The quantum two dimensional Poincaré group from quantum group contraction

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Abstract

A new derivation of the algebra of functions on the two dimensional Euclidean Poincaré group is proposed. It is based on a contraction of the Hopf algebra $\text{Fun}(SO_q(3))$, where the deformation parameter $q$ is sent to one.

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1. Introduction

Various quantum deformations of the Poincaré algebra or group are now available, all relying on some contraction procedure, and all being Hopf algebras. At the algebra level, contracting the deformed universal enveloping algebra $U_q(so(D + 1))$ leads to the deformation of the enveloping Poincaré algebra $U_\kappa(P_D)$ [1] [2] [3]; in these approaches, the original quantum group parameter $q$ is sent to its classical value one, and a dimensionful parameter $\kappa$ appears. At the group level, the quantum group counterpart of the above quantum algebras has been proposed in [4], through the quantization of the induced Poisson structure on the classical Poincaré group. Also, a contraction of $SU_q(2)$ leads to a different definition of the two dimensional Euclidean Poincaré group, where the parameter $q$ survives the contraction [5] [6].

Nevertheless, the duality between the quantum Poincaré groups and the quantum Poincaré algebras is not always established, contrary to what happens when the deformations are based on a semi–simple Lie group or algebra. The lack of an universal $R$–matrix (except in three space–time dimensions) complicates the proof, and only in two dimensions is there a direct proof [7].

In this letter, we propose a construction of the quantum algebra $\text{Fun}(E_q(2))$, as obtained in [4], using a contraction procedure starting directly from the Hopf algebra $\text{Fun}(SO_q(3))$. The parameter $q$ is sent to one as well, thereby mimicking closely the universal enveloping algebra contraction. Apart from increased computational task, there seems to be no obstruction to carry out the same program for other space–time dimensions and space–time signatures [8].

In section two the necessary definitions of quantum group are quickly reviewed, relying essentially on [9]. In section three, a contraction of the classical group and its associated Poisson structure is done, mainly to illustrate the steps that will be used in the next section. Section four deals with the quantum group contraction leading to the definition of $\text{Fun}(E_q(2))$.

2. The Hopf algebra $\text{Fun}(SO_q(3, \mathbb{R}))$

The starting point is $\text{Fun}(SO_q(3))$ [3], the algebra of functions on the quantum group $SO_q(3)$, defined as the non–commutative $\mathbb{C}$–algebra with unity generated by the elements $T = (t_{ij})$ [3] quotiented by the relations

$$R_t T_1 T_2 = T_2 T_1 R_t, \quad (2.1)$$

\footnote{Our convention for indices is that $i, j, k = 1, 2, 3$, whereas $a, b, c = 2, 3$}
where \( T_1 = T \otimes I, T_2 = I \otimes T \). The \( R \)-matrix has the particular expression

\[
R_t = (e_{11} + e_{33}) \otimes e_{22} + q(e_{11} \otimes e_{11} + e_{33} \otimes e_{33}) + q^{-1}(e_{33} \otimes e_{11} + e_{11} \otimes e_{33}) \\
+ (q-q^{-1})(e_{21} \otimes e_{12} + e_{31} \otimes e_{13} + e_{32} \otimes e_{23}) + e_{22} \otimes I \\
- (q-q^{-1})(q^{-\frac{1}{2}}e_{21} \otimes e_{23} + q^{-1}e_{31} \otimes e_{13} + q^{-\frac{1}{2}}e_{32} \otimes e_{12}).
\]

with \( e_{ij} \) the basis of \( 3 \times 3 \) matrices. That this is the orthogonal group is encoded in the supplementary constraints

\[
T C T^T C^{-1} = C T^T C^{-1} T = 1, \quad \text{with} \quad C = \begin{pmatrix}
0 & 0 & q^{-\frac{1}{2}} \\
0 & 1 & 0 \\
q^{\frac{1}{2}} & 0 & 0
\end{pmatrix}.
\]

The Hopf algebra structure is further defined by the homomorphisms \( \Delta, \epsilon \) and anti–homomorphism \( S \):

\[
\Delta(T) = T \otimes T, \quad \epsilon(T) = 1, \quad S(T) = C T^T C^{-1}.
\]

The quantum three dimensional complex plane \( O_q^3(\mathbb{C}) \) is defined as the non–commutative \( \mathbb{C} \)–algebra with unity generated by the elements \( x_1, x_2, x_3 \), subject to the relations

\[
x_1 x_2 = q x_2 x_1, \quad x_2 x_3 = q x_3 x_2, \quad x_1 x_3 - x_3 x_1 = (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) x_2^2.
\]

There is a natural action of the quantum group on this quantum vector space by mean of the algebra homomorphism \( \delta : \text{Fun}(SO_q(3)) \rightarrow \text{Fun}(SO_q(3)) \otimes O_q^3(\mathbb{C}) \), where

\[
\delta(x_i) = t_{ij} \otimes x_j.
\]

It is easy to see that the quadratic form \( x^T C x \) in \( O_q^3(\mathbb{C}) \) is preserved by this action of the quantum group.

The compact real form of \( \text{Fun}(SO_q(3)) \), for real \( q \), is obtained by endowing the algebra with the anti–involution

\[
T^* = C T (C^{-1})^T.
\]

Accordingly, the anti–involution \( x^* = C^T x \) on \( O_q^3(\mathbb{C}) \) defines the real quantum Euclidean plane \( O_q^3(\mathbb{R}) \). Since \( \delta^*(x) = \delta(x^*) \), and the action of \( \text{Fun}(SO_q(3, \mathbb{R})) \) preserves the quadratic form, the definition of the two-dimensional quantum unit sphere as the quotient of \( O_q^3(\mathbb{R}) \) by the relation \( x^* T x = 1 \) is meaningful.
For our purpose, it will be more convenient to work in a real basis for the quantum plane. We choose the new real generators in $O_q^3(\mathbb{R})$:

$$z_1 = \frac{1}{\sqrt{2}}(x_1 + q^{\frac{1}{2}}x_3), \quad z_2 = \frac{-i}{\sqrt{2}}(x_1 - q^{\frac{1}{2}}x_3), \quad z_3 = x_2,$$

or, in short, $z = Mx$. Similarly, we take new generators $V = MTM^{-1}$ in $\text{Fun}(SO_q(3, \mathbb{R}))$, now real $V^* = V$, and define $R_v = (M \otimes M)RT(M^{-1} \otimes M^{-1})$, so that the constraints (2.1) become $R_vV_1V_2 = V_2V_1R_v$, and $\delta(z_i) = v_{ij} \otimes z_j$. Only the orthogonality relations (2.3) are significantly affected

$$V\hat{C}V^T = \hat{C}, \quad \text{with} \quad \hat{C} = MCM^T$$

as well as the antipode $S(V) = \hat{C}V^T\hat{C}$, and the quantum plane relations (2.4)

$$z_3z_2 - qz_2z_3 = i(qz_1z_3 - z_3z_1), \quad z_1z_2 - z_2z_1 = i(1 - q)z_3^2. \quad (2.7)$$

Note that the second relation in (2.4) is nothing but the *-conjugate of the first, so we skip it here.

As $\text{Fun}(SO_q(3, \mathbb{R}))$ is a non-commutative deformation of the algebra of functions on the classical Lie group $SO(3, \mathbb{R})$, the $R$–matrix $R_v$ induces a Poisson structure on that latter algebra through

$$\{\Phi \otimes \Phi\} = [\Phi \otimes \Phi, r], \quad (2.8)$$

where the classical $r$–matrix is defined by $r = h^{-1}(R_v - 1) \mod h$, $\Phi = (\phi_{ij})$ are the matrix elements of the classical group and $h$ is a parameter governing the deformation, $h \rightarrow 0$ being the classical limit \[10\].

Having in mind the contraction procedure of the next sections, we set the parameter $q = \exp(\gamma/R)$, as for the universal enveloping algebra contraction, with $R$ the radius of the sphere (or de Sitter radius), and take $i\gamma$ as our deformation parameter. Since we are dealing with a real algebra, the $i$ factor is necessary if we follow the definition of algebras deformations in \[10\] and require $\{a, b\}^* = \{a^*, b^*\}$.

From the expression of $R_v$, it is not difficult to extract the classical $r$–matrix

$$r = \frac{1}{R}(X_1 \otimes X_2 - X_2 \otimes X_1 + iX_k \otimes X_k) = \frac{1}{R}X_1 \wedge X_2 + (\ldots) \quad (2.9)$$

in the conventional basis for the Lie algebra $so(3, \mathbb{R})$ for the three dimensional representation, $(X_i)_{jk} = \epsilon_{ijk}$. The last term in (2.9) does not contribute to the Poisson bracket (2.8) since it is $SO(3, \mathbb{R})$ ad–invariant.
3. Classical group contraction

Now we perform a $SO(3, \mathbb{R})$ contraction in order to define a Poisson structure on the two dimensional Euclidean Poincaré group. An element of $SO(3, \mathbb{R})$ will be parametrized by the three rotation angles around the respective $z_i$ axis, $\Phi(\theta_k) = \exp(\sum_1^3 \theta_k X_k)$ [11]. Geometrically, one considers the action of $SO(3, \mathbb{R})$ on the two–sphere of radius $R$ in the vicinity of the point $p = (R, 0, 0)$, and let the radius $R \to \infty$. Simultaneously, one imposes that the rotation angles $\theta_2, 3$ go to zero like $1/R$, so that the transformation of a point in the vicinity of $p$ remains at a finite distance from $p$. We therefore define $(\theta_k) = (\theta, \alpha/R, \beta/R)$.

From the explicit expression of $\Phi$ [11], the matrix elements $\phi_{ij}$ have the expansion

$$
\phi_{ij}(\theta_k) = \sum_{n=0}^{\infty} \frac{\phi^i_{n,j}(\theta, \alpha, \beta)}{R^n},
$$

where, depending on the indices $i, j$, the series contains either odd or even powers of $R$. Then, in the limit of infinite radius,

$$
\begin{pmatrix}
1 \\
z'_2 \\
z'_3
\end{pmatrix}
= \lim_{R \to \infty} \frac{z'_1}{R}
= \lim_{R \to \infty} \begin{pmatrix}
\phi^1_{21} & \phi^0_{22} & \phi^0_{23} \\
\phi^1_{31} & \phi^0_{32} & \phi^0_{33}
\end{pmatrix}
\begin{pmatrix}
1 \\
z_2 \\
z_3
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
-\beta s/\theta + \alpha(1 - c)/\theta & c & s \\
\alpha s/\theta + \beta(1 - c/\theta) & -s & c
\end{pmatrix}
\begin{pmatrix}
1 \\
z_2 \\
z_3
\end{pmatrix}
$$

where $s = \sin \theta, c = \cos \theta$. Apart from the awkward parametrization of the translations, this is the usual representation for the classical two dimensional Euclidean Poincaré group, $z_{2,3}$ being the coordinates of the two dimensional space–time.

In order to fully specify the Poisson structure on the Poincaré group, we only need the Poisson brackets for the matrix elements $\phi^n_{ij}$ entering the above equation. This is achieved by plugging the expansion (3.1) in the $SO(3, \mathbb{R})$ Poisson brackets (2.8) and matching terms of equal power in $R$. Dropping the superscripts and setting $\phi_\alpha = \phi_{\alpha 1}$, we get for the non–vanishing Poisson brackets

$$
\begin{align*}
\{ \phi_{22}, \phi_2 \} &= -(\phi_{23})^2 \\
\{ \phi_{23}, \phi_2 \} &= \phi_{22} \phi_{23} \\
\{ \phi_{32}, \phi_2 \} &= -\phi_{23} \phi_{33} \\
\{ \phi_{33}, \phi_2 \} &= \phi_{23} \phi_{32} \\
\{ \phi_2, \phi_3 \} &= -\phi_3
\end{align*}
$$

(3.2)
Using the orthogonality relations (in the limit $R \to \infty$), this can be neatly recast in the compact form
\[
\{ \phi_{ab}, \phi_{cd} \} = 0 \\
\{ \phi_{ab}, \phi_c \} = ((\phi_a^2 - \delta_{a2})\phi_{cb} + \delta_{ac}(\phi_{2b} - \delta_{2b})) \tag{3.3} \\
\{ \phi_2, \phi_3 \} = -\phi_3
\]

This is similar to the expression obtained in [4], with some obvious relabelling.

4. Quantum group contraction

Turning to the quantum case, we will proceed in much the same way as in the previous section. We consider elements of $O_q^2(\mathbb{R})$ living on the two dimensional quantum sphere
\[
\begin{align*}
  z^T \tilde{C} z &= \frac{1 + q^{-1}}{2} (z_1^2 + z_2^2) + \frac{q + q^{-1}}{2} z_3^2 = R^2. 
  \end{align*} \tag{4.1}
\]
It is convenient to absorb an irrelevant factor in $R^2 = 2R^2/(1 + q^{-1})$. As mentioned earlier, the contraction also involves taking $q$ simultaneously to its classical value by letting $q = \exp(\gamma/R)$. Here again, we consider the vicinity of the point $(z_i) = (R, 0, 0)$, $R \to \infty$, and (4.1) allows us to expand $z_1$ as a series in $R$
\[
\begin{align*}
  z_1 &= R \left( 1 - \frac{1}{2R^2} (z_2^2 + z_3^2) + O(R^{-3}) \right), 
  \end{align*} \tag{4.2}
\]
where $z_2, z_3$ are finite (i.e. of order 1). Inserting this expansion in (2.7), we observe that the divergent terms cancel, and the finite part of these expressions yields the constraint
\[
\begin{align*}
  [z_2, z_3] &= -i\gamma z_3. 
  \end{align*} \tag{4.3}
\]
It is worth pointing out that, in contrast to the classical case, the choice of $z_1$ as the diverging coordinate is not arbitrary. This is due to the non–commutativity of the $z$, which for instance links $[z_1, z_2]$ to $z_3$.

Inspired by (3.1), we rewrite the generators of $Fun(SO_q(3, \mathbb{R}))$ as an expansion in the contraction parameter $R$
\[
\begin{align*}
  v_{ij} &= \sum_{n=0}^{\infty} \frac{v_{ij}^n}{R^n}. \tag{4.4}
  \end{align*}
\]
\footnote{Taking $z_3 \to \infty$ with $z_{1,2}$ finite cannot be compensated by our behaviour of $q$ alone, and would require a singular change of variables in (2.7), which would in turn cause troubles in the contraction of $RVV = VVR$.}
From simple requirements we will collect enough informations on the $v_{ij}^n$ to enable us to derive all the necessary relations characterizing the algebra $\text{Fun}(E_q(2))$. First, we require that under the action of $\text{Fun}(SO_q(3, \mathbb{R}))$ by mean of the mapping $\delta$, the elements $z_{2,3}$ remain of order 1 in the limit $R \to \infty$. Since

$$\delta(z_a) = v_{a1} \otimes z_1 + v_{ab} \otimes z_b,$$

this is only possible if $v_{a1}^0 = 0$. Next we apply $\delta$ on both sides of (4.2) to get

$$\frac{1}{R}(v_{11} \otimes z_1 + v_{1a} \otimes z_a) = 1 \otimes 1 - \frac{1}{R^2}(\delta(z_2)^2 + \delta(z_3)^2) + O(R^{-3}), \quad (4.5)$$

which implies that $v_{11}^0 = 1$ and $v_{11}^1 \otimes 1 + v_{a1}^0 \otimes z_a = 0$, since $\delta(z_{2,3})$ are finite. Feeding the first orthogonality relations (2.6) with this partial knowledge allows us conclude, by considering the order 1 term, that $v_{1a}^0 = 0$, and thus $v_{11}^1 = 0$, with the help of (4.5).

Gathering all this, we can take the $R \to \infty$ limit in $\delta(z) = V \otimes z$, provided we divide $z_1$ by $R$, which yields

$$\delta \left( \begin{array}{c} 1 \\ z_2 \\ z_3 \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ v_{21}^1 & v_{22}^0 & v_{23}^0 \\ v_{31}^0 & v_{32}^0 & v_{33}^0 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ z_2 \\ z_3 \end{array} \right). \quad (4.6)$$

It is then natural to take the elements $1, u_{ab} = v_{ab}^0$, and $u_a = v_{a1}^1$ as the generators of $\text{Fun}(E_q(2))$, the algebra of functions on the quantum Euclidean group $E_q(2)$.

We still have to take into account the constraints imposed by the relation $R_v V_1 V_2 = V_2 V_1 R_v$. Expanding the $R_v$-matrix in $R$ and matching the appropriate powers of $R$, we get from the order 1 term that $v_{ab}^0$ commute among themselves. From the order $1/R$ term and the orthogonality relations at order 1, we get relations similar to (3.2), and from the $1/R^2$ term together with orthogonality at order 1 and $1/R$, we get $[v_{31}^1, v_{21}^1] = i\gamma v_{31}^1$. All this is summarized in the compact form

$$[u_{ab}, u_{cd}] = 0$$

$$[u_{ab}, u_c] = i\gamma \left( (u_{a2} - \delta_{a2})u_{c2} + \delta_{ac}(u_{2b} - \delta_{2b}) \right), \quad (4.7)$$

$$[u_2, u_3] = -i\gamma u_3$$

These are all the relations containing only the elements $v_{ij}^{0,1}$. There are of course many other relations involving higher order elements $v_{ij}^n$, but these do not add new constraints.
The limit of the orthogonality relations (2.6) amount to say that the matrix \( U = (u_{ab}) \) is an ordinary orthogonal matrix

\[
U^T U = 1 = UU^T.
\] (4.8)

The algebra \( \text{Fun}(E_q(2)) \) has a Hopf algebra structure inherited from its pre–contracted ancestor \( \text{Fun}(SO_q(3, \mathbb{R})) \), which reads

\[
\begin{align*}
\Delta(U) &= U \otimes U & \epsilon(U) &= 1 & S(U) &= U^T \\
\Delta(u) &= u \otimes 1 + U \otimes u & \epsilon(u) &= 0 & S(u) &= -U^T u
\end{align*}
\] (4.9)

where \( u \) is the column vector \((u_2 u_3)^T\). Note that in deriving these expressions, one frequently encounters elements which are not part of the \( \text{Fun}(E_q(2)) \) algebra as defined in (4.6), like for instance

\[
S(u_2) = S(v_{21}^1) = v_{12}^1 - \frac{i}{2} \gamma (v_{22}^0 - 1).
\]

But with the help of the orthogonality relations, we can always trade these foreign elements for elements belonging to \( \text{Fun}(E_q(2)) \).

We conclude that the algebra \( \text{Fun}(E_q(2)) \) defined by the equations (4.7),(4.8) and (4.9) and equipped with the anti–involution

\[
U^* = U \quad u^* = u,
\]

is a *–Hopf algebra. In addition, the mapping \( \delta \) in (4.6) defines an action of \( \text{Fun}(E_q(2)) \) on the quantum real Poincaré plane generated by the elements \( z_{2,3} = z_{2,3}^* \) subject to the constraint (4.3). This algebra is the same as in [4], but here we derived it only using a contraction on \( \text{Fun}(SO_q(3, \mathbb{R})) \). It differs from [4] in the absence of \( \gamma \) in (3.3), the presentation here being consistent with the definition of algebras deformations in [10].

5. Conclusion

This alternative way of obtaining \( \text{Fun}(E_q(2)) \) shed light on why the direct quantization of [4] works and is linear in \( \gamma \): here this parameter appears only in the ratio \( \gamma/R \), and the contraction amounts essentially to an expansion in \( R \), keeping only the lowest order terms. Therefore, no higher powers of \( \gamma \) arise. Since this approach is very similar in spirit to the method of [2] – we start from a dual structure and take the same limit for \( q \) – we believe that it yields the quantum group dual to the \( \kappa \)–Poincaré algebra, once generalized for higher dimensions. At the moment, this is true for the two dimensional case.

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