Some theoretical aspects of quantum mechanical equations in Rindler space

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Abstract. In this article we have investigated theoretical aspects of the solutions of some of the quantum mechanical problems in Rindler space. We have developed formalisms for the exact analytical solutions for the relativistic equations, along with the approximate form of solutions for the Schrödinger equation. The Hamiltonian operator in Rindler space is found to be non-Hermitian in nature, whereas the energy eigen values are observed to be real in nature. We have noticed that the sole reason behind such real behavior is the $PT$-symmetric form of the Hamiltonian operator. We have also observed that the energy eigen values are negative, linearly quantized and the quantum mechanical system becomes more and more bound with the increase in the strength of gravitational field strength produced by the strongly gravitating objects, e.g., black holes, which is classical in nature.

1 introduction

From the knowledge of literature survey of articles on general theory of relativity and related topics, it has been observed that the principle of equivalence plays a significant role in the studies of various aspects of classical and quantum physics in a uniformly accelerated frame or in Rindler space [1–4], in which the background gravitational field is uniform, at least locally. According to this principle, a frame undergoing uniform accelerated motion in absence of gravity is equivalent to a frame at rest in presence of a constant gravitational field. However, the strength of gravitational field can not be constant throughout the whole space. Within a limited region, i.e. to say that locally it is uniform. Hence the constant acceleration of the frame is also called the local acceleration. It is to be noted further that the principle of equivalence is also applicable for the most general type of motion in curved space-time. Assuming that the uniform acceleration is along the positive $x$-direction, the metric in the Rindler space is then given by

$$g^{\mu\nu} \equiv \text{diag} \left( \left( 1 + \frac{\alpha x}{c^2} \right), -1, -1, -1 \right),$$

whereas for $(1+1)$ dimension it is given by

$$g^{\mu\nu} \equiv \text{diag} \left( \left( 1 + \frac{\alpha x}{c^2} \right), -1 \right),$$

where $\alpha$ is the uniform acceleration [5–9]. The Hamiltonian of the particle in Rindler space, having rest mass $m_0$ is given by

$$H = pv - L = \left( 1 + \frac{\alpha x}{c^2} \right) \left( p^2 c^2 + m_0^2 c^4 \right)^{\frac{1}{2}},$$

where $p$ and $v$ are the particle momentum and velocity respectively along the positive $x$-direction. In the inertial frame with $\alpha = 0$, we get back the results of special theory of relativity. In the case of classical mechanics, $x$, $p$ and $H$ are dynamical variables, whereas in the quantum mechanical picture, $x$, $p$ and $H$ are operators. In the later case $x$ and $p$ are also canonical conjugate of each other, i.e., $[x, p] = i\hbar$. Here it is quite obvious that the Hamiltonian operator represented by eq. (2) is non-Hermitian. However from our subsequent analysis and discussion we will show that the eigen values or the eigen spectra are real in nature. This is found to be solely because of the $PT$-symmetric nature of the Hamiltonian operator [10]. Under $P$ and $T$ operations we have the following relations from $PT$-symmetric quantum
mechanics: \( \exp^{-1} = -x, \, T x T^{-1} = x, \, P p P^{-1} = -p, \, T p T^{-1} = -p, \, P_\alpha P^{-1} = -\alpha, \, T \alpha T^{-1} = \alpha \) and \( T i T^{-1} = -i \).

The last relation is essential for the preservation of canonical quantization relation under \( PT \) operation, i.e., for the validity of \( PT[x, p](PT)^{-1} = i\hbar \).

Hence it is quite obvious to verify that for the Hamiltonian, \( PTH(PT)^{-1} = H \), i.e., the Hamiltonian operator is \( PT \) invariant. We shall show in our subsequent discussion that since eigen functions \( \Psi \) are the functions of the product \( \alpha x \), which is \( PT \)-symmetric, therefore \( PT\Psi(u) = \Psi(u) \), where \( u \) is a function of the product of \( \alpha x \).

To study some of the physical aspects of the quantum mechanical equations in Rindler space, we start with the Hamiltonian, given by eq. (2). In this article along with the exact relativistic solutions for quantum mechanical equations in Rindler space, we have developed a formalism for the solution of Schrödinger equation with a kind of approximation. To the best of our knowledge such studies have not been done before, except some preliminary studies by our group [11, 12] on the exact and some approximate solutions in Rindler space. In [11] we have solved the Schrödinger equation in an exact manner, whereas in [12] we have solved the same equation with some approximation.

We have predicted some new kind of quanta in the exact solution of non-relativistic equation (which has been done for the first time by our group [11]) and in the linear approximation, we have shown a surprising analogy between our result with the cold field emission of electrons from a metal surface under the action of a strong electric field [13] (see also [14]) (this is also done for the first time by our group [12]). In sect. 2, we have developed the formalism for another kind of approximate solution for the Schrödinger equation, which has not been discussed in [12]. In sect. 3, we have presented an exact solution for the relativistic form of quantum mechanical equation. In sect. 4, we have solved the relativistic Klein-Gordon equation. In the present context, the Klein-Gordon equation is not Lorentz invariant. This is because of the choice of Rindler Hamiltonian. But this formalism is also exact in nature. Finally we have given the conclusion of our work. The relativistic calculations and the formalism based on the second-order approximation are reported for the first time in this article. We have observed from the exact solution of the eigen value equation discussed in sect. 3, which is the most important part of this article, the energy eigen values are negative in nature, linearly quantized (\( \alpha l \)) and also \( \propto \alpha \), the strength of gravitational field produced by the strongly gravitating objects, e.g., by a black hole. Therefore the classical gravitational field increases the binding of the particle and remains bound with the black hole. This is of course a new observation. The uniform gravitational field in a local rest frame is treated as a classical background field, which indirectly affects the nature of binding in a black hole.

2 The Schrödinger equation in Rindler space: An approximate solution

Keeping only three terms of the binomial expansion of the factor \((1 + \frac{\alpha x}{c^2})^{-1}\), the Schrödinger equation reduces to

\[
\left( m_0 c^2 + \frac{p^2}{2m_0} \right) \Psi(x, y, z) = \left( 1 + \frac{\alpha x}{c^2} \right)^{-1} E \Psi(x, y, z)
\]

\[
\approx \left( 1 - \frac{\alpha x}{c^2} + \frac{\alpha^2 x^2}{c^4} \right) E \Psi(x, y, z).
\]

After rearranging, the equation may be expressed in the following form:

\[
p^2 \Psi(x, y, z) + \frac{\alpha x}{c^2} E \Psi(x, y, z) - \frac{\alpha^2 x^2}{c^4} E \Psi(x, y, z) = E_k \Psi(x, y, z),
\]

where \( E_k = E - m_0 c^2 \), the kinetic energy of the particle. Now using the separable form of the wave function, given by

\[
\Psi(x, y, z) = N \exp \left( -\frac{ip_y y}{\hbar} \right) \exp \left( -\frac{ip_z z}{\hbar} \right) X(x),
\]

where \( N \) is the normalization constant, we have

\[
-\frac{\hbar^2}{2m_0} \frac{d^2 X}{dx^2} + \frac{\alpha x}{c^2} E X(x) - \frac{\alpha^2 x^2}{c^4} E X(x) = \left( E_k - \frac{p^2}{2m_0} \right) X(x),
\]

where \( p^2 = p_y^2 + p_z^2 \). Since there is no \( y \)- or \( z \)-dependent terms in the Hamiltonian, the \( y \) and \( z \) components of the wave function should behave like plane waves throughout the space. Now rearranging and using \( E_{k\parallel} = E_k - \frac{p_z^2}{2m_0} \), the parallel part of the particle kinetic energy, the above differential equation may be written in the form

\[
\frac{d^2 X}{dx^2} + \frac{2m_0 E}{\hbar^2} \left[ \frac{\alpha^2 x^2}{c^4} - \frac{\alpha x}{c^2} \right] X(x) = -\frac{2m_0}{\hbar^2} E_{k\parallel} X(x),
\]
which may further be expressed in the form

$$\frac{d^2X}{dx^2} + \frac{2m_0E}{\hbar^2} \left[ \left( \frac{\alpha x}{c^2} - \frac{1}{2} \right)^2 - \frac{1}{4} \right] X(x) = -\frac{2m_0}{\hbar^2} E_{k\parallel} X(x).$$

(8)

Now changing the variable from \( x \) to \( p \), where \( p = \frac{\alpha x}{c} - \frac{1}{2} \), the above differential equation can be written as

$$\frac{d^2X}{dp^2} + \frac{q^2}{4} \left( p^2 - \frac{1}{4} \right) X + \frac{q^2}{4} \gamma X = 0,$$

(9)

where

$$q^2 = \frac{8m_0Ee^4}{\alpha^2\hbar^2}$$

(10)

and \( \gamma = \frac{E_{k\parallel}}{E} \) which is \( \leq 1 \). It should be noted that the variable \( p \) is \( PT \) invariant. Using the new variable \( \rho = pq^{1/2} \), we have from the above differential equation,

$$\frac{d^2X}{d\rho^2} + \left( \rho^2 - \frac{1}{4} \right) X(\rho) = 0,$$

(11)

where \( \lambda = \frac{q}{4}(\frac{1}{4} - \gamma) \). Obviously \( \lambda = 0, > 0 \) or \( < 0 \) for \( \gamma = 1/4, \gamma < 1/4 \) or \( \gamma > 1/4 \), respectively. Now for \( \gamma \rightarrow 1 \), that is for the extreme case \( E_{k\parallel} = E \) and \( \lambda = -0.75 \), the minimum value of \( \lambda \). Whereas for \( \gamma \ll 1/4 \), \( i.e., E_{k\parallel} \ll E, \lambda = 0.25 \), the maximum value of \( \lambda \). Both are in units of \( q/4 \). It can very easily be shown that for purely one dimensional case the energy eigen value \( E = \frac{m_0c^2}{\lambda} \). For \( \gamma = 1/4 \), the energy eigen value for purely one-dimensional case is \( E = \frac{3}{4}m_0c^2 \), whereas for \( \gamma < 1/4 \), \( E < \frac{3}{4}m_0c^2 \) and for \( \gamma > 1/4 \), \( E > \frac{3}{4}m_0c^2 \). Since the energy is always finite, we should have \( \gamma < 1 \), which is also obvious from the definition of \( \gamma \). Hence one can very easily show that

$$\frac{q}{4} \approx \frac{m_0c^2}{(1 - \gamma)\hbar\omega},$$

(12)

where we have put \( 2^{1/2} \approx 1 \) and \( \omega = \alpha/c \), some kind of frequency. The solution of the above differential equation for \( \lambda \neq 0 \) is the parabolic cylindrical function \( W(\lambda, \pm \rho) \), given by [15]

$$W(\lambda, \rho) = W(\lambda, 0)\omega_1(\lambda, \rho) + W'(\lambda, 0)\omega_2(\lambda, \rho),$$

(13)

where

$$W(\lambda, 0) = 2^{-3/4} \left| \frac{\Gamma(\frac{1}{4} + \frac{1}{2} i \lambda)}{\Gamma(\frac{1}{4} + \frac{1}{2} i \lambda)} \right|^{1/2} \quad \text{and} \quad W'(\lambda, 0) = -2^{-1/4} \left| \frac{\Gamma(\frac{1}{4} + \frac{1}{2} i \lambda)}{\Gamma(\frac{1}{4} + \frac{1}{2} i \lambda)} \right|^{1/2},$$

(14)

$$\omega_1(\lambda, \rho) = \sum_{n=0}^{\infty} \alpha_n(\lambda) \frac{\rho^{2n}}{2n!} \quad \text{and} \quad \omega_2(\lambda, \rho) = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{\rho^{2n+1}}{(2n+1)!},$$

(15)

where \( \alpha_n(\lambda) \) and \( \beta_n(\lambda) \) satisfy the recursion relations

$$\alpha_{n+2} = \lambda \alpha_{n+1} - \frac{1}{2} (n+1)(2n+1) \alpha_n,$$

$$\beta_{n+2} = \lambda \beta_{n+1} - \frac{1}{2} (n+1)(2n+3) \beta_n \quad \text{and} \quad \alpha_0(\lambda) = \beta_0(\lambda) = 1, \quad \alpha_1(\lambda) = \beta_1(\lambda) = \lambda.$$  

(16)

To show the variation of wave function with \( \rho \), in fig. 1 we have plotted \( |W(\rho, \lambda)|^2 \), the probability density against \( \rho \), for \( \lambda = 0.5 \), curve (a), 0.25, curve (b), 0.05, curve (c), 0.5, curve (d) and 0.75, curve (e) in units of \( q/4 \).
Fig. 1. The variation of $|W(\rho, \lambda)|^2$, the probability density with $\rho$, for $\lambda = 0.5$, curve (a), 0.25, curve (b), −0.05, curve (c), −0.5, curve (d) and −0.75, curve (e) in units of $q/4$.

In the numerical evaluation of $W(\rho, \lambda)$, we have obtained the absolute values for the ratio of $\Gamma$-functions using the formula [15]

$$
\left| \frac{\Gamma(a + ib)}{\Gamma(a)} \right|^2 = \prod_{n=0}^{\infty} \left( 1 + \frac{b^2}{(a + n)^2} \right)^{-1},
$$

where both $a$ and $b$ are real constants. One can conclude from the nature of the curves that the probability densities are of damped oscillatory in nature.

Now for the special case with $\lambda = 0$, the differential equation given by eq. (11) reduces to

$$
\frac{d^2X}{dp^2} + \frac{p^2}{4}X(\rho) = 0.
$$

With a new variable $u = \rho/\sqrt{2}$, this equation becomes

$$
\frac{d^2X}{du^2} + u^2X = 0.
$$

In appendix A.1, we have obtained the solution of this differential equation and is given by

$$
X(\rho) = \frac{\rho^{1/2}}{2^{1/4}}J_{1/4}\left(\frac{\rho^2}{4}\right),
$$

for $\rho \geq 0$. Here $J_n(x)$ is the Bessel function of order $n$. In fig. 2 we have shown the variation of $|X(\rho)|^2$ with $\rho$. The nature of the probability density $|X(\rho)|^2$ is also damped oscillatory type. In this case $\gamma = \frac{1}{4}$, $E = \frac{4}{3}m_0c^2$ and $\frac{q}{4}$ is exactly equal to $\frac{m_0c^2}{\hbar\omega}$.

3 Exact solution

In this section we shall develop a formalism to get exact solution for the relativistic form of quantum mechanical equation. In the present formalism we use natural units, i.e., $\hbar = c = 1$. We begin with the classical Hamiltonian in the Rindler space, given by

$$
H = (1 + ax)(p^2 + m_0^2)^{1/2}.
$$
Then the eigen value equation $H\Psi = E\Psi$ may be written as

$$(1 + \alpha x)(-d_x^2 + m_0^2)^{1/2}\Psi = E\psi,$$

where $d_x = \frac{d}{dx}$ and we assume that the motion is one dimensional and along the positive $x$-direction. Changing the variable from $x$ to $X$, given by $X = 1 + \alpha x$, the above equation reduces to

$$X(-d_X^2 + m^*0^2)^{1/2}\Psi = E\psi,$$

where we have redefined $m_0 \rightarrow m_0/\alpha$ and the energy eigen value $E \rightarrow E/\alpha$. For the new variable $X$, the limit is from 1 to $+\infty$, instead of 0 to $+\infty$. Then we can rewrite the above differential equation as

$$X(-d_X^2 + m^*0^2)^{1/2}\Psi = E\psi.$$

To get an analytical solution, we follow the technique presented in [16–18]. Now using the properties of Dirac delta function, we can write [16–18] the left-hand side of the above equation in the form

$$X(-d_X^2 + m^20^2)^{1/2}\Psi(X) = \int_{-\infty}^{+\infty} X(-d_X^2 + m^20^2)^{1/2}\delta(q - X)\Psi(q)dq.$$

Since $\delta(x-a)f(x) = \delta(x-a)f(a)$, we have

$$X(-d_X^2 + m^20^2)^{1/2}\Psi(X) = \int_{-\infty}^{+\infty} q(-d_X^2 + m^20^2)^{1/2}\delta(q - X)\Psi(q)dq.$$

Using the integral representation of $\delta$-function, given by

$$\delta(q - X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \exp[-i(q - X)p],$$

we have

$$X(-d_X^2 + m^20^2)^{1/2}\Psi(X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(p^2 + m^20^2)^{1/2} \exp[-i(q - X)p]\Psi(q)dq dp.$$

Fig. 2. The variation of $|X(\rho)|^2$ with $\rho$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{The variation of $|X(\rho)|^2$ with $\rho$.}
\end{figure}
Hence we can rewrite the right-hand side in the following form:

\[ X(\alpha^2 + m_0^2)^{1/2}\Psi(X) = \frac{1}{2\pi} (\alpha^2 + m_0^2) \int_{-\infty}^{+\infty} q\Psi(q) dq \]

\[ \int_{-\infty}^{+\infty} dq \exp\left(\frac{-i(q-X)p}{(p^2 + m_0^2)^{1/2}}\right). \tag{29} \]

With some simple algebraic manipulation, the above expression is given by \[15\]

\[ X(\alpha^2 + m_0^2)^{1/2}\Psi(X) = \frac{1}{\pi} (\alpha^2 + m_0^2) \int_{-\infty}^{+\infty} q\Psi(q) dq \int_{0}^{\infty} dp \cos([q-X]p) \]

\[ \frac{1}{(p^2 + m_0^2)^{1/2}} = \frac{1}{\pi} (\alpha^2 + m_0^2) \int_{-\infty}^{+\infty} dq q\Psi(q)K_0(m_0|q-X|). \tag{30} \]

where \(K_0(x)\) is the modified Bessel function of second kind of order zero. On decomposing the \(q\) integral into two parts, we have

\[ X(\alpha^2 + m_0^2)^{1/2}\Psi(X) = \frac{1}{\pi} (\alpha^2 + m_0^2) \]

\[ \times \left[ \int_{-\infty}^{\infty} dq q\Psi(q)K_0[m_0(X-q)] + \int_{X}^{+\infty} dq q\Psi(q)K_0[m_0(q-X)] \right]. \tag{31} \]

Then substituting \(X - q = q_1\) in the first integral and \(q - X = q_2\) in the second integral and redefining \(q_1 = q\) in the first integral, and \(q_2 = q\) in the second integral, we have

\[ X(\alpha^2 + m_0^2)^{1/2}\Psi(X) = \frac{1}{\pi} (\alpha^2 + m_0^2) \]

\[ \times \int_{0}^{\infty} dq K_0(m_0q) [(X+q)\Psi(X+q) + (X-q)\Psi(X-q)]. \tag{32} \]

We seek the series solution for the wave function in the form

\[ \Psi(X) = \sum_{k=1}^{n+1} \gamma_{k,n+1} X^k \exp(-\beta X), \tag{33} \]

where \(\gamma_{k,n+1}\) and \(\beta\) are unknown constants, to be obtained from the recursion relations. Then the \(n\)-th term is given by

\[ X(\alpha^2 + m_0^2)^{1/2} X^n \exp(-\beta X) = \frac{1}{\pi} (\alpha^2 + m_0^2) \exp(-\beta X) \]

\[ \times \int_{0}^{\infty} dq K_0(m_0q) [(X+q)^{n+1} \exp(-\beta q) + \exp(\beta q)(X-q)^{n+1}]. \tag{34} \]

Now expanding \((X+q)^{n+1}\) and \((X-q)^{n+1}\) in Binomial series and then using the standard relation we have \[15\]

\[ \int_{0}^{\infty} x^{n-1} \exp(-\alpha x)K_0(\beta_1 x) dx = \]

\[ \frac{(\pi)^{1/2} (2\beta_1)^{\nu} \Gamma(\mu + \nu)\Gamma(\mu - \nu)}{(\alpha + \beta_1)^{\mu + \nu} \Gamma(\mu + \nu) \Gamma(\mu + \nu - \beta)} F \left( \mu + \nu + \frac{1}{2}; \mu + \frac{1}{2}; \frac{\alpha - \beta_1}{\alpha + \beta_1} \right), \tag{35} \]

where \(F(a, b; c; d)\) is the hypergeometric function and, in our case with \(\nu = 0, \alpha = q, \mu = 1 = k+1, \alpha = \beta\) and \(\beta_1 = m_0\), the integral reduces to

\[ I = \frac{\pi^{1/2} \Gamma(k + 2)^2}{\Gamma(k + 2)(m_0 + \beta)^{k+2}} F \left( k + 2, \frac{1}{2}; k + \frac{5}{2}; \frac{-m_0 - \beta}{m_0 + \beta} \right). \tag{36} \]

Then after a little algebra, we have

\[ X(\alpha^2 + m_0^2)^{1/2} X^n \exp(-\beta X) = \]

\[ \frac{1}{\pi^{1/2}} (\alpha^2 + m_0^2) \exp(-\beta X) \sum_{k=0}^{n+1} \binom{n+1}{k} G_k(m_0, \beta) X^{n+1-k}. \tag{37} \]
From the above expression it is quite obvious that

\[
G_k(m_0, \beta) = \frac{[\Gamma(k + 2)]^2}{\Gamma(k + \frac{3}{2})} \left[ \frac{1}{(m_0 + \beta)^{k+2}} F \left( k + 2, \frac{1}{2}; k + \frac{5}{2}; -m_0 - \beta \right) \right] + \frac{[\Gamma(k + 2)]^2}{\Gamma(k + \frac{3}{2})} (-1)^k \left[ \frac{1}{(m_0 - \beta)^{k+2}} F \left( k + 2, \frac{1}{2}; k + \frac{5}{2}; -m_0 + \beta \right) \right].
\]  

(38)

Now it is a matter of simple algebra to show that

\[
X(-d_X^2 + m_0^2)X^{n+1-k} \exp(-\beta X) = (m_0^2 - \beta^2) \exp(-\beta X)X^{n+1-k} + 2(n+1-k)\beta \exp(-\beta X)X^{n-k} - (n+k-1)(n-k) \exp(-\beta X)X^{n-k-1}.
\]  

(39)

Then

\[
X(-d_X^2 + m_0^2)X^n \exp(-\beta X) = \frac{1}{\pi^{1/2}}(-d_X^2 + m_0^2) \exp(-\beta X) \sum_{k=0}^{n+1} \binom{n+1}{k} \left[ (m_0^2 - \beta^2)G_k(m_0, \beta) + 2\beta kG_{k-1}(m_0, \beta) - k(k-1)G_{k-2}(m_0, \beta) \right] X^{n+1-k}.
\]  

(40)

Hence from the quantum mechanical equation

\[
X(-d_X^2 + m_0^2)^{1/2}\Psi(X) = E\Psi(X),
\]  

(41)

we have from the series solution (polynomial form) of \(\Psi(X)\),

\[
X(-d_X^2 + m_0^2)^{1/2} \sum_{k=1}^{n+1} \gamma_{k,n+1} X^k \exp(-\beta X) = \sum_{k=1}^{n+1} \gamma_{k,n+1} \sum_{p=0}^{k} F_{p,k}(m_0, \beta) X^{k+1-p} = E \sum_{k=1}^{n+1} \gamma_{k,n+1} \exp(-\beta X) X^k.
\]  

(42)

The expressions for \(F_{p,k}(m_0, \beta)\) and \(\gamma_{k,n+1}\) have been derived in appendix A.2 in terms of the parameters \(\beta\) and the rest mass \(m_0\). From the above equation, equating the coefficient of \(x^2\), we have

\[
E_n = \frac{\gamma_{1,n+1}}{\gamma_{2,n+1}} F_{0,n+1}(m_0, \beta),
\]  

(43)

the energy corresponding to the \(n\)-th level of the spectrum. Further equating the coefficients of \(X^l\) from both the sides and putting \(p = 1\), we have

\[
\gamma_{l,n+1}(\beta, m_0)F_{l,l}(m_0, \beta) = E_l(\beta, m_0)\gamma_{l,n+1}(m_0, \beta), \quad \text{or}
\]  

(44)

\[
E_l(m_0, \beta) = F_{l,l}(m_0, \beta),
\]  

(45)

which gives the energy spectrum in terms of the unknown parameter \(\beta\), quantum number \(l\) and the strength of the gravitational field \(\alpha\), where

\[
F_{l,l}(m_0, \beta) = \left[ \frac{-l\beta \alpha}{(2\pi m_0)^{1/2}(m_0^2 - \beta^2)^{3/4}} \right] \Gamma^2 \left( \frac{\beta}{m_0} \right),
\]  

(46)

in which \(l = 1, 2, 3, \ldots\) positive integers. Therefore to obtain the energy spectrum, we have to evaluate the associated Legendre function. It is to be noted further that the energy spectrum is real and linearly quantized \((\propto l)\). Now because of the factor \(\exp(-\beta X)\) the wave functions are bounded and also the energy eigen values are negative in nature. From the above expression it is quite obvious that \(\beta < m_0\). This is also a necessary condition for the argument \(z\) of \(P^\mu_v(z)\) to be real. In fig. 3 we have plotted the variation of \(E_l\) with \(\beta\) (the parameter has been redefined as \(\beta/m_0\)) for \(l = 1\). Since the energy levels are proportional to \(l\) we have not considered other \(l\)-values. We have noticed that the magnitude of the energy eigen value \(|E_l(m_0, \beta)|\) increases with \(\beta\) and the rise is very sharp as \(\beta \rightarrow 1\). However,
Fig. 3. The variation of $E_l$ with $\beta$ for $l = 1$. Here the parameter $\beta$ is redefined as $\beta/m_0$.

it is actually negative throughout and the negativity increases with $\beta$ and also with $\alpha$. Further, the eigen functions are $\propto \exp(-\beta X)$, therefore with the increase of $\beta$ the wave function converges to zero very quickly, whereas the eigen states become more bound because of high negative value of energy for large $\beta$. Or in other words, when $\beta/m_0$ is very close to unity, the nature of wave functions and energy eigen values indicate that the binding of the states are strong enough. Therefore with the increase of negative value of the energy makes the state more bound and simultaneously the spread of wave function in space is reduced. The later is also in agreement with more stronger binding. At this point we would like to emphasise that with the increase in the strength of gravitational field produced by a black, i.e., as we approach more and more close to the event horizon, the absolute value of the energy eigen value increases more and more makes the system more bound. In our study since $\alpha$ is the gravitational field strength of the black hole, we may conclude that the increase in classical background gravitational field, which is assumed to be uniform locally, makes the particle, which is quantum mechanical in nature, more strongly bound. The binding is with the black hole in presence of strong classical backgound field.

4 The Klein-Gordon equation in Rindler space

To obtain the modified form of the Klein-Gordon equation in a uniformly accelerated frame we start with the classical Hamiltonian,

$$H = \left(1 + \frac{\alpha x}{c^2}\right) \left(m_0^2 c^4 + p^2 c^2\right)^{\frac{1}{2}}.$$  (47)

Squaring both the sides, we have

$$H^2 = \left(1 + \frac{\alpha x}{c^2}\right)^2 \left(m_0^2 c^4 + p^2 c^2\right).$$  (48)

Hence the modified form of the Klein-Gordon equation is given by

$$H^2 \Psi(x, y, z) = E^2 \Psi(x, y, z),$$  (49)

where $E$ is the energy eigen value. Writing explicitly the Hamiltonian part and using the separable form of $\Psi(x, y, z)$, (eq. (5)), we have

$$\left(1 + \frac{\alpha x}{c^2}\right)^2 \left(-\hbar^2 \frac{d^2 X}{dx^2} + E^2 X(x)\right) = E^2 X(x),$$  (50)
where \( E_1^2 = (p_y^2 + p_z^2)c^2 + m_0^2c^4 \), the square of the transverse part of particle energy in the relativistic form. Substituting \( 1 + \frac{4a}{\omega^2} = u \), which is PT-symmetric, the above equation may be written in the form

\[
\frac{\hbar^2 \alpha^2}{c^2} \frac{d^2 X}{dw^2} + \frac{E_1^2}{u^2} X(u) = E_1^2 X(u),
\]

which may further be written as

\[
\frac{d^2 X}{dw^2} + \frac{a}{u^2} X(u) = bX(u),
\]

with \( a = \frac{\epsilon^2 E_1^2}{\hbar^2 \alpha^2} \) and \( b = \frac{\epsilon^2 E_1^2}{\hbar^2 \alpha^2} \). Let us put \( w = ub^{1/2} \), then we have

\[
\frac{d^2 X}{dw^2} + \frac{a}{w^2} X(w) - X(w) = 0.
\]

Expressing \( X = w^n Y \), we have

\[
w^n \frac{d^2 Y}{dw^2} + 2nw^{n-1} \frac{dY}{dw} + \left[ n(n - 1) + aw^{n-2} - w^n \right] Y(w) = 0.
\]

Putting \( n = \frac{1}{2} \), i.e., \( X = w^{1/2} Y \), the above equation reduces to

\[
w^2 \frac{d^2 Y}{dw^2} + w \frac{dY}{dw} + \left[ -w^2 - \left( \frac{1}{4} - a \right) \right] Y(w) = 0.
\]

Further on substituting \( w \rightarrow iw \), we have, after rearranging the above equation,

\[
w^2 \frac{d^2 Y}{dw^2} + w \frac{dY}{dw} + \left[ w^2 - \left( \frac{1}{4} - a \right) \right] Y(w) = 0,
\]

which may be written as

\[
w^2 \frac{d^2 Y}{dw^2} + w \frac{dY}{dw} + \left[ w^2 - \gamma^2 \right] Y(w) = 0,
\]

where \( \gamma^2 = \frac{1}{4} - a = \frac{1}{4} - \frac{\epsilon^2 E_1^2}{\hbar^2 \alpha^2} \). For \( \gamma = 0 \),

\[
E = \frac{\hbar \alpha}{2c}.
\]

Defining \( \frac{2}{\epsilon} = \omega_0 \), some frequency, we have

\[
E = \frac{1}{2} \hbar \omega_0,
\]

the zero point energy in some sense. The solution is given by

\[
Y(w) = J_0(iw).
\]

The zeroth-order Bessel function of purely imaginary argument. Then

\[
X(w) = w^{1/2} J_0(iw).
\]

For \( \gamma^2 > 0 \), i.e., \( \frac{1}{4} - \frac{\epsilon^2 E_1^2}{\hbar^2 \alpha^2} > 0 \), or,

\[
E < \frac{1}{2} \hbar \omega_0,
\]

which is less than the zero point energy. Since in quantum mechanics the energy eigen value is always greater than or equal to zero point energy, therefore we have \( \gamma^2 < 0 \) and discarded the \( \gamma^2 > 0 \) solution. Hence the differential equation reduces to

\[
w^2 \frac{d^2 Y}{dw^2} + w \frac{dY}{dw} + \left[ -w^2 + \gamma^2 \right] Y(w) = 0.
\]

Putting \( \gamma \rightarrow i\gamma \) and \( \omega \rightarrow i\omega \), we have, again,

\[
w^2 \frac{d^2 Y}{dw^2} + w \frac{dY}{dw} + \left( w^2 - \gamma^2 \right) Y(w) = 0.
\]
This is the differential equation for Bessel function of order \( i\gamma \) (imaginary order) and argument \( iw \) (imaginary argument). The solution for real order but imaginary argument is well known and is given by

\[
Y(w) = J_{i\gamma}(iw) = -\left(\frac{iw}{2}\right)^{\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(\gamma + \mu + 1)} \left(\frac{w}{2}\right)^{2n}. \tag{65}
\]

Then

\[
X(w) = -\left(\frac{i}{2}\right)^{\gamma} w^{\gamma+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(\gamma + \mu + 1)} \left(\frac{w}{2}\right)^{2n}. \tag{66}
\]

Next we consider \( \gamma^2 < 0 \), i.e., \( \gamma \rightarrow i\gamma \) and \( E > \frac{1}{2}h\omega \), then

\[
Y(w) = J_{\gamma}(iw) \quad \text{and} \quad X(w) = w^{\frac{1}{2}}J_{\gamma}(iw). \tag{67}
\]

Since the most acceptable value of lower limit/ground state of the energy spectrum is \( \geq \frac{1}{2}h\omega \), therefore in the present scenario, the eigen states are represented by the Bessel function of both imaginary orders and imaginary arguments. For the sake of some more physical insight, let us make a detailed analysis of this solution \([19,20]\). To get an analytical solution, we start with a series solution of the form

\[
Y(w) = A(w) \cos(\gamma \ln w) + B(w) \sin(\gamma \ln w), \tag{68}
\]

where

\[
A(w) = a_0 + a_1 w + a_2 w^2 + \ldots + a_n w^n + \ldots \tag{69}
\]

\[
B(w) = b_0 + b_1 w + b_2 w^2 + \ldots + b_n w^n + \ldots \tag{70}
\]

where \( a_0, a_1, \ldots, b_0, b_1, \ldots \) are unknown parameters and are independent of each other. We substitute \( Y(w) \) as given above (eq. (68)) in the differential equation given by eq. (64). Now equating the coefficient of \( w^n \) to zero, we get

\[
\begin{align*}
[a_n, n(n-1) \cos(\gamma \ln \omega) + b_n, n(n-1) \sin(\gamma \ln \omega) - 2a_n, n \sin(\gamma \ln \omega)\gamma \\
+ 2b_n, n \cos(\gamma \ln \omega)\gamma + a_n, n \sin(\gamma \ln \omega)\gamma - b_n, n \cos(\gamma \ln \omega)\gamma \\
+ a_n, n\omega \cos(\gamma \ln \omega) + b_n, n \sin(\gamma \ln \omega) - a_n, n \gamma \sin(\gamma \ln \omega) \\
+ b_n, n \gamma \cos(\gamma \ln \omega)] w^n - \omega^{2+n} [a_n, \cos(\gamma \ln \omega) + b_n, \sin(\gamma \ln \omega)] = 0.
\end{align*}
\]

For \( n = 1 \),

\[
\begin{align*}
\sin(\gamma \ln \omega) [-2a_1 \gamma + b_1 \omega - b_1 \omega^3] \\
+ \cos(\gamma \ln \omega) [2b_1 \omega \gamma + a_1 \omega - a_1 \omega^3] = 0.
\end{align*}
\]

Hence, since \( \sin(\gamma \ln \omega) \) and \( \cos(\gamma \ln \omega) \) are non-zero, we have

\[
-2a_1 \gamma + b_1 - b_1 \omega^2 = 0 \tag{73}
\]

and

\[
2b_1 \gamma + a_1 - a_1 \omega^2 = 0. \tag{74}
\]

These are the simultaneous linear homogeneous equations for \( a_1 \) and \( a_2 \). For the non-trivial solutions of \( a_1 \) and \( a_2 \), we must have

\[
\begin{vmatrix}
-2\gamma & 1 - \omega^2 \\
1 - \omega^2 & 2\gamma
\end{vmatrix} = 0. \tag{75}
\]

But \( \gamma \) is a non-zero real constant, therefore the above determinant cannot be zero. Hence \( a_1 = b_1 = 0 \), which are the trivial solutions. Now it is a matter of simple algebra to show by equating the coefficient of \( \omega^n \) to zero,

\[
a_n = \frac{n a_{n-2} - 2\gamma b_{n-2}}{n(n^2 + 4\gamma^2)} \tag{76}
\]

and

\[
b_n = \frac{n b_{n-2} + 2\gamma a_{n-2}}{n(n^2 + 4\gamma^2)} \tag{77}
\]
Hence it is obvious that the coefficients of all the odd power terms of $\omega$ in eqs. (69) and (70) are zero, i.e., $a_j$ and $b_j$, where $j$ is the odd integer, are zero. Therefore, the odd terms will not contribute to the solution. To get the coefficients of the even power of $\omega$, we seek the solutions in the form

$$Y(w) = Y_1(w) + iY_2(w),$$

where

$$Y_1(w) = C(w) \cos(\gamma \ln iw) + D(w) \sin(\gamma \ln iw)$$

and

$$Y_2(w) = D(w) \cos(\gamma \ln iw) - C(w) \sin(\gamma \ln iw),$$



$$C(w) = \sum_{n=0}^{\infty} C_{2n} \left( \frac{w}{2} \right)^{2n}$$

$$D(w) = \sum_{n=0}^{\infty} D_{2n} \left( \frac{w}{2} \right)^{2n}$$

$$C_{2n} = \frac{nC_{2n-2} - \gamma D_{2n-2}}{n(n^2 + \gamma^2)}$$

and

$$D_{2n} = \frac{nD_{2n-2} + \gamma C_{2n-2}}{n(n^2 + \gamma^2)}.$$  

We use two sets of $(C_0, D_0)$ ($(0, 1)$ and $(1, 0)$). Let us first take $(C_0, D_0) = (0, 1)$. Then

$$C_2 = -\frac{\gamma}{1 + \gamma^2}$$

$$D_2 = \frac{1}{1 + \gamma^2}$$

$$C_4 = -\frac{3\gamma}{2(1^2 + \gamma^2)(2^2 + \gamma^2)}$$

$$D_4 = \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)} - \frac{\gamma^2}{2(1^2 + \gamma^2)(2^2 + \gamma^2)}$$

$$C_6 = -\frac{11\gamma}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} + \frac{\gamma^3}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)}$$

$$D_6 = \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} - \frac{\gamma^2}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)}.$$  

etc., then we can write

$$Y_1^{(1)}(w) = A_1 \sin(\gamma \ln iw) + B_1 \cos(\gamma \ln iw)$$

and

$$Y_2^{(1)}(w) = A_1 \cos(\gamma \ln iw) - B_1 \sin(\gamma \ln iw),$$

where

$$A_1 = 1 + \frac{\gamma}{1 + \gamma^2} \left( \frac{w}{2} \right)^2 + \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)} - \frac{\gamma^2}{2(1^2 + \gamma^2)(2^2 + \gamma^2)} \left( \frac{w}{2} \right)^4$$

$$+ \left\{ \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} - \frac{\gamma^2}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} \right\} \left( \frac{w}{2} \right)^6 + \ldots$$

and

$$B_1 = -\frac{\gamma}{1^2 + \gamma^2} \left( \frac{w}{2} \right)^2 - \frac{3\gamma}{2(1^2 + \gamma^2)(2^2 + \gamma^2)} \left( \frac{w}{2} \right)^4$$

$$+ \left\{ \frac{\gamma^3}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} - \frac{\gamma}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} \right\} \left( \frac{w}{2} \right)^6 + \ldots.$$
Then the final result is
\[ Y(\omega) = Y_2^{(1)}(\omega) + iY_1^{(1)}(\omega). \]  
(95)

For the other set \((C_0, D_0) = (1, 0)\), then we have
\[
C_2 = \frac{1}{1 + \gamma^2}
\]
(96)
\[
D_2 = \frac{\gamma}{1 + \gamma^2}
\]
(97)
\[
C_4 = \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)} - \frac{\gamma^2}{2(1^2 + \gamma^2)(2^2 + \gamma^2)}
\]
(98)
\[
D_4 = \frac{3\gamma}{2(1^2 + \gamma^2)(2^2 + \gamma^2)}
\]
(99)
\[
C_6 = \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} - \frac{\gamma^2}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)}
\]
(100)
\[
D_6 = \frac{11\gamma}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} - \frac{\gamma^3}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)}
\]
(101)

etc., then
\[
Y_1^{(2)}(w) = A_1 \cos(\gamma \ln iw) + A_2 \sin(\gamma \ln iw)
\]
(102)
and
\[
Y_2^{(2)}(w) = A_2 \cos(\gamma \ln iw) - A_1 \sin(\gamma \ln iw),
\]
(103)

with
\[
A_1 = 1 + \frac{1}{1 + \gamma^2} \left(\frac{w}{2}\right)^2 + \left\{ \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)} - \frac{\gamma^2}{2(1^2 + \gamma^2)(2^2 + \gamma^2)} \right\} \left(\frac{w}{2}\right)^4
\]
\[+ \left\{ \frac{1}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} - \frac{\gamma^2}{(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} \right\} \left(\frac{w}{2}\right)^6 + \ldots
\]
(104)
and
\[
A_2 = \frac{\gamma}{1^2 + \gamma^2} \left(\frac{w}{2}\right)^2 + \frac{3\gamma}{2(1^2 + \gamma^2)(2^2 + \gamma^2)} \left(\frac{w}{2}\right)^4
\]
\[+ \left\{ - \frac{\gamma^3}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} + \frac{11\gamma}{6(1^2 + \gamma^2)(2^2 + \gamma^2)(3^2 + \gamma^2)} \right\} \left(\frac{w}{2}\right)^6 + \ldots.
\]
(105)

Then the final result, in this case, is given by
\[
Y(\omega) = Y_1^{(2)}(\omega) + iY_2^{(2)}(\omega)
\]
(106)
and \(A_2 = -A_1\). Now,
\[
\cos(\gamma \ln iw) = \cos\left(\frac{\pi \gamma}{2}\right) \cos(\gamma \ln w) - \sinh\left(\frac{\pi \gamma}{2}\right) \sin(\gamma \ln w)
\]
(107)
\[
\sin(\gamma \ln iw) = \cos\left(\frac{\pi \gamma}{2}\right) \sin(\gamma \ln w) + \sin\left(\frac{\pi \gamma}{2}\right) \cos(\gamma \ln w).
\]
(108)

Then
\[
J_{\gamma}(iw) = Y(\omega),
\]
(109)
where \(w\) and \(\gamma\) are real numbers, and it is easy to show that the probability density is exactly identical for the choice of both the sets of \((C_0, D_0)\). Then the solution can be obtained from eq. (67). In fig. 4 we have plotted the variation of the probability density \(\text{Abs}(X)\) with \(\omega\) for three different values of the parameter \(\gamma\). We have noticed that for \(\gamma > 1.0\), the probability density goes to zero very quickly. The probability density is also like damped oscillator in this case also.
Fig. 4. The variation of $\text{Abs}(X)$ with $\omega$ for three values of the parameter $\gamma$. For $\gamma > 1.0$, the wave function $X$ goes to zero very quickly.

5 Conclusion

In sect. 2, with a kind of quadratic approximation, we have solved the Schrödinger equation and obtained eigen values and the wave functions. We have noticed that the probability densities are damped oscillatory in nature. This type of variation may be interpreted as the decrease in density of created particles as one goes away from the event horizon of a black hole. Or in other wards, the particle production decreases and finally vanishes as the distance from the event horizon becomes large enough.

The exact relativistic version of the quantum mechanical equation has been studied in sect. 3. We have solved the differential equation analytically and obtained the exact solution. We have solved for the eigen functions and also obtained the corresponding energy eigen values as a function of the free parameter $\beta/m_0$. We have noticed that as the ratio approaches unity, the system becomes more and more strongly bound and the single particle energy becomes highly negative, i.e. to say that the binding energy increases and the range of the wave function also decreases. Which also indicates that the eigen state becomes strongly bound.

In sect. 4 we have studied the modified form of the Klein-Gordon equation in Rindler space. We have solved the equation analytically in an exact manner and obtained the eigen functions and the energy eigen values. We have noticed that the wave functions are given by Bessel functions with both imaginary orders and imaginary arguments. However, the energy eigen values are found to be real.

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Appendix A.

Appendix A.1.

For $\lambda = 0$, the reduced form of the differential equation given in eq. (11) is given by

$$\frac{d^2X}{d\rho^2} + \frac{\rho^2}{4} X = 0.$$  \hspace{1cm} (A.1)

To have an analytical solution, we put $u = \rho/2^{1/2}$ Then the above differential equation reduces to

$$\frac{d^2X}{du^2} + u^2 X = 0.$$  \hspace{1cm} (A.2)
Let \( X(u) = u^n \psi(u) \), where \( n \) is an unknown parameter. On substituting \( X(u) \) in the above differential equation, we get
\[
u^2 \frac{d^2 \psi}{du^2} + 2nu \frac{d\psi}{du} + [n(n-1) + \nu^4] \psi = 0. \tag{A.3}
\]
We next put \( u = \beta v^{1/2} \), where \( \beta \) is another unknown quantity and \( v \) is the new variable. Rearranging the above differential equation in terms of the new variable \( v \), we have
\[
u^2 \frac{d^2 \psi}{dv^2} + v \left( n + \frac{1}{2} \right) \frac{d\psi}{dv} + \frac{1}{4} [n(n-1) + \beta^4 v^2] \psi = 0. \tag{A.4}
\]
To reduce this equation to a well-known form of differential equation satisfied by special function, we put \( n = 1/2 \) and \( \beta = 2^{1/2} \). Then we have the final form of the above differential equation,
\[
u^2 \frac{d^2 \psi}{dv^2} + v \frac{d\psi}{dv} + \left( v^2 - \frac{1}{16} \right) \psi = 0. \tag{A.5}
\]
The solution of this equation is \( J_{1/4}(v) \). Hence
\[X(\rho) = \frac{\rho^{1/2}}{214/4} J_{1/4} \left( \frac{\rho^2}{4} \right). \tag{A.6}
\]

**Appendix A.2.**

We know \[15\]
\[
F(a, \frac{1}{2}; a + \frac{1}{2}; -x) = \Gamma \left( a + \frac{1}{2} \right) \frac{x^{1/2-a}}{(1+x)^{1/2}} P_{\frac{1}{2}-a} \left( 1 - \frac{x}{1+x} \right). \tag{A.7}
\]
Then
\[
G_{k-1}(m_0, \beta) = \frac{[\Gamma(k+1)]^2}{(2m_0)^{\frac{1}{2}} (m_0^2 - \beta^2)^{\frac{1+k}{2}}} \left[ P_{\frac{1}{2}-k} \left( \frac{\beta}{m_0} \right) + (-1)^k P_{\frac{1}{2}-k} \left( -\frac{\beta}{m_0} \right) \right] \tag{A.8}
\]
\[
G_{k+2}(m_0, \beta) = \frac{[\Gamma(k+4)]^2}{(2m_0)^{\frac{1}{2}} (m_0^2 - \beta^2)^{\frac{1+k}{2}}} \left[ P_{\frac{1}{2}-k-4} \left( \frac{\beta}{m_0} \right) + (-1)^{k+3} P_{\frac{1}{2}-k-4} \left( -\frac{\beta}{m_0} \right) \right] \tag{A.9}
\]
\[
G_k(m_0, \beta) = \frac{[\Gamma(k+2)]^2}{(2m_0)^{\frac{1}{2}} (m_0^2 - \beta^2)^{\frac{1+k}{2}}} \left[ P_{\frac{1}{2}-k-2} \left( \frac{\beta}{m_0} \right) + (-1)^{k+1} P_{\frac{1}{2}-k-2} \left( -\frac{\beta}{m_0} \right) \right] \tag{A.10}
\]
and
\[
G_{k+1}(m_0, \beta) = \frac{[\Gamma(k+3)]^2}{(2m_0)^{\frac{1}{2}} (m_0^2 - \beta^2)^{\frac{1+k}{2}}} \left[ P_{\frac{1}{2}-k-3} \left( \frac{\beta}{m_0} \right) + (-1)^{k+2} P_{\frac{1}{2}-k-3} \left( -\frac{\beta}{m_0} \right) \right]. \tag{A.11}
\]
Now,
\[
P^{\mu+2}_\nu(x) = -2(\mu + 1) \frac{x}{(1-x^2)^{1/2}} P^{\mu+1}_\nu(x) + (\mu - \nu)(\mu + \nu + 1) p^\nu_\nu(x), \tag{A.12}
\]
where \( \mu = \frac{1}{2} - k - 4, \nu = \frac{1}{2}, x = \pm \frac{\beta}{m_0} \) and \( \frac{x}{(1-x^2)^{1/2}} = \frac{\beta}{(m_0^2 - \beta^2)^{1/2}}. \)
These are the recursion relations for the polynomials $G_i(m_0, \beta)$ for various $i$. To evaluate $G_k(m_0, \beta)$ for any arbitrary $k$ from the above recursion relation, we use

$$G_0(m_0, \beta) = \frac{1}{(2m_0)^{1/2}(m_0^2 - \beta^2)^{3/4}} P_{-1/2}^{-3/2} \left( \frac{\beta}{m_0} \right)$$  \hspace{1cm} (A.13)$$

and

$$G_{k+2}(m_0, \beta) = \frac{1}{(m_0^2 - \beta^2)} \left[ (k + 2)^2 G_k(m_0, \beta) - (2k + 5)\beta G_{k+1}(m_0, \beta) \right].$$ \hspace{1cm} (A.14)$$

Hence

$$G_k(m_0, \beta) = \frac{1}{(m_0^2 - \beta^2)^{3/4}} \left[ k^2 G_{k-2}(m_0, \beta) - (2k + 1)\beta G_{k-1}(m_0, \beta) \right].$$ \hspace{1cm} (A.15)$$

Then

$$X(-d_x^2 + m_0^2)^{1/2} X^n \exp(-\beta X) = M_{n+1}(m_0, \beta, X) \exp(-\beta X) = \sum_{k=0}^{n+1} F_{k,n+1}(m_0, \beta) X^{n+1-k} \exp(-\beta X),$$ \hspace{1cm} (A.18)$$

where

$$F_{k,n+1}(m_0, \beta) = \frac{1}{\pi^{1/2}} \binom{n+1}{k} \left[ kG_{k+2}(m_0, \beta) - \beta G_{k-1}(m_0, \beta) \right].$$ \hspace{1cm} (A.19)$$

Hence we have

$$F_{k,n+2}(m_0, \beta) = \frac{1}{\pi^{1/2}} \frac{(n + 2)!}{k!(n + 2 - k)!}$$

$$\left[ kG_{k+2}(m_0, \beta) - \beta G_{k-1}(m_0, \beta) \right] = \frac{n + 2}{n + 2 - k} F_{k,n+1}(m_0, \beta)$$ \hspace{1cm} (A.20)$$

We also have

$$F_{0,n+1}(m_0, \beta) = \frac{1}{(2\pi m_0)^{1/2}(m_0^2 - \beta^2)^{1/4}} P_{-1/2}^{-1/2} \left( \frac{\beta}{m_0} \right)$$ \hspace{1cm} (A.21)$$

and

$$F_{1,n+1}(m_0, \beta) = \frac{-(n + 1)\beta}{(2\pi m_0)^{1/2}(m_0^2 - \beta^2)^{3/4}} P_{-1/2}^{-3/2} \left( \frac{\beta}{m_0} \right).$$ \hspace{1cm} (A.22)$$

Then for the quantum mechanical equation

$$X(-d_x^2 + m_0^2)^{1/2} X^n \exp(-\beta X) = E\Psi(x),$$ \hspace{1cm} (A.23)$$

with

$$\Psi(X) \propto \sum_{k=1}^{n+1} \gamma_{k,n+1} X^n \exp(-\beta X).$$ \hspace{1cm} (A.24)$$

We have

$$X(-d_x^2 + m_0^2)^{1/2} X^n \exp(-\beta X) = \sum_{k=0}^{n+1} F_{k,n+1}(m_0, \beta) X^{n+1-k} \exp(-\beta X).$$ \hspace{1cm} (A.25)$$
Then

\[
X(-d_x^2 + m_0^2)^{\frac{1}{2}} \sum_{k=1}^{n+1} \gamma_{k,n+1} X^k \exp(-\beta X) = \sum_{k=1}^{n+1} \gamma_{k,n+1} \sum_{p=0}^{k} F_{p,k}(m_0, \beta) X^{k+1-p} \exp(-\beta X) = E \sum_{k=1}^{n+1} \gamma_{k,n+1} X^k \exp(-\beta X). \text{(A.26)}
\]

Hence equating the coefficients of \(X^l\) from both the sides, after putting \(p = 1\), we have

\[
\gamma_{l,n+1}(\beta, m_0) F_{1,l}(m_0, \beta) = E_l(\beta, m_0) \gamma_{l,n+1}(\beta, m_0)
\]

which gives the energy spectrum, provided the parameter \(\beta\) is known, where

\[
F_{1,l}(m_0, \beta) = \frac{-1/\beta}{(2\pi m_0)^{1/2}(m_0^2 - \beta^2)^{3/4}} \int_{-\beta}^{\beta} \left( \frac{\beta}{m_0} \right) F_l(\mu), \text{(A.29)}
\]

with \(l = 1, 2, 3, \ldots\), positive integers. Therefore to obtain the energy spectrum, we have to evaluate the associated Legendre function. It is to be noted further that the energy spectrum is real and linearly quantized and the wave functions are bounded (\(\chi \exp(-\beta X)\)). From the expressions it is quite obvious that \(\beta < m_0\). This is also a necessary condition for the argument \(z\) of \(P^\mu_\nu(z)\). Now [15]

\[
P^\mu_\nu(z) = \frac{1}{\Gamma(1-\mu)} \left( \frac{z + 1}{z - 1} \right)^\mu F\left(-\nu, \nu + 1; 1 - \mu; \frac{1 - z}{2}\right), \text{(A.30)}
\]

with \(|1 - z| < 2\) where [15]

\[
F(a, b; c; d) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} dt, \text{(A.31)}
\]

with \(\text{Re}(c) > \text{Re}(b) > 0\). Here \(\mu = -3/2\), \(\nu = -1/2\) and \(z = \beta/m_0\). Therefore

\[
P_{-3/2}^{-1/2} = \frac{1}{\Gamma(5/2)} \left( \frac{m_0 + \beta}{m_0 - \beta} \right)^{-3/2} F\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{m_0 - \beta}{2m_0}\right). \text{(A.32)}
\]

Now from eq. (A.27), we have

\[
F\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{m_0 - \beta}{2m_0}\right) = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(2)} \int_0^1 t^{-\frac{1}{2}}(1 - t)^{1/2} \left[ 1 - t \left( \frac{m_0 - \beta}{2m_0}\right) \right]^{-1/2} dt. \text{(A.33)}
\]

which may be decomposed into two integers, given by

\[
F\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{m_0 - \beta}{2m_0}\right) = \frac{3(2m_0)^{1/2}}{4}(I_1 - I_2), \text{(A.34)}
\]

where

\[
I_1 = \int_0^1 \frac{dt}{[at^2 + bt]^{1/2}} \text{,} \text{(A.35)}
\]

with \(a = m_0 - \beta\) and \(b = 2m_0\), and

\[
I_2 = \int_0^1 \frac{dt}{[at^2 + bt]^{1/2}} \text{,} \text{(A.36)}
\]

with same \(a\) and \(b\). The integrals \(I_1\) and \(I_2\) can be evaluated analytically [21] and are given by

\[
I_1 = \frac{2(m_0)^{1/2}}{\beta - m_0} \ln \left[ \frac{\beta - m_0 + (m_0 - \beta)^2 + 4m_0^2}{2m_0}\right] \text{,} \text{(A.37)}
\]
and
\[
I_2 = \frac{3(2m_0)^{5/2}}{(\beta - m_0)^2} \ln \left[ \frac{\{(\beta - m_0) + \{(m_0 - \beta)^2 + 4m_0^2\}^{1/2}\}}{2m_0} \right] \\
+ \frac{(2m_0)^{1/2}}{(\beta - m_0)^2} [(\beta - m_0)^2 + 4m_0^2]^{1/2}.
\]

Then we have
\[
P_{-3/2} \left( \frac{\beta}{m_0} \right) = \frac{(2m_0)}{\pi^{1/2}(m_0^2 - \beta^2)^{3/4}} \left[ (2\beta^2 - 4\beta m - 10\beta^2) \ln(P1) - (m_0 - \beta)(\beta^2 - 2\beta m + 5m_0^2)^{1/2} \right],
\]
where
\[
P1 = \frac{m_0 - \beta + \{(m_0 - \beta)^2 + 4m_0^2\}^{1/2}}{2m_0}.
\]
We also need \(P_{-5/2}(\beta/m_0)\), which is given by
\[
P_{-5/2} = \frac{1}{\Gamma(7/2)} \left( \frac{m_0 + \beta}{m_0 - \beta} \right)^{-5/4} F \left( \frac{1}{2}, \frac{3}{2}; \frac{7}{2}; \frac{m_0 - \beta}{2m_0} \right),
\]
where
\[
F \left( \frac{1}{2}, \frac{3}{2}; \frac{7}{2}; \frac{m_0 - \beta}{2m_0} \right) = \frac{\Gamma \left( \frac{7}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma (3)} \int_0^1 (1 - t)^2 \left[ 2m_0 t + (m_0 - \beta) t^2 \right]^{1/2} dt.
\]
The integral may be decomposed into three parts and can easily be evaluated analytically [21]
\[
I_1 = \int_0^1 \frac{dt}{[at^2 + bt]^{1/2}} = \frac{2(2m_0)^{1/2}}{m_0 - \beta} \ln \left[ \frac{m_0 - \beta + \{(m_0 - \beta)^2 + 4m_0^2\}^{1/2}}{2m_0} \right],
\]
\[
I_2 = -2 \int_0^1 \frac{tdt}{[at^2 + bt]^{1/2}} = \frac{6(2m_0)^{5/2}}{(m_0 - \beta)^3} \ln \left[ \frac{(-m_0 - \beta) + \{(m_0 - \beta)^2 + 4m_0^2\}^{1/2}}{2m_0} \right] \\
- \frac{(8m_0)^{1/2}}{(m_0 - \beta)^2} \left[ (m_0 - \beta)^2 + 4m_0^2 \right]^{1/2}
\]
and
\[
I_3 = \int_0^1 \frac{t^2 dt}{[at^2 + bt]^{1/2}} = \frac{2(2m_0)^{9/2}}{(m_0 - \beta)^3} \ln(A1 + A2 + A3),
\]
where \(a\) and \(b\) are the same as before and
\[
A1 = \frac{19}{8} \frac{(m_0 - \beta) + \{(m_0 - \beta)^2 + 4m_0^2\}^{1/2}}{2m_0},
\]
\[
A2 = \frac{11}{8} \frac{(m_0 - \beta)}{2m_0} \left[ (m_0 - \beta)^2 + 4m_0^2 \right]^{1/2}
\]
and
\[
A3 = \frac{(m_0 - \beta)}{4(2m_0)^{3/2}} \left[ (m_0 - \beta)^2 + 4m_0^2 \right]^{3/2}.
\]
Now to obtain the eigen states, we put \(p = 0\) in eq. (A.22) and equate the coefficients of \(x^l\) from both sides and obtain
\[
\gamma_{l,n+1}(\beta, m_0) F_{0,l}(\beta, m_0) = E_{l+1}(\beta, m_0) \gamma_{l+1,n+1}(\beta, m_0).
\]
Since \( F_{0,l}(\beta, m_0) \) is independent of \( l \), we put it as \( F(\beta, m_0) \). Hence

\[
\gamma_{l+1,n+1}(\beta, m_0) = \frac{F(\beta, m_0)}{E_{l+1}(\beta, m_0)} \gamma_{l,n+1}(\beta, m_0). \tag{A.50}
\]

Putting the value of \( E_{l+1}(\beta, m_0) \), we get

\[
\gamma_{l+1,n+1}(\beta, m_0) = \frac{F(\beta, m_0)}{F_{1,l+1}(\beta, m_0)} \gamma_{l,n+1}. \tag{A.51}
\]

This is the recursion relation for \( \gamma_{l,n+1}(\beta, m_0) \). Hence we have

\[
\gamma_{2,n+1}(\beta, m_0) = \frac{F(\beta, m_0)}{F_{1,2}(\beta, m_0)} \gamma_{1,n+1}(\beta, m_0)
\]
\[
\gamma_{3,n+1}(\beta, m_0) = \frac{F(\beta, m_0)}{F_{1,3}(\beta, m_0)} \gamma_{2,n+1}(\beta, m_0)
= \frac{F^2(\beta, m_0)}{F_{1,2}(\beta, m_0)F_{1,3}(\beta, m_0)} \gamma_{1,n+1}(\beta, m_0)
\]
\[
\gamma_{4,n+1}(\beta, m_0) = \frac{F^3(\beta, m_0)}{F_{1,2}(\beta, m_0)F_{1,3}(\beta, m_0)F_{1,4}(\beta, m_0)} \gamma_{1,n+1}(\beta, m_0)
\]
\[
\gamma_{n+1}(\beta, m_0) = \frac{F^{n-1}(\beta, m_0)}{\prod_{i=2} F_{1,i}(\beta, m_0)} \gamma_{1,n+1}(\beta, m_0). \tag{A.54}
\]

The last one is the most general expression. Therefore the wave function can be expressed as

\[
\Psi(X) = \Psi_n(X) = \sum_{l=1}^{n+1} \gamma_{l,n+1} X^l \exp(-\beta X)
\]
\[
= \left[ \sum_{l=1}^{n+1} \frac{F^{l-1}(\beta, m_0)}{\prod_{i=1} F_{1,i}(\beta, m_0)} \right] \gamma_{1,n+1}(\beta, m_0) X^l \exp(-\beta X). \tag{A.56}
\]

The normalization constant \( \gamma_{1,n+1}(\beta, m_0) \) can be obtained from the orthonormality condition

\[
\int_1^\infty \Psi_n^* (X) \Psi_n' (X) \mathrm{d}X = \delta_{nn'}, \tag{A.57}
\]

which gives

\[
\gamma_{1,n+1}(\beta, m_0) = \left[ \sum_{l'=1}^{n+1} \frac{F^{l'-1}(\beta, m_0)F^{l'-1}(\beta, m_0)}{\prod_{i=1} F_{1,i}(\beta, m_0)} \right] \times \left[ \frac{(l+l')! - E_{\gamma}(\beta, l, l')}{2^{l+l'+1} \beta^{l+l'+1}} \right]. \tag{A.58}
\]

Hence the normalization constant is the square root of this quantity. Here,

\[
E_{\gamma}(\beta, l, l') = \int_0^{2\beta} z^{l+l'} \exp(-z) \mathrm{d}z,
\]
the incomplete \( \Gamma \)-function.

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