REIDEMEISTER CLASSES IN SOME WREATH PRODUCTS BY $\mathbb{Z}^k$

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Abstract. Among restricted wreath products $G \wr \mathbb{Z}^k$, where $G$ is a finite Abelian group, we find three large classes of groups admitting an automorphism $\varphi$ with finite Reidemeister number $R(\varphi)$ (number of $\varphi$-twisted conjugacy classes). In other words, groups from these classes do not have the $R_\infty$ property.

If a general automorphism $\varphi$ of $G \wr \mathbb{Z}^k$ has a finite order (this is the case for $\varphi$ detected in the first part of the paper) and $R(\varphi) < \infty$, we prove that $R(\varphi)$ coincides with the number of equivalence classes of finite-dimensional irreducible unitary representations of $G \wr \mathbb{Z}^k$, which are fixed by the dual map $[\rho] \mapsto [\rho \circ \varphi]$ (i.e., we prove the conjecture about finite twisted Burnside-Frobenius theorem, TBFT, for these $\varphi$).

1. Introduction

Suppose, $\Gamma$ is a group and $\phi : \Gamma \to \Gamma$ is an endomorphism. Two elements $x, y \in \Gamma$ are $\phi$-conjugate or twisted conjugate, if and only if there exists an element $g \in \Gamma$ such that $y = gx\phi(g^{-1})$.

The corresponding classes are called Reidemeister or twisted conjugacy classes. The number $R(\phi)$ of them is called the Reidemeister number of $\phi$.

The study of Reidemeister numbers is an important problem related with Topological Dynamics, Number Theory and Representation Theory (see [4]). One of the main problems in the field is to prove or disprove the so-called TBFT (a conjecture about the twisted Burnside-Frobenius theory (or theorem)), which has numerous important consequence for Reidemeister zeta function and for other problems in Topological Dynamics (see a more extended discussion in [14]). Namely the problem is to identify $R(\varphi)$ (when $R(\varphi) < \infty$) in a natural way with the number of fixed points of the induced map $\hat{\varphi}$ of an appropriate dual object. In the initial formulation of the conjecture [6], the dual object was the unitary dual $\hat{\Gamma}$ and $\hat{\varphi} : [\rho] \mapsto [\rho \circ \varphi]$. The conjecture about TBFT was proved in many cases, but failed for an example in [13], which led to the new formulation: TBFT$_f$, where $\hat{\Gamma}$ was replaced by its finite-dimensional part, which is evidently invariant under $\hat{\varphi}$. This is the version, which we will study in this paper for a class of groups. In [14] an example of a group that has neither TBFT nor TBFT$_f$ was presented. The most general proved cases of TBFT$_f$ are the case of polycyclic-by-finite groups [11] and the case of nilpotent torsion-free groups of finite Prüfer rank [10].

Another important problem in the field is to localize the class of groups, where one can consider the TBFT conjecture, i.e., where automorphisms with $R(\varphi) < \infty$ do exist. The

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opposite case is called the $R_\infty$ property. It has some topological consequences itself (see e.g. [18]). A part of recent results about Reidemeister classes and $R_\infty$ can be found in [7, 23, 1, 3, 12, 27, 26, 21] (see also an overview in [9]).

We consider the following restricted wreath product $G \wr \mathbb{Z}^k = \Sigma \rtimes \alpha \mathbb{Z}^k$, where $G$ is a finite Abelian group, $\Sigma$ denotes $\oplus_{x \in \mathbb{Z}^k} G_x$, and $\alpha(x)(g_y) = g_{x+y}$. Here $g_y$ is $g \in G \cong G_x$.

The $R_\infty$ property was completely studied for $k = 1$ in [17], for $G = \mathbb{Z}_p$ with a prime $p$ and arbitrary $k$ in [27], for $G = \mathbb{Z}_m$ and arbitrary $k$ in [15]. In all these cases the TBFT$_f$ was proved.

The complexity of the study increases drastically when we move from $k = 1$ to $k > 1$, because $\mathbb{Z}$ has only one non-trivial automorphism in contrast with $\mathbb{Z}^k$. The groups under consideration can be viewed as generalized lamplighter groups. For a generalization of the lamplighter group in other directions, the twisted conjugacy was considered in [25], [24], and other papers.

In the present paper, we prove (Theorem 3.1) that the groups under consideration do not have the $R_\infty$ property in the following three cases:

1) all prime-power components of $G$ for 2 and 3 have multiplicity at least 2;
2) there is no prime-power components for 2 and $k$ is even;
3) all prime-power components of $G$ for 2 have multiplicity at least 2 and $k = 4s$ for some $s$.

To prove this, we construct corresponding examples, and all of them have a finite order. This motivates us to prove the TBFT$_f$ for all groups of the form $G \wr \mathbb{Z}^k$ and their automorphisms of finite order (Corollary 4.2). The proof is based on a description of Reidemeister classes of $\varphi$ as cylindrical sets (Theorem 4.1).

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2. Preliminaries

We start from some general statements about Reidemeister classes of extensions. Suppose, a normal subgroup $H$ of $G$ is $\varphi$-invariant under an automorphism $\varphi : G \to G$ and $p : G \to G/H$ is the natural projection. Then $\varphi$ induces automorphisms $\varphi' : H \to H$ and $\bar{\varphi} : G/H \to G/H$.

Definition 2.1. Denote $C(\varphi) := \{g \in G : \varphi(g) = g\}$, i.e. $C(\varphi)$ is the subgroup of $G$, formed by $\varphi$-fixed elements.

We will use the notation $\tau_g(x) = gxg^{-1}$ for an inner automorphism as well as for its restriction on a normal subgroup.

The following important properties were obtained in [6, 16], see also [11, 18].

Theorem 2.2. For $G, H, \varphi, \varphi'$, and $\bar{\varphi}$ as above, we have the following.

1. Epimorphism: the projection $G \to G/H$ maps Reidemeister classes of $\varphi$ onto Reidemeister classes of $\bar{\varphi}$, in particular $R(\bar{\varphi}) \leq R(\varphi)$;
2. Estimation by fixed elements: if $|C(\bar{\varphi})| = n$, then $R(\varphi') \leq R(\varphi) \cdot n$;
3. Fixed elements-free case: if $C(\bar{\varphi}) = \{e\}$, then each Reidemeister class of $\varphi'$ is an intersection of the appropriate Reidemeister class of $\varphi$ and $H$;
4. Summation: if $C(\bar{\varphi}) = \{e\}$, then $R(\varphi) = \sum_{j=1}^R R(\tau_{g_j} \circ \varphi')$, where $g_1, \ldots, g_R$ are some elements of $G$ such that $p(g_1), \ldots, p(g_R)$ are representatives of all Reidemeister classes of $\bar{\varphi}$, $R = R(\bar{\varphi})$;
Also we will need the following statement [19] (Lemma 4 and the step (2) in the proof of Theorem A'):

**Lemma 2.3.** Suppose, $\Gamma$ is a residually finite group and \( \varphi : \Gamma \to \Gamma \) is an automorphism with \( R(\varphi) < \infty \). Then \( |C(\varphi)| < \infty \).

One can find in [19] an estimation for \( |C(\varphi)| \), but we will not use it.

Passing to a semidirect product \( \Sigma \rtimes Z^k \), we have by [2] that a couple of automorphisms \( \varphi' : \Sigma \to \Sigma \) and \( \varphi : \Sigma \rtimes Z^k/\Sigma \cong Z^k \to Z^k \cong \Sigma \rtimes Z^k/\Sigma \) define an automorphism \( \varphi \) of \( \Sigma \rtimes Z^k \) (not unique) if and only if

\[
\varphi'(\alpha(m)(h)) = \alpha(\varphi(g))(\varphi'(h)), \quad h \in \Sigma, \quad m \in Z^k.
\]

Since \( \Sigma \) is abelian, by [2, p. 207] the mapping \( \varphi_1 \) defined as \( \varphi' \) on \( \Sigma \) and by \( \varphi \) on \( Z^k \subset \Sigma \times Z^k \) is still an automorphism. Moreover, from the following commutative diagrams

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Sigma & \longrightarrow & \Sigma \rtimes Z^k & \longrightarrow & Z^k & \longrightarrow & 0 \\
\varphi' & & \downarrow \varphi & & \downarrow \varphi_1 & & \downarrow \varphi & & \downarrow \varphi \\
0 & \longrightarrow & \Sigma & \longrightarrow & \Sigma \rtimes Z^k & \longrightarrow & Z^k & \longrightarrow & 0
\end{array}
\]

we have \( R(\varphi) = R(\varphi_1) \). Indeed, if \( R(\varphi) = \infty \) then \( R(\varphi) = R(\varphi_1) = \infty \). If \( R(\varphi) < \infty \) then \( C_{\varphi} = \{0\} \) and by Theorem 2.2

\[
R(\varphi) = \sum_{\text{representatives } m \in Z^k \text{ of Reidemeister classes of } \varphi} R(\tau_m \circ \varphi') = R(\varphi_1).
\]

So, without loss of generality in the \( R_\infty \) questions (not in Section 4) we will assume

\[
(2) \quad Z^k \subset A \rtimes Z^k \text{ is } \varphi\text{-invariant and } \varphi|_{Z^k} = \varphi.
\]

This was discussed briefly in [17, Lemma 3.5] in a particular case.

**Lemma 2.4.** An automorphism \( \varphi : G \rtimes Z^k \to G \rtimes Z^k \) has \( R(\varphi) < \infty \) if and only if \( R(\varphi) < \infty \) and \( R(\tau_m \circ \varphi') < \infty \) for any \( m \in Z^k \) (in fact, it is sufficient to verify this for representatives of Reidemeister classes of \( \varphi \)).

**Proof.** Suppose, \( R(\varphi) < \infty \). By Theorem 2.2, we have \( R(\varphi) < \infty \). Then by Lemma 2.3, we obtain \( |C(\varphi)| < \infty \) (in fact, \( |C(\varphi)| = 1 \), because an automorphism of \( Z^k \) can not have finitely many fixed elements except of 0). So, by Theorem 2.2, \( R(\varphi') < \infty \). Considering \( \tau_z \circ \varphi \), which has \( R(\tau_z \circ \varphi) = R(\varphi) < \infty \), instead of \( \varphi \), we obtain in the same way that \( R(\tau_z \circ \varphi') < \infty \).

Conversely, having \( |C(\varphi)| = 1 \), one can apply the summation formula from Theorem 2.2. \( \square \)

**Lemma 2.5.** Suppose, \( \varphi : Z^k \to Z^k \) is an automorphism and \( F : G \to G \) is an automorphism. Then \( \varphi' \) defined by

\[
(3) \quad \varphi'(a_0) = (Fa)_0, \quad \varphi'(a_x) = (Fa)_{\varphi(x)}
\]

satisfies (1) and so defines an automorphism of \( G \rtimes Z^k \).

Evidently the subgroups \( \oplus G_x \), where \( x \) runs over an orbit of \( \varphi \), are \( \varphi' \)-invariant summands of \( \Sigma \).
Proof. It is sufficient to prove (1) on generating elements of the form $a_x$. Then for any $z \in \mathbb{Z}^k$,

$$
\varphi'(\alpha(z)a_x) = \varphi'(a_{x+z}) = (Fa)_{\varphi(z)} \quad \text{and} \quad \alpha(\varphi(z))(Fa)_{\varphi(z)} = \alpha(\varphi(z))\varphi'(a_x)
$$

and (1) is fulfilled. The first equality in (3) is in fact a particular case of the second one. \qed

It is not difficult to prove (see [5]) that, for $\varphi : \mathbb{Z}^k \to \mathbb{Z}^k$ defined by a matrix $M$, one has

$$
R(\varphi) = \# \text{Coker}(\text{Id} - \varphi) = |\det(E - M)|,
$$

if $R(\varphi) < \infty$, and $|\det(E - M)| = 0$ otherwise.

3. SOME CLASSES OF WREATH PRODUCTS WITHOUT $R_\infty$ PROPERTY

**Theorem 3.1.** Suppose, the prime-power decomposition of $G$ is $\oplus_i (\mathbb{Z}_{p_i^r_i})^{d_i}$. Then under each of the following conditions the corresponding wreath products $G \wr \mathbb{Z}^k$ admit an automorphism $\varphi$ with $R(\varphi) < \infty$, i.e. do not have the property $R_\infty$:

- **Case 1:** for all $p_i = 2$ and $p_i = 3$, we have $d_i \geq 2$ (and is arbitrary for primes $> 3$);
- **Case 2:** there is no $p_i = 2$ and $k$ is even;
- **Case 3:** for all $p_i = 2$, we have $d_i \geq 2$ and $k = 4s$ for some $s$.

**Proof.** In each of these cases we will take an automorphism $\varphi : \mathbb{Z}^k \to \mathbb{Z}^k$ with $R(\varphi) < \infty$ (in fact, of finite order) and define $\varphi' : \Sigma \to \Sigma$ with appropriate properties in accordance with Lemmas 2.4 and 2.5.

**Case 1.** In this case we can take $\varphi = -\text{Id} : \mathbb{Z}^k \to \mathbb{Z}^k$ and construct $\varphi$ similarly to [17]. More specifically, note that $R(\varphi) = 2^k$ and define $\varphi' : \Sigma \to \Sigma$ in the following way. The subgroups $G_x \oplus G_{-x}$ will be invariant subgroups of $\varphi'$ and we define

$$
\varphi' : G_x \oplus G_{-x} \to G_x \oplus G_{-x} \text{ as } \left( \begin{array}{cc} 0 & \Psi \\ \Psi & 0 \end{array} \right),
$$

where $\Psi : G \to G$ is defined as a direct sum of blocks of the following types:

$$
F_2 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) : (\mathbb{Z}_q)^2 \to (\mathbb{Z}_q)^2, \quad F_3 = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) : (\mathbb{Z}_q)^3 \to (\mathbb{Z}_q)^3,
$$

where $q$ are some $(p_i)^{r_i}$ and for each summand $\left( \mathbb{Z}_{(p_i)^{r_i}} \right)^{d_i}$ of $G$ ($d_i \geq 2$, $p_i = 2$ or $p_i = 3$) we have $s$ summands $F_2$, if $d_i = 2s$, or $s - 1$ summands $F_2$ and one summand $F_3$, if $d_i = 2s + 1$. For the remaining summands (i.e. for $p_i > 3$) we do not need to group summands in the above way and we can consider $F_1 : \mathbb{Z}_q \to \mathbb{Z}_q$, $1 \mapsto m(q)$ where $q = (p_i)^{n_i}$. This $m = m(q)$ should be taken in such a way that

$$
m^2 \text{ and } 1 - m^2 \text{ are invertible in } \mathbb{Z}_q.
$$

This can be done for $p_i > 3$: one can take $m = 2$ (and impossible for $p_i = 2$ or $3$).

By Lemma 2.5, we defined an automorphism $\varphi$ of $G \wr \mathbb{Z}^k$ in this way (one may assume (2) to have a unique $\varphi'$).

We claim that $R(\tau_{z_i} \circ \varphi') = R(\alpha(z_i) \circ \varphi') = 1$, $i = 1, \ldots, 2^k$. Consequently, by Theorem 2.2, $R(\varphi) = R(\varphi') = 2^k < \infty$. So we need to prove that $\text{Id}_\Sigma - \alpha(z_i) \circ \varphi'$ is an epimorphism, because, for Abelian groups, this is evidently the same as $R(\alpha(z_i) \circ \varphi') = 1$. This homomorphism has a decomposition of $\Sigma$ into invariant subgroups $G_x \oplus G_{-x+z_i}$, because $\alpha(z_i) : G_x \to G_{x+z_i}$, $\varphi' : G_{-x+z_i} \to G_{x+z_i}$ and $\alpha(z_i) : G_{x-z_i} \to G_x$. Note that the subgroups $G_x$ and $G_{-x+z_i}$
coincide if \( z_i = 2x \) (this corresponds to the case of \( G_0 \) for \( \varphi' \)). Thus it is sufficient to verify the epimorphivity for each \( G_x \oplus G_{-x+z_i} \) and for the exceptional case. Passing to summands of \( G \), it is sufficient to verify the epimorphivity of

\[
\begin{pmatrix}
-E & F_3 \\
F_2 & -E
\end{pmatrix}, \quad \begin{pmatrix}
-E & F_3 \\
F_3 & -E
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
-E & F_1 \\
F_1 & -E
\end{pmatrix} = \begin{pmatrix}
-1 & m \\
m & -1
\end{pmatrix}.
\]

The first two are isomorphisms with the explicit inverses

\[
\begin{pmatrix}
-1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & -1 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}^{-1}, \quad \begin{pmatrix}
-2 & 0 & 1 & 1 & 1 & -1 \\
0 & -1 & 1 & 1 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 1 \\
1 & 1 & -1 & -2 & 0 & 1 \\
1 & 0 & 0 & 0 & -1 & 1 \\
-1 & 0 & 1 & 1 & 1 & -1
\end{pmatrix}^{-1}.
\]

For the third one the invertibility follows from (6). For the exceptional case we formally do not need to verify the epimorphity, because it can add only a finite number to \( R(\varphi') \), but we wish to prove our (more strong) claim (this will be helpful for TBFT). So we have to prove, that

\[ F_2 - E, \quad F_3 - E, \quad m - 1 \]

are epimorphisms. This can be done immediately: \( \det(F_2 - E) = 1 \mod 2, \det(F_3 - E) = 1 \mod 2, \) and \( 1 - m^2 = (1 - m)(1 + m) \).

**Case 2.** Now consider the case of even \( k = 2t \) and \( G \) without 2-subgroup. In this case the construction starts as in [27]: we take \( \varphi : \mathbb{Z}^2 \to \mathbb{Z}^2 \) to be the direct sum of \( t \) copies of \( \mathbb{Z}^2 \to \mathbb{Z}^2 \), \( \left( \begin{array}{c}
u \\ v \end{array} \right) \mapsto M \left( \begin{array}{c}
u \\ v \end{array} \right) \), \( M = \left( \begin{array}{cc}0 & 1 \\ -1 & -1 \end{array} \right) \).

Then \( M \) generates a subgroup of \( GL(2, \mathbb{Z}) \), which is isomorphic to \( \mathbb{Z}_3 \) (see, [22, p. 179]). All orbits of \( M \) have length 3 (except of the trivial one) and the corresponding Reidemeister number \( = \det(E - M) = 3 \). Similarly for \( \varphi' \): the length of any orbit is 3 (except of the zero) and \( R(\varphi) = 3^t \). Also

\[(7) \quad M^2 + M + E = \begin{pmatrix}-1 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix}0 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix}1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Now define \( \varphi' \) as a direct sum of actions for \( \mathbb{Z}_q, q = (p_i)^{r_i}, p_i \geq 3 \).

For \( p_i \geq 3 \) choose \( m = m_i \) such that

\[(8) \quad m^3 \text{ and } 1 - m^3 \text{ are invertible in } \mathbb{Z}_q.
\]

This can be done for \( p_i \geq 3 \): one can take \( m = 3 \) for \( p_i = 7 \) and \( m = 2 \) in the remaining cases (and impossible for \( p_i = 2 \)). Define \( \varphi'(a_0) = (ma)_0 \) and \( \varphi'(a_x) = (ma)_{\varphi}(a) \), where \( a \in \mathbb{Z}_q \subset G \). So, the corresponding subgroup \( \oplus_{g \in \mathbb{Z}_q} (\mathbb{Z}_q)_g \subset \varphi' \)-invariant and decomposed into infinitely many invariant summands \( (\mathbb{Z}_q)_g \oplus (\mathbb{Z}_q)_{\varphi(g)} \oplus (\mathbb{Z}_q)_{\varphi(g)} \text{ isomorphic to } (\mathbb{Z}_q)^3 \) (over generic orbits of \( \varphi \)) and one summand \( (\mathbb{Z}_q)_0 \text{ (over the trivial orbit) } \). Then the corresponding restrictions of \( \varphi' \) and \( 1 - \varphi' \) can be written as multiplication by

\[
\begin{pmatrix}0 & 0 & m \\ m & 0 & 0 \\ 0 & m & 0 \end{pmatrix}, \quad \begin{pmatrix}1 & 0 & -m \\ -m & 1 & 0 \\ 0 & -m & 1 \end{pmatrix}, \quad \text{and } m, \quad 1 - m,
\]
respectively. The three-dimensional mappings are isomorphisms by (8). Since an element $\ell$ is not invertible in $\mathbb{Z}(p^{i})$, if and only if $\ell = u \cdot p_{i}$, the invertibility of one-dimensional mappings follows from (8) and the factorization $1 - m^{3} = (1 - m)(1 + m + m^{2})$. (This construction gives a more explicit presentation of a part of proof of [27, Theorem 4.1])

For $\tau_{z} \circ \varphi'$ we have

$$(\tau_{z} \circ \varphi')(g_{x}) = (mg)\varphi(x) + z, \quad (\tau_{z} \circ \varphi')(g\varphi(x) + z) = (mg)\varphi(x) + \varphi_{z} + z,$$

because $\varphi^3(x) = x$ and $\varphi^2 z + \varphi z = z$ by (7). So $\tau_{z} \circ \varphi'$ has the same matrices as $\varphi'$, but on new invariant summands $(\mathbb{Z}_{q})_{x} \oplus (\mathbb{Z}_{q})_{\varphi(x) + z} \oplus (\mathbb{Z}_{q})_{\varphi(x) + \varphi_{z} + z}$. Similarly for the exceptional orbit. This completes the proof of this case.

**Case 3:** when $d_{i} > 1$ for $p_{i} = 2$ and $k = 4s$. Using the cyclotomic polynomial we can define (similarly to the above $M$) an element of order 5 in $GL(4, \mathbb{Z})$

$$M_{4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(see e.g. [20] for an elementary introduction). For any $k = 4s$, let $M \in GL(k, \mathbb{Z})$ be the direct sum of $s$ copies of $M_{4}$. Let $\varphi: \mathbb{Z}^{k} \rightarrow \mathbb{Z}^{k}$ be defined by $M$. One can calculate

$$\det(M_{4} - E) = 5, \quad \det(M - E) = 5^{s}.$$ 

Hence, by (4), $R(\varphi) = 5^{s} < \infty$. The length of any non-trivial orbit is 5, hence an odd number.

Similarly to $M$, one can verify that

$$(M_{4})^{4} + (M_{4})^{3} + (M_{4})^{2} + M_{4} + E = 0.$$ 

This can be also deduced from the fact that the characteristic polynomial of the “companion matrix” of a polynomial $p$ is just $p$.

For $p$-power components $\mathbb{Z}_{p^{i}}$ with $p > 2$, we define $\varphi'$ (as above) by $a_{0} \mapsto (p - 1)a_{0}$. Then, for an orbit $u, \varphi u, \ldots, \varphi^{r} u$, we need to verify (for finiteness of $R(\varphi)$) that $(p - 1)^{y}$ as a homomorphism $\mathbb{Z}_{p^{i}} \rightarrow \mathbb{Z}_{p^{i}}$ has no non-trivial fixed elements, i.e. $(p - 1)^{y} \not\equiv 1 \mod p$. This is fulfilled because, for an odd $\gamma$, $(p - 1)^{\gamma} - 1 \equiv -2 \not\equiv 0 \mod p$.

For 2-power components $\mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}}$, we define $\varphi'$ by $a_{0} \mapsto F_{2}a_{0}$ (as in (5)). Then, for an orbit $u, \varphi u, \ldots, \varphi^{r} u$, we need to verify that $(F_{2})^{y}$ as a homomorphism $\mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}} \rightarrow \mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}}$ has no non-trivial fixed elements. Here we need to use not only the fact that $\gamma$ is odd, but its more specific form: $\gamma = 5$. In particular it can not be divided by 3 =order of $F_{2} \mod 2$. Hence $(F_{2})^{5} = (F_{2})^{5} = (F_{2})^{2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mod 2$. It has no non-trivial fixed elements mod $2^{i}$ for any $i$.

For 2-power components $\mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}}$, we define $\varphi'$ by $a_{0} \mapsto F_{3}a_{0}$ (as in (5)). Then, for an orbit of $\varphi$ of length $\gamma$, we need to verify that $(F_{3})^{y}$ as a homomorphism $\mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}} \rightarrow \mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}} \oplus \mathbb{Z}_{2^{i}}$ has no non-trivial fixed elements. One can verify, for $i = 1$, i.e. for $2^{i} = 2$, that the order of $F_{3}$ is relatively prime with 5, namely it is equal to 7. Moreover, $(F_{3})^{j}$, $j = 1, \ldots, 6$, has no non-trivial fixed elements. The absence of non-trivial fixed elements is equivalent to $\det((F_{3})^{j} - E) \not\equiv 0 \mod 2$. Then $\det((F_{3})^{j} - E) \not\equiv 0 \mod 2^{i}$. Hence, for $i > 1$ these automorphisms still have no non-trivial fixed elements. The elements $(F_{3})^{i}$,
In particular, this covers all automorphisms, which were considered in [27].

Remark 4.3. number (see [11] and [8]).

Then by Lemma 2.3,

Proof of Theorem. Suppose that \( \varphi \) is an automorphism of the restricted wreath product \( G \wr \mathbb{Z}^k = \oplus_{m \in \mathbb{Z}^k} G_m \rtimes_{\alpha} \mathbb{Z}^k \), where \( G \) is a finite abelian group. Suppose that \( \varphi \) is of finite order. Then \( R(\varphi') \) is 1 or \( \infty \).

Corollary 4.2. In particular, \( \varphi \) has the TBFT property.

Proof of Corollary. By Lemma 2.4, \( R(\varphi) < \infty \) implies \( R(\varphi') < \infty \). Hence, by Theorem 4.1, \( R(\varphi') = 1 \). Considering \( \tau_z \circ \varphi \) instead of \( \varphi \) from the very beginning, we see that \( R(\tau_z \circ \varphi') = 1 \), for any \( z \in \mathbb{Z}^k \). Thus, by Theorem 2.2, Reidemeister classes \( \{g\}_\varphi \) of \( \varphi \) are pull-backs of Reidemeister classes \( \{z\}_\varphi \) of \( \varphi \) under the natural projection \( \pi : G \wr \mathbb{Z}^k \to \mathbb{Z}^k \), i.e. \( \{g\}_\varphi = \pi^{-1}(\{\pi(g)\}_\varphi) \). So, if classes of \( \varphi \) are separated by an epimorphism \( f : \mathbb{Z}^k \to F \) onto a finite group \( F \), then classes of \( \varphi \) are separated by \( f \circ \pi \). It remains to use the equivalence between TBFT and separability of Reidemeister classes in the case of finite Reidemeister number (see [11] and [8]).

Remark 4.3. In particular, this covers all automorphisms, which were considered in [27]. Indeed, it was proved there, that all orbits are finite and their length is equal to the length of orbits of \( \varphi \). But the structure of \( \mathbb{Z}^k \) implies that \( \varphi \) has finite order (consider generators). Hence, \( \varphi' \) and \( \varphi \) are of finite order.

Proof of Theorem. Suppose, \( R(\varphi') > 1 \). Then there exists an element \( \sigma \in \Sigma \) such that \( \sigma \notin \text{Im}(\text{Id} - \varphi') \). Moreover, \( \sigma \notin \text{Im}(\text{Id} - \varphi'_s) \), where \( \varphi'_s \) is the restriction of \( \varphi' \) onto the \( \varphi' \)-invariant subgroup \( \Sigma_s \) generated by \( \sigma \). In particular, \( R(\varphi'_s) > 1 \). By the supposition \( \Sigma_s \) is a finite group with generators \( \sigma, \varphi'_s(\sigma), \ldots, (\varphi')^s(\sigma) \) for some \( s \). Hence, \( \varphi'_s \) has a nontrivial fixed element \( \sigma_0 \), \( \varphi'_s(\sigma_0) = \sigma_0 \) and \( \sigma_0 \neq 0 \). For an element \( m \in \mathbb{Z}^k \) consider the orbit

\[ \alpha(m)\sigma_0, \quad (\varphi')^t(\alpha(m)\sigma_0) = \alpha(\varphi(m))\sigma_0, \quad (\varphi')^t(\alpha(m)\sigma_0) = \alpha(\varphi(m))\sigma_0, \quad \varphi'^{t+1}(m) = m. \]

Then \( (\varphi')^{t+1}(\alpha(m)\sigma_0) = \alpha(m)\sigma_0 \). Passing from \( m \) to \( nm \), \( n \in \mathbb{Z} \), \( m \in \mathbb{Z}^k \), if necessary, we can assume that the supports in \( \mathbb{Z}^k \) of \( \sigma_0, \alpha(\varphi'(nm))\sigma_0, j = 0, \ldots, t \), do not intersect. Then \( \sum_{j=1}^t \alpha(\varphi'(nm))\sigma_0 \) is a fixed element of \( \varphi' \), which is distinct from 0 and \( \sigma_0 \). Increasing \( n \) “in sufficiently large steps” we obtain infinitely many distinct fixed elements in the same way.

Then by Lemma 2.3, \( R(\varphi') = \infty \).

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