Graded PI-exponents of simple Lie superalgebras

Dušan Repovš and Mikhail Zaicev

Abstract. We study $\mathbb{Z}_2$-graded identities of simple Lie superalgebras over a field of characteristic zero. We prove the existence of the graded PI-exponent for such algebras.

1. Introduction

Let $A$ be an algebra over a field $F$ with char $F=0$. A natural way of measuring the polynomial identities satisfied by $A$ is by studying the asymptotic behaviour of its sequence of codimensions $\{c_n(A)\}$, $n=1, 2, \ldots$. If $A$ is a finite dimensional algebra then the sequence $\{c_n(A)\}$ is exponentially bounded. In this case it is natural to ask the question about existence of the limit

$$\lim_{n \to \infty} \sqrt[n]{c_n(A)}$$

called the PI-exponent of $A$. Such question was first asked for associative algebras by Amitsur at the end of 1980’s. A positive answer was given in [6]. Subsequently it was shown that the same problem has a positive solution for finite dimensional Lie algebras [14], for finite dimensional alternative and Jordan algebras [5] and for some other classes. Recently it was shown that in general the limit (1) does not exist even if $\{c_n(A)\}$ is exponentially bounded [15]. The counterexample constructed in [15] is infinite dimensional whereas for finite dimensional algebras the problem of the existence of the PI-exponent is still open. Nevertheless, if dim $A < \infty$ and $A$ is simple then the PI-exponent of $A$ exists as it was proved in [8].

If in addition $A$ has a group grading then graded identities, graded codimensions and graded PI-exponents can also be considered. In this paper we discuss

---

The first author was supported by the SRA grants P1-0292-0101, J1-5435-0101 and J1-6721-0101. The second author was partially supported by RFBR grant 13-01-00234a. We thank the referees for comments and suggestions.
graded codimensions behaviour for finite dimensional simple Lie superalgebras. Graded codimensions of finite dimensional Lie superalgebras were studied in a number of papers (see for example, [11] and [12]). In particular, in [11] an upper bound of graded codimension growth was found for one of the series of simple Lie superalgebras.

In the present paper we prove that the graded PI-exponent of any finite dimensional simple Lie superalgebra always exists. All details concerning numerical PI-theory can be found in [7].

2. Main constructions and definitions

Let \( L = L_0 \oplus L_1 \) be a Lie superalgebra. Elements from the component \( L_0 \) are called \textit{even} and elements from \( L_1 \) are called \textit{odd}. Denote by \( \mathcal{L}(X,Y) \) a free Lie superalgebra with infinite sets of even generators \( X \) and odd generators \( Y \). A polynomial \( f = f(x_1, ..., x_m, y_1, ..., y_n) \in \mathcal{L}(X,Y) \) is said to be a graded identity of Lie superalgebra \( L = L_0 \oplus L_1 \) if \( f(a_1, ..., a_m, b_1, ..., b_n) = 0 \) whenever \( a_1, ..., a_m \in L_0, b_1, ..., b_n \in L_1 \).

Denote by \( \text{Id}^{gr}(L) \) the set of all graded identities of \( L \). Then \( \text{Id}^{gr}(L) \) is an ideal of \( \mathcal{L}(X,Y) \). Given non-negative integers \( 0 \leq k \leq n \), let \( P_{k,n-k} \) be the subspace of all multilinear polynomials \( f = f(x_1, ..., x_k, y_1, ..., y_{n-k}) \in \mathcal{L}(X,Y) \) of degree \( k \) on even variables and of degree \( n-k \) on odd variables. Then \( P_{k,n-k} \cap \text{Id}^{gr}(L) \) is the subspace of all multilinear graded identities of \( L \) of total degree \( n \) depending on \( k \) even variables and \( n-k \) odd variables. Denote also by \( P_{k,n-k}(L) \) the quotient

\[
P_{k,n-k}(L) = \frac{P_{k,n-k}}{P_{k,n-k} \cap \text{Id}^{gr}(L)}.
\]

Then the partial graded \((k, n-k)\)-codimension of \( L \) is

\[
c_{k,n-k}(L) = \dim P_{k,n-k}(L)
\]

and the total graded \( n \)th codimension of \( L \) is

\[
c^{gr}_n(L) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(L).
\]

If the sequence \( \{c^{gr}_n(L)\}_{n \geq 1} \) is exponentially bounded then one can consider the related bounded sequence \( \sqrt[n]{c^{gr}_n(L)} \). The latter sequence has the following lower and upper limits

\[
\exp^{gr}(L) = \liminf_{n \to \infty} \sqrt[n]{c^{gr}_n(L)} \quad \text{and} \quad \exp^{gr}(L) = \limsup_{n \to \infty} \sqrt[n]{c^{gr}_n(L)}
\]
called the \textit{lower} and \textit{upper} PI-exponents of \( L \), respectively. If the ordinary limit exists, it is called the (ordinary) \textit{graded} PI-exponent of \( L \),

\[
\exp^{gr}(L) = \lim_{n \to \infty} n \sqrt[n]{c^{gr}_n(L)}.
\]

Symmetric groups and their representations play an important role in the theory of codimensions. In particular, in the case of graded identities one can consider the \( S_k \times S_{n-k} \)-action on multilinear graded polynomials. Namely, the subspace \( P_{k,n-k} \subseteq \mathcal{L}(X,Y) \) has a natural structure of \( S_k \times S_{n-k} \)-module where \( S_k \) acts on even variables \( x_1, \ldots, x_k \) while \( S_{n-k} \) acts on odd variables \( y_1, \ldots, y_{n-k} \). Clearly, \( P_{k,n-k} \cap \text{Id}^{gr}(L) \) is the submodule under this action and we get an induced \( S_k \times S_{n-k} \)-action on \( P_{k,n-k} \). The character \( \chi_{k,n-k}(L) = \chi(P_{k,n-k}(L)) \) is called \( (k,n-k) \) cocharacter of \( L \). Since \( \text{char} \ F = 0 \), this character can be decomposed into the sum of irreducible characters

\[
\chi_{k,n-k}(L) = \sum_{\lambda \vdash k, \mu \vdash n-k} m_{\lambda,\mu} \chi_{\lambda,\mu}
\]

where \( \lambda \) and \( \mu \) are partitions of \( k \) and \( n-k \), respectively. All details concerning representations of symmetric groups can be found in \cite{9}. An application of \( S_n \)-representations in PI-theory can be found in \cite{1}, \cite{3}, \cite{7}.

Recall that an irreducible \( S_k \times S_{n-k} \)-module with the character \( \chi_{\lambda,\mu} \) is the tensor product of \( S_k \)-module with the character \( \chi_\lambda \) and \( S_{n-k} \)-module with the character \( \chi_\mu \). In particular, the dimension \( \deg \chi_{\lambda,\mu} \) of this module is the product \( d_\lambda d_\mu \) where \( d_\lambda = \deg \chi_\lambda, d_\mu = \deg \chi_\mu \). Taking into account multiplicities \( m_{\lambda,\mu} \) in (3) we get the relation

\[
c_{k,n-k}(L) = \sum_{\lambda \vdash k, \mu \vdash n-k} m_{\lambda,\mu} d_\lambda d_\mu.
\]

A number of irreducible components in the decomposition of \( \chi_{k,n-k}(L) \), i.e. the sum

\[
l_{k,n-k}(L) = \sum_{\lambda \vdash k, \mu \vdash n-k} m_{\lambda,\mu}
\]

is called the \((k,n-k)\)-colength of \( L \). The \textit{total graded colength} \( l^{gr}_n(L) \) is

\[
l^{gr}_n(L) = \sum_{k=0}^{n} l_{k,n-k}(L).
\]

Now let \( L \) be a finite dimensional Lie superalgebra, \( \dim L = d \). Then

\[
c^{gr}_n(L) \leq d^n.
\]
by the results of [2] (see also [4]). On the other hand, there exists a polynomial \( \varphi \) such that
\[
\limsup_{n \to \infty} n^{\varphi(n)} \leq \varphi(n)
\]
for all \( n=1, 2, \ldots \) as it was mentioned in [11]. Note also that \( m_{\lambda, \mu} \neq 0 \) in (3) only if \( \lambda \vdash k, \mu \vdash n - k \) are partitions with at most \( d \) components, that is \( \lambda = (\lambda_1, \ldots, \lambda_p), \mu = (\mu_1, \ldots, \mu_q) \) and \( p, q \leq d = \dim L \).

Since all partitions under our consideration are of the height at most \( d \), we will use the following agreement. If say, \( \lambda \) is a partition of \( k \) with \( p < d \) components then we will write \( \lambda = (\lambda_1, \ldots, \lambda_p) \) anyway, assuming that \( \lambda_p+1 = \ldots = \lambda_d = 0 \).

For studying asymptotic behaviour of codimensions it is convenient to use the following function defined on partitions. Let \( \nu \) be a partition of \( m \), \( \nu = (\nu_1, \ldots, \nu_d) \). We introduce the following function of \( \nu \):
\[
\Phi(\nu) = \frac{1}{(\frac{\nu_1}{m})^{\frac{\nu_1}{\nu}} \ldots (\frac{\nu_d}{m})^{\frac{\nu_d}{\nu}}}
\]
The values \( \Phi(\nu)^m \) and \( d_\nu = \deg \chi_\nu \) are very close in the following sense.

**Lemma 2.1.** [8, Lemma 1] Let \( m \geq 100 \). Then
\[
\frac{\Phi(\nu)^m}{m^{d^2 + d}} \leq d_\nu \leq m \Phi(\nu)^m.
\]

Function \( \Phi \) has also the following useful property. Let \( \nu \) and \( \rho \) be two partitions of \( m \) with the corresponding Young diagrams \( D_\nu, D_\rho \). We say that \( D_\rho \) is obtained from \( D_\nu \) by pushing down one box if there exist \( 1 \leq i < j \leq d \) such that \( \rho_i = \nu_i - 1, \rho_j = \nu_j + 1 \) and \( \rho_t = \nu_t \) for all remaining \( 1 \leq t \leq d \).

**Lemma 2.2.** (see [8, Lemma 3], [16, Lemma 2]) Let \( D_\rho \) be obtained from \( D_\nu \) by pushing down one box. Then \( \Phi(\rho) \geq \Phi(\nu) \).

3. Existence of graded PI-exponents

Throughout this section let \( L = L_0 \oplus L_1 \) be a finite dimensional simple Lie superalgebra, \( \dim L = d \). Then by (5) its upper graded PI-exponent exists,
\[
a = \exp^{gr}(L) = \limsup_{n \to \infty} \sqrt[n]{c_{n}^{gr}(L)}.
\]
Note that the even component \( L_0 \) of \( L \) is not solvable since \( L \) is simple (see [13, Chapter 3, §2, Proposition 2]).

We shall need the following fact.
Remark 3.1. Let $G$ be a non-solvable finite dimensional Lie algebra over a field $F$ of characteristic zero. Then the ordinary PI-exponent of $G$ exists and is an integer not less than 2.

Proof. It is known that $c_n(G)$ is either polynomially bounded or it grows exponentially not slower that $2^n$ (see [10]). The first option is possible only if $G$ is solvable. On the other hand $\exp(G)$ always exists and is an integer [14] therefore we are done. □

By the previous remark $P_{n,0}(L) \gtrsim 2^n$ asymptotically and then

$$a \geq 2.$$  

The following lemma is the key technical step in the proof of our main result.

Lemma 3.2. For any $\varepsilon > 0$ and any $\delta > 0$ there exists an increasing sequence of positive integers $n_0, n_1, \ldots$ such that

(i) $\sqrt[n_{q+1}]{c_{n_q}^{gr}(L)} > (1-\delta)(a-\varepsilon)$ for all $n=q, q=1, 2, \ldots$,

(ii) $n_{q+1} - n_q \leq n_0 + d$.

Proof. Fix $\varepsilon, \delta > 0$. Since $a$ is an upper limit there exist infinitely many indices $n_0$ such that

$$c_{n_0}^{gr}(L) > (a-\varepsilon)^{n_0}.$$  

Fixing one of $n_0$ we can find an integer $0 \leq k_0 \leq n_0$ such that

$$\left(\begin{array}{c} n_0 \\ k_0 \end{array}\right) c_{k_0, n_0-k_0}^{gr}(L) > \frac{1}{n_0+1}(a-\varepsilon)^{n_0} > \frac{1}{2n_0}(a-\varepsilon)^{n_0}$$  

(see (2)). Relation (6) shows that

$$\sum_{\lambda+\mu \vdash n-k} m_{\lambda, \mu} \leq \varphi(n)$$  

for any $0 \leq k \leq n$ where $m_{\lambda, \mu}$ are taken from (3). Then (4) implies the existence of partitions $\lambda \vdash k_0, \mu \vdash n_0 - k_0$ such that

$$\left(\begin{array}{c} n_0 \\ k_0 \end{array}\right) d_{\lambda} d_{\mu} > \frac{1}{2n_0 \varphi(n_0)}(a-\varepsilon)^{n_0}.$$  

The latter inequality means that there exists a multilinear polynomial

$$f = f(x_1, \ldots, x_{k_0}, y_1, \ldots, y_{n_0-k_0}) \in P_{k_0, n_0-k_0}$$
such that $F[S_{k_0} \times S_{n_0-k_0}]$ is an irreducible $F[S_{k_0} \times S_{n_0-k_0}]$-submodule $P_{k_0,n_0-k_0}$ with the character $\chi_{\lambda,\mu}$ and $f \not\in \text{Id}^p(L)$. In particular, there exist $a_1, \ldots, a_{k_0} \in L_0, b_1, \ldots, b_{n_0-k_0} \in L_1$ such that

$$A = f(a_1, \ldots, a_{k_0}, b_1, \ldots, b_{n_0-k_0}) \neq 0$$

in $L$. First we will show how to find $n_1, k_1$ which are approximately equal to $2n_0, 2k_0$, respectively, satisfying the same inequality as (8).

Since $L$ is simple and $A \neq 0$ the ideal generated by $A$ coincides with $L$. Clearly, every simple Lie superalgebra is centerless. Hence one can find $c_1, \ldots, c_{d_1} \in L_0 \cup L_1$ such that

$$[A, c_1, \ldots, c_{d_1}, A] \neq 0$$

and $d_1 \leq d-1$. Here we use the left-normed notation $[[a, b], c] = [a, b, c]$ for nonassociative products. It follows that a polynomial

$$[f_1, z_1, \ldots, z_{d_1}, f_2] = g_2 \in P_{2k_0+p,2n_0-2k_0+r}, \quad p+r = d_1,$$

is also a non-identity of $L$ where $z_1, \ldots, z_{d_1} \in X \cup Y$ are even or odd variables, whereas $f_1$ and $f_2$ are copies of $f$ written on disjoint sets of indeterminates,

$$f_1 = f(x_1^1, \ldots, x_{k_0}^1, y_1^1, \ldots, y_{n_0-k_0}^1),$$

$$f_2 = f(x_1^2, \ldots, x_{k_0}^2, y_1^2, \ldots, y_{n_0-k_0}^2).$$

Consider the $S_{2k_0} \times S_{2n_0-2k_0}$-action on $P_{2k_0+p,2n_0-2k_0+r}$ where $S_{2k_0}$ acts on $x_1^1, \ldots, x_{k_0}^1, x_1^2, \ldots, x_{k_0}^2$ and $S_{2n_0-2k_0}$ acts on $y_1^1, \ldots, y_{n_0-k_0}^1, y_1^2, \ldots, y_{n_0-k_0}^2$. Denote by $M$ the $F[S_{2k_0} \times S_{2n_0-2k_0}]$-submodule generated by $g_2$ and examine its character. It follows from Richardson–Littlewood rule that

$$\chi(M) = \sum_{\nu = 2\lambda \vdash 2k_0 \atop \rho = 2n_0-2k_0} t_{\nu,\rho} \chi_{\nu,\rho}$$

where either $\nu = 2\lambda = (2\lambda_1, \ldots, 2\lambda_d)$ or $\nu$ is obtained from $2\lambda$ by pushing down one or more boxes of $D_{2\lambda}$. Similarly, $\rho$ is either equal to $2\mu$ or $\rho$ is obtained from $2\mu$ by pushing down one or more boxes of $D_{2\mu}$. Then by Lemma 2.2 we have

$$\Phi(\nu) \geq \Phi(2\lambda) = \Phi(\lambda) \quad \text{and} \quad \Phi(\rho) \geq \Phi(2\mu) = \Phi(\mu).$$

By Lemma 2.1 and (9) we have

$$\left( \binom{n_0}{k_0} (\Phi(\lambda)\Phi(\mu)) \right)^{n_0} > \frac{1}{2n_0^3 \varphi(n_0)} (a-\varepsilon)^{n_0}.$$
Now we present the lower bound for binomial coefficients in terms of function \( \Phi \). Clearly, the pair \((k, n-k)\) is a two-component partition of \( n \) if \( k \geq n-k \). Otherwise \((n-k, k)\) is a partition of \( n \). Since \( x^{-x}y^{-y} = y^{-y}x^{-x} \) for all \( x, y \geq 0, x+y = 1 \), we will use the notation \( \Phi(\frac{k}{n}, \frac{n-k}{n}) \) in both cases \( k \geq n-k \) or \( n-k \geq k \). Then it easily follows from the Stirling formula that

\[
\frac{1}{n} \Phi\left(\frac{k}{n}, \frac{n-k}{n}\right)^n \leq \left(\frac{n}{k}\right)^n \leq n \Phi\left(\frac{k}{n}, \frac{n-k}{n}\right)^n,
\]

hence

\[
\left(\frac{qk_0}{qn_0}\right) > \frac{1}{qn_0} \Phi\left(\frac{qk_0}{qn_0}, \frac{qn_0-qk_0}{qn_0}\right)^{qn_0} = \frac{1}{qn_0} \Phi\left(\frac{k_0}{n_0}, \frac{n_0-k_0}{n_0}\right)^{qn_0}
\]

for all integers \( q \geq 2 \) and also

\[
\left(\Phi\left(\frac{k_0}{n_0}, \frac{n_0-k_0}{n_0}\right) \Phi(\lambda) \Phi(\mu)\right)^{n_0} > \frac{1}{2n_0^4\varphi(n_0)(a-\varepsilon)^{n_0}},
\]

by virtue of (10).

Recall that we have constructed earlier a multilinear polynomial \( g_2 = [f_1, z_1, ..., z_{d_1}, f_2] \) which is not a graded identity of \( L \) and \( f_1, f_2 \) are copies of \( f \). Applying the same procedure we can construct a non-identity of the type

\[
g_q = [g_{q-1}, w_1, ..., w_{d_{q-1}}, f_q]
\]

of total degree \( n_{q-1} = n_{q-2} + n_0 + w_1 + ... + w_{d_{q-1}} \) where \( d_{q-1} \leq d \) and \( f_q \) is again a copy of \( f \) for all \( q \geq 2 \).

As in the case \( q = 2 \) the \( F[S_{qk_0} \times S_{qn_0-qk_0}] \)-submodule of \( P_{k_0, n-k}(L) \) (where \( n = n_{q-1} = qn_0 + p', k = k_{q-1} = qk_0 + p'' \)) contains an irreducible summand with the character \( \chi_{\nu, \rho} \) where \( \nu | qk_0, \rho | qn_0 - qk_0, \Phi(\nu) \geq \Phi(\lambda), \Phi(\rho) \geq \Phi(\mu) \). Moreover, for \( n = n_{q-1} \) we have

\[
c^{gr}_n(L) \geq \frac{(qn_0)}{qk_0} d_v d_d > \frac{1}{n^{2d^2+2d}} \left( \frac{qn_0}{qk_0} \right) (\Phi(\lambda) \Phi(\mu))^q \\
> \frac{1}{n^{2d^2+2d+1}} \left( \Phi\left(\frac{k_0}{n_0}, \frac{n_0-k_0}{n_0}\right) \Phi(\lambda) \Phi(\mu)\right)^{q_{n_0}}
\]

by Lemma 2.1 and the inequality (11). Now it follows from (12) that

\[
c^{gr}_n(L) > \frac{1}{n^{2d^2+2d+1}} \left( \frac{1}{2n_0^4\varphi(n_0)} \right)^{q} (a-\varepsilon)^{n_0}.
\]
Note that $qn_0 \leq n \leq qn_0 + qd$. Hence $q/n \leq 1/n_0$ and

$$(a - \varepsilon)^{qn_0} \geq \frac{(a - \varepsilon)^n}{a^{qd}}$$

since $a \geq 2$ (see (7)). Therefore

$$\sqrt[n]{c_n^{gr}(L)} > \frac{(a - \varepsilon)^n}{n^{2d^2 + 2d + 1} (2a^d n_0^4 \varphi(n_0))^{\frac{1}{n_0}}}$$

for all $n = n_q - 1, q = 1, 2, \ldots$ Finally note that the initial $n_0$ can be taken to be arbitrarily large. Hence we can suppose that

$$n^{-\frac{2d^2 + 2d + 1}{n}} (2a^d n_0^4 \varphi(n_0))^{\frac{1}{n_0}} > 1 - \delta$$

for all $n \geq n_0$. Hence the inequality

$$\sqrt[n]{c_n^{gr}(L)} > (1 - \delta)(a - \varepsilon)^n$$

holds for all $n = n_q, q = 0, 1, \ldots$. The second inequality $n_{q+1} - n_q \leq n_0 + d$ follows from the construction of the sequence $n_0, n_1, \ldots$, and we have thus completed the proof. □

Now we are ready to prove the main result of the paper.

**Theorem 3.3.** Let $L$ be a finite dimensional simple Lie superalgebra over a field of characteristic zero. Then its graded PI-exponent

$$\exp^{gr}(L) = \lim_{n \to \infty} \sqrt[n]{c_n^{gr}(L)}$$

exists an is less than or equal to $d = \dim L$.

**Proof.** First note that, given a multilinear polynomial $h = h(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in P_{k,n-k}$, the linear span $M$ of all its values in $L$ is a $L_0$-module since

$$[h, z] = \sum_i h(x_1, \ldots, [x_i, z], \ldots, x_k, y_1, \ldots, y_{n-k})$$

$$+ \sum_j h(x_1, \ldots, x_k, y_1, \ldots, [y_j, z], \ldots, y_{n-k})$$

for any $z \in \mathcal{L}(X, Y)_0$. Hence $ML_1 \neq 0$ in $L$ and $0 \equiv [h, w]$ is not an identity of $L$ for odd variable $w$ as soon as $h \notin \text{Id}^{gr}(L)$. It follows that

$$c_{k,n-k+1}(L) \geq c_{k,n-k}(L)$$
and then
\begin{equation}
(13) \quad c_{n}^{gr}(L) \geq c_{m}^{gr}(L)
\end{equation}
for \( n \geq m \).

Fix arbitrary small \( \varepsilon, \delta > 0 \). By Lemma 3.2 there exists an increasing sequence \( n_{q}, q = 1, 2, \ldots \), such that \( c_{n}^{gr}(L) > ((1-\delta)(a-\varepsilon))^{n} \) for all \( n = n_{q}, q = 0, 1, \ldots \), and \( n_{q+1} - n_{q} \leq n_{0} + d \). Denote \( b = (1-\delta)(a-\varepsilon) \) and take an arbitrary \( n_{q} < n < n_{q+1} \). Then \( c_{n}^{gr}(L) \geq c_{n_{q}}^{gr}(L) \) and \( n - n_{q} \leq n_{0} + d \). Referring to (7) we may assume that \( b > 1 \). Then
\begin{align*}
    b^{n_{q}} &\geq b^{n} b^{-(n_{0} + d)} \\
    c_{n}^{gr}(L) &\geq (b^{1 - \frac{n_{0} + d}{n}})^{n} 
\end{align*}
for all \( n_{q} \leq n \leq n_{q+1} \) and all \( q = 0, 1, \ldots \), that is for all sufficiently large \( n \). The latter inequality means that
\[ \liminf_{n \to \infty} \sqrt[n]{c_{n}^{gr}(L)} \geq (1-\delta)b = (1-\delta)^{2}(a-\varepsilon). \]
Since \( \varepsilon, \delta \) were chosen to be arbitrary, we have thus completed the proof of the theorem. \( \square \)

References

1. Bahturin, Yu. A., *Identical Relations in Lie Algebras*, VNU Science Press, Utrecht, 1987.
2. Bahturin, Yu. and Drensky, V., Graded polynomial identities of matrices, *Linear Algebra Appl.* 357 (2002), 15–34.
3. Drensky, V., *Free Algebras and PI-algebras. Graduate Course in Algebra*, Springer, Singapore, 2000.
4. Giambruno, A. and Regev, A., Wreath products and P.I. algebras, *J. Pure Appl. Algebra* 35 (1985), 133–149.
5. Giambruno, A., Shestakov, I. and Zaicev, M., Finite-dimensional non-associative algebras and codimension growth, *Adv. in Appl. Math.* 47 (2011), 125–139.
6. Giambruno, A. and Zaicev, M., On codimension growth of finitely generated associative algebras, *Adv. Math.* 140 (1998), 145–155.
7. Giambruno, A. and Zaicev, M., *Polynomial Identities and Asymptotic Methods*, Mathematical Surveys and Monographs 122, Am. Math. Soc., Providence, 2005.
8. Giambruno, A. and Zaicev, M., On codimension growth of finite-dimensional Lie superalgebras, *J. Lond. Math. Soc.* (2) 85 (2012), 534–548.
9. James, J. and Kerber, A., *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications 16, Addison-Wesley, London, 1981.
10. Mishchenko, S. P., Growth of varieties of Lie algebras, *Uspekhi Mat. Nauk* 45 (1990), 25–45 (Russian). English transl.: *Russian Math. Surveys* 45 (1990), 27–52.
11. Repovš, D. and Zaicev, M., Graded identities of some simple Lie superalgebras, *Algebr. Represent. Theory* **17** (2014), 1401–1412.
12. Repovš, D. and Zaicev, M., Graded codimensions of Lie superalgebra $b(2)$, *J. Algebra* **422** (2015), 1–10.
13. Scheunert, M., *The Theory of Lie Superalgebras; An Introduction*, Lecture Notes in Math. **716**, Springer, Berlin, 1979.
14. Zaitsev, M. V., Integrality of exponents of growth of identities of finite-dimensional Lie algebras, *Izv. Ross. Akad. Nauk Ser. Mat.* **66** (2002), 23–48 (Russian). English transl.: *Izv. Math.* **66** (2002), 463–487.
15. Zaicev, M., On existence of PI-exponents of codimension growth, *Electron. Res. Announcement Math. Sci.* **21** (2014), 113–119.
16. Zaitsev, M. and Repovš, D., A four-dimensional simple algebra with fractional PI-exponent, *Mat. Zametki* **95** (2014), 538–553 (Russian). English transl.: *Math. Notes* **95** (2014), 487–499.

Dušan Repovš
Faculty of Education, and
Faculty of Mathematics and Physics
University of Ljubljana
SI-1000 Ljubljana
Slovenia
dusan.repovs@guest.arnes.si

Mikhail Zaicev
Department of Algebra,
Faculty of Mathematics and Mechanics
Moscow State University
RU-119992 Moscow
Russia
zaicevmv@mail.ru

Received September 2, 2014
published online July 24, 2015