geq 0$ and $t \geq -1$, then write $A_{c,t} = I_{c,t}(\mathbb{H}_{c,t})$.

**Remark 4.2.** Given and $O$-sequence $\mathcal{H}$ with $-1 < \ell(\mathcal{H}) < \infty$, $c \geq \mathcal{H}(1)$, and $t \geq \ell(\mathcal{H})$, then $\mathcal{H} \leq \mathbb{H}_{c,t}$ and hence $A_{c,t} \subseteq I_{c,t}(\mathcal{H})$. Furthermore, it is easy to see that $(\mathbb{H}_{c,t})_c = \mathbb{H}_{c-1,t}$ and $(\mathbb{H}_{c,t})^c = \mathbb{H}_{c-1,-1}$, so the $A_{c,t}$ follow the same iterative rule as the $I_{c,t}(\mathcal{H})$. That is,

$$A_{c,t} = A_{c-1,t}R_{c,t} + \omega_{c,t}A_{c,t-1}R_{c,t}$$

with $A_{c,-1} = R_{c,-1}$ and for $t > -1$, $A_{0,t} = (0) \subseteq R_{0,t}$. Since the lex ideal attaining $\mathbb{H}_{c,t}$ is $(m)^{t+1}$ it is immediate from Theorem 3.2 that $A_{c,t}$ is generated in degree $t + 1$ (when nonzero).

We also need an ideal in $I_{c,t}(\mathcal{H})$ for which $x_0$ is a nonzero-divisor.
Definition 4.3. Let $c \geq 1$ and $t \geq -1$. Then we write $A'_{c,t}$ to be the ideal in $R_{c,t}$ obtained by removing all minimal generators of $A_{c,t}$ which are divisible by $x_0$ (and take the ideal generated by no elements to be zero).

Example 4.1. For example,

$$A'_{3,2} = (y_0y_1y_2,x_3y_0y_1,x_2x_3y_0,x_1x_2x_3)$$
$$= (y_0y_1y_2 + x_3(y_0y_1,x_2y_0,x_1x_2)) = A'_{2,2} + x_3A'_{3,1}$$

By induction it is easy to show that $A'_{2,t} = (y_0 \cdots y_t)$ for all $t \geq -1$ (where the empty product is taken to be 1 by convention).

Remark 4.3. It is immediate that the conclusions of Propositions 2.1, 2.3, and 2.4 hold for $A'_{c,t}$ and additionally $A'_{c,t}$ is generated in degree $t + 1$ (when nonzero). Following Remark 4.2, it is also easy to see that $A'_{c,t} = A'_{c-1,t} + \omega_{c,t}A'_{c,t-1}$ with initial values $A'_{c,-1} = R_{c,-1} = A_{c,-1}$ and $A'_{1,t} = (0)$ for $t > -1$, so that the conclusion of Proposition 2.5 holds. Although the $A'_{c,t}$ versions of Proposition 3.1 and Theorem 3.1 must be modified slightly (as below), the proofs are nearly identical—we change base cases as well as the statement of the induction hypothesis in each step—and hence are omitted.

Proposition 4.1. Let $c \geq 1$ and $t \geq 0$. Then any minimal prime of $A'_{c,t}$ is generated by $c - 1$ variables, exactly $\lfloor \frac{t}{2} \rfloor$ of which are from $(y_0, \ldots, y_t + \lfloor \frac{c-1}{2} \rfloor)$.

Theorem 4.1. Let $c \geq 1$ and $t \geq 0$. Then $R_{c,t}/A'_{c,t}$ is a dimension $2t + s + 1$ Cohen-Macaulay algebra with projective dimension $c - 1$.

We take here the opportunity to explore the relationship between the $A_{c,t}$ and $A'_{c,t}$.

Proposition 4.2. Let $c \geq 2$ and $t \geq 0$. Then $A'_{c,t-1}R_{c,t}$ is contained in no minimal prime of $A_{c-1,t}R_{c,t}$.

Proof. We do induction on $c$ and $t$. If $t = 0$, then $A'_{c-1}R_{c,0} = R_{c,0}$ so the result follows from Proposition 3.1.

If $c = 2$ and $t > 0$, then the result follows from Propositions 3.1 and 4.1.

Now suppose that $c > 2$ and $t > 0$ and $P$ is a minimal prime of $A_{c-1,t}$. As we saw in the proof of Proposition 3.1 either $P = Q + (\omega_{c-1,t})$ for some $Q \subseteq R_{c-2,t}$ minimal over $A_{c-2,t}$ or $P \subseteq R_{c-1,t-1}$ is minimal over $A_{c-1,t-1}$.

If $P$ is minimal over $A_{c-1,t-1}$ then $A'_{c-1,t-2} \not\subseteq P$ by induction, $\omega_{c-1,t-1} \not\in P$ since $\omega_{c-1,t-1} \not\in R_{c-1,t-1}$, and hence $\omega_{c-1,t-1}A'_{c-1,t-2} \not\subseteq P$. We conclude that $A'_{c,t-1} = A'_{c-1,t-1} + \omega_{c-1,t}A'_{c-1,t-2} \not\subseteq P$ as required.

If $P = Q + (\omega_{c-1,t})$ with $Q$ minimal over $A_{c-2,t}$, then $A'_{c-1,t-1} \not\subseteq Q$ by induction on $c$, hence $A'_{c-1,t-1} \not\subseteq Q + (\omega_{c-1,t}) = P$ since $\omega_{c-1,t} \not\in R_{c-1,t-1}$. We conclude that $A'_{c,t-1} = A'_{c-1,t-1} + \omega_{c-1,t}A'_{c-1,t-2} \not\subseteq P$ as required.

Note that $R_{c,t} = R_{c+2,t-1}$ for $c, t \geq 0$, and thus $A_{c,t}$ and $A'_{c+2,t-1}$ are initially defined over the same ring. In fact, more is true.

Proposition 4.3. Let $c, t \geq 0$. Then $A_{c,t} \subseteq A'_{c+2,t-1}$.

Proof. We do induction on $c$ and $t$, the $c = 0$ and $t = 0$ cases being obvious. So suppose $c, t > 0$. By induction $A_{c-1,t} \subseteq A'_{c+1,t-1}$ and $A_{c-1,t} \subseteq A'_{c+2,t-2}$. Since $\omega_{c,t} = \omega_{c+2,t-1}$ it follows that $A_{c,t} = A_{c-1,t} + \omega_{c,t}A_{c,t-1} \subseteq A'_{c+1,t-1} + \omega_{c+2,t-1}A'_{c+2,t-2} = A'_{c+2,t-1}$ as required.

These facts can be used to give information about the residual of $A_{c,t}$ in $A'_{c,t}$. 
Proposition 4.4. Let $c \geq 2$ and $t \geq 0$. Then $A'_{c,t-1} : A_{c-1,t} = A'_{c,t-1}$.

Proof. We take $t > 0$ since the $t = 0$ case is obvious. Let $\lambda$ be homogeneous such that $\lambda A_{c-1,t} \subseteq A'_{c,t-1}$. Since $A'_{c,t-1}$ is a squarefree monomial ideal (see Remark 4.3), it is the intersection of its minimal primes, and thus if $\lambda \in P$ for all $P$ minimal over $A'_{c,t-1}$, then $\lambda \in A'_{c,t-1}$ as required.

So we suppose that for each $a \in A_{c-1,t}$ there is a minimal prime $P_a$ of $A'_{c,t-1}$ such that $a \in P_a$. It follows that $A_{c-1,t}$ is contained in the union of the minimal primes of $A'_{c,t-1}$ and hence is contained in one of them, call it $P$, by Prime Avoidance.

By Proposition 4.1, $P$ is generated by $c - 1$ variables, and by Proposition 3.1, the same is true for any minimal prime of $A_{c-1,t}$. It follows that $P$ is minimal over $A_{c-1,t}$, but this contradicts Proposition 4.2. \hfill \square

Proposition 4.5. Let $c \geq 2$ and $t \geq -1$. Then $A_{c-2,t} : A_{c-1,t} = A_{c-2,t}$.

Proof. The proof is obvious if $t = -1$, so suppose $t \geq 0$ and let $\lambda$ be homogeneous such that $\lambda A_{c-1,t} \subseteq A_{c-2,t}$. Then $\lambda A_{c-1,t} \subseteq A_{c-2,t} \subseteq A'_{c,t-1}$ by Proposition 4.3 and hence $\lambda \in A'_{c-1,t}$ by Proposition 4.4. Since $A'_{c-1,t} \subseteq A_{c-1,t}$ we have that $\lambda \in A_{c-1,t}$, so $\lambda^2 \in A_{c-2,t}$, and hence $\lambda \in A_{c-2,t}$ since $A_{c-2,t}$ is squarefree (Proposition 2.1).

We can now use the $A_{c,t}$ and $A'_{c,t}$ to form a Gorenstein ideal inside of $I_{c,t}(\mathcal{H})$ with which to form the link.

Definition 4.4. Let $c \geq 1$ and $t, s \geq 0$. Then we define

$$G_{1,t,s} = (x_0 \cdots x_t y_0 \cdots y_{t-1} z_0 \cdots z_{s-1}) \subseteq R_{1,t,s},$$

and for $c \geq 2$,

$$G_{c,t,s} = A_{c-1,t} R_{c,t,s} + \omega_{c,t} z_0 \cdots z_{s-1} A'_{c,t-1} R_{c,t,s}.$$  

When there can be no confusion, we write $\overline{z}$ to denote the product $z_0 \cdots z_{s-1}$, so

$$G_{c,t,s} = A_{c-1,t} + \omega_{c,t} \overline{z} A'_{c,t-1}.$$  

Remark 4.4. Since $\omega_{c,t} \overline{z} A'_{c,t-1} \subseteq A'_{c,t}$, $G_{1,t,s} \subseteq A_{1,t}$, and $A_{c-1,t} A'_{c,t} \subseteq A_{c,t} \subseteq I_{c,t}(\mathcal{H})$, (see Remark 4.2), it is immediate that $G_{c,t,s} \subseteq I_{c,t}(\mathcal{H}) R_{c,t,s}$. Similarly since $A_{c-1,t}$ and $\omega_{c,t} \overline{z} A'_{c,t-1}$ are squarefree (the latter because $A'_{c-1,t} \subseteq R_{c-1,t,0}$) it follows that $G_{c,t,s}$ is a squarefree monomial ideal.

It is not difficult to show that $R / G_{c,t,s}$ is Gorenstein using induction. For the sake of the exposition, we first make one observation in a Lemma.

Lemma 4.1. Let $c \geq 2$ and $t, s \geq 0$. Then

$$A_{c-1,t} R_{c,t,s} \cap \omega_{c,t} \overline{z} A'_{c,t-1} R_{c,t,s} = \omega_{c,t} \overline{z} G_{c-1,t,0} R_{c,t,s}$$

Proof. Note that $\omega_{c,t} \overline{z} \not\in R_{c-1,t}$, so

$$A_{c-1,t} R_{c,t,s} \cap \omega_{c,t} \overline{z} A'_{c,t-1} R_{c,t,s} = \omega_{c,t} \overline{z} (A_{c-1,t} R_{c,t,s} \cap A'_{c,t-1} R_{c,t,s})$$

and thus it is equivalent to show that

$$A_{c-1,t} R_{c,t} \cap A'_{c,t-1} R_{c,t} = G_{c-1,t,0} R_{c,t}.$$  

First we show that $A_{c-1,t} \cap A'_{c,t-1} \subseteq G_{c-1,t,0} R_{c,t}$ by induction on $t$.  

If \( t = 0 \), then
\[
A_{c-1,0}R_{c,0} \cap A'_{c-1,0}R_{c,0} = A_{c-1,0}R_{c,0} \cap R_{c,0}
\]
\[
= A_{c-2,0}R_{c,0} + \omega_{c-1,0}A_{c-1,-1}R_{c,0}
\]
\[
= A_{c-2,0}R_{c,0} + \omega_{c-1,0}A'_{c-1,-1}R_{c,0} = G_{c-1,0,0}R_{c,0}
\]
as required.

Now suppose \( t > 0 \) and we have a monomial \( m \in A_{c-1,t} \cap A'_{c,t-1} \). Of course, \( A_{c-1,t} = A_{c-2,t} + \omega_{c-1,t}A_{c-1,t-1} \) and \( A'_{c,t-1} = A'_{c-1,t-1} + \omega_{c,t-1}A'_{c,t-2} \). If \( m \in A_{c-2,t} \), then we are finished since \( A_{c-2,t} \subseteq G_{c-1,t,0} \), so we suppose that \( m \in \omega_{c-1,t}A_{c-1,t-1} \), and thus that \( \omega_{c-1,t} \mid m \). If \( m \in A'_{c-1,t-1} \), then \( m \in \omega_{c-1,t}A'_{c-1,t-1} \) (since \( \omega_{c-1,t} \not\in R_{c-1,t-1} \)) which is enough since \( \omega_{c-1,t}A'_{c-1,t-1} \subseteq G_{c-1,t,0} \). Thus we conclude that
\[
m \in \omega_{c-1,t}A_{c-1,t-1} \cap \omega_{c,t-1}A'_{c,t-2}
\]
\[
\subseteq A_{c-1,t-1} \cap A'_{c,t-2} \subseteq G_{c-1,t,0}
\]
\[
= A_{c-2,t-1} + \omega_{c-1,t-1}A'_{c-1,t-2}
\]
where the second to last inclusion is by induction. Suppose first that \( m \in A_{c-2,t-1} \). Then \( m \in \omega_{c,t-1}A_{c-2,t-1} \) since \( \omega_{c,t-1} \mid m \) but \( \omega_{c,t-1} \not\in R_{c-2,t-1} \). Noting that \( \omega_{c,t-1} = \omega_{c-2,t} \) we have \( m \in \omega_{c-2,t}A_{c-2,t-1} \subseteq A_{c-2,t} \subseteq G_{c-1,t,0} \) as required. Finally, we suppose that \( m \in \omega_{c-1,t-1}A'_{c-1,t-2} \). But \( \omega_{c-1,t-1}A'_{c-1,t-2} \subseteq A'_{c-1,t-1} \), a case already covered.

The other direction is simpler. We have \( G_{c-1,t,0} = A_{c-2,t} + \omega_{c-1,t}A'_{c-1,t-1} \). But \( A_{c-2,t} \subseteq A_{c-1,t} \cap A'_{c,t-1} \). Also, \( \omega_{c-1,t}A_{c-1,t-1} \subseteq A'_{c-1,t-1} \), \( \omega_{c-1,t}A_{c-1,t-1} \subseteq A'_{c-1,t-1} \), and \( A'_{c-1,t-1} \subseteq A'_{c,t-1} \) while \( \omega_{c-1,t}A_{c-1,t-1} \subseteq A_{c-1,t} \) so \( \omega_{c-1,t}A'_{c-1,t-1} \subseteq A_{c-1,t} \cap A'_{c,t-1} \) as well, which completes the proof.

**Theorem 4.2.** Let \( c \geq 1 \) and \( t, s \geq 0 \). Then \( R/G_{c,t,s} \) is Gorenstein of dimension \( 2t + s \), projective dimension \( c \), and, if \( T \) is a minimal free resolution of \( R/G_{c,t,s} \), then \( T \) is a rank \( 1 \) free module generated in degree \( 2t + s + c \).

**Remark 4.5.** The ideal \( G_{c,t,s} \) is analogous to \( I_{G_{c,t,s}} \) in Migliore and Nagel’s construction, and thus the conclusion of **Theorem 4.2** should be compared with that of **Theorem 4.3** in [9].

**Proof.** The \( c = 1 \) case is immediate and \( c = 2 \) follows because (see Examples 2.1 and 4.1) \( G_{2,t,s} = A_{1,t} + \overline{z}y_{t}A'_{1,t-1} = (x_{0} \cdot \cdot \cdot x_{t}, z_{0} \cdot \cdot \cdot z_{t-1}y_{0} \cdot \cdot \cdot y_{t}) \).

So suppose \( c \geq 3 \). By induction, \( R_{c-1,t,0}/G_{c-1,t,0} \) is Gorenstein with dimension \( 2t \), projective dimension \( c - 1 \), and the generator of the rank \( 1 \) free module at the \( (c - 1) \)st step of a minimal free resolution of \( R_{c-1,t,0}/G_{c-1,t,0} \) has degree \( 2t + c - 1 \). By **Lemma 3.1** and the discussion preceding it, \( R/\omega_{c,t}zG_{c-1,t,0} \) has depth \( 2t + s + 1 \), projective dimension \( c - 1 \), and if \( T_{*} \) is a minimal free resolution of \( R/\omega_{c,t}zG_{c-1,t,0} \), then \( T_{c-1}' \) is rank \( 1 \) and generated in degree \( 2t + s + c \).

By **Proposition 4.1**, \( \omega_{c,t}zG_{c-1,t,0} = A_{c-1,t} \cap \omega_{c,t}zA'_{c,t-1} \), so the long exact sequence in Ext on the short exact sequence
\[
0 \to R \to A_{c-1,t} \oplus R \to R \to 0
\]
shows that
\[
\text{dep } \frac{R}{G_{c,t,s}} = \min \left\{ \text{dep } \frac{R}{\omega_{c,t}zG_{c-1,t,0}}, \text{dep } \frac{R}{A_{c-1,t}}, \text{dep } \frac{R}{\omega_{c,t}zA'_{c,t-1}} \right\}
\]
\[
= \min \{2t + s, 2t + s + 1, 2(t - 1) + s + 2 \} = 2t + s
\]
by Lemma 3.1 and the discussion preceding it, as well as Theorems 3.1 and 4.1.

Now no minimal prime of \(A_{c-1,t}^{c} \) contains \(\omega_{c,t}zA'_{c,t-1}^{c} \) (this follows because \(\omega_{c,t}z \notin R_{c-1,t} \) and by Proposition 4.2) so dim \(R/G_{c,t,s} \leq \dim R/A_{c-1,t} = 2t + s + 1 \). It follows that \(R/G_{c,t,s} \) is Cohen-Macaulay of dimension \(2t + s \) and projective dimension \(c \) (the latter fact by the Auslander-Buchsbaum formula).

So consider the mapping cone resolution obtained from the short exact sequence above. Note that the projective dimensions of each of \(R/\omega_{c,t}zG_{c-1,t,0}, R/A_{c-1,t} \), and \(R/\omega_{c,t}zA'_{c,t-1}^{c} \) is \(c - 1 \) (again see Lemma 3.1 and the comments preceding it, as well as Theorems 3.1 and 4.1), but the projective dimension of \(R/G_{c,t,s} \) is \(c \). It follows by the mapping cone construction that the last term in a minimal free resolution of \(R/G_{c,t,s} \) is a nonzero free submodule of \(T'_{c-1} \), a rank 1 free module generated in degree \(2t + s + c \), and we are finished.

\[
\text{Corollary 4.1. Let } c \geq 1 \text{ and } t, s \geq 0. \text{ Then } \ell(\Delta^{2t+s}H(R/G_{c,t,s})) = 2t + s.
\]

\[
\text{Proof. Since } R/G_{c,t,s} \text{ is dimension } 2t + s \text{ Cohen-Macaulay, } \Delta^{2t+s}H(R/G_{c,t,s}) \text{ is the Hilbert function of an Artinian reduction of } R/G_{c,t,s}, \text{ say } S. \text{ Of course } S \text{ is Gorenstein with the same graded Betti numbers as } R/G_{c,t,s} \text{ (as is well known). But in the dimension zero case, } \ell(\Delta^{2t+s}H(R/G_{c,t,s})) = \ell(H(S)) \text{ equals the socle degree of } S \text{ which is seen to be } 2t + s \text{ by Theorem 4.2.} \]

\[
\text{5. A Gorenstein ideal with Hilbert function } H
\]

We now have all the pieces required to define a Gorenstein ideal with Hilbert function \(H\).

\[
\text{Definition 5.1. Given an SI-sequence } H, \text{ let } c = H(1), t = \ell(\Delta H), \text{ and } s = \ell(H) - 2t + 1. \text{ If } c = 0, \text{ then } t = 0, s = 1, \text{ and we let } J_{0}(H) = 0 \subseteq R_{0,0,1} = k[z_{0}]. \text{ If } c = 1, \text{ then } t = 0, s \geq 2, \text{ and we let } J_{1}(H) = (z_{0}, \ldots, z_{s-1}) \subseteq k[z_0, z_0, \ldots, z_{s-1}].
\]

Finally, for \(c \geq 2\), we let \(J_c(H)\) be the ideal of \(R_{c,t,s}\)
\[
J_{c}(H) = I_{c-1,t}(\Delta H) + G_{c-1,t,s} : I_{c-1,t}(\Delta H).
\]

Recall the classic result of Peskine-Szpiro [11].

\[
\text{Theorem 5.1. Let } G \subseteq I \text{ be ideals of a ring } R \text{ such that } R/G \text{ is Gorenstein and } R/I \text{ is Cohen-Macaulay with } \dim(R/G) = \dim(R/I) = d. \text{ Then } R/(G : I) \text{ is Cohen-Macaulay with } \dim(R/(G : I)) = d \text{ and } G : (G : I) = I. \text{ Additionally, if } I \cap (G : I) = G \text{ and } I \text{ and } (G : I) \text{ share no minimal primes, then } R/(I + (G : I)) \text{ is Gorenstein of dimension } d - 1.
\]

\[
\text{Theorem 5.2. Given an SI-sequence } H, c = H(1), t = \ell(\Delta H) \text{ and } s = \ell(H) - 2t + 1, R_{c,t,s}/J_{c}(H) \text{ is a dimension } 2t + s \text{ Gorenstein } k\text{-algebra.}
\]

\[
\text{Remark 5.1. The ideal } J_{c}(H) \text{ is analogous to } J_{c}(h) \text{ in Migliore and Nagel’s construction so the conclusions of Theorems 5.2, 5.3, and Corollary 5.2 are similar to those of Theorems 6.3 and 8.13 in [9].}
\]

\[
\text{Proof. The result is obvious if } c = 0, 1. \text{ So let } c \geq 2. \text{ By Theorem 3.1 and the comments before Lemma 3.1, } R/I_{c-1,t}(\Delta H) \text{ is Cohen-Macaulay of dimension } 2t + s + 1. \text{ Furthermore, } G_{c-1,t,s} \subseteq I_{c-1,t}(\Delta H) \text{ as observed in Remark 4.4 and dim } R/G_{c-1,t,s} = 2t + s + 1 \text{ by Theorem 4.2. So by Theorem 5.1}
\]

it is enough to show that \( I_{c-1,t}(\Delta H) \cap (G_{c-1,t,s} : I_{c-1,t}(\Delta H)) = G_{c-1,t,s} \) and \( I_{c-1,t}(\Delta H) \) and \( G_{c-1,t,s} \) share no minimal primes (which shows \( G_{c-1,t,s} \subseteq I_{c-1,t}(\Delta H) \)). This is a subscript-free observation, so we use \( I \) and \( G \). If \( m \) is a monomial in \( I \cap (G : I) \), then \( m^2 \in mI \subseteq G \), but \( G \) is squarefree (Remark 4.4) so \( m \in G \) and thus \( I \cap (G : I) \subseteq G \) as required. The other inclusion is obvious.

Now suppose that \( I \) and \( (G : I) \) share a minimal prime. Then there is a \( P \) prime and \( x \not\in I, y \not\in (G : I) \) such that \( P = (I : x) = ((G : I) : y) \). Of course \( x \not\in P \), else \( x^2 \in xP \subseteq I \) implies that \( x \in I \) since \( I \) is squarefree, a contradiction. But \( xyP^2 = xP^2 \subseteq I(G : I) \subseteq G \) and hence \( xyP \subseteq G \) since \( G \) is squarefree. Thus \( xyI \subseteq xyP \subseteq G \) implies \( x \in ((G : I) : y) = P \), a contradiction.

\[ \]

It turns out to be easy to compute the graded Betti numbers of \( I_{c}(H) \) using Corollary 8.2 in [9], which provides the control over the sum of geometrically linked ideals.

**Corollary 5.1.** [9] Suppose \( I_1, I_2 \subseteq S = k[x_1, \ldots, x_n] \) share no minimal primes, \( S/I_1 \) and \( S/I_2 \) are Cohen Macaulay of codimension \( d \), and \( J \subseteq S \) is such that \( I_2 = J : I_1 \) and \( I_1 = J : I_2 \). If \( 2(\reg(I_1)) \leq \reg(J) \), then

\[ \beta_{i,j}^{I_1 + I_2} = \beta_{i,j}^{I_1} + \beta_{i,j}^{I_2} \]

With this in hand the rest is simple.

**Theorem 5.3.** Let \( H \) be an SI-sequence with \( H(1) \geq 1, c = H(1), t = \ell(\Delta H), s = \ell(H) - 2t + 1, \) and let \( L \) be the lex ideal in \( k[\mu_1, \ldots, \mu_{c-1}] \) with Hilbert function \( \Delta H \). Then

\[ \beta_{i,j}^{L(H)} = \begin{cases} \beta_{i,j}^{L} & \text{if } j \leq t + s - 2 \\ \beta_{i,j}^{L} + \beta_{i-c,i-j+2t+s-1-j}^{L} & \text{if } t + s - 1 \leq j \leq t \\ \beta_{i-c,i-j+2t+s-1-j}^{L} & \text{if } j \geq t + 1. \end{cases} \]

Moreover, the regularity of \( R/I_{c}(H) \) is \( 2t + s - 1 \).

**Proof.** The theorem is obvious if \( c = 1 \). For \( c \geq 2 \), note that

\[ 2(\reg(I_{c-1,t}(\Delta H))) = 2(t + 1) \leq 2t + s + 1 = \reg(G_{c-1,t,s}) \]

by Corollary 3.1, Corollary 3.2, Theorem 4.2, and Corollary 4.1 and thus \( I_{c-1,t}(\Delta H) \) and \( G_{c-1,t,s} : I_{c-1,t}(\Delta H) \) satisfy the hypothesis of Corollary 5.1 via Corollary 3.1, Theorem 4.2, and Theorem 5.1. We conclude that

\[ \beta_{i,j}^{L(H)} = \beta_{i,j}^{L - 1}(\Delta H) + \beta_{i-c,i-j+2t+s-1-j}^{L} = \beta_{i,j}^{L} + \beta_{i-c,i-j+2t+s-1-j}^{L}. \]

the first equality by Corollary 5.1 and the last by Theorem 3.2.

Since the regularity of \( T/L \) is \( t \), we have \( \beta_{i,j}^{L} = 0 \) for \( j \geq t + 1 \) and \( \beta_{i-c,i-j+2t+s-1-j}^{L} = 0 \) for \( j \leq t + s - 2 \) which demonstrates the equality required.

Finally, writing \( j = 2t + s - 1 + k \),

\[ \beta_{i-c,i-j+2t+s-1-j}^{L} = \beta_{i-c,i-j-k}^{L} = 0 \]

for all \( i \) and \( k > 0 \) but for \( i = c \) and \( k = 0 \),

\[ \beta_{c-i-c,i-j+2t+s-1-j}^{L} = \beta_{c-i-c,i-k}^{L} = 1 \]

and thus the regularity of \( R/I_{c}(H) \) is \( 2t + s - 1 \) as claimed.

**Remark 5.2.** It’s worth noting that if \( s = 1 \) then Theorem 5.3 collapses to

\[ \beta_{i,j}^{L(H)} = \begin{cases} \beta_{i,j}^{L} & \text{if } j \leq t - 1 \\ \beta_{i,j}^{L} + \beta_{i-c,i-j+t}^{L} & \text{if } j = t \\ \beta_{i-c,i-j+2t-j}^{L} & \text{if } j \geq t + 1, \end{cases} \]
and if $s \geq 2$, then $\beta_{i,i+j} = 0$ for $j \geq t + 1$ and $\beta_{c-i,c-i+2t+s-1-j} = 0$ for $j \leq t + s - 2$ causes the further collapse

$$\beta_{i,i+j}^{L(H)} = \begin{cases} 
\beta_{i,i+j}^L & \text{if } j \leq t \\
\beta_{c-i,c-i+2t+s-1-j}^L & \text{if } j \geq t + s - 1.
\end{cases}$$

Thus Betti diagram we are aiming for consists of two copies of a lex Betti diagram, the second rotated and shifted, then added to the first. If $s = 1$, the shift is such that the last row of the first diagram and the first row of the shifted diagram coincide (and are added, so the middle row of the resulting diagram is the sum of the the last nonzero row of the original Betti diagram with its mirror image). If $s \geq 2$ then the first nonzero row of the shifted diagram sits $s - 1$ rows below the last nonzero row of the original (so for $s \geq 3$ there are $s - 2$ rows of zeros between them).

**Example 5.1.** Consider for example the SI-sequences

$\mathcal{H} = \{1, 4, 6, 4, 1\}$ ($s = 1$),

$\mathcal{G} = \{1, 4, 6, 6, 4, 1\}$ ($s = 2$), and

$\mathcal{K} = \{1, 4, 6, 6, 6, 4, 1\}$ ($s = 3$).

Since the first difference of each of $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{K}$ is $\{1, 3, 2\}$, by Theorem 3.2 $I_{3,2}^{}(\Delta \mathcal{H})$, $I_{3,2}^{}(\Delta \mathcal{G})$, and $I_{3,2}^{}(\Delta \mathcal{K})$, have the same Betti diagram as the lex ideal $L = (\mu_1^2, \mu_1 \mu_2, \mu_1 \mu_3, \mu_2^2, \mu_2 \mu_3^2, \mu_3^3)$, namely

| $\beta^L$ | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 1 | . | . |
| 1 | . | 4 | 4 |
| 2 | . | . | 4 |
| 3 | . | . | . |

The Betti diagrams for $I_4(\mathcal{H})$, $I_4(\mathcal{G})$, and $I_4(\mathcal{K})$ are

| $\beta_{i}^{L_{i}(\mathcal{H})}$ | 1 | 9 | 16 | 9 | 1 |
|---|---|---|---|---|---|
| 0 | 1 | . | . | . |
| 1 | . | 4 | 4 | 1 |
| 2 | . | 4 | 8 | 4 |
| 3 | . | 1 | 4 | 4 |
| 4 | . | . | . | . |

| $\beta_{i}^{L_{i}(\mathcal{G})}$ | 1 | 9 | 16 | 9 | 1 |
|---|---|---|---|---|---|
| 0 | 1 | . | . | . |
| 1 | . | 4 | 4 | 1 |
| 2 | . | 2 | 4 | 2 |
| 3 | . | 2 | 4 | 2 |
| 4 | . | 1 | 4 | 4 |
| 5 | . | . | . | . |

and

| $\beta_{i}^{L_{i}(\mathcal{K})}$ | 1 | 9 | 16 | 9 | 1 |
|---|---|---|---|---|---|
| 0 | 1 | . | . | . |
| 1 | . | 4 | 4 | 1 |
| 2 | . | 4 | 4 | 2 |
| 3 | . | . | . | . |
| 4 | . | 2 | 4 | 2 |
| 5 | . | 1 | 4 | 4 |
| 6 | . | . | . | . |
The Hilbert function of $I_c(H)$ can now be computed from its Betti diagram.

**Corollary 5.2.** Let $H$ be an SI-sequence, $c = H(1)$, $t = \ell(\Delta H)$, and $s = \ell(H) - 2t + 1$. Then

$$\Delta^{2t+s}H(R/I_c(H)) = H.$$ 

**Proof.** The result is obvious if $c = 0, 1$ so we suppose $c \geq 2$. We can compute the Hilbert Series of a graded module over a polynomial ring of dimension $n$ via its graded Betti numbers by Stanley's formula [12]

$$\sum_{\sigma=0}^{\infty} H(M, \sigma)\lambda^\sigma = \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \beta_{i,j}^{M,\lambda} \lambda^{i+j}}{(1 - \lambda)^n}.$$ 

In particular, if $M$ is an Artinian reduction of $R/I_c(H)$, then

$$\Delta^{2t+s}H(R/I_c(H)) = H(M),$$

the graded Betti numbers of $R/I_c(H)$ and $M$ coincide, and, since the regularity of $R/I_c(H)$ is $2t + s - 1$ (Theorem 5.3), then

$$\sum_{\sigma=0}^{2t+s-1} H(M, \sigma)\lambda^\sigma = \frac{\sum_{j=0}^{t} \sum_{i=0}^{c} (-1)^i \beta_{i,j}^{L} \lambda^{i+j}}{(1 - \lambda)^c} + \frac{\sum_{i=0}^{t} \sum_{j=0}^{2t+s-1} (-1)^i \beta_{i,j}^{L} \lambda^{i+j}}{(1 - \lambda)^{c+1}}$$

following the observations in Remark 5.2.

Now the numerator of $\frac{\sum_{j=0}^{2t+s-1} \sum_{i=0}^{c} (-1)^i \beta_{i,j}^{L} \lambda^{i+j}}{(1 - \lambda)^c}$ is divisible by $\lambda^{t+s}$ since $\beta_{i,j}^{L} = 0$ for $t \leq i + j \leq t + s - 1$. It follows that

$$(1 - \lambda) \sum_{\sigma=0}^{2t+s-1} H(M, \sigma)\lambda^\sigma = \frac{\sum_{j=0}^{t} \sum_{i=0}^{c} (-1)^i \beta_{i,j}^{L} \lambda^{i+j}}{(1 - \lambda)^c} + q(\lambda)$$

where $q$ is a polynomial divisible by $\lambda^{t+s}$. Thus $H(M, \sigma) - H(M, \sigma - 1) = (\Delta H)(\sigma)$ and, as is easy to show, $H(M, \sigma) = H(\sigma)$ for $0 \leq \sigma \leq t + s - 1$. But $H(M)$ and $H$ are both symmetric of length $2t + s - 1$, that is, it is enough to show that $H(M, \sigma) = H(\sigma)$ for $\sigma \leq \lfloor \frac{2t+s-1}{2} \rfloor \leq t + s - 1$ and we are finished. \hfill \Box

Our final task is to reduce $R/J$ to a dimension zero Gorenstein ring with the right Hilbert function and the weak Lefschetz property. To do so requires an easy observation.

**Lemma 5.1.** Let $H$ be an SI-sequence with $\ell(H) > 0$, $c = H(1)$, $t = \ell(\Delta H)$, and $s = \ell(H) - 2t + 1$. Then $I_c(H) + (z_0) = I_{c-1,t}(\Delta H) + (z_0)$.

**Proof.** If $c = 1$ or $2$, the result is obvious (again, see Example 2.1) so we suppose $c > 2$. It is enough to show that $G_{c-1,t} : I_{c-1,t}(\Delta H) \supseteq I_{c-1,t}(\Delta H) + (z_0)$. Suppose that $\lambda \in G_{c-1,t} : I_{c-1,t}(\Delta H)$. If there is a minimal generator $m \in I_{c-1,t}(\Delta H)$ such that $\lambda m \in \omega_{c-1,t}[A'_{c-1,t}]$, then since $z_0$ is a non-zero-divisor on $I_{c-1,t}(\Delta H)$ it follows that $z_0$ divides $\lambda$ and we are finished. Thus we may assume that $\lambda A_{c-1,t} \subseteq \lambda I_{c-1,t}(\Delta H) \subseteq A_{c-2,t}$, whence by Proposition 4.5 $\lambda \in A_{c-2,t} \subseteq A_{c-1,t} \subseteq I_{c-1,t}(\Delta H)$ as required. \hfill \Box
Corollary 5.3. Let $\mathcal{H}$ be an SI-sequence with $\mathcal{H}(1) \geq 1$, $c = \mathcal{H}(1)$, $t = \ell_1(\Delta \mathcal{H})$, and $s = \ell_1(\mathcal{H}) - 2t + 1$. Then $R/(J_c(\mathcal{H}) + (z_0))$ is Cohen-Macaulay of dimension $2t + s$.

Proof. The result follows from Lemma 5.1, Theorem 3.1, and the remarks before Lemma 3.1. 

We can now prove the main result of the paper.

Theorem 5.4. Let $\mathcal{H}$ be an SI-sequence. Then there is an Artinian Gorenstein $k$-algebra with the weak Lefschetz property and Hilbert function $\mathcal{H}$ that has unique maximal graded Betti numbers among all Artinian Gorenstein $k$-algebras with the weak Lefschetz property and Hilbert function $\mathcal{H}$.

Proof. If $\mathcal{H} = 1$, then this is obvious, so suppose $\ell_1(\mathcal{H}) > 0$ and let $c = \mathcal{H}(1)$, $t = \ell_1(\Delta \mathcal{H})$, and $s = \ell_1(\mathcal{H}) - 2t + 1$. Since $R/J_c(\mathcal{H})$ and $R/(J_c(\mathcal{H}) + z_0)$ are Cohen Macaulay of dimension $2t + s$ and $k$ is infinite, we can find a length $2t + s$ sequence of linear forms $\ell_1, \ldots, \ell_{2t+s}$ which is regular on both $R/J_c(\mathcal{H})$ and $R/(J_c(\mathcal{H}) + z_0)$. Let $\mathbb{L}$ be the ideal generated by the $\ell_i$, $S = R/\ell_1, \ldots, \ell_s$, and $J_c(\mathcal{H})$ be the image of $J_c(\mathcal{H})$ in $S$. Then $S/J_c(\mathcal{H})$ is an Artinian reduction of $R/J_c(\mathcal{H})$ and hence Gorenstein (Theorem 5.2) with Hilbert function $\mathcal{H}$ (Corollary 5.5). To show that $S/J_c(\mathcal{H})$ has the weak Lefschetz property, it is enough to show that $H(S/J_c(\mathcal{H}) + \mathbb{L}) = \Delta \mathcal{H}$. But $z_0$ is regular on $R/I_{c-1,t}(\Delta \mathcal{H})$ and $J_c(\mathcal{H}) + (z_0) = I_{c-1,t}(\Delta \mathcal{H}) + (z_0)$ (Lemma 5.1), so $\{z_0, \ell_1, \ldots, \ell_{2t+s}\}$ is regular on $R/I_{c-1,t}(\Delta \mathcal{H})$ and thus $H(S/J_c(\mathcal{H}) + \mathbb{L}) = H(R/(I_{c-1,t}(\Delta \mathcal{H}) + \mathbb{L})) = \Delta_2^{2t+s+1}H(R/I_{c-1,t}(\Delta \mathcal{H})) = \Delta \mathcal{H}$ (applying Corollary 3.2).

Finally, since the graded Betti numbers of $S/J_c(\mathcal{H})$ and $R/J_c(\mathcal{H})$ coincide, we are finished by Theorem 5.3 and Migliore and Nagel's bound (Theorem 1.1).

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