Transplanckian Dispersion Relation
and
Entanglement Entropy of Blackhole

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Abstract: The quantum correction to the entanglement entropy of the event horizon is plagued by the UV divergence due to the infinitely blue-shifted near horizon modes. The resolution of this UV divergence provides an excellent window to a better understanding and control of the quantum gravity effects. We claim that the key to resolve this UV puzzle is the transplanckian dispersion relation. We calculate the entanglement entropy using a very general type of transplanckian dispersion relation such that high energy modes above a certain scale are cutoff, and show that the entropy is rendered UV finite. We argue that modified dispersion relation is a generic feature of string theory, and this boundedness nature of the dispersion relation is a general consequence of the existence of a minimal distance in string theory.

1 Introduction and Conclusion

The biggest puzzle in the black hole physics is the origin of the Bekenstein-Hawking entropy [1] and the related information loss problem. There have been various proposals to try to count the microscopic degrees of freedom associated with the Bekenstein-Hawking entropy, in all cases peoples believe that the near horizon fluctuations should play an important role. The most radical one is to single out only the horizon itself by neglecting the other region of the spacetime and then attribute the Bekenstein-Hawking entropy to some kind of dynamics associated with the horizon hypersurface. For example, the stretched horizon in the membrane paradigm [2], the isolated horizon in the context of loop quantum gravity [3] or the conformal field theory associated with the near horizon symmetry algebra [4].

The other common proposal once adopted is that the Bekenstein-Hawking entropy arises from the entanglement of the quantum fluctuations inside and outside the horizon. The reason for this proposal is that the inside region of the horizon is causally disconnected from the outside one, then for the outside observer it is natural to trace over the Hilbert space of the inside region and obtain an entangled state. The entropy associated with these entangled states is called the entanglement entropy of the black hole.
There are various techniques developed to calculate the entanglement entropy of the black hole. One is the brick wall method introduced by ’t Hooft [5]. This method treats the horizon as a brick wall such that the wavefunction strictly vanishes inside the horizon, this is equivalent to the fact that no information associated with the quantum fluctuation inside the horizon can be carried out to the outside observer, and therefore, all the quantum states obeying this boundary condition are entangled. We should emphasize that the above equivalence is only in a heuristic way and a rigorous proof is absent. However, we will take the brick wall method as the starting point in calculating the entanglement entropy without questioning the above underlying assumption of the equivalence.

If one thinks that the fluctuations of the entangled states are the quantized gravitational fluctuations on the background spacetime, then the entanglement entropy could be the origin of the Bekenstein-Hawking entropy. However, this expectation turns out to be too naive. In quantum field theory in flat or curved space, the quantum effect is usually plagued by the UV divergence which is taken care by proper renormalization scheme, however, once gravity is included, the UV divergence will cause enormous back reaction of the background geometry and cannot be justified by the renormalization scheme. The famous example is the well-known cosmological constant problem. Similar problem happens in calculating the entanglement entropy which turns out to be UV divergent so that an infinitesimal brick wall thickness is introduced as an UV cutoff. The brick wall cutoff indicates that the near horizon fluctuations usually get infinitely blue-shifted, which will then cause a large back reaction to themselves and to the background. Once the effect of the back reaction is taken into account, the area law may or may not hold true. If the area law is violated we hope that the entanglement entropy contributes only a small correction to the Bekenstein-Hawking entropy whose area law nature is suggested by the holographic principle, and may be related to some unknown nature of quantum gravity.

This puzzle of UV divergence should be taken seriously because it is a reflection of our lack of understanding and control of quantum gravity effects. From a bottom-up approach, one may be able to get better understanding into the nature of quantum gravity from the possible low energy effective features that can resolve the UV divergence problem of the event horizon. Such feature may point its finger toward the essential aspects of quantum gravity. In this paper, we propose that transplanckian dispersion relations (TDRs) is one such feature.

In our paper [6], we modeled the effect of the back reaction mentioned earlier by modifying the propagator of the fluctuations by the so called transplanckian dispersion relations (TDRs), and then calculate the transplanckian entanglement entropy based on the modified dispersion relations. In principle, one should be able to determine how quantum gravity effects would modify the dispersion relation from string theory. Unfortunately, our present technology in string theory is not powerful enough to allow one to quantize string theory in a blackhole background. Nevertheless, it is extremely natural to expect for TDRs in string theory. For example, in the simpler case of open string in a constant $B$-field background, a modified dispersion relation is resulted due to the IR/UV mixing effect [7]. The resulting dispersion relation respects the symmetry of the background, which is smaller than the full Lorentz symmetry of the flat spacetime. It is clear that in the case of closed string, the dispersion relation will also generally get modified due to string loop effects. See, for example, [8] the string theory calculation in the case of open string in $B$-field.

Although it is still technically impossible at present to derive the TDRs from first principle string theory, it can be argued that [9, 10] due to the existence of a minimal
length scale $l_s$ in string theory, the TDRs is expected to cutoff the high energy mode with energy above $k_0 \sim 1/l_s$. We will thus consider a generic class of the TDRs of such kind instead of some specific one. Our conclusion is that for such a class of TDRs which are bounded by some reasonable energy scale, the entanglement entropy is UV finite. Thus in particular the entanglement entropy is always finite in string theory. We also found that the area law nature is lost with its value negligible in comparison with the Bekenstein-Hawking entropy in the semi-classical limit.

2 Transplanckian Dispersion Relation

The dispersion relation of a particle is a relation between its energy and momentum. At the tree level, it can be obtained from the invariant of the Lorentz group, and for the following background metric considered in this paper

$$ds^2 = g_{00}dt^2 + g_{rr}dr^2 + fd\Omega^2_2,$$  \hspace{1cm} (1)

it takes the following form

$$g^{00}\omega^2 + g^{rr}p_r^2 + p_{\perp}^2 = 0 ,$$  \hspace{1cm} (2)

where $g^{00} = 1/g_{00}, g^{rr} = 1/g_{rr}$ and $p_{\perp}^2 = g^{mn}p_mp_n = \ell(\ell + 1)/f$ is the transverse momentum squared and $\ell$ is the angular momentum quantum number. Note that $g_{00}, g_{rr}$ and $f$ are functions of $r$.

It is convenient to introduce the following definitions

$$\xi^2 := p_{\perp}^2 g_{rr} = \frac{\ell(\ell + 1)g_{rr}}{f}, \quad \rho^2 := -\frac{g_{00}}{g_{rr}},$$  \hspace{1cm} (3)

the dispersion relation (2) takes the form

$$p_r = \sqrt{\frac{\omega^2}{\rho^2} - \xi^2}.$$  \hspace{1cm} (4)

As mentioned in the Introduction, the large back reactions due to the infinitely blue-shifted near horizon modes are supposed to be important in evaluating the entanglement entropy. Moreover, it is quite generally believed that nonlocal effects due to quantum gravity will provide a natural regulator to the UV divergence in quantum field theory by suppressing the contributions of the high energy modes. The simplest proposal to encode the effects of this suppression is to replace the linear dispersion relation by the so-called transplanckian dispersion relation (TDR) \cite{11,12}.

Consider a spherically symmetric background given by the general metric (1), it is reasonable to impose the TDR according to the residual spacetime symmetry preserved by the metric. Since the transplanckian effect will be mostly due to the blue-shift in the near horizon regime in the radial direction, so it is natural to impose TDR along that direction to suppress the blue-shift effect. Therefore we consider TDR of the form

$$g^{00}\omega^2 + g^{rr}H^2(p_r) + p_{\perp}^2 = 0.$$  \hspace{1cm} (5)

The function $H$ describes the transplanckian effects. In terms of the $\rho$ and $\xi$ variables, the TDR (4) then takes the form

$$p_r = H^{-1}(\sqrt{\frac{\omega^2}{\rho^2} - \xi^2}).$$  \hspace{1cm} (6)
For example, Unruh [11] and respectively, Corley and Jacobson (C-J) [12] proposed respectively:

\[ H_n(k) = k_0[\tanh(k/k_0)]^{\frac{1}{n}}; \quad H(k) = \sqrt{k^2 - k^4/(4k_0^2)}, \quad k \leq 2k_0. \]

(7)
The corresponding radial momentum is

\[ p_r = k_0\{\tanh^{-1}[k_0^{-2}(\omega^2/\rho^2 - \xi^2)]^{\frac{1}{2}}\}; \quad p_r^\pm = \sqrt{2k_0}\sqrt{1 \pm \sqrt{1 - k_0^{-2}(\omega^2/\rho^2 - \xi^2)}}. \]

(8)

Note that the Unruh’s proposal does not cutoff the high momentum modes in contrast to the C-J’s case, but just suppresses the high energy modes.

Despite that both Unruh’s and C-J’s proposals have been used generally in the different contexts, it is not quite possible that the TDRs from the higher theory will take those specific forms. As discussed above, string theory suggests generally that the TDRs satisfy the following boundedness condition:

\[ 0 \leq H^2(k) \leq k_0^2, \]

(9)

for some \( k_0 \), and it is natural from string theory that \( k_0 \sim 1/l_s \). We will call these bounded-TDRs. It follows that for this bounded class of TDR

\[ \max(\omega^2/\rho^2 - k_0^2, 0) \leq \xi^2 \leq \omega^2/\rho^2. \]

(10)

The suppression of the high energy modes is effected by the bounded function \( H \), whose explicit form will depend on the details of the quantum gravity effects. In [3] we showed that without knowing explicitly the form of \( H \), the transplanckian entanglement entropy is always rendered UV finite. Thus we conclude that the entanglement entropy is UV finite for the bounded TDRS, in particular in string theory.

### 3 Entanglement Entropy of Black Hole

Using the dispersion relation [2], the entanglement entropy of the black hole was found to be divergent due to the infinitely blue-shifted near horizon region [5]. In the brick wall model, a UV cutoff is introduced to regularize the divergence by imposing the condition on the wavefunction \( \Phi \) so that \( \Phi(r) = 0 \), for \( r \leq r_h + \epsilon \), where \( \epsilon \) is the infinitesimal brick wall thickness [6]. This has the effect of cutting out the near horizon modes.

The total free energy is obtained by summing over the contributions from all the physical modes satisfying the brick wall condition,

\[ \beta F = \int d\omega \ z(\beta \omega) \frac{dg(\omega)}{d\omega} \]

(11)

where the Boltzmann weight \( z(x) := \ln(1 - e^{-x}) \), \( \beta \) is the inverse temperature, and \( g(\omega) \) is the density of states. In the WKB approximation [3], it is

\[ \pi g(\omega) = \int d\rho \mu(\rho)d\xi \ \xi \ p_r, \]

(12)

where \( \mu(\rho) := \frac{2F}{g_{rr} \ d\rho} \) is a “measure factor”. For fixed \( \omega \) and \( \rho \), we require \( \xi \leq \omega/\rho \) to have \( p_r \) real. This condition should be imposed in the \( \xi \)-integration.

Consider the black hole background

\[ ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2 d\Omega_2^2, \]

(13)
where \(0 < h(r) = 1 - 2M/r < 1\). In this case \(\rho = h \) and \(\mu(\rho) = 2(2M)^3\rho/(1 - \rho)^4\). The density of state is

\[
g(\omega) = \frac{2(2M)^3\omega^3}{3\pi} \left( \frac{1}{\epsilon} + \frac{1}{3\delta^3} + \cdots \right),
\]

where we have introduced an UV (resp. IR) cutoff \(\epsilon\) (resp. \(\delta\)) in the \(\rho\)-coordinate, which corresponds to an UV (resp. IR) cutoff \(r = 2M/(1 - \epsilon)\) (resp. \(r = 2M/\delta\)) in the \(r\)-coordinate; and \(\cdots\) are the sub-leading terms in the limit of \(\delta, \epsilon \to 0\). Using (14) and \(\beta = 8\pi M\), we get

\[
F = -\frac{2\pi^3(2M)^3}{45\beta^4} \left( \frac{1}{\epsilon} + \frac{1}{3\delta^3} \right), \quad S = \frac{1}{360} \left( \frac{1}{\epsilon} + \frac{1}{3\delta^3} \right)
\]

which is plagued by both UV and IR divergences. This agrees with the eqn. (3.12) of [5] except that a IR divergent boundary contribution had been dropped there. The IR-divergent piece is independent of \(M\) and represents the contribution from the vacuum. In [5], the UV-divergent piece was shown to give an entropy which obeys the area law

\[
S = \frac{A}{360\pi \epsilon_p^2}
\]

if the UV-cutoff is given in terms of the proper distance, i.e. \(\epsilon_p \approx 4M\sqrt{\epsilon}\). In modern language, the relation between \(\epsilon\) and \(M\) for fixed \(\epsilon_p\) required by the area law indicates a holographic nature of UV/IR connection [13].

We should mention that the UV divergent area law formula (16) is universal, namely, the form is the same for the other event horizons such as the ones in the Rindler space [14] and in the de Sitter space [6]. This indicates the near horizon dynamics dictating the entanglement entropy is universal.

### 4 Transplanckian Black Hole Entropy

Our goal is to determine the UV behavior of \(F\) and \(S\) of the Schwarzschild black hole for a generic bounded TDR with a \(H\) satisfying (9). Combining the condition (10) with the brick wall condition \(\rho \geq \epsilon\), we have the following integration branches contributing to the free energy (up to a factor of \(1/(\beta\pi)\))

\[
\begin{align*}
(1) & \quad \int_{e_{k_0}}^{(1-\delta)k_0} d\omega \int_{\epsilon}^{\omega/k_0} d\rho \mu(\rho) \int_{\sqrt{\omega^2/\rho^2-k_0^2}}^{\omega/\rho} d\xi \xi p_r, \\
(1*) & \quad \int_{(1-\delta)k_0}^{\infty} d\omega \int_{\epsilon}^{1-\delta} d\rho \mu(\rho) \int_{\sqrt{\omega^2/\rho^2-k_0^2}}^{\omega/\rho} d\xi \xi p_r, \\
(2) & \quad \int_{e_{k_0}}^{(1-\delta)k_0} d\omega \int_{\omega/k_0}^{1-\delta} d\rho \mu(\rho) \int_{0}^{\omega/\rho} d\xi \xi p_r, \\
(3) & \quad \int_{0}^{e_{k_0}} d\omega \int_{\epsilon}^{1-\delta} d\rho \mu(\rho) \int_{0}^{\omega/\rho} d\xi \xi p_r.
\end{align*}
\]

Note that by comparing the ranges of the above \(\xi\)-integrations with the ones without transplanckian suppression, we find that the use of TDR leads to a reduction of the allowed angular momentum modes. This leads to a suppression on the density of states and hence eventually to the UV finiteness of the entropy as shown below.
It is convenient to change variable \( x = \sqrt{\omega^2/\rho^2 - \xi^2} \) in order to isolate the \( \omega \)-dependence in \( g(\omega) \). We have

\[
\pi g(\omega) = \int d\rho \, \mu(\rho) \int dx \, x H^{-1}(x). \tag{21}
\]

Note that the integrand does not depend on \( \omega \) and \( \epsilon \), but the integration limits may. Note also that the (1*)-branch only exists for BH and de Sitter due to the additional constraint \( \hbar \leq 1 \). One may try to estimate the UV behavior for the integrals for each branches. However this is rather complicated technically. Surprisingly, without knowing explicitly these integrals, one can show that the sum of the branches (1), (1*), (2) and (3), i.e. the free energy \( F \), is completely UV finite!

To demonstrate this, let us first consider (1*) branch. The integration limits are independent of \( \omega \), i.e. \( \int_{\rho_0}^{1-\delta} d\rho \int_{k_0}^{k} dx \), so is \( g(\omega) \). Therefore (1*) does not contribute. For (1) branch the \( dx \)-integral is independent of \( \omega \), however, there is \( \omega \) dependence for \( g(\omega) \) coming from the integration limits for \( d\rho \)-integral. Using the fundamental theorem of calculus, we can carry out the derivative w.r.t \( \omega \) in (17) and obtain

\[
F_{(1)} = \frac{1}{\pi \beta k_0} \int_{0}^{k_0} dx x H^{-1}(x) \cdot \int_{k_0}^{(1-\delta)k_0} d\omega \, z(\beta \omega) \mu(\frac{\omega}{k_0}). \tag{22}
\]

Similarly, we can carry out the derivative w.r.t. \( \omega \) for branch (2) and (3), and obtain

\[
F_{(2)} = -F_{(1)} + \frac{1}{\pi \beta} \int_{k_0}^{(1-\delta)k_0} d\omega \int_{0}^{k_0} dx \, \mu(\frac{\omega}{x})H^{-1}(x), \tag{23}
\]

\[
F_{(3)} = \frac{1}{\pi \beta} \int_{0}^{k_0} d\omega \int_{k_0}^{\infty} dx \, \mu(\frac{\omega}{x})H^{-1}(x) + \frac{1}{\pi \beta} \int_{0}^{k_0} d\omega \int_{0}^{k_0} dx \, \mu(\frac{\omega}{x})H^{-1}(x). \tag{24}
\]

Physically it is required that the integral

\[
I := \int_{0}^{k_0} dx \, x H^{-1}(x) < \infty \tag{25}
\]

to be finite. The reason is because the density of states for branch (1), in (17), is given by \( \pi g(\omega) = I \cdot \int_{\rho_0}^{\omega/k_0} d\rho \mu(\rho) \), and it should be finite for any sensible physical system. This condition also guarantees our manipulation above for (22), (23) to make sense. Of course (25) is true for both Unruh’s and C-J’s TDRs.

Next we examine the first term of (24), we perform a change of variables by \( \omega = \epsilon k_0 y \) and obtain

\[
\frac{k_0 \epsilon}{\pi \beta} \int_{0}^{1} dy \, z(\beta \epsilon k_0 y) \int_{k_0}^{k_0 y} dx \, \mu(\frac{\epsilon k_0 y}{x})H^{-1}(x). \tag{26}
\]

Using the fact \( z(x) \approx \ln(x) - x/2 + \cdots \) around \( x = 0 \), and assuming the following condition on the near horizon geometry

\[
\epsilon \mu(\epsilon) \approx \epsilon^\alpha \quad \text{for} \quad \alpha > 0, \quad \text{as} \quad \epsilon \to 0, \tag{27}
\]

which is true for Schwarzschild, Rindler and de Sitter metrics, we can manipulate (26) by interchanging the order of integrations over \( dx \) and \( dy \) so that the \( y \)-integration can be carried out, (26) then becomes \( (\pi \alpha k_0)^{-1} \epsilon^{\alpha} \ln \epsilon \int_{0}^{k_0} dx \, x H^{-1}(x) \). This vanishes as \( \epsilon \to 0 \).

Finally, the second term of (24) can be combined with \( F_{(2)} \) of (23), and we obtain for the total free energy

\[
F = \frac{1}{\pi \beta} \int_{0}^{(1-\delta)k_0} d\omega \, z(\beta \omega) \int_{\frac{\omega}{1-\delta}}^{k_0} dx \, \mu(\frac{\omega}{x})H^{-1}(x). \tag{28}
\]
Note that $F$, and hence $S$, is independent of $\epsilon$, i.e. UV finite. This result is quite general and is true as long (i) the transplanckian modification of the radial modes takes the form $[\ref{6}]$ with the suppression condition $[\ref{9}]$, (ii) the metric satisfies a near horizon condition, which expressed in terms of the measure factor as $[\ref{27}]$. We thus see that TDR generally yields a UV finite free energy and entanglement entropy. This is the main result of $[\ref{6}]$.

The explicit form of $[\ref{28}]$ for the Schwarzschild metric gives

$$F \approx -\frac{2k_0^3}{3\pi\beta}(\frac{2M}{\delta})^3 \int_0^1 dy \ y \ln(1 - e^{-\beta k_0 y}) \tanh^{-1} y$$

(29)

where we assume a large IR cutoff $2M/\delta(\gg \beta)$ to extract the leading IR term from the $\rho$-integration. The IR divergence is independent of the background $M$ as in the non-transplanckian case. Moreover, the area law no longer holds for the entropy derived from $[\ref{29}]$; instead, for fixed $k_0$, both the free energy and the entropy are monotonically decreasing functions of $\beta$ so that it can be neglected in the semi-classical limit in comparison with the Bekenstein-Hawking entropy.

References

[1] J. D. Bekenstein, Phys. Rev. D 7 (1973) 2333; S. W. Hawking, Commun. Math. Phys. 43 (1975) 199.

[2] “Black Holes: The Membrane Paradigm,” edited by K. S. Thorne, R. H. Price and D. A. Macdonald, Yale University Press, New Heaven and London, 1986; L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D 48, 3743 (1993).

[3] A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, Phys. Rev. Lett. 80, 904 (1998).

[4] S. Carlip, Phys. Rev. Lett. 82, 2828 (1999); S. N. Solodukhin, Phys. Lett. B 454, 213 (1999); F. L. Lin and Y. S. Wu, Phys. Lett. B 453, 222 (1999).

[5] G. ’t Hooft, Nucl. Phys. B 256 (1985) 727.

[6] D. Chang, C. S. Chu and F. L. Lin, arXiv:hep-th/0306055

[7] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP 0002 (2000) 020.

[8] A. Bilal, C. S. Chu and R. Russo, Nucl. Phys. B 582 (2000) 65; C. S. Chu, R. Russo and S. Sciuto, Nucl. Phys. B 585 (2000) 193.

[9] S. F. Hassan and M. S. Sloth, Nucl. Phys. B 674 (2003) 434.

[10] M. Bastero-Gil, P. H. Frampton and L. Mersini, Phys. Rev. D 65 (2002) 106002.

[11] W. G. Unruh, Phys. Rev. D 51 (1995) 2827.

[12] S. Corley, T. Jacobson, Phys. Rev. D 54 (1996) 1568.

[13] L. Susskind, E. Witten, hep-th/9805114

[14] L. Susskind, J. Uglum, Phys. Rev. D 50 (1994) 2700.