On \( l^p \)-multipliers of functions analytic in the disk

We consider bounded analytic functions in domains generated by sets that have Littlewood–Paley property. We show that each such function is an \( l^p \)-multiplier.

References: 12 items.

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Given a function \( f \) analytic in the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) of the complex plane \( \mathbb{C} \), consider its Taylor expansion:

\[
    f(z) = \sum_{n \geq 0} \hat{f}(n) z^n, \quad z \in D. \tag{1}
\]

For \( 1 \leq p \leq \infty \) let \( A^+_p(D) \) denote the space of all functions (1) such that the sequence of Taylor coefficients \( \hat{f} = \{ \hat{f}(n), \ n = 0, 1, \ldots \} \) belongs to \( l^p \). For \( f \in A^+_p(D) \) we put \( \|f\|_{A^+_p(D)} = \|\hat{f}\|_{l^p} \). A function \( m \) analytic in \( D \) is called an \( l^p \)-multiplier if for every function \( f \) in \( A^+_p(D) \) we have \( m \cdot f \in A^+_p(D) \). We denote the class of all these multipliers by \( M^+_p(D) \). This class is a Banach algebra with respect to the natural norm

\[
    \|m\|_{M^+_p(D)} = \sup_{\|f\|_{A^+_p(D)} \leq 1} \|m \cdot f\|_{A^+_p(D)}
\]

and the usual multiplication of functions. The classes \( M^+_p(D) \) were studied in [1]–[6].

We note that the case when \( p \neq 1, \infty, 2 \) is of a special interest. It is well known that \( M^+_p(D) = M^+_q(D) \) if \( 1/p + 1/q = 1 \), and

\[
    A^+_1(D) = M^+_1(D) = M^+_\infty(D) \subseteq M^+_p(D) \subseteq M^+_2(D) = H^\infty(D),
\]

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2 There is a minor inconsistency in § 6 of the author’s work [6]. Instead of the written “the Poisson integral” there should be “the Riesz projection \( \sum_{k=-\infty}^{\infty} c_k e^{ikt} \to \sum_{k \geq 0} c_k z^k \).
where $H^\infty(D)$ is the Hardy space of bounded analytic functions in $D$.

Let $\Omega \subseteq \mathbb{C}$ be an open domain which contains the disk $D$. We shall present a class of domains $\Omega$ such that each bounded analytic function in $\Omega$ belongs to $M_p^+(D)$. The case when $\Omega$ contains the closure of $D$ is trivial; in this case each bounded analytic function in $\Omega$ belongs to $A_1^+(D)$ and hence belongs to $M_p^+(D)$ for all $p$, $1 \leq p \leq \infty$. The nontrivial case is the case when the boundary of $\Omega$ has common points with the boundary $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ of the disk $D$.

It was shown by Vinogradov [2] that if $r > 1$, $0 \leq \alpha < \pi/2$, and $m$ is a bounded analytic function in the domain

$$\Omega_0 = \{z \in \mathbb{C} : |z| < r, \alpha < \arg(z - 1) < 2\pi - \alpha\}, \quad (2)$$

then $m \in \bigcap_{1 < p < \infty} M_p^+(D)$. Using this result Vinogradov gave the first examples of nontrivial (i.e., infinite) Blaschke products in $M_p^+(D)$. Note that the boundary of a domain (2) has only one common point with the boundary of $D$ (namely the point $z = 1$). As we shall see, a statement similar to Vinogradov’s result holds for domains of a much wider class. Functions analytic in the domains considered below can have uncountable set of singularities on the boundary of the disk.\footnote{We note by the way that the condition $\alpha < \pi/2$ in the Vinogradov theorem is essential. For example, the function $S(z) = \exp\{(z + 1)/(z - 1)\}$ is bounded in the halfplane $\{z \in \mathbb{C} : \Re z < 1\}$, but as it was shown by Verbitskiï [4] $S$ belongs to $M_p^+(D)$ only in the trivial case $p = 2$.}

As is usual, for an arbitrary domain $\Omega \subseteq \mathbb{C}$ by $H^\infty(\Omega)$ we denote the Hardy space of all bounded analytic functions in $\Omega$. For $g \in H^\infty(\Omega)$ we put

$$\|g\|_{H^\infty(\Omega)} = \sup_{z \in \Omega} |g(z)|.$$

Let $J$ be an arc in the boundary circle $\partial D$. Assume that the length $|J|$ of $J$ is strictly less than $\pi$. Let $T_J$ be an arbitrary open isosceles triangle, whose base is the chord that spans the arc $J$, and whose sides lie outside of $D$. Denote by $\theta_T$, the angle between $\partial D$ and a side of $T_J$.

Consider an arbitrary closed set $F \subseteq \partial D$. Let $\tau(F)$ be the family of all arcs complimentary to $F$ (i.e., of all connected components of the compliment $\partial D \setminus F$). We assume that each arc of the family $\tau(F)$ has length strictly less than $\pi$. Consider the domain

$$\Omega_F = D \cup \bigcup_{J \in \tau(F)} T_J,$$
and require in addition that
\[ \inf_{J \in \tau(F)} \theta_{T_J} > 0. \] (3)

We call a domain \( \Omega_F \), obtained in this way, a star-like domain generated by \( F \).

We shall show that under a certain condition imposed on a set \( F \subseteq \partial D \), every function, bounded and analytic in \( \Omega_F \), belongs to \( M^+_p(D) \).

Let \( E \) be a closed set of Lebesque measure zero in the line \( \mathbb{R} \). Consider the family \( \tau(E) \) of all intervals complimentary to \( E \) (i.e., of all connected components of the compliment \( \mathbb{R} \setminus E \)). For an arbitrary interval \( I \subseteq \mathbb{R} \) define the operator \( S_I \) by
\[ \widehat{S_I(f)} = 1_I \hat{f}, \quad f \in L^p \cap L^2(\mathbb{R}), \]
where \( \hat{\cdot} \) is the Fourier transform and \( 1_I \) is the characteristic function of \( I \) (i.e., \( 1_I(t) = 1 \) for \( t \in I \), \( 1_I(t) = 0 \) for \( t \notin I \)). Following [7], we say that a set \( E \) has property \( LP(p) \) \((1 < p < \infty)\) if the corresponding Littlewood–Paley quadratic function
\[ S(f) = \left( \sum_{I \in \tau(E)} |S_I(f)|^2 \right)^{1/2} \]
satisfies
\[ c_1(p)\|f\|_{L^p(\mathbb{R})} \leq \|S(f)\|_{L^p(\mathbb{R})} \leq c_2(p)\|f\|_{L^p(\mathbb{R})}, \quad f \in L^p(\mathbb{R}) \]
(with positive constants \( c_1(p), c_2(p) \) independent of \( f \)). In the case when \( E \) has property \( LP(p) \) for all \( p, \ 1 < p < \infty \), we say that \( E \) has property \( LP \).

Let now \( F \) be a closed set of measure zero in the boundary circle \( \partial D \). We say that \( F \) has property \( LP(p) \) or property \( LP \) if \( F = \{e^{it}, \ t \in E \} \), where \( E \subseteq [0, 2\pi] \) is a set that has property \( LP(p) \) or property \( LP \), respectively.

Remark 1. A classical example of an infinite set \( E \subseteq \mathbb{R} \) that has property \( LP \) is \( E = \{\pm 2^k, \ k \in \mathbb{Z}\} \cup \{0\} \), where \( \mathbb{Z} \) is the set of integers. At the same time there exist uncountable sets that have property \( LP \). This was first established by Hare and Klemes [8]. The existence of such sets was also noted in [9], see details in [10, § 4]. Let us state the corresponding result for sets in \( \partial D \). For each \( p, \ 1 < p < \infty \), there is a constant \( \beta_p \) \((0 < \beta_p < 1)\) such that the following holds. Let \( F \subseteq \partial D \) be a closed set of measure
zero. Suppose that the arcs $J_k, \ k = 1, 2, \ldots,$ complimentary to $F$, being enumerated so that their lengths do not increase, satisfy $|J_{k+1}|/|J_k| \leq \beta_p$ for all sufficiently large $k$. Then $F$ has property LP$(p)$. This in turn implies that if $\lim_{k \to \infty} |J_{k+1}|/|J_k| = 0$, then $F$ has property LP.

The result of this note is the following theorem.

**Theorem.** Suppose that a set $F \subseteq \partial D$ has property LP$(p)$, and $\Omega_F$ is a star-like domain generated by $F$. Then $H^\infty(\Omega_F) \subseteq M^+_p(D)$. If $F$ has property LP, then $H^\infty(\Omega_F) \subseteq \bigcap_{1 < p < \infty} M^+_p(D)$.

**Proof.** Let $G$ be an Abelian group and let $\Gamma$ be the group dual to $G$. Consider a function $m \in L^\infty(\Gamma)$ and the operator $Q$ defined by

$$\hat{Q}f = m\hat{f}, \quad f \in L^p \cap L^2(G),$$

where $\hat{\cdot}$ stands for the Fourier transform on $G$. The function $m$ is called an $L^p$-Fourier multiplier if the corresponding operator $Q$ is a bounded operator from $L^p(G)$ to itself $(1 \leq p \leq \infty)$. Denote the class of all these multipliers by $M_p(\Gamma)$ and put $\|m\|_{M_p(\Gamma)} = \|Q\|_{L^p(G) \to L^p(G)}$. The relation between the multipliers on the line $\mathbb{R}$ and on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is well known [11] (see also [12]). We shall need the Jodeit theorem [12] on the periodic extension of multipliers. According to this theorem, if $f \in M_p(\mathbb{R})$ is a function that vanishes outside of the interval $[0, 2\pi]$ and $g$ is the $2\pi$-periodic function that coincides with $f$ on $[0, 2\pi]$, then $g \in M_p(\mathbb{T})$. Note that there is a direct relation between the spaces $M^+_p(D)$ and $M_p(\mathbb{T})$. Given a function $m \in H^\infty(D)$ consider its (non-tangential) boundary function $m^*(t) = m(e^{it})$. The conditions $m \in M^+_p(D)$ and $m^* \in M_p(\mathbb{T})$ are equivalent [3] (see also [5]).

We shall also need the following statement. Let $E \subseteq \mathbb{R}$ be a set that has property LP$(p)$. Suppose that a function $f \in L^\infty(\mathbb{R})$ is continuously differentiable on each interval complimentary to $E$, and its derivative $f'$ satisfies

$$|f'(t)| \leq \frac{c}{\operatorname{dist}(t, E)}, \quad t \in \mathbb{R} \setminus E,$$

where $\operatorname{dist}(t, E)$ stands for the distance from a point $t$ to the set $E$ and $c > 0$ does not depend on $t$. Then $f \in M_p(\mathbb{R})$. This result of Sjörgen and Sjölin [7] generalizes the well known Mikhlin–Hörmander theorem.
We note now that condition (3) implies the existence of a constant \( c = c(\Omega_F) > 0 \) such that if \( e^{it} \in \partial D \setminus F \), then the circle centered at \( e^{it} \) and of radius \( r(t) = c \cdot \text{dist}(e^{it}, F) \) lies in \( \Omega_F \). Denote this circle by \( \gamma(t) \). Let \( m \in H^\infty(\Omega_F) \). Consider an arc \( J \) complimentary to \( F \). Let \( e^{it} \in J \). Consider the corresponding circle \( \gamma(t) \). For an arbitrary point \( z \) that lies inside \( \gamma(t) \) we have

\[
m'(z) = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{m(\zeta)}{(\zeta - z)^2} d\zeta.
\]

In particular,

\[
m'(e^{it}) = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{m(\zeta)}{(\zeta - e^{it})^2} d\zeta.
\]

Hence, for the derivative \((m^*)'\) of the boundary function \( m^*(t) = m(e^{it}) \) we obtain

\[
|(m^*)'(t)| = |ie^{it}m'(e^{it})| = \left| \frac{1}{2\pi i} \int_{\gamma(t)} \frac{m(\zeta)}{(\zeta - e^{it})^2} d\zeta \right| \leq \frac{1}{2\pi} \int_{\gamma(t)} \frac{|m(\zeta)|}{|\zeta - e^{it}|^2} d\zeta \leq \frac{1}{2\pi} 2\pi r(t) \| m \|_{H^\infty(\Omega_F)} \frac{1}{(r(t))^2} =
\]

\[
= c_1(\Omega_F) \| m \|_{H^\infty(\Omega_F)} \frac{1}{\text{dist}(e^{it}, F)}.
\]

Let \( E \subseteq [0, 2\pi] \) be a set such that \( F = \{e^{it}, t \in E\} \) and \( E \) has property \( \text{LP}(p) \). Without loss of generality we can assume that \( E \) contains the points 0 and 2\( \pi \). Define a function \( f \) on \( \mathbb{R} \) by \( f(t) = 1_{[0,2\pi]}(t)m^*(t), \ t \in \mathbb{R} \). We see that (see (5)) the function \( f \) satisfies (4). Therefore, by the Sjögren–Sjölin theorem, we have \( f \in M_p(\mathbb{R}) \). Hence, using the Jodeit theorem, we obtain \( m^* \in M_p(\mathbb{T}) \). Taking into account the relation between multipliers on \( \mathbb{T} \) and multipliers of functions analytic in the disk \( D \), we obtain \( m \in M^+_p(D) \).

**Remark 2.** As far as the author knows, the question on the existence of a set that has property \( \text{LP}(p) \) for some \( p, p \neq 2 \), but does not have property \( \text{LP} \) is open.

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